Modeling user impatience and its impact on performance in mobile networks

Cheick Sanogo

THÈSE DE DOCTORAT DE
l'UNIVERSITÉ PIERRE ET MARIE CURIE

Spécialité

Télécommunications

École doctorale Informatique, Télécommunications et Électronique (Paris)

Numéro de thèse : 2017TELE0009
Présentée par

Cheick SANOGO

Pour obtenir le grade de
DOCTEUR de l'UNIVERSITÉ PIERRE ET MARIE CURIE

Sujet de la thèse :

Modeling user impatience and its impact on performance in mobile networks

soutenue le 30 novembre 2017

devant le jury composé de :

M. Tijani Chahed	Directeur de thèse
M. Walid Benameur	Directeur de thèse
M. Andre-Luc Beylot	Rapporteur
M. Guillaume Urvoy-Keller	Rapporteur
M. Guy Pujolle	Examinateur
M. Lina Mroueh	Examinateur
M. Sondes Kallel Khemiri	Examinateur

Modeling user impatience and its impact on performance in mobile networks

Cheick Sanogo
Institut Mines-Telecom - TelecomSudParis - UMR CNRS 5157
Thesis directors: Tijani Chahed, Walid Benameur

December 18, 2017

Contents

Résumé

Dans cette thèse nous étudions l'impatience des usagers dans les réseaux mobile et nous quantifions son impact sur la performance du système en présence d'usagers téléchargeant des données dans lequel nous développons deux expressions approximatives de la distribution stationnaire du systéme : un modèle agrégé et un modèle détaillé et nous montrons qu'ils sont très proches

du modèle exact.
Nous étudions la mobilité de l'usager téléchargeant des données et pouvant s'impatienter, et nous quantifions son impact sur la performance des réseaux mobile. Nous considérons le cas de la mobilité due à l'impatience et le cas de la mobilité spontanée des usagers tout en considérant la mobilité intra et inter cellulaire.
Nous étudions également l'impatience de l'usager qui regarde une vidéo streaming durant les phases de pré-chargement de la vidéo et de mise en tampon pendant que la vidéo est arrêtée dès le début de la mise en tampon.
Nous étudions à la fin un système constitué d'usagers pouvant s'impatienter, qui est sous contrôle d'un gestionnaire de système, qui à chaque instant de décisions, choisit une action à exécuter dans le but d'optimiser la performance définie du système.
Nous considérons un système dans lequel les usagers arrivent dans le système à des différents instants et le quittent après la fin de leurs transferts de données, ou plus tôt à l'expiration de leurs durées de patience.
Les applications numériques et les simulations nous ont permis de fournir divers métriques de performance telles que le nombre moyen d'usagers, la proportion d'usagers impatients qui quittent le système avant la fin de leurs transferts de fichiers, le débit, la probabilité d'impatience en tenant compte de la localisation de l'usager dans la cellule, la probabilité d'être impatient durant les phases de mise en tampon, la probabilité de mise en tampon lors d'une séance de vidéo streaming, la politique optimale, la taille limite optimale du système dans le but d'optimiser la performance définie du système, etc...

Chapter 1

Introduction

1.1 Scope and objective

The existence of Fourth Generation or 4G networks is an important advancement in mobile network technology. 4G networks are designed to improve wireless capabilities, network speeds and visual technologies. Users are mainly categorized into elastics flows and streaming flows according to the nature of traffic they produce. A user is called elastic flow when he does transfer of digital documents as files, web pages etc..., while he is called streaming flow when he produces video or audio applications. Nowadays, it is crutial for mobile network operators to improve the user quality of experience (QoE) and the quality of service (QoS).

Congestion reduces system performance and user throughputs. This results to elastic users the increase of the session durations, to streaming users the increase of the number of starvations and the increase of the start-up and rebuffering duration. Users may react to this QoE degradation by aborting their connections before their completion by impatience.

Our aim is to study this phenomenon of impatience in mobile networks and quantify its impact on system performance.

Several works on impatience have been developed.

The first model of impatience was developed by Palm [19] in the 1940's. He introduced an inconvenience function of time $I(t),\ t > 0$ the derivative of which he called irritation. As a plausible form for irritation, Palm proposed $dI(t) = c.t\lambda,\ t > 0$, as being proportional to the hazard rate of user abandonment, our impatience function. This reasoning implies that the distribution of patience (the time a user is willing to wait for service) is Weibull. The special case of exponentially distributed patience, as a M/M/n+M referred as Erlang A, corresponds to $\lambda = 0$, which is irritation (or impatience) that is constant over time.

Barrer in [2] studied impatience under a First-In-First-Out (FIFO) discipline that processes in the order that user arrive in queue.

Authors in [7] used simulations to quantify the impact of user impatience on the system performance for several bandwidth sharing disciplines. They reported that up to 20% of the total carried data traffic are aborded.

Authors in [15] modeled the impatience in the case of M/G/1-Processor sharing (PS) for heavy tailed service by considering that impatience is proportional to service time. They evaluated the reneging probability of users with large service time and they used these results to investigate the impact of admission control on a link of a packet network, and concluded that it globally increases the fraction of users who complete their service.

Authors in [16] and [17] analyzed the performance of a GI/GI/1-PS queue with impatient customers in overload by approximating a bandwidth sharing network with a more tractable

fluid model. They found that user impatience has quite a significant negative impact on system performance, also on finite time scales and they showed that admission control reduces the impact of reneging on system performance in some cases.

In [35] authors studied a M/M/1 queue with an exponential reneging variable. They proved that this system can be approximated by either a reflected Ornstein-Uhlenbeck process or a reflected affine diffusion when the arrival rate exceeds or is close to the service rate and the reneging rate is close to 0. They showed numerically that their proposed diffusion approximations in the context of approximating steady state performance characteristics are accurate.

In [38] authors considered a variant of a M/M/c model in which every user in the queue will abandon the queue after an exponentially patience time. In their model time is only discusssed in discrete terms, which can be seen as short time intervals. More than one event can occur in one time slot. They took as starting point that an arrival at time k occurs during time slot k, service starting at time k starts at the beginning of time slot k and departure at time k occurs at the end of time slot k, an abandonmrnt at time k occurs before any arrival at time k. They called this choice the late-arrival and early abandonment. They provided the steady state probability of their model using the generating function method and an infinite recursion. They provided some performance measures such as the mean queue length, the fraction of users which abandons the system, the throughput.

In [36] authors considered a two phase queueing system with impatient users and multiple vacations where customers arrive according to a Poisson process, and receive a first service as well as a second. User may balk with a certain probability and may leave due to impatience after joining the queue without getting service. The service is assumed to be stopped when the server is on vacation or during the first phase of service. Authors derived the probability generating functions of the number of users in the system for various states of the server, they obtained the closed form expressions for various performance measures as the mean system sizes for various states of the server, the average rate of balking, the average rate of reneging, and the average rate of loss.

Authors in [46] studied the impact of video stream quality on viewer behavior in a data-driven manner. They showed that viewers start to abandon a video if it takes more than 2 seconds to start up, and each incremental delay of 1 second resulting in a 5.8% increase in rate of abandonment.

Author in [47] showed that the abandonment rate of viewers increases when the start-up delay increases, a user who watches a short video is 11.5% more likely to abandon sooner during start-up than a user who watches a long video, a user watching video with a better connectivity is more likely to abandon sooner during start-up.

1.2 Organisation of the manuscript

In chapter 1 we study user impatience and quantify its impact on the performance of mobile networks in the presence of elastic user. We consider a dynamic user setting where users come to the system at different time instants and leave it after a finite duration, either after completion of their data transfers or earlier, at the expiry of some patience duration. We model the stationary distribution of the system and derive several performance metrics such as mean number of users in the system, the probability of impatience, taking into account user location in the cell. We further develop two approximate expressions for the stationary distribution of the system: an aggregate one and a detailed one and show their closeness to the exact model. A transient analysis is also included.

In chapter 2 we study mobility of elastic user who may be impatient and quantify its impact on the performance of the mobile network. We consider the case of mobility due to impatience and the spontaneous mobility of users both intra cell and inter cell. We model the stationary distribution of the system and derive several performance metrics such as mean number of users, the proportion of impatient users who quit the system before completing their file transfers and the throughtput.

In chapter 3 we study the streaming user impatience during the prefetching and the rebuffering phases when starvation happens. We first model the buffer as a M/M/1 queue and we introduce the patience duration of streaming user by considering a packet level analysis in which the video size is assumed to be infinite,for both deterministic and exponential patience durations. Secondly we model the continuous time playback taking into account the flow dynamics in the system constituted of several regions scheduled as a processor sharing and we consider the case of deterministic and exponential patience durations. We derive several the performance metrics such as the probability to be impatient during rebuffering and the probability of starvation.

In chapter 4 we study a system with impatient users controlled by a system manager who has to choose at each decision epoch an action to make in order to optimize the system performance. In the first part the set of actions to be chosen by the system manager is assumed to be finite and the control is dropping and blocking in each region of the system. Classical results of average cost markov decision process for semi markov processes allow us to derive the optimal policy that is the path of optimal decisions to be made by the system manager at each decision epoch in order to optimize the system performance through the value iteration algorithm and the modified policy iteration algorithm. In the second part we study firstly a system with one region and then generalize it to the case of multiple regions using our aggregate model developed in the first chapter of the thesis. The set of actions to be chosen by the system manager is assumed to be a real compact or a compound of real compacts. We provide a theorem that allows us to derive recursively the optimal policy and the optimal system size in order to optimize the system performance.

Finally in chapter 5 we conclude the thesis and present some perspectives for future work.

Chapter 2

Modeling and analysis of user impatience in mobile networks

Abstract

We study in this work user impatience and quantify its impact on the performance of mobile networks, notably 4G LTE, in the presence of data flows experiencing heterogeneous radio conditions. We consider a dynamic user setting where users come to the system at different time instants and leave it after a finite duration, either after completion of their data transfers or earlier, at the expiry of some patience duration. We model the stationary distribution of users in the system and derive several performance metrics such as mean transfer times and the probability of impatience, taking into account user location in the cell. We further develop two approximate expressions for the stationary distribution of the system: an aggregate one and a detailed one and show their closeness to the exact model. We validate our model against simulations and show trends for several performance metrics as a function of impatience rate. A transient analysis is also included; it yields insights on the system performance before reaching steady-state.

2.1 Introduction

With the advent of smartphones and tablets, mobile traffic has exploded and is expected to continue its explosion in the upcoming years. While 3G networks have been for several years under-utilized, they are becoming more and more congested. Even if the deployment of 4G Long Term Evolution (LTE) networks will substantially increase the capacity of mobile networks, it is expected that it will also amplify the phenomenon of mobile traffic increase as more users will be willing to take advantage from the data rates promised by these 4G networks. Congestion is thus inevitable at some point and it is hence important to study its impact on the mobile network performance.

As the large majority of traffic in mobile networks is composed of elastic data applications, congestion reduces user throughputs and extend their session durations. Users may react to this Quality of Experience (QoE) degradation by aborting their connections before their completion. On one hand, the repetition of this situation may lead to user churn, and, on the other hand, impatience of some users alleviates the network load and may enhance the performance of the other patient users who do not choose to quit the system before the completion of their service.

Impatience has been the object of several works dealing with fixed networks.

In [6], a new version of the Erlang formula has been derived taking into account user impatience, resulting in so-called Erlang-I formula (I stands for impatience; other works call it the Erlang-A formula, A standing for abandonment). This formula is applicable for the case of streaming like flows where the service duration is independent of the quantity of resources obtained by the user, unlike our present case of data traffic.

Stanford [3] focuses primarily on exact performance analysis under FIFO discipline. He considers a GI/G/1 queueing system where the nth arrival may renege if his service does not begin before an elapsed random time T_n. An expression for the average fraction of customers who renege from the system, an expression for the waiting time distribution for all arrivals to the system, the waiting time distribution for arrivals who reach the server, the distribution for virtual waiting time in the queue, the distribution of the steady-state number of users in the system and the number of users who leave the system by completing their service are found and written in terms of the distribution of the work seen by an arbitrary arrival to the system.

In [8], the authors modeled data traffic at the flow level and considered impatience of users in the overload regime, when the mean arrival intensity is larger than the mean service rate.

In [18], authors analyze impatience in a call center in which they consider that user's patience is exponentially distributed and the system's capacity is unlimited (M/M/N+M). They outline a method for exact analysis of M/M/N+M model and then proceed with an asymptotic analysis in a regime that is appropriate for large call centers.

In [20] authors considered the problem of sheduling impatient users in a G/GI/1 queue. For that they assumed that every user has a random deadline to begin its service. Given this deadline distribution, a scheduling policy decides the user service order and which user to reject since their dealines have expired and do not leave the queue automatically. They showed that LIFO (last-in first out) is an optimal service order when the deadlines are independent and identically distributed random variables with a concave cumulative distribution function. They also showed that in case of unknown waiting time, the optimal policy for a M/M/1 queue will be the LIFO-PO (push-out) policy.

$N_t = \left(N_t^1, \cdots N_t^r\right)$

In [37] authors studied a M/M/c queue with $c = 1$, $1 < c < \infty$ and $c = \infty$ in a 2-phase fast (1) and slow (0), with impatient users. The service time is assumed to be exponential with parameter μ in fast phase and μ_0 in slow phase with $\mu_0 \leq \mu$. They assumed that each user upon arrival has an individual timer, exponentially distributed. If the system does not change from

phase 0 to phase 1 before user's timer expires, user abandons the queue never to return. For the three models that are the single server case, the multiple server case and the infinite server case, they derived explicit expressions for the probability density function of the number of users in the system, both when the servers are slow and when the system functions normally and they calculated the mean total number of users in the system.

Our aim in this chapter is to study user impatience in the context of mobile cellular networks, with an application to LTE, and quantify its impact on the system performance. We model the system at the flow level for a realistic dynamic setting where users come to the system at different time epochs and leave it after a finite duration, either upon the completion of their data transfers or, in case of impatience, when their patience tolerance is over.

We study the system in steady-state and in transient regimes and obtain analytical expressions for several quality of service metrics reflecting performance, such as mean transfer times, and probability that a user leaves the system because of impatience. And this for several assumptions on the distributions of the file size and patience duration. These metrics take into account the heterogeneity of radio conditions for the different users in the cell, in contrast with existing works on fixed networks where the capacity of the system is constant. Our numerical and simulation results illustrate the trends for these metrics, notably as a function of the mean impatience rate.

The remainder of this chapter is organized as follows. In section 2.2 there are the system description. In section 2.3 we do the steady state analysis. In section 2.4 we derive some QoS metrics. In section 2.5 we do the transient regime analysis. Numerical applications and simulations are done in section 2.6. We conclude the chapter in section 2.7.

2.2 System model

2.2.1 System description

We consider an OFDMA-based homogeneous cellular network and focus on the downlink of one cell with a single base station at its center. With OFDMA, the total bandwidth, which we denote by W, is divided into N orthogonal subcarriers and can be shared between the different users present in the cell in the same time slot.

Due to path loss, the Signal-to-Interference and Noise Ratio (SINR) is lower at the cell edge than at the cell center. This leads to a cell capacity $C(r)$ that depends on the distance r between the user and the base station. In order to obtain this throughput, we make use of a static simulator as described in [5]. This throughput is illustrated in Figure 2.1 for an LTE system in an urban environment.

As can be seen in this figure, the throughput decreases when the user gets further away from the base station. Let $C_1 > C_2 > .... > C_l > .... > C_r$ be the set of throughputs at different positions l in the cell and p_l the probability that the user arrives to the cell in region l (this corresponds to a discretization of Figure 2.1 into r regions where the throughput is almost constant in each region).

2.2.2 System model without user impatience

We assume that data flows arrive to the system following a Poisson process with intensity λ. In the absence of user impatience, the service duration for a single user in position l when he is alone in the system, is given by $T^l = \frac{\sigma}{C_l}$ where σ is the flow size.

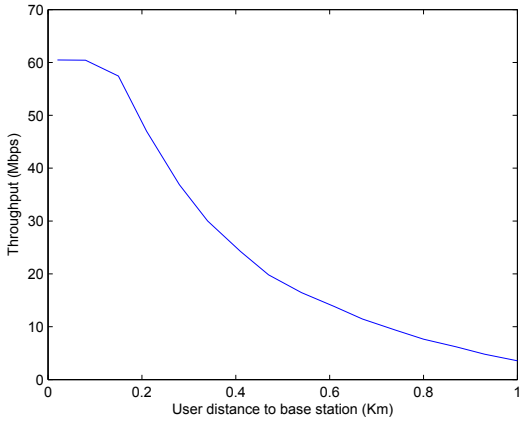

Figure 2.1: Throughput for a user who is alone in the cell, and who is located in different positions. We consider an LTE system using 10 MHZ of spectrum at the 2.6 GHZ band. Cell radius is equal to 1 Km and a MIMO 2*2 scheme is considered.

In the presence of more than one user in the system, the capacity of the cell is shared between them: each user receives a throughput equal to $\gamma_l = \frac{C_l}{n}$ when there are n users in the system. Such a system can be modeled as an M/G/1 - Processor Sharing (PS) queue [4]. In this case, each flow of class l will have a service duration equal to $T^l = \frac{n\sigma}{C_l}$.

2.3 Steady-state analysis

We now study the stationary distribution of the number of flows in the system. We focus on the case where both file size σ and the patience duration that we note by τ are exponentially distributed.

2.3.1 Detailed steady-state analysis

Let us consider the process $N_t = \left(N_t^1, ...N_t^r\right)$, where r is the number of regions in the cell (each with different radio conditions and hence different capacity) and N_t^l is the number of users in region l at time t.

Case of independence between service and patience durations

Let us first assume that the flow size and the patience duration are independent and exponentially distributed with parameters μ_0 and μ, respectively:

$$f_\sigma(t) = \mu e^{-\mu t} 1_{[0,\infty]}(t)$$

$$f_\tau(t) = \mu_0 e^{-\mu_0 t} 1_{[0,\infty]}(t)$$

The service duration in region l with capacity C_l: $T^l = \frac{\sigma}{C_l}$, is also exponentially distributed with parameter $\mu_l = \mu C_l$.

Each flow in region l has a patience duration τ^l exponentially distributed with parameter μ_0^l, and we assume that

$$\mu_0^r \leq \cdots \leq \mu_0^1$$

which means that users at the cell edge become more impatient than users at cell center due to radio conditions.

With these assumptions, the process N_t is an irreducible Markov process with the following intensity matrix:

$$q_{N,N+e_l} = \lambda_l = \lambda p_l$$
$$q_{N,N-e_l} = \frac{n_l}{\sum_{l=1}^{r} n^l}\mu_l + n_l\mu_0^l$$
$$q_{N,N} = -\left(\lambda + \sum_{l=1}^{r} q_{N,N-e^l}\right)$$
$$q_{0,0} = -\lambda$$

where $N = (n_1, ..., n_r)$; n_l being the number of users in region l, $e_l = (0, ...0, 1, 0, ...0)$ and $\lambda = \sum_{l=1}^{r} \lambda_l$; λ_l being the mean arrival rate to region l, and p_l is the probability for the arriving user to be in region l.

The stationary distribution $\pi(N)$ is solution of:

$$\pi Q = 0 \text{ and } \sum_{N\in\mathbb{N}^r} \pi(N) = 1$$

Patience duration function of flow size

It is more realistic to assume that the patience duration depends on the file size, since patience grows with file size, i.e., $\tau = F(\sigma)$.

For simplicity, we assume as in [8] that F is linear:

$$\tau_l = a_l\sigma$$

where $\frac{1}{a_l}$ represents the minimum throughput required to transfer very large documents in region l, referred to as the sustainable throughput; and we assume that the flow size is exponentiel of parameter μ.

Each flow in region l has a patience duration $\tau^l = a_l\sigma$; thus its sojourn time in region l when there are n flows in the system is $T^l_{eff} = \min\left\{nT^l, \tau^l\right\}$ is exponential of parameter $\frac{\mu}{\min\left\{\frac{n}{C_l}, a_l\right\}}$.

The process $N_t = \left(N_t^1, ..., N_t^r\right)_{t>0}$ where N_t^l denotes the number of flows in region l is an irreducible Markov process with the following intensity matrix:

$$q_{N,N+e_l} = \lambda_l = \lambda p_l$$
$$q_{N,N-e_l} = \frac{n_l\mu}{\min\left\{\frac{\sum_{k=1}^{r} n_k}{C_l}, a_l\right\}}$$
$$q_{N,N} = -\left(\lambda + \sum_{l=1}^{r} q_{N,N-e_l}\right)$$

The stationary distribution as previously is the solution to:

$$\pi Q = 0 \text{ and } \sum_{N\in\mathbb{N}^r} \pi(N) = 1$$

2.3.2 Approximate aggregate steady-state analysis

We now derive an approximate closed-form solution for the aggregate steady-state probabilities. We first notice that the actual service duration for a single user in the system located in region l with impatience is given by: $T^l_{eff} = \min\{T^l, \tau^l\}$ where T^l is given by $\frac{\sigma}{C_l}$. In the case of n flows in progress in the system, the actual service duration is given by $T^l_{eff}(n) = \min\{\frac{n\sigma}{C_l}, \tau^l\}$.

Our approximation will be made by assuming that each user in the system is considered to be in region l with probability p_l, its service time will be T_l and its patience duration τ^l, thus its effective mean time in the system when there n active users in the system will be given by:

$$E\left(T_{eff}(n)\right) = \sum_{l=1}^{r} E\left(\min\{\frac{n\sigma}{C_l}, \tau^l\}\right) p_l \tag{2.1}$$

So we assume that the probility that user arrives in a region and the probabity that he is localized in this region are equal.

This resembles to an M/G/1 - State Dependent Processor Sharing (SDPS) queue where the system load, when n users are active, is given by the product of the traffic intensity and the service duration:

$$\rho_{eff}(n) = \lambda E\left(T_{eff}(n)\right) = \lambda \sum_{l=1}^{r} E\left(\min\{\frac{n\sigma}{C_l}, \tau^l\}\right) p_l \tag{2.2}$$

An M/G/1 - SDPS queue whose load is described by this equation is always stable as the sojourn time of the users in the system is always finite (since it is bounded by τ). The number of users in the system can however increase and tend to infinity; in which case, no users will be served at all and they would all leave the system after the impatience duration. This would not happen as long as the following condition holds:

$$\sum_{n=0}^{\infty} \prod_{m=1}^{n} \frac{\rho_{eff}(m)}{m} < \infty \tag{2.3}$$

The stationary distribution of this approximate system is given by:

$$\pi(n) = \frac{\prod_{m=1}^{n} \frac{\rho_{eff}(m)}{m}}{\sum_{i=0}^{\infty} \prod_{j=1}^{i} \frac{\rho_{eff}(j)}{j}} \tag{2.4}$$

Note that the limit after which all users will depart because of impatience without finishing their service is $\pi(0)^{-1} < \infty$.

We give next the derivation of this approximation using some assumptions about the repartition of the users in the cell.

2.3.3 Another proof of Aggregate steady-state analysis

Derivation of equation (2.4)

We now describe the aggregate model for the stationary distribution of the total number of flows in the system without considering the detailed number of users in each region in the system.

We assume Poisson arrivals, the service time and patience duration have a general distribution and there is no assumption of independence between them.

Let us define ξ a type variable which assigns to each user the region in which he is located. ξ is assumed to be a discrete random variable taking values in $\{1, ..., r\}$, and we assume that $P[\xi = l] = p_l$. The time spent by a flow in the system when there are n flows in progress is given by $T_{eff}(n) = \min\{nT, \tau\}$, where T and τ are defined by conditional distributions with respect to ξ:
$(T \mid \xi = l)$ and T^l have the same distribution,
$(\tau \mid \xi = l)$ and τ^l have the same distribution;
where T^l and τ^l are respectively the service time and the patience duration in region l.

Thus

$$P[T < y \mid \xi = l] = P\left[T^l < y\right]$$

$$P[\tau < y \mid \xi = l] = P\left[\tau^l < y\right].$$

Let us recall that p_l is the probability that user arrives in region l. Here we do the approximation that the probability that a user arrives in a region and the probability that he is localized in this region are equal.

For simplicity in the notations we denote T_{eff} by T and $\min\left\{nT^l, \tau^l\right\}$ by T^l.

Let us define the process $X(t) = \left(n(t), T_1, \cdots, T_{n(t)}\right)$, where $n(t)$ is the number of flows in the system at time t, and T_i the spent time by the ith flow. We do so by labeling each flow in the system. If one flow arrives when there are n flows present in the system, all flows will be allocated one of the $(n+1)$ labels with probability $\frac{1}{n+1}$. And when a flow leaves the system, all flows with higher labels in the system have their labels decreased by one.

We denote by g^l and G^l respectively the density function and the distribution function of $\min\left\{nT^l, \tau^l\right\}$. Let $G(y) = P[T < y]$ and $h^l(y) = \frac{g^l(y)}{1-G(y)}$.

Given that the sojourn time T of a flow has lasted for a time y, the probability that he leaves the system within a time interval ϵ is given by:

$$P[T < y + \epsilon / T > y] = \frac{P[T < y + \epsilon, T > y]}{P[T > y]} = \sum_{l=1}^{r} \frac{P\left[T^l < y + \epsilon, T^l > y\right] p_l}{1 - G(y)}$$

$$= \sum_{l=1}^{r} \frac{G^l(y+\epsilon) - G^l(y)}{1 - G(y)} p_l = \sum_{l=1}^{r} h^l(y) p_l \epsilon + o(\epsilon)$$

For simplicity, we add the following notation:

$$P[n(t) = n, T_1 > y_1, \cdots, T_n > y_n] = P[n, y_1, \cdots, y_n : t].$$

The process $X(t)$ is a continuous Markov-process on continuous space. We now look for its stationary distribution if it exists.

We assume that all $y_i > 0$. The event that at time $t + \epsilon$ there are n flows in the system and the different times spent $y_i + \epsilon$ is the union of the following different events:

- at time t there were n flows already present in the system and nothing happens for the duration ϵ;

- at time t there were $n+1$ flows in the system, and one flow has left the system before time $t + \epsilon$;

- at time t there were $n - 1$ flows in the system, and one flow arrives before time $t + \epsilon$.

We note that the probability of a third event is null because of the assumption $y_i > 0$.

$$p\left(n, y_1 + \epsilon, \cdots, y_n + \epsilon : t + \epsilon\right) =$$

$$\left[1 - \left[\lambda + \sum_{l=1}^{r}\sum_{i=1}^{n} h^l(y_i)p_l\right]\epsilon\right] p\left(n, y_1, \cdots, y_n : t\right)$$

$$+ \sum_{l=1}^{r}\sum_{i=1}^{n+1}\int_0^\infty P\left[n+1, y_1, \cdots, y_i, u, y_{i+1}, \cdots, y_n : t\right] h^l(u)p_l\epsilon du + o(\epsilon).$$

The first term on the right hand side reflects the situation in which no arrivals or departure occurs in the time interval $(t, t+\epsilon)$ and all that happens is that the spent service times of the customers who are present at time t age by an amount ϵ, while the summand in the second term reflects the situation in which there are $n+1$ customers present at time t and the one labelled $i+1$ departs, with all higher labels being decreased by one, as stipulated by assumption.

Let us divide this equation by ϵ and take the limit as ϵ tends to zero:

$$\frac{\partial P\left[n, y_1, \cdots, y_n : t\right]}{\partial t} + \sum_{i=1}^{n}\frac{\partial P\left[n, y_1, \cdots, y_n : t\right]}{\partial y_i}$$

$$= -\left[\lambda + \sum_{l=1}^{r}\sum_{i=1}^{n} h_i^l(y_i)p_l\right] P\left[n, y_1, \cdots, y_n : t\right]$$

$$+ \sum_{l=1}^{r}\sum_{i=1}^{n+1}\int_0^\infty P\left[n+1, y_1, \cdots, y_i, u, y_{i+1}, \cdots, y_n : t\right] h^l(u)p_l du$$

Now we need to know what happens at arrival instants. We consider the event where $n-1$ flows were present in the system at time t and a new flow arrives and is allocated position $i+1$ before time $t+\epsilon$:

$$\int_0^\epsilon P\left[n, y_1 + \epsilon, \cdots, y_i + \epsilon, u, \cdots, y_n + \epsilon : t + \epsilon\right] du$$

$$= \frac{\lambda\epsilon}{n} P\left[n-1, y_1, \cdots, y_n : t\right] + o(\epsilon)$$

Dividing by ϵ and letting ϵ tend to zero we obtain:

$$P\left[n, y_1, \cdots, y_i, 0, \cdots, y_n : t\right] = \frac{\lambda}{n} P\left[n-1, y_1, \cdots, y_n : t\right]$$

For the stationary distribution, the first term is null:

$$\frac{\partial \pi\left[n, y_1, \cdots, y_n\right]}{\partial t} = 0$$

and we have:

$$\sum_{i=1}^{n}\frac{\partial \pi\left[n, y_1, \cdots, y_n\right]}{\partial y_i}$$

$$= -\left[\lambda + \sum_{l=1}^{r}\sum_{i=1}^{n} h^l(y_i)p_l\right] \pi\left[n, y_1, \cdots, y_n\right]$$

$$+\sum_{l=1}^{r}\sum_{i=1}^{n+1}\int_0^\infty \pi\left[n+1, y_1, \cdots, y_i, u, y_{i+1}, \cdots, y_n\right] h^l(u) p_l du.$$

And

$$\pi\left[n, y_1, \cdots, y_i, 0, \cdots, y_n\right] = \frac{\lambda}{n}\pi\left[n-1, y_1, \cdots, y_n\right]$$

We note that $G_i(0) = 0$.
By using the definition of derivation as a limit we obtain :

$$\frac{\partial \pi\left[n, y_1, \cdots, y_n\right]}{\partial y_i} = -\left[\sum_{l=1}^{r} h^l(y_i) p_l\right] \pi\left[n, y_1, \cdots, y_n\right]$$

Then the equation becomes:

$$\lambda\pi\left[n, y_1, \cdots, y_n\right] = \sum_{l=1}^{r}\sum_{i=1}^{n+1}\int_0^\infty \pi\left[n+1, y_1, \cdots, y_i, u, y_{i+1}, \cdots, y_n\right] h^l(u) p_l du.$$

These equations have a unique solution that sums to one [9] .
If we integrate all variables y_i, we obtain $\pi(n)$:

$$\pi(n) = \int_0^\infty \cdots \int_0^\infty \pi\left[n, y_1, \cdots, y_n\right] dy_1 \cdots dy_n,$$

The first integration gives:

$$\int_0^\infty \pi\left[n, y_1, \cdots, y_n\right] dy_1$$

$$= \int_0^\infty \pi\left[n, 0, \cdots, y_n / T_1 > u\right] P\left[T_1 > u\right] du$$

By independance

$$= \int_0^\infty \pi\left[n, 0, \cdots, y_n\right] \left[1 - G_1(u)\right] du$$

$$= \frac{\lambda}{n}\pi\left[n-1, y_2, \cdots, y_n\right] \int_0^\infty \left(1 - G_1(u)\right) du$$

$$= \frac{\lambda}{n\mu(n)}\pi\left[n-1, y_2, \cdots, y_n\right]$$

Then by recurrence:

$$\pi(n) = \pi(0) \prod_{k=1}^{n} \frac{\rho(k)}{k}$$

where $\rho(k) = \frac{\lambda}{\mu(k)}$, $E\left[\min\left\{kT, \tau\right\}\right] = \frac{1}{\mu(k)}$ and $\pi(0)^{-1} = \sum_n \prod_{k=1}^{n} \frac{\rho(k)}{k}$.

2.3.4 Approximate detailed steady-state analysis

We now derive an approximate, product-form expression for the steady state distribution of the users in the different locations in the system. Indeed, the Markov chain defined in section 2.3.1 is not reversible. In order to obtain an approximate product expression, we make the assumption

that all users in the region near the base station finish their service before users in the region further away from the base station. It is thus possible to obtain the following:

$$\pi(N) = \pi(0) \prod_{i_1=1}^{n_1} \frac{\rho_1(\sum_{k=1}^{r} n_k - i_1 + 1)}{n_1 - i_1 + 1} \prod_{i_2=1}^{n_2} \frac{\rho_2(\sum_{k=2}^{r} n_k - i_2 + 1)}{n_2 - i_2 + 1} \cdots \prod_{i_r=1}^{n_r} \frac{\rho_r(\sum_{k=r}^{r} n_k - i_r + 1)}{n_r - i_r + 1} \tag{2.5}$$

where $\rho_l(k) = \frac{\lambda_l}{\mu^l(k)}$ and $\frac{1}{\mu^l(k)} = E\left[\min\left\{kT^l, \tau^l\right\}\right]$.

Derivation of equation (2.5)

As in section 2.3.2 we treat the case of general service time T and general patience duration τ, with no assumption of independence between them. Users arrive in region l according a Poisson process of rate λ_l, and $\lambda = \sum_{l=1}^{r} \lambda_l$ is the arriving rate in the system.

Let us denote by $n_l(t)$ the number of flows in region l at time t and T^l_{eff} the time spent by one flow in region l. We recall that $T^l_{eff} = \min\left\{nT^l, \tau^l\right\}$ when there are n flows in the system, where $T^l = \frac{\sigma}{C_l}$ is the service time in region l and τ^l the patience duration in region l. To simplify the notation, we denote T^l_{eff} by T^l the time spent by one flow in region l.

Now let us define the process

$$X(t) = \left(n_1(t), \cdots, n_r(t), T^1_1, \cdots, T^1_{n_1(t)}, \cdots, T^r_1, \cdots, T^r_{n_r(t)}\right);$$

where T^l_i is the time spent by the ith flow in region l. We do so by labeling each flow in each region.

If one flow arrives in region l when there are n_l present flows, all flows in the region will be allocated one of the $(n_l + 1)$ labels with probability $\frac{1}{n_l+1}$. And when a flow departs from the region, all flows of higher labels in the region have their labels decreased by one.

We denote by G^l_i the distribution function of T^l_i, by g^l_i its density function which depends on the number of flows in the system and by $h^l_i(y) = \frac{g(y)^l}{1-G(y)^l}$ the hazard function.

Given that a sojourn time of a flow T^l_i has lasted for a time y, the probability that he leaves the system within a time interval ϵ is then given by:

$$P\left[T^l < y + \epsilon / T^l > y\right] = \frac{G^l(y+\epsilon) - G^l(y)}{1 - G^l(y)} = h^l_i(y)\epsilon + o(\epsilon).$$

For simplicity let us add some notation:

$$N(t) = (n_1(t), \cdots, n_r(t)),$$

$$N = (n_1, \cdots, n_r),$$

$$P\left[N(t) = N, T^1_1 > y^1_1, \cdots, T^1_{n_1} > y^1_{n_1}, \cdots, \cdots, T^r_{n_r} > y^r_{n_r}\right] = P\left[N, y^1_1, \cdots, y^1_{n_1}, \cdots, y^r_{n_r} : t\right].$$

The process $X(t)$ is a continuous Markov-process on continuous space. We look for its stationary distribution, if it exist.

We assume that all $y^l_i > 0$. The event that at time $t + \epsilon$ there are N flows in the system and the different times spent $y^l_i + \epsilon$ are the union of the following different events:
- at time t there were N flows already present in the system and nothing happens for the duration ϵ;
-at time t there were $N + e_l$ flows in the system, and one flow of region l has left the system

before the time $t+\epsilon$;
-at time t there were $N-e_l$ flows in the system, and one flow came in region l before the time $t+\epsilon$.

The probability of third event is small $P\{Third\ event\} = 0$, because of the assumption $y_i^l > 0$.

$$P\left[N, y_1^1+\epsilon, \cdots, y_{n_1}^1+\epsilon, \cdots, y_1^r+\epsilon, \cdots, y_{n_r}^r+\epsilon : t+\epsilon\right]$$

$$= \left[1 - \left[\lambda + \sum_{l=1}^{r}\sum_{i=1}^{n_l} h_i^l(y_i^l)\right]\epsilon\right] P\left[N, y_1^1, \cdots, y_{n_1}^1, \cdots, y_{n_r}^r : t\right] + \sum_{l=1}^{r}\sum_{i=1}^{n_l+1}\int_0^\infty$$

$$P\left[N+e_l, y_1^1, \cdots, y_{n_1}^1, \cdots, y_1^l, \cdots, y_i^l, u, y_{i+1}^l, \cdots, y_{n_l}^l, \cdots, y_{n_r}^r : t\right] h_i^l(u)\epsilon du + o(\epsilon)$$

We divide this equation by ϵ and let ϵ tend to zero, we obtain:

$$\frac{\partial P\left[N, y_1^1, \cdots, y_{n_1}^1, \cdots, y_{n_r}^r : t\right]}{\partial t} + \sum_{l=1}^{r}\sum_{i=1}^{n_l}\frac{\partial P\left[N, y_1^1, \cdots, y_{n_1}^1, \cdots, y_{n_r}^r : t\right]}{\partial y_i^l}$$

$$= -\left[\lambda + \sum_{l=1}^{r}\sum_{i=1}^{n_l} h_i^l(y_i^l)\right] P\left[N, y_1^1, \cdots, y_{n_1}^1, \cdots, y_{n_r}^r : t\right] + \sum_{l=1}^{r}\sum_{i=1}^{n_l+1}\int_0^\infty$$

$$P\left[N+e_l, y_1^1, \cdots, y_{n_1}^1, \cdots, y_1^l, \cdots, y_i^l, u, y_{i+1}^l, \cdots, y_{n_l}^l, \cdots, y_{n_r}^r : t\right] h_i^l(u)du$$

When $N = 0$ the equation becomes:

$$\lambda_l P\left[0 : t\right] = \int_0^\infty P\left[e_l, u : t\right] h^l(u)du$$

Now we have to know what happens at arrival instants. For this we consider the event where $N - e_l$ flows were present in the system at time t and a new flow arrived in region l and was allocated position $i+1$ in region l before time $t+\epsilon$:

$$\int_0^\epsilon P\left[N, y_1^1+\epsilon, \cdots, y_1^l+\epsilon, \cdots, y_i^l+\epsilon, u, \cdots, y_{n_r}^r+\epsilon : t+\epsilon\right] du$$

$$= \frac{\lambda_l \epsilon}{n_l} P\left[N-e_l, y_1^1, \cdots, y_{n_r}^r : t\right] + o(\epsilon)$$

Dividing by ϵ and letting ϵ tend to zero, we obtain:

$$P\left[N, y_1^1, \cdots, y_1^l, \cdots, y_i^l, 0, \cdots, y_{n_r}^r : t\right] = \frac{\lambda_l}{n_l} P\left[N-e_l, y_1^1, \cdots, y_{n_r}^r : t\right]$$

As we look for the stationary distribution, the time derivative is null, and we have:

$$\sum_{l=1}^{r}\sum_{i=1}^{n_l}\frac{\partial \pi\left[N, y_1^1, \cdots, y_{n_1}^1, \cdots, y_{n_r}^r\right]}{\partial y_i^l}$$

$$= -\left[\lambda + \sum_{l=1}^{r}\sum_{i=1}^{n_l} h_i^l(y_i^l)\right] \pi\left[N, y_1^1, \cdots, y_{n_1}^1, \cdots, y_{n_r}^r\right] + \sum_{l=1}^{r}\sum_{i=1}^{n_l+1}\int_0^\infty$$

$$\pi\left[N+e_l, y_1^1, \cdots, y_{n_1}^1, \cdots, y_1^l, \cdots, y_i^l, u, y_{i+1}^l, \cdots, y_{n_l}^l, \cdots, y_{n_r}^r\right] h_i^l(u)du$$

$$\lambda_l \pi\left[0\right] = \int_0^\infty \pi\left[e_l, u\right] h^l(u) du$$

$$\pi\left[N, y_1^1, \cdots, y_1^l, \cdots, y_i^l, 0, \cdots, y_{n_r}^r\right] = \frac{\lambda_l}{n_l} \pi\left[N - e_l, y_1^1, \cdots, y_{n_r}^r\right]$$

We note that $G_i^l(0) = 0$.
We also note that:

$$\frac{d\left(1 - G_i^l(y_i^l)\right)}{dy_i^l} = -g_i^l(y_i^l) = -h_i^l(y_i^l)\left(1 - G_i^l(y_i^l)\right).$$

By definition of derivative:

$$\frac{\partial \pi\left[N, y_1^1, \cdots, y_{n_1}^1, \cdots, y_{n_r}^r\right]}{\partial y_i^l} = \lim_{k \to 0} \frac{1}{k}$$

$$\left[\pi\left[N, y_1^1, \cdots, y_i^l + k, \cdots, y_{n_r}^r\right] - \pi\left[N, y_1^1, \cdots, y_i^l, \cdots, y_{n_r}^r\right]\right]$$

$$= \pi\left[N, y_1^1, \cdots, y_i^l, \cdots, y_{n_r}^r / T_i^l > y_i^l\right] \lim_{k \to 0} \frac{P\left[T_i^l > y_i^l + k\right] - P\left[T_i^l > y_i^l\right]}{k}$$

$$= \pi\left[N, y_1^1, \cdots, y_i^l, \cdots, y_{n_r}^r / T_i^l > y_i^l\right] \lim_{k \to 0} \frac{\left(1 - G_i^l(y_i^l + k)\right) - \left(1 - G_i^l(y_i^l)\right)}{k}$$

$$= -\pi\left[N, y_1^1, \cdots, y_i^l, \cdots, y_{n_r}^r / T_i^l > y_i^l\right] h_i^l(y_i^l)\left(1 - G_i^l(y_i^l)\right)$$

$$= -h_i^l(y_i^l) \pi\left[N, y_1^1, \cdots, y_i^l, \cdots, y_{n_r}^r\right].$$

Then we have equality of the two sums :

$$\sum_{l=1}^{r} \sum_{i=1}^{n_l} \frac{\partial \pi\left[N, y_1^1, \cdots, y_{n_1}^1, \cdots, y_{n_r}^r\right]}{\partial y_i^l} = -\sum_{l=1}^{r} \sum_{i=1}^{n_l} h_i^l(y_i^l) \pi\left[N, y_1^1, \cdots, y_{n_1}^1, \cdots, y_{n_r}^r\right]$$

Thus we have:

$$\lambda \pi\left[N, y_1^1, \cdots, y_{n_1}^1, \cdots, y_{n_r}^r\right]$$

$$= \sum_{l=1}^{r} \sum_{i=1}^{n_l+1} \int_0^\infty \pi\left[N + e_l, y_1^1, \cdots, y_{n_1}^1, \cdots, y_1^l, \cdots, y_i^l, u, y_{i+1}^l, \cdots, y_{n_l}^l, \cdots, y_{n_r}^r\right] h_i^l(u) du$$

These equations have a unique solution that sums to one [9].
By Fubini, we have:

$$\int_0^\infty \left(1 - G^l(y)\right) dy = \int_0^\infty P\left[T^l > y\right] dy = E\left[\int_0^{T^l} dy\right] = E\left[T^l\right] = \frac{1}{\mu^l};$$

Now we use the assumption that all users in region near the base station finish their service before users in the region far from the base station, and we begin by integrating all variables y_k^1 of region 1:

$$\pi(N) = \int_0^\infty \cdots \int_0^\infty \pi\left[N, y_1^1, \cdots, y_{n_1}^1, \cdots, y_{n_r}^r\right] dy_1^1 \cdots dy_{n_r}^r$$

The first integration gives:

$$\int_0^\infty \pi\left[N, y_1^1, \cdots, y_{n_1}^1, \cdots, y_{n_r}^r\right] dy_1^1 = \int_0^\infty \pi\left[N, 0, \cdots, y_{n_1}^1, \cdots, y_{n_r}^r / T_1^1 > u\right] P\left[T_1^1 > u\right] du$$

By independance

$$= \int_0^\infty \pi\left[N, 0, \cdots, y_{n_1}^1, \cdots, y_{n_r}^r\right]\left[1 - G_1^1(u)\right] du$$

$$= \frac{\lambda_1}{n_1}\pi\left[N - e_1, y_2^1, \cdots, y_{n_r}^r\right] \int_0^\infty \left(1 - G_1^1(u)\right) du$$

$$= \frac{\lambda_1}{n_1 \mu^1(\sum_{k=1}^r n_k)}\pi\left[N - e_1, y_2^1, \cdots, y_{n_r}^r\right],$$

We continue the same process for other variables y_k^1 of region 1, then for variable y_k^2 of region 2 and so on (this is our assumption: all users in region near the base station finish their service before users in the region far from the base station). By recurrence we obtain:

$$\pi(N) = \pi(0) \prod_{i_1=1}^{n_1} \frac{\rho_1(\sum_{k=1}^r n_k - i_1 + 1)}{n_1 - i_1 + 1} \prod_{i_2=1}^{n_2} \frac{\rho_2(\sum_{k=2}^r n_k - i_2 + 1)}{n_2 - i_2 + 1} \cdots \prod_{i_r=1}^{n_r} \frac{\rho_r(\sum_{k=r}^r n_k - i_r + 1)}{n_r - i_r + 1}$$

where $\rho_l(k) = \frac{\lambda_l}{\mu^l(k)}$ and $E\left[\min\left\{kT^l, \tau^l\right\}\right] = \frac{1}{\mu^l(k)}$.

