



HAL
open science

Limit theorems for Markov walk conditioned to stay positive

Ronan Lauvergnat

► **To cite this version:**

Ronan Lauvergnat. Limit theorems for Markov walk conditioned to stay positive. Probability [math.PR]. Université de Bretagne Sud, 2017. English. NNT : 2017LORIS451 . tel-01718390

HAL Id: tel-01718390

<https://theses.hal.science/tel-01718390>

Submitted on 27 Feb 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



THESE / UNIVERSITE DE BRETAGNE-SUD
sous le sceau de l'Université Bretagne Loire

pour obtenir le titre de
DOCTEUR DE L'UNIVERSITE DE BRETAGNE-SUD

Mention : Mathématiques

**Ecole doctorale: Santé, Information, Communication,
Mathématiques, Matière**

Présentée par Ronan Lauvergnat

Préparée à l'unité mixte de recherche : UMR CNRS 6205

Etablissement de rattachement : Université de Bretagne Sud

Nom développé de l'unité : Laboratoire de Mathématiques de
Bretagne Atlantique

Théorèmes limites pour des marches aléatoires markoviennes conditionnées à rester positives

Thèse soutenue à Vannes le 8 septembre 2017
devant le jury composé de :

Philippe BOUGEROL
Professeur émérite, Université de Paris VI / Rapporteur

Ion GRAMA
Professeur, Université de Bretagne Sud / Directeur de thèse

Yves GUIVARC'H
Professeur émérite, Université de Rennes 1 / Examineur

Quansheng LIU
Professeur, Université de Bretagne Sud / Examineur

Emile LE PAGE
Professeur émérite, Université de Bretagne Sud / Directeur de thèse

Marc PEIGNÉ
Professeur, Université François Rabelais / Examineur

Françoise PÈNE
Professeure, Université de Bretagne Occidentale / Examinatrice

Vitali WACHTEL
Professor, Institut für Mathematik, Universität Augsburg / Rapporteur

Théorèmes limites pour des marches aléatoires markoviennes conditionnées à rester positives

Résumé

Sur un espace probabilisé $(\Omega, \mathcal{F}, \mathbb{P})$, on se munit d'une chaîne de Markov $(X_n)_{n \geq 0}$ à valeurs dans un espace mesurable abstrait \mathbb{X} . Pour tout point initial de la chaîne $x \in \mathbb{X}$, on désigne par \mathbb{P}_x la probabilité engendrée par les lois fini-dimensionnelles de la chaîne lorsqu'elle est issue de $X_0 = x$. On fixe une fonction réelle $f : \mathbb{X} \rightarrow \mathbb{R}$ et on construit la marche aléatoire associée $S_n = \sum_{k=1}^n f(X_k)$, $n \geq 1$. Pour tout point de départ $y \in \mathbb{R}$ de la marche, on définit le premier temps τ_y pour lequel la marche markovienne $(y + S_n)_{n \geq 1}$ sort de la demi-droite des réels positifs \mathbb{R}_+^* . L'objectif de la présente thèse est d'établir sous des hypothèses assez générales sur la chaîne de Markov $(X_n)_{n \geq 0}$, l'asymptotique de la probabilité de survie de la marche markovienne $\mathbb{P}_x(\tau_y > n)$. On montre également que la loi de la marche markovienne $(y + S_n)_{n \geq 1}$ renormalisée lorsqu'elle est conditionnée à rester positive $\{\tau_y > n\}$ est donnée asymptotiquement par la loi de Rayleigh : $\lim_{n \rightarrow +\infty} \mathbb{P}_x(y + S_n \leq t\sqrt{n}\sigma \mid \tau_y > n) = 1 - e^{-t^2/2}$, $t \in \mathbb{R}$.

Lorsque l'espace \mathbb{X} est fini, on va plus loin et l'on donne des théorèmes locaux pour la marche markovienne conjointement avec le fait qu'elle reste positive. On détermine en particulier $\lim_{n \rightarrow +\infty} \mathbb{P}_x(y + S_n \in [a, b], \tau_y > n)$, où $[a, b]$ est un intervalle fixé de \mathbb{R} .

Enfin, on s'intéresse aux processus de branchement $(Z_n)_{n \geq 0}$ critiques ou sous-critiques soumis à un environnement markovien $(X_n)_{n \geq 0}$ lorsque l'espace d'états \mathbb{X} est fini. On établit les comportements asymptotiques de la probabilité de survie de tels processus de branchement $\mathbb{P}_x(Z_n > 0)$.

Pour information, au début de chaque chapitre, les notations sont redéfinies et les résultats préliminaires sont rappelés. En conséquence la lecture de chacun des chapitres peut se faire indépendamment des autres.

Limit theorems for Markov walks conditioned to stay positive

Abstract

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider a Markov chain $(X_n)_{n \geq 0}$ taking its values in a measurable space \mathbb{X} . For any initial point $x \in \mathbb{X}$ of the Markov chain, let \mathbb{P}_x be the probability generated by the finite dimensional distributions of the Markov chain starting at $X_0 = x$. Fix a real function $f: \mathbb{X} \rightarrow \mathbb{R}$ and consider the associated Markov walk $S_n = \sum_{k=1}^n f(X_k)$, $n \geq 1$. For any starting point $y \in \mathbb{R}$ of the walk, we define the first time τ_y when the Markov walk $(y + S_n)_{n \geq 1}$ exit the real half-line \mathbb{R}_+^* . In this thesis, we establish under general assumptions on the Markov chain $(X_n)_{n \geq 0}$, the asymptotic behaviour of the survival probability of the Markov walk $\mathbb{P}_x(\tau_y > n)$. We prove also that the law of the renormalized Markov walk $(y + S_n)_{n \geq 1}$ conditioned to stay positive $\{\tau_y > n\}$ is given asymptotically by the Rayleigh law: $\lim_{n \rightarrow +\infty} \mathbb{P}_x(y + S_n \leq t\sqrt{n}\sigma \mid \tau_y > n) = 1 - e^{-t^2/2}$, $t \in \mathbb{R}$.

When the space \mathbb{X} is finite, we go further and give local theorems for the Markov chain conjointly with the fact that the Markov walk stays positive. We determine in particular $\lim_{n \rightarrow +\infty} \mathbb{P}_x(y + S_n \in [a, b], \tau_y > n)$, where $[a, b]$ is a fix interval of \mathbb{R} .

Finally, we care about critical and subcritical branching processes $(Z_n)_{n \geq 0}$ under a Markov environment $(X_n)_{n \geq 0}$ when the state space \mathbb{X} is finite. We establish the asymptotic behaviours of the survival probability of such branching processes $\mathbb{P}_x(Z_n > 0)$.

For information, at the beginning of each chapter, notations are redefined and preliminary results are recalled. Consequently, each chapter should be read independently of the others.

Remerciements

Au delà du traditionnel usage qu'on lui attribue, ce paragraphe est pour moi une réelle opportunité de pouvoir exprimer ma très grande reconnaissance aux nombreuses personnes charitables et enthousiastes qui ont rendu cette thèse possible et même agréable. C'est avec une profonde sincérité que je souhaite remercier mon directeur de thèse. Ion, vous m'avez accueilli sans hésiter en stage en licence d'abord puis à nouveau en master. C'est encore vous qui m'avez fortement conseillé de continuer en thèse et c'est sans conteste grâce à votre générosité et votre enthousiasme que je me suis lancé dans ce doctorat pendant lequel j'ai pu mener des travaux au delà de mes ambitions initiales. Votre passion pour votre travail est vraiment admirable. Vos compliments et votre intérêt, très prononcé et toujours joyeux, pour mes questions m'ont permis à bien des égards et sans doute plus que vous ne l'imaginez, de trouver la motivation nécessaire pour continuer à avancer dans mes travaux. Enfin c'est aussi grâce à vous si j'ai pu faire la connaissance d'Emile à qui je souhaite également exprimer ma très vive gratitude. Emile, votre encadrement, nos journées entières passées tous les trois dans la salle de réunion et nos appels téléphoniques réguliers ont insufflé une nouvelle et forte dynamique dans ma thèse. L'étendue et la qualité de votre savoir m'impressionne toujours autant et j'apprécie grandement votre bienveillance naturelle. Merci donc à tous les deux pour avoir rendu cette thèse possible.

Merci aux Professeurs Yves Guivarc'h, Quansheng Liu, Marc Peigné et Françoise Pène pour avoir rapidement accepté de faire partie de mon jury. Merci en particulier aux Professeurs Philippe Bougerol et Vitali Wachtel pour avoir gentiment rapporté mon manuscrit un peu volumineux. Je suis très fier de soutenir devant un jury pour lequel j'ai autant d'estime.

Merci à chacun de mes joyeux compagnons doctorants qui ont rendu le quotidien plein de vie! A Eric, mon double professionnel qui m'a bien aidé de ses conseils à mes débuts (moi aussi un jour j'aurai une prépa ;)), à Jonathan, Salim (et hop Salim te voilà redevenu doctorant) et nos dimanches marvelisés, à Tarik (je vais bientôt te devoir un resto chinois...), à Cuong et ses très beaux progrès en français, à Phu, toujours délicat et attentionné, à KeV qui m'a nourri en livre (très important!) et à nos petites sorties sur Vannes (je te dois toujours deux verres), à Jamjam et à tous nos délires, notamment nos parties de pétanque-miniature à 20h dans les couloirs, à Thao avec qui nous avons affronté ensemble l'épistémologie des sciences, à Du et son romantisme débordant ;), à Thuy et son éternelle bonne humeur, à Xiao et sa sincérité.

Merci également à tous les membres du laboratoire qui m'ont chaleureusement accueilli et avec qui j'ai pu avoir de très beaux échanges. Il y a dans cette université à taille humaine un cadre agréable qui m'a permis de travailler très sereinement. Mais que serait ce cadre sans l'extraordinaire foot du vendredi? Merci énormément Sylvain pour toutes nos parties. Je garderai toujours en mémoire tous les détails qui rendaient ces matchs si amusants : notamment les têtes de JFK, les arrêts de Tini, la rapidité de Barath, les

exploits de Di-Maria, les sprints de Rachid, le tricotage de Romain et j'en passe !

Merci également à tous les étudiants qui ont dû me supporter et avec qui, comme le dit si bien Binet, mes rapports furent aussi divers qu'enrichissants. J'ai eu beaucoup de chance de pouvoir enseigner à l'UBS à Vannes avec un public très appréciable et les heures passées en classe ont toujours, toujours été agréables.

Merci à tous mes amis proches que j'ai parfois un peu négligés sous le prétexte de travailler ma thèse. A Manon, Dada, Vieille Branche, Réminou, Sist, Mon Petit Pote, Je sais qu'ils me pardonneront car ils sont formidables et je souhaite leur exprimer ma profonde reconnaissance.

Enfin et surtout, je tiens à remercier et à exprimer l'immense affection que je porte à ma famille. A toute la cousinade bien entendu mais en particulier à mes deux grandes soeurs Didine et Célia qui je sais, veillent toujours sur moi, à mon beau-frère Cédric qui fait partie de la famille (qu'il le veuille ou non), à mon petit frère Axel, mon complice de toujours, à mes neveux, Eliott et Milo, mes petits princes et enfin à mes parents à qui je dois tout (au moins).

Table des matières

1	Introduction	1
1.1	Contexte	1
1.2	Présentation des objectifs	2
1.3	Résultats antérieurs	4
1.3.1	Le cas brownien	4
1.3.2	Le cas indépendant unidimensionnel	5
1.3.3	Les premiers modèles markoviens	6
1.3.4	Les processus de branchement en environnement aléatoire	8
1.4	Présentation des travaux de thèse	10
1.4.1	Les modèles markoviens considérés	11
1.4.2	Existence d'une fonction harmonique	14
1.4.3	Positivité de la fonction harmonique	17
1.4.4	Théorèmes limites pour des marches markoviennes	19
1.4.5	Théorème local pour des marches markoviennes finies	21
1.4.6	Processus de branchement en environnement markovien	24
1.5	Rappels sur les marches indépendantes	25
1.5.1	Démonstration du Théorème 1.5.1	26
1.5.2	Quelques lemmes d'analyse	35
2	Limit theorems for affine Markov walks conditioned to stay positive	39
2.1	Introduction	39
2.2	Notations and results	41
2.3	Martingale approximation	45
2.4	Bound on the expectation of the killed martingale	47
2.4.1	Preliminary results	47
2.4.2	Bound on the expectation of the killed martingale: the case $\mathbb{E}(a) \geq 0$	49
2.4.3	Bound on the expectation of the killed martingale: the case $\mathbb{E}(a) < 0$	53
2.5	Existence of the harmonic function	56
2.5.1	Existence of the harmonic function: the case $\mathbb{E}(a) \geq 0$	56
2.5.2	Existence of the harmonic function: the case $\mathbb{E}(a) < 0$	58
2.6	Asymptotic for the exit time	63
2.6.1	Auxiliary statements	63
2.6.2	Proof of the claim 2 of Theorem 2.2.2	69
2.6.3	Proof of the claim 1 of Theorem 2.2.2	71
2.6.4	Proof of Corollary 2.2.3	72
2.7	Asymptotic for conditioned Markov walk	72
2.8	The case of non-positive initial point	74
2.9	Appendix	77

2.9.1	Proof of the fact Condition 2.3bis implies Condition 2.3	77
2.9.2	Convergence of recursively bounded monotonic sequences	77
2.9.3	Results on the Brownian case and strong approximation	79
2.9.4	Finiteness of the exit times	80
3	Limit theorems for Markov walks conditioned to stay positive under a spectral gap assumption	83
3.1	Introduction	83
3.2	Main results	86
3.3	Applications	93
3.3.1	Affine random walk in \mathbb{R}^d conditioned to stay in a half-space	93
3.3.2	Two components Markov chains in compact sets under the Doeblin-Fortet condition	94
3.3.3	Markov chains in compact sets under spectral gap assumptions	95
3.4	Preliminary statements	96
3.4.1	Results for the Brownian motion	96
3.4.2	Strong approximation	97
3.5	Martingale approximation and related assertions	97
3.6	Integrability of the killed martingale	102
3.7	Existence and properties of the harmonic function	110
3.8	Positivity of the harmonic function	113
3.9	Asymptotic behaviour of the exit time	122
3.9.1	Preliminary results	122
3.9.2	Proof of Theorem 3.2.3	130
3.9.3	Proof of Theorem 3.2.4	132
3.10	Asymptotic behaviour of the conditioned walk	133
3.11	Appendix: proofs for affine random walks in \mathbb{R}^d	135
3.12	Appendix: proofs for compact Markov chains	142
3.13	A Banach space for the product of matrices	143
3.13.1	Notations	144
3.13.2	The Banach space	145
3.13.3	Proof of M3.1	145
3.13.4	Proof of M3.2 and M3.3	147
3.13.5	Proof of M3.4	153
3.13.6	Proof of M3.5	155
3.13.7	Results	155
3.14	Asymptotic law of the couple $(X_n, S_n)_{n \geq 0}$ conditioned	157
4	Conditioned local limit theorems for random walks defined on finite Markov chains	165
4.1	Introduction	165
4.2	Notations and results	168
4.3	Properties of the dual Markov chain	172
4.4	The perturbed operator	174
4.5	A non asymptotic local limit theorem	178
4.6	Auxiliary bounds	186
4.7	Proof of Theorem 4.2.4	187
4.7.1	Control of E_1	188

4.7.2	Control of E_2	194
4.7.3	Proof of Theorem 4.2.4	198
4.8	Proof of Theorem 4.2.5	199
4.8.1	Preliminary results	199
4.8.2	Proof of Theorem 4.2.5.	203
4.9	Proof of Theorems 4.2.7 and 4.2.8	204
4.9.1	Preliminaries results.	204
4.9.2	Proof of Theorem 4.2.7.	211
4.9.3	Proof of Theorem 4.2.8.	212
4.10	Appendix	214
4.10.1	The non degeneracy of the Markov walk	214
4.10.2	Strong approximation	217
5	The survival probability of critical and subcritical branching processes in finite state space Markovian environment	219
5.1	Introduction	219
5.2	Notations and main results	222
5.3	Preliminary results on the Markov walk	226
5.3.1	The link between the branching process and the associated Markov walk	227
5.3.2	The dual Markov walk	229
5.3.3	Markov walks conditioned to stay positive	231
5.3.4	The change of measure related to the Markov walk	235
5.4	Proofs in the critical case	239
5.5	Proofs in the strongly subcritical case	245
5.6	Proofs in the intermediate subcritical case	248
5.7	Proofs in the weakly subcritical case	254

A Maman,
qui m'affirma que si elle gagnait au loto,
elle s'achèterait bien un volcan.

A Papa,
qui m'expliqua la topologie de l'univers,
avec la queue de mon marsupilami.

Chapitre 1

Introduction

1.1 Contexte

Qu'elles soient dites simples ou à valeurs dans des groupes plus exotiques, les marches aléatoires occupent une place largement privilégiée dans le vaste paysage des probabilités. Formalisme direct de nombreux problèmes concrets (trajectoire d'une particule, évolution d'un capital d'un joueur...), la marche aléatoire peut également apparaître comme un processus plus abstrait mais dont l'étude renseigne de façon cruciale sur le modèle de départ. C'est le cas notamment des processus de branchement en environnement aléatoire qui font l'objet d'un chapitre de ce manuscrit. Cependant, au-delà même de leur immense champ d'application, les marches aléatoires ont ceci de très intéressant : elles soulèvent dans une formulation assez simple des questions non triviales dont la résolution exige de développer des outils et des techniques un peu élaborés. Un premier exemple scolaire est la ruine du joueur dont l'énoncé très simple requiert dans sa résolution l'appareillage des martingales et du théorème de Doob.

La généralisation de l'asymptotique de la probabilité de survivre à n'importe quelle marche aléatoire réelle centrée avec un moment d'ordre 2, dont l'énoncé reste élémentaire, a été démontrée en 1960 par Spitzer [65] comme conséquence de lemmes d'analyse et de combinatoire [64]. Il précise que la loi du processus conditionné est asymptotiquement celle de Rayleigh sans le démontrer. Iglehart apportera une première formulation complète de cette démonstration en 1974 [47] et Bolthausen une version avec des hypothèses plus faibles en 1976 [9]. D'autres auteurs comme par exemple Bertoin et Doney [6] ou Borovkov [10, 11] ont participé à l'émergence et à l'enrichissement de toute la théorie des marches aléatoires réelles centrées et indépendantes qui est désormais bien connue. On trouvera notamment une description complète des principes de base dans les célèbres livres de Spitzer [66] et de Feller [29].

Pourtant, le point clé pour obtenir ces résultats est la factorisation de Wiener-Hopf qui, au moins dans ses conséquences directes, se prête assez mal à des modèles plus élaborés. Malgré d'intéressantes considérations sur la marche lorsque les accroissements sont markoviens comme dans les travaux de Presman [60, 61] par exemple, il faudra attendre plusieurs décennies pour voir apparaître de nouvelles avancées importantes à ce sujet. Motivée en particulier par des modèles physiques importants, l'étude des marches aléatoires en dimensions supérieures a suscité ces dernières années un intérêt croissant et beaucoup de réponses ont été apportées par de nombreux auteurs dont Varopoulos [68, 69, 70], Eichelsbacher et König [27], Garbit [31], Duraj et Wachtel [26] ou encore Denisov et Wachtel [18, 20]. Ces ingénieux travaux sont largement novateurs dans leur

globale approche du problème par rapport au cas unidimensionnel. Dans [18, 20], Denisov et Wachtel développent une méthode articulée sur deux grandes étapes : la construction d'une fonction harmonique d'une part et l'approximation de la marche par un mouvement brownien par le théorème de Komlós-Major-Tusnády [51, 52] d'autre part.

La puissance de cette méthode, affranchie de la factorisation de Wiener-Hopf, suggérait des applications dans de nouveaux modèles, plus généraux que le seul cadre où les accroissements de la marche sont indépendants. La marche intégrée, où l'on considère la somme de la somme de variables aléatoires indépendantes et identiquement distribuées (*i.i.d.* en abrégé), est un premier exemple traité par Dembo, Ding et Gao [15] et perfectionné par Denisov et Wachtel [19]. Cependant pour continuer de développer la méthode en question et afin de l'étendre à d'autres modèles markoviens, il était absolument nécessaire de généraliser le théorème de Komlós-Major-Tusnády dit *KMT* aux chaînes de Markov. Ce remarquable travail a été accompli récemment par mes responsables de thèse Grama et Le Page en collaboration avec Peigné [40] ce qui leur a permis de résoudre l'étude d'un produit de matrices [41].

C'est dans cette conjonction très favorable qu'a débuté ma thèse et les travaux associés. En m'appuyant sur la méthode de Denisov et Wachtel avec le *KMT* établi par Grama, Le Page et Peigné, j'ai principalement pu avec l'aide de mes responsables étendre les résultats sur les marches aléatoires unidimensionnelles à une large classe de marches markoviennes unidimensionnelles. Un exemple de la portée de ces travaux sera donné en généralisant les résultats des processus de branchement critiques et sous-critiques à des environnements markoviens finis.

1.2 Présentation des objectifs

Soit $(\Omega, \mathcal{F}, \mathbb{P})$ un espace probabilisé et $(X_n)_{n \geq 0}$ une suite de variables aléatoires définie sur cet espace et à valeurs dans un espace mesurable $(\mathbb{X}, \mathcal{X})$. Pour une fonction $f : \mathbb{X} \rightarrow \mathbb{R}$ fixée, on considère la marche aléatoire associée, définie par

$$S_0 := 0 \quad \text{et} \quad S_n := \sum_{k=1}^n f(X_k), \quad \forall n \geq 1. \quad (1.2.1)$$

Pour tout réel $y \in \mathbb{R}$, interprété comme le point de départ de la marche, on pose

$$\tau_y := \inf \{k \geq 1 : y + S_k \leq 0\}. \quad (1.2.2)$$

Cette variable aléatoire correspond au premier instant pour lequel la marche $(y + S_n)_{n \geq 1}$ sort de la demi-droite des réels strictement positifs $]0; +\infty[$. Il est facile de voir que pour la filtration naturelle $(\mathcal{F}_n)_{n \geq 0}$ canoniquement associée à la suite $(X_n)_{n \geq 0}$, le temps τ_y est un temps d'arrêt. La première interrogation qui s'offre à nous, au delà d'un calcul explicite de la loi de τ_y qu'il n'est pas raisonnable d'envisager dans un cadre général, est de savoir si τ_y est fini presque sûrement (*p.s.* en abrégé) ou non :

$$\mathbb{P}(\tau_y < +\infty) = 1 ? \quad (\text{Q1/1.2.3})$$

Lorsque c'est le cas, on sait que la marche va nécessairement passer dans les négatifs, que le joueur sera nécessairement ruiné, que la population s'éteindra nécessairement en temps fini. La probabilité de survivre $\mathbb{P}(\tau_y > n)$ jusqu'au temps/à la génération n tend

donc par définition vers 0. La question suivante est de savoir avec quelle vitesse s'effectue cette convergence vers 0, id est trouver $(v_n)_{n \geq 1} \in \mathbb{R}^{\mathbb{N}}$ telle que

$$\mathbb{P}(\tau_y > n) \underset{n \rightarrow +\infty}{\sim} v_n ? \quad (\text{Q2/1.2.4})$$

Le symbole \sim signifie que le rapport des deux termes de chaque côté du symbole tend vers 1 lorsque $n \rightarrow +\infty$. Ensuite, lorsque cette probabilité n'est pas strictement nulle (à n fixé!), on peut s'intéresser à la loi asymptotique des trajectoires, renormalisées, sachant que la marche est restée positive :

$$\forall t \in]0; +\infty[, \quad \lim_{n \rightarrow +\infty} \mathbb{P}(y + S_n \leq t\sqrt{n} \mid \tau_y > n) = ? \quad (\text{Q3/1.2.5})$$

Notons que, si la réponse à (Q1/1.2.3) est affirmative, la probabilité que la marche reste positive infiniment longtemps est nulle. Plutôt que de conditionner par l'évènement $\{\tau_y = +\infty\}$, il nous faut considérer que la marche reste positive jusqu'au temps n , puis faire tendre ce temps vers l'infini. Je donne ici la renormalisation en $\frac{1}{\sqrt{n}}$ car nous nous placerons toujours dans le cadre de modèles dont la loi asymptotique (sans conditionnement) est la loi normale (TCL). Même si l'on interdit aux trajectoires d'intersecter le demi-plan inférieur, la hauteur « typique » de la marche au temps n reste de l'ordre de \sqrt{n} . Si l'on s'intéresse à la probabilité que la marche $y + S_n$ soit dans l'intervalle $[z, z + a]$, avec $z \geq 0$ et $a > 0$, le bon point de vue est de dire que l'on regarde la probabilité que $(y + S_n)/\sqrt{n}$ appartienne à l'intervalle $[z/\sqrt{n}, (z + a)/\sqrt{n}]$ qui est de longueur a/\sqrt{n} . En ce sens, on s'intéresse à la marche *localement*. Si l'on obtient une loi non-dégénérée à la question (Q3/1.2.5), on comprend bien que cette probabilité d'être dans $[z/\sqrt{n}, (z + a)/\sqrt{n}]$ tend vers 0. La question de savoir la vitesse avec laquelle la probabilité tend vers 0 sera également un objet d'intérêt de ce manuscrit : quelle est la vitesse $(w_n)_{n \geq 1} \in \mathbb{R}^{\mathbb{N}}$ pour laquelle on ait

$$\mathbb{P}(y + S_n \in [z, z + a] \mid \tau_y > n) \underset{n \rightarrow +\infty}{\sim} w_n ? \quad (\text{Q4/1.2.6})$$

Cette question est donc plus fine que le résultat dit *intégrale* de (Q3/1.2.5). Cependant lorsqu'elle est résolue pour a fixé (ce qui sera notre cas), elle ne répond pas à la question (Q3/1.2.5) (qui nécessiterait de prendre $z = 0$ et $a = t\sqrt{n}$).

D'autres questions ou des versions différentes (notamment pour le résultat local) peuvent être considérées et en abordant les questions (Q1/1.2.3)-(Q4/1.2.6) nous parlerons également de nombreux autres résultats qui gravitent autour de la même thématique. Pour une plus grande unité de mes propos, les considérations sur les processus de branchement qui découlent des réponses (Q1/1.2.3)-(Q4/1.2.6) seront abordées ultérieurement dans les Sections 1.3.4 et 1.4.6.

Dans la Section 1.3.2, nous allons voir que toutes ces questions ont déjà été résolues en particulier lorsque les accroissements $(X_n)_{n \geq 1}$ sont supposés *i.i.d.*, centrés (de moyenne nulle) avec un moment d'ordre 2 fini. Au sujet de l'hypothèse de moment, précisons que contrairement aux cas multidimensionnels ou markoviens pour lesquels un moment un peu plus grand que 2 est toujours de rigueur, il est possible dans le cas unidimensionnel de formuler des hypothèses plus faibles n'exigeant pas nécessairement un moment d'ordre 2. En dehors du cadre centré, indépendant et unidimensionnel, de nombreuses autres situations plus élaborées (marche avec dérive, en dimension supérieure...) ont également été traitées. Cependant, à l'exception de quelques modèles précis (marche intégrée, produit de matrices), ces questions restaient en suspens lorsque l'on généralise de façon naturelle

l'hypothèse d'accroissements *i.i.d.* par des accroissements markoviens. Insistons sur le fait que si le cadre *i.i.d.* a été posé dans les années 60-70, le cadre markovien lui, même dans sa formulation la plus simple d'un espace d'états fini, avait jusqu'à présent résisté à toute généralisation. En s'appuyant très largement sur les récentes grandes avancées, la présente thèse propose de répondre aux questions (Q1/1.2.3), (Q2/1.2.4) et (Q3/1.2.5) pour une large classe de chaîne de Markov et de répondre à la question (Q4/1.2.6) (plus délicate) dans le cadre d'une chaîne de Markov à espace d'états fini. Nous verrons néanmoins que notre réponse à (Q4/1.2.6) permet de généraliser les résultats sur les processus de branchement critiques et sous-critiques en environnement aléatoire.

1.3 Résultats antérieurs

1.3.1 Le cas brownien

Avant de donner les premières réponses à (Q1/1.2.3)-(Q4/1.2.6) pour des marches aléatoires indépendantes, observons que les homologues continus fournissent une excellente idée des résultats que l'on doit obtenir. Pour le mouvement brownien, les résultats sont bien connus et même plus précis puisque non-asymptotique.

Soit $(B_t)_{t \geq 0}$ un mouvement brownien défini sur un espace probabilisé $(\Omega, \mathcal{F}, \mathbb{P})$ à valeurs dans \mathbb{R} . Pour tout $y > 0$ et $\sigma > 0$, on définit

$$\tau_y^{bm} := \inf \{t \geq 0 : y + \sigma B_t \leq 0\}.$$

On trouvera le résultat suivant dans [57].

Proposition 1.3.1 (Lévy). *Pour tout $y > 0$, $0 \leq a < b$ et $n \geq 1$,*

$$\mathbb{P} \left(\tau_y^{bm} > n, y + \sigma B_n \in [a, b] \right) = \frac{1}{\sqrt{2\pi n\sigma}} \int_a^b \left(e^{-\frac{(s-y)^2}{2n\sigma^2}} - e^{-\frac{(s+y)^2}{2n\sigma^2}} \right) ds.$$

De cette unique formule, on résout nos quatre questions (Q1/1.2.3)-(Q4/1.2.6). Des égalités non-asymptotiques que l'on ne précise pas peuvent également être obtenues.

Corollaire 1.3.2. *Soit $y > 0$ et $\sigma > 0$.*

1. *On a $\tau_y^{bm} < +\infty$ p.s.*
2. *Pour tout $n \geq 1$,*

$$\mathbb{P} \left(\tau_y^{bm} > n \right) = \frac{2}{\sqrt{2\pi n\sigma}} \int_0^y e^{-\frac{s^2}{2n\sigma^2}} ds \underset{n \rightarrow +\infty}{\sim} \frac{2y}{\sqrt{2\pi n\sigma}}.$$

3. *Pour tout $t > 0$,*

$$\lim_{n \rightarrow +\infty} \mathbb{P} \left(y + \sigma B_n \leq t\sqrt{n} \mid \tau_y^{bm} > n \right) = 1 - e^{-\frac{t^2}{2\sigma^2}}.$$

4. *Pour tout $z \geq 0$ et $a > 0$,*

$$\mathbb{P} \left(y + \sigma B_n \in [z, z+a] \mid \tau_y^{bm} > n \right) \underset{n \rightarrow +\infty}{\sim} \frac{(2z+a)a}{n\sigma^2}.$$

Le point 2 nous dit que la probabilité de survivre est de l'ordre d'une constante dépendant de y divisée par \sqrt{n} . Mais si l'on suppose malgré tout que la trajectoire de la marche reste strictement positive alors la loi asymptotique donnée par le point 3 est la loi de Rayleigh. Le point 4 quant à lui nous fournit le théorème local, souvent écrit dans sa formulation de la probabilité conjointe, où la vitesse est alors une constante dépendant du point de départ y et du point d'arrivée z divisé par $n^{3/2}$:

$$\mathbb{P}\left(y + \sigma B_n \in [z, z + a], \tau_y^{bm} > n\right) \underset{n \rightarrow +\infty}{\sim} \frac{2y(2z + a)a}{\sqrt{2\pi}\sigma^3 n^{3/2}}.$$

Tous ces résultats nous guideront pour nos marches aléatoires et nous verrons que dans nos modèles, nous obtenons exactement les mêmes vitesses mais que les constantes se complexifient.

1.3.2 Le cas indépendant unidimensionnel

Revenons au temps discret et supposons que les accroissements $(X_n)_{n \geq 1}$ de la marche définie en (1.2.1) sont *i.i.d.* Dans ce cas, puisque la suite $(f(X_n))_{n \geq 1}$ est aussi *i.i.d.*, on peut supposer sans perte de généralité que $\mathbb{X} = \mathbb{R}$ et que $f = \text{id}$ (l'introduction d'une fonction f est utile dans le cadre des chaînes de Markov). Avec ce formalisme, le résultat suivant répond aux questions (Q1/1.2.3) et (Q2/1.2.4). L'existence de la série donnée dans (1.3.2) ci-dessous est issue du théorème 3.4 de Spitzer [65]. L'asymptotique du temps de sortie lorsque l'on suppose l'existence d'un moment d'ordre 2 est un cas particulier du théorème 1 d'Emery [28] ou encore du théorème 2 de Bingham [7]. On pourra également se référer à Doney [22] ou au théorème 8.9.12 page 381 du livre de Bingham, Goldie et Teugels [8]. Lorsque la marche est à valeurs entières, une version de la proposition suivante est aussi exprimée dans le livre de Spitzer [66] (cf P4 page 382). Soit τ le premier instant pour lequel la marche $(S_n)_{n \geq 1}$ issue de 0 passe dans les positifs :

$$\tau = \min \{k \geq 1 : S_k \leq 0\}.$$

Sous les hypothèses de la proposition suivante, τ est fini *p.s.* et on définit χ_1, \dots, χ_n une suite de variables aléatoires *i.i.d.* de loi commune celle de $-S_\tau$. De cette façon, pour tout $y \geq 0$, la fonction de renouvellement est donnée par,

$$H(y) = 1 + \sum_{k=1}^{+\infty} \mathbb{P}(\chi_1 + \dots + \chi_k \leq y). \quad (1.3.1)$$

On note naturellement \mathbb{E} l'espérance associée à \mathbb{P} .

Proposition 1.3.3 (Spitzer, Emery, Bingham). *Supposons $\mathbb{E}(X_1) = 0$ et $\mathbb{E}(X_1^2) = \sigma^2 \in (0, \infty)$. Alors la série $\sum_{k \geq 1} \frac{1}{k} \left(\mathbb{P}(S_k > 0) - \frac{1}{2} \right)$ converge vers un réel noté*

$$\alpha = \sum_{k=1}^{\infty} \frac{1}{k} \left(\mathbb{P}(S_k > 0) - \frac{1}{2} \right) \in \mathbb{R}. \quad (1.3.2)$$

De plus, pour tout $y \geq 0$,

$$\mathbb{P}(\tau_y > n) \underset{n \rightarrow +\infty}{\sim} \frac{e^\alpha H(y)}{\sqrt{\pi n}}.$$

Pour comprendre dans une certaine mesure l'origine de ce résultat, la Section 1.5 propose de reprendre la démonstration historique de cette proposition lorsque $y = 0$. Cette étape avait été obtenue par Spitzer dans le théorème 3.5 de [65].

Annoncée par Spitzer également, la réponse à la question (Q3/1.2.5) a été démontrée par Iglehart [47] sous la condition d'avoir un moment d'ordre 3 et d'être *non-lattice*. La version ci-dessous avec un moment d'ordre 2 uniquement est due à Bolthausen [9].

Proposition 1.3.4 (Iglehart, Bolthausen). *Supposons que $\mathbb{E}(X_1) = 0$ et que $\mathbb{E}(X_1^2) = \sigma^2 \in (0, \infty)$. Alors, pour tout $t \geq 0$,*

$$\lim_{n \rightarrow +\infty} \mathbb{P} \left(y + S_n \leq t\sqrt{n} \mid \tau_y > n \right) = 1 - e^{-\frac{t^2}{2\sigma^2}}.$$

En ce qui concerne le théorème local et la réponse à la question (Q4/1.2.6), Caravenna [13] apporte un premier résultat. Cependant contrairement à son homologue sans conditionnement (théorème de Stone classique), le théorème obtenu ne fournit un équivalent que pour des points d'arrivée de tailles suffisantes (dans (Q4/1.2.6), z doit être de l'ordre de $c\sqrt{n}$, avec c une constante positive). La proposition suivante provient du théorème 4 de Vatutin et Wachtel [71]. On dit que X_1 est *non-lattice* si sa fonction caractéristique est strictement plus petite que 1 en dehors de 0 : pour tout $\theta \in \mathbb{R}^*$,

$$\mathbb{E} \left(e^{i\theta X} \right) < 1.$$

Proposition 1.3.5 (Vatutin, Wachtel). *Supposons X_1 non-lattice, que $\mathbb{E}(X_1) = 0$ et que $\mathbb{E}(X_1^2) = \sigma^2 \in (0, \infty)$. Alors, pour tout suite $(\delta_n)_{n \geq 1}$ convergant vers 0 et tout réel $a > 0$,*

$$\lim_{n \rightarrow +\infty} \sup_{z \in \left(0, \frac{\delta_n}{\sqrt{n}}\right]} \left| n^{3/2} \mathbb{P} \left(S_n \in [z, z+a], \tau_0 > n \right) - \frac{\int_z^{z+a} H(z') dz'}{\sqrt{2\pi\sigma}} \right| = 0,$$

où H est définie par (1.3.1).

Après ce petit tour d'horizon (non exhaustif) des résultats du cas indépendant unidimensionnel, nous allons nous rapprocher des travaux de cette thèse en introduisant quelques résultats antérieurs traitant de modèles pour lesquels l'indépendance tombe en défaut.

1.3.3 Les premiers modèles markoviens

Des factorisations pour les chaînes de Markov. Quelques généralisations des factorisations, indispensables dans la démonstration du cas indépendant, sont obtenues lorsque l'on remplace l'hypothèse d'indépendance par l'hypothèse selon laquelle la suite $(X_n)_{n \geq 1}$ est une chaîne de Markov à espace d'états fini. Pour de tels travaux, on se reportera à Presman [60, 61] et les références associées.

La marche intégrée. Grâce à leurs travaux sur les marches aléatoires en dimensions supérieures [18, 20], Denisov et Wachtel ont récemment obtenu dans [19] des théorèmes limites pour la marche intégrée conditionnée à rester positive : si $(X_n)_{n \geq 1}$ est une suite de variables aléatoires *i.i.d.* la marche intégrée est définie par

$$S_n^{(2)} = S_1 + \dots + S_n, \quad \forall n \geq 1,$$

où pour tout $n \geq 1$, on a toujours $S_n = X_1 + \dots + X_n$. Naturellement, si les X_n sont indépendants, ce n'est plus le cas pour leurs sommes S_n . Les accroissements $(S_n)_{n \geq 1}$ de la marche intégrée $(S_n^{(2)})_{n \geq 1}$ forment en réalité dans ce cas une chaîne de Markov. La dépendance de S_n par rapport au passé, id est aux états précédents S_1, \dots, S_{n-1} , étant « forte » l'asymptotique du temps de survie possède une vitesse en $n^{-1/4}$, différente de celle en $n^{-1/2}$ présentée dans le Corollaire 1.3.2 et la Proposition 1.3.3. Sans être rigoureux sur la définition d'une « dépendance forte », je précise que les modèles markoviens que nous considérerons auront une dépendance dite « faible » (un trou spectral) et les vitesses associées aux réponses de (Q2/1.2.4)-(Q4/1.2.6) resteront les mêmes que dans le cas indépendant. Dans ce parallèle, notons que la marche intégrée correspond à une récursion stochastique particulière (voir la Section 1.4.1 ou le Chapitre 2 dévolu justement à ce type de chaînes de Markov).

Le produit de matrices. Terminons cette section par un dernier modèle qui a largement influencé mes travaux. Soit $(g_n)_{n \geq 1} \in \text{GL}_d(\mathbb{R})$ une suite de matrices inversibles de taille $d \times d$. Pour tout vecteur de départ $v \in \mathbb{R}^d$, on s'intéresse au premier instant pour lequel le produit $g_n \dots g_1 v$ appartient à la boule unité $\mathbb{B}(0, 1) = \{u \in \mathbb{R}^d : \|u\| \leq 1\}$:

$$\tau_v = \inf \{k \geq 1 : g_k \dots g_1 v \in \mathbb{B}(0, 1)\}.$$

Pour tout $v \in \mathbb{R}^d$, on note \bar{v} son projeté dans l'espace des directions de \mathbb{R}^d , espace noté $\mathbb{P}(\mathbb{R}^d)$. A v fixé, on définit $X_0 = (\text{id}, \bar{v})$ et pour tout $n \geq 1$, $X_n = (g_n, \overline{g_{n-1} \dots g_1 v}) \in \mathbb{X} = \text{GL}_d(\mathbb{R}) \times \mathbb{P}(\mathbb{R}^d)$. On écrit alors que

$$\begin{aligned} \ln(\|g_n \dots g_1 v\|) &= \ln\left(\frac{\|g_n \dots g_1 v\|}{\|g_{n-1} \dots g_1 v\|}\right) + \dots + \ln\left(\frac{\|g_1 v\|}{\|v\|}\right) + \ln(\|v\|) \\ &= f(X_n) + \dots + f(X_1) + y = y + S_n, \end{aligned}$$

avec $y = \ln(\|v\|)$ et f le *cocycle* défini par $f(g, \bar{w}) = \ln\left(\frac{\|gw\|}{\|w\|}\right)$ où w est n'importe quel vecteur non nul de direction \bar{w} . Puisque la suite $(g_n \dots g_1 v)_{n \geq 1}$ reste strictement à l'extérieur de la boule unité si et seulement si le logarithme de sa norme reste strictement positif, on remarque que $\tau_v = \tau_y$ avec τ_y défini en (1.2.2). Dans ce modèle, on voit bien que la suite $(X_n)_{n \geq 0}$ n'est plus *i.i.d.* mais est une chaîne de Markov. En appliquant la démarche de Denisov et Wachtel [20] couplée avec une récente version (voir [40]) du théorème de Komlós-Major-Tusnády [51], Grama, Le Page et Peigné [41] ont montré le résultat suivant.

Proposition 1.3.6 (Grama, Le Page, Peigné). *Sous les hypothèses P1-P5 de [41], il existe une fonction V strictement positive sur le complémentaire de $\mathbb{B}(0, 1)$, telle que*

1. Pour tout $v \notin \mathbb{B}(0, 1)$,

$$\mathbb{P}(\tau_v > n) \underset{n \rightarrow +\infty}{\sim} \frac{2V(v)}{\sqrt{2\pi n\sigma}},$$

où σ est une constante strictement positive.

2. De plus, pour tout $v \notin \mathbb{B}(0, 1)$ et $t \geq 0$,

$$\lim_{n \rightarrow +\infty} \mathbb{P}\left(\ln(\|g_n \dots g_1 v\|) \leq t\sqrt{n} \mid \tau_v > n\right) \underset{n \rightarrow +\infty}{\sim} 1 - e^{-\frac{t^2}{2\sigma^2}}.$$

Tous ces travaux apportent des réponses aux questions (Q1/1.2.3)-(Q3/1.2.5). Pourtant une formulation plus globale restait à poser. La marche intégrée est un modèle assez

précis et puisque la dépendance de la chaîne de Markov $(S_n)_{n \geq 1}$ est explicite, il n'est pas clair que les calculs puissent être étendus directement à d'autres chaînes de Markov. Pour le produit de matrices, la formulation qui concerne la chaîne de Markov semble plus générale. Cependant un examen plus attentif nous montre qu'en réalité les moments des accroissements $(f(X_n))_{n \geq 1}$ par exemple (cf **M4** de [41]) sont bornés uniformément par rapport au point de départ x de la chaîne de Markov $(X_n)_{n \geq 0} = ((g_n, \overline{g_{n-1} \cdots g_1 v}))_{n \geq 0}$. Tout se passe donc comme si la chaîne de Markov vivait dans un espace compact. L'extension des résultats à d'autres chaînes de Markov restait une étape à franchir et des techniques particulières que je détaillerai dans la suite ont dû être mises en place pour obtenir les résultats que je vais présenter dans la Section 1.4.

1.3.4 Les processus de branchement en environnement aléatoire

Avant de présenter mes propres travaux, finissons notre petit tour des résultats antérieurs que nous nous proposons d'affiner. Même si le lien n'est pas explicite au premier abord, l'étude des marches aléatoires a de profondes implications dans les processus de branchement en milieu aléatoire. En quelques mots, un processus de branchement en environnement aléatoire est l'étude d'une population qui se reproduit de façon aléatoire chaque année selon une loi qui est elle-même déterminée par un tirage annuel aléatoire. Puisque c'est ce tirage de l'environnement qui va nous donner la loi de reproduction de tous les individus, l'environnement a un poids très important sur la taille de la population. Ainsi, pour que la population ne s'éteigne pas, il faut que les environnements restent favorables. Or une suite d'environnements *favorables* à la population peut être vue comme une marche qui reste *positive*. Donc si l'on connaît la probabilité que cette marche reste positive on accède à la probabilité que l'environnement reste favorable et donc à la probabilité que la population survive. Ce lien très important entre marche aléatoire et processus de branchement, nous permettra d'illustrer dans la Section 1.4.6 le fait que les réponses aux questions (Q1/1.2.3)-(Q4/1.2.6) ont des conséquences intéressantes.

Présentons ces conséquences à travers les résultats phares et bien connus lorsque les environnements sont indépendants. Soit $(X_n)_{n \geq 0}$ une suite de variables aléatoires sur $(\Omega, \mathcal{F}, \mathbb{P})$ à valeurs dans \mathbb{X} représentant la suite d'environnements dans lesquels évolue un processus de branchement construit de la façon suivante. On fixe une famille de variables aléatoires $(\xi_i^{j,n})_{i \in \mathbb{X}, j, n \geq 1}$ indépendante de la chaîne $(X_n)_{n \geq 0}$ et définie sans perte de généralité sur le même espace probabilisé $(\Omega, \mathcal{F}, \mathbb{P})$. On suppose que pour chaque $i \in \mathbb{X}$ les variables $\xi_i^{j,n}$, $j, n \geq 1$ sont *i.i.d.* et on note f_i leur fonction génératrice commune :

$$\forall s \in [0, 1], \quad f_i(s) = \mathbb{E} \left(s^{\xi_i^{1,1}} \right).$$

Soit alors $(Z_n)_{n \geq 0}$ le processus de branchement associé que l'on construit récursivement par

$$Z_0 = 1 \quad \text{et} \quad Z_{n+1} = \sum_{j=1}^{Z_n} \xi_{X_{n+1}}^{n+1,j}, \quad \forall n \geq 0.$$

Concrètement, on considère que sous l'environnement $X_{n+1} = i$, chaque individu j de la population $\{1, \dots, Z_n\}$ donne naissance, de façon indépendante aux autres individus, à $\xi_i^{n+1,j}$ descendants et que donc la population totale Z_{n+1} l'année suivante est la somme de tous ces descendants. On pose ρ la fonction de \mathbb{X} dans \mathbb{R} définie par

$$\forall i \in \mathbb{X}, \quad \rho(i) = \ln(f'_i(1)) = \ln \left(\mathbb{E} \left(\xi_i^{1,1} \right) \right)$$

et sans plus de détails, on indique que la marche aléatoire associée dont on a fait mention au début du paragraphe est donnée par

$$S_0 = 0 \quad \text{et} \quad S_n = \sum_{k=1}^n \rho(X_k) = \ln \left(f'_{X_1}(1) \cdots f'_{X_n}(1) \right), \quad \forall n \geq 1.$$

Dans le cas d'environnements indépendants, Geiger et Kersting [32] ont montré le résultat suivant.

Proposition 1.3.7 (Geiger et Kersting, cas critique). *Supposons la suite $(X_n)_{n \geq 0}$ i.i.d. et supposons*

$$\mathbb{E}(\rho(X_1)) = 0, \quad 0 < \mathbb{E}(\rho(X_1^2)) < +\infty, \quad \mathbb{E} \left(\frac{f''_{X_1}(1) \left(1 + \ln^+ \left(f'_{X_1}(1) \right) \right)}{\left(f''_{X_1}(1) \right)^2} \right) < +\infty.$$

Alors il existe une constante $0 < c_1 < +\infty$ telle que la probabilité pour que la population Z_n survive jusqu'au temps n est donnée asymptotiquement par

$$\mathbb{P}(Z_n > 0) \underset{n \rightarrow +\infty}{\sim} \frac{c_1}{\sqrt{n}}.$$

Notons que l'hypothèse $\mathbb{E}(\rho(X_1)) = 0$ correspond au cas dit *critique*, lorsque que le nombre d'enfant par personne est « globalement » de 1. Du point de vue de la marche aléatoire associée S_n cela correspond à une marche centrée, sans dérive.

Lorsque cet indicateur $\mathbb{E}(\rho(X_1))$ est strictement négatif on dit être dans un cas *sous-critique*. La population s'éteint alors plus rapidement et la marche aléatoire dérive vers les négatifs. Il est encore possible cependant d'obtenir l'asymptotique de la survie de la population Z_n . L'idée est de recentrer la marche aléatoire S_n à l'aide d'un changement de loi adéquat. Ce centrage fait cependant apparaître « un poids » sur les trajectoires $(S_k)_{n \geq k \geq 1}$ d'ordre e^{-S_n} ce qui signifie que seules les trajectoires dont le point d'arrivée S_n est « petit » auront de l'importance. Suivant cette idée, il apparaît que la réponse à la question (Q4/1.2.6) est utile pour démontrer les résultats du cas sous-critique.

Lorsque $\mathbb{E}(\rho(X_1)) < 0$, il n'y a en réalité pas qu'une seule situation mais trois distinctes chacune donnant un équivalent de la probabilité de survie $\mathbb{P}(Z_n > 0)$ différent :

- si $\mathbb{E} \left[f'_{X_1}(1) \ln \left(f'_{X_1}(1) \right) \right] < 0$ on parle de cas *fortement sous-critique*,
- si $\mathbb{E} \left[f'_{X_1}(1) \ln \left(f'_{X_1}(1) \right) \right] = 0$ on parle de cas *sous-critique intermédiaire*,
- si $\mathbb{E} \left[f'_{X_1}(1) \ln \left(f'_{X_1}(1) \right) \right] > 0$ on parle de cas *faiblement sous-critique*.

On pose

$$\kappa := \mathbb{E} \left(f'_{X_1}(1) \right).$$

On notera que dans les cas sous-critiques non-dégénérés, on a $\kappa \in (0, 1)$.

Dans le cas fortement sous-critique, Guivarc'h et Liu [44] ont établi le résultat suivant. Il avait été précédemment démontré par D'Souza et Hambly [23] sous une condition de moment supplémentaire.

Proposition 1.3.8 (Guivarc'h et Liu, cas fortement sous-critique). *Supposons la suite $(X_n)_{n \geq 0}$ i.i.d. et supposons*

$$\mathbb{E} \left[f'_{X_1}(1) \ln \left(f'_{X_1}(1) \right) \right] < 0, \quad \mathbb{E} \left(Z_1 \ln^+ (Z_1) \right) < +\infty.$$

Alors il existe une constante $0 < c_2 < +\infty$ telle que

$$\mathbb{P}(Z_n > 0) \underset{n \rightarrow +\infty}{\sim} c_2 \kappa^n.$$

Les deux propositions suivantes ont été montrées par Geiger, Kersting et Vatutin [33].

Proposition 1.3.9 (Geiger, Kersting et Vatutin, cas sous-critique intermédiaire). *Supposons la suite $(X_n)_{n \geq 0}$ i.i.d. et supposons*

$$\begin{aligned} \mathbb{E}[\rho(X_1)] &< 0, & \mathbb{E}[f'_{X_1}(1) \ln(f'_{X_1}(1))] &= 0, \\ \mathbb{E}[f'_{X_1}(1) \ln^2(f'_{X_1}(1))] &< +\infty, & \mathbb{E}[f''_{X_1}(1) (1 + \ln^-(f'_{X_1}(1)))] &< +\infty. \end{aligned}$$

Alors il existe une constante $0 < c_3 < +\infty$ telle que

$$\mathbb{P}(Z_n > 0) \underset{n \rightarrow +\infty}{\sim} \frac{c_3 \kappa^n}{\sqrt{n}}.$$

Pour le cas faiblement sous-critique, on pose

$$\gamma := \inf_{0 \leq \theta \leq 1} \mathbb{E}(f'_{X_1}(1)^\theta)$$

et soit $\alpha \in [0, 1]$ tel que

$$\gamma = \mathbb{E}(f'_{X_1}(1)^\alpha).$$

Proposition 1.3.10 (Geiger, Kersting et Vatutin, cas faiblement sous-critique). *Supposons la suite $(X_n)_{n \geq 0}$ i.i.d., que*

$$\begin{aligned} \mathbb{E}[\rho(X_1)] &< 0, & 0 < \mathbb{E}[f'_{X_1}(1) \ln(f'_{X_1}(1))] &< +\infty, \\ \mathbb{E}\left[\frac{f''_{X_1}(1)}{f'_{X_1}(1)^{1-\alpha}}\right] &< +\infty, & \mathbb{E}\left[\frac{f''_{X_1}(1)}{f'_{X_1}(1)^{2-\alpha}}\right] &< +\infty, \end{aligned}$$

et que la loi de $\rho(X_1)$ est non-lattice :

$$\mathbb{P}(\rho(X_1) \in a + b\mathbb{Z}) < 1, \quad \forall 0 < a < b.$$

Alors il existe une constante $0 < c_4 < +\infty$ telle que

$$\mathbb{P}(Z_n > 0) \underset{n \rightarrow +\infty}{\sim} \frac{c_4 \gamma^n}{n^{3/2}}.$$

D'autres situations ont été étudiées, comme par exemple le cas critique par Athreya et Karlin [4, 3] pour des milieux échangeables ou par Le Page et Ye [56] pour des milieux markoviens sous une hypothèse adaptée de densité. Pourtant, à ma connaissance, l'asymptotique exact de la probabilité de survie de processus de branchement en environnement markovien fini n'a jamais été traitée en absence d'hypothèse supplémentaire que le cadre *non-lattice* que nous décrivons.

1.4 Présentation des travaux de thèse

Les principaux travaux de cette thèse ont pour but d'étendre les résultats présentés dans la section précédente à une large classe de marches dont les accroissements sont markoviens. Je l'ai mentionné dans la section précédente pour la marche intégrée, lorsque les accroissements sont markoviens il est possible que la vitesse de convergence répondant à la question (Q2/1.2.4) puisse changer. Si l'on souhaite retrouver les résultats du cas

indépendant, des hypothèses sur la chaîne de Markov constituant les accroissements de la marche sont nécessaires. Le principal objectif de cette thèse est de donner des conditions assez générales sur la chaîne de Markov et d'établir sous ces conditions des résultats analogues au cas indépendant pour répondre aux questions (Q1/1.2.3)-(Q3/1.2.5). Nous verrons également que ces conditions sont satisfaites au moins pour certains exemples concrets. Par la suite, nous constaterons que lorsque la chaîne de Markov est à valeurs dans un espace d'états fini, il nous est possible d'affiner les résultats et de répondre à la question (Q4/1.2.6). Ces développements nous permettront d'établir de nouveaux résultats sur la théorie des processus de branchement critiques et sous-critiques.

1.4.1 Les modèles markoviens considérés

Notre approche s'est faite en deux temps. Même si le cas des chaînes de Markov à espace d'états fini n'avait pas été explicitement traité, le produit de matrices aléatoires [41] suggérait que si l'on pouvait contrôler la dépendance de la chaîne $(X_n)_{n \geq 1}$ uniformément par rapport à son passé $x = X_0$ alors la méthode restait identique. Intuitivement, les questions (Q1/1.2.3)-(Q3/1.2.5) se posaient donc plutôt pour des chaînes de Markov dont la dépendance au passé était « faible » (pour retrouver les mêmes vitesses que dans le cas indépendant) mais pas uniforme. Ainsi, avant de poser un cadre général, nous nous sommes d'abord intéressés à un modèle explicite répondant à ce critère sur la dépendance de la chaîne : la récursion stochastique.

A partir de maintenant et dans la suite, pour tout $x \in \mathbb{X}$, on pose \mathbb{P}_x , respectivement \mathbb{E}_x , la probabilité, respectivement l'espérance, engendrée par les lois fini-dimensionnelles du processus $(X_n)_{n \geq 0}$ sachant que $X_0 = x$.

Les chaînes de Markov affines (CMA). La récursion stochastique que l'on présente ici est le sujet d'étude du Chapitre 2. Soit $(\Omega, \mathcal{F}, \mathbb{P})$ un espace probabilisé et $(a_i, b_i)_{i \geq 1}$ une suite de variables aléatoires *i.i.d.* On note (a, b) une représentation générique dont la loi est celle commune aux $(a_i, b_i)_{i \geq 1}$. On construit récursivement la chaîne de Markov par la transformation affine suivante :

$$X_0 = x \quad \text{et} \quad X_{n+1} = a_{n+1}X_n + b_{n+1}, \quad \forall n \geq 0.$$

On dira alors que $(X_n)_{n \geq 0}$ est une *chaîne de Markov affine (CMA)*. On ne suppose pas a priori que les a_i sont indépendants des b_i , cependant lorsque $a_i = 0$ *p.s.*, on retrouve le cas indépendant. Lorsque $a_i = 1$ *p.s.*, on retrouve le cas de la marche intégrée. Les hypothèses qui suivent contiennent le cas indépendant mais rejettent le cas de la marche intégrée (voir (C1.1/1.4.1) ci-dessous). Présentons brièvement ces hypothèses.

La dépendance « faible » à laquelle j'ai fait allusion au début de cette section se traduit dans ce modèle par l'hypothèse suivante : on suppose qu'il existe $\alpha > 2$ tel que

$$\mathbb{E}(|a|^\alpha) < 1. \tag{C1.1/1.4.1}$$

Ceci correspond à une contraction de la dépendance : intuitivement, l'état suivant de la chaîne X_{n+1} ne dépend en moyenne que d'une fraction de l'état précédent X_n . Une hypothèse de moment est également nécessaire sur les variables $(b_i)_{i \geq 1}$,

$$\mathbb{E}(|b|^\alpha) < +\infty. \tag{C1.2/1.4.2}$$

On suppose la marche non-dégénérée et centrée :

$$\mathbb{P}(b = 0) < 1 \quad \text{et} \quad \mathbb{E}(b) = 0. \tag{C1.3/1.4.3}$$

Une autre condition sera nécessaire pour s'assurer de la stricte positivité de l'équivalent que nous déterminerons dans la réponse à la question (Q2/1.2.4), condition que je ne détaille pas pour l'instant.

Sous les hypothèses (C1.1/1.4.1)-(C1.3/1.4.3), la dépendance de la chaîne X_n en fonction de l'état initial $X_0 = x$ décroît exponentiellement vite au cours du temps n . En effet un simple calcul montre que pour tout $n \geq 1$,

$$\mathbb{E}_x(|X_n|) \leq \mathbb{E}(|a|)^n |x| + \sum_{k=0}^{n-1} \mathbb{E}(|a|)^k \mathbb{E}(|b|).$$

Par la propriété de Jensen, $\mathbb{E}(|a|) < 1$, la série $\sum_{k \geq 0} \mathbb{E}(|a|)^k \mathbb{E}(|b|)$ converge et l'inégalité précédente formalise un peu cette idée de dépendance faible.

Ce modèle est intéressant à deux points de vue. Le premier est celui annoncé en début de section sur le défi que représente la gestion d'une marche aléatoire avec une « réelle » dépendance par rapport à son passé. Le second est le fait que la marche affine est un modèle qui a généré beaucoup d'intérêt en particulier dans sa relation aux modèles ARCH. Pour des résultats de convergence des marches affines (en absence de conditionnement) on pourra se référer notamment aux travaux de Guivarc'h et Le Page [43].

Les Théorèmes 1.4.1, 1.4.3 et 1.4.7 qui suivent dans cette introduction répondent aux questions (Q2/1.2.4) et (Q3/1.2.5) pour ce type de marches markoviennes.

Les chaînes de Markov avec un trou spectral (CMTS). La seconde famille de chaînes de Markov avec laquelle nous allons travailler dans le Chapitre 3 est encore beaucoup plus générale que celui de la récursion stochastique. Le modèle affine précédent nous a permis de dégager les éléments qui étaient essentiels pour assurer les convergences voulues dans (Q2/1.2.4) et (Q3/1.2.5). Ceci étant fait, j'ai pu déterminer un cadre plus abstrait, mais surtout plus général, pour lequel je réponds aux questions (Q2/1.2.4) et (Q3/1.2.5). Pour l'essentiel, le point de vue à adopter était déjà présent dans le formalisme de l'article de Grama, Le Page et Peigné [40]. Sans détailler toutes les hypothèses, disons simplement que l'important est de munir la chaîne de Markov d'un espace de Banach approprié. Cet espace est intrinsèque à la marche et décrit dans un formalisme fonctionnel ses propriétés (croissance et dépendance principalement). Pour $(X_n)_{n \geq 0}$ une chaîne de Markov à valeurs dans \mathbb{X} de noyau \mathbf{P} , l'espace de Banach \mathcal{B} que l'on considère est un sous-ensemble de fonctions de $\mathbb{X} \rightarrow \mathbb{C}$ sur lequel principalement l'opérateur \mathbf{P} possède un trou spectral. Ceci correspond à une hypothèse de mélange pour la chaîne de Markov. Donnons une esquisse du fil directeur de cette hypothèse. Pour toute fonction φ pour laquelle l'intégrale suivante a un sens, on fait agir l'opérateur \mathbf{P} sur φ en définissant

$$\mathbf{P}\varphi(x) = \int_{\mathbb{X}} \varphi(x') \mathbf{P}(x, dx').$$

On dira que \mathbf{P} possède un trou spectral sur \mathcal{B} si \mathbf{P} est un opérateur borné de \mathcal{B} dans \mathcal{B} tel que sur \mathcal{B} on ait

$$\mathbf{P} = \nu(\cdot)e + Q, \quad \nu(Q) = Qe = 0,$$

avec ν une forme linéaire positive, $e : x \mapsto 1$ la fonction constante égale à 1 sur \mathbb{X} et Q un opérateur dont le rayon spectral est strictement plus petit que 1. Une chaîne de Markov vérifiant une telle hypothèse, et d'autres que l'on ne précise pas dans cette introduction (cf Chapitre 3 pour plus de détails) sera dite *chaîne de Markov avec un trou spectral (CMTS)*. On verra que si $(X_n)_{n \geq 0}$ a une mesure invariante avec une condition de

moment alors cette mesure est ν . La fonction e est le vecteur propre de \mathbf{P} associé à la valeur propre 1. Cette propriété de trou spectral nous permet de retrouver un résultat de type Perron-Frobenius soulignant la décroissance rapide de la dépendance de la chaîne $(X_n)_{n \geq 0}$: pour tout $x \in \mathbb{X}$,

$$|\mathbf{P}^n \varphi(x) - \nu(\varphi)| \leq c e^{-cn} \|\delta_x\|_{\mathcal{B}'},$$

avec $\delta_x : \varphi \mapsto \varphi(x)$ la mesure de Dirac en x , \mathcal{B}' le dual de \mathcal{B} et $\|\cdot\|_{\mathcal{B}'}$ une norme sur \mathcal{B}' . Grâce à cette hypothèse de trou spectral, nous allons pouvoir retrouver les résultats du cas indépendant en considérant que si X_{n+1} n'est plus indépendant de X_n , la variable X_{n+p} est, elle, « presque » indépendante de X_n pour p assez grand. Cette idée directrice toute simple demande de nombreuses considérations techniques et le passage du modèle particulier de la récursion stochastique à celui des chaînes de Markov avec un trou spectral a nécessité des modifications d'importance dans notre approche, je vous renvoie au Chapitre 3 pour plus de détails.

Avant de poursuivre et de décrire les principaux résultats de cette thèse, soulignons que ce formalisme associé à l'espace de Banach bien que assez algébriste et abstrait permet, de fait, de couvrir de nombreuses situations : le cas indépendant bien sûr, mais aussi les chaînes de Markov à espace d'états fini ou même compact ainsi que la récursion stochastique réelle $\mathbb{X} = \mathbb{R}$ ou multidimensionnel $\mathbb{X} = \mathbb{R}^d$ (avec une fonction f sur \mathbb{R}^d toujours à valeurs dans \mathbb{R}). L'appendice du Chapitre 3 fournit plus de détails sur la construction des espaces de Banach associés. La Section 3.13 montre également que nos résultats couvrent le cas du produit des matrices aléatoires résolu par Grama, Le Page Peigné [41].

Dans les Théorèmes 1.4.2, 1.4.4 et 1.4.8 des sections suivantes, nous allons répondre aux questions (Q2/1.2.4) et (Q3/1.2.5) dans le cadre de ce type de marches markoviennes.

Les chaînes de Markov à espace d'états finis (CMF). Modèle beaucoup plus réduit mais aussi beaucoup plus simple, le cas où l'espace d'états de la chaîne de Markov est fini (on dira que *la chaîne de Markov est finie (CMF)*) nous sera utile notamment pour affiner nos résultats dans un premier exemple. Ce modèle est élémentaire et ne recouvre plus le modèle indépendant lorsque les valeurs de la suite *i.i.d.* en question sont en nombre infini. Il reste cependant important et l'objectif principal sera de répondre au moins dans cette situation à la question (Q4/1.2.6), ce que l'on développe dans le Chapitre 4. Soit $(X_n)_{n \geq 0}$ une chaîne de Markov sur un espace \mathbb{X} de cardinal fini. On aura besoin de trois hypothèses. On note toujours \mathbf{P} la matrice de transition de $(X_n)_{n \geq 0}$ et on suppose que \mathbf{P} est primitive, c'est-à-dire qu'il existe un entier $k_0 \geq 1$ tel que pour tout $(x, x') \in \mathbb{X}^2$, on a

$$\mathbf{P}^{k_0}(x, x') > 0. \tag{C1/1.4.4}$$

On considère toujours la marche $(y + S_n)_{n \geq 1}$ et le temps de sortie τ_y définis respectivement par (1.2.1) et (1.2.2) ainsi que la fonction f associée. Sous l'hypothèse (C1/1.4.4), nous avons toujours notre dépendance faible au sens où le théorème de Perron-Frobenius assure la convergence exponentielle de $\mathbf{P}^n(x, x')$ vers une unique mesure invariante $\nu(x')$, et ce uniformément en $x \in \mathbb{X}$. L'hypothèse suivante est la condition de centrage :

$$\nu(f) = \sum_{x \in \mathbb{X}} f(x) \nu(x) = 0. \tag{C2/1.4.5}$$

Puisque l'objectif est d'obtenir un théorème local, à l'image de l'article original de Stone [67] et comme classiquement pour un tel résultat, nous aurons besoin d'une hypothèse

de *non-lattice*. Cette condition, formulée ci-dessous, est équivalente à une formulation en terme de rayon spectral strictement plus petit que 1 pour l'opérateur perturbé (voir le Chapitre 4). On suppose que pour tout $(\theta, a) \in \mathbb{R}^2$, il existe une orbite, c'est-à-dire une suite de points x_0, \dots, x_n dans \mathbb{X} communiquant de la façon suivante

$$\mathbf{P}(x_0, x_1) > 0, \mathbf{P}(x_1, x_2) > 0, \dots, \mathbf{P}(x_{n-1}, x_n) > 0, \mathbf{P}(x_n, x_0) > 0,$$

telle que

$$f(x_0) + f(x_1) + \dots + f(x_n) - (n+1)\theta \notin a\mathbb{Z}. \quad (\text{C3/1.4.6})$$

On peut vérifier que cette condition dite *non-lattice* implique en particulier la non-dégénérescence de la marche. Cette idée de non-dégénérescence correspond à $X_1 \neq 0$ *p.s.* dans le cas indépendant, $b \neq 0$ dans le cas d'une marche affine et ici à $\sigma^2 > 0$ où σ^2 est un réel positif d'intérêt correspondant d'une certaine façon à « la variance de la marche ».

Lorsque qu'une chaîne de Markov à espace d'états fini (CMF) satisfait les conditions (C1/1.4.4)-(C3/1.4.6), alors elle répond aux hypothèses M3.1-M3.5 du Chapitre 3 ce qui signifie qu'une chaîne de Markov fini (CMF) est en particulier une chaîne à trou spectral (CMTS). Ainsi les réponses aux questions (Q1/1.2.3)-(Q3/1.2.5) sont toujours données par les Théorèmes 1.4.2, 1.4.4 et 1.4.8. Cependant dans ce cas on peut également répondre à la question (Q4/1.2.6) par les Théorèmes 1.4.9-1.4.11 et appliquer ces résultats pour obtenir les Théorèmes 1.4.13-1.4.16 sur les processus de branchement.

Après cette brève description des principales hypothèses que l'on considèrera sur nos chaînes de Markov, nous allons introduire dans les paragraphes suivants les principaux résultats obtenus durant ces trois années de thèse.

1.4.2 Existence d'une fonction harmonique

Le fait que la réponse à la question (Q1/1.2.3) soit positive et que la probabilité de survivre infiniment longtemps soit nulle pose une difficulté dans la définition d'un processus conditionné à rester positif. Le premier point de vue dont nous avons déjà fait mention propose de conditionner pour un temps fixé $n \geq 1$ par rapport à l'évènement « la marche est restée positive au moins jusqu'au temps n » c'est-à-dire l'évènement $\{\tau_y > n\}$ où y est le point de départ de la marche et τ_y est défini par (1.2.2). Puis dans un second temps on fait tendre n vers l'infini.

Considérons dans l'immédiat, une approche un peu différente. Reprenons les notations de la partie 1.2 et notons que lorsque les accroissements $(X_n)_{n \geq 0}$ ne sont plus indépendants alors la marche $(y + S_n)_{n \geq 0}$ issue de $y \in \mathbb{R}$ n'est plus en général une chaîne de Markov. Si l'on suppose que $(X_n)_{n \geq 0}$ est une chaîne de Markov, c'est le couple $(X_n, y + S_n)_{n \geq 0}$ qu'il faut considérer et qui forme une chaîne de Markov. Son noyau de transition est alors donné par

$$\mathbf{Q}((x, y), A) := \mathbb{P}_x((X_1, y + S_1) \in A),$$

pour tout $(x, y) \in \mathbb{X} \times \mathbb{R}$ et tout A ensemble mesurable de l'espace produit $\mathbb{X} \times \mathbb{R}$. Puisque seules les trajectoires qui sont restées positives nous intéressent, nous définissons la restriction \mathbf{Q}_+ par

$$\mathbf{Q}_+((x, y), A) = \mathbf{Q}((x, y), A)$$

pour tout $(x, y) \in \mathbb{X} \times \mathbb{R}$ et tout A ensemble mesurable de $\mathbb{X} \times \mathbb{R}_+^*$. Alors que $\mathbf{Q}((x, y), \cdot)$ est une probabilité, nous avons perdu de la masse en nous plaçant dans le sous-ensemble

$\mathbb{X} \times \mathbb{R}_+^*$: en général nous avons $\mathbf{Q}_+((x, y), \mathbb{X} \times \mathbb{R}_+^*) < 1$. Une renormalisation du noyau est nécessaire par l'intermédiaire d'une transformée de Doob aussi appelée fonction invariante ou fonction harmonique. On note encore \mathbf{Q}_+ l'opérateur défini sur l'ensemble des fonctions mesurables $\varphi : \mathbb{X} \times \mathbb{R} \rightarrow \mathbb{C}$ par, pour tout $(x, y) \in \mathbb{X} \times \mathbb{R}$,

$$\begin{aligned} \mathbf{Q}_+\varphi(x, y) &= \int_{\mathbb{X} \times \mathbb{R}_+^*} \varphi(x', y') \mathbf{Q}_+((x, y), dx' \times dy') \\ &= \int_{\mathbb{X} \times \mathbb{R}_+^*} \varphi(x', y') \mathbb{P}_x(X_1 \in dx', y + S_1 \in dy') \\ &= \mathbb{E}_x(\varphi(X_1, y + S_1) ; y + S_1 > 0) \\ &= \mathbb{E}_x(\varphi(X_1, y + S_1) ; \tau_y > 1), \end{aligned}$$

où l'on adopte ici et pour tout le reste du document la notation suivante : pour toute variable aléatoire X et tout évènement A :

$$\mathbb{E}(X ; A) := \mathbb{E}(X \mathbb{1}_A).$$

Une fonction V est \mathbf{Q}_+ -harmonique (on se contentera parfois de dire simplement harmonique sans préciser le noyau associé lorsqu'il n'y aura pas d'ambiguïté) si pour tout $(x, y) \in \mathbb{X} \times \mathbb{R}$,

$$\mathbf{Q}_+V(x, y) = \mathbb{E}_x(V(X_1, y + S_1) ; \tau_y > 1) = V(x, y).$$

Notons qu'il n'y a pas unicité de la fonction harmonique, si on multiplie V par une constante c alors la fonction cV est aussi une fonction harmonique. Bien sûr la fonction nulle est toujours une fonction harmonique mais dénuée d'intérêt puisque c'est lorsque que $V(x, y) > 0$ que l'on va pouvoir renormalisée $\mathbf{Q}_+((x, y), \cdot)$. En effet, pour tout couple de points (x, y) tel que $V(x, y) > 0$ et tout A ensemble mesurable de $\mathbb{X} \times \mathbb{R}_+^*$, on définit

$$\begin{aligned} \tilde{\mathbf{Q}}_+((x, y), A) &= \frac{1}{V(x, y)} \mathbf{Q}_+(V \mathbb{1}_A)(x, y) \\ &= \frac{1}{V(x, y)} \mathbb{E}_x(V(X_1, y + S_1) ; (X_1, y + S_1) \in A, \tau_y > 1). \end{aligned}$$

Il est clair que $\tilde{\mathbf{Q}}_+$ est un noyau markovien. De plus si $(\tilde{X}_n, y + \tilde{S}_n)_{n \geq 0}$ est une chaîne de Markov de noyau $\tilde{\mathbf{Q}}_+$ alors sa seconde composante $(y + \tilde{S}_n)_{n \geq 0}$ correspond à une marche aléatoire qui reste positive.

En réalité, ces deux façons différentes d'aborder le problème ne sont que deux présentations d'un même phénomène et la marche aléatoire conditionnée est la même dans les deux constructions. En effet, nous allons voir que lorsque l'on est capable de répondre à la question (Q2/1.2.4), l'asymptotique s'écrit

$$\mathbb{P}_x(\tau_y > n) \underset{n \rightarrow +\infty}{\sim} \frac{2V(x, y)}{\sqrt{2\pi n\sigma}},$$

où V n'est ni plus ni moins qu'une fonction \mathbf{Q}_+ -harmonique. Rappelons que dans le cas indépendant cette fonction harmonique est donnée par la fonction de renouvellement H définie par (1.3.1). Le principe de renouvellement est fondé sur l'observation suivante. Si les accroissements $(X_n)_{n \geq 1}$ sont indépendants et si $(T_i)_{i \geq 1}$ est la suite des indices des records (*ladder epoch*) définie récursivement par

$$T_0 = 0 \quad \text{et} \quad T_{j+1} = \min\{k \geq T_j + 1 : S_k > S_{T_j}\}, \quad \forall j \geq 0,$$

alors les suites $(T_{j+1}-T_j)_{j \geq 0}$ et $(S_{T_{j+1}}-S_{T_j})_{j \geq 0}$ sont *i.i.d.* Cette propriété avantageuse est à la base des développements du principe de renouvellement et en prenant $\chi_{j+1} = S_{T_{j+1}} - S_{T_j}$ on construit la fonction de renouvellement H . Cependant cette propriété tombe en défaut en général lorsque l'on suppose que les accroissements $(X_n)_{n \geq 1}$ forment une chaîne de Markov. La question de construire une fonction harmonique strictement positive mérite alors des considérations nettement différentes.

Dans un esprit de point fixe, pour tout $(x, y) \in \mathbb{X} \times \mathbb{R}$, on définit la suite de réels $V_n(x, y)$ par

$$V_n(x, y) = \mathbb{E}_x(y + S_n; \tau_y > n).$$

Cette définition est motivée par l'observation suivante. Puisque $(X_n, y + S_n)_{n \geq 0}$ est une chaîne de Markov, par la propriété de Markov, on a

$$V_{n+1}(x, y) = \mathbb{E}_x(\mathbb{E}(y + S_{n+1}; \tau_y > n + 1 | X_1)) = \mathbb{E}_x(V_n(X_1, y + S_1); \tau_y > 1).$$

Si l'on suppose que la suite $(V_n(x, y))_{n \geq 1}$ converge vers un réel $V(x, y)$ et si on peut intervertir la limite et l'espérance, cette limite $V(x, y)$ vérifierait alors

$$V(x, y) = \mathbb{E}_x(V(X_1, y + S_1); \tau_y > 1),$$

id est la fonction V est harmonique.

Si cette construction est claire dans son principe, l'existence de la limite et l'hypothèse de domination pour intervertir la limite et l'espérance en invoquant le théorème de convergence dominée de Lebesgue est a contrario une difficulté majeure dans l'élaboration d'une réponse aux questions (Q2/1.2.4) et (Q3/1.2.5). Dans le cas indépendant (pour lequel, le paramètre x n'existe pas) Denisov et Wachtel [18] ont développé une méthode récursive qui permet de montrer que la suite $(V_n(y))_{n \geq 1}$ (ou son équivalent dans la dimension supérieure) est bornée par une constante dépendant uniquement du point de départ y de la marche $(y + S_n)_{n \geq 1}$ (cf Lemme 10 de [18]). Pour les produits de matrices aléatoires, Grama, Le Page et Peigné [41] établissent le même résultat pour leur marche markovienne et montrent que la suite $(V_n(x, y))_{n \geq 1}$ est bornée. Même si la dépendance par rapport au point de départ x de la chaîne de Markov $(X_n)_{n \geq 1}$ est bien présente dans la démonstration, du fait de contrôles initiaux uniformes en x (comme mentionné à la fin de la Section 1.3.3), le majorant de la suite $(V_n(x, y))_{n \geq 1}$ ne dépend en réalité que du point de départ y de la marche $(y + S_n)_{n \geq 1}$ (cf Corollaire 5.7 de [41]).

La question de comment construire une fonction harmonique pour des marches markoviennes (c'est-à-dire dont les accroissements sont markoviens) plus générales restait donc à résoudre. La prise en compte de la dépendance de la chaîne de Markov $(X_n)_{n \geq 1}$ par rapport à son passé x fut l'obstacle majeur qui a motivé mes travaux. Les innovations techniques associées nécessaire pour obtenir les deux théorèmes suivants seront détaillées dans les Chapitres 2 et 3 respectivement.

Théorème 1.4.1 (Chaîne affine). *On suppose que $(X_n)_{n \geq 0}$ est une chaîne de Markov affine (CMA) vérifiant les conditions (C1.1/1.4.1)-(C1.3/1.4.3).*

1. *Pour tout $x \in \mathbb{R}$ et tout $y > 0$, la suite*

$$(V_n(x, y))_{n \geq 1} = (\mathbb{E}_x(y + S_n; \tau_y > n))_{n \geq 1}$$

converge vers un réel noté $V(x, y)$.

2. La fonction V est \mathbf{Q}_+ -harmonique sur $\mathbb{R} \times \mathbb{R}_+^*$: pour tout $x \in \mathbb{R}$ et tout $y > 0$,

$$\mathbf{Q}_+V(x, y) = \mathbb{E}_x(V(X_1, y + S_1) ; \tau_y > 1) = V(x, y).$$

3. Pour tout $x \in \mathbb{R}$, la fonction $y \mapsto V(x, y)$ est positive, croissante et asymptotiquement équivalente à y en l'infini :

$$\lim_{y \rightarrow +\infty} \frac{V(x, y)}{y} = 1.$$

4. Pour tout $\delta > 0$, $p \in (2, \alpha)$, $x \in \mathbb{R}$ et $y > 0$,

$$V(x, y) \leq \left(1 + \delta \left(1 + |x|^{p-1}\right)\right) y + c_{p,\delta} (1 + |x|^p).$$

La principale nouveauté dans la formulation de ce résultat comparé aux résultats antérieurs est dans la dépendance en x , exprimée en particulier dans le point 4. Pour notre second modèle de chaîne de Markov (couvrant le modèle précédent), on a le théorème équivalent suivant.

Théorème 1.4.2 (Chaîne à trou spectral). *Soit $(X_n)_{n \geq 0}$ chaîne de Markov avec un trou spectral (CMTS). Plus précisément, sous les Hypothèses **M3.1-M3.5** du Chapitre 3,*

1. Pour tout $x \in \mathbb{X}$ et $y \in \mathbb{R}$, la suite

$$(V_n(x, y))_{n \geq 1} = (\mathbb{E}_x(y + S_n ; \tau_y > n))_{n \geq 1}$$

converge vers un réel noté $V(x, y)$.

2. La fonction $V : \mathbb{X} \times \mathbb{R} \rightarrow \mathbb{R}$, est \mathbf{Q}_+ -harmonique : pour tout $x \in \mathbb{X}$ et $y \in \mathbb{R}$,

$$\mathbf{Q}_+V(x, y) = \mathbb{E}_x(V(X_1, y + S_1) ; \tau_y > 1) = V(x, y).$$

3. Pour tout $x \in \mathbb{X}$, la fonction $y \mapsto V(x, y)$ est positive, croissante et asymptotiquement équivalente à y en l'infini :

$$\lim_{y \rightarrow +\infty} \frac{V(x, y)}{y} = 1.$$

4. Il existe une fonction $N : \mathbb{X} \rightarrow \mathbb{R}_+$ telle que pour tout $\delta > 0$, $x \in \mathbb{X}$ et $y \in \mathbb{R}$,

$$V(x, y) \leq (1 + \delta) \max(y, 0) + c_\delta (1 + N(x)).$$

1.4.3 Positivité de la fonction harmonique

Comme on l'a vu dans le paragraphe précédent, l'important dans notre construction d'une fonction harmonique est de construire une fonction harmonique qui soit *strictement* positive. Dans cette partie, V , dite la fonction harmonique, désigne la fonction construite dans les Théorèmes 1.4.1 et 1.4.2 comme étant la limite de la suite $(V_n(x, y))_{n \geq 1}$. La stricte positivité de V est un enjeu à ne pas négliger et requiert des hypothèses supplémentaires. Dans le principe général il s'agit de reproduire la méthode récursive de Denisov et Wachtel qui nous a permis d'obtenir les Théorèmes 1.4.1 et 1.4.2 en bornant supérieurement la suite $(V_n(x, y))_{n \geq 1}$. De reproduire donc cette méthode et de la modifier pour construire cette fois-ci une borne inférieure adaptée qui puisse à terme nous permettre de séparer V de 0. Cette borne est donnée par le point 1 du Théorème 1.4.3 et le point 1 du Théorème 1.4.4 ci-dessous.

Dans le cas d'une chaîne de Markov affine (CMA), on donne deux conditions suffisantes pour que la fonction V soit strictement positive pour tous les points $x \in \mathbb{R}$ et tous les points $y > 0$. Sans plus de détail (voir le Chapitre 2), ces conditions sont les suivantes. Pour tout $x \in \mathbb{R}$ et $y > 0$,

$$\mathbb{P}_x(\tau_y > 1) = \mathbb{P}(ax + b > -y) > 0. \quad (\text{C2/1.4.7})$$

Pour tout $x \in \mathbb{R}$ et $y > 0$, il existe $p_0 \in (2, \alpha)$ tel que pour tout $c > 0$, il existe $n_0 \geq 1$ tel que

$$\mathbb{P}_x((X_{n_0}, y + S_{n_0}) \in K_{p_0, c}, \tau_y > n_0) > 0, \quad (\text{C3/1.4.8})$$

où

$$K_{p_0, c} = \{(x, y) \in \mathbb{R} \times \mathbb{R}_+^*, y \geq c(1 + |x|^{p_0})\}.$$

Il est clair que (C3/1.4.8) implique (C2/1.4.7). Cependant la condition (C2/1.4.7) ne permet d'assurer la stricte positivité de V que lorsque $\mathbb{E}(a) \geq 0$. La condition (C3/1.4.8) est, dans une formulation adaptée, déjà présente chez Denisov et Wachtel [20]. Pour plus de détails, je vous renvoie au Chapitre 2.

Théorème 1.4.3 (Chaîne affine). *On suppose que $(X_n)_{n \geq 0}$ est une chaîne de Markov affine (CMA) vérifiant les conditions (C1.1/1.4.1)-(C1.3/1.4.3).*

1. Pour tout $\delta > 0$, $p \in (2, \alpha)$, $x \in \mathbb{R}$ et $y > 0$,

$$V(x, y) \geq (1 - \delta)y + c_{p, \delta}(1 + |x|^p).$$

2. Si on suppose de plus ou bien la condition (C2/1.4.7) et $\mathbb{E}(a) \geq 0$ ou bien la condition (C3/1.4.8), alors la fonction V est strictement positive sur $\mathbb{R} \times \mathbb{R}_+^*$.

Lorsque j'ai traité le cas plus général des chaînes de Markov avec un trou spectral (CMTS), j'ai également changé mon point de vue sur la stricte positivité de V . Plutôt que de chercher des hypothèses assurant la stricte positivité pour tous les points $y > 0$, j'ai désiré décrire le domaine de positivité de V dit aussi support de V ,

$$\text{supp}(V) = \{(x, y) \in \mathbb{X} \times \mathbb{R} : V(x, y) > 0\}.$$

En effet, autant dans le cas indépendant on sent bien que $y = 0$ joue une frontière importante, autant dans le cas markovien la frontière est plus complexe. Bien sûr si $y > 0$ on parle bien pour τ_y d'un temps *de sortie* de la marche $(y + S_n)_{n \geq 1}$ et si $y \leq 0$ plutôt d'un temps *de retour* de cette même marche. Cependant dans le cas markovien, l'impulsion initiale x de la chaîne de Markov possède également son importance. Pour $y > 0$, dans certains cas, il est possible de se munir d'un $x \in \mathbb{X}$ suffisamment défavorable pour que partant de (x, y) on ait $\tau_y = 1$ *p.s.* Ainsi, dans une approche un peu plus étoffée, je n'impose pas à la fonction V d'être positive pour tout point de départ (x, y) mais je donne une description de son support. Il est alors suivant le modèle toujours possible de vérifier si V est strictement positive ou non sur $\mathbb{X} \times \mathbb{R}_+^*$ (elle ne l'est pas nécessairement).

Pour tout $\gamma > 0$, on considère l'ensemble

$$\mathcal{D}_\gamma := \{(x, y) \in \mathbb{X} \times \mathbb{R} : \exists n_0 \geq 1, \mathbb{P}_x(y + S_{n_0} > \gamma(1 + N(X_{n_0})), \tau_y > n_0) > 0\}.$$

On rappelle que l'existence d'une fonction N est donnée par le Théorème 1.4.2. On obtient alors le résultat suivant.

Théorème 1.4.4 (Chaîne à trou spectral). *On suppose que $(X_n)_{n \geq 0}$ est une chaîne de Markov avec un trou spectral (CMTS). Plus précisément, sous les Hypothèses **M3.1-M3.5** du Chapitre 3,*

1. Pour tout $\delta > 0$, $x \in \mathbb{X}$ et $y \in \mathbb{R}$,

$$V(x, y) \geq (1 - \delta) \max(y, 0) - c_\delta (1 + N(x)).$$

2. Il existe $\gamma_0 > 0$ telle que pour tout $\gamma \geq \gamma_0$,

$$\text{supp}(V) = \mathcal{D}_\gamma.$$

1.4.4 Théorèmes limites pour des marches markoviennes

Bien que l'on ait répondu pour l'instant à aucune des questions (Q1/1.2.3)-(Q4/1.2.6) dans le cadre de nos modèles markoviens, les résultats précédents sont d'importance et vont nous permettre, par l'existence d'une fonction harmonique V , de trouver l'équivalent recherché en (Q2/1.2.4). La stricte positivité de V justifiera la non-nullité de cet équivalent. Je passe pour le moment la description de lemmes techniques du Chapitre 2 ou 3 mettant en oeuvre des « trous » dans le processus $(y + S_n)_{n \geq 1}$ pour obtenir des résultats similaires à la marche aléatoire indépendante. Je donne plutôt ci-dessous le principe de départ des démonstrations des théorèmes qui suivent. J'en ai déjà fait rapidement mention, l'idée est de partir d'un *KMT* pour montrer que les résultats liés au mouvement brownien « se transporte » à notre marche markovienne. Ce couplage avec le mouvement brownien a été récemment obtenu pour les chaînes de Markov avec un trou spectral par Grama, Le Page et Peigné [40].

Proposition 1.4.5 (Grama, Le Page, Peigné). *On suppose que $(X_n)_{n \geq 0}$ est une chaîne de Markov affine (CMA) vérifiant les conditions (C1.1/1.4.1)-(C1.3/1.4.3) ou que $(X_n)_{n \geq 0}$ est une chaîne de Markov avec un trou spectral (CMTS) vérifiant les Hypothèses **M3.1-M3.5** du Chapitre 3. Alors, il existe $\varepsilon_0 > 0$ tel que pour tout $\varepsilon \in (0, \varepsilon_0]$, on peut reconstruire sans perte de généralité la chaîne de Markov $(X_n)_{n \geq 0}$ et un mouvement $(B_t)_{t \in \mathbb{R}_+}$ sur un même espace $(\Omega, \mathcal{F}, \mathbb{P})$ de façon à ce que pour tout $x \in \mathbb{X}$ et $n \geq 1$,*

$$\mathbb{P}_x \left(\sup_{0 \leq t \leq 1} |S_{[tn]} - \sigma B_{tn}| > n^{1/2-\varepsilon} \right) \leq \frac{c_\varepsilon}{n^\varepsilon} (1 + N(x)),$$

où σ est un réel positif.

La réponse à la question (Q1/1.2.3) est une conséquence facile de cette Proposition 1.4.5.

Corollaire 1.4.6. *Soit $(X_n)_{n \geq 0}$ une chaîne de Markov affine (CMA) vérifiant les conditions (C1.1/1.4.1)-(C1.3/1.4.3), respectivement une chaîne de Markov avec un trou spectral (CMTS) vérifiant les Hypothèses **M3.1-M3.5** du Chapitre 3. Alors pour tout $x \in \mathbb{R}$ et tout $y \in \mathbb{R}$, respectivement tout $x \in \mathbb{X}$ et tout $y \in \mathbb{R}$, on a*

$$\mathbb{P}_x (\tau_y < +\infty) < +\infty.$$

En ce qui concerne la réponse à la question (Q2/1.2.4) et à la question (Q3/1.2.5), le couplage fonctionne encore mais nécessite de nombreux développements supplémentaires utilisant en particulier les Théorèmes 1.4.1 à 1.4.4. On voit réapparaître l'importance de la fonction harmonique V .

Théorème 1.4.7 (Chaîne affine). *On suppose que $(X_n)_{n \geq 0}$ est une chaîne de Markov affine (CMA) vérifiant les conditions (C1.1/1.4.1)-(C1.3/1.4.3).*

1. Pour tout $p \in (2, \alpha)$, $x \in \mathbb{R}$ et $y > 0$,

$$\sqrt{n} \mathbb{P}_x(\tau_y > n) \leq c_p (1 + y + |x|)^p.$$

2. Si on suppose de plus ou bien la condition (C2/1.4.7) et $\mathbb{E}(a) \geq 0$ ou bien la condition (C3/1.4.8), alors il existe $\sigma > 0$ tel que

i. Pour tout $x \in \mathbb{R}$ et $y > 0$,

$$\mathbb{P}_x(\tau_y > n) \underset{n \rightarrow +\infty}{\sim} \frac{2V(x, y)}{\sqrt{2\pi n \sigma}}.$$

ii. Pour tout $x \in \mathbb{R}$, $y > 0$ et $t > 0$,

$$\mathbb{P}_x \left(\frac{y + S_n}{\sigma \sqrt{n}} \leq t \mid \tau_y > n \right) \xrightarrow{n \rightarrow +\infty} \Phi^+(t),$$

où $\Phi^+(t) = 1 - e^{-\frac{t^2}{2}}$ est la fonction de répartition de la loi de Rayleigh.

Dans le Chapitre 2, on donne également l'asymptotique du temps de retour, c'est-à-dire les mêmes résultats mais pour des points pour lesquels $y \leq 0$.

Dans le Chapitre 3 (pour les chaînes de Markov à trou spectral), notre réponse est plus précise à plusieurs points de vue. On continue de donner des résultats analogues sur le support de V mais on fait également remarquer que sur le complémentaire de V le seul comportement possible est la décroissance très rapide de la probabilité de survie (cf le point 3 du Théorème 1.4.8 ci-dessous). Ce résultat est optimal, je construis dans le Chapitre 3 un exemple d'une chaîne de Markov qui atteint cette borne. Ce dernier comportement tranche avec le cas indépendant pour lequel la marche survie avec une probabilité en C/\sqrt{n} ou ne survie pas (la probabilité est nulle). Les résultats sont plus précis également dans le sens où l'on donne des bornes du terme suivant dans le développement asymptotique lié à chaque réponse. A nouveau, la fonction N est donnée par le Théorème 1.4.2.

Théorème 1.4.8 (Chaîne à trou spectral). *On suppose que $(X_n)_{n \geq 0}$ est une chaîne de Markov avec un trou spectral (CMTS). Plus précisément, sous les Hypothèses M3.1-M3.5 du Chapitre 3, il existe $\sigma > 0$ tel que*

1. Pour tout $(x, y) \in \mathbb{X} \times \mathbb{R}$ et $n \geq 1$,

$$\mathbb{P}_x(\tau_y > n) \leq c \frac{1 + \max(y, 0) + N(x)}{\sqrt{n}}.$$

2. Pour tout $(x, y) \in \text{supp}(V)$,

$$\mathbb{P}_x(\tau_y > n) \underset{n \rightarrow +\infty}{\sim} \frac{2V(x, y)}{\sqrt{2\pi n \sigma}}.$$

3. Pour tout $(x, y) \notin \text{supp}(V)$ et $n \geq 1$,

$$\mathbb{P}_x(\tau_y > n) \leq c e^{-cn} (1 + N(x)).$$

4. Il existe $\varepsilon_0 > 0$ tel que pour tout $\varepsilon \in (0, \varepsilon_0)$, $n \geq 1$ et $(x, y) \in \mathbb{X} \times \mathbb{R}$,

$$\left| \mathbb{P}_x(\tau_y > n) - \frac{2V(x, y)}{\sqrt{2\pi n\sigma}} \right| \leq c_\varepsilon \frac{\max(y, 0) + \left(1 + y\mathbb{1}_{\{y > n^{1/2-\varepsilon}\}} + N(x)\right)^2}{n^{1/2+\varepsilon/16}}.$$

5. Pour tout $(x, y) \in \text{supp}(V)$ and $t \geq 0$,

$$\mathbb{P}_x\left(\frac{y + S_n}{\sigma\sqrt{n}} \leq t \mid \tau_y > n\right) \xrightarrow{n \rightarrow +\infty} \Phi^+(t),$$

où $\Phi^+(t) = 1 - e^{-\frac{t^2}{2}}$ est la fonction de répartition de la loi de Rayleigh.

6. De plus il existe $\varepsilon_0 > 0$ tel que pour tout $\varepsilon \in (0, \varepsilon_0)$, $n \geq 1$, $t_0 > 0$, $t \in [0, t_0]$ et $(x, y) \in \mathbb{X} \times \mathbb{R}$,

$$\left| \mathbb{P}_x\left(y + S_n \leq t\sqrt{n}, \tau_y > n\right) - \frac{2V(x, y)}{\sqrt{2\pi n\sigma}} \Phi^+\left(\frac{t}{\sigma}\right) \right| \leq c_{\varepsilon, t_0} \frac{\max(y, 0) + \left(1 + y\mathbb{1}_{\{y > n^{1/2-\varepsilon}\}} + N(x)\right)^2}{n^{1/2+\varepsilon/16}}.$$

1.4.5 Théorème local pour des marches markoviennes finies

On reprend dans ce paragraphe les principaux résultats du Chapitre 4. On se place dans le cas où la chaîne de Markov est à espace d'états fini (CMF) et l'on souhaite répondre à la question (Q4/1.2.6) à l'aide d'un résultat similaire à la Proposition 1.3.5. La méthode s'inspire très largement de celle que développe Denisov et Wachtel [20] dans le cas *lattice* et se fonde sur l'idée suivante. On reprend toujours les notations (1.2.1) et (1.2.2) et on suppose que les conditions (C1/1.4.4)-(C3/1.4.6) sont satisfaites. On procède en trois étapes, chacune renforçant la vitesse de convergence vers 0 d'un facteur $1/\sqrt{n}$. La première étape est immédiate et consiste à utiliser directement un théorème local pour la marche non-conditionnée. L'inégalité suivante se démontre de la même façon que le théorème 3.4 de Grama et Le Page [39]. Il existe une constante $c > 0$ telle que pour tout $x \in \mathbb{X}$, $y \in \mathbb{R}$, $z \geq 0$, $a > 0$ et $n \geq 1$,

$$I_n(x, y, z) := \mathbb{P}_x(y + S_n \in [z, z + a], \tau_y > n) \leq \mathbb{P}_x(y + S_n \in [z, z + a]) \leq \frac{c(1 + a^2)}{\sqrt{n}}.$$

L'important est que cette majoration ne dépend ni de y ni de z . Dans un deuxième temps, on va en conséquence pouvoir intégrer cette majoration de la façon suivante. Par la propriété de Markov, on écrit que, pour $k = \lfloor n/2 \rfloor$ la partie entière de $n/2$,

$$I_n(x, y, z) = \mathbb{E}_x(I_{n-k}(X_k, y + S_k, z); \tau_y > k) \leq \frac{c(1 + a^2)}{\sqrt{n-k}} \mathbb{P}_x(\tau_y > k).$$

A l'aide du point 1 du Théorème 1.4.8, on obtient une majoration en $1/n$ mais qui dépend cette fois du point de départ y :

$$I_n(x, y, z) \leq \frac{c(1 + a^2)(1 + \max(y, 0))}{n}. \quad (1.4.9)$$

Précisons que la fonction N du Théorème 1.4.8 peut être bornée uniformément en x puisque l'espace \mathbb{X} est fini. La troisième et dernière étape est plus astucieuse et consiste

à « renverser » la marche. Revenons pour un moment au cas indépendant, si X_1, \dots, X_n sont *i.i.d.* alors la chaîne « renversée » X_n, X_{n-1}, \dots, X_1 est également *i.i.d.* Posons $X_1^* = X_n, X_2^* = X_{n-1}, \dots, X_n^* = X_1$. Dans ce cas la marche « renversée » est définie par

$$S_k^* = -f(X_1^*) - \dots - f(X_k^*) = -(S_n - S_{n-k})$$

Plaçons-nous également pour simplifier dans le cas où S_n est une variable discrète. Alors on observe que pour tout $y > 0$ et $z > 0$,

$$\mathbb{P}(y + S_n = z, \tau_y > n) = \mathbb{P}(z + S_n^* = y, \tau_z^* > n)$$

avec $\tau_z^* := \min\{k \geq 1, z + S_k^* \leq 0\}$. Ce changement de regard sur l'évolution de la marche nous permet d'inter-changer les rôles du point de départ y et du point d'arrivée z . Revenons au cas markovien *non-lattice* et supposons pour le moment que l'on puisse également « renverser » la chaîne de Markov et la marche associée. L'inégalité (1.4.9) devient alors,

$$I_n(x, y, z) \leq \frac{c(1 + a^2)(1 + \max(z + a, 0))}{n}.$$

On peut alors à nouveau intégrer cette inégalité comme précédemment et on obtient que

$$I_n(x, y, z) \leq \frac{c(1 + a^3)(1 + \max(z, 0))(1 + \max(y, 0))}{n^{3/2}}.$$

Le processus s'arrête à cette troisième étape puisque cette fois le majorant dépend de y et de z . Ce procédé demande un travail supplémentaire pour obtenir l'asymptotique exact de $I_n(x, y, z)$ mais il nous donne déjà la bonne vitesse en $1/n^{3/2}$. Le fait de devoir « renverser » la chaîne est une difficulté majeure qui nous a poussés à travailler avec des chaînes de Markov à espace d'états fini (*CMF*). Dans ce cas, et sous les hypothèses (C1/1.4.4)-(C3/1.4.6), il existe une unique mesure invariante ν strictement positive sur \mathbb{X} . Il nous est alors possible de définir la chaîne duale et de renverser le processus en loi. On pose

$$\mathbf{P}^*(x, x') = \mathbf{P}(x', x) \frac{\nu(x')}{\nu(x)}, \quad \forall (x, x') \in \mathbb{X}^2. \quad (1.4.10)$$

La matrice \mathbf{P}^* est une matrice markovienne. Dans une situation plus générale que le cas fini, une difficulté majeure après avoir défini l'opérateur dual \mathbf{P}^* est qu'il faut encore montrer que cet opérateur dual \mathbf{P}^* vérifie les mêmes propriétés que l'opérateur initial \mathbf{P} . Cette condition est nécessaire afin de pouvoir étendre tous les résultats précédents connus pour la marche initiale à la marche duale (comme le théorème local sans conditionnement par exemple). Dans le cas fini, on verra que si \mathbf{P} vérifie les conditions (C1/1.4.4)-(C3/1.4.6), alors il n'est pas difficile de montrer que le noyau dual \mathbf{P}^* satisfait également les conditions (C1/1.4.4)-(C3/1.4.6). On définit alors la chaîne duale $(X_n^*)_{n \geq 0}$ comme étant une chaîne de Markov de noyau \mathbf{P}^* et cette chaîne vérifie les mêmes théorèmes que la chaîne initiale $(X_n)_{n \geq 0}$. Pour plus de détails, on renvoie au Chapitre 4 dans lequel je démontre les quatre résultats suivants. Le premier est un résultat de type Gnedenko-Stone. La vitesse donnée est d'ordre n seulement cependant le résultat est uniforme par rapport au point d'arrivée z . On rappelle que ν est la mesure invariante de $(X_n)_{n \geq 0}$, que V est la fonction harmonique définie par le Théorème 1.4.2 et que $\sigma > 0$ est un réel strictement positif décrivant « la variance de la marche ».

Théorème 1.4.9 (Chaîne finie). *On suppose que $(X_n)_{n \geq 0}$ est une chaîne de Markov finie (CMF) vérifiant les conditions (C1/1.4.4)-(C3/1.4.6) et que $a > 0$ est un réel strictement positif. Alors, il existe $\varepsilon_0 \in (0, \frac{1}{4})$ tel que pour tout $\varepsilon \in (0, \varepsilon_0)$, ψ fonction de \mathbb{X} dans \mathbb{R} positive et bornée, $y \in \mathbb{R}$ et $n \geq 2\varepsilon^{-3}$, on a*

$$\begin{aligned} \sup_{x \in \mathbb{X}, z \geq 0} n \left| \mathbb{E}_x(\psi(X_n) ; y + S_n \in [z, z + a], \tau_y > n) - \frac{2a\nu(\psi)V(x, y)}{\sqrt{2\pi n}\sigma^2} \varphi_+\left(\frac{z}{\sqrt{n}\sigma}\right) \right| \\ \leq c(1 + \max(y, 0)) \|\psi\|_\infty \left(\sqrt{\varepsilon} + \frac{c_\varepsilon(1 + \max(y, 0))}{n^\varepsilon} \right), \end{aligned}$$

où $\varphi_+(t) = t e^{-\frac{t^2}{2}} \mathbb{1}_{\{t \geq 0\}}$ est la densité de la loi de Rayleigh.

Le deuxième résultat est le théorème local recherché pour une marche markovienne finie conditionnée à rester positive, théorème qui résout la question (Q4/1.2.6) pour les chaînes de Markov finies. Pour $(X_n^*)_{n \geq 0}$ une chaîne de Markov duale (i.e. de noyau \mathbf{P}^* défini par (1.4.10)), on considère \mathbb{E}_ν^* l'espérance engendrée par les lois fini-dimensionnelles du processus $(X_n^*)_{n \geq 0}$ sachant que la loi initiale de X_0^* est donnée par ν .

Théorème 1.4.10 (Chaîne finie). *On suppose que $(X_n)_{n \geq 0}$ est une chaîne de Markov finie (CMF) vérifiant les conditions (C1/1.4.4)-(C3/1.4.6). Alors pour toute fonction positive et bornée $\psi : \mathbb{X} \rightarrow \mathbb{R}$, $a > 0$, $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z \geq 0$,*

$$\begin{aligned} \lim_{n \rightarrow +\infty} n^{3/2} \mathbb{E}_x(\psi(X_n) ; y + S_n \in [z, z + a], \tau_y > n) \\ = \frac{2V(x, y)}{\sqrt{2\pi}\sigma^3} \int_z^{z+a} \mathbb{E}_\nu^*(\psi(X_1^*) V^*(X_1^*, z' + S_1^*) ; \tau_{z'}^* > 1) dz'. \end{aligned}$$

De plus pour tout $n \geq 1$,

$$\begin{aligned} \sup_{x \in \mathbb{X}} \mathbb{E}_x(\psi(X_n) ; y + S_n \in [z, z + a], \tau_y > n) \\ \leq \frac{c \|\psi\|_\infty}{n^{3/2}} (1 + a^3) (1 + \max(z, 0)) (1 + \max(y, 0)). \end{aligned}$$

Le troisième résultat a été développé pour répondre aux besoins des processus de branchement abordés dans le paragraphe suivant. Il est cependant intéressant en soi et pour le dire un peu grossièrement exprime le fait qu'un processus conditionné à rester positif et à revenir à des valeurs petites (entre z et $z + a$) se comporte asymptotiquement comme le produit indépendant du processus direct et « renversé », tous les deux étant conditionnés à rester positif.

Pour tout $l \geq 1$ on note $\mathcal{C}^+(\mathbb{X}^l \times \mathbb{R})$ l'ensemble des fonctions positives $g : \mathbb{X}^l \times \mathbb{R} \rightarrow \mathbb{R}_+$ vérifiant les deux propriétés suivantes :

- pour tout $(x_1, \dots, x_l) \in \mathbb{X}^l$, la fonction $z \mapsto g(x_1, \dots, x_l, z)$ est continue,
- il existe $\varepsilon > 0$ tel que $\max_{\{(x_1, \dots, x_l) \in \mathbb{X}^l\}} \sup_{z \in \mathbb{R}} g(x_1, \dots, x_l, z) (1 + z)^{2+\varepsilon} < +\infty$.

On suppose que la chaîne duale $(X_n^*)_{n \geq 0}$ est construite de façon à être indépendante de la chaîne directe $(X_n)_{n \geq 0}$ et on note par \mathbb{E}_{x, x^*} l'espérance engendrée par les lois fini-dimensionnelles du processus $(X_n, X_n^*)_{n \geq 0}$ sachant que $(X_0, X_0^*) = (x, x^*)$. On note également V^* la fonction harmonique de la chaîne duale $(X_n^*)_{n \geq 0}$.

Théorème 1.4.11 (Chaîne finie). *On suppose que $(X_n)_{n \geq 0}$ est une chaîne de Markov finie (CMF) vérifiant les conditions (C1/1.4.4)-(C3/1.4.6). Alors pour tout $x \in \mathbb{X}$, $y \in \mathbb{R}$, $l \geq 1$, $m \geq 1$ et toute fonction $g \in \mathcal{C}^+(\mathbb{X}^{l+m} \times \mathbb{R})$,*

$$\begin{aligned} & \lim_{n \rightarrow +\infty} n^{3/2} \mathbb{E}_x (g(X_1, \dots, X_l, X_{n-m+1}, \dots, X_n, y + S_n) ; \tau_y > n) \\ &= \frac{2}{\sqrt{2\pi\sigma^3}} \int_0^{+\infty} \sum_{x^* \in \mathbb{X}} \mathbb{E}_{x, x^*} (g(X_1, \dots, X_l, X_m^*, \dots, X_1^*, z) \\ & \quad \times V(X_l, y + S_l) V^*(X_m^*, z + S_m^*) ; \tau_y > l, \tau_z^* > m) \nu(x^*) dz. \end{aligned}$$

Le quatrième et dernier résultat de cette section découle du Théorème 1.4.11. Il donne le comportement asymptotique de la probabilité que la marche $y + S_n$ passe pour la première fois dans les négatifs au temps n exactement.

Theorem 1.4.12. *On suppose que $(X_n)_{n \geq 0}$ est une chaîne de Markov finie (CMF) vérifiant les conditions (C1/1.4.4)-(C3/1.4.6). Alors pour tout $x \in \mathbb{X}$ et $y \in \mathbb{R}$,*

$$\lim_{n \rightarrow +\infty} n^{3/2} \mathbb{P}_x (\tau_y = n) = \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_0^{+\infty} \mathbb{E}_\nu^* (V^*(X_1^*, z) ; S_1^* \geq z) dz.$$

1.4.6 Processus de branchement en environnement markovien

On reprend le cadre des processus de branchement en environnement aléatoire décrit dans la Section 1.3.4 et qui est l'objet d'étude du Chapitre 5. Le principe est de remplacer l'hypothèse d'indépendance des environnements par le fait que la suite $(X_n)_{n \geq 0}$ est une chaîne de Markov à valeurs dans un espace d'états \mathbb{X} fini (CMF). On suppose toujours (C1/1.4.4), c'est-à-dire que la matrice de transition \mathbf{P} associée est primitive,

$$\exists k_0 \geq 1, \forall (i, j) \in \mathbb{X}^2, \quad \mathbf{P}^{k_0}(i, j) > 0. \quad (\text{C1/1.4.11})$$

Avec les notations de la Section 1.3.4, on suppose également que

$$\forall i \in \mathbb{X}, \quad 0 < \mathbb{E} \left[\left(\xi_i^{1,1} \right)^2 \right] = f_i''(1) < +\infty. \quad (\text{C2/1.4.12})$$

On se place dans un cadre *non-lattice* et on suppose que la condition (C3/1.4.6) est vérifiée pour la fonction ρ : pour tout $(\theta, a) \in \mathbb{R}^2$, il existe une orbite, c'est-à-dire une suite de points i_0, \dots, i_n dans \mathbb{X} communiquant de la façon suivante

$$\mathbf{P}(i_0, i_1) > 0, \mathbf{P}(i_1, i_2) > 0, \dots, \mathbf{P}(i_{n-1}, i_n) > 0, \mathbf{P}(i_n, i_0) > 0,$$

telle que

$$\rho(i_0) + \rho(i_1) + \dots + \rho(i_n) - (n+1)\theta \notin a\mathbb{Z}. \quad (\text{C3/1.4.13})$$

Sous ces hypothèses, les développements précédents des Sections 1.4.2 à 1.4.4 vont nous permettre dans le Chapitre 5 d'étendre les Propositions 1.3.7 à 1.3.10 au cas des environnements markoviens finis. Pour tout $\lambda \in \mathbb{R}$, et tout $i \in \mathbb{X}$, nous verrons que la limite de $\mathbb{E}_i^{1/n} (e^{\lambda S_n})$ quand $n \rightarrow +\infty$ existe et ne dépend pas de i . On définit alors

$$k(\lambda) := \lim_{n \rightarrow +\infty} \mathbb{E}_i^{1/n} (e^{\lambda S_n}).$$

Ce réel $k(\lambda)$ correspond à la valeur propre dominante de l'opérateur perturbé \mathbf{P}_λ qui nous sera utile pour effectuer le changement de loi nécessaire aux cas sous-critiques.

Théorème 1.4.13 (Cas critique). *Supposons les conditions (C1/1.4.11)-(C3/1.4.13) et supposons que*

$$k'(0) = \nu(\rho) = \sum_{i \in \mathbb{X}} \rho(i) \nu(i) = 0.$$

Alors il existe u_1 une fonction sur \mathbb{X} strictement positive telle que pour tout $(i, j) \in \mathbb{X}^2$,

$$\mathbb{P}_i(Z_n > 0, X_n = j) \underset{n \rightarrow +\infty}{\sim} \frac{\nu(j)u_1(i)}{\sqrt{n}}.$$

Théorème 1.4.14 (Cas fortement sous-critique). *Supposons les conditions (C1/1.4.11)-(C3/1.4.13) et supposons que*

$$k'(0) < 0, \quad k'(1) < 0.$$

Alors il existe v_1 et u_2 deux fonctions sur \mathbb{X} strictement positives telles que pour tout $(i, j) \in \mathbb{X}^2$,

$$\mathbb{P}_i(Z_n > 0, X_n = j) \underset{n \rightarrow +\infty}{\sim} k(1)^n v_1(i) u_2(j).$$

On rappelle que sous la condition (C1/1.4.11), il existe une unique mesure invariante pour la chaîne $(X_n)_{n \geq 0}$ que l'on note ν .

Théorème 1.4.15 (Cas critique intermédiaire). *Supposons les conditions (C1/1.4.11)-(C3/1.4.13) et supposons que*

$$k'(0) < 0, \quad k'(1) = 0.$$

Alors il existe v_1 et u_3 deux fonctions sur \mathbb{X} strictement positives telles que pour tout $(i, j) \in \mathbb{X}^2$,

$$\mathbb{P}_i(Z_n > 0, X_n = j) \underset{n \rightarrow +\infty}{\sim} k(1)^n \frac{v_1(i)u_3(j)}{\sqrt{n}}.$$

Théorème 1.4.16 (Cas faiblement sous-critique). *Supposons les conditions (C1/1.4.11)-(C3/1.4.13) et supposons que*

$$k'(0) < 0, \quad k'(1) > 0.$$

Alors il existe un unique $\lambda \in (0, 1)$ tel que $k'(\lambda) = 0$ et il existe u_4 une fonction sur \mathbb{X}^2 strictement positive telle que pour tout $(i, j) \in \mathbb{X}^2$,

$$\mathbb{P}_i(Z_n > 0, X_n = j) \underset{n \rightarrow +\infty}{\sim} k(\lambda)^n \frac{u_4(i, j)}{n^{3/2}}.$$

1.5 Rappels sur les marches indépendantes

Dans cette section je rappelle, et détaille parfois un peu, la démonstration du calcul de l'asymptotique de la probabilité de survie d'une marche aléatoire unidimensionnelle lorsque ses accroissements sont *i.i.d.* De cette façon, on pourra éventuellement constater l'importante différence que possède cette approche historique avec celle que nous nous proposons de suivre dans les chapitres suivants pour traiter nos modèles markoviens. La preuve ci-dessous est très majoritairement due à Spitzer et je paraphrase essentiellement

son livre [66]. Pour une version légèrement différente de cette démonstration on pourra aussi se référer au livre de Feller [29].

Soient $(\Omega, \mathcal{F}, \mathbb{P})$ un espace probabilisé et \mathbb{E} l'espérance associée. On considère $(X_n)_{n \geq 1}$ une suite de variables aléatoires indépendantes et identiquement distribuées définies sur $(\Omega, \mathcal{F}, \mathbb{P})$ et à valeurs dans \mathbb{R} . On définit la marche associée par :

$$\forall n \geq 1, \quad S_n = X_1 + \cdots + X_n \quad \text{et} \quad S_0 = 0.$$

On considère également τ le premier instant strictement positif pour lequel la marche rentre dans la demi-droite des réels négatifs :

$$\tau = \inf \{n \geq 1, S_n \leq 0\}.$$

L'objectif de cette section est de redonner les idées de la démonstration du résultat bien connu de Spitzer [65].

Théorème 1.5.1 (Spitzer). *Supposons que la suite $(X_n)_{n \geq 1}$ est i.i.d. et supposons que*

$$\mathbb{E}(X_1) = 0 \quad \text{et} \quad \sigma^2 = \mathbb{E}(X_1^2) < \infty.$$

1. Alors, la série $\sum_{k \geq 1} \frac{1}{k} (\mathbb{P}(S_k > 0) - \frac{1}{2})$ converge vers un réel noté

$$\alpha = \sum_{k=1}^{+\infty} \frac{1}{k} \left(\mathbb{P}(S_k > 0) - \frac{1}{2} \right) \in \mathbb{R}.$$

2. De plus, la probabilité que la marche survive est donnée asymptotiquement par l'équivalent suivant :

$$\mathbb{P}(\tau > n) \underset{n \rightarrow \infty}{\sim} \frac{e^\alpha}{\sqrt{\pi n}}.$$

Dans la section suivante, on démontre le Théorème 1.5.1. Pour une meilleure lisibilité, les lemmes d'analyse pure sont reportés dans la Section 1.5.2.

1.5.1 Démonstration du Théorème 1.5.1

La première étape est d'obtenir des informations sur la fonction caractéristique du couple (τ, S_τ) . Pour ce faire, introduisons quelques notations.

Définition 1.5.2.

1. On désigne par φ la fonction caractéristique de la loi commune aux X_n , $n \geq 1$: pour tout $\theta \in \mathbb{R}$,

$$\varphi(\theta) := \mathbb{E}(e^{i\theta X_1}).$$

2. On découpe le plan complexe en deux demi-plans :

$$\mathcal{D}^+ := \{z \in \mathbb{C} : \Im(z) > 0\} \quad \text{et} \quad \mathcal{D}^- := \{z \in \mathbb{C} : \Im(z) < 0\} \quad (1.5.1)$$

où $\Im(z)$ est la partie imaginaire du complexe z .

3. Pour tout $n \geq 0$, on pose

$$\begin{aligned} \forall z \in \mathcal{D}^- \cup \mathbb{R}, \quad \varphi_n^-(z) &:= \mathbb{E}(e^{izS_n}; \tau = n) \\ \forall z \in \mathcal{D}^+ \cup \mathbb{R}, \quad \varphi_n^+(z) &:= \mathbb{E}(e^{izS_n}; \tau > n). \end{aligned}$$

4. Pour tout $t \in [0, 1[$, on pose

$$\begin{aligned} \forall z \in \mathcal{D}^- \cup \mathbb{R}, \quad \varphi^-(t, z) &:= \sum_{n=1}^{+\infty} t^n \varphi_n^-(z) = \mathbb{E} \left(t^\tau e^{izS_\tau}; \tau < +\infty \right) \\ \forall z \in \mathcal{D}^+ \cup \mathbb{R}, \quad \varphi^+(t, z) &:= \sum_{n=0}^{+\infty} t^n \varphi_n^+(z) = \mathbb{E} \left(\sum_{n=0}^{\tau-1} t^n e^{izS_n} \right). \end{aligned}$$

Par l'indépendance des accroissements on établit en premier lieu la factorisation suivante :

Lemme 1.5.3 (Factorisation de Wiener-Hopf). *Pour tout $t \in [0, 1[$ et $\theta \in \mathbb{R}$:*

$$1 - \varphi^-(t, \theta) = (1 - t\varphi(\theta)) \varphi^+(t, \theta).$$

Démonstration. Pour tout $n \geq 0$ et $\theta \in \mathbb{R}$,

$$\varphi_{n+1}^-(\theta) + \varphi_{n+1}^+(\theta) = \mathbb{E} \left(e^{i\theta S_{n+1}}; \tau \geq n+1 \right) = \mathbb{E} \left(e^{i\theta S_n} e^{i\theta X_{n+1}}; \tau > n \right).$$

Naturellement, τ est un temps d'arrêt pour la filtration canonique $(\mathcal{F}_n)_{n \geq 0}$ avec $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ et \mathcal{F}_0 la tribu grossière. Ainsi $\{\tau > n\} \in \mathcal{F}_n$ est indépendant de X_{n+1} et

$$\varphi_{n+1}^-(\theta) + \varphi_{n+1}^+(\theta) = \mathbb{E} \left(e^{i\theta S_n}; \tau > n \right) \mathbb{E} \left(e^{i\theta X_{n+1}} \right) = \varphi(\theta) \varphi_n^+(\theta)$$

En sommant on obtient pour tout $t \in [0, 1[$,

$$\varphi^-(t, \theta) + \varphi^+(t, \theta) = 1 + \sum_{n=0}^{+\infty} t^{n+1} \left(\varphi_{n+1}^-(\theta) + \varphi_{n+1}^+(\theta) \right) = 1 + t\varphi(\theta)\varphi^+(t, \theta).$$

□

A l'aide de cette factorisation, on en déduit le résultat suivant.

Lemme 1.5.4. *Pour tout $t \in [0, 1[$ on a*

$$\begin{aligned} \forall z \in \mathcal{D}^- \cup \mathbb{R}, \quad \varphi^-(t, z) &= \mathbb{E} \left(t^\tau e^{izS_\tau}; \tau < +\infty \right) \\ &= 1 - \exp \left(- \sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{E} \left(e^{izS_n}; S_n \leq 0 \right) \right) \end{aligned}$$

et

$$\begin{aligned} \forall z \in \mathcal{D}^+ \cup \mathbb{R}, \quad \varphi^+(t, z) &= \mathbb{E} \left(\sum_{n=0}^{\tau-1} t^n e^{izS_n} \right) \\ &= \exp \left(\sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{E} \left(e^{izS_n}; S_n > 0 \right) \right). \end{aligned}$$

Démonstration. Pour tout $z \in \mathbb{C}$ tel que $|z| < 1$, on a $\exp \left(- \sum_{n=1}^{+\infty} \frac{z^n}{n} \right) = 1 - z$. Donc pour tout $t \in [0, 1[$ et $\theta \in \mathbb{R}$, par indépendance des variables aléatoires X_n , $n \geq 1$,

$$1 - t\varphi(\theta) = \exp \left(- \sum_{n=1}^{+\infty} \frac{t^n}{n} \varphi(\theta)^n \right) = \exp \left(- \sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{E} \left(e^{i\theta S_n} \right) \right).$$

En découpant suivant le signe de S_n , on obtient que

$$1 - t\varphi(\theta) = \exp\left(-\sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{E}\left(e^{i\theta S_n}; S_n > 0\right)\right) \exp\left(-\sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{E}\left(e^{i\theta S_n}; S_n \leq 0\right)\right). \quad (1.5.2)$$

On reprend les notations présentées en (1.5.1) et pour tout $t \in [0, 1[$, on considère ζ_t^- et ζ_t^+ deux fonctions complexes définies sur $\mathcal{D}^- \cup \mathbb{R}$, respectivement $\mathcal{D}^+ \cup \mathbb{R}$ de la façon suivante :

$$\begin{aligned} \forall z \in \mathcal{D}^- \cup \mathbb{R}, \quad \zeta_t^-(z) &:= \left[1 - \sum_{n=1}^{+\infty} t^n \mathbb{E}\left(e^{izS_n}; \tau = n\right)\right] \exp\left(\sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{E}\left(e^{izS_n}; S_n \leq 0\right)\right), \\ \forall z \in \mathcal{D}^+ \cup \mathbb{R}, \quad \zeta_t^+(z) &:= \left[\sum_{n=0}^{+\infty} t^n \mathbb{E}\left(e^{izS_n}; \tau > n\right)\right] \exp\left(-\sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{E}\left(e^{izS_n}; S_n > 0\right)\right). \end{aligned}$$

Il est clair que pour t fixé entre $[0, 1[$, la fonction ζ_t^- est analytique sur \mathcal{D}^- , continue en tout point de \mathbb{R} et bornée sur $\mathcal{D}^- \cup \mathbb{R}$ par

$$\begin{aligned} |\zeta_t^-(z)| &\leq \left[1 + \sum_{n=1}^{+\infty} t^n \mathbb{P}(\tau = n)\right] \exp\left(\sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{P}(S_n \leq 0)\right) \\ &\leq [1 + \mathbb{E}(t^\tau; \tau < +\infty)] \exp\left(\sum_{n=1}^{+\infty} \frac{t^n}{n}\right) \leq \frac{2}{1-t}. \end{aligned} \quad (1.5.3)$$

De même la fonction ζ_t^+ est analytique sur \mathcal{D}^+ , continue en tout point de \mathbb{R} et bornée sur $\mathcal{D}^+ \cup \mathbb{R}$ par

$$|\zeta_t^+(z)| \leq \left[\sum_{n=0}^{+\infty} t^n \mathbb{P}(\tau > n)\right] \exp\left(\sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{P}(S_n > 0)\right) \leq \left(\frac{1}{1-t}\right)^2. \quad (1.5.4)$$

De plus en utilisant la définition de φ^- et φ^+ (voir Définition 1.5.2), le Lemme 1.5.3 et l'égalité (1.5.2), on obtient la relation suivante sur \mathbb{R} entre ζ_t^- et ζ_t^+

$$\begin{aligned} \forall \theta \in \mathbb{R}, \quad \zeta_t^-(\theta) &= [1 - \varphi^-(t, \theta)] \exp\left(\sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{E}\left(e^{i\theta S_n}; S_n \leq 0\right)\right) \\ &= [1 - t\varphi(\theta)] \varphi^+(t, \theta) \exp\left(\sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{E}\left(e^{i\theta S_n}; S_n \leq 0\right)\right) \\ &= \varphi^+(t, \theta) \exp\left(-\sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{E}\left(e^{i\theta S_n}; S_n > 0\right)\right) \\ &= \zeta_t^+(\theta). \end{aligned} \quad (1.5.5)$$

On pose maintenant ζ_t la fonction définie sur \mathbb{C} par

$$\zeta_t(z) = \begin{cases} \zeta_t^-(z) & \text{si } z \in \mathcal{D}^- \cup \mathbb{R}, \\ \zeta_t^+(z) & \text{si } z \in \mathcal{D}^+ \cup \mathbb{R}. \end{cases}$$

D'après (1.5.5), la fonction ζ_t est bien définie sur \mathbb{R} . De plus ζ_t est analytique sur $\mathcal{D}^- \cup \mathcal{D}^+$ et continue sur \mathbb{R} . Donc par le Lemme 1.5.13, ζ_t est analytique sur \mathbb{C} . De plus par (1.5.3) et (1.5.4), ζ_t est bornée sur \mathbb{C} donc par le théorème de Liouville,

$$\exists c_t \in \mathbb{C}, \text{ telle que } \forall z \in \mathbb{C}, \quad \zeta_t(z) = c_t.$$

On détermine la constante en remarquant que, par convergence dominée,

$$\lim_{\substack{\theta \rightarrow +\infty \\ \theta \in \mathbb{R}}} \zeta_t^+(i\theta) = \lim_{\substack{\theta \rightarrow +\infty \\ \theta \in \mathbb{R}}} \left[1 + \sum_{n=1}^{+\infty} t^n \mathbb{E} \left(e^{-\theta S_n} ; \tau > n \right) \right] \exp \left(- \sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{E} \left(e^{-\theta S_n} ; S_n > 0 \right) \right) = 1.$$

Donc $c_t = 1$ et par suite, $\forall z \in \mathbb{C}$, $\zeta_t^-(z) = \zeta_t^+(z) = 1$. Donc par les définitions de ζ_t^- , ζ_t^+ , φ^- et φ^+ , on en conclut que

$$\begin{aligned} \forall z \in \mathcal{D}^- \cup \mathbb{R}, \quad 1 - \varphi^-(t, z) &= \exp \left(- \sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{E} \left(e^{iz S_n} ; S_n \leq 0 \right) \right), \\ \forall z \in \mathcal{D}^+ \cup \mathbb{R}, \quad \varphi^+(t, z) &= \exp \left(\sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{E} \left(e^{iz S_n} ; S_n > 0 \right) \right). \end{aligned}$$

□

Lemme 1.5.5. *Le temps de sortie dans les négatifs τ est fini presque sûrement :*

$$\mathbb{P}(\tau < +\infty) = 1.$$

Démonstration. D'après le Lemme 1.5.4, pour tout $t \in [0, 1[$,

$$\varphi^-(t, 0) = \mathbb{E}(t^\tau ; \tau < +\infty) = 1 - \exp \left(- \sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{P}(S_n \leq 0) \right).$$

Par conséquent, par le théorème de convergence monotone de Lebesgue,

$$\begin{aligned} \mathbb{P}(\tau < +\infty) &= \lim_{\substack{t \rightarrow 1 \\ t < 1}} \mathbb{E}(t^\tau ; \tau < +\infty) \\ &= 1 - \exp \left(- \lim_{\substack{t \rightarrow 1 \\ t < 1}} \sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{P}(S_n \leq 0) \right) \\ &= 1 - \exp \left(- \sum_{n=1}^{+\infty} \frac{1}{n} \mathbb{P}(S_n \leq 0) \right). \end{aligned} \tag{1.5.6}$$

De plus, par le théorème central limite, $\mathbb{P}(S_n \leq 0) \xrightarrow{n \rightarrow +\infty} 1/2$ et donc

$$\sum_{n=1}^{+\infty} \frac{1}{n} \mathbb{P}(S_n \leq 0) = +\infty. \tag{1.5.7}$$

Ce qui, avec (1.5.6), conclut la preuve du lemme. □

Lemme 1.5.6. *Le temps de sortie τ n'est pas intégrable :*

$$\mathbb{E}(\tau) = +\infty.$$

Démonstration. Pour tout $k \geq 1$ et $t \in [0, 1[$, on a $0 \leq (1 - t^k)/(1 - t) \leq k$. Donc pour tout $t \in [0, 1[$,

$$\mathbb{E}(\tau) \geq \frac{1 - \mathbb{E}(t^\tau)}{1 - t} = \frac{1 - \varphi^-(t, 0)}{1 - t}.$$

Par le Lemme 1.5.4,

$$\begin{aligned}\mathbb{E}(\tau) &\geq \exp\left(-\sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{P}(S_n \leq 0)\right) \exp(-\ln(1-t)) \\ &= \exp\left(-\sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{P}(S_n \leq 0) + \sum_{n=1}^{+\infty} \frac{t^n}{n}\right) \\ &= \exp\left(\sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{P}(S_n > 0)\right).\end{aligned}$$

Or de façon analogue à (1.5.7), $\sum_{n=1}^{+\infty} \frac{1}{n} \mathbb{P}(S_n > 0) = +\infty$. D'où, par convergence monotone,

$$\mathbb{E}(\tau) \geq \lim_{\substack{t \rightarrow 1 \\ t < 1}} \exp\left(\sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{P}(S_n > 0)\right) = +\infty.$$

□

Pour établir la convergence de $\sum_{k=1}^{+\infty} \frac{1}{n} \left(\mathbb{P}(S_n > 0) - \frac{1}{2}\right)$ on va avoir besoin des deux lemmes suivants.

