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Nicolás Espitia. Event-based control of networks modeled by a class of infinite dimensional systems. Automatic. Université Grenoble Alpes, 2017. English. NNT : 2017GREAT038 . tel-01693069

HAL Id: tel-01693069

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Communauté UNIVERSITÉ Grenoble Alpes

THÈSE

pour obtenir le grade de

DOCTEUR DE LA COMMUNAUTÉ UNIVERSITÉ DE GRENOBLE ALPES

Spécialité : **Automatique et Productique**

Arrêté ministériel : 25 mai 2016

Présentée par

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préparée au sein du
laboratoire GIPSA-LAB
dans **l'école doctorale Electronique, Electrotechnique, Automatique et Traitement du Signal (EEATS)**

Event-based control of networks modeled by a class of infinite-dimensional systems

Thèse soutenue publiquement le **22 Septembre 2017**,
devant le jury composé de:

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Remerciements

Je tiens tout d'abord à remercier mes encadrants de thèse Antoine Girard, Nicolas Marchand et Christophe Prieur. Ils ont été toujours présents pour m'écouter, me conseiller et m'encourager. Merci pour toute l'aide tout au long de la thèse. Leur vaste connaissance, leur rigueur scientifique et leur humilité scientifique, m'inspirent pour devenir un grand chercheur à l'avenir. Mes remerciements particuliers à Christophe qui, au sein du département, a suivi toutes les étapes de ma formation et avec qui j'ai eu le plaisir de courir deux semi-marathons. Merci pour les encouragements aussi bien pendant la thèse que pendant les courses.

J'exprime ma reconnaissance aux rapporteurs Dr. Thomas Meurer et Dr. Mario Sigalotti. Merci pour les remarques et suggestions précieuses. Également, je remercie les examinateurs Dr. Nicolas Carreno, Dr. Wilfrid Perruquetti et Dr. Alexandre Seuret pour avoir accepté évaluer mon travail.

Je suis également profondément reconnaissant à l'égard du Labex Persyval-lab pour m'avoir soutenu pendant les trois dernières années. Un grand merci spécialement à Anne-Laure Bernardine pour son aide et soutien constants et amicaux.

Je remercie l'ensemble de permanents d'Auto, tout particulièrement avec ceux j'ai eu la chance de discuter et partager: Mirko Fiacchini, John-Jairo Martinez, Nacim Meslem, Gildas Besançon, Emmanuel Wittrant, Olivier Sename, Mazen Alamir, Alain Kibangou. Remerciements spéciaux à Luc Dugard et Christian Commault pour ses conseils, pour leur gentillesse et leur bonne humeur. Ils sont les deux pères Noël pour le Gipsa-Doc!

Un grand merci au professeur Georges Bastin que j'ai eu l'honneur de rencontrer. Merci pour les brefs discussions et pour son travail sur lequel le coeur de ma thèse à été inspirée.

Je tiens aussi à remercier Sophie Tarbouriech et Aneel Tanwani avec qui j'ai commencé une collaboration très fructueuse.

Je remercié l'ensemble du personnel de Gipsa et tout particulièrement des remerciements spéciaux à Sonia Nogueira, Fanny Soler, Patricia Reynier et Martine Brossier.

Je voudrais remercier beaucoup l'ensemble de doctorants et post-doctorants d'Auto: Bruno pour son bon humeur et son "caca fuente", Maëlle, ma première amie au Gipsa qui m'a introduit dans l'association Gipsa-doc. Egalement, Andres, Insaf, Nacim, Rachid, Sebin, Sophie, Sandra, Ronak, Diego, Nadia, Raul, Jonatan, Jean et Jorge Delgado pour tous le moments agréables partagés. Des remerciements spéciaux à Dacheng Zhang, mon pote Chinois, avec qui j'ai partagé de très agréables moments dans notre bureau. À Yashank Gupta, pour être avec/de qui rigoler et son amitié. À Swann Marx, “parce”, pour son amitié et tous les discussion fructueuses. À Miguel Davo, qui s'annonce comme un des meilleurs Automaticiens de la communauté; merci pour son amitié et pour partager avec moi toutes ses vastes connaissances. Finalement, à Juan Tellez - mon partner des courses; merci pour être plus qu'un “compadre":

un frère.

Je tiens également à remercier mes amis rencontrés à Grenoble, aussi bien à mes amis Français, Justine et Marco, que j'ai rencontrés en Colombie et qui font désormais partie de ma famille. Un grand merci à mes potes Cyrielle Varela et Paola Gomez, à mon cousin Juan C Romero "el Loco" et notamment à mes deux potes de la vie, Leonardo Gutierrez et Daniel Giraldo. Merci pour tout le soutien et leur amitié inconditionnelle.

À Valentina pour son amour, son soutien inconditionnel et sa patience tout au long de la thèse. “El que hay pa eso?" sigue... Mon remerciement final est le plus important pour moi. Il va à ma famille. Il n y'a pas de mots qui puissent vraiment exprimer ma plus grande gratitude et l'amour pour eux.

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Notation

Some notations and preliminary definitions used throughout the thesis are as follows:

- $\mathcal{I}_n$ is the set of the number of compartments, numbered from 1 to n;
- $\mathcal{I}_{in} \subset \mathcal{I}_n$ is the index set of input compartments;
- $\tilde{\mathcal{I}}_{in} \subset \mathcal{I}_{in}$ is the index set of input compartments receiving exogenous input flows only;
- $\mathcal{I}_{out} \subset \mathcal{I}_n$ is the index set of outputs compartments;
- $\mathcal{D}_i \subset \mathcal{I}_n$ is the index set of downstream compartments connected directly to compartment i (i.e. those compartments receiving flow from compartment i);
- $\mathcal{U}_i \subset \mathcal{I}_n$ is the index set of upstream compartments connected directly to compartment i (i.e. those compartments sending flow to compartment i);
- $\mathcal{R} \subset \mathcal{I}_n$ is the index set of routing compartments (i.e. those compartments sending flow to two or more downstream compartments);
- $\mathbb{R}^+$ will denote the set of nonnegative real numbers;
- For any vector v, its dth component will be denoted v_d. Given a matrix A, its transpose will be denoted A^T and its component at row i and column j will be denoted by A_{ij}. 0 will denote the zero matrix of suitable dimension;
- The usual Euclidean norm in $\mathbb{R}^n$ is denoted by $|\cdot|$ and the associated matrix norm is denoted $\|\cdot\|$;
- The set of all functions $\phi : [0,1] \to \mathbb{R}^n$ such that $\int_0^1 |\phi(x)|^2 dx < \infty$ is denoted by $L^2([0,1],\mathbb{R}^n)$ that is equipped with the norm $\|\cdot\|_{L^2([0,1],\mathbb{R}^n)}$;
- The restriction of a function $y : I \to J$ on an open interval $(x_1, x_2) \subset I$ is denoted by $y|_{(x_1,x_2)}$;
- Given an interval $I \subseteq \mathbb{R}$ and a set $J \subseteq \mathbb{R}^n$ for some $n \geq 1$, a *piecewise left-continuous function* (resp. a *piecewise right-continuous function*) $y : I \to J$ is a function continuous on each closed interval subset of I except maybe on a finite number of points $x_0 < x_1 < \ldots < x_p$ such that for all $\in \{0, .., p-1\}$ there exists y_l continuous on $[x_l, x_{l+1}]$ and $y_{l|(x_l,x_{l+1})} = y_{|(x_l,x_{l+1})}$. Moreover, at the points $x_0, \cdots, x_p$ the function is continuous from the left (resp. from the right). The set of all piecewise left-continuous functions (resp. piecewise right-continuous functions) is denoted by $\mathcal{C}_{lpw}(I,J)$ (resp. $\mathcal{C}_{rpw}(I,J)$). In addition, we have the following inclusions $\mathcal{C}_{lpw}([0,1],\mathbb{R}^n)$, $\mathcal{C}_{rpw}([0,1],\mathbb{R}^n) \subset L^2([0,1],\mathbb{R}^n)$.

Résumé détaillé

Contexte et motivation

Plusieurs systèmes physiques sont décrits par des systèmes de dimension infinie, notamment par des équations aux dérivées partielles. Une classe très importante ressort: Celle des equations aux dérivées partielles (EDP) hyperboliques. Les systèmes hyperboliques ont été utiles spécifiquement pour la modélisation des systèmes physiques de différentes natures: par exemple des réseaux hydrauliques [6], des réseaux trafic routier [9], des réseaux de transport de gas [30], en mentionnant quelques-uns. Plus d'exemples illustratifs de systèmes régis par les équations hyperboliques peuvent être trouvés dans [3, Chapitre 1].

La commande et la stabilité de tels systèmes ont également attiré beaucoup l'attention au cours de la dernière décennie. Au ce sujet, il y a deux moyens d'agir sur les systèmes: la commande distribuée et la commande frontière. Pour la commande frontière, l'approche backstepping [37, 12] et les techniques de Lyapunov [10, 26, 50, 46] sont les plus généralement utilisées. Quelques applications, dont les actions de contrôle sont aux bornes, peuvent être retrouvées par exemple dans [31, 5, 11, 20, 52] où la stabilité exponentielle des états au régime permanent dépend de la *dissipativité* de la frontière. Plusieurs résultats sur la modélisation de systèmes physiques dans le cadre des systèmes hyperboliques ainsi que la stabilisation de tels systèmes sont largement condensés dans [3].

Alors que les systèmes hyperboliques font partie du coeur de cette thèse, un contexte mathématique bien détaillé n' est pas fourni ici. Nous allons certainement référer les lecteurs à [3] et quelques références qui se trouvent également pour apprécier les contributions principales sur la stabilisation (globale et locale) des systèmes hyperboliques linéaires où quasi-linéaires et leur analyse sous des normes différentes. Dans cette thèse, nous voulons premièrement souligner un exemple motivant pour la modélisation et la commande des réseaux: tout particulièrement, les *réseaux de communication*. Comme nous le verrons dans la suite, étudier les réseaux de communication va nous permettre de proposer une classe de systèmes dont les systèmes hyperboliques peuvent être couplés aux équations aux dérivés ordinaires (EDO), même possiblement avec une structure en cascade.

À ce titre, lorsqu'on traite les réseaux, un moyen pour les décrire est précisément grâce à la modélisation "fluid-flow". À notre connaissance, la modélisation "fluid-flow" des réseaux physiques est une façon de décrire le fluxe de matière à travers des différents éléments du réseaux, qui consiste aussi bien d'une collection des noeuds lesquels se communiquent entre eux, que des lignes qui les connectent. Une des propriétés les plus importantes de la modélisation fluid-flow est la conservation de la masse. Divers modèles dont cette propriété est préservée, peuvent tomber dans le cadre d'une description macroscopique. L'exemple le plus connu est plus étudié est l'évolution du trafic des véhicules dans la route ([27]), dont les vari-

ables macroscopiques sont principalement la densité des voitures et la vitesse moyenne. Le modèle dans ce cadre est donné par une équation hyperbolique de lois de conservation d'une dimension ([9, 67, 3]). Nous présenterons brièvement le modèle ci-dessous:

$$\partial_t \rho(t,x) + \partial_x f(\rho(t,x)) = 0 \tag{1}$$

aussi bien appelé *équation de continuité* où ρ est la densité des voitures (par exemple #voitures/km) à chaque instant de temps t et position x tout au long de la route, et $f(\rho)$ représente le débit des voitures à (t,x). Le débit f est typiquement une fonction de la densité et la vitesse des voitures v, c.a.d $f(\rho) := \rho v(t,x)$. Sous l'hypothèse que les conducteurs adaptent leur vitesse à la densité du trafic local ($v = V(\rho)$), on a le modèle LWR (Lighthill and Whitham in 1955 and then Richards 1956).

$$\partial_t \rho(t,x) + \partial_x (\rho V(\rho)) = 0 \tag{2}$$

D'où, en s'inspirant du flux de trafic sur les réseaux routiers brièvement décrit ci-dessus, plusieurs études portent sur le flux d'information sur les réseaux de communication constitués des lignes de transmission et des noeuds (voir par exemple [16]). Le modèle est exactement comme (2) sauf que ρ est désormais la densité des paquets qui voyagent à travers de la ligne de transmission et le flux est une fonction de la densité et la vitesse moyenne des paquets. Un problème commun sur le trafic routier et les réseaux de communication dans le cadre des EDPs est lié à la congestion. Pour les deux applications, il existe des densités critiques qui divisent le fonctionnement du réseau en deux parties selon le *diagrammme fondamental du flux-densité*: la première partie concerne la zone " *free-flow*" et l'autre est la zone de *congestion*. Cependant, la principale différence du modèle d'un réseau de communication par rapport au réseau routier est que la vitesse moyenne des paquets est censée être constante. Dans ce scénario, le diagramme fondamental est alors donné comme suit: Nous verrons qu'en

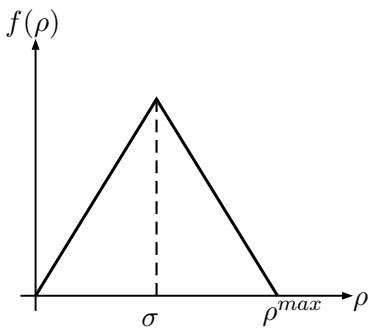

Figure 1: Diagramme fundamental triangulaire de flux-densité pour la loi de conservation scalaire.

raison de la linéarité par morceaux du diagramme fondamental et en supposant des conditions "free-flow", nous finirons par travailler avec des systèmes hyperboliques linéaires de lois de conservation. En plus des modèles macroscopiques, il y a des systèmes de compartiments qui sont également très connus comme un cadre approprié pour décrire des réseaux dynamiques conservatifs. On peut voir par exemple [36] pour l'étude de la dynamique des compartiments, même avec des décalages en tant que retards qui s'avèrent être représentés par des équations

de transport ou tout simplement, lois de conservations linéaires. Dans [4], le contrôle de congestion des réseaux de compartiments est étudié, toujours sous la modélisation “fluid-flow", dans lequel ils exploitent des propriétés des systèmes positifs pour bien établir la stabilité du réseau et ainsi prévenir la congestion, grace aux contrôleurs non linéaires. Dans ces études, chaque noeud représente un compartiment qui contient une quantité variable, par exemple, l'information qui est en train d'être traitée. Conceptuellement, un compartiment c'est une sorte de dispositif de stockage et typiquement modélisé par des EDOs ou par des EDPs lorsque les quantités accumulées sont alors spatialement distribuées. Des travaux similaires à [4], nous pouvons aussi trouver des modèles non linéaires en temps continu en utilisant l'approach “fluid-flow", comme a été introduit dans [44] ou le “fluid-flow" basé sur la conservation pour le contrôle de congestion comme dans [7].

La combinaison des réseaux de compartiments décrits soit par des EDOs ou EDPs, est un sujet qui sera étudié dans cette thèse. Par consequence, quelques questions préliminaires peuvent survenir concernant la stabilité des réseaux de communication:

- *Pour des demandes de debit d'entrée au réseau, est-ce qu'on peut établir des propriétés de stabilité dans un sens approprié pour le modèle linéaire résultant?*

- *En outre, sous des actions de contrôles appropriés, peut-on améliorer de quelque manière la performance du réseau?*

Pour répondre à ces questions, nous nous focaliserons sur le contrôle frontière des systèmes EDP-EDO couplés comme nous regarderons dans le Chapitre 1.

D'autre part, en supposant que les contrôleurs aux bornes en temps continu sont conçus de manière appropriée et sont capables de conduire le système vers un comportement désiré, on peut se demander sur les implémentations possibles dans une plate-forme numérique. Plus précisément, nous étudions le contrôle numérique des EDPs. Dans cette thèse, nous proposons une approche à cette fin: *la commande événementielle.*

À propos de la commande événementielle

La commande événementielle est une stratégie de contrôle numérique qui a pour objectif d'utiliser les ressources informatiques et de communication de façon efficiente tout en mettant à jour les grandeurs de contrôle apériodiquement, seulement lorsqu'il est nécessaire. Plusieurs travaux ont été développés dans ce domaine pour les systèmes de dimension finie (voir par exemple les travaux précurseurs [1, 70] ou les plus récents [33, 49, 65] et les références qui y se trouvent dedans). Il y a deux composantes qui sont essentielles dans le cadre de la commande événementielle. Le premier est une loi de contrôle par retour ou “feedback" qui est conçue pour stabiliser le système. Le deuxième composant est une stratégie de déclenchement ou “trigger" qui détermine les instants quand le contrôleur a besoin d'être mis à jour. La stratégie de trigger le plus souvent utilise une règle statique obtenue par une propriété de stabilité d'entrée-état

(ISS en anglais) comme donné dans [59]. Une extension de cette stratégie est donnée dans [28] où une variable interne dynamique est introduite dans la loi de trigger, pour laquelle il est possible de réduire le nombre des instants de mise à jour du contrôle, par rapport à cela statique. D'autres approaches reposent directement sur la dérivée par rapport au temps de la fonction de Lyapunov ([45, 57]). En plus de l'intérêt de réduire des charges de communication et informatiques, la commande événementielle est aussi connue comme une façon rigoureuse de mettre en oeuvre numériquement les contrôles continus en temps.

En fait, la conception des stratégies de commande événementielle pour les systèmes de dimension infinie est rarement étudiée dans la littérature. L'extension des résultats déjà existant pour les EDOs à systèmes en retard peut être envisagée (dans le cadre et formulation par exemple de [54]) comme a été proposé par exemple dans [21]; néanmoins, ceci est un peu loin du problème que nous attaquerons dans cette thèse. Pour les systèmes paraboliques d'autre part, la commande événementielle est considérée dans [55] et [69]. Beaucoup des difficultés, qui surviennent dans le contexte de la commande événementielle, sont à cause du fait qu'on introduit des discontinuités lorsqu'on met à jour le contrôle. Des contrôleurs de retour de sortie discontinues pour les systèmes de dimension infinie ont été étudiés par exemple dans [48], où un contrôle de retour unitaire et à son tour, une stabilisation globale asymptotique ont été considérés. Bien que le cadre de systèmes hyperboliques à commutation [32, 53, 38] est très inspirant - tout spécialement les travaux dans [38]- pour traiter les caractères bien posés des solutions du système en boucle fermé sous les stratégies de commande événementielle.

En outre, pour la plupart des controlleurs frontière des systèmes hyperboliques, la commande numérique n'a pas été bien étudiée en général. En fait, pour la commande des EDPs, la commande numérique repose généralement sur la réduction du modèle en discrétisant l'espace de façon à ce qu'on obtienne des équations ordinaires. Dans ce cas-là, les approches issues de dimension finie pour la commande numérique peuvent être appliquées. Pourtant, sans une réduction de modèle, ce n'est pas du tout clair à quelle vitesse on doit échantillonner, de façon périodique, avec le but de faire une implementation sur une plateforme numérique. Après, pour des scénarios de grande échelle dont les capteurs et actionneurs sont spatialement distribués, l'information est transmise à travers des canaux de communication numérique. Par consequence le besoin de réduire la consommation d'énergie et sauvegarder les ressources informatiques est aussi un sujet central et clé.

Quelques questions arrivent naturellement lorsqu' on pense à la commande aux bornes des systèmes hyperboliques:

- *À quelle vitesse doit-on échantillonner périodiquement les contrôleurs frontière du temps continu, tout en préservant des propriétés du système EDP?*
- *Pouvons-nous étendre les stratégies de la commande événementielle issues des systèmes de dimension finie pour les systèmes de dimension infinie?*
- *Peut-on garantir aussi bien les propriétés de stabilité que le caractère bien posé du système qu'on considère sous contrôleurs événementiels?*

Soulignons brièvement les contributions principales de cette thèse.

Contributions et structure de la thèse

1. La première contribution de cette thèse concerne la modélisation et la commande frontière des réseaux à grande échelle des compartiments sous la modélisation “fluid-flow" appliqué aux réseaux de communication comme il sera présenté dans le Chapitre 1. Au cours de l'étude dans ce chapitre-là, nous finirons avec le système linéaire couplé EDP-EDO, pour étudier la stabilité et quelques aspects de la commande aux bornes via les techniques de Lyapunov et la formulation des inégalités matricielles (LMIs):

$$\begin{cases} \partial_t y(t,x) + \Lambda \partial_x y(t,x) = 0 \\ \dot{Z}(t) = AZ(t) + G_y y(t,1) + B_w W(t) + D\tilde{d}(t) \end{cases} \tag{3}$$

avec condition aux bornes

$$y(t,0) = G_z Z(t) + B_u U(t) \tag{4}$$

et condition initiale

$$\begin{cases} y(0,x) = y^0(x), \quad x \in [0,1] \\ Z(0) = Z^0 \end{cases} \tag{5}$$

Nous donnons une condition suffisante pour la stabilité ISS en boucle ouverte et nous effectuons la synthèse de contrôle dans le cadre de boucle fermée. Des problèmes d'optimisation sont également abordés.

2. Dans le Chapitre 2 , nous présentons la première approache pour la commande événementielle pour les systèmes hyperboliques de lois de conservation inspirés par des principales stratégies sur les systèmes de dimension finie. On doit dire que les résultats dans ce chapitre sont uniquement appliqués à un cas particulier du système (3)-(5) (réseaux de communication sans buffers, c.a.d couplage EDO et sans des perturbations exogènes). Ainsi, le système qui sera étudié a la forme suivante:

$$\partial_t y(t,x) + \Lambda \partial_x y(t,x) = 0, \qquad x \in [0,1], \quad t \in \mathbb{R}^+ \tag{6}$$

avec la condition aux bornes

$$y(t,0) = Hy(t,1) + Bu(t), \qquad t \in \mathbb{R}^+ \tag{7}$$

la condition initiale

$$y(0,x) = y^0(x), \qquad x \in [0,1] \tag{8}$$

et la fonction de sortie

$$z(t) = y(t,1) \tag{9}$$

En utilisant les techniques de Lyapunov dans la norme L^2, nous proposons trois stratégies de commande événementielle dont la forme $u = \varphi(z)$ (retour de sortie) en impliquant le

temps d'exécution ou trigger qui détermine quand les valeurs du contrôle doivent être mis à jour. Grosso modo, les instants sont donnés par:

$$t_{k+1} = \inf\{t \in \mathbb{R}^+ | t > t_k \wedge \textit{quelque condition trigger appropriée}\}$$

L'analyse de la stabilité exponentielle globale est effectuée et on montre que sous les trois lois de stabilisation événementielle, la solution du système en boucle fermé existe et est unique, alors que le phénomène Zeno est évité.

3. Ensuite, dans le Chapitre 3, nous considérons le problème de stabilisation des EDPs hyperboliques linéaires lorsque les mesures de sortie sont communiquées après avoir été échantillonnées dans le temps et quantifiées dans l'espace. Des contrôleurs statiques et dynamiques sont conçus, ce qui établit la stabilité dans les normes L^2 et H^1 par rapport aux erreurs de mesure. Tout en utilisant des techniques de Lyapunov. Nous montrons que la conception des algorithmes d'échantillonnage garantit une stabilité pratique.

4. Finalement, la dernière contribution concerne la commande événementielle des systèmes hyperboliques linéaires via l'approche Backstepping. Il est prouvé qu'il n' y a pas de phénomène Zeno en raison de l'existence d'un temps minimal entre deux instants trigger (appelé dwell-time) et ainsi le caractère bien posé des solutions et la stabilité sont garantis. La commande événementielle se base sur l'analyse de Lyapunov sur le système cible, ce qui permet de travailler avec une condition de trigger dynamique. Ceci est présenté dans le Chapitre 4.

Introduction

Context and motivation

Several physical systems are described by infinite dimensional systems, namely by partial differential equations (PDEs). A relevant class of infinite dimensional systems stands out: Hyperbolic PDEs. They have been useful in specifically modeling physical networks of different nature: e.g. hydraulic [6], road traffic [9], gas pipeline networks [30], to name a few. More illustrative examples of systems governed by hyperbolic PDEs can be found in a recent book [3, Chapter 1].
The control and stability analysis of such systems have also attracted a lot of attention in the last decade. In this regard, two ways of acting on these systems exist: boundary and in domain control. For boundary control, backstepping [37, 12] and Lyapunov techniques [10, 26, 50, 46] are the most commonly used. Some applications, in which control actions are on the boundary, can be found for instance in [31, 5, 11, 20, 52] where the exponential stability of steady-states depends on the *dissipativity* of the boundary conditions. Several results on the modeling of physical systems in hyperbolic PDE setting along with the stability and boundary stabilization of such systems are widely condensed in [3].

While hyperbolic systems are part of the core of this thesis, a detailed mathematical background of them is not provided. We shall definitely refer the reader to [3] and some references that are also therein to appreciate the main contributions about the stabilization (global and local) of linear and quasi-linear hyperbolic systems and their analysis in different norms. In this thesis however, we want firstly to highlight a motivating example for the modeling and control of networks: particularly *communication networks*. As we will see in the sequel, studying it will allow us to come up with a more general class of systems where hyperbolic equations may be coupled with ordinary differential equations (ODEs), possibly in cascade structure.

To that end, it is worth saying that when dealing with networks, one way to describe them is by means of the *fluid-flow* modeling. To our understanding, fluid-flow modeling of physical networks is a way of describing flow of matter through elements of the network consisting of a finite collection of nodes communicating to each other and links that connect them. One of the most important features of the fluid-flow modeling is the conservation of mass. Several models in which this property is preserved, may result in a macroscopic description. The most traditional and well-studied example is about the evolution of vehicular traffic in roads ([27]), whose macroscopic variables are mainly the density of cars and the averaged velocity. The model in that framework is then given by one dimensional hyperbolic equation of conservation laws ([9, 67, 3]). Let us briefly introduce it as follows:

$$\partial_t \rho(t,x) + \partial_x f(\rho(t,x)) = 0 \tag{1}$$

also called *continuity equation* where ρ is the density of cars (e.g. #cars/km) at time t and

a position x along the road, and $f(\rho)$ represents the flow rate of cars at (t, x). The flow f is typically a function of the density and the velocity of cars v, i.e. $f(\rho) := \rho v(t, x)$. Under the assumption that drivers adapt their speed to the local traffic density ($v = V(\rho)$), one has the so-called LWR model (Lighthill and Whitham in 1955 and then Richards 1956).

$$\partial_t \rho(t, x) + \partial_x(\rho V(\rho)) = 0 \tag{2}$$

Hence, inspired by traffic flow on road networks briefly described above, several studies deal with flow of information on telecommunication networks made up of transmission lines and nodes (see e.g. [16]). The model is exactly as (2) but ρ is now the density of packets traveling through the transmission line and the flow is a function of the density and the average velocity of packets. One common issue on both road traffic and communication networks under PDE setting is related to the congestion. For both applications, there exist critical densities that split the operation of the network in two zones according to the so-called *fundamental diagram of flow-density*: one is the *free-flow* zone and the other is the *congestion* zone. However, a key difference of communication networks model with respect to the road traffic network one is that the averaged velocity of packets is supposed to be constant. In that scenario, the fundamental diagram is then given as follows:

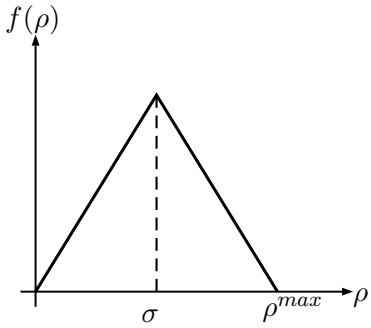

Figure 2: Fundamental triangular diagram of flow-density scalar conservation laws

We shall see that due to the piecewise linearity of the fundamental diagram and by assuming free-flow conditions, we will end up handling linear hyperbolic systems of conservation laws.

In addition to macroscopic models, compartmental systems are also known as suitable framework to describe conservation laws in networks. See for instance [36] for the study of the dynamics of compartmental systems, even with lags representing delays which turn out to be represented by linear transport equations or linear conservation laws. In [4], congestion control of compartmental networks is studied, still under fluid-flow modeling, in which they exploit properties of positive systems to establish stability of the network and prevent congestion by means of nonlinear controls. In those studies, each node represents a compartment which contains a variable quantity, e.g. information being processed. Conceptually, a compartment is a kind of storage device and is typically modeled by ODEs or by PDEs when compartmental networks have the particularity that the accumulated quantities are distributed on space. Related works to [4], one can also find nonlinear continuous-time model using fluid-flow approach,

as introduced in [44] or conservation law-based fluid-flow for network congestion control as in [7]. Combining network of compartments described either by ODEs or PDEs is an issue to be address in this thesis. Therefore some preliminary questions may rise concerning the stability of communication networks:

- *Given input flow demands getting into the network, can we establish stability properties in appropriate sense for the resulting linearized model?*

- *Moreover, under suitable control actions, can we improve somehow the performance of the network?*

To answer those questions, we will focus on boundary control of coupled PDE-ODEs as we will see in Chapter 1.

On the other hand, assuming that continuous-time boundary controls of networks are suitably designed and are able to lead the system to desired behaviors, one may wonder about possible implementations into digital platforms. More precisely, what about digital control of PDEs? In this thesis we propose an approach towards that end: *event-based control*.

About event-based control

Event-based control is a computer control strategy which aims to use communications and computational resources efficiently by updating control inputs aperiodically, only when needed. Several works have been developed in this area for finite-dimensional networked control systems (see for instance the seminal work [1, 70] or the most recent ones [33, 49, 65] and the references therein). Two components are essential in the framework of event-based control. The first one is a feedback control law which has been designed to stabilize the system. The second one is a triggering strategy which determines the time instants when the control needs to be updated. The most commonly triggering strategy uses a static rule obtained by an Input-to-State Stability (ISS) property as in [59]. An extension to this strategy is done in [28] where an internal dynamic is introduced into the triggering rule for which it is possible to reduce the number of control updates with respect to the static policy. Other approaches, among others, rely directly on the time derivative of the Lyapunov function ([45, 57]). Besides the interest of reducing communication and computational loads, event-based control is also known as a rigorous way to digitally implement continuous-time controllers.

Actually, the design of event-based control strategies for infinite-dimensional systems is rarely treated in the literature. Extending existing results for ODEs to time-delay systems can be considered (following the results from e.g. [54]) as proposed for instance in [21]; however, this is quite far from the problem addressed in this thesis. For parabolic PDEs, event-based control strategies are considered in [55] and [69]. Many difficulties that arise in the context of event-based control are due to the introduction of discontinuities when updating the control. Discontinuous output feedback controllers for infinite dimensional systems have

been studied, for instance in [48], where unit feedback controller and in turn global asymptotic stabilization are considered. Although, the framework of switched hyperbolic systems [32, 53, 38] is highly inspiring -especially the work in [38]- for dealing with the well-posedness of the closed-loop solution of such systems under event-based control strategies. In addition, for the most part of boundary controllers for hyperbolic systems, digital control without reducing the model has *not* been studied in general. In fact, for control of PDEs, digital control synthesis commonly relies on reducing the model by discretizing the space so as one gets ordinary differential equations. In that case, finite-dimensional approaches for digital control can be applied. However, without reducing the model, it is not sufficiently clear how fast boundary continuous-time controllers of hyperbolic PDEs must be sampled in a periodic fashion so as to implement them into a digital platform. Besides this, in large scale scenarios where sensors and actuators are distributed, information is transmitted through digital communication channels. Therefore, the need to reduce energy consumption and save communications resources is also a key issue.

Naturally some questions may arise around this subject when thinking about boundary control of hyperbolic PDEs:

- *How fast should we sample in periodic fashion continuous-time boundary controllers while preserving stability properties of the PDE system?*
- *Can we extend event-based control strategies already developed for finite-dimensional systems to infinite dimensional systems?*
- *Can we guarantee both stability properties and the well-posedness of the system under event-based controllers?*

Highly motivated by all aforementioned and in order to answer the above questions, let us briefly highlight the main contributions of this thesis.

Contributions and structure of the thesis

1. The first contribution of this thesis pertains to the modeling and boundary control of large scale networks of compartments under fluid-flow modeling, applied to communication networks as it is going to be presented in Chapter 1. During the study in that chapter, we will end up with the following coupled linear hyperbolic PDE-ODE, to subsequently study stability analysis and boundary control issues by means of Lyapunov techniques and LMI formulation:

$$\begin{cases} \partial_t y(t,x) + \Lambda \partial_x y(t,x) = 0 \\ \dot{Z}(t) = AZ(t) + G_y y(t,1) + B_w W(t) + D\tilde{d}(t) \end{cases} \tag{3}$$

with boundary condition

$$y(t,0) = G_z Z(t) + B_u U(t) \tag{4}$$

and initial condition

$$\begin{cases} y(0,x) = y^0(x), \quad x \in [0,1] \\ Z(0) = Z^0 \end{cases} \tag{5}$$

We give a sufficient condition for ISS stability in open loop, and we perform control synthesis in a closed-loop setting. Optimization issues are also tackled.

2. In Chapter 2 we introduce the first event-based control approach for linear hyperbolic systems of conservations laws inspired by the main strategies of event-based control for finite-dimensional systems. We must say however that the results in that chapter are only applied to a particular case of system (3)-(5) (communication networks without buffers i.e. ODE coupling and without exogenous disturbance). Thus, the system to be studied is of the following form:

$$\partial_t y(t,x) + \Lambda \partial_x y(t,x) = 0, \qquad x \in [0,1], \quad t \in \mathbb{R}^+ \tag{6}$$

with boundary condition

$$y(t,0) = Hy(t,1) + Bu(t), \qquad t \in \mathbb{R}^+ \tag{7}$$

initial condition

$$y(0,x) = y^0(x), \qquad x \in [0,1] \tag{8}$$

and output function

$$z(t) = y(t,1) \tag{9}$$

Using Lyapunov-based techniques in L^2-norm, we propose three event-based control strategies of the form $u = \varphi(z)$ (output feedback) involving the execution time which determines when control values must be updated. Roughly, it has the following form:

$$t_{k+1} = \inf\{t \in \mathbb{R}^+ | t > t_k \wedge \textit{some suitable triggering condition}\}$$

The analysis of global exponential stability is carried out and we prove that under the three event-based stabilization approaches, the solution to the closed-loop system exists and is unique while avoiding Zeno phenomena.

3. Then, in Chapter 3, we consider the problem of stabilization of boundary controlled linear hyperbolic PDEs where the output measurements are communicated after being time-sampled and space-quantized. Static and dynamic controllers are designed, which establish stability in L^2- and H^1-norms with respect to measurement errors using Lyapunov-based techniques. We show that the design of sampling algorithms ensures practical stability.

4. Finally, the last contribution relates to event-based control of linear hyperbolic systems via Backstepping approach. It is proved that no Zeno phenomena is presented because of the existence of a *minimal dwel-time* and then the well-posedness and global exponential stability of the hyperbolic system are guaranteed. The event-based controller is based on Lyapunov analysis on the so-called *target system* that allows to come up with a dynamic triggering condition. This is presented in Chapter 4.

Publications

Peer reviewed international journals

- N. Espitia, A. Girard, N. Marchand, C. Prieur *"Event-based control of linear hyperbolic systems of conservation laws"*. Automatica, Vol 70, pp.275-287, 2016.

Peer reviewed international conferences

- N. Espitia, A. Girard, N. Marchand, C. Prieur *"Dynamic boundary control synthesis of coupled PDE-ODEs for communication networks under fluid flow modeling"*. Accepted for presentation to the 56th IEEE Conference on Decision and Control (CDC) 2017.
- N. Espitia, A.Tanwani, S. Tarbouriech *"Stabilization of boundary controlled hyperbolic PDEs via Lyapunov-based event triggered sampling and quantization"*. Accepted for presentation to the 56th IEEE Conference on Decision and Control (CDC) 2017.
- N. Espitia, A. Girard, N. Marchand, C. Prieur *"Fluid-flow modeling and stability analysis of communication networks"*. IFAC World Congress, Toulouse, France, 2017.
- N. Espitia, A. Girard, N. Marchand, C. Prieur *"Event-based stabilization of linear systems of conservation laws using a dynamic triggering condition"*. Proc. of the 10th IFAC Symposium on Nonlinear Control Systems (NOLCOS), 2016.

Under review international journals

- N. Espitia, A. Girard, N. Marchand, C. Prieur *"Event-based boundary control of* 2×2 *linear hyperbolic systems via Backstepping approach"*. Conditionally accepted as a technical note to IEEE Transactions on Automatic Control.

Chapter 1

Fluid-flow modeling and boundary control of communication networks

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This chapter is made up of two parts. The first part deals with fluid-flow modeling under compartmental representation of networks of conservation laws. The motivating example is a communication network made up of buffers and transmission lines. The resulting model is a coupled linear hyperbolic partial differential equation (PDE) with an ordinary differential equation (ODE) along with a dynamic boundary condition. Two specific control functions with constraints are studied (namely, routing control and access control). The second part is devoted to the dynamic boundary control of communication networks. The boundary control synthesis of the resulting linearized coupled hyperbolic PDE-ODEs is carried by means of Lyapunov techniques and LMIs formulation. Input-to-state stability of the linearized system at an optimal equilibrium is guaranteed while minimizing the asymptotic gain due to the control actions.

1.1 Introduction

We consider that the network studied in this chapter is made up of compartments: one for the fluid dynamics of servers, composed mainly by buffers (modeled by ODEs), the other

to represent delays in transmission (if there are), or simply transmission lines modeled by hyperbolic PDEs. Since we combine both dynamics, the coupling is defined at the boundaries of the hyperbolic PDEs. It results in what we call, coupled PDE-ODE with *dynamic boundary condition*. We aim then at studying the stability of such systems under the assumption that one wants the system to operate in *free-flow* zone to avoid congestion. The main contribution here is the modeling of the network, the study of input-to-state stability properties when operating at some optimal equilibrium point and the dynamic boundary control synthesis that is carried out by Lyapunov analysis which leads to sufficient condition for input-to-state stability under LMIs formulation. In fact, in open-loop setting, one can guarantee ISS without any control action but the throughput of the networks remains quite far from the desired equilibrium point and could lead the network to get congested. Then, by means of suitable control actions, we intend to reduce the impact by allowing the network to operate as close as possible to the optimal equilibrium. Formally, it translates in a minimization of the *asymptotic gain* in the ISS framework. In addition, it turns out that constraints on the control variables must be respected, thus limitation in amplitude of the gains has to be considered when solving the LMIs involved in the synthesis. Optimization issues are also considered for the minimization of the asymptotic gain when performing the control synthesis.

The main results of the first part of this chapter have been condensed in a work accepted for presentation in [24].

1.2 Fluid-flow modeling

In this section, we present a model of communication networks under compartmental fluid-flow dynamics using both partial differential equations and ordinary differential equations. Highly

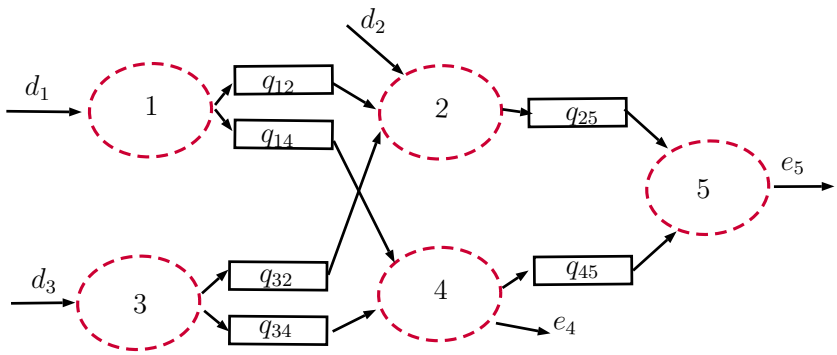

Figure 1.1: Example of a compartmental network.

inspired by [29], [4] and [5], let us consider an example of a general network depicted in Figure 1.1 where each node is illustrated in Figure 1.2 and represents a server in which a buffer stores information to be processed. Variables appearing in Figure 1.2 will be described later on. The other compartments considered in this network are the transmission lines when delays in time may exist. Flow of information will be denoted by q_{ij} that will be properly characterized later on. The fact of considering delays in transmission (not buffer delays) allows to enrich the

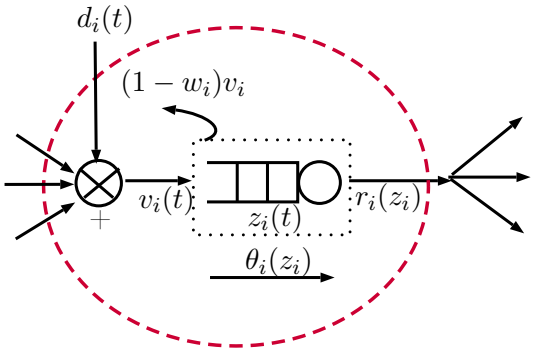

Figure 1.2: Compartment: buffer.

model introduced in [29] where delays are not taken into account. Actually, they assume that the flow transfer is instantaneous between buffers. In this work we will consider rather that traffic flow takes a while in traveling from one buffer to another. We assume that the network is a directed graph in which the directed arcs $i \to j$ of the network represent instantaneous mass transfers between compartments (more precisely, between servers compartments and transmission line compartments). As already mentioned, if there are time delays, they can be enclosed into compartments representing the transmission lines by transport equations with positive velocities. We assume that traffic flow may be routed to different compartments of the network until reaching the destination. That routing mechanism is assumed to be located the end of the server compartments.

Under the proposed general topology of compartments, transmission lines are modeled by the following nonlinear conservation law ([16, 14])

$$\partial_t \rho_{ij}(t,x) + \partial_x f_{ij}(\rho_{ij}(t,x)) = 0, \quad i \in \mathcal{I}_n, j \in \mathcal{D}_i \tag{1.1}$$

for all $x \in [0,1], t \in \mathbb{R}^+$, where ρ_{ij} and $f_{ij}(\rho_{ij})$ are the density and flow of packets respectively and σ_{ij} is a critical density closely related to the probability of loosing packets.
$\mathcal{I}_n$ is the set of the number of compartments, numbered from 1 to n and $\mathcal{D}_i$ is the index set of downstream compartments connected directly to compartment i.

$$f_{ij}(\rho_{ij}) = \begin{cases} \lambda_{ij}\rho_{ij}, & \text{if } 0 \le \rho_{ij} \le \sigma_{ij} \\ \lambda_{ij}(2\sigma_{ij} - \rho_{ij}), & \text{if } \sigma_{ij} \le \rho_{ij} \le \rho_{ij}^{max}, \quad i \in \mathcal{I}_n, j \in \mathcal{D}_i \end{cases} \tag{1.2}$$

Figure 1.3 shows the so-called *fundamental diagram of flow-density.*

For $\rho_{ij} \le \sigma_{ij}$, the network is said to be in *free-flow.* For $\rho_{ij} \ge \sigma_{ij}$, the network is said to be congested. Note that this fundamental diagram is quite similar to the one used in road traffic networks (see e.g. [67, Chapter 8] for CTM models). In communication networks, the averaged velocity of packets is supposed to be constant. The macroscopic model (1.1) has been validated in [15].

In this work, we assume the specific case when each flow is a static monotonic increasing function of the density ρ_{ij}. Under this assumption, the flow $f_{ij}(\rho_{ij})$ equals $\lambda_{ij}\rho_{ij}$, for $0 \le \rho_{ij} \le \sigma_{ij}$, where λ_{ij} is the average velocity of packets among buffers traveling through the

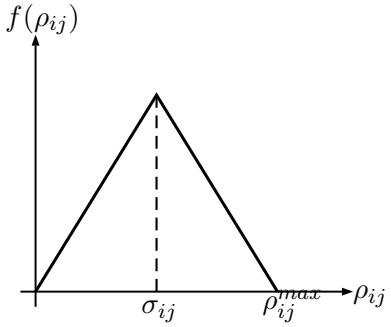

Figure 1.3: Fundamental triangular diagram of flow-density

transmission line. Let us denote the flow $f_{ij}(\rho_{ij}) := q_{ij}$ (as appeared in Figure 1.1). We will focus on the case in which the network operates in free-flow and we will study Lyapunov stability properties on the network when operating in this zone. This case implies that, according to (1.2), $q_{ij} = \lambda_{ij}\rho_{ij}$ for $0 \leq \rho_{ij} \leq \sigma_{ij}$. Therefore

$$\partial_t \rho_{ij}(t,x) = \tfrac{1}{\lambda_{ij}} \partial_t q_{ij}(t,x).$$

Replacing it in (1.1), we obtain the linear hyperbolic system as *kinematic wave equations* (as in [5]), that is

$$\partial_t q_{ij}(t,x) + \lambda_{ij}\partial_x q_{ij}(t,x) = 0, \quad i \in \mathcal{I}_n, \quad j \in \mathcal{D}_i \tag{1.3}$$

Concerning the modeling of buffers (as illustrated in Figure 1.2), let us consider the balance equation for each buffer $i \in \mathcal{I}_n$ as follows:

$$\dot{z}_i(t) = v_i(t) - r_i(z_i(t)) \tag{1.4}$$

where v_i is the sum of all input flows getting into the buffer and r_i is the output flow of the buffer. It will be characterized later as the *processing rate function*. The traffic flow may be routed to different compartments of the network until reaching the destination and can be modulated by suitable actuators. On one hand, we have control actions $u_{ij}(\cdot)$ devoted to route the flow of information through different paths of the network. In this work, they are time-varying and represent continuous time control values, that we call *routing splitting controls*. The routing takes place at the output of the server. On the other hand, we have control actions $w_i(\cdot)$, *access control*, devoted to reject packets (traffic flow) before they enter to the buffers. Therefore, with $v_i(t) = w_i(t)(d_i(t) + \sum_{\substack{k\neq i\\ k\in\mathcal{U}_i}} q_{ki}(t,1))$, we obtain

$$\dot{z}_i(t) = w_i(t)d_i(t) + \sum_{\substack{k\neq i\\ k\in\mathcal{U}_i}} w_i(t)q_{ki}(t,1) - r_i(z_i(t)) \tag{1.5}$$

where

- $d_i(t)$ is the external input flow demands. Note that $d_i \equiv 0$ as long as $i \notin \mathcal{I}_{in}$;
- $\sum_{\substack{k\neq i\\ k\in\mathcal{U}_i}} q_{ki}(t,1)$ are the flows coming from the transmission line connecting upstreams compartments $\mathcal{U}_i$;

- $w_i(t)$ is the access control to buffer i, with $0 \leq w_i(t) \leq 1$,

It represents the rate of accumulation of quantity z_i. Hence, we have that while $d_i(t)$ is the actual input flow demand, then $w_i d_i$ is a fraction of such a demand and $w_i(t)q_{ki}(t,1)$ is a fraction of the incoming flow entering to the buffer i. The output flow $r_i(z_i)$ (processing rate function) of each buffer is considered as the ratio between the quantity z_i and the *residence time*. It is given as follows (see [29, Chapter 2])

$$r_i(z_i) = \frac{z_i}{\theta_i(z_i)}$$

The *residence time* is the averaged time at which packets stay in the server when being processed:

$$\theta_i(z_i) = \frac{1 + z_i}{\epsilon_i} \tag{1.6}$$

with $\epsilon_i > 0$ as the maximal processing capacity of each server. Hence, the processing rate function is given by

$$r_i(z_i) = \frac{\epsilon_i z_i}{1 + z_i} \tag{1.7}$$

It can be can noticed that $r_i(z_i)$ is a positive bounded function of the quantity z_i ($0 \leq r_i(z_i) < \epsilon_i$).

On the other hand, regarding the routing splitting control, the boundary condition for the linear hyperbolic system (1.3) is as follows:

$$q_{ij}(t,0) \quad = \quad u_{ij}(t)r_i(z_i(t)) \tag{1.8}$$

with $0 \leq u_{ji}(t) \leq 1$, $j \in \mathcal{D}_i, i \in \mathcal{I}_n$. In fact $i \in \mathcal{R}$ (index set of routing compartments). Note that the left boundary condition (1.8) depends on the state variable z_i, being this one a solution to the ODE system (1.5). In that sense, we shall consider in the sequel that the boundary condition of the linear hyperbolic PDE is a dynamic boundary one. The left-boundary condition itself is enough to be considered in the whole model for the study of well-posedness and stability issues.

Finally, the output function for each output compartment $i \in \mathcal{I}_{out}$ (index set of outputs compartments) is given by

$$e_i(t) = u_i(t)r_i(z_i(t)) \tag{1.9}$$

with $\sum_{\substack{i \neq j \\ j \in \mathcal{D}_i}} u_{ij}(t) + u_i(t) = 1$ ($u_i(t) \equiv 0$ if $i \notin \mathcal{I}_{out}$). The complete model, for the network as depicted in Figures 1.1 and 1.2, is then:

$$\begin{cases} \partial_t q_{ij}(t,x) + \lambda_{ij}\partial_x q_{ij}(t,x) = 0, \quad i \in \mathcal{I}_n, \quad j \in \mathcal{D}_i \\ \dot{z}_i(t) = w_i(t)d_i(t) + \sum_{\substack{k \neq i \\ k \in \mathcal{U}_i}} w_i(t)q_{ki}(t,1) - r_i(z_i(t)) \end{cases} \tag{1.10}$$

with dynamic boundary condition

$$q_{ij}(t,0) = u_{ij}(t)r_i(z_i(t)), \quad r_i \geq 0 \tag{1.11}$$

output function

$$e_i(t) = u_i(t) r_i(z_i(t)) \tag{1.12}$$

and initial conditions

$$\begin{cases} q_{ij}(0,x) = q_{ij}^0(x), \quad x \in [0,1] \\ z_i(0) = z_i^0 \end{cases} \tag{1.13}$$

1.3 Optimal operating point characterization

On one hand, we want the network to operate at some equilibrium point, to be precise, at some free-flow steady-state. On the other hand, since we are going to deal with input flow demands, we aim at studying the influence of those inputs over the stability of the network. Therefore, input-to-state stability (ISS) property with respect to those inputs by using Lyapunov analysis will be addressed for the corresponding linearized system.

