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Stabilization methods of nonlinear systems with partial measurements and constrained inputs

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UNIVERSITÉ GRENOBLE ALPES

THÈSE

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Présentée par

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Thèse dirigée par **Christophe PRIEUR** et codirigée par **Vincent ANDRIEU**

préparée au sein du
laboratoire GIPSA-LAB

dans l'école doctorale **Électronique, Électrotechnique,
Automatique et Traitement du Signal**

Méthodes de stabilisation de systèmes non-linéaires avec des mesures partielles et des entrées contraintes

Thèse soutenue publiquement le **20 Septembre 2017**,
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"L'excentricité est le grand remède aux grands désespoirs." Vladimir Nabokov

"Muchos años después, frente al pelotón de fusilamiento, el coronel Aureliano Buendía había de recordar aquella tarde remota en que su padre lo llevó a conocer el hielo." *Cien años de soledad*, Gabriel García Marquez.

"Moi j'y connais rien à ces choses-là, mais ce que je sais c'est que ce n'est pas un espace-temps à venir au monde, avec tout le solide en train de valdinguer à blinde sur de l'instable." *La Ballade de Gin & Bobi*, Léo Henry et Stéphane Perger

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Résumé de la thèse

Nombre de phénomènes physiques sont modélisés par des équations différentielles. Dans le cadre de la théorie des systèmes, on parle de systèmes dynamiques. Parmi tous ces systèmes dynamiques, il y en a quelques uns qui comportent des variables sur lesquels on peut agir (une tension, une force, une pression, etc...). Ces variables sont désignées sous le nom de **contrôle**. Le reste des variables, sur lesquelles on ne peut agir, sont désignées sous le nom d'**état**.

L'objet de la théorie du contrôle est d'étudier de construire ce contrôle de manière à donner au système dynamique un comportement désiré établi par un cahier des charges. L'une des propriétés recherchées pour le système dynamique est la stabilité globale asymptotique d'un de ses points d'équilibre. Une telle propriété est en fait la somme des deux propriétés suivantes :

- (*Attractivité*) Un point équilibre est dit globalement attractif si, pour toutes conditions initiales, toutes les trajectoires du système dynamique convergent vers cet équilibre;
- (*Stabilité de Lyapunov*) Un point équilibre est dit stable selon Lyapunov si, plus la condition initiale est proche du point d'équilibre, plus les trajectoires qui en résultent s'approchent du point d'équilibre.

Ce problème s'appelle le **problème de stabilisation**. Cette thèse s'intéresse particulièrement à ce problème.

L'un des moyens d'atteindre un tel objectif est de construire une loi de rétroaction, aussi connue sous le nom de la loi de **feedback**. Une loi de feedback est un contrôle qui dépend de l'état entier ou d'une partie de ce dernier. L'un des avantages de cette méthode est qu'elle permet au contrôle de réagir lorsque le système est sujet à des perturbations. En effet, notons que la stabilité de Lyapunov permet d'obtenir des propriétés de robustesse par rapport à des petites perturbations. En d'autres termes, si un tel contrôle existe, alors il est capable de rejeter des perturbations de petite amplitude.

On parle de **système en boucle ouverte** lorsque le contrôle ne dépend pas de l'état et de **système en boucle fermée** lorsqu'il en dépend.

Il existe aujourd'hui beaucoup de travaux sur la stabilisation de systèmes linéaires de dimension finie, c'est-à-dire de systèmes modélisés par des équations aux dérivées ordinaires. De nombreuses méthodes efficaces et faciles à implémenter permettent aujourd'hui de stabiliser beaucoup de systèmes qui apparaissent dans de nombreux domaines industriels. Pourtant, physiquement, il n'est pas réaliste de s'intéresser à de tels systèmes. La plupart des phénomènes physiques sont non-linéaires. Un nombre non négligeable d'entre eux sont également modélisés par des équations aux dérivées partielles, qui représentent une large classe de système de dimension infinie. L'objectif de cette thèse est d'apporter des méthodes d'analyse ou de stabilisation de systèmes non-linéaires de dimension finie ou infinie.

Nous nous intéresserons dans un premier temps au problème des contrôle contraints. La plupart des contrôles sont en effet sujet à des restrictions. Notons notamment l'exemple des systèmes issus de l'électronique, qui demandent aux contrôles de ne pas dépasser un certain voltage. De tels phénomènes, qui sont purement non-linéaire, peuvent être modélisés notamment par ce que l'on appellera dans cette thèse une **non-linéarité bornée dans un cône**. Il est désormais connu depuis longtemps que négliger de tels phénomènes peut entraîner des instabilités pour le système en boucle fermée. Une bonne introduction à ce problème se trouve dans [94].

Il existe alors deux manières d'aborder ce problème. La première se fait en deux étapes : dans un premier temps, une loi de feedback est construite en ne prenant pas en compte les limitations du contrôle; puis, dans un second temps, on analyse la stabilité asymptotique du système en boucle fermée, lorsque la saturation agit. La deuxième manière consiste à prendre en compte cette contrainte lors de la construction de la loi de feedback.

Dans cette thèse, nous nous intéresserons à la première démarche. Nous l'appliquerons notamment sur des systèmes non-linéaires de dimension infinie. Nous répondrons alors à la question suivante :

Question 1 : *Étant donnée une loi de feedback stabilisante pour un système non-linéaire de dimension infinie, sous quelles conditions le système reste-t-il globalement asymptotiquement stable lorsque l'on modifie la loi de feedback via une nonlinéarité bornée dans un cône ?*

À notre connaissance, les articles pionniers sur ce sujet sont [84] et [81]. Ces articles utilisent la théorie des semigroupes non-linéaires pour prouver la stabilité asymptotique de systèmes de dimension infinie avec une loi de feedback stabilisante modifiée avec une non-linéarité bornée dans un cône. Notons que la théorie des semigroupes non-linéaires vise à généraliser les exponentielles de matrices dans des espaces de Banach ou de Hilbert.

Dans cette thèse, nous donnerons des résultats complémentaires à ces résultats pour des systèmes non-linéaires abstraits et nous étudierons une équation aux dérivées partielles non-linéaire particulière, à savoir l'équation de Korteweg-de Vries, qui modélise les mouvements de vagues dans des canaux de faible profondeur.

Cette thèse apporte également des contributions dans le domaine du problème de la stabilisation par retour de sortie. Récemment, de nombreux outils pour construire des lois de feedback ont été développés. Cependant, beaucoup d'entre eux nécessitent la connaissance de l'état entier. Dans la pratique, les capteurs ne peuvent pas mesurer tout l'état. Cela peut être imputé aussi bien à des coûts de production qu'à des impasses technologiques. On ne peut avoir accès qu'à une mesure partielle de l'état, que l'on nommera **sortie**.

Lorsqu'une loi de feedback dépend de la sortie, on parle alors de **retour de sortie**. On oppose ce terme à celui de **retour d'état**, qui correspond à une loi de feedback qui dépend de l'état tout entier. Pour construire un retour de sortie, il semble naturel de ne faire intervenir dans la loi de feedback que la sortie. Une telle loi de feedback est alors appelée **retour de sortie statique**. Cependant, la construction d'un tel contrôle exige au système de sat-

isfaire des propriétés algébriques contraignantes, et il est donc fructueux de faire appel à un **observateur**.

Dans le cadre de la théorie du contrôle, un observateur se définit comme étant un système dynamique, dépendant de la sortie du système de contrôle, et qui est construit de manière à faire converger son état vers celui du système à contrôler. L'existence de ce système dynamique dépend d'une propriété d'**observabilité** que doit satisfaire le système de contrôle.

Une fois que l'on est capable de construire un retour d'état et un observateur, on doit analyser le système en boucle fermées lorsque l'état de l'observateur est injecté dans la loi de commande par retour d'état. On parle alors d'un **retour de sortie dynamique**, puisque la loi de feedback dépend désormais d'une dynamique supplémentaire.

On peut désormais établir une seconde question :

Question 2 : *Étant donné un observateur et un retour d'état pour une équation de Korteweg-de Vries, le retour de sortie dynamique qui en résulte stabilise-t-il l'origine du système en boucle fermée?*

Cette seconde question est liée au **principe de séparation**, qui est satisfait pour les systèmes linéaires de contrôle à temps invariant et de dimension finie. En d'autres termes, si un tel système est stabilisable et observable, alors un observateur et un retour d'état peuvent être construits séparément pour construire un retour de sortie dynamique stabilisant. Notons qu'il n'existe pas de résultats généraux pour les systèmes linéaires de contrôle de dimension infinie. Un résultat est établi dans [26], où les opérateurs de contrôle et de sortie sont bornés. Cela signifie que le contrôle et la sortie ne se trouvent pas au bord de l'équation aux dérivées partielles.

Il est établi depuis longtemps déjà que le principe de séparation n'est pas satisfait pour les systèmes non-linéaires de dimension finie. L'article [3] rassemble quelques contre-exemples. Cependant, il existe d'autres résultats généraux permettant de construire des retours de sortie dynamique. L'un des plus connus, prouvé dans [97], établit le résultat suivant : si le système de contrôle satisfait une propriété d'observabilité, connue sous le nom d'**observabilité complète uniforme**, and si de plus il satisfait une propriété de stabilisabilité, alors on dit que le système de contrôle est **semi-globalement stabilisable au moyen d'un retour de sortie**. Cette propriété peut se décrire de la manière suivante : si la condition initiale appartient à un compact, un observateur dépendant de ce compact peut être construit conjointement avec un retour d'état dans le but de rendre asymptotiquement stable l'origine du système en boucle fermée avec un bassin d'attraction contenu dans le compact.

Cette thèse apporte un résultat pour le cas des systèmes de contrôle pour lesquels il existe un **retour d'état hybride**. Une telle loi de commande est une loi de commande dynamique qui a un comportement à la fois continu et discret. Ces commandes se sont avérées très utiles, car il existe des restrictions pour certains systèmes de contrôle. L'une des restrictions les plus connues est la condition de Brockett, qui établit que si un système ne satisfait pas certaines conditions, alors il ne peut pas être stabilisé au moyen d'une loi de commande **continue**. La

seule manière de stabiliser de tels systèmes consiste soit à utiliser une loi de commande à temps variant [21, Chapitre 11], soit à utiliser une loi de commande hybride. Nous ne nous intéresserons qu'à la seconde stratégie de commande.

De ces définitions, une troisième question s'impose :

Question 3 : *Si un système de contrôle est complètement uniformément observable et s'il existe un retour d'état hybride stabilisant pour ce système, sous quelles conditions le système est-il semi-globalement stabilisable?*

L'objectif de cette thèse est donc de répondre à ces trois questions. Le manuscrit de thèse est divisé en quatre chapitres. Le Chapitre 1 et 2 répondent à la Question 1. Le Chapitre 3 répond à la Question 2. Enfin, le Chapitre 4 répond à la Question 3.

Voici un résumé précis des chapitres de cette thèse:

Résumé des chapitres

- Le Chapitre 1 traite des systèmes de contrôle abstraits non-linéaires. Il débute par l'introduction de notions essentielles pour la théorie des semigroupes non-linéaires et par l'établissement de quelques résultats connus, utiles pour le développement des résultats obtenus lors de la thèse. Deux théorèmes sont présentés : le premier concerne le caractère bien posé d'un système de contrôle abstrait non-linéaire lorsqu'on le boucle avec une loi de commande stabilisante saturée. Une section est également consacrée au positionnement de nos résultats par rapport à la littérature qui traite de ce sujet. Quelques conditions sur le système en boucle fermée et sur la non-linéarité bornée dans un cône sont données pour prouver le caractère bien posé du système en boucle fermée ainsi que la stabilité asymptotique de l'origine de cette dernière. Deux exemples sont présentés. Le premier est une équation de Korteweg-de Vries linéarisée, tandis que le second est une équation de la chaleur non-linéaire;
- Le Chapitre 2 analyse l'équation de Korteweg-de Vries, dans le cas où son contrôle est saturé par deux types de saturation. Quelques résultats sur la stabilité de l'équation sont donnés. Notons que nous n'avons pas cherché à appliquer ici les résultats du Chapitre 1, car la théorie de semigroupes non-linéaires exige aux non-linéarités considérées de satisfaire des propriétés restrictives que ne satisfait pas la non-linéarité l'équation de Korteweg-de Vries. Nous établissons deux résultats: le premier sur le caractère bien posé du système en boucle fermée, le second sur la stabilité asymptotique de l'origine du système en boucle fermée. Le caractère bien posé est prouvé grâce au théorème du point fixe de Banach et quelques estimées déjà établis dans des articles, tandis que la preuve de la stabilité asymptotique est séparée en deux cas : i) quand le contrôle agit sur tout le domaine, une fonction de Lyapunov ainsi qu'une condition de secteur décrivant l'entrée saturée sont utilisées pour établir la stabilité asymptotique; ii) lorsque le contrôle est localisé, nous prouvons le résultat avec un raisonnement par l'absurde. Un schéma numérique est donné ainsi que des simulations illustrant la stabilité asymptotique du système non-linéaire en boucle fermée de dimension infinie;

- Le Chapitre 3 traite du problème de stabilisation par retour de sortie pour deux équations de Korteweg-de Vries. La première est linéaire, tandis que la seconde est non-linéaire. La construction des retours de sortie est basée sur la méthode de **backstepping**, qui est introduite dans le livre [85]. Cette méthode consiste à trouver une transformation bijective qui lie un système en boucle ouverte instable à **système cible** dont on choisit l'origine globalement asymptotiquement stable. Elle permet notamment de construire des retours d'état et des observateurs. Le caractère bien posé de l'équation non-linéaire est prouvé en appliquant le théorème de point-fixe de Banach, tandis que celui de l'équation linéaire est prouvé en faisant appel à quelques résultats basiques de la théorie des semi-groupes linéaires. La stabilité exponentielle des systèmes en boucle fermée est prouvée avec quelques fonctions de Lyapunov. Des simulations numériques ont été faites pour illustrer nos résultats théoriques;
- Le Chapitre 4 traite de la stabilisation semi-globale de systèmes non-linéaires de dimension finie pour lesquels il existe un retour d'état hybrid régularisant. Une hypothèse de stabilisabilité hybride locale ainsi qu'une hypothèse d'observabilité complète uniforme permettent de construire une loi de commande qui stabilise d'un ensemble d'équilibre avec un retour de sortie hybride. Une propriété de **timer** est exploitée pour la construction explicite de telles lois de commande. Le résultat est illustré avec un système linéaire de dimension finie sujet à des contrôles saturés à reset.

Liste des publications relatives à ce manuscrit

- Journaux

1. S. Marx, V. Andrieu, and C. Prieur. "Semi-global stabilization by an output feedback law from a hybrid state controller." In: *Automatica* **74** (2016), pp. 90-98.
2. S. Marx, E. Cerpa, C. Prieur, and V. Andrieu. "Global stabilization of a Korteweg-de Vries equation with saturating distributed control." *SIAM J. Control Optim.* **55**(3) (2017), pp. 1452-1480.
3. S. Marx, and E. Cerpa. "Output Feedback Control of the Korteweg-de Vries Equation." **To be published in Automatica (2017)**.
4. S. Marx, V. Andrieu, and C. Prieur. "Cone-bounded feedback laws for m -dissipative operators on Hilbert space. **Submitted to Mathematics of Control, Signals and Systems.**

- Conférences

1. S. Marx, V. Andrieu, and C. Prieur. "Using a high-gain observer for a hybrid output feedback: finite-time and asymptotic cases for SISO affine systems." In: *American Control Conference. Portland, OR, 2014*, pp. 4637-4642.
2. S. Marx, and E. Cerpa. "Output Feedback Control of the Linear Korteweg-de Vries equation." In: *Proceedings of the 53rd IEEE Conference on Decision and Control. Los Angeles, CA, 2014*, pp. 2083-2087.

3. S. Marx, E. Cerpa, C. Prieur, and V. Andrieu. "Stabilization of a linear Korteweg-de Vries equation with a saturated internal control." In: *Proceedings of the European Control Conference*. Linz, AU, 2015, pp. 867-872.
4. S. Marx, E. Cerpa, C. Prieur, and V. Andrieu. "Global stabilization of a Korteweg-de Vries equation with a distributed control saturated in L^2 -norm." In: *10th IFAC Symposium on Nonlinear Control Systems (NOLCOS 2016)*, Monterey, CA, 2016.

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Notation

- Let $c \in \mathbb{C}$, $\Re_e(c)$ (resp. $\Im_m(c)$) denotes the real part (resp. the imaginary part) of c ;
- A function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be a class \mathcal{K} function if α is nonnegative, increasing and vanishing at 0. It is said to be a class \mathcal{K}_∞ function if moreover it satisfies

$$\lim_{s \rightarrow +\infty} \alpha(s) = +\infty;$$

- A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be a class \mathcal{KL} function if, for each nonnegative value s , the function $r \rightarrow \beta(r, s)$ is a class \mathcal{K} function and, for every positive value r , the function $s \rightarrow \beta(r, s)$ is strictly decreasing and satisfies

$$\lim_{s \rightarrow +\infty} \beta(r, s) = 0.$$

- The identity operator associated to a Hilbert space Z is denoted by I_Z ;
- Given a Hilbert space Z equipped with the scalar product $\langle \cdot, \cdot \rangle_Z$, a sequence $(z_n)_{n \in \mathbb{N}} \in Z$ *weakly converges to* z if, for all $\tilde{z} \in Z$

$$\lim_{n \rightarrow +\infty} \langle z_n, \tilde{z} \rangle_Z = \langle z, \tilde{z} \rangle_Z. \quad (1)$$

We use the following notation

$$z_n \rightharpoonup_Z z;$$

- Given a positive value L , let $(t, x) \in \mathbb{R}_{\geq 0} \times [0, L] \rightarrow z(t, x)$ be a sufficiently smooth function. The term z_t (respectively z_x) stands for the partial derivative of z with respect to t (respectively with respect to x). Similarly, z_{xxx} stands for the third derivative of z with respect to x . When the function z depends only on x , z' (respectively z''') denotes the first (respectively the third) spatial derivative of z . When the function z depends only on t , $\frac{d}{dt}z$ denotes the time derivative of z ;
- Let Z be a Hilbert space and let $\tilde{Z} \subset Z$ be a Hilbert space. The closure of \tilde{Z} in Z is denoted by $\text{clos}_Z(\tilde{Z})$ with respect to the topology of Z ;
- Let Z and E be Hilbert spaces and $A : E \rightarrow Z$ a linear operator. The term $\text{Ran}(A)$ denotes its range;
- Given a positive value $T \in [0, \infty)$, the space of continuous functions on $[0, T]$ is denoted by $C(0, T)$. Given $k \in \mathbb{N}^*$, a function f is said to be of class $C^k(0, T)$ if $\frac{d^l}{dt^l}f(t)$, where $l \in \{1, \dots, k\}$, belongs to the space $C(0, T)$;
- Let Z be a Hilbert space. Let $T \in [0, \infty]$. The function f belongs to the set $C(0, T; Z)$ if, for all $z \in Z$, the function

$$\begin{aligned} [0, T] &\rightarrow Z \\ t &\mapsto f(t, z) \end{aligned}$$

is continuous;

- Given $n \in \mathbb{N}$, the norm of the space \mathbb{R}^n is denoted by $|\cdot|$;
- Given two Hilbert space Z and \tilde{Z} , the space of continuous linear operators from Z to \tilde{Z} is denoted by $\mathcal{L}(Z, \tilde{Z})$. When $\tilde{Z} = Z$, then the notation that is used is $\mathcal{L}(Z)$;

Introduction

In recent decades, a great deal of effort has been dedicated to the development of tools for the design of stabilizing feedback laws either for finite-dimensional or infinite-dimensional systems. The general problems under consideration in this thesis are the analysis of control systems with restrictions on the input and the design of feedback laws with partial measurement of the state.

The restrictions on the input can be modeled with a *cone-bounded nonlinearity*, which is a bounded and continuous function satisfying a sector condition. One of the most famous is the saturation function (see [94] for a good introduction to this topic). This thesis aims at giving some contributions to this topic in the context of infinite-dimensional systems. From this problem, a question arises.

Question 1: *Given a linear stabilizing feedback law for a nonlinear infinite-dimensional control system, under which conditions does the system remain globally asymptotically stable when modifying the feedback law via a cone-bounded nonlinearity?*

Even for linear finite-dimensional systems, it is already known that saturating a stabilizing feedback law can lead to catastrophic behavior for the stability of the closed-loop system. In fact, in general, the origin of such systems is locally asymptotically stable. The global asymptotic stability of saturated control systems can be obtained if the open-loop system satisfies some stability conditions.

At the best of our knowledge, this topic started in [84], [81], [52] and [44]. These articles use nonlinear semigroup theory to tackle the asymptotic stability of infinite-dimensional systems with a stabilizing feedback modified via a cone-bounded nonlinearity. In this thesis, we aim at giving complementary results for nonlinear abstract control systems and at studying a particular nonlinear partial differential equation, namely the Korteweg-de Vries equation, which models waves on shallow water surfaces.

This thesis aims also at contributing in output feedback stabilization problems. In recent decades, some tools for the design of feedback laws have been found. Some of these feedbacks needs the full-*state* of the system to be known. Note that the state is referred to as the solution to the system. Hence implementing them requires to have a measure of all the variables of the system.

However, in most of the cases, sensors can measure only a part of the state. This partial measurement is called the *output*. In this case, designing a feedback law requires sometimes to build an observer, which is a dynamical system whose state converges to the state of the control system. This thesis aims at contributing in this topic for two Korteweg-de Vries equations for which there exists a stabilizing state feedback law. Hence, a second question arises.

Question 2: *Given an observer and a state feedback law for a Korteweg-de Vries equation,*

does the resulting output feedback law stabilize the origin of the closed-loop system?

This question is referred to as the so-called *separation principle*, which applies for linear time-invariant finite-dimensional control systems. In other words, if such a system is stabilizable and observable, then an observer and a state feedback law can be designed separately to build a stabilizing output feedback law.

We already know that for nonlinear finite-dimensional systems, the separation principle does not hold anymore. Note that the survey [3] collects some counter-examples. If the control system satisfies an observability property, namely the complete uniform observability, and if moreover it satisfies a stabilizability property, then we say that the control system is *semi-globally stabilizable by means of an output feedback*. Roughly speaking, it means the following: if the initial condition belongs to a compact set, an observer depending on this compact set can be designed jointly with the state feedback in order to make asymptotically stable the origin of the closed-loop system with a basin of attraction contained in the compact set. This well-known result is provided in [97].

We aim at contributing in this topic for control systems for which there exists a hybrid state feedback. A hybrid state feedback is a feedback law which has a mixed continuous/discrete behavior. Such controllers are useful for some systems that cannot be stabilized by means of a continuous feedback law. Now, a third and last question can be stated.

Question 3: *If a control system is completely uniformly observable and if there exists a stabilizing hybrid state feedback law for this system, under which conditions is this system semi-globally stabilizable?*

The manuscript is divided into four chapters. Chapters 1 and 2 tackle Question 1. Chapter 3 solves Question 2. Finally, Chapter 4 answers Question 3.

Here is the precise outline of the thesis.

Outline

- Chapter 1 deals with nonlinear abstract control systems. After an introduction to semigroup theory and a short recall of some existing results, some conditions on the open-loop system and on the cone-bounded nonlinearity are given in order to prove the well-posedness and the asymptotic stability of the closed-loop system. Two illustrative examples are provided;
- Chapter 2 tackles the case of the Korteweg-de Vries equation. Two different types of saturated controls are considered. The well-posedness is proven applying a Banach fixed point theorem, using some estimates of this equation and some properties of the saturation function. The proof of the asymptotic stability of the closed-loop system is separated in two cases: i) when the control acts on all the domain, a Lyapunov function together with a sector condition describing the saturating input is used to

conclude on the stability; ii) when the control is localized, we argue by contradiction. Some numerical simulations illustrate the stability of the closed-loop nonlinear partial differential equation;

- Chapter 3 deals with the output feedback stabilization of two Korteweg-de Vries equations. The first one is linear, the second one is nonlinear. The design of the output feedback laws is based on the backstepping method (see [50] for a good introduction on this topic) and the introduction of an appropriate observer. The exponential stability of the closed-loop systems is proven with some Lyapunov functions. Some numerical simulations are performed to illustrate our theoretical results;
- Chapter 4 is about the semi-global stabilization of nonlinear system for which there exists a regularizing hybrid state feedback law. A local hybrid stabilizability and a complete uniform observability are assumed to achieve the stabilization of an equilibrium set with a hybrid output feedback law. Timer property is exploited to propose a design method for such feedbacks. The result is illustrated for a linear system with reset saturated control.

List of Related Publications

- Journals

1. S. Marx, V. Andrieu, and C. Prieur. "Semi-global stabilization by an output feedback law from a hybrid state controller." In: *Automatica* **74** (2016), pp. 90-98.
2. S. Marx, E. Cerpa, C. Prieur, and V. Andrieu. "Global stabilization of a Korteweg-de Vries equation with saturating distributed control." *SIAM J. Control Optim.* **55**(3) (2017), pp. 1452–1480.
3. S. Marx, and E. Cerpa. "Output Feedback Control of the Korteweg-de Vries Equation." **To be published in Automatica (2017)**.
4. S. Marx, V. Andrieu, and C. Prieur. "Cone-bounded feedback laws for m -dissipative operators on Hilbert space. **Submitted to Mathematics of Control, Signals and Systems.**

- Conferences

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3. S. Marx, E. Cerpa, C. Prieur, and V. Andrieu. "Stabilization of a linear Korteweg-de Vries equation with a saturated internal control." In: *Proceedings of the European Control Conference*. Linz, AU, 2015, pp. 867-872.

4. S. Marx, E. Cerpa, C. Prieur, and V. Andrieu. "Global stabilization of a Korteweg-de Vries equation with a distributed control saturated in L^2 -norm." In: *10th IFAC Symposium on Nonlinear Control Systems (NOLCOS 2016)*, Monterey, CA, 2016.

Cone-bounded controllers for m -dissipative operators

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This chapter deals with systems composed by a feedback interconnection of a plant and a cone-bounded nonlinearity. The study of such systems has received considerable attention in recent decades (see e.g., [94], [101], or [45]). Indeed, in most of systems, the control input has a nonlinear dynamic. Nowadays, it is well known that neglecting these nonlinearities can lead to undesirable and even catastrophic behaviors for the stability of closed-loop system. Without any assumption on the open-loop system, only a local stabilization result can be obtained. A classical research line is then to analyze the basin of attraction or to obtain a better one using anti-windup techniques in the case of saturated controls ([37] or [25]).

Tackling this kind of nonlinearities in the case of finite-dimensional systems is already a difficult problem. However, nowadays, numerous techniques are available (see e.g., [94]) and such systems can be analyzed with different techniques: an appropriate Lyapunov function and a sector condition of the saturation map, as introduced in [94] or a frequency approach, leading to the so-called Popov criterion, as reviewed in [45]. In the case of open-loop stable finite-dimensional linear systems with a chain of integrators form, a bounded feedback law can be designed using nested saturations (see e.g., [95], [92], [88] or [51]).

To the best of our knowledge, the study of this topic in the infinite-dimensional case has started with [84, 81, 52]. More recently, some new results have been stated in [44, 27, 64,

76]. This chapter aims at contributing to the study of feedback interconnection of a system (possibly nonlinear) and a cone-bounded nonlinearity in the framework of partial differential equations, more precisely for abstract control systems described using semigroup theory ([72] and [68] are good introductions to linear semigroups and nonlinear semigroups, respectively).

This chapter is organized as follows. Section 1.1 is devoted to the introduction to nonlinear semigroup theory in Hilbert spaces and to some stability results in the infinite-dimensional context. Section 1.2 presents the problem of stabilizing an equilibrium point with a cone-bounded feedback law. Some existing results and some of our contributions are provided. Finally, some concluding remarks are collected in Section 1.3.

Note that the contributions of this chapter, from Section 1.2.2 until the end of the chapter, are based on two of our papers. The first one [64] has been published in a peer-reviewed conference (European Control Conference 2015). The second one [60] is currently being reviewed in the journal *Mathematics of Control, Signals, and Systems*.

1.1 Semigroup theory

1.1.1 Strongly continuous semigroups of contractions and m -dissipative operators

Semigroup theory aims at generalizing matrix exponentials for operators in Hilbert spaces or in Banach spaces. This theory allows also to solve some infinite-dimensional Cauchy problems and some asymptotic stability problems. This section is devoted to the introduction of this theory for systems posed in a Hilbert space¹. Most of the results written below are borrowed from [68].

All along this chapter, Z is a Hilbert space on \mathbb{R} equipped with a scalar product $\langle \cdot, \cdot \rangle_Z$ and a norm $\| \cdot \|_Z$.

Definition 1.1 (Strongly continuous semigroup of contractions)

(i) A family of bounded linear operators $(W(t))_{t \geq 0}$ from Z to Z is said to be a strongly continuous semigroup on Z if it satisfies the following properties

1. $W(0) = I_Z$;
2. $W(t + s) = W(t)W(s)$, $\forall t, s \geq 0$;
3. $\lim_{t \rightarrow 0^+} \|W(t)z - z\|_Z = 0$, $\forall z \in Z$.

(ii) A strongly continuous semigroup on Z is said to be a strongly continuous semigroup of

¹This chapter does not aim at presenting the case of semigroups in Banach spaces. However, the reader may refer to [19] or [68] for an introduction to this topic in this context.

contractions on Z if, for all $t \geq 0$

$$\|W(t)z - W(t)\tilde{z}\|_Z \leq \|z - \tilde{z}\|_Z, \quad \forall z, \tilde{z} \in Z. \quad (1.1)$$

Remark 1.1

Given $n \in \mathbb{N}$, consider a matrix $A \in \mathbb{R}^{n \times n}$. The following operator

$$t \mapsto e^{At}, \quad (1.2)$$

is a strongly continuous semigroup. If moreover

$$\|e^{At}\|_{\mathcal{L}(\mathbb{R}^n)} \leq 1, \quad \forall t \geq 0, \quad (1.3)$$

then $t \mapsto e^{At}$ is a strongly continuous semigroup of contractions. Indeed, for all $z, \tilde{z} \in \mathbb{R}^n$

$$\begin{aligned} |e^{At}z - e^{At}\tilde{z}| &= |e^{At}(z - \tilde{z})|, \\ &\leq \|e^{At}\|_{\mathcal{L}(\mathbb{R}^n)} |z - \tilde{z}|, \\ &\leq |z - \tilde{z}|, \end{aligned}$$

where the linearity of the operator $z \mapsto e^{At}z$, for all $t \in \mathbb{R}_{\geq 0}$, and (1.3) have been used in the second line and the third line, respectively.

Consider an operator (possibly unbounded and nonlinear) $A : D(A) \subseteq Z \rightarrow Z$, where $D(A)$ is a subset of Z defined as follows

$$D(A) := \{z \in Z \mid Az \text{ exists}\}. \quad (1.4)$$

The operator A is said to be closed if its graph $\mathcal{G}(A) := \{(z, Az), z \in Z\}$ is closed in $Z \times Z$. Moreover, if $D(A)$ is dense in Z^2 , then we can define an adjoint $A^* : D(A^*) \subseteq Z \rightarrow Z^3$. The domain $D(A^*)$ is defined by the following set

$$D(A^*) := \{\tilde{z} \in Z \mid \text{there exists } z^* \in Z \text{ such that } \langle Az, \tilde{z} \rangle_Z = \langle z, z^* \rangle_Z \text{ for all } z \in D(A)\}. \quad (1.5)$$

Hence, the adjoint operator A^* is defined as follows

$$\langle Az, \tilde{z} \rangle_Z = \langle z, A^*\tilde{z} \rangle_Z, \quad \forall (z, \tilde{z}) \in D(A) \times D(A^*). \quad (1.6)$$

The following definition links operators and strongly continuous semigroups.

Definition 1.2

The infinitesimal generator $A : D(A) \subseteq Z \rightarrow Z$ of a strongly continuous semigroup $(W(t))_{t \geq 0}$ on Z is defined by

$$Az = \lim_{t \rightarrow 0^+} \frac{W(t)z - z}{t} \quad (1.7)$$

where

$$D(A) := \left\{ z \in Z \mid \lim_{t \rightarrow 0^+} \frac{W(t)z - z}{t} \text{ exists} \right\} \quad (1.8)$$

² $D(A)$ is said to be dense in Z if $\overline{D(A)} = Z$

³The proof of this result is provided in [7, Chapter 2, Page 43]. It uses the so-called Hahn-Banach theorem.

Remark 1.2

Take the same example as in Remark 1.1. Then, we have

$$\lim_{t \rightarrow 0^+} \frac{e^{At}z - z}{t} = Az, \quad (1.9)$$

by definition of the derivative. It means that A is the infinitesimal generator of the strongly continuous semigroup $(e^{At})_{t \geq 0}$.

Let us define the notion of dissipative operator.

Definition 1.3 (Dissipative operator)

An operator $A : D(A) \subseteq Z \rightarrow Z$ is said to be dissipative if the following holds

$$\Re_e \langle Az - A\tilde{z}, z - \tilde{z} \rangle_Z \leq 0, \quad \forall z, \tilde{z} \in D(A). \quad (1.10)$$

Remark 1.3

• Note that, if A is linear, then the following inequality

$$\Re_e \langle Az, z \rangle_Z \leq 0, \quad \forall z \in D(A), \quad (1.11)$$

implies that A is dissipative;

• Given $n \in \mathbb{N}$, let us take the case of the matrix $A \in \mathbb{R}^{n \times n}$. The dissipativity of a matrix implies that

$$z^\top A z \leq 0, \quad \forall z \in \mathbb{R}^n. \quad (1.12)$$

This implies that all the eigenvalues of A are nonpositive. We will see that a similar result can be obtained for infinite-dimensional linear operators.

Thanks to this definition, one can define the m -dissipative operators.

Definition 1.4 (m -dissipative operators)

A dissipative operator $A : D(A) \subseteq Z \rightarrow Z$ is said to be an m -dissipative operator if, for all positive values λ , the following equality holds

$$\text{Ran}(I_Z - \lambda A) = Z. \quad (1.13)$$

The condition (1.13) might be hard to check. In fact, there exists another weaker condition. Let us state it.

Lemma 1.1 ([68], Lemma 2.13, Page 22)

A dissipative operator $A : D(A) \subseteq Z \rightarrow Z$ is an m -dissipative operator if and only if there exists a positive value λ_0 such that

$$\text{Ran}(I_Z - \lambda_0 A) = Z. \quad (1.14)$$

The following theorem gives some conditions for a linear operator to be m -dissipative. Let us state it.

Theorem 1.1 ([72], Corollary 4.4, Chapter 1, page 15)

Let the operator $A : D(A) \subseteq Z \rightarrow Z$ be linear with its domain dense in Z . If A is closed and if A and A^* are both dissipative, then A is an m -dissipative operator.

For linear systems, this theorem is useful, because checking whether an operator is closed and whether this operator and its adjoint are both dissipative is easier than proving (1.14).

Let us introduce the concept of the *resolvent of an operator*. This term comes from the *resolvent set* of an operator $A : D(A) \subseteq Z \rightarrow Z$, which is defined as follows

$$\rho(A) := \{\lambda \in \mathbb{C} \mid I_Z - \lambda A \text{ is injective and } (I_Z - \lambda A)^{-1} \text{ is bounded}\}. \quad (1.15)$$

For *linear* operators, the spectrum of an operator A is the complement of $\rho(A)$ in \mathbb{C} .

Now, we are able to define the resolvent operator of an operator.

Definition 1.5 (Resolvent operator of an operator A)

Let $A : D(A) \subseteq Z \rightarrow Z$ be an operator and $\lambda \in \mathbb{C}$. The resolvent operator of A is defined by

$$\begin{aligned} J_\lambda : D(J_\lambda) &\rightarrow Z, \\ z &\mapsto (I_Z - \lambda A)^{-1}z, \end{aligned} \quad (1.16)$$

where $D(J_\lambda) := \text{Ran}(I_Z - \lambda A)$

The fact that $D(J_\lambda) := \text{Ran}(I_Z - \lambda A)$ comes from [68, Corollary 3.7, page 20].

Remark 1.4

When A is linear, note that the resolvent of A exists if and only if λ does not belong to the spectrum of A .

The following theorem gives a property of the resolvent operator of dissipative operators.

Theorem 1.2 ([68], Corollary 2.10, page 20)

Let Z be a Hilbert space. Let $A : D(A) \subseteq Z \rightarrow Z$ be a dissipative operator with its domain dense in Z . Then, for any positive value λ , the resolvent (1.16) exists and satisfies the following inequality, for all $z, \tilde{z} \in D(J_\lambda)$

$$\|J_\lambda z - J_\lambda \tilde{z}\|_Z \leq \|z - \tilde{z}\|_Z. \quad (1.17)$$

In other words, this means that the set defined by $\{\lambda \in \mathbb{C} \mid \Re_e(\lambda) > 0\}$ belongs to the resolvent set. Hence, for linear operators, this implies that the real parts of the eigenvalues are less or equal to 0. Recall that the particular case of the matrix $A \in \mathbb{R}^{n \times n}$ has been noticed in Remark 1.3.

Thanks to these definitions, we are able to define operators generating semigroups of contraction. Indeed, the following result, borrowed from [68], links nonlinear m -dissipative operators and strongly continuous semigroups of contractions.

Theorem 1.3 ([68], Theorem 4.20, page 103)

The infinitesimal generator $A : D(A) \subseteq Z \rightarrow Z$ of a strongly continuous semigroup of contractions on Z is an m -dissipative operator. Conversely, if A is an m -dissipative operator, then A generates a strongly continuous semigroup of contractions.

This result is crucial. Indeed, this theorem allows to state that a Cauchy problem is well-posed and to prove that the origin of this system is Lyapunov stable. The next section is devoted to the study of Cauchy problems in the framework of abstract nonlinear systems.

1.1.2 Cauchy Problem and Lyapunov stability

Consider the following Cauchy problem

$$\begin{cases} \frac{d}{dt}z = Az, \\ z(0) = z_0, \end{cases} \quad (1.18)$$

where $A : D(A) \subseteq Z \rightarrow Z$ is a (possibly nonlinear) operator. The initial condition z_0 can belong either to Z or to $D(A)$.

An equilibrium point $z_e \in Z$ of (1.18) is defined as follows

$$Az_e = 0. \quad (1.19)$$

One of the aims of control theory is to study the *stability* of such a point. Thus, we need to define what do we mean by stability. Here is the Lyapunov version of the stability.

Definition 1.6 (Lyapunov stability)

An equilibrium point $z_e \in Z$ of (1.18) is said to be Lyapunov stable in Z if, for any $\delta > 0$, there exists $\varepsilon := \varepsilon(\delta)$ such that the following implication holds

$$\|z_0 - z_e\|_Z \leq \varepsilon \Rightarrow \|z(t) - z_e\|_Z \leq \delta, \quad \forall t \geq 0, \quad (1.20)$$

for any mild (or strong) solution $z(t)$ (1.18) with the initial condition z_0 .

Theorem 1.3 is a sufficient condition to obtain the well-posedness of the latter Cauchy Problem. Indeed, the following theorem allows us to obtain a well-posedness for strong and mild solutions.

Theorem 1.4 (Well-posedness and Lyapunov stability [68], Corollary 3.7, page 53)

If A is an m -dissipative operator, then

- if $z_0 \in Z$, (1.18) admits a unique mild solution $z(t) \in C(0, \infty; Z)$;
- if $z_0 \in D(A)$, (1.18) admits a unique strong solution $z(t) \in C(0, \infty; D(A))$.

Moreover, if $z_0 \in D(A)$, then, denoting by $(W(t))_{t \geq 0}$ the strongly continuous semigroup of contractions generated by A , it holds that the two functions

$$t \mapsto \|W(t)z_0\|_Z, \quad t \mapsto \|AW(t)z_0\|_Z \quad (1.21)$$

are non increasing. In addition, it holds that

$$A(0) = 0. \quad (1.22)$$

Hence, we have not only well-posedness, but also Lyapunov stability of the origin, which is an equilibrium point because of (1.22). Indeed, since $(W(t))_{t \geq 0}$ is a strongly continuous semigroup of contraction, then the following holds

$$\|W(t)z_0\|_Z \leq \|z_0\|_Z, \quad \forall t \geq 0. \quad (1.23)$$

The unique solution to (1.18) is bounded by its initial condition. Hence, (1.23) not only implies that the origin is an equilibrium, but also implies that the origin is Lyapunov stable.

Some attractivity results are needed before studying the case of abstract control systems. The next section is devoted to the presentation of some infinite-dimensional versions of LaSalle's Invariance Principle.

1.1.3 Asymptotic stability for abstract systems

1.1.3.1 Attractivity and asymptotic stability

If Theorem 1.4 can be applied, Lyapunov stability of the origin holds. However, in stabilization problem, we want to make the origin of the closed-loop system globally attractive. The definition of this property is provided just below.

Definition 1.7 (Attractivity of an equilibrium)

An equilibrium point $z_e \in Z$ is said to be locally attractive in Z if there exists a positive value δ such that

$$\|z_0 - z_e\|_Z \leq \delta \Rightarrow \lim_{t \rightarrow +\infty} \|z(t) - z_e\|_Z = 0, \quad (1.24)$$

for any mild (or strong) solution $z(t)$ (1.18) with the initial condition z_0 .

The equilibrium point z_e is said to be globally attractive if $\delta = +\infty$.

The aim of the stabilization is to find a feedback law that makes an equilibrium point of the closed-loop system *locally asymptotically stable* or *globally asymptotically stable*. The following definition characterizes this property.

Definition 1.8 (Asymptotic stability)

Assume that (1.18) admits at least one solution and that $z_e \in Z$ is an equilibrium point. This equilibrium point is said to be asymptotically stable in Z if it is Lyapunov stable and attractive. It is said to be globally asymptotically stable if it is Lyapunov stable and globally attractive.

Lyapunov theory exists also for infinite-dimensional systems. A lot of results for finite-dimensional systems apply also for infinite-dimensional systems. Hence, as in the finite-dimensional framework, proving asymptotic stability of equilibrium points of infinite-dimensional systems reduces to finding a Lyapunov function. However, in practice, deriving such a function might be difficult. In this case, a common technique to conclude the attractivity of an equilibrium point is to use the LaSalle's Invariance Principle, which is given in the next section.

1.1.3.2 LaSalle's Invariance Principle

Before stating the LaSalle's Invariance Principle, let us start by giving some definitions.

Definition 1.9 (Positive orbit and ω -limit set)

1. Given a strongly continuous semigroup $(W(t))_{t \geq 0}$ over Z , the positive orbit through $\phi \in Z$ is defined by $\mathcal{O}^+(\phi) := \cup_{t \in \mathbb{R}_{\geq 0}} W(t)\phi$;

2. The weak ω -limit set of φ is the (possibly empty) set defined by

$$\omega_w(\psi) := \{\psi \in Z \mid \text{there exists a sequence } t_n \rightarrow \infty \text{ as } n \rightarrow \infty \text{ such that } W(t_n)\phi \rightarrow \psi \text{ as } n \rightarrow \infty\};$$

3. The strong ω -limit set of ψ is the (possibly empty) set defined by $\omega(\psi) := \bigcap_{\tau \geq 0} \text{clos}_Z \left(\bigcup_{t \geq \tau} W(t)\psi \right)$.

A good introduction to the positive orbit and the ω -limit set for finite-dimensional systems is [43].

These definitions allow us to define the infinite-dimensional version of LaSalle's Invariance Principle. This theorem is given just below.

Theorem 1.5 (LaSalle's Invariance Principle [84], Theorem 3.1 or [38], Theorem 3)

Given a strongly continuous semigroup $(W(t))_{t \geq 0}$ on Z and $\phi \in Z$, one has

(i) If $\mathcal{O}^+(\phi)$ is precompact⁴, then $\omega(\phi)$ is a nonempty, invariant set of Z through $(W(t))_{t \geq 0}$ ⁵;

(ii) If each $W(t)$ is sequentially weakly continuous on Z ⁶ and $\mathcal{O}^+(\phi)$ is bounded, then $\omega_w(\phi)$ is a nonempty, invariant set in Z through $(W(t))_{t \geq 0}$.

⁴A subset in a topological space is said to be precompact if its closure is compact.

⁵A set $\tilde{Z} \subset Z$ is said to be an invariant set of Z through $(W(t))_{t \geq 0}$ if, for any $w \in \tilde{Z}$, $W(t)w \in \tilde{Z}$ for all $t \geq 0$

⁶A function $t \mapsto W$ is said to be sequentially weakly continuous on Z if, for every $\phi_n, \phi \in Z$ such that $\phi_n \rightarrow_Z \phi$, then $W(t)\phi_n \rightarrow_Z W(t)\phi$

Note that the positive orbit has to be precompact if we want to apply the first item of Theorem 1.5. For an m -positive operator A generating a strongly continuous semigroup of contraction $(W(t))_{t \geq 0}$, proving this property reduces to proving the following claim

Assumption 1 (Precompactness of the positive orbit)

The canonical embedding from $D(A)$, equipped with the graph norm, into Z is compact.

Indeed, supposing that Assumption 1 holds and using the fact that the two functions

$$t \mapsto \|W(t)z_0\|_Z, \quad t \mapsto \|AW(t)z_0\|_Z \quad (1.25)$$

are non increasing, it is straightforward that the trajectory of (1.18) defined by $\{z(t) = W(t)z_0, t \geq 0 \text{ and } z_0 \in D(A)\}$ is precompact in Z .

Since all norms in finite-dimensional spaces are equivalent, note that Claim 1 holds for every finite-dimensional systems. However, when infinite-dimensional systems are studied, proving Claim 1 might be difficult to achieve.

1.2 Stabilization of nonlinear systems with a cone-bounded controller

1.2.1 Problem statement and existing results

1.2.1.1 Problem statement

Let U be a Hilbert space on \mathbb{R} equipped with the scalar product $\langle \cdot, \cdot \rangle_U$ and the norm $\|\cdot\|_U$. We consider the stabilization problem of the origin of the following infinite-dimensional control system

$$\frac{d}{dt}z = Az + Bu \quad (1.26)$$

where u in U denotes the controlled input. We assume that the origin is an equilibrium point of (1.26) when $u = 0$ and that $A : D(A) \subseteq Z \rightarrow Z$ is a (possibly nonlinear and unbounded) operator which generates a strongly continuous semigroup of contractions denoted by $(W(t))_{t \geq 0}$. Moreover, we suppose that $B \in \mathcal{L}(U, Z)$.

Assuming that the operator $A - BB^*$ generates a strongly continuous semigroup of contractions $(W_I(t))_{t \geq 0}$ which satisfies, for all $z_0 \in Z$

$$\lim_{t \rightarrow +\infty} \|W_I(t)z_0\|_Z = 0 \quad (1.27)$$

the aim of this chapter is to study the case where the control is given by

$$u = -\sigma(B^*z), \quad (1.28)$$

where $\sigma : U \rightarrow U$ is a mapping which is characterized by the following definition.

Definition 1.10 (Cone-bounded nonlinearities on U)

Let $\sigma : U \rightarrow U$ be a continuous operator such that

1. for all u in U , $\Re_e \{ \langle u, \sigma(u) \rangle_U \} = 0$ implies $u = 0$;
2. there exists a positive value ℓ such that, for all $u \in U$, we have $\|\sigma(u)\|_U \leq \ell \|u\|_U$;
3. for all u, v in U , we have $\Re_e \{ \langle \sigma(u) - \sigma(v), u - v \rangle_U \} \geq 0$.

Such a function σ is called a cone-bounded nonlinearity.

Example 1.1 (Examples of cone-bounded nonlinearities)

Here are some examples of cone-bounded nonlinearities.

1. Any linear mapping $\sigma(u) = \mu u$, where μ is a positive value, is a cone-bounded nonlinearity;
2. The so-called localized saturation (as considered in e.g., [94], [46]) defined by

$$\mathbf{sat}_{loc} : s \in \mathbb{R} \mapsto \mathbf{sat}_{loc}(s) := \begin{cases} -u_s & \text{if } s \leq -u_s, \\ s & \text{if } -u_s \leq s \leq u_s, \\ u_s & \text{if } s \geq u_s, \end{cases} \quad (1.29)$$

with u_s a positive value, is a cone-bounded nonlinearity;

3. For any positive value u_s , the function $s \in \mathbb{R} \mapsto u_s \tanh\left(\frac{s}{u_s}\right)$ is a cone-bounded nonlinearity;
4. The function

$$\tilde{\sigma} : s \in \mathbb{R} \mapsto \mathbf{sat}_{loc}(\varphi(s)), \quad (1.30)$$

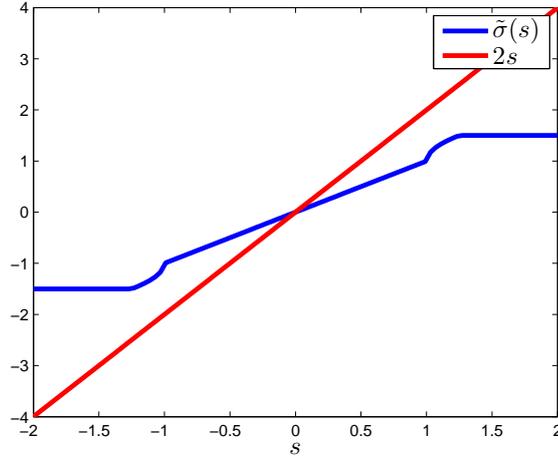
where $u_s > 1$ and where φ is defined as follows

$$\varphi : s \in \mathbb{R} \mapsto \begin{cases} -\sqrt{|s|-1} - 1 & \text{if } s < -1, \\ s & \text{if } s \in [-1, 1], \\ \sqrt{s-1} + 1 & \text{if } s > 1, \end{cases} \quad (1.31)$$

takes values in a bounded set, but it is not globally Lipschitz because of the function $s \mapsto \sqrt{s}$ in the definition of the function φ . Figure 1.1 illustrates the functions $s \mapsto \tilde{\sigma}(s)$ and $s \mapsto 2s$ with $s \in [-2, 2]$ and $u_s = 1.5$. It is clear that this function is a cone-bounded nonlinearity, as introduced in Definition 1.10.

The following sections of this chapter focus on the closed-loop system

$$\begin{cases} \frac{d}{dt} z = Az - B\sigma(B^* z) := A_\sigma z, \\ z(0) = z_0, \end{cases} \quad (1.32)$$

Figure 1.1: Red line: $2s$; Blue line: $\tilde{\sigma}(s)$ with $u_s = 1.5$

where $A_\sigma : D(A_\sigma) \subset Z \rightarrow Z$ is a nonlinear operator for which we assume that $D(A_\sigma) = D(A)$. Note that z_0 can belong to Z or to $D(A)$. Some results dealing with well-posedness and asymptotic stability will be provided.

1.2.1.2 Existing results

Some existing results can be found in the literature. In this section, we will focus in particular on [84] and [81]. These papers study a particular cone-bounded nonlinearity, which is

$$\mathbf{sat}_U(s) := \begin{cases} s & \text{for all } \|s\|_U \leq u_s, \\ \frac{s}{\|s\|_U} u_s & \text{for all } \|s\|_U \geq u_s, \end{cases} \quad (1.33)$$

where u_s is a positive value. In this section only, we will assume that

$$\sigma(s) = \mathbf{sat}_U(s). \quad (1.34)$$

In [84], the following properties are assumed to be satisfied

Assumption 2

1. We have $\sigma(s) = \mathbf{sat}_U(s)$;
2. The operator A is linear and generates a strongly continuous semigroup of contractions denoted by $(W(t))_{t \geq 0}$;
3. The operator $(\lambda I_Z - A)^{-1}$ is compact for all positive values λ ;

4. The operator B is compact;

5. For all $\psi \in Z$, the only solution to

$$B^*W(t)\psi = 0 \quad (1.35)$$

is

$$\psi = 0. \quad (1.36)$$

Items 1., 2. and 3. allow to state the well-posedness of (1.18). Items 4. and 5. allow to apply the weak version of LaSalle's Invariance Principle, i.e. the second item of Theorem 1.5. Note that the item 5. of these assumptions refers to a detectability property.

Under these assumptions, the following theorem is obtained

Theorem 1.6

Assume that Assumption 2 holds. Then, for each $z_0 \in Z$, A_σ generates a strongly continuous semigroup of contractions denoted by $(W_{\text{sat}_U}(t))_{t \geq 0}$ and, for each $z_0 \in Z$, there exists a unique solution to (1.18) defined for all $t \in \mathbb{R}_{\geq 0}$ and given by $z(t) = W_{\text{sat}_U}(t)z_0$. Moreover, the following holds, for all $z_0 \in Z$,

$$z(t) \xrightarrow{Z} 0 \text{ as } t \rightarrow +\infty. \quad (1.37)$$

Only a weak attractivity is obtained on [84]. In fact, since the paper aims at finding result for a particular partial differential equation, i.e. a beam equation, a stronger result is not necessary. The control operator for the partial differential equation belongs to the space $\mathcal{L}(\mathbb{R}, Z)$. Hence, another theorem which takes into account this particular case is stated in [84].