2.4 QoS metrics

Here we use the model defined in section 2.3.1.

Now we focus on one tagged flow which is fixed in region k for example, and we define the process $N_t = \left(N_t^1, ...N_t^r\right)$ where N_t^l is the number of flows in region l seen by the tagged flow.

We add two supplementary states I corresponding to the state in which the tagged flow is impatient and F corresponding to the state in which the tagged flow finishes its service.

The process $(N_t)_t$ is Markovian with the following intensity matrix:

$$q_{N,N+e_l} = \lambda_l = \lambda p_l$$

$$q_{N,N-e_l} = \frac{n_l}{\sum_{i=1}^r n_i + 1}\mu_l + n_l \mu_0^l$$

$$q_{N,I} = \mu_0^k$$

$$q_{N,F} = \frac{\mu_k}{\sum_{i=1}^r n_i + 1}$$

$$q_{N,N} = -\left(\lambda + \mu_0^k + \frac{\mu_k}{\sum_{i=1}^r n_i + 1} + \sum_{i=1}^r q_{N,N-e_i}\right)$$

$$q_N = -q_{N,N}$$

The two states I and F are absorbing, then $(N_t)_t$ is a reducible Makovian process. Let us pose

$$T_I = \inf\left\{t > 0, N_t \in I\right\},$$

$$T_F = \inf\left\{t > 0, N_t \in F\right\},$$

$$T^* = \min\{T_I, T_F\}.$$

The variables T_I, T_F and T^* are stopping times.

Let $I(N) = P[N_{T^*} \in I \mid N_0 = N]$ be the probability that the tagged flow is impatient, and $F(N) = P[N_{T^*} \in F \mid N_0 = N]$ the probability that the tagged flow finishes its service conditioned on initial state $N_0 = N$.

We can see that $I(N) = P[T_I < \infty \mid N_0 = N]$. This is the hitting probability of state I. It can be expressed as $I(N) = E\left[1_{N_{T^*} \in I} \mid Z_0 = N\right] = E[f(N_{T^*}) \mid N_0 = N]$, with $f(n) = 1_{\{n \in I\}}$. Then $I(N)$ can be found as the solution of Dirichlet/FeymannKac problems as follows:

$$Q\ I = 0 \qquad \forall\ N \neq I,\ F$$

$$I(I) = 1$$

$$I(F) = 0$$

Then I is solution of following recurrence equation:

$$I(N) = \sum_{i=1}^{r} \frac{q_{N,N+e_i}}{q_N} I(N+e_i) + \sum_{i=1}^{r} \frac{q_{N,N-e_i}}{q_N} I(N-e_i) + \frac{q_{N,I}}{q_N}.$$

We regroup under matrix form by posing:

$$\mathbb{I} = \begin{pmatrix} I(0) \\ I(e_1) \\ \vdots \\ I(e_r) \\ \vdots \\ I(N-e_1) \\ \vdots \\ I(N-e_r) \\ I(N) \\ I(N+e_1) \\ \vdots \\ I(N+e_r) \\ \vdots \end{pmatrix} \text{ and } \mathbb{E} = \begin{pmatrix} \frac{-q_{0,I}}{q_0} \\ \vdots \\ \vdots \\ \frac{-q_{N,I}}{q_N} \\ \vdots \end{pmatrix}$$

So we have:

$$\mathbb{M}\mathbb{I} = \mathbb{E}$$

where $\mathbb{M}$ is a matrix with coefficient $m_{I,J}$ defined as:

$$M_{0,0} = -1; \quad M_{0,e_i} = \frac{q_{0,e_i}}{q_0};$$

$$M_{N,N-ei} = \frac{q_{N,N-e_i}}{q_N},\ M_{N,N} = -1,\ M_{N,N+ei} = \frac{q_{N,N+e_i}}{q_N}.$$

We solve for F by the same method:

$$F(N) = \sum_{i=1}^{r} \frac{q_{N,N+e_i}}{q_N} F(N+e_i) + \sum_{i=1}^{r} \frac{q_{N,N-e_i}}{q_N} F(N-e_i) + \frac{q_{N,F}}{q_N}.$$

Then we have:

$$\mathbb{M}\mathbb{F} = \mathbb{K}$$

where $\mathbb{F} = \begin{pmatrix} F(0) \\ F(e_1) \\ \vdots \\ F(e_r) \\ \vdots \\ F(N-e_1) \\ \vdots \\ F(N-e_r) \\ F(N) \\ F(N+e_1) \\ \vdots \\ F(N+e_r) \\ \vdots \end{pmatrix}$ and $\mathbb{K} = \begin{pmatrix} \frac{-q_{0,F}}{q_0} \\ \vdots \\ \vdots \\ \frac{-q_{N,F}}{q_N} \\ \vdots \end{pmatrix}$

Please note that $F(N) = 1 - I(N)$.

Let $S(N) = E\left[T^* \mid N_0 = N\right]$ denote the mean sojourn time of the tagged flow in the system conditioned on initial state $N_0 = N$, then S is solution of following equation:

$$S(N) = \frac{1}{q_N} + \sum_{i=1}^{r} \frac{q_{N,N+e_i}}{q_N} S(N+e_i) + \sum_{i=1}^{r} \frac{q_{N,N-e_i}}{q_N} S(N-e_i).$$

We have the following matrix form:

$$\mathbb{M}\mathbb{S} = \mathbb{G}$$

where $\mathbb{S} = \begin{pmatrix} S(0) \\ S(e_1) \\ \vdots \\ S(e_r) \\ \vdots \\ S(N-e_1) \\ \vdots \\ S(N-e_r) \\ S(N) \\ S(N+e_1) \\ \vdots \\ S(N+e_r) \\ \vdots \end{pmatrix}$ and $\mathbb{G} = \begin{pmatrix} -\frac{1}{q_0} \\ \vdots \\ \vdots \\ -\frac{1}{q_N} \\ \vdots \end{pmatrix}$

Let $C(N) = E\left[\int_0^{T_I} \left(N_s^k + 1\right) ds \mid N_0 = N\right]$ be the mean cost of impatience conditioned on $N_0 = N$, then C is solution of the following equation:

$$-QC = g \qquad \forall\ N \neq I$$

$$C(I) = 0$$

where $g(N) = g(n_1, ...n_r) = n_k + 1$.

Then:

$$C(I) = 0$$

$$C(F) = \infty$$

$$C(N) = \sum_{i=1}^{r} \frac{q_{N,N+e_i}}{q_N} C(N+e_i) + \sum_{i=1}^{r} \frac{q_{N,N-e_i}}{q_N} C(N-e_i) + \frac{n_k+1}{q_N}.$$

In matrix form, we have:

$$\mathbb{M}\mathbb{C} = \mathbb{P}$$

where $\mathbb{C} = \begin{pmatrix} C(0) \\ C(e_1) \\ \vdots \\ C(e_r) \\ \vdots \\ C(N-e_1) \\ \vdots \\ C(N-e_r) \\ C(N) \\ C(N+e_1) \\ \vdots \\ C(N+e_r) \\ \vdots \end{pmatrix}$ and $\mathbb{P} = \begin{pmatrix} -\frac{1}{q_0} \\ \vdots \\ \vdots \\ -\frac{n_k+1}{q_N} \\ \vdots \end{pmatrix}$

If we consider that the system is in steady-state regime at the beginning, we obtain the following QoS measures:

$$I = \sum_N I(N)\pi(N)$$

$$F = \sum_N F(N)\pi(N)$$

$$S = \sum_N S(N)\pi(N)$$

$$C = \sum_N C(N)\pi(N)$$

Patience duration function of flow size Now let us consider a tagged flow in region l, and let us define process $N_t = \left(N_t^1, ..., N_t^r\right)_{t>0}$ where N_t^l is the number of flows in region l as seen by the tagged flow. The process N_t is Markovian with intensity matrix given by:

$$q_{N,N+e_l} = \lambda_l = \lambda p_l,$$

$$q_{N,N-e_l} = \frac{n_l \mu}{\min\left\{\frac{\sum_{i=1}^r n_i+1}{C_l}, a_l\right\}},$$

$$q_{N,I} = \frac{\mu}{a_k} 1_{\left\{a_k \leq \frac{\sum_{i=1}^r n_i+1}{c_k}\right\}};$$

$$q_{N,F} = \frac{\mu C_k}{\sum_{i=1}^r n_i + 1} 1_{\left\{\frac{\sum_{i=1}^r n_i+1}{c_k} \leq a_k\right\}};$$

$$q_{N,N} = -\left(\lambda + \sum_{i=1}^{r} q_{N,N-e_i} + q_{N,I} + q_{N,F}\right).$$

We recall that a^l is the coefficient which appears in the expression of patience for a user located in region l: $\tau^l = a^l\sigma$.

The states I and F are absorbing then reccurrent, and the state $\mathbb{N}^r$ is transient. The previous QoS measures are calculated here by replacing $q_{N,M}$ by those in this paragraph.

2.5 Transient regime analyis

Now we study the system without the condition of stability.

We consider the Markov process of tagged flow studied previously in the steady-state analysis part.

Let us pose $B(N,t) = P\left[Z_t \in \{I,F\} \mid Z_0 = N\right]$, the probability that the tagged flow leaves the system at time t conditioned on initial condition $Z_0 = N$. We can show that $B(.,t)$ is solution of the following equation:

$$\frac{\partial B(N,t)}{dt} = QB(N,t),$$

$$B(N,0) = 1_{\{N\in\{I,F\}\}}.$$

Thus:

$$B(N,0) = 0 \quad \forall\ N \neq I,F\ ,$$

$$B(I,0) = B(F,0) = 1\ ,$$

$$\frac{\partial B(N,t)}{dt} = \sum_{i=1}^{r} q_{N,N+e_i}B(N+e_i,t) + q_{N,N}B(N,t) + \sum_{i=1}^{r} q_{N,N-ei}B(N-e_i,t) + q_{N,I}B(I,t) + q_{N,F}B(F,t)$$

$$= \sum_{i=1}^{r} \lambda_i B(N+e_i,t) + \mu_0 + \frac{\mu_k}{\sum_{j=1}^{r} n_j + 1} + q_{N,N}B(N,t) + \sum_{i=1}^{r}\left[\frac{n_i}{\sum_{j=1}^{r} n_j + 1}\mu_i + n_i\mu_0\right] B(N-e_i,t)\ .$$

We apply the Laplace transformation and obtain:

$$\sum_{i=1}^{r} \frac{q_{N,N-e_i}}{q_N}\bar{B}_{N-e_i}(p) - \left[1 + \frac{p}{q_N}\right]\bar{B}_N(p) + \sum_{i=1}^{r} \frac{q_{N,N+e_i}}{q_N}\bar{B}_{N+e_i}(p) = -\left[\frac{B(N,0)}{q_N} + \frac{q_{N,I} + q_{N,F}}{pq_N}\right].$$

which can be regrouped in matrix form:

$$\left\{\mathbb{M} - \frac{p}{q_N}\mathbb{I}\right\}\bar{\mathbb{B}}(p) = \mathbb{D}$$

$$\text{where } \bar{\mathbb{B}}(p) = \begin{pmatrix} \bar{B}_0(p) \\ \bar{B}_{e_1}(p) \\ \vdots \\ \bar{B}_{e_r}(p) \\ \vdots \\ \bar{B}_{N-e_1}(p) \\ \vdots \\ \bar{B}_{N-e_r}(p) \\ \bar{B}_N(p) \\ \bar{B}_{N+e_1}(p) \\ \vdots \\ \bar{B}_{N+e_r}(p) \\ \vdots \end{pmatrix},$$

$$\text{and } \mathbb{D} = \begin{pmatrix} -\frac{pB(0,0)+q_{0,I}+q_{0,F}}{pq_0} \\ \vdots \\ \vdots \\ -\frac{pB(N,0)+q_{N,I}+q_{N,F}}{pq_N} \\ \vdots \end{pmatrix},$$

Let us pose $F(N,t) = P\left[Z_t \in F \mid Z_0 = N\right]$ the probability that the tagged flow finishes its service at instant t conditioned on $Z_0 = N$, and $I(N,t) = P\left[Z_t \in I \mid Z_0 = N\right]$ the probability that the tagged flow leaves the system by impatience at instant t conditioned on $Z_0 = N$. Then by the same method we have:

$$\frac{\partial F(N,t)}{dt} = \sum_{l=1}^{r} q_{N,N+e_l} F(N+e_l,t) + q_{N,N} F(N,t) + \sum_{l=1}^{r} q_{N,N-el} F(N-e_l,t) + q_{N,F},$$

$$F(N,0) = 0 \quad \forall\ N \neq F\ ,$$

$$F(F,0) = 1\ ;$$

With Laplace transformation we obtain:

$$\left\{\mathbb{M} - \frac{p}{q_N}\mathbb{I}\right\}\bar{\mathbb{F}}(p) = \mathbb{L}$$

$$\text{where: } \bar{\mathbb{F}}(p) = \begin{pmatrix} \bar{F}_0(p) \\ \bar{F}_{e_1}(p) \\ \vdots \\ \bar{F}_{e_r}(p) \\ \vdots \\ \bar{F}_{N-e_1}(p) \\ \vdots \\ \bar{F}_{N-e_r}(p) \\ \bar{F}_N(p) \\ \bar{F}_{N+e_1}(p) \\ \vdots \\ \bar{F}_{N+e_r}(p) \\ \vdots \end{pmatrix},$$

$$\text{and } \mathbb{L} = \begin{pmatrix} -\frac{pF(0,0)+q_{0,F}}{pq_0} \\ \vdots \\ \vdots \\ -\frac{pF(N,0)+q_{N,F}}{pq_N} \\ \vdots \end{pmatrix},$$

We also have:

$$\frac{\partial I(N,t)}{dt} = \sum_{l=1}^{r} q_{N,N+e_l} I(N+e_l,t) + q_{N,N} I(N,t) + \sum_{l=1}^{r} q_{N,N-e_l} I(N-e_l,t) + q_{N,I},$$

$$I(N,0) = 0 \quad \forall\ N \neq I\ ,$$

$$I(I,0) = 1\ ;$$

With Laplace transformation we obtain:

$$\left\{ \mathbb{M} - \frac{p}{q_N} \mathbb{I} \right\} \bar{\mathbb{I}}(p) = \mathbb{H}$$

$$\text{where: } \bar{\mathbb{I}}(p) = \begin{pmatrix} \bar{I}_0(p) \\ \bar{I}_{e_1}(p) \\ \vdots \\ \bar{I}_{e_r}(p) \\ \vdots \\ \bar{I}_{N-e_1}(p) \\ \vdots \\ \bar{I}_{N-e_r}(p) \\ \bar{I}_N(p) \\ \bar{I}_{N+e_1}(p) \\ \vdots \\ \bar{I}_{N+e_r}(p) \\ \vdots \end{pmatrix},$$

and $\mathbb{H} = \begin{pmatrix} -\frac{pI(0,0)+q_{0,I}}{pq_0} \\ \vdots \\ \vdots \\ -\frac{pI(N,0)+q_{N,I}}{pq_N} \\ \vdots \end{pmatrix}$,

Now let us consider the Markov process $Z_t = \left(Z_t^1, ...Z_t^r\right)$ (without considering any tagged flow) where Z_t^i is the number of flows in region i. Its intensity matrix is given by:

$$q_{N,N+e_l} = \lambda_l = \lambda p_l,$$
$$q_{N,N-e_l} = \frac{n_l}{\sum_{k=1}^{r} n_k}\mu_l + n_l\mu_0,$$
$$q_{N,N} = -\left(\lambda + \sum_{l=1}^{r} q_{N,N-e_l}\right);$$

Let us denote by $T(N,t) = E\left[\sum_{i=1}^{r} Z_t^i T^l \mid Z_0 = N\right]$ the mean service time in region l at instant t conditioned on $Z_0 = N$, and $S(N,t) = E\left[\min\left\{\sum_{i=1}^{r} Z_t^i T^l, \tau\right\} \mid Z_0 = N\right]$ the mean sojourn time of flow in region l. Then $T(N,t)$ and $S(N,t)$ are solutions of the following equations:

$$\frac{\partial T(N,t)}{dt} = \sum_{i=1}^{r} q_{N,N+e_i} T(N+e_i,t) + \sum_{i=1}^{r} q_{N,N-e_i} T(N-e_i,t) + q_{N,N} T(N,t)\ ,$$
$$T(N,0) = \frac{\sum_{i=1}^{r} n_i}{\mu_l}\ ;$$

and

$$\frac{\partial S(N,t)}{dt} = \sum_{i=1}^{r} q_{N,N+e_i} S(N+e_i,t) + \sum_{i=1}^{r} q_{N,N-e_i} S(N-e_i,t) + q_{N,N} S(N,t)\ ,$$
$$S(N,0) = E\left[\min\left\{\sum_{i=1}^{r} n_i T^l, \tau\right\}\right].$$

2.6 Numerical applications and simulations

2.6.1 Model validation

We first validate our two models (exact and aggregate) by comparing the performance measures obtained analytically versus simulations.

Figure 2.2 shows the mean number of flows in cell center and cell edge as a function of impatience rate, for $\rho = 0.65$. It shows that simulation results and analytical results obtained by the exact model are close.

Figure 2.3 shows the mean number of flows in the system as a function of impatience rate. It shows that simulation results and analytical results obtained by the aggregate model are close.

Figure 2.4 shows the mean number of flows in the system as a function of impatience rate. It compares the simulations results of the aggregate model obtained with three different service time distributions: uniform, exponential and deterministic. The figure shows that the three curves are quite close.

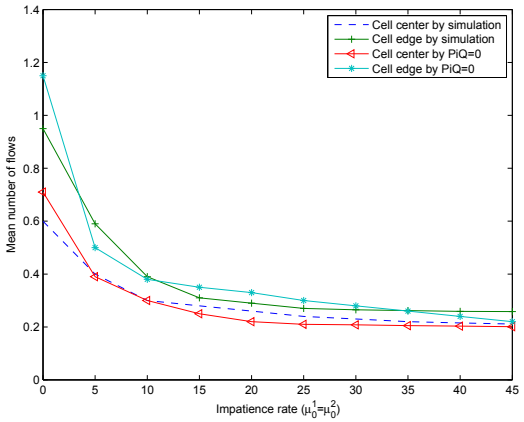

Figure 2.2: Mean number of flows - Simulation versus equation (exact model) $\pi Q = 0$ - $\rho = 0.65$ - $\lambda_1 = \lambda_2 = 8$, $\mu_1 = 32, \mu_2 = 20$, limit of admission = 200, time of simulation= 1500, number of Monte Carlo = 120.

2.6.2 Exact model

We now show some numerical illustrations, in Figures 2.5 and 2.6 for the mean number of flows and mean time spent in the system, respectively, as a function of impatience rate, in the case where flow size is independent of flow duration. And this, considering two regions in the cell: inner and outer, and the following parameters: $\lambda_1 = \lambda_2 = 8$ $(flows/sec)$, $\mu_1 = 32$, $\mu_2 = 20$ $(1/sec)$ for a value of the load $\rho = 0.65$.

We observe that both metrics decrease with increasing impatience rate, as the system now empties faster due to the impatient users who leave it before completing their file transfers. This apparently improved performance is of course counter-balanced by the non-satisfaction of these impatient users, as can be seen in Figure 2.7 where the probability of users who finish their service decreases.

Figures 2.8, 2.9 and 2.10 show the case where the patience duration depends on the flow size for the mean number of flows, the mean sojourn time in the system and the probability to finish service, respectively, as a function of impatience coefficient $1/a$. We observe as for the case of independence that the curves decrease with increasing impatience rate, the system empties faster due to the impatient users who leave before completing their service, so system performance is improved.

Let us remark that in case of patience duration depending on file size, for a user in region l when there are n active users in the system, we have $T_{eff}(n) = \min\left\{n\frac{\sigma}{C_l}, a_l\sigma\right\}$ is exponentially distributed with parameter $\frac{\mu}{\min\left\{\frac{n}{C_l}, a_l\right\}}$ that is not differentiable on a_l as opposed to of the case of independence between patience duration and the flow size in which the above parameter $\frac{\mu_l}{n} + \mu_0^l$ is differentiable on μ_0^l. This fact explains why the curves plotted as a function of impatience rate are smother in case of patience duration independent of file size than the curves plotted in case of patience duration depending on file size.

2.6.3 Approximate aggregate model

We now study how good our aggregate approximation is by comparing it to the performance obtained by the exact detailed one. We plot in Figures 2.11 and 2.12 the mean number of flows

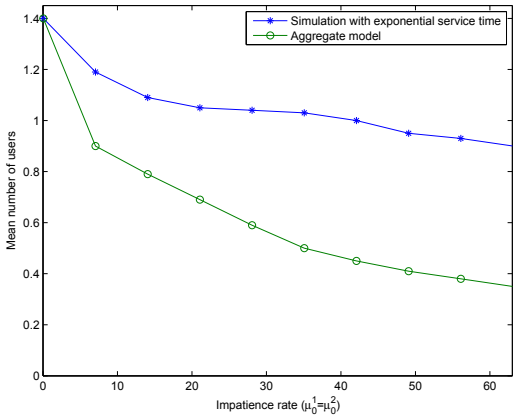

Figure 2.3: Mean number of flows - Simulation aggregate model versus equation (aggregate model) - $\rho = 0.65$ -$\lambda = 16$, $p_1 = 0.5$, $p_2 = 0.5$, $\mu_1 = 32, \mu_2 = 20$, limit of admission = 200, time of simulation = 1500, number of Monte Carlo= 120.

and mean time spent in the system, respectively, as a function of impatience rate, for the same parameters as above, using both approaches. The curves are quite close.

2.6.4 Approximate detailed model

We now turn to the approximate detailed model and compare it to the performance obtained by the exact detailed one. We plot in Figures 2.13 the mean number of flows, as a function of impatience rate, for the same parameters as above, using both approaches. Again, the curves are quite close.

2.7 Conclusion

We modeled user impatience in mobile cellular networks, with an application to 4G LTE, and quantified its impact on system performance in terms of several QoS parameters, such as mean transfer times, mean number of users in the cell and proportion of impatient users.

We observed that impatience results in higher system stability region and lower mean transfer times, at the cost of higher number of users who quit the system before completing their file transfers and hence higher user non-satisfaction.

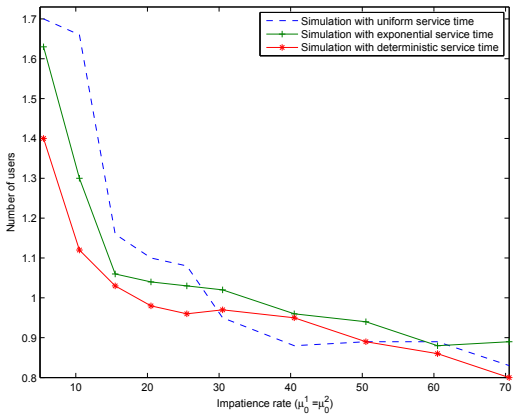

Figure 2.4: Mean number of flows - Simulation aggregate model with different distribution of service time - $\rho = 0.65$ - $\lambda = 16,\ p_1 = 0.5,\ p_2 = 0.5,\ \mu_1 = 32, \mu_2 = 20$, limit of admission = 200, time of simulation = 1500, number of Monte Carlo = 120.

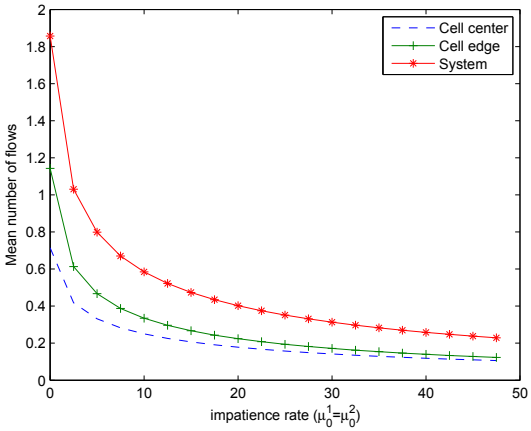

Figure 2.5: Mean number of flows - impatience duration independent of file size - $\rho = 0.65$ -$\lambda_1 = \lambda_2 = 8$, $\mu_1 = 32$, $\mu_2 = 20$, limit of admission = 20.

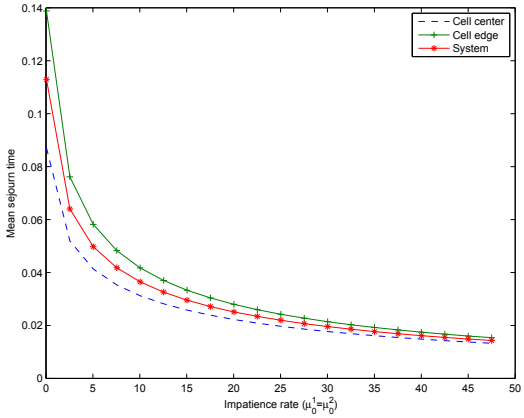

Figure 2.6: Mean sojourn time - impatience duration independent of file size - $\rho = 0.65$- $\lambda_1 = \lambda_2 = 8$, $\mu_1 = 32, \mu_2 = 20$, limit of admission $= 20$.

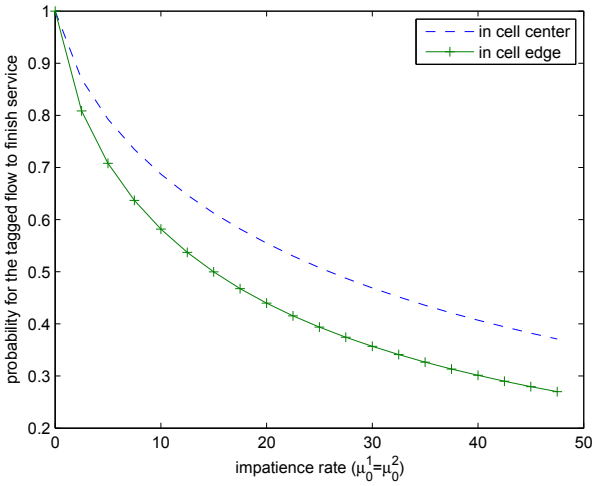

Figure 2.7: Probability of users who finish their service - impatience duration independent of file size - $\rho = 0.65$-$\lambda_1 = \lambda_2 = 8$, $\mu_1 = 32, \mu_2 = 20$, limit of admission $= 20$.

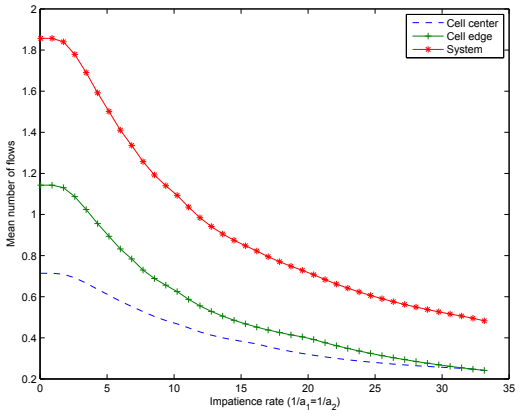

Figure 2.8: Mean number of flows - impatience duration dependent on file size - $\rho = 0.65$-$\lambda_1 = \lambda_2 = 8,\ \mu_1 = 32, \mu_2 = 20$, flow size= 1, limit of admission = 20.

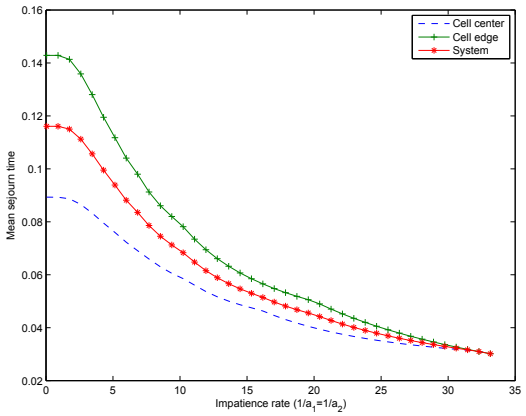

Figure 2.9: Mean sojourn time - impatience duration dependent on file size - $\rho = 0.65$-$\lambda_1 = \lambda_2 = 8,\ \mu_1 = 32, \mu_2 = 20$, flow size= 1, limit of admission = 20.

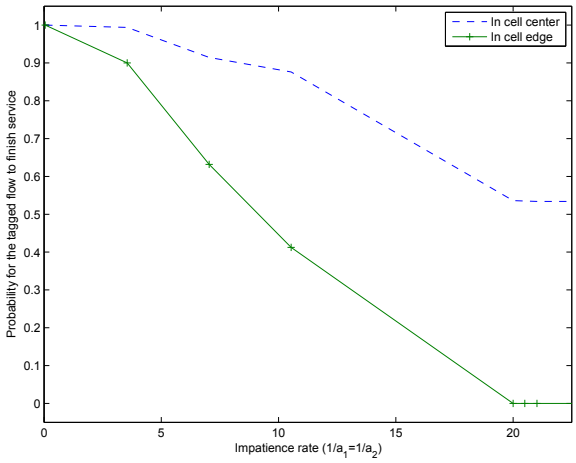

Figure 2.10: Probability of users who finish their service - impatience duration dependent on file size - $\rho = 0.65$-$\lambda_1 = \lambda_2 = 8,\ \mu_1 = 32, \mu_2 = 20$, flow size= 1, limit of admission = 20.

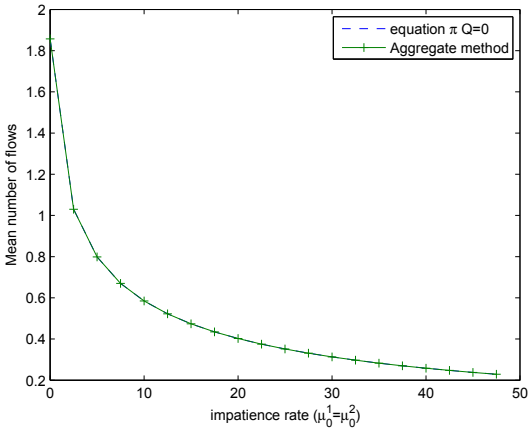

Figure 2.11: Mean number of flows - aggregate versus exact - $\rho = 0.65$-$\lambda_1 = \lambda_2 = 8,\ \lambda = 16, p_1 = p_2 = 0.5,\ \mu_1 = 32, \mu_2 = 20$, limit of admission = 20.

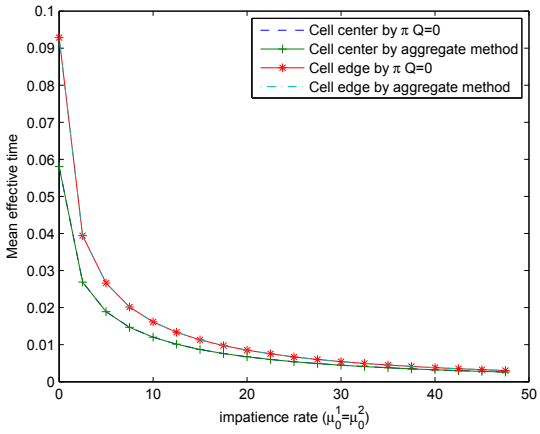

Figure 2.12: Mean sojourn time - aggregate versus exact - $\rho = 0.65$-$\lambda_1 = \lambda_2 = 8,\ \lambda = 16, p_1 = p_2 = 0.5,\ \mu_1 = 32, \mu_2 = 20$, limit of admission $= 20$.

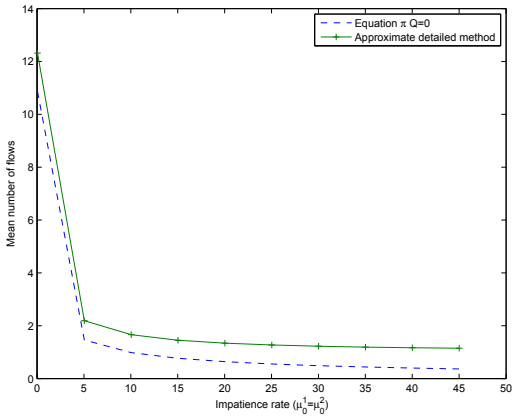

Figure 2.13: Mean number of flows - detailed versus exact - $\rho = 0.97$-$\lambda_1 = \lambda_2 = 12,\ \mu_1 = 32, \mu_2 = 20$, limit of admission $= 30$.

Chapter 3

Modeling Mobility

Abstract

In this chapter we study mobility of users in the case of impatience and quantify its impact on the performance of mobile networks, notably 4G LTE, in the presence of data flows experiencing heterogeneous radio conditions. We consider the mobility due to impatience and the spontaneous mobility of users in the system both intra and inter cell. We derive the stationary distribution several performance metrics such as mean number of users, the proportion of impatient users who quit the system before completing their file transfers and throughtput.

3.1 Introduction

The mobility model describes the movement of mobile users, and how their location change over time. Since mobility plays a significant role in determining the performance of the system, various researchers considered different models of mobility and quantified their impact.

Authors in [21] studied a model of an ad hoc network where n nodes are assumed to be mobile, and examined the per-session throughput for applications with loose delay constraints, such that the topology changes over the time-scale of packet delivery. This assumption allowed them to show that the per-user throughput can increase dramatically when nodes are mobile rather than fixed.

In [24] authors treat the problem of user mobility estimation and prediction to improve the connection reliability and bandwidth efficiency of underlying system architecture. They proceed by developing a hierarchical user mobility model that closely represents the movement behavior of a mobile user, and that, when used with appropriate pattern matching and Kalman filtering techniques, yields an accurate location prediction algorithm.

In [25] authors examine how slower mobility-induced rate variations impact performance at flow level, accounting for the random number of flows sharing the transmission resource. They identify two limit regimes that they denote 'fluid' and 'quasi-stationary' where the rate variations occur on an infinitely fast and an infinitely slow time scale respectively. They show that these limit regimes provide performance bounds depending on calculated load factors, and they prove that for a broad class of Markov-type fading processes, performance varies monotically with the speed of the rate variations.

In [26] authors consider a dynamic setting where users come and go over time as governed by random finite size data transfers, and explicitly allow for users to roam around over the course of their service. They determine the capacity of networks with both intra- and inter-cell mobility. They show that mobility tends to increase the capacity of the system not only in case of globally optimal sheduling, but also when each of the base stations operates according to a fair sharing policy.

In [29] authors focus on high mobility users in LTE networks, they begin by identifying and predicting the future behaviour of user with high mobility. For that they developed an algorithm that identifies users following similar trajectories through a cell. They assume that users that follow similar tajectories through a cell will have similar behaviour regarding mobility and handovers, thus they predict the future behaviour of a user by considering observations made from past users. These predictions allow them to decide which handovers are useful and what is the best destination cell for a handover in order to reduce some unecessary and supoptimal handovers.

In [27] authors proposed some methods for admission capacity planning in OFDMA cellular networks which consider the randomness of the channel gain in formulating the outage radio and the excess capacity ratio. They solved admission capacity planning by three optimization problems that maximize the reduction of the outage ratio, the excess capacity ratio, and the convex combination of them. The proposed planning method provides an attractive means for dimensioning OFDMA cellular networks in which a large fraction of users experience group-mobility.

In [30] authors introduced a method using a simple learning based classification method to recognize the existing moility model in unknown mobility traces that is collected from real motion of mobile Ad-hoc nodes or mobility traces generated by simulators. With simulations they showed significant performance of their method to recognize the mobility model of all unknown traces into one of the supported mobility models.

In [31] authors considered group mobility and evaluated the behaviour of mobile ad hoc

networks. They proposed four different group mobility models: (1) The random waypoint group mobility model that extends the classic random waypoint model by applying mobility to a subset of close by nodes. This allowed them to consider the presence of intra and inter group data traffic. (2) the random direction group mobility model in which the final destination of a group is selected on a border of the movement area. (3) the manhattan group mobility model that forces movement to be only along vertical or horizontal directions. (4) the sequential group mobility model in which groups are ordered and group of label i has to move towards the current position of group of label $i-1$. They showed that the mobility model of a MANET is an important factor to be considered. They showed that the number of groups is more important than the number of nodes and that the impact of the area size is almost negligible.

In [32] author gave an overview and classification of mobility models using simulations. He considered two stochastic principles for speed and direction control in which the new values are correlated to previous values that makes the movement of nodes more smooth than simple approaches of random movement. A speed change occurs following a Poisson process, the times between two direction changes are not assumed to be independent of each other. He finally discussed the impact of the border behavior on the spacial node distribution.

Our aim in this chapter is to study the mobility subject to users impatience in the context of OFDMA networks, with an application to LTE, and quantify its impact on the system performance. We model the system at the flow level for a realistic dynamic setting where users come to the system at different time epochs, move around different regions in the system, and leave it after a finite duration, either upon the completion of their data transfers or, in case of impatience, when their patience tolerance is over.

The remainder of this chapter is organized as follows. In section 3.2 there are the system description and the model of mobility. In section 3.3 we model the mobility as a consequence of impatience, in this part we assume that a user moves only when he is impatient. In section 3.4 we model mobility by a random variable independent of impatience, and we assume that the user can move after its variable of mobility expires. In section 3.5 we consider mobility for general flow size. For that we assume that others users have an exponential flow size, and we track the user with general flow size. Numerical applications and simulations are done in section 3.8. In section 3.7 we treat the case of inter cell mobility and we conclude the chapter in section 3.9.

3.2 System model

As in the previous chapter we consider an OFDMA-based homogeneous cellular network and focus on the downlink of one cell with a single base station at its center. With OFDMA, the total bandwidth, which we denote by W, is divided into N orthogonal subcarriers and can be shared between the different users present in the cell in the same time slot.

Due to path loss, the Signal-to-Interference and Noise Ratio (SINR) is lower at the cell edge than at the cell center. This leads to a cell capacity $C(r)$ that depends on the distance r between the user and the base station.

The throughput decreases when the user gets further away from the base station. Let $C_1 > C_2 > .... > C_l > .... > C_r$ be the set of throughputs at different positions l in the cell and p_l the probability that the user arrives to the cell in region l into r regions where the throughput is almost constant in each region.

We model impatience by introducing a patience duration, denoted by a random variable τ.

Each flow i of region l has its service duration T_i^l and its patience duration τ_i^l and completes its transfer if and only if its service duration is less than its patience duration i.e., $nT_i^l < \tau_i^l$ where n is the total number of flows present in the system.

Let us notice that patience duration may depend on the region: a user near the base station may be less patient than a user at the edge of the cell. In reality users can move from one region to another and/or from one cell to another for different reasons, either because service is long and they hope to get better throughput in another region, or because they are spontaneously moving during the download. By moving they can leave the cell unknowingly or not. Our aim is to model these two cases of mobility.

Let us notice that the inter cell mobility is considered only in section 3.7.

3.3 Mobility due to impatience

Here only the intra cell mobility is considered, our system is a cell consisting of several regions. We assume that leave the cell means leave the system.

We assume that one user moves from one region to another only by impatience. For each user we assume that:

- If the service duration is less than the patience duration, then he leaves the system
- If the patience duration is less than the service duration, so either he moves to another region or he leaves the cell.

We denote by α_{ij} the probability that a user of region i moves to region j, and α_{i0} the probability that a user of region i leaves the cell, with $\forall\ i \in \{1, r\}$, $\sum_{k=0}^{r} \alpha_{ik} = 1$.

Let us introduce the process $N_t = \left(N_t^1, ...N_t^r\right)$, where r is the number of regions in the cell (each with different radio conditions and hence different capacity) and N_t^l is the number of users in region l.

3.3.1 Case of independence between service and patience durations

We assume that the flow size and the patience duration are independent and exponentially distributed with parameters μ_0 and μ respectively.

The service duration in region l with capacity C_l is given by $T^l = \frac{\sigma}{C_l}$, is also exponentially distributed with parameter $\mu^l = \mu C_l$.

Each flow in region l has a patience duration τ^l exponentially distributed with parameter μ_0^l.

With these assumptions, the process N_t is an irreducible Markov process with the following intensity matrix:

$$q_{N,N+e_l} = \lambda_l = \lambda p_l$$

$$q_{N,N-e_l} = \frac{n_l}{\sum_{l=1}^{r} n_l}\mu^l + n_l \mu_0^l \alpha_{l0}$$

$$q_{N,N+e_l-e_k} = n_k \mu_0^k \alpha_{kl}$$

$$q_{N,N} = -\left(\lambda + \sum_{l=1}^{r} q_{N,N-e_l} + \sum_{k=1}^{r}\sum_{l\neq k} q_{N,N+e_l-e_k}\right)$$

$$q_{0,0} = -\lambda$$

where $N = (n_1, ..., n_r)$; n_l being the number of users in region l, $e_l = (0, ...0, 1, 0, ...0)$ is a vector with the l-th component equals to 1 and all others are 0, and $\lambda = \sum_{l=1}^{r} \lambda_l$; λ_l being the mean arrival rate to region l.

Transition from state N to state $N + e_l - e_k$ corresponds to the case in which a user of region k has been impatient during its service and moves to region l.

Transition from state N to state $N - e_l$ corresponds to the case in which a user of region l leaves the cell either after the end of its service or after its patience duration has expired and instead of moving to another region, he leaves the cell (that case occurs with probability α_{l0}).

The stationary distribution $\pi(N)$ is solution of:

$$\pi Q = 0 \text{ and } \sum_{N \in \mathbb{N}^r} \pi(N) = 1.$$

Now let us try to estimate α_{ij}. When a user of some region is impatient there are r possible events: either he moves to one of the other $r - 1$ regions of the system or he leaves the system.

Let us notice that here mobility is not restricted to adjacent regions.

By the formula $\sum_{k=0}^{r} \alpha_{ik} = 1$, we can assume that the distribution of moving is uniform on all regions, including the outside of the system, so $\forall i \in [1, r] \quad \alpha_{ij} = \alpha_{ik}$, $\forall j, k \in [0, r]$. If we denote by P_I^i the probability to be impatient in region i, we can estimate α_{ij} by $\frac{P_I^i}{r}$.

Now let us find P_I^i. For that we focus on one tagged flow located in region i, and we define the process $X_t = \left(X_t^1, ...X_t^r\right)$ where X_t^l is the number of flows in region l seen by the tagged flow.

Let us notice that in this section, mobility occurs only when there is impatience. Thus in order to find the probability of impatience P_I^i, we assume in the study of the process X_t, that users are not allowed to move.

We add two supplementary states I corresponding to the state in which the tagged flow is impatient and F corresponding to the state in which the tagged flow finishes its service.

We assume that the service duration and the patience duration are exponential and independent. With these assumptions the process $(X_t)_t$ is Markovian with the following intensity matrix:

$$q_{N,N+e_l} = \lambda_l = \lambda p_l,$$

$$q_{N,N-e_l} = \frac{n_l}{\sum_{j=1}^{r} n_j + 1} \mu^l + n_l \mu_0^l,$$

$$q_{N,I} = \mu_0^i,$$

$$q_{N,F} = \frac{\mu^i}{\sum_{j=1}^{r} n_j + 1},$$

$$q_{N,N} = -\left(\lambda + \mu_0^i + \frac{\mu^i}{\sum_{j=1}^{r} n_j + 1} + \sum_{l=1}^{r} q_{N,N-e_l}\right),$$

$$q_N = -q_{N,N}.$$

The two states I and F are absorbing, then $(X_t)_t$ is a reducible Makovian process.

Let us pose

$$T_I = \inf\{t > 0, X_t \in I\},$$

$$T_F = \inf\{t > 0, X_t \in F\},$$
$$T^* = \min\{T_I, T_F\}.$$

The variable T_I, T_F and T^* are stopping times.

