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Asymptotic Analysis of Partial Differential Equations Arising in Biological Processes of Anomalous Diffusion

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Asymptotic Analysis of Partial Differential Equations Arising in Biological Processes of Anomalous Diffusion

**Analyse asymptotique d'équations aux dérivées partielles issues
de processus biologiques de diffusion anormale**

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Appendix A

Numerical methods and simulations
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Résumé de la thèse

Cette thèse porte sur l'analyse asymptotique d'équations aux dérivées partielles issues de modèles de déplacement sous-diffusif en biologie cellulaire. Les principaux outils mathématiques utilisés s'inspirent de méthodes d'entropie relative et de techniques liées aux équations de Hamilton-Jacobi.

Elle a été effectuée à mi-temps à l'Unité de Mathématiques Pures et Appliquées, École Normale Supérieure de Lyon, et à mi-temps au Project-Team BEAGLE, Inria Rhône-Alpes, antenne Lyon La Doua, sous la direction de Vincent Calvez, Hugues Berry et Thomas Lepoutre.

Motivation biologique

Certains milieux cellulaires sont très encombrés. Le déplacement aléatoire de protéines peut donc y être entravé. Grâce à des progrès techniques en microscopie, il est possible aujourd'hui d'observer comment le déplacement individuel de certaines protéines dévie de la diffusion normale. En effet, le déplacement quadratique moyen de telles protéines évolue proportionnellement à une puissance du temps plus petite que 1, cas de la diffusion normale :

$$\langle x(t) \rangle \sim t^\mu, \quad 0 < \mu < 1. \quad (1)$$

Ce déplacement aléatoire est dit sous-diffusif. Cette évolution du déplacement quadratique moyen ne satisfait pas la prédiction du théorème de la limite centrale. Un nouveau modèle, qui ne satisfait pas les hypothèses de ce théorème, doit donc être utilisé pour décrire ces comportements.

Équation de renouvellement à sauts

Les marches aléatoires en temps continu (CTRW) sont une classe de modèles microscopiques souvent utilisés pour décrire le déplacement sous-diffusif. Elles décrivent un processus de marche aléatoire couplé à un temps d'attente aléatoire à l'issue de chaque saut de la marche. Dans le cas sous-diffusif, la loi des temps d'attente est à queue lourde :

$$\Phi(a) = \frac{\mu}{(1+a)^{1+\mu}}. \quad (2)$$

Il convient de remarquer que l'espérance de Φ est infinie, même si tout temps d'attente est fini presque-sûrement.

Dans cette thèse, nous étudions des équations mésoscopiques qui traduisent la dynamique décrite par une CTRW tout en tenant compte de la structure d'âge sous-jacente, l'âge étant le temps écoulé depuis le dernier saut. Il s'agit de l'équation aux dérivées partielles structurée en âge à sauts en espace suivante :

$$\begin{cases} \partial_t n(t, x, a) + \partial_a n(t, x, a) + \beta(a)n(t, x, a) = 0, & t \geq 0, \quad a > 0, \quad x \in \mathbb{R} \\ n(t, x, a = 0) = \int_0^\infty \beta(a') \int_{\mathbb{R}} \omega(x - x') n(t, x', a') dx' da' \\ n(t = 0, x, a) = n^0(x, a). \end{cases} \quad (3)$$

Les fonctions β et ω sont respectivement le taux de saut dépendant de l'âge et le noyau de redistribution spatiale des particules. Le taux de saut est relié à la distribution des temps de résidence Φ comme suit :

$$\Phi(a) = \beta(a) \exp\left(-\int_0^a \beta(s) ds\right). \quad (4)$$

Dans le contexte de la sous-diffusion et tout au long de cette thèse, le noyau de redistribution spatiale ω est considéré Gaussien centré en 0, et le taux de saut β satisfait (à perturbations près) :

$$\beta(a) = \frac{\mu}{1+a}. \quad (5)$$

L'originalité de notre approche est de d'étudier directement l'équation intégral-différentielle précédente en se focalisant sur sa structure en âge. Cela diffère de l'approche plus commune au sein de la communauté qui étudie la diffusion anormale, qui consiste à étudier les équations macroscopiques aux dérivées fractionnaires déduites de ces équations mésoscopiques grâce à des transformées de Fourier-Laplace.

Convergence autosimilaire du problème en espace homogène

L'aspect anormal du déplacement aléatoire décrit par (3) provient exclusivement de la queue lourde de la loi des temps d'attente Φ , et non pas de la redistribution spatiale ω . Il est donc utile de commencer par étudier une version plus simple du problème (3) intégrée en espace. Il s'agit de l'équation de renouvellement suivante :

$$\begin{cases} \partial_t n(t, a) + \partial_a n(t, a) + \beta(a)n(t, a) = 0, & t \geq 0, \quad a > 0 \\ n(t, a = 0) = \int_0^\infty \beta(a') \int_{\mathbb{R}} n(t, a') da' \\ n(t = 0, a) = n^0(a). \end{cases} \quad (6)$$

Le premier Chapitre de cette thèse suit l'article publié en collaboration avec Thomas Lepoutre et Hugues Berry [15]. Nous y étudions la décroissance autosimilaire des solutions de l'équation précédente. Nous y prouvons, sous des hypothèses raisonnables, que toute solution de (6) décroît suivant un profil autosimilaire.

Théorème 1 (Berry, Lepoutre, Mateos González).

Soient n^0 à support compact dans $[0, 1)$ et $\beta(a) = \frac{\mu}{1+a}$. Soit

$$N_\infty(t, a) = \begin{cases} \frac{c_\infty}{a^\mu(1+t-a)^{1-\mu}}, & a < 1+t, \\ 0, & a > 1+t. \end{cases}$$

Alors, si $\mu \neq 1/2$, il existe $K > 0$ tel que

$$\|n(t, \cdot) - N_\infty(t, \cdot)\|_1 \leq \frac{K}{(1+t)^\mu} + \frac{K}{(1+t)^{1-\mu}},$$

et si $\mu = 1/2$, il existe $K > 0$ tel que

$$\|n(t, \cdot) - N_\infty(t, \cdot)\|_1 \leq \frac{K(1 + \log(1 + t))}{\sqrt{1 + t}}.$$

La preuve du théorème précédent repose sur un résultat potentiellement plus fort. Il s'agit de la définition du pseudo-équilibre W en Définition 2 et du théorème de convergence 4. Ces nouveaux résultats ont une certaine stabilité par rapport à la perturbation du taux de saut.

Le résultat en question, le Théorème 4, s'énonce plus clairement dans les variables autosimilaires que voici.

$$\begin{cases} \tau = \ln(1 + t), \\ b = \frac{a}{1 + t}. \end{cases} \quad (7)$$

Nous définissons

$$n(t, a) = \frac{1}{1 + t} w(\tau, b). \quad (8)$$

Le problème 6 devient, en variables autosimilaires :

$$\begin{cases} \partial_\tau w + \partial_b((1 - b)w) + e^\tau \beta(e^\tau b)w = 0 \\ w(\tau, 0) = \int_0^1 e^\tau \beta(e^\tau b)w(\tau, b)db \\ w(0, b) = w^0(b). \end{cases} \quad (9)$$

Cette équation non autonome admet un état stationnaire qui en satisfait formellement la limite quand $\tau \rightarrow \infty$:

$$W_\infty(b) = \frac{c_\infty}{b^\mu(1 - b)^{1-\mu}}. \quad (10)$$

Il s'agit de la distribution de Dynkin-Lamperti, ou la loi de l'arc-sinus. La constante de renormalisation c_∞ est définie de sorte que $\|W_\infty\|_1 = 1$. Sous certaines hypothèses, nous pouvons nous attendre à la convergence de $w(\tau, b)$ vers cette distribution.

Le problème en variables autosimilaires (9) n'admet pas d'état stationnaire intégrable. Cela rend impossible l'application directe de méthodes d'entropie relative pour montrer la convergence en entropie de w vers W_∞ . Cela nous mène à introduire une fonction pivot, le pseudo-équilibre $W : (\tau, b) \in \mathbb{R}_+ \times [0, 1) \mapsto W(\tau, b) \in \mathbb{R}_+$, tel que $W(\tau, \cdot)$ converge en norme L^1 vers W_∞ quand $\tau \rightarrow \infty$ avec un taux similaire à celui où $w(\tau, \cdot)$ converge vers $W(\tau, \cdot)$ in L^1 quand τ tend vers l'infini.

Définition 2. Soit W le pseudo-équilibre défini sur $\mathbb{R}_+ \times [0, 1)$ comme suit :

$$W(\tau, b) = \frac{C(\tau)}{e^{B(e^\tau b)}(1 - b)^{1-\mu}}, \quad (11)$$

où $B(a) = \int_0^a \beta(s)ds$ et C est défini de sorte à ce que $\|W(\tau, \cdot)\|_{L^1} = 1$.

En particulier, dans le cas de référence $\beta(a) = \frac{\mu}{1+a}$ – dans lequel nous nous plaçons par la suite –, le pseudo-équilibre prend la forme :

$$W(\tau, b) = \frac{C(\tau)e^{-\mu\tau}}{(e^{-\tau} + b)^\mu(1 - b)^{1-\mu}}.$$

Proposition 3. *Le pseudo-équilibre W converge en norme L^1 vers l'état stationnaire W_∞ avec le taux quantitatif suivant :*

$$\|W(\tau, \cdot) - W_\infty\|_1 \leq K e^{(\mu-1)\tau}, \quad (12)$$

où K est une constante ne dépendant que de μ et de w^0 .

Par une stratégie inspirée de méthodes d'entropie relative [111, 128], nous prouvons une version généralisée du théorème suivant :

Théorème 4 (Berry, Lepoutre, Mateos González).

$$\|w(\tau, \cdot) - W(\tau, \cdot)\|_1 \leq \begin{cases} K(e^{-\mu\tau} + e^{-(1-\mu)\tau}), & \text{if } \mu \neq 1/2 \\ K\tau e^{-\tau/2}, & \text{if } \mu = 1/2. \end{cases} \quad (13)$$

Des simulations numériques suggèrent que le taux réel de convergence d'une solution vers W est meilleur que le taux de convergence de W vers W_∞ . Cela est raisonnable étant donné que la décroissance exponentielle en $e^{-\mu\tau}$ est intrinsèque, au sens où c'est le taux auquel deux solutions se rapprochent ; alors que la décroissance exponentielle en $e^{(\mu-1)\tau}$ est liée à une correction.

Notre contribution principale dans ce travail, à part l'obtention du taux de convergence en soi, est l'utilisation de techniques inspirées d'entropie relative ainsi que l'utilisation du pseudo-équilibre W dans la preuve.

Limite hyperbolique et équation de Hamilton-Jacobi

Les résultats présentés au second Chapitre ont été établis en collaboration avec Vincent Calvez et Pierre Gabriel dans le preprint [28]. Nous étudions la limite hyperbolique de l'équation (3) pour ω distribution Gaussienne et $\beta = \frac{\mu}{1+a}$. Nous prouvons un résultat de stabilité : les solutions des problèmes rééchelonnés à $\varepsilon > 0$ convergent lorsque $\varepsilon \rightarrow 0$ vers la solution de viscosité de l'équation de Hamilton-Jacobi limite des problèmes à $\varepsilon > 0$.

Considérons donc l'équation (3) après rééchelonnement hyperbolique :

$$(t, x, a) \longrightarrow (t/\varepsilon, x/\varepsilon, a)$$

et appliquons une transformation de Hopf-Cole à ce problème :

$$n_\varepsilon(t, x, a) = n(t/\varepsilon, x/\varepsilon, a) = \exp(-\phi_\varepsilon(t, x, a)/\varepsilon). \quad (14)$$

L'étude de ϕ_ε nous permet de quantifier précisément le comportement des queues exponentielles de la fonction densité de probabilité n d'une façon inspirée de la théorie des grandes déviations.

Pour (t, x, a) tel que $\phi_\varepsilon(t, x, a) < \infty$, la fonction ϕ_ε satisfait :

$$\begin{cases} \partial_t \phi_\varepsilon + \frac{1}{\varepsilon} \partial_a \phi_\varepsilon - \beta = 0, & t \geq 0, \quad a > 0, \quad x \in \mathbb{R} \\ \exp(-\phi_\varepsilon(t, x, 0)/\varepsilon) = \int_0^{1+t/\varepsilon} \int_{\mathbb{R}} \beta(a) \omega(z) \exp(-\phi_\varepsilon(t, x - \varepsilon z, a)/\varepsilon) dz da \\ \phi_\varepsilon(0, x, a) = \phi_\varepsilon^0(x, a) = -\varepsilon \ln(n^0(x/\varepsilon, a)). \end{cases} \quad (15)$$

Nommons ψ_ε la condition de bord à $a = 0$. Elle sera notre inconnue principale :

$$\psi_\varepsilon(t, x) = \phi_\varepsilon(t, x, 0). \quad (16)$$

Une idée intéressante dans cet article est celle de démontrer la convergence vers l'équation de Hamilton-Jacobi limite directement pour la condition de bord de notre équation au lieu d'utiliser les fonctions test perturbées introduites par Evans [48] sur la fonction ϕ_ε , qui serait amenée à perdre sa dépendance en âge à la limite.

Nous exprimons la solution de (15) le long des caractéristiques :

$$\phi_\varepsilon(t, x, a) = \begin{cases} \psi_\varepsilon(t - \varepsilon a, x) + \varepsilon \int_0^a \beta(s) ds, & t > 0, \varepsilon a < t \\ \phi_\varepsilon^0(x, a - t/\varepsilon) + \varepsilon \int_{a-t/\varepsilon}^a \beta(s) ds, & t \geq 0, a \geq t/\varepsilon. \end{cases} \quad (17)$$

L'injection de (17) dans la condition de bord à $a = 0$ satisfaite par ϕ_ε dans (15) nous donne :

$$1 = \overbrace{\int_0^{t/\varepsilon} \Phi(a) \int_{\mathbb{R}} \omega(z) \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \psi_\varepsilon(t - \varepsilon a, x - \varepsilon z)]\right) dz da}^{\mathfrak{A}_\varepsilon} + \underbrace{\int_{t/\varepsilon}^{1+t/\varepsilon} \Phi(a) \int_{\mathbb{R}} \omega(z) \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \phi_\varepsilon^0(x - \varepsilon z, a - t/\varepsilon)] + \int_0^{a-t/\varepsilon} \beta\right) dz da}_{\mathfrak{B}_\varepsilon}, \quad (18)$$

où Φ est la distribution des temps d'attente introduite en 4.

La limite formelle de (18) lorsque $\varepsilon \rightarrow 0$ est la suivante :

$$1 = \int_0^\infty \Phi(a) \exp(a \partial_t \psi_0(t, x)) da \int_{\mathbb{R}} \omega(z) \exp(z \partial_x \psi_0(t, x)) dz. \quad (19)$$

Il s'agit d'une équation de Hamilton-Jacobi, équivalente à :

$$\partial_t \psi_0(t, x) + H(\partial_x \psi_0)(t, x) = 0, \quad (20)$$

où H est défini ci-dessous, avec $\hat{\Phi}^{-1}$ la fonction inverse de la transformée de Laplace de Φ :

$$H(p) = \hat{\Phi}^{-1}\left(\frac{1}{\int_{\mathbb{R}} \omega(z) \exp(zp) dz}\right). \quad (21)$$

Le résultat principal du second Chapitre de ma thèse est la preuve de la convergence lorsque ε tend vers 0 de la fonction ψ_ε vers l'unique solution de viscosité ψ_0 de l'équation de Hamilton-Jacobi limite (19), sous les hypothèses raisonnables sur la condition initiale suivantes.

Hypothèse 1 (Condition initiale ϕ_ε^0).

Nous faisons l'Ansatz de type WKB suivant :

$$\phi_\varepsilon^0(x, a) = v_\varepsilon(x) + \varepsilon \eta_\varepsilon(x, a) + \chi_{[0,1)}(a) \quad (22)$$

où $\chi_{[0,1)}(a)$ vaut 0 pour $a \in [0, 1)$ et ∞ pour $a \geq 1$. Le support essentiel de ϕ_ε^0 est donc $[0, 1)$, en accord avec le choix de n^0 . Les fonctions ϕ_ε^0 , v_ε et η_ε satisfont les propriétés suivantes uniformément en ε :

1. v_ε et ϕ_ε^0 sont **Lipschitz** en x uniformément en a .
2. v_ε et ϕ_ε^0 sont **sous-linéaires et bornées en-dessous** : il existe des constantes positives K_0, K_1 telles que pour tout $x \in \mathbb{R}$ et $a \in [0, 1)$, pour tout $\varepsilon > 0$,

$$\begin{cases} -K_0 \leq v_\varepsilon(x) \leq K_0 + K_1|x|, \\ -K_0 \leq \phi_\varepsilon^0(x, a) \leq K_0 + K_1|x|. \end{cases} \quad (23)$$

3. ϕ_ε^0 et v_ε sont **semi-concaves en x uniformément en a** : il existe $\mathfrak{C}_{xx} \in \mathbb{R}$ telle que pour tous $x, h \in \mathbb{R}$ et $a \in [0, 1)$, pour tout $\varepsilon > 0$,

$$\begin{cases} \phi_\varepsilon^0(x+h, a) + \phi_\varepsilon^0(x-h, a) - 2\phi_\varepsilon^0(x, a) \leq \mathfrak{C}_{xx}h^2, \\ v_\varepsilon(x+h) + v_\varepsilon(x-h) - 2v_\varepsilon(x) \leq \mathfrak{C}_{xx}h^2. \end{cases} \quad (24)$$

(Ou de façon équivalente, au sens des distributions, $\partial_x^2 \phi_\varepsilon^0 \leq \mathfrak{C}_{xx}$ et $\partial_x^2 v_\varepsilon \leq \mathfrak{C}_{xx}$.)

4. Les **convergences** suivantes ont lieu, localement uniformément en x :

$$\begin{cases} v_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} v, \\ \varepsilon \eta_\varepsilon(x, a) \xrightarrow{\varepsilon \rightarrow 0} 0. \end{cases} \quad (25)$$

Théorème 5 (Calvez, Gabriel, Mateos González). *Sous les Hypothèses 1, la fonction ψ_ε converge dans L_{loc}^∞ vers ψ_0 , l'unique solution de viscosité de l'équation de Hamilton-Jacobi limite (19) avec pour condition initiale v .*

La stratégie de preuve est la “recette” classique présentée par Barles dans [11, Chapter 2] pour des résultats de stabilité. Il s’agit de montrer des estimations a priori : ψ_ε est localement bornée (dans notre cas, bornée inférieurement et sous-linéaire) et Lipschitz. L’ensemble $(\psi_\varepsilon)_\varepsilon$ est donc compact dans $\mathcal{C}(K)$ pour tout compact K grâce au théorème d’Arzela-Ascoli. À extraction diagonale près, ψ_ε converge vers une fonction ψ_0 . On utilise le fait que ψ_ε est sous- et sur-solution pour prouver que ψ_0 est solution de viscosité. Un résultat d’unicité permet de conclure que toute la suite ψ_ε tend vers l’unique solution de viscosité ψ_0 de l’équation de Hamilton-Jacobi limite.

Malgré l’aspect classique de la preuve, nous y avons rencontré et résolu deux difficultés intéressantes. Je les présente ci-dessous, accompagnées d’heuristiques tirées de la décroissance autosimilaire du problème en espace homogène [15].

Absence d’état stationnaire intégrable

Le problème homogène en espace (6) n’admet pas d’état stationnaire intégrable. Il en résulte que le principe du maximum ne peut pas être appliqué directement pour prouver des bornes sur ψ_ε .

Cependant, des corrections dépendantes du temps nous permettent de prouver les bornes souhaitées. Ces corrections étaient attendues, grâce à l’heuristique suivante. Considérons le problème en espace homogène (6), et, suivant le résultat de décroissance autosimilaire de la Proposition 3 et du Théorème 4, considérons le profil limite $W_\infty(b) = c_\infty b^{-\mu}(1-b)^{\mu-1}$. Nous nous attendons intuitivement à avoir :

$$n(t, a) \simeq \frac{1}{1+t} W_\infty\left(\frac{a}{1+t}\right). \quad (26)$$

Ceci se traduit dans notre contexte en une correction logarithmique en temps de la borne L^∞ de $-\varepsilon \ln(n)$, que l’on retrouve dans le second Chapitre de cette thèse. L’obtention de ces termes correcteurs est la première difficulté abordée dans le second Chapitre.

Contrôle en temps long des sauts issus de la condition initiale

La stratégie de preuve du Théorème 5 conclut sur une procédure de limite de viscosité. Montrer que la limite ψ_0 d’une suite extraite de ψ_ε est une sous-solution de viscosité de l’équation de Hamilton-Jacobi limite (18) est relativement simple. Cependant, la preuve que ψ_0 est une sur-solution est plus complexe.

La preuve requiert de contrôler finement la proportion de protéines – \mathfrak{B}_ε dans la formulation (18) – ayant subi un saut au temps t mais qui n’avait jamais sauté avant. Autrement dit, l’enjeu est de montrer que l’influence de la condition initiale ϕ_ε^0 sur le terme de bord ψ_ε décroît. Or la queue lourde de la distribution Φ des temps d’attente mène à une décroissance très lente de \mathfrak{B}_ε .

En montrant que la semi-concavité de la condition initiale est préservée et en appliquant le principe du maximum de façon ingénieuse, nous récupérons une l’estimation fine localement uniforme en temps suivante :

$$\mathfrak{B}_\varepsilon \lesssim \varepsilon^\mu. \tag{27}$$

Encore une fois, une heuristique utilisant la décroissance autosimilaire prouvée dans le Chapitre 1 mène à un taux semblable.

L’obtention de ce type de raffinement non-local de la borne Lipschitz en temps de ψ_ε est le troisième aspect novateur du Chapitre 2.

Introduction

Motivation and framework

This dissertation treats the asymptotic analysis of age-structured partial differential equations stemming from cellular biology. The main mathematical tools used are inspired by relative entropy methods and Hamilton-Jacobi techniques.

The biological motivation for such equations is the analysis of random motion of proteins inside the cytoplasm of a biological cell in cases when crowding and trapping phenomena cause such motion to deviate from standard diffusion.

I will begin by presenting the modelling, biological and mathematical contexts relevant to my work, and motivate why the study of anomalous diffusion is a current topic of interest in biomathematics.

1 A historical introduction to the modelling of random motion

Let us first introduce “normal” diffusion from a historical point of view, so as to give some insight into the richness of links between different scales of modelling, and so as to explain the reason anomalous diffusion bears that name.

The main message of this Section consists of the following points. Random motion is a very common phenomenon, whose most “immediate” intuitive description is at the microscopic level. The mesoscopic equations of kinetic theory link the microscopic and macroscopic descriptions in a rich way. Mean squared displacement is a key observable magnitude which allows a certain characterisation of random motions. The type of diffusion presented here is “normal” because it is the simplest: memory-less random motion. It is associated to a Gaussian random walk. The reader can find more refined historical introductions at the beginning of the following articles [108, 75, 109], by which this presentation is heavily inspired.

First observations of random motion Many macroscopic systems, from inanimate solid particles in a fluid to animals populations, exhibit some form of random motion. Consequently, the observation – without a proper scientific explanation – of random motion can be traced as far back as one may wish [103, verses 80 – 141].

In a scientifically reproducible context however, the first relevant observations occur in the late 18th and 19th century. In 1785, Jan Ingenhousz presents at the very beginning of [86] a chapter titled *Observations sur l'usage du microscope*, in which he describes how particles of fine charcoal move randomly in a water or alcohol microscope preparation due to the evaporation of the liquid. He states that this is an uninteresting technical issue that can be solved by covering the preparation with a small film of glass. However, he does remark that the effect occurs for both living and inanimate particles, which is one of the key points of Brown's 1828 and

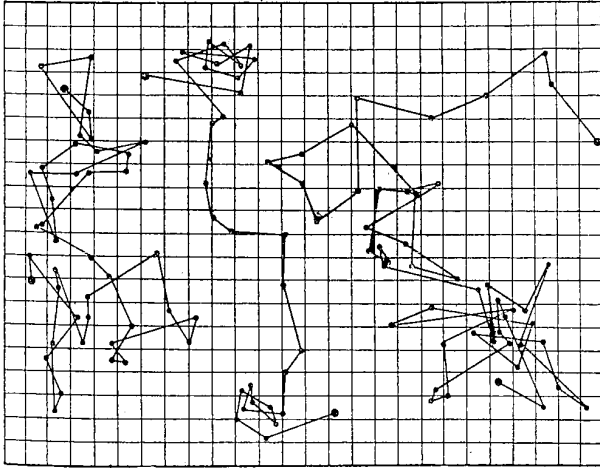


Fig. 7.

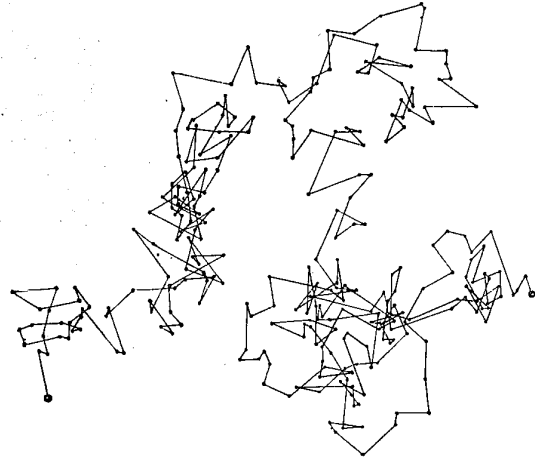


Fig. 8.

Figure 1: Figures 7 and 8 of Perrin’s book [127] describing “the positions occupied by a grain after successive equal time intervals.” In [the left figure] three diagrams are shown, the scale being such that sixteen divisions represent 50 microns. These diagrams were obtained by tracing the horizontal projections of the lines joining consecutive positions occupied by the same mastic grain (radius equal to $.53\mu m$); the positions were marked every 30 seconds.” (Excerpt from the English translation [126].) The right diagram represents “a large number of displacements traced on an arbitrary scale”.

1829 papers [160, 161]. The crucial contribution of Brown is to experimentally refute known mechanistic causes of the motion, and posit the existence of molecules as a possible explanation.

The first systematic recording of individual trajectories of particles can be credited to Jean Perrin, who observed small granules in emulsions – see Figure 1.

Mathematical and physical description Close to the time of publication of Brown’s cited articles, Fourier introduced the heat equation in his 1822 book [64], as a description of heat transfer dynamics:

$$\partial_t u(t, x) = D \Delta_x u(t, x), \quad t \in \mathbb{R}_+, \quad x \in \Omega \subset \mathbb{R}^d. \quad (1)$$

Here, u is a density function depending on the time variable t and the space variable $x \in \Omega$, an open subset of \mathbb{R}^d . However, the first sentence in Fourier’s prologue states that the fundamental causes of heat are unknown¹. This only became a model of diffusion with Fick’s derivation of the heat equation from his diffusion laws [60], thus interpreting u as a density function.

The link between heat transfers and microscopic kinetic energy studied in kinetic theory was introduced in 1857 by Clausius [32]: “On the type of motion that we call heat”. The subsequent works of Maxwell and Boltzmann led to an evolution equation for the velocity probability distribution function for a gas in equilibrium: the Boltzmann equation [19]. It is also in that article that Boltzmann introduced his H -Theorem, to attempt to explain the irreversibility of gaz dynamics by showing that collisions tend to increase entropy. The study of rigorous mean-field limits of

¹“Les causes primordiales ne nous sont point connues ; mais elles sont assujetties à des lois simples et constantes, que l’on peut découvrir par l’observation, et dont l’étude est l’objet de la philosophie naturelle.” [64] – “The original causes [of heat] are unknown to us ; but they are subject to simple and constant laws that we can discover by observation, and whose study is the object of natural philosophy.”

kinetic equations has provided many interesting mathematical problems (see for example [4]) and is a current research topic. We refer the reader to [30] for a historical introduction to kinetic theory, and provide some additional references in the modelling section 3.3.

The Boltzmann equation provides the basis for the random walk approach to Brownian motion, as Einstein writes ² in his 1905 paper [46]. In it, Einstein links the previous “microscopic collisions” description of Brownian molecular motion to the “macroscopic” diffusion description by using kinetic theory and introducing the mean squared displacement. By interpreting the heat equation as describing the evolution of the probability density function of a single particle rather than the density distribution of a substance, Einstein derives an expression of the mean squared displacement which allows the experimental determination of Avogadro’s number (number of molecules in a mole).

Let us define the ensemble-averaged mean squared displacement for microscopic and macroscopic models.

Definition 1 (Microscopic Mean Squared Displacement (MSD)). Consider N particles at respective positions $x_1(t), \dots, x_N(t)$ at time t . Their macroscopic ensemble-averaged mean squared displacement is:

$$\langle x(t)^2 \rangle = \frac{1}{N} \sum_{n=1}^N (x_n(t) - x_n(0))^2. \quad (2)$$

The solution of (1) for a particle initially at $x = 0$ at time $t = 0$ in unbounded space is

$$u(t, x) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right). \quad (3)$$

Interpreting u as a probability density function allows us to give a macroscopic definition of the ensemble-averaged mean squared displacement as the second moment of u :

$$\langle x(t)^2 \rangle_M = \int_{\mathbb{R}^d} x^2 u(t, x) dx. \quad (4)$$

This immediately gives us $\langle x(t)^2 \rangle_M = 2Dt$.

The works of Einstein [46, 45] and Smoluchowski [159] provide the physical and mathematical foundations of Brownian motion, linking the two previous definitions and deriving an expression of Avogadro’s number from the diffusion coefficient D in the second one. We refer to [134] for extensive historical details and to [98] for a course on stochastic processes and Brownian motion. Einstein and Smoluchowski rely on three assumptions [109]: the independence of individual particles, the existence of a time scale below which individual displacements are statistically independent, and the property that particle displacements below that time scale correspond to a typical mean free path in a homogenous and isotropic space. These hypotheses allow the derivation of the heat equation (1) from a random walk by means of the central limit theorem.

In his book *Les atomes* [127, 126], from which Figures 1 and 2 are taken, Perrin gives an experimental validation of the previous work. He observes that mean squared displacement scales linearly with time ². The random walk he observes provides a link between the microscopic

²“In dieser Arbeit soll gezeigt werden, daß nach der molekularkinetischen Theorie der Wärme in Flüssigkeiten suspendierte Körper von mikroskopisch sichtbarer Größe infolge der Molekularbewegung der Wärme Bewegungen von solcher Größe ausführen müssen, daß diese Bewegungen leicht mit dem Mikroskop nachgewiesen werden können.” [46] – “In this work we show, by use of the kinetic theory of heat, that microscopic particles which are suspended in fluids undergo movements of such size that these can be easily detected with a microscope.” – freely translated in [75].

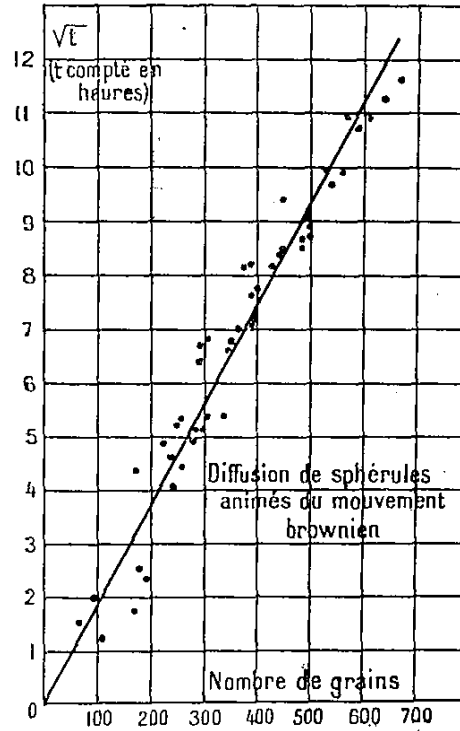


Fig. 10.

Figure 2: Figure 10 of Perrin’s book [127]. Consider an emulsion of gamboge granules in a medium limited by a glass wall to which the granules remain stuck upon contact. The figure represents the square root of the times at which a given number of gamboge granules are stuck to the glass. The squared root of time increases linearly with respect to that number.

dynamics of collisions of molecules with a larger particle, and macroscopic observables such as the diffusion coefficient and the Avogadro number, for which he gives a very accurate experimental estimate (for the time).

Deviation from diffusion However, from 1926 onwards, certain processes were observed whose mean squared displacements do not scale linearly with time. The first of these was turbulent diffusion [136], by Richardson. He observed the relative diffusion of two particles in a turbulent flow could be better described by the following equation than by (1):

$$\partial_t \rho(t, x) = D \partial_x \left(x^{4/3} \partial_x \rho(t, x) \right),$$

which leads to a mean squared displacement with a power-law scaling:

$$\langle x^2(t) \rangle_M \simeq t^3.$$

The following Section will introduce anomalous diffusion as context for my work.

2 Subdiffusion in crowded cellular environments

We wish to focus this Section – as we do throughout the dissertation – on a specific drawback of using normal diffusion to model certain intracellular random motion processes. Namely, that the

macroscopic ensemble-averaged mean squared displacement of Definition 1 does not match that predicted by the model, however we may fit the modelling constants in a way that is consistent with the Definition hereafter. Hence, the observed motion is not standard diffusion.

Definition 2 (Anomalous diffusion). Consider a set of particles undergoing random motion whose mean squared displacement scales as t^μ for some positive μ :

$$\langle x(t)^2 \rangle \simeq t^\mu. \quad (5)$$

1. If $0 < \mu < 1$, the particles are undergoing subdiffusion of exponent μ .
2. If $\mu = 1$, the particles are undergoing diffusion.
3. If $\mu > 1$, the particles are undergoing superdiffusion of exponent μ .
If $\mu = 2$, that superdiffusion is ballistic transport.

This dissertation consists in the study of mathematical tools that describe subdiffusive processes, hence our focus on subdiffusion. We note that the scaling of the mean squared displacement does not fully characterise the type of anomalous motion. The reader may find different subdiffusive processes that share the same MSD scaling in [109, 14] and in [74, Ch. 11].

2.1 Statistical differences between diffusion and subdiffusion

The first remarkable feature of systems exhibiting anomalous diffusion – a consequence of Definition 2 – is that they do not satisfy the Gaussian limit behaviour which is a consequence of the Central Limit Theorem (CLT). It follows that models describing such systems must not satisfy the hypotheses of the CLT. We will expand on this comment in the Modelling sections 3.1 and 3.3.

Several important macroscopic differences between diffusion and subdiffusion ensue, including the following. The fundamental solution of the heat equation is a Gaussian distribution that decays in time. A first order expansion in the Fourier-Laplace space allows the authors of [108] to recover an approximation of the corresponding profile in a subdiffusive problem, whose shape is different as illustrated in Figure 3. Two other macroscopic differences include the dynamics of concentration in inhomogeneous media [55], and differences in front propagation between reaction-diffusion and reaction-subdiffusion systems [158, 119]. The mathematical analysis of such problems is a logical continuation of this thesis, which we sketch in Sections 6.1 and 6.3 respectively.

We have defined subdiffusion as motion whose mean squared displacement is proportional to t^μ . However, that does not identify the process responsible for causing the subdiffusion. The reader may find in [148] an analysis of additional macroscopic statistical indicators of the nature of a random motion useful in the identification of normal diffusion and anomalous dynamics such as diffusion on fractals, continuous time random walks, and fractional Brownian motion. Specifically, the authors propose to use ratios of fourth moment versus the square of the second moment for normal moment statistics and mean maximal excursions statistics.

2.2 Biological instances of anomalous diffusion

Before discussing the biological instances of subdiffusion, we refer the reader to the classical reference [34] for an introduction to cell biology and to [85] for an introduction to mathematical modelling of the cell. The reader may also consult [121], which presents elements of modelling

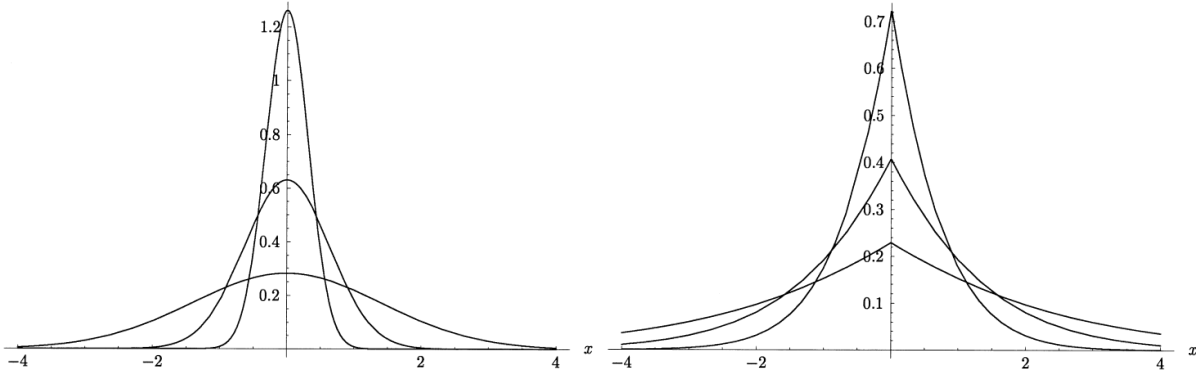


Figure 3: This figure is taken from [108, Fig. 6, 7]. The left hand panel depicts the fundamental solution of the heat equation at times 0.05, 0.2 and 1. The right hand side panel depicts the propagator for subdiffusion with anomalous exponent $1/2$, drawn at times 0.1, 1 and 10. The cusp shapes of the probability density functions differ.

in cell biology, a review of experimental detection techniques for random motion, and biological descriptions of anomalous diffusion.

The occurrence of subdiffusive motion in cellular biology is illustrated in Figure 4, which is taken from [87]. We note that superdiffusive motion also appears in intra-cellular processes involving some type of active transport, *e.g.* in active transport along microtubules [135]. Such studies have taken advantage of recent advances in microscopy that give biologists access to data that becomes progressively finer and more precise. For instance, Figure 4 was made possible thanks to the development of tracking of individual particles inside living cells. The reader may consult a review of experimental techniques in [121, Sections 3 and 4].

In addition to the two previous examples, anomalous random motion occurs in a wide variety of settings. We focus on subdiffusion hereafter, and refer to [51, 1.3] and the bibliography therein for a synthetic presentation of the occurrence of subdiffusion in dendritic spines, cell membranes and morphogenesis.

The reader can find further examples of subdiffusive motion in the references presented hereafter. The article [83] concerns carrier transport in amorphous materials, diffusion in polymers, turbulent systems, flow through porous media, and movement of proteins on cell membranes and inside cells. Concerning Kinetics of protein/DNA interactions, observed through individual mRNA molecule tracking by fluorescence in *E. coli*, the reader can consult [69]. The reader can find an example of the transient subdiffusion of certain telomeres in short time-scales of the order of 10^{-2} to 10^3 seconds in [25]. It is not uncommon for certain subdiffusive processes to correspond to a transient phase between two diffusive phases [146, 14, 87]. Concerning subdiffusion of proteins and lipids in plasma membranes we refer to [141]. Moreover, cells themselves may experience subdiffusive motion [54].

We refer the reader to [107, Ch. 2.3] as well as to the reviews [108, 80] and the bibliography therein for a much more complete landscape of the prevalence of subdiffusion, including examples in porous systems, amorphous semiconductors, subsurface hydrology, proteins and lipids in plasma membranes, mRNA molecules in *Escherichia coli* cells, proteins in the nucleus, etc.

The difference between diffusion and subdiffusion can have clear biological implications in certain situations. For instance, regarding morphogen gradient formation, in [169, 168], the authors describe a model in which morphogens diffuse or subdiffuse from one end of a cell. They

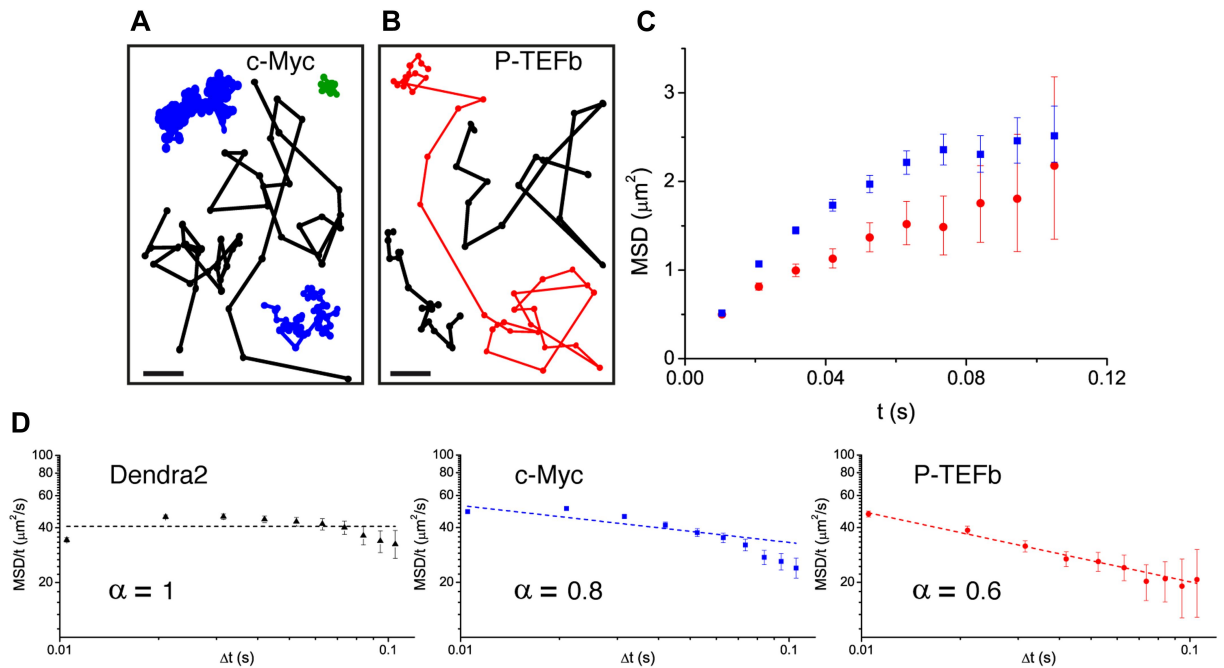


Figure 4: This figure is taken from Izeddin et al. [87]. It describes the intracellular motion of three proteins in the nucleoplasm of a U2OS cell. The mean squared displacement of Dendra-2 is proportional to t , as is the case in diffusive motion. However, c-Myc and P-TEFb exhibit mean squared displacements proportional to t^α , for $\alpha < 1$: their motion, in the long term, is slower than any positive diffusion. These proteins experience subdiffusion.



Figure 5: This figure is taken from [164, Fig. 3 B]. It is a reconstruction based on stimulated emission depletion (STED) [165] imaging of endocytose-related proteins present at a presynaptic (rat) neuron connection. The vertical scale is 200 nm. Around 60 different types of molecules are present. The reader can find a video of presynaptic organisation among the online supplementary materials of the article.

conclude that if the morphogens decay with spatially-dependent rates, the equilibrium concentrations in the diffusive and subdiffusive cases are significantly different. And in what concerns the search of a protein for a nearby target, simulations made in [71] show that subdiffusive motion can achieve faster results than diffusive-mediated searching.

There are different proposed causes of subdiffusion, each leading to a specific type of subdiffusion. They stem from the crowded, heterogeneous and structured nature of cellular media, and the persistence that they induce on random motion taking place in them.

A proposed cause of subdiffusive behaviour in biological media is that of trapping phenomena (*i.e.* binding to immobile traps) [78, 138, 137, 104] resulting from the high heterogeneity and very dense crowding that such media often present [162], [74, Ch. 7.1.1 and 11]. Quantitatively, cell cytoplasm bears densities of macromolecules such as proteins, DNA, RNA, and polysaccharides of the order of hundreds of grams per litre, up to 400 g/L [47]. We note that this highly exceeds the usual concentrations in *in vitro* experiments [9], hence the need for *in vivo* experiments. We refer the reader to [121] for a very complete review of experimental techniques for *in vivo* single molecule tracking. Figure 5 provides a striking illustration of presynaptic neuron crowding. A similar situation is described in [150].

Other proposed causes are linked to the fact that cellular media are structured. Hence, the random motion of larger organelles or molecules can be obstructed by smaller structural elements (see [74, Ch. 7.1.1] and the bibliography therein). This leads to obstructed diffusion.

The glassy properties of the cytoplasm [125] can also play a role. Citing yet again [74, Ch. 7.1.1], the compression of a macromolecule by another one passing close-by induces an elastic

response of the squeezed molecule: hence a memory, and an anomalous diffusion governed by a fractional Brownian motion in this case.

Different causes and models of subdiffusion have different consequences on the coupling of subdiffusion with reaction [146]: locally slowed-down Brownian motion and local hindrance by obstacles improve the apparent affinity of the reaction whereas trapping decreases it.

Since the models we study correspond to continuous time random walks, as we will present in Section 3, the biological phenomena to whose comprehension our work might contribute the most are traps.

We refer the reader to [107, 37] for a thorough introduction to different mathematical models of random motion and tools to analyse them. We introduce those that are relevant to this dissertation in the following Section.

3 Subdiffusion at three modelling scales

In the historical introduction 1 we have seen that describing the same problem at different levels and studying the connections between these different models has been a fecund endeavour. We wish to formalise, in the context of our problem, the nature of these different levels of modelling and of the connections they bear to each other: in a nutshell, microscopic, mesoscopic and macroscopic models. In order to do so let us first present certain reasons linked to experimental technology and modelling strategy that impact the type of model one may prefer in certain situations.

We now present three different mathematical modelling scales, namely microscopic, mesoscopic and macroscopic. We focus on the descriptions of subdiffusive motion at each scale and the links between them. The equations studied in this thesis are presented in the mesoscopic scale section 3.3.

There are more ways of modelling anomalous diffusion than those I present hereafter, such as fractional Brownian motion, generalised diffusion equations, (generalised) Langevin equations, or generalised master equations. The reader may find a detailed description of these models and their differences in the review [109]. Models that share the same evolution of the mean squared displacement can stem from different stochastic mechanisms. For instance, fractional Brownian motion and subdiffusive CTRW share the evolution of their MSDs, but the first is ergodic (read: the ensemble-averaged MSD that we have defined is equal to the time-averaged MSD of one walker) whereas the latter is not [109, Table 1].

A Leitmotiv through the following models is how to deal with the memory terms inherent to anomalous diffusion.

We do note that non-local in time terms do arise in other classes of macroscopic models, for instance in the context of asymptotic limits of transport processes in highly oscillating random media [131, 124, 101].

3.1 Microscopic model: Continuous Time Random Walks

A microscopic model is one that describes the individual behaviour and interactions of a system of discrete particles or agents. Each particle is characterised by its position in an appropriate vector space at a given time.

For instance, in the context of the kinetic theory of gases, one considers a finite system of N particles. Each particle is characterised by its position $X_i(t) \in \mathbb{R}^d$ and speed $V_i(t) \in \mathbb{R}^d$ at time $t \in \mathbb{R}_+$. Particles may be subjected to external forces F_i and interact due to collisions. Noise terms may also be taken into account.

These interactions can be described by a finite system of ordinary differential equations of the form:

$$\begin{cases} \frac{d}{dt}X_i(t) = V_i(t) \\ \frac{d}{dt}V_i(t) = F_i(t) + \text{collisions} + \text{noise} + \dots, \end{cases} \quad (6)$$

for $1 \leq i \leq N$.

The microscopic model for subdiffusion that we follow is of a different type: it is a continuous time random walk (CTRW), a concept introduced in 1965 by Montroll and Weiss [116], generalised by Scher and Montroll [139] in a seminal article which strongly impacted the understanding of photoconductivity in disordered and amorphous semiconductors. Other early important contributions to the development of CTRW include works by Shlesinger [142]. CTRW have played an important role in the description of transport in heterogeneous media. We refer to the extensive bibliography within [108, 2.3] for a detailed list of systems displaying anomalous random motion known before 2000 and to the more recent review [109] for generalisations, updated references, and a comparison of CTRW to other descriptions of anomalous transport.

The reader may find a comprehensive introduction to continuous time random walks in [108] and [107, Ch. 3.2]. The presentation of CTRW that follows is essentially taken from the previous references.

Continuous time random walks are a generalisation of random walks in which the space random variable $X(t)$, describing the position of the walker at time t , experiences discontinuous jumps of random destination $X(t) + Z_i$ at times T_i separated by random waiting times $\Theta_i = T_i - T_{i-1}$. The jumps Z_i are independent identically distributed (iid) random variables, and so are the waiting times Θ_i . In general, Z_i and Θ_i are dependent, and their statistical characteristics are determined by a joint probability density function (pdf) $\Psi(t, x)$. The spatial jump pdf is then given by $\omega(x) = \int_0^\infty \Psi(t, x) dt$, and the waiting time pdf by $\Phi(t) = \int_{\mathbb{R}^d} \Psi(t, x) dx$. However, throughout this dissertation, we will only consider independent jump and waiting time distributions, hence

$$\Psi(t, x) = \Phi(t)\omega(x).$$

Let us define the renewal process that counts the number of jumps that a random walker experiences up to time t :

$$N(t) = \max \{i \in \mathbb{N} \mid T_i \leq t\}. \quad (7)$$

The fact that N is a renewal process plays an important role in the results of Chapter 1. In the state of the art section 4.1, we will comment on results from probability and renewal theory previous and related to ours.

Let $X(0) = 0$. It follows from the definitions that the position X of the walker satisfies

$$X(t) = \sum_{i=1}^{N(t)} Z_i. \quad (8)$$

The process $X(t)$ is called a continuous time random walk. It is a semi-Markov process associated with the two-component Markov chain (X_i, T_i) . We refer the reader to [88] for an introduction to semi-Markov processes and to [52] for a study of semi-Markov time delay and jump processes.