Theorem 1.7

Under Assumption 2 and assuming moreover that $U = \mathbb{R}$, then A_σ generates a strongly continuous semigroup of contraction denoted by $(W_{\text{sat}_U}(t))_{t \geq 0}$ and, for each $z_0 \in Z$, there exists a unique mild solution to (1.18) denoted by $z(t) := W_{\text{sat}_U}(t)z_0$. Moreover, the following holds, for all $z_0 \in Z$

$$\lim_{t \rightarrow +\infty} \|W_{\text{sat}_U}(t)z_0\|_Z = 0 \quad (1.38)$$

Note that in the proof of these two results it is not used the particular form of sat_U , but only the fact that it is globally Lipschitz, monotone and the property 1 of Definition 1.10.

In [81], a better result is stated. The assumptions are weaker than Assumption 2. Let us state them

Assumption 3

1. We have $\sigma(s) = \text{sat}_U(s)$;

1. The operator A is linear and generates a strongly continuous of contractions denoted by $(W(t))_{t \geq 0}$;
2. The operator $A - BB^*$ generates a strongly continuous of contractions denoted by $(W_I(t))_{t \geq 0}$ that satisfies the following, for all $z_0 \in Z$

$$\lim_{t \rightarrow +\infty} \|W_I(t)z_0\|_Z = 0. \quad (1.39)$$

Unlike Assumption 2 provided by [84], neither the operator B nor $(\lambda I_Z - A)^{-1}$ are assumed to be compact. Moreover, instead of assuming a detectability property as in item 5 of Assumption 2, only a stabilizability property is assumed in [81].

A stronger result than Theorem 1.6 is stated in [81]. Here is its statement.

Theorem 1.8

Under Assumption 3, A_σ generates a strongly continuous semigroup of contractions denoted by $(W_{\text{sat}_U}(t))_{t \geq 0}$ and, for each $z_0 \in Z$, (1.18) admits a unique solution denoted by $z(t) := W_{\text{sat}_U}(t)z_0$. Moreover, the following holds, for all $z_0 \in Z$

$$\lim_{t \rightarrow +\infty} \|W_{\text{sat}_U}(t)z_0\|_Z = 0. \quad (1.40)$$

Unlike the proof of Theorem 1.6, the proof of this theorem uses the special structure of sat_U . Moreover, the authors of [81] derive some conditions in order to obtain a similar result for unbounded control operators. Since this chapter is devoted to the case of bounded control operators, this result will not be discussed here.

Papers [84] and [81] have inspired a lot of researchers. Among the results derived from these papers, [52] or [27] can be cited. Note that, even in the context of finite-dimensional systems, these papers have inspired some researchers (see e.g., [56]).

Next section is devoted to our contributions to the topic of infinite-dimensional systems with cone-bounded feedback laws.

1.2.2 Contributions

This section is devoted to some contributions to this topic. It is borrowed from [60]. All along this section, we focus on the system (1.32) and assume that σ is a cone-bounded nonlinearity. Let us recall that A , which generates a strongly continuous semigroup, can either be linear or nonlinear. This makes our results different from the ones written in [84] and in [81]. Note moreover that B is a linear bounded operator, i.e. $B \in \mathcal{L}(Z, U)$.

This section is divided into five parts. Sections 1.2.2.1 and 1.2.2.2 state a well-posedness and an asymptotic stability results, respectively. Sections 2.2.3.1 and 2.2.3.3 are devoted to the proofs of these results. Finally, Section 1.2.4 is the application of these results to a linear Korteweg-de Vries equation.

1.2.2.1 Well-posedness

As it has been noticed at the beginning of the chapter, A is assumed to be an m -dissipative operator. It implies that A is dissipative and that, for all positive value λ , $\text{Ran}(I - \lambda A) = Z$.

Since A is dissipative, from Theorem 1.2, we have that, for all $\lambda > 0$, the operator $J_\lambda : D(J_\lambda) \rightarrow D(A)$ defined by

$$J_\lambda = (I_Z - \lambda A)^{-1}$$

exists and is a contraction. Moreover, we have

$$D(J_\lambda) = \text{Ran}(I_Z - \lambda A).$$

We are now in position to state our well-posedness result.

Theorem 1.9 (Well-posedness and Lyapunov stability)

Assume that σ is a cone-bounded nonlinearity. Moreover, assume that one of the two conditions is fulfilled:

1. σ is globally Lipschitz;
2. There exists a Banach space Z_0 such that $D(A) \subseteq Z_0$ and such that
 - (a) the canonic injection from Z_0 to Z is compact;
 - (b) there exists a positive value N such that, for all $\bar{z} \in Z$,

$$\sup_{z \in Z} \|J_1(\bar{z} - B\sigma(B^*z))\|_{Z_0} < N \quad (1.41)$$

Then, for all z_0 in $D(A)$, there exists a unique strong solution to (1.32) and the operator A_σ generates a strongly continuous semigroup of contractions $(W_\sigma(t))_{t \geq 0}$ such that the two functions

$$t \mapsto \|W_\sigma(t)z_0\|_Z, \quad t \mapsto \|A_\sigma W_\sigma(t)z_0\|_Z$$

are non increasing.

Remark 1.5

Unlike Theorems 1.6 and 1.8, the result provided above refers to strong solutions. However, the function (1.30) in Example 1.1 shows that a cone-bounded nonlinearity does not have to be globally Lipschitz to ensure the well-posedness of the closed-loop system, while Theorem 1.8 needs the cone-bounded nonlinearity to be equal to sat_U .

Remark 1.6

If A is linear, the condition (1.41) may be reduced to the following assumption:

$$\sup_{z \in Z} \|J_1(-B\sigma(B^*z))\|_{Z_0} < \infty. \quad (1.42)$$

Indeed, in that case, (1.42) implies (1.41).

Remark 1.7

Following Lemma 1.1, the condition (1.41) may be rewritten as the following statement: there exists a positive value λ_0 such that, for all $\bar{z} \in Z$,

$$\sup_{z \in Z} \|J_{\lambda_0}(\bar{z} - B\sigma(B^*z))\|_{Z_0} < \infty. \quad (1.43)$$

In order to make easier the reading, we let $\lambda_0 = 1$ as in (1.41), without loss of generality.

Example 1.2

The condition (1.41) imposes a global bound on the mapping σ in a specific norm. As a first illustration, consider the following linear Korteweg-de Vries (for short KdV) equation

$$\begin{cases} z_t(t, x) + z_x(t, x) + z_{xxx}(t, x) + \mathbf{1}_\Omega(z)u(t, x) = 0, & (t, x) \in \mathbb{R}_{\geq 0} \times (0, L), \\ z(t, 0) = z(t, L) = z_x(t, L) = 0, & t \in \mathbb{R}_{\geq 0}, \\ z(0, x) = z_0(x), \end{cases} \quad (1.44)$$

where L is a positive value, $u(t, x)$ is the control, Ω is a nonempty subset of $(0, L)$ and $\mathbf{1}_\Omega$ is defined by

$$\mathbf{1}_\Omega(z) = \begin{cases} 1 & \text{if } z \in \Omega, \\ 0 & \text{otherwise.} \end{cases} \quad (1.45)$$

Setting $Z = L^2(0, L)$ and $U = L^2(\Omega)$, system (1.44) can be written as in (1.26) denoting

$$\begin{aligned} A : D(A) \subset L^2(0, L) &\rightarrow L^2(0, L), \\ z &\mapsto -z' - z''', \end{aligned} \quad (1.46)$$

where

$$D(A) = \{z \in H^3(0, L) \mid z(0) = z(L) = z'(L) = 0\}. \quad (1.47)$$

and

$$\begin{aligned} B : L^2(0, L) &\rightarrow L^2(\Omega), \\ u &\mapsto \mathbf{1}_\Omega(z)u. \end{aligned} \quad (1.48)$$

A straightforward computation, together with some integrations by parts, shows that

$$\begin{aligned} \Re_e \{\langle Az, z \rangle_Z\} &\leq 0, \quad z \in D(A), \\ \Re_e \{\langle \tilde{z}, A^* \tilde{z} \rangle_Z\} &\leq 0, \quad \tilde{z} \in D(A^*). \end{aligned} \quad (1.49)$$

Since A is a closed linear operator and $D(A)$ is dense in Z , according to Theorems 1.1 and 1.3, these latter inequalities imply that A is the infinitesimal generator of a linear strongly continuous semigroup of contractions on Z .

Let $\sigma : U \rightarrow U$ be defined by

$$\sigma(u)(x) = \tilde{\sigma}(u(x)), \quad \forall x \in \Omega,$$

where $\tilde{\sigma} : \mathbb{R} \rightarrow \mathbb{R}$.

Therefore, one has the following proposition:

Proposition 1.1

If $\tilde{\sigma}$ is bounded, then condition (1.42) is satisfied for (1.44).

Proof of Proposition 1.1:

Assume $\tilde{\sigma}$ is bounded by a positive value u_s , that is

$$|\tilde{\sigma}(u(x))| \leq u_s, \quad \forall x \in \Omega. \quad (1.50)$$

Note that if $\tilde{\sigma}$ is bounded, it implies that σ is also bounded as follows:

$$\|\sigma(u)\|_Z \leq Lu_s. \quad (1.51)$$

To prove that (1.41) holds, we follow a strategy similar to the one used in [64] or [76]. First note that

$$Z_0 := H_0^1(0, L) \supset D(A)$$

embeds compactly in Z by the Rellich-Kondrachov theorem (see [7, Theorem 9.16]). This set satisfies item (2)(a) of Theorem 1.9.

The operator A has a compact resolvent (see e.g., [23]), which implies that its spectrum consists only of eigenvalues. Moreover, A generates a linear strongly continuous semigroup of contractions, hence all the eigenvalues of the operator are located in the left half of the complex plane. In particular $1 \in \rho(A)$ and J_1 is invertible. Hence, there exists a unique solution Z to the equation $-(I_Z - A)z(x) = -B\sigma(u)$, where $u \in U$. This latter equation can be rewritten as follows

$$\begin{cases} z(x) + z'(x) + z''''(x) = -B\sigma(u), \\ z(0) = z(L) = z'(L) = 0. \end{cases} \quad (1.52)$$

The unique solution to this equation can be expressed compactly as follows

$$z = -J_1(B\sigma(u)). \quad (1.53)$$

Multiplying the first line of (1.52) by z and integrating between 0 and L leads to

$$\|z\|_{L^2(0,L)}^2 + \int_0^L xz'dx + \int_0^L xz''''dx = - \int_0^L B\sigma(u)zdx \quad (1.54)$$

Integrating by parts this latter inequality twice and using boundary condition in (1.44) lead to

$$\|z\|_{L^2(0,L)}^2 \leq -|z'(0)|^2 - \int_0^L B\sigma(u)zdx \quad (1.55)$$

Applying Young's inequality and using the fact that $\tilde{\sigma}$ is bounded, we obtain

$$\|z\|_{L^2(0,L)}^2 \leq \varepsilon_1 Lu_s^2 + \frac{1}{\varepsilon_1} \|z\|_{L^2(0,L)}^2, \quad (1.56)$$

where $\varepsilon_1 > 1$. Hence, $\|z\|_{L^2(0,L)}^2 \leq \frac{Lu_s^2}{\varepsilon_2}$, with $\varepsilon_2 := 1 - \frac{1}{\varepsilon_1}$.

Now, let us multiply the first line of (1.52) by zx and integrate between 0 and L . After performing some integrations by parts and using the boundary conditions in (1.44), we obtain

$$\begin{aligned} \frac{3}{2} \|z'\|_{L^2(0,L)}^2 &= \frac{1}{2} \|z\|_{L^2(0,L)}^2 - \int_0^L zx^2 dx - \int_0^L zx B\sigma(u) dx \\ &\leq \frac{1}{2} \|z\|_{L^2(0,L)}^2 + \frac{1}{2} \|z\|_{L^2(0,L)}^2 + \frac{L^3}{2} u_s \end{aligned}$$

Therefore, we have

$$\|z'\|_{L^2(0,L)}^2 \leq M, \quad (1.57)$$

where $M := \frac{L^3}{2} u_s + \frac{L u_s^2}{\varepsilon_2}$. By the Poincaré inequality, there is an equivalence between the norm $\|z'\|_{L^2(0,L)}$ and $\|z\|_{H_0^1(0,L)}$. Hence, using the expression (1.53), we can conclude that there exists a positive value c such that, for all $u \in U$

$$\|J_1(B\sigma(u))\|_{H_0^1(0,L)} \leq c. \quad (1.58)$$

Thus, if σ is bounded, the condition (1.41) is satisfied (and more precisely (1.42) in Remark 1.6) for the operator A defined in (1.46) and (1.47), and the operator B defined in (1.48). \square

Example 1.3

As a second illustration, consider the following nonlinear heat equation

$$\begin{cases} z_t(t, x) = z_{xx}(t, x) + \sin(z(t, x)) + u(t, x), & (t, x) \in \mathbb{R}_{\geq 0} \times [0, 1] \\ z(t, 0) = z(t, 1) = 0, & t \in \mathbb{R}_{\geq 0}, \\ z(0, x) = z_0(x), & x \in [0, 1]. \end{cases} \quad (1.59)$$

Setting $Z := L^2(0, 1)$ and $U = L^2(0, 1)$, system (1.59) can be written as in (1.26) denoting

$$\begin{aligned} A : D(A) \subset L^2(0, 1) &\rightarrow L^2(0, 1), \\ z &\mapsto z'' + \sin(z), \end{aligned} \quad (1.60)$$

where

$$D(A) := \{z \in H^2(0, 1), z(0) = z(1) = 0\}, \quad (1.61)$$

and

$$B := I_Z. \quad (1.62)$$

In Appendix B, the operator (1.60) is proved to be m -dissipative. Therefore, it generates a strongly continuous semigroup of contractions.

Let $\sigma : U \rightarrow U$ be defined by

$$\sigma(u)(x) = \tilde{\sigma}(u(x)), \quad x \in [0, L], \quad (1.63)$$

where $\tilde{\sigma} : \mathbb{R} \rightarrow \mathbb{R}$. Following a similar strategy than for the KdV example and using some inequalities proved in Appendix B, the assumption given in (1.41) is satisfied as soon as $\tilde{\sigma}$ is bounded.

1.2.2.2 Asymptotic stability

The second result refers to the global asymptotic stability of the closed-loop system defined by (1.32). Let $(W_I(t))_{t \geq 0}$ be the strongly continuous semigroup of contractions generated by $A - BB^*$, see (1.27)

Theorem 1.10 (Global asymptotic stability)

Assume that σ is a cone-bounded nonlinearity and that, for all z_0 in $D(A)$, there exists a unique strong solution to (1.32). Suppose also that $D(A_\sigma) = D(A)$ and that the operator A_σ generates a strongly continuous semigroup of contractions denoted by $t \mapsto W_\sigma(t)$ such that the two functions

$$t \mapsto \|W_\sigma(t)z_0\|_Z, \quad t \mapsto \|A_\sigma W_\sigma(t)z_0\|_Z$$

are non increasing, for all $z_0 \in D(A)$. Assume moreover that

1. for all z_0 in $D(A)$,

$$\lim_{t \rightarrow +\infty} \|W_I(t)z_0\|_Z = 0;$$

2. $D(A)$ equipped with the graph norm $\|\cdot\|_{D(A)} = \|\cdot\|_Z + \|A \cdot\|_Z$ is a Banach space which is compactly embedded in Z .

Then, the origin of the closed-loop system (1.32) is globally asymptotically stable.

Remark 1.8

Theorem 1.10 is a continuation of the work of [84] and [81]. As in Assumption 3, our result needs only the origin to be stabilizable with the feedback law $u = -B^*z$. However, unlike [81], we are considering that the open-loop system is nonlinear and that the cone-bounded feedback law is more general than sat_U . In [84] it is assumed the compactness of the operator B . This implies that the weak ω -limit set is nonempty and invariant. In this work, we assume an alternative property, that is $D(A)$ is compactly embedded in Z , which implies that the strong ω -limit is nonempty and invariant. Note that this property implies a stronger property for the open-loop system than the property assumed in [84]. However, the operator B does not require to be compact in this work, as assumed in [84].

1.2.3 Proofs of the results

1.2.3.1 Proof of Theorem 1.9: well-posedness

This section aims at proving Theorem 1.9. A Schauder fixed-point theorem will be used. Let us recall it.

Theorem 1.11 (Schauder fixed-point theorem ([21], Theorem B.17))

Let Z be a Banach space and $\mathcal{C} \subseteq Z$ be a convex and compact space. Then, every continuous mapping $f : Z \rightarrow \mathcal{C}$ admits a fixed-point.

The proof of Theorem 1.9 is given just below.

Proof of Theorem 1.9: First, note that $D(A) = D(A_\sigma)$ and A_σ is dissipative in Z . Indeed, for all $z, \tilde{z} \in D(A)$

$$\begin{aligned} \Re_e \{ \langle A_\sigma z - A_\sigma \tilde{z}, z - \tilde{z} \rangle_Z \} &= \Re_e \{ \langle Az - A\tilde{z}, z - \tilde{z} \rangle_Z \} \\ &\quad - \Re_e \{ \langle B\sigma(B^*z) - B\sigma(B^*\tilde{z}), z - \tilde{z} \rangle_Z \}, \\ &\leq - \Re_e \{ \langle \sigma(B^*z) - \sigma(B^*\tilde{z}), B^*(z - \tilde{z}) \rangle_U \}, \\ &\leq 0, \end{aligned} \tag{1.64}$$

where the dissipativity of A and the item 3 of Definition 1.10 have been used to obtain the last two inequalities.

Now, we split our proof into two cases.

First case: item 1 holds. In this case, [82, Lemma 2.1., Part IV, page 165] implies that A_σ is an m -dissipative operator.

From Theorem 1.3, the operator A_σ generates a strongly continuous semigroup of contractions on Z denoted by $(W_\sigma(t))_{t \geq 0}$. Moreover, from Theorem 1.4, it follows that

$$t \mapsto \|A_\sigma W_\sigma(t)z_0\|_Z \tag{1.65}$$

is non increasing. From item 2 of Definition 1.10, it holds $\sigma(0) = 0$ and $W_\sigma(t)0 = 0$. Therefore, the function

$$t \mapsto \|W_\sigma(t)z_0\|_Z \tag{1.66}$$

is a non-increasing function. This concludes the proof of Theorem 1.9 in the case where item 1 holds.

Second case: items 2a and 2b hold. Since A_σ is a dissipative operator, the operator $\tilde{J}_1 = (I_Z - A_\sigma)^{-1}$ exists and is continuous. Moreover, we have $D(\tilde{J}_1) = \text{Ran}(I_Z - A_\sigma)$.

In the second case, in order to apply Theorem 1.4, we must show that

$$Z = \text{Ran}(I_Z - A_\sigma).$$

The inclusion $\text{Ran}(I_Z - A_\sigma) \subset Z$ is obvious. Let us prove that

$$Z \subset \text{Ran}(I_Z - A_\sigma).$$

In other words, for \bar{z} in Z , we must show that there exists \tilde{z} in $D(A)$ such that

$$(I_Z - A)\tilde{z} = \bar{z} - B\sigma(B^*\tilde{z}).$$

Let $\mathcal{T} : Z \rightarrow D(A) \subseteq z_0$ be the mapping

$$\mathcal{T}(z) = J_1[\bar{z} - B\sigma(B^*z)].$$

Let \mathcal{C} be the set defined by

$$\mathcal{C} = \{z \in Z_0 \mid \|z\|_{Z_0} \leq N\},$$

where N comes from (1.41).

By assumption (item 2a of the statement of Theorem 1.9), the canonical injection from Z_0 to Z is compact. Thus, the set \mathcal{C} is pre-compact as a subset of Z and the closure in Z of \mathcal{C} is compact in Z . It is moreover convex since it is a ball of radius N centered at 0. From item 2b in the statement of Theorem 1.9, we compute, for all z in $D(A)$,

$$\begin{aligned} \|\mathcal{T}(z)\|_{Z_0} &= \|J_1[\bar{z} - B\sigma(B^*z)]\|_{Z_0}, \\ &\leq N. \end{aligned}$$

Hence, $\mathcal{T}(Z) \subseteq \mathcal{C}$. Employing Schauder fixed point theorem, it implies that there exists a unique solution to $\mathcal{T}(z) = z$ and thus to (1.32). Therefore, from [48, Theorem 4], it implies that A_σ is an m -dissipative operator. Hence, the result is obtained similarly to the first case. It concludes the proof of Theorem 1.9. \square

1.2.3.2 Proof of Theorem 1.10: asymptotic stability

The proof of Theorem 1.10 relies on the use of an infinite-dimensional version of the LaSalle's Invariance Principle stated in Theorem 1.5.

Before proving it, let us prove the following lemma, that links the attractivity in $D(A)$ and in Z .

Lemma 1.2

Given Z a Hilbert space, let $(W_\sigma(t))_{t \geq 0}$ be a strongly continuous semigroup of contractions on Z . Let $D(A)$ be dense in Z . Then, if for all $z_0 \in D(A)$, the following holds

$$\lim_{t \rightarrow +\infty} \|W_\sigma(t)z_0\|_Z = 0, \quad (1.67)$$

we have, for all $z_0 \in Z$,

$$\lim_{t \rightarrow +\infty} \|W_\sigma(t)z_0\|_Z = 0 \quad (1.68)$$

Proof of Lemma 1.2:

Note that the proof is inspired by [52]. Pick $z_0 \in Z$. Since $D(A)$ is dense in Z , for all positive value ε , there exists $\tilde{z}_0 \in D(A)$ such that

$$\|z_0 - \tilde{z}_0\|_Z \leq \frac{\varepsilon}{2}. \quad (1.69)$$

Since $(W_\sigma(t))_{t \geq 0}$ is a strongly continuous semigroup of contractions, the following holds, for all $t \geq 0$

$$\|W_\sigma(t)x_0 - W_\sigma(t)\tilde{z}_0\|_Z \leq \frac{\varepsilon}{2}. \quad (1.70)$$

Moreover, with (1.67), there exists $t^* := t^*(\varepsilon)$ such that, for all $\tilde{z}_0 \in D(A)$

$$\|W_\sigma(t)\tilde{z}_0\|_Z \leq \frac{\varepsilon}{2}, \quad \forall t \geq t^*. \quad (1.71)$$

Therefore, using a triangle inequality together with (1.70) and (1.71), one is able to prove that

$$\|W_\sigma(t)z_0\|_Z \leq \frac{\varepsilon}{2}, \quad \forall t \geq t^*. \quad (1.72)$$

This concludes the proof of Lemma 1.2. \square

Proof of Theorem 1.10:

The proof is divided into three steps. Given $z \in D(A)$, we first prove that the ω -limit set, denoted by $\omega(z)$, is compact and invariant for the nonlinear strongly continuous semigroup of contractions $(W_\sigma(t))_{t \geq 0}$. Then we prove that, for all initial conditions in $\omega(z)$, the solution to (1.32) converges to 0 in Z . Finally, it is proven that, for all initial conditions in $D(A)$, the solution to (1.32) converges to 0 in Z .

First step: Compactness and invariance of $\omega(z)$. For all Z in $D(A)$,

$$\begin{aligned} \|z\|_Z + \|A_\sigma z\|_Z &= \|z\|_Z + \|Az - B\sigma(B^*z)\|_Z, \\ &\geq c_1 \|B\sigma(B^*z)\|_Z + \|Az - B\sigma(B^*z)\|_Z, \\ &\geq \min\{1, c_1\} \|Az\|_Z, \end{aligned}$$

where the second inequality has been obtained from item 2 of Definition 1.10 and with $c_1 = \frac{1}{\|B\|_{\mathcal{L}(U,Z)} \|B^*\|_{\mathcal{L}(Z,U)}}$. This implies, for all z in $D(A)$,

$$\min\{1, c_1\} (\|z\|_Z + \|Az\|_Z) \leq (1 + c_1) (\|z\|_Z + \|A_\sigma z\|_Z).$$

Since by assumptions, for all z in $D(A)$, the two mappings $t \mapsto \|W_\sigma(t)z\|_Z$ and $t \mapsto \|A_\sigma W_\sigma(t)z\|_Z$ are nonincreasing, the former inequality implies

$$\|W_\sigma(t)z\|_{D(A)} \leq \frac{(1 + c_1)}{\min\{1, c_1\}} (\|z\|_Z + \|A_\sigma z\|_Z), \quad \forall t \geq 0.$$

The set $D(A)$ equipped with the graph norm is compactly embedded in Z , thus the positive orbit $\mathcal{O}^+(x)$ is precompact in Z . Therefore, from Theorem 1.5, for all z in $D(A)$, $\omega(z)$ is not empty, compact and invariant under the nonlinear semigroup $(W_\sigma(t))_{t \geq 0}$, i.e.,

$$W_\sigma(t)w \in \omega(z), \quad \forall (w, t) \in \omega(z) \times \mathbb{R}_+. \quad (1.73)$$

Second step: Asymptotic stability of the origin with initial conditions in $\omega(z)$. Let z be in $D(A)$. For all $t \geq 0$, due to the dissipativity of the operator A ,

$$\frac{1}{2} \frac{d}{dt} \|W_\sigma(t)z\|_Z^2 \leq -\Re_e \{ \langle \sigma(B^*W_\sigma(t)z), B^*W_\sigma(t)z \rangle_U \} \leq 0. \quad (1.74)$$

Since $\mathcal{C} := \text{clos}\{\mathcal{O}^+(x)\}$ is compact in Z and σ is continuous, the function $z \in \mathcal{C} \mapsto \Re_e\{\langle \sigma(B^*z), B^*z \rangle_U\} \in \mathbb{R}$ is uniformly continuous. Let

$$\begin{aligned} S : \mathbb{R}_{\geq 0} &\rightarrow \mathbb{R} \\ t &\mapsto S(t) := \Re_e\{\langle \sigma(B^*W_\sigma(t)z), B^*W_\sigma(t)z \rangle_U\}. \end{aligned} \quad (1.75)$$

The function $t \mapsto W_\sigma(t)z$ is continuous since $(W_\sigma(t))_{t \geq 0}$ is a strongly continuous semigroup of contractions. Moreover, by assumption, its time derivative, i.e. the function $t \mapsto A_\sigma W_\sigma(t)z$, is bounded. Therefore, the function $t \mapsto W_\sigma(t)z_0$ is uniformly continuous. Hence, S is uniformly continuous as a combination of two uniformly continuous functions.

From (1.74), we have, for all $t \geq 0$,

$$\frac{1}{2}\|W_\sigma(t)z\|_Z^2 - \frac{1}{2}\|z\|_Z^2 \leq -\int_0^t S(s)ds. \quad (1.76)$$

Or, rearranging terms, it yields, for all $t \geq 0$,

$$\int_0^t S(s)ds \leq \frac{1}{2}\|z\|_Z^2 - \frac{1}{2}\|W_\sigma(t)z\|_Z^2 \leq \frac{1}{2}\|z\|_Z^2. \quad (1.77)$$

Since S takes positive values, it yields

$$0 \leq \lim_{t \rightarrow +\infty} \int_0^t S(s)ds < \infty. \quad (1.78)$$

From Barb alat's Lemma, we get

$$\lim_{t \rightarrow +\infty} S(t) = 0. \quad (1.79)$$

Thus, from the definition of $\omega(z)$,

$$\Re_e\{\langle \sigma(B^*w), B^*w \rangle_U\} = 0, \quad \forall w \in \omega(z). \quad (1.80)$$

From item 1 in Definition 1.10 of the cone-bounded nonlinearity and (1.73),

$$B^*W_\sigma(t)w = 0, \quad \forall w \in \omega(z), \quad \forall t \geq 0. \quad (1.81)$$

Hence, it implies that for all $w \in \omega(z)$,

$$W_\sigma(t)w = W_I(t)w, \quad \forall t \in \mathbb{R}_+.$$

Therefore, from Assumption 1 of Theorem 1.10, we have, for all $w \in \omega(z)$,

$$\lim_{t \rightarrow +\infty} \|W_\sigma(t)w\|_Z = 0. \quad (1.82)$$

Third step: Asymptotic stability of the origin with initial conditions in $D(A)$.

Let $z \in D(A)$. The aim of this step is to prove that, for all $z \in D(A)$,

$$\lim_{t \rightarrow +\infty} \|W_\sigma(t)z\|_Z = 0. \quad (1.83)$$

Note that, from (1.74), the function $t \mapsto \|W_\sigma(t)z\|_Z^2$ is non-increasing and lower-bounded. Hence, there exists $V_\infty^*(z) \in \mathbb{R}$ such that

$$\lim_{t \rightarrow +\infty} \|W_\sigma(t)z\|_X^2 = V_\infty^*(x) \geq 0. \quad (1.84)$$

Two cases may occur:

1. If $V_\infty^*(z) = 0$, then $\omega(z) = \{0\}$. It means that (1.83) holds;
2. If $V_\infty^*(z) \neq 0$, then $\{0\} \notin \omega(z)$. In this case, (1.83) does not hold.

We will argue by contradiction by assuming that the second item holds and by proving that this case cannot occur. Thus, we assume that

$$V_\infty^*(z) \neq 0. \quad (1.85)$$

Let $w \in \omega(z)$. From (1.82), for all $w \in \omega(z)$, there exists $t(w) > 0$ such that

$$\|W_\sigma(t(w))w\|_Z \leq \frac{1}{6}\|w\|_Z. \quad (1.86)$$

Since $W_\sigma(t(w))$ is a continuous operator and since $\{0\} \notin \omega(z)$, there exists a positive value $\varepsilon_1(w)$ such that, for all $\tilde{z} \in \mathcal{B}(w, \varepsilon_1(w))$,

$$\|W_\sigma(t(w))\tilde{z} - W_\sigma(t(w))w\|_Z \leq \frac{1}{6}\|w\|_Z. \quad (1.87)$$

Therefore, for all $\tilde{z} \in \mathcal{B}(w, \varepsilon_1(w))$,

$$\|W_\sigma(t(w))\tilde{z}\|_Z \leq \|W_\sigma(t(w))\tilde{z} - W_\sigma(t(w))w\|_Z + \|W_\sigma(t(w))w\|_Z \leq \frac{1}{3}\|w\|_Z. \quad (1.88)$$

By reducing $\varepsilon_1(w)$ if needed, we may assume that $\varepsilon_1(w) \leq \frac{1}{3}\|w\|_Z$. Hence, for all $z \in \mathcal{B}(w, \varepsilon_1(w))$,

$$\|w\|_Z - \|\tilde{z}\|_Z \leq \|\tilde{z} - w\|_Z \leq \frac{1}{3}\|w\|_Z. \quad (1.89)$$

Therefore, for all $\tilde{z} \in \mathcal{B}(w, \varepsilon_1(w))$,

$$\|w\|_Z \leq \frac{3}{2}\|\tilde{z}\|_Z, \quad (1.90)$$

and with (1.88), for all $\tilde{z} \in \mathcal{B}(w, \varepsilon_1(w))$,

$$\|W_\sigma(t(w))\tilde{z}\|_Z \leq \frac{1}{2}\|\tilde{z}\|_Z. \quad (1.91)$$

The family $\bigcup \{\mathcal{B}(w, \varepsilon_1(w)), w \in \omega(z)\}$ is a cover by open subsets of $\omega(z)$. Since $\omega(z)$ is a compact set, we can extract a finite cover which we index as follows

$$\omega(z) \subset \bigcup_{i=1}^{N_1} \{\mathcal{B}(w_{1i}, \varepsilon_1(w_{1i}))\}, \quad (1.92)$$

where (w_{1i}) 's are in $\omega(z)$ and for a suitable positive integer N_1 and (1.91) has been used.

By considering

$$t^* := \max_{i \in \{1, \dots, N_1\}} t(w_{1i}), \quad (1.93)$$

together with the fact that the function $t \mapsto \|W_\sigma(t)z\|_Z$ is non increasing for any $\tilde{z} \in \omega(z) \subset D(A)$, we have, for all $\tilde{z} \in \omega(z)$,

$$\|W_\sigma(t^*)\tilde{z}\|_Z \leq \|W_\sigma(t(w_{1i}))\tilde{z}\|_Z \leq \frac{1}{2}\|\tilde{z}\|_Z, \quad (1.94)$$

where $i \in \{1, \dots, N_1\}$ is selected such that $\tilde{z} \in \mathcal{B}(w_{1i}, \varepsilon_1(w_{1i}))$ and (1.94) has been used.

Since the functions $w \mapsto W_\sigma(t^*)w$ and $V : w \rightarrow V(w) = \|w\|_Z^2$ are continuous and since $\{0\} \notin \omega(z)$, for all $w \in \omega(z)$, there exists $\varepsilon_2(w) > 0$ such that, for all $\tilde{z} \in \mathcal{B}(w, \varepsilon_2(w))$,

$$\begin{aligned} |V(\tilde{z}) - V(w)| &\leq \frac{1}{5}V(w), \\ |V(W_\sigma(t^*)\tilde{z}) - V(W_\sigma(t^*)w)| &\leq \frac{1}{4}V(w). \end{aligned} \quad (1.95)$$

Therefore, with (1.94), for all $\tilde{z} \in \mathcal{B}(w, \varepsilon_2(w))$,

$$\begin{aligned} V(W_\sigma(t^*)\tilde{z}) &\leq V(W_\sigma(t^*)w) + \frac{1}{4}V(w), \\ &\leq \frac{1}{4}V(w) + \frac{1}{4}V(w), \\ &\leq \frac{1}{2}V(w). \end{aligned} \quad (1.96)$$

Moreover, the first inequality in (1.95) yields for all $\tilde{z} \in \mathcal{B}(w, \varepsilon_2(w))$,

$$V(w) \leq \frac{6}{5}V(\tilde{z}). \quad (1.97)$$

Finally, with (1.96), it follows, for all $\tilde{z} \in \mathcal{B}(w, \varepsilon_2(w))$,

$$V(W_\sigma(t^*)\tilde{z}) \leq \frac{3}{5}V(\tilde{z}). \quad (1.98)$$

The family $\bigcup \left\{ \mathcal{B}\left(w, \frac{\varepsilon_2(w)}{2}\right), w \in \omega(z) \right\}$ is a cover by open subsets of $\omega(z)$. Since $\omega(z)$ is a compact set, there exists $(w_{21}, \dots, w_{2N_2})$ in $\omega(z)^{N_2}$ such that

$$\omega(z) \subset \bigcup_{i=1}^{N_2} \left\{ \mathcal{B}\left(w_{2i}, \frac{\varepsilon_2(w_{2i})}{2}\right) \right\}. \quad (1.99)$$

Let us pick

$$\varepsilon_{2m} := \min_i \varepsilon_2(w_{2i}). \quad (1.100)$$

Note that, if we assume (1.85) holds, then it implies that there exists $t_1 > 0$ such that, for all $t \geq t_1$,

$$\|W_\sigma(t)z\|_X^2 - V_\infty^*(z) \leq \frac{1}{3}V_\infty^*(z). \quad (1.101)$$

and

$$\|w - W_\sigma(t_1)z\|_Z \leq \frac{\varepsilon_{2m}}{2}, \quad (1.102)$$

where $w \in \omega(z)$.

Since $w \in \omega(z)$, there exists $i \in \{1, \dots, N_2\}$ such that $w \in \mathcal{B}\left(w_{2i}, \frac{\varepsilon_2(w_{2i})}{2}\right)$. Therefore,

$$\begin{aligned} \|w_{2i} - W_\sigma(t_1)z\|_Z &\leq \|w_{2i} - w\|_Z + \|w - W_\sigma(t_1)z\|_Z, \\ &\leq \frac{\varepsilon_2(w_{2i})}{2} + \frac{\varepsilon_{2m}}{2}, \\ &\leq \varepsilon_2(w_{2i}). \end{aligned} \quad (1.103)$$

Since $W_\sigma(t_1)z \in \mathcal{B}(w_{2i}, \varepsilon_2(w_{2i}))$, Equation (1.98) together with the fact that $W_\sigma(t_1 + t^*)z = W_\sigma(t^*)W_\sigma(t_1)z$ imply,

$$\|W_\sigma(t_1 + t^*)z\|_X^2 = V(W_\sigma(t^*)W_\sigma(t_1)z) \leq \frac{3}{5}\|W_\sigma(t_1)z\|_Z^2. \quad (1.104)$$

Therefore, with (1.98) and (1.101), it follows, for all $t \geq t_1$

$$\begin{aligned} \|W_\sigma(t + t^*)z\|_Z^2 - V_\infty^*(z) &\leq \frac{3}{5}\|W_\sigma(t_1)z\|_Z^2 - V_\infty^*(z) \\ &\leq \frac{3}{5}\left(V_\infty^*(z) + \frac{1}{3}V_\infty^*(z)\right) - V_\infty^*(z) \\ &\leq -\frac{1}{3}V_\infty^*(z). \end{aligned} \quad (1.105)$$

Thus, we have

$$\|W_\sigma(t + t^*)z\|_Z^2 \leq \frac{2}{3}V_\infty^*(z) < V_\infty^*(z). \quad (1.106)$$

Since the function $t \mapsto \|W(t)z\|_Z^2$ is nonincreasing, we obtain a contradiction with (1.84). Therefore

$$V_\infty^*(z) = 0.$$

This concludes the proof of the global attractivity of the origin. The Lyapunov stability holds by assumption. This concludes the proof of Theorem 1.10. \square

1.2.4 Application to a linear Korteweg-de Vries equation

In this section, we illustrate Theorems 1.9 and 1.10 with the linear Korteweg-de Vries equation as considered in Example 1.2. In addition, we run some simulations.

Let us note that $B^* : z \in Z \mapsto z|_\Omega \in U$. It one picks the control such that, for all $(t, x) \in [0, \infty) \times [0, L]$,

$$u(t, x) = -B^*z(t, x) := -z(t, x)|_\Omega,$$

then, the origin of (1.44) is $L^2(0, L)$ -globally asymptotically stable (see e.g., [11] or [79] for a proof of such a result). The stabilizability assumption of Theorem 1.10 is satisfied.

Now, let us tackle the case where the feedback law is bounded with the following operator defined, for all $(t, x) \in \mathbb{R}_{\geq 0} \times [0, L]$

$$\sigma : u \in U \mapsto \sigma(u) = \tilde{\sigma}(u)(t, x), \quad (1.107)$$

where $\tilde{\sigma}$ is the function introduced in (1.30). Due to item 4 of Example 1.1, it is a cone-bounded nonlinearity. This particular cone-bounded nonlinearity is illustrated by Figure 1.1.

The feedback law under consideration is as follows

$$u(z) = -B\sigma(B^*z) = -\mathbf{1}_\Omega\sigma(z|_\Omega) = -\sigma(\mathbf{1}_\Omega z). \quad (1.108)$$

Note that with such a feedback law the results of [84] cannot be applied since the function $u \in U \mapsto \sigma(u) \in U$ is not globally Lipschitz.

As stated in Example (1.2), it is known that the conditions of Theorem 1.9 are satisfied. Therefore, Theorem 1.9 applies. Thus, the operator

$$\begin{aligned} A_\sigma : D(A_\sigma) = D(A) \subset L^2(0, L) &\rightarrow L^2(0, L), \\ w &\mapsto -w' - w' - \sigma(\mathbf{1}_\Omega z) \end{aligned} \quad (1.109)$$

generates a strongly continuous semigroup of contractions.

Moreover, using Lemma A.1 given in Appendix A, all the items of Theorem 1.10 are satisfied. Hence, Theorem 1.10 applies and one can conclude that the origin of (1.44) with $u = -\sigma(\mathbf{1}_\Omega z)$ is globally asymptotically stable.

Using a numerical scheme inspired by [70], we performed some numerical simulations. We note z the solution to (1.44) with (1.116) and \tilde{z} the solution to

$$\begin{cases} \tilde{z}_t(t, x) + \tilde{z}_x(t, x) + \tilde{z}_{xxx}(t, x) + \mathbf{1}_\Omega \tilde{z}(t, x) = 0, & (t, x) \in \mathbb{R}_{\geq 0} \times (0, L) \\ \tilde{z}(t, 0) = \tilde{z}(t, L) = \tilde{z}_x(t, L) = 0, & t \in \mathbb{R}_{\geq 0} \\ \tilde{z}(0, z) = \tilde{z}_0(z). \end{cases} \quad (1.110)$$

This latter equation is referred to as the Korteweg-de Vries with a linear feedback law.

We pick $z(0, x) = \tilde{z}(0, x) = 1 - \cos(x)$ and $L = 2\pi$ which is a critical case for the stability of the linear Korteweg-de Vries equation as it is reviewed in [77]. Let us choose $\Omega = [\frac{1}{3}L, \frac{2}{3}L]$. Figure 2.10 illustrates the solution to the system (1.44) with (1.116). We check on the simulation the origin of this equation is attractive. Figure 1.3 illustrates the solution to the system (1.110). It can be checked that the stabilizability assumption of Theorem (1.10) is satisfied. Figure 1.4 illustrates the control $u(t, x) = \sigma(\mathbf{1}_\Omega z)(t, x)$ with respect to the time and the space. We can check that the feedback law is bounded by the constant $u_s = 1.5$. Finally, Figure 1.5 illustrates the time-evolution of the Lyapunov functions $\|z\|_{L^2(0, L)}^2$ and $\|\tilde{z}\|_{L^2(0, L)}^2$. Note that the convergence in $L^2(0, L)$ of \tilde{z} is faster than the convergence in $L^2(0, L)$ of z .

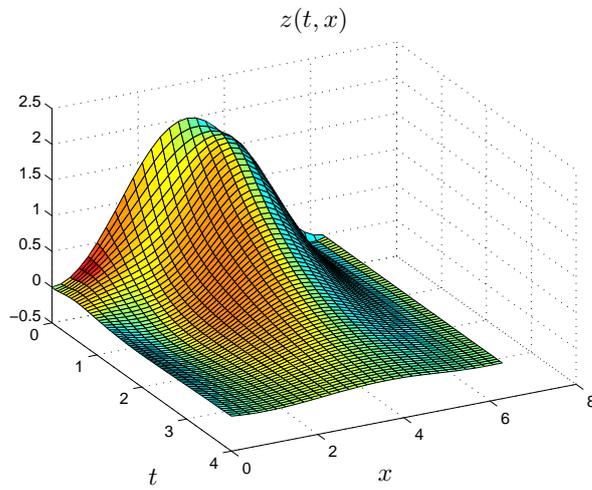


Figure 1.2: Solution $z(t, x)$ with the control $u(t, x) = \sigma(\mathbf{1}_\Omega z)(t, x)$ where $u_s = 1.5$.

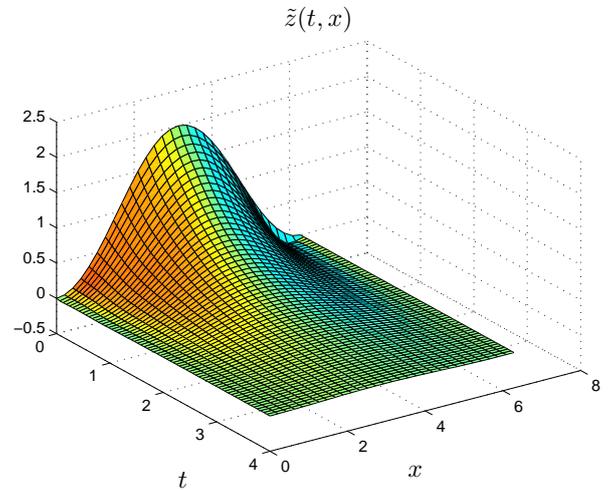


Figure 1.3: Solution $z(t, x)$ with the control $u(t, x) = \sigma(\mathbf{1}_\Omega z)(t, x)$ where $u_s = 1.5$.

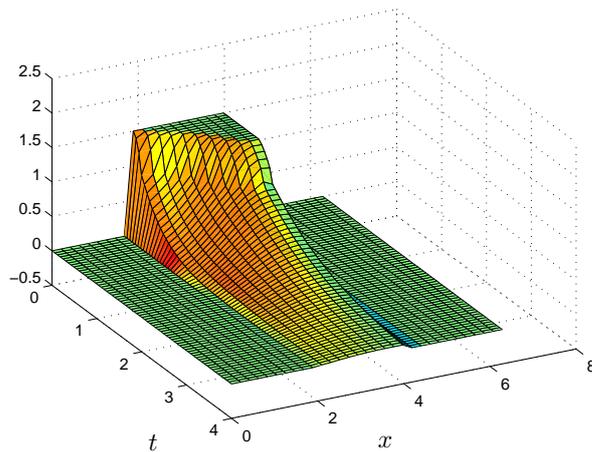


Figure 1.4: Control $u(t, x) = \sigma(\mathbf{1}_\Omega z)(t, x)$ where $u_s = 1.5$.

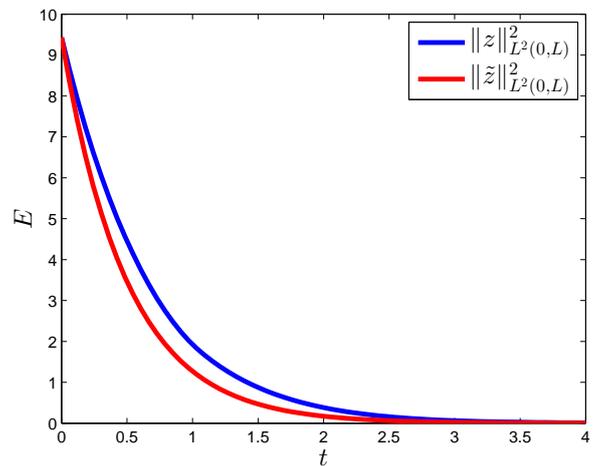


Figure 1.5: Time-evolution of the Lyapunov functions $\|z\|_{L^2(0,L)}^2$ and $\|\tilde{z}\|_{L^2(0,L)}^2$

1.2.5 Application to a nonlinear heat equation

In this section, we illustrate Theorems 1.9 and 1.10 with the linear nonlinear heat equation as considered in Example 1.2.

Let us note that $B^* : z \in Z \mapsto z \in U$. Setting $u(t, x) = -B^*z(t, x) := -z(t, x)$ makes the origin of (1.59) $L^2(0, 1)$ -globally asymptotically stable. Indeed, focus on the following Lyapunov function

$$V(z) = \int_0^1 z(t, x)^2 dx. \quad (1.111)$$

Its derivative along (1.59) yields

$$\frac{d}{dt}V(z) = \int_0^1 z(t, x)z_{xx}(t, x)dx + \int_0^1 \sin(z)(t, x)z(t, x)dx - \int_0^1 z(t, x)^2 dx \quad (1.112)$$

Performing some integrations by parts and using a Poincaré inequality leads to

$$\frac{d}{dt}V(z) \leq - \int_0^1 z_x(t, x)^2 dx + \frac{4}{\pi^2} \int_0^1 z_x(t, x)^2 dx - \int_0^1 z(t, x)^2 dx \quad (1.113)$$

Hence, we have

$$\frac{d}{dt}V(z) \leq -V(z). \quad (1.114)$$

Therefore, the stabilizability assumption of Theorem 1.10 is satisfied.

Now, let us tackle the case where the feedback law is bounded with the following operator defined, for all $(t, x) \in \mathbb{R}_+ \times [0, L]$

$$\sigma : u \in U \mapsto \sigma(u) = \tilde{\sigma}(u)(t, x), \quad (1.115)$$

where $\tilde{\sigma}$ is the function has been introduced in (1.30). Due to item 4 of Example 1.1, it is a cone-bounded nonlinearity. This particular cone-bounded nonlinearity is illustrated by Figure 1.1.

The feedback law under consideration is as follows

$$u = -B\sigma(B^*z) = -\sigma(z). \quad (1.116)$$

Note that with such a feedback law neither the results of [84], nor the ones of [81] cannot be applied since we are considering a nonlinear operator A .

As stated in Example 1.3, it is known that the conditions of Theorem 1.9 are satisfied. Therefore, Theorem 1.9 applies. Thus, the operator

$$A_\sigma : D(A_\sigma) = D(A) \subset L^2(0, L) \rightarrow L^2(0, L), \quad (1.117)$$

$$z \mapsto z'' + \sin(z) - \sigma(z)$$

generates a strongly continuous semigroup of contractions.

Moreover, using the Lemma B.1 given in the Appendix B.2, all the items of Theorem 1.10 are satisfied. Hence, Theorem 1.10 applies and one can conclude that the origin of (1.59) with $u = -\sigma(z)$ is globally asymptotically stable.

1.3 Conclusion

In this chapter, the analysis of a stabilizing controller modified via a cone-bounded nonlinearity has been tackled with various techniques. The well-posedness and the Lyapunov stability are proved using a Schauder fixed-point theorem and some nonlinear semigroup results. Finally, assuming a stabilizability property and precompactness of the trajectories of the solution, an infinite-dimensional version of the LaSalle's Invariance Principle has been used to conclude the asymptotic stability of the origin. These results have been illustrated on a linear Korteweg-de Vries equation and a nonlinear heat equation.

A future research line could be the study of the case of unbounded control operators, which refers to boundary controls. To the best of our knowledge, very few papers deal with such systems in the context of cone-bounded feedback laws. Two of these papers ([44] or [81]) focus on abstract control system and the well-posedness of the closed-loop system with cone-bounded feedback law is assumed. Therefore, even well-posedness is an issue for this topic. Note however that [76] tackles the well-posedness using an approach similar to Section 1.2.2.1.

The next chapter deals with a nonlinear Korteweg-de Vries equation with a saturating distributed control. We will see that the nonlinear semigroup framework might be harder to apply to this nonlinear partial differential equation. Hence, some other tools borrowed from functional analysis and control theory will be presented in order to tackle the global asymptotic stability of the origin of a Korteweg-de Vries equation with saturating distributed control.

Global stabilization of a Korteweg-de Vries equation with saturating distributed control

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In Chapter 1, a method for the analysis of the asymptotic stability of nonlinear abstract systems with a cone-bounded controller has been provided. In order to use classical nonlinear semigroup theory results, the operator describing the open-loop system without control has been assumed to be the infinitesimal generator of a strongly continuous semigroup of contraction.

However, proving this property for a nonlinear partial differential equation might be hard. In fact, for a large class of nonlinear systems, applying classical nonlinear semigroup theory might fail. In such cases, one needs to study the particular structure of the equation. This chapter will aim at studying the nonlinear partial differential equation, namely a Korteweg-de Vries equation, in the context of saturated controllers.

The Korteweg-de Vries equation (KdV for short) is a mathematical model of waves on shallow water surfaces. Its controllability and stabilizability properties have been deeply studied with no constraints on the control, as reviewed in [11, 21, 78].

In the literature, there are some methods stabilizing the KdV equation with boundary [16, 14, 59] or distributed controls [73, 71]. Here we focus on the distributed control case. As proven in [73, 71, 79], a distributed localized feedback law, i.e. a feedback that acts only on a part of the spatial domain, stabilizes globally the origin of the closed-loop system.

In this chapter, two particular cone-bounded nonlinearities will be studied. They are referred to as saturation functions, which is a particular case of cone-bounded nonlinearities. The interested reader can refer to [94] for an introduction to this topic for finite-dimensional systems.

We obtain two main results. Our first main result states that the KdV equation in closed-loop with a saturated control is well-posed. Our second main result states that the origin of the KdV equation in closed loop with a saturated control is globally asymptotically stable.

This chapter is organized as follows. In Section 2.1, the history of the Korteweg-de Vries is provided and some stability results of the equation without control are stated. Section 2.2 deals with the stabilization of the Korteweg-de Vries with a distributed control. Some existing results are provided. In addition, our contributions are given together with their proofs. In Section 2.3, some existings results and some of our results are illustrated with numerical simulations for which we provide a numerical scheme. Section 2.4 collects some concluding remarks and possible research lines.

The contributions of this chapter, collected from Section 2.2.2 until the end, are based on two of our papers. The first one [61] has been published in a peer-reviewed conference (10th IFAC Symposium on Nonlinear Control Systems). The second one [62] has been published in the SIAM Journal on Control and Optimization.

2.1 About the Korteweg-de Vries equation

2.1.1 History

In 1834 John Scott Russell, a Scottish naval engineer, was observing the Union Canal in Scotland when he unexpectedly witnessed a very special physical phenomenon. He saw a particular wave traveling through this channel without losing its shape or velocity. He was so captivated by this event that he focused his attention on these waves for several years and asked the mathematical community to find a specific mathematical model describing them.

