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# Control Sets of Linear Systems and Classification of Almost-Riemannian Structures on Lie Groups

Guilherme Zsigmond Machado

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**UNIVERSIDAD CATÓLICA DEL NORTE**

**FACULTAD DE CIENCIAS**

**Departamento de Matemáticas**

**CONTROL SETS OF LINEAR SYSTEMS AND  
CLASSIFICATION OF ALMOST-RIEMANNIAN  
STRUCTURES ON LIE GROUPS.**

Tesis para optar al grado de Doctor en Ciencias Mención Matemática

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Dr. Philippe Jouan

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et de Universidad Católica del Norte

# Control sets of linear systems and classification of almost-Riemannian structures on Lie groups

Présentée et soutenue par  
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*Vá aonde não haja ninguém, quando regresses pode ser  
que ninguém te reconheça, mas você se reconhecerá.*

# Abstract

This Thesis analyzes the control sets of linear control systems and the isometries of almost-Riemannian structures on Lie groups. The main goal for the first topic is to characterize the properties of control sets such as existence, uniqueness, boundedness and invariance. We study such properties for Lie groups decomposable by eigenvalues of the linear vector field and extend some results to non-compact semi-simple Lie groups with finite center. The second topic main objective is to characterize isometry properties of almost-Riemannian structures. We search for invariants under isometries such as the singular locus and the set of the linear vector field singularities. For nilpotent Lie groups, we prove that all isometries are affine, that is, a composition of a translation with a Lie group automorphisms. To finish this topic we use the obtained results to classify the almost-Riemannian structures on low dimensional Lie groups.

# Summary

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# Chapter 1

## Introduction

This thesis studies two different subjects on Lie groups, control sets of linear control systems and isometries of Almost Riemannian structures.

From geometric control theory, the controllability aspect of a linear control system on Lie groups was studied by several authors such as [4], [15], [16]. In each, the authors proved that property for linear systems on Lie groups doesn't occur frequently. For the restricted linear control systems, there exists the control set concept, i.e., the maximal set where the controllability property holds.

In this work, we study the properties of control set on Lie groups. The obtained results are in Section 1.2 and 1.3.

From the point of view of sub-Riemannian geometry, the classification of geometric structure is a delicate task. In particular, the classification of Almost-Riemannian structures by isometries should preserve some intrinsic invariants to geometric structures.

In this work, we search for geometrical invariant preserved by isometry, a classification of these structures on Lie groups and to determine their groups of isometries. The obtained results reside in Section 1.4

The first section described the basic concepts. Each subsequent sections describe an article of the develop work: one published paper, one submitted and one working in progress. The sections are organized by research area.

## 1.1 Basic definitions

This section contains a brief summary of the essential issues involved in the thesis. Let  $G$  be a connected Lie group with Lie algebra  $\mathfrak{g}$ . First, we introduce the concept of a linear vector field, (that is, an infinitesimal  $G$ -automorphism), its associated derivation and the corresponding decomposition via its Lyapunov spectrum. Next, we define the notion of Linear Control System on  $G$  and review some of the standard facts on accessible sets. Since we intend to study the controllability property for a restricted linear control system, i.e., when the controls are bounded, we include the concept of control sets and some of their general properties. When  $G$  is a non compact semi-simple Lie group it is necessary to include some material about Cartan involutions and Bruhat cell decomposition. Finally, a simple Almost-Riemannian Structure (ARS) on a Lie group is defined by a single linear vector field and  $\dim(G)-1$  left-invariant ones. Since the thesis is concerned with the classification of such structures we set up notation and terminology about the locus, isometries and norms.

### 1.1.1 Linear vector fields and decompositions

We establish here the notion of linear vector fields on Lie groups and some of their basic properties are recalled. In particular, look more closely at the associated derivation  $\mathcal{D}$  and the corresponding  $\mathfrak{g}$ -decomposition induced by its Lyapunov spectrum. The reader is referred to [5],[9],[15],[19].

Let  $G$  be a connected Lie group and  $\mathfrak{g}$  its Lie algebra considered as the set of right-invariant vector fields. A vector field on  $G$  is said to be linear if its flow  $\varphi$  is a 1-parameter group of automorphisms, i.e., for each  $t \in \mathbb{R}$

$$\varphi_t(g \cdot h) = \varphi_t(g) \cdot \varphi_t(h).$$

A linear vector field  $\mathcal{X}$  is analytic and complete. The following characterization is useful. A vector field  $\mathcal{X}$  on a connected Lie group  $G$  is linear if and only if  $\mathcal{X}$  is an affine vector field, i.e., an element of the normalizer

$$\mathfrak{N} = \text{norm}_{V^\omega(G)} \mathfrak{g} = \{F \in V^\omega(G); \forall Y \in \mathfrak{g}, [F, Y] \in \mathfrak{g}\},$$

of  $\mathfrak{g}$  in the algebra  $V^\omega(G)$  of analytic vector fields of  $G$  and  $\mathcal{X}(e) = 0$ .

For a given linear vector field  $\mathcal{X}$  we can associate the derivation  $\mathcal{D}$  of  $\mathfrak{g}$  defined by:

$$\forall Y \in \mathfrak{g} \quad \mathcal{D}Y = -[\mathcal{X}, Y],$$

that is  $\mathcal{D} = -\text{ad}(\mathcal{X})$ . The minus sign in this definition comes from the formula  $[Ax, b] = -Ab$  in  $\mathbb{R}^n$ . It also enables to avoid a minus sign in the useful formula

$$\forall Y \in \mathfrak{g}, \quad \forall t \in \mathbb{R} \quad \varphi_t(\exp Y) = \exp(e^{t\mathcal{D}}Y). \quad (1.1)$$

It can be shown that an affine vector field can be uniquely decomposed into a sum  $F = \mathcal{X} + Z$  where  $\mathcal{X}$  is linear and  $Z$  is right-invariant, see [9] and [19].

Consider the complexification  $\mathfrak{g}_{\mathbb{C}}$  of  $\mathfrak{g}$  and the derivation  $\mathcal{D}_{\mathbb{C}}$  on  $\mathfrak{g}_{\mathbb{C}}$  induced by  $\mathcal{D}$ . For an eigenvalue  $\alpha$  of  $\mathcal{D}$  we defined the  $\alpha$ -generalized eigenspace of  $\mathcal{D}_{\mathbb{C}}$  as

$$(\mathfrak{g}_{\mathbb{C}})_{\alpha} = \{X \in \mathfrak{g}_{\mathbb{C}} : (\mathcal{D}_{\mathbb{C}} - \alpha)^n X = 0 \text{ for some } n \geq 1\}.$$

Since the eigenvalues of  $\mathcal{D}_{\mathbb{C}}$  coincides with the ones of  $\mathcal{D}$ . We have that

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{\alpha \in \text{Spec}(\mathcal{D})} (\mathfrak{g}_{\mathbb{C}})_{\alpha}.$$

Moreover, if  $\beta$  is also an eigenvalue of  $\mathcal{D}$  Proposition 3.1 of [25] implies

$$[(\mathfrak{g}_{\mathbb{C}})_{\alpha}, (\mathfrak{g}_{\mathbb{C}})_{\beta}] \subset (\mathfrak{g}_{\mathbb{C}})_{\alpha+\beta}, \quad (1.2)$$

where  $(\mathfrak{g}_{\mathbb{C}})_{\alpha+\beta} = \{0\}$  when  $\alpha + \beta$  is not an eigenvalue of  $\mathcal{D}$ .

Let

$$\mathfrak{g}_{\mathbb{C}}^{+} = \bigoplus_{\alpha : \text{Re}(\alpha) > 0} (\mathfrak{g}_{\mathbb{C}})_{\alpha}, \quad \mathfrak{g}_{\mathbb{C}}^0 = \bigoplus_{\alpha : \text{Re}(\alpha) = 0} (\mathfrak{g}_{\mathbb{C}})_{\alpha} \quad \text{and} \quad \mathfrak{g}_{\mathbb{C}}^{-} = \bigoplus_{\alpha : \text{Re}(\alpha) < 0} (\mathfrak{g}_{\mathbb{C}})_{\alpha}.$$

Since  $\mathfrak{g}_{\mathbb{C}}^{+}, \mathfrak{g}_{\mathbb{C}}^0$  and  $\mathfrak{g}_{\mathbb{C}}^{-}$  are invariant by conjugation they coincide with the complexification of

$$\mathfrak{g}^{+} := \mathfrak{g}_{\mathbb{C}}^{+} \cap \mathfrak{g}, \quad \mathfrak{g}^0 := \mathfrak{g}_{\mathbb{C}}^0 \cap \mathfrak{g} \quad \text{and} \quad \mathfrak{g}^{-} := \mathfrak{g}_{\mathbb{C}}^{-} \cap \mathfrak{g}.$$

Moreover,  $\mathfrak{g}^{+}, \mathfrak{g}^0$  and  $\mathfrak{g}^{-}$  are  $\mathcal{D}$ -invariant Lie subalgebras of  $\mathfrak{g}$ ;  $\mathfrak{g}^{+}$  and  $\mathfrak{g}^{-}$  are nilpotent and  $\mathfrak{g} = \mathfrak{g}^{+} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^{-}$ . Also,  $\mathfrak{g}^{+}$  and  $\mathfrak{g}^{-}$  are ideals of the Lie subalgebras  $\mathfrak{g}^{+,0} := \mathfrak{g}^{+} \oplus \mathfrak{g}^0$  and  $\mathfrak{g}^{-,0} := \mathfrak{g}^{-} \oplus \mathfrak{g}^0$ , respectively.

The decomposition induced by  $\mathcal{D}$  at level of the Lie algebra extends to the Lie group, we follow [15]. Denote by  $G^{+}, G^0, G^{-}, G^{+,0}, G^{-,0}$  the connected subgroups of  $G$  with Lie algebras  $\mathfrak{g}^{+}, \mathfrak{g}^0, \mathfrak{g}^{-}, \mathfrak{g}^{+,0} = \mathfrak{g}^{+} \oplus \mathfrak{g}^0, \mathfrak{g}^{-,0} = \mathfrak{g}^{-} \oplus \mathfrak{g}^0$  respectively. All these subgroups are closed in  $G$ . The Lie group  $G$  is said to be decomposable if it can be written as  $G = G^{+,0}G^{-} = G^{-,0}G^{+}$ .

### 1.1.2 Linear control systems and control sets

In this section we introduce the notion of linear control systems on  $G$ . We review some of their standard facts, specially general properties of their accessible sets. We also include the notion of control sets and a fundamental result on this subject. For more properties of the control sets the reader can consult [9], [14],[15] and [19].

The next definition extends the notion of linear control systems from Euclidean spaces to Lie groups.

**Definition 1.1.** *A linear control system  $\Sigma$  on a Lie group  $G$  is defined by*

$$\dot{g} = \mathcal{X}(g) + \sum_{i=1}^k u_i Y_i(g), \quad (\Sigma)$$

where  $\mathcal{X}$  is a linear vector field,  $Y_i$  are right-invariant vector fields and the control function  $u = (u_1, \dots, u_k)$  belongs to the set of admissible controls  $\mathcal{U}$ .

In this context  $\mathcal{U} = L^\infty(\mathbb{R}, \Omega \subset \mathbb{R}^k)$ , where  $\Omega$  is a compact and convex subset of  $\mathbb{R}^k$  containing 0 in its interior.

For any  $g \in G$  and  $u \in \mathcal{U}$  the map  $(t, g, u) \mapsto \phi(t, g, u)$  denoted also by  $\phi_{t,u}(g)$  is the solution of the system  $\Sigma$  through the initial condition  $g$  and control  $u$  at time  $t$ . For fixed  $t, u$ , the map  $\phi(t, \cdot, u) : g \in G \mapsto \phi(t, g, u)$  is a diffeomorphism and satisfy the cocycle property

$$\phi(t + s, g, u) = \phi(t, \phi(s, g, u), \Theta_s u), \text{ for } t, s \in \mathbb{R}, g \in G \text{ and } u \in \mathcal{U}$$

where for each  $t \in \mathbb{R}$ ,  $\Theta_t$  denotes the shift flow on  $\mathcal{U}$  defined by  $(\Theta_t u)(s) = u(t + s)$ .

A fundamental statement about the shape of the solution of a linear control system is given in [19], as follows. Let  $g \in G$  an initial condition and  $u \in \mathcal{U}$  an admissible control function. Then, the  $\Sigma$ -solution reads as

$$\phi_{t,u}(g) = L_{\phi_{t,u}(e)}(\varphi_t(g)) = \phi_{t,u} \cdot \varphi_t(g), \quad (1.3)$$

where  $L_g$  denotes the left translation by  $g \in G$ .

For each  $g \in G$  and any positive time  $\tau$

$$\begin{aligned} \mathcal{A}_{< \tau}(g) &= \{h \in G : \exists u \in \mathcal{U}, t \in [0, \tau] \text{ such that } h = \phi_{t,u}(g)\} \\ \mathcal{A}_\tau(g) &= \{h \in G : \exists u \in \mathcal{U}, h = \phi_{\tau,u}(g)\} \\ \mathcal{A}(g) &= \bigcup_{\tau > 0} \mathcal{A}_{\leq \tau}(g) \end{aligned} \quad (1.4)$$

are the set of reachable points from  $g$  up to the time  $\tau$ , the set of reachable points from  $g$  at time  $\tau$  and the reachable set of  $g$ , respectively. In the same way, for any  $\tau > 0$  the sets

$$\begin{aligned}\mathcal{A}_{\leq \tau}^*(g) &= \{h \in G : \exists u \in \mathcal{U}, t \in [0, \tau] \text{ such that } \phi_{t,u}(h) = g\} \\ \mathcal{A}_{\tau}^*(g) &= \{h \in G : \exists u \in \mathcal{U}, \phi_{\tau,u}(h) = g\} \\ \mathcal{A}^*(g) &= \bigcup_{\tau > 0} \mathcal{A}_{\leq \tau}^*(g),\end{aligned}\tag{1.5}$$

are the set of controllable points to  $g$  within time  $\tau$ , the set of controllable points to  $g$  in time  $\tau$  and the controllable set of  $g$ , respectively.

The sets  $\mathcal{A}_{\leq \tau}(e), \mathcal{A}(e), \mathcal{A}_{\leq \tau}^*(e)$  and  $\mathcal{A}^*(e)$  will be denoted such as  $\mathcal{A}_{\leq \tau}, \mathcal{A}, \mathcal{A}_{\leq \tau}^*$  and  $\mathcal{A}^*$  respectively.

Since the system flow satisfies  $\phi_{-t,u}^{-1} = \phi_{-t, \Theta_t(u)}$ . It turns out that

$$\mathcal{A}_{\tau}^* = \varphi_{-\tau}(\mathcal{A}_{\tau}^{-1}).\tag{1.6}$$

**Definition 1.2.** *We say that the system  $\Sigma$*

- i) is locally accessible at  $g$  if for all  $\tau > 0$  the sets  $\mathcal{A}_{\leq \tau}(g)$  and  $\mathcal{A}_{\leq \tau}^*(g)$  have nonempty interior,*
- ii) is locally accessible if it is locally accessible at every  $g \in G$ ,*
- iii) satisfy the Lie algebra rank condition (LARC) if  $\mathcal{L}(g) = T_g G$  for any  $g \in G$ , where*

*$\mathcal{L}$  is the smallest  $\mathfrak{g}$ -subalgebra containing  $\mathcal{X}$  and  $Y^j, j = 1, \dots, k.$ ,*

- iv) is controllable from  $g$  if  $\mathcal{A}(g) = G$ ,*
- v) is controllable if it is controllable from  $g$  for any  $g \in G$ .*

Notice that, the system is locally accessible at  $g \in G$  if it satisfies the Lie algebra rank condition at the point  $g$ .

A more realistic approach to understand the controllability behaviour of a system comes from the notion of control set. Here, we follow [14] and adapt the notions and results from manifolds to Lie groups.

**Definition 1.3.** *A nonempty set  $\mathcal{C} \subset G$  is called a control set of  $\Sigma$  if*

- i) for every  $g \in G$  there exists  $u \in \mathcal{U}$  such that  $\phi(t, g, u) \in \mathcal{C}$  for  $t \geq 0$ ,*

ii)  $\mathcal{C} \subset \text{cl}(\mathcal{A}(g))$  for each  $g \in \mathcal{C}$ ,

iii) it is the maximal set with the properties (i) and (ii).

A control set  $\mathcal{C}$  is said to be *invariant in positive time* (in *negative time*) if  $\phi_{t,u}(\mathcal{C}) \subset \mathcal{C}$  ( $\phi_{-t,u}(\mathcal{C}) \subset \mathcal{C}$ ) for any  $t > 0$  and  $u \in \mathcal{U}$ , respectively.

The next proposition summarizes the main properties of control sets.

**Proposition 1.1.** *Assume that  $\Sigma$  is locally accessible and let  $\mathcal{C}$  be a control set with nonempty interior. It holds:*

1.  $\mathcal{C}$  is connected and  $\text{cl}(\text{int } \mathcal{C}) = \text{cl}(\mathcal{C})$ .
2.  $\text{int } \mathcal{C} \subset \mathcal{A}(g)$  for any  $g \in \mathcal{C}$ . Furthermore, for any  $h \in \text{int } \mathcal{C}$

$$\mathcal{C} = \text{cl}(\mathcal{A}(h)) \cap \mathcal{A}^*(h). \quad (1.7)$$

*In particular, the system is controllable on  $\text{int } \mathcal{C}$  and approximate controllable on  $\mathcal{C}$ .*

3. Assume  $\phi_{t,u}(g)$  is a periodic trajectory, that is,  $\phi_{t+s,u}(g) = \phi_{t,u}(g)$  for some  $s > 0$  and all  $t \in \mathbb{R}$ . Hence, if  $g \in \text{int } \mathcal{C}$  then  $\phi_{t,u}(g) \in \text{int } \mathcal{C}$  for all  $t \in \mathbb{R}$ .
4.  $\mathcal{C}$  is closed  $\Leftrightarrow \mathcal{C}$  is invariant in positive time  $\Leftrightarrow \mathcal{C} = \text{cl}(\mathcal{A}(g))$  for any  $g \in \mathcal{C}$ .
5.  $\mathcal{C}$  is open  $\Leftrightarrow \mathcal{C}$  is invariant in negative time  $\Leftrightarrow \mathcal{C} = \mathcal{A}^*(g)$  for any  $g \in \mathcal{C}$ .

In the sequel, we review some standard facts about semi-simple Lie algebras and its consequences on their corresponding Lie groups. As references we cite [18], [21] and [25].

Let  $G$  be a connected non-compact semi-simple Lie group with Lie algebra  $\mathfrak{g}$  identified with the set of right-invariant vector fields. Let  $\theta$  be a Cartan involution subordinate to a Cartan decomposition of  $\mathfrak{g}$  as the direct sum  $\mathfrak{k} \oplus \mathfrak{s}$ , where  $\mathfrak{k}$  is a compact Lie algebra of  $\mathfrak{g}$  and  $\mathfrak{s}$  is a vectorial subspace. Through the Cartan-Killing form  $\langle X, Y \rangle = \text{tr}(ad_X ad_Y)$  we define the form

$$B_\theta(X, Y) = - \langle X, \theta(Y) \rangle .$$

Choose a maximal Abelian subspace  $\mathfrak{a} \subset \mathfrak{s}$  and a Weyl chamber  $\mathfrak{a}^+ \subset \mathfrak{a}$ . Associated to Lie subalgebra  $\mathfrak{a}$  there exists a set of roots  $\Omega$ . Moreover, for the Weyl chamber  $\mathfrak{a}^+$  there exists a subset of positive roots  $\Omega^+$ . Denotes by  $\Delta$  the simple roots in  $\Omega^+$ , that is,  $\alpha \in \Delta$  if  $\alpha$  can not be written as  $\alpha = \sum_{i=1}^k \beta_i$ , where  $\beta_i$  are positive roots. Let us denote the negative roots  $\Omega^-$  by  $-\Omega^+$ .

Given a root  $\alpha \in \Omega$ , there exists a vector field  $H_\alpha \in \mathfrak{a}$  satisfying  $B_\theta(H, H_\alpha) = \alpha(H)$  for all  $H \in \mathfrak{a}$ . The element  $H_\alpha$  is called coroot.

The Iwasawa decomposition of the Lie algebra is given by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+,$$

where  $\mathfrak{k}, \mathfrak{a}$  are coming from the Cartan decomposition and  $\mathfrak{n}^+(\mathfrak{n}^-)$  is given by  $\sum_{\alpha \in \Omega^+} \mathfrak{g}_\alpha$  ( $\sum_{\alpha \in \Omega^-} \mathfrak{g}_\alpha$ ). Here,  $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$ , that is, the space of roots associated to  $\alpha$ .

The corresponding decompositions at the level of the group read as follows

$$G = KS \text{ and } G = KAN^+,$$

where  $K = \exp \mathfrak{k}$ ,  $S = \exp \mathfrak{s}$ ,  $A = \exp \mathfrak{a}$ ,  $N^+ = \exp \mathfrak{n}^+$ ,  $K \cap AN^+ = \{e\}$ . Furthermore,  $N^+, A$  are simply connected nilpotent Lie groups and  $K$  is compact.

Let  $M^*$  be the normalizer of  $A$  in  $K$ , i.e.,

$$M^* = \{k \in K : Ad_k(H) = H_1 \text{ for } H, H_1 \in \mathfrak{a}\}$$

and denote by  $M$  the centralizer of  $A$  in  $K$ , i.e.,

$$M = \{k \in K : Ad_k(H) = H \text{ for } H \in \mathfrak{a}\}.$$

We denote their Lie algebra as  $\mathfrak{m}^*$  and  $\mathfrak{m}$ , respectively. The Weyl group  $W$  of  $G$  is defined by the quotient  $M^*/M$ .

Let  $\Lambda \subset \Delta$ . It is possible to build the Lie algebra  $\mathfrak{g}(\Lambda)$  using the eigenspaces  $\mathfrak{g}_\alpha$ , for  $\alpha \in \Lambda$ . Define

$$\mathfrak{k}(\Lambda) = \mathfrak{g}(\Lambda) \cap \mathfrak{k}, \mathfrak{a}(\Lambda) = \mathfrak{g}(\Lambda) \cap \mathfrak{a}, \mathfrak{n}^+(\Lambda) = \mathfrak{g}(\Lambda) \cap \mathfrak{n}^+ \text{ and } \mathfrak{n}^-(\Lambda) = \mathfrak{g}(\Lambda) \cap \mathfrak{n}^-.$$

We denote by  $G(\Lambda)$  and  $K(\Lambda)$  the connected Lie groups with Lie algebras  $\mathfrak{g}(\Lambda)$  and  $\mathfrak{k}(\Lambda)$ , respectively. Then  $G(\Lambda)$  is a semi-simple Lie group with finite semi-simple center.

The Iwasawa decomposition for the Lie group  $G(\Lambda)$  reads as

$$G(\Lambda) = K(\Lambda)A(\Lambda)N^+(\Lambda),$$

where  $A(\Lambda) = \exp \mathfrak{a}(\Lambda)$  and  $N^+(\Lambda) = \exp \mathfrak{n}^+(\Lambda)$ .

Let  $\mathfrak{a}_\Lambda$  be the subset of  $\mathfrak{a}$  perpendicular to all  $\alpha \in \Lambda$ , that is,

$$\mathfrak{a}_\Lambda = \{H \in \mathfrak{a} : \alpha(H) = 0, \forall \alpha \in \Lambda\}$$

and the corresponding set  $A_\Lambda = \exp \mathfrak{a}_\Lambda$ .

The parabolic subalgebra of type  $\Lambda$  is defined such as

$$\mathfrak{p}_\Lambda = \mathfrak{n}^-(\Lambda) \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+. \quad (1.8)$$

In particular,  $\Lambda = \{0\}$  is called minimal parabolic, i.e.,  $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$ .

The standard parabolic subgroup  $P_\Lambda$  is the normalizer of the subalgebra  $\mathfrak{p}_\Lambda$  in the Lie group  $G$ . It has the so called Langlands decomposition given by

$$P_\Lambda = K_\Lambda AN^+,$$

where  $K_\Lambda$  decomposes such as  $K_\Lambda = MK(\Lambda)$

In particular, the minimal parabolic subgroup  $P$  is the normalizer of  $\mathfrak{p}$  in  $G$  and its Langlands decomposition is  $P = MAN^+$ . Using previous consideration, is possible to prove the following result

**Theorem 1.2.** (*Bruhat Theorem*) *A semi-simple Lie group  $G$  is broken down as a disjoint union of Bruhat cells as follows*

$$G = PWP = \coprod_{w \in W} PwP,$$

*In particular, the only open cell is given by  $Pw^-P$ , where  $w^-$  stands for the principal involution.*

### 1.1.3 Almost-Riemannian structures

In this subsection we introduce the basic definitions of almost-Riemannian structures on Lie groups. They have been stated in [7], where more details can be found.

For all that concern general sub-Riemannian geometry, including almost-Riemannian one, the reader is referred to [1].

**Definition 1.4.** An almost-Riemannian structure on a  $n$ -dimensional Lie group  $G$  is defined by a set of  $n$  vector fields  $\{\mathcal{X}, Y_1, \dots, Y_{n-1}\}$  where

- (i)  $\mathcal{X}$  is linear,
- (ii)  $Y_1, \dots, Y_{n-1}$  are left-invariant,
- (iii)  $n = \dim G$  and the rank of  $\mathcal{X}, Y_1, \dots, Y_{n-1}$  is full on a nonempty subset of  $G$ ,
- (iv) the set  $\{\mathcal{X}, Y_1, \dots, Y_{n-1}\}$  satisfies the rank condition.

The metric on  $G$  is defined by declaring the frame  $\{\mathcal{X}, Y_1, \dots, Y_{n-1}\}$  to be orthonormal.

Equivalently, we can define the metric by an  $(n - 1)$ -dimensional left-invariant distribution  $\Delta$  provided with a left-invariant Euclidean metric and a linear vector field  $\mathcal{X}$  if they satisfy the conditions (iii) and (iv) of previous definition. The metric of the ARS is then defined by declaring  $\mathcal{X}$  unitary and orthogonal to  $\Delta$ .

**Remark:** The ARSs defined above correspond to what are called "simple ARSs" in [7]. Actually it is possible to define more general ARSs by a set of  $n$  affine vector fields that satisfy suitable properties. However the study of these general ARSs is not begun, except for equivalence (see [7]), and the analysis of their isometries would be beyond the scope of this thesis.

#### Necessary conditions for the rank condition

Notice that if  $[\Delta, \Delta] \subseteq \Delta$  and  $\mathcal{D}(\Delta) \subseteq \Delta$ , then the Lie algebra generated by  $\mathcal{X}, Y_1, \dots, Y_{n-1}$  is equal to  $\mathbb{R}\mathcal{X} \oplus \Delta$ . In both configurations, the rank of this Lie algebra is not full at the identity  $e$ . Consequently, to satisfy the rank condition at least one of the following conditions should hold:

- (i)  $[\Delta, \Delta] \not\subseteq \Delta$
- (ii)  $\mathcal{D}(\Delta) \not\subseteq \Delta$ .

In both cases, the full rank is obtained after one step. According to Corollary 12.15 of [1] this condition implies that there are no strictly abnormal minimizers.

#### Singular locus

The singular locus, denoted by  $\mathcal{Z}$ , is the set of points where the rank of  $\mathcal{X}, Y_1, \dots, Y_{n-1}$  is not full. It is an analytic subset of  $G$ . Assumption (iii)

ensures that  $\mathcal{Z}$  is different from  $G$ , and its analyticity implies that its interior is empty. On the other hand, since  $\mathcal{X}(e) = 0$ , the identity belongs to  $\mathcal{Z}$ . So,  $\mathcal{Z}$  cannot be empty.

Thus, the set  $G \setminus \mathcal{Z}$  is an open, dense and proper subset of  $G$ .

Another important set is the set of singularities of the linear field  $\mathcal{X}$ . It will be denoted by  $\mathcal{Z}_{\mathcal{X}} = \{g \in G : \mathcal{X}(g) = 0\}$ . It is always a subgroup of  $G$  and its Lie algebra is given by  $\ker(\mathcal{D})$ , where  $\mathcal{D}$  is the derivation associated to  $\mathcal{X}$ . The set  $\mathcal{Z}_{\mathcal{X}}$  is included in  $\mathcal{Z}$  but different in general.

The points of  $G \setminus \mathcal{Z}$  will be called the Riemannian points and the ones of  $\mathcal{Z}$  the singular points.

### Norms and isometries

The almost-Riemannian norm on  $T_g G$  is defined for  $X \in T_g G$  as:

$$\|X\| = \min \left\{ \sqrt{v^2 + \sum_1^n u_i^2} : v\mathcal{X}_g + u_1 Y_1(g) + \cdots + u_{n-1} Y_{n-1}(g) = X \right\}.$$

It is infinite if the point  $g$  belongs to the singular locus and  $X$  does not belong to  $\Delta_g$ .

Our purpose is to analyze the isometries of ARSs on Lie groups and to use them to classify these structures. We work with the following definition:

*Let  $(\Sigma)$  and  $(\Sigma')$  be two ARSs on the Lie group  $G$ . An isometry  $\Phi$  from  $(\Sigma)$  onto  $(\Sigma')$  is a diffeomorphism of  $G$  that respects the norms, that is:*

$$\forall g \in G, \forall X \in T_g G \quad \|T_g \Phi.X\|_{\Sigma'} = \|X\|_{\Sigma}.$$

where  $\|\cdot\|_{\Sigma}$  (resp.  $\|\cdot\|_{\Sigma'}$ ) stands for the norm associated to  $(\Sigma)$  in  $T_g G$  (resp. to  $(\Sigma')$  in  $T_{\Phi(g)}(G)$ ).

In particular, an isometry sends orthonormal frames to orthonormal frames.

## 1.2 Control sets of linear control systems on Lie groups

It is very well known that the classical linear control system on the Euclidean space  $\mathbb{R}^d$  is one of the most relevant control systems and it can be written as

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^m u_j(t) b^j, \quad b^j \in \mathbb{R}^d \text{ and } u \in \mathcal{U} \text{ as before.}$$

Here,  $A \in \mathfrak{gl}(d, \mathbb{R})$  the Lie algebra of the real matrices of order  $d$ . Since  $\mathbb{R}^d$  is a commutative Lie group, any constant vector  $b^j$  is an invariant vector field. Moreover, for any  $t \in \mathbb{R}$  the matrix  $e^{tA} \in GL(d, \mathbb{R}) = \text{Aut}(\mathbb{R}^d)$ , showing that the linear control system  $\Sigma$  is a generalization of the classical one from Euclidean spaces to connected Lie groups, [9], see also [22].

In [19] Jouan shows that this class of systems has also its importance on applications to the control theory. The author proves that any affine control systems on a connected manifold  $M$ , whose dynamic generates a finite dimensional Lie algebra, is diffeomorphic to a linear control system on a Lie group or on a homogeneous space. It shows that a better understanding of the system  $\Sigma$  behaviour is in fact relevant.

One of such properties is the matter of controllability. In [15] it is shown that the controllability of  $\Sigma$  is an exceptional property. In fact, assume that  $G$  is nilpotent and the accessibility set  $\mathcal{A}$  is an open set. It turns out that

$$\Sigma \text{ is controllable on } G \Leftrightarrow \text{Spec}_{Ly}(\mathcal{D}) \cap \mathbb{R} = \{0\}.$$

Here,  $\text{Spec}_{Ly}(\mathcal{D})$  denotes the Lyapunov spectrum of the derivation  $\mathcal{D}$ , it means, the sets of the real part of the  $\mathcal{D}$ -eigenvalues induced by the drift vector field  $\mathcal{X}$  of  $\Sigma$ . Recently, the authors in [4] proved that  $\text{Spec}_{Ly}(\mathcal{D}) \cap \mathbb{R} = \{0\}$  implies controllability for any Lie group with the finite semisimple center property, (see Definition 5 below).

To understand the controllability behaviour of restricted linear systems on Lie groups, we need to approach the problem in a more realistic way. We turn our attention to the maximal subsets of  $G$  where approximate controllability of the system holds, means its control sets. Like in the classical linear system in this paper we characterize the control set with nonempty interior of  $\Sigma$  around the identity. As we were expected, topological properties of  $\mathcal{C}$  are intrinsically connected with the eigenvalues of the derivation  $\mathcal{D}$  associated to

the drift  $\mathcal{X}$ . Assume the reachable set  $\mathcal{A}$  is open. Since  $0 \in \text{int}\Omega$ , Corollary 4.5.11 of [14] assures the existence of a control set  $\mathcal{C}$  of  $\Sigma$  that contains the identity element  $e \in G$  in its interior. Thus, our aim here is twofold. First, to study in which cases  $\mathcal{C}$  is in fact the only control set with nonempty interior. Second, to relate topological properties of this control sets with the spectrum of  $\mathcal{D}$ .

The following notion is important for our analysis.

**Definition 1.5.** *A connected Lie group  $G$  has a finite semisimple center property if all semisimple Lie subgroups of  $G$  have finite center.*

Of course, any solvable group and any semisimple Lie group with finite center, like  $\mathfrak{sl}(n, \mathbb{R})$ , has the finite semisimple center property. But also the direct or semidirect product between groups with finite semisimple center have the same property.

