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André Caldeira

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THÈSE

Pour obtenir le grade de

DOCTEUR DE LA COMMUNAUTÉ UNIVERSITÉ GRENOBLE ALPES

Spécialité : AUTOMATIQUE - PRODUCTIQUE

Arrêté ministériel : 25 mai 2016

Présentée par

André CALDEIRA

Thèse dirigée par **Christophe PRIEUR**, Directeur de Recherche
CNRS, CNRS

préparée au sein du **Laboratoire Grenoble Images Parole Signal
Automatique**
dans l'**École Doctorale Electronique, Electrotechnique,
Automatique, Traitement du Signal (EEATS)**

**Techniques d'analyse de stabilité et
synthèse de contrôle pour des systèmes
hyperboliques.**

**Stability analysis techniques and synthesis
of control for hyperbolic systems.**

Thèse soutenue publiquement le **10 mars 2017**,
devant le jury composé de :

Monsieur Edson Roberto DE PIERI

Professeur associé, Université fédérale Santa Catarina- BRESIL,
Examineur

Monsieur Eugênio de Bona CASTELAN NETO

Professeur associé, Université fédérale Santa Catarina- BRESIL,
Président

Monsieur Ubirajara Franco MORENO

Professeur associé, Université fédérale Santa Catarina - BRESIL,
Examineur

Madame Sophie TARBOURIECH

Directeur de recherche, CNRS Délégation Midi-Pyrénées, Rapporteur

Monsieur Leonardo Antônio Borges TÔRRES

Professeur associé, Université Fédérale du Minas Gerais - BRESIL,
Rapporteur

André Francisco Caldeira

**Dynamic Boundary Control of Coupled
PDE-ODE Systems**

Florianópolis, Brazil

March, 2017

André Francisco Caldeira

Dynamic Boundary Control of Coupled PDE-ODE Systems

Thesis submitted to the Graduate Program in Automation and Systems Engineering (PPGEAS/UFSC) and École Doctorale Electronique, Electrotechnique, Automatique, Traitement du Signal - (EEATS-UGA) in partial fulfillment of the requirements for the PhD degree.

Federal University of Santa Catarina - UFSC

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Supervisor: Daniel F. Coutinho (PPGEAS/UFSC) and

Christophe Prieur (GIPSA-lab/UGA)

Co-supervisor: Valter J. S. Leite (PPGEL/CEFET-MG)

Florianópolis, Brazil

March, 2017

PROCES VERBAL DE SOUTENANCE DU 10 mars 2017 à 13h00

ANNEE UNIVERSITAIRE 2016-2017

Doctorant : Monsieur André CALDEIRA né le 27 novembre 1980
 Spécialité : AUTOMATIQUE - PRODUCTIQUE
 Titre de la thèse : Techniques d'analyse de stabilité et synthèse de contrôle pour des systèmes hyperboliques.
 Ecole doctorale : Electronique, Electrotechnique, Automatique, Traitement du Signal (EEATS)
 Unité de recherche : UMR 5216 - Grenoble Images Parole Signal Automatique
 Directeur de thèse : Christophe PRIEUR
 Cotutelle : UNIVERSIDADE FEDERAL DE SANTA CATARINA - BRÉSIL
 Lieu de soutenance : salle : PPGEAS 01 - Department of Automation and Systems, Technological Center, Federal University of Santa Catarina, PO BOX 476, 88040-900, Florianópolis, Brazil.

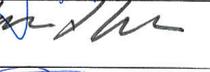
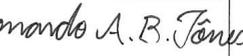
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Membres du Jury :

Nom	Qualité	Etablissement	Rôle	Signature
M. Edson Roberto DE PIERI	Professeur associé	DAS - UFSC	Examineur	
M. Eugênio de Bona CASTELAN NETO	Professeur associé	DAS - UFSC	Examineur	
M. Ubirajara Franco MORENO	Professeur associé	DAS - UFSC	Examineur	
Mme Sophie TARBOURIECH	Directeur de recherche	LAAS-CNRS, Methods and Algorithms in Control, MAC group.	Rapporteur	
M. Leonardo Antônio Borges TÔRRES	Professeur associé	DELT - UFMG	Rapporteur	

Nom et Signature du président du jury :

CASTELAN NETO, Eugênio de Bona



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Candidat : André CALDEIRA

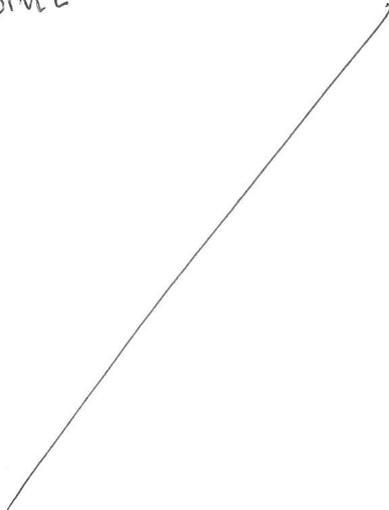
Date de la soutenance : 10 mars 2017

Résultat : Admis

Ajourné

RAPPORT DE SOUTENANCE

Ci-joint



Nom et signature du président du jury :

CASTELAN NETO, Eugénio de Bona

Nom et signature des membres du jury :

TORRES, Leonardo Antonio Borges. Leonardo A.B. Torres

MORENO, Ubirajara Franco. Ubirajara Franco

DE PIERI, Edson Roberto

CASTELAN NETO, Eugénio de Bona

Tarbouriech Sophie

(Vous avez la possibilité de poursuivre la rédaction du rapport de soutenance sur papier libre)

RAPPORT DE SOUTENANCE

Après un exposé d'environ 45 minutes où Monsieur André Francisco Caldeira a présenté de manière concise son travail de doctorat, tout en mettant en évidence les apports et les contributions essentiels de sa thèse, chaque membre du jury a posé des questions concernant divers aspects théoriques et applicatifs du sujet développé. Au cours de ces discussions, il a répondu aux nombreuses questions et commentaires du jury, démontrant une certaine maîtrise et maturité tant au niveau théorique qu'au niveau applicatif, sur un sujet considéré par tous les membres du jury comme étant complexe au niveau des connaissances nécessaires pour son développement. En conclusion, le jury s'est prononcé à l'unanimité pour l'attribution du titre de docteur de la communauté « Université de Grenoble Alpes » à Monsieur André Francisco Caldeira dans la spécialité « Automatique Productique ».



CASTELAN NETO, Eugênio de Bona
Président du jury

Teach is not to transfer knowledge but to create the possibilities for its own production or construction.

(Paulo Freire)

To my parents, José M. Caldeira and Izaneth M. Caldeira

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ABSTRACT

This work studies boundary control strategies for stability analysis and stabilization of first-order hyperbolic system coupled with nonlinear dynamic boundary conditions. The modeling of a flow inside a pipe (fluid transport phenomenon) with boundary control strategy applied in a physical experimental setup is considered as a case study to evaluate the proposed strategies. Firstly, in the context of finite dimension systems, classical control tools are applied to deal with first-order hyperbolic systems having boundary conditions given by the coupling of a heating column dynamical model and a ventilator static model. The tracking problem of this complex dynamics is addressed in a simple manner considering linear approximations, finite difference schemes and an integral action leading to an augmented discrete-time linear system with dimension depending on the step size of discretization in space. Hence, for the infinite dimensional counterpart, two strategies are proposed to address the boundary control problem of first-order hyperbolic systems coupled with nonlinear dynamic boundary conditions. The first one approximates the first-order hyperbolic system dynamics by a pure delay. Then, convex stability and stabilization conditions of uncertain input delayed nonlinear quadratic systems are proposed based on the Lyapunov-Krasovskii (L-K) stability theory which are formulated in terms of Linear Matrix Inequality (LMI) constraints with additional slack variables (introduced by the Finsler's lemma). Thus, strictly Lyapunov functions are used to derive an LMI based approach for the robust regional boundary stability and stabilization of first-order hyperbolic systems with a boundary condition defined by means of a nonlinear quadratic dynamic system. The proposed stability and stabilization LMI conditions are evaluated considering several academic examples and also the flow inside a pipe case study.

Keywords: first-order hyperbolic system, dynamic boundary condition, linear, nonlinear systems, linear approximation, discretization, boundary control, conservation law, fluid flow, Cauchy problem, Lyapunov and

Lyapunov-Krasovskii functionals, Riemann invariants, input-delayed systems, coupled systems, Linear Matrix Inequality.

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ACRONYMS AND NOTATION LIST

Acronyms

DPS	Distributed Parameter Systems
L-K	Lyapunov-Krasovskii
LMI	Linear Matrix Inequality
LPV	Linear Parameter-Varying
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
SeDuMi	Self-Dual-Minimization optimization toolbox
SDP	Semi-Definite Programming

Notation

\triangleq	equal by definition
$:=$	defined as
\square	end of proof
\exists	exists
\in	belong to
\notin	not belong to
\cup	union
\subset	is contained in
\otimes	Kronecker product
z	Advanced operator
\star	indicates symmetrical blocks outside the main diagonal block in LMI constraints
A	notation for matrices (capital letters from the Latin alphabet)
A^T	indicates the transpose operation on a vector or matrix
$A > 0$	indicates that the matrix A is symmetric and positive definite

$A \geq 0$	indicates that the matrix A is symmetric and positive semi-definite
A^H	indicates matrix conjugate transpose of A
$\text{He}\{A\}$	stands for $A + A^T$
\mathcal{A}	Polytope of matrices
$\text{diag}\{\dots\}$	denotes a block-diagonal matrix
\mathbb{R}	set of real numbers
\mathbb{R}^+	set of real non-negative real numbers
\mathbb{N}	set of natural numbers (including zero)
\mathbb{N}_*	set of natural numbers (excluding zero)
Ω	a non-empty connected open subset of \mathbb{R}^n
$t \in [0, +\infty]$ and $x \in [0, L]$	are independent variables representing the time and the space, respectively
$\partial_x f$ and $\partial_t f$ or f_x and f_t	the partial derivatives of a function f with respect to the variables x and t
I	identity matrix of appropriate dimension
$\mathbf{0}$	matrix of zeros having appropriate dimension
$\mathbb{R}^{m \times n}$	is the set of $m \times n$ real matrices
\mathbb{R}^n	is the n -dimensional Euclidean space
$\ \cdot\ $	is the usual Euclidean norm in \mathbb{R}^n
$L^2((0, 1); \mathbb{R}^n)$	is the set of all functions $\phi : (0, 1) \rightarrow \mathbb{R}^n$ such that $\int_0^1 \phi(x)^T \phi(x) dx < \infty$ that is equipped with the norm $\ \cdot\ _{L^2((0,1); \mathbb{R}^n)}$
$\phi : [-d, 0] \rightarrow \mathbb{R}^n$	The space of continuously differentiable functions with finite norm $\ \phi\ _d = \sup_{-d \leq t \leq 0} \ \phi(t)\ $ is denoted by \mathcal{C}_d^n
$x_t \in \mathcal{C}_{[-d, 0]}^n$	is a segment of the function $x_t(s) = x(t + s)$, $\forall s \in [-d, 0]$ and $x_t \in \mathcal{D} \subseteq \mathbb{R}^n$ means that $x(t + s) \in \mathcal{D}$, $\forall s \in [-d, 0]$

1 INTRODUCTION

The transport-reaction, the flow of fluids in open channels or in gas pipelines, the blood flow in the vessels of mammals, the road traffic, the light propagation in optical fibers, the propagation of age-dependent epidemics or the chromatography (BASTIN et al., 2007; DOS SANTOS; PRIEUR, 2008; WITRANT et al., 2010; BANDA; HERTY; KLAR, 2006; CASTILLO et al., 2013), among others, are typical examples of processes that may be represented by hyperbolic partial differential equations (PDEs). In these applications, the system dynamics is represented by one-dimensional hyperbolic systems although the original system dynamics is three-dimensional, because the dominant phenomena evolves in one privileged coordinate dimension while the phenomena in the other directions can be neglected.

This PhD thesis focuses on the study of systems governed by first-order hyperbolic PDEs (commonly used to express the fundamental balance law that occurs in many physical systems when small friction or dissipation effects are neglected (BASTIN et al., 2007)), with boundary conditions modeled by means of ordinary differential equations (ODEs) expressing, for instance, actuator dynamics. The main property of first-order hyperbolic PDEs is the existence of the so-called Riemann coordinates which is a successful tool for demonstrating classical solutions, exponential stability analysis and control design, among other dynamical properties; see, e.g., (BRESSAN, 2000; LI, 1994). In particular, the stability of steady state solutions is a fundamental issue for ODE and PDE systems. Therefore, the exponential stability of steady-state solutions of one-dimensional hyperbolic PDEs coupled to nonlinear ODE system dynamics is studied in this thesis. It is considered conservation laws over a finite space interval, i.e., the spatial domain is an interval of the real line.

The definition of exponential stability is intuitively simple: starting from an arbitrary initial condition, the system time-trajectory has to exponentially converge in spatial-norm to the steady state (globally for linear systems and locally for nonlinear systems). Behind the appar-

ent simplicity of this definition, the stability analysis is however quite challenging. First because this definition is not so easily translated into practical tests of stability. Secondly, there exist various function norms used to measure the deviation with respect to steady state solutions which are not necessarily equivalent and may therefore give rise to different stability tests.

The stability analysis of first-order hyperbolic PDE control systems are subject to sensor and actuator dynamics in two situations: (i) there are sensors and actuators spatially distributed; and (ii) sensors and actuators are only located in the system boundaries (contours). It is considered in this thesis only the situation in which the measurements and actuators are only available at the boundaries. Hence, the exponential stability of steady states closely depends on the so-called *dissipativity* of the boundary conditions which, in many instances, is a natural physical property of the system. This thesis discusses issues where the dissipativity of the boundary conditions (and, consequently the stability) is achieved by applying a boundary feedback control with sensors and actuators located at the boundaries.

A necessary and sufficient stability test for linear homogeneous first-order hyperbolic PDEs (such as systems of linear ODEs) is to verify if the all the roots of the characteristic equation (i.e., the poles) have a negative real part. However, this stability test is not very practical because it is not robust with respect to small variations of the system dynamics. Then, in this thesis, a robust dissipativity test will be proposed derived from the Lyapunov stability approach, which guarantees the existence of a exponentially converging solution for space domain L^2 -norm.

The stability analysis of homogeneous first-order hyperbolic systems has been a research subject for more than thirty years now in the literature. A sufficient condition is that the Jacobian matrix of the boundary conditions has a spectral radius smaller than one, see (SLEMROD, 1983; GREENBERG; TSIEN, 1984; QIN, 1985). The stability analysis can be based on the method of characteristics which can exploit an explicit computation of the ‘reflection’ of the solutions at the

boundaries along the characteristics curves, see for instance (LI, 1994). Other way to perform the stability analysis is by using an appropriate dummy doubling of the system size, (HALLEUX et al., 2003; ??) have shown how the general dissipative boundary condition $\rho|G'(\mathbf{0})| < 1$ that can be established for systems with the general non local boundary condition.

On the other hand, several control practitioners have studied the analysis of dissipative boundary conditions based on the Lyapunov theory. The first attempts employed the system entropy as Lyapunov function candidate as, for instance (CORON; D'ANDRÉA-NOVEL; BASTIN, 1999) and (LEUGERING; SCHMIDT, 2002). The disadvantage of the latter approach was related to the fact that the time derivatives of such entropy-based Lyapunov functions is only *negative semi-definite*. Hence, the LaSalle's invariance principle should be applied to complete the stability analysis. However, the LaSalle's requires the precompactness of the trajectories which is hard to ensure for nonlinear partial differential equations. To overcome this problem, (CORON; BASTIN; D'ANDRÉA-NOVEL (2007) have proposed a strict Lyapunov function whose time derivative is *strictly negative definite*. The advantage of strict conditions is twofold: (i) the proof is simpler than the one using the method of characteristics in view of more direct computations, and (ii) it is easier to be numerically tractable. Another related advantage regards the robustness properties with respect to small uncertainties and disturbances. More recently, (CORON; BASTIN; D'ANDRÉA-NOVEL (2008) have generalized the strict Lyapunov approach to general nonlinear hyperbolic systems.

(CASTILLO et al. (2013) addressed the problem of boundary observer design for one-dimensional first-order linear and quasi-linear strict hyperbolic systems with n rightward convecting transport terms. The stability problem of linear and quasi-linear hyperbolic systems in the presence of dynamic boundary conditions is addressed in (CASTILLO et al., 2012; CASTILLO; WITRANT; DUGARD, 2013; CASTILLO et al., 2013; CASTILLO et al., 2015), while in (DOS SANTOS et al., 2008) a strict Lyapunov function approach is proposed for

the boundary control with integral actions of hyperbolic systems of conservation law that can be diagonalized by means of Riemann invariants.

Various recent contributions and extensions of the previous results worth also to be mentioned.

- [CORON; NGUYEN \(2014\)](#) analyzed dissipative boundary conditions for nonlinear hyperbolic systems in one spatial dimension and proposed sufficient conditions for exponential stability using techniques inspired from the theory of the linear time-delay systems.
- Sufficient conditions for the stability of systems of conservation laws (quasi-linear hyperbolic systems with non-homogeneous terms) were proposed in ([PRIEUR; WINKIN; BASTIN, 2008](#); [PRIEUR, 2009](#)).
- The stability problem of switched linear hyperbolic partial differential equations is considered in ([PRIEUR; GIRARD; WITRANT, 2012](#); [PRIEUR; GIRARD; WITRANT, 2014](#)) by means of Lyapunov techniques.
- A numerical computation of Lyapunov functions for hyperbolic PDEs using LMI formulation and polytopic inclusions is addressed in ([LAMARE; GIRARD; PRIEUR, 2015](#); [LAMARE; GIRARD; PRIEUR, 2016](#)).
- The use of backstepping method for boundary stabilization for non-homogeneous linear systems has been studied in ([KRSTIC et al., 2008](#)) (see, e.g., the text book of [KRSTIC; SMYSHLYAEV \(2008b\)](#) for a complete course on backstepping techniques applied to the boundary control).

Taking the above scenario into account, in this thesis, LMI based techniques are studied to guarantee the dissipativity of the boundary conditions and, consequently, the stability using boundary feedback

control with sensors and actuators located at the boundaries. The control may be implemented with the goal of stabilization (when the system is physically unstable) or simply to achieve an efficient regulation. In some first-order hyperbolic PDEs, the flow of information is usually supplied at the system boundary by a forcing process that provides the information motion along the way by means of wave propagation. In these systems, boundary feedback control often depends on other auxiliary dynamical processes (e.g., sensors and actuators). For instance, a positive-displacement pump may be driven by an electrical motor that imposes a constraint on the pump speed according to the motor torque dynamics and, in this case, the pump speed is a boundary control variable. Thus, the boundary feedback control is studied in situations where first-order hyperbolic PDE systems are coupled to a dynamic boundary system governed by nonlinear ODEs.

In particular, three techniques are considered in this thesis. The first one is a classical time and space finite-dimensional discretization of the first-order hyperbolic PDE which is coupled to a difference equation representing a linear approximation of the nonlinear ODE. In the sequel, the input delay approach approximates the first-order hyperbolic PDE dynamics by a pure delay in the feedback loop of a nonlinear ODE system and the Lyapunov-Krasovckii stability theory of delayed differential equations is applied to derive stability and stabilization conditions guaranteeing the regional stability of the input delayed system. Then, infinite-dimensional tools are used to address the robust boundary control stabilization problem of a coupled first-order hyperbolic PDE with a nonlinear ODE considering strict Lyapunov functionals.

1.1 Motivation

In many applications, such as diffusion, the physical quantity of interest depends on both position and time. These systems are modeled by partial differential equations (PDEs) and the solution evolves on an infinite-dimensional Lebesgue or Hilbert space. For this reason, these systems are often called infinite-dimensional systems. In contrast, the

state of the system modeled by an ordinary differential equation (ODE) evolves on a finite-dimensional system, such as \mathbb{R}^n , and these systems are called finite-dimensional. Since the solution of the PDE reflects the distribution in space of a physical quantity such as the temperature of a rod or the deflection of a beam, these systems are often also called distributed parameter systems (DPS).

Finite-dimensional models are often inadequate for control design for DPS (LASIECKA, 2002). In particular, hyperbolic systems have features such as finite speed of propagation which require approximation by high-order finite dimensional models (PAVEL; CHANG, 2012). While hyperbolic systems of second order describe oscillatory systems such as strings or beams, first-order hyperbolic systems (thesis focus) describe physical problems of transport-reaction type often encountered in many industrial applications such as hydraulic networks (BASTIN et al., 2007; DOS SANTOS; PRIEUR, 2008), gas flow in pipelines (BANDA; HERTY; KLAR, 2006; CASTILLO et al., 2013), flow regulation in deep pits (WITRANT et al., 2010), among others.

The distinct feature of hyperbolic PDEs is that all the eigenmodes of the spatial differential operator typically contain nearly the same amount of energy; as a result a very large number of modes is required to accurately approximate their dynamic behaviour. This feature distinguishes hyperbolic partial differential equations and suggests addressing the control problem on the basis of the infinite-dimensional model itself (CHRISTOFIDES, 2012; LUO; GUO; MORGÜL, 2012).

An example of a control design is performed in (DOWER; FARRELL, 2006), based on a finite-dimensional approximation of the linearized PDE model around the desired steady-state. From a 588th-order finite-difference model a tenth-order reduced model is obtained by open-loop balance truncation, and a tenth-order H_∞ -controller is designed. However, there is no guarantee that the stabilization and regulation properties will hold for the real closed-loop hyperbolic system.

Boundary control has been an area of great interest recently, one reason being the fact that it has addressed the problem on the

infinite-dimensional model itself.

- Results for the stabilization or disturbance rejection for the linear wave equation are developed in (LUO; GUO; MORGÜL, 2012; MORGUL, 1998).
- The backstepping method is an approach that requires numerical computation of the kernel (KRSTIC; SMYSHLYAEV, 2008b; KRSTIC; SMYSHLYAEV, 2008a).
- The zero dynamics-based approach provides a systematic methodology, and typically results in infinite-dimensional controllers (BYRNES; GILLIAM; HU, 2006).
- A Lyapunov-based approach which is related to the recently employed methods in (CORON; BASTIN; D'ANDRÉA-NOVEL, 2007; BASTIN et al., 2007; CORON; BASTIN; D'ANDRÉA-NOVEL, 2008; DOS SANTOS et al., 2008; XU; SALLET, 2002).

1.2 Thesis Scope

Consider the following first-order hyperbolic system of linear conservation laws in Riemann coordinates (TORO, 2013, Chapter 2), (BASTIN; CORON, 2016):

$$\xi_t + \Lambda \xi_x = \mathbf{0}, \quad t \in [0, +\infty), \quad x \in [0, 1], \quad (1.1)$$

where $\xi : [0, +\infty) \times [0, 1] \rightarrow \mathbb{R}^n$, Λ is a diagonal matrix with non-zero diagonal terms such that

$$\Lambda = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\},$$

with

$$\lambda_i > 0, \quad \forall_i \in \{1, \dots, m\},$$

and

$$\lambda_i < 0, \quad \forall_i \in \{m+1, \dots, n\}.$$

The thesis will focus on deriving strategies to ensure the dissipativity of the boundary conditions for the above PDE system. To this end, the following notation is introduced:

$$\boldsymbol{\xi}^+ = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_m \end{bmatrix}, \quad \boldsymbol{\xi}^- = \begin{bmatrix} \xi_{m+1} \\ \vdots \\ \xi_n \end{bmatrix}, \quad \boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\xi}^+ \\ \boldsymbol{\xi}^- \end{bmatrix}, \quad (1.2)$$

$$\Lambda^+ = \text{diag}\{\lambda_1, \dots, \lambda_m\}, \quad \Lambda^- = \text{diag}\{\lambda_{m+1}, \dots, \lambda_n\}, \quad (1.3)$$

where $\boldsymbol{\xi}^+ : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}^m$, $\boldsymbol{\xi}^- : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}^{n-m}$.

In light of the above, the linear hyperbolic system in (1.1) can be recast as follows:

$$\partial_t \begin{bmatrix} \boldsymbol{\xi}^+ \\ \boldsymbol{\xi}^- \end{bmatrix} + \begin{bmatrix} \Lambda^+ & \mathbf{0} \\ \mathbf{0} & \Lambda^- \end{bmatrix} \partial_x \begin{bmatrix} \boldsymbol{\xi}^+ \\ \boldsymbol{\xi}^- \end{bmatrix} = \mathbf{0}. \quad (1.4)$$

The exponential stability of system (1.4) has been the focus of references (CORON; BASTIN; D'ANDRÉA-NOVEL, 2008; PRIEUR; WINKIN; BASTIN, 2008; BASTIN; CORON; D'ANDRÉA-NOVEL, 2009), among others, considering static boundary conditions:

$$\begin{bmatrix} \boldsymbol{\xi}^+(t, 0) \\ \boldsymbol{\xi}^-(t, 1) \end{bmatrix} = \mathbf{K} \begin{bmatrix} \boldsymbol{\xi}^+(t, 1) \\ \boldsymbol{\xi}^-(t, 0) \end{bmatrix}, \quad t \in [0, +\infty), \quad (1.5)$$

where \mathbf{K} is a matrix in $\mathbb{R}^{n \times n}$ having the following structure

$$\mathbf{K} = \begin{bmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{bmatrix}.$$

The initial condition of system (1.4) is given by

$$\boldsymbol{\xi}(0, x) = \bar{\boldsymbol{\xi}}(x) \quad x \in (0, 1). \quad (1.6)$$

Instead of studying the exponential stability of system (1.1) subject to static boundary conditions (1.5), in this thesis, it will be addressed the exponential stability of linear first-order hyperbolic system (1.1) considering the following dynamics for the boundary conditions:

$$\begin{cases} \dot{\mathbf{X}} &= \mathbf{A}\mathbf{X} + \mathbf{B}\boldsymbol{\xi}_{out}(t), \\ \boldsymbol{\xi}_{in}(t) &= \mathbf{C}\mathbf{X} + \mathbf{K}\boldsymbol{\xi}_{out}(t), \end{cases} \quad (1.7)$$

where $A \in \mathbb{R}^{n_\ell \times n_\ell}$, $B \in \mathbb{R}^{n_\ell \times n}$, $C \in \mathbb{R}^{n \times n_\ell}$, $\mathbf{X} \in \mathbb{R}^{n_\ell}$, $n_\ell \leq n$. The vectors ξ_{in} and ξ_{out} are defined as:

$$\xi_{in}(t) \triangleq \begin{bmatrix} \xi^+(t, 0) \\ \xi^-(t, 1) \end{bmatrix}, \quad \xi_{out}(t) \triangleq \begin{bmatrix} \xi^+(t, 1) \\ \xi^-(t, 0) \end{bmatrix}. \quad (1.8)$$

It will be also considered in this thesis that the boundary dynamics is described by a nonlinear ODE where the matrices (A, B, C and K) of (1.7) depends on X (which is coupled with a first-order hyperbolic system). In this case, it will be studied finite-dimensional and infinite-dimensional tools for assessing the coupled PDE-ODE system which are briefly introduced in the sequel. These methods are evaluated considering an experimental setup consisting of a fluid transport in a tube (Poiseuille flow) with a heating nonlinear dynamics at the boundary condition to control the outlet temperature. In particular, the device is constituted by a heating column encasing a resistor, a tube, two ventilators, a gas speed meter and three distributed temperature sensors (described in Chapter 3).

The first proposed solution employs classical finite-dimensional techniques for designing a stabilizing state feedback boundary controller considering the proposed experimental setup which is describe by a coupled finite-dimensional dynamic model of the heating column and a static model of the ventilator. An approximate linear model of the boundary dynamics is considered while a space discretization technique is applied to obtain an augmented discrete linear system with the dimension depending on the step size of discretization in space.

Then, the second proposed solution considers an approximate infinite-dimensional model of the first-order hyperbolic system (1.1) which consists on a pure transport delay leading to the following input delayed nonlinear system:

$$\begin{cases} \dot{x}(t) &= A(x(t), \delta)x(t) + B(x(t), \delta)u(t - \tau(t)), \\ u(t) &= \mathbf{0}, \quad \forall t < 0, \\ u(t) &= K(x(t))x(t), \quad \forall t \geq 0, \\ x(0) &= \bar{x}_0 \end{cases} \quad (1.9)$$

where $x(t) \in \mathcal{X} \subset \mathbb{R}^n$ is the state, $u(t) \in \mathcal{U} \subset \mathbb{R}^{n_u}$ is the control input, $\delta \in \Delta \subset \mathbb{R}^{n_\delta}$ is a vector of uncertain constant parameters, \bar{x}_0 is the initial condition, and $\tau(t) \in \mathbb{R}$ is a time-varying input delay satisfying:

$$0 < \tau(t) \leq d, \quad \dot{\tau}(t) \leq h < 1, \quad \forall t \geq 0, \quad (1.10)$$

with d and h positive scalars. \mathcal{X} , \mathcal{U} and Δ are compact regions defining respectively the state, input and uncertainty domains with \mathcal{X} containing the system origin. The matrices $A(\cdot)$, $B(\cdot)$ and $K(\cdot)$ are affine functions of their arguments, more precisely:

$$A(x(t), \delta) = A_0 + \sum_{i=1}^n x_i(t) A_i + \sum_{i=1}^{n_\delta} \delta_i \check{A}_i, \quad (1.11)$$

$$B(x(t), \delta) = B_0 + \sum_{i=1}^n x_i(t) B_i + \sum_{i=1}^{n_\delta} \delta_i \check{B}_i, \quad (1.12)$$

$$K(x(t)) = K_0 + \sum_{i=1}^n x_i(t) K_i, \quad (1.13)$$

where $x_i(t)$ denotes the i -th entry of $x(t)$; and A_i , \check{A}_j , B_i , \check{B}_j and K_i , for $i = 0, 1, \dots, n$ and $j = 1, \dots, n_\delta$, are given constant matrices with appropriate dimensions.

Since the input delay is assumed to be greater than zero and bounded by d and its time derivative is smaller or equal to one, there exists a unique $t_0 \leq d$ such that $t - \tau(t) < 0$, for all $t \in [0, t_0)$, and $t - \tau(t) \geq 0$, for all $t \geq t_0$. As a consequence, the system operates in open-loop in the interval of time $[0; t_0)$. Hence, the closed-loop system is governed by the following dynamics:

$$\dot{x}(t) = \begin{cases} A(x(t), \delta)x(t), & \text{if } 0 \leq t < t_0; \\ A(x(t), \delta)x(t) + B(x(t), \delta)K(\tilde{x}(t))\tilde{x}(t), & \text{if } t \geq t_0; \end{cases} \quad (1.14)$$

where $x(s) = \bar{x}_0$ for $s \in [-d, 0]$, $\tilde{x}(t) = x(t - \tau(t))$ is the delayed state, $\tau(t) \in (0, d]$ and

$$K(\tilde{x}(t)) = K_0 + \sum_{i=1}^n \tilde{x}_i(t) K_i,$$

with $\tilde{x}_i(t)$ denoting the i -th entry of $\tilde{x}(t)$.

Finally, an infinite-dimensional approach (considering linear conservation laws in Riemann coordinates) with no approximations is applied to the (exponential) stability analysis and stabilization of coupled PDE-ODE systems in a regional context (i.e., local stability with an estimate of the stability region of attraction). In other words, bounded initial state trajectories imply that the state trajectories are bounded and they converge to the system equilibrium point as the time goes to infinity. To this end, it is considered strict Lyapunov functionals and the regional stability is characterized in terms of two sets (the set of boundary initial conditions and the set of ODE initial states). The control law is a nonlinear function of the (inlet and outlet) boundary conditions.

1.3 Methodology

A literature review is performed on the fundamental concepts of the theory of first-order hyperbolic systems in Riemann coordinates, dissipativity of the boundary conditions (static and dynamic), boundary feedback control, nonlinear system with input delay, finite-dimensional tools for hyperbolic PDE system analysis and some mathematical tools needed to deal with the problem in question. Hence, in this thesis, the proposed techniques to address the stability and stabilization problems of first-order linear hyperbolic systems (of conservation laws in Riemann coordinates) with nonlinear dynamic boundary conditions are based on the Lyapunov stability theory and they are numerically solved via the LMI framework. A finite-dimensional approach which is based on classical solutions to coupled PDE-ODE systems is firstly studied. More precisely, it is performed a time and space discretization (considering standard finite difference schemes) to deal with the PDE while a linear approximation technique around the steady state solution is applied to deal with the nonlinear ODE. These approximation schemes lead to an augmented discrete-time system whose dimension depends on the discretization steps. Hence, a discrete-time controller is designed based on the Lyapunov theory in order to impose

the boundary dynamics. This technique is applied to the experimental setup (*Poiseuille* flow system) to control the outlet temperature showing that classical tools can be applied to coupled PDE-ODE systems at the cost of somewhat poor performance.

The two proposed infinite-dimensional techniques are based on Lyapunov functionals with the stability and stabilization conditions formulated in terms of LMI constraints which guarantee the closed-loop stability while providing an estimate of the system region of attraction. The first one models the PDE dynamics as a pure delay leading to an input delayed system in closed-loop (LIU; FRIDMAN, 2014). Subsequently, stability analysis and control design conditions are proposed to guarantee the regional exponential stability of the input delayed nonlinear systems considering a Lyapunov-Krasovskii (L-K) functional (KRASOVSKII, 1963). Two convex optimization problems are proposed to maximize either an estimate of the set of initial states or the admissible delay, with both optimization problems formulated in terms of LMI constraints.

The second infinite-dimensional approach considers that the first-order hyperbolic systems of linear conservation laws is cast in Riemann coordinates. Hence, stability and stabilization conditions with boundary control actions to ensure the dissipativity of the boundary conditions are obtained based on a strict Lyapunov functional whose time derivative is strictly negative definite as in (CORON; BASTIN; D'ANDRÉA-NOVEL, 2007; CASTILLO et al., 2015). Similarly to the former infinite-dimensional technique, two convex optimization problems in terms of LMI constraints are proposed. The first one assuming a given boundary controller and a given set of PDE initial conditions obtains a maximized estimate of the set of ODE initial conditions, while the second one synthesizes the boundary control gains in order to obtain a maximized estimate of the set of the ODE initial conditions for a given set of PDE initial conditions.

An experimental setup (the *Poiseuille* flow) whose control objective is to design a boundary controller to regulate the outlet temperature is also studied in this thesis. The derived model (including

a boundary actuator and an outlet sensor) is considered in numerical simulations to demonstrate the proposed boundary control design methodologies allowing the evaluation and discussion of the results.

1.4 Text organization

The remaining of this thesis is organized as follows:

- **Chapter 2** presents fundamental concepts necessary for the development and understanding of the work, including some basic definitions of first-order hyperbolic systems, characteristic form and Riemann coordinates, linearization around the steady state solution, Riemann coordinates around the steady state, Conservation Laws in Riemann invariants, stability and boundary stabilization, the Cauchy problem in Riemann coordinates, linear systems of Conservation Laws, Exponential Stability for the L^2 -norm, Dissipative boundary conditions.
- **Chapter 3** models the flow inside a pipe experimental setup (i.e., the *Poiseuille* flow) by means of a hyperbolic first-order PDE system in which the inlet boundary condition is imposed by a finite-dimensional dynamic model of heating column assuming a static model for the ventilator. Then, classical finite-dimensional tools are applied to boundary control design considering a linear approximation of the boundary actuator and time and space discretization of the PDE. Thus, an augmented discrete-time linear system is obtained whose dimension depends on the discretization steps. As a result, the tracking problem of the original complex dynamics can be addressed in a simple manner. Hence, a classical pole placement design methodology consisting of a state observer and a state feedback control law (synthesized by means of the LMI framework) is applied to improve the closed-loop dynamics tracking performance.
- **Chapter 4** approximates the PDE dynamics by a pure transport delay leading to a nonlinear input delayed ODE system. For input

delayed systems, the first time interval plays an important role for assessing the local stability of open-loop unstable nonlinear systems, since the boundedness of the state trajectory has to be guaranteed for all time $t \geq 0$ as pointed out in (FRIDMAN; LIU, 2016). Hence, using L-K functionals and the slack variable approach, a set of LMI conditions are derived to the regional stability analysis and control design for uncertain nonlinear quadratic systems subject to a time-varying input delay. The local stability analysis (assuming that a stabilizing control law and bounds on the input delay and its variation are known *a priori*) is characterized in terms of two compact sets \mathcal{R}_0 and \mathcal{R} which are such that the state trajectory $x(t)$ starting in the former set remains inside the latter and converges to the equilibrium point as the time goes to infinity. Then, the stability analysis result is extended for control design in order to determine a quadratic state feedback control law to either maximize the set of admissible initials conditions or maximize the input delay size.

- **Chapter 5** employs strict Lyapunov functionals to devise LMI based conditions to the robust regional boundary stability and stabilization of uncertain first-order hyperbolic systems with boundary condition defined by uncertain nonlinear quadratic dynamic systems (AMATO; COSENTINO; MEROLA, 2007; VALMÓRBIDA; TARBOURIECH; GARCIA, 2010; COUTINHO; DE SOUZA, 2012). Thus, the local boundary stability of coupled linear uncertain first-order hyperbolic PDE systems with nonlinear quadratic dynamic boundary conditions is characterized in terms of compact set \mathcal{B} and \mathcal{R} which are such that the boundary state trajectory starting in the former set remains inside the latter and converges to the equilibrium point as the time goes to infinity. Then, the boundary feedback control design in order to determine a quadratic boundary state feedback control law to either maximize the set of admissible initials conditions or maximize the size of \mathcal{R} .

