# Two-dimensional quantum gravity coupled to non-conformal matter

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## Two-dimensional quantum gravity coupled to non-conformal matter

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# Abstract

Finding a theory of quantum gravity which describes in a consistent way the quantum properties of matter and spacetime geometry is one of the greatest challenges of modern theoretical physics. However after several decades of research it still looks like a wild territory and a lot of conceptual and technical issues need to be resolved. A glimpse of the properties such a theory should have can be granted by the study of simplified toy models that allow for exact computations.

In this thesis we will take this approach from two different points of view.

The first part deals with two-dimensional quantum gravity. In two dimensions quantum gravity is much better understood and many computations can be carried out exactly. Whereas two-dimensional quantum gravity coupled to conformal matter has been widely studied and is now well understood, much less was known until recently about what happens when matter is non-conformal. This is the issue we will focus on in this part of the thesis. First we compute the gravitational action in two cases: a massive scalar field on a Riemann surface with boundaries and a massive Majorana fermion on a manifold without boundary. The last case corresponds to a CFT perturbed by a conformal perturbation and is usually tackled through the DDK ansatz. However as we will see the results do not seem to match. Finally we give a minisuperspace computation of the spectrum of the Mabuchi action, a functional that has been shown to appear in the gravitational action for a massive scalar field.

In the second part we focus on black hole thermal behaviour which provides a lot of insight of how a theory of quantum gravity should look like. In the context of string theory the AdS/CFT correspondence provides powerful tools for understanding the microscopic origin of black holes thermodynamics. Here we will construct a quantum mechanical toy model based on holographic principles to study the dynamics of quantum black holes.

# Chapter 1

# Introduction

## 1.1 Why quantum gravity?

The beginning of the twentieth century has seen the development of two major theories that have completely changed our understanding of Nature: quantum mechanics (and quantum field theory) and general relativity. The first one deals with the fundamental interactions that occur between particles at very short distances: the electromagnetic, weak and strong interactions. On the other hand general relativity describes the large scale structure of the Universe by relating gravitation to the geometry of spacetime. Both of them are very well tested experimentally. Their last successes being respectively the discovery in 2012 by Atlas and CMS detectors at the LHC of the Brout–Englert–Higgs boson predicted in 1964 and the observation of gravitational waves at Ligo and Virgo in 2016.

However these two theories seem to be inconsistent with each other. Indeed their mathematical frameworks are very different and do not look compatible. While general relativity is a classical theory where spacetime is dynamical, quantum field theory needs a fixed spacetime background and, in particular, a fixed time direction. But as all particles interact gravitationally one expects to be able to describe all interactions in a coherent framework. Moreover, general relativity seems to break down at small distances around the Planck scale. Indeed it predicts singularities in spacetime (Big-Bang, black holes) where the curvature becomes infinite and the theory does not make sense. Furthermore semiclassical computations coupling classical general relativity to a quantum field theory show inconsistencies, the most important being Hawking radiation of black holes which questions the unitarity of quantum mechanics, one of its fundamental postulates. As a consequence, the search for a theory of quantum gravity that would unify general relativity and quantum mechanics is a major challenge for theoretical physics of the twenty-first century.

But after almost one century of research quantum gravity remains elusive as there is no such theory related to experiments for the moment. The fact is that one rapidly runs into several issues [1–4]. To begin with, a direct quantization of the Einstein–Hilbert action is not possible as it is perturbatively non-renormalisable: at each order in perturbation theory new divergences occur and one needs an infinity of counterterms to cancel them. A first possibility is that there exists a non-gaussian UV fixed-point which would make the theory non-perturbatively renormalisable (which goes under the name of asymptotic safety). Another one is that general relativity is only an effective low-energy theory which needs to be completed (by higher-derivative terms, specific matter, supersymmetry, strings...) to be made UV finite or renormalisable. An even more difficult issue is to formulate a background independent quantum theory in particular without a specific time. Moreover it is not even clear what should be the observables of a quantum theory of gravity since diffeomorphism invariance of general relativity does not enable us to make sense of local observables. Furthermore an essential postulate of quantum field theory is causality which

means that fields separated by space-like intervals do not influence each other and then commute. But if the metric undergoes quantum fluctuations the notions of time-like, null or space-like separations do not have any meaning anymore. These issues have lead to the development of several frameworks which tackle diverse problems: string theory [5–7], loop quantum gravity [8–10], non-commutative geometry [11–13], dynamical triangulations [14, 15], causal sets [16–19], asymptotic safety [20–22], random matrix and tensor models [23–26]...

There is little doubt that the full understanding of our Universe, in particular concerning the primordial cosmology and black holes, requires a theory of quantum gravity. Another issue that we may hope to solve with a quantum gravity theory is the cosmological constant problem. For the moment we cannot explain why the cosmological constant, though non-zero, is so small, of order $10^{-52}\,\mathrm{m}^{-4}$ while a naive QFT computation would make it at least fifty orders of magnitude bigger [27]. Quantum gravity might also be a solution to remove short-distance divergences that are present in most quantum field theories (even those which are renormalisable but not finite) by introducing a fundamental cut-off at the Planck scale. There is also hope, in particular in string theory or in non-commutative geometry, that this new theory might solve some other problems that plague our current Standard Model of particle physics. First of all the huge number of free parameters (nineteen) in the Standard Model is not satisfying and we would like to be able to describe the physics with as few parameters as possible. In particular we expect that a complete theory would predict the masses of the fundamental particles. This could be achieved by a grand unification theory. A hint for the existence of such a theory is that the three coupling constants of the standard model become almost equal at an energy around $10^{15}\,\mathrm{GeV}$ and the agreement is even better with supersymmetry. Other questions that could be answered by grand unification include the origin of the gauge group $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$, the explanation of the number of families of fermions or the mechanism gives masses to neutrinos. For the moment a grand unified theory is still elusive as no beyond-Standard Model physics has been detected so far.

A major difficulty is that the effects of quantum gravity are expected to be important at energies around the Planck scale ($m_P = \sqrt{\frac{\hbar c}{G}} \sim 10^{19}\,\mathrm{GeV}$) which is far out of reach of modern accelerators which probe energies around 10 TeV. Currently there is no experimental observation that could put us on the right track. Nevertheless our best experimental hopes for the near future come from cosmology [28] (measures of anisotropies in the CMB, detection of primordial black hole radiations, analysis of gravitational waves...).

## 1.2 Two-dimensional quantum gravity

Two directions have been followed in this thesis. The main one is the study of quantum gravity in two dimensions. Indeed one way to tackle the problem of quantum gravity is to study lower-dimensional models: as mentioned earlier there is no available experiment for the moment so it make sense to study a priori akin but simpler systems. One can then hope that quantum gravity in $d < 4$ dimensions is not tremendously different from the four-dimensional one and shares with it some universal properties. This kind of approach has proved very insightful for Yang–Mills theory. For example in two dimensions one can compute exactly various UV and IR properties related to anomalies in the Schwinger model [29, 30]. This model also gives hint of explanation for quark confinement [31]. Two-dimensional quantum gravity shares several peculiar features with two-dimensional Yang–Mills theory. Both have much less degrees of freedom than their four-dimensional counterparts: there is no gluon in two-dimensional Yang–Mills [32] while only the conformal factor (also called the Liouville mode) can propagate in two-dimensions (cf. chapter 2). Topological effects play an important role in both theories. Furthermore symmetry groups are often enlarged in two dimensions: two-dimensional Yang–Mills theory is not only invariant under its local gauge group but also under area preserving diffeomorphisms [32] while locally the two-dimensional conformal algebra is enhanced to the Virasoro algebra which is of infinite dimension. Similarly the

U-duality of the maximal supergravity ($\mathcal{N} = 16$) is given by the affine symmetry $E_{9(9)}$, while in $d \geq 3$ the U-duality is a finite-dimensional Lie group [33].

Following this idea two-dimensional quantum gravity is an interesting toy-model for its four-dimensional counterpart. Indeed it presents several appealing features: many computations can be carried out exactly, in particular quantum corrections due to the interaction between matter and gravity, and it is often renormalisable.

The partition function in two-dimensional quantum gravity is given by the sum of the contributions of all inequivalent surfaces of a given topology, and then by the sum over topologies. Combining these two sums is a highly non-trivial problem. Two classes of models have been developed to tackle this issue. In the discretized approach this sum is replaced by a sum over all inequivalent triangulations of surfaces. These triangulations can be seen as ribbon graphs and then be described as the Feynman graphs of a matrix model. This enables one to transform the complicated functional integral onto a much simpler integral where one integrates over matrix configurations [34–36]. We will not investigate this strategy in this thesis (though in chapter 8 we will come back to matrix models for a completely different reason) and will focus on the other approach, the continuous one. In this approach one treats quantum gravity as a quantum field theory. For this one needs to deal with all the spurious degrees of freedom due to the gauge invariance under diffeomorphisms. To do this we will need to fix a gauge (usually the conformal one), to apply the Faddeev–Popov procedure and then to sum over the remaining degree of freedom, the Liouville mode in conformal gauge.

Quantum gravity in two dimensions is especially simple when one couples it to conformal matter only. Polyakov showed that in this case the Wess–Zumino effective action is given by the Liouville action [37]. This theory is now well defined and understood, in particular the critical exponents [38, 39], the spectrum [40–44] and certain properties of the correlation functions [45–47] have been computed. Using a non-Lagrangian description [48–50] the conformal bootstrap of Liouville theory has demonstrated that it defines a consistent CFT for any complex central charge (see also [51]). Recently a precise mathematical non-perturbative definition of Liouville theory has been established [52–54]. Various reviews on this topic are [49, 55–58].

Despite the fact that four-dimensional gravity is not scale invariant (and even less conformal invariant), the coupling of two-dimensional gravity to non-conformal matter has been mostly ignored in the literature. The case of gravity coupled to a CFT perturbed by primary operators has been studied through the DDK ansatz [44, 59–61], but this approach does not fit in the usual framework of two-dimensional gravity in conformal gauge that will be presented in chapter 4.

Moreover the problem of genuine non-conformal field theories where the perturbation is not a primary operator (such as the mass term of a scalar field) has been tackled only recently. When a massive scalar field is coupled to gravity (possibly with non-minimal coupling, and with a linear term) it was shown that other functionals contribute to the gravitational action [62–64] and in particular an explicit expression was obtained at first order in a small mass expansion. The two functionals that appear are the Mabuchi and the Aubin–Yau actions. They are well-known to mathematicians [65, 66] and appear very naturally in the study of compact Riemann surfaces. They are also encountered in the quantum hall effect [64, 67–69]. More recently an exact expression for the gravitational action has been obtained [70]. In this context two-dimensional gravity is reformulated in the Kähler formalism which also enables one to define rigorously the functional integral [62, 63, 71] and to compute possible deviations from the KPZ formula for the string susceptibility [71–73].

Finally, two-dimensional quantum gravity is also useful to study quantum gravity intrinsically. Indeed string theory, one of the more advanced approach to quantum gravity, can be formulated, in the worldsheet formulation, as matter coupled to two-dimensional quantum gravity. Moreover, several approaches including string theory, loop quantum gravity, dynamical triangulation, non commutative geometry, asymptotic safety and causal sets have found evidences that at high energy (around the Planck scale) quantum gravity undergoes

a dimensional reduction to two dimensions [74, 75].

Our main work was to pursue the study of the gravitational action obtained when coupling massive matter to gravity. As the Mabuchi functional seems to play a major role one needs understand its physical properties and for that to follow the same program as what was done for Liouville theory. Our goal was then to compute the spectrum of the Mabuchi theory. For that we cannot rely on all the apparatus of conformal field theory. We thus opted for the minisuperspace approximation which enables one to compute the spectrum and the 2 and 3-point functions of a theory in the semiclassical limit [76, 77]. Doing this we were lead to study the Mabuchi theory on the cylinder. As it is ill-defined on a genus-1 Riemann surface we had to work with a rescaled version of the action. This problem also made us think about the generalisation of the results of [63] for the computation of the gravitational action for a massive scalar field on Riemann surfaces with boundaries [78].

Whether the same functionals will appear in the gravitational action for other matter fields is still an open question. The remarkable properties of the Mabuchi action described in chapter 4 make it likely to be the case. To shed some light on this issue we began the study of a massive free Majorana fermion minimally coupled to quantum gravity which is the simplest example after the scalar field. Another point of interest of this model is that the mass term is a conformal deformation of the massless theory. Our goal is then to compare our results with the DDK ansatz which as mentioned previously suffers from some inconsistency in this case.

## 1.3 Other research directions during this PhD

The second direction taken in this thesis uses the framework of string theory. The particularity of this theory is that it completely unifies all the four interactions, gauge bosons (including gravitons) and matter appearing as excitations of relativistic extended objects [5–7]. Gauge interactions are described by the degrees of freedom of the end points of open strings while gravitons appear in the spectrum of the closed string. All the groups of grand unification (SU(5), S0(10), $E_6$ ...) are subgroups of $E_8$, one of the gauge group that naturally appear. A very interesting feature of string theory is that it contains all the ingredients of the Standard Model while being UV finite. Moreover it has only one dimensionful parameter, the string length $l_s$ and no dimensionless adjustable parameter.

One way to introduce fermionic matter is to impose supersymmetry on the theory. Superstrings can then live in a ten-dimensional spacetime where some dimensions can be compactified on an appropriate manifold (usually a Calabi–Yau manifold). Moreover it has been realised by Polchinski that superstring theory not only contains strings but also fundamental extended objects of various dimensions called branes. Typical examples are the D-branes where open strings can end. A fundamental characteristic of D-branes is that they are non-perturbative objects. There are five different superstring theories: type I, IIA, IIB, heterotic with SO(32) gauge group and heterotic with $E_8 \times E_8$ gauge group. Each type possesses different kinds of D-branes in addition to the fundamental string. The five theories are related by a network of dualities. In particular some of these dualities like T-duality relate a strongly coupled theory to a weakly coupled one. This is a very interesting feature since for the moment string theory is essentially defined perturbatively. The discovery of these dualities has lead to the realisation that string theory is in fact the low energy limit of an eleven-dimensional theory called M-theory. Though the dualities and the AdS/CFT correspondence give some glimpse on M-theory, it remains for the moment elusive.

String theory is thus a fascinating framework which encompasses quantum gravity and grand unification. Moreover it has lead to a lot of progress in mathematics, for example it lead to the discovery of the mirror symmetry which describes how topologically different Calabi–Yau manifolds are related.

### 1.3.1 A fermionic matrix model for black holes

One of the main achievement of string theory is the conjecture of the AdS/CFT correspondence or holography [79–82]. This states that a gravitational theory in $(d+1)$-dimensional AdS space (the bulk) is dual to a conformal field theory without gravity living in a $d$-dimensional spacetime (called the boundary). This means that there exists a dictionary between the observables of the two theories and that correlators can be computed equivalently in one or the other [83]. This offers great opportunity for quantum gravity since quantum corrections in the bulk may be computed from usual gauge theories in the boundary. However decoding the hologram is highly non-trivial since the boundary gauge theory is always strongly coupled for a bulk gravitational description to exist. Consequently AdS/CFT has mostly been used in the other direction to provide insights on strongly coupled gauge theories from a classical supergravity theory in the bulk. In this context string theory is not considered as the fundamental theory but as a theoretical tool which could provide dual descriptions to interesting systems such as Yang–Mills theory in the IR or at finite temperature.

We will be interested in one of the most studied gauge/gravity duality proposal which relates type II string theory in the near-horizon geometry of an AdS black hole to the maximally supersymmetric $\mathcal{N} = 4$ super Yang–Mills gauge theory. Our goal is to study some properties of black holes with the help of the gauge theory description.

Black holes are classical objects whose gravitational field is so strong that they are surrounded by a surface, called the horizon, from which nothing (not even light) can escape. They have been predicted long ago by general relativity and their existence is supported by numerous (indirect) astrophysical observations, the latest being the detection of the gravitational waves emitted by two colliding black holes. The experimental data fit the theoretical predictions very precisely. Thus observational evidence give very strong support for the existence of black holes, either stellar-mass ones (due to the gravitational collapse of a star, like the X-ray binary star Cygnus X-1) or supermassive ones in the centre of galaxies (like Sagittarius A* in the centre of the Milky Way).

In the 60s and 70s a remarkable link between black holes and thermodynamics has been unravelled. First there is a striking analogy between the laws of black holes mechanics and the laws of thermodynamics. Black holes seem to have an entropy proportional to their area [84]. Then Hawking discovered that when quantum effects are taken into account, black holes have a temperature and emit a corresponding black body radiation which eventually should lead to their evaporation [85]. This thermodynamic interpretation leads to several puzzles. First one can wonder whether black hole entropy does have a statistical interpretation in which case it should count the microstates of the black holes. What is particularly intriguing is the fact that the entropy scales as the area of the horizons while in usual quantum mechanics one expects the number of states to scale as the volume. Microstates counting in string theory and supergravity has successfully recovered the Bekenstein entropy for black holes that preserve some amount of supersymmetry [86, 87]. The story is far less understood for non-supersymmetric black holes in Minkowski space. Second, Hawking radiation is thermal and leads to the complete evaporation of the black hole. Then information which has fallen behind the horizon seems to be lost after the vanishing of the black hole. It looks like this information paradox contradicts the unitarity of quantum mechanics. Since Hawking's discovery there has been a large debate among physicists about whether we should abandon the unitarity of quantum mechanics or whether Hawking computation simply did not take sufficiently into account the entanglement between the interior and exterior of the black hole or would break at the Planck scale [88–92]. Though the debate is far from being closed, the AdS/CFT correspondence sheds some light on the issue. Indeed according to it, the evolution of black holes could be mapped to the evolution of a gauge theory which is completely unitary and thus this property should be mirrored in the BH evolution.

The second project of this thesis takes place in this context. Using insights from holography we can look at black holes from the dual theory perspective which corresponds to a

U($N$) invariant $N \times N$ matrix model and which aims to capture the essential features of the black hole thermal behaviour. Indeed in the gauge theory we can compactify all spatial dimensions to get a quantum mechanical model. Moreover since all objects are in the adjoint representation of U($N$) they are represented by matrices. The states of the Hilbert space of the model are dual to the microstates of the black hole. However even in this zero dimensional version, the quantum mechanical Hilbert space is still infinite-dimensional. In our project, we replace the Grassmann-even matrices by Grassmann-odd (fermionic) matrices, which makes the Hilbert space finite-dimensional. This enables us to get exact numerical results in the sense that we can compute the entire spectrum of the theory. To study the dynamics of thermalisation of the black hole we numerically compute correlation functions between states of this dual theory. Unfortunately the algorithmic complexity of the implementation of Grassmann algebra in a system of many q-bits has prevented us to go beyond the case of $N = 3$ while the AdS/CFT correspondence can only be trusted for large $N$. While some interesting features can already be seen with our results at $N = 3$, others seem to require a larger $N$, at least $N = 4$, which we have not been able to implement so far. In this sense, the results presented in chapter 8 are to be considered as preliminary.

### 1.3.2 String field theory

The standard worldsheet formulation of string theory is a first-quantized description. So many computations could be done only thanks to the power of complex analysis and the fact that the conformal group is infinite-dimensional in two dimensions (some developments also benefited from the non-perturbative dualities). However it suffers from limitations that call for a second-quantized formulation. First the worldsheet formulation of string theory is intrinsically on-shell as a consequence of the BRST and conformal invariances. There are some ways to extend it off-shell, but at the price of ambiguities. In any case renormalization is difficult and can be done only using some specific ad-hoc procedure. Moreover the theory suffers from infrared divergences [93].

Formulating a second-quantization of string theory has proven very challenging. If a complete bosonic string field theory has been constructed since the 90' both for the open [94] and closed [95, 96] string, such a theory was built only recently for the heterotic and type II closed superstrings. Indeed the generalisation to superstrings presents three major difficulties. First it was not known how to deal with the Ramond fields because no kinetic term could be found (because the propagator is not invertible). This problem has been circumvented by introducing a new set of free fields that do not interact with the other ones [97, 98]. Another issue was to deal with the picture changing operators (PCOs) insertions in the (off-shell) correlation functions and the associated spurious poles. A construction of the correlation functions with the correct PCO insertions and free of spurious poles was proposed thanks to vertical integration [99]. Finally one has to take care of the ultraviolet divergences that occur when expressing the correlation functions as a sum over Feynman diagrams.

After solving these problems one gets a quantum field theory for strings and one has to show that it possesses the standard nice properties of a QFT. It has been shown that the action for the string field is real [100]. A $i\varepsilon$ prescription was found [101, 102]. The amplitudes obey the Cutkowski rules and together with Ward identities this leads to a proof of unitarity of string field theory [103, 104]. Then several applications have been made, including the computations of the vacuum shift, mass renormalization [105] and soft theorems [106]. Recent important developments in the explicit construction of the action have been made in [107, 108]. Note that there are other approaches to superstring field theory which offer another perspectives (for example they propose a construction for the open superstring, with super-Riemann surfaces instead of PCO, etc.) – a selected set of references is [109–114].

After contributing to a review on superstring field theory together with Ashoke Sen and collaborators, a project has begun with Harold Erbin, Roji Pius and Ashoke Sen to prove

the crossing symmetry of string field theory. This would close the circle in the history of string theory which started with the study Veneziano amplitude (at a time when one did not know it involves strings!) which arose interest for being manifestly crossing symmetric. In quantum field theory proofs of crossing symmetry have been established for 4 and 5-point amplitudes using analytical properties of the S-matrix [115–117]. Generalising these proofs to string field theory is in fact highly non-trivial since string field theory has been formulated directly in term of Feynman diagrams and not in an axiomatic way. The main issue is the form of the vertices which contain exponentials of the momenta. For these reasons, the project is still at the initial stage and will not be discussed in this thesis.

## 1.4 Outline of the thesis

The outline of the thesis is as follows. In part I we present our work on two-dimensional quantum gravity. Chapter 2 reviews some peculiar properties of two-dimensional gravity at the classical level. Then chapter 3 presents some mathematical tools on spectral analysis that we used in the subsequent chapters. Chapter 4 reviews some known points of two-dimensional quantum gravity particularly in the conformal case. It also presents the Kähler formalism and the Mabuchi and Aubin–Yau functionals. In chapter 5 we derive the generalisation of the Mabuchi action for a scalar field on a Riemann surface with boundaries. In chapter 6 we compute the gravitational action for a two-dimensional massive Majorana fermion. Chapter 7 proposes a computation of the spectrum of the Mabuchi theory from a minisuperspace analysis.

Part II focuses on black holes and holography. We review some main thermodynamical properties of black holes in chapter 8 and introduce the AdS/CFT correspondence. We then show how this correspondence can be used to build quantum mechanical models for black holes, in particular in view to study the information paradox. Finally we present a fermionic matrix model that reproduces some aspects of quantum black holes dynamics in chapter 9.

## 1.5 Publications

The content of the thesis is based on the following papers:

- Introduction
  [118] C. de Lacroix, H. Erbin, S. Kashyap, A. Sen and M. Verma. "Closed Superstring Field Theory and its Applications". arXiv: 1703.06410
  C. de Lacroix, H. Erbin, S. Kashyap, A. Sen and M. Verma. "Lectures Notes on String Field Theory". *Work in progress.*

- Chapter 2
  [119] C. de Lacroix and H. Erbin. "A short note on dynamics and degrees of freedom in $2d$ classical gravity". arXiv: 1612.04097.

- Chapter 5
  [78] A. Bilal and C. de Lacroix. "2D gravitational Mabuchi action on Riemann surfaces with boundaries". Submitted to *Journal of high energy physics* (June 2017), arXiv: arXiv:1703.10541.

- Chapter 6
  A. Bilal, S. Chakrabarti, C. de Lacroix, H. Erbin and V. Lahoche. "Gravitational action for a massive Majorana fermion in $2d$ quantum gravity". *Work in progress.*

- Chapter 7
  [76] C. de Lacroix, H. Erbin and E. Svanes. "Mabuchi spectrum from the minisuperspace". *Physics Letters B* 758 (July. 2016), arXiv: 1511.06150.

[77] C. de Lacroix, H. Erbin and E. Svanes. "Minisuperspace computation of the Mabuchi spectrum". arXiv: 1704.05855.

- Chapter 9
  T. Azeyanagi, A. Bilal, C. de Lacroix, and F. Ferrari. "A fermionic matrix model for black holes". *Work in progress*.

# Part I

# Quantum gravity in two dimensions with massive matter

# Chapter 2

# Classical two-dimensional gravity

In the following of this thesis we will study the coupling of massive matter to two-dimensional quantum gravity. In this chapter we look at the classical behaviour and highlight some simple, but not always well-known, properties. It is mostly based on [119].

Two-dimensional classical gravity is often ignored since it is deemed to be trivial and one is more interested in the quantum effects. However the classical theory presents some peculiarities that need to be kept in mind. In particular one often blames theories for which the semi-classical limit of the quantized theory does not reproduce the classical behaviour, but one should recognize that two-dimensional gravity is such a theory. The most obvious sign of this is that renormalization introduces a cosmological constant even if it is forbidden at the classical level when the matter is conformal.

After a brief review of some general aspects of two-dimensional gravity in section 2.1, we will discuss in this chapter the degrees of freedom and the dynamics when it is coupled to conformal and non-conformal matter. In the latter case gravity displays some particularities that one does not find in higher dimensions – and this is an important point to keep in mind when comparing two and four dimensions. More specifically we show in section 2.2 that for a wide class of Lagrangians there are *less* degrees of freedom when the matter is massive (i.e. when there is no Weyl symmetry). This is a consequence of the invariance of the equations of motion for the metric under the Weyl symmetry, even if the action itself is not invariant. Then in section 2.3 we discuss the fact that unitary matter coupled to gravity does not generically admit dynamics, using a set of scalar fields with an arbitrary (metric-independent) potential as an example. At best only a trivial solution is possible while in some cases there is no solution at all (in similarity with the well-known example of pure gravity with cosmological constant). These conclusions can be easily derived through general formalisms [120, 121] but it is insightful to consider the case of $d = 2$ explicitly.

## 2.1 General considerations

Let $\mathcal{M}$ be a 2-dimensional space with metric $g_{\mu\nu}$ with signature $(-+)$ and whose coordinates are denoted by $x^\mu$. We will denote all the matter fields generically by $\psi$. The total action of the matter coupled to $2d$ gravity is

$$S[g, \psi] = S_{\text{cl}}[g] + S_m[g, \psi] \tag{2.1}$$

where $S_{\text{cl}}$ is the (classical) action for pure gravity. Only the case where the matter action $S_m[g, \psi]$ is obtained by making covariant the action $S_m[\eta, \psi]$ by minimal coupling ($\eta_{\mu\nu}$ being just the flat metric) is considered. The action is required to satisfy the following criteria: renormalizability, invariance under diffeomorphisms and no more than first-order derivatives.

Weyl transformations, which correspond to local rescaling of the metric without changing the coordinates

$$g_{\mu\nu} = \mathrm{e}^{2\sigma(x^\mu)} g'_{\mu\nu}, \tag{2.2}$$

will be a central element of the discussion. In general this transformation is not a symmetry of the action. Note that it should not be conflated with conformal transformations which are diffeomorphisms which leave the metric invariant up to a scale factor. The transformation of the metric is then analogous to (2.2) but the fields transform differently. A theory that is both diffeomorphism and Weyl invariant is conformally invariant as a diffeomorphism that leaves the metric invariant up to a scale factor can always be compensated by a Weyl transformation but the inverse is not necessary true. However for actions that are at most quadratic in the first derivative, which will be the case of the models we will consider, conformal invariance implies Weyl invariance [122, 123]. Hence we assume that the action for conformal matter on curved space is Weyl invariant. We will give more details on conformal field theory in chapter 4.

The action for pure gravity is given by the sum of two terms

$$S_{\mathrm{cl}} = S_{\mathrm{EH}} + S_\mu \tag{2.3}$$

as there are only two invariants which fulfil conditions mentioned above. The first piece $S_{\mathrm{EH}}$ is the Einstein–Hilbert action

$$S_{\mathrm{EH}}[g] = \int \mathrm{d}^2x \sqrt{|g|}\, R = 4\pi\chi, \qquad \chi = 2 - 2h, \tag{2.4}$$

which is a topological invariant in two dimensions and equal to the Euler number $\chi$, $h$ being the genus of the surface. As a consequence it is not dynamical (equivalently the Einstein tensor is identically zero) and it can be ignored as long as one is not interested in topological properties (which we are not). This action is also invariant under Weyl transformation (2.2). The second allowed term is the cosmological constant

$$S_\mu[g] = \mu \int \mathrm{d}^2x \sqrt{|g|} = \mu\, A[g] \tag{2.5}$$

where $A$ is the area of $\mathcal{M}$ associated to the metric $g$. The cosmological constant $\mu$ can be either positive or negative which corresponds respectively (in our current sign convention in the normalization of the action) to anti-de Sitter and de Sitter spaces. This action is not invariant under Weyl transformations (2.2). Its presence has dramatic consequences in two dimensions, as will be exemplified in sections 2.2 and 2.3. In particular in the case of pure gravity the equation of motion reduces to $\mu = 0$ and has no solution since $\mu$ is a fixed parameter of the model.

The classical equations of motion are given by variations of the full action (2.1) with respect to $g_{\mu\nu}$ and $\psi$

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0, \qquad \frac{\delta S}{\delta \psi} = 0. \tag{2.6}$$

Without specifying the action for the matter it is not possible to go further with the second equation. Nonetheless the first equation is dictating a lot of properties of the system and is responsible for its subtleties.

The energy–momentum tensors associated to $S$ and $S_m$ are defined by

$$T_{\mu\nu} = -\frac{4\pi}{\sqrt{|g|}} \frac{\delta S}{\delta g^{\mu\nu}}, \qquad T^{(m)}_{\mu\nu} = -\frac{4\pi}{\sqrt{|g|}} \frac{\delta S_m}{\delta g^{\mu\nu}}. \tag{2.7}$$

Then the metric equation of motion implies that the total energy–momentum tensor (2.7) is vanishing

$$T_{\mu\nu} = T^{(m)}_{\mu\nu} + 2\pi\mu\, g_{\mu\nu} = 0. \tag{2.8}$$

These three independent equations provide constraints on the metric and matter fields since they do not contain derivatives of the metric (see section 2.2 for details on the counting of degrees of freedom). It is convenient to decompose the energy–momentum tensor into its trace $T$ and its traceless components $\bar{T}_{\mu\nu}$

$$\bar{T}_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2}\, T\, g_{\mu\nu}, \qquad T \equiv g^{\mu\nu} T_{\mu\nu}. \tag{2.9}$$

In terms of these variables the equation of motion (2.8) reads

$$T = T^{(m)} + 4\pi\mu = 0, \qquad \bar{T}_{\mu\nu} = \bar{T}^{(m)}_{\mu\nu} = 0. \tag{2.10}$$

An obvious advantage is that the cosmological constant has been decoupled from the traceless tensor, and more generally any matter potential that does not depend on the metric does not appear in $\bar{T}_{\mu\nu}$.

If $S_m[\eta, \psi]$ is conformally invariant, then the action $S_m[g, \psi]$ is Weyl invariant. In this case its energy–momentum tensor (2.7) is traceless $T^{(m)} = 0$. Then the trace of $T_{\mu\nu}$ reduces to the cosmological constant and gives the equation of motion $\mu = 0$. This equation has no solution and solutions for conformally invariant matter coupled to gravity can exist only in the absence of a cosmological constant. Then the equation of motion without cosmological constant is simply $T^{(m)}_{\mu\nu} = 0$ which provides two independent equations because it is symmetric and traceless.

## 2.2 Degrees of freedom

In this section we provide a counting of the degrees of freedom for a general model linear in the inverse metric. As we will recall below, in this case the equations of motion for the metric are Weyl invariant even if the action is not, and as a consequence there are more constraints *without* the Weyl symmetry than with it. At the end we comment the general case.

In two dimensions the naive counting of on-shell degrees of freedom gives $-1$. This negative number is due to the fact that there are no equations of motion for the metric, and as such there are no gauge constraints resulting from them. The counting should be done without cosmological constant: in this case the action is also Weyl invariant which implies that all components of the metric can be fixed (two from diffeomorphisms, one from Weyl invariance) and there is no on-shell degree of freedom.

When matter is present gravity reduces the number of degrees of freedom in 2 dimensions, as already illustrated above and as we will now discuss for general matter fields $\psi$ with a total of $N$ on-shell degrees of freedom (before coupling to gravity). The counting may seem useless in view of the absence of dynamics discussed in 2.3 but it applies to more general situations (such as non-unitary matter).

Since the precise counting is sensitive to the form of the action we will restrict our attention to the simpler case when the matter Lagrangian is linear in the inverse metric (in particular it excludes couplings of the form $R\psi$) and where there is no additional gauge invariance.

$$S_m = \frac{1}{2\pi} \int \mathrm{d}^2x\, \sqrt{|g|}\, \mathcal{L}, \qquad \mathcal{L} = -\frac{1}{2}\big(g^{\mu\nu}\mathcal{L}_{\mu\nu}(\psi) + V(\psi)\big). \tag{2.11}$$

The first term is Weyl invariant because the transformation of $g^{\mu\nu}$ cancels the one of $\sqrt{|g|}$. The traceless and trace components of the energy–momentum tensor are

$$\bar{T}_{\mu\nu} = \mathcal{L}_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\big(g^{\alpha\beta}\mathcal{L}_{\alpha\beta}\big), \qquad T = -V + 4\pi\mu. \tag{2.12}$$

It can be seen that the traceless tensor is the energy–momentum tensor corresponding to the action $S_m$ where all the parameters breaking Weyl invariance have been set to zero.

As a consequence no dynamical component of the metric appear inside and they give two constraints on the matter. The trace is also Weyl invariant since it does not contain the metric and it provides another constraint for the matter. In total the number of on-shell degrees of freedom is reduced by three, giving $N-3$. The $N$ massive scalar fields from 2.3 provides an explicit example. After using the diffeomorphisms, the metric has still one off-shell degree of freedom which should be fixed by the equations of motion of the scalar fields since they are not Weyl invariant.

If the matter action is Weyl invariant, then the trace equation is removed and there is one constraint less, giving $N-2$ on-shell degrees of freedom. As in the case of pure gravity the last off-shell metric component is fixed by a Weyl transformation.

The conclusion is that even if Weyl invariance is not a symmetry of the action it is a symmetry of the equations of motion for $g_{\mu\nu}$. This is similar to the electric-magnetic duality except that the matter equations of motion are breaking this duality. It might be surprising to have a system where adding a gauge symmetry *increases* the number of degrees of freedom, whereas the usual common lore is that a gauge symmetry describes a redundancy among these degrees. This behaviour is very peculiar to two-dimensional gravity because gravity provides constraints together with degrees of freedom. The metric components act as Lagrange multipliers and the Weyl symmetry removes one of those, which implies that one less constraint will be imposed.

## 2.3 Dynamics of unitary matter

The goal of this section is to show that a large class of models of unitary matter coupled to gravity has no dynamics. It is well-known that the energy–momentum tensor can be rewritten as a sum of squares for free scalar fields, which implies that each term vanishes independently. There is a difference between conformal and non-conformal matter since the cosmological constant and the additional potential may provide a negative term in $T_{\mu\nu}$. In fact, using the decomposition (2.10), the equations in both cases can be brought into a similar form and in some cases the behaviour of systems with a cosmological constant is even worse.

We will illustrate this using $N$ scalar fields $X_i$ with a potential $V(X_i)$

$$S_m = -\frac{1}{2\pi}\int \mathrm{d}^2x\,\sqrt{|g|}\,\frac{1}{2}\left(g^{\mu\nu}\partial_\mu X_i\partial_\nu X_i + V(X_i)\right), \qquad i=1,\dots,N, \tag{2.13}$$

where the sum over the index $i$ is implicit and some of the $X_i$ may not appear in the potential. This potential does not contain any constant term which would just correspond to a shift of the cosmological constant. The equations of motion for the metric and the matter are

$$T_{\mu\nu}=0, \qquad -\Delta X_i + \frac{\partial V}{\partial X_i} = 0 \tag{2.14}$$

where $\Delta$ is the curved space scalar Laplacian for the metric $g$

$$\Delta = g^{\mu\nu}\nabla_\mu\nabla_\nu = \frac{1}{\sqrt{|g|}}\left(\partial_\mu\sqrt{|g|}g^{\mu\nu}\partial_\nu\right). \tag{2.15}$$

Then the trace and the traceless part of the energy–momentum tensor read

$$T = -V + 4\pi\mu, \qquad \bar{T}_{\mu\nu} = \partial_\mu X_i\partial_\nu X_i - \frac{1}{2}g_{\mu\nu}(g^{\alpha\beta}\partial_\alpha X_i\partial_\beta X_i). \tag{2.16}$$

One can see that $\bar{T}_{\mu\nu}$ looks like the energy–momentum tensor of $N$ free scalar fields and one can use the usual strategy to solve the associated equation. First one uses the diffeomorphisms to write the metric in the flat conformal gauge

$$g_{\mu\nu} = \mathrm{e}^{2\sigma}\eta_{\mu\nu}. \tag{2.17}$$

Writing explicitly the components of $\bar{T}_{\mu\nu}$ one arrives at the equation

$$2(\bar{T}_{00} \pm \bar{T}_{01}) = (\partial_0 X_i \pm \partial_1 X_i)^2 = 0. \tag{2.18}$$

Since this is a sum of squares the only solution is the one where all terms vanish

$$(\partial_0 \pm \partial_1) X_i = 0 \Longrightarrow \partial_\mu X_i = 0 \Longrightarrow X_i = X_i^0 = \text{cst}. \tag{2.19}$$

In the conformal gauge the matter equation (2.14) and the trace equation give the following constraints on the values of $X_0^i$

$$\frac{\partial V}{\partial X_i}(X_i^0) = 0, \qquad V(X_i^0) = 4\pi\mu. \tag{2.20}$$

It can be noted that no equation allows to fix the value of $\sigma$.

The previous computation shows that unitary matter coupled to $2d$ gravity does not have any dynamics since the solutions of the equations of motion is given by the trivial solution $X_i = \text{cst}$ (with some constraints), even in the presence of interactions (if the potential does not contain the metric). The constraints provided by the vanishing of the trace are too strong and kill all the dynamics (this is not the case for matter without gravity, whether the backgroud is curved or not). Note that this argument does not apply to the $X^\mu$ fields of bosonic string theory since $X^0$ is a timelike boson.

Finally in order to give a specific example and to stress the difference between conformal and non-conformal matter, we consider free massive scalar fields

$$V(X_i) = \frac{1}{2} \sum_i m_i^2 X_i^2 \tag{2.21}$$

where some of the masses can vanish. Then (2.20) implies $X_i^0 = 0$ if $m_i = 0$. Introducing this into the trace equation gives $\mu = 0$ and there is no solution at all. Hence the behaviour is worse if one includes a cosmological constant and non-conformal matter fields since the system of equations is not even consistent, while for conformal matter there is at least the trivial solution $X_i = \text{cst}$ (with the obvious condition that one did not include the cosmological constant).

# Chapter 3

# Spectral theory and heat kernel

In this chapter we present some general results on spectral theory that will be useful for the computation of gravitational actions on Riemann surfaces. We will restrict ourselves to the two-dimensional case to get expressions that will be directly available for latter use. We will focus on operators of Laplace type or which can be derived from covariant derivatives. For now on, except if otherwise mentioned, we will deal with metrics of euclidean signature. We will restrict ourselves for the moment to the case where the Riemann surface is compact. We will see in chapter 5 how some results can be extended to the case of Riemann surfaces with boundaries.

In section 3.1 we first define the operators of Laplace type and give several examples. Then we give a brief review on complex calculus and define generalized Laplacians that can be obtained by the composition of two covariant derivatives. In section 3.2 we define the Green's function, the heat kernel and various types of zeta functions associated to an operator. In section 3.3 we show how to use these spectral functions to compute regularized versions of determinants of operators. This will be crucial for the rest of the thesis.

## 3.1 Laplacians on a Riemann surface

### 3.1.1 Geometric Laplacians and operators of Laplace type

Let $\mathcal{M}$ be a two-dimensional Riemann surface with metric $g$ and covariant derivative $\nabla$. The geometric Laplacian is defined by

$$\Delta = g^{\mu\nu}\nabla_\mu\nabla_\nu. \tag{3.1}$$

For example let's look at the cases that will be of interest for us where the metric is torsion free with an affine connexion which is then equal to the Christoffel symbol

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \tag{3.2}$$

and a spin connexion $\omega$. Then for a scalar field $\phi$, a spinor $\Psi$ and a vector $A^\mu$

$$\nabla_\mu\phi = \partial_\mu\phi \tag{3.3a}$$

$$\nabla_\mu\Psi = \partial_\mu\Psi + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\Psi \tag{3.3b}$$

$$\nabla_\nu A^\mu = \partial_\nu A^\mu + \Gamma^\mu_{\nu\rho}A^\rho. \tag{3.3c}$$

The scalar and spinorial Laplacian are then given by

$$\Delta_{\text{scalar}} = \frac{1}{\sqrt{g}}\partial_\mu \sqrt{g} g^{\mu\nu}\partial_\nu \tag{3.4a}$$

$$\Delta_{\text{spinor}}\Psi = g^{\mu\nu}\left(\left(\partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\right)\nabla_\nu \Psi - \Gamma^\rho_{\mu\nu}\nabla_\rho \Psi\right) = g^{\mu\nu}\partial_\mu\partial_\nu\Psi + v^\mu\partial_\mu\Psi + w\,\Psi \tag{3.4b}$$

with

$$v^\mu = \frac{1}{2}g^{\mu\nu}\omega_{\nu ab}\gamma^{ab} - g^{\sigma\tau}\Gamma^\mu_{\sigma\tau} \tag{3.5a}$$

$$w = \frac{1}{4}g^{\mu\nu}(\partial_\mu\omega_{\nu ab})\gamma^{ab} + \frac{1}{16}g^{\mu\nu}\omega_{\mu ab}\omega_{\nu cd}\gamma^{ab}\gamma^{cd} - \frac{1}{4}g^{\mu\nu}\Gamma^\rho_{\mu\nu}\omega_{\rho ab}\gamma^{ab}. \tag{3.5b}$$

An operator $D$ is said to be of Laplace type if it can be locally expressed as

$$D = -g^{\mu\nu}\partial_\mu\partial_\nu\Psi + a^\mu\partial_\mu\Psi + b. \tag{3.6}$$

Then there exist a unique connexion derived from a covariant derivative $\nabla$ [1] such that [124]

$$D = -\left(g^{\mu\nu}\nabla_\mu\nabla_\nu + Q(x)\right)\Psi. \tag{3.7}$$

It is a special case of elliptic operator. The geometric Laplacians (3.4) are then often defined with a minus sign to put them in this category. This also enables one to work with positive operators ((3.4a) and (3.4b) are negative operators). In the following, if not otherwise mentioned, $\Delta$ will denote the geometric Laplacians (3.4).

The spectrum of an operator of Laplace type is discrete and bounded from below and all its eigenspaces are finite dimensional [125]. We will denote by $\lambda_n$ with $\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_n \leq \ldots$ the eigenvalues of $D$ and by $\varphi_n$ an orthonormal basis of eigenfunctions associated to $\lambda_n$. They form a complete set of the Hilbert space which means that they obey the completeness relation

$$\sum_{n\geq 0}\varphi_n(x)\varphi_n^\dagger(y) = \frac{\delta(x-y)}{\sqrt{g}}. \tag{3.8}$$

The orthonormality condition reads

$$\int \mathrm{d}^2x\sqrt{g}\varphi_m^\dagger(x)\varphi_n(x) = \delta_{mn}. \tag{3.9}$$

### 3.1.2 Complex tensor calculus and generalized Laplacians

In chapter 4 we will encounter determinants of some geometric operators that map a rank $n$ tensor to another one of rank $n \pm 1$. To study them we need to introduce some new differential operators that are generalized Laplacians on Riemann surfaces [126–128]. We place ourselves in the flat conformal gauge where the metric is locally given by

$$\mathrm{d}s^2 = \mathrm{e}^{2\sigma(x)}\delta_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu. \tag{3.10}$$

We can go to complex coordinates

$$\begin{cases} z = x^0 + ix^1 \\ \bar{z} = x^0 - ix^1 \end{cases} \tag{3.11}$$

[1] A covariant derivative is a linear differential operator which maps $(k,l)$ tensors on $(k,l+1)$ tensor fields and which obeys Leibniz composition rule, reduces to partial derivative on scalars and commutes with contractions (for example $\nabla_\mu T^\nu{}_{\nu\rho} = (\nabla T)_\mu{}^\nu{}_{\nu\rho}$). This is achieved by taking the partial derivative and adding a linear correction term which compensates for the transformation of the latter and whose coefficients define the connexion coefficients.

where

$$\mathrm{d}s^2 = \mathrm{e}^{2\sigma(z,\bar{z})}\mathrm{d}z\mathrm{d}\bar{z} \tag{3.12}$$

so that

$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}\mathrm{e}^{2\sigma}, \quad g_{zz} = g_{\bar{z}\bar{z}} = 0. \tag{3.13}$$

The non-zero components of the inverse metric are given by

$$g^{z\bar{z}} = g^{\bar{z}z} = 2\mathrm{e}^{-2\sigma}. \tag{3.14}$$

The change of coordinates (3.11) yields a Jacobian in the integration measure:

$$\mathrm{d}z\mathrm{d}\bar{z} = 2\mathrm{d}x^0\mathrm{d}x^1. \tag{3.15}$$

To stay in the flat conformal gauge under a conformal transformation $z \to f(z)$ we need

$$\mathrm{e}^{2\sigma'(z',\bar{z}')} = \mathrm{e}^{2\sigma(z,\bar{z})}\frac{\partial z}{\partial z'} \tag{3.16}$$

and then $\sigma$ has to transform as

$$2\sigma'(z',\bar{z}') = 2\sigma(z,\bar{z}) + \ln\left|\frac{\partial z}{\partial z'}\right|. \tag{3.17}$$

A basis of forms and vectors is given by

$$\mathrm{d}z = \mathrm{d}x^0 + i\mathrm{d}x^1, \quad \mathrm{d}\bar{z} = \mathrm{d}x^0 - i\mathrm{d}x^1 \tag{3.18a}$$

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x^0} - i\frac{\partial}{\partial x^1}\right), \quad \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x^0} + i\frac{\partial}{\partial x^1}\right). \tag{3.18b}$$

Vector, forms and tensors can be given either in term of their components in the $(x^0, x^1)$ coordinates system or in the $(z, \bar{z})$ system. For example

$$V = V^0\frac{\partial}{\partial x^0} + V^1\frac{\partial}{\partial x^1} = V^z\partial + V^{\bar{z}}\bar{\partial} \tag{3.19}$$

where

$$V^z = V^1 + iV^2, \quad V^{\bar{z}} = V^1 - iV^2. \tag{3.20}$$

Indices are raised an lowered by contracting with the metric which amounts to transform an index $z$ in an index $\bar{z}$. Indeed

$$g_{z\bar{z}}\,\mathrm{d}\bar{z}\wedge\mathrm{d}z\left(V^z\partial + V^{\bar{z}}\bar{\partial}\right) = g_{z\bar{z}}V^z\mathrm{d}\bar{z} + g_{z\bar{z}}V^{\bar{z}}\mathrm{d}z \equiv V_{\bar{z}}\mathrm{d}\bar{z} + V_z\mathrm{d}z. \tag{3.21}$$

For example, $g_{z\bar{z}}T^{\bar{z}z} = T^z_z$. As the metric is anti-diagonal a tensor with indices $z$ and $\bar{z}$ can always be contracted with the metric to be expressed as a tensor with only $z$ indices. We can then restrict ourselves to tensors with $n_+$ upper $z$ indices and $n_-$ lower $z$ indices. Under an analytic change of coordinates such a tensor transforms as

$$T \to T' = \left(\frac{\partial z'}{\partial z}\right)^{n_+}\left(\frac{\partial z}{\partial z'}\right)^{n_-}T = \left(\frac{\partial z}{\partial z'}\right)^{n_- - n_+}T. \tag{3.22}$$

Transformation properties of tensors then only depend on the difference $n = n_- - n_+$ called the helicity. We can thus define a space $\mathcal{T}^n$ of tensors with $n$ indices:

$$\mathcal{T}^n = \left\{T\,|\,T \to \left(\frac{\partial z}{\partial z'}\right)^n T\right\}. \tag{3.23}$$

Moreover a tensor $T \in \mathcal{T}^n$ with upper and lower indices

$$T = T^{z\cdots z}{}_{z\cdots z}\,\partial^{n_+}\mathrm{d}z^{n_-} \tag{3.24}$$

can be rewritten, if $n \geq 0$, with only lower indices as

$$T = t_{z\dots z}\mathrm{d}z^n \tag{3.25}$$

and, if $n < 0$, with only upper indices as

$$T = t^{z\cdots z}\partial^n. \tag{3.26}$$

The covariant derivative of a tensor $T = t_{z\dots z}\mathrm{d}z^n$ of helicity $n > 0$ can also be decomposed (if $n < 0$ the argument still holds with $\mathrm{d}z$ replaced by $\partial$):

$$\nabla_\mu T\,\mathrm{d}x^\mu = \nabla_z t_{z\dots z}\mathrm{d}z^{n+1} + \nabla_{\bar z} t_{z\dots z}\mathrm{d}z^n\mathrm{d}\bar z. \tag{3.27}$$

We then define a first covariant derivative $\nabla^n_z T : \mathcal{T}^n \to \mathcal{T}^{n+1}$ by

$$\begin{aligned}\nabla^n_z t_{z\dots z} \equiv \nabla_z t_{z\dots z} &= (\partial - n\Gamma^z_{zz})\, t_{z\dots z}\\ &= (\partial - 2n\,\partial\sigma)\, t_{z\dots z}\\ &= (g_{z\bar z})^n \partial[(g^{z\bar z})^n t_{z\dots z}].\end{aligned} \tag{3.28}$$

In the same way we define a second covariant derivative $\nabla^z_n T : \mathcal{T}^n \to \mathcal{T}^{n-1}$ by

$$\nabla^z_n t_{z\dots z} \equiv g^{z\bar z}\nabla_{\bar z} t_{z\dots z} = g^{z\bar z}\bar\partial t_{z\dots z} \tag{3.29}$$

since the only non-zero Christoffel symbols are $\Gamma^z_{zz} = 2\partial\sigma$ and $\Gamma^{\bar z}_{\bar z\bar z} = 2\bar\partial\sigma$. The scalar curvature is then given by

$$R = -2\,\mathrm{e}^{-2\sigma}\Delta\sigma. \tag{3.30}$$

A natural scalar product on $\mathcal{T}^n$ is given by

$$\langle T|S\rangle_n = \int \mathrm{d}^2z\sqrt{g}\,(g^{z\bar z})^n T^* S. \tag{3.31}$$

Taking the scalar product of $T \in T^{n+1}$ and $\nabla^z_n S$ with $S \in \mathcal{T}^n$, one gets the adjoint of $\nabla^z_n$:

$$(\nabla^z_n)^\dagger = -\nabla^{n-1}_z. \tag{3.32}$$

We can now define two Laplacians on the spaces $\mathcal{T}^n$ :

$$\Delta^+_n = -2\nabla^z_{n+1}\nabla^n_z = -4\mathrm{e}^{-2\sigma}(\partial\bar\partial - 2n(\partial\sigma)\bar\partial - 2n(\partial\bar\partial\sigma)) \tag{3.33a}$$

$$\Delta^-_n = -2\nabla^{n-1}_z\nabla^z_n = -4\mathrm{e}^{-2\sigma}(\partial\bar\partial - 2n(\partial\sigma)\bar\partial). \tag{3.33b}$$

In flat space we recover the geometric Laplacian with minus sign $-\Delta = -4\partial\bar\partial$. Moreover the scalar Laplacian ($n = 0$) is given by $\Delta_{\text{scalar}} = -\Delta^+_0 = -\Delta^-_0$. These are Laplace type operator: for $\Delta^-_n$, one has

$$a^0 = -4n\mathrm{e}^{-2\sigma}\,\partial\sigma, \quad a^1 = -4i\,n\mathrm{e}^{-2\sigma}\,\partial\sigma, \quad b = 0 \tag{3.34}$$

which implies [124]

$$Q = \frac{n}{2}R \tag{3.35}$$

while for $\Delta^+_n$ one has

$$a^0 = -4n\mathrm{e}^{-2\sigma}\,\partial\sigma, \quad a^1 = -4i\,n\mathrm{e}^{-2\sigma}\,\partial\sigma, \quad b = 2n\partial\bar\partial\sigma \tag{3.36}$$

which implies

$$Q = -\frac{n}{2}R. \tag{3.37}$$

Let $\mathcal{S}^n = \mathcal{T}^n \oplus \mathcal{T}^{-n}$ be the space of traceless symmetric tensors of rank $n$. Covariant derivatives act on this space as

$$\begin{cases} P_n = \nabla^n_z \oplus \nabla^z_{-n} : & \mathcal{S}^n \to \mathcal{S}^{n+1} \\ P_n^\dagger = -(\nabla^z_{n+1} \oplus \nabla_z^{-n-1}) : & \mathcal{S}^{n+1} \to \mathcal{S}^n. \end{cases} \tag{3.38}$$

In particular,

$$P_n^\dagger P_n = \frac{1}{2}(\Delta_n^+ \oplus \Delta^-_{-n}). \tag{3.39}$$

In the next chapter we will encounter the operator $P_1$ which maps vectors on traceless symmetric tensors of rank 2.

Under such an infinitesimal change $\delta\sigma$ of $\sigma$ the covariant derivatives (3.28) and (3.29) transform as

$$\delta\nabla^z_n = -2\delta\sigma\nabla^z_n \tag{3.40a}$$
$$\delta\nabla^n_z = 2n\delta\sigma\nabla^n_z - 2n\nabla^n_z\delta\sigma. \tag{3.40b}$$

This induces the transformations of the Laplacians (3.33);

$$\delta\Delta^-_n = 2(n-1)\delta\sigma\Delta^-_n + 4n\nabla_z^{n-1}\delta\sigma\nabla^z_n \tag{3.41a}$$
$$\delta\Delta^+_n = -2\delta\sigma\Delta^+_n - 4n\nabla^z_{n+1}\delta\sigma\nabla^n_z - 2n\Delta^+_n\delta\sigma. \tag{3.41b}$$

## 3.2 Spectral functions

### 3.2.1 Green's functions

The Green's function of $D$ is the inverse of $D$ in a distributional sense. To define it we have to distinguish between the case where $D$ has zero-modes (i.e. if some $\lambda_i$ are zero) and the case where it does not. If the 0 eigenvalue is degenerate, we write $\lambda_0 \equiv \lambda_{0,i} = 0$ for all of them, and we renumber the other ones such that the first non-zero eigenvalue is $\lambda_1$. Then we index the zero-modes by a new index.

If $D$ does not have zero-modes its Green's function $G$ is the solution of

$$DG(x,y) = \frac{\delta(x-y)}{\sqrt{g}}. \tag{3.42}$$

The solution of $Df(x) = u(x)$ will then be given by the convolution of $u$ by $G$:

$$f(x) = \int \mathrm{d}^2y\sqrt{g}\, G(x,y)u(y). \tag{3.43}$$

The Green's function can be expressed in terms of the eigenfunctions and eigenvalues of $D$ as

$$G(x,y) = \sum_{n\geq 0} \frac{\varphi_n(x)\varphi_n(y)^\dagger}{\lambda_n}. \tag{3.44}$$

If $D$ has zero-modes $\varphi_{0,i}$ (associated to $\lambda_0 = 0$), it is not invertible and one cannot define a Green's function like in (3.42). However we can exclude them from the sum in (3.44) and define

$$\tilde{G}(x,y) = \sum_{n>0} \frac{\varphi_n(x)\varphi_n(y)}{\lambda_n}. \tag{3.45}$$

which satisfies

$$D\tilde{G}(x,y) = \frac{\delta(x-y)}{\sqrt{g}} - \sum_i \varphi_{0,i}\,\varphi^\dagger_{0,i}. \tag{3.46}$$

For example, $D = -\Delta_{\text{scalar}}$ has an unique zero-mode $\varphi_0 = \frac{1}{\sqrt{A}}$ where $A$ is the area of $\mathcal{M}$.

From now on we will put a tilde on all quantities from which the zero-modes have been excluded.

It is well-known that the Green's function of an operator of Laplace type presents a singularity when $y$ goes to $x$. In two dimensions this singularity is given by [6, 129]

$$G_{\text{sing}} = -\frac{1}{4\pi} \ln(\ell(x,y))^2 \tag{3.47}$$

where $\ell(x,y)$ is the geodesic distance between $x$ and $y$. Indeed one has

$$g^{\mu\nu}\partial_\mu\partial_\nu \ln(\ell(x,y))^2 = 4\pi \frac{\delta(x-y)}{\sqrt{g}}. \tag{3.48}$$

If we let $G(x,y) = -\frac{1}{4\pi}\ln(\ell(x,y))^2 + f(x,y)$ we see that $f$ satisfies $-g^{\mu\nu}\partial_\mu\partial_\nu f(x,y) + a^\mu\partial_\mu f(x,y) + bf(x,y) = 0$ and is free from singularity. The same argument works of course for $\tilde{G}$ if we want to exclude the zero-modes. Note that this short-distance singularity remains the same if we subtract any finite number of modes. Indeed, the singularity is related to the large $n$ behaviour of the eigenvalues and eigenvectors.

### 3.2.2 The heat kernel

The heat kernel $K(x,y,t)$ for the operator $D$ is the solution of

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + D\right) K(t,x,y) = 0, \quad K(t,x,y) \sim \frac{\delta(x-y)}{\sqrt{g}} \text{ as } t \to 0. \tag{3.49}$$

In terms of the eigenfunctions and eigenvalues of $D$ it can be expressed as

$$K(t,x,y) = \sum_{n\geq 0} \mathrm{e}^{-\lambda_n t}\, \varphi_n(x)\varphi_n^\dagger(y). \tag{3.50}$$

If $D$ has zero-modes the solution of (3.49) is of course $\tilde{K}$ where the zero-modes have been excluded from the sum. If not precised, when talking about the heat kernel we will think of the solution of (3.49) and, if using the form (3.50) we will always suppose that the possible zero-modes have been excluded.

One defines the integrated heat kernel $K(t)$ by

$$K(t) = \int \mathrm{d}^2x\sqrt{g}\,\mathrm{tr}\,K(t,x,x). \tag{3.51}$$

It can be expressed as

$$K(t) = \sum_{n\geq 0} \mathrm{e}^{-\lambda_n t} = \mathrm{tr}\,\mathrm{e}^{-tD}. \tag{3.52}$$

As is clear from (3.50), for $t > 0$, $K(t,x,y)$ is given by a converging sum and is finite, even as $x \to y$. For $t \to 0$ one recovers various divergences, and, in particular

$$\int_0^\infty \mathrm{d}t\, K(t,x,y) = G(x,y) \tag{3.53}$$

exhibits the short distance singularity of the Green's function which we have shown to be logarithmic in two dimensions (3.47). In the following we will study the asymptotic behaviour of $K$ for $t$ going to 0.

The behaviour of $K$ for small $t$ is related to the asymptotics of the eigenvalues $\lambda_n$ and eigenfunctions $\varphi_n$ for large $n$, which in turn is related to the short-distance properties of

the Riemann surface. The small-$t$ asymptotics is given in terms of local expressions of the curvature and its derivatives. On a compact manifold without boundaries one has [130]

$$K(t,x,y) \sim \frac{1}{4\pi t} \mathrm{e}^{-\ell^2(x,y)/4t} \sum_{k\geq 0} t^k a_k(x,y). \tag{3.54}$$

For small $t$, the exponential forces $\ell^2$ to be small (of order $t$) and one can use normal coordinates around $x$. The expansion coefficients can then be computed recursively [124, 125, 131, 132]. Of major interest for us will be the expression of the first ones at coinciding points:

$$a_0(x,x) = \mathrm{Id} \tag{3.55}$$

$$a_1(x,x) = \frac{R(x)}{6} - Q \tag{3.56}$$

where $R$ is the scalar curvature. For example, for the scalar Laplacian one has the well-known expansion

$$K_{\mathrm{scalar}}(t,x,x) = \frac{1}{4\pi t} + \frac{R}{24\pi} + o(t). \tag{3.57}$$

For the generalized Laplacians we find from (3.35) and (3.37)

$$K_n^{\pm}(t,z,z) = \frac{1}{4\pi t} + \frac{1 \pm 3n}{24\pi} R + o(t). \tag{3.58}$$

In particular, as the Gauss-Bonnet theorem states that $\int \mathrm{d}^2x\sqrt{g}\, R = 4\pi\chi$ we finally get

$$K_n^{\pm}(t) = \frac{A}{4\pi t} + \frac{1 \pm 3n}{6}\chi + \mathcal{O}(t). \tag{3.59}$$

As we will see in chapter 5 if the manifold has boundaries the small-$t$ expansion of the heat kernel involves also half integer powers of $t$.