Let us first characterize the operating point.

1.3.1 Free-flow steady-state characterization

For a given constant input flow demand d_i^*, system (1.10)-(1.13) has infinitely many equilibrium points $\{q_{ij}^*, q_{ki}^*, z_i^*, u_{ij}^*, u_i^*, w_i^*, e_i^*\}$ with q_{ij}^* time and space invariant. Decision variables are u_{ij}^* and w_i^* related to the routing control and the access control respectively. Equilibrium points satisfy the following algebraic equations:

$$\begin{cases} w_i^* d_i^* + \sum_{\substack{k \neq i \\ h \in \mathcal{U}_i}} w_i^* q_{ki}^* - r_i(z_i^*) = 0 \\ q_{ij}^* = u_{ij}^* r_i(z_i^*) \\ e_i^* = u_i^* r_i(z_i^*) \end{cases} \tag{1.14}$$

We assume then that the system admits a *free-flow steady-state.* Among all possible equilibrium points, we choose the free-flow steady-state that meets some performance criterion for the network. Inspired by road traffic networks where two usual performance metrics such as the total travel time (TTT) and total travel distance (TTD) are considered (see e.g. [67]), here we focus on a particular static case:

1) Maximizing the *total output flow rate* of the network. The first optimization objective is as follows:

$$\text{maximize} \quad J_1 = \sum_{i \in \mathcal{I}_{out}} e_i^* \tag{1.15}$$

2) Minimizing the *total mean travel time* (T_{MTT}). In each compartment $i \in \mathcal{I}_n$, information is processed and it takes some time according to the residence time $\theta_i(z_i)$ before it is sent through the transmission line. Besides this, there is a time propagation given by $\frac{1}{\lambda_{ij}}$ due to the transport equation. Let us denote the total travel time in each compartment i by $T_i = \theta_i(z_i^*)$. We do not give any explicit formula of the T_{MTT} because of the complexity that

the network topology might have, but we explain the approach to compute it. The first issue worth remarking is that in this framework, there are no cycles in the network. It implies that there is a finite number of possible paths that the information flow can follow from input compartments until output compartments. Therefore, looking at each input-output path, the sum of total times T_i of compartments involved in along with time propagation between compartments $\frac{1}{\lambda_{ij}}$, is weighted by the effective output flow which travels through them. In order to homogenize, the result is divided by the sum of the output flows of the whole network. Repeating the same procedure with every input-output path and adding the obtained weighted average value, the total mean travel time can be deduced accordingly. In Section 1.6, a specific example is provided to better illustrate the idea. Hence, the second optimization objective is as follows:

$$\text{minimize} \quad J_2 = T_{MTT} \tag{1.16}$$

Let us call $J = \alpha(-J_1) + (1-\alpha)J_2$ the cost function with weighting coefficient $\alpha \in [0,1]$. J is a nonlinear function to be minimized subject to (1.14) along with the following constraints related to:
(1) free-flow conditions over the linear hyperbolic system:

$$q^*_{ij} \leq \sigma_{ij}\lambda_{ij}$$

(2) control variables:

$$0 \leq u^*_{ij} \leq 1, \quad 0 \leq w^*_i \leq 1, \; 0 \leq u^*_i \leq 1, \; \sum_{\substack{i \neq j \\ j \in \mathcal{D}_i}} u^*_{ij} + u^*_i = 1$$

1.3.2 Linearization around the free-flow steady-state

Defining the deviations $y_{ij} = q_{ij} - q^*_{ij}, Z_i = z_i - z^*_i, U_{ij} = u_{ij} - u^*_{ij}, W_i = w_i - w^*_i$ and $\tilde{d}_i = d_i - d^*_i$, $\tilde{e}_i(t) = e_i(t) - e^*_i$ the linearization of the coupled PDE-ODE system in (1.10) with dynamic boundary condition (1.11) and initial condition (1.13) around the optimal free-flow equilibrium is given by

$$\begin{cases} \partial_t y_{ij}(t,x) + \lambda_{ij}\partial_x y_{ij}(t,x) = 0, \quad i \in \mathcal{I}_n, \quad j \in \mathcal{D}_i \\ \dot{Z}_i(t) = d^*_i W_i(t) + w^*_i \tilde{d}_i(t) + \sum_{\substack{k \neq i \\ k \in \mathcal{U}_i}} q^*_{ki} W_i(t) + \sum_{\substack{k \neq i \\ k \in \mathcal{U}_i}} w^*_i y_{ki}(t,1) - r'_i(z^*_i) Z_i(t) \end{cases} \tag{1.17}$$

with dynamic boundary condition

$$y_{ij}(t,0) = u^*_{ij} r'_i(z^*_i) Z_i(t) + r_i(z^*_i) U_{ij}(t), \quad i \in \mathcal{I}_n, \quad j \in \mathcal{D}_i, \tag{1.18}$$

output function

$$\tilde{e}_i(t) = u^*_i r'_i(z^*_i) Z_i(t) + r_i(z^*_i) U_i(t), \tag{1.19}$$

and initial conditions

$$\begin{cases} y_{ij}(0,x) = y^0_{ij}(x), \quad x \in [0,1] \\ Z_i(0) = Z^0_i. \end{cases} \tag{1.20}$$

Recall that $\sum_{\substack{i\neq j\\ j\in\mathcal{D}_i}} u_{ij}(t) + u_i(t) = 1$ and $\sum_{\substack{i\neq j\\ j\in\mathcal{D}_i}} U_{ij}(t) + U_i(t) = 0$. We remark that since $u_i(t) \equiv 0, u_i^* \equiv 0$ if $i \notin \mathcal{I}_{out}$, thus, $U_i(t) \equiv 0$ as well.

The system (1.17)-(1.20) can be written in matrix form as

$$\begin{cases} \partial_t y(t,x) + \Lambda \partial_x y(t,x) = 0 \\ \dot{Z}(t) = AZ(t) + G_y y(t,1) + B_w W(t) + D\tilde{d}(t) \end{cases} \tag{1.21}$$

with boundary condition

$$y(t,0) = G_z Z(t) + B_u U(t) \tag{1.22}$$

and initial condition

$$\begin{cases} y(0,x) = y^0(x), \quad x \in [0,1] \\ Z(0) = Z^0 \end{cases} \tag{1.23}$$

where $y : \mathbb{R}^+ \times [0,1] \to \mathbb{R}^m$ with m given by

$$m := \sum_{i\in\mathcal{I}_n} card(\mathcal{D}_i) \tag{1.24}$$

Λ is a diagonal positive definite matrix in $\mathbb{R}^{m\times m}$ such that $\Lambda = \text{diag}(\lambda_{ij})$, $i \in \mathcal{I}_n$, $j \in \mathcal{D}_i$. $W : \mathbb{R}^+ \to \mathbb{R}^n$, $Z : \mathbb{R}^+ \to \mathbb{R}^n$ and $U : \mathbb{R}^+ \to \mathbb{R}^l$ where l is given by :

$$l := \sum_{i\in\mathcal{R}} (card(\mathcal{D}_i) - 1) + \sum_{i\in\mathcal{R}\cap\mathcal{I}_{out}} card(\mathcal{D}_i)$$

and

- $A := \text{diag}(-r_i'(z_i^*)) \in \mathbb{R}^{n\times n}$;
- $G_y \in \mathbb{R}^{n\times m}$ with $G_y[i,j] = w_i^*$ if $j \in \mathcal{U}_i$ or $G_y[i,j] = 0$ otherwise;
- $B_w := \text{diag}(d_i^* + \sum_{\substack{k\neq i\\ k\in\mathcal{U}_i}} q_{ki}^*)$ in $\mathbb{R}^{n\times n}$ ($d_i \equiv 0$ as long as $i \notin \mathcal{I}_{in}$);
- D is a diagonal matrix in $\mathbb{R}^{n\times n}$ whose diagonal entries are w_i^* if $i \in \mathcal{I}_{in}$ or 0 otherwise.

The rest of matrices are of appropriate dimension and their detailed characterization are given in [24].

Remark 1.1
U in (1.22) can be built by only taking into account certain constraints. For instance, at each routing compartment $i \in \mathcal{R}$, we would have $U_i(t) = -\sum_{\substack{i\neq j\\ j\in\mathcal{D}_i}} U_{ij}(t)$, $j \in \mathcal{D}_i$. This fact will allow to reduce the dimension of the control function which might be helpful for simulation tractability purposes, especially when the network has several routing compartments.

It remains to say that $\tilde{d}$(t) is the input flow that can be viewed in the sequel as an input disturbance. We assume that $\tilde{d}$ is in $\mathcal{C}_{pw}(\mathbb{R}^+;\mathbb{R}^n)$. Finally, $y(0,x) = y^0(x) \in L^2([0,1];\mathbb{R}^m)$ and $Z(0) = Z^0 \in \mathbb{R}^n$.

1.4 Input-to-state stability analysis in open loop

Let us define the notion of input-to-state stability for the system (1.21)-(1.23).

Definition 1.1 (Input-to-state stability ISS)
The system (1.21)-(1.23) *is input-to-state stable (ISS) with respect to* $\tilde{d} \in \mathcal{C}_{pw}(\mathbb{R}^+;\mathbb{R}^n)$*, if there exist* $\nu > 0$*,* $C_1 > 0$ *and* $C_2 > 0$ *such that, for every* $Z^0 \in \mathbb{R}^n$*,* $y^0 \in L^2([0,1];\mathbb{R}^m)$*, the solution to* (1.21)-(1.23) *satisfies, for all* $t \in \mathbb{R}^+$*,*

$$\left(\|Z(t)\|^2 + \|y(t,\cdot)\|^2_{L^2([0,1],\mathbb{R}^m)}\right) \leq C_1 e^{-2\nu t}\left(\|Z^0\|^2 + \|y^0\|^2_{L^2([0,1];\mathbb{R}^m)}\right) + C_2 \sup_{0\leq s\leq t} \|\tilde{d}(s)\|^2 \tag{1.25}$$

C_2 *is called the asymptotic gain.*

Theorem 1.1 (Input-to-state stability analysis in open-loop)
Let $\underline{\lambda} = \min\{\lambda_{ij}\}_{\substack{i\in\mathcal{I}_n \\ j\in\mathcal{D}_i}}$*. Assume that there exist* $\mu, \gamma > 0$*, a symmetric positive definite matrix* $P \in \mathbb{R}^{n\times n}$ *and a diagonal positive matrix* $Q \in \mathbb{R}^{m\times m}$ *such that the following matrix inequality is satisfied:*

$$M_o = \begin{bmatrix} A^TP + PA + G_z^TQ\Lambda G_z + 2\mu\underline{\lambda}P & PG_y & PD \\ \star & -e^{-2\mu}Q\Lambda & 0 \\ \star & \star & -\gamma I \end{bmatrix} \leq 0 \tag{1.26}$$

Then, the system (1.21)-(1.23) *is input-to-state stable (ISS) with respect to inputs* $\tilde{d} \in \mathcal{C}_{pw}(\mathbb{R}^+;\mathbb{R}^n)$*, and the asymptotic gain satisfies*

$$C_2 \leq \frac{\gamma}{2\mu\underline{\lambda}} e^{2\mu}. \tag{1.27}$$

Proof. Let us consider the following Lyapunov function V defined for all $y \in L^2([0,1];\mathbb{R}^n)$ and $Z \in \mathbb{R}^n$ as follows:

$$V(y,Z) = Z^TPZ + \int_0^1 y^Te^{-2\mu x}Qydx \tag{1.28}$$

Computing the time derivative of V along the solutions of (1.21) with the boundary condition

(1.22) yields to:

$$\begin{aligned}\dot{V} =& \dot{Z}^T PZ + Z^T P\dot{Z} + y^T(\cdot,0)Q\Lambda y(\cdot,0) - y^T(\cdot,1)e^{-2\mu}Q\Lambda y(\cdot,1)\\ &- 2\mu\int_0^1 y^T(\Lambda e^{-2\mu x}Q)ydx\\ =&(AZ + G_y y(\cdot,1) + D\tilde{d})^T PZ + Z^T P(AZ + G_y y(\cdot,1) + D\tilde{d}) + Z^T G_z^T Q\Lambda G_z Z\\ &- y^T(\cdot,1)e^{-2\mu}Q\Lambda y(\cdot,1) - 2\mu\int_0^1 y^T(\Lambda e^{-2\mu x}Q)ydx\\ =&Z^T(A^T P + PA + G_z^T Q\Lambda G_z + 2\mu\underline{\lambda}P)Z + y^T(\cdot,1)G_y^T PZ + Z^T PG_y y(\cdot,1)\\ &- y^T(\cdot,1)e^{-2\mu}Q\Lambda y(\cdot,1) + \tilde{d}^T D^T PZ + Z^T PD\tilde{d}\\ &- 2\mu\underline{\lambda}Z^T PZ - 2\mu\int_0^1 y^T(\Lambda e^{-2\mu x}Q)ydx\end{aligned}$$

Since Q is a diagonal positive definite matrix, $\underline{\lambda}Q \leq \Lambda Q$. Taking $\nu = \mu\underline{\lambda}$, it follows

$$\begin{aligned}\dot{V} \leq& -2\nu Z^T PZ - 2\nu\int_0^1 y^T Qye^{-2\mu x}dx\\ &+ Z^T(A^T P + PA + G_z^T Q\Lambda G_z + 2\mu\underline{\lambda}P)Z\\ &- y^T(\cdot,1)e^{-2\mu}Q\Lambda y(\cdot,1) + y^T(\cdot,1)G_y^T PZ + Z^T PG_y y(\cdot,1)\\ &+ \tilde{d}^T D^T PZ + Z^T PD\tilde{d}\end{aligned} \tag{1.29}$$

Adding $\gamma\|\tilde{d}\|^2 - \gamma\|\tilde{d}\|^2$ to (1.29), for some $\gamma > 0$, we have

$$\dot{V} \leq -2\nu V + \begin{bmatrix} Z \\ y(\cdot,1) \\ \tilde{d} \end{bmatrix}^T M_o \begin{bmatrix} Z \\ y(\cdot,1) \\ \tilde{d} \end{bmatrix} + \gamma\|\tilde{d}\|^2$$

where M_o is defined as (1.26). Therefore, as long as $M_o \leq 0$, we have

$$\dot{V} \leq -2\nu V + \gamma\|\tilde{d}\|^2 \tag{1.30}$$

On the other hand, the Lyapunov function (1.28) can be bounded as follows ([61]):

$$\begin{aligned}\lambda_{min}(P)\|Z(t)\|^2 + e^{-2\mu}\lambda_{min}(Q)\|y(t,\cdot)\|^2_{L^2([0,1];\mathbb{R}^n)} \leq& V(y(t,\cdot),Z(t))\\ \leq& \lambda_{max}(P)\|Z(t)\|^2 + \lambda_{max}(Q)\|y(t,\cdot)\|^2_{L^2([0,1];\mathbb{R}^n)}\end{aligned} \tag{1.31}$$

where $\lambda_{min}(\cdot)$ and $\lambda_{max}(\cdot)$, are the minimum and maximum eigenvalues of the matrix under consideration. Furthermore, without loss of generality, P and Q are such that the following inequality holds:

$$\begin{aligned}e^{-2\mu}(\|Z(t)\|^2 + \|y(t,\cdot)\|^2_{L^2([0,1];\mathbb{R}^m)}) \leq& e^{-2\mu}(\lambda_{min}(P)\|Z(t)\|^2 + \lambda_{min}(Q)\|y(t,\cdot)\|^2_{L^2([0,1];\mathbb{R}^m)})\\ \leq& V(y(t,\cdot),Z(t))\\ \leq& \lambda_{max}(P)\|Z(t)\|^2 + \lambda_{max}(Q)\|y(t,\cdot)\|^2_{L^2([0,1];\mathbb{R}^m)}\end{aligned} \tag{1.32}$$

From (1.30) and (1.32), one can claim that the Lyapunov function (1.28) is an ISS-Lyapunov function. This implies global exponential stability as soon as $\tilde{d} \equiv 0$. Moreover, using the *Comparison principle*, from (1.30), we have

$$\begin{aligned} V(t) \leq & e^{-2\nu t}V(0) + \int_0^t e^{-2\nu(t-s)}\gamma\|\tilde{d}(s)\|^2 ds \\ \leq & e^{-2\nu t}V(0) + e^{-2\nu t}\int_0^t e^{2\nu s}\gamma\|\tilde{d}(s)\|^2 ds \\ \leq & e^{-2\nu t}V(0) + \gamma\|\tilde{d}\|_\infty^2 e^{-2\nu t}\int_0^t e^{2\nu s} ds \\ \leq & e^{-2\nu t}V(0) + \tfrac{\gamma}{2\nu}\|\tilde{d}\|_\infty^2 - \tfrac{\gamma}{2\nu}\|\tilde{d}\|_\infty^2 e^{-2\nu t} \end{aligned}$$

with $\|\tilde{d}\|_\infty = \sup_{s\in[0,t]}\|\tilde{d}(s)\|$. Hence,

$$V(t) \leq e^{-2\nu t}V(0) + \tfrac{\gamma}{2\nu}\|\tilde{d}\|_\infty^2 \tag{1.33}$$

Using the bounds of the Lyapunov function as given in (1.32), it is deduced that

$$\begin{aligned} &\|Z(t)\|^2 + \|y(t,\cdot)\|^2_{L^2([0,1];\mathbb{R}^m)} \\ &\quad \leq e^{2\mu}e^{-2\nu t}\big(\lambda_{max}(P)\|Z^0\|^2 + \lambda_{max}(Q)\|y^0\|^2_{L^2([0,1];\mathbb{R}^m)}\big) \\ &\qquad + \tfrac{\gamma}{2\nu}e^{2\mu}\|\tilde{d}\|_\infty^2 \end{aligned} \tag{1.34}$$

Hence,

$$\begin{aligned} &\|Z(t)\|^2 + \|y(t,\cdot)\|^2_{L^2([0,1];\mathbb{R}^m)} \\ &\qquad \leq C_1 e^{-2\nu t}\Big(\|Z^0\|^2 + \|y^0\|^2_{L^2([0,1];\mathbb{R}^m)}\Big) + \tfrac{\gamma}{2\nu}e^{2\mu}\|\tilde{d}\|_\infty^2 \end{aligned} \tag{1.35}$$

with $C_1 = \max\{\lambda_{max}(P), \lambda_{max}(Q)\}e^{2\mu}$. Therefore, the system (1.21) satisfies the ISS property with respect to the disturbance input $\tilde{d}$. The asymptotic gain satisfies

$$C_2 \leq \frac{\gamma}{2\nu}e^{2\mu}. \tag{1.36}$$

It concludes the proof of Theorem 1.1. □

1.5 Closed-loop setting: dynamic boundary control

Since we deal with input flows demands that are viewed as perturbations, it has been provided, in the previous subsection, a sufficient condition for the system (1.21)-(1.23) to be ISS in open-loop with respect to those inputs. In this section, we are interested in designing control actions

in order to minimize the asymptotic gain obtained in open loop. It can be carried out by means of a closed-loop setting using the following control functions,

$$W(t) = \begin{bmatrix} K_z & K_y \end{bmatrix} \begin{bmatrix} Z(t) \\ y(t,1) \end{bmatrix}; \quad U(t) = \begin{bmatrix} L_z & L_y \end{bmatrix} \begin{bmatrix} Z(t) \\ y(t,1) \end{bmatrix}$$

with $K_z \in \mathbb{R}^{n\times n}$, $K_y \in \mathbb{R}^{n\times m}$, $L_z \in \mathbb{R}^{l\times n}$ and $L_y \in \mathbb{R}^{l\times m}$ named as control gains in the sequel. Therefore, the linearized coupled PDE-ODE system (1.21)-(1.23) becomes:

$$\begin{cases} \partial_t y(t,x) + \Lambda \partial_x y(t,x) = 0 \\ \dot{Z}(t) = (A + B_w K_z)Z(t) + (G_y + B_w K_y)y(t,1) + D\tilde{d}(t) \end{cases} \tag{1.37}$$

with dynamic boundary condition

$$y(t,0) = (G_z + B_u L_z)Z(t) + B_u L_y y(t,1) \tag{1.38}$$

and initial condition

$$\begin{cases} y(0,x) = y^0(x), \quad x \in [0,1] \\ Z(0) = Z^0. \end{cases} \tag{1.39}$$

1.5.1 Well-posedness of the system in closed loop

The existence and uniqueness of solutions for a coupled ODE-Hyperbolic system have been studied in [3, Appendix A] in the case when no external inputs are present. In our framework, we deal with external inputs $\tilde{d}$ that are assumed to be in $\mathcal{C}_{pw}$ and that introduce some discontinuities, but the resulting solutions are absolutely continuous. This problem is in fact quite similar to the one in [63] where the well-posedness has been deeply developed using the semi-group approach. Hence, results in [63, Chapter 3] (in turn inspired by [13, Chapter 3]) may be applied to our framework. Without enter in full details about the proof, let us only sketch the main ideas on how the claim may be verified:

- Consider first the operator $\mathfrak{A}$ as follows:

$$dom(\mathfrak{A}) = \left\{ (y,Z) \in H^1([0,1];\mathbb{R}^m) \times \mathbb{R}^n; \begin{bmatrix} y(0) \\ Z \end{bmatrix} = \begin{pmatrix} B_u L_y & (G_z + B_u L_z) \\ 0 & I \end{pmatrix} \begin{bmatrix} y(1) \\ Z \end{bmatrix} \right\}$$

$$\mathfrak{A} \begin{bmatrix} y \\ Z \end{bmatrix} = \begin{pmatrix} -\Lambda y_x \\ A + B_w K_z \end{pmatrix}$$

  along with a perturbation operator $\mathfrak{B} : H^1([0,1];\mathbb{R}^m) \times \mathbb{R}^n \to H^1([0,1];\mathbb{R}^m) \times \mathbb{R}^n$ as follows:

$$\mathfrak{B} \begin{bmatrix} y \\ Z \end{bmatrix} = \begin{pmatrix} 0 & 0 \\ G_y + B_w K_y & 0 \end{pmatrix} \begin{bmatrix} y(1) \\ Z \end{bmatrix}$$

  Setting $\mathcal{X} = (y,Z)^T$ and using these operators, one can write the closed-loop system (1.37)-(1.39) in abstract formulation as follows:

$$\dot{\mathcal{X}}(t) = \mathfrak{A}\mathcal{X}(t) + \mathfrak{B}\mathcal{X}(t) + \begin{bmatrix} 0 \\ D \end{bmatrix} \tilde{d}(t)$$

$$\mathcal{X}^0 \in dom(\mathfrak{A})$$

Having stated that, it can be proved that

- $\mathfrak{A}$ is quasi-dissipative;
- the adjoint $\mathfrak{A}^*$ is quasi-dissipative;
- the operator $\mathfrak{A}$ is closed and $dom(\mathfrak{A})$ is dense in $L^2([0,1];\mathbb{R}^m) \times \mathbb{R}^n$.

Consequently, one can use the *Lumer-Phillips theorem* (see e.g. [13, Corollary 2.2.3]) to conclude that $\mathfrak{A}$ is an infinitesimal generator of a $\mathfrak{C}^0$- semigroup.

- Then, it can be proved that $\mathcal{B}$ is a linear bounded operator. Hence, using [13, Theorem 3.2.1], it can be proved that the operator $\mathfrak{A} + \mathfrak{B}$ is an infinitesimal generator of a $\mathcal{C}^0-$ semigroup as well.
- Finally, using [13, Theorem 3.1.7] and under suitable compatibility conditions, one could end up with solutions $y \in \mathcal{C}^0([0,T]; H^1([0,1];\mathbb{R}^m))$ and $Z \in \mathcal{C}^0([0,T];\mathbb{R}^n)$.

Remark 1.2
In open loop, with $W(t) = 0$ and $U(t) = 0$, the previous result of well-posedness also applies.

1.5.2 ISS control synthesis without constraints

In this subsection we first study the control synthesis without any constraint on the control gains. Next, we will add some optimization objective which results in the minimization of the asymptotic gain.

Theorem 1.2 (Control synthesis)
Let $\underline{\lambda} = \min\{\lambda_{ij}\}_{\substack{i\in\mathcal{I}_n \\ j\in\mathcal{D}_i}}$. Assume that there exist $\mu, \gamma > 0$, a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ a diagonal positive matrix $Q \in \mathbb{R}^{m\times m}$, as well as control gains $K_z \in \mathbb{R}^{n\times n}$, $K_y \in \mathbb{R}^{n\times m}$, $L_z \in \mathbb{R}^{l\times n}$ and $L_y \in \mathbb{R}^{l\times m}$ such that the following matrix inequality, holds :

$$M_c = \begin{pmatrix} M_1 & M_2 & M_3 \\ \star & M_4 & 0 \\ \star & \star & M_5 \end{pmatrix} \leq 0 \tag{1.40}$$

with

- $M_1 := \mathcal{A}^T P + P\mathcal{A} + 2\mu\underline{\lambda}P + \mathcal{G}_z^T Q\Lambda\mathcal{G}_z$;
- $M_2 := P\mathcal{G}_y + \mathcal{G}_z^T Q\Lambda B_u L_y$;
- $M_3 := PD$;
- $M_4 := -e^{-2\mu}Q\Lambda + L_y^T B_u^T Q\Lambda B_u L_y$;

- $M_5 := -\gamma I$.

where

$$\begin{cases} \mathcal{A} := A + B_w K_z \\ \mathcal{G}_z := G_z + B_u L_z \\ \mathcal{G}_y := G_y + B_w K_y \end{cases} \tag{1.41}$$

Then, the closed-loop system (1.37)-(1.39) is input-to-state stable (ISS) with respect to inputs $\tilde{d} \in \mathcal{C}_{pw}(\mathbb{R}^+; \mathbb{R}^n)$, and the asymptotic gain C_2 satisfies

$$C_2 \leq \frac{\gamma}{2\nu} e^{2\mu}. \tag{1.42}$$

with $\nu = \mu\underline{\lambda}$.

Remark 1.3
Particularizing the previous result with $K_z = 0$, $K_y = 0$, $L_z = 0$ and $L_y = 0$, we obtain Theorem 1.1.

Proof. Proceeding similarly as the proof of Theorem 1.1, let us bring back the following Lyapunov function V (as in (1.28)) defined for all $y \in L^2([0,1]; \mathbb{R}^m)$ and $Z \in \mathbb{R}^n$ as follows:

$$V(y, Z) = Z^T P Z + \int_0^1 y^T Q y e^{-2\mu x} dx \tag{1.43}$$

Computing the time derivative of V along the solutions of (1.37), integrating by parts and using the boundary condition (1.38) yields to:

$$\begin{aligned} \dot{V} =& Z^T \big(A^T P + PA + K_z^T B_w^T P + P B_w K_z + 2\mu\underline{\lambda} P + G_z^T Q \Lambda G_z \\ &+ G_z^T Q \Lambda B_u L_z + L_z^T B_u^T Q \Lambda G_z + L_z^T B_u^T Q \Lambda B_u L_z\big) Z \\ &+ y^T(\cdot, 1)\big(-e^{-2\mu} Q\Lambda + L_y^T B_u^T Q \Lambda B_u L_y\big) y(\cdot, 1) \\ &+ y^T(\cdot, 1)\big(G_y^T P + K_y^T B_w^T P + L_y^T B_u^T Q \Lambda G_z + L_y^T B_u^T Q \Lambda B_u L_z\big) Z \\ &+ Z^T \big(P G_y + P B_w K_y + G_z^T Q \Lambda B_u L_y + L_z^T B_u^T Q \Lambda B_u L_y\big) y(\cdot, 1) \\ &+ \tilde{d}^T D^T P Z + Z^T P D \tilde{d} - 2\mu\underline{\lambda} Z^T P Z - 2\mu \int_0^1 y^T (\Lambda Q) y e^{-2\mu x} dx \end{aligned}$$

Since Q is a diagonal positive definite matrix, $\underline{\lambda} Q \leq \Lambda Q$. Taking $\nu = \mu\underline{\lambda}$, it follows

$$\begin{aligned} \dot{V} \leq& -2\nu Z^T P Z - 2\nu \int_0^1 y^T Q y e^{-2\mu x} dx \\ &+ Z^T \big(A^T P + PA + K_z^T B_w^T P + P B_w K_z + 2\mu\underline{\lambda} P + G_z^T Q \Lambda G_z \\ &+ G_z^T Q \Lambda B_u L_z + L_z^T B_u^T Q \Lambda G_z + L_z^T B_u^T Q \Lambda B_u L_z\big) Z \\ &+ y^T(\cdot, 1)\big(-e^{-2\mu} Q\Lambda + L_y^T B_u^T Q \Lambda B_u L_y\big) y(\cdot, 1) \\ &+ y^T(\cdot, 1)\big(G_y^T P + K_y^T B_w^T P + L_y^T B_u^T Q \Lambda G_z + L_y^T B_u^T Q \Lambda B_u L_z\big) Z \\ &+ Z^T \big(P G_y + P B_w K_y + G_z^T Q \Lambda B_u L_y + L_z^T B_u^T Q \Lambda B_u L_y\big) y(\cdot, 1) \\ &+ \tilde{d}^T D^T P Z + Z^T P D \tilde{d} \end{aligned} \tag{1.44}$$

Adding $\gamma\|\tilde{d}\|^2 - \gamma\|\tilde{d}\|^2$ to (1.44), for some $\gamma > 0$, we have

$$\dot{V} \leq -2\nu V + \begin{bmatrix} Z \\ y(\cdot,1) \\ \tilde{d} \end{bmatrix}^T M_c \begin{bmatrix} Z \\ y(\cdot,1) \\ \tilde{d} \end{bmatrix} + \gamma\|\tilde{d}\|^2$$

where M_c is defined as (1.40). Therefore, as long as $M_c \leq 0$, we have

$$\dot{V} \leq -2\nu V + \gamma\|\tilde{d}\|^2$$

The rest of the proof follows the same lines of the proof of Theorem 1.1. Hence, we conclude the proof of Theorem 1.2. □

Remark 1.4
Finding the control gains for Theorem 1.2, may lead to obtain, in the worst case, the same asymptotic gain as in the open-loop case. Let us recall that the control objective is to minimize the asymptotic gain. One way to minimize it, is minimizing an estimate of an upper bound of it. Therefore we could just take as objective function $\frac{\gamma}{2\nu}e^{2\mu}$, subject to $M_c \leq 0$. We do not handle directly this optimization problem but an extension of it while including some constraints on the control gains. We are going to present it in the sequel.

1.5.3 ISS control synthesis with constraints

Due to the nature of the communication network (1.10)-(1.13) where the two control functions are involved, it is important to point out that one must take into account the constraints on those control variables, i.e. $u_{ij}(t) \in [0,1]$ and $w_i(t) \in [0,1]$, $j \in \mathcal{D}_i$, $i \in \mathcal{I}_n$. It implies in turn constraints on the control functions for the linearized system (1.21)-(1.23). That is, $U_{ij}(t) \in [-u^*_{ij}, 1-u^*_{ij}]$ and $W_i(t) \in [-w^*_i, 1-w^*_i]$. Therefore, putting suitable constraints on the norm of the gains K_z, K_y, L_z and L_y, would result on the desired control constraints for some initial conditions. It is important however to point out that we have to impose $L_y = 0$ in order to well handle some technical issues in our analysis as we will see in Proposition 1.1.

Let us first define the set of admissible initial conditions to be considered in the sequel.

Definition 1.2
Let $\mathcal{E}^0$ be the set of all admissible initial conditions and input disturbances for the closed-loop system (1.37)-(1.39) as follows:

$$\begin{aligned}\mathcal{E}^0 = \Big\{ Z^0 \in \mathbb{R}^n; y^0 \in L^2([0,1];\mathbb{R}^m); \tilde{d} \in \mathcal{C}_{pw}(\mathbb{R}^+;\mathbb{R}^n) | \\ \big(e^{2\mu}\lambda_{max}(P)\|Z^0\|^2 + e^{2\mu}\lambda_{max}(Q)\|y^0\|^2_{L^2([0,1],\mathbb{R}^m)}\big) \\ + \tfrac{\gamma}{2\nu}e^{2\mu} \sup_{s\in[0,\infty)} \|\tilde{d}(s)\|^2 \leq \mathcal{M}^2 \Big\}\end{aligned}$$

for a given $\mathcal{M} > 0$.

The problem now is to find the control gains K_z, K_y, L_z with limitation on amplitude such that the control functions for the closed-loop system (1.37)-(1.39) satisfy

$$\begin{cases} \|W_i(t)\| \leq \min\{|-w_i^*|, |1-w_i^*|\} = \delta_i^w \\ \|U_{ij}(t)\| \leq \min\{|-u_{ij}^*|, |1-u_{ij}^*|\} = \delta_{ij}^u \quad i \in \mathcal{I}_n, j \in \mathcal{D}_i \end{cases} \tag{1.45}$$

and that hypothesis of Theorem 1.2 hold.

Proposition 1.1
Let $Z^0, y^0, \tilde{d}$ be in $\mathcal{E}^0$. Let δ_i^w, δ_{ij}^u be given and $p \in [0,1]$. If $\beta_z = \mathcal{M}$, $\beta_y = \sqrt{\sum_{k=1}^m \|G_z^T(k)\|^2}\mathcal{M}$ and $\|K_{zi}\| \leq \frac{p\delta_i^w}{\beta_z}$, $\|K_{yi}\| \leq \frac{(1-p)\delta_i^w}{\beta_y}$, $\|L_{zi}\| \leq \frac{\delta_{ij}^u}{\beta_z}$, then, for all $t \geq 0$, $\|Z(t)\| \leq \beta_z$, $\|y(t,1)\| \leq \beta_y$, and (1.45) holds.

Proof. From (1.34) note that, for all $t \geq 0$,

$$\|Z(t)\|^2 + \|y(t,\cdot)\|^2_{L^2([0,1];\mathbb{R}^m)} \leq e^{2\mu}\lambda_{max}(P)\|Z^0\|^2 + e^{2\mu}\lambda_{max}(Q)\|y^0\|^2_{L^2([0,1];\mathbb{R}^m)} + \frac{\gamma}{2\nu}e^{2\mu}\|\tilde{d}\|^2_\infty$$

Since Z^0, y^0, $\tilde{d}$ are in $\mathcal{E}^0$, it follows from (1.45) that

$$\|Z(t)\|^2 + \|y(t,\cdot)\|^2_{L^2([0,1];\mathbb{R}^m)} \leq \mathcal{M}^2 \tag{1.46}$$

Therefore, from the previous inequality it holds also that, on one hand, for all $t \geq 0$, $\|Z(t)\|^2 \leq \mathcal{M}^2$. In particular, for $t \geq \frac{1}{\underline{\lambda}}$, $\|Z(t - \frac{1}{\underline{\lambda}})\|^2 \leq \mathcal{M}^2$.

On the other hand, using the characteristic solutions of the linear hyperbolic system in (1.37), we have, for $t \geq \frac{1}{\underline{\lambda}}$,

$$y_k(t,1) = y_k(t - \tfrac{1}{\lambda_k}, 0), \quad k = 1, ..., m \tag{1.47}$$

using the boundary condition (1.38) with $L_y = 0$, and recalling the notation in (1.41), we obtain

$$y_k(t,1) = \mathcal{G}_z^T[k]Z(t - \tfrac{1}{\lambda_k}) \quad k = 1, ..., m \tag{1.48}$$

Let us then deduce an upper bound for $\|y(t,1)\|$ as follows. Observe that

$$\|y(t,1)\|^2 = \sum_{k=1}^m y_k^2(t,1) \tag{1.49}$$

Replacing (1.48) into (1.49), we get

$$\begin{aligned} \|y(t,1)\|^2 &= \sum_{k=1}^m \left(\mathcal{G}_z^T[k]Z(t - \tfrac{1}{\lambda_k})\right)^2 \\ &\leq \sum_{k=1}^m \|\mathcal{G}_z^T(k)\|^2\|Z(t - \tfrac{1}{\lambda_k})\|^2 \\ &\leq \sum_{k=1}^m \|\mathcal{G}_z^T(k)\|^2\mathcal{M}^2 \end{aligned} \tag{1.50}$$

Therefore, $\|y(t,1)\| \leq \beta_y$.

Next, by remarking that $\delta_i^w \geq \frac{p\delta_i^w}{\beta_z}\beta_z + \frac{(1-p)\delta_i^w}{\beta_y}\beta_y$ and using the hypothesis, let us consider the following inequalities:

$$\begin{aligned}\frac{p\delta_i^w}{\beta_z}\beta_z + \frac{(1-p)\delta_i^w}{\beta_y}\beta_y &\geq \frac{p\delta_i^w}{\beta_z}\|Z(t)\| + \frac{(1-p)\delta_i^w}{\beta_y}\|y(t,1)\| \\ &\geq \|K_{zi}\|\|Z(t)\| + \|K_{yi}\|\|y(t,1)\| \\ &\geq \|K_{zi}Z(t)\| + \|K_{yi}y(t,1)\| \\ &\geq \|K_{zi}Z(t) + K_{yi}y(t,1)\|\end{aligned} \tag{1.51}$$

Knowing that $W(t) = \begin{bmatrix} K_z & K_y \end{bmatrix} \begin{bmatrix} Z(t) \\ y(t,1) \end{bmatrix}$ and that each component is given by $W_i(t) = K_{zi}Z(t) + K_{yi}y(t,1)$, then, from (1.51) we finally obtain that

$$\|W_i(t)\| \leq \delta_i^w$$

The same analysis applies for $U_{ij}(t)$ to end up with $\|U_{ij}(t)\| \leq \delta_{ij}^u$. It concludes the proof. □

By just extending what was stated in Remark 1.4, the optimization problem can be formulated as a constrained optimization one by putting the conditions on the control gains provided in Proposition 1.1, that is,

Problem 1.1

$$\begin{aligned} &\textit{minimize} && \frac{\gamma}{2\nu}e^{2\mu} \\ &\textit{subject to} && M_c \leq 0; \\ & && \|K_{zi}\| \leq \frac{p\delta_i^w}{\beta_z}; \|K_{yi}\| \leq \frac{(1-p)\delta_i^w}{\beta_y}; \|L_{zi}\| \leq \frac{\delta_{ij}^u}{\beta_z} \end{aligned} \tag{1.52}$$

The constraints of this problem however must be well transformed in order to handle them numerically. This concerns the main result of this subsection. Before we state the main result, let us consider the following proposition which is a variation of the result in [25, Proposition 2.6] or in [51, Section 3.2] and is going to be useful during the proof of Theorem 1.3 .

Proposition 1.2
Let $\tilde{\delta} > 0$ be given. If there exist matrices S and T of appropriate dimension and a positive scalar $\tilde{\eta}$ such that

$$\begin{pmatrix} \tilde{\eta}He(S) - \tilde{\eta}^2 I & T \\ \star & \tilde{\delta}^2 I \end{pmatrix} \geq 0 \tag{1.53}$$

then $\|TS^{-1}\| \leq \tilde{\delta}$. ($He(S) := S^T + S$).

Theorem 1.3
Assume that there exist $\mu, \gamma > 0$, a symmetric positive definite matrix $X \in \mathbb{R}^{n\times n}$, a diagonal positive matrix $Q_3 \in \mathbb{R}^{m\times m}$, as well as matrices $Y_{K_z} \in \mathbb{R}^{n\times n}$, $Y_{K_y} \in \mathbb{R}^{n\times m}$ and $Y_{L_z} \in \mathbb{R}^{l\times n}$ such that:

$$\tilde{M}_c = \begin{pmatrix} XA^T + AX + 2\mu\underline{\lambda}X + Y_{K_z}^T B_w^T + B_w Y_{K_z} & G_y Q_3 + B_w Y_{K_y} & D & XG_z^T + Y_{L_z}^T B_u^T \\ \star & -e^{-2\mu}Q_3 & 0 & 0 \\ \star & \star & -\gamma I & 0 \\ \star & \star & \star & -Q_3 \end{pmatrix} \leq 0 \tag{1.54}$$

and

$$\begin{pmatrix} \eta He(X) - \eta^2 I & Y_{K_z} \\ \star & \left(\frac{p\delta_i^w}{\beta_z}\right)^2 I \end{pmatrix} \geq 0; \begin{pmatrix} \eta He(X) - \eta^2 I & Y_{L_z} \\ \star & \left(\frac{\delta_{ij}^u}{\beta_z}\right)^2 I \end{pmatrix} \geq 0$$
$$\begin{pmatrix} \eta He(Q_3) - \eta^2 I & Y_{K_y} \\ \star & \left(\frac{(1-p)\delta_i^w}{\beta_y}\right)^2 I \end{pmatrix} \geq 0; \tag{1.55}$$

for some $\eta > 0$ and δ_i^w, δ_{ij}^u, β_z, β_y given in Proposition 1.1.

Then, by setting $K_z = Y_{K_z}X^{-1}$, $K_y = Y_{K_y}Q_3^{-1}$ and $L_z = Y_{L_z}X^{-1}$ it holds $\|Y_{K_z}X^{-1}\| \leq \frac{p\delta_i^w}{\beta_z}$, $\|Y_{K_y}Q_3^{-1}\| \leq \frac{(1-p)\delta_i^w}{\beta_y}$ and $\|Y_{L_z}X^{-1}\| \leq \frac{\delta_{ij}^u}{\beta_z}$, the constraints on (1.52) are satisfied and Theorem 1.2 applies.

Proof. The inequality $M_c \leq 0$ in (1.52) is a BMI that can be transformed into a proper LMI (provided μ fixed). To do so, first note that, even with $L_y = 0$ as we have imposed for our analysis, M_c given by (1.40) can be rewritten as follows:

$$M_c = \begin{pmatrix} \mathcal{A}^T P + P\mathcal{A} + 2\mu\underline{\lambda}P & P\mathcal{G}_y & PD \\ \star & -e^{-2\mu}Q\Lambda & 0 \\ \star & \star & -\gamma I \end{pmatrix} + \begin{pmatrix} \mathcal{G}_z^T \\ 0 \\ 0 \end{pmatrix} Q\Lambda \begin{pmatrix} \mathcal{G}_z & 0 & 0 \end{pmatrix} \leq 0 \tag{1.56}$$

Applying the *Schur Complement* on (1.56) and performing the change of variable $Q_2 = Q\Lambda$ (being Q_2 still diagonal), we get that $\mathcal{A}^T P + P\mathcal{A} + 2\mu\underline{\lambda}P \leq 0$ and

$$\begin{pmatrix} \mathcal{A}^T P + P\mathcal{A} + 2\mu\underline{\lambda}P & P\mathcal{G}_y & PD & \mathcal{G}_z^T \\ \star & -e^{-2\mu}Q_2 & 0 & 0 \\ \star & \star & -\gamma I & 0 \\ \star & \star & \star & -Q_2^{-1} \end{pmatrix} \leq 0 \tag{1.57}$$

Multiplying on both sides of the previous matrix by a diagonal matrix $diag(I, Q_2^{-1}, I, I)$, and performing the change of variable $Q_3 = Q_2^{-1}$ (being Q_3 still diagonal), we obtain the following equivalent matrix inequality

$$\begin{pmatrix} \mathcal{A}^T P + P\mathcal{A} + 2\mu\underline{\lambda}P & P\mathcal{G}_y Q_3 & PD & \mathcal{G}_z^T \\ \star & -e^{-2\mu}Q_3 & 0 & 0 \\ \star & \star & -\gamma I & 0 \\ \star & \star & \star & -Q_3 \end{pmatrix} \leq 0 \tag{1.58}$$

Now, multiplying on both sides of the previous matrix by a diagonal matrix $diag(P^{-1}, I, I, I)$, using (1.41) and performing the following change of variables $X = P^{-1}$, $Y_{K_z} = K_z X$, $Y_{L_z} = L_z X$ and $Y_{K_y} = K_y Q_3$, we obtain $\tilde{M}_c$ given by (1.54). Hence, it can be noticed that (1.54) is now a LMI (provided μ fixed). Next, applying Proposition 1.2 we have that if $\begin{pmatrix} \eta He(X)-\eta^2 I & Y_{K_z} \\ \star & (\frac{p\delta_i^w}{\beta_z})^2 I \end{pmatrix} \geq 0$, $\begin{pmatrix} \eta He(X)-\eta^2 I & Y_{L_z} \\ \star & (\frac{\delta_{ij}^u}{\beta_z})^2 I \end{pmatrix} \geq 0$ and $\begin{pmatrix} \eta He(Q_3)-\eta^2 I & Y_{K_y} \\ \star & (\frac{(1-p)\delta_i^w}{\beta_y})^2 I \end{pmatrix} \geq 0$, for some $\eta > 0$, then $\|Y_{K_z} X^{-1}\| \leq \frac{\delta_i^w}{p\beta_z}$, $\|Y_{L_z} X^{-1}\| \leq \frac{\delta_{ij}^u}{\beta_z}$ and $\|Y_{K_y} Q_3^{-1}\| \leq \frac{(1-p)\delta_i^w}{\beta_y}$ respectively. Note that the constraints on (1.52) hold for the original variables. An immediate consequence is that Theorem 1.2 holds. With this, we conclude the proof. □

Remark 1.5
When considering the following problem (Problem 1.2) where the objective function in (1.59) is the same as in Problem 1.1, one can obtain a sub-optimal solution of the Problem 1.1.

Problem 1.2

$$\begin{aligned}
&\textit{minimize} \quad \frac{\gamma}{2\nu} e^{2\mu} \\
&\textit{subject to} \quad \tilde{M}_c \leq 0; \quad (\tilde{M}_c \textit{ as in } (1.54)) \\
&\begin{pmatrix} \eta He(X)-\eta^2 I & Y_{K_z} \\ \star & \left(\frac{p\delta_i^w}{\beta_z}\right)^2 I \end{pmatrix} \geq 0; \begin{pmatrix} \eta He(X)-\eta^2 I & Y_{L_z} \\ \star & \left(\frac{\delta_{ij}^u}{\beta_z}\right)^2 I \end{pmatrix} \geq 0 \\
&\begin{pmatrix} \eta He(Q_3)-\eta^2 I & Y_{K_y} \\ \star & \left(\frac{(1-p)\delta_i^w}{\beta_y}\right)^2 I \end{pmatrix} \geq 0;
\end{aligned} \tag{1.59}$$

Remark 1.6
Note that the objective function in Problem 1.2 is nonlinear, involving variables γ and μ ($\nu = \underline{\lambda}\mu$). However, in order to numerically handle this optimization problem, we combine the line search algorithm on μ. In that case, we deal with an optimization problem within the semi-definite programming framework where the line search on μ leads to successive LMIs.

1.6 Numerical simulations

Let us consider a network under compartmental setting as represented in Figure 1.4 which is made up of 4 buffers along with 5 transmission lines.