- **Chapter 6** some comments on the obtained results, correlated works and future research concludes the thesis.

1.5 General Comments

In this thesis, the model uncertainties (PDE and ODE dynamics) are assumed to belong to convex polytopic regions with known vertices.

The numerical results obtained in all examples presented in this thesis were simulated using the software *MATLAB*[®]. In addition, the LMI conditions are solved by means of the parser YALMIP considering the solvers SeDuMi (STURM, 1999) and *SDPT3* (TÜTÜNCÜ; TOH; TODD, 2003). More details on the parser and the semidefinite programming solvers can be found on the YALMIP website at <https://yalmip.github.io/>.

2 PRELIMINARIES

This chapter introduces some basic and fundamental properties of the class of first-order hyperbolic PDEs. The selected aspects of the equations are those thought to be essential for the stability analysis and boundary stabilization. In short, it is discussed the characteristic form, the Riemann coordinates, steady state solutions, linearization and the boundary stabilization problem. Some typical examples of hyperbolic systems of physical engineering applications are also presented to illustrate some concepts. Necessary and sufficient stability conditions for systems of linear conservation laws under local boundary conditions are presented in time and frequency domains including a review on the notion of exponential stability (in L^2 -norm for time domain and C^0 -norm for frequency domain). In addition, stability conditions of linear conservations laws under dynamic boundary control are presented. The main definitions presented in this chapter were based on the recent text book of [BASTIN; CORON \(2016\)](#), but it was also included concepts taken from other important text books such as ([LAX, 1973](#); [LI, 1994](#); [SERRE, 1999](#); [BRESSAN, 2000](#); [DAFERMOS, 2000](#); [LASIECKA, 2002](#); [TVEITO; WINTHER, 2005](#); [CORON, 2007](#); [SILVA; DATTA; BHATTACHARYYA, 2007](#); [KRSTIC; SMYSHLYAEV, 2008b](#); [TORO, 2013](#); [FRIDMAN, 2014](#)).

2.1 Basic definitions

Although we address only one-dimensional systems of law of conservation, in this first stage, we present the basic definitions for the more general case of one-dimensional systems of balance laws. Let Ω be a non-empty connected open subset of \mathbb{R}^n . A one-dimensional hyperbolic system of n nonlinear balance laws over a finite space interval is a system of PDEs of the form

$$\partial_t e[\mathbf{U}(t,x)] + \partial_x f[\mathbf{U}(t,x)] + g[\mathbf{U}(t,x)] = \mathbf{0}, \quad (2.1)$$

where $t \in [0, +\infty]$ and $x \in [0, L]$ are independent variables representing the time and the space, respectively. $\mathbf{U} : [0, +\infty] \times [0, L] \rightarrow \Omega$ is the vec-

tor of state variables; $e \in \mathcal{C}^2(\Omega; \mathbb{R}^n)$ is the vector of the densities of the balanced quantities; the map e is a diffeomorphism on Ω ; $f \in \mathcal{C}^2(\Omega; \mathbb{R}^n)$ is the vector of the corresponding flux densities; $g \in \mathcal{C}^1(\Omega; \mathbb{R}^n)$ is the vector of source terms representing production or consumption of the balanced quantities inside the system.

In this chapter, the partial derivatives of a function $h(t, x)$ with respect to the variables x and t are indifferently denoted by either $\partial_x h$ and $\partial_t h$ or h_x and h_t .

Under these conditions, system (2.1) can be recast in the following quasi-linear system representation:

$$\mathbf{U}_t + F[\mathbf{U}]\mathbf{U}_x + G[\mathbf{U}] = \mathbf{0}, \quad t \in [0, +\infty), \quad x \in [0, L], \quad (2.2)$$

with $F : \Omega \rightarrow \mathbb{R}^{n \times n}$ and $G : \Omega \rightarrow \mathbb{R}^n$ are of class \mathcal{C}^1 and defined as

$$F[\mathbf{U}] \triangleq \left[\frac{\partial e}{\partial \mathbf{U}} \right]^{-1} \left[\frac{\partial f}{\partial \mathbf{U}} \right], \quad G[\mathbf{U}] \triangleq \left[\frac{\partial e}{\partial \mathbf{U}} \right]^{-1} g[\mathbf{U}].$$

From (2.2) and hereafter, the argument (t, x) will be often omitted in this thesis when not leading to any confusion.

It will be assumed that system (2.2) is *hyperbolic* i.e. that $F[\mathbf{U}]$ has n real eigenvalues (called the *characteristic velocities*) for all $\mathbf{U} \in \Omega$. In the whole of this thesis, it will be also always assumed that these eigenvalues do not vanish in Ω . It follows that a number m of positive eigenvalues (counting multiplicity) is independent of \mathbf{U} . Thus, it will be always possible to consider the following notation for the m positive and $n - m$ negative eigenvalues:

$$\lambda_1[\mathbf{U}], \dots, \lambda_m[\mathbf{U}], -\lambda_{m+1}[\mathbf{U}], \dots, -\lambda_n[\mathbf{U}], \quad \lambda_i[\mathbf{U}] > 0, \quad \forall \mathbf{U} \in \Omega, \quad \forall i.$$

In the particular case where \mathbf{F} is constant (i.e., \mathbf{F} does not depend on \mathbf{U}), the system (2.2) is called *semi-linear*. In this case, the system has constant characteristic velocities denoted:

$$\lambda_1, \dots, \lambda_m, -\lambda_{m+1}, \dots, -\lambda_n, \quad \lambda_i > 0, \quad \forall i.$$

2.1.1 Characteristic form and Riemann coordinates

The class of hyperbolic systems of balance laws can be transformed into a *characteristic form* by defining a set of n *Riemann coordinates*; see for instance, (TORO, 2013, Chapter 2) and (DAFERMOS, 2000, Chapter 7, Section 7.3). The characteristic form is obtained through a change of coordinates $\boldsymbol{\xi} = \boldsymbol{\psi}[\mathbf{U}]$ having the following properties:

- The function $\boldsymbol{\psi} : \Omega \rightarrow \mathcal{R} \subset \mathbb{R}^n$ is a diffeomorphism: $\boldsymbol{\xi} = \boldsymbol{\psi}[\mathbf{U}] \longleftrightarrow \mathbf{U} = \boldsymbol{\psi}^{-1}[\boldsymbol{\xi}]$, with Jacobian matrix $\boldsymbol{\Psi}[\mathbf{U}] \triangleq \frac{\partial \boldsymbol{\psi}}{\partial \mathbf{U}}$.
- The Jacobian matrix $\boldsymbol{\Psi}[\mathbf{U}]$ makes the matrix $F[\mathbf{U}]$ diagonal. That is

$$\boldsymbol{\Psi}[\mathbf{U}]F[\mathbf{U}] = D[\mathbf{U}]\boldsymbol{\Psi}[\mathbf{U}], \quad \mathbf{U} \in \Omega,$$

with $D[\mathbf{U}] = \text{diag}\{\lambda_1[\mathbf{U}], \dots, \lambda_m[\mathbf{U}], -\lambda_{m+1}[\mathbf{U}], \dots, -\lambda_n[\mathbf{U}]\}$.

Then, system (2.2) is equivalent for \mathcal{C}^1 -solutions to the following system in characteristic form expressed in the Riemann coordinates:

$$\boldsymbol{\xi}_t + \Lambda[\boldsymbol{\xi}]\boldsymbol{\xi}_x + \mathbf{C}[\boldsymbol{\xi}] = \mathbf{0}, \quad t \in [0, +\infty), \quad x \in [0, L], \quad (2.3)$$

with $\Lambda[\boldsymbol{\xi}] \triangleq D[\boldsymbol{\psi}^{-1}[\boldsymbol{\xi}]]$ and $\mathbf{C}[\boldsymbol{\xi}] \triangleq \boldsymbol{\Psi}[\boldsymbol{\psi}^{-1}[\boldsymbol{\xi}]]\mathbf{G}[\boldsymbol{\psi}^{-1}[\boldsymbol{\xi}]]$. Clearly, this change of coordinates exists for any system of balance laws with linear flux densities (i.e. with $f[\mathbf{U}] = \mathbf{A}\mathbf{U}$, $\mathbf{A} \in \mathbb{R}^{n \times n}$ constant) when the matrix \mathbf{A} is diagonalizable. For systems with nonlinear flux densities, finding the change of coordinates $\boldsymbol{\xi} = \boldsymbol{\psi}[\mathbf{U}]$ requires to find a solution of the first order partial differential equation $\boldsymbol{\Psi}[\mathbf{U}]F[\mathbf{U}] = D[\mathbf{U}]\boldsymbol{\Psi}[\mathbf{U}]$. As shown in (LAX, 1973, pages 34–35), this partial differential equation can always be solved (at least locally) for systems of size $n = 2$ with distinct characteristic velocities; see to (LI, 1994, page 30)). In contrast, for systems of size larger than 3, the change of coordinates only exists in some particular cases.

2.1.2 Steady state and linearization

A *steady state* (or equilibrium) is a time-invariant space-varying solution $\mathbf{U}(t, x) = \mathbf{U}^*(x) \forall t \in [0, +\infty]$ of system (2.2). It satisfies the ordinary differential equation

$$F[\mathbf{U}^*]\mathbf{U}_x^* + G[\mathbf{U}^*] = \mathbf{0}, \quad x \in [0, L]. \quad (2.4)$$

The linearization of the system about the steady state is then

$$\mathbf{U}_t + A(x)\mathbf{U}_x + B(x)\mathbf{U} = \mathbf{0}, \quad t \in [0, +\infty), \quad x \in [0, L], \quad (2.5)$$

where

$$A(x) \triangleq F[\mathbf{U}^*(x)] \quad \text{and} \quad B(x) \triangleq \left[\frac{\partial}{\partial \mathbf{U}} [F[\mathbf{U}]\mathbf{U}_x^* + G(\mathbf{U})] \right]_{\mathbf{U}=\mathbf{U}^*(x)}. \quad (2.6)$$

In the special case where there is a solution to the algebraic equation $G[\mathbf{U}^*] = \mathbf{0}$, the system has a *constant steady state* (independent of both t and x) and the corresponding linearization is

$$\mathbf{U}_t + A\mathbf{U}_x + B\mathbf{U} = \mathbf{0}, \quad t \in [0, +\infty), \quad x \in [0, L], \quad (2.7)$$

where A and B are constant matrices. In this special case where \mathbf{U}^* is constant, the nonlinear system (2.2) is said to have *uniform steady state*. In general the steady state $\mathbf{U}^*(x)$ is space varying, and the nonlinear system (2.2) is said to have a *nonuniform* steady state.

2.1.3 Riemann coordinates around the steady state

By definition, the steady state of system (2.3) is

$$\xi^*(x) = \psi[\mathbf{U}^*(x)] \text{ such that } \Lambda[\xi^*]\xi_x^* + C[\xi^*] = \mathbf{0}.$$

Then, alternatively, Riemann coordinates may also be defined around this steady state as

$$\xi \triangleq \psi[\mathbf{U}] - \psi[\mathbf{U}^*]. \quad (2.8)$$

With these coordinates the system is now rewritten in characteristic form as

$$\xi_t + \Lambda[\xi, x]\xi_x + C[\xi, x] = \mathbf{0}, \quad t \in [0, +\infty), \quad x \in [0, L], \quad (2.9)$$

with

$$\Lambda[\boldsymbol{\xi}(t,x),x] \triangleq D[\boldsymbol{\Psi}^{-1}[\boldsymbol{\xi}(t,x) + \boldsymbol{\Psi}[\mathbf{U}^*(x)]]].$$

and

$$\begin{aligned} C[\boldsymbol{\xi}(t,x),x] &\triangleq D[\boldsymbol{\Psi}^{-1}[\boldsymbol{\xi}(t,x) + \boldsymbol{\Psi}[\mathbf{U}^*(x)]]] \boldsymbol{\Psi}_x[\mathbf{U}^*(x)] \\ &+ \boldsymbol{\Psi}[\boldsymbol{\Psi}^{-1}[\boldsymbol{\xi}(t,x) + \boldsymbol{\Psi}[\mathbf{U}^*(x)]]] G[\boldsymbol{\Psi}^{-1}[\boldsymbol{\xi}(t,x) + \boldsymbol{\Psi}[\mathbf{U}^*(x)]]]. \end{aligned}$$

The linearization of the system (2.9) gives:

$$\boldsymbol{\xi}_t + \Lambda(x)\boldsymbol{\xi}_x + M(x)\boldsymbol{\xi} = \mathbf{0}, \quad t \in [0, +\infty), \quad x \in [0, L],$$

$$\text{with } \Lambda(x) \triangleq D[\mathbf{U}^*(x)] \text{ and } M(x) \triangleq \left[\frac{\partial C(\boldsymbol{\xi}, x)}{\partial \boldsymbol{\xi}} \right]_{\boldsymbol{\xi}=\mathbf{0}}.$$

Remark that this linear model is also the linearization of system (2.3) around the steady state and that it could be obtained as well by transforming directly the linear system (2.5) into Riemann coordinates. In other words the operations of linearization and Riemann coordinate transformation can be commuted.

Example 2.1. *The Saint-Venant equation (BARRÉ DE SAINT-VENANT, 1871) (also called shallow water equation), which is a simplification of the Navier-Stokes equation, describes the water propagation in a prismatic channel with rectangular cross-section and constant slope as follows:*

$$\partial_t \begin{bmatrix} H \\ V \end{bmatrix} + \partial_x \begin{bmatrix} HV \\ \frac{1}{2}V^2 + gH \end{bmatrix} + \begin{bmatrix} 0 \\ g(CV^2H^{-1} - S) \end{bmatrix} = \mathbf{0}, \quad (2.10)$$

with $H(t,x)$ the water height and $V(t,x)$ the water velocity at time t and location x along the channel. g is the gravity constant, C a friction parameter and S the channel slope. From this equation we have:

$$\begin{aligned} \mathbf{U} &= \begin{bmatrix} H \\ V \end{bmatrix}, \quad A[\mathbf{U}] = \begin{bmatrix} V & H \\ g & V \end{bmatrix}, \\ g[\mathbf{U}] &= \begin{bmatrix} 0 \\ g(CV^2H^{-1} - S) \end{bmatrix}. \end{aligned} \quad (2.11)$$

The eigenvalues of the matrix $\mathbf{A}(\mathbf{U})$ are:

$$V + \sqrt{gH} \quad \text{and} \quad V - \sqrt{gH}.$$

The system is hyperbolic when the so-called Froude's number $F_r = \frac{V}{\sqrt{gH}} < 1$. In such a case, the flow in the channel is said to be fluvial or subcritical.

Under this condition, the system is hyperbolic with characteristic velocities

$$\lambda_1[\mathbf{U}] = V + \sqrt{gH} > 0 > -\lambda_2[\mathbf{U}] = V - \sqrt{gH}.$$

Riemann coordinates may be defined as

$$\xi_1 = V + 2\sqrt{gH}, \quad \xi_2 = V - 2\sqrt{gH},$$

and are inverted as

$$H = \frac{(\xi_1 - \xi_2)^2}{16g}, \quad V = \frac{(\xi_1 + \xi_2)}{2}.$$

With these coordinates, the system is written in characteristic form

$$\xi_t + \Lambda[\xi] \xi_x + C[\xi] = \mathbf{0},$$

with

$$\Lambda[\xi] \triangleq \begin{bmatrix} \lambda_1(\xi) & 0 \\ 0 & \lambda_2(\xi) \end{bmatrix} = \begin{bmatrix} \frac{3\xi_1 + \xi_2}{4} & 0 \\ 0 & \frac{\xi_1 + 3\xi_2}{4} \end{bmatrix},$$

and

$$C[\xi] \triangleq \left[4gC \left(\frac{\xi_1 + \xi_2}{\xi_1 - \xi_2} \right)^2 - gS \right] \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For the **Saint-Venant equation (2.11)** an equilibrium is a constant state H^* , V^* that verifies the relation

$$SH^* = C(V^*)^2.$$

The linearisation of the Saint-Venant equations around the equilibrium in diagonal form with the following characteristic state variables (ξ_1, ξ_2) , characteristic velocities λ_1, λ_2 and the matrix \mathbf{S} :

$$\xi_1 = (V - V^*) + (H - H^*)\sqrt{\frac{g}{H^*}} \quad \xi_2 = (V - V^*) - (H - H^*)\sqrt{\frac{g}{H^*}},$$

$$\lambda_2 = V^* - \sqrt{gH^*} < 0 < \lambda_1 = V^* + \sqrt{gH^*},$$

$$\mathbf{S} = \begin{bmatrix} \gamma & \beta \\ \gamma & \beta \end{bmatrix},$$

$$\text{with } \gamma = \frac{gS}{2} \left[\frac{2}{V^*} - \frac{1}{\sqrt{gH^*}} \right] > 0 \text{ and } \beta = \frac{gS}{2} \left[\frac{2}{V^*} + \frac{1}{\sqrt{gH^*}} \right] > 0.$$

Example 2.2. The Aw-Rascle equation (AW; RASCLE, 2000) is a basic fluid model for the description of road traffic dynamics. It is directly given here in the quasi-linear form. The model is as follows:

$$\begin{aligned} \partial_t \begin{bmatrix} \rho \\ V \end{bmatrix} + \begin{bmatrix} V & \rho \\ 0 & V + \rho V'_o(\rho) \end{bmatrix} \partial_x \begin{bmatrix} \rho \\ V \end{bmatrix} \\ + \begin{bmatrix} 0 \\ \frac{V - V_o(\rho)}{\tau} \end{bmatrix} = \mathbf{0}, \end{aligned} \quad (2.12)$$

with $\rho(t, x)$ the traffic density and $V(t, x)$ the speed of the vehicles at time t and location x along the road. The function $V_o(\rho)$ is the preferential speed function: it is a decreasing function that represents the relation, in the average, between the speed of the vehicles and the traffic density (the higher the density, the lower the speed of the vehicles). The constant parameter τ is a positive time constant. The eigenvalues of the matrix $\mathbf{A}[\mathbf{U}]$ are:

$$\lambda_1[\mathbf{U}] = V \quad \text{and} \quad \lambda_2[\mathbf{U}] = V + \rho V'_o(\rho).$$

For the Aw-Rascle equation (2.12) an equilibrium is a constant state ρ^*, V^* that verifies the relation

$$V^* = V_o(\rho^*).$$

The linearisation of the Aw-Rascle equation around the equilibrium in diagonal form with the following characteristic state variables (ξ_1, ξ_2) , characteristic velocities λ_1, λ_2 and the matrix \mathbf{S} :

$$\xi_1 = V - V^* - V'_o(\rho^*)(\rho - \rho^*) \quad \xi_2 = V - V^*,$$

$$0 < \lambda_2 = V^* + \rho^* V'_o(\rho^*) < \lambda_1 = V^*,$$

$$\mathbf{S} = \begin{bmatrix} \gamma & \beta \\ \gamma & \beta \end{bmatrix},$$

with $\gamma = \frac{V'_o(\rho^*)}{\tau} > 0$ and $\beta = \frac{V^*}{\tau} > 0$.

2.1.4 Conservation laws and Riemann invariants

In special case where there are no source term (i.e $G[\mathbf{U}] = \mathbf{0}$, $\forall \mathbf{U} \in \Omega$), system (2.1) or (2.2) reduces to

$$\begin{aligned} \partial_t e[\mathbf{U}] + \partial_x f[\mathbf{U}] &= \mathbf{0}, \quad \text{or} \quad \mathbf{U}_t + F[\mathbf{U}] \mathbf{U}_x = \mathbf{0}, \\ t &\in [0, +\infty), \quad x \in [0, L]. \end{aligned} \quad (2.13)$$

A system of this form is a hyperbolic system of *conservation laws*, representing a process where the balance quantity is conserved since it can change only by the flux through the boundaries. In that case, it is clear that any constant value \mathbf{U}^* can be a steady state, independently of the value of the coefficient matrix $F[\mathbf{U}]$. Thus such systems have uniform steady states by definition. After transformation in Riemann coordinates (if possible), a system of conservation laws is written in the form

$$\begin{aligned} \partial_t \xi_i + \lambda_i[\boldsymbol{\xi}] \partial_x \xi_i &= 0, \quad i = 1, \dots, m, \\ \partial_t \xi_i - \lambda_i[\boldsymbol{\xi}] \partial_x \xi_i &= 0, \quad i = m + 1, \dots, n. \end{aligned} \quad (2.14)$$

The left-hand sides of these equations are the total time derivatives

$$\frac{d\xi_i}{dt} \triangleq \partial_t \xi_i + \frac{dx}{dt} \partial_x \xi_i,$$

of the Riemann coordinates along the characteristic curves which are the integral curves of the ordinary differential equations

$$\begin{aligned}\frac{dx}{dt} &= \lambda_i[\boldsymbol{\xi}(t, x)], \quad i = 1, \dots, m, \\ \frac{dx}{dt} &= -\lambda_i[\boldsymbol{\xi}(t, x)], \quad i = m + 1, \dots, n,\end{aligned}$$

in the plane (t, x) .

Since $\frac{d\xi_i}{dt} = 0$, it follows that the Riemann coordinates $\xi_i(t, x)$ are constant along the characteristic curves and are therefore called Riemann invariants for systems of conservation laws.

Example 2.3. *The scalar linear advection equation:*

$$\partial_t u + a \partial_x u = 0, \quad -\infty < x < \infty, \quad t > 0, \quad u(x, 0) = u_0(x), \quad (2.15)$$

where $u = u(t, x)$.

Characteristics may be defined as curves $x = x(t)$ in the $t - x$ plane along which the PDE become an ODE. Consider $x = x(t)$ and regard u as a function of t , that is $u = u(x(t), t)$. The rate of change of u along $x = x(t)$ is

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{dx}{dt} \frac{\partial u}{\partial x}. \quad (2.16)$$

If the characteristic curve $x = x(t)$ satisfies the ODE

$$\frac{dx}{dt} = a, \quad (2.17)$$

then the PDE in (2.15), together with (2.16) and (2.17), gives

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0. \quad (2.18)$$

Therefore the rate of change of u along the characteristic curve $x = x(t)$ satisfying (2.17) is zero, that is, u is constant along the curve $x = x(t)$. The speed a in (2.17) is called the characteristic speed and according to (2.17) it is the slope of the curve $x = x(t)$ in the $t - x$ plane. In practice it is more common to use the $x - t$ plane to sketch the characteristics, in which case the slope of the curves in question is $\frac{1}{a}$. The family of characteristic curves $x = x(t)$ given by the ODE (2.17) are shown in Figure 1 for $a > 0$ and they are a one-parameter family of curves.

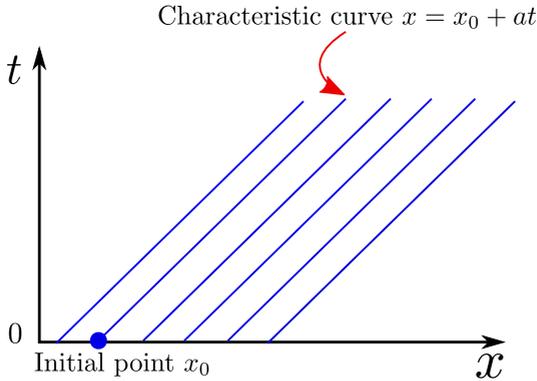


Figure 1 – Characteristics for (2.15) for positive characteristic speed a . Initial condition at time $t = 0$ fixes the initial position x_0

A particular member of this family is determined when initial condition (IC) at time $t = 0$ for the ODE (2.17) is added. Suppose we set

$$x(0) = x_0, \quad (2.19)$$

then the single characteristic curve that passes through the point $(x_0, 0)$, according to (2.17) is

$$x = x_0 + at. \quad (2.20)$$

This is also illustrated in Figure 1. Now we may regard the initial position x_0 as a parameter and in this way we reproduce the full one-parameter family of characteristics. The fact that the curves are parallel is typical of linear hyperbolic PDEs with constant coefficients.

Recall the conclusion from (2.18) that u remains constant along characteristics. Thus, if u is given the initial value $u(x, 0) = u_0(x)$ at time $t = 0$, then along the whole characteristic curve $x(t) = x_0 + at$ that passes through the initial point x_0 on the x -axis, the solution is

$$u(x, t) = u_0(x_0) = u_0(x - at). \quad (2.21)$$

The second equality follows from (2.20). The interpretation of the solution (2.21) of the PDE in (2.15) is this: given an initial profile $u_0(x)$, the PDE

will simply translate this profile with velocity a to the right if $a > 0$ and to the left if $a < 0$. The shape of the initial profile remains unchanged (TORO, 2013).

2.1.5 Stability, boundary stabilization and the associated Cauchy problem

In order to have a unique well defined solution to a quasi-linear hyperbolic system (2.2) over the interval $[0, L]$, initial and boundary conditions must be specified. This thesis address the specific issue of identifying and characterizing dissipative boundary conditions which guarantee bounded solutions converging to an equilibrium.

The feedback control problem is specially interesting when the manipulated control input, the controlled outputs and the measured outputs are physically located at the boundaries. Formally, this means that we consider the system (2.2) under n boundary conditions having the general form

$$\mathcal{B}[\mathbf{U}(t, 0), \mathbf{U}(t, L), \mathbf{Q}(t)] = \mathbf{0}, \quad (2.22)$$

with the map $\mathcal{B} \in \mathcal{C}^1(\Omega \times \Omega \times \mathbb{R}^q, \mathbb{R}^n)$. The dependence of the map \mathcal{B} on $[\mathbf{U}(t, 0), \mathbf{U}(t, L)]$ refers to natural physical constraints on the system. The function $\mathbf{Q}(t) \in \mathbb{R}^q$ represents a set of q exogenous control inputs that can be used for stabilization, output tracking or disturbance rejection.

In the case of static feedback control laws $\mathbf{Q}[\mathbf{U}(t, 0), \mathbf{U}(t, L)]$, one of our main concerns is to analyze the asymptotic convergence of the solutions of the following Cauchy problem

$$\begin{aligned} \text{System } & \mathbf{U}_t + F[\mathbf{U}]\mathbf{U}_x + G[\mathbf{U}] = \mathbf{0}, \quad t \in [0, +\infty), \quad x \in [0, L], \\ \text{B.C. } & \mathcal{B}[\mathbf{U}(t, 0), \mathbf{U}(t, L), \mathbf{Q}[\mathbf{U}(t, 0), \mathbf{U}(t, L)]] = \mathbf{0}, \quad t \in [0, +\infty), \\ \text{I.C. } & \mathbf{U}(0, x) = \mathbf{U}_0(x), \quad x \in [0, L]. \end{aligned}$$

Additional constraints on the initial conditions (I.C.) and the boundary conditions (B.C.) are needed to have a well-posed Cauchy problem. We

examine this issue in the case when the system can be transformed into a characteristic form.

2.1.6 The Cauchy problem in Riemann coordinates

For the physical systems described by hyperbolic equations written in characteristic form (2.3)

$$\xi_t + \Lambda[\xi]\xi_x + C[\xi] = \mathbf{0}, \quad t \in [0, +\infty), \quad x \in [0, L], \quad (2.23)$$

it is a basic property that *at each boundary point the incoming information ξ_{in} is determined by the outgoing information ξ_{out}* (RUSSELL, 1978, Section 3), with the definitions

$$\xi_{in}(t) \triangleq \begin{bmatrix} \xi^+(t, 0) \\ \xi^-(t, L) \end{bmatrix} \quad \text{and} \quad \xi_{out}(t) \triangleq \begin{bmatrix} \xi^+(t, L) \\ \xi^-(t, 0) \end{bmatrix}, \quad (2.24)$$

where ξ^+ and ξ^- are defined as follows:

$$\xi^+ = [\xi_1, \dots, \xi_m]^T, \quad \xi^- = [\xi_{m+1}, \dots, \xi_n]^T.$$

This means that the system (2.3) is subject to boundary conditions having the *nominal form*

$$\xi_{in} = \mathcal{H}[\xi_{out}], \quad (2.25)$$

where the map $\mathcal{H} \in \mathcal{C}^1(\mathbb{R}^n; \mathbb{R}^n)$. Moreover, the initial condition

$$\xi(0, x) = \bar{\xi}(x), \quad x \in [0, L], \quad (2.26)$$

must be specified.

Hence, in Riemann coordinates, the Cauchy problem is formulated as follows:

$$\text{System } \xi_t + \Lambda[\xi]\xi_x + C[\xi] = \mathbf{0}, \quad t \in [0, +\infty), \quad x \in [0, L],$$

$$\text{B.C. } \xi_{in}(t) = \mathcal{H}[\xi_{out}(t)], \quad t \in [0, +\infty),$$

$$\text{I.C. } \xi(0, x) = \bar{\xi}(x), \quad x \in [0, L],$$

For linear system (uniform case), in Riemann coordinates, the Cauchy problem is formulated as follows

$$\text{System } \boldsymbol{\xi}_t + \Lambda \boldsymbol{\xi}_x + \mathbf{C}[\boldsymbol{\xi}] = \mathbf{0}, \quad t \in [0, +\infty), \quad x \in [0, L],$$

$$\text{B.C. } \boldsymbol{\xi}_{in}(t) = K[\boldsymbol{\xi}_{out}(t)], \quad t \in [0, +\infty),$$

$$\text{I.C. } \boldsymbol{\xi}(0, x) = \bar{\boldsymbol{\xi}}(x), \quad x \in [0, L].$$

with $K \in \mathbb{R}^{n \times n}$. The well-posedness of this Cauchy problem may require that the initial condition be compatible with the boundary condition. The compatibility conditions which are necessary for the well-posedness of the Cauchy problem depend on the functional space to which the solutions belong. In this thesis, we will be mainly concerned with solutions $\boldsymbol{\xi}(t, \cdot)$ that may be of class L^2 .

2.2 Systems of Linear Conservation Laws

This section presents the stability of general systems of linear conservation laws under static linear boundary conditions assuming that the hyperbolic system of linear conservation laws is in Riemann coordinates. That is:

$$\boldsymbol{\xi}_t + \Lambda \boldsymbol{\xi}_x = \mathbf{0}, \quad t \in [0, +\infty), \quad x \in [0, L], \quad (2.27)$$

where $\boldsymbol{\xi} : [0, +\infty) \times [0, 1] \rightarrow \mathbb{R}^n$. Without loss of generality, it is assumed that Λ it is diagonal matrix having non-zero diagonal terms such that

$$\Lambda = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_m\},$$

with

$$\lambda_i > 0, \quad \forall_i \in \{1, \dots, m\},$$

and

$$\lambda_i < 0, \quad \forall_i \in \{m+1, \dots, n\}.$$

Hence, the following notation is introduced:

$$\boldsymbol{\xi}^+ = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_m \end{bmatrix}, \quad \boldsymbol{\xi}^- = \begin{bmatrix} \xi_{m+1} \\ \vdots \\ \xi_n \end{bmatrix}, \quad \text{such that } \boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\xi}^+ \\ \boldsymbol{\xi}^- \end{bmatrix},$$

where $\xi^+ : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}^m$ e $\xi^- : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}^{n-m}$ and

$$\begin{cases} \Lambda^+ = \text{diag}\{\lambda_1, \dots, \lambda_m\}, \\ \Lambda^- = \text{diag}\{\lambda_{m+1}, \dots, \lambda_n\}, \quad \lambda_i > 0, \forall i. \end{cases}$$

In light of the above notation, the linear hyperbolic system (2.27) can be cast as follows:

$$\partial_t \begin{bmatrix} \xi^+ \\ \xi^- \end{bmatrix} + \begin{bmatrix} \Lambda^+ & \mathbf{0} \\ \mathbf{0} & \Lambda^- \end{bmatrix} \partial_x \begin{bmatrix} \xi^+ \\ \xi^- \end{bmatrix} = \mathbf{0}. \quad (2.28)$$

One major concern in boundary control schemes is to access the exponential stability of the closed-loop system under boundary conditions of the form

$$\begin{bmatrix} \xi^+(t, 0) \\ \xi^-(t, L) \end{bmatrix} = \mathbf{K} \begin{bmatrix} \xi^+(t, L) \\ \xi^-(t, 0) \end{bmatrix}, \quad t \in [0, +\infty), \quad (2.29)$$

where \mathbf{K} is a matrix in $\mathbb{R}^{n \times n}$ partitioned as follows $\mathbf{K} = \begin{bmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{bmatrix}$, accordingly to (2.28), with the initial condition defined as follows:

$$\xi(0, x) = \bar{\xi}(x) \quad x \in (0, L). \quad (2.30)$$

2.2.1 Exponential Stability Problem

The exponential stability problem of the equilibrium $\xi \equiv \mathbf{0}$ for nonlinear systems of conservation laws $\partial_t \xi + A[\xi] \partial_x \xi = \mathbf{0}$ has been studied by (LI; YU, 1985) in the framework of \mathcal{C}^1 - solutions. For such systems, the issue of finding sufficient dissipative boundary conditions has been addressed in the literature for more than thirty years. To our knowledge, first results were published by (SLEMROD, 1983) and by (GREENBERG; TSIEN, 1984) for the special case of systems of size $n = 2$. A generalization to systems for size $n \geq 3$ was then progressively elaborated by the *Ta-Tsien Li school*, in particular by (QIN, 1985) and by (ZHAO, 1986). All these contributions deal with the particular case of *local* boundary conditions having the specific form

$$\xi^+(t, 0) = G_0[\xi^-(t, 0)], \quad \xi^-(t, 1) = G_1[\xi^+(t, 1)] \quad (2.31)$$

With these boundary conditions, the analysis can be based on the method of characteristics which can exploit an explicit computation of the *reflection* of the solutions at the boundaries along the characteristic curves. This has given rise to the sufficient condition

$$\rho \left[\begin{array}{cc} \mathbf{0} & G_0'(\mathbf{0}) \\ G_1'(\mathbf{0}) & \mathbf{0} \end{array} \right] < 1, \quad (2.32)$$

for the dissipativity of the boundary conditions (2.31) for the \mathcal{C}^1 -norm with $\rho(|M|)$ representing the spectral radius of the matrix M . This result is given for instance by (LI, 1994, Theorem 1.3, page 173) in his seminal book on the stability of the classical solutions of quasi-linear hyperbolic systems. Finally, by using an appropriate dummy doubling of the system size, (HALLEUX et al., 2003, 13, Theorem 4) has shown how the general dissipative boundary condition $\rho[|G'(\mathbf{0})|] < 1$ can be established for systems with the general *non local* boundary condition. This dummy doubling has also been used by (LI; RAO; WANG, 2010) in the framework of \mathcal{C}^1 -solutions.

An alternative approach to the stability analysis of dissipative boundary conditions is based on the use of Lyapunov functions. The first attempts were using entropies as Lyapunov functions as considered by (CORON; D'ANDRÉA-NOVEL; BASTIN, 1999) and (LEUGERING; SCHMIDT, 2002). However, the major drawback of this approach regards the semi-definiteness of the time-derivative of entropy-based Lyapunov functions which are not necessarily *definite negative*. Hence, one has to conclude the analysis applying the LaSalle's invariant set principle. However, this principle requires the precompactness of the trajectories, a property which is difficult to guarantee in the case of nonlinear partial differential equations.

In order to overcome this difficulty, CORON; BASTIN; D'ANDRÉA-NOVEL (2007) have proposed, for systems of size $n = 2$ a strict Lyapunov function whose time derivative is strictly negative definite when $\rho[|G'(\mathbf{0})|] < 1$. The advantage is that the proof is less elaborated than the one using the method of characteristics because it uses more direct computations. Furthermore, another advantage of

Lyapunov analysis is to directly induce robustness properties with respect to small uncertainties and disturbances. CORON; BASTIN; D'ANDRÉA-NOVEL (2008) have generalized the Lyapunov approach to general nonlinear quasi-linear hyperbolic systems. In particular, they emphasize a new weaker dissipative boundary condition which is formulated as follows:

$$\rho_2[G'(\mathbf{0})] \triangleq \text{Inf} \{ \|\Delta G'(\mathbf{0})\Delta^{-1}\|; \Delta \in \mathcal{D}_n^+[\mathbb{R}] \} < 1, \quad (2.33)$$

where $\|\cdot\|$ denotes the usual 2-norm of matrices in $\mathbb{R}^{n \times n}$ and $\mathcal{D}_n^+[\mathbb{R}]$ the set of diagonal matrices whose elements on the diagonal are strictly positive.

Other various contributions and extensions of the previous results deserve also to be mentioned. DOS SANTOS; MASCHKE (2009) addresses a Hamiltonian perspective to the stabilization of systems of two conservation laws. Using the Lyapunov approach, CASTILLO et al. (2013) proposed sufficient conditions for the existence of exponentially stable observers in the case where all eigenvalues Λ_i are positive. Continuing in the Lyapunov stability setting, dynamic boundary conditions stabilization it is proposed in (CASTILLO et al., 2012), (CASTILLO et al., 2013) and (CASTILLO et al., 2015). Using an approach based on delayed differential equations, CORON; NGUYEN have derived sufficient conditions for the exponential stability in the Sobolev norm $W^{2,p}$.