From now we assume that our space state  $G$  is a connected finite semisimple center Lie group. It turns out the following fact, [15].

**Proposition 1.3.** *If  $\mathcal{A}$  is open, therefore*

$$G^{+,0} \subset \mathcal{A} \text{ and } G^{-,0} \subset \mathcal{A}^*.$$

Thus, in the sequel we also assume that the reachable set  $\mathcal{A}$  is an open set.

### 1.2.1 Main results

In the classical Euclidean linear control systems it is well known that under the Kalman rank condition there exists just one control set with non empty interior. We show the same result for  $\Sigma$  on some particular cases.

The main result of this section reads as follows,

**Theorem 3.11 of Chapter 2.**  *$\mathcal{C}$  is the only control set of  $\Sigma$  whose interior intersects*

$$G^{+,0}G^- \text{ and } G^{-,0}G^+.$$

As a consequence, we obtain

**Corollary 3.12 of Chapter 2.** *If  $G$  is decomposable, then  $\mathcal{C}$  is the only control set.*

Furthermore, we also obtain

**Corollary 3.13 of Chapter 2.** *Since any solvable Lie group is decomposable*

$$\mathcal{C} = \text{cl}(\mathcal{A}) \cap \mathcal{A}^*$$

*is the only control set with non empty interior of  $\Sigma$ .*

We prove

**Proposition 3.3 of Chapter 2.** *If  $\mathcal{D}$  is inner and  $G^0$  is compact, then  $G = G^0$ . Moreover, if  $G^0$  is compact it follows that  $G$  decomposable.*

and the following result

**Corollary 3.7 of Chapter 2.** *Assume  $\mathcal{C}$  is invariant. Then,  $\mathcal{C}$  is the unique invariant control set.*

If  $G$  is decomposable the reachable and controllable sets have also such kind of decompositions.

**Lemma 3.5 of Chapter 2.** *If  $G$  is decomposable*

$$\mathcal{A} = \mathcal{A}_{G^-} G^{+,0} \quad \text{and} \quad \mathcal{A}^* = \mathcal{A}_{G^+}^* G^{-,0}, \text{ where } \mathcal{A}_{G^-} = \mathcal{A} \cap G^- \text{ and } \mathcal{A}_{G^+}^* = \mathcal{A}^* \cap G^+.$$

The next result yields information about the relationship between topological properties of control sets and the spectrum of  $\mathcal{D}$ .

**Theorem 3.6 of Chapter 2.** *For the control set  $\mathcal{C}$  containing the identity holds:*

1.  $\mathcal{C}$  is closed if and only if  $\mathcal{A}^* = G$ ,
2.  $\mathcal{C}$  is open if and only if  $\mathcal{A} = G$ ,
3. If  $G$  is nilpotent:
  - i)  $\mathcal{C}$  is closed if and only if  $\mathcal{D}$  has only eigenvalues with nonpositive real part,
  - ii)  $\mathcal{C}$  is open if and only if  $\mathcal{D}$  has only eigenvalues with nonnegative real part,
  - iii)  $\mathcal{C} = G$  if and only if  $\mathcal{D}$  has only eigenvalues with zero real part.

Here, we search for conditions to obtain a bounded control set  $\mathcal{C}$ . As before, we assume that  $G$  has the finite semisimple center property and  $\mathcal{A}$  is open. First, we prove

**Theorem 3.9 of Chapter 2.** *Suppose that  $G$  is semisimple or nilpotent. If  $\text{cl}(\mathcal{A}_{G^-})$ ,  $\text{cl}(\mathcal{A}_{G^+}^*)$  and  $G^0$  are compact subsets of  $G$  then  $\mathcal{C}$  is bounded.*

As a direct consequence we get:

**Corollary 3.10 of Chapter 2.** *Let  $G$  be a nilpotent simply connected Lie group. Then  $\mathcal{C}$  is bounded if, and only if,  $\text{cl}(\mathcal{A}_{G^-})$  and  $\text{cl}(\mathcal{A}_{G^+}^*)$  are compact subsets of  $G$  and  $\mathcal{D}$  is hyperbolic.*

## 1.3 Control set of linear systems on semi-simple Lie groups

Let  $\Sigma$  be a linear control system on a non-compact semi-simple Lie group  $G$ . This section is devoted to the analyze of the control sets with nonempty interior of  $\Sigma$ . As before, the following assumptions will be needed throughout the paper:  $G$  has finite center and the reachable set  $\mathcal{A}$  from the identity is open.

We relate the decomposition of the Lie algebra  $\mathfrak{g}$  through the  $\mathcal{D}$ -eigenvalues induced by the drift vector field  $\mathcal{X}$  with relevant elements of the semi-simple Lie theory. As a references we suggest the books [18], [24] and [25].

Since the group is semi-simple any derivation  $\mathcal{D}$  is inner. Thus, there exists  $X \in \mathfrak{g}$  such that  $\mathcal{D} = ad(X)$ . In particular, its flow  $(\varphi_t)_{t \in \mathbb{R}}$  is given by conjugation as follows

$$\varphi_t(g) = C_{e^{tX}}(g) = e^{-tX} g e^{tX}.$$

The Jordan decomposition of  $X$  is given by  $X = E + H + N$  with  $E, H, N \in \mathfrak{g}$  and such that

$$[X, H] = [X, E] = [X, N] = [H, E] = [H, N] = [E, N] = 0.$$

Moreover, the derivation  $ad_E$  has only pure complex eigenvalue,  $ad_H$  is diagonal with real eigenvalues and  $ad_N$  is nilpotent.

As a direct consequence, the eigenvalues of  $\mathcal{D}$  coincides with the eigenvalues of  $ad_H + ad_E$ . Thus, the  $\mathfrak{g}$ -decomposition depends just on the  $ad_H$ -eigenvalues. Consequently, the subalgebras  $\mathfrak{g}^+$ ,  $\mathfrak{g}^-$  and  $\mathfrak{g}^0$  related to  $ad_H$  are as follows:

$$\mathfrak{g}^+ = \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha, \quad \mathfrak{g}^- = \bigoplus_{\alpha < 0} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{g}^0 = \ker(ad_H) = \mathfrak{z}_H$$

where  $\alpha \in \text{Spec}_{Ly}(ad_H)$ .

### 1.3.1 Main Results

Different from the solvable case discussed in Chapter 3, in the semi-simple case we do not know if there exists only one control set with nonempty interior. This follows from the fact that in the semi-simple case, the subgroup with Lie algebra  $\mathfrak{g}^0$  is not necessarily connected and  $G^0$  is only its

connected component. In fact, in the solvable case the whole group coincide with  $G^{+0}G^-$ , in the semi-simple case such subset is just one component of an open and dense subset of  $G$ . One could expect then, that the number of control sets is at maximum equal to the number of connected components of such subset and it is in fact true for some particular cases as we will see.

Even though uniqueness cannot be ensured, we show that any control set with nonempty interior is related with the control set  $\mathcal{C}_e$  which contains the identity element by a right translation. This is the core of the main result:

**Theorem 3.2 of Chapter 3.** *Let  $\mathcal{P}$  be a control set with nonempty interior of the linear control system  $\Sigma$ , then*

$$\mathcal{P} \subset R_l(\mathcal{C}_e), \text{ for some } l \in K_H$$

where  $K_H = K \cap Z_H$ .

It is possible to complement Theorem 3.2 as follows

**Corollary 3.5 of Chapter 3.** *Let  $\mathcal{P}$  be a control set with nonempty interior of  $\Sigma$ . The following assertions are equivalent*

- i)  $\mathcal{P} = R_l(\mathcal{C}_e)$ , for some  $l \in K_H$
- ii)  $(Z_H)_e \cdot l \subset \text{int}(\mathcal{P})$
- iii)  $(Z_H)_e \cdot l \cap \text{int}(\mathcal{P}) \neq \emptyset$ .

In this context,  $G^0 = (\exp \mathfrak{z}_H)_e = (Z_H)_e$ .

For the special case where the control set in question is invariant, the next statement warranty the uniqueness of control sets.

**Proposition 3.6 of Chapter 3.** *A linear control system admit at most one invariant control set.*

In some special case it is possible to improve Theorem 3.2. In fact, we have

**Theorem 3.8 of Chapter 3.** *Suppose  $\mathcal{D} = ad_E + ad_H$  is semi-simple. Hence, any control set  $\mathcal{P}$  with non empty interior satisfies*

$$\mathcal{P} = R_l(\mathcal{C}_e) \text{ for some } l \in K_H.$$

The above result shows that in this particular case, all the possible control sets with nonempty interior of  $\Sigma$  are just diffeomorphic copies of the control set that contains the identity. In particular they all have the same topological properties and it is possible to give an upper bound for their quantity based on the cardinality of the connected component of the subgroup  $K_H$ .

**Corollary 3.9 of Chapter 3.** *For any linear control system  $\Sigma$  there are at most  $\left| \frac{K_H}{(K_H)_e} \right|$  control sets with non empty interior.*

## 1.4 Isometries of almost-Riemannian structures on Lie groups

The fourth chapter is devoted to isometries of almost-Riemannian structures on Lie groups. The purpose is to classify these structures, to find geometric invariants, and to determine their groups of isometries.

An almost-Riemannian structure (ARS in short) on an  $n$ -dimensional differential manifold can be defined, at least locally, by a set of  $n$  vector fields, considered as an orthonormal frame, that degenerates on some singular set. This geometry goes back to [17], [26] and more recently to [2], [3], [10], [11], [12], [13].

On an  $n$ -dimensional connected Lie group the simplest ARSs are defined by a set of  $n - 1$  left-invariant vector fields and one linear vector field, the rank of which is equal to  $n$  on a proper open and dense subset and that satisfy the rank condition.

These ARSs, among which we find the famous Grushin plane on the Abelian Lie group  $\mathbb{R}^2$ , has been studied in [7]. Among the results of this paper there is a study of the singular locus, that is the set of points where the vector fields fail to be linearly independent. It is an analytic set, but not a subgroup, not even a submanifold, in general. However sufficient conditions for the singular locus to be a submanifold or a subgroup were exhibited in [7] and are recalled in Section 1.1. This locus is very important in what concern the structure of ARSs, in particular in view of a classification. Another important geometric locus is the set of singularities of the linear field. It is always a subgroup.

We deal here with smooth isometries, i.e. diffeomorphisms that respect the Euclidean metric of the tangent space at each point. First of all we show that such an isometry should preserve the singular locus and the group of singularities of the linear field. The main consequence is that the group of isometries of an ARS does not act transitively on  $G$ . Another consequence is that a left translation  $L_g$  is an isometry if and only if  $g$  belongs to the set of singularities of the linear field.

Then we prove that the isometries preserve the left-invariant distribution generated by the  $n - 1$  left-invariant vector fields, and also the linear field (up to the sign), see Theorem 2 of Chapter 4.

These constraints are rather strong. Consider for instance ARSs on the

Heisenberg group, in the case where the distribution generated by the left-invariant vector fields is a subalgebra. Then the group of isometries is generically reduced to the identity.

### 1.4.1 General Theorems about Isometries of ARSs

In this subsection  $(\Sigma)$  and  $(\Sigma')$  stand for two ARSs on the same Lie group  $G$ . An object related to  $(\Sigma')$  will be denoted by the same symbol as the analogous object related to  $(\Sigma)$  but with a prime, that is,  $D'$ ,  $(\varphi'_t)_{t \in \mathbb{R}}$ , and so on.

#### *Translations*

The purpose of this part is to characterize the left translations that are isometries and to show that there exists an isometry from  $(\Sigma)$  onto  $(\Sigma')$  if and only if there exists one such isometry that preserves the identity.

The next proposition proves that the singular locus  $\mathcal{Z}$  and the set singularities of  $\mathcal{Z}_{\mathcal{X}}$  are invariant by isometries.

**Proposition 1 of Chapter 4.** *Let  $\Phi$  be an isometry from  $(\Sigma)$  onto  $(\Sigma')$ . Then  $\Phi$  sends the singular locus of  $(\Sigma)$  onto the one of  $(\Sigma')$  and the set of fixed points of  $\mathcal{X}$  onto the one of  $\mathcal{X}'$ , that is:*

$$\Phi(\mathcal{Z}) = \mathcal{Z}' \quad \text{and} \quad \Phi(\mathcal{Z}_{\mathcal{X}}) = \mathcal{Z}'_{\mathcal{X}'}$$

*In particular  $\mathcal{X}'_{\Phi(e)} = 0$ .*

In difference with classical sub-Riemannian geometry on Lie groups, the left translations are not always isometries.

**Proposition 2 of Chapter 4.** *Let  $g \in G$ . The left translation  $L_g$  is an isometry of  $(\Sigma)$  if and only if  $g$  belongs to the set  $\mathcal{Z}_{\mathcal{X}}$  of fixed points of  $\mathcal{X}$ .*

Thanks to these two propositions we proved the following theorem.

**Theorem 1 of Chapter 4.** *The ARSs  $(\Sigma)$  and  $(\Sigma')$  are isometric if and only if there exists an isometry  $\Phi$  from  $(\Sigma)$  onto  $(\Sigma')$  such that  $\Phi(e) = e$ .*

#### *Preservation of the distribution*

We recall the following notations: if  $Y \in T_e G$  and  $g \in G$  then  $Y_g$  stands for  $TL_g.Y \in T_g G$ .

The key point of Theorem 2 of Chapter 4 is the subsequent lemma that characterizes the elements of  $\Delta_g$  (when  $g$  does not belong to the singular locus): they are the vectors whose norm is invariant under left translations.

**Lemma 1 of Chapter 4.** *Let  $\Sigma$  be an ARS on the Lie group  $G$ . There exists an open and dense subset of  $G \setminus \mathcal{Z}$ , hence of  $G$ , of points  $g$  that satisfy:*

$$\Delta_g = \{Y_g : \|Y_h\|_h = \|Y_g\|_g \text{ in a neighborhood of } g\}.$$

**Theorem 2 of Chapter 4.** *Let  $\Phi$  be an isometry from  $(\Sigma)$  onto  $(\Sigma')$  that preserves the identity. Then:*

1. *Its tangent mapping  $\Phi_*$  sends  $\Delta$  on  $\Delta'$ , that is,  $T_g\Phi.\Delta_g = \Delta'_{\Phi(g)}$  for all  $g \in G$ .*
2. *Either  $\Phi_*\mathcal{X} = \mathcal{X}'$ , and  $T_e\Phi \circ D = D' \circ T_e\Phi$   
or  $\Phi_*\mathcal{X} = -\mathcal{X}'$ , and  $T_e\Phi \circ D = -D' \circ T_e\Phi$ .*

Since the change of  $\mathcal{X}$  to  $-\mathcal{X}$  do not change the ARS, we can always assume that  $\Phi_*\mathcal{X} = \mathcal{X}'$ .

### ***The tangent mapping of an isometry***

The next theorem shows that isometries that preserve the identity  $e$  are completely determined by their differential at  $e$ .

**Theorem 3 of Chapter 4.** *Let  $\Phi_1$  and  $\Phi_2$  be two isometries from  $(\Sigma)$  onto  $(\Sigma')$  that preserve the identity.*

*If  $T_e\Phi_1 = T_e\Phi_2$  then  $\Phi_1 = \Phi_2$ .*

It is of course sufficient to prove that  $\Phi = \Phi_1^{-1} \circ \Phi_2$  is the identity map. The idea of the proof is to show that  $\Phi$  preserves the normal Hamiltonian, and to deduce that it sends normal geodesics onto normal geodesics.

### ***Nilpotent groups***

Kivioja and Le Donne proved in [20] that the isometries of left-invariant metrics on nilpotent groups are affine. Their result cannot be directly applied here since the metric is not left-invariant. It can however be adapted to prove the following result:

**Theorem 4 of Chapter 4.** *If the group  $G$  is nilpotent then the isometries of ARSs of  $G$  that preserve the identity are automorphisms.*

The proof consists in showing that the images of the left-invariant vector fields of the distribution are left-invariant. It is more or less straightforward when  $\Delta$  is not a subalgebra, because in that case there is an underlying left-invariant sub-Riemannian metric, and more complicated when  $\Delta$  is a subalgebra. The conclusion comes from the two following lemmas that have their own interest.

**Lemma 2 of Chapter 4.** *Let  $G$  be a connected Lie group and  $\mathfrak{g}$  its Lie algebra, identified with the set of left-invariant vector fields.*

*Let  $\Phi$  be a diffeomorphism of  $G$  that verifies  $\Phi(e) = e$  and  $\Phi_*Y \in \mathfrak{g}$  for all  $Y \in \mathfrak{g}$ . Then  $\Phi$  is an automorphism.*

**Lemma 3 of Chapter 4.** *Let  $\Delta$  be a subspace of  $\mathfrak{g}$  and  $\mathcal{X}$  be a linear vector field such that the Lie algebra generated by  $\Delta$  and  $\mathcal{X}$  be equal to  $\mathfrak{g} \oplus \mathbb{R}\mathcal{X}$ .*

*If  $\Phi$  is a diffeomorphism of  $G$  such that  $\Phi_*Y \in \mathfrak{g}$  for all  $Y \in \Delta$  and such that  $\Phi_*\mathcal{X}$  is a linear vector field, then  $\Phi$  is an automorphism.*

Theorem 4 of Chapter 4 is no longer true in general if the group is not nilpotent. Counter-examples can be easily built with the help of the Milnor example of the rototranslation group (see [23]) recalled in [20].

**Counter-Example.** The rototranslation group is the universal covering of the group of orientation-preserving isometries of the Euclidean plane. It can be described as  $\mathbb{R}^3$  with the product:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos z & -\sin z & 0 \\ \sin z & \cos z & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}.$$

The Euclidean metric is left-invariant for this product. It can be shown that the group of automorphisms of this group that are isometries is one-dimensional though the group of isometries that preserve the identity is 3-dimensional, which implies that not all isometries are affine.

Let us call this group  $\mathcal{R}$  and let us define an ARS on  $G = \mathcal{R} \times \mathbb{R}^2$  in the following way: the structure of  $\mathcal{R}$  is the previous one, and the one of  $\mathbb{R}^2$  is the Grushin plane. Any diffeomorphism  $\Phi$  that preserves the identity, made of the direct product of an isometry of  $\mathcal{R}$  and an isometry of the Grushin plane is an isometry of this ARS. However if the isometry of  $\mathcal{R}$  is not an automorphism,  $\Phi$  cannot be an automorphism of  $G$ .

Here  $G$  is a solvable, not nilpotent group.

Many different counter-examples can be build, for instance by replacing the Grushin plane by one of the ARSs defined on the Heisenberg group.

**Conclusion**

The results of this subsection show that an isometry that preserves the identity also preserves the left-invariant distribution, the linear field and is characterized by its tangent mapping at the identity. Moreover that last preserves the derivation.

We are consequently interested in diffeomorphisms  $\Phi$  of  $G$  that satisfy:

- (i)  $\Phi(e) = e$
- (ii)  $\Phi_*\Delta = \Delta'$
- (iii)  $\Phi_*\mathcal{X} = \mathcal{X}'$
- (iv)  $T_e\Phi \circ D = D' \circ T_e\Phi$

Since the isometries that preserve the identity are completely determined by their tangent maps at the origin we will first look for invertible linear maps  $P$  on  $T_eG$  that satisfies  $P \circ D = D' \circ P$  and  $P(\Delta_e) = \Delta'_e$ .

If the Lie group  $G$  is simply connected and if such a  $P$  is an automorphism of  $\mathfrak{g}$ , it is the tangent mapping of an automorphism  $\Phi$  of  $G$ . It is easy to see that this automorphism is an isometry. Indeed it transforms any left-invariant vector field into a left-invariant vector field. Since  $\Phi$  satisfies  $P(\Delta_e) = \Delta'_e$ , it satisfies  $\Phi_*\Delta = \Delta'$ . On the other hand it transforms any linear vector field into a linear vector field. Since it satisfies  $P \circ D = D' \circ P$  we have  $\Phi_*\mathcal{X} = \mathcal{X}'$ . So, it preserves the orthonormal frame. Thus  $\Phi$  is an isometry.

If either  $P$  is not an automorphism of  $\mathfrak{g}$  or the Lie group is not simply connected we cannot conclude so easily, and we have to look in each case to the existence of an isometry  $\Phi$  such that  $T_e\Phi = P$ .

In the article we apply the previous conclusion to classify the ARS in two dimensional nonabelian Lie group and three dimensional Heisenberg group.

**1.4.2 Classification of the ARSs on the affine group**

Let  $G$  be the connected 2-dimensional affine group:

$$G = \text{Aff}_+(2) = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}; (x, y) \in \mathbb{R}_+^* \times \mathbb{R} \right\}.$$

Its Lie algebra is solvable and generated by the left-invariant vector fields:

$$gX = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad gY = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \quad \text{where} \quad g = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}.$$

In natural coordinates they write  $X(x, y) = x \frac{\partial}{\partial x}$  and  $Y(x, y) = x \frac{\partial}{\partial y}$ . They verify  $[X, Y] = XY - YX = Y$ .

In the basis  $(X, Y)$ , all derivations  $D$  of the Lie algebra  $\mathfrak{aff}(2)$  have the form

$$D = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}, \quad \text{where } a, b \in \mathbb{R},$$

and the linear vector field  $\mathcal{X}$  associated to such a derivation is

$$\mathcal{X}(g) = \begin{pmatrix} 0 & a(x-1) + by \\ 0 & 0 \end{pmatrix}.$$

In natural coordinates it writes  $\mathcal{X}(x, y) = (a(x-1) + by) \frac{\partial}{\partial y}$ . For more details, see [16].

An ARS on  $\text{Aff}_+(2)$  is defined by a left-invariant vector field  $B = \alpha X + \beta Y$  and a derivation  $D$  such that  $B$  and  $DB$  are linearly independent, in order to satisfy the rank condition. In natural coordinates, the ARS is described as the system

$$\begin{cases} \dot{x} = u\alpha x \\ \dot{y} = v(a(x-1) + by) + u\beta x \end{cases}$$

Though the 2D affine group is not nilpotent, we have the following result:

**Proposition 3 of Chapter 4.** *Let  $\Sigma = (\mathcal{X}, B)$ ,  $\Sigma' = (\mathcal{X}', B')$  be two ARSs on  $\text{Aff}_+(2)$ . If  $\Phi$  is an isometry between  $\Sigma$  and  $\Sigma'$  and  $\Phi(e) = e$  then  $\Phi$  is an automorphism.*

### **Classification by isometries**

This classification is simplified by the facts that isometries fixing the identity are Lie group automorphisms, and that  $\text{Aff}_+(2)$  is simply connected.

**Proposition 4 of Chapter 4.** *Any almost-Riemannian structure on the Lie group  $\text{Aff}_+(2)$  is isometric to one and only one of the structures defined by*

$$D = \begin{pmatrix} 0 & 0 \\ 1 & b \end{pmatrix} \quad \text{and} \quad B = \alpha X,$$

where  $\alpha > 0$  and  $b \geq 0$ .

**Remark:** The singular locus  $\mathcal{Z} = \mathcal{Z}_X$  is a Lie subgroup of  $\text{Aff}_+(2)$ . As we will see it characterizes the ARSs. We have also:

**Proposition 5 of Chapter 4.** *The group of isometries of an ARS on  $\text{Aff}_+(2)$  is the group of left translations by elements of  $\mathcal{Z}_X$ .*

**Global rescaling**

A global rescaling obviously does not change the geometry of an ARS. This allows us to normalize  $\alpha$  to 1.

Therefore any ARS is up to a rescaling isometric to an ARS defined by

$$B = X \text{ and } D = \begin{pmatrix} 0 & 0 \\ 1 & b \end{pmatrix}, \quad \text{with } b \geq 0.$$

These models are completely characterized by their singular locus

$$\mathcal{Z} = \mathcal{Z}_X = \{(x, y) : (x - 1) + by = 0\}.$$

It is always a subgroup but it is normal if and only if  $b = 0$ , that is when the singular locus is the vertical line  $\mathcal{Z} = \mathcal{Z}_X = \{x = 1\}$ .

There are consequently two main models, obtained respectively for  $b = 0$  and for  $b > 0$ .

The first one has been completely studied in [7].

The other one remains to be analyzed.

### 1.4.3 Classification of the ARSs on the Heisenberg group

Let  $G$  be the 3-dimensional Heisenberg group:

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}; (x, y, z) \in \mathbb{R}^3 \right\}.$$

Its Lie algebra  $\mathfrak{g}$  is nilpotent and generated by the left-invariant vector fields:

$$gX = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, gY = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, gZ = \begin{pmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ where } g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

They verify  $[X, Y] = Z$  and the other brackets vanish. In the basis  $(X, Y, Z)$ , all derivations  $D$  of  $\mathfrak{g}$  have the form

$$D = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & a+d \end{pmatrix}, \text{ where } a, b, c, d, e, f \in \mathbb{R},$$

and the associated linear vector field  $\mathcal{X}$  is:

$$\mathcal{X}(x, y, z) = (ax+by)\frac{\partial}{\partial x} + (cx+dy)\frac{\partial}{\partial y} + (ex+fy+(a+d)z + \frac{1}{2}cx^2 + \frac{1}{2}by^2)\frac{\partial}{\partial z}.$$

For more details, see [16].

An ARS on  $G$  is defined by an orthonormal frame  $\{B_1, B_2, \mathcal{X}\}$ , where  $B_1, B_2$  are left-invariant vector fields and  $\mathcal{X}$  is a linear one with associated derivation  $D$ .

To classify the ARSs by isometries, note that Theorem 4 of Chapter 4 states that all isometries that fix the identity are Lie group automorphisms. Since the Heisenberg group is simply connected, is enough to work with Lie algebra automorphisms.

As we will see, there are two very different cases according to whether  $\Delta$  is a subalgebra or not.

#### 1.4.3.1. $\Delta$ is a subalgebra

##### *Classification by isometries*

**Proposition 6 of Chapter 4.** *Any almost-Riemannian structure on  $G$ , whose distribution  $\Delta$  is a subalgebra, is isometric to an almost-Riemannian structure whose orthonormal frame is  $\{X, Z, \mathcal{X}\}$  and the derivation  $D$  has the following form:*

$$D = \begin{pmatrix} 0 & b & 0 \\ c & d & 0 \\ 0 & f & d \end{pmatrix},$$

for some  $c > 0$  and  $d, f \geq 0$ . Moreover, two different ARSs of this form are not isometric.

The complete description of the groups of isometries is a bit technical and can be found in chapter 4. Only the generic result is stated in the next proposition.

**Proposition 7 of Chapter 4.** *Consider an ARS in the form of Proposition 6 of Chapter 4.*

*If  $d \neq 0$   $f \neq 0$  its group of isometries is composed only of left translations by elements of  $\mathcal{Z}_{\mathcal{X}}$ .*

*Moreover, generically  $b, d$  and  $f$  are nonzero and the set  $\mathcal{Z}_{\mathcal{X}}$  is reduced to the identity.*

*Thus in the subalgebra case the group of isometries is generically reduced to the identity.*

### **Global rescaling**

We do not change the geometry of the ARS if we multiply all the vector fields by a common positive constant  $\lambda$ . This global rescaling allows us to normalize  $c$  to 1.

Therefore, any ARS is isometric up to a rescale to one and only one ARS defined by the orthonormal frame  $\{X, Z, \mathcal{X}\}$  where the associated derivation  $D$  is equal to:

$$D = \begin{pmatrix} 0 & b & 0 \\ 1 & d & 0 \\ 0 & f & d \end{pmatrix}$$

with  $d, f \geq 0$ .

The singular locus is here the plane  $\mathcal{Z} = \{x + dy = 0\}$ . It is a subgroup.

#### **1.4.3.2. $\Delta$ is not a subalgebra**

In this case the distribution  $\Delta$  generates the Lie algebra. So, to satisfy the rank condition there is no restriction on the derivation  $D$ . Consequently the only restriction on  $D$  is due to the fact that  $\mathcal{X}$  and  $\Delta$  need be linearly independent in an open and dense set.

### **Classification by isometries**

**Proposition 8 of Chapter 4.** *Any ARS, whose distribution  $\Delta$  is not a subalgebra, is isometric to an ARS whose orthonormal frame is  $\{X, Y, \mathcal{X}\}$ , where the derivation  $D$  has the following form:*

$$D = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & f & a + d \end{pmatrix} \text{ with } c, f \geq 0.$$

*The condition that  $\mathcal{X}(g)$  does not belong to  $\Delta(g)$  everywhere reduces to: If  $b = c = f = 0$  then  $a + d \neq 0$ .*

As in the subalgebra case the description of the group of isometries of a given ARS is here restricted to the generic case. A complete description can be found in chapter 4.

**Proposition 9 of Chapter 4.** *In the nonalgebra case the group of isometries of an ARS is generically reduced to the set of left translations by elements of  $\mathcal{Z}_X$ .*

***Global rescaling***

As previously we can multiply all the vector fields by a common positive constant  $\lambda$  without modifying the geometry.

However there is no particular parameter of  $D$  to normalize in the non-algebra case:

If  $f \neq 0$  we change it to 1. If  $f = 0$  and  $c \neq 0$  we change it to 1. If  $c, f = 0$  and  $b \neq 0$  we change it to  $\pm 1$ . If  $b, c, f = 0$  and  $a + d$  is different from 0 we change it to 1. The singular locus is here

$$\mathcal{Z} = \left\{ fy + (a + d)z - \frac{1}{2}cx^2 + \frac{1}{2}by^2 - dxy = 0 \right\}.$$

It is not a submanifold in general.

***Final remark***

The classification of ARSs of the Heisenberg group can be refined if we forget the left-invariant metric on  $\Delta$ , that is if we consider as equivalent two ARSs that differ only by the left-invariant metric. This is done in Chapter 4.

## Chapter 2

# Control sets of linear control systems on Lie groups



# Control sets of linear systems on Lie groups

Víctor Ayala, Adriano Da Silva and Guilherme Zsigmond

**Abstract.** Like in the classical linear Euclidean system, we would like to characterize for a linear control system on a connected Lie group  $G$  its control set with nonempty interior that contains the identity of  $G$ . We show that many topological properties of this control set are intrinsically connected with the eigenvalues of a derivation associated to the drift of the system. In particular, we prove that if  $G$  is a decomposable Lie group there exists only one control set with nonempty interior for the whole linear system. Furthermore, for nilpotent Lie groups we characterize when this set is bounded.

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**Keywords.** Control sets, Linear systems, Lie groups.

## 1. Introduction

Throughout the paper  $G$  stands for a connected Lie group with Lie algebra  $\mathfrak{g}$  of dimension  $d$ . In [2] the authors introduced the class of linear system on  $G$ , determined by the family of differential equations

$$\dot{g}(t) = \mathcal{X}(g(t)) + \sum_{j=1}^m u_j(t) X^j(g(t)), \quad (\Sigma)$$

here the drift  $\mathcal{X}$  is a linear vector field, i.e., its associated flow  $(\varphi_t)_{t \in \mathbb{R}}$  is such that  $\varphi_t \in \text{Aut}(G)$  for all  $t \in \mathbb{R}$ . The vector fields  $X^j$  are right invariant and  $u \in \mathcal{U} \subset L^\infty(\mathbb{R}, \Omega \subset \mathbb{R}^m)$  is the class of admissible controls where  $\Omega \subset \mathbb{R}^m$  is a compact, convex subset with  $0 \in \text{int}\Omega$ .

It is very well known that the classical linear system on the Euclidean space  $\mathbb{R}^d$  is one of the most relevant control systems and it can be written as

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^m u_j(t) b^j, \quad b^j \in \mathbb{R}^d \text{ and } u \in \mathcal{U}.$$

Here  $A \in \mathfrak{gl}(d, \mathbb{R})$ , the Lie algebra of the real matrices of order  $d$ . Since  $\mathbb{R}^d$  is a commutative Lie group, any constant vector  $b^j$  is an invariant vector field. Moreover,  $e^{tA} \in GL^+(d, \mathbb{R}) = \text{Aut}(\mathbb{R}^d)$ , showing that the linear system

$\Sigma$  is a generalization of the classical linear Euclidean system to an arbitrary connected Lie group  $G$ .

In [9] the author shows that the class of linear systems on Lie groups has also its importance on applications. It is shown that any affine control system on a connected manifold  $M$ , which dynamic generate a finite Lie algebra, is diffeomorphic to a linear control system on a Lie group or on a homogeneous space, showing that the understanding of the behavior of the system  $\Sigma$  is in fact very important.

A relevant property of any control system is the matter of controllability, which means that given any two points on  $G$  it is possible to connect each other through a solution of the system in positive time. In [4] it is shown that the controllability of the linear system  $\Sigma$  is really an exceptional property and it is intrinsically connected with the  $\mathfrak{g}$ -derivation  $\mathcal{D}$  associated with the linear vector field  $\mathcal{X}$ . In fact, assume that  $G$  is nilpotent and the accessibility set from the identity element of  $G$  is open. It turns out that

$$\Sigma \text{ is controllable on } G \Leftrightarrow \text{Spec}(\mathcal{D}) \cap \mathbb{R} = \{0\}.$$

Furthermore, recently the authors in [6] proved that  $\text{Spec}(\mathcal{D}) \cap \mathbb{R} = \{0\}$  implies controllability for any Lie group with finite semisimple center. That is, for any Lie group that admits a maximal semisimple Lie subgroup with finite center.