The index sets involved in the example are: $\mathcal{I}_n = \{1,2,3,4\}$, $\mathcal{I}_{in} = \{1\}$, $\mathcal{I}_{out} = \{4\}$, $\mathcal{U}_1 = \emptyset$, $\mathcal{U}_2 = \{1\}$; $\mathcal{U}_3 = \{1,2\}$, $\mathcal{U}_4 = \{2,3\}$, $\mathcal{D}_1 = \{2,3\}$, $\mathcal{D}_2 = \{3,4\}$, $\mathcal{D}_3 = \{4\}$, $\mathcal{D}_4 = \emptyset$. Consider then the model (1.10)-(1.12) introduced in Section 1.2. We assume that the system admits a *free-flow steady-state* satisfying (1.14) and according to the constrained optimization problem described in Subsection 1.3.1. Recall that the cost function to be minimized is $J = \alpha(-J_1) + (1-\alpha)J_2$ with α chosen according to some decision-maker criteria by means

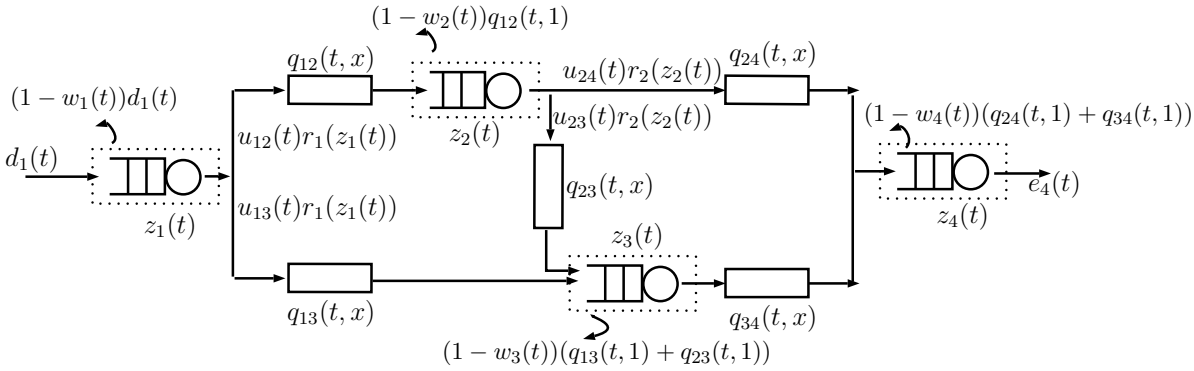

Figure 1.4: Network of compartments made up of 4 buffers and 5 transmission lines.

of the so-called Pareto fronts. Here, $J_1 = e_4^*$ according to (1.15). Following the procedure to compute $J_2 = T_{MTT}$ that we have explained in Subsection 1.3.1, we have

$$\begin{aligned} T_{MTT} =& \frac{(T_4+T_2+T_1+\frac{1}{\lambda_{24}}+\frac{1}{\lambda_{12}})(w_4^*(1-u_{23}^*)w_2^*u_{12}^*w_1^*d_1^*)}{e_4^*} \\ &+ \frac{(T_4+T_3+T_1+\frac{1}{\lambda_{34}}+\frac{1}{\lambda_{13}})(w_4^*w_3^*(1-u_{12}^*)w_1^*d_1^*)}{e_4^*} \\ &+ \frac{(T_4+T_3+T_2+T_1+\frac{1}{\lambda_{34}}+\frac{1}{\lambda_{23}}+\frac{1}{\lambda_{12}})(w_4^*w_3^*u_{23}^*w_2^*u_{12}^*w_1^*d_1^*)}{e_4^*} \end{aligned}$$

with $T_i = \theta_i(z_i^*)$, $i \in \mathcal{I}_n$ (θ_i given by (1.6)) that can be obtained using the minimizers of J, i.e. $(w_1^*, w_2^*, w_3^*, u_{12}^*, u_{23}^*)$ and d_1^*. The linearized system (1.21) around the free-flow equilibrium has the following matrices: $\Lambda = diag(\lambda_{12}, \lambda_{13}, \lambda_{23}, \lambda_{24}, \lambda_{34})$, $A = diag(-r_i^{'}(z_i^*))$ with $r_i^{'}(z_i^*) = \frac{\epsilon_i}{(1+z_i^*)^2}$. $G_z = \begin{pmatrix} u_{12}^*r_1^{'}(z_1^*) & 0 & 0 & 0 \\ (1-u_{12}^*)r_1^{'}(z_1^*) & 0 & 0 & 0 \\ 0 & u_{23}^*r_2^{'}(z_2^*) & 0 & 0 \\ 0 & (1-u_{23}^*)r_2^{'}(z_2^*) & 0 & 0 \\ 0 & 0 & r_3^{'}(z_3^*) & 0 \end{pmatrix}$ $G_y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ w_2^* & 0 & 0 & 0 & 0 \\ 0 & w_3^* & w_3^* & 0 & 0 \\ 0 & 0 & 0 & w_4^* & w_4^* \end{pmatrix}$, $B_w = \begin{pmatrix} d_1^* & 0 & 0 & 0 \\ 0 & q_{12}^* & 0 & 0 \\ 0 & 0 & q_{13}^*+q_{23}^* & 0 \\ 0 & 0 & 0 & q_{24}^*+q_{34}^* \end{pmatrix}$, $D = \begin{pmatrix} w_1^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, and $B_u = \begin{pmatrix} w_1^*d_1^* & 0 \\ -w_1^*d_1^* & 0 \\ 0 & w_2^*u_{12}^*w_1^*d_1^* \\ 0 & -w_2^*u_{12}^*w_1^*d_1^* \\ 0 & 0 \end{pmatrix}$. As initial conditions, we have taken $Z^0 = 0.1z^*$ and $y^0(x) = 0.1q^*$ for all $x \in [0,1]$.

Let us consider the following network parameters for: i) for the processing capacities; ii) the transport velocities; and iii) the critical traffic densities for free-flow condition as reported in Tables 1.1, 1.2 and 1.3. For a given constant input flow demand $d_1^* = 100$, equilibrium

i)

ϵ_1	ϵ_2	ϵ_3	ϵ_4
100	50	80	100

Table 1.1: Processing capacities ϵ_i, $i \in \mathcal{I}_n$.

ii)

λ_{12}	λ_{13}	λ_{23}	λ_{24}	λ_{34}
1	1.2	1.7	0.5	2

Table 1.2: Transport velocities λ_{ij}, $i \in \mathcal{I}_n$, $j \in \mathcal{D}_i$.

iii)

σ_{12}	σ_{13}	σ_{23}	σ_{24}	σ_{34}
50	50	50	30	100

Table 1.3: Critical traffic densities for free-flow condition σ_{ij}, $i \in \mathcal{I}_n$, $j \in \mathcal{D}_i$.

points were found by solving the constrained optimization problem described in Subsection 1.3.1 using the optimization toolbox in Matlab *fgoalattain*. Nevertheless, one is not able to guarantee a global minimizer but a local one because the cost function is not convex. We use a Pareto front of the multi-objective optimization problem J as illustrated in Figure 1.5. We have chosen $\alpha = 0.5$. Therefore, we obtained minimizer decision variables for J as reported

z_1^*	z_2^*	z_3^*	z_4^*	q_{12}^*	q_{13}^*	q_{23}^*	q_{24}^*	q_{34}^*
19	1.98	13.12	5.6	35	60	18.25	15	74.3

Table 1.4: Steady-states z_i^*, q_{ij}^*, $i \in \mathcal{I}_n$, $j \in \mathcal{D}_i$.

in Tables 1.4 and 1.5. With these values, we obtain that $J_1 = 84.8$ and that $J_2 = 2.19$, being J_1 the maximal flow of the network.

We aim first at minimizing the asymptotic gain in open loop subject to (1.26). This can be formulated as an optimization problem involving a bilinear matrix inequality (BMI), which can be solved by a line search algorithm (on μ) to get successive LMIs to be solved using semi-definite programing. At each iteration, hypothesis of Theorem 1.1 are satisfied. Once the optimization problem is solved, Theorem 1.1 holds with optimal matrices $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 20.96 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1.40 \end{pmatrix}$, $Q = \begin{pmatrix} 4.25 & 0 & 0 & 0 & 0 \\ 0 & 3.41 & 0 & 0 & 0 \\ 0 & 0 & 7.023 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ along with optimal values $\mu = 0.11$, $\nu = 0.055$ and $\gamma = 3.59$. Then, a bound of the asymptotic gain obtained is $\frac{\gamma}{2\nu} e^{2\mu} = 40.48$.

Then, in closed loop, when solving Problem 1.2, at each iteration, hypothesis of Theorem 1.3 are satisfied. Once the optimization problem is solved, Theorem 1.2 holds with optimal matrices

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1.55 & 0 & 0 \\ 0 & 0 & 1.13 & 0 \\ 0 & 0 & 0 & 1.11 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1.08 & 0 & 0 \\ 0 & 0 & 0 & 1.07 & 0 \\ 0 & 0 & 0 & 0 & 1.1 \end{pmatrix}$$

along with $\mu = 0.38$ and $\nu = 0.19$. $\gamma = 0.58$. The values of control gains K_z, K_y and L_z are given as follows:

$$K_z = \begin{pmatrix} -7.26\times10^{-3} & 0 & 0 & 0 \\ 0 & -2.22\times10^{-3} & 0 & 0 \\ 0 & 0 & -4.74\times10^{-3} & 0 \\ 0 & 0 & 0 & -1.65\times10^{-3} \end{pmatrix},$$

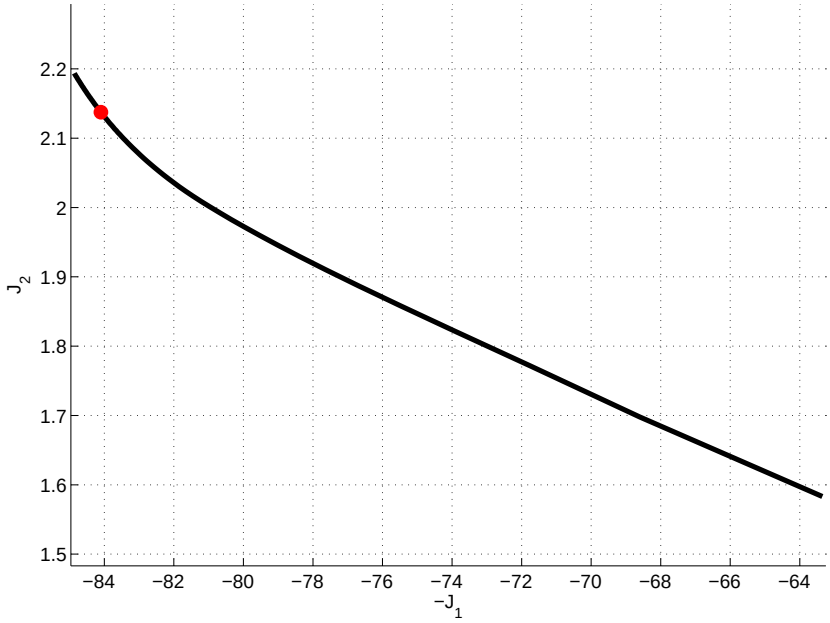

Figure 1.5: Pareto front.

w_1^*	w_2^*	w_3^*	w_4^*	u_{12}^*	u_{23}^*
0.95	0.95	0.95	0.95	0.36	0.54

Table 1.5: decision variables $w_i^*, u_{ij}^*, i \in \mathcal{I}_n, j \in \mathcal{D}_i$.

$$K_y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -8.76\times10^{-3} & 0 & 0 & 0 & 0 \\ 0 & -8.70\times10^{-3} & -8.61\times10^{-3} & 0 & 0 \\ 0 & 0 & 0 & -5.24\times10^{-3} & -4.99\times10^{-3} \end{pmatrix}$$

and

$$L_z = \begin{pmatrix} -4.65\times10^{-4} & 0 & 0 & 0 \\ 0 & -4.25\times10^{-2} & 0 & 0 \end{pmatrix}$$

The asymptotic gain is bounded by $C_2 \leq \frac{\gamma}{2\nu}e^{2\mu} = 3.3$. As we have seen previously, in open loop (i.e. when $L_z = K_y = K_z = 0$) such a bound was given by 40.48. Therefore, it can be observed the benefits of the closed loop setting because we reduced considerably the asymptotic gain, thus the impact of the input flow demands on the behavior of the network while converging to desired equilibrium point in free-flow. On a frame of 40s, we close the loop at $t = \frac{1}{\underline{\lambda}} = 2$. Figure 1.7 shows that, as expected from Theorem 1.5.2, $\dot{V} - 2\nu V - \gamma\|\tilde{d}\|^2$ is less or equal than zero.

Figure 1.8 shows the total output of the network in open loop (black line) and in closed loop (red dashed line). It can be observed that thanks to the control actions, we can guarantee a faster convergence to the equilibrium and it remains closer than the open loop case. For both cases, the input flow demand has the profile depicted in Figure 1.6. In addition, Figure 1.9 shows the control functions *(access control)* devoted to reject traffic flow. It can be noticed that they respect the constraints.

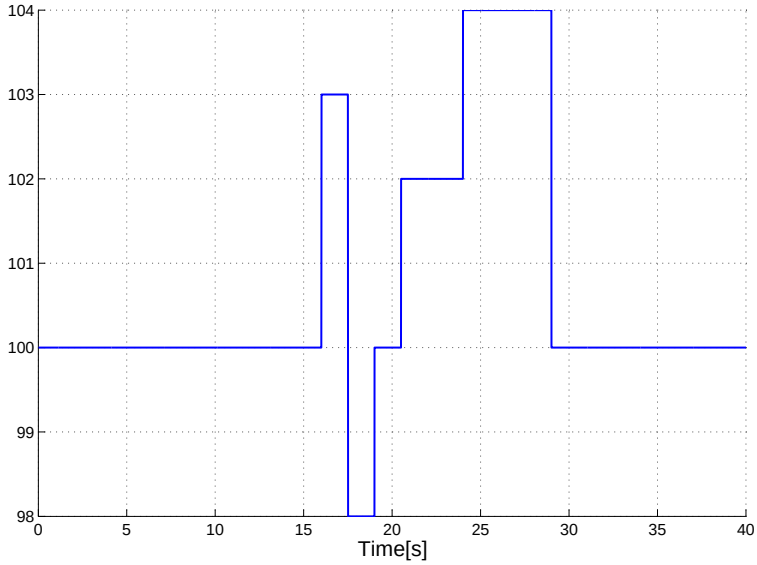

Figure 1.6: Input flow demand $d_1(t)$.

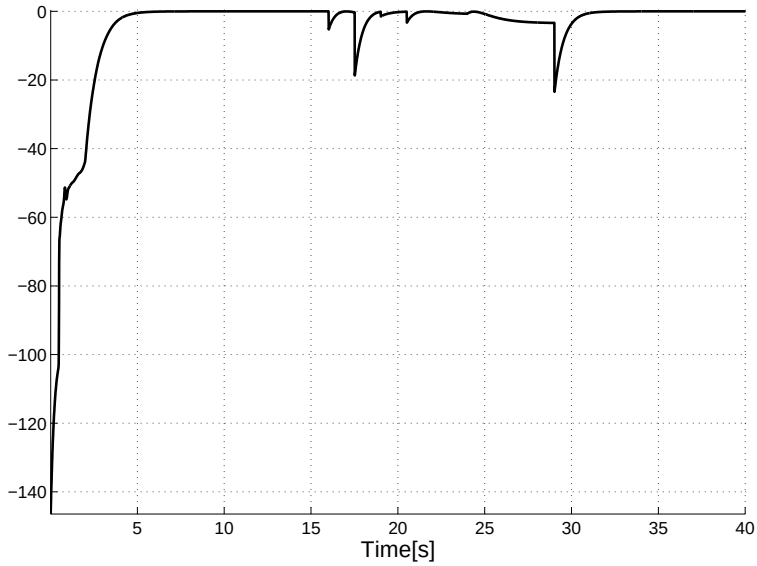

Figure 1.7: Graph of the function $\dot{V} - 2\nu V - \gamma\|\tilde{d}\|^2$.

To finish, let us briefly comment that it turned out that the routing controls do not contribute too much on the minimization of the asymptotic gain. In fact, the gains are too small, besides the fact that we also had to impose $L_y = 0$. Perhaps, when thinking about other control objectives, the routing control could impact better.

Remark 1.7
Algorithms 1 and 2 in the appendix show how to handle the optimization issues treated in this chapter.

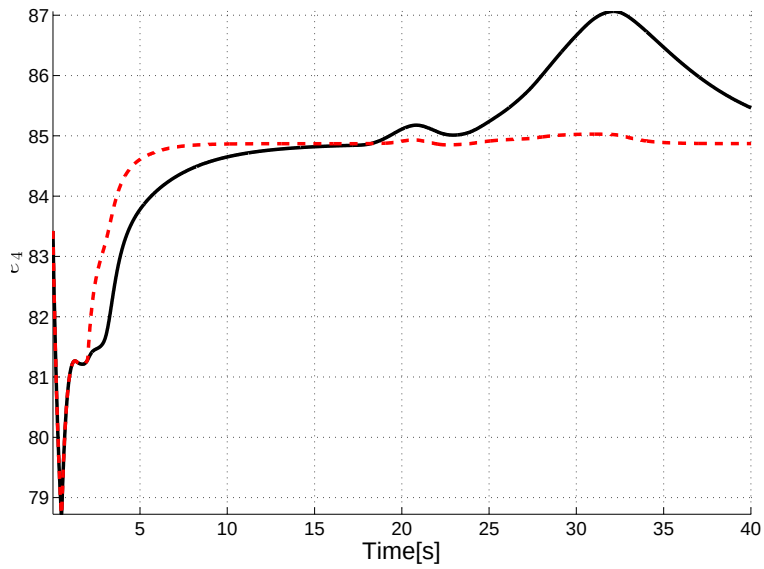

Figure 1.8: Total output flow of the network: in open loop (black line) and in closed loop (red dashed line).

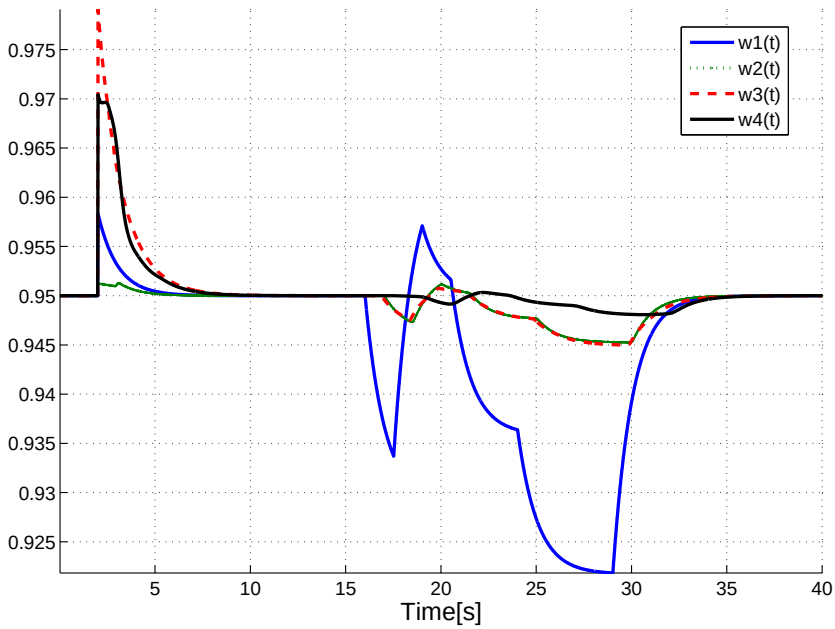

Figure 1.9: Time evolution of control functions w_i.

Chapter 2

Event-based control of linear hyperbolic systems of conservation laws

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In this chapter, we introduce event-based boundary controls for 1-dimensional linear hyperbolic systems of conservation laws. Inspired by event-triggered controls developed for finite-dimensional systems, an extension to the infinite dimensional case by means of Lyapunov techniques, is studied. The main contribution of the chapter lies in the definition of three event-triggering conditions, by which global exponential stability and well-posedness of the system under investigation is achieved. Some numerical simulations are performed for the control of a system describing traffic flow on a roundabout and for a particular instance of the network considered in Section 1.6 when no buffer dynamics are taken into account.

2.1 Introduction

We consider in this chapter a particular case of the system dealt in Chapter 1. As we have mentioned, several example of physical networks modeled by hyperbolic PDEs motivate the use of boundary control. They all motivate the use of boundary control, specially on event-based fashion which is actually a realistic approach for the actuator in those systems. In this chapter we focus on boundary control using Lyapunov techniques where the *dissipativity* property of the boundary conditions is an important issue to be taken into account. We have mentioned also that the framework of switched hyperbolic systems [32, 53, 38] is highly inspiring -especially the work in [38]- for dealing with the well-posedness of the closed-loop solution of such systems under event-based control strategies. The main difference of [38] with respect to the current work is that in [38], no boundary control inputs are considered but rather switching boundary conditions governed by a switching signal, given as a output feedback, that imposes the mode in which the system must evolve.

The main contribution that can be highlighted in this chapter is to propose a rigorous framework for event-based control of linear hyperbolic systems of conservation laws, as well as three event-based stabilization strategies based on the main triggering strategies developed for systems described by ODEs called ISS static event-based stabilization, D^+V event-based stabilization and ISS dynamic event-based stabilization in the sequel. The notion of existence and uniqueness of the solution is treated. It is also established that the number of events in a bounded time interval is necessarily bounded avoiding the well known Zeno phenomena. To the author's knowledge, this work is the first contribution to event-based control for hyperbolic PDE systems proposed in the literature. For PDEs, a well known approach for digital controller synthesis relies on numerical approximations by discretizing the space in order to get an ODE (see e.g. [19]) on which finite dimensional approaches can be applied. In this work, the method is completely different and adresses directly the boundary control without model reduction and the sampling in time of continuous controllers so that implementations on a digital platform may be carried out in an aperiodic fashion.

The work condensed in this chapter was published in [22] and [23].

2.2 Linear Hyperbolic Systems

Let us consider the linear hyperbolic system of conservation laws (given in Riemann coordinates):

$$\partial_t y(t,x) + \Lambda \partial_x y(t,x) = 0 \qquad x \in [0,1], t \in \mathbb{R}^+ \tag{2.1}$$

where $y : \mathbb{R}^+ \times [0,1] \to \mathbb{R}^n$, Λ is a diagonal matrix in $\mathbb{R}^{n\times n}$ such that $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ with $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$. We consider the following boundary condition:

$$y(t,0) = Hy(t,1) + Bu(t), \qquad t \in \mathbb{R}^+ \tag{2.2}$$

where $H \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $u : \mathbb{R}^+ \to \mathbb{R}^m$.
In addition to the partial differential equation (2.1) and the boundary condition (2.2), we

consider the initial condition given by

$$y(0,x) = y^0(x), \qquad x \in [0,1] \tag{2.3}$$

where $y^0 \in \mathcal{C}_{lpw}([0,1],\mathbb{R}^n)$.

Remark 2.1
The results in this chapter can be extended to first order linear hyperbolic systems with both negative and positive speeds ($\lambda_1 < \cdots \lambda_m < 0 < \lambda_{m+1} < \cdots < \lambda_n$) by defining the state description $y = [y_- \quad y_+]^T$, where $y_- \in \mathbb{R}^m$ and $y_+ \in \mathbb{R}^{n-m}$, and applying the change of variable $\tilde{y}(t,x) = [y_-(t,1-x) \quad y_+(t,x)]^T$.

We shall consider possibly discontinuous inputs $u \in \mathcal{C}_{rpw}(\mathbb{R}^+,\mathbb{R}^m)$, therefore solutions of (2.1)-(2.3) may not be differentiable everywhere. Thus, we introduce a notion of weak solutions (generalized ones) (in Subsection 2.2.1) as well as a sufficient condition for the existence and uniqueness of the solution for a class of discontinuous initial conditions and feedback laws (in Subsection 2.2.2).

2.2.1 Solution of the system

We consider solutions of (2.1)-(2.3) in the sense of characteristics [41]. For each component y_d of (2.1), one can define the characteristic curve solution of the differential equation $\dot{x}(t) = \lambda_d$ which is rewritten as $x(t) = x_0 + \lambda_d t$. By doing this, we obtain the following definition (see [38, Definition 4] for a more general case):

Definition 2.1
Let $y^0 \in \mathcal{C}_{lpw}([0,1],\mathbb{R}^n)$ and $u \in \mathcal{C}_{rpw}(\mathbb{R}^+,\mathbb{R}^m)$. A solution to (2.1)-(2.3) is a function $y : \mathbb{R}^+ \times [0,1] \to \mathbb{R}^n$ such that, for all t in $\mathbb{R}^+$ and $x_0 \in [-\lambda_d t, 1-\lambda_d t]$,

$$\frac{d}{dt} y_d(t, x_0 + \lambda_d t) = 0 \tag{2.4}$$

with the initial condition

$$y_d(0,x) = y_d^0(x), \quad \forall x \in [0,1] \tag{2.5}$$

and the boundary condition

$$y_d(t,0) = \sum_{j=1}^{n} H_{dj} y_j(t,1) + \sum_{j=1}^{m} B_{dj} u_j(t), \quad \forall t \in \mathbb{R}^+ \tag{2.6}$$

for all $d = 1,..,n$.

Note that for classical differentiable solutions, (2.1)-(2.3) are equivalent to (2.4)-(2.6) and note that y does not need to be differentiable nor continuous in general but only differentiable along the characteristics as given by (2.4).

In this chapter, we assume that the linear hyperbolic system is only observed at the point $x = 1$ at any time. Therefore we define the output function as follows:

$$z(t) = y(t, 1) \tag{2.7}$$

2.2.2 A sufficient condition for the existence and uniqueness of the solution

Now that solutions intended in this chapter are properly defined, we will consider the following causality assumption:

Assumption 2.1
Let φ be an operator from $\mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^n)$ to $\mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^m)$ satisfying the following causality property: for all s in $\mathbb{R}^+$, for all z, $z^* \in \mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^n)$

$$(\forall t \in [0, s], z(t) = z^*(t)) \implies (\forall t \in [0, s], u(t) = u^*(t))$$

where $u = \varphi(z)$ and $u^* = \varphi(z^*)$.

This assumption enables us to state the following result on existence of solutions:

Proposition 2.1
Let φ satisfy Assumption 2.1 and $y^0 \in \mathcal{C}_{lpw}([0, 1], \mathbb{R}^n)$. Then, there exists a unique solution to the closed-loop system (2.1)-(2.3) with controller $u = \varphi(z)$ where z is defined by (2.7). Moreover, for all $t \in \mathbb{R}^+$ $y(t, \cdot) \in \mathcal{C}_{lpw}([0, 1], \mathbb{R}^n)$ and for all $x \in [0, 1]$ $y(\cdot, x) \in \mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^n)$.

Proof. Let us consider $\overline{\lambda} = \max_{1 \le i \le n}\{\lambda_i\}$ and let $\delta = 1/\overline{\lambda}$ be the minimum time for a characteristic, with velocity $\overline{\lambda}$, to cross the spatial domain [0,1]. For $p \in \mathbb{N}$, let $\Delta_p \subset \mathbb{R}^+$ be defined by $\Delta_p = [p\delta, (p + 1)\delta]$. We will proceed by induction over the interval Δ_p with the following induction property:

- y is defined on $\Delta_p \times [0, 1]$;
- $y(\cdot, x) \in \mathcal{C}_{rpw}(\Delta_p, \mathbb{R}^n)$;
- $y(t, \cdot) \in \mathcal{C}_{lpw}([0, 1], \mathbb{R}^n)$.

Let us consider, for $p = 0$, the interval $\Delta_0 = [0, \delta]$. We first check that y given, for all $(t, x) \in \Delta_0 \times [0, 1]$ and $d \in \{1, \dots, n\}$, by

$$y_d(t, x) = \begin{cases} y_d(t - \frac{x}{\lambda_d}, 0), & \text{if } \lambda_d t \ge x \\ y_d^0\left(x - \lambda_d t\right), & \text{if } \lambda_d t < x \end{cases} \tag{2.8}$$

is the solution to system (2.1)-(2.3). Equation (2.8) can be written in an equivalent way; that is, for all t in Δ_0, x_0 in $[-\lambda_d t, 1-\lambda_d t]$, the solution at each component is given by

$$y_d(t, x_0+\lambda_d t) = \begin{cases} y_d(\frac{-x_0}{\lambda_d}, 0), & \text{if } x_0 \leq 0 \\ y_d^0(x_0), & \text{if } x_0 > 0 \end{cases} \tag{2.9}$$

Since (2.9) does not depend on t, it is clear that $\frac{d}{dt}y_d(t, x_0+\lambda_d t)$ satisfies Definition 2.1, and hence y is solution to (2.1)-(2.3). Conversely, $\frac{d}{dt}y_d(t, x_0+\lambda_d t) = 0$ implies $f(x_0) = y_d(t, x_0+\lambda_d t)$ for all $t \in [0, \delta]$ and $x_0 \in [-\lambda_d t, 1-\lambda_d t]$. In particular, by setting $t = 0$, one gets

$$f(x_0) = y_d(0, x_0) \quad \text{if } x_0 > 0$$

On the other hand, by setting $t = \frac{-x_0}{\lambda_d}$,

$$f(x_0) = y_d(-\tfrac{x_0}{\lambda_d}, 0) \quad \text{if } x_0 \leq 0$$

Therefore, we have obtained equation (2.9) and then, shown that it is the unique solution to (2.1)-(2.3) in the sense of characteristics on $\Delta_0 \times [0,1]$. Furthermore, when $x = 1$, $\lambda_d t < 1$ and for all $t \in \Delta_0$, one gets

$$z_d(t) = y_d(t, 1) = y_d^0(1-\lambda_d t) \tag{2.10}$$

Since z is well defined on Δ_0, using the causality property, one can claim that u is well defined on Δ_0. In addition, using (2.10) and boundary condition (2.6), (2.8) can be rewritten as follows,

$$y_d(t,x) = \begin{cases} \sum_{j=1}^n H_{dj} z_j(t-\frac{x}{\lambda_d}) + \sum_{j=1}^m B_{dj} u_j(t-\frac{x}{\lambda_d}), & \text{if } \lambda_d t \geq x \\ y_d^0(x-\lambda_d t), & \text{if } \lambda_d t < x \end{cases} \tag{2.11}$$

It is worth remarking that $z_j(t-\frac{x}{\lambda_d}) = y_j^0(1-\lambda_d t + x)$. Then, it is proved that y_d depends uniquely on y_d^0 on $\Delta_0 \times [0,1]$.
Since y^0 belongs to $\mathcal{C}_{lpw}([0,1], \mathbb{R}^n)$, z belongs to $\mathcal{C}_{rpw}(\Delta_0, \mathbb{R}^n)$. Therefore, by Assumption 2.1, u belongs to $\mathcal{C}_{rpw}(\Delta_0, \mathbb{R}^m)$. It follows then, from (2.11), that y is defined on $\Delta_0 \times [0,1]$. Moreover,

- $z_j(t-\frac{x}{\lambda_d})$ belongs to $\mathcal{C}_{rpw}$ with respect to t and belongs to $\mathcal{C}_{lpw}$ with respect to x due to the opposite sign in the argument;
- $u_j(t-\frac{x}{\lambda_d})$ belongs to $\mathcal{C}_{rpw}$ with respect to t and belongs to $\mathcal{C}_{lpw}$ with respect to x due to the opposite sign in the argument;
- $y_d^0(x-\lambda_d t)$ belongs to $\mathcal{C}_{lpw}$ with respect to x and belongs to $\mathcal{C}_{rpw}$ with respect to t due to the opposite sign in the argument.

It follows from (2.11) that $y(t,\cdot) \in \mathcal{C}_{lpw}([0,1], \mathbb{R}^n)$ for all t in Δ_0 and that $y(\cdot, x) \in \mathcal{C}_{rpw}(\Delta_0, \mathbb{R}^n)$ for all $x \in [0,1]$. Thus, induction property holds at $p = 0$.

Now, assume that induction property holds for a given $p \in \mathbb{N}$. We are now going to prove the same property for $p+1 > 0$. For that purpose, let us take $y((p+1)\delta, \cdot)$ as the initial

condition of the system. Applying the same arguments as above; and by means of hypothesis of induction, one gets that $z \in \mathcal{C}_{rpw}(\Delta_{p+1}, \mathbb{R}^n)$, $u \in \mathcal{C}_{rpw}(\Delta_{p+1}, \mathbb{R}^m)$. In addition, y is defined on $\Delta_{p+1} \times [0,1]$, $y(t,\cdot)$ exists for all t in Δ_{p+1} and belongs to $\mathcal{C}_{lpw}([0,1], \mathbb{R}^n)$ and $y(\cdot, x)$ belongs to $\mathcal{C}_{rpw}(\Delta_{p+1}, \mathbb{R}^n)$ for every $x \in [0,1]$.

Therefore, we have proved by induction that, for each $p \in \mathbb{N}$, $z \in \mathcal{C}_{rpw}(\Delta_p, \mathbb{R}^n)$ and $y(t,\cdot)$ exists for all t in Δ_p, and belongs to $\mathcal{C}_{lpw}([0,1], \mathbb{R}^n)$ and $y(\cdot, x)$ belongs to $\mathcal{C}_{rpw}(\Delta_p, \mathbb{R}^n)$ for every $x \in [0,1]$. Thus, there exists an unique solution to the closed-loop system (2.1)-(2.3) with $u = \varphi(z)$. Hence, this concludes the proof. □

2.3 Event-based Stabilization

2.3.1 Some issues related to stability

We define the notion of stability considered in the chapter and state one existing result on stability of linear hyperbolic systems of conservation laws.

Definition 2.2
The linear hyperbolic system (2.1)-(2.3),(2.7) with controller $u = \varphi(z)$ is globally exponentially stable (GES) if there exist $\nu > 0$ and $C > 0$ such that, for every $y^0 \in \mathcal{C}_{lpw}([0,1]; \mathbb{R}^n)$, the solution satisfies, for all t in $\mathbb{R}^+$,

$$\|y(t,\cdot)\|_{L^2([0,1]);\mathbb{R}^n)} \leq Ce^{-\nu t}\|y^0\|_{L^2([0,1];\mathbb{R}^n)} \tag{2.12}$$

We want to point out that a particular case studied in literature (see e.g. [31]) is when φ is given by $u = \varphi_0(z)$ as $u(t) = Kz(t)$. This corresponds to continuous time control for which it holds,

$$y(t,0) = Gz(t) \qquad t \in \mathbb{R}^+ \tag{2.13}$$

with $G = H + BK$.

The following assumption is stated in [11] as a sufficient condition, usually called *dissipative boundary condition*, which guarantees that the system (2.1)-(2.3) with boundary condition (2.13) is globally exponentially stable. In this thesis, such a sufficient condition is assumed to be satisfied.

Assumption 2.2
The following inequality holds:

$$\rho_1(G) = Inf\big\{\|\Delta G \Delta^{-1}\|; \Delta \in \mathcal{D}_{n,+}\big\} < 1 \tag{2.14}$$

where $\|\cdot\|$ denotes the usual 2-norm of matrices in $\mathbb{R}^{n\times n}$ and $\mathcal{D}_{n,+}$ denotes the set of diagonal matrices whose elements on the diagonal are strictly positive.

Recall the following result:

Proposition 2.2 ([18])
Under Assumption 2.2, there exist $\mu > 0$, and a diagonal positive definite matrix $Q \in \mathbb{R}^{n\times n}$ (with $Q = \Lambda^{-1}\Delta^2$) such that the following matrix inequality holds

$$G^T Q\Lambda G < e^{-2\mu} Q\Lambda. \tag{2.15}$$

Then, the linear hyperbolic system (2.1)-(2.3),(2.7),(2.13) is GES and (4.29) holds for some $C > 0$ and $\nu = \mu\underline{\lambda}$ where $\underline{\lambda} = \min_{1\le i\le n}\{\lambda_i\}$.

Under the assumption of Proposition 2.2, inspired by [18, Theorem 1], let us recall that the function defined, for all $y(\cdot) \in L^2([0,1],\mathbb{R}^n)$, by

$$V(y) = \int_0^1 y(x)^T Q y(x) e^{-2\mu x} dx \tag{2.16}$$

satisfies, along the classical solutions of (2.1)-(2.3),(2.7) and (2.13), that

$$\dot{V} \le y^T(\cdot,1)\left[G^T Q\Lambda G - e^{-2\mu} Q\Lambda\right] y(\cdot,1) - 2\nu \int_0^1 y^T Q y e^{-2\mu x} dx$$

thus in regard of (2.15), it is a Lyapunov function. The global exponential stability along L^2 solutions follows by density (see [18] for more details).

2.3.2 ISS static event-based stabilization

We introduce in this section a first event-based control scheme for hyperbolic systems of conservation laws and discuss the existence of solutions and their stability under this control strategy. This approach relies on both the Input-to-State Stability property with respect to deviations to sampling and Lyapunov techniques. It is mainly inspired by [59] where the sampling error is restricted to satisfy a state-dependent inequality. It guarantees that the ISS-Lyapunov function is strictly decreasing. In this paper, we will seek for ISS property with respect to a deviation between the continuous controller and the event-based controller, combined with a strict Lyapunov condition using (2.16).

Definition 2.3 (Definition of φ_1)
Let $\varsigma_1, \kappa, \eta, \mu > 0$, K in $\mathbb{R}^{m\times n}$, Q a diagonal positive matrix in $\mathbb{R}^{n\times n}$. Let us define φ_1 the operator which maps z to u as follows:

Let z be in $\mathcal{C}_{rpw}(\mathbb{R}^+,\mathbb{R}^n)$ and let $\tilde{V}$ be given, at $t = \frac{1}{\underline{\lambda}}$, by

$$\tilde{V}(\tfrac{1}{\underline{\lambda}}) = \sum_{i=1}^n Q_{ii} \int_0^1 \left(H_i z(\tfrac{1}{\underline{\lambda}} - \tfrac{x}{\lambda_i})\right)^2 e^{-2\mu x} dx \tag{2.17}$$

and, for all $t > \frac{1}{\underline{\lambda}}$*, by*

$$\tilde{V}(t) = \sum_{i=1}^{n} Q_{ii} \int_0^1 \left(H_i z(t - \tfrac{x}{\lambda_i}) + B_i u(t - \tfrac{x}{\lambda_i})\right)^2 e^{-2\mu x} dx \tag{2.18}$$

Let $\varepsilon_1(t) = \varsigma_1 \tilde{V}(\frac{1}{\underline{\lambda}}) e^{-\eta t}$ *for all* $t \geq \frac{1}{\underline{\lambda}}$*. If* $\tilde{V}(\frac{1}{\underline{\lambda}}) > 0$*, let the increasing sequence of time instants* (t_k^u) *be defined iteratively by* $t_0^u = 0$*,* $t_1^u = \frac{1}{\underline{\lambda}}$*, and for all* $k \geq 1$*,*

$$t_{k+1}^u = \inf\{t \in \mathbb{R}^+ | t > t_k^u \wedge \|BK(-z(t) + z(t_k^u))\|^2 \geq \kappa \tilde{V}(t) + \varepsilon_1(t)\} \tag{2.19}$$

If $\tilde{V}(\frac{1}{\underline{\lambda}}) = 0$*, let the time instants be defined by* $t_0^u = 0$*,* $t_1^u = \frac{1}{\underline{\lambda}}$ *and* $t_2^u = \infty$*.*

Finally, let the control function, $z \mapsto \varphi_1(z)(t) = u(t)$*, be defined by:*

$$\begin{array}{rcll} u(t) & = & 0 & \forall t \in [t_0^u, t_1^u) \\ u(t) & = & Kz(t_k^u) & \forall t \in [t_k^u, t_{k+1}^u), \quad k \geq 1 \end{array} \tag{2.20}$$

Remark 2.2
The boundary condition (2.2) with controller $u = \varphi_1(z)$ *as defined in Definition 2.3 can be rewritten as:*

$$y(t, 0) = Gz(t) + d(t) \qquad t \in \mathbb{R}^+ \tag{2.21}$$

where

$$d(t) = BK(-z(t) + z(t_k^u)) \qquad t \in [t_k^u, t_{k+1}^u) \tag{2.22}$$

which can be seen as a deviation between the continuous controller $u = Kz$ *and the event-based controller of Definition 2.3. Hence, we follow the perturbed system approach as in [59], [39] and [33] that we will call in the sequel ISS static event-based stabilization. The event triggering condition (2.19) ensures that, for all* t*,* $\|d(t)\|^2 \leq \kappa \tilde{V}(t) + \varepsilon_1(t)$*.*

In addition, we point out the possible case when $\tilde{V}(\frac{1}{\underline{\lambda}}) = 0$*, then* $\varepsilon_1(t) = 0$*. From (2.17), it means that for all* $i = 1, .., n$*,* $z(s) = 0$ *with* $s = t - \frac{x}{\lambda_i}$ *for all* $s \in [t - \frac{1}{\lambda_i}, t]$*. In particular, for all* $i = 1, .., n$*,* $z(s) = 0$ *for all* $s \in [t - \frac{1}{\underline{\lambda}}, t]$ *which means that the system has already achieved the steady state in finite time. In that scenario, event-based stabilization would not be required.*

The following proposition shows that $\tilde{V}$ given by (2.18) is an estimate of V.

Proposition 2.3
Let y *be a solution to (2.1)-(2.3). It holds that, for all* $t \geq \frac{1}{\underline{\lambda}}$*,* $V(y(t, \cdot)) = \tilde{V}(t)$*, where* $\tilde{V}(t)$ *is given by (2.18).*

Proof. Along solutions y to the system (2.1)-(2.3) and since Q is diagonal, (2.16) gives:

$$V(y(t, \cdot)) = \sum_{i=1}^{n} Q_{ii} \int_0^1 y_i^2(t, x) e^{-2\mu x} dx$$

Using the first line of (2.8), in particular for all $t \geq \frac{1}{\underline{\lambda}}$, the boundary condition (2.6) and output function (2.7) one has that $y_i(t,x) = y_i(t - \frac{x}{\lambda_i}, 0) = H_i z(t - \frac{x}{\lambda_i}) + B_i u(t - \frac{x}{\lambda_i})$. Therefore, for all $t \geq \frac{1}{\underline{\lambda}}$,

$$\begin{aligned} V(y(t,\cdot)) &= \sum_{i=1}^{n} Q_{ii} \int_0^1 \left[H_i z(t - \tfrac{x}{\lambda_i}) + B_i u(t - \tfrac{x}{\lambda_i}) \right]^2 e^{-2\mu x} dx \\ &= \tilde{V}(t) \end{aligned}$$

This concludes the proof. □

Due to the previous proposition, $\tilde{V}$ can be seen as an estimation of the Lyapunov function V. It is based only on the measured output function and the input value. As a consequence, the triggering condition in (2.19) depends on the measured output function and the input value as well. In addition, it will be discussed in Subsection 2.3.2.1 that ε_1 is to guarantee the existence and uniqueness of the closed-loop system.

In the next section, we will prove that operator φ_1 satisfies Assumption 2.1.

2.3.2.1 Existence and uniqueness of the closed-loop solution

The goal is to prove that $u = \varphi_1(z)$ belongs to $\mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^m)$ provided z is in $\mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^n)$ and that φ_1 is a causal operator.

Lemma 2.1
The operator φ_1 considered in Definition 2.3 satisfies Assumption 2.1.

Proof. Let z in $\mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^n)$ and $u = \varphi_1(z)$ where φ_1 is the operator given in Definition 2.3. Let J be a closed interval subset of $\mathbb{R}^+$. By hypothesis, z has a finite number of discontinuities on J. Let $t_1^z, \cdots, t_M^z \in J$ be the increasing sequence of these discontinuity time instants; and t_0^z and t_{M+1}^z are respectively the lower bound and the upper bound of the interval J. We want to prove that u has a finite number of discontinuities on the time interval $[t_i^z, t_{i+1}^z]$, with $i \in \{0, ..., M\}$. If $\tilde{V}(\frac{1}{\underline{\lambda}}) = 0$, there is only at most one discontinuity which is $t_1^u = \frac{1}{\underline{\lambda}}$. If $\tilde{V}(\frac{1}{\underline{\lambda}}) > 0$, let us remark that it is sufficient to show that there is a finite number of discontinuities on the open time interval (t_i^z, t_{i+1}^z), with $i \in \{0, ..., M\}$.

Let $w_1^i(t)$ be the continuation of $BKz(t)$ on the interval $[t_i^z, t_{i+1}^z]$ with the left limit of $BKz(t)$ in t_{i+1}^z, that is

$$w_1^i(t) = BKz(t), \quad \text{if} \quad t \in [t_i^z, t_{i+1}^z) \tag{2.23}$$

$$w_1^i(t_{i+1}^z) = \lim_{t \to (t_{i+1}^z)^-} BKz(t) \tag{2.24}$$

The definition of $\mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^n)$ ensures that the left limit of $BKz(t)$ exists and that $w_1^i(t)$ is continuous on the closed interval $[t_i^z, t_{i+1}^z]$. Therefore, it is uniformly continuous. It means

that for all $\zeta > 0$, there exists $\tau > 0$ such that

$$\forall t, t' \in [t_i^z, t_{i+1}^z] : |t - t'| < \tau \to \|w_1^i(t) - w_1^i(t')\|^2 < \zeta$$

We denote τ_i the value of τ when $\zeta = \varepsilon_1(t_{i+1}^z)$. Let the sequence (t_k^u) defined by Equation (2.19) in Definition 2.3. Assume first that there exists at least two consecutive discontinuity instants in (t_i^z, t_{i+1}^z) and let t_k^u be the first one of these instants. We will deal later on whether only one time instant exists within this interval.

Let us consider $\| - w_1^i(t) + w_1^i(t_k^u)\|^2$ for all $t \in [t_i^z, t_{i+1}^z]$ where w^i is given in (2.23) and (2.24). Combined with (2.19) of Definition 2.3 and using the continuity of $\tilde{V}$, ε_1 and w_1^i, it holds at time $t = t_{k+1}^u$:

$$\|w_1^i(t_k^u) - w_1^i(t_{k+1}^u)\|^2 \geq \kappa \tilde{V}(t_{k+1}^u) + \varepsilon_1(t_{k+1}^u)$$

Using the non-negativity of $\tilde{V}$, the fact that ε_1 is a decreasing function, the uniform continuity argument and the definition of τ_i, one gets

$$\begin{aligned} &\|w_1^i(t_k^u) - w_1^i(t_{k+1}^u)\|^2 \geq \varepsilon_1(t_{k+1}^u) \geq \varepsilon_1(t_{i+1}^z) \\ &\implies |t_k^u - t_{k+1}^u| \geq \tau_i \end{aligned}$$

Thus, τ_i gives a lower bound for the duration between two input updates, depending only on the interval (t_i^z, t_{i+1}^z).

Finally, an upper bound for the maximal number of input updates on (t_i^z, t_{i+1}^z) is given by:

$$\overline{s}_i = \left\lfloor \frac{t_{i+1}^z - t_i^z}{\tau_i} \right\rfloor$$

If there is at most one element of the sequence (t_k^u) in (t_i^z, t_{i+1}^z) then $\overline{s}_i$ can be chosen equal to 1. To conclude, the number of discontinuities of u on J is bounded by $\overline{S} = \sum_{i=1}^{M} \overline{s}_i + M + 2$ which is finite.

In addition, from (2.20) in Definition 2.3, u is piecewise constant, which yields $u \in \mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^m)$.

Let us now prove that our operator φ_1 satisfies the causality property. Let $s \in \mathbb{R}^+$ and z, $z^* \in \mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^n)$ be given such that

$$z(t) = z^*(t) \quad \forall t \in [0, s] \tag{2.25}$$

Let $u = \varphi_1(z)$ and $u^* = \varphi_1(z^*)$. It will be shown that $u(t) = u^*(t)$ for all $t \in [0, s]$. Let us first consider $t \in [0, s)$. It follows that $u(t) = Kz(t_k^u)$ where t_k^u is the previous triggering time associated to u. It follows also from (2.25) that $z(t_k^u) = z^*(t_k^{u^*})$, and $t_k^u = t_k^{u^*}$ where $t_k^{u^*}$ is the previous triggering time associated to u^*. Therefore $u(t) = Kz^*(t_k^{u^*}) = u^*(t)$ for all $t \in [0, s)$. Let $\tilde{V}$ be defined by (2.18) and $\tilde{V}^*$ be defined similarly replacing z and u by z^* and u^* respectively. Therefore $\tilde{V}(t) = \tilde{V}^*(t)$ for all $t \in [0, s]$.

Let us consider what happens at $t = s$. Three cases are pointed out:

1. Suppose that there is no triggering time at $t = s$. In this case, $u(s) = Kz(t_k^u)$ where $t_k^u = t_k^{u^*}$ is the previous triggering time. Clearly, $u^*(s) = Kz^*(t_k^{u^*}) = Kz(t_k^u) = u(s)$.

2. Suposse that there is a triggering time for both u and u^*, that is at $s = t_{k+1}^u = t_{k+1}^{u^*}$. Then, with (2.25)

$$\begin{aligned} u(t_{k+1}^u) &= Kz(t_{k+1}^u) = Kz(s) \\ &= Kz^*(s) = Kz^*(t_{k+1}^{u^*}) = u^*(t_{k+1}^{u^*}) \end{aligned}$$

3. Suppose that there is a triggering time at time s. Assume without loss of generality that $s = t_{k+1}^u$. Consider $f(t) = \|BK(-z(t) + z(t_k^u))\|^2 - \kappa\tilde{V}(t) - \varepsilon_1(t)$ and $f^*(t) = \|BK(-z^*(t) + z^*(t_k^{u^*}))\|^2 - \kappa\tilde{V}^*(t) - \varepsilon_1(t)$. Then there exists a sequence of time (s_n), $s_n \geq s$ with $s_n \to s$ and $f(s_n) \geq 0$. Since f is in $\mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^n)$, $f(s) \geq 0$. According to (2.25) and using $\tilde{V}(s) = \tilde{V}^*(s)$ it follows that $f(s) = f^*(s) \geq 0$. It means that a trigger happens at the same time for u^* and then $t_{k+1}^{u^*} = t_{k+1}^u = s$. Consequently, one comes back to the previous case and hence $u(s) = u^*(s)$.

Since u is in $\mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^m)$ and the causality property is satisfied, Assumption 2.1 holds. It concludes the proof. □

Combining the previous lemma with Proposition 2.1, we get

Corollary 2.1
For any y^0 in $\mathcal{C}_{lpw}([0,1], \mathbb{R}^n)$, there exists a unique solution to the closed-loop system (2.1)-(2.3),(2.7) and controller $u = \varphi_1(z)$.

2.3.2.2 Stability of the closed-loop system

Let us now state our first main result.

Theorem 2.1
Let K be in $\mathbb{R}^{m\times n}$ such that Assumption 2.2 holds for $G = H + BK$. Let $\mu > 0$, Q a diagonal positive matrix in $\mathbb{R}^{n\times n}$ and $\nu = \mu\underline{\lambda}$ be as in Proposition 2.2. Let σ be in $(0,1)$ and $\alpha > 0$ such that $(1+\alpha)G^TQ\Lambda G \leq e^{-2\mu}Q\Lambda$. Let ρ be the largest eigenvalue of $(1+\frac{1}{\alpha})Q\Lambda$, $\kappa = \frac{2\nu\sigma}{\rho}$, $\eta > 2\nu(1-\sigma)$ and ε_1 and φ_1 be given in Definition 2.3. Let V be given by (2.16). Then the system (2.1)-(2.3),(2.7) with the controller $u = \varphi_1(z)$ has a unique solution and is globally exponentially stable. Moreover, it holds for all $t \geq \frac{1}{\underline{\lambda}}$,

$$D^+V(t) \leq -2\nu(1-\sigma)V(t) + \rho\varepsilon_1(t) \tag{2.26}$$

Proof. The existence and uniqueness of a solution to system (2.1)-(2.3),(2.7) with controller $u = \varphi_1(z)$ is given by Corollary 2.1.