### 3.2.3 Zeta function

The zeta function for the operator $D$ is defined by

$$\zeta(s) = \sum_n \frac{1}{\lambda_n^s}. \tag{3.60}$$

A local version of the zeta function is given by

$$\zeta(s,x,y) = \sum_n \frac{\varphi_n(x)\varphi_n^\dagger(y)}{\lambda_n^s}. \tag{3.61}$$

It is related to the heat kernel by

$$\zeta(s,x,y) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \mathrm{d}t\, t^{s-1} K(t,x,y) \tag{3.62}$$

and

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \mathrm{d}t\, t^{s-1} K(t). \tag{3.63}$$

For the following we will need the values of $\zeta(s)$ and $\zeta(s,x,x)$ for $s = 0$ and their behaviour when $s$ goes to 1.

For $\mathrm{Re}(s) < 1$, the problem of convergence of the integral comes from the region where $t$ is small. Let

$$f_{t_0}(s,x,y) = \int_0^{t_0} \mathrm{d}t\, t^{s-1} K(t,x,y), \quad g_{t_0}(s,x,y) = \int_{t_0}^{+\infty} \mathrm{d}t\, t^{s-1} K(t,x,y) \tag{3.64}$$

with $t_0 > 0$. Then $g_{t_0}$ is an analytic function in the complex plane provided that there is no zero eigenvalue. For small $t_0$ we can use the small $t$ expansion of the heat kernel to study $f_{t_0}$:

$$f_{t_0}(s,x,y) = \int_0^{t_0} \mathrm{d}t\, t^{s-1} \frac{1}{4\pi t} \sum_{k=0}^{\infty} t^k a_k(x,y) = \frac{1}{4\pi} \sum_{k=0}^{\infty} \frac{t_0^{k+s-1}}{k+s-1} a_k(x,y). \tag{3.65}$$

This is an analytic function in $s$ except for $s = 1-k, k \in \mathbb{N}$ where it has poles. One has

$$\zeta(s,x,y) = \frac{1}{\Gamma(s)} \left[ \frac{1}{4\pi} \sum_{k=0}^{\infty} \frac{t_0^{k+s-1}}{k+s-1} a_k(x,y) + g_{t_0}(s,x,y) \right]. \tag{3.66}$$

For small $s$,

$$\Gamma(s) = \frac{1}{s} + \gamma + o(s) \tag{3.67}$$

and

$$\begin{aligned} \zeta(s,x,y) &= \frac{1}{\frac{1}{s} + \gamma + o(s)} \left[ \frac{1}{4\pi} \frac{t_0^{s-1}}{s-1} a_0(x,y) + \frac{1}{4\pi} \frac{t_0^s}{s} a_1(x,y) \right. \\ &\qquad \left. + \frac{1}{4\pi} \sum_{k=2}^{\infty} \frac{t_0^{k+s-1}}{k+s-1} a_k(x,y) + g_{t_0}(s,x,y) \right] \\ &= \frac{1}{4\pi} \frac{t_0^s}{1+\gamma s + o(s^2)} a_1(x,y) + \frac{s}{1+\gamma s + o(s^2)} h_{t_0}(s,x,y) \\ &\xrightarrow[s\to 0]{} \frac{1}{4\pi} a_1(x,y) \end{aligned} \tag{3.68}$$

where

$$h_{t_0}(s,x,y) = g_{t_0}(s,x,y) + \frac{1}{4\pi} \frac{t_0^{s-1}}{s-1} a_0(x,y) + \frac{1}{4\pi} \sum_{k=2}^{\infty} \frac{t_0^{k+s-1}}{k+s-1} a_k(x,y) \tag{3.69}$$

is an analytic function around $s = 0$. Then one can define

$$\zeta(0,x,y) = \frac{1}{4\pi} a_1(x,y). \tag{3.70}$$

In particular

$$\zeta(0) = \frac{1}{4\pi} \int \mathrm{d}^2x \sqrt{g}\, \mathrm{tr} \left( Q(x) - \frac{R(x)}{6} \right) \tag{3.71}$$

In the same way, we see that $\zeta(s,x,y)$ has a pole when $s = 1$ with residue $\frac{a_0(x,y)}{4\pi}$. The residue of $\zeta(s,x,x)$ at $s = 1$ is then $\frac{\mathrm{Id}}{4\pi}$.

In the presence of zero-modes these results are still valid if we consider the function $\tilde{\zeta}$ where the zero-modes have been excluded.

## 3.3 Determinant regularization

### 3.3.1 The path integral formalism

As we will see in chapter 4 the computation of gravitational actions involve the computation of determinants of various operators. Indeed one is interested in the partition function which is given in the functional integral formalism by

$$Z[g] = \int \mathcal{D}\phi\, \mathrm{e}^{-S[g,\phi]} \tag{3.72}$$

where $\phi$ denotes some matter field whose action is given by

$$S[g,\phi] = \int \mathrm{d}^2x\,\sqrt{g}\,\phi^\dagger(x) D\phi(x) \tag{3.73}$$

$D$ being a second order differential operator. For simplicity we will discuss here the case where $\phi$ is a real scalar field. It can be easily generalized for a complex field in which case we also have to integrate over $\phi^\dagger$. The following discussion can also be extended to other kinds of fields by making the necessary changes in the dimensions.

For $S$ to be dimensionless $\phi$ must be also dimensionless (in 2 dimensions). It can be decomposed on the basis of eigenvalues of $D$:

$$\phi(x) = \sum_n a_n \varphi_n(x). \tag{3.74}$$

The normalization condition $\int \mathrm{d}^2x\sqrt{g}\varphi_n^\dagger(x)\varphi_n(x) = 1$ ((3.9)) implies that the modes $\varphi_n$ scales as $\frac{1}{\sqrt{A}}$. This in turn shows that the $a_n$ have the dimension of a length. In the functional integral formalism the integral $\phi$ is defined by a product on integrals on the real numbers $a_n$. For the measure to be dimensionless we have to introduce some energy scale $\mu$ and define $\mathcal{D}\phi = \prod_n \mathrm{d}(\mu a_n)$. Finally (3.72) becomes

$$Z[g] = \int \prod_n \mathrm{d}(\mu a_n)\, \mathrm{e}^{-\sum_n \lambda_n a_n^\dagger a_n}. \tag{3.75}$$

Note that as $D = -g^{\mu\nu}\partial_\mu\partial_\nu + \ldots$ the $\lambda_n$ have the dimension of an inverse length squared and the argument of the exponential is dimensionless.

If $D$ does not have zero-modes the gaussian integrals can be computed and we obtain (formally)

$$Z[g] = \left(\prod_n \frac{\lambda_n}{\mu^2}\right) = \left(\det \frac{D}{\mu^2}\right)^{-\frac{1}{2}}. \tag{3.76}$$

Of course this determinant is infinite and has to be regularized as we will see in the next two sections.

If $D$ has zero-modes $\varphi_{0,i}$ (3.75) becomes

$$Z[g] = \int \prod_{i=1}^N \mathrm{d}(\mu a_{0,i}) \int \prod_{n\geq 0} \mathrm{d}(\mu a_n)\, \mathrm{e}^{-\sum_{n>0} \lambda_n a_n^\dagger a_n}. \tag{3.77}$$

The integration over the zero-modes yields an infinite contribution that can be absorbed in the normalization and we get

$$Z[g] = \Omega[g] \left(\det{}' \frac{D}{\mu^2}\right)^{-\frac{1}{2}}. \tag{3.78}$$

The coefficient $\Omega$ is then an infinite constant. However it can depend on the metric (since the functional measure may depend on it) and one has to study this dependency when computing the variation of the partition function with the metric.

Determinants arise also as Jacobians when changing variables in the path integral. We will encounter this situation in chapter 4 when dealing with the sum over the metrics.

### 3.3.2 Zeta function regularization

In the next chapters we will be interested in the effective action defined by

$$S_{\text{eff}} = -\ln Z. \tag{3.79}$$

As we have seen previously this is proportional to $\ln\det\frac{D}{\mu^2}$ where the energy scale $\mu$ has been introduced for dimensional reasons. It also ensures that the argument of the logarithm is dimensionless (since the $\lambda_n$ scale as an energy square). We can then write (formally)

$$S_{\text{eff}} \propto \ln\det\frac{D}{\mu^2} = \sum_{n>0}\ln\frac{\lambda_n}{\mu^2}. \tag{3.80}$$

To regularize the infinite sum we can do some analytic continuation and write:

$$\sum_{n>0}\ln\frac{\lambda_n}{\mu^2} = -\zeta'(0) - \ln\mu^2\zeta(0). \tag{3.81}$$

As we have shown above $\zeta(0)$ and $\zeta'(0)$ have well-defined finite values. A regularization of the logarithm of the determinant of $D$ is then

$$\ln\det{}_\zeta\frac{D}{\mu^2} = -\zeta'(0) - \ln\mu^2\zeta(0). \tag{3.82}$$

Of course if there are zero-modes they have to be excluded and one actually computes $\det'$.

An alternative derivation of (3.82) based on renormalization can be found in [124]. The link with the usual cutoff regularization is developed in [125, 131, 132].

### 3.3.3 Heat kernel short time regularization procedure

The small $t$ expansion of the heat kernel can also be used directly to regularize the determinant [124, 126, 127]. For that we notice that for $\lambda > 0$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(-\int_\varepsilon^{+\infty}\frac{\mathrm{d}t}{t}\mathrm{e}^{-t\lambda}\right) = \frac{\mathrm{e}^{-\varepsilon\lambda}}{\lambda} \xrightarrow[\varepsilon\to 0]{} \frac{1}{\lambda} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\ln\lambda. \tag{3.83}$$

A regularized version of the logarithm of the determinant of a positive operator $\ln\det' D = \sum_{n>0}\ln\lambda_n$ is then given by

$$\ln\det{}'_\varepsilon D = -\int_\varepsilon^{+\infty}\frac{\mathrm{d}t}{t}\tilde{K}(t). \tag{3.84}$$

This scheme of regularization will be particularly useful in chapter 4 when we will compute the variation of the logarithm of the determinants of the operators $\Delta_n^\pm$.

We now have two ways to regularize $\ln\det\Delta_n^\pm$. Although the two schemes seem to yield different results they are in fact equivalent. Indeed one has

$$\Gamma(s)\zeta_n^\pm(s) = \int_0^\varepsilon \mathrm{d}t\, t^{s-1}K_n^\pm(t) + \int_\epsilon^\infty \mathrm{d}t\, t^{s-1}K_n^\pm(t). \tag{3.85}$$

The small $t$ expansion gives

$$\begin{aligned}\int_0^\varepsilon \mathrm{d}t\, t^{s-1}K_n^\pm(t) &= \frac{A}{2\pi}\frac{\varepsilon^{s-1}}{s-1} + \frac{1\pm 6n}{12}\chi\frac{\varepsilon^s}{s}\\ &= \frac{A}{2\pi(s-1)} + \frac{A}{2\pi}\ln\varepsilon + \frac{1\pm 6n}{12s}\chi + \frac{1\pm 6n}{12}\chi\ln\varepsilon + \mathcal{O}(\varepsilon).\end{aligned}$$

However for small $s$ one has

$$\Gamma(s)\zeta_n^\pm(s) = \frac{1}{s}\zeta_n^\pm(0) + \gamma\zeta_n^\pm(0) + \zeta_n'^\pm(0) + o(s) \tag{3.86}$$

and then

$$-\ln \det{}'_\zeta \frac{\Delta_n^\pm}{\mu^2} = -\ln \det{}'_\varepsilon \Delta_n^\pm + \ln \mu^2 \zeta_n^\pm(0) - \frac{1}{s}\zeta_n^\pm(0) - \gamma \zeta_n^\pm(0) + \frac{A}{2\pi(s-1)} + \frac{A}{2\pi}\ln\varepsilon + \frac{1\pm 6n}{12s}\chi + \frac{1\pm 6n}{12}\chi \ln\varepsilon + \mathcal{O}(\varepsilon) + o(s). \tag{3.87}$$

Note that in $\det_\varepsilon \Delta_n^\pm$ there is an implicit energy scale as $\varepsilon$ has the dimension of an inverse energy square as we can see from (3.83).

Moreover (3.71) shows that $\zeta(0)$ is a local expression. Furthermore as $A = \int \mathrm{d}^2x\sqrt{g}$ and $4\pi\chi = \int \mathrm{d}^2x\sqrt{g}\,R$, terms proportional to $A$ and $\chi$ are local expressions. Up to local counterterms both regularizations give the same results and are then equivalent.

### 3.3.4 Variation of determinants of complex Laplacians

In the next chapter we will need the variation of the complex Laplacian (3.33) under a change in the conformal factor $\sigma$.

The simplest method is to used the regularized version of the determinant of $\Delta_n^\pm$ is given by (3.84):

$$\ln \det{}'_\varepsilon \Delta_n^\pm = -\int_\varepsilon^{+\infty} \frac{\mathrm{d}t}{t} \tilde{K}_n^\pm(t). \tag{3.88}$$

As $\tilde{K}_n^\pm(t) = \operatorname{tr} \mathrm{e}^{-t\Delta_n^\pm} - N_n^\pm$, where $N_n^\pm = \dim \operatorname{Ker} \Delta_n^\pm$ [2], the variation of $\ln \det{}'_\varepsilon \Delta_n^\pm$ under an infinitesimal Weyl transformation $\delta\sigma$ is given by

$$\delta \ln \det{}'_\varepsilon \Delta_n^\pm = \int_\varepsilon^\infty \mathrm{d}t \operatorname{tr}(\delta\Delta_n^\pm \mathrm{e}^{-t\Delta_n^\pm} - \delta N_n^\pm). \tag{3.89}$$

To go further we need to look at the zero-modes of $\Delta_n^\pm$:

- If $\Delta_n^- \phi = 0$, either $\phi \in \operatorname{Ker} \nabla_n^z$ or $\nabla_n^z \phi \in \operatorname{Ker} \nabla_z^{n-1}$. But since $(\nabla_n^z)^\dagger = -\nabla_z^{n-1}$ we have
$$\operatorname{Ker} \nabla_z^{n-1} = \operatorname{Ker}(\nabla_n^z)^\dagger = (\operatorname{Im} \nabla_n^z)^\perp \tag{3.90}$$
so that $\operatorname{Ker} \nabla_n^z \cap \operatorname{Ker} \nabla_z^{n-1} = \{0\}$ and then
$$\operatorname{Ker} \Delta_n^- = \operatorname{Ker} \nabla_n^z. \tag{3.91}$$
The transformation rule (3.40a) then shows that $\delta N_n^- = 0$.

- The same argument shows that
$$\operatorname{Ker} \Delta_n^+ = \operatorname{Ker} \nabla_z^n. \tag{3.92}$$
Let $\psi \in \operatorname{Ker} \nabla_z^n$. Then the transformation rule (3.40b) implies $(\nabla_z^n + \delta\nabla_z^n)(\psi + 2n\delta\sigma\psi) = 0$. This shows that $\delta N_n^+ = 0$.

Then using the transformation rules (3.41) and the identity $\mathrm{e}^{-AB}A = A\mathrm{e}^{-BA}$ we get

$$\begin{aligned}\delta \ln \det{}'_\varepsilon \Delta_n^- &= -2(n-1)\left[\operatorname{tr} \delta\sigma\, \mathrm{e}^{-t\Delta_n^-}\right]_\epsilon^\infty + 2n\left[\operatorname{tr} \delta\sigma\, \mathrm{e}^{-t\Delta_{n-1}^+}\right]_\varepsilon^\infty \\ &= -2(n-1)\int \mathrm{d}z\mathrm{d}\bar{z}\sqrt{g}\,\delta\sigma\left[K_n^-(t,z,z)\right]_\epsilon^\infty + 2n\int \mathrm{d}z\mathrm{d}\bar{z}\sqrt{g}\,\delta\sigma\left[K_{n-1}^+(t,z,z)\right]_\varepsilon^\infty.\end{aligned} \tag{3.93}$$

[2] One can show using Riemann-Roch theorem that $\operatorname{Ker} \Delta_n^\pm$ are finite-dimensional [128].

When $t$ goes to infinity only the zero-modes contribute to the heat kernel as the others are exponentially suppressed. Let $\phi_i$, $i = 1, \dots, N_n^-$, be an orthonormal basis of $\operatorname{Ker} \nabla_n^z = \operatorname{Ker} \Delta_n^-$. Then

$$\lim_{t\to\infty} K_n^-(t,z,z) = \sum_{j=1}^{N_n^-} \phi_j(z)^* \phi_j(z) \tag{3.94}$$

so that

$$\int \mathrm{d}z\mathrm{d}\bar{z}\sqrt{g} \lim_{t\to\infty} \delta\sigma K_n^-(t,z,z) = \sum_{j=1}^{N_n^-} \int \mathrm{d}z\mathrm{d}\bar{z}\sqrt{g}\,\delta\sigma(z)\phi_j(z)^*\phi_j(z) \equiv \sum_{j=1}^{N_n^-} \langle \phi_j|\delta\sigma|\phi_j\rangle. \tag{3.95}$$

Moreover form (3.40a) we see that the $\phi_i$ are still zero-modes of $\nabla_n^z$ after a Weyl transformation $\delta\sigma$. However they do not necessary remain orthonormal since the scalar product in $\mathcal{T}^n$ (3.31) transforms as

$$\delta\langle T|S\rangle = 2(1-n)\langle T|\delta\sigma|S\rangle. \tag{3.96}$$

We then have

$$\delta \ln\det\langle\phi_j|\phi_k\rangle = \delta\,\mathrm{tr}\langle\phi_j|\phi_k\rangle = 2(1-n)\sum_{j=1}^{N_n^-}\langle\phi_j|\delta\sigma|\phi_k\rangle \tag{3.97}$$

so that

$$\big[K_n^-(t,z,z)\big]_\epsilon^\infty = \frac{\delta \ln\det\langle\phi_j|\phi_k\rangle}{2(1-n)} - K_n^-(\varepsilon,z,z). \tag{3.98}$$

Consider now an orthonormal basis $\psi_\alpha$, $\alpha = 1, \dots, N_{n-1}^+$, of $\operatorname{Ker}\Delta_{n-1}^+ = \operatorname{Ker}\nabla_z^{n-1}$. We have seen previously that under a Weyl transformation of parameter $\delta\sigma$ the variation of the $\psi_\alpha$ is given by $\delta\psi_\alpha = 2n\delta\sigma\psi_\alpha$. Combining this with is the transformation of the scalar product in $\mathcal{T}^{n-1}$ we get:

$$\delta \ln\det\langle\psi_\alpha|\psi_\beta\rangle = 2n\sum_{j=1}^{N_{n-1}^+}\langle\psi_\alpha|\delta\sigma|\psi_\beta\rangle. \tag{3.99}$$

Finally we arrive at

$$\begin{aligned}
\delta \ln \frac{\det'_\varepsilon \Delta_n^-}{\det\langle\phi_j|\phi_k\rangle \det\langle\psi_\alpha|\psi_\beta\rangle} &= 2(n-1)\int \mathrm{d}z\mathrm{d}\bar{z}\sqrt{g}\,\delta\sigma K_n^-(\varepsilon,z,z) \\
&\quad - 2n\int \mathrm{d}z\mathrm{d}\bar{z}\sqrt{g}\,\delta\sigma K_{n-1}^+(\varepsilon,z,z) \\
&= -\frac{1}{\pi\varepsilon}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma + \frac{-6n^2+6n-1}{12\pi}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma\, R \\
&= -\frac{1}{\pi\varepsilon}\int \mathrm{d}^2x\sqrt{\hat{g}}\,\mathrm{e}^{2\sigma}\delta\sigma + \frac{-6n^2+6n-1}{12\pi}\int \mathrm{d}^2x\sqrt{\hat{g}}\,\Big(\hat{R} - 2\hat{\Delta}\sigma\Big)\,\delta\sigma
\end{aligned} \tag{3.100}$$

and then

$$\begin{aligned}
\ln \frac{\det'_\varepsilon \Delta_n^-}{\det\langle\phi_j|\phi_k\rangle \det\langle\psi_\alpha|\psi_\beta\rangle} &= \ln \frac{\det'_\varepsilon \hat{\Delta}_n^-}{\det\langle\hat{\phi}_j|\hat{\phi}_k\rangle \det\langle\hat{\psi}_\alpha|\hat{\psi}_\beta\rangle} - \frac{1}{\pi\epsilon}\int \mathrm{d}^2x\sqrt{\hat{g}}\,\mathrm{e}^{2\sigma} \\
&\quad - \frac{6n^2-6n+1}{12\pi}\int \mathrm{d}^2x\sqrt{\hat{g}}\,\Big(\hat{R}\sigma - \sigma\hat{\Delta}\sigma\Big)
\end{aligned} \tag{3.101}$$

where the factor of 2 in front of $\Delta$ has been absorbed through integration by part.

We can do similar computations with $\Delta_n^+$ so that finally

$$\frac{\det{}'\Delta_n^\pm}{\det\langle\phi_j|\phi_k\rangle\det\langle\psi_\alpha|\psi_\beta\rangle} = \frac{\det{}'\hat{\Delta}_n^\pm}{\det\langle\hat{\phi}_j|\hat{\phi}_k\rangle\det\langle\hat{\psi}_\alpha|\hat{\psi}_\beta\rangle}\,\mathrm{e}^{-\frac{6n^2\pm 6n+1}{3}S_L} \tag{3.102}$$

where $S_L$ is the Liouville action with cosmological constant:

$$S_L = \frac{1}{4\pi}\int \mathrm{d}^2x\,\sqrt{\hat{g}}\,\left(-\sigma\hat{\Delta}\sigma + \hat{R}\,\sigma\right) + \mu\int \mathrm{d}^2x\,\sqrt{\hat{g}}\,\mathrm{e}^{2\sigma}. \tag{3.103}$$

The cosmological constant $\mu = -\frac{1}{\pi\varepsilon}$ is infinite and will be absorbed through renormalization. This shows that the cosmological constant always appear in the quantum theory (even if the matter is conformal).

In the case where $n = 0$ (3.93) gives

$$\delta\ln\det{}'_\varepsilon\Delta = 2\int \mathrm{d}z\mathrm{d}\bar{z}\sqrt{g}\,\delta\sigma\left[K_0^-(t,z,z)\right]_\varepsilon^{+\infty}. \tag{3.104}$$

On compact Riemann surfaces the scalar Laplacian has always one an only one zero-mode $\varphi = \frac{1}{\sqrt{A}}$. Then we get

$$\delta\ln\det{}'_\varepsilon\Delta = \frac{\delta A}{A} - \frac{1}{\pi\varepsilon}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma - \frac{1}{6\pi}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma R. \tag{3.105}$$

Finally we get the well-known result:

$$\frac{\det{}'\Delta}{A} = \frac{\det{}'\hat{\Delta}}{\hat{A}}\,\mathrm{e}^{-\frac{1}{3}S_L} \tag{3.106}$$

which yields the gravitational action for the scalar field (see chapter 4)

$$S_{\mathrm{grav}} = -\frac{1}{6}S_L. \tag{3.107}$$

# Chapter 4

# Two-dimensional quantum gravity

We have seen in chapter 2 that at the classical level two-dimensional gravity is trivial. However this is no more the case at the quantum level where the coupling to the matter fields makes the quantum fluctuations of the metric dynamical. We will work in Euclidean signature and study the coupling of some matter $\Psi$ to gravity. This is described by the action

$$S[g, \Psi] = S_\mu[g] + S_m[g, \Psi] \tag{4.1}$$

where $S_\mu$ is the cosmological constant action (2.5) and $S_m$ is the matter action which depends on the considered model (we reduce the action for gravity $S_{\text{cl}}$ to $S_\mu$ as the Einstein–Hilbert action $S_{\text{EH}}$ is just a topological term proportional to the Euler characteristic). The partition function is then defined by

$$Z = \int \mathcal{D}g\, \mathcal{D}\Psi\, \mathrm{e}^{-S[g,\Psi]}. \tag{4.2}$$

To give a sense to this expression the integration measures $\mathcal{D}g$ and $\mathcal{D}\Psi$ have to be defined properly. Moreover to define the path integral we have to deal with the huge gauge invariance of the action under diffeomorphisms. For that we will have to implement the Faddeev–Popov procedure. We will see how to do this in 4.1. In this section we will also define the gravitational action and see how it can be computed. In 4.2 the well known case where the matter is conformal will be reviewed as an example.

In 4.3 we will introduce the Kähler formalism that will be needed in the next chapters. In 4.3.2 we will discuss the most remarkable functionals that can appear in the gravitational action, in particular the Liouville and Mabuchi actions that will be studied in some detail in the subsequent sections 4.4 and 4.5.

## 4.1 Functional integral

In this section we will review the construction of the partition function (4.2) given in [126, 127, 133].

### 4.1.1 Gauge fixing

The first thing we need to do is to define what we mean by the summation over the metrics $g$ implied in (4.2). Indeed the sum has to go only through the independent degrees of freedom. In the formalism of functional integrals the measure is induced by the norm of a small variation of the considered field. This quantity has of course to be invariant under the gauge group of the theory in order to define a measure on the field that is also gauge invariant.

In our case we need invariance under diffeomorphism. Let $\mathcal{M}$ be a two-dimensional Riemann surface endorsed with a metric $g$. For a scalar field $\varphi$ the norm of its variation around its mean value is defined to be

$$\|\delta\varphi\|^2 = \int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}\,\delta\varphi\,\delta\varphi. \tag{4.3}$$

For a vector field $V^\mu$ the simplest way to define the norm is

$$\|\delta V\|^2 = \int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}\,g^{\mu\nu}\,\delta V_\mu \delta V_\nu. \tag{4.4}$$

For a rank 2 tensor we can write more invariant terms. The most general way to define the norm of the variation $\delta g$ of the metric is given by

$$\|\delta g\|^2 = \int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}\left(\frac{g^{\mu\rho}g^{\nu\sigma}}{2} + c\,g^{\mu\nu}g^{\rho\sigma}\right)\delta g_{\mu\nu}\,\delta g_{\rho\sigma}. \tag{4.5}$$

If the trace of $\delta g$ is given by $g^{\mu\nu}\delta g_{\mu\nu} = 4\delta\tau$, then

$$\delta g_{\mu\nu} = 2\delta\tau g_{\mu\nu} + \delta h_{\mu\nu} \tag{4.6}$$

where $\delta h$ is traceless: $g^{\mu\nu}\delta h_{\mu\nu} = 0$. Reporting in (4.5), one gets

$$\|\delta g\|^2 = 8\int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}\,(1+c)(\delta\tau)^2 + \int_M \mathrm{d}^2x\sqrt{g}\,g^{\mu\rho}g^{\nu\sigma}\delta h_{\mu\nu}\delta h_{\rho\sigma}. \tag{4.7}$$

The variations in the directions $\tau$ and $h$ are then decoupled i.e. they are orthogonal. In the path integral we can then factorize the metric measure as

$$\mathcal{D}g = \mathcal{D}\tau\,\mathcal{D}h. \tag{4.8}$$

As (4.7) is gauge invariant (i.e. invariant under diffeomorphisms), the measure (4.8) is also gauge invariant. To deal with the gauge invariance of the action and the measure one needs to follow the Faddeev–Popov procedure.

First we need to fix the gauge. In two dimensions we can choose some reference metric $\hat{g}$ and write any other metric on $\mathcal{M}$ as

$$g = \mathrm{e}^{2\sigma}\hat{g}. \tag{4.9}$$

This is the conformal gauge and $\sigma$ is called the conformal factor or the Liouville mode. In particular on a given chart one may choose to work in the flat conformal gauge where $\hat{g} = \delta$ is the flat metric. However global topology prevents in general the existence of a global flat fiducial metric but the uniformization theorem states that, for a genus $h \geq 2$, we can choose a metric of constant negative curvature.

Under an infinitesimal diffeomorphism generated by $\delta V_\mu$, the variation of the metric is given by

$$\delta g_{\mu\nu} = \nabla_\mu\delta V_\nu + \nabla_\nu\delta V_\mu. \tag{4.10}$$

We can also change the metric by changing the conformal factor $\sigma$ without making any change of coordinates. This Weyl transformation preserves the gauge choice and transforms the metric by

$$\delta g_{\mu\nu} = 2\delta\sigma g_{\mu\nu}. \tag{4.11}$$

The total transformation is then

$$\delta g_{\mu\nu} = \nabla_\mu\delta V_\nu + \nabla_\nu\delta V_\mu + 2\delta\sigma g_{\mu\nu}. \tag{4.12}$$

We can take the trace to come back to the variables $\tau$ and $h$:

$$2\delta\tau = 2\delta\sigma + g^{\mu\nu}\nabla_\mu\delta V_\nu \tag{4.13}$$

and then, as $\delta h_{\mu\nu} = \delta g_{\mu\nu} - 2\delta\tau g_{\mu\nu}$,

$$\delta h_{\mu\nu} = \nabla_\mu \delta V_\nu + \nabla_\nu \delta V_\mu - g_{\mu\nu} g^{\tau\sigma} \nabla_\tau \delta V_\sigma = (P\delta V)_{\mu\nu}. \tag{4.14}$$

The operator $P$ maps a vector on a rank 2 symmetric traceless tensor.

We then have two sets of variables: $(\tau, h)$ and $(\sigma, V)$ so that

$$\mathcal{D}\tau\, \mathcal{D}h = \mathcal{D}\sigma\, \mathcal{D}V |\mathcal{J}| \tag{4.15}$$

with

$$\mathcal{J} = \det \begin{pmatrix} \frac{\partial \tau}{\partial \sigma} & \frac{\partial \tau}{\partial V} \\ \frac{\partial h}{\partial \sigma} & \frac{\partial h}{\partial V} \end{pmatrix} = \det \begin{pmatrix} 1 & \mathcal{O} \\ 0 & P \end{pmatrix}. \tag{4.16}$$

Thus

$$\mathcal{J} = \det P = \sqrt{\det(P^\dagger P)}. \tag{4.17}$$

This is almost the Faddeev–Popov determinant. Indeed if there are vectors $\delta V$ such that $\delta h_{\mu\nu} = 0$ i.e. $P\delta V = 0$ the diffeomorphisms they generate are equivalent to a Weyl transformation since $\delta g_{\mu\nu} = 2\delta\tau g_{\mu\nu}$. The vectors of $\operatorname{Ker} P$ are called the conformal Killing vectors and should not be taken into account to avoid double counting the same transformation. The Faddeev–Popov determinant is then $\sqrt{\det{}'(P^\dagger P)}$ where the zero eigenvalues have been excluded. There will still be a residual gauge invariance under the diffeomorphisms generated by the conformal Killing vectors.

The norm for a vector field (4.4) is invariant under diffeomorphisms and then the measure $\mathcal{D}V$ is invariant too. As the action is diffeomorphism invariant we would like to separate the integral over $V$ from the rest and write it as the (infinite) volume of diffeomorphisms:

$$\int \mathcal{D}V = \Omega_{\text{diff}}. \tag{4.18}$$

Note that the norm (4.4) does not depend on the conformal factor, at least formally.

We should also be aware of another subtlety: on all compact Riemann surfaces except the sphere ($\chi = 0$) there exist deformations that cannot be expressed as $\delta h_{\mu\nu} = (P\delta V)_{\mu\nu}$: there is a finite number of additional degrees of freedom called the Teichmüller deformations.

In the functional integral one has to sum on all metrics that are inequivalent under the gauge group $G = \text{Diff}(\mathcal{M}) \times \text{Weyl}$ for some fixed genus $h$ and then sum over the genus. Let $\mathcal{M}_h$ be the set of all possible metrics for a genus $h$ Riemann surface. We then have to sum over the elements of the moduli space

$$\text{Mod}_h = \mathcal{M}_h / G. \tag{4.19}$$

Let $\text{Diff}_0(\mathcal{M})$ be the set of diffeomorphisms on $\mathcal{M}$ that are connected to the identity that is that can be written as a sum of infinitesimal diffeomorphisms and lead to a transformation of the metric of the form $\delta g_{\mu\nu} = \nabla_\mu \delta V_\nu + \nabla_\nu \delta V_\mu$. The Teichmüller space $\mathcal{T}_h$ is the set of equivalence classes for metrics under Weyl transformations and diffeomorphisms connected to the identity that is:

$$\mathcal{T}_h = \mathcal{M}_h / G_0 \tag{4.20}$$

where $G_0 = \text{Diff}_0(\mathcal{M}) \times \text{Weyl}$.

We saw that infinitesimal diffeomorphisms and Weyl transformations lead to a transformation of the metric of the form

$$\delta g_{\mu\nu} = (2\delta\sigma + \nabla^\rho \delta V_\rho) g_{\mu\nu} + (P\,\delta V)_{\mu\nu} \tag{4.21}$$

where $(P\,\delta V)_{\mu\nu}$ gives the traceless part of the transformation. However one also has to take into account the transformations due to the deformations of the conformal class of the metric called the Teichmüller deformations. Indeed infinitesimal diffeomorphisms and Weyl

transformations do not change the conformal class of the metric (we can still write $g = \mathrm{e}^{2\sigma} g_0$ with the same $g_0$ and a change in $\sigma$).

Any deformation $\delta_{\text{Teich}} g_{\mu\nu}$ can be decomposed as a traceless part and a part with a trace. Equation (4.21) shows that the part with trace can always be compensated by some Weyl transformation (without having to worry about possible change of chart). Finally one can show that the deformations that cannot be reabsorbed by a diffeomorphism (i.e. that cannot be written as $(P\,\delta\tilde{V})_{\mu\nu}$) are the elements of $\operatorname{Ker} P^\dagger$ where $P^\dagger$ acts on symmetric traceless tensors by $(P^\dagger h)_\mu = -2\nabla^\nu \delta h_{\mu\nu}$. $\operatorname{Ker} P^\dagger$ is then the tangent space to Teichmüller space.

Going to complex coordinates and taking $\delta v^z \in \mathcal{T}^{-1}$ and $\delta v_z = g_{z\bar{z}} \delta v^{\bar{z}} \in \mathcal{T}^1$ we see that $P$ acts as

$$(P\delta v)_{zz} = 2\nabla^1_z \delta v_z \tag{4.22a}$$

$$(P\delta v)^{zz} = 2\nabla^z_{-1} \delta v^z. \tag{4.22b}$$

We recognize the operator $P_1$ defined in (3.38):

$$P = 2P_1 = 2\nabla^z_1 \oplus \nabla^{-1}_z : \mathcal{S}^1 \to \mathcal{S}^2. \tag{4.23}$$

For this operator the Riemann–Roch theorem states that [128]

$$\dim \operatorname{Ker} P_1 - \dim \operatorname{Ker} P_1^\dagger = 3\chi = 6(1-h). \tag{4.24}$$

As we have seen $\operatorname{Ker} P_1$ corresponds to the conformal Killing vectors. On the sphere they correspond to transformations of the form

$$z \to \frac{(az+b)}{(cz+d)}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}). \tag{4.25}$$

This gives six conformal Killing vectors on the sphere so $\dim \operatorname{Ker} P_1 = 6$ and $\dim \operatorname{Ker} P_1^\dagger = 0$. So there is no Teichmüller deformation on the sphere. On the torus conformal Killing vectors correspond to translations along the two directions of the torus so that $\dim \operatorname{Ker} P_1 = \dim \operatorname{Ker} P_1^\dagger = 2$. Finally for $h \geq 2$ there is no conformal Killing vector which means that $\dim \operatorname{Ker} P_1^\dagger = 6h - 6$. As $\operatorname{Ker} P_1^\dagger$ is finite dimensional so is $\mathcal{T}_h$ and $\dim \mathcal{T}_h = \dim \operatorname{Ker} P_1^\dagger$. Teichmüller deformations then add a finite number of integration variables in the functional integral.

The partition function is then

$$Z = \Omega^\perp_{\text{diff}} \int \mathcal{D}\sigma \mathcal{D}\Psi [\text{Teich}] \sqrt{\det{}'(P_1^\dagger P_1)} \mathrm{e}^{-S} \tag{4.26}$$

where $\Omega^\perp_{\text{diff}}$ is the volume of diffeomorphisms orthogonal to those generated by the conformal Killing vectors and [Teich] is the volume element of the Teichmüller space which we will now define.

The Teichmüller deformations transform a metric into another one with whom it is not conformally related. In the coordinate system where $g$ can be written as $\mathrm{d}x^2 = \mathrm{e}^{2\sigma} \mathrm{d}z \mathrm{d}\bar{z}$ a metric which is not in the same conformal class will not satisfy $g_{zz} = g_{\bar{z}\bar{z}} = 0$. Instead an infinitesimal Teichmüller deformation is equivalent to an infinitesimal change of coordinates $z \to z + \delta z$ where the fields $\delta z$ and $\delta \bar{z}$ are not globally defined. It transforms $g_{\bar{z}\bar{z}}$ as

$$\delta g_{\bar{z}\bar{z}} = g_{z\bar{z}} \mu^z_{\bar{z}}. \tag{4.27}$$

$\mu^z_{\bar{z}}$ is called a Beltrami differential [7, 127, 128]. If we denote by $\tau_i$ an orthogonal system of coordinates in the Teichmüller space an infinitesimal Teichmüller transformation can be written as

$$\delta g_{\bar{z}\bar{z}} = \sum_{i=1}^{6h-6} \delta\tau_i \mu^z_{i\bar{z}}. \tag{4.28}$$

There is a natural pairing between Beltrami differentials and the elements of $\operatorname{Ker}\nabla_2^z$ (called quadratic differential) which does not involve the metric. Indeed if $\mu_{\bar z}^z$ is a Beltrami differential and $\phi_{zz} \in \operatorname{Ker}\nabla_2^z$ one can define

$$\langle \mu | \phi \rangle = \int \mathrm{d}z\mathrm{d}\bar z\, \mu_{\bar z}^z \phi_{zz}. \tag{4.29}$$

Finally the measure on the Teichmüller space can be written as (to get a factor orthogonal to $\delta\sigma$ and $\delta V$)

$$[\mathrm{Teich}] = \frac{\det\langle \mu_j | \phi_k\rangle}{(\det\langle \phi_j|\phi_k\rangle_g)^{1/2}} \prod_{i=1}^{6h-6} \mathrm{d}\tau_i \tag{4.30}$$

where $\phi_i$, $i = 1, \dots, 6h-6$ is an orthonormal basis of $\operatorname{Ker} P_1^\dagger$. The partition function is then given by

$$Z = \Omega_{\mathrm{diff}}^\perp \int \prod_{i=1}^{6h-6} \mathrm{d}\tau_i \det\langle \mu_j|\phi_k\rangle \int \mathcal{D}\sigma\mathcal{D}\Psi \sqrt{\frac{\det{}'(P_1^\dagger P_1)}{\det\langle\phi_j|\phi_k\rangle_g}} \mathrm{e}^{-S}. \tag{4.31}$$

### 4.1.2 The ghost contribution

Since $\det P_1^\dagger P_1 = \det\Delta_1^+ \det\Delta_{-1}^-$ we have from the results of 3.3.4

$$\left(\frac{\det{}' P_1^\dagger P_1}{\det\langle\phi_j|\phi_k\rangle}\right)^{\frac12} = \left(\frac{\det \hat P_1^\dagger \hat P_1}{\det\langle\hat\phi_j|\hat\phi_k\rangle}\right)^{\frac12} \mathrm{e}^{-\frac{26}{6}S_L}. \tag{4.32}$$

At the end, the partition function is given by

$$Z = \Omega_{\mathrm{diff}}^\perp \int \prod_{i=1}^{6h-6} \mathrm{d}\tau_i \det\langle\mu_j|\phi_k\rangle \sqrt{\frac{\det{}'(\hat P_1^\dagger \hat P_1)}{\det\langle\hat\phi_j|\hat\phi_k\rangle_g}} \int \mathcal{D}\sigma\, \mathrm{e}^{-\frac{26}{6}S_L} \int \mathcal{D}\Psi\, \mathrm{e}^{-S_m[g,\Psi]}. \tag{4.33}$$

A more physical way to compute the Jacobian $(\det P_1^\dagger P_1)^{1/2}$ is to follow the Faddeev–Popov procedure [7, 133]. We introduce the anticommuting ghost fields $c^z$ and its complex conjugate to represent infinitesimal diffeomorphisms and $b_{zz}$ and its complex conjugate to represent infinitesimal deformations orthogonal to the gauge slice. Then the Jacobian can be formally written as

$$(\det P_1^\dagger P_1)^{1/2} = \int \mathcal{D}c\,\mathcal{D}b\, \mathrm{e}^{-S_{\mathrm{gh}}[g,b,c]} \tag{4.34}$$

with

$$S_{\mathrm{gh}}[g,b,c] = \int \mathrm{d}z\mathrm{d}\bar z \sqrt{g}\,(b_{zz}\nabla^z c^z + b_{\bar z\bar z}\nabla^{\bar z} c^{\bar z}). \tag{4.35}$$

The measures are derived from the norms:

$$\|c\|^2 = \int \mathrm{d}^2x\sqrt{g}\, g_{z\bar z}\, c^z c^{\bar z} \tag{4.36a}$$

$$\|b\|^2 = \int \mathrm{d}^2x\sqrt{g}\,(g^{z\bar z})^2 b_{zz} b_{\bar z\bar z}. \tag{4.36b}$$

In conformal gauge the ghost action may be rewritten as

$$S_{\mathrm{gh}}[g,b,c] = \int \mathrm{d}^2z (b\bar\partial c + \bar b\partial\bar c) \tag{4.37}$$

where we have defined

$$b = b_{zz}, \quad \bar b = b_{\bar z\bar z}, \quad c = c^z, \quad \bar c = c^{\bar z}. \tag{4.38}$$

We will see in section 4.2 that the $b, c$ ghost system constitutes a conformal field theory of central charge $c_g = -26$. This give a contribution of $\frac{-26}{6} S_L$ exactly as we found in (4.32). There is some subtlety with the integration over the ghosts fields. Since they are anticommuting the integral over the zero-modes would give zero if we do not insert them in the functional integral (since $\int \mathrm{d}b = 0$ for an anticommuting parameter). However these zero-modes satisfy

$$\bar{\partial} c = g_{z\bar{z}} \nabla^z_{-1} c = 0, \quad \bar{\partial} b = g_{z\bar{z}} \nabla^z_2 b = 0. \tag{4.39}$$

We then see that the zero-modes of the $c$ ghost correspond to the conformal Killing vectors while the zero-modes of the $b$ ghost correspond to the quadratic differentials. The integration over the $c$ zero-modes will then yield the inverse of the volume of the diffeomorphisms generated by conformal Killing vectors while the integration over the $b$ zero-modes will give the factor $\dfrac{\det\langle \mu_j | \phi_k \rangle}{\det\langle \hat{\phi}_j | \hat{\phi}_k \rangle_g^{1/2}}$.

### 4.1.3 Gravitational action

One can define the matter partition function

$$Z_m[g] = \int \mathcal{D}\Psi \, \mathrm{e}^{-S_m[g,\Psi]} \tag{4.40}$$

and the gravitational partition function

$$Z[g] = \mathrm{e}^{-S_\mu[g]} \, Z_m[g] \tag{4.41}$$

such that

$$Z = \int \mathcal{D}g Z[g] = \int \mathcal{D}g \, \mathrm{e}^{-S_\mu[g]} \, Z_m[g]. \tag{4.42}$$

We define the effective action $S_{\mathrm{eff}}[g]$ by

$$Z_m[g] = \mathrm{e}^{-S_{\mathrm{eff}}[g]}. \tag{4.43}$$

If $g$ and $\hat{g}$ are two metrics defined on a Riemann surface, one has

$$Z[g] = \mathrm{e}^{-S_\mu[g]} \frac{Z_m[g]}{Z_m[\hat{g}]} \, Z_m[\hat{g}] \tag{4.44}$$

$$= \mathrm{e}^{-S_\mu[g]} \, \mathrm{e}^{-(S_{\mathrm{eff}}[g] - S_{\mathrm{eff}}[\hat{g}])} \, Z_m[\hat{g}]. \tag{4.45}$$

The gravitational effective action is defined by

$$S_{\mathrm{grav}}[\hat{g}, g] = S_{\mathrm{eff}}[g] - S_{\mathrm{eff}}[\hat{g}] = -\ln \frac{Z_m[g]}{Z_m[\hat{g}]}. \tag{4.46}$$

For most of this thesis we will place ourselves in the conformal gauge (4.9). Similarly the ghost contribution will be lumped inside the matter partition function as it only changes the factor that appears in front of the Liouville part of the gravitational action. In order to distinguish quantities computed in the metric $g$ and $\hat{g}$, the ones associated with the latter will be surmonted by a hat: for example $\hat{A}$ is the area for the metric $\hat{g}$ while $A$ is the area for the metric $g$. In this gauge, the gravitational action $S_{\mathrm{grav}}$ is a functional of $\hat{g}$ and of the Liouville mode $\sigma$ only: $S_{\mathrm{grav}}[\hat{g}, \sigma] \equiv S_{\mathrm{grav}}[\hat{g}, g]$.

We then have

$$Z[g] = \int \mathcal{D}_{\hat{g}} \Psi \, \mathrm{e}^{-S_*[\Psi, \sigma, \hat{g}]} \tag{4.47}$$

where the total action is

$$S_*[\hat{g}, \sigma, \Psi] = S_\mu[\sigma, \hat{g}] + S_{\mathrm{grav}}[\hat{g}, \sigma] + S_m[\hat{g}, \Psi]. \tag{4.48}$$

We see that a major interest of the conformal gauge is that the dynamics of the matter $\Psi$ and of the Liouville mode $\sigma$ are totally decoupled (up to the Teichmüller parameters). One has to deal with two quantum field theories in a fixed curved background. We will see later that the gravitational action $S_{\text{grav}}$ can be computed sometimes more easily even if one does not know how to compute the effective action $S_{\text{eff}}$.

There is a freedom in the decomposition (4.9) of the physical metric into a conformal factor and a background metric: this amounts to the existence of an emergent Weyl symmetry

$$\hat{g} = \mathrm{e}^{2\omega}\hat{g}', \qquad \sigma = \sigma' - \omega. \tag{4.49}$$

The latter is equivalent to the diffeomorphisms in terms of the physical metric and thus should be preserved. The most important consequence is that the total action (Liouville and matter fields) should be a CFT on the background $\hat{g}$.

From (4.46) we see that a gravitational action must satisfy the cocycle identities

$$S_{\text{grav}}[g_1, g_2] = -S_{\text{grav}}[g_2, g_1] \tag{4.50a}$$
$$S_{\text{grav}}[g_1, g_3] = S_{\text{grav}}[g_1, g_2] + S_{\text{grav}}[g_2, g_3]. \tag{4.50b}$$

The stress-energy tensor is the tensor of conserved currents coming from translation invariance. It is defined by

$$T_{\mu\nu} = -\frac{4\pi}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}. \tag{4.51}$$

From the effective action we can obtain the stress-energy tensor with quantum corrections:

$$\langle T_{\mu\nu}\rangle = -\frac{4\pi}{\sqrt{g}}\frac{\delta S_{\text{eff}}}{\delta g^{\mu\nu}}. \tag{4.52}$$

Reversing the argument we see that we can compute the gravitational action by integrating the quantum expectation value of the stress-energy tensor.

The simplest gravitational action which satisfies (4.50) is the cosmological constant action (2.5). In the conformal gauge (4.9), it reads

$$S_\mu = \mu \int \mathrm{d}^2x\, \sqrt{\hat{g}}\, \mathrm{e}^{2\sigma}. \tag{4.53}$$

The cosmological constant $\mu$ can receive quantum corrections and its value may differ from the classical one, but we keep the same symbol. The associated energy–momentum tensor is

$$T^{(\mu)}_{\mu\nu} = 2\pi\mu\, \hat{g}_{\mu\nu}\, \mathrm{e}^{2\sigma}, \qquad T^{(\mu)} = 4\pi\mu\, \mathrm{e}^{2\sigma}, \tag{4.54}$$

while the variation of this action is

$$\frac{1}{\sqrt{\hat{g}}}\frac{\delta S_\mu}{\delta\sigma} = 2\mu\, \mathrm{e}^{2\sigma}. \tag{4.55}$$

## 4.2 The conformal case and the Liouville action

### 4.2.1 Basics of conformal field theory in two dimensions

A conformal transformation is a diffeomorphism $x \to x'$ which leaves the metric invariant up to a scale factor $\Omega(x)$:

$$g_{\mu\nu}(x) \to g_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x). \tag{4.56}$$

Such a transformation preserves the angles $\frac{u\cdot v}{(u^2v^2)^{\frac{1}{2}}}$ between two vectors $u$ and $v$ ($u \cdot v = g_{\mu\nu}(x)u^\mu v^\nu$).

Under an infinitesimal change of coordinates $x^\mu \to x^\mu + \epsilon^\mu(x)$ the metric transforms as

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x) - \nabla_\mu \epsilon_\nu - \nabla_\nu \epsilon_\mu. \tag{4.57}$$

For this to correspond to a conformal transformation we need to have $\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu = K(x) g_{\mu\nu}(x)$. In flat space where $g_{\mu\nu} = \delta_{\mu\nu}$ this implies $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = K(x)\eta_{\mu\nu}$. Taking the trace we get $2\partial^\mu \epsilon_\mu = 2K(x)$ so that the condition is

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \partial^\rho \epsilon_\rho \delta_{\mu\nu}. \tag{4.58}$$

Going to complex coordinates (3.11) and defining $\epsilon(z) = \epsilon_0 + i\epsilon_1$ et $\bar{\epsilon}(\bar{z}) = \epsilon_0 - i\epsilon_1$ this condition becomes

$$\bar{\partial}\epsilon(z) = \partial\bar{\epsilon}(\bar{z}) = \frac{1}{2}(\partial_0 \epsilon_0 - \partial_1 \epsilon_1) = 0. \tag{4.59}$$

We see that conformal transformations correspond to analytic transformations of coordinates: $z \to f(z)$ et $\bar{z} \to \bar{f}(\bar{z})$. The metric transforms as

$$\mathrm{d}s^2 = \mathrm{d}z\mathrm{d}\bar{z} \to \left|\frac{\partial f}{\partial z}\right|^2 \mathrm{d}z\mathrm{d}\bar{z} \tag{4.60}$$

so that

$$\Omega = \left|\frac{\partial f}{\partial z}\right|^2. \tag{4.61}$$

The stress-energy tensor (4.51) is symmetric by definition and is covariantly conserved:

$$\nabla^\mu T_{\mu\nu} = 0. \tag{4.62}$$

For the theory to be conformal we need to have $\delta S = 0$ if $\delta g_{\alpha\beta} = \epsilon g_{\alpha\beta}$. As

$$\delta S = \int \mathrm{d}^2x \frac{\delta S}{\delta g_{\alpha\beta}} \delta g_{\alpha\beta} = -\frac{1}{4\pi} \int \mathrm{d}^2x \sqrt{g}\, \epsilon\, T^\alpha{}_\alpha \tag{4.63}$$

we see that

$$T^\alpha{}_\alpha = 0. \tag{4.64}$$

Classically in a conformal theory the stress-energy tensor is then traceless. In complex coordinates we get

$$T^{z\bar{z}} = T^{\bar{z}z} = T_{z\bar{z}} = T_{\bar{z}z} = 0. \tag{4.65}$$

In flat space the conservation law $\partial^\alpha T_{\alpha\beta}$ and the vanishing of the trace imply

$$\bar{\partial} T_{zz} = \partial T_{\bar{z}\bar{z}} = 0. \tag{4.66}$$

$T(z) = T_{zz}$ is then a holomorphic function while $\bar{T}(\bar{z}) = T_{\bar{z}\bar{z}}$ is anti-holomorphic. As we will see at the quantum level there is an anomaly and the stress-energy tensor is no longer traceless.

An operator $\mathcal{O}$ is said to have conformal weight $(h, \tilde{h})$ if under transformations of the form $\delta z = \epsilon z$ and $\delta\bar{z} = \bar{\epsilon}\bar{z}$ it transforms as

$$\delta\mathcal{O} = -\epsilon(h\mathcal{O} + z\partial\mathcal{O}) - \bar{\epsilon}(\tilde{h}\mathcal{O} + \bar{z}\bar{\partial}\mathcal{O}). \tag{4.67}$$

$h$ and $\bar{h}$ are related to the scaling dimension by $\Delta = h + \bar{h}$ and to the spin by $s = h - \bar{h}$. Its operator product expansion with the stress-energy tensor is then of the form

$$\begin{aligned} \langle T(z)\mathcal{O}(w,\bar{w})\rangle &= \ldots + h\frac{\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \ldots \\ \langle \bar{T}(\bar{z})\mathcal{O}(w,\bar{w})\rangle &= \ldots + \tilde{h}\frac{\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{\bar{z}-\bar{w}} + \ldots. \end{aligned} \tag{4.68}$$

Such an operator is called primary if its operator product with $T$ and $\bar{T}$ does not have singularities of order greater than 2:

$$\begin{aligned}\langle T(z)\mathcal{O}(w,\bar{w})\rangle &= h\frac{\mathcal{O}(w,\bar{w})}{(z-w)^2}+\frac{\partial\mathcal{O}(w,\bar{w})}{z-w}+O(1)\\ \langle \bar{T}(\bar{z})\mathcal{O}(w,\bar{w})\rangle &= \tilde{h}\frac{\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})^2}+\frac{\partial\mathcal{O}(w,\bar{w})}{\bar{z}-\bar{w}}+\mathcal{O}(1).\end{aligned} \tag{4.69}$$

Under a finite conformal transformation $z\to z'(z)$ et $\bar{z}\to\bar{z}'(\bar{z})$, it transforms as

$$\mathcal{O}(z,\bar{z})\to\mathcal{O}'(z',\bar{z}')=\left(\frac{\partial z'}{\partial z}\right)^{-h}\left(\frac{\partial\bar{z}'}{\partial\bar{z}}\right)^{-\tilde{h}}. \tag{4.70}$$

The stress-energy tensor has dimension $\Delta=2$ and spin $s=2$. Then $T$ has conformal weight $(2,0)$ while $\bar{T}$ has $(0,2)$. However it is not primary as its OPE with itself is given by

$$\langle T(z)T(w)\rangle = \frac{c/2}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{z-w} \tag{4.71a}$$

$$\langle \bar{T}(\bar{z})\bar{T}(\bar{w})\rangle = \frac{\bar{c}/2}{(\bar{z}-\bar{w})^4}+\frac{2\bar{T}(\bar{w})}{(\bar{z}-\bar{w})^2}+\frac{\bar{\partial}\bar{T}(\bar{w})}{\bar{z}-\bar{w}}. \tag{4.71b}$$

The constants $c$ and $\bar{c}$ are called the central charges of the theory.

We give below some useful examples [129, 134].

**The free scalar field**

The action for a free scalar field is given by

$$S_{\text{free}}=\int \mathrm{d}^2x\sqrt{g}\,g^{\mu\nu}\partial_\mu X\partial_\nu X. \tag{4.72}$$

It is conformally invariant. However the field $X$ is not primary. Indeed we have seen in (3.47) that the propagator has a logarithmic singularity. This means that the OPE between $X$ and itself is given by

$$\langle X(x)X(y)\rangle = -\frac{1}{4\pi}\ln(\ell(x,y))^2+\text{non-singular}. \tag{4.73}$$

We then see that $X$ does not have a definite scaling dimension and then is not a primary field.

The stress energy tensor is given by

$$T=-\partial X\partial X. \tag{4.74}$$

(in the quantum theory the needed normal ordering will be implied). Computing the OPE with $\partial X$ shows that $\partial X$ is a primary operator of weight $(1,0)$. In the same way, $\mathrm{e}^{ikX}$ is also a primary operator with weight $\left(\frac{k^2}{4},\frac{k^2}{4}\right)$. Finally the OPE of $T$ with itself gives the central charge

$$c=1. \tag{4.75}$$

**The free Majorana fermion**

In chapter 6 we will study the gravitational action for a Majorana fermion. In the Weyl basis (see section 6.1) the action for a free massless Majorana spinor is given by

$$S_{\text{free}}=\int \mathrm{d}^2z\,\sqrt{g}\,(\psi\bar{\partial}\psi-\bar{\psi}\partial\bar{\psi}) \tag{4.76}$$

where $\psi$ is a Grassmann odd function. The OPE of $\psi$ with itself is given by

$$\psi(z)\psi(w) = \frac{1}{2\pi}\frac{1}{z-w} + \text{non-singular}. \tag{4.77}$$

This is an odd function which is consistent with the fact that $\psi$ is anticommuting. The holomorphic component of the stress-energy tensor is

$$T(z) = -\pi\psi(z)\partial\psi(z). \tag{4.78}$$

Computing the OPE of $\psi$ with $T$ shows that this time $\psi$ is a primary field with conformal weight $\left(\frac{1}{2}, \frac{1}{2}\right)$. Then the OPE of $T$ with itself gives the central charge

$$c = \frac{1}{2}. \tag{4.79}$$

**The ghost system**

The ghost action was given by

$$S_{\text{gh}} = \int \mathrm{d}z\mathrm{d}\bar{z}\,(b\bar{\partial}c + \bar{b}\partial\bar{c}). \tag{4.80}$$

The ghosts $b, \bar{b}, c, \bar{c}$ are anticommuting and satisfy

$$b(z)c(w) = \frac{1}{z-w} + \text{non-singular}. \tag{4.81}$$

Moreover the OPE of $b$ with itself and of $c$ with itself are non-singular. The stress-energy tensor is given by

$$T = 2(\partial c)b + c\partial b, \quad \bar{T} = 2(\bar{\partial}\bar{c})\bar{b} + \bar{c}\bar{\partial}\bar{b}. \tag{4.82}$$

The fields $b, \bar{b}, c, \bar{c}$ are primary with weights $(2,0), (0,2), (-1,0)$ and $(0,-1)$. The OPE of $T$ with itself gives the central charge of the ghost system:

$$c_{\text{gh}} = -26. \tag{4.83}$$

### 4.2.2 The conformal anomaly

As we have seen previously conformal symmetry implies that the stress-tensor is classically traceless. However at the quantum level there might be an anomaly. In flat space, using the fact that $T_{\mu\nu}$ is symmetric and conserved one can show [134, sec. 4.3] that the two-point function $\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle$ is traceless which means that $\langle T^\mu_\mu(x)T^\rho_\rho(0)\rangle = 0$ everywhere. This in turn implies that $\langle T^\mu_\mu\rangle = 0$.

But if the background is no longer flat the argument does not hold anymore. However we know that if there is an anomaly it must be a local quantity of dimension 2 which vanishes in flat space the only possibility is that $\langle T^\mu_\mu\rangle$ is proportional to $R$. Using the OPE (4.71a) to compute the variation of $\langle T^\mu_\mu\rangle$ under an infinitesimal Weyl transformation around flat space one can show [6, 129] that the constant of proportionality is just the central charge:

$$\langle T^\mu_\mu\rangle = -\frac{c}{12}R \tag{4.84}$$

Note that one can also use the OPE (4.71b) between $\bar{T}$ and itself to compute $\langle T^\mu_\mu\rangle$. The constant of proportionality would then be $\bar{c}$. This shows that for the theory to be consistent at the quantum level the left and right central charges must be equal $\bar{c} = c$ to avoid the so-called gravitational anomaly.

In conformal gauge

$$R = \mathrm{e}^{-2\sigma}\hat{R} - 2\mathrm{e}^{-2\sigma}\hat{\Delta}\sigma \tag{4.85}$$

This means that if $c \neq 0$ there is at least one physical observable $\langle T^\mu_\mu \rangle$ whose value depends on the gauge choice. This is the conformal (or Weyl) anomaly. Weyl invariance at the quantum level requires $c = 0$.

In conformal gauge the conformal anomaly is directly related to the gravitational action. Indeed one has

$$\langle T^\mu_\mu \rangle = -\frac{4\pi}{\sqrt{g}}\, g^{\mu\nu}\, \frac{\delta S_{\rm eff}}{\delta g^{\mu\nu}} = \frac{2\pi}{\sqrt{g}}\, \frac{\delta S_{\rm eff}}{\delta \sigma} \tag{4.86}$$

so that the integration of $\langle T^\mu_\mu \rangle$ with respect to $\sigma$ gives the gravitational action. Then (4.84) becomes

$$\frac{\delta S_{\rm eff}}{\delta \sigma} = -\frac{c}{24\pi}\sqrt{g}\, R = -\frac{c}{24\pi}\sqrt{\hat{g}}\, (\hat{R} - 2\hat{\Delta}\sigma). \tag{4.87}$$

Finally we see that the gravitational action is given by

$$\begin{aligned} S_{\rm grav}[\hat{g}, g] = S_{\rm eff}[g] - S_{\rm eff}[\hat{g}] &= -\frac{c}{24\pi}\int \mathrm{d}^2x \sqrt{\hat{g}}(\hat{R}\sigma - \sigma\, \hat{\Delta}\sigma) \\ &= -\frac{c}{24\pi}\int \mathrm{d}^2x \sqrt{\hat{g}}(\hat{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + \hat{R}\,\sigma) \\ &\equiv -\frac{c}{6} S_L \end{aligned} \tag{4.88}$$

where $S_L$ is the Liouville action (without cosmological constant) given by [37]

$$S_L = \frac{1}{4\pi}\int \mathrm{d}^2x\, \sqrt{\hat{g}}\, \left(\hat{g}^{\mu\nu}\partial_\mu\sigma\, \partial_\nu\sigma + \hat{R}\,\sigma\right). \tag{4.89}$$

Note that the cosmological constant action will still appear as a counterterm in the functional integral.