Figure 6 gives a 2D illustration of a CTRW $X(t)$ and of the associated renewal process $N(t)$. Continuous time random walks can be characterised by the mean waiting time

$$\bar{T} = \int_0^\infty a\Phi(a) da \quad (9)$$

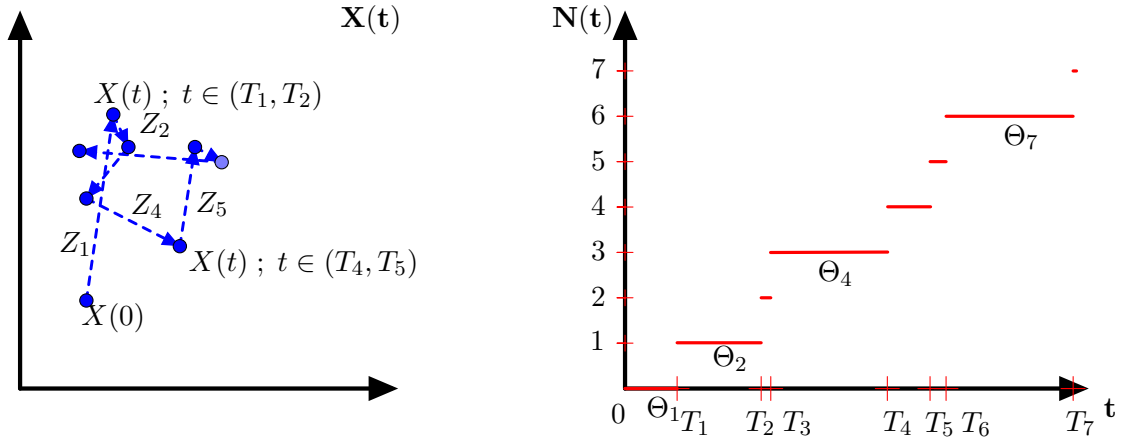


Figure 6: Illustration of a CTRW $X(t)$ and its associated renewal process $N(t)$.

and the variance of the jump destination pdf, which we have assumed to be even:

$$\sigma^2 = \int_{\mathbb{R}^d} z^2 \omega(z) dz. \quad (10)$$

Specifically, CTRW describe different types of motion depending on whether those quantities are finite or not. If \bar{T} and σ^2 are finite, the hypotheses of the central limit theorem are satisfied. This implies that the rescaled walker position $\sqrt{\varepsilon}X(t/\varepsilon)$ converges in distribution to the Brownian motion as $\varepsilon \rightarrow 0$. [107]:

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P} \left(\sqrt{\varepsilon}X \left(\frac{t}{\varepsilon} \right) \leq x \right) = \frac{1}{\sqrt{4\pi Dt}} \exp \left(-\frac{x^2}{4Dt} \right), \quad (11)$$

where $D = \sigma^2 / (2\bar{T})$.

If either \bar{T} or σ^2 is infinite, the central limit theorem does not apply. If \bar{T} is infinite but σ^2 is finite, the described process is asymptotically slower than diffusion. This is the setting of my dissertation, in which I will consider that Φ is a heavy-tailed power-law distribution of the form $\mu(1+a)^{-1-\mu}$, with $0 < \mu < 1$, and ω is a Gaussian distribution. That results in subdiffusive motion of exponent μ .

The cases when σ^2 is infinite due to heavy-tailed jump distributions correspond to Lévy flights [145, 110]. If \bar{T} is finite and σ^2 is infinite, the motion will be superdiffusive. However, if both \bar{T} and σ^2 are infinite due to Φ and ω being two power law distributions, there is a form of “competition” between the two power laws that can lead to “paradoxical diffusion”, as explained in [145] and the bibliography therein. For a detailed introduction to Lévy flights, we also refer the reader to the eponymous chapter of [132].

Continuous time random walks are stochastic processes that exhibit spatial discontinuities, also called “flights” in the biophysical community. This does not mean that the modelled system consist in particles that alternate absolute immobility and teleportation, but rather that such particles are observed at punctual times and can remain trapped close to the locus of observation for a certain time.

There is class of related models usually called “walks” or “runs” in which the discontinuity is not spatial but in velocity. In such models, a particle follows a straight path with a (constant or decaying) velocity during a random time, then changes direction and repeats the process. The

mesoscopic descriptions of such processes are kinetic run and tumble or Boltzmann equations. We will expand on this remark in the appropriate Section 3.3.

3.2 Macroscopic model: time-fractional derivative

A macroscopic model describes a system as a continuous medium characterised by magnitudes that are locally averaged over the microscopic population or set of particles considered. Such quantities can be local population density, mean velocity, temperature, pressure, etc. Examples of macroscopic equations include fluid mechanics equations such as Navier-Stokes, Euler, etc. In the context of mathematical biology, macroscopic models correspond to equations that describe a population by averaging it without accounting for individual deviations from the average, retaining only the time variable. Perthame's book [128] begins by presenting numerous examples of macroscopic population biology models (Fisher, Lotka-Volterra, chemostat equations and other ecology models, Hodgkin-Huxley and other models of neural networks, immunology models, SIR and other epidemiology models, etc.).

In the context of the modelling of subdiffusion, from the microscopic continuous time random walk model formulated in Section 3.1, it is possible to recover macroscopic equations governing the spatiotemporal dynamics of the density of random walkers located at position x at time t .

$$\partial_t \rho(x, t) = D_\mu \mathcal{D}_t^{1-\mu} \Delta \rho(x, t). \quad (12)$$

Here, $D_\mu \in \mathbb{R}_+$ is a generalised diffusion coefficient and $\mathcal{D}_t^{1-\mu}$ is the Riemann-Liouville fractional derivative operator given by

$$\mathcal{D}_t^{1-\mu}(f)(t) = \frac{1}{\Gamma(\mu)} \frac{d}{dt} \int_0^t \frac{f(t')}{(t-t')^{1-\mu}} dt'.$$

Such a fractional dynamics formulation is very attractive for modelling in biology, in particular because of its apparent similarity with the classical diffusion equation. However, contrary to the diffusion equation, the Riemann-Liouville operator is non-local in time. This translates the fact that what happens at a given time t depends on what has happened in the interval $[0, t)$: the underlying CTRW is non-Markovian for most waiting time distributions. Indeed, memory terms play a crucial role in subdiffusive processes. This non-Markovian property becomes a serious obstacle when one wants to couple subdiffusion with chemical reaction [79, 170, 56].

The derivation of the macroscopic fractional equation (12) from CTRW is done in [107, Ch. 2.3.1] as follows (the derivation we present is adapted from the reference). The authors consider a particle that follows a 1 space-dimensional CTRW, described in the previous section. The (non-local) evolution equation for the probability density $\rho(t, x)$ of that particle being at site x at time t given that it was at $x = 0$ at time 0 is given by the following Master equation:

$$\rho(t, x) = \delta(x) \left(1 - \int_0^t \Phi(t') dt' \right) + \int_0^\infty \int_{\mathbb{R}} \Psi(t-t', x-x') \rho(t', x') dx' dt'. \quad (13)$$

The authors denote the time Laplace transform of a function $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\hat{f}(s, x) = \int_0^\infty f(t, x) \exp(-st) dt$$

and the space Fourier transform of the same function f by

$$\tilde{f}(t, k) = \int_{\mathbb{R}} f(t, x) \exp(ikx) dx.$$

They recover, in Laplace-Fourier space:

$$\hat{\rho}(s, k) = \frac{1 - \hat{\Phi}(s)}{s} \frac{\tilde{\rho}_0(k)}{1 - \hat{\Psi}(s, k)}, \quad (14)$$

where $\tilde{\rho}_0(k) = 1$ is the Fourier transform of the initial condition $p_0(x) = \delta(x)$.

The authors then take first non-constant order approximations of $\hat{\Phi}$ and $\tilde{\omega}$ which allow them to simplify expression (14). If the second moment of ω and the mean of Φ are finite, the authors can then invert the Laplace and Fourier transforms and recover that ρ satisfies the diffusion equation.

Let us now consider that the second moment of ω is finite but the mean of Φ is infinite because Φ has a power-law tail of the form $\Phi(t) \sim_{\infty} t^{-(1+\mu)}$. The authors take again an approximation $\hat{\Phi}(s) \sim_{s=0} 1 - (\tau s)^\mu$, which allows them to simplify the expression (14) into:

$$\hat{\rho}(s, k) = \frac{\tilde{\rho}_0(k)}{s + D_\mu s^{1-\mu} k^2}. \quad (15)$$

They invert the expression above thanks to properties of the Riemann-Liouville fractional derivative and recover equation (12) for $0 < \mu < 1$.

We note that the above derivation is not rigorous and lacks certain justifications, such as a control of the error committed by truncating the expansions of $\hat{\Phi}$ and $\tilde{\omega}$. The fractional in time equation (12) belongs to a broader class of equations which also includes fractional in space equations [130, 172].

The macroscopic fractional model is prevalent in the anomalous diffusion literature – see [107, 80, 108, 110, 109, 169, 158, 119] and the references therein. However, it does present certain drawbacks. It is notably complicated to include reaction terms in the fractional formalism, since for anomalous dynamics, reaction and transport are coupled at the macroscopic level [166]. The authors of the cited article add the reaction terms into the mesoscopic non-local in time and space Master equation (13) and derive the macroscopic equation from it. This strongly suggests that the study of the intermediate mesoscopic description is in itself an interesting goal.

3.3 Mesoscopic model: age-structured equations

The mesoscopic scale is that at which the object of study is a probability density function $f(t, x, v) \geq 0$, where $t \in \mathbb{R}_+$ is time, $x \in \mathbb{R}^d$ is space, and $v \in \mathbb{R}^{d'}$ is a structural variable. It is the scale at which statistical physics operates: an intermediate between microscopic and macroscopic scales. In the context of kinetic equations, the structural variable v is a speed variable, and $d = d'$. The reader may refer to the course [20] for an introduction to kinetic equations, to [30, 68] for more advanced material on Boltzmann, and Vlasov and Boltzmann equations (respectively), and to the preprint [27] for propagation phenomena in run-and-tumble equations.

We remark that it is possible to describe a microscopic N particle problem by means of a kinetic equation in which the probability density function $f(t, x, v)$ is a sum of Dirac masses. The scaling limit from kinetic equations to macroscopic ones is more arduous. In the context of fluid mechanics, it consists in defining macroscopic observables by taking averages over the probability density function f . Important contributions to the understanding of such scalings have been made by Bardos, Golse, Levermore, Saint-Raymond, Masmoudi and Lions.

The models we consider bear certain similarities to kinetic equations, but they do not belong to that class. They are structured integro-differential equations closely related to the renewal

McKendrick equation [128, Ch. 3]:

$$\partial_t n(t, a) + \partial_a n(t, a) + d(t, a)n(t, a) = 0, \quad (16)$$

and to the scattering equation [129, Ch.9] (albeit with an added age structure):

$$\partial_t \rho(t, x) + k(x)\rho(t, x) = \int_{\mathbb{R}^d} K(y, x - y)\rho(t, y) dy. \quad (17)$$

In Boltzmann and run-and-tumble equations, the motion of a single particle does not exhibit discontinuities in space, as is the case in scattering equations, but in velocity.

The reader may find related scaling limits from microscopic models to their mesoscopic and macroscopic counterparts in the contexts of coagulation-fragmentation and phase transitions. For instance, the authors of [33] recover the Lifshitz-Slyozov model [99] as a first-order approximation of a limit of the microscopic Becker-Döring model [13], by a process akin to a hydrodynamic limit [70]. Non-local boundary terms are also present in such models [171]. In that context as well, the related Kolmogorov-Johnson-Mehl-Avrami model [53] provides a description of nucleation and growth processes in cristallization [93, 82] and also in DNA replication [92].

In this dissertation, we consider two age-structured equations, corresponding to space-homogenous and space-inhomogeneous versions of the same problem. Since we wish to model a process of random motion in space, let us first introduce the renewal-scattering equation presented in [157, 167]. We take an alternative approach to the macroscopic fractional equation and the mesoscopic non-local Master equation of the previous Section. We associate each random walker with a residence time (age, in short) a which increases at speed 1 with time and is reset when the random walker jumps to another location. In one dimension of space, we note $n(t, x, a)$ the probability density function of walkers at time t that have resided at location x exactly during the last span of time a . The dynamics of the CTRW are then described by means of an age-renewal equation with spatial jumps:

$$\begin{cases} \partial_t n(t, x, a) + \partial_a n(t, x, a) + \beta(a)n(t, x, a) = 0, & t \geq 0, \quad a > 0, \quad x \in \mathbb{R} \\ n(t, x, a = 0) = \int_0^\infty \beta(a') \int_{\mathbb{R}} \omega(x - x')n(t, x', a') dx' da' \\ n(t = 0, x, a) = n^0(x, a). \end{cases} \quad (18)$$

The functions β and ω are respectively the age-dependent jump rate of particles and their space redistribution kernel. The jump rate is related to the residence time distribution Φ defined in the previous section by the following expression:

$$\Phi(a) = \beta(a) \exp\left(-\int_0^a \beta(s) ds\right). \quad (19)$$

The boundary condition on $n(t, x, 0)$ at age $a = 0$ accounts for the particles landing at position x at time t after having “jumped” from position x' , at which they had remained during a time span exactly equal to a' . The fact that the loss term $\beta(a)n(t, x, a)$ is recovered in the boundary condition (and that ω is a probability distribution) leads to the conservation of the total population density $\int_0^\infty \int_{\mathbb{R}} n(\cdot, x, a) dx da$ over time.

We note that this formulation is equivalent to the Master equation (13). However, the treatment it will receive relies fully on the age-structure, which is here explicit: that is their difference. We remark that the renewal-scattering equation (18) is local in time. This corresponds

to a rescue of the Markovian property of the corresponding jump process at the expense of a supplementary structural age variable.

Let us now present the space-homogenous renewal-scattering equation that we study in Chapter 1. Subdiffusive continuous time random walks have heavy-tailed residence times and localised space redistribution kernels. Hence, the anomalous dynamics are a consequence of the underlying renewal dynamics: it is interesting to study the space-homogenous version of equation (18), which is a renewal equation:

$$\begin{cases} \partial_t n(t, a) + \partial_a n(t, a) + \beta(a)n(t, a) = 0, & t \geq 0, \quad a > 0 \\ n(t, a = 0) = \int_0^\infty \beta(a')n(t, a') da' \\ n(t = 0, a) = n^0(a). \end{cases} \quad (20)$$

Throughout this doctoral dissertation, ω will be a Gaussian distribution and $\beta(a)$ will be equal to $\mu/(1+a)$ for $0 < \mu < 1$, or perturbations thereof.

The originality of our approach in the context of the anomalous diffusion community is to study the above integro-differential equations directly. This differs from the common strategy of using the mesoscopic equations only as a means to derive macroscopic fractional equations thanks to Fourier-Laplace tools, and manipulating the macroscopic equations. An interesting advantage of studying the mesoscopic integro-differential equations rather than the fractional PDEs is related to the ability to introduce reaction terms with ease into the equations, which the age-structure allows. On the contrary, the fractional time-derivative in the macroscopic model induces a non-trivial coupling between reaction and anomalous diffusion [166]. Another significant advantage is that there exist many asymptotic analysis tools that can be readily applied to the integro-differential problem.

4 State of the art and mathematical tools

The study of the asymptotic limits of the renewal-scattering equations in this thesis has only been feasible thanks to mathematical tools developed mainly in the last 30 years, which we introduce hereafter.

4.1 Long time asymptotics of renewal equations

Intermediate asymptotics in the integrable case

We follow freely [128, Ch. 3 and 6.4]. Consider the renewal equation (20) defined in the modelling section 3.3, and assume that $\|n^0\|_{L^1} = 1$.

Contrary to the prevalent hypotheses throughout the dissertation, assume that (20) admits an integrable stationary state:

$$N(a) = \frac{\exp\left(-\int_0^a \beta(s) ds\right)}{\int_0^\infty \exp\left(-\int_0^v \beta(s) ds\right) dv}. \quad (21)$$

This is the eigenvector of the direct problem:

$$\begin{cases} \partial_a N(a) + \beta(a)N = 0 \\ N(0) = \int_0^\infty \beta(a)N(a) da \\ \int_0^\infty N(a) da = 1 \\ N(a) \geq 0 \end{cases} \quad (22)$$

The adjoint problem is trivial in this case, so the adjoint eigenvector is $\phi \equiv 1$. Let $H \in \mathcal{C}(\mathbb{R}, \mathbb{R}_+)$ be a convex function that reaches its minimum, 0 at 1. We define the generalised relative entropy following [128, Ch. 3]:

$$\mathcal{H}(t) = \int_0^\infty H\left(\frac{n(t,a)}{N(a)}\right) N(a)\phi(a) da \quad (23)$$

We name

$$u(t,a) = \frac{n(t,a)}{N(a)}.$$

We wish to prove that \mathcal{H} decays in time. We have:

$$\mathcal{H}'(t) = \int_0^\infty H'(u(t,a))\partial_t u(t,a)N(a) da.$$

Dividing the transport equations satisfied by n and N respectively by n and N and subtracting the resulting equations gives us:

$$\partial_t \ln(u) + \partial_a \ln(u) = 0,$$

hence

$$\partial_t u + \partial_a u = 0.$$

It follows that:

$$\begin{aligned} \mathcal{H}(t) &= - \int_0^\infty \partial_a (H(u(t,a))) da \\ &= - \int_0^\infty \beta(a)H(u(t,a))N(a) da - \int_0^\infty \partial_a (H(u(t,a))N(a)) da \\ &= N(0)H(u(t,0)) - \int_0^\infty H(u(t,a))N(a) da. \end{aligned}$$

Hence,

$$\mathcal{H}'(t) = N(0) \left[H\left(\int_0^\infty u(t,a)\beta(a)\frac{N(a)}{N(0)} da\right) - \int_0^\infty H(u(t,a))\beta(a)\frac{N(a)}{N(0)} da \right]. \quad (24)$$

For any probability measure γ , we name $DH(u|\gamma)$ the entropy dissipation with respect to γ and define it as follows:

$$DH(u|\gamma) = \int_0^\infty H(u(t,a))\gamma(a) da - H\left(\int_0^\infty u(t,a)\gamma(a) da\right). \quad (25)$$

With that notation, equation (24) reads:

$$\mathcal{H}'(t) = -N(0)DH(u|\nu), \quad (26)$$

where ν defined as follows is a probability measure:

$$\nu(a) = \beta(a) \frac{N(a)}{N(0)}. \quad (27)$$

Since H is convex, Jensen's inequality gives us that \mathcal{H} is non-increasing. This formalism suffices to prove convergence without a rate [128, Ch. 3.6].

The main idea at this point is to compare \mathcal{H} and DH so as to recover a differential inequality on \mathcal{H} whose integration via Gronwall's Lemma yields an exponential decay rate for \mathcal{H} . Different hypotheses on β allow doing so using techniques described in [128, Ch. 3.7 and 3.9] and in [73]: we refer the reader to those references for detailed computations.

Let us prove a similar result in a case where β is non-increasing and the initial condition n^0 is compactly supported, without loss of generality in $[0, 1]$. Here, we may choose

$$H(x) = |x - 1| \quad (28)$$

so that the entropy functional \mathcal{H} is actually the L^1 distance between n and N . We compute directly:

$$\begin{aligned} N(0)DH(u|\nu) &= \int_0^\infty |n(t, a) - N(a)|\beta(a) da - \left| \int_0^\infty \beta(a)n(t, a) da - N(0) \right| \\ &= \int_0^\infty |n(t, a) - N(a)|\beta(a) da - \left| \int_0^\infty \beta(a) (n(t, a) - N(a)) da \right| \\ &= \int_0^\infty |n(t, a) - N(a)|\beta(a) da - \left| \int_0^\infty (\beta(a) - \beta(1+t)) (n(t, a) - N(a)) da \right| \end{aligned}$$

since the L^1 norms of $n(t, \cdot)$ and of N are conserved in time. Since n satisfies a transport equation, its support is propagated at a known speed – here equal to 1 – so $n(t, \cdot) = 0$ over (t, ∞) . Moreover, $\beta(a) - \beta(1+t)$ is non-negative over $a \in [0, t]$. It follows that:

$$\begin{aligned} N(0)DH(u|\nu) &\geq \int_0^\infty |n(t, a) - N(a)|\beta(a) da - \int_0^\infty (\beta(a) - \beta(1+t)) |n(t, a) - N(a)| da \\ &\geq \beta(1+t) \int_0^\infty |n(t, a) - N(a)| da = \beta(1+t)\mathcal{H}(t). \end{aligned}$$

It follows that

$$\mathcal{H}'(t) \leq -\beta(1+t)\mathcal{H}(t). \quad (29)$$

Gronwall's Lemma allows us to prove that n converges in L^1 with a decay rate depending on β – for instance, if β is bounded below, we recover exponential decay.

We remark that the existence of the eigenvector N is a crucial element of this argumentation. However, no such integrable steady state N exists in the subdiffusive case described in our model Section 3.3. Let us introduce the state of the art in probability theory corresponding to that case.

The Dynkin-Lamperti arc-sine law

Consider the renewal (or counting) process $N(t)$ defined in equation (7), Section 3.1. With the notation therein, we define the spent time $Y(t)$ as the time elapsed since the last renewal and the residual time $R(t)$ as the time remaining before the following renewal:

$$Y(t) = t - T_{N(t)}, R(t) = T_{N(t)+1} - t. \quad (30)$$

We remark that in the context of our age-structured model in Section 3.3, $Y(t)$ corresponds to the age of a particle.

Results on the asymptotic behaviour the processes N and Y for waiting time distributions of infinite expectancy have been known since the works of Dynkin [43, 44] and Lamperti [96, 97] in the late 1950s. Such and related results can be found in reference books [59] tome II, chapters VI.6 and XI (where the reader can find the case of finite expectancy) and [17, Ch. 8.6]. Let us specifically state the Dynkin-Lamperti theorem [44, 97] (see for instance [17, Ch. 8.6.2]).

We say a function l is slowly varying if it satisfies, for any positive λ

$$\frac{f(\lambda x)}{f(x)} \xrightarrow{x \rightarrow \infty} 1. \quad (31)$$

Theorem 3 (Dynkin-Lamperti Theorem). *Assume the waiting time law Φ satisfies, for some $0 < \mu < 1$ and some slowly varying function l :*

$$1 - \int_0^a \Phi(s) ds \sim \frac{l(a)}{a^\mu \Gamma(1 - \mu)}, \quad a \rightarrow \infty.$$

That is a necessary and sufficient condition for the existence of a non-degenerate limit law as $t \rightarrow \infty$ for each of $Y(t)/t$, $R(t)/t$ and $(Y(t), R(t))$. The law for $Y(t)/t$ is the Dynkin-Lamperti or arc-sine law. It is supported over $(0, 1)$ and has density

$$q_\mu(a) = \frac{\sin(\pi\mu)}{\pi} a^{-\mu} (1-a)^{\mu-1}. \quad (32)$$

The limit law for $R(t)/t$ is $q_\mu(a/1+a)$, and that for $(Y(t), R(t))$ is

$$r_\mu(u, v) = \frac{\mu \sin(\pi\mu)}{\pi} (u+v)^{-1-\mu} (1-u)^{\mu-1}.$$

However, no convergence rate is given for our infinite mean waiting time problem in any of these books, and we have been unable to locate such a convergence rate in the subsequent literature up to [84]. Recent developments in Ergodic Theory for mildly related problems (see chapter 8.11 of [17] for an introduction to Darling-Kac theory), have yielded convergence rates, that are optimal in certain cases, as shown in [106] and [149].

We refer the reader to the very thorough study of the existence, regularity and long-time behaviour of renewal equations done in [128, Ch. 3]. Our study in Chapter 1 corresponds to the case stated in [128, Ch. 3.8]: a renewal equation with a death rate and (in our case) an equal birth rate. However, the complications arising from our heavy-tailed waiting time distribution are not addressed in the reference.

The convergence to a self-similar Dynkin-Lamperti profile after a rescaling similar to $Y(t)/t$ is the object of the study of Chapter 1. That work relies on techniques inspired by those sketched in Section 4.1, and also by the entropy techniques used to prove self-similar decay introduced in Section 4.2.

4.2 Relative Entropy

The results of Chapter 1 rely on entropy methods and closely related ideas. The short introduction to entropy methods and the heat equation examples hereafter are based on the course by Dolbeault [41, 40] and some references therein: [4], [114, 111] and [128, Ch. 3.5, 3.6, 4.2.2, 6.3, 6.4]. The first reference presents the accomplishments of a research programme consisting in the application of entropy methods to kinetic equations. The last three introduce general relative entropy to PDEs stemming from biological models.

The illustration of entropy methods on the heat equation presented later follows [16, 5, 12, 41].

General considerations on entropy methods

The mathematical concept of entropy is linked to the physical one, which is a measure of the amount of microscopic states corresponding to a given macroscopic state. The second principle of thermodynamics asserts that (physical) entropy is an increasing function of time in a closed system. Hence Gibbs' principle, which states that the equilibrium distribution of such a system is the one which achieves maximum entropy under certain constraints. Analytically in the context of the Boltzmann equation, this corresponds to Boltzmann's H -Theorem which the reader may find in [18].

In a similar way, mathematical entropy methods allow the establishment of *quantitative* convergence to equilibrium in certain PDE. (Mathematical) Entropy \mathcal{H} is a Lyapunov functional that is non-increasing along the solution, finding its minimum at the equilibrium: its opposite $-\mathcal{H}$ plays the role of the physical entropy.

Specifically, freely quoting [4], an entropy method consists in the following steps³: the identification of an equilibrium state F_∞ ; the definition of a relative entropy $\mathcal{H}[f, F_\infty]$; the definition of an entropy dissipation functional D , which is the opposite of the derivative of the entropy; and the proof that \mathcal{H} decays in time, *i.e.* f converges in entropy to F_∞ .

Such methods apply to a wide variety of problems. For instance, in the context of PDE stemming from biology, [114, 111] and [128, Ch. 3.5, 3.6, 4.2.2, 6.4] treat parabolic and hyperbolic equations and models of growth, chemostats, fragmentation and renewal ; and entropy methods for parabolic equations also appear throughout [129].

The reader can find a description of entropy methods in the context of Perron-Frobenius and Floquet eigenvalue theorems in [128, Ch. 6.3.1, 6.3.2]. For entropy methods in the context of kinetic equations, the reader may refer to [4] and the bibliography therein. The reader can also find a presentation of entropy methods in the context of optimal transport in [156, Ch. 9].

Relative entropy and self-similar decay in the Heat Equation

Following [16, 5, 12, 41], let us present a proof of the self-similar decay of a family of solutions of the heat equation. This will serve both as an illustration in a simpler case and as a first heuristic for the work of Chapter 1. Even though our proof involves mostly L^1 norms, the underlying ideas are heavily inspired by the problem described hereafter.

Consider the heat equation

$$\partial_t \rho(t, x) = \Delta \rho(t, x), \quad t > 0, x \in \mathbb{R}^d \quad (33)$$

coupled with a non-negative, integrable initial condition ρ^0 defined over \mathbb{R}^d . Without loss of generality, we take $\|\rho^0\|_{L^1} = 1$.

Equation (33) can be explicitly solved by convolution with the fundamental solution of the heat equation, the Green kernel G :

$$\rho(t, x) = (G * \rho^0)(t, x) = \int_{\mathbb{R}^d} G(t, x - y) \rho^0(y) dy, \quad (34)$$

where

$$G(t, x) = \frac{\exp\left(-\frac{|x|^2}{4t}\right)}{(4\pi t)^{d/2}}. \quad (35)$$

³In the article [4], the authors define their entropy functional E as the opposite of our \mathcal{H} . As such, their definition corresponds to the physical convention but its manipulation as a Lyapunov functional seems less intuitive.

It follows that ρ decays uniformly to 0 like $O(t^{-d/2})$, since as $t \rightarrow \infty$, $\rho(t, x) \sim G(t, x)$. It is classical to search for the first order term of an asymptotic expansion of the solution as $t \rightarrow \infty$. This amounts to estimating the decay of $u(t, \cdot) - G(t, \cdot)$ in various L^p norms. Such estimates are called intermediate asymptotics.

Since $t^{d/2}G(t, x) = G(1, x/\sqrt{t})$, the decay that we wish to study is self-similar. The reader may find in [41] different self-similar rescalings, including the following time-dependent rescaling:

$$\rho(t, x) = \frac{1}{R^d(t)} v \left(\tau(t), \frac{x}{R(t)} \right) \quad (36)$$

for some positive increasing functions τ, R chosen so that v may satisfy a simple enough equation. For

$$R(t) = \sqrt{1 + 2t}, \quad \tau(t) = \ln(R(t)) = \frac{1}{2} \ln(1 + 2t), \quad (37)$$

the equation satisfied by v is the Fokker-Planck equation:

$$\partial_\tau v(\tau, y) = \delta_y v + \nabla_y (y v(\tau, y)), \quad y \in \mathbb{R}^d, \tau > 0 \quad (38)$$

coupled with the same initial data as for ρ : $v(\tau = 0, y) = \rho^0(y)$. We will hereafter drop the distinction between x and y .

Apart from the preservation of the initial condition, this time-dependent rescaling has the advantage of admitting a stationary solution v_∞ . The study of intermediate asymptotics for ρ now amounts to that of the convergence of v to v_∞ . Indeed, for the Fokker-Planck equation (38),

$$v_\infty(x) = \frac{\exp\left(-\frac{|x|^2}{2}\right)}{(2\pi)^{d/2}}. \quad (39)$$

Let us consider a smooth, convex function H that reaches its minimum at 1. We define the relative entropy:

$$\mathcal{H}[v, v_\infty](\tau) = \int_{\mathbb{R}^d} H\left(\frac{v(\tau, x)}{v_\infty(x)}\right) v_\infty(x) dx, \quad (40)$$

and the relative entropy dissipation D as:

$$D[v, v_\infty](\tau) = -\frac{d}{d\tau} \mathcal{H}[v, v_\infty](\tau). \quad (41)$$

Since v is a solution of the Fokker-Planck equation (38), if we choose

$$H(X) = X \ln X. \quad (42)$$

we recover that the entropy dissipation equals the Fisher information functional:

$$\begin{aligned} D[v, v_\infty](\tau) &= \int_{\mathbb{R}^d} v(\tau, x) \left| \nabla \left(\ln \left(\frac{v(\tau, x)}{v_\infty(x)} \right) \right) \right|^2 dx \\ &= \int_{\mathbb{R}^d} \left| \frac{\nabla v(\tau, x)}{v(\tau, x)} + x \right|^2 v(\tau, x) dx. \end{aligned} \quad (43)$$

The control of the entropy by its dissipation (similar to equation (29)) is achieved in this context via the logarithmic Sobolev inequality (44):

$$\mathcal{H}[v, v_\infty](\tau) \leq \frac{1}{2} D[v, v_\infty](\tau). \quad (44)$$

The reader may find a proof of the inequality in [41], generalisations in the bibliography therein and details on the application of such inequalities in the context of equations related to Fokker-Planck's in [6]. It follows by Gronwall's Lemma that the relative entropy decays exponentially:

$$\mathcal{H}[v, v_\infty](\tau) \leq \mathcal{H}[v^0, v_\infty]e^{-2\tau}. \quad (45)$$

The Csiszár-Kullback inequality (consult e.g. [151] and the bibliography therein) yields:

$$\|v(\tau, \cdot) - v_\infty\|_{L^1}^2 \leq 4\mathcal{H}[v, v_\infty](\tau). \quad (46)$$

We have recovered an exponential convergence rate of v to v_∞ in the L^1 norm:

$$\|v(\tau, \cdot) - v_\infty\|_{L^1} \leq 2\sqrt{\mathcal{H}[v^0, v_\infty]}e^{-\tau}. \quad (47)$$

This amounts to an algebraic convergence rate of ρ towards the asymptotic profile

$$\rho_\infty(t, x) = \frac{1}{R(t)^d} v_\infty \left(\frac{x}{R(t)} \right), \quad (48)$$

in the sense that

$$\|\rho(t, \cdot) - \rho_\infty(t, \cdot)\|_{L^1} \leq \frac{2\sqrt{\mathcal{H}[\rho^0, v_\infty]}}{\sqrt{1+2t}}. \quad (49)$$

The reader can find more general decay estimates proved by means of entropy methods in [12] in the context of the heat equation, or for instance in [118] in that of fast diffusion.

4.3 Viscosity solutions of Hamilton-Jacobi equations

Let us introduce the appropriate formalism on which the results from Chapter 2 rely: the viscosity solutions of Hamilton-Jacobi equations. The main bibliography for this section consists of the following books [11, 10, 49, 2]. A useful uniqueness result is taken from [31], in which Hamilton-Jacobi equations are presented in the context of control theory and convex analysis.

Hamilton-Jacobi equations

Let Ω be an open subset of \mathbb{R}^N and H be a continuous map from $\Omega \times \mathbb{R} \times \mathbb{R}^N$ to \mathbb{R} which we call the Hamiltonian. For any smooth enough function $u : \mathbb{R}_+ \times \mathbb{R}^N \rightarrow \mathbb{R}$, we denote by ∇u the space gradient of u and by $\nabla^2 u$ the space Hessian matrix, respectively:

$$\begin{aligned} \nabla u(t, x) &= (\partial_{x_1} u, \dots, \partial_{x_N} u)(t, x) \\ \nabla^2 u(t, x) &= (\partial_{x_i} \partial_{x_j} u)_{1 \leq i, j \leq N}(t, x). \end{aligned}$$

The following is a first-order evolution Hamilton-Jacobi (HJ) equation:

$$\partial_t u(t, x) + H(x, u(t, x), \nabla u(t, x)) = 0, \quad (t, x) \in \mathbb{R}_+ \times \Omega. \quad (50)$$

The equation can be coupled with initial and/or boundary conditions.

Second order HJ equations are those in which H depends on an additional variable corresponding to $\nabla^2 u$, and stationary HJ equations are those in which u does not depend on t .

Hamilton-Jacobi equations arise naturally in control theory [63, 10, 31], in differential games, in the study of propagation phenomena and large deviation theory [152, 65, 50, 153, 133], in image theory, and phase-transition problems.

Viscosity solutions

Definition 4 (Classical solutions). A classical solution of the HJ equation (50) is a function $u \in \mathcal{C}^1(\mathbb{R}_+ \times \bar{\Omega})$ which satisfies the equation (50) over $\mathbb{R}_+ \times \Omega$ as well as the corresponding initial or boundary condition.

The notion of classical solution is a natural one. However, as is generally the case with non-linear PDEs, HJ equations do not generally admit classical solutions. Indeed, several problems arise even while attempting to define a class of almost-everywhere or weak solutions. Four classical instructive examples of such issues are presented in the introduction of Barles' book [11, Ch. 1]. Namely, the equation or its boundary conditions may not have a well-defined meaning, there may exist an infinite family of solutions (whose adherence, moreover, can include smooth functions that are not weak solutions), and within the framework of almost everywhere solutions, HJ problems may not be stable with respect to single perturbations.

Conversely, viscosity solutions introduced by Crandall and Lions in the early 1980s [36] provide a reasonable framework that guarantees that HJ equations and their boundary conditions have meaning, admit a unique solution under reasonable hypotheses, and satisfy stability results for single perturbation problems.

Definition 5 (Continuous viscosity solutions). A function $u \in \mathcal{C}(\Omega)$ is a viscosity solution of the first order evolution Hamilton-Jacobi equation (50) if and only if:

- for any test function $\phi \in \mathcal{C}^2(\mathbb{R}_+ \times \Omega)$, at any point $(t_0, x_0) \in \mathbb{R}_+ \times \Omega$ at which $u - \phi$ reaches a local maximum,

$$\partial_t \phi(t_0, x_0) + H(x_0, u(t_0, x_0), \nabla \phi(t_0, x_0)) \leq 0, \quad (51)$$

- and for any test function $\phi \in \mathcal{C}^2(\mathbb{R}_+ \times \Omega)$, at any point $(t_0, x_0) \in \mathbb{R}_+ \times \Omega$ at which $u - \phi$ reaches a local minimum,

$$\partial_t \phi(t_0, x_0) + H(x_0, u(t_0, x_0), \nabla \phi(t_0, x_0)) \geq 0. \quad (52)$$

If u satisfies (51) (respectively, (52)), we say that u is a viscosity subsolution of (50) (respectively, a viscosity supersolution).

There exist equivalent definitions of viscosity solutions by means of convex analysis tools. The term “viscosity” solution is used because the solution obtained via the vanishing viscosity method is a viscosity solution. We refer the reader to [11] for the details.

Existence results

There are several existence results for viscosity solutions of Hamilton-Jacobi equations, depending on the properties of the Hamiltonian H . Specifically, if H is convex, it is possible to define the Hopf-Lax value function as we do below, hence building a solution of the HJ equation thanks to its dual control problem. If the Hamiltonian is coercive and satisfies certain additional properties as is the case in [11, Ch. 2.6.1], it is possible to apply Perron's method to construct a solution. In a nutshell, consider the set \mathcal{S} of bounded subsolutions, define a function f that takes at each point x the value: $f(x) = \sup_{v \in \mathcal{S}} v(x)$, and prove f is a viscosity solution. Under weaker hypotheses on the Hamiltonian, an interesting idea is to use a stability result for a class of approached problems. For instance, one may define

$$\mathcal{P}_\varepsilon : \quad \varepsilon \Delta u + \partial_t u + H(\nabla u) = 0,$$

use the regularising properties of the higher-order term to prove existence and uniqueness of the solution to \mathcal{P}_ε , and use a stability result that guarantees that the limit of the solutions to \mathcal{P}_ε is the solution to \mathcal{P}_0 .

In this paragraph, we consider an existence theorem that applies in the context of Chapter 2, where H is convex and coercive, and the initial condition is bounded below and sublinear. Consider the following HJ evolution problem

$$\begin{cases} \partial_t u(t, x) + H(\nabla u(t, x)) = 0, & (t, x) \in (0, T] \times \mathbb{R}^n \\ u(0, x) = u^0(x), & x \in \mathbb{R}^n. \end{cases} \quad (53)$$

Let L be the Lagrangian defined as the Legendre-Fenchel transform of the convex, coercive Hamiltonian H :

$$L(v) = \sup_{p \in \mathbb{R}} \{pv - H(p)\}. \quad (54)$$

We define the value function f via the Hopf-Lax formula:

$$f(t, x) = \min_{y \in \mathbb{R}} \left\{ tL\left(\frac{x-y}{t}\right) + u^0(y) \right\}. \quad (55)$$

Proposition 6. *Let H be a convex, coercive Hamiltonian, and let u^0 be a bounded below, sub-linear, semi-concave and Lipschitz continuous functions over \mathbb{R} . The Hopf-Lax value function f in (55) is well defined for the initial condition u^0 . Moreover, $f(0, \cdot) = u^0$ and f is a viscosity solution of (83).*

The reader may prove the Proposition by following the definitions, methods and computations in [49, Chapters 3.3.2 and 10.3.4] and generalising them from bounded Lipschitz functions to bounded below, sublinear, semi-concave, Lipschitz continuous functions.

In the cited reference [49], boundedness and uniform continuity of u^0 and the solutions are assumed as well, and a unicity result of the viscosity solution in the class of bounded uniformly continuous functions is derived from the existence of a Lipschitz continuous solution. Uniqueness of the viscosity solution of (83) holds under many different sets of hypotheses, among which there are trade-offs between assumptions on u^0 and u on the one hand, and on H on the other.

Uniqueness results

There are many uniqueness results for viscosity solutions of Hamilton-Jacobi equations.

In the context of continuous viscosity solutions, uniqueness is usually proved within the class of uniformly continuous functions that are bounded or grow in a controlled way. Such results can rely on many different yet similar sets of hypotheses, usually involving some combination of coercivity, smoothness and convexity of the Hamiltonian H , and sometimes the existence of a solution in $W^{1,\infty}$. The proofs rely on the application of the maximum principle and the method of the doubling of variables, which consists in comparing a viscosity subsolution u and a viscosity supersolution v of the desired HJ equation taken at different points. The idea is to define a penalised function

$$\phi_\varepsilon(t, s, x, y) = u(t, x) - v(s, y) - \frac{|t-s|}{\varepsilon} - \frac{(x-y)^2}{\varepsilon^2}$$

that reaches its maximum at a certain $(t_\varepsilon, s_\varepsilon, x_\varepsilon, y_\varepsilon)$, and to prove that the penalised maximum converges to the maximum of $u - v$. The end of the proofs rely on an estimate on the differential of ϕ_ε . We refer the reader to [11, 49] for extensive details.

Certain uniqueness results hold in the context of half-relaxed limits for discontinuous viscosity solutions. The proofs follow a strategy similar to that sketched above. However, the convergence of the penalised maximum is harder to obtain. The proofs rely on a strong comparison principle between a locally bounded upper semi-continuous viscosity subsolution and a locally bounded lower semi-continuous viscosity supersolution. The reader may consult [2] for a presentation of the theory of half-relaxed limits.

This section follows the reference [31], from which the existence results that are useful for Chapter 2 are taken.

Consider the following Cauchy problem in which we have a first order evolution Hamilton-Jacobi equation where H does not depend on $u(t, x)$:

$$\begin{cases} \partial_t u(t, x) + H(x, \nabla u(t, x)) = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N \\ u(0, x) = l(x), & x \in \mathbb{R}^N. \end{cases} \quad (56)$$

Let us quote here the theorems taken from [31] on which Theorem 15 from Chapter 2 relies. Namely, we quote Hypothesis 19.10 and a combination of Theorem 19.11 and Corollary 19.17.

Hypothesis 1 (Hypothesis 19.10).

- (a) l is locally Lipschitz and bounded below;
- (b) $H(x, p)$ is locally Lipschitz, and is convex as a function of p for each x ;
- (c) H has superlinear growth in p in the following sense:

$$\lim_{|p| \rightarrow \infty} H(x, p)/|p| = \infty \text{ uniformly for } x \text{ in bounded sets;}$$

- (d) There exist positive constants $C, \kappa > 1$ and $\sigma < \kappa$ such that, for all $(x, p) \in \mathbb{R}^N \times \mathbb{R}^N$,

$$H(x, p) \leq C|p|^\kappa(1 + |x|)^\sigma.$$

Theorem 7 (Corollary 19.17 and Theorem 19.11). *Under Hypothesis 1, there exists a unique viscosity solution u_* of (56) which is bounded below.*

The reader may consult the proofs of these results in [31, Ch. 19]. Such proofs rely on equivalent definitions of viscosity solutions by means of convex analysis tools.

WKB expansion and singular perturbation problems

In 1926, in the context of the high frequency limit of the wave equation and the semi-classical limit of the Schrödinger equation, Wentzel [163], Kramers [95] and Brillouin [24] (WKB) introduced a solution approximation method which now bears their initials ⁴. It consists in making an Ansatz in the form of a series expansion of the fast oscillating complex phase of the solution to such problems, in which the highest order derivative is multiplied by a small parameter. Similar methods apply for hyperbolic scalings of parabolic and hyperbolic equations. The reader may consult [133] for an in depth analysis of such methods. Let us present an application of this method to a parabolic equation.

⁴Precursors include works by Jeffreys [89], and even Liouville and Green.

Fisher-KPP front propagation

Let us illustrate a WKB-inspired technique on a parabolic reaction-diffusion example. Consider the one-dimensional Fisher-KPP equation introduced simultaneously by Fisher [62] and Kolmogorov, Petrovskii and Piskunov [94].

$$\partial_t \rho = D \partial_x^2 \rho + r \rho (1 - \rho), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}. \quad (57)$$

Here, $\rho(t, x)$ is the probability density function associated to some species (animal, bacterial, chemical...), D is its diffusion coefficient, and r its birth rate.

The reader may find in [107, Ch. 4] a synthetic presentation of front propagation for the Fisher equation coupled with different reaction kinetics, sketches of the proofs, and a good physical intuition – as well as an introduction of WKB and Hamilton-Jacobi methods for front propagation in Fisher-KPP.

We remark that $\rho \equiv 1$ is a stable steady state of (57) and $\rho \equiv 0$ is an unstable steady state.

Definition 8 (Travelling wave). A travelling wave solution of (57) is a solution of the form

$$\rho(t, x) = \mathcal{U}(x - ct) \quad (58)$$

where \mathcal{U} is the profile of the wave and c its velocity of propagation.

It has been proved [62, 94, 7, 61] that Fisher-KPP (57) admits positive travelling wave solutions that connect the stable steady state to the unstable steady state:

$$\lim_{-\infty} \mathcal{U} = 1; \quad \lim_{\infty} \mathcal{U} = 0,$$

for any velocity c greater than a minimal velocity:

$$c \geq c^* = 2\sqrt{rD}.$$

Such solutions are unique up to translation.

It is possible to study the travelling wave solutions by considering exponentially decaying initial conditions [8] and a profile \mathcal{U} that is exponentially decaying to the right. A linearization of the problem at the front of the profile for $\rho \ll 1$ yields an ODE from which the existence of travelling wave solutions is recovered by means of a phase portrait study [7].

The geometric optics approach

Following [65, 50], it is possible to decouple the study of the front profile and that of its velocity. In order to do so, we take the hyperbolic rescaling $\rho_\varepsilon(t, x) = \rho(t/\varepsilon, x/\varepsilon)$ of the Fisher equation.

We remark that under this rescaling, the front profile defined in the previous paragraph becomes steeper as ε decreases and converges to the indicatrix function of a set.

$$\rho_\varepsilon(t, x) = \mathcal{U}\left(\frac{x - vt}{\varepsilon}\right) \xrightarrow{\varepsilon \rightarrow 0} \mathbf{1}_{\Omega_t}.$$

In order to study the transition zone between the space sets defined at a given time by $\rho \simeq 1$ and $\rho \ll 1$, we make the WKB Ansatz:

$$\forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \quad \rho_\varepsilon(t, x) = \exp\left(-\frac{\phi_\varepsilon(t, x)}{\varepsilon}\right).$$

The function ϕ_ε is non-negative and satisfies a viscous Hamilton-Jacobi equation:

$$\partial_t \phi_\varepsilon + D|\partial_x \phi_\varepsilon|^2 + r = \varepsilon D \partial_x^2 \phi_\varepsilon + r \phi_\varepsilon, \quad (\mathcal{P}_\varepsilon)$$

which in turn converges formally as $\varepsilon \rightarrow 0$ to the constrained Hamilton-Jacobi equation

$$\min \{ \partial_t \phi_0 + D|\partial_x \phi_0|^2 + r, \phi_0 \} = 0. \quad (\mathcal{P}_0)$$

A stability result is required to conclude: the local uniform convergence of ϕ_ε to ϕ_0 , which is a viscosity solution of the limiting problem \mathcal{P}_0 . It corresponds to the following diagram.

$$\begin{array}{ccc} \phi_\varepsilon & \xrightarrow{\text{solution of}} & (\mathcal{P}_\varepsilon) \\ \downarrow & & \downarrow \text{formal limit} \\ \phi_0 & \xrightarrow{\text{visc. sol.}} & (\mathcal{P}_0). \end{array} \quad (59)$$

A rigorous justification of this result appears in [50]. The interest of this technique is that the theory of Hamilton-Jacobi equations can be applied to analyse the limit equation and Lagrangian dynamics can be used to follow the propagation of its nullset.

The ideas presented in this paragraph are the heuristic foundations of the work we present in Chapter 2. They are also useful in the context of kinetic equations [23].

5 Main results of this thesis

The interaction between mathematics and biology, physics, medicine, etc. bears fruits in two ways: mathematical modelling helps understand certain processes, and certain phenomena inspire mathematical problems whose study is intrinsically interesting. This dissertation is the latter type of fruit: we have taken a model inspired by biological phenomena and have contributed to the mathematical study of the equations in which it consists. Hereafter, we present the results in question in an order that mimics the plan of the dissertation.

5.1 Self-similar decay in a homogenous space setting – entropy methods

The work presented in Chapter 1 is based on the article [15], co-authored with Hugues Berry and Thomas Lepoutre. In it we have investigated the asymptotic behaviour of the age distribution of an age-structured population that undergoes subdiffusive motion as described by our model in Section 3.3, specifically, equation (20). In that model, the space redistribution kernel that governs the destination of particles at each jump is decoupled from the the waiting time distribution. Moreover, the spatial redistribution kernel is Gaussian, and all subdiffusive effects are a consequence of the heavy tailed waiting time distribution. It is therefore of interest to study the space-homogenous version of the problem, described by the age-structured integro-differential renewal equation (20), which we recall.

$$\begin{cases} \partial_t n(t, a) + \partial_a n(t, a) + \beta(a)n(t, a) = 0, & t \geq 0, \quad a > 0 \\ n(t, a = 0) = \int_0^\infty \beta(a')n(t, a')da' \\ n(t = 0, a) = n^0(a). \end{cases} \quad (60)$$

Under suitable hypotheses, we have proved that the solution of (60) decays following a self-similar profile.

Theorem 9 (Berry, Lepoutre, M.G.). Assume n^0 is supported in $[0, 1)$ and $\beta(a) = \frac{\mu}{1+a}$. We denote

$$N_\infty(t, a) = \begin{cases} \frac{c_\infty}{a^\mu(1+t-a)^{1-\mu}}, & a < 1+t, \\ 0, & a > 1+t. \end{cases}$$

Then if $\mu \neq 1/2$, there exists $K > 0$ such that

$$\|n(t, \cdot) - N_\infty(t, \cdot)\|_1 \leq \frac{K}{(1+t)^\mu} + \frac{K}{(1+t)^{1-\mu}},$$

and if $\mu = 1/2$, there exists $K > 0$ such that

$$\|n(t, \cdot) - N_\infty(t, \cdot)\|_1 \leq \frac{K(1 + \log(1+t))}{\sqrt{1+t}}.$$

However, this work contains a result – namely, the definition of the pseudo-equilibrium W in Definition 10 and the subsequent Theorem 12 – that is potentially stronger than that of the previous Theorem, while being instrumental in its proof. This new result holds for perturbed jump rates of the following form.