We are here going to show that the system is globally exponentially stable. Assume for the time-being that $\tilde{V}(\frac{1}{\underline{\lambda}}) > 0$.

First, we establish the following lemma which will be necessary for the proof of Theorem 2.1.

Lemma 2.2
Let y be a solution to (2.1)-(2.3) and let $V(y)$ be given by (2.16). Then, $t \mapsto V(y(t,\cdot))$ is continuous and right differentiable on $\mathbb{R}^+$ and its right time-derivative is given by:

$$\begin{aligned} D^+V = y^T(\cdot,0)Q\Lambda y(\cdot,0) - y^T(\cdot,1)e^{-2\mu}Q\Lambda y(\cdot,1) \\ - 2\mu\int_0^1 y^T(\Lambda e^{-2\mu x}Q)ydx \end{aligned} \tag{2.27}$$

The proof of Lemma 2.2 is given in Appendix 2.5.

Using the boundary condition (2.2) with $u = \varphi_1(z)$, we obtain from its equivalent form (2.21) that (2.27) can be rewritten as follows:

$$\begin{aligned} D^+V &= (Gy(\cdot,1)+d)^TQ\Lambda(Gy(\cdot,1)+d) \\ &\quad -y^T(\cdot,1)e^{-2\mu}Q\Lambda y(\cdot,1) - 2\mu\int_0^1 y^T(\Lambda e^{-2\mu x}Q)ydx \end{aligned}$$

Using the output function given by (2.7), it gives:

$$\begin{aligned} D^+V &= (Gz)^TQ\Lambda Gz + 2(Gz)^TQ\Lambda d + d^TQ\Lambda d \\ &\quad -z^Te^{-2\mu}Q\Lambda z - 2\mu\int_0^1 y^T(\Lambda e^{-2\mu x}Q)ydx \end{aligned} \tag{2.28}$$

By means of a decoupling procedure between d and z using the *Young's inequality* one gets that $(Gz)^TQ\Lambda Gz + 2(Gz)^TQ\Lambda d + d^TQ\Lambda d \leq (1+\alpha)(Gz)^TQ\Lambda Gz + (1+\frac{1}{\alpha})d^TQ\Lambda d$. Since α is such that $(1+\alpha)G^TQ\Lambda G \leq e^{-2\mu}Q\Lambda$, from (2.28) it follows:

$$D^+V \leq -2\mu\int_0^1 y^T\Lambda Qye^{-2\mu x}dx + (1+\tfrac{1}{\alpha})d^TQ\Lambda d$$

Since Q is diagonal positive definite, it holds $\Lambda Q \geq \underline{\lambda}Q$. Thus, taking $\nu = \mu\underline{\lambda}$, it yields,

$$D^+V \leq -2\nu V + (1+\tfrac{1}{\alpha})d^TQ\Lambda d$$

which can be rewritten as follows:

$$\begin{aligned} D^+V &\leq -2\nu V + \rho\|d\|^2 \\ &= -2\nu(1-\sigma)V - 2\nu\sigma V + \rho\|d\|^2, \quad \sigma \in (0,1) \end{aligned}$$

For all $t \geq \frac{1}{\underline{\lambda}}$, Proposition 2.3 implies that

$$D^+V \leq -2\nu(1-\sigma)V - 2\nu\sigma\tilde{V} + \rho\|d\|^2$$

In order to guarantee D^+V is strictly negative, $-2\nu\sigma\tilde{V} + \rho\|d\|^2$ must be strictly negative. Therefore, from the definition of φ_1, events are triggered so as to guarantee for all $t \geq \frac{1}{\underline{\lambda}}$, $\|d\|^2 \leq \kappa\tilde{V} + \varepsilon_1$. Using $\kappa = \frac{2\nu\sigma}{\rho}$, we obtain for all $t \geq \frac{1}{\underline{\lambda}}$,

$$D^+V(t) \leq -2\nu(1-\sigma)V(t) + \rho\varepsilon_1(t)$$

Which gives inequality (2.26) of Theorem 2.1. Then, using the *Comparison principle*, one gets, for all $t \geq \frac{1}{\underline{\lambda}}$,

$$V(y(t,\cdot)) \leq e^{-2\nu(1-\sigma)(t-\frac{1}{\underline{\lambda}})}V(y(\tfrac{1}{\underline{\lambda}},\cdot)) + \rho\int_{\frac{1}{\underline{\lambda}}}^{t} e^{-2\nu(1-\sigma)(t-s)}\varepsilon_1(s)ds$$

and thus,

$$\begin{aligned} V(y(t,\cdot)) \leq & V(y(\tfrac{1}{\underline{\lambda}},\cdot))e^{-2\nu(1-\sigma)(t-\frac{1}{\underline{\lambda}})} + \frac{\rho\varsigma_1\tilde{V}(\frac{1}{\underline{\lambda}})}{2\nu(1-\sigma)-\eta}e^{-\eta t} \\ & - \frac{\rho\varsigma_1\tilde{V}(\frac{1}{\underline{\lambda}})}{2\nu(1-\sigma)-\eta}e^{-2\nu(1-\sigma)(t-\frac{1}{\underline{\lambda}})-\eta\frac{1}{\underline{\lambda}}} \end{aligned}$$

Select $\eta > 2\nu(1-\sigma)$. Thus, we get, for all $t \geq \frac{1}{\underline{\lambda}}$,

$$V(y(t,\cdot)) \leq V(y(\tfrac{1}{\underline{\lambda}},\cdot))e^{-2\nu(1-\sigma)(t-\frac{1}{\underline{\lambda}})} + \frac{\rho\varsigma_1 V(y(\frac{1}{\underline{\lambda}},\cdot))}{\eta-2\nu(1-\sigma)}e^{-2\nu(1-\sigma)(t-\frac{1}{\underline{\lambda}})-\eta\frac{1}{\underline{\lambda}}} \tag{2.29}$$

with $V(y(\frac{1}{\underline{\lambda}},\cdot)) = \tilde{V}(\frac{1}{\underline{\lambda}})$ due to Proposition 2.3. The previous inequality holds even if $\tilde{V}(\frac{1}{\underline{\lambda}}) = 0$ since in this case $V(y(\frac{1}{\underline{\lambda}},\cdot)) = 0$ for all $t \geq \frac{1}{\underline{\lambda}}$.

Let us see what happens for all t in $[0, \frac{1}{\underline{\lambda}})$ so as the right hand side of the inequality depends on the initial condition y^0. For that purpose, let us consider the following function:

$$\mathcal{V}(y) = \int_0^1 y(x)^T Q y(x) e^{2\theta x} dx \tag{2.30}$$

where $\theta > 0$. In addition, from Definition 2.3, $u = 0$ for all $t \in [0, \frac{1}{\underline{\lambda}})$. It implies that the boundary condition given by (2.2) is $y(t,0) = Hy(t,1)$. Computing the right time-derivative of (2.30), it yields

$$D^+\mathcal{V}(y(t,\cdot)) = y^T(t,1)\left[H^TQ\Lambda H - e^{2\theta}Q\Lambda\right]y(t,1) + \int_0^1 2\theta y^T Q\Lambda y e^{2\theta}dx$$

There exists $\theta \geq 0$ such that $H^TQ\Lambda H < e^{2\theta}Q\Lambda$. In addition, since Q is a diagonal positive matrix, $\Lambda Q \leq \overline{\lambda}Q$ holds. Therefore, $D^+\mathcal{V} \leq 2\theta\overline{\lambda}\mathcal{V}$. Hence, the solution of the previous differential inequality thanks to the comparison principle satisfies $\mathcal{V}(y(t,\cdot)) \leq e^{2\theta\overline{\lambda}t}\mathcal{V}(y^0)$. In particular, $\mathcal{V}(y(\frac{1}{\underline{\lambda}},\cdot)) \leq e^{2\theta\frac{\overline{\lambda}}{\underline{\lambda}}}\mathcal{V}(y^0)$. On the other hand, (2.16) and (2.30) imply

$$\begin{aligned} V(y) \leq \mathcal{V}(y) &= \int_0^1 y(x)^T Q y(x) e^{2(\theta+\mu)x} e^{-2\mu x} dx \\ &\leq e^{2(\theta+\mu)}V(y) \end{aligned}$$

and thus, one gets for all $t \in [0, \frac{1}{\underline{\lambda}})$,

$$V(y(t,\cdot)) \leq \mathcal{V}(y(t,\cdot)) \leq e^{2\theta\overline{\lambda}t}\mathcal{V}(y^0) \leq e^{2\theta\overline{\lambda}t}e^{2(\theta+\mu)}V(y^0)$$

In particular, at $t = \frac{1}{\underline{\lambda}}$ we have,

$$V(y(\tfrac{1}{\underline{\lambda}},\cdot)) \leq e^{2\theta\frac{\overline{\lambda}}{\underline{\lambda}}}e^{2(\theta+\mu)}V(y^0) \tag{2.31}$$

Replacing (2.31) in (2.29) we get for all $t \geq \frac{1}{\underline{\lambda}}$,

$$V(y(t,\cdot)) \leq e^{2\theta\frac{\overline{\lambda}}{\underline{\lambda}}}e^{2(\theta+\mu)}e^{-2\nu(1-\sigma)(t-\frac{1}{\underline{\lambda}})}V(y^0) + \frac{\rho\varsigma_1 e^{2\theta\frac{\overline{\lambda}}{\underline{\lambda}}}e^{2(\theta+\mu)}V(y^0)}{\eta - 2\nu(1-\sigma)}e^{-2\nu(1-\sigma)(t-\frac{1}{\underline{\lambda}})-\eta\frac{1}{\underline{\lambda}}}$$

Reorganizing, we finally get,

$$V(y(t,\cdot)) \leq e^{\left(2\theta\frac{\overline{\lambda}}{\underline{\lambda}}+2(\theta+\mu)+2\nu(1-\sigma)\frac{1}{\underline{\lambda}}\right)}\left[1+\rho\frac{\varsigma_1 e^{-\eta\frac{1}{\underline{\lambda}}}}{\eta-2\nu(1-\sigma)}\right]e^{-2\nu(1-\sigma)t}V(y^0)$$

This ends the proof of Theorem 2.1. □

Remark 2.3
We emphasize again that it was crucial that $u \in \mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^m)$ just to be able to apply Proposition 1 and to prove existence and uniqueness of solution.

The decreasing function ε_1 has been added within the triggering law with the aim to prove that the control value, under the triggering condition (2.19), has a finite number of discontinuities as it could be seen in the proof of Lemma 2.1. Indeed, one was able to apply uniform continuity arguments thanks, precisely, to the function ε_1.

2.3.3 D^+V event-based stabilization

Let us consider in this section an event-triggered control similar to the one proposed in [45]. This triggering condition is not based on the existence of an Input-to-State stability assumption but relies on the time derivative of a Lyapunov function. This approach can also be found in an implicit form in [33] for a linear plant, in which it is required that the solution of an event-based implementation satisfies what they call *weaker* inequality. The control value is then only updated when such an inequality is violated. Without entering into further details about the aforementioned approaches for finite-dimensional systems, we will just point out that, in this work, the weaker inequality that causes an event when violated is of the following form:

$$D^+V_{ev} \leq (1-\sigma)D^+V_{cont} + \varepsilon, \quad \sigma \in (0,1)$$

where D^+V_{ev} is the right time-derivative of the Lyapunov function candidate of the system when the control is updated only on events and D^+V_{cont} the right time-derivative of the same Lyapunov function candidate but for a control that is continuously updated and ε a

non-negative decreasing function of the time. Following the same idea, we will consider the Lyapunov function candidate V given by (2.16). Since d will here again denote the deviation between the continuously updated control and the event-triggered ones (as is (2.22)), D^+V_{ev} denotes the right time-derivative of V when $d \neq 0$:

$$D^+V_{ev} = z^T \left[G^TQ\Lambda G - e^{-2\mu}Q\Lambda\right] z + 2(Gz)^TQ\Lambda d \\ +d^TQ\Lambda d + \int_0^1 y^T(-2\mu\Lambda e^{-2\mu x}Q)ydx$$

whereas D^+V_{cont} denotes the right time-derivative of V when $d = 0$:

$$D^+V_{cont} = z^T \left[G^TQ\Lambda G - e^{-2\mu}Q\Lambda\right] z + \int_0^1 y^T(-2\mu\Lambda e^{-2\mu x}Q)ydx \tag{2.32}$$

Now that the main idea has been stated, let us formulate this in a more rigorous way with the definition of the following operator that plays the same role as Definition 2.3 for the ISS static based approach presented in Section 2.3.2. Before we state the definition, let us note that, using (2.32) and (2.32), $D^+V_{ev} \leq (1-\sigma)D^+V_{cont} + \varepsilon$ is equivalent to the following, for all $t \geq 0$ and t_k^u to be defined later on,

$$\begin{aligned}\sigma z(t)^T \left[G^TQ\Lambda G - e^{-2\mu}Q\Lambda\right] z(t) + 2(Gz(t))^TQ\Lambda BK(z(t_k^u) - z(t)) \\ +(BK(z(t_k^u) - z(t)))^TQ\Lambda BK(z(t_k^u) - z(t)) \\ \leq 2\mu\sigma \int_0^1 y^TQ\Lambda y e^{-2\mu x}dx + \varepsilon(t)\end{aligned}$$

Definition 2.4 (Definition of φ_2)
Let $\varsigma_2, \eta > 0$, $\sigma \in (0,1)$, K in $\mathbb{R}^{m\times n}$, $G = H + BK$ in $\mathbb{R}^{n\times n}$, D a diagonal positive definite matrix in $\mathbb{R}^{n\times n}$, $\mu > 0$ such that $G^TDG - e^{-2\mu}D$ is a negative symmetric matrix in $\mathbb{R}^{n\times n}$, V given by (2.16) and $\varepsilon_2(t) = \varsigma_2 V(\frac{1}{\underline{\lambda}})e^{-\eta t}$ for all $t \geq \frac{1}{\underline{\lambda}}$. Let us define φ_2 the operator which maps z to u as follows:

Let z in $\mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^n)$ and the time function $\tilde{V}_2$ be defined similarly to (2.18), for all $t \geq \frac{1}{\underline{\lambda}}$ by:

$$\tilde{V}_2(t) = \sum_{i=1}^n D_{ii} \int_0^1 \left(H_i z(t - \tfrac{x}{\lambda_i}) + B_i u(t - \tfrac{x}{\lambda_i})\right)^2 e^{-2\mu x}dx$$

If $V(\frac{1}{\underline{\lambda}}) > 0$, let the increasing sequence of time instants (t_k^u) be defined iteratively by $t_0^u = 0$, $t_1^u = \frac{1}{\underline{\lambda}}$, and for all $k \geq 1$,

$$\begin{aligned}t_{k+1}^u = \inf\{t \in \mathbb{R}^+ | t > t_k^u \ \wedge \ \sigma z^T(t) \left[G^TDG - e^{-2\mu}D\right] z(t) \\ +2(Gz(t))^TDBK(z(t_k^u) - z(t)) \\ +(BK(z(t_k^u) - z(t)))^TDBK(z(t_k^u) - z(t)) \\ \geq 2\mu\sigma\tilde{V}_2(t) + \varepsilon_2(t)\}\end{aligned} \tag{2.33}$$

If $V(\frac{1}{\underline{\lambda}}) = 0$, let the time instants be defined by $t_0^u = 0$, $t_1^u = \frac{1}{\underline{\lambda}}$ and $t_2^u = \infty$.

Finally, let the control function, $z \mapsto \varphi_2(z)(t) = u(t)$, be defined by:

$$\begin{array}{rcll} u(t) & = & 0 & \forall t \in [t_0^u, t_1^u) \\ u(t) & = & Kz(t_k^u) & \forall t \in [t_k^u, t_{k+1}^u), \quad k \geq 1 \end{array} \tag{2.34}$$

Remark 2.4
Following the same arguments as in the second part of Remark 2.2, whenever $V(\frac{1}{\underline{\lambda}}) = 0$, event-based stabilization would not be required.

The purpose of next two subsections is to prove that system (2.1)-(2.3),(2.7), with the same boundary condition (2.21) as before and under the event-triggered control of Definition 2.4, has a unique solution for any given initial conditions and is globally exponentially stable in the sense of Definition 4.1.

2.3.3.1 Existence and uniqueness of the closed-loop solution

As in Section 2.3.2.1, the goal here is to prove that $u = \varphi_2(z)$ belongs to $\mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^m)$ whenever z is in $\mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^n)$ and that φ_2 is a causal operator. As in Subsection 2.3.2, a decreasing function ε_2 is added into the triggering law which will be helpful to prove the aforementioned assertion.

Lemma 2.3
The operator φ_2 defined in Definition 2.4 satisfies Assumption 2.1.

Proof. The proof follows essentially the same steps as proof of Lemma 2.1. Let z in $\mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^n)$ and $u = \varphi_2(z)$ where φ_2 is the operator given in Definition 2.4. Let J be a closed interval subset of $\mathbb{R}^+$. By assumption, z has a finite number of discontinuities on J. As previously, $t_1^z, \cdots, t_M^z \in J$ will be the increasing sequence of these discontinuity time instants to which we add the the extremities t_0^z and t_{M+1}^z of the interval J. As in the proof of Lemma 2.1, our aim is to prove that u has a finite number of discontinuities on the time interval $[t_i^z, t_{i+1}^z]$, with $i \in \{0, ..., M\}$ and therefore, it is sufficient to show that there is a finite number of discontinuities on the open time interval (t_i^z, t_{i+1}^z), with $i \in \{0, ..., M\}$. If $V(\frac{1}{\underline{\lambda}}) = 0$, there is only at most one discontinuity which is $t_1^u = \frac{1}{\underline{\lambda}}$.

If $V(\frac{1}{\underline{\lambda}}) > 0$, let γ be a strictly positive real number such that the following inequality matrix holds:

$$G^T D G < \frac{\sigma}{\gamma + \sigma} e^{-2\mu} D \tag{2.35}$$

Note that taking γ sufficiently small, this inequality can always be satisfied. Let $w_2 \in \mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^n)$ be defined by:

$$w_2(t) := \sqrt{D} B K z(t) \tag{2.36}$$

where D is as in Definition 2.4 (since D is a diagonal positive definite matrix, D has an unique positive-definite diagonal square root matrix, denoted $\sqrt{D}$, whose diagonal elements are equal to square roots of the diagonal elements of D). Let $w_2^i(t)$ be the continuation of $w_2(t)$ given by (2.36), on the interval $[t_i^z, t_{i+1}^z]$ with the left limit of $w_2(t)$ in t_{i+1}^z, that is:

$$w_2^i(t) = w_2(t), \quad \text{if} \quad t \in [t_i^z, t_{i+1}^z) \tag{2.37}$$

$$w_2^i(t_{i+1}^z) = \lim_{t \to (t_{i+1}^z)^-} w_2(t) \tag{2.38}$$

The definition of $\mathcal{C}_{rpw}(\mathbb{R}^+,\mathbb{R}^n)$ ensures that the left limit of $w_2(t)$ exists and that $w_2^i(t)$ is continuous on the closed interval $[t_i^z, t_{i+1}^z]$. Therefore, it is uniformly continuous. It means that for all $\zeta > 0$, there exists $\tau > 0$ such that

$$\forall t, t' \in [t_i^z, t_{i+1}^z] : |t - t'| < \tau \rightarrow \|w_2^i(t) - w_2^i(t')\|^2 < \zeta$$

As in the proof of Lemma 1, we denote τ_i the value of τ when $\zeta = \frac{\gamma}{\gamma+1}\varepsilon_2(t_{i+1}^z)$. Here again, we assume first that there are at least two consecutive discontinuity instants in (t_i^z, t_{i+1}^z) and let t_k^u be the first one.

Let us consider the triggering condition (2.33) in Definition 2.4 and using the continuity of W_2, ε_2 and w_2^i, it holds at time $t = t_{k+1}^u$:

$$\begin{aligned}\sigma z^T(t_{k+1}^u)\left[G^TDG - e^{-2\mu}D\right]z(t_{k+1}^u) + 2(Gz(t_{k+1}^u))^TDBK(z(t_k^u) - z(t_{k+1}^u))\\ +(BK(z(t_k^u) - z(t_{k+1}^u)))^TDBK(z(t_k^u) - z(t_{k+1}^u))\\ \geq 2\mu\sigma\tilde{V}_2(t_{k+1}^u) + \varepsilon_2(t_{k+1}^u)\end{aligned}$$

Using the *Young's inequality*, one has for γ as in (2.35):

$$\begin{aligned}z^T(t_{k+1}^u)\left[(\gamma+\sigma)G^TDG - \sigma e^{-2\mu}D\right]z(t_{k+1}^u)\\ +(1+\tfrac{1}{\gamma})(BK(z(t_k^u) - z(t_{k+1}^u)))^TDBK(z(t_k^u) - z(t_{k+1}^u))\\ \geq 2\mu\sigma\tilde{V}_2(t_{k+1}^u) + \varepsilon_2(t_{k+1}^u)\end{aligned}$$

Knowing that $(BK(z(t_k^u) - z(t_{k+1}^u)))^TDBK(z(t_k^u) - z(t_{k+1}^u)) = \|\sqrt{D}BK(z(t_k^u) - z(t_{k+1}^u))\|^2$ and according to (2.37) and (2.38), one gets

$$\begin{aligned}\|w_2^i(t_k^u) - w_2^i(t_{k+1}^u)\|^2 \geq \ \tfrac{\gamma}{1+\gamma}\Big(- z^T(t_{k+1}^u)\left[(\gamma+\sigma)G^TDG - \sigma e^{-2\mu}D\right]z(t_{k+1}^u)\\ +2\mu\sigma\tilde{V}_2(t_{k+1}^u) + \varepsilon_2(t_{k+1}^u)\Big)\end{aligned}$$

Moreover, using the non-negativity of $\tilde{V}_2$ and $-z^T(t_{k+1}^u)\left[(\gamma+\sigma)G^TDG - \sigma e^{-2\mu}D\right]z(t_{k+1}^u)$, the fact that ε_2 is a decreasing function, the uniform continuity argument and the definition of τ_i, one gets

$$\begin{aligned}&\|w_2^i(t_{k+1}^u) - w_2^i(t_k^u)\|^2 \geq \tfrac{\gamma}{1+\gamma}\varepsilon_2(t_{k+1}^u) \geq \tfrac{\gamma}{1+\gamma}\varepsilon_2(t_{i+1}^z)\\ &\implies |t_k^u - t_{k+1}^u| \geq \tau_i\end{aligned}$$

Thus, τ_i gives a lower bound for the duration between two input updates, depending only on the interval (t_i^z, t_{i+1}^z). The remaining part of the proof that u is in $\mathcal{C}_{rpw}(\mathbb{R}^+,\mathbb{R}^m)$ follows the lines of Lemma 2.1.

For proving the causality property of the operator φ_2, it is sufficient to follow the same steps as in proof of Lemma 2.1 but considering, at the final step of the proof, $f(t) = \sigma z^T(t)\left[G^TDG - e^{-2\mu}D\right]z(t) + 2(Gz(t))^TDBK(z(t_k^u) - z(t)) + (BK(z(t_k^u) - z(t)))^TDBK(z(t_k^u) - z(t)) - 2\mu\sigma\tilde{V}_2(t) - \varepsilon_2(t)$ and $f^*(t) = \sigma(z^*)^T(t)\left[G^TDG - e^{-2\mu}D\right]z^*(t) + 2(Gz^*(t))^TDBK(z^*(t_k^{u^*}) - z^*(t)) + (BK(z^*(t_k^{u^*}) - z^*(t)))^TDBK(z^*(t_k^{u^*}) - z^*(t)) - 2\mu\sigma\tilde{V}_2^*(t) - \varepsilon_2(t)$.

Since u is in $\mathcal{C}_{rpw}(\mathbb{R}^+,\mathbb{R}^m)$ and the causality property is satisfied, Assumption 2.1 holds. It concludes the proof. □

Combining the previous lemma with Proposition 2.1, we get

Corollary 2.2
For any y^0 in $\mathcal{C}_{lpw}([0,1],\mathbb{R}^n)$, there exists a unique solution to the closed-loop system (2.1)-(2.3),(2.7) and controller $u=\varphi_2(z)$.

2.3.3.2 Stability of the closed-loop system

Let us now state our second main result.

Theorem 2.2
Let K be in $\mathbb{R}^{m\times n}$ such that Assumption 2.2 holds for $G=H+BK$. Let $\mu>0$, Q a diagonal positive matrix in $\mathbb{R}^{n\times n}$ and $\nu=\mu\underline{\lambda}$ be as in Proposition 2.2. Let σ be in $(0,1)$, $\eta>0$ and ε_2 and φ_2 be given in Definition 2.4. Let V be given by (2.16) and d given by (2.22). Then the system (2.1)-(2.3),(2.7) with the controller $u=\varphi_2(z)$ has a unique solution and is globally exponentially stable. Moreover, it holds for all $t\geq\frac{1}{\underline{\lambda}}$,

$$D^+V(t)\leq -2\nu(1-\sigma)V(t)+\varepsilon_2(t) \tag{2.39}$$

Proof. The existence and uniqueness of a solution to system (2.1)-(2.3),(2.7), with $u=\varphi_2(z)$ is given by Corollary 2.2.

We are now going to show that the system is globally exponentially stable. Assume that $V(\frac{1}{\underline{\lambda}})>0$ (the other case is studied as in Theorem 2.1).

From the definition of φ_2, events are triggered so as to guarantee for all $t\geq\frac{1}{\underline{\lambda}}$, $\sigma z^T(t)\left[G^TQ\Lambda G-e^{-2\mu}Q\Lambda\right]z(t)+2(Gz(t))^TQ\Lambda d(t)+d^T(t)Q\Lambda d(t)\leq 2\mu\tilde{V}_2(t)+\varepsilon_2(t)$ and therefore $D^+V_{ev}(t)\leq(1-\sigma)D^+V_{cont}(t)+\varepsilon(t)$ by taking into account that $D=Q\Lambda$ and $\varepsilon=\varepsilon_2$. By construction, it follows that, for all $t\geq\frac{1}{\underline{\lambda}}$,

$$D^+V(t)=D^+V_{ev}(t)\leq(1-\sigma)D^+V_{cont}(t)+\varepsilon_2(t) \tag{2.40}$$

Using $\Lambda Q\geq\underline{\lambda}Q$ and Proposition 2.2, it is clear that $D^+V_{cont}\leq -2\nu V$.

Therefore, in (2.40), for all $t\geq\frac{1}{\underline{\lambda}}$,

$$D^+V(t)\leq -2\nu(1-\sigma)V(t)+\varepsilon_2(t)$$

Now, proceeding exactly as in the proof of Theorem 2.1 when using the *Comparision principle* and analyzing what happens for all $t\in[0,\frac{1}{\underline{\lambda}})$, the final result which proves the global exponential stability, is the following for all $t\geq\frac{1}{\underline{\lambda}}$,

$$V(y(t,\cdot))\leq e^{\left(2\theta\frac{\overline{\lambda}}{\underline{\lambda}}+2(\theta+\mu)+2\nu(1-\sigma)\frac{1}{\underline{\lambda}}\right)}\left[1+\frac{\varsigma_2e^{-\eta\frac{1}{\underline{\lambda}}}}{\eta-2\nu(1-\sigma)}\right]e^{-2\nu(1-\sigma)t}V(y^0)$$

This ends the proof of Theorem 2.2. □

2.3.3.3 Comparison between ISS event-based stabilization and D^+V event-based stabilization

The following proposition links the two event-based control approaches that we have proposed until now.

Proposition 2.4
Let σ be in $(0,1)$, α, ρ, ν, κ and Q be as in Theorem 2.1. Let V be given by (2.16), D^+V_{ev} and D^+V_{cont} be given by (2.32) and (2.32) respectively, d given by (2.22), and $\tilde{V}$ as in Definition 2.3. The following implication holds, for all $t \geq \frac{1}{\underline{\lambda}}$,

$$D^+V_{ev} \geq (1-\sigma)D^+V_{cont} + \varepsilon_2 \Rightarrow \|d\|^2 \geq \frac{2\nu\sigma}{\rho}\tilde{V} + \frac{\varepsilon_2}{\rho} \tag{2.41}$$

Therefore for all solutions to (2.1)-(2.3) and (2.7) in closed-loop, respectively with controllers $u = \varphi_1(z)$ and $u = \varphi_2(z)$, by selecting $\varsigma_1 = \frac{\varsigma_2}{\rho}$ and having the same initial condition, the first triggering time occurs with φ_1 and not with φ_2.

Proof. It holds that $D^+V_{ev} \geq (1-\sigma)D^+V_{cont} + \varepsilon_2$ is equivalent to:

$$\begin{aligned}\sigma z^T\left[G^TQ\Lambda G - e^{-2\mu}Q\Lambda\right]z + 2(Gz)^TQ\Lambda d + d^TQ\Lambda d \\ \geq -\sigma\int_0^1 y^T(-2\mu\Lambda e^{-2\mu x}Q)ydx + \varepsilon_2\end{aligned}$$

Using $\Lambda Q \geq \underline{\lambda}Q$ and the *Young's inequality* we obtain that $D^+V_{ev} \geq (1-\sigma)D^+V_{cont} + \varepsilon_2$ implies,

$$\begin{aligned} & z^T\left[(\sigma+\alpha)G^TQ\Lambda G - \sigma e^{-2\mu}Q\Lambda\right]z + (1+\tfrac{1}{\alpha})d^TQ\Lambda d \\ \geq\ & \sigma z^T\left[G^TQ\Lambda G - e^{-2\mu}Q\Lambda\right]z + 2(Gz)^TQ\Lambda d + d^TQ\Lambda d \\ \geq\ & 2\nu\sigma V + \varepsilon_2\end{aligned}$$

for any $\alpha > 0$ and $\nu = \underline{\lambda}\mu$. Due to Assumption 2.14 and Proposition 2.2, there exists $\alpha > 0$ such that $(\sigma+\alpha)G^TQ\Lambda G - \sigma e^{-2\mu}Q\Lambda < 0$, hence,

$$D^+V_{ev} \geq (1-\sigma)D^+V_{cont} + \varepsilon_2 \rightarrow (1+\tfrac{1}{\alpha})d^TQ\Lambda d \geq 2\nu\sigma V + \varepsilon_2$$

Taking ρ as the largest eigenvalue of matrix $(1+\frac{1}{\alpha})Q\Lambda$ one gets,

$$D^+V_{ev} \geq (1-\sigma)D^+V_{cont} + \varepsilon_2 \rightarrow \rho\|d\|^2 \geq 2\nu\sigma V + \varepsilon_2$$

Finally, using Proposition 2.3, one gets, for all $t \geq \frac{1}{\underline{\lambda}}$,

$$\|d\|^2 \geq \kappa\tilde{V} + \varepsilon_1$$

and thus inequality (2.41) holds, with $\varepsilon_1 = \frac{\varepsilon_2}{\rho}$ and $\kappa = \frac{2\nu\sigma}{\rho}$. □

Remark 2.5
One consequence of Proposition 2.4 is that controller $u = \varphi_1(z)$ generates the first triggering time before the controller $u = \varphi_2(z)$. Of course this does not imply that we have less triggering times with φ_2 than φ_1. However, this property and in turn, larger inter-execution times in average under φ_2 times, are observed on numerical simulations as we will see in the next section.

2.3.4 ISS dynamic event-based stabilization

In this section we introduce a third event-based control strategy relying on the ISS static event-based one introduced in subsection 2.3.2. It is inspired by [28] (for finite dimensional systems) where an internal dynamic variable is added to the event triggering condition in order to reduce the number of triggering times while guaranteeing the exponential stability. We recall that in ISS static event-based stabilization, events are triggered so that $\|d\|^2 - \kappa\tilde{V}$ is always less than ε_1 (see (2.19)). In this new approach, we will rather impose that the *weighted average value* of $\|d\|^2 - \kappa\tilde{V} - \varepsilon_1$ is less than 0. Then, an internal dynamic will be presented under the form $m(t) = e^{-\eta t}\int_{\frac{1}{\underline{\lambda}}}^{t} e^{\eta s}\left(-\kappa\tilde{V}(s) - \varepsilon_1(s) + \|d(s)\|^2\right) ds$ for all $t \geq \frac{1}{\underline{\lambda}}$. Here, d will again denote the deviation between the continuously updated control and the event-triggered one as in (2.22).

Now that the main idea has been stated, let us formulate this in a more rigorous way with the definition of the following operator that plays the same role as Definition 2.3 for the ISS static based approach presented in Section 2.3.2.

Definition 2.5 (Definition of φ_3)
Let σ be in $(0,1)$, $\tilde{V}(t)$, $\varepsilon_1(t)$ given as in Definition 2.3 for all $t \geq \frac{1}{\underline{\lambda}}$, and ρ and κ as in Theorem 2.1. Let us define φ_3 the operator which maps z to u as follows:

Let z be in $\mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^n)$. If $\tilde{V}(\frac{1}{\underline{\lambda}}) > 0$, let the increasing sequence of time instants (t_k^u) be defined iteratively by $t_0^u = 0$, $t_1^u = \frac{1}{\underline{\lambda}}$, and for all $k \geq 1$,

$$t_{k+1}^u = \inf\{t \in \mathbb{R}^+ | t > t_k^u \wedge m(t) \geq 0\} \tag{2.42}$$

where m satisfies the differential equation,

$$\begin{aligned} \dot{m}(t) &= -\eta m(t) + \left(-\kappa\tilde{V}(t) - \varepsilon_1(t) + \|BK(-z(t) + z(t_k^u))\|^2\right) \\ m(\tfrac{1}{\underline{\lambda}}) &= 0 \end{aligned} \tag{2.43}$$

for all $t \in [t_k^u, t_{k+1}^u)$ for a given $\eta > 2\nu(1-\sigma)$.

If $\tilde{V}(\frac{1}{\underline{\lambda}}) = 0$, the time instants are $t_0^u = 0$, $t_1^u = \frac{1}{\underline{\lambda}}$ and $t_2^u = \infty$.

Finally, let the control function be defined by:

$$\begin{aligned} u(t) &= 0 && \forall t \in [t_0^u, t_1^u) \\ u(t) &= Kz(t_k^u) && \forall t \in [t_k^u, t_{k+1}^u), \quad k \geq 1 \end{aligned} \tag{2.44}$$

Note that $m(t_k^u) = 0$ for all $k \geq 1$.

2.3.4.1 Existence and uniqueness of the closed-loop solution

As in both Sections 2.3.2.1 and 2.3.3.1, the goal here is to prove that $u = \varphi_3(z)$ belongs to $\mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^m)$ whenever z is in $\mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^n)$ and that φ_3 is a causal operator. The proof will follow the same steps as in the proofs of Lemmas 2.1 and 2.3. It is worth recalling some of those of steps back here.

Lemma 2.4
The operator φ_3 defined in Definition 2.5 satisfies Assumption 2.1.

Proof. Let us then prove that $u = \varphi_3(z)$ belongs to $\mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^m)$ provided z is in $\mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^n)$. Consider again J a closed interval subset of $\mathbb{R}^+$. Since z is in $\mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^n)$, z has a finite number of discontinuities on J. We denote $t_1^z, \cdots, t_M^z \in J$ as the increasing sequence of these discontinuity time instants to which we add the extremities t_0^z and t_{M+1}^z of the interval J. Once again, the goal is to prove that u has a finite number of discontinuities on the time interval $[t_i^z, t_{i+1}^z]$, with $i \in \{0, ..., M\}$. We have seen that it is sufficient to show that there is a finite number of discontinuities on the open time interval (t_i^z, t_{i+1}^z), with $i \in \{0, ..., M\}$. If $\tilde{V}(\frac{1}{\underline{\lambda}}) = 0$, there is only at most one discontinuity which is $t_1^u = \frac{1}{\underline{\lambda}}$. Let us see the case $\tilde{V}(\frac{1}{\underline{\lambda}}) > 0$.

As in the proof of Lemma 2.1, we define $w^i(t)$ as the continuation of $BKz(t)$ on the interval $[t_i^z, t_{i+1}^z]$ with the left limit of $BKz(t)$ in t_{i+1}^z, that is

$$w^i(t) = BKz(t), \quad \text{if} \quad t \in [t_i^z, t_{i+1}^z) \tag{2.45}$$

$$w^i(t_{i+1}^z) = \lim_{t \to (t_{i+1}^z)^-} BKz(t) \tag{2.46}$$

The definition of $\mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^n)$ ensures that the left limit of $BKz(t)$ exists and that $w^i(t)$ is continuous on the closed interval $[t_i^z, t_{i+1}^z]$. Then, it is uniformly continuous. It means that for all $\zeta > 0$, there exists $\tau > 0$ such that

$$\forall t, t' \in [t_i^z, t_{i+1}^z] : |t - t'| < \tau \Rightarrow \|w^i(t) - w^i(t')\|^2 < \zeta$$

We denote $\overline{\tau}$ the value of τ when $\zeta = \varepsilon_1(t_{i+1}^z)$.

We assume that there are at least two consecutive discontinuity instants in (t_i^z, t_{i+1}^z) and let t_k^u be the first one. Considering (2.42) and (4.42) in Definition 2.5 and using the continuity of m, ε and w^i, it holds at time $t = t_{k+1}^u$:

$$m(t_{k+1}^u) \geq 0 \tag{2.47}$$

Let us prove by contradiction that $|t_k^u - t_{k+1}^u| \geq \overline{\tau}$. To do that, let us assume that $|t_k^u - t_{k+1}^u| < \overline{\tau}$. Then, by uniform continuity, we have $\|w^i(t_k^u) - w^i(s)\|^2 < \varepsilon_1(t_{i+1}^z)$ for all $s \in [t_k^u, t_{k+1}^u]$. Since ε_1

is a decreasing function, it holds also that $\|w^i(t_k^u)-w^i(s)\|^2 < \varepsilon_1(s)$. Due to the non-negativity of $\tilde{V}$, we have

$$\|w^i(t_k^u) - w^i(s)\|^2 < \varepsilon_1(s) + \kappa\tilde{V}(s) \tag{2.48}$$

Multiplying both sides of (2.48) by $e^{\eta s}$ and integrating on $[t_k^u, t_{k+1}^u]$, it yields,

$$\int_{t_k^u}^{t_{k+1}^u} e^{\eta s}\|w^i(t_k^u) - w^i(s)\|^2 ds < \int_{t_k^u}^{t_{k+1}^u} e^{\eta s}\varepsilon_1(s)ds + \int_{t_k^u}^{t_{k+1}^u} e^{\eta s}\kappa\tilde{V}(s)ds$$

Multiplying both sides by $e^{-\eta t_{k+1}^u}$ and re-organizing the previous inequality, one gets,

$$e^{-\eta t_{k+1}^u}\int_{t_k^u}^{t_{k+1}^u} e^{\eta s}(\|w^i(t_k^u) - w^i(s)\|^2 - \kappa\tilde{V}(s) - \varepsilon_1(s))ds < 0 \tag{2.49}$$

Using (4.42) and $m(t_k^u) = 0$, for all $t \geq t_k^u$, we have that

$$m(t) = e^{-\eta t}\int_{t_k^u}^{t} e^{\eta s}\left(-\kappa\tilde{V}(s) - \varepsilon_1(s) + \|w^i(t_k^u) - w^i(s)\|^2\right) ds$$

Then, (2.49) is equivalent to

$$m(t_{k+1}^u) < 0$$

which contradicts (2.47). Hence, $|t_k^u - t_{k+1}^u| \geq \overline{\tau}$. Therefore, $\overline{\tau}$ gives a lower bound for the duration between two input updates, depending only on the interval (t_i^z, t_{i+1}^z). Finally, an upper bound for the maximal number of input updates on (t_i^z, t_{i+1}^z) is given by:

$$\overline{s}_i = \left\lfloor \frac{t_{i+1}^z - t_i^z}{\overline{\tau}} \right\rfloor$$

If there is at most one element of the sequence (t_k^u) in (t_i^z, t_{i+1}^z) then $\overline{s}_i$ can be chosen equal to 1. The number of discontinuities of u on J is bounded by $\overline{S} = \sum_{i=1}^{M}\overline{s}_i + M + 2$ which is finite. To conclude, from (2.44) in Definition 2.5, u is piecewise constant, which yields $u \in \mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^m)$.

For proving the causality property of the operator φ_3, it is sufficient to follow the same steps as in proof of Lemma 2.1

We conclude then that Assumption 2.1 holds. It concludes the proof. □

Combining the previous lemma with Proposition 2.1, we get

Corollary 2.3
For any y^0 in $\mathcal{C}_{lpw}([0,1], \mathbb{R}^n)$, there exists a unique solution to the closed-loop system (2.1)-(2.3),(2.7) and controller $u = \varphi_3(z)$.

Let us now state our third main result of the chapter.

Theorem 2.3
Let K be in $\mathbb{R}^{m\times n}$ such that Assumption 2.2 holds for $G = H + BK$. Let $\mu > 0$, Q a diagonal positive matrix in $\mathbb{R}^{n\times n}$ and $\nu = \mu\underline{\lambda}$ be as in Proposition 2.2. Let σ be in $(0,1)$, η and ε and φ_d be given in Definition 2.5. Let V be given by (2.16) and d given by (2.22). Then the system (2.1)-(2.3),(2.7) with the controller $u = \varphi_3(z)$ has a unique solution and is globally exponentially stable.

Proof. The existence and uniqueness of a solution to system (2.1)-(2.3),(2.7), with $u = \varphi_3(z)$ is given by Corollary 2.3. Let us show that the system is globally exponentially stable. We use the result of Lemma 2.2 and assume first that $\tilde{V}(\frac{1}{\underline{\lambda}}) > 0$. By the same arguments of the proof of Theorem 2.1, here we get also that

$$D^+V \leq -2\nu V + \rho\|d\|^2 \tag{2.50}$$

To show the global exponential stability of the closed-loop system, we consider the following candidate Lyapunov function W for the augmented dynamical system defined, for all $y(\cdot) \in \mathcal{C}_{lpw}([0,1],\mathbb{R}^n)$ and $m \in \mathbb{R}^-$, $\varepsilon_1 \in \mathbb{R}^+$ by

$$W(y,m,\varepsilon_1) = V(y) + \frac{\rho}{\eta - 2\nu(1-\sigma)}\varepsilon_1 - \rho m \tag{2.51}$$

Computing the right time-derivative of (2.51), it yields,

$$D^+W = D^+V - \eta\tfrac{\rho}{\eta-2\nu(1-\sigma)}\varepsilon_1 - \rho(-\eta m - \kappa\tilde{V} - \varepsilon_1 + \|d\|^2) \tag{2.52}$$

Then, replacing (2.50) in (2.52), using $\kappa = \frac{2\sigma\nu}{\rho}$ and applying Proposition 2.3, we obtain for all $t \geq \frac{1}{\underline{\lambda}}$,

$$D^+W(t) \leq -2\nu(1-\sigma)V(t) + \rho\eta m(t) + \rho\varepsilon_1(t) - \eta\tfrac{\rho}{\eta-2\nu(1-\sigma)}\varepsilon_1(t)$$

which can be rewritten as follows:

$$\begin{aligned} D^+W(t) \leq \quad & -2\nu(1-\sigma)(W - \tfrac{\rho}{\eta-2\nu(1-\sigma)}\varepsilon_1 + \rho m) \\ & +\rho\eta m(t) + \rho\varepsilon_1(t) - \eta\tfrac{\rho}{\eta-2\nu(1-\sigma)}\varepsilon_1(t) \end{aligned}$$

Simplifying the previous inequality, one gets,

$$D^+W(t) \quad \leq -2\nu(1-\sigma)W + \rho(-2\nu(1-\sigma)+\eta)m(t)$$

From the definition of φ_3, events are triggered in order to guarantee for all $t \geq \frac{1}{\underline{\lambda}}$, that $m(t) \leq 0$. We obtain accordingly, for all $t \geq \frac{1}{\underline{\lambda}}$,

$$D^+W(t) \leq -2\nu(1-\sigma)W$$

Now, using the *Comparison principle*, for all $t \geq \frac{1}{\underline{\lambda}}$, we have

$$\begin{aligned} V(y(t,\cdot)) \quad &\leq \quad W(y(t,\cdot),m,\varepsilon_1) \\ &\leq \quad e^{-2\nu(1-\sigma)(t-\frac{1}{\underline{\lambda}})}W(y(\tfrac{1}{\underline{\lambda}},\cdot),m,\varepsilon_1) \end{aligned} \tag{2.53}$$

The previous inequality holds even if $\tilde{V}(\frac{1}{\underline{\lambda}}) = 0$ since in this case $W(y(\frac{1}{\underline{\lambda}}, \cdot), m, \varepsilon_1) = 0$ for all $t \geq \frac{1}{\underline{\lambda}}$. Knowing that $m(\frac{1}{\underline{\lambda}}) = 0$ and $\varepsilon_1(\frac{1}{\underline{\lambda}}) = \varsigma_1 V(y(\frac{1}{\underline{\lambda}}, \cdot))e^{-\eta\frac{1}{\underline{\lambda}}}$, inequality (2.53) can be rewritten as follows,

$$V(y(t,\cdot)) \leq e^{-2\nu(1-\sigma)(t-\frac{1}{\underline{\lambda}})}\left(V(y(\tfrac{1}{\underline{\lambda}},\cdot)) + \tfrac{\rho\varsigma_1}{\eta-2\nu(1-\sigma)}V(y(\tfrac{1}{\underline{\lambda}},\cdot))e^{-\eta\frac{1}{\underline{\lambda}}}\right) \tag{2.54}$$

In addition, $V(y(\frac{1}{\underline{\lambda}}, \cdot))$ is given as follows according to (2.31):

$$V(y(\tfrac{1}{\underline{\lambda}},\cdot)) \leq e^{2\theta\frac{\overline{\lambda}}{\underline{\lambda}}}e^{2(\theta+\mu)}V(y^0) \tag{2.55}$$

Therefore, replacing (2.55) in (2.54) we get for all $t \geq \frac{1}{\underline{\lambda}}$,

$$V(y(t,\cdot)) \leq e^{\left(2\theta\frac{\overline{\lambda}}{\underline{\lambda}}+2(\theta+\mu)\right)}\left[1+\frac{\rho\varsigma_1 e^{-\eta\frac{1}{\underline{\lambda}}}}{\eta-2\nu(1-\sigma)}\right]e^{-2\nu(1-\sigma)t}V(y^0)$$

This ends the proof of Theorem 2.3. □

The following proposition states that the first triggering time after $t = \frac{1}{\underline{\lambda}}$ occurs with φ_1 than with φ_3.

Proposition 2.5
Let t^u_{2,φ_1} be given by the rule (2.19) and let t^u_{2,φ_3} be given by the rule (2.42). It holds that after $t = \frac{1}{\underline{\lambda}}$, $t^u_{2,\varphi_1} \leq t^u_{2,\varphi_3}$.

Proof. We want to prove by contradiction that $t^u_{2,\varphi_1} \leq t^u_{2,\varphi_3}$. To do that, let us suppose that the first triggering time after $t = \frac{1}{\underline{\lambda}}$ occurs with φ_3, i.e. $t^u_{2,\varphi_1} > t^u_{2,\varphi_3}$. On one hand, by (2.42), we have

$$m(t^u_{2,\varphi_3}) \geq 0 \tag{2.56}$$

On the other hand, by (2.19), we have

$$\|BK(z(\tfrac{1}{\underline{\lambda}}) - z(s)\|^2 - \kappa\tilde{V}(s) < \varepsilon_1(t^u_{2,\varphi_3}) \leq \varepsilon_1(s) \tag{2.57}$$

for all $s \in [\frac{1}{\underline{\lambda}}, t^u_{2,\varphi_3}]$. Therefore, after some operations on both sides of previous inequality, and integrating on $[\frac{1}{\underline{\lambda}}, t^u_{2,\varphi_3}]$, we obtain, for all $t^u_{2,\varphi_3} \in [\frac{1}{\underline{\lambda}}, \frac{1}{\underline{\lambda}} + \tilde{\tau}]$,

$$\begin{aligned} &e^{-\eta t^u_{2,\varphi_3}}\int_{\frac{1}{\underline{\lambda}}}^{t^u_{2,\varphi_3}} e^{\eta s}(-\kappa\tilde{V}(s) + \|BK(z(\tfrac{1}{\underline{\lambda}}) - z(s))\|^2)ds \\ &< e^{-\eta t^u_{2,\varphi_3}}\int_{\frac{1}{\underline{\lambda}}}^{t^u_{2,\varphi_3}} e^{\eta s}\varepsilon_1(s)ds \end{aligned}$$

where $\tilde{\tau}$ is a lower bound of the duration between t^u_{2,φ_1} and $\frac{1}{\underline{\lambda}}$ i.e. $t^u_{2,\varphi_1} - \frac{1}{\underline{\lambda}} \geq \tilde{\tau}$. It is worth remarking that such inter-execution time exists due to Lemma 2.1. Hence, using (4.42), we finally get,

$$m(t^u_{2,\varphi_3}) < 0 \tag{2.58}$$

leading to a contradiction of (2.56). Then, we must have that $t^u_{2,\varphi_1} \leq t^u_{2,\varphi_3}$. □

2.4 Numerical simulation

Numerical simulations were done by discretizing an example of linear hyperbolic system. For that purpose we have used a two-step variant of the Lax–Friedrichs numerical method presented in [58] and the solver on Matlab in [40]. We selected the parameters of the numerical scheme so that the *Courant-Friedrich-Levy* condition for the numerical stability holds. In addition, the sufficient stability condition is achieved using classical numerical tools for semi-definite programming (see e.g. Yalmip toolbox [43] with SeDuMi solver).