### 4.2.3 Conformal deformations and DDK anstatz

If we want to use two-dimensional quantum gravity as a toy-model for the four-dimensional one we cannot restrict ourselves to the coupling to conformal matter. Indeed the matter of our Universe is not conformal. The first step to go away from conformal matter is to study a CFT which is deformed by a conformal perturbation [60]. This means that the matter action $S_m$ for a set of fields $\psi$ is given by

$$S_m[g, \psi] = S_{\rm cft}[g, \psi] + S_p[g, \psi] \tag{4.90}$$

where $S_{\rm cft}$ is an action which is conformally invariant on flat space and the conformal perturbation $S_p$ is of the form

$$S_p[g, \psi] = \sum_i \lambda^i \int \mathrm{d}^2x \sqrt{g}\, \mathcal{O}_i(\psi), \tag{4.91}$$

where $\mathcal{O}_i(\psi)$ is a set of primary operators (built from the fields $\psi$) with conformal weight $h_i$. While the operators $\mathcal{O}_i$ can depend explicitly on the metric there is also an implicit metric dependence which is due to the regularization needed to remove self-contraction. We will only consider fields without (conformal) spin i.e. such that $\bar{h}_i = h_i$ and then the conformal dimension is given by $\Delta_i = 2h_i$.

A Weyl transformation acts as

$$g = \mathrm{e}^{2\omega}\hat{g}, \qquad \mathcal{O}_i = \mathrm{e}^{-2h_i\omega}\, \hat{\mathcal{O}}_i. \tag{4.92}$$

For the type of theories we are interested in, the action $S_{\rm cft}$ is generically invariant under Weyl transformations [122]. On the other hand the action $S_p$ is not invariant if $h_i \neq 1$ for

at least one operator. If there is no explicit metric dependence, the trace of the energy-momentum tensor for the perturbation is

$$T^{(p)} = 4\pi \sum_i \lambda_i \mathcal{O}_i. \tag{4.93}$$

Let's look at some well-known examples. For a scalar field $X$ we have seen that $X$ is not a primary field. Then the addition of a mass term $m^2 X^2$ is not a conformal perturbation. A conformal perturbation would contain fields of the form $\partial X$ or $\mathrm{e}^{ikX}$. On the contrary for a Majorana spinor $\Psi$ the mass term is given by $m\bar{\Psi}\gamma^*\Psi = 2m\psi\bar{\psi}$. As $\psi$ is a primary field the mass term is now a conformal perturbation. This case will be studied in details in chapter 6.

The DDK construction proposes that the total action in the conformal gauge is

$$S_*[\hat{g},\sigma,\psi] \equiv S_{\mathrm{DDK}}[\hat{g},\sigma,\psi] = Q^2\, S_L[\hat{g},\sigma] + S_{\mathrm{cft}}[\hat{g},\psi] + S^{(p)}_{\mathrm{DDK}}[\hat{g},\sigma,\psi], \tag{4.94a}$$

$$S^{(p)}_{\mathrm{DDK}}[\hat{g},\sigma,\psi] = \sum_i \lambda^i \int \mathrm{d}^2x \sqrt{\hat{g}}\, \mathrm{e}^{2a_i Q\sigma} \hat{\mathcal{O}}_i(\psi). \tag{4.94b}$$

We recall that $Q$ is related to the central charge by $c_m = -6Q^2$ (the ghost contribution is not included since its only contribution to the gravitational action is to shift the coefficient in front of $S_L$ by $-26$). The $a_i$ are chosen such that each term has a conformal weight 1 and is thus invariant under the Weyl transformation

$$\hat{g} = \mathrm{e}^{2\omega}\hat{g}', \qquad \sigma = \sigma' - \omega, \qquad \hat{\mathcal{O}}_i = \mathrm{e}^{-2h_i\omega}\,\hat{\mathcal{O}}_i, \tag{4.95}$$

which leads to the condition

$$a_i(Q - a_i) + h_i = 1 \tag{4.96}$$

(the $a_i^2$ term comes from the regularization of the exponential). The solution is

$$a_i = \frac{Q}{2} - \sqrt{\frac{Q^2}{4} + h_i - 1} \tag{4.97}$$

where the sign is found by matching to the semi-classical solution $Q \to \infty$

$$a_i \sim \frac{1}{Q}(1 - h_i). \tag{4.98}$$

One says that the operator $\mathcal{O}_i$ has been gravitationally dressed.

For example in the case of the massive Majorana fermion the DDK ansatz yields

$$S_{\mathrm{DDK}}[\hat{g},\sigma,\Psi] = i\int \mathrm{d}^2x\,\sqrt{\hat{g}}\,\bar{\Psi}\nabla\!\!\!/\Psi - \frac{1}{12}S_L[\hat{g},\sigma] + m\int \mathrm{d}^2x\sqrt{\hat{g}}\,\mathrm{e}^{\frac{1}{3}\sigma}\bar{\Psi}\gamma^*\Psi. \tag{4.99}$$

To derive (4.94) we note that the matter partition function can be written as [59, sec. 3]

$$Z_m[g] = \int \mathrm{d}_g\psi\, \mathrm{e}^{-S_{\mathrm{cft}}[g,\psi]-S_p[g,\psi]} = Z_{\mathrm{cft}}[g]\left\langle \exp\left(-\lambda\int \mathrm{d}^2\sigma\,\sqrt{g}\,\mathcal{O}\right)\right\rangle_{g,\mathrm{cft}}, \tag{4.100}$$

where $Z_{\mathrm{cft}}[g]$ arises from the normalization of the correlation function. The exponential can be expanded perturbatively in the coupling constant

$$\frac{Z_m[g]}{Z_{\mathrm{cft}}[g]} = \sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}\int \langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle_{g,\mathrm{cft}} \prod_{i=1}^{n}\sqrt{g(x_i)}\,\mathrm{d}^2x_i. \tag{4.101}$$

Since the correlation function is computed in the CFT one can use the relations (4.92) in order to express the quantities in the conformal gauge

$$\frac{Z_m[g]}{Z_{\mathrm{cft}}[g]} = \sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}\int \left\langle \hat{\mathcal{O}}(x_1)\cdots\hat{\mathcal{O}}(x_n)\right\rangle_{\hat{g},\mathrm{cft}} \prod_{i=1}^{n}\mathrm{e}^{2(1-h_i)\sigma(x_i)}\sqrt{\hat{g}(x_i)}\,\mathrm{d}^2x_i. \tag{4.102}$$

Resumming the expansion yields

$$\frac{Z_m[g]}{Z_{\rm cft}[g]} = \frac{1}{Z_{\rm cft}[\hat{g}]} \int \mathcal{D}_{\hat{g}}\psi \, \mathrm{e}^{-S_{\rm cft}[\hat{g},\psi]-S^{(p)}_{\rm DDK}[\hat{g},\sigma,\psi]}. \tag{4.103}$$

Finally (4.94) is recovered by using the relation

$$\ln \frac{Z_{\rm cft}[g]}{Z_{\rm cft}[\hat{g}]} = Q^2 \, S_L[\hat{g},\sigma]. \tag{4.104}$$

The semi-classical result can be recovered by a direct computation with the action

$$S_p[g,\psi] = \sum_i \lambda^i \int \mathrm{d}^2x\sqrt{g}\,\mathcal{O}_i = \sum_i \lambda^i \int \mathrm{d}^2x\sqrt{\hat{g}}\,\mathrm{e}^{2(1-h_i)\sigma}\hat{\mathcal{O}}_i = S^{(p)}_{\rm DDK}[\hat{g},\sigma,\psi]. \tag{4.105}$$

Note that this does not follow from a direct integration of the quantum expectation of the trace (4.93) in the classical limit

$$\left\langle T^{(p)} \right\rangle_g = 4\pi \sum_i \lambda^i \left\langle \mathcal{O}_i \right\rangle_g \sim 4\pi \sum_i \lambda^i \mathcal{O}_{i,\rm cl} \tag{4.106}$$

because one finds an additional factor of $(2-2h_i)^{-1}$.

The DDK ansatz for the gravitational action for a deformed CFT is plagued by several problems:

1. Unlike the gravitational action (4.46) the DDK action is not a Wess–Zumino action since it depends on both the matter fields and the Liouville mode: $S^{(p)}_{\rm DDK}[g,\sigma,\psi]$. This means that the ansatz does not fit directly inside the conformal gauge formalism described in 4.1.3 despite the appearance.

   We recall in the conformal gauge the total action (4.48), in absence of cosmological constant, was given by :

   $$S_*[g,\sigma,\psi] = S_{\rm grav}[\hat{g},\sigma] + S_m[\hat{g},\psi], \qquad S_{\rm grav}[\hat{g},\sigma] = Q^2 S_L[\hat{g},\sigma] + S^{(p)}_{\rm grav}[\hat{g},\sigma]. \tag{4.107}$$

   Comparing with (4.94), the discrepancy between $S_{\rm DDK}$ and $S_{\rm cg}$ can be summarised by the fact that the non-conformal contribution to the gravitational $S^{(p)}_{\rm grav}$ and matter $S_p$ actions have been replaced by a single term $S^{(p)}_{\rm DDK}$.

2. The validity of the conformal gauge approach relies on the conformal invariance of the total action (matter plus gravitational) on the background metric. In the DDK approach this translates into the fact that the gravitationally dressed deformations should be exactly marginal operators, but this is generically not the case beyond the leading order in the coupling [135–141]. This can be understood from the fact that the presence of the operators inside the action modify the renormalization conditions (and) and thus one cannot expect a tree-level condition to hold beyond the semi-classical level.

   Given the OPE coefficients $C^k_{ij}$

   $$V_i(x)V_j(y) \sim C^k_{ij}|x-y|^{2(h_k-h_i-h_j)}V_k(y) \tag{4.108}$$

   the beta functions for operators with $h_i \sim 1$ and for vanishing cosmological constant are

   $$\beta^i = (\Delta^j_i - 2\delta^j_i)\lambda^j + \pi C^i_{jk}\lambda^j\lambda^k + O(\lambda^3) \tag{4.109}$$

   where $\Delta^j_i$ is the normalization of the two-point function (the $O(\lambda^3)$ term has been computed in [139, 140]). One can add a term

   $$S^{(p,2)}_{\rm DDK} = \frac{\pi}{Q(1+4a_k)} C^k_{ij}\lambda^i\lambda^j \int \mathrm{d}^2x\,\sqrt{\hat{g}}\,\sigma\,\mathrm{e}^{2a_kQ\sigma}\hat{\mathcal{O}}_k(\psi) \tag{4.110}$$

to ensure that the beta functions vanish at quadratic order (in practice this implies that $\Delta_i^j$ becomes coupling-dependent). Note that without this additional term the vanishing of the beta functions at $O(\lambda)$ is obtained thanks to the gravitational dressing.

While such an approach may lead to a well-defined gravitational action, computing the higher-order corrections [139] is a formidable task and not more easy than a direct computation of the gravitational action in the conformal gauge (in particular for weights far from 1).

3. As we have seen in (4.106) the DDK action cannot be obtained by a direct computation or by integrating the trace of the quantum matter energy–momentum tensor. In particular the DDK action (4.94b) is linear in the coupling constant while the gravitational action for the massive fermion computed in chapter 6 is not.

4. The derivation from [59] presented before may yield an incorrect result because of the formal manipulations in the functional integral. In particular the correlation functions have logarithmic singularities [137, p. 4] and care is required when resumming an infinite series of those.

This can be summarized by saying that a deformation of the matter CFT does not seem to be equivalent to a deformation of the Liouville plus matter CFT, at least beyond the semi-classical approximation (see also [142, pp. 19–20] for related discussions). A possible explanation for the failure of this construction is that symmetries are not sufficient to determine the form of the action when locality is lost, which is certainly the case in a theory of gravity with massive matter (for example the Mabuchi action is non-local when expressed in terms of the Liouville field).

It is possible that these problems are just apparent and result from a "bad" parametrization of the action and functional integrals. This would amount to prove that

$$\int \mathcal{D}_g\psi\, \mathrm{e}^{-S_{\mathrm{cft}}[\hat{g},\psi]-S^{(p)}_{\mathrm{DDK}}[\hat{g},\sigma,\psi]} = \int \mathcal{D}_g\psi\, \mathrm{e}^{-S_{\mathrm{cft}}[\hat{g},\psi]-S_p[\hat{g},\psi]-S^{(p)}_{\mathrm{grav}}[\hat{g},\psi]}. \tag{4.111}$$

It is also possible that such a relation would hold only after incorporating all the corrections to the DDK action needed to make the beta functions vanish. While establishing directly this identity might be difficult, one may also prove it either by establishing that the correlation functions computed from both sides satisfy the same Ward identities, or either by computing numerically the correlation functions and showing that they agree. This can be already tested in a leading-order approximation in the mass, i.e. in the Liouville–Mabuchi theory. These approaches would not be attempted until one had the (at least tentative) spectrum for the Mabuchi theory: such a proposal was put forward in [76, 77]. Moreover the results of the previous sections provide a concrete example where these ideas can be tested. This justifies our attempt of a derivation from first principles of the gravitational action for a Majorana fermion in 6.

It can be noted that gravitationally dressed conformal deformations are seen as originating in two different contexts: first as the gravitational action of a deformed CFT [59, 60], and secondly as a mean to bring the theory (seen as a statistical system) away from the critical point [38, 61, 143] or as a model for some phenomenon [144]. In particular these systems have been mapped to matrix models [136, sec. 4] and were shown to lead to consistent statistical systems [60, 61].

It should be clear that the second approach gives up the link with gravity and studies the action and the associated functional integral just for themselves: as a consequence any constraint originating from gravity can be relaxed (in particular the Wess–Zumino form and the marginal scaling conditions) and our comments do not apply. Similarly the gravitational dressing of operators in correlation functions such as

$$\left\langle \prod_i \mathrm{e}^{2a_i Q\sigma(x_i)}\mathcal{O}_i(x_i) \right\rangle \tag{4.112}$$

do not pose any problems since the scaling dimensions are not modified in this case [34, 39, 63, 145, 146].

## 4.3 Kähler formalism

### 4.3.1 Kähler parametrization

Since every 2-dimensional manifold is Kähler, another parametrization of the metric $g$ is possible in terms of the Kähler potential $\phi$ [62, 63, 71]. We can then look at another partition of the space of metrics namely the set of Kähler classes [147, 148]. Given a metric $\hat{g}$ defined by

$$\mathrm{d}s^2 = 2\hat{g}_{z\bar{z}}\,\mathrm{d}z\mathrm{d}\bar{z} \tag{4.113}$$

its Kähler potential $\psi$ is locally defined by

$$\hat{g}_{z\bar{z}} = \partial\bar{\partial}\psi(z,\bar{z}). \tag{4.114}$$

The Kähler class of $\hat{g}$ is the set of metrics that satisfy

$$g_{z\bar{z}}(z,\bar{z}) = \hat{g}_{z\bar{z}}(z,\bar{z}) + \partial\bar{\partial}\phi(z,\bar{z}) \tag{4.115}$$

for some function $\phi \in \mathcal{C}^\infty(\mathcal{M})$ (along with $g_{zz} = g_{\bar{z}\bar{z}} = 0$). This amounts to shift the Kähler potential $\psi$ by $\phi$. Note that if $\phi(z,\bar{z}) = f(z) + g(\bar{z})$ where $f$ is holomorphic and $g$ anti-holomorphic (4.115) is just a Kähler transformation which leaves the metric invariant. For compact surfaces holomorphic and anti-holomorphic functions are just the constants so this restricts the invariance of (4.115) to the constant shifts of $\phi$. All metrics in a given class have the same area. Indeed $\sqrt{g} = \sqrt{\hat{g}} + \partial\bar{\partial}\phi$ so that $\int \mathrm{d}z\mathrm{d}\bar{z}\sqrt{g} = \int \mathrm{d}z\mathrm{d}\bar{z}\sqrt{\hat{g}}$ since $\phi$ is $\mathcal{C}^\infty$ on $\mathcal{M}$ (note that $\int \mathrm{d}z\mathrm{d}\bar{z}\,\partial\bar{\partial}\psi \neq 0$ since $\psi$ is not globally defined).

In complex dimension one, the Kähler gauge is equivalent to the conformal gauge at fixed area. Indeed, recalling that $\Delta = 2g^{z\bar{z}}\partial\bar{\partial}$ we see that

$$\hat{g}_{z\bar{z}}\left(1 + \frac{\Delta\phi}{2}\right) = \hat{g}_{z\bar{z}} + \partial\bar{\partial}\phi = g_{z\bar{z}}. \tag{4.116}$$

Defining

$$\mathrm{e}^{2\sigma} = 1 + \frac{\Delta\phi}{2} \tag{4.117}$$

we see that $g = \mathrm{e}^{2\sigma}\hat{g}$. If we want to include area variation in the Kähler parametrization (that is to relate $g$ to a metric $\hat{g}$ of different area) we should rather write

$$\mathrm{e}^{2\sigma} = \frac{A}{\hat{A}}\left(1 + \frac{\hat{A}}{2}\,\hat{\Delta}\phi\right), \tag{4.118}$$

which corresponds to

$$g_{z\bar{z}} = \frac{A}{\hat{A}}\hat{g}_{z\bar{z}} + A\,\partial\bar{\partial}\phi. \tag{4.119}$$

For a given pair $(A,\phi)$ this relation defines $\sigma$ uniquely (up to constant shift of $\phi$), and positivity of the exponential implies the inequality

$$\hat{\Delta}\phi > -\frac{2}{\hat{A}}. \tag{4.120}$$

The Kähler parametrization is very convenient because it can be used to write local actions that would otherwise be non-local in terms of the Liouville field (in the same way that actions non-local in terms of the curvature can be made local in terms of the Liouville field).

The main drawback of this formalism is that it forces to work at fixed area and the subtleties we will find in chapter 7 may originate from this.

The variation of the conformal factor $\delta\sigma$ is related to the one of the Kähler potential $\delta\phi$ and of the area $\delta A$ by

$$\delta\sigma = \frac{\delta A}{2A} - \frac{A}{4}\Delta\delta\phi \ . \tag{4.121}$$

The measure on $\sigma$ is related to the measure on $\phi$ by

$$\|\delta\sigma\|^2 = \int \mathrm{d}^2x\sqrt{g}(\delta\sigma)^2 = \int \mathrm{d}^2x\sqrt{g}\left(\frac{\delta A^2}{4A^2} - \frac{1}{4}\delta A\Delta\delta\phi + \frac{A^2}{16}(\Delta\delta\phi)^2\right) \tag{4.122}$$

The second term vanishes as the integrand is a total derivative. Then

$$\|\delta\sigma\|^2 = \frac{\delta A^2}{4A} + \|\delta\sigma\|_A^2 \tag{4.123}$$

where $\|\delta\sigma\|_A^2$ is the measure on the space of metrics of fixed area $A$:

$$\begin{aligned}\|\delta\sigma\|_A^2 &= \|A\Delta\delta\phi\|^2 \\ &= \frac{1}{16}\int \mathrm{d}^2x\sqrt{g}\,(A\Delta\delta\phi)^2 \\ &= \frac{A}{16\hat{A}}\int \mathrm{d}^2x\sqrt{\hat{g}}\left(1 - \frac{1}{2}\hat{A}\hat{\Delta}\phi\right)^{-1}(\hat{A}\hat{\Delta}\delta\phi)^2\end{aligned} \tag{4.124}$$

The integration on $\sigma$ in the functional integral can be replaced by an integration on $\phi$ if the two integration measures are formally related by

$$\mathcal{D}\sigma = \frac{\mathrm{d}A}{\sqrt{A}}\left(\det\left(\frac{A}{\hat{A}}\left(1 - \frac{1}{2}\hat{A}\hat{\Delta}\phi\right)^{-1}\right)\right)^{\frac{1}{2}}(\det{}'\hat{A}\hat{\Delta})\mathcal{D}_{\hat{g}}\phi \tag{4.125}$$

It is of course purely formal as the determinants are manifestly infinite and have to be regularized [71]. In [149] a definition of $\mathcal{D}\phi$ is given in term of Bergmann metrics.

### 4.3.2 Gravitational functionals

Various functionals appear in the gravitational action (4.46), the most remarkable ones (beside the area functional) are: the Liouville functional, the Mabuchi functional and the Aubin–Yau functional. It is expected that, in general, other functionals are present.

The first one is well-known and describes the effective action when gravity is coupled to conformal matter only while the other two appear when it is coupled to massive matter [62, 63]. Note that, recently, all these functionals have been used in the description of the fractional quantum Hall effect [64, 67].

In the conformal gauge the simplest functional variation is given by the variation of the area (or equivalently of the cosmological constant action):

$$\delta A = 2\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma \tag{4.126}$$

The equivalent in the Kähler formalism is [147]

$$\delta S_{AY} = \frac{1}{A}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma \tag{4.127}$$

where the area has been introduced for dimensional reason. This can be integrated to give the Aubin-Yau action:

$$S_{AY}[\hat{g},\phi] = -\int \mathrm{d}^2x\sqrt{\hat{g}}\left(\frac{1}{4}\phi\hat{\Delta}\phi - \frac{\phi}{\hat{A}}\right) \tag{4.128}$$

In the conformal gauge the variation of the Liouville action was a very natural quantity as it contains only the Ricci scalar $R$:

$$\delta S_L = \frac{1}{4\pi}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)R(x) \tag{4.129}$$

In the Kähler formalism some analogue quantity is given by [147]

$$\delta S = -\frac{1}{4\pi}\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\left(R(x)-\frac{4\pi\chi}{A}\right) \tag{4.130}$$

where $\bar{R}=\frac{4\pi\chi}{A}$ is the average curvature. Integrating this we obtain the Mabuchi action [62, 63, 67, 71]:

$$\begin{aligned} S_M = \frac{1}{4\pi}\int \mathrm{d}^2x\,\sqrt{\hat{g}}\,\Bigg\{ &-\pi\chi\,\hat{g}^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi+\left(\frac{4\pi\chi}{\hat{A}}-\hat{R}\right)\phi \\ &+\frac{2}{\hat{A}}\left(1+\frac{\hat{A}}{2}\hat{\Delta}\phi\right)\ln\left[\frac{A}{\hat{A}}\left(1+\frac{\hat{A}}{2}\hat{\Delta}\phi\right)\right]\Bigg\} \end{aligned} \tag{4.131}$$

where the last term can be expressed in terms of $\sigma$ through (4.118):

$$\frac{2}{\hat{A}}\left(1+\frac{\hat{A}}{2}\hat{\Delta}\phi\right)\ln\left[\frac{A}{\hat{A}}\left(1+\frac{\hat{A}}{2}\hat{\Delta}\phi\right)\right]=\frac{4\sigma}{A}\mathrm{e}^{2\sigma}. \tag{4.132}$$

If $A$ is allowed to vary the total variation of the Mabuchi action is given by

$$\delta S_M = \frac{1}{2\pi}\frac{\delta A}{A}-\frac{1}{4\pi}\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\left(R(x)-\frac{4\pi\chi}{A}\right). \tag{4.133}$$

## 4.4 The Liouville action

Here we study the Liouville action with cosmological constant. To make contact with the literature we work the action obtained from (4.89) by rescaling the conformal factor by $b$: $\sigma\to b\sigma$ and redefining the cosmological constant as $\tilde{\mu}=Q^2\mu$:

$$\tilde{S}_L = \int \mathrm{d}^2x\,\sqrt{\hat{g}}\,\left(\hat{g}^{\mu\nu}\partial_\mu\sigma\,\partial_\nu\sigma+Q\hat{R}\,\sigma+\tilde{\mu}\,\mathrm{e}^{2b\sigma}\right) \tag{4.134}$$

where we have defined

$$Q=\frac{1}{b}. \tag{4.135}$$

### 4.4.1 Basic properties

The variation of the action is

$$\frac{1}{\sqrt{\hat{g}}}\frac{\delta\tilde{S}_L}{\delta\sigma}=Q\hat{R}-2\hat{\Delta}\sigma+2b\,\tilde{\mu}b\mathrm{e}^{2b\sigma}=\mathrm{e}^{2b\sigma}\left(QR+2b\,\tilde{\mu}\right) \tag{4.136}$$

where we have used the fact that the rescaling of the Liouville mode modifies ?:

$$R=\mathrm{e}^{-2b\sigma}(\hat{R}-2b\hat{\Delta}\sigma). \tag{4.137}$$

The equations of motion are then

$$R=-2\tilde{\mu}b^2. \tag{4.138}$$

This means that the extremal points of Liouville theory are the metrics of constant curvature.

For $\tilde{\mu} = 0$ the trace of the energy–momentum tensor reads

$$T^{(L)} = -Q\hat{\Delta}\sigma. \tag{4.139}$$

Using the equations of motion this becomes

$$T^{(L)} = -\frac{Q^2}{2}\hat{R}. \tag{4.140}$$

We see that whereas this vanishes in flat space this is no more the case in curved space. However the equation of motions (4.138) become $R = 0$ which is obviously Weyl invariant. This means that although the action is non conformally invariant the theory is (classically) conformal [150]. Moreover (4.140) shows that $T$ generates a Virasoro algebra with (classical) central charge

$$c_L = 6Q^2. \tag{4.141}$$

As the Liouville action arises when coupling conformal matter to gravity the total action (4.48) must be conformally invariant. This is the case if

$$c_L + c = 0 \tag{4.142}$$

where $c = 26 - c_m$ (central charge of the ghosts minus the one of the matter). This means that

$$Q = \sqrt{\frac{c}{6}}. \tag{4.143}$$

Note that it is only true if the parameters $b$ and $Q$ are related by (4.135).

### 4.4.2 From variable to fixed area

At fixed area the cosmological constant $S_\mu = \mu A$ is fixed and the only contribution to the gravitational action when the matter is conformal is $S_L$. The equation of motion reads [47, sec. 2]

$$\frac{4\pi}{\sqrt{\hat{g}}}\frac{\delta S_L}{\delta_A \sigma} = 0 \quad \Longrightarrow \quad \Delta R = 0 \quad \Longrightarrow \quad R = \frac{4\pi\chi}{A} \tag{4.144}$$

where the subscript $A$ on the variation indicates that only variations of $\sigma$ which keep the area fixed are considered.

To formulate the theory at finite area, we can also write it in the Kähler formalism in terms of $(A, \phi)$. The variation at fixed area is then obtained using (4.121) with $\delta A = 0$:

$$\delta S_L = \frac{A}{16\pi}\int \mathrm{d}^2x\,\sqrt{\hat{g}}\left(-2\hat{\Delta}\sigma + Q\hat{R}\right)\delta\Delta\phi. \tag{4.145}$$

From this we can read the equation of motion

$$0 = \hat{\Delta}R = \mathrm{e}^{2b\sigma}\Delta R. \tag{4.146}$$

implying that the curvature is constant (since any harmonic function on a compact surface is constant) and hence must be given by

$$R = \frac{4\pi\chi}{A}. \tag{4.147}$$

Identifying (4.144) and (4.138) leads to the relation

$$-8\pi\mu b^2 = \frac{4\pi\chi}{A}. \tag{4.148}$$

This relation also results from integrating (4.138) over the manifold and it can be seen in correlation functions upon performing the Laplace transform (which means that it holds not

only on-shell, see [47, sec. 2] for the case $\chi = 2$). In some way this relation encodes how to pass from the fixed to the variable area expressions in the Liouville case, and one may hope that it generalizes to the case of the Mabuchi action. It is then tempting to make the following identification (at least as a rough analogy)

$$2\mu = \frac{\mathrm{sign}\chi}{A}, \qquad b^2 \sim |\chi|. \tag{4.149}$$

This suggests that $b$ and $\chi$ may play analogous roles. This will indeed come back in chapter 7 when comparing the minisuperspace results of the Liouville and Mabuchi actions.

### 4.4.3 Change of the integration measure

The relation (4.141) is only valid is the integration measure $\mathcal{D}\sigma$ is a gaussian one. However this is not the case. Indeed the integration measure $\mathcal{D}\sigma$ in the functional integral is derived from

$$\|\delta\sigma\|^2 = \int \mathrm{d}^2x\sqrt{g}\,(\delta\sigma)^2 = \int \mathrm{d}^2x\sqrt{\hat{g}}\,\mathrm{e}^{2b\sigma}\,(\delta\sigma)^2. \tag{4.150}$$

This is not a gaussian measure since it depends intrinsically of $\sigma$ (i.e. $\mathcal{D}\sigma \equiv \mathcal{D}_\sigma\sigma$). We would like to transform it to the usual gaussian measure in background $\hat{g}$:

$$\|\delta\sigma\|^2 = \int \mathrm{d}^2x\sqrt{\hat{g}}\,(\delta\sigma)^2. \tag{4.151}$$

Such a transformation in general yields a non-trivial Jacobian $\mathcal{J}$. However it is not easy to compute it from first principles. To remedy to this problems DDK proposed to make an ansatz using the fact that this quantity should be local, diffeomorphism and conformally invariant [39, 55, 57]. The more general form for such an action is

$$\mathcal{J} = \int \mathrm{d}^2x\ \sqrt{\hat{g}}\,\left(\hat{g}^{\mu\nu}\partial_\mu\sigma\,\partial_\nu\sigma + \alpha\hat{R}\,\sigma + 4\pi\beta\,\mathrm{e}^{2\gamma\sigma}\right). \tag{4.152}$$

Then the total action will be equivalent to a renormalization of the Liouville action which reads

$$\mathcal{S}_L = \int \mathrm{d}^2x\ \sqrt{\hat{g}}\,\left(\hat{g}^{\mu\nu}\partial_\mu\sigma\,\partial_\nu\sigma + \tilde{Q}\hat{R}\,\sigma + 4\pi\tilde{\mu}\,\mathrm{e}^{2\tilde{b}\sigma}\right). \tag{4.153}$$

The parameters are now independent: $\tilde{b}$ and $\tilde{Q}$ do not need to satisfy (4.135).

If the cosmological constant is set to 0 we want the theory to be conformally invariant. For that we need the total central charge $c_{\mathrm{tot}}$ to vanish i.e.

$$c = c_\sigma + c_m + c_{gh} = 0 \Rightarrow c_\sigma = 26 - c_m. \tag{4.154}$$

However $c_\sigma$ corresponds to the central charge of a scalar field with background charge which is given by [55]

$$c_\sigma = 1 + 6\tilde{Q}^2 \tag{4.155}$$

so that $Q$ is given by

$$\tilde{Q} = \sqrt{\frac{25 - c_m}{6}}. \tag{4.156}$$

Finally we also need the interaction term $\mathrm{e}^{2\tilde{b}\sigma}$ to be a tensor of conformal weight $(1,1)$. The conformal weight $\Delta$ of this exponential is given by

$$\Delta = \tilde{b}(\tilde{Q} - \tilde{b}). \tag{4.157}$$

We then need

$$\tilde{b}(\tilde{Q} - \tilde{b}) = 1 \tag{4.158}$$

which yields

$$\tilde{Q} = \tilde{b} + \frac{1}{\tilde{b}}. \tag{4.159}$$

Note that this approach had been justified by direct computations [146, 151, 152] and should not be confounded with the DDK ansatz for conformal perturbations.

### 4.4.4 States and correlation functions

Several approaches can be used to quantize Liouville theory and compute the spectrum, the correlation functions and other properties [51]: canonical quantization [40, 42, 44], path integral, minisuperspace [41, 44, 153–155], conformal bootstrap, BRST quantization, multiplicative gaussian chaos... We will only review here the simpler approach, the minisuperspace approximation which we will used again for the Mabuchi theory in chapter 7.

The minisuperspace approximation has been widely used in quantum cosmology [156]: the idea consists in restricting the (super)space of metrics under consideration by imposing a symmetry. In two dimensions it is logical to consider the spacetime to be a cylinder, with a non-compact time direction and a periodic spatial direction (used to regularize IR divergences). In this case one can truncate the spatial dependence of the fields by performing a Kaluza–Klein reduction over the circle and by keeping only the (spatial) zero-mode, which results in a quantum-mechanical model. The canonical quantization of this model allows one to determine the spectrum: the resulting Hilbert space can then be interpreted as the one-particle Hilbert space of the full theory from which the Fock space is built by a sum of tensor products [157] (conceptually this is what is presented in many standard textbooks on QFT when one introduces the Fock space of the free scalar field). The wave functions can finally be used to compute correlation functions.

In $d$-dimensional quantum cosmology the validity of such an approximation is not clear, but one can expect it to be well justified in two dimensions. Indeed in two-dimensional gravity with conformal matter the length of the circle (given by the zero-mode $\ell = \mathrm{e}^{\phi}$) specifies completely the spatial geometry up to diffeomorphisms. From this naive argument the superspace and minisuperspace should be equivalent [158] (motivations for this statement using matrix models are also presented in [158]), but it was found in cases where the full theory is under control that there may be corrections to the correlation functions [159]. Nonetheless the spectrum is less sensible to quantum corrections and the minisuperspace in two dimensions is also expected to be equivalent to the semi-classical limit [159] (which can be explicitly checked in the case of the Liouville theory), and hence one may compare the results obtained with the complete theory. In the absence of any other controlled approximations the minisuperspace, by its simplicity, provides a first analysis that may help to refine the questions and give indications about the general features of the model.

From the point of view of two-dimensional CFTs each wave function found in the minisuperspace lifts to a primary operator [160] (additional primary operators can arise due to the fact that the minisuperspace is blind to the winding modes around the spatial direction). This method has been used for numerous CFTs [159–166] and other two-dimensional theories [155, 167–171], and the validity of the minisuperspace approach could be checked in several examples where the full theory is also under control. Additional support for the minisuperspace in the Liouville theory can be found in [153].

Hence we remove the spatial dependence of the Liouville field and we consider the background to be flat (Lorentzian) cylinder:

$$\sigma = \sigma(t), \qquad \hat{g} = \eta. \tag{4.160}$$

With the above conditions the minisuperspace Liouville action reads

$$S_L = \int \mathrm{d}t \left( \frac{\dot{\sigma}^2}{2} - 2\pi\mu\, \mathrm{e}^{2b\sigma} \right). \tag{4.161}$$

The conjugate momentum

$$p = \frac{\delta S_L}{\delta \dot{\sigma}} = \dot{\sigma} \tag{4.162}$$

is used to construct the Hamiltonian

$$H_L = p\dot{\sigma} - L = \frac{p^2}{2} + 2\pi\mu\, \mathrm{e}^{2b\sigma}. \tag{4.163}$$

Note that this Hamiltonian is correctly positive definite.

We can now proceed with the canonical quantization by doing the replacement

$$p = -i\,\frac{\mathrm{d}}{\mathrm{d}\sigma}. \tag{4.164}$$

Then the stationary Schrödinger equation

$$H_L\psi_p = 2p^2\psi_p \tag{4.165}$$

corresponds the differential equation

$$\left(-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\sigma^2} + 2\pi\mu\mathrm{e}^{2b\sigma} - 2p^2\right)\psi_p = 0. \tag{4.166}$$

We then do a change of variable

$$\ell = \mathrm{e}^{b\sigma} \tag{4.167}$$

to bring the equation to the form

$$\left(\ell^2\frac{\mathrm{d}}{\mathrm{d}\ell^2} + \ell\,\frac{\mathrm{d}}{\mathrm{d}\ell} - 4(\hat{\mu}\,\ell^2 - \hat{p}^2)\right)\psi_p = 0 \tag{4.168}$$

where

$$\hat{\mu} = \frac{\pi\mu}{b^2}, \qquad \hat{p} = \frac{p}{b}. \tag{4.169}$$

We recognize the modified Bessel equations whose solutions are

$$\psi_p(\ell) = \alpha_p\,K_{2i\hat{p}}(2\sqrt{\hat{\mu}}\,\ell) + \beta_p\,I_{2i\hat{p}}(2\sqrt{\hat{\mu}}\,\ell). \tag{4.170}$$

The asymptotic behaviour of these functions

$$I_\nu(x) \sim_\infty \frac{\mathrm{e}^x}{\sqrt{2\pi x}}, \qquad K_\nu(x) \sim_\infty \sqrt{\frac{\pi}{2x}}\,\mathrm{e}^{-x} \tag{4.171}$$

requires to set $\beta_p = 0$ in order to remove the exponential growth at infinity. Finally in order to fix $\alpha_p$ the solutions are normalized such that the incoming plane waves have unit coefficient as $\sigma \to -\infty$. Using the formula

$$K_\nu(x) \sim_0 \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu + \frac{\Gamma(-\nu)}{2}\left(\frac{x}{2}\right)^\nu, \tag{4.172}$$

the coefficient of normalization reads

$$\alpha_p = \frac{2\hat{\mu}^{-i\hat{p}}}{\Gamma(-2i\hat{p})}. \tag{4.173}$$

As a consequence the wave functions and its asymptotic form reads

$$\psi_p(\ell) = \frac{2\hat{\mu}^{-i\hat{p}}}{\Gamma(-2i\hat{p})}\,K_{2i\hat{p}}(2\sqrt{\hat{\mu}}\,\ell) \tag{4.174a}$$

$$\sim_0 \mathrm{e}^{2i\hat{p}\sigma} + r_0(p)\mathrm{e}^{-2i\hat{p}\sigma}. \tag{4.174b}$$

The factor $r_0(p)$ corresponds to the reflection coefficient due to the exponential wall

$$r_0(p) = \frac{\Gamma(2i\hat{p})}{\Gamma(-2i\hat{p})}\,\hat{\mu}^{-2i\hat{p}}. \tag{4.175}$$

Because of the wall wave functions with $p$ and $-p$ are not independent

$$\psi_{-p}(\ell) = r_0(-p)\,\psi_p(\ell). \tag{4.176}$$

In order to interpret the spectrum it is necessary to bring the theory back to the plane (we can then do a conformal transformation to go to another background). The Hamiltonian on the latter is given by the dilatation operator $L_0 + \bar{L}_0$ and the associated wave functions are solutions of

$$(L_0 + \bar{L}_0)\psi_\Delta = 2\Delta\,\psi_\Delta \tag{4.177}$$

where $\Delta$ is the conformal weight. Through a conformal transformation the Hamiltonians on the plane and on the cylinder are related by

$$L_0 + \bar{L}_0 - \frac{c}{12} = H_0 - \frac{1}{12}, \tag{4.178}$$

(the last factor corresponds to a zero-point energy), $c$ being the central charge of Liouville theory

$$c = 1 + 6Q^2. \tag{4.179}$$

Comparing these equations with (7.47) teaches that the conformal dimension is related to $p$ by

$$\Delta = \frac{Q^2}{4} + p^2. \tag{4.180}$$

Moreover the states $V_p = \mathrm{e}^{2ip\sigma}$ on the cylinder are mapped to states $V_a = \mathrm{e}^{2a\sigma}$ on the plane where the relation between $a$ and $p$ is

$$a = \frac{Q}{2} + ip. \tag{4.181}$$

Since the Liouville theory is unitary the conformal weights should be positive, i.e. $\Delta \geq 0$, which implies $p \in \mathbb{R}_+$ or $p \in i[0, Q/2)$ (only half of the intervals are considered as a consequence of the reflection).

Until this point there was no restriction on the values of $p$. The first condition that is imposed is unitarity, i.e. $\Delta \geq 0$, which implies $p \in \mathbb{R}_+$ or $ip \in [0, Q/2)$ (only half of the intervals are considered as a consequence of the reflection). Finally the states that form the Hilbert space needs to be (delta-function) normalizable under the canonical inner product. It can be seen that this condition is fulfilled only for $p \in \mathbb{R}$

$$\int_{-\infty}^{\infty} \mathrm{d}\sigma\; \psi_p^*(\sigma)\psi_{p'}(\sigma) = \int_0^\infty \frac{\mathrm{d}\ell}{\ell}\; \psi_p^*(\ell)\psi_{p'}(\ell) = \pi\,\delta(p - p'), \tag{4.182}$$

which moreover form a complete basis. On the other hand the inner product is infinite for the wave functions with $p \in [0, Q/2)$. Hence the states with momentum $p \in \mathbb{R}$ defines Liouville theory: these states are naturally selected in the conformal bootstrap (see [49] for a recent review). On the other hand the non-normalizable states have also a physical interpretation in the context of $2d$ quantum gravity, as was emphasized in [44].

Finally the wave functions can be used to compute a semi-classical approximation to the 3-point structure constant $C(a_1, a_2, a_3)$ (see for example [155, 170]). In particular the limit $b \to 0$ of $C(a_1, a_2, a_3)$ evaluated with the following weights (to simplify the notations we omit the hat on $p_i$ and $\gamma$ but these quantities really correspond to the hatted ones of the previous formulas).

$$a_1 = \frac{Q}{2} + ibp_1, \qquad a_2 = b\gamma, \qquad a_3 = \frac{Q}{2} + ibp_3 \tag{4.183}$$

matches the integral (assuming $\sigma \equiv ip_2 > 0$)

$$C_0(a_1, a_2, a_3) = \int_{-\infty}^{\infty} \mathrm{d}\sigma\; \psi_{bp_1}(\sigma)\mathrm{e}^{2b\gamma\sigma}\psi_{bp_3}(\sigma) \tag{4.184a}$$

$$= \frac{1}{b}\left(\frac{\pi\mu}{b^2}\right)^{-2\tilde{p}} \Gamma(2\tilde{p}) \prod_i \frac{\Gamma\big((-1)^i 2\tilde{p}_i\big)}{\Gamma(2p_i)} \tag{4.184b}$$

where we defined

$$2\tilde{p} = \sum_i p_i, \qquad \tilde{p}_i = \tilde{p} - p_i, \qquad i = 1, 2, 3. \tag{4.185}$$

## 4.5 The Mabuchi action

We recall the expressin of The Mabuchi action (4.131)

$$S_M = \frac{1}{4\pi}\int \mathrm{d}^2x\,\sqrt{\hat{g}}\left\{ -\pi\chi\,\hat{g}^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + \left(\frac{4\pi\chi}{\hat{A}} - \hat{R}\right)\phi \right.$$
$$\left. + \frac{2}{\hat{A}}\left(1+\frac{\hat{A}}{2}\hat{\Delta}\phi\right)\ln\left[\frac{A}{\hat{A}}\left(1+\frac{\hat{A}}{2}\hat{\Delta}\phi\right)\right]\right\}. \tag{4.186}$$

The equation of motion for $\phi$ (or for $\sigma$ at fixed $A$) is

$$R = \frac{4\pi\chi}{A}. \tag{4.187}$$

It is the same equation as the one of Liouville (4.144). This means that both functionals have constant scalar curvature as saddle point.

It was shown in [62, 63] that the Mabuchi action appears in the gravitational action of a massive scalar field (at leading order in a small mass expansion). The properties of this action have been further studied in [71] (see also [64, 67, app. F]).

It is not known whether the Mabuchi action defines a CFT but it seems unlikely to be the case: the non-conformal matter action is not invariant by itself while the total action should be invariant, and hence the non-invariance of the matter action should be compensated by the transformation of the Mabuchi action.

### 4.5.1 Duality between the Liouville and Mabuchi actions

Here we show that the kinetic/potential terms of Liouville (with cosmological constant) (4.89) and Mabuchi (4.131) actions are separately dual under a Legendre transformation (but the full actions are not) [147]. The kinetic and potentials terms of the Liouville action in Lorentzian signature are

$$T_L[\sigma] = -\sigma\hat{\Delta}\sigma + Q\hat{R}\sigma, \qquad V_L[\sigma] = -2\pi\mu\mathrm{e}^{2b\sigma}. \tag{4.188}$$

Let

$$T_M[\sigma,\widehat{\sigma}] = \sigma\widehat{\sigma} - T_L = \sigma\widehat{\sigma} + \sigma\hat{\Delta}\sigma - Q\hat{R}\sigma. \tag{4.189}$$

One has

$$\frac{\delta T_M}{\delta\sigma} = \widehat{\sigma} + 2\hat{\Delta}\sigma - Q\hat{R} \tag{4.190}$$

$T_M$ is then extremal for

$$\hat{\Delta}\sigma = \frac{1}{2}\left(Q\hat{R} - \widehat{\sigma}\right) \Longleftrightarrow \sigma = \frac{1}{2}\left(Q\hat{\Delta}^{-1}\hat{R} - \hat{\Delta}^{-1}\widehat{\sigma}\right) \tag{4.191}$$

Let's do a change of variable:

$$\widehat{\sigma} = 2\hat{\Delta}\varphi + \epsilon\hat{R} \tag{4.192}$$

Then one has

$$\sigma = \frac{1}{2}\left((Q-\epsilon)\hat{\Delta}^{-1}\hat{R} - 2\varphi\right) \tag{4.193}$$

Plugging this value back gives

$$T_M = -\varphi\hat{\Delta}\varphi - \frac{\epsilon+3Q}{2}R\varphi + \frac{Q-\epsilon}{2}\hat{\Delta}^{-1}\hat{R}\hat{\Delta}\varphi + \frac{(Q-\epsilon)(\epsilon+3Q)}{4}\hat{R}\hat{\Delta}^{-1}\hat{R}. \tag{4.194}$$

Taking $\epsilon = \frac{Q}{2}$ gives

$$T_M = -\varphi\hat{\Delta}\varphi - QR\varphi \tag{4.195}$$

Next concerning the potential term one has

$$V_M = \sigma\widehat{\sigma} - V_L = \sigma\widehat{\sigma} + 2\pi\mu \mathrm{e}^{2b\sigma} \tag{4.196}$$

for which the variation reads

$$\widehat{\sigma} = -4\pi\mu b\, \mathrm{e}^{2b\sigma} \quad \Longrightarrow \quad \sigma = \frac{1}{2b} \ln\left(-\frac{\widehat{\sigma}}{4\pi\mu b}\right). \tag{4.197}$$

Plugging back provides the expression

$$V_M = \frac{\widehat{\sigma}}{2b}\left[\ln\left(-\frac{\widehat{\sigma}}{4\pi\mu b}\right) - 1\right] \tag{4.198}$$

Setting

$$\frac{\widehat{\sigma}}{4\pi\mu b} = -\frac{A}{\hat{A}}\left(1 + \frac{\hat{A}}{2\pi\chi}\hat{\Delta}\varphi\right) \tag{4.199}$$

yields finally

$$V_M = -2\pi\mu \frac{A}{\hat{A}}\left(1 + \frac{\hat{A}}{2\pi\chi}\hat{\Delta}\varphi\right)\left[\ln\frac{A}{\hat{A}}\left(1 + \frac{\hat{A}}{2\pi\chi}\hat{\Delta}\varphi\right) - 1\right]. \tag{4.200}$$

Adding together $T_M$ and $V_M$ one recognizes the Mabuchi action (4.131) in Euclidean signature after identifying

$$\varphi = \phi, \qquad -2\mu = \frac{\chi}{A}. \tag{4.201}$$

This is the same relation as in (4.148) with $b = 1$. We will find again this relation in our minisuperspace analysis in chapter 7. The value for $\widehat{\phi}$ corresponding to the Legendre transformations of the kinetic and potential terms are not the same: as a consequence it is not possible to transform the full action.

Note that if one works with the unscaled Mabuchi action then we obtain the following identifications:

$$b^2 = \pi\chi, \qquad 2\mu = \frac{1}{A}. \tag{4.202}$$

which also corresponds to (4.148) (but now with $b \neq 1$).

### 4.5.2 Hamiltonian

In this section we will compute the Hamiltonian of the Mabuchi action. This will be useful for the minisuperspace computations in chapter 7. In this perspective we will introduce a parameter $\alpha$ to enable us to consider both the unscaled and the scaled actions simultaneously: $\alpha = 1$ corresponds to (4.131) (with the boundary term) and $\alpha = (\pi\chi)^{-1}$ to (7.9). Moreover we introduce a parameter $\epsilon = \pm 1$ used to consider both Euclidean and Lorentzian signatures. We consider the Mabuchi action where the Liouville field has been replaced using (4.118) and where we add a boundary term which is a total derivative:

$$\begin{aligned} S_M = \frac{\epsilon}{4\pi}\int \mathrm{d}^2x\,\sqrt{\hat{g}}\,\Big[ &-\pi\chi\alpha\,\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \left(\frac{4\pi\chi}{\hat{A}} - \hat{R}\right)\phi \\ &+ \frac{2}{\hat{A}\alpha}\left(1 + \frac{\hat{A}\alpha}{2}\hat{\Delta}\phi\right)\left(\ln\frac{A}{\hat{A}}\left(1 + \frac{\hat{A}\alpha}{2}\hat{\Delta}\phi\right) - 1\right)\Big], \end{aligned} \tag{4.203}$$

The strategy for computing the Hamiltonian is to perform first an ADM decomposition [172] of the metric in order to extract the time derivative of the Kähler potential before using the Ostrogradski formalism [173] since the action is of second order in time. Note

that the background metric $\hat{g}$ is fixed and for this reason its components are not dynamical. In particular it is not necessary to decompose the curvature $\hat{R}$ and to apply the full ADM formalism.

The ADM decomposition of the metric is

$$\hat{g}_{\mu\nu} = \mathrm{e}^{2\rho} \begin{pmatrix} \epsilon N^2 + M^2 & M \\ M & 1 \end{pmatrix}, \qquad \hat{g}^{\mu\nu} = \frac{\epsilon\, \mathrm{e}^{-2\rho}}{N^2} \begin{pmatrix} 1 & -M \\ -M & \epsilon N^2 + M^2 \end{pmatrix}. \tag{4.204}$$

where $\rho$, $N$ and $M$ are functions of the coordinates (note that the matrix part is flat). This decomposition is valid locally and the topology is hidden in the coordinates [6] and in the values of the conformal factor $\rho$. In particular the latter has to be singular if $\chi \neq 0$ since

$$4\pi\chi = \int \mathrm{d}^2x \sqrt{\hat{g}}\, \hat{R} = \int \mathrm{d}^2x\, \partial^2 \rho \tag{4.205}$$

where $\partial^2$ is the flat Laplacian. Nonetheless we will not work directly with its value and the fact that it contains singularities does not matter.

The square root of the metric determinant is

$$\sqrt{|\hat{g}|} = N\mathrm{e}^{2\rho}. \tag{4.206}$$

The Laplacian is

$$\begin{aligned} \hat{\Delta} = \frac{\epsilon\, \mathrm{e}^{-2\rho}}{N^2} \Big[ \partial_\tau^2 + (\epsilon N^2 + M^2)\, \partial_\sigma^2 - 2M \partial_\tau \partial_\sigma + \left( \frac{MN'}{N} - M' - \frac{\dot{N}}{N} \right) \partial_\tau \\ + \left( 2(\epsilon N' N + M' M) + \frac{M\dot{N}}{N} - (\epsilon N^2 + M^2) \frac{N'}{N} - \dot{M} \right) \partial_\sigma \Big]. \end{aligned} \tag{4.207}$$

The kinetic term of the action (4.203) is

$$-\frac{\epsilon\pi\chi\alpha}{2} \sqrt{\hat{g}}\, \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = -\frac{\pi\chi\alpha}{2N} \left( \dot{\phi}^2 - 2M\dot{\phi}\phi' + (\epsilon N^2 + M^2)\phi'^2 \right). \tag{4.208}$$

It is not needed to decompose the curvature $\hat{R}$ because only $\phi$ is dynamical, not the background metric $\hat{g}$.

Now one can apply the Ostrogradski formalism. The independent variables are $\{\phi, \dot{\phi}\}$ with conjugate momenta $\{\mathcal{P}, P\}$

$$P = \frac{\partial L}{\partial \ddot{\phi}}, \qquad \mathcal{P} = \frac{\partial L}{\partial \dot{\phi}} - \partial_\tau P - \partial_\sigma \frac{\partial L}{\partial \dot{\phi}'} \tag{4.209}$$

and the Hamiltonian reads

$$H = \mathcal{P}\dot{\phi} + P\ddot{\phi} - L \tag{4.210}$$

where the Lagrangian is normalized such that

$$S = \frac{1}{2\pi} \int \mathrm{d}^2x\, L. \tag{4.211}$$

Note that one could also consider $\phi'$ to be an independent variable. Then the last term in $\mathcal{P}$ would correspond to the derivative of its conjugate momentum. The resulting Hamiltonian would then be equivalent to the one obtained below upon integration by part.

The momentum $P$ is

$$P = \frac{1}{2N} \ln \frac{A}{\hat{A}} \left( 1 + \frac{\hat{A}\alpha}{2} \hat{\Delta}\phi \right). \tag{4.212}$$

Using the relation (7.3) one recognizes that the RHS of $P$ is proportional to $\sigma$ and for this reason one can perform a canonical transformation to invert the roles of position and momentum (after having computed the Hamiltonian)

$$NP = \sigma, \qquad \dot{\phi} = -N\Pi. \tag{4.213}$$

Moreover the above expression can be used to solve for $\ddot{\phi}$ in terms of the canonical variables using (4.207).

The second momentum $\mathcal{P}$ is

$$\mathcal{P} = \pi\chi\alpha\,\Pi - \frac{1}{N}\,\dot{\sigma} + \frac{2M}{N}\,\sigma' + \pi\chi\alpha\,\frac{M}{N}\,\phi' - \frac{1}{N}\left(\frac{MN'}{N} - M'\right)\sigma. \tag{4.214}$$

Ultimately one finds the Hamiltonian

$$\begin{aligned} H = {} & \frac{\pi\chi\alpha}{2}\,N\,\Pi^2 - N\,\Pi\mathcal{P} + 2M\,\Pi\sigma' + \left(M' - \frac{MN'}{N} - \frac{\dot{N}}{N}\right)\Pi\sigma + \pi\chi\alpha\,M\,\Pi\phi' \\ & + \frac{\pi\chi\alpha}{2}\,(\epsilon N^2 + M^2)\phi'^2 - \frac{1}{N}\,(\epsilon N^2 + M^2)\phi''\sigma \\ & + \frac{1}{N}\left(\dot{M} + (\epsilon N^2 + M^2)\frac{N'}{N} - 2(\epsilon NN' + MM') - \frac{M\dot{N}}{N}\right)\phi'\sigma \\ & - \frac{\epsilon N}{2}\,\mathrm{e}^{2\rho}\left(\frac{4\pi\chi}{\hat{A}} - \hat{R}\right)\phi + \frac{\epsilon N}{A\alpha}\,\mathrm{e}^{2\rho}\mathrm{e}^{2\sigma} - \frac{2\epsilon N}{\hat{A}\alpha}\,\mathrm{e}^{2\rho}\sigma. \end{aligned} \tag{4.215}$$

In view of the minisuperspace analysis of chapter 7 it is interesting to look at what happens in Lorentzian signature ($\epsilon = -1$) and consider the case where there is no spatial dependence. Going to the flat gauge

$$N = 1, \qquad M = 0, \qquad \rho = 0, \tag{4.216}$$

but keeping $\hat{R}, \chi \neq 0$ for comparison, one finds

$$H = \frac{\pi\chi\alpha}{2}\,\Pi^2 - \Pi\mathcal{P} + \frac{1}{2}\left(\frac{4\pi\chi}{\hat{A}} - \hat{R}\right)\phi + \frac{2}{\hat{A}\alpha}\,\sigma - \frac{1}{A\alpha}\,\mathrm{e}^{2\sigma} \tag{4.217}$$

For $\alpha = 1$ this Hamiltonian has no particularly meaningful limit $\hat{R}, \chi \longrightarrow 0$, and in particular it contains a term linear in $\sigma$ which could lead to an instability. However for $\alpha = (\pi\chi)^{-1}$ one finds

$$H = \frac{\Pi^2}{2} - \mathcal{P}\Pi + \frac{1}{2}\left(\frac{4\pi\chi}{\hat{A}} - \hat{R}\right)\phi + \frac{2\pi\chi}{\hat{A}}\,\sigma - \frac{\pi\chi}{A}\,\mathrm{e}^{2\sigma}. \tag{4.218}$$

an expression we will derive again in (7.41) in the minisuperspace approximation. Taking the limit $\hat{R}, \chi \longrightarrow 0$ gives the minisuperspace Mabuchi Hamiltonian (7.12).

# Chapter 5

# Massive matter on a Riemann surface with boundaries

In this chapter, we study the gravitational action of a massive scalar field $X$ on a two-dimensional Riemann surface with boundaries. The action for the scalar field is given by

$$S_m[g, X] = \frac{1}{2} \int \mathrm{d}^2x \sqrt{g} \left( g^{\mu\nu} \partial_\mu X \partial_\nu X + m^2 X^2 \right) \tag{5.1}$$

As already mentioned earlier, the massive scalar field coupled to gravity on a manifold without boundary has been studied in [63] and [70] and we want to study how these results get modified when the two-dimensional Riemann surface $\mathcal{M}$ has boundaries $\partial\mathcal{M}$. A priori, two things could happen: the corresponding gravitational actions could get additional boundary contributions, and the bulk gravitational Lagrangian at a point $x$ could explicitly depend on the geodesic distances between $x$ and the boundaries. We will indeed observe both of these. These results have been published in [78].

In the following we will place ourselves in conformal gauge

$$g = \mathrm{e}^{2\sigma} \hat{g} \tag{5.2}$$

and study the dynamics of the Liouville mode. We will ignore the contribution of the ghost that arises from this gauge fixing (see chapter 4). We denote by $\Delta$ the negative Laplace-Beltrami operator (so that its eigenvalues are positive):

$$\Delta = -\Delta_{\text{scalar}} = -\frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu) \tag{5.3}$$

It is related to $\hat{\Delta}$ by

$$\Delta = \mathrm{e}^{-2\sigma} \hat{\Delta} \tag{5.4}$$

When the manifold has no boundary the action may be rewritten as

$$S_m[g, X] = \frac{1}{2} \int \mathrm{d}^2x \sqrt{g} X \left( \Delta + m^2 \right) X \tag{5.5}$$

A key point for the following will be to impose conditions under which this equality still holds.

## 5.1 Integration on a surface with boundaries

### 5.1.1 Boundary integrals

Integration by parts on a two-dimensional manifold with boundaries generates boundary terms. The natural way to implement this is via Stoke's theorem for the integration of an exact 2-form.