"I was observing the motion of a boat wich rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour,

preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in windings of the channel. Such, in the month of August 1834, was my first chance interview with a singular and beautiful phenomenon which I have called the Wave of Translation."

We have to wait until 1877 and 1895 before obtaining a model of such a phenomena. The first model is due to Boussinesq and the second one to Korteweg and de Vries. Nowadays, this latter model is more used to describe this phenomena than the Boussinesq model.

The original form of the Korteweg-de Vries equation is written as follows

$$\frac{1}{\sqrt{gh_0}}\eta_t + \eta_x + \frac{h_0^2}{5}\eta_{xxx} + \frac{3}{2h_0}\eta\eta_x = 0, \quad \forall t \geq 0, \forall x \in [0, L_c] \quad (2.1)$$

where η stands for the height of the wave. The terms g , h_0 and L_c denote the gravitational constant the water depth and the canal length, respectively. The derivation of this equation can be found in the Whitham's book [98] which explains also the physical motivation of such an equation.

The KdV equation has several connections to physical problems. Indeed, it can model a lot of physical phenomena, as, for instance, shallow-water waves with weakly non-linear restoring forces or ion acoustic waves in a plasma.

It has also received considerable attention in recent decades in the mathematics community. For instance, its well-posedness and its controllability properties have been intensively studied. Since our contributions deal with stability results, the next section is about stability properties of a KdV equation.

2.1.2 Stability properties of the KdV equation without control

After a suitable scaling, the KdV equation becomes

$$z_t + z_x + z_{xxx} + zz_x = 0, \quad \forall t \geq 0, \forall x \in [0, L], \quad (2.2)$$

where L is a positive value.

The aim of this chapter is to study the case where the boundary conditions are as follows:

$$z(t, 0) = z(t, L) = z_x(t, L) = 0, \quad \forall t \geq 0. \quad (2.3)$$

Hence, the Cauchy problem that will be considered is the following

$$\begin{cases} z_t + z_x + z_{xxx} + zz_x = 0, & (t, x) \in \mathbb{R}_{\geq 0} \times [0, L] \\ z(t, 0) = z(t, L) = z_x(t, L) = 0, & t \in \mathbb{R}_{\geq 0} \\ z(0, x) = z_0(x) \in L^2(0, L), & x \in [0, L] \end{cases} \quad (2.4)$$

Note that the origin is an equilibrium point of this equation. We want to study the stability of this point in the space $L^2(0, L)$.

Before studying the stabilization problem, the case without control should be studied. Indeed, most of the time, the control design depends crucially on the stability of the equilibrium point for the open-loop system.

In the remaining parts of this section, an analysis of the asymptotic stability of the linearized version of the equation around the origin will be provided. Then, some results for stability of the nonlinear equation will be given.

2.1.2.1 Linearized KdV equation

The linearized version of (2.4) around the origin is given by the following equation

$$\begin{cases} z_t + z_x + z_{xxx} = 0, \\ z(t, 0) = z(t, L) = z_x(t, L) = 0, \\ z(0, x) = z_0(x) \in L^2(0, L). \end{cases} \quad (2.5)$$

Note that we know from Chapter 1 that there exists a unique mild solution $z \in C(0, \infty; L^2(0, L))$ to this equation. Moreover, applying Theorem 1.4, we know that the origin of this equation is Lyapunov stable. However, we do not know yet if it is attractive.

Let us perform an energy analysis. Consider the energy

$$E(t) := \frac{1}{2} \|z(t, \cdot)\|_{L^2(0, L)}^2. \quad (2.6)$$

After performing some integrations by parts, its time derivative along (2.5) yields

$$\frac{d}{dt} E(t) = -\frac{1}{2} |z_x(t, 0)|^2. \quad (2.7)$$

From this latter inequality, we can conclude that the energy is nonincreasing. However, we can conclude nothing on the long-time behavior of the energy.

In fact, it depends crucially on the length L . If L belongs to a critical length set that is defined by

$$\mathcal{N} := \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}} \mid k, l \in \mathbb{N}^* \right\}, \quad (2.8)$$

then the energy will remain equal to the initial condition.

Indeed, when L belongs to \mathcal{N} , an additional boundary condition appears when considering some initial conditions. Here is the statement of this result.

Theorem 2.1 ([77])

If $L \in \mathcal{N}$ defined in (2.8), then, for some initial conditions $z_0 \in L^2(0, L)$, there exists solution

to (2.5) which satisfies

$$\begin{cases} z_t + z_x + z_{xxx} = 0, & (t, x) \in \mathbb{R}_{\geq 0} \times [0, L], \\ z(t, 0) = z(t, L) = 0, & t \in \mathbb{R}_{\geq 0}, \\ z_x(t, L) = z_x(t, 0) = 0, & t \geq 0, \\ z(0, x) = z_0(x) \in L^2(0, L), & x \in [0, L]. \end{cases} \quad (2.9)$$

Therefore, if $L \in \mathcal{N}$, for some initial conditions, (2.7) becomes

$$\frac{d}{dt}E(t) = 0. \quad (2.10)$$

Then, in the case where $L \in \mathcal{N}$, we have, for all $t \geq 0$

$$\|z(t, \cdot)\|_{L^2(0, L)} = \|z_0\|_{L^2(0, L)} \quad (2.11)$$

Hence, for all positive values L , the origin of (2.5) is Lyapunov stable, but not attractive. A way to stabilize the origin of (2.5) is to use a control.

Remark 2.1

As it has been mentionned, the set \mathcal{N} defined in (2.8) has been introduced in [77]. However, this paper did not aim at proving stability of the origin of (2.4), but aimed at proving the controllability of the equation with a control located at the right Neumann boundary condition. This system is given by

$$\begin{cases} z_t + z_x + z_{xxx} = 0, & (t, x) \in \mathbb{R}_{\geq 0} \times [0, L], \\ z(t, 0) = z(t, L) = 0, & t \in \mathbb{R}_{\geq 0}, \\ z_x(t, L) = u(t), & t \in \mathbb{R}_{\geq 0}, \\ z(0, x) = z_0(x) \in L^2(0, L), & x \in [0, L] \end{cases} \quad (2.12)$$

where u stands for the control.

The author of [77] proved that the system is exactly controllable¹ if and only if $L \notin \mathcal{N}$.

In some cases, the linearized version of a system may have weaker property than the nonlinear systems. As it will be noticed in the next section, this is the case of the Korteweg-de Vries equation.

2.1.2.2 Nonlinear system

As it has been noticed in Chapter 1 (more precisely by Theorem 1.4), if an operator associated to a partial differential equation is proved to be m -dissipative, then the Lyapunov stability of

¹Given a positive value T , the system (2.12) is said to be exactly controllable in time T if, for every $z_0, z_T \in L^2(0, L)$, there exists a control $u \in L^2(0, T)$ such that the solution to (2.12) satisfies $z(T, \cdot) = z_T(\cdot)$.

the origin of this system is ensured. Hence, before studying the asymptotic stability of the nonlinear system (2.4), we would like to see whether it is possible to apply results provided in Chapter 1. The operator associated to (2.4) is

$$A_{\text{NL}} : D(A_{\text{NL}}) \subset L^2(0, L) \rightarrow L^2(0, L) \quad (2.13)$$

$$z \mapsto -z' - z''' - zz'$$

with

$$D(A_{\text{NL}}) := \{z \in H^3(0, L), z(0) = z(L) = z'(L) = 0\}. \quad (2.14)$$

In order to apply Theorem 1.4, let us check whether the operator A_{NL} is dissipative.

$$\begin{aligned} \langle A_{\text{NL}}z - A_{\text{NL}}\tilde{z}, z - \tilde{z} \rangle_{L^2(0, L)} &:= - \int_0^L (z'(x) - \tilde{z}'(x)) \overline{(z(x) - \tilde{z}(x))} dx \\ &\quad - \int_0^L (z'''(x) - \tilde{z}'''(x)) \overline{(z(x) - \tilde{z}(x))} dx \\ &\quad - \int_0^L (z(x)z'(x) - \tilde{z}(x)\tilde{z}'(x)) \overline{(z(x) - \tilde{z}(x))} dx, \end{aligned} \quad (2.15)$$

where $z, \tilde{z} \in D(A_{\text{NL}})$.

However, it is difficult to determine the sign of the term

$$\int_0^L (z(x)z'(x) - \tilde{z}(x)\tilde{z}'(x)) \overline{(z(x) - \tilde{z}(x))} dx.$$

Hence, applying basic results of nonlinear semigroup theory might be hard for the Korteweg-de Vries equation given in (2.4). Therefore, other mathematical tools than semigroup theory will be used in this chapter to achieve the proof of the well-posedness and of the asymptotic stability of a KdV equation with a saturating distributed control.

Some researchers focused on the nonlinear Korteweg-de Vries to obtain an asymptotic stability results for the origin of the equation. It is worth to mention that there is no hope to obtain global asymptotic stability of the origin for (2.4). Indeed, in [28], an equilibrium with arbitrary large amplitude is built.

Recently, researchers proved the local asymptotic stability of the origin of (2.17)². They aimed at generalizing some results from the finite-dimensional framework to obtain similar results for the Korteweg-de Vries equation (2.4). A result has been obtained for the particular lengths $L = 2\pi$ and $L = 2\pi\sqrt{\frac{7}{3}}$, which belong to the critical length set (2.8). It is written just below.

Theorem 2.2 (Local asymptotic stability of the origin with $L = 2\pi$ and $L = 2\pi\sqrt{\frac{7}{3}}$ [20, 91])

Let us assume that $L = 2\pi$ or $L = 2\pi\sqrt{\frac{7}{3}}$. Then the origin is (locally) asymptotically stable for (2.4).

²The origin of (2.17) is said to be *locally asymptotically stable* if there exists a positive value r such that, for any initial condition $z_0 \in L^2(0, L)$ satisfying $\|z_0\|_{L^2(0, L)} \leq r$, the origin is Lyapunov stable and the state z converges to 0 as t goes to infinity

Surprisingly, the stability result for the origin of (2.4) is better than for its linearized version.

Remark 2.2

In [12] and [15], the controllability of (2.4) has been studied. The system under consideration in these papers is

$$\begin{cases} z_t + z_x + z_{xxx} + zz_x = 0, & (t, x) \in \mathbb{R}_{\geq 0} \times [0, L], \\ z(t, 0) = z(t, L) = 0, & t \in \mathbb{R}_{\geq 0}, \\ z_x(t, L) = u(t), & t \in \mathbb{R}_{\geq 0}, \\ z(0, x) = z_0(x) \in L^2(0, L), & x \in [0, L]. \end{cases} \quad (2.16)$$

The authors of [15] proved that, for every length L , this system is locally exactly controllable for large time ³.

However, these results are only local. With a distributed control, it is possible to make the origin of the closed-loop system globally asymptotically stable. Next sections will consider the case of the Korteweg-de Vries equation with a distributed control. Two cases will be considered: the case where the control is not saturated and the case where the control is saturated.

2.2 Stabilization of a Korteweg-de Vries equation with a distributed control

2.2.1 Case where the control has no constraint: existing results

We suppose that (2.4) has a distributed control. Hence, it can be written as follows

$$\begin{cases} z_t + z_x + z_{xxx} + zz_x + u = 0, & (t, x) \in \mathbb{R}_{\geq 0} \times [0, L], \\ z(t, 0) = z(t, L) = z_x(t, L) = 0, & t \in \mathbb{R}_{\geq 0}, \\ z(0, x) = z_0(x), & x \in [0, L], \end{cases} \quad (2.17)$$

If we let $u(t, x) := a(x)z(t, x)$ in (2.17), where a is a nonnegative and measurable function on $[0, L]$ satisfying

$$\begin{cases} 0 < a_0 \leq a(x) \leq a_1, & \forall x \in \Omega, \\ \text{where } \Omega \text{ is a nonempty open subset of } (0, L), \end{cases} \quad (2.18)$$

then (2.17) becomes

$$\begin{cases} z_t + z_x + z_{xxx} + zz_x + az = 0, & (t, x) \in \mathbb{R}_{\geq 0} \times [0, L], \\ z(t, 0) = z(t, L) = z_x(t, L) = 0, & t \in \mathbb{R}_{\geq 0}, \\ z(0, x) = z_0(x) \in L^2(0, L), & x \in [0, L], \end{cases} \quad (2.19)$$

³The system (2.16) is said to be locally exactly controllable for large time T if there exists two positive values T_M and r such that, for any positive value T satisfying $T > T_M$ and any $z_0, z_T \in L^2(0, L)$ satisfying $\|z_0\|_{L^2(0, L)} \leq r$ and $\|z_T\|_{L^2(0, L)} \leq r$, there exists $u \in L^2(0, T)$ such that the solution to (2.16) satisfies $z(T, \cdot) = z_T(\cdot)$.

Following [79], we get that the origin of (2.19) is globally asymptotically stabilized. More precisely, they obtain the following results

Theorem 2.3 ([79])

The origin for (2.19) is globally asymptotically stable in $L^2(0, L)$. More precisely, there exist a class- \mathcal{K}_∞ function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ and a positive value μ such that, for any initial condition $z_0 \in L^2(0, L)$, the unique solution to (2.19) satisfies the following inequality

$$\|z(t, \cdot)\|_{L^2(0, L)} \leq \alpha(\|z_0\|_{L^2(0, L)})e^{-\mu t}, \quad \forall t \geq 0. \quad (2.20)$$

Remark 2.3

The result obtained [79] is more general than Theorem 2.3. In fact, instead of studying (2.19), the system under consideration in [79] is the following

$$\begin{cases} z_t + z_x + z_{xxx} + b(z)z_x + az = 0, & (t, x) \in \mathbb{R}_{\geq 0} \times [0, L] \\ z(t, 0) = z(t, L) = z_x(t, L) = 0, & t \in \mathbb{R}_{\geq 0} \\ z(0, x) = z_0(x) \in L^2(0, L), & x \in [0, L], \end{cases} \quad (2.21)$$

where a satisfies (2.18) and where b is defined as follows

$$b(0) = 0, \quad |b^{(j)}(\mu)| \leq C(1 + |\mu|^{p-j}), \quad \forall \mu \in \mathbb{R}, \quad (2.22)$$

for $j = 0$ if $1 \leq p < 2$ and for $j = 0, 1, 2$ if $p \geq 2$. Note that, according to this definition, b can be the identity operator, in which case (2.21) would become (2.19). This means that the results obtained in [79] hold also for (2.19). All along this chapter, the results obtained in [79] for this equation will be discussed.

If $\Omega = [0, L]$, the result is better. Indeed, we obtain that the origin of (2.17) is globally exponentially stable in $L^2(0, L)$. Let us give the definition of such a property.

Definition 2.1 (Global exponential stability)

The origin of (2.19) is said to be globally exponentially stable if there exist two positive values C and μ such that, for any initial condition $z_0 \in L^2(0, L)$, the solution to (2.19) satisfies the following inequality

$$\|z(t, \cdot)\|_{L^2(0, L)} \leq Ce^{-\mu t}\|z_0\|_{L^2(0, L)}, \quad \forall t \geq 0. \quad (2.23)$$

Taking the energy E defined by (2.6), its derivative along (2.19)

$$\begin{aligned} E(t) &= -\frac{1}{2}|z_x(t, 0)|^2 - \int_0^L a(x)|z(t, x)|^2 dx \\ &\leq -a_0 \int_0^L |z(t, x)|^2 dx, \end{aligned} \quad (2.24)$$

which ensures an exponential stability with respect to the $L^2(0, L)$ -norm. Note that the decay rate can be selected as large as we want by tuning the parameter a_0 . Such a result is referred to as a rapid stabilization result.

2.2.2 Case where the control is saturated: contributions

Let us consider the KdV equation with a saturated distributed control. It is written as follows

$$\begin{cases} z_t + z_x + z_{xxx} + zz_x + \mathbf{sat}(az) = 0, & (t, x) \in \mathbb{R}_{\geq 0} \times [0, L] \\ z(t, 0) = z(t, L) = z_x(t, L) = 0, & t \in \mathbb{R}_{\geq 0} \\ z(0, x) = z_0(x) \in L^2(0, L), & x \in [0, L], \end{cases} \quad (2.25)$$

where $\mathbf{sat} = \mathbf{sat}_{L^2(0,L)}$ or $\mathbf{sat}_{\text{loc}}$. These two operators have been introduced in the latter chapter (see e.g., Item 2. in Remark 1.1 and (1.33)), but let us recall them. The operator $\mathbf{sat}_{\text{loc}}$ comes from the finite-dimensional theory (see e.g., [94]) and it is defined by, for all $s \in \mathbb{R}$

$$\mathbf{sat}(s) = \begin{cases} -u_s & \text{if } s \leq -u_s, \\ s & \text{if } -u_s \leq s \leq u_s, \\ u_s & \text{if } s \geq u_s, \end{cases} \quad (2.26)$$

for some $u_s > 0$.

As in [76] and [64], we use its extension to infinite dimension called $\mathbf{sat}_{\text{loc}}$. For all sufficiently smooth functions s and for all $x \in [0, L]$, $\mathbf{sat}_{\text{loc}}$ is defined as follows

$$\mathbf{sat}_{\text{loc}}(s)(x) = \mathbf{sat}(s(x)). \quad (2.27)$$

Such a saturation is called localized since its value at x depends only on the value of s at x .

In this work, we also use a saturation operator in $L^2(0, L)$, denoted by $\mathbf{sat}_{L^2(0,L)}$, and defined by

$$\mathbf{sat}_{L^2(0,L)}(s)(x) = \begin{cases} s(x) & \text{if } \|s\|_{L^2(0,L)} \leq u_s, \\ \frac{s(x)u_s}{\|s\|_{L^2(0,L)}} & \text{if } \|s\|_{L^2(0,L)} \geq u_s. \end{cases} \quad (2.28)$$

Note that this definition is borrowed from [84] (see also [81] or [52]) where the saturation is obtained from the norm of the Hilbert space of the control operator.

This saturation seems more natural when studying the stability with respect to an energy, but it is less relevant than $\mathbf{sat}_{\text{loc}}$ for applications. Figure 2.1 illustrates how different these saturations are.

Since these two operators have properties in common, we will use the notation \mathbf{sat} all along this chapter. However, in some cases, we get different results. Therefore, the use of a particular saturation is specified when it is necessary.

Note that, all along this chapter, T is a positive value which refers to a time. We are now able to state the main contributions of this chapter.

Theorem 2.4 (Well posedness)

Given a positive value T , for any initial condition $z_0 \in L^2(0, L)$, there exists a unique mild solution $z \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$ to (2.25).

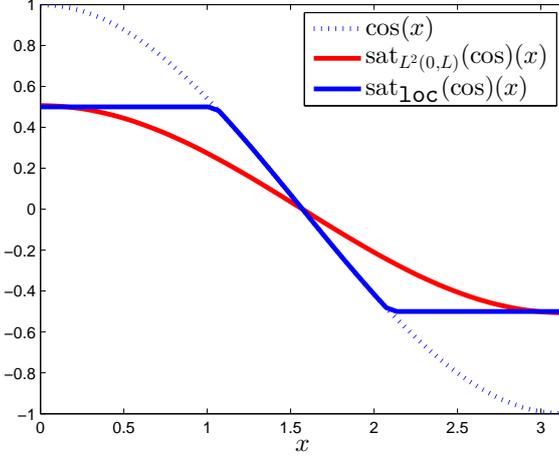


Figure 2.1: $x \in [0, \pi]$. Red: $\mathbf{sat}_{L^2(0,L)}(\cos)(x)$ and $u_s = 0.5$, Blue: $\mathbf{sat}_{loc}(\cos)(x)$ and $u_s = 0.5$, Dotted lines: $\cos(x)$.

Theorem 2.5 (Global asymptotic stability)

Given a nonempty open subset Ω and the positive values a_0 and u_s given by (2.18), and (2.6) and (2.27), respectively, there exist a positive value μ^* and a class \mathcal{K}_∞ function $\alpha_0 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such that, for any $z_0 \in L^2(0, L)$, the mild solution z of (2.25) satisfies

$$\|z(t, \cdot)\|_{L^2(0,L)} \leq \alpha_0(\|z_0\|_{L^2(0,L)})e^{-\mu^*t}, \quad \forall t \geq 0. \tag{2.29}$$

Moreover, in the case where $\Omega = [0, L]$ and $\mathbf{sat} = \mathbf{sat}_{L^2(0,L)}$ we can estimate locally the decay rate of the solution. In other words, for all $r > 0$, for any initial condition $z_0 \in L^2(0, L)$ such that $\|z_0\|_{L^2(0,L)} \leq r$, the mild solution z to (2.25) satisfies

$$\|z(t, \cdot)\|_{L^2(0,L)} \leq \|z_0\|_{L^2(0,L)}e^{-\mu t}, \quad \forall t \geq 0, \tag{2.30}$$

where μ is defined as follows

$$\mu := \min \left\{ a_0, \frac{u_s a_0}{r a_1} \right\}. \tag{2.31}$$

The remaining part of this chapter is devoted to the proof of these results (see Sections 2.2.3.1 and 2.2.3.3, respectively) and to numerical simulations to illustrate Theorem 2.5 (see Section 2.3).

2.2.3 Proofs of Theorems 2.4 and 2.5

2.2.3.1 Well-posedness

Before proving the well-posedness of (2.25), let us recall some useful results on the linear system (2.5). To do that, consider the operator defined by

$$D(A) = \{w \in H^3(0, L), w(0) = w(L) = w'(L) = 0\},$$

$$A : w \in D(A) \subset L^2(0, L) \mapsto (-w' - w''') \in L^2(0, L).$$

Note that it is closed. Moreover, it can be proved that this operator and its adjoint operator defined by

$$\begin{aligned} D(A^*) &= \{w \in H^3(0, L), w(0) = w(L) = w'(0) = 0\}, \\ A^* : w \in D(A^*) \subset L^2(0, L) &\mapsto w' + w''', \end{aligned}$$

are both dissipative.

Therefore, from Theorem 1.1, the operator A generates a strongly continuous semigroup of contractions which we denote by $(W(t))_{t \geq 0}$. We have the following theorem proven in [77] and [11]

Theorem 2.6 (Well-posedness of (2.5), [77],[11])

Let T be any positive value.

- For any initial condition $z_0 \in D(A)$, there exists a unique strong solution $z \in C(0, T; D(A)) \cap C^1(0, T; L^2(0, L))$ to (2.5);
- For any initial condition $z_0 \in L^2(0, L)$, there exists a unique mild solution $z \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$ to (2.5). Moreover, there exists $C_0 > 0$ such that the solution to (2.5) satisfies

$$\|z\|_{C(0, T; L^2(0, L))} + \|z\|_{L^2(0, T; H^1(0, L))} \leq C_0 \|z_0\|_{L^2(0, L)} \quad (2.32)$$

and the extra trace regularity

$$\|z_x(\cdot, 0)\|_{L^2(0, T)} \leq \|z_0\|_{L^2(0, L)}. \quad (2.33)$$

To ease the reading, let us denote the following Banach space, for all $T > 0$,

$$\mathcal{B}(T) := C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$$

endowed with the norm

$$\|z\|_{\mathcal{B}(T)} = \sup_{t \in [0, T]} \|z(t, \cdot)\|_{L^2(0, L)} + \left(\int_0^T \|z(t, \cdot)\|_{H^1(0, L)}^2 dt \right)^{\frac{1}{2}}. \quad (2.34)$$

Before studying the well-posedness of (2.25), we need a well-posedness result with a right-hand side. Given $g \in L^1(0, T; L^2(0, L))$, let us consider z the unique solution ⁴ to the following nonhomogeneous problem:

$$\begin{cases} z_t + z_x + z_{xxx} = g, \\ z(t, 0) = z(t, L) = z_x(t, L) = 0, \\ z(0, \cdot) = z_0. \end{cases} \quad (2.35)$$

Note that we need the following property on the saturation function, which will allow us to state that this type of nonlinearity belongs to the space $L^1(0, T; L^2(0, L))$.

⁴With $g = 0$, the existence and the unicity of z are ensured since A generates a strongly continuous semigroup of contractions. The result follows applying Theorem 1.4. It follows from the semigroup theory the existence and the unicity of z when $g \in L^1(0, T; L^2(0, L))$ (see [72]).

Lemma 2.1

For all $(s, \tilde{s}) \in L^2(0, L) \times L^2(0, L)$, we have

$$\|\mathbf{sat}(s) - \mathbf{sat}(\tilde{s})\|_{L^2(0,L)} \leq 3\|s - \tilde{s}\|_{L^2(0,L)}. \quad (2.36)$$

Proof of Lemma 2.1:

For $\mathbf{sat} = \mathbf{sat}_{L^2(0,L)}$, please refer to [84, Theorem 5.1.] for a proof. For $\mathbf{sat} = \mathbf{sat}_{\text{loc}}$, we know from [46, Page 73] that for all $(s, \tilde{s}) \in L^2(0, L)^2$ and for all $x \in [0, L]$,

$$|\mathbf{sat}_{\text{loc}}(s(x)) - \mathbf{sat}_{\text{loc}}(\tilde{s}(x))| \leq |s(x) - \tilde{s}(x)|.$$

Thus, we get

$$\|\mathbf{sat}_{\text{loc}}(s) - \mathbf{sat}_{\text{loc}}(\tilde{s})\|_{L^2(0,L)} \leq \|s - \tilde{s}\|_{L^2(0,L)},$$

which concludes the proof of Lemma 2.1. \square

We have the following proposition borrowed from [77, Proposition 4.1].

Proposition 2.1 ([77])

If $z \in L^2(0, T; H^1(0, L))$, then $zz_x \in L^1(0, T; L^2(0, L))$ and the map $\psi_1 : z \in L^2(0, T; H^1(0, L)) \mapsto zz_x \in L^1(0, T; L^2(0, L))$ is continuous.

We have also the following proposition.

Proposition 2.2

Assume $a : [0, L] \rightarrow \mathbb{R}$ satisfies (2.18). If $z \in L^2(0, T; H^1(0, L))$, then $\mathbf{sat}(az) \in L^1(0, T; L^2(0, L))$ and the map $\psi_2 : z \in L^2(0, T; H^1(0, L)) \mapsto \mathbf{sat}(az) \in L^1(0, T; L^2(0, L))$ is continuous.

Proof of Proposition 2.2:

Let $z, \tilde{z} \in L^2(0, T; H^1(0, L))$. We have, using Lemma 2.1 and the Hölder inequality

$$\begin{aligned} \|\mathbf{sat}(az) - \mathbf{sat}(a\tilde{z})\|_{L^1(0,T;L^2(0,L))} &\leq 3 \int_0^T \|a(z - \tilde{z})\|_{L^2(0,L)} dt \\ &\leq 3\sqrt{L}a_1\sqrt{T}\|(z - \tilde{z})\|_{L^2(0,T;H^1(0,L))} \end{aligned} \quad (2.37)$$

Plugging $z = 0$ in (2.37) yields $\mathbf{sat}(az) \in L^1(0, T; L^2(0, L))$ and (2.37) implies the continuity of the map ψ_2 . This concludes the proof of Proposition 2.2. \square

Let us study the non-homogenous linear KdV equation with $z_0(x) := 0$. For any

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$g \in L^1(0, T; L^2(0, L))$, it is described with the following equation

$$\begin{cases} z_t + z_x + z_{xxx} + g = 0, \\ z(t, 0) = z(t, L) = z_x(t, L) = 0, \\ z(0, x) = 0. \end{cases} \quad (2.38)$$

It can be rewritten as follows

$$\begin{cases} \dot{y} = Ay + g, \\ z(0) = 0. \end{cases} \quad (2.39)$$

By standard semigroup theory (see [72]), for any positive value t and any function $g \in L^1(\mathbb{R}_{\geq 0}; L^2(0, L))$, the solution to (2.38) can be expressed as follows

$$z(t) = \int_0^t W(t - \tau)g(\tau, x)d\tau. \quad (2.40)$$

Finally, we have the following result borrowed from [79, Lemma 2.2]

Proposition 2.3 ([79])

There exists a positive value C_1 such that for any positive value T and any function $g \in L^1(0, T; L^2(0, L))$ the solution to (2.38) satisfies the following inequality,

$$\left\| \int_0^t W(t - \tau)g(\tau, x)d\tau \right\|_{\mathcal{B}(T)} \leq C_1 \int_0^T \|g(\tau, \cdot)\|_{L^2(0, L)} d\tau, \quad (2.41)$$

where $\|\cdot\|_{\mathcal{B}(T)}$ is given by (2.34).

2.2.3.2 Proof of Theorem 2.4

Let us begin this section with a technical lemma.

Lemma 2.2 ([102])

For any $T > 0$ and $y, z \in \mathcal{B}(T)$,

$$\int_0^T \|(z(t, \cdot)y(t, \cdot))_x\|_{L^2(0, L)} dt \leq 2\sqrt{T}\|z\|_{\mathcal{B}(T)}\|y\|_{\mathcal{B}(T)}. \quad (2.42)$$

The following is a local well-posedness result.

Lemma 2.3 (Local well-posedness)

Let $T > 0$ be given. For any $z_0 \in L^2(0, L)$, there exists $T' \in [0, T]$ depending on $\|z_0\|_{L^2(0, L)}$ such that (2.25) admits a unique mild solution $z \in \mathcal{B}(T')$.

To prove this result, a fixed-point theorem will be used. Let us recall it

Theorem 2.7 (Banach fixed-point theorem [7], Theorem 5.7.)

Let Z be a nonempty complete metric space equipped with the metric d and let $\Gamma : Z \rightarrow Z$ be a strict contraction, i.e., there exists $\ell < 1$ such that

$$d(\Gamma z_1, \Gamma z_2) \leq \ell d(z_1, z_2), \quad \forall z_1, z_2 \in Z. \quad (2.43)$$

Then Γ has a unique fixed-point, $z = \Gamma z$.

Proof of Lemma 2.3:

We follow the strategy of [18] and [79]. We know from (2.38), Proposition 2.1 and Proposition 2.2 that, for all $\tilde{z} \in L^1(0, T; L^2(0, L))$, there exists a unique mild solution to the following system

$$\begin{cases} z_t + z_x + z_{xxx} = -\tilde{z}\tilde{z}_x - \mathbf{sat}(a\tilde{z}), \\ z(t, 0) = z(t, L) = z_x(t, L) = 0, \\ z(0, x) = z_0(x). \end{cases} \quad (2.44)$$

The solution to (2.44) can be written in integral form

$$z(t) = W(t)z_0 - \int_0^t W(t-\tau)(\tilde{z}\tilde{z}_x)(\tau)d\tau - \int_0^t W(t-\tau)\mathbf{sat}(a\tilde{z}(\tau, \cdot))d\tau. \quad (2.45)$$

For given $z_0 \in L^2(0, L)$, let r and T' be positive constants to be chosen later. We define

$$S_{T', r} = \{\tilde{z} \in \mathcal{B}(T'), \|\tilde{z}\|_{\mathcal{B}(T')} \leq r\}, \quad (2.46)$$

which is a closed, convex and bounded subset of $\mathcal{B}(T')$. Consequently, $S_{T', r}$ is a complete metric space in the norm topology induced from $\mathcal{B}(T')$. We define a map Γ on $S_{T', r}$ by, for all $t \in [0, T']$

$$\Gamma(\tilde{z}) := W(t)z_0 - \int_0^t W(t-\tau)(\tilde{z}\tilde{z}_x)(\tau)d\tau - \int_0^t W(t-\tau)\mathbf{sat}(a\tilde{z}(\tau, \cdot))d\tau, \quad \forall \tilde{z} \in S_{T', r}. \quad (2.47)$$

We aim at proving that there exists a unique fixed point to this operator. It follows from Proposition 2.3, Lemma 2.2 and the linear estimates given in Theorem 2.6 that for every $\tilde{z} \in S_{T', r}$, there exists a positive value $C_2 := C_2(a_1, T, L, C_1)$ such that it holds

$$\begin{aligned} \|\Gamma(\tilde{z})\|_{\mathcal{B}(T')} &\leq C_0\|z_0\|_{L^2(0, L)} + C_1 \int_0^T (\|\tilde{z}\tilde{z}_x(\tau, \cdot)\|_{L^2(0, L)} + \|\mathbf{sat}(a\tilde{z}(\tau, \cdot))\|_{L^2(0, L)})d\tau \\ &\leq C_0\|z_0\|_{L^2(0, L)} + 2C_1\sqrt{T'}\|\tilde{z}\|_{\mathcal{B}(T')}^2 + C_2\sqrt{T'}\|\tilde{z}\|_{\mathcal{B}(T')} \end{aligned} \quad (2.48)$$

where the first line has been obtained with the linear estimates given in Theorem 2.6 and the estimate given in Proposition 2.3 and the second line with Lemma 2.2 and Proposition 2.2. We choose $r > 0$ and $T' > 0$ such that

$$\begin{cases} r = 2C_0\|z_0\|_{L^2(0, L)}, \\ 2C_1\sqrt{T'}r + C_2\sqrt{T'} \leq \frac{1}{2}, \end{cases} \quad (2.49)$$

in order to obtain

$$\|\Gamma(\tilde{z})\|_{\mathcal{B}(T')} \leq r, \quad \forall \tilde{z} \in S_{T', r}. \quad (2.50)$$

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Thus, with such r and T' , Γ maps $S_{T',r}$ to $S_{T',r}$. Moreover, one can prove with Proposition 2.3, Lemma 2.2 and the linear estimates given in Theorem 2.6 that

$$\|\Gamma(\tilde{z}_1) - \Gamma(\tilde{z}_2)\|_{\mathcal{B}(T')} \leq \frac{1}{2} \|\tilde{z}_1 - \tilde{z}_2\|_{\mathcal{B}(T')}, \quad \forall \tilde{z}_1, \tilde{z}_2 \in S_{T',r}. \quad (2.51)$$

The existence of mild solutions to the Cauchy problem (2.25) follows by using the Theorem 2.7. \square

Before proving the global well-posedness, we need the following lemma inspired by [23] and [18] which implies that, if there exists a solution for some $T > 0$, then the solution is unique.

Lemma 2.4

Let $T > 0$ and $a : [0, L] \rightarrow \mathbb{R}$ satisfying (2.18). There exists $C_{11} := C_{11}(T, L) > 0$ such that for every $z_0, \tilde{z}_0 \in L^2(0, L)$ for which there exist mild solutions z and \tilde{z} of

$$\begin{cases} z_t + z_x + z_{xxx} + zz_x + \mathbf{sat}(az) = 0, \\ z(t, 0) = z(t, L) = z_x(t, L) = 0, \\ z(0, x) = z_0(x), \end{cases} \quad (2.52)$$

and

$$\begin{cases} \tilde{z}_t + \tilde{z}_x + \tilde{z}_{xxx} + \tilde{z}\tilde{z}_x + \mathbf{sat}(a\tilde{z}) = 0, \\ \tilde{z}(t, 0) = \tilde{z}(t, L) = \tilde{z}_x(t, L) = 0, \\ \tilde{z}(0, x) = \tilde{z}_0(x), \end{cases} \quad (2.53)$$

these solutions satisfy

$$\int_0^T \int_0^L (z_x(t, x) - \tilde{z}_x(t, x))^2 dx dt \leq e^{C_{11}(1 + \|z\|_{L^2(0,T;H^1(0,L))} + \|\tilde{z}\|_{L^2(0,T;H^1(0,L))})} \int_0^L (z_0(x) - \tilde{z}_0(x))^2 dx, \quad (2.54)$$

$$\int_0^T \int_0^L (z(t, x) - \tilde{z}(t, x))^2 dx dt \leq e^{C_{11}(1 + \|z\|_{L^2(0,T;H^1(0,L))} + \|\tilde{z}\|_{L^2(0,T;H^1(0,L))})} \int_0^L (z_0(x) - \tilde{z}_0(x))^2 dx. \quad (2.55)$$

Proof of Lemma 2.4:

We follow the strategy of [23] and [18]. Let us assume that for given $z_0, \tilde{z}_0 \in L^2(0, L)$, there exist $T > 0$ and two different solutions z and \tilde{z} to (2.52) and (2.53), respectively, defined on $[0, T] \times [0, L]$. Then $\Delta := z - \tilde{z}$ defined on $[0, T] \times [0, L]$ is a mild solution of

$$\begin{cases} \Delta_t + \Delta_x + \Delta_{xxx} = -z\Delta_x - \tilde{z}_x\Delta - (\mathbf{sat}(az) - \mathbf{sat}(a\tilde{z})), \\ \Delta(t, 0) = \Delta(t, L) = \Delta_x(t, L) = 0, \\ \Delta(0, x) = z_0(x) - \tilde{z}_0(x). \end{cases} \quad (2.56)$$

Integrating by parts in

$$\int_0^L 2x\Delta(\Delta_t + \Delta_x + \Delta_{xxx} + z\Delta_x + \tilde{z}_x\Delta + \mathbf{sat}(az) - \mathbf{sat}(a\tilde{z})) dx = 0, \quad (2.57)$$

and using the boundary conditions of (2.56), we readily get

$$\begin{aligned} \frac{d}{dt} \int_0^L x \Delta^2 dx + 3 \int_0^L \Delta_x^2 dx &= \int_0^L \Delta^2 dx - 2 \int_0^L xz \Delta \Delta_x dx \\ &+ \int_0^L \tilde{z} \Delta^2 dx + 4 \int_0^L x \tilde{z} \Delta \Delta_x dx - \int_0^L x \Delta (\mathbf{sat}(az) - \mathbf{sat}(a\tilde{z})) dx. \end{aligned} \quad (2.58)$$

By the boundary conditions and the continuous Sobolev embedding $H_0^1(0, L) \subset C([0, T])$, there exists $C_3 = C_3(L) > 0$ such that

$$2 \left| \int_0^L xz \Delta \Delta_x dx \right| \leq C_3 \|z_x\|_{L^2(0, L)} \int_0^L |x \Delta \Delta_x| dx. \quad (2.59)$$

Thus,

$$2 \left| \int_0^L xz \Delta \Delta_x dx \right| \leq \frac{1}{2} \int_0^L \Delta_x^2 dx + \frac{C_3^2}{2} \|z_x\|_{L^2(0, L)}^2 L \int_0^L x \Delta^2 dx. \quad (2.60)$$

Similarly,

$$4 \left| \int_0^L x \tilde{z} \Delta \Delta_x dx \right| \leq \frac{1}{2} \int_0^L \Delta_x^2 dx + 2C_3^2 \|\tilde{z}_x\|_{L^2(0, L)}^2 \int_0^L x \Delta^2 dx. \quad (2.61)$$

Moreover, since \mathbf{sat} is globally Lipschitz with constant 3 (as stated in Lemma 2.1) and for all $x \in [0, L]$, $a(x) \leq a_1$, we use a Hölder inequality to get

$$\begin{aligned} \left| \int_0^L x \Delta (\mathbf{sat}(az) - \mathbf{sat}(a\tilde{z})) dx \right| &\leq \|x \Delta\|_{L^2(0, L)} \|\mathbf{sat}(az) - \mathbf{sat}(a\tilde{z})\|_{L^2(0, L)} \\ &\leq 3 \|a(x) \Delta\|_{L^2(0, L)} \|x \Delta\|_{L^2(0, L)} \\ &\leq 3a_1 \int_0^L x \Delta^2 dx. \end{aligned} \quad (2.62)$$

Note that, from [23, Lemma 16], for every $\phi \in H^1(0, L)$ with $\phi(0) = 0$, and every $d \in [0, L]$,

$$\int_0^L \phi^2 dx \leq \frac{d^2}{2} \int_0^L \phi_x^2 dx + \frac{1}{d} \int_0^L x \phi^2 dx. \quad (2.63)$$

Thus, from (2.63) there exists $C_4 > 0$ such that

$$\int_0^L \Delta^2 dx \leq \frac{1}{2} \int_0^L \Delta_x^2 dx + C_4 \int_0^L x \Delta^2 dx.$$

Moreover, with the boundary conditions of z and the Sobolev embedding $H_0^1(0, L) \subset C([0, T])$, there exists $C_5 = C_5(L) > 0$ such that

$$2 \left| \int_0^L \tilde{z} \Delta^2 dx \right| \leq C_5 \|\tilde{z}_x\|_{L^2(0, L)} \int_0^L \Delta^2 dx.$$

Hence, using the boundary conditions of Δ and (2.63) with $d := \min\{C_5^{-1/2} \|\tilde{z}_x\|_{L^2(0, L)}^{-1/2}, L\}$, there exists $C_6 = C_6(L) > 0$ such that

$$2 \int_0^L \tilde{z} \Delta^2 dx \leq \frac{1}{2} \int_0^L \Delta_x^2 dx + C_6 (1 + \|\tilde{z}_x\|_{L^2(0, L)}^{3/2}) \int_0^L x \Delta^2 dx. \quad (2.64)$$

Finally, there exists $C_7 = C_7(L) > 0$ such that

$$\frac{d}{dt} \int_0^L x \Delta^2 dx + \int_0^L \Delta_x^2 dx \leq C_7(1 + \|z_x\|_{L^2(0,L)}^2 + \|\tilde{z}_x\|_{L^2(0,L)}^2) \int_0^L x \Delta^2 dx. \quad (2.65)$$

In particular,

$$\frac{d}{dt} \int_0^L x \Delta^2 dx \leq C_7(1 + \|z_x\|_{L^2(0,L)}^2 + \|\tilde{z}_x\|_{L^2(0,L)}^2) \int_0^L x \Delta^2 dx. \quad (2.66)$$

Using the Grönwall Lemma, the last inequality and the initial conditions of Δ , we get, for every $t \in [0, T]$,

$$\int_0^L x \Delta^2(t, x) dx \leq e^{C_7(T + \|z\|_{L^2(0,T;H^1(0,L))}^2 + \|\tilde{z}\|_{L^2(0,T;H^1(0,L))}^2)} \int_0^L x (z_0(x) - \tilde{z}_0(x))^2 dx, \quad (2.67)$$

and thus, we obtain the existence of $C_8 = C_8(T, L)$ such that

$$\int_0^T \int_0^L (z_x(t, x) - \tilde{z}_x(t, x))^2 dx dt \leq e^{C_8(\|z\|_{L^2(0,T;H^1(0,L))}^2 + \|\tilde{z}\|_{L^2(0,T;H^1(0,L))}^2)} \int_0^L (z_0(x) - \tilde{z}_0(x))^2 dx. \quad (2.68)$$

Similarly, integrating by parts in

$$\int_0^L \Delta(\Delta_t + \Delta_x + \Delta_{xxx} + z\Delta_x + \tilde{z}_x\Delta + \mathbf{sat}(az) - \mathbf{sat}(a\tilde{z})) dx = 0 \quad (2.69)$$

we get, using the boundary conditions of Δ ,

$$\frac{1}{2} \frac{d}{dt} \int_0^L \Delta^2 dx + \frac{1}{2} \Delta_x^2(t, 0) = - \int_0^L (z\Delta_x - 2\tilde{z}\Delta_x)\Delta dx - \int_0^L \Delta(\mathbf{sat}(az) - \mathbf{sat}(a\tilde{z})) dx. \quad (2.70)$$

Moreover,

$$- \int_0^L (z\Delta_x - 2\tilde{z}\Delta_x)\Delta \leq \int_0^L \Delta_x^2 dx + \int_0^L \left(\frac{1}{2} z^2 + 2\tilde{z}^2 \right) \Delta^2 dx, \quad (2.71)$$

and

$$\left| \int_0^L \Delta(\mathbf{sat}(az) - \mathbf{sat}(a\tilde{z})) dx \right| \leq 3a_1 \int_0^L \Delta^2 dx. \quad (2.72)$$

Thanks to the continuous Sobolev embedding $H_0^1(0, L) \subset C([0, L])$, (2.72) and (2.71), there exists $C_9 = C_9(L) > 0$ such that

$$\frac{1}{2} \frac{d}{dt} \int_0^L \Delta^2 dx \leq \int_0^L \Delta_x^2 dx + C_9 \left(\|z_x\|_{L^2(0,L)}^2 + \|\tilde{z}_x\|_{L^2(0,L)}^2 + 1 \right) \int_0^L \Delta^2 dx. \quad (2.73)$$

Thus applying the Grönwall Lemma, we get the existence of $C_{10} = C_{10}(L) > 0$ such that

$$\int_0^L (z(t, x) - \tilde{z}(t, x))^2 dx \leq e^{C_{10}(1 + \|z\|_{L^2(0,T;H^1(0,L))}^2 + \|\tilde{z}\|_{L^2(0,T;H^1(0,L))}^2)} \int_0^L (z_0(x) - \tilde{z}_0(x))^2 dx. \quad (2.74)$$

With the use of (2.68) and (2.74), this concludes the proof of Lemma 2.4. \square

We aim at removing the smallness condition given by T' in Lemma 2.3, following [18]. Since we have the local well-posedness, we only need to prove the following a priori estimate for any mild solution to (2.25).

Lemma 2.5

For given $T > 0$, there exists $G := G(T) > 0$ such that for any $z_0 \in L^2(0, L)$, for any positive value T' satisfying $0 < T' \leq T$ and for any mild solution $z \in \mathcal{B}(T')$ to (2.25), it holds

$$\|z\|_{\mathcal{B}(T')} \leq G \|z_0\|_{L^2(0,L)}, \quad (2.75)$$

and

$$\|z\|_{L^2(0,L)} \leq \|z_0\|_{L^2(0,L)}. \quad (2.76)$$

Proof of Lemma 2.5:

Let us fix $0 < T' \leq T$. We multiply the first equation in (2.25) by z and integrate on $(0, L)$. Using the boundary conditions in (2.25), we get the following estimates

$$\int_0^L z z_x dx = 0, \quad \int_0^L z z_{xxx} dx = \frac{1}{2} |z_x(t, 0)|^2, \quad \int_0^L z^2 z_x dx = 0.$$

Using the fact that \mathfrak{sat} is odd, we get that

$$\frac{1}{2} \frac{d}{dt} \|z(t, \cdot)\|_{L^2(0,L)}^2 \leq -\frac{1}{2} |z_x(t, 0)|^2 - \int_0^L z \mathfrak{sat}(az) dx \leq 0 \quad (2.77)$$

which implies (2.76). Moreover, using again (2.77), there exists $C_{12} = C_{12}(L) > 0$ such that

$$\|z\|_{L^\infty(0,T';L^2(0,L))} \leq C_{12} \|z_0\|_{L^2(0,L)}. \quad (2.78)$$

It remains to prove a similar inequality for $\|z_x\|_{L^2(0,T';L^2(0,L))}$ to achieve the proof. We multiply (2.25) by xz , integrate on $(0, L)$ and use the following

$$\int_0^L x z z_x dx = -\frac{1}{2} \|z\|_{L^2(0,L)}^2, \quad \int_0^L z z_{xxx} dx = \frac{3}{2} \|z_x\|_{L^2(0,L)}^2,$$

and

$$\begin{aligned} -\int_0^L x z^2 z_x dx &= \frac{1}{3} \int_0^L z^3(t, x) dx \leq \frac{1}{3} \sup_{x \in [0,L]} |z(t, x)| \|z\|_{L^\infty(0,T;L^2(0,L))}^2 \\ &\leq \frac{\sqrt{L}}{3} \|z_x\|_{L^2(0,L)} \|z\|_{L^\infty(0,T;L^2(0,L))}^2 \leq \frac{\sqrt{L}\delta}{6} \|z_x\|_{L^2(0,L)} + \frac{\sqrt{L}}{6\delta} \|z\|_{L^\infty(0,T;L^2(0,L))}^4 \end{aligned} \quad (2.79)$$

where δ is chosen as $\delta := \frac{3}{\sqrt{L}}$. In this way, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_0^L |x^{1/2} z(t, \cdot)|^2 dx - \frac{1}{2} \int_0^L z^2 dx + \frac{3}{2} \int_0^L |z_x|^2 dx - \frac{1}{3} \int_0^L |z|^3 dx = - \int_0^L x \mathfrak{sat}(az) z dx. \quad (2.80)$$

We get, using (2.79) and the fact that \mathfrak{sat} is odd, that

$$\frac{1}{2} \frac{d}{dt} \int_0^L |x^{1/2} z(t, \cdot)|^2 dx + \int_0^L |z_x|^2 dx \leq \frac{1}{2} \|z\|_{L^\infty(0,T;L^2(0,L))}^2 + \frac{L}{18} \|z\|_{L^\infty(0,T;L^2(0,L))}^4. \quad (2.81)$$

Using (2.78) and Grönwall inequality, we get the existence of a positive value $C_{13} = C_{13}(L) > 0$ such that

$$\|z_x\|_{L^2(0,T';L^2(0,L))} \leq C_{13}\|z_0\|_{L^2(0,L)}, \quad (2.82)$$

which concludes the proof of Lemma 2.5. \square

Using a classical extension argument, Lemmas 2.3, 2.5 and 2.4, for any $T > 0$, we can conclude that there exists a unique mild solution in $\mathcal{B}(T)$ to (2.25). Indeed, with Lemma 2.3, we know that there exists $T' \in (0, T)$ such that there exists a unique solution to (2.25) in $\mathcal{B}(T')$. Moreover, Lemma 2.5 allows us to state the existence of a mild solution to (2.25) for every $T > 0$: since the solution z to (2.25) is bounded by its initial condition for every $T' > 0$ belonging to $[0, T]$ as stated in (2.76), we know that there exists a solution to (2.25) in $\mathcal{B}(T)$. Finally, Lemma 2.4 implies that there exists a unique mild solution to (2.25) in $\mathcal{B}(T)$. This concludes the proof of Theorem 2.4.

Remark 2.4

As noticed in Remark 2.3, the authors of [79] studied the stabilization of a KdV equation more general than the one studied in this chapter. It is given by (2.21).

The saturated version of (2.21) is

$$\begin{cases} z_t + z_x + z_{xxx} + b(z)z_x + \mathbf{sat}(az) = 0, \\ z(t, 0) = z(t, L) = z_x(t, L) = 0, \\ z(0, x) = z_0(x). \end{cases} \quad (2.83)$$

The strategy followed in [79] can be followed easily to prove the same result as Theorem 2.4 for (2.83). Note that in [79], provided that the initial condition satisfies some compatibility conditions, the well-posedness is proved for a solutions in $C([0, T]; H^s(0, L)) \cap L^2(0, T; H^{s+1}(0, L))$, where $s \in [0, 3]$. The authors proved this result by looking at $v = z_t$ which solves an equation equivalent to (2.21). In our case, it seems harder to prove such a result. Since the saturation operator introduces some non-smoothness, $v = z_t$ does not solve an equation equivalent to (2.83).

2.2.3.3 Global asymptotic stability

Let us begin by introducing the following definition.

Definition 2.2

System (2.25) is said to be semi-globally exponentially stable in $L^2(0, L)$ if for any $r > 0$ there exists two constants $K := K(r) > 0$ and $\mu := \mu(r) > 0$ such that for any $z_0 \in L^2(0, L)$ such that $\|z_0\|_{L^2(0,L)} \leq r$, the mild solution $y = z(t, x)$ to (2.25) satisfies

$$\|z(t, \cdot)\|_{L^2(0,L)} \leq K\|z_0\|_{L^2(0,L)}e^{-\mu t}, \quad \forall t \geq 0. \quad (2.84)$$

Following [79], we first show that (2.25) is semi-globally exponentially stable in $L^2(0, L)$. From this result, we will be able to prove the global uniform exponential stability of (2.25). To

do that, we state and prove a technical lemma that allows us to bound the saturation function with a linear function as long as the initial condition is bounded. Then we separate our proof into two cases. The first one deals with the case $\Omega = [0, L]$ and $\mathbf{sat} = \mathbf{sat}_{L^2(0,L)}$, while the second one deals with the case $\Omega \subseteq [0, L]$ whatever the saturation is. The tools to tackle these two cases are different. The goal of the next three sections is to prove the following result

Proposition 2.4 (Semi-global exponential stability)

The system (2.25) is semi-globally exponentially stable in $L^2(0, L)$.

Moreover, if $\Omega = [0, L]$ and $\mathbf{sat} = \mathbf{sat}_{L^2(0,L)}$, inequality (2.84) holds with $K = 1$ and μ can be estimated with (2.31) given in Theorem 2.5.

2.2.3.4 Technical Lemma

Before starting the proof of the Proposition 2.4, we will use a property already well-known for saturated control system, namely the sector condition. If the argument of the saturation is bounded, then the saturation can be bounded by a linear function, which allows us to pass from nonlinear control problem to a linear one (see [46]).

Let us state and prove the following lemma.

Lemma 2.6 (Sector Condition)

Let r be a positive value, a function $a : [0, L] \rightarrow \mathbb{R}$ satisfying (2.18) and $k(r)$ defined by

$$k(r) = \min \left\{ \frac{u_s}{a_1 r}, 1 \right\}, \quad (2.85)$$

with u_s given either in (2.6) or (2.27).