For the so-called inhomogeneous quasi-linear hyperbolic systems (i.e with additional zero-order terms):

$$\xi_t + \mathbf{A}[\xi]\xi_x + \mathbf{S}[\xi] = \mathbf{0}, \quad (2.34)$$

the analysis of dissipative boundary conditions is much more complicated and only partial results are known. Using the method of characteristic, PRIEUR; WINKIN; BASTIN (2008) showed that the stability conditions (2.32) holds for inhomogeneous systems when $\|\mathbf{S}(\xi)\|$ is small enough. In (PRIEUR, 2009) this result can be extended to deal with differential or integral boundary errors. Using the Lyapunov approach, it was analyzed the boundary feedback stabilization of gas flow in fan-shaped networks described by isentropic Euler equations in

(GUGAT; HERTY, 2011) and (GUGAT; DICK; LEUGERING, 2011). In this context, PAVEL; CHANG (2012) studied the boundary control of hyperbolic Lotka-Volterra systems with application to pumped Raman amplifiers on optical fibres. For inhomogeneous systems of size $n = 2$ with $m = 1$, CORON et al. (2012) have shown the existence of a full-state feedback law ensuring exponential stability for the H^2 -norm. The proof uses a back-stepping transformation to find new variables for which a strict Lyapunov function can be constructed; see, e.g., (KRSTIC; SMYSHLYAEV, 2008b).

2.2.2 Exponential Stability for the L^2 -norm

In this section, using Lyapunov approach, an explicit condition on the matrix \mathbf{K} is given under which the steady state solution $\boldsymbol{\xi}(t, x) \equiv 0$ of the system (2.28)-(2.30) is globally exponentially stable for the L^2 -norm. Then, the classical definition of solution to the Cauchy problem (2.28)-(2.30) in $L^2((0, L); \mathbb{R}^n)$ is given in the

Definition 2.1. *Let $\boldsymbol{\xi}_0 \in L^2((0, L); \mathbb{R}^n)$. A map $\boldsymbol{\xi} : [0, +\infty) \times (0, L) \rightarrow \mathbb{R}^n$ is a solution of the Cauchy problem (2.28)-(2.30) if $\boldsymbol{\xi} \in C^0([0, +\infty]; L^2((0, L); \mathbb{R}^n))$ is such that, for every $\boldsymbol{\varphi} = (\boldsymbol{\varphi}_-^T, \boldsymbol{\varphi}_+^T)^T \in C^1([0, +\infty) \times [0, L]; \mathbb{R}^n)$ with compact support and satisfying*

$$\begin{bmatrix} \boldsymbol{\varphi}_+(t, L) \\ \boldsymbol{\varphi}_-(t, 0) \end{bmatrix} = \begin{bmatrix} (\Lambda^+)^{-1} \mathbf{K}_{00}^T(\Lambda^+) & (\Lambda^+)^{-1} \mathbf{K}_{10}^T(\Lambda^-) \\ (\Lambda^-)^{-1} \mathbf{K}_{01}^T(\Lambda^+) & (\Lambda^-)^{-1} \mathbf{K}_{11}^T(\Lambda^-) \end{bmatrix} \begin{bmatrix} \boldsymbol{\varphi}_-(t, 0) \\ \boldsymbol{\varphi}_+(t, L) \end{bmatrix}$$

we have

$$\int_0^{+\infty} \int_0^L (\boldsymbol{\varphi}_t^T + \boldsymbol{\varphi}_x^T \Lambda) \boldsymbol{\xi} \, dx \, dt + \int_0^L \boldsymbol{\varphi}^T(0, x) \boldsymbol{\xi}_0(x) \, dx = 0.$$

From the above definition, the following classical result is obtained; see, e.g., (CORON, 2007, Section 2.1 and 2.3).

Proposition 2.1. *For every $\boldsymbol{\xi}_0 \in L^2((0, 1); \mathbb{R}^n)$, the Cauchy problem (2.28)-(2.30) has a unique solution. Moreover, for every $T > 0$, there exists $C(T) > 0$ such that, for every $\boldsymbol{\xi}_0 \in L^2((0, 1); \mathbb{R}^n)$, the solution to the Cauchy problem (2.28)-(2.30) satisfies*

$$\|\boldsymbol{\xi}(t, \cdot)\|_{L^2((0, L); \mathbb{R}^n)} \leq C(T) \|\boldsymbol{\xi}_0\|_{L^2((0, L); \mathbb{R}^n)}, \quad \forall t \in [0, T]. \quad (2.35)$$

It is adopt in this thesis, the following definition for the exponential stability of the linear hyperbolic system as in (2.28)-(2.29)

Definition 2.2. *The linear hyperbolic system (2.28)-(2.29) is exponentially stable if there exist $\nu > 0$ and $C > 0$ such that, for every $\xi_0(x) \in L^2((0, L); \mathbb{R}^n)$, the solution to the Cauchy problem (2.28)-(2.30) satisfies*

$$\|\xi(t, \cdot)\|_{L^2((0, L); \mathbb{R}^n)} \leq C e^{-\nu t} \|\xi_0\|_{L^2((0, L); \mathbb{R}^n)}, \quad \forall t \in [0, +\infty]. \quad (2.36)$$

In order to state the stability condition, it is firstly introduced the functions $\rho_p : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ defined by

$$\rho_p[M] \triangleq \text{Inf} \{ \|\Delta M \Delta^{-1}\|_p, \Delta \in \mathcal{D}_n^+ \}, \quad 1 \leq p \leq \infty, \quad (2.37)$$

where \mathcal{D}_n^+ denotes the set of diagonal $n \times n$ real matrices with strictly positive diagonal entries with

$$\begin{aligned} \|\zeta\|_p &\triangleq \left[\sum_{i=1}^n |\zeta_i|^p \right]^{\frac{1}{p}}, \\ \|\zeta\|_\infty &\triangleq \max \{ |\zeta_1|, \dots, |\zeta_n| \}, \\ \|M\|_p &\triangleq \max_{\|\zeta\|_p=1} \|M\zeta\|_p. \end{aligned} \quad (2.38)$$

for a vector $\zeta \triangleq [\zeta_1 \quad \dots \quad \zeta_n]^T$ and a matrix $M \in \mathbb{R}^{n \times n}$.

Hence, the following stability theorem is given.

Theorem 2.1. (*BASTIN; CORON, 2016, Theorem 3.2*) *The system (2.28)-(2.29) is exponentially stable for the L^2 -norm if $\rho_2[\mathbf{K}] < 1$.*

Proof. We introduce the following candidate Lyapunov function

$$\begin{aligned} \mathbf{v} &= \int_0^L \left[\sum_{i=1}^m \frac{p_i}{\lambda_i} \xi_i^2(t, x) \exp\left(-\frac{\mu x}{\lambda_i}\right) + \sum_{i=m+1}^n \frac{p_i}{\lambda_i} \xi_i^2(t, x) \exp\left(\frac{\mu x}{\lambda_i}\right) \right] dx, \\ &= \int_0^L \left[(\xi^{+T} (\Lambda^+)^{-1} P^+(\mu x) \xi^+) + (\xi^{-T} (\Lambda^-)^{-1} P^-(\mu x) \xi^-) \right] dx, \end{aligned}$$

with

$$P^+(\mu x) \triangleq \text{diag} \left\{ p_1 \exp\left(-\frac{\mu x}{\lambda_1}\right), \dots, p_m \exp\left(-\frac{\mu x}{\lambda_m}\right) \right\}, \quad p_i > 0, \quad (2.39)$$

$$P^-(\mu x) \triangleq \text{diag} \left\{ p_{m+1} \exp\left(+\frac{\mu x}{\lambda_{m+1}}\right), \dots, p_n \exp\left(-\frac{\mu x}{\lambda_n}\right) \right\}, \quad p_i > 0. \quad (2.40)$$

The time derivative of \mathbf{V} along the \mathcal{C}^1 -solutions of (2.28)-(2.29) is

$$\frac{d\mathbf{V}}{dt} = -\mu\mathbf{V} + \mathbf{W},$$

with

$$\mathbf{W} \triangleq - \left[\boldsymbol{\xi}^{+T} P^+(\mu x) \boldsymbol{\xi}^+ \right]_0^L + \left[\boldsymbol{\xi}^{-T} P^-(\mu x) \boldsymbol{\xi}^- \right]_0^L. \quad (2.41)$$

First we will show that the parameters p_i and μ can be selected such that, under the condition $\rho_2[\mathbf{K}] < 1$, \mathbf{W} is negative definite quadratic form in $\boldsymbol{\xi}^-(t, 0)$ and $\boldsymbol{\xi}^+(t, L)$. For this analysis, we introduce the following notations:

$$\boldsymbol{\xi}_0^-(t) \triangleq \boldsymbol{\xi}^-(t, 0), \quad \boldsymbol{\xi}_L^+(t) \triangleq \boldsymbol{\xi}^+(t, L).$$

Using the boundary condition (2.29), we have

$$\begin{aligned} \mathbf{W} &= - \left[\boldsymbol{\xi}^{+T} P^+(\mu x) \boldsymbol{\xi}^+ \right]_0^L + \left[\boldsymbol{\xi}^{-T} P^-(\mu x) \boldsymbol{\xi}^- \right]_0^L \\ &= - \left[\boldsymbol{\xi}_L^{+T} P^+(\mu L) \boldsymbol{\xi}_L^+ + \boldsymbol{\xi}_0^{-T} P^-(0) \boldsymbol{\xi}_0^- \right] \\ &\quad + \left[\boldsymbol{\xi}_L^{+T} K_{00}^T + \boldsymbol{\xi}_0^{-T} K_{01}^T \right] P^+(0) \left[K_{00} \boldsymbol{\xi}_L^+ + K_{01} \boldsymbol{\xi}_0^- \right] \\ &\quad + \left[\boldsymbol{\xi}_L^{+T} K_{10}^T + \boldsymbol{\xi}_0^{-T} K_{11}^T \right] P^1(\mu L) \left[K_{10} \boldsymbol{\xi}_L^+ + K_{11} \boldsymbol{\xi}_0^- \right]. \end{aligned}$$

Since $\rho_2[\mathbf{K}] < 1$ by assumption, there exist $D_0 \in \mathcal{D}_m^+$, $D_1 \in \mathcal{D}_{n-m}^+$ and $\Delta \triangleq \text{diag} \{D_0, D_1\}$ such that

$$\|\Delta \mathbf{K} \Delta^{-1}\| < 1.$$

The parameters p_i are selected such that $P^+(0) = D_0^2$ and $P^-(0) = D_1^2$. With these definitions, regarding \mathbf{W} as a function of μ , we have

$$\mathbf{W}[\mu] = - \begin{bmatrix} \boldsymbol{\xi}_L^{+T} D_0 & \boldsymbol{\xi}_0^{-T} D_1 \end{bmatrix} \boldsymbol{\Omega}[\mu] \begin{bmatrix} D_0 \boldsymbol{\xi}_L^+ \\ D_1 \boldsymbol{\xi}_0^- \end{bmatrix},$$

with

$$\mathbf{\Omega}[\mu] \triangleq \begin{bmatrix} P^+(\mu L)D_0^{-2} & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} D_0K_{00}D_0^{-1} & D_0K_{01}D_1^{-1} \\ D_1K_{10}D_0^{-1} & D_1K_{11}D_1^{-1} \end{bmatrix}^T \\ \begin{bmatrix} D_0K_{00}D_0^{-1} & D_0K_{01}D_1^{-1} \\ P^{-1}(\mu L)D_1^{-1}K_{10}D_0^{-1} & P^{-1}(\mu L)D_1^{-1}K_{11}D_1^{-1} \end{bmatrix},$$

and, for $\mu = 0$,

$$\mathbf{W}(0) = - \begin{bmatrix} \xi_L^{+T} D_0 & \xi_0^{-T} D_1 \end{bmatrix} \text{Pr} I - (\Delta \mathbf{K} \Delta^{-1})^T (\Delta \mathbf{K} \Delta^{-1}) \begin{bmatrix} D_0 \xi_L^+ \\ D_1 \xi_0^- \end{bmatrix}.$$

Since $\|\Delta \mathbf{K} \Delta^{-1}\| < 1$, it follows that $\mathbf{W}(0)$ is strictly negative definite quadratic form in ξ_L^+ and ξ_0^- . Then, by continuity, $\mathbf{W}[\mu]$ remains a strictly negative definite quadratic form for $\mu > 0$ sufficiently small.

Hence, we have

$$\frac{d\mathbf{V}}{dt} = -\mu \mathbf{V} + \mathbf{W} \leq -\mu \mathbf{V},$$

along the system trajectories.

Therefore \mathbf{V} is a strict Lyapunov function and the solutions of the system (2.28)-(2.30) exponentially converge to zero for the L^2 -norm. \square

2.2.3 Dissipative Boundary Conditions

It is notable that the stability condition $\rho_2[\mathbf{K}] < 1$ depends on the value of \mathbf{K} but not on the values of the characteristic velocities λ_i . In other words, the stability condition is independent of the system dynamics (2.28) and depends only on the boundary conditions (2.29). When the matrix \mathbf{K} satisfies such a stability condition, the boundary conditions are said to be **dissipative** and the stability is guaranteed whatever the length L and the time required for solutions to cross the system. Intuitively, this is understood as follows: the solutions, which are moving back and forth between the two boundaries, remain constant along the characteristic lines and are exponentially damped at the boundaries only. This can be also understood using a *small gain principle*. We have observed, that the hyperbolic system (2.28) under the

boundary condition (2.29) can be regarded as a closed loop interconnection of two *causal* input-output systems as represented in Figure 2. It is therefore natural that the stability requires a *small gain* of the feedback loop. The condition relies only on the gain $\rho_2[\mathbf{K}]$ of the system \mathcal{S}_2 since the system \mathcal{S}_1 has a unit gain by definition.

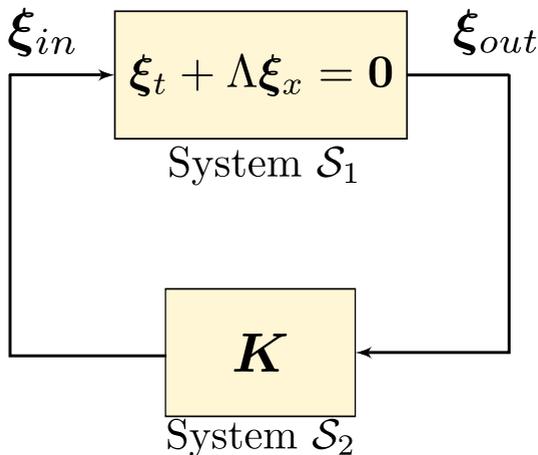


Figure 2 – The linear hyperbolic system (2.28)-(2.29) viewed as a closed loop interconnection of two causal input-output systems

2.3 Exponential stability for the C^0 -norm: analysis in the frequency domain

In this section, we now take the frequency domain viewpoint to analyze the exponential stability of the system (2.28)-(2.29) for the C^0 -norm according to the following definition.

Definition 2.3. *The system (2.28)-(2.29) is exponentially stable for the C^0 -norm if there exist $\nu > 0$ and $C > 0$ such that, for every $\xi_0 \in C^0([0, L], \mathbb{R}^n)$ satisfying the compatibility condition*

$$\begin{bmatrix} \xi^+(t, 0) \\ \xi^-(t, L) \end{bmatrix} = \mathbf{K} \begin{bmatrix} \xi^+(t, L) \\ \xi^-(t, 0) \end{bmatrix}, \quad (2.42)$$

the solution of the Cauchy problem (2.28)-(2.30) satisfies

$$\|\boldsymbol{\xi}(t, \cdot)\|_{C^0((0,L);\mathbb{R}^n)} \leq C e^{vt} \|\boldsymbol{\xi}_0\|_{C^0((0,L);\mathbb{R}^n)}, \quad \forall t \in [0, +\infty). \quad (2.43)$$

The system (2.28) can be regarded as a set of scalar delay systems

$$\begin{aligned} \xi_i(t, L) &= \xi_i(t - \tau_i, 0), \quad i = 1, \dots, m, \\ \xi_j(t, 0) &= \xi_j(t - \tau_j, L), \quad j = m + 1, \dots, n, \quad \tau_k \triangleq \frac{L}{\lambda_k}, \quad k = 1, \dots, n, \end{aligned}$$

which are interconnected by the boundary conditions (2.29). Taking the Laplace transform, it follows that the characteristic function of the system (2.28)-(2.29) is:

$$\det [\mathbf{I}_n - \text{diag} \{e^{-s\tau_1}, \dots, e^{-s\tau_n}\} \mathbf{K}], \quad (2.44)$$

where \mathbf{I}_n is the identity matrix of $\mathbb{R}^{n \times n}$. The roots of this function are called the poles of the system.

Theorem 2.2. (*BASTIN; CORON, 2016, Theorem 3.5*) *The system (2.28)-(2.29) is exponentially stable for the C^0 -norm if and only if the poles of the system are stable.*

Proof. See (HALE; LUNEL, 1993)[Chapter 9, Theorem 3.5]. □

Hence the stability analysis does not require to know the actual location of the poles. It is sufficient to know that they have negative real parts which are bounded away from zero. From the viewpoint of boundary control design, it is obviously of major interest to predict the stability, and therefore the sign of the real parts of the poles, directly from the coefficients of the matrix \mathbf{K} .

2.3.1 Linear Dynamic Boundary Conditions

In this section, we discuss the stability of linear hyperbolic systems with linear dynamic boundary conditions. More precisely, we consider the linear hyperbolic system of conservation laws in Riemann coordinates (2.28) under linear differential boundary conditions of the

following form:

$$\begin{aligned}\dot{\mathbf{X}} &= \mathbf{A}\mathbf{X} + \mathbf{B}\boldsymbol{\xi}_{out}(t), \\ \boldsymbol{\xi}_{in}(t) &= \mathbf{C}\mathbf{X} + \mathbf{K}\boldsymbol{\xi}_{out}(t),\end{aligned}\quad (2.45)$$

where $\mathbf{A} \in \mathbb{R}^{n_\ell \times n_\ell}$, $\mathbf{B} \in \mathbb{R}^{n_\ell \times n}$, $\mathbf{C} \in \mathbb{R}^{n \times n_\ell}$, $\mathbf{X} \in \mathbb{R}^{n_\ell}$, $n_\ell \leq n$. The notations $\boldsymbol{\xi}_{in}$ and $\boldsymbol{\xi}_{out}$ were introduced in (2.24) and stand for

$$\boldsymbol{\xi}_{in}(t) \triangleq \begin{bmatrix} \boldsymbol{\xi}^+(t, 0) \\ \boldsymbol{\xi}^-(t, L) \end{bmatrix} \quad \text{and} \quad \boldsymbol{\xi}_{out}(t) \triangleq \begin{bmatrix} \boldsymbol{\xi}^+(t, L) \\ \boldsymbol{\xi}^-(t, 0) \end{bmatrix}. \quad (2.46)$$

The well-posedness of the Cauchy problem associated to this system is addressed in (BASTIN; CORON, 2016, Appendix A, Theorem A.6).

2.3.1.1 Frequency Domain

Using Laplace transform, the system (2.28)-(2.45) is written in the frequency domain as

$$\boldsymbol{\xi}_{out}(s) = D(s)\boldsymbol{\xi}_{in}(s), \quad \text{with} \quad D(s) \triangleq \text{diag} \{ e^{-s\tau_1}, \dots, e^{-s\tau_n} \}, \quad \tau_i = \frac{L}{\lambda_i},$$

$$(sI - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\boldsymbol{\xi}_{out}(s), \quad \boldsymbol{\xi}_{in}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{K}\boldsymbol{\xi}_{out}(s).$$

Hence the poles of the system are the roots of the characteristic equation

$$\det [I - D(s)(C(sI - A)^{-1}B) + \mathbf{K}] = 0. \quad (2.47)$$

Theorem 2.3. (BASTIN; CORON, 2016, Theorem 3.14) *The steady state $\boldsymbol{\xi}(t, x) \equiv 0$ of the system (2.28)-(2.29) is exponentially stable for the L^∞ -norm if and only if the poles of system are stable (i.e have strictly negative real parts and are bounded away from zero).*

Proof. See (HALE; VERDUYN LUNEL, 2002) and (MICHIELS; NICULESCU, 2007). \square

2.3.1.2 Lyapunov Approach

In the line of the previous developments, consider the following Lyapunov function candidate:

$$\begin{aligned} \mathbf{V} = & \int_0^L \left[\sum_{i=1}^m \frac{p_i}{\lambda_i} \xi_i^2(t, x) \exp\left(-\frac{\mu x}{\lambda_i}\right) + \sum_{i=m+1}^n \frac{p_i}{\lambda_i} \xi_i^2(t, x) \exp\left(\frac{\mu x}{\lambda_i}\right) \right] dx, \\ & + \sum_{j=1}^{\ell} q_j X_j^2, \end{aligned} \quad (2.48)$$

with $\mathbf{X} \triangleq (X_1, \dots, X_{n_\ell})^T$, $p_i > 0$ ($i = 1, \dots, n$), $q_j > 0$ ($j = 1, \dots, n_\ell$).

The time-derivative of this function along the C^1 -solutions of (2.28)-(2.45) is

$$\dot{\mathbf{V}} = -\mu \mathbf{V} + \begin{bmatrix} \boldsymbol{\xi}_{out} \\ \mathbf{X} \end{bmatrix}^T \mathcal{M}(\mu) \begin{bmatrix} \boldsymbol{\xi}_{out} \\ \mathbf{X} \end{bmatrix},$$

with

$$\mathcal{M}(\mu) \triangleq \begin{bmatrix} \mathbf{K}^T P_1(\mu) \mathbf{K} - P_2(\mu) & \mathbf{K}^T P_1(\mu) \mathbf{C} + \mathbf{B}^T \mathbf{Q} \\ \mathbf{C}^T P_1(\mu) \mathbf{K} + \mathbf{Q} \mathbf{B} & \mu \mathbf{Q} + \mathbf{C}^T P_1(\mu) \mathbf{C} + (\mathbf{A}^T \mathbf{Q} + \mathbf{Q} \mathbf{A}) \end{bmatrix},$$

$$P_1(\mu) \triangleq \text{diag} \left\{ p_1, \dots, p_m, p_{m+1} \exp\left(+\frac{\mu L}{\lambda_{m+1}}\right), \dots, p_n \exp\left(+\frac{\mu L}{\lambda_n}\right) \right\},$$

$$P_2(\mu) \triangleq \text{diag} \left\{ p_1 \exp\left(-\frac{\mu L}{\lambda_1}\right), \dots, p_m \exp\left(-\frac{\mu L}{\lambda_m}\right), p_{m+1}, \dots, p_n \right\},$$

$$\mathbf{Q} \triangleq \text{diag} \{ q_1, \dots, q_{n_\ell} \}. \quad (2.49)$$

The exponential stability will hold if there exist $p_i > 0$ and $q_j > 0$ such that the matrix $\mathcal{M}(0)$ is negative-definite; see, e.g., (CASTILLO et al., 2015; CASTILLO et al., 2013; CASTILLO et al., 2012).

Example 2.4. A lossless electrical line connecting an inductive power supply to a capacitive load as shown in Figure 3.

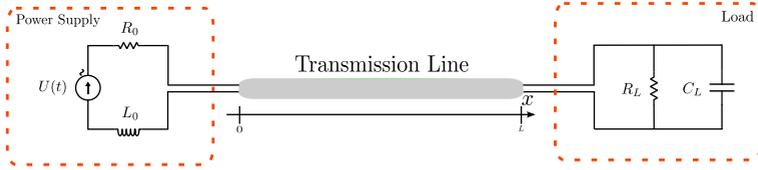


Figure 3 – Transmission line connecting an inductive power supply to a capacitive load

The dynamics of the line are described by the following system of two conservation laws:

$$\begin{aligned} \partial_t I + \frac{1}{L_\ell} \partial_x V &= 0, \\ \partial_t V + \frac{1}{C_\ell} \partial_x I &= 0, \end{aligned} \quad (2.50)$$

with the dynamic boundary conditions:

$$\begin{aligned} L_0 \frac{dI(t, 0)}{dt} + R_0 I(t, 0) + V(t, 0) &= U(t), \\ C_L \frac{dV(t, L)}{dt} + \frac{V(t, L)}{R_L} &= I(t, L). \end{aligned} \quad (2.51)$$

For a given constant input voltage $U(t) = U^*$, the system has a unique constant steady state

$$I^* = \frac{U^*}{R_0 + R_\ell}, \quad V^* = \frac{R_\ell U^*}{R_0 + R_\ell}.$$

The Riemann coordinates are defined as

$$\begin{aligned} \xi_1 &\triangleq (V - V^*) + (I - I^*) \sqrt{\frac{L_\ell}{C_\ell}}, \\ \xi_2 &\triangleq (V - V^*) - (I - I^*) \sqrt{\frac{L_\ell}{C_\ell}} \end{aligned}$$

with the inverse coordinates

$$\begin{aligned} I &= I^* + \frac{\xi_1 - \xi_2}{2} \sqrt{\frac{L_\ell}{C_\ell}}, \\ V &= V^* + \frac{\xi_1 + \xi_2}{2}. \end{aligned}$$

Then, expressing the dynamics (2.50) and the boundary conditions (2.51) in Riemann coordinates, we have

$$\partial_t \xi_1 + \lambda_1 \partial_x \xi_1 = 0, \quad \partial_t \xi_2 - \lambda_2 \partial_x \xi_2 = 0, \quad \lambda_1 = \lambda_2 \triangleq \frac{1}{\sqrt{L_\ell C_\ell}},$$

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{bmatrix}}_A \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & -\beta_1 \\ \beta_2 & 0 \end{bmatrix}}_B \begin{bmatrix} \xi_1(t, L) \\ \xi_2(t, 0) \end{bmatrix},$$

$$\begin{bmatrix} \xi_1(t, 0) \\ \xi_2(t, L) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_C \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_K \begin{bmatrix} \xi_1(t, L) \\ \xi_2(t, 0) \end{bmatrix},$$

with

$$\alpha_1 = \frac{1}{L_0} \sqrt{\frac{C_\ell}{L_\ell}} + \frac{R_0}{L_0}, \quad \alpha_2 = \frac{1}{C_L} \sqrt{\frac{L_\ell}{C_\ell}} + \frac{1}{R_L C_L},$$

$$\beta_1 = \frac{2}{L_0} \sqrt{\frac{C_\ell}{L_\ell}}, \quad \beta_2 = \frac{2}{C_L} \sqrt{\frac{L_\ell}{C_\ell}}. \quad (2.52)$$

The characteristic equation is

$$\underbrace{(s + \alpha_1)(s + \alpha_2)}_{d(s)} + \underbrace{(s + \alpha_1 - \beta_1)(s + \alpha_2 - \beta_2)}_{n(s)} e^{-s\tau} = 0, \quad \tau \triangleq 2L\sqrt{L_\ell C_\ell}.$$

In order to analyze the dependence of the stability of the poles on the length L of the line, it is applied the Walton and Marshall procedure as described in (SILVA; DATTA; BHATTACHARYYA, 2007, Section 5.6). Thus, the first step is to examine the stability when $L = 0$ (i.e. $\tau = 0$) where the characteristic equation reduces to the following second order polynomial with positive coefficients:

$$\tau = 0 \Rightarrow s^2 + \left(\frac{R_0}{L_0} + \frac{1}{R_L C_L} \right) s + \frac{1}{L_0 C_L} \left(1 + \frac{R_0}{R_L} \right) = 0.$$

Obviously, in the above case, the two poles are stable.

The second step consists in computing the following polynomial in ω^2 :

$$\begin{aligned}
 W(\omega^2) &\triangleq d(j\omega)d(-j\omega) - n(j\omega)n(-j\omega) \\
 &= (-\omega^2 + (\alpha_1 + \alpha_2)j\omega + \alpha_1\alpha_2)(-\omega^2 - (\alpha_1 + \alpha_2)j\omega + \alpha_1\alpha_2) \\
 &\quad - (-\omega^2(\alpha_1 + \alpha_2 - \beta_1 - \beta_2)j\omega + (\alpha_1 - \beta_1)(\alpha_2 - \beta_2)) \\
 &\quad (-\omega^2(\alpha_1 + \alpha_2 - \beta_1 - \beta_2)j\omega + (\alpha_1 - \beta_1)(\alpha_2 - \beta_2)) \\
 &= (\alpha_1\alpha_2 - \omega^2)^2 + (\alpha_1\alpha_2)^2\omega^2 - ((\alpha_1 - \beta_1)(\alpha_2 - \beta_2) - \omega^2)^2 \\
 &\quad - (\alpha_1 + \alpha_2 - \beta_1 - \beta_2)^2\omega^2,
 \end{aligned}$$

leading after few computations to

$$W(\omega^2) = (\gamma_1 + \gamma_2)\omega^2 - \gamma_1\gamma_2 + \gamma_1\alpha_2^2 + \gamma_2\alpha_1^2, \quad (2.53)$$

with

$$\begin{aligned}
 \gamma_1 &\triangleq \alpha_1^2 - (\alpha_1 - \beta_1)^2 = \frac{4R_0}{L_0^2} \sqrt{\frac{C_\ell}{L_\ell}}, \\
 \gamma_2 &\triangleq \alpha_2^2 - (\alpha_2 - \beta_2)^2 = \frac{4}{R_L C_L^2} \sqrt{\frac{L_\ell}{C_\ell}}.
 \end{aligned}$$

It follows that the sign of $W(\omega^2)$ for large ω is positive. This means that all the system poles have strictly negative real parts for sufficiently small non-zero values of L .

In the third step, it is noticed that the polynomial (2.53) has a single root:

$$\omega^2 = \frac{\gamma_1\gamma_2 - \gamma_1\alpha_2^2 - \gamma_2\alpha_1^2}{(\gamma_1 + \gamma_2)},$$

which is negative for all positive values of the physical parameters $R_0, R_L, L_0, L_\ell, C_L$ and C_ℓ . Accordingly to physical intuition, it can be concluded that the poles of the system are stable for any line length L .

3 MODELING AND CONTROL OF FLOW WITH DYNAMIC BOUNDARY ACTIONS

In this chapter it is considered fluid transport which is a phenomenon often encountered in many industrial applications such as hydraulic networks (BASTIN et al., 2007; DOS SANTOS; PRIEUR, 2008), gas flow in pipelines (BANDA; HERTY; KLAR, 2006; CASTILLO et al., 2013), flow regulation in deep pits (WITRANT et al., 2010), among others. The fluid transport is modeled by conservation laws, which are first-order hyperbolic Partial Differential Equations (PDEs), commonly used to express the fundamental balance law that occur in many physical systems when small friction or dissipation effects are neglected (BASTIN et al., 2007). It is well-known, that measurements and actuators in distributed parameter systems are not usually available. It is more usually for them to be located at the boundaries.

The stability problem of boundary control in time invariant hyperbolic systems has been attracted attention of academic community for a long time, see, e.g., the references (CORON; BASTIN; D'ANDRÉA-NOVEL, 2007), (CORON; BASTIN; D'ANDRÉA-NOVEL, 2008) and (PRIEUR; WINKIN; BASTIN, 2008). Moreover, a strict Lyapunov function approach is proposed in (DOS SANTOS et al., 2008) for the boundary control with integral actions of hyperbolic systems of conservation law that can be diagonalized by means of Riemann invariants. The stability problem of linear and quasi-linear hyperbolic systems in the presence of dynamic behavior at the boundary conditions is addressed in (CASTILLO et al., 2012; CASTILLO et al., 2015). Reference (CASTILLO et al., 2013) addresses the problem of boundary observer design for one-dimensional first order linear and quasi-linear strict hyperbolic systems with n rightward convecting transport.

The boundary control can have a faster reaction when compared to waves traveling time, because no time response limitation is taken into account at the boundary conditions. In certain applications such as (HALLEUX, 2004; BASTIN; CORON; D'ANDRÉA-NOVEL, 2009),

the wave traveling can be considered much slower than the actuator time response. A static relationship between the control input and the boundary condition can be considered in many cases. However, there are applications where the dynamics associated with the boundary control cannot be neglected (CASTILLO; WITRANT; DUGARD, 2013).

The major concern in this chapter regards the modeling of a flow inside a pipe (fluid transport phenomenon) with boundary control strategy applied in a physical experimental setup. This problem is modeled by a nonlinear hyperbolic system with coupled equations and a boundary condition given by the coupling with a finite-dimensional dynamic model of a heating column and a ventilator static model. The tracking problem of this complex dynamics is addressed in a simple manner by means of classical control tools such as linear approximations, finite difference schemes and integral action, leading to an augmented discrete-time linear system. Hence, a classical pole placement design methodology consisting of a state observer and a state feedback control law is applied to improve the closed-loop dynamics tracking performance. A numerical example considering the Poiseuille flow experimental setup is presented to illustrate the proposed strategy. Some of the results presented in this chapter can be found in (CALDEIRA et al., 2015).

The rest of this chapter can be summarized as follows. Section 3.1 introduces the description of fluid transport systems and the modeling of Poiseuille flow with dynamics at the boundary conditions. In Section 3.2, it is presented an augmented discrete-time linear system obtained by means of linear approximation and spatial discretization with the inclusion of integral action for the steady-state tracking of constant reference signals. Section 3.3 illustrates the proposed approach by means of a numeric example and Section 3.4 ends the chapter.

3.1 System Description and Modeling

The fluid transport system is normally used for industrial applications. Such as the ventilation system in a mining industry

(WITRANT; JOHANSSON; TEAM, 2008) and hydraulic networks (DOS SANTOS; PRIEUR, 2008). To investigate the phenomenon of fluid transport in a Poiseuille flow with dynamics at the boundary conditions, an experimental setup has been designed to test and validate control strategies. Figure 4 shows the schematic of the proposed device.

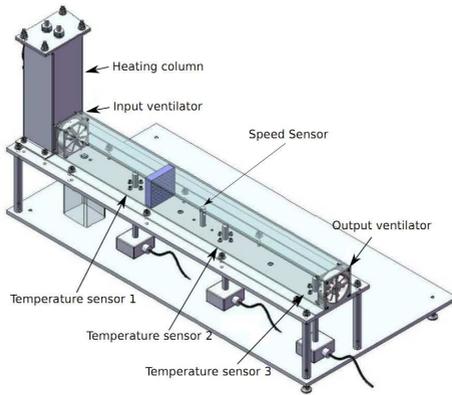


Figure 4 – Experimental setup (Poiseuille Flow)

This device is constituted by a heating column encasing a resistor, a tube, two ventilators, a gas speed meter and three distributed temperature sensors. The control objective is the outlet temperature tracking by driving the power dissipated on the heating resistor at different air flow speeds through the tube (input ventilator). Only the outlet temperature, the outlet flow speed, the outlet pressure and the heating column temperature will be considered as measurements for a closed-loop boundary control strategy. In this work, it is assumed that only the first ventilator (input ventilator) is the actuator and the output ventilator is set to be off.

The modeling of the experimental setup is done by considering three subsystems: the tube, the heating column and the ventilator. One-dimensional transport model is used to describe the gas density, speed and pressure variations in the tube. For the dynamic boundary

conditions, it is considered a zero-dimensional model of control volume approach with heat exchanges coming from the heating resistor in the column. In addition, the Bernoulli's equation is used to relate the pressure, temperature and speed with the ventilator rotation for the static boundary condition, and the perfect gases law is used to relate density with temperature.

3.1.1 Heating Column Model

The control volume is obtained from reference (CASTILLO; WITRANT; DUGARD, 2013). Figure 5 represents a control volume approach for the heating column.

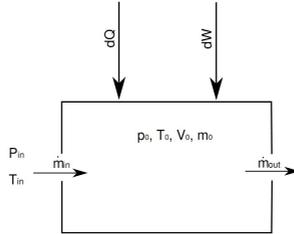


Figure 5 – Control volume approach

Consider the internal energy of a perfect gas:

$$U_0 = C_v m_0 T_0, \quad (3.1)$$

where U_0 is the gas internal energy, T_0 is the gas temperature, m_0 is the mass inside the column and C_v is the specific heat of the gas for constant volume. The time derivative of (3.1) is:

$$\dot{U}_0 = C_v m_0 \dot{T}_0 + C_v T_0 \dot{m}_0. \quad (3.2)$$

Using the first law of thermodynamics, the internal energy of the gas inside the column can be alternatively given by:

$$\dot{U}_0 = h_{in} \dot{m}_{in} + h_{out} \dot{m}_{out} + dQ + dW, \quad (3.3)$$

where h_{in} and h_{out} are the specific enthalpies getting in and out of the volume with mass flow rates \dot{m}_{in} and \dot{m}_{out} , respectively. dQ quantifies

the heat added by the resistor and dW is the work done by the gas. In the case of the heating column, there are two flows interacting with the volume, the input mass flow rate \dot{m}_{in} and the output mass flow rate \dot{m}_{out} . As the gas does not perform any work, then $dW = 0$. In order to write (3.3) in terms of temperature, the specific enthalpy of a gas is used defined by $h = C_p T$ with C_p being the specific heat at constant pressure. Therefore, (3.3) can be redefined as:

$$\dot{U}_0 = C_p T_{in} \dot{m}_{in} - C_p T_0 \dot{m}_{out} + dQ, \quad (3.4)$$

where T_{in} is the heating column input temperature. To simplify the model, consider the following two hypothesis:

H1.1 The pressure dynamics in the heating column is much faster than the temperature dynamics, which allows a quasi-static behavior of the mass and pressure to be considered;

H1.2 $p_0 \approx p_{in}$, where p_{in} is the input pressure and p_0 is the pressure inside the column.