2.4.1 Example 1

As a matter of example, we consider a system of 2×2 hyperbolic conservation laws describing the vehicle traffic flow on a roundabout made up of only two inputs/exits. Inspired by [5], where an example of ramp-meetering control in road traffic networks is treated, we consider the structure of Figure 2.1 for a network in compartmental system representation describing flows on conservative networks. Each compartment or node represents a segment of the roundabout. Based on LWR model (see e.g. [9]), let us consider the special case where the dynamic of the system is written as a set of *kinematic wave equations*,

$$\partial_t q + C(q)\partial_x q = 0 \tag{2.59}$$

where $q = [q_1 \quad q_2]^T$ is the flux. $C(q) = \mathrm{diag}(c_1(q_1) \quad c_2(q_2))$ is the matrix of characteristic velocities. The boundary conditions are

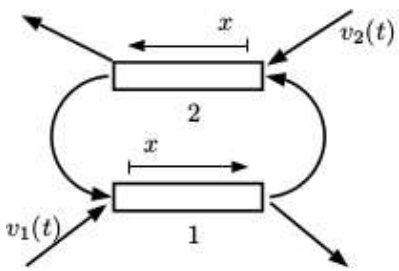

Figure 2.1: Network of 2 conservation laws for a roundabout.

$$\begin{aligned} q_1(t,0) &= \gamma q_2(t,1) + v_1(t) \\ q_2(t,0) &= \beta q_1(t,1) + v_2(t) \end{aligned} \tag{2.60}$$

where γ and β are traffic splitting factors at the two exits of the roundabout and $v_1(t)$ and $v_2(t)$ are the influxes injected from outside into compartments 1 and 2 respectively. They can be viewed as control inputs (e.g. when modulating with traffic lights). With these control actions, one intends to prevent the appearance of traffic jams on the roundabout or to achieve a desired steady-state without congestion. As in [5], we assume free-flow conditions. In that case, characteristic velocities are all positives. In addition, a steady-state for the system

$\{q^*, v^*\}$ is space invariant and satisfies (2.60), that is $q_1^* = \gamma q_2^* + v_1^*$ and $q_2^* = \beta q_1^* + v_2^*$. We select the steady-state $\{q^*, v^*\}$ as a free-flow steady-state. We want to stabilize it under linear output feedback control of the form

$$v_1(t) = v_1^* + k_1(q_2(t,1) - q_2^*)$$
$$v_2(t) = v_2^* + k_2(q_1(t,1) - q_1^*)$$

where k_1, k_2 are tuning control parameters. Defining the deviations $y = q - q^*$ and $u = v - v^*$, the linearization of system (2.59) at the equilibrium is given by

$$\partial_t y + C(q^*)\partial_x y = 0 \tag{2.61}$$

with $y = [y_1 \quad y_2]^T$, $C(q^*) = \mathrm{diag}(c_1(q_1^*) \quad c_2(q_2^*))$ and the boundary condition given by

$$y(t,0) = Hy(t,1) + Bu(t) \tag{2.62}$$

where $H = \begin{pmatrix} 0 & \gamma \\ \beta & 0 \end{pmatrix}$, $B = I_2$ and $u(t) = Ky(t,1)$ with $K = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix}$. We then perform simulations setting $c_1(q_1^*) = 1$, $c_2(q_2^*) = \sqrt{2}$, $\gamma = 0.7$, $\beta = 0.9$. The initial condition is $y(0,x) = \begin{bmatrix} 4x(x-1) & \sin(8\pi x) \end{bmatrix}^T$ for all $x \in [0,1]$.

It is worth remarking that due to the nature of this problem, the system in open-loop converges to the equilibrium, i.e. $\rho_1(H) < 1$ holds, thus the system (2.61) in open-loop is GES. Therefore, the design of the control $u(t)$ would be rather devoted to improve the performance of the network (e.g. to accelerate the time convergence to the steady-state, by decreasing the ρ_1-norm of the boundary condition).

2.4.1.1 Continuous stabilization: controller $u = \varphi_0(z)$

The boundary condition is $y(t,0) = Hy(t,1) + Bu(t)$ where $u(t) = \varphi_0(z)(t) = Kz(t)$ is the continuous controller acting from $t \geq \frac{1}{\underline{\lambda}} = 1$. The initial condition was chosen such that the zero-order compatibility condition is satisfied i.e. $y(0,0) = (H + BK)y(0,1)$. Condition (2.14) holds since $\|\Delta_H H \Delta_H^{-1}\| = 0.8079 < 1$ with $\Delta_H = \begin{pmatrix} 1.2729 & 0 \\ 0 & 1.1426 \end{pmatrix}$ and thus $\rho_1(H) < 1$. Then, K has been designed such that $\rho_1(G) < \rho_1(H) < 1$ with $G = H + BK$. To be more specific, with $K = \begin{pmatrix} 0 & 0.3 \\ -0.9 & 0 \end{pmatrix}$ and $\Delta_G = \begin{pmatrix} 0.9134 & 0 \\ 0 & 1.2580 \end{pmatrix}$, $\|\Delta_G G \Delta_G^{-1}\| = 0.7262 < 1$. It implies that the closed-loop system is GES and the ρ_1-norm of the boundary condition is smaller than the open-loop case. Besides this, condition (2.15) in Proposition 2.2 was also checked getting as a result the existence of scalars $\mu = 0.1$, $\nu = 0.1$ and one symmetric matrix $Q = \begin{pmatrix} 0.8346 & 0 \\ 0 & 1.1191 \end{pmatrix}$.

2.4.1.2 ISS static event-based stabilization: controller $u = \varphi_1(z)$

The boundary condition is now $y(t,0) = Hy(t,1) + Bu(t)$ where $u(t) = \varphi_1(z)(t)$. The parameters for the triggering algorithm were chosen to be $\alpha = 0.5$, $\sigma = 0.9$. Therefore, $\rho = 4.7481$, $\kappa = 0.0379$ and $[(1+\alpha)G^T Q\Lambda G - e^{-2\mu} Q\Lambda] = \begin{pmatrix} -0.6833 & 0 \\ 0 & -0.0439 \end{pmatrix}$ is a symmetric negative definite

matrix. Hence, Theorem 2.1 holds. The function ε_1 used in the triggering condition (2.19) is chosen to be

$$\varepsilon_1(t) = \varsigma_1 V(1) e^{-\eta t}, \quad t \in \mathbb{R}^+ \tag{2.63}$$

with $\eta = 0.4$, $V(1) = 0.6390$, and ς_1 is such that $\varsigma_1 V(1) = 5 \times 10^{-3}$. Figure 2.2 shows the time

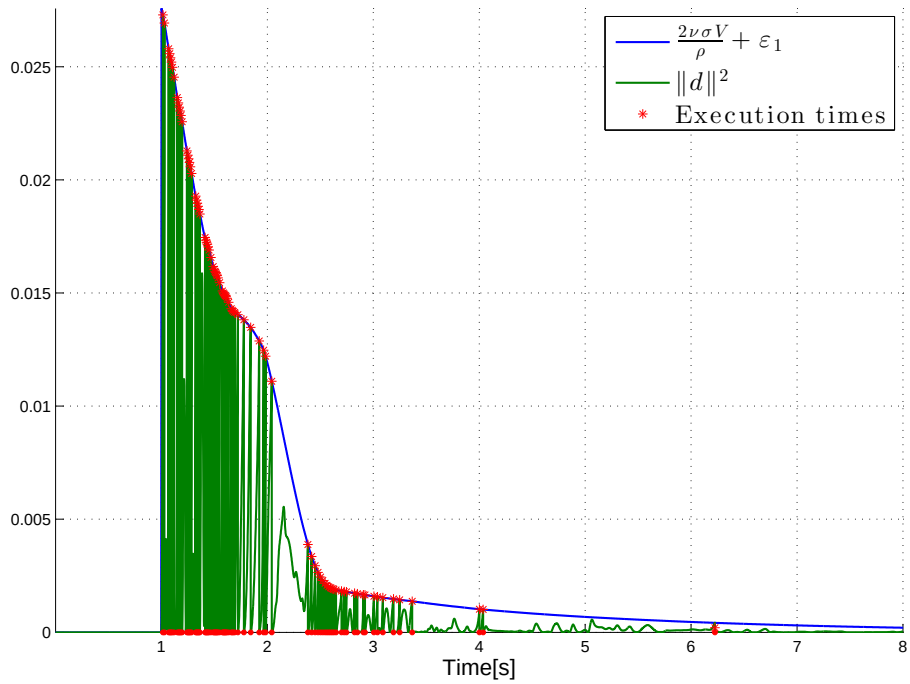

Figure 2.2: Trajectories involved in triggering condition (2.19) for controller $u = \varphi_1(z)$.

evolution of the functions appearing in the triggering condition (2.19). Once the trajectory $\|d\|^2$ reaches the trajectory $\kappa V + \varepsilon_1$, an event is generated, the control value is updated and d is reset to zero. The number of events under this event-based approach was 89, counting them from $t \geq \frac{1}{\underline{\lambda}} = 1$. It is considerable less with respect to continuous stabilization, since the number of discretization points in time is $NT = 8000$ with a discretization step $\Delta t = 1 \times 10^{-3}$.

2.4.1.3 D^+V event-based stabilization: controller $u = \varphi_2(z)$

The boundary condition is now $y(t,0) = Hy(t,1) + Bu(t)$ where $u(t) = \varphi_2(z)(t)$. The only parameter to be chosen here is σ and it is as before, $\sigma = 0.9$. The function ε_2 used in the triggering condition (2.33) is $\varepsilon_2 = \rho\varepsilon_1$ with $\rho = 4.7481$ and ε_1 given by (2.63). The number of events under this event-based approach was 30, counting them from $t \geq 1$.
A comparison of the functions V when stabilizing with φ_0, φ_1 and φ_2 is done as shown in Figure 2.4.

It can be noticed that under the two event-based stabilization approaches, global asymptotic stability is achieved with different observed rates despite similar theoretical guarantees. D^+V event-based stabilization results in slower convergence but leads to larger inter-execution

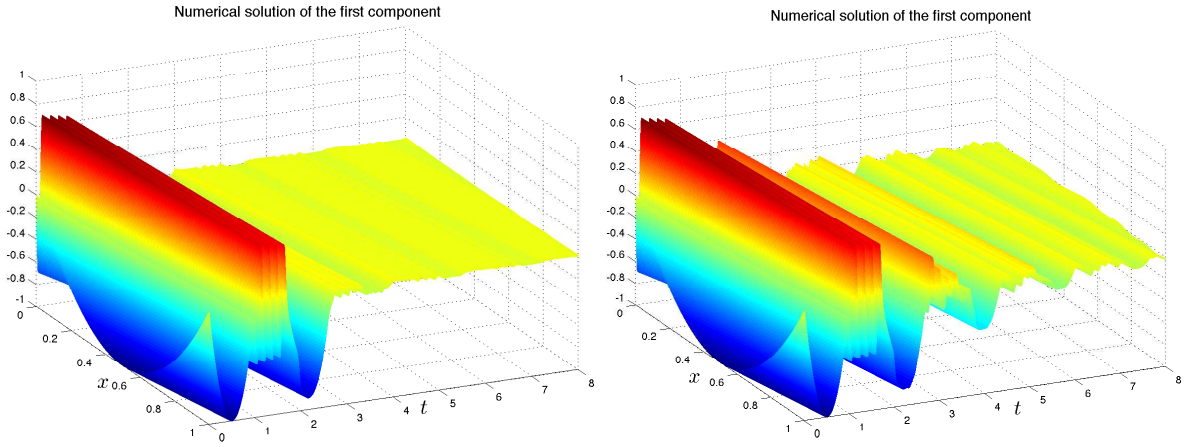

Figure 2.3: Numerical solution of the first component y_1 of the closed-loop system with controller $u = \varphi_1(z)$ (left) and with controller $u = \varphi_2(z)$ (right).

times than the ISS static one which results in faster convergence. Moreover, the first triggering time occurs with φ_1. This is consistent with Proposition 2.4. For both approaches, σ has been chosen to reduce as much as possible the number of triggering times. The closer σ is to zero, the faster triggering is required. Figure 2.3 shows the first component of solution when stabilizing with both φ_1 (left) and φ_2 (right). Note that for both approaches, oscillations are presented near the equilibrium and asymptotic stability is achieved. It is worth remarking that under continuous stabilization with φ_0, it is possible to achieve the convergence to the equilibrium in finite time. Such a time, for this particular illustrating example, is given by $T_F = \frac{1}{\underline{\lambda}} + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} = \frac{4+\sqrt{2}}{2}s$.

2.4.1.4 ISS dynamic event-based stabilization: controller $u = \varphi_3(z)$

The boundary condition is now $y(t,0) = Hy(t,1) + Bu(t)$ where $u(t) = \varphi_3(z)(t)$. The number of events under this event-based approach was 86, counting them from $t \geq 1$. Figure 2.5 shows functions V when stabilizing with φ_1 and φ_3.

Under the event-based controllers φ_1, and φ_3, global asymptotic stability is also achieved with quite different observed rates despite similar theoretical guarantees. Besides this, the first triggering time occurs with φ_1 which is exactly what expected according to Proposition 2.5.

Moreover, for event-based approaches, φ_1, φ_2 and φ_3, we ran simulations for several initial conditions given by $y^0_{a,b}(x) = [ax(1-x) \quad \frac{b}{2}\sin((2a)\pi x)]^T$, $a = 1, ..., 5$ and $b = 1, ..., 10$ on a frame of 8s. Note that these initial conditions satisfy the zero-order compatibility condition. We have computed the duration intervals between two triggering times (inter-execution times). The mean value, standard deviation and the coefficient of variation of inter-execution times for all approaches are reported in Table 2.1 and the density of such inter-execution times is given in Figure 2.6 (for φ_1 and φ_2) and in Figure 2.7 (for φ_1 and φ_3).

The mean value number of triggering times with φ_1 was 121.1 events whereas with φ_2

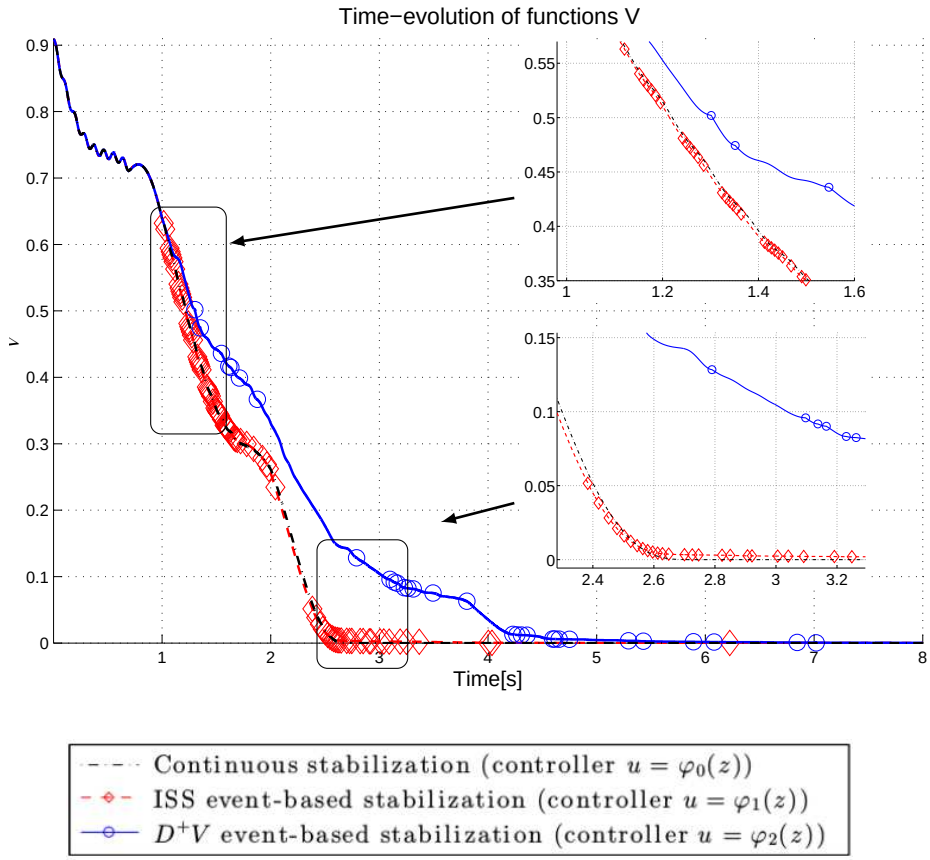

Figure 2.4: Time-evolution of functions V. Legend: Dashed-dotted black line for continuous stabilization with controller $u = \varphi_0(z)$, dashed red line with red diamond marker for ISS event-based stabilization with controller $u = \varphi_1(z)$ and blue line with blue circle marker for D^+V event-based stabilization with controller $u = \varphi_2(z)$.

was 45.72 events and with φ_3 was 109.1 events. They are considerable less with respect to continuous time approach φ_0 since the discretization time for all simulations was $NT = 8000$ with $\Delta t = 1 \times 10^{-3}$. It can be seen that using φ_2 results in larger inter-execution times in average than φ_1. In addition, φ_2 reduces the variability of the inter execution times and with φ_1 it is needed to sample faster than with φ_2. The same comparison can be done for φ_1 and φ_3. Indeed, t can be observed that stabilization with φ_3 results in larger inter-execution times than with φ_1 which was expected because events generated according to φ_3-event-triggered rule, are a weighted average of those generated according to φ_1-event-triggered rule.

2.4.2 Example 2

Fully in line with networks of compartments introduced in Chapter 1, let us consider here the particular case of the network illustrated in Figure 1.4 of Section 1.6 for which there are no buffers as can be seen here in Figure 2.8. It means that we have a network of hyperbolic systems of conservation laws interconnected which is the core of this chapter. Assume that

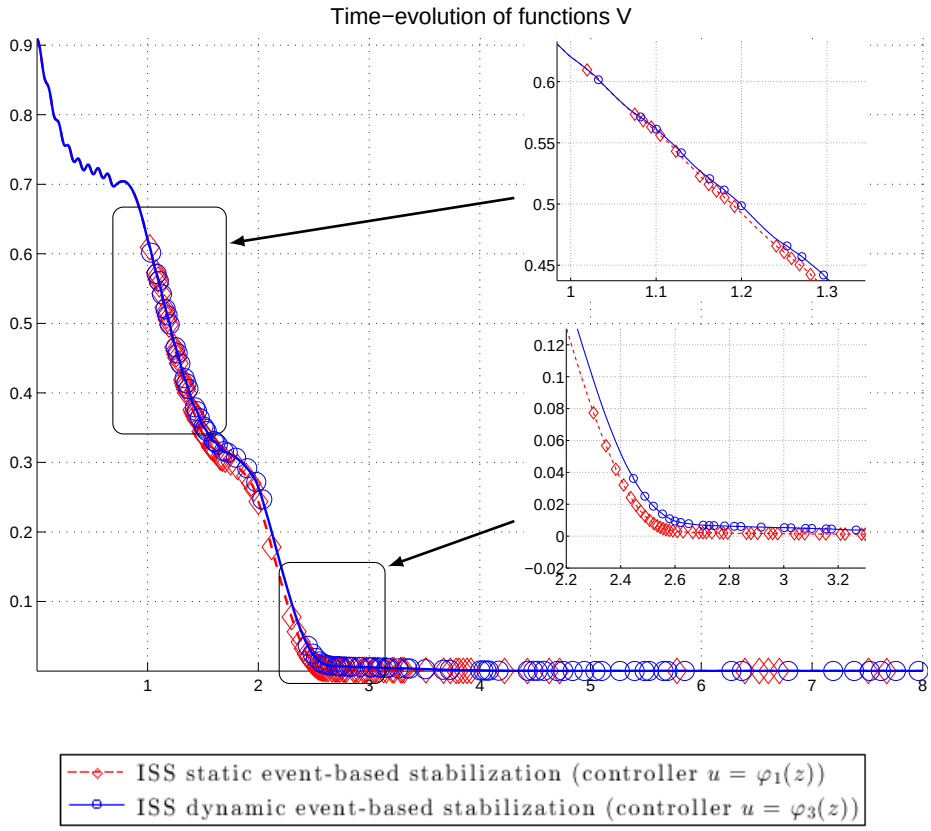

Figure 2.5: Time-evolution of functions V. Legend: Dashed red line with red diamond marker for ISS static event-based stabilization with controller $u = \varphi_1(z)$ and blue line with blue circle marker for ISS dynamic event-based stabilization with controller $u = \varphi_3(z)$.

for a constant input demand, the systems admits a free-flow steady state (we take the same values as in Subsection 1.3.1). The linearized model is as follows:

$$\partial_t y + \Lambda \partial_x y = 0$$

with $y(t,x) \in \mathbb{R}^5$, $\Lambda = diag(1, 1.2, 1.7, 0.5, 2)$. The boundary condition is given by

$$y(t,0) = Hy(t,1) + Bu(t)$$

where $H = \begin{pmatrix} 0 & 0\,0\,0\,0 \\ 0 & 0\,0\,0\,0 \\ u_{23} & 0\,0\,0\,0 \\ 1-u_{23} & 0\,0\,0\,0 \\ 0 & 1\,1\,0\,0 \end{pmatrix}$, where $u_{23} = 0.548$. $B = \begin{pmatrix} d_1^* & 0 \\ -d_1^* & 0 \\ 0 & q_1^* \\ 0 & -q_1^* \\ 0 & 0 \end{pmatrix}$ where $d_1^* = 100$ and $q_1^* = 35$. In addition, $u(t) = Ky(t,1)$ with $K \in \mathbb{R}^{2\times 5}$ (continuous time stabilization φ_0). In order to illustrate the results of this chapter, we assume that in the linearized system, no external disturbance is presented, i.e. $d_1(t) = d_1^*$. For the control synthesis, one can carry out a particular case of the procedure in Section 1.5.3 leading to to find K such that $\rho_1(G) < 1$ with $G = H + BK$.

Remark 2.6

It is important to point out that due to the nature of this network, the system in open-loop is

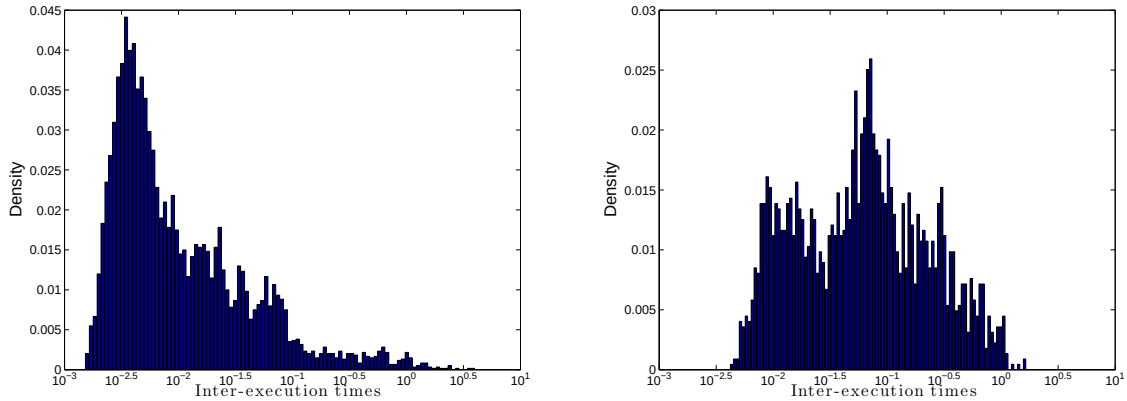

Figure 2.6: Density of the inter-execution times with controller $u = \varphi_1(z)$ (left) and with controller $u = \varphi_2(z)$ (right).

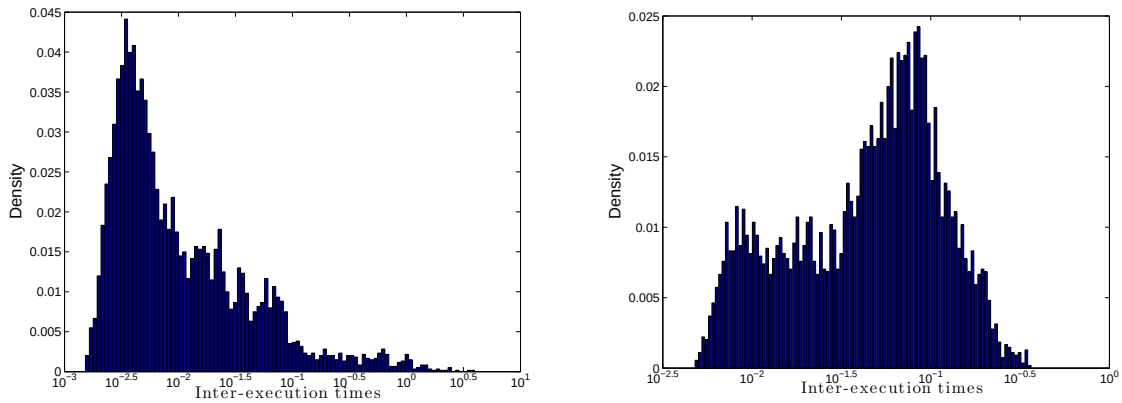

Figure 2.7: Density of the inter-execution times with controller $u = \varphi_1(z)$ (left) and with controller $u = \varphi_3(z)$ (right).

GES (when no external disturbances are present). Indeed, it can be checked that $\rho_1(H) < 1$. *For this network, control actions could be contemplated in order to accelerate the convergence as it has been done in the previous example. Nevertheless, it turned out that when closing the loop with event-based controllers, we fall in the case that no event-based stabilization is required because the systems converges very fast to the origin and the deviation when sampling does not actually affect the dissipativity of the boundary.*

We aim at comparing continuous time control implementation with respect to the event-based control proposed in this chapter. To do so, we propose a control gain K that, as explained, does not improve the performance of the network but allows to appreciate the behaviour of the system under event-based control schemes.

	Mean value	Standard deviation	Coefficient of variation
ISS static event-based	0.0448	0.1702	3.8024
D^+V event-based	0.1361	0.1972	1.4489
ISS dynamic event-based	0.0640	0.0538	0.8411

Table 2.1: Mean value, standard deviation and variability of inter-execution times for ISS static event-based stabilization approach (φ_1), D^+V event-based stabilization approach (φ_2) and ISS dynamic event-based stabilization approach (φ_3).

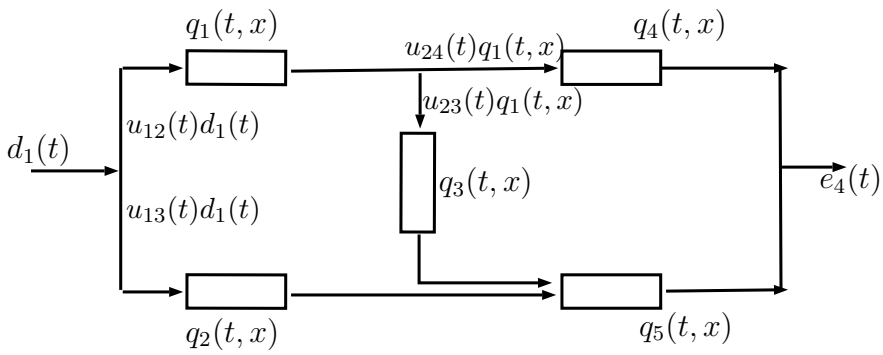

Figure 2.8: Network of compartments made up of 5 transmission lines.

2.4.2.1 Event-based stabilization: controllers (φ_1, φ_2 and φ_3)

Let us take $K = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -0.00285 & 0.00285 & 0.00571 & 0.00714 \end{pmatrix}$. The closed-loop system is GES. We can obtain $Q = diag(4586, 4176, 131.6, 199, 39.3)$ and $\mu = 0.39$. We close the loop from $t_1 = 2$. Figure 2.9 shows the output flow of the network $q_4(t) + q_5(t)$ when closing the loop with the continuous time control $u = \varphi_0(z) = Kz$, and with the three event-based controllers. Under φ_1, φ_2 and φ_3, the number of triggering times obtained are 72, 4 and 41 respectively. The behavior of the output flow is similar and it remains close to the steady-state with different observed rates of convergence. It can also be observed that, the first triggering time after t_1 always happens with φ_1-strategy and that with φ_2 we obtain very long inter-execution times.

2.5 Appendix of Chapter 2

2.5.1 Proof of Lemma 2.2

Let us prove that V is continuous and right differentiable on $\mathbb{R}^+$. From (2.16), it follows that

$$V(y) = \sum_{d=1}^{n} Q_{dd} V_d(y) \tag{2.64}$$

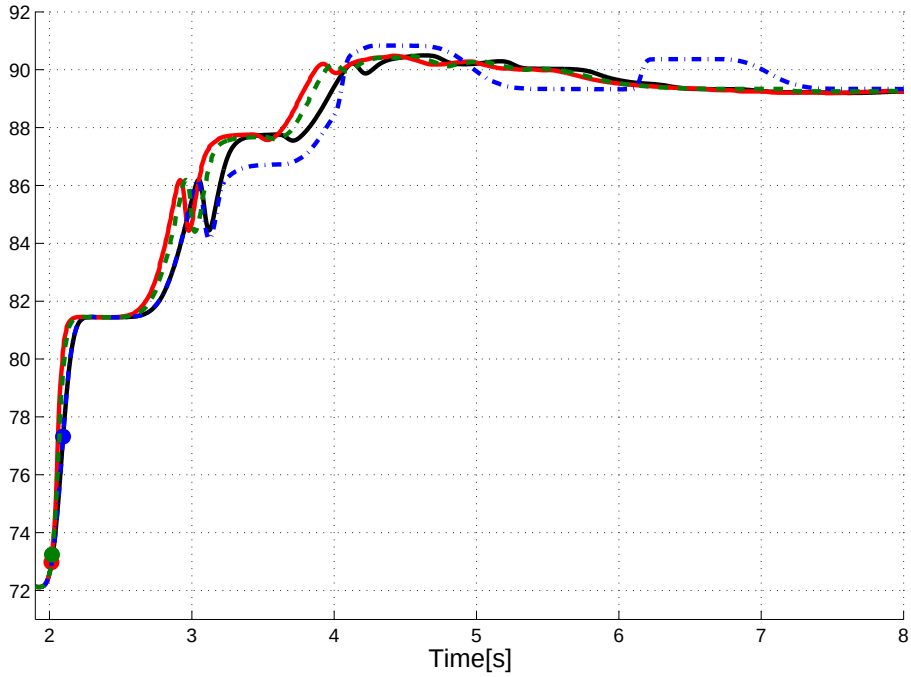

Figure 2.9: Total output flow under continuous time controller φ_0 (black line), ISS-static event-based controller φ_1 (red line), D^+-event-based controller φ_2 (blue dashed-dot line) and ISS-dynamic event-based controller φ_3 (green dashed line).

where $V_d(y) = \int_0^1 y_d^2(x)e^{-2\mu x}dx$. Let y be a solution of (2.1)-(2.3), let t in $\mathbb{R}^+$, $x \in [0,1]$, $h > 0$ then

$$y_d(t+h,x) = \begin{cases} y_d(t+h-\frac{x}{\lambda_d},0), & \text{if } \lambda_d h \geq x \\ y_d\left(t, x-\lambda_d h\right), & \text{if } \lambda_d h < x \end{cases} \tag{2.65}$$

Let $V(t) = V(y(t,\cdot))$ and $V_d(t) = V_d(y_d(t,\cdot))$. Using (2.65), $V_d(t+h) - V_d(t)$ is computed as follows:

$$\begin{aligned} V_d(t+h) - V_d(t) &= \int_0^{\lambda_d h} y_d^2(t+h-\tfrac{x}{\lambda_d},0)e^{-2\mu x}dx \\ &\quad + \int_{\lambda_d h}^1 y_d^2(t,x-\lambda_d h)e^{-2\mu x}dx \\ &\quad - \int_0^1 y_d^2(t,x)e^{-2\mu x}dx \\ &= \int_0^{\lambda_d h} y_d^2(t+h-\tfrac{x}{\lambda_d},0)e^{-2\mu x}dx \\ &\quad + \int_0^{1-\lambda_d h} y_d^2(t,x)e^{-2\mu x}e^{-2\mu\lambda_d h}dx \\ &\quad - \int_0^1 y_d^2(t,x)e^{-2\mu x}dx \end{aligned}$$

Hence,

$$\begin{aligned} V_d(t+h) - V_d(t) = &\int_0^{\lambda_d h} y_d^2(t+h-\tfrac{x}{\lambda_d},0)e^{-2\mu x}dx \\ &- e^{-2\mu\lambda_d h}\int_{1-\lambda_d h}^1 y_d^2(t,x)e^{-2\mu x}dx \\ &+ (e^{-2\mu\lambda_d h}-1)\int_0^1 y_d^2(t,x)e^{-2\mu x}dx \end{aligned}$$

Since $y_d(\cdot,0) \in \mathcal{C}_{rpw}(\mathbb{R}^+,\mathbb{R}^n)$, we have

$$\lim_{h\to 0^+} \tfrac{1}{h}\int_0^{\lambda_d h} y_d^2(t+h-\tfrac{x}{\lambda_d},0)e^{-2\mu x}dx = \lambda_d y_d^2(t,0)$$

Now, due to the fact that $y_d(t,\cdot) \in \mathcal{C}_{lpw}([0,1],\mathbb{R}^n)$, for all $t \geq 0$,

$$\lim_{h\to 0^+} \tfrac{1}{h}e^{-2\mu\lambda_d h}\int_{1-\lambda_d h}^1 y_d^2(t,x)e^{-2\mu x}dx = \lambda_d y_d^2(t,1)e^{-2\mu}$$

In addition, on gets

$$\lim_{h\to 0^+} \tfrac{1}{h}(e^{-2\mu\lambda_d h}-1) = -2\mu\lambda_d$$

therefore,

$$\begin{aligned} \lim_{h\to 0^+} \tfrac{V_d(t+h)-V_d(t)}{h} =&\lambda_d y_d^2(t,0) - \lambda_d y_d^2(t,1)e^{-2\mu} \\ &- 2\mu\lambda_d\int_0^1 y_d^2(t,x)e^{-2\mu x}dx \end{aligned} \tag{2.66}$$

From (2.64) and (2.66) we get,

$$\begin{aligned} \lim_{h\to 0^+} \tfrac{V(t+h)-V(t)}{h} = &\sum_{d=1}^n Q_{dd}\Big(\lambda_d y_d^2(t,0) \\ &- \lambda_d y_d^2(t,1)e^{-2\mu} \\ &- 2\mu\lambda_d\int_0^1 y_d^2(t,x)e^{-2\mu x}dx\Big) \end{aligned}$$

which proves that V is right differentiable and in turn right continuous. Moreover, since $D^+V = \lim_{h\to 0^+}\frac{V(t+h)-V(t)}{h}$, (2.27) holds. In order to prove that V is left continuous, let us consider the case when $h < 0$. Then,

$$y_d(t+h,x) = \begin{cases} y_d(t+h-\frac{(x-1)}{\lambda_d},1), & \text{if } x \geq \lambda_d h+1 \\ y_d\left(t,x-\lambda_d h\right), & \text{if } x < \lambda_d h+1 \end{cases} \tag{2.67}$$

and

$$\begin{aligned} V_d(t+h) - V_d(t) = & \int_{-\lambda_d h}^{1} y_d^2(t,x) e^{-2\mu x} e^{-2\mu\lambda_d h} dx \\ & + \int_{\lambda_d h+1}^{1} y_d^2(t+h-\tfrac{(x-1)}{\lambda_d},1) e^{-2\mu x} dx \\ & - \int_{0}^{1} y_d^2(t,x) e^{-2\mu x} dx \end{aligned}$$

which can be rewritten as follows,

$$\begin{aligned} V_d(t+h) - V_d(t) = & \int_{\lambda_d h+1}^{1} y_d^2(t+h-\tfrac{(x-1)}{\lambda_d},1) e^{-2\mu x} dx \\ & -e^{-2\mu\lambda_d h} \int_{0}^{-\lambda_d h} y_d^2(t,x) e^{-2\mu x} dx \\ & +(e^{-2\mu\lambda_d h}-1) \int_{0}^{1} y_d^2(t,x) e^{-2\mu x} dx \end{aligned} \tag{2.68}$$

From (2.68) it can be noticed that,

$$\lim_{h\to 0^-} V_d(t+h) - V_d(t) = 0$$

Therefore, V is left continuous. Since it is also right continuous, it is continuous on $\mathbb{R}^+$. This concludes the proof of Lemma 2.2.

Chapter 3

Stabilization of boundary controlled linear hyperbolic systems via Lyapunov-based event triggered sampling and quantization

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In this chapter we consider the problem of stabilization of boundary controlled linear hyperbolic partial differential equations where the output measurements are communicated after being time-sampled and space-quantized. Static and dynamic controllers are designed, which establish stability in different norms with respect to measurement errors using Lyapunov-based techniques. For practical purposes, stability in the presence of event-based sampling and quantization errors is analyzed. The design of sampling algorithms ensures practical stability.

3.1 Introduction

We have developed in Chapter 2, event-based sampling algorithms for boundary control of linear hyperbolic systems of conservation laws. The proposed rigorous framework establishes

well-posedness of the closed-loop system and uses Lyapunov techniques for sampling algorithms to ensure exponential stability of the system. On the other hand, in [64, 63], boundary control of linear hyperbolic systems is treated when the output measurements are quantized.

We know that when considering sampling and quantization issues, measurement errors are introduced which in most cases can cause the hyperbolic system to become unstable. Therefore, ISS properties with respect to those errors must be properly addressed. In Chapter 2, more precisely in Section 2.3.2 , a static boundary control yields ISS in L^2-norm by means of Lyapunov analysis. While in [64, 63], ISS in H^1-norm leading to practical stability is obtained. The use of H^1-norm is motivated by the fact that the output function to be quantized, must remain within the range of the quantizer, which is considered to be bounded.

This chapter builds on the ISS notions developed in Chapter 2 and [64] to solve the stabilization problem of linear hyperbolic PDEs when the output is subjected to event-based sampling, and quantization. The main contribution lies on the fact that even under event-triggered sampling of the output, one can still obtain ISS stability in both L^2 and H^1 norms and the well-posedness of the system. In the first instance, assuming the quantizers do not have limitations on data rate, a static control is used and bounds on L^2-norm of the state are obtained. For finite data-rate quantizers, it turns out to be necessary to work with a dynamic controller, and stability in H^1-norm is established.

3.2 Problem formulation

As in Section 3.2, let us consider the linear hyperbolic PDE:

$$\partial_t y(t,x) + \Lambda \partial_x y(t,x) = 0 \tag{3.1}$$

where $x \in [0,1]$, $t \in \mathbb{R}^+$ and $\Lambda = diag(\lambda_i)$ is a diagonal positive definite matrix in $\mathbb{R}^n$. The boundary condition is given by

$$y(t,0) = Hy(t,1) + Bu(t) \tag{3.2}$$

and the initial condition is

$$y(0,x) = y^0(x), \quad x \in (0,1). \tag{3.3}$$

where $y : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$, the input $u : \mathbb{R}^+ \to \mathbb{R}^m$, $H \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. We consider the output of this system to be

$$z(t) = y(t,1). \tag{3.4}$$

The objective is to design the control input u in (3.2) as function of the output measurements such that the resulting closed-loop system is asymptotically stable in appropriate sense. In our setup, we impose certain restrictions on the transmission of output to the controller.

Motivated by the fact that the output is communicated to the controller via a communication channel, we determine the sampling instants, $t_k \in \mathbb{R}_+$, $k \in \mathbb{N}$, such that $y(t_k,1)$ is transmitted to the controller for $t \in [t_k, t_{k+1})$. Additionally, after the sampling instants have

been computed, a quantizer $q : \mathbb{R}^n \to \mathcal{Q}$ encodes each output sample $y(t_k, 1)$ to a discrete alphabet set $\mathcal{Q}$. We consider two cases:

- For $\mathcal{Q}$ countably infinite, where there are no constraints on the domain of the quantizer.
- For quantizers with compact domain, where $\mathcal{Q}$ is possibly finite.

In both cases, we solve the design problem where we compute the sampling instants t_k, $k \in \mathbb{N}$. In order to save communication resources, our objective is to employ event-based strategy for computing the sampling times. We provide the sampling algorithms for both aforementioned cases, and show that the closed-loop system is ISS with respect to the quantization error with appropriate norms.

3.3 Static Control with Infinite Data Rate

To highlight the fundamental ideas behind our approach, we first treat the case where quantization error is assumed to be bounded for all possible values of the output, for example, $q(x) = \lfloor x + 0.5 \rfloor$. In that case, we can talk about stability of y in L^2-norm without requiring any bounds on $z(t)$. We first describe how ISS in L^2-norm is achieved via static output feedback, and then present the sampling algorithm.

3.3.1 ISS via static output feedback

We start by introducing perturbations in the output measurements by letting

$$z_d(t) = y(t, 1) + d(t) \tag{3.5}$$

with $d \in L^\infty(\mathbb{R}^+; \mathbb{R}^n)$. We are interested in designing an output feedback which achieves ISS with respect to d in the following sense:

Definition 3.1 (L^2 input-to-state stability)
The system (3.1)-(3.3),(3.5)*, with controller $u = \varphi(z)$ is input-to-state stable (ISS) with respect to disturbance $d \in \mathcal{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)$, if there exist $\nu > 0$, $C_1 > 0$ and $C_2 > 0$ such that, for every $y^0 \in L^2([0, 1]; \mathbb{R}^n)$, the solution satisfies, for all $t \in \mathbb{R}^+$,*

$$\|y(t, \cdot)\|^2_{L^2([0,1],\mathbb{R}^n)} \leq C_1 e^{-\nu t}(\|y^0\|^2_{L^2([0,1];\mathbb{R}^n)}) + C_2 \|d_{[0,t]}\|^2_\infty. \tag{3.6}$$

In the case there are no perturbation, i.e. $d \equiv 0$, a particular case of φ is a static output feedback control $u(t) = Kz(t)$. Setting $G = H + BK$, the boundary condition (3.2) is

$$y(t, 0) = Gz(t). \tag{3.7}$$

The design of $K \in \mathbb{R}^{m\times n}$ relies on the following assumption which states a sufficient (*dissipative boundary*) condition for the global exponential stability of the system. Let us bring it back here as follows:

Assumption 3.1 (see Assumption 2.2)
The following inequality holds:

$$\rho_2(G) = \inf\left\{\|\Delta G \Delta^{-1}\|;\ \Delta \in \mathcal{D}_{n,+}\right\} < 1 \tag{3.8}$$

where $\|\cdot\|$ denotes the induced Euclidean-norm of matrices in $\mathbb{R}^{n\times n}$ and $\mathcal{D}_{n,+}$ denotes the set of diagonal positive definite matrices.

In case there are perturbations, and $u = Kz_d$, the resulting boundary condition can be expressed as

$$y(t,0) = Gz(t) + BKd(t). \tag{3.9}$$

Under Assumption 3.1, let us recall that the function defined for all $y(\cdot) \in L^2([0,1];\mathbb{R}^n)$, by

$$V(y) = \int_0^1 y(x)^T Q y(x) e^{-2\mu x} dx \tag{3.10}$$

is a Lyapunov function for system (3.1)-(3.3), (3.4),(3.7) where Q is a diagonal positive definite matrix and $\mu > 0$ (see [18]). Even in the presence of perturbations, the system (3.1)-(3.3), (3.5) with control $u = Kz_d$, is ISS in L^2-norm. The result follows using (3.10) as well.

3.3.2 Static control with event-based sampling and quantizer

In this section, we analyze the stability of the closed-loop system when the output is subject to event-triggered sampling and quantization. Highly inspired by [65] and what we have introduced in Chapter 2, we design the sampling algorithm so that L^2-norm of $\|y(t,\cdot)\|$ converges to a bound parameterized by quantization error. In the sequel, we use the boundary controller as $u = \varphi_s(z)$. The operator φ_s encloses the triggering condition, the quantizer and the control function. This requires us to first recall a result from Proposition 2.3 that allows us to express V from (3.10) in terms of measured output.

Denoting $\underline{\lambda} = \min_{1\le i\le n}\{\lambda_i\}$, we define the function $\tilde{V} : [\frac{1}{\underline{\lambda}}, \infty) \to \mathbb{R}^+$, at $t = \frac{1}{\underline{\lambda}}$, by

$$\tilde{V}(t) = \sum_{i=1}^{n} Q_{ii} \int_0^1 \left(H_i z\left(t - \frac{x}{\lambda_i}\right)\right)^2 e^{-2\mu x} dx, \tag{3.11}$$

and for all $t > \frac{1}{\underline{\lambda}}$, by

$$\tilde{V}(t) = \sum_{i=1}^{n} Q_{ii} \int_0^1 \left(H_i z\left(t - \frac{x}{\lambda_i}\right) + B_i u\left(t - \frac{x}{\lambda_i}\right)\right)^2 e^{-2\mu x} dx \tag{3.12}$$

with $Q \in \mathbb{R}^{n\times n}$ a diagonal positive definite matrix.

Proposition 3.1 (see Proposition 2.3)
Let y be a solution to (3.1)-(3.3),(3.4). It holds that for all $t \ge \frac{1}{\underline{\lambda}}$, $\tilde{V}(t) = V(y(t,\cdot))$ with $\tilde{V}$ given by (3.12).

Having stated the above issues, let us now characterize φ_s as follows:

Definition 3.2 (Defintion of φ_s)
Let $\sigma \in (0,1)$, γ_s, ξ, δ, μ, $\nu > 0$, $K \in \mathbb{R}^{m\times n}$. Let $\varepsilon_s(t) = \varepsilon_s(0)e^{-\delta t}$, $\varepsilon_s(0) \leq \xi\tilde{V}(\frac{1}{\underline{\lambda}})$ for all $t \geq \frac{1}{\underline{\lambda}}$.

We define φ_s the operator that maps the output function z to u for which we have:

- *The increasing sequence of time instants (t_k) that is defined iteratively by $t_0 = 0$, $t_1 = \frac{1}{\underline{\lambda}}$, and for all $k \geq 1$,*

$$t_{k+1} = \inf\{t \in \mathbb{R}^+ | t > t_k \ \wedge \gamma_s\|BK(-z(t) + z(t_k))\|^2 \geq 2\nu\sigma\tilde{V}(t) + \varepsilon_s(t)\}. \quad (3.13)$$

  If $\tilde{V}(\frac{1}{\underline{\lambda}}) = 0$, the time instants are $t_0 = 0$, $t_1 = \frac{1}{\underline{\lambda}}$ and $t_2 = \infty$.

- *The quantizer $q : \mathbb{R}^n \to \mathcal{Q}$ has the property that $|q(x) - x| \leq \Delta_q$, for some countable set $\mathcal{Q}$, and a scalar $\Delta_q > 0$.*

- *The static control function φ_s is described by:*

$$\begin{aligned} u(t) &= 0 && \forall t \in [t_0, t_1), \\ u(t) &= Kq(z(t_k)) && \forall t \in [t_k, t_{k+1}), \quad k \geq 1. \end{aligned} \quad (3.14)$$

Remark 3.1
The boundary condition (3.2), with (3.4), under static boundary control, $u = \varphi_s(z)$ as $u(t) = Kq(z(t_k))$, for all $t \geq \frac{1}{\underline{\lambda}}$, can be rewritten as:

$$y(t,0) = (H + BK)z(t) + d_q(t) + d_s(t) \quad (3.15)$$

where

$$\begin{cases} d_q(t) = BK(q(z(t_k)) - z(t_k)) \\ d_s(t) = BK(z(t_k) - z(t)) \quad \forall t \in [t_k, t_{k+1}) \end{cases} \quad (3.16)$$

can be seen as errors related to the quantization and to the sampling respectively.

3.3.2.1 Well-posedness of the closed-loop system

In order to establish the existence and uniqueness of solutions in this section, we use the notion of piecewise continuous solutions as in Chapter 2. As a matter of fact, the controller φ_s introduced in this section has the same nature as the operator defined in Definition 2.3. Using $\mathcal{C}_{rpw}$ (resp. $\mathcal{C}_{lpw}$) to denote piecewise right (resp. left) continuous functions, it follows from the arguments presented in Section 2.2.1 that φ_s satisfies Assumption 2.1, thus $u \in \mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^m)$ provided $z \in \mathcal{C}_{rpw}(\mathbb{R}^+, \mathbb{R}^n)$. It allows us to state the following result on the existence of solutions:

Proposition 3.2 (Existence of solutions)
For any $y^0 \in \mathcal{C}_{lpw}([0,1], \mathbb{R}^n)$, there exists a unique solution to the closed-loop system (3.1)-(3.3), (3.4) with controller $u = \varphi_s(z)$.