(i) Given $\mathbf{sat} = \mathbf{sat}_{L^2(0,L)}$ and $s \in L^2(0, L)$ such that $\|s\|_{L^2(0,L)} \leq r$, we have

$$\left(\mathbf{sat}_{L^2(0,L)}(a(x)s(x)) - k(r)a(x)s(x) \right) s(x) \geq 0, \quad \forall x \in [0, L], \quad (2.86)$$

(ii) Given $\mathbf{sat} = \mathbf{sat}_{loc}$ and $s \in L^\infty(0, L)$ such that, for all $x \in [0, L]$, $|s(x)| \leq r$, we have

$$\left(\mathbf{sat}_{loc}(a(x)s(x)) - k(r)a(x)s(x) \right) s(x) \geq 0, \quad \forall x \in [0, L]. \quad (2.87)$$

Proof of Lemma 2.6:

(i) We first prove item (i) of Lemma 2.6. Two cases may occur

1. $\|as\|_{L^2(0,L)} \geq u_s$;
2. $\|as\|_{L^2(0,L)} \leq u_s$.

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The first case implies that, for all $x \in [0, L]$

$$\mathbf{sat}_{L^2(0,L)}(a(x)s(x)) = \frac{a(x)s(x)}{\|as\|_{L^2(0,L)}} u_s.$$

Thus, for all $x \in [0, L]$,

$$\left(\mathbf{sat}_{L^2(0,L)}(a(x)s(x)) - k(r)a(x)s(x) \right) s(x) = a(x)s(x)^2 \left(\frac{u_s}{\|as\|_{L^2(0,L)}} - k(r) \right).$$

Since

$$\frac{u_s}{\|as\|_{L^2(0,L)}} \geq \frac{u_s}{a_1 \|s\|_{L^2(0,L)}} \geq \frac{u_s}{a_1 r} \geq k(r),$$

we obtain

$$\left(\mathbf{sat}_{L^2(0,L)}(a(x)s(x)) - k(r)a(x)s(x) \right) s(x) \geq 0.$$

Now, let us consider the case $\|as\|_{L^2(0,L)} \leq u_s$. We have, for all $x \in [0, L]$,

$$\mathbf{sat}_{L^2(0,L)}(a(x)s(x)) = a(x)s(x),$$

and then, for all $x \in [0, L]$,

$$\left(\mathbf{sat}_{L^2(0,L)}(a(x)s(x)) - k(r)a(x)s(x) \right) s(x) = a(x)s(x)^2(1 - k(r)) \geq 0.$$

(ii) We now deal with item (ii) of Lemma 2.6.

Let us pick $x \in [0, L]$ and consider the two following cases

1. $|a(x)s(x)| \geq u_s$;
2. $|a(x)s(x)| \leq u_s$.

The first case implies either $a(x)s(x) \geq u_s$ or $a(x)s(x) \leq -u_s$.

Since these two possibilities are symmetric, we just deal with the case $a(x)s(x) \geq u_s$. We have

$$\mathbf{sat}_{1\text{oc}}(a(x)s(x)) = u_s,$$

and then

$$\begin{aligned} \left(\mathbf{sat}_{1\text{oc}}(a(x)s(x)) - k(r)a(x)s(x) \right) s(x) &= u_s s(x) - k(r)a(x)s^2(x) \\ &\geq \left(u_s - k(r)a(x)r \right) s(x) \geq \left(u_s - \frac{u_s}{a_1 r} a(x)r \right) s(x) \geq 0. \end{aligned}$$

The second case implies that $\mathbf{sat}_{1\text{oc}}(a(x)s(x)) = a(x)s(x)$, and then $\left(\mathbf{sat}_{1\text{oc}}(a(x)s(x)) - k(r)a(x)s(x) \right) s(x) = (1 - k(r))a(x)s^2(x) \geq 0$. This concludes the proof of the second item of the Lemma 2.6. \square

2.2.3.5 Proof of Proposition 2.4 when $\Omega = [0, L]$ and $\mathbf{sat} = \mathbf{sat}_{L^2(0,L)}$

Now we are able to prove Proposition 2.4 when $\Omega = [0, L]$ and $\mathbf{sat} = \mathbf{sat}_{L^2(0,L)}$. Let $r > 0$ and $z_0 \in L^2(0, L)$ be such that $\|z_0\|_{L^2(0,L)} \leq r$.

Multiplying (2.25) by z , integrating with respect to x on $(0, L)$ yields

$$\frac{1}{2} \frac{d}{dt} \int_0^L |z(t, x)|^2 dx \leq - \int_0^L \mathbf{sat}_{L^2(0,L)}(az(t, x))z(t, x) dx. \quad (2.88)$$

Note that from (2.76), we get

$$\|z\|_{L^2(0,L)} \leq \|z_0\|_{L^2(0,L)} \leq r. \quad (2.89)$$

Thus, using Lemma 2.6 and (2.88), it follows that

$$\frac{1}{2} \frac{d}{dt} \int_0^L |z(t, x)|^2 dx \leq - \int_0^L k(r)a_0|z(t, x)|^2 dx. \quad (2.90)$$

Applying the Grönwall lemma leads to

$$\|z(t, \cdot)\|_{L^2(0,L)} \leq e^{-\mu t} \|z_0\|_{L^2(0,L)} \quad (2.91)$$

where μ is defined in the statement of Theorem 2.5. This concludes the proof of Proposition 2.4 when $\Omega = [0, L]$ and when $\mathbf{sat} = \mathbf{sat}_{L^2(0,L)}$.

Remark 2.5

The constant μ depends on u_s , r and a_0 . Thus, although we have proven an exponential stability, the rapid stabilization is still an open question. Moreover, in the case $a(x) = a_0 = a_1$ for all $x \in [0, L]$, which is the case where the gain is constant, we obtain that

$$\mu = \min \left\{ a_0, \frac{u_s}{r} \right\}.$$

2.2.3.6 Proof of Proposition 2.4 when $\Omega \subseteq [0, L]$

In this section, we have $\mathbf{sat} = \mathbf{sat}_{L^2(0,L)}$ or $\mathbf{sat} = \mathbf{sat}_{\text{loc}}$. We follow the strategy of [79] and [11]. We use a contradiction argument. It is based on the following unique continuation result.

Theorem 2.8 ([80])

Let $z \in L^2(0, T; H^3(0, L))$ be a solution of

$$z_t + z_x + z_{xxx} + zz_x = 0$$

such that

$$z(t, x) = 0, \forall t \in (t_1, t_2), \forall x \in \Omega,$$

with Ω an open nonempty subset of $(0, L)$. Then

$$z(t, x) = 0, \forall t \in (t_1, t_2), \forall x \in (0, L).$$

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Moreover, the following lemma will be used.

Lemma 2.7 (Aubin-Lions Lemma, [83], Corollary 4)

Let $X_0 \subset X \subset X_1$ be three Banach spaces with X_0, X_1 reflexive spaces. Suppose that X_0 is compactly embedded in X and X is continuously embedded in X_1 . Then

$$\{h \in L^p(0, T; X_0) \mid h_t \in L^q(0, T; X_1)\}$$

embeds compactly in $L^p(0, T; X)$ for any $1 < p, q < \infty$.

Let us now start the proof of Proposition 2.4. Let $r > 0$ and $z_0 \in L^2(0, L)$ be such that

$$\|z_0\|_{L^2(0, L)} \leq r. \quad (2.92)$$

As in the proof of Lemma 2.5, with multiplier techniques applied to (2.25), we obtain

$$\|z(t, \cdot)\|_{L^2(0, L)}^2 = \|z_0\|_{L^2(0, L)}^2 - \int_0^t |z_x(\sigma, 0)|^2 d\sigma - 2 \int_0^t \int_0^L \mathbf{sat}(az) z dx d\sigma, \quad \forall t \in [0, T] \quad (2.93)$$

and

$$\|z\|_{L^2(0, T; H^1(0, L))}^2 \leq \frac{8T + 2L}{3} \|z_0\|_{L^2(0, L)}^2 + \frac{TC}{27} \|z_0\|_{L^2(0, L)}^4. \quad (2.94)$$

Moreover, multiplying (2.25) by $(T - t)y$, we obtain after performing some integrations by parts

$$T \|z_0\|_{L^2(0, L)}^2 \leq \int_0^T \int_0^L |z(t, x)|^2 dx dt + \int_0^T (T - t) |z_x(t, 0)|^2 dt + 2 \int_0^T (T - t) \int_0^L \mathbf{sat}(az) z dx dt. \quad (2.95)$$

Note that, since \mathbf{sat} is an odd function, (2.93) implies that, for all $t \in [0, T]$

$$\|z(t, \cdot)\|_{L^2(0, L)}^2 \leq \|z_0\|_{L^2(0, L)}^2. \quad (2.96)$$

From now on, we will separate the proof into two cases: $\mathbf{sat} = \mathbf{sat}_{L^2(0, L)}$ and $\mathbf{sat} = \mathbf{sat}_{\text{loc}}$.

Case 1: $\mathbf{sat} = \mathbf{sat}_{L^2(0, L)}$.

Using (2.76), we have,

$$\|z(t, \cdot)\|_{L^2(0, L)} \leq r,$$

and we can apply the first item of Lemma 2.6. The inequality (2.93) becomes

$$\|z(t, \cdot)\|_{L^2(0, L)}^2 \leq \|z_0\|_{L^2(0, L)}^2 - \int_0^t |z_x(\sigma, 0)|^2 d\sigma - 2 \int_0^t \int_0^L ak(r)y^2 dx d\sigma. \quad (2.97)$$

Let us state a claim that will be useful in the following.

Claim 2.1

For any $T > 0$ and any $r > 0$ there exists a positive constant $C_{14} = C_{14}(T, r)$ such that for any solution z to (2.25) with an initial condition $z_0 \in L^2(0, L)$ such that $\|z_0\|_{L^2(0, L)} \leq r$, it holds that

$$\|z_0\|_{L^2(0, L)}^2 \leq C_{14} \left(\int_0^T |z_x(t, 0)|^2 dt + 2 \int_0^T \int_0^L k(r)a|z(t, x)|^2 dx dt \right). \quad (2.98)$$

Let us assume Claim 2.1 for the time being. Then (2.93) implies

$$\|z(kT, \cdot)\|_{L^2(0, L)}^2 \leq \gamma^k \|z_0\|_{L^2(0, L)}^2 \quad \forall k \geq 0, \forall t \geq 0, \quad (2.99)$$

where $\gamma \in (0, 1)$. From (2.96), we have $\|z(t, \cdot)\|_{L^2(0, L)} \leq \|z(kT, \cdot)\|_{L^2(0, L)}$ for $kT \leq t \leq (k+1)T$. Thus we obtain, for all $t \geq 0$,

$$\|z(t, \cdot)\|_{L^2(0, L)}^2 \leq \frac{1}{\gamma} \|z_0\|_{L^2(0, L)}^2 e^{\frac{\log \gamma}{T} t}, \quad (2.100)$$

which implies the global asymptotic stability of the mild solution z to (2.25), since $\gamma \in (0, 1)$.

In order to prove Claim 2.1, since the solution to (2.25) satisfies (2.95), it is sufficient to prove that there exists some constant $C_{15} := C_{15}(T, L) > 0$ such that

$$\int_0^T \int_0^L |z|^2 dx dt \leq C_{15} \left(\int_0^T |z_x(t, 0)|^2 dt + 2 \int_0^T \int_0^L k(r)ay^2 dx dt \right) \quad (2.101)$$

provided that $\|z_0\|_{L^2(0, L)} \leq r$. We argue by contradiction to prove the existence of such a constant C_{15} .

Suppose (2.101) fails to be true. Then, there exists a sequence of mild solutions $\{z^n\}_{n \in \mathbb{N}} \subseteq \mathcal{B}(T)$ of (2.25) with

$$\|z^n(0, \cdot)\|_{L^2(0, L)} \leq r \quad (2.102)$$

and such that

$$\lim_{n \rightarrow +\infty} \frac{\|z^n\|_{L^2(0, T; L^2(0, L))}^2}{\int_0^T |z_x^n(t, 0)|^2 dt + 2 \int_0^T \int_0^L k(r)a(z^n)^2 dx dt} = +\infty. \quad (2.103)$$

Note that (2.102) implies with (2.96) that

$$\|z^n(t, \cdot)\|_{L^2(0, L)} \leq r, \quad \forall t \in [0, T]. \quad (2.104)$$

Let $\lambda^n := \|z^n\|_{L^2(0, T; L^2(0, L))}$ and $v^n(t, x) = \frac{z^n(t, x)}{\lambda^n}$. Notice that $\{\lambda^n\}_{n \in \mathbb{N}}$ is bounded, according to (2.104). Hence, there exists a subsequence, that we continue to denote by $\{\lambda^n\}_{n \in \mathbb{N}}$ such that

$$\lambda^n \rightarrow \lambda \geq 0 \quad \text{as } n \rightarrow +\infty.$$

Then v^n fullfills

$$\begin{cases} v_t^n + v_x^n + v_{xxx}^n + \lambda^n v^n v_x^n + \frac{\text{sat}_{L^2(0, L)}(a\lambda^n v^n)}{\lambda^n} = 0, \\ v^n(t, 0) = v^n(t, L) = v_x^n(t, L) = 0, \\ \|v^n\|_{L^2(0, T; L^2(0, L))} = 1, \end{cases} \quad (2.105)$$

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and, due to (2.103), we obtain

$$\int_0^T |v_x^n(t, 0)|^2 dt + 2 \int_0^T \int_0^L ak(r)(v^n)^2 dx dt \rightarrow 0 \text{ as } n \rightarrow +\infty. \quad (2.106)$$

It follows from (2.95) that $\{v^n(0, \cdot)\}_{n \in \mathbb{N}}$ is bounded in $L^2(0, L)$. Note also that from (2.94) $\{v^n\}_{n \in \mathbb{N}}$ is bounded in $L^2(0, T; H^1(0, L))$. Thus we see that $\{v^n v_x^n\}_{n \in \mathbb{N}}$ is a subset of $L^2(0, T; L^1(0, L))$. In fact,

$$\|v^n v_x^n\|_{L^2(0, T; L^1(0, L))} \leq \|v^n\|_{C(0, T; L^2(0, L))} \|v^n\|_{L^2(0, T; H^1(0, L))}. \quad (2.107)$$

Moreover, we have that $\left\{ \frac{\text{sat}_{L^2(0, L)}(a\lambda^n v^n)}{\lambda^n} \right\}_{n \in \mathbb{N}}$ is a bounded sequence in $L^2(0, T; L^2(0, L))$.

Indeed, from Lemma 2.1

$$\left\| \frac{\text{sat}_{L^2(0, L)}(a\lambda^n v^n)}{\lambda^n} \right\|_{L^2(0, T; L^2(0, L))} \leq 3 \|av^n\|_{L^2(0, T; L^2(0, L))} \leq 3a_1 \sqrt{L} \|v^n\|_{L^2(0, T; H^1(0, L))}. \quad (2.108)$$

Thus $\{v^n v_x^n\}_{n \in \mathbb{N}}$ and $\left\{ \frac{\text{sat}_{L^2(0, L)}(a\lambda^n v^n)}{\lambda^n} \right\}_{n \in \mathbb{N}}$ are also subsets of $L^2(0, T; H^{-2}(0, L))$ since $L^2(0, L) \subset L^1(0, L) \subset H^{-1}(0, L) \subset H^{-2}(0, L)$. Combined with (2.105) it implies that $\{v_t^n\}_{n \in \mathbb{N}}$ is a bounded sequence in $L^2(0, T; H^{-2}(0, L))$. Since $\{v^n\}_{n \in \mathbb{N}}$ is a bounded sequence of $L^2(0, T; H^1(0, L))$, then we get with Lemma 2.7 that a subsequence of $\{v^n\}_{n \in \mathbb{N}}$ also denoted by $\{v^n\}_{n \in \mathbb{N}}$ converges strongly in $L^2(0, T; L^2(0, L))$ to a limit v . Moreover, with the last line of (2.105), it holds that $\|v\|_{L^2(0, T; L^2(0, L))} = 1$.

Therefore, having in mind (2.106), we get

$$\|v_x(\cdot, 0)\|_{L^2(0, T)}^2 + \int_0^T \int_0^L ak(r)v^2 dx dt \leq \liminf_{n \rightarrow +\infty} \left\{ \|v_x^n(\cdot, 0)\|_{L^2(0, T)}^2 + \int_0^T \int_0^L ak(r)(v^n)^2 dx dt \right\} = 0. \quad (2.109)$$

Thus,

$$ak(r)v^2(t, x) = 0, \quad \forall x \in [0, L], \forall t \in (0, T), \quad \text{and} \quad v_x(t, 0) = 0, \quad \forall t \in (0, T). \quad (2.110)$$

and therefore

$$v(t, x) = 0, \quad \forall x \in \Omega, \forall t \in (0, T), \quad \text{and} \quad v_x(t, 0) = 0, \quad \forall t \in (0, T). \quad (2.111)$$

We obtain that the limit function v satisfies

$$\begin{cases} v_t + v_x + v_{xxx} + \lambda v v_x = 0, \\ v(t, 0) = v(t, L) = v_x(t, L) = 0, \\ \|v\|_{L^2(0, T; L^2(0, L))} = 1, \end{cases} \quad (2.112)$$

with $\lambda \geq 0$. Let us consider $\tilde{v} = v_t$ which satisfies

$$\begin{cases} \tilde{v}_t + \tilde{v}_x + \tilde{v}_{xxx} + \lambda v_x \tilde{v} + \lambda v \tilde{v}_x = 0, \\ v(t, 0) = v(t, L) = v_x(t, L) = 0, \end{cases} \quad (2.113)$$

with $\tilde{v}(0, \cdot) = -v'(0, \cdot) - v'''(0, \cdot) - \lambda v(0, \cdot)v'(0, \cdot) \in H^{-3}(0, L)$ and

$$u(t, x) = 0, \quad \forall x \in \Omega, \forall t \in (0, T), \quad \text{and} \quad u_x(t, 0) = 0, \quad \forall t \in (0, T).$$

Let us recall the following result.

Lemma 2.8 ([71], Lemma 3.2)

There exists a positive value $C_{16}(T, r) > 0$ such that for any solution \tilde{v} to (2.113) where v is solution to (2.112), it holds

$$\|\tilde{v}_x(\cdot, 0)\|_{L^2(0, T)}^2 + \|\tilde{v}(0, \cdot)\|_{H^{-3}(0, L)}^2 \geq C_{16} \|\tilde{v}(0, \cdot)\|_{L^2(0, L)}^2. \quad (2.114)$$

Applying the result of this lemma, we get $v(0, \cdot) \in L^2(0, L)$ and therefore $\tilde{v} = v_t \in \mathcal{B}(T)$. Since $v, v_t \in L^2(0, T; H^1(0, L))$ and $v \in C([0, T]; H^1(0, L))$, we can conclude that $vv_x \in L^2(0, T; L^2(0, L))$. In this way, $v_{xxx} = -v_t - v_x - \lambda vv_x \in L^2(0, T; L^2(0, L))$ and therefore $v \in L^2(0, T; H^3(0, L))$. Finally, using Theorem 2.8, we obtain

$$v(t, x) = 0, \quad \forall x \in [0, L], \quad t \in [0, T].$$

Thus we get a contradiction with $\|v\|_{L^2(0, T; L^2(0, L))} = 1$. This concludes the proof of Claim 2.1 and also the proof of Lemma 2.4 in the case where $\mathbf{sat} = \mathbf{sat}_{L^2(0, L)}$.

Case 2: $\mathbf{sat} = \mathbf{sat}_{\text{loc}}$.

Following the same strategy as before, we write the following claim.

Claim 2.2

For any $T > 0$ and any $r > 0$, there exists a positive constant $C_{17} = C_{17}(T, r)$ such that for any mild solution z to (2.25) with an initial condition $z_0 \in L^2(0, L)$ such that $\|z_0\|_{L^2(0, L)} \leq r$, it holds that

$$\|z_0\|_{L^2(0, L)}^2 \leq C_{17} \left(\int_0^T |z_x(t, 0)|^2 dt + 2 \int_0^T \int_0^L \mathbf{sat}_{\text{loc}}(az(t, x))z(t, x) dt dx \right). \quad (2.115)$$

If Claim 2.2 holds, we obtain also (2.100) for a suitable choice of γ and we end the proof of Lemma 2.4 when $\mathbf{sat} = \mathbf{sat}_{\text{loc}}$. Due to (2.95), we see that in order to prove Claim 2.2, it is sufficient to obtain the existence of $C_{18} > 0$ such that

$$\int_0^T \int_0^L |z(t, x)|^2 dt dx \leq C_{18} \left(\int_0^T |z_x(t, 0)|^2 dt + 2 \int_0^T \int_0^L \mathbf{sat}_{\text{loc}}(az(t, x))z(t, x) dt dx \right). \quad (2.116)$$

We argue by contradiction to prove (2.116). To this end, we assume that there exists a sequence of mild solutions $\{z^n\}_{n \in \mathbb{N}} \subseteq \mathcal{B}(T)$ to (2.25) with

$$\|z^n(0, \cdot)\|_{L^2(0, L)} \leq r \quad (2.117)$$

and such that

$$\lim_{n \rightarrow +\infty} \frac{\|z^n\|_{L^2(0,T;L^2(0,L))}^2}{\int_0^T |z_x^n(t,0)|^2 dt + 2 \int_0^T \int_0^L \mathbf{sat}_{\text{loc}}(az^n(t,x))z^n(t,x) dt dx} = +\infty. \quad (2.118)$$

Note that (2.117) implies with (2.96) that

$$\|z^n(t, \cdot)\|_{L^2(0,L)} \leq r, \quad \forall t \in [0, T]. \quad (2.119)$$

Note that we have, from (2.92) and (2.94)

$$\|z^n\|_{L^2(0,T;H^1(0,L))}^2 \leq \beta,$$

where

$$\beta := \frac{8T + 2L}{3} r^2 + \frac{TC}{27} r^4.$$

Moreover, due to the Poincaré inequality and the left Dirichlet boundary condition of (2.25), we obtain

$$\sup_{x \in [0,L]} |z^n(t,x)| \leq \sqrt{L} \|z^n(t, \cdot)\|_{H^1(0,L)}, \quad \forall t \in [0, T]. \quad (2.120)$$

Thus, we see that

$$\int_0^T |z^n(t,x)|^2 dt \leq L \|z^n\|_{L^2(0,T;H^1(0,L))}^2 \leq L\beta. \quad (2.121)$$

Now let us consider $\Omega_i \subset [0, T]$ defined as follows

$$\Omega_i = \left\{ t \in [0, T], \sup_{x \in [0,L]} |z(t,x)| > i \right\}. \quad (2.122)$$

In the following, we will denote by Ω_i^c its complement. It is defined by

$$\Omega_i^c = \left\{ t \in [0, T], \sup_{x \in [0,L]} |z(t,x)| \leq i \right\}. \quad (2.123)$$

Since the function $t \mapsto \sup_{x \in [0,L]} |z^n(t,x)|^2$ is a nonnegative function, we have

$$\int_0^T \sup_{x \in [0,L]} |z^n(t,x)|^2 dt \geq \int_{\Omega_i} \sup_{x \in [0,L]} |z^n(t,x)|^2 dt \geq i^2 \nu(\Omega_i), \quad (2.124)$$

where $\nu(\Omega_i)$ denotes the Lebesgue measure of Ω_i . Therefore, with (2.121), we obtain

$$\nu(\Omega_i) \leq \frac{L\beta}{i^2}. \quad (2.125)$$

We deduce from the previous equation that

$$\max\left(T - \frac{L\beta}{i^2}, 0\right) \leq \nu(\Omega_i^c) \leq T. \quad (2.126)$$

Moreover, with the second item of Lemma 2.6, we have, for all $i \in \mathbb{N}$,

$$\begin{aligned} \int_0^T \int_0^L \mathbf{sat}_{\text{loc}}(az^n)z^n dt dx &= \int_{\Omega_i} \int_0^L \mathbf{sat}_{\text{loc}}(az^n)z^n dt dx + \int_{\Omega_i^c} \int_0^L \mathbf{sat}_{\text{loc}}(az^n)z^n dt dx \\ &\geq \int_{\Omega_i^c} \int_0^L \mathbf{sat}_{\text{loc}}(az^n)z^n dt dx \\ &\geq \int_{\Omega_i^c} \int_0^L ak(i)(z^n)^2 dt dx. \end{aligned} \quad (2.127)$$

Let $\lambda^n := \|z^n\|_{L^2(0,T;L^2(0,L))}$ and $v^n(t,x) = \frac{z^n(t,x)}{\lambda^n}$. Notice that $\{\lambda^n\}_{n \in \mathbb{N}}$ is bounded, according to (2.119). Hence, there exists a subsequence, that we continue to denote by $\{\lambda^n\}_{n \in \mathbb{N}}$ such that

$$\lambda^n \rightarrow \lambda \geq 0, \text{ as } n \rightarrow +\infty.$$

Then, v^n fullfills

$$\begin{cases} v_t^n + v_x^n + v_{xxx}^n + \lambda^n v^n v_x^n + \frac{\mathbf{sat}_{\text{loc}}(a\lambda^n v^n)}{\lambda^n} = 0, \\ v^n(t,0) = v^n(t,L) = v_x^n(t,L) = 0, \\ \|v^n\|_{L^2(0,T;L^2(0,L))} = 1, \end{cases} \quad (2.128)$$

and, due to (2.118),

$$\int_0^T |v_x^n(t,0)|^2 dt + 2 \int_0^T \int_0^L \frac{\mathbf{sat}_{\text{loc}}(a\lambda^n v^n)}{\lambda^n} v^n dt dx \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Moreover, due to (2.127), we have, for all $i \in \mathbb{N}$,

$$\int_0^T |v_x^n(t,0)|^2 dt + 2 \int_{\Omega_i^c} \int_0^L ak(i)(v^n)^2 dt dx \rightarrow 0 \text{ as } n \rightarrow +\infty. \quad (2.129)$$

Note that from Lemma 2.1,

$$\left\| \frac{\mathbf{sat}_{\text{loc}}(a\lambda^n v^n)}{\lambda^n} \right\|_{L^2(0,T;L^2(0,L))} \leq 3a_1 \sqrt{L} \|v^n\|_{L^2(0,T;H^1(0,L))}. \quad (2.130)$$

and therefore the sequence $\left\{ \frac{\mathbf{sat}_{\text{loc}}(a\lambda^n v^n)}{\lambda^n} \right\}_{n \in \mathbb{N}}$ is a subset of $L^2(0,T;L^2(0,L))$. In addition, $\{v^n v_x^n\}_{n \in \mathbb{N}}$ is a bounded sequence of $L^2(0,T;L^1(0,L))$. Note that $L^2(0,L) \subset L^1(0,L) \subset H^{-2}(0,L)$, thus $\left\{ \frac{\mathbf{sat}_{\text{loc}}(a\lambda^n v^n)}{\lambda^n} \right\}_{n \in \mathbb{N}}$ and $\{v^n v_x^n\}_{n \in \mathbb{N}}$ are bounded sequences of $L^2(0,T;H^{-2}(0,L))$. Since $v_t^n = -v_x^n - v_{xxx}^n - \lambda^n v^n v_x^n - \frac{\mathbf{sat}_{\text{loc}}(a\lambda^n v^n)}{\lambda^n}$, we know that $\{v_t^n\}_{n \in \mathbb{N}}$ is a subset of $L^2(0,T;H^{-2}(0,L))$. Since $\{v^n\}_{n \in \mathbb{N}}$ is a subset of $L^2(0,T;H^1(0,L))$, we obtain from Lemma 2.7 that $\{v^n\}_{n \in \mathbb{N}}$ converges strongly to a function v in $L^2(0,T;L^2(0,L))$. Futhermore, with (2.129) and due to the non-negativity of k , we have, for all $i \in \mathbb{N}$,

$$ak(i)v(t,x) = 0, \forall x \in [0,L], \forall t \in \Omega_i^c, \text{ and } v_x(t,0) = 0, \forall t \in (0,T). \quad (2.131)$$

Thus, since for all $i \in \mathbb{N}$, $k(i)$ is strictly positive, we have

$$v(t,x) = 0, \forall x \in \Omega, \forall t \in \Omega_i^c, \text{ and } v_x(t,0) = 0, \forall t \in (0,T). \quad (2.132)$$

We obtain

$$v(t, x) = 0, \forall x \in \Omega, \forall t \in \bigcup_{i \in \mathbb{N}} \Omega_i^c, \text{ and } v_x(t, 0) = 0, \forall t \in (0, T). \quad (2.133)$$

Since, with (2.126), we know that $\nu(\bigcup_{i \in \mathbb{N}} \Omega_i^c) = T$, we get that, for almost every $t \in [0, T]$,

$$v(t, x) = 0, \forall x \in \Omega, \text{ and } v_x(t, 0) = 0. \quad (2.134)$$

We obtain that v fullfills

$$\begin{cases} v_t + v_x + v_{xxx} + \lambda v v_x = 0, \\ v(t, 0) = v(t, L) = v_x(t, L) = 0, \end{cases}, \quad \|v\|_{L^2(0, T; L^2(0, L))} = 1. \quad (2.135)$$

Thus v is a solution to a Korteweg-de Vries equation. In particular, it belongs to $\mathcal{B}(T)$ and is consequently in $C(0, T; L^2(0, L))$. Therefore, (2.134) becomes

$$v(t, x) = 0, \forall x \in \Omega, \forall t \in [0, T], \text{ and } v_x(t, 0) = 0, \forall t \in (0, T). \quad (2.136)$$

We are in the same situation as (2.112). Therefore we obtain once again a contradiction. We can conclude that Claim 2 is true. It concludes the proof of Lemma 2.4 when $\mathbf{sat} = \mathbf{sat}_{1oc}$ and completes the proof of Proposition 2.4. \square

Remark 2.6

Since the strategy followed in the last section is to argue by contradiction, we cannot estimate the exponential rate μ . However, such a proof allows us to prove the local exponential stability of the solution whatever the saturation \mathbf{sat} is.

2.2.3.7 Proof of Theorem 2.5

We are now in position to prove Theorem 2.5, following [79]. By Proposition 2.4, there exists a positive value μ^* such that if

$$\|\tilde{z}_0\|_{L^2(0, L)} \leq 1, \quad (2.137)$$

then the corresponding solution \tilde{z} to (2.25) satisfies

$$\|\tilde{z}(t, \cdot)\|_{L^2(0, L)} \leq K_1 \|\tilde{z}_0\|_{L^2(0, L)} e^{-\mu^* t} \quad \forall t \geq 0, \quad (2.138)$$

for some constants $K_1 \geq 1$ which depends only on $\|\tilde{z}_0\|_{L^2(0, L)}$. In addition, for a given $r > 0$, there exist two constants $K_r > 0$ and $\mu_r > 0$ such that if $\|z_0\|_{L^2(0, L)} \leq r$, then any mild solution z to (2.25) satisfies

$$\|z(t, \cdot)\|_{L^2(0, L)} \leq K_r \|z_0\|_{L^2(0, L)} e^{-\mu_r t} \quad \forall t \geq 0. \quad (2.139)$$

Consequently, setting $T_r := \mu_r^{-1} \ln(r K_r)$, we have

$$\|z_0\|_{L^2(0, L)} \leq r \Rightarrow \|z(t_r, \cdot)\|_{L^2(0, L)} \leq 1.$$

Therefore, using (2.138), we obtain

$$\begin{aligned} \|z(t, \cdot)\|_{L^2(0, L)} &\leq K_1 \|z(t_r, \cdot)\|_{L^2(0, L)} e^{-\mu^*(t - T_r)} && \forall t \geq T_r, \\ &\leq K_1 K_r \|z_0\|_{L^2(0, L)} e^{\mu^* T_r} e^{-\mu^* t} && \forall t \geq 0. \end{aligned} \quad (2.140)$$

Thus it concludes the proof of Theorem 2.5. \square

Remark 2.7

As it has been noticed in Remark 2.4, the same result as in Theorem 2.5 can be obtained for (2.21) following the strategy of [79]. Note that in [79], a stabilization in $H^3(0, L)$ is obtained. The authors used a similar strategy as the one described in Remark 2.4. Hence, it seems harder to obtain such a result for (2.83), since the saturation introduces some non-smoothness.

2.3 Simulations

2.3.1 Numerical scheme

In this section we provide some numerical simulations showing the effectiveness of our control design. In order to discretize our KdV equation, we use a finite difference scheme inspired by [70]. The final time is denoted T_{final} . We choose $(N_x + 1)$ points to build a uniform spatial discretization of the interval $[0, L]$ and $(N_t + 1)$ points to build a uniform time discretization of the interval $[0, T_{\text{final}}]$. We pick a space step defined by $dx = L/N_x$ and a time step defined by $dt = T_{\text{final}}/N_t$. We approximate the solution to (2.25) with the following notation $z(t, x) \approx \mathcal{Z}_j^i$, where i denotes the time and j the space discrete variables.

Some used approximations of the derivative are given by

$$\mathcal{D}_- z = \frac{\mathcal{Z}_j^i - \mathcal{Z}_{j-1}^i}{dx} \tag{2.141}$$

and

$$\mathcal{D}_+ z = \frac{\mathcal{Z}_{j+1}^i - \mathcal{Z}_j^i}{dx}. \tag{2.142}$$

As in [70], we choose the numerical scheme $z_x(t, x) \approx \frac{1}{2}(\mathcal{D}_+ + \mathcal{D}_-)(\mathcal{Z}_j^i) := \mathcal{D}(\mathcal{Z}_j^i)$ and $z_t(t, x) \approx \frac{\mathcal{Z}_j^{i+1} - \mathcal{Z}_j^i}{dt}$. For the other differentiation operator, we use $z_{xxx}(t, x) \approx \mathcal{D}_+ \mathcal{D}_+ \mathcal{D}_-(\mathcal{Z}_j^i)$.

Let us introduce a matrix notation. Let us consider the matrices $D_-, D_+, D \in \mathbb{R}^{N_x \times N_x}$ given by

$$D_- = \frac{1}{dx} \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ -1 & 1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & 0 & -1 & 1 \end{bmatrix}, \quad D_+ = \frac{1}{dx} \begin{bmatrix} -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -1 & 1 \\ 0 & \dots & \dots & 0 & -1 \end{bmatrix} \tag{2.143}$$

$$D := \frac{1}{2}(D_+ + D_-) \tag{2.144}$$

and let us define $\mathcal{A} = D_+ D_+ D_- + D$, and $\mathcal{C} = \mathcal{A} + dtI$ where I is the identity matrix in $\mathbb{R}^{N_x \times N_x}$. Note that we choose this forward difference approximation in order to obtain a positive definite matrix \mathcal{C} .

Moreover, for each discrete time i , we denote $\mathcal{Z}^i := [\mathcal{Z}_1^i \ \mathcal{Z}_2^i \ \dots \ \mathcal{Z}_{N_x+1}^i]^\top$.

Thus, inspired by [70], we consider a completely implicit numerical scheme for the approximation of the nonlinear problem (2.25) which reads as follows:

$$\begin{cases} \frac{\mathcal{Z}_j^{i+1} - \mathcal{Z}_j^i}{dt} + (\mathcal{A}\mathcal{Z}^{i+1})_j + \frac{1}{2} (D[(\mathcal{Z}^{i+1})^2])_j + \mathbf{sat}(a_\delta \mathcal{Z}_j^{i+1}) = 0, & j = 1, \dots, N_x, \\ \mathcal{Z}_1^i = \mathcal{Z}_{N_x+1}^i = \mathcal{Z}_{N_x}^i = 0, \\ \mathcal{Z}^1 = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} z_0(x) dx, \end{cases} \quad (2.145)$$

where $x_j = (j + \frac{1}{2})dx$, $x_j = jdx$ and \mathcal{Z}^1 denotes the discretized version of the initial condition $z_0(x)$. Note that a_δ is the approximation of the damping function $a = a(x)$ and is given by $a_\delta = (a_j)_{j=1}^{N_x} \in \mathbb{R}^{N_x}$, where each components a_j is defined by $a_j := \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} a(x) dx$.

Since we have the nonlinearities zz_x and $\mathbf{sat}(az)$, we use an iterative Newton fixed-point method to solve the nonlinear system

$$\mathcal{C}\mathcal{Z}^{i+1} = \mathcal{Z}^i - dt \frac{1}{2} D(\mathcal{Z}^{i+1})^2 - dt \mathbf{sat}(a_\delta \mathcal{Z}^{i+1}).$$

With $N_{iter} = 5$, which denotes the number of iterations of the fixed point method, we get good approximations of the solutions. Note that for sufficiently large N_{iter} the solutions can be approximated with this fixed-point method.

Given \mathcal{Z}^1 satisfying (2.145), the following is the structure of the algorithm used in our simulations.

For $i = 1 : N_t$

- $\mathcal{Z}_1^i = \mathcal{Z}_{N_x}^i = \mathcal{Z}_{N_x+1}^i = 0$;
- Setting $J(1) = \mathcal{Z}^i$, for all $k \in \{1, \dots, N_{iter}\}$, solve

$$J(k+1) = \mathcal{C}^{-1}(\mathcal{Z}^i - dt \frac{1}{2} D(J(k))^2 - dt \mathbf{sat}(a_\delta J(k)))$$

Set $\mathcal{Z}^{i+1} = J(N_{iter})$

end

2.3.2 Simulations

In order to illustrate our theoretical results, we perform some simulations with $L = 2\pi$, for which we know that the linearized KdV equation is not asymptotically stable. To be more specific, letting $z_0(x) = 1 - \cos(x)$ and $f = 0$, it holds that the energy $\|z\|_{L^2(0,L)}^2$ of the

linearized equation (2.5) remains constant for all $t \geq 0$. Let us perform a simulation of (2.25) with these parameters.

We first simulate our system in the case where the damping is not localized. We use the saturation function $\mathbf{sat}_{L^2(0,L)}$. Given $a_0 = 1$, $T_{final} = 6$ and $L = 2\pi$, Figure 2.2 shows the solution to (2.17), denoted by z_w , with the unsaturated control $f = a_0 z_w$ and starting from z_0 . Figure 2.3 illustrates the simulated solution with the same initial condition and a saturated control $f = \mathbf{sat}_{L^2(0,L)}(a_0 z)$ where $u_s = 0.5$. Figure 2.4 gives the evolution of the control with respect to time and space. We check in Figures 2.2 and 2.3 that the solution to (2.25) converges to 0 with the unsaturated and the saturated controls as proven in Theorem 2.5.

The evolution of the L^2 -energy of the solution in these two cases is given by Figure 2.5. With $\|z_0\|_{L^2(0,L)} := 3.07$ and the values of u_s , a_0 and a_1 , the value μ is computed numerically following the formula (2.31) given in Theorem 2.5. It is equal to $\mu = 0.3257$. We deduce from the second point of Theorem 2.5 that the energy function $\|z\|_{L^2(0,L)}^2$ converges exponentially to 0 with an explicit decay rate given by μ as stated in Theorem 2.5.

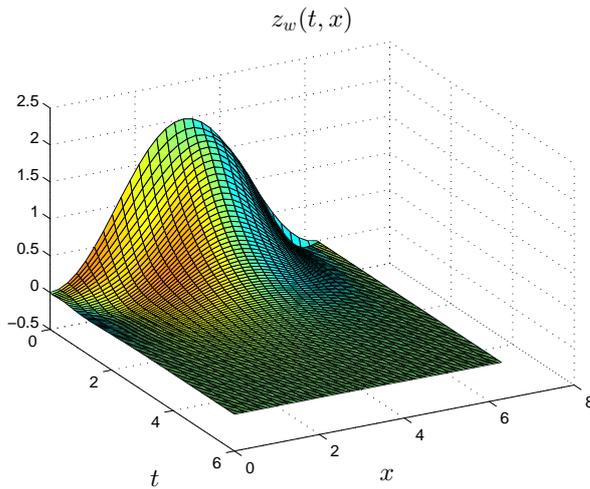


Figure 2.2: Solution $z_w(t, x)$ with the control $f = a_0 z_w$ where $\Omega = [0, L]$

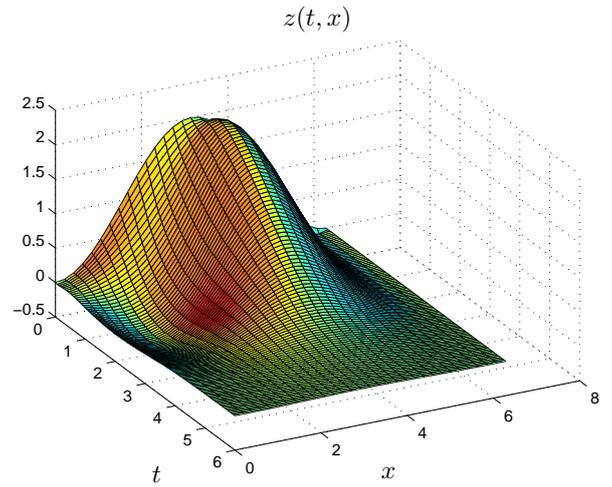


Figure 2.3: Solution $z(t, x)$ with the control $f = \mathbf{sat}_{L^2(0,L)}(a_0 z)$ where $\Omega = [0, L]$, $u_s = 0.5$

We now focus on the case where the damping is localized. We close the loop with the saturated controller $f = \mathbf{sat}_{1_{oc}}(ay)$ where a is defined by $a(x) = a_0 = 1$, for all $x \in \Omega := [\frac{1}{3}L, \frac{2}{3}L]$.

Given $T_{final} = 6$, Figure 2.6 shows the simulated solution of (2.17), denoted by z_w , with a localized control that is not saturated and starting from z_0 . Figure 2.7 illustrates the simulated solution to (2.25) with the same initial condition, but with a localized saturated control whose saturation level is given by $u_s = 0.5$. We check, in Figures 2.6 and 2.7, that the mild solution to (2.25) converges to 0 as stated in Theorem 2.5. Moreover, Figure 2.8 gives the evolution of the control with respect to the time and the space.

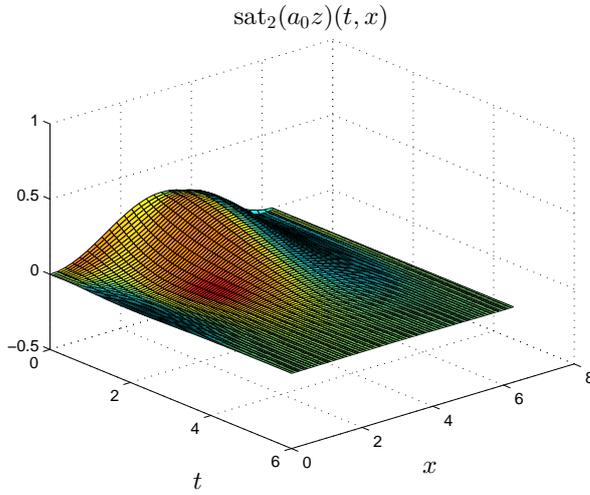


Figure 2.4: Control $f = \text{sat}_{L^2(0,L)}(a_0 z)(t, x)$ where $\Omega = [0, L]$, $u_s = 0.5$

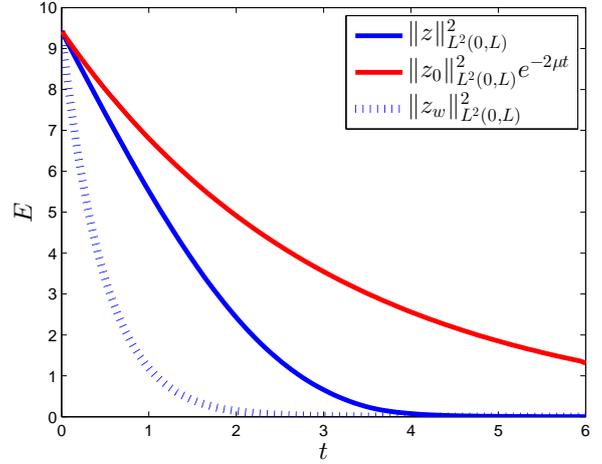


Figure 2.5: Blue: Time evolution of the energy function $\|z\|_{L^2(0,L)}^2$ with a saturation $u_s = 0.5$ and $a_0 = 1$. Red: Time evolution of the theoretical energy $\|z_0\|_{L^2(0,L)}^2 e^{-2\mu t}$. Dotted line: Time evolution of the solution without saturation z_w and $a_0 = 1$.

The evolution of the L^2 -energy of the solution in these two last cases is given by Figure 2.9. We can see that the energy function $\|z\|_{L^2(0,L)}^2$ converges exponentially to 0 as stated in Proposition 2.4. However, in contrary with the case $\text{sat} = \text{sat}_{L^2(0,L)}$ and $\Omega = [0, L]$, we cannot have an estimation of the decay rate since our proof is based on a contradiction argument.

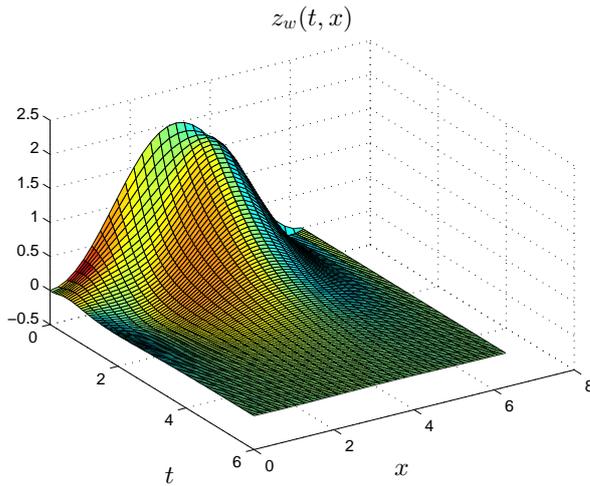


Figure 2.6: Solution $z_w(t, x)$ with a localized feedback law without saturation

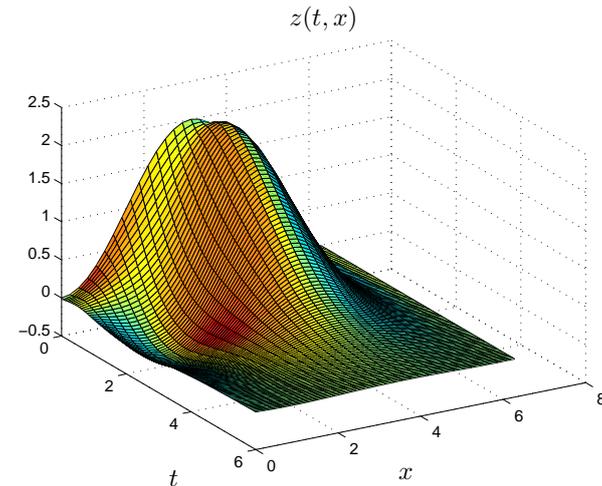


Figure 2.7: Solution $z(t, x)$ with a localized feedback law saturated; $u_s = 0.5$

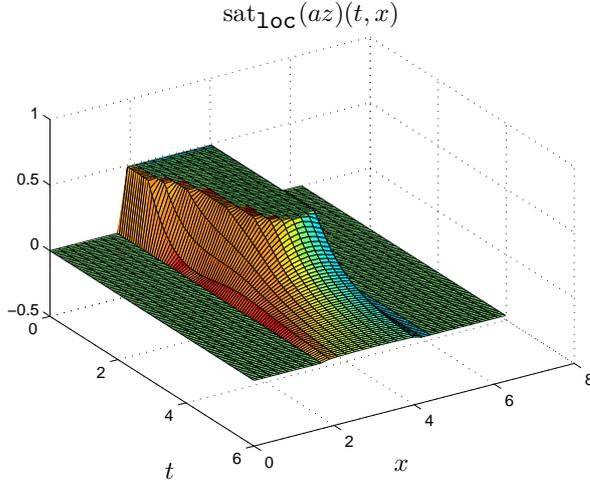


Figure 2.8: Control $f = \text{sat}_{\text{loc}}(az)(t, x)$ where $\Omega = [\frac{1}{3}L, \frac{2}{3}L]$, $u_s = 0.5$

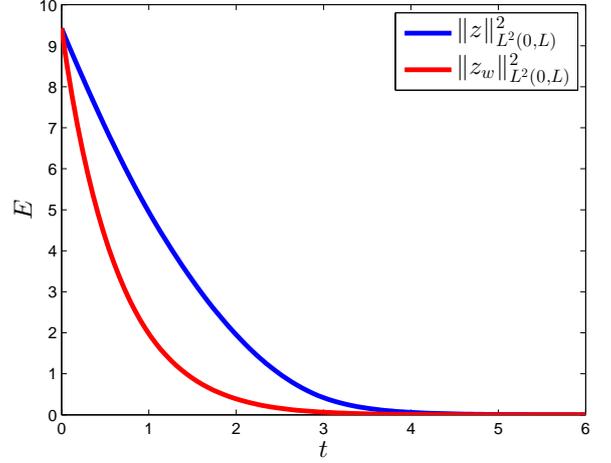


Figure 2.9: Blue: Time evolution of the energy function $\|z\|_{L^2(0,L)}^2$ with a saturation $u_s = 0.5$, $a_0 = 1$ and $\Omega = [\frac{1}{3}L, \frac{2}{3}L]$. Dotted line: Time evolution of the solution without saturation z_w with $a_0 = 1$ and $\Omega = [\frac{1}{3}L, \frac{2}{3}L]$.

2.3.3 Further discussions

2.3.3.1 On the numerical dissipation

In order to illustrate that the origin of the linearized system (2.5) when $L \in \mathcal{N}$ is not attractive, we performed some simulations with a numerical scheme inspired by the one presented above. Keeping the same notation than before, it is provided here

$$\begin{cases} \frac{\mathcal{Z}_j^{i+1} - \mathcal{Z}_j^i}{dt} + (\mathcal{A}\mathcal{Z}^{i+1})_j = 0, & j = 1, \dots, N_x, \\ \mathcal{Z}_1^i = \mathcal{Z}_{N_x+1}^i = \mathcal{Z}_{N_x}^i = 0, \\ \mathcal{Z}^1 = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} z_0(x) dx, \end{cases} \quad (2.146)$$

Since the numerical scheme is linear, studying the numerical stability of this numerical scheme is easier than before. The system given by (2.146) being discrete, the condition to insure the numerical scheme to be stable is the eigenvalues of the inverse of \mathcal{C} to have real parts less or equal to 1.

As in Section 2.3.2, let us pick $L = 2\pi$, $N_x = 30$ and $T_{\text{final}} = 6$. Recall that when $L = 2\pi$, then $L \in \mathcal{N}$, which implies that the origin of (2.5) is not attractive.

In this case, we compute the eigenvalues of \mathcal{C}^{-1} in *Matlab* and we obtain that the eigenvalues which have the highest real part are

$$\lambda_M^1 = 0.9986 + 0.0083i, \quad \lambda_M^2 = 0.9986 - 0.0083i. \quad (2.147)$$

These eigenvalues are close to 1. It means that the origin of (2.146) is close to be marginally stable, which implies that we are close to the continuous case. The asymptotic stability of the origin of (2.146) is due to a *numerical dissipation*, which is necessary for the stability of the numerical scheme.

We pick $z_0(x) = 1 - \cos(x)$. Figure 2.10 illustrates the solution z to (2.5). We can note that the decay rate of the solution is slower than in Figures 2.2, 2.3 and 2.6. Figure 2.11 illustrates the energy function $\|z\|_{L^2(0,L)}^2$, where z is solution to (2.5), and the functions $t \mapsto \|z\|_{L^2(0,L)}^2 e^{-\mu_1 t}$ and $t \mapsto \|z\|_{L^2(0,L)}^2 e^{-\mu_2 t}$ where $\mu_1 = 0.065$ and $\mu_2 = 0.1$. Therefore, we can conclude that the origin of (2.146) is exponentially stable with a decay rate μ satisfying $\mu_1 \leq \mu \leq \mu_2$. These decay rates being very small with respect to the ones in Figure 2.5, this is clear that this stability is due to the numerical dissipativity and not to a natural global asymptotic stability given by the properties of the continuous system. The numerical dissipativity is needed to ensure the stability of the numerical scheme.