To understand the controllability behavior of linear systems on Lie groups, we need to approach the problem in a more realistic way. We turn our attention to the maximal subsets of  $G$  where controllability of the system holds, means, the control sets.

Like in the classical linear system in this paper we characterize the control set with nonempty interior of  $\Sigma$  that contains the identity of  $G$ . As expected, many topological properties of such control set are intrinsically connected with the eigenvalues of the derivation  $\mathcal{D}$ .

In particular, we prove that if  $G$  is decomposable (see Definition 3.2 below) there is exactly one control set with nonempty interior for the linear system  $\Sigma$ . Furthermore, for nilpotent Lie groups we give a necessary and sufficient criterion to determine when  $\mathcal{C}$  is bounded.

The paper is structured as follows: Sect. 2 introduces the general notion of control systems and their control sets on an arbitrary differentiable manifold. We state here basic properties of control sets with nonempty interior. Furthermore, we define linear vector field and linear systems. Associated with any  $\mathfrak{g}$ -derivation there are several Lie algebras and their corresponding Lie groups, connected with the reachable and controllable sets of the system. We take care of this decomposition here. In Sect. 3 we analyze the control sets of  $\Sigma$ . By a general result from [3], it follows that around the identity of  $G$  there exists one of these possible sets. Then we focus our attention on its properties. In particular, we establish necessary and sufficient conditions to decide whenever this set is invariant in positive or negative time. On the other hand, we analyze under which circumstances the control set is the whole  $G$  and when it is bounded. At the end of Sect. 3 we prove our main result. We show that the control set around the identity is the only control set of  $\Sigma$  when the Lie group

$G$  can be decomposable as the product of the subgroups associated with the real parts of the eigenvalues of  $\mathcal{D}$ . At the end of the Sect. 3 we make some comments on further works concerning control sets of linear systems in a more general setup.

## 2. Preliminaries

In this Section we state the definitions and main results concerning to control system, control sets, linear vector field the associated subalgebras and the corresponding subgroups. For more on the subjects the reader could consult [1–4, 7, 8] and [9].

### 2.1. Control systems and its control sets

Let  $M$  be a  $d$  dimensional smooth manifold. By a *control system* in  $M$  we understand the family of ordinary differential equations

$$\dot{x}(t) = f(x(t), u(t)), \quad u \in \mathcal{U}, \quad (1)$$

where  $f : M \times \mathbb{R}^m \rightarrow TM$  is a  $C^\infty$ -map and  $\mathcal{U} \subset L^\infty(\mathbb{R}, \Omega \subset \mathbb{R}^m)$  is the class of restricted admissible control functions. The set  $\Omega \subset \mathbb{R}^m$  is a compact and convex set with  $0 \in \text{int}\Omega$  called the *control range* of the system.

For each control function  $u \in \mathcal{U}$  and each initial value  $x \in M$  there exists an unique solution  $\phi(t, x, u)$  defined on an open interval containing  $t = 0$ , satisfying  $\phi(0, x, u) = x$ . Note that in general  $\phi(t, x, u)$  is only a solution in the sense of Carathéodory, i.e., a locally absolutely continuous curve satisfying the corresponding differential equation almost everywhere. In our context we can assume without lost of generality that any solution can be extended to the whole real line. Hence, we obtain a mapping

$$\phi : \mathbb{R} \times M \times \mathcal{U} \rightarrow M, \quad (t, x, u) \mapsto \phi(t, x, u),$$

satisfying the *cocycle property*

$$\phi(t + s, x, u) = \phi(t, \phi(s, x, u), \Theta_s u)$$

for all  $t, s \in \mathbb{R}$ ,  $x \in M$ ,  $u \in \mathcal{U}$ . Here, for any  $t \in \mathbb{R}$  the map  $\Theta_t$  is the *shift flow* on  $\mathcal{U}$  defined by

$$(\Theta_t u)(s) := u(t + s).$$

Instead of  $\phi(t, x, u)$  we usually write  $\phi_{t,u}(x)$ . Note that the smoothness of  $f$  implies the smoothness of  $\phi_{t,u}$ . Moreover, it follows directly from the cocycle property that

- (i) the inverse of the diffeomorphism  $\phi_{t,u}$  exists and it is given by  $\phi_{-t, \Theta_t u}$
- (ii) the fact that for any  $t > 0$ ,  $\phi_{t,u}(g)$  depends just on  $u|_{[0,t]}$  implies that

$$\phi_{t,u_1}(\phi_{s,u_2}(g)) = \phi_{t+s,u}(g)$$

where  $u \in \mathcal{U}$  is defined through concatenation by

$$u(\tau) = \begin{cases} u_1(\tau) & \text{for } \tau \in [0, s] \\ u_2(\tau - s) & \text{for } \tau \in [s, t + s]. \end{cases}$$

For any  $x \in M$  and  $\tau > 0$  the sets

$$\begin{aligned}\mathcal{A}_{\leq \tau}(x) &:= \{y \in M : \exists u \in \mathcal{U}, t \in [0, \tau] \text{ with } y = \phi_{t,u}(x)\} \\ \mathcal{A}_{\tau}(x) &:= \{y \in M : \exists u \in \mathcal{U}, y = \phi_{\tau,u}(x)\}\end{aligned}\quad (2)$$

$$\mathcal{A}(x) := \bigcup_{\tau > 0} \mathcal{A}_{\leq \tau}(x),$$

are the *set of reachable points from  $x$  up to time  $\tau$* , the *set of reachable points from  $x$  at time  $\tau$*  and the *reachable set of  $x$* , respectively. In the same way, for any  $\tau > 0$  the sets

$$\begin{aligned}\mathcal{A}_{\leq \tau}^*(x) &:= \{y \in M : \exists u \in \mathcal{U}, t \in [0, \tau]; \phi_{t,u}(y) = x\} \\ \mathcal{A}_{\tau}^*(x) &:= \{y \in M : \exists u \in \mathcal{U}, \phi_{\tau,u}(y) = x\}\end{aligned}\quad (3)$$

$$\mathcal{A}^*(x) := \bigcup_{\tau > 0} \mathcal{A}_{\leq \tau}^*(x),$$

are called the *set of controllable points to  $x$  within time  $\tau$* , the *set of controllable points to  $x$  in time  $\tau$*  and the *controllable set of  $x$* , respectively.

**Definition 2.1.** We say that the control system (1)

- (i) is *locally accessible at  $x$*  if for all  $\tau > 0$  the sets  $\mathcal{A}_{\leq \tau}(x)$  and  $\mathcal{A}_{\leq \tau}^*(x)$  have nonempty interior
- (ii) is *locally accessible* if it is locally accessible at every  $x \in M$
- (iii) satisfy the *Lie algebra rank condition (LARC)* if  $\mathcal{L}(x) = T_x M$  for any  $x \in M$ , where

$\mathcal{L}$  is the smallest Lie subalgebra containing any  $f(\cdot, u)$ ,  $u \in \Omega$ .

It is well known that the system is locally accessible at  $x \in M$  if it satisfies the Lie algebra rank condition at the point  $x$ .

On the other hand, a more convenient approach to understand the controllability behavior of a linear system comes from the notion of control set.

**Definition 2.2.** A nonempty set  $\mathcal{C} \subset M$  is called a *control set* of the control system (1) if it is

- (i) controlled invariant, that is, for every  $x \in M$  there exists  $u \in \mathcal{U}$  such that  $\phi(\mathbb{R}, x, u) \subset \mathcal{C}$ ;
- (ii) approximate controllable, that is,  $\mathcal{C} \subset \text{cl}(\mathcal{A}(x))$  for every  $x \in \mathcal{C}$ ;
- (iii) is maximal with properties (i) and (ii).

We also mention the notion of invariant control set as follows:

**Definition 2.3.** Let  $\mathcal{C}$  to be a control set of the control system (1). We say that  $\mathcal{C}$  is invariant in positive time if for any  $t > 0$  and  $u \in \mathcal{U}$  we have that  $\phi_{t,u}(\mathcal{C}) \subset \mathcal{C}$ . In the same way,  $\mathcal{C}$  is invariant in negative time if for any  $t > 0$  and  $u \in \mathcal{U}$  it holds that  $\phi_{-t,u}(\mathcal{C}) \subset \mathcal{C}$ .

The next result summarizes the main properties of control sets with nonempty interior ([3], Theorem 3.1.5).

**Theorem 2.4.** *Assume that the control system (1) is locally accessible and let  $\mathcal{C}$  be a control set with nonempty interior. It holds*

1.  $\mathcal{C}$  is connected and  $\text{cl}(\text{int } \mathcal{C}) = \text{cl}(\mathcal{C})$ ;
2.  $\text{int } \mathcal{C} \subset \mathcal{A}(x)$  for any  $x \in \mathcal{C}$ . For any  $y \in \text{int } \mathcal{C}$

$$\mathcal{C} = \text{cl}(\mathcal{A}(y)) \cap \mathcal{A}^*(y). \quad (4)$$

In particular the system is controllable on  $\text{int } \mathcal{C}$ ;

3. Assume that  $\phi_{t,u}(x)$  is a periodic trajectory, that is,  $\phi_{t+s,u}(x) = \phi_{t,u}(x)$  for some  $s > 0$  and all  $t \in \mathbb{R}$ . Therefore, if  $x \in \text{int } \mathcal{C}$  then  $\phi_{t,u}(x) \in \text{int } \mathcal{C}$  for all  $t \in \mathbb{R}$ ;
4.  $\mathcal{C}$  is closed  $\Leftrightarrow \mathcal{C}$  is invariant in positive time  $\Leftrightarrow \mathcal{C} = \text{cl}(\mathcal{A}(g))$  for any  $g \in \mathcal{C}$ ;
5.  $\mathcal{C}$  is open  $\Leftrightarrow \mathcal{C}$  is invariant in negative time  $\Leftrightarrow \mathcal{C} = \mathcal{A}^*(g)$  for any  $g \in \mathcal{C}$ .

## 2.2. Linear vector fields and decompositions

Let  $G$  be a connected Lie group with Lie algebra  $\mathfrak{g}$  and denote by  $e$  the identity element of  $G$ .

**Definition 2.5.** A vector field  $\mathcal{X}$  on  $G$  is said to be *linear* if its flow  $(\varphi_t)_{t \in \mathbb{R}}$  is a 1-parameter group of  $G$ -automorphisms.

Certainly, the vector field  $\mathcal{X}$  is complete. Furthermore, one can associate to  $\mathcal{X}$  a derivation  $\mathcal{D}$  of  $\mathfrak{g}$  defined by

$$\mathcal{D}Y = -[\mathcal{X}, Y](e), \text{ for all } Y \in \mathfrak{g}. \quad (5)$$

The relation between  $\varphi_t$  and  $\mathcal{D}$  is given by the formula

$$(d\varphi_t)_e = e^{t\mathcal{D}} \quad \text{for all } t \in \mathbb{R}. \quad (6)$$

Moreover, Eq. (6) implies that

$$\varphi_t(\exp Y) = \exp(e^{t\mathcal{D}}Y), \text{ for all } t \in \mathbb{R}, Y \in \mathfrak{g}.$$

On the other hand, if the group is simply connected any derivation  $\mathcal{D}$  has an associated linear vector field  $\mathcal{X} = \mathcal{X}^{\mathcal{D}}$  through the same formula above (see [2]).

Next, we explicitly some decomposition of the Lie algebra  $\mathfrak{g}$  induced by any given derivation  $\mathcal{D}$ . In order to do that, let us consider the generalized eigenspaces of  $\mathcal{D}$  given by

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : (\mathcal{D} - \alpha I)^n X = 0 \text{ for some } n \geq 1\}$$

where  $\alpha$  is an eigenvalue of  $\mathcal{D}$  and  $I$  stands for the identity map on  $\mathfrak{g}$ .

It turns out that  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$  when  $\alpha + \beta$  is an eigenvalue of  $\mathcal{D}$  and zero otherwise. This fact allow to us to decompose  $\mathfrak{g}$  as

$$\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-$$

where

$$\begin{aligned} \mathfrak{g}^+ &= \bigoplus_{\alpha: \operatorname{Re}(\alpha) > 0} \mathfrak{g}_\alpha, & \mathfrak{g}^0 &= \bigoplus_{\alpha: \operatorname{Re}(\alpha) = 0} \mathfrak{g}_\alpha \quad \text{and} \\ \mathfrak{g}^- &= \bigoplus_{\alpha: \operatorname{Re}(\alpha) < 0} \mathfrak{g}_\alpha. \end{aligned}$$

It is easy to see that  $\mathfrak{g}^+, \mathfrak{g}^0, \mathfrak{g}^-$  are Lie algebras and  $\mathfrak{g}^+, \mathfrak{g}^-$  are nilpotent ([11], Proposition 3.1).

At the Lie group level we will denote by  $G^+, G^-, G^0, G^{+,0}$ , and  $G^{-,0}$  the connected Lie subgroups of  $G$  with Lie algebras  $\mathfrak{g}^+, \mathfrak{g}^-, \mathfrak{g}^0, \mathfrak{g}^{+,0} := \mathfrak{g}^+ \oplus \mathfrak{g}^0$  and  $\mathfrak{g}^{-,0} := \mathfrak{g}^- \oplus \mathfrak{g}^0$  respectively.

### 2.3. Linear systems on Lie groups

A linear system on a Lie group  $G$  is determined by the family of ordinary differential equations

$$\dot{g}(t) = \mathcal{X}(g(t)) + \sum_{j=1}^m u_j(t) X^j(g(t)), \quad (\Sigma)$$

where the drift vector field  $\mathcal{X}$  is a linear vector field,  $X^j$  are right invariant vector fields and  $u = (u_1, \dots, u_m) \in \mathcal{U}$  as before.

For a given  $g \in G$ ,  $u \in \mathcal{U}$  and  $t \in \mathbb{R}$  the solution of the linear system  $\Sigma$  starting at  $g$  reads as

$$\phi_{t,u}(g) = \phi_{t,u} \varphi_t(g) = L_{\phi_{t,u}}(\varphi_t(g)), \quad (7)$$

where  $\phi_{t,u} = \phi_{t,u}(e)$  is the solution of  $\Sigma$  starting at the identity element  $e \in G$  and  $L_{\phi_{t,u}}$  is the left translation by  $\phi_{t,u}$  in  $G$  (see for instance [5]).

Let us denote by  $\mathcal{A}_{\leq \tau}$ ,  $\mathcal{A}_\tau$  and  $\mathcal{A}$  the sets  $\mathcal{A}_{\leq \tau}(e)$ ,  $\mathcal{A}_\tau(e)$  and  $\mathcal{A}(e)$ , respectively. For any  $u \in \mathcal{U}$  it follows from Eq. (7) that the solutions of the linear system  $\Sigma$  satisfy  $\phi_{t,u}^{-1} = \phi_{-t, \Theta_t u}$ . Therefore,

$$\mathcal{A}_\tau^* = \varphi_{-\tau}(\mathcal{A}_\tau^{-1}). \quad (8)$$

The next proposition states the main properties of the reachable sets of linear systems ([7], Proposition 2).

**Proposition 2.6.** *With the previous notations it holds:*

1.  $\tau > 0 \Rightarrow \mathcal{A}_\tau = \mathcal{A}_{\leq \tau}$
2.  $0 < \tau_1 \leq \tau_2 \Rightarrow \mathcal{A}_{\tau_1} \leq \mathcal{A}_{\tau_2}$
3.  $g \in G \Rightarrow \mathcal{A}_\tau(g) = \mathcal{A}_\tau \varphi_\tau(g)$
4.  $\tau_1, \tau_2 > 0 \Rightarrow \mathcal{A}_{\tau_1 + \tau_2} = \mathcal{A}_{\tau_1} \varphi_{\tau_1}(\mathcal{A}_{\tau_2}) = \mathcal{A}_{\tau_2} \varphi_{\tau_2}(\mathcal{A}_{\tau_1})$

The next result shows that the accessible set  $\mathcal{A}$  is invariant by right translations of elements whose  $\mathcal{X}$ -orbits are contained in  $\mathcal{A}$  ([4], Lemma 3.1).

**Lemma 2.7.** *Let  $g \in \mathcal{A}$  and assume that  $\varphi_t(g) \in \mathcal{A}$  for any  $t \in \mathbb{R}$ . Then  $\mathcal{A} \cdot g \subset \mathcal{A}$ .*

**Remark 2.8.** Using the above lemma it follows that

$$\text{if } \varphi_t(g) \in \operatorname{cl}(\mathcal{A}) \text{ for any } t \in \mathbb{R} \Rightarrow \operatorname{cl}(\mathcal{A}) \cdot g \subset \operatorname{cl}(\mathcal{A}).$$

In order to extend the controllability results in [4] from solvable groups to more general Lie groups, the authors in [6] introduced the following notion

**Definition 2.9.** Let  $G$  be a connected Lie group. We say that the Lie group  $G$  has finite semisimple center if all semisimple Lie subgroups of  $G$  have finite center.

For such class of Lie groups we have the following result ([6], Theorem 3.9).

**Proposition 2.10.** *Let  $G$  be a connected Lie group with finite semisimple center. If  $\mathcal{A}$  is open, then  $G^{+,0} \subset \mathcal{A}$  and  $G^{-,0} \subset \mathcal{A}^*$ .*

**Remark 2.11.** We should remark that for a group with a finite semisimple center property, the controllability sufficient condition in Theorem 3.9 of [6] reads as

$$e \in \text{int}\mathcal{A}_{\tau_0}, \text{ for some } \tau_0 > 0.$$

However, such condition is equivalent to  $\mathcal{A}$  being open since we are assuming that the control range is compact ([3], Lemma 4.5.2).

**Remark 2.12.** We should also notice that the condition on the openness of  $\mathcal{A}$  implies, in particular, that the system satisfies the Lie algebra rank condition ([2], Theorem 3.3).

The next result states the main controllability properties of linear systems on nilpotent Lie groups (see [4]).

**Theorem 2.13.** *If  $G$  is nilpotent and  $\mathcal{A}$  is open then:*

1.  $\mathcal{A} = G$  if and only if  $G = G^{+,0}$ ;
2.  $\mathcal{A}^* = G$  if and only if  $G = G^{-,0}$ ;
3.  $\mathcal{A} \cap \mathcal{A}^* = G$  if and only if  $G = G^0$ .

### 3. Control sets of a linear system

In this section we prove our main result. We start with a proposition which states the main properties of the subgroups obtained from the  $\mathcal{D}$ -decomposition. As before  $\mathcal{D}$  is the derivation associated with the linear vector field  $\mathcal{X}$  ([4], Proposition 2.9).

**Proposition 3.1.** *It holds:*

1.  $G^{+,0} = G^+G^0 = G^0G^+$  and  $G^{-,0} = G^-G^0 = G^0G^-$ ;
2.  $G^+ \cap G^- = G^{+,0} \cap G^- = G^{-,0} \cap G^+ = \{e\}$ ;
3.  $G^{+,0} \cap G^{-,0} = G^0$ ;
4. All the above subgroups are closed in  $G$ ;
5. If  $G$  is solvable then

$$G = G^{+,0}G^- = G^{-,0}G^+. \quad (9)$$

Moreover, the fixed points of  $\mathcal{X}$  are in  $G^0$ .

**Definition 3.2.** We say that  $G$  is decomposable (by  $\mathcal{D}$ ) if (9) holds.

Next, we obtain some result when the derivation  $\mathcal{D}$  is inner, which means that  $\mathcal{D}$  is completely determined by an element of the Lie algebra  $\mathfrak{g}$ .

**Proposition 3.3.** *If  $\mathcal{D}$  is inner and  $G^0$  is compact, then  $G = G^0$ . Furthermore,  $G$  is decomposable if  $G^0$  is a compact subgroup.*

*Proof.* Let us assume  $\mathcal{D} = \text{ad}(X)$  for some  $X \in \mathfrak{g}$ . Certainly  $X \in \mathfrak{g}^0$  and it holds that

$$\varphi_t(g) = e^{-tX} g e^{tX}, \quad \text{for any } t \in \mathbb{R} \text{ and } g \in G.$$

Consequently, if  $G^0$  is compact, the  $\mathcal{X}$ -orbit

$$\mathcal{O}(g) = \{\varphi_t(g), t \in \mathbb{R}\}$$

is bounded for any  $g \in G$ , since it is contained in the compact set  $K_g = G^0 g G^0$ . In particular, if  $g \in G^-$ , by the  $\mathcal{D}$ -invariance of  $\mathfrak{g}^-$  we obtain that  $\mathcal{O}(g) \subset G^- \cap K_g$  is bounded in  $G^-$ .

Considering that  $\varphi_t|_{G^-}$  is an automorphism of  $G^-$  it follows that

$$\varrho(\varphi_t(g), \varphi_t(h)) \leq \|(d\varphi_t|_{G^-})_e\| \varrho(g, h), \quad g, h \in G^- \text{ and } t \geq 0,$$

for any left invariant Riemannian metric  $\varrho$  on  $G^-$ .

On the other hand, since  $(d\varphi_t)_e = e^{t\mathcal{D}}$  and  $\mathcal{D}|_{\mathfrak{g}^-}$  has only eigenvalues with negative real part there are  $c, \mu > 0$  such that

$$\|(d\varphi_t|_{G^-})_e\| = \left\| e^{t\mathcal{D}|_{\mathfrak{g}^-}} \right\| \leq c^{-1} e^{-\mu t} \text{ for any } t \geq 0$$

implying that

$$\varrho(\varphi_t(g), \varphi_t(h)) \leq c^{-1} e^{-\mu t} \varrho(g, h), \quad g, h \in G^- \text{ and } t \geq 0.$$

Consequently

$$\varrho(\varphi_{-t}(g), \varphi_{-t}(h)) \geq c e^{\mu t} \varrho(g, h), \quad g, h \in G^-, t \geq 0$$

which shows that  $\mathcal{O}(g)$  is bounded in  $G^-$  if and only, if  $g = e$ . Therefore, if  $G^0$  is compact we must have  $G^- = \{e\}$ . Analogously  $G^+ = \{e\}$  and  $G = G^0$  as stated.

Assume now that  $G^0$  is a compact subgroup of  $G$  and  $\mathcal{D}$  is an arbitrary derivation. Let  $\mathfrak{r} = \mathfrak{r}(\mathfrak{g})$  be the solvable radical of  $\mathfrak{g}$  and  $R \subset G$  its associated connected solvable Lie subgroup. Since  $\mathfrak{r}$  is a  $\mathcal{D}$ -invariant ideal of  $\mathfrak{g}$ , we obtain a well induced linear vector field on the semisimple Lie group  $G/R$ .

Now, any derivation on a semisimple Lie algebra is inner. On the other hand, it holds that

$$(G/R)^0 = \pi(G^0) \text{ where } \pi : G \rightarrow G/R$$

is the canonical projection ([4], Lemma 2.3). Therefore, from the compactness of  $G^0$  that we are assuming we get

$$G/R = (G/R)^0 = \pi(G^0).$$

Consequently,  $G = G^0 R$ . But  $R$  is  $\varphi$ -invariant, then item 5. of Proposition 3.1 shows that

$$R = R^{+,0} R^- = R^{-,0} R^+,$$

where  $R^+, R^-, R^0$  are the connected Lie subgroup of  $R$  with Lie algebras

$$\mathfrak{r} \cap \mathfrak{g}^+, \mathfrak{r} \cap \mathfrak{g}^-, \mathfrak{r} \cap \mathfrak{g}^0$$

respectively.

Now, by using item 1. of the same proposition we conclude that

$$G = G^{+,0} \quad G^- = G^{-,0} \quad G^+$$

as stated. □

**Remark 3.4.** Compactness of  $G^0$  can be weakened. In fact, it is enough to ask that the connected components of the Lie subgroup of  $G/R$  given by the singularities of the induced linear vector field on  $G/R$  are compact subsets.

Next we show that if  $G$  is decomposable, the reachable and controllable sets are also decomposables.

**Lemma 3.5.** *Let us assume that  $G$  has finite semisimple center and  $\mathcal{A}$  is open. If  $G$  is decomposable then*

$$\mathcal{A} = \mathcal{A}_{G^-G^{+,0}} \quad \text{and} \quad \mathcal{A}^* = \mathcal{A}_{G^+G^{-,0}}^*, \tag{10}$$

where

$$\mathcal{A}_{G^-} = \mathcal{A} \cap G^- \quad \text{and} \quad \mathcal{A}_{G^+}^* = \mathcal{A}^* \cap G^+.$$

*Proof.* We just show the decomposition for  $\mathcal{A}$  since the other case is analogous. By hypothesis,  $G^{+,0} \subset \mathcal{A}$ . The  $\varphi$ -invariance of  $G^{+,0}$  implies by Lemma 2.7 that

$$\mathcal{A}_{G^-G^{+,0}} \subset \mathcal{A}G^{+,0} \subset \mathcal{A}.$$

Reciprocally, let  $x \in \mathcal{A}$ . Since  $G$  is decomposable there are  $a \in G^-$  and  $b \in G^{+,0}$  such that  $x = ab$ . Moreover,

$$a = xb^{-1} \in \mathcal{A}b^{-1} \subset \mathcal{A} \Rightarrow a \in \mathcal{A}_{G^-}.$$

Therefore,  $\mathcal{A} = \mathcal{A}_{G^-G^{+,0}}$  as desired. □

From now on we assume that  $G$  has finite semisimple center. We notice that there are many Lie groups satisfying this property. Of course, any solvable group and any semisimple Lie group with finite center, like  $\mathfrak{sl}(n, \mathbb{R})$ , has the finite semisimple center property. But also, the direct or semidirect product between groups with finite semisimple center have the same property. Furthermore, according to the Levi Decomposition Theorem given a Lie algebra  $\mathfrak{g}$  there exists a semisimple subalgebra  $\mathfrak{s}$  of  $\mathfrak{g}$  such that

$$\mathfrak{g} = \mathfrak{r}(\mathfrak{g}) \oplus \mathfrak{s}$$

where  $\mathfrak{r}(\mathfrak{g})$  is the solvable radical, i.e., the Lie subalgebra associated with the largest normal solvable subgroup of  $G$ .

In particular, if one Levi subgroup of  $G$  has finite center, then  $G$  has finite semisimple center ([10], Theorem 4.3).

Let us assume from now on that the reachable set  $\mathcal{A}$  is open. Since by our assumptions  $0 \in \mathbb{R}^m$  is in the interior of the control range  $\Omega$  we have by a general result that there exists a control set  $\mathcal{C}$  of  $\Sigma$  that contains the identity element  $e \in G$  in its interior ([3], Corollary 4.5.11). Our aim here is to analyze

the topological properties of this control set and to understand in which cases it is in fact the only control set of  $\Sigma$  with nonempty interior. A first result in this direction is the following

**Theorem 3.6.** *Assume that  $G$  has finite semisimple center and the reachable set  $\mathcal{A}$  of the linear system  $\Sigma$  is open. For the control set  $\mathcal{C}$  it holds that*

1.  $\mathcal{C}$  is closed if and only if  $\mathcal{A}^* = G$
2.  $\mathcal{C}$  is open if and only if  $\mathcal{A} = G$
3. Furthermore, if  $G$  is nilpotent we have
  - (i)  $\mathcal{C}$  is closed if and only if  $\mathcal{D}$  has only eigenvalues with nonpositive real part
  - (ii)  $\mathcal{C}$  is open if and only if  $\mathcal{D}$  has only eigenvalues with nonnegative real part
  - (iii)  $\mathcal{C} = G$  if and only if  $\mathcal{D}$  has only eigenvalues with zero real part

*Proof.* 1. If  $\mathcal{A}^* = G$  we get

$$\mathcal{C} = \text{cl}(\mathcal{A}) \cap \mathcal{A}^* = \text{cl}(\mathcal{A})$$

showing that  $\mathcal{C}$  is closed. Reciprocally, if  $\mathcal{C}$  is closed, by item 4. of Theorem 2.4 it follows that  $\mathcal{C} = \text{cl}(\mathcal{A}(g))$  for any  $g \in G$  and in particular  $\mathcal{C} = \text{cl}(\mathcal{A})$ . By item 2. of the same Proposition we must have  $\mathcal{A} \subset \mathcal{A}^*$ . Since  $G^+ \subset \mathcal{A}$  Lemma 2.7 shows that

$$B = G^+G^{-,0} \subset \mathcal{A}^*G^{-,0} \subset \mathcal{A}^*.$$

Since  $B$  is  $\varphi$ -invariant, Lemma 2.7 implies also that  $B^n \subset \mathcal{A}^*$ . Moreover,  $B$  is a neighborhood of  $e \in G$  and since  $G$  is connected we get

$$G = \bigcup_{n \in \mathbb{N}} B^n \subset \mathcal{A}^* \Rightarrow \mathcal{A}^* = G.$$

2. If  $\mathcal{A} = G$  we obtain

$$\mathcal{C} = \text{cl}(\mathcal{A}) \cap \mathcal{A}^* = \mathcal{A}^*$$

showing in particular that  $\mathcal{C}$  is open. Conversely, let us assume that  $\mathcal{C} = \mathcal{A}^*(g)$  for any  $g \in \mathcal{C}$ . By item 2. of Theorem 2.4 we have that  $\text{int } \mathcal{C} \subset \mathcal{A}(g)$  for any  $g \in \mathcal{C}$ . In particular, since  $\mathcal{C} = \mathcal{A}^*$  and  $\mathcal{A}^*$  is open we get  $\mathcal{A}^* \subset \mathcal{A}$ . Moreover, since  $G^- \subset \mathcal{A}^*$ ,  $G^{+,0} \subset \mathcal{A}$  and both are  $\varphi$ -invariant, we obtain

$$B = G^+G^{-,0} = G^{+,0}G^- \subset \mathcal{A}G^- \subset \mathcal{A}.$$

As before, this fact allows to us to conclude that  $\mathcal{A} = G$  as desired.

3. If  $G$  is nilpotent, Theorem 2.13 items 1. and 2. give us that  $\mathcal{A}^* = G$  if and only if  $G = G^{-,0}$  and  $\mathcal{A} = G$  if and only if  $G = G^{+,0}$  and consequently we have (i) and (ii). By Theorem 2.13 item 3. we obtain that  $G = G^0$  if and only if  $\mathcal{A} \cap \mathcal{A}^* = G$  and so  $\mathcal{C} = G$  if and only if  $G = G^0$  proving (iii).  $\square$

The above theorem implies the uniqueness of the invariant control set. In fact,

**Corollary 3.7.** *Assume that  $G$  has finite semisimple center and the reachable set  $\mathcal{A}$  of the linear system  $\Sigma$  is open. If the control set  $\mathcal{C}$  is invariant, then it is the only invariant one.*

*Proof.* Let us prove the case where  $\mathcal{C}$  is invariant in positive time. By item 4. of Theorem 2.4 that situation happens if and only if  $\mathcal{C}$  is closed and by the theorem above, if and only if  $\mathcal{A}^* = G$ .

If  $\tilde{\mathcal{C}}$  is another closed control set, for any  $g \in \tilde{\mathcal{C}} \subset \mathcal{A}^*$  there are  $\tau > 0, u \in \mathcal{U}$  such that  $\phi_{\tau,u}(g) = e$ . Moreover, by the invariance of  $\tilde{\mathcal{C}}$  in positive time, we must have

$$e = \phi_{\tau,u}(g) \in \phi_{\tau,u}(\tilde{\mathcal{C}}) \subset \tilde{\mathcal{C}}.$$

Showing that  $e \in \mathcal{C} \cap \tilde{\mathcal{C}}$ .

Since two control sets with nonempty intersection must coincide we must have  $\mathcal{C} = \tilde{\mathcal{C}}$ . Then,  $\mathcal{C}$  is the only invariant one.  $\square$

**Remark 3.8.** We should notice that the existence of an invariant control set for a linear system is quite restrictive. In fact, a linear system admits a positive, or negative, invariant control set if and only if the whole group  $G$  is controllable to, or reachable from, the identity respectively.

### 3.1. Bounded control sets

In this section our interest is to search for conditions when the control set  $\mathcal{C}$  is bounded. As before we are assuming that  $G$  is a connected Lie group with finite semisimple center and that the linear system on  $G$  has an open reachable set  $\mathcal{A}$ .

As before, let us consider the sets

$$\mathcal{A}_{G^-} = \mathcal{A} \cap G^- \text{ and } \mathcal{A}_{G^+}^* = \mathcal{A}^* \cap G^+.$$

Since  $\mathcal{A}_{G^-}$ ,  $\mathcal{A}_{G^+}^*$  and  $G^0$  are contained in  $\mathcal{A} \cap \mathcal{A}^*$ , if the control set  $\mathcal{C}$  is bounded we get

$$\text{cl}(\mathcal{A}_{G^-}), \text{cl}(\mathcal{A}_{G^+}^*) \text{ and } G^0 \text{ are compact sets.}$$

In the sequel we show that in some cases the compactness of such sets imply that  $\mathcal{C}$  is bounded.