3.3.2.2 Stability result

Let us state the main result of this section.

Theorem 3.1 (L^2-stability)
Let $K \in \mathbb{R}^{m\times n}$ be such that Assumption 3.1 holds. Let $\mu > 0$, $Q \in \mathbb{R}^{n\times n}$ a diagonal positive matrix and $\nu = \mu\underline{\lambda}$. Let $\sigma \in (0,1)$, $\delta > 2\nu(1-\sigma)$. Let $\varepsilon_s(t)$ be the decreasing function as in Definition 3.2 and assume that there exist γ_q and $\gamma_s > 0$ such that

$$M_c = \begin{pmatrix} G^TQ\Lambda G - Q\Lambda e^{-2\mu} & G^TQ\Lambda & G^TQ\Lambda \\ \star & Q\Lambda - \gamma_q I & Q\Lambda \\ \star & Q\Lambda & Q\Lambda - \gamma_s I \end{pmatrix} \leq 0 \tag{3.17}$$

Then the closed-loop system (3.1)-(3.3),(3.4),(3.15) under controller φ_s, has a unique solution and is ISS in L^2-norm with respect to d_q.

Proof. The existence and uniqueness of solution is given by Proposition 3.2. Let us show that the system is ISS with respect to d_q. Consider the Lyapunov function candidate given by (3.10). The right-time derivative of the Lyapunov function along the characteristic solutions is given as follows (see Lemma 2.2):

$$\begin{aligned} DV^+ = \quad & y^T(\cdot,0)Q\Lambda y(\cdot,0) - y^T(\cdot,1)e^{-2\mu}Q\Lambda y(\cdot,1) \\ & -2\mu\int_0^1 y(\cdot,x)^T(\Lambda e^{-2\mu x}Q)y(\cdot,x)dx. \end{aligned}$$

In the sequel, to write more compact equation, the time and space dependence may be omitted. Using the output function (3.4) and the boundary condition (3.15) under $u = \varphi_s(z)$, we obtain that

$$\begin{aligned} DV^+ \quad &= (Gz + d_q + d_s)^TQ\Lambda(Gz + d_q + d_s) - z^Te^{-2\mu}Q\Lambda z \\ & -2\mu\int_0^1 y^T(\Lambda e^{-2\mu x}Q)ydx. \end{aligned}$$

Since Q is diagonal positive definite, it holds $\underline{\lambda}Q \leq \Lambda Q$ and taking $\nu = \mu\underline{\lambda}$ yields,

$$DV^+ \leq -2\nu V + \begin{bmatrix} z \\ d_q \\ d_s \end{bmatrix}^T \begin{pmatrix} G^TQ\Lambda G - Q\Lambda e^{-2\mu} & G^TQ\Lambda & G^TQ\Lambda \\ \star & Q\Lambda & Q\Lambda \\ \star & Q\Lambda & Q\Lambda \end{pmatrix} \begin{bmatrix} z \\ d_q \\ d_s \end{bmatrix}. \tag{3.18}$$

Adding $\gamma_q\|d_q\|^2 - \gamma_q\|d_q\|^2$ and $\gamma_s\|d_s\|^2 - \gamma_s\|d_s\|^2$ (for suitable and large enough γ_s and $\gamma_q > 0$) to (3.18) we get,

$$DV^+ \leq -2\nu V + \begin{bmatrix} z \\ d_q \\ d_s \end{bmatrix}^T M_c \begin{bmatrix} z \\ d_q \\ d_s \end{bmatrix} + \gamma_q\|d_q\|^2 + \gamma_s\|d_s\|^2 \tag{3.19}$$

As soon as $M_c \leq 0$, we obtain

$$DV^+ \leq -2\nu V + \gamma_q\|d_q\|^2 + \gamma_s\|d_s\|^2 \tag{3.20}$$

which in turn, can be rewritten as follows:

$$DV^+ \leq -2\nu(1-\sigma)V + \gamma_q\|d_q\|^2 - 2\nu\sigma V + \gamma_s\|d_s\|^2$$

for some $\sigma \in (0,1)$. By Proposition 3.1, for all $t \geq \frac{1}{\underline{\lambda}}$, the last two terms of the inequality can be rewritten as $-2\nu\sigma\tilde{V} + \gamma_s\|d_s\|^2$. Using the definition of φ_s, events are triggered in order to guarantee for all $t \geq \frac{1}{\underline{\lambda}}$, $\gamma_s\|d_s\|^2 \leq 2\nu\sigma\tilde{V} + \varepsilon_s$. Therefore, it guarantees that

$$DV^+ \leq -2\nu(1-\sigma)V + \gamma_q\|d_q\|^2 + \varepsilon_s \tag{3.21}$$

Then, using the Comparison principle, for all $t \geq \frac{1}{\underline{\lambda}}$, we get

$$\begin{aligned} V(y(t,\cdot)) \quad &\leq \quad e^{-2\nu(1-\sigma)(t-\frac{1}{\underline{\lambda}})}V(y(\tfrac{1}{\underline{\lambda}},\cdot)) \\ &\quad + \int_{\frac{1}{\underline{\lambda}}}^{t} e^{-2\nu(1-\sigma)(t-s)}\big(\gamma_q\|d_q(s)\|^2 + \varepsilon_s(s)\big)ds \end{aligned}$$

and thus, with $\varepsilon_s(s) = \varepsilon_s(0)e^{-\delta s}$ where $\varepsilon_s(0) \leq \xi\tilde{V}(\frac{1}{\underline{\lambda}})$ (as in Definition 3.2), integrating the last term and selecting $\delta > 2\nu(1-\sigma)$, we get, for all $t \geq \frac{1}{\underline{\lambda}}$,

$$\begin{aligned} V(y(t,\cdot)) \leq & e^{-2\nu(1-\sigma)(t-\frac{1}{\underline{\lambda}})}V(y(\tfrac{1}{\underline{\lambda}},\cdot)) + \tfrac{\gamma_q}{2\nu(1-\sigma)} \sup_{s\in[\frac{1}{\underline{\lambda}},t]} \|d_q(s)\|^2 \\ & + \tfrac{\xi V(y(\frac{1}{\underline{\lambda}},\cdot))}{\delta-2\nu(1-\sigma)} e^{-2\nu(1-\sigma)(t-\frac{1}{\underline{\lambda}})-\delta\frac{1}{\underline{\lambda}}} \end{aligned} \tag{3.22}$$

Analyzing what happens for all $t \in [0, \frac{1}{\underline{\lambda}})$ with the aim that the right-hand side of (3.22) depends on the initial condition y^0, exactly as it has been done in the proof of Theorem 2.1 we obtain that: there exists $\theta > 0$ so that $V(y(\frac{1}{\underline{\lambda}},\cdot))$ is given as follows (see (2.31)),

$$V(y(\tfrac{1}{\underline{\lambda}},\cdot)) \leq e^{2\theta\frac{\overline{\lambda}}{\underline{\lambda}}}e^{2(\theta+\mu)}V(y^0) \tag{3.23}$$

with $\overline{\lambda} = \max_{1\leq i\leq n}\{\lambda_i\}$. Therefore, replacing (3.23) in (3.22) we get for all $t \geq \frac{1}{\underline{\lambda}}$,

$$\begin{aligned} V(y(t,\cdot)) \leq & e^{-2\nu(1-\sigma)(t-\frac{1}{\underline{\lambda}})}e^{2\theta\frac{\overline{\lambda}}{\underline{\lambda}}}e^{2(\theta+\mu)}V(y^0) + \tfrac{\gamma_q}{2\nu(1-\sigma)} \sup_{s\in[\frac{1}{\underline{\lambda}},t]} \|d_q(s)\|^2 \\ & + \tfrac{\xi e^{2\theta\frac{\overline{\lambda}}{\underline{\lambda}}}e^{2(\theta+\mu)}V(y^0)}{\delta-2\nu(1-\sigma)} e^{-2\nu(1-\sigma)(t-\frac{1}{\underline{\lambda}})-\delta\frac{1}{\underline{\lambda}}} \end{aligned}$$

Hence, for all $t \geq \frac{1}{\underline{\lambda}}$,

$$V(y(t,\cdot)) \leq \tilde{C}_1 e^{-2\nu(1-\sigma)t}V(y^0) + \tfrac{\gamma_q}{2\nu(1-\sigma)} \sup_{s\in[\frac{1}{\underline{\lambda}},t]} \|d_q(s)\|^2 \tag{3.24}$$

with

$$\tilde{C}_1 = e^{2\nu(1-\sigma)\frac{1}{\underline{\lambda}}}e^{2\theta\frac{\overline{\lambda}}{\underline{\lambda}}}e^{2(\theta+\mu)}\left(1 + \xi\tfrac{e^{-\delta\frac{1}{\underline{\lambda}}}}{\delta-2\nu(1-\sigma)}\right) \tag{3.25}$$

Since $\sup_{s\in[\frac{1}{\underline{\lambda}},t]} \|d_q(s)\|^2 \leq \sup_{s\in[0,t]} \|d_q(s)\|^2$, we finally get,

$$V(y(t,\cdot)) \leq \tilde{C}_1 e^{-2\nu(1-\sigma)t} V(y^0) + \frac{\gamma_q}{2\nu(1-\sigma)} \sup_{s\in[0,t]} \|d_q(s)\|^2. \tag{3.26}$$

It concludes the proof. □

3.4 Dynamic control with finite data rate

In this section, we consider the case when the sampled-output is subject to a quantizer which has constraints on the domain. We define a finite-rate uniform quantizer $q : \mathbb{R}^n \to \mathcal{Q}$ where $\mathcal{Q} := \{q_1, q_2, ..., q_N\}$ is a set of finite alphabets. It has the following property:

$$|q(x) - x| \leq \Delta_q \quad if \quad |x| \leq M_q \tag{3.27a}$$

and

$$|q(x)| \geq M_q - \Delta_q \quad if \quad |x| > M_q \tag{3.27b}$$

where $\Delta_q > 0$ is the sensitivity of the quantizer and M_q is the range of the quantizer. We refer to [65] or [42] for further details. With the quantizer specified in (3.27), the sampled-output $y(t_k, 1)$ (for some t_k to be defined in the sequel), must be bounded in a proper sense. It turns out that it can be only bounded if the H^1-norm of $y(t,\cdot)$, defined as

$$\|y\|^2_{H^1([0,1];\mathbb{R}^n)} = \|y\|^2_{L^2([0,1];\mathbb{R}^n)} + \|\partial y\|^2_{L^2([0,1];\mathbb{R}^n)}$$

is bounded, as explained in Section 3.4.2. Thus, it is necessary for $y \in H^1([0,1];\mathbb{R}^n)$ to be absolutely continuous. When dealing with quantized and sampled output, a static control would introduce discontinuous inputs at the boundary, which result in y being discontinuous. To overcome this problem, we use a dynamic controller as proposed in [64, 63], which helps in smoothing the discontinuities caused by the quantization and sampling. We introduce then the dynamic $\eta \in \mathbb{R}^n$ satisfying the following ordinary differential equation,

$$\dot{\eta}(t) = -\alpha\eta(t) + \alpha z_d(t) \quad \eta(0) = \eta^0 \tag{3.28}$$

for some $\alpha > 0$ to be chosen later and $z_d(t)$ given by (3.5) where d will be characterized later on. Once again, we shall consider ISS issues with respect to d.

Definition 3.3 (H^1 input-to-state stability)
The system (3.1)-(3.3),(3.5),(3.28) with controller $u = \varphi(z)$ is ISS with respect to disturbance $d \in L^\infty(\mathbb{R}^+, \mathbb{R}^n)$, if there exist $\nu > 0$, $C_1 > 0$ and $C_2 > 0$ such that, for every $y^0 \in H^1([0,1];\mathbb{R}^n)$, $\eta^0 \in \mathbb{R}^n$, the solution satisfies, for all $t \in \mathbb{R}^+$,

$$|\eta(t) - y(t,1)|^2 + \|y(t,\cdot)\|^2_{H^1([0,1],\mathbb{R}^n)} \leq C_1 \|d_{[0,t]}\|^2_\infty + C_2 e^{-\nu t}(|\eta^0 - y(0,1)|^2 + \|y^0\|^2_{H^1([0,1];\mathbb{R}^n)}). \tag{3.29}$$

3.4.1 Event-based and quantized dynamic boundary control

Proceeding similarly as in Section 3.3.2, we will call the dynamic boundary controller as φ_d, where this operator encloses the triggering condition, the quantizer and the dynamic control function. It is rigorously characterized as follows:

Definition 3.4 (Definition of φ_d)
Let $\sigma \in (0,1)$, $\kappa_1 > 0$, γ_s, ξ, $\mu > 0$, $K \in \mathbb{R}^{m\times n}$ and P a symmetric positive definite matrix. Let $\tilde{V}$ be given, at $t = \frac{1}{\underline{\lambda}}$ by (3.11) and for all $t > \frac{1}{\underline{\lambda}}$, by (3.12). Let $\varepsilon_d(t) = \xi(\tilde{V}(\frac{1}{\underline{\lambda}}) + (\eta(\frac{1}{\underline{\lambda}}) - z(\frac{1}{\underline{\lambda}}))^T P(\eta(\frac{1}{\underline{\lambda}}) - z(\frac{1}{\underline{\lambda}})))e^{-\delta t}$ for all $t \geq \frac{1}{\underline{\lambda}}$.

We define φ_d the operator that maps the output function z to u for which we have:

- *The increasing sequence of time instants (t_k) that is defined iteratively by $t_0 = 0$, $t_1 = \frac{1}{\underline{\lambda}}$, and for all $k \geq 1$,*

$$\begin{aligned} t_{k+1} = \inf\{t \in \mathbb{R}^+ | t > t_k \wedge \\ \gamma_s \|\alpha(-z(t) + z(t_k))\|^2 \geq \\ \kappa_1(\eta(t) - z(t))^T P(\eta(t) - z(t)) + \kappa_1 \tilde{V}(t) + \varepsilon_d(t)\}. \end{aligned} \tag{3.30}$$

  where η is obtained from (3.28) by setting $z_d(t) = q(y(t_k, 1))$, with $t \in [t_k, t_{k+1})$, and q defined in (3.27).

  If $\tilde{V}(\frac{1}{\underline{\lambda}}) = 0$ and $(\eta(\frac{1}{\underline{\lambda}}) - z(\frac{1}{\underline{\lambda}}))^T P(\eta(\frac{1}{\underline{\lambda}}) - z(\frac{1}{\underline{\lambda}})) = 0$, the time instants are $t_0 = 0$, $t_1 = \frac{1}{\underline{\lambda}}$ and $t_2 = \infty$.

- *The dynamic control function that is defined by:*

$$\begin{array}{rcll} u(t) & = & \tilde{u}(\eta^0, y^0) & \forall t \in [t_0, t_1) \\ u(t) & = & K\eta(t) & \forall t \in [t_k, t_{k+1}), \quad k \geq 1 \end{array} \tag{3.31}$$

  where, $\tilde{u}_i(\eta^0, y^0) = \sum_{j=1}^n K_{ij} e^{-\alpha t_1} \left(\eta_j^0 + \int_0^{t_1} e^{\alpha s} \alpha y_j^0 (1 - \lambda_j s) ds\right)$, i=1,...,m, with η^0, y^0 satisfying the compatibility condition $y^0(0) = Hy^0(1) + B\tilde{u}(\eta^0, y^0)$ and $u(t_1) = \tilde{u}(\eta^0, y^0) = K\eta(t_1)$.

Remark 3.2
Under $u = \varphi_d(z)$, and (3.4), we can rewrite (3.28), for all $t > t_1$, as follows:

$$\dot{\eta}(t) = -\alpha\eta(t) + \alpha z(t) + d_q(t) + d_s(t) \tag{3.32}$$

where

$$\begin{cases} d_q(t) = \alpha(q(z(t_k)) - z(t_k)) \\ d_s(t) = \alpha(z(t_k) - z(t)) \quad \forall t \in [t_k, t_{k+1}) \end{cases} \tag{3.33}$$

can be seen as the measurement errors resulting from the quantization and sampling, respectively.

3.4.1.1 Well-posedness of the closed-loop system

In order to establish the existence and uniqueness of solutions in this section, we use the notion of *piecewise differentiable solutions* ($\mathcal{C}^1_{pw}$) whose derivatives are piecewise continuous functions (as those treated in Chapter 2). Let us remark that $\mathcal{C}^1_{pw}([0,1],\mathbb{R}^n) \subset H^1([0,1],\mathbb{R}^+)$.

Following an analogue methodology as in Chapter 2 (see Subsection 2.2.2), let us first introduce the following assumption:

Assumption 3.2
Let $\tilde{\varphi}$ be an operator from $\mathcal{C}^1_{pw}(\mathbb{R}^+,\mathbb{R}^n)$ to $\mathcal{C}^1_{pw}(\mathbb{R}^+,\mathbb{R}^m)$ satisfying:

1. *the following causality property: for all s in $\mathbb{R}^+$, for all z, $z^* \in \mathcal{C}^1_{pw}(\mathbb{R}^+,\mathbb{R}^n)$*

$$(\forall t \in [0,s], z(t) = z^*(t)) \implies (\forall t \in [0,s], u(t) = u^*(t))$$

   where $u = \tilde{\varphi}(z)$ and $u^* = \tilde{\varphi}(z^*)$;

2. *the following compatibility condition: $y^0(0) = Hy^0(1) + Bu(0)$.*

It will enable us to state the existence of solutions for the linear hyperbolic system in the following sense:

Proposition 3.3
Let $\tilde{\varphi}$ satisfy Assumption 3.2 and $y^0 \in \mathcal{C}^1_{pw}([0,1],\mathbb{R}^n)$. Then, there exists a unique solution to the closed-loop system (3.1)-(3.3),(3.4) with controller $u = \tilde{\varphi}(z)$. Moreover, for all $t \in \mathbb{R}^+$ $y(t,\cdot) \in \mathcal{C}^1_{pw}([0,1],\mathbb{R}^n)$, and for all $x \in [0,1]$ $y(\cdot,x) \in \mathcal{C}^1_{pw}(\mathbb{R}^+,\mathbb{R}^n)$.

Proof. The proof follows the essentials of proof of Proposition 2.1. It is worth bringing up some of its lines as we adapt it to the current case.

Let us consider $\overline{\lambda} = \max_{1\le i\le n}\{\lambda_i\}$ and let $\delta = 1/\overline{\lambda}$ be the minimum time for a characteristic, with velocity $\overline{\lambda}$, to cross the spatial domain [0,1]. For $p \in \mathbb{N}$, let $\Delta_p \subset \mathbb{R}^+$ be defined by $\Delta_p = [p\delta, (p+1)\delta]$. We will proceed by induction over the interval Δ_p with the following induction property:

- y is defined on $\Delta_p \times [0,1]$;
- $y(\cdot,x) \in \mathcal{C}^1_{pw}(\Delta_p,\mathbb{R}^n)$;
- $y(t,\cdot) \in \mathcal{C}^1_{pw}([0,1],\mathbb{R}^n)$.

Let us consider, for $p = 0$, the interval $\Delta_0 = [0,\delta]$. Following the same arguments as in the first part of proof of Proposition 2.1, we get that y is given, for all $(t,x) \in \Delta_0 \times [0,1]$, by

$$y_i(t,x) = \begin{cases} \sum_{j=1}^n H_{ij} z_j(t - \frac{x}{\lambda_i}) + \sum_{j=1}^m B_{ij} u_j(t - \frac{x}{\lambda_i}), & \text{if } \lambda_i t \geq x \\ y_i^0\left(x - \lambda_i t\right), & \text{if } \lambda_i t < x \end{cases} \tag{3.34}$$

where it has stated that z is well-defined on Δ_0 and by causality property, u is well-defined on Δ_0 as well.

It is worth remarking that $z_j(t - \frac{x}{\lambda_i}) = y_j^0(1 - \lambda_i t + x)$. Then, it is proved that y_i depends uniquely on y_i^0 on $\Delta_0 \times [0,1]$.
Since y^0 belongs to $\mathcal{C}^1_{pw}([0,1],\mathbb{R}^n)$, z belongs to $\mathcal{C}^1_{pw}(\Delta_0,\mathbb{R}^n)$. Therefore, by Assumption 3.2, u belongs to $\mathcal{C}^1_{pw}(\Delta_0,\mathbb{R}^m)$. It follows then, from (3.34), that y is defined on $\Delta_0 \times [0,1]$. Moreover,

- $z_j(t - \frac{x}{\lambda_d})$ belongs to $\mathcal{C}^1_{pw}$;
- $u_j(t - \frac{x}{\lambda_d})$ belongs to $\mathcal{C}^1_{pw}$;
- $y_i^0\left(x - \lambda_d t\right)$ belongs to $\mathcal{C}^1_{pw}$;

Observe that $y_i(t,x)$ is continuous at $x = \lambda_i t$. Indeed, from (3.34), at $x = \lambda_i t$, we have

$$y_i(t,\lambda_i t) = \begin{cases} \sum_{j=1}^n H_{ij} y_j^0(0) + \sum_{j=1}^m B_{ij} u_j(0) \\ y_i^0(0) \end{cases} \tag{3.35}$$

where clearly, the two lines of (3.35) are equivalent due to the compatibility condition.

It follows from (3.34) that $y(t,\cdot) \in \mathcal{C}^1_{pw}([0,1],\mathbb{R}^n)$ for all $t \in \Delta_0$ and that $y(\cdot,x) \in \mathcal{C}^1_{pw}(\Delta_0,\mathbb{R}^n)$ for all $x \in [0,1]$. Thus, induction property holds at $p = 0$.

Now, assume that induction property holds for a given $p \in \mathbb{N}$. We are now going to prove the same property for $p+1 > 0$. For that purpose, let us take $y((p+1)\delta,\cdot)$ as the initial condition of the system. Applying the same arguments as above; and by means of hypothesis of induction, one gets that $z \in \mathcal{C}^1_{pw}(\Delta_{p+1},\mathbb{R}^n)$, $u \in \mathcal{C}^1_{pw}(\Delta_{p+1},\mathbb{R}^m)$. In addition, y is defined on $\Delta_{p+1} \times [0,1]$, $y(t,\cdot)$ exists for all t in Δ_{p+1} and belongs to $\mathcal{C}^1_{pw}([0,1],\mathbb{R}^n)$ and $y(\cdot,x)$ belongs to $\mathcal{C}^1_{pw}(\Delta_{p+1},\mathbb{R}^n)$ for every $x \in [0,1]$.

Therefore, we have proved by induction that, for each $p \in \mathbb{N}$, $z \in \mathcal{C}^1_{pw}(\Delta_p,\mathbb{R}^n)$ and $y(t,\cdot)$ exists for all t in Δ_p, and belongs to $\mathcal{C}^1_{pw}([0,1],\mathbb{R}^n)$ and $y(\cdot,x)$ belongs to $\mathcal{C}^1_{pw}(\Delta_p,\mathbb{R}^n)$ for every $x \in [0,1]$. Thus, there exists an unique solution to the closed-loop system (3.1)-(3.3),(3.4) with $u = \tilde{\varphi}(z)$. Hence, this concludes the proof. □

Lemma 3.1
The operator φ_d considered in Definition 3.4 satisfies Assumption 3.2.

Proof. Let $J = [a,b]$ be a closed interval subset of $\mathbb{R}^+$ and let $z \in \mathcal{C}^1_{pw}(\mathbb{R}^+,\mathbb{R}^n)$. Let us prove that under the triggering condition (3.30) of φ_d, there is a finite number of sampling times on J. If $(\eta(\frac{1}{\underline{\lambda}}) - z(\frac{1}{\underline{\lambda}}))^T P(\eta(\frac{1}{\underline{\lambda}}) - z(\frac{1}{\underline{\lambda}})) = 0$ and $\tilde{V}(\frac{1}{\underline{\lambda}}) = 0$, there is only at most one sampling

time instant and that is t_1. Let us see the case $\tilde{V}(\frac{1}{\underline{\lambda}}) + (\eta(\frac{1}{\underline{\lambda}}) - z(\frac{1}{\underline{\lambda}}))^T P(\eta(\frac{1}{\underline{\lambda}}) - z(\frac{1}{\underline{\lambda}})) > 0$. We define f on J as $f(t) = \alpha z(t)$. Note that $f \in \mathcal{C}^1_{pw}(J, \mathbb{R}^n)$, and also uniformly continuous on J. It means that for all $\zeta > 0$, there exists $\tau > 0$ such that

$$\forall t, t' \in [a, b] : |t - t'| < \tau \implies \|f(t) - f(t')\|^2 \leq \zeta \tag{3.36}$$

We denote $\overline{\tau}$ the value of τ when $\zeta = \frac{1}{\gamma_s}\varepsilon_d(b)$. Assume that there are at least two consecutive sampling time instants in J and let t_k be the first one. Consider $\gamma_s\|f(t) - f(t_k)\|^2$ for all $t \in J$ in such a way that, combined with (3.30), the following holds at time $t = t_{k+1}$:

$$\|f(t_{k+1}) - f(t_k)\|^2 \geq \frac{\kappa_1}{\gamma_s}\tilde{V}(t_{k+1}) + \frac{1}{\gamma_s}\varepsilon_d(t_{k+1}) + \frac{\kappa_1}{\gamma_s}(\eta(t_{k+1}) - z(t_{k+1}))^T P(\eta(t_{k+1}) - z(t_{k+1})).$$

Using the non-negativity of $(\eta - y)^T P(\eta - z)$ and $\tilde{V}$, the fact that ε_d is a decreasing function, the definition of $\overline{\tau}$ and the uniform continuity arguments, we have that

$$\begin{aligned} \|f(t_{k+1}) - f(t_k)\|^2 &\geq \frac{1}{\gamma_s}\varepsilon_d(t_{k+1}) \geq \frac{1}{\gamma_s}\varepsilon_d(b) \\ \implies & \quad |t_{k+1} - t_k| \geq \overline{\tau} \end{aligned}$$

Therefore, $\overline{\tau}$ gives a lower bound for the duration between two sampling times, but depends upon the interval under consideration J. We can conclude that the maximal number of triggering times on J may be given by

$$\overline{s} = \left\lfloor \frac{b - a}{\overline{\tau}} \right\rfloor$$

which is finite.

Moreover, from (3.28), we obtain that $\eta \in \mathcal{C}^1_{pw}(\mathbb{R}^+; \mathbb{R}^n)$. Hence, from (3.31), $u \in \mathcal{C}^1_{pw}(\mathbb{R}^+; \mathbb{R}^m)$ for which, by construction, compatibility conditions are satisfied, i.e. $y^0(0) = Hy^0(1) + Bu(0)$ and $u(t_1) = K\eta(t_1)$.

For proving the causality property of the operator φ_d, it is sufficient to follow the same steps as in proof of Lemma 2.1. We conclude the proof. □

Remark 3.3
It can be noticed that both d_s and d_q from (3.33), are in $\mathcal{C}_{rpw}(J, \mathbb{R}^n)$.

Combining the previous lemma with Proposition 3.3, we get

Corollary 3.1
For any y^0 in $\mathcal{C}^1_{pw}([0, 1], \mathbb{R}^n)$, $\eta^0 \in \mathcal{C}^1_{pw}(\mathbb{R}^+, \mathbb{R}^n)$ satisfying the compatibility conditions, there exists a unique solution to the closed-loop system (3.1)-(3.3),(3.4), (3.32) and controller $u = \varphi_d(z)$.

3.4.1.2 ISS stability result

Remark first that the boundary condition (3.2), with (3.4) under the dynamic boundary controller $u = \varphi_d(z)$ as $u = K\eta$ is then rewritten as follows:

$$\begin{aligned} y(t,0) &= Hz(t) + BK\eta(t) \\ &= Gz(t) + BK(\eta(t) - z(t)) \end{aligned} \tag{3.37}$$

with $G = H + BK$. In order to state the main result, let us introduce some notation. We denote

$$\begin{aligned} F_0 &:= \Lambda^2 G^T \Lambda^{-1} Q_2 G \Lambda^2 - \Lambda^T Q_2 \Lambda^2 e^{-2\mu}; \\ F_1 &:= K^T B^T \Lambda^{-1} Q_2 G \Lambda^2; \\ F_2 &:= K^T B^T \Lambda^{-1} Q_2 BK. \end{aligned} \tag{3.38}$$

for scalars $\mu > 0, \alpha > 0$, $\underline{\lambda} = \min_{1\le i\le n}\{\lambda_i\}$, diagonal positive definite matrices Q_1, Q_2 and a symmetric positive matrix P. In addition, we denote by M_c the matrix in (3.39), and finally,

$$M_c = \begin{pmatrix} G^T Q_1 \Lambda G - Q_1 \Lambda e^{-2\mu} & 0 & G^T Q_1 \Lambda BK \\ 0 & F_0 - F_1 - F_1^T + F_2 & -\alpha F_1^T + \alpha F_2 - P \\ \star & \star & \alpha^2 F_2 - 2\alpha P + 2\mu\underline{\lambda} P + K^T B^T Q_1 \Lambda BK \end{pmatrix} \tag{3.39}$$

define the matrix M_c^d, for some scalars γ_q and $\gamma_s > 0$, as

$$M_c^d = \left(\begin{array}{c|cc} M_c & \begin{matrix} 0 \\ F_1^T - F_2 \\ -\alpha F_2 \end{matrix} & \begin{matrix} 0 \\ F_1^T - F_2 \\ -\alpha F_2 \end{matrix} \\ \hline \star & F_2 - \gamma_q I & F_2 \\ & F_2 & F_2 - \gamma_s I \end{array}\right) \tag{3.40}$$

Let us now present the main result of the second part of this chapter.

Theorem 3.2 (H^1-stability)
Assume that there exist Q_1,Q_2 diagonal positive matrices in $\mathbb{R}^{n\times n}$, P a symmetric positive definite matrix in $\mathbb{R}^{n\times n}$, K in $\mathbb{R}^{m\times n}$, $\alpha > 0$, $\mu > 0$, $\nu = \mu\underline{\lambda}$, $\sigma \in (0,1)$, $\delta > 2\nu(1-\sigma)$ such that the Assumption 3.1 holds. Also, suppose that

$$M_c \le 0 \tag{3.41}$$

and that there exist γ_q and $\gamma_s > 0$ such that

$$M_c^d \le 0. \tag{3.42}$$

Then, the closed-loop system (3.1)-(3.3),(3.4),(3.32) under controller φ_d, has a unique solution and is ISS in H^1 norm with respect to d_q.

Proof. The existence and uniqueness of solution follow from Corollary 3.1. Let us show that the system is ISS with respect to d_q. Consider the Lyapunov function candidate $V : H^1([0,1],\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^+$ given by

$$V := V_1 + V_2 + V_3 \tag{3.43}$$

where $V_1 : H^1([0,1],\mathbb{R}^n) \to \mathbb{R}^+$ is defined as,

$$V_1(y) = \int_0^1 y(x)^T Q_1 y(x) e^{-2\mu x} dx \tag{3.44}$$

$V_2 : H^1([0,1],\mathbb{R}^n) \to \mathbb{R}^+$ is defined as,

$$V_2(y) = \int_0^1 y_x(x)^T Q_2 y_x(x) e^{-2\mu x} dx \tag{3.45}$$

where $y_x := \frac{\partial y}{\partial x}$. Finally, $V_3 : H^1([0,1],\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^+$ is defined as,

$$V_3(y,\eta) = (\eta - y(\cdot,1))^T P(\eta - y(\cdot,1)) \tag{3.46}$$

Let us first analyze V_1. Computing the time-derivative of V_1 along the solutions to (3.1) and integrating by parts yields

$$\begin{aligned}\dot{V}_1 = y^T(\cdot,0)Q_1\Lambda y(\cdot,0) - y^T(\cdot,1)e^{-2\mu}Q_1\Lambda y(\cdot,1)\\ -2\mu\int_0^1 y^T(\Lambda Q_1)ye^{-2\mu x}dx.\end{aligned}$$

Using the boundary condition (3.37) with (3.4), we get

$$\begin{aligned}\dot{V}_1 = \begin{bmatrix} y(\cdot,1) \\ (\eta - y(\cdot,1))\end{bmatrix}^T \begin{pmatrix} G^TQ_1\Lambda G - Q_1\Lambda e^{-2\mu} & G^TQ_1\Lambda BK \\ \star & K^TB^TQ_1\Lambda BK\end{pmatrix}\begin{bmatrix} y(\cdot,1) \\ (\eta - y(\cdot,1))\end{bmatrix}\\ - 2\mu\int_0^1 y^T(\Lambda Q_1)ye^{-2\mu x}dx.\end{aligned}$$

Since Q_1 is diagonal positive definite, it holds $\underline{\lambda}Q_1 \leq \Lambda Q_1$ and taking $\nu = \mu\underline{\lambda}$, yields

$$\dot{V}_1 \leq -2\nu V_1 + \begin{bmatrix} y(\cdot,1) \\ (\eta - y(\cdot,1))\end{bmatrix}^T \begin{pmatrix} G^TQ_1\Lambda G - Q_1\Lambda e^{-2\mu} & G^TQ_1\Lambda BK \\ \star & K^TB^TQ_1\Lambda BK\end{pmatrix}\begin{bmatrix} y(\cdot,1) \\ (\eta - y(\cdot,1))\end{bmatrix}$$

Let us now analyze V_2. To do so, it is important to consider the dynamics of y_x. Therefore, differentiating system (3.1) with respect to x yields

$$\frac{\partial y_x}{\partial t}(t,x) + \Lambda\frac{\partial y_x}{\partial x}(t,x) = 0 \tag{3.47}$$

Differentiating (3.37) with respect to t and using (3.1), the boundary condition is

$$y_x(t,0) = \Lambda^{-1}H\Lambda y_x(t,1) - \Lambda^{-1}BK\dot{\eta}(t) \tag{3.48}$$

Computing the time-derivative of V_2 along the solutions to (3.47) and integrating by parts yields

$$\dot{V}_2 \leq -2\nu V_2 + y_x^T(\cdot,0)Q_2\Lambda y_x(\cdot,0) - y_x^T(\cdot,1)e^{-2\mu}Q_2\Lambda y_x(\cdot,1)$$

Using (3.1), the boundary condition (3.48) can be rewritten as

$$y_x(t,0) = \Lambda^{-1}(H + BK)\Lambda y_x(t,1) - \Lambda^{-1}BK(\dot{\eta} - y_t(t,1)) \tag{3.49}$$

Therefore,

$$\begin{aligned}\dot{V}_2 \leq &- 2\nu V_2 \\ &+ y_x^T(\cdot,1)\Lambda^T G^T(\Lambda^{-1})^T Q_2 G\Lambda y_x(\cdot,1) - y_x(\cdot,1)^T Q_2 \Lambda e^{-2\mu} y_x(\cdot,1) \\ &- (\dot{\eta} - y_t(\cdot,1))^T K^T B^T (\Lambda^{-1})^T Q_2 G\Lambda y_x^T(\cdot,1) \\ &- y_x(\cdot,1)\Lambda^T G^T(\Lambda^{-1})^T Q_2 BK(\dot{\eta} - y_t(\cdot,1))\end{aligned}$$

where we also use (3.1) to write $y_x(t,1) = -\Lambda y_t(t,1)$. Using the η-dynamics from (3.32), and recalling the notation in (3.38), we get the following after some lengthy development,

$$\dot{V}_2 \leq -2\nu V_2 + \begin{bmatrix} y_t(\cdot,1) \\ (\eta - y(\cdot,1)) \\ d_q \\ d_s \end{bmatrix}^T \begin{pmatrix} F_0 - F_1 - F_1^T + F_2 & -\alpha F_1^T + \alpha F_2 & (F_1^T - F_2)[1\ 1] \\ \star & \alpha^2 F_2 & -\alpha F_2[1\ 1] \\ \star & \star & [1\ 1]^T F_2 [1\ 1] \end{pmatrix} \begin{bmatrix} y_t(\cdot,1) \\ (\eta - y(\cdot,1)) \\ d_q \\ d_s \end{bmatrix}.$$

Let us finally analyze V_3. Computing the time-derivative along the solutions yields,

$$\dot{V}_3 = -2\nu V_3 + \begin{bmatrix} y_t(\cdot,1) \\ (\eta - y(\cdot,1)) \\ d_q \\ d_s \end{bmatrix}^T \begin{pmatrix} 0 & -P & 0 \\ \star & -2\alpha P + 2\mu\underline{\lambda}P & P[1\ 1] \\ \star & \star & 0 \end{pmatrix} \begin{bmatrix} y_t(\cdot,1) \\ (\eta - y(\cdot,1)) \\ d_q \\ d_s \end{bmatrix}$$

Combining $\dot{V}_1$, $\dot{V}_2$, $\dot{V}_3$ and adding $\gamma_q\|d_q\|^2 - \gamma_q\|d_q\|^2$ and $\gamma_s\|d_s\|^2 - \gamma_s\|d_s\|^2$, for γ_q, γ_s positive scalars, we obtain that

$$\dot{V} \leq -2\nu V + \begin{bmatrix} y(\cdot,1) \\ y_t(\cdot,1) \\ (\eta - y(\cdot,1)) \\ d_q \\ d_s \end{bmatrix}^T M_c^d \begin{bmatrix} y(\cdot,1) \\ y_t(\cdot,1) \\ (\eta - y(\cdot,1)) \\ d_q \\ d_s \end{bmatrix} + \gamma_q\|d_q\|^2 + \gamma_s\|d_s\|^2 \tag{3.50}$$

with M_c^d given by (3.40).

Assuming that $M_c \leq 0$ (M_c given by (3.39)), there exist γ_q and $\gamma_s > 0$ sufficiently large such that M_c^d is still negative semidefinite. Therefore, from (3.50) we get

$$\dot{V} \leq -2\nu V + \gamma_q\|d_q\|^2 + \gamma_s\|d_s\|^2 \tag{3.51}$$

which in turn, can be rewritten as follows:

$$\dot{V} \leq -2(1-\sigma)\nu V + \gamma_q\|d_q\|^2 - 2\nu\sigma V + \gamma_s\|d_s\|^2 \tag{3.52}$$

for some $\sigma \in (0,1)$. From the definition of φ_d, events are triggered such that for all $t \geq \frac{1}{\underline{\lambda}}$, it holds that

$$\begin{aligned}\dot{V} \leq &-2(1-\sigma)\nu V + \gamma_q\|d_q\|^2 \\ &- 2\nu\sigma(V_1 + V_2 + V_3) + \kappa_1 V_3 + \kappa_1 \tilde{V} + \varepsilon_d.\end{aligned}$$

Using Proposition 3.1, for $t \geq \frac{1}{\underline{\lambda}}$ we have that $V_1(y(t,\cdot)) = \tilde{V}(t)$, and setting $\kappa_1 = 2\sigma\nu$, we get

$$\dot{V} \leq -2\nu(1-\sigma)V + \gamma_q\|d_q\|^2 - 2\sigma\nu V_2 + \varepsilon_d.$$

Due to the non-negativity of V_2, it holds

$$\dot{V} \leq -2\nu(1-\sigma)V + \gamma_q\|d_q\|^2 + \varepsilon_d.$$

The remaining part of the proof follows exactly the same arguments as in the proof of Theorem 3.1 right after using the Comparison principle to end up with,

$$V(y(t,\cdot),\eta(t)) \leq \tilde{C}_1 e^{-2\nu(1-\sigma)t} V(y^0,\eta^0) + \frac{\gamma_q}{2\nu(1-\sigma)} \sup_{s\in[0,t]} \|d_q(s)\|^2 \tag{3.53}$$

with $\tilde{C}_1$ given also by (3.25). Hence, we conclude the proof.

3.4.2 Quantized control and practical stability

Until now we have established that even under event-triggered sampling of the output it is possible to achieve ISS property with respect to d_q. Let us consider now the study of practical stability of the system under quantization errors. (see [64] for further details). As mentioned earlier, in order to apply quantized control with finite data rate, according to rule (3.27), we have to find a bound for $z(t) = y(t,1)$. It would hold also for $z(t_k)$ for the time instants defined in (3.30). The main reason of having used H^1 norm stability analysis, is because a suitable bound for $z(t)$ can be deduced as follows: Observe that

$$\begin{aligned}
|z(t)|^2 &= \left(\left|\int_0^1 s y_x(t,s) + y(t,s)ds\right|\right)^2 \\
&\leq \left(\int_0^1 |s y_x(t,s)|ds + \int_0^1 |y(t,s)|ds|\right)^2 \\
&\leq 2\left(\int_0^1 |s y_x(t,s)|ds\right)^2 + 2\left(\int_0^1 |y(t,s)|ds\right)^2 \\
&\leq 2\left(\int_0^1 |s y_x(t,s)|^2 ds + \int_0^1 |y(t,s)|^2 ds\right) \\
&\leq 2\left(\int_0^1 |y_x(t,s)|^2 ds + \int_0^1 |y(t,s)|^2 ds\right) = 2\|y(t,\cdot)\|^2_{H^1([0,1],\mathbb{R}^n)}.
\end{aligned} \tag{3.54}$$

Moreover, from (3.54), it also holds that

$$|z(t)|^2 \leq 2(\|y_x(t,\cdot)\|^2_{L^2([0,1],\mathbb{R}^n)} + \|y(t,\cdot)\|^2_{L^2([0,1],\mathbb{R}^n)}) + 2|\eta(t) - z(t)|^2. \tag{3.55}$$

Next, for all $y \in H^1([0,1],\mathbb{R}^n)$ and $\eta \in \mathbb{R}^n$, the Lyapunov function V given by (3.43) may be bounded as follows:

$$\begin{aligned}
&e^{-2\mu}c_1\Big(\|y(t,\cdot)\|^2_{L^2} + \|y_x(t,\cdot)\|^2_{L^2} + |\eta(t) - y(t,1)|^2\Big) \\
&\leq V(y(t,\cdot),\eta(t)) \leq \lambda_{max}(Q_1)\|y(t,\cdot)\|^2_{L^2} \\
&+ \lambda_{max}(Q_2)\|y_x(t,\cdot)\|^2_{L^2} + \lambda_{max}(P)|\eta(t) - y(t,1)|^2
\end{aligned} \tag{3.56}$$

where $c_1 = \min\{\lambda_{min}(Q_1), \lambda_{min}(Q_2), \lambda_{min}(P)\}$, and $\lambda_{min}(\cdot)$, $\lambda_{max}(\cdot)$, are the minimum and maximum eigenvalues of the matrix under consideration. Therefore, using (3.56) with (3.55) we obtain that

$$|z(t)|^2 \leq \frac{2e^{2\mu}}{c_1} V(y(t,\cdot), \eta(t)). \tag{3.57}$$

Inequality (3.57) will be useful when determining the ultimate boundedness of the system. To that end, let us prove first that the output remains within the range of the quantizer. Following the same arguments provided in [63, Section 5.2], assume that the initial conditions y^0 and η^0 are such that

$$\frac{2e^{2\mu}}{c_1} V(y^0, \eta^0) \leq M_q^2. \tag{3.58}$$

where M_q is the range of the quantizer defined in (3.27). In addition, suppose that we design the quantizer such that

$$\frac{M_q^2}{\Delta_q^2} \geq \frac{2e^{2\mu}}{c_1} \frac{\gamma_q}{2\nu(1-\sigma)} \tag{3.59}$$

From (3.59), one can always find some $\epsilon > 0$ small enough such that

$$\frac{c_1}{2e^{2\mu}} M_q^2 > \frac{\gamma_q}{2\nu(1-\sigma)} \Delta_q^2 (1+\epsilon) \tag{3.60}$$

Having said that, let us introduce two regions in $H^1([0,1];\mathbb{R}^n) \times \mathbb{R}^n$ as follows:

$$S_{M_q} := \left\{(y,\eta) | V(y,\eta) \leq \frac{c_1}{2e^{2\mu}} M_q^2\right\}$$

$$S_{\Delta_q} := \left\{(y,\eta) | V(y,\eta) \leq \frac{\gamma_q}{2\nu(1-\sigma)} \Delta_q^2 (1+\epsilon)\right\}$$

Note that due to (3.60), $S_{\Delta_q} \subset S_{M_q}$. It has been proved in [63] that S_{M_q} and S_{Δ_q} are forward invariant. Moreover, it has been proved that if for some $t_0 \geq 0$, $(y(t_0), \eta(t_0)) \in S_{M_q} \setminus S_{\Delta_q}$, there exists a time $T_\epsilon > t_0$, such that $(y(T_\epsilon), \eta(T_\epsilon)) \in S_{\Delta_q}$.

Let us focus on S_{M_q}. Since S_{M_q} is invariant and $(y^0, \eta^0) \in S_{M_q}$ due to (3.58), then it follows that $(y, \eta) \in S_{M_q}$. Therefore, using (3.57), we obtain that $|z(t)| \leq M_q$ holds (in particular $|z(t_k)| \leq M_q$ for all $t \in [t_k, t_{k+1})$). Applying the quantization rule (3.27) we obtain that quantization error is always upper bounded by Δ_q, i.e. $|d_q(t)| \leq \Delta_q$.

From (3.53), (3.58) and the what we have just stated above, it follows that

$$V(y(t,\cdot), \eta(t)) \leq \frac{M_q^2 c_1}{2e^{2\mu}} \tilde{C}_1 e^{-2\nu(1-\sigma)t} + \frac{\gamma_q}{2\nu(1-\sigma)} \Delta_q^2$$

Using again the bound on V from (3.56), we get also that

$$\|y(t,\cdot)\|^2_{H^1([0,1],\mathbb{R}^n)} + |\eta - y(t,1)|^2 \leq \tfrac{M_q^2}{2} \tilde{C}_1 e^{-2\nu(1-\sigma)t} + \frac{\gamma_q e^{2\mu}}{2\nu(1-\sigma)c_1} \Delta_q^2$$

Considering finally the behavior for t sufficiently large, we obtain the practical stability with ultimate boundedness of the closed-loop system (3.1)-(3.3),(3.4) with controller φ_d,

$$\overline{\lim}_{t\to\infty} \{\|y(t,\cdot)\|^2_{H^1([0,1],\mathbb{R}^n)} + |\eta - y(t,1)|^2\} \leq \frac{\gamma_q e^{2\mu} \Delta_q^2}{2\nu(1-\sigma)c_1}$$

Comments on the solvability of BMIs (3.17) and (3.40).

For solving (3.17) one can use the line search algorithm on μ which leads to successive LMIs that can be solved under SDP. As a result, we would obtain a diagonal positive matrix Q, $\mu > 0$ (when line search stops when optimization problem is no longer feasible) and γ_q and γ_s sufficiently large.

On the other hand, checking the feasibility of (3.40) is not that straightforward. As we will proceed in the numerical example, we have first to compute K such that the dissipativity condition (Assumption 3.1) holds. Then, fixing α and performing a line search on μ will lead to successive LMIs that can be solved under SDP. The result of the optimization problem gives Q_1, Q_2, P, μ (of the line search before the problem becomes infeasible) and γ_q and γ_s sufficiently large.

Finally, let us briefly comment that as the number of hyperbolic PDEs increases, one might have issues of complexity due to the increasing of number of decision variables. At the moment we do not have a characterization of the complexity which would be a polynomial function of the number of decision variables and the number of lines of the resulting LMIs (see e.g. [66]).

3.5 Simulations

3.5.1 Example 1

We illustrate the results of Section 3.4 by considering the following example of a linear system of 2×2 hyperbolic conservation laws

$$\frac{\partial y}{\partial t}(t,x) + \Lambda \frac{\partial y}{\partial x}(t,x) = 0 \tag{3.61}$$

with $y = [y_1 \quad y_2]^T$, $\Lambda = \mathrm{diag}(1\ ;\ \sqrt{2})$, initial condition $y(0,x) = \left[\begin{smallmatrix}\cos(4\pi x)-1 & \cos(2\pi x)-1\end{smallmatrix}\right]^T$ for all $x \in [0,1]$ and dynamic boundary condition given by $y(t,0) = Hz(t)+Bu(t)$ where $H = \left(\begin{smallmatrix}0 & 1.1\\ 1 & 0\end{smallmatrix}\right)$, $B = I_2$ and $u(t) = K\eta(t)$. Let us consider first the case when stabilization is carried out using a dynamic controller without any measurement error, that is, we set $d_q \equiv 0$ and $d_s \equiv 0$ in (3.33). Therefore, η just satisfies

$$\dot{\eta}(t) = -\alpha\eta(t) + \alpha z(t),$$

where we choose $\alpha = 10$. Furthermore, the gain of the dynamic controller K has been designed such that $\rho_2(G) < 1$ with $G = H + BK$. Indeed with $K = \left(\begin{smallmatrix}0 & -0.7\\ -1 & 0\end{smallmatrix}\right)$, $\rho_2(G) = 3.82 \times 10^{-1}$ and $\Delta = \left(\begin{smallmatrix}9.96\times10^{-1} & 0\\ 0 & 1.04\end{smallmatrix}\right)$. Hence, the dissipativity condition holds, which is a necessary condition for the result in Theorem 3.2. Note that, without control action at the boundary, the system is unstable.