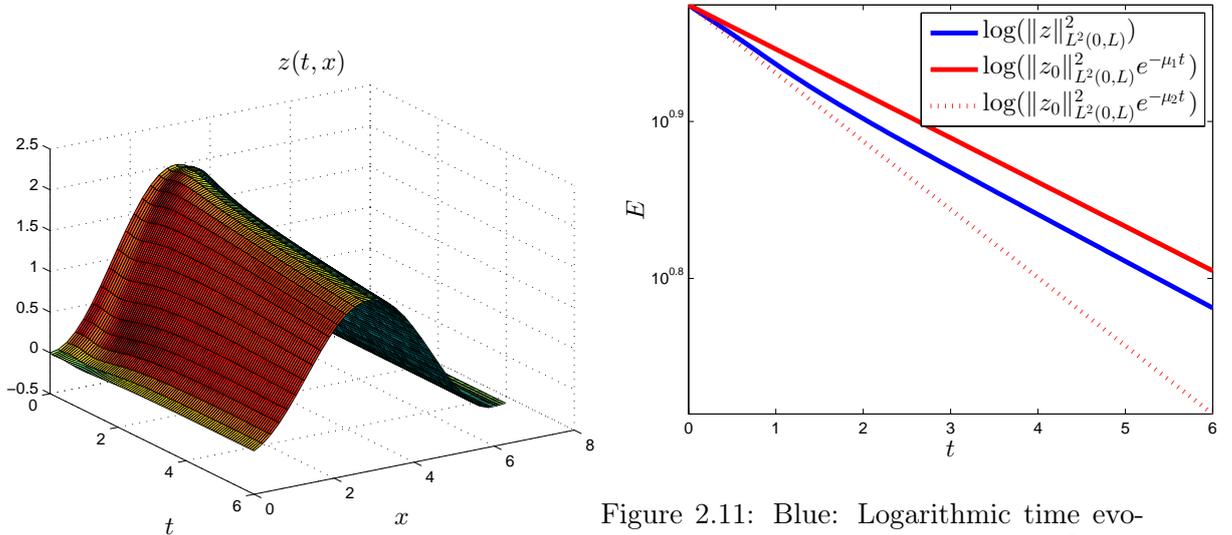


Figure 2.10: Solution $z(t, x)$ of the linearized Korteweg-de Vries equation.

Figure 2.11: Blue: Logarithmic time evolution of the energy function $\|z\|_{L^2(0,L)}^2$ of the linearized Korteweg-de Vries equation; Red: Logarithmic time evolution of $\|z_0\|_{L^2(0,L)}^2 e^{-\mu_1 t}$; Dotted red line: Logarithmic time evolution of $\|z_0\|_{L^2(0,L)}^2 e^{-\mu_2 t}$

2.3.3.2 On the saturation level

Theorem 2.5 states that, in the case where $\mathbf{sat} = \mathbf{sat}_{L^2(0,L)}$ and where $\Omega = [0, L]$, we have an estimation of the decay rate of the solution, which is recalled here

$$\mu = \min \left\{ a_0, \frac{u_s a_0}{r a_1} \right\}, \quad (2.148)$$

where a_0 and a_1 are the bounds of (2.18) and where r is the bound of the initial condition $\|z_0\|_{L^2(0,L)}$.

Note that the case where $\mu = a_0$ is the case where $\|az_0\|_{L^2(0,L)} \leq u_s$. Indeed, in such a case, because of the dissipativity of the solution z to (2.25), the controller will never saturate.

The case where $\mu = \frac{u_s a_0}{r a_1}$ is the case where $\|az_0\|_{L^2(0,L)} \geq u_s$. This implies that, in the particular case where $\mathbf{sat} = \mathbf{sat}_{L^2(0,L)}$ and where $\Omega = [0, L]$, the smaller the saturation level is, the slower is the convergence to 0. In this small section, we would like to show that it is probably the case for either $\mathbf{sat} = \mathbf{sat}$ or $\mathbf{sat} = \mathbf{sat}_{L^2(0,L)}$, whatever is Ω by performing some numerical simulations.

We use the same numerical scheme provided in Section 2.3.1. As in Section 2.3.2, let us pick $L = 2\pi$, $N_x = 30$ and $T_{\text{final}} = 6$. We pick also $z_0(x) = 1 - \cos(x)$. In this case, we have $\|z_0\|_{L^2(0,L)} = 3.07$.

We focus first on the case where $\mathbf{sat} = \mathbf{sat}_{L^2(0,L)}$ and where $\Omega = [0, L]$. The gain (2.18) is such that $a(x) = a_0 = a_1$, for all $x \in [0, L]$. In this case, as it has been noticed just above, we can have an estimation of the decay rate of the solution. Figure 2.12 illustrates the time evolution of the energy function $\|z\|_{L^2(0,L)}^2$ when $u_s = 0.1$ and z is the mild solution to (2.25), the time evolution of the theoretical energy $\|z_0\|_{L^2(0,L)}^2 e^{-2\mu t}$ and the time evolution of the energy function $\|z_w\|_{L^2(0,L)}^2$ where z_w is the mild solution to (2.19). In this case, we have $\mu = 0.0326$. Figure 2.13 illustrates the same energies with $u_s = 1$. In this case, we have $\mu = 0.3257$. Regarding these two figures, it is easy to see that, in the case where $u_s = 1$, the solution is faster than in the case where $u_s = 0.1$. It seems moreover that the estimation μ is worst in the case where $u_s = 0.1$ than in the case where $u_s = 1$.

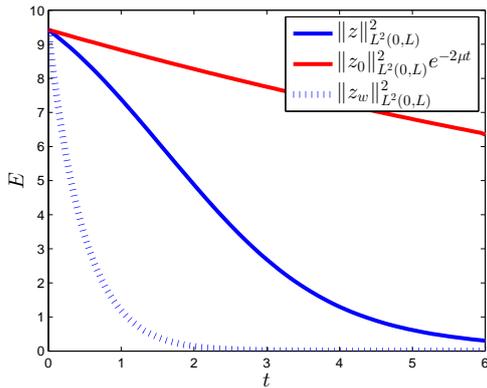


Figure 2.12: Blue: Time evolution of the energy function $\|z\|_{L^2(0,L)}^2$ with a saturation $u_0 = 0.1$ and $a_0 = 1$. Red: Time evolution of the theoretical energy $\|z_0\|_{L^2(0,L)}^2 e^{-2\mu t}$. Dotted line: Time evolution of the solution without saturation z_w and $a_0 = 1$.

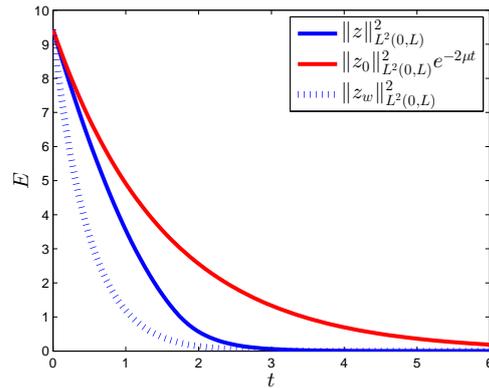


Figure 2.13: Blue: Time evolution of the energy function $\|z\|_{L^2(0,L)}^2$ with a saturation $u_0 = 1$ and $a_0 = 1$. Red: Time evolution of the theoretical energy $\|z_0\|_{L^2(0,L)}^2 e^{-2\mu t}$. Dotted line: Time evolution of the solution without saturation z_w and $a_0 = 1$.

We focus now on the case where $\mathbf{sat} = \mathbf{sat}_{\text{loc}}$ and where $\Omega = [\frac{1}{3}L, \frac{2}{3}L]$. Recall that the decay rate cannot be estimated in the case where $\Omega \neq [0, L]$. The gain (2.18) is such that $a(x) = a_0 = a_1$, for all $x \in \Omega$. Figure 2.12 illustrates the time evolution of the energy function $\|z\|_{L^2(0,L)}^2$ when $u_s = 0.1$ and z is the solution to (2.25) and the time evolution of the energy function $\|z_w\|_{L^2(0,L)}^2$ where z_w is solution to (2.4) without saturation. Figure 2.13 illustrates the same energies with $u_s = 1$. Regarding these two figures, it is easy to see that, in the case where $u_s = 1$, the solution is faster than in the case where $u_s = 0.1$.

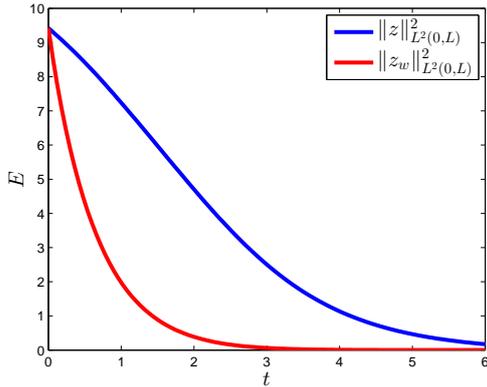


Figure 2.14: Blue: Time evolution of the energy function $\|z\|_{L^2(0,L)}^2$ with a saturation $u_0 = 1$ and $a_0 = 1$. Red: Time evolution of the theoretical energy $\|z_0\|_{L^2(0,L)}^2 e^{-2\mu t}$. Dotted line: Time evolution of the solution without saturation z_w and $a_0 = 1$.

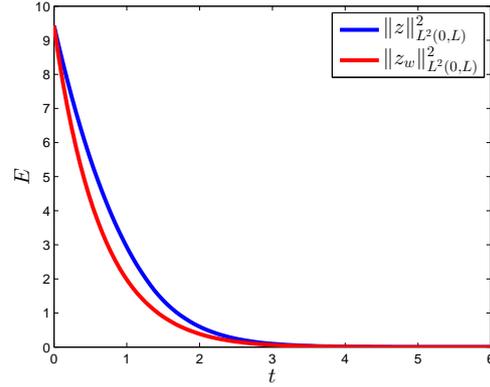


Figure 2.15: Blue: Time evolution of the energy function $\|z\|_{L^2(0,L)}^2$ with a saturation $u_0 = 1$, $a_0 = 1$ and $\Omega = [\frac{1}{3}L, \frac{2}{3}L]$. Dotted line: Time evolution of the solution without saturation z_w with $a_0 = 1$ and $\Omega = [\frac{1}{3}L, \frac{2}{3}L]$.

2.4 Conclusion

In this chapter, we have studied the well-posedness and the asymptotic stability of a Korteweg-de Vries equation with saturated distributed controls. The well-posedness issue has been tackled by using the Banach fixed-point theorem. The stability has been studied with two different methods: in the case where the control acts on all the domain saturated with $\mathbf{sat}_{L^2(0,L)}$, we used a sector condition and Lyapunov theory for infinite-dimensional systems; in the case where the control acts only on a part of the domain saturated with either $\mathbf{sat}_{L^2(0,L)}$ or $\mathbf{sat}_{\text{loc}}$, we argued by contradiction. We have also illustrated our results with some simulations.

To conclude, let us state some questions arising in this context:

1. Can a saturated localized damping stabilize a generalized Korteweg-de Vries equation in $H^3(0, L)$, as it has been shown to be true in the unsaturated case in [79] and [53] ?
2. Is it possible to saturate other damping terms, for instance the one suggested in [73] and used in [66] which dissipates the H^{-1} -norm in the unsaturated case?

3. Another constraint than saturation can be considered. For instance the backlash studied in [93] or the quantization [30].

4. Can we apply the same method for other nonlinear partial differential equations, for instance the Kuramoto-Sivashinsky equation [13, 17] ?

The latter chapters were dealing with distributed controls. Applying such a control needs to have a distributed sensor, i.e. a sensor measuring the solution at each time and at each point of a spatial domain. Moreover, the control has to act on each point of a spatial domain, which might be impossible in practice. To be closer to the physical problem, the control should be located at the boundary. In such cases, we say that the control is a *boundary control*. The next chapter will deal with a Korteweg-de Vries equation controlled from the boundary.

Output feedback stabilization of Korteweg-de Vries equations

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In Chapter 2, some feedback laws for the KdV equation with a distributed controller have been reviewed. As it has been noticed in the conclusion of Chapter 2, for some infinite-dimensional systems, the control is located at the boundary. In this case, computing or proving the existence of a feedback law requires to apply different techniques than the ones presented in the latter chapters.

In this chapter, we focus on the boundary stabilization problem for two KdV equations with a control acting on the left Dirichlet boundary condition. The backstepping method is used in the two cases (for a good introduction to this method, the interested reader may refer to [50]).

Some boundary feedback laws have already been designed in the literature for KdV systems. When the control acts on the right endpoint, we find [16] where a Gramian-based method is applied, and [24] where some suitable integral transforms are used. In [89] and [14], the authors use the backstepping method to design feedback controllers acting on the left

endpoint of the interval. In the case of these two latter papers, note that the feedback law depends on the full state.

However, in most cases, the full state of the system cannot be measured. Thus, it is more realistic to design an output feedback control, i.e., a feedback law depending only on some partial measurements of the state. For some systems, designing such a feedback requires to design also an observer, which is a dynamical system chosen such that its state converges asymptotically to the state of the control system.

For autonomous linear finite-dimensional systems, the separation principle holds. Thus, stabilizability and observability assumptions are sufficient to ensure the stability of the closed-loop system with a dynamic output feedback based on an observer. In other words, if there exists a controller which asymptotically stabilizes the origin of the system and an observer which converges asymptotically to the state of the system, the output feedback built from this observer and this state feedback asymptotically stabilizes the origin of the system.

The aim of this chapter is to present the backstepping method for the design of a state feedback for some Korteweg-de Vries equations and to design an observer, also based on the backstepping method, which will allow us to construct an output feedback law. We will show that in this particular case the separation principle holds.

This chapter is divided into three parts. Section 3.1 presents the output feedback problem for finite-dimensional linear systems and recall some well-known results. The strategy presented in this section will be applied in the next sections. Section 3.2 explains the backstepping method for the design of a state feedback for one of the two Korteweg-de Vries equations we are considering and introduces our main contributions. It also provides the proof of these contributions. Section 3.3 provides some simulations illustrating our results. Finally, Section 3.4 collects some concluding remarks.

The contributions of this chapter, from Section 3.2.2.1 until the end, are based on two of our papers. The first one [59] has been published in a peer-reviewed conference (53rd IEEE Conference on Decision and Control). The second one [58] has been accepted for publication in the journal *Automatica*.

3.1 A short discussion on output feedback laws for finite-dimensional linear systems

3.1.1 Static output feedback

In this section, we will present the output feedback design for linear finite-dimensional systems. Let us consider a linear finite-dimensional system

$$\frac{d}{dt}z = Az + Bu \tag{3.1}$$

where, given $n, m \in \mathbb{N}$, $z \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. The matrix A and B have appropriate dimensions. We assume that the pair (A, B) is controllable¹.

A state feedback law is a control depending on the state of the system. However, in practice, only a partial measurement of the solution is available. It is called the *output* and is given by the following algebraic relation

$$y = Cx, \tag{3.2}$$

where y stands for the output (i.e, the measurement of the system) and where C is a matrix of appropriate dimension. We assume that the pair (A, C) is observable².

Stabilizing (3.1) by means of a feedback law depending only on the measurement y requires to find a function K in the following form

$$u = K(y). \tag{3.3}$$

One may want to find a matrix $K \in \mathbb{R}^{p \times m}$ such that the closed-loop system

$$\frac{d}{dt}z = (A + BKC)z \tag{3.4}$$

is globally exponentially stable. Such a feedback law is called a *static output feedback*, since it does not depend on any additional dynamics.

Finding such a matrix requires the matrix A , B and C to satisfy restrictive spectrum properties. The interested reader may refer to [55] for a precise statement of such a result. The following gives an example of a system that cannot be stabilized by means of a static output feedback.

Example 3.1

Focus on the following matrices

$$A := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1 \quad 0]. \tag{3.5}$$

It is easy to check that the pairs (A, B) and (A, C) are controllable and observable, respectively.

Given $K \in \mathbb{R}$, the closed-loop system with a static output feedback $u = KCz$ is given by

$$\frac{d}{dt}z = \begin{bmatrix} 0 & 1 \\ K & 0 \end{bmatrix} z \tag{3.6}$$

Hence, if K is a positive value, the eigenvalues of the closed-loop system are

$$\lambda_1 = \sqrt{K} \text{ and } \lambda_2 = -\sqrt{K}. \tag{3.7}$$

Therefore, the origin of the closed-loop system is unstable.

¹The pair (A, B) is controllable if and only if $\text{Rank} \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = n$.

²The pair (A, C) is observable if and only if $\text{Rank} \begin{bmatrix} C & CA & \dots & CA^{n-1} \end{bmatrix} = n$

If K is a negative value, then the eigenvalues of the closed-loop system are

$$\lambda_1 = j\sqrt{K} \text{ and } \lambda_2 = -j\sqrt{K}. \quad (3.8)$$

Therefore, the origin of the closed-loop system is stable, but is not attractive. This means that the origin of the system described with the matrix (3.5) cannot be stabilized by means of a static output feedback.

In such cases, another technique should be used. The most known is the *dynamic output feedback* based on an *observer*.

3.1.2 Output feedback based on an observer

3.1.2.1 Observer design

An observer for (3.1) is a dynamic system depending on the measurement y . It is designed such that its state converges to the state of (3.1).

For (3.1), a well-known observer is the Luenberger observer. It is written as follows

$$\frac{d}{dt}\hat{z} = A\hat{z} + Bu + PC(z - \hat{z}), \quad (3.9)$$

where $\hat{z} \in \mathbb{R}^n$ stands for the observer and P is a matrix of appropriate dimension, which is called the *observer gain*. Denoting $\tilde{z} := \hat{z} - z$, the matrix $P \in \mathbb{R}^{p \times n}$ has to be designed such that the origin of the following system

$$\frac{d}{dt}\tilde{z} = (A - PC)\tilde{z}, \quad (3.10)$$

is globally exponentially stable.

Since the pair (A, C) is observable, then there exists a matrix $P \in \mathbb{R}^{p \times n}$ such that the matrix $A - PC$ is Hurwitz³. Hence, the origin of (3.10) is exponentially stable.

Using an observer allows us to have an estimation of the state of the plant, but also to solve the stabilization problem of (3.1). Such a result is referred to as the *Separation principle*.

3.1.2.2 Separation principle

Since the pair (A, B) is controllable, there exists a matrix $K \in \mathbb{R}^{m \times p}$ that makes exponentially stable the origin of following closed-loop system

$$\frac{d}{dt}z = (A - BK)z. \quad (3.11)$$

³Given $n \in \mathbb{N}$, a matrix $A \in \mathbb{R}^{n \times n}$ is said to be *Hurwitz* if every eigenvalue of A has strictly negative real part.

Since we are able to estimate the state by using the observer (3.9), we would like to use it for the closed-loop system as follows

$$\frac{d}{dt}z = Az - BK\hat{z}. \quad (3.12)$$

The following theorem states that the dynamic output feedback $u = K\hat{z}$ makes the origin of the system composed by (3.12) and (3.9) globally asymptotically stable.

Theorem 3.1 (Separation principle)

Consider system (3.1). If the pairs (A, B) and (A, C) are controllable and observable, respectively, then there exist a gain $K \in \mathbb{R}^{m \times n}$ and a gain $P \in \mathbb{R}^{p \times n}$ such that the origin of the closed-loop system

$$\begin{cases} \frac{d}{dt}z = Az - BK\hat{z}, \\ \frac{d}{dt}\hat{z} = A\hat{z} + Bu + PC(z - \hat{z}), \end{cases} \quad (3.13)$$

is globally exponentially stable.

In other words, Theorem 3.1 states that, if the observer and the state feedback law are designed separately, then the resulting output feedback law stabilizes the origin of (3.13).

Remark 3.1

This principle is no longer true in general for linear infinite-dimensional systems and even for nonlinear finite-dimensional systems. In the next chapter, an example of a nonlinear finite-dimensional control system for which the separation does not hold will be provided. In the context of abstract control systems described with linear operators, some results related to the separation principle can be found in [26, Chapter 5, Theorem 5.3.3.]. Note however that this separation principle holds only for bounded control and output operators.

The aim of this chapter is to follow this strategy for two Korteweg-de Vries equations controlled and measured from the boundary. The state feedbacks and the observers are designed separately with a *backstepping method* (a good introduction to this method is [50]). This method will be presented in the remaining parts of this chapter.

3.2 Output feedback law for some Korteweg-de Vries equations

In this section, two Korteweg-de Vries equations will be considered. The first one is a linearized one, which is given by

$$\begin{cases} z_t + z_x + z_{xxx} = 0, & (t, x) \in \mathbb{R}_{\geq 0} \times [0, L], \\ z(t, 0) = u(t), & t \in \mathbb{R}_{\geq 0}, \\ z(t, L) = z_x(t, L) = 0 & t \in \mathbb{R}_{\geq 0}, \\ z(0, x) = z_0(x), & x \in [0, L], \end{cases} \quad (3.14)$$

where u stands for the control. The output is given by

$$y_1(t) = z_{xx}(t, L). \quad (3.15)$$

The second one is nonlinear and is given by

$$\begin{cases} z_t + z_x + z_{xxx} + zz_x = 0, & (t, x) \in \mathbb{R}_{\geq 0} \times [0, L], \\ z(t, 0) = u(t), & t \in \mathbb{R}_{\geq 0}, \\ z_x(t, L) = z_{xx}(t, L) = 0, & t \in \mathbb{R}_{\geq 0}, \\ z(0, x) = z_0(x), & x \in [0, L] \end{cases} \quad (3.16)$$

The output is given by

$$y_2(t) = z(t, L). \quad (3.17)$$

Note that the boundary conditions are different for the two considered Korteweg-de Vries equations.

In [14], a state feedback law has been designed for these two partial differential equations. We will first focus on (3.14) in order to explain the backstepping method for the design of state feedback laws.

3.2.1 State feedback design: backstepping method

3.2.1.1 Main results

There exist different methods to design a state feedback law for boundary control partial differential equations. The Lyapunov method, which reduces to finding a Lyapunov function, has received considerable attention in recent decades. However, it might be hard to derive such a function.

The backstepping method is another technique which is often used for infinite-dimensional boundary control systems. The design is a two-steps design. First, one picks a system, which is called the *target system*, with a structure similar to the control system we aim at stabilizing. The target system is chosen such that its origin is globally asymptotically stable. The target system and the control system we aim at stabilizing are linked with a transformation - which can be a *Volterra* transformation or a *Fredholm* transformation. Then, one proves that this transformation is continuous and that its inverse exists and is also continuous. From this property, the stability and the well-posedness of the closed-loop system is straightforward. A nice introduction to this technique is [50].

Thanks to this method, the authors of [14] stated the following theorem:

Theorem 3.2 ([14], Theorem 1.)

(i) For any positive value λ , there exist a feedback control $u = u(z(t, \cdot))$ and a positive value D such that the strong solution to (3.14) satisfies

$$\|z(t, \cdot)\|_{L^2(0, L)} \leq D e^{-\lambda t} \|z_0\|_{L^2(0, L)}, \quad \forall t \geq 0; \quad (3.18)$$

(ii) For any positive value λ , there exist a feedback control $u = u(z(t, \cdot))$ and two positive values D and r such that the mild solution to (3.16) satisfies

$$\|z(t, \cdot)\|_{L^2(0,L)} \leq D e^{-\lambda t} \|z_0\|_{L^2(0,L)}, \quad \forall t \geq 0, \quad (3.19)$$

provided that

$$\|z_0\|_{L^2(0,L)} \leq r. \quad (3.20)$$

Remark 3.2

Note that the first item (i) of Theorem 3.2 is a global exponential stability result, while the second item (ii) is a local exponential stability result. This is consistent with the facts that (3.14) is a linear equation and that (3.16) is a nonlinear one. Note moreover that the result provided by (i) holds for either strong or mild solutions. This comes from the fact that the system which is considered is linear.

This method is explained in the next section for the design of the state feedback law for (3.14).

3.2.1.2 Backstepping procedure

In the case of (3.14), the authors of [14] chose the transformation $\Pi^1 : L^2(0, L) \rightarrow L^2(0, L)$ that is defined as follows

$$w(t, x) := \Pi^1(z)(t, x) := z(t, x) - \int_x^L k^1(x, s) z(t, s) ds, \quad (3.21)$$

where k^1 is a kernel that will be characterized later and where w solves the following Korteweg-de Vries equation

$$\begin{cases} w_t + w_x + w_{xxx} + \lambda w = 0, & (t, x) \in \mathbb{R}_{\geq 0} \times [0, L], \\ w(t, 0) = w(t, L) = w_x(t, L) = 0 & t \in \mathbb{R}_{\geq 0}, \\ w(0, x) = w_0(x), \end{cases} \quad (3.22)$$

where λ is a positive value. If one uses this transformation, the control $u(t)$ in (3.14) becomes

$$u(t) := \int_0^L k^1(0, s) z(t, s) ds. \quad (3.23)$$

Let us prove that the origin of (3.22) is globally exponentially stable. Focus on the following Lyapunov function

$$V(w) := \frac{1}{2} \int_0^L w(t, x)^2 dx. \quad (3.24)$$

Its derivative along (3.22) yields

$$\frac{d}{dt} V(w) = - \int_0^L w(t, x) w_x(t, x) dx - \int_0^L w(t, x) w_{xxx}(t, x) dx - \lambda \int_0^L w(t, x)^2 dx. \quad (3.25)$$

Performing some integrations by parts, the latter equation becomes

$$\begin{aligned} \frac{d}{dt}V(w) &= -w_x(t,0)^2 - \lambda \int_0^L w(t,x)^2 dx \\ &\leq -\lambda \int_0^L w(t,x)^2 dx \end{aligned} \quad (3.26)$$

Hence, applying the comparison principles yields

$$\|w(t, \cdot)\|_{L^2(0,L)} \leq e^{-\frac{\lambda}{2}t} \|w_0\|_{L^2(0,L)}, \quad \forall t \geq 0. \quad (3.27)$$

Therefore, proving the exponential stability of the origin of (3.14) reduces to prove that Π^1 defined in (3.21) is continuous and that its inverse is also continuous. This proof is achieved by studying the kernel k^1 , that is defined in the triangle $\mathcal{T} := \{(x, s) \in \mathbb{R}_{\geq 0}^2 \mid x \in [0, L], s \in [x, L]\}$. According to [14], this kernel k^1 satisfies the following partial differential equation

$$\begin{cases} k_{xxx}^1 + k_{sss}^1 + k_x^1 + k_s^1 = -\lambda k^1, & x, s \in \mathcal{T}, \\ k^1(x, L) = 0, & x \in [0, L], \\ k^1(x, x) = 0, & x \in [0, L], \\ k_x^1(x, x) = \frac{\lambda}{3}(L - x), & x \in [0, L]. \end{cases} \quad (3.28)$$

Using a method of successive approximations, the authors of [14] proved that k^1 exists and is unique. Appendix C introduces the method of successive approximations and applies it to (3.28).

Then, the authors of [14] focused on the inverse of Π^1 denoted by $\Pi^{1^{-1}} : L^2(0, L) \rightarrow L^2(0, L)$ and which is given by

$$z(t, x) := \Pi^{1^{-1}}(w)(t, x) = w(t, x) + \int_x^L l^1(x, s)w(t, s)ds, \quad (3.29)$$

where l^1 solves the following partial differential equation

$$\begin{cases} l_{xxx}^1 + l_{sss}^1 + l_x^1 + l_s^1 = \lambda l^1, & (x, s) \in \mathcal{T}, \\ l^1(x, L) = 0, & x \in [0, L], \\ l^1(x, x) = 0, & x \in [0, L], \\ l_x^1(x, x) = \frac{\lambda}{3}(L - x), & x \in [0, L]. \end{cases} \quad (3.30)$$

The authors prove that there exists a unique solution to this equation. Finally, we have $\Pi^1, \Pi^{1^{-1}} \in \mathcal{L}(L^2(0, L))$.

Thus, from (3.27) and using the boundedness of Π^1 and $\Pi^{1^{-1}}$, the following holds, for any $t \geq 0$

$$\begin{aligned} \|z(t, \cdot)\|_{L^2(0,L)} &\leq \|\Pi^{1^{-1}}\|_{\mathcal{L}(L^2(0,L), L^2(0,L))} \|w(t, \cdot)\|_{L^2(0,L)} \\ &\leq \|\Pi^{1^{-1}}\|_{\mathcal{L}(L^2(0,L), L^2(0,L))} e^{-\lambda t} \|w_0\|_{L^2(0,L)} \\ &\leq \|\Pi^{1^{-1}}\|_{\mathcal{L}(L^2(0,L), L^2(0,L))} \|\Pi^1\|_{\mathcal{L}(L^2(0,L), L^2(0,L))} e^{-\lambda t} \|z_0\|_{L^2(0,L)}. \end{aligned} \quad (3.31)$$

3.2.1.3 Well-posedness of (3.14)

This section is devoted to the well-posedness of (3.14). Some tools from semigroup theory will be used.

One can state the following theorem:

Theorem 3.3 (Well-posedness of (3.14))

Assume that the control satisfies (3.23). Then, for any initial condition $z_0 \in L^2(0, L)$, there exists a unique mild solution to (3.14).

If moreover z_0 belongs to $H^3(0, L)$ and satisfies the following compatibility condition

$$z_0(0) = \int_0^L k^1(0, s)z_0(s)ds, \quad z_0(L) = z_0'(L) = 0, \quad (3.32)$$

then there exists a unique strong solution $z \in C(0, \infty; H^3(0, L))$ to (3.14) satisfying, for all $t \geq 0$

$$z(t, 0) = \int_0^L k^1(0, s)z(t, s)ds, \quad z(t, L) = z_x(t, L) = 0, \quad (3.33)$$

Proof of Theorem 3.3:

This proof is inspired by [89]. Using semigroup theory, we first prove that the target system is well-posed and then, using the continuity of Π^1 and its inverses, we conclude on the well-posedness of (3.14).

It is easy to see that (3.22) is well-posed. Indeed, one can apply the classical semigroup theory to prove it. This system can be described by the following operator

$$\begin{aligned} \tilde{A} : D(\tilde{A}) \subset L^2(0, L) &\rightarrow L^2(0, L) \\ z &\mapsto -z' - z''' - \lambda z, \end{aligned} \quad (3.34)$$

where

$$D(\tilde{A}) = \{z \in H^3(0, L) \mid z(0) = z(L) = z'(L) = 0\}. \quad (3.35)$$

A straightforward computation, together with some integrations by parts, shows that

$$\begin{aligned} \Re_e \{ \langle \tilde{A}z, z \rangle_{L^2(0, L)} \} &\leq 0, \quad z \in D(\tilde{A}), \\ \Re_e \{ \langle \tilde{z}, \tilde{A}^* \tilde{z} \rangle_{L^2(0, L)} \} &\leq 0, \quad \tilde{z} \in D(\tilde{A}^*). \end{aligned}$$

Since \tilde{A} is a linear operator and $D(\tilde{A})$ is dense in $L^2(0, L)$, according to Theorems 1.1 and 1.3, these latter inequalities imply that \tilde{A} is the infinitesimal generator of a linear strongly continuous semigroup of contractions on $L^2(0, L)$.

Note that we have

$$A\Pi^1 = \Pi^1\tilde{A}, \quad (3.36)$$

where A is defined as follows

$$\begin{aligned} A : D(A) \subset L^2(0, L) &\rightarrow L^2(0, L), \\ z &\mapsto -z' - z'', \end{aligned} \quad (3.37)$$

where

$$D(A) := \left\{ z \in H^3(0, L) \mid z(0) = \int_0^L k(0, s)z(s)ds, z(L) = z'(L) = 0 \right\} \quad (3.38)$$

Therefore, the following holds

$$A = \Pi^1 \tilde{A} \Pi^{1^{-1}}. \quad (3.39)$$

Hence, since the operators Π^1 and Π^{-1} are both bounded, then A generates also a strongly continuous semigroup of contractions on $L^2(0, L)$. From Theorem 1.4, this implies that there exists a unique strong solution $z \in C(0, \infty; D(A))$ to the Cauchy problem (3.14). \square

3.2.2 Dynamic output feedback design for (3.14)

3.2.2.1 Main result

Following the Separation Principle strategy, we aim at finding an observer in order to design an output feedback. Based on [50] and [85], we construct the following observer:

$$\begin{cases} \hat{z}_t + \hat{z}_x + \hat{z}_{xxx} + p_1(x)[y_1(t) - \hat{z}_{xx}(t, L)] = 0, \\ \hat{z}(t, 0) = u(t), \hat{z}(t, L) = \hat{z}_x(t, L) = 0, \\ \hat{u}(0, x) = 0, \end{cases} \quad (3.40)$$

where $p_1 : [0, L] \rightarrow C^3(0, L)$ is a function that will be characterized later. Note that this observer has a similar form as (3.9).

Thanks to this observer, one is able to design an output feedback law for (3.14). Here is the statement of this result.

Theorem 3.4

For any $\lambda_0 > 0$, there exist an output feedback law $u(t) := u(\hat{z}(t, x))$, a function $p_1 = p_1(x)$, and a constant $C > 0$ such that the origin of the coupled system (3.14)-(3.40)

$$\begin{cases} z_t + z_x + z_{xxx} = 0, \\ z(t, 0) = u(t), z(t, L) = z_x(t, L) = 0, \\ \hat{z}_t + \hat{z}_x + \hat{z}_{xxx} + p_1(x)[z_{xx}(t, L) - \hat{z}_{xx}(t, L)] = 0, \\ \hat{z}(t, 0) = u(t), \hat{z}(t, L) = \hat{z}_x(t, L) = 0, \end{cases} \quad (3.41)$$

is globally exponentially stable with a decay rate equal to λ_0 . In other words we have, for any $z_0, \hat{z}_0 \in H^3(0, L)$ satisfying

$$z_0 = \hat{z}_0 = u(0), z_0(L) = \hat{z}_0(L) = z'_0(L) = \hat{z}'_0(L) = 0, \quad (3.42)$$

any strong solution to (3.41) satisfies, for all $t \geq 0$

$$\|z(t, \cdot) - \hat{z}(t, \cdot)\|_{H^3(0,L)} + \|\hat{z}(t, \cdot)\|_{L^2(0,L)} \leq Ce^{-\lambda_0 t} (\|z_0 - \hat{z}_0\|_{H^3(0,L)} + \|\hat{z}_0\|_{L^2(0,L)}), \quad \forall t \geq 0. \quad (3.43)$$

Remark 3.3

Theorem 3.4 does not deal with well-posedness of the closed-loop system. Indeed, since the function p_1 is not clearly defined at this stage, such a result cannot be yet provided. We need more informations on the function p_1 to state a well-posedness result. This result is written in Theorem (3.5) below. The proof of this result is given in Section 3.2.2.4 below.

3.2.2.2 Regularity Result

As we said in the introduction, we consider

$$y_1(t) = z_{xx}(t, L) \quad (3.44)$$

as a partial measurement of the solution. However, since we have the trace of the second derivative with respect to x of z , we need a regularity stronger than in [14]. Indeed, we ask the output $y_1(t)$ to be a continuous function. Thus we have the following lemma.

Lemma 3.1

Let us consider system

$$\begin{cases} z_t + z_x + z_{xxx} = 0, \\ z(t, 0) = u(t), z(t, L) = 0, z_x(t, L) = 0, \\ z(0, x) = z_0(x), \end{cases} \quad (3.45)$$

where $z_0 \in H^3(0, L)$ and satisfies the following compatibility condition

$$z_0(0) = u(0), z_0(L) = z_0'(L) = 0, \quad (3.46)$$

with $u(t) \in H^1(0, T)$. Then $z \in C([0, T], H^3(0, L)) \cap L^2(0, T; H^4(0, L))$ and $z_{xx}(\cdot, L) \in C([0, T])$.

Proof of Lemma 3.1

This proof is based on [33]. Let us consider the following coordinates transformation

$$v = z_t. \quad (3.47)$$

The dynamics of v can be written as follows:

$$\begin{cases} v_t + v_{xxx} + v_x = 0 \\ v(t, 0) = \frac{d}{dt}u(t) \in L^2(0, T), v(t, L) = v_x(t, L) = 0 \\ v(0, x) = (-z_0''' - z_0') \in L^2(0, L) \end{cases} \quad (3.48)$$

By already known well-posedness results for KdV ([33]), we get

$$\begin{aligned}
 & v \in C([0, T], L^2(0, L)) \cap L^2(0, T; H^1(0, L)) \\
 & \Rightarrow z_t \in C([0, T], L^2(0, L)) \cap L^2(0, T; H^1(0, L)) \\
 & \Rightarrow z \in C([0, T], H^3(0, L)) \cap L^2(0, T; H^4(0, L)) \\
 & \Rightarrow z_{xx} \in C([0, T], H^1(0, L)) \cap L^2(0, T; H^2(0, L)) \\
 & \Rightarrow z_{xx} \in C([0, T] \times [0, L]) \\
 & \Rightarrow z_{xx}(\cdot, L) \in C([0, T])
 \end{aligned} \tag{3.49}$$

It concludes the proof of Lemma 3.1. \square

3.2.2.3 Observer design

The structure of the observer (3.40) is based on the finite-dimensional observer that has been recalled in Section 3.1. Since we are dealing with an infinite-dimensional system, note however that the gain $p_1 : [0, L] \rightarrow C^3(0, L)$ is not a matrix as in the finite-dimensional case (3.9), but a function.

In our case, we consider the error $\tilde{z} := z - \hat{z}$, which satisfies:

$$\begin{cases} \tilde{z}_t + \tilde{z}_x + \tilde{z}_{xxx} - p_1(x)\tilde{z}_{xx}(t, L) = 0, \\ \tilde{z}(t, 0) = \tilde{z}(t, L) = \tilde{z}_x(t, L) = 0. \end{cases} \tag{3.50}$$

Given a positive parameter λ , we look for a transformation Π_o^1 defined by:

$$\tilde{z}(t, x) = \Pi_o^1(\tilde{w}(t, x)) = \tilde{w}(t, x) - \int_x^L p^1(x, s)\tilde{w}(t, s)ds \tag{3.51}$$

such that the trajectory \tilde{z} solution to (3.50) is mapped into the trajectory $\tilde{w} = \tilde{w}(t, x)$, solution to the linear system

$$\begin{cases} \tilde{w}_t + \tilde{w}_x + \tilde{w}_{xxx} + \lambda\tilde{w} = 0, \\ \tilde{w}(t, 0) = 0, \tilde{w}(t, L) = 0, \tilde{w}_x(t, L) = 0, \end{cases} \tag{3.52}$$

which is exponentially stable with a decay rate depending on the value of λ as shown in (3.27).

Now, the key step is to find the kernel $p^1 = p^1(x, s)$ such that $\tilde{z}(t, x) = \Pi_o^1(\tilde{w}(t, x))$ satisfies (3.50). By focusing on (3.95) and using the Leibniz rules, we get:

- Differentiation along (3.52)

$$\begin{aligned}
\tilde{z}_t &= \tilde{w}_t(x) - \int_x^L p^1(x, s)[- \tilde{w}_s(s, t) - \tilde{w}_{sss}(s, t) - \lambda \tilde{w}(s, t)] ds \\
&= \tilde{w}_t(x) - \int_x^L (-\lambda p^1(x, s) + p_s^1(x, s) + p_{sss}^1(x, s)) \tilde{w}(s, t) ds \\
&\quad + p^1(x, L) \tilde{w}(L, t) - p^1(x, x) \tilde{w}(x, t) - p^1(x, L) \tilde{w}_{xx}(L, t) \\
&\quad + p^1(x, x) \tilde{w}_{xx}(x, t) - p_s^1(x, x) \tilde{w}_x(x, t) + p_s^1(x, L) \tilde{w}_x(L, t) \\
&\quad - p_s^1(x, L) \tilde{w}(L, t) + p_s^1(x, x) \tilde{w}(x, t)
\end{aligned} \tag{3.53}$$

- Three differentiations with respect to the variable x

$$\tilde{z}_x(x, t) = \tilde{w}_x(x, t) + p^1(x, x) \tilde{w}(x, t) - \int_x^L p_x^1(x, s) \tilde{w}(t, s) ds \tag{3.54}$$

$$\tilde{z}_{xx}(t, x) = \tilde{w}_{xx}(t, x) + \frac{d}{dx} p^1(x, x) \tilde{w}(t, x) + p^1(x, x) \tilde{w}_x(t, x) + p_x^1(x, x) \tilde{w}(t, x) - \int_x^L p_{xx}^1(x, s) \tilde{w}(t, s) ds \tag{3.55}$$

$$\begin{aligned}
\tilde{z}_{xxx}(t, x) &= \tilde{w}_{xxx}(t, x) + \frac{d^2}{dx^2} p^1(x, x) \tilde{w}(t, x) + 2 \frac{d}{dx} p^1(x, x) \tilde{w}_x(t, x) + p^1(x, x) \tilde{w}_{xx}(t, x) \\
&\quad + \frac{d}{dx} p_x^1(x, x) \tilde{w}(t, x) + p_x^1(x, x) \tilde{w}_x(t, x) + p_{xx}^1(x, x) \tilde{w}(t, x) - \int_x^L p_{xxx}^1(x, s) \tilde{w}(t, s) ds
\end{aligned} \tag{3.56}$$

By adding (3.53), (3.54) and (3.56), we get:

$$\begin{aligned}
\tilde{z}_t + \tilde{z}_x + \tilde{z}_{xxx} - p_1(x) \tilde{z}_{xx}(L) &= \tilde{w}_t(t, x) + \tilde{w}_x(t, x) + \tilde{w}_{xxx}(t, x) + \lambda \tilde{w}(t, x) \\
&\quad - \int_x^L (-\lambda p^1(x, s) + p_s^1(x, s) + p_{sss}^1(x, s) + p_{xxx}^1(x, s) + p_x^1(x, s)) \tilde{w}(t, s) ds \\
&\quad + \tilde{w}_x(t, x) \left(2 \frac{d}{dx} p^1(x, x) + p_x^1(x, x) + p_s^1(x, x) \right) \\
&\quad + \tilde{w}(t, x) \left(p_{xx}^1(x, x) + \frac{d^2}{dx^2} p^1(x, x) + \frac{d}{dx} p_x^1(x, x) + \frac{d}{dx} p_x^1(x, x) - p_s^1(x, x) - \lambda \right) \\
&\quad + p^1(x, L) \tilde{w}(L, t) + (p^1(x, L) - p_1(x)) \tilde{w}_{xx}(L, t) - p_s^1(x, L) \tilde{w}_x(L, t)
\end{aligned} \tag{3.57}$$

From this equation, we get four conditions:

1. Equation for $(x, s) \in \mathcal{T}$:

$$p_{sss}^1(x, s) + p_{xxx}^1(x, s) + p_s^1(x, s) + p_x^1(x, s) = \lambda p^1(x, s), \tag{3.58}$$

where we recall that $\mathcal{T} := \{(x, s) \in \mathbb{R}_{\geq 0}^2 \mid x \in [0, L], s \in [x, L]\}$.

2. First boundary condition on (x, x) for $x \in [0, L]$:

$$2 \frac{d}{dx} p^1(x, x) + p_x^1(x, x) + p_s^1(x, x) = 0. \quad (3.59)$$

3. Second boundary condition on (x, x) for $x \in [0, L]$:

$$\frac{d^2}{dx^2} p^1(x, x) + \frac{d}{dx} p_x^1(x, x) + p_{xx}^1(x, x) - p_s^1(x, x) - \lambda = 0. \quad (3.60)$$

4. Appropriate choice of p_1 :

$$p^1(x, L) = p_1(x). \quad (3.61)$$

Moreover, note also that, by setting $x = 0$ in (3.95), we get:

$$p^1(0, s) = 0, \quad \forall s \in [0, L]. \quad (3.62)$$

In addition, we have

$$\tilde{w}(t, 0) = \tilde{w}(t, L) = \tilde{w}_x(t, L) = 0.$$

Finally, the kernel p^1 satisfies the following PDE:

$$\begin{cases} p_{xxx}^1(x, s) + p_{sss}^1(x, s) + p_s^1(x, s) + p_x^1(x, s) = \lambda p^1(x, s), & (x, s) \in \mathcal{T}, \\ p^1(x, x) = 0, & x \in [0, L], \\ p_x^1(x, x) = \frac{\lambda}{3} x, & x \in [0, L], \\ p^1(0, s) = 0, & s \in [0, L]. \end{cases} \quad (3.63)$$

Let us make the following change of variable:

$$\xi = L - s, \quad \eta = L - x, \quad (3.64)$$

and define $F(\xi, \eta) := p^1(x, s)$. Hence:

$$\begin{cases} F_{\xi\xi\xi}(\xi, \eta) + F_{\eta\eta\eta}(\xi, \eta) + F_\eta(\xi, \eta) + F_\xi(\xi, \eta) = -\lambda F(\xi, \eta) & (\xi, \eta) \in \mathcal{T} \\ F(\xi, \xi) = 0 & \xi \in [0, L] \\ F_\xi(\xi, \xi) = \frac{\lambda}{3}(L - \xi) & \xi \in [0, L] \\ F(\xi, L) = 0 & \xi \in [0, L] \end{cases} \quad (3.65)$$

This PDE has already been studied in [14], where no explicit solution has been found, but where the existence of a solution has been proved. A sketch of the proof is provided in Appendix C. Hence, we can conclude that the kernel $p^1 := p^1(x, s)$ exists. Note that the function Π_o^1 defined by (3.95) is linear (by definition) and continuous (because of the existence of p).

3.2.2.4 Well-posedness

We are now in position to prove the well-posedness of the closed-loop system. Let us state the theorem

Theorem 3.5

Suppose that p^1 and k^1 solve (3.63) and (3.28) respectively. Then, for any initial condition $z_0, \hat{z}_0 \in H^3(0, L)$ satisfying the following compatibility condition

$$z_0(0) = \hat{z}_0(0) = \int_0^L k(0, s) \hat{z}_0(s) ds, \quad z_0(L) = \hat{z}_0(L) = 0, \quad z'_0 = \hat{z}'_0(L) = 0, \quad (3.66)$$

there exists a unique strong solution $z, \hat{z} \in C(0, \infty, H^3(0, L)) \times C(0, \infty, H^3(0, L))$ to (3.14)-(3.40) satisfying, for all $t \geq 0$

$$z(t, 0) = \hat{z}(t, 0) = \int_0^L k(0, s) \hat{z}(t, s) ds, \quad z(t, L) = \hat{z}(t, L) = 0, \quad z_x(t, L) = \hat{z}_x(t, L) = 0. \quad (3.67)$$

Proof of Theorem 3.5:

By using the output feedback control

$$u(t) = \int_0^L k^1(0, s) \hat{z}(t, s) ds, \quad (3.68)$$

the transformation Π^1 defined in (3.21) and its inverse, and the transformation Π_o^1 defined in (3.95) and its inverse, we can see that (\tilde{z}, \hat{z}) are mapped into $(\tilde{w}, \hat{w}) = (\Pi_o^{1^{-1}}(\tilde{u}), \Pi^1(\hat{u}))$ solutions to the target systems

$$\begin{cases} \hat{w}_t + \hat{w}_x + \hat{w}_{xxx} + \lambda \hat{w} = - \left\{ p_1(x) - \int_x^L k^1(x, s) p_1(s) ds \right\} \tilde{w}_{xx}(t, L), \\ \hat{w}(0) = \hat{w}(L) = \hat{w}_x(L) = 0, \\ \tilde{w}_t + \tilde{w}_x + \tilde{w}_{xxx} + \lambda \tilde{w} = 0, \\ \tilde{w}_x(0) = \tilde{w}(L) = \tilde{w}_x(L) = 0. \end{cases} \quad (3.69)$$

Note that the parameter λ is the same for the observer and the system itself, without loss of generality.

Hence, as it has been noticed in Section 3.2.1.3, if one is able to prove the well-posedness of this system, then the well-posedness of (3.14)-(3.40) is also proved.

Using classical semigroup, it is clear that the Cauchy problem

$$\begin{cases} \tilde{w}_t + \tilde{w}_x + \tilde{w}_{xxx} + \lambda \tilde{w} = 0, \\ \tilde{w}(t, 0) = \tilde{w}(t, L) = \tilde{w}_x(t, L) = 0, \\ \tilde{w}(0, x) = \tilde{w}_0(x), \end{cases} \quad (3.70)$$

where $\tilde{w}_0(x)$ is in $D(\tilde{A})$, where $D(\tilde{A})$ is defined in (3.35), admits a unique strong solution $\tilde{w} \in C(0, \infty; D(\tilde{A}))$. All the derivations are similar to the ones in Section 3.2.1.3.

Applying Lemma 3.1, one obtains that $\tilde{w}_{xx}(t, L) \in C(0, \infty)$. Since $p_1(x) \in C^3(0, L)$, the following function

$$f : (t, x) \mapsto \left\{ p_1(x) - \int_x^L k^1(x, s)p_1(s)ds \right\} \tilde{w}_{xx}(t, L) \quad (3.71)$$

belongs to the set $C(0, \infty; L^2(0, L))$. From the variation of constants formula, the following Cauchy problem

$$\begin{cases} \hat{w}_t + \hat{w}_x + \hat{w}_{xxx} + \lambda \hat{w} = - \left\{ p_1(x) - \int_x^L k^1(x, s)p_1(s)ds \right\} \tilde{w}_{xx}(t, L), \\ \hat{w}(0) = \hat{w}(L) = \hat{w}_x(L) = 0 \\ \hat{w}(0, x) = \hat{w}_0(x) \in H^3(0, L). \end{cases} \quad (3.72)$$

admits a unique solution $w \in C(0, \infty; D(\tilde{A}))$. Using the continuity of Π^1 and its inverse together with the continuity of Π_o^1 and its inverse, and following a similar proof than in Section (3.2.1.3) it concludes the proof of Theorem (3.5). \square

3.2.2.5 Stability analysis of the closed-loop system

Now, we are on position to prove Theorem 3.4. Let us provide the proof.

Proof of Theorem 3.4:

Instead of dealing directly with the controlled state u and the observer state \hat{z} , we consider the evolution of the couple (\tilde{z}, \hat{z}) where \tilde{z} stands for the error $\tilde{z} = z - \hat{z}$, as introduced in Section 3.2.2.3.

Given that Π^1, Π_o^1 are continuous maps, invertible and their inverse maps are also continuous, the exponential stability of (3.69) would imply the exponential stability of the closed loop system and therefore the proof of Theorem 3.4 will be ended.

In order to prove the exponential stability of (3.69), we use a Lyapunov argument. Let us consider the following function,

$$V(\hat{w}, \tilde{w}, \tilde{w}_t) = V_1(\hat{w}) + V_2(\tilde{w}) + V_3(\tilde{w}_t), \quad (3.73)$$

where

$$V_1(\hat{w}) = \frac{M_1}{2} \int_0^L |\hat{w}(t, x)|^2 dx \quad (3.74)$$

$$V_2(\tilde{w}) = \frac{M_2}{2} \int_0^L |\tilde{w}(t, x)|^2 dx \quad (3.75)$$

$$V_3(\tilde{w}_t) = \frac{M_2}{2} \int_0^L |\tilde{w}_t(t, x)|^2 dx \quad (3.76)$$

with M_1, M_2 two positive values that have to be chosen later.

Remark 3.4

We can prove that this Lyapunov function is equivalent to the one obtained by replacing $V_3(\tilde{w}_t)$ by

$$\tilde{V}_3(\tilde{w}_{xxx}) = \frac{M_2}{2} \int_0^L |\tilde{w}_{xxx}(t, x)|^2 dx \quad (3.77)$$

By this, we mean that the exponential decay of one of them implies the exponential decay of the other one. In fact, we can prove that there exist positive constants d_1, d_2 such that

$$d_1(V_2(\tilde{w}) + \tilde{V}_3(\tilde{w}_{xxx})) \leq V_2(\tilde{w}) + V_3(\tilde{w}_t) \leq d_2(V_2(\tilde{w}) + \tilde{V}_3(\tilde{w}_{xxx})).$$

Taking the time derivative of the function $V(\hat{w}, \tilde{w}, \tilde{w}_t)$, we get after some computations that

$$\begin{aligned} \frac{d}{dt} V_1(\hat{w}) &= M_1 \int_0^L \hat{w}_t(t, x) \hat{w}(t, x) dx \\ &\leq (-M_1 \lambda + D^2) \int_0^L |\hat{w}(t, x)|^2 dx + M_1^2 |\tilde{w}_{xx}(t, L)|^2 \\ &= 2 \left(-\lambda + \frac{D^2}{M_1} \right) V_1(\hat{w}) + M_1^2 |\tilde{w}_{xx}(t, L)|^2 \end{aligned}$$

where $D := \max_{x \in [0, L]} \left\{ p_1(x) - \int_x^L k(x, s) p_1(s) ds \right\}$.