**Theorem 3.9.** *Let us assume that  $G$  is semisimple or nilpotent. If  $\text{cl}(\mathcal{A}_{G^-})$ ,  $\text{cl}(\mathcal{A}_{G^+}^*)$  and  $G^0$  are compact subsets of  $G$  then  $\mathcal{C}$  is bounded.*

*Proof.* If  $G$  is semisimple, the result follows direct from Proposition 3.3 since in this case  $G = G^0$ . For the nilpotent case we prove by induction on the dimension of  $G$ .

If  $\dim G = 1$  then  $G$  is Abelian and by Lemma 3.5 we have that

$$\mathcal{A} \cap \mathcal{A}^* = \mathcal{A}_{G^-} G^{-,0} \cap \mathcal{A}_{G^+}^* G^{+,0} = \mathcal{A}_{G^-} G^0 \mathcal{A}_{G^+}^*.$$

Considering that  $\mathcal{A} \cap \mathcal{A}^*$  is dense in  $\mathcal{C}$  we get

$$\mathcal{C} \subset \text{cl}(\mathcal{A}_{G^-}) G^0 \text{cl}(\mathcal{A}_{G^+}^*).$$

Since the right side set is compact, it follows that  $\mathcal{C}$  is a bounded control set.

Let us assume that  $G$  is a nilpotent Lie group with dimension  $n$ .

Let  $Z_G$  be the center of  $G$  and  $(Z_G)_0$  its connected component of the identity. Since  $(Z_G)_0$  is  $\varphi$ -invariant and Abelian, we have by item 5. of Proposition 3.1 that  $(Z_G)_0 = Z^+ Z^0 Z^-$ . Moreover  $Z^+$ ,  $Z^0$  and  $Z^-$  are  $\varphi$ -invariant normal subgroups of  $G$ . Since  $G$  is nilpotent,  $(Z_G)_0$  is nontrivial and consequently at least one of the subgroups  $Z^+$ ,  $Z^0$  or  $Z^-$  is nontrivial.

Let us analyze the case when  $\{e\} \subsetneq Z^+$ , the other cases are analogous. Consider the connected nilpotent Lie group  $H = G/Z^+$ . By the  $\varphi$ -invariance of  $Z^+$  we get an induced linear system on  $H$  that satisfy

$$\pi(\phi_{t,u}^G(g)) = \phi_{t,u}^H(\pi(g))$$

where  $\pi : G \rightarrow H$  is the canonical projection ([9], Proposition 4).

The above equation gives us that  $\pi(\mathcal{A})$  and  $\pi(\mathcal{A}^*)$  are the reachable and controllable sets of the identity in  $H$  respectively. Considering that

$$\dim H = \dim G - \dim Z^+ < n$$

by hypothesis it follows that

$$\mathcal{C}^H = \text{cl}(\pi(\mathcal{A})) \cap \pi(\mathcal{A}^*) \text{ is a bounded control set.}$$

Since  $\pi$  is an open map, there exists a compact set  $K \subset G$  such that

$$\pi(\mathcal{C}) \subset \mathcal{C}^H \subset \pi(K) \text{ implying that } \mathcal{C} \subset KZ^+.$$

Since  $G$  is decomposable we can assume w.l.o.g that

$$K = K^+ K^{-,0}, \text{ } K^+ \text{ and } K^{-,0} \text{ are compact subsets of } G^+ \text{ and } G^{-,0}$$

respectively.

Take  $x \in \mathcal{A} \cap \mathcal{A}^*$ . There are elements

$$k_1 \in K^+, k_2 \in K^{-,0} \text{ and } z \in Z^+ \text{ such that } x = k_1 k_2 z.$$

But,  $Z^+ \subset (Z_G)_0$  so we obtain  $x = (z k_1) k_2$ . By Lemma 3.5 we know that  $\mathcal{A}^* = \mathcal{A}_{G^+}^* G^{-,0}$  and since  $x \in \mathcal{A}^*$  we get

$$z k_1 \in \mathcal{A}_{G^+}^* \text{ or equivalently } z \in \mathcal{A}_{G^+}^* (K^+)^{-1}.$$

We started with an arbitrary element  $x \in \mathcal{A} \cap \mathcal{A}^*$ , then we can conclude that

$$\mathcal{A} \cap \mathcal{A}^* \subset K \text{cl}(\mathcal{A}_{G^+}^*) (K^+)^{-1}.$$

By hypothesis  $\text{cl}(\mathcal{A}_{G^+}^*)$  is compact and  $\mathcal{A} \cap \mathcal{A}^*$  is dense in  $\mathcal{C}$ , thus  $\mathcal{C}$  is contained in the compact set  $L$ , finishing the proof. □

We will say that the derivation  $\mathcal{D}$  is *hyperbolic* if  $\mathcal{D}$  has just eigenvalues with nonzero real parts. For hyperbolic derivations and simply connected nilpotent Lie groups it is possible to improve Theorem 3.9 as follows:

**Corollary 3.10.** *Let  $G$  be a nilpotent simply connected Lie group. Then  $\mathcal{C}$  is bounded if, and only if,  $\text{cl}(\mathcal{A}_{G^-})$  and  $\text{cl}(\mathcal{A}_{G^+}^*)$  are compact subsets of  $G$  and  $\mathcal{D}$  is hyperbolic.*

*Proof.* In fact, by Theorem 3.9  $\mathcal{C}$  is bounded if and only if  $G^0$ ,  $\text{cl}(\mathcal{A}_{G^-})$  and  $\text{cl}(\mathcal{A}_{G^+}^*)$  are compact subsets. Since  $G$  is simply connected its exponential  $\exp$  is a diffeomorphism which implies that  $G^0$  is compact if and only if  $\mathfrak{g}^0$  is compact if and only if  $\mathfrak{g}^0 = \{0\}$ . □

### 3.2. Uniqueness

In the classical Euclidean Abelian Linear Control Systems it is well known that under the Kalman rank condition there exists just one control set with non empty interior. We will show here that any decomposable Lie group have the same property, i.e., they have at most one control set. We start with,

**Theorem 3.11.** *The set  $\mathcal{C}$  is the only control set of the linear system  $\Sigma$  whose interior intersects  $G^{+,0}G^-$  and  $G^{-,0}G^+$ .*

*Proof.* Let  $\tilde{\mathcal{C}}$  be a control set such that its interior intersects  $G^{+,0}G^-$  and  $G^{-,0}G^+$ . Since two control sets with nonempty intersection must coincide it is enough to show that  $\mathcal{C} \cap \tilde{\mathcal{C}} \neq \emptyset$ . We will divide the proof in three steps:

*Step 1: If  $\text{int } \tilde{\mathcal{C}} \cap G^{+,0}G^- \neq \emptyset$  there are  $\tau_1 > 0$  and  $u_1 \in \mathcal{U}$  such that*

$$a = \phi_{\tau_1, u_1} \in \text{int } \tilde{\mathcal{C}}. \quad (11)$$

Let  $x \in \text{int } \tilde{\mathcal{C}} \cap G^{+,0}G^-$  and let  $g \in G^{+,0}$  and  $h \in G^-$  such that  $x = gh$ . Considering that in  $\text{int } \tilde{\mathcal{C}}$  controllability holds, there are  $\tau > 0$  and  $u \in \mathcal{U}$  such that  $\phi_{\tau, u}(x) = x$ . Let  $\varrho$  be a left invariant Riemannian metric on  $G$ . Since  $\varphi_t(h) \rightarrow e$  as  $t \rightarrow +\infty$  we have that

$$\varrho(\phi_{t, u}(x), \phi_{t, u}(g)) = \varrho(\phi_{t, u}(g)\varphi_t(h), \phi_{t, u}(g)) = \varrho(\varphi_t(h), e) \xrightarrow{t \rightarrow +\infty} 0.$$

Since  $\phi_{n\tau, u}(x) = x$ , for any  $n \in \mathbb{N}$ , we obtain that  $\phi_{t, u}(g) \in \text{int } \tilde{\mathcal{C}}$ , for  $t > 0$  greater enough. Moreover,  $g \in G^{+,0} \subset \mathcal{A}$  which implies by concatenation, that

$$\phi_{\tau_1, u_1} \in \text{int } \tilde{\mathcal{C}} \quad \text{for some } \tau_1 > 0, u_1 \in \mathcal{U},$$

as stated.

*Step 2: If  $\text{int } \tilde{\mathcal{C}} \cap G^{-,0}G^+ \neq \emptyset$  there are  $\tau_2 > 0$  and  $u_2 \in \mathcal{U}$  such that*

$$e \in \phi_{\tau_2, u_2}(\text{int } \tilde{\mathcal{C}}). \quad (12)$$

Let  $x' \in \text{int } \tilde{\mathcal{C}} \cap G^{-,0}G^+$  and  $g'^{-,0}$ ,  $h'^+$  such that  $x' = g'h'$ . Again by controllability in  $\text{int } \tilde{\mathcal{C}}$  there are  $\tau' > 0$ ,  $u' \in \mathcal{U}$  such that  $\phi_{-\tau', u'}(x') = x'$ . Since  $\varphi_{-t}(h') \rightarrow e$  as  $t \rightarrow +\infty$  we have

$$\begin{aligned} \varrho(\phi_{-t, u'}(x'), \phi_{-t, u'}(g')) &= \varrho(\phi_{-t, u'}(g')\varphi_t(h'), \phi_{-t, u'}(g')) \\ &= \varrho(\varphi_{-t}(h'), e) \xrightarrow{t \rightarrow +\infty} 0. \end{aligned}$$

Again, since  $\varphi_{-n\tau', u'}(x') = x'$  for any  $n \in \mathbb{N}$  we get that  $\phi_{-t, u'}(g') \in \text{int } \tilde{\mathcal{C}}$ , for  $t > 0$  greater enough. Moreover,  $g'^{-,0} \subset \mathcal{A}^*$  which gives us by concatenation that for some  $\tau_2 > 0$ ,  $\Theta_{\tau_2} u_2 \in \mathcal{U}$  we have

$$\phi_{-\tau_2, \Theta_{\tau_2} u_2} \in \text{int } \tilde{\mathcal{C}} \Leftrightarrow e \in \phi_{\tau_2, u_2}(\text{int } \tilde{\mathcal{C}}),$$

*Step 3: It holds that  $\mathcal{C} \cap \tilde{\mathcal{C}} \neq \emptyset$ ;*

By (12) there is  $b \in \text{int } \tilde{\mathcal{C}}$  such that  $\phi_{\tau_2, u_2}(b) = e$ . Let  $a = \phi_{\tau_1, u_1} \in \text{int } \tilde{\mathcal{C}}$  given in (11). By controllability in  $\text{int } \tilde{\mathcal{C}}$  there are  $\tau_3 > 0$  and  $u_3 \in \mathcal{U}$  with  $\phi_{\tau_3, u_3}(a) = b$ . If we concatenate  $u_1, u_2$  and  $u_3$  and extend it periodically we obtain an admissible control function  $u \in \mathcal{U}$  with period  $T = \tau_1 + \tau_2 + \tau_3$ .

Moreover,  $\phi_{t,u}(e)$  is a periodic curve of the linear system  $\Sigma$  through  $a$  and  $b$  in int  $\tilde{\mathcal{C}}$ . Since periodic orbits cannot leave the interior of a control set we must have that the curve lies in  $\mathcal{C} \cap \tilde{\mathcal{C}}$  which concludes the proof.  $\square$

As a direct corollary we get.

**Corollary 3.12.** *If  $G$  is decomposable, then  $\mathcal{C}$  is the only control set.*

*Proof.* If  $G = G^{+,0}G^- = G^{-,0}G^+$  it follows that any control set of  $\Sigma$  with nonempty interior is such that its interior intersects  $G^{+,0}G^-$  and  $G^{-,0}G^+$  which by Theorem 3.11 implies that such control set is indeed  $\mathcal{C}$ .  $\square$

In particular, since solvable Lie subgroups are always decomposables we also obtain.

**Corollary 3.13.** *If  $G$  is solvable, then  $\mathcal{C}$  is the only control set.*

### Further work

One should notice that for general Lie groups we do not know if the control set around the identity is the only one. Our next step is to analyze where there are more than one control set for semisimple Lie groups and, together with the solvable case, conclude how many control sets there are for linear systems on any Lie group. The authors believe that for a semisimple Lie group  $G$ , the control sets with nonempty interior are related with the connected components of an open Bruhat cell on  $G$ , since the submanifold  $G^{+,0}G^-$  coincides with the component of the identity of such open cell.

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## References

- [1] Ayala, V., San Martin, L.A.B.: Controllability properties of a class of control systems on Lie groups. In: Lecture Notes in Control and Information Sciences, vol. 258, pp. 83–92 (2001)
- [2] Ayala, V., Tirao, J.: Linear control systems on Lie groups and controllability. In: Ferreyra, G. et al. (eds.) Proceedings of Symposia in Pure Mathematics, Am. Math. Soc., vol. 64, pp. 47–64. 2009. Providence, RI (1999)
- [3] Colonius, F., Kliemann, C.: The Dynamics of Control, Systems & Control: Foundations & Applications. Birkhäuser Boston Inc., Boston, MA (2000)
- [4] Da Silva, A.J.: Controllability of linear systems on solvable Lie groups. SIAM J. Control Optim. **54**(1), 372–390 (2016)

- [5] Da Silva, A.J.: Outer invariance entropy for linear systems on Lie groups. *SIAM J. Control Optim.* **52**(6), 3917–3934 (2014)
- [6] Da Silva, A., Ayala, V.: Controllability of linear systems on Lie groups with finite semisimple center. *SIAM J. Control Optim.* (2015) (**Under review**)
- [7] Jouan, P.: Controllability of linear systems on Lie group. *J. Dyn. Control Syst.* **17**, 591–616 (2011)
- [8] Jouan, P., Dath M.: Controllability of Linear Systems on low dimensional Nilpotent and Solvable Lie groups. *J. Dyn. Control Syst.* **22**(2), 207–225 (2016)
- [9] Jouan, P.: Equivalence of control systems with linear systems on Lie groups and homogeneous spaces. *ESAIM Control Optim. Calc. Var.* **16**, 956–973 (2010)
- [10] Onishchik, A.L., Vinberg, E.B.: *Lie Groups and Lie Algebras III—Structure of Lie Groups and Lie Algebras*. Springer, Berlin (1990)
- [11] San Martin, L.A.B.: *Algebras de Lie*, 2nd edn. Editora Unicamp, Campinas (2010)

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## Chapter 3

# Control sets of linear control systems on semi-simple Lie groups

# Control sets of linear systems on semi-simple Lie groups

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## 1 Introduction

This paper is devoted to characterizing control sets for linear control systems on non-compact semi-simple Lie groups.

In [3] the authors extended the concept of linear control systems as family of

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ordinary differential equations

$$\dot{g}(g) = \mathcal{X}(g(t)) + \sum_{i=1}^k u_i Y_i(g(t)), \quad (\Sigma)$$

where the drift  $\mathcal{X}$  is a linear vector field, that is, its flow  $(\varphi_t)_{t \in \mathbb{R}}$  is a 1-parameter group of  $G$ -automorphisms. The control vector fields  $Y_i$  are right-invariant vector fields and  $u \in \mathcal{U} \subset L^\infty(\mathbb{R}, \Omega \subset \mathbb{R}^k)$  is the admissible control class, where the set  $\Omega$  is a compact, convex subset of  $\mathbb{R}^k$  that contains 0 in its interior.

The control set concept arise from non-controllability of any system, that is, the impossibility of connecting two arbitrary points with a solution of the system  $\Sigma$  in positive time. As shown in [4] and [5] the controllability is an unusual property for linear control system and depends on spectrum of the derivation  $\mathcal{D}$  associated to the system. Suppose that  $G$  is a nilpotent Lie group and the reachable set  $\mathcal{A}$  is open, a linear system  $\Sigma$  is controllable if and only if the real part of all eigenvalue of  $\mathcal{D}$  are zero. For semi-simple Lie groups, suppose it has finite center and the reachable set  $\mathcal{A}$  is open. The real part of all eigenvalue of  $\mathcal{D}$  be zero is also sufficient for the system controllability.

In the previous work [1], many topological properties for control sets were proven. In particular, the authors characterized boundedness of a control set when the Lie group is nilpotent or semi-simple and uniqueness when Lie group is solvable.

This work analyzes control sets for connected non-compact semi-simple Lie groups. This space state can not be decomposed by eigenvalues of the derivation  $\mathcal{D}$  such as the compact semi-simple or solvable Lie groups. Most results in our previous work used this decomposition. To achieve similar results, we complement the proofs with semi-simple properties.

The paper is structured as follows: Section 2 introduces the notions of linear vector fields, linear systems on Lie groups and control sets. The first subsection contains the main properties of linear vector fields and the associated decomposition induced by its derivation. In the sequel, we introduce linear control systems on Lie groups, reachable and controllable sets and their properties. This particular subsection is recursively used during the work. The last subsection is dedicated to the main properties of control sets. It also contains some dynamical concepts which appear in some proofs.

Section 3 contains the main results obtained in this work. The first subsection recall some standard facts of the semi-simple theory and results used further on. In the second subsection we relate the decomposition induced by  $\mathcal{D}$  to parabolic subalgebras and subgroups. In the subsection 'Result' we prove that a control set with non-empty interior is contained in right-translation of the identity control set and also that a linear system on Lie group admits at most one invariant control set with non-empty interior. Moreover, assuming that the derivation  $\mathcal{D}$  is semi-simple, we prove that control sets of the linear system are exactly the right translations of the identity control set, which gives an upper

bound for the quantity of control sets.

## 2 Preliminaries

### 2.1 Linear vector fields and its induced decomposition

In this section, we review standard facts on linear vector fields. The reader is referred to [1], [3],[4],[6].

Let  $G$  be a  $n$ -dimensional connected Lie group and  $\mathfrak{g}$  its Lie algebra (the set of right-invariant vector fields, identified with the tangent space at the identity).

**2.1 Definition** *A linear vector field on  $G$  is an analytic vector field on  $G$ , whose flow  $(\varphi_t)_{t \in \mathbb{R}}$  is a 1-parameter group of  $G$ -automorphisms.*

A natural relation between the associated derivation  $\mathcal{D} = -ad_{\mathcal{X}}$  and the linear vector field flow arise via the equality

$$(d\varphi_t)_e = e^{t\mathcal{D}} \text{ for all } t \in \mathbb{R}. \quad (2.1)$$

In particular, Eq. 2.1 implies that

$$\varphi_t(\exp Y) = \exp(e^{t\mathcal{D}}Y), \text{ for all } t \in \mathbb{R}, Y \in \mathfrak{g}. \quad (2.2)$$

Any derivation decomposes its Lie algebra  $\mathfrak{g}$  as follows: Consider the complexification  $\mathfrak{g}_{\mathbb{C}}$  of  $\mathfrak{g}$  and the derivation  $\mathcal{D}_{\mathbb{C}}$  on  $\mathfrak{g}_{\mathbb{C}}$  induced by  $\mathcal{D}$ . For a  $\mathcal{D}$ -eigenvalue  $\alpha$  we defined the  $\alpha$ -generalized eigenspace of  $\mathcal{D}_{\mathbb{C}}$  as

$$(\mathfrak{g}_{\mathbb{C}})_{\alpha} = \{Y \in \mathfrak{g}_{\mathbb{C}} : (\mathcal{D}_{\mathbb{C}} - \alpha)^n Y = 0 \text{ for some } n \geq 1\}.$$

Since the eigenvalues of  $\mathcal{D}_{\mathbb{C}}$  coincides with the ones of  $\mathcal{D}$  we have that

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{\alpha} (\mathfrak{g}_{\mathbb{C}})_{\alpha},$$

for every  $\alpha$  eigenvalue of  $\mathcal{D}$ . Moreover, if  $\beta$  is also an eigenvalue of  $\mathcal{D}$  Proposition 3.1 of [11] implies

$$[(\mathfrak{g}_{\mathbb{C}})_{\alpha}, (\mathfrak{g}_{\mathbb{C}})_{\beta}] \subset (\mathfrak{g}_{\mathbb{C}})_{\alpha+\beta},$$

where  $(\mathfrak{g}_{\mathbb{C}})_{\alpha+\beta} = \{0\}$  when  $\alpha + \beta$  is not an eigenvalue of  $\mathcal{D}$ . Let

$$\mathfrak{g}_{\mathbb{C}}^+ = \bigoplus_{\alpha: \operatorname{Re}(\alpha) > 0} (\mathfrak{g}_{\mathbb{C}})_{\alpha}, \quad \mathfrak{g}_{\mathbb{C}}^0 = \bigoplus_{\alpha: \operatorname{Re}(\alpha) = 0} (\mathfrak{g}_{\mathbb{C}})_{\alpha}, \quad \mathfrak{g}_{\mathbb{C}}^- = \bigoplus_{\alpha: \operatorname{Re}(\alpha) < 0} (\mathfrak{g}_{\mathbb{C}})_{\alpha}.$$

Since  $\mathfrak{g}_{\mathbb{C}}^+$ ,  $\mathfrak{g}_{\mathbb{C}}^0$  and  $\mathfrak{g}_{\mathbb{C}}^-$  are invariant by conjugation they coincide with the complexification of  $\mathfrak{g}^+ := \mathfrak{g}_{\mathbb{C}}^+ \cap \mathfrak{g}$ ,  $\mathfrak{g}^0 := \mathfrak{g}_{\mathbb{C}}^0 \cap \mathfrak{g}$  and  $\mathfrak{g}^- := \mathfrak{g}_{\mathbb{C}}^- \cap \mathfrak{g}$ . Thus, the decomposition also preserves the Lie algebra structure. Consequently, the subspaces  $\mathfrak{g}^+$ ,  $\mathfrak{g}^-$ ,  $\mathfrak{g}^0$  are Lie subalgebras and  $\mathfrak{g}^+$ ,  $\mathfrak{g}^-$  are nilpotent.

Moreover, this decomposition extends to the Lie group  $G$ . Denote by  $G^+$ ,  $G^0$ ,  $G^-$ ,  $G^{+,0}$ ,  $G^{-,0}$  the connected subgroups of  $G$  with Lie algebras  $\mathfrak{g}^+$ ,  $\mathfrak{g}^0$ ,  $\mathfrak{g}^-$ ,  $\mathfrak{g}^{+,0} := \mathfrak{g}^+ \oplus \mathfrak{g}^0$ ,  $\mathfrak{g}^{-,0} := \mathfrak{g}^- \oplus \mathfrak{g}^0$  respectively.

**2.2 Proposition** *It holds :*

1.  $G^{+,0} = G^+G^0 = G^0G^+$  and  $G^{-,0} = G^-G^0 = G^0G^-$ ,
2.  $G^+ \cap G^- = G^{+,0} \cap G^- = G^{-,0} \cap G^+ = \{e\}$ ,
3.  $G^{+,0} \cap G^{-,0} = G^0$ ,
4. *All the above subgroups are closed in  $G$ ,*
5. *If  $G$  is solvable then  $G = G^{+,0}G^- = G^{-,0}G^+$  Moreover, the fixed points of  $\mathcal{X}$  are in  $G^0$ .*
6. *If  $G^0$  is compact then  $G = G^0$ .*

The proof of above proposition can be found in [1] Proposition 3.3 and [4] Proposition 2.9.

**2.3 Definition** *A Lie group is decomposable by the eigenvalues of the derivation  $\mathcal{D}$  if it is written as the product  $G = G^+G^0G^-$ .*

Further on, we refer to such Lie groups only as decomposable.

## 2.2 Linear systems on Lie groups

The subsection is devoted to linear control systems and their properties. For more on the subject the reader can consult [3], [4], [6] and the book [7].

From the linear vector field generalization arises the extension of linear control systems from Euclidean spaces to Lie groups.

**2.4 Definition** *A linear control system  $\Sigma$  on a Lie group  $G$  is*

$$\dot{g} = \mathcal{X}(g) + \sum_{i=1}^k u_i Y_i(g), \quad (\Sigma)$$

where the  $\mathcal{X}$  is a linear vector field, the  $Y_i$ 's are right-invariant vector fields and the control function  $u = (u_1, \dots, u_k)$  belongs to the set of admissible controls  $\mathcal{U}$ .

Throughout this work we consider

$$\mathcal{U} = \{u : \mathbb{R} \rightarrow \mathbb{R}^k, u \in L^\infty(\mathbb{R}, \mathbb{R}^k), u(t) \subseteq \Omega \text{ a.e.}\},$$

where  $\Omega$  is a compact and convex subset containing 0 in its interior.

For any  $g \in G$ ,  $u \in \mathcal{U}$  the map  $(t, g, u) \mapsto \Phi(t, g, u)$  denotes the system solution for the control  $u$  through the point  $g$ . For fixed time  $t$  and control  $u$ , the solution  $\Phi(t, g, u)$  is a diffeomorphism  $g \in G \mapsto \Phi(t, g, u)$  and will be denoted as  $\Phi_{t,u}(g)$ .

The solution  $\Phi(t, g, u)$  satisfies the cocycle property, that is,

$$\Phi(t + s, g, u) = \Phi(t, \Phi(s, g, u), \Theta_s u)$$

for all  $t, s \in \mathbb{R}$ ,  $g \in G$ ,  $u \in \mathcal{U}$ . Here for any  $t \in \mathbb{R}$  the map  $\Theta_t$  is the shift flow on  $\mathcal{U}$  defined by

$$(\Theta_t u)(s) := u(t + s).$$

Moreover, for a given  $g \in G$ ,  $u \in \mathcal{U}$  and  $t \in \mathbb{R}$ , the linear system solution at point  $g$  can be written as  $\Phi_{t,u}(g) = \Phi_{t,u}(e)\varphi_t(g) = L_{\Phi_{t,u}(e)}(\varphi_t(g))$ . (See [6])

For any  $g \in G$  and real number  $\tau > 0$  the sets

$$\begin{aligned} \mathcal{A}_{\leq \tau}(g) &:= \{h \in G : \exists u \in \mathcal{U}, t \in [0, \tau] \text{ such that } h = \Phi_{t,u}(g)\}, \\ \mathcal{A}_{\tau}(g) &:= \{h \in G : \exists u \in \mathcal{U} \text{ such that } h = \Phi_{\tau,u}(g)\}, \end{aligned} \quad (2.3)$$

$$\mathcal{A}(g) := \bigcup_{\tau > 0} \mathcal{A}_{\leq \tau}(g),$$

are the set of reachable points from  $g$  up to time  $\tau$ , the set of reachable points from  $g$  at time exactly  $\tau$  and the reachable set of  $g$ , respectively. The sets  $\mathcal{A}_{\tau}(e)$ ,  $\mathcal{A}_{\leq \tau}(e)$  and  $\mathcal{A}(e)$  will be denoted as  $\mathcal{A}_{\tau}$ ,  $\mathcal{A}_{\leq \tau}$  and  $\mathcal{A}$  respectively.

Similarly, for any  $g \in G$  and real number  $\tau > 0$  the sets

$$\begin{aligned} \mathcal{A}_{\leq \tau}^*(g) &:= \{h \in G : \exists u \in \mathcal{U}, t \in [0, \tau] \text{ such that } g = \Phi_{t,u}(h)\} \\ \mathcal{A}_{\tau}^*(g) &:= \{h \in G : \exists u \in \mathcal{U} \text{ such that } g = \Phi_{\tau,u}(h)\} \end{aligned} \quad (2.4)$$

$$\mathcal{A}^*(g) := \bigcup_{\tau > 0} \mathcal{A}_{\leq \tau}^*(g),$$

are the set of controllable points to  $g$  up to time  $\tau$ , the set of controllable points to  $g$  at time exactly  $\tau$  and the controllable set of  $g$ , respectively.

Since the flow satisfies  $\Phi_{-t,u}^{-1} = \Phi_{-t, \Theta_t(u)}$  we have

$$\mathcal{A}_{\tau}^* = \varphi_{-\tau}(\mathcal{A}_{\tau}^{-1}). \quad (2.5)$$

**2.5 Proposition** *For a linear control system on a Lie group holds:*

1. if  $0 \leq \tau$  then  $\mathcal{A}_{\leq \tau} = \mathcal{A}_{\tau}$ ,
2. if  $0 \leq \tau_1 \leq \tau_2$  then  $\mathcal{A}_{\tau_1} \subset \mathcal{A}_{\tau_2}$ ,
3. for all  $g \in G$ ,  $\mathcal{A}_{\tau}(g) = \mathcal{A}_{\tau}\varphi_{\tau}(g)$ ,

4. for all  $\tau, \tau' \geq 0$  we have  $\mathcal{A}_{\tau+\tau'} = \mathcal{A}_\tau \varphi_\tau(\mathcal{A}_{\tau'}) = \mathcal{A}_{\tau'} \varphi_{\tau'}(\mathcal{A}_\tau)$ ,
5. for all  $u \in \mathcal{U}$ ,  $g \in \mathbf{G}$  and  $t > 0$ , we have that  $\Phi_{t,u}(\mathcal{A}(g)) \subset \mathcal{A}(g)$ ,
6. the three conditions are equivalent
  - (i)  $e \in \text{int}(\mathcal{A}_t)$  for some  $t > 0$
  - (ii)  $e \in \text{int}(\mathcal{A})$
  - (iii)  $\mathcal{A}$  is open

The proof of items 1. to 3. can be found in [6], Proposition 2. The items 4. in [4] Proposition 2.13. The item 5. in [4] Proposition 2.13 and [7] Lemma 4.5.2.

**2.6 Definition** A control system is locally accessible at  $g \in \mathbf{G}$  if for all  $\tau > 0$  the sets  $\mathcal{A}_\tau(g)$  and  $\mathcal{A}_\tau^*(g)$  has nonempty interior and locally accessible if is locally accessible at all points  $g \in \mathbf{G}$ .

**Remark:** Further on, we assume  $\mathcal{A}$  to be open. Under this assumption, the system  $\Sigma$  satisfies the Lie algebra rank condition and so Theorem A4.4 from [7] assures that  $\Sigma$  is locally accessible.

Since  $\mathcal{A}$  is not a semigroup in general. The following result relates the reachable set with translation by its points

**2.7 Lemma** Let  $g \in \mathcal{A}$  and assume that  $\varphi_t(g) \in \mathcal{A}$  for any  $t \in \mathbb{R}$ . Then  $\mathcal{A} \cdot g \subset \mathcal{A}$ .

The proof can be found in [4] Lemma 3.1.

For semisimple Lie group we deal with subgroups that are not necessarily connected. For such subgroups, we have the following result:

**2.8 Lemma** For  $g \in \mathbf{G}$  it holds:

1. If the orbit  $\{\varphi_t(g), t \in \mathbb{R}\}$  is contained  $\mathcal{A}(g)$  then  $\mathcal{A} \cdot \varphi_t(g) \subset \mathcal{A}(g)$  and similarly the orbit  $\{\varphi_t(g), t \in \mathbb{R}\}$  is contained  $\mathcal{A}^*(g)$  then  $\mathcal{A}^* \cdot \varphi_t(g) \subset \mathcal{A}^*(g)$ .
2. Let  $L \subset \mathbf{G}$  be a  $\varphi$ -invariant subgroup and  $g \in L$ . If the identity connected component  $L_e \subset \mathcal{A}$  then  $\mathcal{A}(g) \subset \mathcal{A} \cdot g$  and  $L_e \subset \mathcal{A}^*$  then  $\mathcal{A}^*(g) \subset \mathcal{A}^* \cdot g$ .

**Proof:** We will only show the inclusions for the reachable sets, since the ones for controllable are analogous.

1. Let  $z \in \mathcal{A}$  and consider  $\tau > 0, u \in \mathcal{U}$  such that  $z = \Phi_{\tau,u}(e)$ . It holds that  $z \cdot \varphi_t(g) = \Phi_{\tau,u}(e) \cdot \varphi_\tau(\varphi_{t-\tau}(g))$ .

If the  $\varphi$ -orbit of  $g$  is contained in  $\mathcal{A}(g)$ , there is  $\tau' > 0$ ,  $u' \in \mathcal{U}$  such that  $\varphi_{t-\tau}(g) = \Phi_{\tau',u'}(g)$  and so

$$z \cdot g = \Phi_{\tau,u}(e) \cdot \varphi_{\tau}(\varphi_{t-\tau}(g)) = \Phi_{\tau,u}(\Phi_{\tau',u'}(g)) = \Phi_{\tau+\tau',u''}(g),$$

$$\text{where } u''(t) = \begin{cases} u'(t) & \text{for } t \in [0, \tau'], \\ u(t - \tau) & \text{for } t \in [\tau', \tau' + \tau]. \end{cases}$$

Therefore  $z \cdot \varphi_t(g) \in \mathcal{A}(g)$  implying that  $\mathcal{A} \cdot \varphi_t(g) \subset \mathcal{A}(g)$  as stated.