Let us pose $I(N) = P[X_{T^*} \in I \mid X_0 = N]$ the probability that the tagged flow be impatient, conditioned on $X_0 = N$.
We can see that $I(N) = P[T_I < \infty \mid X_0 = N]$ which is the hitting probability of state I. The first formulation can be expressed as $I(N) = E\left[1_{X_{T^*} \in I} \mid X_0 = N\right] = E[f(X_{T^*}) \mid X_0 = N]$, with $f(z) = 1_{\{z \in I\}}$. Then $I(N)$ can be found as the solution of Dirichlet/FeymannKac problems as follows:

$$Q\ I = 0 \qquad \forall\ N \neq I,\ F$$
$$I(I) = 1$$
$$I(F) = 0$$

Now let us consider the same process $X_t = \left(X_t^1, ...X_t^r\right)$ without the tagged flow, where X_t^l is the number of flows in region l.

The process X_t is an irreducible Markov process with the following intensity matrix:

$$q_{N,N+e_l} = \lambda_l = \lambda p_l$$
$$q_{N,N-e_l} = \frac{n_l}{\sum_{l=1}^r n_l}\mu^l + n_l\mu_0^l$$
$$q_{N,N} = -\left(\lambda + \sum_{l=1}^r q_{N,N-e_l}\right)$$

The stationary distribution $\pi(N)$ is solution of:

$$\pi Q = 0 \text{ and } \sum_{N \in \mathbb{N}^r} \pi(N) = 1$$

If we consider that the system is in steady-state regime at the beginning, we obtain P_I^i:

$$P_I^i = \sum_N I(N)\pi(N).$$

3.3.2 Patience duration function of flow size

It is more realistic to assume that the patience duration depends on the file size, since patience grows with file size, i.e., $\tau = F(\sigma)$.

As in the previous chapter, we assume that

$$\tau^l = a_l\sigma$$

where $\frac{1}{a_l}$ represents the minimum throughput required to transfer very large documents in region l, referred to as the sustainable throughput; and we assume that the flow size is exponentiel of parameter μ.

Each flow in region l has a patience duration $\tau^l = a_l\sigma$, and a service duration $nT^l = n\frac{\sigma}{C_l}$ when there are n flows in the system.

The process $N_t = \left(N_t^1, ..., N_t^r\right)_{t>0}$ where N_t^l denotes the number of flows in region l is an irreducible Markov process with the following intensity matrix:

$$q_{N,N+e_l} = \lambda_l = \lambda p_l$$

$$q_{N,N-e_l} = \frac{n_l}{\sum_{l=1}^{r} n_l}\mu^l 1_{\left\{\frac{\sum_{l=1}^{r} n_l}{C_l} \leq a_l\right\}} + n_l \frac{\mu}{a_l}\alpha_{i0} 1_{\left\{\frac{\sum_{l=1}^{r} n_l}{C_l} > a_l\right\}}$$

$$q_{N,N+e_j-e_k} = n_k \frac{\mu}{a_k}\alpha_{kj} 1_{\left\{\frac{\sum_{l=1}^{r} n_l}{C_k} > a_k\right\}}$$

$$q_{N,N} = -\left(\lambda + \sum_{l=1}^{r} q_{N,N-e_l} + \sum_{k=1}^{r}\sum_{l\neq k} q_{N,N+e_l-e_k}\right)$$

We derive the probability of impatience in steady state by the similar method used in section 3.3.1.

3.4 Mobility independent of impatience

As in the previous section, only the intra cell mobility is considered, our system is a cell consisting of several regions. We assume that leave the cell means leave the system.

We introduce a random variable for mobility that we denote by M. Each user in the system has its mobility parameter M, its service parameter T and its patience parameter τ. When the mobility parameter of a user of region i expires, he moves to another region j or leaves the system.

Each user in region i has M^{ij} mobility durations for $j \neq i$ and $j \in \{0, 1, \cdots, r\}$, that are assumed to be exponentially distributed and independent with parameter α_{ij}. If the variable M^{ij} expires, then he moves from region i to region j. If the variable M^{i0} expires, then he leaves the cell.

For each user in region i we denote by M^i the minimum of all M^{ij}.

According to our assumptions, M^i is exponentially distributed with parameter $\sum_{j=0, j\neq i}^{r} \alpha_{ij}$.

3.4.1 Case of independence between service and patience durations

We assume that the service duration and the patience duration are exponential and independent with respectively parameters μ, μ_0, and we assume that they are independent of mobility duration that is assumed to be exponentially distributed.

Each user of region l has its sojourn time in this region, denoted by $S^l = \min\left\{nT^l, \tau^l, M^l\right\}$, when there are n users in the system.

- If $S^l = nT^l$, then he leaves the system after finishing service
- If $S^l = \tau^l$, then he leaves the system by impatience
- If $S^l = M^l$, so either he moves to another region or he leaves the system.

Let us consider the process $N_t = \left(N_t^1, ...N_t^r\right)$, where N_t^l is the number of users in region l. The process N_t is an irreducible Markov process with the following intensity matrix:

$$q_{N,N+e_l} = \lambda_l = \lambda p_l$$

$$q_{N,N-e_l} = \frac{n_l}{\sum_{i=1}^{r} n_i}\mu^l + n_l\mu_0^l + n_l\alpha_{l0}$$

$$q_{N,N+e_l-e_k} = n_k\alpha_{kl}$$

$$q_{N,N} = -\left(\lambda + \sum_{l=1}^{r} q_{N,N-e_l} + \sum_{k=1}^{r}\sum_{l\neq k} q_{N,N+e_l-e_k}\right)$$

The stationary distribution $\pi(N)$ is solution of:

$$\pi Q = 0 \text{ and } \sum_{N\in\mathbb{N}^r} \pi(N) = 1.$$

Now we want to study the behavior of a tagged flow in the system. For that we introduce the process $Z_t = \left(Z_t^1, ...Z_t^r\right)$, where Z_t^l is the number of users in region l seen by the tagged flow.

We denote by p_l^* the probability that the tagged flow is localized in region l with $\sum_{l=1}^{r} p_l^* = 1$, that is different from p_l which is the probability that a new user arrives in region l. We assume that p_l^* is constant for all $l \in [1, r]$.

We introduce three supplementary states I, F and M, where:

- I is the state by which the tagged flow leaves the system by impatience
- F is the state by which the tagged flow leaves the system if its service finishes
- M is the state by which the tagged flow leaves the cell by moving

The three states are absorbing, and the process Z_t is a reducible Markov process with the following intensity matrix:

$$q_{N,N+e_l} = \lambda p_l = \lambda_l$$
$$q_{N,N-e_l} = \frac{n_l}{\sum_{i=1}^{r} n_i + 1}\mu^l + n_l\mu_0^l + n_l\alpha_{l0}$$
$$q_{N,N+e_l-e_k} = n_k\alpha_{kl}$$
$$q_{N,I} = \sum_{i=1}^{r} \mu_0^i p_i^*$$
$$q_{N,F} = \sum_{i=1}^{r} \frac{1}{\sum_{i=1}^{r} n_i + 1}\mu^i p_i^*$$
$$q_{N,M} = \sum_{i=1}^{r} \alpha_{i0} p_i^*$$

$$q_{N,N} = -\left(\lambda + \sum_{l=1}^{r} q_{N,N-e_l} + \sum_{k=1}^{r}\sum_{l\neq k} q_{N,N+e_l-e_k} + q_{N,I} + q_{N,F} + q_{N,M}\right)$$

Let us pose

$$T_I = \inf\{t > 0, Z_t \in I\},$$
$$T_F = \inf\{t > 0, Z_t \in F\},$$
$$T_M = \inf\{t > 0, Z_t \in M\},$$
$$T^* = \min\{T_I, T_F, T_M\}.$$

These variables are stopping times. Let us pose $P_I(N) = P\left[Z_{T^*} \in I \mid Z_0 = N\right]$ the probability that the tagged flow leaves the system by impatience, $P_F(N) = P\left[Z_{T^*} \in F \mid Z_0 = N\right]$ the probability that the tagged flow finishes its service, and $P_M(N) = P\left[Z_{T^*} \in M \mid Z_0 = N\right]$ the probability that the tagged flow leaves the cell by moving, all conditioned on $Z_0 = N$.

P_I, P_F, P_M are solution of the Dirichlet/FeymannKac problems as follows:

- P_I is solution of:

$$Q\ P_I = 0 \qquad \forall\ N \neq I,\ F,\ M$$
$$P_I(I) = 1$$
$$P_I(F) = 0$$
$$P_I(M) = 0$$

- P_F is solution of:

$$Q\ P_F = 0 \qquad \forall\ N \neq I,\ F,\ M$$
$$P_F(I) = 0$$
$$P_F(F) = 1$$
$$P_F(M) = 0$$

- P_M is solution of:

$$Q\ P_M = 0 \qquad \forall\ N \neq I,\ F,\ M$$
$$P_M(I) = 0$$
$$P_M(F) = 0$$
$$P_M(M) = 1$$

If we consider that the system is in steady-state regime at the beginning, we obtain the measures of performance in steady state P_I^*, P_F^*, P_M^* as follows:

$$P_I^* = \sum_N P_I(N)\pi(N)$$

$$P_F^* = \sum_N P_F(N)\pi(N)$$

$$P_M^* = \sum_N P_M(N)\pi(N).$$

3.4.2 Patience duration function of flow size

Now let us assume again that

$$\tau^l = a_l \sigma$$

where $\frac{1}{a_l}$ represents the minimum throughput required to transfer very large documents in region l, referred to as the sustainable throughput; and we assume that the flow size exponentiel with parameter μ and independent of mobility.

Each flow in region l has a patience duration $\tau^l = a_l \sigma$, a service duration $nT^l = n\frac{\sigma}{C_l}$ when there are n flows in the system, and a mobility parameter M^l.

The process $N_t = \left(N_t^1, ..., N_t^r\right)_{t>0}$ where N_t^l denotes the number of flows in region l is an irreducible Markov process with the following intensity matrix:

$$\begin{aligned}
q_{N,N+e_l} &= \lambda_l = \lambda p_l \\
q_{N,N-e_l} &= \frac{n_l \mu}{\min\left\{\frac{\sum_{i=1}^r n_i}{C_l}, a_l\right\}} + n_l \alpha_{l0} \\
q_{N,N+e_l-e_k} &= n_k \alpha_{kl} \\
q_{N,N} &= -\left(\lambda + \sum_{l=1}^r q_{N,N-e_l} + \sum_{k=1}^r \sum_{l \neq k} q_{N,N+e_l-e_k}\right)
\end{aligned}$$

The stationary distribution is found by resolving

$$\pi Q = 0 \text{ and } \sum_{N \in \mathbb{N}^r} \pi(N) = 1.$$

To study the behavior of a tagged flow we do as previously, by introducing three absorbing states I, F, M, and by considering the process of the number of flows seen by the tagged flow in each region $Z_t = \left(Z_t^1, ..., Z_t^r\right)_{t>0}$.

Thus the process Z_t is a reducible Markov process with the following intensity matrix:

$$\begin{aligned}
q_{N,N+e_l} &= \lambda_l \\
q_{N,N-e_l} &= \frac{n_l \mu}{\min\left\{\frac{\sum_{i=1}^r n_i + 1}{C_l}, a_l\right\}} + n_l \alpha_{l0} \\
q_{N,N+e_l-e_k} &= n_k \alpha_{kl} \\
q_{N,I} &= \sum_{i=1}^r \frac{\mu}{a_i} 1_{\left\{\frac{\sum_{l=1}^r n_l + 1}{C_i} > a_i\right\}} p_i^* \\
q_{N,F} &= \sum_{i=1}^r \frac{\mu^i}{\sum_{l=1}^r n_l + 1} 1_{\left\{\frac{\sum_{l=1}^r n_l + 1}{C_i} \leq a_i\right\}} p_i^* \\
q_{N,M} &= \sum_{i=1}^r \alpha_{i0} p_i^* \\
q_{N,N} &= -\left(\lambda + \sum_{l=1}^r q_{N,N-e_l} + \sum_{k=1}^r \sum_{l \neq k} q_{N,N+e_l-e_k} + q_{N,I} + q_{N,F} + q_{N,M}\right)
\end{aligned}$$

where p_i^* is the probability that the tagged flow is localized in region i.

We derive the same metrics as in section 3.4.1 by a similar method.

3.5 Mobility for general flow size

Here we consider a system with a tagged user that has a general distribution for the flow size and general patience duration. All other users are assumed to have exponential flow size and patience duration, that we assume independent. We denote by σ the flow size for the tagged flow.

Users arrive in region l according a Poisson process with parameter λ_l, and have a service time T^l, a patience duration τ^l and a mobility variable M^l defined as in the previous section, that are exponentially distributed with parameters μ^l, μ_0^l and α_l respectively.

We consider the process $N_t = (N_t^1, ..., N_t^r)$, where N_t^l is the number of users in region l seen by the tagged flow.

When the tagged flow moves, its throughtput changes because the capacities of different regions are different, so we define the process of mobility of the tagged flow by a process C_t taking values to be shared with other users in $[C_1, ..., C_r]$, where C_l is the capacity in region l.

We assume that the tagged flow moves from region l to region k according to its mobility variable M^{lk} with parameter α_{lk}, as for the other users.

In order to study the behaviour of the tagged flow in the cell, we assume that he can not leave the cell by mobility, that is the reason for which the process of mobility is assumed to take values in $[C_1, ..., C_r]$. The other users however can leave the cell by mobility according to their mobility variable M^{l0}, when they are localized in region l.

Let us notice that the processes N_t and C_t are independent and Markovian.

Let us define by $R(t) = \int_0^t \frac{C_u}{1+\sum_{l=1}^r N_u^l} du$ the file transferred by the tagged flow at time t, so its service time is defined by $T = \inf\{t > 0; R(t) \geq \sigma\}$.

We notice that $R(T) = \sigma$.

Theorem 3.5.1 *The process $Z_t = (N_t, C_t)$ is Markovian with intensity matrix A given by:*
let $Z_l = (N_l, C_l)$ and $Z_k = (N_k, C_k)$ be two states of Z_t
$A_{Z_k,Z_l} = q_{N_k,N_k+e_i} = \lambda_i$ *if* $N_l = N_k + e_i$ *and* $C_l = C_k$;
$A_{Z_k,Z_l} = q_{N_k,N_k-e_i} = \frac{n_i}{\sum_{j=1}^r n_j+1}\mu^i + n_i\mu_0^i + n_i\alpha_{i0}$ *if* $N_l = N_k - e_i$ *and* $C_l = C_k$;
$A_{Z_k,Z_l} = q_{N_k,N_k+e_j-e_k} = n_k\alpha_{kj}$ *if* $N_l = N_k + e_j - e_k$ *and* $C_l = C_k$
$A_{Z_k,Z_l} = \alpha_{kl}$ *if* $N_l = N_k$ *and* $C_l \neq C_k$
$A_{Z_k,Z_l} = 0$ *otherwise,*
where $Q = (q_{N,M})$ is the intensity matrix of the process N_t, and $N_k = (n_1, \cdots, n_r)$.

Proof Let us pose $Z_l = (N_l, C_l)$ and $Z_k = (N_k, C_k)$ two arbitrary states of Z_t. By definition of intensity matrix

$$A_{Z_k,Z_l} = \lim_{h\to 0} \frac{1}{h} P\left[Z_{t+h} = Z_l \mid Z_t = Z_k\right]$$

$$P\left[Z_{t+h} = Z_l \mid Z_t = Z_k\right] = P\left[N_{t+h} = N_l, C_{t+h} = C_l \mid N_t = N_k, C_t = C_k\right]$$

By independence between $(N_t)_t$ and $(C_t)_t$ we have:

$$\begin{aligned} &P\left[N_{t+h} = N_l, C_{t+h} = C_l \mid N_t = N_k, C_t = C_k\right] \\ &= P\left[N_{t+h} = N_l, \mid N_t = N_k, C_t = C_k\right] P\left[C_{t+h} = C_l \mid N_t = N_k, C_t = C_k\right] \\ &= P\left[N_{t+h} = N_l \mid N_t = N_k\right] P\left[C_{t+h} = C_l \mid C_t = C_k\right]. \end{aligned}$$

If $N_l \neq N_k$ and $C_l \neq C_k$, that corresponds to the case in which the tagged flow moves during the transition of the process Z_t, then $A_{Z_l,Z_k} = 0$.

Let us notice that $C_l = C_k$ means that the tagged flow does not move during the transition of the process Z_t.

If $N_l = N_k + e_i$ and $C_l = C_k$ that means that a new user arrives in region i and the tagged flow does not move, then:

$$\begin{aligned} A_{Z_k,Z_l} &= \lim_{h\to 0} \frac{1}{h} P\left[N_{t+h} = N_k + e_i \mid N_t = N_k\right] P\left[C_{t+h} = C_l \mid C_t = C_k\right] \\ &= \lim_{h\to 0} \frac{1}{h} P\left[N_{t+h} = N_k + e_i \mid N_t = N_k\right] P\left[\min_{i\neq k}\left(M^{ki}\right) > h\right] \\ &= \lim_{h\to 0} \frac{1}{h} P\left[N_{t+h} = N_k + e_i \mid N_t = N_k\right] \exp(-h \sum_{i=1, i\neq k}^{r} \alpha_{ki}) \\ &= q_{N_k, N_k + e_i} \end{aligned}$$

where $q_{N,M}$ is the coefficient of the intensity matrix of the process N_t.

If $N_l = N_k - e_i$ and $C_l = C_k$ that means that a user of region i leaves the cell by mobility and the tagged flow does not move, then:

$$\begin{aligned} A_{Z_k,Z_l} &= \lim_{h\to 0} \frac{1}{h} P\left[N_{t+h} = N_k - e_i \mid N_t = N_k\right] P\left[C_{t+h} = C_l \mid C_t = C_k\right] \\ &= \lim_{h\to 0} \frac{1}{h} P\left[N_{t+h} = N_k - e_i \mid N_t = N_k\right] \exp(-h \sum_{i=1, i\neq k}^{r} \alpha_{ki}) \\ &= q_{N_k, N_k - e_i} \end{aligned}$$

If $N_l = N_k + e_j - e_k$ and $C_l = C_k$, that means that a user of region k goes to region j and the tagged flow does not move, then:

$$\begin{aligned} A_{Z_k,Z_l} &= \lim_{h\to 0} \frac{1}{h} P\left[N_{t+h} = N_k + e_j - e_k \mid N_t = N_k\right] P\left[C_{t+h} = C_l \mid C_t = C_k\right] \\ &= \lim_{h\to 0} \frac{1}{h} P\left[N_{t+h} = N_k + e_j - e_k \mid N_t = N_k\right] \exp(-h \sum_{i=1, i\neq k}^{r} \alpha_{ki}) \\ &= q_{N_k, N_k + e_j - e_k} \end{aligned}$$

If $N_l = N_k$ and $C_l \neq C_k$, that means that only the tagged flow moves during the transition of the process Z_t, and he moves from region k to region l, then:

$$\begin{aligned} A_{Z_k,Z_l} &= \lim_{h\to 0} P\left[M^{kl} < h\right] \exp\left(-\left\{q_{N_k,N_k}\right\} h\right) \\ &= \alpha_{kl}. \end{aligned}$$

∎

We denote by τ_e the patience duration of the tagged flow that is assumed to be general.

The probability that the tagged flow finishes its service without being impatient is given by $P\left[T < \tau_e\right]$ which is equal to $P\left[R(\tau_e) > \sigma\right]$.

We can derive the probability that the tagged flows finishes its service through the relationship between Markov process and ordinary differential equation (ODE) as follows. We define a function:

$$g\left(\begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix}, y\right) = \frac{y}{1 + \sum_{l=1}^{r} x_r};$$

The process $R(t)$ can be written as $R(t) = \int_0^t g(Z_u)du$, then by the well-know relationship between Markov process and ODE, $V(t,z) = E\left[R(t) \mid Z_0 = z\right]$ is solution of the following ODE:

$$\frac{\partial V(t,z)}{dt} = AV(t,z) + g(z),$$

$$V(0,z) = 0, \quad \forall z.$$

We can derive the probability that the tagged flows finishes its service through the relationship between Markov process and ordinary differential equation (ODE) as follows. We define a function:

$$g\left(\begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix}, y\right) = \frac{y}{1 + \sum_{l=1}^{r} x_l};$$

The process $R(t)$ can be written as $R(t) = \int_0^t g(Z_u)du$. We can find the Laplace transform of $R(t)$ as follow

$L^R(\lambda) = E\left[\exp\left(-\lambda R(t)\right)\right], \forall \lambda \geqslant 0$. For this let us pose $W(t,z) = E\left[\exp\left(-\lambda R(t)\right) \mid Z_0 = z\right]$. By the well-know relationship between Markov process and PDE, $W(t,z)$ is solution of the following PDE:

$$\frac{\partial W(t,z)}{\partial t} = AW(t,z) - \lambda g\left(z\right) W(t,z),$$

$$W(0,z) = 1, \quad \forall z.$$

We know that the process N_t has a stationnary distribution π^N given by the equation $\pi^N Q = 0$, where Q is its intensity matrix given by:

$$q_{N,N+e_l} = \lambda_l$$

$$q_{N,N-e_l} = \frac{n_l}{\sum_{i=1}^{r} n_i + 1}\mu^l + n_l\mu_0^l + n_l\alpha_{l0}$$

$$q_{N,N+e_l-e_k} = n_k\alpha_{kl}$$

The process C_t is a irreducible, recurent, and reversible Markov process, so it has a stationnary distribution π^C given by the local balance equation:

$$\alpha_{n,n-1}\pi_n^C = \alpha_{n-1,n}\pi_{n-1}^C$$

Thus

$$\pi_n^C = \frac{\alpha_{n-1,n}}{\alpha_{n,n-1}}\pi_{n-1}^C$$

By recurence we have:

$$\pi_n^C = \prod_{l=1}^{n-1} \frac{\alpha_{n-l,n-l+1}}{\alpha_{n-l+1,n-l}}\pi_1^C$$

As π^C is a measure of probability then $\sum_{n=1}^{r} \pi_n^C = 1$, so

$$\pi_1^C = \left\{\sum_{n=1}^{r}\prod_{l=1}^{n-1} \frac{\alpha_{n-l,n-l+1}}{\alpha_{n-l+1,n-l}}\right\}^{-1}$$

Let us denote by E_π the expectation in steady state. By Fubini and by independence between N_t and C_t we have:

$$\begin{aligned} E_\pi\left[R(t)\right] &= E_\pi\left[\int_0^t \frac{C_u}{1+\sum_{l=1}^r N_u^l} du\right] \\ &= \int_0^t E_\pi\left[\frac{C_u}{1+\sum_{l=1}^r N_u^l}\right] du \\ &= \int_0^t E_\pi\left[C_u\right] E_\pi\left[\frac{1}{1+\sum_{l=1}^r N_u^l}\right] du \\ &= tE_\pi\left[C\right] E_\pi\left[\frac{1}{1+\sum_{l=1}^r N^l}\right]. \end{aligned}$$

So we obtain a steady state expression of the throughput $E_\pi\left[\frac{R(t)}{t}\right]$:

$$E_\pi\left[\frac{R(t)}{t}\right] = \left[\sum_{l=1}^r C_l \pi^C(l)\right]\left[\sum_{(n_1,\ldots,n_r)} \frac{\pi(n_1,\ldots,n_r)}{1+\sum_{l=1}^r n_l}\right]$$

3.6 Link between both models

Here we want to see the link between the first and the second models. Let us recall that in the first model the tagged flow has an exponential flow size σ, and its service time is given by: $T = \frac{(1+\sum_{l=1}^r N^l)\sigma}{C}$ where N^l is the number of users in region l, and C is the capacity of the region in which the tagged flow is localized. In the second model the tagged flow has a general service time given by: $T = \inf\{t > 0; R(t) \geq \sigma\}$.

We notice that $R(T) = \sigma$, so in the steady state regime we have:

$$E_\pi\left[\frac{R(T)}{T}\right] = E_\pi\left[\frac{\sigma}{T}\right]$$

In the first model $\frac{\sigma}{T} = \frac{C}{1+\sum_{l=1}^r N^l}$, then

$$E_\pi\left[\frac{R(T)}{T}\right] = E_\pi\left[\frac{C}{1+\sum_{l=1}^r N^l}\right]$$

Thus by independence we have:

$$E_\pi\left[\frac{R(T)}{T}\right] = E_\pi\left[C\right] E_\pi\left[\frac{1}{1+\sum_{l=1}^r N^l}\right]$$

So if we asume for the second model that $E_\pi\left[\frac{R(T)}{T}\right] = E_\pi\left[\frac{R(t)}{t}\right]$ for any given t, then we can conclude that the two expressions of throughput: $E_\pi\left[\frac{R(T)}{T}\right]$ for the second model and $E_\pi\left[\frac{\sigma}{T}\right]$ for the first model are equal.

3.7 Inter cell mobility

In this section we introduce the inter cell mobility. We consider that there are many cells in the system. Each cell contains many regions of different capacity. As previously, users can move from one region to another in the cell; that is called the intra cell mobility. Now users are allowed to move from one cell to another which is called the inter cell mobility.

For geographical reasons, we assume that the inter-cell mobility occurs only between the edge regions, i.e, only user from edge region can move to another cell, and can only arrive to the edge region of the arrival cell.

Let us denote by s the number of cells in the system, $\forall\ i\ \in\ [1,s]$, and we denote by r_i the number of regions in cell i.

Let us introduce the process $Z_t = \left(Z_t^{1,1},...,Z_t^{1,r_1},\cdots,Z_t^{i,1},...,Z_t^{i,r_i},\cdots,Z_t^{s,1},...,Z_t^{1,r_s}\right)$, where $Z_t^{i,l}$ is the number of users in region l of cell i.

The vector $\left(n_1^1,...,n_{r_1}^1,\cdots,n_1^i,...,n_{r_i}^i,\cdots,n_1^s,...,n_{r_s}^s\right)$ is a state of process Z_t. We denote by $n^i = \sum_{l=1}^{r_i} n_l^i$ the total number of users in cell i.

We denote by C_l^i the capacity in region l of cell i.

We assume that user arrives in region l of cell i following a Poisson process of parameter λ_l^i.

A user in region l of cell i having a flow size σ, has a service time given by $\frac{n^i\sigma}{C_l^i}$, when there are n^i active users in cell i. Its patience duration is dented by τ. We denote by M^{intra} its intra mobility variable, and M^{inter} its inter mobility variable.

We assume that the flow size, the patience duration, the intra cell mobility variable and the inter cell mobility variable are independent and exponentially distributed with respectively parameters μ, μ_0, α and ζ.

For a user in region l in cell i, its service time is exponential with parameter $\mu_l^i = C_l^i\mu$, when he is alone in the cell. Its intra cell mobility variable to go to region $l^{'}$ in cell i is exponential with parameter $\alpha_{ll'}^i$. Its inter cell mobility variable to go to edge region r_j of cell j is exponential with parameter ζ^{ij} if l is the edge region of cell i. If l is not the edge region of cell i there are not inter cell mobility.

The interference between different base stations reduces the capacity of each cell, for that we assume that the capacity in each cell is less than if there is no interference. This capacity is shared following a processor sharing as previously.

With these assumptions the process Z_t is Markovian with intensity matrix Q given by:

$$q_{Z,Z+e_l^i} = \lambda_l^i, \quad \forall i \in [1,s]\,,\ \forall l \in [1,r_i]$$

$$q_{Z,Z-e_l^i} = \frac{n_l^i}{n^i}\mu_l^i + n_l^i\mu_0, \quad \forall i \in [1,s]\,,\ \forall l \in [1,r_i]$$

$$q_{Z,Z+e_k^i-e_l^i} = n_l^i\alpha_{lk}^i, \quad \forall i \in [1,s]\,,\ \forall l,k \in [1,r_i]$$

$$q_{Z,Z+e_{r_j}^j-e_{r_i}^i} = n_{r_i}^i\zeta^{ij}, \quad \forall i,j \in [1,s]$$

$$q_{Z,Z} = -\left(\sum_{i=1}^{s}\sum_{l=1}^{r_i}\lambda_l^i + \sum_{i=1}^{s}\sum_{l=1}^{r_i}q_{Z,Z-e_l^i} + \sum_{i=1}^{s}\sum_{l=1}^{r_i}\sum_{k\neq l}q_{Z,Z+e_k^i-e_l^i} + \sum_{i=1}^{s}\sum_{j\neq i}q_{Z,Z+e_{r_j}^j-e_{r_i}^i}\right).$$

The process Z_t is irreducible and its stationnary distribution π is solution of:

$$\pi Q = 0 \text{ and } \sum_{N\in\mathbb{N}^r}\pi(N) = 1.$$

Now let us consider a tagged flow in the system. We consider that the process Z_t as defined previously contains all users in the system except the tagged flow.

We consider that the tagged flow has a flow size denoted by σ, and a patience duration τ.

We introduce the process of mobility of the tagged flow in the system C_t taking values in $\left[C_1^1, ...C_{r1}^1, \cdots, C_1^s, ..., C_{r_s}^s\right]$, where C_l^i is the capacity in region l of cell i. As the other users, we assume that the tagged flow moves in cell from one region to another following its intra cell mobility variable M^{intra} that is exponential with parameter α, he also moves from one cell to another with its inter cell mobility variable M^{inter} that is also exponential with parameter ζ. We assume that the tagged flow, as the other users, can move to another cell only if he is in the edge region and it arrives to the edge region of the arrival cell. If the tagged flow is located in region l of cell i, he can move to region l' in cell i following its intra cell mobility variable that is exponential with parameter $\alpha_{ll'}^i$. He can move from cell i to cell j with its inter cell mobility variable that is exponential with parameter ζ^{ij}.

With these assumptions, the process C_t is an irreducible Markovian process with intensity matrix Q^C given as follows:

$$q_{C_l^i, C_k^i} = \alpha_{lk}^i$$

$$q_{C_{r_i}^i, C_{r_j}^j} = \zeta^{ij}.$$

Its stationnary distribution π^C is solution of:

$$\pi^C Q = 0 \text{ and } \sum_{N \in \mathbb{N}^r} \pi^C(N) = 1.$$

Let recall that all cells are scheduled as a processor sharing, so when the tagged flow is in cell i at time t, its throughtput is divided by $1 + N_t^i$ where $N_t^i = \sum_{l=1}^{r_i} Z_t^{i,l}$ is the total number of users in cell i.

We define $R(t) = \int_0^t \frac{C_u}{1+N_u} du$ the portion of the file transferred by the tagged flow at time t, where N_u is a process that defines the total number of users in the cell where the tagged flow is located.

We notice that N_u is conditioned on C_u, $C_u \in \left[C_1^i, ..., C_{r_i}^i\right]$ then N_u is distributed as N_u^i the total number of users in cell i.

The service time of the tagged flow is given by $T = \inf\{t > 0; R(t) \geq \sigma\}$.

We denote by E_π the expectation in steady state. By Fubini, by conditioning and by independence, we have:

$$E_\pi\left[R(t)\right] = E_\pi\left[\int_0^t \frac{C_u}{1+N_u} du\right]$$

$$= \int_0^t E_\pi\left[\frac{C_u}{1+N_u}\right] du$$

$$= \int_0^t \sum_{i=1}^s E_\pi\left[\frac{C_u}{1+N_u} \mid C_u \in \left[C_1^i, ..., C_{r_i}^i\right]\right] P^\pi\left[C_u \in \left[C_1^i, ..., C_{r_i}^i\right]\right] du$$

$$= t \sum_{i=1}^s E_\pi\left[\frac{C^i}{1+N^i}\right] \pi^C\left[C^i\right]$$

$$= t \sum_{i=1}^s E_\pi\left[C^i\right] E_\pi\left[\frac{1}{1+N^i}\right] \pi^C\left[C^i\right]$$

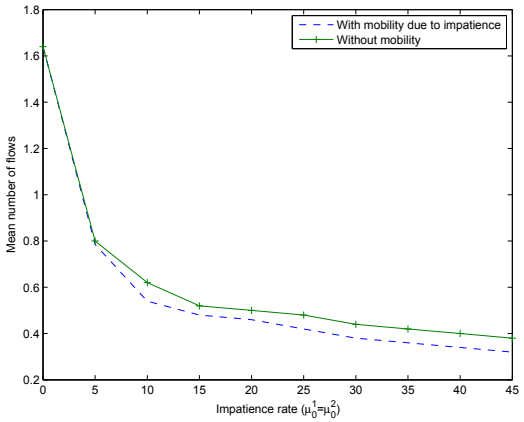

Figure 3.1: Mean number of flows - impatience duration independent of file size - $\rho = 0.65$ - $\lambda_1 = \lambda_2 = 8,\ \mu^1 = 32, \mu^2 = 20$, limit of admission = 15.

$$= t \sum_{i=1}^{s} \left\{ \sum_{l=1}^{r_i} C_l^i \pi^C(C_l^i) \right\} \left\{ \sum_{(n_1^i, \cdots, n_{r_i}^i)} \frac{1}{1 + \sum_{l=1}^{r_i} n_l^i} \pi^i(n_1^i, ..., n_{r_i}^i) \right\} \left\{ \sum_{l=1}^{r_i} \pi^C(C_l^i) \right\}$$

Thus the mean throughtput of the tagged flow in steady state is given by:

$$E_\pi \left[\frac{R(t)}{t} \right] = \sum_{i=1}^{s} \left\{ \sum_{l=1}^{r_i} C_l^i \pi^C(C_l^i) \right\} \left\{ \sum_{(n_1^i, \cdots, n_{r_i}^i)} \frac{1}{1 + \sum_{l=1}^{r_i} n_l^i} \pi^i(n_1^i, ..., n_{r_i}^i) \right\} \left\{ \sum_{l=1}^{r_i} \pi^C(C_l^i) \right\},$$

Where:

$$\pi^i(n_1^i, ..., n_{r_i}^i) = \sum \pi(\cdots, n_1^i, ..., n_{r_i}^i, \cdots)$$

3.8 Numerical applications and simulations

We recall that the mobility variable M is exponential with parameter α. We can notice that $E[M] = \frac{1}{\alpha}$. A user in the system moves a lot when its mobility variable is small, so when its parameter of mobility α is high. High mobility corresponds to high value of α.

3.8.1 Case of intra-cell mobility: exponential flow size

Case of patience duration independent of file size In figure 3.1 we plot the mean number of flows as a function of impatience rate and we compare the case without mobility to the case with mobility due to the impatience. We observe that mobility due to impatience reduces the number of flows in the system but the difference is not very significant.

In figures 3.2 and 3.3, we add the case in which mobility and impatience are independent with two different parameters of mobility $\alpha = 3$ and $\alpha = 10$ respectively. We notice that the mean number of users decreases with mobility. That is confirmed by figure 3.4 in which we plot the mean number of flows as a function of mobility parameter α in the case where mobility is independent of the patience duration that is independent of flow size.

In figure 3.5 we plot the probability to be impatient as a function of impatience in the case independence between mobility, service time and patience duration with different parameters of

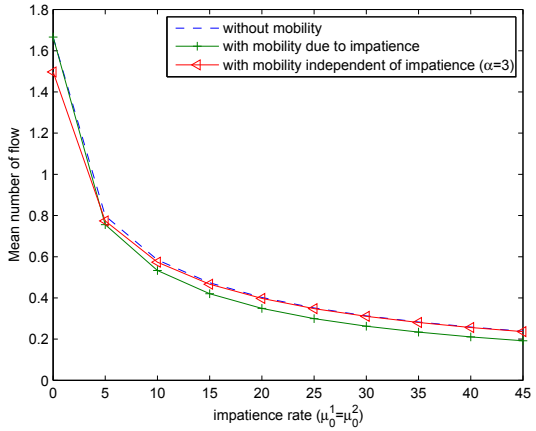

Figure 3.2: Mean number of flows - impatience duration independent of file size - $\rho = 0.65$- $\lambda_1 = \lambda_2 = 8,\ \mu^1 = 32, \mu^2 = 20$, limit of admission = 15.

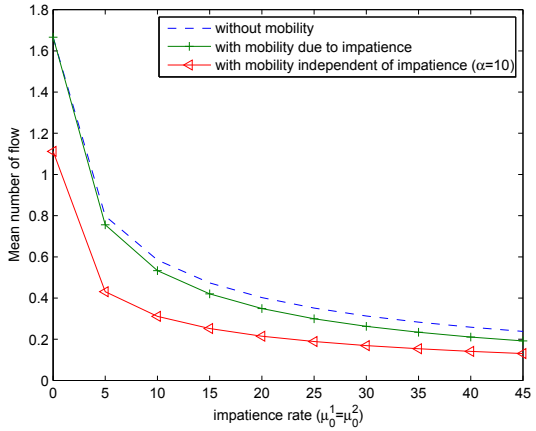

Figure 3.3: Mean number of flows - impatience duration independent of file size - $\rho = 0.65$- $\lambda_1 = \lambda_2 = 8,\ \mu^1 = 32, \mu^2 = 20$, limit of admission = 15.

mobility $\alpha = 0.05$, $\alpha = 0.5$ and $\alpha = 5$. We observe that it increases with impatience rate, and it decreases for high values of mobility rate that corresponds to high mobility. That is confirmed by figure 3.6 in which we plot the probability to be impatient as a function of mobility parameter α in the case of independence between mobility, service time and patience duration.

Case of patience duration dependent on file size Figures 3.7, 3.8, 3.9, 3.10 correspond to the case in which the patience duration depends to the file size. As in the previous case, we observe that the mean number of users and the probability to be impatient decreases when the mobility of users increases.

Let us remark as in the previous chapter, that in the case of patience duration depending on file size, for a user in region l when there are n active users in the system, we have $T_{eff}(n) = \min\left\{n\frac{\sigma}{C_l}, a_l\sigma, M\right\}$ is exponentially distributed with parameter $\frac{\mu}{\min\left\{\frac{n}{C_l}, a_l\right\}} + \alpha$ that is not differentiable on a_l as opposed to the case of independence between patience duration and

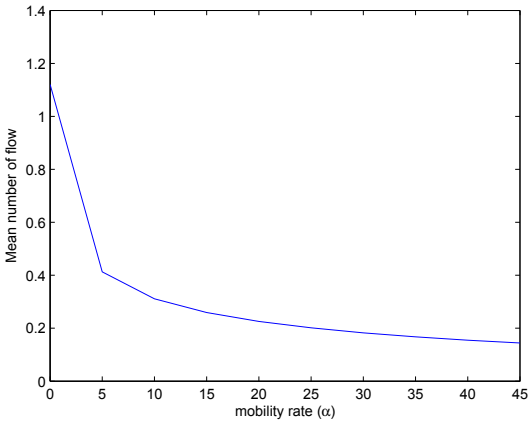

Figure 3.4: Mean number of flows - impatience duration independent of file size - $\rho = 0.65$ - $\lambda_1 = \lambda_2 = 8,\ \mu^1 = 32, \mu^2 = 20, \mu_0^1 = \mu_0^2 = 10$, limit of admission $= 15$.

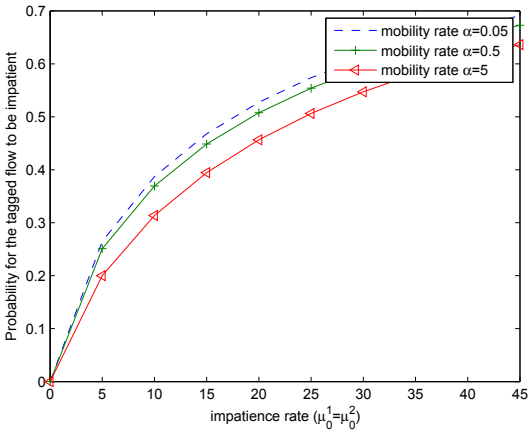

Figure 3.5: Probability of impatience - impatience duration independent of file size - $\rho = 0.65$- $\lambda_1 = \lambda_2 = 8,\ \mu^1 = 32, \mu^2 = 20$, limit of admission $= 15$.

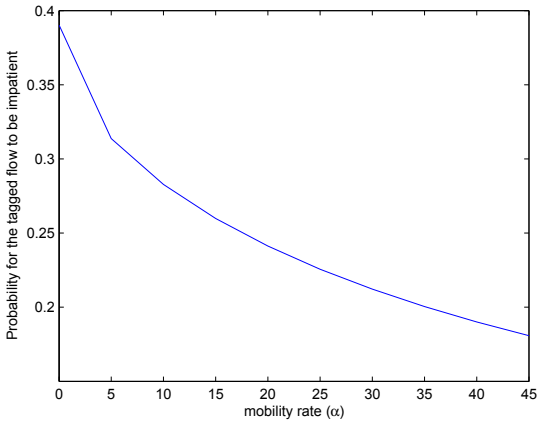

Figure 3.6: Probability of impatience - impatience duration independent of file size - $\rho = 0.65$ - $\lambda_1 = \lambda_2 = 8,\ \mu^1 = 32, \mu^2 = 20, \mu_0^1 = \mu_0^2 = 10$, limit of admission = 15.

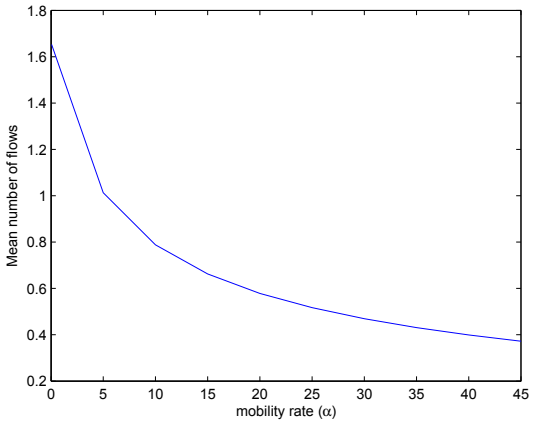

Figure 3.7: Mean number of flows - impatience duration dependent of file size - $\rho = 0.65$ - $\lambda_1 = \lambda_2 = 8,\ \mu^1 = 32, \mu^2 = 20, \frac{1}{a_1} = \frac{1}{a_2} = 10$, flow size = 1, limit of admission = 15.

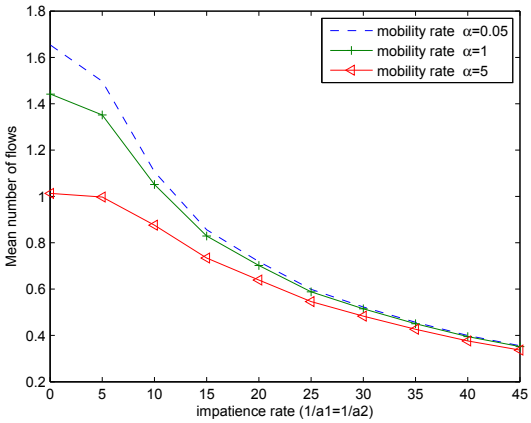

Figure 3.8: Mean number of flows - impatience duration dependent of file size - $\rho = 0.65$ - $\lambda_1 = \lambda_2 = 8,\ \mu^1 = 32, \mu^2 = 20$, flow size $= 1$, limit of admission $= 15$.

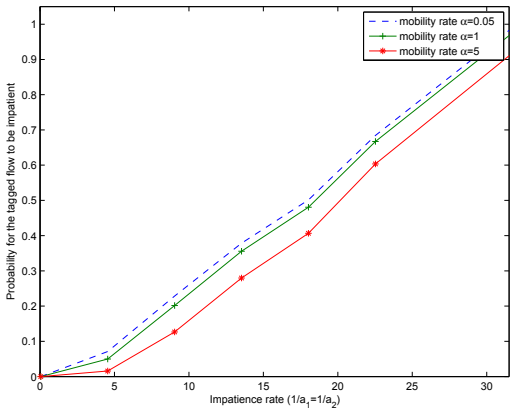

Figure 3.9: Probability of impatience - impatience duration dependent of file size - $\rho = 0.65$ - $\lambda_1 = \lambda_2 = 8,\ \mu^1 = 32, \mu^2 = 20$, flow size $= 1$, limit of admission $= 15$.