Hypothesis 2. Let the jump rate β be a positive, bounded, and non-increasing function satisfying

$$\begin{cases} \lim_{a \rightarrow \infty} a\beta(a) = \mu \in (0, 1), \\ \beta(a) = \frac{\mu}{1+a} + g(a), \end{cases} \quad (61)$$

where $g \in L^1$ is such that there exist $K, \alpha > 0$ satisfying

$$\int_a^\infty |g(s)| ds \leq \frac{K}{(1+a)^\alpha}.$$

The result – Theorem 12 – is better stated in self-similar variables. Let us introduce them.

$$\begin{cases} \tau = \ln(1+t), \\ b = \frac{a}{1+t}. \end{cases} \quad (62)$$

We set

$$n(t, a) = \frac{1}{1+t} w(\tau, b). \quad (63)$$

The problem (60) becomes, in self-similar variables:

$$\begin{cases} \partial_\tau w + \partial_b((1-b)w) + e^\tau \beta(e^\tau b)w = 0 \\ w(\tau, 0) = \int_0^1 e^\tau \beta(e^\tau b)w(\tau, b) db \\ w(0, b) = w^0(b). \end{cases} \quad (64)$$

The previous equation preserves positivity and L^1 norm, but it is not autonomous. Hence it does not admit a classical integrable steady state. However, we may look for a stationary state satisfying the formal limit of the equation, since we consider here $\beta(a) \sim_\infty \frac{\mu}{a}$:

$$\partial_b((1-b)W_\infty) + \frac{\mu}{b}W_\infty = 0.$$

The formal limit of the boundary condition cannot be stated as an equality since W_∞ is expected to blow up at 0, but it can be understood as an equivalence as ε tends to 0 of $W_\infty(\varepsilon)$ and $\int_\varepsilon^1 \frac{\mu}{b} W_\infty(b) db$.

This leads us to define the self-similar equilibrium as:

$$W_\infty(b) = \frac{c_\infty}{b^\mu(1-b)^{1-\mu}} \quad (65)$$

which is called the arc sine distribution, or Dynkin-Lamperti distribution. We recall that the Dynkin-Lamperti Theorem 3 gives the distribution of $Y(t)/t$, which corresponds to a/t at the microscopic level in our problem. The renormalising constant c_∞ is defined such that $\|W_\infty\|_1 = 1$. Under some conditions, we can expect that $w(\tau, b)$ will converge to equation (65) when $\tau \rightarrow \infty$.

The non-existence of an integrable steady state renders impossible the direct use of relative entropy-type techniques: namely, the definition of a Lyapunov functional measuring some distance between $w(t, \cdot)$ and W_∞ that decreases with time. This has led us to introduce a pivot function $W : (\tau, b) \in \mathbb{R}_+ \times [0, 1) \mapsto W(\tau, b) \in \mathbb{R}_+$, such that $W(\tau, \cdot)$ converges in L^1 norm to W_∞ as $\tau \rightarrow \infty$ with a similar rate as $w(\tau, \cdot)$ converges to $W(\tau, \cdot)$ in L^1 as τ tends to ∞ .

Definition 10. We define the pseudo-equilibrium W over $\mathbb{R}_+ \times [0, 1)$ as follows:

$$W(\tau, b) = \frac{C(\tau)}{e^{B(e^\tau b)}(1-b)^{1-\mu}}, \quad (66)$$

where $B(a) = \int_0^a \beta(s) ds$ and C is defined so that $\|W(\tau, \cdot)\|_{L^1} = 1$.

In particular in the reference case $\beta(a) = \frac{\mu}{1+a}$, the pseudo-equilibrium takes the following form:

$$W(\tau, b) = \frac{C(\tau)e^{-\mu\tau}}{(e^{-\tau} + b)^\mu(1-b)^{1-\mu}}.$$

Proposition 11. *Under Hypothesis 2, the pseudo-equilibrium W converges in L^1 norm to the stationary state W_∞ with the following quantitative rate:*

$$\|W(\tau, \cdot) - W_\infty\|_1 \leq K \left(e^{(\mu-1)\tau} + e^{-\alpha\tau} \right), \quad (67)$$

where K is a constant whose value only depends on μ , α and w^0 .

We name \mathcal{H} the L^1 distance between the solution of the rescaled equation (64) and the pseudo-equilibrium:

$$\mathcal{H}(\tau) = \|w(\tau, \cdot) - W(\tau, \cdot)\|_{L^1}. \quad (68)$$

In Chapter 1, we shall define \mathcal{H} more generally as a relative entropy, the L^1 distance being a particular case more suited to our purposes.

Theorem 12 (Berry, Lepoutre, M.G.). *Suppose Hypothesis 2 holds.*

If $\alpha > 1 - \mu$, we recover the optimal rate of convergence

$$\mathcal{H}(\tau) \leq \begin{cases} K(e^{-\mu\tau} + e^{-(1-\mu)\tau}), & \text{if } \mu \neq 1/2 \\ K\tau e^{-\tau/2}, & \text{if } \mu = 1/2. \end{cases} \quad (69)$$

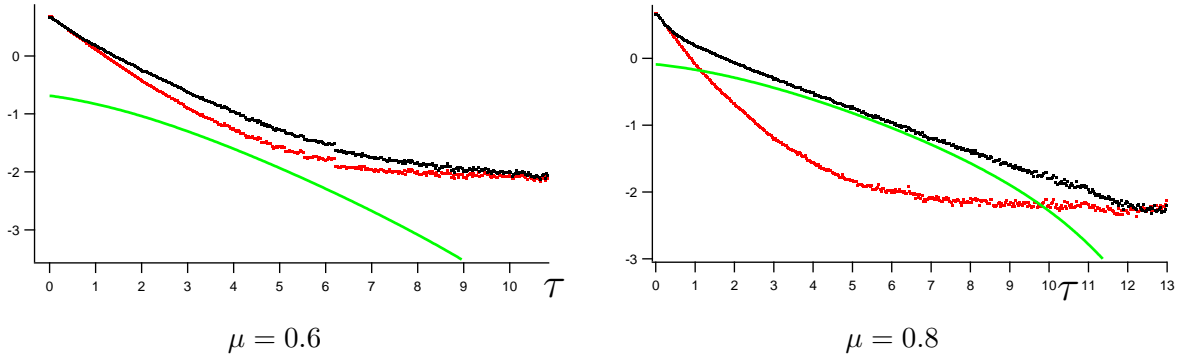


Figure 7: Influence of μ on the decay of $\ln \|w(\tau, \cdot) - W(\tau, \cdot)\|_1$ (red dots) and $\ln \|w(\tau, \cdot) - W_\infty\|$ (black dots). The green curves represent $\ln \|W(\tau, \cdot) - W_\infty\|$. For higher values of μ , w is significantly closer to W than to W_∞ .

If $\alpha \leq 1 - \mu$, we need to distinguish between several cases:

$$\mathcal{H}(\tau) \leq \begin{cases} K(e^{-\alpha\tau} + e^{-\mu\tau}), & \text{if } \mu \neq \alpha < 1 - \mu \\ K(1 + \tau)e^{-\mu\tau}, & \text{if } \alpha = \mu < 1 - \mu, \\ K(\tau e^{-(1-\mu)\tau} + e^{-\mu\tau}), & \text{if } \alpha = 1 - \mu \neq 1/2, \\ K(1 + \tau^2)e^{-\tau/2}, & \text{if } \alpha = \mu = 1 - \mu = 1/2. \end{cases} \quad (70)$$

Here, K denotes some constant whose value only depends on μ , α and w^0 .

Theorem 9 is a Corollary of Proposition 11 and Theorem 12. Considering the convergence rates of Proposition 11 and Theorem 12, we may expect similar rates of convergence of $w(\tau, \cdot)$ to $W(\tau, \cdot)$ and to W_∞ (up to a multiplicative constant). However, interestingly, numerical simulations suggest that for μ close to 1 the pseudo-equilibrium $W(\tau, \cdot)$ provides a much better asymptotic approximation in L^1 norm to $w(\tau, \cdot)$ than the stationary state W_∞ does. Figure 7 is the result of a Monte Carlo simulation of the standard $\beta(a) = \mu/(1+a)$ case. It depicts the L^1 distance between $w(\tau, \cdot)$ and $W(\tau, \cdot)$ in red, and the L^1 distance between $w(\tau, \cdot)$ and W_∞ in black. For larger values of μ , the simulated rates of convergence differ more and more.

Our main contributions in this work are the quantitative convergence rate itself, and the use of techniques inspired by relative entropy methods as well as the pseudo-equilibrium W to prove it.

The corresponding renewal problem in probability theory has been studied extensively since the beginnings of renewal theory. We refer the reader to [102, 57, 42] and the references therein for the foundations of renewal theory, and to the pertinent sections of [59, 117, 72, 84] for more and more recent approaches. Renewal problems in probability theory satisfying hypotheses similar enough to our own exhibit self-similar convergence towards a limiting arc sine (or Dynkin-Lamperti) distribution. It is a result similar to ours, but with an unknown rate of convergence. It is presented in Feller's book [59] tome II, chapter XI, especially in section 5 and onwards. A more recent presentation can be found in [72], in which the bibliography given on the generalised arc sine distribution does not go beyond Feller, and in [84], which cites the Dynkin-Lamperti theorem [117, Theorem 8.6.3.] in that context. We have been unable to find any rate of convergence in the literature. In this context, we believe our quantitative rate of convergence is interesting.

As for the use of a pivot function with respect to which we build a relative entropy, such an idea is of course not new. An application in a similar context can be found for example in [16]. However, we find elegant the specific construction of W and the way it leads to sufficiently simple and general computations. We are also intrigued by the possible sub-optimality of our constants in the rates of convergence for higher μ that numerical simulations detect.

5.2 Large scale asymptotics and limiting Hamilton-Jacobi equation

The work presented in Chapter 2 is based on the preprint [28], co-authored with Vincent Calvez and Pierre Gabriel. In it we have studied the hyperbolic rescaling in time and space of the age-structured jump-renewal problem hereafter – introduced in Section 3.3. It is known that the rescaled problems admit a formal limiting Hamilton-Jacobi equation. In Chapter 2 we show a rigorous stability result: we proved the convergence of the solutions of the rescaled problem to the solution of the limiting Hamilton-Jacobi equation. Let us recall the jump-renewal equation:

$$\begin{cases} \partial_t n(t, x, a) + \partial_a n(t, x, a) + \beta(a)n(t, x, a) = 0, & t \geq 0, \quad a > 0, \quad x \in \mathbb{R} \\ n(t, x, a = 0) = \int_0^\infty \beta(a') \int_{\mathbb{R}} \omega(x - x') n(t, x', a') dx' da' \\ n(t = 0, x, a) = n^0(x, a), & \text{compactly supported in age.} \end{cases} \quad (71)$$

As we have pointed out in the Modeling section 3.3, equation (71) describes an age-structured population of molecules undergoing a subdiffusive process corresponding to a CTRW at the microscopic level. Here, β is an age-dependent jump rate and ω is a space redistribution kernel. The boundary condition on $n(t, x, 0)$ at age $a = 0$ accounts for the particles landing at position x at time t after having “jumped” from position x' , at which they had remained during a time span exactly equal to a' . The fact that the loss term $\beta(a)n(t, x, a)$ is recovered in the boundary condition (and that ω is a probability distribution) leads to the conservation of the total population density $\int_0^\infty \int_{\mathbb{R}} n(\cdot, x, a) dx da$ over time.

Hypothesis 3 (Space jump kernel ω and jump rate β).

Throughout the article, we consider ω a Gaussian probability distribution of mean 0 and variance σ^2 , and β is defined as follows:

$$\begin{cases} \omega(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right), & \sigma > 0 \\ \beta(a) = \frac{\mu}{1+a}, & 0 < \mu < 1 \end{cases} \quad (72)$$

where $\mu \in (0, 1)$ is the subdiffusion exponent.

The stability result we prove follows the heuristics presented on the hyperbolic limit of the Fisher-KPP equation presented in Section 4.3. Here, we consider the hyperbolic rescaling $(t, x, a) \longrightarrow (t/\varepsilon, x/\varepsilon, a)$ of Problem (71), and we apply a Hopf-Cole transform:

$$n_\varepsilon(t, x, a) = n(t/\varepsilon, x/\varepsilon, a) = \exp(-\phi_\varepsilon(t, x, a)/\varepsilon). \quad (73)$$

Studying ϕ_ε enables us to accurately quantify the behaviour of small tails of the probability density function n in a way that is reminiscent of the theory of large deviations.

For (t, x, a) such that $\phi_\varepsilon(t, x, a) < \infty$, the function ϕ_ε satisfies:

$$\begin{cases} \partial_t \phi_\varepsilon + \frac{1}{\varepsilon} \partial_a \phi_\varepsilon - \beta = 0, & t \geq 0, \quad a > 0, \quad x \in \mathbb{R} \\ \exp(-\phi_\varepsilon(t, x, 0)/\varepsilon) = \int_0^{1+t/\varepsilon} \int_{\mathbb{R}} \beta(a) \omega(z) \exp(-\phi_\varepsilon(t, x - \varepsilon z, a)/\varepsilon) dz da \\ \phi_\varepsilon(0, x, a) = \phi_\varepsilon^0(x, a) = -\varepsilon \ln(n^0(x/\varepsilon, a)). \end{cases} \quad (74)$$

Let us denote by ψ_ε the boundary value at $a = 0$, which will be our main unknown:

$$\psi_\varepsilon(t, x) = \phi_\varepsilon(t, x, 0). \quad (75)$$

An interesting idea present in this article is that of proving convergence to the limiting Hamilton-Jacobi equation directly on the boundary condition of our equation (71) instead of using the perturbed test function method introduced in [48] on ϕ_ε .

We compute the solution of equation (74) along characteristic lines:

$$\phi_\varepsilon(t, x, a) = \begin{cases} \psi_\varepsilon(t - \varepsilon a, x) + \varepsilon \int_0^a \beta(s) ds, & t > 0, \quad \varepsilon a < t \\ \phi_\varepsilon^0(x, a - t/\varepsilon) + \varepsilon \int_{a-t/\varepsilon}^a \beta(s) ds, & t \geq 0, \quad a \geq t/\varepsilon. \end{cases} \quad (76)$$

Injecting (76) into the $a = 0$ boundary condition satisfied by ϕ_ε in (74) now yields:

$$\begin{aligned} 1 &= \int_0^{t/\varepsilon} \Phi(a) \int_{\mathbb{R}} \omega(z) \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \psi_\varepsilon(t - \varepsilon a, x - \varepsilon z)]\right) dz da \\ &\quad + \int_{t/\varepsilon}^{1+t/\varepsilon} \Phi(a) \int_{\mathbb{R}} \omega(z) \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \phi_\varepsilon^0(x - \varepsilon z, a - t/\varepsilon)] + \int_0^{a-t/\varepsilon} \beta\right) dz da, \end{aligned} \quad (77)$$

where $\Phi(a) = \beta(a) \exp(\int_0^a \beta(s) ds)$ is the distribution of waiting times, as presented in Section 3.3.

Let us formally derive the limiting Hamilton-Jacobi equation. Taking the formal limit of (77) when $\varepsilon \rightarrow 0$ yields:

$$1 = \int_0^\infty \Phi(a) \exp(a \partial_t \psi_0(t, x)) da \int_{\mathbb{R}} \omega(z) \exp(z \partial_x \psi_0(t, x)) dz. \quad (78)$$

It is a Hamilton-Jacobi equation, since it is equivalent to:

$$\partial_t \psi_0(t, x) + H(\partial_x \psi_0)(t, x) = 0, \quad (79)$$

with H defined as follows, where $\hat{\Phi}^{-1}$ is the inverse function of the Laplace transform of Φ :

$$H(p) = -\hat{\Phi}^{-1}\left(\frac{1}{\int_{\mathbb{R}} \omega(z) \exp(zp) dz}\right). \quad (80)$$

The main result of this paper is the proof of convergence as ε tends to 0 of the boundary value ψ_ε to the unique solution ψ_0 of the limiting Hamilton-Jacobi equation (78), under the suitable hypotheses on the initial condition that follow.

Hypothesis 4 (Initial condition ϕ_ε^0).

Ansatz: we take a WKB expansion of the initial condition:

$$\phi_\varepsilon^0(x, a) = v_\varepsilon(x) + \varepsilon \eta_\varepsilon(x, a) + \chi_{[0,1]}(a) \quad (81)$$

where $\chi_{[0,1]}(a)$ is worth 0 for $a \in [0, 1)$ and ∞ for $a \geq 1$. Hence, the essential support of ϕ_ε^0 is $[0, 1)$, as expected from the choice of n^0 . The functions ϕ_ε^0 , v_ε and η_ε satisfy the following properties uniformly over ε :

1. v_ε is **sublinear** and **bounded below**
2. η_ε is **bounded**.
3. ϕ_ε^0 and v_ε are **semi-concave** in x uniformly in a : there exists $\mathfrak{C}_{xx} \in \mathbb{R}$ such that for any $x, h \in \mathbb{R}$ and $a \in [0, 1)$, for any $\varepsilon > 0$,

$$\begin{cases} \phi_\varepsilon^0(x+h, a) + \phi_\varepsilon^0(x-h, a) - 2\phi_\varepsilon^0(x, a) \leq \mathfrak{C}_{xx}h^2, \\ v_\varepsilon(x+h) + v_\varepsilon(x-h) - 2v_\varepsilon(x) \leq \mathfrak{C}_{xx}h^2. \end{cases} \quad (82)$$

(Or equivalently, in the sense of distributions, $\partial_x^2 \phi_\varepsilon^0 \leq \mathfrak{C}_{xx}$ and $\partial_x^2 v_\varepsilon \leq \mathfrak{C}_{xx}$.)

4. ϕ_ε^0 is **Lipschitz continuous** in x uniformly in $a \in [0, 1)$

We note that semi-concavity is the most restrictive hypothesis.

Theorem 13 (Calvez, Gabriel, Mateos González). *Assume Hypotheses 3 and 4 hold.*

Then the boundary value ψ_ε converges in L_{loc}^∞ to ψ_0 , the unique viscosity solution of the limiting Hamilton-Jacobi equation (78) with an initial condition that is an appropriate limit of the initial conditions at $\varepsilon > 0$.

The proof strategy we follow is the classical “recipe” given by Barles in [11, Chapter 2] to tackle such problems. It consists in the following steps or close variations thereof – namely, a stability result relying on *a priori* estimates and a viscosity limit procedure, and a uniqueness result.

1. Prove that ψ_ε is locally bounded in L^∞ uniformly in $\varepsilon > 0$ — In our case, we prove ψ_ε is bounded below and sublinear.
2. Show that ψ_ε is locally bounded in $W^{1,\infty}$ (or in a Hölder $\mathcal{C}^{0,\alpha}$ space for some $0 \leq \alpha < 1$) uniformly in $\varepsilon > 0$ — In our case, a Lipschitz bound.
3. It follows from the first two steps that the sequence $(\psi_\varepsilon)_\varepsilon$ is compact in $\mathcal{C}(K)$ for any compact set K thanks to the Arzela-Ascoli Theorem.
4. Apply a stability result to a convergent subsequence of $(\psi_\varepsilon)_\varepsilon$ obtained by diagonal extraction: if the ψ_ε are sub- (respectively super-) solutions, then ψ_0 is a viscosity sub- (respectively super-) solution of the limit problem.
5. A comparison result for a viscosity subsolution and supersolution of equation (79) is needed to prove that (79) admits a unique viscosity solution in a suitable class of regularity, for suitable initial conditions.
6. By local compactness of $(\psi_\varepsilon)_\varepsilon$ and Hausdorff separation of $\mathcal{C}(K)$, $\psi_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \psi_0$, the unique viscosity solution of (79).

An alternative proof strategy consists in using the half-relaxed limits methodology described in [11, 2]. However, we did not do so because of two reasons. First, the advantage of half-relaxed limits is that they only require L^∞ *a priori* estimates. In our problem, semi-concavity plays a crucial role in the estimation of the lingering effect of initial conditions on the renewal boundary term. Hence, the natural regularity class for our solutions is at least locally Lipschitz, which greatly reduces the aforementioned advantage. However, and this is the second reason, the trade-off for that advantage is the requirement of a strong comparison result between the upper and lower semi-continuous envelopes of ψ_ε in order to show that the limit ψ_0 of a convergent subsequence of ψ_ε is the unique viscosity solution of the limiting Hamilton-Jacobi equation. Since our Hamiltonian has exponential growth, Lipschitz continuity of the solution is a natural requirement to prove that type of result, and it is unclear whether uniqueness could be obtained without Lipschitz regularity in this problem.

Let us start by Point 5 of the strategy. Several different uniqueness results hold for the evolution Hamilton-Jacobi Cauchy problem:

$$\begin{cases} \partial_t u(t, x) + H(D_x u(t, x)) = 0, & (t, x) \in (0, T] \times \mathbb{R}^n \\ u(0, x) = u^0(x), & x \in \mathbb{R}^n. \end{cases} \quad (83)$$

They rely on different sets of hypotheses on H and on u^0 , with trade-offs between the regularity of the initial condition and the solution of the Hamilton-Jacobi equation on the one hand, and hypotheses on the Hamiltonian on the other. In order to justify our choice of hypotheses imposed on the initial condition, let us present the properties satisfied by H and the uniqueness result that we use.

Proposition 14. *The Hamiltonian H defined in equation (80) satisfies the following properties:*

- (i) $H \in C^\infty(\mathbb{R}, \mathbb{R}_+)$,
- (ii) H is coercive: $\frac{H(p)}{p} \xrightarrow{p \rightarrow \infty} \infty$,
- (iii) H is convex, but not strictly uniformly convex.
- (iv) Behaviour around $p = 0$:

$$H(p) \sim_0 (\sigma p)^{2/\mu} (2\Gamma(1 - \mu))^{1/\mu}. \quad (84)$$

- (v) Behaviour for large p :

$$H(p) \sim_\infty \mu \exp\left(\frac{\sigma^2 p^2}{2}\right). \quad (85)$$

Under such conditions, the uniqueness result – in a suitable class of functions – that we have found most relevant is a modification, for a homogenous Hamiltonian that is not polynomially bounded above, of [31, Theorems 19.11 and 19.17] presented in the State of the Art 4.3.

Theorem 15 (Uniqueness theorem). *Let H be locally Lipschitz, convex and superlinear. Let u^0 be bounded below and Lipschitz continuous. Then there exists a unique viscosity solution of (83) within the class of Lipschitz continuous functions.*

The proof of the result follows Barles’s recipe as stated. The specific *a priori* estimates involved in Points 1 and 2 are lower boundedness, sublinearity and space and time Lipschitz estimates.

Apart from the result of Theorem 13 in itself, the most interesting points of our article concern the way we have dealt with three significant complications on which we have already commented above. In a nutshell, the key aspects of this work are the following.

1. The choice of applying the limit procedure to the boundary value ψ_ε rather than tackling the problem for ϕ_ε by means of perturbed test functions.
2. The non-existence of an integrable stationary state of the space-homogenous problem, the resulting non applicability of the maximum principle to prove bounds on ψ_ε , and the time-dependent corrections that allow us to prove sublinear bounds.
3. The need for an accurate control of the fate of the population that has never jumped before.

Let us further develop these comments.

Boundary hyperbolic limit vs. perturbed test functions

We have a limit problem whose limiting equation has less variables than the equations at $\varepsilon > 0$. The limiting equation is averaged over the fast variable a . Perturbed test functions introduced in [48] allow to tackle such problems by including a WKB-type Ansatz in the test functions themselves in order to deal with the vanishing dependency on the fast variable. The alternative approach we follow in Chapter 2 consists in considering the limit procedure directly on the boundary value ψ_ε , which does not depend on the fast variable a , and dealing with the perturbations that appear as an expression of the initial condition along characteristic lines.

Non-existence of an integrable stationary state.

The proof of the lower boundedness estimate on ψ_ε relies on the preservation of positivity of the transport equation and the boundary integral condition, as well as the lower boundedness of the initial condition in equation (74). The proof of the sublinear upper bound of ψ_ε relies on a comparison of ψ_ε to an expression that only depends on the initial condition ϕ_ε^0 , and on the upper bound on ϕ_ε^0 . None of the bounds on ψ_ε relies on the maximum principle, which does not apply here in a satisfactory way: this is the first main difficulty tackled in this article. It is due to the fact that the space-homogenous version (60) of problem (71) does not admit an integrable stationary measure.

For the sake of clarity, consider a jump rate β different from those we study in this dissertation – *e.g.* $\beta \equiv K$ or $\beta(a) = \mu/(1+a)$ for $\mu > 1$. In such cases, an integrable stationary measure F does exist, and the maximum principle applies as follows.

If there exist constants $0 < c < C$ satisfying

$$cF \leq n^0 \leq CF, \tag{86}$$

then for all positive time t , such bounds are preserved:

$$cF \leq n(t, \cdot) \leq CF. \tag{87}$$

In such a case, the particle mean waiting time equals $\|F\|_1$: indeed, integrating by parts gives us

$$\|F\|_1 = \int_0^\infty a\beta(a) \exp\left(-\int_0^a \beta(s) ds\right) da,$$

and we recover age-integrable estimates on n .

However, returning to our subdiffusive case $\beta(a) = \mu/(1+a)$ for $0 < \mu < 1$ (and perturbations thereof), F is not integrable. Hence, if the previous bounds (87) were to hold, the integrability of n with respect to age would be violated. The maximum principle does not apply in that way.

Despite that, it is still possible to prove suitable bounds by considering time-dependent corrections. Such corrections could be expected thanks to the following heuristic approach inspired by the quantitative self-similar decay shown in [15] in Chapter 1. Taking the expression of the self-similar profile $W_\infty(b) = c_\infty b^{-\mu}(1-b)^{\mu-1}$ presented there, we intuitively expect

$$n(t, a) \simeq \frac{1}{1+t} W_\infty\left(\frac{a}{1+t}\right). \quad (88)$$

This translates in our context as a time logarithmic correction of the L^∞ bound of $-\varepsilon \ln(n)$, which indeed clearly appears in the computations. The recovery of such corrective terms is the first main complication tackled in this paper.

Lipschitz bounds – technical considerations and link to uniqueness proofs

The proof of the space Lipschitz estimates is not overly original, but it does entail certain interesting technical aspects. It relies on applications of the maximum principle to equations involving integrals with respect to a certain probability measure that depends on ψ_ε and the initial condition ϕ_ε^0 , of the increase rates of ψ_ε in x . The main difficulty in these proofs is linked to the fact that the increase rates to which we wish to apply the maximum principle may not reach their extrema. In order to circumvent this problem, we penalise the increase rates, guaranteeing the penalised increase rates reach their extrema at some (t_0, x_0) . We end up with a proof strategy reminiscent of the proof of uniqueness of the solution of a Hamilton-Jacobi equation by the method of doubling the variables. Hence, at one point, it is necessary to bound $|x_0|$. The sublinear hypothesis on the initial condition is not strong enough to guarantee that the space penalisation $-\delta_2 x_0^2$ vanishes as δ_2 tends to 0. However, contrary to the proof of uniqueness of the solution of a Hamilton-Jacobi equation by the method of doubling the variables, we already have uniqueness. Therefore, it is enough to define a sequence of auxiliary functions that satisfy the Lipschitz bound and converge uniformly to ψ_ε , which we do.

The proof of the time Lipschitz bounds relies on the similar tools.

Accurate control of the initial condition-renewal in long time

A viscosity limit procedure corresponding to Points 3 and 4 of the outlined strategy concludes the proof of Theorem 13. Proving that the limit ψ_0 of a subsequence of ψ_ε is a viscosity subsolution of the limiting Hamilton-Jacobi equation (77) is simple enough. However, proving it is a supersolution requires more ingenuity.

The proof requires an accurate control of the fate of the aging particles that come from the initial data ϕ_ε^0 and have never jumped: namely, a sharp enough upper bound on $\mathfrak{B}_\varepsilon(t, x)$, which we define as follows. We name $\mathfrak{B}_\varepsilon(t, x)$ the proportion among particles arriving at a given space location x at a given time t of those that had never jumped before. Due to the heavy tail of the waiting time distribution Φ , \mathfrak{B}_ε decays very slowly in time. By using the preservation of the semi-concavity of the initial condition and applying the maximum principle to the right expressions, we recover a sharp anomalous scaling:

$$\mathfrak{B}_\varepsilon \lesssim \varepsilon^\mu \quad (89)$$

locally uniformly in time. This sort of non-local refinement of the time Lipschitz bound of ψ_ε is the third main difficulty tackled in this work.

6 Perspectives and work in progress

Throughout this dissertation, I have studied structured linear PDE in homogenous space settings by means of methods that can be generalised to non-linear equations. I present hereafter certain logical generalisations of this work, namely the study of space-inhomogeneous settings, the coupling of space and age dynamics, and non-linear problems of propagation in a reaction-subdiffusion setting.

6.1 Multi-compartment model

Relative entropy techniques

Particles undergoing diffusion with a space-dependent diffusion coefficient tend to concentrate more in lower diffusion zones. So do particles undergoing subdiffusion with a space-dependent subdiffusion exponent $\mu(x)$: they concentrate at zones where μ is lower. However, the equilibria differ: in the context of subdiffusion, higher subdiffusion regions seem to be completely depleted, whereas this is not the case for diffusion. The reader can find a derivation of the space-inhomogeneous master equation, heuristics and simulations in a discretised space in [55]

In an ongoing work with Thomas Lepoutre, we propose studying concentration dynamics in the following toy model. We consider particles that jump alternatively between two space compartments, and remain at compartment i during a random time taken according to a law

$$\Phi_i(a) = \frac{\mu_i}{(1+a)^{1+\mu_i}}.$$

We have identified a reasonable Ansatz towards which the concentrations of each compartment seem to decay. However, the convergence by methods similar to those used in Chapter 1 has yet to be proven.

6.2 Coupled age and space dependencies

Hamilton-Jacobi techniques, numerical simulation, hyperbolic shock fronts

Certain intracellular proteins experience a random motion that deviates from normal diffusion. We refer the reader to [81] for a review. There exist significant differences between slow diffusion and subdiffusion [146], for instance in the context of the formation of morphogen gradients [169, 168]. A bounded velocity morphogen movement has consequences that differ from a scenario in which arbitrarily high velocities are possible [154].

Starting from the previous considerations, I wish to study the pertinence and tractability of a generalisation of equation (71) which would incorporate an age dependency of the spatial redistribution kernel, for instance a compact age-dependent support, so as to avoid instantaneous propagation:

$$\begin{cases} \partial_t n + \partial_a n + \beta(a)n(t, x, a) = 0 \\ n(t, x, 0) = \int_{-\infty}^{\infty} \int_0^{\infty} \omega(x - x', a)\beta(a)n(t, x', a) da dx', \end{cases} \quad (90)$$

where $\beta(a) = \mu/(1+a)$ with $0 < \mu < 1$.

I consider the following problems to be of interest in this context.

1. The generalisation the results of convergence to a limiting Hamilton-Jacobi equation proved in [28] (knowing certain hypotheses of the article would not hold any more and half-relaxed limits methods may be necessary).

2. The quantitative study of the effect of the new hypotheses on $\int_0^\infty n(t, x, a) da$.
3. The introduction of reaction terms in equation (90) following the example detailed in 6.3 and the study of the eventual differences between propagation phenomena in the model where ω does not depend on a and in the model above.
4. In [29], the authors study the finite speed propagation of morphogen molecules described by a porous medium equation and a relativist heat equation. They prove the existence of travelling waves and solutions in the form of shocks reminiscent of hyperbolic shock waves. It could be interesting to study the solutions of similar problems where the finite speed displacement is subdiffusive.

6.3 Reaction-subdiffusion integro-differential equations

Hamilton-Jacobi techniques, definition of fronts in an integro-differential setting, numerical simulation, kinetic equations

The Fisher–Kolmogorov–Petrovskii–Piskunov equation is a model of reaction-diffusion which admits travelling wave solutions that propagate at a constant velocity. The reader may consult a very short introduction to front propagation in Fisher-KPP in Section 4.3. However, analogous models in reaction-subdiffusion pose additional definition problems.

In [28], Calvez, Gabriel and I have studied a linear equation with no reaction terms. There exist an extensive literature treating anomalous diffusion-reaction fronts: we refer the reader to two reviews [119], [158] and the references therein. However, the form of the reaction term depends on the studied problem. That dependency has strong consequences on the propagation of fronts in the described systems. By quoting freely section 3.2 of [119],

“[In a reaction-subdiffusion equation structured in time elapsed since the last space jump (= “age”), if the products of the reaction are introduced at an age equal to that of the reactants, the global population] has a tendency of “aging”, which leads to the decrease of the jump probability and hence slowing down of all the processes, including the front propagation (“propagation failure”) [66, 140]. [...] Another approach is adopted in [157, 167] where it is assumed that a molecule born in the course of reaction has zero age, as if it had arrived at the reaction point from somewhere else: the argument of the waiting time distribution is set to zero after the chemical transformation of the molecule. In that case, [they are introduced as a boundary term, and a death term is added in the equation describing the probability density function at positive age].”

I would like to introduce reaction terms following the age structure of equation (71) in order to take into account the non-trivial interaction between reaction and random motion. Several options are sensible, including the following.

Collision-induced renewal

A first interesting step is the study of the following toy model, which describes an $A + A \rightarrow 2A$ reaction whose products are born at age 0, *i.e.* renewal induced by collision.

$$\begin{cases} \partial_t n(t, x, a) + \partial_a n(t, x, a) + \beta(a)n(t, x, a) + Rn(t, x, a) \int_0^\infty n(t, x, a') da' = 0 \\ n(t, x, 0) = \int_0^\infty \int_{\mathbb{R}} \beta(a)\omega(x - x')n(t, x', a) dx' da + R \left(\int_0^\infty n(t, x, a) da \right)^2. \end{cases} \quad (91)$$

Here, the renewal rate $\beta(a)$ is replaced by an effective rate $\beta(a) + R \int_0^\infty n(t, x, a) da$ depending on the local density. I consider interesting to start by comparing the age distributions at fixed x for this problem and for the problem without reaction.

Decoupled reaction and subdiffusion

Let us consider the reaction $A + B \rightarrow 2A$ whose products are born at an age equal to that of the reactants. Let n_1 and n_2 be the respective concentrations of molecules A and B .

$$\left\{ \begin{array}{l} \partial_t n_1(t, x, a) + \partial_a n_1(t, x, a) + \beta(a)n_1(t, x, a) = Rn_2(t, x, a) \int_0^\infty n_1(t, x, a') da' \\ \partial_t n_2(t, x, a) + \partial_a n_2(t, x, a) + \beta(a)n_2(t, x, a) + Rn_2(t, x, a) \int_0^\infty n_1(t, x, a') da' = 0 \\ n_1(t, x, 0) = \int_0^\infty \int_{\mathbb{R}} \beta(a)\omega(x - x')n_1(t, x', a) dx' da \\ n_2(t, x, 0) = \int_0^\infty \int_{\mathbb{R}} \beta(a)\omega(x - x')n_2(t, x', a) dx' da. \end{array} \right. \quad (92)$$

The macroscopic equation describing that reaction is irreversible (see [66], II, B, b – leading edge linearization, non-Markovian case), admits a minimal speed of propagation equal to 0 for a pulled, exponentially decaying, front of A particles. However, the reversible reaction admits a positive minimal speed of propagation of fronts (see [166] – use of Hamilton-Jacobi tools). The mesoscopic equation (92) does not admit a non-0 steady state N that is homogenous in space and such that βN is integrable in age. Therefore, it is not possible to define travelling wave solutions that link two homogenous steady states. However, equation (92) preserves the information on the age structure. I would like to generalise the work done in [28] in order to treat, at first, the invasion by species A of a domain initially containing only molecules of type B .

Chapter 1

Quantitative self-similar decay

Résumé

Dans ce travail en collaboration avec Hugues Berry et Thomas Lepoutre [15], nous décrivons la convergence vers un profil autosimilaire des solutions d'une équation de renouvellement à queue lourde. Une approche inspirée de techniques d'entropie relative nous permet de quantifier explicitement le taux de décroissance. Une difficulté importante apparaît, dû au fait que l'équation en variables auto-similaires n'est pas autonome et que nous ne disposons pas d'une expression analytique d'une solution. Afin de quantifier la convergence, nous estimons l'attraction vers un "pseudo-équilibre" ayant une dépendance en temps. Celui-ci converge à son tour vers un profil stationnaire.

Abstract

In this joint work with Hugues Berry and Thomas Lepoutre [15], we describe the convergence towards a self-similar profile of the solutions of a heavy-tailed renewal equation. An approach inspired by relative entropy techniques allows us to obtain quantitative explicit decay rates. An important difficulty arises from the fact that the equation in self-similar variables is not autonomous and we do not have a specific analytical solution. Therefore, in order to quantify the convergence, we estimate attraction to a time-dependent "pseudo-equilibrium", which in turn converges to a stationary profile.

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1.1 Introduction

1.1.1 Brief model description

Recent methodological advances in cell biology allowed the measurements of the displacement of single molecules (or assemblies thereof) in living single cells. Those investigations have consistently reported that the random displacement of molecules inside cells often deviates from Brownian motion, with the mean squared displacement $\langle \mathbf{r}^2(t) \rangle$ that does not scale linearly with time, as in Brownian motion, but sublinearly, with a power-law behaviour: $\langle \mathbf{r}^2(t) \rangle \propto t^\mu$ [69, 25, 125, 39]. This behaviour is usually referred to as “anomalous” diffusion or “subdiffusion”, since $\mu < 1$ usually, for non-active transport (for a review see e.g. [80]).

Continuous-time random walks (CTRW) are one of the main mechanisms that are recurrently evoked to explain the emergence of subdiffusion in cells. CTRW were introduced fifty years ago by Montroll and Weiss as a generalisation of random walks [116], where the residence time (the time between two consecutive jumps) is a random variable τ with probability distribution $\phi(\tau)$ (see [108] for a review). If the expectation of τ is defined, for instance when τ is dirac-distributed or decays exponentially fast, one recovers the “normal” Brownian motion. However, when the expectation of τ diverges, for instance when $\phi(\tau)$ is heavy-tailed, $\phi(\tau) \propto \tau^{-(1+\mu)}$ with $0 < \mu < 1$, the CTRW describes a subdiffusive behaviour, with $\langle \mathbf{r}^2(t) \rangle \propto t^\mu$.

One great achievement of CTRW is that they can readily be used to derive mean-field equations for the spatio-temporal dynamics of the random walkers. Indeed, starting from $\phi(\tau)$, combinations of Laplace and Fourier transforms lead to a “subdiffusion” equation for the density of random walkers located at position \mathbf{x} at time t : $\partial_t \rho(\mathbf{x}, t) = D_\mu \mathcal{D}_t^{1-\mu} \nabla^2 \rho(\mathbf{x}, t)$ where D_μ is a generalised diffusion coefficient and $\mathcal{D}_t^{1-\mu}$ is the Riemann-Liouville fractional derivative operator [108, 107]. Such a fractional dynamics formulation is very attractive for modelling in biology, in particular because of its apparent similarity with the classical diffusion equation. However, contrarily to the diffusion equation, the Riemann-Liouville operator is non-Markovian. This non-Markovian property becomes a serious obstacle when one wants to couple subdiffusion with chemical reaction [79, 170, 56].

Here, we take an alternative approach to CTRW that maintains the Markovian property of the transport equation at the price of a supplementary independent variable. We associate each random walker with an age a , that is reset when the random walker jumps. In one dimension of space, we note $n(t, x, a)$ the density probability distribution of walkers at time t that have been residing at location x during the last span of time a . The dynamics of the CTRW is then described with an age-renewal equation with spatial jumps that reinitialise the age:

$$\begin{cases} \partial_t n(t, x, a) + \partial_a n(t, x, a) + \beta(a)n(t, x, a) = 0, & t \geq 0, \quad a > 0, \quad x \in \mathbb{R} \\ n(t, x, a = 0) = \int_0^\infty \int_{\mathbb{R}} \beta(a') \omega(x - x') n(t, x', a') dx da' \\ n(t = 0, x, a) = n^0(x, a). \end{cases} \quad (1.1)$$

The kernel ω describes the spatial distribution of jump destinations (typically a Gaussian distribution centered at the origin position), and the function $\beta(a)$ gives the jump rate. Since we are mostly interested here in the subdiffusive case (where the expectation of the residence time diverges), we will focus throughout this Chapter on the case:

$$a\beta(a) \xrightarrow{a \rightarrow \infty} \mu \in (0, 1). \quad (1.2)$$

The precise meaning of the limit will be given later on. The limit μ in eq.(1.1) is the subdiffusion exponent: for $\mu > 1$, eq.(1.1) describes a diffusive process, whereas for $0 < \mu < 1$ the mean

time a particle has to wait between two consecutive renewals diverges and the mean squared displacement exhibits subdiffusion with exponent μ . The distribution of residence time $\phi(\tau)$ evoked above is related to the jump rate as: $\phi(\tau) = \beta(\tau) \exp(-\int_0^\tau \beta(s) ds)$. Note that this age-structured approach is not uncommon in the CTRW literature [107, 55]. Our main contribution here is to use it in conjunction with approaches borrowed from the study of partial differential equations.

In the present work, we restrict our attention to the temporal evolution of the age distribution of the walkers. To this end, we simplify the problem by considering its spatially-homogenous version, namely:

$$\begin{cases} \partial_t n(t, a) + \partial_a n(t, a) + \beta(a)n(t, a) = 0, & t \geq 0, \quad a > 0 \\ n(t, a = 0) = \int_0^\infty \beta(a')n(t, a')da' \\ n(t = 0, a) = n^0(a). \end{cases} \quad (1.3)$$

1.1.2 Self-similar solutions

The only steady state solution of eq.(1.3) in L^1 is 0, which doesn't allow us to describe the dynamics of the system in a satisfactory way. Hence the search for self-similar solutions. An educated guess is that they should be of the following form, with $A(t)$ to be determined:

$$n(t, a) = \frac{1}{A(t)} w\left(\underbrace{\ln(1+t)}_\tau, \underbrace{a/A(t)}_b\right).$$

Let us consider, for the sake of simplicity, an initial condition supported on $[0, 1)$. By injecting the previous expression into eq.(1.3), we find that the natural choice $A(t) = 1 + t$ preserves the initial condition $n(0, b) = n^0(b) = w(0, b) = w^0(b)$, and yields:

$$\begin{cases} \partial_\tau w + \partial_b((1-b)w) + e^\tau \beta(e^\tau b)w = 0 \\ w(\tau, 0) = \int_0^1 e^\tau \beta(e^\tau b)w(\tau, b)db \\ w(0, b) = w^0(b). \end{cases} \quad (1.4)$$

Note that for an initial condition supported on $[0, A_+)$, $A(t) = A_+ + t$ is a better choice and leads to a similar analysis. If the initial condition is not compactly supported, the tail of the age distribution can influence the convergence rate we give below.

It is important to note that the previous system is not autonomous, for the term $e^\tau \beta(e^\tau b)$ depends on τ . This rescaling does not lead to a classical steady state, and we could not find a particular solution of the previous equation. However, we may look for a stationary state satisfying formally the following equation, since we consider here $\beta(a) \sim \frac{\mu}{a}$.

$$\partial_b((1-b)W_\infty) + \frac{\mu}{b}W_\infty = 0$$

where the boundary condition cannot be stated as an equality since W_∞ is expected to blow up at 0, but can be understood as an equivalence as ε tends to 0 of $W_\infty(\varepsilon)$ and $\int_\varepsilon^1 \frac{\mu}{b}W_\infty(b)db$.

This leads us to define the self-similar equilibrium as:

$$W_\infty(b) = \frac{c_\infty}{b^\mu(1-b)^{1-\mu}} \quad (1.5)$$

which is called the arcsine distribution, or Dynkin-Lamperti distribution. c_∞ is defined such that $\|W_\infty\|_1 = 1$. Under some conditions, we can expect that $w(\tau, b)$ will converge to eq.(1.5) when $\tau \rightarrow \infty$.

A similar result in probability theory appears in Feller's book [59] tome II, chapter XI, especially in section 5 and onwards, where the renewal problem is tackled by considering the waiting time before the n^{th} renewal. For an introduction to renewal theory, see the eponymous chapter (8.6) in [17]. However, no convergence rate is given for our infinite mean waiting time problem in any of these books, and we have been unable to locate such a convergence rate in the subsequent literature. Recent developments in Ergodic Theory for mildly related problems (see chapter 8.11 of [17] for an introduction to Darling-Kac theory), have yielded convergence rates, that are optimal in certain cases, as shown in [106] and [149].

1.1.3 Main results

Throughout the Chapter, the following set of hypotheses will intervene. Hypothesis **(H1)** will be used in properties of convergence without a rate while hypothesis **(H2)** will allow convergence rate estimates.

Hypothesis 5 (H1). β is a positive, bounded, and non-increasing function satisfying

$$\lim_{a \rightarrow \infty} a\beta(a) = \mu \in (0, 1).$$

Remark. We will always assume β to be non-increasing for the sake of simplicity (in particular in theorem 20, propositions 18 and 35, and lemmas 28 and 37). The monotonicity can be replaced by the following hypothesis:

β is a positive, bounded function satisfying $\lim_{a \rightarrow \infty} a\beta(a) = \mu \in (0, 1)$, such that $\underline{\beta}$ defined as follows

$$\underline{\beta}(a) = \inf_{x \in [0, a]} \beta(x) \tag{1.6}$$

also satisfies $\lim_{a \rightarrow \infty} a\underline{\beta}(a) = \mu$.

This leads to minor changes in the proofs, the loss of a multiplicative constant in the affected results and replacing β by $\underline{\beta}$ where it corresponds.

Hypothesis 6 (H2). β satisfies **(H1)**. Additionally, $\beta(a) = \frac{\mu}{1+a} + g(a)$, where $g \in L^1$ and there exist $K, \alpha > 0$ such that

$$\int_a^\infty |g(s)| ds \leq \frac{K}{(1+a)^\alpha}.$$

Remark. For the sake of clarity we will investigate separately the particular case $g = 0$, called the "reference case". Then, all our results will be extended to the general case at the expense of the convergence rates.

Due to the specific shape of W_∞ and to the boundary condition, it is difficult to investigate in a direct way the evolution of $\|w - W_\infty\|_1$: the methods we describe subsequently fail to do so. However, we could recover a quantitative explicit convergence rate with respect to a "pseudo-equilibrium" W which will be proved to converge in L^1 to W_∞ .

Definition 16. We define the pseudo-equilibrium W over $\mathbb{R}_+ \times [0, 1)$ as follows:

$$W(\tau, b) = \frac{C(\tau)}{e^{B(e^\tau b)}(1-b)^{1-\mu}}. \tag{1.7}$$

where $B(a) = \int_0^a \beta(s) ds$ and C is defined so that $\|W(\tau, \cdot)\|_{L^1} = 1$.

In particular in the reference case $\beta(a) = \frac{\mu}{1+a}$, it may be written as

$$W(\tau, b) = \frac{C(\tau)e^{-\mu\tau}}{(e^{-\tau} + b)^\mu(1-b)^{1-\mu}}.$$

Note the similarity between this expression and that for W_∞ in eq.(1.5). In the following, we obtain explicit convergence rates of $w(\tau, b)$ to W , developing proofs based on Relative Entropy estimates. Importantly, we show that W converges to W_∞ at the same rate (up to multiplication by a constant) as the rate with which w converges to W . Hence, the convergence to the pseudo-equilibrium W yields a very good estimate of the convergence to the self-similar equilibrium W_∞ . Finally, we carry out Monte-Carlo simulations of zero-dimensional CTRW to illustrate and question the optimality of our main analytical results.

Definition 17. For the moment and for the sake of simplicity, let us define:

$$\mathcal{H}(\tau) = \|w(\tau, \cdot) - W(\tau, \cdot)\|_{L^1([0,1])}.$$

Remark. Later on, we shall define more generally \mathcal{H} as a relative entropy, the L^1 distance being a particular case more suited to our purposes.

In this Chapter, we prove the following propositions:

Proposition 18. Under hypothesis **(H1)**, we have:

$$\mathcal{H}(\tau) \xrightarrow{\tau \rightarrow \infty} 0.$$

Our first quantitative result is a convergence rate for the reference case of hypothesis **(H2)**.

Theorem 19. Let $\beta(a) = \frac{\mu}{1+a}$. Then we have the following convergence rates:

$$\begin{cases} \mathcal{H}(\tau) \leq e^{-\mu\tau} \left[\mathcal{H}(0) \left(\frac{2}{1+e^{-\tau}} \right)^\mu - \frac{8}{2\mu-1} \left(\frac{2}{1+e^{-\tau}} \right)^\mu \right] + e^{-(1-\mu)\tau} \left[\frac{8}{2\mu-1} \left(\frac{2}{1+e^{-\tau}} \right)^\mu \right] & \text{if } \mu \neq \frac{1}{2} \\ \mathcal{H}(\tau) \leq e^{-\frac{\tau}{2}} \left[\left(\frac{2}{1+e^{-\tau}} \right)^{\frac{1}{2}} (\mathcal{H}(0) + 8\tau) \right] & \text{if } \mu = \frac{1}{2}. \end{cases} \quad (1.8)$$

A modified, yet analogous, convergence rate still holds for $g \neq 0$:

Theorem 20. Suppose hypothesis **(H2)** holds.

If $\alpha > 1 - \mu$, we recover the optimal rate of convergence

$$\mathcal{H}(\tau) \leq \begin{cases} K(e^{-\mu\tau} + e^{-(1-\mu)\tau}), & \text{if } \mu \neq 1/2 \\ K\tau e^{-\tau/2}, & \text{if } \mu = 1/2. \end{cases}$$

If $\alpha \leq 1 - \mu$, we need to distinguish between several cases:

$$\mathcal{H}(\tau) \leq \begin{cases} K(e^{-\alpha\tau} + e^{-\mu\tau}), & \text{if } \mu \neq \alpha < 1 - \mu \\ K(1 + \tau)e^{-\mu\tau}, & \text{if } \alpha = \mu < 1 - \mu, \\ K(\tau e^{-(1-\mu)\tau} + e^{-\mu\tau}), & \text{if } \alpha = 1 - \mu \neq 1/2, \\ K(1 + \tau^2)e^{-\tau/2}, & \text{if } \alpha = \mu = 1 - \mu = 1/2. \end{cases}$$

We finally reinterpret our results in terms of non-rescaled variables, for instance in the reference case $\beta(a) = \frac{\mu}{1+a}$.

Corollary 21. *Assume n^0 is supported in $[0, 1)$ and $\beta(a) = \frac{\mu}{1+a}$, then if we denote*

$$N_\infty(t, a) = \begin{cases} \frac{c_\infty}{a^\mu(1+t-a)^{1-\mu}}, & a < 1+t, \\ 0, & a > 1+t. \end{cases}$$

Then if $\mu \neq 1/2$, there exists K such that

$$\|n(t, \cdot) - N_\infty(t, \cdot)\|_1 \leq \frac{K}{(1+t)^\mu} + \frac{K}{(1+t)^{1-\mu}}.$$

If $\mu = 1/2$, then we have

$$\|n(t, \cdot) - N_\infty(t, \cdot)\|_1 \leq \frac{K(1 + \log(1+t))}{\sqrt{1+t}}.$$

Remark. An analogous version in non-rescaled variables can be given for theorem 20.