Thus, H1.1 and H1.2 allow (3.2) and (3.4) to be respectively rewritten as:

$$\dot{U}_0 = C_v m_0 \dot{T}_0, \quad (3.5)$$

$$\dot{U}_0 = C_p \dot{m}_{in} (T_{in} - T_0) + dQ. \quad (3.6)$$

For simplicity, the temperature dynamics can be expressed in terms of the gas density by introducing the following change of variable (perfect gases law):

$$\rho_0 = \frac{p_{in}}{RT_0}, \quad (3.7)$$

where R is the specific ideal gas constant. Taking the time derivative of (3.7) into account, the following holds:

$$\dot{T}_0 = -\frac{R}{p_{in}} T_0^2 \dot{\rho}_0. \quad (3.8)$$

In addition, equating (3.5) and (3.6) and using (3.7)-(3.8) to replace the mass inside the control volume m_0 in terms of the pressure and the

specific gas constant R , it is obtained:

$$\dot{\rho}_0 = -\frac{R\gamma T_{in}\dot{m}_{in}}{p_{in}V_0}\rho_0 - \frac{R}{p_{in}V_0C_v}\rho_0 dQ + \frac{\gamma\dot{m}_{in}}{V_0}, \quad (3.9)$$

where $\rho_0 = \frac{m_0}{V_0}$ is the density inside the heating column, V_0 is the column volume and $\gamma = \frac{C_p}{C_v}$.

3.1.2 Flow model in the tube

Consider the ideal gas flow through a constant section, where all the friction losses and heat transfers are neglected. Hence, the Euler equations can be used to model the system. The equations consist of conservation of mass, momentum and energy and they can be written in terms of the primitive variables density ρ , particle speed v and pressure p . In this case, the flow is assumed to be one-dimensional leading to the following system of PDEs (which are three nonlinear coupled equations):

$$\mathbf{W}_t + \mathbf{A}\mathbf{W}_x = \mathbf{0}, \quad (3.10)$$

where $\mathbf{W} = \begin{bmatrix} \rho \\ v \\ p \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} v & \rho & 0 \\ 0 & v & \frac{1}{\rho} \\ 0 & a^2\rho & v \end{bmatrix}$, with $a = \sqrt{\frac{\gamma p}{\rho}}$ rep-

resenting the sound speed in ideal gas and γ the adiabatic constant. Further, $t \geq 0$ is the time variable and $x \in [0, L]$ is the space variable with $L = 1$ being the tube length.

3.1.3 Ventilator Model

The ventilator is a power driven machine that moves a continuous volume of air by converting rotational mechanical energy to an increase in the total pressure of the moving air. From (MCQUISTON; PARKER; SPITLER, 2005), there are three relationships among fan pressure, speed, power and flow rate, which are stated as ventilator's laws. In this work, it is only considered one ventilator installed at the input side of the tube as the actuator, and a flow rate proportional to

the ventilator speed. Thus, the ventilator rotating speed is the control action.

Bernoulli's equation is used to relate the density, speed and pressure with the ventilator rotation. Using the ventilator law, the boundary condition is modeled, for $x = 0$ and $t \geq 0$, by means of:

$$p_a + \frac{1}{2}\rho(0,t)v(0,t)^2 = K_n C(t)^2 \quad (3.11)$$

where the p_a is the atmospheric pressure, K_n is a constant coefficient, $C(t)$ is the ventilator rotating speed, and $\rho(0,t)$ and $v(0,t)$ represent gas density and velocity at the input side of the tube, respectively.

3.1.4 Outputs $T(L, \cdot)$, $v(L, \cdot)$, $p(L, \cdot)$ and $T(0, \cdot)$

The outputs considered are the particle temperature, speed and pressure at the outlet of the tube, respectively, $T(L, \cdot)$, $v(L, \cdot)$ and $p(L, \cdot)$. The dynamic boundary condition is the temperature in the heating column $T(0, \cdot)$. The outputs $v(L, \cdot)$ and $p(L, \cdot)$ are obtained directly, but the outputs $T(L, \cdot)$ and $T(0, \cdot)$ are obtained by means of the nonlinear relationship (3.7). The pressure inside the tube is considered constant (equal to the atmospheric pressure, p_a). The pressure at the inlet p_{in} and outlet p_{out} of the tube are considered equal, since the pressure differential introduced by the ventilator is supposed to be very small with respect to the pressure inside the tube.

3.2 Augmented Discrete-Time System

This section describes three main steps: (i) the linearization of the coupled nonlinear hyperbolic system (flow model in the tube), the nonlinear ordinary differential equation (ODE) that represents the heating column, the Bernoulli's equation for the ventilator model and the nonlinear relationship between density and temperature; (ii) discretization of the linear system resulting from the first step; and (iii) the representation of the system dynamics in terms of an augmented finite-dimensional discrete-time system, and the development of stability analysis and control design conditions.

3.2.1 Linearization

3.2.1.1 Flow model in the tube

The system (3.10) admits a steady-state (ρ^*, v^*, p^*) and the deviations of the states (ρ, v, p) with respect to their steady-state values are defined as $\bar{\rho} = \rho - \rho^*$, $\bar{v} = v - v^*$, $\bar{p} = p - p^*$.

Thus, the linearization of system (3.10) at this equilibrium is given by

$$\bar{\mathbf{W}}_t + \mathbf{A}^* \bar{\mathbf{W}}_x = \mathbf{0}, \quad (3.12)$$

$$\text{with } \bar{\mathbf{W}} = \begin{bmatrix} \bar{\rho} \\ \bar{v} \\ \bar{p} \end{bmatrix} \text{ and } \mathbf{A}^* = \begin{bmatrix} v^* & \rho^* & 0 \\ 0 & v^* & \frac{1}{\rho^*} \\ 0 & a^{*2}\rho^* & v^* \end{bmatrix}.$$

3.2.1.2 Heating Column Model

Whereas the pressure inside the tube is constant (and equal to the atmospheric pressure), the input mass flow rate will be expressed as

$$\dot{m}_{in} = v(0,t)\rho(0,t)A_t, \quad (3.13)$$

where A_t is the tube cross section area.

Taking (3.13) into account, the boundary condition (3.9) can be expressed as:

$$\begin{aligned} \dot{\rho}(0,t) = & -\frac{R\gamma T_{in}v(0,t)\rho(0,t)A_t}{p_{in}V_0}\rho(0,t) - \frac{R}{p_{in}V_0C_v}\rho(0,t)dQ \\ & + \frac{\gamma v(0,t)\rho(0,t)A_t}{V_0}. \end{aligned} \quad (3.14)$$

Defining

$$\mathcal{A} = -\frac{R\gamma T_{in}A_t}{p_{in}V_0}, \quad \mathcal{B} = -\frac{R}{p_{in}V_0C_v} \text{ and } \mathcal{C} = \frac{\gamma A_t}{V_0},$$

we obtain:

$$\begin{aligned} \dot{\rho}(0,t) = & \mathcal{A}v(0,t)\rho(0,t)^2 + \mathcal{B}dQ\rho(0,t) \\ & + \mathcal{C}v(0,t)\rho(0,t). \end{aligned} \quad (3.15)$$

Then, linearizing (3.15) and defining $dQ = U_2(t)$ yields:

$$\begin{aligned} \dot{\bar{\rho}}(0,t) = & \mathcal{A} \left[\bar{v}(0,t)\rho^{*2} + 2v^*\rho^*\bar{\rho}(0,t) \right] + \mathcal{B} [U_2^*\bar{\rho}(0,t) + \rho^*U_2(t)] \\ & + \mathcal{C} [v^*\bar{\rho}(0,t) + \rho^*\bar{v}(0,t)], \end{aligned} \quad (3.16)$$

where $\bar{\rho}(0,t) = \rho(0,t) - \rho^*$ is a small deviation around the equilibrium point; ρ^* and v^* are the same equilibrium points used for flow model in the tube; and U_2^* is a constant control action at the equilibrium point.

Hence, reorganizing (3.16) leads to

$$\begin{aligned} \dot{\bar{\rho}}(0,t) = & [2v^*\rho^*\mathcal{A} + \mathcal{B}U_2^* + \mathcal{C}v^*]\bar{\rho}(0,t) + [\mathcal{A}\rho^{*2} + \mathcal{C}\rho^*]\bar{v}(0,t) \\ & + \mathcal{B}\rho^*U_2(t). \end{aligned} \quad (3.17)$$

3.2.1.3 Ventilator Model

Linearizing equation (3.11) assuming that p_a is constant, the following boundary condition is derived:

$$\bar{\rho}(0,t)v^{*2} + 2\rho^*v^*\bar{v}(0,t) = 4K_nC^*\bar{C}(t), \quad (3.18)$$

and then defining $\bar{C}(t) = U_1(t)$, $C^* = U_1^*$ and reorganizing (3.18) yields

$$\bar{v}(0,t) = \left[-\frac{v^*}{2\rho^*} \right] \bar{\rho}(0,t) + \left[\frac{4K_nU_1^*}{2\rho^*v^*} \right] U_1(t), \quad (3.19)$$

with U_1^* being a constant control action at the equilibrium point.

3.2.1.4 Outputs $T(L, \cdot)$, $v(L, \cdot)$, $p(L, \cdot)$ and $T(0, \cdot)$

The outputs $\bar{v}(L, \cdot)$ and $\bar{p}(L, \cdot)$ are obtained directly from the linear flow model in the tube. It is required to use the linearization of (3.7) to deal with $\bar{T}(L, \cdot)$ and $\bar{T}(0, \cdot)$, hence the following relationship is obtained

$$\bar{T}(L,t) = -\frac{P_{out}\bar{\rho}(L,t)}{R\rho^{*2}}, \quad \bar{T}(0,t) = -\frac{P_{in}\bar{\rho}(0,t)}{R\rho^{*2}}. \quad (3.20)$$

3.2.2 Discretization

3.2.2.1 Flow model in the tube

In order to spatially discretize the linear hyperbolic system (3.12), we shall make use of the forward and backward difference quotients for \mathbf{W}_t and \mathbf{W}_x , respectively.

Let Δt and Δx be respectively the time and space steps satisfying a Courant-Friedrichs-Lewy (CFL) condition (COURANT; FRIEDRICHS; LEWY, 1967). Then, by routine manipulations over $j = 0, 1, \dots, N$ and $i = 1, 2, \dots, \alpha$ with N being a given integer and $\alpha = \frac{L}{\Delta x}$, the PDE (3.12) is approximated by means of

$$\begin{aligned} \bar{\rho}_d(i, j+1) &= b_1 \bar{\rho}_d(i, j) + b_2 \bar{\rho}_d(i-1, j) \\ &\quad - b_3 \bar{v}_d(i, j) + b_3 \bar{v}_d(i-1, j), \end{aligned} \quad (3.21)$$

$$\begin{aligned} \bar{v}_d(i, j+1) &= b_1 \bar{v}_d(i, j) + b_2 \bar{v}_d(i-1, j) \\ &\quad - b_4 \bar{p}_d(i, j) + b_4 \bar{p}_d(i-1, j), \end{aligned} \quad (3.22)$$

$$\begin{aligned} \bar{p}_d(i, j+1) &= b_1 \bar{p}_d(i, j) + b_2 \bar{p}_d(i-1, j) \\ &\quad - b_5 \bar{v}_d(i, j) + b_5 \bar{v}_d(i-1, j), \end{aligned} \quad (3.23)$$

where $b_1 = \left[1 - \frac{\Delta t}{\Delta x} v^*\right]$, $b_2 = \left[\frac{\Delta t}{\Delta x} v^*\right]$, $b_3 = \left[\frac{\Delta t}{\Delta x} \rho^*\right]$, $b_4 = \left[\frac{\Delta t}{\Delta x \rho^*}\right]$ and $b_5 = \left[\frac{\Delta t}{\Delta x} \gamma p^*\right]$.

3.2.2.2 Ventilator Model

The discretization of (3.19) is obtained by applying a standard discretization $j\Delta t$ leading to

$$\bar{v}_d(0, j) = \left[-\frac{v^*}{2\rho^*}\right] \bar{\rho}_d(0, j) + \left[\frac{4K_n U_1^*}{2\rho^* v^*}\right] U_1(j). \quad (3.24)$$

3.2.2.3 Heating Column Model

Considering $j\Delta t$, the discretization of the heating column approximation in (3.17) is as follows

$$\begin{aligned} \bar{\rho}_d(0, j+1) &= [1 + \Delta t [2v^* \rho^* \mathcal{A} + \mathcal{B}U_2^* + \mathcal{C}v^*]] \bar{\rho}_d(0, j) \\ &\quad + \Delta t [\mathcal{A}\rho^{*2} + \mathcal{C}\rho^*] \bar{v}_d(0, j) + \Delta t [\mathcal{B}\rho^*] U_2(j), \end{aligned} \quad (3.25)$$

with Δt being the period of sampling. Thus, in view of the above, we derive the following expression by replacing $\bar{v}_d(0, j)$ as (3.24) and reorganizing (3.25)

$$\bar{\rho}_d(0, j+1) = \mathcal{Q}_1 \bar{\rho}_d(0, j) + \mathcal{Q}_2 U_1(j) + \mathcal{Q}_3 U_2(j), \quad (3.26)$$

$$\begin{aligned} \text{where } \mathcal{Q}_1 &= \begin{bmatrix} [1 + \Delta t [2v^* \rho^* \mathcal{A} + \mathcal{B}U_2^* + \mathcal{C}v^*]] \\ + \left[\Delta t [\mathcal{A}\rho^{*2} + \mathcal{C}\rho^*] \left[-\frac{v^*}{2\rho^*} \right] \right] \end{bmatrix}, \\ \mathcal{Q}_2 &= \left[\Delta t [\mathcal{A}\rho^{*2} + \mathcal{C}\rho^*] \left[\frac{4K_n U_1^*}{2\rho^* v^*} \right] \right] \text{ and } \mathcal{Q}_3 = \Delta t \mathcal{B}\rho^*. \end{aligned}$$

3.2.2.4 Outputs $T(L, \cdot)$, $v(L, \cdot)$, $p(L, \cdot)$ and $T(0, \cdot)$

The discrete linear approximate outputs $\bar{v}_d(L, \cdot)$ and $\bar{p}_d(L, \cdot)$ are obtained directly from the discrete linear approximate flow model in the tube. For $\bar{T}(L, t)$ and $\bar{T}(0, t)$ as described in (3.20) is applied the standard discretization step $j\Delta t$ leading to:

$$\bar{T}_d(0, j) = -\frac{p_{in}\bar{\rho}_d(0, j)}{R\rho^{*2}}, \quad \bar{T}_d(L, j) = -\frac{p_{out}\bar{\rho}_d(L, j)}{R\rho^{*2}}. \quad (3.27)$$

3.2.3 Discrete-Time Linear Approximation

After the steps of linearization and discretization, an augmented discrete-time linear model can be obtained. To this end, let the augmented state vector be defined by:

$$\begin{aligned} \mathcal{X}(j) &= \begin{bmatrix} \bar{\rho}_d(1, j) & \bar{\rho}_d(2, j) & \cdots & \bar{\rho}_d(\alpha, j) \\ \bar{v}_d(1, j) & \bar{v}_d(2, j) & \cdots & \bar{v}_d(\alpha, j) \\ \bar{p}_d(1, j) & \bar{p}_d(2, j) & \cdots & \bar{p}_d(\alpha, j) \end{bmatrix}^T, \end{aligned} \quad (3.28)$$

and the boundary points as follows

$$\mathcal{X}_c(j) = \begin{bmatrix} \bar{\rho}_d(0, j) & \bar{v}_d(0, j) \end{bmatrix}^T. \quad (3.29)$$

The boundary condition $\bar{\rho}_d(0, j)$ is considered constant and it is not used for the boundary control problem.

Since the boundary control is not directly applied to the states components density $\bar{\rho}_d(0, j)$ and speed $\bar{v}_d(0, j)$, it is employed the ventilator speed $U_1(j)$ (installed in the initial part of the tube) and the power dissipated on the heating resistor $U_2(j)$ (installed in the heating column) as the boundary control actuation. Thus, the control input vector of the discrete-time approximation is defined as:

$$U(j) = \begin{bmatrix} U_1(j) & U_2(j) \end{bmatrix}^T. \quad (3.30)$$

Notice that we have a static boundary condition and a dynamic boundary condition. Hence, to obtain the boundary condition related to the density $\bar{\rho}_d(0, j)$, we use the dynamic model of the heating column. Therefore, the augmented state vector is redefined as follows

$$\mathcal{Z}(j) = \begin{bmatrix} \mathcal{X}(j) \\ \bar{\rho}_d(0, j) \end{bmatrix}. \quad (3.31)$$

Then, the discrete-time linear approximation will be as follows:

$$\Omega: \begin{cases} \mathcal{Z}(j+1) = A\mathcal{Z}(j) + BU(j), \\ y(j) = C\mathcal{Z}(j), \end{cases} \quad (3.32)$$

where j is the sampling time index; $\mathcal{Z}(j) \in \mathbb{R}^{3\alpha+1}$ is the augmented state vector; $U(j) \in \mathbb{R}^m$ represents the control input vector; $y(j) \in \mathbb{R}^p$ represents the output signals; and A , B and C are constant matrices as defined in (3.33) at the top of next page.

$$\begin{aligned}
 A = & \left[\begin{array}{ccc|ccc|ccc|c}
 b_1 & 0 & \dots & 0 & b_3 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & \left(b_2 + \left(b_3 \left(-\frac{v^*}{2\rho^*} \right) \right) \right) \\
 b_2 & \ddots & \ddots & \vdots & b_3 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \dots \\
 0 & \ddots & \ddots & 0 & 0 & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots & \vdots \\
 0 & 0 & b_2 & b_1 & 0 & 0 & b_3 & b_3 & 0 & \dots & \dots & 0 & 0 \\
 \hline
 0 & \dots & \dots & 0 & b_1 & 0 & \dots & 0 & b_4 & 0 & \dots & 0 & b_2 \left(-\frac{v^*}{2\rho^*} \right) \\
 \vdots & \ddots & \ddots & \vdots & b_2 & \ddots & \ddots & \vdots & b_4 & \ddots & \ddots & \vdots & 0 \\
 \vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & 0 & 0 & \ddots & \ddots & 0 & \vdots \\
 0 & \dots & \dots & 0 & 0 & 0 & b_2 & b_1 & 0 & 0 & b_4 & b_4 & 0 \\
 \hline
 0 & \dots & \dots & 0 & b_5 & 0 & \dots & 0 & b_1 & 0 & \dots & 0 & b_5 \left(-\frac{v^*}{2\rho^*} \right) \\
 \vdots & \ddots & \ddots & \vdots & b_5 & \ddots & \ddots & \vdots & b_2 & \ddots & \ddots & \vdots & \vdots \\
 \vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & 0 & 0 & \ddots & \ddots & 0 & \vdots \\
 0 & \dots & \dots & 0 & 0 & 0 & b_5 & b_5 & 0 & 0 & b_2 & b_1 & 0 \\
 0 & \dots & \dots & 0 & 0 & \dots & \dots & 0 & 0 & \dots & \dots & 0 & Q_1
 \end{array} \right] \\
 \\
 B = & \left[\begin{array}{cc|cc}
 b_3 \left(\frac{4K_n U_1^*}{2\rho^* v^*} \right) & 0 & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots \\
 0 & 0 & \vdots & \vdots \\
 \hline
 b_2 \left(\frac{4K_n U_1^*}{2\rho^* v^*} \right) & 0 & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots \\
 0 & 0 & \vdots & \vdots \\
 \hline
 b_5 \left(\frac{4K_n U_1^*}{2\rho^* v^*} \right) & 0 & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots \\
 0 & 0 & \vdots & \vdots \\
 \hline
 Q_2 & Q_3 & &
 \end{array} \right] \\
 \\
 C = & \left[\begin{array}{ccc|ccc|ccc|c}
 0 & \dots & 0 & -\frac{P_{out}}{R\rho^{*2}} & 0 & \dots & \dots & 0 & 0 & \dots & \dots & 0 & 0 \\
 \hline
 0 & \dots & \dots & 0 & 0 & \dots & 0 & 1 & 0 & \dots & \dots & 0 & 0 \\
 \hline
 0 & \dots & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\
 \hline
 0 & \dots & \dots & 0 & 0 & \dots & \dots & 0 & 0 & \dots & \dots & 0 & -\frac{P_{in}}{R\rho^{*2}}
 \end{array} \right]. \tag{3.33}
 \end{aligned}$$

To implement a state feedback control law, a standard Luenberger-like observer is designed. That is, the following control law is considered

$$U(j) = K\hat{z}(j), \quad (3.34)$$

where $\hat{z}(j)$ is the state of the following observer

$$\hat{\Omega} : \{\hat{z}(j+1) = (A - LC)\hat{z}(j) + BU(j) + Ly(j), \quad (3.35)$$

with $K \in \mathbb{R}^{m \times 3\alpha+1}$ and $L \in \mathbb{R}^{3\alpha+1 \times p}$ to be determined such that $[A - BK]$ and $[A - LC]$ are Schur stable. The stability of system (3.35) is analyzed using classical control tools for discrete-time systems (such as pole allocation). Since the original problem is an infinite dimension problem, the numerical stability of the discrete model depends on the discretization steps Δt and Δx and the approximation scheme considered to handle the partial derivatives. The parameters Δt and Δx are adopted to ensure the numerical stability condition (CFL condition, see (COURANT; FRIEDRICHS; LEWY, 1967)). The temporal partial derivative is approximated using forward differences and the spatial partial derivative is approximated using backward differences.

3.3 Numerical Simulations

Let the following system parameters: adiabatic constant $\gamma = 1.4$; molar mass of dry air $M = 28.97$ g/mol.K; ideal gas constant $R = 8.3143$ J/(mol.K); specific heat constant for constant pressure $C_p = 1.005$ KJ/Kg.K for constant volume $C_v = 0.718$ KJ/Kg.K; initial pressure $p_{in} = 1 \times 10^5$ Pa; initial temperature $T_{in} = 304$ K; column volume $V_0 = 4 \times 10^{-3}$ m³; tube cross section area $A_t = 6.4 \times 10^{-3}$ m²; tube length $L = 1$ m; and the constant coefficient of ventilator model $K_n = 1 \times 10^{-4}$. In addition, define the discretization steps as $\Delta t = 0.0075$ and $\Delta x = 0.036$ which ensure the numerical stability condition of (COURANT; FRIEDRICHS; LEWY, 1967) (i.e., $b_1 < |1|$ – with $b_1 = 0.5032$).

Firstly, some simulations are performed aiming to validate the proposed models, i.e., the hyperbolic nonlinear system (3.10) with non-

linear boundary conditions (3.9) and (3.11) referred as *HSNL*, the hyperbolic linear approximation (3.12) with linear boundary conditions (3.17) and (3.19) referred as *HSL* and the augmented discrete-time linear approximate system (3.32) which is referred as *ADLS*. For these three models, it is imposed an initial condition and it is verified the steady-state convergence. To this end, it is considered control inputs $U_1^* = 150$ rpm and $U_2^* = 300$ watts which yield steady-state values $\rho^* = 1.1$ kg/m³ (or, equivalently, $T^* = 320.92$ K), $v^* = 2.02$ m/s and $p^* = 1$ Bar (or, equivalently, 1×10^5 Pa in SI units). Figure 6 shows the

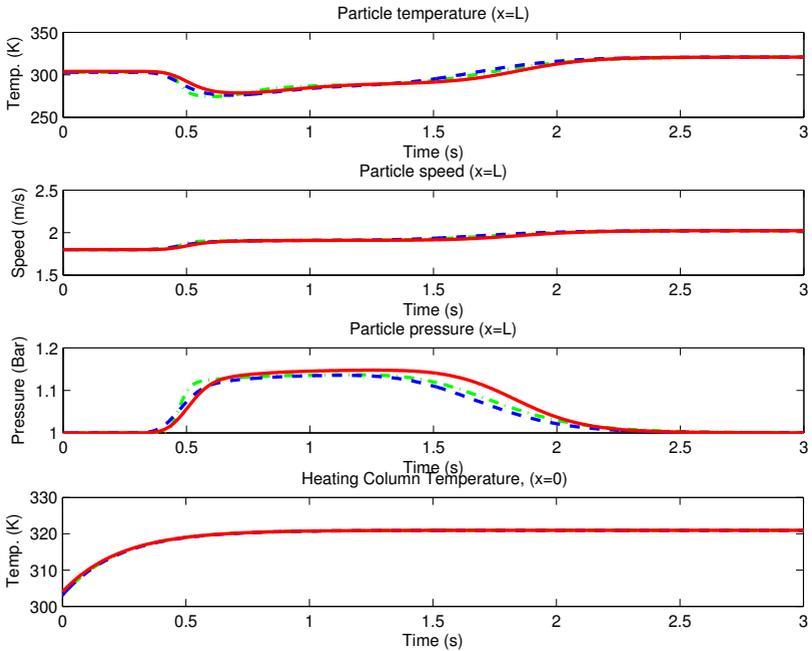


Figure 6 – Temperature, speed and pressure at $x = L$ and temperature at $x = 0$ (i.e., heating column temperature) time responses for the models *HSNL* (—), *HSL* (- - -), and *ADLS* (- · - ·).

time response of temperature, speed and pressure at the tube outlet, and the heating column temperature (tube inlet) for all three models. The initial condition was set to $\rho = 1.1614$ kg/m³, $v = 1.8$ m/s and

$p = 1$ Bar. Notice when converting the value of the initial condition $\rho = 1.1614 \text{ kg/m}^3$ in temperature for *HSNL* using (3.7) that we have obtained 304 K and for the linear approximate models using the linear relationship (3.20) or (3.27), we have obtained 301.4 K . It turns out that all models have converged to the expected values.

Now, a controller for tracking a constant reference temperature at $x = L$ is designed (considering the discrete-time linear approximated model *ADLS*) by adapting the control law proposed in (3.34) with (3.35). To this end, a discrete-time integrator is added to the system dynamics of (3.32) leading to the following state space realization:

$$\tilde{\Omega}: \{\tilde{\mathcal{Z}}(j+1) = \tilde{A}\tilde{\mathcal{Z}}(j) + \tilde{B}U(j) + B_r T_{ref}\} \quad (3.36)$$

where $\tilde{\mathcal{Z}}(j) = [\mathcal{Z}(j)^T \quad \xi(j)]^T$ is the enlarged state vector, $\xi(j)$ is the integrator state, and

$$\tilde{A} = \begin{bmatrix} A & \mathbf{0} \\ -C_r & 1 \end{bmatrix}, \tilde{B} = \begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}, B_r = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix},$$

with C_r and T_{ref} being respectively a constant matrix such that $\tilde{T}_d(L, j) = C_r \mathcal{Z}(j)$ and the desired temperature at $x = L$. Then, the control law (3.34) is modified to be as follows:

$$U(j) = \tilde{K}\tilde{\mathcal{Z}}(j), \quad (3.37)$$

where $\tilde{K} = [K \quad K_r]$ and $\tilde{\mathcal{Z}}(j) = [\hat{\mathcal{Z}}(j)^T \quad \xi(j)]^T$. Figure 7 shows a comparison of the hyperbolic linear PDE model *HSL*, as in (3.12), with linear boundary conditions (3.17)-(3.19) (---) and the discrete-time approximate model *ADLSC* (-·-·-) using in both cases a controller designed for the *ADLSC* model. Notice that the controller is able to track the reference temperature $T_{ref} = 325 \text{ K}$ considering both models showing that the designed control law can be applied for boundary control of hyperbolic PDE systems at the cost of some performance degradation.

3.4 Concluding Remarks

In this chapter, we have presented the boundary control of first order hyperbolic PDE systems associated with dynamic boundary con-

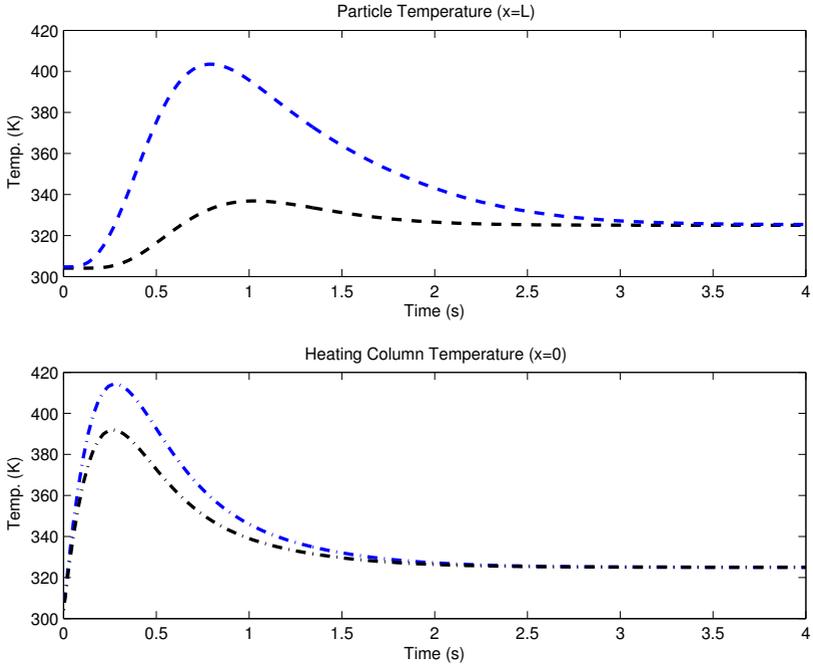


Figure 7 – Time evolution of temperature in $x = L$ and $x = 0$ for *HSL*, using the designed control for *ADLSC* (---) and the *ADLSC*, (---).

ditions. Classical techniques such as linear approximation and discretization have been applied to the model of an experimental setup (consisting of a tube, heating column and ventilator) leading to an augmented discrete-time linear approximate model. Then, an output feedback plus integrator control law is derived to track a constant reference temperature. Simulation results have shown the potentials of the proposed approach. Future research will be concentrated in applying these results on the experimental setup.

4 STABILITY AND STABILIZATION OF INPUT DELAYED NONLINEAR QUADRATIC SYSTEMS

In this chapter we address the dynamic boundary condition system coupled with the first-order hyperbolic system. As we have seen previously, the first-order hyperbolic system can be approximated by a pure delay. Thus, we will address the coupled system of an ODE-PDE by an ODE with input-delay (see Figure 8). In particular, in this chapter, the dynamic boundary is modeled by means of an uncertain nonlinear quadratic system. The stability analysis and control design are devised in a regional setting considering the nonlinear dynamic boundary and approximating the PDE by a pure delay. The link between the PDE-ODE system and the input delayed quadratic system is demonstrated via an experimental setup in Section 4.4.

The stability analysis and control synthesis of dynamical systems subject to delayed inputs is a problem attracting the interest of control practitioners over the last decades due to inherent transport and measurement delays in practical applications and more recently due to induced communication delays in the context of networked control systems (see, for instance, (YUE; HAN, 2005; MAZENC; NICULESCU; KRSTIC, 2012; FRIDMAN, 2010) and references therein). For linear systems, there exist several approaches to deal with input delay in frequency and state-space domains such as the predictor-based approach (MIRKIN; RASKIN, 2003), the reduction technique (AST-STEIN, 1982), and there exist many recent papers mostly employing either Lyapunov-Razumikhin or Lyapunov-Krasovskii stability theories (FRIDMAN, 2014) to take model uncertainties and varying delays into account. On the other hand, there are few results dealing with model uncertainties and/or time-varying input delays for the nonlinear counterpart. For instance, reference (MAZENC; MALISOFF; LIN, 2008) addresses the input-to-state stability problem for the class of input affine nonlinear systems subject to a constant input delay and actuator disturbances, (LI; LIU, 2009) proposes stability and stabilization conditions for uncertain nonlinear systems described by TS-fuzzy models

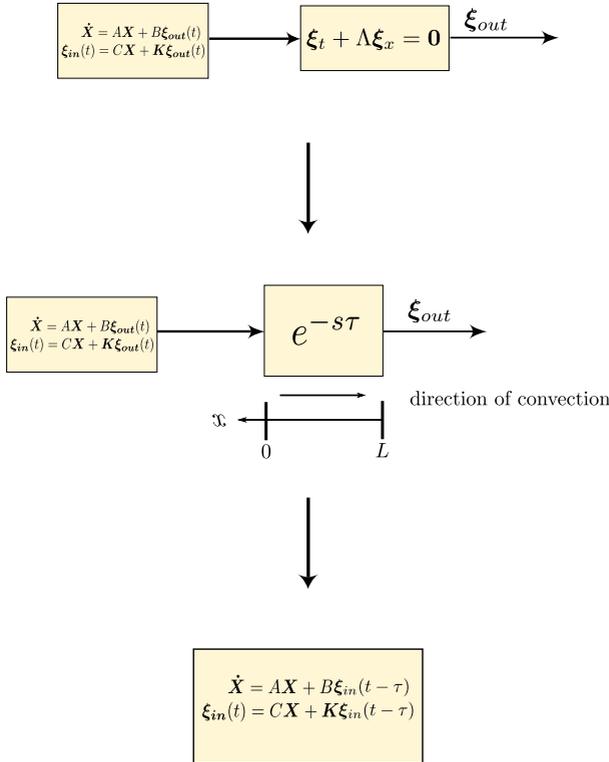


Figure 8 – Approximation of the coupled PDE-ODE system by an ODE system with input delay

with state and input delays, (CHOI; LIM, 2010) consider the stabilization problem of a chain of integrators subject to state-dependent norm bounded uncertainties and input delay, (FISCHER et al., 2012) focuses on the stabilization problem of uncertain Euler-Lagrange systems with a time-varying input delay, and (WANG et al., 2014) studies the stability properties of a class of switched nonlinear systems with input delay by means of Lyapunov-Krasovskii functionals.

An alternative solution to derive stability and stabilization results for input delayed nonlinear systems is to consider a linear approximation of the system around an operating point (e.g., obtained

via the Taylor expansion), and then applying well-established control design tools for input delayed linear systems. If the error introduced by linear approximations is too large, high-order terms of the Taylor expansion can be considered to more precisely describe the system dynamics. For instance, when the model nonlinearities are concentrated in the input channel, it is obtained the class of bilinear models (PARDALOS; YATSENKO, 2008). A natural extension to bilinear models is derived considering the quadratic terms of a Taylor expansion leading to the class of nonlinear quadratic systems (AMATO; COSENTINO; MEROLA, 2007; VALMÓRBIDA; TARBOURIECH; GARCIA, 2010; COUTINHO; DE SOUZA, 2012). It turns out that the latter class of systems can represent a large number of practical processes such as distillation and heating columns (ESPANA; LANDAU, 1978; CASTILLO et al., 2012), induction motors (ARGUELLO-SERRANO; VELEZ-REYES, 1999) and DC–DC converters (THOUNTHONG; PIERFEDERICI, 2010). In view of the good compromise between model accuracy and mathematical complexity, the class of nonlinear quadratic systems has recently attracted the interest of control practitioners (see, e.g., the references (AMATO; COSENTINO; MEROLA, 2007; VALMÓRBIDA; TARBOURIECH; GARCIA, 2010; COUTINHO; DE SOUZA, 2012; MAHOUT; TARBOURIECH; GARCIA, 2002; HUANG; LAM, 2002; LU; FENG; JIANG, 2007; AMATO; COSENTINO; MEROLA, 2010; TARBOURIECH et al., 2009; AMATO et al., 2009)). In particular, the local stabilization problem of open-loop unstable state delayed nonlinear quadratic systems has been addressed by references (DE SOUZA; COUTINHO, 2012) and (DE SOUZA; COUTINHO, 2014) considering respectively *delay-independent* and *delay-dependent* design conditions based on Lyapunov-Razumikhin functions and Lyapunov-Krasovskii functionals.

In the particular case of sampled-data systems, Fridman and co-authors in (FRIDMAN; SEURET; RICHARD, 2004; FRIDMAN, 2010) have proposed the *delay-input* approach to deal with uncertain sampling periods. This approach models aperiodic discretization effects on dynamical systems by means of a time-varying input delay assuming

a bound on the sampling intervals. More recently, the input-delay approach has been extended in (FRIDMAN; LIU, 2016) to cope with the stabilization problem of linear systems subject to actuator saturation and delayed feedback. Inspired by the latter result, this chapter proposes a linear matrix inequality (LMI) based approach to the regional stability analysis and control design for uncertain nonlinear quadratic systems subject to a time-varying input delay considering Lyapunov-Krasovskii functionals. As pointed out in (FRIDMAN; LIU, 2016), the first time interval plays an important role for assessing the local stability of open-loop unstable nonlinear systems, since the boundedness of the state trajectory has to be guaranteed for all time $t \geq 0$. Hence, the local stability analysis (assuming that a stabilizing control law and bounds on the input delay and its variation are known *a priori*) is characterized in terms of two compact sets \mathcal{R}_0 and \mathcal{R} which are such that the state trajectory $x(t)$ starting in the former set remains inside the latter and converges to the equilibrium point as the time goes to infinity. Then, the stability analysis result is extended for control design in order to determine a quadratic state feedback control law to either maximize the set of admissible initials conditions or maximize the input delay size. Some of the results present in this chapter can be found in (CALDEIRA et al., 2016).

The remaining of this chapter is organized as follows. In Section 4.1, it is established the stability and stabilization problems to be addressed and introduced the regional stability notion to be considered. Then, Section 4.2 presents the Lyapunov-Krasovskii based conditions employed to guarantee the state trajectory boundedness in the first time interval as well as the state trajectory convergence to the equilibrium point under analysis. Next, two convex optimization problems are stated in Section 4.3 in order to design a stabilizing delayed state feedback control law to maximize either the set of admissible initial conditions or the input delay size. Ending the chapter, Section 4.5 discusses some concluding remarks and future developments.