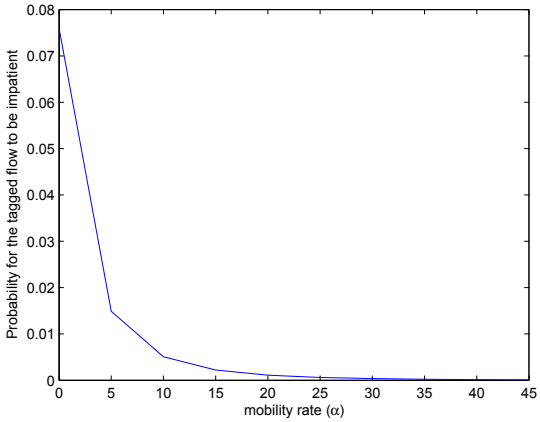

Figure 3.10: Probability of impatience - impatience duration dependent of file size - $\rho = 0.65$ - $\lambda_1 = \lambda_2 = 8,\ \mu^1 = 32, \mu^2 = 20, \frac{1}{a_1} = \frac{1}{a_2} = 10$, flow size = 1, limit of admission = 15.

the flow size in which the above parameter is $\frac{\mu^l}{n} + \mu_0^l + \alpha$ that is differentiable on μ_0^l. This fact explains why the curves plotted as a function of impatience rate, are smother in case of patience duration independent of file size than the curves plotted in case of patience duration depending on file size.

3.8.2 Case of intra-cell mobility: general flow size

Figure 3.11 plots the throughput of the tagged flow as a function of the rate of mobility of the tagged flow going to the cell center in steady state regime; other users mobility rate is fixed. We observe that the throughput increases with increasing mobility rate.

Figure 3.12 plots the throughput of the tagged flow as a function of the rate of mobility of the tagged flow going to the cell edge in steady state regime. We observe that the throughput decreases with the mobility rate. We can explain these obsevations by the fact that the capacity in cell center is higher than in cell edge, thus the throughput is higher in cell center than in cell edge.

Figure 3.13 plots the throughput of the tagged flow as a function of the rate of mobility of the other users in the system; mobility rate of the tagged flow is fixed. We observe that the throughput increases when the mobility rate increases. In fact as observed previously, the number of users decreases when the mobility rate of users increases, thus the throughput of the tagged flow increases.

3.8.3 Case of intra-cell mobility: Simulations

In figure 3.14 we compare the model of exponential service time and the simulation of the model of residual service time with exponential flow size through the mean number of users as function of mobility rate. Simulation is done by considering all users with residual service time as is the case for the tagged flow. The curve obtained by analytical computation of the exponential service time is not constant as indicated in figure 3.14, it is slightly decreasing as a function of mobility rate. We can observe that the average difference between the two curves is less than 3.

In figure 3.15 and 3.16 we use a simulator that considers a tagged flow in the system who is

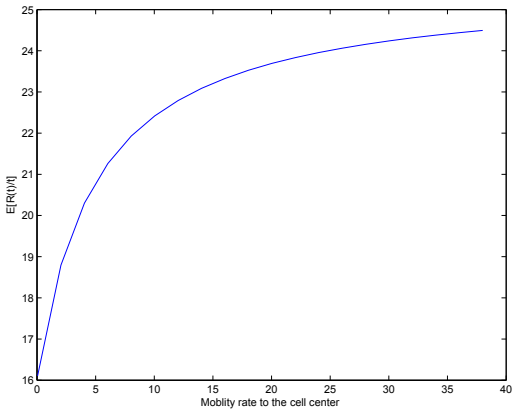

Figure 3.11: Throughput of the tagged flow- Fonction of mobility rate to cell center (α_1) - $\rho = 0.65$ -$\lambda_1 = \lambda_2 = 8,\ \mu^1 = 32, \mu^2 = 20, \mu_0^1 = \mu_0^2 = 3, \alpha_2 = 5, \alpha_{others} = 12$, limit of admission $= 30$.

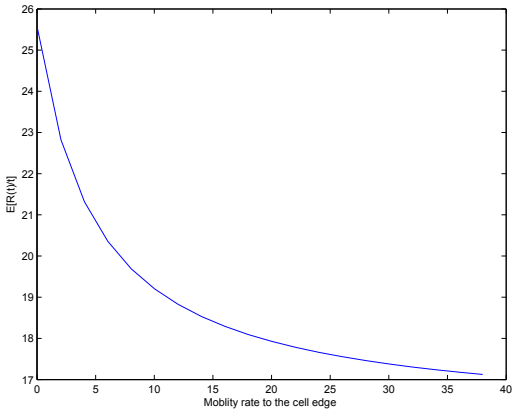

Figure 3.12: Throughput of the tagged flow - Fonction of mobility rate to cell edge (α_2) - $\rho = 0.65$-$\lambda_1 = \lambda_2 = 8,\ \mu^1 = 32, \mu^2 = 20, \mu_0^1 = \mu_0^2 = 3, \alpha_1 = 5, \alpha_{others} = 12$, limit of admission $= 30$.

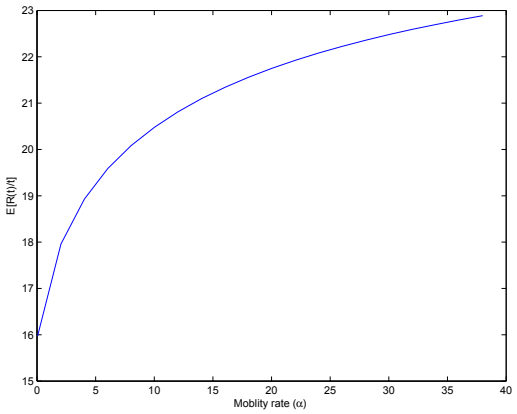

Figure 3.13: Throughput of the tagged flow -Fonction of mobility rate of other users (α_{others}) - $\rho = 0.65$ -$\lambda_1 = \lambda_2 = 8,\ \mu^1 = 32, \mu^2 = 20, \mu_0^1 = \mu_0^2 = 3, \alpha_1 = \alpha_2 = 15$, limit of admission $= 30$.

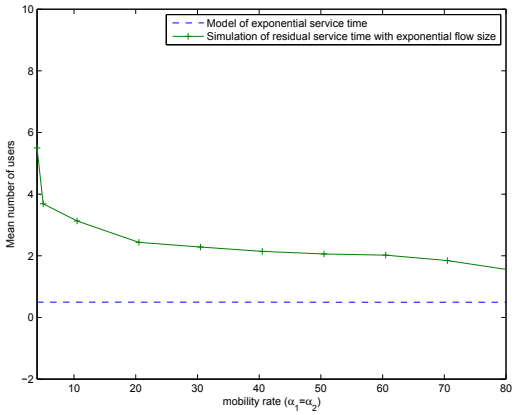

Figure 3.14: Mean number of users of model of exponential service time and simulation of residual service time -Fonction of mobility rate - $\rho = 0.65$ -$\lambda_1 = \lambda_2 = 8,\ \mu^1 = 32, \mu^2 = 20, \mu_0^1 = \mu_0^2 = 29$, limit of admission $= 100$, flow size $= 1$, time of simulation$= 300$, number of Monte Carlo$= 250$.

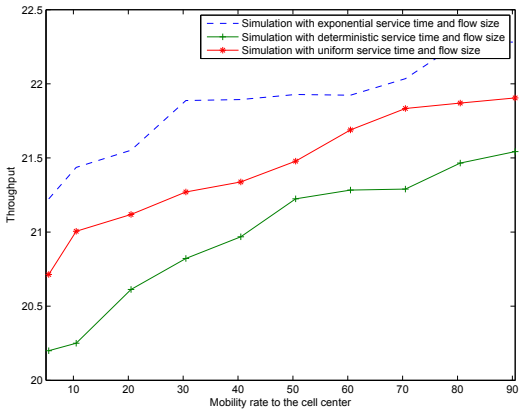

Figure 3.15: Simulation of throughput - Fonction of mobility rate to the cell center - $\rho = 0.65$- $\lambda_1 = \lambda_2 = 8,\ \mu^1 = 32, \mu^2 = 20, \mu_0^1 = \mu_0^2 = 29, \alpha_2 = 25, \alpha_{others} = 15$, limit of admission = 200, flow size = 1, time of simulation= 4700,number of Monte Carlo= 12500.

allowed to move with a residual service time and its variable R defined as previously. All other flows have an exponential service time, a patience duration and a mobility variable that are independent. The simulator gives us the throughput of the tagged flow defined by $\frac{R(t)}{t}$, where t is the time of simulation.

In figures 3.14 and 3.17, we use a simulator that considers all flows with a residual service time. Each flow is allowed to move, has a residual service time variable R, a flow size σ, a patience duration variable τ , a service time defined by $T = \inf\{t > 0; R(t) \geq \sigma\}$. In this simulator, all flows are considered as the tagged flow of the second model. The simulator gives us the number of flows in the system.

We plot in figure 3.15 the throughput of the tagged flow with three different service time distributions: exponential, deterministic and uniform, with the same mean, as a function of the rate of mobility of the tagged flow going to the cell center, and in figure 3.16 we plot it as a function of the rate of moblity of the tagged flow going to the cell edge. We observe that the throughput increases when the rate of mobility going to the cell center increases, and the throughput decreases when the rate of mobility going to the cell edge increases with the three different distributions.

In figure 3.17, we do the simulation of our model by considering all users as we do for the tagged flow with their service account R as defined previously, their flow size, and the same rate of mobility for all users. We plot the number of flows in the system with three diferent distributions for the flow size, the patience duration, and the variable of mobility as a function of the rate of mobility that we assume it to be the same for the cell center and the cell edge. For each of the three distributions: exponential, deterministic or uniform, we assume that the flow size, the patience duration and the variable of mobility have the same distribution. We observe that the number of flows decreases when the rate of mobility increases, and the three curves are close.

3.8.4 Case of inter-cell mobility

In the case of inter cell mobility, numerical applications and simulations are done by considering two cells called cell 1 and cell 2, each cell contains two regions that are the center region and

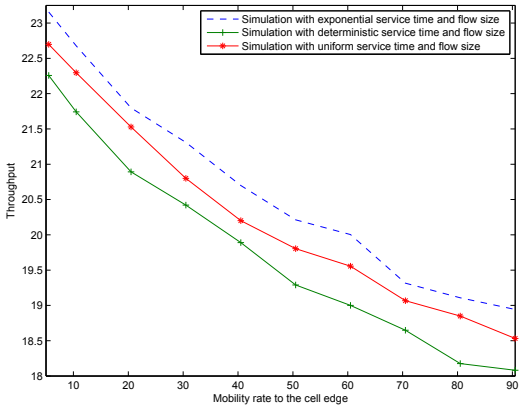

Figure 3.16: Simulation of throughput - Fonction of mobility rate to the cell edge - $\rho = 0.65$ -$\lambda_1 = \lambda_2 = 8,\ \mu^1 = 32, \mu^2 = 20, \mu_0^1 = \mu_0^2 = 29, \alpha_1 = 25, \alpha_{others} = 15$, limit of admission = 200, flow size = 1, time of simulation= 4700,number of Monte Carlo= 12500.

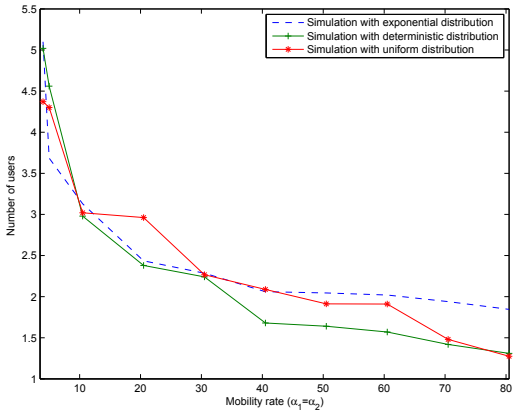

Figure 3.17: Simulation of number of users -Mobility rate - $\rho = 0.65$ -$\lambda_1 = \lambda_2 = 8,\ \mu^1 = 32, \mu^2 = 20, \mu_0^1 = \mu_0^2 = 29$, limit of admission = 100, flow size = 1, time of simulation= 300,number of Monte Carlo= 250.

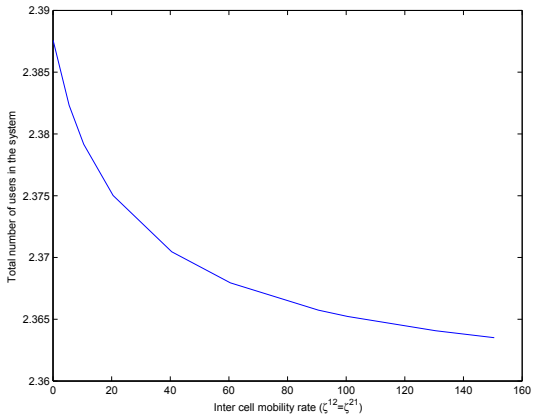

Figure 3.18: Analytical model mean of total number of users -Fonction of inter-cell mobility rate (ζ) - Two cells with two regions -$\lambda_1^1 = \lambda_2^1 = \lambda_1^2 = \lambda_2^2 = 12,\ \mu_1^1 = 32, \mu_2^1 = 20, \mu_1^2 = 40, \mu_2^2 = 18, \mu_0 = 3, \alpha_1^1 = \alpha_2^1 = \alpha_1^2 = \alpha_2^2 = 0.1$, limit of admission $K_1 = K_2 = 15$.

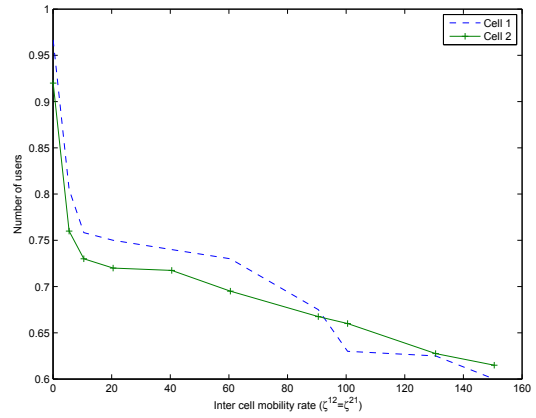

Figure 3.19: Simulation of the mean number of users in each cell -Fonction of inter-cell mobility rate (ζ) - Two Cells of two regions -$\lambda_1^1 = \lambda_2^1 = \lambda_1^2 = \lambda_2^2 = 12,\ \mu_1^1 = 32, \mu_2^1 = 20, \mu_1^2 = 40, \mu_2^2 = 18, \mu_0 = 3, \alpha_1^1 = \alpha_2^1 = \alpha_1^2 = \alpha_2^2 = 0.1$, limit of admission $K_1 = K_2 = 100$, time of simulation = 500, number of Monte Carlo = 200.

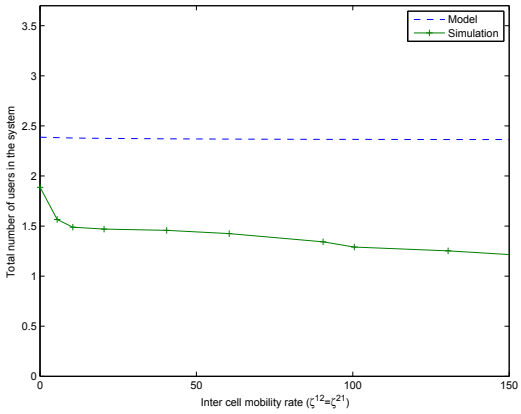

Figure 3.20: Analytical model of mean total number of users and simulation of mean total number of users -Fonction of inter-cell mobility rate (ζ) - Two cells of two regions -$\lambda_1^1 = \lambda_2^1 = \lambda_1^2 = \lambda_2^2 = 12$, $\mu_1^1 = 32, \mu_2^1 = 20, \mu_1^2 = 40, \mu_2^2 = 18, \mu_0 = 3, \alpha_1^1 = \alpha_2^1 = \alpha_1^2 = \alpha_2^2 = 0.1$, limit of admission $K_1 = K_2 = 100$, time of simulation = 500, number of Monte Carlo = 200.

the edge region.

The intra-cell mobility is always considered through parameter α.

In figures 3.19 and 3.20 we use a simulator that considers all flows with exponential service time. Users are assumed to arrive in each region of each cell according to a Poisson process, and each user has a service time, a patience duration variable, an intra-cell mobility variable allowing him to move inside the cell in which he is located, and an inter-cell mobility variable allowing him to move from egde region to egde region of the second cell, that are independent. The simulator gives us the number of users in each cell.

In figure 3.18 we plot the total number of users in the system consisting of two cells, as a function of the inter cell mobility rate ζ. It shows that the total number of users in the system deacreases when the inter cell mobility increases.

Figure 3.19 plots the results of simulations of the number of users in cell 1 and cell 2 as a function of inter cell mobility rate. It shows that the number of users in each cell decreases with an increasing inter cell mobility rate.

Figure 3.20 allows us to make a comparison between the model and simulations. It plots the total number of users in the system as a function of inter cell mobility rate. The curve obtained by analytical computation is not constant as shown in figure 3.20, it is slightly decreasing as indicated in figure 3.18 in which it is plotted alone. We can see that the average difference between the two curves is less than 2.

In figure 3.21, we plot the throughput of the tagged flow as a function of inter cell mobility rate. The curve is plotted assuming that the inter cell mobility rate of the tagged flow and all other users are equal. It shows that the throughput of the tagged flow deacreases when the inter cell mobility rate increases. This can be explained by noticing that the inter-cell mobility occurs only between edge regions of cells, so high inter-cell mobility rate leads users to be in region edge, which decreases their throughtput.

Figure 3.22 plots the throughput of the tagged flow as a function of intra cell mobility α. The curve is plotted by considering that the intra cell mobility rate of the tagged flow and all other users are equal. The curve shows that the throughput of the tagged flow increases when the intra cell mobility rate increases. This confirms the results of the previous studies about

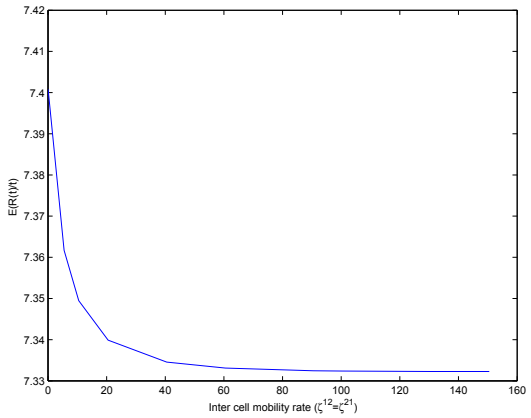

Figure 3.21: Analytical model of throughput of the tagged flow -Fonction of inter-cell mobility rate (ζ) - Two Cells of two regions -$\lambda_1^1 = \lambda_2^1 = \lambda_1^2 = \lambda_2^2 = 12$, $\mu_1^1 = 32, \mu_2^1 = 20, \mu_1^2 = 40, \mu_2^2 = 18, \mu_0 = 3, \alpha_1^1 = \alpha_2^1 = \alpha_1^2 = \alpha_2^2 = 0.1$, limit of admission $K_1 = K_2 = 15$.

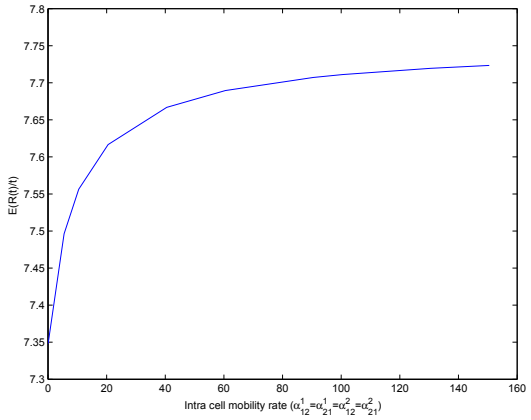

Figure 3.22: Analytical model of throughput of the tagged flow -Fonction of intra-cell mobility rate (ζ) - Two cells of two regions -$\lambda_1^1 = \lambda_2^1 = \lambda_1^2 = \lambda_2^2 = 12$, $\mu_1^1 = 32, \mu_2^1 = 20, \mu_1^2 = 40, \mu_2^2 = 18, \mu_0 = 3$, limit of admission $K_1 = K_2 = 15$.

intra-cell mobility.

3.9 Conclusion

We modeledin this chapter mobility for users subject to impatience in mobile cellular networks, with an application to 4G LTE, and quantified its impact on system performance in terms of several QoS parameters, such as mean number of users, the probability to be impatient in the cell and the throughput.

We observed that mobility results in higher system stability, more users are moving more their number decreases, the more the probability of impatience decreases, and the more the throughput increases.

Chapter 4

Modeling of impatience for streaming flows in mobile networks

Abstract

Our purpose in this part is to obtain the probability of impatience for a streaming user during the prefetching and the rebuffering phase when starvation happens. We first model the buffer as a M/M/1 queue and introduce the patience duration of streaming user by considering a packet level in which the video size is assumed to be infinite, and is composed of packets and we consider the case of deterministic and exponential patience duration. Secondly we model the probability of impatience in the continuous time playback taking into account the flow dynamics in the system constituted of several regions scheduled as a processor sharing and we consider the case of deterministic and exponential patience duration. We derive several performance metrics such as the probability to be impatient during rebuffering and the probability of starvation.

4.1 Introduction

Video services are popular on mobile networks and account for increasing proportion of data transmitted. Among the different online video services, youtube, netflix are the most popular. Services such as social networks enable users to share personal videos thus extending the video streaming audiance.

Nowadays, it is crutial for mobile network operators to improve the user experience of video service or QoE.

In order to define the QoE, it is necessary to know how streaming is played. Video streaming consists of packets, packets are fragments of video. Media player is equipped with a playout buffer that stores arriving packets. Video is played as long as there are packets in the buffer. At the begining the buffer is empty and has to reach a threshold of packets to start video. This phase is called prefetching phase. When the buffer reaches the threshold of prefetching the video starts and the streaming user can watch its video. Once the buffer empties, the video stops, this phenomenon is called the starvation, and the buffer has to reach a threshold to start video. This phase is called rebuffering phase, the streaming user cannot watch its video during this phase until the buffer reaches the threshold of rebuffering.

The prefetching and the rebuffering phases are annoying for the user, so he can leave its video session by impatience. Our aim is to study this phenomenon of impatience.

In [48] authors have demonstrated that impatience can impact Peer-to-peer video on demand (P2P VoD) streaming performance significantly. They showed that impatience can increase interruptions and wasted ressources and they showed that an Earliest-First policy in conjonction with Earliest-Deadline as peer selection strategy is more appropriate in presence of impatience.

In [49] authors studied the impact of video quality on user engagement. For this they used a dataset spanning short video on demand, long video on demand and live content from popular video content providers and they measured join time, rate of buffering events, buffering ratio, rendering quality and average bitrate. They showed that the time spent in buffering has the most impact on the user engagement and the magnitude of the impact depends on the content type. They showed that the impact is large with live content.

Authors in [50] distinguished two sources of energy waste for smarphones during video session: energy wasted during the prefetching when the user closes its video session because of impatience and energy wasted by keeping the wireless interface powered on after receiving a chunk of content that is caused by prefetching chunks that are small. They provided a download scheduling algorithm based on crowd-sourced video viewing statistics that allows them to evaluate the probability for a user to interrupt a video in order to perform the right amount of prefetching. The algorithm balances the amount of the two kinds of energy waste. They showed by simulations that their sheduler reduces the energy waste to half comapared to existing download strategies.

In [12] authors have provided an exact distribution of the number of starvations for a streaming user by considering a finite number of packets that constitutes the video file. They modeled the buffer as an M/M/1 queue and used an approch based on Ballot theorem and an approch based on recursive equations.

Authors in [11] studied the quality of experience for a streaming user in wireless data networks in terms of probability of starvation during the video session. They showed that the flow dynamics are the fundamental cause of starvations. We use their model in section 4.3 in the case of a system constituted of several regions and we introduce the phenomonon of impatience.

This chapter is organized as follows. In section 4.2, we model the buffer at the packet level without considering the impact of other users on the system. We consider that the system is a buffer in which packets arrive according to a Poisson process, and are played with exponential

service time. In section 4.3, we consider a processor sharing system constituted of several regions, in which we consider a tagged streaming user during its video session. In the two sections, we study the cases of deterministic and exponential patience duration.

4.2 Model at packet level

4.2.1 System description

We consider an OFDMA-based homogeneous cellular network and focus on the downlink of one cell with a single base station at its center. We consider a streaming user, who watches its video on Youtube, Dailymotion etc, and we want to model its impatience during the prefetching phase and the rebuffering phase after starvation happens.

Let us assume that the video size is infinite, and the buffer size is infinite. The video size is divided by packets, and as in [12], we assume that packets arrive in the buffer following a Poisson process with intensity λ, and the service rate of packet is assumed to be exponential with parameter μ. So the buffer can be modeled as a M/M/1 queue.

At the beginning of video playing, there are the prefetching phase, in which the user has to wait a start up delay, that is the time at which the buffer has to contain the prefetching threshold that we note by x for the video begins.

The phenomenon of starvation occurs when the buffer is empty, and the playback stops, so the system enters in the rebuffering phase, in which as in prefetching phase the media player waits until the buffer reaches the threshold x_1 of rebuffering x_1. Note that during the rebuffering phase, packets are not served.

We can resume the path of buffer in different steps as follows:

- the prefetching phase: in which user has to wait until the buffer reaches a threshold of prefetching,
- the playout phase: in which user watches its video,
- the starvation: the instant at which the playout stops because the buffer is empty
- the rebuffering phase: this is like the prefetching phase, in which user waits until the buffer reaches a threshold of rebuffering.

The prefetching and the rebuffering phases are annoying to the user, so he can put out the video service by impatience.

Let us notice that the end of the rebuffering phase is the beginning of the playout phase and the end of the playout phase is the beginning of the rebuffering phase.

We introduce the patience duration by a random variable τ, that models the patience of the user. In this work, we consider the cases of exponential and deterministic patience duration.

4.2.2 Exponential patience duration

Here we assume that the patience duration is exponentially distributed with parameter μ_0.

We consider two Markov processes X_t and Y_t to study the buffer path, the first studies the prefetching and the rebuffering phases, and the second studies the playout phase.

As started earlier, we assume that packets arrive in the buffer following a Poisson process with intensity λ.

Let us denote by x the threshold of prefetching phase, and X_t^x the number of packets in the buffer at time t during the prefetching phase.

Note that during the prefetching phase, no packet is played, so there is no service during the prefetching phase. We define two absorbing states I and A, where I is the state in which the user is impatient and puts out the video, and A is the state in which the buffer reaches the threshold x that is the end of the prefetching phase.

With these assumptions, the process X_t^x is Markovian, taking values in $\{0, ..., x-1, A, I\}$, and its intensity matrix Q is given by:

$$q_{n,n+1} = \lambda, \ \forall n \leq x-2,$$

$$q_{x-1,A} = \lambda,$$

$$q_{n,n-1} = 0, \ \forall n,$$

$$q_{n,I} = \mu_0, \ \forall n \leq x-1,$$

$$q_{n,A} = 0, \ \forall n \leq x-2.$$

Let us define $T_I = \inf\{t \geq 0, \ X_t^x = I\}$ and $T_A = \inf\{t \geq 0, \ X_t = A\}$ the time of impatience and the start up delay respectively.

Let us pose $V^x(n) = P\left[X^x_{\min\{T_I,T_A\}} = I / X_0^x = n\right]$, the probability of impatience during the prefetching phase conditioned on the initial state being equal to n, it means that there are n packets in the buffer, and $V^x(0)$ is the probability of impatience with initial empty buffer.

$V^x(n)$ can be computed as a hitting probability of state I, thus $V^x(n)$ is solution of the following equation:

$$(Q \ V^x)_n = 0 \qquad \forall \ n \neq I,$$

$$V^x(I) = 1$$

$$V^x(A) = 0$$

If starvation happens at the end of playout phase, the buffer enters in the rebuffering phase, in which the buffer reaches the threshold of rebuffering x_i, where i is the i^{th} rebuffering phase.

We consider the Markov process $X_t^{x_i}$ by replacing x by x_i, and the probability of impatience will be V^{x_i}.

Now to study the starvation, we assume as previously that packets arrive to the buffer following a Poisson process with mean intensity λ. During the playout phase packets are served following a FIFO Scheduler, with one server, and we assume that the service rate is exponential with parameter μ, and there is no impatience.

We conisder the process Y_t that is the number of packets in the buffer taking values in $\{0, 1, 2..., \}$, and the playout stops when the buffer is empty i.e; when $Y_t = 0$.

With these assumptions, Y_t is a Markovian process with the following intensity matrix Q:

$$q_{n,n+1} = \lambda, \ \forall n,$$

$$q_{n,n-1} = \mu, \forall n > 0.$$

Let us denote by $T_0 = \inf\{t \geq 0, \ Y_t \leq 0\}$, starvation happens when $T_0 < \infty$, and the probability of starvation when the buffer contains n packets is given by $S(n) = P[T_0 < \infty / Y_0 = n]$. This is the hitting probability of state 0, thus S is solution of the following equation:

$$(Q \ S)_n = 0 \qquad \forall \ n \neq 0,$$

$$S(0) = 1.$$

When the prefetching threshold is x, the first starvation probability is $S(x)$, and when the i^{th} rebuffering threshold is x_i, the $i-1^{th}$ starvation probability will be $S(x_i)$.

The probability to be impatient during the l^{th} rebuffering phase V^l is given by:

$$V^l = V^{x_l}(0)\left[1 - V^x(0)\right]\prod_{i=1}^{l-1}\left[1 - V^{x_i}(0)\right]S(x_i)$$

4.2.3 Deterministic patience duration

In this part we assume that the patience duration is deterministic that means $\tau = E[\tau]$.

As previously we consider two Markov process X_t and Y_t, where the first one describes the prefetching and the rebuffering phases, and the second one describes the playout phase.

As the impatience occurs only during the prefetching and the rebuffuring phases, the process Y_t is the same as previously.

We consider an absorbing state A, that is the state in which X_t reaches the threshold of prefetching or rebuffering.

The threshold of prefetching is x, the process X_t^x is Markovian, taking values in $\{0, ..., x-1, A\}$ with intensity matrix Q given by:

$$q_{n,n+1} = \lambda,\ \forall n \leq x-2,$$

$$q_{x-1,A} = \lambda,$$

$$q_{n,n-1} = 0,\ \forall n,$$

$$q_{n,A} = 0,\ \forall n \leq x-2.$$

We define $T_A = \inf\{t \geq 0,\ X_t^x = A\}$ the start up delay of prefetching.

The user is impatient when the start up delay is more than its patience duration: $T_A \geq \tau$. So we define $V_n^x(t) = P[T_A \geq t / X_0^x = n]$, thus $V_0^x(\tau)$ will be the probability to be impatient during the prefetching phase.

By the well-known relationship between Markov process and partial differential equations, $V_n^x(t)$ is solution of:

$$\frac{\partial V_n^x(t)}{\partial t} = QV_n^x(t)$$

Thus:

$$\frac{\partial V_n^x(t)}{\partial t} = \lambda\left[V_{n+1}^x(t) - V_n^x(t)\right]$$

The boundary conditions are: for $t = 0$, obviously $V_n^x(0) = 1\ \forall n$, and if the process starts in state A there is no impatience then $V_A^x(t) = 0$, $\forall t \geq 0$.

We denote by $\bar{V_n^x}(p) = \int_0^\infty e^{-pt}V_n^x(t)\,\mathrm{d}t$ the Laplace transformation of V_n^x. If we apply the Laplace transformation evaluated at the complex number p to the above equation we obtain:

$$\bar{V_n^x}(p) = \frac{1}{p+\lambda}\left[1 + \lambda\bar{V_{n+1}^x}(p)\right]$$

By the boundary condition $V_A^x(t) = 0\ , \forall t \geq 0$, we obtain $\bar{V_{x-1}^x}(p) = \frac{1}{p+\lambda}$: we pose $R = \frac{1}{p+\lambda}$.

Thus by retrograding we deduce:

$$\bar{V_{x-k}^x}(p) = \sum_{i=1}^{k}\lambda^{i-1}R^i.$$

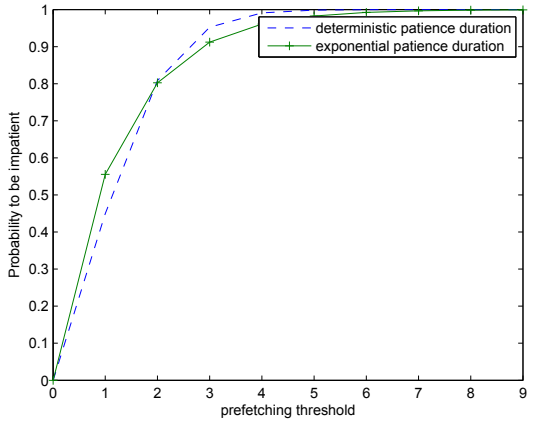

Figure 4.1: Probability to be impatient in prefetching phase- deterministic versus exponential patience duration

By the inverse Laplace transformation, we obtain:

$$V_{x-k}^{x}(t) = \sum_{i=1}^{k} \lambda^{i-1} \frac{t^{i-1}}{(i-1)!} e^{-\lambda t}$$

Thus the probability of impatience is given by:

$$V_{0}^{x}(\tau) = \sum_{i=1}^{x} \lambda^{i-1} \frac{\tau^{i-1}}{(i-1)!} e^{-\lambda \tau}.$$

Let us notice that for infinite threshold x, the probability of impatience is 1:

$$\lim_{x \to \infty} V_{0}^{x}(\tau) = 1, \ \forall \tau \geq 0$$

As for the case of exponential patience duration, the probability of impatience in the ith rebuffering phase with threshold of rebuffering x_i is obtained as in the prefetching phase by replacing x by x_i, it is given by $V_0^{x_i}(\tau)$.

The study of playout phase is the same as in the case of exponential patience duration, because there is no impatience in there. So we obtain the probability of impatience in the l^{th} rebuffering phase by the formula:

$$V^{l}(\tau) = V_{0}^{x_l}(\tau) \left[1 - V_{0}^{x}(\tau)\right] \prod_{i=1}^{l-1} \left[1 - V_{0}^{x_i}(\tau)\right] S(x_i)$$

4.2.4 Numerical applications

We plot in Figures 4.1 and 4.2 the probability to be impatient during the prefetching phase as a function of prefetching threshold and impatience rate respectively for both deterministic and exponential patience duration. We observe that the probability to be impatient increases with increasing prefetching threshold and impatience rate.

In figure 4.3 we observe the probabilty to be impatient during the prefetching phase in the case of exponential patience duration as a function of impatience rate with different prefetching thresolds. It shows that the probability to be impatient increases with increasing impatience rate and increasing prefetching threshold.

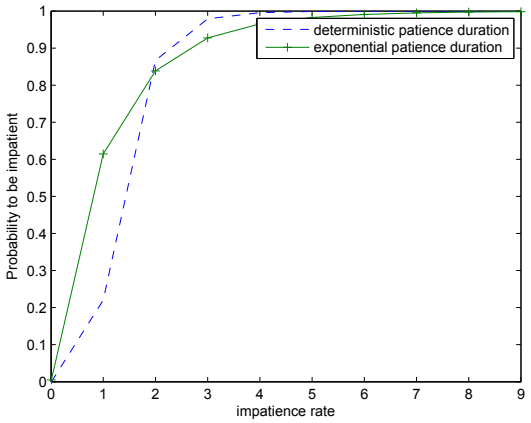

Figure 4.2: Probability to be impatient in prefetching phase - deterministic versus exponential patience duration

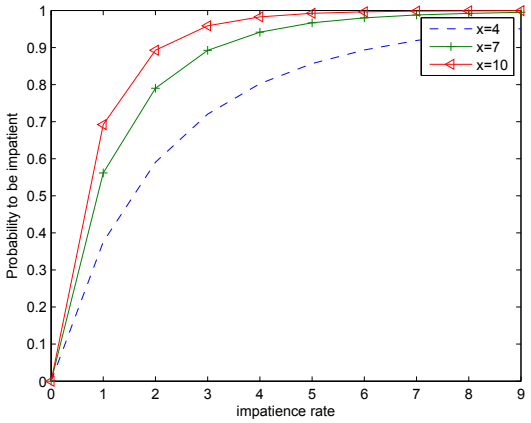

Figure 4.3: Probability to be impatient in prefetching phase- exponential patience duration

4.3 Model at flow level

4.3.1 System description

We consider again an OFDMA-based homogeneous cellular network and focus on the downlink of one cell with a single base station at its center. With OFDMA, the total bandwidth, which we denote by W, is divided into N orthogonal subcarriers and can be shared between the different users present in the cell in the same time slot.

Due to propagation conditions and interference from other cells, the Signal-to-Interference and Noise Ratio (SINR) is lower at the cell edge than at the cell center. This leads to a cell capacity $C(r)$ that depends on the distance r between the user and the base station.

The throughput decreases when the user gets further away from the base station. Let $C_1 > C_2 > .... > C_l > .... > C_r$ be the set of throughputs at different positions l in the cell and p_l the probability that the user arrives to the cell in region l into r regions where the throughput is almost constant in each region.

In this part, we are inspired by the model in [11].

We assume that the system contains only one streaming user and all other users are elastic.

We assume that elastic flows arrive in region l following a Poisson process with intensity λ_l.

We assume that elastic file sizes are exponentially distributed with the same parameter μ. An elastic flow in region l has an exponential service rate with parameter $\mu_l = C_l\mu$ if he is alone in the system, where C_l is the capacity of region l.

In order to study the dynamics of the playout buffer for a streaming flow, we focus on the only streaming flow in the system that we tag, and we assume that he is located in region k.

We define the process $N_t = \left(N_t^1, ..., N_t^r\right)$ where N_t^l is the number of elastics users seen by the tagged flow in region l, we add a supplementary state F that is the state in which the tagged flow finishes its service.

Let us denote by $Bitrate$ the playback speed of video streams measured in bits per seconds.

When the process N_t is at state $N = (n_1, \cdots, n_r)$, the throughput in seconds of the video content of the tagged flow that we denote by b_N^k is given by $b_N^k = \frac{C_k}{Bitrate(n+1)}$, where $n = \sum_{l=1}^r n_l$.

Let us notice that in this study we don't consider impatience for elastic users in the system, our aim is to study the phenomenon of impatience of the tagged flow during its video session.

With these assumptions, the process N_t is Markovian with intensity matrix given by:

$$q_{N,N+e_l} = \lambda p_l = \lambda_l,$$

$$q_{N,N-e_l} = \frac{n_l}{\sum_{i=1}^r n_i + 1}\mu_l,$$

$$q_{N,F} = \frac{\mu_k}{\sum_{i=1}^r n_i + 1},$$

$$q_{N,N} = -\left(\lambda + \frac{\mu_k}{\sum_{i=1}^r n_i + 1} + \sum_{i=l}^r q_{N,N-e_l}\right),$$

$$q_N = -q_{N,N}$$

where $N = (n_1, ..., n_r)$ and $e_i = (0, ...1, ...0)$

Let us denote by $N_e(t)$ the number of state changes of the process N_t at time t, A_l the time that the l^{th} state change takes place with $A_0 = 0$, and by $N_l = N_{A_l}$ the state to which the process changes state after time A_l.

We denote by $Q^k(t)$ the length of playout buffer of tagged flow measured in seconds of video content at time t.

In the prefetching phase, $Q(t)$ is defined as:

$$Q_a^k(t) = \sum_{l=1}^{N_e(t)} b_{N_l}^k \left(A_l - A_{l-1}\right) + b_{N_{N_e(t)}}^k \left(t - A_{N_e(t)}\right).$$

Let us denote by q_a the start-up threshold of prefetching phase, that is the threshold from which the video starts. The start-up delay is defined as $T_a = \inf\left\{t \geq 0, Q_a^k(t) \geq q_a\right\}$.

Let us notice that the prefetching phase is meaningful only when the video duration is longer than start-up threshold.

During the prefetching phase, the tagged flow can be impatient, and closes the media player.

We assume that the tagged flow has a patience duration modeled by a random variable τ. The impatience of the tagged flow occurs when the start up delay is more than its patience duration $T_a > \tau$, in this case he closes the media player, that is the end of his video session.

It is easy to see from the expression of Q_a^k that for an infinitesimal $\epsilon > 0$, $Q_a^k(t+\epsilon) = Q_a^k(t) + b_i^k\epsilon$.

The prefetching phase is equivalent to starvation when the buffer content q_a and the queue is depleted with rate b_i^k, as $Q_a^k(t+\epsilon) = Q_a^k(t) - b_i^k\epsilon$, so the start-up delay can be rewritten as $T_a = \inf\left\{t \geq 0, Q_a^k(t) \leq 0\right\}$.

4.3.2 Deterministic patience duration

In this part, we assume that the tagged flow has a deterministic patience duration which means $E\left[\tau\right] = \tau$.

Let us introduce $V_N(t,q) = P\left[T_a > t/N_0 = N, Q_a^k(0) = q\right]$, then $V_N(\tau, q_a)$ is the probability that the tagged flow is impatient during prefetching phase when the process N_t starts at state N. Note that $V_F(t,q) = 0$.

In order to study $V_N(t,q)$, let us consider an infinitesimal interval $[0,h]$. The queue evolves in $[0,h]$ with four possible events:

- nothing happens
- arrival of one user in the system
- departure of one user
- occurence of more than one event

So we have:

$$V_N(t,q) = (1 + q_{N,N}h)V_N(t-h, q-b_Nh) + \sum_{l=1}^{r} q_{N,N+e_l}hV_{N+e_l}(t-h, q-b_Nh)$$

$$+ \sum_{l=1}^{r} q_{N,N-e_l}hV_{N-e_l}(t-h, q-b_Nh) + o(h).$$

Dividing all by h we have

$$\frac{V_N(t,q) - V_N(t-h, q-b_Nh)}{h} = q_{N,N}V_N(t-h, q-b_Nh) + \sum_{l=1}^{r} q_{N,N+e_l}V_{N+e_l}(t-h, q-b_Nh)$$

$$+\sum_{l=1}^{r} q_{N,N-e_l} V_{N-e_l}(t-h, q-b_N h) + o(1).$$

Letting h tend to zero, we obtain

$$\frac{\partial V_N(t,q)}{\partial t} + b_N \frac{\partial V_N(t,q)}{\partial q} = (QV)_N(t,q) \tag{4.1}$$

where

$$(QV)_N(t,q) = \sum_M q_{N,M} V_M(t,q)$$

$$= q_{N,N} V_N(t,q) + \sum_{l=1}^{r} q_{N,N+e_l} V_{N+e_l}(t,q) + \sum_{l=1}^{r} q_{N,N-e_l} V_{N-e_l}(t,q)$$

The boundary conditions are $V_N(0,q) = 1,\ \forall q > 0, \forall N \neq F$, because the start up delay is always non negative, $V_N(t,0) = 0,\ \forall t > 0$, in this case the start up delay is null then less than t, and $\lim_{q\to\infty} V_N(t,q) = 1$ because the start up delay will be more than all t if the threshold is more enough.

We can write the equation 4.1 in matrix form as:

$$\frac{\partial \mathbb{V}(t,q)}{\partial t} + \mathbb{D}\frac{\partial \mathbb{V}(t,q)}{\partial q} = Q\mathbb{V}(t,q) \tag{4.2}$$

where $\mathbb{V}(t,q) = \begin{bmatrix} V_0(t,q) \\ \vdots \\ V_N(t,q) \\ \vdots \end{bmatrix}$, $\mathbb{D} = Diag\{b_N\}$ is a diagonal matrix.

Equation (4.2) is a system of hyperbolic partial differential equation of first order that can be solved by the method of characteristic by considering the following curve:

$$\frac{dq(t)}{dt} = b_N,$$

$V_N(t,q)$ is solution of the following integral equation:

$$V_N(t,q) = e^{q_{NN}t} V_N(0, q - b_N t) + \int_0^t \sum_{M\neq N} e^{q_{NN}x} q_{NM} V_M(t-x, q-b_N x)\, dx$$

If the tagged flow is not impatient during the prefetching phase, the playback phase begins, the user begins to watch his video and there is no impatience during the playback.

The phenomenon of starvation occurs when the playback stops, and the buffer is empty so it enters in the rebuffering phase in which as for the prefetching phase the tagged flow may be impatient.

In the rebuffering phase, the same queuing process as in the prefetching phase $Q_a^k(t)$ is used, with the threshold of rebuffering q_b.

The probability that the tagged flow is impatient during rebuffering is $V_N(\tau, q_b)$.

Now let us focus on the probability that the prefetching ends in an arbitrary state M.

Let us introduce $R_N^M(q, q_a) = P\left[N_{T_a} = M/N_0 = N, Q_a^k(0) = q\right]$, the probability that the prefetching starts at state N and ends at state M is $R_N^M(0, q_a)$.