Lemme 1.5.7.

$$\lim_{n \rightarrow +\infty} \mathbb{E}\left(\frac{S_n}{\sqrt{n}\sigma}; S_n \geq 0\right) = \frac{1}{\sqrt{2\pi}}.$$

Démonstration. On note F la fonction de répartition de la loi normale centrée et réduite. Pour tout $n \geq 1$, on écrit que

$$\left|\mathbb{E}\left(\frac{S_n}{\sqrt{n}\sigma}; S_n \geq 0\right) - \frac{1}{\sqrt{2\pi}}\right| = \left|\int_0^{+\infty} \mathbb{P}\left(\frac{S_n}{\sqrt{n}\sigma} \geq u\right) - (1 - F(u)) \, du\right|.$$

Fixons $A > 0$, on a alors

$$\begin{aligned}&\left|\mathbb{E}\left(\frac{S_n}{\sqrt{n}\sigma}; S_n \geq 0\right) - \frac{1}{\sqrt{2\pi}}\right| \\ &\leq \int_0^A \left|F(u) - \mathbb{P}\left(\frac{S_n}{\sqrt{n}\sigma} < u\right)\right| \, du + \int_A^{+\infty} \left|1 - F(u) - \mathbb{P}\left(\frac{S_n}{\sqrt{n}\sigma} \geq u\right)\right| \, du \\ &\leq \int_0^A \left|F(u) - \mathbb{P}\left(\frac{S_n}{\sqrt{n}\sigma} < u\right)\right| \, du + \int_A^{+\infty} \frac{1}{u^2} + \frac{\mathbb{E}(S_n^2)}{n\sigma^2 u^2} \, du \\ &= \int_0^A \left|F(u) - \mathbb{P}\left(\frac{S_n}{\sqrt{n}\sigma} < u\right)\right| \, du + \frac{2}{A}.\end{aligned}$$

Ainsi par convergence dominée, pour tout $A > 0$,

$$\lim_{n \rightarrow +\infty} \left|\mathbb{E}\left(\frac{S_n}{\sqrt{n}\sigma}; S_n \geq 0\right) - \frac{1}{\sqrt{2\pi}}\right| \leq \frac{2}{A}$$

et lorsque A tend vers $+\infty$ on obtient le résultat souhaité. □

Lemme 1.5.8.

$$\lim_{\substack{t \rightarrow 1 \\ t < 1}} \sqrt{1-t} \sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{E}(S_n; S_n \geq 0) = \frac{\sigma}{\sqrt{2}}.$$

Démonstration. On commence tout d'abord par observer que, pour tout $t \in [0, 1[$,

$$\frac{1}{\sqrt{1-t}} = \sum_{n=0}^{+\infty} a_n t^n,$$

où $a_n = \frac{(2n)!}{2^{2n}(n!)^2}$, pour tout $n \geq 0$ et par la formule de Stirling,

$$a_n \underset{n \rightarrow +\infty}{\sim} \frac{1}{\sqrt{\pi n}}.$$

Notamment $a_n > 0$ et $\sum_{n \geq 0} a_n t^n$ a un rayon de convergence égal à 1 et diverge en 1. De plus par le Lemme 1.5.7,

$$a_n \underset{n \rightarrow +\infty}{\sim} \frac{\sqrt{2}}{\sigma n} \mathbb{E}(S_n; S_n \geq 0) = b_n$$

Donc par le Lemme 1.5.14,

$$\frac{\sqrt{2}}{\sigma} \sum_{n=0}^{+\infty} \frac{t^n}{n} \mathbb{E}(S_n; S_n \geq 0) \underset{\substack{t \rightarrow 1 \\ t < 1}}{\sim} \frac{1}{\sqrt{1-t}},$$

d'où le résultat. □

Lemme 1.5.9. *La fonction $t \mapsto \sum_{k=1}^{+\infty} \frac{t^k}{k} \left(\mathbb{P}(S_k > 0) - \frac{1}{2} \right)$ admet une limite dans $\mathbb{R} \cup \{+\infty\}$ lorsque $t \rightarrow 1$ et*

$$-\mathbb{E}(S_\tau) = \frac{\sigma}{\sqrt{2}} \exp \left[\lim_{\substack{t \rightarrow 1 \\ t < 1}} \sum_{k=1}^{+\infty} \frac{t^k}{k} \left(\mathbb{P}(S_k > 0) - \frac{1}{2} \right) \right] \in]0, +\infty].$$

Démonstration. Posons $b_n = \mathbb{E}(S_n; \tau > n)$ pour tout $n \geq 0$. Montrons que $(b_n)_{n \geq 0}$ est une suite croissante qui tend vers $-\mathbb{E}(S_\tau)$ quand n tend vers $+\infty$. Soit $n \geq 0$,

$$\begin{aligned} b_{n+1} &= \mathbb{E}(S_{n+1}; \tau > n+1) \\ &= \mathbb{E}(S_{n+1}; \tau > n) - \mathbb{E}(S_{n+1}; \tau = n+1) \\ &= \mathbb{E}(S_n; \tau > n) + \mathbb{E}(X_{n+1}; \tau > n) - \mathbb{E}(S_{n+1}; \tau = n+1). \end{aligned}$$

Or par indépendance des accroissements $(X_n)_{n \geq 1}$ et leur centrage,

$$\mathbb{E}(X_{n+1}; \tau > n) = \mathbb{E}(X_{n+1}) \mathbb{P}(\tau > n) = 0.$$

De plus par définition de τ ,

$$-\mathbb{E}(S_{n+1}; \tau = n+1) = -\mathbb{E}(S_\tau; \tau = n+1) \geq 0.$$

Donc, on en déduit que

$$b_{n+1} \geq \mathbb{E}(S_n; \tau > n) = b_n.$$

Montrons maintenant que $(b_n)_{n \geq 0}$ tend vers $-\mathbb{E}(S_\tau) \in \mathbb{R} \cup \{+\infty\}$. Puisque par le Lemme 1.5.5, τ est fini presque sûrement,

$$-\mathbb{E}(S_\tau) = -\sum_{n=1}^{+\infty} \mathbb{E}(S_n; \tau = n) = \sum_{n=1}^{+\infty} [b_n - \mathbb{E}(S_n; \tau > n-1)].$$

A nouveau par indépendance et centrage des $(X_n)_{n \geq 1}$,

$$-\mathbb{E}(S_\tau) = \sum_{n=1}^{+\infty} [b_n - b_{n-1}] = \lim_{n \rightarrow +\infty} b_n.$$

Comme X_1 est centré et non-identiquement nul, on note au passage que

$$-\mathbb{E}(S_\tau) \geq b_1 = \mathbb{E}(X_1; X_1 > 0) > 0. \quad (1.5.8)$$

Maintenant, puisque $(b_n)_{n \geq 0}$ est croissante, deux possibilités s'offrent à nous : ou bien la suite converge dans \mathbb{R} ou bien la suite diverge vers l'infini. Supposons que $(b_n)_{n \geq 0}$ converge vers $-\mathbb{E}(S_\tau) < +\infty$. Dans ce cas, en invoquant le Lemme 1.5.14 avec $a_n = -\mathbb{E}(S_\tau)$ pour tout $n \geq 1$, on trouve que

$$\sum_{n=1}^{+\infty} b_n t^n = \sum_{n=1}^{+\infty} t^n \mathbb{E}(S_n; \tau > n) \underset{\substack{t \rightarrow 1 \\ t < 1}}{\sim} \frac{-\mathbb{E}(S_\tau)}{1-t}$$

ou encore

$$\lim_{\substack{t \rightarrow 1 \\ t < 1}} (1-t) \sum_{n=1}^{+\infty} t^n \mathbb{E}(S_n; \tau > n) = -\mathbb{E}(S_\tau). \quad (1.5.9)$$

Dans le second cas où $b_n \xrightarrow{n \rightarrow +\infty} +\infty$, d'après le Lemme 1.5.15, l'égalité (1.5.9) reste encore vraie.

Rappelons maintenant que d'après le Lemme 1.5.4, pour tout $t \in [0, 1[$ et tout $\theta \geq 0$,

$$\varphi^+(t, i\theta) = \sum_{n=0}^{+\infty} t^n \mathbb{E}(e^{-\theta S_n}; \tau > n) = \exp\left(\sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{E}(e^{-\theta S_n}; S_n > 0)\right).$$

Puisque $|t^n S_n e^{-\theta S_n} \mathbb{1}_{\{\tau > n\}}| \leq t^n |S_n|$ et que $t^n \mathbb{E}(|S_n|) \leq t^n n \mathbb{E}(|X_1|)$ est sommable en n pour tout $t < 1$, il est possible de dériver terme à terme par rapport à θ le membre de gauche. De même pour le membre de droite, donc pour tout $\theta \geq 0$ et $t \in [0, 1[$,

$$\sum_{n=0}^{+\infty} t^n \mathbb{E}(S_n e^{-\theta S_n}; \tau > n) = \sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{E}(S_n e^{-\theta S_n}; S_n > 0) \exp\left(\sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{E}(e^{-\theta S_n}; S_n > 0)\right).$$

En particulier,

$$\sum_{n=0}^{+\infty} t^n \mathbb{E}(S_n; \tau > n) = \sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{E}(S_n; S_n > 0) \exp\left(\sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{P}(S_n > 0)\right).$$

En utilisant (1.5.9),

$$-\mathbb{E}(S_\tau) = \lim_{\substack{t \rightarrow 1 \\ t < 1}} \sqrt{1-t} \sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{E}(S_n; S_n > 0) \sqrt{1-t} \exp\left(\sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{P}(S_n > 0)\right)$$

En conséquence, par le Lemme 1.5.8,

$$\begin{aligned} -\mathbb{E}(S_\tau) &= \frac{\sigma}{\sqrt{2}} \lim_{\substack{t \rightarrow 1 \\ t < 1}} \sqrt{1-t} \exp\left(\sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{P}(S_n > 0)\right) \\ &= \frac{\sigma}{\sqrt{2}} \lim_{\substack{t \rightarrow 1 \\ t < 1}} \exp\left(-\frac{1}{2} \sum_{n=1}^{+\infty} \frac{t^n}{n} + \sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{P}(S_n > 0)\right) \\ &= \frac{\sigma}{\sqrt{2}} \exp\left(\lim_{\substack{t \rightarrow 1 \\ t < 1}} \sum_{n=1}^{+\infty} \frac{t^n}{n} \left(\mathbb{P}(S_n > 0) - \frac{1}{2}\right)\right), \end{aligned}$$

ce qui avec (1.5.8) établit le lemme. □

Lemme 1.5.10.

$$\sum_{n=1}^{+\infty} \frac{1}{n} \mathbb{P}(S_n = 0) < +\infty.$$

Démonstration. Rappelons que, par le Lemme 1.5.5, le temps τ est fini presque sûrement. Donc, d'après le Lemme 1.5.4, pour tout $t \in [0, 1[$ et tout $\theta \geq 0$,

$$\varphi^-(t, -i\theta) = \mathbb{E}(t^\tau e^{\theta S_\tau}) = 1 - \exp\left(-\sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{E}(e^{\theta S_n}; S_n \leq 0)\right)$$

Par convergence dominée, pour tout $t \in [0, 1[$ lorsque $\theta \rightarrow +\infty$, on obtient que

$$\mathbb{E}(t^\tau; S_\tau = 0) = 1 - \exp\left(-\sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{P}(S_n = 0)\right).$$

Supposons que $\sum_{n=1}^{+\infty} \frac{1}{n} \mathbb{P}(S_n = 0) = +\infty$. Alors, par convergence monotone lorsque $t \rightarrow 1$,

$$\mathbb{P}(S_\tau = 0) = 1.$$

Mais ceci implique notamment que $\mathbb{P}(X_1 < 0) = 0$ ce qui contredit le fait que la loi de X_1 est centrée et non dégénérée. Donc nécessairement $\sum_{n=1}^{+\infty} \frac{1}{n} \mathbb{P}(S_n = 0) < +\infty$. \square

Lemme 1.5.11. La série $\sum_{k \geq 1} \frac{1}{k} \left(\mathbb{P}(S_k > 0) - \frac{1}{2}\right)$ converge vers un réel noté

$$\alpha = \sum_{k=1}^{\infty} \frac{1}{k} \left(\mathbb{P}(S_k > 0) - \frac{1}{2}\right) = \lim_{\substack{t \rightarrow 1 \\ t < 1}} \sum_{k=1}^{+\infty} \frac{t^k}{k} \left(\mathbb{P}(S_k > 0) - \frac{1}{2}\right) \in \mathbb{R}$$

et

$$-\mathbb{E}(S_\tau) = \frac{\sigma}{\sqrt{2}} e^\alpha.$$

Démonstration. Procédons par l'absurde et supposons que

$$\lim_{\substack{t \rightarrow 1 \\ t < 1}} \sum_{n=1}^{+\infty} \frac{t^n}{n} \left(\mathbb{P}(S_n > 0) - \frac{1}{2}\right) = +\infty.$$

Notons que pour tout $t \geq [0, 1[$,

$$\begin{aligned} \sum_{n=1}^{+\infty} \frac{t^n}{n} \left(\mathbb{P}(S_n > 0) - \frac{1}{2}\right) &= \sum_{n=1}^{+\infty} \frac{t^n}{n} \left(\frac{1}{2} - \mathbb{P}(S_n \leq 0)\right) \\ &= -\sum_{n=1}^{+\infty} \frac{t^n}{n} \left(\mathbb{P}(S_n < 0) - \frac{1}{2}\right) - \sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{P}(S_n = 0). \end{aligned}$$

En utilisant le Lemme 1.5.10, la série entière $\sum_{n \geq 1} \frac{t^n}{n} \mathbb{P}(S_n = 0)$ a un rayon de convergence égal à 1 et converge en 1, donc est bornée sur $[0, 1[$. On en déduit que

$$\lim_{\substack{t \rightarrow 1 \\ t < 1}} \sum_{n=1}^{+\infty} \frac{t^n}{n} \left(\mathbb{P}(S_n < 0) - \frac{1}{2}\right) = -\infty. \quad (1.5.10)$$

Or la marche aléatoire $T_n = -X_1 - \dots - X_n$ et $T_0 = 0$ a des accroissements $(-X_n)_{n \geq 1}$ indépendants, identiquement distribués, centrés et dont le moment d'ordre 2 existe. Donc tous les lemmes précédents sont vérifiés pour $(T_n)_{n \geq 0}$. En particulier, par le Lemme 1.5.9,

$$\lim_{\substack{t \rightarrow 1 \\ t < 1}} \sum_{n=1}^{+\infty} \frac{t^n}{n} \left(\mathbb{P}(S_n < 0) - \frac{1}{2} \right) = \lim_{\substack{t \rightarrow 1 \\ t < 1}} \sum_{n=1}^{+\infty} \frac{t^n}{n} \left(\mathbb{P}(T_n > 0) - \frac{1}{2} \right) \in \mathbb{R} \cup \{+\infty\},$$

ce qui contredit (1.5.10). Donc

$$\lim_{\substack{t \rightarrow 1 \\ t < 1}} \sum_{n=1}^{+\infty} \frac{t^n}{n} \left(\mathbb{P}(S_n > 0) - \frac{1}{2} \right) \neq +\infty.$$

Or d'après le Lemme 1.5.9, nécessairement la limite existe dans \mathbb{R} . Notons α cette limite. La série entière $\sum_{n \geq 1} \frac{t^n}{n} \left(\mathbb{P}(S_n > 0) - \frac{1}{2} \right)$ a un rayon de convergence égal à 1 et converge pour $t = 1$, donc d'après le théorème taubérien d'Hardy-Littlewood,

$$\lim_{\substack{t \rightarrow 1 \\ t < 1}} \sum_{n=1}^{+\infty} \frac{t^n}{n} \left(\mathbb{P}(S_n > 0) - \frac{1}{2} \right) = \sum_{n=1}^{+\infty} \frac{1}{n} \left(\mathbb{P}(S_n > 0) - \frac{1}{2} \right) = \alpha \in \mathbb{R}.$$

Par le Lemme 1.5.9 on conclut également que $-\mathbb{E}(S_\tau) = \frac{\sigma}{\sqrt{2}} e^\alpha$. \square

Lemme 1.5.12. *Le comportement asymptotique de la probabilité de survie est donné par*

$$\mathbb{P}(\tau > n) \underset{n \rightarrow +\infty}{\sim} \frac{e^\alpha}{\sqrt{n\pi}}.$$

Démonstration. Considérons la série entière associée, pour tout $t \in [0, 1[$,

$$\begin{aligned} \sum_{n=0}^{+\infty} \mathbb{P}(\tau > n) t^n &= 1 + \sum_{n=1}^{+\infty} \mathbb{P}(\tau > n-1) t^n - \sum_{n=1}^{+\infty} \mathbb{P}(\tau = n) t^n \\ &= 1 + t \sum_{n=0}^{+\infty} \mathbb{P}(\tau > n) t^n - \mathbb{E}(t^\tau). \end{aligned}$$

Donc, d'après le Lemme 1.5.4,

$$(1-t) \sum_{n=0}^{+\infty} \mathbb{P}(\tau > n) t^n = 1 - \varphi^-(t, 0) = \exp \left(- \sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{P}(S_n \leq 0) \right).$$

Par conséquent,

$$\begin{aligned} \sqrt{1-t} \sum_{n=0}^{+\infty} \mathbb{P}(\tau > n) t^n &= \exp \left(-\frac{1}{2} \ln(1-t) - \sum_{n=1}^{+\infty} \frac{t^n}{n} \mathbb{P}(S_n \leq 0) \right) \\ &= \exp \left(\sum_{n=1}^{+\infty} \frac{t^n}{n} \left(\frac{1}{2} - \mathbb{P}(S_n \leq 0) \right) \right) \\ &= \exp \left(\sum_{n=1}^{+\infty} \frac{t^n}{n} \left(\mathbb{P}(S_n > 0) - \frac{1}{2} \right) \right). \end{aligned}$$

Par le Lemme 1.5.11,

$$\lim_{\substack{t \rightarrow 1 \\ t < 1}} \sqrt{1-t} \sum_{n=0}^{+\infty} \mathbb{P}(\tau > n) t^n = e^\alpha.$$

Par le Lemme 1.5.16,

$$\lim_{n \rightarrow +\infty} \frac{1}{\sqrt{n}} \sum_{k=0}^n \mathbb{P}(\tau > k) = \frac{2e^\alpha}{\sqrt{\pi}}.$$

Finalement, puisque $(\mathbb{P}(\tau > n))_{n \geq 0}$ est décroissante, on conclut par le Lemme 1.5.17 que

$$\lim_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}(\tau > n) = \frac{e^\alpha}{\sqrt{\pi}}.$$

□

1.5.2 Quelques lemmes d'analyse

Lemme 1.5.13. *Soient \mathcal{D}^+ et \mathcal{D}^- les demi-plans définis en (1.5.1). Si ζ est une fonction holomorphe sur $\mathcal{D}^+ \cup \mathcal{D}^-$ et continue sur \mathbb{C} alors elle est holomorphe sur \mathbb{C} .*

Inspiré de la proposition P17.3 de [66] page 179, il faut remarquer que l'intégrale de tout triangle entourant un point de l'axe des réels est nulle et que donc par le théorème de Morera, la fonction en question est analytique partout.

On pourra trouver le lemme abélien suivant au théorème 57 page 108 de Hardy [45].

Lemme 1.5.14. *Soient $(a_n)_{n \geq 0}$ et $(b_n)_{n \geq 0}$ deux suites de réels telles que*

1. *pour tout $n \geq 1$, $a_n > 0$,*
2. *la série entière $\sum_{n \geq 0} a_n t^n$ a un rayon de convergence égale à 1 et diverge en $t = 1$,*
3. *les suites $(a_n)_{n \geq 0}$ et $(b_n)_{n \geq 0}$ sont équivalentes : $a_n \underset{n \rightarrow +\infty}{\sim} b_n$.*

Alors les fonctions $t \mapsto \sum_{n=0}^{+\infty} a_n t^n$ et $t \mapsto \sum_{n=0}^{+\infty} b_n t^n$ sont équivalentes en 1 :

$$\sum_{n=0}^{+\infty} a_n t^n \underset{\substack{t \rightarrow 1 \\ t < 1}}{\sim} \sum_{n=0}^{+\infty} b_n t^n.$$

Démonstration. Puisque les suites $(a_n)_{n \geq 0}$ et $(b_n)_{n \geq 0}$ sont équivalentes, il est clair que pour n assez grand, $b_n > 0$ et que la série entière $\sum_{n \geq 0} b_n t^n$ a aussi un rayon de convergence égale à 1 et diverge en $t = 1$. Soit $\varepsilon > 0$, il existe $n_0 \geq 0$ tel que pour tout $n \geq n_0$, on a $|a_n - b_n| \leq \frac{\varepsilon}{2} a_n$. Donc, pour tout $t \in [0, 1[$,

$$\begin{aligned} \left| \sum_{n=0}^{+\infty} a_n t^n - b_n t^n \right| &\leq \sum_{n=0}^{n_0} |a_n - b_n| + \sum_{n=n_0+1}^{+\infty} |a_n - b_n| t^n \\ &\leq \sum_{n=0}^{n_0} |a_n - b_n| + \frac{\varepsilon}{2} \sum_{n=n_0+1}^{+\infty} a_n t^n \end{aligned}$$

Or $\sum_{n=0}^{+\infty} a_n = +\infty$ donc il existe $t_0 < 1$ tel que pour tout $t \in [t_0, 1[$, on a $\sum_{n=0}^{n_0} |a_n - b_n| \leq \frac{\varepsilon}{2} \sum_{n=0}^{+\infty} a_n t^n$. Ainsi, pour tout $t \in [t_0, 1[$,

$$\begin{aligned} \left| \sum_{n=0}^{+\infty} a_n t^n - b_n t^n \right| &\leq \sum_{n=0}^{n_0} |a_n - b_n| + \sum_{n=n_0+1}^{+\infty} |a_n - b_n| t^n \\ &\leq \frac{\varepsilon}{2} \sum_{n=0}^{+\infty} a_n t^n + \frac{\varepsilon}{2} \sum_{n=0}^{+\infty} a_n t^n. \end{aligned}$$

□

Lemme 1.5.15. Soit $(b_n)_{n \geq 0}$ une suite de réels positifs telle que la série entière associée $\sum_{n \geq 0} t^n b_n$ a un rayon de convergence égal à 1 et telle que $b_n \xrightarrow[n \rightarrow +\infty]{} +\infty$. Alors

$$\lim_{\substack{t \rightarrow 1 \\ t < 1}} (1-t) \sum_{n=0}^{+\infty} t^n b_n = +\infty.$$

Démonstration. Puisque $(b_n)_{n \geq 0}$ diverge, pour tout $A > 0$ il existe $n_0 \geq 1$ tel que pour tout $n \geq n_0$, on a $b_n \geq A$. Donc pour tout $t \in [0, 1[$,

$$(1-t) \sum_{n=0}^{+\infty} t^n b_n \geq (1-t) \sum_{n=n_0}^{+\infty} t^n b_n \geq At^{n_0}.$$

D'où, pour tout $A > 0$,

$$\liminf_{\substack{t \rightarrow 1 \\ t < 1}} (1-t) \sum_{n=0}^{+\infty} t^n b_n \geq A$$

et le lemme est vérifié en faisant tendre $A \rightarrow +\infty$. \square

Lemme 1.5.16 (Karamata). Soit $(a_n)_{n \geq 0}$ une suite de réels positifs (ou nuls) telle que la série entière associée $\sum_{n \geq 0} a_n t^n$ a un rayon de convergence égal à 1 et telle que

$$\lim_{\substack{t \rightarrow 1 \\ t < 1}} \sqrt{1-t} \sum_{n=0}^{+\infty} a_n t^n = 1.$$

Alors

$$\lim_{n \rightarrow +\infty} \frac{a_0 + \dots + a_n}{\sqrt{n}} = \frac{2}{\sqrt{\pi}}.$$

Démonstration. Pour toute fonction $h : [0, +\infty[\rightarrow [0, +\infty[$ bornée, on considère, lorsqu'elle existe, la limite suivante :

$$\mathcal{L}(h) = \lim_{\substack{t \rightarrow 1 \\ t < 1}} \sqrt{1-t} \sum_{n=0}^{+\infty} a_n t^n h(t^n).$$

Il est clair que \mathcal{L} est linéaire. De plus, lorsque $h_k : t \mapsto t^k$ avec $k \geq 0$, on obtient :

$$\mathcal{L}(h_k) = \lim_{\substack{t \rightarrow 1 \\ t < 1}} \sqrt{1-t} \sum_{n=0}^{+\infty} a_n t^{n(k+1)} = \lim_{\substack{t \rightarrow 1 \\ t < 1}} \frac{\sqrt{1-t}}{\sqrt{1-t^{k+1}}} \sqrt{1-t^{k+1}} \sum_{n=0}^{+\infty} a_n t^{n(k+1)}.$$

Puisque par hypothèse $\sqrt{1-s} \sum_{n=0}^{+\infty} a_n s^n \rightarrow 1$ quand $s \rightarrow 1$, on trouve donc

$$\mathcal{L}(h_k) = \frac{1}{\sqrt{k+1}}.$$

D'autre part, en notant Γ la fonction gamma : $\Gamma(z) = \int_0^{+\infty} u^{z-1} e^{-u} du$,

$$\int_0^{+\infty} \frac{h_k(e^{-u}) e^{-u}}{\sqrt{u} \Gamma(1/2)} du = \int_0^{+\infty} \frac{e^{-u(k+1)}}{\sqrt{u} \Gamma(1/2)} du = \frac{1}{\sqrt{k+1}} \int_0^{+\infty} \frac{e^{-v}}{\sqrt{v} \Gamma(1/2)} dv = \frac{1}{\sqrt{k+1}}.$$

D'où pour tout $k \geq 0$,

$$\mathcal{L}(h_k) = \int_0^{+\infty} \frac{h_k(e^{-u}) e^{-u}}{\sqrt{u} \Gamma(1/2)} du,$$

et par linéarité, pour tout polynôme P ,

$$\mathcal{L}(P) = \int_0^{+\infty} \frac{P(e^{-u}) e^{-u}}{\sqrt{u}\Gamma(1/2)} du.$$

Fixons désormais $h(t) = \frac{1}{t} \mathbb{1}_{\{t \geq e^{-1}\}}$, pour tout $t \in (0, 1]$. Soit $\varepsilon > 0$ on considère également h_ε^+ et h_ε^- deux fonctions continue sur $[0, 1]$ définie respectivement par

$$h_\varepsilon^+(t) = \begin{cases} h(t) & \text{si } t \in [0, e^{-1} - \varepsilon] \cup [e^{-1}, 1] \\ \frac{e^1}{\varepsilon} (t - e^{-1} + \varepsilon) & \text{si } t \in [e^{-1} - \varepsilon, e^{-1}] \end{cases}$$

et

$$h_\varepsilon^-(t) = \begin{cases} h(t) & \text{si } t \in [0, e^{-1}] \cup [e^{-1} + \varepsilon, 1] \\ \frac{1}{\varepsilon(e^{-1} + \varepsilon)} (t - e^{-1}) & \text{si } t \in [e^{-1}, e^{-1} + \varepsilon] \end{cases}$$

Puisque pour tout $t \in [0, 1]$, on a $h(t) \leq h_\varepsilon^+(t)$, on écrit que

$$\mathcal{L}^+(h) = \limsup_{\substack{t \rightarrow 1 \\ t < 1}} \sqrt{1-t} \sum_{n=0}^{+\infty} a_n t^n h(t^n) \leq \limsup_{\substack{t \rightarrow 1 \\ t < 1}} \sqrt{1-t} \sum_{n=0}^{+\infty} a_n t^n h_\varepsilon^+(t^n).$$

Par le théorème d'approximation de Weierstrass, il existe un polynôme P_ε^+ approchant h_ε^+ : $\sup_{t \in [0, 1]} |h_\varepsilon^+(t) - P_\varepsilon^+(t)| \leq \varepsilon$. En conséquence,

$$\begin{aligned} \mathcal{L}^+(h) &= \mathcal{L}(P_\varepsilon^+) + \varepsilon \\ &= \int_0^{+\infty} \frac{P_\varepsilon^+(e^{-u}) e^{-u}}{\sqrt{u}\Gamma(1/2)} du + \varepsilon \\ &\leq \int_0^{+\infty} \frac{h_\varepsilon^+(e^{-u}) e^{-u}}{\sqrt{u}\Gamma(1/2)} du + 2\varepsilon \\ &\leq \int_0^{+\infty} \frac{h(e^{-u}) e^{-u}}{\sqrt{u}\Gamma(1/2)} du + \int_0^{+\infty} \frac{h_\varepsilon^+(e^{-u}) e^{-u}}{\sqrt{u}\Gamma(1/2)} \mathbb{1}_{\{e^{-u} \in [e^{-1} - \varepsilon, e^{-1}]\}} du + 2\varepsilon \\ &= \int_0^{+\infty} \frac{h(e^{-u}) e^{-u}}{\sqrt{u}\Gamma(1/2)} du + \int_{e^{-1} - \varepsilon}^{e^{-1}} \frac{h_\varepsilon^+(x)}{\sqrt{-\ln(x)}\Gamma(1/2)} dx + 2\varepsilon \\ &\leq \int_0^{+\infty} \frac{h(e^{-u}) e^{-u}}{\sqrt{u}\Gamma(1/2)} du + \frac{e^1 \varepsilon}{\Gamma(1/2)} + 2\varepsilon. \end{aligned}$$

En prenant la limite quand $\varepsilon \rightarrow 0$, on obtient le majorant suivant,

$$\mathcal{L}^+(h) \leq \int_0^{+\infty} \frac{h(e^{-u}) e^{-u}}{\sqrt{u}\Gamma(1/2)} du.$$

De la même façon, on montre que

$$\mathcal{L}^-(h) = \liminf_{\substack{t \rightarrow 1 \\ t < 1}} \sqrt{1-t} \sum_{n=0}^{+\infty} a_n t^n h(t^n) \geq \int_0^{+\infty} \frac{h(e^{-u}) e^{-u}}{\sqrt{u}\Gamma(1/2)} du.$$

Par conséquent,

$$\mathcal{L}(h) = \int_0^{+\infty} \frac{h(e^{-u}) e^{-u}}{\sqrt{u}\Gamma(1/2)} du = \int_0^1 \frac{1}{\sqrt{u}\Gamma(1/2)} du = \frac{2}{\Gamma(1/2)} = \frac{2}{\sqrt{\pi}}.$$

De plus,

$$\begin{aligned}\mathcal{L}(h) &= \lim_{N \rightarrow +\infty} \sqrt{1 - e^{-\frac{1}{N}}} \sum_{n=0}^{+\infty} a_n e^{-\frac{n}{N}} h\left(e^{-\frac{n}{N}}\right) = \lim_{N \rightarrow +\infty} \frac{1}{\sqrt{N}} \sum_{n=0}^{+\infty} a_n \mathbb{1}_{\left\{e^{-\frac{n}{N}} \geq e^{-1}\right\}} \\ &= \lim_{N \rightarrow +\infty} \frac{1}{\sqrt{N}} \sum_{n=0}^N a_n,\end{aligned}$$

ce qui conclut la preuve. \square

Lemme 1.5.17. Soit $(a_n)_{n \geq 0}$ une suite décroissante de réels positifs (ou nuls) telle que

$$\lim_{n \rightarrow +\infty} \frac{a_0 + \cdots + a_n}{\sqrt{n}} = \frac{2}{\sqrt{\pi}}.$$

Alors

$$\lim_{n \rightarrow +\infty} \sqrt{n\pi} a_n = 1.$$

Démonstration. On considère G la fonction définie sur $[0, +\infty[$ par $G(x) = a_{[x]}$ où $[x]$ est la partie entière de $x \in [0, +\infty[$. Pour tout entier $n \geq 1$ et tout réel $\theta > 1$, on commence par remarquer que, par décroissance des a_k et donc de G ,

$$na_{[n\theta]} \leq \frac{1}{\theta - 1} \int_n^{\theta n} G(x) dx \leq na_n.$$

Par hypothèse, $\int_0^n G(x) dx = a_0 + \cdots + a_{n-1} \sim 2\sqrt{n}/\sqrt{\pi}$ quand $n \rightarrow +\infty$. En particulier pour n suffisamment grand, $\int_0^n G(x) dx > 0$. Ainsi,

$$\frac{na_{[n\theta]}}{\int_0^n G(x) dx} \leq \frac{\int_0^{\theta n} G(x) dx - \int_0^n G(x) dx}{(\theta - 1) \int_0^n G(x) dx} \leq \frac{na_n}{\int_0^n G(x) dx}.$$

En passant à la limite lorsque $n \rightarrow +\infty$,

$$\frac{\sqrt{\pi}}{2} \limsup_{n \rightarrow +\infty} \sqrt{n} a_{[n\theta]} \leq \lim_{n \rightarrow +\infty} \frac{\sqrt{\theta n} - \sqrt{n}}{(\theta - 1)\sqrt{n}} = \frac{\sqrt{\theta} - 1}{\theta - 1} \leq \frac{\sqrt{\pi}}{2} \liminf_{n \rightarrow +\infty} \sqrt{n} a_n.$$

Donc, pour tout $\theta > 1$,

$$\frac{\sqrt{\pi}}{2\sqrt{\theta}} \limsup_{p \rightarrow +\infty} \sqrt{p} a_p \leq \frac{\sqrt{\theta} - 1}{\theta - 1} \leq \frac{\sqrt{\pi}}{2} \liminf_{n \rightarrow +\infty} \sqrt{n} a_n.$$

En passant à la limite quand $\theta \rightarrow 1$, on conclut que

$$\limsup_{n \rightarrow +\infty} \sqrt{\pi n} a_n \leq 1 \leq \liminf_{n \rightarrow +\infty} \sqrt{\pi n} a_n.$$

\square

Chapter 2

Limit theorems for affine Markov walks conditioned to stay positive

This chapter is the subject of the article [36] in collaboration with
Ion Grama and Emile Le Page

to appear in

Annales de l'institut Henri Poincaré (B) Probabilités et Statistiques.

RÉSUMÉ. On considère une marche Markovienne réelle $S_n = X_1 + \dots + X_n$ dont les accroissements $(X_n)_{n \geq 1}$ sont définis par une récursion stochastique partant de $X_0 = x$. Pour un point de départ $y > 0$, on note par τ_y le temps de sortie du processus $(y + S_n)_{n \geq 1}$ de la partie positive de la droite des réels. On s'intéresse au comportement asymptotique de la probabilité de l'évènement $\tau_y \geq n$ ainsi qu'à la loi conditionnelle de $y + S_n$ sachant $\tau_y \geq n$ quand $n \rightarrow +\infty$.

ABSTRACT. Consider the real Markov walk $S_n = X_1 + \dots + X_n$ with increments $(X_n)_{n \geq 1}$ defined by a stochastic recursion starting at $X_0 = x$. For a starting point $y > 0$, denote by τ_y the exit time of the process $(y + S_n)_{n \geq 1}$ from the positive part of the real line. We investigate the asymptotic behaviour of the probability of the event $\tau_y \geq n$ and of the conditional law of $y + S_n$ given $\tau_y \geq n$ as $n \rightarrow +\infty$.

2.1 Introduction

Assume that the Markov chain $(X_n)_{n \geq 0}$ is defined by the stochastic recursion

$$X_0 = x \in \mathbb{R}, \quad X_{n+1} = a_{n+1}X_n + b_{n+1}, \quad n \geq 0, \quad (2.1.1)$$

where $(a_i, b_i)_{i \geq 1}$ is a sequence of i.i.d. real random pairs satisfying $\mathbb{E}(|a_1|^\alpha) < 1$ and $\mathbb{E}(|b_1|^\alpha) < +\infty$, for some $\alpha > 2$. Consider the Markov walk $S_n = \sum_{k=1}^n X_k$, $n \geq 1$. Under a set of conditions ensuring the existence of the spectral gap of the transition operator of the Markov chain $(X_n)_{n \geq 0}$, it was established in Guivarc'h and Le Page [43] that there exist constants μ and $\sigma > 0$ such that, for any $t \in \mathbb{R}$,

$$\mathbb{P}_x \left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq t \right) \rightarrow \Phi(t) \quad \text{as } n \rightarrow +\infty, \quad (2.1.2)$$

where Φ is the standard normal distribution function and \mathbb{P}_x is the probability measure generated by $(X_n)_{n \geq 0}$ starting at $X_0 = x$. There are simple expressions of μ and σ in terms of law of the pair (a, b) : in particular $\mu = \frac{\mathbb{E}b}{1 - \mathbb{E}a}$.

For a starting point $y > 0$, define the first time when the affine Markov walk $(y + S_n)_{n \geq 1}$ becomes non-positive by setting

$$\tau_y = \min\{k \geq 1, y + S_k \leq 0\}.$$

In this paper we complete upon the results in [43] by determining the asymptotic of the probability $\mathbb{P}_x(\tau_y > n)$ and proving a conditional version of the limit theorem (2.1.2) for the sum $y + S_n$, given the event $\{\tau_y > n\}$ in the case when $\mu = 0$. The main challenge in obtaining these asymptotics is to prove the existence of a positive harmonic function pertaining to the associated Markov chain $(X_n, y + S_n)_{n \geq 0}$. A positive harmonic function, say V , is defined as a positive solution of the equation $\mathbf{Q}_+ V = V$, where \mathbf{Q}_+ is the restriction on $\mathbb{R} \times \mathbb{R}_+^*$ of the Markov transition kernel \mathbf{Q} of the chain $(X_n, y + S_n)_{n \geq 0}$.

From the more general results of the paper it follows that, under the same hypotheses that ensure the CLT (see Condition 2.1 in Section 2.2), if the pair (a, b) is such that $\mathbb{P}((a, b) \in (0, 1) \times (0, C]) > 0$ and $\mathbb{P}((a, b) \in (-1, 0) \times (0, C]) > 0$, for some $C > 0$, then

$$\mathbb{P}_x(\tau_y > n) \underset{n \rightarrow +\infty}{\sim} \frac{2V(x, y)}{\sqrt{2\pi n\sigma}} \quad (2.1.3)$$

and

$$\mathbb{P}_x\left(\frac{y + S_n}{\sigma\sqrt{n}} \leq t \mid \tau_y > n\right) \underset{n \rightarrow +\infty}{\longrightarrow} \Phi^+(t), \quad (2.1.4)$$

where $\Phi^+(t) = 1 - e^{-t^2/2}$ is the Rayleigh distribution function. In particular, the above mentioned results hold true if a and b are independent and a is such that $\mathbb{P}(a \in (0, 1)) > 0$ and $\mathbb{P}(a \in (-1, 0)) > 0$. Less restrictive assumptions on the pair (a, b) are formulated in our Section 2.2. For example, (2.1.3) and (2.1.4) hold if $a = 0$ and b satisfies Condition 2.1 which covers the case of independent increments.

The above mentioned results are in line with those already known in the literature for random walks with independent increments conditioned to stay in limited areas: the rate $1/\sqrt{n}$ in (2.1.3) and the asymptotic distribution $\Phi^+(t)$ in (2.1.4) are the same. We refer the reader to Iglehart [47], Bolthausen [9], Doney [21], Bertoin and Doney [6], Borovkov [11, 10], Caravenna [13], Eichelsbacher and Köning [27], Garbit [31], Denisov, Vatutin and Wachtel [17], Denisov and Wachtel [18, 20]. More general walks with increments forming a Markov chain have been considered by Presman [60, 61], Varopoulos [68, 69], Dembo [15], Denisov and Wachtel [19] or Grama, Le Page and Peigné [41]. In [60, 61] the case of sums of lattice random variables defined on finite regular Markov chains has been considered. Varopoulos [68, 69] studied Markov chains with bounded increments and obtained lower and upper bounds for the probabilities of the exit time from cones. Some studies take advantage of additional properties: for instance in [19] the Markov walk has a special integrated structure; in [41] the moments of X_n are bounded by some constants not depending on the initial condition. However, to the best of our knowledge, the asymptotic behaviour of the probability $\mathbb{P}_x(\tau_y > n)$ in the case of the stochastic recursion (2.1.1) has not yet been considered in the literature.

Note that the Wiener-Hopf factorization, which usually is employed in the case of independent random variables, cannot be applied in a straightforward manner for Markov chains. Instead, to study the case of the stochastic recursion, we rely upon the developments in [19], [20] and [41]. The main idea of the paper is given below. The existence of the positive harmonic function V is linked to the construction of a martingale approximation for the Markov walk $(S_n)_{n \geq 1}$. While the harmonicity is inherently related to the

martingale properties, the difficulty is to show that the approximating martingale is integrable at the exit time of the Markov walk $(y + S_n)_{n \geq 1}$. In contrast to [20] and [41], our proof of the existence of V employs different techniques according to positivity or not of the values of $\mathbb{E}(a_1)$. The constructed harmonic function allows to deduce the properties of the exit time and the conditional distribution of the Markov walk from those of the Brownian motion using a strong approximation result for Markov chains from Grama, Le Page and Peigné [40].

The technical steps of the proofs are as follows. We first deal with the case when the starting point of the Markov walk $(y + S_n)_{n \geq 0}$ is large: $y > n^{1/2-\varepsilon}$, for some $\varepsilon > 0$. When $y > 0$ is arbitrary, the law of iterated logarithm ensures that the sequence $(|y + S_k|)_{1 \leq k \leq n^{1-\varepsilon}}$ will cross the level $n^{1/2-\varepsilon}$ with high probability. Then, by the Markov property, we are able to reduce the problem to a Markov walk with a large starting point $y' = y + S_{\nu_n}$, where ν_n is the first time when the sequence $|y + S_k|$ exceeds the level $n^{1/2-\varepsilon}$. The major difficulty, compared to [20] and [41], is that, for the affine model under consideration, the sequence $(X_{\nu_n})_{n \geq 1}$ is not bounded in \mathbb{L}^1 . To overcome this we need a control of the moments of X_n in function of the initial state $X_0 = x$ and the lag n .

We end this section by agreeing upon some basic notations. As from now and for the rest of this paper the symbols $c, c_\alpha, c_{\alpha,\beta}, \dots$ denote positive constants depending only on their indices. All these constants are likely to change their values every occurrence. The indicator of an event A is denoted by $\mathbb{1}_A$. For any bounded measurable function f on $\mathbb{X} = \mathbb{R}^d$, $d = 1, 2$, random variable X in \mathbb{X} and event A , the integral $\int_{\mathbb{X}} f(x) \mathbb{P}(X \in dx, A)$ means the expectation $\mathbb{E}(f(X); A) = \mathbb{E}(f(X) \mathbb{1}_A)$.

2.2 Notations and results

Assume that on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we are given a sequence of independent real random pairs (a_i, b_i) , $i \geq 1$, with the same law as the generic random pair (a, b) . Denote by \mathbb{E} the expectation pertaining to \mathbb{P} . Consider the Markov chain $(X_n)_{n \geq 0}$ defined by the affine transformations

$$X_{n+1} = a_{n+1}X_n + b_{n+1}, \quad n \geq 0,$$

where $X_0 = x \in \mathbb{R}$ is a starting point. The partial sum process $(S_n)_{n \geq 0}$ defined by $S_n = \sum_{i=1}^n X_i$ for all $n \geq 1$ and $S_0 = 0$ will be called affine Markov walk. Note that $(S_n)_{n \geq 0}$ itself is not a Markov chain, but the pair $(X_n, S_n)_{n \geq 0}$ forms a Markov chain.

For any $x \in \mathbb{R}$, denote by \mathbb{P}_x and \mathbb{E}_x the probability and the corresponding expectation generated by the finite dimensional distributions of $(X_n)_{n \geq 0}$ starting at $X_0 = x$.

We make use of the following condition which ensures that the affine Markov walk satisfies the central limit theorem (2.1.2) (c.f. [43]):

Condition 2.1. *The pair (a, b) is such that:*

1. *There exists a constant $\alpha > 2$ such that $\mathbb{E}(|a|^\alpha) < 1$ and $\mathbb{E}(|b|^\alpha) < +\infty$.*
2. *The random variable b is non-zero with positive probability, $\mathbb{P}(b \neq 0) > 0$, and centred, $\mathbb{E}(b) = 0$.*

Note that Condition 2.1 is weaker than the conditions required in [43] in the special case $\alpha > 2$. Nevertheless, using the same techniques as in [43] it can be shown that,

under Condition 2.1, the Markov chain $(X_n)_{n \geq 0}$ has a unique invariant measure \mathbf{m} and its partial sum S_n satisfies the central limit theorem (2.1.2) with

$$\mu = \int_{\mathbb{R}} x \mathbf{m}(dx) = \frac{\mathbb{E}(b)}{1 - \mathbb{E}(a)} = 0 \quad (2.2.1)$$

and

$$\sigma^2 = \int_{\mathbb{R}} x^2 \mathbf{m}(dx) + 2 \sum_{k=1}^{\infty} \int_{\mathbb{R}} x \mathbb{E}_x(X_k) \mathbf{m}(dx) = \frac{\mathbb{E}(b^2)}{1 - \mathbb{E}(a^2)} \frac{1 + \mathbb{E}(a)}{1 - \mathbb{E}(a)} > 0. \quad (2.2.2)$$

Moreover, it is easy to see that under Condition 2.1 the Markov chain $(X_n)_{n \geq 0}$ has no fixed point: $\mathbb{P}(ax + b = x) < 1$, for any $x \in \mathbb{R}$. Below we make use of a slightly refined result which gives the rate of convergence in the central limit theorem for S_n with an explicit dependence of the constants on the initial value $X_0 = x$ stated in Section 2.9.3.

For any $y \in \mathbb{R}$ consider the affine Markov walk $(y + S_n)_{n \geq 0}$ starting at y and define its exit time

$$\tau_y = \min\{k \geq 1, y + S_k \leq 0\}.$$

Corollary 2.9.7 implies the finiteness of the stopping time τ_y : under Condition 2.1, it holds $\mathbb{P}_x(\tau_y < +\infty) = 1$, for any $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

The asymptotic behaviour of the probability $\mathbb{P}_x(\tau_y > n)$ is determined by the harmonic function which we proceed to introduce. For any $(x, y) \in \mathbb{R} \times \mathbb{R}$, denote by $\mathbf{Q}(x, y, \cdot)$ the transition probability of the Markov chain $(X_n, y + S_n)_{n \geq 0}$. The restriction of the measure $\mathbf{Q}(x, y, \cdot)$ on $\mathbb{R} \times \mathbb{R}_+^*$ is defined by

$$\mathbf{Q}_+(x, y, B) = \mathbf{Q}(x, y, B)$$

for any measurable set B on $\mathbb{R} \times \mathbb{R}_+^*$ and for any $(x, y) \in \mathbb{R} \times \mathbb{R}$. Let \mathcal{D} be a measurable set in $\mathbb{R} \times \mathbb{R}$ containing $\mathbb{R} \times \mathbb{R}_+^*$. For any measurable $\varphi : \mathcal{D} \rightarrow \mathbb{R}$ set $\mathbf{Q}_+\varphi(x, y) = \int_{\mathbb{R} \times \mathbb{R}_+^*} \varphi(x', y') \mathbf{Q}_+(x, y, dx' \times dy')$. A \mathbf{Q}_+ -harmonic function on \mathcal{D} is any function $V : \mathcal{D} \rightarrow \mathbb{R}$ which satisfies

$$\mathbf{Q}_+V(x, y) = V(x, y), \quad \text{for any } (x, y) \in \mathcal{D}.$$

The existence of a non-negative harmonic function is obvious: $V = 0$ is an example. To ensure the existence of a harmonic function which is positive on a set containing $\mathbb{R} \times \mathbb{R}_+^*$, we need additional assumptions.

Condition 2.2. For all $x \in \mathbb{R}$ and $y > 0$,

$$\mathbb{P}_x(\tau_y > 1) = \mathbb{P}(ax + b > -y) > 0.$$

Condition 2.3. For any $x \in \mathbb{R}$ and $y > 0$, there exists $p_0 \in (2, \alpha)$ such that for any constant $c > 0$, there exists $n_0 \geq 1$ such that,

$$\mathbb{P}_x((X_{n_0}, y + S_{n_0}) \in K_{p_0, c}, \tau_y > n_0) > 0,$$

where

$$K_{p_0, c} = \{(x, y) \in \mathbb{R} \times \mathbb{R}_+^*, y \geq c(1 + |x|^{p_0})\}.$$

It is clear that Condition 2.3 implies Condition 2.2. Moreover under either Condition 2.2 or Condition 2.3, the event $\{\tau_y > n\}$ has positive probability, for any $n \geq 1$, $x \in \mathbb{R}$ and $y > 0$.

The existence of a harmonic function is guaranteed by the following theorem. For any $x \in \mathbb{R}$ consider the process $(M_n)_{n \geq 0}$ defined by

$$M_0 = 0, \quad M_n = S_n + \frac{\mathbb{E}(a)}{1 - \mathbb{E}(a)} (X_n - x), \quad n \geq 1, \quad (2.2.3)$$

and the natural filtration $(\mathcal{F}_n)_{n \geq 0}$ with \mathcal{F}_0 the trivial σ -algebra and \mathcal{F}_n the σ -algebra generated by X_1, X_2, \dots, X_n . It is easy to verify that $(M_n, \mathcal{F}_n)_{n \geq 0}$ is a \mathbb{P}_x -martingale, for any $x \in \mathbb{R}$ (see Gordin [35]).

Theorem 2.2.1. *Assume Condition 2.1.*

1. For any $x \in \mathbb{R}$ and $y > 0$, the random variable M_{τ_y} is integrable,

$$\mathbb{E}_x(|M_{\tau_y}|) < +\infty$$

and the function

$$V(x, y) = -\mathbb{E}_x(M_{\tau_y}), \quad x \in \mathbb{R}, \quad y > 0,$$

is well defined on $\mathbb{R} \times \mathbb{R}_+^*$.

2. The function V has the following properties:

(a) For any $x \in \mathbb{R}$, the function $V(x, \cdot)$ is non-decreasing.

(b) For any $\delta > 0$, $p \in (2, \alpha)$, $x \in \mathbb{R}$ and $y > 0$,

$$\begin{aligned} V(x, y) &\geq \max(0, (1 - \delta)y - c_{p,\delta}(1 + |x|^p)), \\ V(x, y) &\leq (1 + \delta(1 + |x|^{p-1}))y + c_{p,\delta}(1 + |x|^p). \end{aligned}$$

(c) For any $x \in \mathbb{R}$, it holds $\lim_{y \rightarrow +\infty} \frac{V(x, y)}{y} = 1$.

3. The function V is \mathbf{Q}_+ -harmonic on $\mathbb{R} \times \mathbb{R}_+^*$: for any $x \in \mathbb{R}$ and $y > 0$,

$$\mathbf{Q}_+V(x, y) = V(x, y).$$

4. If in addition we assume either Condition 2.2 and $\mathbb{E}(a) \geq 0$, or Condition 2.3, then the function V is positive on $\mathbb{R} \times \mathbb{R}_+^*$.

Using the harmonic function from the previous theorem, we obtain the asymptotic of the tail probability of the exit time τ_y .

Theorem 2.2.2. *Assume Condition 2.1.*

1. For any $p \in (2, \alpha)$, $x \in \mathbb{R}$ and $y > 0$,

$$\sqrt{n}\mathbb{P}_x(\tau_y > n) \leq c_p(1 + y + |x|)^p.$$

2. If in addition we assume either Condition 2.2 and $\mathbb{E}(a) \geq 0$, or Condition 2.3, then for any $x \in \mathbb{R}$ and $y > 0$,

$$\mathbb{P}_x(\tau_y > n) \underset{n \rightarrow +\infty}{\sim} \frac{2V(x, y)}{\sqrt{2\pi n\sigma}}.$$

Corollary 2.2.3. *Assume Condition 2.1. For any $p \in (2, \alpha)$, $x \in \mathbb{R}$, $y > 0$ and $\gamma \in (0, 1/2)$,*

$$\mathbb{E}_x \left(\tau_y^\gamma \right) \leq c_{p,\gamma} (1 + y + |x|)^p.$$

If in addition we assume Condition 2.2 and $\mathbb{E}(a) \geq 0$, or Condition 2.3, then

$$\mathbb{E}_x \left(\tau_y^{1/2} \right) = +\infty.$$

Moreover, we prove that the Markov walk $(y + S_n)_{n \geq 0}$ conditioned to stay positive satisfies the following limit theorem.

Theorem 2.2.4. *Assume either Conditions 2.1, 2.2 and $\mathbb{E}(a) \geq 0$, or Conditions 2.1 and 2.3. For any $x \in \mathbb{R}$, $y > 0$ and $t > 0$,*

$$\mathbb{P}_x \left(\frac{y + S_n}{\sigma \sqrt{n}} \leq t \mid \tau_y > n \right) \xrightarrow{n \rightarrow +\infty} \Phi^+(t),$$

where $\Phi^+(t) = 1 - e^{-t^2}$ is the Rayleigh distribution function.

Theorems 2.2.1, 2.2.2, 2.2.4 can be extended to some non-positive initial points y . Set

$$\mathcal{D}^- := \{(x, y) \in \mathbb{R} \times \mathbb{R}_-, \mathbb{P}_x(\tau_y > 1) = \mathbb{P}(ax + b > -y) > 0\}.$$

Theorem 2.2.5. *Assume Condition 2.1.*

1. *For any $(x, y) \in \mathcal{D}^-$, the random variable M_{τ_y} is integrable and the function $V(x, y) = -\mathbb{E}_x(M_{\tau_y})$, is well defined on \mathcal{D}^- .*
2. *The function V is \mathbf{Q}_+ -harmonic on $\mathcal{D} = \mathcal{D}^- \cup \mathbb{R} \times \mathbb{R}_+^*$.*
3. *If in addition we assume either Condition 2.2 and $\mathbb{E}(a) \geq 0$, or Condition 2.3, then V is positive on $\mathcal{D} = \mathcal{D}^- \cup \mathbb{R} \times \mathbb{R}_+^*$.*
4. *For any $(x, y) \in \mathcal{D}^-$,*

$$\sqrt{n} \mathbb{P}_x(\tau_y > n) \leq c_p (1 + |x|)^p.$$

5. *If in addition we assume either Condition 2.2 and $\mathbb{E}(a) \geq 0$, or Condition 2.3, then*

(a) *For any $(x, y) \in \mathcal{D}^-$,*

$$\mathbb{P}_x(\tau_y > n) \underset{n \rightarrow +\infty}{\sim} \frac{2V(x, y)}{\sqrt{2\pi n\sigma}}.$$

(b) *For any $(x, y) \in \mathcal{D}^-$ and $t > 0$,*

$$\mathbb{P}_x \left(\frac{y + S_n}{\sigma \sqrt{n}} \leq t \mid \tau_y > n \right) \xrightarrow{n \rightarrow +\infty} \Phi^+(t).$$

The study of the asymptotic behaviour of τ_y and $y + S_n$ for $y \leq 0$ can be motivated by the problem of determining the time when the population $y_0 + S_n$, starting at $y_0 > 0$, stays over a fixed level H . When $y = y_0 - H$ is in $(-H, 0]$, the time $\tau_y = \inf\{k \geq 1, y_0 + S_k \leq H\}$ is the return time of the population $y_0 + S_n$ under the level H .

Below we discuss two more restrictive assumptions which, however, are easier to verify than Conditions 2.2 and 2.3, respectively.

Condition 2.2bis. *The law of the pair (a, b) is such that for all $C > 0$,*

$$\mathbb{P}(b \geq C |a|) > 0.$$

Condition 2.3bis. *There exists $C > 0$ such that,*

$$\mathbb{P}((a, b) \in (-1, 0) \times (0, C]) > 0 \quad \text{and} \quad \mathbb{P}((a, b) \in (0, 1) \times (0, C]) > 0.$$

It is straightforward that Condition 2.2bis implies Condition 2.2. This follows from the inequality

$$\mathbb{P}(ax + b > -y) \geq \mathbb{P}(b \geq C |a|),$$

with $C = |x|$. The fact that Condition 2.3bis implies Condition 2.3 is proved in the Appendix 2.9.1.

Under Condition 2.1, it is easy to see that Condition 2.3bis is satisfied, for example, when random variables a and b are independent and $P(a \in (-1, 0)) > 0$ and $P(a \in (0, 1)) > 0$.

Note that, while Condition 2.3 implies Condition 2.2, there is no link between Conditions 2.2bis and 2.3bis. Indeed, if a and b are independent, a is non-negative and the support of b contains \mathbb{R}_+ , then Condition 2.2bis holds true whereas Condition 2.3bis does not. At the opposite, if a and b are independent, b bounded and the support of a equals to $\{-1/2\} \cup \{1/2\}$ then Condition 2.3bis holds true whereas Condition 2.2bis does not.

The outline of the paper is as follows. The martingale approximation $(M_n)_{n \geq 0}$ of the Markov walk $(S_n)_{n \geq 0}$ and some of its properties are given in Section 2.3. In Section 2.4 we prove that the expectation of the killed Markov walk $((y + S_n) \mathbb{1}_{\{\tau_y > n\}})_{n \geq 0}$ is bounded uniformly in n . This allows us to prove the existence of the harmonic function and establish some of its properties in Section 2.5. With the help of the harmonic function and of a strong approximation result for Markov chains we prove Theorems 2.2.2, 2.2.4 and 2.2.5, in Sections 2.6, 2.7 and 2.8 respectively. Section 2.9 is an appendix where we collect some results used in the proofs.

2.3 Martingale approximation

In this section we approximate the Markov walk $(S_n)_{n \geq 0}$ by the martingale defined in (2.2.3) and state some related bounds.

We start by a lemma which shows that there is an exponential decay of the dependence of X_n on the initial state $x = X_0$ as n grows to infinity. This simple fact will be used repeatedly in the sequel.

Lemma 2.3.1. *Assume Condition 2.1. For all $p \in [1, \alpha]$, $x \in \mathbb{R}$, and $n \geq 0$,*

$$\mathbb{E}_x^{1/p} (|X_n|^p) \leq c_p + \left(\mathbb{E}^{1/p} (|a|^p) \right)^n |x| \leq c_p (1 + |x|).$$

Proof. Since $X_n = \sum_{k=1}^n (b_k \prod_{i=k+1}^n a_i) + \prod_{i=1}^n a_i x$, for $n \geq 1$, with the convention $\prod_{i=n+1}^n a_i = 1$, we have by the Minkowski inequality and the independence of $(a_i, b_i)_{i \geq 1}$,

$$\mathbb{E}_x^{1/p} (|X_n|^p) \leq \sum_{k=1}^n \left(\mathbb{E}^{1/p} (|b|^p) \mathbb{E}^{1/p} (|a|^p)^{n-k} \right) + \mathbb{E}^{1/p} (|a|^p)^n |x|.$$

The conclusion of the lemma is thus a direct consequence of Condition 2.1. \square

All over the paper we use the abbreviation

$$\rho = \frac{\mathbb{E}(a)}{1 - \mathbb{E}(a)}. \quad (2.3.1)$$

Using this notation and the martingale $(M_n)_{n \geq 0}$ defined in (2.2.3), for any $x \in \mathbb{R}$ and $y \in \mathbb{R}$, the Markov walk $(y + S_n)_{n \geq 0}$ has the following martingale representation:

$$y + S_n = y + \rho x + M_n - \rho X_n, \quad n \geq 0. \quad (2.3.2)$$

Define the sequence $(X_n^0)_{n \geq 0}$, by

$$X_0^0 = 0 \quad \text{and} \quad X_n^0 = \sum_{k=1}^n b_k \prod_{i=k+1}^n a_i, \quad n \geq 1, \quad (2.3.3)$$

with the convention $\prod_{i=k+1}^n a_i = 1$ for $k = n$. The sequence $(X_n^0)_{n \geq 0}$ corresponds to the stochastic recursion starting at 0. In the same line, we define $M_0^0 = 0$ and $M_n^0 = \sum_{k=1}^n \frac{X_k^0 - \mathbb{E}(a)X_{k-1}^0}{1 - \mathbb{E}(a)}$, for all $n \geq 1$. It is easy to see that the process $(M_n^0, \mathcal{F}_n)_{n \geq 0}$ is a zero mean \mathbb{P}_x -martingale which is related to the martingale $(M_n)_{n \geq 0}$ by the identity

$$M_n = M_n^0 + \Delta_n x, \quad (2.3.4)$$

where

$$\Delta_0 = 0 \quad \text{and} \quad \Delta_n = \sum_{k=1}^n \frac{\prod_{i=1}^{k-1} a_i}{1 - \mathbb{E}(a)} (a_k - \mathbb{E}(a)), \quad n \geq 1.$$

The following two lemmas will be used to control $\mathbb{E}_x(|M_n|^p)$.

Lemma 2.3.2. *Assume Condition 2.1.*

1. *The sequence $(\Delta_n)_{n \geq 0}$ is a centred martingale.*
2. *For all $p \in [1, \alpha]$ and $n \geq 0$,*

$$\mathbb{E}^{1/p}(|\Delta_n|^p) \leq c_p.$$

Proof. The first claim follows from the fact that Δ_n is a difference of two zero mean martingales. Using the Minkowski inequality for $1 \leq p \leq \alpha$, the independence of $(a_i)_{i \geq 1}$ and Condition 2.1 we obtain the second claim. \square

Let us introduce the martingale differences:

$$\xi_k^0 = M_k^0 - M_{k-1}^0 = \frac{X_k^0 - \mathbb{E}(a)X_{k-1}^0}{1 - \mathbb{E}(a)}, \quad k \geq 1.$$

Lemma 2.3.3. *Assume Condition 2.1. For all $p \in [1, \alpha]$ and $n \geq 0$,*

$$\mathbb{E}^{1/p}(|\xi_n^0|^p) \leq c_p \quad \text{and} \quad \mathbb{E}^{1/p}(|M_n^0|^p) \leq c_p \sqrt{n}.$$

Proof. For the increments ξ_n^0 we simply use Lemma 2.3.1 with $x = 0$. For the martingale $(M_n^0)_{n \geq 0}$, the upper bound is obtained by Burkholder inequality: for all $2 < p \leq \alpha$ and all $n \geq 1$,

$$\mathbb{E}^{1/p}(|M_n^0|^p) \leq c_p \mathbb{E}^{1/p} \left(\left(\sum_{k=1}^n (\xi_k^0)^2 \right)^{p/2} \right).$$

By the Hölder inequality with the exponents $u = p/2 > 1$ and $v = \frac{p}{p-2}$, we obtain

$$\mathbb{E}^{1/p} \left(|M_n^0|^p \right) \leq c_p \mathbb{E}^{1/p} \left[\left(\sum_{k=1}^n |\xi_k^0|^{2u} \right)^{\frac{p}{2u}} n^{\frac{p}{2v}} \right] \leq c_p n^{\frac{p-2}{2p}} \left(\sum_{k=1}^n c_p \right)^{1/p} = c_p \sqrt{n}.$$

This proves the claim when $2 < p \leq \alpha$. When $1 \leq p \leq 2$ the assertion follows from the case above since the L^p norm is less than the L^q norm for $q \in (2, \alpha]$. \square

Lemma 2.3.4. *Assume Condition 2.1. For all $p \in [1, \alpha]$ and $n \geq 0$,*

$$\mathbb{E}_x^{1/p} (|M_n|^p) \leq c_p (|x| + \sqrt{n}).$$

Proof. By the Minkowski inequality and equation (2.3.4), for all $1 \leq p \leq \alpha$, $x \in \mathbb{R}$ and $n \geq 1$,

$$\mathbb{E}_x^{1/p} (|M_n|^p) \leq \mathbb{E}^{1/p} (|\Delta_n|^p) |x| + \mathbb{E}^{1/p} (|M_n^0|^p).$$

Then, by the claim 2 of Lemma 2.3.2 and Lemma 2.3.3, the result follows. \square

2.4 Bound on the expectation of the killed martingale

The goal of this section is to prepare the background to prove the integrability of the random variable M_{τ_y} , which is crucial for showing the existence of the harmonic function in Section 2.5. We use different approaches depending on the sign on $\mathbb{E}(a)$: when $\mathbb{E}(a) \geq 0$, in Section 2.4.2 we prove that the expectation of the martingale $(y + \rho x + M_n)_{n \geq 0}$ killed at τ_y is uniformly bounded in n , while, when $\mathbb{E}(a) < 0$, in Section 2.4.3 we prove that the expectation of the same martingale killed at T_y is uniformly bounded in n , where T_y is the exit time of the martingale $(y + \rho x + M_n)_{n \geq 0}$.

2.4.1 Preliminary results

We first state a result concerning the first time when the process $(|y + S_n|)_{n \geq 1}$ (respectively $(|y + \rho x + M_n|)_{n \geq 1}$) crosses the level $n^{1/2-\varepsilon}$. Introduce the following stopping times: for any $n \geq 1$, $\varepsilon \in (0, 1/2)$, $x \in \mathbb{R}$ and $y \in \mathbb{R}$,

$$\nu_n = \nu_{n,\varepsilon,y} = \min \left\{ k \geq 1, |y + S_k| > n^{1/2-\varepsilon} \right\} \quad (2.4.1)$$

and

$$v_n = v_{n,\varepsilon,x,y} = \min \left\{ k \geq 1, |y + \rho x + M_k| > n^{1/2-\varepsilon} \right\}.$$

Lemma 2.4.1. *Assume Condition 2.1. Let $p \in (2, \alpha)$. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, $\delta > 0$, $x \in \mathbb{R}$, $y > 0$ and $n \geq 1$,*

$$\mathbb{P}_x \left(\nu_n > \delta n^{1-\varepsilon} \right) \leq \frac{c_{p,\varepsilon,\delta}}{n^{p/2-p\varepsilon}} + c_{p,\varepsilon,\delta} e^{-c_{p,\varepsilon,\delta} n^{1-2\varepsilon}} |x|^p$$

and

$$\mathbb{P}_x \left(v_n > \delta n^{1-\varepsilon} \right) \leq \frac{c_{p,\varepsilon,\delta}}{n^{p/2-p\varepsilon}} + c_{p,\varepsilon,\delta} e^{-c_{p,\varepsilon,\delta} n^{1-2\varepsilon}} |x|^p.$$

Proof. With $\varepsilon < \min(1/2, \varepsilon_0)$, where ε_0 is defined in Corollary 2.9.6 and $b > 0$ a constant to be chosen below, let $l = \lfloor b^2 \delta n^{1-2\varepsilon} \rfloor$, $K = \lfloor n^\varepsilon / b^2 \rfloor$ and for any $m \geq 1$, $x \in \mathbb{R}$ and $y \in \mathbb{R}$, with $z = y + \rho x$,

$$A_m(x, y) = \left\{ \max_{1 \leq k \leq m} |z + M_{kl}| \leq (1 + 2|\rho|)n^{1/2-\varepsilon} \right\}.$$

Note that by the martingale representation (2.3.2), we have for any $k \geq 2$, $|z + M_k| = |y + S_k + \rho(y + S_k) - \rho(y + S_{k-1})| \leq (1 + |\rho|)|y + S_k| + |\rho||y + S_{k-1}|$. Then, choosing n large enough to have $l \geq 2$,

$$\begin{aligned} \mathbb{P}_x(\nu_n > \delta n^{1-\varepsilon}) &= \mathbb{P}_x \left(\max_{1 \leq k \leq \lfloor \delta n^{1-\varepsilon} \rfloor} |y + S_k| \leq n^{1/2-\varepsilon} \right) \\ &\leq \mathbb{P}_x \left(\max_{2 \leq k \leq \lfloor \delta n^{1-\varepsilon} \rfloor} |z + M_k| \leq (1 + 2|\rho|)n^{1/2-\varepsilon} \right) \\ &\leq \mathbb{P}_x(A_K(x, y)). \end{aligned}$$

Moreover, we have also,

$$\mathbb{P}_x(\nu_n > \delta n^{1-\varepsilon}) \leq \mathbb{P}_x(A_K(x, y)).$$

Since $(X_n, y + S_n)_{n \geq 0}$ is a Markov chain,

$$\begin{aligned} \mathbb{P}_x(A_K(x, y)) &= \int_{\mathbb{R}^2} \mathbb{P}_{x'}(A_1(x', y')) \\ &\quad \times \mathbb{P}_x(X_{(K-1)l} \in dx', y + S_{(K-1)l} \in dy', A_{K-1}(x, y)). \end{aligned} \quad (2.4.2)$$

We use the decomposition (2.3.4) to write that, with $c = 1 + 2|\rho|$,

$$\begin{aligned} \mathbb{P}_{x'}(A_1(x', y')) &\leq \mathbb{P}_{x'}(|z' + M_l^0| \leq 2cn^{1/2-\varepsilon}, |\Delta_l x'| \leq cn^{1/2-\varepsilon}) \\ &\quad + \mathbb{P}_{x'}(|\Delta_l x'| > cn^{1/2-\varepsilon}). \end{aligned}$$

Using (2.3.2) with $x = 0$, we have $M_n^0 = S_n^0 + \rho X_n^0$. By the Markov inequality,

$$\begin{aligned} \mathbb{P}_{x'}(A_1(x', y')) &\leq \mathbb{P}_{x'}(|z' + S_l^0| \leq 3cn^{1/2-\varepsilon}, |\rho| |X_l^0| \leq cn^{1/2-\varepsilon}) \\ &\quad + \mathbb{P}_{x'}(|\rho| |X_l^0| > cn^{1/2-\varepsilon}) + c_p \frac{\mathbb{E}(|\Delta_l|^p)}{n^{p/2-p\varepsilon}} |x'|^p. \end{aligned}$$

Since S_l^0 does not depend on x' , using Lemma 2.3.1 and the claim 2 of Lemma 2.3.2, we obtain

$$\mathbb{P}_{x'}(A_1(x', y')) \leq \sup_{y' \in \mathbb{R}} \mathbb{P}(|y' + S_l^0| \leq 3cn^{1/2-\varepsilon}) + \frac{c_p(1 + |x'|^p)}{n^{p/2-p\varepsilon}}.$$

Inserting this bound in (2.4.2), it follows that

$$\begin{aligned} \mathbb{P}_x(A_K(x, y)) &\leq \mathbb{P}_x(A_{K-1}(x, y)) \sup_{y' \in \mathbb{R}} \mathbb{P}(|y' + S_l^0| \leq 3cn^{1/2-\varepsilon}) \\ &\quad + \frac{c_p}{n^{p/2-p\varepsilon}} \left(1 + \mathbb{E}_x(|X_{(K-1)l}|^p) \right). \end{aligned}$$

Set $r_n = \frac{3cn^{1/2-\varepsilon}}{\sqrt{l}}$. Denote by $\mathbb{B}_{\frac{-y'}{\sqrt{l}}}(r_n)$ the closed ball centred in $\frac{-y'}{\sqrt{l}}$ of radius r_n . The rate of convergence in the central limit theorem from Corollary 2.9.6 (applied with $x = 0$) implies that,

$$\sup_{y' \in \mathbb{R}} \mathbb{P} \left(\frac{S_l^0}{\sqrt{l}} \in \mathbb{B}_{\frac{-y'}{\sqrt{l}}}(r_n) \right) \leq \sup_{y' \in \mathbb{R}} \int_{\mathbb{B}_{\frac{-y'}{\sqrt{l}}}(r_n)} e^{-\frac{u^2}{2\sigma^2}} \frac{du}{\sqrt{2\pi}\sigma} + 2\frac{c_{p,\varepsilon}}{l^\varepsilon}.$$

Moreover,

$$\sup_{y' \in \mathbb{R}} \int_{\mathbb{B}_{\frac{-y'}{\sqrt{l}}}(r_n)} e^{-\frac{u^2}{2\sigma^2}} \frac{du}{\sqrt{2\pi}\sigma} \leq \frac{2r_n}{\sqrt{2\pi}\sigma} \leq \frac{c_\delta}{b}.$$

Let $q < 1$. With b large enough in the definition of l , we have $2\frac{c_{p,\varepsilon}}{l^\varepsilon} \leq \frac{q}{2}$, $\frac{c_\delta}{b} \leq \frac{q}{2}$ and thus

$$\sup_{y' \in \mathbb{R}} \mathbb{P} \left(\frac{S_l^0}{\sqrt{l}} \in \mathbb{B}_{\frac{-y'}{\sqrt{l}}}(r_n) \right) \leq q < 1.$$

Iterating, we get

$$\mathbb{P}_x(A_K(x, y)) \leq q^{K-1} \mathbb{P}_x(A_1(x, y)) + \frac{c_p}{n^{p/2-p\varepsilon}} \sum_{k=0}^{K-2} q^k \left(1 + \mathbb{E}_x \left(|X_{(K-1-k)l}|^p \right) \right).$$

Using the fact that $q^{K-1} \mathbb{P}_x(A_1(x, y)) \leq q^{K-1} = \frac{1}{q} e^{-\lfloor n^\varepsilon/b^2 \rfloor \ln(1/q)} \leq \frac{c_{p,\varepsilon,\delta}}{n^{p/2-p\varepsilon}}$, Lemma 2.3.1 and the fact that $(K-1-k)l \geq c_{\varepsilon,\delta} n^{1-2\varepsilon}$ for all $0 \leq k \leq K-2$, we finally obtain

$$\mathbb{P}_x(A_K(x, y)) \leq \frac{c_{p,\varepsilon,\delta}}{n^{p/2-p\varepsilon}} + c_{p,\varepsilon,\delta} e^{-c_{p,\varepsilon,\delta} n^{1-2\varepsilon}} |x|^p.$$

□

2.4.2 Bound on the expectation of the killed martingale: the case $\mathbb{E}(a) \geq 0$

The difficulty in proving that the expectation $\mathbb{E}_x(y + \rho x + M_n; \tau_y > n)$ is bounded uniformly in n lies in the fact that whereas the killed Markov walk $(y + S_n) \mathbb{1}_{\{\tau_y > n\}}$ is non-negative, the random variable $(y + \rho x + M_n) \mathbb{1}_{\{\tau_y > n\}}$ may be not. In the case when $\mathbb{E}(a) \geq 0$ we take advantage of the properties presented in the next lemma.

Lemma 2.4.2. *Assume Condition 2.1 and $\mathbb{E}(a) \geq 0$.*

1. For all $x \in \mathbb{R}$ and $y > 0$,

$$y + \rho x + M_{\tau_y} \leq 0, \quad \mathbb{P}_x\text{-a.s.}$$

2. For all $x \in \mathbb{R}$ and $y > 0$,

$$\frac{X_{\tau_y}}{1 - \mathbb{E}(a)} < y + \rho x + M_{\tau_y}, \quad \mathbb{P}_x\text{-a.s.}$$

3. For all $x \in \mathbb{R}$ and $y > 0$, the sequence $\left((y + \rho x + M_n) \mathbb{1}_{\{\tau_y > n\}} \right)_{n \geq 0}$ is a submartingale with respect to \mathbb{P}_x .

Proof. Claim 1. Let, for brevity, $z = y + \rho x$. Since, by the definition of τ_y ,

$$X_{\tau_y} = y + S_{\tau_y} - (y + S_{\tau_y-1}) < 0,$$

it follows from (2.3.2) and the bound $\mathbb{E}(a) \geq 0$ that $z + M_{\tau_y} \leq y + S_{\tau_y} \leq 0$.

Claim 2. Rewrite the martingale representation (2.3.2) in the form

$$z + M_n = y + S_{n-1} + \frac{X_n}{1 - \mathbb{E}(a)}. \quad (2.4.3)$$

So, at the exit time τ_y ,

$$\frac{X_{\tau_y}}{1 - \mathbb{E}(a)} = z + M_{\tau_y} - (y + S_{\tau_y-1}) < z + M_{\tau_y}.$$

Claim 3. Using the first claim and the fact that $(M_n)_{n \geq 0}$ is a martingale,

$$\begin{aligned} \mathbb{E}_x(z + M_{n+1}; \tau_y > n + 1 | \mathcal{F}_n) &= z + M_n - \mathbb{E}_x(z + M_{\tau_y}; \tau_y = n + 1 | \mathcal{F}_n) \\ &\quad - \mathbb{E}_x(z + M_{n+1} | \mathcal{F}_n) \mathbb{1}_{\{\tau_y \leq n\}} \\ &\geq (z + M_n) \mathbb{1}_{\{\tau_y > n\}}. \end{aligned}$$

□

In the next lemma we obtain a first bound for the expectation of the killed martingale $((y + \rho x + M_n) \mathbb{1}_{\{\tau_y > n\}})_{n \geq 0}$ which is of order $n^{1/2-2\varepsilon}$, for some $\varepsilon > 0$. Using a recursive procedure we improve it subsequently to a bound not depending on n .

Lemma 2.4.3. *Assume Condition 2.1 and $\mathbb{E}(a) \geq 0$. Let $p \in (2, \alpha)$. For any $\varepsilon \in (0, \frac{p-2}{4p})$, $x \in \mathbb{R}$, $y > 0$ and $n \in \mathbb{N}$, we have*

$$\mathbb{E}_x(y + \rho x + M_n; \tau_y > n) \leq y + \rho x + c|x| + c_p n^{1/2-2\varepsilon}.$$

Proof. By the Doob optional stopping theorem and the claim 2 of Lemma 2.4.2, with $z = y + \rho x$,

$$\mathbb{E}_x(z + M_n; \tau_y > n) \leq z - \mathbb{E}_x\left(\frac{X_{\tau_y}}{1 - \mathbb{E}(a)}; \tau_y \leq n\right).$$

Note that $X_n = \prod_{i=1}^n a_i x + X_n^0$, with X_n^0 given by (2.3.3). Then, with $\varepsilon \in (0, 1/4)$,

$$\begin{aligned} \mathbb{E}_x(z + M_n; \tau_y > n) &\leq z + c \sum_{k=1}^n \prod_{i=1}^k \mathbb{E}(|a_i| |x| + c \mathbb{E}_x(|X_{\tau_y}^0|; \tau_y \leq n, \max_{1 \leq k \leq n} |X_k^0| \leq n^{1/2-2\varepsilon})) \\ &\quad + c \mathbb{E}_x(|X_{\tau_y}^0|; \tau_y \leq n, \max_{1 \leq k \leq n} |X_k^0| > n^{1/2-2\varepsilon}). \end{aligned}$$

By the Markov inequality, for $2 < p < \alpha$,

$$\mathbb{E}_x(z + M_n; \tau_y > n) \leq z + c \sum_{k=1}^n \mathbb{E}^k(|a| |x| + cn^{1/2-2\varepsilon}) + c \mathbb{E}_x\left(\frac{\max_{1 \leq k \leq n} |X_k^0|^p}{n^{\frac{p-1}{2}(1-4\varepsilon)}}\right).$$

By Lemma 2.3.1 (with $x = 0$),

$$\mathbb{E}_x(z + M_n; \tau_y > n) \leq z + c|x| + cn^{1/2-2\varepsilon} + c_p \frac{n}{n^{\frac{p-1}{2}(1-4\varepsilon)}}.$$

Since $\varepsilon \in (0, \frac{p-2}{4p})$, we have $\frac{p-1}{2}(1-4\varepsilon) > 1/2 + 2\varepsilon$ which concludes the proof. □

Now we give an improvement of Lemma 2.4.3 which establishes a bound of the expectation of the killed martingale $((y + \rho x + M_n) \mathbb{1}_{\{\tau_y > n\}})_{n \geq 0}$ depending only on the starting values x, y .

Lemma 2.4.4. *Assume Condition 2.1 and $\mathbb{E}(a) \geq 0$. For any $\delta > 0$, $p \in (2, \alpha)$, $x \in \mathbb{R}$, $y > 0$ and $n \geq 0$,*

$$\mathbb{E}_x(y + \rho x + M_n; \tau_y > n) \leq \left(1 + c_p \delta (1 + |x|)^{p-1}\right) y + c_{p,\delta} (1 + |x|)^p.$$

Moreover, with $\delta = 1$, for any $p \in (2, \alpha)$, $x \in \mathbb{R}$, $y > 0$ and $n \geq 0$,

$$\mathbb{E}_x(y + \rho x + M_n; \tau_y > n) \leq c_p (1 + y + |x|) (1 + |x|)^{p-1}.$$

Proof. Let $\delta > 0$ and $\varepsilon \in (0, \varepsilon_1]$, where $\varepsilon_1 = \min\left(\varepsilon_0, \frac{p-2}{4p}\right)$ and ε_0 is defined in Lemma 2.4.1. Set $z = y + \rho x$. We split the proof following the values of n .

Assume first that $n \leq \delta^{-1/\varepsilon}$. A bound of $\mathbb{E}_x(z + M_n; \tau_y > n)$ is obtained immediately from Lemma 2.4.3: since $z = y + \rho x$, for any $y > 0$,

$$\mathbb{E}_x(z + M_n; \tau_y > n) \leq y + c|x| + c\sqrt{n} \leq y + c_\delta (1 + |x|)$$

and the lemma is proved when $n \leq \delta^{-1/\varepsilon}$.

Assume now that $n \geq \delta^{-1/\varepsilon}$ and $y > n^{1/2-\varepsilon}$. From Lemma 2.4.3, we deduce that,

$$\mathbb{E}_x(y + \rho x + M_n; \tau_y > n) \leq y + \rho x + c|x| + c_p n^{1/2-2\varepsilon} \leq (1 + c_p n^{-\varepsilon})y + c|x|,$$

which proves the lemma when $y > n^{1/2-\varepsilon}$ and n is larger than $\delta^{-1/\varepsilon}$.

Now, we turn to the last case, when $n \geq \delta^{-1/\varepsilon}$ and $0 < y \leq n^{1/2-\varepsilon}$. Introduce the following stopping time:

$$\nu_n^\varepsilon = \nu_n + \lfloor n^\varepsilon \rfloor.$$

We have the following obvious decomposition:

$$\begin{aligned} & \mathbb{E}_x(z + M_n; \tau_y > n) \\ &= \underbrace{\mathbb{E}_x\left(z + M_n; \tau_y > n, \nu_n^\varepsilon > \lfloor n^{1-\varepsilon} \rfloor\right)}_{=: J_1} + \underbrace{\mathbb{E}_x\left(z + M_n; \tau_y > n, \nu_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor\right)}_{=: J_2}. \end{aligned} \quad (2.4.4)$$

Bound of J_1 . Using the Hölder inequality for $1 < p < \alpha$, Lemma 2.3.4 and Lemma 2.4.1, we have

$$J_1 \leq c_{p,\varepsilon} \sqrt{n} (1 + y + |x|) \frac{(1 + |x|)^{p-1}}{n^{(p-1)(\frac{1}{2}-\varepsilon)}}.$$

As $\varepsilon < \frac{p-2}{4p}$, denoting $C_{p,\varepsilon}(x, y) = c_{p,\varepsilon} (1 + y + |x|) (1 + |x|)^{p-1}$, for all $n \geq 1$,

$$J_1 \leq \frac{C_{p,\varepsilon}(x, y)}{n^\varepsilon}. \quad (2.4.5)$$

Bound of J_2 . Using the martingale representation (2.3.2) for the Markov walk $(y + S_n)_{n \geq 1}$, by the Markov property,

$$\begin{aligned} J_2 &= \sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{R} \times \mathbb{R}_+^*} \mathbb{E}_{x'}(y' + \rho x' + M_{n-k}; \tau_{y'} > n - k) \\ &\quad \times \mathbb{P}_x\left(X_{\nu_n^\varepsilon} \in dx', y + S_{\nu_n^\varepsilon} \in dy', \tau_y > \nu_n^\varepsilon, \nu_n^\varepsilon = k\right). \end{aligned}$$

By Lemma 2.4.3,

$$J_2 \leq \mathbb{E}_x \left(z + M_{\nu_n^\varepsilon} + c \left| X_{\nu_n^\varepsilon} \right| + c_p n^{1/2-2\varepsilon}; \tau_y > \nu_n^\varepsilon, \nu_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor \right).$$

For the term $z + M_{\nu_n^\varepsilon}$, we use the fact that $((z + M_n) \mathbb{1}_{\{\tau_y > n\}})_{n \geq 0}$ is a submartingale (claim 3 of Lemma 2.4.2), while for the term $c \left| X_{\nu_n^\varepsilon} \right| = c \left| X_{\nu_n + \lfloor n^\varepsilon \rfloor} \right|$ we use the Markov property at ν_n and Lemma 2.3.1. This gives

$$\begin{aligned} J_2 &\leq \mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor}; \tau_y > \lfloor n^{1-\varepsilon} \rfloor, \nu_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\quad + c_p \mathbb{E}_x \left(n^{1/2-2\varepsilon} + \mathbb{E}^{\lfloor n^\varepsilon \rfloor}(|a|) \left| X_{\nu_n} \right|; \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right). \end{aligned}$$

Since $0 < y \leq n^{1/2-\varepsilon}$ and ν_n is the first time when $(|y + S_n|)_{n \geq 1}$ exceeds $n^{1/2-\varepsilon}$, the jump X_{ν_n} is necessarily positive on the event $\{\tau_y > \nu_n\}$. Therefore, under the condition $\mathbb{E}(a) \geq 0$, by the representation (2.3.2) we have $z + M_{\nu_n} > n^{1/2-\varepsilon}$. Using the last bound, we obtain

$$\begin{aligned} J_2 &\leq \mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor}; \tau_y > \lfloor n^{1-\varepsilon} \rfloor, \nu_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\quad + c_p \mathbb{E}_x \left(\frac{z + M_{\nu_n}}{n^\varepsilon}; \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) + c_p e^{-c_p n^\varepsilon} \mathbb{E}_x \left(\left| X_{\nu_n} \right|; \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right). \end{aligned}$$

Again, using the fact that $((z + M_n) \mathbb{1}_{\{\tau_y > n\}})_{n \geq 0}$ is a submartingale and Lemma 2.3.1, we bound J_2 as follows,

$$\begin{aligned} J_2 &\leq \left(1 + \frac{c_p}{n^\varepsilon} \right) \mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor}; \tau_y > \lfloor n^{1-\varepsilon} \rfloor \right) + c_p e^{-c_p n^\varepsilon} n^{1-\varepsilon} (1 + |x|) \\ &\quad - \underbrace{\mathbb{E}_x \left(\left(z + M_{\lfloor n^{1-\varepsilon} \rfloor} \right) \left(\mathbb{1}_{\{\nu_n^\varepsilon > \lfloor n^{1-\varepsilon} \rfloor\}} + \frac{c_p}{n^\varepsilon} \mathbb{1}_{\{\nu_n > \lfloor n^{1-\varepsilon} \rfloor\}} \right); \tau_y > \lfloor n^{1-\varepsilon} \rfloor \right)}_{=: J_3}. \end{aligned} \quad (2.4.6)$$

We bound J_3 in a same manner as J_1 ,

$$|J_3| \leq c_{p,\varepsilon} \sqrt{\lfloor n^{1-\varepsilon} \rfloor} (1 + y + |x|) c_{p,\varepsilon} \frac{(1 + |x|)^{p-1}}{n^{\frac{p-1}{2} - (p-1)\varepsilon}} \leq \frac{C_{p,\varepsilon}(x, y)}{n^\varepsilon}.$$

Inserting this bound in (2.4.6) and using (2.4.5) and (2.4.4) we find that, for any $n \geq n_0 = \lfloor y^{1/(1/2-\varepsilon)} \rfloor + 1$,

$$\mathbb{E}_x(z + M_n; \tau_y > n) \leq \left(1 + \frac{c_p}{n^\varepsilon} \right) \mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor}; \tau_y > \lfloor n^{1-\varepsilon} \rfloor \right) + \frac{C_{p,\varepsilon}(x, y)}{n^\varepsilon}.$$

Since $((z + M_n) \mathbb{1}_{\{\tau_y > n\}})_{n \geq 0}$ is a submartingale, the sequence $u_n = \mathbb{E}_x(z + M_n; \tau_y > n)$ is non-decreasing. By Lemma 2.9.1 used with $\alpha = c_p$, $\beta = C_{p,\varepsilon}(x, y)$ and $\gamma = 0$ it follows that, for any $n \geq n_0$ and $k_0 \in \{n_0, \dots, n\}$,

$$\mathbb{E}_x(z + M_n; \tau_y > n) \leq \left(1 + \frac{c_{p,\varepsilon}}{k_0^\varepsilon} \right) \mathbb{E}_x(z + M_{k_0}; \tau_y > k_0) + \frac{C_{p,\varepsilon}(x, y)}{k_0^\varepsilon}.$$

By Lemma 2.4.3 and the fact that $z = y + \rho x$, we have

$$\begin{aligned} \mathbb{E}_x(z + M_n; \tau_y > n) &\leq \left(1 + \frac{c_{p,\varepsilon}}{k_0^\varepsilon} \right) y + c_{p,\varepsilon} k_0^{1/2-2\varepsilon} + c_{p,\varepsilon} |x| \\ &\quad + \frac{c_{p,\varepsilon}}{k_0^\varepsilon} (1 + y + |x|) (1 + |x|)^{p-1} \\ &\leq \left(1 + \frac{c_{p,\varepsilon} (1 + |x|)^{p-1}}{k_0^\varepsilon} \right) y + c_{p,\varepsilon, k_0} (1 + |x|)^p. \end{aligned}$$

Choosing $k_0 \geq \delta^{-1/\varepsilon}$, for any $0 < y \leq n^{1/2-\varepsilon}$ and $n \geq \delta^{-1/\varepsilon}$,

$$\mathbb{E}_x(z + M_n; \tau_y > n) \leq \left(1 + c_{p,\varepsilon}\delta(1 + |x|)^{p-1}\right)y + c_{p,\varepsilon,\delta}(1 + |x|)^p.$$

Finally we conclude that the lemma holds true for any $n \in \mathbb{N}$. \square

We can now transfer the bound provided by Lemma 2.4.4 to the Markov walk $(y + S_n)_{n \geq 0}$.

Corollary 2.4.5. *Assume Condition 2.1 and $\mathbb{E}(a) \geq 0$. For any $p \in (2, \alpha)$, $x \in \mathbb{R}$, $y > 0$ and $n \in \mathbb{N}$,*

$$\mathbb{E}_x(y + S_n; \tau_y > n) \leq c_p(1 + y + |x|)(1 + |x|)^{p-1}.$$

Proof. Using equation (2.3.2), the result follows from Lemma 2.4.4 and Lemma 2.3.1. \square

2.4.3 Bound on the expectation of the killed martingale: the case $\mathbb{E}(a) < 0$

We adapt the mainstream of the proof for the case $\mathbb{E}(a) \geq 0$ given in the previous section, highlighting the details that have to be modified.

In the discussion preceding Lemma 2.4.2, we noted that $(y + \rho x + M_n) \mathbb{1}_{\{\tau_y > n\}}$ may not be positive. In the case $\mathbb{E}(a) < 0$, we overcome this by introducing the exit time of the martingale $(y + \rho x + M_n)_{n \geq 0}$: for any $y \in \mathbb{R}$,

$$T_y = \min\{k \geq 1, y + \rho x + M_k \leq 0\}.$$

The importance of this new exit time is stressed by the fact that one can check that when $\mathbb{E}(a) < 0$, the sequence $((y + \rho x + M_n) \mathbb{1}_{\{\tau_y > n\}})_{n \geq 0}$ is not a submartingale (as in Lemma 2.4.2 when $\mathbb{E}(a) \geq 0$) but a supermartingale. Instead we prove that $((y + \rho x + M_n) \mathbb{1}_{\{T_y > n\}})_{n \geq 0}$ is a submartingale (see Lemma 2.4.6 below). This will play an important role in view of obtaining upper bounds. By Corollary 2.9.7 we have $\mathbb{P}_x(T_y < +\infty) = 1$ for any $x \in \mathbb{R}$. The main point is to show the integrability of $y + \rho x + M_{T_y}$. Under the assumption $\mathbb{E}(a) < 0$ we have $\tau_y \leq T_y$ (see Lemma 2.4.6 below), which together with the integrability of $y + \rho x + M_{T_y}$ and the fact $(|y + \rho x + M_n|)_{n \geq 0}$ is a submartingale, will allow us to prove in Section 2.5.2 that $y + \rho x + M_{\tau_y}$ is integrable.

Lemma 2.4.6. *Assume Condition 2.1.*

1. *If $\mathbb{E}(a) < 0$, then for all $x \in \mathbb{R}$ and $y > 0$,*

$$\tau_y \leq T_y \quad \mathbb{P}_x\text{-a.s.}$$

2. *For all $x \in \mathbb{R}$ and $y \in \mathbb{R}$, the sequence $((y + \rho x + M_n) \mathbb{1}_{\{T_y > n\}})_{n \geq 0}$ is a submartingale with respect to \mathbb{P}_x .*

Proof. Claim 1. We note that when $T_y > 1$, by (2.3.2) and (2.4.3), with $z = y + \rho x$,

$$\begin{aligned} y + S_{T_y} &= z + M_{T_y} - \rho X_{T_y} \leq -\rho X_{T_y}, \\ y + S_{T_y-1} &= z + M_{T_y} - \frac{X_{T_y}}{1 - \mathbb{E}(a)} \leq -\frac{X_{T_y}}{1 - \mathbb{E}(a)}. \end{aligned}$$

Since $\rho < 0$, according to the positivity or non-positivity of X_{T_y} , we have respectively $y + S_{T_y} \leq 0$ or $y + S_{T_y-1} \leq 0$. When $T_y = 1$ and $y > 0$ we have $X_1 < 0$ and so $\tau_y = 1 = T_y$.

Claim 2. In a same manner as in the proof of the claim 3 of Lemma 2.4.2, the claim 2 is a consequence of the fact that $z + M_{T_y} \leq 0$ and that $(M_n)_{n \geq 0}$ is a martingale. \square

The following lemma is similar to Lemma 2.4.3 but with T_y replacing τ_y .

Lemma 2.4.7. *Assume Condition 2.1. Let $p \in (2, \alpha)$. For any $\varepsilon \in (0, \frac{p-2}{4p})$, $x \in \mathbb{R}$, $y > -\rho x$ and $n \geq 0$, we have*

$$\mathbb{E}_x(y + \rho x + M_n; T_y > n) \leq y + \rho x + c|x| + c_p n^{1/2-2\varepsilon}.$$

Proof. Note that $z = y + \rho x > 0$. Since at the exit time T_y we have $0 \geq z + M_{T_y} \geq \xi_{T_y} = \frac{X_{T_y} - \mathbb{E}(a)X_{T_y-1}}{1 - \mathbb{E}(a)}$, by the Doob optional stopping theorem,

$$\mathbb{E}_x(z + M_n; T_y > n) \leq z + c\mathbb{E}_x(|X_{T_y}| + |X_{T_y-1}|; T_y \leq n).$$

Since $|X_{T_y}| + |X_{T_y-1}| \leq 2 \max_{1 \leq k \leq n} |X_k| + |x|$ on $\{T_y \leq n\}$, following the proof of Lemma 2.4.3,

$$\begin{aligned} \mathbb{E}_x(z + M_n; T_y > n) &\leq z + c \left(1 + \sum_{k=1}^n \prod_{i=1}^k \mathbb{E}(|a_i|) \right) |x| \\ &\quad + cn^{1/2-2\varepsilon} \mathbb{P} \left(\max_{1 \leq k \leq n} |X_k^0| \leq n^{1/2-2\varepsilon} \right) \\ &\quad + c\mathbb{E} \left(\max_{1 \leq k \leq n} |X_k^0|; \max_{1 \leq k \leq n} |X_k^0| > n^{1/2-2\varepsilon} \right) \\ &\leq z + c|x| + c_p n^{1/2-2\varepsilon}. \end{aligned}$$

□

Lemma 2.4.8. *Assume Condition 2.1. Let $p \in (2, \alpha)$. There exists $\varepsilon_1 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1)$, $x \in \mathbb{R}$, $y \in \mathbb{R}$, $n \geq 0$ and $2 \leq k_0 \leq n$,*

$$\begin{aligned} \mathbb{E}_x(y + \rho x + M_n; T_y > n) &\leq \left(1 + \frac{c_{p,\varepsilon}}{k_0^\varepsilon} \right) \max(y, 0) + c_{p,\varepsilon}|x| + c_{p,\varepsilon}\sqrt{k_0} + c_{p,\varepsilon}e^{-c_{p,\varepsilon}k_0^\varepsilon}|x|^p \\ &\leq c_p(1 + \max(y, 0) + |x|^p). \end{aligned}$$

Proof. We proceed as in the proof of Lemma 2.4.4. Set $\varepsilon_1 = \min(\varepsilon_0, \frac{p-2}{4p})$, where ε_0 is defined in Lemma 2.4.1. Let $\varepsilon \in (0, \varepsilon_1]$. With $z = y + \rho x$ and $v_n^\varepsilon = v_n + \lfloor n^\varepsilon \rfloor$, we have

$$\begin{aligned} \mathbb{E}_x(z + M_n; T_y > n) &= \underbrace{\mathbb{E}_x(z + M_n; T_y > n, v_n^\varepsilon > \lfloor n^{1-\varepsilon} \rfloor)}_{=: J_1} \\ &\quad + \underbrace{\mathbb{E}_x(z + M_n; T_y > n, v_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor)}_{=: J_2}. \end{aligned} \quad (2.4.7)$$

Bound of J_1 . Let $m_\varepsilon = \lfloor n^{1-\varepsilon} \rfloor - \lfloor n^\varepsilon \rfloor$. Since on $\{v_n > m_\varepsilon\}$ it holds $z' = z + M_{m_\varepsilon} \leq n^{1/2-\varepsilon}$, by the Markov property we write that

$$J_1 \leq n^{1/2-\varepsilon} \mathbb{P}_x(v_n > m_\varepsilon) + \int_{\mathbb{R}} \mathbb{E}_{x'}(|M_{n-m_\varepsilon}|) \mathbb{P}_x(X_{m_\varepsilon} \in dx', v_n > m_\varepsilon).$$

By Lemma 2.3.4 and the Hölder inequality,

$$\begin{aligned} J_1 &\leq n^{1/2-\varepsilon} \mathbb{P}_x(v_n > m_\varepsilon) + \mathbb{E}_x \left(c \left(\sqrt{n - m_\varepsilon} + |X_{m_\varepsilon}| \right); v_n > m_\varepsilon \right) \\ &\leq cn^{1/2} \mathbb{P}_x(v_n > m_\varepsilon) + \mathbb{E}_x^{1/p}(|X_{m_\varepsilon}|^p) \mathbb{P}_x^{1/q}(v_n > m_\varepsilon). \end{aligned}$$

By Lemma 2.3.1 and Lemma 2.4.1 (since $m_\varepsilon \geq n^{1-\varepsilon}/c_\varepsilon$),

$$J_1 \leq \frac{c_{p,\varepsilon}}{n^{\frac{p-1}{2}-p\varepsilon}} + c_{p,\varepsilon} e^{-c_{p,\varepsilon}n^{1-2\varepsilon}} |x|^p. \quad (2.4.8)$$

Bound of J_2 . Repeating the arguments used for bounding the term J_2 in Lemma 2.4.4, by the Markov property and Lemma 2.4.7, we get

$$J_2 \leq \mathbb{E}_x \left(z + M_{v_n^\varepsilon} + c |X_{v_n^\varepsilon}| + c_p n^{1/2-2\varepsilon}; T_y > v_n^\varepsilon, v_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor \right).$$

Using the claim 2 of Lemma 2.4.6 and Lemma 2.3.1,

$$\begin{aligned} J_2 &\leq \mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor}; T_y > \lfloor n^{1-\varepsilon} \rfloor, v_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\quad + \mathbb{E}_x \left(c_p n^{1/2-2\varepsilon}; T_y > v_n, v_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) + c_{p,\varepsilon} e^{-c_\varepsilon n^\varepsilon} \mathbb{E}_x \left(|X_{v_n}|; v_n \leq \lfloor n^{1-\varepsilon} \rfloor \right). \end{aligned}$$

On the event $\{T_y > v_n\}$, we have $n^{1/2-\varepsilon} < z + M_{v_n}$. Hence

$$\begin{aligned} J_2 &\leq \mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor}; T_y > \lfloor n^{1-\varepsilon} \rfloor, v_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\quad + c_p \mathbb{E}_x \left(\frac{z + M_{v_n}}{n^\varepsilon}; T_y > v_n, v_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) + c_{p,\varepsilon} e^{-c_\varepsilon n^\varepsilon} \mathbb{E}_x \left(|X_{v_n}|; v_n \leq \lfloor n^{1-\varepsilon} \rfloor \right). \end{aligned}$$

Coupling this with (2.4.8) and (2.4.7) and using again the claim 2 of Lemma 2.4.6, we obtain that

$$\begin{aligned} \mathbb{E}_x(z + M_n; T_y > n) &\leq \left(1 + \frac{c_p}{n^\varepsilon}\right) \mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor}; T_y > \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\quad + \frac{c_{p,\varepsilon}}{n^{\frac{p-1}{2}-p\varepsilon}} + c_{p,\varepsilon} e^{-c_{p,\varepsilon}n^\varepsilon} |x|^p. \end{aligned}$$

Since $((z + M_n) \mathbb{1}_{\{T_y > n\}})_{n \geq 0}$ is a submartingale (claim 2 of Lemma 2.4.6), the sequence $u_n = \mathbb{E}(z + M_n; T_y > n)$ is non-decreasing. By Lemma 2.9.1 with $\alpha = c_p$, $\beta = c_{p,\varepsilon}$, $\gamma = |x|^p$ and $\delta = c_{p,\varepsilon}$, we write that

$$\mathbb{E}_x(z + M_n; T_y > n) \leq \left(1 + \frac{c_{p,\varepsilon}}{k_0^\varepsilon}\right) \mathbb{E}_x(z + M_{k_0}; T_y > k_0) + \frac{c_{p,\varepsilon}}{k_0^\varepsilon} + c_{p,\varepsilon} e^{-c_{p,\varepsilon}k_0^\varepsilon} |x|^p.$$

Using Lemma 2.3.4 and the fact that $z = y + \rho x$, we obtain that

$$\mathbb{E}_x(z + M_n; T_y > n) \leq \left(1 + \frac{c_{p,\varepsilon}}{k_0^\varepsilon}\right) \max(y, 0) + c_{p,\varepsilon} |x| + c_{p,\varepsilon} \sqrt{k_0} + c_{p,\varepsilon} e^{-c_{p,\varepsilon}k_0^\varepsilon} |x|^p.$$

□

To transfer the assertion of Lemma 2.4.8 to the random walk $(y + S_n)_{n \geq 0}$, we need to assume that $\mathbb{E}(a) < 0$.

Corollary 2.4.9. *Assume Condition 2.1 and $\mathbb{E}(a) < 0$. Let $p \in (2, \alpha)$. For any $x \in \mathbb{R}$, $y > 0$ and $n \in \mathbb{N}$,*

$$\mathbb{E}_x(y + S_n; \tau_y > n) \leq c_p (1 + y + |x|^p).$$

Proof. By (2.3.2) and the claim 1 of Lemma 2.4.6, we have

$$\mathbb{E}_x(y + S_n; \tau_y > n) = \mathbb{E}_x(y + \rho x + M_n; T_y \geq \tau_y > n) - \mathbb{E}_x(\rho X_n; \tau_y > n).$$

The result follows from Lemma 2.4.8. □

2.5 Existence of the harmonic function

In this section we prove Theorem 2.2.1. We split the proof into two parts according to the values of $\mathbb{E}(a)$.

2.5.1 Existence of the harmonic function: the case $\mathbb{E}(a) \geq 0$

We start with the following assertion.

Lemma 2.5.1. *Assume Condition 2.1 and $\mathbb{E}(a) \geq 0$. For any $x \in \mathbb{R}$ and $y > 0$, the random variable M_{τ_y} is integrable. Moreover, for any $p \in (2, \alpha)$,*

$$\mathbb{E}_x \left(|M_{\tau_y}| \right) \leq c_p (1 + y + |x|) (1 + |x|)^{p-1}.$$

Proof. Let $z = y + \rho x$. Using the claim 1 of Lemma 2.4.2 and the Doob optional stopping theorem, we have

$$\begin{aligned} \mathbb{E}_x \left(|M_{\tau_y}| ; \tau_y \leq n \right) &\leq -\mathbb{E}_x (z + M_n ; \tau_y \leq n) + y + \rho |x| \\ &= \mathbb{E}_x (z + M_n ; \tau_y > n) - z + y + \rho |x|. \end{aligned}$$

By second bound in Lemma 2.4.4, for all $n \geq 0$,

$$\mathbb{E}_x \left(|M_{\tau_y}| ; \tau_y \leq n \right) \leq c_p (1 + y + |x|) (1 + |x|)^{p-1} =: C_p(x, y).$$

Since $(\{\tau_y \leq n\})_{n \geq 1}$ is a non-decreasing sequence of events and $\mathbb{P}_x(\tau_y < +\infty) = 1$ for any $x \in \mathbb{R}$ (by Corollary 2.9.7), the result follows by the Lebesgue monotone convergence theorem. \square

It follows from Lemma 2.5.1 that the function

$$V(x, y) = -\mathbb{E}_x (M_{\tau_y})$$

is well defined for any $x \in \mathbb{R}$ and $y > 0$, which also proves the claim 1 of Theorem 2.2.1 when $\mathbb{E}(a) \geq 0$.

The following two propositions prove the claims 2 and 3 of Theorem 2.2.1 when $\mathbb{E}(a) \geq 0$.

Proposition 2.5.2. *Assume Condition 2.1 and $\mathbb{E}(a) \geq 0$.*

1. *For any $x \in \mathbb{R}$ and $y > 0$,*

$$V(x, y) = \lim_{n \rightarrow +\infty} \mathbb{E}_x (y + \rho x + M_n ; \tau_y > n) = \lim_{n \rightarrow +\infty} \mathbb{E}_x (y + S_n ; \tau_y > n).$$

2. *For any $x \in \mathbb{R}$, the function $V(x, \cdot)$ is non-decreasing.*

3. *For any $\delta > 0$, $p \in (2, \alpha)$, $x \in \mathbb{R}$ and $y > 0$,*

$$\max(0, y + \rho x) \leq V(x, y) \leq \left(1 + c_p \delta (1 + |x|)^{p-1}\right) y + c_{p, \delta} (1 + |x|)^p.$$

4. *For any $x \in \mathbb{R}$,*

$$\lim_{y \rightarrow +\infty} \frac{V(x, y)}{y} = 1.$$

Proof. We use the notation $z = y + \rho x$.

Claim 1. Since, by Lemma 2.5.1, M_{τ_y} is integrable, we have by the Lebesgue dominated convergence theorem,

$$\mathbb{E}_x(z + M_n; \tau_y > n) = z - \mathbb{E}_x(z + M_{\tau_y}; \tau_y \leq n) \xrightarrow{n \rightarrow +\infty} -\mathbb{E}_x(M_{\tau_y}) = V(x, y).$$

To prove the second equality of the claim 1 we use Lemma 2.3.1 and the fact that $\tau_y < +\infty$:

$$|\mathbb{E}_x(X_n; \tau_y > n)| \leq \mathbb{E}_x^{1/2}(|X_n|^2) \sqrt{\mathbb{P}_x(\tau_y > n)} \leq c_2(1 + |x|) \sqrt{\mathbb{P}_x(\tau_y > n)} \xrightarrow{n \rightarrow +\infty} 0.$$

Using (2.3.2), we obtain the claim 1.

Claim 2. If $y_1 \leq y_2$, then $\tau_{y_1} \leq \tau_{y_2}$ and

$$\mathbb{E}_x(y_1 + S_n; \tau_{y_1} > n) \leq \mathbb{E}_x(y_2 + S_n; \tau_{y_1} > n) \leq \mathbb{E}_x(y_2 + S_n; \tau_{y_2} > n).$$

Taking the limit as $n \rightarrow +\infty$ we get the claim 2.

Claim 3. The upper bound follows from the claim 1 and Lemma 2.4.4. On the event $\{\tau_y > n\}$, we obviously have $y + S_n > 0$ and so by claim 1, $V(x, y) \geq 0$. Moreover, since $z + M_{\tau_y} \leq 0$ (by claim 1 of Lemma 2.4.2), we have, by claim 1,

$$V(x, y) = z - \lim_{n \rightarrow +\infty} \mathbb{E}_x(z + M_{\tau_y}; \tau_y \leq n) \geq z,$$

which proves the lower bound.

Claim 4. By the claim 3, for all $\delta > 0$, $x \in \mathbb{R}$,

$$1 \leq \liminf_{y \rightarrow +\infty} \frac{V(x, y)}{y} \leq \limsup_{y \rightarrow +\infty} \frac{V(x, y)}{y} \leq (1 + c_p \delta (1 + |x|)^{p-1}).$$

Letting $\delta \rightarrow 0$, we obtain the claim 4. □

We now prove that V is harmonic on $\mathbb{R} \times \mathbb{R}_+^*$.

Proposition 2.5.3. *Assume Conditions 2.1 and $\mathbb{E}(a) \geq 0$.*

1. *The function V is \mathbf{Q}_+ -harmonic on $\mathbb{R} \times \mathbb{R}_+^*$: for any $x \in \mathbb{R}$ and $y > 0$,*

$$\mathbf{Q}_+ V(x, y) = V(x, y).$$

2. *If in addition we assume Condition 2.2, then the function V is positive on $\mathbb{R} \times \mathbb{R}_+^*$.*

Proof. *Claim 1.* Denote for brevity $V_n(x, y) = \mathbb{E}_x(y + S_n; \tau_y > n)$. For all $x \in \mathbb{R}$, $y > 0$ and $n \geq 1$, by the Markov property,

$$V_{n+1}(x, y) = \mathbb{E}_x(V_n(X_1, y + S_1); \tau_y > 1).$$

By Corollary 2.4.5, we see that the quantity $V_n(X_1, y + S_1)$ is dominated by the random variable $c_p(1 + y + S_1 + |X_1|)(1 + |X_1|)^{p-1}$ which is integrable with respect to \mathbb{E}_x . Consequently, by the Lebesgue dominated convergence theorem and the claim 1 of Proposition 2.5.2,

$$V(x, y) = \mathbb{E}_x(V(X_1, y + S_1); \tau_y > 1) = \mathbf{Q}_+ V(x, y),$$

where by convention, $V(x, y)\mathbb{1}_{\{y > 0\}} = 0$ if $y \leq 0$ and $x \in \mathbb{R}$.

Claim 2. Fix $x \in \mathbb{R}$ and $y > 0$. Using the claim 1 and the fact that V is non-negative on $\mathbb{R} \times \mathbb{R}_+^*$ (claim 3 of Proposition 2.5.2) we write

$$V(x, y) \geq \mathbb{E}_x \left(V(X_1, y + S_1); \tau_y > 1, X_1 > \frac{-y}{2(1+\rho)} \right).$$

By the lower bound of the claim 3 of Proposition 2.5.2 and (2.3.2),

$$V(x, y) \geq \mathbb{E}_x \left(y + (1+\rho)X_1; \tau_y > 1, X_1 > \frac{-y}{2(1+\rho)} \right) \geq \frac{y}{2} \mathbb{P}_x \left(X_1 > \frac{-y}{2(1+\rho)} \right).$$

By Condition 2.2, we conclude that, $V(x, y) > 0$ for any $x \in \mathbb{R}$ and $y > 0$. \square

2.5.2 Existence of the harmonic function: the case $\mathbb{E}(a) < 0$

In this section we prove the harmonicity and the positivity of the function V in the case $\mathbb{E}(a) < 0$. The following analogue of Lemma 2.5.1 shows that the random variables M_{T_y} and M_{τ_y} are integrable.

Lemma 2.5.4. *Assume Condition 2.1.*

1. For any $x \in \mathbb{R}$ and $y \in \mathbb{R}$,

$$\mathbb{E}_x \left(|M_{T_y}| \right) \leq c_p (1 + |y| + |x|^p).$$

2. If in addition $\mathbb{E}(a) < 0$, then for any $x \in \mathbb{R}$ and $y \in \mathbb{R}$,

$$\mathbb{E}_x \left(|M_{\tau_y}| \right) \leq c_p (1 + |y| + |x|^p).$$

Proof. Claim 1. The proof of the bound of $\mathbb{E}_x \left(|M_{T_y}| \right)$ is similar to that of Lemma 2.5.1 using Lemma 2.4.8 instead of Lemma 2.4.4 and the fact that by Corollary 2.9.7 we have $\mathbb{P}_x (T_y < +\infty) = 1$, $x \in \mathbb{R}$.

Claim 2. By the claim 1 of Lemma 2.4.6, we have $\tau_y \wedge n \leq T_y \wedge n$. Since $(|M_n|)_{n \geq 0}$ is a submartingale, with $z = y + \rho x$,

$$\mathbb{E}_x \left(|M_{\tau_y}|; \tau_y \leq n \right) \leq \mathbb{E}_x \left(|M_{\tau_y \wedge n}| \right) \leq \mathbb{E}_x \left(|M_{T_y \wedge n}| \right) \leq 2|z| + 2\mathbb{E}_x \left(|M_{T_y}|; T_y \leq n \right).$$

The Lebesgue monotone convergence theorem implies the claim 2. \square

It follows from the claim 2 of Lemma 2.5.4 that, under Condition 2.1 and $\mathbb{E}(a) < 0$, the function

$$V(x, y) = -\mathbb{E}_x \left(M_{\tau_y} \right)$$

is well defined for any $x \in \mathbb{R}$ and $y > 0$. This also implies the claim 1 of Theorem 2.2.1 when $\mathbb{E}(a) < 0$. To prove the positivity of the function V on $\mathbb{R} \times \mathbb{R}_+^*$, we also consider the function

$$W(x, y) = -\mathbb{E}_x \left(M_{T_y} \right),$$

which is well defined on $\mathbb{R} \times \mathbb{R}$ by the claim 1 of Lemma 2.5.4. Note that W exists under solely Condition 2.1.

Proposition 2.5.5. *Assume Condition 2.1.*

1. For any $x \in \mathbb{R}$ and $y \in \mathbb{R}$,

$$W(x, y) = \lim_{n \rightarrow +\infty} \mathbb{E}_x (y + \rho x + M_n ; T_y > n).$$

2. For any $x \in \mathbb{R}$, the function $W(x, \cdot)$ is non-decreasing.

3. For any $p \in (2, \alpha)$, there exists $\varepsilon_1 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1]$, $k_0 \geq 2$, $x \in \mathbb{R}$ and $y \in \mathbb{R}$,

$$\max(0, y + \rho x) \leq W(x, y) \leq \left(1 + \frac{c_{p,\varepsilon}}{k_0^\varepsilon}\right) \max(y, 0) + c_{p,\varepsilon} |x| + c_{p,\varepsilon} \sqrt{k_0} + c_{p,\varepsilon} e^{-c_{p,\varepsilon} k_0^\varepsilon} |x|^p.$$

4. For any $x \in \mathbb{R}$,

$$\lim_{y \rightarrow +\infty} \frac{W(x, y)}{y} = 1.$$

5. For any $x \in \mathbb{R}$ and $y \in \mathbb{R}$,

$$W(x, y) = \mathbb{E}_x (W(X_1, y + S_1) ; T_y > 1),$$

and $(W(X_n, y + S_n) \mathbb{1}_{\{T_y > n\}})_{n \geq 0}$ is a martingale.

Proof. The proof is very close to that of Proposition 2.5.2. The upper bound of the claim 3 is obtained taking the limit as $n \rightarrow +\infty$ in Lemma 2.4.8. We prove the claim 4 taking the limit as $y \rightarrow +\infty$ and then as $k_0 \rightarrow +\infty$ in the inequality of the claim 3. The proof of the claim 5 is the same as that of the claim 1 of Proposition 2.5.3. \square

Turning now to V , we have the following proposition.

Proposition 2.5.6. *Assume Condition 2.1 and $\mathbb{E}(a) < 0$.*

1. For any $x \in \mathbb{R}$ and $y > 0$,

$$V(x, y) = \lim_{n \rightarrow +\infty} \mathbb{E}_x (y + \rho x + M_n ; \tau_y > n) = \lim_{n \rightarrow +\infty} \mathbb{E}_x (y + S_n ; \tau_y > n).$$

2. For any $x \in \mathbb{R}$, the function $V(x, \cdot)$ is non-decreasing.

3. For any $p \in (2, \alpha)$, $\delta > 0$, $x \in \mathbb{R}$ and $y > 0$,

$$0 \leq V(x, y) \leq W(x, y) \leq (1 + c_p \delta) y + c_{p,\delta} (1 + |x|^p).$$

4. The function V is \mathbf{Q}_+ -harmonic on $\mathbb{R} \times \mathbb{R}_+^*$: for any $x \in \mathbb{R}$ and $y > 0$,

$$\mathbf{Q}_+ V(x, y) = V(x, y)$$

and $(V(X_n, y + S_n) \mathbb{1}_{\{\tau_y > n\}})_{n \geq 0}$ is a martingale.

Proof. The proofs of the claims 1, 2, 4 and of the lower bound of the claim 3, being similar to that of the previous proposition and of the Proposition 2.5.2, is left to the reader. The upper bound of the claim 3 is a consequence of the fact that $\tau_y \leq T_y$ (claim 1 of Lemma 2.4.6): with $z = y + \rho x$,

$$\begin{aligned} V(x, y) &= \lim_{n \rightarrow +\infty} \mathbb{E}_x (z + M_n ; \tau_y > n) \\ &\leq \lim_{n \rightarrow +\infty} \mathbb{E}_x (z + M_n ; T_y > n) = W(x, y). \end{aligned}$$

\square

Our next goal is to prove that $V(x, y) \geq \max(0, (1 - \delta)y - c_{p,\delta}(1 + |x|^p))$ from which we will deduce the positivity of V . For this we make appropriate adjustments to the proof of Lemmas 2.4.3 and Lemma 2.4.4 where the submartingale $((y + \rho x + M_n) \mathbb{1}_{\{\tau_y > n\}})_{n \geq 0}$ will be replaced by the supermartingale $(W(X_n, y + S_n) \mathbb{1}_{\{\tau_y > n\}})_{n \geq 0}$. Instead of upper bounds in Lemmas 2.4.3 and Lemma 2.4.4 the following two lemmas establish lower bounds.