1.1.4 Outline of the Chapter

The Chapter is organised as follows. In Section 1.2 we set the entropic structure of the equation and the main properties of the pseudo equilibrium W . In particular we establish (non quantitatively) that

$$\lim_{\tau \rightarrow \infty} \|w(\tau, \cdot) - W(\tau, \cdot)\|_1 = 0, \quad \lim_{\tau \rightarrow \infty} \|W_\infty(\tau, \cdot) - W(\tau, \cdot)\|_1 = 0,$$

proving thereby

$$\lim_{\tau \rightarrow \infty} \|w(\tau, \cdot) - W_\infty(\tau, \cdot)\|_1 = 0.$$

Section 1.3 deals with quantitative convergence rates towards the pseudo-equilibrium W , proving Theorems 19 and 20. A convergence rate for $\|w - W_\infty\|_1$, some effects of initial conditions on the convergence rates, and convergence rates in non-rescaled variables are dealt with in Section 1.4. Finally, we show the results of some simulations in the eponymous section.

1.2 Entropic structure

Even if we are mainly estimating L^1 -norms, we see our proof as a specific case of relative entropy inequalities. Rates could be obtained following the lines of our proofs for other entropies.

1.2.1 L^1 contraction for compactly supported solutions

The first evidence of an attractor is the L^1 contraction of compactly supported solutions. If we consider two initial data supported in $[0, 1)$, w_1^0, w_2^0 , non-negative and of mass 1 and the associated solutions w_1, w_2 , then we have the following property (we take $\beta(a) = \frac{\mu}{1+a}$ for this computation)

$$\frac{d}{d\tau} \int_0^1 |w_1(\tau, b) - w_2(\tau, b)| db \leq D(\tau)$$

where

$$D(\tau) = \left| \int_0^1 \frac{\mu}{e^{-\tau} + b} (w_1(\tau, b) - w_2(\tau, b)) db \right| - \int_0^1 \frac{\mu}{e^{-\tau} + b} |w_1(\tau, b) - w_2(\tau, b)| db.$$

Since mass is conserved i.e., $\int_0^1 w_1 - w_2 = 0$, we have easily

$$\begin{aligned} D(\tau) &= \left| \int_0^1 \left(\frac{\mu}{e^{-\tau} + b} - \frac{\mu}{e^{-\tau} + 1} \right) (w_1(\tau, b) - w_2(\tau, b)) db \right| \\ &\quad - \int_0^1 \frac{\mu}{e^{-\tau} + b} |w_1(\tau, b) - w_2(\tau, b)| db \\ &\leq \int_0^1 \left(\frac{\mu}{e^{-\tau} + b} - \frac{\mu}{e^{-\tau} + 1} \right) |w_1(\tau, b) - w_2(\tau, b)| db \\ &\quad - \int_0^1 \frac{\mu}{e^{-\tau} + b} |w_1(\tau, b) - w_2(\tau, b)| db. \end{aligned}$$

Thereby, we obtain

$$\frac{d}{d\tau} \int_0^1 |w_1(\tau, b) - w_2(\tau, b)| db \leq -\frac{\mu}{1 + e^{-\tau}} \int_0^1 |w_1(\tau, b) - w_2(\tau, b)| db.$$

And this leads to

$$\int_0^1 |w_1(\tau, b) - w_2(\tau, b)| db = O(e^{-\mu\tau}).$$

In the next section we identify the attractor towards which solutions converge.

1.2.2 Pseudo equilibrium

We start by recalling the definition of what we call the pseudo equilibrium. We recall Definition 16:

$$W(\tau, b) = \frac{C(\tau)}{(1 - b)^{1 - \mu} e^{B(e^\tau b)}}$$

where $B(a) = \int_0^a \beta(s) ds$ and C is defined so that $\|W(\tau, \cdot)\|_{L^1} = 1$.

Remark. By definition,

$$W(\tau, b = 0) = C(\tau). \tag{1.9}$$

Firstly we establish the fact that $W(\tau, \cdot)$ is an approximation of W_∞ :

Lemma 22. *Assume hypothesis (H1). Then, defining the Dynkin-Lamperti distribution as in (1.5):*

$$W_\infty(b) = \frac{b^{-\mu}(1 - b)^{\mu-1}}{\int_0^1 b^{-\mu}(1 - b)^{\mu-1} db}$$

we have

$$\lim_{\tau \rightarrow +\infty} \|W(\tau, \cdot) - W_\infty\|_1 = 0.$$

Proof. We start with the model case $\beta(a) = \frac{\mu}{1+a}$. In this case, we can write

$$W(\tau, b) = \frac{(e^{-\tau} + b)^{-\mu}(1 - b)^{\mu-1}}{\int_0^1 (e^{-\tau} + b)^{-\mu}(1 - b)^{\mu-1} db},$$

and the result is immediate.

For the general case, we use the following useful bound on W :

Lemma 23. *Under hypothesis (H1), for any $\eta > 0$ satisfying $\eta < \min(\mu, 1 - \mu)$, there exists a constant (depending on β , but not on τ) $C_\eta > 0$ such that*

$$W(\tau, b) \leq C_\eta \frac{(1-b)^{\mu-1}(e^{-\tau} + b)^{-(\mu+\eta)}}{\int_b^1 (1-b')^{\mu-1}(e^{-\tau} + b')^{-(\mu+\eta)} db'}.$$

Proof. We first notice that there always exists a function $g_\eta \geq 0$, compactly supported, such that

$$\beta(a) \leq \frac{\mu + \eta}{1 + a} + g_\eta(a).$$

Thereby, for all $b' \geq b$ and all $\tau \geq 0$, we have

$$B(e^\tau b) - B(e^\tau b') = - \int_{e^\tau b}^{e^\tau b'} \beta(s) ds \geq (\mu + \eta) \ln \left(\frac{1 + e^\tau b}{1 + e^\tau b'} \right) - \|g_\eta\|_1,$$

and

$$e^{B(e^\tau b) - B(e^\tau b')} \geq e^{-\|g_\eta\|_1} (e^{-\tau} + b')^{-(\mu+\eta)} (e^{-\tau} + b)^{\mu+\eta} \quad (1.10)$$

Then we recall that by definition

$$W(\tau, b) = \frac{e^{-B(e^\tau b)}(1-b)^{\mu-1}}{\int_0^1 e^{-B(e^\tau b')} (1-b')^{\mu-1} db'} \leq \frac{(1-b)^{\mu-1}}{\int_b^1 e^{B(e^\tau b) - B(e^\tau b')} (1-b')^{\mu-1} db'}.$$

Inserting (1.10) in the latter, we obtain the result with $C_\eta = e^{\|g_\eta\|_1}$. \square

It is worth noticing that we can establish with the same proof

$$\forall \varepsilon \leq 1/2, \int_0^\varepsilon W(\tau, b) db \leq C'_\eta \varepsilon^{1-(\mu+\eta)}$$

We denote W_{ref} for $\beta(a) = \frac{\mu}{1+a}$ and notice that in our general case

$$\beta(a) = \frac{\mu}{1+a} + g(a).$$

We denote $G(a) = \int_0^a g$ and we can write

$$W(\tau, b) = K(\tau) e^{G(e^\tau) - G(e^\tau b)} W_{ref}(\tau, b).$$

for some $K(\tau) > 0$ that insures the normalisation $\int_0^1 W = 1$. We introduce some $\eta > 0$ as in lemma 23. We already establish in the proof of lemma 23 that for $\varepsilon \leq 1/2$,

$$\int_0^\varepsilon W \leq C_\eta \varepsilon^{1-(\mu+\eta)}.$$

Therefore,

$$1 \geq K(\tau) \int_\varepsilon^1 e^{G(e^\tau) - G(e^\tau b)} W_{ref}(\tau, b) db \geq 1 - C_\eta \varepsilon^{1-(\mu+\eta)}.$$

Let $\varepsilon > 0$ be fixed. Furthermore, since $g(a) = o(\frac{1}{a})$, we have

$$\sup_{b \geq \varepsilon} |G(e^\tau) - G(e^\tau b)| = o \left(\int_{e^\tau \varepsilon}^{e^\tau} \frac{da}{a} \right) = o(\ln \varepsilon) = o(1).$$

In particular, for $\varepsilon > 0$ fixed,

$$e^{G(e^\tau) - G(e^\tau b)} \rightarrow 1,$$

uniformly on $(\varepsilon, 1)$ as $\tau \rightarrow +\infty$. As a consequence, we have for all $\varepsilon > 0$

$$\int_\varepsilon^1 e^{G(e^\tau) - G(e^\tau b)} W_{ref}(\tau, b) db - \int_\varepsilon^1 W_{ref}(\tau, b) db \rightarrow 0.$$

This leads, for any $\varepsilon > 0$, to the bounds:

$$\limsup_{+\infty} K(\tau) \leq \frac{1}{\int_\varepsilon^1 W_\infty}, \quad \liminf_{+\infty} K(\tau) \leq \frac{1 - C_\eta \varepsilon^{1 - (\mu + \eta)}}{\int_\varepsilon^1 W_\infty}.$$

Letting $\varepsilon \rightarrow 0$, we obtain

$$\lim_{+\infty} K(\tau) = 1.$$

What we established proves that, for any $\varepsilon > 0$,

$$\int_\varepsilon^1 |W - W_{ref}| \rightarrow 0.$$

We can conclude using lemma 23 that we have

$$\int_0^1 |W - W_{ref}| \rightarrow 0.$$

Consequently,

$$\lim_{+\infty} \|W - W_\infty\|_1 = 0.$$

This ends the proof of lemma 22. □

The main property of the pseudo equilibrium is the following

Proposition 24. *W satisfies the following system:*

$$\begin{cases} \partial_\tau W(\tau, b) + \partial_b((1 - b)W(\tau, b)) + e^\tau \beta(e^\tau b)W(\tau, b) = W(\tau, b)C(\tau)\delta(\tau) \\ W(\tau, 0)(1 + \delta(\tau)) = \int_0^1 e^\tau \beta(e^\tau b)W(\tau, b)db, \end{cases} \quad (1.11)$$

where $\delta(\tau)$ is defined by the equation

$$\delta(\tau) = \frac{C'(\tau)}{(C(\tau))^2} - \frac{\mu}{C(\tau)}. \quad (1.12)$$

Proof. By computing the partial derivatives of W with respect to b and to τ , we obtain:

$$\partial_b((1 - b)W(\tau, b)) = -W(\tau, b) [\mu + (1 - b)e^\tau \beta(e^\tau b)] \quad (1.13)$$

and:

$$\partial_\tau W(\tau, b) = W(\tau, b) \left[\frac{C'(\tau)}{C(\tau)} - be^\tau \beta(e^\tau b) \right].$$

Therefore, W satisfies:

$$\partial_\tau W(\tau, b) + \partial_b((1 - b)W(\tau, b)) + e^\tau \beta(e^\tau b)W(\tau, b) = W(\tau, b) \left[\frac{C'(\tau)}{C(\tau)} - \mu \right].$$

If we take into account that $\forall \tau \geq 0 \|W(\tau, \cdot)\|_{L^1} = 1$, by integrating the previous equation over $b \in [0, 1]$, we obtain the value of $W(\tau, 0)$, hence the claimed system. □

The next results justify that (1.11) is close to (1.4).

Lemma 25. *Under hypothesis (H1), we have*

$$\lim_{\tau \rightarrow +\infty} C(\tau)\delta(\tau) = 0$$

where by computing the integral over $(0, 1)$ of the first line (1.11) taking into account (1.13) we obtain:

$$C(\tau)\delta(\tau) = \frac{C'(\tau)}{C(\tau)} - \mu = \int_0^1 [be^\tau \beta(e^\tau b) - \mu] W(\tau, b) db. \quad (1.14)$$

Proof. We recall first, by definition

$$|C(\tau)\delta(\tau)| \leq \int_0^1 |e^\tau b \beta(e^\tau b) - \mu| W(\tau, b) db.$$

We can split then the integral into two parts

$$|C(\tau)\delta(\tau)| \leq \underbrace{\int_0^{e^{-\tau/2}} |e^\tau b \beta(e^\tau b) - \mu| W(\tau, b) db}_{I_1(\tau)} + \underbrace{\int_{e^{-\tau/2}}^1 |e^\tau b \beta(e^\tau b) - \mu| W(\tau, b) db}_{I_2(\tau)}.$$

Firstly, we have

$$I_2(\tau) \leq \sup_{(e^{\tau/2}, +\infty)} |a\beta(a) - \mu| \rightarrow 0, \quad \tau \rightarrow +\infty.$$

To estimate $I_1(\tau)$ we notice

$$I_1(\tau) \leq \sup_{\mathbb{R}_+} |a\beta(a) - \mu| \int_0^{e^{-\tau/2}} W \leq \sup_{\mathbb{R}_+} |a\beta(a) - \mu| \left(\|W - W_\infty\|_1 + \int_0^{e^{-\tau/2}} W_\infty \right).$$

We already know from lemma 22 $\|W - W_\infty\|_1 \rightarrow 0$. Furthermore, for large τ we have

$$0 \leq \int_0^{e^{-\tau/2}} W_\infty \leq \frac{c_\infty}{(1 - e^{-\tau/2})^{1-\mu}} \frac{e^{-\tau(1-\mu)/2}}{1 - \mu} \rightarrow 0.$$

Therefore, we have $\lim_{+\infty} I_1 = 0$, which concludes the proof of the lemma. \square

1.2.3 Dissipation of entropy with respect to W

We now introduce the most important tool we will use: the relative entropy (similar to the entropy rate of a stochastic process, or the general relative entropy used in [112, 128]).

Definition 26. Let w be a solution of the equation (1.4) with support included in $[0, 1)$. Let H be a convex, continuous function, C^1 by parts, which reaches its minimum, 0, at 1. We define the generalised relative entropy as:

$$\mathcal{H}(\tau) = \int_0^1 H\left(\frac{w(\tau, b)}{W(\tau, b)}\right) W(\tau, b) db. \quad (1.15)$$

And for a non-negative measure ν on $[0, 1)$ the entropy dissipation $DH(u|\nu)$ is defined by

$$DH(u|\nu) = \int_0^1 H(u(b)) d\nu(b) - H\left(\int_0^1 u(b) d\nu(b)\right).$$

Note that $DH(u|\nu) \geq 0$ if ν is a probability (by Jensen's inequality).

We are now in position to establish a first important inequality on the relative entropy

Proposition 27. *Under (H1), the entropy \mathcal{H} satisfies the following equality:*

$$\mathcal{H}'(\tau) = -C(\tau)DH(u|d\gamma_\tau) + C(\tau)\delta(\tau) \int_0^1 (H(u) - uH'(u)) W(\tau, b)db \quad (1.16)$$

where $d\gamma_\tau(b) = \frac{e^\tau \beta(e^\tau b)W(\tau, b)}{W(\tau, 0)}db$ is a non-negative measure of mass $(1 + \delta(\tau))$ and $u = w/W$.

Proof. The mass of $d\gamma$ is immediately derived from the equation on $W(\tau, 0)$.

We have then:

$$\begin{cases} \partial_\tau w + \partial_b((1-b)w) + e^\tau \beta(e^\tau b)w = 0 \\ \partial_\tau W + \partial_b((1-b)W) + e^\tau \beta(e^\tau b)W = W(\tau, b)C(\tau)\delta(\tau). \end{cases}$$

Denoting $u = w/W$ we arrive at

$$\partial_\tau u + (1-b)\partial_b u = -C\delta u.$$

We multiply this equation by $H'(u)$ and get

$$\begin{aligned} \partial_\tau(H(u)) + (1-b)\partial_b(H(u)) &= -C\delta uH'(u). \\ \begin{cases} W\partial_\tau(H(u)) + (1-b)W\partial_b(H(u)) = -WC\delta uH'(u) \\ H\partial_\tau W + (1-b)H\partial_b W + (e^\tau \beta(e^\tau b) - 1)HW = C\delta HW \end{cases} \\ \partial_\tau(H(u)W) + \partial_b((1-b)H(u)W) + e^\tau \beta(e^\tau b)H(u)W &= \underbrace{WC\delta [H(u) - uH'(u)]}_\eta. \end{aligned}$$

Taking the integral over b of the previous expression yields:

$$\frac{d}{d\tau} \int_0^1 H(u)Wdb - W(\tau, 0)H(u(\tau, 0)) + \int_0^1 e^\tau \beta(e^\tau b)H(u)Wdb = \int_0^1 \eta db$$

and finally,

$$\frac{d}{d\tau} \int_0^1 H(u)Wdb = C(\tau) \left[H \left(\int_0^1 \frac{e^\tau \beta(e^\tau b)w}{W(\tau, 0)}db \right) - \int_0^1 H(u) \frac{e^\tau \beta(e^\tau b)W}{W(\tau, 0)}db \right] + \int_0^1 \eta db$$

which, by definition of the entropy dissipation, proves the proposition. \square

Remark. $C(\tau)\delta(\tau)$ appears naturally as a remainder we will have to estimate in order to prove convergence-related properties for \mathcal{H} , and also for $\|W - W_\infty\|_1$.

1.2.4 L^1 convergence (without a rate) to W

In this section we prove proposition 18. We take therefore $H(x) = |x - 1|$, we have then,

$$DH(u|d\gamma_\tau) = \int_0^1 |u - 1|d\gamma_\tau - \left| \int_0^1 u d\gamma_\tau - 1 \right|.$$

The core of the proof is the following

Lemma 28. *Under hypothesis (H1), we have*

$$\frac{d}{d\tau} \int_0^1 |w - W| db \leq -e^\tau \beta(e^\tau) \int_0^1 |w - W| db + 2|C(\tau)\delta(\tau)|. \quad (1.17)$$

Proof. By definition (abusing the notation),

$$d\gamma_\tau(b) = \frac{e^\tau \beta(e^\tau b) W(\tau, b)}{W(\tau, 0)} \geq \frac{e^\tau \beta(e^\tau)}{C(\tau)} W(\tau, b) = K(\tau) W(\tau, b).$$

Furthermore, since $\int_0^1 (u-1)W = \int_0^1 w - W = 0$, we have

$$\begin{aligned} DH(u|d\gamma_\tau) &= \int_0^1 |u-1| d\gamma_\tau - \left| \int_0^1 u d\gamma_\tau - 1 \right| \\ &= \int_0^1 |u-1| d\gamma_\tau - \left| \int_0^1 (u-1) d\gamma_\tau + \delta(\tau) \right| \\ &= \int_0^1 |u-1| d\gamma_\tau - \left| \int_0^1 (u-1) \underbrace{(d\gamma_\tau - K(\tau)W)}_{\geq 0} + \delta(\tau) \right| \\ &\geq K(\tau) \int_0^1 |u-1| W + \int_0^1 |u-1| (d\gamma_\tau - K(\tau)W) - \left| \int_0^1 (u-1) (d\gamma_\tau - K(\tau)W) \right| - |\delta(\tau)| \\ &\geq K(\tau) \int_0^1 |u-1| W - |\delta(\tau)|. \end{aligned}$$

Since we also have, for $H(x) = |x-1|$,

$$|H(u) - uH'(u)| = |-\text{sign}(u-1)| \leq 1,$$

we obtain for this case

$$\left| \int_0^1 (H(u) - uH'(u)) W \right| \leq 1.$$

And since $C(\tau)K(\tau) = e^\tau \beta(e^\tau)$, we obtain equation (1.17). \square

Since we already have by hypothesis $e^\tau \beta(e^\tau) \rightarrow \mu > 0$, standard ODE arguments yield

$$\limsup_{+\infty} \int_0^1 |w - W| db \leq \frac{2 \limsup_{+\infty} |C(\tau)\delta(\tau)|}{\mu}.$$

We can conclude the proof of proposition 18 using lemma 25.

Remark. We let the reader check that, by defining $\underline{\beta}$ as in (1.6), we may replace the non-increasing β hypothesis by $a\underline{\beta}(a) \xrightarrow{a \rightarrow \infty} \mu$, obtaining the following equation instead of (1.17):

$$\frac{d}{d\tau} \int_0^1 |w - W| db \leq -e^\tau \underline{\beta}(e^\tau) \int_0^1 |w - W| db + 2|C(\tau)\delta(\tau)|. \quad (1.18)$$

Remark. We have now finished developing a framework which allows us to deduce the behaviour of the entropy \mathcal{H} from suitable hypotheses made on β , and showed that, under mild conditions, the entropy tends to 0. The following section will extract a convergence rate from more restrictive hypotheses.

1.3 Rates of convergence to the pseudo equilibrium

1.3.1 The key situation: $\beta(a) = \frac{\mu}{1+a}$

We consider it best to start by presenting this simple case, since the following proofs contain the key innovative elements of the general case while simplifying the presentation of our results.

Here, the pseudo equilibrium W becomes, with $c(\tau) = e^{-\mu\tau}C(\tau)$:

$$W(\tau, b) = \frac{C(\tau)}{(1-b)^{1-\mu}(1+e^\tau b)^\mu} = \frac{c(\tau)}{(1-b)^{1-\mu}(e^{-\tau}+b)^\mu}.$$

We can now compute δ as follows.

Lemma 29.

$$\delta(\tau) = -\frac{e^{-\tau}}{1+e^{-\tau}}. \quad (1.19)$$

Proof.

$$\begin{aligned} C(\tau)\delta(\tau) &= \frac{C'(\tau)}{C(\tau)} - \mu = \frac{c'(\tau)}{c(\tau)} \\ &= \int_0^1 [be^\tau\beta(e^\tau b) - \mu] W(\tau, b)db = \int_0^1 \frac{\mu}{e^{-\tau}+b} [b - (e^{-\tau}+b)] W(\tau, b)db \\ &= -e^{-\tau} \int_0^1 e^\tau\beta(e^\tau b)W(\tau, b)db. \end{aligned}$$

By applying Proposition 24, we obtain:

$$\begin{aligned} C(\tau)\delta(\tau) &= -e^{-\tau}(1+\delta(\tau))W(\tau, 0) \\ &= -e^{-\tau}(1+\delta(\tau))C(\tau) \end{aligned}$$

resulting in the claimed equality since C doesn't vanish. \square

Definition 30. We call c_∞ the limit at ∞ of $c(\tau)$ when such limit exists. Here, it is easy to see c is a decreasing function and c_∞ is well defined.

Lemma 31.

$$-4e^{-(1-\mu)\tau} \leq C(\tau)\delta(\tau) \leq -\frac{c_\infty}{1+e^{-\tau}}e^{-(1-\mu)\tau} \leq 0. \quad (1.20)$$

Proof. We have $c(\tau) = e^{-\mu\tau}C(\tau)$, thus $\frac{C'(\tau)}{C(\tau)} - \mu = \frac{c'(\tau)}{c(\tau)}$, and we establish two trivial bounds on $c(\tau)$. By definition:

$$\frac{1}{c(\tau)} = \int_0^1 \frac{1}{(e^{-\tau}+b)^\mu(1-b)^{1-\mu}}db$$

hence, since $\tau \geq 0$ and $0 \leq \mu \leq 1$, we have:

$$\frac{1}{c(\tau)} \geq \int_0^1 (1+b)^{-\mu}(1-b)^{\mu-1}db \geq \int_0^1 \left(\frac{1-b}{1+b}\right)^\mu db \geq \int_0^1 \frac{1}{2}(1-b)^\mu db = \frac{1}{2(1+\mu)} \geq \frac{1}{4};$$

and likewise:

$$\frac{1}{c(\tau)} \leq \int_0^1 b^{-\mu}(1-b)^{\mu-1}db = \frac{1}{c_\infty}.$$

It follows that:

$$0 \leq c_\infty \leq c(\tau) \leq 2(1+\mu) \leq 4. \quad (1.21)$$

The result of Lemma 29 allows us to conclude. \square

Putting together this lemma and equation (1.17) gives us

Corollary 32. *The following inequality holds*

$$\mathcal{H}'(\tau) \leq -\left(\mu - \frac{\mu}{1+e^\tau}\right) \mathcal{H}(\tau) + 8e^{-(1-\mu)\tau}. \quad (1.22)$$

All is ready to prove Theorem 19 by applying Gronwall's Lemma to the previous inequality.

Proof of Theorem 19. We set $f(s) = -\frac{\mu}{1+e^s}$, which gives $\int_0^\tau f(s)ds = +\mu \ln\left(\frac{1+e^{-\tau}}{2}\right)$. Corollary 32 implies that:

$$\frac{d}{d\tau} \left[\exp\left(\int_0^\tau (\mu + f(s))ds\right) \mathcal{H}(\tau) \right] \leq 8e^{-(1-\mu)\tau} \exp\left(\int_0^\tau (\mu + f(s))ds\right).$$

By integrating over τ , we obtain:

$$\begin{aligned} \mathcal{H}(\tau) &\leq \mathcal{H}(0)e^{-\int_0^\tau \mu+f} + 8 \int_0^\tau e^{-(1-\mu)\tau'} e^{\int_0^{\tau'} \mu+f} e^{-\int_0^{\tau'} \mu+f} d\tau' \\ &\leq e^{-\mu\tau} \mathcal{H}(0) e^{-\int_0^\tau f} + 8e^{-\mu\tau} \int_0^\tau e^{(2\mu-1)\tau'} e^{-\int_{\tau'}^\tau f} d\tau' \\ &\leq e^{-\mu\tau} \mathcal{H}(0) \left(\frac{2}{1+e^{-\tau}}\right)^\mu + 8e^{-\mu\tau} \int_0^\tau e^{(2\mu-1)\tau'} \left(\frac{1+e^{-\tau'}}{1+e^{-\tau}}\right)^\mu d\tau' \\ &\leq e^{-\mu\tau} \left(\frac{2}{1+e^{-\tau}}\right)^\mu \left[\mathcal{H}(0) + 8 \int_0^\tau e^{(2\mu-1)\tau'} d\tau'\right]. \end{aligned}$$

□

1.3.2 A larger class of β

We now consider β satisfying **(H2)**.

Remark. The bound on G is not necessary to prove lemmas 33 and 34: $g \in L^1$ is a strong enough hypothesis. However, that precise bound is necessary for our convergence rate estimates and as such, we assume it holds throughout the section.

We take the following notations:

$$\begin{aligned} G(a) &= \int_a^\infty |g(s)|ds \\ W_{ref}(\tau, b) &= \frac{C_{ref}(\tau)}{(1-b)^{1-\mu}(1+e^\tau b)^\mu} \\ W(\tau, b) &= \frac{C(\tau) \exp\left(-\int_0^{e^\tau b} g(s)ds\right)}{(1-b)^{1-\mu}(1+e^\tau b)^\mu} \end{aligned}$$

where C_{ref} and C ensure $\|W_{ref}(\tau)\|_1 = \|W(\tau)\| = 1$. (We use the same notation as in the proof of lemma 22).

We have:

$$\begin{aligned} C(\tau)\delta(\tau) &= \int_0^1 [e^\tau b \beta(e^\tau b) - \mu] W(\tau, b) db \\ &= \int_0^1 e^\tau b g(e^\tau b) W db + \mu \int_0^1 \left[\frac{e^\tau b}{1+e^\tau b} - 1\right] W(\tau, b) db. \end{aligned}$$

Lemma 33. *Assume **(H2)** holds. Then there exists $K > 0$ such that for any $\tau \geq 0$ and $0 \leq b \leq 1$:*

$$\frac{K^{-1}e^{\mu\tau}}{(1+e^\tau b)^\mu(1-b)^{1-\mu}} \leq W(\tau, b) \leq \frac{Ke^{\mu\tau}}{(1+e^\tau b)^\mu(1-b)^{1-\mu}}. \quad (1.23)$$

Proof. We have:

$$\frac{W(\tau, b)}{W_{ref}(\tau, b)} = \frac{C(\tau)}{C_{ref}(\tau)} \exp\left(-\int_0^{e^\tau b} g(s) ds\right) \in \frac{C(\tau)}{C_{ref}(\tau)} \left[e^{-\|g\|_1}, e^{\|g\|_1}\right].$$

And since $\int_0^1 W(\tau, b) db = \int_0^1 W_{ref}(\tau, b) db = 1$, it follows that:

$$e^{-\|g\|_1} \leq \frac{C(\tau)}{C_{ref}(\tau)} \leq e^{\|g\|_1}$$

which gives us $\frac{W}{W_{ref}} \in L^\infty$ in the sense given above. \square

This result leads, through a proof analogous to that of lemma 31, to the following

Lemma 34. *Under hypothesis (H2), there exists a positive M such that:*

$$\left| \mu \int_0^1 \left[\frac{e^\tau b}{1 + e^\tau b} - 1 \right] W(\tau, b) db \right| \leq M e^{-(1-\mu)\tau}. \quad (1.24)$$

We now give the strategy for estimating the rate of convergence. It is based on the same procedure as before. Consider a non-increasing β and equation (1.17), which measures the dissipation of entropy (or equation (1.18) under the corresponding hypothesis).

$$\mathcal{H}'(\tau) \leq -e^\tau \beta(e^\tau) \mathcal{H}(\tau) + 2|C(\tau)\delta(\tau)|.$$

We have then

$$\left(\mathcal{H} e^{B(e^\tau)} \right)'(\tau) \leq 2|C(\tau)\delta(\tau)| e^{B(e^\tau)}. \quad (1.25)$$

We recall

$$C(\tau)\delta(\tau) = \int_0^1 (e^\tau b \beta(e^\tau b) - \mu) W(\tau, b) db.$$

And also by the definition of W and lemma 33

$$W(\tau, b) \leq \frac{K}{(e^{-\tau} + b)^\mu (1-b)^{1-\mu}}.$$

Therefore, we easily obtain

$$|C(\tau)\delta(\tau)| \leq K \left(\int_0^1 \frac{e^\tau b |g|(e^\tau b)}{(e^{-\tau} + b)^\mu (1-b)^{1-\mu}} db + \left| \mu \int_0^1 \left(\frac{e^\tau b}{1 + e^\tau b} - 1 \right) W(\tau, b) db \right| \right).$$

Lemma 34 then gives us

$$|C(\tau)\delta(\tau)| \leq K \left(\int_0^1 \frac{e^\tau b |g|(e^\tau b)}{(e^{-\tau} + b)^\mu (1-b)^{1-\mu}} db + M e^{(\mu-1)\tau} \right).$$

The constant K may change value from line to line.

We integrate equation (1.25) and get

$$\mathcal{H}(\tau) e^{B(e^\tau)} \leq \mathcal{H}(0) + K \int_0^\tau e^{B(e^{\tau'})} \int_0^1 \frac{e^{\tau'} b |g|(e^{\tau'} b)}{(e^{-\tau'} + b)^\mu (1-b)^{1-\mu}} db d\tau' + K \int_0^\tau e^{(\mu-1)\tau' + B(e^{\tau'})} d\tau'.$$

Using the fact the $B(e^\tau) - \mu\tau$ is bounded from above and below, we can replace $B(e^\tau)$ by $\mu\tau$ with just a change of constants.

$$\mathcal{H}(\tau) e^{\mu\tau} \leq K \left(1 + \int_0^\tau \int_0^1 \frac{e^{\mu\tau'} e^{\tau'} b |g|(e^{\tau'} b)}{(e^{-\tau'} + b)^\mu (1-b)^{1-\mu}} db d\tau' + \int_0^\tau e^{(2\mu-1)\tau'} d\tau' \right). \quad (1.26)$$

Remark. We let the reader check that the non-increasing β hypothesis may be replaced by the following condition on the function defined in (1.6): $a\underline{\beta}(a) \xrightarrow{a \rightarrow \infty} \mu$. Replacing the use of B by that of $\underline{B}(a) = \int_0^a \underline{\beta}$, we still obtain equation (1.26) up to multiplication by a constant, since $\underline{B}(e^\tau) - \mu\tau$ is also bounded from above and below.

Now the work is focused on the estimate of the middle quantity

$$I(\tau) = \int_0^\tau \int_0^1 \frac{e^{\mu\tau'} e^{\tau' b} |g|(e^{\tau' b})}{(e^{-\tau'} + b)^\mu (1-b)^{1-\mu}} db d\tau'. \quad (1.27)$$

We integrate by parts with respect to τ' and recall that $\frac{d}{d\tau} G(e^\tau b) = -e^\tau b |g(e^\tau b)|$. We have

$$\begin{aligned} I(\tau) &= \int_0^1 \left[-G(e^{\tau' b}) \frac{e^{\mu\tau'}}{(e^{-\tau'} + b)^\mu (1-b)^{1-\mu}} \right]_0^\tau db \\ &+ \int_0^1 \int_0^\tau G(e^{\tau' b}) \left(\mu \frac{e^{\mu\tau'}}{(e^{-\tau'} + b)^\mu (1-b)^{1-\mu}} + \frac{\mu e^{-\tau'} e^{\mu\tau'}}{(e^{-\tau'} + b)^{\mu+1} (1-b)^{1-\mu}} \right) d\tau' db. \end{aligned}$$

The first term is bounded from above (since $G \geq 0$) by

$$\int_0^1 \frac{\|g\|_1}{(1+b)^\mu (1-b)^{1-\mu}} db.$$

Finally, since $\frac{e^{-\tau}}{e^{-\tau} + b} \leq 1$,

$$I(\tau) \leq K + 2\mu \int_0^1 \int_0^\tau G(e^{\tau' b}) \frac{e^{\mu\tau'}}{(e^{-\tau'} + b)^\mu (1-b)^{1-\mu}} db d\tau'. \quad (1.28)$$

We need a sharp estimate on the second term. We focus our efforts on the case

$$G(a) \leq \frac{K}{(1+a)^\alpha} \quad \text{for some } \alpha > 0.$$

Proposition 35. *Assume hypothesis (H2).*

Then, if $\mu \neq 1/2$

$$\mathcal{H}(\tau) \leq \begin{cases} K((1+\tau)e^{-\mu\tau}), & \text{if } \mu = \alpha < 1 - \mu \\ K(e^{-\alpha\tau} + e^{-\mu\tau}), & \text{if } \mu \neq \alpha < 1 - \mu \\ K(e^{-(1-\mu)\tau} + e^{-\mu\tau}), & \text{if } \alpha > 1 - \mu \\ K(\tau e^{-(1-\mu)\tau} + e^{-\mu\tau}), & \text{if } \alpha = 1 - \mu. \end{cases}$$

If $\mu = 1/2$, then

$$\mathcal{H}(\tau) \leq \begin{cases} K(e^{-\alpha\tau} + e^{-\mu\tau}), & \text{if } \alpha < 1/2 \\ K(e^{-\tau/2}), & \text{if } \alpha > 1/2 \\ K((1+\tau^2)e^{-\tau/2}), & \text{if } \alpha = 1/2. \end{cases}$$

Note that for $\alpha > (1 - \mu)$ the rate is the same than the one for $g = 0$.

Proof. We need an estimate of

$$\int_0^1 \int_0^\tau \frac{1}{(1 + e^{\tau' b})^\alpha} \frac{e^{\mu\tau'}}{(e^{-\tau'} + b)^\mu (1 - b)^{1-\mu}} d\tau' db = \int_0^1 \int_0^\tau \frac{e^{(\mu-\alpha)\tau'}}{(e^{-\tau'} + b)^{\mu+\alpha} (1 - b)^{1-\mu}} d\tau' db.$$

Putting all together with (1.26), we have

$$H(\tau) \leq K \left(e^{-\mu\tau} + e^{-\mu\tau} \int_0^1 \int_0^\tau \frac{e^{(\mu-\alpha)\tau'}}{(e^{-\tau'} + b)^{\mu+\alpha} (1 - b)^{1-\mu}} d\tau' db + e^{-\mu\tau} \int_0^\tau e^{(2\mu-1)\tau'} d\tau' \right).$$

We essentially need to estimate the middle term.

Lemma 36. *The following holds true:*

$$\int_0^1 \int_0^\tau \mu \frac{e^{(\mu-\alpha)\tau'}}{(e^{-\tau'} + b)^{\mu+\alpha} (1 - b)^{1-\mu}} d\tau' db \leq \begin{cases} K \int_0^\tau e^{(\mu-\alpha)\tau'} d\tau', & \text{if } \alpha < 1 - \mu, \\ K \int_0^\tau e^{(2\mu-1)\tau'} d\tau', & \text{if } \alpha > 1 - \mu, \\ K \int_0^\tau (1 + \tau') e^{(2\mu-1)\tau'} d\tau', & \text{if } \alpha = 1 - \mu. \end{cases}$$

Proof of the lemma. Case 1: $\alpha < 1 - \mu$.

For $\alpha < 1 - \mu$, we simply use the fact that

$$\int_0^1 \frac{1}{(e^{-\tau'} + b)^{\mu+\alpha} (1 - b)^{1-\mu}} db \leq \int_0^1 \frac{1}{b^{\mu+\alpha} (1 - b)^{1-\mu}} db < +\infty.$$

Case 2: $\alpha > 1 - \mu$.

It is obvious that we can restrict to the case $\alpha < 1$. We first need a few intermediate computations. Firstly, for $\gamma > 0$, we have

$$\int_0^1 \frac{\gamma}{(e^{-\tau} + b)^{\gamma+1} (1 - b)^{1-\gamma}} db = \frac{1}{1 + e^{-\tau}} \int_0^1 -\frac{d}{db} e^{f(b)} db.$$

Where

$$f(b) = -\gamma \ln(e^{-\tau} + b) + \gamma \ln(1 - b).$$

Therefore, we have

$$\int_0^1 \frac{\gamma}{(e^{-\tau} + b)^{\gamma+1} (1 - b)^{1-\gamma}} db = \frac{1}{1 + e^{-\tau}} \left[-\left(\frac{(1 - b)}{e^{-\tau} + b} \right)^\gamma \right]_0^1 = \frac{e^{\gamma\tau}}{1 + e^{-\tau}}. \quad (1.29)$$

Using this computation and noticing $\gamma = \mu + \alpha - 1$, we can easily establish that, for $\alpha < 1$,

$$\int_0^1 \frac{1}{(e^{-\tau} + b)^{\mu+\alpha} (1 - b)^{1-\mu}} db = \int_0^1 \frac{(1 - b)^{1-\alpha}}{(e^{-\tau} + b)^{\gamma+1} (1 - b)^{1-\gamma}} db.$$

Applying then (1.29) and using $(1 - b)^{1-\alpha} \leq 1$, we arrive at

$$\int_0^1 \frac{1}{(e^{-\tau} + b)^{\mu+\alpha} (1 - b)^{1-\mu}} db \leq K e^{\gamma\tau}.$$

Injecting, we obtain

$$I_\alpha(\tau) \leq K \int_0^\tau e^{(\gamma+\mu-\alpha)\tau'} d\tau' = K \int_0^\tau e^{(2\mu-1)\tau'} d\tau'.$$

Case 3: $\alpha = 1 - \mu$.

In this case

$$I_\alpha(\tau) = \int_0^\tau \int_0^1 \frac{e^{(2\mu-1)\tau'}}{(e^{-\tau'} + b)(1-b)^{1-\mu}} db d\tau',$$

Cutting the integral on b at $1/2$ for instance, it is easy to establish

$$I_\alpha(\tau) \leq K \left(\int_0^\tau \int_0^1 e^{(2\mu-1)\tau'} \left(1 + \frac{1}{(e^{-\tau'} + b)} \right) db d\tau' \right) \leq K \left(\int_0^\tau e^{(2\mu-1)\tau'} \left(1 + \log(1 + e^{\tau'}) \right) d\tau' \right),$$

thereby, we have

$$I_\alpha(\tau) \leq K \int_0^\tau (2 + \tau') e^{(2\mu-1)\tau'} d\tau'.$$

This ends the proof of lemma 36. \square

To end the proof of proposition 35, we essentially just need to discuss whether the integrals of type $\int_0^\tau e^{\lambda\tau'} d\tau'$ take value $\frac{e^{\lambda\tau}-1}{\lambda}$ or τ , and similarly for integrals of type $\int_0^\tau \tau' e^{\lambda\tau'} d\tau'$. \square

1.4 Rates of convergence towards the equilibrium W_∞ .

1.4.1 Quantitative estimate of $\|W - W_\infty\|_1$

In what follows, we justify how the rate of convergence of w to W can be extended to quantify (up to a multiplicative constant) the rate of convergence towards W_∞ . The main remark is the following.

Lemma 37. *Under hypothesis (H2), we have*

$$\|W - W_\infty\|_1 \leq 2 \int_0^1 |e^\tau b \beta(e^\tau b) - \mu| W(\tau, b) db.$$

Proof. We already know from lemma 22 $\lim_{+\infty} \|W(\tau, \cdot) - W_\infty\|_1 = 0$. Therefore

$$\|W(\tau, \cdot) - W_\infty\|_1 \leq \int_\tau^\infty \left| \frac{d}{d\tau} \int_0^1 |W(\tau', b) - W_\infty(b)| db \right| d\tau' \leq \int_\tau^\infty \int_0^1 |\partial_\tau W(\tau', b)| db d\tau'.$$

Since we have

$$\partial_\tau W = \frac{C'(\tau)}{C(\tau)} W - e^\tau b \beta(e^\tau b) W = \left(\frac{C'(\tau)}{C(\tau)} - \mu \right) W + (\mu - e^\tau b \beta(e^\tau b)) W,$$

it follows that

$$\int_0^1 |\partial_\tau W(\tau, b)| db \leq \left| \frac{C'(\tau)}{C(\tau)} - \mu \right| + \int_0^1 |e^\tau b \beta(e^\tau b) - \mu| W(\tau, b) db.$$

And since

$$\left| \frac{C'(\tau)}{C(\tau)} - \mu \right| \leq \int_0^1 |e^\tau b \beta(e^\tau b) - \mu| W(\tau, b) db,$$

we can conclude. \square

Remark. The bound on G has not been used in the proof of the previous lemma, for which $g \in L^1$ is a strong enough hypothesis.

We encounter yet again the quantity $\int_0^1 |e^\tau b \beta(e^\tau b) - \mu| W$, for which we have already given a time-weighted average estimate in the form of an upper bound (1.28) on (1.27). Let us now provide a pointwise estimate. In the reference case, we have

$$\|W(\tau, \cdot) - W_\infty\|_1 \leq K \int_\tau^\infty \frac{e^{(\mu-1)\tau'}}{1 + e^{-\tau'}} d\tau' \leq K e^{(\mu-1)\tau}.$$

In the situation described by proposition 35, with $\beta = \frac{\mu}{1+a} + g(a)$ and for some $\alpha > 0$, we have,

$$\int_a^\infty |g| \leq \frac{K}{(1+a)^\alpha}.$$

In this case, we can split $\int_0^1 |e^\tau b \beta(e^\tau b) - \mu| W$ into two parts and use the previous arguments of the proof of lemma 34 to claim

$$\begin{aligned} \int_0^1 |e^\tau b \beta(e^\tau b) - \mu| W &\leq \int_\tau^\infty \int_0^1 \left| \frac{\mu b}{e^{-\tau'} + b} - \mu \right| \frac{1}{(e^{-\tau'} + b)^\mu (1-b)^{1-\mu}} db d\tau' \\ &\quad + K \int_\tau^\infty \int_0^1 |e^{\tau'} b g(e^{\tau'} b)| \frac{1}{(e^{-\tau'} + b)^\mu (1-b)^{1-\mu}} db d\tau'. \end{aligned}$$

The first term is already known to be bounded by $K e^{(\mu-1)\tau}$ by lemma 31. The second term satisfies

$$\begin{aligned} \int_\tau^\infty \int_0^1 |e^{\tau'} b g(e^{\tau'} b)| \frac{1}{(e^{-\tau'} + b)^\mu (1-b)^{1-\mu}} db d\tau' &\leq \int_\tau^\infty \int_0^1 |e^{\tau'} b g(e^{\tau'} b)| \frac{1}{b^\mu (1-b)^{1-\mu}} db d\tau' \\ &\leq \int_0^1 \frac{1}{b^\mu (1-b)^{1-\mu}} \left(\int_{e^\tau b}^\infty |g| \right) db \\ &\leq \int_0^1 \frac{1}{b^\mu (1-b)^{1-\mu}} \frac{1}{(1 + e^\tau b)^\alpha} db \\ &\leq K e^{-\alpha\tau}. \end{aligned}$$

Hence the rate of convergence $\|W(\tau, \cdot) - W_\infty\|_1 \leq K (e^{(\mu-1)\tau} + e^{-\alpha\tau})$.

1.4.2 Possible influence of the initial condition

Let us prove a lower bound on the convergence rate of $\ln(\|w(\tau, \cdot) - W(\tau, \cdot)\|_{L^1(0,1)})$ for an initial age distribution $w(0, b) = \delta_0(b)$.

Proposition 38. *Consider the reference case*

$$\beta(a) = \frac{\mu}{1+a}.$$

Suppose the initial age distribution satisfies:

$$w^0(b) = \delta_0(b).$$

We can bound below the total variation:

$$\|W(\tau, b) - w(\tau, b)\|_{TV} \geq e^{-\mu\tau}. \tag{1.30}$$

Proof. For

$$\begin{cases} \phi(\tau) &= 1 - e^{-\tau} \\ \xi(\tau) &= \exp\left(\int_0^\tau e^s \beta(e^s \phi(s)) ds\right) \end{cases}$$

we have:

$$\begin{aligned} \frac{d}{d\tau} [\xi(\tau)w(\tau, \phi(\tau))] &= \xi \left[\frac{\xi'}{\xi} w + \partial_\tau w + \phi' \partial_b w \right] \\ &= \xi [\partial_\tau w + \partial_b ((1 - \phi)w) + e^\tau \beta(e^\tau \phi)w] = 0. \end{aligned}$$

It follows that

$$w(\tau, \phi(\tau)) = \exp\left(-\int_0^\tau e^s \beta(e^s \phi(s)) ds\right) w(0, \phi(0))$$

which, after injecting the corresponding values, yields

$$w(\tau, 1 - e^{-\tau}) = e^{-\mu\tau} \delta$$

with δ a Dirac mass. Therefore:

$$w(\tau, \cdot) = e^{-\mu\tau} \delta_{1-e^{-\tau}} + w_j$$

where $w_j \geq 0$ is the distribution of particles that have jumped at least once over $(0, \tau]$.

Since $W(\tau, b)db$ is an atomless measure, any Dirac mass and Wdb are stranger measures, hence:

$$\|W(\tau, b) - w(\tau, b)\|_{TV} \geq e^{-\mu\tau}.$$

□

For $\mu < \frac{1}{2}$, this lower bound agrees up to multiplication by a constant with the upper bound given in theorem 19: our convergence exponent is optimal for $\mu < \frac{1}{2}$.

Remark. It is worth noting that we have the trivial bound:

$$\mathcal{H}(0) \leq 2. \tag{1.31}$$

($\mathcal{H}(0)$ can be greater than 1 if $w^0(b)db$ has atoms.)

Remark. Our results are proved for compactly-supported initial age distributions, and they will most likely hold for initial age distributions that decrease fast enough. However, if this is not the case, the convergence rates might be affected in a way left for future investigation.

1.4.3 Convergence rates for natural variables

We recall:

$$n(t, a) = e^{-\tau} w(\tau, b)$$

where

$$\begin{cases} \tau &= \ln(1+t) \\ b &= \frac{a}{1+t}. \end{cases}$$

Consider the reference case $\beta(a) = \frac{\mu}{1+a}$ (for which we have set in subsection 1.3.1 $c(\tau) = C(\tau)e^{-\mu\tau}$, the interest being that c tends to a constant). Let us define the natural variables version of W :

Definition 39. We set, for $a < 1+t$:

$$N(t, a) = e^{-\tau} W(\tau, b) = \frac{c(\ln(1+t))}{(1+a)^\mu (1+t-a)^{1-\mu}} \tag{1.32}$$

which leads to the following Proposition.

Proposition 40. *If*

$$\|w(\tau, \cdot) - W(\tau, \cdot)\|_{L^1([0,1])} \leq K_1 e^{-\mu\tau} + K_2 e^{-(1-\mu)\tau}$$

then:

$$\|n(t, \cdot) - N(t, \cdot)\|_{L^1(\mathbb{R}_+)} \leq \frac{K_1}{(1+t)^\mu} + \frac{K_2}{(1+t)^{1-\mu}}. \quad (1.33)$$

Proof. The $e^{-\tau}$ appearing as the Jacobian of the change of integration variables is compensated by the $e^{-\tau}$ in the definition of w and we get the claimed result. \square

Therefore, in the reference case $\beta(a) = \frac{\mu}{1+a}$, the distribution of walkers that have age a at time t converges to $N(t, a)$ algebraically fast, with a rate that is essentially given by $t^{-\min\{\mu, 1-\mu\}}$.

Through a similar proof, keeping in mind $\|W(\tau, \cdot) - W_\infty\|_1 \leq K e^{-(1-\mu)\tau}$ in the reference case, we recover corollary 21, which gives the natural variables version of an L^1 convergence of $w(\tau, \cdot)$ to W_∞ .

1.5 Monte Carlo simulations

In order to illustrate the evolution of the age distribution of the system and check the accuracy of the convergence rates to self-similar equilibrium, we have carried out Monte-Carlo simulations for our reference case $\beta(a) = \frac{\mu}{1+a}$. Consult the Appendix A for references.

In these simulations, we describe explicitly each individual walker i by associating it with an age a_i and a first jumping time τ_i . The initial age of each walker is chosen according to some initial distribution, for instance uniform distribution in $[0, 1]$ or a Dirac distribution at age $a = 0$. The first jumping time of each random walker is sampled from the distribution $\phi(\tau) = \mu/(1+\tau)^{1+\mu}$, that corresponds to our reference jump rate $\beta(a) = \mu/(1+a)$. The simulation then iterates the following steps: (i) find k , the walker with the earlier jump time: $k = \arg \min_i \tau_i$, then (ii) make it jump, i.e. reset its age $a_k = 0$ and finally, (iii) pick its next jump time τ_k according to $\phi(\tau)$. During the simulation, we store the distance between the dynamic equilibrium W at that time and the observed distribution of rescaled ages $b_i = a_i/(1+t)$ of all the walkers i in the simulation: $\|w(\tau, \cdot) - W(\tau, \cdot)\|_{L^1([0,1])}$. We also compute at each time step the L^1 norm of the difference $w(\tau, \cdot) - W_\infty$. Unless stated otherwise we use 20,000 random walkers in each simulation.