4.1 Problem Statement

Consider the following class of input delayed nonlinear control systems:

$$\begin{cases} \dot{x}(t) &= A(x(t), \boldsymbol{\delta})x(t) + B(x(t), \boldsymbol{\delta})u(t - \tau(t)), \\ u(t) &= K(x(t))x(t) \end{cases} \quad (4.1)$$

where $x(t) \in \mathcal{X} \subset \mathbb{R}^n$ is the state, $u(t) \in \mathcal{U} \subset \mathbb{R}^{n_u}$ is the control input, $\boldsymbol{\delta} \in \Delta \subset \mathbb{R}^{n_\delta}$ is a vector of uncertain constant parameters, and $\tau(t) \in \mathbb{R}$ is a time-varying input delay satisfying:

$$0 < \tau(t) \leq d, \quad \dot{\tau}(t) \leq h < 1, \quad \forall t \geq 0, \quad (4.2)$$

with d and h being given positive scalars. \mathcal{X} , \mathcal{U} and Δ are compact regions defining respectively the state, input and uncertainty domains with \mathcal{X} containing the system origin. The matrices $A(\cdot)$, $B(\cdot)$ and $K(\cdot)$ are affine functions of their arguments, more precisely:

$$A(x(t), \boldsymbol{\delta}) = A_0 + \sum_{i=1}^n x_i(t)A_i + \sum_{i=1}^{n_\delta} \delta_i \check{A}_i, \quad (4.3)$$

$$B(x(t), \boldsymbol{\delta}) = B_0 + \sum_{i=1}^n x_i(t)B_i + \sum_{i=1}^{n_\delta} \delta_i \check{B}_i, \quad (4.4)$$

$$K(x(t)) = K_0 + \sum_{i=1}^n x_i(t)K_i, \quad (4.5)$$

where $x_i(t)$ denotes the i -th entry of $x(t)$; and A_i , \check{A}_j , B_i , \check{B}_j and K_i , for $i = 0, 1, \dots, n$ and $j = 1, \dots, n_\delta$, are given constant matrices with appropriate dimensions. Note that system (3.32) with the matrices defined as in (4.3)-(4.5) corresponds to a system with a quadratic dependence on the state and bilinear with respect to the state and control input. Observe that $x = 0$ is an equilibrium point of the closed-loop system and stability analysis will be carried for this point.

Since the input delay is assumed to be bounded by d and its time derivative is smaller than or equal to one, there exists a unique $t_0 \leq d$ such that $t - \tau(t) < 0$, for all $t \in [0, t_0)$, and $t - \tau(t) \geq 0$, for all $t \geq t_0$. As a consequence, the system operates in open-loop in the interval of

time $[0; t_0)$. Hence, the closed-loop system is governed by the following dynamics:

$$\dot{x}(t) = \begin{cases} A(x(t), \delta)x(t) & \text{if } 0 \leq t < t_0 \\ A(x(t), \delta)x(t) + B(x(t), \delta)K(\tilde{x}(t))\tilde{x}(t) & \text{if } t \geq t_0 \end{cases} \quad (4.6)$$

where $x(s) = \bar{x}_0$ for $s \in [-d, 0]$, $\tilde{x}(t) = x(t - \tau(t))$ is the delayed state, $\tau(t) \in (0, d]$ and

$$K(\tilde{x}(t)) = K_0 + \sum_{i=1}^n \tilde{x}_i(t)K_i,$$

with $\tilde{x}_i(t)$ denoting the i -th entry of $\tilde{x}(t)$. For notation simplicity, hereafter, the argument t of $x(t)$, $\tilde{x}(t)$, $u(t)$ and $\tau(t)$ will be often omitted.

The stability analysis of the closed-loop dynamics in (4.6) is particularly interesting for open-loop unstable systems, since for $t \in [0, t_0)$ the trajectory starting in some initial state \bar{x}_0 will move away from the equilibrium point until the feedback signal starts to be applied to the system input at $t = t_0$. Notice in this case that the state trajectory may leave the equilibrium point region of attraction and thus a delayed feedback will not be able to asymptotically stabilize the system origin. In order to prevent this situation and to properly address the stability behavior on the vicinity of the equilibrium point in closed-loop, it will be considered in this paper the following definition of regional stability.

Definition 4.1. *Consider the system (4.1). Let \mathcal{R}_0 and \mathcal{R} be two compact sets such that $\mathcal{R}_0 \subset \mathcal{R} \subset \mathcal{X}$. The system origin ($x = 0$) is robustly regionally stable if, for any $\bar{x}_0 \in \mathcal{R}_0$ and all $\delta \in \Delta$, the state trajectory $x(t)$ remains bounded in \mathcal{R} , for all $t \geq 0$, and it approaches $x = 0$ as $t \rightarrow \infty$.*

Figure 9 illustrates the above notion of regional stability for an open-loop unstable planar system. Notice that the state trajectory starting at $\bar{x}_0 \in \mathcal{R}_0$ diverges from the equilibrium point, but it remains bounded in \mathcal{R} . Then, the state trajectory converges to the equilibrium point when the feedback signal is applied to the system input at $t = \tau$. In view of that, in this chapter, the regions \mathcal{R}_0 and \mathcal{R} will be often re-

ferred to as the set of admissible initial states and the set of reachable states, respectively.

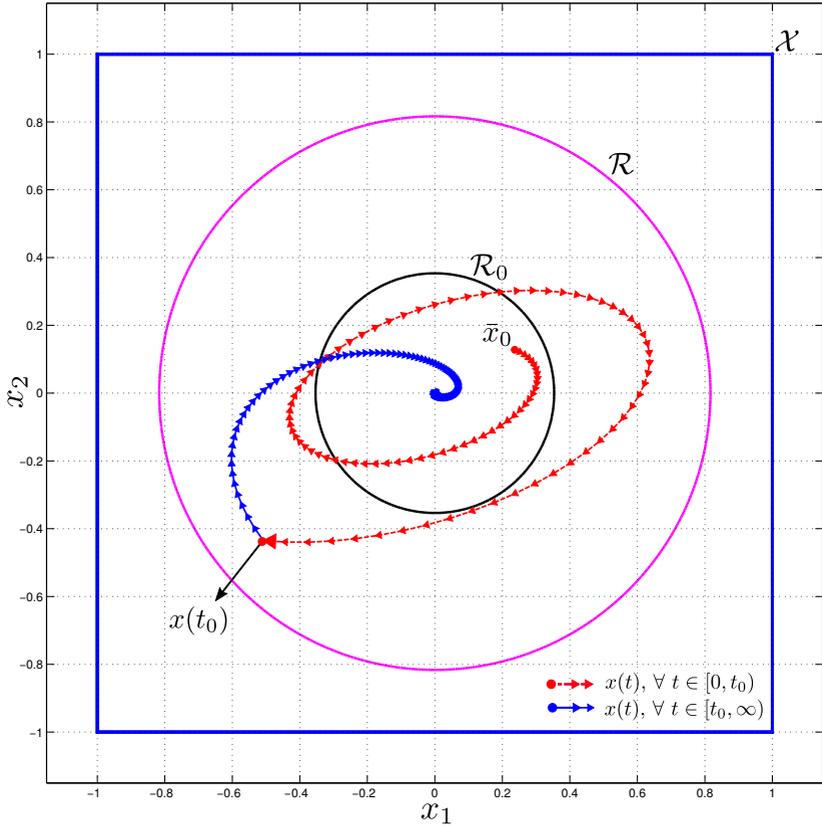


Figure 9 – Illustrative behavior of an open-loop unstable input delayed nonlinear control system.

In light of the above scenario, this chapter is concerned in obtaining numerical and tractable solutions to the regional stability and stabilization problems for uncertain input delayed nonlinear quadratic systems as stated below:

P1 For a given state feedback control matrix $K(\cdot)$ as defined in (4.5),

derive stability analysis conditions ensuring the robust regional stability of the control system in (4.1) while determining either: (i) an upper estimate of the set of initial conditions \mathcal{R}_0 for a given bound d on the input delay; or (ii) an estimate of the maximum admissible delay d for a given set of initial states.

P2 Design a state feedback control matrix $K(\cdot)$ ensuring the robust regional stability of the control system in (4.1) in order to obtain either: (i) a maximized estimate of the set of initial condition \mathcal{R}_0 for a given maximum delay d ; or (ii) an estimate of the maximum admissible delay d for a given set of admissible initial states.

Before ending this section, the following two results taken from specialized literature will be instrumental for deriving the main results of this chapter.

Lemma 4.1. (*BOYD et al., 1994; COUTINHO; DE SOUZA, 2012*) Given matrix functions $N(v) \in \mathbb{R}^{s \times m}$, $S(v) = S(v)' \in \mathbb{R}^{m \times m}$ and $\eta(v) \in \mathbb{R}^m$, with $v \in \mathbb{V} \subseteq \mathbb{R}^v$, then

$$\eta(v)'S(v)\eta(v) < 0, \forall v \in \mathbb{V} : N(v)\eta(v) = 0, \eta(v) \neq 0,$$

if there exists a matrix $L \in \mathbb{R}^{m \times s}$ such that

$$S(v) + \text{He}(LN(v)) < 0, \forall v \in \mathbb{V}.$$

Lemma 4.2. (*DE OLIVEIRA; GEROMEL; BERNUSSOU, 2002*) Let P and G be real square matrices with $P > 0$ and G nonsingular. Then, the following holds:

$$G'PG \geq G + G' - P^{-1}. \quad (4.7)$$

4.2 Stability Analysis

This section develops an LMI approach to the robust regional stability analysis of input delayed quadratic systems as defined in (4.1). It is assumed that a control matrix $K(x)$, which locally stabilizes the closed-loop system with no input delay, is given *a priori*. The solution

to be proposed in the following is based on the Lyapunov-Krasovskii stability theory for input delayed systems (FRIDMAN; LIU, 2016).

Firstly, the notion of local stability for functional differential equations of retarded type is recalled. To this end, consider the following delayed state-space representation:

$$\begin{cases} \dot{x} = f(x_t), & x \in \mathbb{R}^n, x_t \in \mathcal{C}_{[t-d,t]}^n, \\ x(s) = \phi(s), & \forall s \in [-d, 0], \phi \in \mathcal{C}_{[-d,0]}^n. \end{cases} \quad (4.8)$$

It is assumed for the above system that $f : \mathcal{C}_{[t-d,t]}^n \mapsto \mathbb{R}^n$, with $f(0) = 0$, satisfies the conditions of existence and uniqueness of solution for any $\phi \in \mathcal{C}_{[-d,0]}^n$. Hence, the following local version of the Lyapunov-Krasovskii stability theory can be introduced.

Lemma 4.3. (FRIDMAN, 2014; DE SOUZA; COUTINHO, 2014) *Let $u, v, w : \mathbb{R} \mapsto \mathbb{R}$ be continuous, positive definite functions, with $u(\cdot), w(\cdot)$ non-decreasing and $v(\cdot)$ strictly increasing. Suppose there exists a continuously differentiable functional $V : \mathcal{C}_{[-d,0]}^n \mapsto \mathbb{R}$ such that:*

$$(a) \quad u(\|x\|) \leq V(x_t) \leq v(\|x_t\|_{[-d,0]}), \quad \forall x \in \mathcal{X};$$

$$(b) \quad \dot{V}(t, x_t) < -w(\|x\|), \quad \forall x \in \mathcal{X}, t \geq 0.$$

Let $\mathcal{B}(\gamma) := \{\phi \in \mathcal{C}_{[-d,0]}^n : V(\phi) \leq \gamma\}$ with $\gamma > 0$ being such that $\mathcal{D}(\gamma) := \{\phi(s) \in \mathbb{R}^n, s \in [-d, 0] : \phi \in \mathcal{B}(\gamma)\} \subset \mathcal{X}$. Then, the equilibrium point $x = 0$ of (4.8) is locally asymptotically stable. Moreover, $x_t \in \mathcal{B}(\gamma)$, $\forall t \geq 0$, and $\lim_{t \rightarrow \infty} x(t) = 0$ for any $\phi \in \mathcal{B}(\gamma)$.

Since in the interval $[0, t_0)$ the controlled system in (4.6) operates in open-loop, the above result cannot be directly applied to assess the stability of system (4.6) if the open-loop system is unstable. To overcome this difficulty, a procedure inspired by (LIU; FRIDMAN, 2014) will be used. To this end, let the following Lyapunov-Krasovskii functional candidate:

$$V(x_t, \dot{x}_t) = V_1(x(t)) + V_2(x_t) + V_3(\dot{x}_t), \quad (4.9)$$

where

$$\begin{aligned} V_1(x) &= x'P_1x, \quad V_2(x_t) = \int_{t-\tau(t)}^t x'(\alpha)P_2x(\alpha)d\alpha, \\ V_3(\dot{x}_t) &= \int_{-d}^0 \int_{t+\beta}^t \dot{x}'(\alpha)P_3\dot{x}(\alpha)d\alpha d\beta, \end{aligned} \quad (4.10)$$

with P_1 , P_2 and P_3 being symmetric positive-definite matrices to be determined. Hence, the origin of the system in (4.1) is robustly regionally stable if the following three conditions hold for all $\delta \in \Delta$ and some positive scalars ε and φ :

$$\mathbf{C1} \quad \dot{V}(t, x_t, \dot{x}_t, \delta) \leq -\varepsilon \|x\|^2, \quad \forall x(s) \in \mathcal{X}, s \in [t-d, t], t \geq t_0, \delta \in \Delta;$$

$$\mathbf{C2} \quad \dot{V}_1(x(t), \delta) - 2\varphi V_1(x(t)) \leq 0, \quad \forall x(t) \in \mathcal{X}, t \in [0, t_0], \delta \in \Delta;$$

$$\mathbf{C3} \quad \dot{V}(t, x_t, \dot{x}_t, \delta) - 2\varphi V_1(x(t)) \leq 0, \quad \forall x(t) \in \mathcal{X}, t \in [0, t_0], \delta \in \Delta.$$

Notice from Lemma 4.3 that **C1** implies the asymptotic convergence of $x(t)$ for $t \geq d$ provided that the state trajectory $x(t)$, for any $t \in [0, d]$, is confined to a certain neighborhood \mathcal{R} of the system origin satisfying $\mathcal{R} \subset \mathcal{X}$. It will be shown in the sequel that **C2** and **C3** guarantee that $x(t)$, for any $t \in [0, d]$, lies inside \mathcal{R} if x_0 belongs to some set \mathcal{R}_0 , with $\mathcal{R}_0 \subset \mathcal{R}$.

Hence, notice from (KHALIL, 1996, Lemma 2.5) that **C2** implies the following:

$$V_1(x) \leq e^{2\varphi t} V_1(0), \quad \forall t \in [0, t_0], \delta \in \Delta. \quad (4.11)$$

Next, integrating **C3** from 0 to t , $t \in [0, d]$, leads to

$$\int_0^t [\dot{V}(s) - 2\varphi V_1(s)] ds \leq 0, \quad \forall x(t) \in \mathcal{X}, t \in [0, t_0], \delta \in \Delta,$$

which in turn yields:

$$\begin{aligned} V(t) &\leq V(0) + 2\varphi \int_0^t V_1(s) ds \\ &\leq V(0) + 2\varphi V_1(0) \int_0^t e^{2\varphi s} ds \\ &\leq V(0) + V_1(0) [e^{2\varphi t} - 1], \end{aligned} \quad (4.12)$$

where $V(t) = V(x_t)$. Now, for convenience, suppose with out loss of generality that $x(s) = x_0$, $s \in [-d, 0]$. Then, the following holds:

$$V(0) = x_0' P_1 x_0 + \int_{-d}^0 x_0' P_2 x_0 d\alpha \leq x_0' (P_1 + dP_2) x_0 . \quad (4.13)$$

Taking (4.12) and (4.13) into account, $V(t)$ satisfies

$$\begin{aligned} V(t) &\leq x_0' P_1 x_0 + e^{2\varphi t} x_0' P_1 x_0 - x_0' P_1 x_0 + dx_0' P_2 x_0 \\ &\leq e^{2\varphi t} x_0' P_1 x_0 + dx_0' P_2 x_0. \end{aligned} \quad (4.14)$$

Let σ be a positive scalar such that $\sigma P_1 \geq dP_2$ and let \mathcal{R}_0 be the following set:

$$\mathcal{R}_0 := \{x \in \mathbb{R}^n : x' P_1 x \leq 1\} . \quad (4.15)$$

If in addition $x_0 \in \mathcal{R}_0$, then $V(t)$, $t \in [0, d]$, is bounded as follows:

$$V(t) \leq \gamma, \quad \gamma = \sigma + e^{2\varphi d} . \quad (4.16)$$

In other words, the trajectory segment $x_{[0,d]} = x(s)$, for $s \in [0, d]$, is confined to the following set

$$\mathcal{B} := \left\{ x_{[0,d]} \in \mathcal{C}_{[0,d]}^n : V(t) \leq \gamma \right\} \quad (4.17)$$

provided that $\mathcal{D} := \{x(s) \in \mathbb{R}^n, s \in [0, d] : x(s) \in \mathcal{B}\} \subset \mathcal{X}$.

Later in this section, to obtain a numerical and tractable condition for guaranteeing that $\mathcal{D} \subset \mathcal{X}$, it is considered the following set:

$$\mathcal{R} := \{x \in \mathbb{R}^n : V_1(x) = x' P_1 x \leq \gamma\} , \quad (4.18)$$

since $\mathcal{D} \subseteq \mathcal{R}$ by noting that $V(t) \geq V_1(x)$, $\mathcal{R}_0 \subset \mathcal{R}$ for $\gamma \geq 1$. Then, $\mathcal{R} \subset \mathcal{X}$ implies $\mathcal{D} \subset \mathcal{X}$.

4.2.1 Main Result

Before introducing the main result of this section, it will be shown in the sequel how condition **C1** will be handled in this chapter. Thus, in view of Lyapunov-Krasovskii functional in (4.9), the time-

derivative of $V(t)$ is given by:

$$\begin{aligned}\dot{V}(x_t) &= 2x'P_1\dot{x} + x'P_2x - (1 - \hat{\tau})\tilde{x}'P_2\tilde{x} + d\dot{x}'P_3\dot{x} - \int_{t-d}^t \dot{x}(\alpha)'P_3\dot{x}(\alpha) d\alpha \\ &= \frac{1}{d} \int_{t-d}^t [2x'P_1\dot{x} + x'P_2x - (1 - \hat{\tau})\tilde{x}'P_2\tilde{x} + d\dot{x}'P_3\dot{x} - d\dot{x}(\alpha)'P_3\dot{x}(\alpha)] d\alpha\end{aligned}\quad (4.19)$$

Now, let the following notation:

$$\begin{aligned}\eta(t, \alpha) &= \begin{bmatrix} x(t)' & \tilde{x}(t)' & \tau(t)\dot{x}(\alpha)' \end{bmatrix}', \\ \Upsilon(x, \tilde{x}, \delta) &= \begin{bmatrix} A(x, \delta) & B(x, \delta)K(\tilde{x}) & \mathbf{0}_n \end{bmatrix}.\end{aligned}\quad (4.20)$$

Hence, $\dot{V}(t)$ in (4.19) can be cast as follows:

$$\dot{V}(x_t) = \frac{1}{d} \int_{t-d}^t \eta'(t, \alpha) [\Psi(x, \tilde{x}, \delta) + \Upsilon(x, \tilde{x}, \delta)'d P_3\Upsilon(x, \tilde{x}, \delta)] \eta(t, \alpha) d\alpha\quad (4.21)$$

where $\Psi(x, \tilde{x}, \delta) = [\Psi_{i,j}]$, $i, j = 1, 2, 3$, is a symmetric block matrix function whose nonzero blocks are given by:

$$\begin{aligned}\Psi_{11} &= \text{He}\{P_1A(x, \delta)\} + P_2, \quad \Psi_{12} = P_1B(x, \delta)K(\tilde{x}), \\ \Psi_{22} &= -(1 - \hat{\tau})P_2, \quad \Psi_{33} = -\frac{d}{\tau^2}P_3.\end{aligned}\quad (4.22)$$

It turns out that the entries of the auxiliary vector $\eta(t, \alpha)$ satisfy the following equality:

$$x(t) - \tilde{x}(t) - \int_{t-\tau(t)}^t \dot{x}(\alpha) d\alpha = \mathbf{0}.\quad (4.23)$$

To take the above relation into account, let the following notation:

$$\mathcal{J} = \begin{bmatrix} \mathbf{I}_n & -\mathbf{I}_n & -\mathbf{I}_n \end{bmatrix}.\quad (4.24)$$

Thus, the former equality can be written as follows, since $\tau(t) > 0$:

$$\int_{t-\tau(t)}^t \mathcal{J} \eta(t, \alpha) d\alpha = \mathbf{0}.\quad (4.25)$$

Next, note that

$$\int_{t-d}^t \left[\eta'_3(\alpha) \left(\frac{P_3}{\tau(t)^2} \right) \eta_3(\alpha) \right] d\alpha \geq \int_{t-\tau(t)}^t \left[\eta'_3(\alpha) \left(\frac{P_3}{\tau(t)^2} \right) \eta_3(\alpha) \right] d\alpha.\quad (4.26)$$

with $\eta_3(\alpha) = \tau(t)\dot{x}(\alpha)$, since $P_3 > 0$ and $d \geq \tau(t)$. Hence, the following upper-bound on $\dot{V}(x_t)$ holds:

$$\dot{V}(x_t) \leq \frac{1}{\tau(t)} \int_{t-\tau(t)}^t [\eta'(t, \alpha) (\hat{\Psi}(x, \tilde{x}, \delta) + \Upsilon'(x, \tilde{x}, \delta) d P_3 \Upsilon(x, \tilde{x}, \delta)) \eta(t, \alpha)] d\alpha \quad (4.27)$$

by noting that $P_3/\tau(t) \geq P_3/d$, where $\hat{\Psi}(\cdot) = [\hat{\Psi}_{i,j}]$, $i, j = 1, 2, 3$, is a block symmetric matrix whose nonzero blocks are as follows

$$\hat{\Psi}_{11} = \Psi_{11}, \quad \hat{\Psi}_{12} = \Psi_{12}, \quad \hat{\Psi}_{33} = -\frac{P_3}{d}. \quad (4.28)$$

Then, applying Lemma 4.1 to the right-hand side of (4.27) subject to (4.25) leads to:

$$\begin{aligned} \dot{V}(x_t) \leq \frac{1}{\tau(t)} \int_{t-\tau(t)}^t \eta'(t, \alpha) [\hat{\Psi}(x, \tilde{x}, \delta) + \Upsilon'(x, \tilde{x}, \delta) d P_3 \Upsilon(x, \tilde{x}, \delta) \\ + \text{He}\{R J\}] \eta(t, \alpha) d\alpha, \end{aligned} \quad (4.29)$$

where R is a free multiplier to be determined.

Now, it is proposed in the following a result to evaluate if conditions **C1**, **C2** and **C3** hold for all $x \in \mathcal{X}$ and $\delta \in \Delta$ in terms of a finite set of LMI constraints. To this end, it assumed that \mathcal{X} and Δ are given polytopic regions with known vertices with $0 \in \mathcal{X}$. In addition, \mathcal{X} will be either defined in terms of the convex hull of its vertices or in the following equivalent form:

$$\mathcal{X} = \{x \in \mathbb{R}^n : |c'_i x| \leq 1, i = 1, \dots, n_f\} \quad (4.30)$$

with $c_i \in \mathbb{R}^n$, $i = 1, \dots, n_f$, defining the faces of \mathcal{X} .

Theorem 4.1. Consider system (4.1) with (4.2)-(4.5). Let \mathcal{X} and Δ be given polytopic regions defining the state and uncertainty domains. Let d , h , σ and φ be given scalars. Suppose there exist matrices $P_1 > 0$; $P_2 > 0$; $P_3 > 0$; R_{i0} , R_{ij} , \tilde{R}_{ij} , \check{R}_{ik} , S_{i0} , S_{ij} and \check{S}_{ik} , for $i = 1, 2, 3$, $j = 1, \dots, n$ and $k = 1, \dots, n_\delta$; and L having appropriate dimensions and satisfying the following LMIs:

$$\Phi(x, \tilde{x}, \delta) + L\Omega_a(\tilde{x}) + \Omega_a(\tilde{x})L' < 0, \quad \forall (x, \tilde{x}, \delta) \in \mathcal{V}(\mathcal{X} \times \mathcal{X} \times \Delta) \quad (4.31)$$

$$A'(x, \delta)P_1 + P_1A(x, \delta) - 2\varphi P_1 < 0, \forall (x, \delta) \in \mathcal{V}(\mathcal{X} \times \Delta) \quad (4.32)$$

$$\Lambda(x, \delta) < 0, \forall (x, \delta) \in \mathcal{V}(\mathcal{X} \times \Delta) \quad (4.33)$$

$$\begin{bmatrix} \gamma & \gamma c'_i \\ \gamma c_i & P_1 \end{bmatrix} \geq 0, i = 1, \dots, n_f \quad (4.34)$$

$$\sigma P_1 - dP_2 \geq 0 \quad (4.35)$$

where $\Phi(\cdot) = [\Phi_{ij}]$, $i, j = 1, 2, 3, 4$, and $\Lambda(\cdot) = [\Lambda_{ij}]$, $i, j = 1, 2, 3$, are symmetric block matrices; $\gamma = \sigma + e^{2\varphi d}$; and

$$\begin{aligned} \Phi_{11} &= \text{He}\{P_1A(x, \delta) + R_1\} + P_2, \Phi_{12} = (R'_2 - R_1)N + P_1B(x, \delta)K_a, \\ \Phi_{13} &= R'_3 - R_1, \Phi_{14} = dA'(x, \delta)P_3, \Phi_{22} = -N'[R_2 + R'_2 + (1-h)P_2]N, \\ \Phi_{23} &= -N'(R_2 + R'_3), \Phi_{24} = dK'_aB'(x, \delta)P_3, \Phi_{33} = -(P_3/d + R_3 + R'_3), \\ \Phi_{34} &= \mathbf{0}_n, \Phi_{44} = -dP_3; \Lambda_{11} = \text{He}\{P_1A(x, \delta) + S_1\} + P_2 - 2\varphi P_1, \\ \Lambda_{12} &= (S'_2 - S_1), \Lambda_{13} = S'_3 - S_1, \Lambda_{14} = dA'(x, \delta)P_3, \\ \Lambda_{22} &= -(S_2 + S'_2 + (1-h)P_2), \Lambda_{23} = -(S_2 + S'_3), \Lambda_{24} = \mathbf{0}_n, \\ \Lambda_{33} &= -(P_3/d + S_3 + S'_3), \Lambda_{34} = \mathbf{0}_n, \Lambda_{44} = -dP_3, \\ K_a &= \begin{bmatrix} K_0 & K_1 & \cdots & K_n \end{bmatrix}, N = \begin{bmatrix} \mathbf{I}_n \\ \mathbf{0}_{n^2 \times n} \end{bmatrix}'; \Omega_a(\cdot) = \begin{bmatrix} \mathbf{0}_{n^2 \times n} & \Omega & \mathbf{0}_{n^2 \times 2n} \\ \mathbf{0}_{m \times n} & \mathcal{N}_a & \mathbf{0}_{m \times 2n} \end{bmatrix}, \\ \Omega &= \begin{bmatrix} \Pi' \\ -\mathbf{I}_{n^2} \end{bmatrix}', \Pi = \begin{bmatrix} \tilde{x}_1 \otimes \mathbf{I}_n \\ \vdots \\ \tilde{x}_n \otimes \mathbf{I}_n \end{bmatrix}, \mathcal{N}_a = \begin{bmatrix} \mathbf{0}_{m \times n} & \mathcal{N} \end{bmatrix}, \\ \mathcal{N} &= \begin{bmatrix} \tilde{x}_2 \mathbf{I}_n & -\tilde{x}_1 \mathbf{I}_n & \mathbf{0}_n & \cdots & \mathbf{0}_n \\ \mathbf{0}_n & \tilde{x}_3 \mathbf{I}_n & -\tilde{x}_2 \mathbf{I}_n & \cdots & \mathbf{0}_n \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0}_n & \cdots & \mathbf{0}_n & \tilde{x}_n \mathbf{I}_n & -\tilde{x}_{n-1} \mathbf{I}_n \end{bmatrix} \in \mathbb{R}^{m \times n^2}; \\ R_i &= R_{i0} + \sum_{j=1}^n x_j R_{ij} + \sum_{j=1}^n \tilde{x}_j \tilde{R}_{ij} + \sum_{j=1}^{n_\delta} \delta_j \check{R}_{ij}, i = 1, 2, 3; \\ S_i &= S_{i0} + \sum_{j=1}^n x_j S_{ij} + \sum_{j=1}^{n_\delta} \delta_j \check{S}_{ij}, i = 1, 2, 3. \end{aligned} \quad (4.36)$$

Then, the system origin is locally robustly stable. Moreover, for any $x(0) \in \mathcal{R}_0$ and $\delta \in \Delta$, $x(t) \in \mathcal{R}$ for all $t \geq 0$ and $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

Proof. Suppose that (4.31)-(4.35) are satisfied. Then, from convexity arguments, (4.31)-(4.33) are respectively satisfied for all $(x, \tilde{x}, \delta) \in \mathcal{X} \times \mathcal{X} \times \Delta$, $(x, \delta) \in \mathcal{X} \times \Delta$ and $(x, \delta) \in \mathcal{X} \times \Delta$. In the following, it will be shown respectively that **C1**, **C2** and **C3** holds.

C1: note that the matrix inequality in (4.31) pre- and post-multiplied by $\text{diag}\{\mathbf{I}_n, \Pi'_a, \mathbf{I}_n, \mathbf{I}_n\}$ and its transpose, respectively, leads to

$$\begin{bmatrix} \hat{\Psi}(x, \tilde{x}, \delta) + \text{He}\{RJ\} & dY'(x, \tilde{x}, \delta)P_3 \\ dY(x, \tilde{x}, \delta)P_3 & -dP_3 \end{bmatrix} < 0, \\ \forall (x, \tilde{x}, \delta) \in \mathcal{X} \times \mathcal{X} \times \Delta, \quad (4.37)$$

by noting that $K(\tilde{x}) = K_a \Pi_a$, where $\Pi_a = [\mathbf{I}_n \ \Pi']'$ and $R = [R'_1 \ R'_2 \ R'_3]'$.

Applying the Schur's complement to (4.37), and then pre- and post-multiplying it by $\eta'(t, \alpha)$ and $\eta(t, \alpha)$, respectively, yields

$$\eta'(t, \alpha) [\hat{\Psi}(x, \tilde{x}, \delta) + Y'(x, \tilde{x}, \delta) dP_3 Y(x, \tilde{x}, \delta) + \text{He}\{RJ\}] \eta(t, \alpha) < 0, \\ \forall (x, \tilde{x}, \delta) \in \mathcal{X} \times \mathcal{X} \times \Delta, \ \eta(t, \alpha) \neq 0. \quad (4.38)$$

Now, integrate the above with respect to α from $t - \tau(t)$ to t and divide it by $\tau(t)$. Hence, the following is obtained

$$\frac{1}{\tau(t)} \int_{t-\tau(t)}^t \eta'(t, \alpha) [\hat{\Psi}(x, \tilde{x}, \delta) + Y'(x, \tilde{x}, \delta) dP_3 Y(x, \tilde{x}, \delta)] \eta(t, \alpha) d\alpha + \\ \frac{1}{\tau(t)} \int_{t-\tau(t)}^t \eta'(t, \alpha) [RJ + J'R'] \eta(t, \alpha) d\alpha < 0, \\ \forall (x, \tilde{x}, \delta) \in \mathcal{X} \times \mathcal{X} \times \Delta, \ \eta(t, \alpha) \neq 0, \quad (4.39)$$

which in turn yields

$$\dot{V}(t) < 0, \ \forall (x, \delta) \in \mathcal{X} \times \Delta, \ t \geq d, \quad (4.40)$$

taking (4.25), (4.27) and (4.6) into account.

C2: consider the matrix inequality in (4.32). Pre- and post-multiplying it by x' and x results in:

$$\dot{V}_1(x) - 2\varphi V_1(x) < 0, \quad \forall (x, \delta) \in \mathcal{X} \times \Delta, \quad t \in [0, d], \quad (4.41)$$

taking (4.10) into account and by noting that $\dot{x} = A(x, \delta)x$ for $t \in [0, \tau(t))$ with $d \geq \tau(t)$.

C3: note that the matrix inequality in (4.33) can be cast as follows

$$\begin{bmatrix} \Theta(x, \delta) + \text{He}\{S\mathcal{J}\} & d\Xi'(x, \delta)P_3 \\ d\Xi(x, \delta)P_3 & -dP_3 \end{bmatrix} < 0, \\ \forall (x, \delta) \in \mathcal{X} \times \mathcal{X} \times \Delta, \quad (4.42)$$

where $\Xi(x, \delta) = [A(x, \delta) \quad \mathbf{0}_n \quad \mathbf{0}_n]$; $S = [S'_1 \quad S'_2 \quad S'_3]$; and $\Theta(x, \delta)$ is 3×3 block matrix, i.e., $\Theta(\cdot) = [\Theta_{ij}(\cdot)]$, $i, j = 1, 2, 3$, whose nonzero block elements are:

$$\Theta_{11} = \text{He}\{P_1 A(x, \delta)\} + P_2 - 2\varphi P_1, \quad \Theta_{22} = -(1-h)P_2, \quad \Theta_{33} = -P_3/d.$$

Applying the Schur's complement to (4.42), and then pre- and post-multiplying it by $\eta'(t, \alpha)$ and $\eta(t, \alpha)$, respectively, yields

$$\eta'(t, \alpha) [\Theta(x, \delta) + \Xi'(x, \delta) d P_3 \Xi(x, \delta) + \text{He}\{S\mathcal{J}\}] \eta(t, \alpha) < 0, \\ \forall (x, \delta) \in \mathcal{X} \times \Delta, \quad \eta(t, \alpha) \neq 0. \quad (4.43)$$

Then, integrating the above with respect to α from $t - \tau(t)$ to t and dividing it by $\tau(t)$ leads to

$$\frac{1}{\tau(t)} \int_{t-\tau(t)}^t \eta'(t, \alpha) [\Theta(x, \delta) + \Xi'(x, \delta) d P_3 \Xi(x, \delta)] \eta(t, \alpha) d\alpha < 0, \\ \forall (x, \delta) \in \mathcal{X} \times \Delta, \quad \eta(t, \alpha) \neq 0, \quad (4.44)$$

since

$$\int_{t-\tau(t)}^t \eta'(t, \alpha) [S\mathcal{J} + \mathcal{J}'S'] \eta(t, \alpha) d\alpha = 0.$$

Noting that $P_3/\tau(t) \geq P_3/d$, the following is obtained from (4.44):

$$\frac{1}{d} \int_{t-d}^t \eta'(t, \alpha) [\Theta(x, \delta) + \Xi'(x, \delta) d P_3 \Xi(x, \delta)] \eta(t, \alpha) d\alpha < 0, \\ \forall (x, \delta) \in \mathcal{X} \times \Delta, \quad \eta(t, \alpha) \neq 0. \quad (4.45)$$

Next recall that $\dot{x} = A(x, \delta)x$ for $t \in [0, \tau(t))$ with $\tau(t) \leq d$. Thus, in view of (4.10) and (4.35), it follows that:

$$\begin{aligned} \dot{V}(t) - 2\varphi V_1(x) &< 0, \quad \forall x_0, x(s) \in \mathcal{X}, \\ s \in [t-d, t], \quad t &\in [0, d), \quad \delta \in \Delta. \end{aligned} \quad (4.46)$$

Finally, applying the Schur's complement to (4.34) leads to $1 - c'_j(\gamma P_1^{-1})c_j \geq 0$, $j = 1, \dots, n_f$, which from (BOYD et al., 1994) implies $\mathcal{R} \subset \mathcal{X}$ completing the proof. \square

Remark 4.1. *For unstable open-loop systems, observe that φ is an exponential bound on the state trajectory norm over the interval $[0, d)$. An initial estimate of φ might be obtained by considering the largest eigenvalue of a linear approximation of the open-loop system.*

4.2.2 A Convex Solution to Problem P1

In this section, Theorem 4.1 is applied to obtain a solution to problem **P1** assuming that a state feedback gain $K(x)$ which locally stabilizes the closed-loop system with no input delay is known *a priori*. Firstly, it is assumed that a bound on the maximum admissible delay is given. Then, it is addressed the case of a given set of initial conditions (which will be referred to the set of guaranteed initial conditions).

When an upper-bound on the maximal input delay and its variation are given, it is often desirable to estimate the largest set of admissible initial conditions \mathcal{R}_0 assuming a given polytopic state space domain \mathcal{X} . For instance, the volume maximization of \mathcal{R}_0 can be approximately obtained by minimizing the trace of P_1 (since the trace of P_1^{-1} is the sum of the squared semi-axis lengths of \mathcal{R}_0) leading to the following optimization problem:

$$\min_{\varphi, \sigma, P_1, \dots, L} \text{trace}(P_1) \text{ subject to (4.31)-(4.35)}. \quad (4.47)$$

Note that the matrix inequalities in (4.31)-(4.35) become LMIs when σ and φ are given. Thus, the optimization problem in (4.47) can be

numerically solved via semi-definite programming applying a gridding technique over σ and φ .