From the same computation as in (3.27), we see that

$$\frac{d}{dt} V_2(\tilde{w}) \leq -2\lambda V_2(\tilde{w}).$$

Moreover, thanks to the regularity $H^3(0, L)$, the same computation can be applied to $z = \tilde{w}_t$ (see the proof of Lemma 3.1) to obtain

$$\frac{d}{dt} V_3(\tilde{w}_t) \leq -2\lambda V_3(\tilde{w}_t)$$

Thus, we get:

$$\frac{d}{dt} V(\hat{w}, \tilde{w}, \tilde{w}_t) \leq 2 \left(-\lambda + \frac{D^2}{M_1} \right) V_1(\hat{w}) + M_1^2 |\tilde{w}_{xx}(t, L)|^2 - 2\lambda V_2(\tilde{w}) - 2\lambda V_3(\tilde{w}_t).$$

We need to find an upper bound for $|\tilde{w}_{xx}(t, L)|^2$. We multiply

$$\begin{cases} \tilde{w}_t + \tilde{w}_x + \tilde{w}_{xxx} + \lambda \tilde{w} = 0, \\ \tilde{w}(0) = \tilde{w}(L) = \tilde{w}_x(L) = 0, \end{cases} \quad (3.78)$$

by $x\tilde{w}_{xx}$ and after some computations we get

$$|\tilde{w}_{xx}(t, L)|^2 \leq \left(\frac{1}{L} + L\right) \|\tilde{w}_{xx}\|_{L^2(0,L)}^2 + \left(2\lambda + \frac{1}{L}\right) \|\tilde{w}_x\|_{L^2(0,L)}^2 + \frac{1}{L} \|\tilde{w}_t\|_{L^2(0,L)}^2 \quad (3.79)$$

and finally the existence of two positive values a and b such that

$$|\tilde{w}_{xx}(t, L)|^2 \leq a\|\tilde{w}\|_{L^2(0,L)}^2 + b\|\tilde{w}_t\|_{L^2(0,L)}^2 \quad (3.80)$$

Remark 3.5

Here, we have used that the norm $\|f\|_{H^3(0,L)}$ and the norm $\|f\|_{L^2(0,L)} + \|f_{xxx}\|_{L^2(0,L)}$ are equivalent. See also Remark 3.4.

We use the latter inequality to write:

$$\frac{d}{dt}V(\hat{w}, \tilde{w}, \tilde{w}_t) \leq 2\left(-\lambda + \frac{D^2}{M_1}\right)V_1(\hat{w}) + 2a\frac{M_1^2}{M_2}V_2(\tilde{w}) + 2b\frac{M_1^2}{M_2}V_3(\tilde{w}_t) - 2\lambda V_2(\tilde{w}) - 2\lambda V_3(\tilde{w}_t).$$

Therefore,

$$\frac{d}{dt}V(\hat{w}, \tilde{w}, \tilde{w}_t) \leq 2\left(-\lambda + \frac{D^2}{M_1}\right)V_1(\hat{w}) + 2\left(-\lambda + \frac{aM_1^2}{M_2}\right)V_2(\tilde{w}) + 2\left(-\lambda + \frac{bM_1^2}{M_2}\right)V_3(\tilde{w}_t).$$

In this way, by tuning M_1 and M_2 large enough, we get for any $\varepsilon > 0$ that

$$\frac{d}{dt}V(\hat{w}, \tilde{w}, \tilde{w}_t) \leq 2(-\lambda + \varepsilon)V(\hat{w}, \tilde{w}, \tilde{w}_t),$$

which gives an exponential stability with decay rate as close to λ as we want. The rapid stabilization is achieved because the parameter λ can be chosen as large as desired.

This concludes the proof of the stability of the closed loop system with the output feedback control law depending on a boundary measurement of the state. \square

3.2.3 Dynamic output feedback design for (3.16)

3.2.3.1 Main Result

Based on [50] and [85], where the technique provided is similar to Section 3.2.2, we design an observer for (3.16). More precisely, we define, for some appropriate function p_2 to be designed, the following copy of the plant with a term depending on the observation error

$$\begin{cases} \hat{z}_t + \hat{z}_x + \hat{z}_{xxx} + \hat{z}\hat{z}_x + p_2(x)[y_2(t) - \hat{z}(t, L)] = 0, \\ \hat{z}(t, 0) = u(t), \hat{z}_x(t, L) = \hat{z}_{xx}(t, L) = 0, \\ \hat{z}(0, x) = \hat{z}_0(x). \end{cases} \quad (3.81)$$

Our main result is the local stabilization of the KdV equation by using the output $y_2(t) = z(t, L)$, as stated in the following theorem whose proof is given in Section 3.2.3.7.

Theorem 3.6

For any $\lambda_0 > 0$, there exist an output feedback law $u(t) := u(\hat{z}(t, x))$, a function $p_2 = p_2(x)$, and two constants $C > 0$, $r > 0$ such that for any initial conditions $z_0, \hat{z}_0 \in L^2(0, L)$ satisfying

$$\|z_0\|_{L^2(0,L)} \leq r, \quad \|\hat{z}_0\|_{L^2(0,L)} \leq r, \quad (3.82)$$

the mild solution to (3.16)-(3.81) satisfies, for all $t \geq 0$

$$\|z(t, \cdot) - \hat{z}(t, \cdot)\|_{L^2(0,L)} + \|\hat{z}(t, \cdot)\|_{L^2(0,L)} \leq Ce^{-\lambda_0 t} \left(\|z_0 - \hat{z}_0\|_{L^2(0,L)} + \|\hat{z}_0\|_{L^2(0,L)} \right). \quad (3.83)$$

Remark 3.6

Notice that from this theorem we get the exponential decreasing to 0 of the L^2 -norm of the solution $z = z(t, x)$ provided that the L^2 -norm of the initial conditions of the plant and the observer are sufficiently small. Moreover only the mild solutions will be considered in this section.

3.2.3.2 Control Design for the linearized system

The backstepping design applied here is based on the linear part of the equation (3.16). Thus, we consider the control system linearized around the origin

$$\begin{cases} z_t + z_x + z_{xxx} = 0, \\ z(t, 0) = u(t), \quad z_x(t, L) = z_{xx}(t, L) = 0, \end{cases} \quad (3.84)$$

and the linear observer

$$\begin{cases} \hat{z}_t + \hat{z}_x + \hat{z}_{xxx} + p_2(x)[y_2(t) - \hat{z}(t, L)] = 0, \\ \hat{z}(t, 0) = u(t), \quad \hat{z}_x(t, L) = \hat{z}_{xx}(t, L) = 0. \end{cases} \quad (3.85)$$

We follow the same strategy as in Section 3.2.2.

In [14], the following Volterra transformation is introduced

$$w(x) = \Pi^2(z(x)) := z(x) - \int_x^L k^2(x, s) ds. \quad (3.86)$$

The function k^2 is chosen such that $z = z(t, x)$, solution of (3.84) with control

$$u(t) = \int_0^L k^2(0, s) z(t, s) ds, \quad (3.87)$$

is mapped into the trajectory $w = w(t, x)$, solution of the linear system

$$\begin{cases} w_t + w_x + w_{xxx} + \lambda w = 0, \\ w(t, 0) = w_x(t, L) = w_{xx}(t, L) = 0, \end{cases} \quad (3.88)$$

which is exponentially stable for $\lambda > 0$, with a decay rate at least equal to λ .

The kernel function $k^2 : \mathcal{T} \rightarrow \mathbb{R}$ is characterized by

$$\begin{cases} k_{xxx}^2 + k_{sss}^2 + k_x^2 + k_s^2 = -\lambda k^2, & \text{in } \mathcal{T}, \\ k^2(x, L) + k_{ss}(x, L) = 0, & \text{in } [0, L], \\ k^2(x, x) = 0, & \text{in } [0, L], \\ k_x^2(x, x) = \frac{\lambda}{3}(L - x), & \text{in } [0, L], \end{cases} \quad (3.89)$$

where $\mathcal{T} := \{(x, s) \in \mathbb{R}_{\geq 0}^2 \mid x \in [0, L], s \in [x, L]\}$. The solution k^2 to (3.89) exists and belongs to $C^3(\mathcal{T})$. This is proved in [14, Section VI] by using the method of successive approximations. Unlikely the case of heat or wave equations, we do not have an explicit solution.

In [14] it is proved that the transformation (3.86) linking (3.84) and (3.88) is invertible, continuous and with a continuous inverse function. This proof is tackled using the successive approximation method. Therefore, the exponential decay for w , solution to (3.88), implies the exponential decay for the solution z controlled by (3.87). Thus, with this method, the following theorem is proven in [14].

Theorem 3.7

For any $\lambda > 0$, there exists $C > 0$ such that

$$\|z(t, \cdot)\|_{L^2(0,L)} \leq C e^{-\lambda t} \|z_0\|_{L^2(0,L)}, \quad \forall t \geq 0, \quad (3.90)$$

for any solution to (3.84)-(3.87).

Later, we give more details on the observer design. Let us remark that the output feedback law is designed as

$$u(t) := \int_0^L k^2(0, s) \hat{z}(t, s) ds, \quad (3.91)$$

where \hat{z} is the mild solution to (3.85).

Thus we get the following result, which can be compared to Section 3.2.2.1. The proof is given in Section 3.2.3.4.

Theorem 3.8

For any $\lambda_0 > 0$, there exists a positive value C such that for any solution to (3.84)-(3.85)-(3.91) we have, for all $t \geq 0$

$$\|z(t, \cdot) - \hat{z}(t, \cdot)\|_{L^2(0,L)} + \|\hat{z}(t, \cdot)\|_{L^2(0,L)} \leq C e^{-\lambda_0 t} (\|z_0 - \hat{z}_0\|_{L^2(0,L)} + \|\hat{z}_0\|_{L^2(0,L)}). \quad (3.92)$$

Remark 3.7

Notice that from Theorem 3.8, we get the exponential decrease to 0 of the L^2 -norm of the solution $z = z(t, x)$. This result is different from Section 3.2.2, where the equilibrium point 0 is exponentially stable in the domain of the operator. This is due to the fact that the output and the boundary conditions are different.

3.2.3.3 Observer design for the linearized system

The observer (3.85) is based on a Volterra transformation. It transforms the solution $\tilde{z} := z - \hat{z}$ which fullfills the following PDE

$$\begin{cases} \tilde{z}_t(t, x) + \tilde{z}_x(t, x) + \tilde{z}_{xxx}(t, x) - p_2(x)[\tilde{z}(t, L)] = 0, & (t, x) \in \mathbb{R}_{\geq 0} \times [0, L] \\ \tilde{z}(t, 0) = \tilde{z}_x(t, L) = \tilde{z}_{xx}(t, L) = 0, & t \in \mathbb{R}_{\geq 0} \\ \tilde{z}(0, x) = z_0(x) - \hat{z}_0(x) := \tilde{z}_0(x), & x \in [0, L] \end{cases} \quad (3.93)$$

into the following PDE

$$\begin{cases} \tilde{w}_t + \tilde{w}_x + \tilde{w}_{xxx} + \lambda \tilde{w} = 0, \\ \tilde{w}(t, 0) = \tilde{w}_x(t, L) = \tilde{w}_{xx}(t, L) = 0, \\ \tilde{w}(0, x) = \tilde{w}_0(x). \end{cases} \quad (3.94)$$

We choose the same λ than the one used to design the controller given by (3.87). The transformation is given by

$$\tilde{z}(x) := \Pi_o^2(\tilde{w}(x)) = \tilde{w}(x) - \int_x^L p^2(x, s)\tilde{w}(s)ds, \quad (3.95)$$

where p^2 is a kernel that satisfies a partial differential equation and will be defined in the following.

In [90], a similar observer has been computed. Indeed, if we make the following change of coordinates,

$$\bar{x} = L - x \quad (3.96)$$

we obtain a Korteweg-de Vries equation similar to the one studied in [90]. Hence, we know from this latter paper that the kernel $p^2 : \mathcal{T} \rightarrow \mathbb{R}$ solves the following equation

$$\begin{cases} p_{xxx}^2 + p_{sss}^2 + p_x^2 + p_s^2 = \lambda p^2, & \forall (x, s) \in \mathcal{T}, \\ p^2(x, x) = 0, & \forall x \in [0, L], \\ p_x^2(x, x) = \frac{\lambda}{3}(x - L), & \forall x \in [0, L], \\ p^2(0, s) = 0, & \forall s \in [0, L]. \end{cases} \quad (3.97)$$

The solution p^2 to this equation exists and is unique. It belongs to the set $C^3(\mathcal{T})$.

Moreover, once again following [90], we define the following observer gain, for all $x \in [0, L]$

$$p_2(x) = p_{ss}^2(x, L) + p^2(x, L). \quad (3.98)$$

As in [90], we can state that the transformation Π_o^2 is invertible with continuous inverse given by

$$\tilde{w}(x) = \Pi_o^{2^{-1}}(\tilde{z}(x)) = \tilde{z}(x) + \int_x^L m(x, s)\tilde{z}(s)ds \quad (3.99)$$

where $m = m(x, s)$ is also a solution to an equation like (3.97) in the triangular domain \mathcal{T} .

3.2.3.4 Well-posedness and Exponential Stability of the Linear System

Preliminaries

Let us focus on the linearized version of the Korteweg-de Vries equation. The homogeneous equation is given by

$$\begin{cases} z_t + z_x + z_{xxx} = 0, \\ z(t, 0) = z_x(t, L) = z_{xx}(t, L) = 0, \\ z(0, x) = z_0(x). \end{cases} \quad (3.100)$$

The operator associated to this linear PDE is given by:

$$\begin{aligned} A_2 : D(A_2) \subset L^2(0, L) &\rightarrow L^2(0, L), \\ z &\mapsto -z' - z''' \end{aligned} \quad (3.101)$$

whose domain is $D(A_2) := \{z \in H^3(0, L) \mid z(0) = z'(L) = z''(L) = 0\}$. From basic semigroup theory, it is easy to prove that A_2 generates a strongly continuous semigroup of contractions. This semigroup will be denoted by $(W_2(t))_{t \geq 0}$. We also need some results on the non-homogeneous Korteweg-de Vries equation.

Theorem 3.9

(see [49]) Let $T > 0$. For any $z_0 \in L^2(0, L)$ and any $(f, h) \in L^1(0, T; L^2(0, L)) \times H^{\frac{1}{3}}(0, T)^4$, the following Korteweg-de Vries equation

$$\begin{cases} z_t + z_x + z_{xxx} = f, \\ z(t, 0) = h(t), \\ z_x(t, L) = z_{xx}(t, L) = 0, \\ z(0, x) = z_0(x), \end{cases} \quad (3.102)$$

admits a unique solution $z \in \mathcal{B}(T)$. Moreover, there exists a positive value C such that the following inequality holds:

$$\|z\|_{\mathcal{B}(T)} \leq C \left(\|z_0\|_{L^2(0, L)} + \|f\|_{L^1(0, T; L^2(0, L))} + \|h\|_{H^{\frac{1}{3}}(0, T)} \right) \quad (3.103)$$

From this result, the following lemma can be deduced.

Lemma 3.2

Let us suppose all that the assumptions in Theorem 3.9 hold and that in addition $f \in L^2(0, T; L^1(0, L))$. Then, one has the regularity

$$z \in H^{\frac{1}{3}}(0, T; L^2(0, L)). \quad (3.104)$$

⁴The space $H^{\frac{1}{3}}(0, T)$ is defined as follows

$$H^{\frac{1}{3}}(T) := \left\{ h \in L^2(0, T) \mid \left| \frac{h(t) - h(s)}{|t - s|^{\frac{1}{3}}} \right| \in L^2((0, T) \times (0, T)) \right\}$$

Proof of Lemma 3.2:

Note that in particular, $z \in L^2(0, T; H^1(0, L))$. Therefore, since $z_t = -z_x - z_{xxx} + f$, one has

$$z_t \in L^2(0, T; H^{-2}(0, L)).$$

Hence,

$$z \in H^1(0, T; H^{-2}(0, L)).$$

Then, by applying the classical theory of interpolation (see [79, Theorem 2.8.] or [54] for a good introduction to this subject), it is easy to see that $z \in H^{\frac{1}{3}}(0, T; L^2(0, L))$, which concludes the proof of Lemma 3.2. \square

Other result we need is the regularity of the right hand side in the observer.

Lemma 3.3

Let $p_2 \in L^2(0, L)$. Then, given a positive value T , for every $z, \hat{z} \in L^2(0, T; H^1(0, L))$, one has $p_2(x)[z(t, L) - \hat{z}(t, L)] \in L^2(0, T; L^2(0, L))$.

Proof of Lemma 3.3:

Using the Agmon inequality (see [2, Lemma 13.2., page 148]), we obtain that there exists a positive value C_A such that

$$\int_0^T |z(t, L) - \hat{z}(t, L)|^2 dt \leq C_A \int_0^T \|z(t, \cdot) - \hat{z}(t, \cdot)\|_{H^1(0, L)}^2 dt.$$

Since $z, \hat{z} \in L^2(0, T; H^1(0, L))$ and $p_2(x) \in L^2(0, L)$, we conclude the proof. \square

Proof of Theorem 3.8 - Well-posedness:

The closed-loop system (3.84)-(3.85)-(3.91) through the transformations (3.86) and (3.95) can be written as follows

$$\begin{cases} \hat{w}_t + \hat{w}_x + \hat{w}_{xxx} + \lambda \hat{w} = - \left\{ p_2(x) - \int_x^L k^2(x, y) p_2(y) dy \right\} \tilde{w}(t, L), \\ \hat{w}(t, 0) = \hat{w}_x(t, L) = \hat{w}_{xx}(t, L) = 0, \\ \tilde{w}_t + \tilde{w}_x + \tilde{w}_{xxx} + \lambda \tilde{w} = 0, \\ \tilde{w}(t, 0) = \tilde{w}_x(t, L) = \tilde{w}_{xx}(t, L) = 0. \end{cases} \quad (3.105)$$

We only have to prove that system (3.105) is well-posed (with solutions in $\mathcal{B}(T) \times \mathcal{B}(T)$ for any $T > 0$) and that its origin is exponentially stable. In fact, once that is done, by using Lemma 3.2, we conclude that \tilde{w}, \hat{w} belong to $H^{\frac{1}{3}}(0, T; L^2(0, L))$. Thus the control defined by

(3.91) belongs to $H^{\frac{1}{3}}(0, T)$, which is the desired regularity for the input. The existence of solutions and the exponential decay for system (3.84)-(3.85)-(3.91) is obtained by the invertibility of the transformations (3.86) and (3.95).

System (3.105) is in cascade form. We can apply the linear results to get \tilde{w} and then plug it into the equation for \hat{w} as a right hand side by using Theorem 3.9. Thus, we finally get \hat{w} .

With these results in hand we can define the continuous solution map

$$(z_0, \hat{z}_0) \in L^2(0, L)^2 \mapsto \Lambda(z_0, \hat{z}_0) = (z, \hat{z}) \in \mathcal{B}(T)^2 \quad (3.106)$$

where $\mathcal{B}(T)^2 := \mathcal{B}(T) \times \mathcal{B}(T)$ and $z, \hat{z} \in \mathcal{B}(T)$ are the solutions of the linear closed loop system (3.84)-(3.85)-(3.91).

Proof of Theorem 3.8 - Stability:

Let us focus now on the exponential stability. To do that, we consider the following Lyapunov function

$$V(\hat{w}, \tilde{w}) := V_1(\hat{w}) + V_2(\tilde{w}) \quad (3.107)$$

where

$$V_1(\hat{w}) = A \int_0^L \hat{w}(t, x)^2 dx, \quad (3.108)$$

and

$$V_2(\tilde{w}) = B \int_0^L \tilde{w}(t, x)^2 dx. \quad (3.109)$$

The positive values A and B are chosen later. After performing some integrations by parts, we obtain

$$\frac{d}{dt} V_1(\hat{w}) \leq \left(-2\lambda + \frac{D^2}{A} \right) V_1(\hat{w}) + A^2 L |\tilde{w}(t, L)|^2, \quad (3.110)$$

where

$$D = \max_{x \in [0, L]} \left\{ p_2(x) - \int_x^L k^2(x, y) p_2(y) dy \right\}. \quad (3.111)$$

We have also

$$\frac{d}{dt} V_2(\tilde{w}) \leq -2\lambda V_2(\tilde{w}) - B |\tilde{w}(t, L)|^2. \quad (3.112)$$

Therefore, by choosing

$$A > \frac{D^2}{2\lambda} \quad (3.113)$$

and

$$B \geq A^2 L, \quad (3.114)$$

we get

$$\frac{d}{dt} V(\hat{w}, \tilde{w}) \leq -2\mu V(\hat{w}, \tilde{w}), \quad (3.115)$$

where

$$\mu = \left(\lambda - \frac{D^2}{2A} \right) > 0. \quad (3.116)$$

Thus, we obtain the exponential decay of (\tilde{w}, \hat{w}) with a decay rate equal to μ . By using the invertibility and continuity of the transformations Π and Π_o , we conclude the proof of Theorem 3.8 by getting the desired exponential decay for (\tilde{z}, \hat{z}) .

In the next section, we will tackle the well-posedness and the exponential stability of the nonlinear system.

3.2.3.5 Well-posedness and Exponential Stability of the Nonlinear System

Preliminaries

The following results are useful in order to prove the well-posedness of the nonlinear Korteweg-de Vries equation.

Let us recall Proposition 2.1. Some additional informations are provided.

Proposition 3.1 (see [77])

Given a positive value T , let $z \in L^2(0, T; H^1(0, L))$. Then $zz_x \in L^1(0, T; L^2(0, L))$ and the map $z \in L^2(0, T; H^1(0, L)) \mapsto zz_x \in L^1(0, T; L^2(0, L))$ is continuous. Moreover, there exists a positive value C_h such that, for every $u, \tilde{z} \in L^2(0, T; H^1(0, L))$

$$\|zz_x - \tilde{z}\tilde{z}_x\|_{L^1(0,T;L^2(0,L))} \leq C_h \|z + \tilde{z}\|_{L^2(0,T;H^1(0,L))} \|z - \tilde{z}\|_{L^2(0,T;H^1(0,L))} \quad (3.117)$$

Lemma 3.4 (see [49])

Let us note that $(W_2(t))_{t \geq 0}$ denotes the strongly continuous semigroups of contractions generated by the operator (3.101). For any positive value T , there exist two positive values $C_1 := C_1(T)$ and $C_2 := C_2(T)$ such that

(i) For any $v, \hat{v} \in \mathcal{B}(T)$,

$$\int_0^T \|(v(t, \cdot) \hat{v}(t, \cdot))_x\|_{L^2(0,L)} dt \leq C_1 \|v\|_{\mathcal{B}(T)} \|\hat{v}\|_{\mathcal{B}(T)} \quad (3.118)$$

(ii) For $f \in L^1(0, T; L^2(0, L))$, let

$$z = \int_0^t W_2(t-s) f(s) ds,$$

then

$$\|z\|_{\mathcal{B}(T)} \leq C_2 \int_0^T \|f(t, \cdot)\|_{L^2(0,L)} dt. \quad (3.119)$$

Moreover, one can easily prove the following lemma.

Lemma 3.5

Given a positive value T , let $z \in \mathcal{B}(T)$. Then $zz_x \in L^2(0, T; L^1(0, L))$. Moreover, for every $z \in \mathcal{B}(T)$

$$\|zz_x\|_{L^2(0,T;L^1(0,L))} \leq \|z\|_{L^2(0,T;H^1(0,L))} \|z\|_{C([0,T];L^2(0,L))} \quad (3.120)$$

3.2.3.6 Proof of Theorem 3.6 - Well-posedness

We will apply the Banach fixed point theorem in order to prove the well-posedness of the nonlinear closed-loop system.

With u defined by (3.91), and $v, \hat{v} \in \mathcal{B}(T)$, the solutions to

$$\begin{cases} z_t + z_x + z_{xxx} = -vv_x, \\ z(t, 0) = u(t), \\ z_x(t, L) = z_{xx}(t, L) = 0, \\ z(0, x) = z_0(x), \end{cases} \quad (3.121)$$

and

$$\begin{cases} \hat{z}_t + \hat{z}_x + \hat{z}_{xxx} + p_2(x)[z(t, L) - \hat{z}(t, L)] = -\hat{v}\hat{v}_x, \\ \hat{z}(t, 0) = u(t), \\ \hat{z}_x(t, L) = \hat{z}_{xx}(t, L) = 0, \\ \hat{z}(0, x) = \hat{z}_0(x), \end{cases} \quad (3.122)$$

can be written as follows

$$(z, \hat{z})(t) = \Lambda(z_0, \hat{z}_0)(t) - \left(\int_0^t W_2(t-s)v(s)v_x(s)ds, \int_0^t W_2(t-s)\hat{v}(s)\hat{v}_x(s)ds \right) \quad (3.123)$$

where Λ was introduced in (3.106) and $(W_2(t))_{t \geq 0}$ is the strongly continuous semigroup of contractions generated by (3.101). We will be done if we prove that the map Γ defined by the right hand-side of (3.123),

$$\Gamma(v, \hat{v}) = \Lambda(z_0, \hat{z}_0) - \left(\int_0^t W_2(t-s)v(s)v_x(s)ds, \int_0^t W_2(t-s)\hat{v}(s)\hat{v}_x(s)ds \right), \quad (3.124)$$

has a fixed point.

We define, for some $R > 0$ to be chosen later,

$$S_R = \{(v, \hat{v}) \in \mathcal{B}(T) \times \mathcal{B}(T) \mid \|v\|_{\mathcal{B}(T)} + \|\hat{v}\|_{\mathcal{B}(T)} \leq R\}, \quad (3.125)$$

which is a closed, convex and bounded subset of $\mathcal{B}(T)^2$. Consequently, S_R is a complete metric space in the topology induced by $\mathcal{B}(T)^2$. With Theorem 3.9 and previous lemmas, we have the existence of a constant $C > 0$ such that for any $(v, \hat{v}) \in S_R$,

$$\|\Gamma(v, \hat{v})\|_{\mathcal{B}(T)^2} \leq C \left(\|z_0\|_{L^2(0,L)} + \|\hat{z}_0\|_{L^2(0,L)} \right) + C \left(\|v\|_{\mathcal{B}(T)}^2 + \|\hat{v}\|_{\mathcal{B}(T)}^2 \right). \quad (3.126)$$

We consider $r > 0$ and $z_0, \hat{z}_0 \in L^2(0, L)$ such that

$$\|z_0\|_{L^2(0,L)} \leq r, \quad \text{and} \quad \|\hat{z}_0\|_{L^2(0,L)} \leq r.$$

Now we select r, R as follows

$$\begin{cases} 2Cr \leq R/2, \\ 2CR^2 \leq R/2, \end{cases} \quad (3.127)$$

to obtain that for any $(v, \hat{v}) \in S_R$

$$\|\Gamma(v, \hat{v})\|_{\mathcal{B}(T)^2} \leq R. \quad (3.128)$$

Thus, with such r and R , Γ maps S_R into S_R . Moreover, using Lemma 3.1 and Lemma 3.4, we obtain

$$\|\Gamma(v_1, \hat{v}_1) - \Gamma(v_2, \hat{v}_2)\|_{\mathcal{B}^2(T)} \leq C_2 C_h \left(\|v_1 + v_2\|_{\mathcal{B}(T)} \|v_1 - v_2\|_{\mathcal{B}(T)} + \|\hat{v}_1 + \hat{v}_2\|_{\mathcal{B}(T)} \|\hat{v}_1 - \hat{v}_2\|_{\mathcal{B}(T)} \right) \quad (3.129)$$

and thus, choosing R such that

$$2RC_2 C_h \leq \frac{1}{2}$$

we arrive to

$$\|\Gamma(v_1, \hat{v}_1) - \Gamma(v_2, \hat{v}_2)\|_{\mathcal{B}^2(T)} \leq \frac{1}{2} \left(\|v_1 - v_2\|_{\mathcal{B}(T)} + \|\hat{v}_1 - \hat{v}_2\|_{\mathcal{B}(T)} \right) \quad (3.130)$$

for any $(v_1, \hat{v}_1), (v_2, \hat{v}_2) \in S_R$. By using Theorem 2.7, we obtain the existence of a unique fixed point of the map Γ . As previously stated, this fixed point is the solution we are looking for.

3.2.3.7 Exponential stability of the nonlinear system

The aim of this section is to prove Theorem 3.6, i.e., we have to prove the local exponential stability of the nonlinear closed-loop system

$$\begin{cases} z_t(t, x) + z_x(t, x) + z_{xxx}(t, x) + z(t, x)z_x(t, x) = 0, & (t, x) \in \mathbb{R}_{\geq 0} \times [0, L], \\ z(t, 0) = u(t), z_x(t, L) = z_{xx}(t, L) = 0, & t \in \mathbb{R}_{\geq 0} \\ \hat{z}_t(t, x) + \hat{z}_x(t, x) + \hat{z}_{xxx}(t, x) + \hat{z}(t, x)\hat{z}_x(t, x) + p_2(x)[y_2(t) - \hat{z}(t, L)] = 0, & (t, x) \in \mathbb{R}_{\geq 0} \times [0, L], \\ \hat{z}(t, 0) = u(t), \hat{z}_x(t, L) = \hat{z}_{xx}(t, L) = 0, & t \in \mathbb{R}_{\geq 0}, \end{cases} \quad (3.131)$$

where

$$u(t) = \int_0^L k^2(0, s)\hat{z}(t, s)ds, \quad (3.132)$$

and

$$y_2(t) = z(t, L). \quad (3.133)$$

As before, we consider the evolution of the couple (\tilde{z}, \hat{z}) where \tilde{z} stands for the error $\tilde{z} = z - \hat{z}$. Using Π_o^2 and its inverse (see (3.95) and (3.99)), we define $\tilde{w} = \Pi_o^{2^{-1}}(\tilde{z})$. We denote $\hat{w} = \Pi^2(\hat{z})$, where Π^2 is defined in (3.86). The inverse $\Pi^{2^{-1}}$ is given by

$$\hat{z}(x) = \Pi^{2^{-1}}(\hat{w}(x)) = \hat{w}(x) + \int_x^L l^2(x, s)\hat{w}(y)ds, \quad (3.134)$$

where l^2 solves the following equation

$$\begin{cases} l_{xxx}^2 + l_{yyy}^2 + l_x^2 + l_y^2 = \lambda l, & (x, y) \in \mathcal{T}, \\ l^2(x, L) + l_{ss}^2(x, L) = 0, & x \in [0, L], \\ l^2(x, x) = 0, & x \in [0, L], \\ l_x^2(x, x) = \frac{\lambda}{3}(L - x), & x \in [0, L]. \end{cases} \quad (3.135)$$

The existence of such a kernel l^2 has been proven in [14, see Sections IV and VI]. Thus, we can see that (\tilde{z}, \hat{z}) is mapped into (\tilde{w}, \hat{w}) solution to the coupled target system

$$\begin{aligned} \hat{w}_t(t, x) + \hat{w}_x(t, x) + \hat{w}_{xxx}(t, x) + \lambda\hat{w}(t, x) = & \\ - \left\{ p_2(x) - \int_x^L k^2(x, s)p_2(s)ds \right\} \tilde{w}(t, L) & \\ - \left(\hat{w}(t, x) + \int_x^L l^2(x, s)\hat{w}(t, s)ds \right) \cdot \left(\hat{w}_x(t, x) + \int_x^L l_x^2(x, s)\hat{w}(t, s)ds \right) & \\ - \frac{1}{2} \int_x^L |\hat{z}(t, s)|^2 k_s^2(x, s)ds, & \end{aligned} \quad (3.136)$$

$$\begin{aligned} \tilde{w}_t(t, x) + \tilde{w}_x(t, x) + \tilde{w}_{xxx}(t, x) + \lambda\tilde{w}(t, x) = & \\ - \left(\tilde{w}(t, x) - \int_x^L p(x, s)\tilde{w}(t, s)ds \right) \cdot \left(\tilde{w}_x(t, x) - \int_x^L p_x(x, s)\tilde{w}(t, s)ds \right) & \\ - \left(\hat{w}(t, x) + \int_x^L l^2(x, s)\hat{w}(t, s)ds \right) \cdot \left(\tilde{w}_x(t, x) - \int_x^L p_x(x, s)\tilde{w}(t, s)ds \right) & \\ - \left(\tilde{w}(t, x) - \int_x^L p(x, s)\tilde{w}(t, s)ds \right) \cdot \left(\hat{w}_x(t, x) + \int_x^L l_x^2(x, s)\hat{w}(t, s)ds \right) & \\ + \int_x^L \left[\frac{|\tilde{z}(t, s)|^2}{2} + \tilde{z}(t, s)\hat{z}(t, s) \right] m_y(x, s)ds, & \end{aligned} \quad (3.137)$$

with boundary conditions

$$\hat{w}(t, 0) = \hat{w}_x(t, L) = \hat{w}_{xx}(t, L) = 0, \quad (3.138)$$

$$\tilde{w}(t, 0) = \tilde{w}_x(t, L) = \tilde{w}_{xx}(t, L) = 0. \quad (3.139)$$

As in previous section, we will prove the stability of this system by using the same Lyapunov function (3.107). We differentiate (3.108) with respect to time as follows

$$\begin{aligned} \frac{d}{dt} V_1(\hat{w}) = 2M_1 \int_0^L \hat{w}_t(t, x)\hat{w}(t, x)dx & \\ \leq \left(-2\lambda + \frac{D^2}{M_1} \right) V_1(\hat{w}) + M_1^2 L |\tilde{w}(t, L)|^2 - 2M_1 \int_0^L \hat{w}(t, x)F(t, x)dx, & \end{aligned} \quad (3.140)$$

where

$$\begin{aligned} F(t, x) = & \hat{w}(t, x)\hat{w}_x(t, x) + \hat{w}(t, x) \int_x^L l_x^2(x, s)\hat{w}(t, s)ds + \hat{w}_x(t, x) \int_x^L l^2(x, s)\hat{w}(t, s)ds \\ & + \left(\int_x^L l^2(x, s)\hat{w}(t, s)ds \right) \left(\int_x^L l_x^2(x, s)\hat{w}(t, s)ds \right) + \frac{1}{2} \int_x^L |\hat{z}(t, s)|^2 k_s^2(x, s)ds. \end{aligned}$$

By using the same argument as in [14, 24], we can prove the existence of a positive constant $K_1 = K_1(\|l^2\|_{C^1(\mathcal{T})}, \|k^2\|_{C^1(\mathcal{T})})$ such that

$$\left| M_1 \int_0^L \hat{w}(t, x)F(t, x)dx \right| \leq K_1 \left(\int_0^L |\hat{w}(t, x)|^2 dx \right)^{\frac{3}{2}}. \quad (3.141)$$

Then, we estimate $\frac{d}{dt}V_2(\tilde{w})$ as follows

$$\frac{d}{dt}V_2(\tilde{w}) \leq -2\lambda V_2(\tilde{w}) - 2M_2 \int_0^L \tilde{w}(t, x)G(t, x)dx - M_2 |\tilde{w}(t, L)|^2 \quad (3.142)$$

where $G = G(t, x)$ is the right-hand side of (3.137). We can prove the existence of a positive constant $K_2 = K_2(\|l\|_{C^1(\tau)}, \|p\|_{C^1(\tau)}, \|m\|_{C^1(\tau)})$ such that

$$\left| 2M_2 \int_0^L \tilde{w}(t, x)G(t, x)dx \right| \leq K_2 \left(\int_0^L |\hat{w}(t, x)|^2 dx \right)^{\frac{3}{2}} + K_2 \left(\int_0^L |\tilde{w}(t, x)|^2 dx \right)^{\frac{3}{2}}. \quad (3.143)$$

Therefore, for M_1, M_2, μ satisfying (3.113), (3.114) and (3.116), we have

$$\begin{aligned} \frac{d}{dt}V(\hat{w}, \tilde{w}) \leq & -2\mu V(\hat{w}, \tilde{w}) + K_1 \left(\int_0^L |\hat{w}(t, x)|^2 dx \right)^{\frac{3}{2}} + K_2 \left(\int_0^L |\hat{w}(t, x)|^2 dx \right)^{\frac{3}{2}} \\ & + K_2 \left(\int_0^L |\tilde{w}(t, x)|^2 dx \right)^{\frac{3}{2}}. \end{aligned} \quad (3.144)$$

If there exists $t_0 \geq 0$ such that

$$\|\tilde{w}(t_0, \cdot)\|_{L^2(0, L)} \leq \frac{\mu}{K_2} \quad (3.145)$$

and

$$\|\hat{w}(t_0, \cdot)\|_{L^2(0, L)} \leq \frac{\mu}{K_1 + K_2} \quad (3.146)$$

we can conclude

$$\frac{d}{dt}V(\hat{w}, \tilde{w}) \leq -\mu V(\hat{w}, \tilde{w}), \quad \forall t \geq t_0. \quad (3.147)$$

Thus, we get

$$\|\tilde{w}(t, \cdot)\|_{L^2(0, L)} + \|\hat{w}(t, \cdot)\|_{L^2(0, L)} \leq e^{-\frac{\mu}{2}t} (\|\tilde{w}_0\|_{L^2(0, L)} + \|\hat{w}_0\|_{L^2(0, L)}), \quad \forall t \geq 0, \quad (3.148)$$

provided that

$$\begin{aligned} \|\hat{w}_0\|_{L^2(0, L)} & \leq \frac{\mu}{K_1 + K_2}, \\ \|\tilde{w}_0\|_{L^2(0, L)} & \leq \frac{\mu}{K_2}. \end{aligned} \quad (3.149)$$

This concludes the proof of Theorem 3.6 by getting the exponential decay of the system with a smallness condition on the L^2 -norm of the initial data z_0, \hat{z}_0 .

3.3 Simulation

3.3.1 Some notation

In this section we provide some numerical simulations showing the effectiveness of our control design. In order to discretize our KdV equation, we use a finite difference scheme inspired from [70] and similar to the one provided in Chapter 2, Section 2.3.2. The final time for simulations is denoted by T_{final} . We choose $(N_x + 1)$ points to build a uniform spatial discretization of the interval $[0, L]$ and $(N_t + 1)$ points to build a uniform time discretization of the interval $[0, T_{\text{final}}]$. Thus, the space step is $\Delta x = L/N_x$ and the time step $\Delta t = T_{\text{final}}/N_t$. We approximate the solution with the notation $z(t, x) \approx Z_j^i$, where i and j refer to time and space discrete variables, respectively.

The notation for the approximations of the derivative are the same than in Chapter 2, at the beginning of Section 2.3.2. In particular, we note

$$\mathcal{A} := D_+ D_+ D_- + D \quad (3.150)$$

and

$$\mathcal{C} := \mathcal{A} + dtI, \quad (3.151)$$

where I is the identity matrix in \mathbb{R}^{N_x} .

We will denote, for each discrete time i ,

$$\mathcal{Z}^i := [\mathcal{Z}_1^i \quad \mathcal{Z}_2^i \quad \dots \quad \mathcal{Z}_{N_x+1}^i]^T$$

the plant state, and

$$\mathcal{O}^i := [\mathcal{O}_1^i \quad \mathcal{O}_2^i \quad \dots \quad \mathcal{O}_{N_x+1}^i]^T$$

the observer state. For $m = \{1, 2\}$, the state output will be denoted by \mathcal{Y}_Z^m and \mathcal{Y}_O^m stands for the observer output. When $m = 1$, we refer to (3.15) and when $m = 2$, we refer to (3.17). For $m = \{1, 2\}$, the discretized controller gain U^m and observer gain P^m , respectively, are defined by

$$U^m = [K_1^m \quad K_2^m \quad \dots \quad K_{N_x+1}^m]^T$$

and

$$P^m = [P_1^m \quad P_2^m \quad \dots \quad P_{N_x+1}^m]^T.$$

We compute them from a successive approximations method (see [14] and Appendix C). When $m = 1$, we refer to k^1 and p^1 . When $m = 2$, we refer to k^2 and p^2 .

3.3.2 Illustration of the results of Section 3.2.2

Given K^1 and P^1 , the following is the structure of the algorithms used in our simulations. The first algorithm we provide is for the plant (3.16).

$$\begin{cases} \frac{Z_j^{i+1} - Z_j^i}{dt} + (\mathcal{A}Z^{i+1})_j = 0, & j = 1, \dots, N_x, \\ Z_1^i = \sqrt{\sum_{j=1}^{N_x+1} K_j^1 \mathcal{O}_j^i}, \quad Z_{N_x+1}^i = Z_{N_x}^i = 0, \\ Z^1 = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} z_0(x) dx, \end{cases} \quad (3.152)$$

The second numerical scheme we provide is for the observer (3.40)

$$\begin{cases} \frac{\mathcal{O}_j^{i+1} - \mathcal{O}_j^i}{dt} + (\mathcal{A}\mathcal{O}^{i+1})_j + P^1(\mathcal{Y}_Z^1 - \mathcal{Y}_O^1) = 0, & j = 1, \dots, N_x, \\ \mathcal{O}_1^i = \sqrt{\sum_{j=1}^{N_x+1} K_j^1 \mathcal{O}_j^i}, \quad \mathcal{O}_{N_x+1}^i = \mathcal{O}_{N_x}^i = 0, \\ \mathcal{O}^1 = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \hat{z}_0(x) dx, \end{cases} \quad (3.153)$$

where $x_{j+\frac{1}{2}} = j + \frac{1}{2}dx$ and $x_j = jdx$.

In order to illustrate our theoretical results, we perform some simulations on the domain $[0, 2\pi]$. We take $N_x = 30$, $N_t = 167$, $T_{\text{final}} = 8$, $\lambda = 1$, $z_0(x) = 1 - \cos(x)$ and $\hat{z}_0(x) = 0$. Figure 3.1 illustrates the convergence to the origin of the solution to the closed-loop system (3.14), (3.40) and (3.68). Figure 3.2 illustrates the L^2 -norm (3.14) with (3.68) and the L^2 -norm of the solution to the observer (3.81) with (3.68). Finally, Figure 3.3 illustrates the time evolution of the L^2 -norm of the observation error ($z - \hat{z}$) and of $C(z_0 - \hat{z}_0)e^{-\mu t}$ where $C = 2$ and $\mu = 0.4$. Note that the observation error converges to 0 in L^2 -norm. From the simulations, this convergence seems to be exponential as expected.

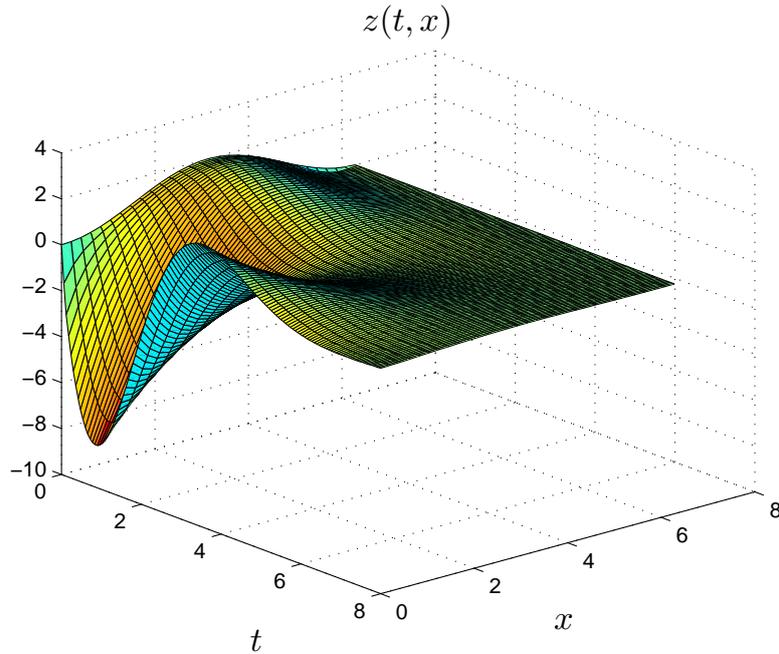


Figure 3.1: Solution to the closed-loop system (3.14)-(3.68)

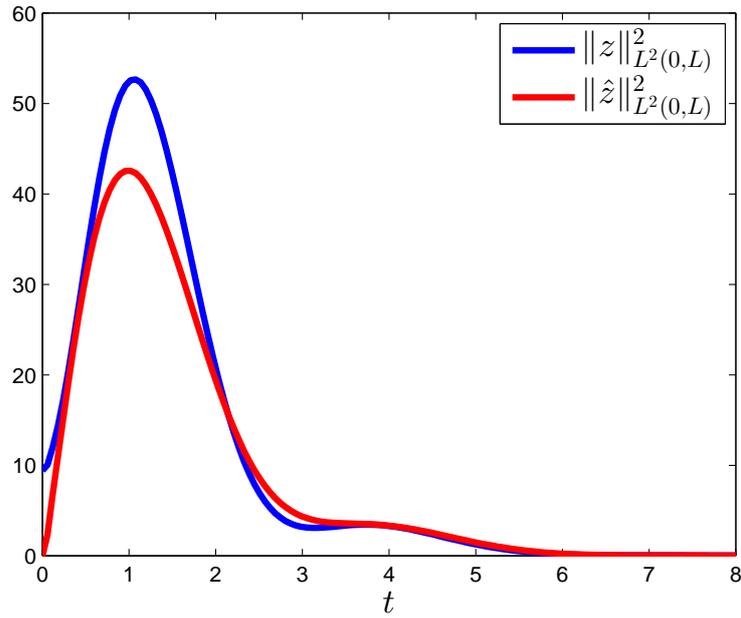


Figure 3.2: Time evolution of the L^2 -norm of the plant (blue line) and of the observer (red line).

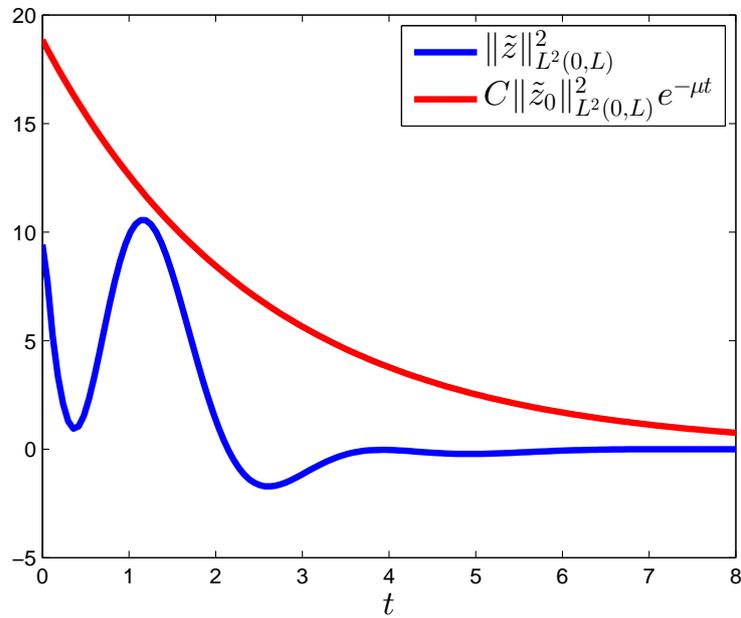


Figure 3.3: Time evolution of the L^2 -norm for the observation error $z - \hat{z}$ and of $C\|\tilde{z}_0\|_{L^2(0,L)}^2 e^{-\mu t}$

3.3.3 Illustration of the results of Section 3.2.3

Given K^2 and P^2 , the following is the structure of the algorithms used in our simulations. The first algorithm we provide is for the plant (3.16).

$$\begin{cases} \frac{\mathcal{Z}_j^{i+1} - \mathcal{Z}_j^i}{dt} + (\mathcal{A}\mathcal{Z}^{i+1})_j + \frac{1}{2} (D[(\mathcal{Z}^{i+1})^2])_j = 0, & j = 1, \dots, N_x, \\ \mathcal{Z}_1^i = \sqrt{\sum_{j=1}^{N_x+1} K_j^2 \mathcal{O}_j^i}, \quad \mathcal{Z}_{N_x+1}^i = 2\mathcal{Z}_{N_x}^i - \mathcal{Z}_{N_x-1}^i = 0, \\ \mathcal{Z}^1 = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} z_0(x) dx, \end{cases} \quad (3.154)$$

The second numerical scheme we provide is for the observer (3.81)

$$\begin{cases} \frac{\mathcal{O}_j^{i+1} - \mathcal{O}_j^i}{dt} + (\mathcal{A}\mathcal{O}^{i+1})_j + \frac{1}{2} (D[(\mathcal{O}^{i+1})^2])_j + P^2(\mathcal{Y}_Z^2 - \mathcal{Y}_O^2) = 0, & j = 1, \dots, N_x, \\ \mathcal{O}_1^i = \sqrt{\sum_{j=1}^{N_x+1} K_j^2 \mathcal{O}_j^i}, \quad \mathcal{O}_{N_x+1}^i = 2\mathcal{O}_{N_x}^i - \mathcal{O}_{N_x-1}^i = 0, \\ \mathcal{O}^1 = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \hat{z}_0(x) dx, \end{cases} \quad (3.155)$$

For both of these equations, since we have the nonlinearities zz_x and $\hat{z}\hat{z}_x$, we will use an iterative fixed point method to solve the nonlinear systems

$$\mathcal{C}\mathcal{Z}^{i+1} = \mathcal{Z}^i - \frac{1}{2}D(\mathcal{Z}^{i+1})^2,$$

and

$$\mathcal{C}\mathcal{O}^{i+1} = \mathcal{O}^i - \frac{1}{2}D(\mathcal{O}^{i+1})^2. \quad (3.156)$$

With $N_{iter} = 5$, which denotes the number of iterations of the fixed point method, we get good approximations of the solutions. The algorithm is similar to the one provided in Chapter 2.

In order to illustrate our theoretical results, we perform some simulations on the domain $[0, 2\pi]$. We take $N_x = 30$, $N_t = 167$, $T_{\text{final}} = 10$, $\lambda = 2$, $u_0(x) = \sin(x)$ and $\hat{u}_0(x) = 0$. Figure 3.4 illustrates the convergence to the origin of the solution to the closed-loop system (3.16) and (3.81) with u given by (3.132). Figure 3.5 illustrates the L^2 -norm of this solution and the L^2 -norm of the solution to the observer (3.81) with u given by (3.132). Finally, Figure 3.6 illustrates the time evolution of the L^2 -norm of the observation error $(z - \hat{z})$ and of $(z_0 - \hat{z}_0)e^{-\mu t}$ where $\mu = 0.4$. Note that the observation error converges to 0 in L^2 -norm. From the simulations, this convergence seems to be exponential as expected.

3.4 Conclusion

In this chapter, the infinite-dimensional backstepping technique has been introduced. This method has been used to design a state feedback law and an observer for two Korteweg-de

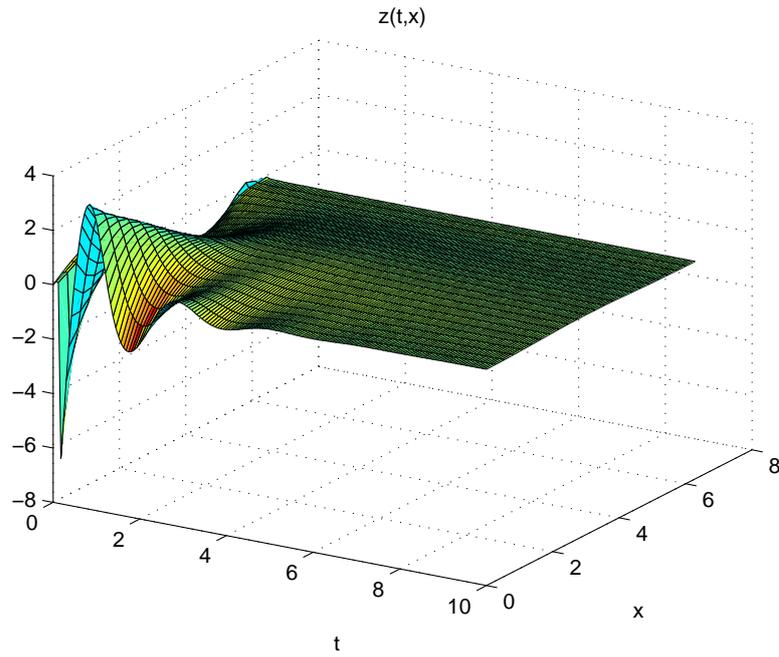


Figure 3.4: Solution to the closed-loop system (3.16) and (3.81) with u given by (3.132).

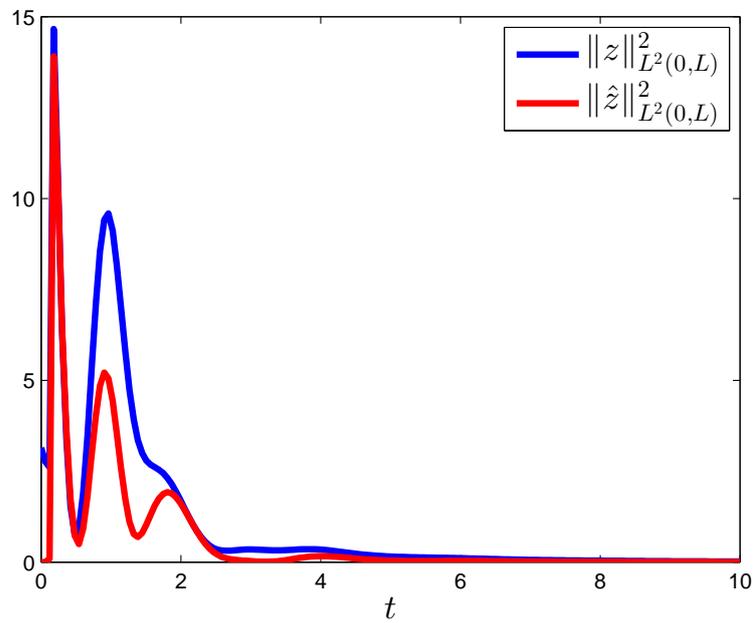


Figure 3.5: Time evolution of the L^2 -norm of the plant (blue line) and of the observer (red line).