We recall that $Q_a^k(t+h) = Q_a^k(t) + b_i h$ then $\frac{d}{dt}\left(Q_a^k(t)\right) = b_i$.

For $h > 0$ and an infinitesimal interval $[0, h]$, when the queue evolves in $[0, h]$, there are four possible events as in the study of V_N, so we have:

$$R_N^M(q, q_a) = (1 + q_{N,N}h)R_N^M(q + b_N h, q_a) + \sum_{l=1}^{r} q_{N,N+e_l} h R_{N+e_l}^M(q + b_N h, q_a)$$

$$+ \sum_{l=1}^{r} q_{N,N-e_l} h R_{N-e_l}^M(q + b_N h, q_a) + o(h).$$

Dividing all by h we have:

$$\frac{R_N^M(q, q_a) - R_N^M(q + b_N h, q_a)}{h} = q_{N,N} R_N^M(q + b_N h, q_a) + \sum_{l=1}^{r} q_{N,N+e_l} R_{N+e_l}^M(q + b_N h, q_a)$$

$$+ \sum_{l=1}^{r} q_{N,N-e_l} R_{N-e_l}^M(q + b_N h, q_a) + o(1).$$

Letting h tend to zero, we obtain:

$$-b_N \frac{\partial R_N^M(q, q_a)}{\partial q} = q_{N,N} R_N^M(q, q_a) + \sum_{l=1}^{r} q_{N,N+e_l} R_{N+e_l}^M(q, q_a) + \sum_{l=1}^{r} q_{N,N-e_l} R_{N-e_l}^M(q, q_a),$$

with the boundary conditions $\forall N \neq M$: $R_N^M(q_a, q_a) = 0$ and $R_N^N(q_a, q_a) = 1$.

We notice that rebuffering happens only if there is starvation.

If the time axis starts at the instant of playing, we define the playback process by:

$$Q_b^k(t) = q_a - t + \sum_{l=1}^{N_e(t)} b_{N_l}^k \left(A_l - A_{l-1}\right) + b_{N_{N_e(t)}}^k \left(t - A_{N_e(t)}\right)$$

Let us define $T_b = \inf\left\{t \geq 0, Q_b^k(t) \leq 0\right\}$, the instant of starvation.

If we pose $d_N = b_N - 1$, then $Q_b^k(t + h) = Q_b^k(t) + d_N h$.

We introduce $S_N^M(q) = P\left[N_{T_b} = M / Q_b^k(0) = q, N_0 = N\right]$ the probability that the playout begins at state N when the buffer content is q and ends at state M, so $S_N^M(q_a)$ is the probability that the playout begins at state N and ends at state M i.e, starvation occurs at state M.

By considering a infinitesimal interval $[0, h]$ and by the same argument as previously, it follows that:

$$-d_N \frac{\partial S_N^M(q)}{\partial q} = \sum_{i=1}^{r} q_{N,N+e_i} S_{N+e_i}^M(q) + \sum_{i=1}^{r} q_{N,N-e_i} S_{N-e_i}^M(q) + q_{N,N} S_N^M(q),$$

The boundary conditions are $S_N^M(0) = 0$ $\forall N \neq M$ if $d_N < 0$, and $S_N^N(0) = 1$.

Now we can compute the probability that the tagged flow is impatient during the first rebuffering phase when the threshold of rebuffering is q_b:

$$I^1 = \sum_L \sum_M \sum_N \left(1 - V_N(\tau, q_a)\right) R_N^M(0, q_a) S_M^L(q_a) V_L(\tau, q_b).$$

In order to introduce the matrix form, we denote by $U_N(\tau, q_a) = 1 - V_N(\tau, q_a)$, and we pose:

$$\mathbb{U}(\tau, q_a) = \begin{bmatrix} U_0(\tau, q_a) \\ \vdots \\ U_N(\tau, q_a) \\ \vdots \end{bmatrix}, \ \mathbb{V}(\tau, q_a) = \begin{bmatrix} V_0(\tau, q_a) \\ \vdots \\ V_N(\tau, q_a) \\ \vdots \end{bmatrix},$$

$$\mathbb{R}(0, q_a) = \begin{bmatrix} R_0^0(0, q_a) \dots R_0^M(0, q_a) & \dots & \\ \vdots & \vdots & \vdots & \vdots \\ R_N^0(0, q_a) \dots R_N^M(0, q_a) & \dots & \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \mathbb{S}(q_a) = \begin{bmatrix} S_0^0(q_a) \dots S_0^L(q_a) & \dots & \\ \vdots & \vdots & \vdots & \vdots \\ S_M^0(q_a) \dots S_M^L(q_a) & \dots & \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

Thus we have the matrix form as:

$$I^1 = \left(\mathbb{U}(\tau, q_a)\right)' \times \mathbb{R}(0, q_a) \times \mathbb{S}(q_a) \times \mathbb{V}(\tau, q_b)$$

By reccurence, we can compute the probability to be impatient during the l^{th} rebuffering phase.

We can derive a metric in steady state for the probability to be impatient during prefetching. For this, we consider the system without the tagged flow, we introduce the process $Z_t = \left(Z_t^1, \cdots, Z_t^r\right)$, where Z_t^l is the number of elastic users in region l. We assume that elastic users arrive in region l following a Poisson process with rate λ_l, and the service rate in region l is exponentially distributed with parameter μ_l. Then the process Z_t is a irreducible Markov process with the following intensity matrix:

$$q_{N,N+e_l} = \lambda_l,$$

$$q_{N,N-e_l} = \frac{n_l}{\sum_{i=1}^r n_i}\mu_l,$$

$$q_{N,N} = -\left(\lambda + \frac{\mu_k}{\sum_{i=1}^r n_i} + \sum_{i=l}^r q_{N,N-e_l}\right),$$

And there exists a steady state probability π given by:

$$\pi Q = 0 \text{ and } \sum_{N \in \mathbb{N}^r} \pi(N) = 1.$$

We can derive the probability for the tagged flow to be impatient in steady state as:

$$V(\tau, q_a) = \sum_N V_N(\tau, q_a)\pi(N).$$

4.3.3 Exponential patience duration

Now let us assume that the patience duration τ is exponentially distributed with parameter μ_0.

Let us denote by $F_N(q) = P\left[T_a > \tau / N_0 = N, Q_a^k(0) = q\right]$, the probability that the tagged flow is impatient during prefetching phase when the process N_t starts at state N and when the buffer has to reach q to start. As q_a is the start-up threshold then $F_N(q_a)$ is the probability that the tagged flow is impatient during prefetching phase when the process N_t starts at state N.

As τ has a density function given by $f_\tau(t) = \mu_0 e^{-\mu_0 t} 1_{[0,\infty[}(t)$, then $F_N(q)$ can be expressed as follows:

$$F_N(q) = P\left[T_a > \tau / N_0 = N, Q_a^k(0) = q\right]$$

$$= \int_0^\infty P\left[T_a > t/N_0 = N, Q_a^k(0) = q\right] \mu_0 e^{-\mu_0 t} dt$$

$$= \int_0^\infty V_N(t,q)\mu_0 e^{-\mu_0 t} dt$$

where $V_N(t,q) = P\left[T_a > t/N_0 = N, Q_a^k(0) = q\right]$ has been studied in the previous section in case of deterministic patience duration.

Theorem 4.3.1 *F_N is solution of the following ordinary differential equation*

$$\frac{dF_N(q)}{dq} + \frac{\mu_0}{b_N} F_N(q) - \frac{1}{b_N}\left(QF\right)_N(q) - \frac{\mu_0}{b_N} = 0 \tag{4.3}$$

$$F_N(0) = 0.$$

where $\left(QF\right)_N(q) = \sum_M q_{N,M} F_M(q)$.

Proof By integration by parts from the expression $F_N(q) = \int_0^\infty V_N(t,q)\mu_0 e^{-\mu_0 t} dt$, we obtain:

$$F_N(q) = \left[-V_N(t,q)e^{-\mu_0 t}\right]_0^\infty + \int_0^\infty \frac{\partial V_N(t,q)}{\partial t} e^{-\mu_0 t} dt$$

Using the fact that $V_N(0,q) = 1$ it follows that:

$$F_N(q) = 1 + \int_0^\infty \frac{\partial V_N(t,q)}{\partial t} e^{-\mu_0 t} dt$$

From 4.1 it follows:

$$F_N(q) = 1 + \int_0^\infty \left\{-b_N \frac{\partial V_N(t,q)}{\partial q} + \left(QV\right)_N(t,q)\right\} e^{-\mu_0 t} dt$$

$$= 1 - \int_0^\infty b_N \frac{\partial V_N(t,q)}{\partial q} e^{-\mu_0 t} dt + \int_0^\infty \sum_M q_{N,M} V_M(t,q) e^{-\mu_0 t} dt$$

$$= 1 - \int_0^\infty b_N \frac{\partial V_N(t,q)}{\partial q} e^{-\mu_0 t} dt + \frac{1}{\mu_0} \sum_M q_{N,M} \int_0^\infty V_M(t,q)\mu_0 e^{-\mu_0 t} dt$$

$$= 1 - \int_0^\infty b_N \frac{\partial V_N(t,q)}{\partial q} e^{-\mu_0 t} dt + \frac{1}{\mu_0} \sum_M q_{N,M} F_M(q)$$

There exists a non negative real M such that $\mid \frac{\partial V_N(t,q)}{\partial q} e^{-\mu_0 t} \mid \leq M e^{-\mu_0 t}$ that is integrable, then we have

$$\int_0^\infty b_N \frac{\partial V_N(t,q)}{\partial q} e^{-\mu_0 t} dt = \frac{\partial}{\partial q} \int_0^\infty b_N V_N(t,q) e^{-\mu_0 t} dt$$

Thus

$$F_N(q) = 1 - \frac{b_N}{\mu_0} \frac{\partial}{\partial q} \int_0^\infty V_N(t,q)\mu_0 e^{-\mu_0 t} dt + \frac{1}{\mu_0} \sum_M q_{N,M} F_M(q)\mu_0 e^{-\mu_0 t} dt$$

$$F_N(q) = 1 - \frac{b_N}{\mu_0} \frac{dF_N(q)}{dq} + \frac{1}{\mu_0}\left(QF\right)_N(q)$$

It follows

$$\frac{dF_N(q)}{dq} + \frac{\mu_0}{b_N} F_N(q) - \frac{1}{b_N}\left(QF\right)_N(q) - \frac{\mu_0}{b_N} = 0.$$

From the expression of F_N we have:

$$F_N(0) = \int_0^\infty V_N(t,0)\mu_0 e^{-\mu_0 t}dt = 0$$

∎

Let us notice that for a large value of threshold $F_N(q) = 1$, since $\lim_{q\to\infty} V_N(t,q) = 1$, then by the theorem of dominated convergence see ([73]) we have:

$$\lim_{q\to\infty} F_N(q) = \lim_{q\to\infty} \int_0^\infty V_N(t,q)\mu_0 e^{-\mu_0 t}dt = \int_0^\infty \lim_{q\to\infty} V_N(t,q)\mu_0 e^{-\mu_0 t}dt = 1.$$

Let us pose $\mathbb{F}(q) = \begin{bmatrix} F_0(q) \\ \vdots \\ F_N(q) \\ \vdots \end{bmatrix}$, $\mathbb{E} = \begin{bmatrix} \frac{\mu_0}{b_0} \\ \vdots \\ \frac{\mu_0}{b_N} \\ \vdots \end{bmatrix}$, and $\mathbb{M} = \begin{bmatrix} \frac{1}{b_0}(q_{0,0} - \mu_0) & \cdots & \frac{1}{b_0}q_{0,N} & \cdots & \\ \vdots & \ddots & \vdots & \ddots & \\ \frac{1}{b_N}q_{N,0} & \cdots & \frac{1}{b_N}(q_{N,N} - \mu_0) & \ddots & \ddots \\ & \ddots & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & \end{bmatrix}$.

It follows from theorem 4.3.1 that:

$$\frac{d\mathbb{F}(q)}{dq} - \mathbb{M}\mathbb{F}(q) - \mathbb{E} = 0,$$

since as noticed in the previous section $V_N(t,0) = 0$.

If the tagged flow is not impatient during prefetching, the user watches his video until a starvation happens, and the rebuffering phase begins. The tagged flow may be impatient during rebuffering. We assume that the patience duration in the rebuffering phase is the same as in the prefetching phase. Then the probability to be impatient during rebuffering is given by $\mathbb{F}(q_b)$ where q_b is the threshold of rebuffering.

The probability for the tagged flow to be impatient in the first rebuffering phase is the same as in the deterministic case by replacing $\mathbb{V}$ by $\mathbb{F}$, and is given by:

$$I^1 = (\mathbb{U}(q_a))' \times \mathbb{R}(0,q_a) \times \mathbb{S}(q_a) \times \mathbb{F}(q_b)$$

where

$$\mathbb{U}(q_a) = \begin{bmatrix} 1 - F_0(q_a) \\ \vdots \\ 1 - F_N(q_a) \\ \vdots \end{bmatrix},$$

So by reccurence we can compute the probability to be impatient during the l^{th} rebuffering phase.

As in the case of deterministic patience duration, we can derive the probability for the tagged flow to be impatient in steady state:

$$F(q_a) = \sum_N F_N(q_a)\pi(N).$$

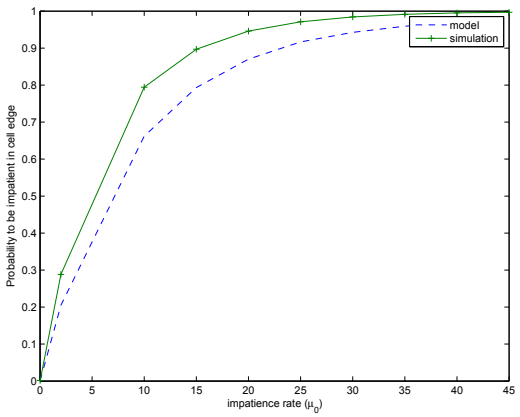

Figure 4.4: Probability to be impatient in prefetching phase in cell edge- exponential patience duration- $\lambda_1 = \lambda_2 = 8, \mu_1 = 32, \mu_2 = 20$, $Bitrate = 4.5$, impatience rate $\mu_0 = 13$, limit of admission 50, time of simulation=50000, number of Monte carlo = 20000.

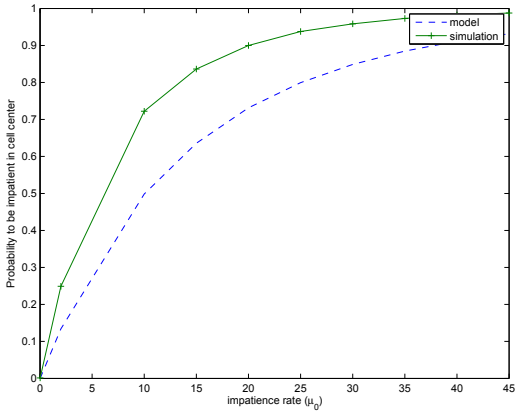

Figure 4.5: Probability to be impatient in prefetching phase in cell center- exponential patience duration- $\lambda_1 = \lambda_2 = 8, \mu_1 = 32, \mu_2 = 20$, $Bitrate = 4.5$, start up threshold $q_a = 0.3$, limit of admission 50, time of simulation=50000, number of Monte carlo = 20000.

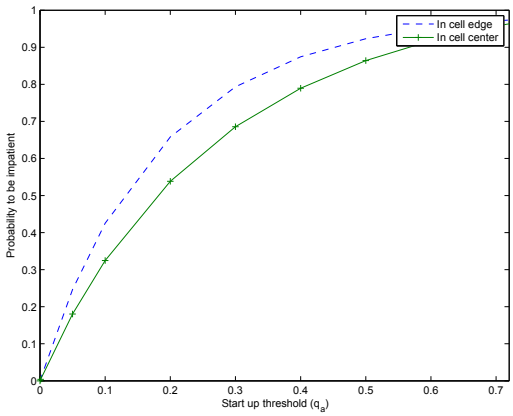

Figure 4.6: Probability to be impatient in prefetching phase- exponential patience duration- $\lambda_1 = \lambda_2 = 8, \mu_1 = 32, \mu_2 = 20$, $Bitrate = 4.5$, impatience rate $\mu_0 = 13$, limit of admission= 10.

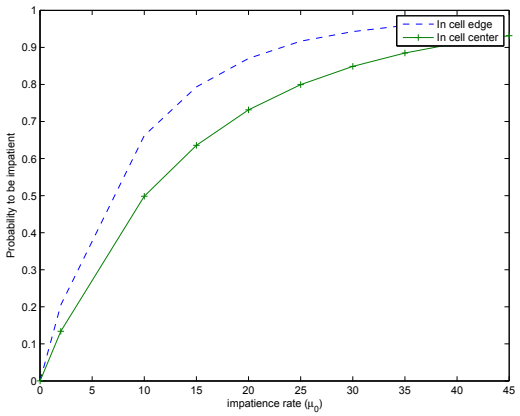

Figure 4.7: Probability to be impatient in prefetching phase- exponential patience duration - $\lambda_1 = \lambda_2 = 8, \mu_1 = 32, \mu_2 = 20$, $Bitrate = 4.5$, start up threshold $q_a = 0.3$, limit of admission= 10.

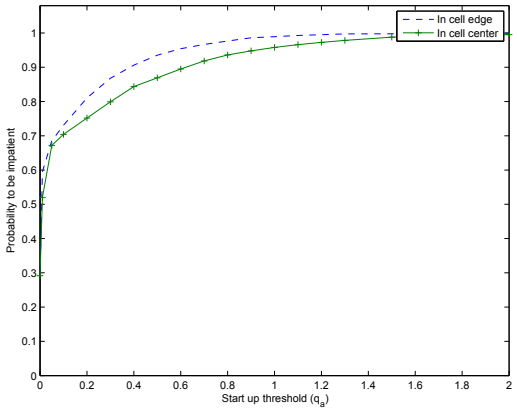

Figure 4.8: Simulation of probability to be impatient in prefetching phase- exponential patience duration- $\lambda_1 = \lambda_2 = 8, \mu_1 = 32, \mu_2 = 20$, $Bitrate = 4.5$, impatience rate $\mu_0 = 13$, limit of admission 50, time of simulation=50000, number of Monte carlo = 20000.

4.3.4 Numerical applications

Figures 4.4 and 4.5 do a comparison between simulation and model. The probability of impatience is plotted as a function of impatience rate in cell edge and cell center, we can observe that the two curves are close.

Figure 4.6 plots the probability to be impatient in prefetching phase as a function of start up threshold. We can observe that the probability to be impatient increases when the start up threshold increases.

In the figure 4.7, we plot the probability to be impatient as a function of the impatience rate, and observe that it increases when the impatience rate increases. We can also observe that the probability to be impatient is higher in cell edge than in cell center, that is due to the fact that the throughput is higher in cell center than in cell edge.

In figures 4.8 and 4.9 we do a discrete event simulation of the system as a function of start up threshold and impatience rate respectively. Results of simulations show that the probability of impatience increases when the start up threshold and the impatience rate increase.

4.4 Conclusion

We modeled in this chapter impatience of a streaming user during its video session in terms of probability of impatience during the prefetching phase and the rebuffering phase at packet level and at flow level for deterministic and exponential patience duration.

Our results showed that the probability of impatience depends on the threshold of prefetching during prefetching and the threshold of rebuffering during rebuffering when starvation occurs: the more the threshold is high, the more the probability of impatience increases.

Our simulations validated the accuracy of our model.

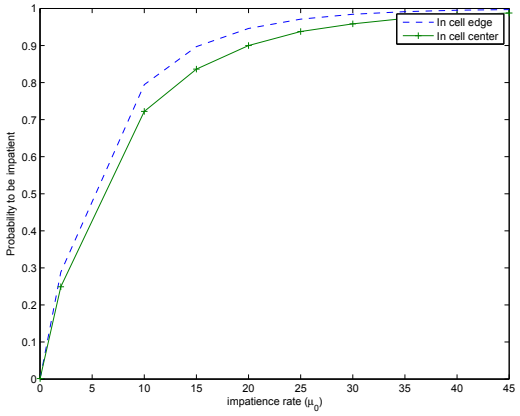

Figure 4.9: Simulation of probability to be impatient in prefetching phase- exponential patience duration- $\lambda_1 = \lambda_2 = 8, \mu_1 = 32, \mu_2 = 20$, $Bitrate = 4.5$, start up threshold $q_a = 0.3$, limit of admission 50, time of simulation=50000, number of Monte carlo = 20000.

Chapter 5

Control

Abstract

In this part we study a system with impatient users controlled by a system manager who has to choose at each decision epoch an action to make in order to optimize the system performance. In the first part the set of actions to be chosen by the system manager is assumed to be finite and the control is dropping and blocking in each region of the system that is scheduled as a processor sharing. Classical results of average cost markov decision process for semi markov process allow us to derive the optimal policy that is the path of optimal decisions to make by the system manager at each decision epoch in order to optimize the system performance through the value iteration algorithm and the modified policy iteration algorithm. In the second part we study firstly a system with one region and then generalize to the case of multiple regions where we use our aggregate model developped in the first chapter. The set of actions to be chosen by the system manager is assumed to be a real compact or a compound of real compacts. We provide a theorem that allows us to derive recursively the optimal policy and the optimal system size in order to optimize the system performance. Our results show that the optimal system size increases when the impatience rate increases.

5.1 Introduction

Congestion and flow control and dynamic bandwidth allocation are new challenging control problems. Admission control is becoming much more complex in heterogeneous traffic types (data, video, voice) than it was in telephony. The ability to control cost dynamically gives to mobile network operators a significant advantage.

The system dynamics can be modeled by the mathematical framework of Markov decision problem (MDP) to optimize the network's desired objectives. A large mathematical theory of MDP is developped in [51, 53].

Authors in [60] have reviewed numerous applications of MDP framework and protocols for wireless sensor networks that consist of autonomous and ressource limited devices for using of MDP in wireless sensor networks.

Authors in [63] studied a system of M/G/1 queue in which they considered a smart customer who is allowed to choose among three strategies: enter in the queue and stay there until its service is finished, leave the system right away, and wait outside the queue. They considered that all other customers join the queue unconditionally and they assumed that if the smart customer enters or leaves the system there are no other option for him and if he chooses to wait, he can make a new decision at the end of its service completion. They showed that for a finite period the optimal strategy for the smart customer is the policy by which he enters a small queue, leaves a large queue and waits when the queue is of an intermediate size.

In [64] the author considered a queueing system with several unbounded servers where customers arrive according to a Poisson process and must join one of the queues that are sheduled as a FIFO. He showed that there are some service time distributions for which it is not optimal to always join the shortest queue.

In [66] authors studied the dispatching problem by considering a distributed server system with several servers operating under a M/D/1 processor sharing queue. They assumed that each arriving task is assigned to one of the available servers and each server processes the given task in parallel as a processor sharing. Their objective is to find an optimal dispatching decision. For this they used an MDP framework in the average cost criterion by taking the mean sojourn time as the function to be minimized and the cost as the number of tasks in the system.

In [67] authors considered the problem of arrival in GI/M/1 queue. They considered a simple limit control policy under which an integer n is considered and a user is admitted to the system if and only if the number of users in the system is less than n. They added an extension that they called conditional acceptance rule that allows the system to conditionally accept a user. They distinguished three categories of states: 1) if the number of users in the system is less than $n-1$, then new arrival is unconditionally accepted, 2) if the number of users in the system is equal to $n-1$, then a user conditionally joins the system but may be rejected later, 3) if the number of users in the system is more than $n-1$, then new arrival is immediately rejected. They considered the following cost-reward structure: 1) there is a reward upon service completion, 2) each user residing in the system incurs waiting time losses, 3) rejecting immediately a user upon arrival results in a cost, 4) rejecting a user that has been conditionally accepted results in a cost. They used the MDP framework in long run cost average and developed some conditions under which conditional acceptance is better than simple control limit rule.

In [68] authors proposed three schemes for call traffic handling with one nonprioritized and two priority oriented by considering a fixed channel assignment. In the nonprioritized scheme the base stations make no distinction between new call attempts and handoff attempts. Attempts which find all channels occupied are cleared. In the two priority schemes, a fixed number of channels in each cell are reserved exclusively for handoff calls, but the difference is that in the second the queueing of handoff attempts is allowed. They derived performance character-

istics such as blocking probability, forced termination probability, and fraction of new calls not completed.

Authors in [69] carried out a performance evaluation for two different classes of channel assignment techniques that are fixed channel allocation and dynamic channel allocation in terms of the call blocking probability, the call dropping probability, the probability of unsuccessful call and the average number of channel rearrangements per call.

In [70] authors proposed an estimation of the dropping probabilities of cellular wireless networks by queuing handoff instead of reserving guard channels. They assumed that inter arrivals are Gamma distributed and service time has a general distribution and they estimated the system performance in terms of the probability of blocking and the probability of forced termination of handoff calls. They showed that the model with handoff requests can be used for optimum system performance instead of the model with guard channels.

In [71]authors proposed an analytical method to calculate the performance of dynamic channel assignment by considering queuing and guard channel combined scheme for handoff prioritisation. With simulations results they showed that their analytical results are accurate.

Authors in [72] provided a closed form expression to the blocking and dropping probabilities in wireless cellular networks where they considered the effect of handoff arrival and guard channels. They developed an algorithm which provides the optimal number of guard channels and the optimal number of channels.

In [52] authors studied the control of service rate in single server queuing system by assuming Poisson arrivals and exponential service time with a state dependent service rate in average cost over an infinite planning horizon. They considered two costs: the cost of congestion that increases with the number of users in the system and the cost associated to the change of service rate that increases with the chosen service rate level. They developped a computation method that proceeds by solving a sequence of approximating problems which are the truncation of the holding cost function. They showed that the optimal policy for the approximation problems converge to a policy that is optimal for the original problem.

Authors in [54] were inspired by [52], they allowed the system manager to control the service rate and the arrival rate. They considered a system of M/M/1 queues with finite system size and state dependent service rate. They provided a theorem that gives an optimal policy and an optimal buffer size. They applied their results to study the price-setting problem where customers are utility maximizing and price and delay-sensitive. They studied a numerical example to compare the social welfare using a dynamic policy and static policies and they showed that the dynamic policy offers significant welfare gains.

We were inspired by [52] and [54] for the second part of this chapter by considering the phenomenon of impatience and by allowing the system manager, in addition to service rate control and arrival rate control, another control which allows him to force a user to leave the system even if its service has not finished or its patience duration is not expired and we add a cost associated to this control. We generalize this to the case of the aggregate model we developped in chapter 1 in which the system is composed of several regions.

This chapter is organized as follows. In section 5.4 we consider the system with several regions with different capacities sheduled as a processor sharing, the set of states and the set actions are assumed to be finite. We pose the related problem of markov decision process in average cost and we derive the optimal policy through the modified policy iteration and the value iteration algorithm. In section 5.5 we consider a system with one region and we assume that the set of actions is a compound of compact of $\mathbb{R}$. We pose the related problem of markov decision process in average cost and we derive a theorem that allows us to compute the optimal policy recursively and the optimal system size. In section 5.6 we generalize the results of section

5.5 to the case of aggregate model studied in chapter 1. We conclude in section 5.7.

5.2 System description

We consider as previously an OFDMA-based homogeneous cellular network and focus on the downlink of one cell with a single base station at its center. With OFDMA, the total bandwidth, which we denote by W, is divided into N orthogonal subcarriers and can be shared between the different users present in the cell in the same time slot.

Due to path loss, the Signal-to-Interference and Noise Ratio (SINR) is lower at the cell edge than at the cell center. This leads to a cell capacity $C(r)$ that depends on the distance r between the user and the base station. In order to obtain this throughput, we make use of a static simulator as described in [5]. This throughput is illustrated in Figure 5.1 for an LTE system in an urban environment.

As can be seen in this figure, the throughput decreases when the user gets further away from the base station. Let $C_1 > .... > C_l > .... > C_r$ be the set of throughputs at different positions l in the cell and P_l the probability that a user is located at position l (this corresponds to a discretization of Figure 5.1 into r regions where the throughput is almost constant).

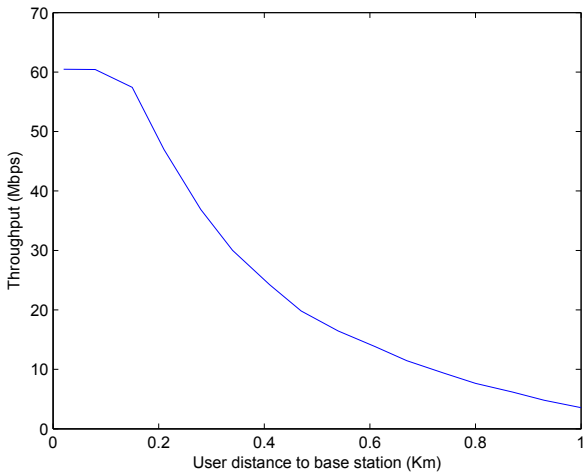

Figure 5.1: Throughput for a user who is alone in the cell, and who is located in different positions. We consider an LTE system using 10 MHZ of spectrum at the 2.6 GHZ band. Cell radius is equal to 1 Km and a MIMO 2*2 scheme is considered.

5.3 Control with countable set of actions

We assume that users arrive in region l following a poisson process with rate λ_l. We model impatience by introducing a patience duration, denoted by random variable τ. Each flow of region l has its service duration T^l and its patience duration τ^l and completes its transfer if and only if its service duration is less than its patience duration i.e., $nT^l < \tau^l$ where n is the total number of flows present in the system.

In this chapter the system is controled by a system manager. We introduce a control or dropping variable D for each user, that allows the system manager to force him to leave the system even if its service is not finished or its patience duration is not expired.

The control variable is assumed to be a exponential random variable of parameter α and independent of service duration and patience duration. Each user of region l has its service time T_l exponentially distributed with parameter μ_l when he is alone in the system, its patience duration τ_l exponentially distributed with parameter μ_0^l, and its control variable D_l also exponentially distributed with parameter α_l. He leaves the system either by end of service, or by impatience, or by contreinte. We denote by $T_{eff}^l(n) = \min\left\{nT^l, \tau^l, D^l\right\}$ its sejourn time in the system when there are n flows present in the system.

5.3.1 Related Markov decision process

There are four main criteria of Markov desion process problem that are: finite horizon criterion, expected total reward criterion, expected total discounted reward criterion and average reward criterion.

Let us denote by $(S_t)_t$ the studied process and R_t the reward associated to an action chosen by the system manager at time t.

In finite horizon criterion the system manager has to make an optimal decision at each decision epoch in order to optimize during a finite time that we denote by T the following value:

$$E\left[\sum_{n=0}^{T} R_n \mid S_0\right]$$

In the case of the three last criteria the system manager has to make an optimal decision at each decision process during an infinte time. In total reward criterion the value to optimize is:

$$E\left[\sum_{n=0}^{\infty} R_n \mid S_0\right]$$

In expected total discounted reward criterion the value to optimize is:

$$E\left[\sum_{n=0}^{\infty} \gamma^n R_n \mid S_0\right]$$

In average reward criterion the value to optimize is:

$$\lim_{T\to\infty} \frac{1}{T} E\left[\sum_{n=0}^{T-1} R_n \mid S_0\right]$$

The average reward criterion is mostly used for problems where the frequency of decisions is high as for queuing problems. In this chapter we consider the average reward criterion.

In this part the system is managed by a system manager whose has objective is to reduce the impact of impatience on the system performance.

The system manager chooses at each decision epoch some actions that allows him to control the system.

The system manager can manage the system through the control variable D and during arrival at each region. Let us introduce the process $N_t = \left(N_t^1, ...N_t^r\right)$ where N_t^l is the number of flows in region l. The process N_t describes the system. With our assumptions N_t is Markovian and changes state either if there is an arrival or a departure. Then the decision epochs can be reduced to the instant when the process changes state, which puts us in the context of Semi Markov decision process problems.

At each decision epoch the system manager makes an action that defines for each user of region l the new parameter of the variable of control α_l, that is the rate of new arrival in region l. We define for each region l, a variable $b_l \in \Xi$, that is controlled by the system manager in order to increase, decrease, limit or block the arrival in region l.

The system manager controls the system through α_l and b_l for $l \in \{1, \cdots, r\}$. Let us define the set of actions by $\mathbb{A} = \{(b_1, \cdots, b_r, \alpha_1, \cdots, \alpha_r), b_l \in \Xi,\ \alpha_l > 0\} \subset \Xi^r \times \mathbb{R}^r$. At each decision epoch the system manager chooses an action $a = (b_1, \cdots, b_r, \alpha_1, \cdots, \alpha_r) \in \mathbb{A}$.

We assume that $\mathbb{A}$ is finite.

Let us recall that the patience duration τ is uncontrolled.

We assume that the system size is K, that means that the total number of users in the system cannot exceed K, so the set of states is $\mathbb{S} = \{(n_1, \cdots, n_r), \sum_{l=1}^{r} n_l \leq K\} \subset \mathbb{N}^r$, that implies that $\mathbb{S}$ is also finite.

When the system is at state N, we denote the set of allowable actions at state N by $\mathbb{A}_N$, that is the set of possible action that the system manager can choose from when the system is at state N.

A policy $\xi = (\xi_t)_t$ specifies the decision rule to be used at all decision epochs, it may be deterministic or randomized.

The deterministic policy is a funtion $\xi_t : \mathbb{S} \to \mathbb{A}$, which defines with certainty the action to choose at time t, $\xi_t(X_t) \in \mathbb{A}$.

The randomized policy $\xi_t(N, a)$ defines the probability to choose the action $a \in \mathbb{A}$ at time t when the system is at state $N \in \mathbb{S}$.

We denote by P^{ξ} the probability joined the policy ξ, and the corresponding expectation will be noted by E^{ξ}. We denote by a_t the process of decisions, i.e the decision to make by the system manager at decision epoch t.

We consider Markovian policy ξ, in which $P^{\xi}[a_t = a \mid N_0, ..., N_t] = P^{\xi}[a_t = a \mid N_t] = \xi_t(N_t, a)$,

When an action $a = (b_1, \cdots, b_r, \alpha_1, \cdots, \alpha_r) \in \mathbb{A}$ is made, the process N_t stays Markovian with intensity matrix Q^a given by:

$$q^a_{N,N+e_l} = \lambda_l b_l$$

$$q^a_{N,N-e_l} = \frac{N_l}{\sum_{i=1}^{r} N_i}\mu_l + N_l\mu_0^l + N_l\alpha_l$$

$$q^a_{N,N} = -\left(\sum_{l=1}^{r}(\lambda_l b_l + q_{N,N-e_l})\right)$$

where $N = (N_1, ..., N_r)$; N_l being the number of users in region l, $e_l = (0, ...0, 1, 0, ...0)$.

For $N, N' \in \mathbb{S}$ and $a \in \mathbb{A}$, we denote by $P(N' \mid N, a)$ the probability of transition from state N to state N' when an action a is made.

We denote by P^a the matrix of transition of the process N_t when the action a is made. P^a is given by Kolmogorov equation by $P^a = \exp(Q^a)$.

For a randomized Markovian policy ξ:

$$P^{\xi}[N_{t+1} \mid N_0, ..., N_t] = \sum_{a \in \mathbb{A}} P^{\xi}[a_t = a \mid N_0, ..., N_t]\, P^{\xi}[N_{t+1} \mid N_0, ..., N_t, a_t = a]$$

$$= \sum_{a \in \mathbb{A}} \xi_t(N_t, a) P^{\xi}[N_{t+1} \mid N_t, a_t = a]$$

$$= P^{\xi}[N_{t+1} \mid N_t],$$

Then it is a Markov process, where the associate Markov-chain has a transition matrix given by:

$$\forall N, N' \in \mathbb{S}, P_{\xi,N,N'} = P^{\xi}\left[N_{t+1} = N' \mid N_t = N\right] = \sum_{a \in \mathbb{A}} \xi_t(s,a) P(N' \mid N, a),$$

and $P(N' \mid N, a)$ is the transition probability from state N to state N' of the Markov process N_t when the decision a is made.

For a policy ξ, we denote by P_ξ the stochastic matrix of transition of the associate Markov process under the policy ξ.

Each decision made by the system manager at each decision epoch has a reward and a cost for the system performance.

Let us consider a tagged flow and the process $X_t = \left(X_t^1, \cdots, X_t^r\right)$, where X_t^l is the number of users in region l seen by the tagged flow.

We consider that the tagged flow is located in region $k \in \{1, \cdots, r\}$.

Let us introduce three supplementary states I^k, F^k and C^k, that are assumed to be absorbing for the process X_t.

As previously we assume that all users have a service time, a patience duration and a control variable that are independent and exponentially distributed.

When an action $a = (b_1, \cdots, b_r, \alpha_1, \cdots, \alpha_r) \in \mathbb{A}$ is made, then the process X_t is Markovian with intensity matrix given by:

$$q_{X,X+e_l} = \lambda_l b_l$$

$$q_{X,X-e_l} = \frac{X_l}{\sum_{i=1}^r X_i + 1}\mu_l + X_l \mu_0^l + X_l \alpha_l$$

$$q_{X,I} = \mu_0^k$$

$$q_{X,F} = \frac{\mu_k}{\sum_{i=1}^r X_i + 1}$$

$$q_{X,C} = \alpha_k$$

$$q_{X,X} = -\left(\mu_0^k + \frac{\mu_k}{\sum_{i=1}^r X_i + 1} + \alpha_k + \sum_{i=1}^r \left(\lambda_l b_l + q_{X,X-e_i}\right)\right)$$

where $X = (X_1, ..., X_r)$ with X_l being the number of users in region l seen by the tagged flow.

Let us pose

$$T_{I^k} = \inf\left\{t > 0, X_t = I^k\right\},$$

$$T_{F^k} = \inf\left\{t > 0, X_t = F^k\right\},$$

$$T_{C^k} = \inf\left\{t > 0, X_t = C^k\right\},$$

Let us define $P_{I^k}(X) = P\left[T_{I^k} < \infty \mid X_0 = X\right]$ the probability that the tagged flow be impatient, $P_{F^k}(N) = P\left[T_{F^k} < \infty \mid X_0 = X\right]$ the probability that the tagged flow finishes its service and $P_{C^k}(N) = P\left[T_{C^k} < \infty \mid X_0 = X\right]$ the probability that the tagged flow leaves the system by force, all conditioned on initial state $X_0 = X$. Then $P_{I^k}(N)$, $P_{F^k}(N)$ and $P_{C^k}(N)$ can be found as the solution of Dirichlet/FeymannKac problems such as:

P_I is solution of:

$$Q\ P_{I^k} = 0 \qquad \forall\ N \neq I^k,\ F^k,\ C^k$$

$$P_{I^k}(I^k) = 1$$

$$P_{I^k}(F^k) = 0$$
$$P_{I^k}(C^k) = 0$$

P_{F^k} is solution of:

$$Q\ P_{F^k} = 0 \qquad \forall\ N \neq I^k,\ F^k,\ C^k$$
$$P_{F^k}(F^k) = 1$$
$$P_{F^k}(I^k) = 0$$
$$P_{F^k}(C^k) = 0$$

P_{C^k} is solution of:

$$Q\ P_{C^k} = 0 \qquad \forall\ N \neq I^k,\ F^k,\ C^k$$
$$P_{C^k}(C^k) = 1$$
$$P_{C^k}(I^k) = 0$$
$$P_{C^k}(F^k) = 0.$$

When an action $a = (b_1, \cdots, b_r, \alpha_1, \cdots, \alpha_r) \in \mathbb{A}$ is made at state N, we define the reward function $R(N, a)$, and the cost function $C(N, a)$ as follows:

$$R(N, a) = \sum_{k=1}^{r} P_{F^k}(N)$$

$$C(N, a) = \sum_{k=1}^{r} P_{C^k}(N)$$

The system manager goal is to choose an action a at each decision epoch, when the system is at state N in order to maximize the quantity $R(N, a) - C(N, a)$.

In the following we only consider deterministic policy because we use a result from Markov decision process that allows us to find optimal deterministic policy.

We consider the problem of average cost criterion in infinite horizon.

For a deterministic policy ξ, let us define

$$\Delta^{\xi}(M) = \lim_{n\to\infty} \frac{1}{n} E^{\xi}\left[\sum_{i=0}^{n-1} \{R(N_i, \xi(N_i)) - C(N_i, \xi(N_i))\} \mid N_0 = M\right]$$

Let us define

$$\Delta^* = \sup_{\xi} \Delta^{\xi}$$

where the supremum is taken over all deterministic policies.

The system manager goal is to find the optimal policy ξ^* over all deterministic policy if it exists that maximizes Δ^* such that:

$$\Delta^* = \Delta^{\xi^*}.$$

and

$$\xi^* \in argmax_{\xi} \Delta^{\xi}.$$

Definition A Markov decision problem is said to be ergodic or recurrent if the transition matrix associated to each deterministic policy consists of a single reccurent class.

A Markov decision problem is said to be communicating if for each pair of states N and $N^{'}$ there exists a deterministic policy ξ under which N is accessible from $N^{'}$ that means there exists an integer $n > 0$, such that $\left(P^{\xi}\right)^n \left(N \mid N^{'}\right) > 0$.

5.3.2 Control with adjustable arrival rate in each region

In this part we assume that $0 \notin \Xi$.

The system manger can adjust the arrival rate through $b_l \in \Xi$ in order to decrease the arrival rate in each region.

In this model we have $\mathbb{A}_N = \mathbb{A}$ for all $N \in S$.

For any action a the process N_t with intensity matrix Q^a is irreducible and reccurent, it then has a stationnary distribution π^a given by:

$$\pi^a Q^a = 0$$

$$\sum_{N \in \mathbb{S}} \pi^a(N) = 1.$$

The process N_t consists of a single recurrent class for every deterministic policy which means that our Markov decision process is ergodic or recurrent.

It is well known for ergodic Markov decision process in the average cost criterion that Δ^ξ is constant $\Delta^\xi(M) = \Delta^\xi, \forall M$ and is given by:

$$\Delta^\xi = \sum_{N \in \mathbb{S}} \pi^\xi(N) \left\{ R(N, \xi(N)) - C(N, \xi(N)) \right\}.$$

Let us denote the set of bounded real functions defined on $\mathbb{S}$ by $\mathbb{V} = \{ f : \mathbb{S} \longrightarrow \mathbb{R},\ f < \infty \}$

Now let us enounce somes results about ergodic Markov decision process (see [53]).

Theorem 5.3.1 *Let us assume that the set of state $\mathbb{S}$ is countable. If there exists a $g \in \mathbb{R}$ and an $h \in \mathbb{V}$, such that for each state $N \in \mathbb{S}$, the following optimality equation is satisfied*

$$\max_{a \in \mathbb{A}_N} \left\{ R(N,a) - C(N,a) - g + \sum_{M \in \mathbb{S}} P(M \mid N, a)\, h(M) - h(N) \right\} = 0 \tag{5.1}$$

Then $g = \Delta^*$.

Proof See [53] ∎

Let us notice that the above theorem is applied only for ergodic Markov decision process.

In our model $\mathbb{S}$ is finite, it is then countable which verifies the assumption in the above theorem.

The following theorem gives us existence and uniqueness of the optimal solution (See [53]).

Theorem 5.3.2 *Assume $\mathbb{S}$ and $\mathbb{A}_N$ for each state N are finite, for each state $N \in \mathbb{S}$ and each action $a \in \mathbb{A}_N$ the reward function $R(N,a)$ is bounded, and the model is ergodic, then*

There exists a $g \in \mathbb{R}$ and an $h \in \mathbb{V}$ such that for each $N \in \mathbb{S}$, the optimality equation is verified

$$\max_{a \in \mathbb{A}_N} \left\{ R(N,a) - C(N,a) - g + \sum_{M \in \mathbb{S}} P(M \mid N, a)\, h(M) - h(N) \right\} = 0$$

If any other solution g' and h' of 5.1 exists, then $g = g'$.