Lemma 2.5.7. *Assume Condition 2.1 and $\mathbb{E}(a) < 0$. For any $p \in (2, \alpha)$, there exists $\varepsilon_1 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1]$, $x \in \mathbb{R}$, $y > 0$ and $n \in \mathbb{N}$,*

$$\mathbb{E}_x(W(X_n, y + S_n); \tau_y > n) \geq W(x, y) - c_{p,\varepsilon} n^{1/2-2\varepsilon} - c_{p,\varepsilon} |x|^p.$$

Proof. By the claim 1 of Lemma 2.4.6 and the claim 5 of Lemma 2.5.5, as in the proof of Lemma 2.4.3,

$$\mathbb{E}_x(W(X_n, y + S_n); \tau_y > n) = W(x, y) - \mathbb{E}_x(W(X_{\tau_y}, y + S_{\tau_y}); T_y > \tau_y, \tau_y \leq n).$$

Using the claim 3 of Proposition 2.5.5 and the fact that $y + S_{\tau_y} \leq 0$,

$$\begin{aligned} \mathbb{E}_x(W(X_{\tau_y}, y + S_{\tau_y}); T_y > \tau_y, \tau_y \leq n) &\leq \\ &\mathbb{E}_x\left(c_{p,\varepsilon} |X_{\tau_y}| + c_{p,\varepsilon} \sqrt{k_0} + c_{p,\varepsilon} e^{-c_{p,\varepsilon} k_0^\varepsilon} |X_{\tau_y}|^p; \tau_y \leq n\right). \end{aligned}$$

Taking $k_0 = \lfloor n^{1-4\varepsilon} \rfloor$, the end of the proof is the same as the proof of Lemma 2.4.3. \square

Lemma 2.5.8. *Assume Condition 2.1 and $\mathbb{E}(a) < 0$. For any $p \in (2, \alpha)$ there exists $\varepsilon_1 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1]$, $k_0 \geq 2$, $x \in \mathbb{R}$ and $y > 0$,*

$$\mathbb{E}_x(W(X_n, y + S_n); \tau_y > n) \geq y \left(1 - \frac{c_{p,\varepsilon}}{k_0^\varepsilon}\right) - c_{p,\varepsilon} k_0^2 (1 + |x|^p).$$

Proof. The proof is similar to that of Lemma 2.4.4. With $v_n^\varepsilon = v_n + \lfloor n^\varepsilon \rfloor$, we have

$$J_0 = \mathbb{E}_x(W(X_n, y + S_n); \tau_y > n) \geq \mathbb{E}_x(W(X_n, y + S_n); \tau_y > n, v_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor).$$

Using the Markov property, Lemma 2.5.7 and the fact that $n - v_n^\varepsilon \leq n$,

$$J_0 \geq \mathbb{E}_x\left(W(X_{v_n^\varepsilon}, y + S_{v_n^\varepsilon}) - c_{p,\varepsilon} n^{1/2-2\varepsilon} - c_{p,\varepsilon} |X_{v_n^\varepsilon}|^p; \tau_y > v_n^\varepsilon, v_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor\right).$$

By the claim 1 of Lemma 2.4.6, on $\{\tau_y > v_n\}$ we have $z + M_{v_n} > n^{1/2-\varepsilon}$, where $z = y + \rho x$. Moreover, using the fact that $(W(X_n, y + S_n) \mathbb{1}_{\{\tau_y > n\}})_{n \geq 1}$ is a non-negative martingale (claim 3 and 5 of Proposition 2.5.5) and the fact that $\tau_y \leq T_y$ a.s. (claim 1 of Lemma 2.4.6) we can see that $(W(X_n, y + S_n) \mathbb{1}_{\{\tau_y > n\}})_{n \geq 1}$ is a supermartingale. From this and as in the bound of the term J_2 of Lemma 2.4.4, we obtain that

$$\begin{aligned} J_0 &\geq \mathbb{E}_x\left(W(X_{\lfloor n^{1-\varepsilon} \rfloor}, y + S_{\lfloor n^{1-\varepsilon} \rfloor}); \tau_y > \lfloor n^{1-\varepsilon} \rfloor\right) \\ &\quad - \mathbb{E}_x\left(W(X_{\lfloor n^{1-\varepsilon} \rfloor}, y + S_{\lfloor n^{1-\varepsilon} \rfloor}); \tau_y > \lfloor n^{1-\varepsilon} \rfloor, v_n^\varepsilon > \lfloor n^{1-\varepsilon} \rfloor\right) \\ &\quad - \frac{c_{p,\varepsilon}}{n^\varepsilon} \mathbb{E}_x\left(z + M_{v_n}; T_y > v_n, v_n \leq \lfloor n^{1-\varepsilon} \rfloor\right) - c_{p,\varepsilon} e^{-c_{p,\varepsilon} n^\varepsilon} (1 + |x|^p). \end{aligned} \tag{2.5.1}$$

Using the claim 3 of Proposition 2.5.5 with $k_0 = n$ and the martingale representation (2.3.2), the absolute value of the second term in the r.h.s. of (2.5.1) does not exceed

$$c_{p,\varepsilon} \mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor} + \sqrt{n} + |X_{\lfloor n^{1-\varepsilon} \rfloor}| + e^{-c_{p,\varepsilon} n^\varepsilon} |X_{\lfloor n^{1-\varepsilon} \rfloor}|^p ; \right. \\ \left. \tau_y > \lfloor n^{1-\varepsilon} \rfloor, v_n^\varepsilon > \lfloor n^{1-\varepsilon} \rfloor \right).$$

Since $\left((z + M_n) \mathbb{1}_{\{T_y > n\}} \right)_{n \geq 0}$ is a submartingale, by claim 2 of Lemma 2.4.6, the absolute value of the third term is less than

$$\frac{c_{p,\varepsilon}}{n^\varepsilon} \mathbb{E}_x (z + M_n ; T_y > n).$$

These bounds imply

$$J_0 \geq \mathbb{E}_x \left(W \left(X_{\lfloor n^{1-\varepsilon} \rfloor}, y + S_{\lfloor n^{1-\varepsilon} \rfloor} \right) ; \tau_y > \lfloor n^{1-\varepsilon} \rfloor \right) \\ - c_{p,\varepsilon} \mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor} + \sqrt{n} + |X_{\lfloor n^{1-\varepsilon} \rfloor}| ; \tau_y > \lfloor n^{1-\varepsilon} \rfloor, v_n^\varepsilon > \lfloor n^{1-\varepsilon} \rfloor \right) \\ - c_{p,\varepsilon} e^{-c_{p,\varepsilon} n^\varepsilon} \mathbb{E}_x \left(|X_{\lfloor n^{1-\varepsilon} \rfloor}|^p ; \tau_y > \lfloor n^{1-\varepsilon} \rfloor, v_n^\varepsilon > \lfloor n^{1-\varepsilon} \rfloor \right) \\ - \frac{c_{p,\varepsilon}}{n^\varepsilon} \mathbb{E}_x (z + M_n ; T_y > n) - c_{p,\varepsilon} e^{-c_{p,\varepsilon} n^\varepsilon} (1 + |x|^p). \quad (2.5.2)$$

Using the Markov property with the intermediate time $m_\varepsilon = \lfloor n^{1-\varepsilon} \rfloor - \lfloor n^\varepsilon \rfloor$, Lemmas 2.3.4 and 2.3.1 and the fact that $v_n^\varepsilon = v_n + \lfloor n^\varepsilon \rfloor$, the absolute value of the second term in the r.h.s. of (2.5.2) is bounded by

$$c_{p,\varepsilon} \mathbb{E}_x \left(|z + M_{m_\varepsilon}| + c n^{\varepsilon/2} + c |X_{m_\varepsilon}| + \sqrt{n} + c(1 + |X_{m_\varepsilon}|) ; \tau_y > m_\varepsilon, v_n > m_\varepsilon \right),$$

which, in turn, using the fact that $z + M_{m_\varepsilon} \leq n^{1/2-\varepsilon}$ on $\{v_n > m_\varepsilon\}$, is less than

$$c_{p,\varepsilon} \mathbb{E}_x \left(\sqrt{n} + |X_{m_\varepsilon}| ; \tau_y > m_\varepsilon, v_n > m_\varepsilon \right).$$

The absolute value of the third term in the r.h.s. of (2.5.2) is bounded using Lemma 2.3.1 by $c_{p,\varepsilon} e^{-c_{p,\varepsilon} n^\varepsilon} (1 + |x|^p)$. The fourth term is bounded by Lemma 2.4.8. Collecting these bounds, we obtain

$$J_0 \geq \mathbb{E}_x \left(W \left(X_{\lfloor n^{1-\varepsilon} \rfloor}, y + S_{\lfloor n^{1-\varepsilon} \rfloor} \right) ; \tau_y > \lfloor n^{1-\varepsilon} \rfloor \right) \\ - c_{p,\varepsilon} \mathbb{E}_x \left(\sqrt{n} + |X_{m_\varepsilon}| ; \tau_y > m_\varepsilon, v_n > m_\varepsilon \right) - \frac{c_{p,\varepsilon}}{n^\varepsilon} (1 + y + |x|^p). \quad (2.5.3)$$

Coupling the Hölder inequality with Lemma 2.3.1 and Lemma 2.4.1, we find that the second term in the r.h.s. of (2.5.3) does not exceed

$$c_{p,\varepsilon} \left(\sqrt{n} + \mathbb{E}_x^{1/p} (|X_{m_\varepsilon}|^p) \right) \mathbb{P}_x^{1/q} \left(v_n > \frac{n^{1-\varepsilon}}{c_\varepsilon} \right) \leq c_{p,\varepsilon} \left(\sqrt{n} + |x| \right) \frac{c_{p,\varepsilon} (1 + |x|)^{p-1}}{n^{\frac{p-1}{2} - (p-1)\varepsilon}}.$$

Implementing this into (2.5.3),

$$J_0 \geq \mathbb{E}_x \left(W \left(X_{\lfloor n^{1-\varepsilon} \rfloor}, y + S_{\lfloor n^{1-\varepsilon} \rfloor} \right) ; \tau_y > \lfloor n^{1-\varepsilon} \rfloor \right) - \frac{c_{p,\varepsilon}}{n^\varepsilon} (1 + y + |x|^p).$$

Since $\left(W(X_n, y + S_n) \mathbb{1}_{\{\tau_y > n\}} \right)_{n \geq 1}$ is a supermartingale, Lemma 2.9.2 implies that

$$J_0 \geq \mathbb{E}_x \left(W(X_{k_0}, y + S_{k_0}) ; \tau_y > k_0 \right) - \frac{c_{p,\varepsilon}}{k_0^\varepsilon} (1 + y + |x|^p).$$

Using the lower bound of the claim 3 of Proposition 2.5.5 and Lemma 2.3.4, we deduce that

$$\mathbb{E}_x(W(X_n, y + S_n); \tau_y > n) \geq y \mathbb{P}_x(\tau_y > k_0) - y \frac{c_{p,\varepsilon}}{k_0^\varepsilon} - c_{p,\varepsilon} \sqrt{k_0} - c_{p,\varepsilon} |x|^p.$$

Now, when $y \rightarrow +\infty$, one can see that $\mathbb{P}_x(\tau_y > k_0) \rightarrow 1$: more precisely,

$$\mathbb{P}_x(\tau_y > k_0) \geq \mathbb{P}_x\left(\max_{1 \leq k \leq k_0} |X_k| < \frac{y}{k_0}\right) \geq 1 - c \frac{k_0^2(1 + |x|)}{y}.$$

Finally,

$$\mathbb{E}_x(W(X_n, y + S_n); \tau_y > n) \geq y \left(1 - \frac{c_{p,\varepsilon}}{k_0^\varepsilon}\right) - c_{p,\varepsilon} k_0^2 (1 + |x|^p).$$

□

Under Condition 2.3 we use Lemma 2.5.8 to prove that V is positive on $\mathbb{R} \times \mathbb{R}_+^*$.

Proposition 2.5.9. *Assume Conditions 2.1 and $\mathbb{E}(a) < 0$.*

1. For any $\delta > 0$, $p \in (2, \alpha)$, $x \in \mathbb{R}$, $y > 0$,

$$V(x, y) \geq (1 - \delta)y - c_{p,\delta} (1 + |x|^p).$$

2. For any $x \in \mathbb{R}$,

$$\lim_{y \rightarrow +\infty} \frac{V(x, y)}{y} = 1.$$

3. If in addition we assume Condition 2.3, then the function V is positive on $\mathbb{R} \times \mathbb{R}_+^*$.

Proof. Claim 1. Using the claim 1 of Lemma 2.4.6 and the claims 3 and 5 of Proposition 2.5.5, with $z = y + \rho x$, we write

$$\begin{aligned} & \mathbb{E}_x(z + M_n; \tau_y > n) \\ & \geq \mathbb{E}_x(z + M_n; T_y > n) - \mathbb{E}_x(W(X_n, y + S_n); T_y > n, \tau_y \leq n) \\ & = \mathbb{E}_x(z + M_n; T_y > n) - W(x, y) + \mathbb{E}_x(W(X_n, y + S_n); \tau_y > n). \end{aligned}$$

Using Lemma 2.5.8, the claim 1 of Proposition 2.5.5 and the claim 1 of Proposition 2.5.6, we obtain

$$V(x, y) \geq y \left(1 - \frac{c_{p,\varepsilon}}{k_0^\varepsilon}\right) - c_{p,\varepsilon} k_0^2 (1 + |x|^p).$$

Taking k_0 large enough, the claim 1 is proved.

Claim 2. Taking the limit as $y \rightarrow +\infty$ and as $\delta \rightarrow 0$ in the claim 1, we obtain first that $\liminf_{y \rightarrow +\infty} V(x, y)/y \geq 1$. By the claim 3 of Proposition 2.5.6, we obtain also that $\limsup_{y \rightarrow +\infty} V(x, y)/y \leq 1$.

Claim 3. Fix $x \in \mathbb{R}$, $y > 0$ and $\delta_0 > 0$. By Condition 2.3, there exists $p_0 \in (2, \alpha)$ such that for any $c > 0$ there exists $n_0 \geq 1$ such that $\mathbb{P}_x((X_{n_0}, y + S_{n_0}) \in K_{p_0, c}, \tau_y > n_0) > 0$. Thus, using the claim 4 of Proposition 2.5.6,

$$V(x, y) \geq \mathbb{E}_x(V(X_{n_0}, y + S_{n_0}); (X_{n_0}, y + S_{n_0}) \in K_{p_0, c}, \tau_y > n_0).$$

Using the claim 1 with $p = p_0$ and $\delta = 1/2$ and choosing the constant $c = 2c_{p_0, \delta} + 2\delta_0$, there exists n_0 such that

$$V(x, y) \geq \delta_0 \mathbb{P}_x((X_{n_0}, y + S_{n_0}) \in K_{p_0, c}, \tau_y > n_0) > 0.$$

□

2.6 Asymptotic for the exit time

The aim of this section is to prove Theorem 2.2.2. The asymptotic for the exit time of the Markov walk $(y + S_n)_{n \geq 0}$ will be deduced from the asymptotic of the exit time for the Brownian motion in Corollary 2.9.4 using the functional approximation in Proposition 2.9.5.

2.6.1 Auxiliary statements

We start by proving an analogue of Corollaries 2.4.5 and 2.4.9, where n is replaced by the stopping time ν_n defined by (2.4.1).

Lemma 2.6.1. *Assume Condition 2.1. For any $p \in (2, \alpha)$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, $x \in \mathbb{R}$, $y > 0$ and $n \geq 1$,*

$$E_1 = \mathbb{E}_x \left(y + S_{\nu_n}; \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) \leq c_{p,\varepsilon} (1 + y + |x|) (1 + |x|)^{p-1}.$$

Proof. When $\tau_y > \nu_n > 1$, we note that

$$0 < X_{\nu_n} < y + S_{\nu_n}. \quad (2.6.1)$$

Using the martingale representation (2.3.2) and (2.6.1), we have

$$y + S_{\nu_n} \leq z + M_{\nu_n} + \max(0, -\rho) X_{\nu_n} \leq z + M_{\nu_n} + \max(0, -\rho) (y + S_{\nu_n}),$$

with $z = y + \rho x$, and so

$$0 < y + S_{\nu_n} \leq \frac{1}{1 - \max(0, -\rho)} (z + M_{\nu_n}) \leq 2(z + M_{\nu_n}).$$

Consequently, using Lemma 2.3.1 when $\nu_n = 1$,

$$\begin{aligned} E_1 &\leq c(1 + y + |x|) + c \mathbb{E}_x \left(z + M_{\nu_n}; \tau_y > \nu_n, 1 < \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\leq c(1 + y + |x|) + \underbrace{c \mathbb{E}_x \left(z + M_{\nu_n}; \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right)}_{E'_1}. \end{aligned} \quad (2.6.2)$$

Now, denoting $\nu_n \wedge \lfloor n^{1-\varepsilon} \rfloor = \min(\nu_n, \lfloor n^{1-\varepsilon} \rfloor)$, we write

$$\begin{aligned} E'_1 &= c \mathbb{E}_x \left(z + M_{\nu_n \wedge \lfloor n^{1-\varepsilon} \rfloor} \right) - c \mathbb{E}_x \left(z + M_{\nu_n \wedge \lfloor n^{1-\varepsilon} \rfloor}; \tau_y \leq \nu_n \wedge \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\quad - c \mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor}; \tau_y > \lfloor n^{1-\varepsilon} \rfloor, \nu_n > \lfloor n^{1-\varepsilon} \rfloor \right). \end{aligned}$$

Since $(M_n)_{n \geq 0}$ is a centred martingale, using Lemma 2.5.1 when $\mathbb{E}(a) \geq 0$ and the claim 2 of Lemma 2.5.4 when $\mathbb{E}(a) < 0$, Lemmas 2.3.4, 2.4.1 and Hölder inequality, we obtain

$$E'_1 \leq c_{p,\varepsilon} (1 + y + |x|) (1 + |x|)^{p-1}.$$

Implementing this into (2.6.2), it concludes the proof. \square

Now, we can prove an upper bound of order $1/n^{1/2-\varepsilon}$ of the probability of survival $\mathbb{P}_x(\tau_y > n)$.

Lemma 2.6.2. *Assume Condition 2.1. For any $p \in (2, \alpha)$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, $x \in \mathbb{R}$, $y > 0$ and $n \geq 1$,*

$$\mathbb{P}_x(\tau_y > n) \leq c_{p,\varepsilon} \frac{(1+y+|x|)(1+|x|)^{p-1}}{n^{1/2-\varepsilon}}.$$

Moreover, summing these bounds, we have

$$\sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{P}_x(\tau_y > k) \leq c_{p,\varepsilon} (1+y+|x|)(1+|x|)^{p-1} n^{1/2+\varepsilon}.$$

Proof. We write

$$\mathbb{P}_x(\tau_y > n) \leq \mathbb{E}_x \left(\frac{y + S_{\nu_n}}{n^{1/2-\varepsilon}} ; \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) + \mathbb{P}_x(\nu_n > n^{1-\varepsilon}).$$

Using Lemma 2.6.1 and Lemma 2.4.1, the claim follows. \square

Before to proceed with the proof of Theorem 2.2.2, we need two additional technical lemmas. Recall the notation $\nu_n^{\varepsilon/6} = \nu_n + \lfloor n^{\varepsilon/6} \rfloor$.

Lemma 2.6.3. *Assume Condition 2.1. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, $x \in \mathbb{R}$ and $y > 0$,*

$$E_2 = \mathbb{E}_x \left(y + S_{\nu_n^{\varepsilon/6}} ; \tau_y > \nu_n^{\varepsilon/6}, \nu_n^{\varepsilon/6} \leq \lfloor n^{1-\varepsilon} \rfloor \right) \xrightarrow{n \rightarrow +\infty} V(x, y).$$

Proof. Using the martingale approximation (2.3.2),

$$\begin{aligned} E_2 &= \underbrace{-\rho \mathbb{E}_x \left(X_{\nu_n^{\varepsilon/6}} ; \tau_y > \nu_n^{\varepsilon/6}, \nu_n^{\varepsilon/6} \leq \lfloor n^{1-\varepsilon} \rfloor \right)}_{=: E_{21}} \\ &\quad + \underbrace{\mathbb{E}_x \left(z + M_{\nu_n^{\varepsilon/6}} ; \tau_y > \nu_n^{\varepsilon/6}, \nu_n^{\varepsilon/6} \leq \lfloor n^{1-\varepsilon} \rfloor \right)}_{=: E_{22}}. \end{aligned} \quad (2.6.3)$$

Bound of E_{21} . By the Markov property, Lemma 2.3.1 and the fact that $(y + S_{\nu_n})/n^{1/2-\varepsilon} > 1$,

$$\begin{aligned} |E_{21}| &\leq c \mathbb{E}_x \left(1 + e^{-cn^{\varepsilon/6}} |X_{\nu_n}| ; \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\leq \frac{c}{n^{1/2-\varepsilon}} E_1 + c e^{-cn^{\varepsilon/6}} \sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{E}_x(|X_k|). \end{aligned}$$

By Lemma 2.6.1, we obtain

$$|E_{21}| \leq c_{p,\varepsilon} \frac{(1+y+|x|)(1+|x|)^{p-1}}{n^{1/2-\varepsilon}}. \quad (2.6.4)$$

Bound of E_{22} . We proceed in the same way as for bounding E'_1 defined in (2.6.2):

$$\begin{aligned} E_{22} &= z - \mathbb{E}_x \left(z + M_{\tau_y} ; \tau_y \leq \nu_n^{\varepsilon/6} \wedge \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\quad - \mathbb{E}_x \left(z + M_{\nu_n^{\varepsilon/6} \wedge \lfloor n^{1-\varepsilon} \rfloor} ; \tau_y > \nu_n^{\varepsilon/6} \wedge \lfloor n^{1-\varepsilon} \rfloor, \nu_n^{\varepsilon/6} > \lfloor n^{1-\varepsilon} \rfloor \right). \end{aligned}$$

By the Hölder inequality, Lemma 2.3.4 and Lemma 2.4.1,

$$E_{22} \leq z - \mathbb{E}_x \left(z + M_{\tau_y}; \tau_y \leq \nu_n^{\varepsilon/6} \wedge \lfloor n^{1-\varepsilon} \rfloor \right) + c_{p,\varepsilon} \frac{(1+y+|x|)(1+|x|)^{p-1}}{n^{\frac{p-2}{2}-c_p\varepsilon}}. \quad (2.6.5)$$

Since $\nu_n^{\varepsilon/6} \geq \lfloor n^{\varepsilon/6} \rfloor \rightarrow +\infty$ as $n \rightarrow +\infty$ and M_{τ_y} is integrable (using Lemma 2.5.1 when $\mathbb{E}(a) \geq 0$ and the claim 2 of Lemma 2.5.4 when $\mathbb{E}(a) < 0$), by the Lebesgue dominated convergence we deduce that

$$\lim_{n \rightarrow +\infty} E_{22} = -\mathbb{E}_x \left(M_{\tau_y} \right) = V(x, y).$$

Coupling this with equations (2.6.3) and (2.6.4), we conclude that $E_2 \xrightarrow[n \rightarrow +\infty]{} V(x, y)$. \square

Lemma 2.6.4. *Assume Condition 2.1. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, $x \in \mathbb{R}$ and $y > 0$,*

$$E_3 = \mathbb{E}_x \left(y + S_{\nu_n^{\varepsilon/6}}; y + S_{\nu_n^{\varepsilon/6}} > n^{1/2-\varepsilon/6}, \tau_y > \nu_n^{\varepsilon/6}, \nu_n^{\varepsilon/6} \leq \lfloor n^{1-\varepsilon} \rfloor \right) \xrightarrow[n \rightarrow +\infty]{} 0.$$

Proof. The first step of the proof consists in proving that we can replace the time $\nu_n^{\varepsilon/6}$ in the definition of E_3 by the time ν_n . More precisely, we shall prove that the following bound holds true:

$$\begin{aligned} E_3 &\leq cn^{\varepsilon/6} \underbrace{\mathbb{E}_x \left(y + S_{\nu_n}; y + S_{\nu_n} > n^{1/2-\varepsilon/2}, \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right)}_{=:E_{31}} \\ &\quad + c_{p,\varepsilon} \frac{(1+y+|x|)(1+|x|)^{p-1}}{n^{\varepsilon/6}}. \end{aligned} \quad (2.6.6)$$

To this end, we bound E_3 as follows:

$$\begin{aligned} E_3 &\leq E_{31} + \underbrace{\mathbb{E}_x \left(\left| S_{\nu_n^{\varepsilon/6}} - S_{\nu_n} \right|; y + S_{\nu_n} > n^{1/2-\varepsilon/2}; \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right)}_{=:E_{32}} \\ &\quad + \underbrace{\mathbb{E}_x \left(y + S_{\nu_n}; y + S_{\nu_n} \leq n^{1/2-\varepsilon/2}, y + S_{\nu_n^{\varepsilon/6}} > n^{1/2-\varepsilon/6}, \right.}_{=:E_{33}} \\ &\quad \left. \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right)}_{=:E_{34}} \\ &\quad + \underbrace{\mathbb{E}_x \left(\left| S_{\nu_n^{\varepsilon/6}} - S_{\nu_n} \right|; y + S_{\nu_n} \leq n^{1/2-\varepsilon/2}, y + S_{\nu_n^{\varepsilon/6}} > n^{1/2-\varepsilon/6}, \right.}_{=:E_{34}} \\ &\quad \left. \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right)}_{=:E_{34}}. \end{aligned} \quad (2.6.7)$$

Bound of E_{32} . By the Markov property and Lemma 2.3.1,

$$\begin{aligned} E_{32} &\leq \int_{\mathbb{R} \times \mathbb{R}_+^*} \mathbb{E}_{x'} \left(\left| S_{\lfloor n^{\varepsilon/6} \rfloor} \right| \right) \mathbb{P}_x \left(X_{\nu_n} \in dx', y + S_{\nu_n} \in dy' \right) \\ &\quad y + S_{\nu_n} > n^{1/2-\varepsilon/2}, \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \\ &\leq \mathbb{E}_x \left(cn^{\varepsilon/6} (1 + |X_{\nu_n}|); y + S_{\nu_n} > n^{1/2-\varepsilon/2}, \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right). \end{aligned}$$

If $\tau_y > \nu_n > 1$, by (2.6.1), we have $|X_{\nu_n}| = X_{\nu_n} < y + S_{\nu_n}$. Using this bound when $\nu_n > 1$ and the Markov inequality when $\nu_n = 1$,

$$\begin{aligned} E_{32} &\leq \mathbb{E}_x \left(cn^{\varepsilon/6} (1 + |X_1|) ; y + X_1 > n^{1/2-\varepsilon/2}, \nu_n = 1 \right) + cn^{\varepsilon/6} E_{31} \\ &\leq c \frac{(1 + y + |x|)(1 + |x|)}{n^{1/2-c\varepsilon}} + cn^{\varepsilon/6} E_{31}. \end{aligned} \quad (2.6.8)$$

Bound of E_{33} . By the Markov property,

$$\begin{aligned} E_{33} &\leq \int_{\mathbb{R} \times \mathbb{R}_+^*} y' \mathbb{P}_{x'} \left(y' + S_{\lfloor n^{\varepsilon/6} \rfloor} > n^{1/2-\varepsilon/6} \right) \mathbb{P}_x (X_{\nu_n} \in dx', y + S_{\nu_n} \in dy', \\ &\quad y + S_{\nu_n} \leq n^{1/2-\varepsilon/2}, \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor). \end{aligned}$$

When $y' \leq n^{1/2-\varepsilon/2}$, by the Markov inequality, we have,

$$\mathbb{P}_{x'} \left(y' + S_{\lfloor n^{\varepsilon/6} \rfloor} > n^{1/2-\varepsilon/6} \right) \leq \mathbb{P}_{x'} \left(\left| S_{\lfloor n^{\varepsilon/6} \rfloor} \right| > \frac{n^{1/2-\varepsilon/6}}{c_\varepsilon} \right) \leq \frac{c_\varepsilon n^{\varepsilon/6} (1 + |x'|)}{n^{1/2-\varepsilon/6}}.$$

On the event $\{y + S_{\nu_n} \leq n^{1/2-\varepsilon/2}, \tau_y > \nu_n\}$, we obviously have $x' = X_{\nu_n} \leq n^{1/2-\varepsilon/2}$. From these bounds, using the positivity of X_{ν_n} for $\nu_n > 1$, see (2.6.1), we obtain

$$E_{33} \leq \mathbb{E}_x \left((y + S_1) \frac{c_\varepsilon (1 + |X_1|)}{n^{1/2-\varepsilon/3}} ; \nu_n = 1 \right) + \frac{c_\varepsilon}{n^{\varepsilon/2-\varepsilon/3}} E_1.$$

By Lemma 2.6.1, we obtain

$$E_{33} \leq c_{p,\varepsilon} \frac{(1 + y + |x|)(1 + |x|)^{p-1}}{n^{\varepsilon/6}}. \quad (2.6.9)$$

Bound of E_{34} . Again, by the Markov property,

$$\begin{aligned} E_{34} &\leq \int_{\mathbb{R} \times \mathbb{R}_+^*} \mathbb{E}_{x'} \left(\left| S_{\lfloor n^{\varepsilon/6} \rfloor} \right| ; y' + S_{\lfloor n^{\varepsilon/6} \rfloor} > n^{1/2-\varepsilon/6} \right) \mathbb{P}_x (X_{\nu_n} \in dx', \\ &\quad y + S_{\nu_n} \in dy', y + S_{\nu_n} \leq n^{1/2-\varepsilon/2}, \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor). \end{aligned}$$

When $y' \leq n^{1/2-\varepsilon/2}$, using the Markov inequality and Lemma 2.3.1, we have

$$\mathbb{E}_{x'} \left(\left| S_{\lfloor n^{\varepsilon/6} \rfloor} \right| ; y' + S_{\lfloor n^{\varepsilon/6} \rfloor} > n^{1/2-\varepsilon/6} \right) \leq \mathbb{E}_{x'} \left(\frac{c_\varepsilon^{p-1} \left| S_{\lfloor n^{\varepsilon/6} \rfloor} \right|^p}{n^{\frac{p-1}{2}-(p-1)\varepsilon/6}} \right) \leq c_{p,\varepsilon} \frac{(1 + |x'|)^p}{n^{\frac{p-1}{2}-c_p\varepsilon}}.$$

Then, using Lemma 2.3.1 again and the Markov property for the terms in the last sum below,

$$\begin{aligned} E_{34} &\leq \frac{c_{p,\varepsilon}}{n^{\frac{p-1}{2}-c_p\varepsilon}} + \frac{c_{p,\varepsilon}}{n^{\frac{p-1}{2}-c_p\varepsilon}} \sum_{k=1}^{\lfloor n^\varepsilon \rfloor} \mathbb{E}_x (|X_k|^p) + \frac{c_{p,\varepsilon}}{n^{\frac{p-1}{2}-c_p\varepsilon}} \sum_{k=\lfloor n^\varepsilon \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{E}_x (|X_k|^p ; \tau_y > k) \\ &\leq \frac{c_{p,\varepsilon} (1 + |x|^p)}{n^{\frac{p-1}{2}-c_p\varepsilon}} + \frac{c_{p,\varepsilon}}{n^{\frac{p-1}{2}-c_p\varepsilon}} \sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor - \lfloor n^\varepsilon \rfloor} \mathbb{E}_x \left(1 + e^{-c_p n^\varepsilon} |X_k|^p ; \tau_y > k \right) \\ &\leq \frac{c_{p,\varepsilon} (1 + |x|^p)}{n^{\frac{p-1}{2}-c_p\varepsilon}} + c_{p,\varepsilon} e^{-c_p \varepsilon n^\varepsilon} (1 + |x|^p) + \frac{c_{p,\varepsilon}}{n^{\frac{p-1}{2}-c_p\varepsilon}} \sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{P}_x (\tau_y > k). \end{aligned}$$

Using the second bound in Lemma 2.6.2, and taking $\varepsilon > 0$ small enough, we obtain

$$E_{34} \leq c_{p,\varepsilon} \frac{(1+y+|x|)(1+|x|)^{p-1}}{n^{\frac{p-2}{2}-c_p\varepsilon}} \xrightarrow{n \rightarrow +\infty} 0. \quad (2.6.10)$$

Inserting (2.6.8), (2.6.9) and (2.6.10) in (2.6.7), we conclude the proof of (2.6.6).

Bound of $cn^{\varepsilon/6}E_{31}$. Note that, when $\nu_n > 1$ and $y + S_{\nu_n} > n^{1/2-\varepsilon/2}$, we have $X_{\nu_n} = y + S_{\nu_n} - (y + S_{\nu_n-1}) > n^{1/2-\varepsilon/2} - n^{1/2-\varepsilon} \geq \frac{n^{1/2-\varepsilon/2}}{c_\varepsilon}$. Consequently,

$$\begin{aligned} cn^{\varepsilon/6}E_{31} &\leq \underbrace{cn^{\varepsilon/6}\mathbb{E}_x(y + S_{\nu_n}; \nu_n \leq \lfloor n^\varepsilon \rfloor)}_{=:E_{35}} \\ &\quad + \underbrace{cn^{\varepsilon/6}\mathbb{E}_x\left(y + S_{\nu_n}; X_{\nu_n} > \frac{n^{1/2-\varepsilon/2}}{c_\varepsilon}, \tau_y > \nu_n, \lfloor n^\varepsilon \rfloor < \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor\right)}_{=:E_{36}}. \end{aligned} \quad (2.6.11)$$

Bound of E_{35} . Using the definition of ν_n , the Markov inequality and Lemma 2.3.1,

$$\begin{aligned} E_{35} &\leq cn^{\varepsilon/6}\mathbb{E}_x\left(\max_{k \leq \lfloor n^\varepsilon \rfloor} |y + S_k|; \max_{k \leq \lfloor n^\varepsilon \rfloor} |y + S_k| > n^{1/2-\varepsilon}\right) \\ &\leq \frac{c_p(1+y+|x|)^2}{n^{1/2-c_p\varepsilon}}. \end{aligned} \quad (2.6.12)$$

Bound of E_{36} . The idea is based on the observation that, according to the first bound in Lemma 2.3.1, the random variables $y + S_{\nu_n - \lfloor n^\varepsilon \rfloor}$ and X_{ν_n} are "almost" independent. In this line, summing over the values of ν_n and bounding the indicators $\mathbb{1}_{\{\nu_n=k\}}$ by 1, we write

$$\begin{aligned} E_{36} &\leq cn^{\varepsilon/6} \sum_{k=\lfloor n^\varepsilon \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{E}_x\left(y + S_{k-\lfloor n^\varepsilon \rfloor}; X_k > \frac{n^{1/2-\varepsilon/2}}{c_\varepsilon}, \tau_y > k\right) \\ &\quad + cn^{\varepsilon/6} \sum_{k=\lfloor n^\varepsilon \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{E}_x\left(|S_k - S_{k-\lfloor n^\varepsilon \rfloor}|; X_k > \frac{n^{1/2-\varepsilon/2}}{c_\varepsilon}, \tau_y > k\right). \end{aligned}$$

By the Markov property,

$$\begin{aligned} E_{36} &\leq cn^{\varepsilon/6} \sum_{k=\lfloor n^\varepsilon \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{R} \times \mathbb{R}_+^*} y' \mathbb{P}_{x'}\left(X_{\lfloor n^\varepsilon \rfloor} > \frac{n^{1/2-\varepsilon/2}}{c_\varepsilon}\right) \\ &\quad \times \mathbb{P}_x\left(X_{k-\lfloor n^\varepsilon \rfloor} \in dx', y + S_{k-\lfloor n^\varepsilon \rfloor} \in dy', \tau_y > k - \lfloor n^\varepsilon \rfloor\right) \\ &\quad + cn^{\varepsilon/6} \sum_{k=\lfloor n^\varepsilon \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{E}_x\left(n^\varepsilon \max_{k-\lfloor n^\varepsilon \rfloor \leq i \leq k} |X_i|; X_k > \frac{n^{1/2-\varepsilon/2}}{c_\varepsilon}, \tau_y > k\right). \end{aligned} \quad (2.6.13)$$

Recall that, under $\mathbb{P}_{x'}$, by (2.3.3), $X_{\lfloor n^\varepsilon \rfloor} = \prod_{i=1}^{\lfloor n^\varepsilon \rfloor} a_i x' + X_{\lfloor n^\varepsilon \rfloor}^0$. Then, since a_i 's are independent and identically distributed, by claim 1 of Condition 2.1 and Lemma 2.3.1,

$$\begin{aligned} \mathbb{P}_{x'}\left(X_{\lfloor n^\varepsilon \rfloor} > \frac{n^{1/2-\varepsilon/2}}{c_\varepsilon}\right) &\leq \mathbb{P}\left(\prod_{i=1}^{\lfloor n^\varepsilon \rfloor} a_i x' > \frac{n^{1/2-\varepsilon/2}}{2c_\varepsilon}\right) + \mathbb{P}\left(|X_{\lfloor n^\varepsilon \rfloor}^0| > \frac{n^{1/2-\varepsilon/2}}{2c_\varepsilon}\right) \\ &\leq c_\varepsilon e^{-c_\varepsilon n^\varepsilon} |x'| + \frac{c_{p,\varepsilon}}{n^{\frac{p}{2}-c_p\varepsilon}}. \end{aligned} \quad (2.6.14)$$

Inserting (2.6.14) into (2.6.13) and using Cauchy-Schwartz inequality, by Corollaries 2.4.5 and 2.4.9,

$$E_{36} \leq \sum_{j=1}^{\lfloor n^{1-\varepsilon} \rfloor} \left(c_\varepsilon e^{-c_\varepsilon n^\varepsilon} \mathbb{E}_x^{1/2} (|y + S_j|^2) \mathbb{E}_x^{1/2} (|X_j|^2) + \frac{c_{p,\varepsilon}}{n^{\frac{p}{2}-c_p\varepsilon}} (1+y+|x|)(1+|x|)^{p-1} \right) \\ + cn^{\varepsilon+\varepsilon/6} \sum_{k=\lfloor n^\varepsilon \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{E}_x \left(\frac{\max_{k-\lfloor n^\varepsilon \rfloor \leq i \leq k} |X_i|^p}{n^{\frac{p-1}{2}-c_p\varepsilon}} ; \tau_y > k - \lfloor n^\varepsilon \rfloor \right).$$

Using the decomposition (2.3.2) and Lemmas 2.3.1 and 2.3.4

$$E_{36} \leq c_{p,\varepsilon} \frac{(1+y+|x|)(1+|x|)^{p-1}}{n^{\frac{p-2}{2}-c_p\varepsilon}} \\ + \frac{c_p}{n^{\frac{p-1}{2}-c_p\varepsilon}} \sum_{k=\lfloor n^\varepsilon \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{E}_x \left(n^\varepsilon (1+|X_{k-\lfloor n^\varepsilon \rfloor}|^p) ; \tau_y > k - \lfloor n^\varepsilon \rfloor \right).$$

Re-indexing $j = k - \lfloor n^\varepsilon \rfloor$, after some elementary transformations, we get

$$E_{36} \leq c_{p,\varepsilon} \frac{(1+y+|x|)(1+|x|)^{p-1}}{n^{\frac{p-2}{2}-c_p\varepsilon}} + \frac{c_p}{n^{\frac{p-1}{2}-c_p\varepsilon}} \sum_{j=1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{P}_x (\tau_y > j) \\ + \frac{c_p}{n^{\frac{p-1}{2}-c_p\varepsilon}} \sum_{j=1}^{\lfloor n^\varepsilon \rfloor} \mathbb{E}_x (|X_j|^p) + \frac{c_p}{n^{\frac{p-1}{2}-c_p\varepsilon}} \sum_{j=\lfloor n^\varepsilon \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{E}_x (|X_j|^p ; \tau_y > j - \lfloor n^\varepsilon \rfloor).$$

Again using the Markov property, Lemma 2.3.1 and Lemma 2.6.2, we have

$$E_{36} \leq c_{p,\varepsilon} \frac{(1+y+|x|)(1+|x|)^{p-1}}{n^{\frac{p-2}{2}-c_p\varepsilon}} + \frac{c_p}{n^{\frac{p-1}{2}-c_p\varepsilon}} \sum_{j=1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{P}_x (\tau_y > j) \\ + c_p e^{-c_p n^\varepsilon} \sum_{j=1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{E}_x (|X_j|^p ; \tau_y > j) \\ \leq c_{p,\varepsilon} \frac{(1+y+|x|)(1+|x|)^{p-1}}{n^{\frac{p-2}{2}-c_p\varepsilon}}.$$

Inserting this bound and (2.6.12) into (2.6.11), we obtain

$$cn^{\varepsilon/6} E_{31} \leq \frac{c_{p,\varepsilon} (1+y+|x|)^p}{n^{\frac{p-2}{2}-c_p\varepsilon}}.$$

Together with (2.6.6), this bound implies that

$$E_3 \leq \frac{c_p (1+y+|x|)^p}{n^{\varepsilon/6}} \xrightarrow{n \rightarrow +\infty} 0. \quad (2.6.15)$$

□

2.6.2 Proof of the claim 2 of Theorem 2.2.2

Assume either Conditions 2.1, 2.2 and $\mathbb{E}(a) \geq 0$, or Conditions 2.1 and 2.3. Introducing the stopping time $\nu_n^{\varepsilon/6} = \nu_n + \lfloor n^{\varepsilon/6} \rfloor$, we have

$$\mathbb{P}_x(\tau_y > n) = \mathbb{P}_x(\tau_y > n, \nu_n^{\varepsilon/6} \leq \lfloor n^{1-\varepsilon} \rfloor) + \mathbb{P}_x(\tau_y > n, \nu_n^{\varepsilon/6} > \lfloor n^{1-\varepsilon} \rfloor). \quad (2.6.16)$$

We bound the second term by Lemma 2.4.1: for $2 < p < \alpha$,

$$\mathbb{P}_x(\tau_y > n, \nu_n^{\varepsilon/6} > \lfloor n^{1-\varepsilon} \rfloor) \leq \mathbb{P}_x\left(\nu_n > \frac{n^{1-\varepsilon}}{c_\varepsilon}\right) \leq c_{p,\varepsilon} \frac{(1+|x|)^p}{n^{p/2-c_p\varepsilon}} = o\left(\frac{1}{\sqrt{n}}\right). \quad (2.6.17)$$

To bound the first term, we introduce more notations. Let $(B_t)_{t \geq 0}$ be the Brownian motion from Proposition 2.9.5, A_k be the event $A_k = \{\max_{0 \leq t \leq 1} |S_{\lfloor tk \rfloor} - \sigma B_{tk}| \leq k^{1/2-2\varepsilon}\}$ where σ is defined by (2.2.2), and \bar{A}_k be its complement. Using the Markov property, we have

$$\begin{aligned} \mathbb{P}_x(\tau_y > n, \nu_n^{\varepsilon/6} \leq \lfloor n^{1-\varepsilon} \rfloor) &= \sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{R} \times \mathbb{R}_+^*} \underbrace{\mathbb{P}_{x'}(\tau_{y'} > n-k, \bar{A}_{n-k}) \mathbb{P}_x(X_k \in dx', \\ &\quad \underbrace{y + S_k \in dy', \tau_y > k, \nu_n^{\varepsilon/6} = k}_{=: J_1})}_{=: J_1} \\ &+ \sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{R} \times \mathbb{R}_+^*} \underbrace{\mathbb{P}_{x'}(\tau_{y'} > n-k, A_{n-k}) \mathbb{P}_x(X_k \in dx', \\ &\quad \underbrace{y + S_k \in dy', \tau_y > k, \nu_n^{\varepsilon/6} = k}_{=: J_2})}_{=: J_2}. \end{aligned} \quad (2.6.18)$$

Bound of J_1 . Taking into account that $n-k \geq \frac{n}{c_\varepsilon}$ for any $k \leq \lfloor n^{1-\varepsilon} \rfloor$, by Proposition 2.9.5 with ε small enough, we find

$$\mathbb{P}_{x'}(\tau_{y'} > n-k, \bar{A}_{n-k}) \leq \mathbb{P}_{x'}(\bar{A}_{n-k}) \leq c_{p,\varepsilon} (1+|x'|)^p n^{-2\varepsilon}.$$

By the Markov property and the first bound in Lemma 2.3.1,

$$J_1 \leq \mathbb{E}_x \left(c_{p,\varepsilon} e^{-c_{p,\varepsilon} n^{\varepsilon/6}} |X_{\nu_n}|^p + \frac{c_{p,\varepsilon}}{n^{2\varepsilon}}; \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right).$$

Since $\frac{y+S_{\nu_n}}{n^{1/2-\varepsilon}} > 1$, using Lemma 2.6.1,

$$J_1 \leq c_{p,\varepsilon} e^{-c_{p,\varepsilon} n^{\varepsilon/6}} (1+|x|)^p + \frac{c_{p,\varepsilon}}{n^{1/2-\varepsilon+2\varepsilon}} E_1 \leq \frac{c_{p,\varepsilon} (1+y+|x|)(1+|x|)^{p-1}}{n^{1/2+\varepsilon}}. \quad (2.6.19)$$

Bound of J_2 . The idea is as follows. When $y' \leq \theta_n \sqrt{n}$, with $\theta_n = n^{-\varepsilon/6}$, we are going to control the probability $\mathbb{P}_{x'}(\tau_{y'} > n-k, A_{n-k})$ in J_2 by the claim 2 of Corollary 2.9.4. When $y' > \theta_n \sqrt{n}$ we shall apply Lemma 2.6.4. Accordingly, we split J_2 into two terms as

follows:

$$\begin{aligned}
J_2 &= \underbrace{\sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{R} \times \mathbb{R}_+^*} \mathbb{P}_{x'}(\tau_{y'} > n-k, A_{n-k}) \mathbb{P}_x(X_k \in dx', y + S_k \in dy', \\
&\quad y + S_k > n^{1/2-\varepsilon/6}, \tau_y > k, \nu_n^{\varepsilon/6} = k)}_{=: J_3} \\
&+ \underbrace{\sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{R} \times \mathbb{R}_+^*} \mathbb{P}_{x'}(\tau_{y'} > n-k, A_{n-k}) \mathbb{P}_x(X_k \in dx', y + S_k \in dy', \\
&\quad y + S_k \leq n^{1/2-\varepsilon/6}, \tau_y > k, \nu_n^{\varepsilon/6} = k)}_{=: J_4}. \tag{2.6.20}
\end{aligned}$$

Bound of J_3 . Let τ_y^{bm} be the exit time of the Brownian motion defined by (2.9.10) and $y'_+ = y' + (n-k)^{1/2-2\varepsilon}$. Since

$$\mathbb{P}_{x'}(\tau_{y'} > n-k, A_{n-k}) \leq \mathbb{P}_{x'}(\tau_{y'_+}^{bm} > n-k), \tag{2.6.21}$$

using the claim 1 of Corollary 2.9.4 with $y'_+ > 0$, we get

$$J_3 \leq \sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{E}_x \left(c \frac{y + S_k + (n-k)^{1/2-2\varepsilon}}{\sqrt{n-k}}; y + S_k > n^{1/2-\varepsilon/6}, \tau_y > k, \nu_n^{\varepsilon/6} = k \right).$$

Since $\frac{c}{\sqrt{n-k}} \leq \frac{c_\varepsilon}{\sqrt{n}}$ and $y + S_k + (n-k)^{1/2-2\varepsilon} \leq 2(y + S_k)$ on the event $\{y + S_k > n^{1/2-\varepsilon/6}\}$, using Lemma 2.6.4, we have

$$J_3 \leq \frac{c_\varepsilon}{\sqrt{n}} E_3 = o\left(\frac{1}{\sqrt{n}}\right). \tag{2.6.22}$$

Upper bound of J_4 . Since $\frac{n}{c_\varepsilon} \leq n-k \leq n$, we have $y'_+ \leq c_\varepsilon(n-k)^{1/2-\varepsilon/6}$ when $y' \leq n^{1/2-\varepsilon/6}$. Using (2.6.21), from the claim 2 of Corollary 2.9.4 with $\theta_m = c_\varepsilon m^{-\varepsilon/6}$, we deduce that

$$\begin{aligned}
J_4 &\leq \sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{E}_x \left(\frac{2}{\sqrt{2\pi(n-k)\sigma}} (y + S_k + (n-k)^{1/2-2\varepsilon}) (1 + c\theta_{n-k}^2); \right. \\
&\quad \left. y + S_k \leq n^{1/2-\varepsilon/6}, \tau_y > k, \nu_n^{\varepsilon/6} = k \right). \tag{2.6.23}
\end{aligned}$$

Taking into account that $\frac{1}{\sqrt{n-k}} \leq \frac{1}{\sqrt{n}} \left(1 + \frac{c_\varepsilon}{n^\varepsilon}\right)$, $\theta_{n-k} \leq \frac{c_\varepsilon}{n^{\varepsilon/6}}$ and $1 < \frac{y+S_k}{n^{1/2-\varepsilon}}$, we obtain

$$J_4 \leq \frac{2}{\sqrt{2\pi n \sigma}} \left(1 + \frac{c_\varepsilon}{n^{\varepsilon/3}}\right) E_2 + \frac{c_\varepsilon}{n^{1/2+\varepsilon}} E_1. \tag{2.6.24}$$

Using Lemma 2.6.1 and Lemma 2.6.3, we get the following upper bound,

$$J_4 \leq \frac{2V(x, y)}{\sqrt{2\pi n \sigma}} (1 + o(1)). \tag{2.6.25}$$

Lower bound of J_4 . In the same way as for the upper bound of J_4 , with $y'_- = y + S_{\nu_n^{\varepsilon/6}} - (n - \nu_n^{\varepsilon/6})^{1/2-2\varepsilon} > 0$ on the event $\{(n - \nu_n^{\varepsilon/6})^{1/2-2\varepsilon} < y + S_{\nu_n^{\varepsilon/6}}\}$, we have

$$J_4 \geq \frac{2}{\sqrt{2\pi n\sigma}} \left(1 - \frac{c_\varepsilon}{n^{\varepsilon/3}}\right) \mathbb{E}_x \left(y'_- ; (n - \nu_n^{\varepsilon/6})^{1/2-2\varepsilon} < y + S_{\nu_n^{\varepsilon/6}} \leq n^{1/2-\varepsilon/6}, \right. \\ \left. \tau_y > \nu_n^{\varepsilon/6}, \nu_n^{\varepsilon/6} \leq \lfloor n^{1-\varepsilon} \rfloor \right) \quad (2.6.26) \\ - \sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{R}} \mathbb{P}_{x'}(\bar{A}_{n-k}) \mathbb{P}_x(X_k \in dx', \tau_y > k, \nu_n^{\varepsilon/6} = k).$$

Using the fact that $-y'_- \geq 0$ on $\{(n - \nu_n^{\varepsilon/6})^{1/2-2\varepsilon} \geq y + S_{\nu_n^{\varepsilon/6}}\}$, we obtain in a same way as for the upper bound of J_1 ,

$$J_4 \geq \frac{2}{\sqrt{2\pi n\sigma}} \left(1 - \frac{c_\varepsilon}{n^{\varepsilon/3}}\right) E_2 - \frac{2}{\sqrt{2\pi n\sigma}} \mathbb{E}_x \left(n^{1/2-2\varepsilon} \frac{y + S_{\nu_n}}{n^{1/2-\varepsilon}} ; \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ - \frac{2}{\sqrt{2\pi n\sigma}} E_3 - \frac{c_{p,\varepsilon}(1+y+|x|)(1+|x|)^{p-1}}{n^{1/2+\varepsilon}} \\ \geq \frac{2}{\sqrt{2\pi n\sigma}} \left(1 - \frac{c_\varepsilon}{n^{\varepsilon/3}}\right) E_2 - \frac{c}{n^{1/2+\varepsilon}} E_1 - \frac{c}{\sqrt{n}} E_3 - \frac{c_{p,\varepsilon}(1+y+|x|)(1+|x|)^{p-1}}{n^{1/2+\varepsilon}}.$$

Consequently, using the results of Lemma 2.6.3, Lemma 2.6.1 and Lemma 2.6.4 we conclude that

$$J_4 \geq \frac{2V(x,y)}{\sqrt{2\pi n\sigma}} (1 - o(1)). \quad (2.6.27)$$

Coupling the obtained lower bound with the upper bound in (2.6.25) we obtain $J_4 \sim \frac{2V(x,y)}{\sqrt{2\pi n\sigma}}$. With the decomposition of J_2 in (2.6.20) and the bound of J_3 in (2.6.22) we get $J_2 \sim \frac{2V(x,y)}{\sqrt{2\pi n\sigma}}$. Finally, the claim 2 of Theorem 2.2.2 follows from (2.6.16), (2.6.17), (2.6.18) and (2.6.19).

2.6.3 Proof of the claim 1 of Theorem 2.2.2

Assume Condition 2.1. All the necessary bounds are obtained in the previous section 2.6.2. It is easy to see that they hold under solely Condition 2.1. We highlight how to gather them. By (2.6.16), (2.6.17), (2.6.18) and (2.6.20), we have,

$$\mathbb{P}_x(\tau_y > n) \leq c_{p,\varepsilon} \frac{(1+|x|^p)}{\sqrt{n}} + J_1 + J_3 + J_4.$$

Then, by (2.6.19), (2.6.22), and (2.6.24),

$$\mathbb{P}_x(\tau_y > n) \leq c_{p,\varepsilon} \frac{(1+y+|x|)(1+|x|)^{p-1}}{\sqrt{n}} + \frac{c_\varepsilon}{\sqrt{n}} E_3 + \frac{c_\varepsilon}{\sqrt{n}} (E_2 + E_1).$$

Now, by Lemma 2.6.1, (2.6.3) and (2.6.15),

$$\mathbb{P}_x(\tau_y > n) \leq c_{p,\varepsilon} \frac{(1+y+|x|)^p}{\sqrt{n}} + \frac{c_\varepsilon}{\sqrt{n}} (E_{21} + E_{22}).$$

Finally, using (2.6.4), (2.6.5) and Lemmas 2.5.1 and 2.5.4 we have,

$$\begin{aligned} \mathbb{P}_x(\tau_y > n) &\leq \frac{c_\varepsilon}{\sqrt{n}} \left(z - \mathbb{E}_x \left(z + M_{\tau_y}; \tau_y \leq \nu_n^{\varepsilon/6} \wedge \lfloor n^{1-\varepsilon} \rfloor \right) \right) + c_{p,\varepsilon} \frac{(1+y+|x|)^p}{\sqrt{n}} \\ &\leq \frac{c_\varepsilon}{\sqrt{n}} \mathbb{E}_x(|M_{\tau_y}|) + c_{p,\varepsilon} \frac{(1+y+|x|)^p}{\sqrt{n}} \\ &\leq c_p \frac{(1+y+|x|)^p}{\sqrt{n}}. \end{aligned}$$

2.6.4 Proof of Corollary 2.2.3

For any $p > 0$,

$$\mathbb{E}_x(\tau_y^p) = \sum_{k=0}^{+\infty} \mathbb{P}_x(\tau_y > k) ((k+1)^p - k^p).$$

Now the first and the second assertions of the corollary follow respectively from the claim 1 and 2 of Theorem 2.2.2.

2.7 Asymptotic for conditioned Markov walk

In this section we prove Theorem 2.2.4. We will deduce the asymptotic of the Markov walk $(y + S_n)_{n \geq 0}$ conditioned to stay positive from the corresponding result for the Brownian motion given by Proposition 2.9.3. As in Section 2.6, we will use the functional approximation of Proposition 2.9.5. We will refer frequently to Section 2.6 in order to shorten the exposition.

Proof of Theorem 2.2.4. Introducing $\nu_n^{\varepsilon/6} = \nu_n + \lfloor n^{\varepsilon/6} \rfloor$ and taking into account Condition 2.2 or 2.3, we have

$$\begin{aligned} \mathbb{P}_x(y + S_n \leq t\sqrt{n} \mid \tau_y > n) &= \underbrace{\frac{\mathbb{P}_x(y + S_n \leq t\sqrt{n}, \tau_y > n, \nu_n^{\varepsilon/6} > \lfloor n^{1-\varepsilon} \rfloor)}{\mathbb{P}_x(\tau_y > n)}}_{=:L_1} \\ &\quad + \underbrace{\frac{\mathbb{P}_x(y + S_n \leq t\sqrt{n}, \tau_y > n, \nu_n^{\varepsilon/6} \leq \lfloor n^{1-\varepsilon} \rfloor)}{\mathbb{P}_x(\tau_y > n)}}_{=:L_2}. \end{aligned} \tag{2.7.1}$$

Bound of L_1 . Using Lemma 2.4.1 and Theorem 2.2.2,

$$L_1 \leq \frac{\mathbb{P}_x(\nu_n > \frac{n^{1-\varepsilon}}{c_\varepsilon})}{\mathbb{P}_x(\tau_y > n)} \leq \frac{c_{p,\varepsilon} (1+|x|)^p}{n^{\frac{p}{2}-c_p\varepsilon} \mathbb{P}_x(\tau_y > n)} \xrightarrow{n \rightarrow +\infty} 0. \tag{2.7.2}$$

Bound of L_2 . As in Section 2.6, setting $A_k = \left\{ \max_{0 \leq t \leq 1} |S_{\lfloor tk \rfloor} - \sigma B_{tk}| \leq k^{1/2-2\varepsilon} \right\}$, by the

Markov property,

$$\begin{aligned}
& \mathbb{P}_x(\tau_y > n) L_2 \\
&= \underbrace{\sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{R} \times \mathbb{R}_+^*} \mathbb{P}_{x'}(y' + S_{n-k} \leq t\sqrt{n}, \tau_{y'} > n-k, \bar{A}_{n-k}) \mathbb{P}_x(X_k \in dx', \\
&\quad y + S_k \in dy', \tau_y > k, \nu_n^{\varepsilon/6} = k)}_{=:\mathbb{P}_x(\tau_y > n)L_3} \\
&+ \underbrace{\sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{R} \times \mathbb{R}_+^*} \mathbb{P}_{x'}(y' + S_{n-k} \leq t\sqrt{n}, \tau_{y'} > n-k, A_{n-k}) \mathbb{P}_x(X_k \in dx', \\
&\quad y + S_k \in dy', y + S_k > n^{1/2-\varepsilon/6}, \tau_y > k, \nu_n^{\varepsilon/6} = k)}_{=:\mathbb{P}_x(\tau_y > n)L_4} \\
&+ \underbrace{\sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{R} \times \mathbb{R}_+^*} \mathbb{P}_{x'}(y' + S_{n-k} \leq t\sqrt{n}, \tau_{y'} > n-k, A_{n-k}) \mathbb{P}_x(X_k \in dx', \\
&\quad y + S_k \in dy', y + S_k \leq n^{1/2-\varepsilon/6}, \tau_y > k, \nu_n^{\varepsilon/6} = k)}_{=:\mathbb{P}_x(\tau_y > n)L_5}. \tag{2.7.3}
\end{aligned}$$

Bound of L_3 . Using the bound of J_1 in (2.6.19) and Theorem 2.2.2,

$$L_3 \leq \frac{J_1}{\mathbb{P}_x(\tau_y > n)} \leq \frac{c_{p,\varepsilon}(1+y+|x|)(1+|x|)^{p-1}}{n^{1/2+\varepsilon}\mathbb{P}_x(\tau_y > n)} \xrightarrow{n \rightarrow +\infty} 0. \tag{2.7.4}$$

Bound of L_4 . Using the bound of J_3 in (2.6.22) and Theorem 2.2.2, we have

$$L_4 \leq \frac{J_3}{\mathbb{P}_x(\tau_y > n)} = o(1). \tag{2.7.5}$$

Upper bound of L_5 . Define $t_+ = t + \frac{2}{(n-k)^{2\varepsilon}}$ and $y'_+ = y' + (n-k)^{1/2-2\varepsilon}$. By Proposition 2.9.3,

$$\begin{aligned}
& \mathbb{P}_{x'}(y' + S_{n-k} \leq t\sqrt{n}, \tau_{y'} > n-k, A_{n-k}) \\
& \leq \mathbb{P}(y'_+ + \sigma B_{n-k} \leq t_+\sqrt{n}, \tau_{y'_+}^{bm} > n-k) \\
& = \frac{1}{\sqrt{2\pi(n-k)}\sigma} \int_0^{t_+\sqrt{n}} e^{-\frac{(s-y'_+)^2}{2(n-k)\sigma^2}} - e^{-\frac{(s+y'_+)^2}{2(n-k)\sigma^2}} ds.
\end{aligned}$$

Note that for any $y' \leq n^{1/2-\varepsilon/6}$ we have $y'_+/\sqrt{n} \leq \frac{2}{n^{\varepsilon/6}}$ and for any $k \leq \lfloor n^{1-\varepsilon} \rfloor$ we have $n(1 - \frac{1}{n^\varepsilon}) \leq n-k \leq n$. Using these remarks with the fact that $\text{sh}(x) \leq x(1 + \frac{x^2}{6} \text{ch}(x))$ for any $x \geq 0$, we obtain after some calculations that

$$\begin{aligned}
& \mathbb{P}_{x'}(y' + S_{n-k} \leq t\sqrt{n}, \tau_{y'} > n-k, A_{n-k}) \\
& \leq \frac{2y'_+}{\sqrt{2\pi n}\sigma} \left(1 + \frac{c_\varepsilon}{n^\varepsilon}\right) \int_0^{t_+\sqrt{n}} \frac{s e^{-\frac{s^2+(y'_+)^2}{2(n-k)\sigma^2}}}{(n-k)\sigma^2} \left[1 + \frac{s^2 (y'_+)^2}{6(n-k)^2\sigma^4} \text{ch}\left(\frac{sy'_+}{(n-k)\sigma^2}\right)\right] ds \\
& \leq \frac{2y'_+}{\sqrt{2\pi n}\sigma} \left(1 + \frac{c_{t,\varepsilon}}{n^{\varepsilon/3}}\right) \left(1 - e^{-\frac{t^2}{2\sigma^2}}\right).
\end{aligned}$$

Consequently, using the same arguments as in the proof of Theorem 2.2.2 in Section 2.6 (see the developments from (2.6.23) to (2.6.25)), we obtain, with $\Phi_\sigma^+(t) = 1 - e^{-\frac{t^2}{2\sigma^2}}$,

$$L_5 \leq \left(1 + \frac{c_{t,\varepsilon}}{n^{\varepsilon/3}}\right) \Phi_\sigma^+(t) \frac{2V(x,y)}{\sqrt{2\pi n}\sigma \mathbb{P}_x(\tau_y > n)} (1 + o(1)),$$

which by the claim 2 of Theorem 2.2.2 implies that

$$L_5 = \Phi_\sigma^+(t) (1 + o(1)). \quad (2.7.6)$$

Lower bound of L_5 . In the same way as for the upper bound, with $y'_- = y' - (n-k)^{1/2-2\varepsilon}$ and $t_- = t - \frac{2}{(n-k)^{2\varepsilon}}$, we have

$$\begin{aligned} & \mathbb{P}_x(\tau_y > n) L_5 \\ & \geq \sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{R}_+^*} \mathbb{P}(y'_- + \sigma B_{n-k} \leq t_- \sqrt{n}, \tau_{y'_-}^{bm} > n-k) \mathbb{P}_x(y + S_k \in dy'), \\ & \quad (n-k)^{1/2-2\varepsilon} < y + S_k \leq n^{1/2-\varepsilon/6}, \tau_y > k, \nu_n^{\varepsilon/6} = k) \\ & - \sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{R}} \mathbb{P}_{x'}(\bar{A}_{n-k}) \mathbb{P}_x(X_k \in dx', \tau_y > k, \nu_n^{\varepsilon/6} = k). \end{aligned}$$

Using Proposition 2.9.3 with y'_- , which is positive when $(n-k)^{1/2-2\varepsilon} < y' \leq n^{1/2-\varepsilon/6}$, we obtain after calculation that

$$\mathbb{P}(y'_- + \sigma B_{n-k} \leq t_- \sqrt{n}, \tau_{y'_-}^{bm} > n-k) \geq \frac{2y'_-}{\sqrt{2\pi n}\sigma} \left(1 - \frac{c_{t,\varepsilon}}{n^{\varepsilon/3}}\right) \Phi_\sigma^+(t).$$

Copying the proof of the bound of J_1 in (2.6.19) and using the same arguments as in the proof of Theorem 2.2.2 in Section 2.6 (see the developments from (2.6.26) to (2.6.27)), we get

$$L_5 \geq \Phi_\sigma^+(t) \frac{2V(x,y)}{\sqrt{2\pi n}\sigma \mathbb{P}_x(\tau_y > n)} (1 - o(1)) = \Phi_\sigma^+(t) (1 - o(1)).$$

Coupling this with (2.7.6) we obtain that

$$L_5 = \Phi_\sigma^+(t) (1 + o(1)).$$

Inserting this and (2.7.4) and (2.7.5) into (2.7.3), we deduce that $L_2 \underset{n \rightarrow +\infty}{\sim} \Phi_\sigma^+(t)$. By (2.7.1) and (2.7.2), we finally have

$$\mathbb{P}_x(y + S_n \leq t\sqrt{n} \mid \tau_y > n) \xrightarrow{n \rightarrow +\infty} \Phi_\sigma^+(t).$$

Changing t into $t\sigma$, this concludes the proof.

2.8 The case of non-positive initial point

In this section, we prove Theorem 2.2.5.

Lemma 2.8.1. *Assume Condition 2.1. For any $(x,y) \in \mathcal{D}^-$, the random variable M_{τ_y} is integrable and the function $V(x,y) = -\mathbb{E}_x(M_{\tau_y})$, is well defined on \mathcal{D}^- .*

Proof. If $\mathbb{E}(a) \geq 0$, by the Markov inequality, with $z = y + \rho x$,

$$\begin{aligned} \mathbb{E}_x(z + M_n; \tau_y > n) &= \int_{\mathbb{R} \times \mathbb{R}_+^*} \mathbb{E}_{x'}(y' + \rho x' + M_{n-1}; \tau_{y'} > n-1) \\ &\quad \times \mathbb{P}_x(X_1 \in dx', y + S_1 \in dy', \tau_y > 1). \end{aligned}$$

Since $y + S_1 > 0$ on $\{\tau_y > 1\}$, by Lemma 2.4.4,

$$\begin{aligned} \mathbb{E}_x(z + M_n; \tau_y > n) &\leq c_p \mathbb{E}_x\left((1 + y + S_1 + |X_1|)(1 + |X_1|)^{p-1}; \tau_y > 1\right) \\ &\leq c_p \mathbb{E}_x\left((1 + |X_1|)^p\right) \\ &\leq c_p(1 + |x|)^p. \end{aligned} \tag{2.8.1}$$

Moreover

$$\begin{aligned} \mathbb{E}_x\left(|M_{\tau_y}|; \tau_y \leq n\right) &\leq |z| + \sum_{k=2}^n \int_{\mathbb{R} \times \mathbb{R}_+^*} \mathbb{E}_{x'}(|y' + \rho x' + M_{k-1}|; \tau_{y'} = k-1) \\ &\quad \times \mathbb{P}_x(X_1 \in dx', y + S_1 \in dy', \tau_y > 1) \\ &\quad + \mathbb{E}_x(|M_1|; \tau_y = 1). \end{aligned}$$

Since $y + S_1 > 0$ on $\{\tau_y > 1\}$, by Lemma 2.4.2,

$$\begin{aligned} \mathbb{E}_x\left(|M_{\tau_y}|; \tau_y \leq n\right) &\leq c(1 + |y| + |x|) - \mathbb{E}_x(z + M_{\tau_y}; \tau_y \leq n) \\ &\leq c(1 + |y| + |x|) + \mathbb{E}_x(z + M_n; \tau_y > n). \end{aligned}$$

Using (2.8.1), we deduce that $\mathbb{E}_x\left(|M_{\tau_y}|; \tau_y \leq n\right) \leq c_p(1 + |y| + |x|^p)$. Consequently, by the Lebesgue monotone convergence theorem, the assertion is proved when $\mathbb{E}(a) \geq 0$. When $\mathbb{E}(a) < 0$, the assertion follows from Lemma 2.5.4. \square

Lemma 2.8.2. *Assume Condition 2.1. The function V is \mathbf{Q}_+ -harmonic on $\mathcal{D} = \mathcal{D}^- \cup \mathbb{R} \times \mathbb{R}_+^*$. If in addition we assume either Condition 2.2 and $\mathbb{E}(a) \geq 0$, or Condition 2.3, then the function V is positive on $\mathcal{D} = \mathcal{D}^- \cup \mathbb{R} \times \mathbb{R}_+^*$.*

Proof. Note that by Corollary 2.9.7, we have $\mathbb{P}_x(\tau_y < +\infty) = 1$, for any $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Therefore, by the Lebesgue dominated convergence theorem,

$$V(x, y) = -\mathbb{E}_x(M_{\tau_y}) = z - \lim_{n \rightarrow \infty} \mathbb{E}_x(z + M_{\tau_y}; \tau_y \leq n) = \lim_{n \rightarrow \infty} \mathbb{E}_x(z + M_n; \tau_y > n),$$

for any $(x, y) \in \mathcal{D}^-$. The fact that V is \mathbf{Q}_+ -harmonic on \mathcal{D} can be proved in the same way as in the proof of Proposition 2.5.3. Therefore, for any $(x, y) \in \mathcal{D}^-$,

$$V(x, y) = \mathbb{E}_x(V(X_1, y + S_1); \tau_y > 1). \tag{2.8.2}$$

By the claim 2 of Proposition 2.5.3 and the claim 3 of Proposition 2.5.9, on $\{\tau_y > 1\}$, the random variable $V(X_1, y + S_1)$ is positive almost surely. Since by the definition of \mathcal{D}^- , we have $\mathbb{P}_x(\tau_y > 1) > 0$, we conclude that $V(x, y) > 0$ for any $(x, y) \in \mathcal{D}^-$. \square

Lemma 2.8.3. *Assume Condition 2.1.*

1. For any $(x, y) \in \mathcal{D}^-$,

$$\sqrt{n} \mathbb{P}_x(\tau_y > n) \leq c_p(1 + |x|)^p.$$

2. If in addition we assume either Condition 2.2 and $\mathbb{E}(a) \geq 0$, or Condition 2.3, then for any $(x, y) \in \mathcal{D}^-$,

$$\mathbb{P}_x(\tau_y > n) \underset{n \rightarrow +\infty}{\sim} \frac{2V(x, y)}{\sqrt{2\pi n\sigma}}.$$

Proof. By the Markov property,

$$\sqrt{n}\mathbb{P}_x(\tau_y > n) = \int_{\mathbb{R} \times \mathbb{R}_+^*} \sqrt{n}\mathbb{P}_{x'}(\tau_{y'} > n-1) \mathbb{P}_x(X_1 \in dx', y + S_1 \in dy', \tau_y > 1).$$

By Theorem 2.2.2, for any $y' > 0$, we have $\sqrt{n}\mathbb{P}_{x'}(\tau_{y'} > n-1) \leq c_p(1 + y' + |x'|)^p$ and moreover, for any $y \leq 0$,

$$\mathbb{E}_x(c_p(1 + y + S_1 + |X_1|)^p; \tau_y > 1) \leq c_p(1 + |x|)^p.$$

Then, we obtain the claim 1 and by the Lebesgue dominated convergence theorem and the claim 2 of Theorem 2.2.2,

$$\lim_{n \rightarrow \infty} \sqrt{n}\mathbb{P}_x(\tau_y > n) = \mathbb{E}_x\left(\frac{2V(X_1, y + S_1)}{\sqrt{2\pi\sigma}}; \tau_y > 1\right).$$

Using (2.8.2) we conclude the proof. \square

Lemma 2.8.4. Assume either Conditions 2.1, 2.2 and $\mathbb{E}(a) \geq 0$, or Conditions 2.1 and 2.3. For any $(x, y) \in \mathcal{D}^-$ and $t > 0$,

$$\mathbb{P}_x\left(\frac{y + S_n}{\sigma\sqrt{n}} \leq t \mid \tau_y > n\right) \underset{n \rightarrow +\infty}{\longrightarrow} 1 - e^{-\frac{t^2}{2}}.$$

Proof. Similarly as in the proof of Lemma 2.8.3, we write,

$$\begin{aligned} & \mathbb{P}_x\left(\frac{y + S_n}{\sigma\sqrt{n}} \leq t \mid \tau_y > n\right) \\ &= \frac{1}{\mathbb{P}_x(\tau_y > n)} \int_{\mathbb{R} \times \mathbb{R}_+^*} \mathbb{P}_{x'}\left(\frac{y' + S_{n-1}}{\sigma\sqrt{n-1}} \leq t; \tau_{y'} > n-1\right) \\ & \quad \times \mathbb{P}_x(X_1 \in dx', y + S_1 \in dy', \tau_y > 1) \\ &= \frac{1}{\sqrt{n}\mathbb{P}_x(\tau_y > n)} \int_{\mathbb{R} \times \mathbb{R}_+^*} \mathbb{P}_{x'}\left(\frac{y' + S_{n-1}}{\sigma\sqrt{n-1}} \leq t \mid \tau_{y'} > n-1\right) \sqrt{n}\mathbb{P}_{x'}(\tau_{y'} > n-1) \\ & \quad \times \mathbb{P}_x(X_1 \in dx', y + S_1 \in dy', \tau_y > 1). \end{aligned}$$

Since, by Lemma 2.8.3, $\sqrt{n}\mathbb{P}_{x'}(\tau_{y'} > n-1) \leq c_p(1 + |x'|)^p$, applying the Lebesgue dominated convergence theorem, Theorem 2.2.2, Theorem 2.2.4 and Lemma 2.8.3, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{P}_x\left(\frac{y + S_n}{\sigma\sqrt{n}} \leq t \mid \tau_y > n\right) \\ &= \frac{\sqrt{2\pi}\sigma}{2V(x, y)} \int_{\mathbb{R} \times \mathbb{R}_+^*} \left(1 - e^{-\frac{t^2}{2}}\right) \frac{2V(x', y')}{\sqrt{2\pi}\sigma} \mathbb{P}_x(X_1 \in dx', y + S_1 \in dy', \tau_y > 1). \end{aligned}$$

Using (2.8.2) concludes the proof. \square

2.9 Appendix

2.9.1 Proof of the fact Condition 2.3bis implies Condition 2.3

We suppose that Condition 2.3bis holds. Then, there exists $\delta > 0$ such that

$$\mathbb{P}((a, b) \in [-1 + \delta, 0] \times [\delta, C]) > 0 \quad (2.9.1)$$

and

$$\mathbb{P}((a, b) \in [0, 1 - \delta] \times [\delta, C]) > 0. \quad (2.9.2)$$

For any $x \in \mathbb{R}$, set $C_x = \max(|x|, \frac{C}{\delta})$ and

$$\mathcal{A}_n = \left\{ \delta \leq X_1 \leq C_x, \delta \leq X_2 \leq C_{X_1}, \dots, \delta \leq X_n \leq C_{X_{n-1}} \right\}.$$

Using (2.9.1) for $x < 0$ and (2.9.2) for $x \geq 0$, we obtain that $\mathbb{P}_x(\mathcal{A}_1) > 0$. By the Markov property, we deduce that $\mathbb{P}_x(\mathcal{A}_n) > 0$. Moreover, it is easy to see that, on \mathcal{A}_n , we have $y + S_k \geq y + k\delta > 0$, for all $k \leq n$, and $|X_n| \leq C_x$. Taking $n = n_0$ large enough, we conclude that Condition 2.3 holds under Condition 2.3bis.

2.9.2 Convergence of recursively bounded monotonic sequences

The following two lemmas give sufficient conditions for a monotonic sequence to be bounded.

Lemma 2.9.1. *Let $(u_n)_{n \geq 1}$ be a non-decreasing sequence of reals such that there exist $n_0 \geq 2$, $\varepsilon \in (0, 1)$, $\alpha, \beta, \gamma \geq 0$ and $\delta > 0$ such that for any $n \geq n_0$,*

$$u_n \leq \left(1 + \frac{\alpha}{n^\varepsilon}\right) u_{\lfloor n^{1-\varepsilon} \rfloor} + \frac{\beta}{n^\varepsilon} + \gamma e^{-\delta n^\varepsilon}. \quad (2.9.3)$$

Then, for any $n \geq n_0$ and any integer $k_0 \in \{n_0, \dots, n\}$,

$$\begin{aligned} u_n &\leq \exp\left(\frac{\alpha}{k_0^\varepsilon} \frac{2^\varepsilon 2^{\varepsilon^2}}{2^{\varepsilon^2} - 1}\right) \left(u_{k_0} + \frac{\beta}{k_0^\varepsilon} \frac{2^\varepsilon 2^{\varepsilon^2}}{2^{\varepsilon^2} - 1} + \gamma \frac{\exp\left(-\delta \frac{k_0^\varepsilon}{2^\varepsilon}\right)}{1 - e^{-\delta(2^{\varepsilon^2} - 1)}} \right) \\ &\leq \left(1 + \frac{C_{\alpha, \varepsilon}}{k_0^\varepsilon}\right) u_{k_0} + \beta \frac{C_{\alpha, \varepsilon}}{k_0^\varepsilon} + \gamma C_{\alpha, \delta, \varepsilon} e^{-c_{\alpha, \delta, \varepsilon} k_0^\varepsilon}. \end{aligned}$$

In particular, choosing k_0 constant, it follows that $(u_n)_{n \geq 1}$ is bounded.

Proof. Fix $n \geq n_0$ and $k_0 \in \{n_0, \dots, n\}$ and consider for all $j \geq 0$,

$$p_j = \lfloor n^{(1-\varepsilon)^j} \rfloor.$$

The sequence $(p_j)_{j \geq 0}$ starts at $n_0 = n$, is non-increasing and converge to 1. So there exists $m = m(k_0) \in \mathbb{N}$ such that $p_m \geq k_0 \geq p_{m+1}$. Since $n^{(1-\varepsilon)^j} / 2 \geq k_0 / 2 \geq 1$, for all $j \in \{0, \dots, m\}$, we have

$$n^{(1-\varepsilon)^j} \geq p_j \geq n^{(1-\varepsilon)^j} - 1 \geq \frac{n^{(1-\varepsilon)^j}}{2}. \quad (2.9.4)$$

Using (2.9.3) and the fact that $(u_n)_{n \geq 2}$ is non-decreasing, we write for all $j = 0, \dots, m$,

$$u_{p_j} \leq \left(1 + \frac{\alpha}{p_j^\varepsilon}\right) u_{p_{j+1}} + \frac{\beta}{p_j^\varepsilon} + \gamma e^{-\delta p_j^\varepsilon} \leq \left(1 + \frac{\alpha}{p_j^\varepsilon}\right) \left(u_{p_{j+1}} + \frac{\beta}{p_j^\varepsilon} + \gamma e^{-\delta p_j^\varepsilon}\right).$$

Iterating, we obtain that

$$u_n \leq A_m \left(u_{p_{m+1}} + \beta B_m + \gamma C_m\right),$$

where $A_m = \prod_{j=0}^m \left(1 + \frac{\alpha}{p_j^\varepsilon}\right)$, $B_m = \sum_{j=0}^m \frac{1}{p_j^\varepsilon}$ and $C_m = \sum_{j=0}^m e^{-\delta p_j^\varepsilon}$. Since $p_{m+1} \leq k_0$ and since $(u_n)_{n \geq 2}$ is non-decreasing,

$$u_n \leq A_m (u_{k_0} + \beta B_m + \gamma C_m). \quad (2.9.5)$$

Now, we bound A_m as follows,

$$A_m \leq \prod_{j=0}^m e^{\frac{\alpha}{p_j^\varepsilon}} = e^{\alpha B_m}. \quad (2.9.6)$$

Denoting $\eta_j = n^{-(1-\varepsilon)^j \varepsilon}$, using (2.9.4), we have $B_m \leq 2^\varepsilon \sum_{j=0}^m \eta_j$. Moreover, for all $j \leq m$, we note that $\frac{\eta_j}{\eta_{j+1}} = \frac{1}{n^{\varepsilon^2(1-\varepsilon)^j}} \leq \frac{1}{k_0^{\varepsilon^2}} \leq \frac{1}{2^{\varepsilon^2}} < 1$ and so

$$\eta_j \leq \frac{\eta_m}{2^{\varepsilon^2(m-j)}} \leq \frac{1}{p_m^\varepsilon 2^{\varepsilon^2(m-j)}} \leq \frac{1}{k_0^\varepsilon 2^{\varepsilon^2(m-j)}}. \quad (2.9.7)$$

Therefore, B_m is bounded as follows:

$$B_m \leq \frac{2^\varepsilon}{k_0^\varepsilon} \sum_{k=0}^m \left(\frac{1}{2^{\varepsilon^2}}\right)^k \leq \frac{1}{k_0^\varepsilon} \frac{2^\varepsilon 2^{\varepsilon^2}}{2^{\varepsilon^2} - 1}. \quad (2.9.8)$$

Using (2.9.4) and (2.9.7), we have

$$C_m \leq \sum_{j=0}^m e^{-\frac{\delta}{2^{\varepsilon^2} \eta_j}} \leq \sum_{j=0}^m \exp\left(-\frac{\delta k_0^\varepsilon 2^{\varepsilon^2(m-j)}}{2^\varepsilon}\right).$$

Since for any $u \geq 0$ and $k \in \mathbb{N}$, we have $(1+u)^k \geq 1+ku$, it follows that

$$C_m \leq e^{-\frac{\delta k_0^\varepsilon}{2^\varepsilon}} \sum_{k=0}^m \exp\left(-\delta k \left(2^{\varepsilon^2} - 1\right)\right) \leq \frac{e^{-\frac{\delta k_0^\varepsilon}{2^\varepsilon}}}{1 - e^{-\delta(2^{\varepsilon^2} - 1)}}. \quad (2.9.9)$$

Putting together (2.9.6), (2.9.8) and (2.9.9) into (2.9.5) proves the lemma. \square

Lemma 2.9.2. *Let $(u_n)_{n \geq 1}$ be a non-increasing sequence of reals such that there exist $n_0 \geq 2$, $\varepsilon \in (0, 1)$ and $\beta \geq 0$ such that for any $n \geq n_0$,*

$$u_n \geq u_{\lfloor n^{1-\varepsilon} \rfloor} - \frac{\beta}{n^\varepsilon}.$$

Then, for any $n \geq n_0$ and any integer $k_0 \in \{n_0, \dots, n\}$,

$$u_n \geq u_{k_0} - \frac{\beta}{k_0^\varepsilon} \frac{2^\varepsilon 2^{\varepsilon^2}}{2^{\varepsilon^2} - 1} = u_{k_0} - c_\varepsilon \frac{\beta}{k_0^\varepsilon}.$$

In particular, choosing k_0 constant, it follows that $(u_n)_{n \geq 1}$ is bounded.

Proof. For the proof it is enough to use Lemma 2.9.1 with u_n replaced by $-u_n$. \square

2.9.3 Results on the Brownian case and strong approximation

Consider the standard Brownian motion $(B_t)_{t \geq 0}$ living on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define the exit time

$$\tau_y^{bm} = \inf\{t \geq 0, y + \sigma B_t \leq 0\}, \quad (2.9.10)$$

where $\sigma > 0$. The following assertions are due to Lévy [57].

Proposition 2.9.3. *For any $y > 0$, $0 \leq a \leq b$ and $n \geq 1$,*

$$\mathbb{P}(\tau_y^{bm} > n) = \frac{2}{\sqrt{2\pi n\sigma}} \int_0^y e^{-\frac{s^2}{2n\sigma^2}} ds.$$

and

$$\mathbb{P}(\tau_y^{bm} > n, y + \sigma B_n \in [a, b]) = \frac{1}{\sqrt{2\pi n\sigma}} \int_a^b \left(e^{-\frac{(s-y)^2}{2n\sigma^2}} - e^{-\frac{(s+y)^2}{2n\sigma^2}} \right) ds.$$

From this one can deduce easily:

Corollary 2.9.4.

1. *For any $y > 0$,*

$$\mathbb{P}(\tau_y^{bm} > n) \leq c \frac{y}{\sqrt{n}}.$$

2. *For any sequence of real numbers $(\theta_n)_{n \geq 0}$ such that $\theta_n \xrightarrow{n \rightarrow +\infty} 0$,*

$$\sup_{y \in [0; \theta_n \sqrt{n}]} \left(\frac{\mathbb{P}(\tau_y^{bm} > n)}{\frac{2y}{\sqrt{2\pi n\sigma}}} - 1 \right) = O(\theta_n^2).$$

To transfer the results from the Brownian motion to the Markov walk, we use a functional approximation given in Theorem 3.3 from Grama, Le Page and Peigné [40]. We have to construct an adapted Banach space \mathcal{B} and verify the hypotheses **M1** – **M5** in [40] which are necessary to apply Theorem 3.3. Fix $p \in (2, \alpha)$ and let ε, θ, c_0 and δ be positive numbers such that $c_0 + \varepsilon < \theta < 2c_0 < \alpha - \varepsilon$ and $2 < 2 + 2\delta < (2 + 2\delta)\theta \leq p$. Define the Banach space $\mathcal{B} = \mathcal{L}_{\varepsilon, c_0, \theta}$ as the set of continuous function f from \mathbb{R} to \mathbb{C} such that $\|f\| = |f|_\theta + [f]_{\varepsilon, c_0} < +\infty$, where

$$|f|_\theta = \sup_{x \in \mathbb{R}} \frac{|f(x)|}{1 + |x|^\theta}, \quad [f]_{\varepsilon, c_0} = \sup_{\substack{(x, y) \in \mathbb{R}^2 \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\varepsilon (1 + |x|^{c_0}) (1 + |y|^{c_0})}.$$

For example, one can take $\varepsilon < \min(\frac{p-2}{4}, \frac{1}{2})$, $c_0 = 1$, $\theta = 1 + 2\varepsilon$ and $2 + 2\delta = \frac{p}{1+2\varepsilon}$. Using the techniques from [43] one can verify that, under Condition 2.1, the Banach space \mathcal{B} and the perturbed operator $\mathbf{P}_t f(x) = \mathbb{E}_x(f(X_1) e^{itX_1})$ satisfy Hypotheses **M1** – **M5** in [40]. The hypothesis **M1** is verified straightforwardly. In particular the norm of the Dirac measure δ_x is bounded: $\|\delta_x\|_{\mathcal{B} \rightarrow \mathcal{B}} \leq 1 + |x|^\theta$, for each $x \in \mathbb{R}$. We refer to Proposition 4 and Corollary 3 of [43] for **M2** – **M3**. For **M4**, we have

$$\mu_\delta(x) = \sup_{k \geq 1} \mathbb{E}_x^{1/2+2\delta} (|X_n|^{2+2\delta}) \leq c_\delta (1 + |x|).$$

Hypothesis **M5** follows from Proposition 1 of [43] and Lemma 2.3.1.

With these considerations, the $C(x) = C_1(1 + \mu_\delta(x) + \|\delta_x\|)^{2+2\delta}$ in Theorem 3.3 established in [40] is less than $c_p(1 + |x|)^p$, where C_1 is a constant. Therefore Theorem 3.3 can be reformulated in the case of the stochastic recursion as follows.

Proposition 2.9.5. *Assume Condition 2.1. For any $p \in (2, \alpha)$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, $x \in \mathbb{R}$ and $n \geq 1$, without loss of generality (on an extension of the initial probability space) one can reconstruct the sequence $(S_n)_{n \geq 0}$ with a continuous time Brownian motion $(B_t)_{t \in \mathbb{R}_+}$, such that*

$$\mathbb{P}_x \left(\sup_{0 \leq t \leq 1} |S_{[tn]} - \sigma B_{tn}| > n^{1/2-\varepsilon} \right) \leq \frac{c_{p,\varepsilon}}{n^\varepsilon} (1 + |x|)^p,$$

where σ is given by (2.2.2).

This proposition plays the crucial role in the proof of Theorem 2.2.2 and Theorem 2.2.4 (cf. Sections 2.6 and 2.7). The following straightforward consequence of Proposition 2.9.5 is used in the proof of Lemma 2.4.1 in Section 2.4. Set $\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-\frac{u^2}{2}} du$.

Corollary 2.9.6. *Assume Condition 2.1. For any $p \in (2, \alpha)$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, $x \in \mathbb{R}$ and $n \geq 1$,*

$$\sup_{u \in \mathbb{R}} \left| \mathbb{P}_x \left(\frac{S_n}{\sqrt{n}} \leq u \right) - \Phi \left(\frac{u}{\sigma} \right) \right| \leq \frac{c_{p,\varepsilon}}{n^\varepsilon} (1 + |x|)^p.$$

Proof. Let $\varepsilon \in (0, 1/2)$ and $A_n = \left\{ \sup_{0 \leq t \leq 1} |S_{[tn]} - \sigma B_{tn}| > n^{1/2-\varepsilon} \right\}$. For any $x \in \mathbb{R}$ and any $u \in \mathbb{R}$,

$$\mathbb{P}_x \left(\frac{S_n}{\sqrt{n}} \leq u \right) \leq \mathbb{P}_x(A_n) + \mathbb{P}_x \left(\frac{\sigma B_n}{\sqrt{n}} \leq u + \frac{1}{n^\varepsilon} \right),$$

where the last probability does not exceed $\Phi\left(\frac{u}{\sigma}\right) + c_\varepsilon n^{-\varepsilon}$. Using Proposition 2.9.5, we conclude that there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ and $x \in \mathbb{R}$,

$$\mathbb{P}_x \left(\frac{S_n}{\sqrt{n}} \leq u \right) \leq \Phi \left(\frac{u}{\sigma} \right) + \frac{c_{p,\varepsilon}}{n^\varepsilon} (1 + |x|)^p.$$

In the same way we obtain a lower bound and the assertion follows. \square

2.9.4 Finiteness of the exit times

Corollary 2.9.7. *Assume Condition 2.1. For any $x \in \mathbb{R}$ and $y \in \mathbb{R}$,*

$$\mathbb{P}_x(\tau_y < +\infty) = 1 \quad \text{and} \quad \mathbb{P}_x(T_y < +\infty) = 1.$$

Proof. Let $y > 0$ and $\varepsilon \in (0, 1/2)$. Set $A_n = \left\{ \sup_{0 \leq t \leq 1} |S_{[tn]} - \sigma B_{tn}| \leq n^{1/2-\varepsilon} \right\}$. Using Proposition 2.9.5, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, $x \in \mathbb{R}$ and $y > 0$,

$$\begin{aligned} \mathbb{P}_x(\tau_y > n) &\leq \mathbb{P}_x(\tau_y > n, A_n) + \mathbb{P}_x(\bar{A}_n) \\ &\leq \mathbb{P}(\tau_{y+n^{1/2-\varepsilon}}^{bm} > n) + \frac{c_{p,\varepsilon}}{n^\varepsilon} (1 + |x|)^p. \end{aligned}$$

Since, by the claim 1 of Corollary 2.9.4, $\mathbb{P}(\tau_{y+n^{1/2-\varepsilon}}^{bm} > n) \leq c \frac{y+n^{1/2-\varepsilon}}{\sqrt{n}} \leq (1+y) \frac{c}{n^\varepsilon}$, taking the limit as $n \rightarrow +\infty$ we conclude that $\mathbb{P}_x(\tau_y < +\infty) = 1$.

Let $D_n = \left\{ \max_{1 \leq k \leq n} |S_k - M_k| \leq n^{1/2-\varepsilon} \right\}$. Obviously

$$\begin{aligned} \mathbb{P}_x(T_y > n) &\leq \mathbb{P}_x(T_y > n, A_n, D_n) + \mathbb{P}_x(\overline{A}_n) + \mathbb{P}_x(\overline{D}_n) \\ &\leq \mathbb{P}\left(\tau_{y+2n^{1/2-\varepsilon}}^{bm} > n\right) + \frac{c_{p,\varepsilon}}{n^\varepsilon} (1 + |x|)^p + \mathbb{P}_x\left(\max_{1 \leq k \leq n} |\rho X_k| > n^{1/2-\varepsilon}\right). \end{aligned}$$

Using the claim 1 of Corollary 2.9.4, the Markov inequality and Lemma 2.3.1, for any $\varepsilon \in (0, \varepsilon_0]$, $x \in \mathbb{R}$ and $y > 0$,

$$\mathbb{P}_x(T_y > n) \leq (1 + y) \frac{c}{n^\varepsilon} + \frac{c_{p,\varepsilon}}{n^\varepsilon} (1 + |x|)^p + c_p \frac{1 + |x|^p}{n^{\frac{p-2}{2}-p\varepsilon}}.$$

Choosing ε small enough and taking the limit as $n \rightarrow +\infty$ we conclude the second assertion when $y > 0$.

When $y \leq 0$, the results follow since the applications $y \mapsto \tau_y$ and $y \mapsto T_y$ are non-decreasing. \square

Acknowledgements. The authors would like to thank the two anonymous referees for their thorough reviews of the original manuscript and helpful comments and corrections which led to a substantial improvement in the presentation.

Chapter 3

Limit theorems for Markov walks conditioned to stay positive under a spectral gap assumption

This chapter is the subject of the article [38] written in collaboration with Ion Grama and Emile Le Page to appear in *The Annals of Probability*.

RÉSUMÉ. On considère une chaîne de Markov $(X_n)_{n \geq 0}$ à valeurs dans un espace d'états \mathbb{X} . Pour f une fonction réelle définie sur \mathbb{X} , on pose $S_n = \sum_{i=1}^n f(X_i)$, $n \geq 1$. Soit \mathbb{P}_x la probabilité engendrée par la chaîne de Markov lorsque l'état initial est donné par $X_0 = x$. Pour tout point de départ $y \in \mathbb{R}$, on définit τ_y comme étant le premier instant pour lequel la marche markovienne $(y + S_n)_{n \geq 1}$ devient négative ou nulle. Sous la condition que la marche S_n soit sans dérive, on détermine l'asymptotique de $\mathbb{P}_x(\tau_y > n)$ ainsi que celui de la loi de la marche conditionnée $\mathbb{P}_x(y + S_n \leq \cdot \sqrt{n} \mid \tau_y > n)$ as $n \rightarrow +\infty$.

ABSTRACT. Consider a Markov chain $(X_n)_{n \geq 0}$ with values in the state space \mathbb{X} . Let f be a real function on \mathbb{X} and set $S_n = \sum_{i=1}^n f(X_i)$, $n \geq 1$. Let \mathbb{P}_x be the probability measure generated by the Markov chain starting at $X_0 = x$. For a starting point $y \in \mathbb{R}$ denote by τ_y the first moment when the Markov walk $(y + S_n)_{n \geq 1}$ becomes non-positive. Under the condition that S_n has zero drift, we find the asymptotics of the probability $\mathbb{P}_x(\tau_y > n)$ and of the conditional law $\mathbb{P}_x(y + S_n \leq \cdot \sqrt{n} \mid \tau_y > n)$ as $n \rightarrow +\infty$.

3.1 Introduction

Assume that on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we are given a sequence of random variables $(X_n)_{n \geq 1}$ with values in a measurable space \mathbb{X} . Let f be a real function on \mathbb{X} . Suppose that the random walk $S_n = \sum_{i=1}^n f(X_i)$, $n \geq 1$ has zero drift. For a starting point $y \in \mathbb{R}$ denote by τ_y the time at which $(y + S_n)_{n \geq 1}$ first passes into the interval $(-\infty, 0]$. We are interested in the asymptotic behaviour of the probability $\mathbb{P}(\tau_y > n)$ and of the conditional law of $\frac{y + S_n}{\sqrt{n}}$ given the event $\{\tau_y > n\} = \{S_1 > 0, \dots, S_n > 0\}$ as $n \rightarrow +\infty$.

The case when f is the identity function and $(X_n)_{n \geq 1}$ are i.i.d. in $\mathbb{X} = \mathbb{R}$ has been extensively studied in the literature. We refer to Spitzer [66], Iglehart [47, 48], Bolthausen [9], Doney [22], Bertoin and Doney [6], Borovkov [10, 11], Caravenna [13], Vatutin and

Wachtel [71] to cite only a few. Recent progress has been made for random walks with independent increments in $\mathbb{X} = \mathbb{R}^d$, see Eichelbacher and König [27], Denisov and Wachtel [20, 18] and Duraj [25]. However, to the best of our knowledge, the case of the Markov chains has been treated only in some special cases. Upper and lower bounds for $\mathbb{P}(\tau_y > n)$ have been obtained in Varopoulos [68], [69] for Markov chains with bounded jumps and in Dembo, Ding and Gao [15] for integrated random walks based on independent increments. An approximation of $\mathbb{P}(\tau_y > n)$ by the survival probability of the Brownian motion for Markov walk under moment conditions is given in Varopoulos [70]. Exact asymptotic behaviour was determined in Presman [60, 61] in the case of sums of random variables defined on a finite Markov chain under the additional assumption that the distributions have an absolute continuous component and in Denisov and Wachtel [19] for integrated random walks. The case of products of i.i.d. random matrices which reduces to the study of a particular Markov chain defined on a merely compact state space was considered in [41] and the case of affine walks in \mathbb{R} has been treated in [36] (Chapter 2). We also point out the work of Denisov, Korshunov and Wachtel [16] where a constructive analysis of harmonic functions for Markov chains with values in \mathbb{N} is performed.

In this paper we determine the limit of the probability of the exit time τ_y and of the law of $y + S_n$ conditioned to stay positive for a Markov chain under the assumption that its transition operator has a spectral gap. In particular our results cover the case of Markov chains with compact state spaces and the affine random walks in \mathbb{R} (see [36]/Chapter 2) and \mathbb{R}^d (see Gao, Guivarc'h and Le Page [30]). Our results apply also to the case of sums of i.i.d. random variables.

To present briefly the main results of the paper denote by \mathbb{P}_x and \mathbb{E}_x the probability and the corresponding expectation generated by the trajectories of a Markov chain $(X_n)_{n \geq 1}$ with the initial state $X_0 = x \in \mathbb{X}$. Let \mathbf{Q} be the transition operator of the Markov chain $(X_n, y + S_n)_{n \geq 1}$ and let \mathbf{Q}_+ be the restriction of \mathbf{Q} on $\mathbb{X} \times \mathbb{R}_+^*$. We show that under appropriate assumptions, there exists a \mathbf{Q}_+ -harmonic function V with non-empty support $\text{supp}(V)$ in $\mathbb{X} \times \mathbb{R}$ such that, for any $(x, y) \in \text{supp}(V)$,

$$\mathbb{P}_x(\tau_y > n) \underset{n \rightarrow +\infty}{\sim} \frac{2V(x, y)}{\sqrt{2\pi n\sigma}} \quad (3.1.1)$$

and

$$\mathbb{P}_x \left(\frac{y + S_n}{\sigma\sqrt{n}} \leq t \mid \tau_y > n \right) \underset{n \rightarrow +\infty}{\longrightarrow} \Phi^+(t),$$

where $\Phi^+(t) = 1 - e^{-\frac{t^2}{2}}$ is the Rayleigh distribution function and σ is a positive real. Moreover, we complete this result by giving the behaviour of $\mathbb{P}_x(\tau_y > n)$ on the complement of $\text{supp}(V)$: for any $(x, y) \notin \text{supp}(V)$,

$$\mathbb{P}_x(\tau_y > n) \leq c_x e^{-cn}, \quad (3.1.2)$$

where c_x depends on x and c is a constant. This is different from the case of sums of i.i.d. real random variables, where instead of (3.1.2), on $\text{supp}(V)^c$ it holds $\mathbb{P}_x(\tau_y > n) = 0$. We give an example of a Markov chain for which the bound (3.1.2) is attained and state uniform versions of (3.1.1) and (3.1.2). A characterization of the $\text{supp}(V)$ is given in point 4 of Theorem 3.2.2. For details we refer to Section 3.2.

The study of the asymptotic behaviour of the probability $\mathbb{P}(\tau_y > n)$ for walks on the real line \mathbb{R} is usually based on the Wiener-Hopf factorization (see Feller [29]). Unfortunately the Wiener-Hopf factorisation is not well suited for more general walks, as for

example those with values in \mathbb{R}^d or for walks with dependent increments. For random walks with dependent increments and for random walks with independent increments in \mathbb{R}^d , Varopoulos [70], Eichelbacher and König [27] and Denisov and Wachtel [20] have developed an alternative approach based on the existence of the harmonic function. Using the particular structure of the underlying models such extensions were performed in Denisov and Wachtel [19] for integrated random walks, in [41] for products of random matrices and in [36] (Chapter 2) for affine random walks in \mathbb{R} . Despite these advances, there are still some major difficulties in transferring the harmonic function approach to the case of more general Markov chains. In this paper we extend it to Markov chains under spectral gap assumptions. Let us highlight below the key points of the proofs.

We begin with the construction of a martingale approximation $(M_n)_{n \geq 1}$ for $(S_n)_{n \geq 1}$ following the approach of Gordin [35]. One of the delicate points of the proof is to control the difference $S_n - M_n$. We make use of the spectral gap property of the transition operator \mathbf{P} of the Markov chain $(X_n)_{n \geq 1}$ relatively to some Banach space \mathcal{B} (for details we refer to Section 3.2). Our martingale approximation is such that

$$(z + M_n) - (y + S_n) = r(X_n),$$

where $r(x) = \Theta(x) - f(x)$ is the coboundary, $z = y + r(x)$ and Θ is the solution of the Poisson equation $\Theta - \mathbf{P}\Theta = f$. Under Hypothesis M3.4 we can control $|r(x)|$ by $c(1 + N(x))$ where $N \in \mathcal{B}$ has bounded moments $\mathbb{E}_x^{1/\alpha}(N(X_n)^\alpha) \leq c(1 + N(x))$, for some $\alpha > 2$. Note that in the case of products of random matrices [41] the coboundary is bounded, so that $\sup_{n \geq 1} |S_n - M_n|$ is bounded by a constant \mathbb{P}_x -a.s. for any $x \in \mathbb{X}$, which simplifies greatly the proofs. The extension to the case of unbounded coboundary turns out to be quite laborious even for particular examples. We refer to the case of affine Markov walks considered in [36] (Chapter 2), where the authors have benefited from the special structure of the model.

The next step is the proof of the existence of a positive harmonic function. The starting idea is very simple. Let $V_n(x, y) := \mathbb{E}_x((y + S_n)\mathbb{1}_{\{\tau_y > n\}})$ be the expectation of the Markov walk $(y + S_n)_{n \geq 1}$ killed at τ_y . Since by the Markov property, $V_{n+1}(x, y) = \mathbf{Q}_+ V_n(x, y)$, taking the limit as $n \rightarrow +\infty$ under appropriate assumptions, yields that the function $V(x, y) = \lim_{n \rightarrow +\infty} V_n(x, y)$ is \mathbf{Q}_+ -harmonic. Using the approximating martingale, the function V can be identified as $V(x, y) = -\mathbb{E}_x(M_{\tau_y})$. To justify this approach, it is important to control uniformly in n the expectation $w_n := \mathbb{E}_x((z + M_n)\mathbb{1}_{\{\tau_y > n\}})$. Our key idea (in contrast to [41] and [36]/Chapter 2) is the introduction of two extra stopping times T_z and \widehat{T}_z : the first time when $(z + M_n)_{n \geq 1}$ leaves \mathbb{R}_+^* and the first time larger than τ_y when $(z + M_n)_{n \geq 1}$ leaves \mathbb{R}_+^* , respectively, where as before $z = y + r(x)$. Clearly, \widehat{T}_z depends on τ_y and dominates both, τ_y and T_z . The relation of the time \widehat{T}_z to the exit times τ_y and T_z is explicitly given in Lemma 3.5.3 which is an application of the Markov property to \widehat{T}_z . This property is useful to control uniformly in n the expectation $u_n := \mathbb{E}_x((z + M_n)\mathbb{1}_{\{\widehat{T}_z > n\}})$, which is one of the crucial points of the proof. To establish this we note that the sequence $(u_n)_{n \geq 0}$ is increasing, since $((z + M_n)\mathbb{1}_{\{\widehat{T}_z > n\}})_{n \geq 1}$ is a submartingale. In addition we show that it satisfies a recurrence equation, which implies its boundedness. Using the previous arguments we obtain a uniform control on the expectation w_n . All the details can be found in Sections 3.6 and 3.7. The proof of the (strict) positivity of V is also rather involved but uses similar arguments based on the subharmonicity of the function $\widehat{W}(x, z) = -\mathbb{E}_x(M_{\widehat{T}_z})$. (see Section 3.8).

Now we can turn to the tail behaviour of the exit time τ_y . It is inferred from that of the exit time τ_y^{bm} of the Brownian motion, using the Donsker invariance principle for

sums defined on Markov chains with a the rate of convergence, recently proved in [40]. The result in [40] gives the explicit dependence of the constants on the norm $\|\delta_x\|_{\mathcal{B}'}$ of the Dirac measure δ_x and on the absolute moments $\mu_\alpha(x) = \sup_{n \geq 1} \mathbb{E}_x^{1/\alpha} (|f(X_n)|^\alpha)$ for some initial state $x \in \mathbb{X}$ and some $\alpha > 2$. To have a control on the constants we make use of Hypothesis M3.4. Note that for products of random matrices [41], $\|\delta_x\|_{\mathcal{B}'}$ and $\mu_\alpha(x)$ are bounded uniformly in the initial state $x \in \mathbb{X}$, so that the rate of convergence invariance principle does not depend on the initial state. The case of when $\|\delta_x\|_{\mathcal{B}'}$ and $\mu_\alpha(x)$ are not bounded was studied in details in [36] (Chapter 2) for affine Markov walks.

The paper is organized as follows. In Section 3.2 we introduce the necessary notations and state our main results. In Section 3.3 we give applications of the results of the paper to stochastic recursions in \mathbb{R}^d and Markov chains with compact state space. In Section 3.4 we collect some preliminary results. In Section 3.5 we construct the approximating martingale and state some of its properties and of the associated exit times. In Section 3.6 we prove that the expectations $\mathbb{E}_x((y + S_n)\mathbb{1}_{\{\tau_y > n\}})$ are bounded uniformly in n . Using the results of Sections 3.5 and 3.6, we establish in Section 3.7 the existence of a \mathbf{Q}_+ -harmonic function and prove in Section 3.8 that this function is not identically zero. We determine the limit of the probability $\mathbb{P}_x(\tau_y > n)$ in Section 3.9 and that of the conditioned law of $(y + S_n)/(\sigma\sqrt{n})$ given the event $\{\tau_y > n\}$ in Section 3.10.

We end this section by setting some basic notations. For the rest of the paper the symbol c denotes a positive constant depending on the all previously introduced constants. Sometimes, to stress the dependence of the constants on some parameters α, β, \dots we shall use the notations $c_\alpha, c_{\alpha, \beta}, \dots$. All these constants are likely to change their values every occurrence. For any real numbers u and v , denote by $u \wedge v = \min(u, v)$ the minimum between u and v . The indicator of an event A is denoted by $\mathbb{1}_A$. For any bounded measurable function f on \mathbb{X} , random variable X in \mathbb{X} and event A , the integral $\int_{\mathbb{X}} f(x)\mathbb{P}(X \in dx, A)$ means the expectation $\mathbb{E}(f(X); A) = \mathbb{E}(f(X)\mathbb{1}_A)$.

3.2 Main results

Let $(X_n)_{n \geq 0}$ be a Markov chain taking values in the measurable state space $(\mathbb{X}, \mathcal{X})$, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For any given $x \in \mathbb{X}$, denote by $\mathbf{P}(x, \cdot)$ its transition probability, to which we associate the transition operator

$$\mathbf{P}g(x) = \int_{\mathbb{X}} g(x')\mathbf{P}(x, dx'),$$

for any complex bounded measurable function g on \mathbb{X} . Denote by \mathbb{P}_x and \mathbb{E}_x the probability and the corresponding expectation generated by the finite dimensional distributions of the Markov chain $(X_n)_{n \geq 0}$ starting at $X_0 = x$. We remark that $\mathbf{P}g(x) = \mathbb{E}_x(g(X_1))$ and $\mathbf{P}^n g(x) = \mathbb{E}_x(g(X_n))$ for any g complex bounded measurable, $x \in \mathbb{X}$ and $n \geq 1$.

Let f be a real valued function defined on the state space \mathbb{X} and let \mathcal{B} be a Banach space of complex valued functions on \mathbb{X} endowed with the norm $\|\cdot\|_{\mathcal{B}}$. Let $\|\cdot\|_{\mathcal{B} \rightarrow \mathcal{B}}$ be the operator norm on \mathcal{B} and let $\mathcal{B}' = \mathcal{L}(\mathcal{B}, \mathbb{C})$ be the topological dual of \mathcal{B} endowed with the norm $\|\varphi\|_{\mathcal{B}'} = \sup_{h \in \mathcal{B}} \frac{|\varphi(h)|}{\|h\|_{\mathcal{B}}}$, for any $\varphi \in \mathcal{B}'$. Denote by e the unit function of \mathbb{X} : $e(x) = 1$, for any $x \in \mathbb{X}$ and by δ_x the Dirac measure at $x \in \mathbb{X}$: $\delta_x(g) = g(x)$, for any $g \in \mathcal{B}$.

Following [40], we assume the following hypotheses.

Hypothesis M3.1 (Banach space).

1. The unit function e belongs to \mathcal{B} .
2. For any $x \in \mathbb{X}$, the Dirac measure δ_x belongs to \mathcal{B}' .
3. The Banach space \mathcal{B} is included in $L^1(\mathbf{P}(x, \cdot))$, for any $x \in \mathbb{X}$.
4. There exists a constant $\kappa \in (0, 1)$ such that for any $g \in \mathcal{B}$, the function $e^{itf} g$ is in \mathcal{B} for any t satisfying $|t| \leq \kappa$.

Under the point 3 of **M3.1**, $\mathbf{P}g(x)$ exists for any $g \in \mathcal{B}$ and $x \in \mathbb{X}$.

Hypothesis M3.2 (Spectral gap).

1. The map $g \mapsto \mathbf{P}g$ is a bounded operator on \mathcal{B} .
2. There exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$\mathbf{P} = \Pi + Q,$$

where Π is a one-dimensional projector and Q is an operator on \mathcal{B} satisfying $\Pi Q = Q\Pi = 0$ and for any $n \geq 1$,

$$\|Q^n\|_{\mathcal{B} \rightarrow \mathcal{B}} \leq c_1 e^{-c_2 n}.$$

Since Π is a one-dimensional projector and e is an eigenvector of \mathbf{P} , there exists a linear form $\nu \in \mathcal{B}'$, such that for any $g \in \mathcal{B}$,

$$\Pi g = \nu(g)e. \tag{3.2.1}$$

When Hypotheses **M3.1** and **M3.2** hold, we set $\mathbf{P}_t g := \mathbf{P}(e^{itf} g)$ for any $g \in \mathcal{B}$ and $t \in [-\kappa, \kappa]$. In particular $\mathbf{P}_0 = \mathbf{P}$.

Hypothesis M3.3 (Perturbed transition operator).

1. For any $|t| \leq \kappa$ the map $g \mapsto \mathbf{P}_t g$ is a bounded operator on \mathcal{B} .
2. There exists a constant $C_{\mathbf{P}} > 0$ such that, for any $n \geq 1$ and $|t| \leq \kappa$,

$$\|\mathbf{P}_t^n\|_{\mathcal{B} \rightarrow \mathcal{B}} \leq C_{\mathbf{P}}.$$

The following hypothesis will be important for establishing the main results.

Hypothesis M3.4 (Local integrability). *The Banach space \mathcal{B} contains a sequence of real non-negative functions N, N_1, N_2, \dots such that:*

1. There exist $\alpha > 2$ and $\gamma > 0$ such that, for any $x \in \mathbb{X}$,

$$\max \left\{ |f(x)|^{1+\gamma}, \|\delta_x\|_{\mathcal{B}'}, \mathbb{E}_x^{1/\alpha} (N(X_n)^\alpha) \right\} \leq c(1 + N(x))$$

and

$$N(x) \mathbb{1}_{\{N(x) > l\}} \leq N_l(x), \quad \text{for any } l \geq 1.$$

2. There exists $c > 0$ such that, for any $l \geq 1$,

$$\|N_l\|_{\mathcal{B}} \leq c.$$

3. There exist $\beta > 0$ and $c > 0$ such that, for any $l \geq 1$,

$$|\nu(N_l)| \leq \frac{c}{l^{1+\beta}}.$$

A comment on Hypothesis **M3.4** seems to be appropriate. Although the function N belongs to the Banach space \mathcal{B} , the truncated function $x \mapsto N(x)\mathbb{1}_{\{N(x)>l\}}$ may not belong to \mathcal{B} . Fortunately, in many interesting cases, there exists an element N_l in \mathcal{B} dominating it. We refer to Section 3.3, where we verify Hypothesis **M3.4** for stochastic recursions in \mathbb{R}^d and for Markov chains with compact state space. Note also that the function f need not belong to the Banach space \mathcal{B} .

Under Hypotheses **M3.1**, **M3.2** and **M3.4**, we have, for any $x \in \mathbb{X}$ and $n \geq 1$,

$$\begin{aligned} \mathbb{E}_x(N(X_n)) &= \nu(N) + Q^n N(x) \\ &\leq |\nu(N)| + \|Q^n\|_{\mathcal{B} \rightarrow \mathcal{B}} \|N\|_{\mathcal{B}} \|\delta_x\|_{\mathcal{B}'} \\ &\leq c(1 + e^{-cn} N(x)) \end{aligned} \quad (3.2.2)$$

and, in the same way, for any $x \in \mathbb{X}$, $l \geq 1$ and $n \geq 1$,

$$\mathbb{E}_x(N_l(X_n)) \leq \frac{c}{l^{1+\beta}} + c e^{-cn} (1 + N(x)). \quad (3.2.3)$$

Moreover, from the point 1 of **M3.4**, one can easily verify that, for any $x \in \mathbb{X}$,

$$\mu_\alpha(x) := \sup_{n \geq 1} \mathbb{E}_x^{1/\alpha}(|f(X_n)|^\alpha) \leq c \left(1 + N(x)^{\frac{1}{1+\gamma}}\right). \quad (3.2.4)$$

The following proposition is proved in [40], where the bounds on the right follow from (3.2.4) and again **M3.4**.

Proposition 3.2.1. *Assume that the Markov chain $(X_n)_{n \geq 0}$ and the function f satisfy Hypotheses **M3.1-M3.4**.*

1. *There exists a constant μ such that, for any $x \in \mathbb{X}$ and $n \geq 1$,*

$$|\mathbb{E}_x(f(X_n)) - \mu| \leq c e^{-cn} \left(1 + \mu_\alpha(x)^{1+\gamma} + \|\delta_x\|_{\mathcal{B}'}\right) \leq c e^{-cn} (1 + N(x)).$$

2. *There exists a constant $\sigma \geq 0$ such that, for any $x \in \mathbb{X}$ and $n \geq 1$,*

$$\sup_{m \geq 0} \left| \text{Var}_x \left(\sum_{k=m+1}^{m+n} f(X_k) \right) - n\sigma^2 \right| \leq c \left(1 + \mu_\alpha(x)^{2+2\gamma} + \|\delta_x\|_{\mathcal{B}'}\right) \leq c \left(1 + N(x)^2\right),$$

where Var_x is the variance under \mathbb{P}_x .

We do not assume the existence of the stationary probability measure. If a stationary probability measure ν' satisfying $\nu'(N^2) < +\infty$ exists then, under Hypotheses **M3.1-M3.4**, we have that $\nu' = \nu$ is necessarily unique and it holds (see [40])

$$\nu(f) = \mu \quad \text{and} \quad \sigma^2 = \nu(f^2) - \nu(f)^2 + 2 \sum_{n=1}^{+\infty} \left[\nu(fP^n f) - \nu(f)^2 \right]. \quad (3.2.5)$$

Hypothesis M3.5 (Centring and non-degeneracy). *We suppose that the constants μ and σ defined in Proposition 3.2.1 satisfy $\mu = 0$ and $\sigma > 0$.*

Under **M3.5** it follows from Proposition **3.2.1** that, for any $x \in \mathbb{X}$ and $n \geq 1$,

$$|\mathbb{E}_x(f(X_n))| \leq c e^{-cn} (1 + N(x)). \quad (3.2.6)$$

Let $y \in \mathbb{R}$ be a starting point and $(y + S_n)_{n \geq 0}$ be the Markov walk defined by $S_n := \sum_{k=1}^n f(X_k)$, $n \geq 1$ with $S_0 = 0$. Denote by τ_y the first moment when $y + S_n$ becomes non-positive:

$$\tau_y := \inf \{k \geq 1 : y + S_k \leq 0\}.$$

It is shown in Lemma **3.5.5** that for any $y \in \mathbb{R}$ and $x \in \mathbb{X}$, the stopping time τ_y is \mathbb{P}_x -a.s. finite. The asymptotic behaviour of the probability $\mathbb{P}_x(\tau_y > n)$ is determined by the harmonic function which we proceed to introduce. For any $(x, y) \in \mathbb{X} \times \mathbb{R}$, denote by $\mathbf{Q}(x, y, \cdot)$ the transition probability of the Markov chain $(X_n, y + S_n)_{n \geq 0}$. The restriction of the measure $\mathbf{Q}(x, y, \cdot)$ on $\mathbb{X} \times \mathbb{R}_+^*$ is defined by

$$\mathbf{Q}_+(x, y, B) = \mathbf{Q}(x, y, B)$$

for any measurable set B on $\mathbb{X} \times \mathbb{R}_+^*$ and for any $(x, y) \in \mathbb{X} \times \mathbb{R}$. For any bounded measurable function $\varphi : \mathbb{X} \times \mathbb{R} \rightarrow \mathbb{R}$ set $\mathbf{Q}_+\varphi(x, y) = \int_{\mathbb{X} \times \mathbb{R}_+^*} \varphi(x', y') \mathbf{Q}_+(x, y, dx' \times dy')$, where $(x, y) \in \mathbb{X} \times \mathbb{R}$. A function $V : \mathbb{X} \times \mathbb{R} \rightarrow \mathbb{R}$ is said to be \mathbf{Q}_+ -harmonic if

$$\mathbf{Q}_+V(x, y) = V(x, y), \quad \text{for any } (x, y) \in \mathbb{X} \times \mathbb{R}.$$

We shall deal only with non-negative harmonic functions V . Denote by $\text{supp}(V)$ the support of such a function V ,

$$\text{supp}(V) := \{(x, y) \in \mathbb{X} \times \mathbb{R} : V(x, y) > 0\}.$$

On the complement of $\text{supp}(V)$, the function V is 0. For any $\gamma > 0$, consider the set

$$\mathcal{D}_\gamma := \{(x, y) \in \mathbb{X} \times \mathbb{R} : \exists n_0 \geq 1, \mathbb{P}_x(y + S_{n_0} > \gamma(1 + N(X_{n_0})), \tau_y > n_0) > 0\}.$$

The following assertion proves the existence of a non-identically zero harmonic function.

Theorem 3.2.2. *Assume Hypotheses **M3.1-M3.5**.*

1. For any $x \in \mathbb{X}$, $y \in \mathbb{R}$, the sequence $(\mathbb{E}_x(y + S_n; \tau_y > n))_{n \geq 0}$ converges to a real number $V(x, y)$:

$$\mathbb{E}_x(y + S_n; \tau_y > n) \xrightarrow{n \rightarrow +\infty} V(x, y).$$

2. The function $V : \mathbb{X} \times \mathbb{R} \rightarrow \mathbb{R}$, defined in the previous point is \mathbf{Q}_+ -harmonic, i.e. for any $x \in \mathbb{X}$, $y \in \mathbb{R}$,

$$\mathbf{Q}_+V(x, y) = \mathbb{E}_x(V(X_1, y + S_1); \tau_y > 1) = V(x, y).$$

3. For any $x \in \mathbb{X}$, the function $V(x, \cdot)$ is non-negative and non-decreasing on \mathbb{R} and

$$\lim_{y \rightarrow +\infty} \frac{V(x, y)}{y} = 1.$$

Moreover, for any $\delta > 0$, $x \in \mathbb{X}$ and $y \in \mathbb{R}$,

$$(1 - \delta) \max(y, 0) - c_\delta (1 + N(x)) \leq V(x, y) \leq (1 + \delta) \max(y, 0) + c_\delta (1 + N(x)).$$

4. There exists $\gamma_0 > 0$ such that, for any $\gamma \geq \gamma_0$,

$$\text{supp}(V) = \mathcal{D}_\gamma.$$

The following result gives the asymptotic of the exit probability for fixed $(x, y) \in \mathbb{X} \times \mathbb{R}$.

Theorem 3.2.3. *Assume Hypotheses M3.1-M3.5.*

1. For any $(x, y) \in \text{supp}(V)$,

$$\mathbb{P}_x(\tau_y > n) \underset{n \rightarrow +\infty}{\sim} \frac{2V(x, y)}{\sqrt{2\pi n\sigma}}.$$

2. For any $(x, y) \notin \text{supp}(V)$ and $n \geq 1$,

$$\mathbb{P}_x(\tau_y > n) \leq c e^{-cn} (1 + N(x)).$$

Now we complete the point 1 of the previous theorem by some estimations.

Theorem 3.2.4. *Assume Hypotheses M3.1-M3.5.*

1. There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $n \geq 1$ and $(x, y) \in \mathbb{X} \times \mathbb{R}$,

$$\left| \mathbb{P}_x(\tau_y > n) - \frac{2V(x, y)}{\sqrt{2\pi n\sigma}} \right| \leq c_\varepsilon \frac{\max(y, 0) + \left(1 + y \mathbb{1}_{\{y > n^{1/2-\varepsilon}\}} + N(x)\right)^2}{n^{1/2+\varepsilon/16}}.$$

2. Moreover, for any $(x, y) \in \mathbb{X} \times \mathbb{R}$ and $n \geq 1$,

$$\mathbb{P}_x(\tau_y > n) \leq c \frac{1 + \max(y, 0) + N(x)}{\sqrt{n}}.$$

Finally, we give the asymptotic of the conditional law of $y + S_n$.

Theorem 3.2.5. *Assume Hypotheses M3.1-M3.5.*

1. For any $(x, y) \in \text{supp}(V)$ and $t \geq 0$,

$$\mathbb{P}_x \left(\frac{y + S_n}{\sigma\sqrt{n}} \leq t \mid \tau_y > n \right) \xrightarrow{n \rightarrow +\infty} \Phi^+(t),$$

where $\Phi^+(t) = 1 - e^{-\frac{t^2}{2}}$ is the Rayleigh distribution function.

2. Moreover there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $n \geq 1$, $t_0 > 0$, $t \in [0, t_0]$ and $(x, y) \in \mathbb{X} \times \mathbb{R}$,

$$\begin{aligned} \left| \mathbb{P}_x \left(y + S_n \leq t\sqrt{n}, \tau_y > n \right) - \frac{2V(x, y)}{\sqrt{2\pi n\sigma}} \Phi^+ \left(\frac{t}{\sigma} \right) \right| \\ \leq c_{\varepsilon, t_0} \frac{\max(y, 0) + \left(1 + y \mathbb{1}_{\{y > n^{1/2-\varepsilon}\}} + N(x)\right)^2}{n^{1/2+\varepsilon/16}}. \end{aligned}$$

We now comment on Theorems 3.2.2 and 3.2.3.

Remark 3.2.6. If we assume that there exist $\delta > 0$ and $M > 0$ such that for any $x \in \mathbb{X}$, $\mathbb{P}_x(f(X_1) > \delta, N(X_1) \leq M) > 0$, then one can see that the set $\mathbb{X} \times [0, +\infty)$ is included in $\text{supp}(V)$.

Remark 3.2.7. The sets $(\mathcal{D}_\gamma)_{\gamma>0}$ are nested and become equal to $\text{supp}(V)$ for large γ : we have $\mathcal{D}_{\gamma_1} \supseteq \mathcal{D}_{\gamma_2} \supseteq \mathcal{D}_\gamma = \text{supp}(V)$, for $\gamma_1 \leq \gamma_2 \leq \gamma$, where γ is large enough (see Proposition 3.8.8).

Remark 3.2.8. The set $\text{supp}(V)$ is not empty. More precisely there exists $\gamma_1 > 0$ such that

$$\{(x, y) \in \mathbb{X} \times \mathbb{R} : y > \gamma_1 (1 + N(x))\} \subseteq \text{supp}(V),$$

see Proposition 3.8.8. Example 3.2.11 and Figure 3.1 illustrate this property.

Remark 3.2.9. When $(X_n)_{n \geq 1}$ are i.i.d., it is well known that $\mathbb{P}_x(\tau_y > n) = 0$ for any $(x, y) \notin \text{supp}(V)$. When the sequence $(X_n)_{n \geq 1}$ is a Markov chain, instead of this, we have an exponential bound, see the point 2 of Theorem 3.2.3. We show that this bound is attained for some Markov walk. We refer for details to Example 3.2.12.

Example 3.2.10 (Random walks in \mathbb{R}). Suppose that $(X_n)_{n \geq 1}$ are i.i.d. real random variables of mean 0 and positive variance with finite absolute moments of order $p > 2$. In this case, one can take $N = N_l = 0$, $l \geq 0$. Therefore,

$$\mathcal{D}_\gamma := \{y \in \mathbb{R} : \exists n_0 \geq 1, \mathbb{P}(y + S_{n_0} > \gamma, \tau_y > n_0) > 0\}.$$

Since the walk $(y + S_n)_{n \geq 1}$ can increase at each step with positive probability, it follows that $\mathbb{P}(y + S_{n_0} > \gamma, \tau_y > n_0) > 0$ if and only if $\mathbb{P}(\tau_y > 1) = \mathbb{P}(y + X_1 > 0) > 0$. Thus, $[0, +\infty) \subseteq (-\max \text{supp}(\boldsymbol{\mu}), +\infty) = \mathcal{D}_\gamma = \text{supp}(V)$, for every $\gamma > 0$, where $\boldsymbol{\mu}$ is the common law of X_n and $\text{supp}(\boldsymbol{\mu})$ is its support.

The following example is intended to illustrate Remark 3.2.8.

Example 3.2.11. Consider the following special case of the one dimensional stochastic recursion: $X_{n+1} = a_{n+1}X_n + b_{n+1}$ where $(a_i)_{i \geq 1}$ and $(b_i)_{i \geq 1}$ are two independent sequences of i.i.d. random variables. In this example we consider that the law of a_i is $\frac{1}{2}\boldsymbol{\delta}_{\{-1/2\}} + \frac{1}{2}\boldsymbol{\delta}_{\{1/2\}}$ and that of b_i is uniform on $[-1, 1]$. The state space \mathbb{X} is \mathbb{R} . The functions N and N_l are given by $N(x) = |x|^{1+\varepsilon}$ for some $\varepsilon > 0$, and $N_l(x) = N(x)\phi_l(|x|)$ with ϕ_l defined by (3.11.4). The Banach space satisfying M3.1-M3.5 is constructed in Section 3.11 (see also [36]/Chapter 2). One can verify that the domain of positivity of the function V is $\text{supp}(V) = \{(x, y) \in \mathbb{R}^2 : y > -\frac{|x|}{2} - 1\} = \mathcal{D}_\gamma$, for all $\gamma > 0$. Obviously, $\{(x, y) \in \mathbb{X} \times \mathbb{R} : y > \frac{1}{2}(1 + |x|^{1+\varepsilon})\} \subseteq \text{supp}(V)$, see Figure 3.1.

The next example is intended to show that the inequality of the point 2 of Theorem 3.2.3 is attained.

Example 3.2.12. Consider the Markov walk $(X_n)_{n \geq 0}$ living on the finite state space $\mathbb{X} := \{-1; 1; -3; 7/6\}$ with the transition probabilities given in Figure 3.2. Suppose that f is the identity function on \mathbb{X} . It is easy to see that the assumptions stated in Remark 3.3.10 of Section 3.3.3 are satisfied and thereby so are Hypotheses M3.1-M3.5. In particular, M3.4 holds with $N = N_l = 0$ for any $l \geq 1$. Now, when $x = 1$ and $y \in (1, 3]$ or when $x = -1$ and $y \in (-1, 2]$, one can check that the Markov walk $y + S_n$ stays positive if and only if the values of the variables X_i alternate between 1 and -1 and therefore, for such starting points (x, y) , we have $\mathbb{P}_x(\tau_y > n) = \left(\frac{1}{2}\right)^n$. This shows that, when the random variables $(X_n)_{n \geq 1}$ form a Markov chain, the survival probability $\mathbb{P}_x(\tau_y > n)$ has an asymptotic behaviour different from that in the independent case where it can be either equivalent to $\frac{c_{x,y}}{\sqrt{n}}$ or 0.

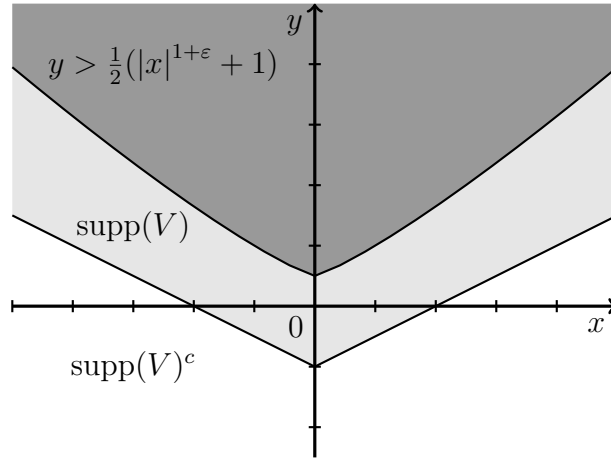


Figure 3.1 –

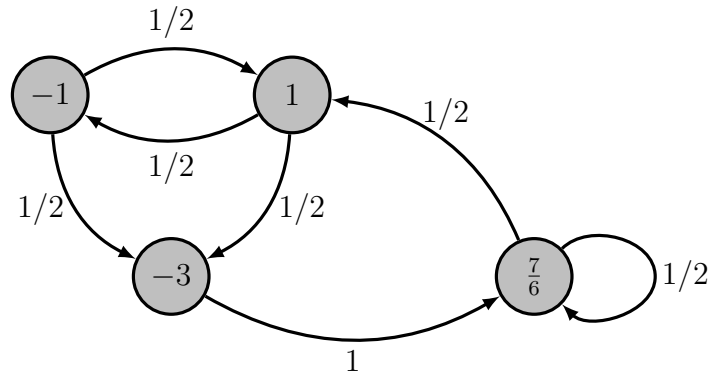


Figure 3.2 –

In this example we can make explicit the support of V . Since $N = 0$, the function V is positive if and only if there exists an integer $n \geq 1$ such that $\mathbb{P}_x(y + S_n > \gamma, \tau_y > n) > 0$ for a γ large enough. This is possible only if the chain can reach the state $X_n = 7/6$ within a trajectory of $(y + S_k)_{n \geq k \geq 1}$ which stays positive, *i.e.* $\mathbb{P}_x(X_n = 7/6, \tau_y > n) > 0$. Consequently

$$\begin{aligned} \text{supp}(V) &= \{-1\} \times (2, +\infty) \cup \{1\} \times (3, +\infty) \cup \{-3, 7/6\} \times (-7/6, +\infty) \\ &= \mathcal{D}_3 = \{(x, y) \in \mathbb{X} \times \mathbb{R} : \exists n \geq 1, \mathbb{P}_x(y + S_n > 3, \tau_y > n) > 0\}. \end{aligned}$$

To sum up, this model presents the three possible asymptotic behaviours of the probability $\mathbb{P}_x(\tau_y > n)$: for any $(x, y) \in \text{supp}(V) = \{-1\} \times (2, +\infty) \cup \{1\} \times (3, +\infty) \cup \{-3, 7/6\} \times (-7/6, +\infty)$,

$$\mathbb{P}_x(\tau_y > n) \underset{n \rightarrow +\infty}{\sim} \frac{2V(x, y)}{\sqrt{2\pi n\sigma}},$$

for any $(x, y) \in \{-1\} \times (-1, 2] \cup \{1\} \times (1, 3]$ and $n \geq 1$,

$$\mathbb{P}_x(\tau_y > n) = \left(\frac{1}{2}\right)^n,$$

for any $(x, y) \in \{-1\} \times (-\infty, -1] \cup \{1\} \times (-\infty, 1] \cup \{-3, 7/6\} \times (-\infty, -7/6]$ and $n \geq 1$,

$$\mathbb{P}_x(\tau_y > n) = 0.$$

3.3 Applications

We illustrate the results of Section 3.2 by considering three particular models.