Let $\alpha = \alpha_\mu \mathrm{d}x^\mu$ be a 1-form on $\mathcal{M}$ and $\widehat{\alpha}$ its restriction to $\partial\mathcal{M}$ (i.e. the pull-back of $\alpha$ under the inclusion map of $\partial\mathcal{M}$ into $\mathcal{M}$): $\widehat{\alpha} = \alpha_\mu \mathrm{d}\widehat{x}^\mu$ where the $\mathrm{d}\widehat{x}^\mu$ are the "projections" of the $\mathrm{d}x^\mu$ on the tangent space to $\partial\mathcal{M}$. We define the (not necessarily normalized) tangent vector $t^\mu$ as

$$\mathrm{d}\widehat{x}^\mu = t^\mu \mathrm{d}l \tag{5.6}$$

where $\mathrm{d}l$ is the proper length one-form on $\partial\mathcal{M}$. Thus

$$\widehat{\alpha} = \alpha_\mu t^\mu \mathrm{d}l. \tag{5.7}$$

As an example, let $\mathcal{M}$ be the unit sphere $S^2$, with standard coordinates $(\theta, \varphi)$, with the polar cap $\theta \leq \theta_0$ removed. Then $\partial\mathcal{M}$ is the circle at $\theta = \theta_0$, so that

$$\mathrm{d}\widehat{\theta} = 0, \qquad \mathrm{d}\widehat{\varphi} = \mathrm{d}\varphi. \tag{5.8}$$

As $\mathrm{d}l = \sin\theta \, \mathrm{d}\varphi$ the tangent vector components are

$$t^\theta = 0, \qquad t^\varphi = \frac{1}{\sin\theta}. \tag{5.9}$$

Integrating the 1-form $\widehat{\alpha}$ over $\partial\mathcal{M}$ then yields

$$\int_{\partial\mathcal{M}} \widehat{\alpha} = \int_{\partial\mathcal{M}} \alpha_\mu \mathrm{d}\widehat{x}^\mu = \int_{\partial\mathcal{M}} \alpha_\mu t^\mu \, \mathrm{d}l \ , \tag{5.10}$$

without the need to introduce a metric. However, sometimes, the 1-form $\alpha$ is the Hodge dual of some other 1-form $\beta$, i.e. $\alpha = {}^*\beta$, and this requires a metric. Indeed, we have

$${}^*\mathrm{d}x^\mu = g^{\mu\nu}\epsilon_{\nu\rho}\mathrm{d}x^\rho = g^{\mu\nu}\sqrt{g}\,\widehat{\epsilon}_{\nu\rho}\mathrm{d}x^\rho, \tag{5.11}$$

where $\widehat{\epsilon}_{12} = 1$, $\widehat{\epsilon}_{21} = -1$. Then ${}^*\beta = \beta_\mu {}^*\mathrm{d}x^\mu = \beta_\mu g^{\mu\nu}\epsilon_{\nu\rho}\,\mathrm{d}x^\rho$ and

$$\widehat{{}^*\beta} = \beta_\mu g^{\mu\nu}\epsilon_{\nu\rho}\,\mathrm{d}\widehat{x}^\rho = \beta_\mu g^{\mu\nu}\epsilon_{\nu\rho}t^\rho\,\mathrm{d}l. \tag{5.12}$$

One then defines the not necessarily normalized normal vector $n^\mu$ as

$$n^\mu = g^{\mu\nu}\epsilon_{\nu\rho}t^\rho = g^{\mu\nu}\sqrt{g}\,\widehat{\epsilon}_{\nu\rho}t^\rho. \tag{5.13}$$

which implies $\widehat{{}^*\beta} = \beta_\mu n^\nu \mathrm{d}l$. Note that $g_{\mu\nu}n^\mu t^\nu = \epsilon_{\mu\nu}t^\mu t^\nu = 0$, as expected. For the above example of the sphere with the polar cap removed we have $g^{\varphi\varphi} = \frac{1}{\sin^2\theta}$, $\sqrt{g} = \sin\theta$ and

$$n^\theta = \sin\theta_0, \qquad t^\varphi = 1, \qquad n^\varphi = 0. \tag{5.14}$$

Integration by parts follows from Stoke's theorem,

$$\int_{\mathcal{M}} \mathrm{d}\gamma = \int_{\partial\mathcal{M}} \widehat{\gamma} \ , \tag{5.15}$$

for any one-form $\gamma$. We are mostly interested in $\mathrm{d}\gamma$ being a kinetic term, i.e. $\gamma = \phi\,{}^*\mathrm{d}\phi = {}^*(\phi\,\mathrm{d}\phi)$. Then $\mathrm{d}\gamma = \mathrm{d}\phi\wedge{}^*\mathrm{d}\phi + \phi\,\mathrm{d}\,{}^*\mathrm{d}\phi$. We have $\mathrm{d}\phi\wedge{}^*\mathrm{d}\phi = \mathrm{d}x^1\mathrm{d}x^2\,\sqrt{g}\,g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, so that

$$\phi\,\mathrm{d}\,{}^*\mathrm{d}\phi = \phi\,\partial_\mu(g^{\mu\nu}\sqrt{g}\,\partial_\nu\phi)\mathrm{d}x^1\wedge\mathrm{d}x^2 = -\mathrm{d}x^1\wedge\mathrm{d}x^2\sqrt{g}\,\phi\Delta\phi \ , \tag{5.16}$$

Then Stoke's theorem gives the following integration by parts formula

$$\int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}\,g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}\,\phi\Delta\phi + \int_{\partial\mathcal{M}} \mathrm{d}l\,n^\mu\phi\,\partial_\mu\phi \ . \tag{5.17}$$

$\mathrm{d}l$ and $n^a$ are related to $\mathrm{d}\hat{l}$ and $\hat{n}^a$ by

$$\mathrm{d}l = \mathrm{e}^\sigma\mathrm{d}\hat{l}, \qquad n^a = \mathrm{e}^{-\sigma}\hat{n}^a. \tag{5.18}$$

We then also have

$$\partial_n \equiv n^a \partial_a = \mathrm{e}^{-\sigma} \partial_{\hat{n}} \tag{5.19}$$

One sees that

$$\mathrm{d}l\, \partial_n \delta\sigma = \mathrm{e}^{\sigma} \mathrm{d}l_0\, \mathrm{e}^{-\sigma} \partial_n^0 \delta\sigma = \mathrm{d}l_0\, \partial_n^0 \delta\sigma = \delta(\mathrm{d}l_0\, \partial_n^0 \sigma) = \delta(\mathrm{d}l\, \partial_n \sigma) \tag{5.20}$$

which is useful to obtain the expression of the Liouville action on a Riemann surface with boundaries.

### 5.1.2 Boundary conditions

In the presence of boundaries, we have to impose some boundary conditions. Our choice will be guided by two requirements: we want $\Delta + m^2$ to be hermitian and we want to preserve the fact that

$$\int_{\mathcal{M}} \mathrm{d}^2x \sqrt{g}\, \Delta f \equiv \int_{\mathcal{M}} \mathrm{d}^2x \sqrt{\hat{g}}\, \hat{\Delta} f = 0. \tag{5.21}$$

The hermiticity condition

$$(\varphi_1, \Delta\varphi_2) \equiv \int_{\mathcal{M}} \mathrm{d}^2x \sqrt{g}\, \overline{\varphi}_1 \Delta\varphi_2 = \int_{\mathcal{M}} \mathrm{d}^2x \sqrt{g}\, \overline{\Delta\varphi_1}\, \varphi_2 \equiv (\Delta\varphi_1, \varphi_2) \tag{5.22}$$

implies that the boundary term

$$\int_{\partial\mathcal{M}} \mathrm{d}l\, n^\mu \left(\overline{\partial_\mu \varphi_1} \varphi_2 - \bar{\varphi}_1 \partial_\mu \varphi_2\right) , \tag{5.23}$$

must vanish. As usual, this leads to two possible choices of boundary conditions: either $\varphi = 0$ (Dirichlet) or $n^a \partial_a \varphi = 0$ (Neumann) on the boundary. Actually, the modified Neumann (Robin) conditions $n^a \partial_a \varphi = c\, \varphi$ with real $c$ are also possible. Our second condition reads

$$0 = \int_{\mathcal{M}} \mathrm{d}^2x \sqrt{g}\, \Delta f = \int_{\partial\mathcal{M}} \mathrm{d}l\, n^\mu \partial_\mu f \ , \tag{5.24}$$

selecting the Neumann boundary conditions. In particular, if the massive matter field(s) $X$ obey these boundary conditions, one may freely integrate by parts in the matter action and (5.5) still holds for a manifold $\mathcal{M}$ with boundaries. From now on, we will always assume that the matter field(s) obey Neumann boundary conditions. What about the Kähler field $\phi$ and the conformal factor $\sigma$? It follow from (4.118) that $\phi$ also must satisfy Neumann conditions (in the metric $\hat{g}$). Indeed, the area should be given by $A = \int \sqrt{\hat{g}}\, \mathrm{e}^{2\sigma}$ which, by (4.118) implies that $0 = \int \sqrt{\hat{g}}\, \hat{\Delta}\phi$ which is possible only if $\hat{n}^a \partial_a \phi = 0$ on $\partial\mathcal{M}$. This same relation (4.118) also implies $\partial_n \sigma = -\frac{A}{4} e^{-2\sigma} \partial_n(\hat{\Delta}\phi)$, showing that it is not compatible to impose Neumann boundary conditions also on $\sigma$.

## 5.2 The gravitational action

Following the notations of chapter 3 we call $\varphi_n$ and $\lambda_n$ the orthonormalised eigenfunctions and eigenvalues of the hermitian (thanks to the boundary condition) differential operator appearing in $S_m$:

$$(\Delta + m^2)\varphi_n = \lambda_n \varphi_n \, , \quad \int \mathrm{d}^2x \sqrt{g}\, \varphi_n \varphi_m = \delta_{nm} \, , \quad n^\mu \partial_\mu \varphi_m = 0 \text{ on } \partial\mathcal{M}. \tag{5.25}$$

Since $\Delta + m^2$ is real, one may choose the eigenfunctions $\varphi_n$ to be real, which we will always assume (unless an obvious complex choice has been made, like the standard spherical harmonics on the round sphere). We take the indices $n$ to be $n \geq 0$ with $n = 0$ referring to

the lowest eigenvalue. In particular, the Laplace operator always has a constant zero-mode, $\varphi_0 = \frac{1}{\sqrt{A}}$ and thus $\lambda_0 = m^2$, since this constant obviously obeys the Neumann boundary condition.

As explained in chapter 3 the gravitational action is given by

$$S_{\text{grav}}[\hat{g}, g] = \frac{1}{2} \ln \frac{\det(\Delta + m^2)}{\det(\hat{\Delta} + m^2)}. \tag{5.26}$$

In the massless case, we need to exclude the zero-mode. Using the regularization of determinants in terms of the zeta function (3.82) we finally arrive at

$$S_{\text{grav}}[\hat{g}, g] = -\frac{1}{2} \left( \zeta_g'(0) + \zeta_g(0) \ln \widehat{\mu}^2 \right) + \frac{1}{2} \left( \zeta_{\hat{g}}'(0) + \zeta_{\hat{g}}(0) \ln \widehat{\mu}^2 \right). \tag{5.27}$$

where $\widehat{\mu}$ is some arbitrary mass scale. It is important to notice that this formula expressing the gravitational action in terms of the zeta function is true whether the Riemann surface has a boundary or not. Of course, the zeta function for a manifold with boundary will have some properties that differ from the case without boundary. Formally, the zeta functions are always defined by (3.60), but the properties of the manifold are encoded in the eigenvalues $\lambda_n$ that appear in the sum.

The strategy of [63] and [70], that we will also follow here, was to determine the infinitesimal change of the zeta functions from the infinitesimal change of the eigenvalues $\lambda_n$ under an infinitesimal change of the metric, and then to integrate this relation to get $S_{\text{grav}}[\hat{g}, g]$. The change of the eigenvalues is obtained from (almost) standard quantum mechanical perturbation theory, as we discuss next.

### 5.2.1 Perturbation theory

We want to study how the eigenvalues $\lambda_n$ and eigenfunctions $\varphi_n$ change under an infinitesimal change of the metric. Since $g = \mathrm{e}^{2\sigma}\hat{g}$, the Laplace operator $\Delta$ and hence also $\Delta + m^2$ only depend on the conformal factor $\sigma$ and on $\hat{g}$: $\Delta = \mathrm{e}^{-2\sigma}\hat{\Delta}$ and thus under a variation $\delta\sigma$ of $\sigma$ one has

$$\delta\Delta = -2\delta\sigma\hat{\Delta} \quad \Rightarrow \quad \langle\varphi_k|\delta\Delta|\varphi_n\rangle = -2(\lambda_n - m^2)\langle\varphi_k|\delta\sigma|\varphi_n\rangle, \tag{5.28}$$

where, of course, $\langle\varphi_k|\delta\sigma|\varphi_n\rangle = \int \mathrm{d}^2x\sqrt{g}\,\varphi_k\delta\sigma\varphi_n$. One can then apply standard quantum mechanical perturbation theory. The only subtlety comes from the normalisation condition in (5.25) which also gets modified when varying $\sigma$ [63, 132]: the scalar product in the metric $g + \delta g$ where $\delta g = 2\delta\sigma g$ is related to the one in the metric $g$ by

$$\langle\chi|\psi\rangle_{g+\delta g} = \langle\chi|\psi\rangle_g + 2\langle\chi|\delta\sigma|\psi\rangle_g. \tag{5.29}$$

One finds

$$\delta\lambda_n = -2(\lambda_n - m^2)\langle\varphi_n|\delta\sigma|\varphi_n\rangle\ , \tag{5.30}$$

$$\delta\varphi_n = -\langle\varphi_n|\delta\sigma|\varphi_n\rangle\,\varphi_n - 2\sum_{k\neq n}\frac{\lambda_n - m^2}{\lambda_n - \lambda_k}\langle\varphi_k|\delta\sigma|\varphi_n\rangle\,\varphi_k\ . \tag{5.31}$$

The first term in (5.31) comes from the modification of the normalisation condition.

Let us insists that this is first-order perturbation theory in $\delta\sigma$, but it is exact in $m^2$. Note the trivial fact that, since $\lambda_0 = m^2$, one consistently has

$$\delta\lambda_0 = 0. \tag{5.32}$$

### 5.2.2 Variation of the determinant

As mentioned above, in order to compute $S_{\rm grav}[\hat{g}, g]$ as given by (5.27), we will compute $\delta\zeta'(0) \equiv \delta\zeta'_g(0)$ and $\delta\zeta(0) \equiv \delta\zeta_g(0)$ and express them as "exact differentials" so that one can integrate them and obtain the finite differences $\zeta'_{g_2}(0) - \zeta'_{g_1}(0)$ and $\zeta_{g_2}(0) - \zeta_{g_1}(0)$.

From (5.30) one immediately gets, to first order in $\delta\sigma$,

$$\zeta_{g+\delta g}(s) = \sum_{n\geq 0} \frac{1}{(\lambda_n + \delta\lambda_n)^s} = \zeta_g(s) + 2s \sum_{n\geq 0} \frac{\lambda_n - m^2}{\lambda_n^{s+1}} \langle\varphi_n|\delta\sigma|\varphi_n\rangle \ , \tag{5.33}$$

As noted before, $\delta\lambda_0 = 0$ and, hence, there is no zero-mode contribution to the second term and one could just equally well rewrite the following results in terms of the $\widetilde{\zeta}$-functions defined by excluding the zero-mode [70]. We get

$$\delta\zeta(s) = 2s \int {\rm d}^2x\sqrt{g}\,\delta\sigma(x) \left[\zeta(s,x,x) - m^2\zeta(s+1,x,x)\right] \ , \tag{5.34}$$

As we have seen in chapter 3, $\zeta(s,x,x)$ has a pole at $s=1$ for every $x$. For $x$ in the bulk, this pole is the only singularity. However, as $x$ goes to the boundary there could be, a priori, additional singularities for other values of $s$, in particular for $s=0$. Keeping this in mind we find

$$\begin{aligned}
\delta\zeta'(0) &= 2\left\{ \lim_{s\to 0}\left[1 + s\frac{\rm d}{{\rm d}s}\right] - m^2 \lim_{s\to 1}\left[1 + (s-1)\frac{\rm d}{{\rm d}s}\right] \right\} \int {\rm d}^2x\sqrt{g}\,\delta\sigma(x)\zeta(s,x,x) \\
\delta\zeta(0) &= 2\left\{ \lim_{s\to 0} s - m^2 \lim_{s\to 1}(s-1) \right\} \int {\rm d}^2x\sqrt{g}\ \delta\sigma(x)\zeta(s,x,x)\,.
\end{aligned} \tag{5.35}$$

Our goal now will be to compute $\int {\rm d}^2x\sqrt{g}\ \delta\sigma(x)\zeta(s,x,x)$ on a Riemann surface with boundaries and extract its behaviour as $s\to 0$ and $s\to 1$. More generally, we will compute $\int \sqrt{g}\, f(x)\zeta(s,x,x)$ where $f$ is some sort of "test function". Once we have determined these quantities, we will get the variation of the gravitational action under an infinitesimal change of metric as

$$\begin{aligned}
\delta S_{\rm grav} &= -\frac{1}{2}\delta\zeta'(0) - \frac{1}{2}\delta\zeta(0)\,\ln\widehat{\mu}^2 \\
&= -\left\{ \lim_{s\to 0}\left[1 + s\left(\frac{\rm d}{{\rm d}s} + \ln\widehat{\mu}^2\right)\right] - m^2 \lim_{s\to 1}\left[1 + (s-1)\left(\frac{\rm d}{{\rm d}s} + \ln\widehat{\mu}^2\right)\right] \right\} \\
&\qquad \int {\rm d}^2x\sqrt{g}\,\delta\sigma(x)\zeta(s,x,x)
\end{aligned} \tag{5.36}$$

## 5.3 Green's functions and heat kernel on a manifold with boundaries

In this section we discuss how the heat kernel and Green's functions discussed in chapter 3 are modified when we add boundaries to a two-dimensional Riemann surface. We assume that the Riemann surface $\mathcal{M}$ has a boundary $\partial\mathcal{M}$ and that we have imposed Neumann boundary conditions. We notationally distinguish between the quantities on compact manifolds without boundaries and those on manifolds with boundaries by writing the former in curved letter ($\mathcal{K}$, $\mathcal{G}$). We first see how it works on simple examples before generalizing.

### 5.3.1 Examples

Before going on, it is useful to discuss some very simple examples of manifolds with boundaries: the one-dimensional interval, the two-dimensional cylinder which is the product of the interval and a circle, and the two-dimensional half-sphere.

**The one-dimensional interval**

The simplest example is a one-dimensional manifold that is just the interval $\mathcal{M} = [0, \pi]$ with trivial metric and $\Delta = -\partial_x^2$. We also take $m = 0$. The normalized eigenfunctions that satisfy the Neumann boundary conditions are

$$\varphi_0 = \frac{1}{\sqrt{\pi}}, \qquad \varphi_n(x) = \sqrt{\frac{2}{\pi}} \cos nx, n \in \mathbb{N}^* \tag{5.37}$$

and the eigenvalues are $\lambda_n = n^2$. Then we formally have for any function $f(\lambda)$:

$$\begin{aligned} \sum_{n=0}^{\infty} f(n^2)\varphi_n(x)\varphi_n(y) &= \frac{1}{\pi}\left[f(0) + \sum_{n=1}^{\infty} f(n^2) 2\cos nx\, \cos ny\right] \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} f(n^2)\left[\mathrm{e}^{in(x-y)} + \mathrm{e}^{in(x+y)}\right] \end{aligned} \tag{5.38}$$

For $f = 1$ this is just the completeness relation (3.8) with the right-hand side equal to $\delta(x-y)+\delta(x+y)$ where the $\delta$ are $2\pi$-periodic Dirac distributions, i.e. defined on the circle $S^1 = [0, 2\pi]$. With $x, y \in \mathcal{M} \setminus \partial\mathcal{M} = ]0, \pi[$, $x + y$ never is $0 \mod 2\pi$ and the $\delta(x + y)$ never contributes. Actually, $x + y = x - y_C$, where $y_C = -y$ is the image point of $y$ due to the boundary at $y = 0$. One would also expect additional image points due to the second boundary, but because of the $2\pi$ periodicity, these additional image points are equivalent to $y$ and $y_C$. If $y = y_B$ is on the boundary, say $y = 0$, then the image point $y_C$ coincides with $y$ (possibly $\mod 2\pi$) and we get $2\delta(x - y_B) = 2\delta(x)$. But $\int_0^\pi \mathrm{d}x\, \delta(x) = \frac{1}{2}$ and in any case the integral of the right-hand side of (3.8) correctly gives 1.

If we let $f(n^2) = \frac{1}{n^2}$, the relation (5.38) expresses the Green's function $G_I$ on the interval with Neumann boundary conditions in terms of a sum of Green's functions $\mathcal{G}_{S^1}$ on the circle

$$G_I(x, y) = \mathcal{G}_{S^1}(x, y) + \mathcal{G}_{S^1}(x, -y)\ , \tag{5.39}$$

a construction well-known as the method of images.

Similarly, if we let $f(n^2) = \mathrm{e}^{-tn^2}$ we get a relation that expresses the heat kernel on the interval $K_I(t, x, y)$ as the sum of two heat kernels $\mathcal{K}_{S^1}$ on the circle, one at $x, y$ and the other at $x, -y$:

$$K_I(t, x, y) = \mathcal{K}_{S^1}(t, x, y) + \mathcal{K}_{S^1}(t, x, -y)\ . \tag{5.40}$$

Actually, the sums can be expressed in terms of the theta function

$$\theta_3(\nu|\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} \mathrm{e}^{2\pi i n\nu} \quad , \quad q = \mathrm{e}^{i\pi\tau} \tag{5.41}$$

as

$$\mathcal{K}_{S^1}(t, x, y) = \frac{1}{2\pi}\,\theta_3\left(\frac{x-y}{2\pi}|i\frac{t}{\pi}\right)\ . \tag{5.42}$$

The small-$t$ asymptotics is obtained by applying Poisson resummation, or equivalently the modular transformation of $\theta_3$ under $\tau \to -\frac{1}{\tau}$,

$$\theta_3(\nu|\tau) = (-i\tau)^{-1/2}\mathrm{e}^{-i\pi\nu^2/\tau}\theta_3\left(\frac{\nu}{\tau}| - \frac{1}{\tau}\right)\ , \tag{5.43}$$

which yields

$$\mathcal{K}_{S^1}(t, x, y) = \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-(x-y+2\pi n)^2/(4t)}\ , \tag{5.44}$$

and which expresses the heat kernel on the circle as a sum over all geodesics going from $x$ to $y$ winding $n$ times around the circle and having length squared $(x - y + 2\pi n)^2$. This is

of course the expected result for the diffusion (Brownian motion) on a circle. For small $t$, the leading term in $\mathcal{K}_{S^1}(t,x,y)$ always is the $n=0$ term. For $\mathcal{K}_{S^1}(t,x,-y)$, however, the leading term is $n=0$ if $x+y<\pi$, while it is $n=-1$ if $\pi<x+y$. Thus

$$\begin{aligned} K_I(t,x,y) &\sim_{t\to 0} \frac{\mathrm{e}^{-(x-y)^2/(4t)}}{\sqrt{4\pi t}} + \frac{\mathrm{e}^{-(x+y)^2/(4t)}}{\sqrt{4\pi t}}\,, \quad \text{for } \ 0\le x+y\le \pi, \\ K_I(t,x,y) &\sim_{t\to 0} \frac{\mathrm{e}^{-(x-y)^2/(4t)}}{\sqrt{4\pi t}} + \frac{\mathrm{e}^{-(2\pi-x-y)^2/(4t)}}{\sqrt{4\pi t}}\,, \quad \text{for } \ \pi\le x+y\le 2\pi\,. \end{aligned} \tag{5.45}$$

Of course, only the coefficient $\frac{a_0}{\sqrt{4\pi}} = \frac{1}{\sqrt{4\pi}}$ appears, since all other $a_r$ involve the curvature and vanish in our present example. In any case, we see that for small $t$, the first term is exponentially small unless $x$ is close to $y$ within a distance of order $\sqrt{t}$. Similarly, the second term is exponentially small unless $x+y$ (or $2\pi-x-y$) is of order $\sqrt{t}$ which is possible only if $x$ and $y$ both are close to the boundary at 0 (or at $\pi$), and thus also close to each other, within a distance of order $\sqrt{t}$. Thus, for $x$ or $y$ in the bulk, the second term does not contribute to the small-$t$ expansion. It is only if both points go to one and the same boundary that the second term becomes important. We see that we can just as well write this small-$t$ asymptotic expansion as

$$K_I(t,x,y) \sim_{t\to 0} \frac{\mathrm{e}^{-(x-y)^2/(4t)}}{\sqrt{4\pi t}} + \sum_{\partial\mathcal{M}_i} \frac{\mathrm{e}^{-(x-y_C^{(i)})^2/(4t)}}{\sqrt{4\pi t}}\,, \tag{5.46}$$

where the sum is over the different boundary components and $y_C^{(i)}$ denotes the image ("conjugate") point of $y$ with respect to the boundary component $\partial\mathcal{M}_i$, i.e. $y_C^{(1)} = -y$ and $y_C^{(2)} = 2\pi - y$. While the use of image points is familiar from solving the Laplace equation for simple geometries in the presence of boundaries, we have seen that we should actually think of $(x-y_C^{(i)})^2$ as the length squared of the geodesic from $x$ to $y$ that is reflected once at the boundary $\partial\mathcal{M}_i$. Geodesics with multiple reflections necessarily are much longer and give exponentially subleading contributions. Of course, if one uses the exact expression (5.44) for $\mathcal{K}_{S^1}(t,x,y)+\mathcal{K}_{S^1}(t,x,-y)$ the heat kernel of the interval is expressed as a sum over all geodesic paths from $x$ to $y$ being reflected an arbitrary number of times at the two boundaries.

### The cylinder

The two-dimensional cylinder is just an interval times a circle, $I\times S^1$. Thus, if we choose the interval of length $a$ and the circle of circumference $2b$, the normalized eigenfunctions of the Laplace operator satisfying the Neumann boundary conditions are

$$\varphi_{0,m}(x_1,x_2) = \frac{\mathrm{e}^{i\pi m x_2/b}}{\sqrt{2ab}}\,, \quad \varphi_{n,m}(x_1,x_2) = \frac{\mathrm{e}^{i\pi m x_2/b}}{\sqrt{ab}}\cos\frac{n\pi x_1}{a}\,, \quad m\in\mathbf{Z}\ ,\ n\in\mathbb{N}^*\ . \tag{5.47}$$

The heat kernel for the Laplace operator then simply is the product of the heat kernel for the circle and the heat kernel of the interval as just given in the previous example, with the obvious replacements $\pi\to a,b$ :

$$\begin{aligned} K_{\mathrm{cyl}}(t,x_1,x_2,y_1,y_2) &= \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} \exp\left(-t\left(\frac{\pi^2n^2}{a^2}+\frac{\pi^2m^2}{b^2}\right)\right)\varphi_{n,m}(x_1,x_2)\,\varphi^*_{n,m}(y_1,y_2) \\ &= \frac{1}{4ab}\,\theta_3\left(\frac{x_2-y_2}{2b}\Big|i\frac{\pi t}{b^2}\right)\left[\theta_3\left(\frac{x_1-y_1}{2a}\Big|i\frac{\pi t}{a^2}\right)+\theta_3\left(\frac{x_1+y_1}{2a}\Big|i\frac{\pi t}{a^2}\right)\right] \\ &= \mathcal{K}_{\mathrm{torus}}(t,x_1,x_2,y_1,y_2)+\mathcal{K}_{\mathrm{torus}}(t,x_1,x_2,-y_1,y_2)\ , \end{aligned} \tag{5.48}$$

with the corresponding torus obviously having periods $2a$ and $2b$. Poisson resummation or equivalently the modular transformation formula for $\theta_3$ yields

$$K_{\text{cyl}}(t, x_1, x_2, y_1, y_2) = \frac{1}{4\pi t} \sum_{n,m\in\mathbb{Z}}^{\infty} \exp\left(-\frac{(x_1 - y_1 + 2na)^2 + (x_2 - y_2 + 2mb)^2}{4t}\right) + (y_1 \to -y_1). \tag{5.49}$$

Again, this expresses the heat kernel as a sum over all geodesics going from $x$ to $y$ winding $m$ times around the circle direction of the cylinder and being reflected $2n$ times (for the first term) or $2n+1$ times (for the second term) at the boundaries of the cylinder.

**The upper half sphere**

Our last example involves a curved two-dimensional manifold with a boundary: let $\mathcal{M}$ be the upper half of the standard round sphere of unit radius, i.e. $\mathcal{M} = S^2_+$, parametrized by $\theta \in [0, \frac{\pi}{2}]$ and $\varphi \in [0, 2\pi]$. Then the boundary $\partial\mathcal{M}$ is just the circle at $\theta = \frac{\pi}{2}$ and the normal derivative is $n^a\partial_a = \partial_\theta$. The eigenfunctions of the Laplace operator $\Delta$ on the sphere $S^2$ are the spherical harmonics $Y_l^m$ and they still satisfy

$$\Delta Y_l^m = l(l+1)Y_l^m \tag{5.50}$$

on $S^2_+$. However, not all of them satisfy the Neumann boundary condition. The parity of the $Y_l^m$ is $(-1)^l$, so that

$$Y_l^m(\theta, \varphi) = (-1)^l Y_l^m(\pi - \theta, \varphi + \pi) = (-1)^{l-m} Y_l^m(\pi - \theta, \varphi). \tag{5.51}$$

It follows that $Y_l^m$ is even (resp. odd) under reflection by the equator at $\theta = \frac{\pi}{2}$ if $l - m$ is even (resp. odd), and hence satisfies Neumann (resp. Dirichlet) conditions at $\theta = \frac{\pi}{2}$. Thus, for each $l$, there are $l + 1$ allowed values of $m$. It follows for even $l - m$ that

$$\int_{S^2_+} \bar{Y}_{l_1}^{m_1} Y_{l_2}^{m_2} = \frac{1}{2}\int_{S^2} \bar{Y}_{l_1}^{m_1} Y_{l_2}^{m_2} = \frac{1}{2}\delta_{l_1 l_2}\delta_{m_1 m_2}. \tag{5.52}$$

We see that the orthonormal eigenfunctions $\varphi_n$ of the Laplace operator on $\mathcal{M}$ obeying the boundary conditions simply are the $\sqrt{2}Y_l^m$ with $l - m$ even. It also follows from (5.51) that

$$Y_l^m(\theta, \varphi) + Y_l^m(\pi - \theta, \varphi) = \begin{cases} 0 & \text{if } l - m \text{ odd} \\ 2Y_l^m(\theta, \varphi) & \text{if } l - m \text{ even.} \end{cases} \tag{5.53}$$

Thus we have for any function $f(\lambda)$ the formal relation

$$\sum_{\substack{l,m \\ l-m \text{ even}}} f\left(l(l+1)\right) \overline{\sqrt{2}Y_l^m(\theta, \varphi)}\sqrt{2}Y_l^m(\theta', \varphi') = \sum_{l,m} f\left(l(l+1)\right) \overline{Y_l^m(\theta, \varphi)} \times \left(Y_l^m(\theta', \varphi') + Y_l^m(\pi - \theta', \varphi')\right) \tag{5.54}$$

where it is of course understood that $l \geq 0$ and $|m| \leq l$. If we simply take $f = 1$, this is just the completeness relation, with it's right-hand side being

$$\delta(\cos\theta - \cos\theta')\delta(\varphi - \varphi') + \delta(\cos\theta + \cos\theta')\delta(\varphi - \varphi'). \tag{5.55}$$

Here, the first term is just $\frac{1}{\sqrt{g}}\delta(x - y)$ while the second $\delta$ is $\frac{1}{\sqrt{g}}\delta(x - y_C)$, where $y_C = (\pi - \theta', \varphi')$ is the "image point" of $y = (\theta', \varphi')$. As for the interval and the cylinder, this image point is always outside of $\mathcal{M}$, except if $y$ is on the boundary. In the latter case both delta functions contribute equally and one has $2\delta(\cos\theta)\delta(\varphi - \varphi')$ which correctly gives 1

when integrated over $S^2_+$. This shows again that the completeness relation (3.8) continues to hold for $x$ or $y$ on the boundary.

If we let $f\left(l(l+1)\right) = \frac{1}{l(l+1)+M^2}$ in (5.54) (we temporarily denote the mass by $M$ to avoid any confusion with the quantum number $m$) this relation expresses the Green's functions of $\Delta + M^2$ on the upper half sphere $S^2_+$ in terms of a sum of two Green's function on the sphere, one at $x$ and $y$ and the other at $x$ and $y_C$:

$$G_{S^2_+}(\theta,\varphi;\theta',\varphi') = \mathcal{G}_{S^2}(\theta,\varphi;\theta',\varphi') + \mathcal{G}_{S^2}(\theta,\varphi;\pi-\theta',\varphi'). \tag{5.56}$$

An analogous relation holds for the $\widetilde{G}$ when the zero-mode is excluded, as well as for $\widetilde{G}^{(0)}$ when $M = 0$ and the zero-mode is excluded. It is interesting to study the short-distance singularity of this Green's function. In two dimensions, the short-distance singularity of the Green's function is logarithmic, and one has e.g. $\widetilde{G}^{(0)}_{S^2}(\theta,\varphi;\theta',\varphi) \sim -\frac{1}{4\pi}\ln(\theta-\theta')^2$ as $\theta \to \theta'$. Then, on the half sphere, the singularity as $\theta \to \theta'$ for any $(\theta',\varphi') \notin \partial\mathcal{M}$ is given by this same logarithmic singularity, since $\widetilde{G}^{(0)}_{S^2}(\theta,\varphi;\pi-\theta',\varphi)$ is non-singular. However, if $(\theta',\varphi') \in \partial\mathcal{M}$, i.e. $\theta' = \frac{\pi}{2}$, then the short-distance singularity of $\widetilde{G}^{(0)}_{S^2_+}$ is twice as large, i.e. $-\frac{1}{2\pi}\ln(\theta-\theta')^2$, in agreement with the factor 2 that accompanied the $\delta(x-y_B)$.

Finally, taking $f(\lambda) = e^{-t\lambda}$, we get the corresponding relation between the heat kernels:

$$K_{S^2_+}(t,\theta,\varphi;\theta',\varphi') = \mathcal{K}_{S^2}(t,\theta,\varphi;\theta',\varphi') + \mathcal{K}_{S^2}(t,\theta,\varphi;\pi-\theta',\varphi'). \tag{5.57}$$

### 5.3.2 The heat kernel continued

As it appeared from the previous examples, in simple geometries, the Green's functions and the heat kernel can be obtained from the corresponding Green's functions or heat kernels on a "bigger" manifold without boundary, by a method of images. In all three cases we have seen that

$$K(t,x,y) = \mathcal{K}(t,x,y) + \mathcal{K}(t,x,y_C)\ , \tag{5.58}$$

where $\mathcal{K}$ is the heat kernel on the "bigger" compact manifold and $y_C$ the "image point" of $y$. However, we have also seen in the example of the interval that the leading term in the asymptotic small-$t$ expansion to be used for $\mathcal{K}(t,x,y_C)$ differs depending on whether $x$ and $y$ are close to one or the other boundary. Thus the small-$t$ asymptotic expansion has the following form

$$K(t,x,y) \sim \frac{\mathrm{e}^{-m^2 t}}{4\pi t}\left[\mathrm{e}^{-\ell^2(x,y)/4t}\sum_{k\geq 0} t^k a_k(x,y) + \sum_{\partial\mathcal{M}_i}\mathrm{e}^{-\ell^2(x,y_C^{(i)})/4t}\sum_{k\geq 0} t^k \widetilde{a}_k^{(i)}(x,y_C^{(i)})\right]. \tag{5.59}$$

Indeed, for $m = 0$, the heat kernel describes the diffusion (Brownian motion) of a particle on the manifold from $x$ to $y$. On a flat manifold, this is given as a sum over the geodesic paths from $x$ to $y$ as $\frac{1}{4\pi t}\mathrm{e}^{-\ell^2_{(r)}(x,y)/(4t)}$, where $\ell_{(r)}(x,y)$ is the geodesic length of the $r^{\text{th}}$ path. In particular, if the manifold has boundaries, there are (possibly infinitely) many geodesic paths that involve one or several reflections at the boundaries. We write $\ell_i(x,y)$ for the length of the geodesic path from $x$ to $y$ that involves exactly one reflection at the boundary component $\partial\mathcal{M}_i$. Moreover, on a curved manifold, each $\frac{1}{4\pi t}\mathrm{e}^{-\ell^2_{(r)}(x,y)/(4t)}$ gets multiplied by a power series in $t$ with coefficients that can be determined order by order from the differential equation (3.49), see e.g. [174]. For small $t$ the leading terms can involve at most one reflection, resulting indeed in the form (5.59), with $\ell^2(x,y_C^{(i)})$ replaced by $\mathrm{e}^{-\ell_i^2(x,y)/4t}$. However, for small $t$, the terms involving $\mathrm{e}^{-\ell_i^2(x,y)/4t}$ with one reflection at $\partial\mathcal{M}_i$ can only contribute if the points $x$ and $y$ are close to the boundary $\partial\mathcal{M}_i$, and close to each other, within a distance $\simeq \sqrt{t}$. As $t \to 0$, one zooms in close to the boundary which thus becomes flat. Now for a flat boundary, the length of the geodesic from $x$ to $y$ involving one reflection

at $\partial\mathcal{M}_i$ is the same as the length of the geodesic from $x$ to the "mirror" image point $y_C^{(i)}$ and $\mathrm{e}^{-\ell_1^2(x,y)/4t} \simeq \mathrm{e}^{-\ell^2(x,y_C^{(i)})/4t}$, with any differences at finite $t$ being included in a redefinition of the coefficients $\widetilde{a}_k^{(i)}(x, y_C^{(i)}) \to a_k^{(i)}(x,y)$. Thus, we can rewrite (5.59) equivalently as

$$K(t,x,y) \sim \frac{\mathrm{e}^{-m^2 t}}{4\pi t}\left[\mathrm{e}^{-\ell^2(x,y)/4t}\sum_{k\geq 0} t^k a_k(x,y) + \sum_{\partial\mathcal{M}_i} \mathrm{e}^{-\ell_i^2(x,y)/4t}\sum_{k\geq 0} t^k a_k^{(i)}(x,y)\right] . \quad (5.60)$$

For $x = y$ we have, in particular, $\ell_1^2(x,x) = 4\ell^2(x,\partial\mathcal{M}_i)$, where $\ell(x,\partial\mathcal{M}_i)$ denotes the geodesic distance of the point $x$ to the boundary $\partial\mathcal{M}_i$. Thus

$$K(t,x,x) \sim \frac{\mathrm{e}^{-m^2 t}}{4\pi t}\left[\left(1+\sum_{k\geq 1} t^k a_k(x,x)\right) + \sum_{\partial\mathcal{M}_i} \mathrm{e}^{-\ell^2(x,\partial\mathcal{M}_i)/t}\left(1+\sum_{k\geq 1} t^k a_k^{(i)}(x,x)\right)\right] . \quad (5.61)$$

Here, the local expressions $a_r(x,x)$ are the same as on a compact Riemann surface without boundary, e.g. $a_1(x,x) = \frac{R(x)}{6}$.

If we are going to take the $t \to 0$ limit, we will find that the terms involving the boundaries drop out, unless the point $x$ is on the boundary $\partial\mathcal{M}_i$. In this case the corresponding boundary terms diverge for $t \to 0$ (as do the bulk terms). Thus these boundary terms behave as a Dirac delta concentrated on the boundary. To be more precise, let us look at the heat kernel evaluated at $x = y$ and integrated over the manifold against a "test function" $f$:

$$\int_{\mathcal{M}} \mathrm{d}^2x\,\sqrt{g}\,f(x)K(t,x,x) . \quad (5.62)$$

Then the first term in (5.61) just gives the usual bulk result, while each of the boundary terms yields

$$\frac{\mathrm{e}^{-m^2 t}}{4\pi t}\int_{\mathcal{M}} \mathrm{d}^2x\,\sqrt{g}\,f(x)\mathrm{e}^{-\ell^2(x,\partial\mathcal{M})/t}\left(1+\sum_{k\geq 1} t^k a_k^{(i)}(x,x)\right) . \quad (5.63)$$

Again, for small $t$, the exponential forces $x$ to be close to the boundary. We may then view the integral as an integral over the boundary and an integral normal to the boundary. For a given boundary point $x_B$ we can Taylor expand all quantities around this point and do the integral in the normal direction. The leading small-$t$ term of this normal integral then simply is given by (using Riemann normal coordinates around $x_B$)

$$\mathrm{e}^{-m^2 t}\int_0^\infty \mathrm{d}x_n\sqrt{\widetilde{g}(x_B)}f(x_B)\frac{\mathrm{e}^{-(x-x_B)^2/t}}{4\pi t} = \frac{\mathrm{e}^{-m^2 t}}{8\pi}\sqrt{\frac{\pi}{t}}\sqrt{\widetilde{g}(x_B)}f(x_B) , \quad (5.64)$$

where $\widetilde{g}$ is the metric induced on the boundary, so that $\mathrm{d}x_B\sqrt{\widetilde{g}(x_B)} = \mathrm{d}l$. The $\mathcal{O}(t^0)$-corrections to this expression involve the normal derivatives of $\sqrt{g}$ and of $f$. As a result the small-$t$ asymptotic expansion of (5.62) has the form

$$\int_{\mathcal{M}} \mathrm{d}^2x\,\sqrt{g}\,f(x)K(t,x,x) = \frac{1}{4\pi t}\left[\int \mathrm{d}^2x\,\sqrt{g(x)}f(x) + \frac{\sqrt{\pi t}}{2}\sum_i \int_{\partial\mathcal{M}_i} \mathrm{d}l\,f(x) + \mathcal{O}(t)\right] . \quad (5.65)$$

The leading small-$t$ singularity $\sim t^{-1}$ is given by the usual bulk term, while the boundary-terms yield subleading singularities $\sim t^{-1/2}$.

The expansion (5.65) can be generalised to all Laplace type operators using similar arguments [124, 130].

## 5.4 Local zeta functions and Green's function at coinciding points

Recall that local versions of the zeta functions were defined in (3.61) as

$$\zeta(s,x,y) = \sum_{n\geq 0} \frac{\varphi_n(x)\varphi_n(y)}{\lambda_n^s}.$$

Note that $\zeta(1,x,y) = G(x,y)$. They are related to the heat kernel by

$$\zeta(s,x,y) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} K(t,x,y) \ . \tag{5.66}$$

Of course, this formula involves the heat kernel for all values of $t$, not just the small-$t$ asymptotics. However, for $s = 0, -1, -2, \ldots$, $\frac{1}{\Gamma(s)}$ has zeros and the value of $\zeta(s,x,y)$ is entirely determined by the singularities of the integral over $t$ that arise from the small-$t$ asymptotics of $K$. As shown above, the latter is given by local quantities on the Riemann surface. In particular, for any point not on the boundary of $\mathcal{M}$, we have

$$\zeta(0,x,x) = \frac{R(x)}{24\pi} - \frac{m^2}{4\pi} \quad , \quad x \neq \partial\mathcal{M} \ . \tag{5.67}$$

On the other hand, the values for $s = 1, 2, 3, \ldots$ or the derivative at $s = 0$ cannot be determined just from the small-$t$ asymptotics and require the knowledge of the full spectrum of $\Delta + m^2$, i.e. they contain global information about the Riemann surface.

Clearly, $\zeta(1,x,y) = G(x,y)$ is singular as $x \to y$. For $s \neq 1$, $\zeta(s,x,y)$ provides a regularization of the propagator. It will be useful to study in more details the singularities of $\zeta(s,x,y)$ which occur for $s \to 1$ and $x \to y$. More generally, as is clear from (5.66), any possible singularities of $\zeta(s,x,y)$ for $s \leq 1$ come from the region of the integral where $t$ is small. Thus we define

$$\zeta_{\mathrm{sing}}(s,x,y) \simeq \frac{1}{\Gamma(s)} \int_0^{\mu^{-2}} \mathrm{d}t\, t^{s-1} K(t,x,y) \ , \tag{5.68}$$

where $\mu$ is some (arbitrary) large scale we introduce to separate the singular and non-singular parts, so that $\zeta - \zeta_{\mathrm{sing}}$ is free of singularities. For large $\mu^2$, say $\mu^2 A \gg 1$, where $A$ is the area of our manifold, we can use the small-$t$ asymptotics (5.59) or (5.60) of $K$ to evaluate $\zeta_{\mathrm{sing}}$. With $t$ small, the $e^{-\ell^2/4t}$ are exponentially small unless $\ell^2 \lesssim t$. This means that in the first sum we must have $y = x + \mathcal{O}(\sqrt{t})$ and in the second sum $y_C^{(i)} = x + \mathcal{O}(\sqrt{t})$. Since $a_0(x,y) = a_0(x,x) + \mathcal{O}(\ell^2(x,y)\, R) = 1 + \mathcal{O}(t\, R)$, and similarly for $a_0^i(x, y_C^{(i)})$, and since the $\mathcal{O}(tR)$ terms do not contribute to the singularity at $s \to 1$, we define $\zeta_{\mathrm{sing}}(s,x,y)$ more precisely as

$$\begin{aligned}
\zeta_{\mathrm{sing}}(s,x,y) &= \frac{1}{4\pi\,\Gamma(s)} \int_0^{\mu^{-2}} \mathrm{d}t\, t^{s-2} \left( \mathrm{e}^{-\ell^2(x,y)/4t} + \sum_{\partial\mathcal{M}_i} \mathrm{e}^{-\ell_i^2(x,y)/4t} \right) \\
&= \frac{\mu^{2-2s}}{4\pi\Gamma(s)} \left[ E_s\left( \frac{\ell^2(x,y)\mu^2}{4} \right) + \sum_{\partial\mathcal{M}_i} E_s\left( \frac{\ell_i^2(x,y)\mu^2}{4} \right) \right] \ ,
\end{aligned} \tag{5.69}$$

where the exponential integral (or incomplete gamma) function is defined by

$$E_r(z) = \int_1^\infty \mathrm{d}u\, u^{-r} \mathrm{e}^{-zu} \ , \quad \frac{\mathrm{d}}{\mathrm{d}z} E_r(z) = -E_{r-1}(z) \ . \tag{5.70}$$

As $z \to 0$, the $E_r(z)$ are regular for $r > 1$ and have a logarithmic singularity for $r = 1$:

$$E_1(z) = -\gamma - \ln z + \mathcal{O}(z) \quad \text{as } z \to 0 \ . \tag{5.71}$$

For $x \neq y$, the exponential integral functions are non-singular and we can set $s = 1$ in (5.69),

$$\zeta_{\text{sing}}(1, x, y) = \frac{1}{4\pi}\left[E_1\left(\frac{\ell^2(x,y)\mu^2}{4}\right) + \sum_{\partial\mathcal{M}_i} E_1\left(\frac{\ell_i^2(x,y)\mu^2}{4}\right)\right] , \tag{5.72}$$

with the singularity appearing as the short-distance singularity for $x \to y$. We have :

$$\text{as } x \to y \ : \quad \zeta_{\text{sing}}(1, x, y) \simeq \frac{1}{4\pi}\left[-\gamma - \ln\left(\frac{\ell^2(x,y)\mu^2}{4}\right) + \sum_{\partial\mathcal{M}_i} E_1\left(\frac{\ell_i^2(x,y)\mu^2}{4}\right)\right] , \tag{5.73}$$

up to terms that vanish for $x = y$. If moreover $x \to y \to \partial\mathcal{M}_i$, i.e. they go to one of the boundaries, one has (again, up to terms that vanish in this limit)

$$\text{as } x \to y \to \partial\mathcal{M}_i \ : \quad \zeta_{\text{sing}}(1, x, y) \simeq \frac{1}{4\pi}\left[-2\gamma - \ln\left(\frac{\ell^2(x,y)\mu^2}{4}\right) - \ln\left(\frac{\ell_i^2(x,y)\mu^2}{4}\right)\right] . \tag{5.74}$$

On the other hand, for $s \neq 1$, we can set $x = y$ directly in (5.69). More precisely, we assume $\operatorname{Re} s > 1$ and analytically continue in the end. Then (recall $\ell_i^2(y, y) = 4\ell^2(y, \partial\mathcal{M}_i)$)

$$\zeta_{\text{sing}}(s, y, y) = \frac{\mu^{2-2s}}{4\pi\Gamma(s)}\left[\frac{1}{s-1} + \sum_{\partial\mathcal{M}_i} E_s\left(\ell^2(y, \partial\mathcal{M}_i)\mu^2\right)\right] . \tag{5.75}$$

If $y \notin \partial\mathcal{M}$, only the first term yields a pole at $s = 1$, while for $y \in \partial\mathcal{M}$ the second term also yields the same pole and, hence, the residue is doubled:

$$\zeta_{\text{sing}}(s, y, y) = \frac{\mu^{2-2s}}{4\pi\Gamma(s)}\,\frac{2}{s-1} , \quad (y \in \partial\mathcal{M}) . \tag{5.76}$$

Thus, we see that $\zeta(s, x, x)$ has a pole at $s = 1$ with residue $\frac{a_0(x,x)}{4\pi} = \frac{1}{4\pi}$ for $x \notin \partial\mathcal{M}$ and residue $\frac{a_0(x,x)}{2\pi} = \frac{1}{2\pi}$ for $x \in \partial\mathcal{M}_i$.

Just as for the heat kernel itself, we will actually encounter expressions where $\zeta_{\text{sing}}(s, y, y)$ is multiplied by some $f(y)$ and integrated over the manifold. Proceeding similarly to the derivation of (5.65) we find

$$\begin{aligned}\int_{\mathcal{M}} \mathrm{d}^2y\sqrt{g}f(y)\zeta_{\text{sing}}(s, y, y) = {}& \frac{\mu^{2-2s}}{4\pi\Gamma(s)}\frac{1}{s-1}\int_{\mathcal{M}} \mathrm{d}^2y\sqrt{g}f(y) + \frac{\mu^{1-2s}}{8\sqrt{\pi}\Gamma(s)}\frac{1}{s-\frac{1}{2}}\int_{\partial\mathcal{M}} \mathrm{d}l\, f(y) \\ & + \frac{\mu^{-2s}}{8\pi\Gamma(s)}\frac{1}{s}\int_{\partial\mathcal{M}} \mathrm{d}l\, \partial_n f(y) + \frac{\mu^{-1-2s}}{32\sqrt{\pi}\Gamma(s)}\frac{1}{s+\frac{1}{2}}\int_{\partial\mathcal{M}} \mathrm{d}l\, \partial_n^2 f(y) + \dots\end{aligned} \tag{5.77}$$

This integrated expression exhibits poles at $s = 1, \frac{1}{2}, -\frac{1}{2}, -1, -\frac{3}{2}, \dots$ and no pole at $s = 0$. This infinite series of poles translates the discontinuous behaviour between (5.75) and (5.76) due to the fact that the limits $s \to 1$ and $z \to 0$ of $E_s(z)$ do not commute. In particular, one has

$$\begin{aligned}&\lim_{s\to 1}\left[1 + (s-1)\left(\frac{\mathrm{d}}{\mathrm{d}s} + \ln\widehat{\mu}^2\right)\right]\int_{\mathcal{M}} \mathrm{d}^2y\sqrt{g}\, f(y)\, \zeta_{\text{sing}}(s, y, y) \\ &= \frac{1}{4\pi}\left(\gamma - \ln\frac{\mu^2}{\widehat{\mu}^2}\right)\int_{\mathcal{M}} \mathrm{d}^2y\sqrt{g}\, f(y) + \frac{1}{4\sqrt{\pi}\mu}\int_{\partial\mathcal{M}} \mathrm{d}l\, F(y, \mu) ,\end{aligned} \tag{5.78}$$

where

$$F(y, \mu) = f(y) + \frac{1}{2\sqrt{\pi}\mu}\partial_n f(y) + \frac{1}{12\mu^2}\partial_n^2 f(y) + \dots . \tag{5.79}$$

All boundary terms are at least $\sim \frac{1}{\mu}$ and we can thus restate the previous relation as

$$\lim_{s\to 1}\left[1+(s-1)\left(\frac{\mathrm{d}}{\mathrm{d}s}+\ln\widehat{\mu}^2\right)\right]\int_{\mathcal{M}}\mathrm{d}^2y\sqrt{g}\,f(y)\,\zeta_{\mathrm{sing}}(s,y,y)$$
$$=\frac{1}{4\pi}\left(\gamma-\ln\frac{\mu^2}{\widehat{\mu}^2}\right)\int_{\mathcal{M}}\mathrm{d}^2y\sqrt{g}\,f(y)+\mathcal{O}\left(\frac{1}{\mu}\right) \tag{5.80}$$

In any case,

$$\zeta_{\mathrm{R}}(s,x,y)=\zeta(s,x,y)-\zeta_{\mathrm{sing}}(s,x,y) \tag{5.81}$$

is free of singularities and, in particular, has finite limits as $s\to 1$ and $x\to y$, in one order or the other, i.e. $\zeta_{\mathrm{R}}(1,x,x)$ is finite and well-defined. We then let

$$G_\zeta(y)=\zeta_{\mathrm{R}}(1,y,y)+\frac{\gamma}{4\pi}=\lim_{s\to 1}\left(\zeta(s,y,y)-\zeta_{\mathrm{sing}}(s,y,y)\right)+\frac{\gamma}{4\pi} \tag{5.82}$$

This is an important quantity, called the "Green's function at coinciding points". Note that $G_\zeta(y)$ contains global information about the Riemann surface and cannot be expressed in terms of local quantities only. Combining (5.78) and (5.82) we get

$$\lim_{s\to 1}\left[1+(s-1)\left(\frac{\mathrm{d}}{\mathrm{d}s}+\ln\widehat{\mu}^2\right)\right]\int_{\mathcal{M}}\mathrm{d}^2y\sqrt{g}\,f(y)\,\zeta(s,y,y)$$
$$=\int_{\mathcal{M}}\mathrm{d}^2y\sqrt{g}\,f(y)\left(G_\zeta(y)-\frac{1}{4\pi}\ln\frac{\mu^2}{\widehat{\mu}^2}\right)+\frac{1}{4\sqrt{\pi}\mu}\int_{\partial\mathcal{M}}\mathrm{d}l\,F(y,\mu). \tag{5.83}$$

Note that the precise definition of $G_\zeta$ depends on our choice of $\mu$, as is also obvious from this last relation since its left-hand side is $\mu$-independent.

The other ingredient needed for computing the variation of the gravitational action was

$$\lim_{s\to 0}\left[1+s\left(\frac{\mathrm{d}}{\mathrm{d}s}+\ln\widehat{\mu}^2\right)\right]\int\mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\zeta(s,x,x). \tag{5.84}$$

Replacing $\zeta_{\mathrm{sing}}(s,x,x)$ by $\zeta(s,x,x)$ we see from (5.77) that $\int\mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\zeta(s,x,x)$ actually has poles for $s=1,\frac{1}{2},-\frac{1}{2},-1,\ldots$ but *not* for $s=0$, since the would-be pole is cancelled by the $\frac{1}{\Gamma(s)}$. Hence, $\int\mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\zeta(s,x,x)$ is regular at $s=0$ and, adding the bulk contribution (5.67) and the boundary contribution read from (5.77), we get

$$\lim_{s\to 0}\left[1+s\left(\frac{\mathrm{d}}{\mathrm{d}s}+\ln\widehat{\mu}^2\right)\right]\int\mathrm{d}^2x\sqrt{g}\,f(x)\zeta(s,x,x)$$
$$=\int\mathrm{d}^2x\sqrt{g}\,f(x)\zeta(0,x,x)$$
$$=\frac{1}{4\pi}\int\mathrm{d}^2x\sqrt{g}\,f(x)\left(\frac{R}{6}-m^2\right)+\frac{1}{8\pi}\int_{\partial\mathcal{M}}\mathrm{d}l\,\partial_n f(x). \tag{5.85}$$

Let us relate $G_\zeta(y)$ to the Green's function $G(x,y)$ at coinciding points with the short-distance singularity subtracted. Since $\zeta_{\mathrm{R}}(s,x,y)=\zeta(s,x,y)-\zeta_{\mathrm{sing}}(s,x,y)$ is free of singularities, we may change the order of limits. If we first let $s=1$, so that $\zeta(1,x,y)=G(x,y)$ and $\zeta_{\mathrm{sing}}(1,x,y)$ is given by (5.73), we find

$$G_\zeta(y)=\lim_{x\to y}\left[G(x,y)+\frac{1}{4\pi}\left(\ln\left(\frac{\ell^2(x,y)\mu^2}{4}\right)+2\gamma-\sum_{\partial\mathcal{M}_i}E_1\left(\frac{\ell_i^2(x,y)\mu^2}{4}\right)\right)\right]. \tag{5.86}$$

We know that $G_\zeta(y)$ is a non-singular quantity for all $y\in\mathcal{M}$, in particular also on the boundary. The logarithm subtracts the generic short-distance singularity of $G(x,y)$, while the $E_1$ subtract the additional singularities present whenever $y\in\partial\mathcal{M}_i$.

If, as before, we multiply this relation by some smooth $f(y)$ and integrate over the manifold, we get in particular for these $E_1$-terms:

$$\lim_{x\to y}\int_{\mathcal{M}} \mathrm{d}^2y\sqrt{g}\, f(y) E_1\left(\frac{\ell_i^2(x,y)\mu^2}{4}\right) = \int_0^{\mu^{-2}} \frac{\mathrm{d}t}{t}\int_{\mathcal{M}} \mathrm{d}^2y\sqrt{g}\, f(y) e^{-\ell^2(y,\partial\mathcal{M}_i)/t}$$
$$= \frac{\sqrt{\pi}}{\mu}\int_{\partial\mathcal{M}_i} \mathrm{d}l\, F(y,\mu) \tag{5.87}$$

with $F(y,\mu)$ defined in (5.79). It follows that we may rewrite (5.83) as

$$\lim_{s\to 1}\left[1+(s-1)\left(\frac{\mathrm{d}}{\mathrm{d}s}+\ln\widehat{\mu}^2\right)\right]\int_{\mathcal{M}} \mathrm{d}^2y\sqrt{g}\, f(y)\,\zeta(s,y,y) = \int_{\mathcal{M}} \mathrm{d}^2y\sqrt{g}\, f(y)\, G_{\mathrm{R,bulk}}(y)\ , \tag{5.88}$$

where

$$G_{\mathrm{R,bulk}}(y) = \lim_{x\to y}\left[G(x,y)+\frac{1}{4\pi}\left(\ln\left(\frac{\ell^2(x,y)\widehat{\mu}^2}{4}\right)+2\gamma\right)\right] \tag{5.89}$$

is the Green's function at coinciding points with its bulk singularity subtracted. If necessary, (5.88) again shows that this does not depend on the arbitrarily introduced $\mu$ (although it does depend on $\widehat{\mu}$ which was part of our definition of the functional integral). While the quantity $G_{\mathrm{R,bulk}}(x)$ has the advantage of being $\mu$-independent, it has a (logarithmic) singularity as $x$ approaches the boundary. However, we know that these singularities must be integrable as is clear from the equality of (5.88) with (5.83) which is finite, independently of the arbitrary choice of $\mu$. As will become clear next, while $G_\zeta(x)$ satisfies Neumann boundary conditions, this is not the case of $G_{\mathrm{R,bulk}}(x)$.

To study the boundary condition satisfied by $G_\zeta(y)$ we only need its behaviour in the immediate vicinity of the relevant boundary component which can be read from (5.86):

$$G_\zeta(y) \simeq \lim_{x\to y}\left[G(x,y)+\frac{1}{4\pi}\left(\ln\left(\frac{\ell^2(x,y)\mu^2}{4}\right)+\ln\left(\frac{\ell_i^2(x,y)\mu^2}{4}\right)+3\gamma\right)\right] \text{ as } y\to\partial\mathcal{M}_i \tag{5.90}$$

Now, $G(x,y)$ satisfies the Neumann condition in both its arguments. The same is true for the sum of the logarithms, up to terms that vanish as $x$ and $y$ approach the boundary. Indeed as one zooms in close to the boundary, the boundary becomes flat and the geometry locally Euclidean, and using Riemann normal coordinates in the normal and tangential directions around the relevant boundary point (such that the boundary is at zero normal coordinate), one has $\ell^2(x,y)\simeq(x_t-y_t)^2+(x_n-y_n)^2$ as well as $\ell_i^2(x,y)\simeq(x_t-y_t)^2+(x_n+y_n)^2$. Then

$$\partial_{x^n}\ell^2(x,y)|_{x^n=0} \simeq -2y_n = -\partial_{x^n}\ell_i^2(x,y)|_{x^n=0} \tag{5.91}$$

as well as

$$\ell^2(x,y)|_{x^n=0} \simeq \ell_i^2(x,y)|_{x^n=0}. \tag{5.92}$$

It follows that

$$\partial_{x^n}\left[\ln\left(\frac{\ell^2(x,y)m^2}{4}\right)+\ln\left(\frac{\ell_i^2(x,y)\mu^2}{4}\right)\right]|_{x^n=0} = 0 \tag{5.93}$$

i.e. the sum of the logarithms satisfies the Neumann condition in $x$ up to terms that vanish as $x$ and $y$ approach the boundary. Since $\ell_i^2(x,y)$ is symmetric in $x$ and $y$, the same is true in $y$. Now if any function $h(x,y)$ satisfies the Neumann condition in both arguments, the function $H(y)=\lim_{x\to y}h(x,y)=\lim_{\mathrm{e}\to 0}h(y+\mathrm{e},y)$ then obviously also satisfies the Neumann condition. We conclude that $G_\zeta$ satisfies the Neumann boundary condition on every boundary component $\partial\mathcal{M}_i$, i.e.

$$n^a\partial_a G_\zeta(y) = 0\ , \quad \text{for } y\in\partial\mathcal{M}\ . \tag{5.94}$$

It follows that, if $\phi$ is any smooth function that also satisfies Neumann conditions, one has

$$\int_{\mathcal{M}} \mathrm{d}^2y\sqrt{g}\,\Delta\phi\,G_\zeta(y) = \int_{\mathcal{M}} \mathrm{d}^2y\sqrt{g}\,\phi\,\Delta G_\zeta(y)\ . \tag{5.95}$$

It is now also clear that $G_{\mathrm{R,bulk}}$ does not satisfy the Neumann condition since its definition lacks the crucial third term in (5.90).