First, we note that in all cases, the simulated L^1 distance between w and the pseudo-equilibrium W is indeed bounded above by the expression given in theorem 19 (except at very high τ , when our bound becomes lower than the numerical error of the simulation). The example given in figure 1.1 corresponds to $\mu = 0, 4$ and $\mu = 0, 8$, and an initial age distribution Dirac at 0 for the red dots, and uniform on $[0, 1]$ for the blue dots, the black curve representing the upper bound proved in theorem 19 taken for $\mathcal{H}(0) = 2$, which is an upper bound for $\mathcal{H}(0)$. As we see, the multiplicative constant we lose (the overestimation of K in theorem 19 corresponding to the losses throughout the inequalities used to prove our bound) is not too high.

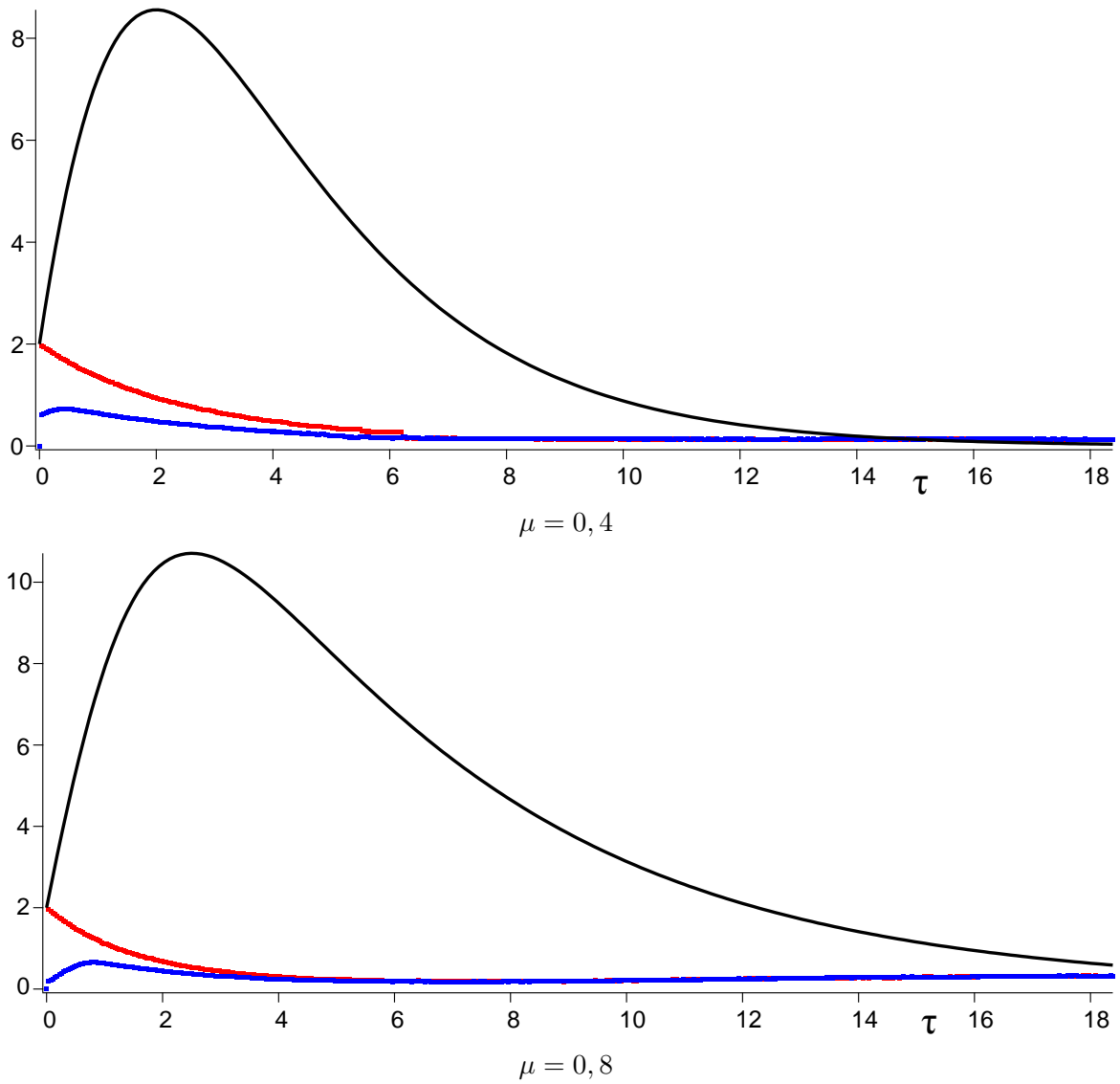
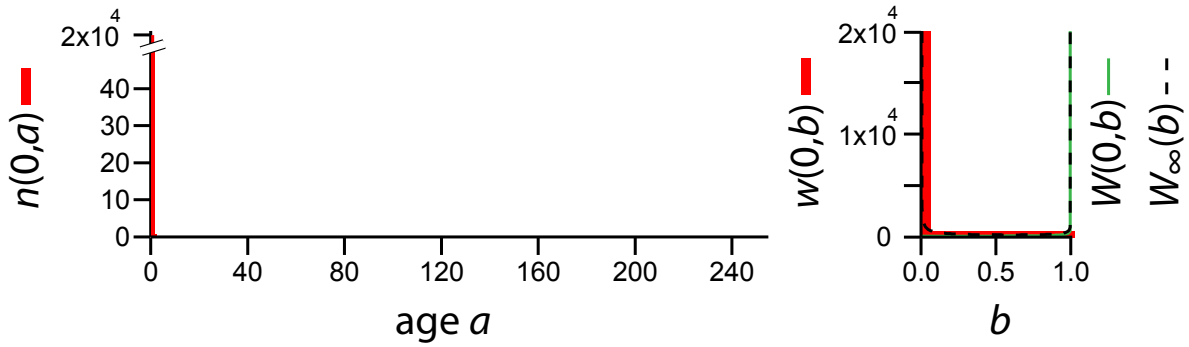


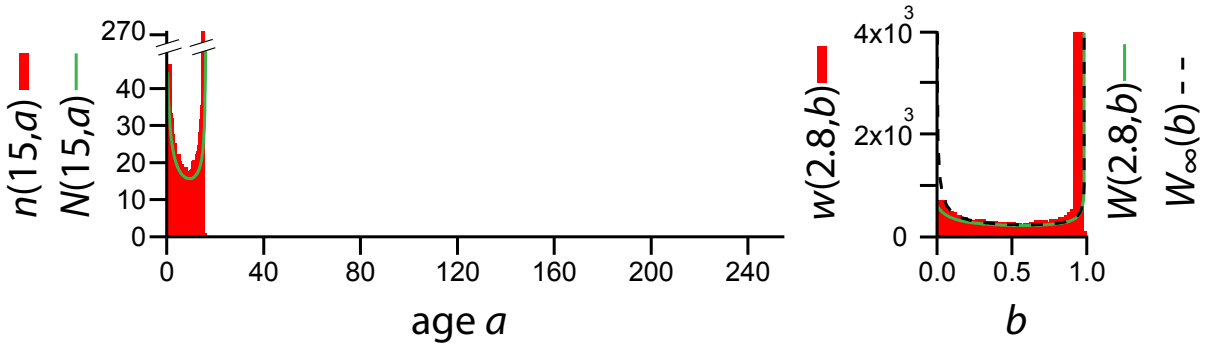
Figure 1.1: $\|w(\tau, \cdot) - W(\tau, \cdot)\|_1$ lies under the theoretical bound (black curve) for an initial age distribution δ_0 (red dots) and $\mathcal{U}(0, 1)$ (blue dots).

In order to illustrate graphically the behaviour of the solution to our equations and its convergence towards the pseudo-equilibrium, Figure 1.2 displays, for $\mu = 0,6$ and an initial age distribution $n^0 = w^0 = \delta_0$, the time evolution of the simulation results expressed either in the original variables $n(t, \cdot)$ (histograms), $N(t, \cdot)$ (full line) on the left-hand side column or in the rescaled variables $w(\tau, \cdot)$ (histograms), $W(\tau, \cdot)$ (full line) on the right-hand side column. Moreover, the rescaled variables panels also show as grey dotted lines the equilibrium W_∞ , to which W converges.

a) $t = 0 ; \tau = 0$



b) $t = 15 ; \tau = 2.8$



c) $t = 250 ; \tau = 5.5$

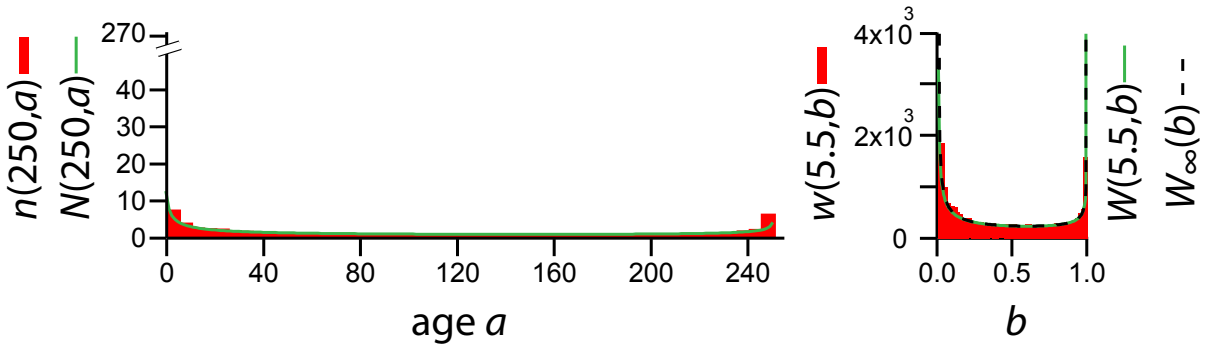


Figure 1.2: Evolution of n , N , w , W and W_∞ along time, for $\mu = 0,6$ and an initial age distribution $n^0 = w^0 = \delta_0$.

From visual inspection of the is figure, it is clear that $n(t, a)$ largely flattens as $t \rightarrow \infty$ (note the difference in the y-axis scale between the panels). The figure depicts a pointwise convergence of the simulated w to the pseudo-equilibrium W which in turn converges pointwise to W_∞ . Moreover, it illustrates how rescaling allows a better description of the self-similar behaviour, which is difficult to grasp in natural variables since n converges pointwise to 0. The next sections quantify the simulated convergence rates.

1.5.1 Exponential fit of $\|w(\tau, \cdot) - W(\tau, \cdot)\|_1$

To quantify the convergence rates in the simulations, we fit the distance $\|w(\tau, \cdot) - W(\tau, \cdot)\|$ by the following function:

$$f(\lambda, A, B) = Ae^{-\lambda\tau} + Be^{-(1-\lambda)\tau} + C. \quad (1.34)$$

Remark (Heuristic estimate of the error term). C is a simulation error, that we evaluate to $C \approx 0,1$. This is consistent both with empirical evidence and with a simple heuristic overevaluation of C as $\sqrt{\#bins/\#particles}$, which is roughly 0,16.

Remark. According to the above analysis one expects $\lambda = \mu$. A and B are multiplicative parameters: we expect A around 2 and $|B|$ close to 0, since $\mathcal{H}(0) = 2$ and our upper boundary is of the form $\mathcal{H}(\tau) \leq [\mathcal{H}(0) - (\leq 0)]e^{-\mu\tau} + ke^{-(1-\mu)\tau}$, with k small.

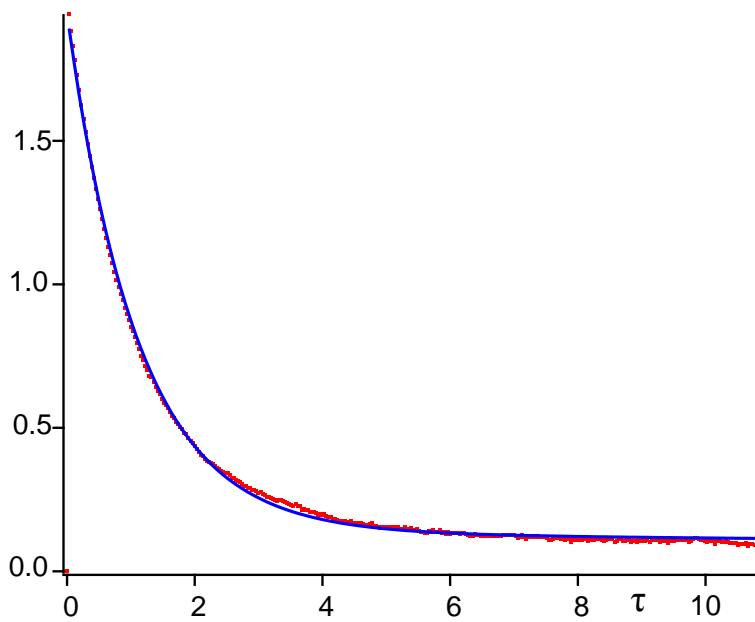
Remark. Another possible explanation of the predominance of $e^{-\mu\tau}$ over $e^{-(1-\mu)\tau}$ in the convergence rate is linked to the fact that, for a given β , two solutions w_1 and w_2 corresponding to different, compactly supported initial conditions, satisfy, for a certain constant K (see Subsection 1.2.1):

$$\|w(\tau, \cdot) - w_1(\tau, \cdot)\|_1 \leq Ke^{-\mu\tau}.$$

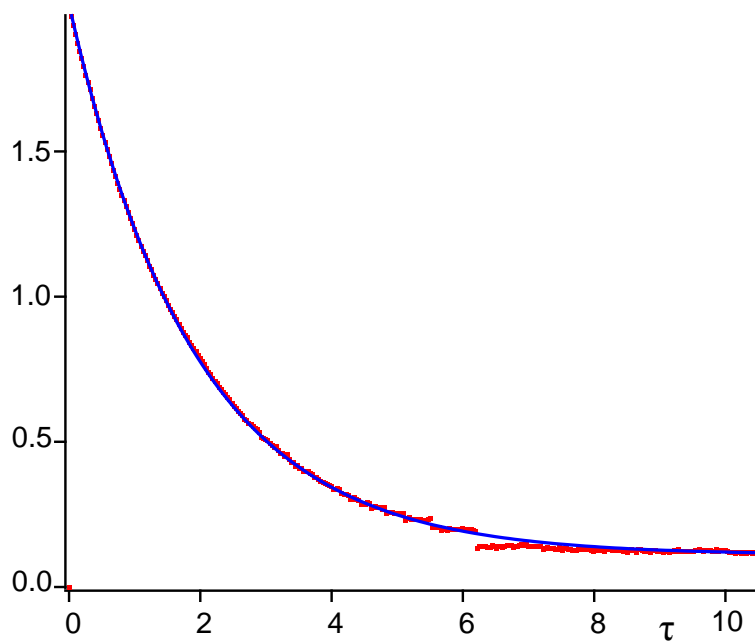
Figure 1.3 presents as examples three cases that exhibit a certain diversity: $\mu = 0,9$, $\mu = 0,5$ and $\mu = 0,2$. We plot in red dots the evolution along τ of the simulated value of $\|w(\tau, \cdot) - W(\tau, \cdot)\|_1$ and use function f defined in equation (1.34) to fit the results (blue curves). The fit results are given in the figures, \pm one standard deviation.

We first note that in the three panels of figure 1.3, $C \approx 0.1$ as expected and our estimates for λ are very close to μ . Note that in the second panel, with $\mu = 0.5$, the values of A and B cannot be estimated independently thus the large inaccuracy/variance on their determination. Finally, the third panel shows a marked discontinuity around $\tau = 6$. This is due to the discretisation of the age distribution: with small values of μ , the number of random walkers that have never experienced a single renewal during the simulation period becomes large. Since, according to our initial conditions, all walkers have the same initial age, many walkers will enter the last age bin simultaneously thus causing the observed discontinuity. However even in this case, we obtain a very good fit for λ by restricting the fit to the values before the discontinuity and fixing C to 0.8.

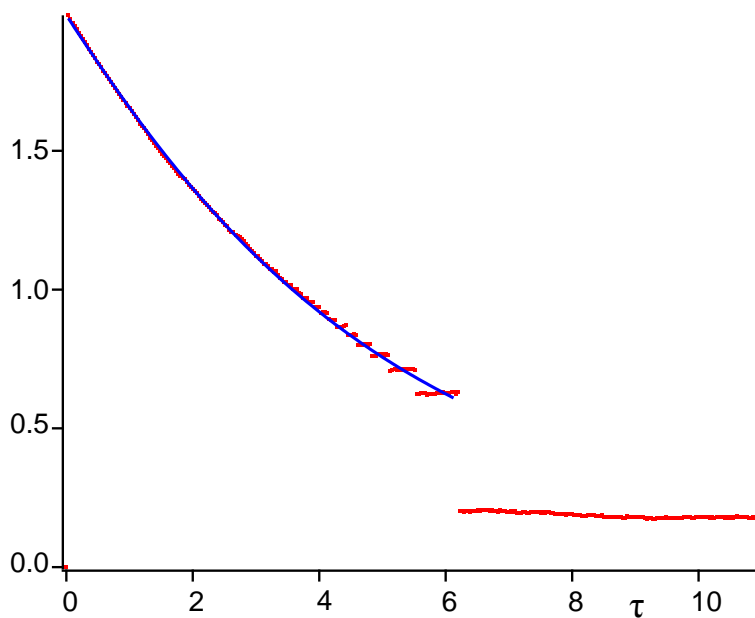
Figure 1.4 summarises the values of λ determined from Monte-Carlo simulations identical to those shown in Fig.1.3 (red crosses), together with the diagonal line $\lambda = \mu$ (blue). For all the values of μ tested, the simulations confirm that w tends to W with a sum of exponential rates given by μ and $1 - \mu$. Therefore, taken together, those simulation results, while agreeing with our analytical estimations, suggest that our estimate of $\|w - W\|_1$ may not be optimal, in particular for larger values of μ .



$\mu = 0.9.$
 $\lambda = 0.899 \pm 0.007$
 $A = 1.795 \pm 0.007$
 $B = 0.050 \pm 0.008$
 $C = 0.098 \pm 0.004$



$\mu = 0.5.$
 $\lambda = 0.523 \pm 0.033$
 $A = 1.895 \pm 1.240$
 $B = -7.560 \times 10^{-5} \pm 1.240$
 $C = 0.110 \pm 0.001.$



$\mu = 0.2.$
 $\lambda = 0.219 \pm 0.001$
 $A = 2.030 \pm 0.013$
 $B = -0.121 \pm 0.015$
 $C = 0.08 \pm 0$

Figure 1.3: Fit by f defined in equation (1.34) (blue curves), for different μ , of the simulated $\|w(\tau \cdot) - W(\tau \cdot)\|_1$ (red dots). Initial age distribution: $w^0 = \delta_0$

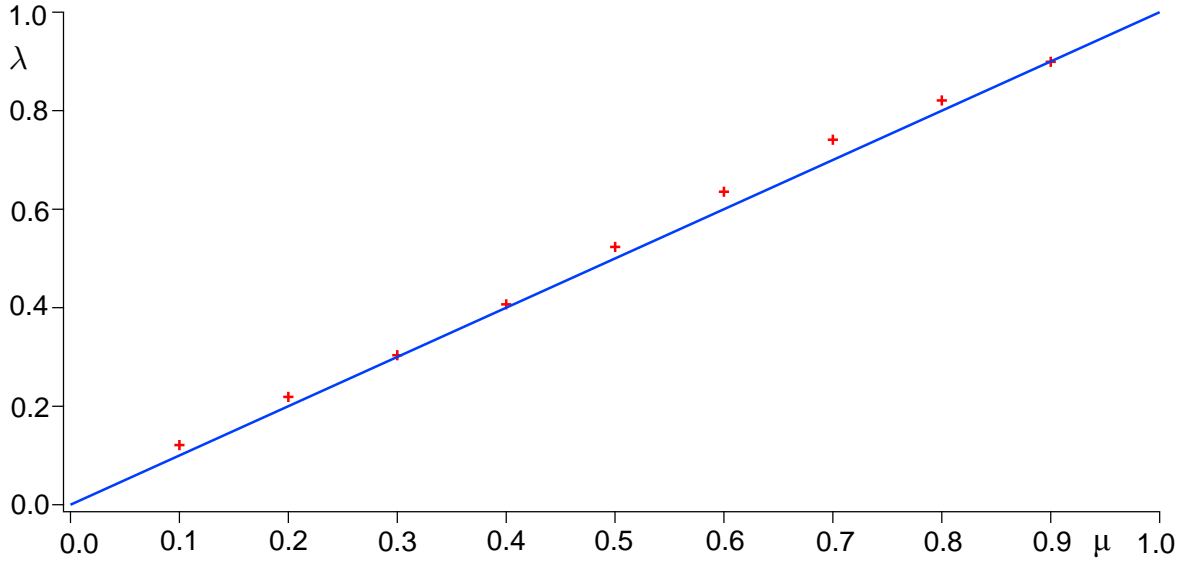


Figure 1.4: Values of the exponent λ found by using function f from equation (1.34) to fit the simulated values of $\|w(\tau, \cdot) - W(\tau, \cdot)\|_1$ for $\mu \in \{0, 1; 0, 2; \dots; 0, 9\}$.

1.5.2 For large μ , W provides a better asymptotic approximation of w than W_∞

Figure 1.5 compares the distances between w and W (red dots) on one hand, and w and W_∞ (black dots) on the other, for three values of μ . For $\mu \leq 0.5$, these two distances are very similar (actually, $W(\tau, \cdot)$ and W_∞ are much closer to each other than to w). However, as μ increases, this trend reverses: this figure shows that for large enough τ , w becomes significantly closer to W than to W_∞ : the distance between w and W converges much faster to 0. Therefore, according to those simulation results W is a much better asymptotic approximation w for $\mu > 0.5$, thus justifying further its utility here.

1.6 Future developments

Throughout the Chapter we have estimated L^1 norms, but we have presented the estimates in the context of an entropic structure. It is indeed possible by means analogous to ours to prove entropy inequalities for dissipations corresponding to other H functions than $|1 - \cdot|$. For instance, the classical $H(x) = x \ln x - x + 1$ also allows us to prove a convergence rate of the corresponding entropy to 0: it is also $K(e^{-\mu\tau} + e^{(\mu-1)\tau})$. Thanks to the Csiszár-Kullback inequality, it is also possible to prove a rate of convergence of $\|w(\tau, \cdot) - W(\tau, \cdot)\|_1$ to 0, albeit one worse than that obtained in theorems 19 and 20.

We may encounter inequalities such as that of proposition 27, bounding the derivative of an entropy with respect to a probability measure Wdb by an entropy dissipation with respect to another measure (which we can compare to the dissipation with respect to a probability measure $d\gamma_\tau$). When the comparison of $DH(u|Wdb)$ and $DH(u|d\gamma)$ does not follow calculations as straightforward as ours, an alternative may be to rely on a precise Jensen estimate comparing the entropy dissipations with respect to two absolutely continuous probability measures.

Here, we have considered a spatially-homogeneous (zero-dimensional), age-dependent renewal probability $\beta(a)$. We believe the ideas we have exposed may be used to tackle the problem with

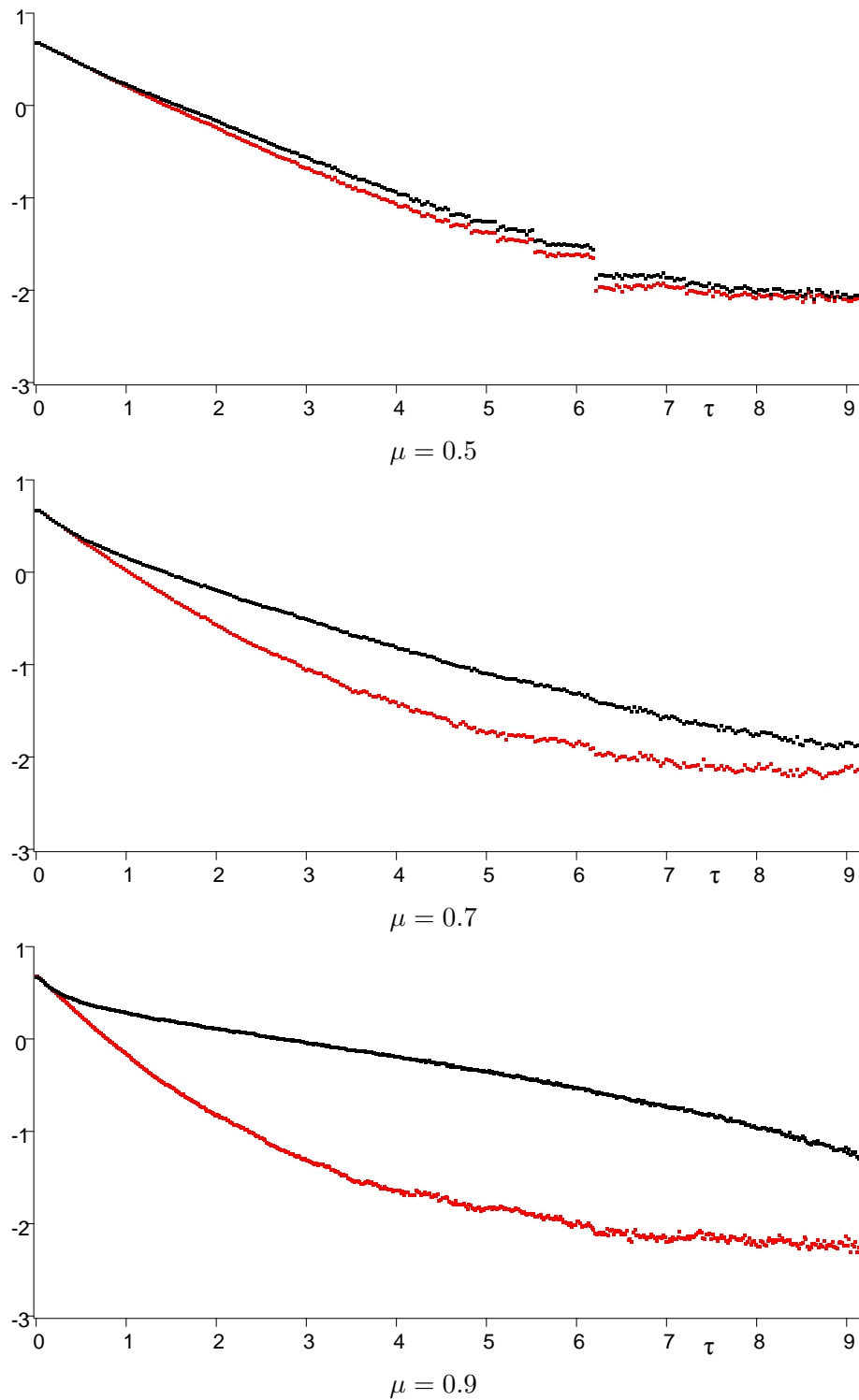


Figure 1.5: Influence of μ on $\ln \|w(\tau, \cdot) - W(\tau, \cdot)\|_1$ (red dots) and $\ln \|w - W_\infty\|$ (black dots): for higher values of μ , w is significantly closer to W than to W_∞ .

a spatial extension, for instance in a discrete space setting.

One major interest of our age-structure approach of CTRW is that the dynamics remain Markovian. We believe that keeping Markovian properties will be crucially helpful when introducing the coupling between sub-diffusive CTRW and reaction, since the coupling should simply consist in the addition of the reaction and the subdiffusion terms (contrarily to the case of fractional dynamics). However, the extent to which the supplementary age variable will make this process more complex remains to be evaluated.

1.7 Appendix

The case $\mu = 1$

It is quite interesting to notice that even if the behaviour is not really self-similar, our method gives a precise asymptotic for the case $\mu = 1$. To illustrate this, we focus on the reference case: $\beta(a) = \frac{1}{1+a}$. In this case the 'pseudo equilibrium reads'

$$W(\tau, b) = \frac{1}{(e^{-\tau} + b) \log(1 + e^\tau)}.$$

This pseudo equilibrium tends to a Dirac mass but gives still quantitative information. Indeed, following the same computation than for equation (1.17) for the case $\mu < 1$, we obtain easily

$$\frac{d}{d\tau} \int_0^1 |w - W| \leq -\frac{e^\tau}{1 + e^\tau} \int_0^1 |w - W| + 2 |C(\tau)\delta(\tau)|.$$

Where we also have

$$C(\tau)\delta(\tau) = \int_0^1 \left(\frac{e^\tau b}{1 + e^\tau b} - 1 \right) W(\tau, b) db.$$

This leads to

$$C(\tau)\delta(\tau) = - \int_0^1 \frac{e^{-\tau}}{(e^{-\tau} + b)^2 \log(1 + e^\tau)} = \frac{e^{-\tau}}{\log(1 + e^\tau)} \left(\frac{1}{e^{-\tau} + 1} - \frac{1}{e^{-\tau}} \right).$$

And finally,

$$C(\tau)\delta(\tau) = -\frac{e^\tau}{(1 + e^\tau) \log(1 + e^\tau)} \rightarrow 0.$$

And we can still claim that $\int_0^1 |w - W| \rightarrow 0$. We can give a (rough) estimate for a rate of convergence. Integrating, we have

$$\int_0^1 |w - W| \leq \frac{1}{1 + e^\tau} \int_0^1 |w - W|(\tau = 0) + 2 \frac{1}{1 + e^\tau} \int_0^\tau \frac{e^{\tau'}}{\log(1 + e^{\tau'})} d\tau'.$$

We estimate the second term

$$\frac{1}{1 + e^\tau} \int_0^\tau \frac{e^{\tau'}}{\log(1 + e^{\tau'})} d\tau' = \frac{1}{1 + e^\tau} \int_1^{e^\tau} \frac{1}{\log(1 + u)} du.$$

This term behaves as $1/\tau$. Indeed, we have easily (splitting the integral at $e^{\alpha\tau}$ for $\alpha < 1$):

$$\frac{1}{\log(1 + e^\tau)} \leq \frac{1}{1 + e^\tau} \int_1^{e^\tau} \frac{1}{\log(1 + u)} du \leq \frac{e^{(\alpha-1)\tau}}{1 + e^{-\tau}} + \frac{1}{\log(1 + e^{\alpha\tau})}$$

$$\int_0^1 |w - W| \leq \frac{1}{1 + e^\tau} \int_0^1 |w - W|(\tau = 0) + \frac{K}{1 + \tau} \leq \frac{K'}{1 + \tau}.$$

Acknowledgements

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We wish to thank Sergei Fedotov for many valuable discussions. This work could not have been written without the help of Vincent Calvez.

Chapter 2

A limiting Hamilton-Jacobi equation

Résumé

Les échelles de temps caractéristiques de déplacements sous-diffusifs sont beaucoup plus lentes que celles de déplacements diffusifs. Dans ce travail en collaboration avec Vincent Calvez et Pierre Gabriel [28], nous considérons la limite hyperbolique $(t, x) \rightarrow (t/\varepsilon, x/\varepsilon)$ d'une équation structurée en âge décrivant le déplacement sousdiffusif de, par exemple, une protéine à l'intérieur d'une cellule biologique. Il est connu que les solutions des équations rééchelonnées satisfont une équation de Hamilton-Jacobi à la limite formelle quand $\varepsilon \rightarrow 0$. Dans ce travail, nous prouvons des estimations Lipschitz uniformes et établissons la convergence vers la solution de viscosité de l'équation de Hamilton-Jacobi limite. Les deux principaux obstacles auxquels nous avons fait face sont la non-existence d'une mesure stationnaire intégrable, et l'importance des termes de mémoires dans les déplacements sousdiffusif.

Abstract

Subdiffusive motion takes place at a much slower timescale than diffusive motion. As a preliminary step to studying reaction-subdiffusion pulled fronts, we consider in this joint work with Vincent Calvez and Pierre Gabriel [28] the hyperbolic limit $(t, x) \rightarrow (t/\varepsilon, x/\varepsilon)$ of an age-structured equation describing the subdiffusive motion of, *e.g.*, some protein inside a biological cell. Solutions of the rescaled equations are known to satisfy a Hamilton-Jacobi equation in the formal limit $\varepsilon \rightarrow 0$. In this work we derive uniform Lipschitz estimates, and establish the convergence towards the viscosity solution of the limiting Hamilton-Jacobi equation. The two main obstacles overcome in this work are the non-existence of an integrable stationary measure, and the importance of memory terms in subdiffusion.

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2.1 Introduction

2.1.1 Brief model description

Consistent experimental evidence stemming from recent methodological advances in cell biology such as *in vivo* single molecule tracking, report that the intra-cellular random motion of certain molecules often deviates from Brownian motion. Macroscopically, their mean squared displacement does not scale linearly with time, but as a power law t^μ for some exponent $0 < \mu < 1$ [69, 25, 125, 39, 87]. This behaviour, due to crowding and trapping phenomena, is usually referred to as “anomalous” diffusion or “subdiffusion”. The reader may consult [80] for a review.

One of the standard mechanisms used to describe the emergence of subdiffusion in cells is continuous time random walks (CTRW), a generalisation of random walks that couples a waiting time random process at each “jump” of the random walk [116]. CTRW can be used [108, 110, 107] to derive macroscopic equations governing the spatiotemporal dynamics of the density of random walkers located at position x at time t :

$$\partial_t \rho(x, t) = D_\mu \mathcal{D}_t^{1-\mu} \Delta \rho(x, t).$$

Here, D_μ is a generalised diffusion coefficient and $\mathcal{D}_t^{1-\mu}(f)(t) = \frac{1}{\Gamma(\mu)} \frac{d}{dt} \int_0^t \frac{f(t')}{(t-t')^{1-\mu}} dt'$ is the Riemann-Liouville fractional derivative operator. Such a fractional dynamics formulation is very attractive for modelling in biology, in particular because of its apparent similarity with the classical diffusion equation. However, contrary to the diffusion equation, the Riemann-Liouville operator is non-local in time. This is the “trace” of the non-Markovian property of the underlying CTRW process. Indeed, memory terms play a crucial role in subdiffusive processes. This non-Markovian property becomes a serious obstacle when one wants to couple subdiffusion with chemical reaction [79, 170, 56].

In this work, following [107], we take an alternative approach that rescues the Markovian property of the jump process at the expense of a supplementary age variable. We associate each random walker with a residence time (age, in short) a , which is reset when the random walker jumps to another location. In one dimension of space, we denote by $n(t, x, a)$ the probability density function of walkers at time t that have been located at x exactly during the last span of time a . The dynamics of the CTRW are then described [157, 167, 107, 55] by means of an age-renewal equation with spatial jumps:

$$\begin{cases} \partial_t n(t, x, a) + \partial_a n(t, x, a) + \beta(a)n(t, x, a) = 0, & t \geq 0, \quad a > 0, \quad x \in \mathbb{R} \\ n(t, x, a = 0) = \int_0^\infty \beta(a') \int_{\mathbb{R}} \omega(x - x') n(t, x', a') dx' da' \\ n(t = 0, x, a) = n^0(x, a). \end{cases} \quad (2.1)$$

The boundary condition on $n(t, x, 0)$ at age $a = 0$ accounts for the particles landing at position x at time t after having “jumped” from position x' , at which they had remained during a time span exactly equal to a' . Here, β is the age-dependent rate of jump, and ω is the distribution of jump distances. They are chosen in the following way.

Hypothesis 7 (Space jump kernel ω and jump rate β).

We assume that ω is a Gaussian probability distribution of mean 0 and variance σ^2 , and that β

is decaying for large age in a precise way:

$$\begin{cases} \omega(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right), & \sigma > 0 \\ \beta(a) = \frac{\mu}{1+a}, & 0 < \mu < 1 \end{cases} \quad (2.2)$$

The assumption of ω could be relaxed. However, we believe this would induce unnecessary technicality. The specific choice of the rate of jump β is crucial. Only the case $\mu \in (0, 1)$ yields subdiffusion. This could be relaxed to an asymptotic equivalence – see Chapter 1 –, however we will stick to (2.2) for the sake of clarity.

The fact that the loss term $\beta(a)n(t, x, a)$ is recovered in the boundary condition (and that ω will be taken to be a probability distribution) leads to the conservation of the total population density $\int_0^\infty \int_{\mathbb{R}} n(\cdot, x, a) dx da$ along time.

For mathematical reasons, we restrict to initial conditions compactly supported in age. Without loss of generality, we assume:

$$(\forall x) \quad \text{supp}(n^0(x, \cdot)) = [0, 1]. \quad (2.3)$$

Further technical hypotheses will be made later on.

The probability that a particle reaches age a without jumping is $\exp\left(-\int_0^a \beta(s) ds\right)$. On the other hand, the jump rate of particles at age a is $\beta(a)$. Hence, the distribution of residence times $\Phi(a)$ (meaning the distribution of the age of particles when they jump) is given by

$$\Phi(a) = \beta(a) \exp\left(-\int_0^a \beta(s) ds\right) = \frac{\mu}{(1+a)^{1+\mu}}. \quad (2.4)$$

A noteworthy observation is that the mean residence time of particles $\int_0^\infty a\Phi(a) da$ is infinite since $\mu \in (0, 1)$. This is a signature of subdiffusion at a larger scale [107].

The motivation for our current work is the construction of tools that may allow us to better understand the behaviour of pulled reaction-subdiffusion fronts, covered by an extensive literature [66, 67, 158, 1, 147, 119, 113]. The reader may find a short and comprehensive review in [158]. We consider here the large scale asymptotics of equation (2.1) in the hyperbolic rescaling $(t/\varepsilon, x/\varepsilon, a)$, suitable for the study of constant speed fronts. Note that the mean field subdiffusive effects appear at a scale $\left(\frac{t}{\varepsilon^{2/\mu}}, \frac{x}{\varepsilon}, a\right)$ (the reader may find such results in [107]) and will not be captured by our analysis. This is consistent with the large deviations approach used to study rare events in probability theory. In the reaction-subdiffusion setting, pulled fronts (as opposed to pushed fronts driven by reaction kinetics) are indeed driven by the few particles which jump ahead of the front and not by the mean movement of particles. We refer to the seminal articles by Freidlin [65] and by Evans and Souganidis [50] for the introduction of PDE tools inspired by large deviation methods in order to study geometric optics approximations for solutions of certain reaction-diffusion equations containing a small parameter.

2.1.2 Hyperbolic limit and derivation of the Hamilton-Jacobi equation.

Let us study the large scale asymptotics of the probability density function n in a hyperbolic scaling. We make the following Ansatz (Hopf-Cole transformation):

$$n_\varepsilon(t, x, a) = n(t/\varepsilon, x/\varepsilon, a) = \exp(-\phi_\varepsilon(t, x, a)/\varepsilon). \quad (2.5)$$

This enables us to accurately measure the behaviour of small tails of the probability density function n , reminiscent of large deviation principle theory. The function n_ε satisfies the following equation,

$$\begin{cases} \partial_t n_\varepsilon + \frac{1}{\varepsilon} \partial_a n_\varepsilon + \frac{1}{\varepsilon} \beta n_\varepsilon = 0, & t \geq 0, \quad a > 0, \quad x \in \mathbb{R} \\ n_\varepsilon(t, x, 0) = \int_0^{1+t/\varepsilon} \int_{\mathbb{R}} \beta(a) \omega(z) n_\varepsilon(t, x - \varepsilon z, a) \, dz \, da \\ n_\varepsilon(0, x, a) = n_\varepsilon^0(x, a) = n^0(x/\varepsilon, a). \end{cases} \quad (2.6)$$

Remark (Evolution of the support in age of n_ε). We recall that $\text{supp } n^0(x, \cdot) \subseteq [0, 1]$, and note that the term $\partial_t n_\varepsilon + \partial_a n_\varepsilon / \varepsilon$ in equation (2.6) corresponds to transport in age at speed $1/\varepsilon$. The consequence of this transport is that the support in age of n_ε at time t is $[0, 1 + t/\varepsilon)$, whence the upper limit $1 + t/\varepsilon$ for the integral giving the boundary condition.

For (t, x, a) such that $\phi_\varepsilon(t, x, a) < \infty$, ϕ_ε satisfies:

$$\begin{cases} \partial_t \phi_\varepsilon + \frac{1}{\varepsilon} \partial_a \phi_\varepsilon - \beta = 0, & t \geq 0, \quad a > 0, \quad x \in \mathbb{R} \\ \exp(-\phi_\varepsilon(t, x, 0)/\varepsilon) = \int_0^{1+t/\varepsilon} \int_{\mathbb{R}} \beta(a) \omega(z) \exp(-\phi_\varepsilon(t, x - \varepsilon z, a)/\varepsilon) \, dz \, da \\ \phi_\varepsilon(0, x, a) = \phi_\varepsilon^0(x, a) = -\varepsilon \ln(n^0(x/\varepsilon, a)). \end{cases} \quad (2.7)$$

Let us denote by ψ_ε the boundary value at $a = 0$, which will be our main unknown:

$$\psi_\varepsilon(t, x) = \phi_\varepsilon(t, x, 0). \quad (2.8)$$

We compute the solution of equation (2.7) along characteristic lines:

$$\phi_\varepsilon(t, x, a) = \begin{cases} \psi_\varepsilon(t - \varepsilon a, x) + \varepsilon \int_0^a \beta(s) \, ds, & t > 0, \quad \varepsilon a < t \\ \phi_\varepsilon^0(x, a - t/\varepsilon) + \varepsilon \int_{a-t/\varepsilon}^a \beta(s) \, ds, & t \geq 0, \quad a \geq t/\varepsilon. \end{cases} \quad (2.9)$$

Injecting (2.9) into the $a = 0$ boundary condition satisfied by ϕ_ε in (2.7) now yields:

$$\begin{aligned} 1 &= \int_0^{t/\varepsilon} \Phi(a) \int_{\mathbb{R}} \omega(z) \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \psi_\varepsilon(t - \varepsilon a, x - \varepsilon z)]\right) \, dz \, da \\ &\quad + \int_{t/\varepsilon}^{1+t/\varepsilon} \Phi(a) \int_{\mathbb{R}} \omega(z) \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \phi_\varepsilon^0(x - \varepsilon z, a - t/\varepsilon)] + \int_0^{a-t/\varepsilon} \beta\right) \, dz \, da. \end{aligned} \quad (2.10)$$

Let us formally derive the limiting Hamilton-Jacobi equation. Taking the formal limit of (2.10) when $\varepsilon \rightarrow 0$ yields:

$$1 = \int_0^\infty \Phi(a) \exp(a \partial_t \psi_0(t, x)) \, da \int_{\mathbb{R}} \omega(z) \exp(z \partial_x \psi_0(t, x)) \, dz. \quad (2.11)$$

It is a Hamilton-Jacobi equation, since it is equivalent to:

$$\partial_t \psi_0(t, x) + H(\partial_x \psi_0)(t, x) = 0, \quad (2.12)$$

with H defined as follows, where $\hat{\Phi}^{-1}$ is the inverse function of the Laplace transform of Φ :

$$H(p) = \hat{\Phi}^{-1}\left(\frac{1}{\int_{\mathbb{R}} \omega(z) \exp(zp) \, dz}\right). \quad (2.13)$$

Remark (Alternative choice of β). The limiting equation makes sense for a large class of functions β , including constant functions $\beta \equiv K$. Indeed, the scaling considered here does not depend on the diffusive regime, whether it is anomalous or not. The reader may find interesting situations for which the large deviation scaling is problem-dependent in [23, 115].

2.1.3 Properties of the Hamiltonian

We will now prove that the Hamiltonian H satisfies some properties often encountered in the literature.

Proposition 41. *The Hamiltonian H defined in (2.13) has the following properties:*

- (i) $H \in C^\infty(\mathbb{R}, \mathbb{R}_+)$,
- (ii) F is coercive: $\frac{H(p)}{p} \xrightarrow{p \rightarrow \infty} \infty$,
- (iii) H is convex, but not strictly uniformly convex.

Proof.

- (i) Let

$$F(p, h) = \int_0^\infty \Phi(a) \exp(-ah) \, da - \left(\int_{\mathbb{R}} \omega(z) \exp(zp) \, dz \right)^{-1}.$$

F is strictly decreasing with respect to its second variable over \mathbb{R} . For all $p \in \mathbb{R} \setminus \{0\}$, since ω is a Gaussian centred at 0 and Φ is a probability measure, it follows that $F(p, 0) < 0$. For any $p \in \mathbb{R}$, we have $\lim_{h \rightarrow \infty} F(p, h) = 0$. Hence for each $p \in \mathbb{R}$ there exists a unique $H \in \mathbb{R}_+$ such that $F(p, H) = 0$. This condition is equivalent to equation (2.11), hence H is well defined.

The function F is C^∞ , and $F(0, 0) = 0$. Strict monotonicity and an implicit functions result end the proof.

- (ii) We have:

$$\int_{\mathbb{R}} \omega(z) e^{zp} \, dz = \exp\left(\frac{\sigma^2 p^2}{2}\right). \quad (2.14)$$

It follows from equation (2.11) that

$$\int_0^\infty \Phi(a) e^{-aH(p)} \, da \geq \int_0^1 \Phi(a) e^{-H(p)} \, da \geq C e^{-H(p)},$$

hence

$$C e^{-H(p)} \leq \exp\left(-\frac{\sigma^2 p^2}{2}\right),$$

which implies $H(p) \gtrsim 1 + p^2$. Hence H is coercive.

- (iii) Differentiating equation (2.11) with respect to p yields:

$$0 = \int_0^\infty \int_{\mathbb{R}} \left(\nabla_p H(p) - \frac{z}{a} \right) a \Phi(a) \exp(-aH(p)) \omega(z) \exp(zp) \, dz \, da,$$

after which a second differentiation gives us:

$$0 = \int_0^\infty \int_{\mathbb{R}} a D_p^2 H(p) d\gamma(z, a) - \int_0^a \int_{\mathbb{R}} a^2 \left(\nabla_p H(p) - \frac{z}{a} \right)^2 d\gamma(z, a),$$

where $d\gamma(z, a) = \Phi(a)\omega(z) \exp(-aH(p)) \exp(zp) dz da$ is a non-negative measure. It follows that $D_p^2 H \geq 0$.

However, the Hamiltonian H is not strictly uniformly convex, since $D_p^2 H(0) = 0$. This is proved as follows. We first remark, from (2.13) and (2.11), that $H(0) = 0$. We recover the following expression for $\nabla_p H(p)$:

$$\nabla_p H(p) = \frac{\int_0^\infty \int_{\mathbb{R}} z \Phi(a) \omega(z) \exp(zp) dz da}{\int_0^\infty \int_{\mathbb{R}} a \Phi(a) \omega(z) \exp(zp) dz da}.$$

Since $\int_0^\infty a \Phi(a) da = \infty$, we deduce that $\nabla_p H(0) = 0$ and recover:

$$D_p^2 H(0) = \frac{\int_{\mathbb{R}} z^2 \omega(z) dz}{\int_0^\infty a \Phi(a) da} = 0.$$

□

Proposition 42 (Behaviour of H around 0). *Suppose ω is a Gaussian function of variance σ^2 . Around $p = 0$, we have*

$$H(p) \sim_0 (\sigma p)^{2/\mu} (2\Gamma(1 - \mu))^{1/\mu}. \quad (2.15)$$

Proof. We have $\Phi(a) = \mu(1+a)^{-1-\mu}$, hence, thanks to equations (2.11) and (2.14):

$$\int_0^\infty \frac{\mu}{(1+a)^{1+\mu}} (\exp(-aH) - 1) da = \hat{\Phi}(H) - 1 = \exp(-(\sigma p)^2/2) - 1 \sim_{p=0} -(\sigma p)^2/2.$$

Denoting $b = aH$, since $H(0) = 0$ the left hand side becomes:

$$\begin{aligned} & \frac{1}{H} \int_0^\infty \frac{\mu}{(1+b/H)^{1+\mu}} (e^{-b} - 1) db \\ &= H^\mu \int_0^\infty \frac{\mu}{(H/b+1)^{1+\mu}} b^{-1-\mu} (e^{-b} - 1) db \\ &\sim_{H=0} H^\mu \int_0^\infty \mu b^{-1-\mu} (e^{-b} - 1) db. \end{aligned}$$

Integrating that last expression by parts ends the proof. □

Proposition 43 (Behaviour of H for large p). *Suppose ω is a Gaussian function of variance σ^2 . Around ∞ , we have*

$$H(p) \sim_\infty \mu \exp\left(\frac{\sigma^2 p^2}{2}\right). \quad (2.16)$$

Proof. Retaking the computations of Proposition 42, we recover:

$$\frac{1}{H(p)} \int_0^\infty \frac{\mu}{\left(1 + \frac{b}{H(p)}\right)^{1+\mu}} e^{-b} db = \exp\left(-\frac{\sigma^2 p^2}{2}\right).$$

It follows that

$$\frac{1}{H(p)} \int_0^\infty \mu \left[1 - (1 + \mu) \frac{b}{H(p)} \right] e^{-b} db \leq \exp\left(-\frac{\sigma^2 p^2}{2}\right),$$

which by integration leads to

$$\begin{aligned} \frac{\mu}{H(p)} \left(1 - \frac{1 + \mu}{H(p)} \right) &\leq \exp\left(-\frac{\sigma^2 p^2}{2}\right) \\ H(p) &= \mu \exp\left(\frac{\sigma^2 p^2}{2}\right) \left(1 - \frac{1 + \mu}{H(p)} \right). \end{aligned}$$

The divergence of H (e.g. coercivity in Proposition 41) concludes the proof. \square

Remark. Boundedness above by a polynomial can be an important property in the calculus of variations approach to Hamilton-Jacobi equations. It implies that the Lagrangian defined by convex duality in equation (2.18) is coercive in the sense that there exist $C > 0$ and $r > 1$ such that $L(v) \geq C|v|^r$, allowing the application of Tonelli's direct method to the problem of minimising the corresponding action functional. However, as is often the case, it can be replaced by additional regularity hypotheses on the class of solutions. We will not elaborate further on this claim in this Remark, but we refer the reader to [31, Chapter 19, Theorems 19.11 and 19.17] for a calculus of variations and proximal analysis approach to Hamilton-Jacobi equations. We state in Theorem 45 a variation of [31, Theorems 19.11 and 19.17] that applies to our case.