In addition, condition (3.41) is verified for suitable Q_1, Q_2 and P. In fact, by choosing properly K and α and performing a line search on μ, one leads to several LMIs (linear in variables P, Q_1, and Q_2) to be solved via semi-definite programming. With K and α given as before,

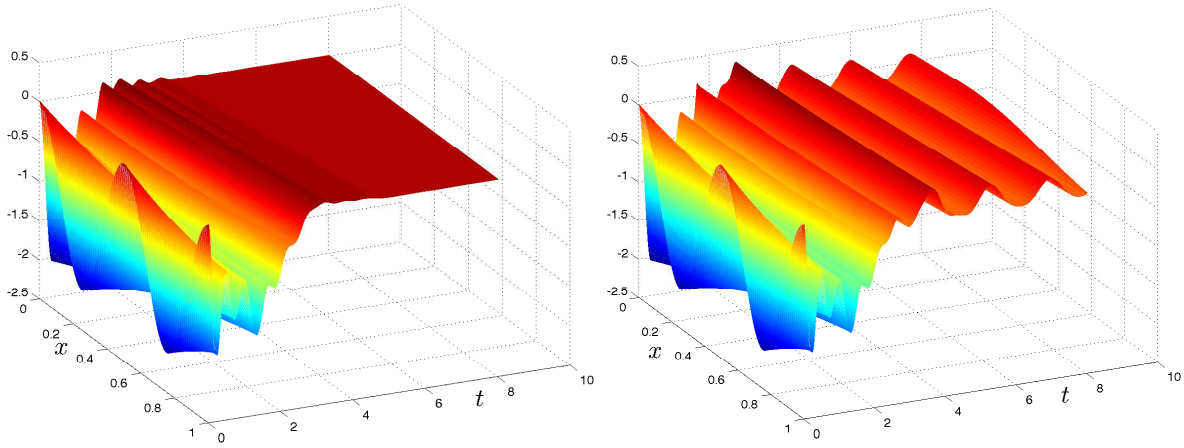

Figure 3.1: Evolution of y_1 with $d_s = d_q = 0$ (left). Evolution of y_1 with quantized samples (right).

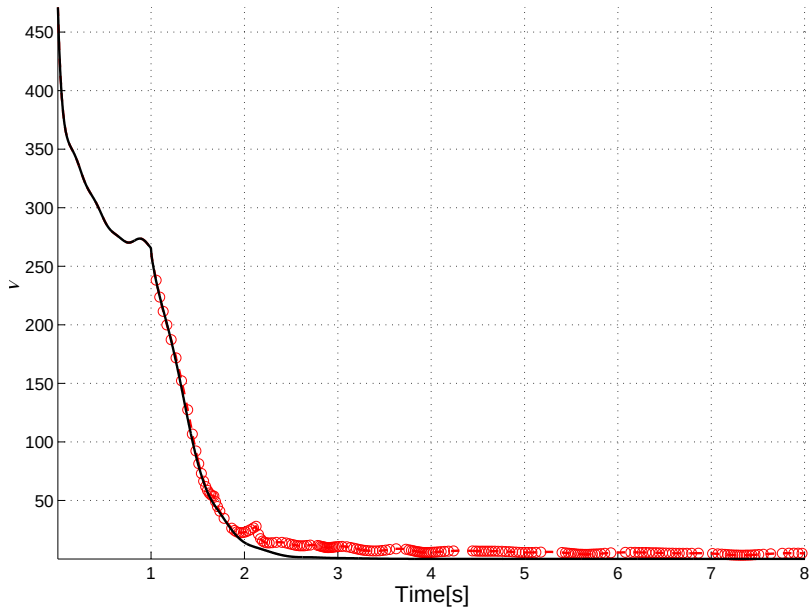

Figure 3.2: Time-evolution of functions V.

we obtain $P = \begin{pmatrix} 3.64\times10^1 & 0 \\ 0 & 2.51\times10^1 \end{pmatrix}$, $Q_1 = \begin{pmatrix} 1.24\times10^2 & 0 \\ 0 & 1.34\times10^2 \end{pmatrix}$, $Q_2 = \begin{pmatrix} 4.41\times10^{-2} & 0 \\ 0 & 4.67\times10^{-2} \end{pmatrix}$ and scalars $\mu = \nu = 1.4 \times 10^{-1}$. If the output was neither sampled nor quantized, we would obtain global exponential stability under $u = K\eta$.

Consider now the case of event-triggered sampling and quantized output. We use the following uniform quantizer

$$q(x) = \Delta_q \left\lfloor \frac{x}{\Delta_q} + 0.5 \right\rfloor$$

whose sensitivity is given by Δ_q. We choose $\Delta_q = 1$ and, for the sake of simplicity, we assume the range of the quantizer to be large. The parameters for the triggering condition are $\sigma = 0.9$, $\varepsilon(0) = 0.1$ and $\delta = 0.28$. Condition $M_c^d \leq 0$ in (3.42) is verified with $\gamma_s = 22.4$ and

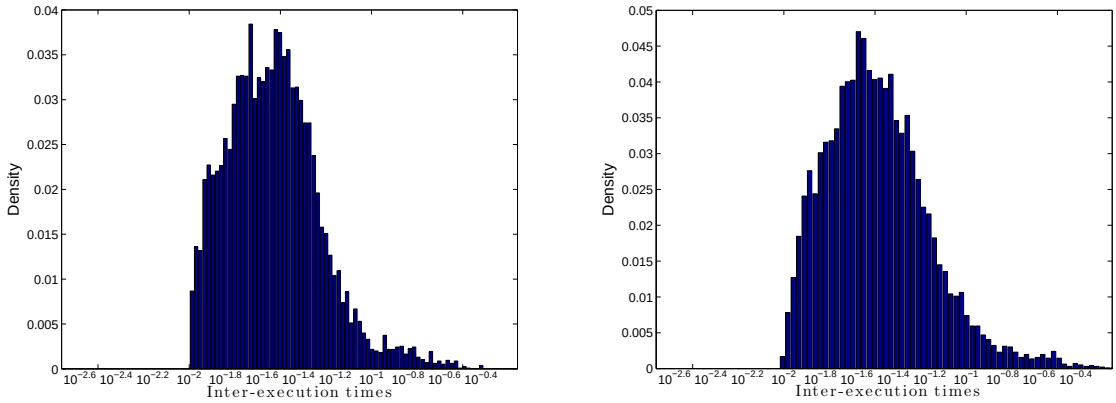

Figure 3.3: Density of the inter-execution times with $\sigma = 0.9$, $\Delta_q = 0.1$ (left) and $\Delta_q = 0.5$ (right)

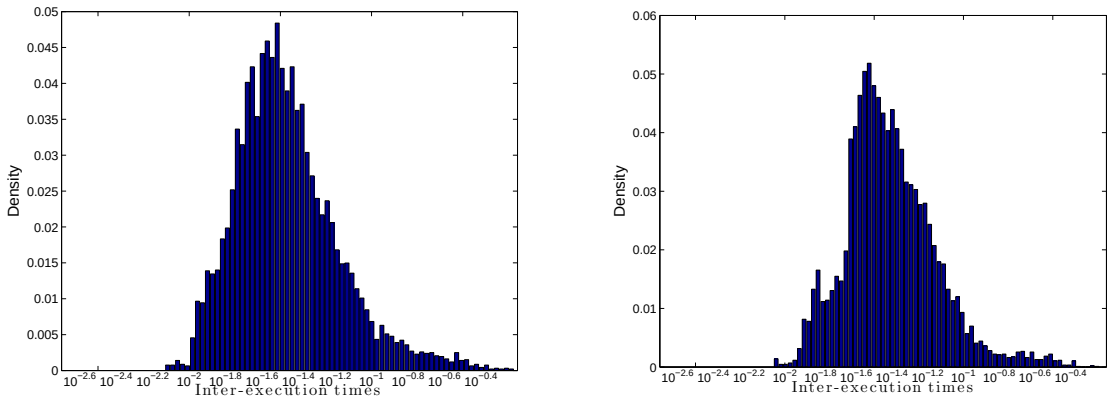

Figure 3.4: Density of the inter-execution times with $\sigma = 0.9$, $\Delta_q = 1$ (left) and $\Delta_q = 2$ (right)

$\gamma_q = 67.4$. Then, Theorem 3.2 applies. We close the loop from $t_1 = \frac{1}{\underline{\lambda}} = 1$. Figure 3.1 shows the plot of y_1 with a dynamic controller without measurement errors on the left, and with the same controller in the presence of sampling and quantization errors on the right. Moreover, Figure 3.2 shows the time-evolution of function V given by (3.43) with $d_s = d_q = 0$ (black line) and with under φ_d (red dashed line with circle markers) using sampled and quantized measurements. It can be noticed that in the later case, no convergence to the origin is achieved but to a ball characterized by the quantization error.

In order to observe whether there is a relationship between the sampling parameter with the quantization error in terms of sampling speed and length of inter-execution times, we performed simulations on a frame of $8s$ for 60 different initial conditions satisfying the compatibility condition. We compute then the inter-execution times by fixing $\sigma \in \{0.9, 0.1\}$ (event-triggered parameter in (3.30)) and varying $\Delta_q \in \{0.1, 0.5, 1, 2\}$. Figures 3.3, 3.4, 3.5 and 3.6 show the density of inter-execution times for the eight possible cases. In addition,

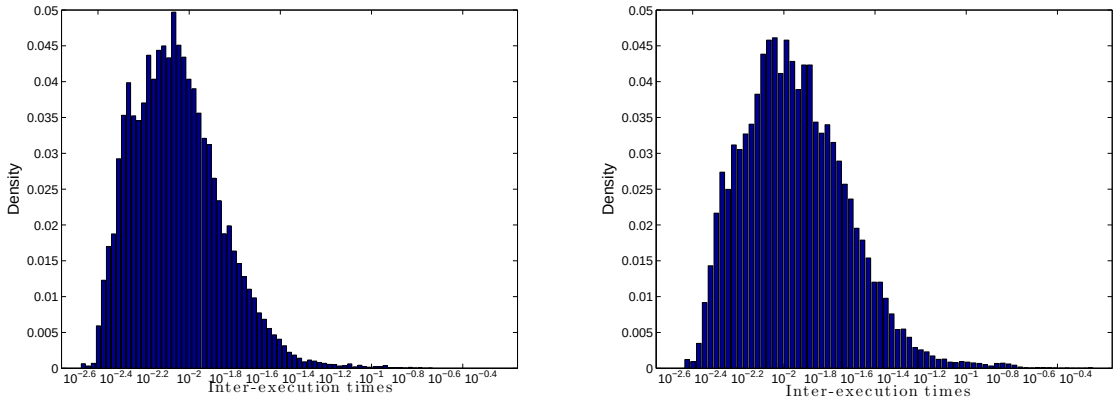

Figure 3.5: Density of the inter-execution times with $\sigma = 0.1$, $\Delta_q = 0.1$ (left) and $\Delta_q = 0.5$ (right)

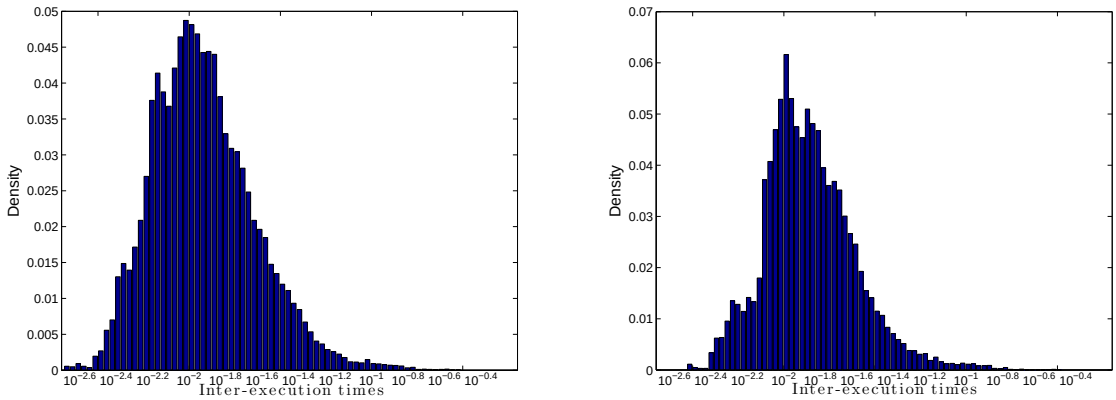

Figure 3.6: Density of the inter-execution times with $\sigma = 0.1$, $\Delta_q = 1$ (left) and $\Delta_q = 2$ (right)

Table 3.1 reports the mean value of the number of sampling times or events. Hence, what it is interesting to point out from the observed simulations results is that, when fixing σ and increasing Δ_q, one can reduce slightly the number of executions times, and that the size of inter-execution times remains quite the same. The reduction of events is not prominent, therefore, we believe that Δ_q does not affect too much the sampling speed and the inter-execution times. Note however that, as expected, σ is indeed the key parameter that impacts the sampling speed. The closer σ is to zero, the faster we should sample. It results in shorter inter-execution times as it can be compared from Figures 3.3 and 3.4 with respect to Figures 3.5 and 3.6.

Finally, we obtained from the numerical simulations, an estimative of the ultimate boundedness of the Lyapunov function as reported in Table 3.2. It can be observed that we obtained bigger ultimate boundedness when $\sigma = 0.1$ for different values of Δ_q than $\sigma = 0.9$. In other words, when sampling faster (as in the case $\sigma = 0.1$), the Lyapunov function converges to a

Mean value of number of sampling times (events)	$\Delta_q = 0.1$	$\Delta_q = 0.5$	$\Delta_q = 1$	$\Delta_q = 2$
$\sigma = 0.9$	193	160	154	144
$\sigma = 0.1$	679	466	452	412

Table 3.1: Mean value of number of events.

bigger attracting ball. It seems then that the influence of the sensitivity of the quantizer is more relevant in this case, thus in the presence of quantized measurements, sampling faster would not always mean that the system will converge to a smaller ball around the origin. It could be interesting to continue studying this issue by changing initial conditions, system dynamics as well as event-trigger parameters, with the aim to provide better and more clear conclusions about the relationship between event-triggered parameters along with the sensitivity of quantizer and the ultimate boundedness.

Estimation of the attracting ball	$\Delta_q = 0.1$	$\Delta_q = 0.5$	$\Delta_q = 1$	$\Delta_q = 2$
$\sigma = 0.9$	0.16	1.78	7.28	25.2
$\sigma = 0.1$	0.23	4.6	7.94	33.19

Table 3.2: Size of the attracting ball of the Lyapunov function.

3.5.2 Example 2

In this example we take again the particular case of communication networks just like (2.4.2) considered in Chapter 2 as depicted in Figure 3.7. Let us recall the linearized model:

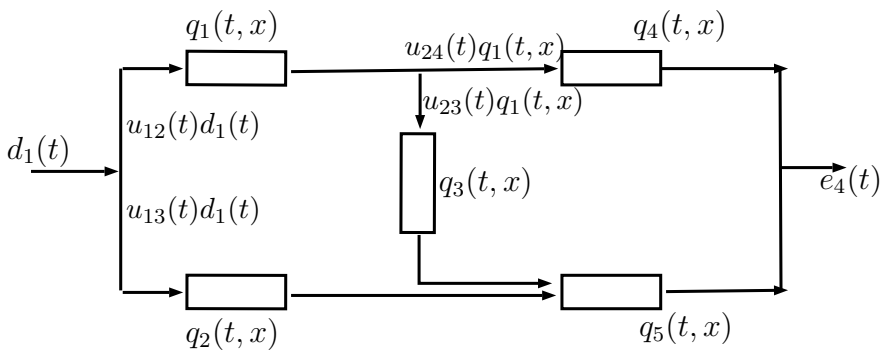

Figure 3.7: Network of compartments made up of 5 transmission lines.

$$\partial_t y + \Lambda \partial_x y = 0$$

with $y(t,x) \in \mathbb{R}^5$, $\Lambda = diag(1, 1.2, 1.7, 0.5, 2)$. The boundary condition is given by

$$y(t,0) = Hy(t,1) + Bu(t)$$

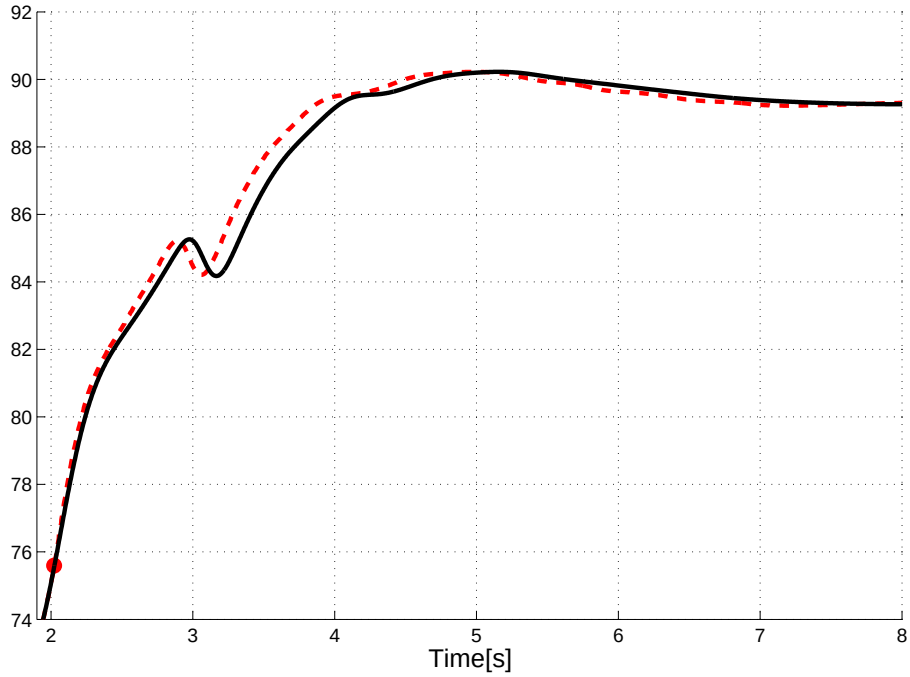

Figure 3.8: Total output flow without measurements errors (black line) and in the presence of event-based sampling and quantization (red dashed line).

where $H = \begin{pmatrix} 0 & 0\,0\,0\,0 \\ 0 & 0\,0\,0\,0 \\ u_{23} & 0\,0\,0\,0 \\ 1-u_{23} & 0\,0\,0\,0 \\ 0 & 1\,1\,0\,0 \end{pmatrix}$, where $u_{23} = 0.548$. $B = \begin{pmatrix} d_1^* & 0 \\ -d_1^* & 0 \\ 0 & q_1^* \\ 0 & -q_1^* \\ 0 & 0 \end{pmatrix}$ where $d_1^* = 100$ and $q_1^* = 35$.

In addition, $u(t) = K\eta(t)$ with $K \in \mathbb{R}^{2\times 5}$ (dynamic controller) where η satisfies, without measurements errors,

$$\dot{\eta}(t) = -\alpha\eta(t) + \alpha z(t),$$

We take K as in example 2.4.2 such that $G = H + BK$ is dissipative. We choose also $\alpha = 10$. Hence, condition (3.41) is verified and we obtain $P = \begin{pmatrix} 17.1 & 0 & 0 & 0 & 0 \\ 0 & 21.4 & -4.32 & -8.64 & -10.8 \\ 0 & -4.32 & 21.4 & 8.64 & 10.8 \\ 0 & -8.64 & 8.64 & 34.4 & 21.6 \\ 0 & -10.8 & 10.8 & 21.6 & 44.1 \end{pmatrix}$, $Q1 = diag(331.4, 332.7, 157.2, 299.4, 49.5)$, $Q_2 = diag(214.6, 164.9, 0.87, 0, 58, 0.62)$ and $\mu = 0.12$. We close the loop from $t_1 = 2$. Using the same triggering parameters as in the previous example and the same quantization function, we get that $M_c^d \leq 0$ in (3.42) is also verified $\gamma_q = 133$, $\gamma_s = 44.3$, thus Theorem 3.2 applies.

Figure 3.8 shows the output flow $q_4(t) + q_5(t)$ of the network when closing the loop with the dynamic continuous time control $u = K\eta$ without measurements errors (black line) and in the presence of event-based sampling and quantization (red dashed line).

Chapter 4

Event-based boundary control of a linear 2×2 hyperbolic system via backstepping approach

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4.4.1 Event-based stabilization 111

In this chapter, we introduce an event-based boundary control for a 2×2 coupled linear hyperbolic system. We use a well established backstepping controller which stabilizes the system along with a dynamic triggering condition which determines when the controller must be updated. The main contributions rely on the definition of an event-based controller under which global exponential stability of the system is achieved and furthermore, the existence of a minimal dwell-time between two triggering times is guaranteed. Since no Zeno phenomenon is presented, the well-posedness of the system under the event-based controller is stated. A simulation example is presented to illustrate the results.

4.1 Introduction

In Chapter 2 we have studied event-based controllers using output feedback by following Lyapunov techniques and taking into account the *dissipativity condition* on the boundary for stability of systems of conservations laws. In this chapter, we will consider 2×2 coupled linear hyperbolic system of balance laws, or simply named as non-uniform linear hyperbolic systems.

The linear hyperbolic system of balance law in open loop that we will consider in the sequel is unstable. As it is clearly stated in [3, Chapter 7], this kind of systems has an intrinsic limit of stabilizability and can not be stabilized by means of linear boundary output feedback laws

because there is no control Lyapunov function of the diagonal form (as (2.16) in Chapter 2) that allows to obtain a static output feedback. This important issue has been established in [2].

Therefore, in this chapter, we use rather a full state-feedback control which is designed following the backstepping approach for stabilizing a linear hyperbolic system of balance laws.

The main contribution of this work relies on the study of an event-based controller using a dynamic triggering condition. We introduce such a triggering policy using the Lyapunov function candidate for the so-called *target system* along with the deviation between continuous time controller and the event-based one when sampling. We prove then that there exists a minimal dwell-time between triggering times and no Zeno phenomena is presented. Consequently, we prove the well-posedness of the system and finally the global exponential stability of the closed-loop system.

This work has been condensed in IEEE-TAC paper which is conditionally accepted.

4.2 Preliminaries on backstepping boundary control of 2×2 linear hyperbolic PDEs

Let us consider the linear hyperbolic system

$$u_t(t,x) + \lambda_1 u_x(t,x) = c_1 v(t,x) \tag{4.1}$$
$$v_t(t,x) - \lambda_2 v_x(t,x) = c_2 u(t,x) \tag{4.2}$$

along with the following boundary conditions:

$$u(t,0) = qv(t,0) \tag{4.3}$$
$$v(t,1) = U(t) \tag{4.4}$$

where $u, v : \mathbb{R}^+ \times [0,1] \to \mathbb{R}$ are the system states with $x \in [0,1]$, $t \geq 0$, $U(t)$ is the control input and $\lambda_1 > 0$, $\lambda_2 > 0$. In addition, for technical issues related to the existence of solutions, we assume that $c_1, c_2 \neq 0$, $q \neq 0$, $\cos(w) - q\frac{\lambda_1}{c_1} w \sin(w) \neq 0$ if $c_1 c_2 > 0$ and $\cosh(w) + q\frac{\lambda_1}{c_1} w \sinh(w) \neq 0$ if $c_1 c_2 < 0$, where $w = \sqrt{\frac{|c_1 c_2|}{\lambda_1 \lambda_2}}$.

In order to stabilize this system, the backstepping method has been considered for instance in [68] and [12]. Roughly, the idea of the backstepping method is to use an invertible Volterra integral transformation to convert the unstable linear hyperbolic PDE (4.1)-(4.4) into a stable linear hyperbolic of conservation laws, which is usually called *target system* and is given as follows:

$$\alpha_t(t,x) + \lambda_1 \alpha_x(t,x) = 0 \tag{4.5}$$
$$\beta_t(t,x) - \lambda_2 \beta_x(t,x) = 0 \tag{4.6}$$

with the following boundary conditions:

$$\alpha(t,0) = q\beta(t,0) \tag{4.7}$$
$$\beta(t,1) = 0 \tag{4.8}$$

where $\alpha, \beta : \mathbb{R}^+ \times [0,1] \to \mathbb{R}$. Hence, $U(t)$ can be chosen to realize the transformation.

4.2.1 Backstepping transformation and kernel equations

The following backstepping Volterra transformation has been used to map the system (4.1)-(4.4) into the system (4.5)-(4.8):

$$\alpha(t,x) = u(t,x) - \int_0^x K^{uu}(x,\xi)u(t,\xi)d\xi - \int_0^x K^{uv}(x,\xi)v(t,\xi)d\xi \tag{4.9}$$

$$\beta(t,x) = v(t,x) - \int_0^x K^{vu}(x,\xi)u(t,\xi)d\xi - \int_0^x K^{vv}(x,\xi)v(t,\xi)d\xi \tag{4.10}$$

It has been shown that by introducing (4.9)-(4.10) into (4.5)-(4.6), integrating by parts and using the boundary conditions, the original system is transformed to the target system with the kernel $K = \begin{pmatrix} K^{uu}(x,\xi) & K^{uv}(x,\xi) \\ K^{vu}(x,\xi) & K^{vv}(x,\xi) \end{pmatrix}$, of the Volterra transformation, satisfying the following linear hyperbolic equations:

$$\lambda_1 K_x^{uu}(x,\xi) + \lambda_1 K_\xi^{uu}(x,\xi) = -c_2 K^{uv}(x,\xi) \tag{4.11}$$
$$\lambda_1 K_x^{uv}(x,\xi) - \lambda_2 K_\xi^{uv}(x,\xi) = -c_1 K^{uu}(x,\xi) \tag{4.12}$$
$$\lambda_2 K_x^{vu}(x,\xi) - \lambda_1 K_\xi^{vu}(x,\xi) = c_2 K^{vv}(x,\xi) \tag{4.13}$$
$$\lambda_2 K_x^{vv}(x,\xi) + \lambda_2 K_\xi^{vv}(x,\xi) = c_1 K^{vu}(x,\xi) \tag{4.14}$$

with boundary conditions

$$K^{uu}(x,0) = \tfrac{\lambda_2}{q\lambda_1} K^{uv}(x,0) \tag{4.15}$$
$$K^{uv}(x,x) = \tfrac{c_1}{\lambda_1+\lambda_2} \tag{4.16}$$
$$K^{vu}(x,x) = -\tfrac{c_2}{\lambda_1+\lambda_2} \tag{4.17}$$
$$K^{vv}(x,0) = \tfrac{q\lambda_1}{\lambda_2} K^{vu}(x,0) \tag{4.18}$$

The kernel equations evolve in a triangular domain given by $\mathcal{T} = \{(x,\xi) : 0 \le \xi \le x \le 1\}$. It is known that there exists a unique solution to (4.11)-(4.18), that the transformation is invertible, and that the inverse transformation, which maps the target system into the original system (4.1)-(4.4), is given by [12]:

$$u(t,x) = \alpha(t,x) + \int_0^x L^{\alpha\alpha}(x,\xi)\alpha(t,\xi)d\xi + \int_0^x L^{\alpha\beta}(x,\xi)\beta(t,\xi)d\xi \tag{4.19}$$

$$v(t,x) = \beta(t,x) + \int_0^x L^{\beta\alpha}(x,\xi)\alpha(t,\xi)d\xi + \int_0^x L^{\beta\beta}(x,\xi)\beta(t,\xi)d\xi \tag{4.20}$$

Moreover, the kernel $L = \begin{pmatrix} L^{\alpha\alpha}(x,\xi) & L^{\alpha\beta}(x,\xi) \\ L^{\beta\alpha}(x,\xi) & L^{\beta\beta}(x,\xi) \end{pmatrix}$ of this transformation satisfies the following linear hyperbolic equations whose solution exists and is unique:

$$\lambda_1 L_x^{\alpha\alpha}(x,\xi) + \lambda_1 L_\xi^{\alpha\alpha}(x,\xi) = c_1 L^{\beta\alpha}(x,\xi) \tag{4.21}$$
$$\lambda_1 L_x^{\alpha\beta}(x,\xi) - \lambda_2 L_\xi^{\alpha\beta}(x,\xi) = c_1 L^{\beta\beta}(x,\xi) \tag{4.22}$$
$$\lambda_2 L_x^{\beta\alpha}(x,\xi) - \lambda_1 L_\xi^{\beta\alpha}(x,\xi) = -c_2 L^{\alpha\alpha}(x,\xi) \tag{4.23}$$
$$\lambda_2 L_x^{\beta\beta}(x,\xi) + \lambda_2 L_\xi^{\beta\beta}(x,\xi) = -c_2 L^{\alpha\beta}(x,\xi) \tag{4.24}$$

with boundary conditions

$$L^{\alpha\alpha}(x,0) = \tfrac{\lambda_2}{q\lambda_1} L^{\alpha\beta}(x,0) \tag{4.25}$$
$$L^{\alpha\beta}(x,x) = \tfrac{c_1}{\lambda_1+\lambda_2} \tag{4.26}$$
$$L^{\beta\alpha}(x,x) = -\tfrac{c_2}{\lambda_1+\lambda_2} \tag{4.27}$$
$$L^{\beta\beta}(x,0) = \tfrac{q\lambda_1}{\lambda_2} L^{\beta\alpha}(x,0) \tag{4.28}$$

Definition 4.1 (L^2-norm stability)
The linear hyperbolic system (4.1)-(4.4) with controller U is globally exponentially stable (GES) if there exist $\tilde{\upsilon} > 0$ and $C > 0$ such that, for every $(u^0, v^0)^T \in L^2([0,1];\mathbb{R}^2)$, the solution satisfies, for all t in $\mathbb{R}^+$,

$$\|(u(t,\cdot), v(t,\cdot))^T\|_{L^2([0,1];\mathbb{R}^2)} \leq Ce^{-\tilde{\upsilon}t}\|(u^0, v^0)^T\|_{L^2([0,1];\mathbb{R}^2)} \tag{4.29}$$

As it can be seen in [12], $U(t)$ is a continuous full-state feedback control which is designed to ensure that the closed-loop system is GES in L^2 norm. The aforementioned backstepping transformation is used to get $U(t)$ under the form

$$U(t) = \int_0^1 K^{vu}(1,\xi)u(t,\xi)d\xi + \int_0^1 K^{vv}(1,\xi)v(t,\xi)d\xi \tag{4.30}$$

Equivalently, (4.30) can be expressed as follows:

$$U(t) = \int_0^1 L^{\beta\alpha}(1,\xi)\alpha(t,\xi)d\xi + \int_0^1 L^{\beta\beta}(1,\xi)\beta(t,\xi)d\xi \tag{4.31}$$

Note that the gains of the controller are the kernels satisfying (4.21)-(4.28).

Furthermore, in [12], the following Lyapunov function candidate is considered to show that the system (4.5)-(4.8) is GES:

$$V(\alpha,\beta) = \int_0^1 (A\alpha^2(x)\frac{e^{-\mu x}}{\lambda_1} + B\beta^2(x)\frac{e^{\mu x}}{\lambda_2})dx \tag{4.32}$$

with $A = e^\mu$, $B = q^2 e^\mu + 1$ and $\mu > 0$. Since the system (4.5)-(4.8) is GES, so is the system (4.1)-(4.4). Indeed, since the transformation (4.9)-(4.10) is invertible, when applying either the continuous control (4.30) or (4.31), the original system has the same stability properties as the target system.

4.3 Event-based stabilization

In this section, we introduce an event-based control scheme for stabilization of the hyperbolic system (4.1)-(4.2). It relies on both the backstepping continuous-time control (4.31) that will be sampled on events and a triggering condition which determines when the event should occur. For that, we slightly modify the boundary conditions in both systems (4.1)-(4.4) and (4.5)-(4.7) by considering a perturbation on one of the boundaries. More precisely, let us consider the following linear hyperbolic system,

$$u_t(t,x) + \lambda_1 u_x(t,x) = c_1 v(t,x) \tag{4.33}$$
$$v_t(t,x) - \lambda_2 v_x(t,x) = c_2 u(t,x) \tag{4.34}$$
$$u(t,0) = q v(t,0) \tag{4.35}$$
$$v(t,1) = U_d(t) \tag{4.36}$$

where $U_d(t) = U(t) + d(t)$ with $U(t)$ given by (4.31) and $d(t)$ can be seen as a disturbance that will be rigorously characterized later on. It is worth remarking that here, d will not be an external disturbance (as considered for instance in [60] where the equations considered there are similar to (4.33)-(4.40) but the problem statement is quite different to the one in this chapter) and is not intended to be rejected. Here, d can be viewed as a deviation between a continuous controller and an event-based one.

Then, applying the backstepping transformation (4.9)-(4.10), one has the equivalent system (*Target perturbed system*):

$$\alpha_t(t,x) + \lambda_1 \alpha_x(t,x) = 0 \tag{4.37}$$
$$\beta_t(t,x) - \lambda_2 \beta_x(t,x) = 0 \tag{4.38}$$
$$\alpha(t,0) = q\beta(t,0) \tag{4.39}$$
$$\beta(t,1) = d(t) \tag{4.40}$$

In addition, the function (4.32) will be used in the sequel in order to introduce the triggering condition. In fact, the event triggering law can be achieved using a strict Lyapunov condition along with an ISS property with respect to a deviation between the continuous controller and the event-based one, as introduced in Chapter 2. Actually, developing ideas from that chapter, we can end up with a triggering condition which depends only on the current state and the deviation between controllers. For that reason, it can be called static triggering condition. However, in the present framework, it turned out that it is very difficult to find a minimal dwell-time between two event times when considering a static triggering condition. To overcome this problem, we will propose a dynamic triggering condition for which we are able to prove the existence of a minimal dwell-time and in turn, the well-posedness of the system under investigation.

It is worth mentioning that guaranteeing the existence of a minimal dwell-time avoids the so-called Zeno phenomenon that means infinite triggering times in a finite-time interval. In practice, Zeno phenomenon would represent infeasible implementation into digital platforms since one would require to sample infinitely fast.

Therefore, inspired by [28] and [23] (see Chapter 2), let us define the event-based controller considered in this chapter. In the sequel we will call it φ and it encloses both the triggering condition and the backstepping feedback controller and Lyapunov analysis will be carried out for the target perturbed system.

Definition 4.2 (Definition of φ)
Let $\sigma \in (0,1)$, $\theta \geq 0$ $\eta > 0$, $\mu > 0$, $\upsilon = \mu \min\{\lambda_1, \lambda_2\}$, $\kappa_1, \kappa_2 > 0$, $m^0 \in \mathbb{R}^-$, $B = q^2 e^{\mu} + 1$. Let L the kernel of the inverse backstepping transformation (4.19)-(4.20) which is solution to the system (4.21)-(4.28). Let $t \mapsto V(\alpha(t,\cdot), \beta(t,\cdot))$ be given by (4.32).

We define φ the functional from $\mathcal{C}^0(\mathbb{R}^+; L^2([0,1];\mathbb{R}^2))$ to $\mathcal{C}_{pw}(\mathbb{R}^+, \mathbb{R})$ that maps $(\alpha, \beta)^T$ to U_d as follows:

- *Let the increasing sequence of time instants (t_k) be defined iteratively by $t_0 = 0$, and for all $k \geq 1$,*

$$\begin{aligned} t_{k+1} = \inf\{t \in \mathbb{R}^+ | t > t_k \wedge \theta B e^{\mu} \Big(\int_0^1 & L^{\beta\alpha}(1,\xi)(\alpha(t_k,\xi) - \alpha(t,\xi)) d\xi \\ & + \int_0^1 L^{\beta\beta}(1,\xi)(\beta(t_k,\xi) - \beta(t,\xi)) d\xi \Big)^2 \\ & \geq \theta \sigma \upsilon V(t) - m(t)\} \end{aligned} \tag{4.41}$$

  where m satisfies the ordinary differential equation,

$$\begin{aligned} \dot{m}(t) = -\eta m(t) + \Bigg(B e^{\mu} \Big(& \int_0^1 L^{\beta\alpha}(1,\xi)(\alpha(t_k,\xi) - \alpha(t,\xi)) d\xi \\ & + \int_0^1 L^{\beta\beta}(1,\xi)(\beta(t_k,\xi) - \beta(t,\xi)) d\xi \Big)^2 \\ & - \sigma \upsilon V(t) - \kappa_1 \alpha^2(t,1) - \kappa_2 \beta^2(t,0) \Bigg) \end{aligned}$$

  for a given $\eta \geq \upsilon(1-\sigma)$ and $m(0) = m^0$.

- *Let the control function be defined by:*

$$U_d(t) = \int_0^1 L^{\beta\alpha}(1,\xi)\alpha(t_k,\xi) d\xi + \int_0^1 L^{\beta\beta}(1,\xi)\beta(t_k,\xi) d\xi \tag{4.42}$$

  for all $t \in [t_k, t_{k+1})$.

Remark 4.1
Let us remark that d in (4.40), given by

$$\begin{aligned} d(t) \quad &= \textstyle\int_0^1 L^{\beta\alpha}(1,\xi)\alpha(t_k,\xi) d\xi + \int_0^1 L^{\beta\beta}(1,\xi)\beta(t_k,\xi) d\xi \\ &\quad - \textstyle\int_0^1 L^{\beta\alpha}(1,\xi)\alpha(t,\xi) d\xi - \int_0^1 L^{\beta\beta}(1,\xi)\beta(t,\xi) d\xi \end{aligned} \tag{4.43}$$

for all $t \in [t_k, t_{k+1})$, can be seen as a deviation between the continuous controller (4.31) and the event-based controller (4.42). As in [22], we follow the perturbed approach inspired by [59], [39] and [33] from finite-dimensional systems. In this setting, the event triggering condition ensures that, for all $t \geq 0$, $\theta B e^{\mu} d^2(t) \leq \theta\sigma\kappa V(t) - m(t)$ which in turn guarantees $m(t) \leq 0$ as stated in the following lemma. In addition, $m(t)$ can be seen as a weighted averaged value of $Be^{\mu}d^2 - \sigma\upsilon V - \kappa_1\alpha^2(\cdot,1) - \kappa_2\beta^2(\cdot,0)$.

Lemma 4.1
Under the definition of φ, it holds that $\theta B e^{\mu} d^2(t) - \theta\sigma\upsilon V(t) + m(t) \leq 0$ and $m(t) \leq 0$.

Proof. By construction, from Definition 4.2, with (4.43), events are triggered to guarantee, for all $t \geq 0$,

$$\theta B e^{\mu} d^2(t) - \theta\sigma\upsilon V(t) \leq -m(t) \tag{4.44}$$

If $\theta = 0$, we obtain $m(t) \leq 0$. In the case $\theta > 0$, it follows from (4.44) that

$$B e^{\mu} d^2(t) - \sigma\upsilon V(t) \leq -\frac{1}{\theta} m(t) \tag{4.45}$$

Then, using (4.42), we have that for all $t \geq 0$,

$$\dot{m} \leq -\eta m - \frac{1}{\theta} m - \kappa_1\alpha^2(\cdot,1) - \kappa_2\beta^2(\cdot,0)$$

Hence, by the Comparison principle, we conclude that $m(t) \leq 0$, for all $t \geq 0$. □

Proposition 4.1
There exists a unique solution $(u,v)^T \in \mathcal{C}^0([t_k, t_{k+1}]; L^2([0,1];\mathbb{R}^2))$ to the system (4.33)-(4.36) between two time instants t_k and t_{k+1}.

Proof. For a constant input $U_d(t) = U(t_k)$ for all $t \in [t_k, t_{k+1})$, the system admits a unique equilibrium point $\{u^*, v^*\}$ satisfying:

$$u^*_x = \tfrac{c_1}{\lambda_1} v^*$$

$$v^*_x = \tfrac{-c_2}{\lambda_2} u^*$$

$$u^*(0) = q v^*(0) \tag{4.46}$$

$$v^*(1) = U_d = U(t_k) \tag{4.47}$$

Let us consider $u^*_{xx}(x) = -w^2 u^*(x)$, with $w = \sqrt{\frac{|c_1 c_2|}{\lambda_1\lambda_2}}$, whose solution is given, in the case when $c_1c_2 > 0$, by $u^*(x) = a\cos(wx) + b\sin(wx)$. Similarly, we can obtain that $v^*(x) = \frac{\lambda_1}{c_1}(-aw\sin(wx) + bw\cos(wx))$. Using (4.46) and (4.47) one can uniquely obtain a and b, that is, $a = q\frac{U(t_k)}{\cos w - q\frac{\lambda_1}{c_1} w\sin w}$ and $b = \frac{c_1}{\lambda_1 w}\frac{U(t_k)}{\cos w - q\frac{\lambda_1}{c_1} w\sin w}$. In the case when $c_1c_2 < 0$, we would obtain $u^*(x) = a\cosh(wx) + b\sinh(wx)$ and $v^*(x) = \frac{\lambda_1}{c_1}(aw\sinh(wx) + bw\cosh(wx))$ with $a = q\frac{U(t_k)}{\cosh w + q\frac{\lambda_1}{c_1} w\sinh w}$ and $b = \frac{c_1}{\lambda_1 w}\frac{U(t_k)}{\cosh w + q\frac{\lambda_1}{c_1} w\sinh w}$.

By performing the change of variable $\tilde{u} = u - u^*$ and $\tilde{v} = v - v^*$, we obtain the following hyperbolic system of balance laws, for all $t \in [t_k, t_{k+1})$:

$$\tilde{u}_t(t,x) + \lambda_1 \tilde{u}_x(t,x) = c_1 \tilde{v}(t,x) \tag{4.48}$$
$$\tilde{v}_t(t,x) - \lambda_2 \tilde{v}_x(t,x) = c_2 \tilde{u}(t,x) \tag{4.49}$$
$$\tilde{u}(t,0) = q\tilde{v}(t,0) \tag{4.50}$$
$$\tilde{v}(t,1) = 0 \tag{4.51}$$

This system is a particular case of the system considered in [18]. Therefore, the classical definition of solution in L^2 can be applied, thus $(\tilde{u}, \tilde{v})^T \in \mathcal{C}^0([t_k, t_{k+1}]; L^2([0,1]; \mathbb{R}^2))$ (see [18, Definition 1]). Hence, for the original variables, it holds that $(u, v)^T \in \mathcal{C}^0([t_k, t_{k+1}]; L^2([0,1]; \mathbb{R}^2))$. It concludes the proof.

□

Using (4.9)-(4.10), it follows straightforwardly that there exists a unique solution $(\alpha, \beta)^T \in \mathcal{C}^0([t_k, t_{k+1}]; L^2([0,1]; \mathbb{R}^2))$ to the system (4.37)-(4.40) between two time instants t_k and t_{k+1}. This allows to state the following result which will be useful for the sequel.

Proposition 4.2
The function d given by (4.43) and the function V given by (4.32), are continuous on $[t_k, t_{k+1}]$.

Proof. One one hand, by the definition of the inner product, it can be noticed that d in (4.43) is as follows:

$$d(t) = \left\langle \begin{pmatrix} L^{\beta\alpha}(1,\cdot) \\ L^{\beta\beta}(1,\cdot) \end{pmatrix}, \begin{pmatrix} \alpha(t_k,\cdot) \\ \beta(t_k,\cdot) \end{pmatrix} \right\rangle_{L^2([0,1];\mathbb{R}^2)} - \left\langle \begin{pmatrix} L^{\beta\alpha}(1,\cdot) \\ L^{\beta\beta}(1,\cdot) \end{pmatrix}, \begin{pmatrix} \alpha(t,\cdot) \\ \beta(t,\cdot) \end{pmatrix} \right\rangle_{L^2([0,1];\mathbb{R}^2)}$$

for all $t \in [t_k, t_{k+1})$. Since $\alpha(t,\cdot)$ and $\beta(t,\cdot)$ are continuous with respect to time due to Proposition 4.1, and the inner product preserves the continuity, it follows that d is in $\mathcal{C}^0([t_k, t_{k+1}], \mathbb{R})$. On the other hand, V given by (4.32), can be viewed as

$$V(\alpha(t,\cdot), \beta(t,\cdot)) = \left\| \begin{pmatrix} \sqrt{\frac{Ae^{-\mu\cdot}}{\lambda_1}}\alpha(t,\cdot) \\ \sqrt{\frac{Be^{\mu\cdot}}{\lambda_2}}\beta(t,\cdot) \end{pmatrix} \right\|^2_{L^2([0,1];\mathbb{R}^2)}$$

Again, due to continuity arguments for $\alpha(t,\cdot)$ and $\beta(t,\cdot)$, and the L^2-norm preserving the continuity, we conclude that $V(\alpha(t,\cdot), \beta(t,\cdot))$ is a continuous function with respect to t. □

Lemma 4.2
For d given by (4.43) and V given by (4.32), it holds that

$$(\dot{d}(t))^2 \leq \varepsilon_1 \alpha^2(t,1) + \varepsilon_2 d^2(t) + \varepsilon_3 V(t) \tag{4.52}$$

for $\varepsilon_1, \varepsilon_2$ and $\varepsilon_3 > 0$ and for all $t \in (t_k, t_{k+1})$.

Proof. From (4.43), let us take its time derivative as follows:

$$\dot{d}(t) = -\int_0^1 L^{\beta\alpha}(1,\xi)\alpha_t(t,\xi)d\xi - \int_0^1 L^{\beta\beta}(1,\xi)\beta_t(t,\xi)d\xi$$

Using the dynamics (4.37) and (4.38), it clearly follows that

$$\dot{d}(t) = \lambda_1 \int_0^1 L^{\beta\alpha}(1,\xi)\alpha_x(t,\xi)d\xi - \lambda_2 \int_0^1 L^{\beta\beta}(1,\xi)\beta_x(t,\xi)d\xi$$

Integrating by parts, one gets

$$\begin{aligned}\dot{d}(t) = \lambda_1\alpha(t,1)L^{\beta\alpha}(1,1) - \lambda_1\alpha(t,0)L^{\beta\alpha}(1,0) - \lambda_1\int_0^1 L_x^{\beta\alpha}(1\xi)\alpha(t,\xi)d\xi - \lambda_2\beta(t,1)L^{\beta\beta}(1,1)\\ +\lambda_2\beta(t,0)L^{\beta\beta}(1,0) + \lambda_2\int_0^1 L_x^{\beta\beta}(1,\xi)\beta(t,\xi)d\xi\end{aligned}$$

Due to (4.39), we have

$$\begin{aligned}\dot{d}(t) = \lambda_1\alpha(t,1)L^{\beta\alpha}(1,1) - \lambda_2\beta(t,1)L^{\beta\beta}(1,1) + \beta(t,0)(-\lambda_1 qL^{\beta\alpha}(1,0) + \lambda_2 L^{\beta\beta}(1,0))\\ -\lambda_1\int_0^1 L_x^{\beta\alpha}(1\xi)\alpha(t,\xi)d\xi + \lambda_2\int_0^1 L_x^{\beta\beta}(1,\xi)\beta(t,\xi)d\xi\end{aligned} \tag{4.53}$$

Recalling from (4.27)-(4.28) that $L^{\beta\alpha}(1,1) = -\frac{c_2}{\lambda_1+\lambda_2}$ and $L^{\beta\beta}(1,0) = q\frac{\lambda_1}{\lambda_2}L^{\beta\alpha}(1,0)$, we replace them into (4.53), thus

$$\begin{aligned}\dot{d}(t) &= \lambda_1\alpha(t,1)\frac{-c_2}{\lambda_1+\lambda_2} - \lambda_2\beta(t,1)L^{\beta\beta}(1,1)\\ &\quad -\lambda_1\int_0^1 L_x^{\beta\alpha}(1\xi)\alpha(t,\xi)d\xi + \lambda_2\int_0^1 L_x^{\beta\beta}(1,\xi)\beta(t,\xi)d\xi\end{aligned} \tag{4.54}$$

Now, taking the square of $\dot{d}$ and using the *Young's inequality*, we can bound it as follows:

$$\begin{aligned}(\dot{d}(t))^2 &\le 2\left(\frac{\lambda_1 c_2}{\lambda_1+\lambda_2}\alpha(t,1) + \lambda_2 L^{\beta\beta}(1,1)\beta(t,1)\right)^2\\ &\quad +2\Big(-\lambda_1\int_0^1 L_x^{\beta\alpha}(1,\xi)\alpha(t,\xi)d\xi + \lambda_2\int_0^1 L_x^{\beta\beta}(1,\xi)\beta(t,\xi)\Big)^2\\ &\le 4(\tfrac{\lambda_1 c_2}{\lambda_1+\lambda_2}\alpha(t,1))^2 + 4(\lambda_2 L^{\beta\beta}(1,1)\beta(t,1))^2\\ &\quad +4\lambda_1^2\left(\int_0^1 L_x^{\beta\alpha}(1,\xi)\alpha(t,\xi)d\xi\right)^2 + 4\lambda_2^2\left(\int_0^1 L_x^{\beta\beta}(1,\xi)\beta(t,\xi)d\xi\right)^2\end{aligned}$$

By the *Cauchy Schwarz inequality*, one gets

$$\begin{aligned}(\dot{d}(t))^2 &\le 4(\tfrac{\lambda_1 c_2}{\lambda_1+\lambda_2})^2\alpha^2(t,1) + 4\lambda_2^2(L^{\beta\beta}(1,1))^2\beta^2(t,1)\\ &\quad +4\lambda_1^2\int_0^1\left(L_x^{\beta\alpha}(1,\xi)\right)^2 d\xi\int_0^1\alpha^2(t,\xi)d\xi + 4\lambda_2^2\int_0^1\left(L_x^{\beta\beta}(1,\xi)\right)^2 d\xi\int_0^1\beta^2(t,\xi)d\xi\end{aligned}$$

Let us remark that, $\int_0^1 (L_x^{\beta\alpha}(1,\xi))^2 d\xi$ and $\int_0^1 (L_x^{\beta\beta}(1,\xi))^2 d\xi$ exist and let us call them $L_x^{\tilde{\beta}\alpha}$ and $L_x^{\tilde{\beta}\beta}$ respectively. In fact, this is due to the regularity of the Kernels on the domain $\mathcal{T}$ as proved in [68, Theorem 5]. Therefore,

$$\begin{aligned}(\dot{d}(t))^2 \quad &\leq \quad (\tfrac{2\lambda_1 c_2}{\lambda_1+\lambda_2})^2\alpha^2(t,1) + (2\lambda_2 L^{\beta\beta}(1,1))^2\beta^2(t,1) \\ &\quad +4\max\{\lambda_1^2 L_x^{\tilde{\beta}\alpha}, \lambda_2^2 L_x^{\tilde{\beta}\beta}\}\left(\int_0^1 \alpha^2(t,\xi)+\beta^2(t,\xi)d\xi\right)\end{aligned}$$

In addition, let us remark that for (4.32), there exists $r_1 > 0$ (depending on μ) such that $\frac{1}{r_1}\int_0^1(\alpha^2(t,x)+\beta^2(t,x))dx \leq V(\alpha(t,\cdot),\beta(t,\cdot)) \leq r_1\int_0^1(\alpha^2(t,x)+\beta^2(t,x))dx$ (see e.g. [62] for a more general quadratic Lyapunov function candidate). Hence $(\dot{d})^2$ is finally bounded as follows:

$$(\dot{d}(t))^2 \leq (\tfrac{2\lambda_1 c_2}{\lambda_1+\lambda_2})^2\alpha^2(t,1) + (2\lambda_2 L^{\beta\beta}(1,1))^2 d^2(t) + 4\max\{\lambda_1^2 L_x^{\tilde{\beta}\alpha}, \lambda_2^2 L_x^{\tilde{\beta}\beta}\} r_1 V \tag{4.55}$$

with $d^2 = \beta^2(t,1)$ due to (4.40). Setting $\varepsilon_1 = (\frac{2\lambda_1 c_2}{\lambda_1+\lambda_2})^2$, $\varepsilon_2 = (2\lambda_2 L^{\beta\beta}(1,1))^2$ and $\varepsilon_3 = 4\max\{\lambda_1^2 L_x^{\tilde{\beta}\alpha}, \lambda_2^2 L_x^{\tilde{\beta}\beta}\} r_1$, we finish the proof. □

Theorem 4.1
Under the event-based controller φ in Definiton 4.2, with positive scalars θ, σ, μ, υ, B, κ_1, κ_2 and ε_1 (from Lemma 4.2) satisfying the following conditions,

$$\kappa_1 \geq \max\{2\theta B e^{\mu}\varepsilon_1, 2\theta\sigma\upsilon\} \tag{4.56}$$

$$\kappa_2 \geq 2\theta\sigma\upsilon \tag{4.57}$$

There exists a minimal dwell-time $\tau > 0$ between two triggering times, i.e. $t_{k+1} - t_k \geq \tau$, for all $k \geq 0$.