We can now evaluate (5.83) for $f = \Delta\phi$ and use (5.95) to get

$$\begin{aligned}\lim_{s\to 1}\left[1+(s-1)\left(\frac{\mathrm{d}}{\mathrm{d}s}+\ln\widehat{\mu}^2\right)\right]\int_{\mathcal{M}} \mathrm{d}^2y\sqrt{g}\,\Delta\phi(y)\,\zeta(s,y,y)\\ = \int_{\mathcal{M}} \mathrm{d}^2y\sqrt{g}\,\phi(y)\,\Delta G_\zeta(y) + \frac{1}{4\sqrt{\pi}\mu}\int_{\partial\mathcal{M}}\mathrm{d}l\,\Phi(y,\mu)\end{aligned} \tag{5.96}$$

where

$$\Phi(y,\mu) = \Delta\phi + \frac{1}{2\sqrt{\pi}\mu}\partial_n\Delta\phi + \frac{1}{12\mu^2}\partial_n^2\Delta\phi + \dots \tag{5.97}$$

At this point one might be tempted to take $\mu\to\infty$ to get rid of the last term but, of course, one must remember that $G_\zeta$ also depends on $\mu$. However, this relation shows that, since the left-hand side does not depend on $\mu$, the quantity $\int_{\mathcal{M}} \mathrm{d}^2y\sqrt{g}\,\phi(y)\,\Delta G_\zeta(y)$ has a finite limit as $\mu\to\infty$ and we arrive at the two following equivalent expressions:

$$\begin{aligned}\lim_{s\to 1}\left[1+(s-1)\left(\frac{\mathrm{d}}{\mathrm{d}s}+\ln\widehat{\mu}^2\right)\right]\int_{\mathcal{M}} \mathrm{d}^2y\sqrt{g}\,\Delta\phi(y)\,\zeta(s,y,y)\\ = \lim_{\mu\to\infty}\int_{\mathcal{M}} \mathrm{d}^2y\sqrt{g}\,\phi(y)\,\Delta G_\zeta(y)\\ = \int_{\mathcal{M}} \mathrm{d}^2y\sqrt{g}\,\Delta\phi(y)\,G_{\mathrm{R,bulk}}(y)\ .\end{aligned} \tag{5.98}$$

Both ways of writing require a comment: While the $\mu\to 0$ limit of the integral involving $\Delta G_\zeta$ exists, this is not the case of $\Delta G_\zeta(y)$ itself for $y$ on the boundary. On the other hand, in the integral involving $G_{\mathrm{R,bulk}}$, even though $G_{\mathrm{R,bulk}}$ does not satisfy the Neumann condition, one might want to integrate by parts generating a boundary term:

$$\int_{\mathcal{M}} \mathrm{d}^2y\sqrt{g}\,\Delta\phi(y)\,G_{\mathrm{R,bulk}}(y)\ ?= ?\int_{\mathcal{M}} \mathrm{d}^2y\sqrt{g}\,\phi(y)\,\Delta G_{\mathrm{R,bulk}}(y) + \int_{\partial\mathcal{M}}\mathrm{d}l\,\phi(y)\partial_n G_{\mathrm{R,bulk}}(y). \tag{5.99}$$

However, this is not possible : both terms on the r.h.s. are meaningless since $\Delta G_{\mathrm{R,bulk}}(y)$ has a non-integrable singularity as $y$ approaches the boundary (expected to be $\sim 1/\ell^2(y,\partial\mathcal{M})$), and $\partial_n G_{\mathrm{R,bulk}}$ is infinite everywhere on the boundary.

## 5.5 The Mabuchi action on a manifold with boundaries

We are now in position to assemble our results and determine the gravitational action on a Riemann surface with boundaries. As already explained, the strategy is to use the infinitesimal variation of $S_{\mathrm{grav}}$ under an infinitesimal change of the metric as given by (5.36), and then to integrate $\delta S_{\mathrm{grav}}$ to obtain $S_{\mathrm{grav}}[\hat{g},g]$.

Inserting (5.85) and (5.88) into (5.36), we immediately get

$$\begin{aligned}\delta S_{\mathrm{grav}} &= -\frac{1}{24\pi}\left[\int_{\mathcal{M}}\sqrt{g}\,\delta\sigma\,R + 3\int_{\partial\mathcal{M}}\mathrm{d}l\,\partial_n\delta\sigma\right]\\ &\quad + m^2\left[\int_{\mathcal{M}}\sqrt{g}\,\delta\sigma\left(G_\zeta + \frac{1}{4\pi} - \frac{1}{4\pi}\ln\frac{\mu^2}{\hat{\mu}^2}\right) + \frac{1}{4\sqrt{\pi}\mu}\int_{\partial\mathcal{M}}\mathrm{d}l\,F(y,\mu)\right]\\ &= -\frac{1}{24\pi}\left[\int_{\mathcal{M}}\sqrt{g}\,\delta\sigma\,R + 3\int_{\partial\mathcal{M}}\mathrm{d}l\,\partial_n\delta\sigma\right] + m^2\int_{\mathcal{M}}\sqrt{g}\,\delta\sigma\left(G_{\mathrm{R,bulk}} + \frac{1}{4\pi}\right)\ .\end{aligned} \tag{5.100}$$

Note that this is not an expansion in powers of $m^2$ but an exact result. Our perturbation theory was a first order perturbation in $\delta\sigma$ not in $m^2$. Indeed, $G_\zeta$ and $G_{\mathrm{R,bulk}}$ still depend on $m^2$ and we get exactly the first two terms in an expansion in powers of $m^2$ if we replace them by the corresponding quantities $G_\zeta^{(0)}$ and $G_{\mathrm{R,bulk}}^{(0)}$ defined for the massless case. However there is a subtlety here, since in the massless case the zero-mode must be excluded from the sum over eigenvalues defining the Green's function. If we denote with a tilde all quantities lacking the zero-mode contribution we have

$$G(x,y) = \frac{1}{m^2 A} + \widetilde{G}(x,y), \quad G_\zeta(x) = \frac{1}{m^2 A} + \widetilde{G}_\zeta(x), \quad G_{\mathrm{R,bulk}}(x) = \frac{1}{m^2 A} + \widetilde{G}_{\mathrm{R,bulk}}(x) \ . \tag{5.101}$$

The quantities $\widetilde{G}$, $\widetilde{G}_\zeta$ and $\widetilde{G}_{\mathrm{R,bulk}}$ all have a smooth limit as $m \to 0$. Thus, the expansion in powers of $m^2$ reads

$$\begin{aligned}
\delta S_{\mathrm{grav}} &= -\frac{1}{24\pi}\left[\int_{\mathcal{M}} \sqrt{g}\,\delta\sigma\, R + 3\int_{\partial\mathcal{M}} \mathrm{d}l\, \partial_n \delta\sigma\right] + \frac{\delta A}{2A} \\
&\quad + m^2\left[\int_{\mathcal{M}} \sqrt{g}\,\delta\sigma\,\left(\widetilde{G}_\zeta^{(0)} + \frac{1}{4\pi} - \frac{1}{4\pi}\ln\frac{\mu^2}{\hat{\mu}^2}\right) + \frac{1}{4\sqrt{\pi}\mu}\int_{\partial\mathcal{M}} \mathrm{d}l\, F(y,\mu)\right] + \mathcal{O}(m^4) \\
&= -\frac{1}{24\pi}\left[\int_{\mathcal{M}} \sqrt{g}\,\delta\sigma\, R + 3\int_{\partial\mathcal{M}} \mathrm{d}l\, \partial_n \delta\sigma\right] + \frac{\delta A}{2A} \\
&\quad + m^2\left[\int_{\mathcal{M}} \sqrt{g}\,\delta\sigma\,\left(\widetilde{G}_{\mathrm{R,bulk}}^{(0)} + \frac{1}{4\pi}\right)\right] + \mathcal{O}(m^4) \ .
\end{aligned}$$

The first term is independent of $m$ and corresponds to $-\frac{1}{6}$ times the variation of the Liouville action on a manifold with boundary [175, 176]

$$S_L = \frac{1}{4\pi}\left(\int_{\mathcal{M}} \sqrt{\hat{g}}\,(\hat{g}^{ab}\partial_a\sigma\partial_b\sigma + \hat{R}\sigma) + 3\int_{\partial\mathcal{M}} \mathrm{d}l\,\partial_n\sigma\right) \ , \tag{5.102}$$

while the $\frac{\delta A}{2A}$-term contributes a piece $\frac{1}{2}\ln\frac{A}{\hat{A}}$ to $S_{\mathrm{grav}}$. Recall that, contrary to $\phi$ or $\delta\phi$, the field $\sigma$ and its variation $\delta\sigma$ do not satisfy the Neumann condition.

### 5.5.1 Variation of $\widetilde{G}^{(0)}_{\mathrm{R,bulk}}$

To go further, we need the variation of $\widetilde{G}_\zeta^{(0)}$ and of $\int_{\partial\mathcal{M}} \mathrm{d}l\,\Sigma$ or the variation of $\widetilde{G}^{(0)}_{\mathrm{R,bulk}}$ under an infinitesimal variation of the metric corresponding to $\delta\sigma$. At this point it turns out to be easier to study the variation of $\widetilde{G}^{(0)}_{\mathrm{R,bulk}}$ which is obtained exactly as for a manifold without boundary (we will see an alternative derivation using the variation of $\widetilde{G}_\zeta^{(0)}$ in the next section). The simplest derivation just uses the differential equation satisfied by $G(x,y)$ to obtain

$$\delta G(x,y) = -2m^2 \int \mathrm{d}^2 z \sqrt{g}\, G(x,z)\,\delta\sigma(z)\, G(z,y) \ , \tag{5.103}$$

which satisfies the Neumann conditions. Alternatively, one can use the perturbation theory formulae (5.30) and (5.31) to obtain

$$\begin{aligned}
\delta G(x,y) &= \sum_n \frac{\delta\varphi_n(x)\varphi_n(y) + \varphi_n(x)\delta\varphi_n(y)}{\lambda_n} - \frac{\varphi_n(x)\varphi_n(y)\delta\lambda_n}{\lambda_n^2} \\
&= -2m^2 \sum_{n,k} \frac{\langle\varphi_k|\delta\sigma|\varphi_n\rangle\varphi_k(x)\varphi_n(y)}{\lambda_n\lambda_k} = -2m^2\int \mathrm{d}^2 z\sqrt{g}\, G(x,z)\delta\sigma(z)G(z,y) \ ,
\end{aligned} \tag{5.104}$$

in agreement with (5.103). Next, the variation of $\ell^2(x,y)$ was given e.g. in [63, 70, 132]. In the limit $x \to y$ one simply has $\ell^2(x,y) \simeq g_{ab}\mathrm{d}x^a\mathrm{d}x^b = e^{2\sigma(y)}\hat{g}_{ab}\mathrm{d}x^a\mathrm{d}x^b$ which shows that

one has $\delta\ell^2(x,y) \simeq 2\delta\sigma(y)\,\ell^2(x,y)$ and, hence,

$$\lim_{x\to y} \delta \ln\left(\ell^2(x,y)\mu^2\right) = 2\delta\sigma(y)\ . \tag{5.105}$$

It follows that

$$\delta G_{\mathrm{R,bulk}}(x) = -2m^2 \int \mathrm{d}^2z\sqrt{g}\,(G(x,z))^2\ \delta\sigma(z) + \frac{\delta\sigma(x)}{2\pi}\ . \tag{5.106}$$

Separating the zero-mode parts $\frac{1}{m^2A}$, this is rewritten as

$$\delta\widetilde{G}_{\mathrm{R,bulk}}(x) = -\frac{4}{A}\int \mathrm{d}^2z\sqrt{g}\,\widetilde{G}(x,z)\delta\sigma(z) - 2m^2\int \mathrm{d}^2z\sqrt{g}\left(\widetilde{G}(x,z)\right)^2\,\mathrm{d}\sigma(z) + \frac{\delta\sigma(x)}{2\pi}\ . \tag{5.107}$$

Since $\delta\sqrt{g} = 2\sqrt{g}\,\delta\sigma$, it follows that

$$\delta\int\sqrt{g}\,\widetilde{G}_{\mathrm{R,bulk}}(x) = \int\sqrt{g}\,2\delta\sigma\,\left(\widetilde{G}_{\mathrm{R,bulk}}(x) + \frac{1}{4\pi}\right) + \mathcal{O}(m^2)\ . \tag{5.108}$$

(Note that the first term in (5.107) integrates to zero and does not contribute in (5.108).) Thus

$$\delta S_{\mathrm{grav}} = -\frac{1}{6}\delta S_L + \frac{1}{2}\frac{\delta A}{A} + \frac{m^2}{2}\ \delta\ \int\sqrt{g}\,\widetilde{G}^{(0)}_{\mathrm{R,bulk}}(x) + \mathcal{O}(m^4)\ , \tag{5.109}$$

where $\widetilde{G}^{(0)}_{\mathrm{R,bulk}}$ is computed from the Green's function without zero-mode of the massless theory. Thus the order $m^2$ term in (5.109) is given by the variation of the functional

$$\Phi_G[g] = \int\sqrt{g}\,\widetilde{G}^{(0)}_{\mathrm{R,bulk}}(x;g)\ , \tag{5.110}$$

where we explicitly indicated the dependence of $\widetilde{G}^{(0)}_{\mathrm{R,bulk}}$ on the metric $g$. Thus

$$S_{\mathrm{grav}}[\hat{g},g] = -\frac{1}{6}S_L[\hat{g},g] + \frac{1}{2}\ln\frac{A}{\hat{A}} + \frac{m^2}{2}\ (\Phi_G[g] - \Phi_G[\hat{g}]) + \mathcal{O}(m^4). \tag{5.111}$$

In order to express $\Phi_G[g] - \Phi_G[\hat{g}]$ as a local functional of $\sigma$ and $\phi$, we use again (5.107) in the zero-mass limit and replace $\delta\sigma$ in the first term by $\frac{\delta A}{2A} - \frac{A}{4}\Delta\delta\phi$ according to (4.121):

$$\begin{aligned}\delta\widetilde{G}^{(0)}_{\mathrm{R,bulk}}(x) &= \int \mathrm{d}^2z\sqrt{g}\,\widetilde{G}^{(0)}(x,z)\Delta\delta\phi(z) + \frac{\delta\sigma(x)}{2\pi}\\ &= \delta\phi(x) - \frac{1}{A}\int \mathrm{d}^2z\sqrt{g}\,\delta\phi(z) + \frac{\delta\sigma(x)}{2\pi}\ .\end{aligned} \tag{5.112}$$

Note that we integrated the Laplace operator by parts without generating boundary terms since both $\widetilde{G}$ and $\delta\phi$ satisfy the Neumann boundary conditions (which is not the case for $\delta\sigma$) and then we used the differential equation (3.45). Equation (5.112) can be integrated as

$$\widetilde{G}^{(0)}_{\mathrm{R,bulk}}(x,g) - \widetilde{G}^{(0)}_{\mathrm{R,bulk}}(x,\hat{g}) = \phi(x) + \frac{\sigma(x)}{2\pi} - S_{\mathrm{AY}}[\hat{g},g]\ , \tag{5.113}$$

with

$$S_{\mathrm{AY}}[\hat{g},g] = -\int\sqrt{\hat{g}}\,\left(\frac{1}{4}\hat{g}^{ab}\,\partial_a\phi\,\partial_b\phi - \frac{\phi}{\hat{A}}\right)\ . \tag{5.114}$$

Finally we obtain

$$\begin{aligned}\Phi_G[g] - \Phi_G[\hat{g}] &= \int\sqrt{\hat{g}}\,\left(\frac{A}{\hat{A}} - \frac{A}{2}\hat{\Delta}\phi\right)\left(\widetilde{G}^{(0)}_{\mathrm{R,bulk}}[\hat{g}] + \phi + \frac{\sigma}{2\pi} - S_{AY}[\hat{g},g]\right) - \Phi_G[\hat{g}]\\ &= \frac{A-\hat{A}}{\hat{A}}\Phi_G[\hat{g}] - \frac{A}{2}\int\sqrt{\hat{g}}\,\left(\frac{1}{2}\phi\hat{\Delta}\phi - \frac{1}{\pi A}\sigma\mathrm{e}^{2\sigma} + \hat{\Delta}\phi\,\widetilde{G}^{(0)}_{\mathrm{R,bulk}}[\hat{g}]\right)\ .\end{aligned} \tag{5.115}$$

As already emphasized, contrary to $G(x,y)$ or $G_\zeta(x)$, the quantities $G_{\mathrm{R,bulk}}$ and $\widetilde{G}^{(0)}_{\mathrm{R,bulk}}$ do not satisfy the Neumann condition. Moreover, $\partial_n G_{\mathrm{R,bulk}}$ and $\partial\widetilde{G}^{(0)}_{\mathrm{R,bulk}}$ are singular on the boundary.

Thus, we arrive at

$$S_{\mathrm{grav}}[\hat{g},g] = -\frac{1}{6}S_L[\hat{g},g] + \frac{1}{2}\ln\frac{A}{\hat{A}} + \frac{m^2\,(A-\hat{A})}{2\hat{A}}\Phi_G[\hat{g}] + \frac{m^2 A}{4}\left[\int\sqrt{\hat{g}}\left(-\frac{1}{2}\phi\hat{\Delta}\phi + \frac{1}{\pi A}\sigma \mathrm{e}^{2\sigma} - \hat{\Delta}\phi\,\widetilde{G}^{(0)}_{\mathrm{R,bulk}}[\hat{g}]\right)\right] + \mathcal{O}(m^4)\ . \quad (5.116)$$

The first line contains the usual Liouville action along with a factor $+\frac{1}{2}\ln\frac{A}{\hat{A}}$, as well as a contribution to the cosmological constant action. The cosmological constant action is required in any case to act as a counterterm to cancel the divergence that accompanies the $\zeta'(0)$ when properly evaluating the determinant, e.g. with the spectral cut-off regularization as was done in [132]. The terms in the second line are the genuine order $m^2$ corrections. Using the second equality in (5.98), we can rewrite the latter using $\widetilde{G}^{(0)}_\zeta \equiv \widetilde{G}^{(0)}_\zeta[\hat{g},\mu]$ instead of $\widetilde{G}^{(0)}_{\mathrm{R,bulk}}$. In particular, this allows us to integrate by parts the Laplacian, but it requires to take the $\mu\to\infty$ limit :

$$S_{\mathrm{grav}}[\hat{g},g] = -\frac{1}{6}S_L[\hat{g},g] + \frac{1}{2}\ln\frac{A}{\hat{A}} + \frac{m^2\,(A-\hat{A})}{2\hat{A}}\Phi_G[\hat{g}] + \frac{m^2 A}{4}\lim_{\mu\to\infty}\left[\int\sqrt{\hat{g}}\left(-\frac{1}{2}\phi\hat{\Delta}\phi + \frac{1}{\pi A}\sigma \mathrm{e}^{2\sigma} - \phi\,\hat{\Delta}\widetilde{G}^{(0)}_\zeta[\hat{g},\mu]\right)\right] + \mathcal{O}(m^4)\ . \quad (5.117)$$

Written this way, the order $m^2$-terms looks like the usual Mabuchi plus Aubin–Yau actions found for manifolds without boundary [63, 70]. However, here the function $\hat{\Delta}\widetilde{G}^{(0)}_\zeta(x)$ is no longer a simple expression but depends non-trivially on the point $x$ and in particular on the distances from the various boundary components. Of course, the same is true for $\widetilde{G}^{(0)}_{\mathrm{R,bulk}}(x)$.

### 5.5.2 Variation of $\widetilde{G}^{(0)}_\zeta$

We can also derive the gravitation action directly in terms of $\tilde{G}^{(0)}_\zeta$ instead of $\tilde{G}_{\mathrm{R,bulk}}$. For this we need the variation of $\tilde{G}^{(0)}_\zeta$ for finite $\mu$ which requires the use of some additional variational formulae we will derive here.

First, the variation of $\tilde{G}_\zeta$ also involves the variation of $E_1\left(\frac{\ell^2(y,y^i_C)\mu^2}{4}\right)$. For $x\to y$ and close to the boundary one has

$$\delta\ell^2(y,y^i_C) \simeq 2\delta\sigma(y_B)\,\ell^2(y,y^i_C). \quad (5.118)$$

It follows that

$$\delta E_1\left(\frac{\ell^2(y,y^i_C)\mu^2}{4}\right) = E_1'\left(\frac{\ell^2(y,y^i_C)\mu^2}{4}\right)\frac{\ell^2(y,y^i_C)\mu^2}{2}\,\delta\sigma(y^i_B) = -2\,\mathrm{e}^{-\frac{\ell^2(y,y^i_C)\mu^2}{4}}\delta\sigma(y^i_B) \quad (5.119)$$

where we used the fact that

$$\frac{\mathrm{d}}{\mathrm{d}x}E_s(x) = -E_{s-1}(x), \qquad \frac{\mathrm{d}}{\mathrm{d}x}E_1(x) = -E_0(x) = -\frac{\mathrm{e}^{-x}}{x}. \quad (5.120)$$

Thus we get

$$\delta G_\zeta(x) = -2m^2\int \mathrm{d}^2z\sqrt{g}\,(G(x,z))^2\,\delta\sigma(z) + \frac{\delta\sigma(x)}{2\pi} + \sum_{\partial\mathcal{M}_i}\mathrm{e}^{-\ell^2(x,x^i_C)\mu^2/4}\,\frac{\delta\sigma(x^i_B)}{2\pi}. \quad (5.121)$$

One also encounters $\sum_i \int_{\partial\mathcal{M}_i} \mathrm{d}l\,\delta\sigma$ :

$$\sum_i \int_{\partial\mathcal{M}_i} \mathrm{d}l\,\delta\sigma(x_B) = \sum_i \delta\left(\int_{\partial\mathcal{M}_i} \mathrm{d}l\right) = \delta L(\partial\mathcal{M}) \tag{5.122}$$

where $L(\partial\mathcal{M})$ is the total length of the boundary. The finite variation of $\tilde{G}^{(0)}_\zeta$ follows from (5.121) in the zero-mass limit as

$$\begin{aligned}\tilde{G}^{(0)}_\zeta(x,g) - \tilde{G}^{(0)}_\zeta(x,\hat{g}) = \phi(x) + \frac{\sigma(x)}{2\pi} - S_{\mathrm{AY}}[\hat{g},g] \\ - \frac{1}{4\pi}\sum_{\partial\mathcal{M}_i}\left(E_1\left(\frac{\ell_g^2(x,x_C^i)\mu^2}{4}\right) - E_1\left(\frac{\ell_{\hat{g}}^2(x,x_C^i)\mu^2}{4}\right)\right).\end{aligned} \tag{5.123}$$

Finally, at finite $\mu$ the gravitational action is given by

$$\begin{aligned}S_{\mathrm{grav}}[\hat{g},g] = -\frac{1}{6}S_L[\hat{g},g] + \frac{1}{2}\ln\frac{A}{\hat{A}} + \frac{m^2\,(A-\hat{A})}{2\hat{A}}\left(\Phi_G[\hat{g}] + \frac{1}{4\sqrt{\pi}\,\mu}L(\partial\mathcal{M},\hat{g})\right) \\ + \frac{m^2 A}{4}\int\sqrt{\hat{g}}\left(-\frac{1}{2}\phi\hat{\Delta}\phi + \frac{1}{\pi A}\sigma \mathrm{e}^{2\sigma} - \phi\,\hat{\Delta}\tilde{G}^{(0)}_\zeta[\hat{g}]\right) \\ - \frac{m^2 A}{16\sqrt{\pi}}\left(\frac{1}{\mu}\int_{\partial\mathcal{M}}\mathrm{d}\hat{l}\,\hat{\Delta}\phi + \mathcal{O}\left(\frac{1}{\mu^2}\right)\right) + \mathcal{O}(m^4)\end{aligned} \tag{5.124}$$

which gives back (5.117) when taking $\mu \to +\infty$.

## 5.6 The cylinder

In this section we work out the gravitational action for the simplest two-dimensional manifold with a boundary : the cylinder. As we have seen in section 5.3.1 the heat kernel and hence also the Green's function on the cylinder are obtained from the corresponding quantities on the torus by a method of images. Thus to get the Green's function of the Laplace operator on the cylinder of length $T$ and circumference $2\pi R$ we first determine the Green's function on the torus with periods $2T$ and $2\pi R$.

With respect to our general notation, throughout this section we consider a fixed reference metric $\hat{g}$ and corresponding Laplacian $\hat{\Delta}$ and Green's functions $G(z_1,z_2;\hat{g})$ although we will mostly drop the reference to $\hat{g}$.

### 5.6.1 Green's function on the torus

To get the Green's function on the torus with periods $2a$ and $2b$ , in principle, one could take the heat kernel $\mathcal{K}_{\mathrm{torus}}(t,x_1,x_2,y_1,y_2)$ as constructed from the eigenfunctions (5.47) and eigenvalues of the Laplace operator, cf (5.48) and integrate over $t$ form 0 to $\infty$. However, we have not been able to find any useful formula for $\int_0^\infty \mathrm{d}t\,\theta_3\left(i\frac{\pi t}{b^2}\middle|\frac{x_2-y_2}{2b}\right)\theta_3\left(i\frac{\pi t}{a^2}\middle|\frac{x_1-y_1}{2a}\right)$. Instead, we will follow the usual approach to identify a suitable doubly periodic solution of the Laplace equation with the correct singularity at the origin. It will be convenient to use a complex coordinate $z$. Thus, in this section we will change our notation with respect to the previous one and call $x$ and $y$ the real and imaginary parts of $z$:

$$z = x + iy \quad , \quad x \simeq x + 2a \quad , \quad y \simeq y + 2b\ . \tag{5.125}$$

When we need to label two points we will use $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. We thus have a square torus with modular parameter $\tau = i\frac{b}{a}$. The reference metric $\hat{g}$ is just the standard metric $\mathrm{d}s^2 = \mathrm{d}z\mathrm{d}\bar{z}$ and $\hat{\Delta} = -4\partial\bar{\partial}$.

The Green's function $\mathcal{G}(z_1, z_2)$ must be doubly periodic in both $z_1$ and $z_2$ with periods $2a$ and $2b\,i$. By the translational invariance of the torus, it can only depend on the difference $z_1 - z_2$, and it must exhibit the appropriate $-\frac{1}{4\pi}\ln|z_1 - z_2|^2$ singularity as $z_1 \to z_2$ in order to satisfy

$$\hat{\Delta}\mathcal{G}(z_1, z_2) = \delta^{(2)}(z_1 - z_2) - \frac{1}{\hat{A}}. \tag{5.126}$$

Define the function $g$ (not to be confused with the metric) as

$$\begin{aligned} g(z) &= \frac{(\mathrm{Im}\, z)^2}{8ab} - \frac{1}{4\pi}\ln\left[\theta_1\left(\frac{z}{2a}\Big|i\frac{b}{a}\right)\theta_1\left(\frac{\bar{z}}{2a}\Big|i\frac{b}{a}\right)\right] \\ &= \frac{(\mathrm{Im}\, z)^2}{8ab} - \frac{1}{4\pi}\ln\left|\theta_1\left(\frac{z}{2a}\Big|i\frac{b}{a}\right)\right|^2 , \end{aligned} \tag{5.127}$$

where the elliptic theta function is defined as [177]

$$\begin{aligned} \theta_1(\nu|\tau) &= 2q^{1/4}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\sin(2n+1)\pi\nu \\ &= 2q^{1/4}\sin\pi\nu\prod_{n=1}^{\infty}(1-q^{2n})(1-2q^{2n}\cos 2\pi\nu + q^{4n}) , \quad q = \mathrm{e}^{i\pi\tau} . \end{aligned} \tag{5.128}$$

The latter satisfies $\theta_1(\nu+1|\tau) = -\theta_1(\nu|\tau)$ and $\theta_1(\nu+\tau|\tau) = -\mathrm{e}^{-i\pi(2\nu+\tau)}\theta_1(\nu|\tau)$. It follows that

- $g(x+iy)$ is periodic under $x \to x+2a$ and under $y \to y+2b$.
- It is obvious from the factorization of the logarithm that for $z \neq 0$

$$-4\partial\bar{\partial}g = -\frac{1}{4ab} = -\frac{1}{\hat{A}} \tag{5.129}$$

  where $\hat{A}$ is the area of the torus.
- As $z \to 0$, $g(z) \sim -\frac{1}{4\pi}\ln|z|^2$, which together with the previous relation ensures that

$$\hat{\Delta}g(z) = -4\partial\bar{\partial}g(z) = \delta^{(2)}(z) - \frac{1}{\hat{A}} . \tag{5.130}$$

- We have the symmetry properties

$$g(\bar{z}) = g(z), \qquad g(-z) = g(z), \qquad g'(-z) = -g'(z). \tag{5.131}$$

Thus

$$\mathcal{G}(z_1, z_2) = g(z_1 - z_2) \tag{5.132}$$

is the appropriate Green's function on the torus. It would be satisfying to show that this coincides with the expression for the Green's function obtained by integrating the heat kernel one gets from the eigenfunction expansion but, as already mentioned, we have not been able to find a corresponding identity in the literature.

One can then define the renormalized Green's function at coinciding points $\mathcal{G}_R(z)$ on the torus, after subtracting the short-distance singularity as

$$\begin{aligned} \mathcal{G}_R(z) &= \lim_{z_1 \to z_2 \equiv z}\left(\mathcal{G}(z_1, z_2) + \frac{1}{4\pi}\ln|z_1 - z_2|^2\right) \\ &= -\frac{1}{2\pi}\ln\left(\frac{\pi}{a}\, q^{1/4}\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)}\right) \end{aligned} \tag{5.133}$$

with $q = \mathrm{e}^{-\pi b/a}$. As was expected from the isometries of the torus $\mathcal{G}_R$ is a constant.

### 5.6.2 Green's function and Green's functions at coinciding points on the cylinder

We now construct the Green's function on the cylinder of length $T$ (coordinate $x$) and circumference $2\pi R$ (coordinate $y$). We impose Neumann boundary conditions at $x = 0$ and $x = T$. Let $g$ be the function defined in (5.127) with $a = T$ and $b = \pi R$:

$$g(z) = \frac{(\mathrm{Im}\, z)^2}{8\pi RT} - \frac{1}{4\pi} \ln \left[ \theta_1 \left( \frac{z}{2T} | i\pi \frac{R}{T} \right) \theta_1 \left( \frac{\bar{z}}{2T} | i\pi \frac{R}{T} \right) \right] . \tag{5.134}$$

Again, the Neumann boundary conditions are achieved by adding to $g(z_1 - z_2)$ the same functions with $z_2 = x_2 + iy_2$ replaced by the appropriate image points. The boundary at $x = 0$ requires the image point $z_2^C = -\bar{z}_2 = -x_2 + iy_2$, while the boundary at $x = T$ would require to add the image points of $z_2$ and $z_2^C$, i.e. $T + (T - x_2 + iy_2) = 2T + z_2^C$ and $T + (T + x_2 + iy_2) = 2T + z_2$. However, due to the $2T$-periodicity these points are equivalent to $z_2$ and $z_2^C$ and adding $g$ at these points would result in an over-counting. Of course, this is in agreement with the relation between the heat kernels of the torus and the cylinder, from which the corresponding Green's functions could be obtained by integration over $t$. Thus we let

$$G^{\mathrm{cyl}}(z_1, z_2) = \mathcal{G}(z_1, z_2) + \mathcal{G}(z_1, \bar{z}_2) = g(z_1 - z_2) + g(z_1 + \bar{z}_2) . \tag{5.135}$$

Using the symmetry properties (5.131), one easily verifies that this indeed satisfies Neumann conditions at $x_1 = 0$ and $T$ as well as at $x_2 = 0$ and $T$, e.g.

$$\begin{aligned} \partial_{n,1} G^{\mathrm{cyl}}(z_1, z_2)\big|_{z_1 = T} &= g'(T + iy_1 - x_2 - iy_2) + g'(T + iy_1 + x_2 - iy_2) \\ &= g'(T - iy_1 - x_2 + iy_2) - g'(-T - iy_1 - x_2 + iy_2) = 0 . \end{aligned} \tag{5.136}$$

From (5.130) we see that $G^{\mathrm{cyl}}$ satisfies, for any $x_2 \neq 0, T$,

$$\hat{\Delta}_{z_1} G^{\mathrm{cyl}}(z_1, z_2) = \delta^{(2)}(z_1 - z_2) - \frac{1}{2\pi RT} \tag{5.137}$$

where the term $-\frac{1}{2\pi RT}$ arises as $-\frac{2}{A^{\mathrm{torus}}}$ and equals $-\frac{1}{A^{\mathrm{cyl}}}$. Integrating the right-hand side of (5.137) over the cylinder then correctly yields 0.

Next, we need to determine the various Green's functions at coinciding points that played an important role for formulating the gravitational action, i.e. $G_\zeta^{\mathrm{cyl}}(z)$ and $G_{\mathrm{R,bulk}}^{\mathrm{cyl}}(z)$. In the present specific case of the cylinder it is useful to first define yet another function $G_{\mathrm{R}}^{\mathrm{cyl}}(z)$ by

$$\begin{aligned} G_{\mathrm{R}}^{\mathrm{cyl}}(z) = \lim_{z_1 \to z_2 \equiv z} \Bigg( & G^{\mathrm{cyl}}(z_1, z_2) \\ & + \frac{1}{4\pi} \ln \left[ \sin \frac{\pi(z_1 - z_2)}{2T} \sin \frac{\pi(\bar{z}_1 - \bar{z}_2)}{2T} \sin \frac{\pi(z_1 + \bar{z}_2)}{2T} \sin \frac{\pi(\bar{z}_1 + z_2)}{2T} \right] \Bigg) . \end{aligned} \tag{5.138}$$

The additional terms subtract the bulk singularity at $z_1 \to z_2$, as well as the boundary singularities that occur as $z_1 \to z_2 \to 0$ or $T$. Explicitly we find that $G_{\mathrm{R}}^{\mathrm{cyl}}$ only depends on $x = \mathrm{Re}\, z$ (as well as on $q = e^{-\pi^2 R/T}$, of course):

$$G_{\mathrm{R}}^{\mathrm{cyl}}(z) \equiv G_{\mathrm{R}}^{\mathrm{cyl}}(x) = -\frac{1}{2\pi} \ln \left[ \frac{\theta_1(\frac{x}{T} | i\pi \frac{R}{T})}{\sin \frac{\pi x}{T}} \right] - \frac{1}{2\pi} \ln \left[ 2q^{1/4} \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)} \right] . \tag{5.139}$$

One sees again, that this is non-singular, even as $x \to 0$ or $x \to T$. It is clear from its definition that $G_{\mathrm{R}}^{\mathrm{cyl}}$ satisfies Neumann boundary conditions, as follows also from the explicit expression just given.

However, it is not this quantity $G_{\rm R}^{\rm cyl}$ which enters the gravitational action, but rather $G_\zeta^{\rm cyl}$ or $G_{\rm R,bulk}^{\rm cyl}$. These quantities differ from $G_{\rm R}^{\rm cyl}$ by the following terms:

$$\begin{aligned}\delta G^{\rm cyl}_{\zeta/R}(z) &\equiv G^{\rm cyl}_\zeta(z) - G^{\rm cyl}_R(z)\\ &= \frac{1}{4\pi}\left[2\ln\frac{T\mu}{\pi} - E_1(x^2\mu^2) - E_1((T-x)^2\mu^2) - 2\ln\sin\frac{\pi x}{T} + 2\gamma\right]\ ,\\ \delta G^{\rm cyl}_{\rm R,bulk/R}(z) &\equiv G^{\rm cyl}_{\rm R,bulk}(z) - G^{\rm cyl}_R(z)\\ &= \frac{1}{4\pi}\left[2\ln\frac{T\widehat{\mu}}{\pi} - 2\ln\sin\frac{\pi x}{T} + 2\gamma\right]\ ,\end{aligned} \tag{5.140}$$

The expression $\delta G^{\rm cyl}_{\zeta/R}$ is non-singular, even as $x\to 0$ or $x\to T$, as it obviously should be, since all singularities have been removed in the definition of $G_\zeta$, as well as in the one of $G_R$. On the other hand, $\delta G^{\rm cyl}_{\zeta/{\rm R,bulk}}$ is singular as $x\to 0$ or $x\to T$, and in particular it cannot satisfy the Neumann condition. On the other hand, $\delta G^{\rm cyl}_{\zeta/R}$ satisfies the Neumann condition at $x=0$ and $x=T$ for any finite $\mu$. Explicitly we have

$$G^{\rm cyl}_{\rm R,bulk}(x) = -\frac{1}{2\pi}\ln\theta_1\left(\frac{x}{T}\middle|i\pi\frac{R}{T}\right) + \frac{1}{2\pi}\left(\ln\frac{T\widehat{\mu}}{\pi} + \gamma - \ln\left[2q^{1/4}\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)}\right]\right) \tag{5.141}$$

with only the first term depending on $x$.

### 5.6.3 Laplacian of $G_R$ and $G_\zeta$ on the cylinder

We also need the Laplacian of the Green's function at coinciding points $G_R^{\rm cyl}$ and $G_\zeta^{\rm cyl}$. Recall that $\hat\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$, so that $\hat\Delta G_R^{\rm cyl}(x) = -\frac{\partial^2}{\partial x^2}G_R^{\rm cyl}(x)$ and we find from (5.139), using a formula from p 358 of [177], and writing $q = {\rm e}^{-\pi^2 R/T}$ :

$$\begin{aligned}\hat\Delta G_R^{\rm cyl}(x) &= \left(\frac{1}{2\pi}\ln\left[\frac{\theta_1(\frac{x}{T}|i\pi\frac{R}{T})}{\sin\frac{\pi x}{T}}\right]\right)''\\ &= \frac{1}{2\pi T}\left(\frac{\theta_1'\left(\frac{x}{T}\middle|i\pi\frac{R}{T}\right)}{\theta_1\left(\frac{x}{2T}\middle|i\pi\frac{R}{T}\right)} - \pi\cot\frac{\pi x}{T}\right)'\\ &= \frac{2}{T}\left(\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin\frac{2\pi n x}{T}\right)'\\ &= \frac{4\pi}{T^2}\sum_{n=1}^{\infty}\frac{n\,q^{2n}}{1-q^{2n}}\cos\frac{2\pi n x}{T}.\end{aligned} \tag{5.142}$$

This sum converges for all $x$, as well as all $R\neq 0$ and all finite $T$. To obtain $\hat\Delta G_\zeta^{\rm cyl}$ we need to add $\hat\Delta\delta G_{\zeta/R}^{\rm cyl}$ which, using (5.140) and (5.120), is given by

$$\hat\Delta\,\delta G^{\rm cyl}_{\zeta/R}(x) = \frac{1}{2\pi}\left[\frac{1}{x^2}+2\mu^2\right]{\rm e}^{-x^2\mu^2} + \frac{1}{2\pi}\left[\frac{1}{(T-x)^2}+2\mu^2\right]{\rm e}^{-(T-x)^2\mu^2} - \frac{\pi}{2T^2}\frac{1}{\sin^2\frac{\pi x}{T}}\ , \tag{5.143}$$

and thus

$$\begin{aligned}\hat\Delta G^{\rm cyl}_\zeta(x) &= \frac{1}{2\pi}\left[\frac{1}{x^2}+2\mu^2\right]{\rm e}^{-x^2\mu^2} + \frac{1}{2\pi}\left[\frac{1}{(T-x)^2}+2\mu^2\right]{\rm e}^{-(T-x)^2\mu^2} - \frac{\pi}{2T^2}\frac{1}{\sin^2\frac{\pi x}{T}}\\ &\quad + \frac{4\pi}{T^2}\sum_{n=1}^{\infty}\frac{n\,q^{2n}}{1-q^{4n}}\cos\frac{2\pi n x}{T}\end{aligned} \tag{5.144}$$

where $q = \mathrm{e}^{-\pi^2 R/T}$.

Note that this is non-singular for all *finite* $\mu$, even as $x \to 0$ or $x \to T$.

If one thinks of the cylinder as a simple Euclidean version of one compact space and one time dimension, one would like to study the limit where the cylinder becomes infinitely long, i.e. $T \to \infty$. However, as $\frac{T}{R} \to \infty$, one has $q \to 1$ and the sum over $n$ diverges, hence this expression ceases to be valid. To study the behaviour as $\frac{T}{R} \to \infty$, one must first do the modular transformation $\tau \equiv i\pi\frac{R}{T} \to \widetilde{\tau} = -\frac{1}{\tau} = i\frac{T}{\pi R}$. This reads for $\theta_1$, as well as for $\theta_2$ (which we will need below),

$$\begin{aligned}\theta_1(\nu|\tau) &= \frac{i}{(-i\tau)^{1/2}} \exp\left(-i\pi\frac{\nu^2}{\tau}\right) \theta_1\left(\frac{\nu}{\tau}\Big| -\frac{1}{\tau}\right) \ , \\ \theta_2(\nu|\tau) &= \frac{1}{(-i\tau)^{1/2}} \exp\left(-i\pi\frac{\nu^2}{\tau}\right) \theta_4\left(\frac{\nu}{\tau}\Big| -\frac{1}{\tau}\right) \ . \end{aligned} \tag{5.145}$$

This will allow us to write the theta functions as sums of powers of $\widetilde{q} = e^{i\pi\widetilde{\tau}} = e^{-T/R}$. Note that the first argument $\frac{\nu}{\tau}$ now is imaginary which will turn the sin and cos in (5.142) into sinh and cosh. We get, through similar manipulations as above,

$$\begin{aligned}\hat{\Delta}G_R^{\mathrm{cyl}}(x) &= \left(\frac{1}{2\pi}\ln\frac{\theta_1\left(-\frac{ix}{\pi R}\middle| i\frac{T}{\pi R}\right)}{\sin\frac{\pi x}{T}} - \frac{x^2}{2\pi RT} + \frac{1}{2\pi}\ln i\sqrt{\frac{T}{\pi R}}\right)'' \\ &= \frac{\pi}{2T^2}\frac{1}{\sin^2\frac{\pi x}{T}} - \frac{1}{\pi RT} - \frac{1}{2\pi R^2}\frac{1}{\sinh^2\frac{x}{R}} - \frac{4}{\pi R^2}\sum_{n=1}^{\infty}\frac{n\,\widetilde{q}^{2n}}{1-\widetilde{q}^{2n}}\cosh\frac{2nx}{R} \ , \end{aligned} \tag{5.146}$$

where $\widetilde{q} = \mathrm{e}^{i\pi\widetilde{\tau}} = \mathrm{e}^{-T/R}$.

Note that the poles at $x = 0$ cancel and that this representation as an infinite sum is convergent and finite for all $|x| < T$. Note also that, although not obvious, within this interval $(-T, T)$ these functions are periodic under $x \to x + T$. Then the finiteness at $x = 0$ implies finiteness at $x = T$, too.

To study the $T \to \infty$ limit, we set

$$x = \frac{T}{2} + t \quad , \quad t \in \left[-\frac{T}{2}, \frac{T}{2}\right] . \tag{5.147}$$

so that, in particular, finite $t$ corresponds to values in the "middle" of the cylinder. Of course, we could use (5.146), but we certainly get a better convergence if we first re-express the $\theta_1\left(\frac{1}{2} + \frac{t}{T}\middle| i\pi\frac{R}{T}\right)$ appearing in (5.142) in terms of $\theta_2\left(\frac{t}{T}\middle| i\pi\frac{R}{T}\right)$ and then use the modular transformation to $\theta_4$. The corresponding sums for the derivatives will converge much faster:

$$\begin{aligned}\hat{\Delta}G_R^{\mathrm{cyl}}\left(\frac{T}{2}+t\right) &= \left(\frac{1}{2\pi}\ln\frac{\theta_4\left(-\frac{it}{\pi R}\middle| i\frac{T}{\pi R}\right)}{\cos\frac{\pi t}{T}} - \frac{t^2}{2\pi RT} + \frac{1}{4\pi}\ln\frac{T}{\pi R}\right)'' \\ &= -\frac{1}{\pi RT} + \frac{\pi}{2T^2}\frac{1}{\cos^2\frac{\pi t}{T}} - \frac{4}{\pi R^2}\sum_{n=1}^{\infty}\frac{n\,\widetilde{q}^{n}}{1-\widetilde{q}^{2n}}\cosh\frac{2nt}{R} \ . \end{aligned} \tag{5.148}$$

Upon adding (5.143), we also get

$$\begin{aligned}\hat{\Delta}G_\zeta^{\mathrm{cyl}}\left(\frac{T}{2}+t\right) &= -\frac{1}{\pi RT} + \frac{1}{2\pi}\left[\frac{1}{(\frac{T}{2}+t)^2} + 2\mu^2\right]\mathrm{e}^{-(\frac{T}{2}+t)^2\mu^2} \\ &\quad + \frac{1}{2\pi}\left[\frac{1}{(\frac{T}{2}-t)^2} + 2\mu^2\right]\mathrm{e}^{-(\frac{T}{2}-t)^2\mu^2} - \frac{4}{\pi R^2}\sum_{n=1}^{\infty}\frac{n\,\widetilde{q}^{n}}{1-\widetilde{q}^{2n}}\cosh\frac{2nt}{R} \end{aligned} \tag{5.149}$$

Again, we have a finite and convergent expression for all $|t| < \frac{T}{2}$. On the other hand, as $t \to \pm\frac{T}{2}$, both the $\frac{1}{2\pi(\frac{T}{2}\pm t)^2}$ and the sum diverge. However, we know that $\hat{\Delta}G_R^{\rm cyl}$ and $\hat{\Delta}G_\zeta^{\rm cyl}$ are finite for all $t \in [-\frac{T}{2}, \frac{T}{2}]$.

It is now easy to see what happens for finite $t$ and $T \to \infty$. In this limit $\widetilde{q} = \mathrm{e}^{-T/R} \to 0$ and the sum does not contribute. Also, $\mathrm{e}^{-(\frac{T}{2}\pm t)^2\mu^2}$ vanishes exponentially for all finite $\mu$. Thus, for $t$ finite and $T \to \infty$

$$\hat{\Delta}G_\zeta^{\rm cyl}\left(\frac{T}{2}+t\right) \simeq -\frac{1}{\pi RT} + \mathcal{O}\left(e^{-T^2\mu^2}, \mathrm{e}^{-T/R}\right) = -\frac{2}{\hat{A}^{\rm cyl}} + \mathcal{O}\left(\mathrm{e}^{-T^2\mu^2}, \mathrm{e}^{-T/R}\right) \ . \quad (5.150)$$

### 5.6.4 The gravitational action on the cylinder

We are now in position to explicitly give the gravitational action on the cylinder. The Liouville part has a universal form, while the purely cosmological term is not of much interest since it gets combined with the corresponding counterterm. Hence, we will only display the genuine order $m^2A$-term that generalises the Mabuchi (and Aubin–Yau) actions. From (5.116) we get

$$S_{\rm grav}^{\rm cyl}[\hat{g}, g]\Big|_{m^2A} = \frac{m^2A}{4}\int_0^T \mathrm{d}x \int_0^{2\pi R} \mathrm{d}y \left(-\frac{1}{2}\phi\hat{\Delta}\phi + \frac{1}{\pi A}\sigma \mathrm{e}^{2\sigma} + \frac{1}{2\pi}\hat{\Delta}\phi \, \ln\theta_1\left(\frac{x}{T}\middle| i\pi\frac{R}{T}\right)\right) \quad (5.151)$$

with $\hat{\Delta} = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ (since $\ln\theta_1$ does not depend on $y$, the $y$-derivative is a total derivative and gives a vanishing result). Alternatively, we can use (5.117) and obtain

$$\begin{aligned} S_{\rm grav}^{\rm cyl}[\hat{g}, g]\Big|_{m^2A} = \frac{m^2A}{4} \lim_{\mu\to\infty} \int_0^T \mathrm{d}x \int_0^{2\pi R} \mathrm{d}y \Bigg\{ & -\frac{1}{2}\phi\hat{\Delta}\phi + \frac{1}{\pi A}\sigma\mathrm{e}^{2\sigma} \\ & + \phi\left[\frac{\pi}{2T^2}\frac{1}{\sin^2\frac{\pi x}{T}} - \frac{4\pi}{T^2}\sum_{n=1}^{\infty}\frac{n\,\mathrm{e}^{-2n\pi^2R/T}}{1-q^{-4n\pi^2R/T}}\cos\frac{2\pi n x}{T}\right. \\ & \left. - \frac{1}{2\pi}\left[\frac{1}{x^2}+2\mu^2\right]\mathrm{e}^{-x^2\mu^2} - \frac{1}{2\pi}\left[\frac{1}{(T-x)^2}+2\mu^2\right]\mathrm{e}^{-(T-x)^2\mu^2}\right]\Bigg\}. \end{aligned} \quad (5.152)$$

One can show that with $\phi$ obeying Neumann boundary conditions, the limit $\mu \to \infty$ indeed exists (and equals (5.151), of course).

In order to study the limit of an infinitely long cylinder, we have seen that one has to set $x = \frac{T}{2} + t$ and use (5.149) in order to obtain the limit $T \to \infty$ as given by (5.150). Thus

$$S_{\rm grav}^{\infty\ {\rm cyl}}[\hat{g}, g]\Big|_{m^2A} = \frac{m^2A}{4} \lim_{T\to\infty} \int_{-T/2}^{T/2} \mathrm{d}t \int_0^{2\pi R} \mathrm{d}y \left[-\frac{1}{2}\phi\hat{\Delta}\phi + \frac{1}{\pi A}\sigma\mathrm{e}^{2\sigma} + \frac{2}{\hat{A}}\phi\right]. \quad (5.153)$$

Comparing with the Mabuchi action of a manifold without boundary as defined in (4.131), we see that this corresponds to the Mabuchi action with $h = 0$ and $\hat{R} = 0$:

$$S_{\rm grav}^{\infty\ {\rm cyl}}[\hat{g}, g]\Big|_{m^2A} = \frac{m^2A}{16\pi} S_{\rm M}[\hat{g}, g]\Big|_{h=0,\ \hat{R}=0}. \quad (5.154)$$

This equality is to be understood as an equality of the Lagrangian densities, rather than of the actions. Note that, of course, $\hat{R} = 0$ was to be expected for a cylinder, but what is interesting is the replacement $\chi = 2(1-h) \to 2$.

# Chapter 6

# Gravitational action for massive fermionic matter

In this chapter we compute the gravitational action for a massive Majorana fermion $\Psi$. The goal of this computation is to compare the result with the DDK ansatz (4.99). In section 6.1 we review the main properties of Euclidean spinors in two-dimensions and define the Majorana condition. In section 6.2 we define the action for the massive Majorana fermion and construct the functional integral. We also recall the spectral functions that we will be used to compute the gravitational action in section 6.3.

## 6.1 Two-dimensional spinors

In this section we summarize the main properties of Euclidean spinors and gamma matrices in two dimensions [7, app. 7.5, 8.5, 178, chap. 2, 3, 179, 180, sec. 13.2]. We will work with local indices denoted by the first letters of the alphabet $a, b, \ldots$

### 6.1.1 Clifford algebra and gamma matrices

The SO(2) Clifford algebra is an algebra generated by the two gamma matrices $\gamma^a$ satisfying the anti-commutation relation

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab}. \tag{6.1}$$

Both matrices are taken to be unitary, and as a consequence Hermitian

$$(\gamma^a)^\dagger = \gamma^a. \tag{6.2}$$

The last element of the algebra corresponds to the antisymmetric product

$$\gamma^{ab} = \frac{1}{2}\,[\gamma^a, \gamma^b] = -i\,\epsilon^{ab}\gamma_* \tag{6.3}$$

which is proportional to the chirality matrix

$$\gamma_* = i\,\gamma^0\gamma^1 = \frac{i}{2}\,\epsilon_{ab}\gamma^{ab}. \tag{6.4}$$

It corresponds to the generator of the SO(2) group

$$M^{ab} = \frac{i}{2}\,\gamma^{ab} = \epsilon^{ab}\gamma_*. \tag{6.5}$$

The chirality matrix is Hermitian, unitary and anticommutes will other gamma matrices

$$\gamma_*^2 = 1, \qquad (\gamma_*)^\dagger = \gamma_*, \qquad \gamma_*\gamma^a = -\gamma^a\gamma^*. \tag{6.6}$$

### 6.1.2 Dirac and Majorana spinors

A Dirac spinor $\Psi$ is a 2-dimensional complex vector with anti-commuting components that forms a reducible representation of the Clifford algebra. Such a spinor transforms under a Lorentz transformation as

$$\delta\Psi = -\frac{i}{4}\,\omega_{ab}M^{ab}\,\Psi = -i\frac{\omega}{4}\,\gamma_*\Psi, \qquad \omega \equiv \omega_{ab}\epsilon^{ab}. \tag{6.7}$$

The Dirac conjugation corresponds to Hermitian conjugation

$$\bar\Psi = \Psi^\dagger. \tag{6.8}$$

This object transforms as

$$\delta\bar\Psi = i\frac{\omega}{4}\,\bar\Psi\,\gamma_* \tag{6.9}$$

such that $\bar\Psi\Psi$ is a scalar.

Introducing the charge conjugation matrix $C$ such that

$$(\gamma^\mu)^* = C\gamma^\mu C^{-1}, \qquad (\gamma^\mu)^t = C\gamma^\mu C^{-1}, \tag{6.10}$$

the charge conjugated spinor and its Dirac conjugate (giving the Majorana conjugate) are defined by

$$\Psi^c = C^{-1}\Psi^*, \qquad \bar\Psi^c = \Psi^t C. \tag{6.11}$$

Both spinors transform respectively as $\Psi$ and $\bar\Psi$. Note that $C$ is unitary and symmetric.

From a Dirac spinor one can obtain two different irreducible representations: a Weyl (or chiral) spinor or a Majorana (or real) spinor. The latter is given by the reality condition

$$\Psi^c = \Psi \quad\Longrightarrow\quad \Psi^* = C\Psi. \tag{6.12}$$

This implies in particular that the Dirac and Majorana conjugations coincide. A Weyl spinor is obtained from a Dirac spinor by projecting it on its positive or negative chirality

$$\Psi = P_\pm\Psi, \qquad P_\pm = \frac{1}{2}(1 \pm \gamma_*). \tag{6.13}$$

Note that it is not possible to have Majorana–Weyl fermion in Euclidean signature (contrary to what happens in Lorentzian signature).

Given two Majorana spinors $\Psi_1$ and $\Psi_2$, the Majorana flip relation reads

$$\bar\Psi_1\gamma^{\mu_1}\cdots\gamma^{\mu_n}\Psi_2 = \bar\Psi_2\gamma^{\mu_n}\cdots\gamma^{\mu_1}\Psi_1. \tag{6.14}$$

For $\Psi_1 = \Psi_2 = \Psi$ this formula implies

$$\bar\Psi\gamma^{\mu\nu}\Psi = 0. \tag{6.15}$$

### 6.1.3 Gamma matrix representations

#### Majorana basis

In the Majorana representation the Dirac matrices read

$$\gamma^0 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma_* = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{6.16}$$

which implies that the charge conjugation is the identity

$$C = 1. \tag{6.17}$$

In this basis a Majorana spinor has real components

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad \psi_1^* = \psi_1, \quad \psi_2^* = \psi_2, \tag{6.18}$$

The scalar bilinears are

$$\bar{\Psi}\Psi = (\psi_1)^2 + (\psi_2)^2 = 0, \qquad \bar{\Psi}\gamma_*\Psi = -2i\,\psi_1\psi_2, \tag{6.19}$$

while the kinetic operator reads

$$i\,\gamma^\mu\partial_\mu = i \begin{pmatrix} \partial_1 & \partial_0 \\ \partial_0 & -\partial_1 \end{pmatrix}. \tag{6.20}$$

**Weyl basis**

In the Weyl representation the Dirac matrices read

$$\Gamma^0 = \sigma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \Gamma^1 = \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Gamma_* = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6.21}$$

which implies that the charge conjugation is

$$C = \Gamma^1. \tag{6.22}$$

In this basis a Majorana spinor has complex components conjugated to each other

$$\Psi = \begin{pmatrix} \psi^* \\ \psi \end{pmatrix}. \tag{6.23}$$

The relation with the Weyl components is

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2). \tag{6.24}$$

The scalar bilinears are

$$\bar{\Psi}\Psi = \psi\psi^* + \psi^*\psi = 0, \qquad \bar{\Psi}\gamma_*\Psi = 2\,\psi\psi^*, \tag{6.25}$$

while the kinetic operator reads

$$i\,\gamma^\mu\partial_\mu = \begin{pmatrix} 0 & \bar{\partial} \\ -\partial & 0 \end{pmatrix}. \tag{6.26}$$

## 6.2 Majorana fermion field theory

### 6.2.1 Action

The action for a massive two-dimensional Majorana fermion $\Psi$ coupled to gravity is [181]

$$S = \frac{1}{4\pi}\int \mathrm{d}^2x\,\sqrt{g}\,\bar{\Psi}(i\not{\nabla} + m\gamma_*)\Psi. \tag{6.27}$$

The fact that the mass term contains the chirality matrix originates from the fact that the standard bilinear $\bar{\Psi}\Psi$ vanishes ((6.19) and (6.25)). Moreover the important flip relation (6.14) for a Majorana fermion indicates that the connection term vanishes. Nonetheless one needs to keep it in order to define the functional integral since one has to work with a covariant object.

In the Weyl basis, the action in components $\Psi = (\psi^*, \psi)$ reads

$$S = \frac{1}{4\pi}\int \mathrm{d}^2z\,\sqrt{g}\,(\psi\bar{\partial}\psi - \psi^*\partial\psi^* + 2m\,\psi\psi^*) \tag{6.28}$$

where $\partial = \frac{1}{2}(\partial_0 - i\partial_1)$ and $\bar{\partial} = \frac{1}{2}(\partial_0 + i\partial_1)$, and one recognizes the standard form for the massive Ising model [182, sec. 2.2].

### 6.2.2 Functional integral

We now construct the functional integral for the Majorana field theory. Let $D = i\nabla\!\!\!/ + m\gamma^*$. First we notice that

$$D\Psi = \lambda\Psi \tag{6.29}$$

does not have any real solution. However it admits a complete basis of complex unitary eigenvectors $\psi_\lambda$ associated with eigenvalues $\lambda \in \mathbb{R}$.

If $\lambda > 0$, then $\psi_\lambda^*$ is an eigenvalue of $D$ associated to $-\lambda$. We will then denote the positive eigenvalues by $\lambda_n$, $n \in \mathbb{N}$ and the negative ones by $\lambda_{-n} = -\lambda_n$. The associated eigenvectors will be denoted by $\psi_n$ and $\psi_{-n} = \psi_n^*$. Then

$$(i\nabla\!\!\!/ + m\gamma^*)\psi_n = \lambda_n\psi_n, \quad \lambda_n \geq 0. \tag{6.30}$$

We can now take

$$\chi_n = \frac{1}{\sqrt{2}}(\psi_n + \psi_n^*) \tag{6.31}$$

$$\phi_n = \frac{1}{i\sqrt{2}}(\psi_n - \psi_n^*). \tag{6.32}$$

These are real (Majorana) spinors that satisfy

$$\begin{cases} (i\nabla\!\!\!/ + m\gamma^*)\chi_n &= i\lambda_n\phi_n \\ (i\nabla\!\!\!/ + m\gamma^*)\phi_n &= -i\lambda_n\chi_n \end{cases} \tag{6.33}$$

and then

$$\begin{cases} \left(-\Delta + \frac{R}{4} + m^2\right)\chi_n &= \lambda_n^2\chi_n \\ \left(-\Delta + \frac{R}{4} + m^2\right)\phi_n &= \lambda_n^2\phi_n. \end{cases} \tag{6.34}$$

As the basis $(\psi_n, \psi_n^*)$ is orthonormal, we see that $\langle\chi_n|\chi_m\rangle = \delta_{mn}$, $\langle\phi_n|\phi_m\rangle = \delta_{mn}$ and $\langle\chi_n|\phi_m\rangle = 0$ where the scalar product is defined by

$$\langle\phi|\chi\rangle = \int \mathrm{d}^2x\sqrt{g}\,\phi^\dagger(x)\chi(x). \tag{6.35}$$

Note that the spinors $\psi_n$, $\chi_n$ and $\phi_n$ are commuting classical functions.

We can then decompose a Majorana spinor quantum field $\Psi$ as

$$\Psi = \sum_{n\geq 0}(b_n\chi_n + c_n\phi_n) \tag{6.36}$$

where the $b_n$ and $c_n$ are Grassmannian.

We then have

$$(i\nabla\!\!\!/ + m\gamma^*)\Psi = i\sum_{n\geq 0}\lambda_n(b_n\phi_n - c_n\chi_n) \tag{6.37}$$

and

$$\left\langle\Psi|(i\nabla\!\!\!/ + m\gamma^*)\Psi\right\rangle = i\sum_{n\geq 0}\lambda_n(c_nb_n - b_nc_n). \tag{6.38}$$

Define

$$a_n = (c_n + ib_n), \quad a_n^\dagger = c_n - ib_n \tag{6.39}$$

so that

$$a_n^\dagger a_n = i(c_nb_n - b_nc_n). \tag{6.40}$$

Then the partition function is given by

$$Z[g] = \int \mathcal{D}\Psi \mathrm{e}^{-S} = \int \prod_{n\geq 0}\mathrm{d}a_n^\dagger \mathrm{d}a_n \mathrm{e}^{-\sum_{n\geq 0}\lambda_n a_n^\dagger a_n} = \prod_{n>0}\lambda_n. \tag{6.41}$$

But (formally)

$$\prod_{n>0}\lambda_n=\left(\prod_{n>0}\lambda_n^2\right)^{1/2}=\left(\prod_{n\neq 0}\lambda_n^2\right)^{1/4}=\det\left(-\Delta+\frac{R}{4}+m^2\right)^{1/4}. \tag{6.42}$$

Of course if $m=0$ $D$ may have zero-modes which must be excluded from the product and which yield an infinite normalization factor which may depend on the metric.

Now $D^2=-\Delta+\frac{R}{4}+m^2$ is an operator of Laplace type and we can use the standard heat kernel and zeta function methods seen in chapter 3 to compute the gravitational action. The spinor Laplacian was given in (3.4b).