Proof See [53] ∎

The optimality equation 5.1 can be rewritten as:

$$g + h(N) = \max_{a \in \mathbb{A}_N} \left\{ R(N,a) - C(N,a) + \sum_{M \in \mathbb{S}} P\left(M \mid N, a\right) h(M) \right\}$$

For an $h \in \mathbb{V}$, a policy ξ^h is called h-improving if for each $N \in \mathbb{S}$

$$R(N, \xi^h(N)) - C(N, \xi^h(N)) + \sum_{M \in \mathbb{S}} P\left(M \mid N, \xi^h(N)\right) h(M)$$

$$= \max_{a \in \mathbb{A}_N} \left\{ R(N,a) - C(N,a) + \sum_{M \in \mathbb{S}} P\left(M \mid N, a\right) h(M) \right\}$$

This means that for each $N \in \mathbb{S}$,

$$\xi^h(N) \in argmax_{a \in \mathbb{A}_N} \left\{ R(N,a) - C(N,a) + \sum_{M \in \mathbb{S}} P\left(M \mid N, a\right) h(M) \right\}$$

The following theorem gives us the conditions under which there is a solution to the equation 5.1 (see [53]).

Theorem 5.3.3 *Assume $\mathbb{S}$ and $\mathbb{A}_N$ for each state N, are finite, for each state $N \in \mathbb{S}$ and each action $a \in \mathbb{A}_N$ the reward function $R(N,a)$ is bounded, and the model is ergodic, then*

There exists a $g \in \mathbb{R}$ and an $h \in \mathbb{V}$ such that equation (5.1) is satisfied

$g = \Delta^*$

Any h-improving policy is an optimal policy.

Proof See [53] ∎

Modified Policy Iteration Algorithm

For $\epsilon > 0$, a policy ξ is said to be $\epsilon-$optimal if

$$\mid \Delta^{\xi} - \Delta^* \mid < \epsilon.$$

In this part we expose an algorithm that allows us to obtain a $\epsilon-$optimal policy, for an arbitrary $\epsilon > 0$.

We use the Modified policy iteration algorithm that is derived in [53].

Let us introduce the norm sp, defined on $\mathbb{V}$, as;

$$\forall V \in \mathbb{V}, \quad sp(V) = \max_{s \in \mathbb{S}} V(s) - \min_{s \in \mathbb{S}} V(s),$$

Let us set a sequence of non-negative integers $(m_n)_n$.

The modified policy iteration algorithm is described as follows:

1. Set $n = 0$, for each $N \in \mathbb{S}$, select $v^0(N) \in \mathbb{R}$, specify $\epsilon > 0$.
2. For each $N \in \mathbb{S}$, choose $\xi_{n+1}(N)$ which satisfies

$$R(N, \xi_{n+1}(N)) - C(N, \xi_{n+1}(N)) + \sum_{M \in \mathbb{S}} P\left(M \mid N, \xi_{n+1}(N)\right) v^n(M)$$

$$= \max_{a \in \mathbb{A}_N} \left\{ R(N,a) - C(N,a) + \sum_{M \in \mathbb{S}} P\left(M \mid N, a\right) v^n(M) \right\}$$

This means

$$\xi_{n+1}(N) \in argmax_{a \in \mathbb{A}_N} \left\{ R(N,a) - C(N,a) + \sum_{M \in \mathbb{S}} P\left(M \mid N, a\right) v^n(M) \right\}.$$

Set $\xi_{n+1}(N) = \xi_n(N)$ if possible.

3. a. Set $k = 0$, and for each $N \in \mathbb{S}$, set

$$u_n^0(N) = R(N, \xi_{n+1}(N)) - C(N, \xi_{n+1}(N)) + \sum_{M \in \mathbb{S}} P\left(M \mid N, \xi_{n+1}(N)\right) v^n(M).$$

b. If

$$sp\left(u_n^0 - v^n\right) < \epsilon,$$

go to step (4). Otherwise go to step (c).

c. If $k = m_n$, go to (e). Othetwise, for each $N \in \mathbb{S}$, compute $u_n^{k+1}(N)$ as follows

$$u_n^{k+1}(N) = R(N, \xi_{n+1}(N)) - C(N, \xi_{n+1}(N)) + \sum_{M \in \mathbb{S}} P\left(M \mid N, \xi_{n+1}(N)\right) u_n^k(M).$$

d. Increment k by 1, and return to (c).

e. Set $v^{n+1} = u_n^{m_n}$, go to (2).

4. For each $N \in \mathbb{S}$, choose $\xi_\epsilon(N)$, such that

$$R(N, \xi_\epsilon(N)) - C(N, \xi_\epsilon(N)) + \sum_{M \in \mathbb{S}} P\left(M \mid N, \xi_\epsilon(N)\right) v^n(M)$$

$$= \max_{a \in \mathbb{A}_N} \left\{ R(N,a) - C(N,a) + \sum_{M \in \mathbb{S}} P\left(M \mid N, a\right) v^n(M) \right\}$$

This means

$$\xi_\epsilon(N) \in argmax_{a \in \mathbb{A}_N} \left\{ R(N,a) - C(N,a) + \sum_{M \in \mathbb{S}} P\left(M \mid N, a\right) v^n(M) \right\}$$

and stop.

The following theorem ensures the convergence of the above algorithm ([53]).

Theorem 5.3.4 *Assume that the Markov decision process is ergodic, and assume that under all deterministic policies, the associate Markov process is aperiodic.*

Then for any sequence of non-negative integers (m_n) and for any $\epsilon > 0$, there exists an integer n in which the modified policy iteration algorithm stops.

Proof See [53] ∎

In order to apply the above theorem that ensures that the modified policy iteration algorithm terminates, aperiodicity is required.

There exists a simple transformation called aperiodicity transformation, that allows, under all policies, to make the associate Markov process aperiodic, see [53].

Let us define the transformed Markov process with components indicated by " $\bar{}$ ".

The aperiodicity transformation consists in choosing a fixed real θ satisfying $0 < \theta < 1$, and defining the set of states by $\bar{\mathbb{S}} = \mathbb{S}$, the set of accessible actions by $\bar{\mathbb{A}}_N = \mathbb{A}_N$ for all states N,

$$\bar{R}(N,a) = \theta R(N,a),\ \bar{C}(N,a) = \theta C(N,a)\ \ \forall N \in \mathbb{S},\ \forall a \in \mathbb{A}_N,$$

and

$$\bar{P}(M \mid N,a) = (1-\theta)\,\delta(M \mid N) + \theta P(M \mid N,a),\ \ \forall N \in \mathbb{S},\ \forall a \in \mathbb{A}_N$$

where $\delta(M \mid N)$ is equal to 1 if $M = N$ and is equal to 0 otherwise.

The transformed Markov process is aperiodic under all policies.

In [53] it is shown that the sets of optimal stationary policies for the original and the transformed problems are identical, and $\bar{\Delta}^* = \theta\Delta^*$.

5.3.3 Control with admission limit in each region

In this part we assume that instead of controlling the rate of entry in each region, the system manager controls the number of admissible users in each region, at each decision epoch.

Here we pose $b_l = 1_{\{N_l < K_l\}}$, where K_l is the number of admissible users in region l and N_l is the number of active users in region l. Thus we can rewrite the new set of actions as $\mathbb{A} = \{(K_1, \cdots, K_r, \alpha_1, \cdots, \alpha_r)\} \subset \mathbb{N}^r \times \mathbb{R}^r$.

As previously we assume that $\mathbb{A}$ is finite, and that the system has a finite size K, that means that the total number of users in the system is assumed to not exceed K.

Let us notice that the system manger cannot choose a number of admissible users K_l in region l, less than the current number of users N_l in region l. So when the system is at state N, the set of admissible actions $\mathbb{A}_N = \{(K_1, \cdots, K_r, \alpha_1, \cdots, \alpha_r),\ K_l \geq N_l,\ \forall l, \sum_{l=1}^r K_l \leq K\} \subset \mathbb{A}$.

When the action $a = (K_1, \cdots, K_r, \alpha_1, \cdots, \alpha_r) \in \mathbb{A}$ is choosen by the system manager, then the process N_t stays Markovian with intensity matrix Q^a given by:

$$\begin{aligned}
q^a_{N,N+e_l} &= \lambda_l, \quad if\ N_l \leq K_l - 1,\\
q^a_{N,N+e_l} &= 0, \quad if\ N_l = K_l,\\
q^a_{N,N-e_l} &= \frac{N_l}{\sum_{i=1}^r N_i}\mu_l + N_l\mu_0^l + N_l\alpha_l,\\
q^a_{N,N} &= -\left(\sum_{l=1}^r q_{N,N+e_l} + \sum_{l=1}^r q_{N,N-e_l}\right).
\end{aligned}$$

It's obvious to see that for two arbitrary states N and M there exists a deterministic policy ξ under which M is accessible from N, that means that there exists an integer $n > 0$ such that $\left(P^\xi\right)^n (M \mid N) > 0$, our Markov decision process is said to be communicating.

In a communicating model in the average cost criterion the expression Δ^ξ:

$$\Delta^\xi(M) = \lim_{n\to\infty} \frac{1}{n} E^\xi \left[\sum_{i=0}^{n-1} \{R(N_i, \xi(N_i)) - C(N_i, \xi(N_i))\} \mid N_0 = M\right]$$

is not constant as in the ergodic model.

The following theorem gives us the optimality equation for the communicating model (See [53]).

Theorem 5.3.5 *If the set of states $\mathbb{S}$ is finite, and if there exists a $g \in \mathbb{V}$ and an $h \in \mathbb{V}$, such that the following optimality equations are satisfied for each $N \in \mathbb{S}$:*

$$\max_{a \in \mathbb{A}_N} \left\{ \sum_{M \in \mathbb{S}} P\left(M \mid N, a\right) g(M) - g(N) \right\} = 0 \tag{5.2}$$

and

$$\max_{a \in \mathbb{B}_N} \left\{ R(N,a) - C(N,a) - g(N) + \sum_{M \in \mathbb{S}} P\left(M \mid N, a\right) h(M) - h(N) \right\} = 0, \tag{5.3}$$

where $\mathbb{B}_N = \left\{ a' \in \mathbb{A}_N,\ \sum_{M \in \mathbb{S}} P\left(M \mid N, a\right) g(M) - g(N) = 0 \right\}$,

Then $g = \Delta^*$.

Proof See [53] ∎

It is sometimes difficult to explicit the set $\mathbb{B}_N$, so we introduce the modified optimality equation as follows:

$$\max_{a \in \mathbb{A}_N} \left\{ \sum_{M \in \mathbb{S}} P\left(M \mid N, a\right) g(M) - g(N) \right\} = 0,$$

and

$$\max_{a \in \mathbb{A}_N} \left\{ R(N,a) - C(N,a) - g(N) + \sum_{M \in \mathbb{S}} P\left(M \mid N, a\right) h(M) - h(N) \right\} = 0, \tag{5.4}$$

The following theorem enables us to join the modified and the unmodified optimality equation (See [53]).

Theorem 5.3.6 *Let us assume that $\mathbb{S}$ and $\mathbb{A}_N$ are finite for each state $N \in \mathbb{S}$, and that there exists a $g \in \mathbb{V}$ and an $h \in \mathbb{V}$ satisfying the optimality equations 5.2 and 5.3, then there exists an $M > 0$ such that g and $h + Mg$ satisfy the modified optimality equations 5.2 and 5.4.*

Proof See [53] ∎

The next theorem is analogous to theorem 5.3.5, and prooves the optimality of the modified optimality equations (See [53]).

Theorem 5.3.7 *Let us assume $\mathbb{S}$ to be countable.*

If there exists a $g \in \mathbb{V}$ and an $h \in \mathbb{V}$, such that the modified optimality equations are satisfied: for each $N \in \mathbb{S}$:

$$\max_{a \in \mathbb{A}_N} \left\{ \sum_{M \in \mathbb{S}} P\left(M \mid N, a\right) g(M) - g(N) \right\} = 0$$

and

$$\max_{a \in \mathbb{B}_N} \left\{ R(N,a) - C(N,a) - g(N) + \sum_{M \in \mathbb{S}} P\left(M \mid N, a\right) h(M) - h(N) \right\} = 0,$$

Then $g = \Delta^*$.

Proof See [53] ∎

The next theorem establishes the existence of solutions to the optimality equations and the modified optimality equations.

Theorem 5.3.8 *Let us assume $\mathbb{S}$ and $\mathbb{A}_N$ to be finite for each state,*
There exists g that satisfy $g = \Delta^*$
There exists an $h \in \mathbb{V}$, such that g and h verify the optimality equations (5.2) and (5.3),
There exists an $h' \in \mathbb{V}$, such that g and h' verify the modified optimality equations (5.2) and (5.4),

Proof See [53] ∎

Now let us give the following theorem which establishes the optimal policy.

Theorem 5.3.9 *Let us assume $\mathbb{S}$ and $\mathbb{A}_N$ to be finite for each state N,*
Then there exists a deterministic optimal policy.
If g and h satisfy the optimality equations (5.2) and (5.3), then the policy ξ defined such that:

$$g(N) = \left\{ \sum_{M \in \mathbb{S}} P\left(M \mid N, \xi(N)\right) g(M) \right\}$$

and

$$\left\{ R(N, \xi(N)) - C(N, \xi(N)) + \sum_{M \in \mathbb{S}} P\left(M \mid N, \xi(N)\right) h(M) \right\}$$
$$= \max_{a \in \mathbb{B}_N} \left\{ R(N, a) - C(N, a) + \sum_{M \in \mathbb{S}} P\left(M \mid N, a\right) h(M) \right\},$$

This is equivalent to

$$\xi(N) \in argmax_{a \in \mathbb{B}_N} \left\{ R(N, a) - C(N, a) + \sum_{M \in \mathbb{S}} P\left(M \mid N, a\right) h(M) \right\}$$

where $\mathbb{B}_N = \left\{ a' \in \mathbb{A}_N,\ \sum_{M \in \mathbb{S}} P\left(M \mid N, a\right) g(M) - g(N) = 0 \right\}$,
is the optimal policy.
If $g^{'}$ and $h^{'}$ satisfy the modified optimality equations (5.2) and (5.4), then the policy $\xi^{'}$ defined such that:

$$g(N) = \left\{ \sum_{M \in \mathbb{S}} P\left(M \mid N, \xi^{'}(N)\right) g(M) \right\}$$

and

$$\left\{ R(N, \xi^{'}(N)) - C(N, \xi^{'}(N)) + \sum_{M \in \mathbb{S}} P\left(M \mid N, \xi^{'}(N)\right) h(M) \right\}$$
$$= \max_{a \in \mathbb{A}_N} \left\{ R(N, a) - C(N, a) + \sum_{M \in \mathbb{S}} P\left(M \mid N, a\right) h(M) \right\},$$

This is equivalent to

$$\xi^{'}(N) \in argmax_{a \in \mathbb{A}_N} \left\{ R(N, a) - C(N, a) + \sum_{M \in \mathbb{S}} P\left(M \mid N, a\right) h(M) \right\}$$

is the optimal policy.

Proof See [53] ∎

Theorem 5.3.10 *Assume that the model is communicating and let ξ be a stationary policy*

1. *Suppose there exist two states N and $M \in \mathbb{S}$, such that $\Delta^{\xi}(N) < \Delta^{\xi}(M)$. Then there exists a stationary policy $\xi^{'}$ for which $\Delta^{\xi^{'}}(N) = \Delta^{\xi^{'}}(M) \geq \Delta^{\xi}(M)$*
2. *Suppose ξ is an optimal policy. Then there exists a stationary optimal policy $\xi^{'}$ with optimal gain that is constant at each state, $\Delta^{\xi^{'}}(N) = \Delta^{*}\ \forall N \in \mathbb{S}$.*

Proof See [53] ∎

The above theorem shows that the optimal gain is constant as in an ergodic model.
Let us notice that the gain Δ^{ξ} associated to an arbitrary policy ξ is not necessary constant.
We use the value iteration algorithm (See [53]) to compute the optimal policy. The value iteration algorithm for ergodic model can be used in communicating model because its convergence needs the optimal gain to be constant as said in the above theorem.

Value Iteration Algorithm

1. Set $n = 0$, for each $N \in \mathbb{S}$, select $v^{0}(N) \in \mathbb{R}$, specify $\epsilon > 0$.
2. For each $N \in \mathbb{S}$, compute $v^{n+1}(N)$ by
$$v^{n+1}(N) = \max_{a \in \mathbb{A}_N} \left\{ R(N,a) - C(N,a) + \sum_{M \in \mathbb{S}} P\left(M \mid N, a\right) v^{n}(M) \right\}.$$
3. If
$$sp\left(v^{n+1} - v^{n}\right) < \epsilon$$
Go to (4). Otherwise increment n by 1 and go back to (2).
4. For each $N \in \mathbb{S}$, choose $\xi_{\epsilon}(N)$, such that
$$R(N, \xi_{\epsilon}(N)) - C(N, \xi_{\epsilon}(N)) + \sum_{M \in \mathbb{S}} P\left(M \mid N, \xi_{\epsilon}(N)\right) v^{n}(M)$$
$$= \max_{a \in \mathbb{A}_N} \left\{ R(N,a) - C(N,a) + \sum_{M \in \mathbb{S}} P\left(M \mid N, a\right) v^{n}(M) \right\}$$
This means
$$\xi_{\epsilon}(N) \in argmax_{a \in \mathbb{A}_N} \left\{ R(N,a) - C(N,a) + \sum_{M \in \mathbb{S}} P\left(M \mid N, a\right) v^{n}(M) \right\}$$
and stop.

The following theorem ensures the convergence of the Value Iteration Algorithm (See [53]).

Theorem 5.3.11 *Assume that the model is such that the optimal gain Δ^{*} is constant on $\mathbb{S}$, and assume that under all deterministic policies the associated Markov process is aperiodic. Then there exists an integer n in which the Value Iteration Algorithm terminates.*

Proof See [53] ∎

As our model is communicating, then the optimal gain is constant which satisfies the first assumption of the above theorem. The second assumption can be satisfied by using the aperiodicity transformation exposed previously.

Let us notice that the theorem is also applicable to Ergodic model.

5.3.4 Control with region blocking

In this part the system manager is able to block an arrival in each region.

Here we assume $\Xi = \{0, 1\}$, $b_l = 0$ means that all arrivals in region l are rejected and $b_l = 1$ means that all arrivals in region l are accepted unless the total number of users in the system exceeds K.

When the action $a = (b_1, \cdots, b_r, \alpha_1, \cdots, \alpha_r) \in \mathbb{A}$ is chosen by the system manager, then the process N_t stays Markovian with intensity matrix Q^a given by:

$$q^a_{N,N+e_l} = \lambda_l, \quad if\ b_l = 1,\ and\ \sum_{i=1}^{r} N_i \leq K - 1,$$

$$q^a_{N,N+e_l} = 0, \quad if\ b_l = 0,\ or\ \sum_{i=1}^{r} N_i \geq K$$

$$q^a_{N,N-e_l} = \frac{N_l}{\sum_{i=1}^{r} N_i}\mu_l + N_l\mu_0^l + N_l\alpha_l,$$

$$q^a_{N,N} = -\left(\sum_{l=1}^{r} q_{N,N+e_l} + \sum_{l=1}^{r} q_{N,N-e_l}\right).$$

where $N = (N_1, \cdots, N_r)$.

It's obvious to see that for two arbitrary states N and M there exists a deterministic policy ξ under which M is accessible from N, that means that there exists an integer $n > 0$ such that $\left(P^{\xi}\right)^n (M \mid N) > 0$, our Markov decision process is then communicating.

Thus all results about communicating model as exposed in the previous section can be used here.

5.3.5 Numerical applications

In this part numerical applications we consider a system consisting of two regions.

We first begin by the case of adjustable arrival rate in each region corresponding to the model in section 5.4.1.

We consider that set of actions is defined as follows:

$$b_1, b_2 \in \{1, 2, \cdots, 5\}$$

$$\alpha_1, \alpha_2 \in \{5, 10, \cdots, 25\},$$

and we take $\mu_1 = 32$, $\mu_2 = 20$, $\lambda_1 = \lambda_2 = 8$, $\mu_0 = 4$ and the system size $K = 15$.

For $\epsilon = 0.55$, the modified policy iteration algorithm says that the ϵ-optimal policy is the same at each state and is given by:

$$\xi(N) = (b_1, b_2, \alpha_1, \alpha_2) = (5, 1, 5, 5), \quad \forall N \in \mathbb{S}.$$

For the case of control with admission limit in each region (model in section 5.4.2) we consider the following set of admissible actions at state $N = (N_1, N_2) \in \mathbb{S}$:

$$\alpha_1, \alpha_2 \in \{5, 10, \cdots, 25\},$$

$$K_1 \in \langle N_1, N_1 + 1, N_1 + 2, \cdots \}$$

$$K_2 \in \langle N_2, N_2 + 1, N_2 + 2, \cdots \}$$

We choose $\mu_1 = 32$, $\mu_2 = 20$, $\lambda_1 = \lambda_2 = 8$, $\mu_0 = 4$ and the system size $K = 15$.

For $\epsilon = 0.55$, the value iteration algorithm gives us the ϵ-optimal policy shown in the table 5.1.

Table 5.1: Value iteration results

System state	K_1	K_2	α_1	α_2
(0,0)	1	1	5	5
(0,1)	1	1	5	5
(0,2)	1	2	5	5
(0,3)	1	3	5	5
(0,4)	1	4	5	5
(0,5)	1	5	5	5
(1,0)	1	1	5	5
(1,1)	1	1	5	5
(1,2)	1	2	5	5
(1,3)	1	3	5	5
(1,4)	1	4	5	5
(2,0)	2	1	5	5
(2,1)	2	1	5	5
(2,2)	2	2	5	5
(2,3)	2	3	5	5
(3,0)	3	1	5	5
(3,1)	3	1	5	5
(3,2)	3	2	5	5
(4,0)	4	1	5	5
(4,1)	4	1	5	5
(5,0)	5	1	5	5
⋮	⋮	⋮	⋮	⋮
(15,0)	15	1	5	5

We finish with the case of control with region blocking (model in section 5.4.3), in which we consider the following set of actions:

$$b_1, b_2 \in \{0, 1\}$$

$$\alpha_1, \alpha_2 \in \{5, 10, \cdots, 25\}.$$

We choose $\mu_1 = 32$, $\mu_2 = 20$, $\lambda_1 = \lambda_2 = 8$, $\mu_0 = 4$ and the system size $K = 15$.

For $\epsilon = 0.55$, the value iteration algorithm says that the ϵ-optimal policy is the same at each state and is given by:

$$\xi(N) = (b_1, b_2, \alpha_1, \alpha_2) = (1, 1, 5, 5), \quad \forall N \in \mathbb{S}.$$

5.4 Control with compact set of actions

In this part we develop another aproach of control. We consider that the set of actions is compact and not countable as it was the case in the previous section. We derive some results that allow us to find optimal policy recursively without using value iteration and modified policy iteration algorithms as we did in the previous section. In addition our results provide the system manager with an optimal system size for admission control.

5.4.1 Model

Let us first consider a system with one cell composed of one region. Then the case of multiple regions will be studied in section 5.6.

Users are assuming to arrive according to a Poisson process with rate λ. Users in the system are served as a processor sharing. When there are n active users in the system, each user has a service time given by $T = n\frac{\sigma}{C}$, where σ is the flow size, and C is the system capacity. The flow size is assumed to be exponentially distributed, then the service time $\frac{\sigma}{C}$ if he is alone in the system, is also exponentially distributed with parameter that we denote by μ. If there are n active users in the system, because of the processor sharing sheduler, the service time will be exponentially distributed with parameter $\frac{\mu}{n}$.

Each user in the system has a patience duration denoted by τ that is assumed to be exponentially distributed with parameter μ_0. We introduce a variable of control for dropping D, that allows the system manager to force a user to leave the system even if he does not finish his service. The variable of control is also assumed to be exponentially distributed with parameter denoted by α.

Each user in the system has its service time T, its patience duration τ, and its variable of control D, that we assume independent. So we can define for each user its effective time in the system given by $T_{eff} = \min\left\{\frac{n\sigma}{C}, \tau, D\right\}$, that by assumption of independence, is exponentially distributed with parameter $\frac{\mu}{n} + \mu_0 + \alpha$, when there are n active users in the system. In this section the system is controled by the system manager, who chooses at each decision epoch some actions to make in order to optimize the system performance which means in our work to maximize some function that we will define later.

Let us introduce the process N_t which is the number of users in the system at instant t taking values in $\{0, 1, 2, ...\}$. Under our assumption the process is an irreductible Markov process with intensity matrix Q given by:

$$q_{n,n+1} = \lambda,$$

$$q_{n,n-1} = \mu + n\left(\mu_0 + \alpha\right), \quad \forall n > 0.$$

This Markov process N_t has a steady state probabilty π for the number of users in the system that verifies the local balance equation:

$$\left(\mu + n\left(\mu_0 + \alpha\right)\right)\pi_n = \lambda\pi_{n-1}. \tag{5.5}$$

5.4.2 Related Markov decision process problem

In this section we study the problem of choosing the arrival rate λ, the service rate by the parameter μ, and the parameter of the control variable or dropping variable α. As the process is Markovian, we consider that the decision epochs are the instants at which the process changes state, that falls in the problem of semi-Markov decision problems.

In this part we only consider deterministic actions. The system manager has to choose an action $a = (\lambda, \mu, \alpha) \in A$ after either an arrival or departure instance. We assume that the set of

actions $A = [0, M] \times [0, S] \times [0, L]$, where M, S and L are respectively the maximum value that can reach the arrival rate λ, the service rate μ and the dropping rate α.

Let us define $c^1(\mu)$ the cost rate associated to the change of service rate μ, $c^2(\alpha)$ the cost rate associated to the change of parameter of control variable α, and $b(\lambda)$ the value rate associated with arrival rate λ. The system manager incurs a holding cost at each state of the system that we denote by h_n, when the state is n. h_n can be seen as the cost for the system to contain n active users or the cost of congestion.

Let us assume that $c^1(0) = 0$, c^1 is a non-decreasing and convex function on $[0, S]$, $c^2(0) = 0$, c^2 is a non-decreasing and convex function on $[0, L]$, and b is assumed to be non-decreasing, strictly concave, continuous differentiable on $[0, M]$ with $b(0) = 0$, we also assume that $b^{'}(0) < \infty$.

Let us notice that the assumption c^1 and c^2 are non-decreasing is natural because increasing the service rate and the dropping rate can't be done without high cost value.

It is natural to assume that the holding cost or the cost of congestion h_n is non-decreasing in n, $h_0 = 0$ and $\lim_{n\to\infty} h_n = \infty$. When the number of users in the system increases, congestion increases then the cost of congestion increases. When the system is empty, there is no congestion then the cost of congestion is null.

We consider the problem of long run average generated per unit time unit over infinite planning horizon.

We define a policy as a triplet of vectors $\left(\vec{\lambda}, \vec{\mu}, \vec{\alpha}\right)$, where $\vec{\lambda} = (\lambda_0, \lambda_1, \lambda_2, \cdots)$, $\vec{\mu} = (\mu_0, \mu_1, \mu_2, \cdots)$ and $\vec{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \cdots)$. $a_n = (\lambda_n, \mu_n, \alpha_n)$ would be the action chosen by the system manager when the system is at state n. We assume by convention that $\mu_0 = \alpha_0 = 0$.

A policy $\left(\vec{\lambda}, \vec{\mu}, \vec{\alpha}\right)$ is said to be ergodic if under this policy the process N_t has a unique steady state probability $\pi\left(\vec{\lambda}, \vec{\mu}, \vec{\alpha}\right)$ that satisfies the above local balance equation. We define the long run average generated under an ergodic policy $\left(\vec{\lambda}, \vec{\mu}, \vec{\alpha}\right)$ by:

$$Z\left(\vec{\lambda}, \vec{\mu}, \vec{\alpha}\right) = \sum_{n=0}^{\infty} \pi_n\left(\vec{\lambda}, \vec{\mu}, \vec{\alpha}\right)\left[b(\lambda_n) - c^1(\mu_n) - c^2(\alpha_n) - h_n\right]. \tag{5.6}$$

We define the optimal long run average by:

$$Z^* = \sup_{\left(\vec{\lambda}, \vec{\mu}, \vec{\alpha}\right)} Z\left(\vec{\lambda}, \vec{\mu}, \vec{\alpha}\right), \tag{5.7}$$

where the supremum is taken over all ergodic policies.

A policy $\left(\vec{\lambda}, \vec{\mu}, \vec{\alpha}\right)$ is said to be optimal if $Z^* = Z\left(\vec{\lambda}, \vec{\mu}, \vec{\alpha}\right)$.

The system manager objective is to find an optimal policy in order to maximize the long run average.

We first consider our system with a finite system size, which we denote by K. We derive the optimal system size later.

5.4.3 Optimality equation

We assume that the system has a finite state with K the system size, so the process N_t takes values in $\{0, 1, 2, \cdots, K\}$.

By convention we assume that $\lambda_K = 0$ and $\mu_0 = 0$.

According to the optimality equations for a semi-Markov decision process problem with long run average cost criterion we have the following equations:

$$v_0 = \sup_{\lambda \in [0,M]} \left\{ \frac{b(\lambda) - z}{\lambda} + v_1 \right\} \tag{5.8}$$

$$v_n = \sup_{(\lambda,\mu,\alpha)\in A} \left\{ \frac{b(\lambda) - c^1(\mu) - c^2(\alpha) - h_n - z}{\lambda + [\mu + n(\mu_0 + \alpha)]} + \frac{\lambda}{\lambda + [\mu + n(\mu_0 + \alpha)]} v_{n+1} + \frac{[\mu + n(\mu_0 + \alpha)]}{\lambda + [\mu + n(\mu_0 + \alpha)]} v_{n-1} \right\}, \tag{5.9}$$

$$\forall n \in \{1, \cdots, K-1\}$$

$$v_K = \sup_{\mu\in[0,S],\alpha\in[0,L]} \left\{ \frac{-c^1(\mu) - c^2(\alpha) - h_K - z}{\mu + K(\mu_0 + \alpha)} + v_{K-1} \right\} \tag{5.10}$$

z is interpreted as a guess at the supremum average value.

Let us notice that in equation (5.8), $\frac{1}{\lambda}$ represents the expectation until the next state that is 1, and the probability for the process to leave the state 0 and go to state 1 is $\frac{\lambda}{\lambda} = 1$. In equation (5.9), $\frac{1}{\lambda+[\mu+n(\mu_0+\alpha)]}$ represents the expectation for the process to change state being at state n, $\frac{\lambda}{\lambda+[\mu+n(\mu_0+\alpha)]}$ represents the probability for the process to leave state n and go to state $n+1$, and $\frac{\mu+n(\mu_0+\alpha)}{\lambda+[\mu+n(\mu_0+\alpha)]}$ the probability to leave state n and go to state $n-1$. In equation (5.10) $\frac{1}{\mu+K(\mu_0+\alpha)}$ is the expectation for the process to change state, the only state in which the process can go while at state K is the state $K-1$ that has a probability equal to 1.

$v_0, v_1, v_2, \cdots, v_K$ are called relative value functions in average cost value, and are only determined up to an additive constant. So we define the relative value differences:

$$y_n = v_{n-1} - v_n, \ \forall n \in \{1, \cdots, K\}.$$

Then equations (5.8), (5.9), (5.10) can be written as:

$$z = \sup_{\lambda\in[0,M]} \{b(\lambda) - \lambda y_1\} \tag{5.11}$$

$$z = \sup_{\lambda\in[0,M]} \{b(\lambda) - \lambda y_{n+1} +\} + \sup_{\mu\in[0,S]} \left\{\mu y_n - c^1(\mu)\right\} + \sup_{\alpha\in[0,L]} \left\{n\alpha y_n - c^2(\alpha)\right\} + n\mu_0 y_n - h_n \tag{5.12}$$

$$z = \sup_{\mu\in[0,S]} \left\{\mu y_K - c^1(\mu)\right\} + \sup_{\alpha\in[0,L]} \left\{K\alpha y_K - c^2(\alpha)\right\} + K\mu_0 y_K - h_K \tag{5.13}$$

Let us define ζ, ϕ^1 and ϕ^2 as follows:

$$\zeta(y) = \sup_{\lambda\in[0,M]} \{b(\lambda) - \lambda y\}, \quad \forall y \in \left[0, b'(0)\right] \tag{5.14}$$

$$\phi^1(y) = \sup_{\mu\in[0,S]} \left\{\mu y - c^1(\mu)\right\}, \quad \forall y \geq 0 \tag{5.15}$$

$$\phi^2(y, n) = \sup_{\alpha\in[0,L]} \left\{n\alpha y - c^2(\alpha)\right\}, \quad \forall y \geq 0 \tag{5.16}$$

Let us notice that ζ is only defined on $\left[0, b'(0)\right]$, this will be clear in the analysis. Thus equations (5.11), (5.12), (5.13) can be rewritten as follows:

$$z = \zeta(y_1) \tag{5.17}$$

$$z = \zeta(y_{n+1}) + \phi^1(y_n) + \phi^2(y_n, n) + n\mu_0 y_n - h_n, \ \forall n \in \{0, \cdots, K-1\} \tag{5.18}$$

$$z = \phi^1(y_K) + \phi^2(y_K, K) + K\mu_0 y_K - h_K. \tag{5.19}$$

Let us notice that for a fixed y, the function $b(\lambda) - \lambda y$ is continuous with λ since b is continuous by assumption, and the set $[0, M]$ is compact, then the maximizer exists and takes values in $[0, M]$. We denote by $\eta(y)$ the smallest maximizer of supremum in the expression of $\zeta(y)$ such that:

$$\zeta(y) = b(\eta(y)) - \eta(y)y.$$

Similary the functions $\mu y - c^1(\mu)$ and $n\alpha y - c^2(\alpha)$ are continuous with μ and α respetively and the sets $[0, S]$ and $[0, L]$ are compact, then the maximizers in the expression of ϕ^1 and ϕ^2 exist and take values in $[0, S]$ and $[0, L]$ respectively . We denote by $\psi^1(y)$ the smallest maximizer of supremum in the expression of $\phi^1(y)$ and $\psi^2(y, n)$ the smallest maximizer of supremum in the expression of $\phi^2(y)$ such that:

$$\phi^1(y) = \psi^1(y)y - c^1(\psi^1(y))$$

$$\phi^2(y, n) = n\psi^2(y, n)y - c^2(\psi^2(y, n))$$

Theorem 5.4.1 *If there exists an integer K, a real z and a vector $(y_1, y_2, \cdots, y_K)$ such that:*

- *z and $(y_1, y_2, \cdots, y_K)$ solve the optimality equation (5.17), (5.18), (5.19)*
- *$h_n + z \geq \phi^1(b'(0)) + \phi^2(b'(0), n) + n\mu_0 b'(0)$, $\forall n \geq K + 1$*
- *$0 \leq y_n \leq b'(0)$, $\forall n \in \{1, \cdots, K\}$*

Then $z \geq Z\left(\vec{\lambda}, \vec{\mu}, \vec{\alpha}\right)$, for all ergodic policies $\left(\vec{\lambda}, \vec{\mu}, \vec{\alpha}\right)$, thus $z \geq Z^*$.
Let there be the following policy $\left(\vec{\lambda^*}, \vec{\mu^*}, \vec{\alpha^*}\right)$ given by:

$$\lambda_n^* = \eta(y_{n+1}),\ \forall n \in \{0, \cdots, K-1\}$$

$$\lambda_n^* = 0,\ \forall n \geq K$$

$$\mu_n^* = \psi^1(y_n),\ \forall n \in \{0, \cdots, K\}$$

$$\mu_n^* = S,\ \forall n \geq K+1$$

$$\alpha_n^* = \psi^2(y_n, n),\ \forall n \in \{0, \cdots, K\}$$

$$\alpha_n^* = L,\ \forall n \geq K+1$$

If the policy $\left(\vec{\lambda^*}, \vec{\mu^*}, \vec{\alpha^*}\right)$ is ergodic, then it is optimal among all ergodic policy that truncate the system size to K:

$$Z\left(\vec{\lambda^*}, \vec{\mu^*}, \vec{\alpha^*}\right) = Z^*.$$

Proof Let us consider an arbitrary ergodic policy $\left(\vec{\lambda}, \vec{\mu}, \vec{\alpha}\right)$. Then it has steady state probability π satisfying the local balance equation:

$$(\mu_n + n(\mu_0 + \alpha_n))\pi_n = \lambda_{n-1}\pi_{n-1},\ \forall n \geq 1$$

and

$$\sum_{n=0}^{\infty} \pi_n = 1.$$

By definition of ζ, ϕ^1 and ϕ^2 we have:

$$\zeta(y_{n+1}) = \sup_{\lambda\in[0,M]} \{b(\lambda) - \lambda y_{n+1}\} \geq b(\lambda_n) - \lambda_n y_{n+1},\ \forall n \in \{1,\cdots,K-1\} \tag{5.20}$$

$$\phi^1(y_n) = \sup_{\mu\in[0,S]} \left\{\mu y_n - c^1(\mu)\right\} \geq \mu_n y_n - c^1(\mu_n),\ \forall n \in \{1,\cdots,K-1\} \tag{5.21}$$

$$\phi^2(y_n, n) = \sup_{\alpha\in[0,L]} \left\{n\alpha y_n - c^2(\alpha)\right\} \geq n\alpha_n y_n - c^2(\alpha_n),\ \forall n \in \{1,\cdots,K-1\} \tag{5.22}$$

Using equation (5.18), and summing equations (5.20), (5.21), (5.22), we get,

$$z + h_n = \zeta(y_{n+1}) + \phi^1(y_n) + \phi^2(y_n, n) + n\mu_0 y_n$$

$$\geq b(\lambda_n) - \lambda_n y_{n+1} + \mu_n y_n - c^1(\mu_n) + n\alpha_n y_n - c^2(\alpha_n) + n\mu_0 y_n,\ \forall n \in \{1,\cdots,K-1\}.$$

Thus

$$z \geq b(\lambda_n) - \lambda_n y_{n+1} + \mu_n y_n - c^1(\mu_n) + n\alpha_n y_n - c^2(\alpha_n) + n\mu_0 y_n - h_n$$

Let us multiply both sides by π_n and let us sum over $n \in \{1,\cdots,K-1\}$, we obtain:

$$z\sum_{n=1}^{K-1}\pi_n \geq \sum_{n=1}^{K-1}\pi_n\left[b(\lambda_n) - c^1(\mu_n) - c^2(\alpha_n) - h_n\right] + \sum_{n=1}^{K-1}\left(\pi_n y_n\left[\mu_n + n\alpha_n + n\mu_0\right] - \lambda_n\pi_n y_{n+1}\right)$$

The equations of local balance give $\lambda_n\pi_n = \left(\mu_{n+1} + (n+1)\left(\mu_0 + \alpha_{n+1}\right)\right)\pi_{n+1}$,
Thus

$$\sum_{n=1}^{K-1}\left(\pi_n y_n\left[\mu_n + n\alpha_n + n\mu_0\right] - \lambda_n\pi_n y_{n+1}\right)$$

$$= \sum_{n=1}^{K-1}\left(\pi_n y_n\left[\mu_n + n\left(\alpha_n + \mu_0\right)\right] - \pi_{n+1}y_{n+1}\left[\mu_{n+1} + (n+1)\left(\mu_0 + \alpha_{n+1}\right)\right]\right)$$

$$= \pi_1 y_1\left[\mu_1 + \alpha_1 + \mu_0\right] - \pi_N y_K\left[\mu_N + N\left(\alpha_K + \mu_0\right)\right]$$

So we obtain

$$z\sum_{n=1}^{K-1}\pi_n \geq \sum_{n=1}^{K-1}\pi_n\left[b(\lambda_n) - c^1(\mu_n) - c^2(\alpha_n) - h_n\right] + \pi_1 y_1\left[\mu_1 + \alpha_1 + \mu_0\right] - \pi_K y_K\left[\mu_K + K\left(\alpha_K + \mu_0\right)\right] \tag{5.23}$$

Similary by (5.17) and (5.19) and using the fact that $c^1(\mu_0) = c^2(\mu_0) = 0$ we have:

$$z\pi_0 \geq \pi_0 b(\lambda_0) - \pi_0\lambda_0 y_1 - \pi_0 c^1(\mu_0) - \pi_0 c^2(\mu_0) \tag{5.24}$$

$$z\pi_K \geq \pi_K\left[\mu_K + K\left(\alpha_K + \mu_0\right)\right]y_K - \pi_K c^1(\mu_K) - \pi_K c^2(\mu_K) - \pi_K h_K \tag{5.25}$$

Summing equations (5.23), (5.24) and (5.25), and using the fact that $\pi_0\lambda_0 y_1 = \pi_1\left[\mu_1 + \alpha_1 + \mu_0\right]y_1$, we obtain:

$$z\sum_{n=0}^{K}\pi_n \geq \sum_{n=0}^{K-1}\pi_n\left[b(\lambda_n) - c^1(\mu_n) - c^2(\alpha_n) - h_n\right] - \pi_K c^1(\mu_K) - \pi_K c^2(\alpha_K) - \pi_K h_K \tag{5.26}$$

Let us consider the case of $n \geq K+1$.

By definition of ϕ^1 and ϕ^2 we have $\forall n \geq K+1$:

$$\phi^1(b^{'}(0)) \geq b^{'}(0)\mu_n - c^1(\mu_n)$$

$$\phi^2(b^{'}(0), n) \geq nb^{'}(0)\alpha_n - c^2(\alpha_n)$$

The assumption in the theorem says that $h_n + z \geq \phi^1(b^{'}(0)) + \phi^2(b^{'}(0), n) + n\mu_0 b^{'}(0)$, $\forall n \geq K+1$

Then we obtain:

$$z \geq \phi^1(b^{'}(0)) + \phi^2(b^{'}(0), n) + n\mu_0 b^{'}(0) - h_n, \ \forall n \geq K+1$$

Thus

$$z \geq b^{'}(0)\mu_n - c^1(\mu_n) + nb^{'}(0)\alpha_n - c^2(\alpha_n) + n\mu_0 b^{'}(0) - h_n, \ \forall n \geq K+1$$

Multiplying both sides of this inequality by π_n, for $n \geq K+1$, we obtain:

$$\pi_n z \geq b^{'}(0)\pi_n\mu_n - \pi_n c^1(\mu_n) + nb^{'}(0)\pi_n\alpha_n - \pi_n c^2(\alpha_n) + n\pi_n\mu_0 b^{'}(0) - \pi_n h_n, \ \forall n \geq K+1$$

$$\pi_n z \geq b^{'}(0)\pi_n \left[\mu_n + n\left(\alpha_n + \mu_0\right)\right] - \pi_n c^1(\mu_n) - \pi_n c^2(\alpha_n) - \pi_n h_n, \ \forall n \geq K+1$$

This gives by using the equations of local balance:

$$\pi_n z \geq b^{'}(0)\pi_{n-1}\lambda_{n-1} - \pi_n c^1(\mu_n) - \pi_n c^2(\alpha_n) - \pi_n h_n, \ \forall n \geq K+1$$

By assumption b is concave, it then satisfies $b^{'}(0)\lambda_{n-1} \geq b(\lambda_{n-1})$. Then we obtain:

$$\pi_n z \geq b(\lambda_{n-1})\pi_{n-1} - \pi_n c^1(\mu_n) - \pi_n c^2(\alpha_n) - \pi_n h_n, \ \forall n \geq K+1 \tag{5.27}$$

Let us pose an integer N that is larger than K. Let us sum equation (5.27) over N and let us add it to equation (5.26), we obtain:

$$z\sum_{n=0}^{N}\pi_n \geq \sum_{n=0}^{N}\pi_n\left[b(\lambda_n) - c^1(\mu_n) - c^2(\alpha_n) - h_n\right] - \pi_N b(\lambda_N) \tag{5.28}$$

Passing to the limit as $N \to \infty$, and using the fact that $\pi_N b(\lambda_N) \to 0$, it follows that:

$$z \geq \sum_{n=0}^{\infty}\pi_n\left[b(\lambda_n) - c^1(\mu_n) - c^2(\alpha_n) - h_n\right] = Z\left(\vec{\lambda}, \vec{\mu}, \vec{\alpha}\right)$$

This implies that:

$$z \geq Z^*$$

Now let us consider the candidate policy $\left(\vec{\lambda^*}, \vec{\mu^*}, \vec{\alpha^*}\right)$ given in the theorem. So the inequality in equation (5.26) is an equality. For an ergodic policy that truncates the buffer size to K, we have $\sum_{n=0}^{K}\pi_n = 1$, and $b(\lambda_K) = 0$, then we conclude:

$$z\sum_{n=0}^{K}\pi_n^* = \sum_{n=0}^{K-1}\pi_n^*\left[b(\lambda_n^*) - c^1(\mu_n^*) - c^2(\alpha_n^*) - h_n\right] - \pi_K^* c^1(\mu_K^*) - \pi_K c^2(\alpha_K^*) - \pi_K^* h_K$$

Thus

$$z = \sum_{n=0}^{K}\pi_n^*\left[b(\lambda_n^*) - c^1(\mu_n^*) - c^2(\alpha_n^*) - h_n\right]$$

That implies that the policy $\left(\vec{\lambda^*}, \vec{\mu^*}, \vec{\alpha^*}\right)$ is optimal among all ergodic policies that truncate the system size to K. ∎

5.4.4 Some properties of ζ, ϕ^1 and ϕ^2

In this part we derive some important properties of ζ, ϕ^1 and ϕ^2 and their associated maximizer that we need in our analysis.