3.3.1 Affine random walk in \mathbb{R}^d conditioned to stay in a half-space

Let $d \geq 1$ be an integer and $(g_n)_{n \geq 1} = (A_n, B_n)_{n \geq 1}$ be a sequence of i.i.d. random elements in $\text{GL}(d, \mathbb{R}) \times \mathbb{R}^d$ following the same distribution μ . Let $(X_n)_{n \geq 0}$ be the Markov chain on \mathbb{R}^d defined by

$$X_0 = x \in \mathbb{R}^d, \quad X_{n+1} = A_{n+1}X_n + B_{n+1}, \quad n \geq 1.$$

Set $S_n = \sum_{k=1}^n f(X_k)$, $n \geq 1$, where the function $f(x) = \langle u, x \rangle$ is the projection of the vector $x \in \mathbb{R}^d$ on the direction defined by the vector $u \in \mathbb{R}^d \setminus \{0\}$. For any $y \in \mathbb{R}$, consider the first time when the random walk $(y + S_n)_{n \geq 1}$ becomes non-positive:

$$\tau_y = \inf\{k \geq 1 : y + S_k \leq 0\}.$$

This stopping time coincides with the entry time of the affine walk $(\sum_{k=1}^n X_k)_{n \geq 0}$ in the closed half-subspace $\{s \in \mathbb{R}^d : \langle u, s \rangle \leq -y\}$.

Introduce the following hypothesis.

Hypothesis 3.3.1.

1. There exists a constant $\delta > 0$, such that

$$\mathbb{E}(\|A_1\|^{2+2\delta}) < +\infty, \quad \mathbb{E}(|B_1|^{2+2\delta}) < +\infty$$

and

$$k(\delta) = \lim_{n \rightarrow +\infty} \mathbb{E}^{1/n}(\|A_n A_{n-1} \dots A_1\|^{2+2\delta}) < 1.$$

2. There is no proper affine subspace of \mathbb{R}^d which is invariant with respect to all the elements of the support of μ .

3. For any vector $v_0 \in \mathbb{R}^d \setminus \{0\}$,

$$\mathbb{P}({}^t A_1^{-1} v_0 = {}^t A_2^{-1} v_0) < 1,$$

where ${}^t A$ is the transpose of A , for any $A \in \text{GL}(d, \mathbb{R})$.

4. The vector B_1 is centred: $\mathbb{E}(B_1) = 0$.

Proposition 3.3.2. Under Hypothesis 3.3.1, Theorems 3.2.2–3.2.5 hold true.

Proposition 3.3.2 is proved in Appendix 3.11 where we construct an appropriate Banach space \mathcal{B} and show that Hypotheses M3.1–M3.5 are satisfied with $N(x) = |x|^{1+\varepsilon}$, for some $\varepsilon > 0$ and with $N_l(x) = N(x)\phi_l(|x|)$, where ϕ_l is defined by (3.11.4).

Remark 3.3.3. The set $\text{supp}(V)$ depends on the law of (A_i, B_i) . In the case when A_i are independent of B_i and the support of the law of $\langle u, B_i \rangle$ contains a sequence converging to $+\infty$, one can verify that $\text{supp}(V) = \mathbb{R}^d \times \mathbb{R}$.

3.3.2 Two components Markov chains in compact sets under the Doeblin-Fortet condition

Let (X, d_X) be a compact metric space, $\mathcal{C}(X)$ and $\mathcal{L}(X)$ be the spaces of continuous and Lipschitz complex functions on X , respectively. Define

$$|h|_\infty = \sup_{x \in X} |h(x)|, \quad \forall h \in \mathcal{C}(X)$$

and

$$[h]_X = \sup_{\substack{(x,y) \in X \\ x \neq y}} \frac{|h(x) - h(y)|}{d_X(x, y)}, \quad \forall h \in \mathcal{L}(X).$$

We endow $\mathcal{C}(X)$ with the uniform norm $|\cdot|_\infty$ and $\mathcal{L}(X)$ with the norm $|\cdot|_{\mathcal{L}} = |\cdot|_\infty + [\cdot]_X$, respectively. Consider the space $\mathbb{X} := X \times X$ with the metric $d_{\mathbb{X}}$ on \mathbb{X} defined by $d_{\mathbb{X}}((x_1, x_2), (y_1, y_2)) = d_X(x_1, y_1) + d_X(x_2, y_2)$, for any (x_1, x_2) and (y_1, y_2) in \mathbb{X} . Denote by $\mathcal{L}(\mathbb{X})$ the space of the Lipschitz complex function on \mathbb{X} endowed with the norm $\|\cdot\|_{\mathcal{L}} = \|\cdot\|_\infty + [\cdot]_{\mathbb{X}}$, where

$$\|h\|_\infty = \sup_{x \in \mathbb{X}} |h(x)|, \quad \forall h \in \mathcal{C}(\mathbb{X})$$

and

$$[h]_{\mathbb{X}} = \sup_{\substack{(x,y) \in \mathbb{X} \\ x \neq y}} \frac{|h(x) - h(y)|}{d_{\mathbb{X}}(x, y)}, \quad \forall h \in \mathcal{L}(\mathbb{X}).$$

Following Guivarc'h and Hardy [42], consider a Markov chain $(\chi_n)_{n \geq 0}$ on X with transition probability P . Let $(X_n)_{n \geq 0}$ be the Markov chain on \mathbb{X} defined by $X_n = (\chi_{n-1}, \chi_n)$, $n \geq 1$ and $X_0 = (0, \chi_0)$: its transition probability is given by

$$\mathbf{P}((x_1, x_2), dy_1 \times dy_2) = \delta_{x_2}(dy_1) P(x_2, dy_2).$$

For a fixed real function f on \mathbb{X} , let $S_n := \sum_{k=1}^n f(X_k)$ be the associated Markov walk and, for any $y \in \mathbb{R}$, let $\tau_y := \inf \{n \geq 1 : y + S_n \leq 0\}$ be the associated exit time.

In order to apply the results stated in the previous section, we need some hypotheses on the function f and the operator P on $\mathcal{C}(X)$ defined by $Ph(x) = \int_X h(y)P(x, dy)$ for any $x \in X$ and any $h \in \mathcal{C}(X)$.

Hypothesis 3.3.4.

1. For any h in $\mathcal{C}(X)$, respectively in $\mathcal{L}(X)$, the function Ph is an element of $\mathcal{C}(X)$, respectively of $\mathcal{L}(X)$.
2. There exist constants $n_0 \geq 1$, $0 < \rho < 1$ and $C > 0$ such that, for any function $h \in \mathcal{L}(X)$, we have

$$|P^{n_0} h|_{\mathcal{L}} \leq \rho |h|_{\mathcal{L}} + C |h|_\infty$$

3. The unique eigenvalue of P of modulus 1 is 1 and the associated eigenspace is generated by the function $e: x \mapsto 1$, i.e. if there exist $\theta \in \mathbb{R}$ and $h \in \mathcal{L}(X)$ such that $Ph = e^{i\theta} h$, then h is constant and $e^{i\theta} = 1$.

Under Hypothesis 3.3.4, one can check that conditions (a), (b), (c) and (d) of Chapter 3 in Norman [59] hold true and we can apply the theorem of Ionescu Tulcea and Marinescu [49] (see also [42]). Coupling this theorem with the point 3 of Hypothesis 3.3.4 we obtain the following proposition.

Proposition 3.3.5.

1. There exists a unique P -invariant probability ν on X .
2. For any $n \geq 1$ and $h \in \mathcal{L}(X)$,

$$P^n h = \nu(h) + R^n h,$$

where R is an operator on $\mathcal{L}(X)$ with a spectral radius $r(R) < 1$.

Suppose that f and ν satisfy the following hypothesis.

Hypothesis 3.3.6.

1. The function f belongs to $\mathcal{L}(\mathbb{X})$.
2. The function f is centred, in the sense that

$$\int_{\mathbb{X}} f(x, y) P(x, dy) \nu(dx) = 0.$$

3. The function f is non-degenerated, that means that there is no function $h \in \mathcal{L}(X)$ such that

$$f(x, y) = h(x) - h(y),$$

for P_ν -almost all (x, y) , where $P_\nu(dx \times dy) = P(x, dy) \nu(dx)$.

Assuming Hypotheses 3.3.4 and 3.3.6, Guivarc'h and Hardy [42] have established that the sequence $(S_n/\sqrt{n})_{n \geq 1}$ converges weakly to a centred Gaussian random variable of variance $\sigma^2 > 0$, under the probability \mathbb{P}_x generated by the finite dimensional distributions of the Markov chain $(X_n)_{n \geq 0}$ starting at $X_0 = x$, for any $x \in X$. Moreover, under the same hypotheses, we show in Appendix 3.12 that **M3.1-M3.5** are satisfied with $N = N_l = 0$, thereby proving the following assertion.

Proposition 3.3.7. *Under Hypotheses 3.3.4 and 3.3.6, Theorems 3.2.2–3.2.5 hold true.*

3.3.3 Markov chains in compact sets under spectral gap assumptions

In this section we give sufficient conditions in order that a Markov chain with values in a compact set satisfy conditions **M3.1-M3.5**.

Let (\mathbb{X}, d) be a compact metric space and $(X_n)_{n \geq 0}$ be a Markov chain living in \mathbb{X} . Denote by \mathbf{P} the transition probability of $(X_n)_{n \geq 0}$ and by $\mathcal{C}(\mathbb{X})$ the Banach algebra of the continuous complex functions on \mathbb{X} endowed with the uniform norm

$$|h|_\infty = \sup_{x \in \mathbb{X}} |h(x)|, \quad h \in \mathcal{C}(\mathbb{X}).$$

Consider a real function f defined on \mathbb{X} , the transition operator \mathbf{P} on $\mathcal{C}(\mathbb{X})$ associated to the transition probability of $(X_n)_{n \geq 0}$ and the unit function e defined on \mathbb{X} by $e(x) = 1$, for any $x \in \mathbb{X}$.

Hypothesis 3.3.8.

1. For any $h \in \mathcal{C}(\mathbb{X})$, the function $\mathbf{P}h$ is an element of $\mathcal{C}(\mathbb{X})$.
2. The operator \mathbf{P} has a unique invariant probability ν .

3. For any $n \geq 1$,

$$\mathbf{P}^n = \Pi + Q^n,$$

where Π is the one-dimensional projector on $\mathcal{C}(\mathbb{X})$ defined by $\Pi(h) = \nu(h)e$, for any $h \in \mathcal{C}(\mathbb{X})$, Q is an operator on $\mathcal{C}(\mathbb{X})$ of spectral radius $r(Q) < 1$ satisfying $\Pi Q = Q\Pi = 0$.

4. The function f belongs to $\mathcal{C}(\mathbb{X})$ and is ν -centred, i.e. $\nu(f) = 0$.

5. The function f is non-degenerated, that is there is no function $h \in \mathcal{C}(\mathbb{X})$ such that

$$f(X_1) = h(X_0) - h(X_1), \quad \mathbb{P}_\nu\text{-a.s.},$$

where \mathbb{P}_ν is the probability generated by the finite dimensional distributions of the Markov chain $(X_n)_{n \geq 0}$ when the initial law of X_0 is ν .

Consider the Markov walk $S_n = \sum_{k=1}^n f(X_k)$. It is well known, that under Hypothesis 3.3.8 the normalized sum S_n/\sqrt{n} converges in law to a centred normal distribution of variance $\sigma^2 > 0$ with respect to the probability \mathbb{P}_x generated by the finite dimensional distributions of the Markov chain $(X_n)_{n \geq 0}$ starting at $X_0 = x$, for any $x \in \mathbb{X}$.

Proposition 3.3.9. *Under Hypothesis 3.3.8, Theorems 3.2.2–3.2.5 hold true.*

All the elements of the proof are contained in the proof of Proposition 3.3.7 (see Appendix 3.12), which therefore is left to the reader. In particular Hypothesis M3.4 holds with $N = N_l = 0$.

Remark 3.3.10. As a special example of the compact case, consider the Markov chain $(X_n)_{n \geq 1}$ taking values in a finite space \mathbb{X} . Assume that $(X_n)_{n \geq 1}$ is aperiodic and irreducible with transition matrix \mathbf{P} . Let f be a finite function on \mathbb{X} . We shall verify Hypothesis 3.3.8. The Banach space \mathcal{B} consists of all finite real functions on \mathbb{X} , therefore condition 1 is obvious. Moreover, there is a unique invariant measure ν , which proves condition 2. According to Perron-Frobenius theorem, the transition matrix \mathbf{P} admits 1 as the only simple eigenvalue of modulus 1, which implies condition 3. Assume in addition that $\nu(f) = 0$ (which is condition 4) and that there exists a path x_0, \dots, x_n in \mathbb{X} such that $\mathbf{P}(x_0, x_1) > 0, \dots, \mathbf{P}(x_{n-1}, x_n) > 0, \mathbf{P}(x_n, x_0) > 0$ and $f(x_0) + \dots + f(x_n) \neq 0$ (which implies condition 5). Thus all the conclusions of Theorems 3.2.2–3.2.5 hold true.

3.4 Preliminary statements

3.4.1 Results for the Brownian motion

Let $(B_t)_{t \geq 0}$ be the standard Brownian motion with values in \mathbb{R} living on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define the exit time

$$\tau_y^{bm} = \inf\{t \geq 0 : y + \sigma B_t \leq 0\}, \tag{3.4.1}$$

where $\sigma > 0$. The following affirmations are due to Lévy [57].

Lemma 3.4.1. *For any $y > 0$, $0 \leq a \leq b$ and $n \geq 1$,*

$$\mathbb{P}\left(\tau_y^{bm} > n, y + \sigma B_n \in [a, b]\right) = \frac{1}{\sqrt{2\pi n\sigma}} \int_a^b \left(e^{-\frac{(s-y)^2}{2n\sigma^2}} - e^{-\frac{(s+y)^2}{2n\sigma^2}}\right) ds.$$

Lemma 3.4.2.

1. For any $y > 0$,

$$\mathbb{P}\left(\tau_y^{bm} > n\right) \leq c \frac{y}{\sqrt{n}}.$$

2. For any sequence of real numbers $(\theta_n)_{n \geq 0}$ such that $\theta_n \xrightarrow[n \rightarrow +\infty]{} 0$,

$$\sup_{y \in [0; \theta_n \sqrt{n}]} \left(\frac{\mathbb{P}\left(\tau_y^{bm} > n\right)}{\frac{2y}{\sqrt{2\pi n \sigma}}} - 1 \right) = O(\theta_n^2).$$

3.4.2 Strong approximation

Under hypotheses **M3.1-M3.5** it is proved in [40] that there is a version of the Markov walk $(S_n)_{n \geq 0}$ and of the standard Brownian motion $(B_t)_{t \geq 0}$ living on the same probability space which are close enough in the following sense:

Proposition 3.4.3. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0]$, without loss of generality one can reconstruct the sequence $(S_n)_{n \geq 0}$ together with a continuous time Brownian motion $(B_t)_{t \in \mathbb{R}_+}$, such that for any $x \in \mathbb{X}$ and $n \geq 1$,*

$$\mathbb{P}_x \left(\sup_{0 \leq t \leq 1} |S_{[tn]} - \sigma B_{tn}| > n^{1/2-\varepsilon} \right) \leq \frac{c_\varepsilon}{n^\varepsilon} (1 + N(x)), \quad (3.4.2)$$

where σ is defined in the point 2 of Proposition 3.2.1.

In the original result the right-hand side in (3.4.2) is $c_\varepsilon n^{-\varepsilon} (1 + \mu_\alpha(x) + \|\delta_x\|_{\mathcal{G}'})^\alpha \leq c_\varepsilon n^{-\varepsilon} (1 + N(x))^\alpha$ with $\alpha > 2$, by the point 1 of the Hypothesis **M3.5**. To obtain the result of Proposition 3.4.3 it suffices to take the power $1/\alpha$ on the both sides and to use the obvious inequality $p < p^{1/\alpha}$, for $p \in [0, 1]$.

Using Proposition 3.4.3 we easily deduce the following result.

Corollary 3.4.4. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{R}$ and $n \geq 1$,*

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}_x \left(\frac{S_n}{\sqrt{n}} \leq t \right) - \int_{-\infty}^t e^{-\frac{u^2}{2\sigma^2}} \frac{du}{\sqrt{2\pi\sigma}} \right| \leq \frac{c_\varepsilon}{n^\varepsilon} (1 + N(x)).$$

3.5 Martingale approximation and related assertions

In this section we construct an approximating martingale for the Markov walk $(S_n)_{n \geq 0}$, which will be used subsequently to define the harmonic function. We also state some useful properties.

Consider Θ the real valued function defined on \mathbb{X} by:

$$\Theta(x) = f(x) + \sum_{k=1}^{+\infty} \mathbf{P}^k f(x), \quad \forall x \in \mathbb{X}.$$

It is well known that Θ is the solution of the Poisson equation

$$\Theta - \mathbf{P}\Theta = f.$$

For any $x \in \mathbb{X}$, let

$$r(x) = \mathbf{P}\Theta(x) = \Theta(x) - f(x) = \sum_{k=1}^{+\infty} \mathbf{P}^k f(x).$$

Following Gordin [35], define the process $(M_n)_{n \geq 0}$ by setting $M_0 = 0$ and, for any $n \geq 1$,

$$M_n = \sum_{k=1}^n [\Theta(X_k) - \mathbf{P}\Theta(X_{k-1})] = \sum_{k=1}^n [\Theta(X_k) - r(X_{k-1})].$$

For any $x \in \mathbb{X}$, we have that $(M_n)_{n \geq 0}$ is a zero mean \mathbb{P}_x -martingale with respect to the natural filtration $(\mathcal{F}_n)_{n \geq 0}$. Denote by ξ_n the increments of the martingale $(M_n)_{n \geq 0}$: for any $n \geq 1$,

$$\xi_n := \Theta(X_n) - r(X_{n-1}).$$

In the sequel it will be convenient to consider the martingale $(z + M_n)_{n \geq 1}$ starting at

$$z = y + r(x).$$

The reason for this is the following approximation which is an easy consequence of the definition of the martingale $(z + M_n)_{n \geq 1}$: for any $x \in \mathbb{X}$ and $y \in \mathbb{R}$, we have

$$z + M_n = y + S_n + r(X_n). \tag{3.5.1}$$

From (3.2.6) we deduce the following assertion.

Lemma 3.5.1. *The functions Θ and r exist on \mathbb{X} and for any $x \in \mathbb{X}$,*

$$|\Theta(x)| \leq c(1 + N(x)) \quad \text{and} \quad |r(x)| \leq c(1 + N(x)).$$

We show that the moments of order $p \in [1, \alpha]$ of the martingale $(M_n)_{n \geq 0}$ are bounded.

Lemma 3.5.2.

1. For any $p \in [1, \alpha]$, $x \in \mathbb{X}$ and $n \geq 1$,

$$\mathbb{E}_x^{1/p} (|M_n|^p) \leq c_p \sqrt{n} (1 + N(x)).$$

2. For any $x \in \mathbb{X}$ and $n \geq 1$,

$$\mathbb{E}_x (|M_n|) \leq c (\sqrt{n} + N(x)).$$

Proof. First we control the increments ξ_n . By Lemma 3.5.1, for any $n \geq 1$,

$$|\xi_n| \leq c(1 + N(X_n) + N(X_{n-1})). \tag{3.5.2}$$

So, using the point 1 of Hypothesis M3.4 and (3.2.2), for any $n \geq 1$,

$$\mathbb{E}_x^{1/p} (|\xi_n|^p) \leq c_p (1 + N(x)) \quad \forall p \in [1, \alpha], \tag{3.5.3}$$

$$\mathbb{E}_x (|\xi_n|) \leq c + c e^{-cn} N(x). \tag{3.5.4}$$

Proof of the claim 1. By Burkholder's inequality, for $2 < p \leq \alpha$,

$$\mathbb{E}_x^{1/p} (|M_n|^p) \leq c_p \mathbb{E}_x^{1/p} \left(\left(\sum_{k=1}^n \xi_k^2 \right)^{p/2} \right).$$

Using Hölder's inequality with the exponents $u = p/2 > 1$ and $v = \frac{p}{p-2}$, we obtain

$$\mathbb{E}_x^{1/p} (|M_n|^p) \leq c_p \mathbb{E}_x^{1/p} \left[\left(\sum_{k=1}^n \xi_k^{2u} \right)^{\frac{p}{2u}} n^{\frac{p}{2v}} \right] = c_p n^{\frac{p-2}{2p}} \left(\sum_{k=1}^n \mathbb{E}_x [|\xi_k|^p] \right)^{1/p}.$$

From (3.5.3), for any $p \in (2, \alpha]$,

$$\mathbb{E}_x^{1/p} (|M_n|^p) \leq c_p n^{\frac{p-2}{2p}} \left(\sum_{k=1}^n c_p (1 + N(x))^p \right)^{1/p} \leq c_p \sqrt{n} (1 + N(x)). \quad (3.5.5)$$

Using the Jensen inequality for $p \in [1, 2]$, we obtain the claim 1.

Proof of the claim 2. Consider $\varepsilon \in (0, 1/2)$. By (3.5.4),

$$\begin{aligned} \mathbb{E}_x (|M_n|) &\leq \sum_{k=1}^{\lfloor n^\varepsilon \rfloor} \mathbb{E}_x (|\xi_k|) + \mathbb{E}_x (|M_n - M_{\lfloor n^\varepsilon \rfloor}|) \\ &\leq cn^\varepsilon + cN(x) + \mathbb{E}_x (|M_n - M_{\lfloor n^\varepsilon \rfloor}|). \end{aligned}$$

Since $(X_n, M_n)_{n \geq 0}$ is a Markov chain, by the Markov property, the claim 1 and (3.2.2),

$$\begin{aligned} \mathbb{E}_x (|M_n|) &\leq cn^\varepsilon + cN(x) + \mathbb{E}_x \left(\mathbb{E} (|M_n - M_{\lfloor n^\varepsilon \rfloor}| \mid \mathcal{F}_{\lfloor n^\varepsilon \rfloor}) \right) \\ &\leq cn^\varepsilon + cN(x) + \mathbb{E}_x \left[c(n - \lfloor n^\varepsilon \rfloor)^{1/2} (1 + N(X_{\lfloor n^\varepsilon \rfloor})) \right] \\ &\leq c\sqrt{n} + c_\varepsilon N(x). \end{aligned}$$

□

A key point in the proof of the existence and of the positivity of the harmonic function is the introduction of the following stopping times. Let T_z be the first time when the martingale $(z + M_n)_{n \geq 1}$ becomes non-positive, and let \hat{T}_z be the first time, after the time τ_y , when the martingale $(z + M_n)_{n \geq 1}$ becomes non-positive. Precisely, for any $x \in \mathbb{X}$, $z \in \mathbb{R}$ and $y = z - r(x)$, set

$$T_z := \inf \{k \geq 1 : z + M_k \leq 0\} \quad \text{and} \quad \hat{T}_z := \inf \{k \geq \tau_y : z + M_k \leq 0\}. \quad (3.5.6)$$

The finiteness of the stopping times τ_y , T_z and \hat{T}_z is proved in Lemmas 3.5.5, 3.5.6 and 3.5.7 below. Now we point out some elementary facts which will be helpful in the sequel. First, the stopping time \hat{T}_z is such that $\tau_y \leq \hat{T}_z$ and $T_z \leq \hat{T}_z$. Since τ_y is the exit time of $(y + S_n)_{n \geq 0}$, by the Markov property,

$$\mathbb{P}_x (\tau_y > n) = \int_{\mathbb{X} \times \mathbb{R}} \mathbb{P}_{x'} (\tau_{y'} > n - k) \mathbb{P}_x (X_k \in dx', y + S_k \in dy', \tau_y > k). \quad (3.5.7)$$

A similar expression holds true for T_z . Unfortunately, (3.5.7) does not hold for \hat{T}_z . Instead we have a more sophisticated expression given by the following lemma. We shall use repeatedly the same trick for more complicated functionals, as for example $\mathbb{E}_x (z + M_n; \hat{T}_z > n)$.

Lemma 3.5.3. *For any $x \in \mathbb{X}$, $z \in \mathbb{R}$, $n \geq 1$, $k \leq n$ and $y = z - r(x)$,*

$$\begin{aligned} \mathbb{P}_x (\hat{T}_z > n) &= \int_{\mathbb{X} \times \mathbb{R}} \mathbb{P}_{x'} (\hat{T}_{z'} > n - k) \mathbb{P}_x (X_k \in dx', z + M_k \in dz', \tau_y > k) \\ &\quad + \int_{\mathbb{X} \times \mathbb{R}} \mathbb{P}_{x'} (T_{z'} > n - k) \mathbb{P}_x (X_k \in dx', z + M_k \in dz', \tau_y \leq k, \hat{T}_z > k). \end{aligned}$$

Proof. Since $\widehat{T}_z \geq \tau_y$, for any $k \leq n$, we have

$$\mathbb{P}_x(\widehat{T}_z > n) = \mathbb{P}_x(\tau_y > n) + \sum_{i=1}^{n-k} \mathbb{P}_x(\tau_y = i + k, \widehat{T}_z > n) + \mathbb{P}_x(\tau_y \leq k, \widehat{T}_z > n).$$

By the Markov property and (3.5.1), with $y' = z' - r(x')$,

$$\begin{aligned} \mathbb{P}_x(\widehat{T}_z > n) &= \int_{\mathbb{X} \times \mathbb{R}} \mathbb{P}_{x'}(\tau_{y'} > n - k) \mathbb{P}_x(X_k \in dx', z + M_k \in dz', \tau_y > k) \\ &\quad + \sum_{i=1}^{n-k} \int_{\mathbb{X} \times \mathbb{R}} \mathbb{P}_{x'}(\tau_{y'} = i, z' + M_i > 0, \dots, z' + M_{n-k} > 0) \\ &\quad \quad \quad \times \mathbb{P}_x(X_k \in dx', z + M_k \in dz', \tau_y > k) \\ &\quad + \int_{\mathbb{X} \times \mathbb{R}} \mathbb{P}_{x'}(T_{z'} > n - k) \mathbb{P}_x(X_k \in dx', z + M_k \in dz', \tau_y \leq k, \\ &\quad \quad \quad z + M_{\tau_y} > 0, \dots, z + M_k > 0). \end{aligned}$$

Putting together the first two terms we get the result. \square

The following lemma will be useful in the next sections.

Lemma 3.5.4. *For any $x \in \mathbb{X}$ and $z \in \mathbb{R}$, the sequence $\left((z + M_n) \mathbb{1}_{\{\widehat{T}_z > n\}}\right)_{n \geq 0}$ is a \mathbb{P}_x -submartingale.*

Proof. Let $x \in \mathbb{X}$, $z \in \mathbb{R}$. For any $n \geq 0$,

$$\begin{aligned} &\mathbb{E}_x \left((z + M_{n+1}) \mathbb{1}_{\{\widehat{T}_z > n+1\}} \middle| \mathcal{F}_n \right) \\ &= \mathbb{E}_x \left((z + M_{n+1}) \mathbb{1}_{\{\widehat{T}_z > n\}} \middle| \mathcal{F}_n \right) - \mathbb{E}_x \left((z + M_{n+1}) \mathbb{1}_{\{\widehat{T}_z = n+1\}} \middle| \mathcal{F}_n \right) \\ &= (z + M_n) \mathbb{1}_{\{\widehat{T}_z > n\}} - \mathbb{E}_x \left((z + M_{\widehat{T}_z}) \mathbb{1}_{\{\widehat{T}_z = n+1\}} \middle| \mathcal{F}_n \right). \end{aligned}$$

By the definition of \widehat{T}_z we have $z + M_{\widehat{T}_z} \leq 0$ \mathbb{P}_x -a.s. and the result follows. \square

We end this section by proving the finiteness of τ_y , T_z and \widehat{T}_z .

Lemma 3.5.5. *For any $x \in \mathbb{X}$ and $y \in \mathbb{R}$,*

$$\tau_y < +\infty \quad \mathbb{P}_x\text{-a.s.}$$

Proof. Let $x \in \mathbb{X}$. Assume first that $y > 0$. Since $\{\tau_y > n\}$ is a non-increasing sequence of events,

$$\mathbb{P}_x(\tau_y = +\infty) = \lim_{n \rightarrow +\infty} \mathbb{P}_x(\tau_y > n) = \lim_{n \rightarrow +\infty} \mathbb{P}_x(y + S_k > 0, \forall k \leq n).$$

Using Proposition 3.4.3,

$$\mathbb{P}_x(y + S_k > 0, \forall k \leq n) \leq \frac{c_\varepsilon}{n^\varepsilon} (1 + N(x)) + \mathbb{P}\left(\tau_{y+n^{1/2-\varepsilon}}^{bm} > n\right).$$

Thus, by the point 1 of Lemma 3.4.2,

$$\mathbb{P}_x(\tau_y > n) \leq \frac{c_\varepsilon}{n^\varepsilon} (1 + N(x)) + c \frac{y + n^{1/2-\varepsilon}}{\sqrt{n}} \leq \frac{c_\varepsilon}{n^\varepsilon} (1 + y + N(x)). \quad (3.5.8)$$

When $y \leq 0$, we have, for any $y' > 0$, $\mathbb{P}_x(\tau_y > n) \leq \mathbb{P}_x(\tau_{y'} > n)$. Taking the limit when $y' \rightarrow 0$, we obtain that

$$\mathbb{P}_x(\tau_y > n) \leq \frac{c_\varepsilon}{n^\varepsilon} (1 + N(x)). \quad (3.5.9)$$

From (3.5.8) and (3.5.9) it follows that, for any $y \in \mathbb{R}$,

$$\mathbb{P}_x(\tau_y > n) \leq \frac{c_\varepsilon}{n^\varepsilon} (1 + \max(y, 0) + N(x)). \quad (3.5.10)$$

Taking the limit as $n \rightarrow +\infty$, we conclude that $\tau_y < +\infty$ \mathbb{P}_x -a.s. \square

The same result can be obtained for the exit time T_z of the martingale $(z + M_n)_{n \geq 0}$.

Lemma 3.5.6. *For any $x \in \mathbb{X}$ and $z \in \mathbb{R}$,*

$$T_z < +\infty \quad \mathbb{P}_x\text{-a.s.}$$

Proof. Let $x \in \mathbb{X}$, $z \in \mathbb{R}$ and $y = z - r(x)$. Assume first that $y = z - r(x) > 0$. Following the proof of Lemma 3.5.5,

$$\mathbb{P}_x(T_z = +\infty) = \lim_{n \rightarrow +\infty} \mathbb{P}_x(z + M_k > 0, \forall k \leq n).$$

By (3.5.1) the martingale $(z + M_n)_{n \geq 0}$ is related to the Markov walk $(y + S_n)_{n \geq 0}$, which gives

$$\begin{aligned} \mathbb{P}_x(z + M_k > 0, \forall k \leq n) &\leq \mathbb{P}_x(y + S_k > -n^{1/2-\varepsilon}, \forall k \leq n) \\ &\quad + \mathbb{P}_x\left(\max_{1 \leq k \leq n} |r(X_k)| > n^{1/2-\varepsilon}\right). \end{aligned} \quad (3.5.11)$$

On the one hand, in the same way as in the proof of Lemma 3.5.5,

$$\mathbb{P}_x(y + S_k > -n^{1/2-\varepsilon}, \forall k \leq n) \leq \frac{c_\varepsilon}{n^\varepsilon} (1 + N(x)) + \mathbb{P}_x(\tau_{y+2n^{1/2-\varepsilon}}^{bm} > n). \quad (3.5.12)$$

On the other hand, using Lemma 3.5.1, for n large enough,

$$\mathbb{P}_x\left(\max_{1 \leq k \leq n} |r(X_k)| > n^{1/2-\varepsilon}\right) \leq \sum_{k=1}^{\lfloor n^\varepsilon \rfloor} \mathbb{E}_x\left(\frac{cN(X_k)}{n^{1/2-\varepsilon}}\right) + \sum_{k=\lfloor n^\varepsilon \rfloor+1}^n \mathbb{E}_x\left(\frac{cN_l(X_k)}{n^{1/2-\varepsilon}}\right),$$

where $l = cn^{1/2-\varepsilon}$. So, using (3.2.3) and taking $\varepsilon \leq \min\left(\frac{1}{6}, \frac{\beta}{2(3+\beta)}\right)$, we obtain

$$\mathbb{P}_x\left(\max_{1 \leq k \leq n} |r(X_k)| > n^{1/2-\varepsilon}\right) \leq \frac{c_\varepsilon}{n^\varepsilon} (1 + N(x)). \quad (3.5.13)$$

Putting together (3.5.11), (3.5.12) and (3.5.13) and using the point 1 of Lemma 3.4.2, we have, for $z > r(x)$,

$$\mathbb{P}_x(T_z > n) \leq \frac{c_\varepsilon}{n^\varepsilon} (1 + N(x)) + c \frac{y + 2n^{1/2-\varepsilon}}{\sqrt{n}} \leq \frac{c_\varepsilon}{n^\varepsilon} (1 + \max(z, 0) + N(x)).$$

Since $z \mapsto T_z$ is non-decreasing, we obtain the same bound for any $z \in \mathbb{R}$,

$$\mathbb{P}_x(T_z > n) \leq \frac{c_\varepsilon}{n^\varepsilon} (1 + \max(z, 0) + N(x)). \quad (3.5.14)$$

Taking the limit as $n \rightarrow +\infty$ we conclude that $T_z < +\infty$ \mathbb{P}_x -a.s. \square

Lemma 3.5.7. *For any $x \in \mathbb{X}$ and $z \in \mathbb{R}$,*

$$\widehat{T}_z < +\infty \quad \mathbb{P}_x\text{-a.s.}$$

Proof. In order to apply Lemmas 3.5.5 and 3.5.6, we write, with $y = z - r(x)$,

$$\begin{aligned} \mathbb{P}_x(\widehat{T}_z > n) &\leq \mathbb{P}_x(\tau_y > \lfloor n/2 \rfloor) + \int_{\mathbb{X} \times \mathbb{R}} \mathbb{P}_{x'}(T_{z'} > n - \lfloor n/2 \rfloor) \mathbb{P}_x(X_{\lfloor n/2 \rfloor} \in dx', \\ &\quad z + M_{\lfloor n/2 \rfloor} \in dz', \tau_y \leq \lfloor n/2 \rfloor, \widehat{T}_z > \lfloor n/2 \rfloor). \end{aligned}$$

Using (3.5.10), (3.5.14) and the definition of y , we have

$$\begin{aligned} \mathbb{P}_x(\widehat{T}_z > n) &\leq \frac{c_\varepsilon}{n^\varepsilon} (1 + \max(y, 0) + N(x)) \\ &\quad + \frac{c_\varepsilon}{n^\varepsilon} \mathbb{E}_x(1 + z + M_{\lfloor n/2 \rfloor} + N(X_{\lfloor n/2 \rfloor}); \tau_y \leq \lfloor n/2 \rfloor, \widehat{T}_z > \lfloor n/2 \rfloor). \end{aligned}$$

By the point 1 of Hypothesis M3.4,

$$\begin{aligned} \mathbb{P}_x(\widehat{T}_z > n) &\leq \frac{c_\varepsilon}{n^\varepsilon} (1 + \max(y, 0) + N(x)) + \frac{c_\varepsilon}{n^\varepsilon} \mathbb{E}_x(z + M_{\lfloor n/2 \rfloor}; \widehat{T}_z > \lfloor n/2 \rfloor) \\ &\quad - \frac{c_\varepsilon}{n^\varepsilon} \mathbb{E}_x(z + M_{\lfloor n/2 \rfloor}; \tau_y > \lfloor n/2 \rfloor). \end{aligned}$$

Using (3.5.1), we see that on the event $\{\tau_y > \lfloor n/2 \rfloor\}$ we have $z + M_{\lfloor n/2 \rfloor} > r(X_{\lfloor n/2 \rfloor})$. Then, by Lemma 3.5.1 and the point 1 of Hypothesis M3.4,

$$\mathbb{P}_x(\widehat{T}_z > n) \leq \frac{c_\varepsilon}{n^\varepsilon} (1 + \max(y, 0) + N(x)) + \frac{c_\varepsilon}{n^\varepsilon} \mathbb{E}_x(z + M_{\lfloor n/2 \rfloor}; \widehat{T}_z > \lfloor n/2 \rfloor).$$

Using Lemma 3.6.4, we have

$$\mathbb{P}_x(\widehat{T}_z > n) \leq \frac{c_\varepsilon}{n^\varepsilon} (1 + \max(y, 0) + N(x)).$$

Finally, we conclude that

$$\mathbb{P}_x(\widehat{T}_z = +\infty) = \lim_{n \rightarrow +\infty} \mathbb{P}_x(\widehat{T}_z > n) = 0.$$

□

3.6 Integrability of the killed martingale

The goal of this section is to show that the expectations of the martingale $(z + M_n)_{n \geq 0}$ killed at \widehat{T}_z and of the Markov walk $(y + S_n)_{n \geq 0}$ killed at τ_y are bounded uniformly in n .

We start by establishing two auxiliary bounds of order $n^{1/2-2\varepsilon}$ for the expectations of the martingale $(z + M_n)_{n \geq 0}$ killed at T_z or at \widehat{T}_z .

Lemma 3.6.1. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$, $z \in \mathbb{R}$ and $n \geq 1$, it holds*

$$\mathbb{E}_x(z + M_n; T_z > n) \leq \max(z, 0) + c_\varepsilon (n^{1/2-2\varepsilon} + N(x)).$$

Proof. Using the fact that $(M_n)_{n \geq 0}$ is a zero mean martingale and the optional stopping theorem,

$$\mathbb{E}_x(z + M_n; T_z > n) = z - \mathbb{E}_x(z + M_n; T_z \leq n) = z - \mathbb{E}_x(z + M_{T_z}; T_z \leq n).$$

By the definition of T_z , on the event $\{T_z > 1\}$, we have

$$\xi_{T_z} = z + M_{T_z} - (z + M_{T_z-1}) < z + M_{T_z} \leq 0.$$

Using this inequality and (3.5.2), we obtain

$$\begin{aligned} \mathbb{E}_x(z + M_n; T_z > n) &\leq z\mathbb{P}_x(T_z > 1) + \mathbb{E}_x(|\xi_1|; T_z = 1) + \mathbb{E}_x(|\xi_{T_z}|; 1 < T_z \leq n) \\ &\leq \max(z, 0) + c\mathbb{E}_x(1 + N(X_{T_z}) + N(X_{T_z-1}); T_z \leq n). \end{aligned} \quad (3.6.1)$$

We bound $\mathbb{E}_x(N(X_{T_z}); T_z \leq n)$ as follows. Let ε be a real number in $(0, 1/6)$ and set $l = \lfloor n^{1/2-2\varepsilon} \rfloor$. Using the point 1 of Hypothesis M3.4 we write

$$\begin{aligned} \mathbb{E}_x(N(X_{T_z}); T_z \leq n) &\leq n^{1/2-2\varepsilon} + \mathbb{E}_x(N(X_{T_z}); N(X_{T_z}) > n^{1/2-2\varepsilon}, T_z \leq n) \\ &\leq n^{1/2-2\varepsilon} + \sum_{k=1}^{\lfloor n^\varepsilon \rfloor} \mathbb{E}_x(N(X_k)) + \sum_{k=\lfloor n^\varepsilon \rfloor+1}^n \mathbb{E}_x(N_l(X_k)). \end{aligned}$$

By (3.2.2) and (3.2.3),

$$\mathbb{E}_x(N(X_{T_z}); T_z \leq n) \leq cn^{1/2-2\varepsilon} + cN(x) + \frac{cn}{l^{1+\beta}} + ce^{-cn^\varepsilon}(1 + N(x)).$$

Choosing $\varepsilon < \min(\frac{\beta}{4(2+\beta)}, \frac{1}{6})$, we find that

$$\mathbb{E}_x(N(X_{T_z}); T_z \leq n) \leq c_\varepsilon n^{1/2-2\varepsilon} + c_\varepsilon N(x). \quad (3.6.2)$$

In the same manner, we obtain that $\mathbb{E}_x(N(X_{T_z-1}); T_z \leq n) \leq c_\varepsilon n^{1/2-2\varepsilon} + c_\varepsilon N(x)$. Consequently, from (3.6.2) and (3.6.1), we conclude the assertion of the lemma. \square

Lemma 3.6.2. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$, $z \in \mathbb{R}$ and $n \geq 1$, we have*

$$\mathbb{E}_x(z + M_n; \widehat{T}_z > n) \leq \max(z, 0) + c_\varepsilon (n^{1/2-2\varepsilon} + n^{2\varepsilon} N(x)).$$

Proof. Let ε be a real number in $(0, 1/4)$ and $n \geq 1$. Denoting $z_+ := z + n^{1/2-2\varepsilon}$ we have,

$$\begin{aligned} \mathbb{E}_x(z + M_n; \widehat{T}_z > n) &= \underbrace{\mathbb{E}_x(z + M_n; T_{z_+} \leq n, \widehat{T}_z > n)}_{=: J_1} \\ &\quad + \underbrace{\mathbb{E}_x(z + M_n; T_{z_+} > n, \widehat{T}_z > n)}_{=: J_2}. \end{aligned} \quad (3.6.3)$$

Bound of J_1 . Recall that $y = z - r(x)$. Using the definition of \widehat{T}_z , we can see that on the event $\{\tau_y \leq k, \widehat{T}_z > k\}$ it holds $z_+ + M_k > z + M_k > 0$. So, we have

$\mathbb{P}_x(\tau_y \leq k, \widehat{T}_z > k, T_{z_+} = k) = 0$. Using this fact and the Markov property, in the same way as in the proof of Lemma 3.5.3,

$$J_1 = \sum_{k=1}^n \int_{\mathbb{X} \times \mathbb{R}} \mathbb{E}_{x'}(z' + M_{n-k}; \widehat{T}_{z'} > n - k) \\ \times \mathbb{P}_x(X_k \in dx', z + M_k \in dz', \tau_y > k, T_{z_+} = k).$$

Since $z + M_{T_{z_+}} < 0$, using the point 2 of Lemma 3.5.2, we have

$$J_1 \leq c \mathbb{E}_x(\sqrt{n} + N(X_{T_{z_+}}); \tau_y > T_{z_+}, T_{z_+} \leq n).$$

By the approximation (3.5.1), on the event $\{\tau_y > T_{z_+}\}$, it holds

$$r(X_{T_{z_+}}) = z + M_{T_{z_+}} - (y + S_{T_{z_+}}) < -n^{1/2-2\varepsilon}.$$

Therefore, by Lemma 3.5.1,

$$J_1 \leq cn^{2\varepsilon} \mathbb{E}_x(|r(X_{T_{z_+}})| + N(X_{T_{z_+}}); |r(X_{T_{z_+}})| > n^{1/2-2\varepsilon}, T_{z_+} \leq n) \\ \leq cn^{2\varepsilon} + cn^{2\varepsilon} \mathbb{E}_x(N(X_{T_{z_+}}); T_{z_+} \leq n).$$

Choosing ε small enough, by (3.6.2),

$$J_1 \leq cn^{2\varepsilon} + c_\varepsilon n^{2\varepsilon} (n^{1/2-4\varepsilon} + N(x)) \leq c_\varepsilon n^{1/2-2\varepsilon} + c_\varepsilon n^{2\varepsilon} N(x). \quad (3.6.4)$$

Bound of J_2 . By Lemma 3.6.1, there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$J_2 \leq \mathbb{E}_x(z_+ + M_n; T_{z_+} > n) \leq \max(z, 0) + c_\varepsilon n^{1/2-2\varepsilon} + c_\varepsilon N(x).$$

Inserting this bound and (3.6.4) into (3.6.3), for any $\varepsilon \in (0, \varepsilon_0)$, we deduce the assertion of the lemma. \square

Let ν_n be the first time when the martingale $z + M_n$ exceeds $n^{1/2-\varepsilon}$: for any $n \geq 1$, $\varepsilon \in (0, 1/2)$ and $z \in \mathbb{R}$,

$$\nu_n = \nu_{n,\varepsilon,z} := \min \{k \geq 1 : z + M_k > n^{1/2-\varepsilon}\}. \quad (3.6.5)$$

The control on the joint law of ν_n and \widehat{T}_z is given by the following lemma.

Lemma 3.6.3. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $\delta > 0$, $x \in \mathbb{X}$, $z \in \mathbb{R}$ and $n \geq 1$,*

$$\mathbb{P}_x(\nu_n > \delta n^{1-\varepsilon}, \widehat{T}_z > \delta n^{1-\varepsilon}) \leq c_{\varepsilon,\delta} e^{-c_{\varepsilon,\delta} n^\varepsilon} (1 + N(x)).$$

Proof. Let $\varepsilon \in (0, 1/4)$ and $\delta > 0$. Without loss of generality, we assume that $n \geq c_{\varepsilon,\delta}$, where $c_{\varepsilon,\delta}$ is large enough. Set $K := \lfloor n^\varepsilon/2 \rfloor$. We split the interval $[1, \delta n^{1-\varepsilon}]$ by subintervals of length $l := \lfloor \delta n^{1-2\varepsilon} \rfloor$. For any $k \in \{1, \dots, K\}$, introduce the event $A_{k,z} := \{\max_{1 \leq k' \leq k} (z + M_{k'l}) \leq n^{1/2-\varepsilon}\}$. Then

$$\mathbb{P}_x(\nu_n > \delta n^{1-\varepsilon}, \widehat{T}_z > \delta n^{1-\varepsilon}) \leq \mathbb{P}_x(A_{2K,z}, \widehat{T}_z > 2Kl). \quad (3.6.6)$$

By the Markov property, as in the proof of Lemma 3.5.3, with $y = z - r(x)$, we have

$$\begin{aligned}
 & \mathbb{P}_x \left(A_{2K,z}, \widehat{T}_z > 2Kl \right) \\
 &= \int_{\mathbb{X} \times \mathbb{R}} \mathbb{P}_{x'} \left(A_{2,z'}, \widehat{T}_{z'} > 2l \right) \mathbb{P}_x \left(X_{2(K-1)l} \in dx', z + M_{2(K-1)l} \in dz', \right. \\
 & \qquad \qquad \qquad \left. A_{2(K-1),z}, \tau_y > 2(K-1)l \right) \\
 &+ \int_{\mathbb{X} \times \mathbb{R}} \mathbb{P}_{x'} \left(A_{2,z'}, T_{z'} > 2l \right) \mathbb{P}_x \left(X_{2(K-1)l} \in dx', z + M_{2(K-1)l} \in dz', \right. \\
 & \qquad \qquad \qquad \left. A_{2(K-1),z}, \tau_y \leq 2(K-1)l, \widehat{T}_z > 2(K-1)l \right). \quad (3.6.7)
 \end{aligned}$$

Moreover, with $y' = z' - r(x')$, we write also that

$$\begin{aligned}
 & \mathbb{P}_{x'} \left(A_{2,z'}, \widehat{T}_{z'} > 2l \right) \\
 &= \int_{\mathbb{X} \times \mathbb{R}} \mathbb{P}_{x''} \left(A_{1,z''}, \widehat{T}_{z''} > l \right) \mathbb{P}_{x'} \left(X_l \in dx'', z' + M_l \in dz'', A_{1,z'}, \tau_{y'} > l \right) \\
 & \quad + \int_{\mathbb{X} \times \mathbb{R}} \mathbb{P}_{x''} \left(A_{1,z''}, T_{z''} > l \right) \\
 & \quad \quad \quad \times \mathbb{P}_{x'} \left(X_l \in dx'', z' + M_l \in dz'', A_{1,z'}, \tau_{y'} \leq l, \widehat{T}_{z'} > l \right). \quad (3.6.8)
 \end{aligned}$$

Bound of $\mathbb{P}_{x''} \left(A_{1,z''}, \widehat{T}_{z''} > l \right)$. Note that on the event $\{\tau_{y'} > l\}$ we have $z' + M_l - r(X_l) = y' + S_l > 0$. Consequently, in the first integral of the right-hand side of (3.6.8), the integration over $\mathbb{X} \times \mathbb{R}$ can be replaced by the integration over $\{(x'', z'') \in \mathbb{X} \times \mathbb{R} : z'' - r(x'') > 0\}$. Therefore it is enough to bound $\mathbb{P}_{x''} \left(A_{1,z''}, \widehat{T}_{z''} > l \right)$ for x'' and z'' satisfying $y'' = z'' - r(x'') > 0$. Using (3.5.1) we have,

$$\begin{aligned}
 \mathbb{P}_{x''} \left(A_{1,z''}, \widehat{T}_{z''} > l \right) &\leq \mathbb{P}_{x''} \left(y'' + S_l \leq 2n^{1/2-\varepsilon}, |r(X_l)| \leq n^{1/2-\varepsilon} \right) \\
 & \quad + \mathbb{P}_{x''} \left(|r(X_l)| > n^{1/2-\varepsilon} \right).
 \end{aligned}$$

Therefore, there exists a constant $c_{\varepsilon,\delta}$ such that

$$\mathbb{P}_{x''} \left(A_{1,z''}, \widehat{T}_{z''} > l \right) \leq \mathbb{P}_{x''} \left(\frac{S_l}{\sqrt{l}} \leq c_{\varepsilon,\delta} \right) + \mathbb{E}_{x''} \left(\frac{|r(X_l)|}{n^{1/2-\varepsilon}} \right).$$

Using Corollary 3.4.4 and Lemma 3.5.1, there exists $\varepsilon_0 \in (0, 1/4)$, such that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$\mathbb{P}_{x''} \left(A_{1,z''}, \widehat{T}_{z''} > l \right) \leq \int_{-\infty}^{c_{\varepsilon,\delta}} e^{-\frac{u^2}{2\sigma^2}} \frac{du}{\sqrt{2\pi}\sigma} + \frac{c_{\varepsilon}}{l^\varepsilon} (1 + N(x'')) + \frac{c}{n^{1/2-\varepsilon}} \mathbb{E}_{x''} (1 + N(X_l)).$$

Using the point 1 of Hypothesis M3.4 and the fact that $l^\varepsilon \geq n^{\varepsilon/2}/c_{\varepsilon,\delta}$ for $\varepsilon < 1/4$, we have,

$$\mathbb{P}_{x''} \left(A_{1,z''}, \widehat{T}_{z''} > l \right) \leq q_{\varepsilon,\delta} + \frac{c_{\varepsilon,\delta}}{n^{\varepsilon/2}} (1 + N(x'')), \quad (3.6.9)$$

with $q_{\varepsilon,\delta} := \int_{-\infty}^{c_{\varepsilon,\delta}} e^{-\frac{u^2}{2\sigma^2}} \frac{du}{\sqrt{2\pi}\sigma} < 1$.

Bound of $\mathbb{P}_{x''} \left(A_{1,z''}, T_{z''} > l \right)$. On the event $\{T_{z''} > l\}$ we have $z'' + M_l > 0$. Using (3.5.1) and Corollary 3.4.4, in the same way as in the proof of the bound (3.6.9), we

obtain

$$\begin{aligned}
 \mathbb{P}_{x''} (A_{1,z''}, T_{z''} > l) &\leq \mathbb{P}_{x''} (0 < z'' + M_l \leq n^{1/2-\varepsilon}) \\
 &\leq \int_{\frac{-y''}{\sqrt{l}} - c_{\varepsilon,\delta}}^{\frac{-y''}{\sqrt{l}} + c_{\varepsilon,\delta}} e^{-\frac{u^2}{2\sigma^2}} \frac{du}{\sqrt{2\pi}\sigma} + \frac{c_{\varepsilon,\delta}}{n^{\varepsilon/2}} (1 + N(x'')) \\
 &\leq q_{\varepsilon,\delta} + \frac{c_{\varepsilon,\delta}}{n^{\varepsilon/2}} (1 + N(x'')). \tag{3.6.10}
 \end{aligned}$$

Bound of $\mathbb{P}_{x'} (A_{2,z'}, \widehat{T}_{z'} > 2l)$. Inserting (3.6.9) and (3.6.10) into (3.6.8) and using (3.2.2), we have

$$\begin{aligned}
 \mathbb{P}_{x'} (A_{2,z'}, \widehat{T}_{z'} > 2l) &\leq q_{\varepsilon,\delta} + \frac{c_{\varepsilon,\delta}}{n^{\varepsilon/2}} + \frac{c_{\varepsilon,\delta}}{n^{\varepsilon/2}} \mathbb{E}_{x'} (N(X_l)) \\
 &\leq q_{\varepsilon,\delta} + \frac{c_{\varepsilon,\delta}}{n^{\varepsilon/2}} + c_{\varepsilon,\delta} e^{-c_{\varepsilon,\delta} n^{1-2\varepsilon}} N(x'). \tag{3.6.11}
 \end{aligned}$$

Bound of $\mathbb{P}_{x'} (A_{2,z'}, T_{z'} > 2l)$. By the Markov property,

$$\begin{aligned}
 \mathbb{P}_{x'} (A_{2,z'}, T_{z'} > 2l) &= \int_{\mathbb{X} \times \mathbb{R}} \mathbb{P}_{x''} (A_{1,z''}, T_{z''} > l) \\
 &\quad \times \mathbb{P}_{x'} (X_l \in dx'', z' + M_l \in dz'', A_{1,z'}, T_{z'} > l).
 \end{aligned}$$

Using (3.6.10) to bound the probability inside the integral, we get

$$\mathbb{P}_{x'} (A_{2,z'}, T_{z'} > 2l) \leq q_{\varepsilon,\delta} + \frac{c_{\varepsilon,\delta}}{n^{\varepsilon/2}} + c_{\varepsilon,\delta} e^{-c_{\varepsilon,\delta} n^{1-2\varepsilon}} N(x'). \tag{3.6.12}$$

Inserting the bounds (3.6.11) and (3.6.12) into (3.6.7), we find that

$$\begin{aligned}
 \mathbb{P}_x (A_{2K,z}, \widehat{T}_z > 2Kl) \\
 &\leq \left(q_{\varepsilon,\delta} + \frac{c_{\varepsilon,\delta}}{n^{\varepsilon/2}} \right) \mathbb{P}_x (A_{2(K-1),z}, \widehat{T}_z > 2(K-1)l) + c_{\varepsilon,\delta} e^{-c_{\varepsilon,\delta} n^{1-2\varepsilon}} (1 + N(x)).
 \end{aligned}$$

Iterating this inequality, we get

$$\mathbb{P}_x (A_{2K,z}, \widehat{T}_z > 2Kl) \leq \left(q_{\varepsilon,\delta} + \frac{c_{\varepsilon,\delta}}{n^{\varepsilon/2}} \right)^K + c_{\varepsilon,\delta} e^{-c_{\varepsilon,\delta} n^{1-2\varepsilon}} (1 + N(x)) \sum_{k=0}^{K-1} \left(q_{\varepsilon,\delta} + \frac{c_{\varepsilon,\delta}}{n^{\varepsilon/2}} \right)^k.$$

As $K = \lfloor n^\varepsilon/2 \rfloor$ and $q_{\varepsilon,\delta} < 1$ it follows that, for n large enough, $\left(q_{\varepsilon,\delta} + \frac{c_{\varepsilon,\delta}}{n^{\varepsilon/2}} \right)^K \leq c_{\varepsilon,\delta} e^{-c_{\varepsilon,\delta} n^\varepsilon}$, which, in turn, implies

$$\mathbb{P}_x (A_{2K,z}, \widehat{T}_z > 2Kl) \leq c_{\varepsilon,\delta} e^{-c_{\varepsilon,\delta} n^\varepsilon} (1 + N(x)).$$

□

Lemma 3.6.4. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$, $z \in \mathbb{R}$, $n \geq 2$ and any integer $k_0 \in \{2, \dots, n\}$,*

$$\mathbb{E}_x (z + M_n; \widehat{T}_z > n) \leq \left(1 + \frac{c_\varepsilon}{k_0^\varepsilon} \right) (\max(z, 0) + cN(x)) + c_\varepsilon k_0^{1/2}.$$

Proof. Set for brevity $u_n := \mathbb{E}_x(z + M_n; \widehat{T}_z > n)$. By Lemma 3.5.4, the sequence $(u_n)_{n \geq 1}$ is non-decreasing. Let $\varepsilon \in (0, 1/2)$. We shall prove below that, for $n \geq 2$,

$$u_n \leq \left(1 + \frac{c_\varepsilon}{n^\varepsilon}\right) u_{\lfloor n^{1-\varepsilon} \rfloor} + c_\varepsilon e^{-c_\varepsilon n^\varepsilon} (1 + N(x)). \quad (3.6.13)$$

Using Lemma 9.1 of [36] (Lemma 2.9.1 in Chapter 2), we obtain that for any $n \geq 2$ and $k_0 \in \{2, \dots, n\}$,

$$u_n \leq \left(1 + \frac{c_\varepsilon}{k_0^\varepsilon}\right) u_{k_0} + c_\varepsilon e^{-c_\varepsilon k_0^\varepsilon} (1 + N(x)).$$

Next, by the point 2 of Lemma 3.5.2, $u_{k_0} \leq \mathbb{E}_x(|M_{k_0}|) \leq c(\sqrt{k_0} + N(x))$, so that

$$u_n \leq \left(1 + \frac{c_\varepsilon}{k_0^\varepsilon}\right) (\max(z, 0) + cN(x)) + c_\varepsilon k_0^{1/2},$$

which proves Lemma 3.6.4.

Establishing (3.6.13) is rather tedious. In the proof we make use of Lemmas 3.6.2 and 3.6.1. Consider the stopping time $\nu_n^\varepsilon := \nu_n + \lfloor n^\varepsilon \rfloor$. Then,

$$\begin{aligned} u_n &\leq \underbrace{\mathbb{E}_x(z + M_n; \widehat{T}_z > n, \nu_n^\varepsilon > \lfloor n^{1-\varepsilon} \rfloor)}_{=: J_1} \\ &\quad + \underbrace{\mathbb{E}_x(z + M_n; \widehat{T}_z > n, \nu_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor)}_{=: J_2}. \end{aligned} \quad (3.6.14)$$

Bound of J_1 . Set $m_\varepsilon = \lfloor n^{1-\varepsilon} \rfloor - \lfloor n^\varepsilon \rfloor$ and recall that $y = z - r(x)$. Using the fact that $\{\nu_n^\varepsilon > \lfloor n^{1-\varepsilon} \rfloor\} = \{\nu_n > m_\varepsilon\}$ and the Markov property, as in the proof of Lemma 3.5.3,

$$\begin{aligned} J_1 &= \int_{\mathbb{X} \times \mathbb{R}} \mathbb{E}_{x'}(z' + M_{n-m_\varepsilon}; \widehat{T}_{z'} > n - m_\varepsilon) \\ &\quad \times \mathbb{P}_x(X_{m_\varepsilon} \in dx', z + M_{m_\varepsilon} \in dz', \tau_y > m_\varepsilon, \nu_n > m_\varepsilon) \\ &\quad + \int_{\mathbb{X} \times \mathbb{R}} \mathbb{E}_{x'}(z' + M_{n-m_\varepsilon}; T_{z'} > n - m_\varepsilon) \\ &\quad \times \mathbb{P}_x(X_{m_\varepsilon} \in dx', z + M_{m_\varepsilon} \in dz', \tau_y \leq m_\varepsilon, \widehat{T}_z > m_\varepsilon, \nu_n > m_\varepsilon). \end{aligned}$$

On the event $\{\nu_n > m_\varepsilon\}$, we have $z' = z + M_{m_\varepsilon} \leq n^{1/2-\varepsilon} \leq n^{1/2}$. Moreover by the point 2 of Lemma 3.5.2, $\mathbb{E}_{x'}(|M_{n-m_\varepsilon}|) \leq cn^{1/2} + cN(x')$. Therefore,

$$J_1 \leq c\mathbb{E}_x(n^{1/2} + N(X_{m_\varepsilon}); \widehat{T}_z > m_\varepsilon, \nu_n > m_\varepsilon).$$

Set $m'_\varepsilon = m_\varepsilon - \lfloor n^\varepsilon \rfloor = \lfloor n^{1-\varepsilon} \rfloor - 2\lfloor n^\varepsilon \rfloor$. Using the Markov property and (3.2.2),

$$\begin{aligned} J_1 &\leq c \int_{\mathbb{X}} [n^{1/2} + \mathbb{E}_{x'}(N(X_{\lfloor n^\varepsilon \rfloor}))] \mathbb{P}_x(X_{m'_\varepsilon} \in dx', \widehat{T}_z > m'_\varepsilon, \nu_n > m'_\varepsilon) \\ &\leq cn^{1/2} \mathbb{P}_x(\widehat{T}_z > m'_\varepsilon, \nu_n > m'_\varepsilon) + ce^{-cn^\varepsilon} \mathbb{E}_x(N(X_{m'_\varepsilon})). \end{aligned}$$

By Lemma 3.6.3 and the point 1 of Hypothesis M3.4,

$$J_1 \leq c_\varepsilon n^{1/2} e^{-c_\varepsilon n^\varepsilon} (1 + N(x)) + ce^{-cn^\varepsilon} (1 + N(x)) \leq c_\varepsilon e^{-c_\varepsilon n^\varepsilon} (1 + N(x)). \quad (3.6.15)$$

Bound of J_2 . By the Markov property, as in the proof of Lemma 3.5.3, we have

$$\begin{aligned}
 J_2 &= \sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{X} \times \mathbb{R}} \mathbb{E}_{x'} \left(z' + M_{n-k}; \widehat{T}_{z'} > n-k \right) \\
 &\quad \times \mathbb{P}_x \left(X_k \in dx', z + M_k \in dz', \tau_y > k, \nu_n^\varepsilon = k \right) \\
 &\quad + \int_{\mathbb{X} \times \mathbb{R}} \mathbb{E}_{x'} \left(z' + M_{n-k}; T_{z'} > n-k \right) \\
 &\quad \times \mathbb{P}_x \left(X_k \in dx', z + M_k \in dz', \tau_y \leq k, \widehat{T}_z > k, \nu_n^\varepsilon = k \right).
 \end{aligned}$$

By Lemmas 3.6.2 and 3.6.1,

$$\begin{aligned}
 J_2 &\leq \underbrace{c_\varepsilon \mathbb{E}_x \left(n^{1/2-2\varepsilon} + n^{2\varepsilon} N \left(X_{\nu_n^\varepsilon} \right); \widehat{T}_z > \nu_n^\varepsilon, \nu_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor \right)}_{=: J_{21}} \\
 &\quad + \underbrace{\mathbb{E}_x \left(\max \left(z + M_{\nu_n^\varepsilon}, 0 \right); \widehat{T}_z > \nu_n^\varepsilon, \nu_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor \right)}_{=: J_{22}}. \tag{3.6.16}
 \end{aligned}$$

Bound of J_{21} . Using the Markov property and (3.2.2),

$$\begin{aligned}
 J_{21} &\leq c_\varepsilon \int_{\mathbb{X}} \mathbb{E}_{x'} \left(n^{1/2-2\varepsilon} + n^{2\varepsilon} N \left(X_{\lfloor n^\varepsilon \rfloor} \right) \right) \mathbb{P}_x \left(X_{\nu_n} \in dx', \widehat{T}_z > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\
 &\leq c_\varepsilon \mathbb{E}_x \left(n^{1/2-2\varepsilon} + e^{-c_\varepsilon n^\varepsilon} N \left(X_{\nu_n} \right); \widehat{T}_z > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right).
 \end{aligned}$$

Again by (3.2.2),

$$\begin{aligned}
 \mathbb{E}_x \left(e^{-c_\varepsilon n^\varepsilon} N \left(X_{\nu_n} \right); \widehat{T}_z > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) &\leq e^{-c_\varepsilon n^\varepsilon} \sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{E}_x \left(N \left(X_k \right); \nu_n = k \right) \\
 &\leq c_\varepsilon e^{-c_\varepsilon n^\varepsilon} n^{1-\varepsilon} (1 + N(x)). \tag{3.6.17}
 \end{aligned}$$

Therefore,

$$J_{21} \leq \underbrace{c_\varepsilon \mathbb{E}_x \left(n^{1/2-2\varepsilon}; \widehat{T}_z > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right)}_{=: J'_{21}} + c_\varepsilon e^{-c_\varepsilon n^\varepsilon} (1 + N(x)). \tag{3.6.18}$$

By the definition of ν_n , we have $n^{1/2-2\varepsilon} < \frac{z+M_{\nu_n}}{n^\varepsilon}$. So

$$J'_{21} \leq \frac{c_\varepsilon}{n^\varepsilon} \mathbb{E}_x \left(z + M_{\nu_n}; \widehat{T}_z > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right).$$

Using Lemma 3.5.4,

$$\begin{aligned}
 J'_{21} &\leq \frac{c_\varepsilon}{n^\varepsilon} \mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor}; \widehat{T}_z > \lfloor n^{1-\varepsilon} \rfloor \right) \\
 &\quad - \underbrace{\frac{c_\varepsilon}{n^\varepsilon} \mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor}; \widehat{T}_z > \lfloor n^{1-\varepsilon} \rfloor, \nu_n > \lfloor n^{1-\varepsilon} \rfloor \right)}_{=: J''_{21}}. \tag{3.6.19}
 \end{aligned}$$

Note that on the event $\{\tau_y > \lfloor n^{1-\varepsilon} \rfloor\}$, by (3.5.1), we have $z + M_{\lfloor n^{1-\varepsilon} \rfloor} > r \left(X_{\lfloor n^{1-\varepsilon} \rfloor} \right)$ while on the event $\{\tau_y \leq \lfloor n^{1-\varepsilon} \rfloor, \widehat{T}_z > \lfloor n^{1-\varepsilon} \rfloor\}$ we have $z + M_{\lfloor n^{1-\varepsilon} \rfloor} > 0$. Therefore, by the definition of \widehat{T}_z ,

$$\begin{aligned}
 -J''_{21} &\leq -\mathbb{E}_x \left(r \left(X_{\lfloor n^{1-\varepsilon} \rfloor} \right); \tau_y > \lfloor n^{1-\varepsilon} \rfloor, \nu_n > \lfloor n^{1-\varepsilon} \rfloor \right) \\
 &\leq c \mathbb{E}_x \left(1 + N \left(X_{\lfloor n^{1-\varepsilon} \rfloor} \right); \widehat{T}_z > \lfloor n^{1-\varepsilon} \rfloor, \nu_n > \lfloor n^{1-\varepsilon} \rfloor \right).
 \end{aligned}$$

Using the Markov property and (3.2.2),

$$\begin{aligned} -J''_{21} &\leq c\mathbb{E}_x \left(1 + e^{-cn^\varepsilon} N(X_{m_\varepsilon}) ; \widehat{T}_z > m_\varepsilon, \nu_n > m_\varepsilon \right) \\ &\leq c\mathbb{P}_x \left(\nu_n > m_\varepsilon, \widehat{T}_z > m_\varepsilon \right) + ce^{-cn^\varepsilon} (1 + N(x)). \end{aligned}$$

By Lemma 3.6.3,

$$-J''_{21} \leq c_\varepsilon e^{-c_\varepsilon n^\varepsilon} (1 + N(x)). \quad (3.6.20)$$

Putting together (3.6.20) and (3.6.19),

$$J'_{21} \leq \frac{c_\varepsilon}{n^\varepsilon} \mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor} ; \widehat{T}_z > \lfloor n^{1-\varepsilon} \rfloor \right) + c_\varepsilon e^{-c_\varepsilon n^\varepsilon} (1 + N(x)). \quad (3.6.21)$$

From (3.6.21) and (3.6.18) it follows that

$$J_{21} \leq \frac{c_\varepsilon}{n^\varepsilon} \mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor} ; \widehat{T}_z > \lfloor n^{1-\varepsilon} \rfloor \right) + c_\varepsilon e^{-c_\varepsilon n^\varepsilon} (1 + N(x)). \quad (3.6.22)$$

Bound of J_{22} . On the event $\{\widehat{T}_z > \nu_n^\varepsilon, \tau_y \leq \nu_n^\varepsilon\}$ we have $z + M_{\nu_n^\varepsilon} > 0$. Consequently

$$\begin{aligned} J_{22} &= \mathbb{E}_x \left(z + M_{\nu_n^\varepsilon} ; \widehat{T}_z > \nu_n^\varepsilon, \nu_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\quad + \mathbb{E}_x \left(\max(z + M_{\nu_n^\varepsilon}, 0) - (z + M_{\nu_n^\varepsilon}) ; \tau_y > \nu_n^\varepsilon, \nu_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor \right). \end{aligned}$$

By Lemma 3.5.4,

$$\begin{aligned} J_{22} &\leq \mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor} ; \widehat{T}_z > \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\quad - \underbrace{\mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor} ; \widehat{T}_z > \lfloor n^{1-\varepsilon} \rfloor, \nu_n^\varepsilon > \lfloor n^{1-\varepsilon} \rfloor \right)}_{=: J''_{22}} \\ &\quad - \underbrace{\mathbb{E}_x \left(z + M_{\nu_n^\varepsilon} ; z + M_{\nu_n^\varepsilon} < 0, \tau_y > \nu_n^\varepsilon, \nu_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor \right)}_{=: J'_{22}}. \end{aligned} \quad (3.6.23)$$

In the same way as in the proof of the bound of J''_{21} , replacing ν_n by ν_n^ε , one can prove that

$$-J''_{22} \leq c_\varepsilon e^{-c_\varepsilon n^\varepsilon} (1 + N(x)). \quad (3.6.24)$$

Moreover, using (3.5.1), on the event $\{\tau_y > \nu_n^\varepsilon\}$, we have $-(z + M_{\nu_n^\varepsilon}) < -r(X_{\nu_n^\varepsilon})$. So, by Lemma 3.5.1 and the Markov property

$$\begin{aligned} J'_{22} &\leq \mathbb{E}_x \left(|r(X_{\nu_n^\varepsilon})| ; \widehat{T}_z > \nu_n^\varepsilon, \nu_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\leq \mathbb{E}_x \left(c(1 + N(X_{\nu_n^\varepsilon})) ; \widehat{T}_z > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &= c \int_{\mathbb{X}} \mathbb{E}_{x'} \left(1 + N(X_{\lfloor n^\varepsilon \rfloor}) \right) \mathbb{P}_x \left(X_{\nu_n} \in dx', \widehat{T}_z > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right). \end{aligned}$$

Using (3.2.2),

$$J'_{22} \leq c\mathbb{E}_x \left(1 + e^{-cn^\varepsilon} N(X_{\nu_n}) ; \widehat{T}_z > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right).$$

Therefore, from (3.6.17) with the notation J'_{21} from (3.6.18),

$$J'_{22} \leq J'_{21} + c_\varepsilon e^{-c_\varepsilon n^\varepsilon} (1 + N(x)). \quad (3.6.25)$$

With (3.6.21), (3.6.23) and (3.6.24) we obtain,

$$J_{22} \leq \left(1 + \frac{c_\varepsilon}{n^\varepsilon}\right) u_{\lfloor n^{1-\varepsilon} \rfloor} + c_\varepsilon e^{-c_\varepsilon n^\varepsilon} (1 + N(x)). \quad (3.6.26)$$

Inserting (3.6.26) and (3.6.22) into (3.6.16),

$$J_2 \leq \left(1 + \frac{c_\varepsilon}{n^\varepsilon}\right) u_{\lfloor n^{1-\varepsilon} \rfloor} + c_\varepsilon e^{-c_\varepsilon n^\varepsilon} (1 + N(x)). \quad (3.6.27)$$

Now, inserting (3.6.15) and (3.6.27) into (3.6.14), we find (3.6.13). \square

Corollary 3.6.5. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$, $y \in \mathbb{R}$, $n \geq 2$ and any integer $k_0 \in \{2, \dots, n\}$,*

$$\mathbb{E}_x(y + S_n; \tau_y > n) \leq \left(1 + \frac{c_\varepsilon}{k_0^\varepsilon}\right) (\max(y, 0) + cN(x)) + c_\varepsilon k_0^{1/2}.$$

Proof. First, using the definition of \widehat{T}_z and Lemma 3.6.4, with $z = y + r(x)$,

$$\begin{aligned} \mathbb{E}_x(z + M_n; \tau_y > n) &= \mathbb{E}_x(z + M_n; \widehat{T}_z > n) - \mathbb{E}_x(z + M_n; \tau_y \leq n, \widehat{T}_z > n) \\ &\leq \mathbb{E}_x(z + M_n; \widehat{T}_z > n) \end{aligned} \quad (3.6.28)$$

$$\leq \left(1 + \frac{c_\varepsilon}{k_0^\varepsilon}\right) (\max(z, 0) + cN(x)) + c_\varepsilon k_0^{1/2}. \quad (3.6.29)$$

Now, using (3.5.1), Lemma 3.5.1 and (3.2.2),

$$\begin{aligned} \mathbb{E}_x(y + S_n; \tau_y > n) &= \mathbb{E}_x(z + M_n; \tau_y > n) - \mathbb{E}_x(r(X_n); \tau_y > n) \\ &\leq \mathbb{E}_x(z + M_n; \tau_y > n) + c(1 + e^{-cn} N(x)) \\ &\leq \left(1 + \frac{c_\varepsilon}{k_0^\varepsilon}\right) (\max(z, 0) + cN(x)) + c_\varepsilon k_0^{1/2}. \end{aligned}$$

Using the definition of z concludes the proof. \square

3.7 Existence and properties of the harmonic function

The idea is very simple. Set for brevity $V_n(x, y) := \mathbb{E}_x(y + S_n; \tau_y > n)$. By the Markov property $V_{n+1}(x, y) = \mathbf{Q}_+ V_n(x, y)$. We show that $\lim_{n \rightarrow \infty} V_n(x, y)$ exists and is equal to $V(x, y) := -\mathbb{E}_x(M_{\tau_y})$. Then the harmonicity of V follows by the Lebesgue dominated convergence theorem. The key point of the proof is the integrability of the random variable M_{τ_y} . To justify the applicability of the Lebesgue dominated convergence theorem we use Lemma 3.6.4. We also shall establish some properties of V . They will be deduced from those of the following two functions: $W(x, z) := -\mathbb{E}_x(M_{T_z})$ and $\widehat{W}(x, z) := -\mathbb{E}_x(M_{\widehat{T}_z})$. The strict positivity of V is technically more delicate and therefore is deferred to the next section.

Lemma 3.7.1. *Let $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z = y + r(x)$. The random variables $M_{\widehat{T}_z}$, M_{T_z} and M_{τ_y} are integrable and*

$$\max \left\{ \mathbb{E}_x \left(\left| M_{\widehat{T}_z} \right| \right), \mathbb{E}_x \left(\left| M_{T_z} \right| \right), \mathbb{E}_x \left(\left| M_{\tau_y} \right| \right) \right\} \leq c(1 + |z| + N(x)) < +\infty.$$

In particular, the following functions are well defined, for any $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z \in \mathbb{R}$,

$$V(x, y) := -\mathbb{E}_x \left(M_{\tau_y} \right), \quad W(x, z) := -\mathbb{E}_x \left(M_{T_z} \right) \quad \text{and} \quad \widehat{W}(x, z) := -\mathbb{E}_x \left(M_{\widehat{T}_z} \right).$$

Proof. Let $n \geq 1$. The stopping times $\tau_y \wedge n$, $T_z \wedge n$ and $\widehat{T}_z \wedge n$ are bounded and satisfy $\tau_y \wedge n \leq \widehat{T}_z \wedge n$ and $T_z \wedge n \leq \widehat{T}_z \wedge n$. Since $(|M_n|)_{n \geq 0}$ is a submartingale, we have

$$\max \left\{ \mathbb{E}_x \left(\left| M_{\tau_y \wedge n} \right| \right), \mathbb{E}_x \left(\left| M_{T_z \wedge n} \right| \right) \right\} \leq \mathbb{E}_x \left(\left| M_{\widehat{T}_z \wedge n} \right| \right). \quad (3.7.1)$$

Using the optional stopping theorem,

$$\begin{aligned} \mathbb{E}_x \left(\left| M_{\widehat{T}_z \wedge n} \right| \right) &\leq -\mathbb{E}_x \left(z + M_{\widehat{T}_z}; \widehat{T}_z \leq n \right) + \mathbb{E}_x \left(|z + M_n|; \tau_y > n \right) \\ &\quad + \mathbb{E}_x \left(z + M_n; \tau_y \leq n, \widehat{T}_z > n \right) + |z| \\ &= -\mathbb{E}_x \left(z + M_n; \widehat{T}_z \leq n \right) - 2\mathbb{E}_x \left(z + M_n; z + M_n \leq 0, \tau_y > n \right) \\ &\quad + \mathbb{E}_x \left(z + M_n; \tau_y > n \right) + \mathbb{E}_x \left(z + M_n; \tau_y \leq n, \widehat{T}_z > n \right) + |z| \\ &= -z + 2\mathbb{E}_x \left(z + M_n; \widehat{T}_z > n \right) \\ &\quad - 2\mathbb{E}_x \left(z + M_n; z + M_n \leq 0, \tau_y > n \right) + |z|. \end{aligned}$$

On the event $\{z + M_n \leq 0, \tau_y > n\}$, by (3.5.1), it holds $|z + M_n| \leq |r(X_n)|$. Therefore, by Lemma 3.5.1 and the point 1 of Hypothesis M3.4, we have

$$-2\mathbb{E}_x \left(z + M_n; z + M_n \leq 0, \tau_y > n \right) \leq c(1 + N(x)),$$

Using Lemma 3.6.4,

$$\mathbb{E}_x \left(\left| M_{\widehat{T}_z} \right|; \widehat{T}_z \leq n \right) \leq \mathbb{E}_x \left(\left| M_{\widehat{T}_z \wedge n} \right| \right) \leq c(1 + |z| + N(x)). \quad (3.7.2)$$

By the Lebesgue monotone convergence theorem and the fact that $\widehat{T}_z < +\infty$, we deduce that $M_{\widehat{T}_z}$ is \mathbb{P}_x -integrable and

$$\mathbb{E}_x \left(\left| M_{\widehat{T}_z} \right| \right) \leq c(1 + |z| + N(x)).$$

In the same manner, using (3.7.1), (3.7.2) and Lemmas 3.5.5 and 3.5.6, we conclude that M_{τ_y} and M_{T_z} are \mathbb{P}_x -integrable and

$$\max \left\{ \mathbb{E}_x \left(\left| M_{\tau_y} \right| \right), \mathbb{E}_x \left(\left| M_{T_z} \right| \right) \right\} \leq c(1 + |z| + N(x)).$$

The assertion of the lemma follows obviously from the last two inequalities. \square

Proposition 3.7.2.

1. *Let $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z = y + r(x)$. Then*

$$V(x, y) = \lim_{n \rightarrow +\infty} \mathbb{E}_x \left(z + M_n; \tau_y > n \right) = \lim_{n \rightarrow +\infty} \mathbb{E}_x \left(y + S_n; \tau_y > n \right)$$

and

$$\begin{aligned} W(x, z) &= \lim_{n \rightarrow +\infty} \mathbb{E}_x \left(z + M_n; T_z > n \right), \\ \widehat{W}(x, z) &= \lim_{n \rightarrow +\infty} \mathbb{E}_x \left(z + M_n; \widehat{T}_z > n \right). \end{aligned}$$

2. For any $x \in \mathbb{X}$, the functions $y \mapsto V(x, y)$, $z \mapsto W(x, z)$ and $z \mapsto \widehat{W}(x, z)$ are non-decreasing on \mathbb{R} .
3. There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$, $z \in \mathbb{R}$ and any integer $k_0 \geq 2$,

$$\widehat{W}(x, z) \leq \left(1 + \frac{c_\varepsilon}{k_0^\varepsilon}\right) (\max(z, 0) + cN(x)) + c_\varepsilon k_0^{1/2} \quad (3.7.3)$$

and, for any $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z = y + r(x)$,

$$0 \leq \min\{V(x, y), W(x, z)\} \leq \max\{V(x, y), W(x, z)\} \leq \widehat{W}(x, y). \quad (3.7.4)$$

In particular, for any $x \in \mathbb{X}$ and $y \in \mathbb{R}$,

$$0 \leq V(x, y) \leq c(1 + \max(y, 0) + N(x)). \quad (3.7.5)$$

4. For any $x \in \mathbb{X}$ and $y \in \mathbb{R}$,

$$V(x, y) = \mathbf{Q}_+ V(x, y) := \mathbb{E}_x(V(X_1, y + S_1); \tau_y > 1)$$

and $(V(X_n, y + S_n) \mathbb{1}_{\{\tau_y > n\}})_{n \geq 0}$ is a \mathbb{P}_x -martingale.

Proof. Claim 1. Let v be any of the stopping times τ_y, T_z , or \widehat{T}_z . By the martingale property, for $n \geq 1$,

$$\mathbb{E}_x(z + M_n; v > n) = z\mathbb{P}_x(v > n) - \mathbb{E}_x(M_n; v \leq n).$$

Using Lemmas 3.5.5, 3.5.6, 3.5.7, 3.7.1 and the Lebesgue dominated convergence theorem,

$$\mathbb{E}_x(z + M_n; v > n) = -\mathbb{E}_x(M_n).$$

Moreover, by (3.5.1),

$$\mathbb{E}_x(y + S_n; \tau_y > n) = \mathbb{E}_x(z + M_n; \tau_y > n) - \mathbb{E}_x(r(X_n); \tau_y > n).$$

Since, by Lemma 3.5.1, the point 1 of Hypothesis M3.4 and Lemma 2.9.7, we have

$$\begin{aligned} |\mathbb{E}_x(r(X_n); \tau_y > n)| &\leq c\mathbb{E}_x^{1/2}((1 + N(X_n))^2) \mathbb{P}_x^{1/2}(\tau_y > n) \\ &\leq c(1 + N(x)) \mathbb{P}_x^{1/2}(\tau_y > n) \xrightarrow{n \rightarrow +\infty} 0, \end{aligned} \quad (3.7.6)$$

the claim 1 follows.

Proof of the claim 2. Let $x \in \mathbb{X}$. For any $y' \leq y$, we obviously have $\tau_{y'} \leq \tau_y$. Therefore, for $n \geq 1$,

$$\mathbb{E}_x(y' + S_n; \tau_{y'} > n) \leq \mathbb{E}_x(y + S_n; \tau_{y'} > n) \leq \mathbb{E}_x(y + S_n; \tau_y > n).$$

Taking the limit as $n \rightarrow +\infty$ and using the claim 1, it follows that $V(x, y') \leq V(x, y)$. In the same way $W(x, z') \leq W(x, z)$ for $z' \leq z$. To prove the monotonicity of \widehat{W} , we note that, for any $z' \leq z$, $y' = z' - r(x)$ and $y = z - r(x)$, we have $\widehat{T}_{z'} = \min\{k \geq \tau_{y'} : z' + M_k \leq 0\} \leq \min\{k \geq \tau_y : z' + M_k \leq 0\} \leq \widehat{T}_z$. So

$$\begin{aligned} \mathbb{E}_x(z' + M_n; \widehat{T}_{z'} > n) &\leq \mathbb{E}_x(z + M_n; \widehat{T}_{z'} > n, \widehat{T}_z > n) \\ &\leq \mathbb{E}_x(y + S_n; \tau_y > n) + \mathbb{E}_x(|r(X_n)|; \tau_y > n) \\ &\quad + \mathbb{E}_x(z + M_n; \tau_y \leq n, \widehat{T}_z > n) \\ &\leq \mathbb{E}_x(z + M_n; \widehat{T}_z > n) + 2\mathbb{E}_x(|r(X_n)|; \tau_y > n). \end{aligned}$$

As in (3.7.6), taking the limit as $n \rightarrow +\infty$, by the claim 1, we have $\widehat{W}(x, z') \leq \widehat{W}(x, z)$.

Proof of the claim 3. The inequality (3.7.3) is a direct consequence of the claim 1 and Lemma 3.6.4. Moreover, taking the limit as $n \rightarrow \infty$ in (3.6.28), we get $V(x, y) \leq \widehat{W}(x, z)$.

To bound W , we write, for $n \geq 1$,

$$\begin{aligned} \mathbb{E}_x(z + M_n; T_z > n) &= \mathbb{E}_x(z + M_n; \tau_y \leq n, \widehat{T}_z > n, T_z > n) \\ &\quad + \mathbb{E}_x(z + M_n; z + M_n > 0, \tau_y > n, T_z > n). \end{aligned}$$

Since $z + M_n > 0$ on the event $\{\tau_y \leq n, \widehat{T}_z > n\}$,

$$\begin{aligned} \mathbb{E}_x(z + M_n; T_z > n) &\leq \mathbb{E}_x(z + M_n; \tau_y \leq n, \widehat{T}_z > n) \\ &\quad + \mathbb{E}_x(z + M_n; z + M_n > 0, \tau_y > n) \\ &= \mathbb{E}_x(z + M_n; \widehat{T}_z > n) \\ &\quad - \mathbb{E}_x(z + M_n; z + M_n \leq 0, \tau_y > n). \end{aligned}$$

Using the approximation (3.5.1),

$$\mathbb{E}_x(z + M_n; T_z > n) \leq \mathbb{E}_x(z + M_n; \widehat{T}_z > n) + \mathbb{E}_x(|r(X_n)|; \tau_y > n). \quad (3.7.7)$$

As in (3.7.6), using the claim 1,

$$W(x, z) \leq \widehat{W}(x, z).$$

Now, since $y + S_n$ is positive on the event $\{\tau_y > n\}$, by the claim 1, we see that $V(x, y) \geq 0$ and in the same way, $W(x, z) \geq 0$. This proves (3.7.4).

Inequality (3.7.5) follows from (3.7.3) and (3.7.4).

Proof of the claim 4. By the Markov property, for $n \geq 1$,

$$\begin{aligned} V_{n+1}(x, y) &:= \mathbb{E}_x(y + S_{n+1}; \tau_y > n + 1) \\ &= \int_{\mathbb{X} \times \mathbb{R}} V_n(x', y') \mathbb{P}_x(X_1 \in dx', y + S_1 \in dy', \tau_y > 1), \end{aligned} \quad (3.7.8)$$

where, by Corollary 3.6.5, $V_n(x', y') \leq c(1 + |y'| + N(x'))$ and by the point 1 of Hypothesis M3.4,

$$\mathbb{E}_x(1 + |y + S_1| + N(X_1)) \leq c(1 + |y| + N(x)) < +\infty.$$

Taking the limit in (3.7.8), by the Lebesgue dominated convergence theorem, we have

$$V(x, y) = \mathbf{Q}_+ V(x, y) := \mathbb{E}_x(V(X_1, y + S_1); \tau_y > 1).$$

□

3.8 Positivity of the harmonic function

The aim of this section is to prove that the harmonic function V is non-identically zero and to precise its support.

For any $x \in \mathbb{X}$, $z \in \mathbb{R}$ and $n \geq 0$, denote for brevity,

$$\widehat{W}_n(x, z) = \widehat{W}(X_n, z + M_n) \mathbb{1}_{\{\widehat{T}_z > n\}}. \quad (3.8.1)$$

Although it is easy to verify that $\widehat{W}(x, z) \geq z$ (see Lemma 3.8.1) which, in turn, ensures that $\widehat{W}(x, z) > 0$ for any $z > 0$, it is not straightforward to give a lower bound for the function V . We show that $V(x, y) = \lim_{n \rightarrow +\infty} \mathbb{E}_x(\widehat{W}_n(x, z); \tau_y > n)$ (Lemma 3.8.2) and use the fact that $(\widehat{W}_n(x, z) \mathbb{1}_{\{\tau_y > n\}})_{n \geq 0}$ is a \mathbb{P}_x -supermartingale (Lemma 3.8.1). By a recurrent procedure similar to that used in Lemma 3.6.4, we obtain a lower bound for V (Lemma 3.8.6) which subsequently is used to prove the positivity of V (Lemma 3.8.8).

Lemma 3.8.1.

1. For any $x \in \mathbb{X}$ and $z \in \mathbb{R}$,

$$\widehat{W}(x, z) \geq z.$$

2. For any $x \in \mathbb{X}$,

$$\lim_{z \rightarrow +\infty} \frac{\widehat{W}(x, z)}{z} = 1.$$

3. The function \widehat{W} is subharmonic, i.e. for any $x \in \mathbb{X}$, $z \in \mathbb{R}$ and $n \geq 0$,

$$\mathbb{E}_x(\widehat{W}_n(x, z)) \geq \widehat{W}(x, z).$$

4. For any $x \in \mathbb{X}$ and $z \in \mathbb{R}$, $(\widehat{W}_n(x, z) \mathbb{1}_{\{\tau_y > n\}})_{n \geq 0}$ is a \mathbb{P}_x -supermartingale.

Proof. *Claim 1.* By the Doob optional theorem and the definition of \widehat{T}_z , for any $n \geq 1$,

$$\mathbb{E}_x(z + M_n; \widehat{T}_z > n) = z - \mathbb{E}_x(z + M_{\widehat{T}_z}; \widehat{T}_z \leq n) \geq z.$$

Taking the limit as $n \rightarrow +\infty$ and using the point 1 of Proposition 3.7.2 proves the claim 1.

Proof of the claim 2. By the claim 1, $\liminf_{z \rightarrow +\infty} \widehat{W}(x, z)/z \geq 1$. Moreover, by (3.7.3), for any $k_0 \geq 2$,

$$\limsup_{z \rightarrow \infty} \frac{\widehat{W}(x, z)}{z} \leq \left(1 + \frac{c_\varepsilon}{k_0^\varepsilon}\right).$$

Taking the limit as $k_0 \rightarrow +\infty$, the claim follows.

Proof of the claim 3. Recall the notation $y = z - r(x)$. Using the Markov property, as in the proof of Lemma 3.5.3, for any $k \geq 1$,

$$\begin{aligned} \mathbb{E}_x(z + M_{n+k}; \widehat{T}_z > n + k) &= \int_{\mathbb{X} \times \mathbb{R}} \mathbb{E}_{x'}(z' + M_n; \widehat{T}_{z'} > n) \\ &\quad \times \mathbb{P}_x(X_k \in dx', z + M_k \in dz', \tau_y > k) \\ &\quad + \int_{\mathbb{X} \times \mathbb{R}} \mathbb{E}_{x'}(z' + M_n; T_{z'} > n) \\ &\quad \times \mathbb{P}_x(X_k \in dx', z + M_k \in dz', \tau_y \leq k, \widehat{T}_z > k). \end{aligned} \quad (3.8.2)$$

We shall find the limits as $n \rightarrow +\infty$ of the two terms in the right hand side of (3.8.2). By Lemmas 3.6.4 and 3.5.1, $\mathbb{E}_{x'}(z' + M_n; \widehat{T}_{z'} > n) \leq c(1 + |y'| + N(x'))$, with $y' = z' - r(x')$. Moreover by the point 1 of Hypothesis M3.4, $\mathbb{E}_x(1 + |y + S_k| + N(X_k)) \leq ck(1 + |y| + N(x)) < +\infty$. So, by the Lebesgue dominated convergence theorem and the point 1 of Proposition 3.7.2,

$$\begin{aligned} \int_{\mathbb{X} \times \mathbb{R}} \mathbb{E}_{x'}(z' + M_n; \widehat{T}_{z'} > n) \mathbb{P}_x(X_k \in dx', z + M_k \in dz', \tau_y > k) \\ \xrightarrow{n \rightarrow +\infty} \mathbb{E}_x(\widehat{W}(X_k, z + M_k); \tau_y > k). \end{aligned} \quad (3.8.3)$$

Moreover, using (3.7.7), Lemmas 3.6.4 and 3.5.1 and the point 1 of Hypothesis M3.4,

$$\mathbb{E}_{x'}(z' + M_n; T_{z'} > n) \leq c(1 + |z'| + N(x')).$$

Again, by the Lebesgue dominated convergence theorem and the point 1 of Proposition 3.7.2, we have

$$\begin{aligned} \int_{\mathbb{X} \times \mathbb{R}} \mathbb{E}_{x'}(z' + M_n; T_{z'} > n) \mathbb{P}_x(X_k \in dx', z + M_k \in dz', \tau_y \leq k, \widehat{T}_z > k) \\ \xrightarrow{n \rightarrow +\infty} \mathbb{E}_x(W(X_k, z + M_k); \tau_y \leq k, \widehat{T}_z > k). \end{aligned} \quad (3.8.4)$$

Putting together (3.8.2), (3.8.3), (3.8.4) and using the point 1 of Proposition 3.7.2,

$$\begin{aligned} \widehat{W}(x, z) &= \mathbb{E}_x(\widehat{W}(X_k, z + M_k); \tau_y > k) \\ &\quad + \mathbb{E}_x(W(X_k, z + M_k); \tau_y \leq k, \widehat{T}_z > k). \end{aligned} \quad (3.8.5)$$

Now, taking into account (3.7.4) and the identity $\{\tau_y > k\} = \{\tau_y > k, \widehat{T}_z > k\}$, we obtain the claim 3.

Proof of the claim 4. By the point 3 of Proposition 3.7.2, W is a non-negative function. Therefore, using (3.8.5),

$$\widehat{W}(x, z) \geq \mathbb{E}_x(\widehat{W}(X_1, z + M_1); \tau_y > 1),$$

which implies that $(\widehat{W}_n(x, z) \mathbb{1}_{\{\tau_y > n\}})_{n \geq 0}$ is a supermartingale. \square

Lemma 3.8.2. For any $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z = y + r(x)$,

$$V(x, y) = \lim_{n \rightarrow +\infty} \mathbb{E}_x(\widehat{W}_n(x, z); \tau_y > n).$$

Proof. For any $n \geq 1$, $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z = y + r(x)$,

$$\mathbb{E}_x(z + M_n; \tau_y > n) = \mathbb{E}_x(z + M_n; \widehat{T}_z > n) - \mathbb{E}_x(z + M_n; \tau_y \leq n, \widehat{T}_z > n).$$

By the point 1 of Lemma 3.8.1, on the event $\{\widehat{T}_z > n\}$ we have $z + M_n \leq \widehat{W}_n(x, z)$ and therefore

$$\begin{aligned} \mathbb{E}_x(z + M_n; \tau_y > n) &\geq \mathbb{E}_x(z + M_n; \widehat{T}_z > n) - \mathbb{E}_x(\widehat{W}_n(x, z)) \\ &\quad + \mathbb{E}_x(\widehat{W}_n(x, z); \tau_y > n). \end{aligned} \quad (3.8.6)$$

Moreover, by (3.7.3), for any $\delta > 0$,

$$\begin{aligned} \mathbb{E}_x(\widehat{W}_n(x, z)) &\leq (1 + \delta) \mathbb{E}_x(z + M_n; \widehat{T}_z > n) + c_\delta \mathbb{E}_x(1 + N(X_n); \widehat{T}_z > n) \\ &\quad - (1 + \delta) \mathbb{E}_x(z + M_n; z + M_n < 0, \tau_y > n). \end{aligned}$$

On the event $\{z + M_n < 0, \tau_y > n\}$, by (3.5.1), it holds $r(X_n) < z + M_n < 0$. Therefore, using Lemma 3.5.1,

$$\mathbb{E}_x(\widehat{W}_n(x, z)) \leq (1 + \delta) \mathbb{E}_x(z + M_n; \widehat{T}_z > n) + c_\delta \mathbb{E}_x(1 + N(X_n); \widehat{T}_z > n).$$

By the Markov property and (3.2.2),

$$\begin{aligned} \mathbb{E}_x \left(1 + N(X_n) ; \widehat{T}_z > n \right) &\leq c \mathbb{E}_x \left(1 + e^{-cn/2} N(X_{\lfloor n/2 \rfloor}) ; \widehat{T}_z > \lfloor n/2 \rfloor \right) \\ &\leq c \mathbb{P}_x \left(\widehat{T}_z > \lfloor n/2 \rfloor \right) + c e^{-cn} (1 + N(x)). \end{aligned}$$

By Lemma 3.5.7 and the point 1 of Proposition 3.7.2,

$$\lim_{n \rightarrow +\infty} \mathbb{E}_x \left(\widehat{W}_n(x, z) \right) \leq (1 + \delta) \widehat{W}(x, z). \quad (3.8.7)$$

Taking the limit as $n \rightarrow +\infty$ in (3.8.6) and using the previous bound, we obtain that

$$V(x, y) \geq -\delta \widehat{W}(x, z) + \lim_{n \rightarrow +\infty} \mathbb{E}_x \left(\widehat{W}_n(x, z) ; \tau_y > n \right).$$

Since this inequality holds true for any $\delta > 0$ small enough, we obtain the bound

$$\lim_{n \rightarrow +\infty} \mathbb{E}_x \left(\widehat{W}_n(x, z) ; \tau_y > n \right) \leq V(x, y). \quad (3.8.8)$$

Now, by the point 1 of Lemma 3.8.1,

$$\mathbb{E}_x(z + M_n ; \tau_y > n) \leq \mathbb{E}_x \left(\widehat{W}(X_n, z + M_n) ; \tau_y > n \right).$$

Taking the limit as $n \rightarrow +\infty$ and using the point 1 of Proposition 3.7.2, we obtain that

$$V(x, y) \leq \lim_{n \rightarrow +\infty} \mathbb{E}_x \left(\widehat{W}_n(x, z) ; \tau_y > n \right).$$

Together with (3.8.8), this concludes the proof. \square

Remark 3.8.3. Taking the limit in the point 3 of Lemma 3.8.1, we can deduce that

$$\lim_{n \rightarrow +\infty} \mathbb{E}_x \left(\widehat{W}_n(x, z) \right) \geq \widehat{W}(x, z).$$

Coupling this result with (3.8.7), it follows that

$$\lim_{n \rightarrow +\infty} \mathbb{E}_x \left(\widehat{W}_n(x, z) \right) = \widehat{W}(x, z).$$

Lemma 3.8.4. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $n \geq 1$, $x \in \mathbb{X}$, $z \in \mathbb{R}$ and $y = z - r(x)$, we have*

$$\mathbb{E}_x \left(\widehat{W}_n(x, z) ; \tau_y > n \right) \geq \widehat{W}(x, z) + c \min(z, 0) - c_\varepsilon \left(n^{1/2-2\varepsilon} + n^{2\varepsilon} N(x) \right).$$

Proof. Using the point 3 of Lemma 3.8.1, the bound (3.7.3) and the point 1 of Hypothesis M3.4, we have, for any $n \geq 1$,

$$\begin{aligned} \mathbb{E}_x \left(\widehat{W}_n(x, z) ; \tau_y > n \right) &= \mathbb{E}_x \left(\widehat{W}_n(x, z) \right) - \mathbb{E}_x \left(\widehat{W}_n(x, z) ; \tau_y \leq n \right) \\ &\geq \widehat{W}(x, z) - c \mathbb{E}_x \left(z + M_n ; \tau_y \leq n, \widehat{T}_z > n \right) - c(1 + N(x)). \end{aligned}$$

Again by the point 1 of M3.4, Lemma 3.6.2 and the Doob optional stopping theorem,

$$\begin{aligned} \mathbb{E}_x \left(\widehat{W}_n(x, z) ; \tau_y > n \right) &\geq \widehat{W}(x, z) - c \left[\mathbb{E}_x \left(z + M_n ; \widehat{T}_z > n \right) - \mathbb{E}_x \left(z + M_n ; \tau_y > n \right) \right] \\ &\quad - c(1 + N(x)) \\ &\geq \widehat{W}(x, z) - c \left[\max(z, 0) - z + \mathbb{E}_x \left(z + M_{\tau_y} ; \tau_y \leq n \right) \right] \\ &\quad - c_\varepsilon \left(n^{1/2-2\varepsilon} + n^{2\varepsilon} N(x) \right) - c(1 + N(x)). \end{aligned}$$

By (3.5.1), $z + M_{\tau_y} \leq r(X_{\tau_y})$. Therefore, in the same way as in the proof of (3.6.2),

$$\mathbb{E}_x(z + M_{\tau_y}; \tau_y \leq n) \leq c\mathbb{E}_x(1 + N(X_{\tau_y}); \tau_y \leq n) \leq c_\varepsilon n^{1/2-2\varepsilon} + c_\varepsilon N(x).$$

Together with the previous bound, this implies that

$$\mathbb{E}_x(\widehat{W}_n(x, z); \tau_y > n) \geq \widehat{W}(x, z) + c \min(z, 0) - c_\varepsilon (n^{1/2-2\varepsilon} + n^{2\varepsilon} N(x)).$$

□

Lemma 3.8.5. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $n \geq 2$, $k_0 \in \{2, \dots, n\}$, $x \in \mathbb{X}$ and $z \in \mathbb{R}$, with $y = z - r(x)$, we have*

$$\mathbb{E}_x(\widehat{W}_n(x, z); \tau_y > n) \geq \mathbb{E}_x(\widehat{W}_{k_0}(x, z); \tau_y > k_0) - \frac{c_\varepsilon}{k_0^\varepsilon} (\max(z, 0) + 1 + N(x)).$$

Proof. Let $\varepsilon \in (0, 1)$. Set for brevity $u_n := \mathbb{E}_x(\widehat{W}_n(x, z); \tau_y > n)$ for $n \geq 1$. By the point 4 of Lemma 3.8.1, the sequence $(u_n)_{n \geq 1}$ is non-increasing. We shall prove that

$$u_n \geq u_{\lfloor n^{1-\varepsilon} \rfloor} - \frac{c_\varepsilon}{n^\varepsilon} (\max(z, 0) + 1 + N(x)). \quad (3.8.9)$$

By Lemma 9.2 of [36] (Lemma 2.9.2 in Chapter 2) on the convergence of recursively bounded non-increasing sequences, we conclude that, for any $n \geq 2$ and $k_0 \in \{2, \dots, n\}$,

$$u_n \geq u_{k_0} - \frac{c_\varepsilon}{k_0^\varepsilon} (\max(z, 0) + 1 + N(x)),$$

which proves the assertion of the lemma.

It remains to establish (3.8.9). Consider the stopping time $\nu_n^\varepsilon = \nu_n + \lfloor n^\varepsilon \rfloor$. By the Markov property, with $y' = z' - r(x')$,

$$\begin{aligned} u_n &\geq \mathbb{E}_x(\widehat{W}_n(x, z); \tau_y > n, \nu_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor) \\ &= \sum_{k=\lfloor n^\varepsilon \rfloor + 1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{X} \times \mathbb{R}} \mathbb{E}_{x'}(\widehat{W}_{n-k}(x', z'); \tau_{y'} > n - k) \\ &\quad \mathbb{P}_x(X_k \in dx', z + M_k \in dz', \tau_y > k, \nu_n^\varepsilon = k). \end{aligned}$$

Using Lemma 3.8.4, we obtain,

$$\begin{aligned} u_n &\geq \mathbb{E}_x(\widehat{W}_{\nu_n^\varepsilon}(x, z); \tau_y > \nu_n^\varepsilon, \nu_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor) \\ &\quad + c\mathbb{E}_x(\min(z + M_{\nu_n^\varepsilon}, 0); \tau_y > \nu_n^\varepsilon, \nu_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor) \\ &\quad - c_\varepsilon \mathbb{E}_x(n^{1/2-2\varepsilon} + n^{2\varepsilon} N(X_{\nu_n^\varepsilon}); \tau_y > \nu_n^\varepsilon, \nu_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor). \end{aligned}$$

On the event $\{z + M_{\nu_n^\varepsilon} \leq 0, \tau_y > \nu_n^\varepsilon\}$, by (3.5.1), we have $0 \geq z + M_{\nu_n^\varepsilon} \geq r(X_{\nu_n^\varepsilon})$. Therefore, by Lemma 3.5.1,

$$\begin{aligned} \mathbb{E}_x(\min(z + M_{\nu_n^\varepsilon}, 0); \tau_y > \nu_n^\varepsilon, \nu_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor) \\ \geq -c\mathbb{E}_x(1 + N(X_{\nu_n^\varepsilon}); \tau_y > \nu_n^\varepsilon, \nu_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor). \end{aligned}$$

Consequently, using the point 4 of Lemma 3.8.1 and (3.2.2),

$$\begin{aligned} u_n &\geq \mathbb{E}_x \left(\widehat{W}_{\lfloor n^{1-\varepsilon} \rfloor}(x, z); \tau_y > \lfloor n^{1-\varepsilon} \rfloor, \nu_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\quad - c_\varepsilon \mathbb{E}_x \left(n^{1/2-2\varepsilon} + e^{-c_\varepsilon n^\varepsilon} N(X_{\nu_n}); \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right). \end{aligned}$$

By the definition of ν_n , we have $n^{1/2-2\varepsilon} \leq (z + M_{\nu_n})/n^\varepsilon$. Then as in (3.6.17),

$$\begin{aligned} u_n &\geq \mathbb{E}_x \left(\widehat{W}_{\lfloor n^{1-\varepsilon} \rfloor}(x, z); \tau_y > \lfloor n^{1-\varepsilon} \rfloor, \nu_n^\varepsilon \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\quad - \frac{c_\varepsilon}{n^\varepsilon} \mathbb{E}_x \left(z + M_{\nu_n}; \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\quad - c_\varepsilon e^{-c_\varepsilon n^\varepsilon} (1 + N(x)). \end{aligned}$$

Rearranging the terms, we have

$$\begin{aligned} u_n &\geq u_{\lfloor n^{1-\varepsilon} \rfloor} - c_\varepsilon e^{-c_\varepsilon n^\varepsilon} (1 + N(x)) \\ &\quad - \frac{c_\varepsilon}{n^\varepsilon} \underbrace{\mathbb{E}_x \left(z + M_{\nu_n}; \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right)}_{=: I_1} \\ &\quad - \underbrace{\mathbb{E}_x \left(\widehat{W}_{\lfloor n^{1-\varepsilon} \rfloor}(x, z); \tau_y > \lfloor n^{1-\varepsilon} \rfloor, \nu_n^\varepsilon > \lfloor n^{1-\varepsilon} \rfloor \right)}_{=: I_2}. \end{aligned} \tag{3.8.10}$$

Bound of I_1 . To bound I_1 we use the facts that, by the definition of ν_n , $z + M_{\nu_n} > n^{1/2-\varepsilon} > 0$ and that $\widehat{T}_z \geq \tau_y$. Taking into account Lemma 3.5.4, we have

$$\begin{aligned} I_1 &\leq \mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor}; \widehat{T}_z > \lfloor n^{1-\varepsilon} \rfloor, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &= \mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor}; \widehat{T}_z > \lfloor n^{1-\varepsilon} \rfloor \right) - J''_{21}, \end{aligned}$$

where J''_{21} is defined in (3.6.19). Now, it follows from Lemma 3.5.4 and the point 1 of Proposition 3.7.2, that $(\mathbb{E}_x(z + M_{\lfloor n^{1-\varepsilon} \rfloor}; \widehat{T}_z > \lfloor n^{1-\varepsilon} \rfloor))_{n \geq 0}$ is a non-decreasing sequence which converges to $\widehat{W}(x, z)$ and so $\mathbb{E}_x(z + M_{\lfloor n^{1-\varepsilon} \rfloor}; \widehat{T}_z > \lfloor n^{1-\varepsilon} \rfloor) \leq \widehat{W}(x, z)$. Using (3.6.20), we find that

$$I_1 \leq \widehat{W}(x, z) + c_\varepsilon e^{-c_\varepsilon n^\varepsilon} (1 + N(x)). \tag{3.8.11}$$

Bound of I_2 . By (3.8.1) and (3.7.3),

$$\begin{aligned} I_2 &\leq c \mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor} \left(1 - \mathbb{1}_{\{z + M_{\lfloor n^{1-\varepsilon} \rfloor} < 0\}} \right); \widehat{T}_z > \lfloor n^{1-\varepsilon} \rfloor, \nu_n^\varepsilon > \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\quad + c \mathbb{E}_x \left(1 + N(X_{\lfloor n^{1-\varepsilon} \rfloor}); \widehat{T}_z > \lfloor n^{1-\varepsilon} \rfloor, \nu_n^\varepsilon > \lfloor n^{1-\varepsilon} \rfloor \right). \end{aligned}$$

On the event $\{z + M_{\lfloor n^{1-\varepsilon} \rfloor} < 0, \widehat{T}_z > \lfloor n^{1-\varepsilon} \rfloor\} = \{z + M_{\lfloor n^{1-\varepsilon} \rfloor} < 0, \tau_y > \lfloor n^{1-\varepsilon} \rfloor\}$, it holds $z + M_{\lfloor n^{1-\varepsilon} \rfloor} > r(X_{\lfloor n^{1-\varepsilon} \rfloor})$. Therefore, using Lemma 3.5.1,

$$I_2 \leq c \mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor} + 1 + N(X_{\lfloor n^{1-\varepsilon} \rfloor}); \widehat{T}_z > \lfloor n^{1-\varepsilon} \rfloor, \nu_n^\varepsilon > \lfloor n^{1-\varepsilon} \rfloor \right).$$

By Lemma 3.5.4,

$$\mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor}; \widehat{T}_z > \lfloor n^{1-\varepsilon} \rfloor, \nu_n^\varepsilon > \lfloor n^{1-\varepsilon} \rfloor \right) \leq J_1,$$

where J_1 is defined in (3.6.14). Using inequalities (3.6.15), (3.2.2) and Lemma 3.6.3, with $m_\varepsilon = \lfloor n^{1-\varepsilon} \rfloor - \lfloor n^\varepsilon \rfloor$, we obtain

$$\begin{aligned} I_2 &\leq c_\varepsilon e^{-c_\varepsilon n^\varepsilon} (1 + N(x)) + c \mathbb{E}_x \left(1 + e^{-cn^\varepsilon} N(X_{m_\varepsilon}) ; \widehat{T}_z > m_\varepsilon, \nu_n > m_\varepsilon \right) \\ &\leq c_\varepsilon e^{-c_\varepsilon n^\varepsilon} (1 + N(x)). \end{aligned} \tag{3.8.12}$$

Putting together (3.8.12), (3.8.11) and (3.8.10) and using (3.7.3), we obtain (3.8.9), which completes the proof of the lemma. \square

Proposition 3.8.6.

1. For any $\delta \in (0, 1)$, $x \in \mathbb{X}$ and $y > 0$,

$$V(x, y) \geq (1 - \delta)y - c_\delta (1 + N(x)).$$

2. For any $x \in \mathbb{X}$,

$$\lim_{y \rightarrow +\infty} \frac{V(x, y)}{y} = 1.$$

Proof. Claim 1. By Lemmas 3.8.5 and 3.8.2, we immediately have, with $z = y + r(x)$,

$$V(x, y) \geq \mathbb{E}_x \left(\widehat{W}_{k_0}(x, z) ; \tau_y > k_0 \right) - \frac{c_\varepsilon}{k_0^\varepsilon} (\max(z, 0) + 1 + N(x)).$$

Using the point 1 of Lemma 3.8.1 and the point 2 of Lemma 3.5.2,

$$\begin{aligned} V(x, y) &\geq \mathbb{E}_x (z + M_{k_0} ; \tau_y > k_0) - \frac{c_\varepsilon}{k_0^\varepsilon} (\max(z, 0) + 1 + N(x)) \\ &\geq z \mathbb{P}_x (\tau_y > k_0) - c \left(\sqrt{k_0} + N(x) \right) - \frac{c_\varepsilon}{k_0^\varepsilon} (\max(z, 0) + 1 + N(x)). \end{aligned}$$

Since, by the union bound and the Markov inequality,

$$\mathbb{P}_x (\tau_y > k_0) \geq \mathbb{P}_x \left(\max_{1 \leq k \leq k_0} |f(X_k)| < \frac{y}{k_0} \right) \geq 1 - \frac{ck_0^2 (1 + N(x))}{y},$$

we obtain that, by the definition of z ,

$$V(x, y) \geq \left(1 - \frac{c_\varepsilon}{k_0^\varepsilon} \right) y - c_\varepsilon k_0^2 (1 + N(x)). \tag{3.8.13}$$

Let $\delta \in (0, 1)$. Taking k_0 large enough, we obtain the desired inequality.

Proof of the claim 2. By the claim 1, for any $\delta \in (0, 1)$ and $x \in \mathbb{X}$, we have that $\liminf_{y \rightarrow +\infty} V(x, y)/y \geq 1 - \delta$. Taking the limit as $\delta \rightarrow 0$, we obtain the lower bound. Now by (3.7.4) and (3.7.3), for any integer $k_0 \geq 2$, $y \in \mathbb{R}$ and $z = y + r(x)$,

$$V(x, y) \leq \widehat{W}(x, z) \leq \left(1 + \frac{c_\varepsilon}{k_0^\varepsilon} \right) (\max(z, 0) + cN(x)) + c_\varepsilon k_0^{1/2}.$$

Using the definition of z , we conclude that

$$\limsup_{y \rightarrow +\infty} \frac{V(x, y)}{y} \leq \lim_{k_0 \rightarrow +\infty} \left(1 + \frac{c_\varepsilon}{k_0^\varepsilon} \right) = 1.$$

\square

Now, for any $\gamma > 0$, consider the stopping time:

$$\zeta_\gamma := \inf \{k \geq 1 : |y + S_k| > \gamma(1 + N(X_k))\}. \quad (3.8.14)$$

The control on the tail of ζ_γ is given by the following Lemma.

Lemma 3.8.7. *For any $\gamma > 0$, $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $n \geq 1$,*

$$\mathbb{P}_x(\zeta_\gamma > n) \leq c e^{-c_\gamma n} (1 + N(x)).$$

Proof. The reasoning is very close to that of the proof of the Lemma 3.6.3. Let $\gamma > 0$. Consider the integer $l \geq 1$ which will be chosen later. Define $K := \lfloor \frac{n}{2l} \rfloor$ and introduce the event $A_{k,y}^\gamma := \bigcap_{k' \in \{1, \dots, k\}} \{|y + S_{k'}| \leq \gamma(1 + N(X_{k'}))\}$. We have

$$\mathbb{P}_x(\zeta_\gamma > n) \leq \mathbb{P}_x(A_{2K,y}^\gamma).$$

By the Markov property,

$$\begin{aligned} \mathbb{P}_x(A_{2K,y}^\gamma) &= \int_{\mathbb{X} \times \mathbb{R}} \int_{\mathbb{X} \times \mathbb{R}} \mathbb{P}_{x''}(A_{1,y''}^\gamma) \mathbb{P}_{x'}(X_l \in dx'', y' + S_l \in dy'', A_{1,y'}^\gamma) \\ &\quad \times \mathbb{P}_x(X_{2(K-1)l} \in dx', y + S_{2(K-1)l} \in dy', A_{2(K-1),y}^\gamma). \end{aligned} \quad (3.8.15)$$

We write

$$\begin{aligned} \mathbb{P}_{x''}(A_{1,y''}^\gamma) &\leq \mathbb{P}_{x''}(|y'' + S_l| \leq 2\gamma\sqrt{l}) + \mathbb{P}_{x''}(N(X_l) > \sqrt{l}) \\ &\leq \mathbb{P}_{x''}\left(\frac{-y''}{\sqrt{l}} - 2\gamma \leq \frac{S_l}{\sqrt{l}} \leq \frac{-y''}{\sqrt{l}} + 2\gamma\right) + \mathbb{E}_{x''}\left(\frac{N(X_l)}{\sqrt{l}}\right). \end{aligned}$$

By Corollary 3.4.4 and the point 1 of Hypothesis M3.4, there exists $\varepsilon_0 \in (0, 1/4)$ such that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$\mathbb{P}_{x''}(A_{1,y''}^\gamma) \leq \int_{\frac{-y''}{\sqrt{l}} - 2\gamma}^{\frac{-y''}{\sqrt{l}} + 2\gamma} e^{-\frac{u^2}{2\sigma^2}} \frac{du}{\sqrt{2\pi}\sigma} + \frac{2c_\varepsilon}{l^\varepsilon} (1 + N(x'')) + \frac{c}{\sqrt{l}} (1 + N(x'')).$$

Set $q_\gamma := \int_{-2\gamma}^{2\gamma} e^{-\frac{u^2}{2\sigma^2}} \frac{du}{\sqrt{2\pi}\sigma} < 1$. From (3.8.15), we obtain

$$\begin{aligned} \mathbb{P}_x(A_{2K,y}^\gamma) &\leq \int_{\mathbb{X} \times \mathbb{R}} \left(q_\gamma + \frac{c_\varepsilon}{l^\varepsilon} + \frac{c_\varepsilon}{l^\varepsilon} \mathbb{E}_{x'}(N(X_l)) \right) \\ &\quad \times \mathbb{P}_x(X_{2(K-1)l} \in dx', y + S_{2(K-1)l} \in dy', A_{2(K-1),y}^\gamma) \\ &\leq \left(q_\gamma + \frac{c_\varepsilon}{l^\varepsilon} \right) \mathbb{P}_x(A_{2(K-1),y}^\gamma) + c_\varepsilon e^{-c_\varepsilon l} \mathbb{E}_x(N(X_{2(K-1)l}); A_{2(K-1),y}^\gamma). \end{aligned}$$

For brevity, set $p_K = \mathbb{P}_x(A_{2K,y}^\gamma)$ and $E_K = \mathbb{E}_x(N(X_{2Kl}); A_{2K,y}^\gamma)$. Then, the previous inequality can be rewritten as

$$p_K \leq \left(q_\gamma + \frac{c_\varepsilon}{l^\varepsilon} \right) p_{K-1} + c_\varepsilon e^{-c_\varepsilon l} E_{K-1}. \quad (3.8.16)$$

Moreover, from (3.2.2), we have

$$E_K \leq c p_{K-1} + c e^{-c_{2l}} E_{K-1}. \quad (3.8.17)$$

Using (3.8.16) and (3.8.17), we write that

$$\begin{pmatrix} p_K \\ E_K \end{pmatrix} \leq A_l \begin{pmatrix} p_{K-1} \\ E_{K-1} \end{pmatrix} \quad (3.8.18)$$

where

$$A_l := \begin{pmatrix} q_\gamma + \frac{c_\varepsilon}{l^\varepsilon} & c_\varepsilon e^{-c_\varepsilon l} \\ c & c e^{-cl} \end{pmatrix} \xrightarrow{l \rightarrow +\infty} A = \begin{pmatrix} q_\gamma & 0 \\ c & 0 \end{pmatrix}.$$

Since the spectral radius q_γ of A is less than 1, we can choose $l = l(\varepsilon, \gamma)$ large enough such that the spectral radius $\rho_{\varepsilon, \gamma}$ of A_l is less than 1. Iterating (3.8.18), we get

$$p_K \leq c \rho_{\varepsilon, \gamma}^K \max(p_1, E_1) \leq c \rho_{\varepsilon, \gamma}^K (1 + N(x)).$$

Taking into account that $K \geq c_{\varepsilon, \gamma} n$, we obtain

$$\mathbb{P}_x \left(A_{2K, y}^\gamma \right) \leq c e^{-c_\gamma n} (1 + N(x)).$$

□

Now we shall establish some properties of the set \mathcal{D}_γ introduced in Section 3.2. It is easy to see that, for any $\gamma > 0$,

$$\mathcal{D}_\gamma = \{(x, y) \in \mathbb{X} \times \mathbb{R} : \exists n_0 \geq 1, \mathbb{P}_x(\zeta_\gamma \leq n_0, \tau_y > n_0) > 0\},$$

where ζ_γ is defined by (3.8.14).

Proposition 3.8.8.

1. For any $\gamma_1 \leq \gamma_2$, it holds $\mathcal{D}_{\gamma_1} \supseteq \mathcal{D}_{\gamma_2}$.
2. For any $\gamma > 0$, there exists $c_\gamma > 0$ such that

$$\mathcal{D}_\gamma^c \subseteq \{(x, y) \in \mathbb{X} \times \mathbb{R} : \mathbb{P}_x(\tau_y > n) \leq e^{-c_\gamma n} (1 + N(x)), n \geq 1\}.$$

3. For any $\gamma > 0$, the domain of positivity of the function V is included in \mathcal{D}_γ :

$$\text{supp}(V) = \{(x, y) \in \mathbb{X} \times \mathbb{R} : V(x, y) > 0\} \subseteq \mathcal{D}_\gamma.$$

4. There exists $\gamma_0 > 0$ such that for any $\gamma \geq \gamma_0$,

$$\text{supp}(V) = \mathcal{D}_\gamma.$$

Moreover,

$$\left\{ (x, y) \in \mathbb{X} \times \mathbb{R}_+^* : y > \frac{\gamma_0}{2} (1 + N(x)) \right\} \subseteq \text{supp}(V).$$

Proof. **Claim 1.** For any $\gamma_1 \leq \gamma_2$, we have $\zeta_{\gamma_1} \leq \zeta_{\gamma_2}$ and the claim 1 follows.

Claim 2. Fix $\gamma > 0$. By the definition of \mathcal{D}_γ , for any $(x, y) \in \mathcal{D}_\gamma^c$ and $n \geq 1$,

$$0 = \mathbb{P}_x(\zeta_\gamma \leq n, \tau_y > n) = \mathbb{P}_x(\tau_y > n) - \mathbb{P}_x(\zeta_\gamma > n, \tau_y > n).$$

From this, using Lemma 3.8.7, we obtain

$$\mathbb{P}_x(\tau_y > n) = \mathbb{P}_x(\zeta_\gamma > n, \tau_y > n) \leq \mathbb{P}_x(\zeta_\gamma > n) \leq e^{-c_\gamma n} (1 + N(x)).$$

Claim 3. Fix $\gamma > 0$. Using the claim 2 and Lemma 3.5.2, we have, for any $(x, y) \in \mathcal{D}_\gamma^c$, $z = y + r(x)$ and $n \geq 1$,

$$\begin{aligned} \mathbb{E}_x(z + M_n; \tau_y > n) &\leq |z| \mathbb{P}_x(\tau_y > n) + \mathbb{E}_x^{1/2}(|M_n|^2) \mathbb{P}_x^{1/2}(\tau_y > n) \\ &\leq |z| (1 + N(x)) e^{-c_\gamma n} + c\sqrt{n} (1 + N(x))^{3/2} e^{-c_\gamma n}. \end{aligned}$$

Taking the limit when $n \rightarrow +\infty$, by the point 1 of Proposition 3.7.2, we get

$$V(x, y) = 0,$$

and we conclude that $\mathcal{D}_\gamma^c \subseteq \text{supp}(V)^c$.

Claim 4. By the point 1 of Proposition 3.8.6, taking $\delta = 1/2$, there exists $\gamma_0 > 0$ such that, for any $x \in \mathbb{X}$ and $y > 0$,

$$V(x, y) \geq \frac{y}{2} - \frac{\gamma_0}{4} (1 + N(x)). \tag{3.8.19}$$

Now, fix $(x, y) \in \mathcal{D}_{\gamma_0}$ and let $n_0 \geq 1$ be an integer such that $\mathbb{P}_x(\zeta_{\gamma_0} \leq n_0, \tau_y > n_0) > 0$. By the point 4 of Proposition 3.7.2,

$$\begin{aligned} V(x, y) &= \mathbb{E}_x(V(X_{n_0}, y + S_{n_0}); \tau_y > n_0) \\ &\geq \mathbb{E}_x(V(X_{n_0}, y + S_{n_0}); \tau_y > n_0, \zeta_{\gamma_0} \leq n_0). \end{aligned}$$

By the Doob optional stopping theorem, (3.8.19) and the definition of ζ_{γ_0} (see (3.8.14)),

$$\begin{aligned} V(x, y) &\geq \mathbb{E}_x(V(X_{\zeta_{\gamma_0}}, y + S_{\zeta_{\gamma_0}}); \tau_y > \zeta_{\gamma_0}, \zeta_{\gamma_0} \leq n_0) \\ &\geq \frac{1}{2} \mathbb{E}_x\left(y + S_{\zeta_{\gamma_0}} - \frac{\gamma_0}{2} (1 + N(X_{\zeta_{\gamma_0}})); \tau_y > \zeta_{\gamma_0}, \zeta_{\gamma_0} \leq n_0\right) \\ &\geq \frac{1}{2} \mathbb{E}_x\left(\frac{\gamma_0}{2} (1 + N(X_{\zeta_{\gamma_0}})); \tau_y > \zeta_{\gamma_0}, \zeta_{\gamma_0} \leq n_0\right) \\ &\geq \frac{\gamma_0}{4} \mathbb{P}_x(\tau_y > n_0, \zeta_{\gamma_0} \leq n_0). \end{aligned}$$

Now, since n_0 has been chosen such that the last probability is strictly positive, we get that $V(x, y) > 0$. This proves that $\mathcal{D}_{\gamma_0} \subseteq \text{supp}(V)$. Using the claims 1 and 3, for any $\gamma \geq \gamma_0$, we obtain that $\mathcal{D}_\gamma \subseteq \mathcal{D}_{\gamma_0} \subseteq \text{supp}(V) \subseteq \mathcal{D}_\gamma$ and so $\mathcal{D}_\gamma = \mathcal{D}_{\gamma_0} = \text{supp}(V)$. Using (3.8.19) proves the second assertion of the claim 4. \square

Proof of Theorem 3.2.2. The claim 1 is proved by the point 1 of Proposition 3.7.2 ; the claim 2 is proved by the point 4 of Proposition 3.7.2 ; the claim 3 is proved by the points 2 and 3 of Proposition 3.7.2 and by Proposition 3.8.6 ; the claim 4 is proved by the point 4 of Proposition 3.8.8.