Vries equations. The first one is linear, the second one is nonlinear. The well-posedness of

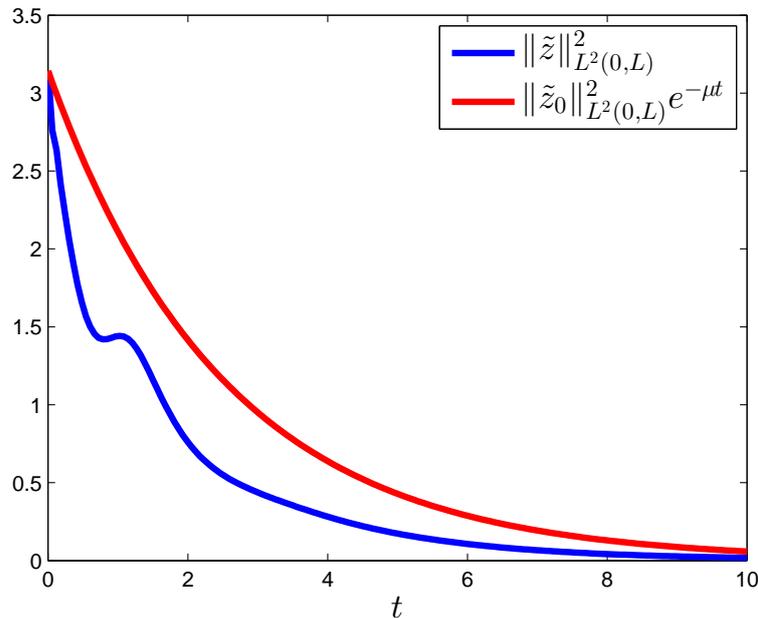


Figure 3.6: Time evolution of the L^2 -norm for the observation error $z - \hat{z}$ and of $\|\tilde{z}_0\|_{L^2(0,L)}^2 e^{-\mu t}$

these two equations has been tackled with semigroup theory and with the Banach-fixed point theorem in the case of the nonlinear equation. The exponential stability of the origin of these two systems has been proved using some Lyapunov functions. Finally, our results have been illustrated with some numerical simulations.

It could be interesting to design another observer for the KdV equation. One which might be useful is the high-gain observer, which is an observer for finite-dimensional systems satisfying a special structure. This technique will be introduced in details in the next chapter.

Semi-global stabilization by an output feedback law from a hybrid state controller

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This chapter deals with hybrid finite-dimensional systems, i.e. finite-dimensional systems with a mixed discrete/continuous dynamics. As it has been noticed in Chapter 3, separation principle may fail for nonlinear finite-dimensional system. The aim of this chapter is to present a design method for output feedback laws for nonlinear systems for which there exists a hybrid state controller.

In recent years, many techniques for designing a stabilizing control law for nonlinear dynamical systems have been developed. It is now possible to achieve stabilization of equilibria for a large class of models. However, due to Brockett's necessary condition for stabilizability, it is well known that some systems cannot be stabilized by a continuous controller. Some of these systems can however be stabilized with a hybrid state feedback law, i.e. a discrete/continuous controller (see e.g. [75], where the Brockett integrator is stabilized with a quasi optimal hybrid control). Moreover, the use of hybrid control laws may be interesting to address performance issues (see e.g. [74]). This explains the great interest of the control community in the synthesis of hybrid control laws (see [35], [39, 40, 31, 99]).

The output feedback stabilization problem has also attracted the attention of numerous researchers. Indeed, employing a state feedback law is most of the cases impossible, since the

sensors can only access to partial measurements of the state. Output feedback laws may be designed from a separation principle. More precisely, two tools are designed separately: a stabilizing state feedback law and an asymptotic state observer. However, if this approach is fruitful for linear systems, the separation principle does not hold in general for nonlinear systems. For instance, there exist stabilizable and observable systems for which the global asymptotic stabilization by output feedback is impossible ([67]). Nevertheless, from weak stabilizability and observability assumptions, some semi-global results may be obtained (see e.g. [97] or [42, Pages 125-172]). However, in this case the observer and the state feedback have to be jointly designed (not separately) (see also [3] for some global results).

The aim of this chapter is to address the stabilization by hybrid output feedback law. In [96], a local separation principle is stated. However, the construction of the observer is not explicit. Here, from a hybrid state feedback controller and an observability property, an algorithm is provided to build hybrid output feedback laws which stabilize semi-globally the equilibrium plant. If moreover a robust Lyapunov function is known, the feedback law design becomes explicit.

This chapter is organized as follows. Section 4.1 introduces an observability notion and motivates the use of hybrid controller. Section 4.2 provides our contributions together with the proof of our results. An illustrative example is also given. Finally, Section 4.3 collects some concluding remarks.

The contributions of this chapter, from Section 4.2 until the end, are based on two of our papers. The first one [65] has been published in a peer-reviewed conference (American Control Conference 2014). The second one [63] has been published in *Automatica*.

4.1 Complete uniform observability and hybrid controllers

4.1.1 Complete uniform observability

4.1.1.1 Why may the separation principle fail ?

As it has been noticed in Chapter 3, the separation principle given in Theorem 3.1 is generally not true for nonlinear finite-dimensional systems. The interested reader can refer to the survey [3] where some examples of control systems for which the separation principle fails are provided. Let us start this section with one of them, which is borrowed from [47].

Example 4.1

Consider the following system

$$\begin{cases} \frac{d}{dt}z_{p1} = -z_{p1} + z_{p2}z_{p1}^2 + u, \\ \frac{d}{dt}z_{p2} = -z_{p2} + z_{p1}^2, \end{cases} \quad (4.1)$$

where $z = [z_{p_1} \ z_{p_2}]^\top \in \mathbb{R}^2$ stands for the state. We have the following output

$$y = z_{p_1}. \quad (4.2)$$

The feedback law $u = -z_{p_2}z_{p_1}^2$ stabilizes globally the origin of (4.1)¹. Indeed, let us consider the function

$$V(z_{p_1}, z_{p_2}) := \frac{1}{2}z_{p_1}^2 + \frac{1}{2}(z_{p_2} + z_{p_1}^2)^2. \quad (4.3)$$

It is a positive definite function and a proper function. Hence, it is a candidate Lyapunov function.

Its derivative along the solution to (4.1) yields

$$\begin{aligned} \frac{d}{dt}V(z_{p_1}, z_{p_2}) &= -z_{p_1}^2 + (-z_{p_2} + z_{p_1}^2 - 2z_{p_1}^2)(z_{p_2} + z_{p_1}^2) \\ &= -V(z_{p_1}, z_{p_2}) \end{aligned} \quad (4.4)$$

This implies the global asymptotic stability of the origin of (4.1) when using the state feedback law $u = -z_{p_2}z_{p_1}^2$.

The following dynamical system

$$\frac{d}{dt}\hat{z}_{p_2} = -\hat{z}_{p_2} + y^2 \quad (4.5)$$

is a converging observer for the state z_{p_2} . Indeed,

$$\frac{d}{dt}(\hat{z}_{p_2} - z_{p_2}) = -(\hat{z}_{p_2} - z_{p_2}). \quad (4.6)$$

Hence, we have

$$|\hat{z}_{p_2} - z_{p_2}| \leq e^{-t}|\hat{z}_{p_2}^\# - z_{p_2}^\#|, \quad \forall t \geq 0, \quad (4.7)$$

where $\hat{z}_{p_2}^\# := \hat{z}_{p_2}(0)$ and $z_{p_2}^\# := z_{p_2}(0)$. Therefore, the state of the observer converges to the state z_{p_2} of the plant (4.1). Note that z_{p_1} does not have to be estimated, since it is already measured.

If one uses the following output feedback law

$$u = -\hat{z}_{p_2}y^2 \quad (4.8)$$

there exist solutions to the following closed-loop system

$$\begin{cases} \frac{d}{dt}z_{p_1} = -z_{p_1} + z_{p_2}z_{p_1}^2 - \hat{z}_{p_2}y^2, \\ \frac{d}{dt}z_{p_2} = -z_{p_2} + z_{p_1}^2, \\ \frac{d}{dt}\hat{z}_{p_2} = -\hat{z}_{p_2} + y^2, \\ y = z_{p_1}. \end{cases} \quad (4.9)$$

¹The static output feedback $u = -y^5$ stabilizes the origin of (4.1). The feedback law used in this example has only an educational interest.

that escape to infinity in finite time. Indeed, denoting by $\tilde{z}_{p_2}(t) := z_{p_2} - \hat{z}_{p_2}(t) := \tilde{z}_{p_2}^\# e^{-t}$, where $\tilde{z}_{p_2}^\# := z_{p_2}^\# - \hat{z}_{p_2}^\#$, the first line of the latter system can be written as follows

$$\frac{d}{dt} z_{p_1} = -z_{p_1} + z_{p_1}^2 \tilde{z}_{p_2}^\# e^{-t}. \quad (4.10)$$

Following [47], we know that there exists an explicit solution to this equation, which is

$$z_{p_1}(t) := \frac{2z_{p_1}^\#}{(2 - z_{p_1}^\# \tilde{z}_{p_2}^\#)e^t + z_{p_1}^\# \tilde{z}_{p_2}^\# e^{-t}}, \quad (4.11)$$

where $z_{p_1}^\# := z_{p_1}(0)$. From this solution, we can see that if

$$z_{p_1}^\# \tilde{z}_{p_2}^\# > 2, \quad (4.12)$$

then the state z_{p_1} tends to infinity in finite time.

Finally, even if there exists a global observer and a global stabilizing state feedback for a finite-dimensional nonlinear control system, the corresponding output feedback may not stabilize globally the origin of the closed-loop system. Hence, the notion of observability for nonlinear systems should be given in details.

4.1.1.2 Observability notions

In [32], the authors introduce an observer, namely the *high-gain observer*, that is purely nonlinear, together with an observability notion. Let us introduce them.

Consider the following controlled system

$$\frac{d}{dt} z_p = f_p(z_p) + g_p(z_p)u, \quad y = h_p(z_p), \quad (4.13)$$

where $z_p \in \mathbb{R}^{n_p}$, $y \in \mathbb{R}$, $u \in \mathcal{U} \subset \mathbb{R}$. Note that $f_p : \mathbb{R}^{n_p} \rightarrow \mathbb{R}^{n_p}$ and $g_p : \mathbb{R}^{n_p} \rightarrow \mathbb{R}^{n_p}$, $h_p : \mathbb{R}^{n_p} \rightarrow \mathbb{R}$ are $n_p + 1$ times continuously differentiable². The set \mathcal{U} can be bounded (it yields a saturated control problems). We assume that the origin of (4.13) is an equilibrium point.

Define the C^1 mapping $\phi : \mathbb{R}^{n_p} \rightarrow \mathbb{R}^{n_p}$ as follows

$$\phi(z_p) = \left[h_p(z_p) \quad L_{f_p} h_p(z_p) \quad \dots \quad L_{f_p}^{n_p-1} h_p(z_p) \right]^\top, \quad (4.14)$$

where $L_{f_p}^i h_p(z_p)$ denotes the i -th Lie derivative of h_p along f_p ³.

The observability notion employed all along this chapter can be now stated.

²These mappings are sufficiently smooth so that the mapping ϕ defined in (4.14) is C^1 and so that the function B defined in (4.36) is locally Lipschitz.

³Given $i \in \mathbb{N}$, the i -th Lie derivative of h_p along f_p is defined as follows

$$L_{f_p}^i h_p(z_p) := \frac{\partial^i}{\partial z_p^i} h_p(z_p) f_p(z_p). \quad (4.15)$$

Definition 4.1 ((Global) Complete Uniform Observability ([32]))

System (4.13) is completely uniformly observable, that is

- (i) The mapping $\phi : \mathbb{R}^{n_p} \rightarrow \phi(\mathbb{R}^{n_p}) = \mathbb{R}^{n_p}$ is a diffeomorphism;
- (ii) System (4.13) is observable for any input $u(t)$, i.e. on any finite time interval $[0, T]$, for any measurable bounded input $u(t)$ defined on $[0, T]$, the initial state is uniquely determined on the basis of the output $y(t)$ and the input $u(t)$.

Remark 4.1

Note that there exist other type of observability notions for nonlinear systems. For uncontrolled systems, one can cite [4], where some conditions for the existence of a nonlinear Luenberger observer for nonlinear systems are derived.

Example 4.2 • The following system

$$\begin{cases} \frac{d}{dt} z_{p1} = z_{p2}, \\ \frac{d}{dt} z_{p2} = z_{p2}^2 + u, \\ y = z_{p1} \end{cases} \quad (4.16)$$

is completely uniformly observable. Indeed, since

$$L_{f_p} h_p(z_p) = [1 \ 0] \begin{bmatrix} z_{p2} \\ z_{p2}^2 \end{bmatrix} = z_{p2} \quad (4.17)$$

then $\phi(z_p)$ is defined as follows

$$\phi(z_p) : [z_{p1} \ z_{p2}]^\top \in \mathbb{R}^2 \mapsto [z_{p1} \ z_{p2}]^\top \in \mathbb{R}^2. \quad (4.18)$$

It is clear that it defines a diffeomorphism. Moreover, the item (ii) of Definition 4.1 is also satisfied.

- The system given in Example 4.1 is not completely uniformly observable. Indeed, the following holds

$$L_{f_p} h_p(z_p) = [1 \ 0] \begin{bmatrix} -z_{p1} + z_{p2} z_{p1}^2 \\ -z_{p2} + z_{p1}^2 \end{bmatrix} = -z_{p1} + z_{p2} z_{p1}^2. \quad (4.19)$$

Hence, $\phi(z_p)$ is defined as follows

$$\phi(z_p) : [z_{p1} \ z_{p2}]^\top \in \mathbb{R}^2 \mapsto z_{p1} \quad -z_{p1} + z_{p2} z_{p1}^2. \quad (4.20)$$

It is easy to check that this function does not define a diffeomorphism in \mathbb{R}^2 . Indeed, let us focus on its jacobian matrix, which is defined as follows

$$J\phi(z_p) := \begin{bmatrix} 1 & -1 + 2z_{p2} \\ 0 & z_{p1}^2 \end{bmatrix} \quad (4.21)$$

Its determinant is given by

$$\det J\phi(z_p) = z_{p1}^2, \quad (4.22)$$

which is equal to 0 when $z_{p1} = 0$. Hence, the function $z_p \mapsto \phi(z_p)$ does not define a diffeomorphism and the system given in Example 4.1 is not completely uniformly observable.

If (4.13) satisfies the properties given by Definition (4.1) and if moreover there exists a *continuous* stabilizing state feedback for (4.13), then we know from [97] that the origin of (4.13) is *semi-globally stabilizable by means of a dynamic output feedback*. Let us provide the definition of such a stability.

Definition 4.2 (Semi-global stabilizability [97])

The origin of system (4.13) is said to be semi-globally stabilizable by means of a dynamic output feedback if, for each compact set D_p , there exist a dynamic output feedback $u = \theta_c(\hat{z}_p)$, $\hat{z}_p = \theta_p(y, \hat{z}_p)$ and a compact set D_o such that the equilibrium $(0, 0) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_p}$ of the closed-loop system is asymptotically stable with basin of attraction containing $D_p \times D_o$.

Remark 4.2

This property is different from the one provided in Definition 2.2. Indeed, the semi-global stabilizability implies that the controller θ_c depends on the compact set D_p , while the semi-global exponential stability is a property that a uncontrolled system can satisfy.

The semi-global stabilizability is satisfied if there exists a *continuous* stabilizing state feedback for (4.13) and if (4.13) is completely uniformly observable. However, as it will be noticed in the next subsection, some control systems cannot be stabilized by means of a continuous state feedback.

4.1.2 Hybrid controllers

As it has been noticed in the introduction and in the latter section, there exist some conditions which do not allow to stabilize systems with continuous state feedback law. One of these conditions is the Brockett condition.

Consider the following control system

$$\frac{d}{dt}z_p = f(z_p, u), \quad (4.23)$$

where, for $n_p, m \in \mathbb{N}$, $z_p \in \mathbb{R}^{n_p}$ and $u \in \mathbb{R}^m$. The vector field $f : \mathbb{R}^{n_p} \times \mathbb{R}^m \rightarrow \mathbb{R}^{n_p}$ is sufficiently smooth.

The following result gives a necessary condition for the stabilizability by means of a continuous stationary feedback law of such a system

Theorem 4.1 (Brockett's condition [8])

If the control system (4.23) can be locally asymptotically stabilized by means of a continuous

stationary feedback law, then the image by f of every neighborhood of $[0 \ 0]^\top \in \mathbb{R}^{n_p} \times \mathbb{R}^m$ is a neighborhood of $0 \in \mathbb{R}^{n_p}$

Example 4.3

A classical system to illustrate this restriction is the Brockett’s integrator

$$\begin{cases} \frac{d}{dt}z_{p1} = u_1, \\ \frac{d}{dt}z_{p2} = u_2, \\ \frac{d}{dt}z_{p3} = z_{p1}u_2 - z_{p2}u_1 \end{cases} \tag{4.24}$$

This system is not stabilizable by means of a continuous time-invariant feedback law. Indeed, for every $\eta \in \mathbb{R} \setminus \{0\}$, the equation

$$\begin{cases} u_1 = 0, \\ u_2 = 0, \\ z_{p1}u_2 - z_{p2}u_1 = \eta \end{cases} \tag{4.25}$$

where the unknown are $[z_{p1} \ z_{p2} \ u_1 \ u_2]^\top \in \mathbb{R}^4$, has no solution. Therefore, the Brockett’s integrator does not satisfy the Brockett’s condition and there does not exist any continuous stationary feedback laws that stabilize the origin of this equation.

To solve this problem, researchers in control theory derived several methods. The two main ones are

1. Using a time-varying state feedback (see e.g., [21, Chapter 11] and [22]);
2. Using a hybrid state feedback, i.e. a feedback mixed continuous/discrete (see e.g., [74]).

In this chapter, we will focus on the second strategy. Note that a good introduction to hybrid systems is [35]. This book provides a framework for the well-posedness and the stability of equilibrium sets of systems mixing continuous/discrete behavior. We do not aim at presenting this theory here.

4.2 Semi-global stabilization by an output feedback law from a hybrid state controller

4.2.1 Main results

4.2.1.1 Assumptions

The system under consideration is (4.13). Inspired by [75] and [86], the origin, which is an equilibrium point for (4.13), is assumed to be stabilizable by a hybrid state feedback.

Assumption 4.1

(Persistent Flow Stabilizability) *There exists a hybrid controller defined by $(\mathcal{F}_c, \mathcal{J}_c, f_c, g_c, \theta_c)$, where \mathcal{F}_c and \mathcal{J}_c are closed sets, $\mathcal{F}_c \cup \mathcal{J}_c = \mathbb{R}^{n_p+n_c}$, $g_c : \mathbb{R}^{n_p+n_c} \rightarrow \mathbb{R}^{n_c}$, $f_c : \mathbb{R}^{n_p+n_c} \rightarrow \mathbb{R}^{n_c}$ and $\theta_c : \mathbb{R}^{n_p+n_c} \rightarrow \mathcal{U}$ are continuous functions and a positive value λ in $(0, 1)$ such that the set $\{0\} \times [0, 1]$ in $\mathbb{R}^{n_p+n_c} \times \mathbb{R}$ is asymptotically stable for the system:*

$$\begin{cases} \frac{d}{dt} z_p = f_p(z_p) + g_p(z_p)\theta_c(z_p, z_c) \\ \frac{d}{dt} z_c = f_c(z_p, z_c) \\ \frac{d}{dt} \tau = 1 - \tau \end{cases} \quad (z_p, z_c, \tau) \in \mathcal{F}_c \times \mathbb{R}_{\geq 0} \quad (4.26a)$$

$$\begin{cases} z_p^+ = z_p \\ z_c^+ = g_c(z_p, z_c) \\ \tau^+ = 0 \end{cases} \quad (z_p, z_c, \tau) \in \mathcal{J}_c \times \mathbb{R}_{\geq \lambda} \quad (4.26b)$$

with basin of attraction $\mathcal{B} \times \mathbb{R}_{\geq 0}$, where \mathcal{B} is an open subset of $\mathbb{R}^{n_p+n_c}$.

The sets $\mathcal{F}_c \times \mathbb{R}_{\geq 0}$ and $\mathcal{J}_c \times \mathbb{R}_{\geq \lambda}$ are called respectively the flow and jump sets associated to the continuous and discrete dynamics. The notion of solutions and of asymptotic stability discussed all along the chapter are borrowed from [35].

Note that the set \mathcal{B} allows us to consider cases where the stabilizability of the origin of the closed-loop system is only local. In the case where $\mathcal{B} := \mathbb{R}^{n_p+n_c}$, the stabilizability property becomes global.

Remark 4.3

An important feature of the hybrid state feedback control law is that its dynamics include a timer τ . It implies that there exists a dwell time between two consecutive jumps and consequently it prevents the existence of Zeno solutions. In the case in which this property is not satisfied for the state feedback, a timer can be added as presented in [10, Part V, C.]. Such a technique is called a temporal regularization. However, in this case, only semi-global practical stabilizability is obtained.

The problem under consideration in this chapter is to design a stabilizing output feedback law based on this hybrid state feedback. The design presented in this chapter requires the following complete uniform observability assumption.

Assumption 4.2 ((Global) Complete Uniform Observability)

System (4.13) is completely uniformly observable.

Remark 4.4

In [65], from a weaker observability assumption, i.e. an observability property holding for just one control, a finite-time convergent observer and a hybrid state feedback controller has been used to design an output feedback law. Such a strategy does not need a persistent flow

stabilizability assumption. However only a weak stability property is obtained for the closed-loop system.

4.2.1.2 First main result

Inspired by the approach of [97], from Assumptions 4.1 and 4.2, a semi-global output feedback result may be obtained.

Theorem 4.2 (Semi-global asymptotic stability)

Assume Assumptions 4.1 and 4.2 hold. Assume moreover that g_c satisfies that, for all (z_p, z_c) in $\mathcal{B} \cap \mathcal{J}_c$, the set

$$\{(z_p, g_c(w, z_c)), w \in \mathbb{R}^{n_p}\} \quad (4.27)$$

is a compact subset of \mathcal{B} , then the origin of system (4.13) is semi-globally asymptotically stabilizable by a hybrid output feedback. In other words, for all compact sets Γ contained in $\mathcal{B}^p := \{z_p \in \mathbb{R}^{n_p}, (z_p, 0) \in \mathcal{B}\}$, there exist a C^1 function $\Psi_p : \mathbb{R}^{n_p} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and a positive real number c_x such that the set $\{0\} \times [0, 1]$ in $\mathbb{R}^{2n_p+n_c} \times [0, 1]$ is asymptotically stable for the system

$$\begin{cases} \frac{d}{dt} z_p = f_p(z_p) + g_p(z_p)u \\ \frac{d}{dt} z_c = f_c(\tilde{z}_p, z_c) \\ \frac{d}{dt} \tau = 1 - \tau \\ y = h_p(z_p), \quad u = \theta_c(\tilde{z}_p, z_c) \end{cases} \quad (\tilde{z}_p, z_c, \tau) \in \mathcal{F}_c \times \mathbb{R}_{\geq 0} \quad (4.28a)$$

$$\begin{cases} z_p^+ = z_p \\ \hat{x}_p^+ = \hat{x}_p \\ z_c^+ = g_c(\tilde{z}_p, z_c) \\ \tau^+ = 0 \end{cases} \quad (\tilde{z}_p, z_c, \tau) \in \mathcal{J}_c \times \mathbb{R}_{\geq \lambda}. \quad (4.28b)$$

where \tilde{z}_p is defined by ⁴

$$\tilde{z}_p = \mathbf{sat}_{c_x}(\hat{z}_p), \quad (4.29)$$

with basin of attraction containing $\Gamma \times \{0\} \times \{0\} \times \mathbb{R}_{\geq 0}$ (which is a subset of $\mathbb{R}^{n_p} \times \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \times \mathbb{R}_{\geq 0}$).

The design of the output feedback law which proves this theorem is based on a Lyapunov inverse theorem. However, two datas miss in the output feedback law given in Theorem 4.2: the positive real number c_x , the saturation level for the feedback law, and the observer dynamics Ψ_p . In order to give an explicit result, the existence of a robust Lyapunov function is assumed, in the next section.

⁴ Given a positive real number c , $\mathbf{sat}_c : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the saturating vector function defined by $\mathbf{sat}_c(0) = 0$ and $\mathbf{sat}_c(z_p) := z_p \min \left\{ 1, \frac{c}{|z_p|} \right\}, \forall x \neq 0$.

Remark 4.5

An important feature of this theorem is that an assumption needs to be imposed on the function g_c (see equation (4.27)). This is due to a design of a set in which the solution should stay for a suitable duration. In the particular case in which \mathcal{B} is $\mathbb{R}^{n_p+n_c}$, the previous condition is trivially satisfied if g_c is such that

$$|g_c(w, z_c)| \leq \gamma(|z_c|), \quad \forall (w, z_c) \in \mathbb{R}^{n_p+n_c} \cap \mathcal{J}_c$$

where $\gamma \in \mathcal{K}$. Moreover, note that there is a large class of systems that satisfy such a condition. For instance, switch systems or reset systems, as the one considered in Section 4.2.3 below.

4.2.1.3 Second main result

In this section an explicit result is introduced. It is based on the following assumption, where it is denoted $\mathcal{Z} = [z_p^\top \ z_c^\top \ \tau^\top]^\top$.

Assumption 4.3 (Robust Lyapunov function)

Let \mathcal{B} be an open subset of $\mathbb{R}^{n_p} \times \mathbb{R}^{n_c}$ and let denote $\mathcal{A} := \{0\} \times [0, 1] \subset \mathcal{B} \times \mathbb{R}_{\geq 0}$. There exist a hybrid controller defined by $(\mathcal{F}_c, \mathcal{J}_c, f_c, g_c, \theta_c)$, where \mathcal{F}_c and \mathcal{J}_c are closed sets, $\mathcal{F}_c \cup \mathcal{J}_c = \mathbb{R}^{n_p+n_c}$, $g_c : \mathbb{R}^{n_p+n_c} \rightarrow \mathbb{R}^{n_c}$, $f_c : \mathbb{R}^{n_p+n_c} \rightarrow \mathbb{R}^{n_c}$ and $\theta_c : \mathbb{R}^{n_p+n_c} \rightarrow \mathcal{U}$ are continuous functions, a positive value λ in $(0, 1)$, positive values α_1 and $\alpha_2 \in (0, 1)$ and a C^1 proper⁵ function $V : \mathcal{B} \times \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ satisfying $\{\mathcal{Z} \in \mathcal{B} \times \mathbb{R}, V(\mathcal{Z}) = 0\} = \mathcal{A}$. For all positive real numbers l , the level set of V defined as

$$D_l := \{(z_p, z_c, \tau) \in \mathcal{B} \times \mathbb{R}_{\geq 0} : V(z_p, z_c, \tau) \leq l\}, \quad (4.30)$$

is a compact subset of $\mathcal{B} \times \mathbb{R}$.

Moreover, there exists a positive real number ε_r and an increasing C^0 function $\rho : [0, \varepsilon_r] \rightarrow \mathbb{R}_+$ with $\rho(0) = 0$ such that for all (\mathcal{Z}, e) in $D_l \times \mathbb{R}^{n_p}$ such that $|e| \leq \varepsilon_r$, the following inequalities hold.

- If $(z_p + e, z_c, \tau) \in (\mathcal{B} \cap \mathcal{F}_c \times \mathbb{R}_{\geq 0})$,

$$\frac{\partial V}{\partial \mathcal{Z}}(\mathcal{Z})F(\mathcal{Z}, z_p + e) \leq -\alpha_1 V(z_p, z_c, \tau) + \rho(|e|) \quad (4.31)$$

- If $(z_p + e, z_c, \tau) \in (\mathcal{B} \cap \mathcal{J}_c \times \mathbb{R}_{\geq \lambda})$

$$V(G(\mathcal{Z}, z_p + e)) - V(\mathcal{Z}) \leq -\alpha_2 V(z_p, z_c, \tau) + \rho(|e|), \quad (4.32)$$

where F is defined by

$$F(\mathcal{Z}, \cdot) := [(f_p(z_p) + g_p(z_p)\theta_c(\cdot, z_c))^\top \ f_c(\cdot, z_c)^\top \ 1 - \tau]^\top$$

and G is defined by

$$G(\mathcal{Z}, \cdot) := [z_p^\top \ g_c(\cdot, z_c)^\top \ 0]^\top.$$

⁵A map is called proper if inverse images of compact sets are compact.

This assumption allows to obtain an explicit result.

Theorem 4.3 (Design of an output feedback law)

Under Assumptions 4.2 and 4.3, assume that the set defined by (4.27) is a compact subset of \mathcal{B} . Then the set $\{0\} \times [0, 1]$ in $\mathbb{R}^{n_p+n_c} \times \mathbb{R}$ is semi-globally asymptotically stabilizable. In other words, the conclusion of Theorem 4.2 holds. Moreover c_x is computed in Section 4.2.2.3 and ψ_p is computed in Section 4.2.2.4 from the Lyapunov function V together with the robustness margin ε_r and the positive value λ of Assumption 4.3, and from the function ϕ of Assumption 4.2.

Let us note that it is difficult to be more explicit since the derivations of c_x and Ψ_p are quite long and require several steps. This is already the case in [97]. Moreover, continuing what has been stated in Remark 4.3, it is crucial to have a Persistent-Flow Stabilizability in order to design explicitly our observer.

In a first step, c_x is computed in (4.35) in order to force the solutions to remain in a compact set for a certain amount of (flow) time. The function Ψ_p is a high-gain observer which is tuned in Lemma 4.2. It forces the error to reach the robustness margin obtained from Assumption 4.3 before the solution escapes the compact set.

The explicit construction of these two data and the proof that this output feedback law is a solution to Theorem 4.3 is reported in Section 4.2.2.

4.2.2 Proofs of the main results

4.2.2.1 Proof of Theorem 4.2 from Theorem 4.3

In order to prove Theorem 4.2 from Theorem 4.3 it is sufficient to prove that Assumption 4.1 implies Assumption 4.3. This can be obtained from an inverse Lyapunov result. First, from [35, Corollary 7.32] there exists a positive value $\alpha \in (0, 1)$ and a smooth proper function $V : \mathcal{B} \times \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ satisfying

$$\begin{aligned} \{\mathcal{Z} \in \mathcal{B} \times \mathbb{R} : V(\mathcal{Z}) = 0\} &= \mathcal{A} \\ \frac{\partial V}{\partial \mathcal{Z}}(\mathcal{Z})F(\mathcal{Z}, z_p) &\leq -V(\mathcal{Z}), \quad \forall (z_p, z_c, \tau) \in (\mathcal{B} \cap \mathcal{F}_c) \times [0, \lambda] \\ V(G(\mathcal{Z}, z_p)) - V(\mathcal{Z}) &\leq -\alpha V(\mathcal{Z}), \quad \forall (z_p, z_c, \tau) \in (\mathcal{B} \cap \mathcal{J}_c) \times \mathbb{R}_{\geq 0} \end{aligned} \tag{4.33}$$

Let l be a positive real number such that the level set D_l is a compact subset of $\mathcal{B} \times \mathbb{R}$. Consider the two functions r_1 and r_2 defined as

$$\begin{aligned} r_1(s) &= \max_{|e| \leq s, (z_p+e, z_c, \tau) \in D_l} \frac{\partial V(\mathcal{Z})}{\partial \mathcal{Z}} F(\mathcal{Z}, z_p) + \frac{1}{2} V(\mathcal{Z}) \\ r_2(s) &= \max_{|e| \leq s, (z_p+e, z_c, \tau) \in D_l} V(G(\mathcal{Z}, z_p)) - \left(1 - \frac{1}{2}\alpha\right) V(\mathcal{Z}) \end{aligned}$$

Since F, G are continuous and V is smooth, r_1 and r_2 are also continuous functions. Moreover $r_1(0) < 0$ and $r_2(0) < 0$. Therefore there exist ε_r^1 and ε_r^2 such that $r_1(s) < 0$ for all $s \leq \varepsilon_r^1$ and $r_2(s) < 0$ for all $s \leq \varepsilon_r^2$. Let $\varepsilon_r = \min(\varepsilon_r^1, \varepsilon_r^2)$. For all $|e| \leq \varepsilon_r$ and $(z_p, z_c, \tau) \in D_l$ it yields the following:

- If $(z_p + e, z_c, \tau)$ in $(\mathcal{B} \cap \mathcal{F}_c \times \mathbb{R}_{\geq 0})$,

$$\frac{\partial V}{\partial \mathcal{Z}}(\mathcal{Z})F(\mathcal{Z}, z_p) \leq -\frac{1}{2}V(\mathcal{Z}), \forall \mathcal{Z} \in \mathcal{B} \cap \mathcal{F}_c \times \mathbb{R}_{\geq 0}$$

- If $(z_p + e, z_c, \tau)$ in $(\mathcal{B} \cap \mathcal{J}_c \times \mathbb{R}_{\geq \lambda})$

$$V(G(\mathcal{Z}, z_p)) - V(\mathcal{Z}) \leq -\frac{1}{2}\alpha V(\mathcal{Z}), \forall \mathcal{Z} \in \mathcal{B} \cap \mathcal{J}_c \times \mathbb{R}_{\geq \lambda}.$$

Hence this Lyapunov function is the same than the one introduced in Assumption 4.3 with $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = \frac{\alpha}{2}$.

Consider now the increasing function $\rho : [0, \varepsilon_r] \rightarrow [0, +\infty)$ defined as follows⁶

$$\rho(s) \geq \max \left\{ \max_{(z_p+e, z_c, \tau) \in (D_{l_4} \cap \mathcal{F}_c \times \mathbb{R}_{\geq 0}), |e| \leq s} \nu_1(\mathcal{Z}, e), \max_{(z_p+e, z_c, \tau) \in (D_{l_4} \cap \mathcal{J}_c \times \mathbb{R}_{\geq 0}), |e| \leq s} \nu_2(\mathcal{Z}, e) \right\}$$

where

$$\nu_1(\mathcal{Z}, e) = \left| \frac{\partial V}{\partial \mathcal{Z}}(\mathcal{Z}) (F(\mathcal{Z}, z_p + e) - F(\mathcal{Z}, z_p)) \right|,$$

and

$$\nu_2(\mathcal{Z}, e) = |V(G(\mathcal{Z}, z_p + e)) - V(G(\mathcal{Z}, z_p))|.$$

With this function, Assumption 4.3 is satisfied. This ends the proof of Assumption 4.3 from Assumption 4.1. Therefore, as soon as Theorem 4.3 is valid, Theorem 4.2 holds under Assumptions 4.1 and 4.2.

4.2.2.2 Construction of the output feedback law

In the next sections, we follow a similar approach to [97]. We first compute a saturation level c_x , a time of existence T_{\min} and a compact subset of $\mathcal{B} \times \mathbb{R}_{\geq 0}$ denoted by D_{l_4} such that, when saturating the controller with c_x , the solution starting from $\mathcal{B} \times \mathbb{R}_{\geq 0}$ remains in D_{l_4} for all time less than T_{\min} . Then, with T_{\min} and the margin of robustness c_e from Assumption 4.3, we design an observer such that the error dynamics converges to 0 asymptotically and such that, for all time higher than T_{\min} , the error dynamics belongs to the margin of robustness. Finally, we prove the attractiveness and the stability of the closed-loop system with the output feedback law.

⁶This function is well defined due to the fact that \mathcal{F}_c and \mathcal{J}_c are closed sets.

4.2.2.3 Selection of c_x and minimal time of existence of solutions

This section differs from the strategy employed in [97]. Indeed, since we have a hybrid dynamics, the solution of the closed-loop system can jump. Therefore, the computation of T_{\min} becomes difficult. However, thanks to the timer dynamics, we can assert that between two jumps the solution of the closed-loop system belongs to the flow set. This allows us to compute T_{\min} and thus c_x .

In the remaining part of this subsection, we consider the system defined by

$$\begin{cases} \frac{d}{dt}z_p = f_p(z_p) + g_p(z_p)\theta_c(\omega, z_c) \\ \frac{d}{dt}z_c = f_c(\omega, z_c) \\ \frac{d}{dt}\tau = 1 - \tau \\ (\omega, z_c, \tau) \in \mathcal{F}_c \times \mathbb{R}_{\geq 0}, \end{cases} \quad (4.34a)$$

$$\begin{cases} z_p^+ = z_p \\ z_c^+ = g_c(\omega, z_c) \\ \tau^+ = 0 \end{cases} \quad (\omega, z_c, \tau) \in \mathcal{J}_c \times \mathbb{R}_{\geq \lambda}, \quad (4.34b)$$

where ω is an external perturbation function in \mathbb{R}^{n_p} . Such a system is not a classical hybrid system as the ones introduced in [35] since the flow and jump sets are defined with an external disturbance. Note that in the particular case in which $\omega = \hat{z}_p$ defined in (4.29), the solution to system (4.34) by adding the dynamics of \hat{z}_p is also solution to system (4.28). Hence, this implies the well-posedness of the closed-loop system as considered in [35, Chapter 2].

Two cases may be distinguished to construct the sets: i) Solution to (4.34) does not jump; ii) Solution to (4.34) jumps at least one time. The first case is similar to the continuous case (and thus to [97]). The second case takes into account the hybrid behavior of the system under consideration. Under Assumption 4.3, let $l_1 = \max_{z_p \in \Gamma, \tau \in [0, \lambda]} V(z_p, 0, \tau)$. Note that D_{l_1} is a compact subset of $\mathcal{B} \times \mathbb{R}_{\geq 0}$ (see the notation employed in equation (4.30)). Let $l_2 > l_1$ such that $D_{l_2} \subset \mathcal{B} \times \mathbb{R}_{\geq 0}$. To deal with the jump that can occur, we consider $D_{l_2}^+ = \bigcup_{(z_p, z_c, \tau) \in D_{l_2}} \{(z_p, g_c(w, z_c), 0), w \in \mathbb{R}^{n_p}\}$. Since it is assumed that the set defined in (4.27) is a compact subset of $\mathcal{B} \times \mathbb{R}_{\geq 0}$, it yields that $D_{l_2}^+$ is also a compact subset of $\mathcal{B} \times \mathbb{R}_{\geq 0}$. Let l_3 be such that D_{l_3} is a compact subset which satisfies $D_{l_2}^+ \subset D_{l_3} \subset \mathcal{B} \times \mathbb{R}_{\geq 0}$. Finally, let $l_4 > l_3$ so that D_{l_4} is a compact subset which contains D_{l_3} .

With these sets in hands, the positive real number c_x can be selected as

$$c_x = \max_{(z_p, z_c, \tau) \in D_{l_4}} \{|z_p|\} \quad (4.35)$$

Let us now establish the following property for solutions to system (4.34) initiated from D_{l_1} .

Lemma 4.1 (Minimal existence time of solution in D_{l_4})

There exists $T_{\min} > 0$ such that for all ω in $\mathcal{L}_{loc}^\infty([0, +\infty); \mathbb{R}^{n_p})$ with $|\omega(t)| \leq c_x$ for all t in $[0, T_{\min}]$, and all $\mathcal{Z}^\# := (z_p^\#, z_c^\#, \tau^\#)$ in D_{l_1} , all solutions $\mathcal{Z}(\cdot, \cdot)$ to (4.34) with $\mathcal{Z}(0, 0) = \mathcal{Z}^\#$ and all (t, j) in $\text{dom}(\mathcal{Z})$ ⁷ then $\mathcal{Z}(t, j) \in D_{l_4}$ for all $0 \leq t \leq T_{\min}$.

Proof of Lemma 4.1:

Let \bar{V} the positive real number defined by

$$\bar{V} = \max_{\mathcal{Z} \in D_{l_4}, |\omega| \leq c_x} \left| \frac{\partial V}{\partial \mathcal{Z}}(\mathcal{Z}) F(\mathcal{Z}, \omega) \right|.$$

In the remaining part of the proof, we show that Lemma 4.1 holds with T_{\min} chosen as any positive real number satisfying

$$T_{\min} < \min \left\{ -\ln(1 - \lambda), \frac{l_2 - l_1}{\bar{V}}, \frac{l_4 - l_3}{\bar{V}} \right\}.$$

Let $\mathcal{Z}^\#$ be in D_{l_1} and let \mathcal{Z} be a solution to system (4.34) whose initial condition is $\mathcal{Z}^\#$. For all (t, j) in $\text{dom}(\mathcal{Z})$. To ease the notation we denote $V(t, j) = V(\mathcal{Z}(t, j))$.

Let (t, j) in $\text{dom}(\mathcal{Z})$ such that $0 \leq t \leq T_{\min}$. To prove the lemma, we need to show that $\mathcal{Z}(t, j)$ is in D_{l_4} . First of all, we show that $j \leq 1$. Indeed, assume $j \geq 2$. This implies that there exist t_0 and t_1 such that $0 \leq t_0 < t_1 \leq t$ such that $(t_0, 0), (t_0, 1), (t_1, 1), (t_1, 2)$ are in $\text{dom}(\mathcal{Z})$. Note that $\tau(t_0, 1) = 0$ and $\tau(t_1, 1) = \lambda$. Moreover, for all s in $[t_0, t_1]$, $(s, 1)$ is in $\text{dom}(\mathcal{Z})$ and $\frac{d}{dt}\tau(s, 1) = 1 - \tau(s, 1)$. Hence, integrating this equation between t_0 and t_1 , we get that $t \geq t_1 - t_0 > -\ln(1 - \lambda) \geq T_{\min}$. This is impossible, and therefore $j \leq 1$.

So two cases may be distinguished.

$j = 0$ This case is illustrated by Figure 4.1. $j = 0$ implies that $s \in [0, t] \mapsto \mathcal{Z}(s, 0)$ is a continuous mapping with $\mathcal{Z}(0, 0)$ in $D_{l_1} \subset D_{l_2}$. Hence we can define t^* , the largest time in $[0, t]$ such that $\mathcal{Z}(s, 0)$ is in D_{l_2} (i.e. $t^* = \max_{s \in [0, t], x(\ell, 0) \in D_{l_2}, \forall \ell \in [0, s]} \{s\}$). Note that if $t^* = t$ then this implies that $z(t, 0)$ is in D_{l_2} , hence the result. Assume $t^* < t$. This implies that for all s in $[0, t^*]$ we have $\frac{d}{dt}V(s, 0) = \frac{\partial V}{\partial \mathcal{Z}}(\mathcal{Z}(s, 0))F(\mathcal{Z}(s, 0), \omega(s)) \leq \bar{V}$. This gives $V(t^*, 0) \leq \bar{V}t^* + V(0, 0) \leq \bar{V}T_{\min} + l_1 < l_2$. Hence $\mathcal{Z}(t^*, 0)$ is in the interior of D_{l_2} . It yields that there exists $\varepsilon > 0$ such that $\mathcal{Z}(t^* + \varepsilon, 0)$ is in the interior of D_{l_2} which contradicts the fact that t^* is an extremum.

$j = 1$ This case is illustrated by Figure 4.2. $j = 1$ implies that there exists t_0 in $[0, t]$ such that $(t_0, 0)$ and $(t_0, 1)$ are in $\text{dom}(\mathcal{Z})$ and $(w(t_0, 1), z_c(t_0, 1))$ is in \mathcal{J}_c . Following the first case study, it is possible to show that $\mathcal{Z}(t_0, 0)$ is in D_{l_2} . Moreover, we have $z_p(t_0, 1) = z_p(t_0, 0)$ and $z_c(t_0, 1) = g_c(w(t_0), z_c(t_0, 0))$. This implies that $\mathcal{Z}(t_0, 1) \in D_{l_2}^+$. Note that $[t_0, t] \mapsto \mathcal{Z}(s, 1)$ is a continuous mapping with $\mathcal{Z}(t_0, 1)$ in $D_{l_3} \subset D_{l_4}$. As in the previous case, we define t^* , the largest time in $[t_0, t]$ such that $\mathcal{Z}(s, 1)$ is in D_{l_4} (i.e.

⁷The definition of $\text{dom}(\mathcal{Z})$ is borrowed from [35, Definition 2.3].

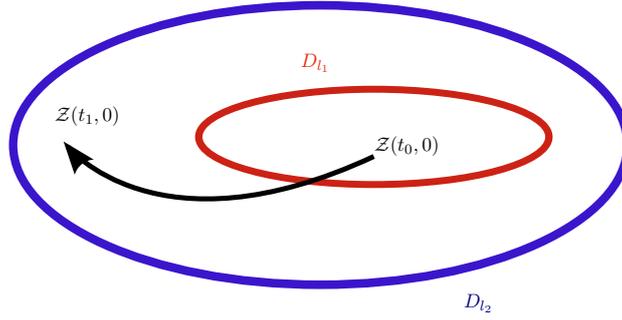


Figure 4.1: Case 1: $j = 0$

$t^* = \max_{s \in [t_0, t], \mathcal{Z}(\ell, 1) \in D_{l_4}, \forall \ell \in [t_0, s]} \{s\}$. Note that if $t^* = t$ then this implies that $\mathcal{Z}(t, 1)$ is in D_{l_4} , hence the result. Assume $t^* < t$. This implies that, for all s in $[t_0, t^*]$, it holds $\frac{d}{dt}V(s, 1) = \frac{\partial V}{\partial \mathcal{Z}}(\mathcal{Z}(s, 1))F(\mathcal{Z}(s, 1), \omega(s)) \leq \bar{V}$. This implies $V(t^*, 1) \leq \bar{V}t^* + V(t_0, 1) \leq \bar{V}T_{\min} + l_3 < l_4$. Hence $\mathcal{Z}(t^*, 1)$ is in the interior of D_{l_4} and following the previous case, we get a contradiction.

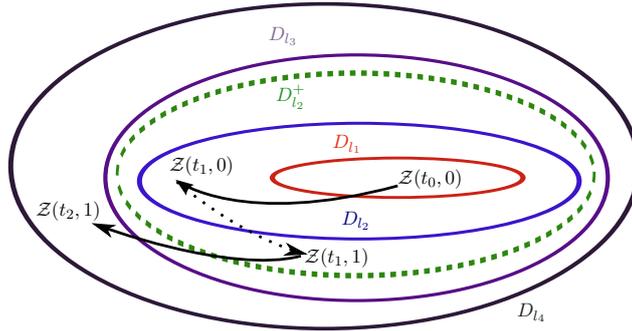


Figure 4.2: Case 1: $j = 1$. Continuous lines stand for flows. Dotted lines stand for jumps.

This concludes the proof of Lemma 4.1. □

4.2.2.4 Construction of Ψ_p

In this subsection we design a high-gain observer for the system (4.13). Since only continuous time dynamics are considered in this subsection, to simplify the notation, only the flow components of the time domain are considered. The function Ψ_p is selected to ensure that \hat{z}_p estimates the state z_p along solutions initiated in the projection of D_{l_1} on \mathbb{R}^{n_p} that is

$$\Pi_p(D_{l_1}) = \{z_p \in \mathbb{R}^{n_p}, \exists(z_c, \tau) \in \mathbb{R}^{n_c} \times \mathbb{R}_{\geq 0} \mid (z_p, z_c, \tau) \in D_{l_1}\}$$

and as long as these solutions remain in the projection of D_{l_4} on \mathbb{R}^{n_p} that is

$$\Pi_p(D_{l_4}) = \{z_p \in \mathbb{R}^{n_p}, \exists(z_c, \tau) \in \mathbb{R}^{n_c} \times \mathbb{R}_{\geq 0} \mid (z_p, z_c, \tau) \in D_{l_4}\}.$$

The estimation property has to be obtained for all input $u(\cdot)$ taking values in $\mathcal{U} \cap [-\bar{u}, \bar{u}]$ with

$$\bar{u} = \max_{|z_p| \leq c_x, (z_c, \tau) \in \Pi_c(D_{l_4})} \theta_c(z_p, z_c),$$

where

$$\Pi_c(D_{l_4}) = \{(z_c, \tau) \in \mathbb{R}^{n_c+1}, \exists z_p \in \mathbb{R}^{n_p} \mid (z_p, z_c, \tau) \in D_{l_4}\},$$

which is the projection of D_{l_4} on \mathbb{R}^{n_c} . Moreover, the observer has to be designed such that the estimation error is smaller than the stability margin of the controller after T_{\min} . More precisely, with Assumption 4.3, and letting $l = l_4$ yields the positive real number ε_r and the increasing mapping ρ . The observer has to be designed such that the set

$$\{(z_p, \hat{z}_p) \in \mathbb{R}^{2n_p} \mid |z_p - \hat{z}_p| < c_e\},$$

where

$$c_e := \min \left\{ \rho^{-1} \left(\frac{\alpha_1 l_4}{2} \right), \rho^{-1} \left(\frac{\alpha_2 l_4}{2} \right), \varepsilon_r \right\},$$

is reached after T_{\min} .

A possible approach to design such observer is to use high-gain observer methodology. With Assumption 4.2, from [32, Theorem 2], setting $w = \phi(z_p) \in \mathbb{R}^{n_p}$, the system (4.13) can be rewritten as follows:

$$\frac{d}{dt} w = A(w) + B(w)u \tag{4.36}$$

with w in \mathbb{R}^{n_p} and

$$\begin{aligned} A(w) &= \frac{\partial \phi(z_p)}{\partial z_p} f_p(z_p) = [w_2 \quad \dots \quad w_{n_p} \quad \chi(w)]^\top \\ B(w) &= \frac{\partial \phi(z_p)}{\partial z_p} g_p(z_p) \\ &= [b_1(w_1) \quad \dots \quad b_{n_p-1}(w_1, \dots, w_{n_p-1}) \quad b_{n_p}(w_1, \dots, w_{n_p})]^\top. \end{aligned}$$

where $\chi := L_f h(\phi^{-1}(w))$ and b_i are locally Lipschitz functions as soon as f_p is smooth enough. From there, the following lemma may be obtained.

Lemma 4.2 (Tunable observer)

Let $\Psi_p : \mathbb{R}^{n_p} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^{n_p}$ be defined by

$$\Psi_p(\hat{z}_p, y, u) = \left(\frac{\partial \phi}{\partial z_p}(\hat{z}_p, u) \right)^{-1} \left[\tilde{A}(\phi(\hat{z}_p)) + \tilde{B}(\phi(\hat{z}_p))u + \mathcal{L}K[y - h(\hat{z}_p)] \right] \tag{4.37}$$

where $\tilde{A} : \mathbb{R}^{n_p} \rightarrow \mathbb{R}^{n_p}$ and $\tilde{B} : \mathbb{R}^{n_p} \rightarrow \mathbb{R}^{n_p}$ are functions

$$\tilde{A}(w) = [w_2 \quad \dots \quad w_{n_p} \quad \tilde{\chi}(w)]^\top$$

$$\tilde{B}(w) = [\tilde{b}_1(w_1) \dots \tilde{b}_{n_p-1}(w_1, \dots, w_{n_p-1}) \tilde{b}_{n_p}(w_1, \dots, w_{n_p})]^\top$$

and

$$\mathcal{L} = \text{diag}(\ell, \dots, \ell^{n_p}) \quad (4.38)$$

where ℓ is a positive real number larger than 1 and $K = [-k_1 \dots -k_{n_p}]^\top$ is such that the matrix

$$\mathcal{G} := \begin{bmatrix} -k_1 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ -k_{n_p} & 0 & \dots & \dots & 0 \end{bmatrix}, \quad (4.39)$$

is Hurwitz. Note that the functions ⁸ $\tilde{b}_i : \mathbb{R}^i \rightarrow \mathbb{R}$, $i = 1, \dots, n_p$ and $\tilde{\chi} : \mathbb{R}^{n_p} \rightarrow \mathbb{R}$ are such that

1. For all w in $\phi(D_{l_4})$, $\tilde{\chi}(w) = \chi(w)$ and $\tilde{b}_i(w) = b_i(w)$ where χ and (b_i) 's are defined in (4.36);
2. $\tilde{\chi}$ and \tilde{b}_i are globally Lipschitz functions.