We let $\Lambda_n=\lambda_n^2$ and we will put a subscript (0) to all quantities where $m=0$. We have

$$\Lambda_n=\Lambda_n^{(0)}+m^2. \tag{6.43}$$

We will use a zeta function regularization of the determinant (3.82):

$$\ln\det{}_\zeta\frac{D}{\mu^2}=-\zeta'(0)-\ln\mu^2\zeta(0). \tag{6.44}$$

The gravitational action is then given by

$$S_{\rm grav}[\hat g,g]=-\ln\frac{Z_m[g]}{Z_m[\hat g]}=\frac{1}{4}(\zeta_g'(0)+\ln\mu^2\zeta_g(0))-\frac{1}{4}(\zeta_{\hat g}'(0)+\ln\mu^2\zeta_{\hat g}(0)). \tag{6.45}$$

The infinitesimal variation is

$$\delta S_{\rm grav}[g]=\frac{1}{4}(\delta\zeta_g'(0)+\ln\mu^2\delta\zeta_g(0)). \tag{6.46}$$

### 6.2.3 Spectral functions

We will denote collectively $\chi_n$ and $\phi_n$ by $\Psi_n$ as they are associated to the same eigenvalue of $D^2=-\Delta+R/4+m^2$. We recall briefly the expressions of the spectral functions defined in 3.2.

The Green's function for the operator $D^2$ is given by:

$$D^2G(x,y)=\frac{\delta(x-y)}{\sqrt{g}}\mathrm{I}_2\Leftrightarrow G(x,y)=\sum_n\frac{\Psi_n(x)\Psi_n^\dagger(y)}{\Lambda_n} \tag{6.47}$$

where n is running over positive and negative integers. Taking the hermitian conjugate we see that

$$G(y,x)=G(x,y)^\dagger. \tag{6.48}$$

The Green's function without the zero-modes $\Psi_{0,i}$ is defined by

$$D^2\tilde G(x,y)=\frac{\delta(x-y)}{\sqrt{g}}\mathrm{I}_2-\sum_i\Psi_{0,i}(x)\Psi_{0,i}^\dagger(y)\Leftrightarrow\tilde G(x,y)=\sum_{n>0}\frac{\Psi_n(x)\Psi_n^\dagger(y)}{\Lambda_n}. \tag{6.49}$$

$G$ and $\tilde G$ are related by

$$G(x,y)=\frac{1}{m^2}\sum_i\Psi_{0,i}(x)\Psi_{0,i}^\dagger(y)+\tilde G(x,y). \tag{6.50}$$

The heat kernel is the solution of

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}+D^2\right)K(t,x,y)=0,\quad K(t,x,y)\sim_{t\to 0}\frac{\delta(x-y)}{\sqrt{g}}\mathrm{I}_2 \tag{6.51}$$

which can be expressed as

$$K(t,x,y) = \sum_n \mathrm{e}^{-\Lambda_n t} \Psi_n(x) \Psi_n^\dagger(y) \tag{6.52}$$

while the integrated heat kernel is given by

$$K(t) = \int \mathrm{d}^2x \sqrt{g}\, \mathrm{tr}_D\, K(t,x,x) = \sum_n \mathrm{e}^{-\Lambda_n t}. \tag{6.53}$$

In this case the coefficients of the small $t$ expansion (3.55) becomes

$$a_0(x,x) = \mathrm{I}_2 \tag{6.54a}$$

$$a_1(x,x) = \left(-\frac{R}{12} - m^2\right) \mathrm{I}_2 \tag{6.54b}$$

so that

$$K(t) = \frac{A}{2\pi t} - \frac{1}{24\pi} \int \mathrm{d}^2x \sqrt{g} R(x) - \frac{m^2 A}{2\pi} + o(t). \tag{6.55}$$

The zeta functions are defined by

$$\zeta(s,x,y) = \sum_n \frac{\Psi_n(x) \Psi_n^\dagger(y)}{\Lambda_n^s}, \quad \zeta(s) = \sum_n \frac{1}{\Lambda_n^s}. \tag{6.56}$$

We can compute $\zeta(0)$ using (6.54b):

$$\zeta(0) = -\frac{1}{24\pi} \int \mathrm{d}^2x \sqrt{g} R(x) - \frac{m^2 A}{2\pi}. \tag{6.57}$$

Finally we note that for $s \neq 1$, $\mathrm{tr}_D\, \zeta(s,x,x)$ is a scalar and we have

$$\mathrm{tr}_D\, \Delta \zeta(s,x,x) = \Delta\, \mathrm{tr}_D\, \zeta(s,x,x) \tag{6.58}$$

where in the l.h.s. $\Delta$ is the spinor Laplacian (3.4b) while in the r.h.s. $\Delta$ is the scalar Laplacian (3.4a). Of course this is true only for the zeta function at coincident points (not for $\zeta(s,x,y)$).

## 6.3 Gravitational action

### 6.3.1 Conformal variations

In conformal gauge $g_{\mu\nu} = \mathrm{e}^{2\sigma} \hat{g}_{\mu\nu}$, the vielbein is given by

$$e_\mu^a = \mathrm{e}^\sigma\, \hat{e}_\mu^a. \tag{6.59}$$

Then the affine et spin connections are

$$\Gamma_{\nu\rho}^\mu = \hat{\Gamma}_{\nu\rho}^\mu + \delta_\nu^\mu \partial_\rho \sigma + \delta_\rho^\mu \partial_\nu \sigma - \hat{g}_{\nu\rho}\, \partial^\mu \sigma \tag{6.60}$$

$$\omega_{\mu ab} = \hat{\omega}_{\mu ab} + \hat{e}_{\mu a}\, \hat{e}_b^\nu\, \partial_\nu \sigma - \hat{e}_{\mu b}\, \hat{e}_a^\nu\, \partial_\nu \sigma. \tag{6.61}$$

This gives the covariant derivative for a spinor:

$$\nabla_\mu \Psi = \hat{\nabla}_\mu \Psi + \frac{1}{2}\, \hat{e}_{\mu a}\, \hat{e}_b^\nu\, \partial_\nu \sigma\, \gamma^{ab}\, \Psi \tag{6.62}$$

so that

$$\not{\nabla} = \mathrm{e}^{-\sigma} \hat{\not{\nabla}} + \frac{1}{2} \mathrm{e}^{-\sigma} \partial_\mu \sigma \hat{\gamma}^\mu. \tag{6.63}$$

The spinor Laplacian $\Delta$ is

$$\Delta\Psi = \mathrm{e}^{-2\sigma}\hat{\Delta}\Psi + \mathrm{e}^{-2\sigma}\,\partial_\nu\sigma\,\hat{\gamma}^{\mu\nu}\,\hat{\nabla}_\mu\Psi - \frac{1}{4}\,\mathrm{e}^{-2\sigma}\,\partial_\mu\sigma\,\partial^\mu\sigma\,\Psi. \tag{6.64}$$

As

$$R = \mathrm{e}^{-2\sigma}\hat{R} - 2\,\mathrm{e}^{-2\sigma}\hat{\Delta}\sigma \tag{6.65}$$

we have

$$D^2 = \mathrm{e}^{-2\sigma}\left(-\hat{\Delta} + \frac{1}{4}\,\hat{R}\right) - \mathrm{e}^{-2\sigma}\,\partial_\nu\sigma\,\hat{\gamma}^{\mu\nu}\,\hat{\nabla}_\mu + \frac{1}{4}\,\mathrm{e}^{-2\sigma}\,\partial_\mu\sigma\,\partial^\mu\sigma - \frac{1}{2}\mathrm{e}^{-2\sigma}\hat{\Delta}\sigma + m^2 \tag{6.66}$$

and

$$\delta D^2 = -2\delta\sigma\left(-\Delta + \frac{1}{4}R\right) - \partial_\nu(\delta\sigma)\gamma^{\mu\nu}\nabla_\mu - \frac{1}{2}\Delta(\delta\sigma). \tag{6.67}$$

### 6.3.2 Perturbation theory

To compute the variation of the zeta function we need the variation of the eigenvalues of $D^2$ under a Weyl transformation. For this we distinguish between the zero-modes and the other ones.

For $m = 0$, $-\Delta + \frac{R}{4}$ is a positive operator which may have zero-modes which are the zero-modes of $\not{\nabla}$. A Weyl transformation does not change the number of zero-modes. Indeed if $\hat{\chi}$ satisfies $\hat{\not{\nabla}}\hat{\chi} = 0$ then (6.63) implies $\not{\nabla}\left(\mathrm{e}^{-\sigma/2}\hat{\chi}\right) = 0$. On the sphere we can take for $\hat{g}$ the round metric which has $R > 0$ constant. As $-\Delta$ is a positive operator we see that $-\Delta + \frac{R}{4}$ cannot have any zero-mode. On the torus we can use the flat metric $\hat{g} = \delta$ for which there are two constant zero-modes:

$$\hat{\phi}_0 = \frac{1}{\sqrt{A}}\begin{pmatrix}1\\0\end{pmatrix}, \qquad \hat{\chi}_0 = \frac{1}{\sqrt{A}}\begin{pmatrix}0\\1\end{pmatrix}. \tag{6.68}$$

For genus $g \geq 2$, we can take for $\hat{g}$ a metric of constant negative curvature and one typically has zero-modes.

The variation of the non-zero eigenvalues $\Lambda_n$ can be obtained through usual perturbation theory:

$$\delta\Lambda_n = \langle\Psi_n|\delta D^2|\Psi_n\rangle = -2(\Lambda_n - m^2)\langle\Psi_n|\delta\sigma|\Psi_n\rangle - \langle\Psi_n|\partial_\nu(\delta\sigma)\gamma^{\mu\nu}\nabla_\mu|\Psi_n\rangle - \frac{1}{2}\langle\Psi_n|\Delta(\delta\sigma)|\Psi_n\rangle. \tag{6.69}$$

### 6.3.3 Variation of the zeta function

To obtain the gravitational action we need $\delta\zeta(0)$ and $\delta\zeta'(0)$. One has:

$$\delta\zeta(s) = -s\sum_n \frac{\delta\Lambda_n}{\Lambda_n^{s+1}}. \tag{6.70}$$

As there is no zero-mode contribution to this equation we can work with the function $\tilde{\zeta}$ instead of $\zeta$:

$$\delta\zeta(s) = \delta\tilde{\zeta}(s). \tag{6.71}$$

Doing this we do not have to distinguish between the massive and the massless case. Using (5.29) we get

$$\begin{aligned}\delta\tilde{\zeta}(s) = {} & 2s\sum_{n\neq 0}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\frac{\Psi_n^\dagger(x)\Psi_n(x)}{\Lambda_n^s} - 2m^2 s\sum_{n\neq 0}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\frac{\Psi_n^\dagger(x)\Psi_n(x)}{\Lambda_n^{s+1}} \\ & + s\sum_{n\neq 0}\int \mathrm{d}^2x\sqrt{g}\,\frac{\partial_\nu(\delta\sigma)\Psi_n^\dagger(x)\gamma^{\mu\nu}\nabla_\mu\Psi_n(x)}{\Lambda_n^{s+1}} + \frac{s}{2}\sum_{n\neq 0}\int \mathrm{d}^2x\sqrt{g}\,\Delta(\delta\sigma)\frac{\Psi_n^\dagger(x)\Psi_n(x)}{\Lambda_n^{s+1}}.\end{aligned} \tag{6.72}$$

The first two terms can be expressed in terms of trace of the local zeta function:

$$\sum_{n\neq 0}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\frac{\Psi_n^\dagger(x)\Psi_n(x)}{\Lambda_n^s}=\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\,\tilde{\zeta}(s,x,x). \tag{6.73}$$

To evaluate the third term one needs to integrate by parts and use $\gamma^{\mu\nu}=\gamma^\mu\gamma^\nu-g^{\mu\nu}$:

$$\begin{aligned}\int \mathrm{d}^2x\sqrt{g}\,\partial_\nu(\delta\sigma)\Psi_n^\dagger(x)\gamma^{\mu\nu}\nabla_\mu\Psi_n(x)=&-\frac{1}{4}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)R(x)\Psi_n^\dagger(x)\Psi_n(x)\\&+\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)(\not{\nabla}\Psi_n)^\dagger(x)\not{\nabla}\Psi_n(x)\\&-\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\nabla_\mu(\Psi_n^\dagger(x))\nabla^\mu\Psi_n(x).\end{aligned} \tag{6.74}$$

Note that for $n\neq 0$ $\not{\nabla}\Psi_n$ is also an eigenvector of $D^2$ associated to the eigenvalue $\Lambda_n$ and

$$\int \mathrm{d}^2x\sqrt{g}\,(\not{\nabla}\Psi_n)^\dagger\not{\nabla}\Psi_n=-\int \mathrm{d}^2x\sqrt{g}\,\Psi_n^\dagger\not{\nabla}^2\Psi_n=\Lambda_n^{(0)}=\Lambda_n-m^2. \tag{6.75}$$

As the heat kernel is uniquely defined by (6.51), (3.62) implies that the zeta function is also uniquely defined and does not depend on the basis of eigenvectors one has chosen. Then one has

$$\tilde{\zeta}(s,x,y)=\sum_{n\neq 0}\frac{\not{\nabla}\Psi_n(x)\not{\nabla}\Psi_n^\dagger(y)}{\Lambda_n^{(0)}\Lambda_n^s}=\sum_{n\neq 0}\frac{\not{\nabla}\Psi_n(x)\not{\nabla}\Psi_n^\dagger(y)}{\Lambda_n^s(\Lambda_n-m^2)}. \tag{6.76}$$

One has $\Lambda_n=\Lambda_n^{(0)}+m^2$ and then $\Lambda_n\geq m^2$ for all $n\neq 0$. Then, for all $m$, one has

$$\tilde{\zeta}(s,x,y)=\sum_{n\neq 0}\frac{\not{\nabla}\Psi_n(x)\not{\nabla}\Psi_n^\dagger(y)}{\Lambda_n^{s+1}}\sum_k\left(\frac{m^2}{\Lambda_n}\right)^k=\sum_{n\neq 0,k}\frac{\not{\nabla}\Psi_n(x)\not{\nabla}\Psi_n^\dagger(y)}{\Lambda_n^{s+1+k}}m^{2k} \tag{6.77}$$

then

$$\begin{aligned}\sum_{n\neq 0}\frac{\not{\nabla}\Psi_n(x)\not{\nabla}\Psi_n^\dagger(y)}{\Lambda_n^{s+1}}&=\tilde{\zeta}(s,x,y)-m^2\sum_{n\neq 0}\frac{\not{\nabla}\Psi_n(x)\not{\nabla}\Psi_n^\dagger(y)}{\Lambda_n^{s+2}}-\sum_{n\neq 0;k\geq 2}\frac{\not{\nabla}\Psi_n(x)\not{\nabla}\Psi_n^\dagger(y)}{\Lambda_n^{s+1+k}}m^{2k}\\&=\tilde{\zeta}(s,x,y)-m^2\tilde{\zeta}(s+1,x,y)+\sum_{n\neq 0;k\geq 1}\frac{\not{\nabla}\Psi_n(x)\not{\nabla}\Psi_n^\dagger(y)}{\Lambda_n^{s+2+k}}m^{2k+2}\\&\quad-\sum_{n\neq;k\geq 2}\frac{\not{\nabla}\Psi_n(x)\not{\nabla}\Psi_n^\dagger(y)}{\Lambda_n^{s+1+k}}m^{2k}\\&=\tilde{\zeta}(s,x,y)-m^2\tilde{\zeta}(s+1,x,y).\end{aligned} \tag{6.78}$$

To compute the last term in (6.74), one sees that

$$\nabla_\mu\tilde{\zeta}(s,x,x)=\sum_{n\neq 0}\frac{(\nabla_\mu\Psi_n(x))\Psi_n^\dagger(x)}{\Lambda_n^s}+\sum_{n\neq 0}\frac{\Psi_n(x)\nabla_\mu\Psi_n^\dagger(x)}{\Lambda_n^s} \tag{6.79}$$

then

$$\begin{aligned}\nabla^\mu\nabla_\mu\tilde{\zeta}(s,x,x)&=\sum_{n\neq 0}\frac{(\Delta\Psi_n(x))\Psi_n^\dagger(x)}{\Lambda_n^s}+2\sum_{n\neq 0}\frac{(\nabla_\mu\Psi_n(x))\nabla^\mu\Psi_n^\dagger(x)}{\Lambda_n^s}+\sum_{n\neq 0}\frac{\Psi_n(x)\Delta\Psi_n^\dagger(x)}{\Lambda_n^s}\\&=-2\tilde{\zeta}(s-1,x,x)+2\left(\frac{R}{4}+m^2\right)\tilde{\zeta}(s,x,x)+2\sum_{n\neq 0}\frac{(\nabla_\mu\Psi_n(x))\nabla^\mu\Psi_n^\dagger(x)}{\Lambda_n^s}.\end{aligned} \tag{6.80}$$

Then

$$\sum_{n\neq 0}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\frac{\nabla_\mu(\Psi_n^\dagger(x))\nabla^\mu\Psi_n(x)}{\Lambda_n^{s+1}} = \frac{1}{2}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D(\Delta\tilde{\zeta}(s+1,x,x)) + \int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\,\tilde{\zeta}(s,x,x) - \int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\left(\frac{R}{4}+m^2\right)\mathrm{tr}_D\,\tilde{\zeta}(s+1,x,x). \tag{6.81}$$

Finally

$$\sum_{n\neq 0}\int \mathrm{d}^2x\sqrt{g}\,\frac{\partial_\nu(\delta\sigma)\Psi_n^\dagger(x)\gamma^{\mu\nu}\nabla_\mu\Psi_n(x)}{\Lambda_n^{s+1}} = -\frac{1}{2}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D(\Delta\tilde{\zeta}(s+1,x,x)). \tag{6.82}$$

The last term in (6.72) is

$$\sum_{n\neq 0}\int \mathrm{d}^2x\sqrt{g}\,\Delta(\delta\sigma)\frac{\Psi_n^\dagger(x)\Psi_n(x)}{\Lambda_n^{s+1}} = \int \mathrm{d}^2x\sqrt{g}\,\delta\sigma\,\mathrm{tr}_D(\Delta\tilde{\zeta}(s+1,x,x)). \tag{6.83}$$

Finally the total variation of the zeta function is given by

$$\delta\zeta(s) = 2s\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\,\tilde{\zeta}(s,x,x) - 2m^2 s\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\,\tilde{\zeta}(s+1,x,x) \tag{6.84}$$

and

$$\delta\zeta'(s) = 2s\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\,\tilde{\zeta}'(s,x,x) + 2\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\,\tilde{\zeta}(s,x,x) - 2m^2\left(s\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\,\tilde{\zeta}'(s+1,x,x) + \int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\,\tilde{\zeta}(s+1,x,x)\right). \tag{6.85}$$

We then have

$$\begin{aligned}\delta\zeta(0) &= \lim_{s\to 0}\left[2s\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\,\tilde{\zeta}(s,x,x) - 2m^2 s\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\,\tilde{\zeta}(s+1,x,x)\right]\\ &= -2m^2\lim_{s\to 0} s\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\,\tilde{\zeta}(s+1,x,x)\\ &= -2m^2\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\,\frac{a_0(x,x)}{4\pi}\\ &= -\frac{m^2}{\pi}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\end{aligned} \tag{6.86}$$

and

$$\delta\zeta'(0) = 2\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\,\tilde{\zeta}(0,x,x) - 2m^2\lim_{s\to 0}\left[\left(1+s\frac{\mathrm{d}}{\mathrm{d}s}\right)\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\,\tilde{\zeta}(s+1,x,x)\right]. \tag{6.87}$$

It is exactly the same formula as the one of [70] for the case of the massive scalar field.

### 6.3.4 The massless case

In the massless case we expect that the gravitational action is just the Liouville action with a factor $1/2$ compared to the scalar case.

When $m = 0$ one has

$$\delta\tilde{\zeta}(0) = 0 \tag{6.88}$$

and

$$\begin{aligned}\delta\tilde{\zeta}'(0) &= 2\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\,\tilde{\zeta}(0,x,x)\\ &= 2\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\frac{1}{4\pi}\,\mathrm{tr}_D\,a_1(x,x)\\ &= -\frac{1}{12\pi}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)R(x)\end{aligned} \tag{6.89}$$

and

$$\delta S_{\mathrm{grav}} = \frac{1}{4}\delta\tilde{\zeta}'(0) = -\frac{1}{48\pi}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)R(x) = -\frac{1}{12}\delta S_L \tag{6.90}$$

so that the gravitational action is

$$S_{\mathrm{grav}}[\hat{g},\sigma] = -\frac{1}{12}S_L[\hat{g},\sigma] \tag{6.91}$$

which is what was expected from (4.88) since the central charge is $c = \frac{1}{2}$.

### 6.3.5 The massive case

Using the fact that

$$\tilde{\zeta}(0,x,x) = \zeta(0,x,x) - \sum_i \Psi_{0,i}(x)\Psi_{0,i}^\dagger(x) = \frac{a_1(x,x)}{4\pi} - \sum_i \Psi_{0,i}(x)\Psi_{0,i}^\dagger(x) \tag{6.92}$$

we find that

$$\begin{aligned}\delta\zeta'(0) = &-\frac{1}{12\pi}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)R(x) - 2\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\sum_i \Psi_{0,i}^\dagger(x)\Psi_{0,i}(x)\\ &-\frac{m^2}{\pi}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x) - 2m^2\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\,\tilde{\zeta}_{\mathrm{reg}}(1,x,x)\end{aligned} \tag{6.93}$$

where we have defined

$$\tilde{\zeta}_{\mathrm{reg}}(s,x,x) = \tilde{\zeta}(s,x,x) - \frac{1}{4\pi(s-1)}. \tag{6.94}$$

Then

$$\begin{aligned}\delta S_{\mathrm{grav}} = &-\frac{1}{12}\delta S_L - \frac{1}{2}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\sum_i \Psi_{0,i}^\dagger(x)\Psi_{0,i}(x)\\ &-\frac{m^2}{2}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\left(\tilde{\zeta}_{\mathrm{reg}}(1,x,x) + \frac{\ln\mu^2}{4\pi} + \frac{1}{4\pi}\right)\end{aligned} \tag{6.95}$$

The zeta function gives a regularized version of the Green's function at coincident point:

$$G_\zeta(x) = \lim_{s\to 1}\left(\mu^{2s-2}\zeta(s,x,x) - \frac{a_0(x)}{4\pi(s-1)}\right) = \lim_{s\to 1}\left(\mu^{2s-2}\zeta(s,x,x) - \frac{\mathrm{I}_2}{4\pi(s-1)}\right) \tag{6.96}$$

and the same for $\tilde{\zeta}$ and $\tilde{G}_\zeta$ where $G_\zeta$ and $\tilde{G}_\zeta$ are related by

$$G_\zeta(x) = \tilde{G}_\zeta(x) + \sum_i \Psi_{0,i}(x)\Psi_{0,i}^\dagger(x). \tag{6.97}$$

We then have

$$\tilde{\zeta}_{\rm reg}(1,x,x) = \tilde{G}_\zeta(x) - \frac{1}{4\pi}\ln\mu^2 \tag{6.98}$$

so that (6.95) becomes

$$\begin{aligned}\delta S_{\rm grav} &= -\frac{1}{12}\delta S_L - \frac{1}{2}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\sum_i \Psi^\dagger_{0,i}(x)\Psi_{0,i}(x)\\ &\quad - \frac{m^2}{2}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\left(\tilde{G}_\zeta(x)+\frac{1}{4\pi}\right)\\ &= -\frac{1}{12}\delta S_L - \frac{m^2}{2}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\left(G_\zeta(x)+\frac{1}{4\pi}\right).\end{aligned} \tag{6.99}$$

The last term appears with a factor $-m^2/2$ comparing to the case of the scalar field where it is $+m^2$ [70]. In particular due to the trace over the identity which gives a factor of 2 the second term in the parenthesis cancels for a system with one Majorana and one scalar field of the same mass, and one is left with

$$\delta S_{\rm grav} = -\frac{1}{12}\left(1+\frac{1}{2}\right)\delta S_L + m^2\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\left(G_\zeta^{\rm scalar}(x) - \frac{1}{2}\,\mathrm{tr}_D\,G_\zeta^{\rm Majorana}(x)\right). \tag{6.100}$$

Now we want to express $\delta S_{\rm grav}$ as the variation of some functional. For this we note that

$$\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma\,\mathrm{tr}_D\,G_\zeta(x) = \frac{1}{2}\delta\left(\int \mathrm{d}^2x\sqrt{g}\;\mathrm{tr}_D\,G_\zeta(x)\right) - \frac{1}{2}\int \mathrm{d}^2x\sqrt{g}\,\mathrm{tr}_D\,\delta G_\zeta(x). \tag{6.101}$$

We will then compute $\int \mathrm{d}^2x\sqrt{g}\,\mathrm{tr}_D\,\delta G_\zeta(x)$. To do this we define another regularization of the Green's function:

$$G_R(x) = \lim_{y\to x}\left(G(x,y) + \frac{1}{4\pi}\ln\left(\mu^2\ell^2(x,y)\right)\right). \tag{6.102}$$

It is linked to $G_\zeta$ by

$$G_\zeta(x) = G_R(x) + \alpha \tag{6.103}$$

where $\alpha$ is a constant [132]. This implies

$$\delta G_\zeta(x) = \delta G_R(x) = \delta\lim_{y\to x}\left(G(x,y) + \frac{1}{4\pi}\ln\left(\mu^2\ell^2(x,y)\right)\right). \tag{6.104}$$

Under an infinitesimal Weyl transformation (6.47) becomes

$$\left(D^2+\delta D^2\right)\left(G(x,y)+\delta G(x,y)\right) = \frac{\delta(x-y)}{\sqrt{g}}(1-2\delta\sigma). \tag{6.105}$$

Then, at first order in $\delta\sigma$, $\delta G$ is solution of

$$D^2\delta G(x,y) = -2m^2\delta\sigma(x)G(x,y) + \partial_\nu\delta\sigma(x)\gamma^{\mu\nu}\nabla_\mu G(x,y) + \frac{1}{2}\Delta(\delta\sigma(x))G(x,y). \tag{6.106}$$

This solution is obtained by convolution with $G$:

$$\begin{aligned}\delta G(x,y) &= -2m^2\int \mathrm{d}^2z\sqrt{g}\,G(x,z)\delta\sigma(z)G(z,y) + \int \mathrm{d}^2z\sqrt{g}\,G(x,z)\partial_\nu\delta\sigma(z)\gamma^{\mu\nu}\nabla_\mu G(z,y)\\ &\quad + \frac{1}{2}\int \mathrm{d}^2z\sqrt{g}\,G(x,z)\Delta(\delta\sigma(z))G(z,y).\end{aligned} \tag{6.107}$$

Using the fact that

$$\delta\ell^2(x,y) = \ell^2(x,y)(\delta\sigma(x)+\delta\sigma(y)) + \mathcal{O}(\ell^4(x,y)) \tag{6.108}$$

which implies that

$$\lim_{y\to x} \delta \ln(\mu^2 \ell^2(x,y)) = 2\delta\sigma(x) \tag{6.109}$$

and some computations similar to what we did before, we find that

$$\int \mathrm{d}^2x\sqrt{g}\,\mathrm{tr}_D\,\delta G_\zeta(x) = -2m^2\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\,\zeta(2,x,x) + \frac{1}{2\pi}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x). \tag{6.110}$$

This gives:

$$\delta S_{\mathrm{grav}} = -\frac{1}{12}\delta S_L - \frac{m^2}{4}\,\delta\left(\int \mathrm{d}^2x\sqrt{g}\;\mathrm{tr}_D\,G_\zeta(x)\right) - \frac{m^4}{2}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\,\zeta(2,x,x). \tag{6.111}$$

To deal with the last term of the r.h.s. we will express it in term of the heat kernel:

$$\begin{aligned}\delta K(t) = \delta\tilde{K}(t) &= -t\sum_{n\neq 0}\mathrm{e}^{-\Lambda_n t}\delta\Lambda_n\\ &= -2t\left(\frac{\mathrm{d}}{\mathrm{d}t}+m^2\right)\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\,\tilde{K}(t,x,x)\\ &\quad - t\sum_{n\neq 0}\mathrm{e}^{-\Lambda_n t}\int \mathrm{d}^2x\sqrt{g}\,\Psi_n^\dagger(x)\partial_\nu(\delta\sigma)\gamma^{\mu\nu}\nabla_\mu\Psi_n(x)\\ &\quad - \frac{t}{2}\int \mathrm{d}^2x\sqrt{g}\,\Delta(\delta\sigma(x))\,\mathrm{tr}_D\,\tilde{K}(t,x,x).\end{aligned} \tag{6.112}$$

To compute the second term we introduce the generalised heat kernel

$$K(s,t,x,y) = \sum_n \mathrm{e}^{-\Lambda_n t}\frac{\Psi_n(x)\Psi_n(y)}{\Lambda_n^s}. \tag{6.113}$$

In particular

$$K(0,t,x,y) = K(t,x,y). \tag{6.114}$$

We can then redo the computations we have done for the variation of the zeta function. In particular (6.82) can be generalised as

$$\sum_{n\neq 0}\mathrm{e}^{-\Lambda_n t}\int \mathrm{d}^2x\sqrt{g}\,\frac{\partial_\nu(\delta\sigma)\Psi_n^\dagger(x)\gamma^{\mu\nu}\nabla_\mu\Psi_n(x)}{\Lambda_n^{s+1}} = -\frac{1}{2}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D(\Delta\tilde{K}(s+1,t,x,x)). \tag{6.115}$$

Taking $s=-1$ we find that

$$\sum_{n\neq 0}\mathrm{e}^{-\Lambda_n t}\int \mathrm{d}^2x\sqrt{g}\,\partial_\nu(\delta\sigma)\Psi_n^\dagger(x)\gamma^{\mu\nu}\nabla_\mu\Psi_n(x) = -\frac{1}{2}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D(\Delta\tilde{K}(t,x,x)). \tag{6.116}$$

We then have

$$\delta K(t) = -2t\left(\frac{\mathrm{d}}{\mathrm{d}t}+m^2\right)\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\,\tilde{K}(t,x,x). \tag{6.117}$$

Note that the differential equations satisfied by $K$ and $K^{(0)}$ imply that

$$K(t,x,y) = \mathrm{e}^{-m^2 t}K^{(0)}(t,x,y). \tag{6.118}$$

The eigenfunctions expansions show that the same is true between $\tilde{K}$ and $\tilde{K}^{(0)}$ which implies

$$\delta K(t) = -2t\mathrm{e}^{-m^2 t}\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\,\tilde{K}^{(0)}(t,x,x). \tag{6.119}$$

We then have

$$\begin{aligned}\int_0^\infty \mathrm{d}t\,\frac{\mathrm{e}^{m^2t}-m^2t-1}{t}\delta\tilde{K}(t) &= -2\left[\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma\,\mathrm{tr}_D\,\tilde{K}^{(0)}(t,x,x)\right]_0^{+\infty}\\ &\quad+2\int_0^{+\infty}\mathrm{d}t\,(m^2t+1)\left(m^2+\frac{\mathrm{d}}{\mathrm{d}t}\right)\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma\,\mathrm{tr}_D\,\tilde{K}(t,x,x)\\ &= -2\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma\,\mathrm{tr}_D\left[\tilde{K}^{(0)}(t,x,x)-\tilde{K}(t,x,x)\right]_0^{+\infty}\\ &\quad+2m^2\int \mathrm{d}t\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma\,\mathrm{tr}_D\,\tilde{K}(t,x,x)\\ &\quad+2m^2\int \mathrm{d}t\,t\,\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma\,\mathrm{tr}_D\,\tilde{K}(t,x,x)\\ &\quad+2m^4\int \mathrm{d}t\,t\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma\,\mathrm{tr}_D\,\tilde{K}(t,x,x)\\ &= 4m^4\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\,\tilde{\zeta}(2,x,x)\\ &= 4m^4\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\,\zeta(2,x,x)\\ &\quad-4\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\sum_i \Psi_{0,i}^\dagger(x)\Psi_{0,i}(x).\end{aligned}\tag{6.120}$$

Reporting in (6.111) we get

$$\begin{aligned}\delta S_{\mathrm{grav}} = &-\frac{1}{12}\delta S_L-\frac{m^2}{4}\,\delta\left(\int \mathrm{d}^2x\sqrt{g}\;\mathrm{tr}_D\,G_\zeta(x)\right)-\frac{1}{8}\,\delta\left(\int_0^\infty \mathrm{d}t\,\frac{\mathrm{e}^{m^2t}-m^2t-1}{t}\tilde{K}(t)\right)\\ &-\frac{1}{2}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\sum_i \Psi_{0,i}^\dagger(x)\Psi_{0,i}(x).\end{aligned}\tag{6.121}$$

We then have an expression for $\delta S_{\mathrm{grav}}$ as a variation of a local functional up to a term involving the projection on the subspace of zero-modes.

To obtain a small mass expansion we note that

$$\delta\left(\int \mathrm{d}^2x\sqrt{g}\;\mathrm{tr}_D\,G_\zeta(x)\right)=\delta\left(\int \mathrm{d}^2x\sqrt{g}\;\mathrm{tr}_D\,\tilde{G}_\zeta(x)\right)\tag{6.122}$$

since the normalization condition of the zero-modes do not change:

$$\delta\left(\sum_i\int \mathrm{d}^2x\sqrt{g}\,\Psi_{0,i}^\dagger(x)\Psi_{0,i}(x)\right)=0.\tag{6.123}$$

Furthermore

$$\int_0^\infty \mathrm{d}t\,\frac{\mathrm{e}^{m^2t}-m^2t-1}{t}K(t)=\mathcal{O}((m^2A)^2)\tag{6.124}$$

so that

$$\begin{aligned}\delta S_{\mathrm{grav}} = &-\frac{1}{12}\delta S_L-\frac{m^2}{4}\,\delta\left(\int \mathrm{d}^2x\sqrt{g}\;\mathrm{tr}_D\,\tilde{G}_\zeta(x)\right)\\ &-\frac{1}{2}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\sum_i \Psi_{0,i}^\dagger(x)\Psi_{0,i}(x)+\mathcal{O}(m^4).\end{aligned}\tag{6.125}$$

To get the gravitational action from (6.121) or (6.125) we need to express the third term as the variation of some functional. For the moment we were not able to do this generically so we present the two simplest cases: the sphere and the torus.

**The sphere**

On the sphere there is no zero-mode ($G(x,y) = \tilde{G}(x,y)$) and (6.121) reduces to

$$\delta S_{\rm grav} = \delta\left(-\frac{1}{12}S_L - \frac{m^2}{4}\int \mathrm{d}^2x\sqrt{g}\ \mathrm{tr}_D\, G_\zeta(x) + \frac{1}{8}\int_0^\infty \mathrm{d}t\frac{\mathrm{e}^{m^2t} - m^2t - 1}{t}K(t)\right). \quad (6.126)$$

We then obtain the gravitational action

$$\begin{aligned} S_{\rm grav}[\hat{g},g] = &-\frac{1}{12}S_L[\hat{g},g] - \frac{m^2}{4}\int \mathrm{d}^2x\left(\sqrt{g}\ \mathrm{tr}_D\, G_\zeta(x;g) - \sqrt{\hat{g}}\,\mathrm{tr}_D\, G_\zeta(x;\hat{g})\right) \\ &+ \frac{1}{8}\int_0^\infty \mathrm{d}t\frac{\mathrm{e}^{m^2t} - m^2t - 1}{t}(K_g(t) - K_{\hat{g}}(t)). \end{aligned} \quad (6.127)$$

When $m$ goes to zero $G_\zeta$ has a finite limit and, at first order in $m^2$ one has

$$S_{\rm grav}[\hat{g},g] = -\frac{1}{12}S_L[\hat{g},g] - \frac{m^2}{4}\left(A\Psi_G[g] - \hat{A}\Psi_G[\hat{g}]\right) + \mathcal{O}(m^4) \quad (6.128)$$

where

$$\Psi_G[g] = \frac{1}{A}\int \mathrm{d}^2x\sqrt{g}\ \mathrm{tr}_D\, G_\zeta(x;g). \quad (6.129)$$

The gravitational action can then be written as

$$\begin{aligned} S_{\rm grav}[\hat{g},g] &= -\frac{1}{12}S_L[\hat{g},g] - \frac{m^2}{4}\left(A\Psi_G[g] - \hat{A}\Psi_G[\hat{g}]\right) + \mathcal{O}(m^4) \\ &= -\frac{1}{12}S_L[\hat{g},g] - \frac{Am^2}{4}\left(\Psi_G[g] - \Psi_G[\hat{g}]\right) + \frac{m^2}{4}(A-\hat{A})\Psi_G[\hat{g}] + \mathcal{O}(m^4) \end{aligned} \quad (6.130)$$

where the last term only contributes to the cosmological constant.

**The torus**

On the torus there are two zero-modes given, for the flat torus by (6.68). For a small deformation around the flat torus one then has

$$\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\sum_i \Psi^\dagger_{0,i}(x)\Psi_{0,i}(x) = \frac{2}{A}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x) = 2\frac{\delta A}{A} = 2\delta\ln\frac{A}{\hat{A}}. \quad (6.131)$$

Then the gravitational action is given by

$$S_{\rm grav}[\hat{g},g] = -\frac{1}{12}S_L[\hat{g},g] - \ln\frac{A}{\hat{A}} - \frac{Am^2}{4}\left(\Psi_G[g] - \Psi_G[\hat{g}]\right) + \frac{m^2}{4}(A-\hat{A})\Psi_G[\hat{g}] + \mathcal{O}(m^4). \quad (6.132)$$

### 6.3.6 Outlook

To obtain the gravitational action the next step is to express $\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\sum_i \Psi^\dagger_{0,i}(x)\Psi_{0,i}(x)$ as the variation of some functional for Riemann surfaces of genus $\geq 2$. An idea would be to use the uniformization theorem to write the metric on such a surface as

$$\mathrm{d}s^2 = \frac{\mathrm{d}z\mathrm{d}\bar{z}}{(\mathrm{Im}\, z)^2}. \quad (6.133)$$

Next we need to find an explicit expression for $(\Psi_G[g] - \Psi_G[\hat{g}])$ for the small mass expansion. It is likely that we will encounter again the Mabuchi and Aubin–Yau functionals which appear quite naturally, but the combination will probably be different from the scalar case.

Finally we see that nor (6.121) neither (6.125) can give the DDK action (4.99). Indeed the former are a given by a complicated development in the mass whose first term (after $m^0$) is in $m^2$ and not in $m$.

# Chapter 7

# Spectrum of the Mabuchi action from a minisuperspace analysis

As we have seen, when conformal matter is coupled to two-dimensional quantum gravity the resulting gravitational action is the Liouville one. Liouville theory is now well understood. In particular its spectrum has been computed in [40–42, 44, 155]. We now need to follow the same program when the matter is non conformal. An interesting step would be to find the spectrum of the pure Mabuchi theory. In order to tackle this problem we rely on the minisuperspace approximation where the quantum field theory is reduced to the quantum mechanics of a point particle. In general the dynamics of the zero-mode is sufficient to build the Hilbert space which is made from normalizable wave functions.

We find that, in this limit, the Mabuchi Hamiltonian coincides with the Liouville Hamiltonian. As a consequence the spectra are identical in both theories. Using the results for the spectrum we can provide an expression for the 3-point function in the semi-classical limit. The results of this chapter have been published in [76, 77].

We first give our ansatz for the computation of the minisuperspace action. Indeed a rigorous derivation of the minisuperspace action and of the associated Hamiltonian requires a variable-area action, but the latter is not known for the Mabuchi theory. We provide several independent derivations relying on different assumptions of the minisuperspace action for the Mabuchi action: the fact that the results agree in all cases gives a strong support to our proposal. Finally we stress that this computation considers the Mabuchi action in isolation, and in particular without the area-dependent coupling constant arising from the matter, the reason being that this piece is not universal.

## 7.1 Rescaling of the Mabuchi action

In order to prepare the study of the minisuperspace it is necessary to rescale the Kähler potential and the Mabuchi action

$$\phi = \frac{\tilde{\phi}}{\pi\chi}, \qquad S_M = \frac{\tilde{S}_M}{\pi\chi} \tag{7.1}$$

such that the action reads

$$\tilde{S}_M = \frac{1}{4\pi}\int \mathrm{d}^2x\,\sqrt{\hat{g}}\,\left(-\hat{g}^{\mu\nu}\partial_\mu\tilde{\phi}\,\partial_\nu\tilde{\phi} + \left(\frac{4\pi\chi}{\hat{A}} - \hat{R}\right)\tilde{\phi} + \frac{4\pi\chi}{A}\,\sigma\,\mathrm{e}^{2\sigma}\right) \tag{7.2}$$

and the relation (4.118) between the Liouville mode and the Kähler potential becomes

$$\mathrm{e}^{2\sigma} = \frac{A}{\hat{A}}\left(1 + \frac{\hat{A}}{2\pi\chi}\,\hat{\Delta}\tilde{\phi}\right). \tag{7.3}$$

In the following we will omit the tildes on $\phi$ and $S_M$. Note that, introducing the dependence in the area, the above action can also be written as

$$S_M[A] = S_M[\hat{A}] + \frac{\chi}{2} \ln \frac{A}{\hat{A}}. \tag{7.4}$$

This rescaling requires explanations since it is singular for $\chi = 0$ (genus 1 surfaces), which is precisely the case we will be looking at in the following sections. We want to argue that this rescaling is necessary in order to get a consistent result:

1. The first point is that the kinetic term of (4.131) vanishes for $\chi = 0$ which indicates possible pathologies. On the other hand the action (7.2) is canonically normalized (up to a minus sign that we expect to be also an artifact of the Kähler parametrization).

2. Despite the fact that the relation between the fields is singular for $\chi \to 0$, the equations of motion, the Hamiltonian and the spectrum are well-defined even in the limit $\chi \to 0$.

3. The Kähler formalism itself presents other oddities. For example in [63] it was found that in the gravitational action the factors multiplying the Mabuchi (and Aubin–Yau) actions depend on $A$. As a consequence the equation of motion for the area (necessary to recover the full dynamics with respect to the Liouville field) contains the actions themselves, which is odd. More generally it is strange that what would be coupling constants in standard cases depend on a parameter that is integrated over in the functional integral. Another difficulty is to compute the energy–momentum tensor: taking the Liouville mode and the background metric as the independent variables, the relation (4.118) implies that the variation of $\phi$ in terms of $\hat{g}$ does not vanish (and similarly for $A$ and $\hat{A}$) and thus the variation of the action in terms of $\hat{g}$ is involved.

4. Adding the cosmological constant term $\mu A$ and using the expression (7.4) one directly finds the equation of motion for the area to be

$$\mu + \frac{\chi}{2A} = 0, \tag{7.5}$$

   which corresponds to (4.148). Then by plugging this result into (4.187) one finds the same equation than (4.138), in the same way that (4.187) was matching (4.138). Due to the comments below (4.148) it is possible that this relation holds at the level of the functional integral. This will be used in the next section to infer the possible minisuperspace action, where we will find other support for this procedure.

5. The action (7.2) contains only the geometric quantities

$$\bar{R} = \frac{4\pi\chi}{A}, \qquad \hat{\bar{R}} = \frac{4\pi\chi}{\hat{A}}, \tag{7.6}$$

   and this is also true of the factors in front of the Mabuchi action in [63] (in agreement with the comment at the bottom of p. 21 of [63]). According to the previous point this would mean that every instance of $\chi/A$ could be replaced by $-\mu$ and this would remove the ambiguities described above (with this interpretation the apparent divergences discussed below (4.21) of [63] would be an artifact of the formulation).

6. There are various instances where the action and/or the fields are rescaled by a parameter that tends to zero. This procedure is used to extract meaningful information when the details of the system are smeared in the limit we are taking such that one needs to "zoom".

   A simple example is the case of a dimensional reduction where the volume of the additional dimensions are taken to zero: since it multiplies the full action it is necessary to rescale the latter to obtain a non-trivial result. More generally this happens also

when one rescales some variables by the volume of a non-compact manifold or by the number of degrees of freedom when the latter is taken to be infinite (e.g. in matrix models when scaling the coupling constants or the fields, see for example [183]).

A closer example to our problem is the Liouville theory. Usually the Liouville action with a cosmological constant is written as (in particular when it is studied by itself)

$$S'_L = \frac{1}{4\pi}\int \mathrm{d}^2x\sqrt{\hat{g}}\left((\partial\sigma)^2 + Q\hat{R}\sigma + 4\pi\mu\,\mathrm{e}^{2b\sigma}\right) \tag{7.7}$$

where $Q = 1/b$ or $Q = 1/b + b$ depending on whether one is working with the action of (4.89) or with the DDK/bootstrap action [38, 39, 47]. The above action is not well-defined in the semi-classical limit $b \to 0$: for this reason one needs to perform the rescaling

$$\sigma = \frac{\sigma_c}{b}, \qquad \mu = \frac{\mu_c}{b^2}, \qquad S_L = b^2 S_c. \tag{7.8}$$

This should be compared with (7.1). Moreover it should be noted that the semi-classical limit is part of the minisuperspace approximation [47, sec. 5].

Even though the analogy is not complete since the parameter $b$ is continuous while $\chi$ is discrete, it does provide some additional support to this rescaling. Moreover, the form of the minisuperspace approximation of the unscaled action (4.118) does not give a meaningful result.

7. One could have considered to rescale by $-\pi\chi$ instead in order to make the kinetic term positive definite, but one would find that the Hamiltonian is not positive definite – see (4.215) – and the identification (4.148) would not hold for the potential. Moreover as argued in point 5) one should not take too seriously the negative sign in front of the kinetic term since the coupling constant is also negative for $\chi < 0$: then the replacement (4.148) would make the combination positive for all genus.

Even if none of these arguments is sufficiently rigorous to prove alone that the rescaling is well-defined, the convergence of these arguments gives support to this idea and points more toward the fact that the various pathologies are not genuine but rather due to the formalism. Since there is no other formalism at our disposition we will use the action (7.3) as our starting point.

Finally the action (7.3) will be modified a last time to

$$S_M = \frac{1}{4\pi}\int \mathrm{d}^2x\,\sqrt{\hat{g}}\left[-\hat{g}^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + \left(\frac{4\pi\chi}{\hat{A}} - \hat{R}\right)\phi + \frac{4\pi\chi}{A}\,(\sigma-1)\,\mathrm{e}^{2\sigma}\right] \tag{7.9}$$

where a trivial term has been added. In terms of the Liouville mode it means that one can shift the field $\sigma$ by a constant term without changing anything, while in terms of the Kähler potential it becomes a constant term and a boundary term. Moreover the addition of this term makes the variation of the action better defined since it cancels a boundary term proportional to the normal derivative of $\delta\phi$, which does not vanish (in the same way that one is adding a Gibbons–Hawking–York term in general relativity).

## 7.2 Computations of the minisuperspace Hamiltonian

In this section we will argue that a good minisuperspace action for the Mabuchi functional (in Lorentzian signature) is

$$S_M = -\frac{1}{2}\int \mathrm{d}t\left[\dot{\phi}^2 - \ddot{\phi}\ln\left(\frac{\ddot{\phi}}{4\pi\mu}\right) + \ddot{\phi}\right] \tag{7.10}$$

together with the relation between $\sigma$ and $\phi$

$$\mathrm{e}^{2\sigma} = \frac{\ddot{\phi}}{4\pi\mu}, \tag{7.11}$$

Note that for $\varpi = \frac{\ddot{K}}{8\pi\mu}$ the second term of this action corresponds to the one of the flat (or BMS) Liouville theory in the minisuperspace approximation. The latter corresponds to the asymptotic theory of $3d$ Minkowski $M_3$ in the same sense that the usual Liouville theory is the asymptotic theory of $\mathrm{adS}_3$ [184, 185]. We will argue that the Hamiltonian of (7.10) is equal to the one of the Liouville theory (in the minisuperspace approximation)

$$H_M = \frac{\Pi^2}{2} + 2\pi\mu\,\mathrm{e}^{2\sigma}, \tag{7.12}$$

$\Pi$ being the conjugate momentum of $\sigma$.

Due to various pathologies of the formalism (explained in 7.1 and 7.2.1) we are not able to give a rigorous derivation of this result. Nonetheless we present three computations of the Hamiltonian (7.12) that all rely on different (mild) assumptions and for this reason we believe that together they provide a support for our conjecture. Moreover since the action (7.10) reproduces the main characteristics of the Mabuchi action (4.131) (standard kinetic term for $\phi$ and potential in $\sigma\,\mathrm{e}^{2\sigma}$) it is expected to capture the main features of the zero-mode dynamics in the Mabuchi theory. Hence even if a rigorous derivation can be performed only by starting with a variable area action which is not known one is still able to make progress.

The first subsection explains the various subtleties of the minisuperspace approximation while the other ones present the different derivations.

### 7.2.1 Minisuperspace approximation

The minisuperspace approximation consists in studying only time-dependent Kähler potential and Liouville mode

$$\sigma(t,x) = \sigma(t), \qquad \phi(t,x) = \phi(t). \tag{7.13}$$

In order to single out a globally defined time direction (global hyperbolicity) for the Hamiltonian formalism, the background spacetime is taken to be a cylinder $I \times S^1$ where $I$ is an interval of length $T$. This cylinder is obtained from the torus $T^2$ by unwrapping one of its dimension. This direct product structure implies that the spacetime is flat

$$\chi = 0, \qquad g_0 = \eta, \qquad t \in \left[-\frac{T}{2}, \frac{T}{2}\right], \qquad x \in [0, 2\pi). \tag{7.14}$$

The physically-relevant case is when time is non-compact with $T \to \infty$ and $I = \mathbb{R}$ (the infinite cylinder can also be obtained by taking the radius of one of the torus circle to infinity). Unfortunately several difficulties arise from the fact that the Mabuchi action is formulated at fixed area, and that it depends on $\chi$ and $\hat{A} = 2\pi\,T$. For this reason one needs to be careful when taking the limits.

The dynamical variables in the fixed area formalism are $\phi$ and $A$ and do not include $\hat{A}$. Since the background metric $\hat{g}$ results from a gauge choice, it can be chosen such that $\hat{A}$ have some specific values (in particular the area $\hat{A}$ does not appear in the equation of motion). For these different reasons it is expected that one can take $\hat{A} \to \infty$ (corresponding to $T \to \infty$), independently of the value of $A$ which can be kept finite. Note that this limit is taken by changing the ranges of the coordinates as described above and not the components of the metric: for this reason $\hat{g}$ is kept fixed while taking the limit. This point is similar to the question of whether the moduli describing a Riemann surfaces appear in the definition of the coordinate ranges or in the metric [6]. This may introduce some spurious singularity in $\sigma$ but this will be of no importance for our study. In any case working at variable

area is necessary in order to have a non-trivial dynamics in the minisuperspace: starting with the Liouville action (4.89) one finds that the minisuperspace Hamiltonian reduces to a free field Hamiltonian. Below we will find the same result for the Mabuchi minisuperspace Hamiltonian. Moreover the Hamiltonian formalism is usually formulated in terms of a time taking value over the real axis and for this reason it is more natural to consider variable (non-compact) area for the $\hat{g}$ metric in order to have a time that ranges over the full real axis.

In a second step we impose that the curvature vanishes since the spacetime is flat, $\hat{R} = 0$. A potential problem can arise because $\chi = 0$ for flat space, but the Lagrangian is singular in this case (this translates into the vanishing of the kinetic term in (4.131) before the rescaling of the action). Two different solutions are possible. The first one is to consider a singular $\sigma$ such that

$$4\pi\chi = \int \mathrm{d}^2x \sqrt{|g|}\, R \neq \int \mathrm{d}^2x \sqrt{|\hat{g}|}\, \hat{R}, \tag{7.15}$$

in which case $\hat{R} = 0$ does not imply $\chi = 0$. One is then forced to work with patches in order to deal with the singularity of $\sigma$: as explained above we do not work directly with the value of $\phi$ in this paper and this should be of no consequence. The second solution is simpler: from the relation

$$\hat{R}\hat{A} = 4\pi\chi, \tag{7.16}$$

which is valid for constant $R_0$, one sees that $\chi$ can take a non-zero value if one takes the limits $\hat{A} \to \infty$ and $\hat{R} \to 0$ simultaneously such that the product is constant (this is a form of double scaling limit). We will adopt this view in 7.2.4 and we will formally work with $\chi \neq 0$.

Another motivation for keeping $\chi$ arbitrary until the end is the fact that the operations of taking a limit in the Lagrangian or in some quantity computed from it may not commute. The Liouville theory provides again an example: it is well-known that one should not set $\hat{R} = 0$ in the Lagrangian before computing the energy–momentum tensor for the flat space case $\hat{g} = \eta$ since the variation of this term gives a non-vanishing contribution in the limit $\hat{R} \to 0$. So one should avoid to take limits directly in the Lagrangian if one is not sure of the effect this will have when computing other quantities.

Finally the question of the Wick rotation needs to be addressed since a positive definite Hamiltonian requires the signature to be Lorentzian. Since the background spacetime is non-compact it is expected that the Wick rotation can be performed without any problem. Let us comment the case of finite $T$: then it is not clear if one can perform the Wick rotation since analyticity can be lost for a variable defined on a segment. On the other hand no pathological behaviour is seen when performing the Wick rotation for the Liouville theory defined with finite time $t \in I$. Note that compact spacetimes with Lorentzian signature are perfectly well-defined and it can be convenient to use them at intermediate stages of computations. For example it is frequent in QFT to consider "spacetime in a box" in order to regulate IR divergences, before taking the infinite limit volume. Moreover the equation of motion for the Liouville mode has the same form (4.138) in both Euclidean and Lorentzian signatures. Hence the relation (4.148) also holds and indicates that classical solutions have finite area $A$ even in Lorentzian signature except possibly if $\chi = 0$ at the same time. For these reasons it is fine to first perform the Wick rotation of the action and later to consider the infinite area limit.

### 7.2.2 First derivation: infinite area and flat limits

Gathering all the previous elements, the relation (7.3) between the Liouville and Kähler fields becomes

$$\mathrm{e}^{2\sigma} = -\frac{A}{2\pi\chi}\,\ddot{\phi}. \tag{7.17}$$

and the minisuperspace action of the Mabuchi action (7.9) reads

$$S_M = -\frac{1}{2}\int \mathrm{d}t \left[\dot{\phi}^2 - \ddot{\phi}\ln\left(-\frac{A}{2\pi\chi}\,\ddot{\phi}\right) + \ddot{\phi}\right]. \tag{7.18}$$

The overall minus sign comes from the Lorentzian signature and we have set $\hat{R} = 0$ while the integration over the spatial direction has provided a factor $2\pi$. It is straightforward to check that the variation of (7.18) agrees with the minisuperspace approximation of (4.187)

$$\ddot{\sigma} = \frac{2\pi\chi}{A}\,\mathrm{e}^{2\sigma}. \tag{7.19}$$

Since the action (7.18) does not depend on $\phi$ it is possible to reduce it to a first order action by considering $\dot{K}$ as the canonical variable (in fact the condition $\hat{R} = \text{cst}$ is sufficient for this to happen in view of the relation (7.16)). The conjugate momenta $P$ reads

$$P = \frac{\delta S_M}{\delta\ddot{\phi}} = \frac{1}{2}\ln\left(-\frac{A}{2\pi\chi}\,\ddot{\phi}\right). \tag{7.20}$$

It is necessary to invert this expression

$$\ddot{\phi} = -\frac{2\pi\chi}{A}\,\mathrm{e}^{2P}. \tag{7.21}$$

in order to compute the Hamiltonian

$$H_M = P\ddot{\phi} - L_M = \frac{\dot{\phi}^2}{2} - \frac{\pi\chi}{A}\,\mathrm{e}^{2P}. \tag{7.22}$$

Comparing the relations (7.21) and (7.17) shows that $P$ can be identified with $\sigma$. Performing a canonical transformation to exchange position and momentum

$$P = \sigma, \qquad \dot{\phi} = -\Pi, \tag{7.23}$$

where $\Pi$ is the canonical momentum associated to $\phi$, provides the Hamiltonian

$$H_M = \frac{\Pi^2}{2} - \frac{\pi\chi}{A}\,\mathrm{e}^{2\sigma}. \tag{7.24}$$

It is straightforward to check that the equations of motion (7.19) follow from this Hamiltonian. At this point it is possible to set $\chi = 0$ (which is well-defined) and to add the cosmological constant term to find (7.12)

$$H_M = \frac{\Pi^2}{2} + 2\pi\mu\,\mathrm{e}^{2\sigma}. \tag{7.25}$$

This is equivalent to insert the cosmological constant term in the path integral and to replace the integration over $(\phi, A)$ by the one over $\sigma$. Note that the same effect is achieved using the arguments in section 4.4 and the replacement (4.148).

In this form the Hamiltonian is explicitly positive definite and it is nothing else but the Liouville Hamiltonian (4.163) in the minisuperspace approximation (with $b = 1$ corresponding to the case where the Liouville mode has not been rescaled). The equation of motion for this Hamiltonian is

$$\ddot{\sigma} = -4\pi\mu\,\mathrm{e}^{2\sigma}. \tag{7.26}$$

which corresponds to the minisuperspace approximation of (4.138) which is also expected for the Mabuchi theory following the comments in 7.1.

One may be surprised to start with a Lagrangian (7.10) containing a negative-definite kinetic term and to end with a positive-definite Hamiltonian (7.25). This is a consequence

of the presence of higher-derivatives: the $\Pi^2$ term comes entirely from the $-L_M$ term which explains why it has the correct sign, compared to the standard computation in the absence of higher-derivatives where the first term contributes typically with an opposite sign and is twice bigger. In particular one can see here that rescaling with $-\pi\chi$ to get a positive-definite kinetic term in the Lagrangian would have lead to a negative-definite Hamiltonian.

The consistency of these computations can be checked by following the same approach with the Liouville action (4.89) at fixed area: the minisuperspace Hamiltonian of the latter is simply $H_L = p^2/2$ which coincides with (7.24) when $\chi = 0$, and the full Hamiltonian (4.163) is recovered by adding the cosmological constant.

### 7.2.3 Second derivation: Legendre transformation

It has been observed in [147] that the kinetic and potential terms of the Mabuchi action are respectively Legendre dual to the kinetic term of the Liouville action and to the cosmological constant potential (the kinetic terms including the linear piece). Despite the fact that this relation did not receive any explanation it gives a simple consistency check of the other derivations: we will apply it in the minisuperspace approximation and show that the resulting action corresponds to (7.10).

Let's define from (4.89) and (4.131)

$$T_L = \frac{\dot\sigma^2}{2}, \qquad V_L = -2\pi\mu\, \mathrm{e}^{2\sigma} \tag{7.27}$$

(the $2\pi$ in the second term comes from integrating over $S^1$, since the cosmological constant is not normalized) along with the Legendre transforms of these functions

$$T_M = \sigma\hat\sigma - T_L = \sigma\hat\sigma - \frac{\dot\sigma^2}{2}, \qquad V_M = \sigma\hat\sigma - V_L = \sigma\hat\sigma + 2\pi\mu\, \mathrm{e}^{2\sigma}. \tag{7.28}$$

We need to extremize the above functions with respect to $\sigma$ and plug back the result.

Let's start with $T_L$:

$$\frac{\delta T_M}{\delta\sigma} = \hat\sigma + \ddot\sigma = 0. \tag{7.29}$$

Defining $\hat\sigma = -\ddot\varphi$ one obtains the solution, and plugging back gives (under the integral)

$$T_M = -\ddot\varphi\ -\frac{\ddot\varphi^2}{2} = \frac{\ddot\varphi^2}{2}. \tag{7.30}$$

Let's apply the same procedure to $V_L$:

$$\frac{\delta V_M}{\delta\sigma} = \hat\sigma + 4\pi\mu\, \mathrm{e}^{2\sigma} = 0. \tag{7.31}$$

The solution reads

$$\sigma = \frac{1}{2}\ln\left(-\frac{\hat\sigma}{4\pi\mu}\right), \tag{7.32}$$

and defining again $\hat\sigma = -\ddot\varphi$ one obtains

$$V_M = \frac{\hat\sigma}{2}\left(\ln\left(-\frac{\hat\sigma}{4\pi\mu}\right) - 1\right) = -\frac{\ddot\varphi}{2}\left(\ln\left(\frac{\ddot\varphi}{4\pi\mu}\right) - 1\right). \tag{7.33}$$

Note the presence of a boundary term.

The final action that we obtain by gathering both terms is

$$S_M = \frac{1}{2}\int \mathrm{d}t \left[\dot\varphi^2 - \ddot\varphi \ln\left(\frac{\ddot\varphi}{4\pi\mu}\right) + \ddot\varphi\right]. \tag{7.34}$$

Note that the Liouville terms were in Lorentzian signature whereas the resulting Mabuchi action is in Euclidean signature. The explanation may be that one should rescale with $-\pi\chi$ instead of $\pi\chi$ in (7.1) (this can be supported by the fact that, according to the discussion of 7.1, the coupling constants at variable area would get a minus sign without this). This action is the same as (7.10) upon the identification $\psi = \phi$ and thus it will yield the Hamiltonian (7.25). Note that it naturally incorporates the boundary term from (7.9) and the relation (4.148). Other forms of the Mabuchi actions could have been obtained by considering a different $\hat{\phi}$ in the transformation of $V_L$, but the above choice is more natural since $\hat{\phi}$ is the same for both $T_L$ and $V_L$.

### 7.2.4 Third derivation: Ostrogradski formalism

In order to generalize the computation of 7.2.2 we will consider the case where $\hat{A} = 2\pi T$ and $\hat{R}$ are kept finite (generalizing the idea that one should not set terms to zero directly in the Lagrangian). In this case the Lagrangian is of higher order in the derivatives and one needs to use the Ostrogradski formalism [173] in the same way as what we did when computing the Hamiltonian of the full action in 4.5.2.