On the other hand, it might be of interest to estimate the maximum admissible delay d ensuring robust regional stability when a set of guaranteed initial conditions $\tilde{\mathcal{R}}_0$ is given. In this chapter, the set $\tilde{\mathcal{R}}_0$ will be characterized as follows

$$\tilde{\mathcal{R}}_0 = \{x \in \mathbb{R}^n : x' P_0 x \leq 1\}, \quad (4.48)$$

where $P_0 > 0$ is a given matrix defining the size and shape of $\tilde{\mathcal{R}}_0$ such that $\tilde{\mathcal{R}}_0 \subset \mathcal{X}$. In this scenario, it is proposed the following optimization problem:

$$\max_{d, \varphi, \sigma, P_1, \dots, L} d \text{ subject to (4.31)-(4.35) and } P_0 - P_1 \geq 0. \quad (4.49)$$

Notice that the constraint $P_0 - P_1 \geq 0$ implies $\tilde{\mathcal{R}}_0 \subset \mathcal{R}_0$. In addition, similarly to the former optimization problem, the constraints (4.31)-(4.35) become LMIs if a gridding technique over σ , φ and d is applied.

4.2.3 Numerical Example 1

Consider the following uncertain input delayed nonlinear quadratic system:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 + 0.2x_2 \\ 2x_1 + x_2 & x_1 + \delta \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 + \delta \end{bmatrix} u(t - \tau(t)), \\ x &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \end{aligned} \quad (4.50)$$

where $x \in \mathbb{R}^2$ is the state, $\delta \in \mathbb{R}$ is a bounded parameter and $u \in \mathbb{R}$ is the control input being subject to a time-varying delay $\tau(t)$.

The origin of the above system is open-loop unstable and the following locally stabilizing control law has been proposed in reference (COUTINHO; DE SOUZA, 2012):

$$u = K(x)x, \quad K(x) = K_0 + \sum_{i=1}^2 x_i K_i, \quad (4.51)$$

considering an uncertainty free system with no input delay, where

$$\begin{aligned} K_0 &= \begin{bmatrix} -6.86 & -5.38 \end{bmatrix}, K_1 = \begin{bmatrix} -1.54 & -0.94 \end{bmatrix}, \\ K_2 &= \begin{bmatrix} -1.04 & -0.09 \end{bmatrix}. \end{aligned} \quad (4.52)$$

This example focus on evaluating the local stability of the closed loop system assuming that:

- (i) The uncertain parameter δ is bounded to $\Delta = [-0.35 \ 0.35]$;
- (ii) The time-varying delay $\tau(t)$ satisfies $0 < \tau(t) \leq d$ and $\dot{\tau}(t) \leq h$, $\forall t \geq 0$, where d and h are given parameters; and
- (iii) The state-space domain is defined by

$$\mathcal{X} = \{x \in \mathbb{R}^2 : |x_i| \leq 0.5, i = 1, 2\}. \quad (4.53)$$

Firstly, optimization problem (4.47) is applied to determine maximized estimates of the set of admissible initial conditions \mathcal{R}_0 as defined in (4.15) for different values of d and h leading to results show in Table 3. The values of φ and σ showed in this table were obtained in order to minimize the trace of P_1 (i.e., approximately maximizing the size of \mathcal{R}_0). Notice, in this case, that larger input delays yield smaller regions of admissible initial conditions. In particular, Fig. 10 shows the set of admissible initial conditions \mathcal{R}_0 , a stable trajectory starting at the boundary of \mathcal{R}_0 , the set of reachable states \mathcal{R} as well as the state-space domain \mathcal{X} considering the parameters given in the last row of Table 3.

Then, optimization problem (4.49) is applied to obtain a lower bound on the maximal admissible delay d considering a given bound h on the delay variation and different sets of guaranteed initial conditions $\tilde{\mathcal{R}}_0$ (defined in terms of the matrix P_0) leading to the results given in Table 4. It turns out in this case that larger input delays are admissible at the cost of smaller sets of guaranteed initial conditions. Fig. 11 shows the obtained sets \mathcal{R}_0 and \mathcal{R} as well as the given sets $\tilde{\mathcal{R}}_0$ and \mathcal{X}

d	h	σ	φ	$\mathbf{trace}(P_1)$
0.01	1.0	0.01	2.0	13.39
0.05	1.0	0.50	2.0	24.86
0.05	0.5	0.01	2.0	17.81
0.10	0.5	0.20	3.0	34.49
0.17	0.5	0.01	2.7	95.31

Table 3 – Trace of P_1 for different values of d , h , σ and φ .

considering the parameters values in the 2nd row of Table 4. These solutions were obtained by performing a grid search over the parameters h , σ , and φ .

P_0	d	h	σ	φ	$\mathbf{trace}(P_1)$
$10I$	0.01	1.0	0.01	2.7	13.55
$15I$	0.05	0.5	0.01	2.7	19.09
$20I$	0.05	1.0	0.30	2.7	23.25
$40I$	0.10	0.5	0.30	2.7	33.50
$60I$	0.15	1.0	0.01	2.7	60.47

Table 4 – Lower bounds on d for different values of P_0 , h , σ and φ .

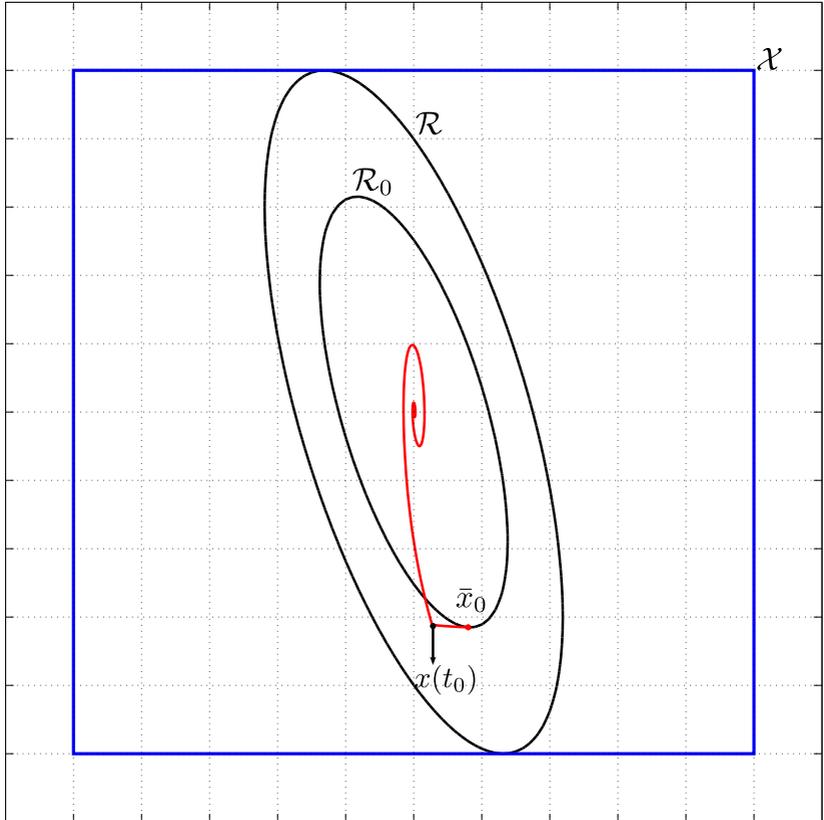


Figure 10 – A stable trajectory and the regions \mathcal{R}_0 , \mathcal{R} and \mathcal{X} considering 4.47, $d = 0.17$ and $h = 0.5$.

4.3 Regional Stabilization

This section addresses the design of a state feedback gain $K(\bar{x})$ which is an affine matrix function of \bar{x} such that the closed-loop system of (4.6) is regionally stable for all $\delta \in \Delta$. The proposed design will be based on Theorem 4.1 together with a parametrization $K(\bar{x}) = F(\bar{x})P_1$, with $F(\bar{x})$ being an affine matrix function of \bar{x} to be determined.

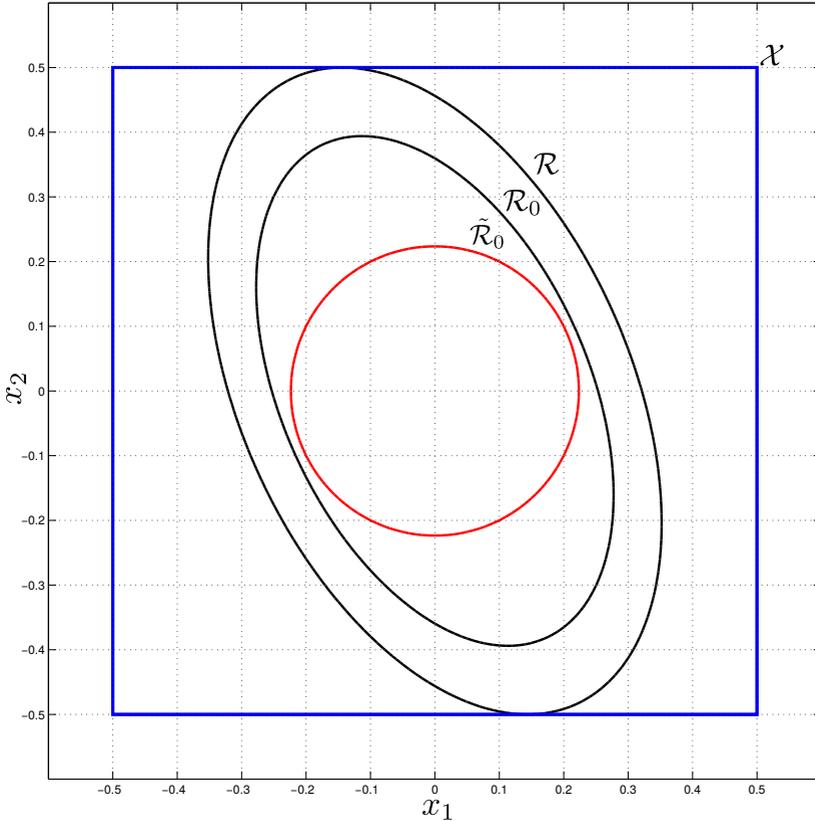


Figure 11 – Estimates of \mathcal{R}_0 and \mathcal{R} for given sets $\tilde{\mathcal{R}}_0$ and \mathcal{X} .

Thus, consider the following similarity transformations:

$$\xi_1(t) = P_1 x(t), \quad \xi_2(t) = P_1 \bar{x}(t), \quad \xi_3(\alpha) = \tau P_1 \dot{x}(\alpha), \quad (4.54)$$

and let

$$\xi(t, \alpha) = \begin{bmatrix} \xi_1'(t) & \xi_2'(t) & \xi_3'(\alpha) \end{bmatrix}', \quad Q_1 = P_1^{-1}, \quad Q_2 = Q_1 P_2 Q_1. \quad (4.55)$$

In view of above definitions, observe that (4.29) can be recast as

follows:

$$\begin{aligned} \dot{V}(t) \leq & \frac{1}{\tau(t)} \int_{t-\tau(t)}^t \xi'(t, \alpha) [\check{\Psi}(x, \tilde{x}, \delta) + \check{Y}'(x, \tilde{x}, \delta) d P_3 \check{Y}(x, \tilde{x}, \delta) \\ & + \text{He}\{R \mathcal{J}\}] \xi(t, \alpha) d\alpha, \end{aligned} \quad (4.56)$$

where R is a free multiplier added as a null term that will make it possible to use Lemma 4.1 at the end. $\check{\Psi}(\cdot) = [\check{\Psi}_{ij}]$, $i, j = 1, 2, 3$; and

$$\begin{aligned} \check{\Psi}_{11} &= \text{He}\{A(x, \delta)Q_1\} + Q_2, \quad \check{\Psi}_{12} = B(x)F(\tilde{x}), \\ \check{\Psi}_{13} &= \mathbf{0}_n, \quad \check{\Psi}_{22} = -(1-h)Q_2, \quad \check{\Psi}_{23} = \mathbf{0}_n, \quad \check{\Psi}_{33} = -\frac{Q_1 P_3 Q_1}{d}; \\ \check{Y}(x, \tilde{x}, \delta) &= \begin{bmatrix} A(x, \delta)Q_1 & B(x, \delta)F(\tilde{x}) & \mathbf{0}_n \end{bmatrix}. \end{aligned} \quad (4.57)$$

Next, to obtain a numerical and tractable condition, notice in light of Lemma 4.2 that

$$P_3 \leq (G + G' - G'P_3G)^{-1}$$

for any nonsingular G and $P_3 > 0$. Hence, the following holds for $G = Q_1$ and $Q_3 = Q_1 P_3 Q_1$:

$$P_3 \leq (2Q_1 - Q_3)^{-1}. \quad (4.58)$$

Then, the following upper-bound on the right-hand side of (4.56) is obtained:

$$\begin{aligned} \dot{V}(x_t) \leq & \frac{1}{\tau(t)} \int_{t-\tau(t)}^t \xi'(t, \alpha) \left[\check{\Psi}(x, \tilde{x}, \delta) + d \check{Y}'(x, \tilde{x}, \delta) (2Q_1 - Q_3)^{-1} \check{Y}(x, \tilde{x}, \delta) \right. \\ & \left. + \text{He}\{R \mathcal{J}\} \right] \xi(t, \alpha) d\alpha. \end{aligned} \quad (4.59)$$

Now, it is proposed in the sequel an LMI-based result for designing a state feedback gain $K(\tilde{x})$ such that the closed-loop system of (4.6) is regionally stable for all $\delta \in \Delta$.

Theorem 4.2. *Consider system (4.1) with (4.2)-(4.4). Let \mathcal{X} and Δ be given polytopic regions defining the state and uncertainty domains. Let d , h , σ and φ be given scalars. Suppose there exist matrices $Q_1 > 0$; $Q_2 > 0$;*

$Q_3 > 0$; $F_0, F_j, R_{i0}, R_{ij}, \tilde{R}_{ij}, \tilde{R}_{ik}, S_{i0}, S_{ij}$ and \check{S}_{ik} , for $i = 1, 2, 3, j = 1, \dots, n$ and $k = 1, \dots, n_\delta$; and L satisfying the following LMIs:

$$\check{\Phi}(x, \tilde{x}, \delta) + L\Omega_a(\tilde{x}) + \Omega_a(\tilde{x})'L' < 0, \quad \forall (x, \tilde{x}, \delta) \in \mathcal{V}(\mathcal{X} \times \mathcal{X} \times \Delta) \quad (4.60)$$

$$Q_1A'(x, \delta) + A(x, \delta)Q_1 - 2\varphi Q_1 < 0, \quad \forall (x, \delta) \in \mathcal{V}(\mathcal{X} \times \Delta) \quad (4.61)$$

$$\check{\Lambda}(x, \delta) < 0, \quad \forall (x, \delta) \in \mathcal{V}(\mathcal{X} \times \Delta) \quad (4.62)$$

$$1 - \gamma c_i' Q_1 c_i \geq 0, \quad i = 1, \dots, n_f \quad (4.63)$$

$$\sigma Q_1 - dQ_2 \geq 0 \quad (4.64)$$

where $\check{\Phi}(\cdot) = [\check{\Phi}_{ij}]$, $i, j = 1, 2, 3, 4$, and $\check{\Lambda}(\cdot) = [\check{\Lambda}_{ij}]$, $i, j = 1, 2, 3$, are symmetric block matrices; $\gamma = \sigma + e^{2\varphi d}$; and

$$\begin{aligned} \check{\Phi}_{11} &= \text{He}\{A(x, \delta)Q_1 + \check{R}_1\} + Q_2, \quad \check{\Phi}_{12} = (\check{R}'_2 - \check{R}_1)N + B(x, \delta)F_a, \\ \check{\Phi}_{13} &= \check{R}'_3 - \check{R}_1, \quad \check{\Phi}_{14} = Q_1A'(x, \delta), \quad \check{\Phi}_{22} = -N'[\check{R}_2 + \check{R}'_2 + (1-h)Q_2]N, \\ \check{\Phi}_{23} &= -N'(\check{R}_2 + \check{R}'_3), \quad \check{\Phi}_{24} = F_a'B'(x, \delta), \quad \check{\Phi}_{33} = -(Q_3/d + \check{R}_3 + \check{R}'_3), \\ \check{\Phi}_{34} &= \mathbf{0}_n, \quad \check{\Phi}_{44} = (Q_3 - 2Q_1)/d; \quad \check{\Lambda}_{11} = \text{He}\{A(x, \delta)Q_1 + \check{S}_1\} + Q_2 - 2\varphi Q_1, \\ \check{\Lambda}_{12} &= \check{S}'_2 - \check{S}_1, \quad \check{\Lambda}_{13} = \check{S}'_3 - \check{S}_1, \quad \check{\Lambda}_{14} = Q_1A'(x, \delta), \\ \check{\Lambda}_{22} &= -[\check{S}_2 + \check{S}'_2 + (1-h)Q_2], \quad \check{\Lambda}_{23} = -(\check{S}_2 + \check{S}'_3), \quad \check{\Lambda}_{24} = \check{\Lambda}_{34} = \mathbf{0}_n, \\ \check{\Lambda}_{33} &= -(Q_3/d + \check{S}_3 + \check{S}'_3), \quad \check{\Lambda}_{44} = (Q_3 - 2Q_1)/d, \\ F_a &= \begin{bmatrix} F_0 & F_1 & \cdots & F_n \end{bmatrix}, \\ \check{R}_i &= R_{i0} + \sum_{j=1}^n x_j R_{ij} + \sum_{j=1}^n \tilde{x}_j \tilde{R}_{ij} + \sum_{j=1}^{n_\delta} \delta_j \check{R}_{ij}, \quad i = 1, 2, 3; \\ \check{S}_i &= S_{i0} + \sum_{j=1}^n x_j S_{ij} + \sum_{j=1}^{n_\delta} \delta_j \check{S}_{ij}, \quad i = 1, 2, 3. \end{aligned} \quad (4.65)$$

Then, the origin of the closed-loop system in (4.6), with

$$K(\tilde{x}) = K_0 + \sum_{i=1}^n \tilde{x}_i K_i, \quad K_j = F_j P_1, \quad P_1 = Q_1^{-1}, \quad j = 0, 1, \dots, n, \quad (4.66)$$

is locally robustly stable. Moreover, for any $x(0) \in \mathcal{R}_0$ and $\delta \in \Delta$, $x(t) \in \mathcal{R}$ for all $t \geq 0$ and $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

Proof. Suppose that (4.60)-(4.62) are satisfied, then from convexity arguments they are respectively satisfied for all $(x, \tilde{x}, \delta) \in \mathcal{X} \times \mathcal{X} \times \Delta$, $(x, \delta) \in \mathcal{X} \times \Delta$ and $(x, \delta) \in \mathcal{X} \times \Delta$.

Firstly, it will be shown that (4.60) implies $\dot{V}(t) < 0$, $\forall (x, \delta) \in \mathcal{X} \times \Delta$. Hence, pre- and post-multiplying (4.60) by $\text{diag}\{\mathbf{I}_n, \Pi'_a, \mathbf{I}_n, \mathbf{I}_n\}$ and its transpose respectively yields:

$$\bar{\Phi}(x, \tilde{x}, \delta) < 0, (x, \tilde{x}, \delta) \in \mathcal{X} \times \mathcal{X} \times \Delta, \quad (4.67)$$

where $\Pi_a = [\mathbf{I}_n \quad \Pi']'$; $\bar{\Phi} = [\bar{\Phi}_{ij}]$, $i, j = 1, 2, 3, 4$, and

$$\begin{aligned} \bar{\Phi}_{11} &= \check{\Phi}_{11}, \quad \bar{\Phi}_{12} = (\check{R}'_2 - \check{R}'_1) + B(x, \delta)F_a\Pi_a, \quad \bar{\Phi}_{13} = \check{\Phi}_{13}, \quad \bar{\Phi}_{14} = \check{\Phi}_{14}, \\ \bar{\Phi}_{22} &= -[\check{R}_2 + \check{R}'_2 + (1-h)Q_2], \quad \bar{\Phi}_{23} = -(\check{R}_2 + \check{R}'_3), \quad \bar{\Phi}_{24} = \Pi'_a F'_a B'(x, \delta), \\ \bar{\Phi}_{33} &= \check{\Phi}_{33}, \quad \bar{\Phi}_{34} = \mathbf{0}_n, \quad \bar{\Phi}_{44} = \check{\Phi}_{44}. \end{aligned}$$

Let $R = [\check{R}'_1 \quad \check{R}'_2 \quad \check{R}'_3]'$. Then, applying the Schur's complement to $\bar{\Phi}(x, \tilde{x}, \delta) < 0$ leads to:

$$\begin{aligned} \check{\Psi}(x, \tilde{x}, \delta) + d\check{\Upsilon}'(x, \tilde{x}, \delta)(2Q_1 - Q_3)^{-1}\check{\Upsilon}(x, \tilde{x}, \delta) + \text{He}\{R\mathcal{J}\} < 0, \\ \forall (x, \tilde{x}, \delta) \in \mathcal{X} \times \mathcal{X} \times \Delta, \end{aligned} \quad (4.68)$$

by noting that $F_a\Pi_a = F(x)$. Hence, taking (4.59) into account, the following is obtained:

$$\dot{V}(t) < 0, \forall (x, \tilde{x}, \delta) \in \mathcal{X} \times \mathcal{X} \times \Delta. \quad (4.69)$$

Next, it is shown in the sequel that (4.61)-(4.64) imply (4.32)-(4.35).

- (i) (4.61) \rightarrow (4.32): pre- and post-multiplying (4.61) by $Q_1^{-1} = P_1$ yields (4.32).
- (ii) (4.62) \rightarrow (4.33): let $\bar{\Lambda}(x, \delta)$ be equal to $\check{\Lambda}(x, \delta)$ except by the (4,4) block which is replaced by $-(dP_3)^{-1}$. By noting that $Q_3 -$

$2Q_1 \geq -P_3^{-1}$ (from Lemma 4.2 and $Q_3 = Q_1 P_3 Q_1$), it follows that $\bar{\Lambda}(x, \delta) < 0, \forall (x, \delta) \in \mathcal{X} \times \Delta$. Thus, pre- and post-multiplying $\bar{\Lambda}(x, \delta) < 0$ by $\text{diag}\{P_1, P_1, P_1, d P_3\}$ implies that (4.33) holds for all $(x, \delta) \in \mathcal{X} \times \Delta$, with $S_i = P_1 \check{S}_i P_1, i = 1, 2, 3$.

1. (4.63) \rightarrow (4.34): multiplying (4.63) by γ and then applying the Schur's complement yields (4.34).
2. (4.64) \rightarrow (4.35): pre- and post-multiplying (4.64) by $Q_1^{-1} = P_1$ leads to (4.35), since by definition $Q_2 = Q_1 P_2 Q_1$.

The rest of this proof follows directly from Theorem 4.1. □

4.3.1 Solving Problem P2

Theorem 4.2 can be applied for designing the state feedback gain $K(x)$, as defined in (4.5), in order to locally stabilize the closed-loop system of (4.6) while either maximizing the size of the set of admissible initial conditions (for a given admissible delay) or to maximize a bound on the maximal admissible delay (for a given set of guaranteed initial conditions). It is assumed in both cases, that a polytopic state space domain \mathcal{X} is known *a priori*.

Firstly, assume that the maximal input delay d and its variation h are given. Then, in order to obtain a maximized estimate of the set of admissible initial conditions \mathcal{R}_0 , the following optimization problem is proposed:

$$\max_{\varphi, \sigma, Q_1, \dots, L} \log \det(Q_1) \text{ subject to (4.60)-(4.64)}. \quad (4.70)$$

The above optimization problem can be numerically solved via semi-definite programming applying a gridding technique over σ and φ . In addition, notice that the maximization of $\log \det(Q_1)$ implies the volume maximization of \mathcal{R}_0 .

Now, assume that the set of guaranteed initial conditions $\tilde{\mathcal{R}}_0$ as defined in (4.48) is known. Thus, the following optimization problem is

proposed to obtain a lower-bound on the maximal admissible delay:

$$\max_{d, \varphi, \sigma, Q_1, \dots, L} d \text{ subject to (4.60)-(4.64) and } \begin{bmatrix} P_0 & \mathbf{I}_n \\ \mathbf{I}_n & Q_1 \end{bmatrix} \geq 0. \quad (4.71)$$

Notice that the latter matrix inequality implies $\tilde{\mathcal{R}}_0 \subset \mathcal{R}_0$. In addition, similarly to the optimization problem of (4.70), a numerical solution in terms of LMI constraints can be searched for by means of a gridding technique over σ , φ and d .

4.3.2 Numerical Example 2

Consider the uncertain input delayed nonlinear quadratic system of (4.50). This system is open-loop unstable and a local stability analysis has been performed in Example 1, Section 4.2.3, considering the locally stabilizing control law proposed in reference (COUTINHO; DE SOUZA, 2012) which did not consider neither input delay nor parameter uncertainty.

In this example, optimization problems (4.70) and (4.71) are applied for control purposes considering the same setup of Example 1. That is: (i) the state space domain \mathcal{X} is as defined in (4.53); (ii) the uncertain parameter δ is bounded to $\Delta = [-0.035 \ 0.035]$; and (iii) the time-varying delay $\tau(t)$ satisfies $0 < \tau(t) \leq d$ and $\dot{\tau}(t) \leq h$, $\forall t \geq 0$, where d and h are supposed to be given parameters. Then, these results are compared to the ones obtained in Example 1 which have considered the stabilizing controller of (4.51) and (4.52).

Firstly, the optimization problem (4.70) is applied to determine a maximized estimate of the set admissible initial conditions \mathcal{R}_0 as defined in (4.15) considering the largest admissible delay d considered in Example 1. A gridding technique has been applied to determine the values of φ and σ in order to maximize $\log \det(Q_1)$ (i.e., the volume maximization of \mathcal{R}_0). Fig. 12 shows the estimates of the set of admissible initial conditions \mathcal{R}_0 and of the set of reachable states \mathcal{R} considering:

$$d = 0.17, \quad h = 0.5, \quad \sigma = 0.01, \quad \varphi = 2.7. \quad (4.72)$$

Notice that the size of admissible initial conditions \mathcal{R}_0 is much larger than the one given in Fig. 10. For comparison purposes, the controller designed by means of optimization problem (4.70) has led to trace($P_1 = Q_1^{-1}$) = 20.21 contrasting with the value 95.31 obtained in Example 1.

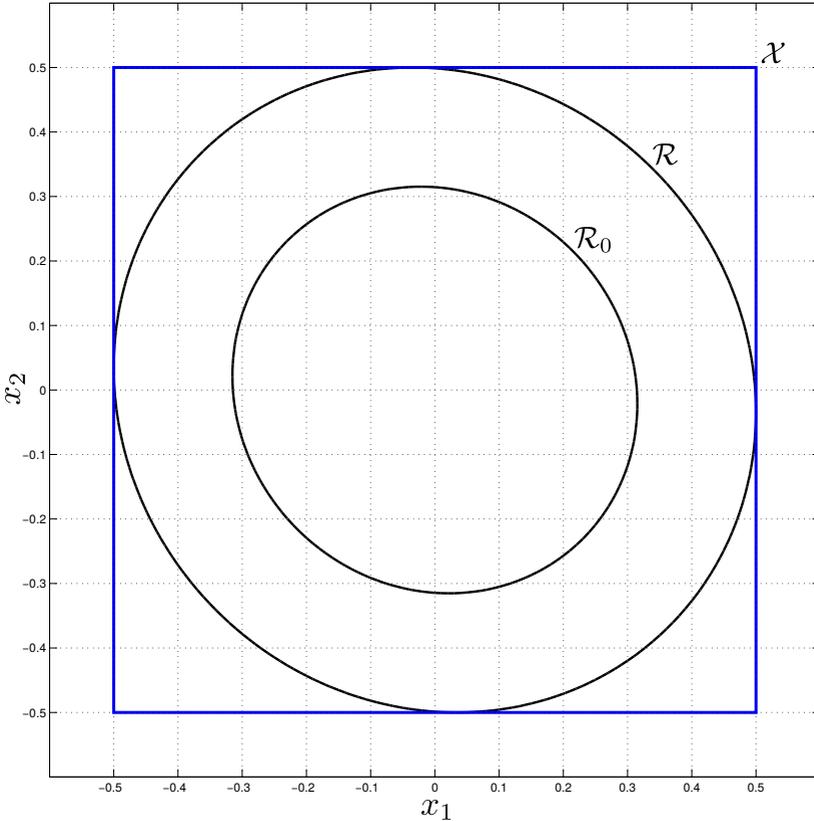


Figure 12 – Estimates of the sets \mathcal{R}_0 and \mathcal{R} for $d = 0.17$ considering a controller designed by means of (4.70).

In addition, it is possible to find feasible solutions for the optimization problem (4.70) considering a larger bound on the delay size or a larger state space domain \mathcal{X} . For instance, Fig. 13 shows estimates of \mathcal{R}_0 and \mathcal{R} considering a maximum delay of $d = 0.72$ which is much

larger than the maximum admissible delay of $d = 0.17$ obtained in Example 1 for the same scenario. Observe as expected that larger values of d yields smaller admissible initial conditions. On the other hand, considering a bound $d = 0.17$ on the delay size and the following state space domain:

$$\mathcal{X} = \{x \in \mathbb{R}^2 : |x_i| \leq 0.92, i = 1, 2\}, \quad (4.73)$$

it is obtained a larger estimate of the set of admissible initial conditions as illustrate in Fig. 14. It is also shown in the latter figure a closed-loop state trajectory starting in $x_0 = [-0.3 \ 0.56]'$ at the boundary of \mathcal{R}_0 . Then, optimization problem (4.71) is applied to estimate a lower bound on the maximal admissible delay d considering a given bound h on the delay variation, the state space domain defined in (4.73) and the following set of guaranteed initial conditions:

$$\tilde{\mathcal{R}}_0 = \{x \in \mathbb{R}^2 : x'P_0x \leq 1\}, \quad P_0 = 20\mathbf{I}_2. \quad (4.74)$$

Fig. 15 shows the obtained estimates of \mathcal{R}_0 and \mathcal{R} as well as the given sets $\tilde{\mathcal{R}}_0$ and \mathcal{X} , considering $d = 0.18$, $h = 1.00$, $\sigma = 0.30$ and $\varphi = 2.70$.

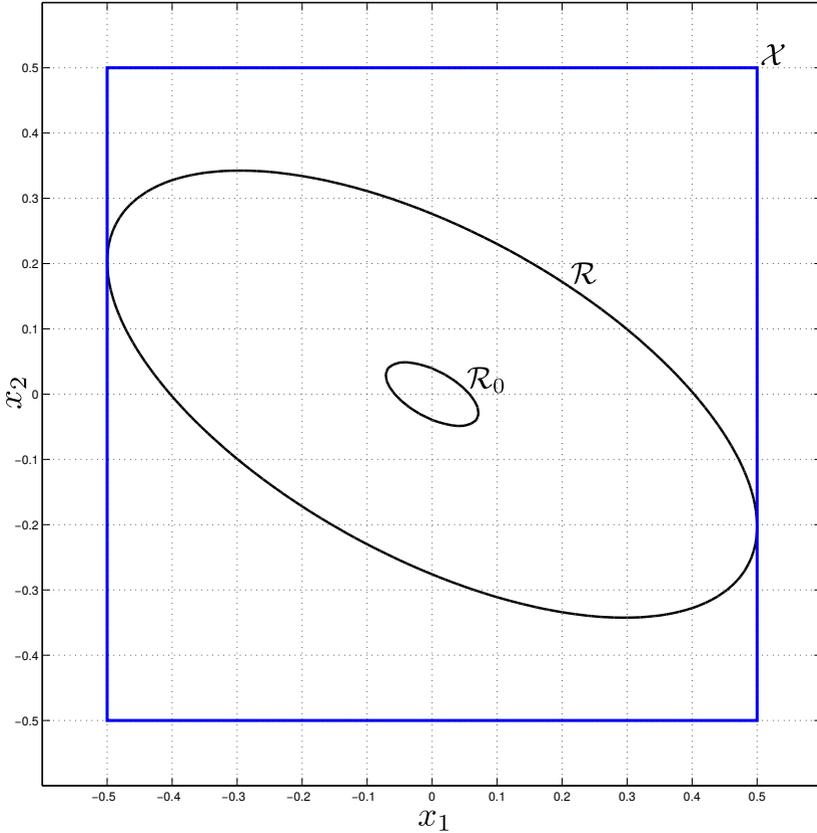


Figure 13 – Estimates of the sets \mathcal{R}_0 and \mathcal{R} for $d = 0.72$ considering a controller designed by means of (4.70).

4.4 Application to a Poiseuille Flow

A detailed system description and modeling are presented in Chapter 3, Section 3.1. In this section, we present only some details of the modeling as well as some simplifications.

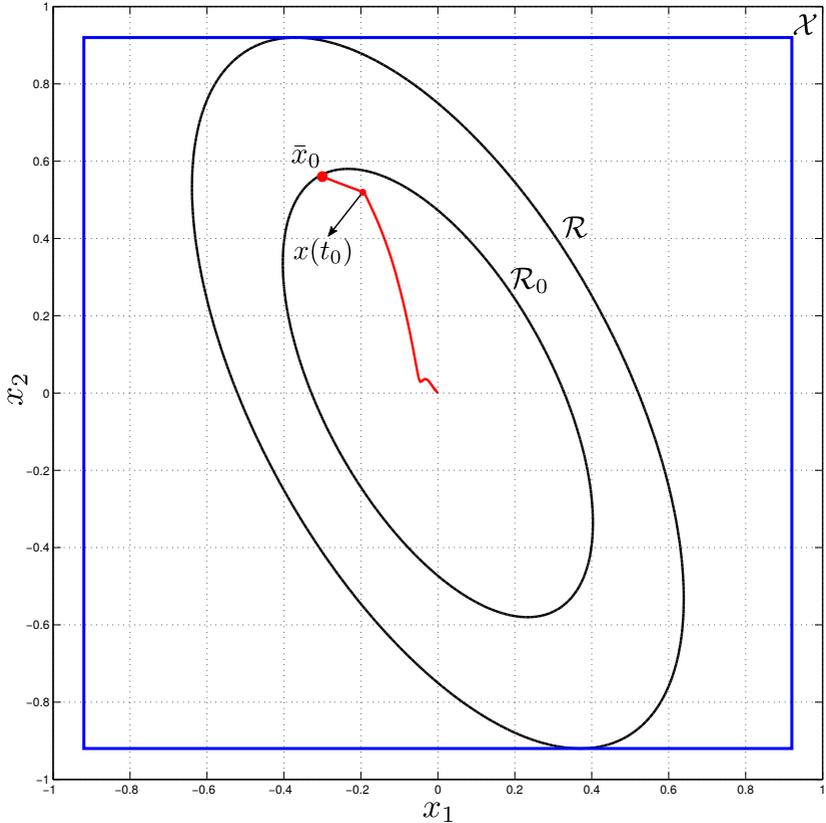


Figure 14 – Estimates of the sets \mathcal{R}_0 and \mathcal{R} , for $d = 0.17$ and \mathcal{X} as in (4.73), and a stable closed-loop trajectory for a controller obtained by means of (4.70).

4.4.1 Flow model in the tube

Consider the ideal gas flow through a constant section, where all the friction losses and heat transfers are neglected. Hence, the Euler equations can be used to model the system. The equations consist of conservation of mass, momentum and energy and they can be written in terms of the primitive variables density ρ , particle speed v and pressure

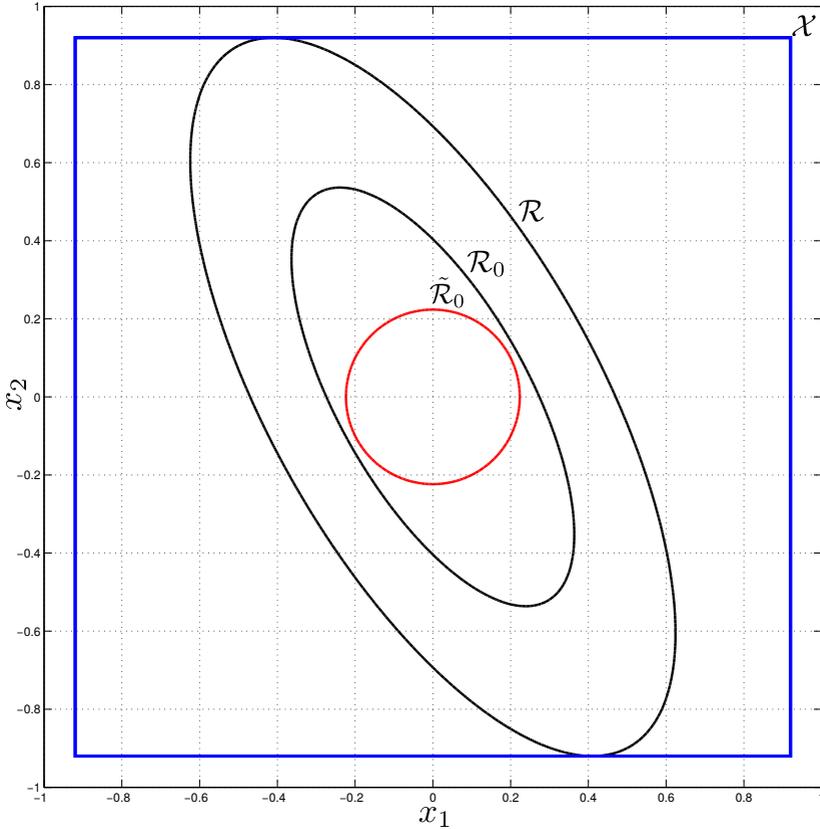


Figure 15 – Estimates of \mathcal{R}_0 and \mathcal{R} for \mathcal{X} and $\tilde{\mathcal{R}}_0$ as in (4.73) and (4.74), respectively, obtained by means of (4.71).

p (see (WINTERBONE; PEARSON, 2000)). In this case, the flow is assumed to be one-dimensional leading to the following system of PDEs (which are three nonlinear coupled equations):

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{A}[\mathbf{W}] \frac{\partial \mathbf{W}}{\partial x} + \mathbf{C}[\mathbf{W}] = 0, \quad (4.75)$$

$$\mathbf{W} = \begin{bmatrix} \rho \\ v \\ p \end{bmatrix}; \mathbf{A} = \begin{bmatrix} v & \rho & 0 \\ 0 & v & \frac{1}{\rho} \\ 0 & a^2\rho & v \end{bmatrix}; \mathbf{C} = \begin{bmatrix} 0 \\ G \\ (\gamma-1)\rho(q+vG) \end{bmatrix}.$$

where $a = \sqrt{\frac{\gamma p}{\rho}}$ representing the sound speed in ideal gas, G is a term associated with the friction losses and q is a term associated with the wall heat exchanges and γ the adiabatic constant. Further, $t \geq 0$ is the time variable and $x \in [0, L]$ is the space variable with $L = 1$ being the tube length.