3.9 Asymptotic behaviour of the exit time

3.9.1 Preliminary results

Lemma 3.9.1. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z = y + r(x)$,*

$$\begin{aligned} E_1 &:= \mathbb{E}_x(z + M_{\nu_n}; \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor) \leq c_\varepsilon (1 + \max(y, 0) + N(x)), \quad n \geq 1, \\ E_2 &:= \mathbb{E}_x(z + M_{\nu_n^{\varepsilon^2}}; \tau_y > \nu_n^{\varepsilon^2}, \nu_n^{\varepsilon^2} \leq \lfloor n^{1-\varepsilon} \rfloor) \xrightarrow{n \rightarrow \infty} V(x, y). \end{aligned}$$

Moreover, for any $n \geq 1$, $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$ and $y \in \mathbb{R}$,

$$|E_2 - V(x, y)| \leq \frac{c_\varepsilon}{n^{\varepsilon/8}} (1 + \max(y, 0) + N(x)).$$

Proof. Using the fact $\{\tau_y > \nu_n\} \subseteq \{\widehat{T}_z > \nu_n\}$ and Lemma 3.5.4, for $n \geq 1$,

$$E_1 \leq \mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor}; \widehat{T}_z > \lfloor n^{1-\varepsilon} \rfloor \right) - J''_{21},$$

where J''_{21} is defined in (3.6.19) and by (3.6.20) the quantity $-J''_{21}$ does not exceed $c_\varepsilon e^{-c_\varepsilon n^\varepsilon} (1 + N(x))$. Again, by Lemma 3.5.4 and the point 1 of Proposition 3.7.2, we have that $(\mathbb{E}_x(z + M_n; \widehat{T}_z > n))_{n \geq 0}$ is a non-decreasing sequence which converges to $\widehat{W}(x, z)$. So, using the point 3 of Proposition 3.7.2 and the fact that $z = y + r(x)$,

$$E_1 \leq \widehat{W}(x, z) + c_\varepsilon e^{-c_\varepsilon n^\varepsilon} (1 + N(x)) \leq c_\varepsilon (1 + \max(y, 0) + N(x)). \quad (3.9.1)$$

By the point 4 of Proposition 3.7.2, we have

$$\begin{aligned} V(x, y) &= \mathbb{E}_x \left(V(X_n, y + S_n); \tau_y > n, \nu_n^{\varepsilon^2} \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\quad + \mathbb{E}_x \left(V(X_n, y + S_n); \tau_y > n, \nu_n^{\varepsilon^2} > \lfloor n^{1-\varepsilon} \rfloor \right). \end{aligned}$$

Using the point 3 of Proposition 3.7.2, for any $k_0 \geq 2$,

$$\begin{aligned} V(x, y) &\leq \mathbb{E}_x \left(V(X_{\nu_n^{\varepsilon^2}}, y + S_{\nu_n^{\varepsilon^2}}); \tau_y > \nu_n^{\varepsilon^2}, \nu_n^{\varepsilon^2} \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\quad + c \mathbb{E}_x \left(\max(z + M_n, 0) + 1 + N(X_n); \tau_y > n, \nu_n^{\varepsilon^2} > \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\leq \left(1 + \frac{c_\varepsilon}{k_0^\varepsilon} \right) E_2 + c_\varepsilon \mathbb{E}_x \left(\sqrt{k_0} + N(X_{\nu_n^{\varepsilon^2}}); \tau_y > \nu_n^{\varepsilon^2}, \nu_n^{\varepsilon^2} \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\quad - \underbrace{c_\varepsilon \mathbb{E}_x \left(z + M_{\nu_n^{\varepsilon^2}}; z + M_{\nu_n^{\varepsilon^2}} < 0, \tau_y > \nu_n^{\varepsilon^2}, \nu_n^{\varepsilon^2} \leq \lfloor n^{1-\varepsilon} \rfloor \right)}_{=J'_{22}(\varepsilon^2)} \\ &\quad + c \mathbb{E}_x \left(z + M_n + |r(X_n)| + 1 + N(X_n); \tau_y > n, \nu_n^{\varepsilon^2} > \lfloor n^{1-\varepsilon} \rfloor \right). \end{aligned}$$

From the previous bound, using the Markov property, the bound (3.2.2) and the approximation (3.5.1), we get

$$\begin{aligned} V(x, y) &\leq \left(1 + \frac{c_\varepsilon}{k_0^\varepsilon} \right) E_2 + J'_{22}(\varepsilon^2) + \underbrace{c \mathbb{E}_x \left(z + M_n; \widehat{T}_z > n, \nu_n^{\varepsilon^2} > \lfloor n^{1-\varepsilon} \rfloor \right)}_{=J_1(\varepsilon^2)} \\ &\quad + c_\varepsilon \mathbb{E}_x \left(\sqrt{k_0} + e^{-cn^{\varepsilon^2}} N(X_{\nu_n}); \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\quad + c \mathbb{E}_x \left(1 + e^{-c_\varepsilon n} N(X_{\lfloor n^{1-\varepsilon} \rfloor}); \tau_y > \lfloor n^{1-\varepsilon} \rfloor, \nu_n^{\varepsilon^2} > \lfloor n^{1-\varepsilon} \rfloor \right). \end{aligned}$$

Proceeding in the same way as for the bound (3.6.25),

$$\begin{aligned} J'_{22}(\varepsilon^2) &\leq c_\varepsilon \mathbb{E}_x \left(1 + e^{-cn^{\varepsilon^2}} N(X_{\nu_n}); \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\leq \frac{c_\varepsilon}{n^{1/2-\varepsilon}} E_1 + c_\varepsilon e^{-c_\varepsilon n^{\varepsilon^2}} (1 + N(x)). \end{aligned}$$

Moreover, similarly as for the bound (3.6.15), we have

$$J_1(\varepsilon^2) \leq c_\varepsilon e^{-c_\varepsilon n^{\varepsilon^2}} (1 + N(x)).$$

Taking into account these bounds and using Lemma 3.6.3,

$$V(x, y) \leq \left(1 + \frac{c_\varepsilon}{k_0^\varepsilon}\right) E_2 + \frac{c_\varepsilon \sqrt{k_0}}{n^{1/2-\varepsilon}} E_1 + c_\varepsilon e^{-c_\varepsilon n^{\varepsilon^2}} (1 + N(x)). \quad (3.9.2)$$

Analogously, by (3.8.13) and (3.5.1), we have the lower bound

$$\begin{aligned} V(x, y) &\geq \mathbb{E}_x \left(V \left(X_{\nu_n^{\varepsilon^2}}, y + S_{\nu_n^{\varepsilon^2}} \right); \tau_y > \nu_n^{\varepsilon^2}, \nu_n^{\varepsilon^2} \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\geq \left(1 - \frac{c_\varepsilon}{k_0^\varepsilon}\right) E_2 - c_\varepsilon k_0^2 \mathbb{E}_x \left(1 + N \left(X_{\nu_n^{\varepsilon^2}} \right); \tau_y > \nu_n^{\varepsilon^2}, \nu_n^{\varepsilon^2} \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\geq \left(1 - \frac{c_\varepsilon}{k_0^\varepsilon}\right) E_2 - \frac{c_\varepsilon k_0^2}{n^{1/2-\varepsilon}} E_1 - c_\varepsilon k_0^2 e^{-c_\varepsilon n^{\varepsilon^2}} (1 + N(x)). \end{aligned} \quad (3.9.3)$$

Taking $k_0 = n^{1/4-\varepsilon}$ in (3.9.3) and (3.9.2), we conclude that, for any $\varepsilon \in (0, 1/8)$,

$$|V(x, y) - E_2| \leq \frac{c_\varepsilon}{n^{\varepsilon/8}} E_2 + \frac{c_\varepsilon}{n^\varepsilon} (E_1 + 1 + N(x)).$$

Again, using (3.9.3),

$$|V(x, y) - E_2| \leq \frac{c_\varepsilon}{n^{\varepsilon/8}} V(x, y) + \frac{c_\varepsilon}{n^\varepsilon} (E_1 + 1 + N(x)).$$

Finally, employing (3.9.1) and (3.7.5),

$$|V(x, y) - E_2| \leq \frac{c_\varepsilon}{n^{\varepsilon/8}} (1 + \max(y, 0) + N(x)).$$

□

Lemma 3.9.2. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $n \geq 1$,*

$$\mathbb{P}_x(\tau_y > n) \leq \frac{c_\varepsilon}{n^{1/2-\varepsilon}} (1 + \max(y, 0) + N(x)).$$

Moreover, summing this bound, for any $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $n \geq 1$, we have

$$\sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{P}_x(\tau_y > k) \leq c_\varepsilon (1 + \max(y, 0) + N(x)) n^{1/2+\varepsilon/2}.$$

Proof. Using Lemma 3.6.3 and Lemma 3.9.1, with $z = y + r(x)$ and $n \geq 1$,

$$\begin{aligned} \mathbb{P}_x(\tau_y > n) &\leq \mathbb{P}_x \left(\tau_y > n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) + \mathbb{P}_x \left(\widehat{T}_z > n, \nu_n > \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\leq \mathbb{E}_x \left(\frac{z + M_{\nu_n}}{n^{1/2-\varepsilon}}; \tau_y > n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) + c_\varepsilon e^{-c_\varepsilon n^\varepsilon} (1 + N(x)) \\ &\leq \frac{c_\varepsilon}{n^{1/2-\varepsilon}} (1 + \max(y, 0) + N(x)). \end{aligned}$$

□

Lemma 3.9.3. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z = y + r(x)$,*

$$E_3 := \mathbb{E}_x \left(z + M_{\nu_n}; z + M_{\nu_n} > n^{1/2-\varepsilon/2}, \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) \xrightarrow{n \rightarrow +\infty} 0.$$

More precisely, for any $n \geq 1$, $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z = y + r(x)$,

$$E_3 \leq c_\varepsilon \frac{\max(y, 0) + \left(1 + y \mathbb{1}_{\{y > n^{1/2-2\varepsilon}\}} + N(x)\right)^2}{n^\varepsilon}.$$

Proof. Notice that when $\nu_n \neq 1$ the following inclusion holds:

$$\{z + M_{\nu_n} > n^{1/2-\varepsilon/2}\} \subseteq \{\xi_{\nu_n} > n^{1/2-\varepsilon/2} - n^{1/2-\varepsilon} \geq c_\varepsilon n^{1/2-\varepsilon/2}\}.$$

Therefore,

$$\begin{aligned} E_3 &\leq \underbrace{\mathbb{E}_x(z + M_{\nu_n}; \nu_n \leq 2 \lfloor n^\varepsilon \rfloor)}_{=:E_{30}} \\ &\quad + \underbrace{\sum_{k=2\lfloor n^\varepsilon \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{E}_x(z + M_k; \xi_k > c_\varepsilon n^{1/2-\varepsilon/2}, \tau_y > k, \nu_n = k)}_{=:E_{31}}. \end{aligned} \quad (3.9.4)$$

Bound of E_{30} . For $y \leq n^{1/2-2\varepsilon}$, by (3.6.5), the Markov inequality and Lemma 3.5.2,

$$\mathbb{P}_x(\nu_n \leq 2 \lfloor n^\varepsilon \rfloor) \leq \sum_{k=1}^{2\lfloor n^\varepsilon \rfloor} \mathbb{P}_x(r(x) + M_k > n^{1/2-\varepsilon} - y) \leq \frac{c_\varepsilon(1 + N(x))}{n^{1/2-3\varepsilon}}.$$

For $y > n^{1/2-2\varepsilon}$, in the same way, we have $\mathbb{P}_x(\nu_n \leq 2 \lfloor n^\varepsilon \rfloor) \leq \frac{c_\varepsilon(1+y+N(x))}{n^{1/2-3\varepsilon}}$. Putting together these bounds, we get, for any $y \in \mathbb{R}$,

$$\mathbb{P}_x(\nu_n \leq 2 \lfloor n^\varepsilon \rfloor) \leq \frac{c_\varepsilon(1 + y\mathbb{1}_{\{y > n^{1/2-2\varepsilon}\}} + N(x))}{n^{1/2-3\varepsilon}}. \quad (3.9.5)$$

Using Lemma 3.5.2,

$$\begin{aligned} E_{30} &\leq z\mathbb{P}_x(\nu_n \leq 2 \lfloor n^\varepsilon \rfloor) + \sum_{k=1}^{2\lfloor n^\varepsilon \rfloor} \mathbb{E}_x^{1/2}(|M_k|^2) \mathbb{P}_x^{1/2}(\nu_n \leq 2 \lfloor n^\varepsilon \rfloor) \\ &\leq \frac{c_\varepsilon(1 + y\mathbb{1}_{\{y > n^{1/2-2\varepsilon}\}} + N(x))^2}{n^\varepsilon}. \end{aligned} \quad (3.9.6)$$

Bound of E_{31} . Changing the index of summation ($j = k - \lfloor n^\varepsilon \rfloor$) and using the Markov property,

$$\begin{aligned} E_{31} &\leq \sum_{j=\lfloor n^\varepsilon \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{X} \times \mathbb{R}} \max(z', 0) \mathbb{P}_{x'}(\xi_{\lfloor n^\varepsilon \rfloor} > c_\varepsilon n^{1/2-\varepsilon/2}) \\ &\quad \underbrace{\times \mathbb{P}_x(X_j \in dx', z + M_j \in dz', \tau_y > j)}_{=:E_{32}} \\ &\quad + \sum_{j=\lfloor n^\varepsilon \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{X} \times \mathbb{R}} \mathbb{E}_{x'}^{1/2}(|M_{\lfloor n^\varepsilon \rfloor}|^2) \mathbb{P}_{x'}^{1/2}(\xi_{\lfloor n^\varepsilon \rfloor} > c_\varepsilon n^{1/2-\varepsilon/2}) \\ &\quad \underbrace{\times \mathbb{P}_x(X_j \in dx', z + M_j \in dz', \tau_y > j)}_{=:E_{33}}. \end{aligned} \quad (3.9.7)$$

Bound of E_{32} . Using (3.5.2), the Markov inequality and (3.2.3) with $l = \lfloor c_\varepsilon n^{1/2-\varepsilon/2} \rfloor$,

$$\begin{aligned} \mathbb{P}_{x'}(\xi_{\lfloor n^\varepsilon \rfloor} > c_\varepsilon n^{1/2-\varepsilon/2}) &\leq \mathbb{P}_{x'}(N(X_{\lfloor n^\varepsilon \rfloor}) > c_\varepsilon n^{1/2-\varepsilon/2}) \\ &\quad + \mathbb{P}_{x'}(N(X_{\lfloor n^\varepsilon \rfloor-1}) > c_\varepsilon n^{1/2-\varepsilon/2}) \\ &\leq \frac{1}{l} \mathbb{E}_{x'}(N_l(X_{\lfloor n^\varepsilon \rfloor})) + \frac{1}{l} \mathbb{E}_{x'}(N_l(X_{\lfloor n^\varepsilon \rfloor-1})) \\ &\leq \frac{c}{l^{2+\beta}} + \frac{c}{l} e^{-cn^\varepsilon} (1 + N(x')). \end{aligned}$$

Choosing $\varepsilon > 0$ small enough we find that

$$\mathbb{P}_{x'} \left(\xi_{\lfloor n^\varepsilon \rfloor} > c_\varepsilon n^{1/2-\varepsilon/2} \right) \leq \frac{c_\varepsilon}{n^{1+\beta/4}} + c_\varepsilon e^{-c_\varepsilon n^\varepsilon} N(x'). \quad (3.9.8)$$

By the definition of E_{32} in (3.9.7),

$$\begin{aligned} E_{32} &\leq \frac{c_\varepsilon}{n^{1+\beta/4}} \sum_{j=\lfloor n^\varepsilon \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} [\mathbb{E}_x(z + M_j; \tau_y > j) + \mathbb{E}_x(|r(X_j)|)] \\ &\quad + c_\varepsilon e^{-c_\varepsilon n^\varepsilon} \sum_{j=\lfloor n^\varepsilon \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \left[\max(z, 0) \mathbb{E}_x(N(X_j)) + \mathbb{E}_x^{1/2}(|M_j|^2) \mathbb{E}_x^{1/2}(N(X_j)^2) \right]. \end{aligned}$$

Using (3.6.29), Lemma 3.5.2 and the point 1 of Hypothesis M3.4, we find that

$$E_{32} \leq c_\varepsilon \frac{\max(y, 0) + \left(1 + y \mathbb{1}_{\{y > n^{1/2-2\varepsilon}\}} + N(x)\right) (1 + N(x))}{n^{\beta/4}}. \quad (3.9.9)$$

Bound of E_{33} . Using (3.9.8) and Lemma 3.5.2, we have

$$E_{33} \leq \sum_{j=\lfloor n^\varepsilon \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{E}_x \left(n^{\varepsilon/2} (1 + N(X_j)) \left(\frac{c_\varepsilon}{n^{1/2+\beta/8}} + c_\varepsilon e^{-c_\varepsilon n^\varepsilon} N(X_j)^{1/2} \right); \tau_y > j \right).$$

By the Markov property,

$$E_{33} \leq c_\varepsilon e^{-c_\varepsilon n^\varepsilon} (1 + N(x))^{3/2} + \frac{c_\varepsilon}{n^{1/2+\beta/8-\varepsilon/2}} \sum_{j=1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{E}_x \left(1 + e^{-cn^\varepsilon} N(X_j); \tau_y > j \right).$$

Using Lemma 3.9.2,

$$E_{33} \leq c_\varepsilon \frac{\max(y, 0) + (1 + N(x))^{3/2}}{n^{\beta/8-3\varepsilon/2}}. \quad (3.9.10)$$

With (3.9.10), (3.9.9) and (3.9.7), for $\varepsilon > 0$ small enough, we find that

$$E_{31} \leq c_\varepsilon \frac{\max(y, 0) + \left(1 + y \mathbb{1}_{\{y > n^{1/2-2\varepsilon}\}} + N(x)\right) (1 + N(x))}{n^\varepsilon}.$$

This bound, together with (3.9.6) and (3.9.4), proves the lemma. \square

Lemma 3.9.4. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z = y + r(x)$,*

$$E_4 := \mathbb{E}_x \left(z + M_{\nu_n^{\varepsilon^2}}; z + M_{\nu_n^{\varepsilon^2}} > n^{1/2-\varepsilon/4}, \tau_y > \nu_n^{\varepsilon^2}, \nu_n^{\varepsilon^2} \leq \lfloor n^{1-\varepsilon} \rfloor \right) \xrightarrow{n \rightarrow +\infty} 0.$$

More precisely, for any $n \geq 1$, $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z = y + r(x)$,

$$E_4 \leq c_\varepsilon \frac{\max(y, 0) + \left(1 + y \mathbb{1}_{\{y > n^{1/2-2\varepsilon}\}} + N(x)\right)^2}{n^{\varepsilon/2}}.$$

Proof. We shall apply Lemma 3.9.3. For this we write, for any $n \geq 1$,

$$\begin{aligned}
 E_4 &= \mathbb{E}_x \left(z + M_{\nu_n^{\varepsilon^2}} ; z + M_{\nu_n^{\varepsilon^2}} > n^{1/2-\varepsilon/4}, z + M_{\nu_n} > n^{1/2-\varepsilon/2}, \right. \\
 &\quad \left. \tau_y > \nu_n^{\varepsilon^2}, \nu_n^{\varepsilon^2} \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\
 &\quad \underbrace{\hspace{15em}}_{=:E_{41}} \\
 &+ \mathbb{E}_x \left(z + M_{\nu_n^{\varepsilon^2}} ; z + M_{\nu_n^{\varepsilon^2}} > n^{1/2-\varepsilon/4}, z + M_{\nu_n} \leq n^{1/2-\varepsilon/2}, \right. \\
 &\quad \left. \tau_y > \nu_n^{\varepsilon^2}, \nu_n^{\varepsilon^2} \leq \lfloor n^{1-\varepsilon} \rfloor \right). \tag{3.9.11} \\
 &\quad \underbrace{\hspace{15em}}_{=:E_{42}}
 \end{aligned}$$

Bound of E_{41} . By the Markov property,

$$\begin{aligned}
 E_{41} &= \sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor - \lfloor n^{\varepsilon^2} \rfloor} \int_{\mathbb{X} \times \mathbb{R}} \mathbb{E}_{x'} \left(z' + M_{\lfloor n^{\varepsilon^2} \rfloor} ; z' + M_{\lfloor n^{\varepsilon^2} \rfloor} > n^{1/2-\varepsilon/4}, \tau_{y'} > \lfloor n^{\varepsilon^2} \rfloor \right) \\
 &\quad \times \mathbb{P}_x \left(X_k \in dx', z + M_k \in dz', z + M_k > n^{1/2-\varepsilon/2}, \tau_y > k, \nu_n = k \right),
 \end{aligned}$$

where $y' = z' - r(x')$. Moreover, for any $x' \in \mathbb{X}$, $z' \in \mathbb{R}$, using (3.6.29), we have

$$\begin{aligned}
 &\mathbb{E}_{x'} \left(z' + M_{\lfloor n^{\varepsilon^2} \rfloor} ; z' + M_{\lfloor n^{\varepsilon^2} \rfloor} > n^{1/2-\varepsilon/4}, \tau_{y'} > \lfloor n^{\varepsilon^2} \rfloor \right) \\
 &\leq \mathbb{E}_{x'} \left(z' + M_{\lfloor n^{\varepsilon^2} \rfloor} ; z' + M_{\lfloor n^{\varepsilon^2} \rfloor} > 0, \tau_{y'} > \lfloor n^{\varepsilon^2} \rfloor \right) \\
 &\leq \mathbb{E}_{x'} \left(z' + M_{\lfloor n^{\varepsilon^2} \rfloor} ; \tau_{y'} > \lfloor n^{\varepsilon^2} \rfloor \right) + \mathbb{E}_{x'} \left(|r(X_{n^{\varepsilon^2}})| \right) \\
 &\leq c_\varepsilon \max(z', 0) + c_\varepsilon (1 + N(x')).
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 E_{41} &\leq c_\varepsilon E_3 + c_\varepsilon \mathbb{E}_x \left(1 + N(X_{\nu_n}) ; z + M_{\nu_n} > n^{1/2-\varepsilon/2}, \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\
 &\leq 2c_\varepsilon E_3 + c_\varepsilon \mathbb{E}_x \left(N(X_{\nu_n}) ; N(X_{\nu_n}) > n^{1/2-\varepsilon}, \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\
 &\quad + c_\varepsilon \mathbb{E}_x \left(n^{1/2-\varepsilon} ; N(X_{\nu_n}) \leq n^{1/2-\varepsilon}, z + M_{\nu_n} > n^{1/2-\varepsilon/2}, \right. \\
 &\quad \left. \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\
 &\leq 3c_\varepsilon E_3 + c_\varepsilon \underbrace{\mathbb{E}_x \left(N(X_{\nu_n}) ; N(X_{\nu_n}) > n^{1/2-\varepsilon}, \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right)}_{=:E'_{41}}. \tag{3.9.12}
 \end{aligned}$$

Denoting $l = \lfloor n^{1/2-\varepsilon} \rfloor$ and using the point 1 of M3.4 and (3.2.3), we have

$$\begin{aligned}
 E'_{41} &\leq \mathbb{E}_x \left(\frac{N(X_{\nu_n})^2}{n^{1/2-\varepsilon}} ; \nu_n \leq \lfloor n^\varepsilon \rfloor \right) + \sum_{k=\lfloor n^\varepsilon \rfloor + 1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{E}_x (N_l(X_k) ; \tau_y > k, \nu_n = k) \\
 &\leq \frac{cn^\varepsilon (1 + N(x))^2}{n^{1/2-\varepsilon}} + \sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \left[\frac{c}{l^{1+\beta}} \mathbb{P}_x (\tau_y > k) + ce^{-cn^\varepsilon} \mathbb{E}_x (1 + N(X_k)) \right].
 \end{aligned}$$

Using Lemma 3.9.2 and taking $\varepsilon > 0$ small enough,

$$E'_{41} \leq c_\varepsilon \frac{\max(y, 0) + (1 + N(x))^2}{n^{\min(1, \beta)/4}}. \tag{3.9.13}$$

In conjunction with Lemma 3.9.3, from (3.9.12) we obtain that, for some $\varepsilon > 0$,

$$E_{41} \leq c_\varepsilon \frac{\max(y, 0) + \left(1 + y \mathbb{1}_{\{y > n^{1/2-2\varepsilon}\}} + N(x)\right)^2}{n^\varepsilon}. \quad (3.9.14)$$

Bound of E_{42} . For any $z' \in (0, n^{1/2-\varepsilon/2}]$, we have

$$\left(z' + M_{\lfloor n^{\varepsilon^2} \rfloor}\right) \mathbb{P}_{x'}(z' + M_{\lfloor n^{\varepsilon^2} \rfloor} > n^{1/2-\varepsilon/4}) \leq z' \mathbb{P}_{x'}(M_{\lfloor n^{\varepsilon^2} \rfloor} > c_\varepsilon n^{1/2-\varepsilon/4}) + \left|M_{\lfloor n^{\varepsilon^2} \rfloor}\right|.$$

Therefore, by the Markov property,

$$\begin{aligned} E_{42} &\leq \underbrace{\int_{\mathbb{X} \times \mathbb{R}} z' \mathbb{P}_{x'} \left(M_{\lfloor n^{\varepsilon^2} \rfloor} > c_\varepsilon n^{1/2-\varepsilon/4} \right) \mathbb{P}_x (X_{\nu_n} \in dx', z + M_{\nu_n} \in dz', \\ &\quad z + M_{\nu_n} \leq n^{1/2-\varepsilon/2}, \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor)}_{=: E_{43}} \\ &\quad + \underbrace{\int_{\mathbb{X} \times \mathbb{R}} \mathbb{E}_{x'} \left(\left| M_{\lfloor n^{\varepsilon^2} \rfloor} \right| \right) \mathbb{P}_x (X_{\nu_n} \in dx', z + M_{\nu_n} \in dz', \\ &\quad z + M_{\nu_n} \leq n^{1/2-\varepsilon/2}, \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor)}_{=: E_{44}}. \end{aligned} \quad (3.9.15)$$

Bound of E_{43} . Using Lemma 3.5.2,

$$\mathbb{P}_{x'} \left(M_{\lfloor n^{\varepsilon^2} \rfloor} > c_\varepsilon n^{1/2-\varepsilon/4} \right) \leq \frac{c_\varepsilon n^{\varepsilon^2} (1 + N(x'))}{n^{1/2-\varepsilon/4}}.$$

Therefore, we have

$$\begin{aligned} E_{43} &\leq \mathbb{E}_x \left(\frac{c_\varepsilon}{n^{3\varepsilon/4-\varepsilon^2}} (z + M_{\nu_n}) \mathbb{1}_{\{N(X_{\nu_n}) \leq n^{1/2-\varepsilon}\}} + \frac{c_\varepsilon}{n^{\varepsilon/4-\varepsilon^2}} N(X_{\nu_n}) \mathbb{1}_{\{N(X_{\nu_n}) > n^{1/2-\varepsilon}\}}; \right. \\ &\quad \left. z + M_{\nu_n} \leq n^{1/2-\varepsilon/2}, \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\leq \frac{c_\varepsilon}{n^{3\varepsilon/4-\varepsilon^2}} E_1 + \frac{c_\varepsilon}{n^{\varepsilon/4-\varepsilon^2}} E'_{41}. \end{aligned}$$

By Lemma 3.9.1 and (3.9.13), we obtain for some small $\varepsilon > 0$,

$$E_{43} \leq c_\varepsilon \frac{\max(y, 0) + (1 + N(x))^2}{n^{\varepsilon/2}}. \quad (3.9.16)$$

Bound of E_{44} . Again by Lemma 3.5.2, $\mathbb{E}_{x'} \left(\left| M_{\lfloor n^{\varepsilon^2} \rfloor} \right| \right) \leq n^{\varepsilon^2} (1 + N(x'))$. Consequently,

$$\begin{aligned} E_{44} &\leq \frac{c_\varepsilon}{n^{\varepsilon-\varepsilon^2}} \mathbb{E}_x \left(z + M_{\nu_n}; N(X_{\nu_n}) \leq n^{1/2-2\varepsilon}, \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\quad + c_\varepsilon n^{\varepsilon^2} \mathbb{E}_x \left(N(X_{\nu_n}); N(X_{\nu_n}) > n^{1/2-2\varepsilon}, \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right). \end{aligned}$$

Proceeding exactly as in the proof of the bound of E'_{41} but with $l = \lfloor n^{1/2-2\varepsilon} \rfloor$, we obtain, by Lemma 3.9.1,

$$E_{44} \leq c_\varepsilon \frac{\max(y, 0) + (1 + N(x))^2}{n^{\varepsilon/2}}.$$

Putting together this bound with (3.9.16) and (3.9.15), we find that

$$E_{42} \leq c_\varepsilon \frac{\max(y, 0) + (1 + N(x))^2}{n^{\varepsilon/2}}.$$

So, using (3.9.11) and (3.9.14), we obtain the second assertion. The first one is an easy consequence of the second one. \square

The following results are similar to that provided by Lemmas 3.9.1 and 3.9.4 (see E_2 and E_4 respectively).

Lemma 3.9.5. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$ and $y \in \mathbb{R}$,*

$$\begin{aligned} F_2 &:= \mathbb{E}_x \left(y + S_{\nu_n^{\varepsilon^2}} ; \tau_y > \nu_n^{\varepsilon^2}, \nu_n^{\varepsilon^2} \leq \lfloor n^{1-\varepsilon} \rfloor \right) \xrightarrow{n \rightarrow \infty} V(x, y), \\ F_4 &:= \mathbb{E}_x \left(y + S_{\nu_n^{\varepsilon^2}} ; y + S_{\nu_n^{\varepsilon^2}} > n^{1/2-\varepsilon/8}, \tau_y > \nu_n^{\varepsilon^2}, \nu_n^{\varepsilon^2} \leq \lfloor n^{1-\varepsilon} \rfloor \right) \xrightarrow{n \rightarrow +\infty} 0. \end{aligned}$$

More precisely, for any $n \geq 1$, $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$ and $y \in \mathbb{R}$,

$$|F_2 - V(x, y)| \leq \frac{c_\varepsilon}{n^{\varepsilon/8}} (1 + \max(y, 0) + N(x))$$

and

$$F_4 \leq c_\varepsilon \frac{\max(y, 0) + \left(1 + y \mathbb{1}_{\{y > n^{1/2-2\varepsilon}\}} + N(x)\right)^2}{n^{\varepsilon/2}}.$$

Proof. By (3.5.1), for any $n \geq 1$,

$$|F_2 - E_2| \leq \underbrace{\mathbb{E}_x \left(\left| r \left(X_{\nu_n^{\varepsilon^2}} \right) \right| ; \tau_y > \nu_n^{\varepsilon^2}, \nu_n^{\varepsilon^2} \leq \lfloor n^{1-\varepsilon} \rfloor \right)}_{=: F'_2}.$$

Using the Markov property, the definition of ν_n and Lemma 3.9.1,

$$\begin{aligned} F'_2 &\leq c \mathbb{E}_x \left(1 + e^{-cn\varepsilon^2} N(X_{\nu_n}) ; \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\leq \frac{c}{n^{1/2-\varepsilon}} E_1 + c e^{-cn\varepsilon^2} (1 + N(x)) \\ &\leq \frac{c_\varepsilon}{n^{1/2-\varepsilon}} (1 + \max(y, 0) + N(x)). \end{aligned} \tag{3.9.17}$$

Therefore, by Lemma 3.9.1,

$$|F_2 - V(x, y)| \leq |E_2 - V(x, y)| + F'_2 \leq \frac{c_\varepsilon}{n^{\varepsilon/8}} (1 + \max(y, 0) + N(x)).$$

Now we shall control F_4 . Recall the notation $z = y + r(x)$. By equation (3.5.1), we note that on the event

$$\left\{ z + M_{\nu_n^{\varepsilon^2}} \leq n^{1/2-\varepsilon/4} \right\} \cap \left\{ y + S_{\nu_n^{\varepsilon^2}} > n^{1/2-\varepsilon/8} \right\}$$

we have $\left| r \left(X_{\nu_n^{\varepsilon^2}} \right) \right| > c_\varepsilon n^{1/2-\varepsilon/8}$. Therefore,

$$y + S_{\nu_n^{\varepsilon^2}} \leq n^{1/2-\varepsilon/4} - r \left(X_{\nu_n^{\varepsilon^2}} \right) \leq \left(\frac{c_\varepsilon}{n^{\varepsilon/8}} + 1 \right) \left| r \left(X_{\nu_n^{\varepsilon^2}} \right) \right|,$$

which implies that

$$F_4 \leq \mathbb{E}_x \left(y + S_{\nu_n^{\varepsilon^2}} ; z + M_{\nu_n^{\varepsilon^2}} > n^{1/2-\varepsilon/4}, \tau_y > \nu_n^{\varepsilon^2}, \nu_n^{\varepsilon^2} \leq \lfloor n^{1-\varepsilon} \rfloor \right) + c_\varepsilon F'_2.$$

By (3.5.1), Lemma 3.9.4 and (3.9.17), we conclude that

$$F_4 \leq E_4 + F'_2 + c_\varepsilon F'_2 \leq c_\varepsilon \frac{\max(y, 0) + \left(1 + y \mathbb{1}_{\{y > n^{1/2-2\varepsilon}\}} + N(x)\right)^2}{n^{\varepsilon/2}}.$$

\square

3.9.2 Proof of Theorem 3.2.3

Assume that $(x, y) \in \mathbb{X} \times \mathbb{R}$. Let $(B_t)_{t \geq 0}$ be the Brownian motion defined by Proposition 3.4.3. For any $k \geq 1$, consider the event

$$A_k = \left\{ \sup_{0 \leq t \leq 1} |S_{[tk]} - \sigma B_{tk}| \leq k^{1/2-2\varepsilon} \right\} \quad (3.9.18)$$

and denote by \bar{A}_k its complement. Let $n \geq 1$ and remind that $\nu_n^{\varepsilon^2} = \nu_n + \lfloor n^{\varepsilon^2} \rfloor > \lfloor n^{\varepsilon^2} \rfloor$. With the previous notation, we write

$$\begin{aligned} \mathbb{P}_x(\tau_y > n) &= \mathbb{P}_x\left(\tau_y > n, \nu_n^{\varepsilon^2} > \lfloor n^{1-\varepsilon} \rfloor\right) \\ &+ \underbrace{\sum_{k=\lfloor n^{\varepsilon^2} \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{X} \times \mathbb{R}} \mathbb{P}_{x'}(\tau_{y'} > n-k, \bar{A}_{n-k}) \mathbb{P}_x(X_k \in dx', y+S_k \in dy', \tau_y > k, \nu_n^{\varepsilon^2} = k)}_{=: J_1} \\ &+ \underbrace{\sum_{k=\lfloor n^{\varepsilon^2} \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{X} \times \mathbb{R}} \mathbb{P}_{x'}(\tau_{y'} > n-k, A_{n-k}) \mathbb{P}_x(X_k \in dx', y+S_k \in dy', \tau_y > k, \nu_n^{\varepsilon^2} = k)}_{=: J_2}. \end{aligned} \quad (3.9.19)$$

Bound of J_1 . Since $n-k \geq c_\varepsilon n$, for any $k \leq \lfloor n^{1-\varepsilon} \rfloor$, by Proposition 3.4.3, we have

$$\mathbb{P}_{x'}(\tau_{y'} > n-k, \bar{A}_{n-k}) \leq \mathbb{P}_{x'}(\bar{A}_{n-k}) \leq \frac{c_\varepsilon(1+N(x'))}{n^{2\varepsilon}}.$$

So, using the fact that $n^{1/2-\varepsilon} \leq z + M_{\nu_n}$ and Lemma 3.9.1,

$$\begin{aligned} J_1 &\leq \frac{c_\varepsilon}{n^{2\varepsilon}} \mathbb{E}_x \left(1 + e^{-cn^{\varepsilon^2}} N(X_{\nu_n}); \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\leq \frac{c_\varepsilon}{n^{1/2+\varepsilon}} E_1 + c_\varepsilon e^{-c_\varepsilon n^{\varepsilon^2}} (1+N(x)) \\ &\leq \frac{c_\varepsilon(1+\max(y,0)+N(x))}{n^{1/2+\varepsilon}}. \end{aligned} \quad (3.9.20)$$

Bound of J_2 . We split J_2 into two terms:

$$\begin{aligned} J_2 &= \sum_{k=\lfloor n^{\varepsilon^2} \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{X} \times \mathbb{R}} \mathbb{P}_{x'}(\tau_{y'} > n-k, A_{n-k}) \\ &\quad \times \underbrace{\mathbb{P}_x(X_k \in dx', y+S_k \in dy', y+S_k > n^{1/2-\varepsilon/8}, \tau_y > k, \nu_n^{\varepsilon^2} = k)}_{=: J_3} \\ &+ \sum_{k=\lfloor n^{\varepsilon^2} \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{X} \times \mathbb{R}} \mathbb{P}_{x'}(\tau_{y'} > n-k, A_{n-k}) \\ &\quad \times \underbrace{\mathbb{P}_x(X_k \in dx', y+S_k \in dy', y+S_k \leq n^{1/2-\varepsilon/8}, \tau_y > k, \nu_n^{\varepsilon^2} = k)}_{=: J_4}. \end{aligned} \quad (3.9.21)$$

Bound of J_3 . With $y'_+ = y' + (n - k)^{1/2-2\varepsilon}$, we have

$$\mathbb{P}_{x'}(\tau_{y'} > n - k, A_{n-k}) \leq \mathbb{P}_{x'}(\tau_{y'_+}^{bm} > n - k), \quad (3.9.22)$$

where τ_y^{bm} is defined in (3.4.1). By the point 1 of Lemma 3.4.2 and Lemma 3.9.5,

$$\begin{aligned} J_3 &\leq \frac{c_\varepsilon}{\sqrt{n}} \mathbb{E}_x \left(y + S_{\nu_n^{\varepsilon^2}} + n^{1/2-2\varepsilon}; y + S_{\nu_n^{\varepsilon^2}} > n^{1/2-\varepsilon/8}, \tau_y > \nu_n^{\varepsilon^2}, \nu_n^{\varepsilon^2} \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\leq \frac{2c_\varepsilon}{\sqrt{n}} F_4 \\ &\leq c_\varepsilon \frac{\max(y, 0) + \left(1 + y \mathbb{1}_{\{y > n^{1/2-2\varepsilon}\}} + N(x)\right)^2}{n^{1/2+\varepsilon/2}}. \end{aligned} \quad (3.9.23)$$

Upper bound of J_4 . For $y' \leq n^{1/2-\varepsilon/8}$ and any $k \leq \lfloor n^{1-\varepsilon} \rfloor$, it holds $y'_+ \leq 2n^{1/2-\varepsilon/8} \leq c_\varepsilon(n-k)^{1/2-\varepsilon/8}$. Therefore, by (3.9.22) and the point 2 of Lemma 3.4.2 with $\theta_m = c_\varepsilon m^{-\varepsilon/8}$ and $m = n - k$, we have

$$\begin{aligned} J_4 &\leq \sum_{k=\lfloor n^{\varepsilon^2} \rfloor + 1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{X} \times \mathbb{R}} \frac{2(1 + \theta_{n-k}^2)}{\sqrt{2\pi(n-k)\sigma}} \mathbb{E}_x \left(y + S_k + (n-k)^{1/2-2\varepsilon}; \right. \\ &\quad \left. y + S_k \leq n^{1/2-\varepsilon/8}, \tau_y > k, \nu_n^{\varepsilon^2} = k \right). \end{aligned}$$

Since $\frac{2(1+\theta_{n-k}^2)}{\sqrt{2\pi(n-k)\sigma}} \leq \frac{2}{\sqrt{2\pi n\sigma}} \left(1 + \frac{c_\varepsilon}{n^{\varepsilon/4}}\right)$ and $n^{1/2-\varepsilon} \leq z + M_{\nu_n}$, we get

$$\begin{aligned} J_4 &\leq \frac{2}{\sqrt{2\pi n\sigma}} \left(1 + \frac{c_\varepsilon}{n^{\varepsilon/4}}\right) \mathbb{E}_x \left(y + S_{\nu_n^{\varepsilon^2}} + n^{1/2-2\varepsilon}; y + S_{\nu_n^{\varepsilon^2}} \leq n^{1/2-\varepsilon/8}, \right. \\ &\quad \left. \tau_y > \nu_n^{\varepsilon^2}, \nu_n^{\varepsilon^2} \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\ &\leq \frac{2}{\sqrt{2\pi n\sigma}} \left(1 + \frac{c_\varepsilon}{n^{\varepsilon/4}}\right) F_2 + \frac{c_\varepsilon}{n^{1/2+\varepsilon}} E_1. \end{aligned}$$

By Lemmas 3.9.1, 3.9.5 and (3.7.5),

$$J_4 \leq \frac{2V(x, y)}{\sqrt{2\pi n\sigma}} + \frac{c_\varepsilon(1 + \max(y, 0) + N(x))}{n^{1/2+\varepsilon/8}}. \quad (3.9.24)$$

Lower bound of J_4 . With $y'_- = y' - (n - k)^{1/2-2\varepsilon}$, we have $\mathbb{P}_{x'}(\tau_{y'} > n - k, A_{n-k}) \geq \mathbb{P}_{x'}(\tau_{y'_-}^{bm} > n - k) - \mathbb{P}_{x'}(\bar{A}_{n-k})$. Considering the event $\{y + S_k > (n - k)^{1/2-2\varepsilon}\}$ and repeating the arguments used to bound J_1 (see (3.9.20)), we obtain

$$\begin{aligned} J_4 &\geq \sum_{k=\lfloor n^{\varepsilon^2} \rfloor + 1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{X} \times \mathbb{R}} \mathbb{P}_{x'}(\tau_{y'_-}^{bm} > n - k) \mathbb{P}_x(X_k \in dx', y + S_k \in dy', \\ &\quad y + S_k \leq n^{1/2-\varepsilon/8}, y + S_k > (n - k)^{1/2-2\varepsilon}, \tau_y > k, \nu_n^{\varepsilon^2} = k) \\ &\quad - \frac{c_\varepsilon(1 + \max(y, 0) + N(x))}{n^{1/2+\varepsilon}}. \end{aligned}$$

Using the point 2 of Lemma 3.4.2 and Proposition 3.4.3,

$$\begin{aligned}
 J_4 &\geq \frac{2}{\sqrt{2\pi n\sigma}} \left(1 - \frac{c_\varepsilon}{n^{\varepsilon/4}}\right) \mathbb{E}_x \left(y + S_{\nu_n^{\varepsilon^2}} - (n - \nu_n^{\varepsilon^2})^{1/2-2\varepsilon}; \right. \\
 &\quad \left. y + S_{\nu_n^{\varepsilon^2}} > (n - \nu_n^{\varepsilon^2})^{1/2-2\varepsilon}, y + S_{\nu_n^{\varepsilon^2}} \leq n^{1/2-\varepsilon/8}, \tau_y > \nu_n^{\varepsilon^2}, \nu_n^{\varepsilon^2} \leq \lfloor n^{1-\varepsilon} \rfloor \right) \\
 &\quad - \frac{c_\varepsilon (1 + \max(y, 0) + N(x))}{n^{1/2+\varepsilon}} \\
 &\geq \frac{2}{\sqrt{2\pi n\sigma}} \left(1 - \frac{c_\varepsilon}{n^{\varepsilon/4}}\right) F_2 - \frac{c_\varepsilon}{\sqrt{n}} F_4 - \frac{c_\varepsilon}{n^{1/2+\varepsilon}} E_1 - \frac{c_\varepsilon (1 + \max(y, 0) + N(x))}{n^{1/2+\varepsilon}}.
 \end{aligned}$$

By Lemmas 3.9.1, 3.9.5 and (3.7.5),

$$J_4 \geq \frac{2V(x, y)}{\sqrt{2\pi n\sigma}} - c_\varepsilon \frac{\max(y, 0) + \left(1 + y \mathbb{1}_{\{y > n^{1/2-2\varepsilon}\}} + N(x)\right)^2}{n^{1/2+\varepsilon/8}}. \quad (3.9.25)$$

Putting together (3.9.25), (3.9.24), (3.9.23) and (3.9.21),

$$\left| J_2 - \frac{2V(x, y)}{\sqrt{2\pi n\sigma}} \right| \leq c_\varepsilon \frac{\max(y, 0) + \left(1 + y \mathbb{1}_{\{y > n^{1/2-2\varepsilon}\}} + N(x)\right)^2}{n^{1/2+\varepsilon/8}}.$$

Taking into account (3.9.20), (3.9.19) and Lemma 3.6.3, we conclude that, for any $(x, y) \in \mathbb{X} \times \mathbb{R}$,

$$\left| \mathbb{P}_x(\tau_y > n) - \frac{2V(x, y)}{\sqrt{2\pi n\sigma}} \right| \leq c_\varepsilon \frac{\max(y, 0) + \left(1 + y \mathbb{1}_{\{y > n^{1/2-2\varepsilon}\}} + N(x)\right)^2}{n^{1/2+\varepsilon/8}}. \quad (3.9.26)$$

Taking the limit as $n \rightarrow +\infty$ in (3.9.26), we obtain the point 1 of Theorem 3.2.3. The point 2 of Theorem 3.2.3 is an immediate consequence of the points 2 and 4 of Proposition 3.8.8.

3.9.3 Proof of Theorem 3.2.4

The point 1 of Theorem 3.2.4 is exactly (3.9.26). In order to prove the point 2 of Theorem 3.2.4, we shall first establish a bound for $\mathbb{P}_x(\tau_y > n)$ when $z = y + r(x) \geq n^{1/2-\varepsilon}$, $n \geq 1$. Set $m_\varepsilon = n - \lfloor n^\varepsilon \rfloor$. By the Markov property,

$$\begin{aligned}
 \mathbb{P}_x(\tau_y > n) &= \int_{\mathbb{X} \times \mathbb{R}} \mathbb{P}_{x'}(\tau_{y'} > m_\varepsilon) \\
 &\quad \times \mathbb{P}_x \left(X_{\lfloor n^\varepsilon \rfloor} \in dx', y + S_{\lfloor n^\varepsilon \rfloor} \in dy', \tau_y > \lfloor n^\varepsilon \rfloor \right). \quad (3.9.27)
 \end{aligned}$$

For any $x' \in \mathbb{X}$ and $y' > 0$, using A_{m_ε} defined by (3.9.18), we have

$$\mathbb{P}_{x'}(\tau_{y'} > m_\varepsilon) \leq \mathbb{P}_{x'}(\tau_{y'_+}^{bm} > m_\varepsilon) + \mathbb{P}_{x'}(\bar{A}_{m_\varepsilon}),$$

where $\tau_{y'_+}^{bm}$ is defined by (3.4.1) and $y'_+ = y' + m_\varepsilon^{1/2-2\varepsilon}$. By the point 1 of Lemma 3.4.2 and Proposition 3.4.3,

$$\mathbb{P}_{x'}(\tau_{y'} > m_\varepsilon) \leq \frac{cy'_+}{\sqrt{m_\varepsilon}} + \frac{c_\varepsilon}{m_\varepsilon^{2\varepsilon}} (1 + N(x')) \leq \frac{c_\varepsilon y'}{\sqrt{n}} + \frac{c_\varepsilon}{n^{2\varepsilon}} + \frac{c_\varepsilon}{n^{2\varepsilon}} N(x').$$

Introducing this bound in (3.9.27), we get

$$\mathbb{P}_x(\tau_y > n) \leq \frac{c_\varepsilon}{\sqrt{n}} \mathbb{E}_x(y + S_{\lfloor n^\varepsilon \rfloor}, \tau_y > \lfloor n^\varepsilon \rfloor) + \frac{c_\varepsilon}{n^{2\varepsilon}} + \frac{c_\varepsilon}{n^{2\varepsilon}} \mathbb{E}_x(N(X_{\lfloor n^\varepsilon \rfloor})).$$

Using Corollary 3.6.5, the inequality (3.2.2) and the fact that $n^{1/2-\varepsilon} \leq z$, we find

$$\mathbb{P}_x(\tau_y > n) \leq \frac{c_\varepsilon(z + N(x))}{\sqrt{n}}. \tag{3.9.28}$$

Now, for any $x \in \mathbb{X}$, $z \in \mathbb{R}$ and $y = z - r(x)$, using the Markov property, (3.9.28) and the fact that $\sqrt{n - \nu_n} \geq c_\varepsilon \sqrt{n}$ on the event $\{\nu_n \leq \lfloor n^{1-\varepsilon} \rfloor\}$, we have

$$\begin{aligned} \mathbb{P}_x(\tau_y > n) &\leq \frac{c_\varepsilon}{\sqrt{n}} \mathbb{E}_x(z + M_{\nu_n} + N(X_{\nu_n}); \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor) \\ &\quad + \mathbb{P}_x(\tau_y > n, \nu_n > \lfloor n^{1-\varepsilon} \rfloor). \end{aligned}$$

Using Lemma 3.6.3 and the fact that $N(X_{\nu_n}) \leq z + M_{\nu_n}$ on the event $\{N(X_{\nu_n}) \leq n^{1/2-\varepsilon}\}$, with $l = \lfloor n^{1/2-\varepsilon} \rfloor$, it holds

$$\begin{aligned} \mathbb{P}_x(\tau_y > n) &\leq \frac{c_\varepsilon}{\sqrt{n}} \mathbb{E}_x\left((z + M_{\nu_n}) \left(1 + \mathbb{1}_{\{N(X_{\nu_n}) \leq n^{1/2-\varepsilon}\}}\right); \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor\right) \\ &\quad + \frac{c_\varepsilon}{\sqrt{n}} \mathbb{E}_x(N_l(X_{\nu_n}); \tau_y > \nu_n, \nu_n \leq \lfloor n^{1-\varepsilon} \rfloor) + c_\varepsilon e^{-c_\varepsilon n^\varepsilon} (1 + N(x)) \\ &\leq \frac{2c_\varepsilon}{\sqrt{n}} E_1 + \frac{c_\varepsilon}{\sqrt{n}} \sum_{k=1}^{\lfloor n^\varepsilon \rfloor} \mathbb{E}_x(N_l(X_k)) \\ &\quad + \frac{c_\varepsilon}{\sqrt{n}} \sum_{k=\lfloor n^\varepsilon \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{E}_x(N_l(X_k); \tau_y > k) + c_\varepsilon e^{-c_\varepsilon n^\varepsilon} (1 + N(x)). \end{aligned}$$

By (3.2.3) and the Markov property,

$$\begin{aligned} \mathbb{P}_x(\tau_y > n) &\leq \frac{c_\varepsilon}{\sqrt{n}} E_1 + \frac{c_\varepsilon}{\sqrt{n}} \left(\frac{cn^\varepsilon}{l^{1+\beta}} + (1 + N(x)) \right) + c_\varepsilon e^{-c_\varepsilon n^\varepsilon} (1 + N(x)) \\ &\quad + \frac{c_\varepsilon}{\sqrt{n}} \sum_{j=1}^{\lfloor n^{1-\varepsilon} \rfloor - \lfloor n^\varepsilon \rfloor} \left[\frac{c}{l^{1+\beta}} \mathbb{P}_x(\tau_y > j) + c e^{-cn^\varepsilon} \mathbb{E}_x((1 + N(X_j))) \right] \\ &\leq \frac{c_\varepsilon}{\sqrt{n}} E_1 + \frac{c_\varepsilon(1 + N(x))}{\sqrt{n}} + \frac{c_\varepsilon}{\sqrt{n}} \frac{c}{l^{1+\beta}} \sum_{j=1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{P}_x(\tau_y > j). \end{aligned}$$

Using Lemmas 3.9.1 and 3.9.2, we deduce the point 2 of Theorem 3.2.4.

3.10 Asymptotic behaviour of the conditioned walk

In this section, we prove Theorem 3.2.5. The arguments are similar to those given in Section 3.9. We also keep the same notations. Assume that $(x, y) \in \mathbb{X} \times \mathbb{R}$ and let $t_0 > 0$

be a positive real. For any $t \in [0, t_0]$ and $n \geq 1$, we write

$$\begin{aligned}
 & \mathbb{P}_x \left(y + S_n \leq t\sqrt{n}, \tau_y > n \right) \\
 &= \mathbb{P}_x \left(y + S_n \leq t\sqrt{n}, \tau_y > n, \nu_n^{\varepsilon^2} > \lfloor n^{1-\varepsilon} \rfloor \right) \\
 &+ \underbrace{\sum_{k=\lfloor n^{\varepsilon^2} \rfloor + 1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{X} \times \mathbb{R}} \mathbb{P}_{x'} \left(y' + S_{n-k} \leq t\sqrt{n}, \tau_{y'} > n-k, \bar{A}_{n-k} \right)}_{=:L_1} \\
 &\quad \times \underbrace{\mathbb{P}_x \left(X_k \in dx', y + S_k \in dy', \tau_y > k, \nu_n^{\varepsilon^2} = k \right)}_{=:L_1} \\
 &+ \underbrace{\sum_{k=\lfloor n^{\varepsilon^2} \rfloor + 1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{X} \times \mathbb{R}} \mathbb{P}_{x'} \left(y' + S_{n-k} \leq t\sqrt{n}, \tau_{y'} > n-k, A_{n-k} \right)}_{=:L_2} \\
 &\quad \times \underbrace{\mathbb{P}_x \left(X_k \in dx', y + S_k \in dy', \tau_y > k, \nu_n^{\varepsilon^2} = k \right)}_{=:L_2}. \tag{3.10.1}
 \end{aligned}$$

Bound of L_1 . With J_1 defined in (3.9.19) and with the bound (3.9.20), we have,

$$L_1 \leq J_1 \leq \frac{c_\varepsilon (1 + \max(y, 0) + N(x))}{n^{1/2+\varepsilon}}. \tag{3.10.2}$$

Bound of L_2 . According to whether $y + S_k \leq n^{1/2-\varepsilon/8}$ or not, we write

$$\begin{aligned}
 L_2 &= \sum_{k=\lfloor n^{\varepsilon^2} \rfloor + 1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{X} \times \mathbb{R}} \mathbb{P}_{x'} \left(y' + S_{n-k} \leq t\sqrt{n}, \tau_{y'} > n-k, A_{n-k} \right) \\
 &\quad \times \underbrace{\mathbb{P}_x \left(X_k \in dx', y + S_k \in dy', y + S_k > n^{1/2-\varepsilon/8}, \tau_y > k, \nu_n^{\varepsilon^2} = k \right)}_{=:L_3} \\
 &+ \sum_{k=\lfloor n^{\varepsilon^2} \rfloor + 1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{X} \times \mathbb{R}} \mathbb{P}_{x'} \left(y' + S_{n-k} \leq t\sqrt{n}, \tau_{y'} > n-k, A_{n-k} \right) \\
 &\quad \times \underbrace{\mathbb{P}_x \left(X_k \in dx', y + S_k \in dy', y + S_k \leq n^{1/2-\varepsilon/8}, \tau_y > k, \nu_n^{\varepsilon^2} = k \right)}_{=:L_4}. \tag{3.10.3}
 \end{aligned}$$

Bound of L_3 . With J_3 defined in (3.9.21) and with the bound (3.9.23), we have

$$L_3 \leq J_3 \leq c_\varepsilon \frac{\max(y, 0) + \left(1 + y \mathbb{1}_{\{y > n^{1/2-2\varepsilon}\}} + N(x)\right)^2}{n^{1/2+\varepsilon/2}}. \tag{3.10.4}$$

Bound of L_4 . We start with the upper bound. Set $y'_+ = y' + (n-k)^{1/2-2\varepsilon}$ and $t_+ = t + \frac{2}{n^{2\varepsilon}}$. Note that on the event $\{y' + S_{n-k} \leq t\sqrt{n}, \tau_{y'} > n-k, A_{n-k}\}$ we have $y'_+ + \sigma B_{n-k} \leq t_+ \sqrt{n}$ and $\tau_{y'_+}^{bm} > n-k$. Therefore, by Lemma 3.4.1,

$$\begin{aligned}
 & \mathbb{P}_{x'} \left(y' + S_{n-k} \leq t\sqrt{n}, \tau_{y'} > n-k, A_{n-k} \right) \\
 & \leq \frac{2}{\sqrt{2\pi}} \int_0^{\frac{t_+ \sqrt{n}}{\sigma \sqrt{n-k}}} e^{-s^2/2} \operatorname{sh} \left(s \frac{y'_+}{\sqrt{n-k}\sigma} \right) ds.
 \end{aligned}$$

We shall use the following bounds:

$$\begin{aligned} \operatorname{sh}(u) &\leq u \left(1 + \frac{u^2}{6} \operatorname{ch}(u) \right), & \text{for } u \geq 0, \\ \frac{y'_+}{\sigma\sqrt{n-k}} &\leq \frac{y'_+}{\sigma\sqrt{n}} \left(1 + \frac{c_\varepsilon}{n^\varepsilon} \right) \leq \frac{c_\varepsilon}{n^{\varepsilon/8}}, & \text{for } y' \leq n^{1/2-\varepsilon/8} \text{ and } k \leq \lfloor n^{1-\varepsilon} \rfloor, \\ \frac{t_+\sqrt{n}}{\sigma\sqrt{n-k}} &\leq \frac{t}{\sigma} + \frac{c_{\varepsilon,t_0}}{n^\varepsilon} \leq c_{\varepsilon,t_0}, & \text{for } k \leq \lfloor n^{1-\varepsilon} \rfloor. \end{aligned}$$

Consequently,

$$\begin{aligned} \mathbb{P}_{x'} \left(y' + S_{n-k} \leq t\sqrt{n}, \tau_{y'} > n-k, A_{n-k} \right) &\leq \frac{2y'_+}{\sqrt{2\pi n\sigma}} \left(1 + \frac{c_\varepsilon}{n^\varepsilon} \right) \int_0^{\frac{t_+\sqrt{n}}{\sigma\sqrt{n-k}}} s e^{-s^2/2} \left(1 + \frac{c_\varepsilon s^2}{n^{\varepsilon/4}} \operatorname{ch}(c_\varepsilon s) \right) ds \\ &\leq \frac{2y'_+}{\sqrt{2\pi n\sigma}} \left(1 + \frac{c_\varepsilon}{n^\varepsilon} \right) \left(1 + \frac{c_{\varepsilon,t_0}}{n^{\varepsilon/4}} \right) \left(\int_0^{\frac{t}{\sigma}} s e^{-s^2/2} ds + \int_{\frac{t}{\sigma}}^{\frac{t_+\sqrt{n}}{\sigma\sqrt{n-k}}} s e^{-s^2/2} ds \right) \\ &\leq \frac{2y'_+}{\sqrt{2\pi n\sigma}} \left(1 + \frac{c_{\varepsilon,t_0}}{n^{\varepsilon/4}} \right) \left(1 - e^{-\frac{t^2}{2\sigma^2}} + \frac{c_{\varepsilon,t_0}}{n^\varepsilon} \right). \end{aligned}$$

This implies the upper bound (with F_2 and E_1 from Lemmas 3.9.5 and 3.9.1, respectively)

$$\begin{aligned} L_4 &\leq \frac{2}{\sqrt{2\pi n\sigma}} \left(1 + \frac{c_{\varepsilon,t_0}}{n^{\varepsilon/4}} \right) \left(1 - e^{-\frac{t^2}{2\sigma^2}} + \frac{c_{\varepsilon,t_0}}{n^\varepsilon} \right) F_2 + \frac{c_{\varepsilon,t_0}}{n^{1/2+\varepsilon}} E_1 \\ &\leq \frac{2V(x,y)}{\sqrt{2\pi n\sigma}} \left(1 - e^{-\frac{t^2}{2\sigma^2}} \right) + \frac{c_{\varepsilon,t_0} (1 + \max(y,0) + N(x))}{n^{1/2+\varepsilon/8}}. \end{aligned}$$

The proof of the lower bound of L_4 , being similar, is left to the reader:

$$L_4 \geq \frac{2V(x,y)}{\sqrt{2\pi n\sigma}} \left(1 - e^{-\frac{t^2}{2\sigma^2}} \right) - c_{\varepsilon,t_0} \frac{\max(y,0) + \left(1 + y\mathbb{1}_{\{y > n^{1/2-2\varepsilon}\}} + N(x) \right)^2}{n^{1/2+\varepsilon/8}}.$$

Combining the upper and the lower bounds of L_4 and (3.10.4) with (3.10.3) we obtain an asymptotic development of L_2 . Implementing this development and the bound (3.10.2) into (3.10.1) and using Lemma 3.6.3, we conclude that

$$\begin{aligned} \left| \mathbb{P}_x \left(y + S_n \leq t\sqrt{n}, \tau_y > n \right) - \frac{2V(x,y)}{\sqrt{2\pi n\sigma}} \left(1 - e^{-\frac{t^2}{2\sigma^2}} \right) \right| &\leq c_{\varepsilon,t_0} \frac{\max(y,0) + \left(1 + y\mathbb{1}_{\{y > n^{1/2-2\varepsilon}\}} + N(x) \right)^2}{n^{1/2+\varepsilon/8}}. \end{aligned}$$

Using the asymptotic of $\mathbb{P}_x(\tau_y > n)$ provided by Theorem 3.2.3 finishes the proof of Theorem 3.2.5.

3.11 Appendix: proofs for affine random walks in \mathbb{R}^d

In this section we prove Proposition 3.3.2. For this we verify that Hypotheses M3.1-M3.5 hold true on an appropriate Banach space which we proceed to introduce. Let

$\delta > 0$ be the constant from Hypothesis 3.3.1. Denote by $\mathcal{C}(\mathbb{R}^d)$ the space of continuous complex valued functions on \mathbb{R}^d . Let ε and θ be two positive numbers satisfying

$$1 + \varepsilon < \theta < 2 < 2 + 2\varepsilon < 2 + 2\delta.$$

For any function $h \in \mathcal{C}(\mathbb{R}^d)$ introduce the norm $\|h\|_{\theta,\varepsilon} = |h|_\theta + [h]_\varepsilon$, where

$$|h|_\theta = \sup_{x \in \mathbb{R}^d} \frac{|h(x)|}{(1 + |x|)^\theta}, \quad [h]_\varepsilon = \sup_{x \neq y} \frac{|h(x) - h(y)|}{|x - y|^\varepsilon (1 + |x|) (1 + |y|)}$$

and consider the Banach space

$$\mathcal{B} := \mathcal{L}_{\theta,\varepsilon} = \left\{ h \in \mathcal{C}(\mathbb{R}^d) : \|h\|_{\theta,\varepsilon} < +\infty \right\}.$$

Proof of M3.1. Conditions 1, 2 and 3 of M3.1 can be easily verified under the point 1 of Hypothesis 3.3.1 and the fact that $\|\delta_x\|_{\mathcal{B}} \leq (1 + |x|)^\theta$, for any $x \in \mathbb{R}^d$.

We verify the point 4 of Hypothesis M3.1. For any $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ and $t \in \mathbb{R}$, we have $|e^{itf(x)} - e^{itf(y)}| \leq |t| |f(x) - f(y)| \leq |t| |u| |x - y|$ and $|e^{itf(x)} - e^{itf(y)}| \leq 2$. Therefore, we write

$$|e^{itf(x)} - e^{itf(y)}| \leq 2^{1-\varepsilon} |t|^\varepsilon |u|^\varepsilon |x - y|^\varepsilon.$$

Supposing that $|x| \leq |y|$, we obtain, for any $h \in \mathcal{L}_{\theta,\varepsilon}$,

$$|e^{itf(x)} h(x) - e^{itf(y)} h(y)| \leq |e^{itf(x)} - e^{itf(y)}| |h|_\theta (1 + |x|)^\theta + |h(x) - h(y)|.$$

Since $\theta < 2$, we have $[e^{itf} h - e^{itf} h]_\varepsilon \leq 2^{1-\varepsilon} |t|^\varepsilon |u|^\varepsilon |h|_\theta + [h]_\varepsilon$. Consequently, $\|e^{itf} h\|_{\theta,\varepsilon} \leq (1 + 2^{1-\varepsilon} |t|^\varepsilon |u|^\varepsilon) \|h\|_{\theta,\varepsilon}$ and the point 4 is verified.

Proof of M3.2 and M3.3. We shall verify that the conditions of the theorem of Ionescu-Tulcea and Marinescu are satisfied (see [59] and [49]). We start by establishing two lemmas.

Lemma 3.11.1. *Assume Hypothesis 3.3.1.*

1. *There exists a constant $c > 0$ such that, for any $t \in \mathbb{R}$, $n \geq 1$, and $h \in \mathcal{L}_{\theta,\varepsilon}$,*

$$|\mathbf{P}_t^n h|_\theta \leq c |h|_\theta.$$

2. *There exist constants c_1, c_2 and $\rho < 1$ such that, for any $n \geq 1$, $h \in \mathcal{L}_{\theta,\varepsilon}$ and $t \in \mathbb{R}$,*

$$[\mathbf{P}_t^n h]_\varepsilon \leq c_1 \rho^n [h]_\varepsilon + c_2 |t|^\varepsilon |h|_\theta.$$

3. *For any $t \in \mathbb{R}$, the operator \mathbf{P}_t is compact from $(\mathcal{B}, \|\cdot\|_{\theta,\varepsilon})$ to $(\mathcal{C}(\mathbb{R}^d), |\cdot|_\theta)$.*

Proof. Claim 1. For any $x \in \mathbb{R}^d$,

$$|\mathbf{P}_t^n h(x)| = \left| \mathbb{E}_x \left(e^{itS_n} h(X_n) \right) \right| \leq 3^\theta |h|_\theta \left(1 + \mathbb{E} \left(\|\Pi_n\|^\theta \right) |x|^\theta + \mathbb{E} \left(|X_n^0|^\theta \right) \right),$$

with $\Pi_n = A_n A_{n-1} \dots A_1$ and $X_n^0 = g_n \dots g_1 \cdot 0 = \sum_{k=1}^n A_n \dots A_{k+1} B_k$. By the point 1 of Hypothesis 3.3.1, there exist $c(\delta) > 0$ and $0 < \rho(\delta) < 1$ such that, for any $n \geq 1$,

$$\mathbb{E}^{\frac{2+2\delta}{\theta}} \left(\|\Pi_n\|^\theta \right) \leq \mathbb{E} \left(\|\Pi_n\|^{2+2\delta} \right) \leq c(\delta) \rho(\delta)^n \xrightarrow{n \rightarrow +\infty} 0,$$

from which it follows that

$$\mathbb{E} \left(|X_n^0|^\theta \right) \leq \left(\sum_{k=1}^n \mathbb{E}^{1/\theta} \left(\|\Pi_n\|^\theta \right) \mathbb{E}^{1/\theta} \left(|B_1|^\theta \right) \right)^\theta < +\infty.$$

This proves the claim 1.

Proof of the claim 2. For any $x \neq y \in \mathbb{R}^d$, with $|x| \leq |y|$, we have

$$\begin{aligned} & |\mathbf{P}_t^n h(x) - \mathbf{P}_t^n h(y)| \\ & \leq \mathbb{E} \left(2^{1-\varepsilon} |t|^\varepsilon |u|^\varepsilon \left(\sum_{k=1}^n \|\Pi_k\| \right)^\varepsilon |x - y|^\varepsilon |h|_\theta \left(1 + \|\Pi_n\| |x| + |X_n^0| \right)^\theta \right) \\ & \quad + \mathbb{E} \left([h]_\varepsilon \|\Pi_n\|^\varepsilon |x - y|^\varepsilon \left(1 + \|\Pi_n\| |x| + |X_n^0| \right) \left(1 + \|\Pi_n\| |y| + |X_n^0| \right) \right). \end{aligned}$$

Since $\theta < 2$, we obtain that

$$[\mathbf{P}_t^n h]_\varepsilon \leq 2^{1-\varepsilon} |t|^\varepsilon |u|^\varepsilon C_2(n) |h|_\theta + C_1(n) [h]_\varepsilon,$$

where

$$C_1(n) = \mathbb{E} \left(\|\Pi_n\|^\varepsilon \left(1 + \|\Pi_n\| + |X_n^0| \right)^2 \right)$$

and

$$C_2(n) = \mathbb{E} \left(\left(\sum_{k=1}^n \|\Pi_k\| \right)^\varepsilon \left(1 + \|\Pi_n\| + |X_n^0| \right)^\theta \right).$$

Since $2 + 2\varepsilon < 2 + 2\delta = p$, by the Hölder inequality,

$$\begin{aligned} C_1(n) & \leq \mathbb{E}^{\frac{\varepsilon}{1+\varepsilon}} \left(\|\Pi_n\|^{1+\varepsilon} \right) \mathbb{E}^{\frac{1}{1+\varepsilon}} \left(\left(1 + \|\Pi_n\| + |X_n^0| \right)^{2+2\varepsilon} \right) \\ & \leq c(\delta)^{\frac{\varepsilon}{p}} \rho(\delta)^{\frac{n\varepsilon}{p}} 3^2 \left(1 + c(\delta)^{\frac{2}{p}} + \left(\frac{c(\delta)^{\frac{1}{p}} \mathbb{E}^{\frac{1}{p}} (|B_1|^p)}{1 - \rho(\delta)^{\frac{1}{p}}} \right)^2 \right), \end{aligned}$$

which shows that $C_1(n)$ converges exponentially fast to 0. In the same way, taking into account that $\theta < 2$ we show that $C_2(n)$ is bounded:

$$\begin{aligned} C_2(n) & \leq \left(\sum_{k=1}^n \mathbb{E}^{\frac{1}{1+\varepsilon}} \left(\|\Pi_k\|^{1+\varepsilon} \right) \right)^\varepsilon \mathbb{E}^{\frac{1}{1+\varepsilon}} \left(\left(1 + \|\Pi_n\| + |X_n^0| \right)^{2+2\varepsilon} \right) \\ & \leq \left(\frac{c(\delta)^{\frac{1}{p}}}{1 - \rho(\delta)^{\frac{1}{p}}} \right)^\varepsilon 3^2 \left(1 + c(\delta)^{\frac{2}{p}} + \left(\frac{c(\delta)^{\frac{1}{p}} \mathbb{E}^{\frac{1}{p}} (|B_1|^p)}{1 - \rho(\delta)^{\frac{1}{p}}} \right)^2 \right). \end{aligned}$$

Proof of the claim 3. Let B be a bounded subset of \mathcal{B} , $(h_n)_{n \geq 0}$ be a sequence in B and K be a compact of \mathbb{R}^d . Using the claim 1, it follows that, for any $x \in K$ and $n \geq 0$,

$$|\mathbf{P}_t h_n(x)| \leq c |h_n|_\theta (1 + |x|)^\theta \leq c_K,$$

which implies that the set $\mathcal{A} = \{\mathbf{P}_t h_n : n \geq 0\}$ is uniformly bounded in $(\mathcal{C}(K), |\cdot|_\infty)$, where $|\cdot|_\infty$ is the supremum norm. By the claims 1 and 2, we have that, for any $x, y \in K$ and $n \geq 0$,

$$|\mathbf{P}_t h_n(x) - \mathbf{P}_t h_n(y)| \leq [\mathbf{P}_t h_n]_\varepsilon |x - y|^\varepsilon (1 + |x|)^\theta (1 + |y|)^\theta \leq c_K \|h_n\|_{\mathcal{B}} |x - y|^\varepsilon$$

and, thereby, the set \mathcal{A} is uniformly equicontinuous. By the theorem of Arzelà-Ascoli, we conclude that \mathcal{A} is relatively compact in $(\mathcal{C}(K), |\cdot|_\infty)$. Using a diagonal extraction, we deduce that there exist a subsequence $(n_k)_{k \geq 1}$ and a function $\varphi \in \mathcal{C}(\mathbb{R}^d)$ such that, for any compact $K \subset \mathbb{R}^d$,

$$\sup_{x \in K} |P_t h_{n_k}(x) - \varphi(x)| \xrightarrow{n \rightarrow +\infty} 0.$$

Moreover, by the claims 1 and 2, for any $n \geq 1$ and $x \in \mathbb{R}^d$,

$$|P_t h_n(x)| \leq |P_t h_n(0)| + [P_t h_n]_\varepsilon |x|^\varepsilon (1 + |x|) \leq c |h_n|_\theta + c \|h_n\|_{\mathcal{B}} |x|^\varepsilon (1 + |x|).$$

Since B is bounded, we have $|P_t h_n(x)| \leq c(1 + |x|)^{1+\varepsilon}$, for any $x \in \mathbb{R}^d$, as well as $\varphi(x) \leq c(1 + |x|)^{1+\varepsilon}$, for any $x \in \mathbb{R}^d$. Consequently, for any $k \geq 1$ and $A > 0$,

$$\sup_{x \in \mathbb{R}^d} \frac{|P_t h_{n_k}(x) - \varphi(x)|}{(1 + |x|)^\theta} \leq \sup_{|x| \leq A} |P_t h_{n_k}(x) - \varphi(x)| + 2c \sup_{|x| > A} \frac{(1 + |x|)^{1+\varepsilon}}{(1 + |x|)^\theta}.$$

Taking the limit as $k \rightarrow +\infty$ and then the limit as $A \rightarrow +\infty$, we can conclude that $\lim_{k \rightarrow +\infty} |P_t h_{n_k} - \varphi|_\theta = 0$. □

Lemma 3.11.2. *Assume Hypothesis 3.3.1.*

1. *The operator \mathbf{P} has a unique invariant probability ν which coincides with the distribution of the \mathbb{P} -a.s. convergent series $Z := \sum_{k=1}^{+\infty} A_1 \dots A_{k-1} B_k$. Moreover, the unique eigenvalue of modulus 1 of the operator \mathbf{P} on \mathcal{B} is 1 and the associated eigenspace is generated by the function $e: x \mapsto 1$.*

2. *Let $t \in \mathbb{R}^*$. If $h \in \mathcal{B}$ and $z \in \mathbb{C}$ of modulus 1 are such that*

$$\mathbf{P}_t h(x) = zh(x), \quad x \in \text{supp}(\nu),$$

then $h = 0$ on $\text{supp}(\nu)$.

Proof. We proceed as in Guivarc'h and Le Page [43] and Buraczewski, Damek and Guivarc'h [12]. For any $g = (A, B) \in \text{GL}(d, \mathbb{R}) \times \mathbb{R}^d$ and $x \in \mathbb{R}^d$, we set $g \cdot x = Ax + B$.

Proof of claim 1. Since $k(\delta) < 1$, the series $\sum_k \mathbb{E}^{\frac{1}{2+2\delta}}(|A_1 \dots A_{k-1} B_k|^{2+2\delta})$ converges and so the sequence $g_1 \dots g_n \cdot x = A_1 \dots A_n x + \sum_{k=1}^n A_1 \dots A_{k-1} B_k$ converges almost surely to $Z = \sum_{k=1}^{+\infty} A_1 \dots A_{k-1} B_k$ as $n \rightarrow +\infty$. Therefore, for any $\varphi \in \mathcal{B}$, the sequence $\varphi(g_1 \dots g_n \cdot x)$ converges to $\varphi(Z)$ almost surely as $n \rightarrow +\infty$. Moreover, since $|\varphi(x)| \leq |\varphi|_\theta (1 + |x|)^\theta$ and $\theta < 2 + 2\delta$, the sequence $(\varphi(g_1 \dots g_n \cdot x))_{n \geq 1}$ is uniformly integrable. So $\mathbf{P}^n \varphi(x)$ converges to $\mathbb{E}(\varphi(Z))$ as $n \rightarrow +\infty$. This proves that the distribution ν of Z is the only invariant probability of \mathbf{P} .

Fix $z \in \mathbb{C}$ such that $|z| = 1$ and let $h \neq 0$ belonging to \mathcal{B} be an eigenfunction of \mathbf{P} , so that $\mathbf{P}h = zh$. From the previous argument, it follows that, for any $x \in \mathbb{R}^d$,

$$z^n h(x) = \mathbf{P}^n h(x) \xrightarrow{n \rightarrow +\infty} \nu(h).$$

Since there exists $x \in \mathbb{R}^d$ such that $h(x) \neq 0$, the sequence $(z^n)_{n \geq 1}$ should be convergent which is possible only if $z = 1$. From this, we deduce that for any $x \in \mathbb{R}^d$, $h(x) = \mathbb{E}(h(Z))$ which implies that h is constant.

Proof of the claim 2. Our argument is by contradiction. Let $t \in \mathbb{R}^*$, $h \in \mathcal{B}$ and $z \in \mathbb{C}$ of modulus 1 be such that $\mathbf{P}_t h(x) = zh(x)$, for any $x \in \text{supp}(\nu)$ and suppose that there exists $x_0 \in \text{supp}(\nu)$ such that $h(x_0) \neq 0$.

First we establish that $|h|$ is constant on the support of the distribution ν . Since ν is μ -invariant, for any $(g, x) \in \text{supp}(\mu) \times \text{supp}(\nu)$ we have $g \cdot x \in \text{supp}(\nu)$. From this fact it follows that $\mathbf{P}_t^n h(x) = z^n h(x)$, for any $n \geq 1$ and $x \in \text{supp}(\nu)$. This implies that $|h|(x) \leq \mathbf{P}^n |h|(x)$, for any $x \in \text{supp}(\nu)$. Note also that $|h|$ belongs to \mathcal{B} . Therefore, as we have seen in the proof of the first claim, we have, $\lim_{n \rightarrow +\infty} \mathbf{P}^n |h|(x) = \nu(|h|) = \mathbb{E}(|h|(Z)) < +\infty$, for any $x \in \text{supp}(\nu)$. So $|h|(x) \leq \int_{x' \in \mathbb{R}^d} |h|(x') \nu(dx')$, for any $x \in \text{supp}(\nu)$. Since $|h|$ is continuous, this implies that $|h|$ is constant on the support of ν . In particular, this means that $h(x) \neq 0$ for any $x \in \text{supp}(\nu)$.

Since the support of ν is stable by all the elements of the support of μ , we deduce that the random variable $\xi_n(x) = \exp(it \langle u, \sum_{k=1}^n g_k \dots g_1 \cdot x \rangle) h(g_n \dots g_1 \cdot x)$ takes values on the sphere $\mathbb{S}_{\nu(|h|)} = \{a \in \mathbb{C} : |a| = \nu(|h|)\}$, for all x in the support of ν . Moreover, the mean $z^n h(x)$ of $\xi_n(x)$ is also on $\mathbb{S}_{\nu(|h|)}$, which is possible only if $\xi_n(x)$ is a constant, for any $x \in \text{supp}(\nu)$. Consequently, for any pair $x, y \in \text{supp}(\nu)$, there exists an event $\Omega_{x,y}$ of \mathbb{P} -probability one such that on $\Omega_{x,y}$ it holds, for any $n \geq 1$,

$$\exp\left(it \left\langle u, \sum_{k=1}^n g_k \dots g_1 \cdot v \right\rangle\right) h(g_n \dots g_1 \cdot v) = z^n h(v),$$

with $v \in \{x, y\}$, from which we get

$$\frac{h(g_n \dots g_1 \cdot y)}{h(g_n \dots g_1 \cdot x)} = \frac{h(y)}{h(x)} \exp\left(it \left\langle \sum_{k=1}^n {}^t A_k u, x - y \right\rangle\right). \quad (3.11.1)$$

In addition, for any $n \geq 1$,

$$\mathbb{E}\left(\left|\frac{h(g_n \dots g_1 \cdot y)}{h(g_n \dots g_1 \cdot x)} - 1\right|\right) = \mathbb{E}\left(\left|\frac{h(g_1 \dots g_n \cdot y)}{h(g_1 \dots g_n \cdot x)} - 1\right|\right).$$

Since, for $v \in \{x, y\}$, the sequence $h(g_1 \dots g_n \cdot v)$ converges a.s. to $h(Z)$ and since h is bounded with a constant modulus, we have by (3.11.1),

$$\begin{aligned} 0 &= \lim_{n \rightarrow +\infty} \mathbb{E}\left(\left|\frac{h(g_n \dots g_1 \cdot y)}{h(g_n \dots g_1 \cdot x)} - 1\right|\right) \\ &= \lim_{n \rightarrow +\infty} \mathbb{E}\left(\left|\frac{h(y)}{h(x)} \exp\left(it \left\langle \sum_{k=1}^n {}^t A_k u, x - y \right\rangle\right) - 1\right|\right). \end{aligned}$$

Taking into account that the series $\sum_{k=1}^n {}^t A_k$ converges a.s. to a random variable Z' , we have for any $x, y \in \text{supp}(\nu)$,

$$\mathbb{E}\left(\left|\frac{h(y)}{h(x)} e^{it \langle Z' u, x - y \rangle} - 1\right|\right) = 0. \quad (3.11.2)$$

Since the support of ν is invariant by all the elements of the support of μ , by the point 2 of Hypothesis 3.3.1, we deduce that the support of ν is not contained in an affine subspace of \mathbb{R}^d , i.e. for any $1 \leq j \leq d$, there exist $x_j, y_j \in \text{supp}(\nu)$, such that the family $(v_j)_{1 \leq j \leq d} = (x_j - y_j)_{1 \leq j \leq d}$ generates \mathbb{R}^d . From (3.11.2), we conclude that for any $1 \leq j \leq d$,

$$\frac{h(y_j)}{h(x_j)} e^{it \langle Z' u, v_j \rangle} = 1, \quad \mathbb{P}\text{-a.s.}$$

Let θ_j be such that $\frac{h(x_j)}{h(y_j)} = e^{i\theta_j}$. Denoting by $\boldsymbol{\eta}_u$ the distribution of $Z'u$, we obtain that $\langle Z'u, v_j \rangle \in \frac{\theta_j + 2\pi\mathbb{Z}}{t}$ \mathbb{P} -a.s. and so the support of $\boldsymbol{\eta}_u$ is discrete. Moreover, the measure $\boldsymbol{\eta}_u$ is invariant for the Markov chain $X'_{n+1} = {}^tA_{n+1}(X'_n + u)$ and so, for any Borel set B of \mathbb{R}^d ,

$$\boldsymbol{\eta}_u(B) = \mathbb{E} \left(\int_{v \in \mathbb{R}^d} \mathbb{1}_B({}^tA_1(v + u)) \boldsymbol{\eta}_u(dv) \right). \quad (3.11.3)$$

Since $\boldsymbol{\eta}_u$ is discrete, the set $E_{max} = \{x \in \mathbb{R}^d : \boldsymbol{\eta}_u(\{x\}) = \max_{y \in \mathbb{R}^d} \boldsymbol{\eta}_u(\{y\})\}$ is non-empty and finite. Moreover, using (3.11.3) with $B = \{x\}$ and $x \in E_{max}$, we can see that the image ${}^tA_1^{-1}x - u$ belongs to E_{max} \mathbb{P} -a.s. Denoting by v_0 the barycentre of E_{max} , we find that

$$\mathbb{P}({}^tA_1^{-1}v_0 - u = v_0) = 1.$$

The fact that $u \neq 0$ implies that $v_0 \neq 0$. The latter implies that ${}^tA_1^{-1}v_0 = v_0 + u = {}^tA_2^{-1}v_0$ almost surely, which contradicts the point 3 of Hypothesis 3.3.1. \square

The conditions (b), (c) and (d) of the theorem of Ionescu-Tulcea and Marinescu as stated in Chapter 3 of Norman [59] follow from points 1-3 of Lemma 3.11.1 respectively. It remains to show the condition (a). Let $(h_n)_{n \geq 0}$ be a sequence in $\mathcal{L}_{\theta, \varepsilon}$ satisfying $\|h_n\|_{\theta, \varepsilon} \leq K$, for any $n \geq 0$ and some constant K and suppose that there exists $h \in \mathcal{C}(\mathbb{R}^d)$ such that $\lim_{n \rightarrow +\infty} |h_n - h|_{\theta} = 0$. For any $x, y, z \in \mathbb{R}^d$ and $n \geq 0$,

$$\begin{aligned} & \frac{|h(x) - h(y)|}{|x - y|^\varepsilon (1 + |x|)(1 + |y|)} + \frac{|h(z)|}{(1 + |z|)^\theta} \\ & \leq |h_n - h|_{\theta} \left(\frac{(1 + |x|)^\theta + (1 + |y|)^\theta}{|x - y|^\varepsilon (1 + |x|)(1 + |y|)} + 1 \right) + [h_n]_\varepsilon + |h_n|_{\theta}. \end{aligned}$$

Taking the limit as $n \rightarrow +\infty$, shows that $h \in \mathcal{L}_{\theta, \varepsilon}$ and $\|h\|_{\theta, \varepsilon} \leq K$.

The theorem of Ionescu-Tulcea and Marinescu and the unicity of the one-dimensional projector proved in the point 1 of Lemma 3.11.2 imply Hypothesis M3.2. Hypothesis M3.3 is obtained easily from Lemma 3.11.1.

The point 2 of Lemma 3.11.2 will be used latter to prove that $\sigma^2 > 0$.

Proof of M3.4. By the hypothesis $\alpha = \frac{2+2\delta}{1+\varepsilon} > 2$. Consider the function $N: \mathbb{R}^d \rightarrow \mathbb{R}_+$ defined by $N(x) = |x|^{1+\varepsilon}$. For any $x, y \in \mathbb{R}^d$ satisfying $|x| \leq |y|$,

$$|N(x) - N(y)| \leq (1 + \varepsilon) |y|^\varepsilon |x - y|.$$

Using the fact that $|N(x) - N(y)| \leq 2|y|^{1+\varepsilon}$, we have

$$|N(x) - N(y)| \leq (1 + \varepsilon)^\varepsilon 2^{1-\varepsilon} |y|^{\varepsilon^2 + (1+\varepsilon)(1-\varepsilon)} |x - y|^\varepsilon = c_\varepsilon |y| |x - y|^\varepsilon.$$

Together with $|N|_{\theta} < +\infty$, this proves that the function N is in $\mathcal{B} = \mathcal{L}_{\theta, \varepsilon}$.

Obviously $|f(x)|^{1+\varepsilon} = |\langle u, x \rangle|^{1+\varepsilon} \leq |u|^{1+\varepsilon} (1 + N(x))$. Moreover, for any $h \in \mathcal{L}_{\theta, \varepsilon}$,

$$|h(x)| \leq [h]_\varepsilon |x|^\varepsilon (1 + |x|) + |h(0)| \leq 2 \|h\|_{\theta, \varepsilon} (1 + N(x))$$

and so $\|\boldsymbol{\delta}_x\|_{\mathcal{B}} \leq 2(1 + N(x))$. Note that for any $p \in [1, \alpha]$,

$$\mathbb{E}^{1/p} (N(g_n \dots g_1 \cdot x)^p) \leq 2^{1+\varepsilon} \left(\mathbb{E}^{1/p} \left(\|\Pi_n\|^{p(1+\varepsilon)} \right) N(x) + \mathbb{E}^{1/p} \left(|g_n \dots g_1 \cdot 0|^{p(1+\varepsilon)} \right) \right).$$

Since $p(1+\varepsilon) \leq 2 + 2\delta$, the previous inequality proves that $\mathbb{E}_x^{1/p} (N(X_n)^p) \leq c(1 + N(x))$. Thus, we proved the first inequality of the point 1 of M3.4.

For any $l \geq 1$, we consider the function ϕ_l on \mathbb{R}_+ defined by:

$$\phi_l(t) = \begin{cases} 0 & \text{if } t \leq l^{\frac{1}{1+\varepsilon}} - 1, \\ t - \left(l^{\frac{1}{1+\varepsilon}} - 1\right) & \text{if } t \in \left[l^{\frac{1}{1+\varepsilon}} - 1, l^{\frac{1}{1+\varepsilon}}\right], \\ 1 & \text{if } t \geq l^{\frac{1}{1+\varepsilon}}. \end{cases} \quad (3.11.4)$$

Define N_l on \mathbb{R}^d by $N_l(x) = \phi_l(|x|)N(x)$. For any $x \in \mathbb{R}^d$, we have $N(x)\mathbb{1}_{\{N(x)>l\}} \leq N_l(x) \leq N(x)$ which implies that $|N_l|_\theta \leq |N|_\theta < +\infty$. Moreover, for any $x, y \in \mathbb{R}^d$ satisfying $|x| \leq |y|$, we have

$$|\phi_l(|y|) - \phi_l(|x|)| \leq \min(|y| - |x|, 1).$$

So

$$|N_l(y) - N_l(x)| \leq [N]_\varepsilon |x - y|^\varepsilon (1 + |x|)(1 + |y|) + |x|^{1+\varepsilon} |y - x|^\varepsilon.$$

Since $|x| \leq |y|$, we obtain that $[N_l]_\varepsilon \leq [N]_\varepsilon + 1 < +\infty$. Therefore, the function N_l belongs to $\mathcal{B} = \mathcal{L}_{\theta, \varepsilon}$, which finishes the proof of the point 1 of **M3.4**.

Moreover, $\|N_l\|_{\theta, \varepsilon} \leq \|N\|_{\theta, \varepsilon} + 1$ and, so the point 2 of **M3.4** is also established.

Since $\int_{\mathbb{X}} |x|^p \nu(dx) < +\infty$, for any $p \leq 2 + 2\delta$, we find that

$$\nu(N_l) \leq \int_{\mathbb{X}} |x|^{1+\varepsilon} \mathbb{1}_{\{|x| \geq l^{\frac{1}{1+\varepsilon}} - 1\}} \nu(dx) \leq \frac{\int_{\mathbb{X}} |x|^{2+2\delta} \nu(dx)}{\left(l^{\frac{1}{1+\varepsilon}} - 1\right)^{2+2\delta-(1+\varepsilon)}}.$$

Choosing $\beta = \alpha - 2 > 0$, we obtain the point 3 of **M3.4**.

Proof of M3.5. Using (3.2.5) and the point 4 of Hypothesis 3.3.1,

$$\mu = \int_{\mathbb{R}^d} \langle u, x \rangle \nu(dx) = \left\langle u, \mathbb{E} \left(\sum_{k=1}^{+\infty} A_1 \dots A_{k-1} B_k \right) \right\rangle = 0. \quad (3.11.5)$$

Now we prove that $\sigma^2 > 0$. For this, suppose the contrary: $\sigma^2 = 0$. One can easily check that the function f belongs to \mathcal{B} . Using **M3.2** and the fact that $\nu(f) = \mu = 0$, we deduce that $\sum_{n \geq 0} \|\mathbf{P}^n f\|_{\theta, \varepsilon} = \sum_{n \geq 0} \|Q^n f\|_{\theta, \varepsilon} < +\infty$ and therefore the series $\sum_{n \geq 0} \mathbf{P}^n f$ converges in $(\mathcal{B}, \|\cdot\|_{\theta, \varepsilon})$. We denote by $\Theta \in \mathcal{B}$ its limit and notice that the function Θ satisfies the Poisson equation: $\Theta - \mathbf{P}\Theta = f$.

Using the bound (3.2.6), we have that $\left| \sum_{n=1}^N f(x) \mathbf{P}^n f(x) \right| \leq c(1 + N(x))^2$. By the Lebesgue dominated convergence theorem, from (3.2.5), we obtain

$$\begin{aligned} \sigma^2 &= \int_{\mathbb{R}^d} f(x) (2\Theta(x) - f(x)) \nu(dx) \\ &= \int_{\mathbb{R}^d} \left(\Theta^2(x) - (\mathbf{P}\Theta)^2(x) \right) \nu(dx) \\ &= \int_{\text{GL}(d, \mathbb{R}) \times \mathbb{R}^d \times \mathbb{R}^d} \left(\Theta(g_1 \cdot x) - \mathbf{P}\Theta(x) \right)^2 \mu(dg_1) \nu(dx). \end{aligned}$$

As $\sigma^2 = 0$, we have $\Theta(g_1 \cdot x) = \mathbf{P}\Theta(x)$, i.e. $f(g_1 \cdot x) = \mathbf{P}\Theta(x) - \mathbf{P}\Theta(g_1 \cdot x)$, $\mu \times \nu$ -a.s. Consequently, there exists a Borel subset B_0 of \mathbb{R}^d such that $\nu(B_0) = 1$ and for any $t \in \mathbb{R}$ and $x \in B_0$,

$$\int_{\text{GL}(d, \mathbb{R}) \times \mathbb{R}^d} e^{it\langle u, g_1 \cdot x \rangle} e^{it\mathbf{P}\Theta(g_1 \cdot x)} \mu(dg_1) = e^{it\mathbf{P}\Theta(x)}.$$

Since the functions in the both sides are continuous, this equality holds for every $x \in \text{supp}(\nu)$. Since $\Theta \in \mathcal{L}_{\theta, \varepsilon}$, the function $x \mapsto e^{it\mathbf{P}\Theta(x)}$ belongs to $\mathcal{L}_{\theta, \varepsilon} \setminus \{0\}$. This contradicts the point 2 of Lemma 3.11.2 and we conclude that $\sigma^2 > 0$ and so **M3.5** holds true.

3.12 Appendix: proofs for compact Markov chains

In this section we prove Proposition 3.3.7. For this we show that M3.1-M3.5 hold true with $N = N_l = 0$, for the Markov chain $(X_n)_{n \geq 1}$, the function f and the Banach space $\mathcal{L}(\mathbb{X})$ given in Section 3.3.2.

Proof of M3.1. Obviously the Dirac measure belongs to $\mathcal{L}(\mathbb{X})'$ and $\|\delta_x\|_{\mathcal{L}(\mathbb{X})'} \leq 1$ for any $x \in \mathbb{X}$. For any $h \in \mathcal{L}(\mathbb{X})$ and $t \in \mathbb{R}$ the function $e^{itf} h$ belongs to $\mathcal{L}(\mathbb{X})$ and

$$\|e^{itf} h\|_{\mathcal{L}} \leq |t| [f]_{\mathbb{X}} \|h\|_{\infty} + \|h\|_{\mathcal{L}} \leq (|t| [f]_{\mathbb{X}} + 1) \|h\|_{\mathcal{L}}. \quad (3.12.1)$$

Proof of M3.2. Let (x_1, x_2) and (y_1, y_2) be two elements of \mathbb{X} and $h \in \mathcal{L}(\mathbb{X})$. Since

$$\mathbf{P}h(x_1, x_2) = \int_X h(x_2, x') P(x_2, dx'),$$

we have $\|\mathbf{P}h\|_{\infty} \leq \|h\|_{\infty}$. Denote by h_{x_2} the function $z \mapsto h(x_2, z)$, which is an element of $\mathcal{L}(X)$. Since $[h_{x_2}]_X \leq [h]_{\mathbb{X}}$ and $|h_{x_2}|_{\infty} \leq \|h\|_{\infty}$, we obtain also that

$$\begin{aligned} |\mathbf{P}h(x_1, x_2) - \mathbf{P}h(y_1, y_2)| &= |Ph_{x_2}(x_2) - Ph_{y_2}(y_2)| \\ &\leq [Ph_{x_2}]_X d_X(x_2, y_2) + [h]_{\mathbb{X}} d_X(x_2, y_2) \\ &\leq (|P|_{\mathcal{L} \rightarrow \mathcal{L}} \|h\|_{\mathbb{X}} + [h]_{\mathbb{X}}) d_X(x_2, y_2), \end{aligned}$$

where $|P|_{\mathcal{L} \rightarrow \mathcal{L}}$ is the norm of the operator $P: \mathcal{L}(X) \rightarrow \mathcal{L}(X)$. Therefore \mathbf{P} is a bounded operator on $\mathcal{L}(\mathbb{X})$ and $\|\mathbf{P}\|_{\mathcal{L} \rightarrow \mathcal{L}} \leq (1 + |P|_{\mathcal{L} \rightarrow \mathcal{L}})$. Now, for any $h \in \mathcal{L}(\mathbb{X})$, we define the function F_h by

$$F_h(x_2) := \int_X h(x_2, x') P(x_2, dx') = \mathbf{P}h(x_1, x_2).$$

Notice that F_h belongs to $\mathcal{L}(X)$ and $|F_h|_{\mathcal{L}} \leq \|\mathbf{P}h\|_{\mathcal{L}}$. So by Proposition 3.3.5, for any $n \geq 2$, $(x_1, x_2) \in \mathbb{X}$ and $h \in \mathcal{L}(\mathbb{X})$,

$$\mathbf{P}^n h(x_1, x_2) = P^{n-1} F_h(x_2) = \nu(F_h) + R^{n-1} F_h(x_2) = \nu(h)e(x_1, x_2) + Q^n h(x_1, x_2),$$

where the probability ν is defined on \mathbb{X} by

$$\nu(h) = \nu(F_h) = \int_{X \times X} h(x', x'') P(x', dx'') \nu(dx'),$$

the function e is the unit function on \mathbb{X} , $e(x_1, x_2) = 1, \forall (x_1, x_2) \in \mathbb{X}$ and Q is the linear operator on $\mathcal{L}(\mathbb{X})$ defined by $Qh = R(F_h) = \mathbf{P}h - \nu(h)$. By Proposition 3.3.5, the operator Q is bounded and for any $n \geq 1$, $\|Q^n\|_{\mathcal{L} \rightarrow \mathcal{L}} \leq |R^{n-1}|_{\mathcal{L} \rightarrow \mathcal{L}} \|\mathbf{P}\|_{\mathcal{L} \rightarrow \mathcal{L}} \leq c e^{-cn}$. Since ν is invariant by P , one can easily verify that $\Pi Q = Q \Pi = 0$, where Π is the one-dimensional projector defined on $\mathcal{L}(\mathbb{X})$ by $\Pi h = \nu(h)e$.

Proof of M3.3. For any $t \in \mathbb{R}$, $h \in \mathcal{L}(\mathbb{X})$ and $(x_1, x_2) \in \mathbb{X}$,

$$\mathbf{P}_t h(x_1, x_2) = \int_X e^{itf(x_2, x')} h(x_2, x') P(x_2, dx') = \sum_{n=0}^{+\infty} \frac{i^n t^n}{n!} L_n(h)(x_1, x_2),$$

where $L_n(h) = \mathbf{P}(f^n h)$. Since $(\mathcal{L}(\mathbb{X}), \|\cdot\|_{\mathcal{L}})$ is a Banach algebra, it follows that L_n is a bounded operator on $\mathcal{L}(\mathbb{X})$ and $\|L_n\|_{\mathcal{L} \rightarrow \mathcal{L}} \leq \|\mathbf{P}\|_{\mathcal{L} \rightarrow \mathcal{L}} \|f\|_{\mathcal{L}}^n$. Consequently, the

application $t \mapsto \mathbf{P}_t$ is analytic on \mathbb{R} and so, by the analytic perturbation theory of linear operators (see [50]), there exists $\varepsilon_0 > 0$ such that, for any $|t| \leq \kappa$,

$$\mathbf{P}_t^n = \lambda_t^n \Pi_t + Q_t^n,$$

where λ_t is an eigenvalue of \mathbf{P}_t , Π_t is the projector on the one-dimensional eigenspace of λ_t and Q_t is an operator of spectral radius $r(Q_t) < |\lambda_t|$ such that $\Pi_t Q_t = Q_t \Pi_t = 0$. The functions $t \mapsto \lambda_t$, $t \mapsto \Pi_t$ and $t \mapsto Q_t$ are analytic on $[-\kappa, \kappa]$. Furthermore, for any $h \in \mathcal{L}(\mathbb{X})$ and $(x_1, x_2) \in \mathbb{X}$,

$$|\mathbf{P}_t h|(x_1, x_2) = \left| \int_X e^{itf(x_2, x')} h(x_2, x') P(x_2, dx') \right| \leq \|h\|_\infty$$

and necessarily $|\lambda_t| \leq 1$, for any $|t| \leq \kappa$. Consequently

$$\sup_{|t| \leq \kappa, n \geq 1} \|\mathbf{P}_t^n\|_{\mathcal{L} \rightarrow \mathcal{L}} \leq c.$$

Proof of M3.4 and M3.5. Since for any $x \in \mathbb{X}$, $|f(x)| \leq |f|_\infty$ and $\|\delta_x\|_{\mathcal{L}(\mathbb{X})'} \leq 1$, we can choose $N = 0$ and $N_l = 0$ for any $l \geq 1$ and Hypothesis M3.4 is obviously satisfied.

Finally, Hypothesis 3.3.6 ensures that M3.5 holds true.

Acknowledgements

The authors are very grateful to the anonymous referee for valuable comments and corrections which helped them to improve the first version of this work.

The two next sections are not a part of the article *Limit theorems for Markov walks conditioned to stay positive under a spectral gap assumption* [38]. However, they are interesting developments: in Section 3.13, I show how to construct a Banach space which satisfies Hypotheses M3.1-M3.5 and in Section 3.14, I prove that it is possible to improve Theorem 3.2.5 giving the asymptotic behaviour of the couple $(X_n, y + S_n)_{n \geq 1}$ knowing that the Markov walk stay positive, $\tau_y > n$.

3.13 A Banach space for the product of matrices

The purpose of this section is to prove that Theorems 3.2.2-3.2.5 can be applied for the product of random matrices considered by Grama, Le Page and Peigné [41]. This additional example and the fact that Hypotheses M3.1-M3.5 are satisfied in previous models stress the global nature of Theorem 3.2.2-3.2.5.

Note that the Banach space constructed in [41] does not satisfy entirely Hypotheses M3.1-M3.5. Indeed the tricky point is that in Hypothesis M3.4 we suppose that the function f is bounded by a function belonging to the Banach space. However in the article [41] the Banach space is included in the set of bounded function whereas the considered function $f = \rho$ is not bounded. Consequently in this section, we construct a new Banach space and show that, with this Banach space, Hypotheses M3.1-M3.5 are satisfied. We start by introducing again some notations of [41].

3.13.1 Notations

Let $\mathbb{G} = \text{GL}_d(\mathbb{R})$ be the set of invertible matrices of size $d \times d$, with $d \geq 1$. We endow \mathbb{R}^d with the euclidean norm, $\|v\| = \sqrt{\sum_{i=1}^d v_i^2}$, for any $v = (v_1, \dots, v_d) \in \mathbb{R}^d$ and \mathbb{G} with the associated operator norm, $\|g\| = \sup_{v \in \mathbb{R}^d \setminus \{0\}} \|gv\| / \|v\|$, for any $g \in \mathbb{G}$. Denote by $\mathbb{P}(\mathbb{R}^d)$ the associated projective space and for any $v \in \mathbb{R}^d$, let $\bar{v} \in \mathbb{P}(\mathbb{R}^d)$ be the direction of v . We endow $\mathbb{P}(\mathbb{R}^d)$ with the angular distance $d(\bar{u}, \bar{v}) = \|u \wedge v\| / (\|u\| \|v\|)$, where $u \wedge v$ is the vector product of u and v . The group \mathbb{G} act on the projective space $\mathbb{P}(\mathbb{R}^d)$ by multiplication: for any $\bar{v} \in \mathbb{P}(\mathbb{R}^d)$, denote by $g \cdot \bar{v} = \overline{gv}$ the action of g on the direction \bar{v} . Finally, for any $g \in \mathbb{G}$, we define

$$N(g) = \max \left(\|g\|, \|g^{-1}\| \right).$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, \mathbb{E} be the associated expectation and $(g_n)_{n \geq 1}$ a sequence of random variable *i.i.d.* defined on Ω and taking its values in \mathbb{G} where the common law is denoted by μ . We now recall the assumptions of [41].

P3.1. *There exists $\delta_0 > 0$ such that*

$$\mathbb{E} \left(N(g_1)^{\delta_0} \right) = \int_{\mathbb{G}} \exp(\delta_0 \log(N(g))) \mu(g) < +\infty.$$

P3.2 (Strongly irreducibility). *The action of the support of μ on \mathbb{R}^d is strongly irreducible i.e. there is no proper finite union of subspaces of \mathbb{R}^d which is invariant by Γ_μ , the smallest closed semigroup containing the support of μ .*

P3.3 (Propriété de contraction). *The semigroup Γ_μ contains a contacting sequence.*

Let ρ be the cocycle defined by

$$\rho(g, \bar{v}) := \log \left(\frac{\|gv\|}{\|v\|} \right), \quad \forall (g, \bar{v}) \in \mathbb{G} \times \mathbb{P}(\mathbb{R}^d).$$

Under conditions **P3.1-P3.3**, it is well-known that there exists a unique measure ν which is μ -invariant on $\mathbb{P}(\mathbb{R}^d)$.

P3.4. *The upper Lyapunov exponent is equal to 0 : $\int_{\mathbb{G} \times \mathbb{P}(\mathbb{R}^d)} \rho(g, \bar{v}) \mu(dg) \nu(d\bar{v}) = 0$.*

The condition **P5** of Grama, Le Page and Peigné [41] ensure that the harmonic function is positive for any $y > 0$. This assumption is not necessary in our case and Proposition 3.13.11 will make explicit the exact domain of positivity of this harmonic function.

For more details on the conditions **P3.1-P3.4**, we refer to the article [41]. We introduce now the random walk associated to the product of random matrices. For any $n \geq 1$, we define

$$G_n := g_n \dots g_1 \quad \text{and} \quad G_0 = \text{Id}.$$

Let \mathbb{B} be the closed unit ball of \mathbb{R}^d . To study the first time when the product $G_n v$, for $v \notin \mathbb{B}$ goes into the unit ball \mathbb{B} , on consider the logarithm of its norm

$$\log(\|G_n v\|) = \sum_{k=1}^n \rho(g_k, G_{k-1} \cdot \bar{v}) + \log(\|v\|).$$

Let $\mathbb{X} = \mathbb{G} \times \mathbb{P}(\mathbb{R}^d)$. For $x = (g, \bar{v}) \in \mathbb{X}$, we consider $(X_n)_{n \geq 0}$ the Markov chain on Ω taking its values in \mathbb{X} defined by $X_0 = x$ and

$$X_n = (g_n, G_{n-1} g \cdot \bar{v}), \quad \forall n \geq 1.$$

The associated Markov walk is given by $S_n = \rho(X_1) + \dots + \rho(X_n)$.

3.13.2 The Banach space

We denote by $\mathcal{C}(\mathbb{X}, \mathbb{C})$ the set of the continuous functions from \mathbb{X} to \mathbb{C} . We fix the following parameters

$$\varepsilon = \frac{\delta_0}{8}, \quad \theta = 3\varepsilon = \frac{3\delta_0}{8}, \quad \alpha = 5\varepsilon = \frac{5\delta_0}{8}, \quad \beta = 7\varepsilon = \frac{7\delta_0}{8},$$

where δ_0 is defined by **P3.1**. For any function $h \in \mathcal{C}(\mathbb{X}, \mathbb{C})$, we set

$$|h|_\theta = \sup_{(g, \bar{u}) \in \mathbb{X}} \frac{|h(g, \bar{u})|}{N(g)^\theta},$$

$$k_{\varepsilon, \alpha}(h) = \sup_{\substack{g \in \mathbb{G} \\ \bar{u} \neq \bar{v}}} \frac{|h(g, \bar{u}) - h(g, \bar{v})|}{d(\bar{u}, \bar{v})^\varepsilon N(g)^\alpha}, \quad k'_{\varepsilon, \beta}(h) = \sup_{\substack{g \neq g' \\ \bar{u} \in \mathbb{P}(\mathbb{R}^d)}} \frac{|h(g, \bar{u}) - h(g', \bar{u})|}{\|g - g'\|^\varepsilon N(g)^\beta N(g')^\beta},$$

and we define the norm

$$\|h\|_{\mathcal{B}} := |h|_\theta + k_{\varepsilon, \alpha}(h) + k'_{\varepsilon, \beta}(h),$$

and the associated Banach space

$$\mathcal{B} := \{h \in \mathcal{C}(\mathbb{X}, \mathbb{C}) : \|h\|_{\mathcal{B}} < +\infty\}.$$

3.13.3 Poof of M3.1

For the ease of the reader, we gradually recall Hypotheses **M3.1-M3.5** by respectively Propositions **3.13.1**, **3.13.6**, **3.13.7**, **3.13.8** and **3.13.9**.

Proposition 3.13.1 (Banach space). *Assume **P3.1**. Then,*

1. *The constant function equal to 1, denoted by e belongs to \mathcal{B} .*
2. *For any $x \in \mathbb{X}$, the Dirac measure δ_x belongs to the dual of \mathcal{B} , denoted by \mathcal{B}' .*
3. *The Banach space \mathcal{B} is included in $L^1(\mathbf{P}(x, \cdot))$, for any $x \in \mathbb{X}$.*
4. *For any $t \in \mathbb{R}$ and $h \in \mathcal{B}$, the function $e^{it\rho} h$ is in \mathcal{B} .*

Proof. *Point 1.* Since $N(g) \geq 1$ for any $g \in \mathbb{G}$, it is clear that $e \in \mathcal{B}$.

Point 2. For any $x = (g, \bar{u}) \in \mathbb{X}$ and $h \in \mathcal{B}$,

$$|\delta_x(h)| = |h(x)| \leq N(g)^\theta |h|_\theta \leq N(g)^\theta \|h\|_{\mathcal{B}}.$$

So $\delta_x \in \mathcal{B}'$ and

$$\|\delta_x\|_{\mathcal{B}'} \leq N(g)^\theta. \tag{3.13.1}$$

Point 3. For any $x = (g, \bar{u}) \in \mathbb{X}$ and $h \in \mathcal{B}$,

$$\mathbf{P} |h|(x) = \int_{\mathbb{G}} |h(g_1, g \cdot \bar{u})| \boldsymbol{\mu}(dg_1) \leq |h|_\theta \int_{\mathbb{G}} N(g_1)^\theta \boldsymbol{\mu}(dg_1),$$

where \mathbf{P} is the Markovian operator associated to $(X_n)_{n \geq 0}$. Since $\theta = 3\delta_0/8 \leq \delta_0$, by **P3.1**, we have

$$\mathbf{P} |h|(x) \leq c |h|_\theta < +\infty.$$

Point 4. Fix $h \in \mathcal{B}$ and $t \in \mathbb{R}$. First we note that

$$\left| e^{it\rho} h \Big|_{\theta} \right| = |h|_{\theta}. \quad (3.13.2)$$

Next, for any (g, \bar{u}) and (g, \bar{v}) in \mathbb{X} , we write that

$$\left| e^{it\rho(g, \bar{u})} h(g, \bar{u}) - e^{it\rho(g, \bar{v})} h(g, \bar{v}) \right| \leq |h|_{\theta} N(g)^{\theta} \left| e^{it\rho(g, \bar{u})} - e^{it\rho(g, \bar{v})} \right| + |h(g, \bar{u}) - h(g, \bar{v})|. \quad (3.13.3)$$

Let u and v be two vectors of direction \bar{u} and \bar{v} respectively, of norm equal to 1 and such that $\langle u, v \rangle \geq 0$. We have

$$\left| e^{it\rho(g, \bar{u})} - e^{it\rho(g, \bar{v})} \right| \leq |t| |\rho(g, \bar{u}) - \rho(g, \bar{v})| = |t| \left| \log \left(\frac{\|gu\|}{\|gv\|} \right) \right|.$$

Using the fact that $|\log(s)| \leq |1 - s|$ for any $s > 0$,

$$|\rho(g, \bar{u}) - \rho(g, \bar{v})| \leq \frac{\|g(u - v)\|}{\|gv\|} \leq \|g\| \|g^{-1}\| \|u - v\|.$$

We recall that

$$\|u - v\| \leq \sqrt{2}d(\bar{u}, \bar{v}) \quad \text{and that} \quad d(\bar{u}, \bar{v}) \leq \|u - v\|. \quad (3.13.4)$$

So

$$|\rho(g, \bar{u}) - \rho(g, \bar{v})| \leq \sqrt{2}N(g)^2 d(\bar{u}, \bar{v}). \quad (3.13.5)$$

Consequently

$$\left| e^{it\rho(g, \bar{u})} - e^{it\rho(g, \bar{v})} \right| \leq \sqrt{2} |t| N(g)^2 d(\bar{u}, \bar{v}).$$

Moreover $\left| e^{it\rho(g, \bar{u})} - e^{it\rho(g, \bar{v})} \right| \leq 2$, so

$$\left| e^{it\rho(g, \bar{u})} - e^{it\rho(g, \bar{v})} \right| \leq 2^{1-\varepsilon+\varepsilon/2} |t|^{\varepsilon} N(g)^{2\varepsilon} d(\bar{u}, \bar{v})^{\varepsilon}.$$

From (3.13.3), we deduce that

$$\left| e^{it\rho(g, \bar{u})} h(g, \bar{u}) - e^{it\rho(g, \bar{v})} h(g, \bar{v}) \right| \leq 2 |t|^{\varepsilon} |h|_{\theta} N(g)^{\theta+2\varepsilon} d(\bar{u}, \bar{v})^{\varepsilon} + k_{\varepsilon, \alpha}(h) d(\bar{u}, \bar{v})^{\varepsilon} N(g)^{\alpha}.$$

Since $\alpha = \theta + 2\varepsilon$, we have

$$k_{\varepsilon, \alpha}(e^{it\rho} h) \leq 2 |t|^{\varepsilon} |h|_{\theta} + k_{\varepsilon, \alpha}(h) < +\infty. \quad (3.13.6)$$

We proceed in the same way for $k'_{\varepsilon, \beta}(e^{it\rho} h)$. Fix (g, \bar{u}) and (g', \bar{u}) in \mathbb{X} . We have

$$\left| e^{it\rho(g, \bar{u})} h(g, \bar{u}) - e^{it\rho(g', \bar{u})} h(g', \bar{u}) \right| \leq |h|_{\theta} N(g)^{\theta} \left| e^{it\rho(g, \bar{u})} - e^{it\rho(g', \bar{u})} \right| + |h(g, \bar{u}) - h(g', \bar{u})|.$$

As previously,

$$\left| e^{it\rho(g, \bar{u})} - e^{it\rho(g', \bar{u})} \right| \leq |t| \frac{\|(g' - g)u\|}{\|g'u\|} \leq |t| N(g') \|g - g'\|.$$

Since $\left| e^{it\rho(g, \bar{u})} - e^{it\rho(g', \bar{u})} \right| \leq 2$, we deduce that

$$\left| e^{it\rho(g, \bar{u})} - e^{it\rho(g', \bar{u})} \right| \leq 2 |t|^{\varepsilon} N(g')^{\varepsilon} \|g - g'\|^{\varepsilon}.$$

Consequently,

$$\begin{aligned} \left| e^{it\rho(g, \bar{u})} h(g, \bar{u}) - e^{it\rho(g', \bar{u})} h(g', \bar{u}) \right| &\leq 2 |t|^\varepsilon |h|_\theta N(g)^\theta N(g')^\varepsilon \|g - g'\|^\varepsilon \\ &\quad + k'_{\varepsilon, \beta}(h) \|g - g'\|^\varepsilon N(g)^\beta N(g')^\beta. \end{aligned}$$

Since $\varepsilon \leq \beta$ and $\theta \leq \beta$,

$$k'_{\varepsilon, \beta}(e^{it\rho} h) \leq 2 |t|^\varepsilon |h|_\theta + k'_{\varepsilon, \beta}(h) < +\infty. \quad (3.13.7)$$

Using (3.13.2), (3.13.6) and (3.13.7), we conclude that $e^{it\rho} h \in \mathcal{B}$ for any $t \in \mathbb{R}$. Moreover,

$$\|e^{it\rho} h\|_{\mathcal{B}} \leq \|h\|_{\mathcal{B}} + 4 |t|^\varepsilon |h|_\theta.$$

□

3.13.4 Proof of M3.2 and M3.3

We recall that the perturbed operator is given by $\mathbf{P}_t h(x) = \mathbf{P}(e^{it\rho} h)(x)$ for any $t \in \mathbb{R}$, $h \in \mathcal{B}$ and $x \in \mathbb{X}$. We will prove that the perturbed operator \mathbf{P}_t , $t \in \mathbb{R}$ satisfies the hypotheses of the theorem of Ionescu-Tulcea and Marinescu [49]. This will imply in particular that \mathbf{P} have a spectral gap and from this, we will establish Proposition 3.13.6. Previously, we recall a result of Le Page [54] (Theorem 1). One can also see an expression of this result in Grama, Le Page and Peigné [41] (Proposition 8.6).

Proposition 3.13.2. *Assume conditions P3.1-P3.3. Then there exist $\varepsilon_0 > 0$ and $r_{\varepsilon_0} \in (0, 1)$ such that*

$$\lim_{n \rightarrow +\infty} \sup_{\bar{u} \neq \bar{v}} \mathbb{E} \left(\frac{d(G_n \cdot \bar{u}, G_n \cdot \bar{v})^{\varepsilon_0}}{d(\bar{u}, \bar{v})^{\varepsilon_0}} \right)^{1/n} = r_{\varepsilon_0}.$$

In the following lemma we give a control of the norm of \mathbf{P}_t .

Lemme 3.13.3. *Assume conditions P3.1-P3.3. For any $t \in \mathbb{R}$, $n \geq 1$ and $h \in \mathcal{B}$, the function $\mathbf{P}_t^n h$ belongs to \mathcal{B} . Moreover,*

$$\|\mathbf{P}_t^n h\|_{\mathcal{B}} \leq c_\varepsilon (1 + |t|^\varepsilon) |h|_\theta + c_\varepsilon k_{\varepsilon, \alpha}(h) r_\varepsilon^n.$$

Proof. Fix $t \in \mathbb{R}$ and $n \geq 1$. Observe that for any $h \in \mathcal{B}$, $x = (g, \bar{u}) \in \mathbb{X}$ and $n \geq 1$

$$\mathbf{P}_t^n h(x) = \mathbb{E}_x \left(e^{itS_n} h(X_n) \right).$$

Since $S_n \in \mathbb{R}$, by the definition of X_n ,

$$|\mathbf{P}_t^n h(x)| \leq \mathbb{E}_x (|h(g_n, G_{n-1}g \cdot \bar{u})|) \leq |h|_\theta \mathbb{E} \left(N(g_n)^\theta \right). \quad (3.13.8)$$

Consequently, using the fact that $\theta = 3\delta_0/8 \leq \delta_0$ and P3.1,

$$|\mathbf{P}_t^n h|_\theta \leq |h|_\theta \mathbb{E} \left(N(g_1)^\theta \right) < +\infty. \quad (3.13.9)$$

For any $x \in \mathbb{X}$, denote by X_n^x the Markov chain starting to $X_0 = x$ and by S_n^x the associated Markov chain. For any (g, \bar{u}) and (g, \bar{v}) in \mathbb{X} ,

$$\begin{aligned} \Delta_n &:= |\mathbf{P}_t^n h(g, \bar{u}) - \mathbf{P}_t^n h(g, \bar{v})| = \left| \mathbb{E} \left(e^{itS_n^{(g, \bar{u})}} h(X_n^{(g, \bar{u})}) - e^{itS_n^{(g, \bar{v})}} h(X_n^{(g, \bar{v})}) \right) \right| \\ &\leq \mathbb{E} (|h(g_n, G_{n-1}g \cdot \bar{u}) - h(g_n, G_{n-1}g \cdot \bar{v})|) \\ &\quad + \mathbb{E} \left(|h(g_n, G_{n-1}g \cdot \bar{u})| \left| e^{itS_n^{(g, \bar{u})}} - e^{itS_n^{(g, \bar{v})}} \right| \right) \\ &\leq k_{\varepsilon, \alpha}(h) \mathbb{E} (d(G_{n-1}g \cdot \bar{u}, G_{n-1}g \cdot \bar{v})^\varepsilon N(g_n)^\alpha) \\ &\quad + |h|_\theta \mathbb{E} \left(N(g_n)^\theta \left| e^{itS_n^{(g, \bar{u})}} - e^{itS_n^{(g, \bar{v})}} \right| \right) \end{aligned}$$

Using the fact that

$$\left| e^{itS_n^{(g, \bar{u})}} - e^{itS_n^{(g, \bar{v})}} \right| \leq \min \left(|t| |S_n^{(g, \bar{u})} - S_n^{(g, \bar{v})}|, 2 \right) \leq 2^{1-\varepsilon} |t|^\varepsilon |S_n^{(g, \bar{u})} - S_n^{(g, \bar{v})}|^\varepsilon$$

and the independence of g_i , $i \geq 1$, we deduce that

$$\begin{aligned} \Delta_n &\leq k_{\varepsilon, \alpha}(h) \mathbb{E} (d(G_{n-1}g \cdot \bar{u}, G_{n-1}g \cdot \bar{v})^\varepsilon) \mathbb{E} (N(g_n)^\alpha) \\ &\quad + 2^{1-\varepsilon} |t|^\varepsilon |h|_\theta \mathbb{E} \left(N(g_n)^\theta |\rho(g_n, G_{n-1}g \cdot \bar{u}) - \rho(g_n, G_{n-1}g \cdot \bar{v})|^\varepsilon \right) \\ &\quad + 2^{1-\varepsilon} |t|^\varepsilon |h|_\theta \sum_{k=1}^{n-1} \mathbb{E} \left(N(g_n)^\theta \right) \mathbb{E} (|\rho(g_k, G_{k-1}g \cdot \bar{u}) - \rho(g_k, G_{k-1}g \cdot \bar{v})|^\varepsilon). \end{aligned}$$

By (3.13.5) and the fact that $\theta + 2\varepsilon = 5\varepsilon = \alpha$,

$$\begin{aligned} \Delta_n &\leq \left(k_{\varepsilon, \alpha}(h) + 2^{1-\varepsilon/2} |t|^\varepsilon |h|_\theta \right) \mathbb{E} (d(G_{n-1}g \cdot \bar{u}, G_{n-1}g \cdot \bar{v})^\varepsilon) \mathbb{E} (N(g_1)^\alpha) \\ &\quad + 2^{1-\varepsilon/2} |t|^\varepsilon |h|_\theta \mathbb{E} \left(N(g_1)^\theta \right) \sum_{k=1}^{n-1} \mathbb{E} \left(N(g_1)^{2\varepsilon} \right) \mathbb{E} (d(G_{k-1}g \cdot \bar{u}, G_{k-1}g \cdot \bar{v})^\varepsilon). \end{aligned} \tag{3.13.10}$$

Now we will use the fact that the sequence $(G_k)_{k \geq 0}$ is contracting on the directions. Without loss of generality, we can assume that $\delta_0/8 \leq \varepsilon_0$. So, by Proposition 3.13.2, there exist n_0 and $r_\varepsilon \in (0, 1)$ such that for any $n \geq n_0$ and $(\bar{u}, \bar{v}) \in \mathbb{P}(\mathbb{R}^d)^2$,

$$\mathbb{E} (d(G_n g \cdot \bar{u}, G_n g \cdot \bar{v})^\varepsilon) \leq r_\varepsilon^n d(g \cdot \bar{u}, g \cdot \bar{v})^\varepsilon.$$

By (3.13.4), with u and v two vectors of directions \bar{u} and \bar{v} respectively, with norm equal to 1 and satisfying $\langle u, v \rangle \geq 0$, we have

$$\begin{aligned} d(g \cdot \bar{u}, g \cdot \bar{v}) &\leq \left\| \frac{gu}{\|gu\|} - \frac{gv}{\|gv\|} \right\| \leq \frac{\|g(u-v)\|}{\|gu\|} + \|gv\| \left| \frac{1}{\|gu\|} - \frac{1}{\|gv\|} \right| \\ &\leq N(g)^2 \|u-v\| + \frac{\| \|gv\| - \|gu\| \|}{\|gu\|} \\ &\leq 2N(g)^2 \|u-v\| \\ &\leq 2\sqrt{2}N(g)^2 d(\bar{u}, \bar{v}). \end{aligned}$$

So, for any $n \geq n_0$

$$\mathbb{E} (d(G_n g \cdot \bar{u}, G_n g \cdot \bar{v})^\varepsilon) \leq 2^{3\varepsilon/2} r_\varepsilon^n N(g)^{2\varepsilon} d(\bar{u}, \bar{v})^\varepsilon,$$

In the same way, for any $n \leq n_0$,

$$\begin{aligned} \mathbb{E} (d(G_n g \cdot \bar{u}, G_n g \cdot \bar{v})^\varepsilon) &\leq 2^{3\varepsilon/2} \mathbb{E} \left(N(G_n g)^{2\varepsilon} \right) d(\bar{u}, \bar{v})^\varepsilon \leq 2^{3\varepsilon/2} \mathbb{E} \left(N(g_1)^{2\varepsilon} \right)^n N(g)^{2\varepsilon} d(\bar{u}, \bar{v})^\varepsilon \\ &\leq c_\varepsilon N(g)^{2\varepsilon} d(\bar{u}, \bar{v})^\varepsilon, \end{aligned}$$

where hereafter c_ε is a constant depending only on ε which its value is likely to change every occurrence. We obtain that for any $n \geq 1$,

$$\mathbb{E} (d(G_n g \cdot \bar{u}, G_n g \cdot \bar{v})^\varepsilon) \leq c_\varepsilon r_\varepsilon^n N(g)^{2\varepsilon} d(\bar{u}, \bar{v})^\varepsilon.$$

Putting together this last inequality with (3.13.10),

$$\Delta_n \leq c_\varepsilon k_{\varepsilon, \alpha}(h) r_\varepsilon^{n-1} N(g)^{2\varepsilon} d(\bar{u}, \bar{v})^\varepsilon + c_\varepsilon |t|^\varepsilon |h|_\theta N(g)^{2\varepsilon} d(\bar{u}, \bar{v})^\varepsilon \sum_{k=1}^n r_\varepsilon^{k-1}.$$

Since $2\varepsilon \leq \alpha = 5\varepsilon$, we deduce that

$$k_{\varepsilon, \alpha}(\mathbf{P}_t^n h) \leq c_\varepsilon |t|^\varepsilon |h|_\theta + c_\varepsilon k_{\varepsilon, \alpha}(h) r_\varepsilon^n < +\infty. \quad (3.13.11)$$

In the same way, for any (g, \bar{u}) and (g', \bar{u}) in \mathbb{X} ,

$$\begin{aligned} \Delta'_n &:= |\mathbf{P}_t^n h(g, \bar{u}) - \mathbf{P}_t^n h(g', \bar{u})| \\ &\leq k_{\varepsilon, \alpha}(h) \mathbb{E} \left(d(G_{n-1} g \cdot \bar{u}, G_{n-1} g' \cdot \bar{u})^\varepsilon \right) \mathbb{E} (N(g_n)^\alpha) \\ &\quad + 2^{1-\varepsilon} |t|^\varepsilon |h|_\theta \mathbb{E} \left(N(g_n)^\theta |\rho(g_n, G_{n-1} g \cdot \bar{u}) - \rho(g_n, G_{n-1} g' \cdot \bar{u})|^\varepsilon \right) \\ &\quad + 2^{1-\varepsilon} |t|^\varepsilon |h|_\theta \sum_{k=1}^{n-1} \mathbb{E} \left(N(g_k)^\theta \right) \mathbb{E} \left(|\rho(g_k, G_{k-1} g \cdot \bar{u}) - \rho(g_k, G_{k-1} g' \cdot \bar{u})|^\varepsilon \right). \end{aligned}$$

Using again (3.13.5),

$$\Delta'_n \leq c_\varepsilon k_{\varepsilon, \alpha}(h) \mathbb{E} \left(d(G_{n-1} g \cdot \bar{u}, G_{n-1} g' \cdot \bar{u})^\varepsilon \right) + c_\varepsilon |t|^\varepsilon |h|_\theta \sum_{k=1}^n \mathbb{E} \left(d(G_{k-1} g \cdot \bar{u}, G_{k-1} g' \cdot \bar{u})^\varepsilon \right).$$

As previously, by (3.13.4),

$$\begin{aligned} d(g \cdot \bar{u}, g' \cdot \bar{u}) &\leq \left\| \frac{gu}{\|gu\|} - \frac{g'u}{\|g'u\|} \right\| \leq \frac{\|(g-g')u\|}{\|gu\|} + \|g'u\| \left| \frac{1}{\|gu\|} - \frac{1}{\|g'u\|} \right| \\ &\leq 2N(g) \|g - g'\|. \end{aligned}$$

So using Proposition 3.13.2, for any $n \geq 1$,

$$\mathbb{E} (d(G_n g \cdot \bar{u}, G_n g' \cdot \bar{v})^\varepsilon) \leq c_\varepsilon r_\varepsilon^n N(g)^\varepsilon \|g - g'\|^\varepsilon.$$

Thereby,

$$\Delta'_n \leq c_\varepsilon k_{\varepsilon, \alpha}(h) r_\varepsilon^n N(g)^\varepsilon \|g - g'\|^\varepsilon + c_\varepsilon |t|^\varepsilon |h|_\theta N(g)^\varepsilon \|g - g'\|^\varepsilon.$$

Since $\beta \geq \varepsilon$ and $N(g') \geq 1$, we get

$$k'_{\varepsilon, \beta}(\mathbf{P}_t^n h) \leq c_\varepsilon k_{\varepsilon, \alpha}(h) r_\varepsilon^n + c_\varepsilon |t|^\varepsilon |h|_\theta < +\infty. \quad (3.13.12)$$

Putting together with (3.13.9), (3.13.11) and (3.13.12), it completes the proof. \square

We now show that the conditions of theorem of Ionescu-Tulcea and Marinescu [49] are satisfied under **P3.1-P3.3**. For more details on the theorem of Ionescu-Tulcea and Marinescu we refer to the book of Norman [59]. We define $\mathcal{C}_\theta := \{h \in \mathcal{C}(\mathbb{X}, \mathbb{C}) : |h|_\theta < +\infty\}$.