We have computed the Hamiltonian for $p \in [0, 0.5]$, different values of μ , and $\sigma = 0.5$, as depicted in Figure 2.1. Figure 2.2 shows how H and the asymptotic behaviour close to 0 proved in Proposition 42 grow apart for p large enough, as expected from the result of Proposition 43. For a visual representation of the evolution in time of the solution ψ_0 of the Hamilton-Jacobi equation (2.10), we refer to Figure 2.3, which is the result of a weighted essentially non-oscillatory (WENO) scheme of order 5, Lax-Friedrichs numerical scheme. We refer the reader to the Appendix A and to [155] for a review of such numerical methods. In Figure 2.3, the initial data taken for the first and third subfigures is the same, in order to illustrate how subdiffusion slows down significantly as time advances. The initial conditions in the second and third subfigures are such that $H(\partial_x \psi) = \lambda(t)\psi$ for $\partial_x \psi$ close to 0 for the respective Hamiltonians, hence the preserved shape of the decaying profiles, most noticeable in the log scale. Indeed, injecting the Ansatz $\psi_0(t, x) = c(t)x^\alpha$ into $\partial_t \psi + \tilde{H}(\partial_x \psi)$, with \tilde{H} given by the approached expression at 0 of H (2.15), leads to defining such initial conditions.

We recall that, under suitable hypotheses on H and on g , classical existence and uniqueness results hold for the evolution Hamilton-Jacobi Cauchy problem:

$$\begin{cases} \partial_t u(t, x) + H(D_x u(t, x)) = 0, & (t, x) \in (0, T] \times \mathbb{R}^n \\ u(0, x) = g(x), & x \in \mathbb{R}^n. \end{cases} \quad (2.17)$$

Let L be the Lagrangian defined as the Legendre-Fenchel transform of the convex, coercive Hamiltonian H :

$$L(v) = \sup_{p \in \mathbb{R}} \{pv - H(p)\}. \quad (2.18)$$

We define the value function f via the Hopf-Lax formula:

$$f(t, x) = \min_{y \in \mathbb{R}} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}. \quad (2.19)$$

We recall the Proposition 6 from the state of the art Section 4.3, which applies to our problem.

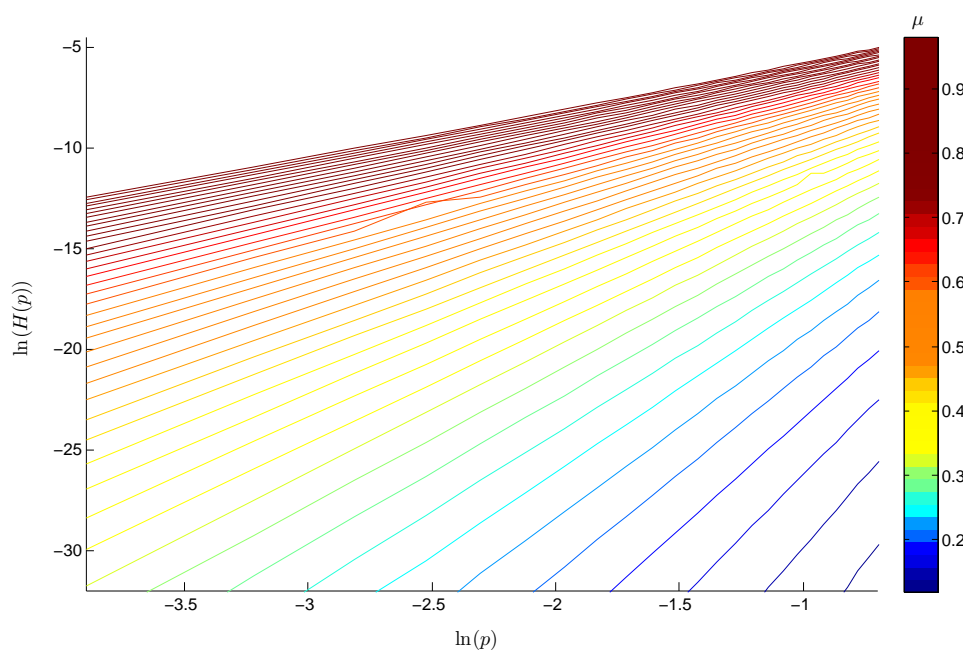


Figure 2.1: $\ln(H(p))$ plotted against $\ln(p)$ for $p \in [0, 0.5]$ for values of μ ranging from 0.12 (lower line) to 0.98 (upper line) (the successive graph positions rise monotonically with respect to μ). $H(p)$ behaves as a power of p for $p \ll 1$. This illustrates Proposition 42.

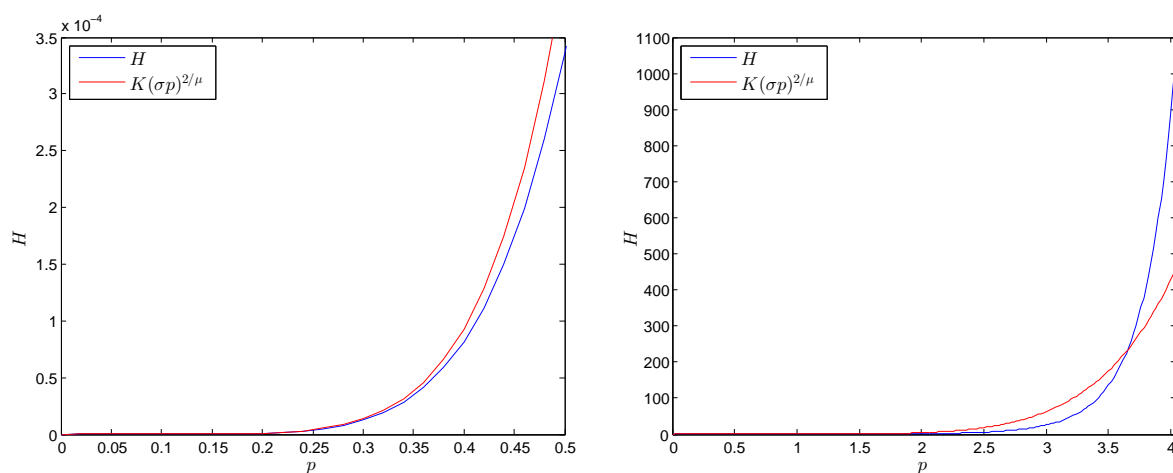
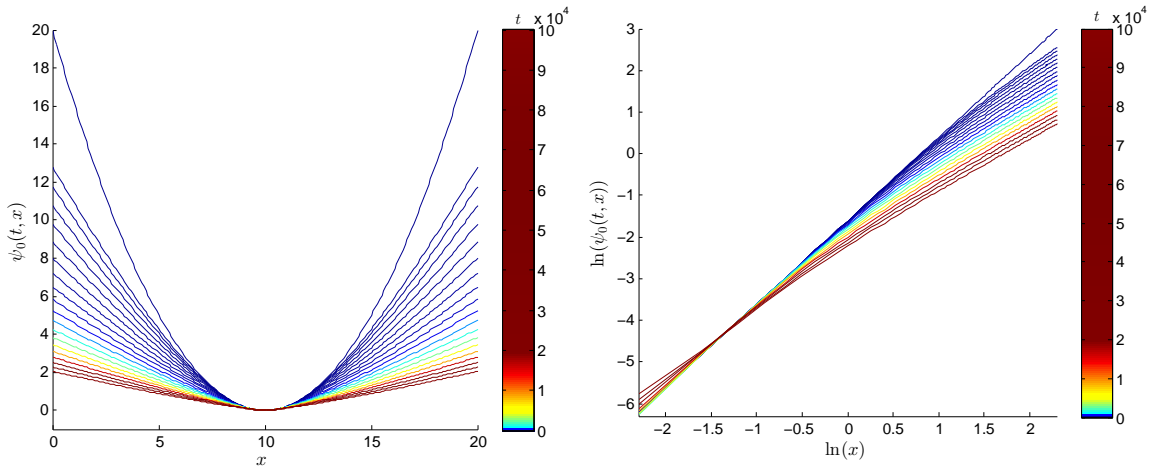
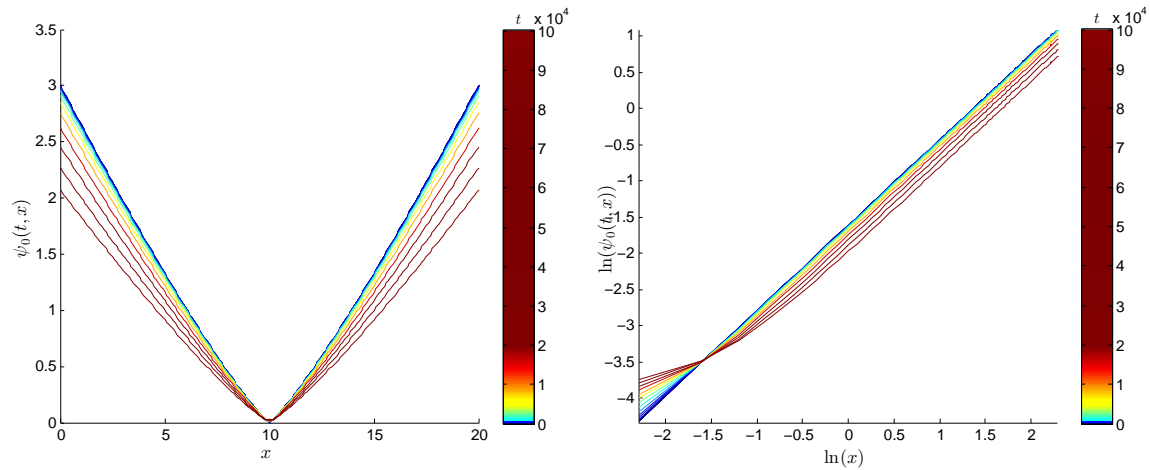


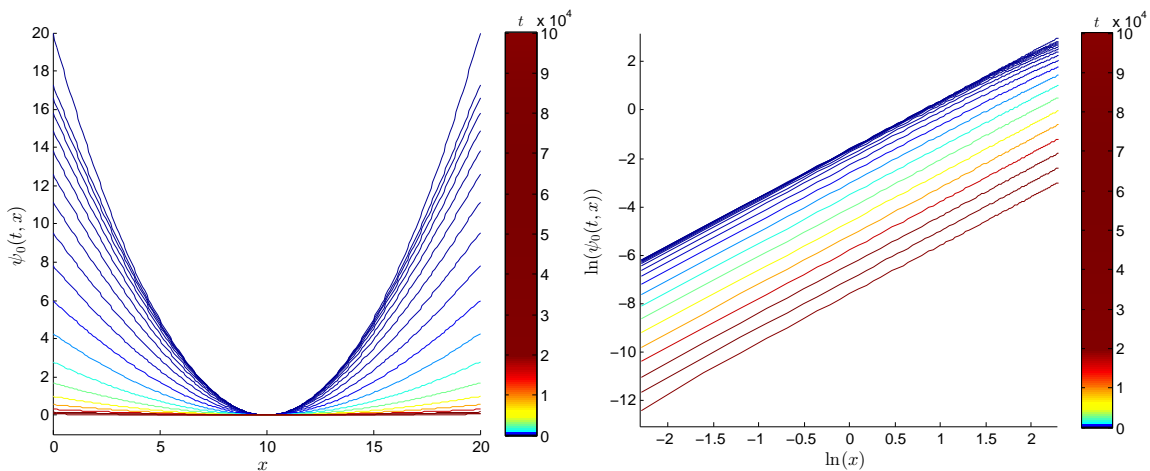
Figure 2.2: Comparison of H and its asymptotic behaviour near 0 for $\mu = 0.3$. Far from 0, they do not match, as expected from Proposition 43. Here $K = (2\Gamma(1 - \mu))^{-1/\mu}$.



Subdiffusive case with $\mu = 0.3$ and $\beta(a) = \mu/(1 + a)$. $\psi_0(0, x) = 0.2(x - 10)^2$.



Subdiffusive case with $\mu = 0.3$ and $\beta(a) = \mu/(1 + a)$. $\psi_0(0, x) = 0.2(x - 10)^{2/2-\mu}$.



Diffusive case with $D = 0.01$ and $\beta(a) = D$. $\psi_0(0, x) = 0.2(x - 10)^2$.

Figure 2.3: Decay of $\psi_0(t, \cdot)$ (left) and $\ln(\psi_0(t, \cdot))$ (right) for $\sigma = 1$, $t \in [0, 100000]$ (shown in the color bar) and $x \in [0, 20]$ with periodic boundary conditions. The presented plots are taken at 20 regular intervals in $\ln(t) \in [0, 11.5]$. As t increases, each successive graph lies below the previous one for the larger values of $|x|$.

Proposition 44. *Let H be a convex, coercive Hamiltonian, and let g be a bounded below, sub-linear, semi-concave and Lipschitz continuous function over \mathbb{R} . The Hopf-Lax value function f in (2.19) is well defined for the initial condition g . Moreover, $f(0, \cdot) = g$ and f is a viscosity solution of (2.17).*

We state hereafter a relevant uniqueness theorem in a suitable class of functions: a version of [31, Theorems 19.11 and 19.17] for a homogenous Hamiltonian that is not polynomially bounded above.

Theorem 45 (Uniqueness theorem). *Let H be locally Lipschitz, convex and superlinear. Let g be bounded below and Lipschitz continuous. Then there exists a unique viscosity solution of (2.17) within the class of Lipschitz continuous functions.*

This uniqueness theorem is a corollary of [31, Corollary 19.17], which follows from [31, Theorem 19.11]. In that last theorem it is assumed that H has polynomial growth for $|p| \rightarrow \infty$, which is not our case, as stated in Proposition 43. We overcome this issue by assuming that u is globally Lipschitz continuous so that H takes values only over a compact set. The reader can find the proof of the theorems we have cited in [31, Chapter 19.4].

2.1.4 Main hypotheses and results

Throughout this Chapter, we will work over the set $[0, T] \times \mathbb{R}$ for some $T > 0$, and we will denote by C any real constant whose value is irrelevant.

The main result of this Chapter is that, under suitable hypotheses on the initial condition ϕ_ε^0 , the boundary value ψ_ε converges to the unique viscosity solution ψ_0 of the limiting Hamilton-Jacobi Cauchy problem (2.11) for an appropriate initial condition v . In this subsection we will state and justify which initial conditions we have chosen, justify our choice of proof strategy, and describe what we consider the crucial features of our work.

We will make the following hypotheses on the initial condition.

Hypothesis 8 (Initial condition ϕ_ε^0).

We assume that the initial condition has the following form:

$$\phi_\varepsilon^0(x, a) = v_\varepsilon(x) + \varepsilon \eta_\varepsilon(x, a) + \chi_{[0,1]}(a) \quad (2.20)$$

where $\chi_{[0,1]}(a)$ is worth 0 for $a \in [0, 1)$ and ∞ for $a \geq 1$. Hence, ϕ_ε^0 takes finite value in $[0, 1)$ only, according to the assumption on n^0 (2.3). The functions ϕ_ε^0 , v_ε and η_ε satisfy the following properties uniformly over ε :

1. v_ε is bounded below, and has at most linear growth from above: there exist positive constants K_0, K_1 such that for any $x \in \mathbb{R}$ and $a \in [0, 1)$, for any $\varepsilon > 0$,

$$-K_0 \leq v_\varepsilon(x) \leq K_0 + K_1|x|. \quad (2.21)$$

2. η_ε is uniformly bounded.

3. ϕ_ε^0 is semi-concave in x uniformly in a : there exists $\mathfrak{C}_{xx} \in \mathbb{R}$ such that for any $x, h \in \mathbb{R}$ and $a \in [0, 1)$, for any $\varepsilon > 0$,

$$\phi_\varepsilon^0(x+h, a) + \phi_\varepsilon^0(x-h, a) - 2\phi_\varepsilon^0(x, a) \leq \mathfrak{C}_{xx}h^2, \quad (2.22)$$

(Or equivalently, in the sense of distributions, $\partial_x^2 \phi_\varepsilon^0 \leq \mathfrak{C}_{xx}$.)

4. ϕ_ε^0 is Lipschitz continuous in x uniformly in $a \in [0, 1]$: there exists C_L such that, for any $a \in [0, 1]$, for any $x_1, x_2 \in \mathbb{R}$ and for any $\varepsilon > 0$,

$$|\phi_\varepsilon^0(x_1, a) - \phi_\varepsilon^0(x_2, a)| \leq C_L |x_1 - x_2|. \quad (2.23)$$

5. We assume that there exists a limit function v such that $v_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} v$, locally uniformly in x, a .

Remark. When working at fixed ε , since the previous hypotheses hold uniformly over $\varepsilon \in (0, 1)$, we will abuse the notation and drop the ε subindices from v_ε and η_ε .

The following theorem is our main result.

Theorem 46. Under Hypotheses 7 and 8, $\psi_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{L_{\text{loc}}^\infty} \psi_0$, which is the unique viscosity solution of the limiting Hamilton-Jacobi equation (2.11) with initial condition $v(x)$ among the class of bounded below, Lipschitz continuous and semi-concave functions.

Remark (Initial conditions – interpretation). We may interpret the additive perturbation η in the Ansatz (2.20) as a multiplicative perturbation in terms of n :

$$n_\varepsilon^0(x, a) = \tilde{n}_\varepsilon(x, a) \exp(-v_\varepsilon(x)/\varepsilon) \mathbb{1}_{[0,1]}(a),$$

where $\mathbb{1}_A$ is worth 1 over the set A and 0 elsewhere, and $\tilde{n}_\varepsilon(x, a) = \exp(-\eta_\varepsilon(x, a))$. We make the strong assumption that the support in age is uniformly bounded for $x \in \mathbb{R}$. This is for technical reasons which are commented in Subsection 2.1.5.

Remark (Lack of compatibility of the initial condition). The initial condition that we take is smooth enough in x (Lipschitz continuous). However, we do not require for it to be compatible in the sense that the influx relation at age $a = 0$ is satisfied at time $t = 0$ in (2.1). As a consequence, we allow discontinuities along $t = \varepsilon a$. This means that in general, we may have:

$$n_\varepsilon(t, x, 0) = \int_0^1 \int_{\mathbb{R}} \omega(z) \Phi(a) n_\varepsilon^0(x - \varepsilon z, a) dz da \neq n_\varepsilon^0(x, 0). \quad (2.24)$$

Such compatibility is assumed in [128, Chapter 3.4] to infer regularity with respect to a in the space-homogenous setting. Such regularity is not required in the present contribution.

Remark (Initial age profile). We make the strong assumption that the support in age is uniformly bounded for $x \in \mathbb{R}$. This is for technical reasons on which we comment later on in Subsection 2.1.5. The discontinuity that results from $\chi_{[0,1]}$ does not play a significant role in the a priori estimates since it lies outside of the essential domain of ϕ_ε^0 .

Remark. The second strong assumption is the semi-concavity of the initial condition. It is required for technical reasons, to handle the contribution of initial data (remaining) at time t . This implies local Lipschitz continuity. Therefore, we have deemed reasonable to assume global Lipschitz continuity. This is also in accordance with the uniqueness result that we use.

Remark (Half-relaxed limits). The global proof strategies of this work follow the basic methodology described in [11, Chapter 2]. This strategy relies on the proof of Lipschitz a priori estimates, the use of the Arzela-Ascoli theorem to extract a convergent subsequence $\psi_{\varepsilon_n} \rightarrow \psi_0$, a stability result to show ψ_0 is a viscosity solution of the limiting problem, and a comparison result to show ψ_0 is its unique viscosity solution; hence by compactness and Hausdorff separation arguments,

the whole (ψ_ε) sequence converges to ψ_0 . An alternative toolkit by means of which Hamilton-Jacobi stability results are often tackled is the half-relaxed limits theory described in [11, 2]. It presents the advantage of only requiring L^∞ a priori estimates, rather than Lipschitz continuity. In our case, this is no big improvement since the natural regularity of our initial condition is at least locally Lipschitz in x as stated in Remark 2.1.4. A drawback of this method is that one needs to prove a strong comparison result between the upper and lower semi-continuous envelopes of ψ_ε in order to show that the limit ψ_0 of a convergent subsequence of ψ_ε is the unique viscosity solution of the limiting Hamilton-Jacobi equation. Since our Hamiltonian H has exponential growth, such a strong comparison result is likely to require Lipschitz estimates as is already the case in the uniqueness Theorem 45 that we use. This seems to be the case as well in a related work in the context of evolutionary biology [120]. Consequently, even in the event that half-relaxed limits methods would allow us to reach results similar to ours, it is not clear that they would provide any improvement.

Apart from the stability result of Theorem 46, the most interesting points of the present work concern the way we have dealt with two significant complications, on which we will proceed to comment.

2.1.5 Main ideas and difficulties

Usually, similar limit problems for which the limit equation is averaged with respect to the fast variable (age a) are handled with the perturbed test function method introduced in [48], see for instance [21, 22, 26]. However, in our setting the perturbed function would be naturally unbounded. Here, we by-pass this issue by working directly on the boundary value of our solution (2.1). Namely, we reduce the solution $\phi_\varepsilon(t, x, a)$ to the knowledge of $\psi_\varepsilon(t, x) = \phi_\varepsilon(t, x, 0)$. Note that the reconstruction of ϕ_ε from ψ_ε along characteristic lines makes the problem non local in time. That is the first main idea in this work. The two following subsections describe the two major difficulties that we have encountered.

Corrected maximum principle

While defining the waiting time distribution Φ in (2.4) in the model description subsection 2.1.1, we noted that the mean residence time of particles $\int_0^\infty a\Phi(a) da$ is infinite in the subdiffusive case $\beta(a) = \mu/(1+a)$ with $\mu \in (0, 1)$.

It is equivalent to say that $N_\infty(a) = \exp(-\int_0^a \beta(s) ds)$ is not an integrable function. This induces additional mathematical difficulties in the proof of uniform bounds in Section 2.2, since $N_\infty(a)$ would be the stationary age distribution of the space homogeneous problem (2.25), if integrable (as it is the case for $\mu > 1$), see [128], Chapter 1 and classical renewal theory [58].

In order to illustrate the consequence this has on the use of the maximum principle to prove bounds on ψ_ε , let us first consider the integrable case $N_\infty \in L^1$ for the space homogeneous problem:

$$\begin{cases} \partial_t n(t, a) + \partial_a n(t, a) + \beta(a)n(t, a) = 0, & t \geq 0, \quad a > 0 \\ n(t, a = 0) = \int_0^\infty \beta(a')n(t, a') da' \\ n(t = 0, a) = n^0(a). \end{cases} \quad (2.25)$$

In the integrable case, provided n/N_∞ is initially bounded:

$$cN_\infty \leq n^0 \leq CN_\infty,$$

an application of the maximum principle shows that the bounds are propagated for all positive time t :

$$cN_\infty \leq n(t, \cdot) \leq CN_\infty.$$

However, the fact that N_∞ is not integrable precludes any use of this reference function in the subdiffusive case. Besides, it is well known that the long-time asymptotics of the space homogeneous problem (2.25) follow a self-similar scaling [58]:

$$n(t, a) \simeq \frac{1}{1+t} W_\infty \left(\frac{a}{1+t} \right), \quad (2.26)$$

where W_∞ is the Dynkin-Lamperti arc-sine law. The precise description of the intermediate asymptotics (*i.e.* the estimate of the distance between the solution at time t and the self-similar profile) was the subject of [15]. As a side remark, we expect a time correction in any L^∞ estimate of $\ln n$, as it is the case in Proposition 48, equation (2.32).

Remark (Renormalisation by a non-stationary measure). It is a common idea to study a vanishing multiplicative perturbation of a steady state distribution in order to prove convergence to it. We refer the reader to an article related to our work, [21], and to the WKB-type Ansatz the authors use to prove convergence to the Maxwellian of their equation. Following that idea while taking into account the non-existence of an integrable stationary age distribution N_∞ , it is tempting to make the Ansatz

$$n \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, a \right) = N \left(\frac{t}{\varepsilon}, a \right) \exp \left(-\frac{\phi_\varepsilon(t, x, a)}{\varepsilon} \right),$$

where N corresponds to the “pseudo-equilibrium” of Chapter 1: an auxiliary function that approaches the formal steady state. Section 2.4.1 shows that such an Ansatz leads to the same limiting Hamilton-Jacobi equation and explains the drawbacks of such an approach by sketching the required computations. Such computations are very close to those present in this Chapter, but additional terms have to be estimated and proved to vanish. That is the reason why we have chosen not to renormalise n while defining ϕ_ε in equation (2.5).

Contribution of initial data at time t

The second main difficulty we have tackled appears in Section 2.3.2, in the proof that the limit ψ_0 of a subsequence of ψ_ε is a viscosity super-solution of the limiting Hamilton-Jacobi equation (2.11). It stems from the long persistence of the initial condition in the renewal flux term.

In our proof, the renewal flux term at age $a = 0$ is split into 2 relative contributions: that of the particles which have already jumped before, and that of the particles which have never jumped before (respectively, \mathfrak{A}_ε and \mathfrak{B}_ε in equation (2.76)). Due to the heavy tail of the waiting time distribution, the latter decays very slowly. Thus, it has a relatively high contribution, that we must bound above in order to prove that ψ_0 is a viscosity supersolution. We solve this issue by a refined estimate of the relative contribution which expresses an anomalous exponent:

$$\mathfrak{B}_\varepsilon = \mathcal{O}(\varepsilon^\mu). \quad (2.27)$$

We can connect this anomalous behaviour to the long-time asymptotics of the space-homogenous problem (2.25) through the following heuristics.

Let $\mathfrak{p}(t)$ be the proportion, among the particles that jump at a given time t , of particles that had never jumped before – hence \mathfrak{p} plays the role of \mathfrak{B}_ε defined in equation (2.76). Introducing

the characteristic lines in a similar way to that which led to equation (2.9), we recover:

$$\mathbf{p}(t) = \frac{\int_t^{1+t} \beta(a)n_0(a-t) \exp\left(\int_{a-t}^a \beta(s) ds\right) da}{n(t,0)}.$$

Assume the initial condition n_0 is uniformly distributed over $a \in [0,1)$. Suppose that the denominator $n(t,0)$ behaves exactly like the self-similar profile as in (2.26). We recover that, up to multiplication by constants, the following quantities have the same asymptotic behaviour for large t :

$$n(t,0) = \mathcal{O}\left((1+t)^{\mu-1}\right).$$

Taking into account $\beta(a) = \mu/(1+a)$, it follows that we expect \mathbf{p} to decay in a polynomial way, as $(1+t)^{-2\mu}$. It is very interesting to note that this expression is homogenous in t to equation (2.84), in which ε is homogenous to t^{-1} and T is homogenous to t .

The proof of the bound $\mathcal{O}(\varepsilon^\mu)$ involves a semi-concavity assumption which is propagated for any $t \in [0, T]$.

2.1.6 Organization of the Chapter

Section 2.2 deals with the proof of Theorem 47, namely, a uniform in ε bound on ψ_ε in $W_{\text{loc}}^{1,\infty}([0, T] \times \mathbb{R})$, each subsection corresponding to the bounds on ψ_ε and its first partial derivatives. The proofs mainly involve comparison results stemming from the maximum principle. Section 2.3 proves that ψ_0 is a viscosity solution of the limiting Hamilton-Jacobi equation (2.11), which is the result of Theorem 46, and that it is its unique viscosity solution, thanks to Theorem 45. The first subsection shows it is a subsolution, the second, a supersolution.

During the revision stage of this manuscript, the authors became aware of a preprint by Nordmann, Perthame, and Taing, which addresses similar questions, in the context of evolutionary biology. Our model is simpler as it is conservative, and jump rates are homogeneous with respect to the space variable. On the other hand, our results are stronger as we obtain the rigorous limit of the problem as $\varepsilon \rightarrow 0$.

2.2 Uniform local boundedness and Lipschitz continuity of ψ_ε

Remark. When working at fixed ε , since the previous hypotheses hold uniformly over $\varepsilon \in (0, 1)$, we will abuse the notation and drop the ε subindices from v_ε and η_ε .

This whole section deals with the proof of the following Theorem.

Theorem 47. *a Let $T > 0$ and $0 < \varepsilon < 1$. Under hypotheses 7 and 8, ψ_ε is bounded in $W_{\text{loc}}^{1,\infty}([0, T] \times \mathbb{R})$ uniformly in ε , with the following quantitative bounds, where $(t, x) \in [0, T] \times \mathbb{R}$:*

1.

$$\psi_\varepsilon(t, x) \geq \inf v - \beta(0) - \sup_{\varepsilon \in (0,1)} \left| \ln \left\| \exp\left(-\inf_x \eta_\varepsilon(x, \cdot)\right) \right\|_{L^1} \right|, \quad (2.28)$$

$$\psi_\varepsilon(t, x) \leq K_0 + K_1|x| + C_\eta + 2\sigma^2 K_1^2 + (1 + \mu)T. \quad (2.29)$$

2.

$$\text{Lip}(\psi_\varepsilon(t, \cdot)) \leq C_L. \quad (2.30)$$

3.

$$\text{Lip } \psi_\varepsilon(\cdot, x) \leq \max \left(\mu(1 + \mu), \int_{\mathbb{R}} \omega(z) \exp(C_L|z|) dz \right). \quad (2.31)$$

Subsection 2.2.1 proves the more accurate ε -dependant bounds of Propositions 48 from which the uniform bounds of Theorem 47.1 follow. The Lipschitz continuity results of Theorem 47.2 and Theorem 47.3 are proved in Subsections 2.2.2 and 2.2.3 respectively.

Proposition 48. *Let $T > 0$ and $\varepsilon > 0$. Under hypotheses 7 and 8, $\psi_\varepsilon \in L^\infty([0, T] \times \mathbb{R})$, with the quantitative bounds stated below: for any $(t, x) \in [0, T] \times \mathbb{R}$,*

$$\psi_\varepsilon(t, x) \geq \mathfrak{c} = \inf v - \varepsilon \ln \beta(0) - \varepsilon \ln \left\| \exp \left(- \inf_x \eta(x, a) \right) \right\|_{L^1} \quad (2.32)$$

$$\psi_\varepsilon(t, x) \leq \mathfrak{C}_T = K_0 + C_\eta + K_1|x| + \varepsilon(1 + \mu) \ln \left(1 + \frac{t}{\varepsilon} \right) + \varepsilon 2\sigma^2 K_1^2 \quad (2.33)$$

2.2.1 Local boundedness of ψ_ε

This subsection deals with the proof of Proposition 48.

Remark. As mentioned previously in Subsection 2.1.5, the space-homogenous problem does not admit an integrable stationary measure. Moreover, it has been shown in Chapter 1 that it exhibits a self-similar decay. The side effect in our context is a time-dependent correction term on the upper bound, of the following form:

$$-\varepsilon \ln \left(\int_0^1 \int_{\mathbb{R}} \Phi(a + T/\varepsilon) e^{\int_0^a \beta} \omega(z) \exp(-\eta(x - \varepsilon z, a)) dz da \right).$$

Let us start by bounding ψ_ε from below, proving the first half of the Proposition.

Proof of the lower bound (2.32) of Proposition 48.

From equation (2.7) and the Ansatz (2.20) made on the initial condition in Hypothesis 8 we gather that we may define n^0 as follows:

$$n^0(x, a) = n_\varepsilon^0(\varepsilon x, a) = \exp \left(- \frac{v(\varepsilon x)}{\varepsilon} - \eta(\varepsilon x, a) \right) \mathbb{1}_{[0,1]}(a).$$

Remark. Bear in mind that ε is fixed and that we have dropped the ε subindex from $v_\varepsilon = v$ and $\eta_\varepsilon = \eta$, since our hypotheses on the initial condition hold uniformly over $\varepsilon \in (0, 1)$.

Let us define $\bar{n} : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ as the solution of the following equation:

$$\begin{cases} \partial_t \bar{n}(t, a) + \partial_a \bar{n}(t, a) + \beta(a) \bar{n}(t, a) = 0 \\ \bar{n}(t, 0) = \int_0^\infty \beta(a) \bar{n}(t, a) da \\ \bar{n}^0(a) = \exp \left(- \inf_x v(\varepsilon x)/\varepsilon \right) \exp \left(- \inf_x \eta(\varepsilon x, a) \right) \mathbb{1}_{[0,1]}(a) \geq \sup_x n^0(x, a). \end{cases} \quad (2.34)$$

Since β is a non-increasing function, ω is a probability measure, n is the solution of equation (2.1) for an initial condition n^0 and $\bar{n}^0(a) \geq \sup_x n^0(x, a)$, it follows that for any $t, a \geq 0$:

$$\bar{n}(t/\varepsilon, a) \geq \sup_x n(t/\varepsilon, x/\varepsilon, a) = \sup_x n_\varepsilon(t, x, a).$$

Moreover, since β is non-increasing and the L^1 norm of \bar{n} is preserved,

$$\bar{n}(t/\varepsilon, 0) = \int_0^\infty \beta(a) \bar{n}(t/\varepsilon, a) \, da \leq \beta(0) \int_0^\infty \bar{n}(t/\varepsilon, a) \, da = \beta(0) \|\bar{n}^0\|_{L^1}.$$

It follows that:

$$\psi_\varepsilon(t, x) = -\varepsilon \ln n_\varepsilon(t, x, 0) \geq -\varepsilon \ln(\beta(0) \|\bar{n}^0\|_{L^1}).$$

After computing the L^1 norm of \bar{n}^0 :

$$\|\bar{n}^0\|_{L^1} = \exp(-\inf_x v/\varepsilon) \int_0^\infty \exp(-\inf_x \eta(x, a)) \, da,$$

and since η is bounded (Hypothesis 8.2), we obtain the claimed result (2.32). \square

The rest of this subsection is devoted to proving the upper bound (2.33) in Proposition 48.

Proof of the upper bound (2.33) of Proposition 48.

From equation (2.10) we recover:

$$\begin{aligned} \exp\left(-\frac{1}{\varepsilon} \psi_\varepsilon(t, x)\right) &= \int_0^{t/\varepsilon} \int_{\mathbb{R}} \Phi(a) \omega(z) \exp\left(-\frac{1}{\varepsilon} \psi_\varepsilon(t - \varepsilon a, x - \varepsilon z)\right) \, dz \, da \\ &\quad + \int_0^1 \int_{\mathbb{R}} \Phi\left(a + \frac{t}{\varepsilon}\right) \omega(z) \exp\left(-\frac{1}{\varepsilon} \phi_\varepsilon^0(x - \varepsilon z, a)\right) \exp\left(\int_0^a \beta\right) \, dz \, da. \end{aligned}$$

Since the first right-hand side term is non-negative and v is sublinear by Hypothesis 8.1,

$$\begin{aligned} e^{-\psi_\varepsilon(t, x)/\varepsilon} &\geq \int_0^1 \int_{\mathbb{R}} \Phi\left(a + \frac{t}{\varepsilon}\right) \omega(z) e^{-\frac{1}{\varepsilon}(K_0 + K_1|x - \varepsilon z|)} e^{-\eta(x - \varepsilon z, a)} e^{\int_0^a \beta} \, dz \, da \quad (2.35) \\ &\geq \frac{\Phi\left(1 + \frac{t}{\varepsilon}\right)}{\Phi(0)} \int_0^1 \int_{\mathbb{R}} \Phi(a) e^{\int_0^a \beta} \omega(z) \exp\left(-\frac{K_0}{\varepsilon} - \frac{K_1|x|}{\varepsilon} - K_1|z|\right) e^{-\eta(x - \varepsilon z, a)} \, dz \, da \\ &\geq e^{-\frac{K_0}{\varepsilon}} e^{-\frac{K_1|x|}{\varepsilon}} \frac{\Phi\left(1 + \frac{t}{\varepsilon}\right)}{\Phi(0)} \int_0^1 \int_{\mathbb{R}} \Phi(a) e^{\int_0^a \beta} \frac{\exp\left(-\frac{z^2}{2\sigma^2} - K_1|z|\right)}{\sigma\sqrt{2\pi}} e^{-\eta(x - \varepsilon z, a)} \, dz \, da. \end{aligned}$$

Moreover,

$$\begin{aligned} &\int_0^1 \int_{\mathbb{R}} \Phi(a) e^{\int_0^a \beta} \frac{\exp\left(-\frac{z^2}{2\sigma^2} - K_1|z|\right)}{\sigma\sqrt{2\pi}} e^{-\eta(x - \varepsilon z, a)} \, dz \, da \\ &= \int_0^1 \int_{\mathbb{R}} \Phi(a) e^{\int_0^a \beta} e^{-\eta(x - \varepsilon z, a)} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\left[-\frac{z^2}{2\sigma^2} - K_1z\right] \mathbb{1}_{z \geq 0} + \left[-\frac{z^2}{2\sigma^2} + K_1z\right] \mathbb{1}_{z < 0}\right) \, dz \, da. \end{aligned}$$

Let us bound the quantity inside the exponential in the previous equation.

$$\begin{aligned} &\left[-\frac{z^2}{2\sigma^2} - K_1z\right] \mathbb{1}_{z \geq 0} + \left[-\frac{z^2}{2\sigma^2} + K_1z\right] \mathbb{1}_{z < 0} \\ &= -\frac{1}{2\sigma^2} [(z + 2\sigma^2 K_1)^2 \mathbb{1}_{z \geq 0} + (z - 2\sigma^2 K_1)^2 \mathbb{1}_{z < 0}] + 2\sigma^2 K_1^2 \\ &\geq -\frac{1}{2\sigma^2} [(z + 2\sigma^2 K_1)^2 + (z - 2\sigma^2 K_1)^2] + 2\sigma^2 K_1^2 \\ &\geq -\frac{2z^2}{2\sigma^2} - 2\sigma^2 K_1^2. \end{aligned}$$

The previous computations and the boundedness of η (Hypothesis 8.2) give us the existence of some $C_\eta \in \mathbb{R}$ satisfying:

$$\begin{aligned} e^{-\psi_\varepsilon(t,x)/\varepsilon} &\geq e^{-\frac{K_0}{\varepsilon}} e^{-\frac{K_1|x|}{\varepsilon}} e^{-2\sigma^2 K_1^2} \frac{\Phi\left(1 + \frac{t}{\varepsilon}\right)}{\Phi(0)} \int_0^1 \int_{\mathbb{R}} \Phi(a) e^{\int_0^a \beta \omega(\sqrt{2}z)} e^{-\eta(x-\varepsilon z, a)} dz da \\ &\geq \exp\left(-\frac{K_0}{\varepsilon}\right) \exp\left(-\frac{K_1|x|}{\varepsilon}\right) \exp\left(-2\sigma^2 K_1^2\right) \frac{\Phi\left(1 + \frac{t}{\varepsilon}\right)}{\Phi(0)} \exp\left(-\frac{C_\eta}{\varepsilon}\right). \end{aligned}$$

Taking the logarithm of the above expression yields the desired upper bound of equation (2.33):

$$\psi_\varepsilon(t, x) \leq K_0 + C_\eta + K_1|x| + \varepsilon(1 + \mu) \ln\left(1 + \frac{t}{\varepsilon}\right) + \varepsilon 2\sigma^2 K_1^2. \quad (2.36)$$

□

2.2.2 Lipschitz continuity in x of ψ_ε

This subsection deals with the proof of Theorem 47.2, which we recall as Proposition 49.

Proposition 49. *Let $T > 0$ and $\varepsilon > 0$. Under hypotheses 7 and 8, ψ_ε is Lipschitz continuous in x over \mathbb{R} uniformly in $t \in [0, T]$ with a Lipschitz constant less than or equal to that of ϕ_ε^0 in x :*

$$\text{Lip}(\psi_\varepsilon(t, \cdot)) \leq C_L. \quad (2.37)$$

The keystone of our proof is an application of the maximum principle to the increase rate of ψ_ε . Let us set useful notations. Let $h \in (0, 1)$. We name the following differences:

$$\left\{ \begin{array}{l} Z(t, x, a, z) = \psi_\varepsilon(t, x) - \psi_\varepsilon(t - \varepsilon a, x - \varepsilon z) \\ \quad Z_h(t, x, a, z) = Z(t, x + h, a, z) \\ Y(t, x, a, z) = \psi_\varepsilon(t, x) - \phi_\varepsilon^0(x - \varepsilon z, a - t/\varepsilon) \\ \quad Y_h(t, x, a, z) = Y(t, x + h, a, z). \end{array} \right. \quad (2.38)$$

We define the increase rates

$$\left\{ \begin{array}{l} u_h(t, x) = \frac{1}{h} (\psi_\varepsilon(t, x + h) - \psi_\varepsilon(t, x)), \\ w_h^0(x, a) = \frac{1}{h} (\phi_\varepsilon^0(x + h, a) - \phi_\varepsilon^0(x, a)). \end{array} \right. \quad (2.39)$$

Let us illustrate the application of the maximum principle in a situation with no added technical complications. In order to do so, we will suppose the increase rate u_h reaches its extrema.

Proposition 50 (Lipschitz bounds for reached extrema). *Assume u_h reaches its extrema over $[0, T] \times \mathbb{R}$. Then u_h is bounded as follows:*

$$\|u_h\|_{L^\infty([0, T] \times \mathbb{R})} \leq C_L, \quad (2.40)$$

where C_L is the Lipschitz constant in x of ϕ_ε^0 uniformly in $a \in [0, 1)$ defined in Hypothesis 8.4.

Proof. Let us begin by proving u_h is bounded by specific increase rates of the initial condition:

$$\inf_{\mathbb{R} \times [0,1]} w_h^0 \leq u_h \leq \sup_{\mathbb{R} \times [0,1]} w_h^0. \quad (2.41)$$

By subtracting equation (2.10) taken at (t, x) from the same equation taken at $(t, x + h)$ we recover:

$$\begin{aligned} 0 &= \int_0^{\frac{t}{\varepsilon}} \int_{\mathbb{R}} \Phi(a) \omega(z) [\exp(Z_h(t, x, a, z)/\varepsilon) - \exp(Z(t, x, a, z)/\varepsilon)] dz da \\ &\quad + \int_0^1 \int_{\mathbb{R}} \Phi\left(a + \frac{t}{\varepsilon}\right) \omega(z) e^{\int_0^a \beta} [\exp(Y_h(t, x, a, z)/\varepsilon) - \exp(Y(t, x, a, z)/\varepsilon)] dz da. \end{aligned}$$

We will abuse the notation and write Z instead of $Z(t, x, a, z)$ hereafter. By factoring, we recover:

$$\begin{aligned} 0 &= \int_0^{\frac{t}{\varepsilon}} \int_{\mathbb{R}} \Phi(a) \omega(z) \exp\left(\frac{Z}{\varepsilon}\right) \left[\exp\left(\frac{h}{\varepsilon} [u_h(t, x) - u_h(t - \varepsilon a, x - \varepsilon z)]\right) - 1 \right] dz da \\ &+ \int_0^1 \int_{\mathbb{R}} \Phi\left(a + \frac{t}{\varepsilon}\right) \omega(z) e^{\int_0^a \beta} \exp\left(\frac{Y}{\varepsilon}\right) \left[\exp\left(\frac{h}{\varepsilon} [u_h(t, x) - w_h^0(t - \varepsilon a, x - \varepsilon z)]\right) - 1 \right] dz da. \end{aligned} \quad (2.42)$$

We have supposed u_h reaches its extrema over $[0, T] \times \mathbb{R}$. Let us prove the upper bound of the Proposition by contradiction.

Assume u_h reaches its maximum at (t_0, x_0) and suppose that the upper bound does not hold:

$$u_h(t_0, x_0) > \sup_{\mathbb{R} \times [0,1]} w_h^0(x, a).$$

Let us take the previous equation at (t_0, x_0) and apply the maximum principle. We obtain

$$\begin{aligned} 0 &= \int_0^{\frac{t_0}{\varepsilon}} \int_{\mathbb{R}} \Phi(a) \omega(z) \exp\left(\frac{Z}{\varepsilon}\right) \underbrace{\left[\exp\left(\frac{h}{\varepsilon} [u_h(t_0, x_0) - u_h(t_0 - \varepsilon a, x_0 - \varepsilon z)]\right) - 1 \right]}_{\geq 0} dz da \\ &+ \int_0^1 \int_{\mathbb{R}} \Phi\left(a + \frac{t_0}{\varepsilon}\right) \omega(z) e^{\int_0^a \beta} \exp\left(\frac{Y}{\varepsilon}\right) \underbrace{\left[\exp\left(\frac{h}{\varepsilon} [u_h(t_0, x_0) - w_h^0(t_0 - \varepsilon a, x_0 - \varepsilon z)]\right) - 1 \right]}_{> 0} dz da. \end{aligned} \quad (2.43)$$

Since over $(a, z) \in (0, 1) \times \mathbb{R}$,

$$\Phi\left(a + \frac{t_0}{\varepsilon}\right) \omega(z) e^{\int_0^a \beta} e^{\frac{Y}{\varepsilon}} > 0,$$

the right-hand side of equation (2.43) is positive: we have reached the desired contradiction. This proves the upper bound of equation (2.41). The lower bound in that equation is proved in an analogous way.

By Lipschitz continuity with constant C_L of ϕ_ε^0 (Hypothesis 8.4), it follows that

$$\|w_h^0\|_\infty \leq C_L,$$

hence the claimed bound (2.40). \square

However, the above expression u_h does not necessarily reach its bounds. The strategy we follow to prove the bounds of Proposition 49 consists in the following truncation of ψ_ε .

1. Let $R > 0$. We define $\phi_\varepsilon^{0,R}$, which coincides with ϕ_ε^0 over $[-R, R] \times \mathbb{R}_+$ and is such that v_ε^R is bounded over \mathbb{R} by

$$-K_0 \leq v_\varepsilon^R(x) \leq K_0 + K_1 R.$$

We prove the corresponding ψ_ε^R are bounded: they satisfy a modified, simpler version of bound (2.36).

2. We then define u_h^R : the versions of u_h corresponding to the ψ_ε^R . We add penalising terms to u_h^R and ensure that the penalised functions reach their extrema.
3. This allows us to apply the maximum principle to the penalised functions. The boundedness of u_h^R allows us to appropriately bound the space-coordinate at which the penalised modified functions reach their extrema. We conclude that the perturbed functions u_h^R are bounded by the expressions stated in Proposition 50. This means the perturbed functions ψ_ε^R are C_L -Lipschitz continuous in x uniformly over $a \in [0, 1]$.
4. Finally, we show that $(\psi_\varepsilon^R)_R$ converges to ψ_ε uniformly over any compact $[0, T] \times [-M, M]$. The uniqueness of the solution of the transport problem at fixed ε is crucial. It follows that ψ_ε satisfies the Lipschitz bounds of Proposition 49 over every compact, hence over $[0, T] \times \mathbb{R}$.

Let $R > 0$, and let us define the following modified initial conditions $\phi_\varepsilon^{0,R}$:

$$\begin{cases} \phi_\varepsilon^{0,R}(t, x) = v_\varepsilon^R(x) + \varepsilon \eta_\varepsilon(x, a) + \chi_{[0,1]}(a), & \text{where} \\ v_\varepsilon^R(x) = \min(v_\varepsilon(x), K_0 + K_1 R). \end{cases} \quad (2.44)$$

It follows that $\phi_\varepsilon^{0,R} \leq \phi_\varepsilon^0$ over $\mathbb{R} \times \mathbb{R}_+$, with equality over $[-R, R] \times \mathbb{R}_+$. Moreover, v_ε^R is bounded. The modified initial condition $\phi_\varepsilon^{0,R}$ also satisfies Hypothesis 8.

Let ψ_ε^R be the solution of equation (2.10) corresponding to the initial condition $\phi_\varepsilon^{0,R}$. Let us prove ψ_ε^R is bounded by adapting the proof of Proposition 48.

Lemma 51 (L^∞ bounds of the perturbed function). *With the definitions above, ψ_ε^R satisfies, for any $(t, x) \in [0, T] \times \mathbb{R}$:*

$$\begin{cases} \psi_\varepsilon^R(t, x) \geq \mathfrak{c}^R := -K_0 - \varepsilon \ln \beta(0) - \varepsilon \ln \left\| \exp \left(- \inf_x \eta(x, a) \right) \right\|_{L^1} \\ \psi_\varepsilon^R(t, x) \leq \mathfrak{c}_T^R := K_0 + K_1 R + C_\eta + \varepsilon (1 + \mu) \ln \left(1 + \frac{t}{\varepsilon} \right). \end{cases} \quad (2.45)$$

Proof. The lower bound follows from that of Proposition 48, since Hypothesis 8 is satisfied by the perturbed initial condition $\phi_\varepsilon^{0,R}$.

As for the upper bound, it follows from a modification of the proof of the upper bound of Proposition 48. Specifically, equation (2.35) becomes:

$$e^{-\psi_\varepsilon^R(t,x)/\varepsilon} \geq \exp\left(-\frac{1}{\varepsilon}[K_0 + K_1 R]\right) \frac{\Phi(1+t/\varepsilon)}{\Phi(0)} \exp\left(-\frac{C_\eta}{\varepsilon}\right). \quad (2.46)$$

Taking the logarithm concludes the proof of the modified upper bound. \square

Let us give the perturbed versions of equation (2.38)

$$\begin{cases} Z^R(t, x, a, z) = \psi_\varepsilon^R(t, x) - \psi_\varepsilon^R(t - \varepsilon a, x - \varepsilon z) \\ Z_h^R(t, x, a, z) = Z^R(t, x + h, a, z) \\ Y^R(t, x, a, z) = \psi_\varepsilon^R(t, x) - \phi_\varepsilon^{0,R}(x - \varepsilon z, a - t/\varepsilon) \\ Y_h^R(t, x, a, z) = Y^R(t, x + h, a, z), \end{cases} \quad (2.47)$$

and of the increase rates (2.39)

$$\begin{cases} u_h^R(t, x) = \frac{1}{h} (\psi_\varepsilon^R(t, x + h) - \psi_\varepsilon^R(t, x)), \\ w_h^{0,R}(t, x) = \frac{1}{h} (\phi_\varepsilon^{0,R}(x + h, a) - \phi_\varepsilon^{0,R}(x, a)). \end{cases} \quad (2.48)$$

Since $\psi_\varepsilon^R \in L^\infty$, it follows from Lemma 51 that its increase rate is bounded as follows over $(t, x) \in [0, T] \times \mathbb{R}$:

$$|u_h^R(t, x)| \leq \frac{1}{h} [\mathfrak{C}_T^R - \mathfrak{c}^R], \quad (2.49)$$

where \mathfrak{C}_T^R and \mathfrak{c}^R are the upper and lower bounds of ψ_ε^R over $[0, T] \times \mathbb{R}$ defined in equation (2.45). However, the increase rate u_h^R may not reach its bounds.