In order to simplify (4.75) the following hypothesis are considered:

H.1 the propagation speed of the entropy wave (average energy and mass) inside the tube is much slower than the sound speed $u \ll a$;

H.2 the pressure inside the tube is considered constant (equal to the atmospheric pressure), because the pressure differential introduced by the ventilator is very small;

H.3 the heat exchanges and the friction in the tube are neglected: $q = 0$ and $G = 0$, i.e. $\mathbf{C}[\mathbf{W}] = 0$.

Hypothesis *H.1* and *H.2*, imply $\frac{\partial v(x,t)}{\partial x} \approx 0$ and $\frac{\partial p(x,t)}{\partial x} = 0$. This reduces system (4.75) to the following convection equation

$$\frac{\partial \rho(x,t)}{\partial t} + v(t) \frac{\partial \rho(x,t)}{\partial x} = 0, \quad (4.76)$$

where $v(t)$ is the time-varying convection parameter of (4.76). The gas speed $v(t)$ in the tube is measured. Using *H.1*, *H.2* and *H.3* allows the input mass flow rate to be expressed as

$$\dot{m}_{in} = v(t)\rho(0,t)A_t, \quad (4.77)$$

where A_t is the tube cross section area. With (4.77), the boundary condition of (4.76) can be expressed as:

$$\dot{\rho}(0,t) = -\frac{R\gamma T_{in}v(t)A_t}{p_{in}V_0}\rho(0,t)^2 + \frac{\gamma v(t)A_t}{V_0}\rho(0,t) - \frac{R}{p_{in}V_0C_v}\rho(0,t)dQ. \quad (4.78)$$

Define the control input as:

$$dQ = C_v \gamma v(t) A_t \left(\frac{pin}{R} - T_{in} \rho_{ref} \right) + K \rho(L, t) \quad (4.79)$$

4.4.2 Delay systems Derived from Distributed Parameter Systems

Distributed parameter systems are distinguished by the fact that the states, controls and outputs may depend on spatial position. Thus the natural description for the system model is a PDE. For the PDE-ODE cascades systems (see (SUSTO; KRSTIC, 2010; KRSTIC; SMYSHLYAEV, 2008a)), with the nonlinear dynamic boundary system defined in (4.78) and the convection equation (4.76), where the interconnections are of Dirichlet type. The convection equation (4.76) can be considered an pure varying time-delay. For a flow speed $v(t)$ given by:

$$\underline{v} \leq v^* \leq \bar{v} \quad (4.80)$$

with \underline{v} , \bar{v} representing the minimum and maximum values for v^* respectively.

Often one may act on system at $x = 0$. This leads to a dynamic boundary condition defined by (4.78), and the sensor is located in $\rho(L, t)$ and no density measurement is taken inside the heating column. Then the value $\rho(0, t)$ can be given by $\rho(0, t) = \rho(L, t - \frac{L}{v^*})$.

Therefore, the PDE-ODE cascades system, can be considered a input delay quadratic system, defined as:

$$\dot{\rho}(0, t) = -\frac{R \gamma T_{in} v^* A_t}{p_{in} V_0} \rho(0, t)^2 + \frac{\gamma v^* A_t}{V_0} \rho(0, t) - \frac{R}{p_{in} V_0 C_v} \rho(0, t) dQ. \quad (4.81)$$

Let define the control input as:

$$dQ = C_v \gamma v(t) A_t \left(\frac{pin}{R} - T_{in} \rho_{ref} \right) + u(t - \tau^*),$$

$$u(t - \tau^*) = K \rho(0, t - \tau^*), \quad \tau^* = \frac{L}{v^*}. \quad (4.82)$$

Now, a controller for tracking a constant reference density $\rho(L, t)$ is designed. To this end, an integral action is added to the system dynamics

(4.81) leading to the following state space realization:

$$\begin{bmatrix} \dot{\rho} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \left(-\frac{R\gamma T_{in} v^* A_t}{\rho_{in} V_0} \rho + \frac{\gamma v^* A_t}{V_0} \right) & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \rho \\ \xi \end{bmatrix} + \begin{bmatrix} \left(-\frac{R}{\rho_{in} V_0 C_v} \right) \rho \\ 0 \end{bmatrix} dQ + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rho_{ref}, \quad (4.83)$$

with $\rho = \rho(0, t)$, dQ defined in (4.82) and $\xi = \xi(t)$ is the integrator state. For this example we use a linear control law, and the control law (4.5) is modified to be as follows:

$$u = (K_0 + \rho K_1 + \xi K_2) \begin{bmatrix} \rho \\ \xi \end{bmatrix}, \quad K_1 = \mathbf{0}, \quad K_2 = \mathbf{0}. \quad (4.84)$$

4.4.3 Output Temperature Boundary Control

The output temperature boundary control is designed for system (4.83). Let define the error vector as:

$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad e_1 = \rho - \rho_{ref}, \quad e_2 = \xi - \xi_{ref}, \quad (4.85)$$

where ρ_{ref} is the desired output density. It is easy to show that system (4.76) with boundary conditions (4.78) can be expressed as follows:

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} \left(-\frac{R\gamma T_{in} v^* A_t}{\rho_{in} V_0} (e_1 + \rho_{ref}) + \frac{\gamma v^* A_t}{V_0} \right) & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} e_1 + \rho_{ref} \\ e_2 + \xi_{ref} \end{bmatrix} + \begin{bmatrix} \left(-\frac{R}{\rho_{in} V_0 C_v} \right) (e_1 + \rho_{ref}) \\ 0 \end{bmatrix} dQ, \quad (4.86)$$

with dQ defined as (4.82).

Note that the nonlinear input-delay system (4.86) can be seen as an approximation of the coupled PDE-ODE system with the nonlinear ODE defined in (4.78) and the convection equation (4.76). Thus, we can apply Theorem 1 or 2 proposed in this work. In order to operate

the experimental setup between the temperatures of 300 K and 325 K and the flow speed between 1 m/s and 3 m/s .

In order to design the control for nonlinear input-delay system (4.86), consider the control architecture presented in Figure 16.

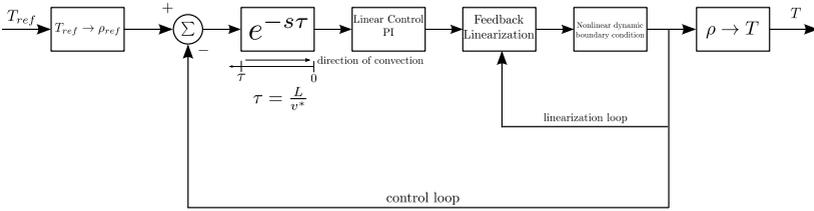


Figure 16 – Flow tube control Architecture using the input-delay system

Consider the following system parameters: adiabatic constant $\gamma = 1.4$; molar mass of dry air $M = 28.97 \text{ g/mol.K}$; ideal gas constant $R = 8.3143 \text{ J/(mol.K)}$; specific heat constant for constant pressure $C_p = 1.005 \text{ KJ/Kg.K}$ for constant volume $C_v = 0.718 \text{ KJ/Kg.K}$; initial pressure $p_{in} = 1 \text{ Bar}$ (or, equivalently, $1 \times 10^5 \text{ Pa}$ in SI units); initial temperature $T_{in} = 300 \text{ K}$; initial density $\rho = 1.1768 \text{ Kg/m}^3$ column volume $V_0 = 4 \times 10^{-3} \text{ m}^3$; tube cross section area $A_t = 6.4 \times 10^{-3} \text{ m}^2$; and the tube length $L = 1 \text{ m}$. The optimization problem (4.70) is applied to determine a maximized estimate of the set admissible initial conditions \mathcal{R}_0 as defined in (4.15) considering the largest admissible delay d which will be defined in terms of constant flow speed (convection parameter) and the tube length $d = \frac{L}{v^*}$. Applying the optimization problem (4.70) leads to the following control gains:

- $v^* = 3 \text{ m/s}$

$$K_0 = [-1.8737 \quad 0.7859], \quad K_1 = \mathbf{0}, \quad K_2 = \mathbf{0}, \quad (4.87)$$

- $v^* = 2 \text{ m/s}$

$$K_0 = [-1.5248 \quad 0.7221], \quad K_1 = \mathbf{0}, \quad K_2 = \mathbf{0}, \quad (4.88)$$

- $v^* = 1 \text{ m/s}$

$$K_0 = [-0.7937 \ 0.3445], \quad K_1 = \mathbf{0}, \quad K_2 = \mathbf{0}, \quad (4.89)$$

which ensure that the equilibrium $e = \mathbf{0}$ is exponentially stable. In order to illustrate the effectiveness of the proposed approach, some simulation results of the coupled PDE-ODE system with the nonlinear ODE defined in (4.78), the convection equation (4.76) and the controls (4.87)-(4.89) are presented for different flow speeds. A change of temperature reference from 300 K to 320 K (which can be transformed into a density reference using the perfect gases law) is shown.

As depicted in Figure 17, the system effectively follows the change of reference for different flow speeds. The faster the flow speed, the faster the convergence, as the fluid transport time is smaller. For the flow speed $v^* = 1 \text{ m/s}$ the delay between sensor information temperature and action in the power of the resistor heating column is 1 s. Figure 18 shows the respective control inputs obtained for the simulation results. The power dissipated by the heating resistor has to be greater as the flow speed increases. This is due to the fact that in this case, the gas residence time inside the heating column is smaller and the amount of energy absorbed by the gas is smaller. The Figure 19 shows the simulation results obtained when using the proposed control (4.87)-(4.89) to coupled PDE-ODE system with the nonlinear ODE defined in (4.78), and the convection equation (4.76). We consider a tube length $L = 1 \text{ m}$. The space was divided in 200 parts and the space derivatives were written using a finite difference scheme. An ODE solver was used to obtain the solution. Figure 20 shows the region of admissible initial conditions \mathcal{R}_0 for the state-space domain defined by $\mathcal{X} = \{x \in \mathbb{R}^2 : |x_i| \leq 0.5, i = 1, 2\}$. The optimization problem (4.47) is applied to determine maximized estimates of the set of admissible initial conditions \mathcal{R}_0 as defined in (4.15) for different values of flow speed that consequently implies that different values of d and h . The values of φ and σ were obtained in order to minimize the trace of P_1 (i.e., approximately maximizing the size of \mathcal{R}_0). As expected, for smaller values

of flow speed, which results in a greater delay the region of admissible initial conditions \mathcal{R}_0 is lower.

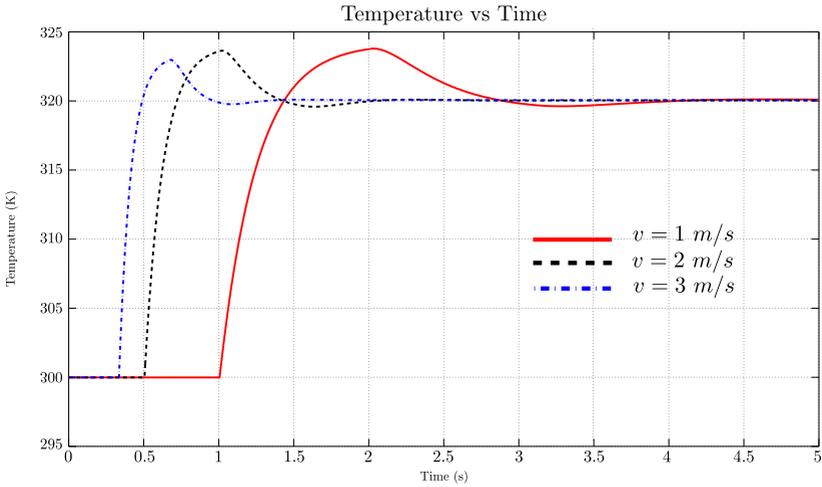


Figure 17 – Temperature evolution for the input delay system (4.86) with the controls (4.87)-(4.89)

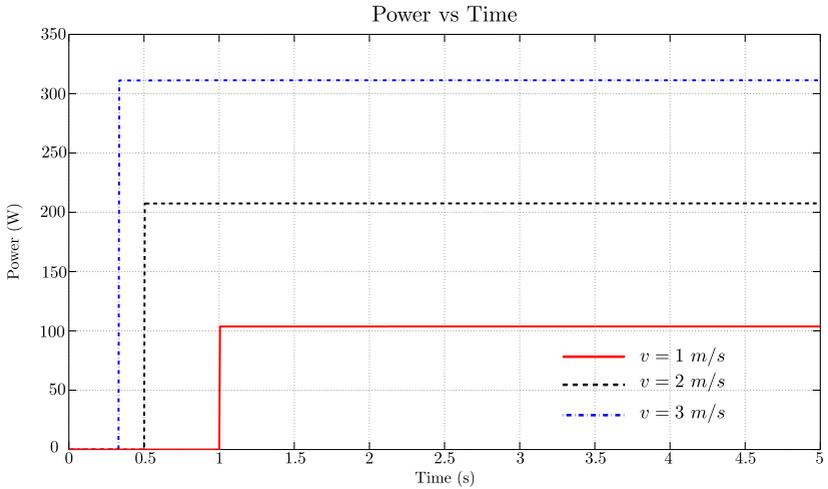


Figure 18 – Control Input for different flow speeds

4.5 Concluding Remarks

This chapter has addressed the stability and stabilization problems of open-loop unstable nonlinear quadratic input-delayed system subject to parameter uncertainty in a local stability setting. The proposed results are based on Lyapunov-Krasovskii stability theory of standard functional differential equations, but taking into account that the initial trajectory is constant. The first one proposes LMI based conditions to ensure the local stability of the closed-loop system assuming a bounded nonzero input delay while estimating the set of reachable states. Then, these results are extended to the synthesis of a stabilizing quadratic state feedback control law in terms of a finite set of LMI constraints. Optimization problems are also proposed to either maximize the set of admissible initial states (assuming a given bound on the input delay) or maximize a lower bound on the input delay for a given set of guaranteed initial conditions. Numerical examples have shown the potential of the proposed approach as a tool for the stability analysis and control design for input delayed nonlinear quadratic systems. Future

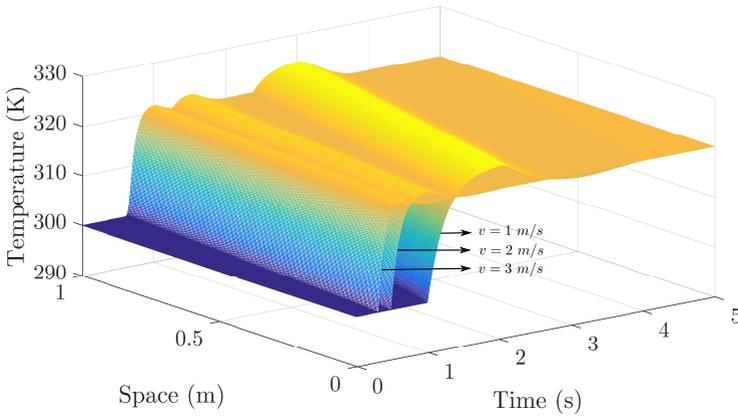


Figure 19 – Output temperature boundary control results using the proposed control (4.87)-(4.89) to coupled PDE-ODE system with the nonlinear ODE defined in (4.78) and the convection equation (4.76).

research will be concentrated in extending the approach to a broader class of nonlinear systems and to deal with nonlinear networked control systems under aperiodic sampling.

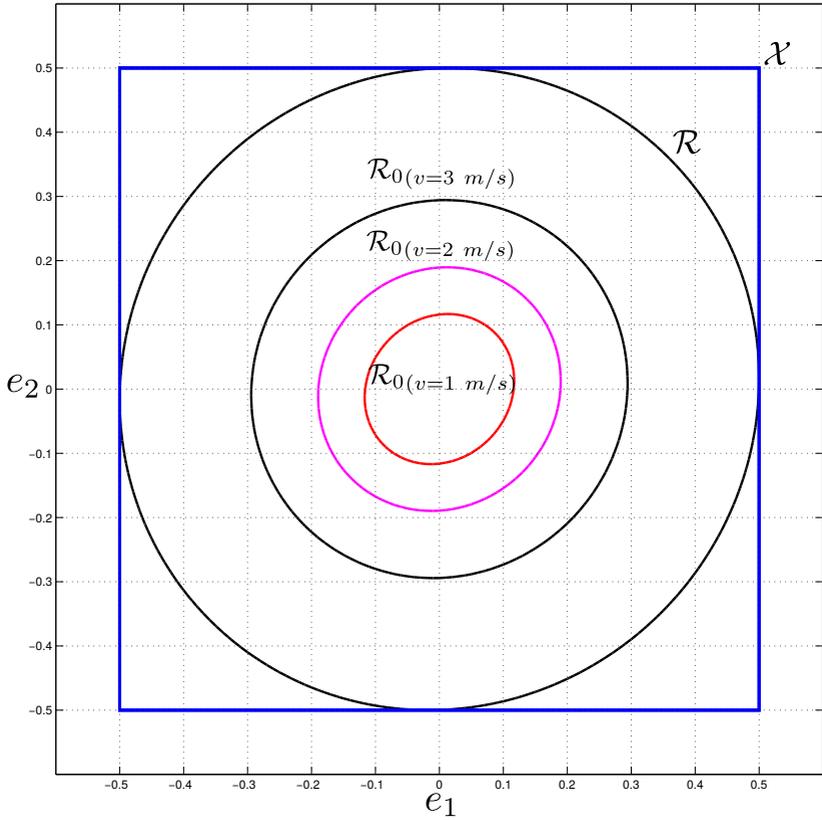


Figure 20 – Estimates of \mathcal{R}_0 and \mathcal{R} for given set of \mathcal{X} .

5 BOUNDARY CONTROL OF COUPLED PDE-ODE SYSTEMS

In industrial processes, a large variety of physical systems is governed by hyperbolic partial differential equations (PDEs) such as hydraulic networks (BASTIN; CORON, 2011; DIAGNE; BASTIN; CORON, 2012) and gas devices (CASTILLO et al., 2013; CASTILLO et al., 2015). In particular, fluid transport is often modeled by balance laws (or conservation laws when additive and dissipative terms are disregarded) which are hyperbolic PDEs normally used to express the fundamental dynamics of open conservative systems. However, infinite dimensional systems introduce variable time-delays that make the closed loop control much more challenging and, moreover, distributed measurements and actuators are not usually available. As a consequence, it is more common that actuators and measurements are located at the boundaries which is desired in practical applications as, for instance, in the references previously cited. In addition, a large number of numerical techniques based on finite-dimensional tools, which are often used to the stability analysis of PDE systems, provides only approximate solutions.

On the other hand, Lyapunov theory has been largely applied for several decades to deal with the stability analysis and control design of finite dimension systems described by ordinary differential equations (EDOs) (VIDYASAGAR, 2002). In the particular case of linear dynamical systems, a large number of stability and stabilization results are cast in terms of linear matrix inequality (LMI) constraints (BOYD et al., 1994), which are numerically solved using dedicated software (LOFBERG, 2004). The LMI framework is a powerful tool for linear systems, since it can deal with a large diversity of control and systems theory problems such as robust stability, input-to-output performance, state- and dynamic output-feedback, and state estimation (DUAN; YU, 2013). More recently, LMI tools were also applied to deal with nonlinear finite-dimensional systems, for instance in the references (COUTINHO et al., 2008), (PAPACHRISTODOULOU; PRAJNA, 2005) and (CHESI, 2010) among others.

In the context of infinite dimensional systems, the stability problem of boundary control for first-order hyperbolic systems using quadratic strict Lyapunov function was more recently stated; see, for instance, (CORON; BASTIN; D'ANDRÉA-NOVEL, 2007; CORON; BASTIN; D'ANDRÉA-NOVEL, 2008; PRIEUR; GIRARD; WITRANT, 2014) among others. A common assumption in most of available results is that the boundary action is faster than the wave travel making possible to establish a static relationship between the control input and the boundary condition. Nevertheless, there are applications where the dynamics associated to the boundary control action cannot be neglected as for instance in the temperature control of an airflow in a heating column. To deal with dynamic boundary actions, the discretization of the infinite-dimensional system and the use of finite-dimensional tool is often employed. On the other hand, recently, several approaches considering infinite-dimensional based techniques have been proposed such as: (CASTILLO et al., 2012) that proposes some sufficient conditions for the exponential stability of hyperbolic systems with linear time invariant dynamic boundary conditions, (KRSTIC; SMYSHLYAEV, 2008a) that considers the backstepping technique for the boundary control of hyperbolic and time-delayed systems, (CASTILLO et al., 2015) that studies the dynamic boundary stabilization of linear parameter-varying (LPV) hyperbolic systems using the LMI framework, (LAMARE; GIRARD; PRIEUR, 2016) that considers the problem of stability analysis and control synthesis for first-order hyperbolic linear PDEs over a bounded interval with spatially varying coefficients, and (DOS SANTOS MARTINS; RODRIGUES; DIAGNE, 2012) that proposes a multimodel approach with a bilinear matrix inequality to deal with the stability of a hyperbolic system representing the flow in a open channel. It turns out that the majority of the latter references consider only dynamic boundary actions described by linear differential equations.

This chapter is focused on the boundary stabilization of uncertain first-order hyperbolic systems with boundary action governed by an uncertain nonlinear quadratic differential equation. The class

of quadratic systems can describe nonlinear dynamics having a state-space model containing quadratic nonlinearities in the state variables and bilinear terms involving the state and control signal. This class of systems can represent a large number of processes such as distillation and heating columns (ESPANA; LANDAU, 1978; CASTILLO et al., 2012), induction motors (ARGUELLO-SERRANO; VELEZ-REYES, 1999) and DC–DC converters (THOUNTHONG; PIERFEDERICI, 2010), and it has attracted recurring interest of the control practitioners; see, for instance, (AMATO; COSENTINO; MEROLA, 2007; AMATO et al., 2009; VALMÓRBIDA; TARBOURIECH; GARCIA, 2010; COUTINHO; DE SOUZA, 2012) and references therein. Hence, local boundary stabilization conditions for the coupled linear first-order hyperbolic system and nonlinear quadratic dynamic boundary actuation are proposed in terms of a finite set of LMI constraints based on the Lyapunov stability theory. The local stability is characterized in a regional context, that is, bounded initial state trajectories will imply that the state trajectories remain bounded and they converge to the equilibrium point as the time goes to infinity. In addition, an optimization problem is proposed for the boundary feedback control design in order to determine a quadratic boundary state feedback control law to maximize the set of admissible boundary initial conditions while guaranteeing the stability of the coupled ODE-PDE system. Some of the results present in this chapter can be found in (CALDEIRA et al., 2017).

The remaining of this chapter is organized as follows. In Section 5.2 the problem of interest is stated. Then, Section 5.2 presents the main stability results for uncertain first-order linear hyperbolic system subject to nonlinear boundary conditions. Next, in Section 5.3, the stability results are adapted to control design by means of an appropriate similarity transformation where a stabilizing nonlinear boundary state feedback control law is designed in order to maximize the region of admissible initial conditions. Then, an uncertain first-order hyperbolic system is used to model the fluid transport in a Poiseuille flow with nonlinear dynamics at the boundary conditions illustrating the results

of Sections 5.2 and 5.3. Section 5.5 ends the chapter by providing some concluding remarks.

5.1 Problem Statement

Let n be a positive integer, Ω an open non-empty subset of \mathbb{R}^n and Δ a non empty convex set of $\mathbb{R}^{n\delta}$. Consider the general class of first-order uncertain hyperbolic systems of order n defined as follows:

$$\partial_t \boldsymbol{\xi}(t, x) + \mathbf{A}(\boldsymbol{\delta}) \partial_x \boldsymbol{\xi}(t, x) = \mathbf{0}, \quad t \in \mathbb{R}, \quad x \in [0, 1], \quad (5.1)$$

where $\boldsymbol{\xi} : \mathbb{R}^+ \times [0, 1] \rightarrow \Omega$, $\boldsymbol{\delta} \in \Delta$ is a continuous vector function of time with bounded magnitude, $\mathbf{A}(\boldsymbol{\delta}) : \Delta \mapsto \mathbb{R}^{n \times n}$ is a diagonal and continuous matrix function (called the characteristic matrix), i.e., $\mathbf{A}(\boldsymbol{\delta}) = \text{diag}\{\lambda_1(\boldsymbol{\delta}), \lambda_2(\boldsymbol{\delta}), \dots, \lambda_n(\boldsymbol{\delta})\}$ with $\lambda_i(\boldsymbol{\delta}) \neq 0$ for all $\boldsymbol{\delta} \in \Delta$, ∂_t and ∂_x denote the partial derivatives with respect to time and space, respectively.

Assumption 1. *The diagonal elements of $\mathbf{A}(\boldsymbol{\delta})$ satisfy the following inequalities for all $\boldsymbol{\delta} \in \Delta$:*

$$0 < \lambda_1(\boldsymbol{\delta}) < \dots < \lambda_n(\boldsymbol{\delta}).$$

Associated to (5.1), consider the following nonlinear quadratic dynamic boundary action:

$$\begin{cases} \dot{\boldsymbol{\xi}}_{in}(t) &= A(\boldsymbol{\xi}_{in}(t), \boldsymbol{\delta}) \boldsymbol{\xi}_{in}(t) + B(\boldsymbol{\xi}_{in}(t), \boldsymbol{\delta}) u(t), \\ u(t) &= G(\boldsymbol{\xi}_{in}(t)) \boldsymbol{\xi}_{in}(t) + K(\boldsymbol{\xi}_{out}(t)) \boldsymbol{\xi}_{out}(t), \end{cases} \quad (5.2)$$

where

$$\boldsymbol{\xi}_{in}(t) = \boldsymbol{\xi}(0, t), \quad \boldsymbol{\xi}_{out}(t) = \boldsymbol{\xi}(1, t) \quad (5.3)$$

are the boundary conditions of (5.1) which interconnect the dynamics of (5.1) and (5.2), $\boldsymbol{\xi}_{in} \in \Xi \subset \mathbb{R}^n$ is the state vector of the dynamic boundary condition, $\boldsymbol{\xi}_{out} \in \bar{\Xi} \subset \mathbb{R}^n$ is the measurement, $u(t) \in \mathbb{R}^{n_u}$ is the boundary control input, Ξ is some compact region of the boundary state-space containing $\boldsymbol{\xi}_{in} = \mathbf{0}$ and to be specified later in this chapter.

The matrices $A(\cdot) \in \mathbb{R}^{n \times n}$, $B(\cdot) \in \mathbb{R}^{n \times n_u}$, $G(\cdot) \in \mathbb{R}^{n_u \times n}$ and $K(\cdot) \in \mathbb{R}^{n_u \times n}$ are affine matrix functions on their arguments, that is:

$$[A(\boldsymbol{\xi}_{in}, \delta) \quad B(\boldsymbol{\xi}_{in}, \delta)] = [A_0 \quad B_0] + \sum_{i=1}^n \boldsymbol{\xi}_{in}^{[i]} [A_i \quad B_i] \quad (5.4)$$

$$+ \sum_{i=1}^{n_\delta} \delta^{[i]} [\check{A}_i \quad \check{B}_i],$$

$$[G(\boldsymbol{\xi}_{in}) \quad K(\boldsymbol{\xi}_{out})] = [G_0 \quad K_0] + \sum_{i=1}^n [\boldsymbol{\xi}_{in}^{[i]} G_i \quad \boldsymbol{\xi}_{out}^{[i]} K_i], \quad (5.5)$$

with $A_i, B_i, G_i, K_i, i = 0, 1, \dots, n$, and $\check{A}_j, \check{B}_j, j = 1, \dots, n_\delta$, being given constant real matrices with appropriate dimensions. It is assumed that the pair (A_0, B_0) is stabilizable and the unforced system of (5.2) is allowed to be unstable.

The initial conditions of the coupled PDE-ODE system of (5.1) and (5.2), namely

$$\boldsymbol{\xi}(0, x) = \bar{\boldsymbol{\xi}}(x) \quad \text{and} \quad \boldsymbol{\xi}_{in}(0) = \bar{\boldsymbol{\xi}}_{in}, \quad (5.6)$$

are assumed to satisfy:

$$\bar{\boldsymbol{\xi}} \in \mathcal{A} \quad \text{and} \quad \bar{\boldsymbol{\xi}}_{in} \in \mathcal{B}, \quad (5.7)$$

where the sets \mathcal{A} and \mathcal{B} are defined as follows

$$\mathcal{A} := \left\{ \bar{\boldsymbol{\xi}} \in \mathbb{R}^n : \int_0^1 \bar{\boldsymbol{\xi}}(x)' \bar{\boldsymbol{\xi}}(x) dx \leq \sigma \right\}, \quad (5.8)$$

$$\mathcal{B} := \left\{ \bar{\boldsymbol{\xi}}_{in} \in \mathbb{R}^n : \bar{\boldsymbol{\xi}}_{in}' P_1 \bar{\boldsymbol{\xi}}_{in} \leq 1 \right\},$$

with the scalar $\sigma > 0$ and matrix $P_1 > 0$ defining the sizes of \mathcal{A} and \mathcal{B} , respectively. In the particular case of linear dynamic boundary conditions, that is when A_i, B_i, \check{A}_i and \check{B}_i are null matrices for $i = 1, \dots, n$, the existence and uniqueness of solutions to (5.2), (5.3) and (5.6) for initial conditions satisfying (5.7) is ensured, see e.g. (CORON; BASTIN; D'ANDRÉA-NOVEL, 2008). As these solutions may not be differentiable everywhere, the concept of weak solutions of partial differential

equations has to be used (see (DIAGNE; BASTIN; CORON, 2012) and references therein for further details).

This chapter is concerned in obtaining sufficient conditions to the regional stability and stabilization problems for coupled PDE-ODE system of (5.1) and (5.2) as stated below:

- P1** For given state feedback gain matrices $G(\xi_{in})$ and $K(\xi_{out})$ as defined in (5.5), devise stability analysis conditions ensuring the robust local stability of the coupled PDE-ODE system while determining a maximized set of initial boundary states \mathcal{B} for a given set \mathcal{A} .
- P2** Design the state feedback gain matrices $G(\xi_{in})$ and $K(\xi_{out})$ as defined in (5.5), ensuring the robust local stability of the coupled PDE-ODE system while maximizing the set of initial boundary states \mathcal{B} for a given set \mathcal{A} .

Before ending this section, we introduce the notion of exponential stability for the coupled PDE-ODE system of (5.1) and (5.2) to be considered in this chapter.

Definition 5.1. *The coupled PDE-ODE system of (5.1) and (5.2), with initial conditions $\bar{\xi}$ and $\bar{\xi}_{in}$ satisfying (5.7), is said to be locally robustly exponentially stable if there exist positive scalars α and β such that the following holds:*

$$\left(\|\xi_{in}(t)\| + \|\xi(t, \cdot)\|_{L^2((0,1);\mathbb{R}^n)} \right) \leq \beta e^{-\alpha t} \left(\|\bar{\xi}_{in}\| + \|\bar{\xi}\|_{L^2((0,1);\mathbb{R}^n)} \right),$$

$$\forall t \in \mathbb{R}^+, \delta \in \Delta. \quad (5.9)$$

5.2 Local Stability Analysis

In this section, it is developed an LMI approach to derive a numerical and tractable solution to the robust regional stability analysis problem **P1** as defined in Section 5.1. To this end, consider the following

Lyapunov function candidate defined for all continuously differentiable functions $\xi : \mathbb{R}^+ \times [0, 1] \rightarrow \Omega$:

$$V(\xi) = \xi'_{in} P_1 \xi_{in} + \int_0^1 e^{-\mu x} \xi(x)' P_2 \xi(x) dx, \quad (5.10)$$

where $P_1, P_2 \in \mathbb{R}^{n \times n}$ are positive definite diagonal matrices and μ is a positive scalar.

Then, evaluating the time derivative of $V(\cdot)$ along the solutions of (5.1) and (5.2) with (5.7) leads to:

$$\begin{aligned} \dot{V}(\xi(t, \cdot)) &= 2\xi'_{in}(t)' P_1 (A(\xi_{in}, \delta) + B(\xi_{in}, \delta)G(\xi_{in})) \xi_{in}(t) \\ &+ 2\xi'_{in}(t)' P_1 B(\xi_{in}, \delta)K(\xi_{out}) \xi_{out}(t) - [e^{-\mu x} \xi(t, x)' \Lambda(\delta) P_2 \xi(t, x)]_{x=0}^{x=1} \\ &\quad - \mu \int_0^1 e^{-\mu x} \xi(t, x)' \Lambda(\delta) P_2 \xi(t, x) dx \end{aligned} \quad (5.11)$$

In view of the above and taking (5.3) into account, $\dot{V}(\xi(t, \cdot))$ can be cast as follows:

$$\dot{V}(\xi(t, \cdot)) = -m_1(\xi, \delta) + m_2(\xi_{in}, \xi_{out}, \delta) \quad (5.12)$$

where

$$\begin{aligned} m_1(\xi, \delta) &= \mu \left(\xi'_{in} \Lambda(\delta) P_1 \xi_{in} + \int_0^1 e^{-\mu x} \xi(t, x)' \Lambda(\delta) P_2 \xi(t, x) dx \right), \\ m_2(\xi_{in}, \xi_{out}, \delta) &= \xi'_a \Phi(\xi_{in}, \xi_{out}, \delta) \xi_a, \quad \xi_a = \begin{bmatrix} \xi_{in} \\ \xi_{out} \end{bmatrix}, \end{aligned} \quad (5.13)$$

$$\Phi(\xi_{in}, \xi_{out}, \delta) = \left[\begin{array}{c|c} \begin{matrix} A(\xi_{in}, \delta)' P_1 + P_1 A(\xi_{in}, \delta) + G(\xi_{in})' B(\xi_{in}, \delta)' P_1 \\ + P_1 B(\xi_{in}, \delta) G(\xi_{in}) + \Lambda(\delta) P_2 + \mu \Lambda(\delta) P_1 \end{matrix} & P_1 B(\xi_{in}, \delta) K(\xi_{out}) \\ \hline K(\xi_{out})' B(\xi_{in}, \delta)' P_1 & -e^{-\mu} \Lambda(\delta) P_2 \end{array} \right].$$

Due to Assumption 1 and since Λ is a continuous function on the compact set Δ , there exists a sufficiently small positive value ε such that $\varepsilon I_n \leq \Lambda(\delta)$ for all $\delta \in \Delta$. Then, if

$$m_2(\xi_{in}, \xi_{out}, \delta) < 0, \quad \forall (\xi_{in}, \xi_{out}, \delta) \in \Xi \times \Xi \times \Delta, \quad (5.14)$$

the following holds

$$\dot{V}(\xi(t, \cdot)) \leq -\mu \varepsilon V(\xi(t)), \quad \forall t \in \mathbb{R}^+, \quad (5.15)$$

from the fact that $m_1(\xi(t, \cdot), \delta) \geq \mu \varepsilon V(\xi(t, \cdot))$.

Notice that the inequality (5.15) implies that the coupled PDE-ODE system is locally exponentially stable if the condition (5.14) is satisfied along the solutions of (5.1) and (5.2) for all $\bar{\xi} \in \mathcal{A}$ and $\bar{\xi}_{in} \in \mathcal{B}$. Hence, we have to guarantee that the solution of the coupled PDE-ODE system is confined to a region

$$\mathcal{R}(\gamma) := \{\xi : V(\xi) \leq \gamma\} \quad (5.16)$$

for a certain $\gamma > 0$.

Firstly, from (5.15), the following holds:

$$V(\xi(t, \cdot)) \leq e^{-\mu \varepsilon t} V(\bar{\xi}), \quad \forall \delta \in \Delta.$$

Next, provided that $\bar{\xi} \in \mathcal{A}$, $\bar{\xi}_{in} \in \mathcal{B}$ and $\mu > 0$, notice that the following holds:

$$\int_0^1 e^{-\mu x} x' P_2 x \, dx \leq \int_0^1 x' P_2 x \, dx \leq \rho \int_0^1 x' x \, dx \leq \rho \sigma.$$

For some scalar ρ such that $\rho I_n - P_2 \geq 0$, we obtain:

$$V(\bar{\xi}) \leq 1 + \rho \sigma.$$

Hence, in view of the above, the following is satisfied:

$$V(\xi(t, \cdot)) \leq \gamma, \quad \gamma = 1 + \rho \sigma, \quad \forall t \in \mathbb{R}^+, \delta \in \Delta. \quad (5.17)$$

Now, let $V_1(\xi_{in}) = \xi_{in}' P_1 \xi_{in}$, then it follows from (5.10) that:

$$V_1(\xi_{in}(t)) \leq V(\xi(t, \cdot)) \leq \gamma, \quad \forall t \in \mathbb{R}^+, \delta \in \Delta. \quad (5.18)$$

Let the following set:

$$\mathcal{D} := \{\xi_{in} : V_1(\xi_{in}) = \xi_{in}' P_1 \xi_{in} \leq \gamma\}. \quad (5.19)$$

Thus, the condition in (5.18) guarantees that $\xi_{in} \in \mathcal{D}$ for all $t \in \mathbb{R}^+$ and $\delta \in \Delta$, since $V_1(\xi_{in}(t)) \leq V(\xi(t, \cdot)) \leq V(\bar{\xi}) \leq \gamma$ for all $t \in \mathbb{R}^+$ and $\delta \in \Delta$. Then, provided that $\mathcal{D} \subset \mathfrak{E}$, the condition in (5.14) holds along the solutions of (5.1) and (5.2) for all $\bar{\xi} \in \mathcal{A}$ and $\bar{\xi}_{in} \in \mathcal{B}$.