Proposition 5.4.2 *The function ϕ^1 is convex, differentiable almost everywhere with y, for each fixed $n \geq 1$, $\phi^2(\ ,n)$ is convex, differentiable almost everywhere with y, and we have:*

- $\phi'^1(\ ) = \psi^1(\ )$, $\phi^1(y) = \int_0^y \psi^1(u)du$
- $\phi'^2(\ ,n) = n\psi^2(\ ,n)$, $\phi^2(y,n) = n\int_0^y \psi^2(u,n)du$, $\forall n \geq 1$

It follows that $\phi^1(\ )$ and $\phi^2(\ ,n)$ are nondecreasing with y.

Proof Let $y_0 \geq 0$, and let $x_0 = \psi^1(y_0)$ such that:

$$\phi^1(y_0) = x_0 y_0 - c^1(x_0)$$

Let an arbitrary $y \geq 0$,

$$\phi^1(y) = \sup_{x\in[0,S]} \left\{xy - c^1(x)\right\} \geq yx_0 - c^1(x_0)$$

Then

$$\phi^1(y) - \phi^1(y_0) \geq x_0\,(y - y_0)\,,\ \forall y \geq 0$$

This implies that ϕ^1 is convex. A finite convex function is continuous and differentiable almost everywhere, and the above inequality implies that its derivative at y_0 is $x_0 = \psi^1(y_0)$, thus we have $\phi'^1() = \psi^1()$ wherever the derivative exists. Then we have the following Rieman-integral form:

$$\phi^1(y) = \int_0^y \psi^1(u)du$$

Similary for ϕ^2, let $y_0 \geq 0$, and $n \geq 1$, let $x_0 = \psi^2(y_0, n)$ such that:

$$\phi^2(y_0, n) = nx_0 y_0 - c^2(x_0)$$

Let an arbitrary $y \geq 0$,

$$\phi^2(y, n) = \sup_{x\in[0,L]} \left\{nxy - c^2(x)\right\} \geq nyx_0 - c^2(x_0)$$

Thus

$$\phi^2(y, n) - \phi^2(y_0, n) \geq nx_0\,(y - y_0)\,,\ \forall y \geq 0$$

This implies that $\phi^2(\ ,n)$ is convex with y, so it is continuous and differentiable almost everywhere, and its derivative at y_0 is $nx_0 = n\psi^2(y_0, n)$, thus we have the following Rieman-integral form:

$$\phi^2(y, n) = n\int_0^y \psi^2(u, n)du.$$

As $\psi^1 \in [0, S]$ and $\psi^2 \in [0, L]$, then they are positive, this implies that $\phi^1(\ )$ and $\phi^2(\ ,n)$ are nondecreasing. ∎

Proposition 5.4.3 *Let $(c'^1)^{-1}$ and $(c'^2)^{-1}$ be the inverse function of the derivative function of c^1 and c^2 respectively, we have*

- *The function* $\psi^1(y) = 0, \forall y \leq c'^1(0)$, *and if* c'^1 *is differentiable then* $\psi^1(y) = (c'^1)^{-1}(y), \ \forall y > c'^1(0)$
- *For each* $n \geq 1$, $\psi^2(y,n) = 0, \ \forall y \leq \frac{c'^2(0)}{n}$, *and if* c'^2 *is differentiable, then* $\psi^2(y,n) = (c'^2)^{-1}(ny), \ \forall y > \frac{c'^2(0)}{n}$

Proof Let us recall that we assumed that c^1 and c^2 are strictly convex and continuous differentiable, with $c^1(0) = c^2(0) = 0$, then their respective derivative functions c'^1 and c'^2 are strictly increasing, thus their inverse functions $(c'^1)^{-1}$ and $(c'^2)^{-1}$ exist. Let us define $p^1 = \sup\left\{y \geq 0, \phi^1(y) = 0\right\}$ and $p_n^2 = \sup\left\{y \geq 0, \phi^2(y,n) = 0\right\}$. Since ϕ^1 and ϕ^2 are positive (that can be seen by their Rieman-integral form), it follows that:

$$\psi^1(y) = 0, \forall y \leq p^1$$

and

$$\psi^2(y,n) = 0, \forall y \leq p_n^2,$$

So we have to prove that $p^1 = c'^1(0)$ and $p_n^2 = \frac{c'^2(0)}{n}$.

p^1 can be written as:

$$p^1 = \sup\left\{y \geq 0, \sup_{x\in[0,S]}\left\{xy - c^1(x)\right\} = 0\right\}$$

$$= \sup\left\{y \geq 0, xy - c^1(x) \leq 0, \ \forall x \in [0,S]\right\}$$

$$= \inf\left\{\frac{c^1(x)}{x} \leq 0, \ \forall x \in [0,S]\right\}$$

By convexity of c^1 and the fact that $c^1(0) = 0$ it follows that

$$\frac{c^1(x)}{x} \geq c'^1(0), \ \forall x \in [0,S]$$

This implies that $p^1 = c'^1(0)$.

p_n^2 can be written as:

$$p_n^2 = \sup\left\{y \geq 0, \sup_{x\in[0,L]}\left\{nxy - c^2(x)\right\} = 0\right\}$$

$$= \sup\left\{y \geq 0, nxy - c^2(x) \leq 0, \ \forall x \in [0,L]\right\}$$

$$= \inf\left\{\frac{c^2(x)}{nx} \leq 0, \ \forall x \in [0,L]\right\}$$

By convexity of c^2 and the fact that $c^2(0) = 0$ it follows that

$$\frac{c^2(x)}{x} \geq c'^2(0), \ \forall x \in [0,S]$$

This implies that $p_n^2 = \frac{c'^2(0)}{n}$.

Let us define $f(x) = xy - c^1(x)$, for $y > p^1$, defined on $[0,S]$, and $g_n(x) = nxy - c^2(x)$ for $y \geq p_n^2$, defined on $[0,L]$, and $n \geq 1$.

The real function f is continuous differentiable, so its maximizer and its minimizer satisfy $f^{'}(x) = y - c^{'1}(x) = 0$, since $f^{''}(x) = -c^{''1}(x) < 0$ because c^1 is assumed to be strictly convex, then $c^{''1} > 0$, then there are no minimizer for f. Now let us assume that f has more than two maximizers, and let x_1 and x_2 be two maximizers of f, with $x_1 < x_2$. We have $f^{'}(x_1) = f^{'}(x_2) = 0$. Then by Rolle theorem there exists x_3 such as $x_1 < x_3 < x_2$, satisfying $f^{''}(x_3) = 0 = -c^{''1}(x_3)$. By assumption c^1 is stricly convex then $c^{''1}(x_3) > 0$ which is absurd. So the maximizer is unique and is equal to $\psi^1(y)$.

We do a similar proof for g, and it follows that $\psi^2(y,n) = (c^{'2})^{-1}(ny)$. ∎

Proposition 5.4.4 *The function ζ is decreasing, concave, differentiable almost everywhere on $\left[0, b^{'}(0)\right]$ with $\zeta^{'}(y) = -\eta(y)$, and we have:*

$$\zeta(y) = b(M) - \int_0^y \eta(u)du$$

Proof Let $y_0 \leq b^{'}(0)$, and let $x_0 = \eta(y_0)$ such that:

$$\zeta(y_0) = b(x_0) - y_0 x_0,$$

Let an arbitrary $y \leq b^{'}(0)$, so

$$\zeta(y) \geq b(x_0) - y x_0,$$

Thus:

$$\zeta(y) - \zeta(y_0) \geq x_0\,(y_0 - y)$$

Then

$$[-\zeta(y_0)] - \left[-\zeta(y_)\right] \geq x_0\,(y_0 - y)$$

This implies that $[-\zeta]$ is convex with derivative in y_0 equal to $x_0 = \eta(y_0)$. Then we conclude that ζ is concave with $\zeta^{'} = -\eta \in [0, M]$. This implies that ζ is also decreasing. It follows that its Rieman-integral form is given by

$$\zeta(y) = \zeta(0) - \int_0^y \eta(u)du.$$

By assumption b is increasing then:

$$\zeta(0) = \sup_{x \in [0,M]} b(x) = b(M).$$

∎

Proposition 5.4.5 *The function ζ is nonnegative, strictly decreasing on $\left[0, b^{'}(0)\right]$, its inverse ζ^{-1} is well defined on $\left[0, b^{'}(0)\right]$ and $\zeta(b^{'}(0) = 0$.*

Moreover

$$\eta(y) = M,\ \forall y \in \left[0, b^{'}(M)\right]$$

$$\eta(y) = \left[b^{'}\right]^{-1}(y),\ \forall y \in \left[b^{'}(M), b^{'}(0)\right]$$

Proof Let us define $a = \inf\{y \geq 0, \zeta(y) \leq 0\}$

Then:

$$a = \inf\{y \geq 0, b(x) - xy \leq 0, \ \forall x \in [0, M]\}$$

$$a = \inf\left\{y \geq 0, y \geq \frac{b(x)}{x}, \ \forall x \in [0, M]\right\}$$

$$a = \sup\left\{\frac{b(x)}{x}, \ \forall x \in [0, M]\,.\right\}$$

Moreover b is concave on $[0, M]$ and $b(0) = 0$, then $\frac{b(x)}{x} \leq b^{'}(0), \ \forall x \in [0, M]$. This implies that $a = b^{'}(0)$. Thus $\forall y \in \left[0, b^{'}(0)\right]$, $\zeta(y) \geq 0$ and $\zeta(b^{'}(0)) = 0$.

As b is assumed to be strictly concave, then its derivative $b^{'}$ is strictly decreasing and continous, then its inverse $\left[b^{'}\right]^{-1}$ exists, and is continuous and strictly decreasing.

Let $y \in \left[0, b^{'}(M)\right]$, let $f(x) = b(x) - xy$, $\forall x \in [0, M]$, then $f^{'}(x) = b^{'}(x) - y \geq 0$. This implies that f is increasing, then the $\sup_{x\in[0,M]} f(x) = f(M)$, thus $\eta(y) = M$.

Let $y \in \left[b^{'}(M), b^{'}(0)\right]$, so $f^{'}(x) = b^{'}(x) - y = 0$ implies $y = b^{'}(x)$, this implies $x = \eta(y) = \left[b^{'}\right]^{-1}(y)$. As $\zeta^{'} = -\eta < 0$ on $\left[0, b^{'}(0)\right]$ then ζ is strictly decreasing. Then its inverse exists. ∎

5.4.5 Computation of the solution of the optimality equation

In this part we derive an explicit solution for the optimality equations (5.17), (5.18), (5.19).

If we have $\phi^1(b^{'}(0)) + \phi^2(b^{'}(0), 1) + \mu_0 b^{'}(0) \leq h_1$, it follows under the above theorem 5.4.1 that it is optimal to not accept any user.

In order to derive the optimal system size that is more than 0, we assume that

$$\phi^1(b^{'}(0)) + \phi^2(b^{'}(0), 1) + \mu_0 b^{'}(0) > h_1. \tag{5.29}$$

In order to have a finite optimal system size, we assume that:

$$\lim_{n\to\infty} \frac{h_n - \phi^1(b^{'}(0)) - \phi^2(b^{'}(0), n)}{nb^{'}(0)} \geq \mu_0. \tag{5.30}$$

Theorem 5.4.6 *There exists* $[a_n, b_n]$, *and the functions* $y_n(\ )$ *on* $[a_n, b_n]$, *for* $n \in \langle 1, 2, \cdots, K+1\}$, *defined inductively as follows:*

- $0 = a_1 < a_2 < \cdots a_K < a_{K+1} < b_{K+1} < b_K < \cdots b_K < b_1 = b(M)$,
- $h_{n-1} + a_n - \phi^1(y_{n-1}(a_n)) - \phi^2(y_{n-1}(a_n), n-1) - (n-1)\mu_0 y_{n-1}(a_n) = 0$ *and* $h_{n-1} + b_n - \phi^1(y_{n-1}(b_n)) - \phi^2(y_{n-1}(b_n), n-1) - (n-1)\mu_0 y_{n-1}(b_n) = b(M)$, $\forall n \in \langle 2, \cdots, K+1\}$
- $y_n(z) = \zeta^{-1}\left(z + h_{n-1} - \phi^1(y_{n-1}(z)) - \phi^2(y_{n-1}(z), n-1) - (n-1)\mu_0 y_{n-1}(z)\right)$, $\forall z \in [a_n, b_n]$ *and* $\forall n \in \langle 2, \cdots, K+1\}$
- $y_n(\ )$ *is continuous and strictly decreasing with* $y_n(a_n) = b^{'}(0)$ *and* $y_n(b_n) = 0$, $\forall n \in \langle 1, \cdots, K+1\}$
- $h_n + a_n < \phi^1(b^{'}(0)) + \phi^2(b^{'}(0), n) + n\mu_0 b^{'}(0)$, $\forall n \in \langle 1, \cdots, K\}$
- $h_{K+1} + a_{K+1} \geq \phi^1(b^{'}(0)) + \phi^2(b^{'}(0), K+1) + (K+1)\mu_0 b^{'}(0)$

There exists an optimal system size given by

$$K = \max\left\{n \geq 1, h_n + a_n < \phi^1(b^{'}(0)) + \phi^2(b^{'}(0), n) + n\mu_0 b^{'}(0)\right\}.$$

Proof Let us notice that the last assertion in the theorem is a stopping condition.

We prove the first four assertions by induction. We assume that these assertions are true at step $K = 1$. Let us begin by proving that they are true for $K = 2$. We first begin by constructing a_2 and b_2.

Let us define $f_1(z) = z + h_1 - \phi^1(y_1(z)) - \phi^2(y_1(z), 1) - y_1(z)\mu_0$, for $z \in [a_1, b_1]$.

As y_1 is assumed to be strictly decreasing by the step $K = 1$ of induction, then it follows that f_1 is strictly increasing and continuous. Then by the assumption in equation (5.29), we have:

$$f_1(a_1) = h_1 - \phi^1(y_1(z)) - \phi^2(y_1(z), 1) - y_1(z)\mu_0 < 0$$

and

$$f_1(b_1) = b(M) + h_1 - \phi^1(y_1(b_1)) - \phi^2(y_1(b_1), 1) - y_1(b_1)\mu_0 = b(M) + h_1 > 0$$

Then by the intermediate value theorem there exists $a_2 \in [a_1, b_1]$ such that $f_1(a_2) = 0$. It follows that $a_2 + h_1 - \phi^1(y_1(a_2)) - \phi^2(y_1(a_2), 1) - y_1(a_2)\mu_0 = 0$.

Let us pose $g(z) = f_1(z) - b(M)$, we have $g(a_2) = -b(M) < 0$ and $g(b_1) = f_1(b_1) - b(M) = h_1 > 0$, then by the intermediate value theorem there exists $b_2 \in [a_2, b_1]$ such that $g(b_2) = 0$. That is $h_1 + b_2 - \phi^1(y_1(b_2)) - \phi^2(y_1(b_2), 1) - y_1(b_2)\mu_0 = b(M)$.

As $f_1(a_2) = 0$ and $f_1(b_2) = b(M)$, and f_1 is strictly increasing, then $f_1(\ ) \in [0, b(M)]$. Then we can define y_2 as follows:

$$y_2(z) = \zeta^{-1}\left(z + h_1 - \phi^1(y_1(z)) - \phi^2(y_1(z), 1) - \mu_0 y_1(z)\right),\ \forall z \in [a_2, b_2]$$

This is continuous and strictly decreasing, with $y_2(a_2) = \zeta^{-1}(0) = b^{'}(0)$ and $y_2(b_2) = \zeta^{-1}(b(M)) = 0$ since $\zeta(0) = b(M)$. Then the induction is true at step $K = 2$.

Now let us assume that the induction is true at step $j - 1$.

So if $h_j + a_j \geq \phi^1(b^{'}(0)) + \phi^2(b^{'}(0), j) + j\mu_0 b^{'}(0)$, then we set $K = j - 1$ and $a_n = a_{j-1}$, $\forall n > K + 1$, the induction terminates.

Or we have $h_j + a_j < \phi^1(b^{'}(0)) + \phi^2(b^{'}(0), j) + j\mu_0 b^{'}(0)$.

In this case we define $f_j(z) = z + h_j - \phi^1(y_j(z)) - \phi^2(y_j(z), j) - jy_j(z)\mu_0$, for $z \in [a_j, b_j]$. It follows that f_j is strictly increasing and continuous and we have:

$$f_j(a_j) = a_j + h_j - \phi^1(y_j(a_j)) - \phi^2(y_j(a_j), j) - jy_j(a_j)\mu_0 = a_j + h_j - \phi^1(b^{'}(0)) - \phi^2(b^{'}(0), j) - jb^{'}(0)\mu_0 < 0$$

and

$$f_j(b_j) = b_j + h_j - \phi^1(y_j(b_j)) - \phi^2(y_j(b_j), 1) - jy_j(b_j)\mu_0 > b_j + h_{j-1} > b(M).$$

Then by the intermediate value theorem there exists $a_{j+1} \in [a_j, b_j]$ such that $f_j(a_{j+1}) = 0$. It follows that $a_{j+1} + h_j - \phi^1(y_j(a_{j+1})) - \phi^2(y_j(a_{j+1}), j) - jy_j(a_{j+1})\mu_0 = 0$.

Let us pose $g(z) = f_j(z) - b(M)$, we have $g(a_{j+1}) = -b(M) < 0$ and $g(b_j) = f_1(b_j) - b(M) > 0$, then by the intermediate value theorem there exists $b_{j+1} \in [a_{j+1}, b_j]$ such that $g(b_{j+1}) = 0$. That is $h_j + b_{j+1} - \phi^1(y_j(b_{j+1})) - \phi^2(y_j(b_{j+1}), j) - jy_j(b_{j+1})\mu_0 = b(M)$.

As $f_j(a_{j+1}) = 0$ and $f_j(b_{j+1}) = b(M)$, and f_j is strictly increasing, then $f_j(\ ) \in [0, b(M)]$. So we can define y_{j+1} as follows

$$y_{j+1}(z) = \zeta^{-1}\left(z + h_j - \phi^1(y_j(z)) - \phi^2(y_j(z), j) - j\mu_0 y_j(z)\right),\ \forall z \in [a_{j+1}, b_{j+1}].$$

This is continuous and strictly decreasing, with $y_{j+1}(a_{j+1}) = \zeta^{-1}(0) = b^{'}(0)$ and $y_{j+1}(b_{j+1}) = \zeta^{-1}(b(M)) = 0$.

This proves the induction at step j.

To continue at step $j+1$, we have as above two cases. We define

$$K = \max\left\{n \geq 1, h_n + a_n < \phi^1(b^{'}(0)) + \phi^2(b^{'}(0), n) + n\mu_0 b^{'}(0)\right\}.$$

Assumption (5.30) allows us to say that K is finite.

Then we conclude by definition of K that:

$$h_{K+1} + a_{K+1} \geq \phi^1(b^{'}(0)) + \phi^2(b^{'}(0), K+1) + (K+1)\mu_0 b^{'}(0).$$

∎

Let us notice that the optimal system size provided by the above theorem is an optimal admission control for the system manager in order to maximize the long run average function. This means that it is optimal to block all new arrivals when the number of users in the system reaches this optimal system size.

The above theorem allows us to construct an optimal solution.

Theorem 5.4.7 *Let $z^* = a_{K+1}$, where K is the optimal system size provided by the previous theorem, and let $y_n^* = y_n(z^*)$, for $n \in \{1, \cdots, K\}$.*

Then z^* and $(y_1^*, \cdots, y_K^*)$ solve the optimal equations such that:

$$z^* = \zeta(y_1^*) \tag{5.31}$$

$$z^* = \zeta(y_{n+1}^*) + \phi^1(y_n^*) + \phi^2(y_n^*, n) + n\mu_0 y_n^* - h_n, \ \forall n \in \{0, \cdots, K-1\} \tag{5.32}$$

$$z^* = \phi^1(y_K^*) + \phi^2(y_K^*, K) + K\mu_0 y_K^* - h_K. \tag{5.33}$$

and $0 < y_n^* < b^{'}(0), \ n \in \{1, \cdots, K\}$.

Moreover, the following policy $\left(\vec{\lambda^*}, \vec{\mu^*}, \vec{\alpha^*}\right)$ given by:

$$\lambda_n^* = \eta(y_{n+1}^*), \ \forall n \in \{0, \cdots, K-1\}$$

$$\lambda_n^* = 0, \ \forall n \geq K$$

$$\mu_n^* = \psi^1(y_n^*), \ \forall n \in \{0, \cdots, K\}$$

$$\mu_n^* = S, \ \forall n \geq K+1$$

$$\alpha_n^* = \psi^2(y_n^*, n), \ \forall n \in \{0, \cdots, K\}$$

$$\alpha_n^* = L, \ \forall n \geq K+1$$

is optimal.

Proof Let us notice that $y_1(z) = \zeta^{-1}(z)$ defined on $[a_1, a_2]$. Then it follows that $\zeta(y_1^*) = \zeta(y_1(z^*)) = z^*$ which proves equation (5.31).

Next consider $n \in \{0, \cdots, K-1\}$, it follows by construction of y_{n+1} that:

$$y_{n+1}^* = y_{n+1}(z^*) = \zeta^{-1}\left(z^* + h_n - \phi^1(y_n(z^*)) - \phi^2(y_n(z^*), n) - n\mu_0 y_n(z^*)\right)$$

This implies equation (5.32). For the last equation of optimality we have $y^*_{K+1} = y_{K+1}(a_{K+1}) = b^{'}(0)$, it follows that:

$$b^{'}(0) = \zeta^{-1}\left(z^* + h_K - \phi^1(y_K(z^*)) - \phi^2(y_K(z^*), K) - K\mu_0 y_K(z^*)\right)$$

Using $\zeta(b^{'}(0)) = 0$, it follows equation (5.33).

Now let us prove $0 < y^*_n < b^{'}(0)$.

Let us recall that $y_n(b_n) = 0$, and y_n is strictly decreasing, since $z^* = a_{K+1} < b_n$, then $y^*_n = y_n(z^*) > y_n(b_n) = 0\ \forall n \in \{1, \cdots, K\}$.

We recall that y_n is strictly decreasing, $a_{K+1} > a_n$, and $y_n(a_n) = b^{'}(0)$, it follows that $y^*_n = y_n(z^*) < y_n(a_n) = b^{'}(0)$.

From Theorem (5.4.6) and by definition of K, for $n \geq K+1$:

$$h_n + z^* \geq \phi^1(b^{'}(0)) + \phi^2(b^{'}(0), n) + n\mu_0 b^{'}(0)$$

All assumptions of Theorem (5.4.1) are verified, then z^* is optimal, and it follows that the given candidate policy is optimal. ∎

5.4.6 Numerical applications

In this part we do some numerical applications with a simple example.

Let us pose $b(\lambda) = \frac{\lambda}{1+\lambda}$, $c^1(\mu) = \mu^2$ and $c^2(\alpha) = \alpha^2$.
We can see that b is strictly concave, non-decreasing with $b(0) = 0$, then b satisfies all assumptions, and $b^{'}(0) = 1$.

The functions c^1 and c^2 are non-decreasing, convex, and C^2- continuous differentiable, so they verify all required assumptions.

With some calculations we obtain:

$$\zeta(y) = (\sqrt{y} - 1)^2,$$
$$\zeta^{-1}(x) = (1 - \sqrt{x})^2,$$
$$\eta(y) = \frac{1}{\sqrt{y}} - 1,$$
$$\phi^1(y) = \frac{y^2}{4},$$
$$\phi^2(y, n) = \frac{(ny)^2}{4},$$
$$\psi^1(y) = \frac{y}{2},$$
$$\psi^2(y, n) = \frac{ny}{2},$$

Let us pose $h_n = n^3$, and assume that b is defined on $[0, 10]$.

In order to make an illustration of the recursive construction method given in theorem 5.4.6 and theorem 5.4.7, we pose $\mu_0 = 10$, for which the optimal system size K is equal to 3, a_n and b_n of theorem 5.4.6 are given in the table 5.2

By theorem 5.4.7, we have $z^* = a_4 = 0.4132$. It follows that $y^*_1 = 0.1273$, $y^*_2 = 0.4033$ and $y^*_3 = 0.3855$.

The optimal policy is given in the table 5.3.

In figures 5.2, 5.3 and 5.4 we plot the optimal system size as a function of impatience rate for three different congestion costs: $h_n = n^2$, $h_n = n^3$ and $h_n = n^4$ respectively. The figures show that the optimal system size increases when the impatience rate increases, and that the optimal system size decreases when the congestion cost increases.

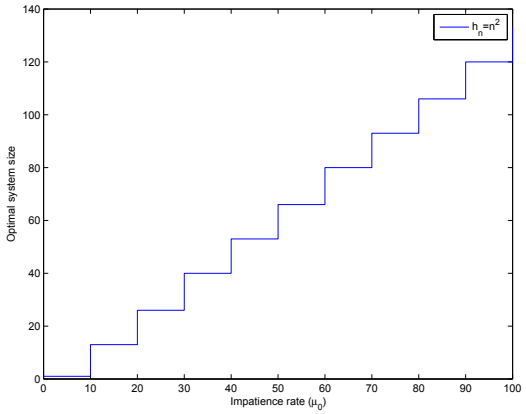

Figure 5.2: Optimal system size-$h_n = n^2$

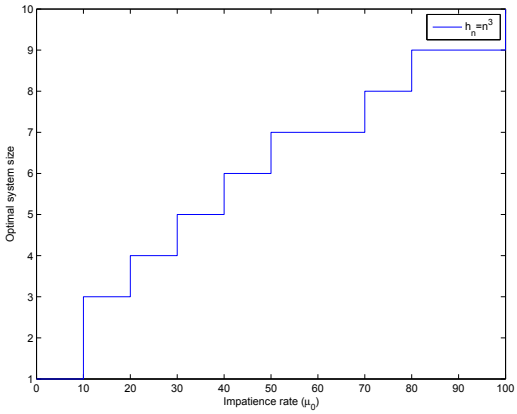

Figure 5.3: Optimal system size-$h_n = n^3$

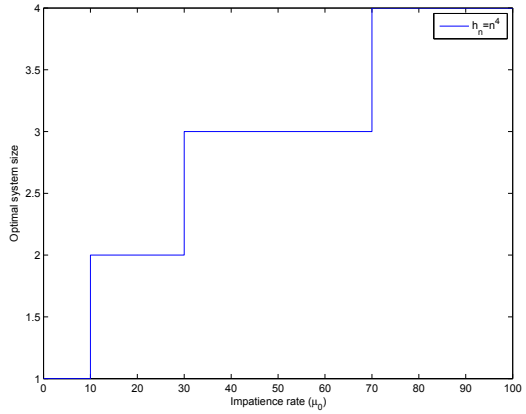

Figure 5.4: Optimal system size-$h_n = n^4$

Table 5.2: Theorem 5.4.6 results

Steps	a	b
1	0	0.9090
2	0.3942	0.5565
3	0.4132	0.4172
4	0.4132	

Table 5.3: Theorem 5.4.7 results

System state	λ	μ	α
0	1.8028	0	0
1	0.5747	0.0636	0.0636
2	0.6106	0.2016	0.4033
3	0	0.1928	0.5783

5.5 Control with compact set of actions: multiple region case.

In this section we generalize the results of the previous section to the case of one cell with multiple regions. We use the aggregate approximation model we developped in chapter 1.

Let us recall that in this model we assume that there are r regions in the system. Users arrive to the cell according to a Poisson process with rate λ. Users arrive in region l according to a Poisson process with rate $\lambda_l = \lambda p_l$, where p_l is the probability that user arrives in region l, thus $\sum_{l=1}^{r} p_l = 1$.

We assume that service is exponential in region l, with parameter μ_l. Each user of region l has a patience duration assumed to be exponential with parameter μ_0^l. To simplify our analysis we assume that $\mu_0^l = \mu_0,\ \forall l \in \{1, r\}$. We assume that service time and patience duration are independent.

In chapter 1 we derived a approximate, aggregate steady state distribution for the number of users in the system is given by:

$$\pi(n) = \frac{\prod_{l=1}^{n} \frac{\rho_{eff}(l)}{l}}{\sum_{i=0}^{\infty} \prod_{j=1}^{i} \frac{\rho_{eff}(j)}{j}} \tag{5.34}$$

with

$$\rho_{eff}(n) = \lambda \sum_{l=1}^{r} E\left(\min\left\{nT^l, \tau^l\right\}\right) p_l \tag{5.35}$$

In this part we introduce the control or dropping variable D, allowing the manager to expulse a user after a finite time in the system. A user of region l has a control or dropping variable D^l assumed to be exponential with parameter α_l. We assume that service time, patience duration and control variable are independent.

With the control variable the new steady state probability is then given by:

$$\pi(n) = \frac{\prod_{l=1}^{n} \frac{\rho_{eff}(l)}{l}}{\sum_{i=0}^{\infty} \prod_{j=1}^{i} \frac{\rho_{eff}(j)}{j}} \tag{5.36}$$

$$\rho_{eff}(n) = \lambda \sum_{l=1}^{r} E\left(\min\left\{nT^l, \tau^l, D^l\right\}\right) p_l = n\lambda \sum_{l=1}^{r} \frac{p_l}{\mu_l + n\left(\mu_0 + \alpha_l\right)} \tag{5.37}$$

Let us notice that this steady state probability satisfies the local balance equations given by:

$$\left\{\sum_{l=1}^{r}\frac{p_l}{\mu_l+n\left(\mu_0+\alpha_l\right)}\right\}^{-1}\pi_n=\lambda\pi_{n-1}. \tag{5.38}$$

Then in steady state our model is equivalent to a birth-death process where the service time at state n is exponential with parameter $\left\{\sum_{l=1}^{r}\frac{p_l}{\mu_l+n(\mu_0+\alpha_l)}\right\}^{-1}$.

5.5.1 Related Markov decision process problem

As in the case of one region studied in the previous section, in this section we study the problem of choosing the arrival rate in the system λ, the service rate in each region by the parameters $(\mu_1,\cdots,\mu_r)$, and the parameters of the control or dropping variable in each region $(\alpha_1,\cdots,\alpha_r)$.

We consider that the decision epochs are the instants at which the process changes state.

Only deterministic actions will be considered.

The system manager has to choose an action $a=(\lambda,\mu_1,\cdots,\mu_r,\alpha_1,\cdots,\alpha_r)\in\mathbb{A}$ after either an arrival or departure instance, where the set of actions $\mathbb{A}$ is assumed to be $\mathbb{A}=[0,M]\times[0,S]^r\times[0,L]^r$.

As in the previous section let us define $c^1(\mu_1,\cdots,\mu_r)$ the cost rate associated to the change of service rate to $(\mu_1,\cdots,\mu_r)$ in each region, this is a real function defined on $[0,S]^r$, $c^2(\alpha_1,\cdots,\alpha_r)$ the cost rate associated to the change of parameters of control or dropping variables to $(\alpha_1,\cdots,\alpha_r)$ in each region, this is a real function defined on $[0,L]^r$, and $b(\lambda)$ the value rate associated with arrival rate λ, this is a real function defined on $[0,M]$. The system manager incurs a holding cost at each state of the system that we denote by h_n, when the state is n. h_n can be seen as the cost for the system to contain n active users or the cost of congestion.

Let us assume that $c^1(0,\cdots,0)=0$, c^1 is non-decreasing in each of its r variables on $[0,S]$, and continuous, convex function on $[0,S]^r$, $c^2(0,\cdots,0)=0$, c^2 is non-decreasing in each of its r variables on $[0,L]$ and continuous, convex function on $[0,L]^r$. The function b is assumed to be non-decreasing, strictly concave, continuous differentiable on $[0,M]$ with $b(0)=0$, we also assume that $b'(0)<\infty$. It is natural to assume that the holding cost h_n is non-decreasing in n, with $h_0=0$ and $\lim_{n\to\infty}h_n=\infty$.

We consider the problem of long run average generated per unit time unit over infinite planning horizon.

We define a policy as a triplet of vector $\left(\vec{\lambda},\vec{\mu},\vec{\alpha}\right)$, where

$$\vec{\lambda}=(\lambda_0,\lambda_1,\lambda_2,\cdots)$$

,

$$\vec{\mu}=\left((\mu_1,\cdots,\mu_r)_0,(\mu_1,\cdots,\mu_r)_1,(\mu_1,\cdots,\mu_r)_2,\cdots\right)$$

and

$$\vec{\alpha}=\left((\alpha_1,\cdots,\alpha_r)_0,(\alpha_1,\cdots,\alpha_r)_1,(\alpha_1,\cdots,\alpha_r)_2,\cdots\right).$$

$a_n=(\lambda_n,(\mu_1,\cdots,\mu_r)_n,(\alpha_1,\cdots,\alpha_r)_n)$ would be the action chosen by the system manager when the system is at state n.

We assume by convention that $(\mu_1,\cdots,\mu_r)_0=(\alpha_1,\cdots,\alpha_r)_0=0$.

A policy $\left(\vec{\lambda},\vec{\mu},\vec{\alpha}\right)$ is said to be ergodic if under this policy the process N_t has a unique steady state probability $\pi\left(\vec{\lambda},\vec{\mu},\vec{\alpha}\right)$ that satisfies the local balance equation.

We define the long run average generated under an ergodic policy $\left(\vec{\lambda},\vec{\mu},\vec{\alpha}\right)$ by:

$$Z\left(\vec{\lambda},\vec{\mu},\vec{\alpha}\right)=\sum_{n=0}^{\infty}\pi_n\left(\vec{\lambda},\vec{\mu},\vec{\alpha}\right)\left[b(\lambda_n)-c^1\left(\mu_1,\cdots,\mu_r\right)-c^2\left(\alpha_1,\cdots,\alpha_r\right)-h_n\right]. \tag{5.39}$$

We define the optimal long run average by:

$$Z^*=\sup_{\left(\vec{\lambda},\vec{\mu},\vec{\alpha}\right)} Z\left(\vec{\lambda},\vec{\mu},\vec{\alpha}\right), \tag{5.40}$$

where the supremum is taken over all ergodic policies.

A policy $\left(\vec{\lambda},\vec{\mu},\vec{\alpha}\right)$ is said to be optimal if $Z^*=Z\left(\vec{\lambda},\vec{\mu},\vec{\alpha}\right)$.

The system manager objective is to find an optimal policy in order to maximize the long run average.

We first consider that the system has a finite size, we denote it by K. We derive the optimal system size later.

5.5.2 Optimality equation

We assume that the system size is K, so the process N_t takes values in $\mathbb{S}=\{0,1,2,\cdots,K\}$.

By convention we assume that $\lambda_K=0$.

According to the optimality equations for a semi-Markov decision process problem with long run average cost criterion we have the following equations:

$$v_0=\sup_{\lambda\in[0,M]}\left\{\frac{b(\lambda)-z}{\lambda}+v_1\right\} \tag{5.41}$$

$$\begin{aligned} v_n=\sup_{(\lambda,\mu_1,\cdots,\mu_r,\alpha_1,\cdots,\alpha_r)\in A}&\frac{b(\lambda)-c^1\left(\mu_1,\cdots,\mu_r\right)-c^2\left(\alpha_1,\cdots,\alpha_r\right)-h_n-z}{\lambda+\left\{\sum_{l=1}^r\frac{p_l}{\mu_l+n(\mu_0+\alpha_l)}\right\}^{-1}} \\ &+\frac{\lambda}{\lambda+\left\{\sum_{l=1}^r\frac{p_l}{\mu_l+n(\mu_0+\alpha_l)}\right\}^{-1}}v_{n+1}+\frac{\left\{\sum_{l=1}^r\frac{p_l}{\mu_l+n(\mu_0+\alpha_l)}\right\}^{-1}}{\lambda+\left\{\sum_{l=1}^r\frac{p_l}{\mu_l+n(\mu_0+\alpha_l)}\right\}^{-1}}v_{n-1}, \\ &\forall n\in\{1,\cdots,K-1\} \end{aligned} \tag{5.42}$$

$$v_K=\sup_{(\mu_1,\cdots,\mu_r)\in[0,S]^r,(\alpha_1,\cdots,\alpha_r)\in[0,L]^r}\left\{\frac{-c^1\left(\mu_1,\cdots,\mu_r\right)-c^2\left(\alpha_1,\cdots,\alpha_r\right)-h_K-z}{\left\{\sum_{l=1}^r\frac{p_l}{\mu_l+K(\mu_0+\alpha_l)}\right\}^{-1}}+v_{K-1}\right\} \tag{5.43}$$

z is interpreted as a guess at the supremum average value.

Let us notice that in equation (5.41), $\frac{1}{\lambda}$ represents the expectation until to the next state which is 1, and the probability for the process to leave the state 0 and go to state 1 is $\frac{\lambda}{\lambda}=1$. In equation (5.42) $\frac{1}{\lambda+\left\{\sum_{l=1}^r\frac{p_l}{\mu_l+n(\mu_0+\alpha_l)}\right\}^{-1}}$ represents the expectation for the process to change state while at state n, $\frac{\lambda}{\lambda+\left\{\sum_{l=1}^r\frac{p_l}{\mu_l+n(\mu_0+\alpha_l)}\right\}^{-1}}$ represents the probability for the process to leave

state n and go to state $n+1$, and $\frac{\left\{\sum_{l=1}^{r}\frac{p_l}{\mu_l+n(\mu_0+\alpha_l)}\right\}^{-1}}{\lambda+\left\{\sum_{l=1}^{r}\frac{p_l}{\mu_l+n(\mu_0+\alpha_l)}\right\}^{-1}}$ the probability to leave state n and go to state $n-1$. In equation (5.43) $\frac{1}{\left\{\sum_{l=1}^{r}\frac{p_l}{\mu_l+K(\mu_0+\alpha_l)}\right\}^{-1}}$ is the expectation for the process to change state, the only state in which the process can go while at state K is state $K-1$ which has a probability equal to 1.

$v_0, v_1, v_2, \cdots, v_K$ are called relative value functions in average cost value, and are only determined up to an additive constant.

We define the relative value differences by:

$$y_n = v_{n-1} - v_n, \ \forall n \in \{1, \cdots, K\}.$$

Thus equations (5.41), (5.42), (5.43) can be rewritten as:

$$z = \sup_{\lambda\in[0,M]} \{b(\lambda) - \lambda y_1\} \tag{5.44}$$

$$z = \sup_{\lambda\in[0,M]} \{b(\lambda) - \lambda y_{n+1}\} \tag{5.45}$$

$$+ \sup_{(\mu_1,\cdots,\mu_r)\in[0,S]^r,(\alpha_1,\cdots,\alpha_r)\in[0,L]^r} \left\{\left\{\sum_{l=1}^{r}\frac{p_l}{\mu_l+n(\mu_0+\alpha_l)}\right\}^{-1} y_n - c^1(\mu_1,\cdots,\mu_r) - c^2(\alpha_1,\cdots,\alpha_r)\right\} - h_n$$

$$\forall n \in \{1, \cdots, K-1\}$$

$$z = \sup_{(\mu_1,\cdots,\mu_r)\in[0,S]^r,(\alpha_1,\cdots,\alpha_r)\in[0,L]^r} \left\{\left\{\sum_{l=1}^{r}\frac{p_l}{\mu_l+K(\mu_0+\alpha_l)}\right\}^{-1} y_K - c^1(\mu_1,\cdots,\mu_r) - c^2(\alpha_1,\cdots,\alpha_r)\right\} - h_K. \tag{5.46}$$

Let us define ζ and ϕ as follows:

$$\zeta(y) = \sup_{\lambda\in[0,M]} \{b(\lambda) - \lambda y\}, \quad \forall y \in \left[0, b'(0)\right] \tag{5.47}$$

$$\phi(y,n) = \sup_{(\mu_1,\cdots,\mu_r)\in[0,S]^r,(\alpha_1,\cdots,\alpha_r)\in[0,L]^r} \left\{\left\{\sum_{l=1}^{r}\frac{p_l}{\mu_l+n(\mu_0+\alpha_l)}\right\}^{-1} y - c^1(\mu_1,\cdots,\mu_r) - c^2(\alpha_1,\cdots,\alpha_r)\right\}, \tag{5.48}$$

$$\forall y \geq 0$$

Let us notice that ζ has already been studied in the case of one region in the previous section, and is assumed to be defined on $\left[0, b'(0)\right]$.

Equations (5.44), (5.45), and (5.46) can be rewritten as follows:

$$z = \zeta(y_1) \tag{5.49}$$

$$z = \zeta(y_{n+1}) + \phi(y_n, n) - h_n, \ \forall n \in \{0, \cdots, K-1\} \tag{5.50}$$

$$z = \phi(y_K, K) - h_K. \tag{5.51}$$

As established in the previous section, we denote by $\eta(y)$ the smallest maximizer of the supremum in the expression of $\zeta(y)$ such that:

$$\zeta(y) = b(\eta(y)) - \eta(y)y.$$

As by assumption c^1 and c^2 are continuous, so for a fixed y and n, the function $\left\{\sum_{l=1}^{r} \frac{p_l}{\mu_l + n(\mu_0+\alpha_l)}\right\}^{-1} y - c^1(\mu_1, \cdots, \mu_r) - c^2(\alpha_1, \cdots, \alpha_r)$ is continuous on $[0, S]^r \times [0, L]^r$, which is compact, then the maximizers in the expression of ϕ exist and take values in $[0, S]^r \times [0, L]^r$. We denote by $\psi(y, n) = \left(\psi^1(y, n), \psi^2(y, n)\right)$ the smallest maximizer in each component of $[0, S]$ and $[0, L]$, of the supremum in the expression of $\phi(y, n)$, where $\psi^1(y, n)$ and $\psi^2(y, n)$ take values in $[0, S]^r$ and $[0, L]^r$ respectively.

Now let us derive the analog of theorem 5.4.1.

Theorem 5.5.1 *If there exists an integer K, a real z and a vector $(y_1, y_2, \cdots, y_K)$ such that:*

- *z and $(y_1, y_2, \cdots, y_K)$ solve the optimality equations (5.49), (5.50), (5.51)*
- *$h_n + z \geq \phi(b'(0), n)$, $\forall n \geq K + 1$*
- *$0 \leq y_n \leq b'(0)$, $\forall n \in \{1, \cdots, K\}$*

Then $z \geq Z\left(\vec{\lambda}, \vec{\mu}, \vec{\alpha}\right)$, for all ergodic policies $\left(\vec{\lambda}, \vec{\mu}, \vec{\alpha}\right)$, thus $z \geq Z^*$.

Let there be the following policy $\left(\vec{\lambda^*}, \vec{\mu^*}, \vec{\alpha^*}\right)$ given by:

$$\lambda_n^* = \eta(y_{n+1}),\ \forall n \in \{0, \cdots, K-1\}$$

$$\lambda_n^* = 0,\ \forall n \geq K$$

$$(\mu_1^*, \cdots, \mu_r^*)_n = \psi^1(y_n, n),\ \forall n \in \{0, \cdots, K\}$$

$$(\mu_1^*, \cdots, \mu_r^*)_n = (S \cdots, S),\ \forall n \geq K + 1$$

$$(\alpha_1^*, \cdots, \alpha_r^*)_n = \psi^2(y_n, n),\ \forall n \in \{0, \cdots, K\}$$

$$(\alpha_1^*, \cdots, \alpha_r^*)_n = (L \cdots, L),\ \forall n \geq K + 1$$

If the policy $\left(\vec{\lambda^*}, \vec{\mu^*}, \vec{\alpha^*}\right)$ is ergodic, then it is optimal among all ergodic policies that truncate the system size to K:

$$Z\left(\vec{\lambda^*}, \vec{\mu^*}, \vec{\alpha^*}\right) = Z^*.$$

Proof Let us consider an arbitrary ergodic policy $\left(\vec{\lambda}, \vec{\mu}, \vec{\alpha}\right)$.