Let us first prove that $u_h^R \leq C_L$ over $[0, T] \times \mathbb{R}$ by applying the maximum principle. In order to do so, we define a penalised function that, as we prove in Lemma 52, reaches its upper bound. Let $\delta_1, \delta_2 \in (0, 1)$. We define the penalised increase rate

$$\tilde{u}_h^R(t, x) = u_h^R(t, x) - \delta_1 t - \delta_2 x^2. \quad (2.50)$$

Lemma 52. *The penalised function $\tilde{u}_h^R(t, x)$ reaches its maximum over $[0, T] \times \mathbb{R}$ at some point (t_0, x_0) satisfying, for some positive constant C :*

$$|x_0| \leq R_2(\delta_2) := \delta_2^{-1/2} h^{-1/2} \sqrt{2[\mathfrak{C}_T^R - \mathfrak{c}^R]}. \quad (2.51)$$

Proof. The function \tilde{u}_h^R is continuous, bounded above over $[0, T] \times \mathbb{R}$, and tends to $-\infty$ as $|x|$ tends to ∞ , uniformly over $t \in [0, T]$. It follows that \tilde{u}_h^R reaches its maximum over $[0, T] \times \mathbb{R}$ at some point (t_0, x_0) . By definition,

$$u_h^R(t_0, x_0) - \delta_1 t_0 - \delta_2 x_0^2 = \tilde{u}_h^R(t_0, x_0) \geq \tilde{u}_h^R(0, 0) = u_h^R(0, 0).$$

Therefore,

$$\begin{aligned} \delta_2 x^2 &\leq u_h^R(t_0, x_0) - u_h^R(0, 0) - \delta_1 t_0 \\ &\leq \frac{1}{h} [\psi_\varepsilon^R(t_0, x_0 + h) - \psi_\varepsilon^R(t_0, x_0) - \psi_\varepsilon^R(0, h) + \psi_\varepsilon^R(0, 0)] - \delta_1 t_0 \\ &\leq \frac{2}{h} [\mathfrak{C}_T^R - \mathfrak{c}^R], \end{aligned}$$

hence the upper bound on $|x_0|$. \square

Remark. It follows that $\delta_2|x_0(\delta_1, \delta_2)| \xrightarrow{\delta_2 \rightarrow 0} 0$ uniformly over $\delta_1 \in [0, 1]$. This allows us to control the penalising term in x as $\delta_2 \rightarrow 0$. That is the reason why we have taken a bounded approximation of the initial conditions.

Let $\delta_1, \delta_2 \in (0, 1)$, and let $(t_0, x_0) \in [0, T] \times \mathbb{R}$ be a point at which \tilde{u}_h^R reaches its maximum over $[0, T] \times \mathbb{R}$.

Proposition 53 (Upper bound for the perturbed penalised increase rate).

There exists a positive constant C (depending on T, h, ε, R , which are fixed in this Proposition) such that, if $\delta_2 < \min(\delta_1, C\delta_1^2)$, the perturbed penalised increase rate \tilde{u}_h^R is bounded above as follows:

$$\tilde{u}_h^R(t_0, x_0) \leq C_L. \quad (2.52)$$

The proof of the Proposition relies on Lemmas 54 and 55. It will be presented after that of the latter.

Lemma 54. *Assume (2.52) does not hold, so the perturbed penalised function satisfies instead at its maximum point:*

$$\tilde{u}_h^R(t_0, x_0) > C_L. \quad (2.53)$$

Then

$$0 > \mathfrak{A}_x(\delta_1, \delta_2, t_0, x_0) + \mathfrak{B}_x(\delta_1, \delta_2, t_0, x_0), \quad (2.54)$$

where, evaluated at $\mathfrak{P} = (\delta_1, \delta_2, t_0, x_0)$, \mathfrak{A}_x and \mathfrak{B}_x are defined as follows:

$$\mathfrak{A}_x(\mathfrak{P}) = \int_0^{\frac{t_0}{\varepsilon}} \int_{\mathbb{R}} \Phi(a) \omega(z) \exp\left(\frac{Z^R}{\varepsilon}\right) [\exp(\delta_1 h a + \delta_2 h(2x_0 - \varepsilon z)) - 1] dz da, \quad (2.55)$$

$$\mathfrak{B}_x(\mathfrak{P}) = \int_0^1 \int_{\mathbb{R}} \Phi\left(a + \frac{t_0}{\varepsilon}\right) \omega(z) e^{\int_0^a \beta} \exp\left(\frac{Y^R}{\varepsilon}\right) \left[\exp\left(\frac{h}{\varepsilon} [\delta_1 t_0 + \delta_2 x_0^2]\right) - 1\right] dz da. \quad (2.56)$$

Proof. Recall equation (2.42) in the proof of Proposition 50. Its perturbed version, which we take at (t_0, x_0) , holds as well:

$$\begin{aligned} 0 &= \int_0^{\frac{t_0}{\varepsilon}} \int_{\mathbb{R}} \Phi(a) \omega(z) e^{\frac{Z^R}{\varepsilon}} \left[\exp\left(\frac{h}{\varepsilon} [u_h^R(t_0, x_0) - u_h^R(t_0 - \varepsilon a, x_0 - \varepsilon z)]\right) - 1 \right] dz da \\ &+ \int_0^1 \int_{\mathbb{R}} \Phi\left(a + \frac{t_0}{\varepsilon}\right) \omega(z) e^{\int_0^a \beta} e^{\frac{Y^R}{\varepsilon}} \left[\exp\left(\frac{h}{\varepsilon} [u_h^R(t_0, x_0) - w_h^{0,R}(t_0 - \varepsilon a, x_0 - \varepsilon z)]\right) - 1 \right] dz da. \end{aligned} \quad (2.57)$$

Let us inject into the previous equation, for $(t, x) \in \{(t_0, x_0), (t_0 - \varepsilon a, x_0 - \varepsilon z)\}$:

$$u_h^R(t, x) = \tilde{u}_h^R(t, x) + \delta_1 t + \delta_2 x^2.$$

We also recall that for any $(z, \varepsilon, a) \in \mathbb{R} \times (0, 1) \times [0, t_0/\varepsilon]$, we have:

$$\begin{aligned} \tilde{u}_h^R(t_0, x_0) &\geq \tilde{u}_h^R(t_0 - \varepsilon a, x_0 - \varepsilon z), \\ \tilde{u}_h^R(t_0, x_0) &> w_h^{0,R}(t_0 - \varepsilon a, x_0 - \varepsilon z). \end{aligned}$$

The expressions above allow us to recover the following inequality:

$$\begin{aligned}
 0 &> \int_0^{\frac{t_0}{\varepsilon}} \int_{\mathbb{R}} \Phi(a) \omega(z) \exp\left(\frac{Z^R}{\varepsilon}\right) \left[\exp\left(\frac{h}{\varepsilon} [\delta_1 t_0 + \delta_2 x_0^2 - \delta_1(t_0 - \varepsilon a) - \delta_2 |x_0 - \varepsilon z|^2]\right) - 1 \right] dz da \\
 &\quad + \int_0^1 \int_{\mathbb{R}} \Phi\left(a + \frac{t_0}{\varepsilon}\right) \omega(z) e^{\int_0^a \beta} \exp\left(\frac{Y^R}{\varepsilon}\right) \left[\exp\left(\frac{h}{\varepsilon} [\delta_1 t_0 + \delta_2 x_0^2]\right) - 1 \right] dz da.
 \end{aligned} \tag{2.58}$$

It follows that

$$0 > \mathfrak{A}_x(\delta_1, \delta_2, t_0, x_0) + \mathfrak{B}_x(\delta_1, \delta_2, t_0, x_0).$$

□

Let us now prove that $(\mathfrak{A}_x + \mathfrak{B}_x)(\delta_1, \delta_2, t_0, x_0) \geq 0$ for δ_1, δ_2 small enough in a sense that we will quantify in the Lemma below. This will allow us to reach a desired contradiction, which will hold as we pass to the limit when δ_1 and δ_2 tend to 0 in a way that we will specify. We recall that $(t_0, x_0) \in [0, T] \times [-R_2(\delta_2), R_2(\delta_2)]$ depends on (δ_1, δ_2) (where $R_2(\delta_2)$ is the upper bound of $|x_0|$ proved in Lemma 52). Since that relationship is not explicit, the result we prove takes the following form.

Lemma 55. *For any $\delta_1 \in (0, 1)$ there exists $\bar{\delta}_2 \in (0, \delta_1)$ such that for any $\delta_2 \in (0, \bar{\delta}_2)$,*

$$(\mathfrak{A}_x + \mathfrak{B}_x)(\delta_1, \delta_2, t_0, x_0) \geq 0.$$

Proof. Let us begin by providing rough yet sufficiently accurate lower bounds of \mathfrak{A}_x and \mathfrak{B}_x defined in equations (2.55) and (2.56).

Since $|e^X - 1| \geq X$, we have:

$$\begin{aligned}
 \mathfrak{A}_x(\mathfrak{P}) &\geq \int_0^{\frac{t_0}{\varepsilon}} \int_{\mathbb{R}} \Phi(a) \omega(z) \exp\left(\frac{Z^R}{\varepsilon}\right) [\delta_1 h a + \delta_2 h (2x_0 - \varepsilon z) z] dz da \\
 &\geq - \int_0^{\frac{t_0}{\varepsilon}} \int_{\mathbb{R}} \Phi(a) \omega(z) \exp\left(\frac{Z^R}{\varepsilon}\right) \left[\delta_2 h \varepsilon z^2 + 2\sqrt{\delta_2} \sqrt{h} \sqrt{2(\mathfrak{C}_T^R - \mathfrak{C}^R)} |z| \right] dz da
 \end{aligned}$$

thanks to the bound on $|x_0|$, equation (2.51). The difference Z_ε^R is also bounded over $[0, T] \times \mathbb{R}$:

$$|Z_\varepsilon^R| \leq \mathfrak{C}_T^R - \mathfrak{C}^R.$$

It follows that:

$$\mathfrak{A}_x(\mathfrak{P}) \geq -\sqrt{\delta_2} \int_0^{\frac{t_0}{\varepsilon}} \Phi(a) da \exp\left(\frac{1}{\varepsilon} [\mathfrak{C}_T^R - \mathfrak{C}^R]\right) \int_{\mathbb{R}} \omega(z) \left[\sqrt{\delta_2} h \varepsilon z^2 + 2\sqrt{2h(\mathfrak{C}_T^R - \mathfrak{C}^R)} |z| \right] dz. \tag{2.59}$$

Since the first and second moments of the Gaussian ω are bounded and $t_0 \in [0, T]$, the following expression is bounded uniformly in δ_2 :

$$\int_{\mathbb{R}} \omega(z) \left[\sqrt{\delta_2} h \varepsilon z^2 + 2\sqrt{2h(\mathfrak{C}_T^R - \mathfrak{C}^R)} |z| \right] dz.$$

Since Φ is bounded, it follows that for some constant C_A that only depends on T, ε, h, R (which are fixed throughout this proof),

$$\mathfrak{A}_x \geq -C_A t_0 \sqrt{\delta_2}. \tag{2.60}$$

In order to bound \mathfrak{B}_x below, we take into account its positivity and use in a similar way as above the inequalities $|e^X - 1| \geq X$ and $Y_\varepsilon \geq c^R - K_0 - K_1 R$ (following from the definition of $\phi_\varepsilon^{0,R}$). We recover:

$$\begin{aligned} \mathfrak{B}_x(\mathfrak{P}) &= \int_0^1 \int_{\mathbb{R}} \Phi \left(a + \frac{t_0}{\varepsilon} \right) \omega(z) e^{\int_0^a \beta} \exp \left(\frac{Y^R}{\varepsilon} \right) \left[\exp \left(\frac{h}{\varepsilon} [\delta_1 t_0 + \delta_2 x_0^2] \right) - 1 \right] dz da \\ &\geq \exp \left(\frac{1}{\varepsilon} [c^R - K_0 - K_1 R] \right) \int_0^1 \int_{\mathbb{R}} \Phi \left(a + \frac{t_0}{\varepsilon} \right) \omega(z) e^{\int_0^a \beta} \left[\frac{h}{\varepsilon} (\delta_1 t_0 + \delta_2 x_0^2) \right] dz da \\ &\geq \delta_1 t_0 \frac{h}{\varepsilon} \exp \left(\frac{1}{\varepsilon} [c^R - K_0 - K_1 R] \right) \int_0^1 \Phi \left(a + \frac{t_0}{\varepsilon} \right) e^{\int_0^a \beta} da \\ &\geq \delta_1 t_0 \frac{h}{\varepsilon} \exp \left(\frac{1}{\varepsilon} [c^R - K_0 - K_1 R] \right) \int_0^1 \Phi \left(a + \frac{T}{\varepsilon} \right) e^{\int_0^a \beta} da \end{aligned}$$

We recover, here again, that for some constant C that only depends on T, ε, h, R (which are fixed throughout this proof),

$$\mathfrak{B}_x \geq C_B t_0 \delta_1. \quad (2.61)$$

Let us now distinguish two cases:

1. *Case 1: $t_0 = 0$.*

In this case, for any $\delta_1, \delta_2 \in (0, 1)$, $\mathfrak{A}_x(\delta_1, \delta_2, t_0, \cdot) = 0$ uniformly over \mathbb{R} and $\mathfrak{B}_x(\delta_1, \delta_2, t_0, x) \geq 0$ uniformly over $x \in \mathbb{R}$, with equality if and only if $x = 0$. Hence for any $\delta_2 \in (0, \delta_1)$, $(\mathfrak{A}_x + \mathfrak{B}_x)(\delta_1, \delta_2, t_0, x_0) \geq 0$.

2. *Case 2: $0 < t \leq T$.*

The bounds from equations (2.60) and (2.61) hold. Hence $(\mathfrak{A}_x + \mathfrak{B}_x)(\delta_1, \delta_2, t_0, x_0)$ is strictly positive for any $0 < \delta_2 < \delta_1$ satisfying:

$$\delta_2 < \left(\frac{C_B}{C_A} \right)^2 \delta_1^2. \quad (2.62)$$

□

Proof of Proposition 53. For $\delta_2 < \min \left(\delta_1, \left(\frac{C_B}{C_A} \right)^2 \delta_1^2 \right)$, the negation of equation (2.52) allows us to reach the contradiction $0 > \mathfrak{A}_x + \mathfrak{B}_x \geq 0$. The left inequality is the result of Lemma 54, and the second is that of Lemma 55. Therefore, equation 2.52 holds. □

Proposition 56 (Bounds for the perturbed increase rate). *The perturbed increase rate u_h^R is bounded over $(t, x) \in [0, T] \times \mathbb{R}$ as follows:*

$$|u_h^R(t, x)| \leq C_L. \quad (2.63)$$

Proof. Let $(t, x) \in [0, T] \times \mathbb{R}$. By definition of \tilde{u}_h^R and thanks to Proposition 53, we have for any δ_1, δ_2 small enough:

$$u_h^R(t, x) = \tilde{u}_h^R(t, x) + \delta_1 t + \delta_2 x^2 \leq C_L + \delta_1 t + \delta_2 x.$$

Taking the limit as δ_1 and $\delta_2 < \min\left(\delta_1, \left(\frac{C_B}{C_A}\right)^2 \delta_1^2\right)$ converge to 0 allows us to recover the upper bound $u_h^R(t, x) \leq C_L$.

The lower bound is recovered by using the following perturbed function that reaches its minimum:

$$\hat{u}_h^R(t, x) = u_h^R(t, x) + \delta_1 t + \delta_2 x^2,$$

and following an analogous proof strategy. \square

Let $M > 0$. We denote by K_M the compact set $[0, T] \times [-M, M]$.

Proposition 57 (Uniform convergence over every compact). *The perturbed functions ψ_ε^R converge to ψ_ε uniformly over K_M as R tends to infinity.*

Proof. Both the transport problem and the integral boundary condition of equation (2.7) preserve positivity. For $R' > R$, it follows that the initial condition inequality $\phi_\varepsilon^{0,R} \leq \phi_\varepsilon^{0,R'} \leq \phi_\varepsilon^0$ is propagated: for all $(t, x) \in [0, T] \times \mathbb{R}$, we have:

$$\psi_\varepsilon^R(t, x) \leq \psi_\varepsilon^{R'}(t, x) \leq \psi_\varepsilon(t, x).$$

The perturbed function ψ_ε^R is bounded below uniformly in $R \in \mathbb{N}$, and ψ_ε is bounded above over K . It follows that ψ_ε^R is bounded uniformly in R over K . By monotonicity and boundedness, ψ_ε^R converges to a function $\tilde{\psi}_\varepsilon$ as $R \rightarrow \infty$ over K_M . This convergence is uniform over K_M by compactness.

Since this holds for all $M > 0$, it follows that ψ_ε^R converges to $\tilde{\psi}_\varepsilon$ pointwise over $[0, T] \times \mathbb{R}$. Beppo Levi's monotone convergence theorem ensures that $\tilde{\psi}_\varepsilon$ is a solution of the integral formulation of the transport problem (2.10) for the initial condition ϕ_ε^0 , which is the limit of $\phi_\varepsilon^{0,R}$ as $R \rightarrow \infty$. At fixed $\varepsilon > 0$, that problem admits a unique solution: ψ_ε . It follows that $\tilde{\psi}_\varepsilon = \psi_\varepsilon$, and that ψ_ε^R converges uniformly over K_M towards ψ_ε as R tends to infinity. \square

We are now ready to prove Proposition 49.

Proof of the Proposition 49. Let us begin by bounding u_h above: for $(t, x) \in [0, T] \times \mathbb{R}$,

$$u_h(t, x) \leq C_L. \tag{2.64}$$

From Proposition 56 we gather that for any positive R the modified increase rate u_h^R satisfies, for $(t, x) \in [0, T] \times \mathbb{R}$:

$$u_h^R(t, x) \leq C_L.$$

Moreover, Proposition 57 gives us the uniform convergence of ψ_ε^R to ψ_ε over any compact $K_M = [0, T] \times [-M, M]$, hence $(u_h^R)_R$ also converges to u_h uniformly over all compact subsets of $[0, T] \times \mathbb{R}$. It follows that u_h is bounded above by C_L over K_M . This holds for any positive M , hence the upper bound (2.64).

An analogous lower bound holds for u_h : for any $(t, x) \in [0, T] \times \mathbb{R}$,

$$u_h(t, x) \geq -C_L.$$

This can be proved by using the following perturbed function that reaches its minimum:

$$\hat{u}_h^R(t, x) = u_h^R(t, x) + \delta_1 t + \delta_2 x^2,$$

and by modifying step by step the pertinent computations in the proofs of this Section.

It follows that $\|u_h\|_{L^\infty([0, T] \times \mathbb{R})} \leq C_L$. Since this holds for any positive h , it follows that ψ_ε is C_L -Lipschitz continuous, as claimed. \square

2.2.3 Lipschitz continuity in t of ψ_ε

This subsection will deal with the proof of Theorem 47.3, which we recall as Proposition 58.

Proposition 58. *Let $T > 0$ and $\varepsilon > 0$. Under Hypotheses 7 and 8, ψ_ε is Lipschitz continuous in t over $(t, x) \in [0, T] \times \mathbb{R}$, with the quantitative bound stated below:*

$$\text{Lip } \psi_\varepsilon(\cdot, x) \leq \max \left(\mu(1 + \mu), \int_{\mathbb{R}} \omega(z) \exp(C_L |z|) dz \right). \quad (2.65)$$

Remark. The proof we present relies heavily on the methods introduced in the previous Subsection. A more powerful version of the proof can be achieved by direct adaptation of that of the previous Subsection. Indeed, considering the increase rate in time instead of the derivative allows for weaker hypotheses on the waiting time distribution. However, doing so leads to merely technical complications the resolution of which is not of great interest. The proof we present is shorter and clearer.

Let $R > 0$. We will prove that ψ_ε^R defined above Lemma 51 is Lipschitz continuous in t over $[0, T] \times \mathbb{R}$ with a Lipschitz constant that does not depend on R . Then, by uniform convergence over compact sets – Proposition 57 – we will recover the same Lipschitz estimates for ψ_ε .

Let us start by showing ψ_ε^R is Lipschitz continuous in t with a suboptimal Lipschitz constant that does depend on R .

Lemma 59. *There exists a positive constant C_R depending on R such that for all $h \in [0, 1]$,*

$$|\psi_\varepsilon^R(t + h, x) - \psi_\varepsilon^R(t, x)| \leq C_R \frac{h}{\varepsilon}.$$

Proof. From equation (2.10) we gather:

$$\begin{aligned} & \exp \left(-\frac{1}{\varepsilon} \psi_\varepsilon^R(t + h, x) \right) - \exp \left(-\frac{1}{\varepsilon} \psi_\varepsilon^R(t, x) \right) = \\ & \int_0^{t/\varepsilon} \int_{\mathbb{R}} [\Phi(a + h/\varepsilon) - \Phi(a)] \omega(z) \exp \left(-\frac{1}{\varepsilon} \psi_\varepsilon^R(t - \varepsilon a, x - \varepsilon z) \right) dz da \\ & + \int_0^{h/\varepsilon} \Phi(a) \omega(z) \exp \left(-\frac{1}{\varepsilon} \psi_\varepsilon^R(t + h - \varepsilon a, x - \varepsilon z) \right) dz da \\ & + \int_{t/\varepsilon}^{1+t/\varepsilon} \int_{\mathbb{R}} [\Phi(a + h/\varepsilon) - \Phi(a)] \omega(z) \exp \left(-\frac{1}{\varepsilon} \phi_\varepsilon^{0,R}(x - \varepsilon z, a - t/\varepsilon) \right) \exp \left(\int_0^{a-t/\varepsilon} \beta \right) dz da. \end{aligned}$$

We recall that $t \in [0, T]$, ε is fixed, and $h \in [0, 1]$. Φ is bounded and non-negative. It follows that for some $C > 0$,

$$\left| \frac{\Phi(a + h/\varepsilon)}{\Phi(a)} - 1 \right| \leq C \frac{h}{\varepsilon}.$$

That allows us to bound the first and third right-hand-side terms as follows:

$$\begin{aligned} & \left| \int_0^{t/\varepsilon} \int_{\mathbb{R}} [\Phi(a + h/\varepsilon) - \Phi(a)] \omega(z) \exp \left(-\frac{1}{\varepsilon} \psi_\varepsilon^R(t - \varepsilon a, x - \varepsilon z) \right) dz da \right| \\ & \leq C \frac{h}{\varepsilon} \int_0^{t/\varepsilon} \int_{\mathbb{R}} \Phi(a) \omega(z) \exp \left(-\frac{1}{\varepsilon} \psi_\varepsilon^R(t - \varepsilon a, x - \varepsilon z) \right) dz da \end{aligned}$$

and

$$\begin{aligned} & \left| \int_{t/\varepsilon}^{1+t/\varepsilon} \int_{\mathbb{R}} [\Phi(a+h/\varepsilon) - \Phi(a)] \omega(z) \exp\left(-\frac{1}{\varepsilon} \phi_\varepsilon^{0,R}(x-\varepsilon z, a-t/\varepsilon)\right) \exp\left(\int_0^{a-t/\varepsilon} \beta\right) dz da \right| \\ & \leq C \frac{h}{\varepsilon} \int_{t/\varepsilon}^{1+t/\varepsilon} \int_{\mathbb{R}} \Phi(a) \omega(z) \exp\left(-\frac{1}{\varepsilon} \phi_\varepsilon^{0,R}(x-\varepsilon z, a-t/\varepsilon)\right) \exp\left(\int_0^{a-t/\varepsilon} \beta\right) dz da. \end{aligned}$$

Thanks to equation (2.10), the sum of the two previous upper bounds is

$$\frac{Ch}{\varepsilon} \exp\left(-\frac{1}{\varepsilon} \psi_\varepsilon^R(t, x)\right),$$

and ψ_ε^R is bounded below over $[0, T+h/\varepsilon] \times \mathbb{R}$ thanks to Proposition 48, so there exists some uniform-in- ε constant C such that their sum is less than or equal to Ch/ε . The lower boundedness of ψ_ε^R also yields a bound on the second right-hand-side term:

$$\left| \int_0^{h/\varepsilon} \int_{\mathbb{R}} \Phi(a) \omega(z) \exp\left(-\frac{1}{\varepsilon} \psi_\varepsilon^R(t+h-\varepsilon a, x-\varepsilon z)\right) dz da \right| \leq C' \frac{h}{\varepsilon}$$

with $C' = \|\Phi\|_\infty \|\omega\|_\infty \exp\left(-\frac{1}{\varepsilon} \inf_{[0, T+h/\varepsilon] \times \mathbb{R}} \psi_\varepsilon^R\right)$. It follows that:

$$\left| \exp\left(-\frac{1}{\varepsilon} \psi_\varepsilon^R(t+h, x)\right) - \exp\left(-\frac{1}{\varepsilon} \psi_\varepsilon^R(t, x)\right) \right| \leq \frac{h}{\varepsilon} [C + C'].$$

The function \ln is Lipschitz continuous over $[\exp(-\sup_{[0, T+h] \times \mathbb{R}} \psi_\varepsilon^R/\varepsilon), \exp(-\inf_{[0, T+h] \times \mathbb{R}} \psi_\varepsilon^R/\varepsilon)]$ with Lipschitz constant $\exp(\sup_{[0, T+h] \times \mathbb{R}} \psi_\varepsilon^R/\varepsilon) \leq \exp(\mathfrak{C}_T^R/\varepsilon) = L$, where \mathfrak{C}_T^R defined in equation (2.45) depends on R . Therefore,

$$|\psi_\varepsilon^R(t+h, x) - \psi_\varepsilon^R(t, x)| \leq L \left| \exp\left(-\frac{1}{\varepsilon} \psi_\varepsilon^R(t+h, x)\right) - \exp\left(-\frac{1}{\varepsilon} \psi_\varepsilon^R(t, x)\right) \right| \leq \frac{h}{\varepsilon} [C + C'] L.$$

□

The Lipschitz bound in Lemma 59 implies, thanks to Rademacher's theorem, that the function $\partial_t \psi_\varepsilon^R$ is defined almost everywhere, and bounded over $[0, T] \times \mathbb{R}$. Without loss of generality, we may prolong by 0 the function $\partial_t \psi_\varepsilon^R$ at the points where it is not defined. We do not modify the notation for this prolongation. We may now apply the maximum principle directly to that derivative. This eliminates the need of using an increase rate as we did in the previous Subsection.

Differentiating (2.10) with respect to t and multiplying by $e^{\frac{1}{\varepsilon} \psi_\varepsilon^R(t, x)} \neq 0$ yields:

$$\begin{aligned} 0 &= \Phi\left(\frac{t}{\varepsilon}\right) \left[\int_{\mathbb{R}} \omega(z) e^{-\frac{1}{\varepsilon} \psi(0, x-\varepsilon z)} dz - \int_{\mathbb{R}} \omega(z) e^{-\frac{1}{\varepsilon} \phi_\varepsilon^{0,R}(x-\varepsilon z, 0)} dz + \int_{\mathbb{R}} \omega(z) e^{-\frac{1}{\varepsilon} \phi_\varepsilon^{0,R}(x-\varepsilon z, 1)} e^{\int_0^1 \beta} dz \right] \\ &+ \int_0^{t/\varepsilon} \int_{\mathbb{R}} \Phi \omega [\partial_t \psi_\varepsilon^R(t, x) - \partial_t \psi_\varepsilon^R(t-\varepsilon a, x-\varepsilon z)] e^{-\frac{1}{\varepsilon} \psi_\varepsilon^R(t-\varepsilon a, x-\varepsilon z)} dz da \\ &+ \int_{t/\varepsilon}^{1+t/\varepsilon} \int_{\mathbb{R}} \Phi \omega \left[\partial_t \psi_\varepsilon^R(t, x) - \left[\beta\left(a-\frac{t}{\varepsilon}\right) - \frac{1}{\varepsilon} \partial_a \phi_\varepsilon^{0,R}\left(x-\varepsilon z, a-\frac{t}{\varepsilon}\right) \right] \right] e^{-\frac{1}{\varepsilon} \phi_\varepsilon^{0,R}(x-\varepsilon z, a-\frac{t}{\varepsilon})} e^{\int_0^{a-t/\varepsilon} \beta} dz da. \end{aligned} \tag{2.66}$$

where $\int_{\mathbb{R}} \omega(z) \exp\left(-\frac{1}{\varepsilon} \phi_\varepsilon^{0,R}(x-\varepsilon z, 1)\right) \exp\left(\int_0^1 \beta\right) dz = 0$ since $n^0(x-\varepsilon z, 1) = 0$, and $\partial_a \phi_\varepsilon^{0,R}$ is defined in the sense of distributions.

We may now prove precise Lipschitz bounds on ψ_ε^R .

Proposition 60. *The function ψ_ε^R is Lipschitz continuous in t over $(t, x) \in [0, T] \times \mathbb{R}$, with the quantitative bounds stated below:*

$$\partial_t \psi_\varepsilon^R(t, x) \geq - \int_{\mathbb{R}} \omega(z) \exp(C_L |z|) dz \quad (2.67)$$

$$\partial_t \psi_\varepsilon^R(t, x) \leq \mu(1 + \mu). \quad (2.68)$$

Proof of the upper bound (2.68).

Let us define the following penalised function for $\delta_1, \delta_2 \in (0, 1)$:

$$\xi_+(t, x) = \partial_t \psi_\varepsilon^R(t, x) - \delta_1 t - \delta_2 x^2.$$

Since $\partial_t \psi_\varepsilon^R$ is bounded over $[0, T] \times \mathbb{R}$, the penalised function reaches its maximum over $[0, T] \times \mathbb{R}$ at some point (t_0, x_0) . Injecting the expression of ξ_+ into equation 2.66 yields:

$$\begin{aligned} 0 = & \Phi \left(\frac{t}{\varepsilon} \right) \left[\int_{\mathbb{R}} \omega(z) \exp \left(-\frac{1}{\varepsilon} \psi_\varepsilon^R(0, x - \varepsilon z) \right) dz - \int_{\mathbb{R}} \omega(z) \exp \left(-\frac{1}{\varepsilon} \phi_\varepsilon^{0,R}(x - \varepsilon z, 0) \right) dz \right] \\ & + \int_0^{\frac{t}{\varepsilon}} \int_{\mathbb{R}} \Phi \omega [\xi_+(t, x) - \xi_+(t - \varepsilon a, x - \varepsilon z) + \delta_1 [t - (t - \varepsilon a)] + \delta_2 [x^2 - (x - \varepsilon z)^2]] e^{-\frac{1}{\varepsilon} \psi_\varepsilon^R(t - \varepsilon a, x - \varepsilon z)} dz da \\ & + \int_0^1 \int_{\mathbb{R}} \Phi \left(a + \frac{t}{\varepsilon} \right) \omega(z) \left[\xi_+(t, x) + \delta_1 t + \delta_2 x^2 - \left[\beta(a) - \frac{1}{\varepsilon} \partial_a \phi_\varepsilon^{0,R}(x - \varepsilon z, a) \right] \right] \\ & \exp \left(-\frac{1}{\varepsilon} \phi_\varepsilon^{0,R}(x - \varepsilon z, a) \right) \exp \left(\int_0^a \beta \right) dz da. \end{aligned} \quad (2.69)$$

At (t_0, x_0) , the previous expression becomes:

$$\begin{aligned} 0 \geq & \Phi \left(\frac{t_0}{\varepsilon} \right) \left[\int_{\mathbb{R}} \omega(z) \exp \left(-\frac{1}{\varepsilon} \psi_\varepsilon^R(0, x_0 - \varepsilon z) \right) dz - \int_{\mathbb{R}} \omega(z) \exp \left(-\frac{1}{\varepsilon} \phi_\varepsilon^{0,R}(x_0 - \varepsilon z, 0) \right) dz \right] \\ & + \int_0^1 \int_{\mathbb{R}} \Phi \left(a + \frac{t_0}{\varepsilon} \right) \omega(z) \left[\xi_+(t_0, x_0) - \left[\beta(a) - \frac{1}{\varepsilon} \partial_a \phi_\varepsilon^{0,R}(x_0 - \varepsilon z, a) \right] \right] \\ & \exp \left(-\frac{1}{\varepsilon} \phi_\varepsilon^{0,R}(x_0 - \varepsilon z, a) \right) \exp \left(\int_0^a \beta \right) dz da \\ & + \int_0^{\frac{t_0}{\varepsilon}} \int_{\mathbb{R}} \Phi \omega [\delta_1 \varepsilon a + \delta_2 \varepsilon z (2x_0 - \varepsilon z)] e^{-\frac{1}{\varepsilon} \psi_\varepsilon^R(t - \varepsilon a, x - \varepsilon z)} dz da \\ & + \int_0^1 \int_{\mathbb{R}} \Phi \left(a + \frac{t}{\varepsilon} \right) \omega(z) [\delta_1 t + \delta_2 x^2] \exp \left(-\frac{1}{\varepsilon} \phi_\varepsilon^{0,R}(x - \varepsilon z, a) \right) \exp \left(\int_0^a \beta \right) dz da. \end{aligned} \quad (2.70)$$

It follows that:

$$\begin{aligned} \xi_+(t_0, x_0) \leq & \frac{\int_0^1 \int_{\mathbb{R}} \Phi \left(a + \frac{t_0}{\varepsilon} \right) \omega(z) \left[\beta(a) - \partial_a \phi_\varepsilon^{0,R}(x_0 - \varepsilon z, a) / \varepsilon \right] \exp \left(\int_0^a \beta(s) ds - \frac{1}{\varepsilon} \phi_\varepsilon^{0,R}(x_0 - \varepsilon z, a) \right) dz da}{\int_0^1 \int_{\mathbb{R}} \Phi \left(a + \frac{t_0}{\varepsilon} \right) \omega(z) \exp \left(\int_0^a \beta \right) \exp \left(-\frac{1}{\varepsilon} \phi_\varepsilon^{0,R}(x_0 - \varepsilon z, a) \right) dz da} \\ & + \frac{\Phi \left(\frac{t_0}{\varepsilon} \right) \int_{\mathbb{R}} \omega(z) \left[\exp \left(-\frac{1}{\varepsilon} \phi_\varepsilon^{0,R}(x_0 - \varepsilon z, 0) \right) - \exp \left(-\frac{1}{\varepsilon} \psi_\varepsilon^R(0, x_0 - \varepsilon z) \right) \right] dz}{\int_0^1 \int_{\mathbb{R}} \Phi \left(a + t_0 / \varepsilon \right) \omega(z) \exp \left(\int_0^a \beta \right) \exp \left(-\frac{1}{\varepsilon} \phi_\varepsilon^{0,R}(x_0 - \varepsilon z, a) / \varepsilon \right) dz da} + \mathfrak{A}_t^0 + \mathfrak{B}_t^0, \end{aligned} \quad (2.71)$$

where:

$$\mathfrak{A}_t^0 = \frac{\int_0^{\frac{t_0}{\varepsilon}} \int_{\mathbb{R}} \Phi \omega [\delta_1 \varepsilon a + \delta_2 \varepsilon z (2x_0 - \varepsilon z)] e^{-\frac{1}{\varepsilon} \psi_\varepsilon^R(t_0 - \varepsilon a, x_0 - \varepsilon z)} dz da}{\int_0^1 \int_{\mathbb{R}} \Phi(a + t_0/\varepsilon) \omega(z) \exp(\int_0^a \beta) \exp\left(-\phi_\varepsilon^{0,R}(x_0 - \varepsilon z, a)/\varepsilon\right) dz da} \quad (2.72)$$

$$\mathfrak{B}_t^0 = \frac{\int_0^1 \int_{\mathbb{R}} \Phi(a + \frac{t_0}{\varepsilon}) \omega(z) [\delta_1 t_0 + \delta_2 x_0^2] \exp\left(-\frac{1}{\varepsilon} \phi_\varepsilon^{0,R}(x_0 - \varepsilon z, a)\right) \exp(\int_0^a \beta) dz da}{\int_0^1 \int_{\mathbb{R}} \Phi(a + t_0/\varepsilon) \omega(z) \exp(\int_0^a \beta) \exp\left(-\phi_\varepsilon^{0,R}(x_0 - \varepsilon z, a)/\varepsilon\right) dz da}. \quad (2.73)$$

Integrating the numerator of the first term by parts yields:

$$\partial_t \psi_\varepsilon^R(t_0, x_0) \leq \mathfrak{A}_t^+ + \mathfrak{B}_t^+ + \mathfrak{A}_t^0 + \mathfrak{B}_t^0, \quad (2.74)$$

with:

$$\mathfrak{A}_t^+ = -\frac{\Phi\left(\frac{t_0}{\varepsilon}\right) \int_{\mathbb{R}} \omega(z) \exp\left(-\frac{1}{\varepsilon} \psi_\varepsilon^R(0, x_0 - \varepsilon z)\right) dz}{\int_0^1 \int_{\mathbb{R}} \Phi\left(a + \frac{t_0}{\varepsilon}\right) \omega(z) \exp(\int_0^a \beta) \exp\left(-\frac{1}{\varepsilon} \phi_\varepsilon^{0,R}(x_0 - \varepsilon z, a)\right) dz da}$$

and

$$\mathfrak{B}_t^+ = -\frac{\int_0^1 \int_{\mathbb{R}} \frac{\Phi'(a + \frac{t_0}{\varepsilon})}{\Phi(a + \frac{t_0}{\varepsilon})} \Phi\left(a + \frac{t_0}{\varepsilon}\right) \omega(z) \exp(\int_0^a \beta) \exp\left(-\frac{1}{\varepsilon} \phi_\varepsilon^{0,R}(x_0 - \varepsilon z, a)\right) dz da}{\int_0^1 \int_{\mathbb{R}} \Phi\left(a + \frac{t_0}{\varepsilon}\right) \omega(z) \exp(\int_0^a \beta) \exp\left(-\frac{1}{\varepsilon} \phi_\varepsilon^{0,R}(x_0 - \varepsilon z, a)\right) dz da}.$$

The expression \mathfrak{A}_t^+ is non-positive, and $-\frac{\Phi'(a + \frac{t_0}{\varepsilon})}{\Phi(a + \frac{t_0}{\varepsilon})} = \frac{\mu(1+\mu)}{1+a+\frac{t_0}{\varepsilon}}$, so $\mathfrak{B}_t^+ \leq \mu(1+\mu)$. Those bounds are uniform in $\delta_1, \delta_2 \in (0, 1)$. As for \mathfrak{A}_t^0 and \mathfrak{B}_t^0 , their denominators are bounded uniformly in δ_1, δ_2 and we will prove their numerators tend to 0 as δ_1 and δ_2 tend to 0. Indeed, by definition of (t_0, x_0) , we have $\xi_+(0, 0) \leq \xi_+(t_0, x_0)$. Hence, the uniform-in- x Lipschitz bound of Lemma 59 allows us to bind $|x_0|$ as follows :

$$\begin{aligned} \partial_t \psi_\varepsilon^R(0, 0) &\leq \partial_t \psi_\varepsilon^R(t_0, x_0) - \delta_1 t_0 - \delta_2 x_0^2, \\ \delta_2 x_0^2 &\leq \partial_t \psi_\varepsilon^R(t_0, x_0) - \partial_t \psi_\varepsilon^R(0, 0) \leq 2 \frac{C_R h}{\varepsilon}, \\ |x_0| &\leq (\delta_2)^{-1/2} \left(2 \frac{C_R h}{\varepsilon}\right)^{1/2}. \end{aligned}$$

Hence $\delta_2 x_0^2 \xrightarrow{\delta_2 \rightarrow 0} 0$. By boundedness of the factors under the numerator integrals, it follows that \mathfrak{A}_t^0 and \mathfrak{B}_t^0 converge to 0 as δ_1 and δ_2 tend to 0. Taking that limit concludes the proof of the upper bound (2.68). \square

Proof of the lower bound (2.67).

The same arguments as in the proof of the upper bound allow us to define the penalised function

$$\xi_-(t, x) = \partial_t \psi_\varepsilon^R(t, x) + \delta_1 t + \delta_2 x^2$$

which reaches its minimum over $[0, T] \times \mathbb{R}$ at a point we name (t_1, x_1) . Computations analogous to those above lead to the following inequality (after the same integration by parts):

$$\partial_t \psi_\varepsilon^R(t_1, x_1) \geq \mathfrak{A}_t^- + \mathfrak{B}_t^- - \mathfrak{A}_t^0 - \mathfrak{B}_t^0, \quad (2.75)$$

with:

$$\mathfrak{A}_t^- = - \frac{\Phi\left(\frac{t_1}{\varepsilon}\right) \int_{\mathbb{R}} \omega(z) \exp\left(-\frac{1}{\varepsilon} \psi_{\varepsilon}^R(0, x_1 - \varepsilon z)\right) dz}{\int_0^1 \int_{\mathbb{R}} \Phi\left(a + \frac{t_1}{\varepsilon}\right) \omega(z) \exp\left(\int_0^a \beta\right) \exp\left(-\frac{1}{\varepsilon} \phi_{\varepsilon}^{0,R}(x_1 - \varepsilon z, a)\right) dz da}$$

and

$$\mathfrak{B}_t^- = - \frac{\int_0^1 \int_{\mathbb{R}} \frac{\Phi'(a + \frac{t_1}{\varepsilon})}{\Phi(a + \frac{t_1}{\varepsilon})} \Phi\left(a + \frac{t_1}{\varepsilon}\right) \omega(z) e^{\int_0^a \beta} e^{-\frac{1}{\varepsilon} \phi_{\varepsilon}^{0,R}(x_1 - \varepsilon z, a)} dz da}{\int_0^1 \int_{\mathbb{R}} \Phi\left(a + \frac{t_1}{\varepsilon}\right) \omega(z) \exp\left(\int_0^a \beta\right) \exp\left(-\frac{1}{\varepsilon} \phi_{\varepsilon}^{0,R}(x_1 - \varepsilon z, a)\right) dz da}.$$

The expressions \mathfrak{A}_t^0 and \mathfrak{B}_t^0 defined in the proof of the upper bound are taken here at (t_1, x_1) . The waiting time distribution Φ is a decreasing function, so $\mathfrak{B}_t^- \geq 0$. In order to bound \mathfrak{A}_t^- , we may notice:

$$\Phi(a) \Phi\left(\frac{t}{\varepsilon}\right) = \Phi\left(a + \frac{t}{\varepsilon}\right) \mu\left(\frac{1 + a + \frac{t}{\varepsilon}}{1 + a + \frac{t}{\varepsilon} + a \frac{t}{\varepsilon}}\right)^{1+\mu} \leq \Phi\left(a + \frac{t}{\varepsilon}\right).$$

Since at $t = 0$ we have

$$\exp\left(-\frac{1}{\varepsilon} \psi_{\varepsilon}^R(0, x_1 - \varepsilon z)\right) = \int_0^1 \int_{\mathbb{R}} \Phi(a) \omega(y) \exp\left(-\frac{1}{\varepsilon} \phi_{\varepsilon}^{0,R}(x_1 - \varepsilon z - \varepsilon y, a)\right) \exp\left(\int_0^a \beta\right) dy da,$$

we recover:

$$\mathfrak{A}_t^- \geq - \frac{\int_0^1 \int_{\mathbb{R}} \Phi\left(a + \frac{t_1}{\varepsilon}\right) \omega(z) e^{\int_0^a \beta} e^{-\frac{1}{\varepsilon} \phi_{\varepsilon}^{0,R}(x_1 - \varepsilon z, a)} \left(\int_{\mathbb{R}} \omega(y) e^{\frac{1}{\varepsilon} [\phi_{\varepsilon}^{0,R}(x_1 - \varepsilon z, a) - \phi_{\varepsilon}^{0,R}(x_1 - \varepsilon z - \varepsilon y, a)]} dy\right) dz da}{\int_0^1 \int_{\mathbb{R}} \Phi\left(a + \frac{t_1}{\varepsilon}\right) \omega(z) e^{\int_0^a \beta} e^{-\frac{1}{\varepsilon} \phi_{\varepsilon}^{0,R}(x_1 - \varepsilon z, a)} dz da}.$$

Under Hypothesis 8.4, ϕ_{ε}^0 is Lipschitz continuous in x with Lipschitz constant C_L , and so are the functions $\phi_{\varepsilon}^{0,R}$, with the same Lipschitz continuity constant uniformly over R . It follows that

$$\int_{\mathbb{R}} \omega(y) e^{\frac{1}{\varepsilon} [\phi_{\varepsilon}^{0,R}(x_1 - \varepsilon z, a) - \phi_{\varepsilon}^{0,R}(x_1 - \varepsilon z - \varepsilon y, a)]} dy \leq \int_{\mathbb{R}} \omega(y) \exp(C_L |y|) dy,$$

which is bounded above since ω is a Gaussian. As we have proved before, the expressions \mathfrak{A}_t^0 and \mathfrak{B}_t^0 converge to 0 as δ_1 and δ_2 tend to 0. Taking that limit concludes the proof of the lower bound (2.67). \square

Proof of Proposition 58. Thanks to Proposition 57, ψ_{ε}^R converges uniformly over compact sets to ψ_{ε} . Therefore, the uniform-in- R bounds from Proposition 60 are preserved as R tends to infinity. We recover the claimed Lipschitz estimate for ψ_{ε} . \square

2.3 Viscosity limit procedure

In this section, we continue to work over $[0, T] \times \mathbb{R}$.

We deduce from the Lipschitz estimates that there exists a Lipschitz function ψ_0 such that $\psi_{\varepsilon} \rightarrow \psi_0$ up to extraction. We do not specify the extracted sequence (ε_n) , for the sake of clarity. We shall prove that ψ_0 is the unique viscosity solution of the Hamilton-Jacobi equation

$$1 = \int_0^{\infty} \Phi(a) \exp(a \partial_t \psi_0(t, x)) da \int_{\mathbb{R}} \omega(z) \exp(z \partial_x \psi_0(t, x)) dz \quad (\text{HJ})$$

with initial condition $\psi_0(0, x) = v(x)$.

Equation (2.10) is equivalent to the following, which allows us to define \mathfrak{A}_ε and \mathfrak{B}_ε and is better suited for the following proofs:

$$\begin{aligned}
 1 &= (\mathfrak{A}_\varepsilon + \mathfrak{B}_\varepsilon)(\psi_\varepsilon)(t, x), \quad \text{where:} \\
 \mathfrak{A}_\varepsilon(\psi_\varepsilon)(t, x) &= \int_0^{t/\varepsilon} \int_{\mathbb{R}} \omega(z) \Phi(a) \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \psi_\varepsilon(t - \varepsilon a, x - \varepsilon z)]\right) dz da, \\
 \mathfrak{B}_\varepsilon(\psi_\varepsilon)(t, x) &= \int_0^1 \int_{\mathbb{R}} \omega(z) \Phi(a + t/\varepsilon) \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \phi_\varepsilon^0(x - \varepsilon z, a)]\right) \exp\left(\int_0^a \beta\right) dz da.
 \end{aligned} \tag{2.76}$$

2.3.1 Viscosity subsolution

Proposition 61. *Under hypotheses 7 and 8, ψ_0 is a viscosity subsolution of (HJ).*

Proof. Let $\Psi \in \mathcal{C}^2(\mathbb{R}^+ \times \mathbb{R})$ be a test function such that $\psi_0 - \Psi$ admits a maximum at (t_0, x_0) . By compactness in $W_{\text{loc}}^{1, \infty}([0, T] \times \mathbb{R})$, thanks to the a priori estimates, we obtain for a subsequence of $\varepsilon \rightarrow 0$ which we will not rename: $(t_\varepsilon, x_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} (t_0, x_0)$, where $(t_\varepsilon, x_\varepsilon)$ is a point at which $\psi_\varepsilon - \Psi$ reaches its maximum. We have then:

$$\forall \varepsilon > 0 \quad \forall (z, a) \in \mathbb{R} \times [0, \frac{t_\varepsilon}{\varepsilon}],$$

$$\psi_\varepsilon(t_\varepsilon, x_\varepsilon) - \Psi(t_\varepsilon, x_\varepsilon) \geq \psi_\varepsilon(t_\varepsilon - \varepsilon a, x_\varepsilon - \varepsilon z) - \Psi(t_\varepsilon - \varepsilon a, x_\varepsilon - \varepsilon z).$$

Since \mathfrak{B}_ε is non-negative, it follows that:

$$1 \geq \mathfrak{A}_\varepsilon(\psi_\varepsilon)(t_\varepsilon, x_\varepsilon) \geq \mathfrak{A}_\varepsilon(\Psi)(t_\varepsilon, x_\varepsilon).$$

However:

$$\begin{aligned}
 \Psi(t_\varepsilon, x_\varepsilon) - \Psi(t_\varepsilon - \varepsilon a, x_\varepsilon - \varepsilon z) &= \varepsilon a \partial_t \Psi(t_\varepsilon, x_\varepsilon) + \varepsilon z \partial_x \Psi(t_\varepsilon, x_\varepsilon) \\
 &+ \frac{1}{2} \varepsilon^2 \int_0^1 (1-s)^2 [a^2 \partial_t^2 \Psi(t_\varepsilon - \varepsilon s a, x_\varepsilon - \varepsilon s z) + 2 a z \cdot \partial_t \partial_x \Psi(t_\varepsilon - \varepsilon s a, x_\varepsilon - \varepsilon s z) \\
 &+ z^2 \partial_x^2 \Psi(t_\varepsilon - \varepsilon s a, x_\varepsilon - \varepsilon s z)] ds.
 \end{aligned} \tag{2.77}$$

Therefore we have, for all $A > 0$:

$$\begin{aligned}
 1 &\geq \int_0^A \int_{-A}^A \Phi(a) \omega(z) \exp\left\{a \partial_t \Psi(t_\varepsilon, x_\varepsilon) + z \partial_x \Psi(t_\varepsilon, x_\varepsilon) \right. \\
 &+ \frac{1}{2} \varepsilon \int_0^1 (1-s)^2 [a^2 \partial_t^2 \Psi(t_\varepsilon - \varepsilon s a, x_\varepsilon - \varepsilon s z) + 2 a z \cdot \partial_t \partial_x \Psi(t_\varepsilon - \varepsilon s a, x_\varepsilon - \varepsilon s z) \\
 &\left. + z^2 \partial_x^2 \Psi(t_\varepsilon - \varepsilon s a, x_\varepsilon - \varepsilon s z)] ds \right\} dz da.
 \end{aligned}$$

Since Ψ is \mathcal{C}^2 , the previous expression tends, for fixed A , when $\varepsilon \rightarrow 0$, to:

$$1 \geq \int_0^A \int_{-A}^A \Phi(a) \omega(z) \exp[a \partial_t \Psi(t_0, x_0) + z \partial_x \Psi(t_0, x_0)] dz da.$$

It follows that:

$$1 \geq \int_0^\infty \int_{\mathbb{R}} \Phi(a) \omega(z) \exp[a \partial_t \Psi(t_0, x_0) + z \partial_x \Psi(t_0, x_0)] dz da.$$

Therefore ψ_0 is a viscosity subsolution of (HJ). □

2.3.2 Viscosity supersolution

In order to prove that ψ_0 is a viscosity supersolution of (HJ), we need to control the \mathfrak{B}_ε term in equation (2.76), whose positivity sufficed in the previous subsection. This is tantamount to controlling the fate of the aging particles that come from the initial data and have never jumped.