In the sequel, it is introduced the main result of this section which proposes a numerical and tractable solution to problem **P1** in terms of a finite set of LMI constraints. To this end, it is assumed that Ξ is a given polytopic region containing $\xi_{in} = \mathbf{0}$ with known vertices. Moreover, Ξ can be also equivalently defined in terms of its faces as below

$$\Xi = \{\xi_{in} \in \mathbb{R}^n : |c_i' \xi_{in}| \leq 1, i = 1, \dots, n_f\}, \quad (5.20)$$

with $c_i \in \mathbb{R}^n, i = 1, \dots, n_f$, defining the faces of Ξ .

Theorem 5.1 (Stability Analysis). *Consider the PDE-ODE system (5.1) and (5.2), with the initial conditions defined by (5.7) and (5.8), and given control gains $G(\xi_{in})$ and $K(\xi_{out})$. Let Ξ and Δ be given polytopes, and ρ, σ and μ be given positive scalars. Suppose there exist diagonal matrices P_1 and P_2 , and a matrix L , with appropriate dimensions, satisfying the following:*

$$P_1 > 0, P_2 > 0, \rho \mathbf{I}_n - P_2 \geq 0, \quad (5.21)$$

$$\begin{bmatrix} \gamma & \gamma c_j' \\ \gamma c_j & P_1 \end{bmatrix} \geq 0, j = 1, \dots, n_f, \quad (5.22)$$

$$\begin{aligned} \Theta(\xi_{in}, \xi_{out}, \delta, \mu) + \text{He}\{LM(\xi_{in}, \xi_{out})\} &< 0, \\ \forall (\xi_{in}, \xi_{out}, \delta) &\in \mathcal{V}(\mathcal{X} \times \mathcal{X} \times \Delta), \end{aligned} \quad (5.23)$$

where $\gamma = 1 + \rho\sigma$ and

$$\Theta(\xi_{in}, \xi_{out}, \delta, \mu) = \left[\begin{array}{c|c} \begin{array}{l} N'(A(\xi_{in}, \delta)'P_1 + P_1A(\xi_{in}, \delta)) + \mathcal{G}'B(\xi_{in}, \delta)'P_1N \\ + N'P_1B(\xi_{in}, \delta)\mathcal{G} + N'(\Lambda(\delta)P_2 + \mu\Lambda(\delta)P_1)N \end{array} & N'P_1B(\xi_{in}, \delta)\mathcal{K} \\ \hline \mathcal{K}'B(\xi_{in}, \delta)'P_1N & -N'e^{-\mu}\Lambda(\delta)P_2N \end{array} \right],$$

$$N = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n \times n^2} \end{bmatrix}, \mathcal{G} = \begin{bmatrix} G_0 & G_1 & \dots & G_n \end{bmatrix}, \mathcal{K} = \begin{bmatrix} K_0 & K_1 & \dots & K_n \end{bmatrix},$$

$$\mathcal{M}(\xi_{in}, \xi_{out}) = \begin{bmatrix} \Pi(\xi_{in}) & -\mathbf{I}_{n^2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathcal{N}(\xi_{in}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Pi(\xi_{out}) & -\mathbf{I}_{n^2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{N}(\xi_{out}) \end{bmatrix}, \Pi(\xi) = \begin{bmatrix} \xi^{[1]} \otimes \mathbf{I}_n \\ \vdots \\ \xi^{[n]} \otimes \mathbf{I}_n \end{bmatrix},$$

$$N(\xi) = \begin{bmatrix} \xi^{[2]} \mathbf{I}_n & -\xi^{[1]} \mathbf{I}_n & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \xi^{[3]} \mathbf{I}_n & -\xi^{[2]} \mathbf{I}_n & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \xi^{[n]} \mathbf{I}_n & -\xi^{[n-1]} \mathbf{I}_n \end{bmatrix}. \quad (5.24)$$

Then, the origin of the coupled PDE-ODE system of (5.1) and (5.2) is locally robustly exponentially stable in the sense of Definition 5.1. Moreover, for any $\bar{\xi} \in \mathcal{A}$ and $\bar{\xi}_{in} \in \mathcal{B}$, the system trajectories remain bounded to \mathcal{R} as defined in (5.19), for all $t \geq 0$, and vanish to zero as time goes to infinity.

Proof. Suppose the conditions of Theorem 5.1 are satisfied, then from convexity arguments the condition in (5.23) is also satisfied for all $(\bar{\xi}_{in}, \bar{\xi}_{out}, \delta) \in \Xi \times \Xi \times \Delta$.

Firstly, it will be shown that (5.23) implies that (5.14) holds. Then, let

$$\Psi(\bar{\xi}_{in}, \bar{\xi}_{out}) = \begin{bmatrix} \Pi_a(\bar{\xi}_{in}) & \mathbf{0} \\ \mathbf{0} & \Pi_a(\bar{\xi}_{out}) \end{bmatrix}, \quad \Pi_a(\xi) = \begin{bmatrix} \mathbf{I}_n \\ \Pi(\xi) \end{bmatrix}.$$

Hence, pre- and post-multiplying (5.23) by $\Psi(\bar{\xi}_{in}, \bar{\xi}_{out})'$ and $\Psi(\bar{\xi}_{in}, \bar{\xi}_{out})$, respectively, yields

$$\Phi(\bar{\xi}_{in}, \bar{\xi}_{out}, \delta) < 0, \quad \forall (\bar{\xi}_{in}, \bar{\xi}_{out}, \delta) \in \Xi \times \Xi \times \Delta,$$

since $\mathcal{M}(\bar{\xi}_{in}, \bar{\xi}_{out})\Psi(\bar{\xi}_{in}, \bar{\xi}_{out}) = 0$ by construction, and thus

$$m_2(\bar{\xi}_{in}, \bar{\xi}_{out}, \delta) < 0, \quad \forall (\bar{\xi}_{in}, \bar{\xi}_{out}, \delta) \in \Xi \times \Xi \times \Delta.$$

Next, in light of the definition of $V(\bar{\xi})$ in (5.10) and taking (5.12)-(5.14) into account, the following holds:

$$\dot{V}(\bar{\xi}(t, \cdot)) \leq -\mu \varepsilon V(\bar{\xi}(t)), \quad \forall t \in \mathbb{R}^+.$$

Now, notice that (5.22) implies $\mathcal{D} \subset \Xi$; see, e.g., (BOYD et al., 1994). Hence, for all $\bar{\xi} \in \mathcal{A}$ and $\bar{\xi}_{in} \in \mathcal{B}$, it follows that $V_1(\bar{\xi}_{in}(t)) \leq$

$V(\boldsymbol{\xi}(t, \cdot)) \leq V(\bar{\boldsymbol{\xi}}) \leq \gamma$ for $t \in \mathbb{R}^+$ and $\delta \in \Delta$, which completes the proof. \square

Theorem 5.1 can be applied to obtain an estimate of the largest set \mathcal{B} of admissible initial boundary states assuming that \mathcal{A} and Ξ are given a priori. For instance, the volume of \mathcal{B} can be approximately maximized by minimizing the trace of P_1 (since the trace of P_1^{-1} is the sum of the squared semi-axis lengths of \mathcal{B}) leading to the following optimization problem

$$\min_{\mu, \rho, P_1, P_2, L} \text{trace}\{P_1\} \text{ subject to (5.21)-(5.23)}. \quad (5.25)$$

Notice that the matrix inequalities in (5.21)-(5.23) are LMIs when ρ and μ are given. Hence, in order to obtain a solution to the optimization problem in (5.25) via semi-definite programming we can apply a gridding technique over ρ and μ .

5.3 Regional Stabilization

This section addresses the problem of designing the state feedback gain matrices $G(\cdot)$ and $K(\cdot)$ such that the coupled PDE-ODE system of (5.1) and (5.2) with (5.7) is regionally exponentially stable for all $\delta \in \Delta$. The proposed design will be based on Theorem 5.1 together with matrix parametrization involving the Lyapunov matrix P_1 and the control gains.

Thus, let the following similarity transformations:

$$\begin{aligned} \boldsymbol{\eta}_{in} &= P_1 \boldsymbol{\xi}_{in}, \quad \boldsymbol{\eta}_{out} = P_1 \boldsymbol{\xi}_{out}, \quad \boldsymbol{\eta} = P_1 \boldsymbol{\xi}, \quad Q_1 = P_1^{-1}, \quad Q_2 = Q_1 P_2 Q_1, \\ G_p(\boldsymbol{\xi}_{in}) &= G(\boldsymbol{\xi}_{in}) Q_1, \quad K_p(\boldsymbol{\xi}_{out}) = K(\boldsymbol{\xi}_{out}) Q_1, \end{aligned} \quad (5.26)$$

with $Q_1 > 0$.

In view of above definitions, observe that $m_1(\boldsymbol{\xi}, \delta)$ and

$m_2(\xi_{in}, \xi_{out}, \delta)$ in (5.12) can be respectively recast as follows:

$$m_1(\xi, \delta) = \mu \left(\eta'_{in} Q_1 \Lambda(\delta) \eta_{in} + \int_0^1 e^{-\mu x} \eta(t, x)' Q_2 \Lambda(\delta) \eta(t, x) dx \right), \quad (5.27)$$

$$m_2(\xi_{in}, \xi_{out}, \delta) = \eta'_a \Phi_p(\xi_{in}, \xi_{out}, \delta) \eta_a, \quad (5.28)$$

where

$$\eta_a = \begin{bmatrix} \eta_{in} \\ \eta_{out} \end{bmatrix},$$

$$\Phi_p(\xi_{in}, \xi_{out}, \delta) = \left[\begin{array}{c|c} \frac{Q_1 A(\xi_{in}, \delta)' + A(\xi_{in}, \delta) Q_1 + G_p(\xi_{in})' B(\xi_{in})'}{B(\xi_{in}) G_p(\xi_{in}) + Q_2 \Lambda(\delta) + \mu Q_1 \Lambda(\delta)} & B(\xi_{in}, \delta) K_p(\xi_{out}) \\ \hline K_p(\xi_{out})' B(\xi_{in}, \delta)' & -e^{-\mu} \Lambda(\delta) Q_2 \end{array} \right].$$

Hence, using similar arguments to Section 5.2, the condition $\dot{V}(\xi(t, \cdot)) \leq -\mu \varepsilon V(\xi(t, \cdot))$ will hold for $t \in \mathbb{R}^+$ if $m_2(\xi_{in}, \xi_{out}, \delta)$ as defined in (5.28) satisfies

$$m_2(\xi_{in}, \xi_{out}, \delta) < 0, \quad \forall (\xi_{in}, \xi_{out}, \delta) \in \Xi \times \Xi \times \Delta. \quad (5.29)$$

In addition, notice that the condition $\rho \mathbf{I}_n - P_2 \geq 0$ was considered in Section 5.2 to obtain a bound γ on $V(\xi(t, \cdot))$ as in (5.18) holds for all $t \in \mathbb{R}^+$ and $\delta \in \Delta$. To this end, the following result will be instrumental to obtain a numerical and tractable solution.

Lemma 5.1 ((DE OLIVEIRA; GEROMEL; BERNUSSOU, 2002)). *Let P and R be real square matrices with $P > 0$ and R nonsingular. Then, the following holds:*

$$R' P^{-1} R \geq R + R' - P \quad (5.30)$$

Now, taking the parametrization of P_2 in (5.26) into account, note that $\rho \mathbf{I}_n - P_2 \geq 0$ can be equivalently written as follows:

$$Q_1 Q_2^{-1} Q_1 - \varphi \mathbf{I}_n \geq 0, \quad \varphi = \frac{1}{\rho}, \quad (5.31)$$

since $P_2 = Q_1^{-1} Q_2 Q_1^{-1}$.

Hence, the following inequality is a sufficient condition for (5.31) to hold:

$$2Q_1 - Q_2 - \varphi \mathbf{I}_n \geq 0, \quad (5.32)$$

by applying Lemma 5.1 with $R = Q_1$ and $P = Q_2$.

In view of the above developments, it is proposed in the sequel an LMI-based result for designing the state feedback gain matrices $G(\boldsymbol{\xi}_{in})$ and $K(\boldsymbol{\xi}_{out})$ such that the origin of the coupled PDE-ODE of (5.1) and (5.2) is robustly regionally stable in closed-loop.

Theorem 5.2 (Regional Stabilization). *Consider the PDE-ODE system (5.1) and (5.2), with the initial conditions defined by (5.7) and (5.8). Let Ξ and Δ be given polytopes, and φ , σ and μ be given positive scalars. Suppose there exist diagonal matrices Q_1 and Q_2 , and matrices L , F_i , S_i , $i = 0, 1, \dots, n$, with appropriate dimensions, satisfying the following:*

$$Q_1 > 0, Q_2 > 0, 2Q_1 - Q_2 - \varphi \mathbf{I}_n \geq 0, \quad (5.33)$$

$$1 - \gamma c_j' Q_1 c_j \geq 0, j = 1, \dots, n_f, \quad (5.34)$$

$$\begin{aligned} \Theta_p(\boldsymbol{\xi}_{in}, \boldsymbol{\xi}_{out}, \delta, \mu) + \text{He}\{LM(\boldsymbol{\xi}_{in}, \boldsymbol{\xi}_{out})\} < 0, \\ \forall (\boldsymbol{\xi}_{in}, \boldsymbol{\xi}_{out}, \delta) \in \mathcal{V}(\mathcal{X} \times \mathcal{X} \times \Delta), \end{aligned} \quad (5.35)$$

where $\gamma = 1 + \frac{\sigma}{\varphi}$ and

$$\Theta_p(\boldsymbol{\xi}_{in}, \boldsymbol{\xi}_{out}, \delta, \mu) = \left[\begin{array}{c|c} \begin{array}{l} N'(Q_1 A(\boldsymbol{\xi}_{in}, \delta)' + A(\boldsymbol{\xi}_{in}, \delta) Q_1) + \mathcal{F}' B(\boldsymbol{\xi}_{in}, \delta)' N \\ + N' B(\boldsymbol{\xi}_{in}, \delta) \mathcal{F} + N' (\Lambda(\delta) Q_2 + \mu \Lambda(\delta) Q_1) N \end{array} & N' B(\boldsymbol{\xi}_{in}, \delta) \mathcal{S} \\ \hline S' B(\boldsymbol{\xi}_{in}, \delta)' N & -e^{-\mu} N' \Lambda(\delta) Q_2 N \end{array} \right],$$

$$\mathcal{F} = \begin{bmatrix} F_0 & F_1 & \cdots & F_n \end{bmatrix}, \quad \mathcal{S} = \begin{bmatrix} S_0 & S_1 & \cdots & S_n \end{bmatrix}.$$

Then, the origin of the coupled PDE-ODE system (5.1) and (5.2), with

$$\begin{aligned} G(\boldsymbol{\xi}_{in}) &= F_0 Q_1^{-1} + \sum_{i=1}^n \boldsymbol{\xi}_{in}^{[i]} F_i Q_1^{-1}, \\ K(\boldsymbol{\xi}_{out}) &= S_0 Q_1^{-1} + \sum_{i=1}^n \boldsymbol{\xi}_{out}^{[i]} S_i Q_1^{-1}, \end{aligned} \quad (5.36)$$

is locally robustly exponentially stable in the sense of Definition 5.1. Moreover, for any $\bar{\xi} \in \mathcal{A}$ and $\bar{\xi}_{in} \in \mathcal{B}$, the solution of the coupled PDE-ODE system remains bounded to \mathcal{R} as defined in (5.16), with $P_1 = Q_1^{-1}$ and $P_2 = Q_1^{-1}Q_2Q_1^{-1}$, for all $t \geq 0$, and vanish to zero as the time goes to infinity.

Proof. Suppose the conditions of Theorem 5.2 are satisfied, then from convexity arguments the condition in (5.35) is also satisfied for all $(\xi_{in}, \xi_{out}, \delta) \in \Xi \times \Xi \times \Delta$.

Firstly, it be shown that (5.35) implies that (5.29) holds. Then, let

$$\Psi(\xi_{in}, \xi_{out}) = \begin{bmatrix} \Pi_a(\xi_{in}) & \mathbf{0} \\ \mathbf{0} & \Pi_a(\xi_{out}) \end{bmatrix}, \quad \Pi_a(\xi) = \begin{bmatrix} \mathbf{I}_n \\ \Pi(\xi) \end{bmatrix}.$$

Hence, pre- and post-multiplying (5.35) by $\Psi(\xi_{in}, \xi_{out})'$ and $\Psi(\xi_{in}, \xi_{out})$, respectively, yields

$$\Phi_p(\xi_{in}, \xi_{out}, \delta, \mu) < 0, \quad \forall (\xi_{in}, \xi_{out}, \delta) \in \Xi \times \Xi \times \Delta,$$

since $\mathcal{M}(\xi_{in}, \xi_{out})\Psi(\xi_{in}, \xi_{out}) = 0$ by construction.

Now, pre- and post multiplying the above by η'_a and η_a , respectively, leads to (5.14) taking the definitions of η_{in} and η_{out} in (5.26) into account and the fact that:

$$G(\xi_{in}) = \mathcal{F}\Pi_a(\xi_{in})\eta_{in}, \quad K(\xi_{out}) = \mathcal{S}\Pi_a(\xi_{out})\eta_{out}.$$

Next, notice that (5.34) implies $\mathcal{D} \in \Xi$; see, e.g., (BOYD et al., 1994). Then, consider the condition $2Q_1 - Q_2 - \varphi\mathbf{I}_n \geq 0$ on the right-hand side of (5.33). From Lemma 5.1, it follows that:

$$P_2^{-1} - \varphi\mathbf{I}_n \geq 0,$$

since $P_2 = Q_1^{-1}Q_2Q_1^{-1}$. Hence, multiplying the above by $P_2\rho$ with $\rho = \varphi^{-1}$, the following is obtained

$$\rho\mathbf{I}_n - P_2 \geq 0$$

Thus, the rest of this proof follows straightforwardly from Theorem 5.1. \square

Similarly to the stability analysis counterpart, Theorem 5.2 can be applied for designing the state feedback gain matrices $G(\xi_{in})$ and $K(\xi_{out})$, as defined in (5.5), in order to robustly regionally stabilize the coupled PDE-ODE system of (5.1) and (5.2) in closed-loop while maximizing the volume of \mathcal{B} for a given set \mathcal{A} . Thus, the following optimization problem solves problem **P2**:

$$\max_{\mu, \varphi, Q_1, Q_2, F_0, \dots, S_n, L} \log \det(Q_1) \text{ subject to (5.33)-(5.35)}. \quad (5.37)$$

Notice that the matrix inequalities in (5.33)-(5.35) are LMIs when φ and μ are given *a priori*. Hence, a solution to the above optimization problem is searched for via semi-definite programming by applying a gridding technique over φ and μ .

5.4 Application to a Poiseuille Flow

The fluid transport system is normally used for industrial applications such as the ventilation system of mining industry (WITRANT; JOHANSSON; TEAM, 2008) and hydraulic networks (DOS SANTOS; PRIEUR, 2008). To investigate the phenomenon of fluid transport in a Poiseuille flow subject to dynamic boundary conditions, an experimental setup has been designed to test and validate the proposed results.

The experimental setup consists on a heating column encasing a resistor, a tube, two ventilators, a gas speed meter and three distributed temperature sensors. The control objective is tracking the outlet temperature by driving the power dissipated on the heating resistor at different air flow speeds through the tube. Only the outlet temperature and the flow speed will be considered as measurements for a closed-loop boundary control strategy and thus $G(\xi_{in})$ will be set to be zero. In this work, it is assumed that the output ventilator is set to be off.

The modeling of the experimental setup is done by considering two subsystems: the heating column and the tube. One-dimensional transport model is used to describe the gas density, speed and pressure variations in the tube. For the dynamic boundary conditions, it is considered a zero-dimensional model of control volume approach with heat

exchanges coming from the heating resistor in the column. The perfect gases law is used to convert density on temperature.

System description and modeling are presented in Chapter 3, Section 3.1. Applying some simplifications (according Chapter 4, Section 4.4) the system of PDEs reduces to the following convection equation

$$\frac{\partial \rho(x,t)}{\partial t} + v(t) \frac{\partial \rho(x,t)}{\partial x} = 0, \quad (5.38)$$

where $v(t)$ is the time-varying convection parameter of (5.38). The gas speed $v(t)$ in the tube is measured.

For the dynamic boundary condition, considering that the input mass flow rate to be expressed as $\dot{m}_{in} = v(t)\rho(0,t)A_t$. Thus, the dynamic boundary condition can be expressed as:

$$\begin{aligned} \dot{\rho}(0,t) = & -\frac{R\gamma T_{in}v(t)A_t}{p_{in}V_0} \rho(0,t)^2 + \frac{\gamma v(t)A_t}{V_0} \rho(0,t) \\ & - \frac{R}{p_{in}V_0 C_v} \rho(0,t) dQ, \end{aligned} \quad (5.39)$$

where A_t is the tube cross section area.

5.4.1 Temperature Boundary Control

The output temperature boundary control is designed for (5.38) with boundary conditions (5.39). Define the density error as:

$$\xi = \rho - \rho_{ref}, \quad (5.40)$$

where ρ_{ref} is the desired output density. It is easy to show that system (5.38) with boundary conditions (5.39) can be expressed as follows:

$$\frac{\partial \xi(x,t)}{\partial t} + \Lambda(\delta) \frac{\partial \xi(x,t)}{\partial x} = 0, \quad (5.41)$$

with the boundary conditions:

$$\dot{\xi}_{in} = -A(\delta)\xi_{in}^2 + a(\delta)\xi_{in} - B\xi_{in}dQ, \quad \xi_0 = \xi_{in}, \quad (5.42)$$

where $A(\delta) = \frac{R\gamma T_{in}v(\delta)A_t}{p_{in}V_0}$, $a(\delta) = \frac{\gamma v(\delta)A_t}{V_0}$ and $B = \frac{R}{p_{in}V_0 C_v}$

The speed $v(\delta)$ is measured and is considered an uncertain parameter. The temperature inside the heating column is not measured. Define the control input using a feedback (partial) and feedforward linearization as:

$$dQ = F(\delta, \rho_{ref}) + K(\xi_{out})\xi_{out}, \quad \xi_{out} = \xi(L, t), \quad (5.43)$$

with $F(\delta, \rho_{ref}) = C_v \gamma v(\delta) A_t \left(\frac{p_{in}}{R} - T_{in} \rho_{ref} \right)$.

This yields to the system (5.41) with boundary condition (5.42), which corresponds to the system considered in Theorem 5.1 and Theorem 5.2. Define the convex set $\Delta = [1, 3]$ as the flow speed 1 *m/s* and 3 *m/s* in order to operate the experimental setup between the temperatures of 300 *K* and 325 *K*.

In order to design the boundary control for system (5.41) with boundary condition (5.42), consider the control architecture present in Figure 21

Consider the following system parameters: adiabatic constant $\gamma = 1.4$; molar mass of dry air $M = 28.97$ *g/mol.K*; ideal gas constant $R = 8.3143$ *J/(mol.K)*; specific heat constant for constant pressure $C_p = 1.005$ *KJ/Kg.K* for constant volume $C_v = 0.718$ *KJ/Kg.K*; initial pressure $p_{in} = 1$ Bar (or, equivalently, 1×10^5 Pa in SI units); initial temperature $T_{in} = 300$ *K*; initial density $\rho = 1.1768$ *Kg/m³* column volume $V_0 = 4 \times 10^{-3}$ *m³*; tube cross section area $A_t = 6.4 \times 10^{-3}$ *m²*; and the tube length $L = 1$ *m*.

In this scenario, we have applied Theorem 5.1 with (5.25) considering a linear control law $u_L = K\xi_{out}$ as well as a nonlinear one $u_{NL} = (K_0 + \xi_{out}K_1)\xi_{out}$ with

$$\begin{aligned} K &= -0.8 \\ K_0 &= -0.8, \quad K_1 = 8, \end{aligned} \quad (5.44)$$

and an initial temperature error equal to 25 *K*. Figure 22 displays simulation results for the two control laws as above, $\mu = 0.3$, where $e_L(t)$ denotes $e(t)$ for $u = u_L$ and $e_{NL}(t)$ corresponds to the case where $u = u_{NL}$. It is observed that the convergence of the temperature error with the non-linear gain outperformed the linear counterpart.

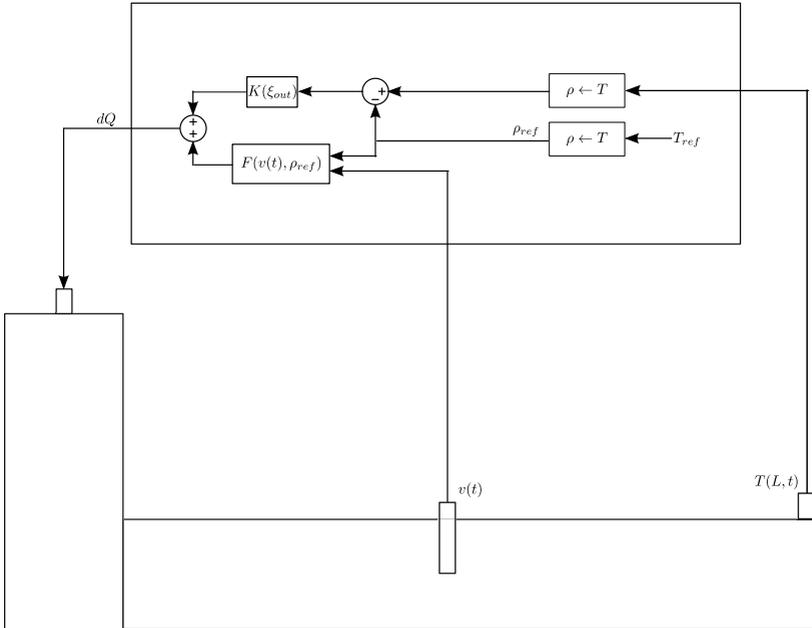


Figure 21 – Flow tube control architecture.

Now, in order to illustrate the effectiveness of the proposed non-linear boundary control design strategy, some simulation results of system (5.41) with boundary condition (5.42) are given considering a controller designed by applying Theorem 5.2 with (5.37) for $\mu = 0.5$. Thus, the following gains are obtained:

$$K_0 = -0.9, \quad K_1 = 9.5. \quad (5.45)$$

The closed-loop results obtained are presented in Figures 23 and 24. It is supposed a change on the temperature reference from 300 K to 325 K (transformed into a density reference using the perfect gases law) for the simulations.

As depicted in Figure 23, the system effectively follows the change of the reference for different flow speeds. The faster is the flow speed, the faster is the convergence time of the fluid transport.

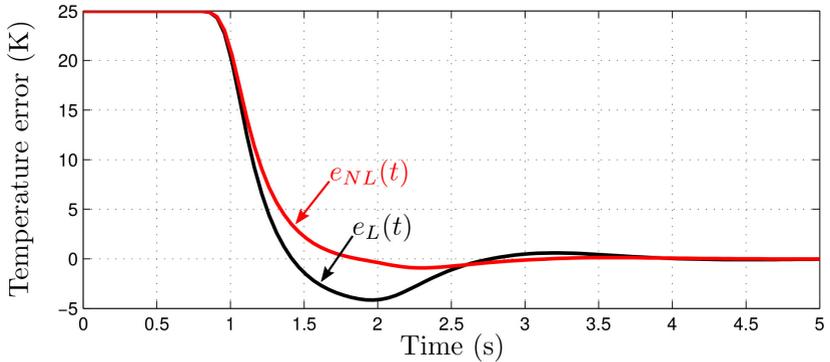


Figure 22 – Temperature error trajectories $e_L(t)$ and $e_{NL}(t)$ for flow speed 1 m/s .

Figure 24 shows the respective control inputs obtained for the latter simulations.

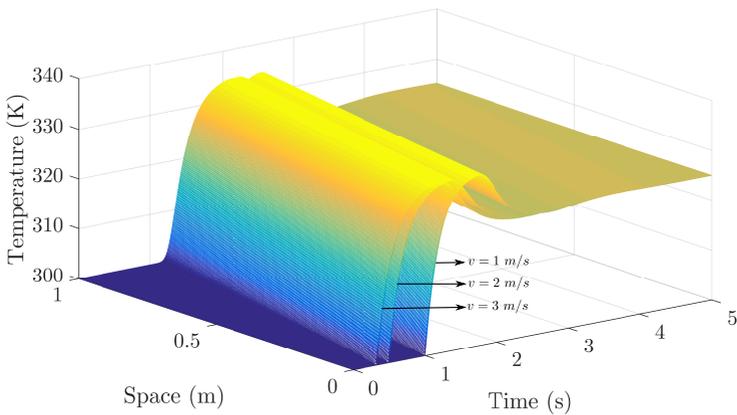


Figure 23 – Output temperature boundary control results using the proposed control (5.45) to the coupled PDE-ODE experimental setup.

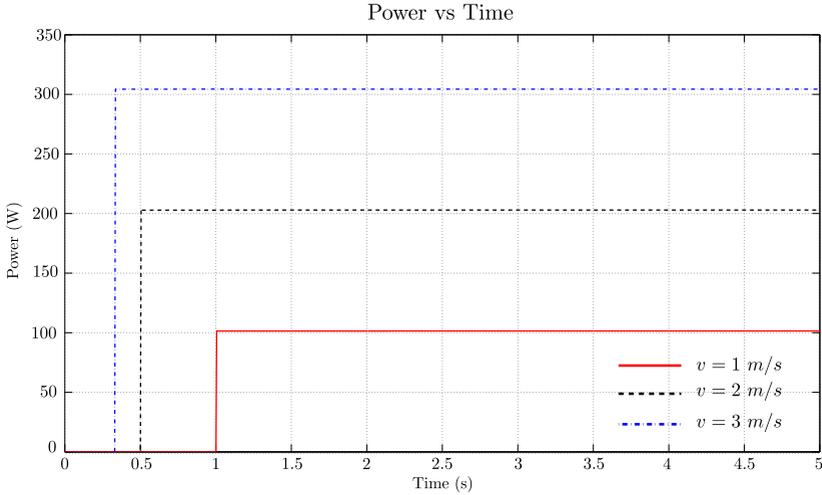


Figure 24 – Control inputs for $v = 1 \text{ m/s}$, $v = 2 \text{ m/s}$ and $v = 3 \text{ m/s}$.

5.5 Concluding Remarks

This chapter was concerned with the boundary control of uncertain first-order hyperbolic system subject to nonlinear boundary actuation dynamics. An LMI based result has been proposed for assessing the regional robust exponential stability of the closed-loop system. The stability result is then extended to cope with nonlinear boundary control design based on strict Lyapunov functions. An experimental setup was given to illustrate the approach where simulation results have shown the effectiveness of the proposed strategy.

6 CONCLUSION

This thesis has focused on the study of boundary control for coupled first-order hyperbolic systems with nonlinear dynamic boundary conditions. To the author's opinion, the main thesis contributions rely on the development of two novel results in the infinite-dimensional setting for the stability analysis (i.e., the dissipativity of the boundary conditions) and control design for this class of systems. In the sequel, the main aspects considered in this thesis are summarized.

In the first part of this thesis, a literature review on the thesis subject and a study on the background concepts of hyperbolic PDE systems in Riemann coordinates including boundary feedback control are performed.

Then, finite-dimensional analysis and control tools are applied to the boundary control design of a first-order hyperbolic system with a nonlinear boundary dynamic action. The classical approach is applied to an experimental setup consisting of a tube on a pipe, a heating column, a ventilator and a sensor to measure the outlet temperature. The experimental setup is modeled in terms of a first-order hyperbolic PDE with a nonlinear boundary dynamics (i.e., the heating column and the ventilator). Hence, the outlet temperature tracking problem of the experimental setup is addressed in a simple manner considering linear approximations around the equilibrium point, finite difference schemes and an integral action leading to an augmented discrete-time linear system whose dimension depends on the step size of discretization in space. It can be concluded that in some cases that the tracking problem via boundary control of infinite-dimensional systems can be based on a finite-dimensional model and standard classical control tools, however at the cost of some poor performance. Most part of these results were presented at the 2015 IEEE Multi-Conference on Systems and Control (MSC); see (CALDEIRA et al., 2015).

In the second part of this thesis, two infinite-dimensional approaches have been proposed to the boundary of coupled (first order hyperbolic) PDE and nonlinear ODE systems. In the first one, the

first-order hyperbolic PDE system is approximated by a pure delay that together with an uncertain nonlinear quadratic boundary dynamics led to an input delayed nonlinear quadratic system in closed-loop. Strict conditions for the regional stability and stabilization of uncertain input delayed nonlinear systems based on Lyapunov-Krasovskii (L-K) method with slack variables are formulated in terms of Linear Matrix Inequality (LMI) constraints. This approach is then applied to the proposed experimental setup showing a significative increase in the closed-loop performance compared to the classical finite-dimensional approach. The better performance occurs at the cost of a larger numerical complexity. These results have been submitted for a possible publication in the International Journal of Robust and Nonlinear Control; see (CALDEIRA et al., 2016).

In the sequel, an infinite-dimensional approach based on strict Lyapunov conditions is proposed to the regional stability and stabilization of first-order hyperbolic systems with an uncertain nonlinear quadratic dynamic boundary condition. The proposed result ensures the local exponential stability of the closed-loop system while determining an estimate of the set of admissible initial conditions. When applied to the proposed experimental setup, this approach has also outperformed the performance of the closed-loop system considering the classical approach. The stability and stabilization results have been summarized in (CALDEIRA et al., 2017) which is currently under construction and to be submitted to a peer reviewed journal.

To the author's knowledge, until the end of this thesis, this is the first work to address the robust regional stability and stabilization problems of a first-order hyperbolic system coupled with a nonlinear dynamic boundary condition.

6.1 Correlated work

During the thesis development, the thesis author have devised some correlated works which are summarized in the sequel.

- \mathcal{H}_∞ ε -guaranteed cost of uncertain discrete-time systems with time-varying state delays (CALDEIRA et al., 2012) – this paper has proposed an alternative approach to the robust \mathcal{H}_∞ performance analysis of discrete time systems with a time-varying delay in the state vector subject to polytopic uncertainties. In this proposal, sufficient conditions are combined for robust \mathcal{H}_∞ performance analysis in terms of feasibility of convex problems described as LMIs and a technique of polytope partition combined with branch-and-bound (BnB) algorithm. This allows to calculate the \mathcal{H}_∞ guaranteed cost with a prescribed precision. The procedure consists in the successive partition of the polytope containing the matrices that represent the system and, at each partition, the \mathcal{H}_∞ performance index of the system is evaluated by means of functions that give upper and lower bounds for such index. The possibility of determining the precision of the calculation of \mathcal{H}_∞ guaranteed cost comes from the fact that the BnB algorithm uses such upper and lower bound functions that converge to the optimum value. Numerical examples are presented to illustrate the efficiency of the proposal.
- *Numerically Tractable Stability Tests for 2-D Singular Discrete-Time Systems* (CALDEIRA et al., 2013) – this paper presents numerically tractable LMI conditions to the stability analysis of 2-D singular discrete-time systems. Two classes of 2-D singular systems were considered. The 1st one is the general singular Fornasini-Marchesini model and the 2nd one is a singular Roesser model. In addition, the proposed conditions also guarantee that the 2-D singular system is acceptable and jump modes free. Numerical examples are presented to illustrate the proposed approach. This result has been developed in the early stages of this thesis in order to derive a reduced-order finite-dimensional representation of PDE systems. However, this approach showed to be quite conservative for analyzing the stability of hyperbolic PDEs.

6.2 Future Research

This thesis has proposed numerical tractable solutions to the regional stability and stabilization of first-order hyperbolic systems coupled to nonlinear dynamic boundary conditions. However, there are still several points to be further investigated in the context of nonlinear systems. The thesis author will concentrated on investigating some of the following issues:

- Investigate the dissipativity of boundary conditions for linear inhomogeneous hyperbolic systems.
- Devise state observer design conditions for first-order linear hyperbolic system coupled with dynamic boundary conditions in order to obtain output feedback boundary controllers.
- Study the stability and stabilization problems for linear hyperbolic system coupled with switched boundary conditions.
- Practical implementation of the experimental setup with the application of the proposed solutions.

In particular, the delay input approach is directly applicable to nonlinear input delayed systems which has a vast number of practical applications besides the application on the boundary control of first-order hyperbolic systems. In this context, the following subject will be further investigated:

- Application of state and parameter dependent L-K functionals in order to obtain less conservative results.
- Development of \mathcal{H}_2 and \mathcal{H}_∞ control design conditions for this class of systems.
- Extend the methodology to a larger class of nonlinear input delayed systems.

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