2. Consider  $z \in \mathcal{A}(g)$ ,  $\tau > 0$ , and  $u \in \mathcal{U}$  such that

$$z = \Phi_{\tau,u}(g) = \Phi_{\tau,u} \cdot \varphi_{\tau}(g).$$

The  $\varphi$ -invariance of the connected component  $L_e \cdot g$  implies

$$L_e \cdot g = \varphi_{\tau}(L_e \cdot g) = \varphi_{\tau}(L_e) \cdot \varphi_{\tau}(g) = L_e \cdot \varphi_{\tau}(g).$$

So,  $\varphi_{\tau}(g) = l_{\tau} \cdot g$  where  $l_{\tau} \in L_e$ . In consequence,

$$z \in \mathcal{A}_{\tau} \cdot l_{\tau}g \subset \mathcal{A} \cdot l_{\tau}g \subset \mathcal{A} \cdot g,$$

for the last inclusion use Lemma 2.7, since  $l_{\tau} \in L_e$ . Since  $z \in \mathcal{A}(g)$  is arbitrary, the reachable set  $\mathcal{A}(g)$  is contained in  $\mathcal{A} \cdot g$ , as stated. ■

### 2.3 Control sets

In a more general context, in [7] the authors define control sets for arbitrary systems on connected manifolds. We specify this definition for linear control system on Lie groups.

**2.9 Definition** *A nonempty set  $C \subset G$  is called a control set of the linear control system  $\Sigma$  if it is*

1. *controlled invariant, that is, for every  $g \in C$ , there exists  $u \in \mathcal{U}$  such that  $\Phi_{t,u}(g) \in C$  for some  $t \geq 0$ ,*
2. *approximate controllable, that is,  $C \subset \text{cl}(\mathcal{A}(g))$  for every  $g \in C$ ,*
3.  *$C$  is maximal with properties (i) and (ii).*

A condition for the existence of control set is given by Proposition 3.2.4, in [7]. It states that if a set  $C$  is maximal with respect to the property (2.) and  $\text{int}(C) \neq \emptyset$ , then  $C$  is a control set.

A set  $C$  is said to be invariant in positive, negative time if for any  $t > 0$  and  $u \in \mathcal{U}$

$$\Phi_{t,u}(C) \subset C, \Phi_{-t,u}(C) \subset C \text{ respectively.}$$

The following proposition states the main properties of a control set with nonempty interior.

**2.10 Proposition** *Let  $\Sigma$  be a linear control system and assume that it is locally accessible. For a given control set  $C$  with nonempty interior, it holds:*

1.  $C$  is connected and  $\text{cl}(\text{int}(C)) = \text{cl}(C)$ ;
2. If  $g \in C$  then  $\text{int}(C) \subset \mathcal{A}(g)$ . If  $h \in \text{int}(C)$  the control set is

$$C = \text{cl}(\mathcal{A}(h)) \cap \mathcal{A}^*(h);$$

*In particular the system is controllable on  $\text{int}(C)$ .*

3. Assume that  $\Phi_{t,u}(g)$  is a periodic trajectory, that is,  $\Phi_{t+s,u}(g) = \Phi_{t,u}(g)$  for some  $s > 0$  and all  $t \in \mathbb{R}$ . Thus, if  $g \in \text{int}(C)$  then  $\Phi_{t,u}(g) \in \text{int}(C)$ , for all  $t \in \mathbb{R}$ .
4.  $C$  is closed  $\Leftrightarrow C$  is invariant in positive time  $\Leftrightarrow C = \text{cl}(\mathcal{A}(g))$  for any  $g \in C$ .
5.  $C$  is open  $\Leftrightarrow C$  is invariant in negative time  $\Leftrightarrow C = \mathcal{A}^*(g)$  for any  $g \in C$ .

The proof can be found in [7], Theorem 3.1.5.

Since  $G$  has semi-simple center finite, the following result relates the Lie group to the reachable and controllable set of the linear system.

**2.11 Proposition** *Let  $G$  be a connected Lie group with finite semisimple center. If  $\mathcal{A}$  is open, then  $G^{+,0} \subset \mathcal{A}$  and  $G^{-,0} \subset \mathcal{A}^*$ .*

For more details see [4].

**2.12 Theorem** *Suppose that  $G$  has finite semisimple center and the reachable set  $\mathcal{A}$  is open. For a control set  $C$  it holds that:*

1.  $C$  is closed if and only if  $\mathcal{A}^* = G$ ;
2.  $C$  is open if and only if  $\mathcal{A} = G$ .

For more details, see Theorem 3.6 in [1].

Next we introduce the notion of chain transitivity that will be needed ahead.

For  $g, h \in G$  and  $\epsilon, T > 0$ . An  $(\epsilon, T)$ -chain from  $g$  to  $h$  is given by a natural number  $n \in \mathbb{N}$ , together with points  $g = g_0, g_1, \dots, g_n = h \in G$ , controls  $u_0, \dots, u_{n-1}$  and times  $t_0, \dots, t_{n-1} \geq T$ , such that  $d(\varphi_{t_i, u_i}(g_i), g_{i+1}) < \epsilon$  for  $i = 0, 1, \dots, n-1$ .

A subset  $H \subset G$  is chain transitive if for all  $g, h \in H$  and all  $\epsilon, T > 0$ , there exists a  $(\epsilon, T)$ -chain from  $g$  to  $h$ .

**2.13 Proposition** *Let  $K \subset G$  be a compact invariant set for the uncontrolled system  $\Sigma$  (that is using the control  $u = 0$ ) such that the corresponding flow is chain transitive on  $K$  and the accessibility rank condition holds on  $K$ , i.e., the Lie algebra generated by  $\Sigma$  is  $n$ -dimensional in each point of  $K$ .*

*Then the following assertions are equivalent:*

- (i) *There is a control set  $D$  with  $K \subset \text{int}(D)$*
- (ii) *for all  $g \in K$   $(0, g) \in \mathcal{U} \times G$  is an inner pair, i.e., there is  $T > 0$  such that  $\Phi_{T,0}(g) = \varphi_T(g) \in \text{int}\mathcal{A}(g)$ .*

The proof can be found in [7], Corollary 4.5.11.

The  $\omega$ -limit set of a subset  $H \subset G$  is defined as

$$\omega(H) = \{g \in G : \exists t_k \rightarrow \infty, h_k \in H \text{ and } u \in \mathcal{U} \text{ such that } \Phi_{t_k, u}(h_k) \rightarrow g\}.$$

**2.14 Proposition** *The flow restricted to an  $\omega$ -limit set  $\omega(g)$  for  $g \in G$  is chain transitive.*

For more details see [7], Proposition B. 2.28.

## 3 Control sets on semi-simple Lie groups

### 3.1 General semi-simple Lie theory

In this subsection we review some facts about semi-simple Lie groups theory. As reference we cite [5] [9], [8],[12].

Consider the Lie group  $G$  as a connected non-compact semi-simple with finite center. Let  $\theta$  be a Cartan involution on  $\mathfrak{g}$ , subordinate to a Cartan decomposition of  $\mathfrak{g}$  as  $\mathfrak{k} \oplus \mathfrak{s}$ , where  $\mathfrak{k}$  is a subalgebra and  $\mathfrak{s}$  is a vectorial subspace. From the Cartan-Killing form  $\langle X, Y \rangle = \text{tr}(ad_X ad_Y)$  define the form

$$B_\theta(X, Y) = - \langle X, \theta(Y) \rangle .$$

Choose a maximal Abelian subspace  $\mathfrak{a} \subset \mathfrak{s}$  and a Weyl chamber  $\mathfrak{a}^+ \subset \mathfrak{a}$ . Subordinate to the Abelian subspace  $\mathfrak{a}$  exists a set of roots  $\Omega$  and to the Weyl chamber  $\mathfrak{a}^+$  exists the subset of positive roots  $\Omega^+$ . Denote by  $\Delta$  the simple roots in  $\Omega^+$ , that is,  $\alpha \in \Delta$  if it cannot be written as  $\alpha = \sum_{i=1}^k \beta_i$  where all  $\beta_i$ 's are positive roots. Define the negative roots as  $\Omega^- = -\Omega^+$ .

Given a root  $\alpha \in \Omega$ , there exists a vector field  $H_\alpha$  belonging to  $\mathfrak{a}$  satisfying  $B_\theta(H, H_\alpha) = \alpha(H)$ , for all  $H \in \mathfrak{a}$ . The vector field  $H_\alpha$  is called coroot.

The Iwasawa decomposition for the Lie algebra  $\mathfrak{g}$  is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+,$$

where  $\mathfrak{k}, \mathfrak{a}$  are inherited from Cartan decomposition and  $\mathfrak{n}^+$  is given by  $\sum_{\alpha \in \Omega^+} \mathfrak{g}_\alpha$ , where  $\mathfrak{g}_\alpha$  is the root space associated to  $\alpha$ .

The respective global decompositions for the Lie group are written as

$$G = KS \text{ and } G = KAN^+,$$

where  $K = \exp \mathfrak{k}$ ,  $S = \exp \mathfrak{s}$ ,  $A = \exp \mathfrak{a}$ ,  $N^+ = \exp \mathfrak{n}^+$ ,  $K \cap AN^+ = \{e\}$ ,  $N^+, A$  are simply connected nilpotent Lie subgroups and  $K$  is a compact Lie subgroups.

Let  $M^*$  be the normalizer of  $A$  in  $K$ , i.e.,

$$M^* = \{g \in K : Ad_g(a) = a_1, \text{ with } a, a_1 \in A\}$$

and  $M$  be the centralizer of  $A$  in  $K$ , i.e.,

$$M = \{g \in K : Ad_g(a) = a \forall a \in A\}.$$

Denote its Lie algebras as  $\mathfrak{m}^*$  and  $\mathfrak{m}$  respectively.

The Weyl group  $W$  of  $G$  is  $M^*/M$ . In particular, it possess an element  $w^- \in W$ , which send the positive roots  $\Omega^+$  to  $\Omega^-$ , such element is unique and called the principal involution.

Let  $\Lambda$  be a subset of  $\Delta$ . We can construct the Lie algebra  $\mathfrak{g}(\Lambda)$  using the eigenspaces  $\mathfrak{g}_\alpha$ , for  $\alpha \in \Lambda$ . Define the Lie algebras as  $\mathfrak{k}(\Lambda) = \mathfrak{g}(\Lambda) \cap \mathfrak{k}$ ,  $\mathfrak{a}(\Lambda) = \mathfrak{g}(\Lambda) \cap \mathfrak{a}$ , and  $\mathfrak{n}^+(\Lambda) = \mathfrak{g}(\Lambda) \cap \mathfrak{n}^+$ .

Let  $G(\Lambda)$  and  $K(\Lambda)$  be the connected Lie groups with Lie algebras  $\mathfrak{g}(\Lambda)$  and  $\mathfrak{k}(\Lambda)$ , respectively. The Lie subgroup  $G(\Lambda)$  is a connected semi-simple with finite center and its Iwasawa decomposition is

$$G(\Lambda) = K(\Lambda)A(\Lambda)N^+(\Lambda),$$

where  $A(\Lambda) = \exp \mathfrak{a}(\Lambda)$ ,  $N^+(\Lambda) = \exp \mathfrak{n}^+(\Lambda)$ .

Let  $\mathfrak{a}_\Lambda$  be the subset of  $\mathfrak{a}$  perpendicular for all  $\alpha \in \Lambda$ , that is

$$\mathfrak{a}_\Lambda = \{H \in \mathfrak{a} : \alpha(H) = 0, \forall \alpha \in \Lambda\}.$$

Define  $A_\Lambda = \exp \mathfrak{a}_\Lambda$ .

The parabolic subalgebra of type  $\Lambda$  is

$$\mathfrak{p}_\Lambda = \mathfrak{n}^-(\Lambda) \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+. \quad (3.6)$$

In particular,  $\Lambda = \{0\}$  is called minimal parabolic, that is,  $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$ .

The standard parabolic subgroup  $P_\Lambda$  is the normalizer of the subalgebra  $\mathfrak{p}_\Lambda$  in the Lie group  $G$ . It has a Langlands decomposition given by

$$P_\Lambda = K_\Lambda AN^+,$$

where  $K_\Lambda$  decomposes as  $K_\Lambda = MK(\Lambda)$

In particular, the minimal parabolic subgroup  $P$  is the normalizer of  $\mathfrak{p}$  in  $G$  and its Iwasawa decomposition is  $P = MAN^+$ .

**3.1 Theorem (Bruhat Theorem)** *The Lie group  $G$  is equal to*

$$G = PWP = \coprod_{w \in W} PwP,$$

where the second term is a disjoint union and  $PwP$  is a Bruhat cell. In particular, the only open cell is given  $Pw^-P$ , where  $w^-$  stand for the principal involution.

## 3.2 Jordan decomposition

In this subsection we relate the Lie algebra decomposition with elements of the semi-simple theory. As references we suggest the books [8], [10] and [12].

The derivation  $\mathcal{D}$  associated to the drift  $\mathcal{X}$  is inner, that is,  $\mathcal{D}$  is equal to  $ad_X$  for some right-invariant vector field  $X \in \mathfrak{g}$  and its flow  $(\varphi_t)_{t \in \mathbb{R}}$  is given by

$$\varphi_t(g) = \exp^{-tX} g \exp^{tX} = C_{\exp^{tX}}(g).$$

The adjoint  $ad : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$  defines a natural representation of  $\mathfrak{g}$  in the linear group  $\mathfrak{gl}(\mathfrak{g})$ . The Jordan decomposition applied to  $X$  results in the decomposition of  $X$  in  $E, H, N \in \mathfrak{g}$  such that  $X = E + H + N$  and

$$[X, H] = [X, E] = [X, N] = [H, E] = [H, N] = [E, N] = 0.$$

Moreover, the  $ad_E$ -eigenvalues are pure complex, the derivation  $ad_H$  is diagonal with only real eigenvalues and  $ad_N$  is nilpotent. Consequently, the eigenvalues of  $\mathcal{D}$  are equal to the eigenvalues of  $(ad_H + ad_E)$ . Thus, the decomposition by eigenvalues of  $\mathfrak{g}$  depends only on the  $ad_H$ -eigenvalues. Consequently, the subalgebras  $\mathfrak{g}^+, \mathfrak{g}^-$  and  $\mathfrak{g}_0$  relate to  $ad_H$  as follows:

$$\begin{aligned} \mathfrak{g}^+ &= \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha, \\ \mathfrak{g}^- &= \bigoplus_{\alpha < 0} \mathfrak{g}_\alpha, \\ \mathfrak{g}^0 &= \ker(ad_H) = \mathfrak{z}_H, \end{aligned}$$

where  $\alpha \in \text{Spec}(ad_H)$ .

For  $X \in \mathfrak{g}$ , there exists an Iwasawa decomposition such that  $H \in \text{cl } \mathfrak{a}^+$  and  $E \in \mathfrak{k}_H = \{Y \in \mathfrak{k} : [Y, H] = 0\}$ .

Indeed, the first part is guaranteed by Proposition 4.6, pg.420 in [8] and by the positivity choice. For the second statement, note that the eigenvalues of  $ad_E$  are pure complex and by Jordan decomposition  $ad_E$  commute with  $ad_H$ .

Denote  $\mathfrak{a}_H$  the maximal Abelian subspace such that  $H \in \text{cl } \mathfrak{a}^+$ . For  $H \in \text{int}(\mathfrak{a}^+)$ , take  $\mathfrak{n}_H^+ = \bigoplus_{\lambda \in \Omega^+} \mathfrak{g}_\lambda$  and  $\mathfrak{n}_H^- = \bigoplus_{\lambda \in \Omega^-} \mathfrak{g}_\lambda$ .

For  $H \in \text{bd}(\mathfrak{a}^+)$ , define

$$\Lambda(H) = \{\alpha \in \Lambda : \alpha(H) = 0\},$$

consequently  $\mathfrak{z}_H = \mathfrak{z}_{\Lambda(H)}$ ,  $\mathfrak{k}_H = \mathfrak{k}_{\Lambda(H)}$  and  $\mathfrak{n}^\pm(\Lambda(H)) = \mathfrak{n}^\pm \cap \mathfrak{z}_{\Lambda(H)}$ .

Let  $\mathfrak{n}_\Lambda^\pm = \sum_{\alpha \in \Omega^\pm - \langle \Lambda \rangle} \mathfrak{g}_\alpha$  and  $N_\Lambda^\pm = \exp(\mathfrak{n}_\Lambda^\pm)$ . Notice  $N^+$  decomposes as  $N^+ = N(\Lambda)^+ N_\Lambda^+$ , where  $N(\Lambda)^+ = \exp(\mathfrak{n}^\pm(\Lambda(H)))$  normalizes  $N_\Lambda^+$  and  $N(\Lambda)^+ \cap N_\Lambda^+ = e$ . In this case, take  $\mathfrak{n}_H^\pm = \mathfrak{n}_\Lambda^\pm$  and  $\mathfrak{z}_H = \mathfrak{z}_{\Lambda(H)}$ .

Define  $\mathfrak{g}^+ = \mathfrak{n}_H^+$ ,  $\mathfrak{g}^- = \mathfrak{n}_H^-$ ,  $\mathfrak{g}^0 = \mathfrak{z}_H$  and the connected subgroups of  $G$  :

$$G^+ = \exp(\mathfrak{n}_H^+) = N_H^+,$$

$$G^- = \exp(\mathfrak{n}_H^-) = N_H^-,$$

$$G^0 = (\exp(\mathfrak{z}_H))_e = (Z_H)_e.$$

In difference from solvable Lie groups, these subgroups do not decompose  $G$ .

Take

$$\mathfrak{p}_H = \sum_{\alpha(H) \geq 0} g_\alpha,$$

where  $\alpha \in \Omega$  and  $P_H$  is the subgroups of  $G$  such that  $Ad_{P_H}(\mathfrak{p}_H) = \mathfrak{p}_H$ .

Define  $P_H^- = C_{w^-}(P_H)$ , equivalently  $Ad_{P_H^-}(\mathfrak{p}_H^-) = \mathfrak{p}_H^-$ , where

$$\mathfrak{p}_H^- = \sum_{\alpha(H) \leq 0} g_\alpha = Ad_{w^-}(\mathfrak{p}_H).$$

In particular the connected components  $(P_H)_e$  and  $(P_H^-)_e$  are equal to  $G^0 G^+$  and  $G^0 G^-$  respec. They are an open sets contained in the translated open cell of the Bruhat decomposition. Moreover, consider the sets  $B_k$  as  $N_H^+(Z_H)_e \cdot k N_H^-$ . The union  $\bigcup_{k \in K} B_k$  is the translated open cell.

Indeed, the translated open cell is

$$(\overline{w^-})^{-1} P w^- P = (\overline{w^-})^{-1} N^+ w^- M A N^+ = (\overline{w^-})^{-1} N^+ (\overline{w^-}) M A N^+ = N^- M A N^+,$$

where  $(\overline{w^-}) = w^-$  is a representation of the principal involution  $w^-$  in  $K$ .

On the other hand, the subgroups  $N^\pm$  decompose as  $N(\Lambda)^\pm N_H^\pm$ . Since  $N(\Lambda)^\pm, A, M$  are contained in  $Z_H$  and  $Z_H$  can be written as  $\bigcup_{k \in K} (Z_H)_e \cdot k$ , we obtain

$$N^-MAN^+ = \bigcup_{k \in K} N_H^-(Z_H)_e \cdot kN_H^+$$

In particular, we have  $\bigcup_{k \in K_H} B_k$  is the open Bruhat cell in  $G$ .

### 3.3 Results

The results are placed in this subsection. As before, we consider  $G$  as connected non-compact semi-simple Lie group with finite center and that the reachable set from identity  $\mathcal{A}$  is open.

We start by proving the existence of a control set containing the identity. Since the compact set  $\{e\}$  is chain transitive by the flow of the linear vector field, Proposition 2.13 ensures the existence of a control set containing  $\{e\}$  if and only if  $(0, e)$  is an inner pair. On the other hand, by Proposition 2.5 item 6. the identity belongs to  $\text{int}\mathcal{A}_t$  for some  $t > 0$  if and only if  $\mathcal{A}$  is open. Denote by  $C_e$  the control set containing the identity.

**3.2 Theorem** *If  $D$  is a control set with nonempty interior of the linear control system  $\Sigma$ , then  $D \subset R_l(C_e)$ , for some  $l \in K_H$ .*

Before proving the previous theorem, we need the following lemmas.

**3.3 Lemma** *The linear vector field flow preserves  $Z_H$ , that is,  $\varphi_t(Z_H) = Z_H$ .*

**Proof:** For a given  $g \in Z_H$ , the linear vector field flow is

$$\begin{aligned} \varphi_t(g) &= C_{\exp(tX)}(g) \\ &= \exp(-t(N+E)) \exp(-tH)g \exp(tH) \exp(t(E+N)) \end{aligned}$$

Since  $g \in Z_H$ , we have  $\exp(-tH)g \exp(tH) = g$ . As consequence, the flow is

$$\varphi_t(g) = \exp(-t(N+E))g \exp(t(E+N)).$$

Apply  $Ad_{\varphi_t(g)}$  to the vector field  $H$ , we obtain

$$\begin{aligned} Ad_{\varphi_t(g)}(H) &= Ad_{\exp(-t(N+E))g \exp(t(E+N))}(H) \\ &= Ad_{\exp(-t(N+E))}Ad_gAd_{\exp(t(E+N))}(H) \\ &= H. \end{aligned}$$

■

**3.4 Lemma** *If  $k$  belong to  $K_H \cap \text{cl}(\mathcal{A})$  then  $k^{-1} \in \text{cl}(\mathcal{A})$ .*

**Proof:** Consider  $k \in K_H \cap \text{cl}(\mathcal{A})$ . Lemma 3.3 ensures that  $Z_H$  is  $\varphi$ -invariant and therefore its connected components are also  $\varphi$ -invariant. Proposition 2.11 implies that  $G^0 = (Z_H)_e$  is contained in  $\mathcal{A}$ . As consequence

$$\varphi_t(k \cdot (Z_H)_e) = k \cdot (Z_H)_e = k \cdot \varphi_t((Z_H)_e) \subset \text{cl}(\mathcal{A})\varphi_t((Z_H)_e) \subset \text{cl}(\mathcal{A}).$$

It proves that  $\varphi_t(k) \in \text{cl}(\mathcal{A})$  for all  $t \in \mathbb{R}$ .

Since  $\{\varphi_t(k) : t \in \mathbb{R}\} \subset \text{cl}(\mathcal{A})$ , we have  $\text{cl}(\mathcal{A}) \cdot k \subset \text{cl}(\mathcal{A})$ .

Let  $h = \Phi_{t,u} \in \mathcal{A}$ . Follows

$$h \cdot g = \Phi_{t,u} \cdot \varphi_t(\varphi_{-t}(g)) = \Phi_{t,u}(\varphi_{-t}(g)). \quad (3.7)$$

For  $\varphi_{-t}(g) \in \text{cl}(\mathcal{A})$ , there is a sequence  $z_n \in \mathcal{A}_{t_n}$  such that  $z_n \rightarrow \varphi_{-t}(g)$ . By continuity,  $\Phi_{t,u}(z_n) \rightarrow h \cdot g$ . On the other hand  $\Phi_{t,u}(z_n) \in \mathcal{A}_{t+t_n} \subset \mathcal{A}$ , thus  $h \cdot g \in \text{cl}(\mathcal{A})$ .

Choosing  $h$  and  $g$  as  $k$ , we have  $\text{cl}\{k^n : n \in \mathbb{N}\}$  is contained  $\text{cl}(\mathcal{A})$ . However,  $K_H$  is a compact subgroup,  $\text{cl}\{k^n : n \in \mathbb{N}\} \subset \text{cl}(\mathcal{A})$  is a subgroup. Therefore  $k^{-1} \in \text{cl}(\mathcal{A})$  as stated.  $\blacksquare$

Now we return to the proof of Theorem 3.2

**Proof:** This proof is divided in 4 steps:

**Step 1.** There is  $a \in (P_H)_e$  and  $k \in K_H$  such that  $\text{int}(D) \subset \mathcal{A}(ak)$ .

Since  $\cup_{k \in K_H} B_k$  is open and dense in the Lie group, there exists  $k \in K_H$  such that  $\text{int}(D) \cap B_k \neq \emptyset$ . Theorem 3.1 ensure the existence of  $a \in (P_H)_e$  and  $b \in N_H^-$  such that  $x = akb \in \text{int}(D) \cap B_k$ .

Since in  $\text{int}(D)$  the exactly controllability holds, there exists  $\tau > 0$  and  $u \in \mathcal{U}$  such that  $x = \Phi_{n\tau,u}(x)$  for any  $n \in \mathbb{N}$ .

Let  $\rho$  be a left invariant Riemannian metric on  $G$ , we have

$$\begin{aligned} \rho(x, \Phi_{n\tau,u}(ak)) &= \rho(\Phi_{n\tau,u}(x), \Phi_{n\tau,u}(ak)) \\ &= \rho(\Phi_{n\tau,u}(ak)\varphi_{n\tau}(b), \Phi_{n\tau,u}(ak)) \\ &= \rho(\varphi_{n\tau}(b), e) \rightarrow 0. \end{aligned}$$

Therefore, for  $t > 0$  large enough, it holds that  $\Phi_{t,u}(ak) \in \text{int}(D)$ . Since  $\text{int}(D)$  is exactly controllable we have  $\text{int}(D) \subset \mathcal{A}(ak)$  as stated.

**Step 2.** There is  $b \in (P_H^-)_e$  and  $l \in K_H$  such that  $D \subset \mathcal{A}^*(bl)$ .

Similarly to Step 1. Note  $\cup_{k \in K_H} B_k^{-1}$  is also open and dense in  $G$ . In consequence, exist  $b \in (P_H^-)_e$  and  $l \in K_H$  such that  $bl \in \text{int}(D)$ . By Proposition 2.10 item 2, we have that  $D = \text{cl}(\mathcal{A}(bl)) \cap \mathcal{A}^*(bl)$  and so  $D \subset \mathcal{A}^*(bl)$ .

**Step 3.** It holds  $k \in \text{cl}(\mathcal{A}) \cdot 1$ .

From steps 1 and 2, we have  $\text{int}(D) \subset \mathcal{A}(ak) \cap \mathcal{A}^*(bl)$ . Therefore  $bl \in \mathcal{A}(ak)$  and so

$$bl \in \mathcal{A}(ak) \subset \mathcal{A} \cdot ak \subset \mathcal{A} \cdot k.$$

The first inclusion follows from Lemma 2.8 item 2 and the second one from Lemma 2.7.

Since  $b \in (P_H^-)_e$ , there exists  $c \in N_H^-$  and  $d \in (Z_H)_e$  such that  $b = cd$ . Thus,  $cl(l^{-1}dl) = bl \in \mathcal{A} \cdot k$ , which implies that  $cl \in \mathcal{A} \cdot k(l^{-1}dl)^{-1}$ . Since  $(kl^{-1})d^{-1}(kl^{-1})^{-1} \in (Z_H)_e$  we obtain

$$\mathcal{A} \cdot k(l^{-1}dl)^{-1} = \mathcal{A} \cdot (kl^{-1})d^{-1}(kl^{-1})^{-1} \cdot k \subset \mathcal{A} \cdot k, \quad (3.8)$$

proving that  $cl \in \mathcal{A} \cdot k$ .

Moreover, the  $\varphi$ -invariance of the connected component  $(Z_H)_e \cdot k$  implies that  $\varphi_t(\mathcal{A} \cdot k) \subset \mathcal{A} \cdot k$  and so  $\varphi_t(cl) \in \mathcal{A} \cdot k$ , for all  $t \geq 0$ .

Also, the  $\varphi$ -invariance of  $l \cdot (Z_H)_e$  implies the existence of  $z_t^{-1} \in (Z_H)_e$  for any  $t \in \mathbb{R}$  such that  $\varphi_t(l) = lz_t^{-1}$  and so

$$\varphi_t(c)lz_t^{-1} = \varphi_t(cl) \in \mathcal{A} \cdot k \Rightarrow \varphi_t(c)l \in \mathcal{A} \cdot kz_t = \mathcal{A} \cdot (kz_tk^{-1})k \subset \mathcal{A} \cdot k$$

because  $kz_tk^{-1} \in (Z_H)_e$ .

Thus,  $\varphi_t(c)l \in \mathcal{A} \cdot k$  for  $t \geq 0$  and since  $\varphi_t(c) \rightarrow e$  as  $t \rightarrow \infty$ , we get  $l \in \text{cl}(\mathcal{A}) \cdot k$ . As consequence,  $lk^{-1} \in \text{cl}(\mathcal{A}) \cap K_H$ . By Lemma 3.4 we obtain  $kl^{-1} = (lk^{-1})^{-1} \in \text{cl}(\mathcal{A})$  and so  $k \in \text{cl}(\mathcal{A}) \cdot l$  as stated.

**Step 4.** There exists  $l \in K_H$  such that  $D \subset R_l(C_e)$ .

The first step ensures that  $\text{int}(D) \subset \mathcal{A}(ak)$ , and the third step ensures that

$$ak = k(k^{-1}ak) \in \text{cl}(\mathcal{A}) \cdot l(k^{-1}ak) = \text{cl}(\mathcal{A}) \cdot (lk^{-1})a(lk^{-1})^{-1} \cdot l \subset \text{cl}(\mathcal{A}) \cdot l,$$

since  $(lk^{-1})a(lk^{-1})^{-1} \in (P_H)_e$ .

The  $\varphi$ -invariance of the connected component  $(P_H)_e \cdot l$  implies  $\text{cl}(\mathcal{A}(ak)) \subset \text{cl}(\mathcal{A}) \cdot l$  proving  $D \subset \text{cl}(\mathcal{A}) \cdot l$ .

Moreover, by Lemma 2.7 and Lemma 2.8, it also holds that  $\mathcal{A}^*(bl) \subset \mathcal{A}^* \cdot bl \subset \mathcal{A}^* \cdot l$ . Therefore  $D \subset \text{cl}(\mathcal{A}(ak)) \cap \mathcal{A}^*(bl) \subset \text{cl}(\mathcal{A}) \cdot l \cap \mathcal{A}^* \cdot l = R_l(C_e)$ .  $\blacksquare$

From the previous theorem we obtain two equivalent conditions for the inverse inclusion.

**3.5 Corollary** *Let  $D$  be a control set with nonempty interior of the linear system  $\Sigma$ . The following assertions are equivalent:*

1.  $D = R_l(C_e)$ , for some  $l \in K_H$
2.  $(Z_H)_e \cdot l \subset \text{int}(D)$ ,
3.  $(Z_H)_e \cdot l \cap \text{int}(D) \neq \emptyset$ .

**Proof:** (1  $\Leftrightarrow$  2) Notice  $(Z_H)_e \subset \mathcal{A}$  and  $(Z_H)_e \subset \mathcal{A}^*$ . So  $(Z_H)_e \subset \text{int}(C_e)$ . By equality  $D = R_l(C_e)$  implies

$$(Z_H)_e \cdot l \subset R_l(\text{int}(C_e)) \subset \text{int}(D). \quad (3.9)$$

Reciprocally, the  $(Z_H)_e \cdot l$  invariance by  $\varphi$  together with Lemma 2.8 imply that  $R_l(\mathcal{A}) = \mathcal{A}(l)$  and  $R_l(\mathcal{A}^*) = \mathcal{A}^*(l)$ .

Moreover, since  $l \in \text{int}D$  it follows from Proposition 2.10 that

$$D = \text{cl}(\mathcal{A}(l)) \cap \mathcal{A}^*(l) = \text{cl}(R_l(\mathcal{A})) \cap R_l(\mathcal{A}) = R_l(C_e).$$

(2  $\Rightarrow$  3) It is trivial.

(3  $\Rightarrow$  1) It is enough to prove  $(Z_H)_e \cdot l \cap \text{int}(D) \neq \emptyset$  implies  $D = R_l(C_e)$ .

However, for any  $c \in (Z_H)_e \cdot l \cap \text{int}(D)$  applying by Lemma 2.8 we obtain  $D = R_c(C_e)$ .

Since  $c = zl$  for some  $z \in (Z_H)_e$  and  $R_z(C_e) = C_e$ , we have that

$$R_l(C_e) = R_l(R_z(C_e)) = (R_l \circ R_z)(C_e) = R_{zl}(C_e) = R_c(C_e) = D.$$

■

The next result shows the invariant control sets are rare.

**3.6 Proposition** *A linear control system  $\Sigma$  admits at most one invariant in positive(negative) time control set with non-empty interior.*

**Proof:** We will prove the invariant in positive time case. This proof is divided in two steps:

**Step 1.** If  $D$  is an invariant control set then  $C_e$  is also invariant.

Since  $\text{int}(D) \neq \emptyset$ , exist  $a \in N_H^-, b \in (Z_H)_e \cdot l$  and  $c \in N_H^+$  such that  $x = cba \in \text{int}(D)$ .

The exactly controllability implies the existence of a  $\tau > 0$ ,  $u \in \mathcal{U}$  such that for all  $n \in \mathbb{N}$ , it holds that

$$x = \Phi_{-n\tau, u}(x) = \Phi_{-n\tau, u}(ab)\varphi_{-n\tau}(c).$$

Since  $\varphi_{-t}(c) \rightarrow e$  as  $t \rightarrow \infty$ , for  $n \in \mathbb{N}$  sufficiently large the flow  $\Phi_{-n\tau, u}(ab)$  belong to  $\text{int}(D)$ . By hypothesis, the invariance of the control set  $D$  ensures

$$ab \in \Phi_{n\tau, \Theta_{-n\tau}u}(\text{int}(D)) \subset \text{int}(D).$$

Also, together with Lemmas 2.7 and 2.8 we have

$$D \subset \mathcal{A}^*(ab) \subset \mathcal{A}^* \cdot ab \subset \mathcal{A}^* \cdot b.$$

Therefore, exist a  $\tau' > 0$ ,  $u' \in U$  such that  $\Phi_{-\tau', u'} \cdot b \in \text{int}(D)$  and since  $\Phi_{-\tau', u'} \cdot b = \Phi_{-\tau', u'}(\varphi_{\tau'}(b))$ , we obtain that  $\varphi_{\tau'}(b) \in \text{int}(D)$ .