Proof. From the definition of φ, events are triggered to guarantee, for all $t \geq 0$,

$$\theta B e^{\mu} d^2(t) \leq \theta\sigma\upsilon V(t) - m(t) \tag{4.58}$$

Let us consider the following function involving the functions in (4.58).

$$\psi = \frac{\theta B e^{\mu} d^2 + \frac{1}{2}m}{\theta\sigma\upsilon V - \frac{1}{2}m}$$

A lower bound for the inter-execution times according to (4.41) is given by the time it takes for the function ψ to go from $\psi(t_k)$ to $\psi(t_{k+1}) = 1$, where $\psi(t_k) \leq 0$ (virtue of $m(t_k) \leq 0$ due to Lemma 4.1 and $d(t_k) = 0$). Note that ψ is a continuous function on $[t_k, t_{k+1}]$ thanks to Proposition 4.2 and the fact that $m \in \mathcal{C}^0(\mathbb{R}^+, \mathbb{R}^-)$. Then, by the intermediate value theorem, there exists $t_k^{'} > t_k$ such that for all $t \in [t_k^{'}, t_{k+1}]$, $\psi(t) \in [0,1]$. We have then that for all $t \in [t_k^{'}, t_{k+1}]$, the time derivative of ψ is given as follows:

$$\dot{\psi} = \frac{2\theta B e^{\mu} d\dot{d} + \frac{1}{2}\dot{m}}{\theta\sigma\upsilon V - \frac{1}{2}m} - \frac{(\theta\sigma\upsilon\dot{V} - \frac{1}{2}\dot{m})}{\theta\sigma\upsilon V - \frac{1}{2}m}\psi$$

Using the *Young's inequality* as $2d\dot{d} \leq d^2 + (\dot{d})^2$, and from (4.42) we have that

$$\begin{aligned}\dot{\psi} \leq\ & \frac{\theta B e^{\mu} d^2}{\theta\sigma\upsilon V - \frac{1}{2}m} + \frac{\theta B e^{\mu}(\dot{d})^2}{\theta\sigma\upsilon V - \frac{1}{2}m} \\ & + \frac{\frac{1}{2}\left(-\eta m + B e^{\mu} d^2 - \sigma\upsilon V - \kappa_1\alpha^2(\cdot,1) - \kappa_2\beta^2(\cdot,0)\right)}{\theta\sigma\upsilon V - \frac{1}{2}m} \\ & - \frac{\theta\sigma\upsilon\dot{V}}{\theta\sigma\upsilon V - \frac{1}{2}m}\psi + \frac{\frac{1}{2}\left(-\eta m + B e^{\mu} d^2 - \sigma\upsilon V\right)}{\theta\sigma\upsilon V - \frac{1}{2}m}\psi \\ & + \frac{\frac{1}{2}\left(-\kappa_1\alpha^2(\cdot,1) - \kappa_2\beta^2(\cdot,0)\right)}{\theta\sigma\upsilon V - \frac{1}{2}m}\psi \end{aligned} \tag{4.59}$$

where $\dot{V}$ in (4.59) is the time derivative of (4.32) along the solutions (4.37)-(4.38). Indeed, by integrating by parts and using the boundary conditions (4.39)-(4.40), $\dot{V}$ is given as follows:

$$\begin{aligned}\dot{V} =\ & -\alpha^2(t,1)Ae^{-\mu} + \beta^2(t,0)(q^2A - B) \\ & + Be^{\mu}d^2(t) - \mu\int_0^1 (\alpha^2(x)Ae^{-\mu x} + \beta^2(x)Be^{\mu x})dx\end{aligned}$$

with $A = e^{\mu}$ and $B = q^2e^{\mu} + 1$. Replacing $\dot{V}$ in (4.59) and using (4.52) we obtain

$$\begin{aligned}\dot{\psi} \leq\ & \frac{\theta B e^{\mu} d^2}{\theta\sigma\upsilon V - \frac{1}{2}m} + \frac{\theta B e^{\mu}\varepsilon_1\alpha^2(\cdot,1)}{\theta\sigma\upsilon V - \frac{1}{2}m} + \frac{\theta B e^{\mu}\varepsilon_2 d^2}{\theta\sigma\upsilon V - \frac{1}{2}m} + \frac{\theta B e^{\mu}\varepsilon_3 V}{\theta\sigma\upsilon V - \frac{1}{2}m} \\ & + \frac{\frac{1}{2}\left(-\eta m + B e^{\mu} d^2 - \sigma\upsilon V - \kappa_1\alpha^2(\cdot,1) - \kappa_2\beta^2(\cdot,0)\right)}{\theta\sigma\upsilon V - \frac{1}{2}m} \\ & - \frac{\theta\sigma\upsilon\Big(-(\alpha^2(\cdot,1) + \beta^2(\cdot,0))\Big)}{\theta\sigma\upsilon V - \frac{1}{2}m}\psi \\ & - \frac{\theta\sigma\upsilon\Big(B e^{\mu} d^2 - \mu\int_0^1(\alpha^2 A e^{-\mu x} + \beta^2 B e^{\mu x})dx\Big)}{\theta\sigma\upsilon V - \frac{1}{2}m}\psi \\ & + \frac{\frac{1}{2}\left(-\eta m + B e^{\mu} d^2 - \sigma\upsilon V\right)}{\theta\sigma\upsilon V - \frac{1}{2}m}\psi + \frac{\frac{1}{2}\left(-\kappa_1\alpha^2(\cdot,1) - \kappa_2\beta^2(\cdot,0)\right)}{\theta\sigma\upsilon V - \frac{1}{2}m}\psi \end{aligned} \tag{4.60}$$

Re-organizing terms and knowing that $\mu\int_0^1(\alpha^2Ae^{-\mu x} + \beta^2Be^{\mu x}) \leq \mu\max\{\lambda_1,\lambda_2\}V$, (4.60) is rewritten as follows

$$\begin{aligned}\dot{\psi} \leq\ & \frac{\theta B e^{\mu}(1 + \varepsilon_2 + \frac{1}{2\theta})d^2}{\theta\sigma\upsilon V - \frac{1}{2}m} + \frac{(\theta B e^{\mu}\varepsilon_1 - \frac{1}{2}\kappa_1)\alpha^2(\cdot,1)}{\theta\sigma\upsilon V - \frac{1}{2}m} \\ & + \frac{(\theta B e^{\mu}\varepsilon_3 - \frac{1}{2}\sigma\upsilon)V}{\theta\sigma\upsilon V - \frac{1}{2}m} - \frac{\frac{1}{2}\eta m}{\theta\sigma\upsilon V - \frac{1}{2}m} - \frac{\frac{1}{2}\kappa_2\beta^2(\cdot,0)}{\theta\sigma\upsilon V - \frac{1}{2}m} \\ & + \frac{\left(\theta\sigma\upsilon\alpha^2(\cdot,1) - \frac{1}{2}\kappa_1\alpha^2(\cdot,1)\right)}{\theta\sigma\upsilon V - \frac{1}{2}m}\psi \\ & + \frac{\left(\theta\sigma\upsilon\beta^2(\cdot,0) - \frac{1}{2}\kappa_2\beta^2(\cdot,0)\right)}{\theta\sigma\upsilon V - \frac{1}{2}m}\psi + \frac{(-\theta\sigma\upsilon + \frac{1}{2})B e^{\mu} d^2}{\theta\sigma\upsilon V - \frac{1}{2}m}\psi \\ & + \frac{(\theta\sigma\upsilon\mu\max\{\lambda_1,\lambda_2\} - \frac{1}{2}\sigma\upsilon)V}{\theta\sigma\upsilon V - \frac{1}{2}m}\psi - \frac{\frac{1}{2}\eta m}{\theta\sigma\upsilon V - \frac{1}{2}m}\psi \end{aligned}$$

Setting $\kappa_1 \geq \max\{2\theta Be^{\mu}\varepsilon_1, 2\theta\sigma\upsilon\}$ and $\kappa_2 \geq 2\theta\sigma\upsilon$ in light of (4.56)-(4.57), we have

$$\begin{aligned}\dot{\psi} \leq\ & \frac{\theta Be^{\mu}(1+\varepsilon_2+\frac{1}{2\theta})d^2}{\theta\sigma\upsilon V-\frac{1}{2}m}+\frac{(\theta Be^{\mu}\varepsilon_3-\frac{1}{2}\sigma\upsilon)V}{\theta\sigma\upsilon V-\frac{1}{2}m}-\frac{\frac{1}{2}\eta m}{\theta\sigma\upsilon V-\frac{1}{2}m} \qquad (4.61)\\ & +\frac{(-\theta\sigma\upsilon+\frac{1}{2})Be^{\mu}d^2}{\theta\sigma\upsilon V-\frac{1}{2}m}\psi+\frac{(\theta\sigma\upsilon\mu\max\{\lambda_1,\lambda_2\}-\frac{1}{2}\sigma\upsilon)V}{\theta\sigma\upsilon V-\frac{1}{2}m}\psi\\ & -\frac{\frac{1}{2}\eta m}{\theta\sigma\upsilon V-\frac{1}{2}m}\psi \qquad (4.62)\end{aligned}$$

By remarking that $\left(-\frac{\frac{1}{2}\eta m}{\theta\sigma\upsilon V-\frac{1}{2}m}\right) \leq \eta$, $\left(\frac{(\theta Be^{\mu}\varepsilon_3-\frac{1}{2}\sigma\upsilon)V}{\theta\sigma\upsilon V-\frac{1}{2}m}\right) \leq \frac{\theta Be^{\mu}\varepsilon_3-\frac{1}{2}\sigma\upsilon}{\theta\sigma\upsilon}$ and $\frac{(\theta\sigma\upsilon\mu\max\{\lambda_1,\lambda_2\}-\frac{1}{2}\sigma\upsilon)V}{\theta\sigma\upsilon V-\frac{1}{2}m} \leq \frac{(\theta\sigma\upsilon\mu\max\{\lambda_1,\lambda_2\}-\frac{1}{2}\sigma\upsilon)}{\theta\sigma\upsilon}$, (4.61) yields

$$\begin{aligned}\dot{\psi} \leq\ & \frac{\theta Be^{\mu}(1+\varepsilon_2+\frac{1}{2\theta})d^2}{\theta\sigma\upsilon V-\frac{1}{2}m}+\frac{\theta Be^{\mu}\varepsilon_3-\frac{1}{2}\sigma\upsilon}{\theta\sigma\upsilon}+\eta\\ & +\frac{(-\theta\sigma\upsilon+\frac{1}{2})Be^{\mu}d^2}{\theta\sigma\upsilon V-\frac{1}{2}m}\psi+\frac{(\theta\sigma\upsilon\mu\max\{\lambda_1,\lambda_2\}-\frac{1}{2}\sigma\upsilon)}{\theta\sigma\upsilon}\psi+\eta\psi\end{aligned}$$

which is rewritten as follows,

$$\begin{aligned}\dot{\psi} \leq\ & \frac{(1+\varepsilon_2+\frac{1}{2\theta})(\theta Be^{\mu}d^2+\frac{1}{2}m-\frac{1}{2}m)}{\theta\sigma\upsilon V-\frac{1}{2}m}+\left(\frac{\theta Be^{\mu}\varepsilon_3-\frac{1}{2}\sigma\upsilon}{\theta\sigma\upsilon}+\eta\right)\\ & +\frac{(-\theta\sigma\upsilon+\frac{1}{2})(\theta Be^{\mu}d^2+\frac{1}{2}m-\frac{1}{2}m)}{\theta(\theta\sigma\upsilon V-\frac{1}{2}m)}\psi+\left(\frac{\theta\sigma\upsilon\mu\max\{\lambda_1,\lambda_2\}-\frac{1}{2}\sigma\upsilon}{\theta\sigma\upsilon}+\eta\right)\psi\end{aligned}$$

By remarking that $\frac{-\frac{1}{2}m(1+\varepsilon_2+\frac{1}{2\theta})}{\theta\sigma\upsilon V-\frac{1}{2}m} \leq (1+\varepsilon_2+\frac{1}{2\theta})$, $\frac{-\frac{1}{2}m}{\theta\sigma\upsilon V-\frac{1}{2}m}\frac{(-\theta\sigma\upsilon+\frac{1}{2})}{\theta} \leq \frac{(-\theta\sigma\upsilon+\frac{1}{2})}{\theta}$ and that ψ is given by $\frac{\theta Be^{\mu}d^2+\frac{1}{2}m}{\theta\sigma\upsilon V-\frac{1}{2}m}$, it can be finally deduced that

$$\begin{aligned}\dot{\psi} \leq\ & \left(\frac{\theta Be^{\mu}\varepsilon_3-\frac{1}{2}\sigma\upsilon}{\theta\sigma\upsilon}+\eta+(1+\varepsilon_2+\frac{1}{2\theta})\right)\\ & +\left(\frac{(-\theta\sigma\upsilon+\frac{1}{2})}{\theta}+\frac{\theta\sigma\upsilon\mu\max\{\lambda_1,\lambda_2\}-\frac{1}{2}\sigma\upsilon}{\theta\sigma\upsilon}+\eta+(1+\varepsilon_2+\frac{1}{2\theta})\right)\psi\\ & +\frac{(-\theta\sigma\upsilon+\frac{1}{2})}{\theta}\psi^2\end{aligned}$$

This differential inequality has the form

$$\dot{\psi} \leq a_0+a_1\psi+a_2\psi^2$$

where, after some simplifications,

$$\begin{aligned}a_0 &= \frac{Be^{\mu}\varepsilon_3}{\sigma\upsilon}+\eta+\varepsilon_2+1\\ a_1 &= -\sigma\upsilon+\mu\max\{\lambda_1,\lambda_2\}+\eta+\varepsilon_2+1+\frac{1}{2\theta}\\ a_2 &= -\sigma\upsilon+\frac{1}{2\theta}\end{aligned}$$

where a_0, a_1 are a_2 turn out to be positive scalars (as soon as $\theta < \frac{1}{2\sigma\nu}$).

Then, by the Comparison principle, it follows that the time needed by ψ to go from $\psi(t_k^{'}) = 0$ to $\psi(t_{k+1}) = 1$ is at least

$$\tau = \int_0^1 \frac{1}{a_0 + a_1 s + a_2 s^2} ds$$

which is well-defined due to the positivity of a_0, a_1 and a_2.

Thus, $t_{k+1} - t_k^{'} \geq \tau$. Consequently, as $t_{k+1} - t_k \geq t_{k+1} - t_k^{'}$, we achieve that $t_{k+1} - t_k \geq \tau$, being then τ a lower bound of the inter-execution times or minimal dwell-time. It concludes the proof. □

Now that we have proved that there is a minimal dwell-time, no Zeno solution can appear. Therefore we are able to state the following result on the the existence of solutions of the system (4.33)-(4.36) for all $t \in \mathbb{R}^+$.

Corollary 4.1
There exists a unique solution $(u, v)^T \in \mathcal{C}^0(\mathbb{R}^+, L^2([0,1]; \mathbb{R}^2))$ to the system (4.33)-(4.36).

Proof. This is an immediate consequence of Proposition 4.1 and Theorem 4.1. The solution is iteratively built between successive triggering times. □

Remark 4.2
Due to the backstepping transformation (4.9)-(4.10), the well-posedness of the target perturbed system (4.37)-(4.40) immediately follows as well.

Let us state the main result of this chapter.

Theorem 4.2
Let $\sigma \in (0,1)$, $\mu > 0$, $\upsilon = \mu \min\{\lambda_1, \lambda_2\}$, $A = e^{\mu}$, $B = e^{\mu} q^2 + 1$, ε_1 (from Lemma 4.2). Let $\eta \geq \upsilon(1-\sigma)$ and $0 < \theta \leq \min\{\frac{1}{2\sigma\upsilon}, \frac{1}{2Be^{\mu}\varepsilon_1}\}$, κ_1 and κ_2 such that

$$\max\{2\theta B e^{\mu} \varepsilon_1, 2\theta\sigma\upsilon\} \leq \kappa_1 \leq 1 \tag{4.63}$$

$$2\theta\sigma\upsilon \leq \kappa_2 \leq 1 \tag{4.64}$$

holds. Let V be given by (4.32) and d given by (4.43). Then the system (4.33)-(4.36) with event-based controller $U_d = \varphi$ has a unique solution and is globally exponentially stable.

Proof. The existence and uniqueness of a solution to the system (4.33)-(4.36) with controller φ is given by Corollary 4.1. Let us show that the system is globally exponential stable.

Consider the following Lyapunov function candidate for the augmented system (4.37)-(4.40) with (4.42), defined for all $(\alpha(t,\cdot), \beta(t,\cdot)) \in L^2([0,1]; \mathbb{R}^2)$ and $m \in \mathbb{R}^-$ by

$$W(\alpha, \beta, m) = V(\alpha, \beta) - m \tag{4.65}$$

Taking the time derivative of (4.65) along the solutions, it yields,

$$\begin{aligned}\dot{W} &= -\alpha^2(\cdot,1)Ae^{-\mu}+\beta^2(\cdot,0)(q^2A-B)+Be^{\mu}d^2\\ &\quad -\mu\int_0^1(\alpha^2(x)Ae^{-\mu x}+\beta^2(x)Be^{\mu x})dx-\dot{m}\end{aligned} \tag{4.66}$$

Setting $\upsilon = \mu\min\{\lambda_1,\lambda_2\}$, note that $-\mu\int_0^1(\alpha^2(x)Ae^{-\mu x}+\beta^2(x)Be^{\mu x})dx < -\upsilon\int_0^1(\alpha^2(x)A\frac{e^{-\mu x}}{\lambda_1}+\beta^2(x)B\frac{e^{\mu x}}{\lambda_2})dx$. Moreover, setting $A=e^{\mu}$, $B=q^2e^{\mu}+1$, and using (4.42), from (4.66) one gets,

$$\begin{aligned}\dot{W} &\le -\upsilon V-\alpha^2(\cdot,1)-\beta^2(\cdot,0)+Be^{\mu}d^2+\eta m-Be^{\mu}d^2+\sigma\upsilon V\\ &\quad +\kappa_1\alpha^2(\cdot,1)+\kappa_2\beta^2(\cdot,0)\end{aligned} \tag{4.67}$$

which can be rewritten as follows:

$$\begin{aligned}\dot{W} &\le -\upsilon(1-\sigma)W+(-\upsilon(1-\sigma)+\eta)m\\ &\quad +(\kappa_1-1)\alpha^2(\cdot,1)+(\kappa_2-1)\beta^2(\cdot,0)\end{aligned}$$

Setting κ_1 and κ_2 in light of (4.63)-(4.64) we have that $\kappa_1\le 1$ and $\kappa_2\le 1$ and that meet the constraints (4.56)-(4.57) i.e. $\kappa_1\ge\max\{2\theta Be^{\mu}\varepsilon_1, 2\theta\sigma\upsilon\}$ and $\kappa_2\ge 2\theta\sigma\upsilon$ (conditions to be satisfied to guarantee the existence of a minimal dwell-time).

Therefore, it follows that

$$\dot{W}\le -\upsilon(1-\sigma)W+(-\upsilon(1-\sigma)+\eta)m$$

From the definition of φ, events are triggered to guarantee, for all $t>0$, $\theta Be^{\mu}d^2(t)\le\theta\sigma\upsilon V(t)-m(t)$. Then, by Lemma 4.1, we guarantee also that $m\le 0$. Recalling that $\eta\ge\upsilon(1-\sigma)$, we obtain

$$\dot{W}\le -\upsilon(1-\sigma)W$$

By the Comparison principle, and remarking that $V(\alpha,\beta)\le W(\alpha,\beta,m)$ we have, for all $t\ge 0$,

$$V(\alpha(t,\cdot),\beta(t,\cdot))\le e^{-\upsilon(1-\sigma)t}W(\alpha^0,\beta^0,m^0)$$

With $m^0=0$, we just obtain

$$V(\alpha(t,\cdot),\beta(t,\cdot))\le e^{-\upsilon(1-\sigma)t}V(\alpha^0,\beta^0) \tag{4.68}$$

which in fact proves that the system (4.37)-(4.40) is GES in L^2 norm. Therefore, as it has been well established in backstepping approach for hyperbolic PDEs, using the inverse transformation of (4.9)-(4.10) (i.e. (4.19)-(4.20)), the system (4.33)-(4.36) is also GES in L^2 norm. More precisely, an estimate of the the L^2 norm of system (4.33)-(4.36) in terms of the L^2 norm of system (4.37)-(4.40) can be done as follows (see e.g. [17] for further details):

$$\|(u(t,\cdot),v(t,\cdot))^T\|^2_{L^2([0,1];\mathbb{R}^2)}\le(1+2\|L\|_\infty)^2(1+2\|K\|_\infty)^2r_1^2e^{-\upsilon(1-\sigma)t}\|(u^0(\cdot),v^0(\cdot))^T\|^2_{L^2([0,1];\mathbb{R}^2)}$$

where $\|K\|_\infty=\max_{(x,\xi)\in\mathcal{T}}|K(x,\xi)|$, $\|L\|_\infty=\max_{(x,\xi)\in\mathcal{T}}|L(x,\xi)|$. Hence, this concludes the proof. □

Comments on the choice of parameters.
Note that while υ and B are given by stability issues, and σ is related to the decay rate, θ is a free parameter to be properly chosen as given in hypothesis of Theorem 4.2, then one can set κ_1 and κ_2 meeting (4.63)-(4.64). Let us remark however that in this work, an optimal choice of parameters regarding conservatism or sampling speed, is not tackled. We leave the study of the influence of parameters to the performance of the system for future investigations. In this chapter we namely focus on the stability result and well-posedness.

Final remarks

Remark 4.3
Let us remark that if a periodic sampling scheme is intended to be applied to the system (4.33)-(4.36) *instead of an event-based scheme as presented throughout the chapter, one suitable period could be the minimal dwell-time τ obtained from Theorem 4.1.*

Remark 4.4
Results in this chapter may be extended to systems with space-varying coefficients (based on e.g. [68] for the computation of Kernels L to be used in Definition 4.2) or even to $m+n$ hyperbolic equations (inspired by e.g. [35]). However, the result on the existence of a minimal dwell-time provided in Theorem 4.1 must be carefully addressed due to complexity of technical details and some assumptions that may be given in terms of matrix inequalities.

Remark 4.5
Both the triggering condition (4.41) *and the control function* (4.42) *given in Definition 4.2 depend on the target system states. This fact is helpful in proving the global exponential stability using Lyapunov analysis as well as in stating the existence of the minimal dwell-time. Note however that the triggering condition and the control function may of course be reformulated in terms of the actual (u, v)- system states thanks to the backstepping transformation.*

4.4 Numerical simulations

Consider the system (4.33)-(4.36) with $\lambda_1 = 1$, $\lambda_2 = \sqrt{2}$, $c_1 = 1.5$, $c_2 = 2$ and $q = 1/4$. The initial conditions are $u^0(x) = qv^0(x)$ with $v^0(x) = 10(1-x)$ for all $x \in [0, 1]$.

4.4.1 Event-based stabilization

The boundary conditions are $u(t, 0) = qv(t, 0)$ and $v(t, 1) = U_d(t)$ where $U_d(t) = U(t) + d(t)$. In addition, $\upsilon = 0.1$, $\mu = 0.0707$ and $B = 0.533$, $\varepsilon_1 = 2.745$. Concerning the triggering algorithm, we choose the following parameters: $\sigma = 0.9$, $\theta = 8\times 10^{-3}$, $\eta = 0.1$, $\kappa_1 = 2.75\times 10^{-2}$ and $\kappa_2 = 7.8723\times 10^{-4}$. They satisfy the constraints (4.56)-(4.57).

The number of events under this approach is 9 on a frame of 4s meaning that the control value needed to be updated only 9 times.

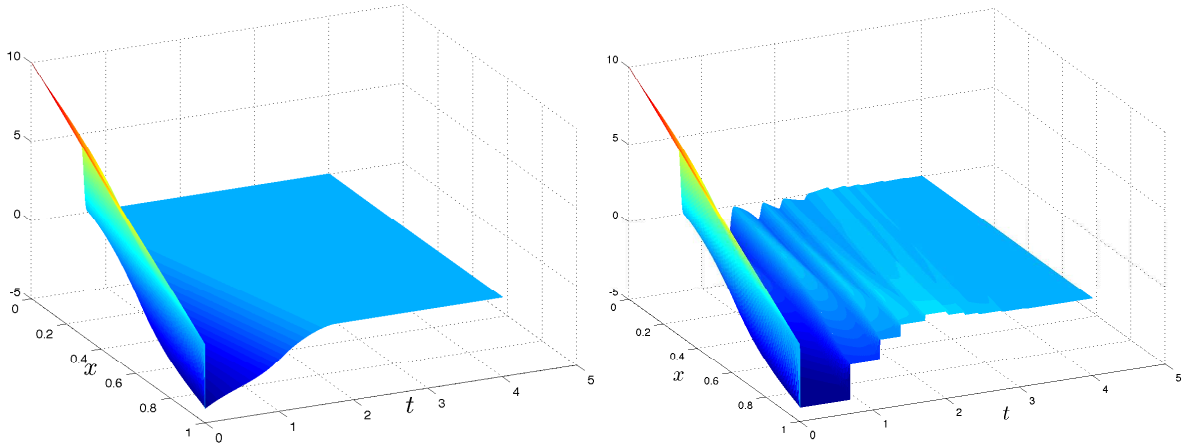

Figure 4.1: Numerical solution of the second component v of the closed-loop system with continuous time controller U (left) and with event-based controller U_d (right).

Figure 4.1 shows the second component of solution $v(t, x)$ when stabilizing with continuous time controller U (left) and the event-based controller U_d (right). Note that attractivity to the origin is achieved and the overall behavior for both solutions is similar. Nevertheless, for the continuous case, it is well known that the system converges to the origin in finite time. In the event-based case, no conclusion in this issue can be provided yet. Note also the discontinuities introduced on the right boundary according to U_d and the propagation from the right to the left across the spacial domain. Figure 4.2 shows the time evolution of the functions appearing in the triggering condition (4.41). Once the trajectory $\theta B e^{\mu} d^2$ reaches the trajectory $\theta\sigma\nu V - m$, an event is generated, the control value is updated and d is reset to zero.

Figure 4.3 shows the continuous-time backstepping controller U and the discontinuous backstepping controller (event-based one) U_d.

Finally, we run simulations for 100 different initial conditions given by $u^0(x) = qv^0(x)$ with $v^0(x) = 4\sin(\frac{4\pi}{a\sqrt{b}}x)$, $a = 1, ..., 10$ and $b = 1, ..., 10$ on a frame of 4s. As in Section 2.4 of Chapter 2, we have computed the inter-execution times between two triggering times. We compare the cases when $\sigma = 0.9$ and $\sigma = 0.1$. Figure 4.4 shows the density of inter-execution times. It is worth mentioning that the discretization step in time for all simulations was 0.001, therefore, it can be noticed from Figure 4.4 that the observed minimal-inter execution time for both cases is much longer that such a value. In addition, as expected, with $\sigma = 0.1$ one has to sample faster and one gets shorter inter-execution times.

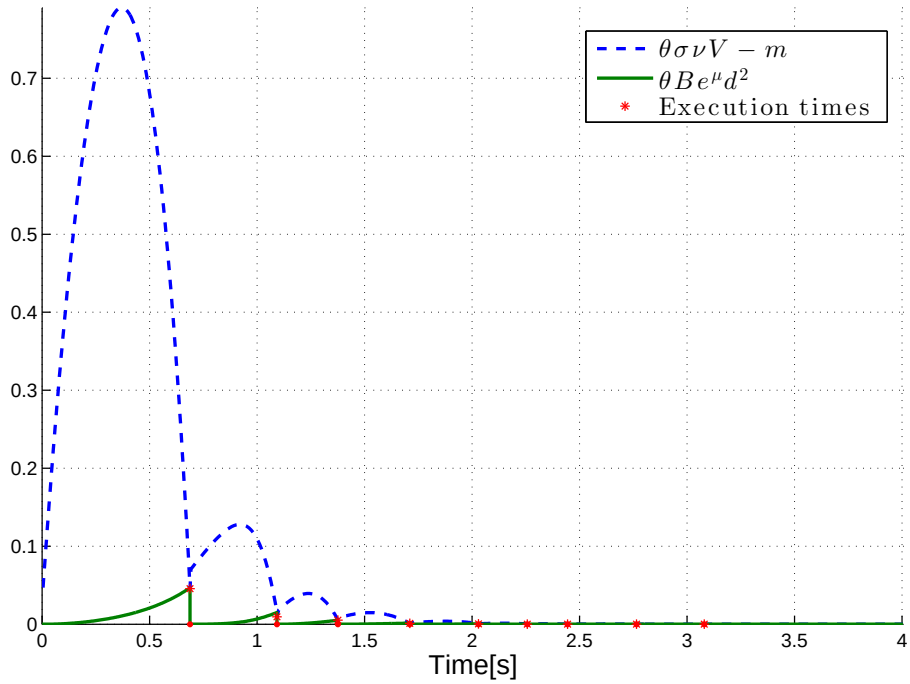

Figure 4.2: Trajectories involved in triggering condition (4.41) for controller $U_d = \varphi(\alpha, \beta)$.

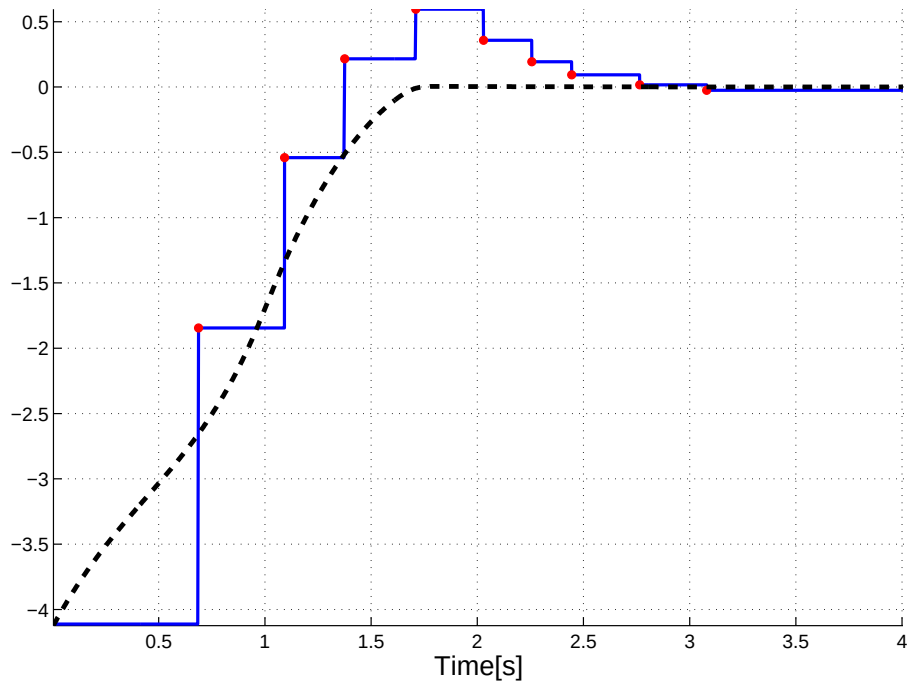

Figure 4.3: Time-evolution of the continuous-time control U (black dashed line) and the event-based control U_d (blue line with red circle marker)

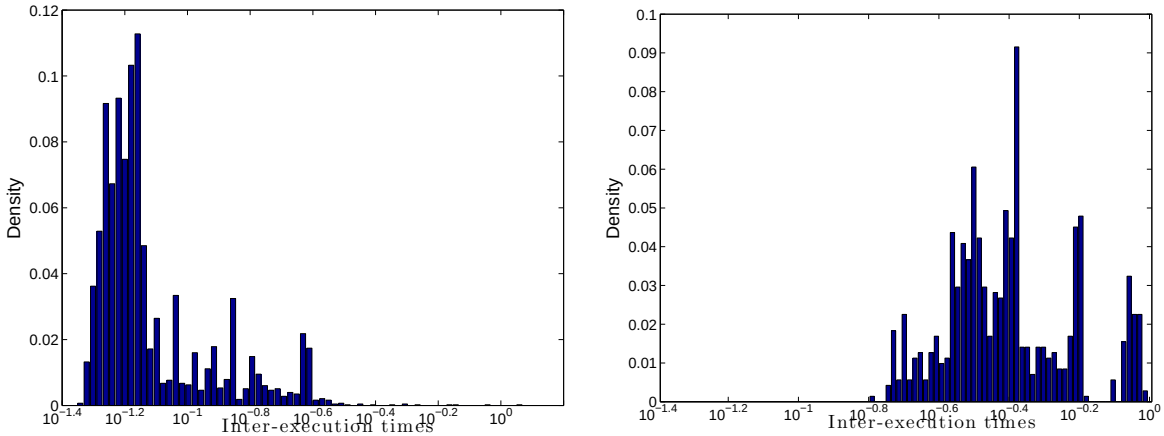

Figure 4.4: Density of the inter-execution times, with $\sigma = 0.1$ (left) and $\sigma = 0.9$ (right).

Conclusion and perspectives

Conclusion

In this thesis, we have studied the modeling and boundary control of networks described by a class of infinite dimensional systems: networks of hyperbolic partial differential equations of conservation laws. Highly inspired by macroscopic models in communication networks and under fluid-flow modeling, we dealt with a coupled PDE-ODE, where the nodes (servers) are modeled by ODEs whereas transmission lines are described by hyperbolic equations when communication delays may be taken into account. For the resulting linearized system around an optimal equilibrium point, input-to state stability (ISS) analysis as well as asymptotic gain control synthesis were carried out by means of Lyapunov techniques and LMI formulation.

We have then dealt with a particular case of the previous model: considering only the linear hyperbolic system of conservation laws (without the ODE coupling and without exogenous disturbance), preliminary results on event-based boundary control were introduced. To the best of our knowledge, digital control, in particular, sampled-data control for PDEs is not well developed and determining the sampling period while guaranteeing asymptotic stability is still a challenge. Therefore, we combined boundary control - quite well established for linear hyperbolic systems- with some of the main strategies of event-based control of finite-dimensional systems. We considered also the case when the output-state is subject to quantization. It turns out that event-based control is a suitable approach to sample aperiodically continuous time boundary controllers. We considered that, under event-based controllers, one can guarantee both the well-posedness and stability (in appropriate sense) of the linear hyperbolic PDE.

More precisely we addressed some of the questions that have risen during the work and that have motivated this thesis. As a matter of example:

- In Chapter 1, we came up with control-oriented model of networks and with dynamic boundary control, namely *access control* and *routing control* to minimize the asymptotic gain while guaranteeing ISS properties. It translates into the fact that, even in the presence of input flow disturbances, the system converges fast enough to an optimal equilibrium point and remains close to it, thus good performance of the network without congestion can be guaranteed;
- In Chapter 2 we have introduced three event-based controllers highly inspired by the event-based strategies from finite-dimensional systems. We confirm that it is possible to sample aperiodically continuous-time boundary controllers of linear hyperbolic PDEs. Actuating on event-based fashion would be a realistic approach of the actuation on the systems;
- In Chapter 3 we introduced quantization issues. Static and dynamic controllers are

designed, which establish stability in L^2- and H^1- norms with respect to measurement errors using Lyapunov-based techniques;

- Finally, in Chapter 4, we combined the Backstepping approach, well-developed tool for boundary control of PDEs, with the event-based notions that we developed in the previous chapters. We believe that this is the first contribution on event-based control for hyperbolic systems under Backtepping approach.

Perspectives

A lot of work on both boundary control of PDEs and event-based control needs still to be done. Let us mention some perspectives as follows:

- Event-based controllers could be introduced for the whole coupled PDE-ODE model that we have studied in Chapter 1. In fact, extending the ISS static event-based approach, could turn out natural. Roughly, the sequence of triggering times would be as follows: $t_0^u = 0$, $t_1^u = \frac{1}{\underline{\lambda}}$ and for all $k \geq 1$,

$$\begin{aligned} t_{k+1} = \inf\{t \in \mathbb{R}^+ | t > t_k \wedge \rho_1 \|B_w K_z(-Z(t) + Z(t_k))\|^2 + \rho_2 \|B_w K_y(-y(t,1) + y(t_k,1))\|^2 \\ + \rho_3 \|B_u L_z(-Z(t) + Z(t_k))\|^2 + \rho_4 \|B_u L_y(-y(t,1) + y(t_k,1))\|^2 \\ \geq 2\sigma\nu\tilde{V}(t) + \varepsilon(t)\} \end{aligned}$$

  Nevertheless, some technical difficulties came up when dealing again with the well-posedness of solutions. This is an issue worth to continue working on;

- Another interesting point is to apply the event-based control strategies to open channels modeled by the Saint-Venant equations (see e.g. [31]). Indeed, actuation might be expensive due to the actuator inertia when regulating the water level and the water flow rate by using gates opening as the control actions. Then, event-based control would suggest to modulate efficiently the gates opening, only when needed. Another interesting application could be flow control on vehicle highway traffic networks with junctions as considered in [9]. It would generalize what has been done in Section 4. The rate inflow might be controlled throughout traffic lights modulation in strategies such as ramp-metering on event-based fashion which is actually a realistic approach for the actuator in the system;

- Regarding the Backstepping approach studied in Chapter 4, since in more realistic scenarios, backstepping controllers are designed using observed states, for event-based control under backstepping, triggering laws should also include an estimate of the state. Based on [68], the output feedback control can be used as a continuous control to be sampled on events. It is important however to guarantee that under the triggering condition depending only on the observed states, there is no Zeno phenomenon.

- It could be interesting to extend many of tools used in sampled-data finite-dimensional systems for the stabilization with aperiodic sampling as provided in [34] and references

therein. In particular, we could be inspired by *looped functionals* introduced in [56] in order to establish a period that can be used to force a dwell-time when proposing event-triggered strategies for hyperbolic PDEs.

- It could be interesting to study other classes of infinite dimensional systems, for instance, *parabolic* partial differential equations. In fact, when diffusion phenomena might be present in networks, we could consider the approach addressed in [8] for the modeling of large networks. Then, stabilization and control of these systems can be tackled following for instance well-known analysis tools via semi-group theory as in [13];

- Finally, inspired by the results of Chapter 4, it could be interesting to introduce event-based boundary control for parabolic PDEs, via Backstepping and flatness based method, as addressed for instance in [47].

Appendix A

Algorithms

Solving BMIs involved in Chapter 1

In this section, we illustrate the algorithms used to solve optimization issues in Chapter 1. First, Algorithm 1 is related to Theorem 1.1 for the open-loop case. Performing line search on μ, leads to successive LMIs that can be solve under Semi-Definite Programming.

```
Input: Λ, A, G_y, B_w, B_u, D, G_z.
Output: P_opt, Q_opt, μ_opt, Asymtp gain opt.
begin
    for μ in line search do
        Setting variables: P, Q, γ ;
        Solve SDP convex optimization problem ;
        minimize γ/(2μ λ̲) e^{2μ};
        s.t M_o (1.26) ≤ 0;
        P ≥ I;
        Q ≥ I;
        if Status = "Solved" then
            Gain_opt = γ/(2μ λ̲) e^{2μ};
            P_opt = P;
            Q_opt = Q;
            μ_opt = μ
        else
            break;
        end
    end
    Asymtp gain opt =Gain_opt;
end
return P_opt, Q_opt, μ_opt, Asymtp gain opt.
```

Algorithm 1: Solving MIs for ISS stability (minimizing the asymptotic gain) in Theorem 1.1

Then, Algorithm 2 is related to Theorems 1.2 and 1.3 for the closed-loop case.

Input: Λ, A, G_y, B_w, B_u, D, G_z, $\mathcal{M}$, β_z, β_y, η, δ_i^w and δ_{ij}^u.
Output: P_{opt}, Q_{opt}, K_{zopt}, K_{yopt}, L_{zopt}, μ_{opt}, *Asymtp gain opt.*
begin
  for μ *in line search* **do**
    Setting variables: X, Y_{K_z}, Y_{K_y}, Y_{L_y}, Q_3, γ;
    Solve SDP convex optimization problem ;
    minimize $\frac{\gamma}{2\mu\underline{\lambda}}e^{2\mu}$;
    s.t $\tilde{M}_c$ (1.54) ≤ 0;
    $X \leq I$;
    $Q_3\Lambda \leq I$;
    and (1.55) ;
    if *Status* = *"Solved"* **then**
      $Gain_{opt} = \frac{\gamma}{2\mu\underline{\lambda}}e^{2\mu}$;
      $X_{opt} = X$;
      $Q_{3opt} = Q_3$;
      $Y_{K_z opt}$;
      $Y_{K_y opt}$;
      $Y_{L_y opt}$;
      $\mu_{opt} = \mu$
    else
      break;
    end
  end
  Asymtp gain opt $=Gain_{opt}$;
  Setting original variables $P_{opt} = X^{-1}$;
  $Q_{opt} = \Lambda^{-1}Q_{3opt}^{-1}$;
  $K_{zopt} = Y_{K_z opt}X_{opt}^{-1}$;
  $K_{yopt} = Y_{K_y opt}X_{opt}^{-1}$;
  $L_{zopt} = Y_{L_z opt}X_{opt}^{-1}$;
end
return P_{opt}, Q_{opt}, K_{zopt}, K_{yopt}, L_{zopt}, μ_{opt}, *Asymtp gain opt.*

Algorithm 2: Solving MIs for control synthesis (minimizing the asymptotic gain) in Theorems 1.2 and 1.3

Event-triggered algorithm related to event-based controllers φ_1, φ_2 and φ_3 in Chapter 2

In this section, we illustrate the algorithm used to solve numerically the linear hyperbolic system on events when closing the loop with event-based controllers φ_1, φ_2 and φ_3. We refer the reader to [40] to see the details about the setting of the PDE solver. Here, we use the fact that the PDE solver integrates the solution via *sol = hpde(sol,howfar, time step)* after defining the problem to be solved *sol=setup(form,pdfun,t,x,y,method,bcd,Newman)* along with boundary condition set as *[YR, YL]=bcfn(t,y, YLex, Yrex)*.

```
Input: y(0,x) (Initial conditions), Λ, H, B, K, Q, μ, Thorizon, mesh discretization,
       other parameters depending of the triggering condition.
Output: y(t,x) (solution), V(y) (Lyapunov function), ev (event function)
begin
    Setting the system to be solved;
    sol = setup(form,@pdefun,t,x,Y,method,[],@bcfun);
    y_t + Λ y_x = 0;
    while t ≤ Thorizon do
        if there is no event then
            for i = j to NT(length time mesh) do
                if there is no event then
                    Solving pde before event-detection;
                    Using @bcfun for boundary condition;
                    y(t,1) = (H + BK)y(t,0) + d (Dissipative boundary condition with
                    disturbance);
                    sol = hpde(sol,howfar,timestep);
                    t = sol.t;
                    y = sol.y;
                    Computing deviations when sampling;
                    d(t)=BK * (y(t_i,1) − y(t,1)) ;
                    computing Lyapunov function;
                    V ((2.18));
                    monitoring event-triggering condition;
                    ev ;
                    either from φ1, φ2 or φ3 ((2.19), (2.33) and (2.42) respectively) ;
                else
                    there is an event;
                    j=i;
                    stop the integration;
                    break;
                end
            end
        end
        Updating;
        d(t_i) = 0 ;
        Using @bcfun for boundary condition;
        y(t,1) = (H + BK)y(t,0) ;
        monitoring event-triggering condition;
        ev;
        either from φ1, φ2 or φ3 ((2.19), (2.33) and (2.42) respectively) ;
    end
    return y(t,x) V(y) ev
end
```

Algorithm 3: Event-Trigger algorithm for linear hyperbolic systems

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Résumé – Cette thèse propose des contributions sur la commande événementielle pour des réseaux modélisés par une classe des systèmes de dimension infinie. Premièrement nous nous focalisons sur la modélisation et contrôle frontière des réseaux qui sont décrits par des systèmes hyperboliques de lois de conservation. En nous inspirant de modèles macroscopiques dans le cadre des réseaux de communications, nous traitons des systèmes couplés EDP-EDO, dont les noeuds (les serveurs) sont modélisés par des EDO non-linéaires alors que des lignes de transmission sont décrites par des systèmes hyperboliques lorsque des retards peuvent être pris en compte. Pour le système linéarisé resultant, autour d'un point d'équilibre optimal, on effectue aussi bien une analyse de stabilité "Input-to-state stable" que de la synthèse du contrôle pour le gain asymptotique grâce à une analyse de fonction de Lyapunov et une formulation LMI.

Ensuite, nous considérons des aspects théoriques de la commande évènementielle aux frontières pour les systèmes hyperboliques. D'un côté, avec cette stratégie de contrôle, nous ciblons la réduction de la consommation d' énergie en traitant les contraintes de communication et de calcul. D' autre part, nous utilisons cette stratégie comme une manière rigoureuse pour échantillonner temporellement lorsqu' on a besoin de mettre en oeuvre les contrôleurs continus sur une plateforme numérique. Une étude mathématique sur l'existence et l' unicité des solutions ainsi que sur les aspects de stabilité est réalisée.

Mots clés : Réseaux de lois de conservation , equations aux dérivées partielles hyperboliques linéaires, commande évènementielle, techniques de Lyapunov, conditions de trigger.

Abstract — This thesis provides contributions on event-based control of networks modeled by a class of infinite dimensional systems. We first focus on the modeling and boundary control of networks described by hyperbolic systems of conservation laws. Highly inspired by macroscopic models in communication networks, we deal with a coupled PDE-ODE, where the nodes (servers) are modeled by nonlinear ODEs whereas transmission lines are described by hyperbolic equations when communication delays may be taken into account. For the resulting linearized system around an optimal equilibrium point, Input-to state stability (ISS) analysis as well as asymptotic gain control synthesis are carried out by means of Lyapunov techniques and LMI formulation.

We then address some theoretical aspects of event-based boundary control of hyperbolic systems. One one hand, with this computer control strategy, we intend to reduce energy consumption when dealing with communication and computational constraints. On the other hand, we use this strategy as a rigorous way of sampling in time when implementation of

continuous time controllers on a digital platform is required. A mathematical study regarding well-posedness of the solutions as well as stability issues is conducted.

Keywords: Networks of conservation laws, linear hyperbolic partial differential equations, event-based control, Lyapunov techniques, triggering conditions

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