Lemme 3.13.4. *Assume **P3.1-P3.3**. The Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is included in the set $(\mathcal{C}_\theta, |\cdot|_\theta)$ which is also a Banach space.*

1. Let $(h_n)_{n \geq 0} \in \mathcal{B}^{\mathbb{N}}$ and $h \in \mathcal{C}_\theta$ be such that $|h_n - h|_\theta \rightarrow 0$ when $n \rightarrow +\infty$ and such that for any $n \geq 0$, $\|h_n\|_{\mathcal{B}} \leq C$. Therefore $h \in \mathcal{B}$ and $\|h\|_{\mathcal{B}} \leq C$.
2. For any $t \in \mathbb{R}$ and $h \in \mathcal{B}$, we have

$$\sup_{n \geq 0} \frac{|\mathbf{P}_t^n h|_\theta}{|h|_\theta} < +\infty.$$

3. For any $t \in \mathbb{R}$, there exist $k \geq 1$, $r \in (0, 1)$ and $c > 0$ such that for any $h \in \mathcal{B}$,

$$\|\mathbf{P}_t^k h\|_{\mathcal{B}} \leq r \|h\|_{\mathcal{B}} + c |h|_\theta.$$

4. For any $t \in \mathbb{R}$, the operator \mathbf{P}_t from $(\mathcal{B}, \|\cdot\|)$ to $(\mathcal{C}_\theta, |\cdot|_\theta)$ is compact : for any bounded subspace B of \mathcal{B} , the set $\mathbf{P}_t B$ is relatively compact.

Proof. *Point 1.* Let $(h_n)_{n \geq 0} \in \mathcal{B}^{\mathbb{N}}$ and $h \in \mathcal{C}_\theta$. We suppose that $|h_n - h|_\theta \rightarrow 0$ as $n \rightarrow +\infty$ and that $\|h_n\|_{\mathcal{B}} \leq C$, for any $n \geq 0$. For any $(g_1, g_2, g_3, g'_3) \in \mathbb{G}^4$, $(\bar{u}_1, \bar{u}_2, \bar{u}'_2, \bar{u}_3) \in \mathbb{P}(\mathbb{R}^d)^4$ and $n \geq 0$, we write that

$$\begin{aligned} & \frac{|h(g_1, \bar{u}_1)|}{N(g_1)^\theta} + \frac{|h(g_2, \bar{u}_2) - h(g_2, \bar{u}'_2)|}{d(\bar{u}_2, \bar{u}'_2)^\varepsilon N(g_2)^\alpha} + \frac{|h(g_3, \bar{u}_3) - h(g'_3, \bar{u}_3)|}{\|g_3 - g'_3\|^\varepsilon N(g_3)^\beta N(g'_3)^\beta} \\ & \leq \frac{|h_n(g_1, \bar{u}_1)|}{N(g_1)^\theta} + \frac{|h_n(g_2, \bar{u}_2) - h_n(g_2, \bar{u}'_2)|}{d(\bar{u}_2, \bar{u}'_2)^\varepsilon N(g_2)^\alpha} + \frac{|h_n(g_3, \bar{u}_3) - h_n(g'_3, \bar{u}_3)|}{\|g_3 - g'_3\|^\varepsilon N(g_3)^\beta N(g'_3)^\beta} \\ & \quad + |h - h_n|_\theta \left(1 + \frac{2N(g_2)^\theta}{d(\bar{u}_2, \bar{u}'_2)^\varepsilon N(g_2)^\alpha} + \frac{N(g_3)^\theta + N(g'_3)^\theta}{\|g_3 - g'_3\|^\varepsilon N(g_3)^\beta N(g'_3)^\beta} \right) \end{aligned}$$

Therefore

$$\begin{aligned} & \frac{|h(g_1, \bar{u}_1)|}{N(g_1)^\theta} + \frac{|h(g_2, \bar{u}_2) - h(g_2, \bar{u}'_2)|}{d(\bar{u}_2, \bar{u}'_2)^\varepsilon N(g_2)^\alpha} + \frac{|h(g_3, \bar{u}_3) - h(g'_3, \bar{u}_3)|}{\|g_3 - g'_3\|^\varepsilon N(g_3)^\beta N(g'_3)^\beta} \\ & \leq C + |h - h_n|_\theta \left(1 + \frac{2N(g_2)^\theta}{d(\bar{u}_2, \bar{u}'_2)^\varepsilon N(g_2)^\alpha} + \frac{N(g_3)^\theta + N(g'_3)^\theta}{\|g_3 - g'_3\|^\varepsilon N(g_3)^\beta N(g'_3)^\beta} \right) \end{aligned}$$

Taking the limit as $n \rightarrow +\infty$, we conclude that

$$\|h\|_{\mathcal{B}} \leq C.$$

Point 2. It is a straightforward consequence of (3.13.9),

$$\sup_{n \geq 0} \frac{|\mathbf{P}_t^n h|_\theta}{|h|_\theta} \leq \mathbb{E} \left(N(g_1)^\theta \right) < +\infty.$$

Point 3. By Lemma 3.13.3, for any $t \in \mathbb{R}$, $n \geq 1$ and $h \in \mathcal{B}$

$$\|\mathbf{P}_t^n h\|_{\mathcal{B}} \leq c_\varepsilon (1 + |t|^\varepsilon) |h|_\theta + c_\varepsilon k_{\varepsilon, \alpha}(h) r_\varepsilon^n.$$

Since $r_\varepsilon \in (0, 1)$, there exists $n_0 \geq 1$ such that $c_\varepsilon r_\varepsilon^n \leq r < 1$, for any $n \geq n_0$, which proves the point 3.

Point 4. Let B be a bounded subset of \mathcal{B} . We will show that $\mathbf{P}_t B$ est relatively compact: for a fixed sequence $(h_n)_{n \geq 0}$ in B , we will construct a subsequence of $\mathbf{P}_t h_n$ which converges in $(\mathcal{C}_\theta, |\cdot|_\theta)$. Fix K a compact of \mathbb{X} . For any $x = (g, \bar{u}) \in K$ and $n \geq 0$, by (3.13.8),

$$|\mathbf{P}_t h_n(x)| \leq \mathbb{E} \left(N(g_1)^\theta \right) |h_n|_\theta. \quad (3.13.13)$$

Since $(h_n)_{n \geq 0}$ is bounded, we deduce that $(\mathbf{P}_t h_n(x))_{n \geq 0}$ is bounded in \mathbb{C} and so relatively compact. Let us show that $(\mathbf{P}_t h_n)_{n \geq 0}$ is equicontinuous in $x \in K$. For any $y = (g', \bar{v}) \in K$ and $n \geq 0$,

$$\begin{aligned} |\mathbf{P}_t h_n(x) - \mathbf{P}_t h_n(y)| &\leq |\mathbf{P}_t h_n(g, \bar{u}) - \mathbf{P}_t h_n(g, \bar{v})| + |\mathbf{P}_t h_n(g, \bar{v}) - \mathbf{P}_t h_n(g', \bar{v})| \\ &\leq k_{\varepsilon, \alpha} (\mathbf{P}_t h_n) d(\bar{u}, \bar{v})^\varepsilon N(g)^\alpha + k_{\varepsilon, \beta} (\mathbf{P}_t h_n) \|g - g'\|^\varepsilon N(g)^\beta N(g')^\beta. \end{aligned}$$

Since K is compact, there exists c_K such that $N(g) \leq c_K$ for any $g \in \mathbb{G}$. So using (3.13.11) and (3.13.12),

$$|\mathbf{P}_t h_n(x) - \mathbf{P}_t h_n(y)| \leq c_{\varepsilon, K} (|t|^\varepsilon + 1) \|h_n\|_{\mathcal{B}} \left(d(\bar{u}, \bar{v})^\varepsilon + \|g - g'\|^\varepsilon \right).$$

Since $(|h_n|_\theta)_{n \geq 0}$ is bounded, we deduce that the sequence $(\mathbf{P}_t h_n)_{n \geq 0}$ is equicontinuous. Therefore, by the theorem of Ascoli-Arzelà, the set $\{\mathbf{P}_t h_n, n \geq 0\}$ is relatively compact in $(\mathcal{C}(K, \mathbb{C}), |\cdot|_\infty)$. By a diagonalisation argument, there exist a subsequence $(\mathbf{P}_t h_{n_k})_{k \geq 0}$ and a function $\varphi \in \mathcal{C}(\mathbb{X}, \mathbb{C})$ such that for any compact K in \mathbb{X} , we have

$$\sup_{x \in K} |\mathbf{P}_t h_{n_k}(x) - \varphi(x)| \xrightarrow[k \rightarrow +\infty]{} 0.$$

Moreover, by (3.13.13), $|\mathbf{P}_t h_n(x)| \leq c_B$, for any $x \in \mathbb{X}$ and $n \geq 0$. So, for any $x \in \mathbb{X}$, $\varphi(x) \leq 1 + c_B$. We deduce that, for any $A > 0$ and $n \geq 0$,

$$\sup_{x \in \mathbb{X}} \frac{|\mathbf{P}_t h_{n_k}(x) - \varphi(x)|}{N(g)^\theta} \leq \sup_{\substack{N(g) \leq A \\ \bar{u} \in \mathbb{P}(\mathbb{R}^d)}} |\mathbf{P}_t h_{n_k}(g, \bar{u}) - \varphi(g, \bar{u})| + \sup_{\substack{N(g) > A \\ \bar{u} \in \mathbb{P}(\mathbb{R}^d)}} \frac{1 + 2c_B}{N(g)^\theta}.$$

Since $\{(g, \bar{u}) \in \mathbb{X} : N(g) \leq A\}$ is compact, we have for any $A > 0$,

$$\limsup_{k \rightarrow +\infty} \sup_{x \in \mathbb{X}} \frac{|\mathbf{P}_t h_{n_k}(x) - \varphi(x)|}{N(g)^\theta} \leq \frac{1 + 2c_B}{A^\theta}.$$

Taking the limit as $A \rightarrow +\infty$, we conclude that the sequence $(\mathbf{P}_t h_{n_k})_{k \geq 0}$ converges in $(\mathcal{C}_\theta, |\cdot|_\theta)$ to φ and so, finally, the subset $\mathbf{P}_t B$ is relatively compact in $(\mathcal{C}_\theta, |\cdot|_\theta)$. \square

To establish **M3.2**, we need one more result from Le Page [54] (Corollary 1) also stated in Grama, Le Page and Peigné [41]. We recall that under **P3.1-P3.3**, there exists a unique measure ν on $\mathbb{P}(\mathbb{R}^d)$ which is μ -invariant, i.e. such that for any continuous function $\varphi : \mathbb{P}(\mathbb{R}^d) \rightarrow \mathbb{C}$,

$$(\mu * \nu)(\varphi) = \int_{\mathbb{X}} \varphi(g \cdot \bar{u}) \nu(d\bar{u}) \mu(dg) = \int_{\mathbb{P}(\mathbb{R}^d)} \varphi(\bar{u}) \nu(d\bar{u}) = \nu(\varphi). \quad (3.13.14)$$

Proposition 3.13.5. *Assume conditions **P3.1-P3.3**. For any continuous function $\varphi : \mathbb{P}(\mathbb{R}^d) \rightarrow \mathbb{C}$, we have*

$$\lim_{n \rightarrow +\infty} \sup_{\bar{u} \in \mathbb{P}(\mathbb{R}^d)} |\mathbb{E}(\varphi(G_n \cdot \bar{u})) - \nu(\varphi)| = 0.$$

Proposition 3.13.6 (Spectral gap). *Assume **P3.1-P3.3**. Then,*

1. *The map $h \mapsto \mathbf{P}h$ is a bounded operator on \mathcal{B} .*
2. *There exist two constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\mathbf{P} = \Pi + Q,$$

where Π is a one-dimensional projector and Q is an operator on \mathcal{B} such that $\Pi Q = Q\Pi = 0$. Moreover for any $n \geq 1$,

$$\|Q^n\|_{\mathcal{B} \rightarrow \mathcal{B}} \leq c_1 e^{-c_2 n}.$$

Proof. Point 1. It is a straightforward consequence of Lemma 3.13.3 for $n = 1$ and $t = 0$.

Point 2. From Lemma 3.13.4 and the theorem of Ionescu-Tulcea and Marinescu [49], we know that there exists a finite number of eigenvalues of modulus 1, say $\lambda_1, \dots, \lambda_p$ and operators Π_1, \dots, Π_p, Q such that $\mathbf{P} = \sum_{i=1}^p \lambda_i \Pi_i + Q$ with Π_i orthogonal projectors satisfying $\Pi_i Q = Q\Pi_i = 0$ and the spectral radius of Q is strictly less than 1 and so

$$\|Q^n\|_{\mathcal{B} \rightarrow \mathcal{B}} \leq c_1 e^{-c_2 n}.$$

It remains to prove that 1 is the unique eigenvalue of modulus 1 and that the associated eigenspace is one-dimensional. Let $\lambda \in \mathbb{C}$ be an eigenvalue of \mathbf{P} of modulus 1 and let $h \in \mathcal{B}$ be an associated eigenvector. For any $x = (g, \bar{u}) \in \mathbb{X}$ and $n \geq 1$,

$$\lambda^n h(x) = \mathbf{P}^n h(x) = \mathbb{E}(h(g_n, G_{n-1}g \cdot \bar{u})).$$

Consider $\tilde{h}(\bar{v}) = \mathbb{E}(h(g_1, \bar{v}))$, for any $\bar{v} \in \mathbb{P}(\mathbb{R}^d)$. By the independence of g_i ,

$$\lambda^n h(x) = \mathbb{E}(\tilde{h}(G_{n-1}g \cdot \bar{u}))$$

Since $h \in \mathcal{B}$, \tilde{h} is ε -Hölder: for any \bar{v} and $\bar{w} \in \mathbb{P}(\mathbb{R}^d)$,

$$|\tilde{h}(\bar{v}) - \tilde{h}(\bar{w})| \leq k_{\varepsilon, \alpha}(h) \mathbb{E}(N(g_1)^\alpha) d(\bar{v}, \bar{w})^\varepsilon,$$

we deduce that the function \tilde{h} is continuous and so by Proposition 3.13.5,

$$\lambda^n h(x) \xrightarrow{n \rightarrow +\infty} \nu(\tilde{h}) = \int_{\mathbb{X}} h(g_1, \bar{v}) \nu(d\bar{v}) \mu(dg_1).$$

Since h is an eigenvector, by definition, there exists $x_0 \in \mathbb{X}$ such that $h(x_0) \neq 0$. So $\lambda = 1$ and $h(x) = \nu(\tilde{h})e$ is collinear to the constant function equal to 1, for any $x \in \mathbb{X}$. This proves that 1 is the unique eigenvalue of modulus 1 and that its associated eigenspace is one-dimensional, which concludes the proof of the point 2. \square

Proposition 3.13.7 (Perturbed transition operator). *Assume **P3.1-P3.3** and set $\kappa > 0$.*

1. *For any $|t| \leq \kappa$, the map $g \mapsto \mathbf{P}_t g$ is a bounded operator on \mathcal{B} .*
2. *There exists a constant $C_{\mathbf{P}} > 0$ such that, for any $n \geq 1$ and $|t| \leq \kappa$,*

$$\|\mathbf{P}_t^n\|_{\mathcal{B} \rightarrow \mathcal{B}} \leq C_{\mathbf{P}}.$$

Proof. It is a straightforward consequence of Lemma 3.13.3. \square

3.13.5 Proof of M3.4

For any $h \in \mathcal{B}$, we define

$$\tilde{\nu}(h) := \int_{\mathbb{X}} h(g_1, \bar{v}) \nu(d\bar{v}) \mu(dg_1). \quad (3.13.15)$$

Proposition 3.13.8 (Local integrability). *Assume P3.1. The Banach space \mathcal{B} contains a sequence of non-negative function $\tilde{N}, \tilde{N}_1, \tilde{N}_2, \dots$ such that:*

1. *There exists $p_{max} > 2$ and $\gamma > 0$ such that, for any $x \in \mathbb{X}$,*

$$\max \left\{ |\rho(x)|^{1+\gamma}, \|\delta_x\|_{\mathcal{B}'}, \mathbb{E}_x^{1/p_{max}} \left(\tilde{N}(X_n)^{p_{max}} \right) \right\} \leq c \left(1 + \tilde{N}(x) \right)$$

and

$$\tilde{N}(x) \mathbb{1}_{\{\tilde{N}(x) > l\}} \leq \tilde{N}_l(x), \quad \text{for any } l \geq 1.$$

2. *There exists $c > 0$ such that, for any $l \geq 1$,*

$$\|\tilde{N}_l\|_{\mathcal{B}} \leq c.$$

3. *There exist $\delta > 0$ and $c > 0$ such that, for any $l \geq 1$,*

$$|\tilde{\nu}(\tilde{N}_l)| \leq \frac{c}{l^{1+\delta}}.$$

Proof. For any $x = (g, \bar{u}) \in \mathbb{X}$ we define $\tilde{N}(x) = (\|g\| + \|g^{-1}\|)^\theta$. For any $t \geq 0$ and $l \geq 1$, we consider

$$\phi_l(t) = \begin{cases} 0 & \text{if } t \leq l-1 \\ t - (l-1) & \text{if } t \in [l-1, l] \\ 1 & \text{if } t \geq l \end{cases}$$

and we define $\tilde{N}_l(x) = \phi_l(\tilde{N}(x)) \tilde{N}(x)$, for any $x \in \mathbb{X}$ and $l \geq 1$, .

The function \tilde{N} belongs to \mathcal{B} : it is easy to see that $|\tilde{N}|_\theta \leq 2^\theta$ and that $k_{\varepsilon, \alpha}(\tilde{N}) = 0$. Moreover, for any $(g, g') \in \mathbb{G}$ and $\bar{u} \in \mathbb{P}(\mathbb{R}^d)$,

$$|\tilde{N}(g, \bar{u}) - \tilde{N}(g', \bar{u})| \leq \theta \sup_{\xi \geq 1} \xi^{\theta-1} \left| \|g\| + \|g^{-1}\| - \|g'\| - \|(g')^{-1}\| \right|$$

Without loss of generality, we can assume that $\delta_0 \leq 8/3$ i.e $\theta \leq 1$ and so

$$|\tilde{N}(g, \bar{u}) - \tilde{N}(g', \bar{u})| \leq \left(\|g - g'\| + \|g^{-1}(g - g')(g')^{-1}\| \right) \leq 2 \|g - g'\| N(g) N(g').$$

Moreover

$$|\tilde{N}(g, \bar{u}) - \tilde{N}(g', \bar{u})| \leq 2^\theta N(g)^\theta + 2^\theta N(g')^\theta \leq 2^{\theta+1} N(g)^\theta N(g')^\theta.$$

Consequently,

$$|\tilde{N}(g, \bar{u}) - \tilde{N}(g', \bar{u})| \leq 2^{(\theta+1)(1-\varepsilon)+\varepsilon} \|g - g'\|^\varepsilon N(g)^{\varepsilon+(1-\varepsilon)\theta} N(g')^{\varepsilon+(1-\varepsilon)\theta}. \quad (3.13.16)$$

Since $\varepsilon + (1 - \varepsilon)\theta \leq 4\varepsilon \leq \beta$, we obtain that

$$k'_{\varepsilon, \beta}(\tilde{N}) \leq 2^{\theta+1}.$$

Thereby $\tilde{N} \in \mathcal{B}$ and $\|\tilde{N}\|_{\mathcal{B}} \leq 2^{\theta+2}$.

We now show that $\tilde{N}_l \in \mathcal{B}$, for any $l \geq 1$. Fix $l \geq 1$. We note that $\tilde{N}_l(x) \leq \tilde{N}(x)$, for any $x \in \mathbb{X}$, and so $|\tilde{N}_l|_{\theta} \leq |\tilde{N}|_{\theta} \leq 2^{\theta}$. Since for any $g \in \mathbb{G}$, the function $\bar{u} \mapsto \tilde{N}(g, \bar{u})$ is constant, the function $\bar{u} \mapsto \tilde{N}_l(g, \bar{u})$ is also constant and so $k_{\varepsilon, \alpha}(\tilde{N}_l) = 0$. For any $(g, g') \in \mathbb{G}$ and $\bar{u} \in \mathbb{P}(\mathbb{R}^d)$,

$$\begin{aligned} |\tilde{N}_l(g, \bar{u}) - \tilde{N}_l(g', \bar{u})| &\leq \tilde{N}(g, \bar{u}) \left| \phi_l(\tilde{N}(g, \bar{u})) - \phi_l(\tilde{N}(g', \bar{u})) \right| \\ &\quad + \|\phi_l\|_{\infty} |\tilde{N}(g, \bar{u}) - \tilde{N}(g', \bar{u})| \\ &\leq (2^{\theta} N(g)^{\theta} + 1) |\tilde{N}(g, \bar{u}) - \tilde{N}(g', \bar{u})|. \end{aligned}$$

Using (3.13.16),

$$\begin{aligned} |\tilde{N}_l(g, \bar{u}) - \tilde{N}_l(g', \bar{u})| &\leq (2^{\theta} + 1) N(g)^{\theta} 2^{\theta+1} \|g - g'\|^{\varepsilon} N(g)^{\varepsilon+\theta} N(g')^{\varepsilon+\theta} \\ &\leq 2^{2\theta+2} \|g - g'\|^{\varepsilon} N(g)^{\varepsilon+2\theta} N(g')^{\varepsilon+2\theta}. \end{aligned}$$

Finally, since $\varepsilon + 2\theta = 7\varepsilon = \beta$, we conclude that $k'_{\varepsilon, \beta}(\tilde{N}_l) \leq 2^{2\theta+2}$, that $\tilde{N}_l \in \mathcal{B}$ and that

$$\|\tilde{N}_l\|_{\mathcal{B}} \leq 2^{2\theta+3}. \quad (3.13.17)$$

Point 1. Recall that $\rho(x) = \log(\|gu\|)$, for any $x = (g, \bar{u}) \in \mathbb{X}$ and fix $\gamma > 0$ ($\gamma = 1$ for example). If $\|gu\| \geq 1$, then $|\rho(x)|^{1+\gamma} \leq \log(\|g\|)^{1+\gamma} \leq c_{\gamma, \theta} \|g\|^{\theta}$. If $\|gu\| \leq 1$, then $|\rho(x)|^{1+\gamma} \leq (-\log(\|g^{-1}\|^{-1}))^{1+\gamma} \leq c_{\gamma, \theta} \|g^{-1}\|^{\theta}$. In every case,

$$|\rho(x)|^{1+\gamma} \leq c_{\gamma, \theta} \tilde{N}(x).$$

By (3.13.1), we have seen that $\|\delta_x\|_{\mathcal{B}} \leq N(g)^{\theta} \leq \tilde{N}(x)$, for any $x = (g, \bar{u}) \in \mathbb{X}$. Choosing $p_{\max} = 8/3 > 2$, we get $\theta p_{\max} = \delta_0$ and so

$$\mathbb{E}_x^{1/p_{\max}} \left(\tilde{N}(X_n)^{p_{\max}} \right) \leq 2^{\theta} \mathbb{E}^{1/p_{\max}} \left(N(g_n)^{\theta p_{\max}} \right) = c_{\delta_0} < +\infty.$$

By the definition of ϕ_l , it is clear that $\tilde{N}(x) \mathbb{1}_{\{\tilde{N}(x) > l\}} \leq \tilde{N}_l(x)$, for any $l \geq 1$ and $x \in \mathbb{X}$, which proves the point 1.

Point 2. This point is proved by (3.13.17).

Point 3. Fix $\delta = 2/3$. By the definition of \tilde{N}_l , we have, for any $l \geq 2$,

$$\begin{aligned} \tilde{\nu}(\tilde{N}_l) &= \int_{\mathbb{X}} \tilde{N}_l(g_1, \bar{v}) \nu(d\bar{v}) \mu(dg_1) \leq \int_{\mathbb{X}} \tilde{N}(g_1, \bar{v}) \mathbb{1}_{\{\tilde{N}(g_1, \bar{v}) \geq l-1\}} \nu(d\bar{v}) \mu(dg_1) \\ &\leq \int_{\mathbb{X}} \frac{\tilde{N}(g_1, \bar{v})^{2+\delta}}{(l-1)^{1+\delta}} \nu(d\bar{v}) \mu(dg_1) \\ &\leq \frac{2^{\theta(2+\delta)}}{(l-1)^{1+\delta}} \int_{\mathbb{G}} N(g_1)^{\theta(2+\delta)} \mu(dg_1). \end{aligned}$$

Since $\theta(2 + \delta) = \delta_0$, we conclude that $\tilde{\nu}(\tilde{N}_l) \leq c_{\delta_0}/l^{5/3}$, for any $l \geq 1$. \square

3.13.6 Proof of M3.5

Let μ and σ^2 be defined by the point 1 and 2, respectively, of the Proposition 3.2.1. The measure $\tilde{\nu}$ defined by (3.13.15) is \mathbf{P} -invariant. Indeed for any continuous and bounded function $h : \mathbb{X} \rightarrow \mathbb{C}$,

$$\begin{aligned} \tilde{\nu}(\mathbf{P}h) &= \int_{\mathbb{X}} \mathbf{P}h(g, \bar{u}) \nu(d\bar{u}) \mu(dg) = \int_{\mathbb{X} \times \mathbb{G}} h(g_1, g \cdot \bar{u}) \mu(dg_1) \mu(dg) \nu(d\bar{u}) \\ &= \int_{\mathbb{X}} \tilde{h}(g \cdot \bar{u}) \mu(dg) \nu(d\bar{u}). \end{aligned}$$

where \tilde{h} is defined by $\tilde{h}(\bar{v}) = \int_{\mathbb{G}} h(g_1, \bar{v}) \mu(dg_1)$, for any $\bar{v} \in \mathbb{P}(\mathbb{R}^d)$. Since ν is $bs\mu$ -invariant (see (3.13.14)),

$$\tilde{\nu}(\mathbf{P}h) = \int_{\mathbb{P}(\mathbb{R}^d)} \tilde{h}(\bar{u}) \nu(d\bar{u}) = \tilde{\nu}(h).$$

Note that $\tilde{\nu}(\tilde{N}^2) \leq 2^{2\theta} \mu(N^{2\theta})$ and since $2\theta \leq \delta_0$, we deduce that $\tilde{\nu}(\tilde{N}^2) < +\infty$. Thereby, the equation (3.2.5) is satisfied.

Proposition 3.13.9 (Centring and non-degeneracy). *Assume P3.1-P3.4. Then, the walk $(S_n)_{n \geq 1}$ est centred :*

$$\tilde{\nu}(\rho) = \mu = 0,$$

and non-degenerated:

$$\sigma^2 = \text{Var}_{\tilde{\nu}}(\rho(X_1)) + 2 \sum_{k=2}^{+\infty} \text{Cov}_{\tilde{\nu}}(\rho(X_1), \rho(X_k)) > 0.$$

Proof. The hypothesis P3.4 implies that $\tilde{\nu}(\rho) = 0$. Moreover, by the theorem 2 of Le Page [54] and under Hypotheses P3.1-P3.4, we know that $\frac{1}{n} \mathbb{E}_x(S_n^2)$ converges to a positive number, say $\tilde{\sigma}^2 > 0$. So using Proposition 3.2.1, one can see that $\sigma^2 = \tilde{\sigma}^2 > 0$. \square

3.13.7 Results

By Propositions 3.13.1, 3.13.6, 3.13.7, 3.13.8 and 3.13.9, we see that Hypotheses M3.1-M3.5 are satisfied and so Theorem 3.2.2-3.2.5 precise the behaviour of the associated random walk $(S_n)_{n \geq 1}$.

Proposition 3.13.10. *Assume P3.1-P3.4. Theorem 3.2.2-3.2.5 hold true for the random walk constructed by the product of random matrices $S_n = \log(\|G_n g \cdot x\|)$.*

We detail here some points implied by the previous proposition. For any $\gamma > 0$, we define

$$\mathcal{D}'_{\gamma} := \{(x, y) \in \mathbb{X} \times \mathbb{R} : \exists n_0 \geq 1, \mathbb{P}_x(y + S_{n_0} > \gamma, \tau_y > n_0) > 0\}$$

Proposition 3.13.11. *Assume P3.1-P3.4.*

1. *The function V defined by Theorem 3.2.2 satisfies, for any $y \in \mathbb{R}$, $x \in \mathbb{X}$ and $\delta > 0$,*

$$(1 - \delta) \max(y, 0) - c_{\delta} \leq V(x, y) \leq (1 + \delta) \max(y, 0) + c_{\delta}.$$

2. *There exists $\gamma'_0 > 0$ such that, for any $\gamma > \gamma'_0$,*

$$\text{supp}(V) = \mathcal{D}'_{\gamma}.$$

Proof. Point 1. By the point 2 of Theorem 3.2.2, for any $(x, y) \in \mathbb{X} \times \mathbb{R}$, we have $V(x, y) = \mathbb{E}_x(V(X_1, y + S_1); \tau_y > 1)$. Therefore, using the point 3 of Theorem 3.2.2,

$$\begin{aligned} V(x, y) &\leq \mathbb{E}_x\left((1 + \delta)(y + S_1) + c_\delta(1 + \tilde{N}(X_1)); \tau_y > 1\right) \\ &\leq (1 + \delta) \max(y, 0) + c_\delta \mathbb{E}_x\left(1 + |\rho(X_1)| + \tilde{N}(X_1)\right). \end{aligned}$$

By the point 1 of Proposition 3.13.8,

$$V(x, y) \leq (1 + \delta) \max(y, 0) + c_{\delta, \theta} \mathbb{E}\left(N(g_1)^\theta\right) \leq (1 + \delta) \max(y, 0) + c_{\delta, \theta}.$$

We proceed in a similar way to obtain the lower bound:

$$\begin{aligned} V(x, y) &\geq (1 - \delta) \mathbb{E}_x(y + S_1; y + S_1 > 0) - c_\delta \mathbb{E}_x\left(1 + \tilde{N}(X_1); \tau_y > 1\right) \\ &\geq (1 - \delta)y + (1 - \delta) \mathbb{E}_x(S_1) - (1 - \delta) \mathbb{E}_x(y + S_1; y + S_1 \leq 0) - c_{\delta, \theta} \\ &\geq (1 - \delta)y - c_{\delta, \theta}. \end{aligned}$$

Since $V \geq 0$, we conclude that

$$V(x, y) \geq (1 - \delta) \max(y, 0) - c_{\delta, \theta}$$

Point 2. Taking $\delta = 1/2$ in the point 1, there exists $\gamma'_0 = 4c_\delta > 0$ such that for any $(x, y) \in \mathbb{X} \times \mathbb{R}$,

$$V(x, y) \geq \frac{y}{2} - \frac{\gamma'_0}{4}.$$

We recall that the sets \mathcal{D}_γ are defined before Theorem 3.2.2 by

$$\mathcal{D}_\gamma := \left\{ (x, y) \in \mathbb{X} \times \mathbb{R} : \exists n_0 \geq 1, \mathbb{P}_x\left(y + S_{n_0} > \gamma\left(1 + \tilde{N}(X_{n_0})\right), \tau_y > n_0\right) > 0 \right\}.$$

Since $\tilde{N} \geq 0$, pour tout $\gamma > 0$, we see that $\mathcal{D}_\gamma \subseteq \mathcal{D}'_\gamma$. Thereby, using the point 3 of Proposition 3.8.8, we deduce that for any $\gamma > 0$,

$$\text{supp}(V) \subseteq \mathcal{D}'_\gamma. \quad (3.13.18)$$

Moreover, for any $\gamma > 0$, we consider $\zeta'_\gamma := \inf\{k \geq 1 : |y + S_k| > \gamma\}$ and fix $(x, y) \in \mathcal{D}'_{\gamma'_0}$. There exists $n_0 \geq 1$ such that $\mathbb{P}_x(y + S_{n_0} > \gamma, \tau_y > n_0) > 0$. Consequently, in a similar way as in the proof of the point 4 of Proposition 3.8.8,

$$\begin{aligned} V(x, y) &\geq \mathbb{E}_x\left(V(X_{n_0}, y + S_{n_0}); \tau_y > n_0, \zeta'_{\gamma'_0} \leq n_0\right) \\ &\geq \frac{1}{2} \mathbb{E}_x\left(y + S_{\zeta'_{\gamma'_0}} - \frac{\gamma'_0}{2}; \tau_y > \zeta'_{\gamma'_0}, \zeta'_{\gamma'_0} \leq n_0\right) \\ &\geq \frac{\gamma'_0}{4} \mathbb{P}_x\left(\tau_y > \zeta'_{\gamma'_0}, \zeta'_{\gamma'_0} \leq n_0\right) > 0. \end{aligned}$$

Therefore $\mathcal{D}'_{\gamma'_0} \subseteq \text{supp}(V)$. Moreover, it is easy to see that $\mathcal{D}'_{\gamma_1} \subseteq \mathcal{D}'_{\gamma_2}$, for any $\gamma_1 \geq \gamma_2$. So $\mathcal{D}'_\gamma \subseteq \text{supp}(V)$, for any $\gamma \geq \gamma'_0$, which, together with (3.13.18), concludes the proof. \square

Proposition 3.13.12. *Assume P3.1-P3.4. Then, for any $(x, y) \notin \text{supp}(V)$,*

$$\mathbb{P}_x(\tau_y > n) \leq c e^{-cn}.$$

Proof. Fix $(x, y) \notin \text{supp}(V)$. By the point 2 of Theorem 3.2.2, we note that $0 = V(x, y) = \mathbb{E}_x(V(X_1, y + S_1); \tau_y > 1)$. Consequently, on the event $\{\tau_y > 1\}$, we have $(X_1, y + S_1) \notin \text{supp}(V)$. Using the Markov property, and the point 2 of Theorem 3.2.3, for any $n \geq 2$, we deduce that

$$\mathbb{P}_x(\tau_y > n) \leq c e^{-c(n-1)} \mathbb{E}_x\left(1 + \tilde{N}(X_1); \tau_y > 1\right) \leq c e^{-cn} \mathbb{E}\left(N(g_1)^\theta\right).$$

□

Using the same ideas as in the proofs of Propositions 3.13.11 and 3.13.12 and using Theorem 3.2.4 and 3.2.5, we obtain the following proposition.

Proposition 3.13.13. *Assume P3.1-P3.4.*

1. There exists $\varepsilon'_0 > 0$ such that, for any $\varepsilon' \in (0, \varepsilon'_0)$, $n \geq 1$ and $(x, y) \notin \text{supp}(V)$,

$$\left| \mathbb{P}_x(\tau_y > n) - \frac{2V(x, y)}{\sqrt{2\pi n\sigma}} \right| \leq c_{\varepsilon'} \frac{(1 + \max(y, 0))^2}{n^{1/2+\varepsilon'}}.$$

2. For any $(x, y) \in \mathbb{X} \times \mathbb{R}$ and $n \geq 1$,

$$\mathbb{P}_x(\tau_y > n) \leq c \frac{1 + \max(y, 0)}{\sqrt{n}}.$$

3. There exists $\varepsilon'_0 > 0$ such that for any $\varepsilon' \in (0, \varepsilon'_0)$, $n \geq 1$, $t_0 > 0$, $t \in [0, t_0]$ and $(x, y) \in \mathbb{X} \times \mathbb{R}$,

$$\left| \mathbb{P}_x\left(y + S_n \leq t\sqrt{n}, \tau_y > n\right) - \frac{2V(x, y)}{\sqrt{2\pi n\sigma}} \Phi^+\left(\frac{t}{\sigma}\right) \right| \leq c_{\varepsilon', t_0} \frac{(1 + \max(y, 0))^2}{n^{1/2+\varepsilon'}}.$$

3.14 Asymptotic law of the couple $(X_n, S_n)_{n \geq 0}$ conditioned

The goal of this section is to improve Theorem 3.2.5, giving the law of $(X_n, y + S_n)_{n \geq 0}$ conditionally to the fact that the random walk stays positive $\{\tau_y > n\}$. Roughly speaking, by the law of the iterated logarithm, the Markov walk at the time n is of order \sqrt{n} . So its "inertia" increases with n whereas the Markov chain converges exponentially fast to the stationary measure. The "effect" of X_n on the walk $y + S_n$ decreases with n . This motivates the fact that, intuitively, the chain and the walk are asymptotically independent. More precisely, our goal is to prove the following theorem:

Theorem 3.14.1. *Assume Hypotheses M3.1-M3.5.*

1. For any non-negative bounded function $h : \mathbb{X} \rightarrow \mathbb{R}_+$ belonging to \mathcal{B} , any $(x, y) \in \text{supp}(V)$ and $t \geq 0$, we have

$$\mathbb{E}_x\left(h(X_n); y + S_n \leq t\sigma\sqrt{n} \mid \tau_y > n\right) \xrightarrow{n \rightarrow +\infty} \nu(h) \Phi^+(t),$$

where Φ^+ is the distribution function of the Rayleigh law, $\Phi^+(t) = 1 - e^{-\frac{t^2}{2}}$.

2. Moreover, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, non-negative bounded function $h : \mathbb{X} \rightarrow \mathbb{R}_+$ belonging to \mathcal{B} , any $n \geq 1$, $t_0 > 0$, $t \in [0, t_0]$ and $(x, y) \in \mathbb{X} \times \mathbb{R}$,

$$\begin{aligned} & \left| \mathbb{E}_x \left(h(X_n) ; y + S_n \leq t\sigma\sqrt{n}, \tau_y > n \right) - \nu(h) \frac{2V(x, y)}{\sqrt{2\pi n\sigma}} \left(1 - e^{-\frac{t^2}{2}} \right) \right| \\ & \leq c_{\varepsilon, t_0} (\|h\|_\infty + \|h\|_{\mathcal{B}}) \frac{\max(y, 0) + \left(1 + y \mathbb{1}_{\{y > (n-k)^{1/2-\varepsilon}\}} + N(x) \right)^2}{n^{1/2+\varepsilon/16}}. \end{aligned}$$

Proof. To prove that the asymptotic law of $(X_n)_{n \geq 0}$ is the invariant measure even if we condition by the fact that the walk stays positive, we make a "gap" between the index of $h(X_n)$ and of $\{\tau_y > n - k\}$ of size $k = \lfloor n^\varepsilon \rfloor$, negligible in comparison with n and we make use the fact that the speed of the convergence of $(X_n)_{n \geq 0}$ to the stationary measure is exponentially fast.

For any $n \geq 0$ and $\varepsilon > 0$ we define $k = \lfloor n^\varepsilon \rfloor$ and $t_n = \frac{1}{n^\varepsilon}$. Set $t_0 > 0$. For any $t \in [0, t_0]$, we will give a lower and upper control of the following expectation:

$$I_0 := \mathbb{E}_x \left(h(X_n) ; y + S_n \leq t\sigma\sqrt{n}, \tau_y > n \right). \quad (3.14.1)$$

We denote also

$$I_0(x, y, n, u) := \mathbb{E}_x \left(h(X_n) ; y + S_n \leq u, \tau_y > n \right). \quad (3.14.2)$$

Upper bound of I_0 . By the Markov property,

$$I_0 = \int_{\mathbb{X} \times \mathbb{R}} I_0(x', y', k, t\sigma\sqrt{n}) \mathbb{P}_x(X_{n-k} \in dx', y + S_{n-k} \in dy', \tau_y > n - k).$$

Inserting the event $\{y' = y + S_{n-k} \leq \sigma(t + t_n)\sqrt{n}\}$, we write

$$\begin{aligned} I_0 &= \int_{\mathbb{X} \times \mathbb{R}} I_0(x', y', k, t\sigma\sqrt{n}) \mathbb{P}_x(X_{n-k} \in dx', y + S_{n-k} \in dy', \\ & \quad \underbrace{y + S_{n-k} \leq \sigma(t + t_n)\sqrt{n}, \tau_y > n - k}_{=: I_1}) \\ & \quad + \int_{\mathbb{X} \times \mathbb{R}} I_0(x', y', k, t\sigma\sqrt{n}) \mathbb{P}_x(X_{n-k} \in dx', y + S_{n-k} \in dy', \\ & \quad \underbrace{y + S_{n-k} > \sigma(t + t_n)\sqrt{n}, \tau_y > n - k}_{=: I_2}). \end{aligned} \quad (3.14.3)$$

Bound of I_1 . Since h is non-negative,

$$I_0(x', y', k, t\sigma\sqrt{n}) = \mathbb{E}_{x'} \left(h(X_k) ; y' + S_k \leq t\sigma\sqrt{n}, \tau_{y'} > n \right) \leq \mathbb{E}_{x'} \left(h(X_k) \right).$$

Using the spectral gap property **M3.2** and the fact that $h \in \mathcal{B}$, we know that

$$\mathbb{E}_{x'} \left(h(X_k) \right) = \mathbf{P}^k h(x') = \nu(h) + Q^k h(x'). \quad (3.14.4)$$

with,

$$\left| Q^k h(x') \right| = \left| \delta_{x'} \left(Q^k h \right) \right| \leq \|\delta_{x'}\|_{\mathcal{B}'} \left\| Q^k \right\|_{\mathcal{B} \rightarrow \mathcal{B}} \|h\|_{\mathcal{B}} \leq c \|h\|_{\mathcal{B}} e^{-ck} \|\delta_{x'}\|_{\mathcal{B}'}.$$

Consequently, by the point 1 of Hypothesis **M3.4**

$$|Q^k h(x')| \leq c \|h\|_{\mathcal{B}} e^{-ck} (1 + N(x')). \quad (3.14.5)$$

Thereby,

$$I_0(x', y', k, t\sigma\sqrt{n}) \leq \mathbb{E}_{x'}(h(X_k)) \leq \nu(h) + c \|h\|_{\mathcal{B}} e^{-ck} (1 + N(x'))$$

and so

$$\begin{aligned} I_1 &\leq \nu(h) \mathbb{P}_x \left(y + S_{n-k} \leq \sigma(t + t_n)\sqrt{n}, \tau_y > n - k \right) \\ &\quad + c \|h\|_{\mathcal{B}} e^{-ck} \mathbb{E}_x (1 + N(X_{n-k})). \end{aligned} \quad (3.14.6)$$

By the inequality (3.2.2), we have

$$\mathbb{E}_x (N(X_{n-k})) \leq c (1 + N(x)). \quad (3.14.7)$$

Moreover, using the point 2 of Theorem 3.2.5 with $t'_0 = \sup_{n \geq 2} \sigma(t + t_n) \frac{\sqrt{n}}{\sqrt{n-k}} \leq c_\varepsilon(t_0 + 1)$,

$$\begin{aligned} I'_1 &:= \mathbb{P}_x \left(y + S_{n-k} \leq \sigma(t + t_n) \frac{\sqrt{n}}{\sqrt{n-k}} \sqrt{n-k}, \tau_y > n - k \right) \\ &\leq \frac{2V(x, y)}{\sqrt{2\pi(n-k)\sigma}} \left(1 - e^{-\frac{(t+t_n)^2 n}{2(n-k)}} \right) + c_{\varepsilon, t_0} \frac{\max(y, 0) + \left(1 + y \mathbb{1}_{\{y > (n-k)^{1/2-\varepsilon}\}} + N(x) \right)^2}{(n-k)^{1/2+\varepsilon/16}}. \end{aligned}$$

Note that $n - k \geq n \left(1 - \frac{1}{n^{1-\varepsilon}} \right)$ and that $t + t_n = t + \frac{1}{n^\varepsilon}$. Therefore

$$I'_1 \leq \frac{2V(x, y)}{\sqrt{2\pi n \sigma}} \left(1 + \frac{c}{n^{1-\varepsilon}} \right) \left(1 - e^{-\frac{t^2}{2} \left(1 + \frac{c}{n^{1-\varepsilon}} \right) - \frac{c(1+t_0)}{n^\varepsilon}} \right) + R_n(x, y),$$

where

$$R_n(x, y) = c_{\varepsilon, t_0} \frac{\max(y, 0) + \left(1 + y \mathbb{1}_{\{y > n^{1/2-\varepsilon}\}} + N(x) \right)^2}{n^{1/2+\varepsilon/16}}. \quad (3.14.8)$$

So,

$$I'_1 \leq \frac{2V(x, y)}{\sqrt{2\pi n \sigma}} \left(1 + \frac{c}{n^{1-\varepsilon}} \right) \left(1 - e^{-\frac{t^2}{2} + \frac{c t_0}{n^\varepsilon}} \right) + R_n(x, y).$$

Using the upper bound of the point 3 of Theorem 3.2.2 and changing the constant c_{ε, t_0} in $R_n(x, y)$, we obtain that

$$I'_1 \leq \frac{2V(x, y)}{\sqrt{2\pi n \sigma}} \left(1 - e^{-\frac{t^2}{2}} \right) + R_n(x, y). \quad (3.14.9)$$

Inserting (3.14.9) and (3.14.7) in (3.14.6),

$$\begin{aligned} I_1 &\leq \nu(h) \frac{2V(x, y)}{\sqrt{2\pi n \sigma}} \left(1 - e^{-\frac{t^2}{2}} \right) + \nu(h) R_n(x, y) + c \|h\|_{\mathcal{B}} e^{-cn^\varepsilon} (1 + N(x)) \\ &\leq \nu(h) \frac{2V(x, y)}{\sqrt{2\pi n \sigma}} \left(1 - e^{-\frac{t^2}{2}} \right) + (\|h\|_\infty + \|h\|_{\mathcal{B}}) R_n(x, y). \end{aligned} \quad (3.14.10)$$

Bound of I_2 . Since h is bounded, by (3.14.2) we have

$$I_0(x', y', k, t\sigma\sqrt{n}) \leq \|h\|_\infty \mathbb{P}_{x'}(y' + S_k \leq t\sigma\sqrt{n}).$$

When $y' > \sigma(t + t_n)\sqrt{n}$, it is clear that

$$I_0(x', y', k, t\sigma\sqrt{n}) \leq \|h\|_\infty \mathbb{P}_{x'}(S_k < -t_n\sigma\sqrt{n}) \leq \|h\|_\infty \mathbb{P}_{x'}(|S_k| > t_n\sigma\sqrt{n}). \quad (3.14.11)$$

This last inequality tends to 0 as $n \rightarrow +\infty$: by the Markov inequality,

$$\mathbb{P}_{x'}(|S_k| > t_n\sigma\sqrt{n}) \leq \sum_{i=1}^k \mathbb{E}_{x'}\left(\frac{|f(X_i)|}{t_n\sigma\sqrt{n}}\right).$$

By the point 1 of Hypothesis **M3.4** and (3.14.7),

$$\mathbb{P}_{x'}(|S_k| > t_n\sigma\sqrt{n}) \leq c(1 + N(x')) \frac{k}{t_n\sigma\sqrt{n}} = \frac{c}{n^{1/2-2\varepsilon}}(1 + N(x')). \quad (3.14.12)$$

Using (3.14.12), (3.14.11) and the definition of I_2 in (3.14.3), we deduce that

$$I_2 \leq \frac{c\|h\|_\infty}{n^{1/2-2\varepsilon}} \underbrace{\mathbb{E}_x(1 + N(X_{n-k}); \tau_y > n - k)}_{=: I'_2}. \quad (3.14.13)$$

By the Markov property,

$$I'_2 \leq \int_{\mathbb{X}} \mathbb{E}_{x'}(1 + N(X_k)) \mathbb{P}_x(X_{n-2k} \in dx', \tau_y > n - 2k).$$

Using (3.2.2), we obtain that

$$I'_2 \leq c\mathbb{P}_x(\tau_y > n - 2k) + ce^{-ck} \mathbb{E}_x(N(X_{n-2k})).$$

By the point 2 of Theorem 3.2.4 and the equation (3.14.7), we find that

$$I'_2 \leq c \frac{1 + \max(y, 0) + N(x)}{\sqrt{n} - 2k} + ce^{-cn\varepsilon} (1 + N(x)) \leq c_\varepsilon \frac{1 + \max(y, 0) + N(x)}{\sqrt{n}}. \quad (3.14.14)$$

Therefore, by (3.14.13),

$$I_2 \leq c_\varepsilon \|h\|_\infty \frac{1 + \max(y, 0) + N(x)}{n^{1-2\varepsilon}}. \quad (3.14.15)$$

Finally, by (3.14.3), (3.14.10) and (3.14.15), we obtain the following upper bound:

$$I_0 \leq \nu(h) \frac{2V(x, y)}{\sqrt{2\pi n\sigma}} \left(1 - e^{-\frac{t^2}{2}}\right) + (\|h\|_\infty + \|h\|_{\mathcal{B}}) R_n(x, y). \quad (3.14.16)$$

Lower bound of I_0 . As in the upper bound, by the Markov property,

$$\begin{aligned} I_0 &= \int_{\mathbb{X} \times \mathbb{R}} I_0(x', y', k, t\sigma\sqrt{n}) \mathbb{P}_x(X_{n-k} \in dx', y + S_{n-k} \in dy', \tau_y > n - k) \\ &\geq \int_{\mathbb{X} \times \mathbb{R}} I_0(x', y', k, t\sigma\sqrt{n}) \mathbb{P}_x(X_{n-k} \in dx', y + S_{n-k} \in dy', \\ &\quad y + S_{n-k} \leq \sigma(t - t_n)\sqrt{n}, \tau_y > n - k). \end{aligned}$$

We introduce the following notation:

$$\begin{aligned}
 I_0 &\geq \underbrace{\int_{\mathbb{X} \times \mathbb{R}} \mathbb{E}_{x'}(h(X_k)) \mathbb{P}_x(X_{n-k} \in dx', y + S_{n-k} \in dy', \\
 &\quad y + S_{n-k} \leq \sigma(t - t_n)\sqrt{n}, \tau_y > n - k)}_{=: I_3} \\
 &- \underbrace{\int_{\mathbb{X} \times \mathbb{R}} \mathbb{E}_{x'}(h(X_k); y' + S_k > t\sigma\sqrt{n}) \mathbb{P}_x(X_{n-k} \in dx', y + S_{n-k} \in dy', \\
 &\quad y + S_{n-k} \leq \sigma(t - t_n)\sqrt{n}, \tau_y > n - k)}_{=: I_4} \quad (3.14.17) \\
 &- \underbrace{\int_{\mathbb{X} \times \mathbb{R}} \mathbb{E}_{x'}(h(X_k); \tau_{y'} \leq k) \mathbb{P}_x(X_{n-k} \in dx', y + S_{n-k} \in dy', \\
 &\quad y + S_{n-k} \leq \sigma(t - t_n)\sqrt{n}, \tau_y > n - k)}_{=: I_5}.
 \end{aligned}$$

Bound of I_3 . The term I_3 is the main term. Using (3.14.4) and (3.14.5), we have

$$\mathbb{E}_{x'}(h(X_k)) \geq \nu(h) - c \|h\|_{\mathcal{B}} e^{-ck} (1 + N(x')).$$

Therefore

$$I_3 \geq \underbrace{\nu(h) \mathbb{P}_x(y + S_{n-k} \leq \sigma(t - t_n)\sqrt{n}, \tau_y > n - k)}_{=: I'_3} - c \|h\|_{\mathcal{B}} e^{-ck} \mathbb{E}_x(1 + N(X_{n-k})). \quad (3.14.18)$$

We bound I'_3 in a similar way as in the bound of I'_1 . Using the point 2 of Theorem 3.2.5 with $t'_0 = \sup_{n \geq 2} \sigma(t - t_n) \frac{\sqrt{n}}{\sqrt{n-k}} \leq c_\varepsilon t_0$ and the notation $R_n(x, y)$ defined by (3.14.8),

$$\begin{aligned}
 I'_3 &\geq \frac{2V(x, y)}{\sqrt{2\pi(n-k)}\sigma} \left(1 - e^{-\frac{(t-t_n)^2 n}{2(n-k)}}\right) - R_n(x, y) \\
 &\geq \frac{2V(x, y)}{\sqrt{2\pi n}\sigma} \left(1 - e^{-\left(\frac{t^2}{2} - \frac{c(1+t_0)}{n^\varepsilon}\right)}\right) - R_n(x, y) \\
 &\geq \frac{2V(x, y)}{\sqrt{2\pi n}\sigma} \left(1 - e^{-\frac{t^2}{2} - \frac{c t_0}{n^\varepsilon}}\right) - R_n(x, y)
 \end{aligned}$$

By the upper bound of the point 3 of Theorem 3.2.2,

$$I'_3 \geq \frac{2V(x, y)}{\sqrt{2\pi n}\sigma} \left(1 - e^{-\frac{t^2}{2}}\right) - R_n(x, y) \quad (3.14.19)$$

So, as in bound (3.14.10) of I_1 and using (3.14.18) and (3.14.19), on obtain that

$$I_3 \geq \nu(h) \frac{2V(x, y)}{\sqrt{2\pi n}\sigma} \left(1 - e^{-\frac{t^2}{2}}\right) - (\|h\|_\infty + \|h\|_{\mathcal{B}}) R_n(x, y). \quad (3.14.20)$$

Bound of I_4 . The bound of I_4 is very similar to the bound of I_2 . Since h is bounded, we have

$$\mathbb{E}_{x'}(h(X_k); y' + S_k > t\sigma\sqrt{n}) \leq \|h\|_\infty \mathbb{P}_{x'}(y' + S_k > t\sigma\sqrt{n}).$$

When $y' \leq \sigma(t - t_n)\sqrt{n}$, we write that

$$\mathbb{E}_{x'}(h(X_k); y' + S_k > t\sigma\sqrt{n}) \leq \|h\|_\infty \mathbb{P}_{x'}(S_k > t_n\sigma\sqrt{n}) \leq \|h\|_\infty \mathbb{P}_{x'}(|S_k| > t_n\sigma\sqrt{n}).$$

By (3.14.12) and the definition of I_4 in (3.14.17),

$$I_4 \leq \frac{c \|h\|_\infty}{n^{1/2-2\varepsilon}} \mathbb{E}_x(1 + N(X_{n-k}); \tau_y > n - k),$$

which is the same upper bound as in (3.14.13). Using the bound of I'_2 in (3.14.14), we obtain that

$$I_4 \leq c_\varepsilon \|h\|_\infty \frac{1 + \max(y, 0) + N(x)}{n^{1-2\varepsilon}}. \quad (3.14.21)$$

Bound of I_5 . To control the term I_5 , we write:

$$\begin{aligned} I_5 &\leq \underbrace{\int_{\mathbb{X} \times \mathbb{R}} \mathbb{E}_{x'}(h(X_k); \tau_{y'} \leq k) \mathbb{P}_x(X_{n-k} \in dx', y + S_{n-k} \in dy', \\ &\quad y + S_{n-k} > n^{3\varepsilon}, \tau_y > n - k)}_{=: I_{51}} \\ &+ \underbrace{\int_{\mathbb{X} \times \mathbb{R}} \mathbb{E}_{x'}(h(X_k); \tau_{y'} \leq k) \mathbb{P}_x(X_{n-k} \in dx', y + S_{n-k} \in dy', \\ &\quad y + S_{n-k} \leq n^{3\varepsilon}, \tau_y > n - k)}_{=: I_{52}}. \end{aligned}$$

Bound of I_{51} . When $y' > n^{3\varepsilon}$, we have

$$\mathbb{E}_{x'}(h(X_k); \tau_{y'} \leq k) \leq \|h\|_\infty \sum_{i=1}^k \mathbb{P}_x(y' + S_i \leq 0) \leq \|h\|_\infty \sum_{i=1}^k \mathbb{P}_x(|S_i| > n^{3\varepsilon})$$

By the Markov inequality and (3.14.7),

$$\mathbb{E}_{x'}(h(X_k); \tau_{y'} \leq k) \leq \|h\|_\infty \sum_{i=1}^k \sum_{j=1}^i \frac{\mathbb{E}_{x'}(|f(X_j)|)}{n^{3\varepsilon}} \leq \frac{c \|h\|_\infty k^2}{n^{3\varepsilon}} (1 + N(x')).$$

So,

$$I_{51} \leq \frac{c \|h\|_\infty}{n^\varepsilon} \mathbb{E}_x(1 + N(X_{n-k}); \tau_y > n - k) = \frac{c \|h\|_\infty}{n^\varepsilon} I'_2,$$

where I'_2 is defined by (3.14.13). Therefore by (3.14.14),

$$I_{51} \leq c_\varepsilon \|h\|_\infty \frac{1 + \max(y, 0) + N(x)}{n^{1/2+\varepsilon}}. \quad (3.14.22)$$

Bound of I_{52} . The idea is to prove that the event $\{y + S_{n-k} \leq n^{3\varepsilon}\}$ has a probability which tends to 0 when n tends to infinity. We have

$$\begin{aligned} I_{52} &\leq \|h\|_\infty \mathbb{P}_x(y + S_{n-k} \leq n^{3\varepsilon}, \tau_y > n - k) \\ &= \|h\|_\infty \mathbb{P}_x\left(y + S_{n-k} \leq \frac{n^{3\varepsilon}}{\sqrt{n-k}} \sqrt{n-k}, \tau_y > n - k\right). \end{aligned}$$

Using the point 2 of Theorem 3.2.5, with $t' = \frac{n^{3\varepsilon}}{\sqrt{n-k}}$ and $t'_0 = c_\varepsilon = \sup_{n \geq 2} \frac{n^{3\varepsilon}}{\sqrt{n-k}} < +\infty$,

$$\begin{aligned} I_{52} &\leq \|h\|_\infty \frac{2V(x, y)}{\sqrt{2\pi n\sigma}} \left(1 - e^{-\frac{n^{6\varepsilon}}{2(n-k)\sigma^2}} \right) \\ &\quad + \|h\|_\infty c_\varepsilon \frac{\max(y, 0) + \left(1 + y \mathbb{1}_{\{y > (n-k)^{1/2-\varepsilon}\}} + N(x)\right)^2}{(n-k)^{1/2+\varepsilon/16}} \\ &\leq \|h\|_\infty \frac{2V(x, y)}{\sqrt{2\pi n\sigma}} \left(1 - e^{-\frac{c_\varepsilon}{n^{1-6\varepsilon}}} \right) + \|h\|_\infty R_n(x, y), \end{aligned}$$

where $R_n(x, y)$ is defined by (3.14.8). By the point 3 of Theorem 3.2.2, we obtain that

$$I_{52} \leq c_\varepsilon \|h\|_\infty \frac{1 + \max(y, 0) + N(x)}{n^{3/2-6\varepsilon}} + \|h\|_\infty R_n(x, y) \leq \|h\|_\infty R_n(x, y). \quad (3.14.23)$$

By (3.14.22) and (3.14.23), we conclude the proof of the bound of I_5 :

$$I_5 \leq \|h\|_\infty R_n(x, y). \quad (3.14.24)$$

Finally, by (3.14.17), (3.14.20), (3.14.21) and (3.14.24), we have the following lower bound:

$$I_0 \geq \nu(h) \frac{2V(x, y)}{\sqrt{2\pi n\sigma}} \left(1 - e^{-\frac{t^2}{2}} \right) - (\|h\|_\infty + \|h\|_{\mathcal{B}}) R_n(x, y). \quad (3.14.25)$$

Putting together (3.14.16) and (3.14.25), with the definition of I_0 in (3.14.1) and of $R_n(x, y)$ in (3.14.8), it proves the point 2 of Theorem 3.14.1. The point 1 of Theorem 3.14.1 is a consequence of the point 1 of Theorem 3.2.3 and of the point 2 of Theorem 3.14.1. □

Chapter 4

Conditioned local limit theorems for random walks defined on finite Markov chains

RÉSUMÉ. Soit $(X_n)_{n \geq 0}$ une chaîne de Markov à valeurs dans un espace d'états fini \mathbb{X} partant de $X_0 = x \in \mathbb{X}$ et soit f une fonction à valeurs réelles définie sur \mathbb{X} . On pose $S_n = \sum_{k=1}^n f(X_k)$, $n \geq 1$. Pour tout $y \in \mathbb{R}$ on considère τ_y le premier instant pour lequel la marche $y + S_n$ devient négatif. Nous étudions le comportement asymptotique de la probabilité $\mathbb{P}_x(y + S_n \in [z, z + a], \tau_y > n)$ lorsque $n \rightarrow +\infty$. Nous établissons en premier lieu une version conditionnelle du théorème local de Stone pour cette probabilité. Ensuite nous déterminons un équivalent d'ordre $n^{3/2}$. Nous décrivons également le comportement asymptotique de la probabilité $\mathbb{P}_x(\tau_y = n)$ quand $n \rightarrow +\infty$ et donnons des généralisations très utiles dans les applications.

ABSTRACT. Let $(X_n)_{n \geq 0}$ be a Markov chain with values in a finite state space \mathbb{X} starting at $X_0 = x \in \mathbb{X}$ and let f be a real function defined on \mathbb{X} . Set $S_n = \sum_{k=1}^n f(X_k)$, $n \geq 1$. For any $y \in \mathbb{R}$ denote by τ_y the first time when $y + S_n$ becomes non-positive. We study the asymptotic behaviour of the probability $\mathbb{P}_x(y + S_n \in [z, z + a], \tau_y > n)$ as $n \rightarrow +\infty$. We first establish for this probability a conditional version of the local limit theorem of Stone. Then we find for it an asymptotic equivalent of order $n^{3/2}$. We also describe the asymptotic behaviour of the probability $\mathbb{P}_x(\tau_y = n)$ as $n \rightarrow +\infty$ and give some generalizations which are useful in applications.

4.1 Introduction

Assume that on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we are given a sequence of real valued random variables $(X_n)_{n \geq 1}$. Consider the random walk $S_n = \sum_{k=1}^n X_k$, $n \geq 1$. Suppose first that $(X_n)_{n \geq 1}$ are independent identically distributed of zero mean and finite variance. For any $y > 0$ denote by τ_y the first time when $y + S_n$ becomes non-positive. The study of the asymptotic behaviour of the probability $\mathbb{P}(\tau_y > n)$ and of the law of $y + S_n$ conditioned to stay positive (i.e. given the event $\{\tau_y > n\}$) has been initiated by Spitzer [66] and developed subsequently by Iglehart [47], Bolthausen [9], Doney [22], Bertoin and Doney [6], Borovkov [10, 11], to cite only a few. Important progress has been achieved by employing a new approach based on the existence of the harmonic function in Varopoulos [68], [69], Eichelbacher and König [27] and recently by Denisov and Wachtel [18, 19, 20]. In

this line Grama, Le Page and Peigné [41] and the authors in [36], [38] (Chapters 2 and 3) have studied sums of functions defined on Markov chains under spectral gap assumptions. The goal of the present paper is to complete these investigations by establishing local limit theorems for random walks defined on finite Markov chains and conditioned to stay positive.

Local limit theorems for the sum of independent random variables without conditioning have attracted much attention, since the pioneering work of Gnedenko [34] and Stone [67]. The first local limit theorem for a random walk conditioned to stay positive has been established in Iglehart [48] in the context of walks with negative drift $\mathbb{E}X_1 < 0$. Caravenna [13] studied conditioned local limit theorems for random variables in the domain of attraction of the normal law and Vatutin and Wachtel [71] for random variables X_k in the domain of attraction of the stable law. Denisov and Wachtel [20] obtained a local limit theorem for random walks in \mathbb{Z}^d conditioned to stay in a cone based on the harmonic function approach.

The ordinary and conditioned local limit theorems in the case of Markov chains are less studied in the literature. Le Page [54] stated a local limit theorem for products of random matrices and Guivarc'h and Hardy [42] have considered a local limit theorem for sums $S_n = \sum_{k=1}^n f(X_k)$ where $(X_n)_{n \geq 0}$ is a Markov chain under spectral gap assumptions and f a real function defined on the state space of the chain. In the conditional case we are aware only of the results of Presman [60] and [61] who has considered the case of finite Markov chains in a more general setting but which, because of rather stringent assumptions, does not cover the results of this paper. We note also the work of Le Page and Peigné [55] who have proved a conditioned local limit theorem for the stochastic recursion.

Let us briefly review main results of the paper concerning conditioned local limit behaviour of the walk $S_n = \sum_{k=1}^n f(X_k)$ defined on a finite Markov chain $(X_n)_{n \geq 0}$. From more general statement of Theorem 4.2.4, under the conditions that the underlying Markov chain is irreducible and aperiodic and that $(S_n)_{n \geq 0}$ is centred and non-lattice, for fixed $x \in \mathbb{X}$ and $y \in \mathbb{R}$, it follows that, uniformly in $z \geq 0$,

$$\lim_{n \rightarrow \infty} \left(n \mathbb{P}_x(y + S_n \in [z, z + a], \tau_y > n) - \frac{2aV(x, y)}{\sqrt{2\pi}\sigma^2} \varphi_+ \left(\frac{z}{\sqrt{n}\sigma} \right) \right) = 0, \quad (4.1.1)$$

where $\varphi_+(t) = t e^{-\frac{t^2}{2}} \mathbb{1}_{\{t \geq 0\}}$ is the Rayleigh density. The relation (4.1.1) is an extension of the classical local limit theorem by Stone [67] to the case of Markov chains. We refer to Caravenna [13] and Vatutin and Wachtel [71], where the corresponding result has been obtained for independent random variables in the domain of attraction of the normal law.

We note that while (4.1.1) is consistent for large z , it is not informative for z in a compact set. A meaningful local limit behaviour for fixed values of z can be obtained from our Theorem 4.2.5. Under the same assumptions, for any fixed $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z \geq 0$,

$$\lim_{n \rightarrow +\infty} n^{3/2} \mathbb{P}_x(y + S_n \in [z, z + a], \tau_y > n) = \frac{2V(x, y)}{\sqrt{2\pi}\sigma^3} \int_z^{z+a} \int_{\mathbb{X}} V^*(x', z') \nu(dx') dz'. \quad (4.1.2)$$

For sums of independent random variables similar limit behaviour was found in Vatutin and Wachtel [71]. It should be noted that (4.1.1) and (4.1.2) complement each other: the main term in (4.1.1) is meaningful for large z such that $z \sim n^{1/2}$ as $n \rightarrow \infty$, while (4.1.2) holds for z in compact sets.

We also state extensions of (4.1.1) and (4.1.2) to the joint law of X_n and $y + S_n$. These extensions are useful in applications, in particular, for determining the exact asymptotic behaviour of the survival time for branching processes in a Markovian environment. They also allow us to infer the local limit behaviour of the exit time τ_y (see Theorem 4.2.8): under the assumptions mentioned before, for any $x \in \mathbb{X}$ and $y \in \mathbb{R}$,

$$\lim_{n \rightarrow +\infty} n^{3/2} \mathbb{P}_x(\tau_y = n) = \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_0^{+\infty} \mathbb{E}_\nu^*(V^*(X_1^*, z); S_1^* \geq z) dz.$$

The approach employed in this paper is different from that in [60], [61] and [55] which all are based on Wiener-Hopf arguments. Our technique is close to that in Denisov and Wachtel [20], however, in order to make it work for a random walk $S_n = \sum_{k=1}^n f(X_k)$ defined on a Markov chain $(X_n)_{n \geq 0}$, we have to overcome some essential difficulties. One of them is related to the problem of the reversibility of the Markov walk $(S_n)_{n \geq 0}$. Let us explain this point in more details. When $(X_n)_{n \geq 1}$ are \mathbb{Z} -valued independent identically distributed random variables, let $(S_n^*)_{n \geq 1}$ be the reverse walk given by $S_n^* = \sum_{k=1}^n X_k^*$, where $(X_n^*)_{n \geq 1}$ is a sequence of independent identically distributed random variables of the same law as $-X_1$. Denote by τ_z^* the first time when $(z + S_k^*)_{k \geq 0}$ becomes non-positive. Then, due to exchangeability of the random variables $(X_n)_{n \geq 1}$, we have

$$\mathbb{P}(y + S_n = z, \tau_y > n) = \mathbb{P}(z + S_n^* = y, \tau_z^* > n). \tag{4.1.3}$$

This relation does not hold any more for the walk $S_n = \sum_{k=1}^n f(X_k)$, where $(X_n)_{n \geq 0}$ is a Markov chain. Even though $(X_n)_{n \geq 0}$ takes values on a finite state space \mathbb{X} and there exists a dual chain $(X_n^*)_{n \geq 0}$, the main difficulty is that the function $f : \mathbb{X} \mapsto \mathbb{R}$ can be arbitrary and therefore the Markov walk $(S_n)_{n \geq 0}$ is not necessarily lattice valued. In this case the Markov chain formed by the couple $(X_n, y + S_n)_{n \geq 0}$ cannot be reversed directly as in (4.1.3). We cope with this by altering the arrival interval $[z, z + h]$ in the following two-sided bound

$$\begin{aligned} \sum_{x^* \in \mathbb{X}} \mathbb{E}_{x^*}^* \left(\psi_{x^*}^*(X_n^*) \mathbb{1}_{\{z + S_n^* \in [y-h, y], \tau_z^* > n\}} \right) \nu(x^*) \\ \leq \mathbb{P}_x(y + S_n \in [z, z + h], \tau_y > n) \\ \leq \sum_{x^* \in \mathbb{X}} \mathbb{E}_{x^*}^* \left(\psi_{x^*}^*(X_n^*) \mathbb{1}_{\{z+h + S_n^* \in [y, y+h], \tau_{z+h}^* > n\}} \right) \nu(x^*), \end{aligned} \tag{4.1.4}$$

where ν is the invariant probability of the Markov chain $(X_n)_{n \geq 1}$, $\psi_x^* : \mathbb{X} \mapsto \mathbb{R}_+$ is a function such that $\nu(\psi_x^*) = 1$ (see (4.6.2) for a precise definition) and $S_n^* = -\sum_{k=1}^n f(X_k^*)$, $\forall n \geq 1$. Following this idea, for a fixed $a > 0$ we split the interval $[z, z + a]$ into p subintervals of length $h = a/p$ and we determine the exact upper and lower bounds for the corresponding expectations in (4.1.4). We then patch up the obtained bounds to obtain a precise asymptotic as $n \rightarrow +\infty$ for the probabilities $\mathbb{P}_x(y + S_n \in [z, z + a], \tau_y > n)$ for a fixed $a > 0$ and let then p go to $+\infty$. This resumes very succinctly how we suggest generalizing (4.1.3) to the non-lattice case. Together with some further developments in Sections 4.7 and 4.8, this allows us to establish Theorems 4.2.4 and 4.2.5.

The outline of the paper is as follows:

- Section 4.2: We give the necessary notations and formulate the main results.
- Section 4.3: Introduce the dual Markov chain and state some of its properties.
- Section 4.4: Introduce and study the perturbed transition operator.

- Section 4.5: We prove a non-asymptotic local limit theorem for sums defined on Markov chains.
- Section 4.6: We collect some auxiliary bounds.
- Sections 4.7, 4.8 and 4.9 : Proofs of Theorems 4.2.4, 4.2.5 and 4.2.7, 4.2.8, respectively.
- Section 4.10. We state auxiliary assertions which are necessary for the proofs.

Let us end this section by fixing some notations. The symbol c will denote a positive constant depending on the all previously introduced constants. Sometimes, to stress the dependence of the constants on some parameters α, β, \dots we shall use the notations $c_\alpha, c_{\alpha, \beta}, \dots$. All these constants are likely to change their values every occurrence. The indicator of an event A is denoted by $\mathbb{1}_A$. For any bounded measurable function f on \mathbb{X} , random variable X in \mathbb{X} and event A , the integral $\int_{\mathbb{X}} f(x) \mathbb{P}(X \in dx, A)$ means the expectation $\mathbb{E}(f(X); A) = \mathbb{E}(f(X) \mathbb{1}_A)$.

4.2 Notations and results

Let $(X_n)_{n \geq 0}$ be a homogeneous Markov chain on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in the finite state space \mathbb{X} . Denote by \mathcal{C} the set of complex functions defined on \mathbb{X} endowed with the norm $\|\cdot\|_\infty$: $\|g\|_\infty = \sup_{x \in \mathbb{X}} |g(x)|$, for any $g \in \mathcal{C}$. Let \mathbf{P} be the transition kernel of the Markov chain $(X_n)_{n \geq 0}$ to which we associate the following transition operator: for any $x \in \mathbb{X}$ and $g \in \mathcal{C}$,

$$\mathbf{P}g(x) = \sum_{x' \in \mathbb{X}} g(x') \mathbf{P}(x, x').$$

For any $x \in \mathbb{X}$, denote by \mathbb{P}_x and \mathbb{E}_x the probability, respectively the expectation, generated by the finite dimensional distributions of the Markov chain $(X_n)_{n \geq 0}$ starting at $X_0 = x$. We assume that the Markov chain is irreducible and aperiodic, which is equivalent to the following hypothesis.

Hypothesis M4.1. *The matrix \mathbf{P} is primitive: there exists $k_0 \geq 1$ such that for any $x \in \mathbb{X}$ and any non-negative and non identically zero function $g \in \mathcal{C}$,*

$$\mathbf{P}^{k_0} g(x) > 0.$$

Let f be a real valued function defined on \mathbb{X} and let $(S_n)_{n \geq 0}$ be the process defined by

$$S_0 = 0 \quad \text{and} \quad S_n = f(X_1) + \dots + f(X_n), \quad \forall n \geq 1.$$

For any starting point $y \in \mathbb{R}$ we consider the Markov walk $(y + S_n)_{n \geq 0}$ and we denote by τ_y the first time when the Markov walk becomes non-positive:

$$\tau_y := \inf \{k \geq 1, y + S_k \leq 0\}.$$

Under **M4.1**, by the Perron-Frobenius theorem, there is a unique positive invariant probability ν on \mathbb{X} satisfying the following property: there exist $c_1 > 0$ and $c_2 > 0$ such that for any function $g \in \mathcal{C}$ and $n \geq 1$,

$$\sup_{x \in \mathbb{X}} |\mathbb{E}_x(g(X_n)) - \nu(g)| = \|\mathbf{P}^n g - \nu(g)\|_\infty \leq \|g\|_\infty c_1 e^{-c_2 n}, \tag{4.2.1}$$

where $\nu(g) = \sum_{x \in \mathbb{X}} g(x) \nu(x)$.

The following two hypotheses ensure that the Markov walk has no-drift and is non-lattice, respectively.

Hypothesis M4.2. *The function f is centred:*

$$\nu(f) = 0.$$

Hypothesis M4.3. *For any $(\theta, a) \in \mathbb{R}^2$, there exists a sequence x_0, \dots, x_n in \mathbb{X} such that*

$$\mathbf{P}(x_0, x_1) \cdots \mathbf{P}(x_{n-1}, x_n) \mathbf{P}(x_n, x_0) > 0$$

and

$$f(x_0) + \cdots + f(x_n) - (n + 1)\theta \notin a\mathbb{Z}.$$

Under Hypothesis **M4.1**, it is shown in Section 4.4 that Hypothesis **M4.3** is equivalent to the condition that the perturbed operator \mathbf{P}_t has a spectral radius less than 1 for $t \neq 0$; for more details we refer to Section 4.4. Furthermore, in the Appendix (see Lemma 4.10.3, Section 4.10), we show that Hypotheses **M4.1-M4.3** imply that the following number σ^2 , which is the limit of $\mathbb{E}_x(S_n^2)/n$ as $n \rightarrow +\infty$ for any $x \in \mathbb{X}$, is not zero:

$$\sigma^2 := \nu(f^2) + 2 \sum_{n=1}^{+\infty} \nu(f\mathbf{P}^n f) > 0. \tag{4.2.2}$$

Under spectral gap assumptions, the asymptotic behaviour of the survival probability $\mathbb{P}_x(\tau_y > n)$ and of the conditional law of the Markov walk $\frac{y+S_n}{\sqrt{n}}$ given the event $\{\tau_y > n\}$ have been studied in [38] (Chapter 3). It is easy to see that under **M4.1**, **M4.2** and (4.2.2) the conditions of [38] (Chapter 3) are satisfied (see Section 4.10). We summarize the main results of [38] (Chapter 3) in the following propositions.

Proposition 4.2.1 (Preliminary results, part I). *Assume Hypotheses **M4.1-M4.3**. There exists a non-degenerate non-negative function V on $\mathbb{X} \times \mathbb{R}$ such that*

1. *For any $(x, y) \in \mathbb{X} \times \mathbb{R}$ and $n \geq 1$,*

$$\mathbb{E}_x(V(X_n, y + S_n) ; \tau_y > n) = V(x, y).$$

2. *For any $x \in \mathbb{X}$, the function $V(x, \cdot)$ is non-decreasing and for any $(x, y) \in \mathbb{X} \times \mathbb{R}$,*

$$V(x, y) \leq c(1 + \max(y, 0)).$$

3. *For any $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $\delta \in (0, 1)$,*

$$(1 - \delta) \max(y, 0) - c_\delta \leq V(x, y) \leq (1 + \delta) \max(y, 0) + c_\delta.$$

Since the function V satisfies the point 1, it is said to be harmonic.

Proposition 4.2.2 (Preliminary results, part II). *Assume Hypotheses **M4.1-M4.3**.*

1. *For any $(x, y) \in \mathbb{X} \times \mathbb{R}$,*

$$\lim_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_x(\tau_y > n) = \frac{2V(x, y)}{\sqrt{2\pi\sigma}},$$

where σ is defined by (4.2.2).

2. *For any $(x, y) \in \mathbb{X} \times \mathbb{R}$ and $n \geq 1$,*

$$\mathbb{P}_x(\tau_y > n) \leq c \frac{1 + \max(y, 0)}{\sqrt{n}}.$$

Define the support of V by

$$\text{supp}(V) := \{(x, y) \in \mathbb{X} \times \mathbb{R} : V(x, y) > 0\}. \quad (4.2.3)$$

Note that from property 3 of Proposition 4.2.1, for any fixed $x \in \mathbb{X}$, the function $y \mapsto V(x, y)$ is positive for large y . For further details on the properties of $\text{supp}(V)$ we refer to [38] (Chapter 3).

Proposition 4.2.3 (Preliminary results, part III). *Assume Hypotheses M4.1-M4.3.*

1. For any $(x, y) \in \text{supp}(V)$ and $t \geq 0$,

$$\mathbb{P}_x \left(\frac{y + S_n}{\sigma\sqrt{n}} \leq t \mid \tau_y > n \right) \xrightarrow{n \rightarrow +\infty} \Phi^+(t),$$

where $\Phi^+(t) = 1 - e^{-\frac{t^2}{2}}$ is the Rayleigh distribution function.

2. There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $n \geq 1$, $t_0 > 0$, $t \in [0, t_0]$ and $(x, y) \in \mathbb{X} \times \mathbb{R}$,

$$\left| \mathbb{P}_x \left(y + S_n \leq t\sqrt{n}\sigma, \tau_y > n \right) - \frac{2V(x, y)}{\sqrt{2\pi n}\sigma} \Phi^+(t) \right| \leq c_{\varepsilon, t_0} \frac{(1 + \max(y, 0)^2)}{n^{1/2+\varepsilon}}.$$

In the point 1 of Proposition 4.2.2 and the point 2 of Proposition 4.2.3, the function V can be zero, so that for all pairs (x, y) satisfying $V(x, y) = 0$ it holds

$$\lim_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_x (\tau_y > n) = 0$$

and

$$\lim_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_x (y + S_n \leq t\sqrt{n}\sigma, \tau_y > n) = 0.$$

Now we proceed to formulate the main results of the paper. Our first result is an extension of Gnedenko-Stone local limit theorem originally stated for sums of independent random variables. The following theorem generalizes it to the case of sums of random variables defined on Markov chains conditioned to stay positive.

Theorem 4.2.4. *Assume Hypotheses M4.1-M4.3. Let $a > 0$ be a positive real. Then there exists $\varepsilon_0 \in (0, 1/4)$ such that for any $\varepsilon \in (0, \varepsilon_0)$, non-negative function $\psi \in \mathcal{C}$, $y \in \mathbb{R}$ and $n \geq 3\varepsilon^{-3}$, we have*

$$\begin{aligned} & \sup_{x \in \mathbb{X}, z \geq 0} n \left| \mathbb{E}_x (\psi(X_n) ; y + S_n \in [z, z + a], \tau_y > n) - \frac{2a\nu(\psi)V(x, y)}{\sqrt{2\pi}\sigma^2 n} \varphi_+ \left(\frac{z}{\sqrt{n}\sigma} \right) \right| \\ & \leq c(1 + \max(y, 0)) \|\psi\|_\infty \left(\sqrt{\varepsilon} + \frac{c_\varepsilon(1 + \max(y, 0))}{n^\varepsilon} \right), \end{aligned}$$

where $\varphi_+(t) = t e^{-\frac{t^2}{2}} \mathbb{1}_{\{t \geq 0\}}$ is the Rayleigh density and the constants c and c_ε may depend on a .

Note that Theorem 4.2.4 is meaningful only for large values of z such that $z \sim n^{1/2}$ as $n \rightarrow \infty$. Indeed, the remainder term is of order $n^{-1-\varepsilon}$, with some small $\varepsilon > 0$, while for a fixed z the leading term is of order $n^{-3/2}$. When $z = cn^{1/2}$ the leading term becomes of order n^{-1} while the remainder is still $o(n^{-1})$. To deal with the case of z in compact sets a

more refined result will be given below. We will deduce it from Theorem 4.2.4, however for the proof we need the concept of duality.

Let us introduce the dual Markov chain and the corresponding associated Markov walk. Since ν is positive on \mathbb{X} , the following dual Markov kernel \mathbf{P}^* is well defined:

$$\mathbf{P}^*(x, x^*) = \frac{\nu(x^*)}{\nu(x)} \mathbf{P}(x^*, x), \quad \forall (x, x^*) \in \mathbb{X}^2. \tag{4.2.4}$$

It is easy to see that ν is also \mathbf{P}^* -invariant. The dual of $(X_n)_{n \geq 0}$ is the Markov chain $(X_n^*)_{n \geq 0}$ with values in \mathbb{X} and transition probability \mathbf{P}^* . Without loss of generality we can consider that the dual Markov chain $(X_n^*)_{n \geq 0}$ is defined on an extension of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and that it is independent of the Markov chain $(X_n)_{n \geq 0}$. We define the associated dual Markov walk by

$$S_0^* = 0 \quad \text{and} \quad S_n^* = \sum_{k=1}^n -f(X_k^*), \quad \forall n \geq 1. \tag{4.2.5}$$

For any $z \in \mathbb{R}$, define also the exit time

$$\tau_z^* := \inf \{k \geq 1 : z + S_k^* \leq 0\}. \tag{4.2.6}$$

For any $x \in \mathbb{X}$, denote by \mathbb{P}_x^* and \mathbb{E}_x^* the probability, respectively the expectation, generated by the finite dimensional distributions of the Markov chain $(X_n^*)_{n \geq 0}$ starting at $X_0^* = x$. It is shown in Section 4.3 that the dual Markov chain $(X_n^*)_{n \geq 0}$ satisfies Hypotheses **M4.1-M4.3** as do the original chain $(X_n)_{n \geq 0}$. Thus, Propositions 4.2.1-4.2.3 hold also for $(X_n^*)_{n \geq 0}$ with $V, \tau, (S_n)_{n \geq 0}$ and \mathbb{P}_x replaced by $V^*, \tau^*, (S_n^*)_{n \geq 0}$ and \mathbb{P}_x^* . Note also that both chains have the same invariant probability ν . Denote by $\mathbb{E}_\nu, \mathbb{E}_\nu^*$ the expectations generated by the finite dimensional distributions of the Markov chains $(X_n)_{n \geq 0}$ and $(X_n^*)_{n \geq 0}$ in the stationary regime.

Our second result is a conditional version of the local limit theorem for fixed x, y and z .

Theorem 4.2.5. *Assume Hypotheses **M4.1-M4.3**.*

1. *For any non-negative function $\psi \in \mathcal{C}$, $a > 0$, $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z \geq 0$,*

$$\begin{aligned} & \lim_{n \rightarrow +\infty} n^{3/2} \mathbb{E}_x(\psi(X_n) ; y + S_n \in [z, z + a], \tau_y > n) \\ &= \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_z^{z+a} \mathbb{E}_\nu^*(\psi(X_1^*) V^*(X_1^*, z' + S_1^*) ; \tau_{z'}^* > 1) dz'. \end{aligned}$$

2. *Moreover, there exists $c > 0$ such that for any $a > 0$, non-negative function $\psi \in \mathcal{C}$, $y \in \mathbb{R}$, $z \geq 0$ and $n \geq 1$,*

$$\begin{aligned} & \sup_{x \in \mathbb{X}} \mathbb{E}_x(\psi(X_n) ; y + S_n \in [z, z + a], \tau_y > n) \\ & \leq \frac{c \|\psi\|_\infty}{n^{3/2}} (1 + a^3) (1 + z) (1 + \max(y, 0)). \end{aligned}$$

In the particular case when $\psi = 1$, the previous theorem rewrites as follows:

Corollary 4.2.6. *Assume Hypotheses **M4.1-M4.3**.*

1. For any $a > 0$, $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z \geq 0$,

$$\begin{aligned} \lim_{n \rightarrow +\infty} n^{3/2} \mathbb{P}_x (y + S_n \in [z, z + a], \tau_y > n) \\ = \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_z^{z+a} \int_{\mathbb{X}} V^*(x', z') \nu(dx') dz'. \end{aligned}$$

2. Moreover, there exists $c > 0$ such that for any $a > 0$, $y \in \mathbb{R}$, $z \geq 0$ and $n \geq 1$,

$$\sup_{x \in \mathbb{X}} \mathbb{P}_x (y + S_n \in [z, z + a], \tau_y > n) \leq \frac{c}{n^{3/2}} (1 + a^3) (1 + z) (1 + \max(y, 0)).$$

Note that the assertion 1 of Theorem 4.2.5 and assertion 1 of Corollary 4.2.6 hold for fixed $a > 0$, $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z \geq 0$ and that these results are no longer true when z is not in a compact set, for instance when $z \sim n^{1/2}$.

The following result extends Theorem 4.2.5 to some functionals of the trajectories of the chain $(X_n)_{n \geq 0}$. For any $(x, x^*) \in \mathbb{X}^2$, the probability generated by the finite dimensional distributions of the two dimensional Markov chain $(X_n, X_n^*)_{n \geq 0}$ starting at $(X_0, X_0^*) = (x, x^*)$ is given by $\mathbb{P}_{x, x^*} = \mathbb{P}_x \times \mathbb{P}_{x^*}^*$. Let \mathbb{E}_{x, x^*} be the corresponding expectation. For any $l \geq 1$, denote by $\mathcal{C}^+(\mathbb{X}^l \times \mathbb{R}_+)$ the set of non-negative functions $g: \mathbb{X}^l \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying the following properties:

- for any $(x_1, \dots, x_l) \in \mathbb{X}^l$, the function $z \mapsto g(x_1, \dots, x_l, z)$ is continuous,
- there exists $\varepsilon > 0$ such that $\max_{x_1, \dots, x_l \in \mathbb{X}} \sup_{z \geq 0} g(x_1, \dots, x_l, z)(1 + z)^{2+\varepsilon} < +\infty$.

Theorem 4.2.7. Assume Hypotheses **M4.1-M4.3**. For any $x \in \mathbb{X}$, $y \in \mathbb{R}$, $l \geq 1$, $m \geq 1$ and $g \in \mathcal{C}^+(\mathbb{X}^{l+m} \times \mathbb{R}_+)$,

$$\begin{aligned} \lim_{n \rightarrow +\infty} n^{3/2} \mathbb{E}_x (g(X_1, \dots, X_l, X_{n-m+1}, \dots, X_n, y + S_n); \tau_y > n) \\ = \frac{2}{\sqrt{2\pi\sigma^3}} \int_0^{+\infty} \sum_{x^* \in \mathbb{X}} \mathbb{E}_{x, x^*} (g(X_1, \dots, X_l, X_m^*, \dots, X_1^*, z) \\ \times V(X_l, y + S_l) V^*(X_m^*, z + S_m^*); \tau_y > l, \tau_z^* > m) \nu(x^*) dz. \end{aligned}$$

As a consequence of Theorem 4.2.7 we deduce the following asymptotic behaviour of the probability of the event $\{\tau_y = n\}$ as $n \rightarrow +\infty$.

Theorem 4.2.8. Assume Hypotheses **M4.1-M4.3**. For any $x \in \mathbb{X}$ and $y \in \mathbb{R}$,

$$\lim_{n \rightarrow +\infty} n^{3/2} \mathbb{P}_x (\tau_y = n) = \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_0^{+\infty} \mathbb{E}_\nu^* (V^*(X_1^*, z); S_1^* \geq z) dz.$$

4.3 Properties of the dual Markov chain

In this section we establish some properties of the dual Markov chain and of the corresponding Markov walk.

Lemma 4.3.1. Suppose that the operator \mathbf{P} satisfies Hypotheses **M4.1-M4.3**. Then the dual operator \mathbf{P}^* satisfies also **M4.1-M4.3**.

Proof. By the definition of \mathbf{P}^* , for any $x^* \in \mathbb{X}$,

$$\sum_{x \in \mathbb{X}} \nu(x) \mathbf{P}^*(x, x^*) = \sum_{x \in \mathbb{X}} \mathbf{P}(x^*, x) \nu(x) = \nu(x^*),$$

which proves that ν is also \mathbf{P}^* -invariant. Thus Hypothesis **M4.2**, $\nu(f) = \nu(-f) = 0$, is satisfied for both chains. Moreover, it is easy to see that for any $n \geq 1$, $(x, x^*) \in \mathbb{X}^2$,

$$(\mathbf{P}^*)^n(x, x^*) = \mathbf{P}^n(x^*, x) \frac{\nu(x^*)}{\nu(x)}.$$

This shows that \mathbf{P}^* satisfies **M4.1** and **M4.3**. □

Note that the operator \mathbf{P}^* is the adjoint operator of \mathbf{P} in the space $L^2(\nu)$: for any functions g and h on \mathbb{X} ,

$$\nu(g(\mathbf{P}^*)^n h) = \nu(h \mathbf{P}^n g).$$

In particular for any $n \geq 1$, $\nu(f(\mathbf{P}^*)^n f) = \nu(f \mathbf{P}^n f)$ and we note that

$$\sigma^2 = \nu((-f)^2) + \sum_n \nu((-f)(\mathbf{P}^*)^n(-f)).$$

The following assertion plays a key role in the proofs.

Lemma 4.3.2 (Duality). *For any probability measure \mathbf{m} on \mathbb{X} , any $n \geq 1$ and any function F from \mathbb{X}^n to \mathbb{R} ,*

$$\mathbb{E}_{\mathbf{m}}(F(X_1, \dots, X_{n-1}, X_n)) = \mathbb{E}_{\nu}^* \left(F(X_n^*, X_{n-1}^*, \dots, X_1^*) \frac{\mathbf{m}(X_{n+1}^*)}{\nu(X_{n+1}^*)} \right).$$

Proof. We write

$$\begin{aligned} & \mathbb{E}_{\mathbf{m}}(F(X_1, \dots, X_{n-1}, X_n)) \\ &= \sum_{x_0, x_1, \dots, x_{n-1}, x_n, x_{n+1} \in \mathbb{X}} F(x_1, \dots, x_{n-1}, x_n) \mathbf{m}(x_0) \\ & \quad \mathbb{P}_{x_0}(X_1 = x_1, X_2 = x_2, \dots, X_{n-1} = x_{n-1}, X_n = x_n, X_{n+1} = x_{n+1}). \end{aligned}$$

By the definition of \mathbf{P}^* , we have

$$\begin{aligned} & \mathbb{P}_{x_0}(X_1 = x_1, X_2 = x_2, \dots, X_{n-1} = x_{n-1}, X_n = x_n, X_{n+1} = x_{n+1}) \\ &= \mathbf{P}(x_0, x_1) \mathbf{P}(x_1, x_2) \dots \mathbf{P}(x_{n-1}, x_n) \mathbf{P}(x_n, x_{n+1}) \\ &= \mathbf{P}^*(x_1, x_0) \frac{\nu(x_1)}{\nu(x_0)} \mathbf{P}^*(x_2, x_1) \frac{\nu(x_2)}{\nu(x_1)} \dots \mathbf{P}^*(x_n, x_{n-1}) \frac{\nu(x_n)}{\nu(x_{n-1})} \mathbf{P}^*(x_{n+1}, x_n) \frac{\nu(x_{n+1})}{\nu(x_n)} \\ &= \frac{\nu(x_{n+1})}{\nu(x_0)} \mathbb{P}_{x_{n+1}}^* (X_1^* = x_n, X_2^* = x_{n-1}, \dots, X_n^* = x_1, X_{n+1}^* = x_0) \end{aligned}$$

and the result of the lemma follows. □

4.4 The perturbed operator

For any $t \in \mathbb{R}$, denote by \mathbf{P}_t the perturbed transition operator defined by

$$\mathbf{P}_t g(x) = \mathbf{P} \left(e^{itf} g \right) (x) = \mathbb{E}_x \left(e^{itf(X_1)} g(X_1) \right), \quad \text{for any } g \in \mathcal{C}, x \in \mathbb{X},$$

where \mathbf{i} is the complex $\mathbf{i}^2 = -1$. Let also r_t be the spectral radius of \mathbf{P}_t . Note that for any $g \in \mathcal{C}$, $\|\mathbf{P}_t g\|_\infty \leq \|e^{itf} g\|_\infty = \|g\|_\infty$ and so

$$r_t \leq 1. \tag{4.4.1}$$

We introduce the two following definitions:

— A sequence $x_0, x_1, \dots, x_n \in \mathbb{X}$, is a *path* (between x_0 and x_n) if

$$\mathbf{P}(x_0, x_1) \cdots \mathbf{P}(x_{n-1}, x_n) > 0.$$

— A sequence $x_0, x_1, \dots, x_n \in \mathbb{X}$, is an *orbit* if $x_0, x_1, \dots, x_n, x_0$ is a path.

Note that under Hypothesis **M4.1**, for any $x_0, x \in \mathbb{X}$ it is always possible to connect x_0 and x by a path x_0, x_1, \dots, x_n, x in \mathbb{X} .

Lemma 4.4.1. *Assume Hypothesis **M4.1**. The following statements are equivalent:*

1. *There exists $(\theta, a) \in \mathbb{R}^2$ such that for any orbit x_0, \dots, x_n in \mathbb{X} , we have*

$$f(x_0) + \cdots + f(x_n) - (n+1)\theta \in a\mathbb{Z}.$$

2. *There exist $t \in \mathbb{R}^*$, $h \in \mathcal{C} \setminus \{0\}$ and $\theta \in \mathbb{R}$ such that for any $(x, x') \in \mathbb{X}^2$,*

$$h(x') e^{itf(x')} \mathbf{P}(x, x') = h(x) e^{it\theta} \mathbf{P}(x, x').$$

3. *There exists $t \in \mathbb{R}^*$ such that*

$$r_t = 1.$$

Proof. The point 1 implies the point 2. Suppose that the point 1 holds. Fix $x_0 \in \mathbb{X}$ and set $h(x_0) = 1$. For any $x \in \mathbb{X}$, define $h(x)$ in the following way: for any path x_0, \dots, x_n, x in \mathbb{X} we set

$$h(x) = e^{it\theta(n+1)} e^{-it(f(x_1) + \cdots + f(x_n) + f(x))},$$

where $t = \frac{2\pi}{a}$. Note that if $a = 0$, then the point 1 holds also for $a = 1$ and so, without lost of generality, we assume that $a \neq 0$. We first verify that h is well defined on \mathbb{X} . Recall that under Hypothesis **M4.1**, for any $x \in \mathbb{X}$ it is always possible to connect x_0 and x by a path. We have to check that the value of $h(x)$ does not depend on the choice of the path. Let $p, q \geq 1$ and x_0, x_1, \dots, x_p, x in \mathbb{X} and x_0, y_1, \dots, y_q, x in \mathbb{X} be two paths between x_0 and x . We complete these paths to orbits as follows. Under Hypothesis **M4.1**, there exist $n \geq 1$ and z_1, \dots, z_n in \mathbb{X} such that

$$\mathbf{P}(x, z_1) \cdots \mathbf{P}(z_n, x_0) > 0,$$

i.e. the sequence x, z_1, \dots, z_n, x_0 is a path. So, the sequences $x_0, x_1, \dots, x_p, x, z_1, \dots, z_n$ and $x_0, y_1, \dots, y_q, x, z_1, \dots, z_n$ are orbits. By the point 1, there exist $l_1, l_2 \in \mathbb{Z}$ such that

$$\begin{aligned} f(x_1) + \cdots + f(x_p) + f(x) &= al_1 - (f(z_1) + \cdots + f(z_n) + f(x_0)) + (p+n+2)\theta \\ &= al_1 - al_2 + (f(y_1) + \cdots + f(y_q) + f(x)) \\ &\quad - (q+n+2)\theta + (p+n+2)\theta. \end{aligned}$$

Therefore,

$$e^{it\theta(p+1)} e^{-it(f(x_1)+\dots+f(x_p)+f(x))} = e^{-it(al_1-al_2)} e^{it\theta(q+1)} e^{-it(f(y_1)+\dots+f(y_q)+f(x))}$$

and since $ta = 2\pi$ it proves that h is well defined. Now let $(x, x') \in \mathbb{X}^2$ be such that $\mathbf{P}(x, x') > 0$. There exists a path x_0, x_1, \dots, x_n, x between x_0 and x and so

$$h(x) = e^{it\theta(n+1)} e^{-it(f(x_1)+\dots+f(x_n)+f(x))}.$$

Since $x_0, x_1, \dots, x_n, x, x'$ is a path between x_0 and x' , we have also

$$h(x') = e^{it\theta(n+2)} e^{-it(f(x_1)+\dots+f(x_n)+f(x)+f(x'))} = h(x) e^{it\theta} e^{-itf(x')}.$$

Note that since the modulus of h is 1, this function belongs to $\mathcal{C} \setminus \{0\}$.

The point 2 implies the point 1. Suppose that the point 2 holds and let x_0, \dots, x_n be an orbit. Using the point 2 repeatedly, we have

$$h(x_0) = h(x_n) e^{it\theta} e^{-itf(x_0)} = \dots = h(x_0) e^{it\theta(n+1)} e^{-it(f(x_0)+\dots+f(x_n))}.$$

Since h is a non-identically zero function with a constant modulus, necessarily, h is never equal to 0 and so $f(x_0) + \dots + f(x_n) - (n+1)\theta \in \frac{2\pi}{t}\mathbb{Z}$.

The point 2 implies the point 3. Suppose that the point 2 holds. Summing on x' we have, for any $x \in \mathbb{X}$,

$$\mathbf{P}(h e^{itf})(x) = \mathbf{P}_t h(x) = h(x) e^{it\theta}.$$

Therefore h is an eigenvector of \mathbf{P}_t associated to the eigenvalue $e^{it\theta}$ which implies that $r_t \geq |e^{it\theta}| = 1$ and by (4.4.1), $r_t = 1$.

The point 3 implies the point 2. Suppose that the point 3 holds. There exist $h \in \mathcal{C} \setminus \{0\}$ and $\theta \in \mathbb{R}$ such that $\mathbf{P}_t h = h e^{it\theta}$. Without loss of generality, we suppose that $\|h\|_\infty = 1$. Since $\mathbf{P}_t^n h = h e^{itn\theta}$ for any $n \geq 1$, by (4.2.1), for any $x \in \mathbb{X}$, we have

$$|h(x)| = |\mathbf{P}_t^n h(x)| \leq \mathbf{P}^n |h|(x) \xrightarrow{n \rightarrow +\infty} \nu(|h|). \tag{4.4.2}$$

From (4.4.2), letting $x_0 \in \mathbb{X}$ be such that $|h(x_0)| = \|h\|_\infty = 1$, it is easy to see that

$$|h(x_0)| \leq \sum_{x \in \mathbb{X}} |h(x)| \nu(x) \leq |h(x_0)|.$$

From this it follows that the modulus of h is constant on \mathbb{X} : $|h(x)| = |h(x_0)| = 1$ for any $x \in \mathbb{X}$. Consequently, there exists $\alpha: \mathbb{X} \rightarrow \mathbb{R}$ such that for any $x \in \mathbb{X}$,

$$h(x) = e^{i\alpha(x)}. \tag{4.4.3}$$

With (4.4.3) the equation $\mathbf{P}_t h = h e^{it\theta}$ can be rewritten as

$$\forall x \in \mathbb{X}, \quad \sum_{x' \in \mathbb{X}} e^{i\alpha(x')} e^{itf(x')} \mathbf{P}(x, x') = e^{i\alpha(x)} e^{it\theta}.$$

Since $e^{i\alpha(x)} e^{it\theta} \in \{z \in \mathbb{C} : |z| = 1\}$ and $e^{i\alpha(x')} e^{itf(x')} \in \{z \in \mathbb{C} : |z| = 1\}$, for any $x' \in \mathbb{X}$, the previous equation holds only if $h(x') e^{itf(x')} = e^{i\alpha(x')} e^{itf(x')} = e^{i\alpha(x)} e^{it\theta} = h(x) e^{it\theta}$ for any $x' \in \mathbb{X}$ such that $\mathbf{P}(x, x') > 0$. \square

Define the operator norm $\|\cdot\|_{\mathcal{C} \rightarrow \mathcal{C}}$ on \mathcal{C} as follows: for any operator $R: \mathcal{C} \rightarrow \mathcal{C}$, set

$$\|R\|_{\mathcal{C} \rightarrow \mathcal{C}} := \sup_{g \in \mathcal{C} \setminus \{0\}} \frac{\|R(g)\|_{\infty}}{\|g\|_{\infty}}.$$

Lemma 4.4.2. *Assume Hypotheses [M4.1](#) and [M4.3](#). For any compact set K included in \mathbb{R}^* there exist constants $c_K > 0$ and $c'_K > 0$ such that for any $n \geq 1$,*

$$\sup_{t \in K} \|\mathbf{P}_t^n\|_{\mathcal{C} \rightarrow \mathcal{C}} \leq c_K e^{-c'_K n}.$$

Proof. By Lemma [4.4.1](#), under Hypotheses [M4.1](#) and [M4.3](#), we have $r_t \neq 1$ for any $t \neq 0$ and hence, using [\(4.4.1\)](#),

$$r_t < 1, \quad \forall t \in \mathbb{R}^*.$$

It is well known that

$$r_t = \lim_{n \rightarrow +\infty} \|\mathbf{P}_t^n\|_{\mathcal{C} \rightarrow \mathcal{C}}^{1/n}.$$

Since $t \mapsto \mathbf{P}_t$ is continuous, the function $t \mapsto r_t$ is the infimum of the sequence of upper semi-continuous functions $t \mapsto \|\mathbf{P}_t^n\|_{\mathcal{C} \rightarrow \mathcal{C}}^{1/n}$ and therefore is itself upper semi-continuous. In particular, for any compact set K included in \mathbb{R}^* , there exists $t_0 \in K$ such that

$$\sup_{t \in K} r_t = r_{t_0} < 1.$$

We deduce that for $\varepsilon = (1 - \sup_{t \in K} r_t)/2 > 0$ there exists $n_0 \geq 1$ such that for any $n \geq n_0$,

$$\|\mathbf{P}_t^n\|_{\mathcal{C} \rightarrow \mathcal{C}}^{1/n} \leq \sup_{t \in K} r_t + \varepsilon < 1.$$

Choosing $c_{K'} = -\ln(\sup_{t \in K} r_t + \varepsilon)$ and $c_K = \max_{n \leq n_0} \|\mathbf{P}_t^n\|_{\mathcal{C} \rightarrow \mathcal{C}} e^{c_{K'} n} + 1$, the lemma is proved. \square

In the proofs we make use of the following assertion which is a consequence of the perturbation theory of linear operator (see for example [\[50\]](#)). The point [5](#) is proved in Lemma 2 of Guivarc'h and Hardy [\[42\]](#).

Proposition 4.4.3. *Assume Hypotheses [M4.1](#) and [M4.2](#). There exist a real $\varepsilon_0 > 0$ and operator valued functions Π_t and Q_t acting from $[-\varepsilon_0, \varepsilon_0]$ to the set of operators onto \mathcal{C} such that*

1. the maps $t \mapsto \Pi_t$, $t \mapsto Q_t$ and $t \mapsto \lambda_t$ are analytic at 0,
2. the operator \mathbf{P}_t has the following decomposition,

$$\mathbf{P}_t = \lambda_t \Pi_t + Q_t, \quad \forall t \in [-\varepsilon_0, \varepsilon_0],$$

3. for any $t \in [-\varepsilon_0, \varepsilon_0]$, Π_t is a one-dimensional projector and $\Pi_t Q_t = Q_t \Pi_t = 0$,
4. there exist $c_1 > 0$ and $c_2 > 0$ such that, for any $n \in \mathbb{N}^*$,

$$\sup_{t \in [-\varepsilon_0, \varepsilon_0]} \|Q_t^n\|_{\mathcal{C} \rightarrow \mathcal{C}} \leq c_1 e^{-c_2 n},$$

5. the function λ_t has the following expansion at 0: for any $t \in [-\varepsilon_0, \varepsilon_0]$,

$$\left| \lambda_t - 1 + \frac{t^2 \sigma^2}{2} \right| \leq c |t|^3.$$

Note that $\lambda_0 = 1$ and $\Pi_0(\cdot) = \Pi(\cdot) = \nu(\cdot)e$, where e is the unit function of \mathbb{X} : $e(x) = 1$, for any $x \in \mathbb{X}$.

Lemma 4.4.4. *Assume Hypotheses [M4.1](#) and [M4.2](#). There exists $\varepsilon_0 > 0$ such that for any $n \geq 1$ and $t \in [-\varepsilon_0\sqrt{n}, \varepsilon_0\sqrt{n}]$,*

$$\left\| \mathbf{P}_{\frac{t}{\sqrt{n}}}^n - e^{-\frac{t^2\sigma^2}{2}} \Pi \right\|_{\mathcal{C} \rightarrow \mathcal{C}} \leq \frac{c}{\sqrt{n}} e^{-\frac{t^2\sigma^2}{4}} + c e^{-cn}.$$

Proof. By the points [2](#) and [3](#) of Proposition [4.4.3](#), for any $t/\sqrt{n} \in [-\varepsilon_0, \varepsilon_0]$,

$$\mathbf{P}_{\frac{t}{\sqrt{n}}}^n = \lambda_{\frac{t}{\sqrt{n}}}^n \Pi_{\frac{t}{\sqrt{n}}} + Q_{\frac{t}{\sqrt{n}}}^n.$$

By the points [1](#) and [4](#) of Proposition [4.4.3](#), for $n \geq 1$,

$$\left\| \Pi_{\frac{t}{\sqrt{n}}} - \Pi \right\|_{\mathcal{C} \rightarrow \mathcal{C}} \leq \sup_{u \in [-\varepsilon_0, \varepsilon_0]} \|\Pi'_u\|_{\mathcal{C} \rightarrow \mathcal{C}} \frac{|t|}{\sqrt{n}} \leq c \frac{|t|}{\sqrt{n}}, \tag{4.4.4}$$

$$\sup_{t \in [-\varepsilon_0, \varepsilon_0]} \left\| Q_{\frac{t}{\sqrt{n}}}^n \right\|_{\mathcal{C} \rightarrow \mathcal{C}} \leq c e^{-cn}. \tag{4.4.5}$$

Let α be the complex valued function defined on $[-\varepsilon_0, \varepsilon_0]$ by $\alpha(t) = \frac{1}{t^3} \left(\lambda_t - 1 + \frac{t^2\sigma^2}{2} \right)$ for any $t \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}$ and $\alpha(0) = 0$. By the point [5](#) of Proposition [4.4.3](#), there exists $c > 0$ such that

$$\forall t \in [-\varepsilon_0, \varepsilon_0], \quad |\alpha(t)| \leq c. \tag{4.4.6}$$

With this notation, we have for any $t/\sqrt{n} \in [-\varepsilon_0, \varepsilon_0]$,

$$\begin{aligned} \left| \lambda_{\frac{t}{\sqrt{n}}}^n - e^{-\frac{t^2\sigma^2}{2}} \right| &\leq \underbrace{\left| \left(1 - \frac{t^2\sigma^2}{2n} + \frac{t^3}{n^{3/2}} \alpha \left(\frac{t}{\sqrt{n}} \right) \right)^n - \left(1 - \frac{t^2\sigma^2}{2n} \right)^n \right|}_{=: I_1} \\ &\quad + \underbrace{\left| \left(1 - \frac{t^2\sigma^2}{2n} \right)^n - e^{-\frac{t^2\sigma^2}{2}} \right|}_{=: I_2}. \end{aligned} \tag{4.4.7}$$

Without loss of generality, the value of $\varepsilon_0 > 0$ can be chosen such that $\varepsilon_0^2\sigma^2 \leq 1$ and so for any $t/\sqrt{n} \in [-\varepsilon_0, \varepsilon_0]$, we have $1 - \frac{t^2\sigma^2}{2n} \geq 1/2$. Therefore,

$$\begin{aligned} I_1 &\leq \left(1 - \frac{t^2\sigma^2}{2n} \right)^n \left| \left(1 + \frac{t^3}{n^{3/2} \left(1 - \frac{t^2\sigma^2}{2n} \right)} \alpha \left(\frac{t}{\sqrt{n}} \right) \right)^n - 1 \right| \\ &\leq \left(1 - \frac{t^2\sigma^2}{2n} \right)^n \sum_{k=1}^n \binom{n}{k} \left| \frac{t^3}{n^{3/2} \left(1 - \frac{t^2\sigma^2}{2n} \right)} \alpha \left(\frac{t}{\sqrt{n}} \right) \right|^k \\ &= \left(1 - \frac{t^2\sigma^2}{2n} \right)^n \left[\left(1 + \frac{|t|^3}{n^{3/2} \left(1 - \frac{t^2\sigma^2}{2n} \right)} \left| \alpha \left(\frac{t}{\sqrt{n}} \right) \right| \right)^n - 1 \right]. \end{aligned}$$

Using the inequality $1 + u \leq e^u$ for $u \in \mathbb{R}$, the fact that $1 - \frac{t^2\sigma^2}{2n} \geq 1/2$ and the bound [\(4.4.6\)](#), we have

$$I_1 \leq e^{-\frac{t^2\sigma^2}{2}} \left(e^{\frac{c|t|^3}{\sqrt{n}}} - 1 \right).$$

Next, using the inequality $e^u - 1 \leq u e^u$ for $u \geq 0$ and the fact that $|t|/\sqrt{n} \leq \varepsilon_0$,

$$I_1 \leq e^{-\frac{t^2\sigma^2}{2}} \frac{c}{\sqrt{n}} |t|^3 e^{c\varepsilon_0 t^2}. \quad (4.4.8)$$

Again, without loss of generality, the value of $\varepsilon_0 > 0$ can be chosen such that $c\varepsilon_0^2 \leq \sigma^2/8$ (this has no impact on (4.4.6) which holds for any $[-\varepsilon'_0, \varepsilon'_0] \subseteq [-\varepsilon_0, \varepsilon_0]$). Thus, from (4.4.8) it follows that

$$I_1 \leq \frac{c}{\sqrt{n}} e^{-\frac{t^2\sigma^2}{4}}. \quad (4.4.9)$$

Using the inequalities $1 - u \leq e^{-u}$ for $u \in \mathbb{R}$ and $\ln(1 - u) \geq -u - u^2$ for $u \leq 1$, we have

$$I_2 = e^{-\frac{t^2\sigma^2}{2}} - \left(1 - \frac{t^2\sigma^2}{2n}\right)^n \leq e^{-\frac{t^2\sigma^2}{2}} - e^{-\frac{t^2\sigma^2}{2} - \frac{t^4\sigma^4}{4n}} \leq \frac{t^4\sigma^4}{4n} e^{-\frac{t^2\sigma^2}{2}} \leq \frac{c}{\sqrt{n}} e^{-\frac{t^2\sigma^2}{4}}. \quad (4.4.10)$$

Putting together (4.4.7), (4.4.9) and (4.4.10), we obtain that, for any $t/\sqrt{n} \in [-\varepsilon_0, \varepsilon_0]$,

$$\left| \lambda_{\frac{t}{\sqrt{n}}}^n - e^{-\frac{t^2\sigma^2}{2}} \right| \leq \frac{c}{\sqrt{n}} e^{-\frac{t^2\sigma^2}{4}}. \quad (4.4.11)$$

In the same way, one can prove that

$$|t| \left| \lambda_{\frac{t}{\sqrt{n}}}^n \right| \leq e^{-\frac{t^2\sigma^2}{4}}. \quad (4.4.12)$$

The right hand side in the assertion of the lemma can be bounded as follows:

$$\left\| \mathbf{P}_{\frac{t}{\sqrt{n}}}^n - e^{-\frac{t^2\sigma^2}{2}} \Pi \right\|_{\mathcal{C} \rightarrow \mathcal{C}} \leq \left| \lambda_{\frac{t}{\sqrt{n}}}^n \right| \left\| \Pi_{\frac{t}{\sqrt{n}}} - \Pi \right\|_{\mathcal{C} \rightarrow \mathcal{C}} + \left| \lambda_{\frac{t}{\sqrt{n}}}^n - e^{-\frac{t^2\sigma^2}{2}} \right| \|\Pi\|_{\mathcal{C} \rightarrow \mathcal{C}} + \left\| Q_{\frac{t}{\sqrt{n}}}^n \right\|_{\mathcal{C} \rightarrow \mathcal{C}}.$$

Using (4.4.4), (4.4.5), (4.4.11) and (4.4.12), we obtain that, for any $t/\sqrt{n} \in [\varepsilon_0, \varepsilon_0]$,

$$\left\| \mathbf{P}_{\frac{t}{\sqrt{n}}}^n - e^{-\frac{t^2\sigma^2}{2}} \Pi \right\|_{\mathcal{C} \rightarrow \mathcal{C}} \leq \frac{c}{\sqrt{n}} e^{-\frac{t^2\sigma^2}{4}} + c e^{-cn}.$$

□

4.5 A non asymptotic local limit theorem

In this section we establish a local limit theorem for the Markov walk jointly with the Markov chain. Our result is similar to that in Grama and Le Page [39] where the case of sums of independent random variables is considered under the Cramér condition. We refer to Guivarc'h and Hardy [42] for local limit theorem for a Markov chain with compact state space. In contrast to [42] our local limit theorem gives a control of the remainder term.

We first establish a local limit theorem for integrable functions with Fourier transforms with compact supports. For any integrable function $h: \mathbb{R} \rightarrow \mathbb{R}$ denote by \hat{h} its Fourier transform:

$$\hat{h}(t) = \int_{\mathbb{R}} e^{-itu} h(u) du, \quad \forall t \in \mathbb{R}.$$

When \hat{h} is integrable, by the inversion formula,

$$h(u) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{itu} \hat{h}(t) dt, \quad \forall u \in \mathbb{R}.$$

For any integrable functions h and g , let

$$h * g(u) = \int_{\mathbb{R}} h(v)g(u - v) dv$$

be the convolution of h and g . Denote by φ_{σ} the density of the centred normal law with variance σ^2 :

$$\varphi_{\sigma}(u) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{u^2}{2\sigma^2}}, \quad \forall u \in \mathbb{R}. \tag{4.5.1}$$

Lemma 4.5.1. *Assume Hypotheses **M4.1-M4.3**. For any $A > 0$, any integrable function h on \mathbb{R} whose Fourier transform \widehat{h} has a compact support included in $[-A, A]$, any real function ψ defined on \mathbb{X} and any $n \geq 1$,*

$$\begin{aligned} & \sup_{y \in \mathbb{R}} \sqrt{n} \left| \mathbb{E}_x (h(y + S_n) \psi(X_n)) - h * \varphi_{\sqrt{n}\sigma}(y) \nu(\psi) \right| \\ & \leq \|\psi\|_{\infty} \left(\frac{c}{\sqrt{n}} \|h\|_{L^1} + \|\widehat{h}\|_{L^1} c_A e^{-cAn} \right). \end{aligned}$$

Proof. By the inversion formula and the Fubini theorem,

$$\begin{aligned} I_0 & := \sqrt{n} \left| \mathbb{E}_x (h(y + S_n) \psi(X_n)) - h * \varphi_{\sqrt{n}\sigma}(y) \nu(\psi) \right| \\ & = \frac{\sqrt{n}}{2\pi} \left| \mathbb{E}_x \left(\int_{\mathbb{R}} e^{it(y+S_n)} \widehat{h}(t) dt \psi(X_n) \right) - \int_{\mathbb{R}} \widehat{h}(t) \widehat{\varphi}_{\sqrt{n}\sigma}(t) e^{ity} dt \nu(\psi) \right| \\ & = \frac{\sqrt{n}}{2\pi} \left| \int_{\mathbb{R}} e^{ity} \left(\mathbf{P}_t^n \psi(x) - e^{-\frac{t^2 \sigma^2 n}{2}} \nu(\psi) \right) \widehat{h}(t) dt \right|. \end{aligned}$$

Since $\widehat{h}(t) = 0$ for any $t \notin [-A, A]$, we write

$$\begin{aligned} I_0 & \leq \underbrace{\frac{\sqrt{n}}{2\pi} \left| \int_{\varepsilon_0 \leq |t| \leq A} e^{ity} \left(\mathbf{P}_t^n \psi(x) - e^{-\frac{t^2 \sigma^2 n}{2}} \nu(\psi) \right) \widehat{h}(t) dt \right|}_{=: I_1} \\ & \quad + \underbrace{\frac{\sqrt{n}}{2\pi} \left| \int_{|t| \leq \varepsilon_0} e^{ity} \left(\mathbf{P}_t^n \psi(x) - e^{-\frac{t^2 \sigma^2 n}{2}} \nu(\psi) \right) \widehat{h}(t) dt \right|}_{=: I_2}, \end{aligned} \tag{4.5.2}$$

where ε_0 is defined by Lemma 4.4.4.

Bound of I_1 . By Lemma 4.4.2, for any $\varepsilon_0 \leq |t| \leq A$, we have

$$\|\mathbf{P}_t^n \psi\|_{\infty} \leq \|\psi\|_{\infty} c_{A, \varepsilon_0} e^{-c_{A, \varepsilon_0} n}.$$

Consequently,

$$\begin{aligned} I_1 & \leq \frac{\sqrt{n}}{2\pi} \left(\|\psi\|_{\infty} c_{A, \varepsilon_0} e^{-c_{A, \varepsilon_0} n} + e^{-\frac{\varepsilon_0^2 \sigma^2 n}{2}} |\nu(\psi)| \right) \|\widehat{h}\|_{L^1} \\ & \leq \|\psi\|_{\infty} \|\widehat{h}\|_{L^1} c_{A, \varepsilon_0} e^{-c_{A, \varepsilon_0} n}. \end{aligned} \tag{4.5.3}$$

Bound of I_2 . Substituting $s = t\sqrt{n}$, we write

$$\begin{aligned} I_2 & = \frac{1}{2\pi} \left| \int_{|s| \leq \varepsilon_0 \sqrt{n}} e^{i \frac{sy}{\sqrt{n}}} \left(\mathbf{P}_{\frac{s}{\sqrt{n}}}^n \psi(x) - e^{-\frac{s^2 \sigma^2}{2}} \nu(\psi) \right) \widehat{h} \left(\frac{s}{\sqrt{n}} \right) ds \right| \\ & \leq \frac{1}{2\pi} \int_{|s| \leq \varepsilon_0 \sqrt{n}} \left| \mathbf{P}_{\frac{s}{\sqrt{n}}}^n \psi(x) - e^{-\frac{s^2 \sigma^2}{2}} \nu(\psi) \right| \left| \widehat{h} \left(\frac{s}{\sqrt{n}} \right) \right| ds. \end{aligned}$$

By Lemma 4.4.4, for any $|s| \leq \varepsilon_0 \sqrt{n}$, we have

$$\begin{aligned} \left| \mathbf{P}_{\frac{s}{\sqrt{n}}}^n \psi(x) - e^{-\frac{s^2 \sigma^2}{2}} \nu(\psi) \right| &\leq \left\| \mathbf{P}_{\frac{s}{\sqrt{n}}}^n(\psi) - e^{-\frac{s^2 \sigma^2}{2}} \Pi(\psi) \right\|_{\infty} \\ &\leq \|\psi\|_{\infty} \left\| \mathbf{P}_{\frac{s}{\sqrt{n}}}^n - e^{-\frac{s^2 \sigma^2}{2}} \Pi \right\|_{\mathcal{C} \rightarrow \mathcal{C}} \\ &\leq \|\psi\|_{\infty} \left(\frac{c}{\sqrt{n}} e^{-\frac{s^2 \sigma^2}{4}} + c e^{-cn} \right). \end{aligned}$$

Therefore,

$$\begin{aligned} I_2 &\leq \|\psi\|_{\infty} \left(\frac{c}{\sqrt{n}} \int_{\mathbb{R}} e^{-\frac{s^2 \sigma^2}{4}} \|\widehat{h}\|_{\infty} ds + c e^{-cn} \|\widehat{h}\|_{L^1} \right) \\ &\leq \|\psi\|_{\infty} \left(\frac{c}{\sqrt{n}} \|h\|_{L^1} + c e^{-cn} \|\widehat{h}\|_{L^1} \right). \end{aligned} \quad (4.5.4)$$

Putting together (4.5.2), (4.5.3) and (4.5.4), concludes the proof. \square

We extend the result of Lemma 4.5.1 for any integrable function (with not necessarily integrable Fourier transform). As in Stone [67], we introduce the kernel κ defined on \mathbb{R} by

$$\kappa(u) = \frac{1}{2\pi} \left(\frac{\sin\left(\frac{u}{2}\right)}{\frac{u}{2}} \right)^2, \quad \forall u \in \mathbb{R}^* \quad \text{and} \quad \kappa(0) = \frac{1}{2\pi}.$$

The function κ is integrable and its Fourier transform is given by

$$\widehat{\kappa}(t) = 1 - |t|, \quad \forall t \in [-1, 1], \quad \text{and} \quad \widehat{\kappa}(t) = 0 \quad \text{otherwise.}$$

Note that

$$\int_{\mathbb{R}} \kappa(u) du = \widehat{\kappa}(0) = 1 = \int_{\mathbb{R}} \widehat{\kappa}(t) dt.$$

For any $\varepsilon > 0$, we define the function κ_{ε} on \mathbb{R} by

$$\kappa_{\varepsilon}(u) = \frac{1}{\varepsilon} \kappa\left(\frac{u}{\varepsilon}\right).$$

Its Fourier transform is given by $\widehat{\kappa}_{\varepsilon}(t) = \widehat{\kappa}(\varepsilon t)$. Note also that, for any $\varepsilon > 0$, we have

$$\int_{|u| \geq \frac{1}{\varepsilon}} \kappa(u) du \leq \frac{1}{\pi} \int_{\frac{1}{\varepsilon}}^{+\infty} \frac{4}{u^2} du = \frac{4}{\pi} \varepsilon. \quad (4.5.5)$$

For any non-negative and locally bounded function h defined on \mathbb{R} and any $\varepsilon > 0$, let $\overline{h}_{\varepsilon}$ and $\underline{h}_{\varepsilon}$ be the "thickened" functions: for any $u \in \mathbb{R}$,

$$\overline{h}_{\varepsilon}(u) = \sup_{v \in [u-\varepsilon, u+\varepsilon]} h(v) \quad \text{and} \quad \underline{h}_{\varepsilon}(u) = \inf_{v \in [u-\varepsilon, u+\varepsilon]} h(v).$$

For any $\varepsilon > 0$, denote by $\mathcal{H}_{\varepsilon}$ the set of non-negative and locally bounded functions h such that h , $\overline{h}_{\varepsilon}$ and $\underline{h}_{\varepsilon}$ are measurable from $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ to $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ and Lebesgue-integrable (where $\mathcal{B}(\mathbb{R})$, $\mathcal{B}(\mathbb{R}_+)$ are the Borel σ -algebras).