The  $(Z_H)_e \cdot l$  invariance by the flow  $\varphi$  implies  $(Z_H)_e \cdot l \cap \text{int}(D) \neq \emptyset$ . As consequence of Corollary 3.5 we have that  $D = R_l(C_e)$ .

However, Proposition 2.10 ensures that a control set with nonempty interior is invariant if and only if it is closed.

Therefore  $C_e$  is closed and so it is invariant as stated.

**Step 2.** The control set  $C_e$  is the unique control set.

Theorem 2.12 item 1. states that a control set is positively invariant if and only if  $G = \mathcal{A}^*$ . So, for any  $l \in K_H$  there exists  $\tau > 0$  such that  $l^{-1} \in \mathcal{A}_\tau^*$ .

Since  $\mathcal{A}_\tau^* = \varphi_{-\tau}(\mathcal{A}_\tau^{-1})$ , we have that  $\varphi_{-\tau}(l) \in \mathcal{A}_\tau$ . Moreover,

$$(Z_H)_e \cdot l = \varphi_{-\tau}((Z_H)_e \cdot l) = (Z_H)_e \cdot \varphi_{-\tau}(l) \subset C_e.$$

In particular,  $l \in C_e$  which implies  $R_l(C_e) = C_e$ .

Since  $l \in K_H$  is arbitrary, Theorem 3.2 implies that  $D \subset C_e$  for any control set  $D$  with nonempty interior. Therefore  $C_e$  is the unique control set as stated. ■

### 3.3.1 Semi-simple derivation

Suppose that the derivation  $\mathcal{D}$  is semi-simple, that is,  $\mathcal{D} = ad_E + ad_H$ . This assumption allows to prove the Theorem 3.2 inverse inclusion. We start proving a necessary Lemma.

**3.7 Lemma** *If the derivation  $\mathcal{D}$  is semi-simple, then the linear vector field flow preserves  $K_H$ .*

**Proof:** Notice  $\mathfrak{k}_H$  is a subalgebra. Since  $\mathcal{D}$  is inner for  $k \in K_H$  the linear vector field flow of the linear fields is  $\varphi_t(k) = C_{\exp(t(H+E))}(k) = C_{\exp(tE)}(k)$ , for  $E \in \mathfrak{k}_H$ . ■

**3.8 Theorem** *Suppose the derivation  $\mathcal{D}$  is semi-simple. Any control set  $D$  with nonempty interior satisfies  $D = R_l(C_e)$ , for some  $l \in K_H$ .*

**Proof:** We only have to prove that,  $R_l(C_e)$  is a control set for any  $l \in K_H$ . However, since the derivation  $\mathcal{D}$  is semi-simple, Lemma 3.7 implies that  $K_H$  is  $\varphi$ -invariant.

Since  $K_H$  is  $\varphi$ -invariant, any of its connected components are also  $\varphi$ -invariant. Therefore, for any  $l \in K_H$  the set  $\omega(l) \subset (K_H)_e \cdot l$  and  $\omega(l) \neq \emptyset$ , since  $(K_H)_e \cdot l$  is compact.

By Proposition 2.13 exists a control set containing the  $\omega$ -limit set if and only if  $\Phi_{T,0}(g)$  is a inner pair for all  $g$  belonging to the  $\omega(l)$ .

On the other hand,  $\Phi_{t,0}(g)$  is equal to  $\varphi_t(g)$ . Since  $\mathcal{A}$  is open Proposition 2.5 item 5. states  $e \in \text{int}\mathcal{A}_T$ , for some  $T > 0$  and  $\mathcal{A}(g) = \mathcal{A} \cdot g$  by Lemma 2.8. Thus  $\Phi_{T,0}(g)$  belong to  $\mathcal{A}(g)$  is a inner pair for all  $g \in \omega(K_H)$

Therefore  $(Z_H)_e \cdot l \cap \text{int}(D) \neq \emptyset$  and by Corollary 3.5  $D = R_l(C_e)$ . ■

**3.9 Corollary** *There are at most  $\left| \frac{K_H}{(K_H)_e} \right|$  control sets with nonempty interior in  $G$ .*

**3.10 Proposition** *If  $\mathcal{D}$  is a semi-simple, there is no bounded control sets with non empty interior.*

**Proof:** By Theorem 3.8 is sufficient to prove that  $C_e$  is unbounded. Notice that Proposition 2.10 item 2. and Proposition 2.11 ensures that  $G^0$  is contained in  $C_e$ . In particular,  $G^0$  contains the Abelian part of the Iwasawa decomposition,  $A = \exp(\mathfrak{a})$ , which is a simply connected Abelian Lie group. Consequently  $l^{-1}A \subset D$ . ■

## References

- [1] AYALA, V., DA SILVA, A. J., ZSIGMOND G. *Control sets of linear systems on Lie groups*. Nonlinear Differential Equations and Applications, Vol. 24:8 (2017)
- [2] AYALA, V., SAN MARTIN, L.A.B., *Controllability properties of a class of control systems on Lie groups*, Lecture Notes in Control and Information Sciences 258 (2001), pp. 83-92.
- [3] AYALA, V., TIRAO, J., *Linear Control Systems on Lie Groups and Controllability*. American Mathematical Society, Series: Symposia in Pure Mathematics, Vol. 64, pp. 47-64, 1999.
- [4] DA SILVA, A. J., *Controllability of linear systems on solvable Lie groups*. To appear in SIAM, 2016.
- [5] DA SILVA, A. J. , AYALA, V., *Controllability of linear systems on Lie groups with finite semisimple center*. Submitted to the SIAM Journal of Control and Optimization, December 2015.
- [6] JOUAN, PH., *Equivalence of Control Systems with Linear Systems on Lie Groups and Homogeneous Spaces*, ESAIM: Control Optimization and Calculus of Variations, 16 (2010) 956-973.
- [7] COLONIUS, F., KLIEMANN, W., **The Dynamics of Control**. Birkhäuser, 2000.
- [8] HELGASON, S., **Differential Geometry, Lie Groups and Symmetric Spaces**, Academic Press, (1978).

- [9] KNAPP, A. W., **Lie Groups Beyond an Introduction**, Second Edition, Birkhäuser, Berlin, (2004).
- [10] ONISHCHIK, A. L. AND E. B. VINBERG, **Lie Groups and Lie Algebras III - Structure of Lie Groups and Lie Algebras**, Berlin: Springer (1990).
- [11] SAN MARTIN, L.A.B., **Àlgebra de Lie**, Ed. Unicamp,1999
- [12] WARNER, G., **Harmonic analysis on semi-simple Lie groups I**, Springer-Verlag (1972).

## Chapter 4

# Isometries of almost-Riemannian structures on Lie groups

# Isometries of almost-Riemannian structures on Lie groups

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## Abstract

A simple Almost-Riemannian Structure on a Lie group  $G$  is defined by a linear vector field (that is an infinitesimal automorphism) and  $\dim(G) - 1$  left-invariant ones.

It is first proven that two different ARSs are isometric if and only if there exists an isometry between them that fixes the identity. Such an isometry preserves the left-invariant distribution and the linear field. If the Lie group is nilpotent it is an automorphism.

These results are used to state a complete classification of the ARSs on the 2D affine and the Heisenberg groups.

Keywords: Lie groups; Linear vector fields; Almost-Riemannian geometry; Isometries.

## 1 Introduction

This paper is devoted to isometries of Almost-Riemannian structures on Lie groups. The purpose is to classify these structures, to find geometric invariants, and to determine their groups of isometries.

An almost-Riemannian structure (ARS in short) on an  $n$ -dimensional differential manifold can be defined, at least locally, by a set of  $n$  vector fields, considered as an orthonormal frame, that degenerates on some singular set. This geometry goes back to [13] and [21]. It appears as a part of sub-Riemannian geometry, and has aroused some interest, as shown by the recent papers [2], [3], [7], [8], [9], [10], [11].

On an  $n$ -dimensional connected Lie group the simplest ARSs are defined by a set of  $n - 1$  left-invariant vector fields and one linear vector field, the rank of which is equal to  $n$  on a proper open and dense subset and that satisfy the rank condition (a vector field on a Lie group is linear if its flow is a one parameter group of automorphisms, see Section 2, and [5], [6], [12], [15], [16] about linear systems on Lie groups).

These ARSs, among which we find the famous Grushin plane on the Abelian Lie group  $\mathbb{R}^2$ , has been studied in [4]. Among the results of this paper there is a study of the singular locus, that is the set of points where the vector fields fail to be independent. It is an analytic set, but not a subgroup, not even a submanifold, in general. However sufficient conditions for the singular locus to be a submanifold or a subgroup were exhibited and are recalled in Section 2. This locus is very important in what concern the structure of ARSs, in particular in view of a classification. Another important geometric locus is the set of singularities of the linear field. It is always a subgroup.

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In this paper we deal with smooth isometries, i.e. diffeomorphisms that respect the Euclidean metric of the tangent space at each point. First of all we show that such an isometry should preserve the singular locus and the group of singularities of the linear field. The main consequence is that the group of isometries of an ARS does not act transitively on  $G$ . Another consequence is that a left translation  $L_g$  is an isometry if and only if  $g$  belongs to the set of singularities of the linear field.

Then we prove that the isometries preserve the left-invariant distribution generated by the  $n - 1$  left-invariant vector fields, and also the linear field (up to the sign), see Theorem 2.

These constraints are rather strong. Consider for instance ARSs on the Heisenberg group, in the case where the distribution generated by the left-invariant vector fields is a subalgebra. Then the group of isometries is generically reduced to the identity.

Kivioja and Le Donne proved in [18] that the isometries of left-invariant metrics on nilpotent groups are affine, that is composed of an automorphism and a left translation. Their result cannot be directly applied here since the metric is not left-invariant. It can however be adapted, and it is shown in Theorem 4 that the isometries of ARSs on nilpotent Lie groups are affine.

The paper is organized as follows. In Section 2 the basic definitions and notations are stated, together with some useful results.

Section 3 is devoted to the general theorems quoted above.

In Sections 4 and 5 the ARSs of the 2D affine group and the 3D Heisenberg group are completely classified. We begin by a classification by isometries, up to a global rescaling of the metric that does not modify the geometry. In order to reduce the number of parameters we then identify ARSs that differ only by the left-invariant metric on the left-invariant distribution. The typical ARSs that we obtain are characterized by the following geometric invariants: the singular locus, the set of singularities of the linear field, the tangency points and the eigenvalues of the derivation associated to the linear field.

## 2 Basic definitions

### 2.1 Linear vector fields

In this section the definition of linear vector fields and some of their properties are recalled. More details can be found in [14].

Let  $G$  be a connected Lie group and  $\mathfrak{g}$  its Lie algebra (the set of left-invariant vector fields, identified with the tangent space at the identity). A vector field on  $G$  is said to be *linear* if its flow is a one-parameter group of automorphisms. Notice that a linear vector field is consequently analytic and complete. The flow of a linear vector field  $\mathcal{X}$  will be denoted by  $(\varphi_t)_{t \in \mathbb{R}}$ .

The following characterization will be useful in the sequel.

#### **Characterization of linear vector fields**

*A vector field  $\mathcal{X}$  on a connected Lie group  $G$  is linear if and only if  $\mathcal{X}$  belongs to the normalizer of  $\mathfrak{g}$  in the algebra  $V^\omega(G)$  of analytic vector fields of  $G$  and verifies  $\mathcal{X}(e) = 0$ , that is*

$$\forall Y \in \mathfrak{g} \quad [\mathcal{X}, Y] \in \mathfrak{g} \quad \text{and} \quad \mathcal{X}(e) = 0. \quad (1)$$

According to (1) one can associate to a given linear vector field  $\mathcal{X}$  the derivation  $D$  of  $\mathfrak{g}$  defined by:

$$\forall Y \in \mathfrak{g} \quad DY = -[\mathcal{X}, Y],$$

that is  $D = -\text{ad}(\mathcal{X})$ . The minus sign in this definition comes from the formula  $[Ax, b] = -Ab$  in  $\mathbb{R}^n$ . It also enables to avoid a minus sign in the useful formula:

$$\forall Y \in \mathfrak{g}, \quad \forall t \in \mathbb{R} \quad \varphi_t(\exp Y) = \exp(e^{tD}Y). \quad (2)$$

An *affine vector field* is an element of the normalizer  $\mathfrak{N}$  of  $\mathfrak{g}$  in  $V^\omega(G)$ , that is

$$\mathfrak{N} = \text{norm}_{V^\omega(G)}\mathfrak{g} = \{F \in V^\omega(G); \forall Y \in \mathfrak{g}, [F, Y] \in \mathfrak{g}\},$$

so that an affine vector field is linear if and only if it vanishes at the identity.

It can be shown (see [6] or [14]) that an affine vector field can be uniquely decomposed into a sum  $F = \mathcal{X} + Z$  where  $\mathcal{X}$  is linear and  $Z$  is right-invariant.

## 2.2 Almost-Riemannian structures

For all that concern general sub-Riemannian geometry, including almost-Riemannian one, the reader is referred to [1]. About Almost-Riemannian structures on Lie groups more details can be found in [4].

**Definition 1** *An almost-Riemannian structure on an  $n$ -dimensional Lie group  $G$  is defined by a set of  $n$  vector fields  $\{\mathcal{X}, Y_1, \dots, Y_{n-1}\}$  where*

- (i)  $\mathcal{X}$  is linear;
- (ii)  $Y_1, \dots, Y_{n-1}$  are left-invariant;
- (iii)  $n = \dim G$  and the rank of  $\mathcal{X}, Y_1, \dots, Y_{n-1}$  is full on a nonempty subset of  $G$ ;
- (iv) the set  $\{\mathcal{X}, Y_1, \dots, Y_{n-1}\}$  satisfies the rank condition.

*The metric is defined by declaring the frame  $\{\mathcal{X}, Y_1, \dots, Y_{n-1}\}$  to be orthonormal.*

**Equivalent definition.** *An ARS on a Lie group can as well be defined by an  $(n-1)$ -dimensional left-invariant distribution  $\Delta$  (that is  $\Delta = \text{Span}\{Y_1, \dots, Y_{n-1}\}$ ), a left-invariant Euclidean metric on  $\Delta$  and a linear vector field  $\mathcal{X}$  assumed to satisfy the conditions (iii) and (iv) of Definition 1. The metric of the ARS is then defined by declaring  $\mathcal{X}$  unitary and orthogonal to  $\Delta$ .*

### Necessary conditions for the rank condition

First notice that if  $[\Delta, \Delta] \subseteq \Delta$  and  $D(\Delta) \subseteq \Delta$ , then the Lie algebra generated by  $\mathcal{X}, Y_1, \dots, Y_{n-1}$  is equal to  $\mathbb{R}\mathcal{X} \oplus \Delta$ . But the rank of that Lie algebra is not full at the identity  $e$ . Consequently the rank condition implies that at least one of the following conditions hold:

- (i)  $[\Delta, \Delta] \not\subseteq \Delta$
- (ii)  $D(\Delta) \not\subseteq \Delta$

In both cases, the full rank is obtained after one step.

### Singular locus

The set where the rank of  $\mathcal{X}, Y_1, \dots, Y_{n-1}$  is not full will be referred to as the *singular locus* and denoted by  $\mathcal{Z}$ . It is an analytic subset of  $G$ . By Assumption (iii) it is not equal to  $G$ , and by analyticity its interior is empty. On the other hand  $\mathcal{X}(e) = 0$  and it cannot be empty. Finally  $G \setminus \mathcal{Z}$  is an open, dense and proper subset of  $G$ .

The *set of singularities of the linear field  $\mathcal{X}$*  will be denoted by  $\mathcal{Z}_\mathcal{X} = \{g \in G; \mathcal{X}(g) = 0\}$ . It is a subgroup of  $G$ , the Lie algebra of which is  $\ker(D)$ , where  $D$  is the derivation associated to  $\mathcal{X}$ . It is included in  $\mathcal{Z}$  but different in general.

The points of  $G \setminus \mathcal{Z}$  will be called *the Riemannian points* and the ones of  $\mathcal{Z}$  *the singular points*.

### Norms and isometries

The almost-Riemannian norm on  $T_gG$  is defined by:

$$\text{For } X \in T_gG, \quad \|X\| = \min \left\{ \sqrt{v^2 + \sum_1^n u_i^2}; v\mathcal{X}_g + u_1Y_1(g) + \cdots + u_{n-1}Y_{n-1}(g) = X \right\}.$$

It is infinite if the point  $g$  belongs to the singular locus and  $X$  does not belong to  $\Delta_g$ .

**Definition 2** Let  $(\Sigma)$  and  $(\Sigma')$  be two ARSs on the Lie group  $G$ . An isometry  $\Phi$  from  $(\Sigma)$  onto  $(\Sigma')$  is a diffeomorphism of  $G$  that respects the norms, that is:

$$\forall g \in G, \forall X \in T_gG \quad \|T_g\Phi.X\|_{\Sigma'} = \|X\|_{\Sigma}.$$

where  $\|\cdot\|_{\Sigma}$  (resp.  $\|\cdot\|_{\Sigma'}$ ) stands for the norm associated to  $(\Sigma)$  in  $T_gG$  (resp. to  $(\Sigma')$  in  $T_{\Phi(g)}G$ ).

### 2.3 Summary of some results

The following results are proved in [4].

**Theorem 1 of [4]** If  $\Delta$  is a subalgebra of  $\mathfrak{g}$  then the singular locus  $\mathcal{Z}$  is an analytic, embedded, codimension one submanifold of  $G$ . Its tangent space at the identity is  $D^{-1}\Delta$ .

**Theorem 2 of [4]** If the Lie algebra  $\mathfrak{g}$  is solvable and  $\Delta$  is a subalgebra of  $\mathfrak{g}$ , then the singular locus  $\mathcal{Z}$  is a codimension one subgroup of  $G$  whose Lie algebra is  $\mathfrak{z} = D^{-1}\Delta$ .

### 2.4 Notations

In the sequel the following notations will be used:

1. Let  $Y \in \mathfrak{g}$ . Then  $Y_g$  stands for  $TL_g.Y$ , where  $L_g$  is the left translation by  $g$ , and  $TL_g$  its differential. The Euclidean norm in  $T_gG$  is denoted by  $\|\cdot\|_g$ , or simply by  $\|\cdot\|$  if no confusion is possible. Notice that if  $g \in \mathcal{Z}$  then  $\|\cdot\|_g$  is only defined on  $\Delta_g$ .
2. To the linear vector field  $\mathcal{X}$  we associate  $F(g) = TL_{g^{-1}}.\mathcal{X}_g \in \mathfrak{g}$ . In some expressions we also write  $F_g$  for  $F(g)$  when it is lighter, for instance in  $\text{ad}(F_g)$ .

Finally the following formula, proven in [4], will be used in some proofs.

$$T_gF = (D + \text{ad}(F_g)) \circ TL_{g^{-1}}. \quad (3)$$

Notice that here  $F_g$  is an element of  $\mathfrak{g}$ .

## 3 General Theorems about Isometries of ARSs

In this section  $(\Sigma)$  and  $(\Sigma')$  stand for two simple ARSs on the same Lie group  $G$ . An object related to  $(\Sigma')$  will be denoted by the same symbol as the analogous object related to  $(\Sigma)$  but with a prime, that is,  $D'$ ,  $(\varphi'_t)_{t \in \mathbb{R}}$ , and so on.

### 3.1 Translations

The purpose of this subsection is to characterize the left translations that are isometries and to show that there exists an isometry from  $(\Sigma)$  onto  $(\Sigma')$  if and only if there exists one such isometry that preserves the identity.

**Proposition 1** *Let  $\Phi$  be an isometry from  $(\Sigma)$  onto  $(\Sigma')$ . Then  $\Phi$  sends the singular locus of  $(\Sigma)$  onto the one of  $(\Sigma')$  and the set of fixed points of  $\mathcal{X}$  onto the one of  $\mathcal{X}'$ , that is:*

$$\Phi(\mathcal{Z}) = \mathcal{Z}' \quad \text{and} \quad \Phi(\mathcal{Z}_{\mathcal{X}}) = \mathcal{Z}'_{\mathcal{X}'}$$

*In particular  $\mathcal{X}'_{\Phi(e)} = 0$ .*

*Proof.* Since the singular locus is the set of points of  $G$  where the rank of  $\text{Span}\{\mathcal{X}, Y_1, \dots, Y_{n-1}\}$  is not full, it is clear that  $\Phi(\mathcal{Z}) = \mathcal{Z}'$ . We are left to prove that  $\Phi(\mathcal{Z}_{\mathcal{X}}) = \mathcal{Z}'_{\mathcal{X}'}$ .

Let  $V \in \Delta_e$ ,  $V \neq 0$ . It can be written in an unique way as  $V = \sum_{i=1}^{n-1} a_i Y_i$  and we may assume

$$\text{without lost of generality that } \|V\|_e = \sqrt{\sum_{i=1}^{n-1} a_i^2} = 1.$$

Let  $g \in \mathcal{Z}_{\mathcal{X}}$  and  $V_g = TL_g.V$ . Since  $\mathcal{X}(g) = 0$  the vector  $V_g$  writes  $V_g = \sum_{i=1}^{n-1} a_i Y_i(g)$  in an unique way, and consequently  $\|V_g\|_g = 1 = \|V\|_e$ .

Let now  $g \in \mathcal{Z} \setminus \mathcal{Z}_{\mathcal{X}}$ . At such a point  $\mathcal{X}(g) = \sum_{i=1}^{n-1} b_i Y_i(g) \neq 0$ . Let  $b = \sqrt{\sum_{i=1}^{n-1} b_i^2} > 0$  and let us

choose the particular  $V = \sum_{i=1}^{n-1} a_i Y_i$  where  $a_i = b^{-1} b_i$ . Then  $V_g = TL_g.V$  writes  $V_g = \sum_{i=1}^{n-1} a_i Y_i(g)$  but also as

$$V_g = v\mathcal{X}_g + \sum_{i=1}^{n-1} u_i Y_i(g) \quad \text{with} \quad vb_i + u_i = a_i, \quad i = 1, \dots, n-1.$$

The square of the norm of  $V_g$  is  $\inf\{v^2 + \sum_{i=1}^{n-1} u_i^2; vb_i + u_i = a_i \text{ for } i = 1, \dots, n-1\}$ . A straight-

forward computation shows that the minimum is attained for  $v = \frac{b}{1+b^2}$  and consequently that

$$\|V_g\|_g^2 = \frac{1}{1+b^2} < 1 \text{ though } \|V\|_e = 1.$$

This shows that the function  $g \mapsto \|TL_g.V\|$  is not continuous at  $g$ .

Therefore the sub-Riemannian metric is continuous at points of  $\mathcal{Z}_{\mathcal{X}}$  (and at Riemannian points of course) but discontinuous at the points of  $\mathcal{Z} \setminus \mathcal{Z}_{\mathcal{X}}$ . This implies that the image of the set  $\mathcal{Z} \setminus \mathcal{Z}_{\mathcal{X}}$  by the isometry  $\Phi$  should be  $\mathcal{Z}' \setminus \mathcal{Z}'_{\mathcal{X}'}$ , which is the desired conclusion. ■

**Proposition 2** *Let  $g \in G$ . The left translation  $L_g$  is an isometry of  $(\Sigma)$  if and only if  $g$  belongs to the set  $\mathcal{Z}_{\mathcal{X}}$  of fixed points of  $\mathcal{X}$ .*

*Proof.*

According to Proposition 1 the condition is necessary.

Conversely let  $g \in \mathcal{Z}_{\mathcal{X}}$ . Since  $Y_1, \dots, Y_{n-1}$  are left-invariant we have  $(L_g)_* Y_i = Y_i$  for  $i = 1, \dots, n-1$ . On the other hand  $(L_g)_* \mathcal{X} = \mathcal{X} + Z$  for some right-invariant vector field  $Z$ . Indeed

$$\forall Y \in \mathfrak{g} \quad [Y, \mathcal{X}] = (L_g)_*[Y, \mathcal{X}] = [(L_g)_* Y, (L_g)_* \mathcal{X}] = [Y, (L_g)_* \mathcal{X}].$$

This proves that  $(L_g)_* \mathcal{X}$  is an affine vector field whose adjoint action on  $\mathfrak{g}$  is the same as the one of  $\mathcal{X}$ . According to [14] it is equal to  $\mathcal{X}$  up to a right-invariant vector field  $Z$ .

Moreover  $(\mathcal{X} + Z)(g) = TL_g \mathcal{X}_e = 0$ . But  $\mathcal{X}_g = 0$  and necessarily  $Z = 0$ . This proves that  $(L_g)_* \mathcal{X} = \mathcal{X}$ , and together with  $(L_g)_* Y_i = Y_i$  for  $i = 1, \dots, n-1$ , that  $L_g$  is an isometry. ■

**Theorem 1** *The ARSs  $(\Sigma)$  and  $(\Sigma')$  are isometric if and only if there exists an isometry  $\Phi$  from  $(\Sigma)$  onto  $(\Sigma')$  such that  $\Phi(e) = e$ .*

*Proof.*

Let  $\Psi$  be an isometry from  $(\Sigma)$  onto  $(\Sigma')$ . Firstly  $\Psi(e) \in \mathcal{Z}'_{\mathcal{X}'}$ , according to Proposition 1. According to Proposition 2 the left translation  $L_{\Psi(e)^{-1}}$  is an isometry of  $(\Sigma')$ . Then  $\Phi = L_{\Psi(e)^{-1}} \circ \Psi$  suits. ■

### 3.2 Preservation of the distribution

We recall the following notations: if  $Y \in T_e G$  and  $g \in G$  then  $Y_g$  stands for  $TL_g Y \in T_g G$ .

The key point of this subsection is the following lemma that characterizes the elements of  $\Delta_g$  (when  $g$  does not belong to the singular locus): they are the vectors whose norm is invariant under (small) left translations.

**Lemma 1** *Let  $\Sigma$  be an ARS on the Lie group  $G$ . There exists an open and dense subset of  $G \setminus \mathcal{Z}$ , hence of  $G$ , of points  $g$  that satisfy:*

$$\Delta_g = \{Y_g; \|Y_h\|_h = \|Y_g\|_g \text{ in a neighborhood of } g\}.$$

*Proof.* Let  $g \in G \setminus \mathcal{Z}$  and  $Y \in T_e G$ . If  $Y = \sum_{i=1}^{n-1} a_i Y_i \in \Delta$ , then  $Y_h$  writes uniquely as  $Y_h = \sum_{i=1}^{n-1} a_i Y_i(h)$  and  $\|Y_h\|_h$  is constant in a neighborhood of  $g$ .

We have to show that this is no longer true if  $Y \notin \Delta$ .

Since the singular locus is an analytic set, we can find  $X \in T_e G$  such that  $\exp(tX) \notin \mathcal{Z}$  for  $t$  in some open interval  $(0, \tau)$ . Recall that  $F$  stands for the mapping from  $G$  to  $T_e G$  defined by  $F(g) = TL_{g^{-1}} \mathcal{X}_g$ . The previous condition is equivalent to  $F(\exp(tX)) \notin \Delta$  for  $t \in (0, \tau)$ , or better to:

$$\langle \omega, F(\exp(tX)) \rangle \neq 0 \quad \text{for } t \in (0, \tau) \quad (4)$$

where  $\omega$  stands for a nonvanishing one-form on  $\mathfrak{g}$  orthogonal to  $\Delta$ . But Condition (4) together with  $\langle \omega, F(e) \rangle = 0$  imply

$$\frac{d}{dt} \langle \omega, F(\exp(tX)) \rangle \neq 0$$

on  $(0, \tau)$  (except may be at isolated points). Consequently the set

$$\Omega = \{g \in G; g \notin \mathcal{Z} \text{ and } \frac{d}{dt}|_{t=0} \langle \omega, F(g \exp(tX)) \rangle \neq 0\}$$

is not empty. This set being defined by analytic conditions is open and dense in  $G$ .

Let  $g \in \Omega$ , let us consider the path  $\gamma(t) = F(g \exp(tX))$  in  $T_e G$  and let  $Y = \gamma(0) = F(g)$ . Since  $g \notin \mathcal{Z}$  the vector  $F(g)$  does not belong to  $\Delta_e$ , the  $n$ -uple  $(F(g), Y_1, \dots, Y_{n-1})$  is a basis of  $T_e G$ , and the derivative  $\gamma'(0)$  can be written as  $\gamma'(0) = vF(g) + \sum_{i=1}^{n-1} u_i Y_i = vY + \sum_{i=1}^{n-1} u_i Y_i$ . Then

we have

$$\begin{aligned} \gamma(t) &= Y + t\gamma'(0) + O(t^2) && \text{hence} \\ Y &= \gamma(t) - t\gamma'(0) + O(t^2) \\ &= \gamma(t) - t(vY + \sum_{i=1}^{n-1} u_i Y_i) + O(t^2) && \text{and} \\ (1 + tv)Y &= \gamma(t) - t \sum_{i=1}^{n-1} u_i Y_i + O(t^2). \end{aligned}$$

Let us denote by  $\|\cdot\|_t$  the Euclidean norm in  $T_eG$  defined by the moving orthonormal frame  $(\gamma(t), Y_1, \dots, Y_{n-1})$ . We have

$$(1 + tv)^2 \|Y\|_t^2 = 1 + O(t^2) \quad \text{and} \quad \|Y\|_t^2 = \frac{1}{(1 + tv)^2} + O(t^2).$$

From that we obtain

$$\frac{d}{dt}\Big|_{t=0} \|Y\|_t^2 = -2v.$$

Since  $g \in \Omega$  the vector  $\gamma'(0)$  does not belong to  $\Delta$  and  $v \neq 0$ . Moreover we have obviously  $\|Y_{\gamma(t)}\|_{\gamma(t)} = \|Y\|_t$ , which proves that in any neighborhood of  $g$ ,  $\|Y_h\|_h$  is not constant.

The same happens for any  $Z$  that does not belong to  $\Delta$  because it is the sum of an element of  $\Delta$  and of  $\alpha Y$  for some nonzero real number  $\alpha$ . ■

**Theorem 2** *Let  $\Phi$  be an isometry from  $(\Sigma)$  onto  $(\Sigma')$  that preserves the identity. Then:*

1. *Its tangent mapping  $\Phi_*$  sends  $\Delta$  on  $\Delta'$ , that is,  $T_g\Phi.\Delta_g = \Delta_{\Phi(g)}$  for all  $g \in G$ .*
2. *Either  $\Phi_*\mathcal{X} = \mathcal{X}'$ , and  $T_e\Phi \circ D = D' \circ T_e\Phi$   
or  $\Phi_*\mathcal{X} = -\mathcal{X}'$ , and  $T_e\Phi \circ D = -D' \circ T_e\Phi$ .*

*Proof.* According to Lemma 1 there is an open and dense subset  $\Omega$  of points  $g$  of  $G \setminus \mathcal{Z}$  where the distribution is characterized by:

$$\Delta_g = \{Y_g; \|Y_h\|_h = \|Y_g\|_g \text{ in a neighborhood of } g\}.$$

This characterization should be preserved by the isometry  $\Phi$  hence  $T_g\Phi(\Delta_g) = \Delta'_{\Phi(g)}$ . The density of  $\Omega$  implies that this equality actually holds at all points.

Let now  $g \in G \setminus \mathcal{Z}$ . The image by  $\Phi$  of the orthonormal frame  $(\mathcal{X}(g), Y_1(g), \dots, Y_{n-1}(g))$  is the orthonormal frame  $(T_g\Phi.\mathcal{X}(g), T_g\Phi.Y_1(g), \dots, T_g\Phi.Y_{n-1}(g))$ . But the fact that  $T_g\Phi.\Delta_g = \Delta'_{\Phi(g)}$  implies that  $T_g\Phi.\mathcal{X}(g)$  is an unitary vector orthogonal to  $\Delta'_{\Phi(g)}$ , that is,

$$T_g\Phi.\mathcal{X}(g) = \pm\mathcal{X}'(\Phi(g)). \tag{5}$$

In this equality the sign is constant on any connected subset of  $G$  where  $\mathcal{X}$  does not vanish. On the other hand the set  $\mathcal{Z}_{\mathcal{X}}$  of singularities of  $\mathcal{X}$  is a subgroup of  $G$  whose Lie algebra is  $\ker(D)$ . The set  $G \setminus \mathcal{Z}_{\mathcal{X}}$  can be disconnected only if the dimension of  $\ker(D)$  is  $n - 1$ . In that case let  $Y$  be any element of  $T_eG$  that does not belong to  $\ker(D)$ . Then  $\exp(tY)$  does not belong to  $\mathcal{Z}_{\mathcal{X}}$ , for  $t$  in some interval  $(0, T)$ , with  $T > 0$ . The sign of the equality  $T_{\exp(tY)}\Phi.\mathcal{X}(\exp(tY)) = \pm\mathcal{X}'(\Phi(\exp(tY)))$  being constant for  $t \in (0, T)$ , we denote it by  $\epsilon(Y) = \pm 1$ . We push this equality to  $T_eG$  and we derivate it at  $t = 0$ . On the one hand we get (here  $\varphi_t$  stands for the flow of  $\mathcal{X}$  and we use Formula (2)):

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} TL_{(\Phi(\exp(tY)))^{-1}}.T_{\Phi(\exp(tY))}.\mathcal{X}(\Phi(\exp(tY))) &= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} (\Phi(\exp(tY)))^{-1}\Phi(\varphi_s(\exp(tY))) \\ &= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} (\Phi(\exp(tY)))^{-1}\Phi(\exp(e^{sD}tY)) = \frac{d}{ds}\Big|_{s=0} (-T_e\Phi.Y + T_e\Phi e^{sD}Y) \\ &= T_e\Phi DY, \end{aligned}$$

and on the other hand:

$$\frac{d}{dt}\Big|_{t=0} TL_{(\Phi(\exp(tY)))^{-1}} \cdot \mathcal{X}'_{\Phi(\exp(tY))} = \frac{d}{dt}\Big|_{t=0} F'(\Phi(\exp(tY))) = D'T_e\Phi.Y$$

according to Formula (3) recalled in Section 2.