We proceed in several steps. The key idea is to compare the relative weights of \mathfrak{A}_ε and \mathfrak{B}_ε , by means of the quantity $\psi_\varepsilon(t, x) - \psi_\varepsilon(0, x)$. Because the sum of the two contributions equals one, we shall deduce that $\mathfrak{A}_\varepsilon \rightarrow 1$, and $\mathfrak{B}_\varepsilon \rightarrow 0$. Interestingly enough, we get a quantitative estimate on the convergence rate.

Step 1: A crude estimate on \mathfrak{B}_ε . The following Lemma boils down the estimate on \mathfrak{B}_ε to some estimate on time increments of ψ_ε . Here, the boundedness of the age support is crucial.

Lemma 62 (Simple bounds for \mathfrak{B}_ε).

$$\frac{\Phi(t/\varepsilon)}{\Phi(0)} \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \psi_\varepsilon(0, x)]\right) \leq \mathfrak{B}_\varepsilon(\psi_\varepsilon)(t, x) \leq \frac{\Phi(1+t/\varepsilon)}{\Phi(1)} \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \psi_\varepsilon(0, x)]\right) \quad (2.78)$$

Proof. This is a consequence of the following claim: for all $h > 0$,

$$a \mapsto \Phi(a+h)/\Phi(a)$$

is an increasing function. Indeed,

$$\begin{aligned} \frac{d}{da} \frac{\Phi(a+h)}{\Phi(a)} &= \frac{\Phi'(a+h)\Phi(a) - \Phi'(a)\Phi(a+h)}{(\Phi(a))^2} \\ &= \frac{\exp\left(\int_0^a \beta\right) \exp\left(\int_0^{a+h} \beta\right)}{(\Phi(a))^2} \left(\beta'(a+h)\beta(a) - \beta'(a)\beta(a+h) + \underbrace{\beta(a)\beta(a+h)[\beta(a) - \beta(a+h)]}_{\geq 0} \right), \end{aligned}$$

which is positive since, $\beta(a) = \mu/(1+a)$ being non-increasing and convex, $\beta'(a+h) \geq \beta'(a)$ by convexity and $\beta(a) \geq \beta(a+h) \geq 0$. This proves the claim.

We now write \mathfrak{B}_ε as follows:

$$\mathfrak{B}_\varepsilon(\psi_\varepsilon)(t, x) = \int_0^1 \frac{\Phi(a+t/\varepsilon)}{\Phi(a)} \Phi(a) \int_{\mathbb{R}} \omega(z) \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \phi_\varepsilon^0(x - \varepsilon z, a)]\right) \exp\left(\int_0^a \beta\right) dz da$$

and recover the lower and upper bounds by monotonicity and thanks to (2.10). \square

Step 2: A lower bound for \mathfrak{A}_ε . The goal of the following Lemma is to remove the x variations from the contribution in \mathfrak{A}_ε . Hence, the problem will be reduced to estimate for a given x . This strongly relies on semi-concavity.

Semi-concavity is a natural regularity for Hamilton-Jacobi equations. It can result either from the propagation of regularity on the initial data, or on regularization property of the Hamilton-Jacobi equation [49, Chapter 3.3]. The latter usually relies on uniform convexity of the Hamiltonian, which is not the case here. Below, we derive propagation estimates for $\varepsilon > 0$.

Lemma 63 (Lower bound for \mathfrak{A}_ε). *For ε small enough and $(t, x) \in [0, T] \times \mathbb{R}$,*

$$\mathfrak{A}_\varepsilon(\psi)(t, x) \geq \left[1 - \varepsilon \frac{\mathfrak{C}_{xx}}{2} \int_{\mathbb{R}} \omega(z) z^2 dz\right] \int_0^{t/\varepsilon} \Phi(a) \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \psi_\varepsilon(t - \varepsilon a, x)]\right) da \quad (2.79)$$

where \mathfrak{C}_{xx} is the upper bound of $\partial_x^2 \phi_\varepsilon^0$ from Hypothesis 8.3.

Proof. Firstlt, let us prove that the semi-concavity of the initial condition is preserved. By differentiating (2.10) twice with respect to x , we obtain:

$$\begin{aligned}
 0 &= \int_0^{t/\varepsilon} \int_{\mathbb{R}} \omega(z) \Phi(a) \left[(\partial_x \psi_\varepsilon(t, x) - \partial_x \psi_\varepsilon(t - \varepsilon a, x - \varepsilon z))^2 + \right. \\
 &\quad \left. \frac{1}{\varepsilon} (\partial_x^2 \psi_\varepsilon(t, x) - \partial_x^2 \psi_\varepsilon(t - \varepsilon a, x - \varepsilon z)) \right] \exp \left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \psi_\varepsilon(t - \varepsilon a, x - \varepsilon z)] \right) dz da \\
 &\quad + \int_0^1 \int_{\mathbb{R}} \omega(z) \Phi(a + t/\varepsilon) \left[(\partial_x \psi_\varepsilon(t, x) - \partial_x \phi_\varepsilon^0(x - \varepsilon z, a))^2 + \right. \\
 &\quad \left. \frac{1}{\varepsilon} (\partial_x^2 \psi_\varepsilon(t, x) - \partial_x^2 \phi_\varepsilon^0(x - \varepsilon z, a)) \right] \exp \left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \phi_\varepsilon^0(x - \varepsilon z, a)] \right) \exp \left(\int_0^a \beta \right) dz da,
 \end{aligned} \tag{2.80}$$

Since ψ_ε and ϕ_ε^0 are Lipschitz continuous in x and thanks to Rademacher's theorem they are almost everywhere differentiable, the squared terms are well defined and non-negative. Moreover, Hypothesis 8.3 gives us $\partial_x^2 \phi_\varepsilon^0 \leq \mathfrak{C}_{xx}$ in the sense of distributions. We recover an upper bound for $\partial_x^2 \psi_\varepsilon$. Indeed, at $(t_0, x_0) = \arg \max \partial_x^2 \psi_\varepsilon$ ⁵, an application of the maximum principle allows us to recover:

$$\partial_x^2 \psi_\varepsilon(t_0, x_0) \leq \mathfrak{C}_{xx}. \tag{2.81}$$

Secondly, we deduce the following simple Taylor estimate,

$$\begin{aligned}
 \exp \left(-\frac{1}{\varepsilon} [\psi_\varepsilon(t - \varepsilon a, x - \varepsilon z) - \psi_\varepsilon(t - \varepsilon a, x)] \right) &\geq 1 - \frac{1}{\varepsilon} [\psi_\varepsilon(t - \varepsilon a, x - \varepsilon z) - \psi_\varepsilon(t - \varepsilon a, x)] \\
 &\geq 1 - \frac{1}{\varepsilon} \left[-\varepsilon z \partial_x \psi_\varepsilon(t - \varepsilon a, x) + \frac{\mathfrak{C}_{xx}}{2} \varepsilon^2 z^2 \right].
 \end{aligned}$$

Then, since $\int_{\mathbb{R}} z \omega(z) dz = 0$, we have,

$$\int_{\mathbb{R}} \omega(z) \exp \left(-\frac{1}{\varepsilon} [\psi_\varepsilon(t - \varepsilon a, x - \varepsilon z) - \psi_\varepsilon(t - \varepsilon a, x)] \right) dz \geq 1 - \varepsilon \frac{\mathfrak{C}_{xx}}{2} \int_{\mathbb{R}} \omega(z) z^2 dz. \tag{2.82}$$

The result of the Lemma follows. \square

Step 3: An upper bound on $\psi_\varepsilon(t, x) - \psi_\varepsilon(0, x)$. We are now ready to apply the maximum principle on the time increment for a fixed x .

Lemma 64 (Upper bound). *Let us fix $x \in \mathbb{R}$. Let m be the maximum over $t \in [0, T]$ of $\psi_\varepsilon(t, x) - \psi_\varepsilon(0, x)$. For $K = (\mathfrak{C}_{xx}/2) \int_{\mathbb{R}} \omega(z) z^2 dz$, we have:*

$$e^{m/\varepsilon} \leq 1 + \frac{T}{\varepsilon} + K\varepsilon \left(1 + \frac{T}{\varepsilon} \right)^{1+\mu}. \tag{2.83}$$

Proof. We deduce from the identity $\mathfrak{A}_\varepsilon + \mathfrak{B}_\varepsilon = 1$, from Lemma 63 and from the lower bound in Lemma 62, that,

$$1 \geq [1 - K\varepsilon] \int_0^{t/\varepsilon} \Phi(a) \exp \left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \psi_\varepsilon(t - \varepsilon a, x)] \right) da + \frac{\Phi(t/\varepsilon)}{\Phi(0)} \exp \left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \psi_\varepsilon(0, x)] \right).$$

⁵ $\partial_x^2 \psi_\varepsilon$ may not reach its maximum. The bounds over $\partial_x \psi_\varepsilon$ of Proposition 49 allow us to define a modified function that does reach its maximum and to proceed as in subsection 2.2.2.

Applying the maximum principle and denoting $t_0 = \arg \max_{[0, T]} \psi_\varepsilon(\cdot, x) - \psi_\varepsilon(0, x)$ results in

$$1 \geq [1 - K\varepsilon] \int_0^{t_0/\varepsilon} \Phi(a) da + \frac{\Phi(t_0/\varepsilon)}{\Phi(0)} e^{m/\varepsilon},$$

hence:

$$\begin{aligned} e^{m/\varepsilon} &\leq \frac{\Phi(0)}{\Phi(t_0/\varepsilon)} \left[1 - (1 - K\varepsilon) \int_0^{t_0/\varepsilon} \Phi(a) da \right] \\ &\leq \left(1 + \frac{t_0}{\varepsilon} \right)^{1+\mu} \left[\int_{t_0/\varepsilon}^\infty \Phi(a) da + K\varepsilon \int_0^{t_0/\varepsilon} \Phi(a) da \right] \\ &\leq 1 + \frac{t_0}{\varepsilon} + K\varepsilon \left[\left(1 + \frac{t_0}{\varepsilon} \right)^{1+\mu} - \left(1 + \frac{t_0}{\varepsilon} \right) \right], \end{aligned}$$

hence the result. \square

Step 4: An upper bound on \mathfrak{B}_ε . Back to the upper bound in Lemma 62, we are in position to conclude.

Proposition 65 (Upper bound for \mathfrak{B}_ε). *Under hypotheses 7 and 8, for any $(t, x) \in [0, T] \times \mathbb{R}$, \mathfrak{B}_ε decays in the following way as $\varepsilon \rightarrow 0$:*

$$\mathfrak{B}_\varepsilon \leq \varepsilon^\mu \frac{2^{1+\mu}}{t^{1+\mu}} [T + K_1 \varepsilon^{1-\mu} + K_2 \varepsilon], \quad (2.84)$$

for some explicit constants K_1, K_2 .

Proof. Lemma 64 and the upper bound in Lemma 62 give us:

$$\begin{aligned} \mathfrak{B}_\varepsilon(\psi_\varepsilon)(t, x) &\leq \frac{\Phi(1 + t/\varepsilon)}{\Phi(1)} \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \psi_\varepsilon(0, x)]\right) \\ &\leq \frac{2^{1+\mu}}{(2 + t/\varepsilon)^{1+\mu}} e^{m/\varepsilon} \\ &\leq 2^{1+\mu} \left(\frac{\varepsilon}{t}\right)^{1+\mu} \left(1 + \frac{T}{\varepsilon} + K\varepsilon \left(1 + \frac{T}{\varepsilon}\right)^{1+\mu}\right). \end{aligned}$$

\square

Step 5: Conclusion of the proof. The accurate upper bound on \mathfrak{B}_ε that we have just proved allows us to proceed to the crucial result of this section.

Proposition 66. *Under hypotheses 7 and 8, ψ_0 is a viscosity supersolution of (HJ).*

Proof. Let $\Psi \in \mathcal{C}^2(\mathbb{R}^+ \times \mathbb{R})$ be a test function such that $\psi_0 - \Psi$ admits a strict local minimum at (t_0, x_0) , with $t_0 > 0$. We make the distinction between two cases:

If $\partial_t \Psi(t_0, x_0) \geq 0$, then we get immediately

$$1 \leq \int_0^\infty \Phi(a) \exp(a \partial_t \Psi(t_0, x_0)) da \int_{\mathbb{R}} \omega(z) \exp(z \partial_x \Psi(t_0, x_0)) dz.$$

Indeed, if $\partial_t \Psi(t_0, x_0) > 0$ then the right hand side is infinite. Whereas, if $\partial_t \Psi(t_0, x_0) = 0$, this equality follows from the symmetry of ω , see also (2.14).

If $\partial_t \Psi(t_0, x_0) < 0$, then there exists $\nu > 0$, and a ball of radius $0 < 2h < t_0/10$, $B((t_0, x_0), 2h)$ such that $\partial_t \Psi(t, x) < -\nu$ over the ball. On the other hand, by uniform convergence of ψ_ε to ψ_0 , there exists $(t_\varepsilon, x_\varepsilon)$ such that $\psi_\varepsilon - \Psi$ reaches a local minimum at $(t_\varepsilon, x_\varepsilon)$. We assume that ε is small enough such that $(t_\varepsilon, x_\varepsilon) \in B((t_0, x_0), h)$.

The contribution \mathfrak{B}_ε is handled thanks to Proposition 65, uniformly in $t \in [t_0/2, T]$:

$$\forall \delta > 0, \exists \varepsilon_\delta > 0 \mid \forall \varepsilon \in (0, \varepsilon_\delta), \mathfrak{B}_\varepsilon(\Psi)(t_\varepsilon, x_\varepsilon) < \delta.$$

The contribution \mathfrak{A}_ε is handled by splitting the time integral into two contributions: those ages which are smaller than h/ε , and those ages which are greater. The small ages are dealt with thanks to the local minimum property:

$$\forall (z, a) \in B((t_\varepsilon, x_\varepsilon), h/\varepsilon) \quad \psi_\varepsilon(t_\varepsilon, x_\varepsilon) - \Psi(t_\varepsilon, x_\varepsilon) \leq \psi_\varepsilon(t_\varepsilon - \varepsilon a, x_\varepsilon - \varepsilon z) - \Psi(t_\varepsilon - \varepsilon a, x_\varepsilon - \varepsilon z).$$

Recalling the identity $\mathfrak{A}_\varepsilon + \mathfrak{B}_\varepsilon = 1$, we deduce

$$1 - \delta \leq A_\varepsilon(\psi_\varepsilon)(t_\varepsilon, x_\varepsilon) \leq I + II + III, \quad (2.85)$$

where we set $h > 0$ and define:

$$\begin{aligned} I &= \int_0^{\frac{h}{\varepsilon}} \int_{-\frac{h}{\varepsilon}}^{\frac{h}{\varepsilon}} \Phi(a) \omega(z) \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t_\varepsilon, x_\varepsilon) - \psi_\varepsilon(t_\varepsilon - \varepsilon a, x_\varepsilon - \varepsilon z)]\right) dz da, \\ II &= \int_0^{h/\varepsilon} \int_{\mathbb{R} \setminus [-\frac{h}{\varepsilon}, \frac{h}{\varepsilon}]} \Phi(a) \omega(z) \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t_\varepsilon, x_\varepsilon) - \psi_\varepsilon(t_\varepsilon - \varepsilon a, x_\varepsilon - \varepsilon z)]\right) dz da, \\ III &= \int_{h/\varepsilon}^{1+\frac{t}{\varepsilon}} \int_{\mathbb{R}} \Phi(a) \omega(z) \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t_\varepsilon, x_\varepsilon) - \psi_\varepsilon(t_\varepsilon - \varepsilon a, x_\varepsilon - \varepsilon z)]\right) dz da. \end{aligned} \quad (2.86)$$

- Limit of I – small ages and spaces.

Thanks to the local maximum property we have:

$$I \leq \int_0^{\frac{h}{\varepsilon}} \int_{-\frac{h}{\varepsilon}}^{\frac{h}{\varepsilon}} \Phi(a) \omega(z) \exp\left(\frac{1}{\varepsilon} [\Psi(t_\varepsilon, x_\varepsilon) - \Psi(t_\varepsilon - \varepsilon a, x_\varepsilon - \varepsilon z)]\right) dz da,$$

on which we perform the same Taylor expansion as in (2.77), which yields:

$$\begin{aligned} I &\leq \int_0^{\frac{h}{\varepsilon}} \int_{-\frac{h}{\varepsilon}}^{\frac{h}{\varepsilon}} \Phi(a) \omega(z) \exp\left\{ a \partial_t \Psi(t_\varepsilon, x_\varepsilon) + z \partial_x \Psi(t_\varepsilon, x_\varepsilon) \right. \\ &\quad + \frac{1}{2} \int_0^1 (1-s)^2 [a^2 \partial_t^2 \Psi(t_\varepsilon - \varepsilon s a, x_\varepsilon - \varepsilon s z) + 2 a z \cdot \partial_t \partial_x \Psi(t_\varepsilon - \varepsilon s a, x_\varepsilon - \varepsilon s z) \\ &\quad \left. + z^2 \partial_x^2 \Psi(t_\varepsilon - \varepsilon s a, x_\varepsilon - \varepsilon s z)] ds \right\} dz da. \end{aligned}$$

Since $\Psi \in \mathcal{C}^2$ only takes values over $B((t_0, x_0), 2h)$ in the expression above, uniformly in ε , a domination argument allows us to pass to the limit $\varepsilon \rightarrow 0$ and recover the following limit for the right hand side:

$$\int_0^\infty \int_{\mathbb{R}} \Phi(a) \omega(z) \exp([a \partial_t \Psi + z \partial_x \Psi](t_0, x_0)) dz da.$$

- Limit of II – small ages, large spaces.

Since ψ_ε is Lipschitz continuous in x with some constant L , we can localise the expression of II at x_ε at a price:

$$II \leq \int_0^{h/\varepsilon} \int_{\mathbb{R} \setminus [-\frac{h}{\varepsilon}, \frac{h}{\varepsilon}]} \Phi(a) \omega(z) e^{L|z|} \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t_\varepsilon, x_\varepsilon) - \psi_\varepsilon(t_\varepsilon - \varepsilon a, x_\varepsilon)]\right) dz da.$$

Thanks to the local maximum property,

$$II \leq \int_0^{h/\varepsilon} \int_{\mathbb{R} \setminus [-\frac{h}{\varepsilon}, \frac{h}{\varepsilon}]} \Phi(a) \omega(z) e^{L|z|} \exp\left(\frac{1}{\varepsilon} [\Psi(t_\varepsilon, x_\varepsilon) - \Psi(t_\varepsilon - \varepsilon a, x_\varepsilon)]\right) dz da.$$

And by negativity of $\partial_t \Psi$ around (t_0, x_0) ,

$$II \leq \int_0^{h/\varepsilon} \int_{\mathbb{R} \setminus [-\frac{h}{\varepsilon}, \frac{h}{\varepsilon}]} \Phi(a) \omega(z) e^{L|z|} e^{-\nu a} dz da,$$

which converges to 0 as $\varepsilon \rightarrow 0$.

- Limit of III – large ages.

Since ψ_ε is Lipschitz continuous with some Lipschitz constant L , we recover:

$$III \leq \int_{\frac{h}{\varepsilon}}^{1+\frac{t}{\varepsilon}} \int_{\mathbb{R}} \Phi(a) \omega(z) e^{|z|L} \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t_\varepsilon, x_\varepsilon) - \psi_\varepsilon(t_\varepsilon - \varepsilon a, x_\varepsilon)]\right) dz da.$$

We have:

$$\begin{aligned} & \psi_\varepsilon(t_\varepsilon, x_\varepsilon) - \psi_\varepsilon(t_\varepsilon - \varepsilon a, x_\varepsilon) = \\ & = \psi_\varepsilon(t_\varepsilon, x_\varepsilon) - \psi_\varepsilon(t_\varepsilon - h, x_\varepsilon) + \Psi(t_\varepsilon - h, x_\varepsilon) - \Psi(t_\varepsilon, x_\varepsilon) + \Psi(t_\varepsilon, x_\varepsilon) - \Psi(t_\varepsilon - h, x_\varepsilon) + (t_\varepsilon - h, x_\varepsilon) - \psi_\varepsilon(t_\varepsilon - \varepsilon a, x_\varepsilon) \end{aligned}$$

Thanks to the local maximum property, the sum of the four first terms is non-positive. It follows that:

$$III \leq \int_{\frac{h}{\varepsilon}}^{1+\frac{t}{\varepsilon}} \int_{\mathbb{R}} \Phi(a\varepsilon) \omega(z) e^{|z|L} \exp\left(\frac{1}{\varepsilon} [\Psi(t_\varepsilon, x_\varepsilon) - \Psi(t_\varepsilon - h, x_\varepsilon)]\right) \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t_\varepsilon - h, x_\varepsilon) - \psi_\varepsilon(t_\varepsilon - \varepsilon a, x_\varepsilon)]\right) dz da$$

Since $\partial_t \Psi \leq -\nu$ over $B((t_\varepsilon, x_\varepsilon), h/\varepsilon)$, we can bound III as follows:

$$\begin{aligned} III & \leq \int_{\frac{h}{\varepsilon}}^{1+\frac{t}{\varepsilon}} \int_{\mathbb{R}} \Phi(a) \omega(z) e^{|z|L} e^{-\frac{\nu h}{\varepsilon}} \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t_\varepsilon - h, x_\varepsilon) - \psi_\varepsilon(t_\varepsilon - \varepsilon a, x_\varepsilon)]\right) dz da \\ & \leq \int_0^{1+\frac{t-h}{\varepsilon}} \int_{\mathbb{R}} \Phi\left(a + \frac{h}{\varepsilon}\right) \omega(z) e^{|z|L} e^{-\frac{\nu h}{\varepsilon}} \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t_\varepsilon - h, x_\varepsilon) - \psi_\varepsilon(t_\varepsilon - h - \varepsilon a, x_\varepsilon)]\right) dz da. \end{aligned}$$

We conclude thanks to the following semi-group property, which is proved *mutatis mutandis* in the same way as Lemma 64:

$$\exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t_\varepsilon, x_\varepsilon) - \psi_\varepsilon(t_\varepsilon - l, x_\varepsilon)]\right) \leq 1 + \frac{l}{\varepsilon} + K\varepsilon \left[1 + \frac{l}{\varepsilon}\right]^{1+\mu}. \quad (2.87)$$

It follows that

$$III \leq \int_0^{1+\frac{t-h}{\varepsilon}} \exp\left(-\frac{\nu h}{\varepsilon}\right) \Phi\left(a + \frac{h}{\varepsilon}\right) [1 + a + o(a)] da \int_{\mathbb{R}} \omega(z) e^{|z|L} dz.$$

Since ω is a Gaussian distribution, the integral in z (right factor) is finite. Since $\nu h > 0$, t is bounded and Φ is algebraic, the integral in a (left factor) converges to 0 as $\varepsilon \rightarrow 0$.

Passing to the limit $\varepsilon \rightarrow 0$ in (2.85) now gives us:

$$1 - \delta \leq \int_0^\infty \int_{\mathbb{R}} \Phi(a) \omega(z) \exp([a\partial_t \Psi + z\partial_x \Psi](t_0, x_0)) dz da.$$

By taking the limit when $\delta \rightarrow 0$ we recover:

$$1 \leq \int_0^\infty \Phi(a) \exp(a\partial_t \Psi(t_0, x_0)) da \int_{\mathbb{R}} \omega(z) \exp(z\partial_x \Psi(t_0, x_0)) dz.$$

Therefore, ψ_0 is a viscosity supersolution of (HJ). □

Proof of Theorem 46. Propositions 61 and 66 prove ψ_0 is a viscosity solution of the Hamilton-Jacobi equation (2.11). Since ψ_0 is bounded below and Lipschitz continuous, and the Hamiltonian H satisfies the pertinent hypotheses, Theorem 45 proves that ψ_0 is the unique viscosity solution of (2.11). Local compactness of $(\psi_\varepsilon)_\varepsilon$ and standard Hausdorff separation arguments prove that the whole sequence ψ_ε tends to ψ_0 . □

2.4 Discussion and Perspectives

There are two main aspects we would like to discuss in this section. First, we will support and elaborate on the claim we made in Remark 2.1.5, that equation (2.11) is the same as the limiting Hamilton-Jacobi equation derived after renormalising n by a non-stationary measure inspired by Chapter 1 that approaches a meaningful self-similar profile. Second, we will discuss a setting in which the jump rate β depends not only on age but also on space.

2.4.1 Renormalising by a non-stationary measure

The idea of renormalising the solution of a kinetic equation by a stationary measure and studying some multiplicative perturbation term is classical. However, as has been shown in Chapter 1, it cannot be applied here in a straightforward way because, were a steady state to exist in self-similar variables for our equation, it would be infinite at age 0, rendering the boundary condition a meaningless “ $\infty = \infty$ ” equality. Let us attempt to remain as close as possible to the underlying principle by using a function that corresponds to the pseudo-equilibrium of Chapter 1.

For any $t > 0$ and $0 < a < 1 + t$, let

$$N(t, a) = (1 + a)^{-\mu} (1 + t - a)^{\mu-1}. \tag{2.88}$$

We also set, for any $x \in \mathbb{R}$, $t > 0$ and $0 < a < 1 + t$:

$$u(t, x, a) = \frac{n(t, x, a)}{N(t, a)}. \tag{2.89}$$

We define the following measure, for $t > 0$ and $0 < a < 1 + t$:

$$\nu_t(a) = \beta(a) \frac{N(t, a)}{N(t, 0)} = \frac{\mu(1 + t)^{1-\mu}}{(1 + a)^{1+\mu} (1 + t - a)^{1-\mu}}. \tag{2.90}$$

Direct computation gives us

$$\partial_t \ln N + \partial_a \ln N + \beta(a) = 0,$$

which is also satisfied by n . Hence, u satisfies:

$$\begin{cases} \partial_t u(t, x, a) + \partial_a u(t, x, a) = 0, & t \geq 0, \quad a > 0, \quad x \in \mathbb{R} \\ u(t, x, 0) = \int_0^{1+t} \int_{\mathbb{R}} \nu_t(a) \omega(x - x') u(t, x', a) dx' da' \\ u(0, x, a) = u^0(x, a) = n^0(x, a) N(0, a) \end{cases} \quad \text{with } \text{supp}(u^0(x, \cdot)) = [0, 1]. \quad (2.91)$$

Let us take a hyperbolic time - space scaling and a Hopf-Cole transform:

$$u_\varepsilon(t, x, a) = u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, a\right) = \exp\left(-\frac{1}{\varepsilon} \tilde{\varphi}_\varepsilon^0(t, x, a)\right). \quad (2.92)$$

Characteristic flow of (2.91) leads us to define:

$$\tilde{\varphi}_\varepsilon(t, x, a) = \begin{cases} \tilde{\varphi}_\varepsilon^0(x, a - t/\varepsilon), & a > t/\varepsilon \\ \tilde{\psi}_\varepsilon(t - \varepsilon a, x), & a \leq t/\varepsilon. \end{cases} \quad (2.93)$$

Let us also set, in agreement with the Ansatz (2.20) in Hypothesis 8:

$$\begin{aligned} \tilde{\varphi}_\varepsilon^0(x, a) &= v(x) + \varepsilon \xi(x, a) + \chi_{[0,1]}(a) \\ &= v(x) + \varepsilon [\eta(x, a) - (1 + \mu) \ln(1 + a) - (1 - \mu) \ln(1 - a)] + \chi_{[0,1]}(a), \end{aligned} \quad (2.94)$$

where χ_A is worth 0 over the set A and $+\infty$ outside of A .

With the previous definitions, $\tilde{\varphi}_\varepsilon$ satisfies the following equation, which is analogous to (2.10):

$$\begin{aligned} 1 &= \int_0^{t/\varepsilon} \int_{\mathbb{R}} \exp\left(\frac{1}{\varepsilon} \left[\tilde{\psi}_\varepsilon(t, x) - \tilde{\psi}_\varepsilon(t - \varepsilon a, x - \varepsilon z) \right]\right) \nu_{t/\varepsilon}(a) \omega(z) dz da \\ &\quad + \int_0^1 \int_{\mathbb{R}} \exp\left(\frac{1}{\varepsilon} \left[\tilde{\psi}_\varepsilon(t, x) - \tilde{\varphi}_\varepsilon^0(x - \varepsilon z, a) \right]\right) \nu_{t/\varepsilon}(a + t/\varepsilon) \omega(z) dz da. \end{aligned} \quad (2.95)$$

Remark. For any positive t ,

$$\int_0^{1+t/\varepsilon} \nu_{t/\varepsilon}(a) da = \frac{1}{1 + \frac{1}{1+t/\varepsilon}} \xrightarrow{\varepsilon \rightarrow 0} 1.$$

Assuming sufficient regularity, (2.95) gives us:

$$1 = \int_0^{t/\varepsilon} \int_{\mathbb{R}} \exp\left(a \partial_t \tilde{\psi}_\varepsilon(t, x)\right) \exp\left(z \partial_x \tilde{\psi}_\varepsilon(t, x)\right) \exp(o(1)) \omega(z) \Phi(a) \left(\frac{1 + t/\varepsilon}{1 - a + t/\varepsilon}\right)^{1-\mu} dz da.$$

Hence the formal limit of (2.95) is the same Hamilton-Jacobi equation as (2.11):

$$1 = \int_0^\infty \Phi(a) \exp\left(a \partial_t \tilde{\psi}_0(t, x)\right) da \int_{\mathbb{R}} \omega(z) \exp\left(z \partial_x \tilde{\psi}_0(t, x)\right) dz,$$

with the same initial condition v .

Remark. In order to prove convergence of this newly defined $\tilde{\psi}_\varepsilon$ to $\tilde{\psi}_0$, solution of the limiting Hamilton-Jacobi equation, the computations required are more or less the same as those presented in this Chapter, but they contain an additional term that must be estimated:

$$\begin{aligned} \frac{1}{2 + t/\varepsilon} &= \int_0^{t/\varepsilon} \int_{\mathbb{R}} \left[\exp\left(\frac{1}{\varepsilon} \left[\tilde{\psi}_\varepsilon(t, x) - \tilde{\psi}_\varepsilon(t - \varepsilon a, x - \varepsilon z) \right]\right) - 1 \right] \nu_{t/\varepsilon}(a) \omega(z) dz da \\ &\quad + \int_0^1 \int_{\mathbb{R}} \left[\exp\left(\frac{1}{\varepsilon} \left[\tilde{\psi}_\varepsilon(t, x) - \tilde{\varphi}_\varepsilon^0(x - \varepsilon z, a) \right]\right) - 1 \right] \nu_{t/\varepsilon}(a + t/\varepsilon) \omega(z) dz da. \end{aligned}$$

This is due to the fact that $\nu_{t/\varepsilon}$ is not a probability measure over $[0, 1 + t/\varepsilon]$. Since $\nu_{t/\varepsilon}$ does approach a probability measure for any $t > 0$ as $\varepsilon \rightarrow 0$, this is not a major problem.

2.4.2 Space-dependent jump rate

Our study, as briefly mentioned in the Introduction, has a biological motivation. The random motion we model takes place in cellular media in which heterogeneities are often prevalent. Hence the relevance of considering a space-dependant jump rate $\beta(x, a)$. There are different pertinent ways of defining the jump rate, depending on what we intend to model. Here, we will only consider the simple case of a slow space variation of the jump rate, in the sense that follows. We define

$$\beta(x, a) = \frac{\mu(x)}{1+a}, \quad (2.96)$$

where $0 < \mu < 1$ is Lipschitz continuous, and consider the following problem:

$$\begin{cases} \partial_t n_\varepsilon(t, x, a) + \frac{1}{\varepsilon} \partial_a n_\varepsilon(t, x, a) + \frac{1}{\varepsilon} \beta(x, a) n_\varepsilon(t, x, a) = 0, & t \geq 0, \quad a > 0, \quad x \in \mathbb{R} \\ n_\varepsilon(t, x, 0) = \int_0^{1+t/\varepsilon} \int_{\mathbb{R}} \beta(x - \varepsilon z, a) \omega(z) n_\varepsilon(t, x - \varepsilon z, a) dz da \\ n_\varepsilon(0, x, a) = n_\varepsilon^0(x, a) = n^0(x/\varepsilon, a). \end{cases} \quad (2.97)$$

Remark. It follows that $n(t, x, a) = n_\varepsilon(\varepsilon t, \varepsilon x, a)$ satisfies the problem below, with a jump rate that varies slowly in space:

$$\begin{cases} \partial_t n(t, x, a) + \partial_a n(t, x, a) + \beta(\varepsilon x, a) n(t, x, a) = 0, & t \geq 0, \quad a > 0, \quad x \in \mathbb{R} \\ n(t, x, 0) = \int_0^{1+t} \int_{\mathbb{R}} \beta(\varepsilon x', a) \omega(x - x') n(t, x', a) dx' da \\ n(0, x, a) = n^0(x, a). \end{cases} \quad (2.98)$$

Since μ is Lipschitz continuous,

$$\beta(x - \varepsilon z, a) = \frac{\mu(x) + \mathcal{O}(\varepsilon z)}{1+a}. \quad (2.99)$$

The formulation of (2.97) along characteristic lines allows us to recover, for ψ_ε and ϕ_ε^0 defined as in (2.9),

$$\begin{aligned} 1 &= \int_0^{t/\varepsilon} \int_{\mathbb{R}} \omega(z) \Phi(x - \varepsilon z, a) \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \psi_\varepsilon(t - \varepsilon a, x - \varepsilon z)]\right) dz da \\ &+ \int_{t/\varepsilon}^{1+t/\varepsilon} \int_{\mathbb{R}} \omega(z) \Phi(x - \varepsilon z, a) \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \phi_\varepsilon^0(x - \varepsilon z, a - t/\varepsilon)] + \int_0^{a-t/\varepsilon} \beta(x, s) ds\right) dz da, \end{aligned} \quad (2.100)$$

where

$$\Phi(x, a) = \beta(x, a) \exp\left(-\int_0^a \beta(x, s) ds\right). \quad (2.101)$$

Thanks to (2.99) and since ω is a Gaussian, it follows that (2.100) admits a formal limiting Hamilton-Jacobi equation, similar to the space-independent case (2.11). Here however, the Hamiltonian depends on space:

$$1 = \int_0^\infty \Phi(x, a) \exp(a \partial_t \psi_0(t, x)) da \int_{\mathbb{R}} \omega(z) \exp(z \partial_x \psi_0(t, x)) dz. \quad (2.102)$$

Yet again, that is a Hamilton-Jacobi equation, since it is equivalent to:

$$\partial_t \psi_0(t, x) + H(x, \partial_x \psi_0)(t, x) = 0, \quad (2.103)$$

with H defined as follows, where $(\hat{\Phi}(x, \cdot))^{-1}$ is the inverse function of the Laplace transform of $\Phi(x, \cdot)$:

$$H(x, p) = - \left(\hat{\Phi}(x, \cdot) \right)^{-1} \left(\frac{1}{\int_{\mathbb{R}} \omega(z) \exp(zp) dz} \right). \quad (2.104)$$

Passing to the limit rigorously is left for further work.

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Appendix A

Numerical methods and simulations

Within this thesis I have used several numerical simulations. However, I do not present them within the chapters where they are used. I will give here some background and bibliography.

A.1 Monte Carlo simulations

Monte Carlo simulations consist in the (computer-assisted) random sampling of many occurrences of a random process and rely on the expectation that the results satisfy the convergence predicted by the Central Limit Theorem. Buffon's needle experiment (the approximation of π by means of repeated random throws of a needle on a surface patterned by parallel lines) can be considered a precursor of such methods, which were formalised by Fermi, Ulam and von Neumann and are today useful in the context of physical, biological, engineering and economic problems that exhibit many coupled degrees of freedom.

In the context of our work, the renewal-scattering equations that we study rely on a microscopic stochastic continuous time random walk model. Hence, our Monte Carlo simulations are relatively straightforward. We refer the reader to [38] for a detailed introduction to Monte Carlo methods and to the article that introduced the Mersenne twister random number generator that we use [105].

A.2 Finite Volume schemes

Certain simulations that I have done within the context of Chapter 1 use upwind finite volume schemes. Let us illustrate the concept on a 1D transport equation:

$$\partial_t \rho(t, x) + v(t, x) \partial_x \rho(t, x) = 0,$$

which we consider to have periodic boundary conditions. The velocity v is supposed known. We discretise the equation over a space grid:

$$(x_0, x_1, \dots, x_{N-1}, x_N = x_0),$$

in which we identify x_N to x_0 , with a discretization step Δx . We also consider a time discretisation of step Δt : for $j \geq 0$, we set $t_j = j \Delta t$.

We denote by the subindex i, j the approximation at (t_j, x_i) given by the considered numerical scheme for the corresponding function.

The upwind approach consists in approximating the space derivatives of ρ at each x_i in a way that depends on the direction in which the density ρ is transported at that point. The idea is to use the values of ρ on the stencils upwind from x_i to approximate $\partial_\rho(t, x_i)$. We refer by the superindices $-$ and $+$ to the respective discretisations using the left and the right stencils with respect to the point at which we discretise. In a nutshell,

$$\left\{ \begin{array}{ll} \text{If } v_{i,j} > 0 & \text{then } \partial_x \rho_{i,j} = \partial_x \rho_{i,j}^- \\ \text{If } v_{i,j} < 0 & \text{then } \partial_x \rho_{i,j} = \partial_x \rho_{i,j}^+ \\ \text{If } v_{i,j} = 0 & \text{then choice is indifferent.} \end{array} \right. \quad (\text{A.1})$$

In a first order explicit upwind finite volume numerical scheme, $\partial_x \rho_{i,j}^\pm$ are given by:

$$\left\{ \begin{array}{l} \partial_x \rho_{i,j}^- = \frac{1}{\Delta x} (\rho_{i,j} - \rho_{i-1,j}) \\ \partial_x \rho_{i,j}^+ = \frac{1}{\Delta x} (\rho_{i+1,j} - \rho_{i,j}). \end{array} \right. \quad (\text{A.2})$$

The reader may consult [3] for an introduction to numerical simulations and for an analysis of numerical schemes related to the above.

A.3 WENO discretisation of Hamilton-Jacobi equations

This section is taken and summarized from Paul Vigneaux's Doctoral Dissertation [155], which elaborates much further than is here required and contains useful bibliography on the discretisation of Hamilton-Jacobi equations by means of Essentially Non Oscillating (ENO) and Weighted Essentially Non Oscillating (WENO) schemes. A more complete source can be found in Chapters 3 and 5 of [122].

Essentially Non Oscillating (ENO) schemes were first developed by Harten and Osher [77] and Engquist, Osher, Chakravarty [76] for the numerical simulation of conservation laws. They were simplified and perfected by Shu and Osher [143, 144]. The first application of ENO schemes to Hamilton-Jacobi equations is due to Osher and Sethian [123]. Even though the idea of applying ENO schemes to Hamilton-Jacobi equations stemmed from the link that such equations share only in one space dimension, the resulting numerical schemes can be extended to higher space dimensions.

ENO schemes rely on a polynomial interpolation of the approximate solution in order to compute the boundary flux on the discretisation cells. A choice is made in ENO schemes to interpolate over the region in which the polynomial is the smoothest so as to have a high order precision and to avoid oscillations near shock profiles.

ENO schemes are robust and high order uniformly up to shocks. However, since they choose the stencil over which they approximate the space differential of the solution of the studied equation, rounding errors around points at which the solution or its derivatives vanish may lead to abrupt changes in the stencil choice. This can reduce the precision of ENO schemes over sets on which the solution is smooth. We refer to [155, Ch. 2.1.1] and the bibliography therein for a concrete description of ENO schemes.

WENO schemes were introduced by Liu, Osher and Chan [100] as an improvement on ENO schemes. They avoid the instabilities of ENO schemes over regions where the solution is smooth by using a convex combination of the three different ENO discretisation choices ponderated by ad hoc coefficients. Such coefficients take into account the local smoothness of the solution so

as to be roughly equal over smooth regions and roughly 0 near discontinuities. Thus, an ENO scheme behaves as a centred scheme over regions where the solution is smooth, and as an ENO scheme at around discontinuities.

Let us present the order 5 WENO approach (WENO5) for the discretisation of the Hamilton-Jacobi equation below. The approach – taken from [155] – is based on [90, 91] and is explained in detail in [122]. It is the approach that we use in Chapter 1.

Consider the 1D evolution Hamilton-Jacobi equation

$$\partial_t u(t, x) + H(\partial_x u(t, x)) = 0, \quad (\text{A.3})$$

and keep the notation introduced in the previous paragraph for the discretisations.

We define the left and right discretizations of $\partial_x u$ at (t_j, x_i) as follows:

$$\partial_x u_{i,j}^{+,-} = w_1 \left(\frac{v_1}{3} - \frac{7v_2}{6} + \frac{11v_3}{6} \right) + w_2 \left(\frac{-v_2}{6} + \frac{5v_3}{6} + \frac{v_4}{3} \right) + w_3 \left(\frac{v_3}{3} + \frac{5v_4}{6} - \frac{v_5}{6} \right), \quad (\text{A.4})$$

where the coefficients are those given by the expressions (A.5), (A.6), (A.7), and (A.8) below. Note that the expressions differ for the left and right versions. However, in order to lighten the notation, we will not add superindices. We consider the discretisation at time t_j and also omit the subindex j in the following expressions.

For $\partial_x u_i^-$, we let:

$$\begin{aligned} v_1 &= \frac{(u_{i-2} - u_{i-3})}{\Delta x}, & v_2 &= \frac{(u_{i-1} - u_{i-2})}{\Delta x}, & v_3 &= \frac{(u_i - u_{i-1})}{\Delta x}, \\ v_4 &= \frac{(u_{i+1} - u_i)}{\Delta x}, & v_5 &= \frac{(u_{i+2} - u_{i+1})}{\Delta x}. \end{aligned} \quad (\text{A.5})$$

For $\partial_x u_i^+$, we let:

$$\begin{aligned} v_1 &= \frac{(u_{i+3} - u_{i+2})}{\Delta x}, & v_2 &= \frac{(u_{i+2} - u_{i+1})}{\Delta x}, & v_3 &= \frac{(u_{i+1} - u_i)}{\Delta x}, \\ v_4 &= \frac{(u_i - u_{i-1})}{\Delta x}, & v_5 &= \frac{(u_{i-1} - u_{i-2})}{\Delta x}. \end{aligned} \quad (\text{A.6})$$

We define the regularity coefficients:

$$\begin{aligned} S_1 &= \frac{13}{12} (v_1 - 2v_2 + v_3)^2 + \frac{1}{4} (v_1 - 4v_2 + 3v_3)^2, \\ S_2 &= \frac{13}{12} (v_2 - 2v_3 + v_4)^2 + \frac{1}{4} (v_2 - v_4)^2, \\ S_3 &= \frac{13}{12} (v_3 - 2v_4 + v_5)^2 + \frac{1}{4} (3v_3 - 4v_4 + v_5)^2. \end{aligned} \quad (\text{A.7})$$

We take a small enough ε , for instance $\varepsilon = 10^{-6}$ for data of order of magnitude 1, and we define the weights:

$$\begin{aligned} a_1 &= \frac{1}{10} \frac{1}{(\varepsilon + S_1)^2}, & w_1 &= \frac{a_1}{a_1 + a_2 + a_3}, \\ a_2 &= \frac{6}{10} \frac{1}{(\varepsilon + S_2)^2}, & w_2 &= \frac{a_2}{a_1 + a_2 + a_3}, \\ a_3 &= \frac{3}{10} \frac{1}{(\varepsilon + S_3)^2}, & w_3 &= \frac{a_3}{a_1 + a_2 + a_3}. \end{aligned} \quad (\text{A.8})$$

A.4 Lax-Friedrichs scheme for Hamilton-Jacobi equations

Lax-Friedrichs numerical schemes were introduced by Crandall and Lions in [35]. The reader may consult [122, Ch. 5.3] for a much more detailed presentation than that given hereafter.

Consider the 1D evolution Hamilton-Jacobi equation (A.3). The Lax-Friedrichs numerical scheme is the following approximation of the Hamiltonian H :

$$\hat{H}_{i,j} = H\left(\frac{\partial_x u_{i,j}^- + \partial_x u_{i,j}^+}{2}\right) - \alpha_x \left(\frac{\partial_x u_{i,j}^+ - \partial_x u_{i,j}^-}{2}\right) \quad (\text{A.9})$$

where α_x is a dissipation coefficient that controls the amount of numerical viscosity, taken to be worth:

$$\alpha_x = \max |H'(\partial_x u)|. \quad (\text{A.10})$$

Finding the maximum can be subtle.

Increasing the dissipation coefficient α_x increases the amount of artificial dissipation, which decreases the quality of the solution. However, taking α_x too low may induce oscillations. It follows that the maximisation over the whole domain used for the classical Lax-Friedrichs scheme is not optimal. Refinements of the Lax-Friedrichs (LF) scheme that seek to take α_x as low as feasible include the Stencil LF scheme, in which the minimisation is done over the grid nodes used to evaluate $u^{+,-}$, and the Local LF scheme, where only u^+ and u^- are used to compute α_x . For extensive details the reader may consult [122, Ch. 5.3.1], and for a synthetic presentation, [155, Ch. 3.1.3].

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Résumé

Cette thèse est consacrée à l'analyse asymptotique d'équations aux dérivées partielles issues de modèles de déplacement sous-diffusif en biologie cellulaire. Notre motivation biologique est fondée sur les nombreuses observations récentes de protéines cytoplasmiques dont le déplacement aléatoire dévie de la diffusion Fickienne normale.

Dans la première partie, nous étudions la décroissance auto-similaire de la solution d'une équation de renouvellement à queue lourde vers un état stationnaire. Les idées mises en jeu sont inspirées de méthodes d'entropie relative.

Nos principaux apports sont la preuve d'un taux de décroissance en norme L^1 vers la loi de l'arc-sinus et l'introduction d'une fonction pivot spécifique dans une méthode d'entropie relative.

La seconde partie porte sur la limite hyperbolique d'une équation de renouvellement structurée en âge et à sauts en espace. Nous y prouvons un résultat de "stabilité" : les solutions des problèmes rééchelonnés à $\varepsilon > 0$ convergent lorsque $\varepsilon \rightarrow 0$ vers la solution de viscosité de l'équation de Hamilton-Jacobi limite des problèmes à $\varepsilon > 0$. Les outils mis en jeu proviennent de la théorie des équations de Hamilton-Jacobi.

Ce travail présente trois idées intéressantes. La première est celle de prouver le résultat de convergence sur la condition de bord du problème plutôt que d'utiliser des fonctions test perturbées. La deuxième consiste en l'introduction de termes correcteurs logarithmiques en temps dans des estimations a priori ne découlant pas directement du principe du maximum. Cela est dû à la non-existence d'un équilibre du problème homogène en espace. La troisième est une estimation précise de la décroissance de l'influence de la condition initiale sur le terme de renouvellement. Elle correspond à une estimation fine d'une version non-locale de la dérivée temporelle de la solution.

Au cours de cette thèse, des simulations numériques de type Monte Carlo, schémas volumes finis, Lax-Friedrichs et Weighted Essentially Non Oscillating ont été réalisées.

Mots-clés : Analyse asymptotique, équations aux dérivées partielles, diffusion anormale, équations structurées, entropie relative, équations de Hamilton-Jacobi, sous-diffusion en biologie cellulaire

Abstract

This thesis is devoted to the asymptotic analysis of partial differential equations modelling subdiffusive random motion in cell biology. The biological motivation for this work is the numerous recent observations of cytoplasmic proteins whose random motion deviates from normal Fickian diffusion.

In the first part, we study the self-similar decay towards a steady state of the solution of a heavy-tailed renewal equation. The ideas therein are inspired from relative entropy methods.

Our main contributions are the proof of an L^1 decay rate towards the arc-sine distribution and the introduction of a specific pivot function in a relative entropy method.

The second part treats the hyperbolic limit of an age-structured space-jump renewal equation. We prove a “stability” result: the solutions of the rescaled problems at $\varepsilon > 0$ converge as $\varepsilon \rightarrow 0$ towards the viscosity solution of the limiting Hamilton-Jacobi equation of the $\varepsilon > 0$ problems. The main mathematical tools used come from the theory of Hamilton-Jacobi equations.

This work presents three interesting ideas. The first is that of proving the convergence result on the boundary condition of the studied problem rather than using perturbed test functions. The second consists in the introduction of time-logarithmic correction terms in a priori estimates that do not follow directly from the maximum principle. That is due to the non-existence of a suitable equilibrium for the space-homogenous problem. The third is a precise estimate of the decay of the influence of the initial condition on the renewal term. This is tantamount to a refined estimate of a non-local version of the time derivative of the solution.

Throughout this thesis, we have performed numerical simulations of different types: Monte Carlo, finite volume schemes, Lax-Friedrichs schemes and Weighted Essentially Non Oscillating schemes.

Keywords: Asymptotic analysis, partial differential equations, anomalous diffusion, structured equations, relative entropy, Hamilton-Jacobi equations, subdiffusion in cell biology