The Mabuchi action (in Lorentzian signature) in the minisuperspace approximation (no spatial dependence) is

$$S_M = -\frac{1}{2}\int \mathrm{d}t \left[\dot{\phi}^2 + \left(\frac{4\pi\chi}{\hat{A}} - \hat{R}\right)\phi + \frac{2\pi\chi}{\hat{A}}\left(1 - \frac{\hat{A}}{2\pi\chi}\ddot{\phi}\right)\left(\ln\frac{A}{\hat{A}}\left(1 - \frac{\hat{A}}{2\pi\chi}\ddot{\phi}\right) - 1\right)\right] \tag{7.35}$$

with the relation

$$\mathrm{e}^{2\sigma} = \frac{A}{\hat{A}}\left(1 - \frac{\hat{A}}{2\pi\chi}\ddot{\phi}\right). \tag{7.36}$$

The canonical variables are taken to be $(\phi, \mathcal{P})$ and $(\dot{\phi}, P)$ where the conjugate momenta are

$$P = \frac{\partial L}{\partial \ddot{\phi}} = \frac{1}{2}\ln\frac{A}{\hat{A}}\left(1 - \frac{\hat{A}}{2\pi\chi}\ddot{\phi}\right), \tag{7.37a}$$

$$\mathcal{P} = \frac{\partial L}{\partial \dot{\phi}} - \frac{\mathrm{d}P}{\mathrm{d}t} = -\dot{\phi} - \frac{1}{2}\frac{\mathrm{d}P}{\mathrm{d}t}. \tag{7.37b}$$

In particular we can invert the first relation to find $\ddot{\phi}$ in terms of the canonical variable

$$\ddot{\phi} = \frac{2\pi\chi}{A}\left(\frac{A}{\hat{A}} - \mathrm{e}^{2P}\right). \tag{7.38}$$

Moreover comparing this expression with (7.36) one finds $P = \sigma$. The Hamiltonian reads

$$H = \mathcal{P}\dot{\phi} + P\ddot{\phi} - L = \mathcal{P}\dot{\phi} + \frac{2\pi\chi}{\hat{A}}P + \frac{\dot{\phi}^2}{2} + \frac{1}{2}\left(\frac{4\pi\chi}{\hat{A}} - \hat{R}\right)\phi - \frac{\pi\chi}{A}\mathrm{e}^{2P}. \tag{7.39}$$

The canonical transformation

$$P = \sigma, \qquad \dot{\phi} = -\Pi \tag{7.40}$$

can be performed in order to express the Hamiltonian (7.39) in terms of the Liouville field

$$H = \frac{\Pi^2}{2} - \mathcal{P}\Pi + \frac{1}{2}\left(\frac{4\pi\chi}{\hat{A}} - \hat{R}\right)\phi + \frac{2\pi\chi}{\hat{A}}\sigma - \frac{\pi\chi}{A}\mathrm{e}^{2\sigma}. \tag{7.41}$$

Note that we can obtain this minisuperspace Hamiltonian as a limit from the full Hamiltonian computed through an ADM parametrization of the metric in 4.5.2.

The Hamiltonian is well-defined for $\chi = 0$ and $\hat{R} = 0$ and it reduces to

$$H = \frac{\Pi^2}{2} - \mathcal{P}\Pi, \tag{7.42}$$

noting that in this case it is not necessary to take the limit $\hat{A} \to \infty$. After performing the canonical transformation

$$\tilde{\Pi} = \Pi - \mathcal{P}, \qquad \tilde{\mathcal{P}} = \mathcal{P}, \qquad \tilde{\sigma} = \sigma, \qquad \tilde{\phi} = \sigma + \phi, \tag{7.43}$$

the Hamiltonian reads (omitting the tildes)

$$H = \frac{\Pi^2}{2} - \frac{\mathcal{P}^2}{2}. \tag{7.44}$$

Adding the cosmological constant gives finally

$$H = \frac{\Pi^2}{2} + 2\pi\mu\, \mathrm{e}^{2\sigma} - \frac{\mathcal{P}^2}{2}. \tag{7.45}$$

Hence one recovers Liouville Hamiltonian plus a free (ghost) term. The interpretation of this additional ghost field with respect to the other methods is not clear but it can be expected that it is just an artifact of the fixed area formalism.

## 7.3 Minisuperspace canonical quantization of the Mabuchi theory

The minisuperspace approximation is well-suited to determine the Hilbert space of the theory as the latter can be found by studying the dynamics of the zero-mode only which simplifies greatly the canonical quantization of the action. Through several changes of variables we were are able to show that the Mabuchi and Liouville Hamiltonians are equal, implying that the Mabuchi spectrum is identical to the Liouville spectrum.

The spectrum of Mabuchi theory is determined through the canonical quantization

$$\Pi \longrightarrow -i\,\frac{\mathrm{d}}{\mathrm{d}\sigma}. \tag{7.46}$$

It coincides with the minisuperspace quantization of the Liouville theory [41, 44, 153] that we reviewed in 4.4.4. The stationary Schrödinger equation reads (if the action had not been rescaled by $\pi\chi$ before, it would be equivalent to rescale the eigenvalues here).

$$H_M \psi_p = 2p^2\, \psi_p, \tag{7.47}$$

where the definition of the eigenvalue is conventional, and this provides the differential equation

$$\left(-\frac{1}{2}\,\frac{\mathrm{d}^2}{\mathrm{d}\sigma^2} + 2\pi\mu\, \mathrm{e}^{2\sigma} - 2p^2\right) \psi_p(\sigma) = 0. \tag{7.48}$$

It corresponds to the modified Bessel equation whose solutions are

$$\psi_p(\sigma) = \frac{2(\pi\mu)^{-ip}}{\Gamma(-2ip)}\, K_{2ip}(2\sqrt{\pi\mu}\, \mathrm{e}^{\sigma}) \tag{7.49a}$$

$$\sim_0 \mathrm{e}^{2ip\sigma} + r_0(p)\mathrm{e}^{-2ip\sigma}. \tag{7.49b}$$

The second linearly independent solution has been removed because it blows up at $\sigma \to \infty$ and the normalization has been chosen such that the incoming plane waves have unit coefficient as $\sigma \to -\infty$. The factor

$$r_0(p) = \frac{\Gamma(2ip)}{\Gamma(-2ip)}\, (\pi\mu)^{-2ip} \tag{7.50}$$

$r_0(p)$ was interpreted as a reflection coefficient in the Liouville theory, but its interpretation in terms of the Mabuchi action is not clear. Moreover it can be seen that wave functions with $\pm p$ are not independent.

$$\psi_{-p}(\sigma) = r_0(-p)\,\psi_p(\sigma). \tag{7.51}$$

Additional constraints such as normalisability are needed in order to restrict the eigenvalues. As was the case for Liouville only the states with $p \in \mathbb{R}$ are (delta-function) normalizable under the canonical inner product and belong then to the physical Hilbert space. Moreover these states form a complete basis.

Since it is not clear if the Mabuchi theory defines a CFT we do not link these eigenvalues to conformal weights as we did in 4.4.4 for the Liouville theory. In particular the unitarity condition is not clear and hence we do not comment the status of the states with $p \in i\mathbb{R}$ (note that some of those play a physical role in $2d$ Liouville gravity).

As a consequence the operators and the associated eigenvalues are identical in the Mabuchi and Liouville theories. On one hand it is not surprising due to the fact that the classical equations of motion are identical, but on the other hand it is highly non-trivial that the very complicated action (7.10) reduces to the Liouville Hamiltonian after performing suitable changes of variables. Without the identification of the momentum $P$ to the Liouville mode it would have been very difficult to extract the wave functions for $\phi$.

Finally the semi-classical limit of the 3-point function can be read from the minisuperspace

$$C_0(p_1,p_2,p_3) = \int_{-\infty}^{\infty} \mathrm{d}\sigma\ \psi_{p_1}(\sigma)\mathrm{e}^{-2ip_2\sigma}\psi_{p_3}(\sigma) \tag{7.52a}$$

$$= (\pi\mu)^{-2\tilde{p}}\,\Gamma(2\tilde{p})\prod_i \frac{\Gamma\big((-1)^i 2\tilde{p}_i\big)}{\Gamma(2p_i)} \tag{7.52b}$$

where we defined

$$2\tilde{p} = \sum_i p_i, \qquad \tilde{p}_i = \tilde{p} - p_i, \qquad i = 1,2,3. \tag{7.53}$$

Of course this result agrees with the Liouville theory in the minisuperspace approximation, but discrepancies will certainly appear beyond the semi-classical limit.

# Part II

# A quantum mechanical model for black holes from holography

# Chapter 8

# Black holes and holography

Very soon after the construction of general relativity it appeared that some solutions of Einstein equations display singularities i.e. points where the curvature of spacetime becomes infinite. It has been shown by Penrose and Hawking that these singularities appear in quite generic situations [186–188]. Black holes constitute a major class of solutions which display this behaviour. A black hole is a region of spacetime where the gravitational force is so strong that nothing can escape from it. In the centre of the black hole lies a singularity. This singularity is hidden behind an event horizon which is the boundary of the region from which no causal signal can reach an observer at infinity. It is not clear whether the singularity (and even the horizon) still appears when the quantum effects are taken into account. Indeed when the curvature becomes very large general relativity cannot be trusted anymore and one has to take into account quantum gravity effects. This is why black holes seem to offer a window on the elusive theory of quantum gravity.

At first sight black holes are just classical solutions of general relativity. Their structure is very simple and only depends on their mass, angular momentum and charge. In four-dimensional Minkowski spacetime there are then only four possible black hole solutions: Schwarzschild (non-rotating and uncharged), Kerr (rotating and uncharged), Reissner–Nordström (non-rotating and charged) and Kerr–Newman (rotating and charged). However as developed in section 8.1 black holes display some puzzling thermal behaviour: it has an entropy, a temperature and then emits thermal radiation. When analysing the thermodynamics of black holes one rapidly encounters strange phenomena, the most intriguing being the information paradox which highlights some strong contradiction between general relativity and quantum mechanics.

A powerful tool to study quantum black holes is given by the AdS/CFT correspondence which is reviewed in 8.2. This duality proposes to study the properties of a gravitational system from the one of a standard quantum field theory without gravity. This enables one to build simpler quantum mechanical models to study the quantum dynamics of black holes.

## 8.1 The thermal behaviour of black holes

### 8.1.1 Thermodynamics of black holes

In this section we review some fundamental aspects of black hole thermodynamics. We will momentarily reintroduce the dimensionful constants $G$, $\hbar$ and $c$.

The fact that nothing can escape from a black hole raises a thermodynamics issue: if an object with an entropy falls into the black hole then the entropy of the outside will seem to decrease which contradicts the second law of thermodynamics. There is no problem with the first law since the black hole has an energy, its mass, which will increase if an object carrying energy falls inside the horizon. We see that here what preserves the first law is the

fact that the black hole has been assigned with an energy. This suggests that the second law might be preserved if we assign some kind of entropy to the black hole.

This idea is pushed forward when we notice that some characteristics of black holes look very much alike those of thermodynamical systems [87, 88, 189, 190]. First a black hole can be completely characterised by a few macroscopic parameters namely its mass $M$, its charge $Q$ and its spin (or angular momentum) $J$. Correspondingly a thermodynamical system is entirely described by a few macroscopic parameters like its temperature, pressure, chemical potential... What is even more striking is that black holes mechanics follows laws that are very similar to the laws for thermodynamics:

- The zeroth law states that the temperature is constant throughout a body at thermal equilibrium. In the same way the surface gravity $\kappa = \frac{1}{4GM}$ is constant on the horizon of stationary black holes (the surface gravity might be thought as the limiting force one needs to exert from infinity to keep an object stationary at the horizon of a static black hole).

- The first law is concerned with energy conservation: $\mathrm{d}E = T\mathrm{d}S - p\mathrm{d}V + \mu\mathrm{d}n$ where $E$ is the energy, $V$ the volume and $n$ the number of particles. Correspondingly for black holes one has
$$\mathrm{d}M = \frac{\kappa}{8\pi G}\mathrm{d}A + \mu\mathrm{d}Q + \Omega\,\mathrm{d}J \tag{8.1}$$
(for a Schwarzschild black hole, $\mu = \Omega = 0$ since there is no charge nor spin).

- The second law states that there exists an extensive quantity, the entropy $S$ who can never decrease in an isolated system: $\Delta S \geq 0$. For black holes the area theorem states that the area $A$ of the horizon cannot decrease in any process: $\Delta A \geq 0$. For example when two Schwarzschild black holes of masses $M_1$ and $M_2$ coalesce they form a black hole of mass $M_1 + M_2$. Since the area of the horizon is proportional to the mass square we see that the area indeed increases ($(M_1 + M_2)^2 \geq M_1^2 + M_2^2$).

These observations lead Bekenstein to propose that black holes have an entropy proportional to there area [84].

However if black holes do have an entropy there must also exist a quantity to play the role of the temperature:
$$\frac{1}{T} = \frac{\partial S}{\partial E}. \tag{8.2}$$
But a thermodynamical object at non-zero temperature emits thermal radiation which seems paradoxal for a black hole that is supposed to absorb everything without remitting anything. However, by a semiclassical computation treating the black hole metric as a fixed background where evolve quantum fields, Hawking showed that black holes do actually radiate thermally at a temperature $T$ given by the Hawking temperature [85]:
$$T = \frac{\hbar}{8\pi GM}. \tag{8.3}$$
Reporting in (8.2) gives the Bekenstein–Hawking entropy:
$$S_{\mathrm{BH}} = \frac{Ac^3}{4G\hbar}. \tag{8.4}$$

In statistical systems the entropy is the quantity which counts the number of states of the system. Therefore it is natural to ask whether (8.4) has a statistical interpretation and if yes which states it is actually counting.

### 8.1.2 The information paradox

However the thermodynamics of black holes raises more questions than it gives answers. The main issue is that if black holes radiate they will evaporate. But no matter in which state the black hole was when it formed, Hawking radiation is always (approximately) thermal [89, 191]. This means that if initially the black hole was in a pure state $\Psi$, which means that the density matrix takes the form $\rho = |\Psi\rangle\langle\Psi|$, the final state will be a mixed state i.e. described by a density matrix of the form $\rho = \sum_n p_n |\Psi_n\rangle\langle\Psi_n|$ where the $\Psi_n$ are an orthonormal basis of states and the $p_n$ are (almost thermal) probability weights (satisfying $\sum_n p_n = 1$), more than one being non-zero. This is in contradiction with the unitarity of quantum mechanics which implies that the evolution of a quantum system preserves pure states.

This also means that the information that was present in the initial state is not recovered in the radiation and then disappears at the end of the evaporation of the black hole. This is the information paradox. A measure of the missing information is given by the Von Neumann entropy $S = -\operatorname{tr}(\rho \ln \rho) = -\sum_n p_n \ln p_n$.

There are several proposals concerning what happens to this missing information [88–92].

1. The information is lost and we have to give up the unitarity of quantum mechanics. This was the first conclusion of Hawking but we will see that AdS/CFT tends to discard it [192].

2. At the last stage of the evaporation when the curvature of the horizon reaches the Planck scale, Hawking's semiclassical computation cannot be trusted. The black hole leaves a stable remnant of planckian size in which the initial information is concentrated. But for the total state to remain pure this remnant must have a huge entanglement entropy, much larger than that of a black hole of the same mass. The remnant would have a very low mass (since most of the black hole would have evaporated) but an enormous number of states to achieve for the high entropy. But objects of this kind have a huge production rate which is not consistent with observations. Even supposing that the remnants slowly evaporate and disappear does not solve the problem since the transmission of so much information with so little energy would take a time so long that the remnant would still look stable.

3. All the information is in fact returned in the radiation which is actually not a mixed state. The information is contained in the entanglement of the fields of the radiation. Every small subsystems would look thermal but the entire state would be pure. This implies a breakdown of causality and locality.

4. Other ideas have been proposed. For example [193] argues that the information loss comes from the impossibility for a distant observer to measure the state of a black hole with arbitrary accuracy.

The information paradox opens a window on what could be some essential features of quantum gravity. In any case it seems that we will have to give up some postulates of quantum mechanics, either unitarity or locality. As we will see the holographic proposal suggests that we should rather look at the third solution of the paradox and that quantum gravity would be essentially non-local.

## 8.2 Studying black holes from holography

### 8.2.1 The AdS/CFT correspondence

As we have seen in the previous section the entropy of black holes scales as the area $A$ of the horizon. Bekenstein [194] then conjectured that the maximum entropy that can fit into

some region of space with boundary area $A$ is equal to the entropy of the black hole whose horizon has area $A$ (for now on we come back to natural units):

$$S_{\max} = \frac{A}{4}. \tag{8.5}$$

Indeed if the entropy $S$ of the region is superior to the entropy $S_{\mathrm{BH}}$ of the black hole then we could add some matter to the region to form a black hole. But then the total entropy would decrease which contradicts the second law of thermodynamics. This has lead to the holographic proposal [74, 195–197] which states that the number of degrees of freedom of a gravitating system in some region of space is the same as the one of a system on the boundary of this region. The holographic hypothesis requires a major transformation in the description of fields and interactions (and maybe of the structure of spacetime itself). Indeed in holographic systems the number of degrees of freedom and their growth in terms of the size of the system are much lower than what is expected in standard (non-gravitational) systems. In the latter the number of degrees of freedom is proportional to the volume while in holographic systems it scales as the area. It seems that these particularities are the sign of the non-locality of the underlying theory of quantum gravity. This is not completely surprising since diffeomorphism invariance prevents us from building local observables in quantum gravity (however what is puzzling there is that the Universe looks local at the scales that are currently reachable by experiments).

The holographic proposal found a concrete realisation with the conjecture of the AdS/CFT correspondence by Maldacena [79]. It states that non-perturbative string theory in asymptotic AdS background is dual to a conformal field theory living on the boundary. This means that there is a dictionary between the observables in some gravity theory and those in a gauge theory that enables to use one picture or the other to compute correlation functions. What is surprising with this construction is that it relates a theory of gravity with a gauge theory without any kind of gravity. Moreover as CFTs are UV complete the holographic correspondence provides a non-perturbative description of a UV complete theory of quantum gravity. Here we will just review the fundamentals ideas of AdS/CFT, more details can be found in [80–82, 91, 198–200].

In the original formulation of the conjecture the bulk theory consists of type IIB string theory on $\mathrm{AdS}_5 \times S_5$ where the AdS and 5-sphere radii are equal. The flux of the 5-form through $S_5$ is an integer $N$ and is related to the AdS radius $r_{\mathrm{AdS}}$ by

$$r_{\mathrm{AdS}}^4 = 4\pi g_s N \alpha'^2 \tag{8.6}$$

where $\alpha'$ is the universal Regge slope of string theory related to the string length $l_s$ by $\alpha' = l_s^2$ and $g_s$ is the string coupling. The CFT side of the correspondence is given by four-dimensional $\mathcal{N} = 4$ super-Yang–Mills theory with $\mathrm{SU}(N)$ gauge group and coupling $g_{\mathrm{YM}}^2 = g_s$. The idea of the AdS/CFT correspondence comes from the study of a stack of $N$ D3-branes in type IIB string theory in 10 dimensions. At large $N$ the geometry near the stack of branes approaches the one of $\mathrm{AdS}_5 \times S_5$. However at any finite $N$ the system admits $\mathcal{N} = 4$ four-dimensional SYM with $\mathrm{SU}(N)$ gauge group as a low-energy description. Then in their overlapping area of validity both theories must give an equivalent description of the system.

The elaboration of the AdS/CFT conjecture relies on perturbation theory on both sides. On the SYM side this can be trusted as long as it is weakly coupled i.e. as long as $g_{\mathrm{YM}}^2 N \ll 1$. On the other hand the classical description of the geometry of $\mathrm{AdS}_5$ is valid only when the radius of AdS is large compared to the string length given by $\alpha'$ which is the case, according to (8.6), when $g_{\mathrm{YM}}^2 N \gg 1$. For the two theory to be equivalent one then needs to be strongly coupled when the other is weakly coupled and vice-versa: AdS/CFT is a strong-weak duality.

The t'Hooft limit consists in taking $N$ to infinity while keeping $\lambda = g_{\mathrm{YM}}^2 N$ fixed. In the gauge theory we can then do a $1/N$ perturbative expansion which corresponds to a topological expansion of the Feynman diagrams. The bulk side is now classical type IIB string

theory (i.e. without $\alpha'$ corrections) and the $1/N$ expansion corresponds to a perturbative loop expansion in $g_S$.

A signal of the duality is that the symmetry groups of $\mathrm{AdS}_5 \times S_5$ and of $\mathcal{N} = 4$ four-dimensional SYM match. Indeed by definition, the symmetry group of $\mathrm{AdS}_5$ is $\mathrm{SO}(4,2)$ which corresponds also to the conformal group in four dimensions. Moreover $\mathcal{N} = 4$ supersymmetry gives rise to a $\mathrm{SU}(4)$ R-symmetry which is isomorphic to the $\mathrm{S0}(6)$ symmetry group of $S^5$. There is also an $\mathrm{SL}(2,\mathbb{R})$ weak-strong duality on both sides, in the four-dimensional $\mathcal{N} = 4$ SYM and in type IIB string theory.

The first dictionary between the two sides of the correspondence has been established shortly after Maldacena proposal [201, 202]. To each light field $\Psi_i$ in the bulk theory corresponds a local operator $\mathcal{O}_i$ with the same spin in the CFT. For example massive scalar fields in the bulk are dual to scalar fields in the CFT whose conformal dimension is related to the mass. The graviton in AdS is mapped to the stress-energy tensor of the CFT. If the gravity theory has a spin-1 vector field $A_\mu$, then the dual CFT has a spin-1 operator $J_\mu$. If $A_\mu$ is massless the current $J_\mu$ is conserved. This constitutes a central feature of the AdS/CFT correspondence: gauge symmetries in the bulk correspond to global symmetries in the CFT.

The partition function $Z_{\mathrm{ST}}$ of the string theory depends on the value $\Psi_{B,i}$ of the fields $\Psi_i$ on the boundary $\mathcal{B}$. The AdS/CFT correspondence states that this is related to an expectation value in the CFT by

$$Z_{\mathrm{ST}} = \left\langle \mathrm{e}^{\int_{\mathcal{B}} \sum_i \Psi_{B,i} \mathcal{O}_i} \right\rangle_{\mathrm{cft}} . \tag{8.7}$$

The fields $\Psi_i$ act as sources for the boundary operators $\mathcal{O}_i$. The correlators of the $\Psi_i$ in the gravity theory match the ones of the $\mathcal{O}_i$ computed in the CFT.

Nowadays the AdS/CFT correspondence is strongly believed to hold in much more general cases than what was thought at the beginning, and even if the gauge theory is not supersymmetric. Indeed it has been conjectured that any relativistic conformal field theory on $\mathbb{R} \times S^{d-1}$ is dual to a theory of quantum gravity in an asymptotically $\mathrm{AdS} \times M$ spacetime where $M$ is some compact manifold (which can be trivial or not) [203]. However we do not know in general how to construct the bulk geometry from the dual gauge theory. Moreover as the gauge theory is strongly coupled, the duality has mostly been used to provide understanding on non-perturbative aspects of Yang–Mills theory from the weakly coupled gravity side rather than the other way around.

### 8.2.2 Black holes in AdS/CFT

As we have seen the study of thermodynamics properties of black holes has played a mayor role in the discovery of the AdS/CFT correspondence. It also gave some consistency verifications for example with the computation of the quasinormal modes. When a black hole is perturbed it emits gravitational waves that put it back in its initial state according to the no-hair theorem. These oscillations decay exponentially with time which is consistent with the picture the perturbation will eventually be absorbed by the black hole. The frequencies and damping times of these oscillations called the quasinormal modes are independent of the perturbation and depend only on the black hole. In term of the thermal behaviour of black holes this means that small perturbations of the black hole thermalize and the correlation functions vanish exponentially. AdS/CFT states that a large static AdS black hole is dual to an (almost) thermal state in some CFT. Perturbations of the black hole correspond to perturbations of this thermal QFT state and the quasinormal modes describe the thermalization of the system [204]. For the $\mathrm{AdS}_3$ black hole this has provided a good check of AdS/CFT [205]. Note that the exponential decay of correlation functions is only valid at large $N$. Indeed at finite $N$ the spectrum is finite and correlation functions will present Poincaré recurrences. This also implies that the spectrum of black holes is continuous oth-

erwise the Fourier transform of the correlation functions would be a sum of delta functions which does not describe an exponentially decaying function.

Rapidly AdS/CFT has revealed to be a powerful tool to deepen our understanding of black holes through gauge theory technology (in particular large $N$ methods). Of course a caveat is needed there: for the moment the duality still relies on the AdS geometry of the bulk theory even though there are attempts to enlarge it to de Sitter or Minkowski spaces. Then AdS/CFT mostly gives us information on AdS black holes which are different from the Minkowski ones (in particular because AdS boundary conditions look like putting the black hole in a box). However they still present the main thermodynamical characteristics in which we are interested for Minkowski black hole (an entropy given by $A/4$, thermal radiation, ...). For example the quasinormal modes of Minkowski black holes have a long-time behaviour slightly different from the AdS ones (a power-law tail) but the main characteristics remain. So it seems reasonable to hope that we can gain some useful insights thanks to AdS/CFT.

First of all AdS/CFT seems to give us an answer to the information paradox: as the black hole geometry is dual to some CFT which is unitary, the process of evaporation of the black hole must be described by some evolution in the CFT which is governed by a unitary S-matrix. Then black hole evaporation is unitary and the information is conserved [192]. However as argued in [92, 206] AdS/CFT tells us that the information is conserved but not by which process and does not explain us where Hawking's computation gets wrong.

A main characteristic of AdS is that it has finite energy and has an infinite potential wall at asymptotic infinity. This gives boundary conditions that look like those of a box. There are two kinds of black holes in AdS: small ones that evaporate and large ones that can be stable. Indeed large black hole can be in thermal equilibrium with the radiation. These black holes undergo a first order phase transition called the Hawking–Page transition [207]. Under the Hawking–Page temperature pure radiation is more stable while black hole becomes more stable for $T > T_{\mathrm{HP}}$. There are also two other critical temperatures $T_1 < T_{\mathrm{HP}} < T_2$: for $T < T_1$ black holes are not possible at all while for $T > T_2$ all radiation will necessarily collapse to form a black hole. Such phase transition needs to be recovered in any CFT supposed to be dual to a black hole. It has been argued that the Hawking–Page transition corresponds in the CFT side to a confinement-deconfinement transition [83]. Note that in this case the CFT is strongly coupled ($\lambda \gg 1$) since it is dual to a classical gravitational system.

### 8.2.3 Building quantum models for black holes from holography

The most natural way to construct models from holography is to look at D-branes construction from string theory. A stack of $N$ coincident D$p$-branes is described by $N \times N$ matrices which depend on $p+1$ coordinates. If we compactify all spatial dimensions this reduces to a simple quantum mechanical problem. Another way to build a purely quantum mechanical model is to look at a system of $N$ D0-branes described by the BFSS model [208]. In $d$ dimensions the action for the bosonic part is given by

$$S = \frac{1}{2g^2} \int \mathrm{d}t \, \mathrm{tr} \left( (D_t \phi^\mu)^2 + [\phi^\mu, \phi^\nu]^2 \right) \tag{8.8}$$

where the $\phi^\mu$ are $d$ $N \times N$ matrices. For $d = 10$ this action can also be obtained by dimensional reduction of ten-dimensional $\mathcal{N} = 1$ SYM or of four-dimension $\mathcal{N} = 4$ SYM. This action is clearly $\mathrm{SO}(d)$ invariant. At large $N$ the D0-brane model has been argued to be dual to M-theory at low energy [208] and to type IIA string theory at higher energy [209]. Recently it has been shown that this model was chaotic [210, 211]. This has lead to a model for black hole evaporation [211, 212].

More general models can be built which need to be $\mathrm{U}(N)$ invariant. Most of the time an additional $\mathrm{O}(D)$ invariance is imposed where $D$ is the number of matrices and is then interpreted as the dimension of the bulk. The matrices transform as vectors under this $\mathrm{O}(D)$ [183]. Large $N$ matrix models are unfortunately difficult to solve since in many cases we do

not know how to perform the sum over planar diagrams. To make progress it is natural to try to understand even simpler models. Doing that we should be careful not to oversimplify and to conserve the black hole like properties like continuous spectrum and quasinormal behaviour...

A first strategy is to notice that a large class of gauge theories exhibits black hole like behaviour in the large $N$ limit even at weak coupling. A first hint of this is that $\mathcal{N} = 4$ SYM exhibits even at weak coupling a deconfinement transition between a low temperature confining phase and a high temperature deconfined one. This has been shown to correspond to a Hagedorn transition [213, 214] where the partition function diverges due to an exponential growth of the number of states with the energy. It has been proposed that this transition could be dual to some kind of Hawking–Page transition of a stringy black hole [214, 215]. As we have seen gauge theories can describe black holes only at large $N$. At finite $N$ the spectrum is finite and there is no thermalization. This means that the large $N$ limit acts in some sense as a thermodynamical limit. Moreover the fact that the spectrum becomes continuous in this limit is highly non-trivial. It has been shown in [216] that, for a large class of models, this is a non-perturbative effect. Even at small t'Hooft coupling perturbation theory cannot be trusted. Indeed turning on a small perturbation will lift the enormous degeneracy of the free theory (of order $e^{N^2}$ for states of energy of order $N^2$). This is the reason why the system is mixing and thermalizes. This thermalization appears even for the weakly coupled theory which means that it could have some interpretation as some kind of stringy black hole in the dual theory. This strategy has been followed successfully for example for the Iizuka–Polchinski [217] and Iizuka–Okuda–Polchinski [218] models which find a qualitative change of behaviour at weak coupling between finite $N$ and large $N$ where they observe information loss. They deduce that the $1/N$ corrections dual to quantum gravity corrections in the bulk are crucial to resolve the information paradox. However these models are very simple and to not implement the singlet constraint so their results need to be generalised.

An other path of research, the one we will follow in the rest of this chapter, is to build simplified models where non-perturbative computations can be performed even if they seem quite different from the usual models of gauge theory. A recent proposal that has lead to a lot of interesting insights is the Sachdev–Ye–Kitaev (SYK) model. This is a much simpler model consisting of $N$ Majorana fermions with random couplings involving a few of these fermions at a time (this can be viewed as a fermionic vector model). Amazingly it has been shown to display some main properties of quantum black holes like chaos, quasinormal behaviour, emergent reparametrization symmetry... [219–221]. The Feynman graphs of these models were shown to be equivalent to the ones of fermionic random tensor models [222]. The latter were in turn related to matrix models at large $d$ in [183]. Interestingly one can also address the properties of $1/N$ corrections in this context [223].

In the following we will construct a quantum mechanical model that still preserves the matrix structure of (8.8) but is much simpler since the matrices are fermionic. Then the spectrum is finite and can be computed (numerically) exactly without relying on perturbation theory.

# Chapter 9

# A fermionic matrix model for black holes

In this chapter we describe a quantum mechanical model for black holes which is built using the ideas of 8.2.3. The main feature of this model is that it uses fermionic matrices which enables to keep the matrix structure of gauge theories while restricting the Hilbert space to be finite-dimensional. The model is described in section 9.1. We build the Hilbert space in section 9.2 and explain in section 9.3 how to compute the Hamiltonian. Finally we present some preliminary results in section 9.4.

## 9.1 Description of the model

We consider a model with two fermionic matrices of size $N$:

$$X = (a_{ij}^\dagger), \qquad Y = (b_{ij}^\dagger) \tag{9.1}$$

where $a_{ij}^\dagger$ and $b_{ij}^\dagger$ are fermionic creation operators. We set $\bar{X}_{ij} = a_{ji}$ and $\bar{Y}_{ij} = b_{ji}$. We will denote the set of the two matrices by $X^\mu = (X, Y)$. As explained in 8.2.3 we want a model for which high energy levels are highly degenerated so that a perturbation lifts the degeneracy and produces a quasi-continuum spectrum when $N$ is large. This forces us to work with at least two matrices since this cannot happen in a model of one matrix upon which a Hamiltonian is constructed. Indeed a perturbation will commute with the free Hamiltonian and will then never lift the degeneracy (since it can be diagonalized simultaneously with the free Hamiltonian). In this case the spectrum will remain discrete even in the large $N$ limit.

The Hilbert space is constructed by acting with the creation matrices on the vacuum $|0\rangle$. The dynamics of the model is determined by a Hamiltonian $H$ with a $\mathrm{U}(N) \times \mathrm{SO}(2)$ invariance, where the $\mathrm{U}(N)$ acts on $X$ and $Y$ by conjugation and the $\mathrm{SO}(2)$ rotates the pair $(X, Y)$. Invariants under $\mathrm{SO}(2)$ can then be built using the invariant tensors $\delta_{\mu\nu}$ and $\epsilon_{\mu\nu}$. We can replace the $\mathrm{SO}(2)$ by an $\mathrm{U}(1)$ acting on the complex combinations

$$Z = \frac{X + iY}{\sqrt{2}}, \quad \bar{Z} = \frac{X - iY}{\sqrt{2}}, \quad Z^\dagger = \frac{X^\dagger - iY^\dagger}{\sqrt{2}}, \quad \bar{Z}^\dagger = \frac{X^\dagger + iY^\dagger}{\sqrt{2}}. \tag{9.2}$$

We can consider that $Z$ creates one unit of $\mathrm{U}(1)$ charge while $\bar{Z}$ creates minus one unit. Similarly, $Z^\dagger$ destroys one unit of charge while $\bar{Z}^\dagger$ destroys minus one unit, i.e. increases the charge by one unit. We will denote by $l$ the corresponding conserved quantum number: it is the number of $Z$ minus the number of $\bar{Z}$ used to excite a state from the vacuum. For example, the state $\mathrm{tr}\, Z\bar{Z}Z|0\rangle$ has charge $l = 1$. The elements of the matrices $Z$ and $\bar{Z}$ will be denoted by $\theta_{ij}^\dagger$ and $\bar{\theta}_{ij}^\dagger$.

It follows from the gauge symmetry of any action of the type (8.8) that a U($N$) singlet constraint is imposed on the states. Hence the Fock will only contain states that can be written as multi-traces of the form

$$\mathrm{tr}(Z^{\alpha_1}\bar{Z}^{\beta_1}\cdots Z^{\alpha_n}\bar{Z}^{\beta_n})\ldots\mathrm{tr}(Z^{\rho_1}\bar{Z}^{\sigma_1}\cdots Z^{\rho_m}\bar{Z}^{\sigma_m})|0\rangle \tag{9.3}$$

We will say that a state is at level $n$ if it contains products of $n$ oscillators. The maximum level is $2^{2N^2}$ when all the oscillators are excited. Using group theory, we can compute the dimension of the Fock space and of the subspaces associated with a given level $n$. There is a symmetry between the level $n$ and the level $2^{2N^2}-n$ because there is an arbitrariness in the choice of the vacuum: one could have chosen level $2^{2N^2}$ as the vacuum and exchange the role of $a^\dagger, b^\dagger$ with $a, b$. Consequently, we only have to do computations up to level $2^{N^2}$. For $N=2$ and $N=3$ the dimensions are given in the following table:

| $N$ \ $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 2 | 2 | 6 | 10 | | | | | |
| 3 | 1 | 2 | 2 | 6 | 14 | 26 | 40 | 50 | 71 | 88 |

The most general quadratic, $U(N)$ and $U(1)$ invariant hermitian Hamiltonian is given by

$$\begin{aligned} H_2 = {} & \alpha\,\mathrm{tr}\,ZZ^\dagger + \beta\,\mathrm{tr}\,\bar{Z}\bar{Z}^\dagger + \gamma\,\mathrm{tr}(Z\bar{Z}+\bar{Z}^\dagger Z^\dagger) \\ & + \tilde{\alpha}\,\mathrm{tr}\,Z\,\mathrm{tr}\,Z^\dagger + \tilde{\beta}\,\mathrm{tr}\,\bar{Z}\,\mathrm{tr}\,\bar{Z}^\dagger + \tilde{\gamma}(\mathrm{tr}\,Z\,\mathrm{tr}\,\bar{Z}+\mathrm{tr}\,\bar{Z}^\dagger\,\mathrm{tr}\,Z^\dagger) \end{aligned} \tag{9.4}$$

$\mathrm{tr}\,ZZ^\dagger$ counts the number of $Z$-creation operators used to create a state from the vacuum while $\mathrm{tr}\,\bar{Z}\bar{Z}^\dagger$ counts the number of $\bar{Z}$-creation operators. Thus $\mathrm{tr}\,ZZ^\dagger + \mathrm{tr}\,\bar{Z}\bar{Z}^\dagger$ measures the total number $n$ of creation operators used to create a given state from the vacuum i.e. the level of the state. For simplicity we will take $\alpha=\beta=1$ and $\tilde{\alpha}=\tilde{\beta}=\tilde{\gamma}=0$. The quadratic Hamiltonian is then the sum of a free Hamiltonian

$$H_0 = \mathrm{tr}\left(ZZ^\dagger + \bar{Z}\bar{Z}^\dagger\right) \tag{9.5}$$

and an interaction term

$$H_{\mathrm{int},2} = \gamma\,\mathrm{tr}(Z\bar{Z}+\bar{Z}^\dagger Z^\dagger). \tag{9.6}$$

Note that the term $H_{\mathrm{int},2}$ does not change the degeneracy of the spectrum. This can be seen by doing a Bogolubov-type transformation. Indeed, defining

$$U = \cos(\vartheta)Z + \sin(\vartheta)\bar{Z}^\dagger, \quad V = \cos(\vartheta)\bar{Z} - \sin(\vartheta)Z^\dagger \tag{9.7}$$

we see that $U^i_j$ and $(U^\dagger)^k_l$ on one side and $V^i_j$ and $(V^\dagger)^k_l$ on an another satisfy standard anticommutation relations while $U$ and $U^\dagger$ anticommute with $V$ and $V^\dagger$. We have

$$\mathrm{tr}\left(UU^\dagger + VV^\dagger\right) = \cos(2\vartheta)\,\mathrm{tr}\left(ZZ^\dagger + \bar{Z}\bar{Z}^\dagger\right) + \sin(2\vartheta)\,\mathrm{tr}\left(Z\bar{Z}+\bar{Z}^\dagger Z^\dagger\right) + 2N^2\sin^2(\vartheta). \tag{9.8}$$

Choosing $\vartheta$ such that $\tan(2\vartheta)=\gamma$ we have

$$H_2 = \sqrt{1+\gamma^2}\,\mathrm{tr}\left(UU^\dagger + VV^\dagger\right) - N^2\left(\sqrt{1+\gamma^2}-1\right). \tag{9.9}$$

The spectrum of $H_2$ has then the same degeneracy as the one of $H_0$, the eigenvalues being shifted now to $\sqrt{1+\gamma^2}\,n - N^2\left(\sqrt{1+\gamma^2}-1\right)$ instead of $n$. Consequently we do not expect that turning on $\gamma$ will change the qualitative behaviour of the system.

As U(1) invariance forbids any cubic interaction the simplest interactions terms with more than two oscillators are at least quartic. As there are around a hundred possible quartic terms we need to choose some particular combination. A natural choice is motivated by the

interaction term in $\mathcal{N}=4$ SYM or in the D0-brane (8.8) action. It is $\mathrm{tr}[\phi^\mu,\phi^\nu][\phi_\mu,\phi_\nu]$ where we identify $\phi^\mu$ with a matrix-valued position in the $\mu$ direction, i.e. in terms of the creation and annihilation matrices as $\phi^1 = X + X^\dagger$ and $\phi^2 = Y + Y^\dagger$. One finds

$$H_{\mathrm{int},4} = \mathrm{tr}[\phi^\mu,\phi^\nu][\phi_\mu,\phi_\nu] = 4\,\mathrm{tr}(Z+\bar{Z}^\dagger)(\bar{Z}+Z^\dagger)(\bar{Z}+Z^\dagger)(Z+\bar{Z}^\dagger) - 8N^3 \tag{9.10}$$

Interestingly other natural terms such as $\mathrm{tr}\{\phi^\mu,\phi^\nu\}\{\phi_\mu,\phi_\nu\}$ or $\mathrm{tr}(\phi^\mu\phi_\mu)^2$ can be also expressed with the same expression $\mathrm{tr}(Z+\bar{Z}^\dagger)(\bar{Z}+Z^\dagger)(\bar{Z}+Z^\dagger)(Z+\bar{Z}^\dagger)$:

$$\begin{cases} \mathrm{tr}\{\phi^\mu,\phi^\nu\}\{\phi_\mu,\phi_\nu\} = -4\,\mathrm{tr}(Z+\bar{Z}^\dagger)(\bar{Z}+Z^\dagger)(\bar{Z}+Z^\dagger)(Z+\bar{Z}^\dagger) + 12N^3 + 2N \\ \mathrm{tr}(\phi^\mu\phi_\mu)^2 = 2\,\mathrm{tr}(Z+\bar{Z}^\dagger)(\bar{Z}+Z^\dagger)(\bar{Z}+Z^\dagger)(Z+\bar{Z}^\dagger) + 4N^3 \end{cases} \tag{9.11}$$

The total Hamiltonian is then

$$H = H_0 + \gamma H_{\mathrm{int},2} + g H_{\mathrm{int},4}. \tag{9.12}$$

The fact that the matrices are fermionic implies some relations:

$$\mathrm{tr}(X^{2p}) = 0, \forall p \in \mathbb{N}^*, \qquad X^{2N} = 0. \tag{9.13}$$

Moreover we know by Cayley–Hamilton theorem that the power $N$ of a bosonic matrix (i.e. a product of an even number of fermionic ones) is a linear combination of lower powers of this matrix. The traces (9.3) are then not independent: they form an overcomplete set. To construct the Hilbert space we need to take into these relations. To get a basis of the Hilbert space we will then use Gram-Schmidt orthonormalization procedure.

## 9.2 Construction of the Hilbert space

There are two parts of the basis: the part with level $\leq N^2$ and the part with level $> N^2$. As already mentioned it is sufficient to construct only the first part.

The first step is to generate all inequivalent single-traces operators. This requires several precautions. First as in the case with bosonic matrices traces that are related by a circular permutation of their arguments have to be included only once. Doing this amounts to solve the Polya problem, which is already implemented in Mathematica. Secondly we need to take care of the relations (9.13) that are specific to fermionic matrices in order to avoid carrying states that are trivially vanishing. Finally we also have to discard states that contain a bosonic matrix to the power $N$ since this state is a linear combination of over states. To be sure to take into account all these constraints and to obtain a minimal list of single-traces the simplest is to build it level by level.

Next we use this list of single-traces to construct a basis containing all single-trace and multi-trace states. We first have to delete states which contain the product of an even number of some fermionic trace since they automatically vanish. Then we implement Gram-Schmidt orthogonalization scheme. The scalar product is defined by the usual one on Grassmann algebras:

$$\langle 0|\theta\ldots\theta\theta^\dagger\ldots\theta^\dagger|O\rangle = \begin{cases} 0 \text{ if all the } \theta \text{ match all the } \theta^\dagger \\ 1 \text{ otherwise.} \end{cases} \tag{9.14}$$

With this process we get a basis of the Hilbert space classed by level and by charge. It is then easy to change the order to get a basis in which in will be simpler to compute the Hamiltonian that is classed by charge and then by level.

A main difficulty is that comparing all the factors in a Grassmannian expression takes a time which is exponential in the length of this expression. For $N=3$ we only need to go to level $n=9$ which is still doable by this method but the level $n=16$ needed for $N=4$ and beyond is clearly out of reach. Another idea would be to use Wick theorem to

compute the scalar products. However although it is much faster for levels smaller than 10 it is much slower for higher levels since the complexity is factorial in $n$. Then it seems that to do computations with higher-dimensional matrices we need to think of another way to construct the Hilbert space. Analytical expressions for the basis do exist in the bosonic case and can be extended to the fermionic one [224] but the expressions are much too complicated to be used for numerical simulations.

As an example we give an orthogonal basis of the Hilbert space in the case $N = 2$ (for $N = 3$ there are several hundreds of states and we did not find relevant to insert them there). We group the states by charge $l$ and then by level $n$. Indeed U(1) invariance of the Hamiltonian means that the interactions will always preserve $l$ while the levels $n$ will be mixed. We only write the results for $l \leq 0$ since the ones for $l > 0$ are obtained by exchanging $Z$ and $\bar{Z}$

- $l = 0$
  - $n = 0$: the vacuum $|0\rangle$
  - $n = 2$: $\operatorname{tr} Z\bar{Z}|0\rangle$, $\left(-\frac{1}{2}\operatorname{tr} Z\bar{Z} + \operatorname{tr} Z \operatorname{tr} \bar{Z}\right)|0\rangle$
  - $n = 4$: $\operatorname{tr} Z^2\bar{Z}^2|0\rangle$, $\operatorname{tr} Z\bar{Z}Z\bar{Z}|0\rangle$, $\left(-\frac{1}{2}\operatorname{tr} Z\bar{Z}Z\bar{Z} + \operatorname{tr} Z \operatorname{tr} Z\bar{Z}^2\right)|0\rangle$, $\operatorname{tr} Z \operatorname{tr} \bar{Z} \operatorname{tr} Z\bar{Z}|0\rangle$
  - $n = 6$: $\operatorname{tr} Z^\dagger\bar{Z}^\dagger|\Omega\rangle$, $\left(-\frac{1}{2}\operatorname{tr} Z^\dagger\bar{Z}^\dagger + \operatorname{tr} Z^\dagger \operatorname{tr} \bar{Z}^\dagger\right)|\Omega\rangle$
  - $n = 8$: the Dirac sea : $|\Omega\rangle = \theta^\dagger_{11}\theta^\dagger_{12}\theta^\dagger_{21}\theta^\dagger_{22}\bar{\theta}^\dagger_{11}\bar{\theta}^\dagger_{12}\bar{\theta}^\dagger_{21}\bar{\theta}^\dagger_{22}|0\rangle$
- $l = -1$
  - $n = 1$: $\operatorname{tr} \bar{Z}|0\rangle$
  - $n = 3$: $\operatorname{tr} Z\bar{Z}^2|0\rangle$, $\operatorname{tr} \bar{Z} \operatorname{tr} Z\bar{Z}|0\rangle$
  - $n = 5$: $\operatorname{tr} Z^\dagger(\bar{Z}^\dagger)^2|\Omega\rangle$, $\operatorname{tr} \bar{Z}^\dagger \operatorname{tr} Z^\dagger\bar{Z}^\dagger|\Omega\rangle$
  - $n = 7$: $\operatorname{tr} \bar{Z}^\dagger|\Omega\rangle$
- $l = -2$
  - $n = 2$: no state
  - $n = 4$: $\operatorname{tr} Z^3 \operatorname{tr} \bar{Z}|0\rangle$ and $\operatorname{tr} Z^2\bar{Z} \operatorname{tr} Z|0\rangle$
  - $n = 6$: no state
- $l = -3$
  - $n = 3 \operatorname{tr} \bar{Z}^3$
  - $n = 5 \operatorname{tr} (Z^\dagger)^3$
- $l = -4$
  - $n = 4 \operatorname{tr} \bar{Z} \operatorname{tr} \bar{Z}^3$

## 9.3 Computation of Hamiltonians

The computation of the Hamiltonians (9.6) and (9.10) requires to do the scalar product of states of different levels and in particular states that are written as created from the vacuum and states written as annihilation operators action on the Dirac sea.

A first difficulty is that a typical state involves a sum of many products of Grassmannian. It is then much slower to compute the scalar product of two such states than to compute the scalar product of a state with an element of the Grassmann basis. To reduce as much as possible the number and the complexity of the scalar products we need to compute we will use a projection of the space of singlets on a subspace of the Grassmannians. Let's denote by

$|e_I\rangle, I = 1, \ldots, 2^{N^2}$ the elements of the Grassmann basis (that is product of Grassmannian of the form $\theta^\dagger_{i_1 j_1} \ldots \theta^\dagger_{i_p j_p}$) at level $n \leq N^2$ and by $|\phi_\alpha\rangle, \alpha = 1, \ldots, d$ the elements of the orthogonal basis of singlets ($d$ is the dimension of one subspace of interest, for example the subspace of states with $l = 0$). We have $\langle\phi_\alpha|\phi_\beta\rangle = c_\alpha \delta_{\alpha\beta}$. We then write

$$|\phi_\alpha\rangle = \sum_{I=1}^{2^{2N^2}} {M_\beta}^I |e_I\rangle = \sum_{I=1}^{d} {M_\beta}^I |e_I\rangle + \sum_{\tilde{J}=d}^{2^{2N^2}} {M_\beta}^{\tilde{I}} |e_{\tilde{I}}\rangle \tag{9.15}$$

where the $|e_I\rangle, I = 1, \ldots, d$ are chosen such that ${M_\alpha}^I$ is invertible. Of course for the states at level $n > N^2$, we will take the same Grassmannians for the projection but now considering them as annihilation operators acting on the Dirac sea and replacing the $\theta^\dagger$ by $\theta$.

Our goal is to compute the matrix elements $H_{\alpha\beta}$ of an Hamiltonian $H$ that is the $H_{\beta\alpha}$ defined by

$$H|\phi_\alpha\rangle = H_{\beta\alpha}|\phi_\beta\rangle. \tag{9.16}$$

We have

$$H|\phi_\alpha\rangle = H_{\beta\alpha}|\phi_\beta\rangle = H_{\beta\alpha}\left(\sum_{J=1}^{d} {M_\beta}^J |e_J\rangle + \sum_{\tilde{J}=d}^{2^{2N^2}} {M_\beta}^{\tilde{J}} |e_{\tilde{J}}\rangle\right) \tag{9.17}$$

thus, for $I = 1, \ldots, d$

$$\langle e_I|H|\phi_\alpha\rangle = H_{\beta\alpha}{M_\beta}^I. \tag{9.18}$$

If we note $\hat{h}$ the matrix whose elements are $\hat{h}_{I\alpha} = \langle e_I|H|\phi_\alpha\rangle$, then

$$\hat{h}^T = (H^T M)^T \Leftrightarrow H = (M^T)^{-1}\hat{h}. \tag{9.19}$$

We then see that we can compute the scalar products $\langle e_I|H|\phi_\alpha\rangle$ instead of $\langle\phi_\alpha|H|\phi_\beta\rangle$ which is much simpler.

Let's consider a subspace of the Hilbert space consisting of states of a given $l$ and $\tilde{d}$ be the number of states with level $n \leq N^2$. The matrix $\hat{h}_{I\alpha}$ can be written by block as

$$\hat{h} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{9.20}$$

where, denoting by $\tilde{e}_I$ (resp. $\tilde{\phi}_\alpha$) the state $e_I$ (resp. $\phi_\alpha$) where creation operators are replaced by annihilation operators, the blocks are constituted by matrix elements of the form :

- $A_{ij} = \langle 0|e_I H \phi_\alpha|0\rangle$, $I \leq \tilde{d}$, $\alpha \leq \tilde{d}$
- $B_{ij} = \langle 0|e_I H \tilde{\phi}_\alpha|\Omega\rangle$, $I \leq \tilde{d}$, $\alpha > \tilde{d}$
- $C_{ij} = \langle \Omega|\tilde{e}_I H \phi_\alpha|0\rangle$, $I > \tilde{d}$, $\alpha \leq \tilde{d}$
- $D_{ij} = \langle \Omega|\tilde{e}_I H \tilde{\phi}_\alpha|\Omega\rangle$, $I > \tilde{d}$, $\alpha > \tilde{d}$

At this point we only know how to compute scalar products between states written in the same language: creation operators acting on the vacuum on both side or annihilation operators acting on the Dirac sea on both sides. To compute matrix elements of the form $\langle 0|e_I H \tilde{\phi}_\alpha|\Omega\rangle$ or $\langle \Omega|\tilde{e}_I H \phi_\alpha|0\rangle$, we just have to know how to transform $e_I|0\rangle$ to $\mathcal{O}|\Omega\rangle$ and $\tilde{e}_I|\Omega\rangle$ to $\tilde{\mathcal{O}}|0\rangle$ where $\mathcal{O}$ is an operator which contains only annihilation operators and $\tilde{\mathcal{O}}$ contains only creation operators. We have

$$|\Omega\rangle = \theta^\dagger_{11}\theta^\dagger_{12}\theta^\dagger_{21}\theta^\dagger_{22}\bar{\theta}^\dagger_{11}\bar{\theta}^\dagger_{12}\bar{\theta}^\dagger_{21}\bar{\theta}^\dagger_{22}|0\rangle \equiv c^\dagger_1 \ldots c^\dagger_{2N^2}|0\rangle. \tag{9.21}$$

We will always take the $e_I$ to be ordered chains of creation operators. We then just have to know that if $i_1 < \ldots < i_k$:

$$c_{i_1} \ldots c_{i_k} |\Omega\rangle = (-1)^{i_1 - 1} \ldots (-1)^{i_k - 1} c_1^\dagger \ldots c_{i_1-1}^\dagger c_{i_1+1}^\dagger \ldots c_{i_k-1}^\dagger c_{i_k+1}^\dagger \ldots c_{2N^2}^\dagger |0\rangle. \tag{9.22}$$

The same formula works to go from annihilation operators acting on the Dirac sea to creation operators on the vacuum. We now just have to compute matrix elements of the form $\langle 0|e_I H \phi_\alpha|0\rangle$ or $\langle \Omega|\tilde{e}_I H \tilde{\phi}_\alpha|\Omega\rangle$. For the first one, we have to put $H$ in anti-normal order (annihilation operators on the left, creation on the right) while for the second one, we have to put it in normal order.

## 9.4 Results

The first step of the analysis is to compute the spectrum of the theory. As expected the qualitative features are not modified when turning on $\gamma$ which will then be set to 0. We see in figure 9.1 that as soon as we turn on a small coupling constant $g$ for the quartic interaction the degeneracy is lifted and the spectrum becomes quasi-continuous. This is encouraging since it was not obvious that taking $N$ as small as 3 will enable us to see such a drastic effect.

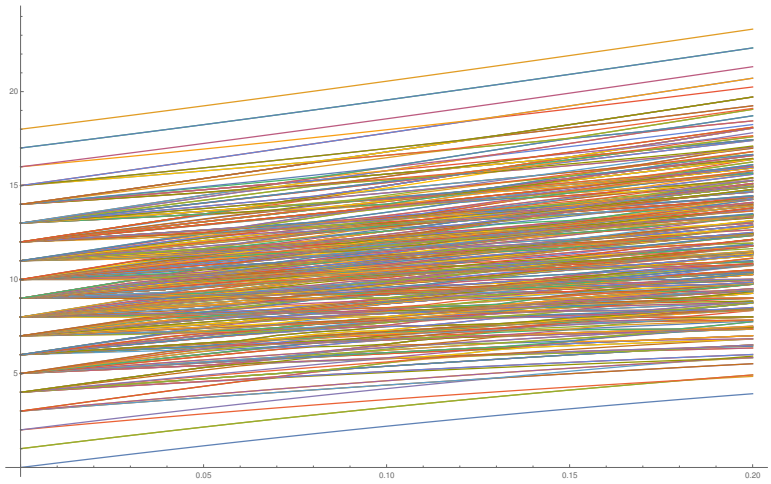


Figure 9.1 – Evolution of the spectrum for coupling $g$ from 0 to 0.2.

We also observe that when the coupling grows gaps appear in the spectrum (figure 9.2). This may be due to the fact that our interaction term is not sufficiently generic when the matrices are fermionic and so the mixing is not complete.

The continuous aspect of the spectrum encourages to investigate whether we can observe a black hole like phase transition. This indeed seems to be the case as we see in figure 9.3 where we plot the free energy as a function of the temperature. For each curve there appears to be a critical temperature $T_c$ below which the free energy is almost constant and above which the free energy decreases rapidly with the temperature. This critical temperature decreases when the coupling grows and seems to disappear for $g \approx 0.8$. We recover this phase transition in the Green's function

$$G(t) = \frac{1}{Z} \operatorname{tr} \mathrm{e}^{-H/T} \, \mathrm{e}^{itH} \, A \, \mathrm{e}^{-itH} A - \left( \frac{1}{Z} \operatorname{tr} \mathrm{e}^{-H/T} \, A \right)^2 \tag{9.23}$$

of the operator $A = \operatorname{tr}(Z\bar{Z} + \bar{Z}^\dagger Z^\dagger)$ ($Z$ denotes the partition function). We see in figure 9.4 that for $T < T_c$ (blue curve) the Green's function just oscillates while for $T \geq T_c$ (orange

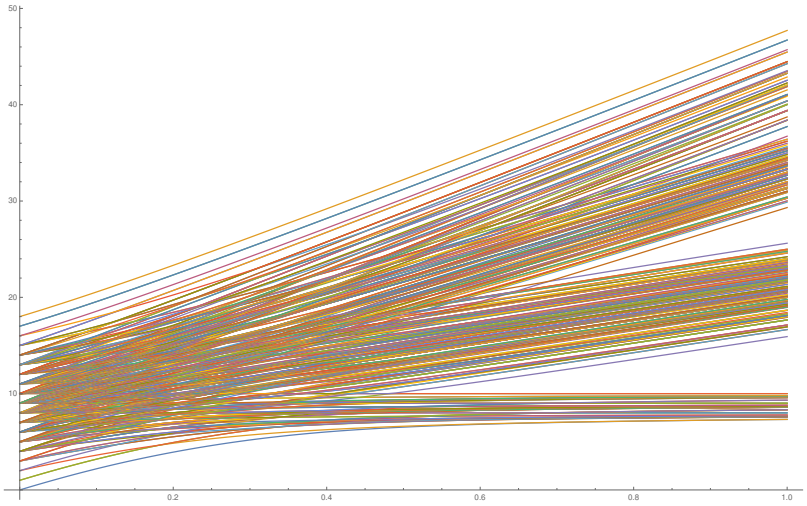


Figure 9.2 – Evolution of the spectrum for coupling $g$ from 0 to 1.

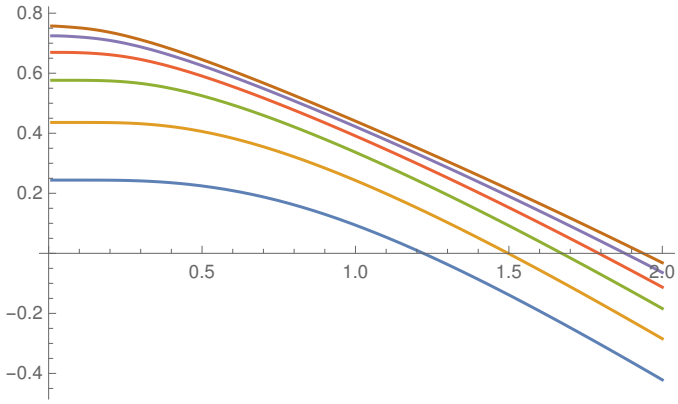


Figure 9.3 – Evolution of the free energy with the temperature (the coupling increases with the origin point of the free energy).

and green curves) we begin to see an exponential decrease which could indeed indicate thermalization. It appears desirable to increase $N$ to see this effect more clearly.

Despite its simplicity this model presents encouraging features for modelling quantum black holes. However the study needs to be developed further. In particular it would be interesting to see whether a more generic interaction term modifies the thermal behaviour of the system. Another interesting point would be to explore whether we can see some trace of quantum chaos. Of course being able to extend the computations to $N = 4$ would lead significant progress since the number of states that mix in this case is by far much higher.

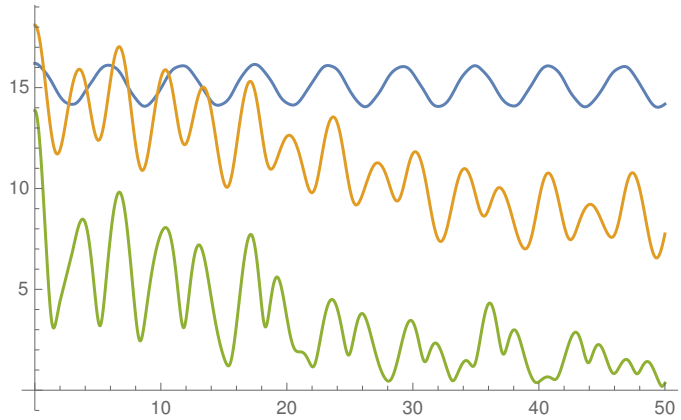


Figure 9.4 – Time evolution of the absolute value of the Green's function for $g = 0.2$.

# Part III

# Appendices

# Annexe A

# Résumé en français

## A.1 Nécessité d'une théorie de gravité quantique

Au début du vingtième siècle deux théories majeures ont changé radicalement notre compréhension des lois de la physique : la mécanique quantique (et la théorie quantique des champs) et la relativité générale. La première concerne les interactions fondamentales entre particules à très courtes distances : l'électromagnétisme et les interactions faible et forte. La relativité générale, quant à elle, décrit les structures de l'Univers à grande échelle en reliant la gravitation à la géométrie de l'espace-temps. Les deux théories sont très bien vérifiées expérimentalement, leurs derniers succès étant respectivement la découverte en 2012 par les détecteurs Atlas et CMS au LHC du boson de Brout–Englert–Higgs prédit en 1964 et l'observation des ondes gravitationnelles à Ligo et Virgo en 2016.