Finally we have obtained  $T_e\Phi DY = \epsilon(Y)D'T_e\Phi.Y$ . By linearity we have also  $T_e\Phi D(-Y) = \epsilon(Y)D'T_e\Phi.(-Y)$ , and consequently  $T_{\exp(-tY)}\Phi.\mathcal{X}'(\exp(tY)) = \epsilon(Y)\mathcal{X}'(\Phi(\exp(-tY)))$  for  $t$  small enough. But  $\exp(tY)$  and  $\exp(-tY)$  are located in different connected components of  $G \setminus \mathcal{Z}_{\mathcal{X}}$  (if different connected components exist), and this proves that the sign is constant in Formula (5).  $\blacksquare$

**Remark.** Since the replacement of  $\mathcal{X}'$  by  $-\mathcal{X}'$  does not modify the ARS  $(\Sigma')$  we can always assume that  $\Phi_*\mathcal{X} = \mathcal{X}'$ .

### 3.3 The tangent mapping of an isometry

**Theorem 3** *Let  $\Phi_1$  and  $\Phi_2$  be two isometries from  $(\Sigma)$  onto  $(\Sigma')$  that preserve the identity. If  $T_e\Phi_1 = T_e\Phi_2$  then  $\Phi_1 = \Phi_2$ .*

Before proving the theorem let us recall that the normal Hamiltonian of an ARS is

$$\mathcal{H} = \frac{1}{2}\langle \lambda_g, \mathcal{X}(g) \rangle^2 + \frac{1}{2} \sum_{i=1}^{n-1} \langle \lambda_g, Y_i(g) \rangle^2,$$

where  $\lambda_g \in T_g^*G$ . At Riemannian points the geodesics are the projections of the integral curves of the Hamiltonian vector field associated to  $\mathcal{H}$  (see [1] for more information).

*Proof.* The diffeomorphism  $\Phi_2^{-1} \circ \Phi_1$  being an isometry of  $(\Sigma)$ , it is sufficient to prove that an isometry of  $(\Sigma)$  that preserves the identity  $e$  and whose tangent mapping at  $e$  is the identity of  $T_eG$  is itself the identity mapping of  $G$ .

So let  $\Phi$  be such an isometry of  $(\Sigma)$ .

In order to simplify the notation let  $\Psi$  be the lift of  $\Phi$  to the cotangent space, that is

$$\Psi(g, \lambda_g) = (\Phi(g), \lambda_g \circ T_{\Phi(g)}\Phi^{-1}).$$

It is a classical fact that  $\Psi$  is a symplectomorphism. According to Proposition 4.51 of [1] we have

$$\Psi_* \vec{\mathcal{H}} = \overline{\mathcal{H} \circ \Psi^{-1}}$$

where  $\mathcal{H}$  stands for the normal sub-Riemannian Hamiltonian and  $\vec{\mathcal{H}}$  for the associated Hamiltonian vector field.

Let  $g$  be a Riemannian point of  $G$ . A straightforward computation shows that:

$$\mathcal{H} \circ \Psi^{-1}(g, \lambda_g) = \frac{1}{2}\langle \lambda_g, T_{\Phi^{-1}(g)}\Phi.\mathcal{X}(\Phi^{-1}(g)) \rangle^2 + \frac{1}{2} \sum_{i=1}^{n-1} \langle \lambda_g, T_{\Phi^{-1}(g)}\Phi.Y_i(\Phi^{-1}(g)) \rangle^2. \quad (6)$$

Since  $\Phi$  is an isometry,  $(T_{\Phi^{-1}(g)}\Phi.\mathcal{X}(\Phi^{-1}(g)), T_{\Phi^{-1}(g)}\Phi.Y_1(\Phi^{-1}(g)), \dots, T_{\Phi^{-1}(g)}\Phi.Y_{n-1}(\Phi^{-1}(g)))$  is an orthonormal basis of  $T_gG$  and (6) is half the sum of the squares of the coordinates of  $\lambda_g$  in the dual basis. It is a standard fact of linear algebra that this is invariant by isometry. Consequently

$$\mathcal{H} \circ \Psi^{-1}(g, \lambda_g) = \frac{1}{2}\langle \lambda_g, \mathcal{X}(g) \rangle^2 + \frac{1}{2} \sum_{i=1}^{n-1} \langle \lambda_g, Y_i(g) \rangle^2 = \mathcal{H}(g, \lambda_g).$$

The set of Riemannian points being dense in  $G$ , this implies  $\Psi_* \overrightarrow{\mathcal{H}} = \overline{\mathcal{H} \circ \Psi^{-1}} = \overrightarrow{\mathcal{H}}$ .

Let  $(g(t), \lambda(t))$  be a normal extremal with initial conditions  $g(0) = e$ , and  $\lambda(0) = \lambda_g \in T_e^*G$ . Then  $\Psi(g(t), \lambda(t))$  is also a normal extremal with initial conditions  $e$  and  $(T_e\Phi^{-1})^*\lambda_g$ . Since  $T_e\Phi = Id$  we get  $\Phi(g(t)) = g(t)$  for all  $t$  for which  $g(t)$  is defined. This is enough to prove that  $\Phi = Id$ . ■

### 3.4 Nilpotent groups

**Theorem 4** *If the group  $G$  is nilpotent then the isometries of ARSs of  $G$  that preserve the identity are automorphisms.*

*Proof.* Let  $(\Sigma)$  and  $(\Sigma')$  be two ARSs on  $G$ , and  $\Phi$  an isometry from  $(\Sigma)$  onto  $(\Sigma')$  that preserves the identity.

(i)  $\Delta$  and  $\Delta'$  are not subalgebras. In that case  $\Delta$  and  $\Delta'$  together with their left-invariant metrics define left-invariant sub-Riemannian metrics on  $G$ . Since  $T_g\Phi$  is an isometry from  $\Delta_g$  onto  $\Delta_{\Phi(g)}$  for all  $g \in G$ , the diffeomorphism  $\Phi$  is an isometry of these left-invariant structures. According to the theorem of Kivioja-Le Donne ([18]),  $\Phi$  is an automorphism.

(ii)  $\Delta$  and  $\Delta'$  are subalgebras. In that case  $\Delta$  and  $\Delta'$  define left-invariant integrable distributions on  $G$ . If we denote by  $H$  (resp. by  $H'$ ) the connected Lie subgroup generated by  $\Delta$  (resp. by  $\Delta'$ ) then the leaves of the induced foliations are the cosets  $gH$  (resp. the cosets  $gH'$ ).

Let  $g \in G$ , and let  $M = \Phi(gH)$  be the image of the coset  $gH$  by  $\Phi$ . Since  $\Phi$  is a diffeomorphism,  $M$  is a submanifold of  $G$ , and for any  $h \in H$  the tangent space at the point  $\Phi(gh)$  is  $T_{gh}\Phi \cdot \Delta_{gh} = \Delta'_{\Phi(gh)}$ . Consequently  $M$  is the leaf through the point  $\Phi(g)$  of the distribution  $\Delta'$ , hence equal to  $\Phi(g)H'$ , and  $\Phi$  is an isometry from  $gH$  onto  $\Phi(g)H'$  (because  $T_g\Phi$  respects the metric between  $\Delta_g$  and  $\Delta_{\Phi(g)}$  for all  $g \in G$ ).

But the left translations respect the metric of  $\Delta$ , and the one of  $\Delta'$  as well, so that  $L_{\Phi(g)^{-1}} \circ \Phi \circ L_g$  is an isometry from  $H$  onto  $H'$ . These two groups are nilpotent, and according to The Kivioja-Le Donne's Theorem again, the restriction of  $\Phi$  to  $H$  is an automorphism onto  $H'$ .

Let  $Y \in \Delta$ , and  $\bar{Y}$  be the associated left-invariant vector field of  $H$ . Since  $L_{\Phi(g)^{-1}} \circ \Phi \circ L_g$  is an isometry from  $H$  onto  $H'$ , we get that  $(L_{\Phi(g)^{-1}} \circ \Phi \circ L_g)_* \bar{Y}$  is a left-invariant vector field  $\bar{Z}$  of  $H'$ .

Let  $h \in H$ . Then  $\Phi(gh)$  belongs to  $\Phi(g)H'$ , and is equal to  $\Phi(g)h'$  for  $h' = L_{\Phi(g)^{-1}} \circ \Phi \circ L_g(h)$ . The equality

$$T_h(L_{\Phi(g)^{-1}} \circ \Phi \circ L_g)_* \bar{Y}_h = \bar{Z}_{h'}$$

implies

$$\begin{aligned} T_{gh}\Phi \cdot Y_{gh} &= T_{gh}\Phi \cdot T_e L_{gh} \cdot Y_e = T_{gh}\Phi \cdot T_h L_g \cdot \bar{Y}_h \\ &= T_{h'} L_{\Phi(g)} \cdot \bar{Z}_{h'} = T_{h'} L_{\Phi(g)} T_e L_{h'} \cdot Z_e = T_e L_{\Phi(g)h'} \cdot Z_e = T_e L_{\Phi(gh)} \cdot Z_e. \end{aligned}$$

Since  $g$  and  $h$  are arbitrary, this proves that the image by  $\Phi_*$  of the **left-invariant vector field  $Y$  of  $G$**  is a left-invariant vector field of  $G$ .

This being true for all  $Y \in \Delta$ , and according to the forthcoming lemma 3 the isometry  $\Phi$  is an automorphism. ■

**Remark.** In the algebra case, the proof uses only that the groups  $H$  and  $H'$  are nilpotent, not that the group  $G$  itself is nilpotent.

**Lemma 2** *Let  $G$  be a connected Lie group and  $\mathfrak{g}$  its Lie algebra, identified with the set of left-invariant vector fields.*

Let  $\Phi$  be a diffeomorphism of  $G$  that verifies  $\Phi(e) = e$  and  $\Phi_*Y \in \mathfrak{g}$  for all  $Y \in \mathfrak{g}$ . Then  $\Phi$  is an automorphism.

*Proof.* Let  $Y \in \mathfrak{g}$  be a left-invariant vector field and  $Z = \Phi_*Y$ . The flow of  $Y$  is  $(t, g) \mapsto g \exp(tY)$  and the one of  $\Phi_*Y$  is  $(t, g) \mapsto \Phi(\Phi^{-1}(g) \exp(tY))$ . On the other hand  $Z$  is left-invariant by assumption and we get:

$$\forall g \in G \quad \forall t \in \mathbb{R} \quad \Phi(\Phi^{-1}(g) \exp(tY)) = g \exp(tZ). \quad (7)$$

Formula (7) applied at  $g = e$  gives  $\Phi(\exp(tY)) = \exp(tZ)$  for all  $t \in \mathbb{R}$ .

Let  $V$  be a neighborhood of  $e$  in  $G$  and  $U$  be a neighborhood of  $0$  in  $\mathfrak{g}$  such that  $\exp$  be a diffeomorphism from  $U$  onto  $V$ . If  $Y \in U$ , then  $y = \exp(Y) \in V$  and for all  $g \in G$  holds:

$$\Phi(\Phi^{-1}(g)y) = \Phi(\Phi^{-1}(g) \exp(Y)) = g \exp(Z) = g\Phi(\exp(tY)) = g\Phi(y).$$

In other words we have  $\Phi(z\Phi(y)) = \Phi(z)\Phi(y)$  for all  $z = \Phi^{-1}(g) \in G$  and all  $y \in V$ . Since  $V$  generates  $G$  this equality is true for all  $z, y \in G$  and  $\Phi$  is an automorphism. ■

**Lemma 3** *Let  $\Delta$  be a subspace of  $\mathfrak{g}$  and  $\mathcal{X}$  be a linear vector field such that the Lie algebra generated by  $\Delta$  and  $\mathcal{X}$  be equal to  $\mathfrak{g} \oplus \mathbb{R}\mathcal{X}$ .*

*If  $\Phi$  is a diffeomorphism of  $G$  such that  $\Phi_*Y \in \mathfrak{g}$  for all  $Y \in \Delta$  and such that  $\Phi_*\mathcal{X}$  is a linear vector field, then  $\Phi$  is an automorphism.*

*Proof.* Since the Lie bracket of a left-invariant vector field with another left-invariant vector field or a linear field is left-invariant, the Lie algebra generated by the  $\Phi_*Y$ ,  $Y \in \Delta$ , and  $\Phi_*\mathcal{X}$  is obviously equal to  $\mathfrak{g} \oplus \mathbb{R}\Phi_*\mathcal{X}$ . According to Lemma 2 the diffeomorphism  $\Phi$  is an automorphism. ■

Theorem 2 is no longer true in general if the group is not nilpotent. Counter-examples can be easily built with the help of the Milnor example of the rototranslation group (see [19]) recalled in [18].

**Counter-Example.** The rototranslation group is the universal covering of the group of orientation-preserving isometries of the Euclidean plane. It can be described as  $\mathbb{R}^3$  with the product:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos z & -\sin z & 0 \\ \sin z & \cos z & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

The Euclidean metric is left-invariant for this product. It can be shown that the group of automorphisms of this group that are isometries is one-dimensional though the group of isometries that preserve the identity is 3-dimensional, which implies that not all isometries are affine.

Let us call this group  $\mathcal{R}$  and let us define an ARS on  $G = \mathcal{R} \times \mathbb{R}^2$  in the following way: the structure of  $\mathcal{R}$  is the previous one, and the one of  $\mathbb{R}^2$  is the Grushin plane. Any diffeomorphism  $\Phi$  that preserves the identity, made of the direct product of an isometry of  $\mathcal{R}$  and an isometry of the Grushin plane is an isometry of this ARS. However if the isometry of  $\mathcal{R}$  is not an automorphism,  $\Phi$  cannot be an automorphism of  $G$ .

Here  $G$  is a solvable, not nilpotent group.

Many different counter-examples can be build, for instance by replacing the Grushin plane by one of the ARSs defined later on the Heisenberg group.

### 3.5 Conclusion

The results of this section show that an isometry that preserves the identity also preserves the left-invariant distribution, the linear field and is characterized by its tangent mapping at the identity. Moreover that last should preserve the derivation.

We are consequently interested in diffeomorphisms  $\Phi$  of  $G$  that satisfy:

- (i)  $\Phi(e) = e$
- (ii)  $\Phi_*\Delta = \Delta'$
- (iii)  $\Phi_*\mathcal{X} = \mathcal{X}'$
- (iv)  $T_e\Phi \circ D = D' \circ T_e\Phi$

Since the isometries that preserve the identity are completely determined by their tangent maps at the origin we will first look for invertible linear maps  $P$  on  $T_eG$  that verifies  $P \circ D = D' \circ P$  and  $P(\Delta_e) = \Delta'_e$ .

If the Lie group  $G$  is simply connected and if such a  $P$  is an automorphism of  $\mathfrak{g}$ , it is the tangent mapping of an automorphism  $\Phi$  of  $G$ . It is easy to see that this automorphism is an isometry. Indeed it transforms any left-invariant vector field into a left-invariant vector field. Since  $\Phi$  satisfies  $P(\Delta_e) = \Delta'_e$ , it satisfies  $\Phi_*\Delta = \Delta'$ . On the other hand it transforms any linear vector field into a linear vector field. Since it satisfies  $P \circ D = D' \circ P$  we have  $\Phi_*\mathcal{X} = \mathcal{X}'$ .

If either  $P$  is not an automorphism of  $\mathfrak{g}$  or the Lie group is not simply connected we cannot conclude so easily, and we have to look in each case to the existence of an isometry  $\Phi$  such that  $T_e\Phi = P$ .

#### Classification.

In the two following sections we classify the ARSs on the 2D affine group and the Heisenberg group. This classification is done in three steps:

1. The ARSs are at first classified by isometry.
2. It is clear that a global rescaling of the metric does not modify the geometry. We consequently normalize the ARSs by rescaling in a second step.
3. In order to emphasize the main geometric structures we then accept to modify the left-invariant metric in the left-invariant distribution  $\Delta$  (but we keep it Euclidean and left-invariant). This amounts to "forget" the metric in  $\Delta$ . The typically different geometries are thus exhibited.

## 4 Classification of the ARSs on the affine group

Let  $G$  be the connected 2-dimensional affine group:

$$G = Aff_+(2) = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}; (x, y) \in \mathbb{R}_+^* \times \mathbb{R} \right\}.$$

Its Lie algebra is solvable<sup>1</sup> and generated by the left-invariant vector fields:

$$gX = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad gY = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \quad \text{where} \quad g = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}.$$

---

<sup>1</sup>A Lie algebra  $\mathfrak{g}$  is solvable if its derived series terminates in the zero Lie algebra: define  $\mathcal{D}^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$  and by induction  $\mathcal{D}^{n+1}\mathfrak{g} = [\mathcal{D}^n\mathfrak{g}, \mathcal{D}^n\mathfrak{g}]$ , then  $\mathfrak{g}$  is solvable if  $\mathcal{D}^n\mathfrak{g}$  vanishes for some integer  $n$ .

In natural coordinates they write  $X(x, y) = x \frac{\partial}{\partial x}$  and  $Y(x, y) = x \frac{\partial}{\partial y}$ . They verify  $[X, Y] = XY - YX = Y$ .

In the basis  $(X, Y)$ , all derivations  $D$  of the Lie algebra  $\mathfrak{aff}(2)$  have the form

$$D = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}, \text{ where } a, b \in \mathbb{R}.$$

The linear vector field  $\mathcal{X}$  associated to such a derivation is

$$\mathcal{X}(g) = \begin{pmatrix} 0 & a(x-1) + by \\ 0 & 0 \end{pmatrix}.$$

In natural coordinates it writes  $\mathcal{X}(x, y) = (a(x-1) + by) \frac{\partial}{\partial y}$ . For more details, see [12].

An ARS on  $Aff_+(2)$  is defined by a left-invariant vector field  $B = \alpha X + \beta Y$  and a derivation  $D$  such that  $B$  and  $DB$  are linearly independent, in order to satisfy the rank condition. In natural coordinates, the ARS is described as the system

$$\begin{cases} \dot{x} = u\alpha x \\ \dot{y} = v(a(x-1) + by) + u\beta x \end{cases}$$

**Proposition 3** *Let  $\Sigma = (\mathcal{X}, B)$ ,  $\Sigma' = (\mathcal{X}', B')$  be two ARSs on  $Aff_+(2)$ . If  $\Phi$  is an isometry between  $\Sigma$  and  $\tilde{\Sigma}$  and  $\Phi(e) = e$  then  $\Phi$  is an automorphism.*

*Proof.* Note that Theorem 2 ensures that  $\Phi_*(\Delta) = \Delta'$ ,  $\Phi_*(\mathcal{X}) = \epsilon \mathcal{X}'$ , where  $\epsilon = \pm 1$ .

Since  $\Phi_*$  preserves the Euclidean metric in  $\Delta$ , which is 1-dimensional, we have

$$\Phi_*(B) = \epsilon' B', \quad \text{with } \epsilon' = \pm 1.$$

In other words  $\Phi_* Y$  is a left-invariant vector field for all  $Y \in \Delta$ . Together with  $\Phi(e) = e$ , and according to Lemma 3, we obtain that  $\Phi$  is an automorphism. ■

We are now in position to state a complete classification of ARSs on  $Aff_+(2)$ , and to characterize them by geometric invariants.

#### 4.1 Classification by isometries

The almost-Riemannian structures classification by isometries is simplified by two facts. First, Proposition 3 states that isometries fixing the identity are Lie group automorphism, and second  $Aff_+(2)$  is simply connected. Thus, instead of searching for Lie groups automorphism, is enough to search for Lie algebra automorphisms.

Let  $P$  be an automorphism of the Lie algebra  $\mathfrak{aff}(2)$ . It is easy to see that  $P$  has form

$$P = \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \text{ for } c \in \mathbb{R} \text{ and } d \in \mathbb{R}^*$$

and is associated to the Lie group automorphism  $\Phi(x, y) = (x, c(x-1) + dy)$ , via the equality  $P = T_e \Phi$ .

**Proposition 4** *Any almost-Riemannian structure on the Lie group  $Aff_+(2)$  is isometric to one and only one of the structures defined by*

$$D = \begin{pmatrix} 0 & 0 \\ 1 & b \end{pmatrix} \quad \text{and} \quad B = \alpha X,$$

where  $\alpha > 0$  and  $b \geq 0$ .

*Proof.* First note that the change of the vector fields  $\mathcal{X}$  to  $-\mathcal{X}$  or  $B$  to  $-B$  does not modify the metric.

The ARS is defined by  $B = \alpha X + \beta Y$  and  $D = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$ , and we have

$$DB = D(\alpha X + \beta Y) = \alpha DX + \beta DY = (\alpha a + \beta b)Y.$$

Thus, the 2x2-matrix whose first (resp. second) column contains the coefficients of  $B$  (resp.  $DB$ ) in the basis  $\{X, Y\}$  is given by

$$(B \ DB) = \begin{pmatrix} \alpha & 0 \\ \beta & (\alpha a + b\beta) \end{pmatrix}.$$

The rank condition is satisfied if and only if  $\alpha(\alpha a + b\beta) \neq 0$ .

Since  $B$  and  $DB$  are linearly independent, we can define an isomorphism  $P$  of the vector space  $\mathfrak{g}$  by

$$P(B) = \alpha X \quad \text{and} \quad P(DB) = \alpha Y.$$

This isomorphism turns out to be a Lie algebra automorphism on  $\mathfrak{g}$ . Indeed, the equality

$$[B, DB] = [\alpha X + \beta Y, (\alpha a + b\beta)Y] = \alpha(\alpha a + b\beta)Y = \alpha DB,$$

implies

$$P[B, DB] = P(\alpha DB) = \alpha^2 Y = \alpha^2 [X, Y] = [\alpha X, \alpha Y] = [P(B), P(DB)].$$

Using Theorem 2 the derivation  $D$  is transformed to the derivation  $PDP^{-1}$ , which is characterized by :

$$PDP^{-1}X = Y \quad \text{and} \quad PDP^{-1}Y = bY.$$

Therefore, any ARS is isometric to

$$PDP^{-1} = \begin{pmatrix} 0 & 0 \\ 1 & b \end{pmatrix} \quad \text{and} \quad PB = \alpha X$$

with  $\alpha > 0$  and  $b \geq 0$ . It is easy to check that two different such ARSs are not isometric. ■

**Remark:** The singular locus  $\mathcal{Z} = \mathcal{Z}_{\mathcal{X}}$  is a Lie subgroup of  $Aff_+(2)$ .

**Proposition 5** *The group of isometries of an ARS on  $Aff_+(2)$  is the group of left translations by elements of  $\mathcal{Z}_{\mathcal{X}}$ .*

*Proof.* Let  $\Phi$  be an isometry. Suppose that  $\Phi$  fixes the identity. Proposition 3 ensures that any isometry  $\Phi$  on  $Aff_+(2)$  is an automorphism. Since  $\Phi_*(\Delta) = \Delta$ , we obtain  $T_e\Phi = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$  and using  $\Phi_*(\mathcal{X}) = \pm\mathcal{X}$ , we conclude  $PDP^{-1} = D$  if and only if  $d = 1$ , i.e. if and only if  $\Phi$  is the identity map.

Thus, the isometry group of an ARS is the group of left translations by elements of  $g \in \mathcal{Z}_{\mathcal{X}}$ . ■

## 4.2 Global rescaling

We do not change the geometry of the ARS, if we multiply all the vector fields by a common positive constant  $\lambda$ . Choosing  $\lambda$  as  $\frac{1}{\alpha}$ , we rescale  $\alpha X$  to  $X$  and the derivation  $D$  to

$$\lambda D = \begin{pmatrix} 0 & 0 \\ \frac{1}{\alpha} & \frac{b}{\alpha} \end{pmatrix}. \text{ The ARS is isometric to } \begin{pmatrix} 0 & 0 \\ 1 & \frac{b}{\alpha} \end{pmatrix},$$

for some  $\frac{b}{\alpha} \geq 0$ . Therefore, any ARS is up to a rescaling isometric to an ARS defined by

$$B = X \text{ and } D = \begin{pmatrix} 0 & 0 \\ 1 & b \end{pmatrix}, \text{ with } b \geq 0.$$

Equivalently, it is defined by

$$B = X \text{ and the singular locus } \mathcal{Z}_{\mathcal{X}} = \{(x, y) : (x - 1) + by = 0, \text{ for some } b \geq 0\}.$$

Moreover, the singular locus is normal if and only if  $b = 0$ , that is,  $\mathcal{Z}_{\mathcal{X}} = \{(1, y), y \in \mathbb{R}\}$ . For the other cases, that is, if  $b > 0$  the singular locus  $\mathcal{Z}_{\mathcal{X}} = \{(x, \frac{x-1}{b}), x \in \mathbb{R}\}$  is not normal.

## 4.3 Deformation of the Euclidean metric in $\Delta$

Deforming the left-invariant metric, we can assume  $B = bX$ , (if  $b > 0$ ) and the derivation

$$D = \begin{pmatrix} 0 & 0 \\ b & b \end{pmatrix}, \text{ which are isometric up to a rescale to } B = bX \text{ and } \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Therefore, any almost-Riemannian structure on the Lie group  $Aff_+(2)$  is related by  $B = X$  and the derivation

$$D = \begin{pmatrix} 0 & 0 \\ 1 & b \end{pmatrix}, \text{ where } b = 0, 1.$$

Moreover, the singular locus  $\mathcal{Z}$  is a normal Lie subgroup if and only if  $b = 0$  and not normal otherwise.

### 4.3.1 Conclusion

There two main models, the ones obtained above after rescaling and modification of the left-invariant metric, are completely characterized by the singular locus.

The case where the singular locus is a normal subgroup, that is the the case  $b = 0$ , has been completely studied in [4].

The other one remains to be analyzed.

## 5 Classification of the ARSs on the Heisenberg group

Let  $G$  be the 3-dimensional Heisenberg group:

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}; (x, y, z) \in \mathbb{R}^3 \right\}.$$

Its Lie algebra  $\mathfrak{g}$  is nilpotent<sup>2</sup> and generated by the left-invariant vector fields:

$$gX = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad gY = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } gZ = \begin{pmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ where } g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

They verify  $[X, Y] = Z$  and the other brackets vanish. In the basis  $(X, Y, Z)$ , all derivations  $D$  of  $\mathfrak{g}$  have the form

$$D = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & a+d \end{pmatrix}, \text{ where } a, b, c, d, e, f \in \mathbb{R}.$$

The linear vector field  $\mathcal{X}$  associated to such a derivation is

$$\mathcal{X}(x, y, z) = (ax + by) \frac{\partial}{\partial x} + (cx + dy) \frac{\partial}{\partial y} + (ex + fy + (a+d)z + \frac{1}{2}cx^2 + \frac{1}{2}by^2) \frac{\partial}{\partial z}.$$

For more details, see [12].

An ARS on  $G$  is defined by an orthonormal frame  $\{B_1, B_2, \mathcal{X}\}$ , where  $B_1, B_2$  are left-invariant vector fields and  $\mathcal{X}$  is a linear one with associated derivation  $D$ .

To classify the ARSs by isometries, note that Theorem 4 states that all isometries that fix the identity are Lie group automorphisms. Since the Heisenberg group is simply connected, is enough to work with Lie algebra automorphisms.

As we will see, there are two very different cases according to whether  $\Delta$  is a subalgebra or not.

## 5.1 $\Delta$ is a subalgebra

### 5.1.1 Classification by isometries

An automorphism of the Lie algebra  $\mathfrak{g}$ , in the basis  $(X, Y, Z)$ , has the form:

$$P = \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ \epsilon & \zeta & \det(A) \end{pmatrix}, \text{ for } A \text{ equal to } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \text{ and } \det(A) \neq 0.$$

Note that  $Z$  belongs to  $\Delta$ . Indeed, suppose  $B_1 = a_1X + b_1Y + c_1Z$  and  $B_2 = a_2X + b_2Y + c_2Z$ , the first two coordinates are linearly dependent, because  $\Delta$  is a subalgebra. Thus there exists  $\lambda \in \mathbb{R}$  such that  $(a_1, b_1) = \lambda(a_2, b_2)$ . If  $\lambda = 0$  than  $B_2 = c_2Z$ , otherwise take the linear combination  $X - \lambda Y = (c_1 - \lambda c_2)Z$ .

So, we can choose the orthonormal frame of  $\Delta$  as  $\{B_1, \eta Z\}$ , for some  $\eta \in \mathbb{R}^*$ .

**Proposition 6** *Any almost-Riemannian structure on  $G$ , whose distribution  $\Delta$  is a subalgebra, is isometric to an almost-Riemannian structure whose orthonormal frame is  $\{X, Z, \mathcal{X}\}$  and the derivation  $D$  has the following form:*

$$D = \begin{pmatrix} 0 & b & 0 \\ c & d & 0 \\ 0 & f & d \end{pmatrix},$$

for some  $c > 0$  and  $d, f \geq 0$ . Moreover, two different ARSs of this form are not isometric.

<sup>2</sup>A Lie algebra  $\mathfrak{g}$  is nilpotent if its central series terminates in the zero Lie algebra: define  $\mathcal{C}^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$  and by induction  $\mathcal{C}^{n+1}\mathfrak{g} = [\mathcal{C}^n\mathfrak{g}, \mathfrak{g}]$ , then  $\mathfrak{g}$  is nilpotent if  $\mathcal{C}^n\mathfrak{g}$  vanishes for some integer  $n$ .

*Proof.* Let  $P$  be the isomorphism defined by  $P(B_1) = X$ ,  $P(DB_1) = \frac{a_1}{\eta}Y$  and  $P(\eta Z) = Z$ . The isomorphism  $P$  is an automorphism because

$$P[B_1, Z] = 0 = [X, \eta Z] = [P(B_1), P(Z)].$$

$$P[B_1, Y] = P(a_1 Z) = \frac{a_1}{\eta}Z = [X, \frac{a_1}{\eta}Y] = [P(B_1), P(DB_1)].$$

The derivation  $\tilde{D} = PDP^{-1}$  associated to  $X$  verifies  $\tilde{D}X = PDP^{-1}X = PDB = \frac{a_1}{\eta}Y$ . Hence, in relation to the basis  $(X, Y, Z)$  it has the form

$$\tilde{D} = \begin{pmatrix} 0 & b & 0 \\ c & d & 0 \\ 0 & f & d \end{pmatrix} \quad \text{where } c = \frac{a_1}{\eta} \neq 0.$$

The changes  $\mathcal{X}$  to  $-\mathcal{X}$  and  $B$  to  $-B$  allow to assume  $d \geq 0$ . Take

$$P_{\epsilon\epsilon'} = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon^{-1}\epsilon' & 0 \\ 0 & 0 & \epsilon' \end{pmatrix}, \quad \text{where } \epsilon, \epsilon' = \pm 1.$$

We can apply  $P_{\epsilon\epsilon'}$  to change  $b$  to  $\epsilon\epsilon'b$ ,  $c$  to  $\epsilon\epsilon'c$  and  $f$  to  $\epsilon f$ . Thus, we can assume  $c > 0$  and  $d, f \geq 0$ . It is easy to see that two ARSs related to different derivations of this form are not isometric. ■

**Proposition 7** *The group of isometries of an ARS in the form of Proposition 6 is:*

*If  $d \neq 0$   $f \neq 0$  the ARS group of isometries is composed only of left translations by elements of  $\mathcal{Z}_{\mathcal{X}}$ .*

*If  $d \neq 0$   $f = 0$  the ARS group of isometries is composed of left translations by elements of  $\mathcal{Z}_{\mathcal{X}}$  and infinitesimal isometries as  $P_{\epsilon\epsilon'}$  for  $\epsilon = \pm 1$  and  $\epsilon' = 1$ .*

*If  $d = 0$   $f \neq 0$  the ARS group of isometries is composed of left translations by elements of  $\mathcal{Z}_{\mathcal{X}}$  and infinitesimal isometries as  $P_{\epsilon\epsilon'}$  for  $\epsilon = \epsilon' = \pm 1$ .*

*If  $d = f = 0$  the ARS group of isometries is composed of left translations by elements of  $\mathcal{Z}_{\mathcal{X}}$  and infinitesimal isometries as  $P_{\epsilon\epsilon'}$  for  $\epsilon, \epsilon' = \pm 1$ .*

*Generically  $b, d$  and  $f$  are nonzero and the set  $\mathcal{Z}_{\mathcal{X}}$  is reduced to the identity. Thus in the subalgebra case, the group of isometries is generically reduced to the identity.*

*Proof.* Let  $\Phi$  be an isometry fixing the identity. It is an automorphism. Let  $T_e\Phi = P_{\epsilon\epsilon'}$ . Note that  $P_{\epsilon\epsilon'}(X) = \epsilon X$  and  $P_{\epsilon\epsilon'}(Z) = \epsilon' Z$ . We obtain that  $P_{\epsilon\epsilon'}$  changes  $D$  to

$$P_{\epsilon\epsilon'}DP_{\epsilon\epsilon'}^{-1} = \begin{pmatrix} 0 & \epsilon'b & 0 \\ \epsilon'c & d & 0 \\ 0 & \epsilon f & d \end{pmatrix}. \quad (8)$$

Since  $\Phi_*(\mathcal{X}) = \pm\mathcal{X}$  we have  $P_{\epsilon\epsilon'}D(P_{\epsilon\epsilon'})^{-1} = \pm D$ .