Lemma 4.5.2. *For any function $h \in \mathcal{H}_{\varepsilon}$, $\varepsilon \in (0, 1/4)$ and $u \in \mathbb{R}$,*

$$\underline{h}_{\varepsilon} * \kappa_{\varepsilon^2}(u) - \int_{|v| \geq \varepsilon} \underline{h}_{\varepsilon}(u-v) \kappa_{\varepsilon^2}(v) dv \leq h(u) \leq (1 + 4\varepsilon) \overline{h}_{\varepsilon} * \kappa_{\varepsilon^2}(u).$$

Proof. Note that for any $|v| \leq \varepsilon$ and $u \in \mathbb{R}$, we have $u \in [u - v - \varepsilon, u - v + \varepsilon]$. So,

$$\underline{h}_\varepsilon(u - v) \leq h(u) \leq \bar{h}_\varepsilon(u - v). \quad (4.5.6)$$

Using the fact that $\int_{\mathbb{R}} \kappa_{\varepsilon^2}(u) du = 1$ and (4.5.5), we write

$$\begin{aligned} h(u) &= \int_{|v| \leq \varepsilon} h(u) \kappa_{\varepsilon^2}(v) dv + h(u) \int_{|v| \geq \varepsilon} \kappa_{\varepsilon^2}(v) dv \\ &\leq \int_{|v| \leq \varepsilon} \bar{h}_\varepsilon(u - v) \kappa_{\varepsilon^2}(v) dv + h(u) \frac{4}{\pi} \varepsilon. \end{aligned}$$

Therefore,

$$h(u) \left(1 - \frac{4}{\pi} \varepsilon\right) \leq \int_{\mathbb{R}} \bar{h}_\varepsilon(u - v) \kappa_{\varepsilon^2}(v) dv = \bar{h}_\varepsilon * \kappa_{\varepsilon^2}(u).$$

For any $\varepsilon \in (0, 1/4)$,

$$h(u) \leq \frac{1}{1 - 2\varepsilon} \bar{h}_\varepsilon * \kappa_{\varepsilon^2}(u) \leq (1 + 4\varepsilon) \bar{h}_\varepsilon * \kappa_{\varepsilon^2}(u).$$

Moreover, from (4.5.6),

$$\begin{aligned} h(u) &\geq \int_{|v| \leq \varepsilon} h(u) \kappa_{\varepsilon^2}(v) dv \\ &\geq \int_{|v| \leq \varepsilon} \underline{h}_\varepsilon(u - v) \kappa_{\varepsilon^2}(v) dv \\ &= \underline{h}_\varepsilon * \kappa_{\varepsilon^2}(u) - \int_{|v| \geq \varepsilon} \underline{h}_\varepsilon(u - v) \kappa_{\varepsilon^2}(v) dv. \end{aligned}$$

□

Lemma 4.5.3. *Let $\varepsilon > 0$ and $h \in \mathcal{H}_\varepsilon$.*

1. *For any $y \in \mathbb{R}$ and $n \geq 1$,*

$$\sqrt{n} \left(\bar{h}_\varepsilon * \kappa_{\varepsilon^2} \right) * \varphi_{\sqrt{n}\sigma}(y) \leq \sqrt{n} \left(h * \varphi_{\sqrt{n}\sigma} \right) (y) + c \left\| \bar{h}_{2\varepsilon} - h \right\|_{L^1} + c\varepsilon \|h\|_{L^1},$$

where $\varphi_{\sqrt{n}\sigma}(\cdot)$ is defined by (4.5.1).

2. *For any $y \in \mathbb{R}$ and $n \geq 1$,*

$$\sqrt{n} \left(\bar{h}_\varepsilon * \kappa_{\varepsilon^2} \right) * \varphi_{\sqrt{n}\sigma}(y) \leq c \left\| \bar{h}_\varepsilon \right\|_{L^1}.$$

3. *For any $y \in \mathbb{R}$ and $n \geq 1$,*

$$\sqrt{n} \left(\underline{h}_\varepsilon * \kappa_{\varepsilon^2} \right) * \varphi_{\sqrt{n}\sigma}(y) \geq \sqrt{n} \left(h * \varphi_{\sqrt{n}\sigma} \right) (y) - c \|h - \underline{h}_{2\varepsilon}\|_{L^1} - c\varepsilon \|h\|_{L^1}.$$

Proof. For any $\varepsilon > 0$, $|v| \leq \varepsilon$ and $u \in \mathbb{R}$ it holds $[u - v - \varepsilon, u - v + \varepsilon] \subset [u - 2\varepsilon, u + 2\varepsilon]$. Therefore,

$$\underline{h}_\varepsilon(u - v) \geq \underline{h}_{2\varepsilon}(u) \quad \text{and} \quad \bar{h}_\varepsilon(u - v) \leq \bar{h}_{2\varepsilon}(u). \quad (4.5.7)$$

Consequently, for any $u \in \mathbb{R}$,

$$\begin{aligned} \bar{h}_\varepsilon * \kappa_{\varepsilon^2}(u) &\leq \bar{h}_{2\varepsilon}(u) \int_{|v| \leq \varepsilon} \kappa_{\varepsilon^2}(v) dv + \int_{|v| \geq \varepsilon} \bar{h}_\varepsilon(u - v) \kappa_{\varepsilon^2}(v) dv \\ &\leq \bar{h}_{2\varepsilon}(u) + \int_{|v| \geq \varepsilon} \bar{h}_\varepsilon(u - v) \kappa_{\varepsilon^2}(v) dv. \end{aligned}$$

From this, using the bound $\sqrt{n}\varphi_{\sqrt{n}\sigma}(\cdot) \leq 1/(\sqrt{2\pi}\sigma)$ and (4.5.5), we obtain that

$$\begin{aligned} \sqrt{n} \left(\bar{h}_\varepsilon * \kappa_{\varepsilon^2} \right) * \varphi_{\sqrt{n}\sigma}(y) &\leq \sqrt{n} \left(\bar{h}_{2\varepsilon} * \varphi_{\sqrt{n}\sigma} \right) (y) \\ &\quad + \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} \int_{|v| \geq \varepsilon} \bar{h}_\varepsilon(u-v) \kappa_{\varepsilon^2}(v) \, dv \, du \\ &= \sqrt{n} \left(\bar{h}_{2\varepsilon} * \varphi_{\sqrt{n}\sigma} \right) (y) + \frac{2\sqrt{2}}{\pi^{3/2}\sigma} \varepsilon \|\bar{h}_\varepsilon\|_{L^1}. \end{aligned}$$

Using again the bound $\sqrt{n}\varphi_{\sqrt{n}\sigma}(\cdot) \leq 1/(\sqrt{2\pi}\sigma)$, we get

$$\begin{aligned} \sqrt{n} \left(\bar{h}_\varepsilon * \kappa_{\varepsilon^2} \right) * \varphi_{\sqrt{n}\sigma}(y) &\leq \sqrt{n} \left(h * \varphi_{\sqrt{n}\sigma} \right) (y) + \int_{\mathbb{R}} |\bar{h}_{2\varepsilon}(u) - h(u)| \frac{du}{\sqrt{2\pi}\sigma} + c\varepsilon \|\bar{h}_\varepsilon\|_{L^1} \\ &\leq \sqrt{n} \left(h * \varphi_{\sqrt{n}\sigma} \right) (y) + c \|\bar{h}_{2\varepsilon} - h\|_{L^1} + c\varepsilon \|\bar{h}_{2\varepsilon}\|_{L^1} \\ &\leq \sqrt{n} \left(h * \varphi_{\sqrt{n}\sigma} \right) (y) + (c + c\varepsilon) \|\bar{h}_{2\varepsilon} - h\|_{L^1} + c\varepsilon \|h\|_{L^1}, \end{aligned}$$

which proves the claim 1.

In the same way,

$$\sqrt{n} \left(\bar{h}_\varepsilon * \kappa_{\varepsilon^2} \right) * \varphi_{\sqrt{n}\sigma}(y) \leq \frac{1}{\sqrt{2\pi}\sigma} \|\bar{h}_\varepsilon * \kappa_{\varepsilon^2}\|_{L^1} = \frac{1}{\sqrt{2\pi}\sigma} \|\bar{h}_\varepsilon\|_{L^1},$$

which establishes the claim 2.

By (4.5.7) and (4.5.5),

$$\underline{h}_\varepsilon * \kappa_{\varepsilon^2}(u) \geq \underline{h}_{2\varepsilon}(u) \int_{|v| \leq \varepsilon} \kappa_{\varepsilon^2}(v) \, dv \geq \left(1 - \frac{4}{\pi}\varepsilon\right) \underline{h}_{2\varepsilon}(u).$$

Integrating this inequality and using once again the bound $\sqrt{n}\varphi_{\sqrt{n}\sigma}(\cdot) \leq \frac{1}{\sqrt{2\pi}\sigma}$, we have

$$\begin{aligned} \sqrt{n} \left(\underline{h}_\varepsilon * \kappa_{\varepsilon^2} \right) * \varphi_{\sqrt{n}\sigma}(y) &\geq \sqrt{n} \left(1 - \frac{4}{\pi}\varepsilon\right) \underline{h}_{2\varepsilon} * \varphi_{\sqrt{n}\sigma}(y) \\ &\geq \sqrt{n} \left(\underline{h}_{2\varepsilon} * \varphi_{\sqrt{n}\sigma} \right) (y) - \frac{4}{\pi} \varepsilon \frac{1}{\sqrt{2\pi}\sigma} \|\underline{h}_{2\varepsilon}\|_{L^1}. \end{aligned}$$

Inserting h , we conclude that

$$\begin{aligned} \sqrt{n} \left(\underline{h}_\varepsilon * \kappa_{\varepsilon^2} \right) * \varphi_{\sqrt{n}\sigma}(y) &\geq \sqrt{n} \left(h * \varphi_{\sqrt{n}\sigma} \right) (y) - \frac{1}{\sqrt{2\pi}\sigma} \|h - \underline{h}_{2\varepsilon}\|_{L^1} - c\varepsilon \|\underline{h}_{2\varepsilon}\|_{L^1} \\ &\geq \sqrt{n} \left(h * \varphi_{\sqrt{n}\sigma} \right) (y) - c \|h - \underline{h}_{2\varepsilon}\|_{L^1} - c\varepsilon \|h\|_{L^1}. \end{aligned}$$

□

We are now equipped to prove a non-asymptotic theorem for a large class of functions h .

Lemma 4.5.4. *Assume Hypotheses [M4.1-M4.3](#). Let $\varepsilon \in (0, 1/4)$. For any function $h \in \mathcal{H}_\varepsilon$, any non-negative function $\psi \in \mathcal{C}$ and any $n \geq 1$,*

$$\begin{aligned} &\sup_{x \in \mathbb{X}, y \in \mathbb{R}} \sqrt{n} \left| \mathbb{E}_x \left(h(y + S_n) \psi(X_n) \right) - h * \varphi_{\sqrt{n}\sigma}(y) \nu(\psi) \right| \\ &\leq c \|\psi\|_\infty \left(\|h - \underline{h}_{2\varepsilon}\|_{L^1} + \|\bar{h}_{2\varepsilon} - h\|_{L^1} \right) + c \|\psi\|_\infty \|\bar{h}_{2\varepsilon}\|_{L^1} \left(\frac{1}{\sqrt{n}} + \varepsilon + c_\varepsilon e^{-c_\varepsilon n} \right), \end{aligned}$$

where $\varphi_{\sqrt{n}\sigma}(\cdot)$ is defined by (4.5.1). Moreover,

$$\sup_{x \in \mathbb{X}, y \in \mathbb{R}} \sqrt{n} \mathbb{E}_x \left(h(y + S_n) \psi(X_n) \right) \leq c \|\psi\|_\infty \|\bar{h}_{2\varepsilon}\|_{L^1} \left(1 + c_\varepsilon e^{-c_\varepsilon n} \right).$$

Proof. We prove upper and lower bounds for $\sqrt{n}\mathbb{E}_x(h(y+S_n)\psi(X_n))$ from which the claim will follow.

The upper bound. By Lemma 4.5.2, we have, for any $x \in \mathbb{X}$, $n \geq 1$, $y \in \mathbb{R}$ and $\varepsilon \in (0, 1/4)$,

$$\mathbb{E}_x(h(y+S_n)\psi(X_n)) \leq (1+4\varepsilon)\mathbb{E}_x(\bar{h}_\varepsilon * \kappa_{\varepsilon^2}(y+S_n)\psi(X_n))$$

Since \bar{h}_ε is integrable, the function $u \mapsto \bar{h}_\varepsilon * \kappa_{\varepsilon^2}(u)$ is integrable and its Fourier transform $u \mapsto \widehat{\bar{h}_\varepsilon * \kappa_{\varepsilon^2}}(u)$ has a support included in $[-1/\varepsilon^2, 1/\varepsilon^2]$. Consequently, by Lemma 4.5.1,

$$\begin{aligned} I_0 &:= \sqrt{n}\mathbb{E}_x(h(y+S_n)\psi(X_n)) \\ &\leq \sqrt{n}(1+4\varepsilon)\left(\bar{h}_\varepsilon * \kappa_{\varepsilon^2}\right) * \varphi_{\sqrt{n}\sigma}(y)\nu(\psi) \\ &\quad + 2\|\psi\|_\infty\left(\frac{c}{\sqrt{n}}\|\bar{h}_\varepsilon * \kappa_{\varepsilon^2}\|_{L^1} + \|\widehat{\bar{h}_\varepsilon * \kappa_{\varepsilon^2}}\|_{L^1} c_\varepsilon e^{-c_\varepsilon n}\right). \end{aligned}$$

Using the points 1 and 2 of Lemma 4.5.3 and the fact that $|\nu(\psi)| \leq \|\psi\|_\infty$, we deduce that

$$\begin{aligned} I_0 &\leq \sqrt{n}\left(h * \varphi_{\sqrt{n}\sigma}\right)(y)\nu(\psi) + \|\psi\|_\infty\left(c\|\bar{h}_{2\varepsilon} - h\|_{L^1} + c\varepsilon\|h\|_{L^1}\right) + 4\varepsilon c\|\bar{h}_\varepsilon\|_{L^1}\|\psi\|_\infty \\ &\quad + 2\|\psi\|_\infty\left(\frac{c}{\sqrt{n}}\|\bar{h}_\varepsilon * \kappa_{\varepsilon^2}\|_{L^1} + \|\widehat{\bar{h}_\varepsilon * \kappa_{\varepsilon^2}}\|_{L^1} c_\varepsilon e^{-c_\varepsilon n}\right). \end{aligned}$$

Note that $\|\bar{h}_\varepsilon * \kappa_{\varepsilon^2}\|_{L^1} = \|\bar{h}_\varepsilon\|_{L^1}$ and

$$\|\widehat{\bar{h}_\varepsilon * \kappa_{\varepsilon^2}}\|_{L^1} \leq \|\bar{h}_\varepsilon\|_{L^1} \int_{\mathbb{R}} \widehat{\kappa}_{\varepsilon^2}(t) dt = \|\bar{h}_\varepsilon\|_{L^1} \int_{\mathbb{R}} \widehat{\kappa}(\varepsilon^2 t) dt = \frac{1}{\varepsilon^2} \|\bar{h}_\varepsilon\|_{L^1}.$$

Consequently,

$$\begin{aligned} I_0 &\leq \sqrt{n}\left(h * \varphi_{\sqrt{n}\sigma}\right)(y)\nu(\psi) + c\|\psi\|_\infty\|\bar{h}_{2\varepsilon} - h\|_{L^1} \\ &\quad + c\|\psi\|_\infty\|\bar{h}_\varepsilon\|_{L^1}\left(\frac{1}{\sqrt{n}} + \varepsilon + c_\varepsilon e^{-c_\varepsilon n}\right). \end{aligned} \tag{4.5.8}$$

From (4.5.8), taking into account that $\sqrt{n}\left(h * \varphi_{\sqrt{n}\sigma}\right)(y) \leq c\|h\|_{L^1}$, we deduce, in addition, that

$$I_0 \leq c\|\psi\|_\infty\|\bar{h}_{2\varepsilon}\|_{L^1}\left(1 + c_\varepsilon e^{-c_\varepsilon n}\right). \tag{4.5.9}$$

The lower bound. By Lemma 4.5.2, we write that

$$\begin{aligned} I_0 &\geq \underbrace{\sqrt{n}\mathbb{E}_x(\underline{h}_\varepsilon * \kappa_{\varepsilon^2}(y+S_n)\psi(X_n))}_{=: I_1} \\ &\quad - \underbrace{\sqrt{n}\mathbb{E}_x\left(\int_{|v| \geq \varepsilon} \underline{h}_\varepsilon(y+S_n-v)\kappa_{\varepsilon^2}(v)dv\psi(X_n)\right)}_{=: I_2}. \end{aligned} \tag{4.5.10}$$

Bound of I_1 . The Fourier transform of $\underline{h}_\varepsilon * \kappa_{\varepsilon^2}$ has a compact support included in $[-1/\varepsilon^2, 1/\varepsilon^2]$. So by Lemma 4.5.1,

$$I_1 \geq \sqrt{n}(\underline{h}_\varepsilon * \kappa_{\varepsilon^2}) * \varphi_{\sqrt{n}\sigma}(y)\nu(\psi) - \|\psi\|_\infty\left(\frac{c}{\sqrt{n}}\|\underline{h}_\varepsilon * \kappa_{\varepsilon^2}\|_{L^1} + \|\widehat{\underline{h}_\varepsilon * \kappa_{\varepsilon^2}}\|_{L^1} c_\varepsilon e^{-c_\varepsilon n}\right),$$

Using the point 3 of Lemma 4.5.3 and the fact that $|\nu(\psi)| \leq \|\psi\|_\infty$,

$$I_1 \geq \sqrt{n} \left(h * \varphi_{\sqrt{n}\sigma} \right) (y) \nu(\psi) - c \|\psi\|_\infty (\|h - \underline{h}_{2\varepsilon}\|_{L^1} + \varepsilon \|h\|_{L^1}) \\ - \|\psi\|_\infty \left(\frac{c}{\sqrt{n}} \|\underline{h}_\varepsilon * \kappa_{\varepsilon^2}\|_{L^1} + \|\widehat{\underline{h}_\varepsilon * \kappa_{\varepsilon^2}}\|_{L^1} c_\varepsilon e^{-c_\varepsilon n} \right).$$

Since $\|\underline{h}_\varepsilon * \kappa_{\varepsilon^2}\|_{L^1} = \|\underline{h}_\varepsilon\|_{L^1} \leq \|h\|_{L^1}$ and since $\|\widehat{\underline{h}_\varepsilon * \kappa_{\varepsilon^2}}\|_{L^1} \leq \|\underline{h}_\varepsilon\|_{L^1} \|\widehat{\kappa_{\varepsilon^2}}\|_{L^1} = \frac{1}{\varepsilon^2} \|\underline{h}_\varepsilon\|_{L^1} \leq \frac{1}{\varepsilon^2} \|h\|_{L^1}$, we deduce that

$$I_1 \geq \sqrt{n} \left(h * \varphi_{\sqrt{n}\sigma} \right) (y) \nu(\psi) - c \|\psi\|_\infty \|h - \underline{h}_{2\varepsilon}\|_{L^1} \\ - c \|\psi\|_\infty \|h\|_{L^1} \left(\frac{1}{\sqrt{n}} + \varepsilon + c_\varepsilon e^{-c_\varepsilon n} \right). \quad (4.5.11)$$

Bound of I_2 . With the notation $g_{\varepsilon,v}(u) = \underline{h}_\varepsilon(u - v)$, we have

$$I_2 = \int_{|v| \geq \varepsilon} \sqrt{n} \mathbb{E}_x (g_{\varepsilon,v}(y + S_n) \psi(X_n)) \kappa_{\varepsilon^2}(v) dv.$$

Consequently, using (4.5.9), we find that

$$I_2 \leq c \|\psi\|_\infty \left(1 + c_\varepsilon e^{-c_\varepsilon n} \right) \int_{|v| \geq \varepsilon} \|\overline{(g_{\varepsilon,v})_{2\varepsilon}}\|_{L^1} \kappa_{\varepsilon^2}(v) dv.$$

Note that, for any u and $v \in \mathbb{R}$,

$$\overline{(g_{\varepsilon,v})_{2\varepsilon}}(u) = \sup_{w \in [u-2\varepsilon, u+2\varepsilon]} \underline{h}_\varepsilon(w - v) \leq \sup_{w \in [u-2\varepsilon, u+2\varepsilon]} h(w - v) = \overline{h}_{2\varepsilon}(u - v).$$

So, $\|\overline{(g_{\varepsilon,v})_{2\varepsilon}}\|_{L^1} \leq \|\overline{h}_{2\varepsilon}\|_{L^1}$ and

$$I_2 \leq c \|\psi\|_\infty \|\overline{h}_{2\varepsilon}\|_{L^1} \left(1 + c_\varepsilon e^{-c_\varepsilon n} \right) \int_{|v| \geq \varepsilon} \kappa_{\varepsilon^2}(v) dv.$$

By (4.5.5),

$$I_2 \leq c \|\psi\|_\infty \|\overline{h}_{2\varepsilon}\|_{L^1} \left(\varepsilon + c_\varepsilon e^{-c_\varepsilon n} \right). \quad (4.5.12)$$

Putting together (4.5.10), (4.5.11) and (4.5.12), we obtain that

$$I_0 \geq \sqrt{n} \left(h * \varphi_{\sqrt{n}\sigma} \right) (y) \nu(\psi) - c \|\psi\|_\infty \|h - \underline{h}_{2\varepsilon}\|_{L^1} \\ - c \|\psi\|_\infty \|\overline{h}_{2\varepsilon}\|_{L^1} \left(\frac{1}{\sqrt{n}} + \varepsilon + c_\varepsilon e^{-c_\varepsilon n} \right). \quad (4.5.13)$$

Putting together the upper bound (4.5.8) and the lower bound (4.5.13), the first inequality of the lemma follows. The second inequality is proved in (4.5.9). \square

We now apply Lemma 4.5.4 when the function h is an indicator of an interval.

Corollary 4.5.5. *Assume Hypotheses M4.1-M4.3. For any $a > 0$, $\varepsilon \in (0, 1/4)$, any non-negative function $\psi \in \mathcal{C}$ and any $n \geq 1$,*

$$\sup_{x \in \mathbb{X}, y \in \mathbb{R}, z \geq 0} \sqrt{n} \left| \mathbb{E}_x (\psi(X_n) ; y + S_n \in [z, z + a]) - a \varphi_{\sqrt{n}\sigma}(z - y) \nu(\psi) \right| \\ \leq c(a + \varepsilon) \|\psi\|_\infty \left(\frac{1}{\sqrt{n}} + \frac{a}{n} + \varepsilon + c_\varepsilon e^{-c_\varepsilon n} \right),$$

where $\varphi_{\sqrt{n}\sigma}(\cdot)$ is defined by (4.5.1). In particular, there exists $c > 0$ such that for any $a > 0$,

$$\sup_{x \in \mathbb{X}, y \in \mathbb{R}, z \geq 0} \sqrt{n} \mathbb{E}_x(\psi(X_n); y + S_n \in [z, z + a]) \leq c(1 + a^2) \|\psi\|_\infty. \quad (4.5.14)$$

Proof. Let $z \geq 0, a > 0, \varepsilon \in (0, 1/4)$. For any $y \in \mathbb{R}$ set

$$h(y) = \mathbb{1}_{[z, z+a]}(y).$$

It is clear that

$$\bar{h}_\varepsilon(y) = \mathbb{1}_{[z-\varepsilon, z+a+\varepsilon]}(y) \quad \text{and} \quad \underline{h}_\varepsilon(y) = \mathbb{1}_{[z+\varepsilon, z+a-\varepsilon]}(y),$$

where by convention $\mathbb{1}_{[z+\varepsilon, z+a-\varepsilon]}(y) = 0$ when $a \leq 2\varepsilon$. It is also easy to see that

$$\|h - \underline{h}_{2\varepsilon}\|_{L^1} = \|\bar{h}_{2\varepsilon} - h\|_{L^1} = 4\varepsilon \quad \text{and} \quad \|\bar{h}_{2\varepsilon}\|_{L^1} = a + 4\varepsilon.$$

Taking into account these last equalities and using Lemma 4.5.4, we find that

$$\begin{aligned} & \left| \mathbb{E}_x(\psi(X_n); y + S_n \in [z, z + a]) - \mathbb{1}_{[z, z+a]} * \varphi_{\sqrt{n}\sigma}(y) \nu(\psi) \right| \\ & \leq c(a + \varepsilon) \|\psi\|_\infty \left(\frac{1}{\sqrt{n}} + \varepsilon + c_\varepsilon e^{-c_\varepsilon n} \right). \end{aligned} \quad (4.5.15)$$

Moreover, the convolution $\mathbb{1}_{[z, z+a]} * \varphi_{\sqrt{n}\sigma}$ is equal to

$$\mathbb{1}_{[z, z+a]} * \varphi_{\sqrt{n}\sigma}(y) = \int_{\mathbb{R}} \mathbb{1}_{\{z \leq y-u \leq z+a\}} \frac{e^{-\frac{u^2}{2n\sigma^2}}}{\sqrt{2\pi n\sigma}} du = \Phi_{\sqrt{n}\sigma}(y - z) - \Phi_{\sqrt{n}\sigma}(y - z - a),$$

where $\Phi_{\sqrt{n}\sigma}(t) = \int_{-\infty}^t \frac{e^{-\frac{u^2}{2n\sigma^2}}}{\sqrt{2\pi n\sigma}} du$ is the distribution function of the centred normal law of variance $n\sigma^2$. By the Taylor-Lagrange formula, there exists $\xi \in (y - z - a, y - z)$ such that

$$\Phi_{\sqrt{n}\sigma}(y - z - a) = \Phi_{\sqrt{n}\sigma}(y - z) - a\varphi_{\sqrt{n}\sigma}(y - z) + \frac{a^2}{2}\varphi'_{\sqrt{n}\sigma}(\xi).$$

Using the fact that $\sup_{u \in \mathbb{R}} |u| e^{-u^2} \leq c$,

$$\left| \mathbb{1}_{[z, z+a]} * \varphi_{\sqrt{n}\sigma}(y) - a\varphi_{\sqrt{n}\sigma}(z - y) \right| \leq \frac{ca^2}{n}. \quad (4.5.16)$$

Putting together (4.5.15) and (4.5.16), we conclude that

$$\begin{aligned} & \left| \mathbb{E}_x(\psi(X_n); y + S_n \in [z, z + a]) - a\varphi_{\sqrt{n}\sigma}(z - y) \nu(\psi) \right| \\ & \leq c(a + \varepsilon) \|\psi\|_\infty \left(\frac{1}{\sqrt{n}} + \frac{a}{n} + \varepsilon + c_\varepsilon e^{-c_\varepsilon n} \right). \end{aligned}$$

□

4.6 Auxiliary bounds

We state two bounds on the expectation $\mathbb{E}_x(\psi(X_n); y + S_n \in [z, z + a], \tau_y > n)$. The first one is of order $1/n$ and independent of z . Then we reverse the Markov chain to improve it to a bound of order $1/n^{3/2}$. We refer to Denisov and Wachtel [20] for related results in the case of lattice valued independent random variables.

Lemma 4.6.1. *Assume Hypotheses [M4.1-M4.3](#). There exists $c > 0$ such that for any $a > 0$, non-negative function $\psi \in \mathcal{C}$, $y \in \mathbb{R}$ and $n \geq 1$*

$$\sup_{x \in \mathbb{X}, z \geq 0} \mathbb{E}_x(\psi(X_n); y + S_n \in [z, z + a], \tau_y > n) \leq \frac{c}{n} \|\psi\|_\infty (1 + a^2) (1 + \max(y, 0)).$$

Proof. We split the time n into two parts $k := \lfloor n/2 \rfloor$ and $n - k$. By the Markov property,

$$\begin{aligned} E_0 &:= \mathbb{E}_x(\psi(X_n); y + S_n \in [z, z + a], \tau_y > n) \\ &= \sum_{x' \in \mathbb{X}} \int_0^{+\infty} \mathbb{E}_{x'}(\psi(X_k); y' + S_k \in [z, z + a], \tau_{y'} > k) \\ &\quad \times \mathbb{P}_x(X_{n-k} = x', y + S_{n-k} \in dy', \tau_y > n - k) \\ &\leq \sum_{x' \in \mathbb{X}} \int_0^{+\infty} \mathbb{E}_{x'}(\psi(X_k); y' + S_k \in [z, z + a]) \\ &\quad \times \mathbb{P}_x(X_{n-k} = x', y + S_{n-k} \in dy', \tau_y > n - k). \end{aligned}$$

Using the uniform bound (4.5.14) in Corollary 4.5.5, we obtain that

$$E_0 \leq \frac{c \|\psi\|_\infty}{\sqrt{k}} (1 + a^2) \mathbb{P}_x(\tau_y > n - k).$$

By the point 2 of Proposition 4.2.2, we get

$$E_0 \leq \frac{c \|\psi\|_\infty (1 + a^2) (1 + \max(y, 0))}{\sqrt{k} \sqrt{n - k}}.$$

Since $n - k \geq n/2$ and $k \geq n/4$ for any $n \geq 4$, the lemma is proved (the case when $n \leq 4$ is trivial). \square

Lemma 4.6.2. *Assume Hypotheses [M4.1-M4.3](#). There exists $c > 0$ such that for any $a > 0$, non-negative function $\psi \in \mathcal{C}$, $y \in \mathbb{R}$, $z \geq 0$ and $n \geq 1$*

$$\sup_{x \in \mathbb{X}} \mathbb{E}_x(\psi(X_n); y + S_n \in [z, z + a], \tau_y > n) \leq \frac{c \|\psi\|_\infty}{n^{3/2}} (1 + a^3) (1 + z) (1 + \max(y, 0)).$$

Proof. Set again $k = \lfloor n/2 \rfloor$. By the Markov property

$$\begin{aligned} E_0 &:= \mathbb{E}_x(\psi(X_n); y + S_n \in [z, z + a], \tau_y > n) \\ &= \sum_{x' \in \mathbb{X}} \int_0^{+\infty} \underbrace{\mathbb{E}_{x'}(\psi(X_k); y' + S_k \in [z, z + a], \tau_{y'} > k)}_{=: E'_0} \\ &\quad \times \mathbb{P}_x(X_{n-k} = x', y + S_{n-k} \in dy', \tau_y > n - k). \end{aligned} \tag{4.6.1}$$

Using Lemma 4.3.2 with $\mathbf{m} = \delta_{x'}$ and

$$F(x_1, \dots, x_k) = \psi(x_k) \mathbb{1}_{\{y' + f(x_1) + \dots + f(x_k) \in [z, z + a], \forall i \in \{1, \dots, k\}, y' + f(x_1) + \dots + f(x_i) > 0\}},$$

we have

$$E'_0 = \mathbb{E}_{\nu}^* \left(\psi(X_1^*) \frac{\mathbb{1}_{\{x'\}}(X_{k+1}^*)}{\nu(X_{k+1}^*)}; y' + f(X_k^*) + \cdots + f(X_1^*) \in [z, z + a], \right. \\ \left. \forall i \in \{1, \dots, k\}, y' + f(X_k^*) + \cdots + f(X_{k-i+1}^*) > 0 \right).$$

By the Markov property,

$$E'_0 = \mathbb{E}_{\nu}^* (\psi(X_1^*) \psi_{x'}^*(X_k^*); y' + f(X_k^*) + \cdots + f(X_1^*) \in [z, z + a], \\ \forall i \in \{1, \dots, k\}, y' + f(X_k^*) + \cdots + f(X_{k-i+1}^*) > 0).$$

where

$$\psi_{x'}^*(x^*) = \mathbb{E}_{x^*}^* \left(\frac{\mathbb{1}_{\{x'\}}(X_1^*)}{\nu(X_1^*)} \right) = \frac{\mathbf{P}^*(x^*, x')}{\nu(x')} = \frac{\mathbf{P}(x', x^*)}{\nu(x^*)} \leq \frac{1}{\inf_{x \in \mathbb{X}} \nu(x)}. \quad (4.6.2)$$

On the event $\{y' + f(X_k^*) + \cdots + f(X_1^*) \in [z, z + a]\} = \{z + a + S_k^* \in [y', y' + a]\}$, we have

$$\left\{ \forall i \in \{1, \dots, k\}, y' + f(X_k^*) + \cdots + f(X_{k-i+1}^*) > 0, y' > 0 \right\} \\ \subset \left\{ \forall i \in \{1, \dots, k-1\}, z + a - f(X_{k-i}^*) - \cdots - f(X_1^*) > 0, z + a + S_k^* > 0 \right\} \\ = \left\{ \tau_{z+a}^* > k \right\}.$$

So, for any $y' > 0$,

$$E'_0 \leq c \|\psi\|_{\infty} \mathbb{P}_{\nu}^* (z + a + S_k^* \in [y', y' + a], \tau_{z+a}^* > k).$$

Using Lemma 4.6.1 we have uniformly in $y' > 0$,

$$E'_0 \leq \frac{c \|\psi\|_{\infty}}{k} (1 + a^2) (1 + \max(z + a, 0)) \leq \frac{c \|\psi\|_{\infty}}{k} (1 + a^3) (1 + z). \quad (4.6.3)$$

Putting together (4.6.3) and (4.6.1) and using the point 2 of Proposition 4.2.2,

$$E_0 \leq \frac{c \|\psi\|_{\infty}}{k} (1 + a^3) (1 + z) \mathbb{P}_x (\tau_y > n - k) \leq \frac{c \|\psi\|_{\infty}}{k \sqrt{n - k}} (1 + a^3) (1 + z) (1 + \max(y, 0)).$$

Since $n - k \geq n/2$ and $k \geq n/4$ for any $n \geq 4$, the lemma is proved. \square

4.7 Proof of Theorem 4.2.4

The aim of this section is to bound

$$E_0 := \mathbb{E}_x (\psi(X_n); y + S_n \in [z, z + a], \tau_y > n) \quad (4.7.1)$$

uniformly in the end point z . The point is to split the time n into $n = n_1 + n_2$, where $n_2 = \lfloor \varepsilon^3 n \rfloor$ and $n_1 = n - \lfloor \varepsilon^3 n \rfloor$, and $\varepsilon \in (0, 1)$. Using the Markov property, we shall bound the process between n_1 and n by the local limit theorem (Corollary 4.5.5) and

between 1 and n_1 by the integral theorem (Proposition 4.2.3). Following this idea we write

$$E_0 = \underbrace{\mathbb{E}_x(\psi(X_n); y + S_n \in [z, z + a], \tau_y > n_1)}_{=: E_1} - \underbrace{\mathbb{E}_x(\psi(X_n); y + S_n \in [z, z + a], n_1 < \tau_y \leq n)}_{=: E_2}. \quad (4.7.2)$$

For the ease of reading the bounds of E_1 and E_2 are given in separate sections.

4.7.1 Control of E_1

Lemma 4.7.1. *Assume Hypotheses M4.1-M4.3. For any $a > 0$ and $\varepsilon \in (0, 1/4)$ there exist $c = c_a > 0$ depending only on a and $c_\varepsilon > 0$ such that for any non-negative function $\psi \in \mathcal{C}$, any $y \in \mathbb{R}$ and $n \in \mathbb{N}$, such that $\varepsilon^3 n \geq 1$ we have*

$$\sup_{x \in \mathbb{X}, z \geq 0} n \left| E_1 - \frac{a}{\sqrt{n_2} \sigma} \nu(\psi) \mathbb{E}_x \left(\varphi \left(\frac{y - z + S_{n_1}}{\sqrt{n_2} \sigma} \right); \tau_y > n_1 \right) \right| \leq c (1 + \max(y, 0)) \|\psi\|_\infty \left(\varepsilon + \frac{c_\varepsilon}{\sqrt{n}} \right).$$

where $E_1 = \mathbb{E}_x(\psi(X_n); y + S_n \in [z, z + a], \tau_y > n_1)$, $n_2 = \lfloor \varepsilon^3 n \rfloor$, $n_1 = n - \lfloor \varepsilon^3 n \rfloor$ and $\varphi(t) = e^{-t^2/2} / \sqrt{2\pi}$.

Proof. By the Markov property,

$$E_1 = \sum_{x' \in \mathbb{X}} \int_0^{+\infty} \underbrace{\mathbb{E}_{x'}(\psi(X_{n_2}); y' + S_{n_2} \in [z, z + a])}_{=: E'_1} \times \mathbb{P}_x(y + S_{n_1} \in dy', X_{n_1} = x', \tau_y > n_1). \quad (4.7.3)$$

From now on we consider that the real $a > 0$ is fixed. By Corollary 4.5.5, for any $\varepsilon^{5/2} \leq \varepsilon \in (0, 1/4)$,

$$\sqrt{n_2} \left| E'_1 - a \varphi_{\sqrt{n_2} \sigma}(z - y') \nu(\psi) \right| \leq c \|\psi\|_\infty \left(\frac{1}{\sqrt{n_2}} + \varepsilon^{5/2} + c_\varepsilon e^{-c_\varepsilon n_2} \right),$$

with c depending only on a . Consequently, using (4.7.3) and the fact that $n_2 = \lfloor \varepsilon^3 n \rfloor \geq c_\varepsilon n$,

$$\begin{aligned} & \left| E_1 - a \nu(\psi) \mathbb{E}_x \left(\varphi_{\sqrt{n_2} \sigma}(y - z + S_{n_1}); \tau_y > n_1 \right) \right| \\ & \leq \frac{c \|\psi\|_\infty}{\sqrt{n_2}} \left(\frac{c_\varepsilon}{\sqrt{n}} + \varepsilon^{5/2} + c_\varepsilon e^{-c_\varepsilon n} \right) \mathbb{P}_x(\tau_y > n_1). \end{aligned}$$

Therefore, by (4.5.1) and the point 2 of Proposition 4.2.2, we obtain that

$$\begin{aligned} & \left| E_1 - \frac{a}{\sqrt{n_2} \sigma} \nu(\psi) \mathbb{E}_x \left(\varphi \left(\frac{y - z + S_{n_1}}{\sqrt{n_2} \sigma} \right); \tau_y > n_1 \right) \right| \\ & \leq c \|\psi\|_\infty \frac{1 + \max(y, 0)}{\sqrt{n_2} \sqrt{n_1}} \left(\frac{c_\varepsilon}{\sqrt{n}} + \varepsilon^{5/2} \right). \end{aligned}$$

Since $n_2 \geq \varepsilon^3 n \left(1 - \frac{1}{\varepsilon^{3n}}\right)$ and $n_1 \geq \frac{n}{2}$, we have

$$\begin{aligned} c \|\psi\|_\infty \frac{1 + \max(y, 0)}{\sqrt{n_2} \sqrt{n_1}} \left(\frac{c_\varepsilon}{\sqrt{n}} + \varepsilon^{5/2} \right) &\leq c \|\psi\|_\infty \frac{1 + \max(y, 0)}{\varepsilon^{3/2} n} \left(1 + \frac{c_\varepsilon}{n} \right) \left(\frac{c_\varepsilon}{\sqrt{n}} + \varepsilon^{5/2} \right) \\ &\leq c \|\psi\|_\infty \frac{1 + \max(y, 0)}{n} \left(\varepsilon + \frac{c_\varepsilon}{\sqrt{n}} \right) \end{aligned}$$

and the lemma follows. \square

To find the limit behaviour of E_1 , we will develop $\frac{1}{\sqrt{n_2}} \mathbb{E}_x \left(\varphi \left(\frac{y + S_{n_1} - z}{\sqrt{n_2} \sigma} \right); \tau_y > n_1 \right)$. To this aim, we prove the following lemma which we will apply first with the standard normal density function φ , and later on with the Rayleigh density φ_+ .

Lemma 4.7.2. *Assume Hypotheses [M4.1-M4.3](#). Let $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative derivable function such that $\Psi(t) \rightarrow 0$ as $t \rightarrow +\infty$. Moreover we suppose that Ψ' is a continuous function on \mathbb{R} such that $\max(|\Psi(t)|, |\Psi'(t)|) \leq c e^{-\frac{t^2}{4}}$. There exists $\varepsilon_0 \in (0, 1/2)$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $y \in \mathbb{R}$, $m_1 \geq 1$ and $m_2 \geq 1$, we have*

$$\begin{aligned} \sup_{x \in \mathbb{X}, z \geq 0} \left| \mathbb{E}_x \left(\Psi \left(\frac{y + S_{m_1} - z}{\sqrt{m_2} \sigma} \right); \tau_y > m_1 \right) \right. \\ \left. - \frac{2V(x, y)}{\sqrt{2\pi m_1} \sigma} \int_0^{+\infty} \Psi \left(\sqrt{\frac{m_1}{m_2}} t - \frac{z}{\sqrt{m_2} \sigma} \right) \varphi_+(t) dt \right| \\ \leq c_\varepsilon \frac{(1 + \max(y, 0))^2}{m_1^\varepsilon \sqrt{m_2}} + c \frac{1 + \max(y, 0)}{\sqrt{m_1}} \left(e^{-c \frac{m_1}{m_2}} + \varepsilon^4 \right), \end{aligned}$$

where $\varphi_+(t) = t e^{-\frac{t^2}{2}}$.

Proof. Let $x \in \mathbb{X}$, $y \in \mathbb{R}$, $z \geq 0$, $m_1 \geq 1$ and $m_2 \geq 1$ and fix $\varepsilon_1 \in (0, 1)$. We consider two cases. Assume first that $z \leq \sqrt{m_1} \sigma / \varepsilon_1$. Using the regularity of the function Ψ , we note that

$$\begin{aligned} J_0 &:= \mathbb{E}_x \left(\Psi \left(\frac{y + S_{m_1} - z}{\sqrt{m_2} \sigma} \right); \tau_y > m_1 \right) \\ &= - \int_0^{+\infty} \sqrt{\frac{m_1}{m_2}} \Psi' \left(\sqrt{\frac{m_1}{m_2}} t - \frac{z}{\sqrt{m_2} \sigma} \right) \mathbb{P}_x \left(\frac{y + S_{m_1}}{\sqrt{m_1} \sigma} \leq t, \tau_y > m_1 \right) dt. \end{aligned}$$

Denote by J_1 the following integral:

$$J_1 := - \frac{2V(x, y)}{\sqrt{2\pi m_1} \sigma} \int_0^{+\infty} \sqrt{\frac{m_1}{m_2}} \Psi' \left(\sqrt{\frac{m_1}{m_2}} t - \frac{z}{\sqrt{m_2} \sigma} \right) \left(1 - e^{-\frac{t^2}{2}} \right) dt. \quad (4.7.4)$$

Using the point 2 of Proposition 4.2.3, with $t_0 = 2/\varepsilon_1$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$,

$$\begin{aligned} |J_0 - J_1| &\leq c_{\varepsilon, \varepsilon_1} \frac{(1 + \max(y, 0))^2}{m_1^{1/2+\varepsilon}} \int_0^{\frac{2}{\varepsilon_1}} \sqrt{\frac{m_1}{m_2}} \left| \Psi' \left(\sqrt{\frac{m_1}{m_2}} t - \frac{z}{\sqrt{m_2} \sigma} \right) \right| dt \\ &\quad + \left(\frac{2V(x, y)}{\sqrt{2\pi m_1} \sigma} + \mathbb{P}_x(\tau_y > m_1) \right) \int_{\frac{2}{\varepsilon_1}}^{+\infty} \sqrt{\frac{m_1}{m_2}} \left| \Psi' \left(\sqrt{\frac{m_1}{m_2}} t - \frac{z}{\sqrt{m_2} \sigma} \right) \right| dt. \end{aligned}$$

Using the point 2 of Proposition 4.2.1 and the point 2 of Proposition 4.2.2, with $\|\Psi'\|_\infty = \sup_{t \in \mathbb{R}} |\Psi'(t)|$,

$$\begin{aligned} |J_0 - J_1| &\leq c_{\varepsilon, \varepsilon_1} \frac{(1 + \max(y, 0))^2}{m_1^\varepsilon \sqrt{m_2}} \|\Psi'\|_\infty + c \frac{1 + \max(y, 0)}{\sqrt{m_1}} \sqrt{\frac{m_1}{m_2}} \int_{\frac{2}{\varepsilon_1}}^{+\infty} e^{-\frac{\left(\sqrt{\frac{m_1}{m_2}}t - \frac{z}{\sqrt{m_2}\sigma}\right)^2}{4}} dt \\ &\leq c_{\varepsilon, \varepsilon_1} \frac{(1 + \max(y, 0))^2}{m_1^\varepsilon \sqrt{m_2}} + c \frac{1 + \max(y, 0)}{\sqrt{m_1}} \int_{\sqrt{\frac{m_1}{m_2}}\left(\frac{2}{\varepsilon_1} - \frac{z}{\sqrt{m_1}\sigma}\right)}^{+\infty} e^{-\frac{s^2}{4}} ds. \end{aligned}$$

Since $z \leq \frac{\sqrt{m_1}\sigma}{\varepsilon_1}$, we have $\frac{2}{\varepsilon_1} - \frac{z}{\sqrt{m_1}\sigma} \geq \frac{1}{\varepsilon_1} \geq 1$ and so

$$|J_0 - J_1| \leq c_{\varepsilon, \varepsilon_1} \frac{(1 + \max(y, 0))^2}{m_1^\varepsilon \sqrt{m_2}} + c \frac{1 + \max(y, 0)}{\sqrt{m_1}} e^{-\frac{m_1}{8m_2}} \int_{\mathbb{R}} e^{-\frac{s^2}{8}} ds. \quad (4.7.5)$$

Moreover, by the definition of J_1 in (4.7.4), we have

$$\begin{aligned} J_1 &= \frac{2V(x, y)}{\sqrt{2\pi m_1}\sigma} \left[-\Psi \left(\sqrt{\frac{m_1}{m_2}}t - \frac{z}{\sqrt{m_2}\sigma} \right) \left(1 - e^{-\frac{t^2}{2}} \right) \right]_{t=0}^{t=+\infty} \\ &\quad + \frac{2V(x, y)}{\sqrt{2\pi m_1}\sigma} \int_0^{+\infty} \Psi \left(\sqrt{\frac{m_1}{m_2}}t - \frac{z}{\sqrt{m_2}\sigma} \right) t e^{-\frac{t^2}{2}} dt \\ &= \frac{2V(x, y)}{\sqrt{2\pi m_1}\sigma} \int_0^{+\infty} \Psi \left(\sqrt{\frac{m_1}{m_2}}t - \frac{z}{\sqrt{m_2}\sigma} \right) \varphi_+(t) dt. \end{aligned} \quad (4.7.6)$$

Now, assume that $z > \frac{\sqrt{m_1}\sigma}{\varepsilon_1}$. We write

$$\begin{aligned} J_0 &\leq c \mathbb{E}_x \left(e^{-\frac{(y+S_{m_1}-z)^2}{4m_2\sigma^2}}; y + S_{m_1} \leq \frac{\sqrt{m_1}\sigma}{2\varepsilon_1}, \tau_y > m_1 \right) \\ &\quad + \|\Psi\|_\infty \mathbb{P}_x \left(y + S_{m_1} > \frac{\sqrt{m_1}\sigma}{2\varepsilon_1}, \tau_y > m_1 \right) \\ &\leq c e^{-\frac{m_1}{16m_2\varepsilon_1^2}} \mathbb{P}_x(\tau_y > m_1) + \|\Psi\|_\infty \frac{2\varepsilon_1}{\sqrt{m_1}\sigma} \mathbb{E}_x(y + S_{m_1}; \tau_y > m_1). \end{aligned}$$

Using the points 3 and 1 of Proposition 4.2.1, we can verify that

$$\mathbb{E}_x(y + S_{m_1}; \tau_y > m_1) \leq \mathbb{E}_x(2V(y + S_{m_1}, X_{m_1}) + c; \tau_y > m_1) \leq 2V(x, y) + c.$$

So by the point 2 of Proposition 4.2.2 and the point 2 of Proposition 4.2.1,

$$J_0 \leq c \frac{1 + \max(y, 0)}{\sqrt{m_1}} e^{-\frac{cm_1}{m_2}} + \frac{c\varepsilon_1}{\sqrt{m_1}} (1 + \max(y, 0)).$$

In the same way,

$$\begin{aligned} J_1 &= \frac{2V(x, y)}{\sqrt{2\pi m_1}\sigma} \int_0^{+\infty} \Psi \left(\sqrt{\frac{m_1}{m_2}}t - \frac{z}{\sqrt{m_2}\sigma} \right) \varphi_+(t) dt \\ &\leq \frac{c(1 + \max(y, 0))}{\sqrt{m_1}} \left[\int_0^{\frac{1}{2\varepsilon_1}} e^{-\frac{m_1}{4m_2}\left(t - \frac{z}{\sqrt{m_1}\sigma}\right)^2} \varphi_+(t) dt + \|\Psi\|_\infty \int_{\frac{1}{2\varepsilon_1}}^{+\infty} t e^{-\frac{t^2}{2}} dt \right] \\ &\leq \frac{c(1 + \max(y, 0))}{\sqrt{m_1}} \left[e^{-\frac{m_1}{16m_2\varepsilon_1^2}} \int_0^{+\infty} \varphi_+(t) dt + \|\Psi\|_\infty e^{-\frac{1}{16\varepsilon_1^2}} \int_0^{+\infty} t e^{-\frac{t^2}{4}} dt \right] \\ &\leq \frac{c(1 + \max(y, 0))}{\sqrt{m_1}} \left(e^{-\frac{cm_1}{m_2}} + e^{-\frac{c}{\varepsilon_1^2}} \right). \end{aligned}$$

From the last two bounds it follows that for any $z > \frac{\sqrt{m_1}\sigma}{\varepsilon_1}$,

$$|J_0 - J_1| \leq J_0 + J_1 \leq \frac{c(1 + \max(y, 0))}{\sqrt{m_1}} \left(e^{-\frac{cm_1}{m_2}} + \varepsilon_1 \right). \quad (4.7.7)$$

Putting together (4.7.6), (4.7.7) and (4.7.5) and taking $\varepsilon_1 = \varepsilon^4$, we obtain the desired inequality for any $z \geq 0$,

$$|J_0 - J_1| \leq c_\varepsilon \frac{(1 + \max(y, 0))^2}{m_1^\varepsilon \sqrt{m_2}} + \frac{c(1 + \max(y, 0))}{\sqrt{m_1}} \left(e^{-\frac{cm_1}{m_2}} + \varepsilon^4 \right).$$

□

Lemma 4.7.3. *Assume Hypotheses M4.1-M4.3. There exists $\varepsilon_0 \in (0, 1/2)$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $y \in \mathbb{R}$, $n \in \mathbb{N}$ such that $\varepsilon^3 n \geq 1$, we have*

$$\begin{aligned} \sup_{x \in \mathbb{X}, z \geq 0} \left| \frac{n}{\sqrt{n_2}} \mathbb{E}_x \left(\varphi \left(\frac{y + S_{n_1} - z}{\sqrt{n_2}\sigma} \right); \tau_y > n_1 \right) - \frac{2V(x, y)}{\sqrt{2\pi}\sigma} \varphi_+ \left(\frac{z}{\sqrt{n_1}\sigma} \right) \right| \\ \leq c_\varepsilon \frac{(1 + \max(y, 0))^2}{n^\varepsilon} + c(1 + \max(y, 0)) \varepsilon, \end{aligned}$$

where $\varphi(t) = e^{-\frac{t^2}{2}} / \sqrt{2\pi}$, $\varphi_+(t) = t e^{-\frac{t^2}{2}} \mathbb{1}_{\{t \geq 0\}}$, $n_2 = \lfloor \varepsilon^3 n \rfloor$ and $n_1 = n - \lfloor \varepsilon^3 n \rfloor$.

Proof. Denote

$$J_0 := \mathbb{E}_x \left(\varphi \left(\frac{y + S_{n_1} - z}{\sqrt{n_2}\sigma} \right); \tau_y > n_1 \right)$$

and

$$\begin{aligned} J_1 &:= \frac{2V(x, y)}{\sqrt{2\pi n_1}\sigma} \int_0^{+\infty} \varphi \left(\sqrt{\frac{n_1}{n_2}} t - \frac{z}{\sqrt{n_2}\sigma} \right) \varphi_+(t) dt \\ &= \frac{2V(x, y)}{\sqrt{2\pi n_1}\sigma} \int_0^{+\infty} \sqrt{\frac{n_2}{n_1}} \varphi \sqrt{\frac{n_2}{n_1}} \left(t - \frac{z}{\sqrt{n_1}\sigma} \right) \varphi_+(t) dt \\ &= \frac{2V(x, y)}{\sqrt{2\pi}\sigma} \frac{\sqrt{n_2}}{n_1} \varphi \sqrt{\frac{n_2}{n_1}} * \varphi_+ \left(\frac{z}{\sqrt{n_1}\sigma} \right), \end{aligned} \quad (4.7.8)$$

where $\varphi_{\{\cdot\}}(\cdot)$ is defined in (4.5.1). By Lemma 4.7.2 we have

$$\frac{n_1}{\sqrt{n_2}} |J_0 - J_1| \leq c_\varepsilon n_1 \frac{(1 + \max(y, 0))^2}{n_1^\varepsilon n_2} + c n_1 \frac{1 + \max(y, 0)}{\sqrt{n_1}\sqrt{n_2}} \left(e^{-\frac{cn_1}{n_2}} + \varepsilon^4 \right).$$

Since $\frac{n}{2} \leq n_1 \leq n$ and $\varepsilon^3 n - 1 \leq n_2 \leq \varepsilon^3 n$,

$$\begin{aligned} \frac{n}{\sqrt{n_2}} |J_0 - J_1| &\leq c_\varepsilon \frac{(1 + \max(y, 0))^2}{n^\varepsilon} + c \frac{1 + \max(y, 0)}{\varepsilon^{3/2}} \left(1 + \frac{c_\varepsilon}{n} \right) \left(e^{-\frac{c}{\varepsilon^3}} + \varepsilon^4 \right) \\ &\leq c_\varepsilon \frac{(1 + \max(y, 0))^2}{n^\varepsilon} + c(1 + \max(y, 0)) \varepsilon. \end{aligned} \quad (4.7.9)$$

Let J_2 be the following term:

$$J_2 := \frac{2V(x, y)}{\sqrt{2\pi}\sigma} \frac{\sqrt{n_2}}{n_1} \varphi_+ \left(\frac{z}{\sqrt{n_1}\sigma} \right). \quad (4.7.10)$$

Using (4.7.8),

$$|J_1 - J_2| \leq \frac{2V(x, y)}{\sqrt{2\pi}\sigma} \frac{\sqrt{n_2}}{n_1} \int_{\mathbb{R}} \varphi \sqrt{\frac{n_2}{n_1}}(t) \left| \varphi_+ \left(\frac{z}{\sqrt{n_1}\sigma} - t \right) - \varphi_+ \left(\frac{z}{\sqrt{n_1}\sigma} \right) \right| dt.$$

By the point 2 of Proposition 4.2.1, we write

$$\begin{aligned} \frac{n}{\sqrt{n_2}} |J_1 - J_2| &\leq c(1 + \max(y, 0)) \|\varphi'_+\|_{\infty} \int_{\mathbb{R}} \varphi \sqrt{\frac{n_2}{n_1}}(t) |t| dt \\ &\leq c(1 + \max(y, 0)) \sqrt{\frac{n_2}{n_1}} \int_{\mathbb{R}} \varphi(s) |s| ds \\ &\leq c(1 + \max(y, 0)) \varepsilon^{3/2}. \end{aligned} \quad (4.7.11)$$

Putting together (4.7.9) and (4.7.11), we obtain that

$$\sup_{x \in \mathbb{X}, z \geq 0} \frac{n}{\sqrt{n_2}} |J_0 - J_2| \leq c_{\varepsilon} \frac{(1 + \max(y, 0))^2}{n^{\varepsilon}} + c(1 + \max(y, 0)) \varepsilon. \quad (4.7.12)$$

It remains to link J_2 from (4.7.10) to the desired equivalent. We distinguish two cases. If $\frac{z}{\sigma} \leq \frac{\sqrt{n}}{\varepsilon}$,

$$\begin{aligned} \left| \frac{n}{\sqrt{n_2}} J_2 - \frac{2V(x, y)}{\sqrt{2\pi}\sigma} \varphi_+ \left(\frac{z}{\sqrt{n}\sigma} \right) \right| &\leq cV(x, y) \left| \frac{n}{n_1} \varphi_+ \left(\frac{z}{\sqrt{n_1}\sigma} \right) - \varphi_+ \left(\frac{z}{\sqrt{n}\sigma} \right) \right| \\ &\leq cV(x, y) \left(\|\varphi_+\|_{\infty} \left| \frac{n}{n_1} - 1 \right| + \left| \frac{1}{\sqrt{n_1}} - \frac{1}{\sqrt{n}} \right| \left| \frac{z}{\sigma} \right| \|\varphi'_+\|_{\infty} \right) \\ &\leq cV(x, y) \left(\frac{n_2}{n_1} + \frac{1}{\sqrt{n_1}} \left| 1 - \sqrt{1 - \frac{n_2}{n}} \right| \frac{\sqrt{n}}{\varepsilon} \right) \\ &\leq cV(x, y) \left(\varepsilon^3 + \frac{\varepsilon^3}{\varepsilon} \right). \end{aligned}$$

If $\frac{z}{\sigma} > \frac{\sqrt{n}}{\varepsilon} \geq \frac{\sqrt{n_1}}{\varepsilon}$, we have

$$\left| \frac{n}{\sqrt{n_2}} J_2 - \frac{2V(x, y)}{\sqrt{2\pi}\sigma} \varphi_+ \left(\frac{z}{\sqrt{n}\sigma} \right) \right| \leq cV(x, y) \sup_{u \geq \frac{1}{\varepsilon}} \varphi_+(u) \leq cV(x, y) e^{-\frac{c}{\varepsilon^2}}.$$

Therefore, using the point 2 of Proposition 4.2.1, we obtain that in each case

$$\left| \frac{n}{\sqrt{n_2}} J_2 - \frac{2V(x, y)}{\sqrt{2\pi}\sigma} \varphi_+ \left(\frac{z}{\sqrt{n}\sigma} \right) \right| \leq c(1 + \max(y, 0)) \varepsilon^2. \quad (4.7.13)$$

Putting together (4.7.12) and (4.7.13), proves the lemma. \square

Another consequence of Lemma 4.7.2 is the following lemma which will be used in Section 4.8.

Lemma 4.7.4. *Assume Hypotheses M4.1-M4.3. There exists $\varepsilon_0 \in (0, 1/2)$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $y \in \mathbb{R}$, $n \in \mathbb{N}$ such that $\varepsilon^3 n \geq 2$, we have*

$$\begin{aligned} \sup_{x \in \mathbb{X}} \left| \frac{n^{3/2}}{n_2 - 1} \mathbb{E}_x \left(\varphi_+ \left(\frac{y + S_{n_1}}{\sqrt{n_2 - 1}\sigma} \right); \tau_y > n_1 \right) - \frac{V(x, y)}{\sigma} \right| \\ \leq c_{\varepsilon} \frac{(1 + \max(y, 0))^2}{n^{\varepsilon}} + c(1 + \max(y, 0)) \varepsilon, \end{aligned}$$

where $\varphi_+(t) = t e^{-\frac{t^2}{2}} \mathbb{1}_{\{t \geq 0\}}$ is the Rayleigh density function, $n_1 = n - \lfloor \varepsilon^3 n \rfloor$ and $n_2 = \lfloor \varepsilon^3 n \rfloor$.

Proof. Using Lemma 4.7.2 with $\Psi = \varphi_+$, $m_1 = n_1$, $m_2 = n_2 - 1$ and $z = 0$,

$$\begin{aligned} \frac{n^{3/2}}{n_2 - 1} |J_0 - J_1| &\leq c_\varepsilon \frac{(1 + \max(y, 0))^2 n^{3/2}}{(n_2 - 1)^{3/2} n_1^\varepsilon} + c \frac{(1 + \max(y, 0)) n^{3/2}}{(n_2 - 1) \sqrt{n_1}} \left(e^{-c \frac{n_1}{(n_2 - 1)}} + \varepsilon^4 \right) \\ &\leq c_\varepsilon \frac{(1 + \max(y, 0))^2}{n^\varepsilon} + c \frac{(1 + \max(y, 0))}{\varepsilon^3} \left(1 + \frac{c_\varepsilon}{n} \right) \left(e^{-\frac{c}{\varepsilon^3}} + \varepsilon^4 \right) \\ &\leq c_\varepsilon \frac{(1 + \max(y, 0))^2}{n^\varepsilon} + c (1 + \max(y, 0)) \varepsilon, \end{aligned} \tag{4.7.14}$$

where

$$J_0 := \mathbb{E}_x \left(\varphi_+ \left(\frac{y + S_{n_1}}{\sqrt{n_2 - 1} \sigma} \right); \tau_y > n_1 \right)$$

and

$$\begin{aligned} \frac{n^{3/2}}{n_2 - 1} J_1 &:= \frac{n^{3/2}}{n_2 - 1} \frac{2V(x, y)}{\sqrt{2\pi n_1} \sigma} \int_0^{+\infty} \varphi_+ \left(\sqrt{\frac{n_1}{n_2 - 1}} t \right) \varphi_+(t) dt \\ &= \frac{n^{3/2}}{n_2 - 1} \frac{2V(x, y)}{\sqrt{2\pi n_1} \sigma} \sqrt{\frac{n_1}{n_2 - 1}} \int_0^{+\infty} t^2 e^{-\frac{(\frac{n_1}{n_2 - 1} + 1)t^2}{2}} dt \\ &= \frac{n^{3/2}}{(n_2 - 1)^{3/2}} \frac{2V(x, y)}{\sqrt{2\pi} \sigma} \int_0^{+\infty} t^2 \sqrt{\frac{2\pi(n_2 - 1)}{n - 1}} \varphi_{\sqrt{\frac{n_2 - 1}{n - 1}}}(t) dt \end{aligned}$$

where $\varphi_{\{\cdot\}}(\cdot)$ is defined in (4.5.1). So,

$$\begin{aligned} \frac{n^{3/2}}{n_2 - 1} J_1 &= \frac{n^{3/2}}{\sqrt{n - 1} (n_2 - 1)} \frac{2V(x, y)}{\sigma} \frac{n_2 - 1}{2(n - 1)} \\ &= \frac{n^{3/2}}{(n - 1)^{3/2}} \frac{V(x, y)}{\sigma}. \end{aligned}$$

By the point 2 of Proposition 4.2.1,

$$\left| \frac{n^{3/2}}{n_2 - 1} J_1 - \frac{V(x, y)}{\sigma} \right| \leq \frac{c}{n} (1 + \max(y, 0)). \tag{4.7.15}$$

The lemma follows from (4.7.14) and (4.7.15). \square

Thanks to Lemmata 4.7.1 and 4.7.3 we can bound E_1 from (4.7.2) as follows.

Lemma 4.7.5. *Assume Hypotheses M4.1-M4.3. For any $a > 0$ there exists $\varepsilon_0 \in (0, 1/4)$ such that for any $\varepsilon \in (0, \varepsilon_0)$, any non-negative function $\psi \in \mathcal{C}$, any $y \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $\varepsilon^3 n \geq 1$, we have*

$$\begin{aligned} \sup_{x \in \mathbb{X}, z \geq 0} n \left| E_1 - \frac{2a\nu(\psi) V(x, y)}{\sqrt{2\pi} \sigma^2} \varphi_+ \left(\frac{z}{\sqrt{n} \sigma} \right) \right| \\ \leq c (1 + \max(y, 0)) \|\psi\|_\infty \left(\varepsilon + \frac{c_\varepsilon (1 + \max(y, 0))}{n^\varepsilon} \right), \end{aligned}$$

where $E_1 = \mathbb{E}_x(\psi(X_n); y + S_n \in [z, z + a], \tau_y > n_1)$, $n_1 = n - \lfloor \varepsilon^3 n \rfloor$ and φ_+ is the Rayleigh density function: $\varphi_+(t) = t e^{-\frac{t^2}{2}} \mathbb{1}_{\{t \geq 0\}}$.

Proof. From Lemmas 4.7.1 and 4.7.3, it follows that

$$\begin{aligned}
& n \left| E_1 - \frac{2a\nu(\psi)V(x,y)}{\sqrt{2\pi}\sigma^2} \varphi_+ \left(\frac{z}{\sqrt{n}\sigma} \right) \right| \\
& \leq c(1 + \max(y, 0)) \|\psi\|_\infty \left(\varepsilon + \frac{c_\varepsilon}{\sqrt{n}} \right) \\
& \quad + \left| \frac{a\nu(\psi)}{\sigma} \right| \left(c_\varepsilon \frac{(1 + \max(y, 0))^2}{n^\varepsilon} + c(1 + \max(y, 0)) \varepsilon \right) \\
& \leq c(1 + \max(y, 0)) \|\psi\|_\infty \left(\varepsilon + \frac{c_\varepsilon(1 + \max(y, 0))}{n^\varepsilon} \right).
\end{aligned}$$

□

4.7.2 Control of E_2

In this section we bound the term E_2 defined by (4.7.2). To this aim let us recall and introduce some notations: for any $\varepsilon \in (0, 1)$, we consider $n_2 = \lfloor \varepsilon^3 n \rfloor$, $n_1 = n - n_2 = n - \lfloor \varepsilon^3 n \rfloor$, $n_3 = \lfloor \frac{n_2}{2} \rfloor$ and $n_4 = n_2 - n_3$. We define also

$$E_{21} := \mathbb{E}_x \left(\psi(X_n) ; y + S_n \in [z, z + a], y + S_{n_1} \leq \varepsilon\sqrt{n}, n_1 < \tau_y \leq n \right) \quad (4.7.16)$$

$$E_{22} := \mathbb{E}_x \left(\psi(X_n) ; y + S_n \in [z, z + a], y + S_{n_1} > \varepsilon\sqrt{n}, n_1 < \tau_y \leq n_1 + n_3 \right) \quad (4.7.17)$$

$$E_{23} := \mathbb{E}_x \left(\psi(X_n) ; y + S_n \in [z, z + a], y + S_{n_1} > \varepsilon\sqrt{n}, n_1 + n_3 < \tau_y \leq n \right) \quad (4.7.18)$$

and we note that

$$E_2 = E_{21} + E_{22} + E_{23}. \quad (4.7.19)$$

Lemma 4.7.6. *Assume Hypotheses M4.1-M4.3. For any $a > 0$ there exists $\varepsilon_0 \in (0, 1/4)$ such that for any $\varepsilon \in (0, \varepsilon_0)$, any non-negative function $\psi \in \mathcal{C}$, any $y \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $\varepsilon^3 n \geq 1$, we have*

$$\sup_{x \in \mathbb{X}, z \geq 0} nE_{21} \leq c \|\psi\|_\infty (1 + \max(y, 0)) \left(\sqrt{\varepsilon} + \frac{c_\varepsilon(1 + \max(y, 0))}{n^\varepsilon} \right)$$

where E_{21} is given as in (4.7.16) by

$$E_{21} = \mathbb{E}_x \left(\psi(X_n) ; y + S_n \in [z, z + a], y + S_{n_1} \leq \varepsilon\sqrt{n}, n_1 < \tau_y \leq n \right)$$

and $n_1 = n - \lfloor \varepsilon^3 n \rfloor$.

Proof. Using the Markov property and the uniform bound (4.5.14) of Corollary 4.5.5, with $n_2 = \lfloor \varepsilon^3 n \rfloor$,

$$\begin{aligned}
E_{21} &= \sum_{x' \in \mathbb{X}} \int_0^{+\infty} \mathbb{E}_{x'} \left(\psi(X_{n_2}) ; y' + S_{n_2} \in [z, z + a], \tau_{y'} \leq n_2 \right) \\
& \quad \times \mathbb{P}_x \left(X_{n_1} = x', y + S_{n_1} \in dy', y + S_{n_1} \leq \varepsilon\sqrt{n}, \tau_y > n_1 \right) \\
&\leq \frac{c \|\psi\|_\infty}{\sqrt{n_2}} \mathbb{P}_x \left(y + S_{n_1} \leq \varepsilon\sqrt{n}, \tau_y > n_1 \right).
\end{aligned}$$

We note that $\frac{\varepsilon\sqrt{n}}{\sigma\sqrt{n_1}} \leq \frac{\varepsilon}{\sigma\sqrt{1-\varepsilon^3}} \leq \frac{2}{\sigma}\varepsilon$ and so by the point 2 of Proposition 4.2.3 with $t_0 = 2\varepsilon/\sigma$:

$$E_{21} \leq \frac{c \|\psi\|_\infty}{\sqrt{n_2}} \left(\frac{cV(x, y)}{\sqrt{n_1}} \Phi^+ \left(\frac{\varepsilon\sqrt{n}}{\sigma\sqrt{n_1}} \right) + \frac{c_\varepsilon (1 + \max(y, 0)^2)}{n_1^{1/2+\varepsilon}} \right).$$

Using the point 2 of Proposition 4.2.1 and taking into account that $n_2 \geq \varepsilon^3 n \left(1 - \frac{c_\varepsilon}{n}\right)$, $n_1 \geq n/2$ and that $\Phi^+(t) \leq \Phi^+(t_0) \leq \frac{t_0^2}{2}$ for any $t \in (0, t_0)$,

$$\begin{aligned} nE_{21} &\leq \frac{c \|\psi\|_\infty}{\varepsilon^{3/2}} \left(1 + \frac{c_\varepsilon}{n}\right) (1 + \max(y, 0)) \left(\varepsilon^2 + \frac{c_\varepsilon (1 + \max(y, 0))}{n^\varepsilon}\right) \\ &\leq c \|\psi\|_\infty (1 + \max(y, 0)) \left(\sqrt{\varepsilon} + \frac{c_\varepsilon (1 + \max(y, 0))}{n^\varepsilon}\right), \end{aligned}$$

which implies the assertion of the lemma. □

Lemma 4.7.7. *Assume Hypotheses M4.1-M4.3. For any $a > 0$ there exists $\varepsilon_0 \in (0, 1/4)$ such that for any $\varepsilon \in (0, \varepsilon_0)$, any non-negative function $\psi \in \mathcal{C}$, any $y \in \mathbb{R}$, and $n \in \mathbb{N}$ satisfying $\varepsilon^3 n \geq 2$, we have*

$$\sup_{x \in \mathbb{X}, z \geq 0} nE_{22} \leq c \|\psi\|_\infty (1 + \max(y, 0)) \left(e^{-\frac{c}{\varepsilon}} + \frac{c_\varepsilon}{n^\varepsilon} \right),$$

where E_{22} is given as in (4.7.17) by

$$E_{22} = \mathbb{E}_x \left(\psi(X_n) ; y + S_n \in [z, z + a], y + S_{n_1} > \varepsilon\sqrt{n}, n_1 < \tau_y \leq n_1 + n_3 \right)$$

and $n_1 = n - \lfloor \varepsilon^3 n \rfloor$, $n_2 = \lfloor \varepsilon^3 n \rfloor$ and $n_3 = \lfloor \frac{n_2}{2} \rfloor$.

Proof. By the Markov property,

$$\begin{aligned} E_{22} &= \sum_{x' \in \mathbb{X}} \int_0^{+\infty} \underbrace{\mathbb{E}_{x'}(\psi(X_{n_2}) ; y' + S_{n_2} \in [z, z + a], \tau_{y'} \leq n_3)}_{E'_{22}} \\ &\quad \times \mathbb{P}_x(X_{n_1} = x', y + S_{n_1} \in dy', y + S_{n_1} > \varepsilon\sqrt{n}, \tau_y > n_1). \end{aligned} \tag{4.7.20}$$

Bound of E'_{22} . By the Markov property and the uniform bound (4.5.14) in Corollary 4.5.5, with $n_4 = n_2 - n_3 = n - n_1 - n_3$,

$$\begin{aligned} E'_{22} &= \sum_{x'' \in \mathbb{X}} \int_{\mathbb{R}} \mathbb{E}_{x''}(\psi(X_{n_4}) ; y'' + S_{n_4} \in [z, z + a]) \\ &\quad \times \mathbb{P}_{x'}(X_{n_3} = x'', y' + S_{n_3} \in dy'', \tau_{y'} \leq n_3) \\ &\leq \frac{c \|\psi\|_\infty}{\sqrt{n_4}} \mathbb{P}_{x'}(\tau_{y'} \leq n_3). \end{aligned}$$

Let $(B_t)_{t \geq 0}$ be the Brownian motion defined by Proposition 4.10.4. Denote by A_n the following event:

$$A_n = \left\{ \sup_{t \in [0, 1]} |S_{\lfloor tn \rfloor} - \sigma B_{tn}| \leq n^{1/2-\varepsilon} \right\},$$

and by \overline{A}_n its complement. We have

$$E'_{22} \leq \frac{c \|\psi\|_\infty}{\sqrt{n_4}} \left[\mathbb{P}_{x'}(\tau_{y'} \leq n_3, A_{n_3}) + \mathbb{P}_{x'}(\tau_{y'} \leq n_3, \overline{A}_{n_3}) \right]. \quad (4.7.21)$$

Note that for any $x' \in \mathbb{X}$ and any $y' > \varepsilon\sqrt{n}$,

$$\mathbb{P}_{x'}(\tau_{y'} \leq n_3, A_{n_3}) \leq \mathbb{P}\left(\tau_{y'-n_3}^{bm, 1/2-\varepsilon} \leq n_3\right),$$

where, for any $y'' > 0$, $\tau_{y''}^{bm}$ is the exit time of the Brownian motion starting at y'' defined by (4.10.7). Since $y' > \varepsilon\sqrt{n}$, it implies that

$$\begin{aligned} \mathbb{P}_{x'}(\tau_{y'} \leq n_3, A_{n_3}) &\leq \mathbb{P}\left(\inf_{t \in [0,1]} \sigma B_{tn_3} \leq n_3^{1/2-\varepsilon} - y'\right) \\ &\leq \mathbb{P}\left(\inf_{t \in [0,1]} \sigma B_{tn_3} \leq \left(\frac{\varepsilon^3 n}{2}\right)^{1/2-\varepsilon} - \varepsilon\sqrt{n}\right) \\ &\leq \mathbb{P}\left(\inf_{t \in [0,1]} \sigma B_{tn_3} \leq -\varepsilon\sqrt{n} \left(1 - \frac{\varepsilon^{1/2-3\varepsilon}}{n^\varepsilon}\right)\right). \end{aligned}$$

Since $\sqrt{n}/\sqrt{n_3} \geq \sqrt{2}/\varepsilon^{3/2}$,

$$\begin{aligned} \mathbb{P}_{x'}(\tau_{y'} \leq n_3, A_{n_3}) &\leq \mathbb{P}\left(\left|\frac{B_{n_3}}{\sqrt{n_3}}\right| \geq \frac{\varepsilon\sqrt{n}}{\sigma\sqrt{n_3}} \left(1 - \frac{1}{n^\varepsilon}\right)\right) \\ &\leq \mathbb{P}\left(|B_1| \geq \frac{\sqrt{2}}{\sigma\sqrt{\varepsilon}} \left(1 - \frac{1}{n^\varepsilon}\right)\right) \\ &\leq c e^{-\frac{\varepsilon}{\varepsilon} \left(1 - \frac{c}{n^\varepsilon}\right)}. \end{aligned} \quad (4.7.22)$$

Therefore, putting together (4.7.21) and (4.7.22) and using Proposition 4.10.4,

$$E'_{22} \leq \frac{c \|\psi\|_\infty}{\sqrt{n_4}} \left(c e^{-\frac{\varepsilon}{\varepsilon} \left(1 - \frac{c}{n^\varepsilon}\right)} + \mathbb{P}_{x'}(\overline{A}_{n_3}) \right) \leq \frac{c \|\psi\|_\infty}{\sqrt{n_4}} \left(e^{-\frac{\varepsilon}{\varepsilon} \left(1 - \frac{c}{n^\varepsilon}\right)} + \frac{c_\varepsilon}{n_3^\varepsilon} \right).$$

Since $n_4 \geq n_2/2 \geq \frac{\varepsilon^3 n}{2} \left(1 - \frac{c_\varepsilon}{n}\right)$ and $n_3 \geq n_2/2 - 1 \geq \frac{\varepsilon^3 n}{2} \left(1 - \frac{c_\varepsilon}{n}\right)$, we have

$$E'_{22} \leq \frac{c \|\psi\|_\infty}{\varepsilon^{3/2} \sqrt{n}} \left(1 + \frac{c_\varepsilon}{n}\right) \left(e^{-\frac{\varepsilon}{\varepsilon}} e^{\frac{c_\varepsilon}{n^\varepsilon}} + \frac{c_\varepsilon}{n^\varepsilon} \right) \leq \frac{c \|\psi\|_\infty}{\sqrt{n}} \left(e^{-\frac{\varepsilon}{\varepsilon}} + \frac{c_\varepsilon}{n^\varepsilon} \right). \quad (4.7.23)$$

Inserting (4.7.23) in (4.7.20) and using the point 2 of Proposition 4.2.2 and the fact that $n_1 \geq n/2$, we conclude that

$$E_{22} \leq \frac{c \|\psi\|_\infty (1 + \max(y, 0))}{n} \left(e^{-\frac{\varepsilon}{\varepsilon}} + \frac{c_\varepsilon}{n^\varepsilon} \right).$$

□

Lemma 4.7.8. *Assume Hypotheses M4.1-M4.3. For any $a > 0$ there exists $\varepsilon_0 \in (0, 1/4)$ such that for any $\varepsilon \in (0, \varepsilon_0)$, any non-negative function $\psi \in \mathcal{C}$, any $y \in \mathbb{R}$, and $n \in \mathbb{N}$ such that $\varepsilon^3 n \geq 3$, we have*

$$\sup_{x \in \mathbb{X}, z \geq 0} n E_{23} \leq c \|\psi\|_\infty (1 + \max(y, 0)) \left(\varepsilon + \frac{c_\varepsilon}{n^\varepsilon} \right),$$

where E_{23} is given as in (4.7.18) by

$$E_{23} = \mathbb{E}_x \left(\psi(X_n) ; y + S_n \in [z, z + a], y + S_{n_1} > \varepsilon\sqrt{n}, n_1 + n_3 < \tau_y \leq n \right)$$

and $n_1 = n - \lfloor \varepsilon^3 n \rfloor$, $n_2 = \lfloor \varepsilon^3 n \rfloor$ and $n_3 = \lfloor \frac{n_2}{2} \rfloor$.

Proof. By the Markov property,

$$E_{23} \leq \sum_{x' \in \mathbb{X}} \int_0^{+\infty} \underbrace{\mathbb{E}_{x'}(\psi(X_{n_2}) ; y' + S_{n_2} \in [z, z + a], n_3 < \tau_{y'} \leq n_2)}_{=: E'_{23}} \mathbb{P}_x(X_{n_1} = x', y + S_{n_1} \in dy', y + S_{n_1} > \varepsilon\sqrt{n}, \tau_y > n_1). \quad (4.7.24)$$

We consider two cases: when $z \leq \frac{\varepsilon\sqrt{n}}{2}$ and when $z > \frac{\varepsilon\sqrt{n}}{2}$.

Fix first $0 \leq z \leq \frac{\varepsilon\sqrt{n}}{2}$. Using Corollary 4.5.5, we have for any $y' > \varepsilon\sqrt{n}$,

$$\begin{aligned} E'_{23} &\leq \mathbb{E}_{x'}(\psi(X_{n_2}) ; y' + S_{n_2} \in [z, z + a]) \\ &\leq \frac{a\nu(\psi)}{\sqrt{2\pi n_2}\sigma} e^{-\frac{(z-y')^2}{2n_2\sigma^2}} + \frac{c\|\psi\|_\infty}{\sqrt{n_2}} \left(\frac{1}{\sqrt{n_2}} + \varepsilon^{5/2} + c_\varepsilon e^{-c_\varepsilon n_2} \right) \\ &\leq \frac{c\|\psi\|_\infty}{\varepsilon^{3/2}\sqrt{n}} \left(1 + \frac{c_\varepsilon}{n} \right) \left(e^{-\frac{\varepsilon^2 n}{8n_2\sigma^2}} + \frac{c_\varepsilon}{\sqrt{n}} + \varepsilon^{5/2} + c_\varepsilon e^{-c_\varepsilon n} \right) \\ &\leq \frac{c\|\psi\|_\infty}{\varepsilon^{3/2}\sqrt{n}} \left(1 + \frac{c_\varepsilon}{n} \right) \left(e^{-\frac{\varepsilon}{8}} + \frac{c_\varepsilon}{\sqrt{n}} + \varepsilon^{5/2} \right). \end{aligned}$$

So, when $0 \leq z \leq \frac{\varepsilon\sqrt{n}}{2}$, we have

$$E'_{23} \leq \frac{c\|\psi\|_\infty}{\sqrt{n}} \left(\frac{c_\varepsilon}{\sqrt{n}} + \varepsilon \right). \quad (4.7.25)$$

Now we consider that $z > \frac{\varepsilon\sqrt{n}}{2}$. Using Lemma 4.3.2 with $\mathbf{m} = \delta_{x'}$ and

$$\begin{aligned} F(x_1, \dots, x_{n_2}) &= \psi(x_{n_2}) \mathbb{1}_{\{y' + f(x_1) + \dots + f(x_{n_2}) \in [z, z + a], \exists k \in \{n_3 + 1, \dots, n_2 - 1\}, y' + f(x_1) + \dots + f(x_k) \leq 0\}}, \end{aligned}$$

we obtain

$$\begin{aligned} E'_{23} &:= \mathbb{E}_{x'}(\psi(X_{n_2}) ; y' + S_{n_2} \in [z, z + a], n_3 < \tau_{y'} \leq n_2) \\ &\leq \mathbb{E}_\nu^* \left(\psi(X_1^*) \frac{\mathbb{1}_{\{x'\}}(X_{n_2+1}^*)}{\nu(X_{n_2+1}^*)} ; y' + f(X_{n_2}^*) + \dots + f(X_1^*) \in [z, z + a], \right. \\ &\quad \left. \exists k \in \{n_3 + 1, \dots, n_2 - 1\}, y' + f(X_{n_2}^*) + \dots + f(X_{n_2-k+1}^*) \leq 0 \right). \end{aligned}$$

By the Markov property,

$$\begin{aligned} E'_{23} &\leq \|\psi\|_\infty \mathbb{E}_\nu^* \left(\psi_{x'}^*(X_{n_2}^*) ; y' + f(X_{n_2}^*) + \dots + f(X_1^*) \in [z, z + a], \right. \\ &\quad \left. \exists k \in \{n_3 + 1, \dots, n_2 - 1\}, y' + f(X_{n_2}^*) + \dots + f(X_{n_2-k+1}^*) \leq 0 \right). \end{aligned}$$

where $\psi_{x'}^*$ is a function defined on \mathbb{X} by the equation (4.6.2). We note that, on the event $\{y' + f(X_{n_2}^*) + \dots + f(X_1^*) \in [z, z + a]\} = \{z + S_{n_2}^* \in [y' - a, y']\}$, we have

$$\begin{aligned} & \{\exists k \in \{n_3 + 1, \dots, n_2 - 1\}, y' + f(X_{n_2}^*) + \dots + f(X_{n_2-k+1}^*) \leq 0\} \\ & \subset \{\exists k \in \{n_3 + 1, \dots, n_2 - 1\}, z - f(X_{n_2-k}^*) - \dots - f(X_1^*) \leq 0\} \\ & = \{\tau_z^* \leq n_2 - n_3 - 1\}. \end{aligned}$$

Consequently,

$$E'_{23} \leq c \|\psi\|_\infty \mathbb{P}_\nu^* \left(z + S_{n_2}^* \in [y' - a, y'], \tau_z^* \leq n_4 - 1 \right),$$

with $n_4 = n_2 - n_3 = \lfloor \varepsilon^3 n \rfloor - \lfloor \frac{\varepsilon^3 n}{2} \rfloor \geq \frac{\varepsilon^3 n}{2} \left(1 - \frac{c_\varepsilon}{n}\right)$. Proceeding in the same way as for the term E'_{22} in (4.7.23) and using the fact that z is larger than $c\varepsilon\sqrt{n}$, we have

$$E'_{23} \leq \frac{c \|\psi\|_\infty}{\sqrt{n}} \left(e^{-\frac{c}{\varepsilon}} + \frac{c_\varepsilon}{n^\varepsilon} \right). \quad (4.7.26)$$

Putting together (4.7.25) and (4.7.26), for any $z \geq 0$, we obtain

$$E'_{23} \leq \frac{c \|\psi\|_\infty}{\sqrt{n}} \left(\varepsilon + \frac{c_\varepsilon}{n^\varepsilon} \right).$$

Inserting this bound in (4.7.24) and using the point 2 of Proposition 4.2.2, we conclude that

$$E_{23} \leq \frac{c \|\psi\|_\infty (1 + \max(y, 0))}{n} \left(\varepsilon + \frac{c_\varepsilon}{n^\varepsilon} \right).$$

□

Putting together Lemmas 4.7.6, 4.7.7 and 4.7.8, by (4.7.19), we obtain the following bound for E_2 :

Lemma 4.7.9. *Assume Hypotheses **M4.1-M4.3**. For any $a > 0$ there exists $\varepsilon_0 \in (0, 1/4)$ such that for any $\varepsilon \in (0, \varepsilon_0)$, any non-negative function $\psi \in \mathcal{C}$, any $y \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $\varepsilon^3 n \geq 3$, we have*

$$\sup_{x \in \mathbb{X}, z \geq 0} nE_2 \leq c \|\psi\|_\infty (1 + \max(y, 0)) \left(\sqrt{\varepsilon} + \frac{c_\varepsilon (1 + \max(y, 0))}{n^\varepsilon} \right),$$

where E_2 is given as in (4.7.2) by

$$E_2 = \mathbb{E}_x (\psi(X_n) ; y + S_n \in [z, z + a], n_1 < \tau_y \leq n)$$

and $n_1 = n - \lfloor \varepsilon^3 n \rfloor$.

4.7.3 Proof of Theorem 4.2.4

By (4.7.1) and (4.7.2),

$$\mathbb{E}_x (\psi(X_n) ; y + S_n \in [z, z + a], \tau_y > n) = E_1 + E_2.$$

Lemma 4.7.5 estimates E_1 and Lemma 4.7.9 bounds E_2 . Taking into account these two lemmas, Theorem 4.2.4 follows.

4.8 Proof of Theorem 4.2.5

4.8.1 Preliminary results

Lemma 4.8.1. *Assume Hypotheses [M4.1-M4.3](#). For any $a > 0$ and $p \in \mathbb{N}^*$, there exists $\varepsilon_0 \in (0, 1/4)$ such that for any $\varepsilon \in (0, \varepsilon_0)$ there exists $n_0(\varepsilon) \geq 1$ such that any non-negative function $\psi \in \mathcal{C}$, any $y' > 0$, $z \geq 0$, $k \in \{0, \dots, p-1\}$ and $n \geq n_0(\varepsilon)$, we have*

$$\begin{aligned} \sup_{x' \in \mathbb{X}} E'_k &\leq \frac{2a}{\sqrt{2\pi p}(n_2-1)\sigma^2} \varphi_+ \left(\frac{y'}{\sigma\sqrt{n_2-1}} \right) \\ &\quad \times \mathbb{E}_\nu^* \left(\psi(X_1^*) V^* \left(X_1^*, z_k + \frac{a}{p} + S_1^* \right); \tau_{z_k + \frac{a}{p}}^* > 1 \right) \\ &\quad + \frac{c \|\psi\|_\infty (1+z)}{n} \left(\varepsilon + \frac{c_\varepsilon (1+z)}{n^{\varepsilon^8}} \right) \end{aligned}$$

and

$$\begin{aligned} \inf_{x' \in \mathbb{X}} E'_k &\geq \frac{2a}{\sqrt{2\pi p}(n_2-1)\sigma^2} \varphi_+ \left(\frac{y'}{\sigma\sqrt{n_2-1}} \right) \mathbb{E}_\nu^* \left(\psi(X_1^*) V^* (X_1^*, z_k + S_1^*); \tau_{z_k}^* > 1 \right) \\ &\quad - \frac{c \|\psi\|_\infty (1+z)}{n} \left(\varepsilon + \frac{c_\varepsilon (1+z)}{n^{\varepsilon^8}} \right) \end{aligned}$$

where $E'_k = \mathbb{E}_{x'} \left(\psi(X_{n_2}); y' + S_{n_2} \in \left(z_k, z_k + \frac{a}{p} \right], \tau_{y'} > n_2 \right)$, $z_k = z + \frac{ka}{p}$ and $n_2 = \lfloor \varepsilon^3 n \rfloor$.

Proof. Using Lemma [4.3.2](#) with $\mathbf{m} = \delta_{x'}$ and

$$F(x_1, \dots, x_{n_2}) = \psi(x_{n_2}) \mathbb{1}_{\{y' + f(x_1) + \dots + f(x_{n_2}) \in (z_k, z_k + \frac{a}{p}], \forall i \in \{1, \dots, n_2\}, y' + f(x_1) + \dots + f(x_i) > 0\}},$$

we have

$$\begin{aligned} E'_k &= \mathbb{E}_\nu^* \left(\psi(X_1^*) \psi_{x'}^*(X_{n_2}^*); y' + f(X_{n_2}^*) + \dots + f(X_1^*) \in \left(z_k, z_k + \frac{a}{p} \right], \right. \\ &\quad \left. \forall i \in \{1, \dots, n_2\}, y' + f(X_{n_2}^*) + \dots + f(X_{n_2-i+1}^*) > 0 \right). \end{aligned}$$

where $\psi_{x'}^*$ is the function defined on \mathbb{X} by [\(4.6.2\)](#).

The upper bound. Note that, on the event

$$\left\{ y' + f(X_{n_2}^*) + \dots + f(X_1^*) \in \left(z_k, z_k + \frac{a}{p} \right] \right\} = \left\{ z_k + \frac{a}{p} + S_{n_2}^* \in \left[y', y' + \frac{a}{p} \right) \right\},$$

we have

$$\begin{aligned} &\left\{ \forall i \in \{1, \dots, n_2\}, y' + f(X_{n_2}^*) + \dots + f(X_{n_2-i+1}^*) > 0, y' > 0 \right\} \\ &\subset \left\{ \forall i \in \{1, \dots, n_2-1\}, z_k + \frac{a}{p} - f(X_{n_2-i}^*) - \dots - f(X_1^*) > 0, \right. \\ &\quad \left. z_k + \frac{a}{p} + S_{n_2}^* > 0 \right\} \\ &= \left\{ \tau_{z_k + \frac{a}{p}}^* > n_2 \right\}. \end{aligned} \tag{4.8.1}$$

So, for any $y' > 0$,

$$\begin{aligned} E'_k &\leq \mathbb{E}_\nu^* \left(\psi(X_1^*) \psi_{x'}^*(X_{n_2}^*) ; z_k + \frac{a}{p} + S_{n_2}^* \in \left[y', y' + \frac{a}{p} \right], \tau_{z_k + \frac{a}{p}}^* > n_2 \right) \\ &\leq \sum_{x'' \in \mathbb{X}} \int_0^{+\infty} \psi(x'') \mathbb{E}_{x''}^* \left(\psi_{x'}^*(X_{n_2-1}^*) ; z'' + S_{n_2-1}^* \in \left[y', y' + \frac{a}{p} \right], \tau_{z''}^* > n_2 - 1 \right) \\ &\quad \times \mathbb{P}_\nu^* \left(X_1^* = dx'', z_k + \frac{a}{p} + S_1^* \in dz'', \tau_{z_k + \frac{a}{p}}^* > 1 \right). \end{aligned}$$

Using Theorem 4.2.4 for the reverse chain with $\varepsilon' = \varepsilon^8$, we obtain that

$$\begin{aligned} E'_k &\leq \frac{2a\nu(\psi_{x'}^*)}{\sqrt{2\pi}(n_2-1)p\sigma^2} \varphi_+ \left(\frac{y'}{\sqrt{n_2-1}\sigma} \right) \sum_{x'' \in \mathbb{X}} \int_0^{+\infty} \psi(x'') V^*(x'', z'') \\ &\quad \times \mathbb{P}_\nu^* \left(X_1^* = dx'', z_k + \frac{a}{p} + S_1^* \in dz'', \tau_{z_k + \frac{a}{p}}^* > 1 \right) \\ &\quad + \frac{c \|\psi_{x'}^*\|_\infty \|\psi\|_\infty}{n_2-1} \mathbb{E}_\nu^* \left(\left(1 + \max \left(z_k + \frac{a}{p} + S_1^*, 0 \right) \right) \right. \\ &\quad \left. \times \left(\sqrt{\varepsilon^8} + \frac{c_\varepsilon \left(1 + \max \left(z_k + \frac{a}{p} + S_1^*, 0 \right) \right)}{(n_2-1)\varepsilon^8} \right) \right), \tau_{z_k + \frac{a}{p}}^* > 1 \Big). \end{aligned}$$

Note that by (4.6.2), $\nu(\psi_{x'}^*) = 1$ and $\|\psi_{x'}^*\|_\infty \leq c$. So,

$$\begin{aligned} E'_k &\leq \frac{2a}{\sqrt{2\pi}(n_2-1)p\sigma^2} \varphi_+ \left(\frac{y'}{\sqrt{n_2-1}\sigma} \right) \mathbb{E}_\nu^* \left(\psi(X_1^*) V^* \left(X_1^*, z_k + \frac{a}{p} + S_1^* \right), \tau_{z_k + \frac{a}{p}}^* > 1 \right) \\ &\quad + \frac{c \|\psi\|_\infty}{\varepsilon^3 n} \left(1 + \frac{c_\varepsilon}{n} \right) (1+z) \left(\varepsilon^4 + \frac{c_\varepsilon(1+z)}{n\varepsilon^8} \right) \end{aligned}$$

and the upper bound of the lemma is proved.

The lower bound. Similarly as in the proof of the upper bound we note that, on the event $\{y' + f(X_{n_2}^*) + \dots + f(X_1^*) \in (z_k, z_k + \frac{a}{p})\} = \{z_k + S_{n_2}^* \in [y' - \frac{a}{p}, y']\}$, we have

$$\begin{aligned} &\{\forall i \in \{1, \dots, n_2\}, y' + f(X_{n_2}^*) + \dots + f(X_{n_2-i+1}^*) > 0\} \\ &\quad \supset \{\forall i \in \{1, \dots, n_2-1\}, z_k - f(X_{n_2-i}^*) - \dots - f(X_1^*) > 0\} \\ &\quad = \{\tau_{z_k}^* > n_2 - 1\} \supset \{\tau_{z_k}^* > n_2\}. \end{aligned} \tag{4.8.2}$$

Let $y'_+ := \max(y' - a/p, 0)$ and $a' := \min(y', a/p) \in (0, a]$. For any $\eta \in (0, a')$,

$$\begin{aligned} E'_k &\geq \mathbb{E}_\nu^* \left(\psi(X_1^*) \psi_{x'}^*(X_{n_2}^*) ; z_k + S_{n_2}^* \in \left[y' - \frac{a}{p}, y' \right], \tau_{z_k}^* > n_2 \right) \\ &\geq \sum_{x'' \in \mathbb{X}} \int_0^{+\infty} \psi(x'') \mathbb{E}_{x''}^* \left(\psi_{x'}^*(X_{n_2-1}^*) ; z'' + S_{n_2-1}^* \in [y'_+, y'_+ + a' - \eta], \tau_{z''}^* > n_2 - 1 \right) \\ &\quad \times \mathbb{P}_\nu^* \left(X_1^* = dx'', z_k + S_1^* \in dz'', \tau_{z_k}^* > 1 \right). \end{aligned}$$

Using Theorem 4.2.4,

$$\begin{aligned}
E'_k &\geq \frac{2(a' - \eta)\nu(\psi_{x'}^*)}{\sqrt{2\pi}(n_2 - 1)\sigma^2} \varphi_+ \left(\frac{y'_+}{\sqrt{n_2 - 1}\sigma} \right) \sum_{x'' \in \mathbb{X}} \int_0^{+\infty} \psi(x'') V^*(x'', z'') \\
&\quad \times \mathbb{P}_\nu^* \left(X_1^* = dx'', z_k + S_1^* \in dz'', \tau_{z_k}^* > 1 \right) \\
&\quad - \frac{c \|\psi_{x'}^*\|_\infty \|\psi\|_\infty}{n_2 - 1} \mathbb{E}_\nu^* \left((1 + \max(z_k + S_1^*, 0)) \right. \\
&\quad \quad \left. \times \left(\sqrt{\varepsilon^8} + \frac{c_\varepsilon (1 + \max(z_k + S_1^*, 0))}{(n_2 - 1)\varepsilon^8} \right), \tau_{z_k}^* > 1 \right) \\
&\geq \frac{2(a' - \eta)}{\sqrt{2\pi}(n_2 - 1)\sigma^2} \varphi_+ \left(\frac{y'_+}{\sqrt{n_2 - 1}\sigma} \right) \mathbb{E}_\nu^* \left(\psi(X_1^*) V^*(X_1^*, z_k + S_1^*), \tau_{z_k}^* > 1 \right) \\
&\quad - \frac{c \|\psi\|_\infty}{\varepsilon^3 n} \left(1 + \frac{c_\varepsilon}{n} \right) (1 + z) \left(\varepsilon^4 + \frac{c_\varepsilon (1 + z)}{n\varepsilon^8} \right).
\end{aligned}$$

Note that, if $y' \geq a/p$ we have

$$\begin{aligned}
(a' - \eta)\varphi_+ \left(\frac{y'_+}{\sqrt{n_2 - 1}\sigma} \right) &= \left(\frac{a}{p} - \eta \right) \varphi_+ \left(\frac{y' - \frac{a}{p}}{\sqrt{n_2 - 1}\sigma} \right) \\
&\geq \left(\frac{a}{p} - \eta \right) \varphi_+ \left(\frac{y'}{\sqrt{n_2 - 1}\sigma} \right) - \|\varphi'_+\|_\infty \frac{a^2}{p^2 \sqrt{n_2 - 1}\sigma}
\end{aligned}$$

and if $0 < y' \leq a/p$ we have

$$\begin{aligned}
(a' - \eta)\varphi_+ \left(\frac{y'_+}{\sqrt{n_2 - 1}\sigma} \right) &= 0 \geq \left(\frac{a}{p} - \eta \right) \varphi_+ \left(\frac{y'}{\sqrt{n_2 - 1}\sigma} \right) - \|\varphi'_+\|_\infty \frac{ay'}{p\sqrt{n_2 - 1}\sigma} \\
&\geq \left(\frac{a}{p} - \eta \right) \varphi_+ \left(\frac{y'}{\sqrt{n_2 - 1}\sigma} \right) - \|\varphi'_+\|_\infty \frac{a^2}{p^2 \sqrt{n_2 - 1}\sigma}.
\end{aligned}$$

Moreover, using the points 1 and 2 of Proposition 4.2.1, we observe that

$$\mathbb{E}_\nu^* \left(\psi(X_1^*) V^*(X_1^*, z_k + S_1^*), \tau_{z_k}^* > 1 \right) \leq c \|\psi\|_\infty (1 + z).$$

Consequently, for any $y' > 0$,

$$\begin{aligned}
E'_k &\geq \frac{2\left(\frac{a}{p} - \eta\right)}{\sqrt{2\pi}(n_2 - 1)\sigma^2} \varphi_+ \left(\frac{y'}{\sqrt{n_2 - 1}\sigma} \right) \mathbb{E}_\nu^* \left(\psi(X_1^*) V^*(X_1^*, z_k + S_1^*), \tau_{z_k}^* > 1 \right) \\
&\quad - \frac{c_\varepsilon \|\psi\|_\infty}{n^{3/2}} (1 + z) - \frac{c \|\psi\|_\infty}{n} (1 + z) \left(\varepsilon + \frac{c_\varepsilon (1 + z)}{n\varepsilon^8} \right).
\end{aligned}$$

Taking the limit as $\eta \rightarrow 0$, the lower bound of the lemma follows. \square

Lemma 4.8.2. *Assume Hypotheses [M4.1-M4.3](#). For any $a > 0$ and $p \in \mathbb{N}^*$, there exists $\varepsilon_0 \in (0, 1/4)$ such that for any $\varepsilon \in (0, \varepsilon_0)$ there exists $n_0(\varepsilon) \geq 1$ such that any non-negative function $\psi \in \mathcal{C}$, any $y \in \mathbb{R}$, $z \geq 0$ and $n \geq n_0(\varepsilon)$, we have*

$$\begin{aligned}
\sup_{x \in \mathbb{X}} n^{3/2} E_0 &\leq \frac{2aV(x, y)}{p\sqrt{2\pi}\sigma^3} \sum_{k=0}^{p-1} \mathbb{E}_\nu^* \left(\psi(X_1^*) V^* \left(X_1^*, z_k + \frac{a}{p} + S_1^* \right); \tau_{z_k + \frac{a}{p}}^* > 1 \right) \\
&\quad + pc \|\psi\|_\infty (1 + z) (1 + \max(y, 0)) \left(\varepsilon + \frac{c_\varepsilon (1 + z + \max(y, 0))}{n\varepsilon^8} \right)
\end{aligned}$$

and

$$\inf_{x \in \mathbb{X}} n^{3/2} E_0 \geq \frac{2aV(x, y)}{p\sqrt{2\pi\sigma^3}} \sum_{k=0}^{p-1} \mathbb{E}_{\nu}^* \left(\psi(X_1^*) V^*(X_1^*, z_k + S_1^*) ; \tau_{z_k}^* > 1 \right) \\ - pc \|\psi\|_{\infty} (1+z) (1 + \max(y, 0)) \left(\varepsilon + \frac{c_{\varepsilon} (1+z + \max(y, 0))}{n^{\varepsilon^8}} \right)$$

where $E_0 = \mathbb{E}_x(\psi(X_n) ; y + S_n \in (z, z+a], \tau_y > n)$ and for any $k \in \{0, \dots, p-1\}$, $z_k = z + \frac{ka}{p}$.

Proof. Set $n_1 = n - \lfloor \varepsilon^3 n \rfloor$ and $n_2 = \lfloor \varepsilon^3 n \rfloor$. By the Markov property, for any $p \geq 1$,

$$E_0 = \sum_{x' \in \mathbb{X}} \int_0^{+\infty} \mathbb{E}_{x'}(\psi(X_{n_2}) ; y' + S_{n_2} \in (z, z+a], \tau_{y'} > n_2) \\ \times \mathbb{P}_x(X_{n_1} = dx', y + S_{n_1} \in dy', \tau_y > n_1) \\ = \sum_{x' \in \mathbb{X}} \int_0^{+\infty} \sum_{k=0}^{p-1} E'_k \times \mathbb{P}_x(X_{n_1} = dx', y + S_{n_1} \in dy', \tau_y > n_1),$$

where for any $k \in \{0, \dots, p-1\}$,

$$E'_k = \mathbb{E}_{x'} \left(\psi(X_{n_2}) ; y' + S_{n_2} \in \left(z_k, z_k + \frac{a}{p} \right], \tau_{y'} > n_2 \right)$$

and $z_k = z + \frac{ka}{p}$.

The upper bound. By Lemma 4.8.1,

$$E_0 \leq \frac{2a}{p(n_2 - 1)\sqrt{2\pi\sigma^2}} \sum_{k=0}^{p-1} \mathbb{E}_x \left(\varphi_+ \left(\frac{y + S_{n_1}}{\sigma\sqrt{n_2 - 1}} \right) ; \tau_y > n_1 \right) J_1(k) \\ + \sum_{k=0}^{p-1} \frac{c \|\psi\|_{\infty}}{n} (1+z) \left(\varepsilon + \frac{c_{\varepsilon} (1+z)}{n^{\varepsilon^8}} \right) \mathbb{P}_x(\tau_y > n_1),$$

where $J_1(k) = \mathbb{E}_{\nu}^* \left(\psi(X_1^*) V^*(X_1^*, z_k + \frac{a}{p} + S_1^*) ; \tau_{z_k + \frac{a}{p}}^* > 1 \right)$, for any $k \in \{0, \dots, p-1\}$.

By Lemma 4.7.4 and the point 2 of Proposition 4.2.2,

$$n^{3/2} E_0 \leq \frac{2a}{p\sqrt{2\pi\sigma^2}} \sum_{k=0}^{p-1} J_1(k) \frac{V(x, y)}{\sigma} \\ + \frac{1}{p} \sum_{k=0}^{p-1} J_1(k) \left(\frac{c_{\varepsilon} (1 + \max(y, 0))^2}{n^{\varepsilon}} + c(1 + \max(y, 0)) \varepsilon \right) \\ + pc \|\psi\|_{\infty} (1+z) \left(\varepsilon + \frac{c_{\varepsilon} (1+z)}{n^{\varepsilon^8}} \right) (1 + \max(y, 0)).$$

Note that, using the points 1 and 2 of Proposition 4.2.1, we have

$$\frac{1}{p} \sum_{k=0}^{p-1} J_1(k) \leq c \|\psi\|_{\infty} (1+z).$$

Therefore

$$n^{3/2} E_0 \leq \frac{2aV(x, y)}{p\sqrt{2\pi}\sigma^3} \sum_{k=0}^{p-1} J_1(k) + pc \|\psi\|_\infty (1+z) (1+\max(y, 0)) \left(\varepsilon + \frac{c_\varepsilon (1+z+\max(y, 0))}{n^{\varepsilon^8}} \right)$$

and the upper bound of the lemma is proved.

The lower bound. The proof of the lower bound is similar to the proof of the upper bound and therefore will not be detailed. \square

4.8.2 Proof of Theorem 4.2.5.

The second point of Theorem 4.2.5 was proved by Lemma 4.6.2. It remains to prove the first point. Let $\psi \in \mathcal{C}$, $a > 0$, $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z \geq 0$. Suppose first that $z > 0$. For any $n \geq 1$ and $\eta \in (0, \min(z, 1))$,

$$\mathbb{E}_x(\psi(X_n) ; y + S_n \in [z, z+a], \tau_y > n) \leq E_0(\eta), \quad (4.8.3)$$

where $E_0(\eta) = \mathbb{E}_x(\psi(X_n) ; y + S_n \in (z-\eta, z+a], \tau_y > n)$. Taking the limit as $n \rightarrow +\infty$ in Lemma 4.8.2, we have, for any $p \in \mathbb{N}^*$ and $\varepsilon \in (0, \varepsilon_0(p))$,

$$\begin{aligned} & \limsup_{n \rightarrow +\infty} n^{3/2} E_0(\eta) \\ & \leq \frac{2(a+\eta)V(x, y)}{\sqrt{2\pi}p\sigma^3} \sum_{k=0}^{p-1} \mathbb{E}_\nu^* \left(\psi(X_1^*) V^* \left(X_1^*, z_{k,\eta} + \frac{a+\eta}{p} + S_1^* \right) ; \tau_{z_{k,\eta} + \frac{a+\eta}{p}}^* > 1 \right) \\ & \quad + pc \|\psi\|_\infty (1+z-\eta) (1+\max(y, 0)) \varepsilon, \end{aligned}$$

with $z_{k,\eta} = z - \eta + \frac{k(a+\eta)}{p}$ for $k \in \{0, \dots, p-1\}$. Taking the limit as $\varepsilon \rightarrow 0$,

$$\begin{aligned} & \limsup_{n \rightarrow +\infty} n^{3/2} E_0(\eta) \\ & \leq \frac{2(a+\eta)V(x, y)}{\sqrt{2\pi}p\sigma^3} \sum_{k=0}^{p-1} \mathbb{E}_\nu^* \left(\psi(X_1^*) V^* \left(X_1^*, z_{k,\eta} + \frac{a+\eta}{p} + S_1^* \right) ; \tau_{z_{k,\eta} + \frac{a+\eta}{p}}^* > 1 \right). \end{aligned}$$

By the point 2 of Proposition 4.2.1, the function $u \mapsto V^*(x^*, u - f(x^*)) \mathbb{1}_{\{u-f(x^*)>0\}}$ is monotonic and so is Riemann integrable. Since \mathbb{X} is finite, we have

$$\begin{aligned} & \lim_{p \rightarrow +\infty} \frac{a+\eta}{p} \sum_{k=0}^{p-1} \mathbb{E}_\nu^* \left(\psi(X_1^*) V^* \left(X_1^*, z_{k,\eta} + \frac{a+\eta}{p} + S_1^* \right) ; \tau_{z_{k,\eta} + \frac{a+\eta}{p}}^* > 1 \right) \\ & = \mathbb{E}_\nu^* \left(\psi(X_1^*) \int_{z-\eta}^{z+a} V^*(X_1^*, z' + S_1^*) \mathbb{1}_{\{z'+S_1^*>0\}} dz' \right) \\ & = \int_{z-\eta}^{z+a} \mathbb{E}_\nu^* (\psi(X_1^*) V^*(X_1^*, z' + S_1^*) ; \tau_{z'}^* > 1) dz'. \end{aligned}$$

Therefore,

$$\limsup_{n \rightarrow +\infty} n^{3/2} E_0(\eta) \leq \frac{2V(x, y)}{\sqrt{2\pi}\sigma^3} \int_{z-\eta}^{z+a} \mathbb{E}_\nu^* (\psi(X_1^*) V^*(X_1^*, z' + S_1^*) ; \tau_{z'}^* > 1) dz'.$$

Taking the limit as $\eta \rightarrow 0$ and using (4.8.3), we obtain that, for any $z > 0$,

$$\begin{aligned} \limsup_{n \rightarrow +\infty} n^{3/2} \mathbb{E}_x(\psi(X_n); y + S_n \in [z, z + a], \tau_y > n) \\ = \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_z^{z+a} \mathbb{E}_\nu^*(\psi(X_1^*) V^*(X_1^*, z' + S_1^*); \tau_{z'}^* > 1) dz'. \end{aligned} \quad (4.8.4)$$

If $z = 0$, we have

$$\mathbb{E}_x(\psi(X_n); y + S_n \in [0, a], \tau_y > n) = \mathbb{E}_x(\psi(X_n); y + S_n \in (0, a], \tau_y > n).$$

Using Lemma 4.8.2 and the same arguments as before, it is easy to see that (4.8.4) holds for $z = 0$.

Since $[z, z + a] \supset (z, z + a]$ we have obviously

$$\mathbb{E}_x(\psi(X_n); y + S_n \in [z, z + a], \tau_y > n) \geq \mathbb{E}_x(\psi(X_n); y + S_n \in (z, z + a], \tau_y > n).$$

Using this and Lemma 4.8.2 we obtain (4.8.4) with \liminf instead of \limsup , which concludes the proof of the theorem.

4.9 Proof of Theorems 4.2.7 and 4.2.8

4.9.1 Preliminaries results.

Lemma 4.9.1. *Assume Hypotheses M4.1-M4.3. For any $x \in \mathbb{X}$, $y \in \mathbb{R}$, $z \geq 0$, $a > 0$, any non-negative function $\psi: \mathbb{X} \rightarrow \mathbb{R}_+$ and any non-negative and continuous function $g: [z, z + a] \rightarrow \mathbb{R}_+$, we have*

$$\begin{aligned} \lim_{n \rightarrow +\infty} n^{3/2} \mathbb{E}_x(g(y + S_n) \psi(X_n); y + S_n \in [z, z + a], \tau_y > n) \\ = \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_z^{z+a} g(z') \mathbb{E}_\nu^*(\psi(X_1^*) V^*(X_1^*, z' + S_1^*); \tau_{z'}^* > 1) dz'. \end{aligned}$$

Proof. Fix $x \in \mathbb{X}$, $y \in \mathbb{R}$, $z \geq 0$, $a > 0$, and let $\psi: \mathbb{X} \rightarrow \mathbb{R}_+$ be a non-negative function and $g: [z, z + a] \rightarrow \mathbb{R}_+$ be a non-negative and continuous function. For any measurable non-negative and bounded function $\varphi: \mathbb{R} \rightarrow \mathbb{R}_+$, we define

$$I_0(\varphi) := n^{3/2} \mathbb{E}_x(\psi(X_n) \varphi(y + S_n); \tau_y > n).$$

We first prove that for any $0 \leq \alpha < \beta$ we have

$$I_0(\mathbb{1}_{[\alpha, \beta)}) \xrightarrow{n \rightarrow +\infty} \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_\alpha^\beta \mathbb{E}_\nu^*(\psi(X_1^*) V^*(X_1^*, z' + S_1^*); \tau_{z'}^* > 1) dz'. \quad (4.9.1)$$

Since $[\alpha, \beta) \subset [\alpha, \beta]$, the upper limit is a straightforward consequence of Theorem 4.2.5:

$$\begin{aligned} \limsup_{n \rightarrow +\infty} I_0(\mathbb{1}_{[\alpha, \beta)}) &\leq \limsup_{n \rightarrow +\infty} n^{3/2} \mathbb{E}_x(\psi(X_n); y + S_n \in [\alpha, \beta], \tau_y > n) \\ &= \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_\alpha^\beta \mathbb{E}_\nu^*(\psi(X_1^*) V^*(X_1^*, z' + S_1^*); \tau_{z'}^* > 1) dz'. \end{aligned}$$

and for the lower limit, we write for any $\eta \in (0, \beta - \alpha)$,

$$\begin{aligned} \liminf_{n \rightarrow +\infty} I_0 \left(\mathbb{1}_{[\alpha, \beta]} \right) &\geq \liminf_{n \rightarrow +\infty} n^{3/2} \mathbb{E}_x \left(\psi \left(X_n \right) ; y + S_n \in [\alpha, \beta - \eta], \tau_y > n \right) \\ &= \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_{\alpha}^{\beta - \eta} \mathbb{E}_{\nu}^* \left(\psi \left(X_1^* \right) V^* \left(X_1^*, z' + S_1^* \right) ; \tau_{z'}^* > 1 \right) dz'. \end{aligned}$$

Taking the limit as $\eta \rightarrow 0$, it proves (4.9.1).

From (4.9.1), it is clear that by linearity, for any non-negative step function $\varphi = \sum_{k=1}^N \gamma_k \mathbb{1}_{[\alpha_k, \beta_k]}$, where $N \geq 1$, $\gamma_1, \dots, \gamma_N \in \mathbb{R}_+$ and $0 < \alpha_1 < \beta_1 = \alpha_2 < \dots < \beta_N$, we have

$$\lim_{n \rightarrow +\infty} I_0(\varphi) = \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_{\alpha_1}^{\beta_N} \varphi(z') \mathbb{E}_{\nu}^* \left(\psi \left(X_1^* \right) V^* \left(X_1^*, z' + S_1^* \right) ; \tau_{z'}^* > 1 \right) dz'.$$

Since g is continuous on $[z, z + a]$, for any $\varepsilon \in (0, 1)$ there exists $\varphi_{1, \varepsilon}$ and $\varphi_{2, \varepsilon}$ two stepwise functions on $[z, z + a)$ such that $g - \varepsilon \leq \varphi_{1, \varepsilon} \leq g \leq \varphi_{2, \varepsilon} \leq g + \varepsilon$. Consequently,

$$\begin{aligned} &\left| \lim_{n \rightarrow +\infty} I_0(g) - \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_z^{z+a} g(z') \mathbb{E}_{\nu}^* \left(\psi \left(X_1^* \right) V^* \left(X_1^*, z' + S_1^* \right) ; \tau_{z'}^* > 1 \right) dz' \right| \\ &\leq \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \varepsilon \int_z^{z+a} \mathbb{E}_{\nu}^* \left(\psi \left(X_1^* \right) V^* \left(X_1^*, z' + S_1^* \right) ; \tau_{z'}^* > 1 \right) dz'. \end{aligned}$$

Taking the limit as $\varepsilon \rightarrow 0$, concludes the proof of the lemma. \square

For any $l \geq 1$ we denote by $\mathcal{C}_b^+(\mathbb{X}^l \times \mathbb{R})$ the set of measurable non-negative functions $g: \mathbb{X}^l \times \mathbb{R} \rightarrow \mathbb{R}_+$ bounded and such that for any $(x_1, \dots, x_l) \in \mathbb{X}^l$, the function $z \mapsto g(x_1, \dots, x_l, z)$ is continuous.

Lemma 4.9.2. *Assume Hypotheses M4.1-M4.3. For any $x \in \mathbb{X}$, $y \in \mathbb{R}$, $z \geq 0$, $a > 0$, $l \geq 1$, any non-negative functions $\psi: \mathbb{X} \rightarrow \mathbb{R}_+$ and $g \in \mathcal{C}_b^+(\mathbb{X}^l \times \mathbb{R})$, we have*

$$\begin{aligned} &\lim_{n \rightarrow +\infty} n^{3/2} \mathbb{E}_x \left(g \left(X_1, \dots, X_l, y + S_n \right) \psi \left(X_n \right) ; y + S_n \in [z, z + a), \tau_y > n \right) \\ &= \frac{2}{\sqrt{2\pi\sigma^3}} \int_z^{z+a} \mathbb{E}_x \left(g \left(X_1, \dots, X_l, z' \right) V \left(X_l, y + S_l \right) ; \tau_y > l \right) \\ &\quad \times \mathbb{E}_{\nu}^* \left(\psi \left(X_1^* \right) V^* \left(X_1^*, z' + S_1^* \right) ; \tau_{z'}^* > 1 \right) dz'. \end{aligned}$$

Proof. We reduce the proof to the previous case using the Markov property. Fix $x \in \mathbb{X}$, $y \in \mathbb{R}$, $z \geq 0$, $a > 0$, $l \geq 1$, $\psi: \mathbb{X} \rightarrow \mathbb{R}_+$ and $g \in \mathcal{C}_b^+(\mathbb{X}^l \times \mathbb{R})$. For any $n \geq l + 1$, by the Markov property,

$$\begin{aligned} I_0 &:= n^{3/2} \mathbb{E}_x \left(g \left(X_1, \dots, X_l, y + S_n \right) \psi \left(X_n \right) ; y + S_n \in [z, z + a), \tau_y > n \right) \\ &= \mathbb{E}_x \left(n^{3/2} J_{n-l} \left(X_1, \dots, X_l, y + S_l \right), \tau_y > l \right), \end{aligned}$$

where for any $(x_1, \dots, x_l) \in \mathbb{X}^l$, $y' \in \mathbb{R}$ and $k \geq 1$,

$$J_k(x_1, \dots, x_l, y') = \mathbb{E}_{x_l} \left(g(x_1, \dots, x_l, y' + S_k) \psi(X_k) ; y' + S_k \in [z, z + a), \tau_{y'} > k \right).$$

By the point 2 of Theorem 4.2.5,

$$n^{3/2} J_{n-l}(X_1, \dots, X_l, y + S_l) \leq c \|g\|_{\infty} \|\psi\|_{\infty} (1 + z) (1 + \max(y + S_l, 0)).$$

Consequently, by the Lebesgue dominated convergence theorem (in fact the expectation \mathbb{E}_x is a finite sum) and Lemma 4.9.1,

$$\begin{aligned} \lim_{n \rightarrow +\infty} I_0 &= \frac{2}{\sqrt{2\pi}\sigma^3} \int_z^{z+a} \mathbb{E}_x (g(X_1, \dots, X_l, z') V(X_l, y + S_l) ; \tau_y > l) \\ &\quad \times \mathbb{E}_\nu^* (\psi(X_1^*) V^*(X_1^*, z' + S_1^*) ; \tau_{z'}^* > 1) dz'. \end{aligned}$$

□

Lemma 4.9.2 can be reformulated for the dual Markov walk as follows:

Lemma 4.9.3. *Assume Hypotheses M4.1-M4.3. For any $x' \in \mathbb{X}$, $z \geq 0$, $y' \geq 0$, $a > 0$, $m \geq 1$ and any function $g \in \mathcal{C}_b^+(\mathbb{X}^m \times \mathbb{R})$, we have*

$$\begin{aligned} \lim_{n \rightarrow +\infty} n^{3/2} \mathbb{E}_\nu^* \left(g(X_m^*, \dots, X_1^*, y' - S_n^*) \frac{\mathbb{1}_{\{X_{n+1}^* = x'\}}}{\nu(X_{n+1}^*)} ; z + S_n^* \in [y', y' + a), \tau_z^* > n \right) \\ = \frac{2}{\sqrt{2\pi}\sigma^3} \int_{y'}^{y'+a} \mathbb{E}_\nu^* (g(X_m^*, \dots, X_1^*, y' - y'' + z) V^*(X_m^*, z + S_m^*) ; \tau_z^* > m) \\ \quad \times V(x', y'') dy''. \end{aligned}$$

Proof. Fix $x' \in \mathbb{X}$, $z \geq 0$, $y' \geq 0$, $a > 0$, $m \geq 1$ and $g \in \mathcal{C}_b^+(\mathbb{X}^m \times \mathbb{R})$. Let $\psi_{x'}^*$ be the function defined on \mathbb{X} by (4.6.2) and consider for any $n \geq m + 1$,

$$I_0 := n^{3/2} \mathbb{E}_\nu^* (g(X_m^*, \dots, X_1^*, y' - S_n^*) \psi_{x'}^*(X_n^*) ; z + S_n^* \in [y', y' + a), \tau_z^* > n).$$

By Lemma 4.9.2 applied to the dual Markov walk, we have

$$\begin{aligned} I_0 \xrightarrow{n \rightarrow +\infty} \frac{2}{\sqrt{2\pi}\sigma^3} \sum_{x^* \in \mathbb{X}} \int_{y'}^{y'+a} \mathbb{E}_{x^*}^* (g(X_m^*, \dots, X_1^*, y' + z - y'') V^*(X_m^*, z + S_m^*) ; \tau_z^* > m) \\ \quad \times \mathbb{E}_\nu (\psi_{x'}^*(X_1) V(X_1, y'' + S_1) ; \tau_{y''} > 1) \nu(x^*) dy''. \end{aligned}$$

Moreover, using (4.6.2) and the fact that ν is \mathbf{P} -invariant, for any $x' \in \mathbb{X}$, $y'' \geq 0$,

$$\begin{aligned} \mathbb{E}_\nu (\psi_{x'}^*(X_1) V(X_1, y'' + S_1) ; \tau_{y''} > 1) \\ = \sum_{x_1 \in \mathbb{X}} \frac{\mathbf{P}(x', x_1)}{\nu(x_1)} V(x_1, y'' + f(x_1)) \mathbb{1}_{\{y'' + f(x_1) > 0\}} \nu(x_1) \\ = \mathbb{E}_{x'} (V(X_1, y'' + S_1) ; \tau_{y''} > 1). \end{aligned}$$

By the point 1 of Proposition 4.2.1, the function V is harmonic and so

$$\begin{aligned} \lim_{n \rightarrow +\infty} I_0 &= \frac{2}{\sqrt{2\pi}\sigma^3} \int_{y'}^{y'+a} \mathbb{E}_\nu^* (g(X_m^*, \dots, X_1^*, y' - y'' + z) V^*(X_m^*, z + S_m^*) ; \tau_z^* > m) \\ &\quad \times V(x', y'') dy''. \end{aligned}$$

□

Lemma 4.9.4. *Assume Hypotheses M4.1-M4.3. For any $x \in \mathbb{X}$, $y \in \mathbb{R}$, $z \geq 0$, $a > 0$, $m \geq 1$ and any function $g \in \mathcal{C}_b^+(\mathbb{X}^m \times \mathbb{R})$, we have*

$$\begin{aligned} \lim_{n \rightarrow +\infty} n^{3/2} \mathbb{E}_x (g(X_{n-m+1}, \dots, X_n, y + S_n) ; y + S_n \in (z, z + a], \tau_y > n) \\ = \frac{2V(x, y)}{\sqrt{2\pi}\sigma^3} \int_z^{z+a} \mathbb{E}_\nu^* (g(X_m^*, \dots, X_1^*, z') V^*(X_m^*, z' + S_m^*) ; \tau_{z'}^* > m) dz'. \end{aligned}$$

Proof. Fix $x \in \mathbb{X}$, $y \in \mathbb{R}$, $z \geq 0$, $a > 0$, $m \geq 1$ and $g \in \mathcal{C}_b^+(\mathbb{X}^m \times \mathbb{R})$. For any $n \geq m$, consider

$$I_n(x, y) := \mathbb{E}_x(g(X_{n-m+1}, \dots, X_n, y + S_n) ; y + S_n \in (z, z + a], \tau_y > n). \quad (4.9.2)$$

For any $l \geq 1$ and $n \geq l + m$, by the Markov property, we have

$$n^{3/2} I_n(x, y) = \mathbb{E}_x(n^{3/2} I_{n-l}(X_l, y + S_l) ; \tau_y > l). \quad (4.9.3)$$

For any $p \geq 1$ and $0 \leq k \leq p$ we define $z_k := z + \frac{ak}{p}$. For any $x' \in \mathbb{X}$, $y' > 0$, $n \geq l + m$ and $p \geq 1$, we write

$$n^{3/2} I_{n-l}(x', y') = \sum_{k=0}^{p-1} n^{3/2} \mathbb{E}_{x'}(g(X_{n-l-m+1}, \dots, X_{n-l}, y' + S_{n-l}) ; y' + S_{n-l} \in (z_k, z_{k+1}], \tau_{y'} > n - l).$$

Using Lemma 4.3.2, we get

$$n^{3/2} I_{n-l}(x', y') = \sum_{k=0}^{p-1} n^{3/2} \mathbb{E}_{\nu}^*(g(X_m^*, \dots, X_1^*, y' - S_{n-l}^*) \psi_{x'}^*(X_{n-l}^*) ; y' - S_{n-l}^* \in (z_k, z_{k+1}], \forall i \in \{1, \dots, n - l\}, y' + f(X_{n-l}^*) + \dots + f(X_{n-l-i+1}^*) > 0),$$

where $\psi_{x'}^*$ is defined by (4.6.2).