Cependant, ces deux théories ne semblent pas cohérentes l'une avec l'autre. En effet, les formalismes mathématiques qui les décrivent sont très différents et paraissent incompatibles. Tandis que la relativité générale est une théorie classique où l'espace-temps est dynamique, la théorie quantique des champs se place dans un espace-temps fixe avec, en particulier, une direction de temps fixée. Mais, comme toutes les particules interagissent gravitationnellement, on s'attend à pouvoir décrire toutes les interactions avec un formalisme cohérent. De plus, la relativité générale ne semble pas valable à courtes distances (de l'ordre de l'échelle de Planck). En effet, elle prédit des singularités dans l'espace-temps (Big-Bang, trous noirs) où la courbure devient infinie et où la théorie ne fait plus sens. En outre, des calculs semi-classiques couplant la relativité générale classique à une théorie quantique des champs révèlent des incohérences, la plus importante étant la radiation de Hawking des trous noirs qui remet en question l'unitarité de la mécanique quantique, l'un de ses postulats fondamentaux. Par conséquent, la recherche d'une théorie de gravité quantique qui unifie la relativité générale et la mécanique quantique est l'un des défis majeurs de la physique théorique du XXI$^{e}$ siècle.

Mais après presque un siècle de recherche, la gravité quantique nous échappe toujours et il n'existe pour le moment aucune théorie vérifiée expérimentalement. Il faut dire que le physicien en quête de gravité quantique se heurte rapidement à de nombreuses difficultés [1–4]. En premier lieu, une quantification directe de l'action d'Einstein–Hilbert n'est pas possible car elle est perturbativement non-renormalisable : à chaque ordre dans la théorie des perturbations, de nouvelles divergences apparaissent et il faut introduire une infinité de contre-termes pour les supprimer. Un premier remède serait l'existence d'un point fixe UV non-gaussien qui rendrait la théorie renormalisable de façon non-perturbative (sécurité asymptotique). Un autre serait que la relativité générale n'est qu'une théorie effective valide uniquement à basse énergie et qui doit être complétée (par des termes avec des dérivées d'ordre supérieur, d'autres types de matière, de la supersymétrie, des cordes…) pour être rendue finie dans l'ultraviolet ou renormalisable. Un point encore plus difficile est la formu-

lation d'une théorie des champs indépendante de l'espace-temps de fond, et en particulier sans temps fixé. En outre, on ne sait toujours pas quels devraient être les observables d'une théorie de gravité quantique puisque l'invariance par difféomorphismes de la relativité générale ne permet pas de donner un sens à des observables locaux. De plus, un postulat essentiel de la théorie quantique des champs est la causalité, ce qui signifie que des champs séparés par un intervalle de genre espace ne s'influencent pas mutuellement et commutent. Mais si la métrique est soumise à des fluctuations quantiques, les notions d'intervalles de genre temps ou de genre espace n'ont plus de sens. Ces différents problèmes ont conduit au développement de différents formalismes qui résolvent diverses questions : la théorie des cordes [5–7], la gravité quantique à boucles [8–10], la géométrie non-commutative [11–13], les triangulations dynamiques [14, 15], les ensembles causaux [16–19], la sécurité asymptotique [20–22], les modèles de matrices et les tenseurs aléatoires [23–26]...

Il y a peu de doutes que la compréhension complète de l'Univers, en particulier en ce qui concerne la cosmologie primordiale et les trous noirs, nécessite une théorie de gravité quantique. Une telle théorie pourrait aussi permettre de résoudre le problème de la constante cosmologique. Actuellement, on ne peut pas expliquer pourquoi la constante cosmologique, bien que non nulle, est si petite, de l'ordre de $10^{-52}\,\mathrm{m}^{-4}$ alors qu'un calcul naïf de théorie quantique des champs trouve une valeur plus grande d'une cinquantaine d'ordres de grandeur [27]. La gravité quantique pourrait aussi être une solution pour supprimer les divergences à courtes distances présentes dans la plupart des théories quantiques des champs (y compris celles qui sont renormalisables mais pas finies) en introduisant une coupure fondamentale à l'échelle de Planck. Il semble aussi, en particulier en théorie des cordes ou en géométrie non-commutative, que cette nouvelle théorie pourrait résoudre d'autres problèmes de l'actuel Modèle standard de la physique des particules. Premièrement, le nombre important de paramètres libres (dix-neuf) du Modèle standard n'est pas satisfaisant et l'on souhaiterait pouvoir décrire la physique avec le plus petit nombre possible de paramètres. En particulier, une théorie complète devrait pouvoir prédire les masses des particules fondamentales, ce qui peut être réalisé par une théorie de grande unification. Le fait que les trois constantes de couplage du Modèle standard deviennent presque égales à une énergie autour de $10^{15}\,\mathrm{GeV}$ est un indice fort pour l'existence d'une telle théorie (et l'accord est encore meilleur si l'on inclut la supersymétrie). Parmi les questions qui pourraient trouver une réponse grâce à la grande unification, on trouve, entre autres, l'origine du groupe de jauge $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$, l'explication du nombre de familles de fermions ou le mécanisme qui donne leurs masses aux neutrinos. A cette heure, la grande unification reste un problème ouvert et aucune physique au-delà du Modèle standard n'a encore été détectée.

La gravité quantique n'est pour l'instant pas accessible expérimentalement ce qui constitue une difficulté majeure. En effet, on s'attend à ce que les effets de la gravité quantique deviennent importants à des énergies autour de l'échelle de Planck ($m_P = \sqrt{\frac{\hbar c}{G}} \sim 10^{19}\,\mathrm{GeV}$) ce qui est loin des énergies actuellement à portée des accélérateurs de l'ordre de $10\,\mathrm{TeV}$. Il n'y a donc pour le moment pas d'observation expérimentale qui puisse nous guider sur le bon chemin. Néanmoins, les meilleurs espoirs expérimentaux pour le futur viennent de la cosmologie [28] (mesures des anisotropies dans le CMB, détection des radiations de trous noirs primordiaux, analyse des ondes gravitationnelles...).

## A.2 Gravité quantique à deux dimensions

### A.2.1 Motivations

Dans cette thèse, deux directions ont été suivies. La plus importante est l'étude de la gravité quantique à deux dimensions. En effet, une façon de s'attaquer au problème de la gravité quantique est d'étudier des modèles en dimension inférieure : comme mentionné plus haut, aucun résultat expérimental n'est disponible, il fait donc sens d'étudier des modèles semblables mais plus simples. On peut en effet espérer que la gravité quantique à $d < 4$ di-

mensions n'est pas fondamentalement différente de celle à quatre dimensions et partage avec cette dernière des propriétés universelles. Ce type d'approche s'est montré fructueux dans le cas de la théorie de Yang–Mills. A deux dimensions, on peut par exemple calculer exactement diverses propriétés UV et IR reliées aux anomalies dans le modèle de Schwinger [29, 30]. Ce modèle donne aussi une ébauche d'explication pour le confinement des quarks [31]. On retrouve en gravité quantique à deux dimensions plusieurs caractéristiques particulières de la théorie de Yang–Mills à deux dimensions. Toutes deux ont beaucoup moins de degrés de liberté que leurs analogues à quatre dimensions : il n'y a pas de gluon dans Yang–Mills à deux dimensions [32] tandis que seul le facteur conforme (aussi appelé mode de Liouville) peut se propager à deux dimensions (cf. chapitre 2). Les effets topologiques jouent des rôles importants dans les deux théories. De plus, les groupes de symétrie sont souvent élargis à deux dimensions : Yang–Mills bidimensionnel n'est pas seulement invariant sous son groupe de jauge local mais aussi sous les difféomorphismes préservant l'aire [32] tandis que l'algèbre conforme locale à deux dimensions devient l'algèbre de Virasoro de dimension infinie. De même la U-dualité de la supergravité maximale ($\mathcal{N} = 16$) est donnée par la symétrie affine $E_{9(9)}$, tandis que pour $d \geq 3$ la U-dualité est un groupe de Lie de dimension finie [33].

Selon ce principe, la gravité quantique bidimensionnelle est un modèle jouet intéressant pour son analogue à quatre dimensions. En effet, elle présente plusieurs caractéristiques qui la rendent plus aisément manipulable : de nombreux calculs peuvent être faits exactement, en particulier les corrections quantiques dues à l'interaction entre la matière et la gravité, et elle est souvent renormalisable.

En gravité quantique à deux dimensions, la fonction de partition est donnée par la somme des contributions de toutes les surfaces inéquivalentes d'une topologie donnée, puis par la somme sur les topologies. Combiner ces deux sommes est un problème hautement non-trivial. Pour le résoudre, deux classes de modèles ont été développés. Dans l'approche discrète, cette somme est remplacée par une somme sur toutes les triangulations inéquivalentes de surfaces. Ces triangulations peuvent être interprétées comme des graphes en ruban et décrites comme des graphes de Feynman pour un modèle de matrices. Cela permet de transformer une intégrale fonctionnelle compliquée en une intégrale beaucoup plus simple où l'on intègre sur les configurations matricielles [34–36]. Nous ne développerons pas cette stratégie dans cette thèse (bien que nous reviendrons au chapitre 8 sur des modèles de matrices mais dans un but complètement différent) et nous nous concentrerons sur la deuxième approche, celle des modèles dans le continu qui traitent la gravité quantique comme une théorie quantique des champs. Cela nécessite de faire attention aux degrés de liberté redondants provenant de l'invariance de jauge sous les difféomorphismes. Pour cela, il faut fixer une jauge (généralement la jauge conforme), appliquer la procédure de Faddeev–Popov puis sommer sur le degré de liberté restant, le mode de Liouville dans le cas de la jauge conforme.

La gravité quantique à deux dimensions est particulièrement simple quand elle est couplée uniquement à de la matière conforme. Polyakov a montré que, dans ce cas, l'action effective de Wess–Zumino est donnée par l'action de Liouville [37]. Cette théorie est maintenant bien définie et bien comprise, en particulier les exposants critiques [38, 39], le spectre [40–44] et certaines propriétés des fonctions de corrélations [45–47] ont été calculés. En utilisant une description non-lagrangienne [48–50] le bootstrap conforme de la théorie de Liouville a montré qu'il s'agissait bien d'une théorie des champs conforme cohérente pour toute charge centrale complexe (voir aussi [51]). Récemment, une définition mathématique non-perturbative de la théorie de Liouville a été établie [52–54]. Voir [49, 55–58] pour des revues sur le sujet.

Bien que la gravité à quatre dimensions ne soit pas invariante d'échelle (et encore moins invariante conforme), le couplage de la gravité bidimensionnelle à de la matière non-conforme a été peu investigué dans la littérature. Le cas de la gravité couplée à une théorie conforme perturbée par des opérateurs primaires a été étudié grâce à l'ansatz de DDK [44, 59–61], mais cette approche ne s'intègre pas dans le formalisme de la gravité quantique à deux dimensions dans la jauge conforme présenté au chapitre 4.

En outre, l'étude de théories véritablement non-conformes, où la perturbation n'est pas

un opérateur primaire (comme le terme de masse pour un champ scalaire) n'a commencé que récemment. Quand un champ scalaire massif est couplé à la gravité (éventuellement pas couplage non-minimal et avec un terme linéaire) d'autres fonctionnelles contribuent à l'action gravitationnelle [62–64]. En particulier, une expression explicite a été obtenue au premier ordre dans un développement à petite masse. Les deux fonctionnelles qui apparaissent sont les actions de Mabuchi et Aubin–Yau, bien connues des mathématiciens [65, 66] qui étudient les surfaces de Riemann compactes. On les rencontre également dans la description de l'effet Hall quantique [64, 67–69]. Plus récemment, une expression exacte pour l'action gravitationnelle a été obtenue [70]. Dans ce contexte, la gravité à deux dimensions est reformulée dans le formalisme de Kähler ce qui permet de définir rigoureusement l'intégrale fonctionnelle [62, 63, 71] et de calculer de possibles déviations de la formule de KPZ pour la susceptibilité de la corde [71–73].

Pour finir, la gravité quantique bidimensionnelle a aussi des applications dans l'étude intrinsèque de la gravité quantique. En effet, la théorie des cordes, l'une des approches les plus abouties de gravité quantique, peut être formulée, dans le formalisme de la surface d'univers, comme de la matière couplée à la gravité quantique bidimensionnelle. En outre, plusieurs approches, dont la théorie des cordes, la gravité quantique à boucles, les triangulations dynamiques, la géométrie non-commutative, la sécurité asymptotique et les ensembles causaux, semblent montrer que la gravité quantique subit, à haute énergie (autour de l'échelle de Planck) une réduction dimensionnelle à deux dimensions [74, 75].

### A.2.2 Jauge conforme et action gravitationnelle

Le travail principal de cette thèse a consisté à poursuivre l'étude de l'action gravitationnelle obtenue quand de la matière massive est couplée à la gravité. Comme la fonctionnelle de Mabuchi a l'air de jouer un rôle essentiel, il est important de comprendre ses propriétés physiques et de suivre le même programme que ce qui a été fait pour la théorie de Liouville. Notre objectif a donc été de calculer le spectre de l'action de Mabuchi. Pour cela, les techniques des théories conformes ne peuvent pas être utilisées. Nous avons donc opté pour l'approximation du minisuperespace qui permet de calculer le spectre et les fonctions à 2 et 3 points d'une théorie dans la limite semi-classique [76, 77]. Ce faisant, nous avons été amenés à étudier la théorie de Mabuchi sur le cylindre. Comme cette théorie est mal définie sur une surface de Riemann de genre 1, nous avons dû travailler avec une version redimensionnée. Ce problème nous a fait réfléchir à la généralisation des résultats de [63] au calcul de l'action gravitationnelle pour un champ scalaire massif sur une surface de Riemann avec bords [78].

Considérons une surface de Riemann $\mathcal{M}$ munie d'une métrique $g$ sur laquelle vivent des champs de matière dénotés génériquement par $\Psi$. La gravité bidimensionnelle est décrite par l'action

$$S[g,\Psi] = S_{\mathrm{EH}}[g] + S_{\mu}[g] + S_{m}[g,\Psi] \tag{A.1}$$

où

- l'action de Einstein–Hilbert

$$S_{\mathrm{EH}}[g] = \int \mathrm{d}^2x\sqrt{g}\,R = 4\pi\chi \tag{A.2}$$

  est un terme topologique, non-dynamique à deux dimensions qui n'intervient que dans la somme sur les genres ;

- $S_{\mu}$ est le terme de constante cosmologique

$$S_{\mu}[g] = \mu\int \mathrm{d}^2x\sqrt{g} = \mu A[g]; \tag{A.3}$$

- $S_m$ est l'action pour la matière.

On peut alors définir la fonction de partition pour la matière

$$Z_m[g] = \int \mathcal{D}\Psi \, \mathrm{e}^{-S_m[g,\Psi]} \tag{A.4}$$

et celle pour la gravité

$$Z[g] = \mathrm{e}^{-S_\mu[g]} \, Z_m[g]. \tag{A.5}$$

La fonction de partition totale est alors donnée par

$$Z = \int \mathcal{D}g \, Z[g] = \int \mathcal{D}g \, \mathrm{e}^{-S_\mu[g]} \, Z_m[g]. \tag{A.6}$$

L'action effective est définie par

$$Z_m[g] = \mathrm{e}^{-S_{\mathrm{eff}}[g]}. \tag{A.7}$$

Si $g$ et $\hat{g}$ sont deux métriques sur $\mathcal{M}$ on a alors ce qui implique

$$Z[g] = \mathrm{e}^{-S_\mu[g]} \, \frac{Z_m[g]}{Z_m[\hat{g}]} \, Z_m[\hat{g}] = \mathrm{e}^{-S_\mu[g]} \, \mathrm{e}^{-(S_{\mathrm{eff}}[g] - S_{\mathrm{eff}}[\hat{g}])} \, Z_m[\hat{g}]. \tag{A.8}$$

On définit alors l'action effective gravitationnelle (action de Wess–Zumino) comme la différence des actions effectives entre les métriques $g$ et $\hat{g}$ (voir le chapitre 4 pour plus de détails) :

$$S_{\mathrm{grav}}[\hat{g}, g] = S_{\mathrm{eff}}[g] - S_{\mathrm{eff}}[\hat{g}] = -\ln \frac{Z_m[g]}{Z_m[\hat{g}]}. \tag{A.9}$$

Le calcul de l'intégrale fonctionnelle nécessite de ne prendre en compte que les degrés de liberté physiques et donc de ne pas compter les degrés de liberté redondants dus à l'invariance de jauge. Pour cela, il faut fixer une jauge pour les difféomorphismes. On choisit ici la jauge conforme :

$$g = \mathrm{e}^{2\sigma} \hat{g} \tag{A.10}$$

où $\sigma$ est le mode de Liouville et $\hat{g}$ une métrique de fond fixée. La fonction de partition gravitationnelle s'écrit alors

$$Z[\sigma] = \mathrm{e}^{-S_{\mathrm{grav}}[\hat{g},\sigma]} Z_m[\hat{g}, \Psi]. \tag{A.11}$$

L'intérêt de la jauge conforme réside dans le fait que la matière est découplée du mode de Liouville (modulo le couplage aux déformations de Teichmüller) :

$$Z[g] = \int \mathcal{D}_{\hat{g}}\Psi \, \mathrm{e}^{-S_*[\Psi,\sigma,\hat{g}]} \tag{A.12}$$

où

$$S_*[\hat{g}, \sigma, \Psi] = S_\mu[\sigma, \hat{g}] + S_{\mathrm{grav}}[\hat{g}, \sigma] + S_m[\hat{g}, \Psi]. \tag{A.13}$$

L'action effective est donnée par la valeur moyenne quantique de la trace du tenseur énergie-impulsion :

$$\langle T^\mu_\mu \rangle = -\frac{4\pi}{\sqrt{g}} \, g^{\mu\nu} \, \frac{\delta S_{\mathrm{eff}}}{\delta g^{\mu\nu}} = \frac{2\pi}{\sqrt{g}} \, \frac{\delta S_{\mathrm{eff}}}{\delta \sigma}. \tag{A.14}$$

Dans le cas où la matière est conforme, l'intégration de l'anomalie conforme

$$\langle T^\mu_\mu \rangle = -\frac{c}{12} R \tag{A.15}$$

donne immédiatement l'action effective gravitationnelle

$$S_{\mathrm{grav}}[\hat{g}, g] = S_{\mathrm{eff}}[g] - S_{\mathrm{eff}}[\hat{g}] = -\frac{c}{6} S_L \tag{A.16}$$

en terme de l'action de Liouville

$$S_L = \frac{1}{4\pi}\int \mathrm{d}^2x\ \sqrt{\hat{g}}\ \left(\hat{g}^{\mu\nu}\partial_\mu\sigma\,\partial_\nu\sigma + \hat{R}\,\sigma\right). \tag{A.17}$$

Si la matière n'est pas conforme, il peut être difficile de calculer la valeur moyenne du tenseur énergie-impulsion et il faut recourir à d'autres méthodes.

Dans les cas qui nous intéressent, le formalisme de Kähler offre une paramétrisation pratique des métriques. La classe conforme est en effet équivalente à la classe de Kähler :

$$\mathrm{e}^{2\sigma} = \frac{A}{\hat{A}}\left(1 + \frac{\hat{A}}{2}\hat{\Delta}\phi\right), \quad \hat{\Delta}\phi > -\frac{2}{\hat{A}} \tag{A.18}$$

où $\phi$ est le potentiel de Kähler. Une variation infinitésimale du facteur conforme se décompose en une variation de l'aire et une variation du potentiel de Kähler :

$$\delta\sigma = \frac{\delta A}{2A} - \frac{A}{4}\Delta\delta\phi\ . \tag{A.19}$$

Ce formalisme permet d'écrire en terme de $\phi$ des actions locales qui seraient non-locales en terme de $\sigma$.

Plusieurs fonctionnelles remarquables apparaissent fréquemment dans les actions gravitationnelles :

- la constante cosmologique (A.3)
- l'action de Aubin–Yau

$$S_{AY}[\hat{g},\phi] = -\int \mathrm{d}^2x\sqrt{\hat{g}}\left(\frac{1}{4}\phi\hat{\Delta}\phi - \frac{\phi}{\hat{A}}\right) \tag{A.20}$$

- l'action de Liouville (A.17)
- l'action de Mabuchi

$$\begin{aligned} S_M = \frac{1}{4\pi}\int \mathrm{d}^2x\ \sqrt{\hat{g}}\ \Bigg\{ &-\pi\chi\,\hat{g}^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + \left(\frac{4\pi\chi}{\hat{A}} - \hat{R}\right)\phi \\ &+ \frac{2}{\hat{A}}\left(1 + \frac{\hat{A}}{2}\hat{\Delta}\phi\right)\ln\left[\frac{A}{\hat{A}}\left(1 + \frac{\hat{A}}{2}\hat{\Delta}\phi\right)\right]\Bigg\}. \end{aligned} \tag{A.21}$$

Par exemple, pour un champ scalaire massif sur une surface de Riemann décrit par l'action de matière

$$S_m[g,X] = \frac{1}{2}\int \mathrm{d}^2x\sqrt{g}\left(g^{\mu\nu}\partial_\mu X\partial_\nu X + m^2X^2\right) = \frac{1}{2}\int \mathrm{d}^2x\sqrt{g}X\left(\Delta + m^2\right)X \tag{A.22}$$

l'action gravitationnelle est donnée par

$$S_{\mathrm{grav}}[\hat{g},g] = \frac{1}{2}\ln\frac{A}{\hat{A}} + S_\mu + \frac{c}{6}S_L + \alpha S_M + \beta\left(S_{\mathrm{AY}} - \int \mathrm{d}^2x\sqrt{g_c}\phi\right) + \mathcal{O}(m^2) \tag{A.23}$$

où $\alpha = \frac{m^2A}{4}$ et $\beta = \frac{m^2Ah}{2}$ et $g_c$ est la métrique canonique sur la surface [63, 70].

Le Laplacien sur une surface de Riemann est défini par

$$\Delta = g^{\mu\nu}\nabla_\mu\nabla_\nu \tag{A.24}$$

où $\nabla$ désigne la dérivée covariante qui dépend du type de champ considéré. Un opérateur $D$ est dit de type Laplace s'il est de la forme

$$D = g^{\mu\nu}\partial_\mu\partial_\nu + a^\mu\partial_\mu + b. \tag{A.25}$$

Pour un tel opérateur, on peut définir les fonctions spectrales suivantes ($\lambda_n$ désignent les valeurs propres de $D$, $\varphi_n$ une base orthonormale de vecteurs propres et $\varphi_{0,i}$ les éventuels modes zéros) :

- Fonction de Green

$$\tilde{G}(x,y) = \sum_{n>0} \frac{\varphi_n(x)\varphi_n(y)}{\lambda_n} \Leftrightarrow D\tilde{G}(x,y) = \frac{\delta(x-y)}{\sqrt{g}} - \sum_i \varphi_{0,i}\,\varphi_{0,i}^\dagger$$

- Noyau de la chaleur

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + D\right) K(t,x,y) = 0, \quad K(t,x,y) \sim \frac{\delta(x-y)}{\sqrt{g}} \text{ as } t \to 0 \Leftrightarrow K(t,x,y) = \sum_{n\geq 0} \mathrm{e}^{-\lambda_n t}\,\varphi_n(x)\varphi_n^\dagger(y).$$

$$K(t) = \int \mathrm{d}^2x\sqrt{g}\,\mathrm{tr}\,K(t,x,x) = \sum_{n\geq 0} \mathrm{e}^{-\lambda_n t} = \mathrm{tr}\,\mathrm{e}^{-tD}.$$

- Fonctions zêta

$$\zeta(s) = \sum_n \frac{1}{\lambda_n^s} = \frac{1}{\Gamma(s)} \int_0^{+\infty} \mathrm{d}t\, t^{s-1} K(t)$$

$$\zeta(s,x,y) = \sum_n \frac{\varphi_n(x)\varphi_n^\dagger(y)}{\lambda_n^s} = \zeta(s,x,y) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \mathrm{d}t\, t^{s-1} K(t,x,y)$$

On note avec un tilde les fonctions où les modes zéros ont été supprimés de la somme.

La fonction zêta permet de donner un sens à des déterminants à priori infinis. Le calcul de l'intégrale fonctionnelle fait souvent intervenir de tels déterminants :

$$\ln\det\frac{D}{\mu^2} = \sum_{n>0} \ln\frac{\lambda_n}{\mu^2}. \tag{A.26}$$

où $\mu$ est un paramètre d'échelle nécessaire pour avoir des quantités sans dimension. Une continuation analytique permet d'écrire la somme infinie sous la forme

$$\sum_{n>0} \ln\frac{\lambda_n}{\mu^2} = -\zeta'(0) - \ln\mu^2\zeta(0). \tag{A.27}$$

Le développement aux temps courts bien connu du noyau de la chaleur permet de montrer que $\zeta(0)$ et $\zeta'(0)$ ont des valeurs finies bien définies. On peut ainsi définir une régularisation du logarithme du déterminant de $D$ par

$$\ln\det{}_\zeta\frac{D}{\mu^2} = -\zeta'(0) - \ln\mu^2\zeta(0). \tag{A.28}$$

### A.2.3 Action gravitationnelle pour un champ scalaire massif sur une surface de Riemann avec bords

Le chapitre 5 explique comment généraliser le résultat de (A.23) sur une surface avec bords. Dans ce cas il faut d'abord imposer des conditions aux bords. Si l'on veut conserver l'hermiticité de $\Delta + m^2$ et la condition $\int_{\mathcal{M}} \mathrm{d}^2x\sqrt{g}\,\Delta f \equiv \int_{\mathcal{M}} \mathrm{d}^2x\sqrt{\hat{g}}\,\hat{\Delta} f = 0$ (afin que l'égalité (A.22) soit toujours valable), on s'aperçoit qu'il faut imposer les conditions aux bords de Neumann pour le champ scalaire : $n^a\partial_a X = 0$. Comme $A = \int \sqrt{\hat{g}}\,\mathrm{e}^{2\sigma}\,\phi$, la relation $\mathrm{e}^{2\sigma} = \frac{A}{\hat{A}}\left(1 + \frac{\hat{A}}{2}\hat{\Delta}\phi\right)$ impose à $\phi$ de vérifier aussi les conditions aux bords de Neumann. Par contre, le champ $\sigma$ ne peut vérifier ces conditions en même temps que $\phi$.

Le comportement des fonctions spectrales sur une surface avec bords soulève quelques subtilités. En premier lieu, le développement aux temps courts du noyau de la chaleur contient une singularité sous-dominante en $t^{-1/2}$ qui n'apparait pas sur une surface sans

bord. Si l'on intègre le noyau de la chaleur à points coïncidents avec une fonction test $f$ on obtient alors

$$\int_{\mathcal{M}} \mathrm{d}^2x\,\sqrt{g}\,f(x)K(t,x,x) = \frac{1}{4\pi t}\int \mathrm{d}^2x\,\sqrt{g(x)}f(x) + \frac{1}{8\sqrt{\pi t}}\sum_i \int_{\partial\mathcal{M}_i} \mathrm{d}l\,f(x) + \mathcal{O}(1). \quad \text{(A.29)}$$

Deuxièmement, le comportement de la fonction zêta à points coïncidents dépend de la localisation du point considéré : sur la bord ou à l'intérieur de la surface. Ainsi, $\zeta(s,x,x)$ a un pôle en $s=1$ avec résidu $\frac{1}{4\pi}$ si $x \notin \partial\mathcal{M}$ et $\frac{1}{2\pi}$ si $x \in \partial\mathcal{M}$. La fonction de Green à point coïncident, qui joue un rôle majeur dans le calcul de l'action gravitationnelle est définie par

$$G_\zeta(y) = \lim_{s\to 1}\left(\zeta(s,y,y) - \zeta_{\mathrm{sing}}(s,y,y)\right) + \frac{\gamma}{4\pi} \quad \text{(A.30)}$$

où $\zeta_{\mathrm{sing}}(s,y,y)$ est définie différemment selon que $y$ est ou non sur un bord :

$$\zeta_{\mathrm{sing}}(s,y,y) = \begin{cases} \frac{\mu^{2-2s}}{4\pi\Gamma(s)}\,\frac{1}{s-1} \text{ si } y \in M\setminus\partial\mathcal{M} \\ \frac{\mu^{2-2s}}{4\pi\Gamma(s)}\,\frac{2}{s-1} \text{ si } y \in \partial\mathcal{M} \end{cases} \quad \text{(A.31)}$$

Nous montrons dans ce chapitre 5 comment obtenir une expression générale pour l'action gravitationnelle. Cependant, il n'a pas encore été possible d'écrire une expression analytique en termes de fonctions usuelles. Il est alors plus intéressant de regarder le développement à petite masse :

$$\begin{aligned} S_{\mathrm{grav}}[\hat{g},g] = &-\frac{1}{6}S_L[\hat{g},g] + \frac{1}{2}\ln\frac{A}{\hat{A}} + \frac{m^2\,(A-\hat{A})}{2\hat{A}}\left(\Phi_G[\hat{g}] + \frac{1}{4\sqrt{\pi}\,\mu}L(\partial\mathcal{M},\hat{g})\right) \\ &+ \frac{m^2A}{4}\int\sqrt{\hat{g}}\left(-\frac{1}{2}\phi\hat{\Delta}\phi + \frac{1}{\pi A}\sigma\mathrm{e}^{2\sigma} - \phi\,\hat{\Delta}\tilde{G}^{(0)}_\zeta[\hat{g}]\right) \\ &- \frac{m^2A}{16\sqrt{\pi}}\left(\frac{1}{\mu}\int_{\partial\mathcal{M}}\mathrm{d}\hat{l}\,\hat{\Delta}\phi + \mathcal{O}\left(\frac{1}{\mu^2}\right)\right) + \mathcal{O}(m^4) \end{aligned} \quad \text{(A.32)}$$

En particulier, sur le cylindre

$$S^{\mathrm{cyl}}_{\mathrm{grav}}[\hat{g},g]\Big|_{m^2A} = \frac{m^2A}{4}\int_0^T \mathrm{d}x\int_0^{2\pi R}\mathrm{d}y\,\left(-\frac{1}{2}\phi\hat{\Delta}\phi + \frac{1}{\pi A}\sigma\mathrm{e}^{2\sigma} + \frac{1}{2\pi}\hat{\Delta}\phi\,\ln\theta_1\left(\frac{x}{T}\Big|i\pi\frac{R}{T}\right)\right) \quad \text{(A.33)}$$

et, si on fait tendre la longueur vers l'infini

$$\begin{aligned} S^{\infty\ \mathrm{cyl}}_{\mathrm{grav}}[\hat{g},g]\Big|_{m^2A} &= \frac{m^2A}{4}\lim_{T\to\infty}\int_{-T/2}^{T/2}\mathrm{d}t\int_0^{2\pi R}\mathrm{d}y\,\left[-\frac{1}{2}\phi\hat{\Delta}\phi + \frac{1}{\pi A}\sigma\mathrm{e}^{2\sigma} + \frac{2}{\hat{A}}\phi\right] \\ &= \frac{m^2A}{16\pi}S_{\mathrm{M}}[\hat{g},g]\Big|_{h=0,\ \hat{R}=0}. \end{aligned} \quad \text{(A.34)}$$

Bien qu'on s'attendait à $\hat{R}=0$ sur le cylindre, ce qui est intéressant est le remplacement $\chi = 2(1-h)$ par 2.

### A.2.4 Action gravitationnelle pour un spineur de Majorana

On ne sait pas actuellement si les mêmes fonctionnelles apparaitraient dans l'action gravitationnelle pour d'autres champs de matière. Les propriétés remarquables de l'action de Mabuchi décrites au chapitre 4 font que l'on s'attend intuitivement à ce que ce soit le cas. Pour éclairer la question, nous avons commencé au chapitre 6 l'étude de l'exemple le plus simple après le champ scalaire : un spineur de Majorana libre et massif couplé par couplage minimal à la gravité quantique. Ce modèle présente aussi l'intérêt que le terme de masse est cette fois une déformation conforme de la théorie sans masse. L'objectif est alors de comparer

les résultats avec l'ansatz de DDK qui, comme mentionné précédemment, présente quelques incohérences dans ce cas.

On étudie un spineur de Majorana $\Psi$ (ce qui revient à dire que $\Psi$ est réel) décrit par l'action

$$S = \frac{1}{4\pi} \int \mathrm{d}^2x \, \sqrt{g} \, \bar{\Psi}(i\not{\nabla} + m\gamma_*)\Psi. \tag{A.35}$$

L'opérateur $i\not{\nabla} + m\gamma^*$ n'a pas de vecteur propre réel. Par contre les parties réelles et imaginaires de ses vecteurs propres complexes satisfont

$$\begin{cases} (i\not{\nabla} + m\gamma^*)\chi_n &= i\lambda_n\phi_n \\ (i\not{\nabla} + m\gamma^*)\phi_n &= -i\lambda_n\chi_n \end{cases} \tag{A.36}$$

ce qui, en mettant au carré, donne

$$\begin{cases} \left(-\Delta + \frac{R}{4} + m^2\right)\chi_n &= \lambda_n^2\chi_n \\ \left(-\Delta + \frac{R}{4} + m^2\right)\phi_n &= \lambda_n^2\phi_n. \end{cases} \tag{A.37}$$

En décomposant $\Psi$ sur la base orthonormale des $\phi_n$ et des $\chi_n$, on trouve que la fonction de partition est donnée par

$$Z[g] = \prod_{n>0} \lambda_n = \left(\prod_{n\neq 0} \lambda_n^2\right)^{1/4} = \det\left(-\Delta + \frac{R}{4} + m^2\right)^{1/4}. \tag{A.38}$$

Introduisant une échelle $\mu$, la régularisation par la fonction zêta donne

$$\delta S_{\mathrm{grav}}[g] = \frac{1}{4}(\delta\zeta_g'(0) + \ln\mu^2\delta\zeta_g(0)). \tag{A.39}$$

En utilisant la théorie des perturbations, on obtient la variation de la fonction zêta puis celle de l'action gravitationnelle :

$$\delta S_{\mathrm{grav}} = -\frac{1}{12}\delta S_L - \frac{m^2}{2}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\,\mathrm{tr}_D\left(G_\zeta(x) + \frac{1}{4\pi}\right) \tag{A.40}$$

où $G_\zeta$ est la fonction de Green à points coïncidents définie par

$$G_\zeta(x) = \lim_{s\to 1}\left(\mu^{2s-2}\zeta(s,x,x) - \frac{\mathrm{I}_2}{4\pi(s-1)}\right) = \tilde{G}_\zeta(x) + \sum_i \Psi_{0,i}(x)\Psi_{0,i}^\dagger(x). \tag{A.41}$$

On peut alors effectuer un développement en $m^2$ :

$$\begin{aligned} \delta S_{\mathrm{grav}} = &-\frac{1}{12}\delta S_L - \frac{m^2}{4}\,\delta\left(\int \mathrm{d}^2x\sqrt{g}\;\mathrm{tr}_D\,\tilde{G}_\zeta(x)\right) \\ &-\frac{1}{2}\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\sum_i \Psi_{0,i}^\dagger(x)\Psi_{0,i}(x) + \mathcal{O}(m^4). \end{aligned} \tag{A.42}$$

Pour obtenir une expression explicite de l'action gravitationnelle, il est nécessaire d'exprimer $\int \mathrm{d}^2x\sqrt{g}\,\delta\sigma(x)\sum_i \Psi_{0,i}^\dagger(x)\Psi_{0,i}(x)$ comme la variation d'une certaine fonctionnelle. Sur la sphère, il n'y a pas de mode zéro et le problème ne se pose pas. On peut aussi facilement obtenir une expression pour le tore. Par contre, pour les surfaces de genre $\geq 2$ nous n'avons pas encore trouvé une telle expression (une idée pourrait être d'utiliser le théorème d'uniformisation et d'écrire la métrique sur la surface sous la forme $\mathrm{d}s^2 = \frac{\mathrm{d}z\mathrm{d}\bar{z}}{(\mathrm{Im}\,z)^2}$). Ensuite, il faudra trouver une expression analytique de la variation de l'intégrale de la fonction de Green. Il est probable qu'on retrouverait les actions de Mabuchi et Aubin–Yau mais la combinaison sera sûrement différente de celle du cas scalaire.

Finalement, il est clair que (A.42) ne peut pas redonner l'action de DDK pour un fermion de Majorana massif donnée par

$$S_*[\hat{g},\sigma,\Psi] = S_{\mathrm{DDK}}[\hat{g},\sigma,\Psi] = i\int \mathrm{d}^2x\,\sqrt{\hat{g}}\,\bar{\Psi}\hat{\nabla\!\!\!/}\Psi - \frac{1}{12}S_L[\hat{g},\sigma] + m\int \mathrm{d}^2x\sqrt{\hat{g}}\,\mathrm{e}^{\frac{1}{2}\sigma}\bar{\Psi}\gamma^*\Psi. \tag{A.43}$$

En effet (A.42) est un développement compliqué dans la masse dont le terme d'ordre le plus bas (après $m^0$) est en $m^2$ et pas en $m$.

### A.2.5 Spectre de l'action de Mabuchi dans l'approximation du minisuperespace

Les équations du mouvement pour l'action de Mabuchi (A.21) sont les mêmes que pour l'action de Liouville à aire fixée :

$$R = \frac{4\pi\chi}{A}. \tag{A.44}$$

L'action (A.21) est mal définie sur une surface de genre 1 qui est justement ce qui nous intéresse pour le minisuperespace. On est donc amené à redéfinir le potentiel de Kähler et l'action :

$$\phi = \frac{\tilde{\phi}}{\pi\chi}, \qquad S_M = \frac{\tilde{S}_M}{\pi\chi} \tag{A.45}$$

où

$$\tilde{S}_M = \frac{1}{4\pi}\int \mathrm{d}^2x\,\sqrt{\hat{g}}\,\left(-\hat{g}^{\mu\nu}\partial_\mu\tilde{\phi}\,\partial_\nu\tilde{\phi} + \left(\frac{4\pi\chi}{\hat{A}} - \hat{R}\right)\tilde{\phi} + \frac{4\pi\chi}{A}\,\sigma\,\mathrm{e}^{2\sigma}\right). \tag{A.46}$$

La relation entre le mode de Liouville et le potentiel de Kähler devient

$$\mathrm{e}^{2\sigma} = \frac{A}{\hat{A}}\left(1 + \frac{\hat{A}}{2\pi\chi}\,\hat{\Delta}\tilde{\phi}\right). \tag{A.47}$$

Dans la suite, on supprime les tildes sur le champ et l'action redéfinis.

L'approximation du minisuperespace consiste à se restreindre à des potentiels de Kähler et des modes de Liouville qui ne dépendent que du temps :

$$\sigma(t,x) = \sigma(t), \qquad \phi(t,x) = \phi(t). \tag{A.48}$$

La relation entre $\sigma$ et $\phi$ devient

$$\mathrm{e}^{2\sigma} = -\frac{A}{2\pi\chi}\,\ddot{\phi}. \tag{A.49}$$

Pour pouvoir utiliser le formalisme hamiltonien, il faut définir une direction de temps globale. L'espace de fond est alors un cylindre $I \times S^1$ où $I$ est un intervalle de longueur $T$. Cela implique que l'espace-temps est plat :

$$\chi = 0, \qquad g_0 = \eta, \qquad t \in \left[-\frac{T}{2},\frac{T}{2}\right], \qquad x \in [0,2\pi). \tag{A.50}$$

Cependant, plusieurs méthodes (prise de limites, formalisme d'Ostrogradski, transformation de Legendre) permettent de ne pas se restreindre à un espace physique plat ($\chi \neq \hat{\chi} = 0$) et de conjecturer que, dans l'approximation du minisuperespace, l'action de Mabuchi devient

$$S_M = -\frac{1}{2}\int \mathrm{d}t\,\left[\dot{\phi}^2 - \ddot{\phi}\ln\left(-\frac{A}{2\pi\chi}\,\ddot{\phi}\right) + \ddot{\phi}\right] \tag{A.51}$$

qui reproduit bien les équations du mouvement dans cette approximation

$$\ddot{\sigma} = \frac{2\pi\chi}{A}\,\mathrm{e}^{2\sigma}. \tag{A.52}$$

On peut alors appliquer le formalisme hamiltonien à cette action. Le moment conjugué de $\dot{\phi}$ est donné par

$$P = \frac{\delta S_M}{\delta \ddot{\phi}} = \frac{1}{2} \ln \left( -\frac{A}{2\pi\chi} \ddot{\phi} \right). \tag{A.53}$$

Après une transformation canonique (qui échange la position et le moment)

$$P = \sigma, \qquad \dot{\phi} = -\Pi, \tag{A.54}$$

on trouve le Hamiltonien

$$H_M = \frac{\Pi^2}{2} - \frac{\pi\chi}{A} \mathrm{e}^{2\sigma}. \tag{A.55}$$

On peut maintenant prendre $\chi = 0$ et insérer la constante cosmologique ce qui donne

$$H_M = \frac{\Pi^2}{2} + 2\pi\mu \, \mathrm{e}^{2\sigma}. \tag{A.56}$$

Il coïncide avec le Hamiltonien de la théorie de Liouville dans l'approximation du minisuperespace.

Le spectre

$$H_M \psi_p = 2p^2 \, \psi_p \tag{A.57}$$

s'obtient par quantification canonique : le remplacement usuel

$$\Pi \longrightarrow -i \frac{\mathrm{d}}{\mathrm{d}\sigma} \tag{A.58}$$

transforme l'équation aux valeurs propres en une équation de Bessel modifiée :

$$\left( -\frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}\sigma^2} + 2\pi\mu \, \mathrm{e}^{2\sigma} - 2p^2 \right) \psi_p(\sigma) = 0. \tag{A.59}$$

Les solutions de cette équation donnent les fonctions d'onde

$$\psi_p(\sigma) = \frac{2(\pi\mu)^{-ip}}{\Gamma(-2ip)} K_{2ip}(2\sqrt{\pi\mu}\, \mathrm{e}^{\sigma}) \sim_0 \mathrm{e}^{2ip\sigma} + r_0(p) \mathrm{e}^{-2ip\sigma}. \tag{A.60}$$

qui forment une base orthonormale pour $p \in \mathbb{R}$.

On peut aussi lire la fonction à 3 points dans l'approximation du minisuperespace :

$$C_0(p_1, p_2, p_3) = \int_{-\infty}^{\infty} \mathrm{d}\sigma \; \psi_{p_1}(\sigma) \mathrm{e}^{-2ip_2\sigma} \psi_{p_3}(\sigma) \tag{A.61}$$

$$= (\pi\mu)^{-2\tilde{p}} \, \Gamma(2\tilde{p}) \prod_i \frac{\Gamma\big((-1)^i 2\tilde{p}_i\big)}{\Gamma(2p_i)} \tag{A.62}$$

où

$$2\tilde{p} = \sum_i p_i, \qquad \tilde{p}_i = \tilde{p} - p_i, \qquad i = 1, 2, 3. \tag{A.63}$$

## A.3 Autres directions de recherche explorées pendant cette thèse

Le second point de vue abordé dans cette thèse se place dans le cadre de la théorie des cordes. Cette théorie a pour particularité d'unifier complètement les quatre interactions, les bosons de jauge (y compris le graviton) et la matière apparaissant comme les excitations d'objets étendus relativistes. [5–7]. Les interactions de jauge sont décrites par les degrés de liberté des extrémités des cordes ouvertes tandis que les gravitons apparaissent dans le spectre de

la corde fermée. De plus, tous les groupes de grande unification (SU(5), S0(10), $E_6$ ...) sont des sous-groupes de $E_8$, l'un des groupes de jauge qui apparait naturellement. En outre, la théorie des cordes contient tous les ingrédients du Modèle standard tout en étant finie dans l'ultraviolet. Et, cerise sur le gâteau, elle n'a qu'un unique paramètre dimensionné, la longueur de la corde $l_s$ et aucun paramètre sans dimension ajustable.

Pour introduire de la matière fermionique, une possibilité consiste à rendre la théorie supersymétrique. Les supercordes peuvent vivre dans un espace à dix dimensions, certaines pouvant être compactifiées sur une variété appropriée (habituellement une variété de Calabi–Yau). Polchinski s'est rendu compte que les théories de supercordes ne contiennent pas seulement des cordes mais aussi des objets fondamentaux étendus de dimensions variées, nommés branes. L'exemple typique est les D-branes sur lesquelles les cordes ouvertes peuvent s'accrocher. Les D-branes sont fondamentalement des objets non-perturbatifs. Il y a cinq types différents de théories de supercordes : type I, IIA, IIB, hétérotique avec groupe de jauge SO(32) et hétérotique avec groupe de jauge $E_8 \times E_8$. Chaque type possède différentes sortes de D-branes en plus de la corde fondamentale. Les cinq théories sont reliées par un réseau de dualités. En particulier, certaines d'entre elles comme la T-dualité, relient une théorie fortement couplée à une faiblement couplée. Il s'agit là d'un trait très intéressant car, pour le moment, la théorie des cordes est essentiellement définie perturbativement. La découverte de ces dualités a conduit à réaliser que la théorie des cordes est en fait la limite à basse énergie d'une théorie à onze dimensions appelée théorie M. Bien que ces dualités et la correspondance AdS/CFT donnent quelques aperçus de la théorie M, elle reste actuellement peu comprise.

La théorie des cordes est donc une théorie fascinante qui englobe la gravité quantique et la grande unification. En outre, elle a donné lieu à de nombreux progrès en mathématiques, comme la découverte de la symétrie miroir qui décrit comment sont reliées des variétés de Calabi–Yau de topologies différentes.

### A.3.1 Un modèle de matrices fermioniques pour décrire des trous noirs

L'un des résultats majeurs de la théorie des cordes est la conjecture de la correspondance AdS/CFT ou holographie [79–82] qui établit une dualité entre une théorie de gravitation dans un espace AdS à $d+1$ dimensions (le bulk) et une théorie des champs conforme sans gravité vivant dans un espace-temps à $d$ dimensions (appelé le bord). Cela signifie qu'il existe un dictionnaire entre les observables des deux théories et que les corrélateurs peuvent être calculés de manière équivalente dans une théorie ou dans l'autre [83]. La correspondance AdS/CFT est un outil formidable pour la gravité quantique puisque les corrections quantiques dans le bulk peuvent être calculées à partir d'une théorie de jauge usuelle sur le bord. Cependant, le décodage de l'hologramme est hautement non-trivial car la théorie de jauge sur le bord est toujours fortement couplée si une description gravitationnelle classique existe pour le bulk. Par conséquent, AdS/CFT a surtout été utilisée dans l'autre direction pour fournir des indications sur les théories de jauge fortement couplées à partir d'une théorie classique de supergravité dans le bulk. Dans ce contexte, la théorie des cordes n'est plus considérée comme la théorie fondamentale mais comme un outil théorique qui offre des descriptions duales de systèmes intéressants comme la théorie de Yang–Mills dans l'infrarouge ou à température finie.

Nous nous intéresserons ici à l'une des propositions les plus étudiées de dualité théorie de jauge / gravité qui relie la théorie des cordes de type II dans une géométrie de proche horizon d'un trou noir AdS à la théorie de super Yang–Mills maximalement supersymétrique $\mathcal{N}=4$. L'objectif est d'étudier certaines propriétés des trous noirs grâce à la description en terme de théorie de jauge.

Les trous noirs sont des objets classiques dont le champ gravitationnel est si fort qu'ils sont entourés d'une surface, appelée horizon, d'où rien ne peut s'échapper, même pas la

lumière. Leur existence a été prédite depuis longtemps par la relativité générale et est supportée par de nombreuses (mais indirectes) observations astrophysiques, la dernière en date étant la détection des ondes gravitationnelles émises par la coalescence de deux trous noirs. Les données expérimentales s'accordent aux prédictions théoriques avec une grande précision. Ainsi, les données observationnelles donnent un fort crédit à l'existence des trous noirs, qu'ils soient de masse stellaire (causés par l'effondrement gravitationnel d'une étoile, comme pour l'étoile binaire à rayons X Cygnus X-1) ou supermassifs dans le centre des galaxies (comme Sagittarius A* au centre de la Voie Lactée).

Un lien remarquable entre les trous noirs et la thermodynamique a été mis à jour dans les années 60 et 70. Il y a une analogie frappante entre les lois de la mécanique des trous noirs et celles de la thermodynamique. Les trous noirs semblent avoir une entropie proportionnelle à leur aire [84]. Puis, Hawking a montré que, quand les effets quantiques sont pris en compte, les trous noirs ont une température et émettent une radiation de corps noir qui devrait conduire à leur évaporation [85]. Cette interprétation thermodynamique soulève plusieurs problèmes. En premier lieu, on peut se demander si l'entropie des trous noirs a vraiment une interprétation statistique, auquel cas elle devrait compter les micro-états du trou noir. Il est particulièrement intriguant de constater que l'entropie des trous noirs varie comme l'aire de l'horizon alors qu'en mécanique quantique usuelle, on s'attend à ce que le nombre d'états varie comme le volume. Le comptage des micro-états en théorie des cordes et supergravité a permis de retrouver avec succès l'entropie de Bekenstein pour les trous noirs qui préservent certaines supersymétries [86, 87] mais la question est bien moins comprise pour les trous noirs non-supersymétriques dans un espace temps Minkowskien. Deuxièmement, la radiation de Hawking est thermale et conduit à l'évaporation totale du trou noir. L'information tombée derrière l'horizon semble ainsi perdue après la disparition du trou noir. Ce paradoxe de l'information parait contredire l'unitarité de la mécanique quantique. Depuis la découverte de Hawking, le débat fait rage parmi les physiciens quant à savoir si l'unitarité de la mécanique quantique doit être abandonnée ou si le calcul de Hawking ne tient simplement pas assez compte de l'intrication entre l'intérieur et l'extérieur du trou noir voire s'il ne serait en fait plus valable à l'échelle de Planck [88–92]. Bien que la discussion soit loin d'être close, la correspondance AdS/CFT fournit quelque éclairage sur le sujet. En effet, selon elle, l'évolution des trous noirs peut être décrite par l'évolution d'une théorie de jauge qui est complètement unitaire et, par conséquent, devrait être elle-même unitaire.

Le second projet de cette thèse, présenté au chapitre 9 se déroule dans ce contexte. D'après la correspondance AdS/CFT, les trous noirs peuvent être étudiés d'après la théorie de jauge duale qui correspond à un modèle de matrices $N \times N$ invariant sous $\mathrm{U}(N)$ censé capturer les caractéristiques essentielles de leur comportement thermal. En effet, on peut compactifier toutes les dimensions spatiales de la théorie de jauge pour obtenir un modèle de mécanique quantique. Comme tous les objets sont dans la représentation adjointe de $\mathrm{U}(N)$, ils sont représentés par des matrices. Les états de l'espace de Hilbert de ce modèle sont duaux des micro-états du trou noir. Cependant, même dans cette version à zéro dimension, l'espace de Hilbert quantique est toujours de dimension infinie. Dans notre projet, on remplace les matrices usuelles par des matrices fermioniques, ce qui rend l'espace de Hilbert de dimension finie. Cela permet d'obtenir des résultats numériques exacts, dans le sens où l'on peut calculer l'intégralité du spectre de la théorie. Pour étudier la dynamique de la thermalisation du trou noir, on calcule numériquement les fonctions de corrélation entre les états de la théorie duale.

Le modèle contient deux matrices fermioniques de taille $N$

$$X = (a_{ij}^\dagger), \qquad Y = (b_{ij}^\dagger) \tag{A.64}$$

où $a_{ij}^\dagger$ et $b_{ij}^\dagger$ sont des opérateurs de création fermioniques. Pour reproduire le comportement de thermalisation d'un trou noir, il nous faut un modèle dont les niveaux d'énergie sont hautement dégénérés afin qu'une perturbation lève la dégénérescence et produise un spectre quasi-continu quand $N$ est grand. L'espace de Hilbert contient les états créés en agissant sur le vide avec les matrices de création. La dynamique du modèle est déterminée par un

Hamiltonien $H$ invariant sous $\mathrm{U}(N) \times \mathrm{SO}(2)$, où $\mathrm{U}(N)$ agit sur $X$ et $Y$ par conjugaison tandis que $\mathrm{SO}(2)$ effectue une rotation de la paire $(X, Y)$. On peut remplacer $\mathrm{SO}(2)$ par un $\mathrm{U}(1)$ qui agit sur les combinaisons complexes

$$Z = \frac{X + iY}{\sqrt{2}}, \quad \bar{Z} = \frac{X - iY}{\sqrt{2}}, \quad Z^\dagger = \frac{X^\dagger - iY^\dagger}{\sqrt{2}}, \quad \bar{Z}^\dagger = \frac{X^\dagger + iY^\dagger}{\sqrt{2}}. \tag{A.65}$$

La construction du modèle s'inspire de modèles classiques utilisés en AdS/CFT, en particulier de l'action des D0-branes. Pour cela, on impose que les états soient des singlets sous $\mathrm{U}(N)$. L'espace de Fock ne contient donc que des états qui peuvent être écrits comme des multi-traces de la forme

$$\mathrm{tr}(Z^{\alpha_1} \bar{Z}^{\beta_1} \cdots Z^{\alpha_n} \bar{Z}^{\beta_n}) \ldots \mathrm{tr}(Z^{\rho_1} \bar{Z}^{\sigma_1} \cdots Z^{\rho_m} \bar{Z}^{\sigma_m}) |0\rangle. \tag{A.66}$$

Le Hamiltonien choisit s'écrit

$$H = H_0 + \gamma H_{\mathrm{int},2} + g H_{\mathrm{int},4}. \tag{A.67}$$

où

- $H_0$ est le Hamiltonien libre
$$H_0 = \mathrm{tr}\left(Z Z^\dagger + \bar{Z} \bar{Z}^\dagger\right) \tag{A.68}$$
- $H_{\mathrm{int},2}$ est un terme d'interaction quadratique
$$H_{\mathrm{int},2} = \gamma \,\mathrm{tr}(Z \bar{Z} + \bar{Z}^\dagger Z^\dagger). \tag{A.69}$$
- le terme d'interaction quartique $H_{\mathrm{int},4}$ vient du terme d'interaction $\mathrm{tr}[\phi^\mu, \phi^\nu][\phi_\mu, \phi_\nu]$ de $\mathcal{N} = 4$ SYM ou de l'action pour les D0-branes où l'on identifie $\phi^\mu$ avec une position à valeur matricielle dans la direction $\mu$, i.e $\phi^1 = X + X^\dagger$ et $\phi^2 = Y + Y^\dagger$. On obtient
$$H_{\mathrm{int},4} = \mathrm{tr}[\phi^\mu, \phi^\nu][\phi_\mu, \phi_\nu] = 4\,\mathrm{tr}(Z + \bar{Z}^\dagger)(\bar{Z} + Z^\dagger)(\bar{Z} + Z^\dagger)(Z + \bar{Z}^\dagger) - 8N^3 \tag{A.70}$$

Le calcul du spectre montre que la dégénérescence est levée dès que l'on ajoute un faible couplage $g$. Le spectre devient alors quasi-continu. Ce résultat est encourageant car il n'était pas évident qu'un $N$ aussi petit que 3 permettrait d'observer un effet aussi marqué. Quand le couplage augmente, on observe que des plages vides apparaissent dans le spectre. Cela peut être dû au fait que notre terme d'interaction n'est pas suffisamment générique quand les matrices sont fermioniques et que le mélange n'est pas complet.

Cet aspect continu du spectre incite à se demander si l'on peut observer une transition de phase de type trou noir. Cela semble en effet être le cas, que ce soit pour l'énergie libre ou pour la fonction de Green

$$G(t) = \frac{1}{Z} \,\mathrm{tr}\, \mathrm{e}^{-H/T}\, \mathrm{e}^{itH}\, A\, \mathrm{e}^{-itH} A - \left(\frac{1}{Z} \,\mathrm{tr}\, \mathrm{e}^{-H/T}\, A\right)^2 \tag{A.71}$$

de l'opérateur $A = \mathrm{tr}(Z \bar{Z} + \bar{Z}^\dagger Z^\dagger)$ ($Z$ dénote la fonction de partition). La température critique semble décroître quand on augmente le couplage jusqu'à la disparition de la transition de phase. Il serait souhaitable de pouvoir augmenter $N$ pour voir ces effets plus clairement.

Malheureusement, la complexité algorithmique de l'implémentation de l'algèbre de Grassmann pour un système comportant de nombreux q-bits ne nous a pas permis d'aller plus loin que le cas $N = 3$ alors que la correspondance AdS/CFT n'est valable que pour $N$ grand. Bien que certains traits intéressants puissent déjà être observés dans nos résultat pour $N = 3$, d'autres semblent nécessiter un $N$ plus grand, au moins $N = 4$ que nous n'avons pas réussi à implémenter jusqu'à présent. Dans ce sens, ces résultats doivent être considérés comme préliminaires.

### A.3.2 Théorie des champs de cordes

La formulation standard de la théorie des cordes en terme de surface d'univers est une description qui correspond à une première quantification. De nombreux calculs ont pu être faits grâce à la puissance de l'analyse complexe et au fait que le groupe conforme est de dimension infinie à deux dimensions (certains développements ont également bénéficié des dualités non-perturbatives). Cependant, la théorie souffre de limitations qui nécessitent une deuxième quantification. Premièrement, à cause des invariances BRST et conforme, la formulation en terme de surface d'univers est intrinsèquement on-shell. Il existe des méthodes pour l'étendre off-shell mais au prix d'ambiguïtés. Dans tous les cas, la renormalisation est difficile et ne peut être conduite qu'en utilisant des procédures ad-hoc. En outre, la théorie comporte des divergences infrarouges [93].

La deuxième quantification de la théorie des cordes s'est révélée être un véritable défi. Bien qu'une théorie des champs de cordes bosonique ait été développée depuis les années 90 à la fois pour les cordes ouvertes [94] et fermées [95, 96], ce n'est que récemment qu'une telle théorie a été construite pour les supercordes fermées hétérotiques et de type II. En effet, la généralisation aux supercordes se heurte à trois difficultés majeures. Premièrement, on ne savait pas comment traiter les champs de Ramond car il n'était pas possible d'écrire un terme cinétique (puisque le propagateur n'est pas inversible). Ce problème a été contourné en introduisant un nouvel ensemble de champs libres qui n'interagissent pas avec les autres [97, 98]. Deuxièmement l'insertion des opérateurs de changement d'image (PCOs) dans les fonctions de corrélation (off-shell) entrainait des pôles non-physiques. Une construction des fonctions de corrélation avec le bon choix d'insertions de PCOs et libre de pôles non-physiques a été proposée grâce à l'intégration verticale [99]. Finalement, il fallait également régler le problème des divergences ultraviolettes qui apparaissent lorsqu'on exprime les fonctions de corrélation comme une somme de diagrammes de Feynman.

Après avoir résolu ces problèmes, on obtient une théorie quantique des champs pour les cordes et il s'agit de montrer qu'elle en possède les bonnes propriétés usuelles. Il a été prouvé que l'action pour le champ de corde est réelle [100] et une prescription $i\varepsilon$ a été trouvée [101, 102]. Les amplitudes satisfont les règles de Cutkowski ce qui, combiné avec les identités de Ward, permet d'établir l'unitarité de la théorie des champs de cordes [103, 104]. La théorie a conduit à plusieurs applications dont les calculs du décalage du vide, de la renormalisation de la masse [105] et l'établissement de théorèmes "softs" [106]. Des développements importants dans la construction explicite de l'action ont été réalisés dans [107, 108]. Il existe d'autres approches de la théorie des champs de cordes qui offrent d'autres perspectives (par exemple, certaines proposent une construction pour la supercorde ouverte ou utilisent des super-surfaces de Riemann au lieu des PCOs, etc.) – une sélection de références est [109–114].

Après une contribution à une revue sur la théorie des champs de cordes avec Ashoke Sen et collaborateurs, un projet a commencé avec Harold Erbin, Roji Pius et Ashoke Sen visant à démontrer la symétrie de croisement de la théorie des champs de cordes. Cela fermerait le cercle dans l'histoire de la théorie des cordes qui a commencé avec l'étude de l'amplitude de Veneziano (à une époque où l'on ne savait même pas qu'elle décrivait des cordes !) dont l'intérêt était justement de présenter de façon manifeste la symétrie de croisement. En théorie quantique des champs, des preuves de la symétrie de croisement ont été établies pour les amplitudes à 4 et 5 points en utilisant les propriétés analytiques de la matrice S [115–117]. La généralisation de ces preuves à la théorie des champs de cordes s'est révélée hautement non-triviale car cette théorie est formulée directement en terme de diagrammes de Feynman et pas de façon axiomatique. Le problème majeur vient du fait que les vertex contiennent des exponentielles dans les moments. Pour ces raisons, le projet n'en est qu'à son commencement et ne sera pas discuté dans cette thèse.

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