Generically  $b, c, d$  and  $f$  are non zero, so  $\epsilon$  and  $\epsilon'$  need to be equal to 1. Moreover a straightforward computation proves that  $\mathcal{Z}_{\mathcal{X}}$  is reduced to  $\{e\}$ . As a consequence the group of isometries is generically reduced to the identity.

Considering Equation (8) it is easy to conclude in the other cases. ■

### 5.1.2 Global rescaling

We do not change the geometry of the ARS if we multiply all the vector fields by a common positive constant  $\lambda$ . This global rescaling allows us to use

$$P = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \alpha\epsilon \end{pmatrix} \text{ for } \alpha \in \mathbb{R}^* \text{ and } \epsilon = \pm 1, \text{ to changes } D \text{ into } \tilde{D} = \begin{pmatrix} 0 & \alpha\epsilon^{-1}b & 0 \\ \alpha^{-1}\epsilon c & d & 0 \\ 0 & \alpha f & d \end{pmatrix}.$$

Therefore, any ARS is isometric up to a rescale to one and only one ARS defined by the orthonormal frame  $\{X, Z, \mathcal{X}\}$  where the associated derivation  $D$  is equal to:

$$D = \begin{pmatrix} 0 & b & 0 \\ 1 & d & 0 \\ 0 & f & d \end{pmatrix}$$

### 5.1.3 Deformation of the Euclidean metric in $\Delta$

In order to obtain a classification with less parameters, we consider as equivalent two ARSs that have the same linear fields, same distributions but provided with a different left-invariant metrics.

In other words, we forget the left-invariant metrics. Let  $\Sigma$  be the ARS defined by the orthonormal frame  $\{X, Z, \mathcal{X}\}$ , where the associated derivation is  $D = \begin{pmatrix} 0 & b & 0 \\ 1 & d & 0 \\ 0 & f & d \end{pmatrix}$ . We consider as equivalent to  $\Sigma$ , any ARS obtained by rescaling and conjugation by automorphisms that preserve  $\Delta$ , that is:

$$P = \begin{pmatrix} \alpha & \beta & 0 \\ 0 & \delta & 0 \\ \epsilon & \zeta & \alpha\delta \end{pmatrix}.$$

Thus, we can conclude that, any almost-Riemannian structure on  $G$ , whose left-invariant distribution  $\Delta$  is a Lie subalgebra, is related to one and only one ARS whose orthonormal frame is  $\{X, Z, \mathcal{X}\}$ , and as previously the associated derivation  $D$  has the form:

$$D = \begin{pmatrix} 0 & b & 0 \\ 1 & d & 0 \\ 0 & f & d \end{pmatrix},$$

but the constants  $b, d$  and  $f$  are normalized as follows:

case	d	b	f	singular locus $\mathcal{Z}$	$\mathcal{Z}_{\mathcal{X}}$	eigenvalues $l_1, l_2$
(i)	1	$> \frac{-1}{4}$	0	$x+y=0$	$(0,0,0)$	$l_1 = \frac{1+\sqrt{1+4b}}{2}$ $l_2 = \frac{1-\sqrt{1+4b}}{2}$
(ii)	1	$< \frac{-1}{4}$	0	$x+y=0$	$(0,0,0)$	$l_1 = \frac{1+\sqrt{-(1+4b)i}}{2}$ $l_2 = \frac{1-\sqrt{-(1+4b)i}}{2}$
(iii)	1	0	1	$x+y=0$	$(-y, y, -y - \frac{1}{2}y^2)$	$l_1 = 0$ $l_2 = 1$
(iv)	1	0	0	$x+y=0$	$(-y, y, -\frac{1}{2}y^2)$	$l_1 = 0$ $l_2 = 1$
(v)	0	1	0	$x=0$	$x=y=0$	$l_1 = 1$ $l_2 = -1$
(vi)	0	-1	0	$x=0$	$x=y=0$	$l_1 = i$ $l_2 = -i$
(vii)	0	0	1	$x=0$	$x=y=0$	$l_1 = l_2 = 0$
(viii)	0	0	0	$x=0$	$x=0$	$l_1 = l_2 = 0$

**Remark:** For this case, we know that  $\mathcal{Z}$  is always a Lie subgroup and its Lie algebra  $T_e\mathcal{Z} = D^{-1}\Delta$ . The Lie subalgebra  $\Delta$  is not tangent to the singular locus. Therefore, there are no tangency points when  $\Delta$  is a subalgebra.

## 5.2 $\Delta$ is not a subalgebra

In this case the distribution  $\Delta$  generates the Lie algebra. So, to satisfy the rank condition there are no restriction on the derivation  $D$ . On the other hand, Condition (iii) of Definition 1 states that  $\mathcal{X}$  and  $\Delta$  need be linearly independent in an open and dense set.

### 5.2.1 Classification by isometries

**Proposition 8** *Any ARS, whose distribution  $\Delta$  is not a subalgebra, is isometric to an ARS whose orthonormal frame is  $\{X, Y, \mathcal{X}\}$ , where the derivation  $D$  has the following form:*

$$D = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & f & a+d \end{pmatrix} \text{ with } c, f \geq 0.$$

The condition that  $\mathcal{X}(g)$  does not belong to  $\Delta(g)$  everywhere reduces to: If  $b = c = f = 0$  then  $a + d \neq 0$ .

*Proof.* If  $\Delta$  is not a subalgebra then

$$[B^1, B^2] = (a_1b_2 - a_2b_1)Z \notin \Delta.$$

Let us choose the isomorphism  $P$  defined by

$$P(B^1) = X, \quad P(B^2) = Y \quad \text{and} \quad PZ = \frac{1}{(a_1b_2 - a_2b_1)}Z.$$

Actually, the isomorphism  $P$  is a Lie algebra automorphism

$$[P(B^1), P(B^2)] = [X, Y] = Z = P((a_1b_2 - a_2b_1)Z) = P[B^1, B^2].$$

To finish, we can apply a rotation within  $\Delta$  in order to simplify the derivation  $D$ . Consider

$$P_{\theta, \epsilon} = \begin{pmatrix} R_{\theta\epsilon} & 0 \\ 0 & \det(R_{\theta\epsilon}) \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\epsilon \sin(\theta) & 0 \\ \sin(\theta) & \epsilon \cos(\theta) & 0 \\ 0 & 0 & \epsilon \end{pmatrix},$$

where  $R_{\theta\epsilon}$  is an orthogonal transformation of  $\Delta$  and  $P_{\epsilon, \epsilon'}$  is defined in previous section.

Case (i)  $e, f \neq 0, c \in \mathbb{R}$ . Apply  $P_{\theta, 1}$  for some  $\theta$  to vanish  $e$  then apply  $P_{\epsilon, \epsilon'}$  to transform  $f$  to  $\epsilon f$  and  $c$  to  $\epsilon'c$ . So, we can assume  $c, f \geq 0$ .

Case (ii)  $e = 0, f \neq 0$  and  $c \in \mathbb{R}$ . Apply  $P_{\epsilon, \epsilon'}$  to transform  $f$  to  $\epsilon f$  and  $c$  to  $\epsilon'c$ . So, we can assume  $c, f \geq 0$ .

Case (iii)  $e, f = 0$  and  $c \in \mathbb{R}$ . Apply  $P_{\epsilon, \epsilon'}$  to transform  $c$  to  $\epsilon'c$  so we can assume  $c \geq 0$ . ■

**Proposition 9** *The group of isometries of such an ARS is generated by the left translations by elements of  $\mathcal{Z}_{\mathcal{X}}$  and some of the automorphisms  $P_{\theta, \epsilon}$  defined above. Generically, the group of isometries is reduced to left translations.*

*Proof.* Let  $\Phi$  be an isometry fixing the identity. Theorem 4 ensures that  $\Phi$  is an automorphism. Since  $\Phi$  preserves the Euclidean metric in  $\Delta$ , we have  $\Phi_*(X) = \alpha X + \epsilon\beta Y$ ,  $\Phi_*(Y) = -\beta X + \epsilon\alpha Y$  and  $\Phi_*(\mathcal{X}) = \epsilon'\mathcal{X}$ , with  $\alpha^2 + \beta^2 = 1$ , and  $\epsilon, \epsilon' = \pm 1$ .

Note that the differential  $\Phi_*$  at the identity has the form of the Lie algebra automorphism  $P_{\theta, \epsilon}$ . The derivation

$$D = \begin{pmatrix} A & 0 \\ (e, f) & \text{tr}(A) \end{pmatrix} \text{ is changed by } P_{\theta\epsilon} \text{ to } P_{\theta\epsilon} D P_{\theta\epsilon}^{-1} = \begin{pmatrix} R_{\theta\epsilon} A R_{\theta\epsilon}^{-1} & 0 \\ \epsilon(e, f) R_{\theta\epsilon}^{-1} & \text{tr}(A) \end{pmatrix}.$$

Note that for  $e, f \neq 0$ ,  $P_{\theta\epsilon'}$  alters  $(e, f)$ , except when  $\theta = 2k\pi$  for  $k \in \mathbb{Z}$  and the automorphism  $P_{\epsilon\epsilon'}$  change  $b$  to  $\epsilon'b$ ,  $c$  to  $\epsilon'c$ ,  $e$  to  $\epsilon\epsilon'c$  and  $f$  to  $\epsilon f$ . If  $(e, f) \neq (0, 0)$  to preserve  $D$  we need  $\epsilon = \epsilon' = 1$ . Generically  $\text{tr}(A)$ ,  $e, f \neq 0$  and the only solution is  $P_{\theta\epsilon'} = I$ . ■

### 5.2.2 Global rescaling

We do not change the geometry of the ARS, if we multiply all the vector fields by a common positive constant  $\lambda$ . This global rescaling allows us to use

$$P = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix} \text{ where } \alpha \in \mathbb{R}^*, \text{ to changes } D \text{ into } \tilde{D} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & \alpha^{-1}f & a+d \end{pmatrix}.$$

Therefore, any ARS is defined up to a rescale by the orthonormal frame  $\{X, Z, \mathcal{X}\}$ , where the derivation  $D$  associated to  $\mathcal{X}$  has one of the following form: If  $f \neq 0$  we change it to 1. If  $f = 0$  and  $c \neq 0$  we change it to 1. If  $c, f = 0$  and  $b \neq 0$  we change it to  $\pm 1$ . If  $b, c, f = 0$  and  $a + d$  is different from 0, so we change it to 1.

### 5.2.3 Deformation of the Euclidean metric in $\Delta$

In order to get a classification with less parameters, we consider as equivalent ARS that have the same linear fields, same distributions but provided with a different left-invariant metrics.

In other words, we forget the left-invariant metric on  $\Delta$ . Consider the orthonormal frame of the previous section.

We consider as equivalent to  $\Sigma$  any ARS obtained by rescaling and conjugation by automorphisms that preserve  $\Delta$ , that is:

$$P = \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & \alpha\delta - \beta\gamma \end{pmatrix}.$$

The Lie algebra automorphism  $P$  transforms the ARS  $(\Sigma)$  to the ARS  $(\tilde{\Sigma})$  defined by the orthonormal frame  $\{X, Z, \tilde{\mathcal{X}}\}$  where  $\tilde{\mathcal{X}}$  is associated to the derivation

$$\tilde{D} = \frac{1}{(\alpha\delta - \beta\gamma)} \begin{pmatrix} \delta(\alpha a + \beta c) - \gamma(\alpha b + \beta d) & -\beta(\alpha a + \beta c) + \alpha(\alpha b + \beta d) & 0 \\ \delta(\gamma a + \delta c) - \gamma(\gamma b + \delta d) & -\beta(\gamma a + \delta c) + \alpha(\gamma b + \delta d) & 0 \\ \delta e - \gamma f & -\beta e + \alpha f & (\alpha\delta - \beta\gamma)(a + d) \end{pmatrix}.$$

A short computation proves the impossibility to vanish  $e$  and  $f$  simultaneously. We use the diagonalization process to classify by eigenvalues. By doing so, we may alter the entries  $e$  and  $f$  again. Thus, first we need to put the derivation  $D$  in a block form and then apply the previous computations to obtain the following forms:

$$(1) \begin{pmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ e & f & l_1 + l_2 \end{pmatrix}, (2) \begin{pmatrix} l_1 & 1 & 0 \\ 0 & l_1 & 0 \\ e & f & 2l_1 \end{pmatrix}, (3) \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ e & f & 2a \end{pmatrix}, \quad (9)$$

where  $l_1, l_2, a, b, e, f$  are real numbers,  $b \neq 0$  and  $a^2 + b^2 > 0$ .

For  $\Delta = \{X, Y\}$  and  $D = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & a+d \end{pmatrix}$  the singular locus is described by the following formula

$$\mathcal{Z} = \{ex + fy + (a+d)z - \frac{1}{2}cx^2 + \frac{1}{2}by^2 - dxy = 0\}.$$

**The case (1)** have four distinct subcases with distinct characteristics:

case	$l_1$	$l_2$	$l_1 + l_2$	singular locus $\mathcal{Z}$	$\mathcal{Z}_{\mathcal{X}}$
(i)	$\neq 0$	$\neq 0$	$\neq 0$	$(l_1 + l_2)z = l_2xy - ex - fy$	$x = y = z = 0$
(ii)	$\neq 0$	$-l_1$	0	$0 = l_1xy + ex + fy$	$x = y = 0$
(iii)	$\neq 0$	0	$l_1$	$l_1z = -ex - fy$	$(0, y, \frac{f}{l_1}y)$
(iv)	0	0	0	$0 = ex + fy$	$0 = ex + fy$

The case (2) has two different behaviours:

cases	$l_1$	singular locus $\mathcal{Z}$	$\mathcal{Z}_{\mathcal{X}}$
(i)	$\neq 0$	$ex + fy + 2l_1z + \frac{1}{2}y^2 - l_1xy = 0$	$(0, 0, 0)$
(ii)	0	$ex + fy + \frac{1}{2}y^2 = 0$	$(0, 0, z)$

The case (3) has two different behaviours and a restriction  $a^2 + b^2 > 0$ :

cases	a	singular locus $\mathcal{Z}$	$\mathcal{Z}_{\mathcal{X}}$
(i)	$\neq 0$	$ex + fy + 2az + \frac{1}{2}bx^2 + \frac{1}{2}by^2 - axy = 0$	$(0, 0, 0)$
(ii)	0	$ex + fy + \frac{1}{2}bx^2 + \frac{1}{2}by^2 = 0$	$(0, 0, z)$

*Tangency points.* It is clear that an isometry sends tangency points to tangency points and that to find them it is necessary to describe the singular locus.

Notice in case (1.i), the singular locus  $\mathcal{Z}$  is  $g^{-1}(0)$ , where the function  $g$  is

$$g(x, y, z) = -(l_1 + l_2)z + l_2xy - ex - fy.$$

The differential  $dg$  is surjective at all points of the locus that is, 0 is a regular value of the function  $g$ . So, the inverse image  $g^{-1}(0)$  is a connected two dimensional manifold. This case has one tangency point  $(x, y, z) = (\frac{-f}{l_1}, \frac{e}{l_2}, \frac{-ef}{l_2(l_1+l_2)})$ .

**In case (1.ii)**, the function  $g$  is

$$g(x, y, z) = l_1xy - ex - fy. \quad (10)$$

The differential  $dg$  is not surjective at all points of the singular locus when  $f$  or  $e$  is equal to zero.

If  $e, f \neq 0$ , the singular locus  $\mathcal{Z}$  is a nonconnected two dimensional manifold. If  $f = 0$ , the singular locus is the union two planes intersecting in the line  $(0, \frac{e}{l_1}, z)$ . If  $e = 0$ , the singular locus is the union two planes intersecting in the line  $(\frac{f}{l_1}, 0, z)$ . If  $e, f = 0$  the orthonormal frame  $\{X, Y, \mathcal{X}\}$  does not describe an ARS.

*Tangency points.* This case has three different behaviours. If  $e$  and  $f$  are both nonzero, there are no tangency points, if  $e = 0$  tangency points form a line  $(\frac{f}{l_1}, 0, z)$ , if  $f = 0$  tangency points form a line  $(0, \frac{e}{l_1}, z)$ . In these cases the respective lines are the intersection of the two planes.

**In case (1.iii)** the function  $g$  is

$$g(x, y, z) = l_1z + ex + fy. \quad (11)$$

Note that the function  $g$  describes a plane. So, the singular locus is a connected two dimensional manifold.

*Tangency points.* This case has two possible behaviours. If  $e$  is zero, the tangency points form a 1-dimensional manifold described as  $(\frac{-f}{l_1}, y, \frac{-fy}{l_1})$ ,  $y \in \mathbb{R}$ . If  $e$  is not zero, there are no tangency points.

**In case (1.iv)** the function  $g$  is

$$g(x, y, z) = -ex - fy. \quad (12)$$

Note that the function  $g$  describes a plane. So, the singular locus is a connected two dimensional manifold. If  $e$  and  $f$  are both zero then  $D$  is the zero derivation, which do not describe an ARS. There are no tangency points.

**In case (2.i)** the function  $g$  is

$$g(x, y, z) = ex + fy + 2l_1z + \frac{1}{2}y^2 - l_1xy,$$

and 0 is a regular value. So, the singular locus is a connected two dimensional manifold. This ARS has one tangency point given by  $(x, y, z) = (-\frac{f}{l_1} - \frac{e}{l_1^2}, \frac{e}{l_1}, -\frac{e^2}{4l_1^2} - \frac{ef}{2l_1})$ .

**In case (2.ii)**, the function  $g$  is

$$g(x, y, z) = ex + fy + \frac{1}{2}y^2. \quad (13)$$

The differential  $dg$  is surjective at all points only if  $e$  is different from zero. Under this assumption the singular locus  $\mathcal{Z}$  is a connected two dimensional manifold.

If  $e$  is zero, there are many points where the application  $dg$  is not surjective. In this case, the set  $g^{-1}(0)$  has different behaviours according to the value of  $f$ . If  $f \neq 0$ , the set  $g^{-1}(0)$  is the union two planes perpendicular to the  $y$ -axis, one through the point  $(0, 0, 0)$  and the other through the point  $(0, -2f, 0)$ . If  $f = 0$ , the set  $g^{-1}(0)$  is a plane perpendicular to the  $y$ -axis passing through the point  $(0, 0, 0)$ .

Thus, if  $e \neq 0$ , there are no tangency points. If  $e = 0$ , the tangency points need to satisfy the equation  $f + y = 0$ . So, for  $f \neq 0$ , there are no tangency points. If  $f = 0$  all the singular locus is equal to the set of tangency points.

**In case (3.i)** the function  $g$  is

$$g(x, y, z) = ex + fy + 2az + \frac{1}{2}bx^2 + \frac{1}{2}by^2 - axy. \quad (14)$$

The differential  $dg$  is surjective at all points of the singular locus. So,  $\mathcal{Z}$  is a connected two dimensional manifold. Since the following linear system solution is unique

$$e + bx - ay = 0 \quad \text{and} \quad f + by + ax = 0,$$

there is one tangency point.

**In case (3.ii)**, the function  $g$  is

$$g(x, y, z) = ex + fy + \frac{1}{2}bx^2 + \frac{1}{2}by^2. \quad (15)$$

The differential  $dg$  is not surjective, but the equality  $ex + fy + \frac{1}{2}bx^2 + \frac{1}{2}by^2 = 0$  represents an elliptic cylinder when  $e$  or  $f$  is different from 0. If  $e = f = 0$ , it is a line through the identity. If  $e$  or  $f$  is different from 0, there are no tangency points. If  $e$  and  $f$  both vanish, the tangency points forms the line  $(0, 0, z) \quad z \in \mathbb{R}$ .

In order to simplify even more the derivation, is possible to apply one or both of the following two kinds of automorphisms:

$$P_{\alpha, \beta} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha\beta \end{pmatrix} \quad \text{and} \quad P_{\theta} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

depending on each case, together with a global rescaling. We obtain that any almost-Riemannian structure on  $G$ , whose left-invariant distribution  $\Delta$  is not a Lie subalgebra, is related to only one ARS whose the orthonormal frame is  $\{X, Y, \mathcal{X}\}$  where the derivation  $D$  has the following form:

$$D_1 = \begin{pmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ e & f & l_1 + l_2 \end{pmatrix} \quad D_2 = \begin{pmatrix} l_1 & 1 & 0 \\ 0 & l_1 & 0 \\ e & f & l_1 \end{pmatrix} \quad D_3 = \begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ e & f & 2a \end{pmatrix}$$

case	$l_1$	$l_2$	$(e, f)$	singular locus $\mathcal{Z}$	tangency points	number of cc of $G - \mathcal{Z}$
1.i.1	1	$\mathbb{R} - \{0, -1\}$	(1, 1)	submanifold	$(-1, \frac{1}{l_2}, \frac{-1}{l_2(1+l_2)})$	two
1.i.2	1	$\mathbb{R} - \{0, -1\}$	(1, 0)	submanifold	$(0, \frac{1}{l_2}, 0)$	two
1.i.3	1	$\mathbb{R} - \{0, -1\}$	(0, 0)	submanifold	$(0, 0, 0)$	two
1.ii.1	1	-1	(1, 1)	submanifold	no tangency points	three
1.ii.2	1	-1	(1, 0)	not submanifold	$(0, 1, z), z \in \mathbb{R}$ .	four
1.iii.1	1	0	(1, 1)	submanifold	no tangency points	two
1.iii.2	1	0	(0, 1)	submanifold	$(-1, y, -y), y \in \mathbb{R}$ .	two
1.iii.3	1	0	(1, 0)	submanifold	no tangency points	two
1.iii.4	1	0	(0, 0)	submanifold	$x = z = 0$	two
1.iv.1	0	0	(0, 1)	Lie subgroup	no tangency points	two

\* cc means connected components

case	$l_1$	$(e, f)$	singular locus $\mathcal{Z}$	tangency points	number of cc of $G - \mathcal{Z}$
2.i.1	$\mathbb{R}^*$	(1, 1)	submanifold	$(-2, 1, -\frac{3}{4})$	two
2.i.2	1	(0, 1)	submanifold	$(-1, 0, 0)$	two
2.i.3	1	(1, 0)	submanifold	$(-1, 1, -\frac{1}{4})$	two
2.i.4	1	(0, 0)	submanifold	$(0, 0, 0)$	two
2.ii.1	$\mathbb{R}^*$	(1, 1)	submanifold	no tangency points	two
2.ii.2	1	(0, 1)	submanifold	no tangency points	three
2.ii.3	1	(1, 0)	submanifold	no tangency points	two
2.ii.4	1	(0, 0)	Lie subgroup	tangency point set is equal to $\mathcal{Z}$	two

case	a	b	$(e, f)$	singular locus $\mathcal{Z}$	tangency points	number of cc of $G - \mathcal{Z}$
3.i.1	1	$\mathbb{R}^*$	(0, 1)	submanifold	one tangency point	two
3.i.2	1	$\mathbb{R}^*$	(0, 0)	submanifold	$(0, 0, 0)$	two
3.ii.1	0	1	(0, 1)	submanifold	no tangency points	two
3.ii.2	0	1	(0, 0)	Lie subgroup	the line $x = y = 0$ .	one

## References

- [1] A. Agrachev, D. Barilari, U. Boscain, *Introduction to Riemannian and Sub-Riemannian Geometry*, <http://webusers.imj-prg.fr/~davide.barilari/Notes.php>
- [2] A. Agrachev, U. Boscain, M. Sigalotti, *A Gauss-Bonnet like formula on two-dimensional almost-Riemannian manifolds*, Discrete Contin. Dyn. Syst. 20 (4) (2008) 801-822.
- [3] A. Agrachev, U. Boscain, G. Charlot, R. Ghezzi, and M. Sigalotti, *Two dimensional almost-Riemannian structures with tangency points*. Ann. Inst. H. Poincaré. Anal. Non Linéaire 27 (2010), 793-807.
- [4] V. Ayala, Ph. Jouan *Almost-Riemannian Geometry on Lie groups*, SIAM J. Control and Optimization 54 (2016), no.5, 2919-2947.
- [5] L. San Martin and V. Ayala *Controllability properties of a class of control systems on Lie Groups*, Nonlinear control in the year 2000, Vol. 1, 83-92, L.N. in Control and I.S., 258, Springer, 2001.

- [6] V. Ayala and J. Tirao *Linear control systems on Lie groups and Controllability*, Proceedings of Symposia in Pure Mathematics, Vol 64, AMS, 1999, 47-64.
- [7] B. Bonnard, J.-B. Caillau, R. Sinclair, M. Tanaka, *Conjugate and cut loci of a two-sphere of revolution with application to optimal control*, Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (4) (2009) 1081-1098.
- [8] B. Bonnard, G. Charlot, R. Ghezzi, G. Janin, *The sphere and the cut locus at a tangency point in two-dimensional almost-Riemannian geometry*, J. Dyn. Control Syst. 17 (1) (2011) 141-161.
- [9] U. Boscain, G. Charlot, M. Gaye, P. Mason *Local properties of almost-Riemannian structures in dimension 3*, Discrete and Continuous Dynamical Systems-A, Volume 35, Issue 9 (2015), pp. 4115-4147.
- [10] U. Boscain, G. Charlot, R. Ghezzi, M. Sigalotti, *Lipschitz classification of almost-Riemannian distances on compact oriented surfaces*, J. Geom. Anal., January 2013, Volume 23, Issue 1, pp 438-455.
- [11] U. Boscain, G. Charlot, R. Ghezzi *Normal forms and invariants for 2-dimensional almost-Riemannian structures*, Differential Geometry and its Applications 31 (2013) 41-62.
- [12] M. Dath, Ph. Jouan *Controllability of linear systems on low dimensional nilpotent and solvable Lie groups*, accepted for publication in JDCS, DOI 10.1007/s10883-014-9258-z.
- [13] V.V. Grushin *A certain class of hypoelliptic operators*, Mat. Sb.(N.S.)83 (125) (1970) 456-473.
- [14] Ph. Jouan *Equivalence of Control Systems with Linear Systems on Lie Groups and Homogeneous Spaces* ESAIM: Control Optimization and Calculus of Variations, 16 (2010) 956-973.
- [15] Ph. Jouan *Controllability of linear system on Lie groups*, Journal of Dynamical and control systems, Vol. 17, No 4 (2011) 591-616.
- [16] Ph. Jouan *Invariant measures and controllability of finite systems on compact manifolds*, ESAIM: COCV 18 (2012) 643-655.
- [17] V. Jurdjevic *Geometric control theory*, Cambridge university press, 1997.
- [18] V. Kivioja, E. Le Donne *Isometries of nilpotent metric groups*, arxiv.org/abs/1601.08172.
- [19] J. Milnor *Curvatures of left invariant metrics on Lie groups*, Advances in Math. 21 (1976), no. 3, 293-329.
- [20] Yu. L. Sachkov *Control Theory on Lie groups*, Journal of Mathematical Sciences, Vol. 156, No. 3, 2009.
- [21] T. Takasu *Generalized Riemannian Geometry I*, The journal of the Yokohama Municipal University. Series D (1956).

# Bibliography

- [1] AGRACHEV A., BARILARI D., BOSCAIN U., *Introduction to Riemannian and Sub-Riemannian Geometry*, <http://webusers.imj-prg.fr/~david.barilari/Notes.php>
- [2] AGRACHEV A., BOSCAIN U., CHARLOT G., GHEZZI R., AND SIGALOTTI M., *Two dimensional almost-Riemannian structures with tangency points*. Ann. Inst. H. Poincaré . Anal. Non Lin. 27 (2010), 793-807.
- [3] AGRACHEV A., BOSCAIN U., SIGALOTTI M., *A Gauss-Bonnet like formula on two-dimensional almost-Riemannian manifolds*, Discrete Contin. Dyn. Syst. 20 (4) (2008) 801-822.
- [4] AYALA, V., DA SILVA, A., *Controllability of linear systems on Lie groups with finite semisimple center*. Submitted, December 2015.
- [5] AYALA, V., DA SILVA, A. J., ZSIGMOND G., *Control sets of linear systems on Lie groups*. Nonlinear Differential Equations and Applications, Vol. 24:8 (2017)
- [6] AYALA, V., JOUAN, PH., DA SILVA, A. AND ZSIGMOND G., *Control sets of linear systems on semi-simple Lie groups*. To be submitted.
- [7] AYALA V., JOUAN PH., *Almost-Riemannian Geometry on Lie groups*, SIAM J. Control and Optimization 54 (2016), no.5, 2919-2947.
- [8] AYALA V., JOUAN PH., ZSIGMOND G., *Isometries of almost-Riemannian structure on Lie groups*, To be submitted.
- [9] AYALA V., AND TIRAO J., *Linear control systems on Lie groups and Controllability*, American Mathematical Society, Series: Symposia in Pure Mathematics, Vol.64 pp 47-64, 1999.

- [10] BONNARD B., CAILLAU J.B., SINCLAIR R., TANAKA M., *Conjugate and cut loci of a two-sphere of revolution with application to optimal control*, Ann. Inst. H. Poincaré Anal. Non Lin. 26 (4) (2009) 1081-1098.
- [11] BONNARD B., CHARLOT G., GHEZZI R., JANIN G., *The sphere and the cut locus at a tangency point in two-dimensional almost-Riemannian geometry*, J. Dyn. Control Syst. 17 (1) (2011) 141-161.
- [12] BOSCAIN U., CHARLOT G., GHEZZI R., *Normal forms and invariants for 2-dimensional almost-Riemannian structures*, Differential Geometry and its Applications 31 (2013) 41-62.
- [13] BOSCAIN U., CHARLOT G., GHEZZI R., SIGALOTTI M., *Lipschitz classification of almost-Riemannian distances on compact oriented surfaces*, J. Geom. Anal., January 2013, Volume 23, Issue 1, pp 438-455.
- [14] COLONIUS F., KLIEMANN W., *The Dynamics of Control*, Systems & Control: Foundations & Applications. Birkäuser Boston, Inc., Boston, MA, 2000.
- [15] DA SILVA A. J., *Controllability of linear systems on solvable Lie groups*. SIAM Journal on Control and Optimization 54 No. 1 (2016), pp. 372-390.
- [16] DATH M., JOUAN, PH., *Controllability of Linear Systems on low dimensional Nilpotent and Solvable Lie groups*, Journal of Dynamics and Control Systems (2014).
- [17] GRUSHIN V.V., *A certain class of hypoelliptic operators*, Mat. Sb.(N.S.)83 (125) (1970) 456-473.
- [18] HELGASON S., *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, (1978).
- [19] JOUAN PH., *Equivalence of Control Systems with Linear Systems on Lie Groups and Homogeneous Spaces*, ESAIM: Control Optimization and Calculus of Variations, 16 (2010) 956-973.
- [20] KIVIOJA V., LE DONNE E., *Isometries of nilpotent metric groups*, arxiv.org/abs/1601.08172.

- [21] KNAPP A. W., *Lie Groups Beyond an Introduction*, Second Edition, Birkhäuser, Berlin, (2004).
- [22] MARKUS L., *Controllability of Multi-trajectories on Lie Groups*. Proceedings of Dynamical Systems and Turbulence, Lecture Notes in Mathematics 898 (1980), pp. 250-265, Warwick.
- [23] MILNOR J., *Curvatures of left invariant metrics on Lie groups*, Advances in Math. 21 (1976), no. 3, 293-329.
- [24] ONISHCHIK A. L., VINBERG E. B., *Lie Groups and Lie Algebras III - Structure of Lie Groups and Lie Algebras*, Berlin: Springer (1990).
- [25] SAN MARTIN L. A. B., *Algebras de Lie*, Second Edition, Editora Unicamp, (2010).
- [26] TAKASU T., *Generalized Riemannian Geometry I*, The journal of the Yokohama Municipal University. Series D (1956).