In order to simplify the notations, the action chosen by the system manger at state $n \in \mathbb{S}$, is denoted by $a_n = \left(\lambda_n, \mu_n^1, \cdots, \mu_n^r, \alpha_n^1, \cdots, \alpha_n^r\right)_n$.

Under the policy $\left(\vec{\lambda}, \vec{\mu}, \vec{\alpha}\right)$ the system has a steady state probability π satisfying the local balance equations:

$$\left\{\sum_{l=1}^{r} \frac{p_l}{\mu_n^l + n(\mu_0 + \alpha_n^l)}\right\}^{-1} \pi_n = \lambda_{n-1}\pi_{n-1},\ \forall n \geq 1$$

and

$$\sum_{n=0}^{\infty} \pi_n = 1.$$

By definition of ζ andϕ we have :

$$\zeta(y_{n+1}) \geq b(\lambda_n) - \lambda_n y_{n+1}, \ \forall n \in \{1, \cdots, K-1\} \tag{5.52}$$

$$\phi(y_n, n) \geq \left\{ \left\{ \sum_{l=1}^{r} \frac{p_l}{\mu_n^l + n\left(\mu_0 + \alpha_n^l\right)} \right\}^{-1} y_n - c^1\left(\mu_n^1, \cdots, \mu_n^r\right) - c^2\left(\alpha_n^1, \cdots, \alpha_n^r\right) \right\} \tag{5.53}$$

Using equation (5.50), and summing equations (5.52), (5.53), we get,

$$z + h_n = \zeta(y_{n+1}) + \phi(y_n, n)$$

$$\geq b(\lambda_n) - \lambda_n y_{n+1} + \left\{ \left\{ \sum_{l=1}^{r} \frac{p_l}{\mu_n^l + n\left(\mu_0 + \alpha_n^l\right)} \right\}^{-1} y_n - c^1\left(\mu_n^1, \cdots, \mu_n^r\right) - c^2\left(\alpha_n^1, \cdots, \alpha_n^r\right) \right\}.$$

Thus

$$z \geq b(\lambda_n) - \lambda_n y_{n+1} + \left\{ \left\{ \sum_{l=1}^{r} \frac{p_l}{\mu_n^l + n\left(\mu_0 + \alpha_n^l\right)} \right\}^{-1} y_n - c^1\left(\mu_n^1, \cdots, \mu_n^r\right) - c^2\left(\alpha_n^1, \cdots, \alpha_n^r\right) \right\} - h_n$$

Let us multiply both sides by π_n and let us sum over $n \in \{1, \cdots, K-1\}$, we obtain:

$$z \sum_{n=1}^{K-1} \pi_n \geq \sum_{n=1}^{K-1} \pi_n \left[b(\lambda_n) - c^1\left(\mu_n^1, \cdots, \mu_n^r\right) - c^2\left(\alpha_n^1, \cdots, \alpha_n^r\right) - h_n \right]$$

$$+ \sum_{n=1}^{K-1} \left(\pi_n y_n \left\{ \sum_{l=1}^{r} \frac{p_l}{\mu_n^l + n\left(\mu_0 + \alpha_n^l\right)} \right\}^{-1} - \lambda_n \pi_n y_{n+1} \right)$$

The equations of local balance give $\lambda_n \pi_n = \left\{ \sum_{l=1}^{r} \frac{p_l}{\mu_{n+1}^l + (n+1)\left(\mu_0 + \alpha_{n+1}^l\right)} \right\}^{-1} \pi_{n+1}$,

Thus

$$\sum_{n=1}^{K-1} \left(\pi_n y_n \left\{ \sum_{l=1}^{r} \frac{p_l}{\mu_n^l + n\left(\mu_0 + \alpha_n^l\right)} \right\}^{-1} - \lambda_n \pi_n y_{n+1} \right)$$

$$= \sum_{n=1}^{K-1} \left(\pi_n y_n \left\{ \sum_{l=1}^{r} \frac{p_l}{\mu_n^l + n\left(\mu_0 + \alpha_n^l\right)} \right\}^{-1} - \pi_{n+1} y_{n+1} \left\{ \sum_{l=1}^{r} \frac{p_l}{\mu_{n+1}^l + (n+1)\left(\mu_0 + \alpha_{n+1}^l\right)} \right\}^{-1} \right)$$

$$= \pi_1 y_1 \left\{ \sum_{l=1}^{r} \frac{p_l}{\mu_1^l + \left(\mu_0 + \alpha_1^l\right)} \right\}^{-1} - \pi_K y_K \left\{ \sum_{l=1}^{r} \frac{p_l}{\mu_K^l + K\left(\mu_0 + \alpha_K^l\right)} \right\}^{-1}$$

So we obtain

$$z \sum_{n=1}^{K-1} \pi_n \geq \sum_{n=1}^{K-1} \pi_n \left[b(\lambda_n) - c^1\left(\mu_n^1, \cdots, \mu_n^r\right) - c^2\left(\alpha_n^1, \cdots, \alpha_n^r\right) - h_n \right] \tag{5.54}$$

$$+\pi_1 y_1 \left\{ \sum_{l=1}^{r} \frac{p_l}{\mu_1^l + (\mu_0 + \alpha_1^l)} \right\}^{-1} - \pi_K y_K \left\{ \sum_{l=1}^{r} \frac{p_l}{\mu_K^l + K(\mu_0 + \alpha_K^l)} \right\}^{-1}$$

Similary by equations (5.49) and (5.51) and using the fact that $c^1\left(\mu_0^1, \cdots, \mu_0^r\right) = c^2\left(\alpha_0^1, \cdots, \alpha_0^r\right) = 0$ we have:

$$z\pi_0 \geq \pi_0 b(\lambda_0) - \pi_0 \lambda_0 y_1 - \pi_0 c^1\left(\mu_0^1, \cdots, \mu_0^r\right) - \pi_0 c^2\left(\alpha_0^1, \cdots, \alpha_0^r\right) \tag{5.55}$$

$$z\pi_K \geq \pi_K \left\{ \sum_{l=1}^{r} \frac{p_l}{\mu_K^l + K(\mu_0 + \alpha_K^l)} \right\}^{-1} y_K - \pi_K c^1\left(\mu_K^1, \cdots, \mu_K^r\right) - \pi_K c^2\left(\alpha_K^1, \cdots, \alpha_K^r\right) - \pi_K h_K \tag{5.56}$$

Summing equations (5.54),(5.55)and (5.56), and using the fact that $\pi_0\lambda_0 y_1 = \pi_1 \left\{ \sum_{l=1}^{r} \frac{p_l}{\mu_1^l + (\mu_0 + \alpha_1^l)} \right\}^{-1} y_1$ we obtain:

$$z\sum_{n=0}^{K} \pi_n \geq \sum_{n=1}^{K-1} \pi_n \left[b(\lambda_n) - c^1\left(\mu_n^1, \cdots, \mu_n^r\right) - c^2\left(\alpha_n^1, \cdots, \alpha_n^r\right) - h_n \right] \tag{5.57}$$

$$-\pi_K c^1\left(\mu_K^1, \cdots, \mu_K^r\right) - \pi_K c^2\left(\alpha_K^1, \cdots, \alpha_K^r\right) - \pi_K h_K$$

Let us consider the case of $n \geq K+1$.
By definition of ϕ we have $\forall n \geq K+1$:

$$\phi(b'(0), n) \geq \left\{ \left\{ \sum_{l=1}^{r} \frac{p_l}{\mu_n^l + n(\mu_0 + \alpha_n^l)} \right\}^{-1} b'(0) - c^1\left(\mu_n^1, \cdots, \mu_n^r\right) - c^2\left(\alpha_n^1, \cdots, \alpha_n^r\right) \right\}$$

The assumption in the theorem says that $h_n + z \geq \phi(b'(0), n)$, $\forall n \geq K+1$, then we obtain:

$$z \geq \phi(b'(0), n) - h_n, \ \forall n \geq K+1$$

Thus

$$z \geq \left\{ \sum_{l=1}^{r} \frac{p_l}{\mu_n^l + n(\mu_0 + \alpha_n^l)} \right\}^{-1} b'(0) - c^1\left(\mu_n^1, \cdots, \mu_n^r\right) - c^2\left(\alpha_n^1, \cdots, \alpha_n^r\right) - h_n, \ \forall n \geq K+1$$

Multiplying both sides of this inequality by π_n, for $n \geq K+1$ we obtain:

$$\pi_n z \geq \pi_n \left\{ \sum_{l=1}^{r} \frac{p_l}{\mu_n^l + n(\mu_0 + \alpha_n^l)} \right\}^{-1} b'(0) - \pi_n c^1\left(\mu_n^1, \cdots, \mu_n^r\right) - \pi_n c^2\left(\alpha_n^1, \cdots, \alpha_n^r\right) - \pi_n h_n, \ \forall n \geq K+1$$

This gives by using the equations of local balance:

$$\pi_n z \geq b'(0)\pi_{n-1}\lambda_{n-1} - \pi_n c^1\left(\mu_n^1, \cdots, \mu_n^r\right) - \pi_n c^2\left(\alpha_n^1, \cdots, \alpha_n^r\right) - \pi_n h_n, \ \forall n \geq K+1$$

By assumption b is concave, it then satisfies $b'(0)\lambda_{n-1} \geq b(\lambda_{n-1})$. Then we obtain:

$$\pi_n z \geq b(\lambda_{n-1})\pi_{n-1} - \pi_n c^1\left(\mu_n^1, \cdots, \mu_n^r\right) - \pi_n c^2\left(\alpha_n^1, \cdots, \alpha_n^r\right) - \pi_n h_n, \ \forall n \geq K+1 \tag{5.58}$$

Let us pose an integer N that is larger than K. Let us sum equation (5.58) over N and let us add it to equation (5.57), we obtain:

$$z\sum_{n=0}^{N}\pi_n \geq \sum_{n=0}^{N}\pi_n\left[b(\lambda_n) - c^1\left(\mu_n^1,\cdots,\mu_n^r\right) - c^2\left(\alpha_n^1,\cdots,\alpha_n^r\right) - h_n\right] - \pi_K b(\lambda_K) \tag{5.59}$$

Passing to the limit as $N \to \infty$, and using the fact that $\pi_N b(\lambda_N) \to 0$, it follows that:

$$z \geq \sum_{n=0}^{\infty}\pi_n\left[b(\lambda_n) - c^1\left(\mu_n^1,\cdots,\mu_n^r\right) - c^2\left(\alpha_n^1,\cdots,\alpha_n^r\right) - h_n\right] = Z\left(\vec{\lambda},\vec{\mu},\vec{\alpha}\right)$$

This implies that:

$$z \geq Z^*$$

Now let us consider the candidate policy $\left(\vec{\lambda^*},\vec{\mu^*},\vec{\alpha^*}\right)$ given in the theorem. So the inequality in equation (5.57) is an equality. For an ergodic policy that truncates the system size to K, we have $\sum_{n=0}^{K}\pi_n = 1$, and $b(\lambda_K) = 0$. Then we conclude:

$$z = \sum_{n=0}^{K}\pi_n^*\left[b(\lambda_n^*) - c^1\left(\mu_n^{*1},\cdots,\mu_n^{*r}\right) - c^2\left(\alpha_n^{*1},\cdots,\alpha_n^{*r}\right) - h_n\right]$$

This implies that the policies $\left(\vec{\lambda^*},\vec{\mu^*},\vec{\alpha^*}\right)$ is optimal among all ergodic policy that truncate the system size to K. ∎

Now let us derive somes properties of ϕ.

For an integer n, let us define a real function F_n defined on $[0,S]^r \times [0,L]^r$ by:

$$F_n\left(\mu_1,\cdots,\mu_r,\alpha_1,\cdots,\alpha_r\right) = \left\{\sum_{l=1}^{r}\frac{p_l}{\mu_l + n\left(\mu_0+\alpha_l\right)}\right\}^{-1}.$$

Proposition 5.5.2 *For each fixed $n \geq 1$, the function $\phi(\ ,n)$ is convex, differentiable almost everywhere with y, and we have:*

- $\phi^{'}(y,n) = F_n\left(\psi(y,n)\right)$,
- $\phi(y,n) = \int_0^y F_n\left(\psi(u,n)\right)du$

It follows that $\phi(\ ,n)$ is nondecreasing with y.

Proof Let $y_0 \geq 0$, and let $x_0 = \left(x_0^1,x_0^2\right) = \left(\psi^1(y_0,n),\psi^2(y_0,n)\right) = \psi(y_0,n)$, such that:

$$\phi(y_0,n) = F_n\left(\psi(y_0,n)\right)y_0 - c^1(x_0^1) - c^2(x_0^2)$$

Let an arbitrary $y \geq 0$,

$$\phi(y,n) \geq F_n\left(\psi(y_0,n)\right)y - c^1(x_0^1) - c^2(x_0^2)$$

Then

$$\phi(y,n) - \phi(y_0,n) \geq F_n\left(\psi(y_0,n)\right)\left(y-y_0\right),$$

This implies that ϕ is convex. A finite convex function is continuous and differentiable almost everywhere, and the above inequality implies that its derivative at y_0 is $F_n\left(\psi(y_0,n)\right)$.

Then we have the following Rieman-integral form:

$$\phi(y,n) = \int_0^y F_n\left(\psi(u,n)\right)du.$$

As F_n is non-negative, this implies that $\phi(\ n)$ is nondecreasing. ∎

5.5.3 Computation of the solution of the optimality equations

In this part we derive an explicit solution of the optimality equations.

If we have $\phi(b^{'}(0), 1) \leq h_1$ it follows under the above theorem that it is optimal to not accept any user.

In order to derive the optimal system size that is more than 0, we assume that

$$\phi(b^{'}(0), 1) > h_1. \tag{5.60}$$

In order to have a finite optimal system size, we assume that:

$$\lim_{n\to\infty} h_n \geq \phi(b^{'}(0), n). \tag{5.61}$$

Theorem 5.5.3 *There exists* $[a_n, b_n]$*, and the functions* $y_n(\ )$ *on* $[a_n, b_n]$ *, for* $n \in \langle 1, 2, \cdots, K+1\}$*, defined inductively as follows:*

- $0 = a_1 < a_2 < \cdots a_K < a_{K+1} < b_{K+1} < b_K < \cdots b_K < b_1 = b(M)$,
- $h_{n-1} + a_n - \phi(y_{n-1}(a_n), n-1) = 0$ *and* $h_{n-1} + b_n - \phi(y_{n-1}(b_n), n-1) = b(M)$, $\forall n \in \langle 2, \cdots, K+1\}$
- $y_n(z) = \zeta^{-1}\left(z + h_{n-1} - \phi(y_{n-1}(z), n-1)\right)$, $\forall z \in [a_n, b_n]$ *and* $\forall n \in \langle 2, \cdots, K+1\}$
- $y_n(\ )$ *is continuous and strictly decreasing with* $y_n(a_n) = b^{'}(0)$ *and* $y_n(b_n) = 0$, $\forall n \in \langle 1, \cdots, K+1\}$
- $h_n + a_n < \phi(b^{'}(0), n)$, $\forall n \in \langle 1, \cdots, K\}$
- $h_{K+1} + a_{K+1} \geq \phi(b^{'}(0), K+1)$

There exists an optimal system size given by

$$K = \max\left\{n \geq 1, h_n + a_n < \phi(b^{'}(0), n)\right\}.$$

Proof Let us notice that the last assertion in the theorem is a stopping condition.

We prove the first four assertions by induction. We assume that these assertions are true at step $K = 1$. Let us begin by proving that it is true for $K = 2$. So we first begin by constructing a_2 and b_2.

Let us define $f_1(z) = z + h_1 - \phi(y_1(z), 1)$, for $z \in [a_1, b_1]$.

As y_1 is assumed to be strictly decreasing by the step $K = 1$ of the induction, then it follows that f_1 is strictly increasing and continuous. Then by the assumption in equation (5.60), we have:

$$f_1(a_1) = h_1 - \phi(y_1(a_1), 1) < 0$$

and

$$f_1(b_1) = b(M) + h_1 - \phi(y_1(b_1), 1) = b(M) + h_1 > 0$$

Then by the intermediate value theorem there exists $a_2 \in [a_1, b_1]$ such that $f_1(a_2) = 0$. It follows that $a_2 + h_1 - \phi(y_1(a_2), 1) = 0$.

Let us pose $g(z) = f_1(z) - b(M)$, we have $g(a_2) = -b(M) < 0$ and $g(b_1) = f_1(b_1) - b(M) = h_1 > 0$, then by the intermediate value theorem there exists $b_2 \in [a_2, b_1]$ such that $g(b_2) = 0$. That is $h_1 + b_2 - \phi(y_1(b_2), 1) = b(M)$.

As $f_1(a_2) = 0$ and $f_1(b_2) = b(M)$, and f_1 is strictly increasing, then $f_1(z) \in [0, b(M)]\,,\ \forall z \in [a_2, b_2]\,.$

We can define y_2 as follows:

$$y_2(z) = \zeta^{-1}\left(z + h_1 - \phi(y_1(z), 1)\right),\ \forall z \in [a_2, b_2]$$

This is continuous and strictly decreasing, with $y_2(a_2) = \zeta^{-1}(0) = b^{'}(0)$ and $y_2(b_2) = \zeta^{-1}(b(M)) = 0$ since $\zeta(0) = b(M)$.

Then the induction is true at step $K = 2$.

Now let us assume that the induction is true at step $j - 1$.

If $h_j + a_j \geq \phi(b^{'}(0), j)$, then we set $K = j - 1$ and $a_n = a_{j-1}$, $\forall n > K + 1$, the induction terminates.

Otherwise we have $h_j + a_j < \phi(b^{'}(0), j)$.

In this case we define $f_j(z) = z + h_j - \phi(y_j(z), j)$, for $z \in [a_j, b_j]$. It follows that f_j is strictly increasing and continuous. We have:

$$f_j(a_j) = a_j + h_j - \phi(y_j(a_j), j) = a_j + h_j - \phi(b^{'}(0), j) < 0$$

and

$$f_j(b_j) = b_j + h_j - \phi(y_j(b_j), j) > b_j + h_{j-1} > b(M).$$

Then by the intermediate value theorem there exists $a_{j+1} \in [a_j, b_j]$ such that $f_j(a_{j+1}) = 0$.

It follows that $a_{j+1} + h_j - \phi(y_j(a_{j+1}), j) = 0$.

Let us pose $g(z) = f_j(z) - b(M)$, we have $g(a_{j+1}) = -b(M) < 0$ and $g(b_j) = f_j(b_j) - b(M) > 0$, then by the intermediate value theorem there exists $b_{j+1} \in [a_{j+1}, b_j]$ such that $g(b_{j+1}) = 0$. That is $h_j + b_{j+1} - \phi(y_j(b_{j+1}), j) = b(M)$.

As $f_j(a_{j+1}) = 0$ and $f_j(b_{j+1}) = b(M)$, and f_j is strictly increasing, then $f_j(z) \in [0, b(M)]\,,\ \forall z \in [a_{j+1}, b_{j+1}]\,.$

We can define y_{j+1} as follows

$$y_{j+1}(z) = \zeta^{-1}\left(z + h_j - \phi(y_j(z), j)\right),\ \forall z \in [a_{j+1}, b_{j+1}]\,.$$

This is continuous and strictly decreasing, with $y_{j+1}(a_{j+1}) = \zeta^{-1}(0) = b^{'}(0)$ and $y_{j+1}(b_{j+1}) = \zeta^{-1}(b(M)) = 0$.

This proves the induction at step j.

To continue at step $j + 1$ we have as above two cases. So we can define

$$K = \max\left\{n \geq 1, h_n + a_n < \phi(b^{'}(0), n)\right\}.$$

Assumption (5.61) allows us to say that K is finite.

By definition of K we have:

$$h_{K+1} + a_{K+1} \geq \phi(b^{'}(0), K + 1).$$

∎

We can now derive a construction of the optimal solution.

Theorem 5.5.4 *Let $z^* = a_{K+1}$, where K is the optimal system size provided by the previous theorem, and let $y_n^* = y_n(z^*)$, for $n \in \{1, \cdots, K\}$.*

Then z^* and $(y_1^*, \cdots, y_K^*)$ solve the optimal equations such that:

$$z^* = \zeta(y_1^*) \tag{5.62}$$

$$z^* = \zeta(y^*_{n+1}) + \phi(y^*_n, n) - h_n,\ \forall n \in \{0, \cdots, K-1\} \tag{5.63}$$

$$z^* = \phi(y^*_K, K) - h_K. \tag{5.64}$$

and $0 < y^*_n < b^{'}(0),\ n \in \{1, \cdots, K\}$.

Moreover, the following policy $\left(\vec{\lambda^*}, \vec{\mu^*}, \vec{\alpha^*}\right)$ *given by:*

$$\lambda^*_n = \eta(y_{n+1}),\ \forall n \in \{0, \cdots, K-1\}$$

$$\lambda^*_n = 0,\ \forall n \geq K$$

$$(\mu^*_1, \cdots, \mu^*_r)_n = \psi^1(y_n, n),\ \forall n \in \{0, \cdots, K\}$$

$$(\mu^*_1, \cdots, \mu^*_r)_n = (S \cdots, S),\ \forall n \geq K+1$$

$$(\alpha^*_1, \cdots, \alpha^*_r)_n = \psi^2(y_n, n),\ \forall n \in \{0, \cdots, K\}$$

$$(\alpha^*_1, \cdots, \alpha^*_r)_n = (L \cdots, L),\ \forall n \geq K+1$$

is optimal.

Proof Let us notice that $y_1(z) = \zeta^{-1}(z)$ defined on $[a_1, a_2]$. Then it follows that $\zeta(y^*_1) = \zeta(y_1(z^*)) = z^*$ which proves equation (5.62).

Next consider $n \in \{0, \cdots, K-1\}$, it follows by construction of y_{n+1} that:

$$y^*_{n+1} = y_{n+1}(z^*) = \zeta^{-1}\left(z^* + h_n - \phi(y_n(z^*), n)\right)$$

which implies equation (5.63).

For the last equation of optimality we have $y^*_{K+1} = y_{K+1}(a_{K+1}) = b^{'}(0)$, it follows that:

$$b^{'}(0) = \zeta^{-1}\left(z^* + h_K - \phi(y_K(z^*), K)\right)$$

Using $\zeta(b^{'}(0)) = 0$, it follows equation (5.64).

Now let us prove $0 < y^*_n < b^{'}(0)$.

Let us recall that $y_n(b_n) = 0$, and y_n is strictly decreasing, since $z^* = a_{K+1} < b_n$, then $y^*_n = y_n(z^*) > y_n(b_n) = 0\ \forall n \in \{1, \cdots, K\}$.

We recall that y_n is strictly decreasing, $a_{K+1} > a_n$, and $y_n(a_n) = b^{'}(0)$, it follows that $y^*_n = y_n(z^*) < y_n(a_n) = b^{'}(0)$.

From Theorem (5.5.3) and by definition of K, for $n \geq K+1$:

$$h_n + z^* \geq \phi(b^{'}(0), n)$$

All assumptions of Theorem (5.5.1) are verified, then z^* is optimal, and it follows that the given candidate policy is optimal. ∎

5.5.4 Numerical applications

In this part we do some numerical applications with a simple example. For this conisder a system consisting of two regions, in which we assume that the system manger cannot change the capacity in each region. In our numerical applications we assume that $\mu_1 = 32$ and $\mu_2 = 20$. As the system manager is not able to change μ_1 and μ_2, we assume that the cost associated to the change of capacity is null $c^1(\mu_1, \mu_2) = 0$ which satisfies all required assumptions about c^1.

We pose $b(\lambda) = \frac{\lambda}{1+\lambda}$ and $c^2(\alpha_1, \alpha_2) = \alpha_1^2 + \alpha_2^2$.

We can see that b is strictly concave, non-decreasing with $b(0) = 0$, then b satisfies all assumptions, and $b^{'}(0) = 1$.

The function c^2 is non-decreasing in each of its variables, convex, and C^2- continuous differentiable, so it verifies all required assumptions.

With some calculations we obtain:

$$\zeta(y) = (\sqrt{y} - 1)^2,$$

$$\zeta^{-1}(x) = (1 - \sqrt{x})^2,$$

$$\eta(y) = \frac{1}{\sqrt{y}} - 1,$$

Let us pose $h_n = n^3$, and assume that b is defined on $[0, 10]$.

In order to make an illustration of the recursive construction method given in theorem 5.5.3 and theorem 5.5.4, we pose $\mu_0 = 7$, for which the optimal system size K is equal to 3, a_n and b_n of theorem 5.5.3 are given in the table 5.4

Table 5.4: Theorem 5.5.3 results

Steps	a	b
1	0	0.9090
2	0.6021	0.7085
3	0.6311	0.6340
4	0.631229	

By theorem 5.5.4, we have $z^* = a_4 = 0.631229$. It follows that $y_1^* = 0.0422$, $y_2^* = 0.2187$ and $y_3^* = 0.5893$.

The optimal policy is given in the table 5.5

Table 5.5: Theorem 5.5.4 results

System state	λ	α_1	α_2
0	3.8679	0	0
1	1.1383	0.0071	0.0071
2	0.3027	0.0795	0.1440
3	0	0.3441	0.5521

In figures 5.5, 5.6 and 5.7 we plot the optimal system size as a function of impatience rate for three different congestion costs: $h_n = n^2$, $h_n = n^3$ and $h_n = n^4$ respectively. The figures show that the optimal system size increases when the impatience rate increases, and that the optimal system size decreases when the congestion cost increases.

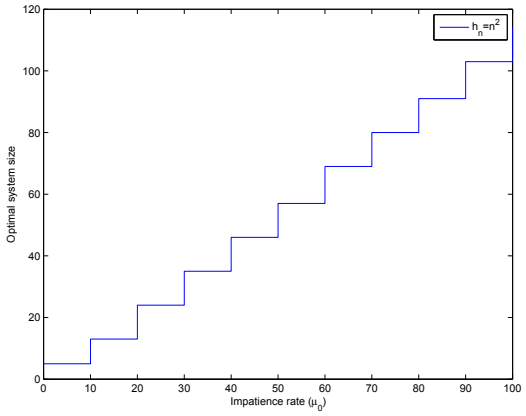

Figure 5.5: Optimal system size-$h_n = n^2$

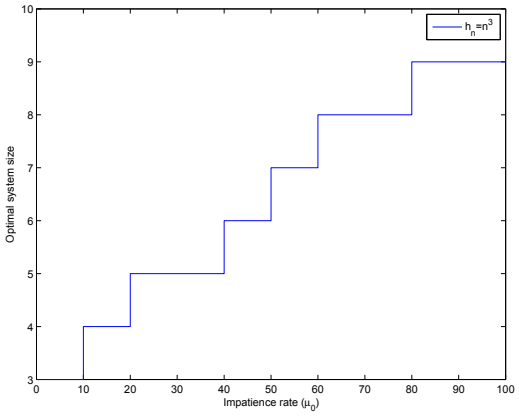

Figure 5.6: Optimal system size-$h_n = n^3$

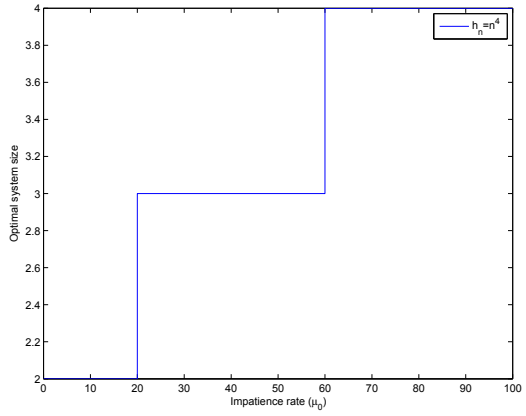

Figure 5.7: Optimal system size-$h_n = n^4$

5.6 Conclusion

We first modeled a system consisting of one cell with several regions taking into account user impatience where control is made through blocking and dropping. We used a Markov decision process framework to find an optimal policy through value iteration and modified iteration policy algorithms in order to reduce the impact of impatience on the system performance. We then developed another approach of control considering first the case of one region and then extended it to the case of multiple regions where set of actions is compact. We derived some results that allow us to find the optimal policy recursively and derive optimal system size for blocking. Our results showed that the optimal system size increases when the impatience rate increases.

Chapter 6

Conclusion and perspectives

We modeled in this thesis impatience of users in mobile cellular networks, with an application to 4G LTE, and quantified its impact on system performance in terms of several QoS parameters, such as mean transfer times, mean number of users in the cell and proportion of impatient users. We observed that impatience results in higher system stability region and lower mean transfer times, at the cost of higher number of users who quit the system before completing their file transfers and hence higher user non-satisfaction.

We also considered impatience of users who can move through different regions of a cell (intra cell cell mobility) and through different cells of the system (inter-mobility). We observed that mobility results in higher system stability, the more the users are moving the more their number decreases, and the more the probability of impatience decreases, and the more the throughput increases.

We modeled moreover impatience of streaming users during their video sessions in terms of probability of impatience during the prefetching phase and the rebuffering phase at the packet level and at the flow level for deterministic and exponential patience durations. We showed that the probability of impatience depends on the threshold of prefetching during prefetching and the threshold of rebuffering during rebuffering when starvation occurs, the more the threshold is high, the more the probability of impatience gets.

Eventually we modeled a system under control where control is made through blocking and dropping. We used a Markov decision process framework to find the optimal policy through value iteration and modified iteration policy algorithms in order to reduce the impact of impatience on the system performance. We developed a second aproach of control where the set of actions is compact. We obtained some results that allow us to find the optimal policy recursively and derive the optimal system size for blocking. Our results showed that the optimal system size increases when the impatience rate increases.

In chapter 1 we mainly modeled user impatience when the system is stable. An interesting perspective would be to consider user impatience when the system is in overload state. We could consider an eventual diffusion approximation taking into account the phenomenon of impatience.

In chapter 2 we modeled mobility of user subject to impatience. We considered a tagged user with a general flow size and we assumed that all other users have a exponential flow size. An interesting perspective would be to consider that all users have a general flow size.

In chapter 3 we modeled impatience of streaming users for exponential patience duration and deterministic patience duration case. An interesting perspective woud be to consider the case for general distribution of patience duration.

In chapter 4 we modeled a system under control in which the system is assumed to be stable. An interesting perspective would be to control the system when it is in overload. We could

consider a diffusion approximation in order to control the obtained diffusion process.

Bibliography

[1] C. Sanogo, T. Chahed and S.E. Elayoubi *Modeling user impatience and its impact on performance in mobile networks*, PIMRC'13 - Mobile and Wireless Networks, 8-11 Sept 2013.

[2] D. Barrer, *Queueing with impatient customers and ordered service*, Operations Research 5 (1957), 650-656.

[3] R.E. Stanford, *Reneging phenomena in single channel queues*, Mathematics of Operations Research 4 (1979), 162-178.

[4] T. Bonald and A. Proutiere, *Wireless downlink data channels*, ACM Mobicom 2003.

[5] L. Rong, S.E. Elayoubi and O. Ben Haddada, *Performance Evaluation of Cellular Networks Offering TV Services*, IEEE Transactions on Vehicular Technology, 2010.

[6] F. Baccelli and G. Hebuterne, *On queues with impatient customers*, Performance '81, pages 159-179. North-Holland, 1981.

[7] S.C. Yang and G. de Veciana, *Bandwidth Sharing: The Role of User Impatience*, Globecom'2001, November 2001.

[8] T. Bonald, J. Roberts, *Congestion at flow level and the impact of user behaviour*, Computer Networks, Vol. 42, pp. 521-536, 2003.

[9] M. Miyazawa. and G. Yamazaki, *The Basic Equations for a Supplemented GSMP and its Applications To Queues*, Journal of Applied Probability, 25 (1988), 565-578.

[10] M.K. Karray, *Analytical evaluation of QoS in the downlink OFDMA wireless cellular networks serving streaming and elastic traffic*, IEEE Trans. on Wireless Commun, 2010.

[11] Y.D. Xu, S.E. Elayoubi, E. Altman and R.E. Azouzi, *Impact of Flow-level Dynamics on QoE of Video Streaming in Wireless Networks*, IEEE INFOCOM, 2013.

[12] Y.D. Xu, E. Altman et. al, *Probabilistic Analysis of Buffer Starvation in Markovian Queues*, IEEE INFOCOM, 2012.

[13] S. Asmussen, *Ruin Probabilities*, World Scientific Pub.Vol 2, 2000.

[14] Y. Lu,S.M. Li, *On the probability of ruin in Markov-modulated risk model*, Insurance: Mathematics and Economics, No.37, pp:522-532, 2005.

[15] J. Boyer, F. Guillemin, P. Robert, B. Zwart, *Heavy tailed M/G/1-PS queues with impatience and admission control in packet networks*, IEEE INFOCOM 2003.

[16] H. Christian Gromoll, P. Robert, B. Zwart, R. Bakker, *The impact of Reneging in Processor Sharing Queues*, June 26-30, 2006,

[17] H. Christian Gromoll, P. Robert, B. Zwart, R. Bakker, *Fluid limits for processor sharing queues with impatience*, Inria-00001202 version 1-5 April 2006.

[18] O. Garnett, A. Mandelbaum, M. Reiman, *Designing a Call Center with Impatient Customers*, Manufacturing and service operations management Vol. 4, No. 3, Summer 2002.

[19] C. Palm, *Intensitatsschwankungen im Fernsprechverkehr*, Ericsson Technics, 44:1-189, 1943.(English translation by North-Holland, Amsterdam, 1988.)

[20] Z.X. Zhao, S.S. Panwar and D. Towsley *Queuing Performancen with Impatient Customers*, IEEE 1991.

[21] M. Grossglauser and D. Tse, *Mobility increases the capacity of ad-hoc wireless networks* in IEEE INFOCOM, 2001, pp. 1360-1369.

[22] S. Diggavi, M. Grossglauser and D. Tse, *Even one-dimensional mobility Mobility increases ad-hoc wireless capacity,* Laussane, Switzerland, June 2002.

[23] N. Bansal and Z. Liu, *Capacity, Delay and Mobility in Wireless Ad-Hoc Networks*, in IEEE INFOCOM 2003, San Francisco, CA, April 1-3 2003.

[24] T. Lui and P. Bahl, *Mobility Modeling, Location Tracking, and Trajectory Prediction in Wireless ATM Networks*, in IEEE journal on selected areas in communication, vol. 16, NO, 6, August 1998.

[25] T. Bonald, S.C. Borst and A. Proutiere, *How Mobility Impacts the Flow-Level Performance of Wireless Data Systems*, In: Proc. IEEE Infocom 2004.

[26] S. Borst, A. Proutiere, and N. Hegde, *Capacity of Wireless Data Networks with Intra and Inter-Cell Mobility*, in IEEE Communications Socoety subject matter experts for publication in the proceedings IEEE Infocom, April 2006.

[27] K. Lee and V. C.M. Leung, *Capacity Planning for Group-Mobility Users in OFDMA Wireless Networks*, in EURASIP journal on Wireless Communications and Networking, Volume 2006, Article ID 75820, Pages 1-12.

[28] N.O. Oyie, P.K. Lang'at and S. Musyoki, *Femtocell Cluster-Based Mobility-Aware Resource Allocation Scheme OFDMA Network*, in Proceeding of 2014 International Conference on Sustainable Research and Innovation, Volume 5, 7th-9th May 2014.

[29] B. Sas, K. Spaey and C. Blondia, *Classifying Users Based on Their Mobility Behavior in LTE Networks*, in The Tenth International Conference on Wireless and Mobile Communications, ICWMC 2014.

[30] S.M. Mousavi, H.R. Rabiee, M. Moshref and A. Dabirmoghaddam, *Mobility Pattern Recognition in Mobile Ad-Hoc Networks*, in MC'o7(Mobility'07), September 10-12,2007, Singapore.

[31] J.C. Cano, C. Calafate and P. Manzoni, *Modeling of mobility and groups in inter-vehicular MANET-based networks*, IEEE, C-K, Toh Senior Member, IEEE, 2007.

[32] C. Bettstetter, *Mobility Modeling in Wireless Networks: Categorization, Smooth Movement, and Border Effects*,Mobile Computing and Communications Review, Volume 5, Number 3, July 2001.

[33] H. Abou-Zeid, H.S. Hassanein and R. Atawia, *Towards Mobility-Aware Predictive Radio Access: Modeling, Simulation, and Evaluation in LTE Networks* Network Architecture and Design-Wireless Communication, C.25, 2014.

[34] P. Gurusanthosh, A. Rostami and R. Manivasakan, *SDMA:A Semi-Distributed Mobility Anchroring in LTE Networks*, 2013 International Conference on Selected Topics in Mobile and Wireless Networking (MoWNeT), pages 133-139, 19-21 August 2013.

[35] A.R. Ward and P.W. Glynn, *A Diffusion Approximation for a Markovian Queue with Reneging* In Queueing Systems 43,103-128, 2003 Kluwer Academic Publishers.

[36] D. Yue, W. Yue and X. Li, *Analysis of a Two-Phase Queueing System with Impatient Customers and Multiple Vacations* The Tenth International Symposium on Operations Research and Its Applications (ISORA 2011), pp. 292-298, August 28-31, 2011.

[37] N. Perel and U. Yechiali, *Queues with slow servers and impatient customers*In European Journal of Operational Research 201 (2010) 247-258.

[38] R. Nobel and S.V.D. Ster, *A Discrete-Time Queueing Model with Abandonments* in AENORM vol.18(68) october 2010.

[39] Y.M. Ko and N. Gautam, *Critically loaded multi-server queues with abandonments, retrials, and time-varying parameters* arXiv:0922.2436v1, 12 Nov 2009.

[40] S. Weber and G. Veciana, *Flow-Level QoS for a Dynamic Load of Rate Adaptive Sessions Sharing a Bottleneck Link* Computer Networks, volume 51, issue 8, pages 1981-1997, 6 June 2007.

[41] T. Bonald, S. Elayoubi and Y. Lin G. Veciana, *A Flow-Level Performance Model for Mobile Networks Carrying Adaptive Streaming Traffic* HAL archives-ouvertes.fr, 17 Dec 2015.

[42] A. Mansy, M. Fayed, M. Ammar and G. Veciana, *Networks-layer Fairness for Adaptive Video Streams*, IFIP Networking Conference, 20-22 May 2015.

[43] W. Ashmawi, R. Guerin, S. Wolf and M. Pinson, *On the Impact of Policing and Rate Guarantees in Diff-Serv Networks: A Video Streaming Application Perspective* Association for Computing Machinery, 27-31 August 2001.

[44] M.Z. Shafiq, J. Erman, L. Ji, A. Liu, J. Pang and J. Wang, *Understanding the Impact of Network Dynamics on Mobile Video User Engagement* The 2014 ACM international conference on Measurement and modeling of computer systems, pages 367-379, 16-20 June 2014.

[45] W. Song and W. Zhuang, *Performance Analysis of Probabilistic Multipath Transmission of Video Streaming Traffic over Multi-radio Wireless Devices* Wireless Communications, IEEE Transactions on Volume 11 issue 4, pages 1554-1564, 20 Feb 2012 .

[46] S.S. Krishnan and R. Sitaraman, *Video Stream Quality Impact Viewer Behavior: Inferring Causality Using Quasi-Experimental Designs* IMC'12, Boston Massachusetts,USA, 14-16 November 2012.

[47] R. Sitaraman, *Network Performance: Does It Really Matter To Users And By How Much?*, 2013 IEEE.

[48] K.W. Hwang, V. Gopalakrishnan, R. Jana, S. Lee, V. Misra and K.K. Ramakrishnan *Abandonment and its impact on P2P VoD streaming* 9-11 Sept 2013 IEEE.

[49] F. Dobrian, V. Sekar, A. Awan, I. Stoica, D. Joseph, A. Ganjam, J. Zhan and H. Zhang *Understanding the impact of video quality on user engagement.*, In Proceedings of the ACM SIGCOMM Conference on Applications, Technologies, Architectures, and Protocols for Computer Communication, pages 362-373, New York, NY, USA, 2011. ACM.

[50] M.A. Hoque, M. Siekkinen and J.K. Nurminen *Using Crowd-Sourced Viewing Statistics to Save Energy in Wireless Video Streaming* MobiCom'13 Sept 30-October 04 2013, Miami, FI. USA.

[51] D.P. Bertsekas *Dynamic Programming and Optimal Control*, Vol. 2. Athena Scientific, 4th edition, Belmont, MA, June 18, 2012.

[52] J.M. George and J.M. Harrison *Dynamic control of a queue with adjustable service rate*, Operations Research, vol 49, pp. 720-731, 23 May 2000.

[53] M.L. Puterman *Markov Decision Processes.* Wiley-Interscience, New York, 1994.

[54] B. Ata and S. Shneorson *Dynamic Control of an M/M/1 Service System with Adjustable Arrival and Service Rates.* 2006 INFORMS.Vol. 52, No. 11, November 2006, pp. 1778-1791.

[55] S. Karlin and H. M. Taylor *A First Course in Stochastic Processes.* 2nd ed. Academic Press, New York, April 11, 1975.

[56] D.W. Low *Optimal dynamic pricing policies for an M/M/s queue.* Oper.Res.22545-561, June 1, 1974.

[57] C. Maglaras and A. Zeevi *Pricing and capacity sizing for sys- tems with shared resources: Approximate solutions and scaling relations.* Management Sci. 49 1018-1038. 2003.

[58] S. Chafik and C. Daoui *A Modified Policy Iteration Algorithm for Discounted Reward Markov Decision Processes.* International journal of computer applications (0975-8887. Volume 133-No.10, January 2016.

[59] E.D. Vecchia, S.D. Marco and A.J. Marie *Illustrated review of convergence conditions of the value iteration algorithm and the rolling horizon procedure for average-cost MDPs.* INRIA, Rapport de recherche no. 7710, August 2011.

[60] M.A. Alsheikh, D.T. Hoang, D. Niyato, H.P. Tan and S. Lin *Markov Decision Processes with Applications in Wireless Sensor Networks: A Survey.*arXiv:1501.00644v1, January 2015.

[61] M.J. Sobel *Making short-run changes in production when employment level is fixed.*Opns Res.18, 35-51, 1970.

[62] D.J. White *An example of loosely coupled stages in dynamic programming,* Mgmi Sci.19, 739-746, 1973.

[63] A. Mandelbaum and U. Yechiall *Optimal entering rules for a customer with wait option at an M/G/1 queue,* Mgmi Sci.29, 174-187, 1983.

[64] W. Whitt *Deciding which queue to join: Some counterexamples,* Informs Operations Research, vol.34, No.1, 55-62. Jan-Feb, 1986.

[65] S.A. Lippman and S. Stidham *Individual versus social optimizationin exponential congestion systems,* Operations Research 25, 1977.

[66] E. Hyytia, A. Penttinen, S. Aalto and J. Virtamo *Dispatching problem with fixed size jobs and processor sharing discipline.*in 23rd Int. Teletraffic Congress (ITC'23), San Fransisco, USA, Sep. 2011, pp. 190-197.

[67] H. Mendelson and U. Yechiali *Controlling the GI/M/1 queue by conditional acceptance of customers.*European Journal of Operational Research. Volume 7, issue 1, May 1981, pages 77-85.

[68] D. Hong and S. Rappaport *Traffic model and performance analysis for cellular mobile radio telephone systems with prioritized handoff procedures,* IEEE Transactions on Vehicular Technology 35(3), September 1986.

[69] E.D. Re, R. Fantacci and G. Giambene *Performance evaluation of different resource management strategies in mobile cellular networks,* Journal Telecommunications Systems, Volume 12 Issue 4, December 1999 Pages 315-340.

[70] R.K. Samanta, P. Bhattacharjee and G. Sanyal *Performance Analysis of Cellular Wireless Network by Queuing Priority Handoff calls,* International Journal of Electrical and Electronics Engineering 3:8 2009.

[71] Z.H. Zheng and W.H. Lam *Performance Analysis of dynamic channel assignment queuing ang guard chnnel combined scheme for handoff prioritisation,* Electronics letters 5th December 2002 Vol. 38 No.2.

[72] K.S. Trivedi, S. Dharmaraja and X. Ma *Analytic modeling of handoffs in wireless cellular networks,* Information Sciences 148 (2002) 155-166.

[73] O.H. Lerma and J.B. Lasserre *Fatou's lemma and Lebesgue's convergence theorem for measures.* Journal of Applied Mathematics and Stochastics Analysis, pp.137-146, January 2000.