Moreover there exists a class \mathcal{KL} function β such that for all solutions z_p to (4.13) and all solutions \hat{z}_p to

$$\frac{d}{dt} \hat{z}_p = \Psi_p(\hat{z}_p, y, u) \quad (4.40)$$

with initial condition $(z_p^\#, \hat{z}_p^\#) \in \Pi_p(D_{l_4}) \times \Pi_p(D_{l_1})$, input $u(\cdot)$ taking value in $\mathcal{U} \cap [-\bar{u}, \bar{u}]$, and for all t such that $z_p(s) \in \Pi_p(D_{l_4})$, where $s \in [0, t]$, the following holds:

1. $|z_p(s) - \hat{z}_p(s)| \leq \beta(|z_p^\# - \hat{z}_p^\#|, s)$, $\forall s \in [0, t]$;
2. If $t \geq T_{\min}$, $|z_p(s) - \hat{z}_p(s)| \leq c_e$, $\forall s \in [T_{\min}, t]$.

Proof of Lemma 4.2:

There exists c_χ such that, for all $(z^1, z^2) \in (\mathbb{R}^{n_p})^2$, $|\chi(z^1) - \chi(z^2)| \leq c_\chi |z^1 - z^2|$. Note also that from the definition of ϕ we have $y = h_p(\phi(z_p)) = z_1$. With the Lipschitz property, the \tilde{b}_i terms satisfy, for all $(z^1, z^2) \in \Pi_p(D_{l_4})^2$, $|\tilde{b}_i(z^1) - \tilde{b}_i(z^2)| \leq c_{b_i} \sum_{j=1}^i |z_j^1 - z_j^2|$. Moreover, setting $\hat{w} := \phi(\hat{z}_p)$, it yields along the solutions to (4.40):

$$\frac{d}{dt} \hat{w} = \tilde{A}(\hat{w}) + \tilde{B}(\hat{w})u + \mathcal{L}K[y - \hat{w}_1], \quad (4.41)$$

As standard in high-gain observer design, let \tilde{e} be the scaled error components:

$$\tilde{e}_i = \frac{\hat{w}_i - (\phi(z_p))_i}{\ell^{i-1}}. \quad (4.42)$$

⁸These functions exist from the Kirszbraun extension theorem, see e.g. [29, Definition 2.10.43] and from the fact that $\phi(D_{l_4})$ is a compact subset of \mathbb{R}^{n_p} .

Therefore, as soon as $z_p \in \Pi_p(D_{l_4})$, it holds

$$\frac{d}{dt}\tilde{e} = \ell\mathcal{G}\tilde{e} + \Delta(\hat{w}, z_p, u), \quad (4.43)$$

where $\Delta(\hat{w}, z_p, u)$ is obtained from $\tilde{\chi}$ and \tilde{b}_i for all $i = 1, \dots, n_p$. From the structure of the observation error, there exists a positive real number c_Δ such that for all (\hat{z}, z_p, u) in $\mathbb{R}^{n_d} \times \Pi_p(D_{l_4}) \times \mathcal{U} \cap [-\bar{u}, \bar{u}]$ and for all $\ell > 1$ $|\Delta(\hat{w}, z_p, u)| \leq c_\Delta|\tilde{e}|$. The matrix \mathcal{G} being Hurwitz, there exist a positive definite matrix P and a positive value d satisfying

$$P\mathcal{G} + \mathcal{G}^\top P \leq -dP. \quad (4.44)$$

Let $U : \mathbb{R}^{n_p} \rightarrow \mathbb{R}_{\geq 0}$ be the following (Lyapunov) function $U(\tilde{e}) = \tilde{e}^\top P\tilde{e}$. Its time derivative along the solutions to the system (4.43) with (\hat{z}, z_p, u) in $\mathbb{R}^{n_d} \times \Pi_p(D_{l_4}) \times \mathcal{U} \cap [-\bar{u}, \bar{u}]$ and $\ell > 1$ satisfies the following inequalities.

$$\begin{aligned} \frac{d}{dt}U(\tilde{e}) &\leq -\ell dU(\tilde{e}) + 2\tilde{e}^\top P\Delta(\hat{w}, z_p, u) \\ &\leq -\ell dU(\tilde{e}) + 2c_\Delta|P\tilde{e}||\tilde{e}| \\ &\leq -\ell dU(\tilde{e}) + 2c_\Delta \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}U(\tilde{e}) \end{aligned} \quad (4.45)$$

Consider a solution z_p to (4.13) and a solution \hat{z}_p to (4.40) with initial condition $(z_p^\#, \hat{z}_p^\#) \in \Pi_p(D_{l_1})^2$, input $u(\cdot)$ taking value in $\mathcal{U} \cap [-\bar{u}, \bar{u}]$, and t such that $z_p(s)$ is in $\Pi_p(D_{l_4})$ for all s in $[0, t]$. With Grönwall lemma, we get along this solution

$$U(\tilde{e}(t)) \leq \exp\left(-\left(\ell d - 2c_\Delta \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right)t\right)U(\tilde{e}(0)) \quad (4.46)$$

Moreover, for all (z_p, \hat{w}) , it holds that

$$|\phi(z_p) - \hat{w}|^2 \frac{\lambda_{\min}(P)}{\ell^{2n_p}} \leq U(\tilde{e}) \leq \frac{\lambda_{\max}(P)}{\ell^2} |\phi(z_p) - \hat{w}|^2,$$

hence, for all $t \geq 0$

$$|\phi(z_p(t)) - \hat{w}(t)| \leq \exp\left(-\left(\frac{\ell}{2}d - c_\Delta \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right)t\right) \ell^{n_p-1} \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} |\phi(z_p^\#) - \phi(\hat{z}_p^\#)|. \quad (4.47)$$

Note that $\Pi_p(D_{l_4})$ and $\phi(\Pi_p(D_{l_4}))$ are compact subsets of \mathbb{R}^{n_p} and that the functions $\phi : \mathbb{R}^{n_p} \rightarrow \mathbb{R}^{n_p}$ and $\phi^{-1} : \mathbb{R}^{n_p} \rightarrow \mathbb{R}^{n_p}$ are continuous. Hence, the two class \mathcal{K} functions ν and ν^* can be defined as

$$\begin{aligned} \nu(s) &= \max_{z_p \in \Pi_p(D_{l_4}), |z_p^* - z_p| \leq s} |\phi(z_p) - \phi(z_p^*)| \\ \nu^*(s) &= \max_{z \in \phi(\Pi_p(D_{l_4})), |w^* - w| \leq s} |\phi^{-1}(w) - \phi^{-1}(w^*)|. \end{aligned}$$

These functions satisfy that, for all w in $\phi(\Pi_p(D_{l_4}))$ and \hat{w} in \mathbb{R}^{n_p} ,

$$|\phi^{-1}(w) - \phi^{-1}(\hat{w})| \leq \nu^*(|w - \hat{w}|), \quad (4.48)$$

$$|w - \hat{w}| \leq \nu(|\phi^{-1}(w) - \phi^{-1}(\hat{w})|). \quad (4.49)$$

Let now $\ell > 1$ be any positive real number such that

$$\Omega(\ell) := \frac{\ell}{2}d - c_\Delta \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} > 0$$

$$\nu \left(\exp(-\Omega(\ell)T_{\min}) \ell^{n_p-1} \mathfrak{M} \right) \leq c_\epsilon, \quad (4.50)$$

where $\mathfrak{M} := 2 \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \max_{z \in \phi(\Pi_p(D_{l_1}))} |z|$. We can rewrite (4.50) as follows:

$$\exp\left(-\ell \frac{T_{\min}d}{2}\right) \exp\left(T_{\min}c_\Delta \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right) \ell^{n_p-1} \leq \frac{\nu^{-1}(c_\epsilon)}{\mathfrak{M}}$$

Since the exponential function dominates the polynomial function, the condition (4.50) is satisfied, for ℓ sufficiently large. From (4.47), (4.48) and (4.49) the first item of Lemma 4.2 holds with the function

$$\beta(s, t) = \nu^* \left(\exp(-\Omega(\ell)t) \ell^{n_p-1} \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \nu(s) \right).$$

The second item of Lemma 4.2 is deduced from (4.47), (4.48) and (4.50) and the fact that $(z_p^\#, \hat{z}_p^\#) \in \Pi_p(D_{l_1}) \times \Pi_p(D_{l_1})$. \square

4.2.2.5 Proof of Theorem 4.3

Consider the positive value c_x obtained in (4.35) and the function Ψ_p obtained in (4.37). In the first part of the proof, we show attractivity of the set $\{0\} \times [0, 1]$ in $\mathbb{R}^{2n_p+n_c} \times [0, 1]$ along the solutions to system (4.28) whose initial condition is in $\tilde{\Gamma} := \Gamma \times \{0\} \times \mathbb{R}_{\geq 0} \subset \mathcal{B}^p \times \{0\} \times \mathbb{R}_{\geq 0}$. The stability is shown in a second part.

Note that the hybrid system (4.28) satisfies the basic assumptions for hybrid systems [34, Assumption 6.5.] with the flow set $\mathcal{F}_c^o = \{(z_p, \hat{z}_p, z_c, \tau) \in \mathbb{R}^{2n_p+n_c} \times \mathbb{R}_{\geq 0} : \mathfrak{F}(z_p, \hat{z}_p, z_c, \tau) \in \mathbb{R}^{n_p} \times \mathcal{F}_c \times \mathbb{R}_{\geq 0}\}$ and the jump set $\mathcal{J}_c^o = \{(z_p, \hat{z}_p, z_c, \tau) \in \mathbb{R}^{2n_p+n_c} \times \mathbb{R}_{\geq 0} : \mathfrak{F}(z_p, \hat{z}_p, z_c, \tau) \in \mathbb{R}^{n_p} \times \mathcal{J}_c \times \mathbb{R}_{\geq \lambda}\}$, where $\mathfrak{F} : (z_p, \hat{z}_p, z_c, \tau) \mapsto (z_p, \text{sat}_{c_x}(\hat{z}_p), z_c, \tau)$.

First Part : Attractivity. Let $\mathcal{Z}^\# := (z_p^\#, z_c^\#, \hat{z}_p^\#, \tau^\#)$ be in $\tilde{\Gamma}$ and consider a solution $\mathcal{Z} = (z_p, z_c, \hat{z}_p, \tau)$ to (4.28) whose initial condition is $\mathcal{Z}^\#$ and defined on its time domain denoted $\text{dom}(\mathcal{Z})$.

Note that the system (4.28) can be rewritten as the hybrid system (4.34) with $\omega = \text{sat}_{c_x}(\hat{z}_p)$ and \hat{z}_p is given with the observer (4.37).

With Lemma 4.1 and with the τ dynamics (persistent flow), we know that there exists j_0 such that (T_{\min}, j_0) is in $\text{dom}(\mathcal{Z})$ and for all (t, j) in $\text{dom}(\mathcal{Z})$ with $t \leq T_{\min}$ then $(z_p(t, j), z_c(t, j), \tau(t, j))$ is in D_{l_4} . Thus, for all (t, j) in $\text{dom}(\mathcal{Z})$ with $t \leq T_{\min}$, the control input satisfies $|u(t, j)| \leq \bar{u}$. Let $\text{dom}_{D_{l_4}}(\mathcal{Z})$ be the time domain of the solution restricted to

the set D_{l_4} . With Lemma 4.2, for all (t, j) in $\text{dom}_{D_{l_4}}(\mathcal{Z})$ with $t \geq T_{\min}$, $|z_p(t, j) - \hat{z}_p(t, j)| \leq c_e$. Moreover, for all (t, j) in $\text{dom}_{D_{l_4}}(\mathcal{Z})$ with $t \geq T_{\min}$, we have $|z_p(t, j)| \leq c_x$. It implies

$$\begin{aligned} |z_p(t, j) - \tilde{z}_p(t, j)| &= |\text{sat}_{c_x}(z_p(t, j)) - \text{sat}_{c_x}(\hat{z}_p(t, j))| \\ &\leq |z_p(t, j) - \hat{z}_p(t, j)| \leq c_e . \end{aligned}$$

It is now possible to show that for all (t, j) in $\text{dom}(\mathcal{Z})$, $\mathcal{Z}(t, j)$ is in D_{l_4} . We will argue by contradiction to prove this assertion. By assuming that it is false, two cases may occur.

- The solution escapes D_{l_4} when flowing. Hence, there exists (t_0, j_0) in $\text{dom}(\mathcal{Z})$ such that $(z_p, z_c, \tau)(t_0, j_0)$ is in D_{l_4} and for all $\varepsilon > 0$, there exists $\delta < \varepsilon$ such that $(t_0 + \delta, j_0)$ is in $\text{dom}(\mathcal{Z})$ and $(z_p, z_c, \tau)(t_0 + \delta, j_0)$ is not in D_{l_4} . Note that this implies that $(z_p, z_c)(t_0, j_0)$ is at the boundary of D_{l_4} . Consequently, this implies $V(t_0, j_0) = l_4$. Note moreover, keeping in mind that $|z_p(t_0, j_0) - \tilde{z}_p(t_0, j_0)| \leq |z_p(t_0, j_0) - \hat{z}_p(t_0, j_0)| \leq c_e \leq \varepsilon_r$ we get, from Assumption 4.3, $\frac{d}{dt}V(t_0, j_0) \leq -\alpha_1 V(t_0, j_0) + \rho(c_e) \leq -\frac{\alpha_1 l_4}{2}$. This implies that the function $s \mapsto V(t_0 + s, j_0)$ is strictly decreasing. It contradicts the existence of small ε .

- The solution escapes D_{l_4} when jumping. Hence, there exists (t_0, j_0) in $\text{dom}(\mathcal{Z})$ such that $(z_p, z_c, \tau)(t_0, j_0)$ is in D_{l_4} and $(z_p, z_c, \tau)(t_0, j_0 + 1)$ is not in D_{l_4} . Since $|z_p(t_0, j_0) - \hat{z}_p(t_0, j_0)| \leq c_e \leq \varepsilon_r$, with Assumption 4.3, it follows $V(t_0, j_0 + 1) \leq (1 - \alpha_2)V(t_0, j_0) + \rho(c_e) \leq (1 - \frac{\alpha_2}{2})l_4 < l_4$. This is a contradiction with the escape of the solution from D_{l_4} .

Consequently, for all (t, j) in $\text{dom}(\mathcal{Z})$, we have $\mathcal{Z}(t, j)$ is in D_{l_4} . Note that the timer forces the t direction of the time domain to be unbounded. Hence, thanks to the Lemma 4.2, $\lim_{t+j \rightarrow +\infty} |\hat{z}_p(t, j) - z_p(t, j)| = 0$. We get the result employing the triangular structure of the system with the ISS property in D_{l_4} (i.e. Assumption 4.3) (see e.g. [9, Theorem 3.1.]).

Second Part: Stability. To conclude the proof, let us prove the stability property. Let S_ε be defined by $S_\varepsilon = \{(z_p, z_c, \hat{z}_p, \tau) : V(z_p, z_c, \tau) \leq \varepsilon \text{ and } |z_p - \hat{z}_p| \leq \varepsilon\}$, where $\varepsilon < l_1$. Moreover, let \mathcal{N}_l be an open neighborhood of \mathcal{A} defined as

$$\mathcal{N}_l = \left\{ (z_p, z_c, \hat{z}_p, \tau) \mid V(z_p, z_c, \tau) < \frac{\varepsilon}{4}, \beta(|z_p - \hat{z}_p|, 0) < \min \left\{ \rho^{-1}\left(\frac{\alpha_1 \varepsilon}{4}\right), \rho^{-1}\left(\frac{\alpha_2 \varepsilon}{4}\right), \frac{\varepsilon}{2} \right\} \right\},$$

where (β, ρ) and (α_1, α_2) are the two functions and the two positive real numbers given in Lemma 4.2 and Assumption 4.3. Note that $\mathcal{N}_l \subset S_\varepsilon$.

Now consider \mathcal{Z} a solution to the closed-loop system (4.28) starting from any point of \mathcal{N}_l with time domain denoted $\text{dom}(\mathcal{Z})$. Let $\text{dom}_{S_\varepsilon}(\mathcal{Z})$ denote the hybrid time domain of the solution restricted to the set S_ε .

Let us show that this solution \mathcal{Z} remains in S_ε . Note that $\hat{z}_p(0, 0)$ and $z_p(0, 0)$ are in $\Pi_p(D_{l_1})$. With Lemma 4.2, it yields that for all (t, j) in $\text{dom}_{S_\varepsilon}(\mathcal{Z})$,

$$\begin{aligned} |z_p(t, j) - \hat{z}_p(t, j)| &\leq \beta(|z_p(0, 0) - \hat{z}_p(0, 0)|, 0) \\ &\leq \min \left\{ \rho^{-1}\left(\frac{\alpha_1 \varepsilon}{4}\right), \rho^{-1}\left(\frac{\alpha_2 \varepsilon}{4}\right), \frac{\varepsilon}{2} \right\} . \end{aligned}$$

Moreover, $(z_p(t, j), z_c(t, j), \tau(t, j))$ is in $D_{l_1} \subset D_{l_4}$, hence from the ISS inequalities that hold in D_{l_4} (see Assumption 4.3), we get for all $t_2 > t_1$ and j such that (t_1, j) and (t_2, j) are in

$\text{dom}_{S_\varepsilon}(\mathcal{Z})$

$$\frac{d}{dt}V(z_p(t, j), z_c(t, j), \tau(t, j)) \leq -\alpha_1 V(z_p(t, j), z_c(t, j), \tau(t, j)) + \frac{\alpha_1 \varepsilon}{4}$$

which implies with $V(\mathcal{Z}(t_1, j)) \leq \frac{\varepsilon}{4}$

$$V(z_p(t, j), z_c(t, j), \tau(t, j)) \leq \frac{\varepsilon}{4}, \quad t \in [t_1, t_2]$$

Hence, it can not escape S_ε by flowing since it does not reach the boundary of S_ε . Moreover, for all (t, j) in $\text{dom}_{S_\varepsilon}(\mathcal{Z})$ such that $(t, j+1)$ is in $\text{dom}(\mathcal{Z})$, from the ISS inequalities of Assumption 4.3 once again, it yields, if $V(\mathcal{Z}(t, j)) \leq \frac{\varepsilon}{4}$,

$$V(z_p(t, j+1), z_c(t, j+1), \tau(t, j+1)) \leq (1 - \alpha_2)V(z_p(t, j), z_c(t, j), \tau(t, j)) + \frac{\alpha_2 \varepsilon}{4} < \frac{\varepsilon}{4}$$

Hence, $\mathcal{Z}(t, j+1) \in S_\varepsilon$.

Thus, if the initial condition $\mathcal{Z}^\#$ is in \mathcal{N}_l , then the solution remains in S_ε . Since S_ε can be made as closed as wanted from \mathcal{A} , this proves stability of the set \mathcal{A} and concludes the proof of Theorem 4.2. \square

4.2.3 Example

To illustrate Theorem 4.2, the following second order system is considered:

$$\frac{d}{dt}z_p = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} z_p + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \text{sat}_{u_s}(v), \quad y = [1 \quad 0] z_p. \quad (4.51)$$

With $u = \text{sat}_{u_s}(v)$, $\mathcal{U} = [-u_s, u_s]$ and $u_s = 3$, it is a system of the form (4.13). Note that the set of control \mathcal{U} is bounded and that $y = x_{p_1}$. Consequently, despite the fact that the dynamics are linear, the stabilization problem for this system is not an easy task although recent works [100, 41] propose design methods for saturated actuator. The strategy presented in this chapter can be also efficient for purely nonlinear systems as it is shown in [65] where it is proven the existence of a hybrid output feedback for a chain of integrators with a nonlinearity from a state feedback uniting a global and a local controllers. Following [57], we compute the hybrid controller such that $\mathcal{A} = \{0\} \times \{0\} \times \{0\} \times [0, 1]$ is locally asymptotically stable for the FORE system

$$\begin{cases} \frac{d}{dt}z_p = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} z_p + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \text{sat}_{u_s}(kz_c) \\ \frac{d}{dt}z_c = \begin{bmatrix} -2 & 0 \\ 0 & -12 \end{bmatrix} z_c - z_p \\ \frac{d}{dt}\tau = 1 - \tau \end{cases} \quad (4.52a)$$

$$(z_p^\top M z_c \geq 0) \text{ and } \tau \in \mathbb{R}_{\geq 0}$$

$$\begin{cases} z_p^+ = z_p \\ z_c^+ = 0 \\ \tau^+ = 0 \end{cases} \quad (z_p^\top M z_c \leq 0) \text{ and } \tau \in \mathbb{R}_{\geq \lambda} \quad (4.52b)$$

with $k = [-13 \ -30]$, $\lambda = 0.01$, $M = \begin{bmatrix} 0 & -I_1 \\ -I_1 & 0 \end{bmatrix}$ and

$$\mathcal{E}(P_V) = \left\{ (z_p, z_c) \in \mathbb{R}^4, [z_p; z_c]P_V[z_p; z_c]^\top \leq 1 \right\} \subset \mathcal{B},$$

where P_V is the positive definite matrix defined as $P_V = \begin{bmatrix} 4.0560 & 0.6868 & 6.6342 & 0.0000 \\ * & 6.4905 & 0.0000 & 6.6342 \\ * & * & 49.4184 & 6.6162 \\ * & * & * & 55.9387 \end{bmatrix}$. The matrix P_V has been computed to satisfy a set of linear matrix inequalities (LMIs) given in [57, Theorem 3.] employing the LMI solver [87]. Note that employing an equivalent continuous feedback law $u_c = \text{sat}_{u_s}(kz_c)$, without any reset, leads to a smaller estimation of the basin of attraction (i.e. the matrix P_V^c that describes the lower approximation of the basin of attraction has bigger eigenvalues than P_V). The set \mathcal{B} denotes the basin of attraction and $\mathcal{E}(P_V)$ is a lower approximation of this one. Moreover, letting $\alpha_1 = 0.4017$, $\alpha_2 = 0.6078$, $\varepsilon_r = 0.0550$ and the function $\rho(|e|) := 13.3418|e|$, Assumption 4.3 is satisfied.

Since (4.51) satisfies the well-known Kalman observability rank condition for the pair (C, A) , Assumption 4.2 holds for the system (4.51). Moreover, denoting the projection $\mathcal{E}(P_V)$ on \mathbb{R}^{n_p} by $\Pi_p(\mathcal{E}(P_V)) = \{z_p \in \mathbb{R}^2 : 4.0560z_{p_1}^2 + 6.4905z_{p_2}^2 + 1.3736z_{p_1}z_{p_2} \leq 1\}$ and letting $\Gamma = \{z_p \in \mathbb{R}^2 : 4.0560z_{p_1}^2 + 6.4905z_{p_2}^2 + 1.3736z_{p_1}z_{p_2} \leq 0.6\} \subset \Pi_p(\mathcal{E}(P_V))$, hence, it is obtained from Theorem 4.2 that the set $\{0\} \times \{0\} \times [0, 1]$ is asymptotically stable for the system (4.52) with basin of attraction containing $\Gamma \times \{0\} \times \{0\} \times \mathbb{R}_{\geq 0}$.

Following Lemmas 4.1 and 4.2, we aim at computing all the variables. Here, from the definition of Γ , we let $l_1 = 0.6$ and $l_2 = 0.7$. Due to (4.52), $g_c = 0$ and thus $D_{l_2}^+ \subset D_{l_2}$. It follows that $l_3 = 0.8$ and $l_4 = 0.9$ are suitable values. Thus, we can compute $c_x := \sqrt{P^{-1/2}QP^{-1/2}l_4} = 0.6624$ where $Q = \text{diag}(1, 1, 0, 0)$. We check that $k_1 = -6$ and $k_2 = -9$ are such that \mathcal{G} defined in Lemma 4.2 is Hurwitz. With all these parameters, we numerically compute $c_e = 0.0136$, $T_{\min} = 0.01$, $d = 2$, $P = \begin{bmatrix} 0.1000 & 0.0077 \\ 0.0077 & 0.0635 \end{bmatrix}$ and $\mathfrak{M} = 1.2070$. Then we check that $\ell = 1000$ satisfies (4.50). Finally we can build the observer satisfying the structure of (4.37). Therefore we can apply Theorem 4.3 and state that closing the loop of (4.51) with the observer of the form (4.40) and the output feedback law coming from (4.52) leads to a semi-global stability.

With $(z_p^\#, z_c^\#, \hat{z}_p^\#, \tau^\#) = (0.15, 0, 0, 0, 0, 0, 0) \in \mathcal{E}(P) \times [0, 1]$, The time evolution of z_{p_1} and \hat{z}_{p_1} satisfying respectively (4.51) and (4.40) are given by Figure 4.3. The first figure illustrates the convergence of the observer, the second the asymptotic stability of the closed-loop system. Figure 4.4 illustrates the phase portrait of z_p .

4.3 Conclusion

An output feedback law has been designed for SISO affine systems for which there exists a hybrid state feedback under an observability assumption. Some special assumptions (for instance the existence of a timer) have been useful to deal with the hybrid case. The result has been applied to design a hybrid output feedback law from a FORE controller.

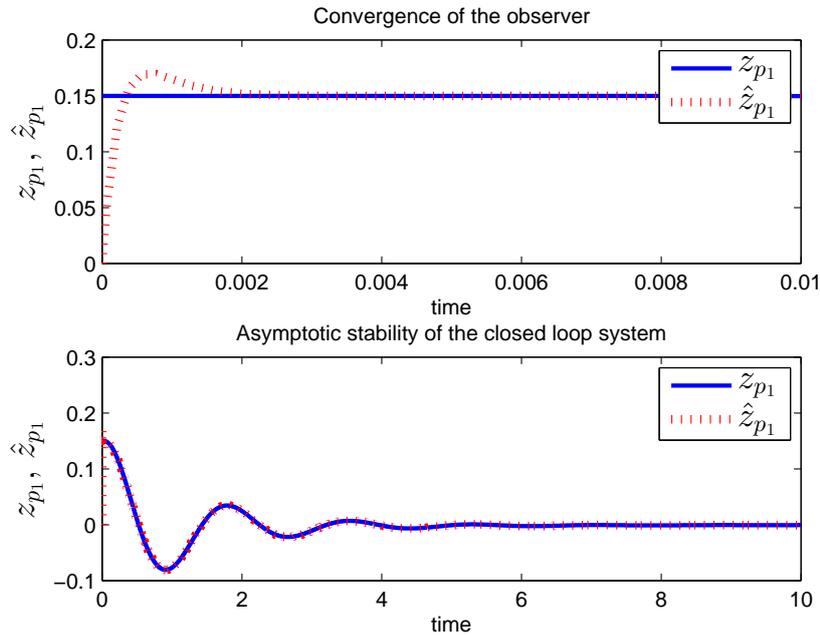


Figure 4.3: Time evolution of z_{p1} (blue plain line) and \hat{z}_{p1} (red dashed line).

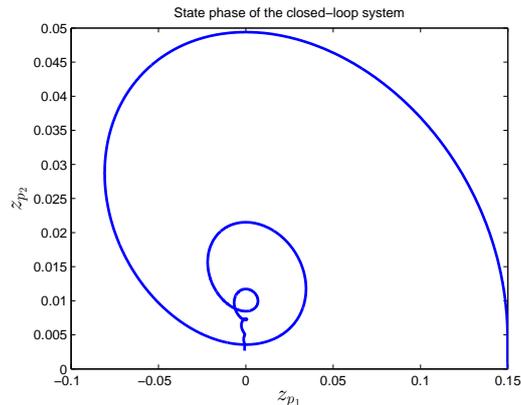


Figure 4.4: Phase portrait of the state z_p .

Some open questions follow. Does an output feedback exist even with a less strong assumption than the Assumption 4.1, such as an assumption which does not state any knowledge of the time between two consecutive jumps? Another technical assumption is given by (4.27). Is it possible to weaken this assumption and obtain a similar result as the one given by Theorem 4.2? In [3], some output feedback laws are presented to achieve global stabilization of equilibrium point. Assuming stronger observability property, can such output feedback laws be achieved for hybrid case? Moreover, designing an observer that converges in finite time with a stability property, and thus adapting [5] for the hybrid systems case, could be also an interesting research line.

Conclusion and perspectives

In this thesis, some stabilization problems for nonlinear dynamical systems have been introduced. It contributes to the general theory of control nonlinear systems and stabilization of equilibrium points by means of output feedback laws. Both finite and infinite-dimensional systems have been studied. To be more specific, we tackled Questions 1, 2 and 3 collected in the Introduction using various techniques from finite and infinite-dimensional theory of stability and stabilization.

Answer to Question 1

In Chapter 1, nonlinear abstract control systems with a bounded linear control operators have been studied. Assuming that

- the nonlinear operator A is m -dissipative,
- there exists a stabilizing linear feedback law,
- the positive orbit of the closed-loop system is precompact,

we proved that when modifying this feedback via a cone-bounded nonlinearity the origin of the closed-loop system remains globally asymptotically stable. The Schauder fixed-point theorem has been used to conclude on the well-posedness and an infinite-dimensional version of LaSalle's invariance principle has been applied to tackle the asymptotic stability of the origin of the closed-loop system.

In Chapter 2, a particular nonlinear partial differential equation, namely the Korteweg-de Vries equation, has been studied. We proved that when modifying a distributed stabilizing feedback law with two types of saturations the origin of the closed-loop system remains globally asymptotically stable. A numerical scheme has been provided in order to illustrate our results and some existing results for the Korteweg-de Vries equation.

Answer to Question 2

Chapter 3 introduces the separation principle strategy for linear finite-dimensional systems. We followed this strategy for two Korteweg-de Vries equations for which there exist state feedback laws designed with a backstepping method. Thanks to the backstepping method and to a Lyapunov analysis, we proved that designing observers separately from state feedback laws yields stabilizing output feedback laws. Some simulations have been performed to illustrate our theoretical results.

Answer to Question 3

Finally, in Chapter 4, the semi-global output feedback stabilization of nonlinear finite-dimensional systems for which there exists a local regularizing hybrid controller has been tackled. Indeed, if the system is completely uniformly observable and if there exists a regularizing hybrid state feedback, then we proved that the resulting output feedback (locally) stabilizes the closed-loop system. A high-gain observer strategy has been followed in order to design the observer. The hybrid system theory has been used to conclude on the asymptotic stability of the closed-loop system. An illustrative example has been provided.

Contributions of the thesis

The main contributions of this thesis are

- the study of abstract nonlinear control systems stabilized with a cone-bounded controller;
- the study of the global asymptotic stability of the Korteweg-de Vries equation with a saturating distributed feedback law with two different types of saturation;
- the numerical scheme of the nonlinear Korteweg-de Vries equation with a saturating distributed feedback law;
- the output feedback stabilization problem of two Korteweg-de Vries equations using the backstepping method;
- the output feedback stabilization problem for nonlinear finite-dimensional systems for which there exists a hybrid controller.

Tools used along the thesis

- Nonlinear semigroup theory for abstract control systems;
- Nonlinear partial differential equation theory;
- Fixed-point theorems;
- Stability analysis with Lyapunov functions for both finite and infinite-dimensional systems;
- Saturated control systems theory;
- Backstepping method for partial differential equations;
- Hybrid systems theory;
- High-gain observer theory for nonlinear finite-dimensional systems.

Perspectives

The main remaining challenges are

- extending our results for abstract nonlinear control systems with unbounded control operators. At the best of our knowledge, numerous problems remain open;
- studying other nonlinearities than the cone-bounded ones for a abstract control system or a particular partial differential equation (the Korteweg-de Vries equation, for example). The backlash nonlinearity, studied in [93], could be interesting;
- studying the influence of the saturation on the stabilizing backstepping controller designed for the Korteweg-de Vries equation. Some tools from saturation control systems could be used;
- studying the influence of the saturation on a boundary stabilizing feedback law for hyperbolic systems (see e.g. [6]). It could be also interesting to follow in this case the anti-windup strategy [101];
- extending the high-gain observer theory for the Korteweg-de Vries equation. In order to be close to the finite-dimensional case, the nonlinearity should be studied at the boundary;
- developing new observer strategies for infinite-dimensional systems. The Kazantzis-Kravaris/Luenberger one, introduced in [4], could be interesting. In particular, it could be used to solve some parametric estimation problems, as is has been done in the linear finite-dimensional case in [1];
- studying the output feedback stabilization for hybrid nonlinear finite-dimensional systems. Indeed, instead of studying a continuous plant, it could be interesting to study a hybrid plant;
- solving the output feedback stabilization for the Brockett's integrator for which there exists a hybrid state feedback law. The design of this state feedback law is provided in [36].

Precompactness of the operator given by the KdV equation with a cone-bounded feedback law

This section is devoted to the proof of the precompactness of the canonical embedding from $D(A_\sigma) = D(A)$, defined in (1.47), into $Z := L^2(0, L)$. Let us state the lemma and prove it.

Lemma A.1

The canonical embedding from $D(A_\sigma)$, equipped with the graph norm, into $Z := L^2(0, L)$ is compact.

Proof of Lemma A.1: We follow the strategy of [64], [76] and [69]. Let us recall the definition of the graph norm

$$\begin{aligned} \|z\|_{D(A_\sigma)}^2 &:= \|z\|_{L^2(0,L)}^2 + \|A_\sigma z\|_{L^2(0,L)}^2 \\ &= \int_0^L (|z(x)|^2 + |-z''''(x) - z'(z) - \sigma(\mathbf{1}_\Omega z)(z)|^2) dx \\ &= \int_0^L (|z(x)|^2 + |z''''(x) + z'(z) + \sigma(\mathbf{1}_\Omega z)(z)|^2) dx. \end{aligned} \quad (\text{A.1})$$

Note that

$$\|\sigma(\mathbf{1}_\Omega z)\|_{L^2(0,L)} \leq 2\|z\|_{L^2(0,L)}. \quad (\text{A.2})$$

From the definition of the graph norm, we get the following two inequalities

$$\|z\|_{D(A_\sigma)}^2 \geq \|z\|_{L^2(0,L)}^2 \quad (\text{A.3})$$

and, since, for all $(s, \tilde{s}) \in \mathbb{C}^2$, it holds $|s + \tilde{s}|^2 \leq 2|s|^2 + 2|\tilde{s}|^2$, we have

$$\begin{aligned} \|z\|_{D(A_\sigma)}^2 &\geq \frac{1}{2} \int_0^L |-\sigma(\mathbf{1}_\Omega z)(z)|^2 dx \\ &\quad + \frac{1}{2} \int_0^L |z''''(x) + z'(z) + \sigma(\mathbf{1}_\Omega z)(z)|^2 dx \\ &\geq \frac{1}{4} \int_0^L |z''''(x) + z'(z)|^2 dx. \end{aligned} \quad (\text{A.4})$$

Noticing that $\|z'''\|_{L^2(0,L)}^2 = \|z''' + z' - z'\|_{L^2(0,L)}^2$, we have

$$\|z'''\|_{L^2(0,L)}^2 \leq 2\|z''' + z'\|_{L^2(0,L)}^2 + 2\|z'\|_{L^2(0,L)}^2, \quad (\text{A.5})$$

and using that $\|z'\|_{L^2(0,L)}^2 = \|z' + z''' - z''' + zx - xz\|_{L^2(0,L)}^2$, we obtain

$$\begin{aligned} \|z'\|_{L^2(0,L)}^2 &\leq 2\|z' + z'''\|_{L^2(0,L)}^2 + 2\|z''' - zx + xz\|_{L^2(0,L)}^2 \\ &\leq 2\|z' + z'''\|_{L^2(0,L)}^2 + 4\|z''' - xz\|_{L^2(0,L)}^2 + 4\|xz\|_{L^2(0,L)}^2 \\ &\leq 2\|z' + z'''\|_{L^2(0,L)}^2 + 4\|z'''\|_{L^2(0,L)}^2 - 8 \int_0^L z z''''(x)z(x)dx + 8\|xz\|_{L^2(0,L)}^2. \end{aligned}$$

Deriving some integrations by parts, we get

$$\int_0^L z z''''(x)z(x)dx = \frac{3}{2}\|z'\|_{L^2(0,L)}^2,$$

and therefore

$$\|z'\|_{L^2(0,L)}^2 \leq 2\|z' + z'''\|_{L^2(0,L)}^2 + 4\|z'''\|_{L^2(0,L)}^2 - 12\|z'\|_{L^2(0,L)}^2 + 8\|xz\|_{L^2(0,L)}^2. \quad (\text{A.6})$$

Hence,

$$13\|z'\|_{L^2(0,L)}^2 \leq 2\|z' + z'''\|_{L^2(0,L)}^2 + 4\|z'''\|_{L^2(0,L)}^2 + 8L^2\|z\|_{L^2(0,L)}^2. \quad (\text{A.7})$$

Plugging inequality (A.5) in (A.7), we have

$$\begin{aligned} 13\|z'\|_{L^2(0,L)}^2 &\leq 2\|z' + z'''\|_{L^2(0,L)}^2 + 4\left(2\|z''' + z'\|_{L^2(0,L)}^2 + 2\|z'\|_{L^2(0,L)}^2\right) \\ &\quad + 8L^2\|z\|_{L^2(0,L)}^2 \\ &\leq 10\|z' + z'''\|_{L^2(0,L)}^2 + 8\|z'\|_{L^2(0,L)}^2 + 8L^2\|z\|_{L^2(0,L)}^2. \end{aligned}$$

Therefore,

$$\|z'\|_{L^2(0,L)}^2 \leq 2\|z' + z'''\|_{L^2(0,L)}^2 + \frac{8L^2}{5}\|z\|_{L^2(0,L)}^2. \quad (\text{A.8})$$

Considering Equations (A.3) and (A.4), it leads us to the following inequality, for all $z \in D(A)$,

$$\|z'\|_{L^2(0,L)}^2 \leq \Delta\|z\|_{D(A_\sigma)}^2 \quad (\text{A.9})$$

where Δ is a term which depends only on L .

Thus, if we consider now a sequence $\{z_n\}_{n \in \mathbb{N}}$ in $D(A_\sigma)$ bounded for the graph norm of $D(A_\sigma)$, we have from (A.9) that this sequence is bounded in $H_0^1(0, L)$. Since the canonical embedding from $H_0^1(0, L)$ to $L^2(0, L)$ is compact, there exists a subsequence still denoted $\{z_n\}_{n \in \mathbb{N}}$ such that $z_n \rightarrow z$ in $L^2(0, L)$. Thus Z belongs to $L^2(0, L)$ which allows us to state that $D(A_\sigma)$ embeds compactly in Z . It concludes the proof of Lemma A.1. \square

Nonlinear heat equation: m -dissipativity and precompactity

B.1 Proof of the m -dissipativity of the nonlinear heat equation

This subsection is devoted to the proof of the following theorem.

Theorem B.1

The operator defined by (1.60) is m -dissipative

Proof of Theorem B.1:

The proof of Theorem B.1 is divided into two steps. First, the operator A is proved to be dissipative. Secondly, we prove that, for all $f \in L^2(0, L)$, there exist $z \in D(A)$ such that

$$z - Az = f. \quad (\text{B.1})$$

Let us recall that the dissipativity and the existence of $z \in D(A)$ such that (B.1) holds imply that A is an m -dissipative operator (see Definition 1.1).

First step: Dissipativity of the operator A .

Note that we have

$$\langle Az - A\tilde{z}, z - \tilde{z} \rangle_{L^2(0,1)} = \int_0^1 (z - \tilde{z})(z'' - \tilde{z}'') dx + \int_0^L (z - \tilde{z})(\sin(z) - \sin(\tilde{z})) dx. \quad (\text{B.2})$$

Performing some integrations by parts leads to

$$\int_0^L (z - \tilde{z})(z'' - \tilde{z}'') dx = - \int_0^1 (z' - \tilde{z}')^2 dz. \quad (\text{B.3})$$

Moreover, using the fact that \sin is Lipschitz together with a Poincaré inequality, one has

$$\int_0^L (z - \tilde{z})(\sin(z) - \sin(\tilde{z})) dx \leq \int_0^L (z - \tilde{z})^2 dx \leq \frac{4}{\pi^2} \int_0^1 (z' - \tilde{z}')^2 dx. \quad (\text{B.4})$$

Hence, it is easy to see that

$$\langle Az - A\tilde{z}, z - \tilde{z} \rangle_{L^2(0,1)} \leq 0. \quad (\text{B.5})$$

Second step: Existence of $z \in D(A)$ such that (B.1) holds

To prove the existence of $z \in D(A)$ such that (B.1) holds, one has to prove that there exists a solution to the following nonlinear ODE

$$\begin{cases} z - z'' + \sin(z) = f, \\ z(0) = z(1) = 0. \end{cases} \quad (\text{B.6})$$

We aim at applying the Schauder fixed-point theorem to the following nonhomogeneous linear ODE

$$\begin{cases} z - z'' = -\sin(\tilde{z}) + f, \\ z(0) = z(1) = 0, \end{cases} \quad (\text{B.7})$$

where $\tilde{z} \in L^2(0, 1)$. It is easy to see that there exists a unique solution to (B.7).

Focus on the map

$$\begin{aligned} \mathcal{T} : L^2(0, 1) &\rightarrow L^2(0, 1) \\ \tilde{z} &\mapsto z = \mathcal{T}(\tilde{z}) \end{aligned} \quad (\text{B.8})$$

where $z = \mathcal{T}(\tilde{z})$ is the unique solution to (B.7).

We define

$$C := \{z \in H_0^1(0, 1) \mid \|z\|_{H_0^1(0,1)} \leq M\}. \quad (\text{B.9})$$

From the theorem of Rellich, the injection of $H_0^1(0, 1)$ in $L^2(0, 1)$ is compact, then C is bounded in $H_0^1(0, 1)$ and is relatively compact in $L^2(0, 1)$. Moreover, it is a closed subset of $L^2(0, 1)$. Thus C is a compact subset of $L^2(0, 1)$. In order to apply the Schauder theorem, we have to prove that $\mathcal{T}(L^2(0, 1)) \subset C$ for a suitable choice of M and λ . Let us multiply the first line of (B.7) by z and then integrate between 0 and 1. After some integrations by parts, one has

$$\|z\|_{L^2(0,1)}^2 + \|z'\|_{L^2(0,1)}^2 = \int_0^1 f z dx - \int_0^1 \sin(\tilde{z}) dz. \quad (\text{B.10})$$

Hence, applying Cauchy Schwarz inequality leads to

$$\|z'\|_{L^2(0,1)}^2 \leq \frac{1}{2} \|f\|_{L^2(0,1)}^2 + \frac{1}{2} - \|z\|_{L^2(0,1)}^2 + \|z\|_{L^2(0,1)}^2. \quad (\text{B.11})$$

Therefore, since $\|z'\|_{L^2(0,1)}^2$ and $\|z\|_{H_0^1(0,1)}$ are equivalent by the Poincaré inequality, one has

$$\|z\|_{H_0^1(0,1)} \leq M, \quad (\text{B.12})$$

where

$$M := \sqrt{\|f\|_{L^2(0,1)}^2 + 1}.$$

Hence, applying Theorem 1.11, it concludes the proof of Theorem B.1. \square

B.2 Precompactness of the nonlinear heat equation with a cone-bounded nonlinearity

This subsection is devoted to the proof of the following lemma.

Lemma B.1

The canonical embedding from $D(A_\sigma)$, equipped with the graph norm, into $Z := L^2(0, 1)$ is compact.

Proof of Lemma B.1:

We follow the strategy of [64], [76] and [69]. Let us recall the definition of the graph norm

$$\begin{aligned} \|z\|_{D(A_\sigma)}^2 &:= \|z\|_{L^2(0,1)}^2 + \|A_\sigma z\|_{L^2(0,1)}^2 \\ &= \int_0^1 (|z(x)|^2 + |z''(x) + \sin(z)(x) - \sigma(z)(x)|^2) dx \end{aligned} \quad (\text{B.13})$$

Note that we have

$$\|z\|_{D(A_\sigma)}^2 \geq \|z\|_{L^2(0,1)}^2 \quad (\text{B.14})$$

and

$$\begin{aligned} \|z\|_{D(A_\sigma)}^2 &\geq \frac{1}{4} \int_0^1 |\sigma(z)(x)|^2 dx + \frac{1}{4} \int_0^1 |-\sin(z)(x)|^2 dx \\ &\quad + \frac{1}{4} \int_0^1 |z''(x) + \sin(z)(x) - \sigma(z)(x)|^2 dx \\ &\geq \frac{1}{8} \int_0^1 |z''(x)|^2 dx. \end{aligned} \quad (\text{B.15})$$

Hence,

$$\|z\|_{D(A_\sigma)}^2 \geq \frac{1}{8} \|z''\|_{L^2(0,1)}^2. \quad (\text{B.16})$$

Noticing that $\|z\|_{L^2(0,1)}^2 = \|z - z'' + z''\|_{L^2(0,1)}^2$, we have

$$\begin{aligned} \|z\|_{L^2(0,1)}^2 &= \|z + z''\|_{L^2(0,1)}^2 + \|z''\|_{L^2(0,1)}^2 \\ &= \|z\|_{L^2(0,1)}^2 + \|z''\|_{L^2(0,1)}^2 + 2 \int_0^1 z(x)z''(x)dx + \|z''\|_{L^2(0,1)}^2. \end{aligned} \quad (\text{B.17})$$

Therefore, we have

$$- \int_0^1 z(x)z''(x)dx = \|z''\|_{L^2(0,1)}^2. \quad (\text{B.18})$$

Performing an integration by parts, we obtain

$$\int_0^1 z(x)z''(x)dx = -\|z'\|_{L^2(0,1)}^2. \quad (\text{B.19})$$

Hence, using (B.16), the following inequality holds

$$\|z'\|_{L^2(0,1)}^2 \leq 8\|z\|_{D(A_\sigma)}^2. \quad (\text{B.20})$$

Thus, if we consider now a sequence $\{z_n\}_{n \in \mathbb{N}}$ in $D(A_\sigma)$ bounded for the graph norm of $D(A_\sigma)$, we have from (A.9) that this sequence is bounded in $H_0^1(0, L)$. Since the canonical embedding from $H_0^1(0, L)$ to $L^2(0, L)$ is compact, there exists a subsequence still denoted $\{z_n\}_{n \in \mathbb{N}}$ such that $z_n \rightarrow z$ in $L^2(0, L)$. Thus z belongs to $L^2(0, L)$ which allows us to state that $D(A_\sigma)$ embeds compactly in Z . It concludes the proof of Lemma B.1. \square

Successive approximation method. Application to the kernel existence problem

C.1 Presentation of the method

Let us consider the following initial problem:

$$\begin{cases} \frac{d}{dt}z = f(t, z) \\ z(t_0) = z_0 \end{cases} \quad (\text{C.1})$$

By observing this problem, we can assume that:

$$\phi(t) = z_0 + \int_{t_0}^t f(s, \phi(s))ds \quad (\text{C.2})$$

is a solution of (C.1). In other words, the solutions of (C.1) are also solutions of (C.2). However, we have to prove the existence of such solutions. The successive approximation method proposes to consider a sequence of approximations that has to get closer and closer to the solutions of (C.2). These approximations are all based on (C.2) as follows:

$$\begin{aligned} \phi_0(t) &= z_0 \\ \phi_1(t) &= z_0 + \int_{t_0}^t f(s, \phi_0(s))ds \\ \phi_2(t) &= z_0 + \int_{t_0}^t f(s, \phi_1(s))ds \\ &\vdots \\ \phi_k(t) &= z_0 + \int_{t_0}^t f(s, \phi_{k-1}(s))ds \end{aligned} \quad (\text{C.3})$$

If one can demonstrate that sequence of solutions $(\phi_k)_{k \in \mathbb{N}}$ converges to some function $\phi(t)$ that satisfies (C.2), then the argument goes, one has not only proven the existence of a

solution to the initial value problem, but has also constructed a formula for the solution. This fact is based on the following theorem:

Theorem C.1

(Picard Lindlöf existence theorem) Let a and b be real positive constants, and assume that $f(t, x) : \mathbb{R}^2 \rightarrow \mathbb{R}$ is bounded (with least upper bound M), continuous and Lipschitz, with Lipschitz constant K on the rectangle:

$$R = \{(t, x) \in \mathbb{R}^2 \mid |t - t_0| \leq a, |z - z_0| \leq b\} \tag{C.4}$$

Then the successive approximations:

$$\phi_0(t) = z_0, \quad \phi_k(t) = z_0 + \int_{t_0}^t f(s, \phi_{k-1}(s)) ds \tag{C.5}$$

converge to the solution of the initial value problem $\{\frac{d}{dt}z = f(t, z), z(t_0) = z_0\}$ on the interval:

$$I = \left\{ t \in \mathbb{R} \mid |t - t_0| \leq \alpha = \min \left(a, \frac{b}{M} \right) \right\} \tag{C.6}$$

C.2 Application of the method [14]

Let us consider the (3.28) and let us make a variable chagement with the following formula

$$t = s - x, \quad \eta = x + s$$

Now, define $G(\eta, t) := k(x, s)$. With the variable chagement, we get

$$\begin{cases} G_{tt\eta} + 2G_{\eta\eta\eta} + 2G_{\eta} = -\lambda G & (t, s) \in \mathcal{T}_0, \\ G(\eta, 2L - \eta) = 0 & s \in [L, 2L], \\ G(\eta, 0) = 0 & s \in [0, 2L], \\ G_t(\eta, 0) = \frac{\lambda}{6}(\eta - 2L) & s \in [0, 2L], \end{cases} \tag{C.7}$$

where $\mathcal{T}_0 = \{(\eta, t) \mid t \in [0, L], \eta \in [t, 2L - y]\}$.

Let us transform this system into an integral one. We write the equation in variables (χ, ξ) , integrate ξ between 0 and τ and use that $G_t(\eta, 0) = \frac{\lambda}{6}(\eta - 2L)$. Next, we integrate τ between 0 and t and use that $G_{\eta}(\chi, 0) = 0$. Finally, we integrate χ between η and $2L - t$ and use that $G(\eta, 2L - \eta) = 0$. Then, we get an integral expression of (C.7) written as follows

$$G(\eta, t) = -\frac{\lambda t}{6}(2L - t - \eta) + \frac{1}{6} \int_{\eta}^{2L-t} \int_0^t \int_0^{\tau} (2G_{\eta} + 2G_{\eta} + \lambda G)(\chi, \xi) d\xi d\tau d\chi \tag{C.8}$$

We consider the following sequence of solutions

$$\begin{aligned}
G^1(\eta, t) &= -\frac{\lambda t}{6}(2L - t - s) \\
G^2(\eta, t) &= -\frac{\lambda t}{6}(2L - t - \eta) + \frac{1}{6} \int_{\eta}^{2L-t} \int_0^t \int_0^{\tau} (2G_{\eta\eta\eta}^1 + 2G_{\eta}^1 + \lambda G^1)(\chi, \xi) d\xi d\tau d\chi \\
&\vdots \\
G^{n+1}(\eta, t) &= -\frac{\lambda t}{6}(2L - t - \eta) + \frac{1}{6} \int_{\eta}^{2L-t} \int_0^t \int_0^{\tau} (2G_{\eta\eta\eta}^n + 2G_{\eta}^n + \lambda G^n)(\chi, \xi) d\xi d\tau d\chi
\end{aligned} \tag{C.9}$$

After some computations, we get the formula

$$G^k(\eta, t) = \sum_{i=1}^k \left(a_k^i t^{2k-1} + b_k^i t^{2k} \right) [(2L - t)^i - \eta^i] \tag{C.10}$$

where the coefficients satisfy $b_k^k = 0$. By a recursive reasonment, we can state the existence of positive constants M and B such that, for any $k \geq 1$ and any $(s, t) \in \mathcal{T}_0$

$$|G^k(\eta, t)| \leq M \frac{B^k}{(2k)!} (t^{2k+1} + t^{2k}) \tag{C.11}$$

Thanks to this formula, we get the uniform convergence of the series $\sum_{n=1}^{\infty} G^n(\eta, t)$ and we get a solution of our integral equation. Indeed, we can write

$$\begin{aligned}
6(G - G^1)(\eta, t) &= 6 \sum_{n=1}^{\infty} G^{n+1}(\eta, t) \\
&= \sum_{n=1}^{\infty} \int_{\eta}^{2L-t} \int_0^t (2G_{\eta\eta\eta}^n + 2G_{\eta}^n + \lambda G^n)(\chi, \xi) d\xi d\tau d\chi \\
&= \int_{\eta}^{2L-t} \int_0^t \left(2 \sum_{n=1}^{\infty} G_{\eta\eta\eta}^n + 2 \sum_{n=1}^{\infty} G_{\eta}^n + \lambda \sum_{n=1}^{\infty} G^n \right) d\xi d\tau d\chi \\
&= \int_{\eta}^{2L-t} \int_0^t (2G_{\eta\eta\eta}^n + 2G_{\eta}^n + \lambda G)(\chi, \xi) d\xi d\tau d\chi,
\end{aligned} \tag{C.12}$$

where we have used that the corresponding series $\sum_{n=1}^{\infty} G_{\eta}^n$ and $\sum_{n=1}^{\infty} G_{\eta\eta\eta}^n$ are also uniformly convergent.

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