The upper bound. Using (4.8.1), we have

$$n^{3/2} I_{n-l}(x', y') \leq \sum_{k=0}^{p-1} n^{3/2} \mathbb{E}_{\nu}^*(g(X_m^*, \dots, X_1^*, y' - S_{n-l}^*) \psi_{x'}^*(X_{n-l}^*) ; z_{k+1} + S_{n-l}^* \in [y', y' + a/p], \tau_{z_{k+1}}^* > n - l).$$

By Lemma 4.9.3,

$$\limsup_{n \rightarrow +\infty} n^{3/2} I_{n-l}(x', y') \leq \frac{2}{\sqrt{2\pi}\sigma^3} \sum_{k=0}^{p-1} \int_{y'}^{y'+a/p} J_k(y' - y'') V(x', y'') dy'',$$

where for any $k \geq 0$ and $t \in \mathbb{R}$,

$$J_k(t) := \mathbb{E}_{\nu}^*(g(X_m^*, \dots, X_1^*, t + z_{k+1}) V^*(X_m^*, z_{k+1} + S_m^*) ; \tau_{z_{k+1}}^* > m).$$

Note that for any $t \in [-a/p, 0]$

$$J_k(t) \leq \underbrace{\mathbb{E}_{\nu}^* \left(\sup_{t \in [-a/p, 0]} g(X_m^*, \dots, X_1^*, t + z_{k+1}) V^*(X_m^*, z_{k+1} + S_m^*) ; \tau_{z_{k+1}}^* > m \right)}_{=: J_k^p}. \quad (4.9.4)$$

Since $y'' \mapsto V(x', y'')$ is non-decreasing (see the point 2 of Proposition 4.2.1), we have

$$\limsup_{n \rightarrow +\infty} n^{3/2} I_{n-l}(x', y') \leq \frac{a}{p} \sum_{k=0}^{p-1} \frac{2J_k^p}{\sqrt{2\pi}\sigma^3} V\left(x', y' + \frac{a}{p}\right).$$

Moreover, by (4.9.2) and the point 2 of Theorem 4.2.5,

$$n^{3/2}I_{n-l}(X_l, y + S_l) \leq \|g\|_\infty c(1+z)(1+\max(y+S_l, 0)).$$

Consequently, by (4.9.3) and the Lebesgue dominated convergence theorem (or using just the fact that \mathbb{X} is finite),

$$\limsup_{n \rightarrow +\infty} n^{3/2}I_n(x, y) \leq \frac{a}{p} \sum_{k=0}^{p-1} \frac{2J_k^p}{\sqrt{2\pi\sigma^3}} \mathbb{E}_x \left(V \left(X_l, y + S_l + \frac{a}{p} \right); \tau_y > l \right).$$

Using the point 3 of Proposition 4.2.1, for any $\delta \in (0, 1)$,

$$\limsup_{n \rightarrow +\infty} n^{3/2}I_n(x, y) \leq \frac{a}{p} \sum_{k=0}^{p-1} \frac{2J_k^p}{\sqrt{2\pi\sigma^3}} \mathbb{E}_x \left((1+\delta) \left(y + S_l + \frac{a}{p} \right) + c_\delta; \tau_y > l \right)$$

and again using the point 3 of Proposition 4.2.1, for any $\delta \in (0, 1)$,

$$\limsup_{n \rightarrow +\infty} n^{3/2}I_n(x, y) \leq \frac{a}{p} \sum_{k=0}^{p-1} \frac{2J_k^p}{\sqrt{2\pi\sigma^3}} \mathbb{E}_x \left(\frac{1+\delta}{1-\delta} V(X_l, y + S_l) + 2\frac{a}{p} + c_\delta; \tau_y > l \right).$$

Using the point 1 of Proposition 4.2.1 and the point 2 of Proposition 4.2.2 and taking the limit as $l \rightarrow +\infty$,

$$\limsup_{n \rightarrow +\infty} n^{3/2}I_n(x, y) \leq \frac{a}{p} \sum_{k=0}^{p-1} \frac{2J_k^p}{\sqrt{2\pi\sigma^3}} \frac{1+\delta}{1-\delta} V(x, y).$$

Taking the limit as $\delta \rightarrow 0$,

$$\limsup_{n \rightarrow +\infty} n^{3/2}I_n(x, y) \leq \frac{a}{p} \sum_{k=0}^{p-1} \frac{2J_k^p}{\sqrt{2\pi\sigma^3}} V(x, y). \quad (4.9.5)$$

For any $(x_1^*, \dots, x_m^*) \in \mathbb{X}^m$ and $u \in \mathbb{R}$, let

$$\begin{aligned} g_m(u) &:= g(x_m^*, \dots, x_1^*, u), \\ V_m^*(u) &:= V^*(x_m^*, u - f(x_1^*) - \dots - f(x_m^*)) \mathbb{1}_{\{u - f(x_1^*) > 0, \dots, u - f(x_1^*) - \dots - f(x_m^*) > 0\}}. \end{aligned} \quad (4.9.6)$$

The function $u \mapsto g_m(u)$ is uniformly continuous on $[z, z+a]$. Consequently, for any $\varepsilon > 0$, there exists $p_0 \geq 1$ such that for any $p \geq p_0$,

$$\frac{a}{p} \sum_{k=0}^{p-1} \sup_{t \in [-a/p, 0]} g_m(t + z_{k+1}) V_m^*(z_{k+1}) \leq \frac{a}{p} \sum_{k=0}^{p-1} (g_m(z_{k+1}) + \varepsilon) V_m^*(z_{k+1}).$$

Moreover, using the point 2 of Proposition 4.2.1, it is easy to see that the function $u \mapsto V_m^*(u)$ is non-decreasing and so is Riemann-integrable. Therefore, as $p \rightarrow +\infty$, we have

$$\limsup_{p \rightarrow +\infty} \frac{a}{p} \sum_{k=0}^{p-1} \sup_{t \in [-a/p, 0]} g_m(t + z_{k+1}) V_m^*(z_{k+1}) \leq \int_z^{z+a} (g_m(z') + \varepsilon) V_m^*(z') dz'.$$

Thus, when $\varepsilon \rightarrow 0$,

$$\limsup_{p \rightarrow +\infty} \frac{a}{p} \sum_{k=0}^{p-1} \sup_{t \in [-a/p, 0]} g_m(t + z_{k+1}) V_m^*(z_{k+1}) \leq \int_z^{z+a} g_m(z') V_m^*(z') dz'. \quad (4.9.7)$$

Moreover, since $u \mapsto V_m^*(u)$ is non-decreasing,

$$\frac{a}{p} \sum_{k=0}^{p-1} \sup_{t \in [-a/p, 0]} g_m(t + z_{k+1}) V_m^*(z_{k+1}) \leq \|g\|_\infty V_m^*(z + a)a.$$

Consequently, by the Lebesgue dominated convergence theorem, (4.9.4), (4.9.7) and the Fubini theorem,

$$\begin{aligned} & \limsup_{p \rightarrow +\infty} \frac{a}{p} \sum_{k=0}^{p-1} \frac{2J_k^p}{\sqrt{2\pi\sigma^3}} V(x, y) \\ &= \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \mathbb{E}_\nu^* \left(\limsup_{p \rightarrow +\infty} \frac{a}{p} \sum_{k=0}^{p-1} \sup_{t \in [-a/p, 0]} g(X_m^*, \dots, X_1^*, t + z_{k+1}) \right. \\ & \quad \left. \times V^*(X_m^*, z_{k+1} + S_m^*) ; \tau_{z_{k+1}}^* > m \right) \\ &\leq \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_z^{z+a} \mathbb{E}_\nu^* (g(X_m^*, \dots, X_1^*, z') V^*(X_m^*, z' + S_m^*) ; \tau_{z'}^* > m) dz'. \end{aligned}$$

By (4.9.5), we obtain that,

$$\begin{aligned} & \limsup_{n \rightarrow +\infty} n^{3/2} I_n(x, y) \\ &\leq \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_z^{z+a} \mathbb{E}_\nu^* (g(X_m^*, \dots, X_1^*, z') V^*(X_m^*, z' + S_m^*) ; \tau_{z'}^* > m) dz'. \end{aligned}$$

The lower bound. Repeating similar arguments as in the upper bound, by (4.8.2), we have for any $x' \in \mathbb{X}$, $y' > 0$, $l \geq 1$, $n \geq l + m + 1$, $p \geq 1$,

$$\begin{aligned} n^{3/2} I_{n-l}(x', y') &\geq \sum_{k=0}^{p-1} n^{3/2} \mathbb{E}_\nu^* \left(g(X_m^*, \dots, X_1^*, y' - S_{n-l}^*) \psi_{x'}^*(X_{n-l}^*) ; \right. \\ & \quad \left. z_k + S_{n-l}^* \in [y' - a/p, y'] , \tau_{z_k}^* > n - l \right) \\ &= \sum_{k=0}^{p-1} n^{3/2} \mathbb{E}_\nu^* \left(g(X_m^*, \dots, X_1^*, y'_+ + a' - S_{n-l}^*) \psi_{x'}^*(X_{n-l}^*) ; \right. \\ & \quad \left. z_k + S_{n-l}^* \in [y'_+, y'_+ + a'] , \tau_{z_k}^* > n - l \right), \end{aligned}$$

where $y'_+ = \max(y' - a/p, 0)$ and $a' = \min(y', a/p) \in (0, a/p)$. Using Lemma 4.9.3,

$$\liminf_{n \rightarrow +\infty} n^{3/2} I_{n-l}(x', y') \geq \sum_{k=0}^{p-1} \frac{2}{\sqrt{2\pi\sigma^3}} \int_{y'_+}^{y'_+ + a'} L_k(y'_+ + a' - y'') V(x', y'') dy'',$$

where, for any $t \in \mathbb{R}$,

$$L_k(t) := \mathbb{E}_\nu^* \left(g(X_m^*, \dots, X_1^*, t + z_k) V^*(X_m^*, z_k + S_m^*) ; \tau_{z_k}^* > m \right).$$

Since $y'' \mapsto V(x', y'')$ is non-decreasing (see the point 2 of Proposition 4.2.1), we have

$$\liminf_{n \rightarrow +\infty} n^{3/2} I_{n-l}(x', y') \geq a' \sum_{k=0}^{p-1} \frac{2L_k^p}{\sqrt{2\pi\sigma^3}} V(x', y'_+),$$

where

$$L_k^p := \mathbb{E}_\nu^* \left(\inf_{t \in [0, a/p]} g(X_m^*, \dots, X_1^*, t + z_k) V^*(X_m^*, z_k + S_m^*) ; \tau_{z_k}^* > m \right). \quad (4.9.8)$$

Moreover, by the point 3 of Proposition 4.2.1, for any $\delta \in (0, 1)$,

$$a'V(x', y'_+) \geq (1-\delta)a'y'_+ - c_\delta \geq (1-\delta) \left(y' - \frac{a}{p} \right) \frac{a}{p} - c_\delta \geq \frac{a}{p} \frac{1-\delta}{1+\delta} V(x', y') - \frac{a}{p} c_\delta - \left(\frac{a}{p} \right)^2 - c_\delta.$$

Consequently, using (4.9.3) and the Fatou Lemma,

$$\liminf_{n \rightarrow +\infty} n^{3/2} I_n(x, y) \geq \sum_{k=0}^{p-1} \frac{2L_k^p}{\sqrt{2\pi}\sigma^3} \mathbb{E}_x \left(\frac{a}{p} \frac{1-\delta}{1+\delta} V(X_l, y + S_l) - c_\delta (1+a^2) ; \tau_y > l \right).$$

Using the point 1 of Proposition 4.2.1 and the point 2 of Proposition 4.2.2 and taking the limit as $l \rightarrow +\infty$ and then as $\delta \rightarrow 0$,

$$\liminf_{n \rightarrow +\infty} n^{3/2} I_n(x, y) \geq \frac{a}{p} \sum_{k=0}^{p-1} \frac{2L_k^p}{\sqrt{2\pi}\sigma^3} V(x, y). \quad (4.9.9)$$

Using the notation from (4.9.6) and the fact that $u \mapsto g_m(u)$ is uniformly continuous on $[z, z+a]$, for any $\varepsilon > 0$,

$$\liminf_{p \rightarrow +\infty} \frac{a}{p} \sum_{k=0}^{p-1} \inf_{t \in [0, a/p]} g_m(t + z_k) V_m^*(z_k) \geq \int_z^{z+a} (g_m(z') - \varepsilon) V_m^*(z') dz'.$$

Taking the limit as $\varepsilon \rightarrow 0$,

$$\liminf_{p \rightarrow +\infty} \frac{a}{p} \sum_{k=0}^{p-1} \inf_{t \in [0, a/p]} g_m(t + z_k) V_m^*(z_k) \geq \int_z^{z+a} g_m(z') V_m^*(z') dz'.$$

By the Fatou lemma, (4.9.8) and (4.9.9), we conclude that

$$\begin{aligned} \liminf_{n \rightarrow +\infty} n^{3/2} I_n(x, y) &\geq \frac{2V(x, y)}{\sqrt{2\pi}\sigma^3} \mathbb{E}_\nu^* \left(\liminf_{p \rightarrow +\infty} \frac{a}{p} \sum_{k=0}^{p-1} \inf_{t \in [0, a/p]} g(X_m^*, \dots, X_1^*, t + z_k) \right. \\ &\quad \left. \times V^*(X_m^*, z_k + S_m^*) ; \tau_{z_k}^* > m \right) \\ &\geq \frac{2V(x, y)}{\sqrt{2\pi}\sigma^3} \int_z^{z+a} \mathbb{E}_\nu^* (g(X_m^*, \dots, X_1^*, z') V^*(X_m^*, z' + S_m^*) ; \tau_{z'}^* > m) dz'. \end{aligned}$$

□

From now on, we consider that the dual Markov chain $(X_n^*)_{n \geq 0}$ is *independent* of $(X_n)_{n \geq 0}$. Recall that its transition probability \mathbf{P}^* is defined by (4.2.4) and that, for any $z \geq 0$, the associated Markov walk $(z + S_n^*)_{n \geq 0}$ and the associated exit time τ_z^* are defined by (4.2.5) and (4.2.6) respectively. Recall also that for any $(x, x^*) \in \mathbb{X}^2$, we denote by \mathbb{P}_{x, x^*} and \mathbb{E}_{x, x^*} the probability and the expectation generated by the finite dimensional distributions of the Markov chains $(X_n)_{n \geq 0}$ and $(X_n^*)_{n \geq 0}$ starting at $X_0 = x$ and $X_0^* = x^*$ respectively.

Lemma 4.9.5. *Assume Hypotheses M4.1-M4.3. For any $x \in \mathbb{X}$, $y \in \mathbb{R}$, $z \geq 0$, $a > 0$, $l \geq 1$, $m \geq 1$ and any function $g \in \mathcal{C}_b^+(\mathbb{X}^{l+m} \times \mathbb{R})$, we have*

$$\begin{aligned} & \lim_{n \rightarrow +\infty} n^{3/2} \mathbb{E}_x (g(X_1, \dots, X_l, X_{n-m+1}, \dots, X_n, y + S_n) ; y + S_n \in (z, z + a], \tau_y > n) \\ &= \frac{2}{\sqrt{2\pi\sigma^3}} \int_z^{z+a} \sum_{x^* \in \mathbb{X}} \mathbb{E}_{x, x^*} (g(X_1, \dots, X_l, X_m^*, \dots, X_1^*, z') \\ & \quad \times V(X_l, y + S_l) V^*(X_m^*, z' + S_m^*) ; \tau_y > l, \tau_{z'}^* > m) dz' \nu(x^*). \end{aligned}$$

Proof. Fix $x \in \mathbb{X}$, $y \in \mathbb{R}$, $z \geq 0$, $a > 0$, $l \geq 1$, $m \geq 1$ and $g \in \mathcal{C}_b^+(\mathbb{X}^{l+m} \times \mathbb{R})$. For any $n \geq l + m$, by the Markov property,

$$\begin{aligned} I_0 &:= n^{3/2} \mathbb{E}_x (g(X_1, \dots, X_l, X_{n-m+1}, \dots, X_n, y + S_n) ; y + S_n \in (z, z + a], \tau_y > n) \\ &= \sum_{x_1, \dots, x_l \in \mathbb{X}^l} n^{3/2} \mathbb{E}_{x_l} (g(x_1, \dots, x_l, X_{n-l-m+1}, \dots, X_{n-l}, y_l + S_{n-l}) ; \\ & \quad y_l + S_{n-l} \in (z, z + a], \tau_{y_l} > n - l) \times \mathbb{P}_x (X_1 = x_1, \dots, X_l = x_l, \tau_y > l), \end{aligned}$$

where $y_l = x_1 + \dots + x_l$. Using the Lebesgue dominated convergence theorem (or simply the fact that \mathbb{X}^l is finite) and Lemma 4.9.4, we conclude that

$$\begin{aligned} \lim_{n \rightarrow +\infty} I_0 &= \frac{2}{\sqrt{2\pi\sigma^3}} \sum_{x_1, \dots, x_l \in \mathbb{X}^l} V(x_l, y_l) \mathbb{P}_x (X_1 = x_1, \dots, X_l = x_l, \tau_y > l) \\ & \quad \times \int_z^{z+a} \mathbb{E}_{\nu}^* (g(x_1, \dots, x_l, X_m^*, \dots, X_1^*, z') V^*(X_m^*, z' + S_m^*) ; \tau_{z'}^* > m) dz'. \end{aligned}$$

□

4.9.2 Proof of Theorem 4.2.7.

For any $l \geq 1$, denote by $\mathcal{C}^+(\mathbb{X}^l \times \mathbb{R}_+)$ the set of non-negative functions $g: \mathbb{X}^l \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying the following properties:

- for any $(x_1, \dots, x_l) \in \mathbb{X}^l$, the function $z \mapsto g(x_1, \dots, x_l, z)$ is continuous,
- there exists $\varepsilon > 0$ such that $\max_{x_1, \dots, x_l \in \mathbb{X}} \sup_{z \geq 0} g(x_1, \dots, x_l, z)(1 + z)^{2+\varepsilon} < +\infty$.

Fix $x \in \mathbb{X}$, $y \in \mathbb{R}$, $l \geq 1$, $m \geq 1$ and $g \in \mathcal{C}^+(\mathbb{X}^{l+m} \times \mathbb{R})$. For brevity, denote

$$g_{l,m}(y + S_n) = g(X_1, \dots, X_l, X_{n-m+1}, \dots, X_n, y + S_n).$$

Set

$$\begin{aligned} I_0 &:= n^{3/2} \mathbb{E}_x (g_{l,m}(y + S_n) ; \tau_y > n) \\ &= \sum_{k=0}^{+\infty} \underbrace{n^{3/2} \mathbb{E}_x (g_{l,m}(y + S_n) ; y + S_n \in (k, k + 1], \tau_y > n)}_{=: I_k(n)}. \end{aligned}$$

Since $g \in \mathcal{C}^+(\mathbb{X}^{l+m} \times \mathbb{R})$, we have

$$I_k(n) \leq \frac{N(g)}{(1 + k)^{2+\varepsilon}} n^{3/2} \mathbb{P}_x (y + S_n \in (k, k + 1], \tau_y > n),$$

where $N(g) = \max_{x_1, \dots, x_{l+m} \in \mathbb{X}} \sup_{z \geq 0} g(x_1, \dots, x_{l+m}, z)(1+z)^{2+\varepsilon} < +\infty$. By the point 2 of Theorem 4.2.5, we have

$$I_k(n) \leq \frac{cN(g)(1 + \max(y, 0))}{(k+1)^{1+\varepsilon}}.$$

Consequently, by the Lebesgue dominated convergence theorem,

$$\lim_{n \rightarrow +\infty} I_0 = \sum_{k=0}^{+\infty} \lim_{n \rightarrow +\infty} n^{3/2} \mathbb{E}_x (g_{l,m}(y + S_n); y + S_n \in (k, k+1], \tau_y > n),$$

By Lemma 4.9.5,

$$\lim_{n \rightarrow +\infty} I_0 = \frac{2}{\sqrt{2\pi\sigma^3}} \sum_{k=0}^{+\infty} \int_k^{k+1} \sum_{x^* \in \mathbb{X}} \mathbb{E}_{x,x^*} (g_{l,m}(z') V(X_l, y + S_l) V^*(X_m^*, z' + S_m^*); \tau_y > l, \tau_{z'}^* > m) dz' \nu(x^*),$$

which establishes Theorem 4.2.7.

4.9.3 Proof of Theorem 4.2.8.

Theorem 4.2.8 will be deduced from Theorem 4.2.7.

Let $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $n \geq 1$. Since \mathbb{X} is finite we note that $\|f\|_\infty = \sup_{x \in \mathbb{X}} |f(x)|$ exists. This implies

$$\mathbb{P}_x (\tau_y = n+1) = \mathbb{P}_x (y + S_n + f(X_{n+1}) \leq 0, y + S_n \in [0, \|f\|_\infty], \tau_y > n).$$

By the Markov property,

$$\mathbb{P}_x (\tau_y = n+1) = \mathbb{E}_x (g(X_n, y + S_n); \tau_y > n),$$

where, for any $(x', y') \in \mathbb{X} \times \mathbb{R}$,

$$g(x', y') = \mathbb{P}_{x'} (y' + f(X_1) \leq 0) \mathbb{1}_{\{y' \in [0, \|f\|_\infty]\}} = \mathbb{1}_{\{y' \in [0, \|f\|_\infty]\}} \sum_{x_1 \in \mathbb{X}} \mathbf{P}(x', x_1) \mathbb{1}_{\{y' + f(x_1) \leq 0\}}.$$

Since $g(x', \cdot)$ is a staircase function, for any $\varepsilon > 0$ there exist two functions φ_ε and ψ_ε on $\mathbb{X} \times \mathbb{R}$ and $N \subset \mathbb{X} \times \mathbb{R}$ such that

- for any $x' \in \mathbb{X}$, the functions $\varphi_\varepsilon(x', \cdot)$ and $\psi_\varepsilon(x', \cdot)$ are continuous and have a compact support included in $[-1, \|f\|_\infty + 1]$,
- for any $(x', y') \in (\mathbb{X} \times \mathbb{R}) \setminus N$, it holds $\varphi_\varepsilon(x', y') = g(x', y') = \psi_\varepsilon(x', y')$,
- for any $(x', y') \in \mathbb{X} \times \mathbb{R}$, it holds $0 \leq \varphi_\varepsilon(x', y') \leq g(x', y') \leq \psi_\varepsilon(x', y') \leq 1$,
- the set N is sufficiently small:

$$\int_{-1}^{\|f\|_\infty + 1} \mathbb{E}_\nu^* (V^*(X_1, z + S_1^*); \tau_z^* > 1, (X_1, z) \in N) dz \leq \varepsilon. \quad (4.9.10)$$

The upper bound. For any $\varepsilon > 0$, using Theorem 4.2.7, we have

$$\begin{aligned} I^+ &:= \limsup_{n \rightarrow +\infty} n^{3/2} \mathbb{P}_x (\tau_y = n+1) \\ &\leq \limsup_{n \rightarrow +\infty} n^{3/2} \mathbb{E}_x (\psi_\varepsilon(X_n, y + S_n); \tau_y > n) \\ &= \frac{2}{\sqrt{2\pi\sigma^3}} \int_0^{+\infty} \sum_{x^* \in \mathbb{X}} \mathbb{E}_{x,x^*} (\psi_\varepsilon(X_1^*, z) V(X_1, y + S_1) \\ &\quad V^*(X_1^*, z + S_1^*); \tau_y > 1, \tau_z^* > 1) \nu(x^*) dz. \end{aligned}$$

Using the point 1 of Proposition 4.2.1,

$$\begin{aligned}
 I^+ &\leq \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_0^{\|f\|_\infty+1} \mathbb{E}_\nu^*(\psi_\varepsilon(X_1^*, z) V^*(X_1^*, z + S_1^*); \tau_z^* > 1) dz \\
 &\leq \underbrace{\frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_0^{\|f\|_\infty} \mathbb{E}_\nu^*(g(X_1^*, z) V^*(X_1^*, z + S_1^*); \tau_z^* > 1) dz}_{=: I_1} \\
 &\quad + \underbrace{\frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_0^{\|f\|_\infty+1} \mathbb{E}_\nu^*(V^*(X_1^*, z + S_1^*); \tau_z^* > 1, (X_1^*, z) \in N) dz}_{=: I_2}. \tag{4.9.11}
 \end{aligned}$$

Since ν is \mathbf{P}^* -invariant, we have

$$\begin{aligned}
 I_1 &= \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_0^{\|f\|_\infty} \sum_{x^* \in \mathbb{X}} g(x^*, z) V^*(x^*, z - f(x^*)) \mathbb{1}_{\{z - f(x^*) > 0\}} \nu(x^*) dz \\
 &= \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_0^{\|f\|_\infty} \sum_{x^*, x_1 \in \mathbb{X}} \mathbb{1}_{\{z + f(x_1) \leq 0\}} \mathbf{P}(x^*, x_1) \nu(x^*) V^*(x^*, z - f(x^*)) \mathbb{1}_{\{z - f(x^*) > 0\}} dz \\
 &= \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_0^{\|f\|_\infty} \sum_{x^*, x_1 \in \mathbb{X}} \mathbb{1}_{\{z + f(x_1) \leq 0\}} \mathbf{P}^*(x_1, x^*) \nu(x_1) V^*(x^*, z - f(x^*)) \mathbb{1}_{\{z - f(x^*) > 0\}} dz \\
 &= \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_0^{\|f\|_\infty} \sum_{x_1 \in \mathbb{X}} \mathbb{1}_{\{z + f(x_1) \leq 0\}} \nu(x_1) \mathbb{E}_{x_1}^*(V^*(X_1^*, z + S_1^*); \tau_z^* > 1) dz.
 \end{aligned}$$

Using the point 1 of Proposition 4.2.1,

$$I_1 = \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_0^{\|f\|_\infty} \mathbb{E}_\nu^*(V^*(X_1^*, z); S_1^* \geq z) dz. \tag{4.9.12}$$

Moreover, by (4.9.10), we get

$$I_2 \leq \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \varepsilon. \tag{4.9.13}$$

Putting together (4.9.11), (4.9.12) and (4.9.13) and taking the limit as $\varepsilon \rightarrow 0$, we obtain that

$$I^+ \leq \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_0^{\|f\|_\infty} \mathbb{E}_\nu^*(V^*(X_1^*, z); S_1^* \geq z) dz. \tag{4.9.14}$$

Lower bound. In a similar way, using Theorem 4.2.7, we write

$$\begin{aligned}
 I^- &:= \liminf_{n \rightarrow +\infty} n^{3/2} \mathbb{P}_x(\tau_y = n + 1) \\
 &\geq \liminf_{n \rightarrow +\infty} n^{3/2} \mathbb{E}_x(\varphi_\varepsilon(X_n, y + S_n); \tau_y > n) \\
 &= \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_0^{\|f\|_\infty+1} \mathbb{E}_\nu^*(\varphi_\varepsilon(X_1^*, z) V^*(X_1^*, z + S_1^*); \tau_z^* > 1) dz \\
 &\geq I_1 - I_2.
 \end{aligned}$$

Using (4.9.12) and (4.9.13) and taking the limit as $\varepsilon \rightarrow 0$, we obtain that

$$I^- \geq \frac{2V(x, y)}{\sqrt{2\pi\sigma^3}} \int_0^{\|f\|_\infty} \mathbb{E}_\nu^*(V^*(X_1^*, z); S_1^* \geq z) dz,$$

which together with (4.9.14) concludes the proof.

4.10 Appendix

4.10.1 The non degeneracy of the Markov walk

In [38] (Chapter 3), it is proved that the statements of Propositions 4.2.1-4.2.3 hold under more general assumptions (see Hypotheses M1-M5 of [38]/Chapter 3). We will link these assumptions to our Hypotheses **M4.1-M4.3**. The assumptions M1-M3 in [38] (Chapter 3), with the Banach space \mathcal{C} , are well known consequences of Hypothesis **M4.1** of this paper. Hypothesis M4 in [38] (Chapter 3) is also obvious with $N = N_1 = \dots = 0$. By Hypothesis **M4.2**, to obtain Hypothesis M5 of [38] (Chapter 3), it remains only to prove that σ defined by (4.2.2) is strictly positive. First we give a necessary and sufficient condition. Recall that the words *path* and *orbit* are defined in Section 4.4.

Lemma 4.10.1. *Assume Hypothesis **M4.1**. The following statements are equivalent:*

1. *The Cesàro mean of f on the orbits is constant: there exists $m \in \mathbb{R}$ such that for any orbit x_0, \dots, x_n we have*

$$f(x_0) + \dots + f(x_n) = (n + 1)m.$$

2. *There exist a constant $m \in \mathbb{R}$ and a function $h \in \mathcal{C}$ such that for any $(x, x') \in \mathbb{X}^2$,*

$$\mathbf{P}(x, x')f(x') = \mathbf{P}(x, x')(h(x) - h(x') + m).$$

3. *The following real $\tilde{\sigma}^2$ is equal to 0*

$$\tilde{\sigma}^2 = \nu(f^2) - \nu(f)^2 + 2 \sum_{n=1}^{+\infty} [\nu(f\mathbf{P}^n f) - \nu(f)^2] = 0.$$

Proof. *The point 1 implies the point 2.* Suppose that the point 1 holds. Fix $x_0 \in \mathbb{X}$ and set $h(x_0) = 0$. For any $x \in \mathbb{X}$, we define $h(x)$ in the following way: for any path x_0, x_1, \dots, x_n, x in \mathbb{X} , we set

$$h(x) = -f(x) - f(x_n) - \dots - f(x_1) + (n + 1)m.$$

We shall verify that h is well defined. By Hypothesis **M4.1**, we can find at least a path to define $h(x)$. Now we have to check that this definition does not depend on the choice of the path. Let x_0, x_1, \dots, x_p, x and x_0, y_1, \dots, y_q, x be two paths. By Hypothesis **M4.1**, there exists a path x, z_1, \dots, z_n, x_0 in \mathbb{X} between x and x_0 . Since $x_0, x_1, \dots, x_p, x, z_1, \dots, z_n$ and $x_0, y_1, \dots, y_q, x, z_1, \dots, z_n$ are two orbits, by the point 1, we have

$$\begin{aligned} -f(x) - f(x_p) - \dots - f(x_1) + (p + 1)m &= f(x_0) + f(z_1) + \dots + f(z_n) - (n + 1)m \\ &= -f(x) - f(y_q) - \dots - f(y_1) + (q + 1)m \end{aligned}$$

and so the function h is well defined on \mathbb{X} . Now let $(x, x') \in \mathbb{X}^2$ such that $\mathbf{P}(x, x') > 0$. By Hypothesis **M4.1**, there exists x_0, x_1, \dots, x_n, x a path between x_0 and x . Since $\mathbf{P}(x_0, x_1) \cdots \mathbf{P}(x_n, x)\mathbf{P}(x, x') > 0$, by the definition of h , we have

$$\begin{aligned} h(x) &= -f(x) - f(x_n) - \dots - f(x_1) + (n + 1)m \\ h(x') &= -f(x') - f(x) - f(x_n) - \dots - f(x_1) + (n + 2)m. \end{aligned}$$

In particular

$$h(x') = -f(x') + h(x) + m.$$

The point 2 implies the point 1. Suppose that the point 2 holds and let x_0, \dots, x_n be an orbit. Using the point 2,

$$h(x_0) = h(x_n) - f(x_0) + m = \dots = h(x_0) - f(x_0) - f(x_n) - \dots - f(x_1) + (n+1)m,$$

and the point 1 follows.

The point 2 implies the point 3. Suppose that the point 2 holds. Denote by \tilde{f} the ν -centred function:

$$\tilde{f}(x) = f(x) - \nu(f), \quad \forall x \in \mathbb{X}. \quad (4.10.1)$$

By the point 2, for any $x \in \mathbb{X}$,

$$\mathbf{P}\tilde{f}(x) = h(x) - \mathbf{P}h(x) + m - \nu(f). \quad (4.10.2)$$

Using the fact that ν is \mathbf{P} -invariant, we obtain that $\nu(\tilde{f}) = 0 = m - \nu(f)$ and so,

$$m = \nu(f). \quad (4.10.3)$$

Consequently, by (4.10.2), $\mathbf{P}^n \tilde{f} = \mathbf{P}^{n-1}h - \mathbf{P}^n h$ for any $n \geq 1$ and therefore,

$$\sum_{k=1}^n \mathbf{P}^k \tilde{f} = h - \mathbf{P}^n h. \quad (4.10.4)$$

Let

$$\tilde{\Theta} := \sum_{k=0}^{+\infty} \mathbf{P}^k \tilde{f}$$

be the solution of the Poisson equation $\tilde{\Theta} - \mathbf{P}\tilde{\Theta} = \tilde{f}$, which by (4.2.1), is well defined. Taking the limit as $n \rightarrow +\infty$ in (4.10.4) and using (4.2.1),

$$\mathbf{P}\tilde{\Theta} = \tilde{\Theta} - \tilde{f} = h - \nu(h).$$

Therefore, for any $(x, x') \in \mathbb{X}^2$,

$$\tilde{\Theta}(x') - \mathbf{P}\tilde{\Theta}(x) = \tilde{f}(x') + \mathbf{P}\tilde{\Theta}(x') - \mathbf{P}\tilde{\Theta}(x) = \tilde{f}(x') + h(x') - h(x).$$

Using the point 2 and (4.10.3), it follows that

$$\tilde{\Theta}(x') - \mathbf{P}\tilde{\Theta}(x) = 0, \quad (4.10.5)$$

for any $(x, x') \in \mathbb{X}^2$ such that $\mathbf{P}(x, x') > 0$. Moreover,

$$\tilde{\sigma}^2 = \nu(\tilde{f}^2) + 2 \sum_{n=1}^{+\infty} \nu(\tilde{f}\mathbf{P}^n \tilde{f}) = \nu(\tilde{f}(\tilde{f} + 2\mathbf{P}\tilde{\Theta})) = \nu((\tilde{\Theta} - \mathbf{P}\tilde{\Theta})(\tilde{\Theta} + \mathbf{P}\tilde{\Theta})).$$

Since ν is \mathbf{P} -invariant,

$$\begin{aligned} \tilde{\sigma}^2 &= \nu(\mathbf{P}(\tilde{\Theta}^2)) - 2\nu((\mathbf{P}\tilde{\Theta})^2) + \nu((\mathbf{P}\tilde{\Theta})^2) \\ &= \sum_{(x, x') \in \mathbb{X}} \left[\tilde{\Theta}(x')^2 - 2\tilde{\Theta}(x')\mathbf{P}\tilde{\Theta}(x) + (\mathbf{P}\tilde{\Theta}(x))^2 \right] \mathbf{P}(x, x')\nu(x) \\ &= \sum_{(x, x') \in \mathbb{X}} (\tilde{\Theta}(x') - \mathbf{P}\tilde{\Theta}(x))^2 \mathbf{P}(x, x')\nu(x). \end{aligned} \quad (4.10.6)$$

By (4.10.5), we conclude that $\tilde{\sigma}^2 = 0$.

The point 3 implies the point 2. Suppose that the point 3 holds. By (4.10.6), for any $(x, x') \in \mathbb{X}$ such that $\mathbf{P}(x, x') > 0$ we have

$$\tilde{\Theta}(x') - \mathbf{P}\tilde{\Theta}(x) = 0.$$

Let $h = \mathbf{P}\tilde{\Theta}$. Since $\tilde{\Theta}$ is the solution of the Poisson equation,

$$\tilde{f}(x') + h(x') - h(x) = 0.$$

By the definition of \tilde{f} in (4.10.1), for any $(x, x') \in \mathbb{X}$ such that $\mathbf{P}(x, x') > 0$,

$$f(x') = h(x) - h(x') + m,$$

with $m = \nu(f)$. □

Note that under Hypothesis M4.2, Lemma 4.10.1 can be rewritten as follows.

Lemma 4.10.2. *Assume Hypotheses M4.1 and M4.2. The following statements are equivalent:*

1. *The mean of f on the orbits is equal to zero: for any orbit x_0, \dots, x_n , we have*

$$f(x_0) + \dots + f(x_n) = 0.$$

2. *There exists a function $h \in \mathcal{C}$ such that for any $(x, x') \in \mathbb{X}^2$,*

$$\mathbf{P}(x, x')f(x') = \mathbf{P}(x, x')(h(x) - h(x')).$$

3. *The real σ^2 is equal to 0:*

$$\sigma^2 = \nu(f^2) + 2 \sum_{n=1}^{+\infty} \nu(f\mathbf{P}^n f) = 0.$$

Now we prove that the Hypothesis M4.3 ("non-lattice condition"), implies that the Markov walk is non-degenerated.

Lemma 4.10.3. *Under Hypotheses M4.1-M4.3, we have*

$$\sigma^2 = \nu(f^2) + 2 \sum_{n=1}^{+\infty} \nu(f\mathbf{P}^n f) > 0$$

Proof. We proceed by *reductio ad absurdum*. Suppose that $\sigma^2 = 0$. By Lemma 4.10.2, for any orbit x_0, \dots, x_n , we have

$$f(x_0) + \dots + f(x_n) = 0,$$

which implies the negation of Hypothesis M4.3 with $\theta = a = 0$. □

4.10.2 Strong approximation

Let $(B_t)_{t \geq 0}$ be the standard Brownian motion on \mathbb{R} defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the exit time

$$\tau_y^{bm} := \inf\{t \geq 0, y + \sigma B_t \leq 0\}, \quad (4.10.7)$$

where σ is defined by (4.2.2). It is proved in Grama, Le Page and Peigné [40] that there is a version of the Markov walk $(S_n)_{n \geq 0}$ and of the standard Brownian motion $(B_t)_{t \geq 0}$ living on the same probability space which are close enough in the following sense:

Proposition 4.10.4. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0]$, $x \in \mathbb{X}$ and $n \geq 1$, without loss of generality (on an extension of the initial probability space) one can reconstruct the sequence $(S_n)_{n \geq 0}$ with a continuous time Brownian motion $(B_t)_{t \in \mathbb{R}_+}$, such that*

$$\mathbb{P}_x \left(\sup_{0 \leq t \leq 1} |S_{[tn]} - \sigma B_{tn}| > n^{1/2-\varepsilon} \right) \leq \frac{C_\varepsilon}{n^\varepsilon}.$$

Chapter 5

The survival probability of critical and subcritical branching processes in finite state space Markovian environment

RÉSUMÉ. Soit $(Z_n)_{n \geq 0}$ un processus de branchement en environnement aléatoire défini par une chaîne de Markov $(X_n)_{n \geq 0}$ prenant ses valeurs dans un espace d'états fini \mathbb{X} et partant de $X_0 = i \in \mathbb{X}$. Nous étudions le comportement asymptotique de la probabilité que $Z_n > 0$ lorsque $n \rightarrow +\infty$. Nous montrons que l'ordre de convergence dépend de la fonction $k(\lambda) := \lim_{n \rightarrow +\infty} \mathbb{E}_i^{1/n} (e^{\lambda S_n})$, $\lambda \in \mathbb{R}$ et $i \in \mathbb{X}$, où $(S_n)_{n \geq 0}$ correspond à la marche markovienne associée. Dans la classification qui en découle, nous étudions quatre cas différents : critique, fortement, intermédiaire et faiblement sous-critique.

ABSTRACT. Let $(Z_n)_{n \geq 0}$ be a branching process in a random environment defined by a Markov chain $(X_n)_{n \geq 0}$ with values in a finite state space \mathbb{X} starting at $X_0 = i \in \mathbb{X}$. We study the asymptotic behaviour of the probability that $Z_n > 0$ as $n \rightarrow +\infty$. We found that it depends on the values of the function $k(\lambda) := \lim_{n \rightarrow +\infty} \mathbb{E}_i^{1/n} (e^{\lambda S_n})$, $\lambda \in \mathbb{R}$ and $i \in \mathbb{X}$, where $(S_n)_{n \geq 0}$ is the associated Markov walk. The function k permits to give a classification of types of the asymptotic behaviours of the survival probability. In particular we analyse four different cases: critical and strongly, intermediate and weakly subcritical regimes.

5.1 Introduction

Galton-Watson branching process is one of the most used models in the dynamic of populations. It has numerous applications in different areas such as biology, medicine, physics, economics etc; for an introduction we refer to Harris [46] or Athreya and Ney [5] and to the references therein. A significant advancement in the theory and practice was made with the introduction of the branching process in which the offspring distributions vary according to a random environment, see Smith and Wilkinson [62] and Athreya and Karlin [4, 3]. This allowed a more adequate modeling and turned out to be very fruitful from the practical as well as from the mathematical points of view. The recent advances in the study of conditioned limit theorems for sums of functions defined on Markov chains in [41], [36], [38] and [37] (Chapters 2, 3 and 4) open the way to treat some unsolved

questions in the case of Markovian environments. The problem we are interested here is to study the asymptotic behaviour of the survival probability.

Assume first that on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we are given a branching process $(Z_n)_{n \geq 0}$ in a random environment represented by the i.i.d. sequence $(X_n)_{n \geq 0}$ with values in the space \mathbb{X} . Let $f_i(\cdot)$ be the probability generating function of the offspring distributions of $(Z_n)_{n \geq 0}$, provided the value of the environment is $i \in \mathbb{X}$. In a remarkable series of papers Afanasyev [1], Dekking [14], Kozlov [53], Liu [58], D'Souza and Hambly [23], Geiger and Kersting [32], Guivarc'h and Liu [44] and Geiger, Kersting and Vatutin [33] under various assumptions have determined the asymptotic behaviour as $n \rightarrow +\infty$ of the survival probability $\mathbb{P}(Z_n > 0)$. Let $\phi(\lambda)$ be the Laplace transform of the random variable $\ln f'_{X_1}(1)$: $\phi(\lambda) = \mathbb{E} \left(e^{\lambda \ln f'_{X_1}(1)} \right)$, $\lambda \in \mathbb{R}$, where \mathbb{E} is the expectation pertaining to \mathbb{P} . In function of the values of the derivatives $\phi'(0) = \mathbb{E}(\ln f'_{X_1}(1))$ and $\phi'(1) = \mathbb{E}(f'_{X_1}(1) \ln f'_{X_1}(1))$ and under some additional moment assumptions on the variables $\ln f'_{X_1}(1)$ and Z_1 , the following asymptotic results have been found. In the critical case, $\phi'(0) = 0$, it was shown in [53] and [32] that $\mathbb{P}(Z_n > 0) \sim \frac{c}{\sqrt{n}}$; hereafter c stands for a constant and \sim means equivalence of sequences as $n \rightarrow +\infty$. The behaviour in the subcritical case, $\phi'(0) < 0$, turns out to depend on the value $\phi'(1)$. The strongly subcritical case, $\phi'(0) < 0$ & $\phi'(1) < 0$, has been studied in [23] and [44] where it was shown that $\mathbb{P}(Z_n > 0) \sim c\phi(1)^n$, with $0 < \phi(1) = \mathbb{E}f'_{X_1}(1) < 1$. In the intermediate and weakly subcritical cases, $\phi'(0) < 0$ & $\phi'(1) = 0$ and $\phi'(0) < 0$ & $\phi'(1) > 0$, respectively, it was shown in [33] that $\mathbb{P}(Z_n > 0) \sim cn^{-1/2}\phi(1)^n$ and $\mathbb{P}(Z_n > 0) \sim cn^{-3/2}\phi(\lambda)^n$, where λ is the unique critical point of ϕ : $\phi'(\lambda) = 0$.

The goal of the present paper is to determine the asymptotic behaviour as $n \rightarrow +\infty$ of the survival probability $\mathbb{P}_i(Z_n > 0)$ when the environment $(X_n)_{n \geq 0}$ is a Markov chain with values in a finite state space \mathbb{X} . Hereafter \mathbb{P}_i and \mathbb{E}_i are the probability and expectation generated by the trajectories of $(X_n)_{n \geq 0}$ starting at $X_0 = i \in \mathbb{X}$. Set $\rho(i) = \ln f'_i(1)$, $i \in \mathbb{X}$. Consider the associated Markov walk $S_n = \sum_{k=1}^n \rho(X_k)$, $n \geq 0$. In the case of a Markovian environment the behaviour of the survival probability $\mathbb{P}_i(Z_n > 0)$ depends on the function

$$k(\lambda) := \lim_{n \rightarrow +\infty} \mathbb{E}_i^{1/n} \left(e^{\lambda S_n} \right),$$

which is well defined, analytic in $\lambda \in \mathbb{R}$ and does not depend on $i \in \mathbb{X}$ (see Section 5.3.4). In some sense the function k plays the same role that the function ϕ in the case of i.i.d. environment.

Let us present briefly the main results of the paper. Under appropriate conditions, we show the asymptotic behaviour of the survival probability $\mathbb{P}_i(Z_n > 0)$ in function of the following classification:

- Critical case: if $k'(0) = 0$, then, for any $i, j \in \mathbb{X}$,

$$\mathbb{P}_i(Z_n > 0, X_n = j) \underset{n \rightarrow +\infty}{\sim} \frac{\nu(j)u(i)}{\sqrt{n}},$$

where $u(i)$ is a constant depending on i and ν is the stationary probability measure of the Markov chain $(X_n)_{n \geq 0}$.

- Strongly subcritical case: if $k'(0) < 0$ and $k'(1) < 0$, then, for any $i, j \in \mathbb{X}$,

$$\mathbb{P}_i(Z_n > 0, X_n = j) \underset{n \rightarrow +\infty}{\sim} v_1(i)u(j)k(1)^n.$$

where $u(j)$ and $v_1(i)$ are depending only on j and i respectively.

— Intermediate subcritical case: if $k'(0) < 0$ and $k'(1) = 0$, then, for any $i, j \in \mathbb{X}$,

$$\mathbb{P}_i(Z_n > 0, X_n = j) \underset{n \rightarrow +\infty}{\sim} v_1(i)u(j) \frac{k(1)^n}{\sqrt{n}}.$$

where $u(i)$ depends only on i .

— Weakly subcritical case: if $k'(0) < 0$ and $k'(1) > 0$, then, for any $i, j \in \mathbb{X}$,

$$\mathbb{P}_i(Z_n > 0, X_n = j) \underset{n \rightarrow +\infty}{\sim} k(\lambda)^n \frac{u(i, j)}{n^{3/2}},$$

where $u(i, j)$ depends only on i and j and λ is the critical point of k : $k'(\lambda) = 0$.

The critical case has been considered in Le Page and Ye [56] in a more general setting. However, the conditions in their paper do not cover the present situation and the employed method is different from ours.

From the results of Section 5.3.4 it follows that the classification stated above coincides with the usual classification for branching processes when the environment is i.i.d. Indeed, Lemma 5.3.15 implies that $k'(0) = \mathbb{E}_\nu(\ln f'_{X_1}(1))$, where \mathbb{E}_ν is the expectation generated by the finite dimensional distributions of the Markov chain $(X_n)_{n \geq 0}$ in the stationary regime. For an i.i.d. environment this is exactly $\mathbb{E}(\ln f'_{X_1}(1)) = \phi'(0)$. The value $k'(1)$ can also be related to the first moment of the random variable $\ln f'_{X_1}(1)$. For this we need the transfer operator P_λ related to the Markov chain $(X_n)_{n \geq 0}$, see Section 5.3.4 for details. The normalized transfer operator $\tilde{\mathbf{P}}_\lambda$ generates a Markov chain whose invariant probability is denoted by $\tilde{\nu}_\lambda$. Again by Lemma 5.3.15, it holds $\frac{k'(1)}{k(1)} = \tilde{\mathbb{E}}_{\tilde{\nu}_\lambda}(\ln f'_{X_1}(1))$, where $\tilde{\mathbb{E}}_{\tilde{\nu}_\lambda}$ is the expectation generated by the finite dimensional distributions of the Markov chain $(X_n)_{n \geq 0}$ with transition probabilities $\tilde{\mathbf{P}}_\lambda$ in the stationary regime. For an i.i.d. environment, we have $\frac{k'(1)}{k(1)} = \mathbb{E}(f'_{X_1}(1) \ln f'_{X_1}(1)) = \phi'(1)$, which shows that both classifications are equivalent.

Now we shall shortly explain the approach of the paper. We start with a well known relation between the survival probability $\mathbb{P}_i(Z_n > 0)$ and the associated random walk $(S_n)_{n \geq 0}$ which goes back to Agresti [2] and which is adapted it to the Markov environment as follows: for any initial state $X_0 = i$,

$$\mathbb{P}_i(Z_n > 0) = \mathbb{E}_i(q_n), \quad \text{where} \quad q_n^{-1} = e^{-S_n} + \sum_{k=0}^{n-1} e^{-S_k} \eta_{k+1, n} \tag{5.1.1}$$

and under the assumptions of the paper the random variables $\eta_{k+1, n}$ are bounded. Our proof is essentially based on three tools: conditioned limit theorems for Markov chains which have been obtained recently in [38] and [37] (Chapters 3 and 4), the exponential change of measure which is defined with the help of the transfer operator, see Guivarc'h and Hardy [42], and the duality for Markov chains which we develop in Section 5.3.2.

Let us first consider the critical case. Let τ_y be the first moment when the random walk $(y + S_n)_{n \geq 0}$ becomes negative. In the critical case, one can show that only the trajectories that stay positive (i.e. when $\tau_y > n$) have impact on the survival probability, so that the probability $\sqrt{n}\mathbb{P}(Z_n > 0, \tau_y \leq n)$ is negligible as $n \rightarrow +\infty$ and $y \rightarrow +\infty$. This permits to replace the expectation $\sqrt{n}\mathbb{E}_i(q_n)$ by $\sqrt{n}\mathbb{E}_i(q_n; \tau_y > n) = \sqrt{n}\mathbb{E}_i(q_n | \tau_y > n) \mathbb{P}_i(\tau_y > n)$. The asymptotic behaviour of $\sqrt{n}\mathbb{P}_i(\tau_y > n)$ is given in [38] (Chapter 3) and using the local limit theorem from [37] (Chapter 4) we show that $\mathbb{E}_i(q_n | \tau_y > n)$ converges to a positive constant.

The subcritical case is much more delicate. Using the normalized transfer operator $\tilde{\mathbf{P}}_\lambda$ we apply a change of the probability measure, say $\tilde{\mathbb{P}}_i$, under which (5.1.1) reduces to the study of the expectation $k(\lambda)^n \tilde{\mathbb{E}}_i(e^{-\lambda S_n} q_n)$. Choosing $\lambda = 1$, we have $\tilde{\mathbb{E}}_i(e^{-S_n} q_n) = \tilde{\mathbb{E}}_i^*(q_n^*)$, where $\tilde{\mathbb{E}}_i^*$ is the expectation generated by the dual Markov walk $(S_n^*)_{n \geq 0}$,

$$(q_n^*)^{-1} = 1 + \sum_{k=1}^n e^{-S_k^*} \eta_k^* \tag{5.1.2}$$

and the random variables η_k^* are bounded. In the strongly subcritical case the series in (5.1.2) converges by the law of large numbers for $(S_n^*)_{n \geq 0}$, so the resulting rate of convergence is determined only by $k(1)^n$. To find the asymptotic behaviour of the expectation $\tilde{\mathbb{E}}_i^*(q_n^*)$ in the intermediate subcritical case we proceed basically in the same way as in the critical case which explains the apparition of the factor $n^{-1/2}$. In the weakly subcritical case we choose λ to be the critical point of k : $k'(\lambda) = 0$. We make use of the conditioned local limit theorem which, in addition to $k(\lambda)^n$, contributes with the factor $n^{-3/2}$.

The outline of the paper is as follows:

- Section 5.2: We give the necessary notations and formulate the main results.
- Section 5.3: Introduce the associated Markov chain and relate it to the survival probability. Introduce the dual Markov chain. State some useful assertions for walks on Markov chains conditioned to stay positive and on the transfer operator.
- Sections 5.4, 5.5, 5.6 and 5.7: Proofs in the critical, strongly subcritical, intermediate subcritical and weakly subcritical cases, respectively.

Let us end this section by fixing some notations. The symbol c will denote a positive constant depending on the all previously introduced constants. Sometimes, to stress the dependence of the constants on some parameters α, β, \dots we shall use the notations $c_\alpha, c_{\alpha, \beta}, \dots$. All these constants are likely to change their values every occurrence. The indicator of an event A is denoted by $\mathbb{1}_A$. For any bounded measurable function f on \mathbb{X} , random variable X in some measurable space \mathbb{X} and event A , the integral $\int_{\mathbb{X}} f(x) \mathbb{P}(X \in dx, A)$ means the expectation $\mathbb{E}(f(X); A) = \mathbb{E}(f(X) \mathbb{1}_A)$.

5.2 Notations and main results

Assume that $(X_n)_{n \geq 0}$ is a homogeneous Markov chain defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in the finite state space \mathbb{X} . Let \mathcal{C} be the set of functions from \mathbb{X} to \mathbb{C} . Denote by \mathbf{P} the transition operator of the chain $(X_n)_{n \geq 0}$: $\mathbf{P}g(i) = \mathbb{E}_i(g(X_1))$, for any $g \in \mathcal{C}$ and $i \in \mathbb{X}$. Set $\mathbf{P}(i, j) = \mathbf{P}(\delta_j)(i)$, where $\delta_j(i) = 1$ if $i = j$ and $\delta_j(i) = 0$ else. Note that the iterated operator \mathbf{P}^n , $n \geq 0$ is given by $\mathbf{P}^n g(i) = \mathbb{E}_i(g(X_n))$. Let \mathbb{P}_i be the probability on (Ω, \mathcal{F}) generated by the finite dimensional distributions of the Markov chain $(X_n)_{n \geq 0}$ starting at $X_0 = i$. Denote by \mathbb{E} and \mathbb{E}_i the corresponding expectation associated to \mathbb{P} and \mathbb{P}_i .

We assume in the sequel that $(X_n)_{n \geq 0}$ is irreducible and aperiodic. This is known to be equivalent to the following condition:

Condition 5.1. *The matrix \mathbf{P} is primitive, which means that there exists $k_0 \geq 1$ such that, for any non-negative and non-identically zero function $g \in \mathcal{C}$ and $i \in \mathbb{X}$,*

$$\mathbf{P}^{k_0} g(i) > 0.$$

By the Perron-Frobenius theorem, under Condition 5.1, there exist positive constants c_1 and c_2 , a unique positive \mathbf{P} -invariant probability ν on \mathbb{X} and an operator Q on \mathcal{C} such that for any $g \in \mathcal{C}$ and $n \geq 1$,

$$\mathbf{P}g(i) = \nu(g) + Q(g)(i) \quad \text{and} \quad \|Q^n(g)\|_\infty \leq c_1 e^{-c_2 n} \|g\|_\infty,$$

where $\nu(g) := \sum_{i \in \mathbb{X}} g(i)\nu(i)$, $Q(1) = \nu(Q(g)) = 0$ and $\|g\|_\infty = \max_{i \in \mathbb{X}} |g(i)|$. In particular, for any $(i, j) \in \mathbb{X}^2$, we have

$$|\mathbf{P}^n(i, j) - \nu(j)| \leq c_1 e^{-c_2 n}. \tag{5.2.1}$$

The branching process in the Markov environment $(X_n)_{n \geq 0}$ is defined with the help of a collection of generating functions

$$f_i(s) := \mathbb{E}(s^{\xi_i}), \quad \forall i \in \mathbb{X}, \quad s \in [0, 1], \tag{5.2.2}$$

where the random variable ξ_i takes its values in \mathbb{N} and means the total offspring of one individual when the environment is $i \in \mathbb{X}$. For any $i \in \mathbb{X}$, let $(\xi_i^{n,j})_{j,n \geq 1}$ be independent and identically distributed random variables with the same generating function f_i living on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that the sequence $(\xi_i^{n,j})_{j,n \geq 1}$ is independent of the Markov chain $(X_n)_{n \geq 0}$.

Assume that the offspring distribution satisfies the following moment constraints.

Condition 5.2. *For any $i \in \mathbb{X}$, the random variable ξ_i is non-identically zero and has a finite variance:*

$$0 < \mathbb{E}(\xi_i) \quad \text{and} \quad \mathbb{E}(\xi_i^2) < +\infty, \quad \forall i \in \mathbb{X}.$$

Note that, under Condition 5.2 we have,

$$\forall i \in \mathbb{X}, \quad 0 < \mathbb{E}(\xi_i) = f'_i(1) < +\infty.$$

and

$$\forall i \in \mathbb{X}, \quad f''_i(1) = \mathbb{E}(\xi_i^2) - \mathbb{E}(\xi_i) < +\infty.$$

Define the branching process $(Z_n)_{n \geq 0}$ iteratively: for each time $n = 1, 2, \dots$, given the environment $X_n = i$, the total offspring of each individual $j \in \{1, \dots, Z_{n-1}\}$ is given by the random variable $\xi_i^{n,j}$, so that the total population is

$$Z_0 = 1 \quad \text{and} \quad Z_n = \sum_{j=1}^{Z_{n-1}} \xi_{X_n}^{n,j}, \quad \forall n \geq 1. \tag{5.2.3}$$

We shall consider branching processes $(Z_n)_{n \geq 0}$ in one of the following two regimes: critical or subcritical (see below for the precise definition). In both cases the probability that the population survives until the n -th generation tends to zero, $\mathbb{P}(Z_n > 0) \rightarrow 0$ as $n \rightarrow +\infty$, see Smith and Wilkinson [63]. As noted in the introduction, when the environment is i.i.d., the question of determining the speed of this convergence was answered in [32], [44] and [33]. The key point in establishing their results is a close relation between the branching process and the associated random walk. Let us introduce the associated Markov walk corresponding to our setting. Define the real function ρ on \mathbb{X} by

$$\rho(i) = \ln f'_i(1), \quad \forall i \in \mathbb{X}. \tag{5.2.4}$$

The associated Markov walk $(S_n)_{n \geq 0}$ is defined as follows:

$$S_0 := 0 \quad \text{and} \quad S_n := \ln \left(f'_{X_1}(1) \cdots f'_{X_n}(1) \right) = \sum_{k=1}^n \rho(X_k), \quad \forall n \geq 1. \quad (5.2.5)$$

In order to state the precise results we need one more condition, namely that the Markov walk $(S_n)_{n \geq 0}$ is non-lattice:

Condition 5.3. For any $(\theta, a) \in \mathbb{R}^2$, there exist x_0, \dots, x_n in \mathbb{X} such that

$$\mathbf{P}(x_0, x_1) \cdots \mathbf{P}(x_{n-1}, x_n) \mathbf{P}(x_n, x_0) > 0$$

and

$$\rho(x_0) + \cdots + \rho(x_n) - (n+1)\theta \notin a\mathbb{Z}.$$

The following function plays an important role in determining the asymptotic behaviour of the branching processes when the environment is Markovian. It will be shown in Section 5.3.4 that under Conditions 5.1 and 5.3, for any $\lambda \in \mathbb{R}$ and any $i \in \mathbb{X}$, the following limit exists and does not depend on the initial state of the Markov chain $X_0 = i$:

$$k(\lambda) := \lim_{n \rightarrow +\infty} \mathbb{E}_i^{1/n} \left(e^{\lambda S_n} \right).$$

Let us recall some facts on the function k which will be discussed in details in Section 5.3.4 and which are used here for the formulation of the main results. The function k is closely related to the so-called transfer operator \mathbf{P}_λ which is defined for any $\lambda \in \mathbb{R}$ on \mathcal{C} by the relation

$$\mathbf{P}_\lambda g(i) := \mathbf{P} \left(e^{\lambda \rho} g \right) (i) = \mathbb{E}_i \left(e^{\lambda S_1} g(X_1) \right), \quad \text{for } g \in \mathcal{C}, i \in \mathbb{X}. \quad (5.2.6)$$

In particular, $k(\lambda)$ is an eigenvalue of the operator \mathbf{P}_λ corresponding to an eigenvector v_λ and is equal to its spectral radius. Moreover, the function $k(\lambda)$ is analytic on \mathbb{R} , see Lemma 5.3.15. Note also that the transfer operator \mathbf{P}_λ is not Markov, but it can be easily normalized so that the operator $\tilde{\mathbf{P}}_\lambda g = \frac{\mathbf{P}_\lambda(gv_\lambda)}{k(\lambda)v_\lambda}$ is Markovian. We shall denote by $\tilde{\nu}_\lambda$ its unique invariant probability measure.

The branching process in Markovian environment is said to be *subcritical* if $k'(0) < 0$, *critical* if $k'(0) = 0$ and *supercritical* if $k'(0) > 0$. This definition at first glance may appear different from what is expected in the case of branching processes with i.i.d. environment. With a closer look, however, the relation to the usual i.i.d. classification becomes clear from the following identity, which is established in Lemma 5.3.15:

$$k'(0) = \nu(\rho) = \mathbb{E}_\nu(\rho(X_1)) = \mathbb{E}_\nu \left(\ln f'_{X_1}(1) \right), \quad (5.2.7)$$

where \mathbb{E}_ν is the expectation generated by the finite dimensional distributions of the Markov chain $(X_n)_{n \geq 0}$ in the stationary regime, i.e. when the starting point X_0 is a random variable distributed according to the \mathbf{P} -invariant measure ν . In particular, when the environment $(X_n)_{n \geq 0}$ is just an i.i.d. sequence of random variables with common law ν , it follows from (5.2.7) that the two classifications coincide.

We proceed to formulate our main result in the critical case.

Theorem 5.2.1 (Critical case). *Assume Conditions 5.1-5.3 and*

$$k'(0) = 0.$$

Then, there exists a positive function u on \mathbb{X} such that for any $(i, j) \in \mathbb{X}^2$,

$$\mathbb{P}_i(Z_n > 0, X_n = j) \underset{n \rightarrow +\infty}{\sim} \frac{\nu(j)u(i)}{\sqrt{n}}.$$

The asymptotic for the probability that $Z_n > 0$ in the case of i.i.d. environment has been established earlier by Geiger and Kersting [32] under some moment assumptions on the random variable $\rho(X_1) = \ln(f'_{X_1}(1))$, which are weaker than our assumption on finiteness of the state space \mathbb{X} . Since we deal with dependent environment, Theorem 5.2.1 is not covered by the results in [32].

Now we consider the subcritical case. The classification of the asymptotic behaviours of the survival time of a branching process $(Z_n)_{n \geq 0}$ in the subcritical case $k'(0) < 0$ is made in function of the values of $k'(1)$. We say that the branching process in Markovian environment is *strongly subcritical* if $k'(0) < 0, k'(1) < 0$, *intermediately subcritical* if $k'(0) < 0, k'(1) = 0$ and *weakly subcritical* if $k'(0) < 0, k'(1) > 0$. In order to relate these definitions to the values of some moments of the random variable $\ln f'_{X_1}(1)$, we note that, again by Lemma 5.3.15,

$$\frac{k'(1)}{k(1)} = \tilde{\nu}_1(\rho) = \mathbb{E}_{\tilde{\nu}_1}(\rho(X_1)) = \mathbb{E}_{\tilde{\nu}_1}(\ln f'_{X_1}(1)), \tag{5.2.8}$$

where $\mathbb{E}_{\tilde{\nu}_\lambda}$ is the expectation generated by the finite dimensional distributions of the Markov chain $(X_n)_{n \geq 0}$ with transition probabilities $\tilde{\mathbf{P}}_\lambda$ in the stationary regime, i.e. when the starting point X_0 is a random variable distributed according to the unique positive $\tilde{\mathbf{P}}_\lambda$ -invariant probability $\tilde{\nu}_\lambda$. Since $k(1) > 0$, the equivalent classification can be done according to the value of the expectation $\mathbb{E}_{\tilde{\nu}_1}(\ln(f'_{X_1}(1)))$. When the environment is an i.i.d. sequence of common law $\tilde{\nu}$ we have in addition

$$\frac{k'(1)}{k(1)} = \mathbb{E}_{\tilde{\nu}_1}(\ln f'_{X_1}(1)) = \mathbb{E}_\nu(f'_{X_1}(1) \ln f'_{X_1}(1)) = \phi'_\nu(1), \tag{5.2.9}$$

where $\phi_\nu(\lambda) = \mathbb{E}_\nu(e^{\lambda \ln f'_{X_1}(1)})$, $\lambda \in \mathbb{R}$. This shows that both classifications (the one according to the values of $k'(1)$ and the other according to the values of $\phi'_\nu(1)$) for branching processes with i.i.d. environment are equivalent. We would like to stress that, in general, the identity (5.2.9) is not fulfilled for a Markovian environment and therefore the function $\phi_\nu(\lambda)$ is not the appropriate one for the classification. For a Markovian environment the classification equally can be done using the function $K'(\lambda)$, where $K(\lambda) = \ln k(\lambda)$, $\lambda \in \mathbb{R}$.

Note that by Lemma 5.3.15 the function $\lambda \mapsto K(\lambda)$ is strictly convex. In the strongly and intermediate subcritical cases, this implies that $0 < k(\lambda) < 1$.

The following theorem gives the asymptotic behaviour of the survival probability jointly with the state of the Markov chain in the strongly subcritical case.

Theorem 5.2.2 (Strongly subcritical case). *Assume Conditions 5.1-5.3 and*

$$k'(0) < 0, \quad k'(1) < 0.$$

Then, there exists a positive function u on \mathbb{X} such that for any $(i, j) \in \mathbb{X}^2$,

$$\mathbb{P}_i(Z_n > 0, X_n = j) \underset{n \rightarrow +\infty}{\sim} k(1)^n v_1(i)u(j).$$

Recall that v_1 is the eigenfunction of the transfer operator \mathbb{P}_1 (see also Section 5.3.4 eq. (5.3.31) for details). Note also that in the formulation of the Theorem 5.2.2 we can drop the assumption $k'(0) < 0$, since it is implied by the assumption $k'(1) < 0$, by strict convexity of $K(\lambda)$. The corresponding result in the case when the environment is i.i.d. has been established by Guivarc'h and Liu [44] under some moment assumptions on the random variable $\rho(X_1) = \ln(f'_{X_1}(1))$. Our result extends [44] to finite dependent environments.

A break trough in determining the behaviour of the survival probability for intermediate subcritical and weakly subcritical cases for branching processes with i.i.d. environment was made by Geiger, Kersting and Vatutin [33]. Note that the original results in [33] have been established under some moment assumptions on the random variable $\rho(X_1) = \ln(f'_{X_1}(1))$. For these two cases and finite Markovian environments we give below the asymptotic of the survival probability jointly with the state of the Markov chain.

Theorem 5.2.3 (Intermediate subcritical case). *Assume Conditions 5.1-5.3 and*

$$k'(0) < 0, \quad k'(1) = 0.$$

Then, there exists a positive function u on \mathbb{X} such that for any $(i, j) \in \mathbb{X}^2$,

$$\mathbb{P}_i(Z_n > 0, X_n = j) \underset{n \rightarrow +\infty}{\sim} k(1)^n \frac{v_1(i)u(j)}{\sqrt{n}}.$$

As in the previous Theorem 5.2.2, $k'(1) = 0$ implies the assumption $k'(0) < 0$, since the function $\lambda \mapsto K(\lambda) = \ln(k(\lambda))$ is strictly convex (see Lemma 5.3.15).

Theorem 5.2.4 (Weakly subcritical case). *Assume Conditions 5.1-5.3 and*

$$k'(0) < 0, \quad k'(1) > 0.$$

Then, there exist a unique $\lambda \in (0, 1)$ satisfying $k'(\lambda) = 0$ and a positive function u on \mathbb{X}^2 such that for any $(i, j) \in \mathbb{X}^2$,

$$\mathbb{P}_i(Z_n > 0, X_n = j) \underset{n \rightarrow +\infty}{\sim} k(\lambda)^n \frac{u(i, j)}{n^{3/2}}.$$

The existence and the unicity of $\lambda \in (0, 1)$ satisfying $k'(\lambda) = 0$ and $0 < k(\lambda) < 1$ in Theorem 5.2.4 is an obvious consequence of the strict convexity of K . Note that Theorems 5.2.1, 5.2.2, 5.2.3 and 5.2.4 give the asymptotic behaviour of the joint probabilities $\mathbb{P}_i(Z_n > 0, X_n = j)$. By summing both sides of the corresponding equivalences in j we obtain the asymptotic behaviour of the survival probability $\mathbb{P}_i(Z_n > 0)$. The corresponding results for the survival probability when the Markovian environment is in the stationary regime are easily obtained by integrating the previous ones with respect to the invariant measure ν .

5.3 Preliminary results on the Markov walk

The aim of this section is to provide necessary assertions on the Markov chain $(X_n)_{n \geq 0}$ and on the associated Markov walk $(S_n)_{n \geq 0}$ defined by (5.2.5) and to relate them to the survival probability of $(Z_n)_{n \geq 0}$ at generation n . For the ease of the reader we recall the outline of the section:

- Subsection 5.3.1: Relate the branching process $(Z_n)_{n \geq 0}$ to the associated Markov walk $(S_n)_{n \geq 0}$.
- Subsection 5.3.2: Construct the dual Markov chain $(X_n^*)_{n \geq 0}$.
- Subsection 5.3.3: Recall results on the Markov walks conditioned to stay positive.
- Subsection 5.3.4: Introduce the transfer operator of the Markov chain $(X_n)_{n \geq 0}$ and the change of the probability measure. State the properties of the associated Markov walk $(S_n)_{n \geq 0}$ under the changed measure.

5.3.1 The link between the branching process and the associated Markov walk

In this section we recall some identities on the branching process. Some of them are stated for the commodity of the reader and are merely adaptations to the Markovian environments of the well-known statements in the i.i.d. case.

The first one is a representation of the conditioned probability generating function given the environment:

Lemma 5.3.1 (Conditioned generating function). *For any $s \in [0, 1]$ and $n \geq 1$,*

$$\mathbb{E}_i \left(s^{Z_n} \mid X_1, \dots, X_n \right) = f_{X_1} \circ \dots \circ f_{X_n}(s).$$

Proof. For all $s \in [0, 1]$, $n \geq 1$, $(z_1, \dots, z_{n-1}) \in \mathbb{N}^{n-1}$ and $(i_1, \dots, i_n) \in \mathbb{X}^n$, by (5.2.3), we have

$$\mathbb{E}_i \left(s^{Z_n} \mid Z_1 = z_1, \dots, Z_{n-1} = z_{n-1}, X_1 = i_1, \dots, X_n = i_n \right) = \mathbb{E} \left(s^{\sum_{j=1}^{z_{n-1}} \xi_{i_n}^{n,j}} \right).$$

Since $(\xi_{i_n}^{n,j})_{j \geq 1}$ are i.i.d., by (5.2.2),

$$\mathbb{E}_i \left(s^{Z_n} \mid Z_1 = z_1, \dots, Z_{n-1} = z_{n-1}, X_1 = i_1, \dots, X_n = i_n \right) = f_{i_n}(s)^{z_{n-1}}.$$

From this we get,

$$\mathbb{E}_i \left(s^{Z_n} \mid X_1 = i_1, \dots, X_n = i_n \right) = \mathbb{E}_i \left(f_{i_n}(s)^{Z_{n-1}} \mid X_1 = i_1, \dots, X_{n-1} = i_{n-1} \right).$$

By induction, for any $(i_1, \dots, i_n) \in \mathbb{X}^n$,

$$\mathbb{E}_i \left(s^{Z_n} \mid X_1 = i_1, \dots, X_n = i_n \right) = f_{i_1} \circ \dots \circ f_{i_n}(s).$$

and the assertion of the lemma follows. \square

For any $n \geq 1$ and $s \in [0, 1]$ set

$$q_n(s) := 1 - f_{X_1} \circ \dots \circ f_{X_n}(s) \quad \text{and} \quad q_n := q_n(0). \quad (5.3.1)$$

Lemma 5.3.1 implies that

$$\mathbb{P}_i(Z_n > 0 \mid X_1, \dots, X_n) = q_n. \quad (5.3.2)$$

Taking the expectation in (5.3.2), we obtain the well-known equality, which will be the starting point for our study:

$$\mathbb{P}_i(Z_n > 0) = \mathbb{E}_i(q_n). \quad (5.3.3)$$

Under Condition 5.2, for any $i \in \mathbb{X}$ and $s \in [0, 1)$, we have $f_i(s) \in [0, 1)$. Therefore $f_{X_1} \circ \dots \circ f_{X_n}(s) \in [0, 1)$ and in particular

$$q_n \in (0, 1], \quad \forall n \geq 1. \quad (5.3.4)$$

Introduce some additional notations, which will be used all over the paper:

$$f_{k,n} := f_{X_k} \circ \dots \circ f_{X_n}, \quad \forall n \geq 1, \forall k \in \{1, \dots, n\}, \quad (5.3.5)$$

$$f_{n+1,n} := \text{id}, \quad \forall n \geq 1, \quad (5.3.6)$$

$$g_i(s) := \frac{1}{1 - f_i(s)} - \frac{1}{f'_i(1)(1 - s)}, \quad \forall i \in \mathbb{X}, \forall s \in [0, 1), \quad (5.3.7)$$

$$\eta_{k,n}(s) := g_{X_k}(f_{k+1,n}(s)), \quad \forall n \geq 1, \forall k \in \{1, \dots, n\}, \forall s \in [0, 1), \quad (5.3.8)$$

$$\eta_{k,n} := \eta_{k,n}(0) = g_{X_k}(f_{k+1,n}(0)), \quad \forall n \geq 1, \forall k \in \{1, \dots, n\}. \quad (5.3.9)$$

The key point in proving our main results is the following assertion which relies the random variable $q_n(s)$ to the associated Markov walk $(S_n)_{n \geq 0}$, see (5.2.5). This relation is known from Agresti [2] in the case of linear fractional generating functions. It turned out to be very useful for studying general branching processes and was generalized in Geiger and Kersting [32]. We adapt their argument to the case when the environment is Markovian.

Lemma 5.3.2. *For any $s \in [0, 1)$ and $n \geq 1$,*

$$q_n(s)^{-1} = \frac{e^{-S_n}}{1 - s} + \sum_{k=0}^{n-1} e^{-S_k} \eta_{k+1,n}(s).$$

Proof. With the notations (5.3.6)-(5.3.9) we write for any $s \in [0, 1)$ and $n \geq 1$,

$$\begin{aligned} q_n(s)^{-1} &:= \frac{1}{1 - f_{X_1} \circ \dots \circ f_{X_n}(s)} \\ &= \frac{1}{1 - f_{1,n}(s)} \\ &= g_{X_1}(f_{2,n}(s)) + \frac{f'_{X_1}(1)^{-1}}{1 - f_{2,n}(s)} \\ &= \dots \\ &= \frac{(f'_{X_1}(1) \cdots f'_{X_n}(1))^{-1}}{1 - s} + g_{X_1}(f_{2,n}(s)) \\ &\quad + \sum_{k=2}^n (f'_{X_1}(1) \cdots f'_{X_{k-1}}(1))^{-1} g_{X_k}(f_{k+1,n}(s)) \\ &= \frac{e^{-S_n}}{1 - s} + \sum_{k=0}^{n-1} e^{-S_k} \eta_{k+1,n}(s). \end{aligned}$$

□

Taking $s = 0$ in Lemma 5.3.2 we obtain the following identity which will play the central role in the proofs:

$$q_n^{-1} = e^{-S_n} + \sum_{k=0}^{n-1} e^{-S_k} \eta_{k+1,n}, \quad \forall n \geq 1. \quad (5.3.10)$$

Since f_i is convex on $[0, 1]$ for all $i \in \mathbb{X}$, the function g_i is non-negative,

$$\forall s \in [0, 1), \quad g_i(s) = \frac{f'_i(1)(1-s) - (1-f_i(s))}{(1-f_i(s))f'_i(1)(1-s)} \geq 0, \tag{5.3.11}$$

which, in turn, implies that the random variables $\eta_{k+1,n}$ are non-negative for any $n \geq 1$ and $k \in \{0, \dots, n-1\}$.

Lemma 5.3.3. *Assume Condition 5.2. For any $n \geq 2$, $(i_1, \dots, i_n) \in \mathbb{X}^n$ and $s \in [0, 1)$, we have*

$$0 \leq g_{i_1}(f_{i_2} \circ \dots \circ f_{i_n}(s)) \leq \eta := \max_{i \in \mathbb{X}} \frac{f''_i(1)}{f'_i(1)^2} < +\infty.$$

Moreover, for any $(i_n)_{n \geq 1} \in \mathbb{X}^{\mathbb{N}^*}$, and any $k \geq 1$,

$$\lim_{n \rightarrow +\infty} g_{i_k}(f_{i_{k+1}} \circ \dots \circ f_{i_n}(0)) \in [0, \eta]. \tag{5.3.12}$$

Proof. Fix $(i_n)_{n \geq 1} \in \mathbb{X}^{\mathbb{N}^*}$. For any $i \in \mathbb{X}$ and $s \in [0, 1)$, we have $f_i(s) \in [0, 1)$. So $f_{i_2} \circ \dots \circ f_{i_n}(s) \in [0, 1)$. In addition, by (5.3.11), g_i is non-negative on $[0, 1)$ for any $i \in \mathbb{X}$, therefore $g_{i_1}(f_{i_2} \circ \dots \circ f_{i_n}(s)) \geq 0$. Moreover by the lemma 2.1 of [32], for any $i \in \mathbb{X}$ and any $s \in [0, 1)$,

$$g_i(s) \leq \frac{f''_i(1)}{f'_i(1)^2}. \tag{5.3.13}$$

By Condition 5.2, $\eta < +\infty$ and so $g_{i_1}(f_{i_2} \circ \dots \circ f_{i_n}(s)) \in [0, \eta]$, for any $s \in [0, 1)$.

Since f_i is increasing on $[0, 1)$ for any $i \in \mathbb{X}$, it follows that for any $k \geq 1$ and any $n \geq k+1$,

$$0 \leq f_{i_{k+1}} \circ \dots \circ f_{i_n}(0) \leq f_{i_{k+1}} \circ \dots \circ f_{i_n} \circ f_{i_{n+1}}(0) \leq 1,$$

and the sequence $(f_{i_{k+1}} \circ \dots \circ f_{i_n}(0))_{n \geq k+1}$ converges to a limit, say $l \in [0, 1]$. For any $i \in \mathbb{X}$, the function g_i is continuous on $[0, 1)$ and we have

$$\begin{aligned} \lim_{\substack{s \rightarrow 1 \\ s < 1}} g_i(s) &= \lim_{\substack{s \rightarrow 1 \\ s < 1}} \frac{f'_i(1)(1-s) - (1-f_i(s))}{f'_i(1)(1-f_i(s))(1-s)} = \lim_{\substack{s \rightarrow 1 \\ s < 1}} \frac{1}{f'_i(1)} \frac{f_i(s) - 1 - f'_i(1)(s-1)}{(s-1)^2} \frac{1-s}{1-f_i(s)} \\ &= \frac{1}{f'_i(1)} \frac{f''_i(1)}{2} \frac{1}{f'_i(1)} = \frac{f''_i(1)}{2f'_i(1)^2} < +\infty. \end{aligned} \tag{5.3.14}$$

Denoting $g_i(l) = \frac{f''_i(1)}{2f'_i(1)^2}$ if $l = 1$, we conclude that $g_{i_k}(f_{i_{k+1}} \circ \dots \circ f_{i_n}(0))$ converges to $g_{i_k}(l)$ as $n \rightarrow +\infty$. By (5.3.11) and (5.3.13), we obtain that $g_{i_k}(l) \in [0, \eta]$. \square

5.3.2 The dual Markov walk

We will introduce the dual Markov chain $(X_n^*)_{n \geq 0}$ and the associated dual Markov walk $(S_n^*)_{n \geq 0}$ and state some of their properties.

Since ν is positive on \mathbb{X} , the following dual Markov kernel \mathbf{P}^* is well defined:

$$\mathbf{P}^*(i, j) = \frac{\nu(j)}{\nu(i)} \mathbf{P}(j, i), \quad \forall (i, j) \in \mathbb{X}^2. \tag{5.3.15}$$

Let $(X_n^*)_{n \geq 0}$ be a dual Markov chain, independent of the chain $(X_n)_{n \geq 0}$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$, living on \mathbb{X} and with transition probability \mathbf{P}^* . We define the dual Markov walk by

$$S_0^* = 0 \quad \text{and} \quad S_n^* = - \sum_{k=1}^n \rho(X_k^*), \quad \forall n \geq 1. \tag{5.3.16}$$

For any $z \in \mathbb{R}$, let τ_z^* be the associated exit time

$$\tau_z^* := \inf \{k \geq 1 : z + S_k^* \leq 0\}. \tag{5.3.17}$$

For any $i \in \mathbb{X}$, denote by \mathbb{P}_i^* and \mathbb{E}_i^* the probability, respectively the expectation generated by the finite dimensional distributions of the Markov chain $(X_n^*)_{n \geq 0}$ starting at $X_0^* = i$.

It is easy to see that ν is also \mathbf{P}^* -invariant and for any $n \geq 1$, $(i, j) \in \mathbb{X}^2$,

$$(\mathbf{P}^*)^n(i, j) = \mathbf{P}^n(j, i) \frac{\nu(j)}{\nu(i)}.$$

This last formula implies in particular the following result.

Lemma 5.3.4. *Assume Conditions 5.1 and 5.3 for the Markov kernel \mathbf{P} . Then Conditions 5.1 and 5.3 hold also for dual kernel \mathbf{P}^* .*

Similarly to (5.2.1), we have for any $(i, j) \in \mathbb{X}^2$,

$$|(\mathbf{P}^*)^n(i, j) - \nu(j)| \leq c e^{-cn}. \tag{5.3.18}$$

Note that the operator \mathbf{P}^* is the adjoint of \mathbf{P} in the space $L^2(\nu)$: for any functions f and g on \mathbb{X} ,

$$\nu(f(\mathbf{P}^*)^n g) = \nu(g\mathbf{P}^n f).$$

For any measure \mathbf{m} on \mathbb{X} , let $\mathbb{E}_{\mathbf{m}}$ (respectively $\mathbb{E}_{\mathbf{m}}^*$) be the expectation associated to the probability generated by the finite dimensional distributions of the Markov chain $(X_n)_{n \geq 0}$ (respectively $(X_n^*)_{n \geq 0}$) with the initial law \mathbf{m} .

Lemma 5.3.5 (Duality). *For any probability measure \mathbf{m} on \mathbb{X} , any $n \geq 1$ and any function $g: \mathbb{X}^n \rightarrow \mathbb{C}$,*

$$\mathbb{E}_{\mathbf{m}}(g(X_1, \dots, X_n)) = \mathbb{E}_{\nu}^* \left(g(X_n^*, \dots, X_1^*) \frac{\mathbf{m}(X_{n+1}^*)}{\nu(X_{n+1}^*)} \right).$$

Moreover, for any $n \geq 1$ and any function $g: \mathbb{X}^n \rightarrow \mathbb{C}$,

$$\mathbb{E}_i(g(X_1, \dots, X_n); X_{n+1} = j) = \mathbb{E}_j^* \left(g(X_n^*, \dots, X_1^*); X_{n+1}^* = i \right) \frac{\nu(j)}{\nu(i)}.$$

Proof. The first equality is proved in Lemma 3.2 of [37] (Lemma 4.3.2 of Chapter 4). The second can be deduced from the first as follows. Taking $\mathbf{m} = \delta_i$ and $\tilde{g}(i_1, \dots, i_n, i_{n+1}) = g(i_1, \dots, i_n) \mathbb{1}_{\{i_{n+1}=j\}}$, from the first equality of the lemma, we see that

$$\begin{aligned} \mathbb{E}_i(g(X_1, \dots, X_n); X_{n+1} = j) &= \mathbb{E}_{\nu}^* \left(\tilde{g}(X_{n+1}^*, \dots, X_1^*); X_{n+2}^* = i \right) \frac{1}{\nu(i)} \\ &= \mathbb{E}_{\nu}^* \left(g(X_{n+1}^*, \dots, X_2^*); X_1^* = j, X_{n+2}^* = i \right) \frac{1}{\nu(i)}. \end{aligned}$$

Since ν is \mathbf{P}^* -invariant, we obtain

$$\begin{aligned} \mathbb{E}_i(g(X_1, \dots, X_n); X_{n+1} = j) &= \sum_{i_1 \in \mathbb{X}} \mathbb{E}_{i_1}^* \left(g(X_n^*, \dots, X_1^*); X_{n+1}^* = i \right) \frac{1}{\nu(i)} \mathbb{1}_{\{i_1=j\}} \nu(i_1) \\ &= \mathbb{E}_j^* \left(g(X_n^*, \dots, X_1^*); X_{n+1}^* = i \right) \frac{\nu(j)}{\nu(i)}. \end{aligned}$$

□

5.3.3 Markov walks conditioned to stay positive

In this section we recall the main results from [38] and [37] (Chapters 3 and 4) for Markov walks conditioned to stay positive. We complement these results by some new assertions which will be used in the proofs.

For any $y \in \mathbb{R}$ define the first time when the Markov walk $(S_n)_{n \geq 0}$ becomes non-positive by setting

$$\tau_y := \inf \{k \geq 1 : y + S_k \leq 0\}.$$

Under Conditions 5.1, 5.3 and $\nu(\rho) = 0$ the stopping time τ_y is well defined and finite \mathbb{P}_i -almost surely for any $i \in \mathbb{X}$.

The following three assertions deal with the existence of the harmonic function, the limit behaviour of the probability of the exit time and of the law of the random walk $y + S_n$, conditioned to stay positive and are taken from [38] (Chapter 3).

Proposition 5.3.6 (Preliminary results, part I). *Assume Conditions 5.1, 5.3 and $\nu(\rho) = 0$. There exists a non-negative function V on $\mathbb{X} \times \mathbb{R}$ such that*

1. For any $(i, y) \in \mathbb{X} \times \mathbb{R}$ and $n \geq 1$,

$$\mathbb{E}_i(V(X_n, y + S_n) ; \tau_y > n) = V(i, y).$$

2. For any $i \in \mathbb{X}$, the function $V(i, \cdot)$ is non-decreasing and for any $(i, y) \in \mathbb{X} \times \mathbb{R}$,

$$V(i, y) \leq c(1 + \max(y, 0)).$$

3. For any $i \in \mathbb{X}$, $y > 0$ and $\delta \in (0, 1)$,

$$(1 - \delta)y - c_\delta \leq V(i, y) \leq (1 + \delta)y + c_\delta.$$

We define

$$\sigma^2 := \nu(\rho^2) - \nu(\rho)^2 + 2 \sum_{n=1}^{+\infty} [\nu(\rho \mathbf{P}^n \rho) - \nu(\rho)^2]. \tag{5.3.19}$$

It is known that under Conditions 5.1 and 5.3 we have $\sigma^2 > 0$, see Lemma 10.3 in [37] (Lemma 4.10.3 of Chapter 4).

Proposition 5.3.7 (Preliminary results, part II). *Assume $\nu(\rho) = 0$, Conditions 5.1 and 5.3.*

1. For any $(i, y) \in \mathbb{X} \times \mathbb{R}$,

$$\lim_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(\tau_y > n) = \frac{2V(i, y)}{\sqrt{2\pi\sigma}},$$

where σ is defined by (5.3.19).

2. For any $(i, y) \in \mathbb{X} \times \mathbb{R}$ and $n \geq 1$,

$$\mathbb{P}_i(\tau_y > n) \leq c \frac{1 + \max(y, 0)}{\sqrt{n}}.$$

We denote by $\text{supp}(V) = \{(i, y) \in \mathbb{X} \times \mathbb{R} : V(i, y) > 0\}$ the support of the function V . Note that from property 3 of Proposition 5.3.6, for any fixed $i \in \mathbb{X}$, the function $y \mapsto V(i, y)$ is positive for large y . For more details on the properties of $\text{supp}(V)$ see [38] (Chapter 3).

Proposition 5.3.8 (Preliminary results, part III). *Assume Conditions 5.1, 5.3 and $\nu(\rho) = 0$.*

1. For any $(i, y) \in \text{supp}(V)$ and $t \geq 0$,

$$\mathbb{P}_i \left(\frac{y + S_n}{\sigma\sqrt{n}} \leq t \mid \tau_y > n \right) \xrightarrow{n \rightarrow +\infty} \Phi^+(t),$$

where $\Phi^+(t) = 1 - e^{-\frac{t^2}{2}}$ is the Rayleigh distribution function.

2. There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $n \geq 1$, $t_0 > 0$, $t \in [0, t_0]$ and $(i, y) \in \mathbb{X} \times \mathbb{R}$,

$$\left| \mathbb{P}_i \left(y + S_n \leq t\sqrt{n}\sigma, \tau_y > n \right) - \frac{2V(i, y)}{\sqrt{2\pi}n\sigma} \Phi^+(t) \right| \leq c_{\varepsilon, t_0} \frac{(1 + \max(y, 0)^2)}{n^{1/2+\varepsilon}}.$$

The next assertions are two local limit theorems for the associated Markov walk $y + S_n$ from [37] (Chapter 4).

Proposition 5.3.9 (Preliminary results, part IV). *Assume Conditions 5.1, 5.3 and $\nu(\rho) = 0$.*

1. For any $i \in \mathbb{X}$, $a > 0$, $y \in \mathbb{R}$, $z \geq 0$ and any non-negative function $\psi: \mathbb{X} \rightarrow \mathbb{R}_+$,

$$\begin{aligned} \lim_{n \rightarrow +\infty} n^{3/2} \mathbb{E}_i (\psi(X_n); y + S_n \in [z, z + a], \tau_y > n) \\ = \frac{2V(i, y)}{\sqrt{2\pi}\sigma^3} \int_z^{z+a} \mathbb{E}_{\nu}^* (\psi(X_1^*) V^*(X_1^*, z' + S_1^*); \tau_{z'}^* > 1) dz'. \end{aligned}$$

2. Moreover, for any $a > 0$, $y \in \mathbb{R}$, $z \geq 0$, $n \geq 1$ and any non-negative function $\psi: \mathbb{X} \rightarrow \mathbb{R}_+$,

$$\sup_{i \in \mathbb{X}} \mathbb{E}_i (\psi(X_n); y + S_n \in [z, z + a], \tau_y > n) \leq \frac{c(1 + a^3)}{n^{3/2}} \|\psi\|_{\infty} (1 + z) (1 + \max(y, 0)).$$

Recall that the dual chain $(X_n^*)_{n \geq 0}$ is constructed independently of the chain $(X_n)_{n \geq 0}$. For any $(i, j) \in \mathbb{X}^2$, the probability generated by the finite dimensional distributions of the two dimensional Markov chain $(X_n, X_n^*)_{n \geq 0}$ starting at $(X_0, X_0^*) = (i, j)$ is given by $\mathbb{P}_{i,j} = \mathbb{P}_i \times \mathbb{P}_j$. Let $\mathbb{E}_{i,j}$ be the corresponding expectation. For any $l \geq 1$ we define $\mathcal{C}^+(\mathbb{X}^l \times \mathbb{R}_+)$ the set of non-negative function $g: \mathbb{X}^l \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying the following properties:

- for any $(i_1, \dots, i_l) \in \mathbb{X}^l$, the function $z \mapsto g(i_1, \dots, i_l, z)$ is continuous,
- there exists $\varepsilon > 0$ such that $\max_{i_1, \dots, i_l \in \mathbb{X}} \sup_{z \geq 0} g(i_1, \dots, i_l, z)(1 + z)^{2+\varepsilon} < +\infty$.

Proposition 5.3.10 (Preliminary results, part V). *Assume Conditions 5.1, 5.3 and $\nu(\rho) = 0$. For any $i \in \mathbb{X}$, $y \in \mathbb{R}$, $l \geq 1$, $m \geq 1$ and $g \in \mathcal{C}^+(\mathbb{X}^{l+m} \times \mathbb{R}_+)$,*

$$\begin{aligned} \lim_{n \rightarrow +\infty} n^{3/2} \mathbb{E}_i (g(X_1, \dots, X_l, X_{n-m+1}, \dots, X_n, y + S_n); \tau_y > n) \\ = \frac{2}{\sqrt{2\pi}\sigma^3} \int_0^{+\infty} \sum_{j \in \mathbb{X}} \mathbb{E}_{i,j} (g(X_1, \dots, X_l, X_m^*, \dots, X_1^*, z) \\ \times V(X_l, y + S_l) V^*(X_m^*, z + S_m^*); \tau_y > l, \tau_z^* > m) \nu(j) dz. \end{aligned}$$

We complete these results by determining the asymptotic behaviour of the law of the Markov chain $(X_n)_{n \geq 1}$ jointly with $\{\tau_y > n\}$.

Lemma 5.3.11. *Assume Conditions 5.1, 5.3 and $\nu(\rho) = 0$. Then, for any $(i, y) \in \mathbb{X} \times \mathbb{R}$ and $j \in \mathbb{X}$, we have*

$$\lim_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(X_n = j, \tau_y > n) = \frac{2V(i, y)\nu(j)}{\sqrt{2\pi\sigma}}.$$

Proof. Fix $(i, y) \in \mathbb{X} \times \mathbb{R}$ and $j \in \mathbb{X}$. We will prove that

$$\begin{aligned} \frac{2V(i, y)\nu(j)}{\sqrt{2\pi\sigma}} &\leq \liminf_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(X_n = j, \tau_y > n) \\ &\leq \limsup_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(X_n = j, \tau_y > n) \leq \frac{2V(i, y)\nu(j)}{\sqrt{2\pi\sigma}}. \end{aligned}$$

The upper bound. By the Markov property, for any $n \geq 1$ and $k = \lfloor n^{1/4} \rfloor$ we have

$$\mathbb{P}_i(X_n = j, \tau_y > n) \leq \mathbb{P}_i(X_n = j, \tau_y > n - k) = \mathbb{E}_i(\mathbf{P}^k(X_{n-k}, j); \tau_y > n - k).$$

Using (5.2.1), we obtain that

$$\mathbb{P}_i(X_n = j, \tau_y > n) \leq (\nu(j) + ce^{-ck}) \mathbb{P}_i(\tau_y > n - k).$$

Using the point 1 of Proposition 5.3.7 and the fact that $k = \lfloor n^{1/4} \rfloor$,

$$\limsup_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(X_n = j, \tau_y > n) \leq \frac{2V(i, y)\nu(j)}{\sqrt{2\pi\sigma}}. \tag{5.3.20}$$

The lower bound. Again, let $n \geq 1$ and $k = \lfloor n^{1/4} \rfloor$. We have

$$\mathbb{P}_i(X_n = j, \tau_y > n) \geq \mathbb{P}_i(X_n = j, \tau_y > n - k) - \mathbb{P}_i(n - k < \tau_y \leq n). \tag{5.3.21}$$

As for the upper bound, using the Markov property and (5.2.1),

$$\begin{aligned} \mathbb{P}_i(X_n = j, \tau_y > n - k) &= \mathbb{E}_i(\mathbf{P}^k(X_{n-k}, j); \tau_y > n - k) \\ &\geq (\nu(j) - ce^{-ck}) \mathbb{P}_i(\tau_y > n - k). \end{aligned}$$

Using the point 1 of Proposition 5.3.7 and using the fact that $k = \lfloor n^{1/4} \rfloor$,

$$\liminf_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(X_n = j, \tau_y > n - k) \geq \frac{2V(i, y)\nu(j)}{\sqrt{2\pi\sigma}}. \tag{5.3.22}$$

Furthermore, on the event $\{n - k < \tau_y \leq n\}$, we have

$$0 \geq \min_{n-k < i \leq n} y + S_i \geq y + S_{n-k} - k \|\rho\|_\infty,$$

where $\|\rho\|_\infty$ is the maximum of $|\rho|$ on \mathbb{X} . Consequently,

$$\begin{aligned} \mathbb{P}_i(n - k < \tau_y \leq n) &\leq \mathbb{P}_i(y + S_{n-k} \leq ck, \tau_y > n - k) \\ &= \mathbb{P}_i\left(y + S_{n-k} \leq \frac{ck}{\sqrt{n-k}} \sqrt{n-k}, \tau_y > n - k\right). \end{aligned}$$

Now, using the point 2 of Proposition 5.3.8 with $t_0 = \max_{n \geq 1} \frac{ck}{\sqrt{n-k}}$, we obtain that, for $\varepsilon > 0$ small enough,

$$\mathbb{P}_i(n-k < \tau_y \leq n) \leq \frac{2V(i, y)}{\sqrt{2\pi(n-k)\sigma}} \left(1 - e^{-\frac{ck^2}{2(n-k)}}\right) + c_\varepsilon \frac{(1+y^2)}{(n-k)^{1/2+\varepsilon}}.$$

Therefore, since $k = \lfloor n^{1/4} \rfloor$,

$$\lim_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(n-k < \tau_y \leq n) = 0. \quad (5.3.23)$$

Putting together (5.3.21), (5.3.22) and (5.3.23), we conclude that

$$\liminf_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(X_n = j, \tau_y > n) \geq \frac{2V(i, y)\nu(j)}{\sqrt{2\pi}\sigma},$$

which together with (5.3.20) concludes the proof of the lemma. \square

Now, with the help of the function V from Proposition 5.3.6, for any $(i, y) \in \text{supp}(V)$, we define a new probability $\mathbb{P}_{i,y}^+$ on $\sigma(X_n, n \geq 1)$ and the corresponding expectation $\mathbb{E}_{i,y}^+$, which are characterized by the following property: for any $n \geq 1$ and any $g: \mathbb{X}^n \rightarrow \mathbb{C}$,

$$\mathbb{E}_{i,y}^+(g(X_1, \dots, X_n)) := \frac{1}{V(i, y)} \mathbb{E}_i(g(X_1, \dots, X_n) V(X_n, y + S_n); \tau_y > n). \quad (5.3.24)$$

The fact that $\mathbb{P}_{i,y}^+$ is a probability measure and that it does not depend on n follows easily from the point 1 of Proposition 5.3.6. The probability $\mathbb{P}_{i,y}^+$ is extended obviously to the hole probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The corresponding expectation is again denoted by $\mathbb{E}_{i,y}^+$.

Lemma 5.3.12. *Assume Conditions 5.1, 5.3 and $\nu(\rho) = 0$. Let $m \geq 1$. For any $n \geq 1$, bounded measurable function $g: \mathbb{X}^m \rightarrow \mathbb{C}$, $(i, y) \in \text{supp}(V)$ and $j \in \mathbb{X}$,*

$$\lim_{n \rightarrow +\infty} \mathbb{E}_i(g(X_1, \dots, X_m); X_n = j | \tau_y > n) = \mathbb{E}_{i,y}^+(g(X_1, \dots, X_m)) \nu(j).$$

Proof. For the sake of brevity, for any $(i, j) \in \mathbb{X}^2$, $y \in \mathbb{R}$ and $n \geq 1$, set

$$J_n(i, j, y) := \mathbb{P}_i(X_n = j, \tau_y > n).$$

Fix $m \geq 1$ and let g be a function $\mathbb{X}^m \rightarrow \mathbb{C}$. By the point 1 of Proposition 5.3.7, it is clear that for any $(i, y) \in \text{supp}(V)$ and n large enough, $\mathbb{P}_i(\tau_y > n) > 0$. By the Markov property, for any $j \in \mathbb{X}$ and $n \geq m+1$ large enough,

$$\begin{aligned} I_0 &:= \mathbb{E}_i(g(X_1, \dots, X_m); X_n = j | \tau_y > n) \\ &= \mathbb{E}_i\left(g(X_1, \dots, X_m) \frac{J_{n-m}(X_m, j, y + S_m)}{\mathbb{P}_i(\tau_y > n)}; \tau_y > m\right). \end{aligned}$$

Using Lemma 5.3.11 and the point 1 of Proposition 5.3.7, by the Lebesgue dominated convergence theorem,

$$\begin{aligned} \lim_{n \rightarrow +\infty} I_0 &= \mathbb{E}_i\left(g(X_1, \dots, X_m) \frac{V(X_m, y + S_m)}{V(i, y)}; \tau_y > m\right) \nu(j) \\ &= \mathbb{E}_{i,y}^+(g(X_1, \dots, X_m)) \nu(j). \end{aligned}$$

\square

Lemma 5.3.13. *Assume Conditions 5.1, 5.3 and $\nu(\rho) = 0$. For any $(i, y) \in \text{supp}(V)$, we have, for any $k \geq 1$,*

$$\mathbb{E}_{i,y}^+ (e^{-S_k}) \leq \frac{c(1 + \max(y, 0)) e^y}{k^{3/2} V(i, y)}.$$

In particular,

$$\mathbb{E}_{i,y}^+ \left(\sum_{k=0}^{+\infty} e^{-S_k} \right) \leq \frac{c(1 + \max(y, 0)) e^y}{V(i, y)}.$$

Proof. By (5.3.24), for any $k \geq 1$,

$$\mathbb{E}_{i,y}^+ (e^{-S_k}) = \mathbb{E}_i \left(e^{-S_k} \frac{V(X_k, y + S_k)}{V(i, y)} ; \tau_y > k \right).$$

Using the point 2 of Proposition 5.3.6,

$$\begin{aligned} \mathbb{E}_{i,y}^+ (e^{-S_k}) &\leq e^y \mathbb{E}_i \left(e^{-(y+S_k)} \frac{c(1 + \max(0, y + S_k))}{V(i, y)} ; \tau_y > k \right) \\ &= e^y \sum_{p=0}^{+\infty} \mathbb{E}_i \left(e^{-(y+S_k)} \frac{c(1 + \max(0, y + S_k))}{V(i, y)} ; y + S_k \in (p, p + 1], \tau_y > k \right) \\ &\leq e^y \sum_{p=0}^{+\infty} e^{-p} \frac{c(1 + p)}{V(i, y)} \mathbb{P}_i (y + S_k \in [p, p + 1], \tau_y > k). \end{aligned}$$

By the point 2 of Proposition 5.3.9,

$$\begin{aligned} \mathbb{E}_{i,y}^+ (e^{-S_k}) &\leq \frac{c}{k^{3/2}} \sum_{p=0}^{+\infty} e^{-p} (1 + p)^2 \frac{e^y (1 + \max(0, y))}{V(i, y)} \\ &= \frac{c(1 + \max(0, y)) e^y}{k^{3/2} V(i, y)}. \end{aligned}$$

This proves the first inequality of the lemma. Summing both sides in k and using the Lebesgue monotone convergence theorem, it proves also the second inequality of the lemma. \square

5.3.4 The change of measure related to the Markov walk

In this section we shall establish some useful properties of the Markov chain under the exponential change of the probability measure, which will be crucial in the proofs of the results of the paper.

For any $\lambda \in \mathbb{R}$, let \mathbf{P}_λ be the transfer operator defined on \mathcal{C} by, for any $g \in \mathcal{C}$ and $i \in \mathbb{X}$,

$$\mathbf{P}_\lambda g(i) := \mathbf{P} (e^{\lambda \rho} g) (i) = \mathbb{E}_i (e^{\lambda S_1} g(X_1)). \quad (5.3.25)$$

From the Markov property, it follows easily that, for any $g \in \mathcal{C}$, $i \in \mathbb{X}$ and $n \geq 0$,

$$\mathbf{P}_\lambda^n g(i) = \mathbb{E}_i (e^{\lambda S_n} g(X_n)). \quad (5.3.26)$$

For any non-negative function $g \geq 0$, $\lambda \in \mathbb{R}$, $i \in \mathbb{X}$ and $n \geq 1$, we have

$$\mathbf{P}_\lambda^n g(i) \geq \min_{x_1, \dots, x_n \in \mathbb{X}^n} e^{\lambda(\rho(x_1) + \dots + \rho(x_n))} \mathbf{P}^n g(i). \quad (5.3.27)$$

Therefore the matrix \mathbf{P}_λ is primitive *i.e.* satisfies the Condition 5.1. By the Perron-Frobenius theorem, there exists a positive number $k(\lambda) > 0$, a positive function $v_\lambda : \mathbb{X} \rightarrow \mathbb{R}_+^*$, a positive linear form $\nu_\lambda : \mathcal{C} \rightarrow \mathbb{C}$ and a linear operator Q_λ on \mathcal{C} such that for any $g \in \mathcal{C}$, and $i \in \mathbb{X}$,

$$\mathbf{P}_\lambda g(i) = k(\lambda)\nu_\lambda(g)v_\lambda(i) + Q_\lambda(g)(i), \tag{5.3.28}$$

$$\nu_\lambda(v_\lambda) = 1 \quad \text{and} \quad Q_\lambda(v_\lambda) = \nu_\lambda(Q_\lambda(g)) = 0, \tag{5.3.29}$$

where the spectral radius of Q_λ is strictly less than $k(\lambda)$:

$$\frac{\|Q_\lambda^n(g)\|_\infty}{k(\lambda)^n} \leq c_\lambda e^{-c_\lambda n} \|g\|_\infty. \tag{5.3.30}$$

Note that, in particular, $k(\lambda)$ is equal to the spectral radius of \mathbf{P}_λ , and, moreover, $k(\lambda)$ is an eigenvalue associated to the eigenvector v_λ :

$$\mathbf{P}_\lambda v_\lambda(i) = k(\lambda)v_\lambda(i). \tag{5.3.31}$$

From (5.3.28) and (5.3.29), we have for any $n \geq 1$,

$$\mathbf{P}_\lambda^n g(i) = k(\lambda)^n \nu_\lambda(g)v_\lambda(i) + Q_\lambda^n(g)(i). \tag{5.3.32}$$

By (5.3.30), for any $g \in \mathcal{C}$ and $i \in \mathbb{X}$,

$$\lim_{n \rightarrow +\infty} \frac{\mathbf{P}_\lambda^n g(i)}{k(\lambda)^n} = \nu_\lambda(g)v_\lambda(i)$$

and so for any non-negative and non-identically zero function $g \in \mathcal{C}$ and $i \in \mathbb{X}$,

$$k(\lambda) = \lim_{n \rightarrow +\infty} (\mathbf{P}_\lambda^n g(i))^{1/n} = \lim_{n \rightarrow +\infty} \mathbb{E}_i^{1/n} \left(e^{\lambda S_n} g(X_n) \right). \tag{5.3.33}$$

Note that when $\lambda = 0$, we have $k(0) = 1$, $v_0(i) = 1$ and $\nu_0(i) = \nu(i)$, for any $i \in \mathbb{X}$. However, in general case, the operator \mathbf{P}_λ is no longer a Markov operator and we define $\tilde{\mathbf{P}}_\lambda$ for any $\lambda \in \mathbb{R}$ by

$$\tilde{\mathbf{P}}_\lambda g(i) = \frac{\mathbf{P}_\lambda(gv_\lambda)(i)}{k(\lambda)v_\lambda(i)} = \frac{\mathbf{P} \left(e^{\lambda \rho} gv_\lambda \right) (i)}{k(\lambda)v_\lambda(i)} = \frac{\mathbb{E}_i \left(e^{\lambda S_1} g(X_1)v_\lambda(X_1) \right)}{k(\lambda)v_\lambda(i)}, \tag{5.3.34}$$

for any $g \in \mathcal{C}$ and $i \in \mathbb{X}$. It is clear that $\tilde{\mathbf{P}}_\lambda$ is a Markov operator: by (5.3.31),

$$\tilde{\mathbf{P}}_\lambda v_0(i) = \frac{\mathbf{P}_\lambda(v_\lambda)(i)}{k(\lambda)v_\lambda(i)} = 1,$$

where for any $i \in \mathbb{X}$, $v_0(i) = 1$. Iterating (5.3.34) and using (5.3.26), we see that for any $n \geq 1$, $g \in \mathcal{C}$ and $i \in \mathbb{X}$.

$$\tilde{\mathbf{P}}_\lambda^n g(i) = \frac{\mathbf{P}_\lambda^n(gv_\lambda)(i)}{k(\lambda)^n v_\lambda(i)} = \frac{\mathbb{E}_i \left(e^{\lambda S_n} g(X_n)v_\lambda(X_n) \right)}{k(\lambda)^n v_\lambda(i)}. \tag{5.3.35}$$

In particular, as in (5.3.27),

$$\tilde{\mathbf{P}}_\lambda^n g(i) \geq \min_{x_1, \dots, x_n \in \mathbb{X}^n} e^{\lambda(\rho(x_1) + \dots + \rho(x_n))} v_\lambda(x_n) \frac{\mathbf{P}^n g(i)}{k(\lambda)^n v_\lambda(i)}.$$

The following lemma is an easy consequence of this last inequality.

Lemma 5.3.14. *Assume Conditions 5.1 and 5.3 for the Markov kernel \mathbf{P} . Then for any $\lambda \in \mathbb{R}$, Conditions 5.1 and 5.3 hold also for the operator $\tilde{\mathbf{P}}_\lambda$.*

Using (5.3.32) and (5.3.35), the spectral decomposition of $\tilde{\mathbf{P}}_\lambda$ is given by

$$\tilde{\mathbf{P}}_\lambda^n g(i) = \nu_\lambda(gv_\lambda) v_0(i) + \frac{Q_\lambda^n(gv_\lambda)(i)}{k(\lambda)^n v_\lambda(i)} = \tilde{\nu}_\lambda(g) v_0(i) + \tilde{Q}_\lambda^n(g)(i),$$

with, for any $\lambda \in \mathbb{R}$, $g \in \mathcal{C}$ and $i \in \mathbb{X}$,

$$\tilde{\nu}_\lambda(g) := \nu_\lambda(gv_\lambda) \quad \text{and} \quad \tilde{Q}_\lambda(g)(i) := \frac{Q_\lambda(gv_\lambda)(i)}{k(\lambda)v_\lambda(i)}. \tag{5.3.36}$$

By (5.3.29),

$$\tilde{\nu}_\lambda(\tilde{Q}_\lambda(g)) = \nu_\lambda\left(\frac{Q_\lambda(gv_\lambda)}{k(\lambda)}\right) = 0 \quad \text{and} \quad \tilde{Q}_\lambda(v_0) = \frac{Q_\lambda(v_\lambda)(i)}{k(\lambda)v_\lambda(i)} = 0.$$

Consequently, $\tilde{\nu}_\lambda$ is the positive invariant measure of $\tilde{\mathbf{P}}_\lambda$ and since by (5.3.30),

$$\|\tilde{Q}_\lambda^n(g)\|_\infty \leq \frac{\|Q_\lambda^n(gv_\lambda)\|_\infty}{k(\lambda)^n \min_{i \in \mathbb{X}} v_\lambda} \leq c_\lambda e^{-c_\lambda n} \|g\|_\infty,$$

we can conclude that for any $(i, j) \in \mathbb{X}^2$,

$$|\tilde{\mathbf{P}}_\lambda^n(i, j) - \tilde{\nu}_\lambda(j)| \leq c_\lambda e^{-c_\lambda n}.$$

Fix $\lambda \in \mathbb{R}$ and let $\tilde{\mathbb{P}}_i$ and $\tilde{\mathbb{E}}_i$ be the probability, respectively the expectation, generated by the finite dimensional distributions of the Markov chain $(X_n)_{n \geq 0}$ with transition operator $\tilde{\mathbf{P}}_\lambda$ and starting at $X_0 = i$. For any $n \geq 1$, $g: \mathbb{X}^n \rightarrow \mathbb{C}$ and $i \in \mathbb{X}$,

$$\tilde{\mathbb{E}}_i(g(X_1, \dots, X_n)) := \frac{\mathbb{E}_i(e^{\lambda S_n} g(X_1, \dots, X_n) v_\lambda(X_n))}{k(\lambda)^n v_\lambda(i)}. \tag{5.3.37}$$

We are now interested in establishing some properties of the function $\lambda \mapsto k(\lambda)$ which are important to distinguish between the different subcritical cases.

Lemma 5.3.15. *Assume Conditions 5.1 and 5.3. The function $\lambda \mapsto k(\lambda)$ is analytic on \mathbb{R} . Moreover the function $K: \lambda \mapsto \ln(k(\lambda))$ is strictly convex and satisfies for any $\lambda \in \mathbb{R}$,*

$$K'(\lambda) = \frac{k'(\lambda)}{k(\lambda)} = \tilde{\nu}_\lambda(\rho), \tag{5.3.38}$$

and

$$K''(\lambda) = \tilde{\nu}_\lambda(\rho^2) - \tilde{\nu}_\lambda(\rho)^2 + 2 \sum_{n=1}^{+\infty} [\tilde{\nu}_\lambda(\rho \tilde{\mathbf{P}}_\lambda^n \rho) - \tilde{\nu}_\lambda(\rho)^2] =: \tilde{\sigma}_\lambda^2. \tag{5.3.39}$$

Proof. It is clear that $\lambda \mapsto \mathbf{P}_\lambda$ is analytic on \mathbb{R} and consequently, by the perturbation theory for linear operators (see for example [50] or [24]) $\lambda \rightarrow k(\lambda)$, $\lambda \rightarrow v_\lambda$ and $\lambda \mapsto \nu_\lambda$ are also analytic on \mathbb{R} . In particular we write for any $h \in \mathbb{R}$,

$$\begin{aligned} \mathbf{P}_{\lambda+h} &= \mathbf{P}_\lambda + h\mathbf{P}'_\lambda + \frac{h^2}{2}\mathbf{P}''_\lambda + o(h^2), \\ v_{\lambda+h} &= v_\lambda + hv'_\lambda + \frac{h^2}{2}v''_\lambda + o(h^2), \\ k(\lambda+h) &= k(\lambda) + hk'(\lambda) + \frac{h^2}{2}k''(\lambda) + o(h^2), \end{aligned}$$

where for any $h \in \mathbb{R}$, $o(h^2)$ refers to an operator, a function or a real such that $o(h^2)/h^2 \rightarrow 0$ as $h \rightarrow 0$. Since $v_{\lambda+h}$ is an eigenvector of $\mathbf{P}_{\lambda+h}$ we have $\mathbf{P}_{\lambda+h}v_{\lambda+h} = k(\lambda+h)v_{\lambda+h}$ and its development gives

$$\mathbf{P}_\lambda v_\lambda = k(\lambda)v_\lambda,$$

$$\mathbf{P}_\lambda v'_\lambda + \mathbf{P}'_\lambda v_\lambda = k(\lambda)v'_\lambda + k'(\lambda)v_\lambda, \quad (5.3.40)$$

$$\frac{1}{2}\mathbf{P}_\lambda v''_\lambda + \mathbf{P}'_\lambda v'_\lambda + \frac{1}{2}\mathbf{P}''_\lambda v_\lambda = \frac{1}{2}k(\lambda)v''_\lambda + k'(\lambda)v'_\lambda + \frac{1}{2}k''(\lambda)v_\lambda. \quad (5.3.41)$$

Since ν_λ is an invariant measure, $\nu_\lambda(\mathbf{P}_\lambda g) = k(\lambda)\nu_\lambda(g)$ and (5.3.40) implies that

$$k(\lambda)\nu_\lambda(v'_\lambda) + \nu_\lambda(\mathbf{P}'_\lambda v_\lambda) = k(\lambda)\nu_\lambda(v'_\lambda) + k'(\lambda).$$

In addition, by (5.3.25), $\mathbf{P}'_\lambda v_\lambda = \mathbf{P}_\lambda(\rho v_\lambda)$. Therefore,

$$k(\lambda)\nu_\lambda(\rho v_\lambda) = k'(\lambda),$$

which, with the definition of $\tilde{\nu}_\lambda$ in (5.3.36), proves (5.3.38).

From (5.3.41) and the fact that $\nu_\lambda(\mathbf{P}_\lambda g) = k(\lambda)\nu_\lambda(g)$, we have

$$\frac{k(\lambda)}{2}\nu_\lambda(v''_\lambda) + k(\lambda)\nu_\lambda(\rho v'_\lambda) + \frac{k(\lambda)}{2}\nu_\lambda(\rho^2 v_\lambda) = \frac{1}{2}k(\lambda)\nu_\lambda(v''_\lambda) + k'(\lambda)\nu_\lambda(v'_\lambda) + \frac{1}{2}k''(\lambda).$$

So,

$$\frac{k''(\lambda)}{k(\lambda)} = \nu_\lambda(\rho^2 v_\lambda) + 2 \left[\nu_\lambda(\rho v'_\lambda) - \frac{k'(\lambda)}{k(\lambda)}\nu_\lambda(v'_\lambda) \right].$$

By (5.3.38), we obtain that

$$\begin{aligned} K''(\lambda) &= \frac{k''(\lambda)}{k(\lambda)} - \left(\frac{k'(\lambda)}{k(\lambda)} \right)^2 \\ &= \nu_\lambda(\rho^2 v_\lambda) - \nu_\lambda^2(\rho v_\lambda) + 2[\nu_\lambda(\rho v'_\lambda) - \nu_\lambda(\rho v_\lambda)\nu_\lambda(v'_\lambda)]. \end{aligned} \quad (5.3.42)$$

It remains to determine v'_λ . By (5.3.40), we have

$$v'_\lambda - \frac{\mathbf{P}_\lambda v'_\lambda}{k(\lambda)} = \frac{\mathbf{P}_\lambda(\rho v_\lambda)}{k(\lambda)} - \frac{k'(\lambda)}{k(\lambda)}v_\lambda$$

and for any $n \geq 0$, using (5.3.38),

$$\frac{\mathbf{P}_\lambda^n v'_\lambda}{k(\lambda)^n} - \frac{\mathbf{P}_\lambda^{n+1} v'_\lambda}{k(\lambda)^{n+1}} = \frac{\mathbf{P}_\lambda^{n+1}(\rho v_\lambda)}{k(\lambda)^{n+1}} - \nu_\lambda(\rho v_\lambda)v_\lambda. \quad (5.3.43)$$

Note that

$$\frac{\mathbf{P}_\lambda^{n+1}(\rho v_\lambda)}{k(\lambda)^{n+1}} - \nu_\lambda(\rho v_\lambda)v_\lambda = \frac{Q_\lambda^{n+1}(\rho v_\lambda)}{k(\lambda)^{n+1}}.$$

By (5.3.30),

$$\left\| \frac{\mathbf{P}_\lambda^{n+1}(\rho v_\lambda)}{k(\lambda)^{n+1}} - \nu_\lambda(\rho v_\lambda)v_\lambda \right\|_\infty \leq c_\lambda e^{-c_\lambda(n+1)} \|\rho v_\lambda\|_\infty = c_\lambda e^{-c_\lambda(n+1)}.$$

Consequently, by (5.3.43), the series $\sum_{n \geq 0} \left[\frac{\mathbf{P}_\lambda^n v'_\lambda}{k(\lambda)^n} - \frac{\mathbf{P}_\lambda^{n+1} v'_\lambda}{k(\lambda)^{n+1}} \right]$ converges absolutely and we deduce that

$$v'_\lambda = \sum_{n=0}^{+\infty} \left[\frac{\mathbf{P}_\lambda^{n+1}(\rho v_\lambda)}{k(\lambda)^{n+1}} - \nu_\lambda(\rho v_\lambda) v_\lambda \right].$$

In particular,

$$\nu_\lambda(v'_\lambda) = \sum_{n=0}^{+\infty} [\nu_\lambda(\rho v_\lambda) - \nu_\lambda(\rho v_\lambda)] = 0,$$

and

$$\nu_\lambda(\rho v'_\lambda) = \sum_{n=0}^{+\infty} \left[\frac{\nu_\lambda(\rho \mathbf{P}_\lambda^{n+1}(\rho v_\lambda))}{k(\lambda)^{n+1}} - \nu_\lambda(\rho v_\lambda)^2 \right].$$

Therefore (5.3.42) becomes

$$K''(\lambda) = \nu_\lambda(\rho^2 v_\lambda) - \nu_\lambda^2(\rho v_\lambda) + 2 \sum_{n=0}^{+\infty} \left[\frac{\nu_\lambda(\rho \mathbf{P}_\lambda^{n+1}(\rho v_\lambda))}{k(\lambda)^{n+1}} - \nu_\lambda(\rho v_\lambda)^2 \right].$$

To conclude the proof of the lemma, we establish that $K''(\lambda) > 0$, from which the strict convexity of K follows. By (5.3.36),

$$K''(\lambda) = \tilde{\nu}_\lambda(\tilde{\rho}_\lambda^2) + 2 \sum_{n=1}^{+\infty} [\tilde{\nu}_\lambda(\tilde{\rho}_\lambda \tilde{\mathbf{P}}_\lambda^n \tilde{\rho}_\lambda)], \tag{5.3.44}$$

where for any $\lambda \in \mathbb{R}$, $\tilde{\rho}_\lambda = \rho - \tilde{\nu}_\lambda(\rho)v_0$. Moreover, Conditions 5.1 and 5.3 and Lemma 5.3.14 imply that the normalized transfer operator $\tilde{\mathbf{P}}_\lambda$ together with the function $\tilde{\rho}_\lambda$ satisfies Conditions 5.1 and 5.3. In conjunction with (5.3.44) and Lemma 10.3 of [37], this proves that (5.3.44) and so (5.3.39) is positive. \square

5.4 Proofs in the critical case

In this section we prove Theorem 5.2.1. By (5.3.3) and (5.3.10), the survival probability of the branching process is related to the study of the sum

$$q_n^{-1} = e^{-S_n} + \sum_{k=0}^{n-1} e^{-S_k} \eta_{k+1,n}$$

where $(S_n)_{n \geq 0}$ is a Markov walk defined by (5.2.5). Very roughly speaking, the sum q_n^{-1} converges mainly when the walk stays positive: $S_k > 0$ for any $k \geq 1$ and we will see that (at least in the critical case) only positive trajectories of the Markov walk $(S_n)_{n \geq 0}$ count for the survival of the branching process.

Recall that the hypotheses of Theorem 5.2.1 are Conditions 5.1-5.3 and $k'(0) = \nu(\rho) = 0$. Under these assumptions the conclusions of all the theorems of Section 5.3.3 hold for the probability \mathbb{P}_i , for any $i \in \mathbb{X}$. Recall also that $\mathbb{E}_{i,y}^+$ is the expectation corresponding to the probability measure (5.3.24). We carry out the proof through a series of lemmata.

Lemma 5.4.1. *Assume conditions of Theorem 5.2.1. For any $m \geq 1$, $(i, y) \in \text{supp}(V)$, and $j \in \mathbb{X}$, we have*

$$\lim_{n \rightarrow +\infty} \mathbb{P}_i(Z_m > 0; X_n = j \mid \tau_y > n) = \mathbb{E}_{i,y}^+(q_m) \nu(j).$$

Proof. Fix $m \geq 1$, $(i, y) \in \text{supp}(V)$, and $j \in \mathbb{X}$. By (5.3.2), for any $n \geq m + 1$,

$$\begin{aligned} \mathbb{P}_i(Z_m > 0, X_n = j, \tau_y > n) &= \mathbb{E}_i(\mathbb{P}_i(Z_m > 0 \mid X_1, \dots, X_n); X_n = j, \tau_y > n) \\ &= \mathbb{E}_i(\mathbb{E}_i(q_m \mid X_1, \dots, X_n); X_n = j, \tau_y > n) \\ &= \mathbb{E}_i(q_m; X_n = j, \tau_y > n). \end{aligned}$$

Thereby, using Lemma 5.3.12, we conclude that

$$\lim_{n \rightarrow +\infty} \mathbb{P}_i(Z_m > 0; X_n = j \mid \tau_y > n) = \lim_{n \rightarrow +\infty} \mathbb{E}_i(q_m; X_n = j \mid \tau_y > n) = \mathbb{E}_{i,y}^+(q_m) \nu(j).$$

□

By Lemma 5.3.3, we have for any $(i, y) \in \text{supp}(V)$, $k \geq 1$ and $n \geq k + 1$,

$$0 \leq \eta_{k,n} \leq \eta := \max_{x \in \mathbb{X}} \frac{f_x''(1)}{f_x'(1)^2} < +\infty \quad \mathbb{P}_{i,y}^+\text{-a.s.} \quad (5.4.1)$$

By (5.3.11) and (5.3.13), this equation holds also when $n = k$. Moreover, by Lemma 5.3.3,

$$\eta_{k,\infty} := \lim_{n \rightarrow +\infty} \eta_{k,n} \in [0, \eta] \quad \mathbb{P}_{i,y}^+\text{-a.s.} \quad (5.4.2)$$

Let q_∞ be the following random variable:

$$q_\infty := \left[\sum_{k=0}^{+\infty} e^{-S_k} \eta_{k+1,\infty} \right]^{-1} \in [0, +\infty]. \quad (5.4.3)$$

The random variable q_∞^{-1} is $\mathbb{P}_{i,y}^+$ -integrable for any $(i, y) \in \text{supp}(V)$: indeed by (5.4.2),

$$q_\infty^{-1} \leq \sum_{k=0}^{+\infty} e^{-S_k} \eta.$$

Using Lemma 5.3.13, for any $(i, y) \in \text{supp}(V)$

$$\mathbb{E}_{i,y}^+(q_\infty^{-1}) \leq \eta \mathbb{E}_{i,y}^+ \left(\sum_{k=0}^{+\infty} e^{-S_k} \right) \leq \eta \frac{c(1 + \max(y, 0)) e^y}{V(i, y)} < +\infty. \quad (5.4.4)$$

Lemma 5.4.2. *Assume conditions of Theorem 5.2.1. For any $(i, y) \in \text{supp}(V)$,*

$$\lim_{m \rightarrow +\infty} \mathbb{E}_{i,y}^+ \left(|q_m^{-1} - q_\infty^{-1}| \right) = 0, \quad (5.4.5)$$

and

$$\lim_{m \rightarrow +\infty} \mathbb{E}_{i,y}^+ (|q_m - q_\infty|) = 0. \quad (5.4.6)$$

Proof. Let $(i, y) \in \text{supp}(V)$ and fix $l \geq 1$. By (5.3.10) and (5.4.3), we have for all $m \geq l + 2$,

$$\begin{aligned} \mathbb{E}_{i,y}^+ \left(|q_m^{-1} - q_\infty^{-1}| \right) &= \mathbb{E}_{i,y}^+ \left(\left| e^{-S_m} + \sum_{k=0}^{m-1} e^{-S_k} \eta_{k+1,m} - \sum_{k=0}^{+\infty} e^{-S_k} \eta_{k+1,\infty} \right| \right) \\ &\leq \mathbb{E}_{i,y}^+ \left(e^{-S_m} \right) + \mathbb{E}_{i,y}^+ \left(\sum_{k=0}^l e^{-S_k} |\eta_{k+1,m} - \eta_{k+1,\infty}| \right) \\ &\quad + \mathbb{E}_{i,y}^+ \left(\sum_{k=l+1}^{m-1} e^{-S_k} |\eta_{k+1,m} - \eta_{k+1,\infty}| \right) + \mathbb{E}_{i,y}^+ \left(\sum_{k=m}^{+\infty} e^{-S_k} \eta_{k+1,\infty} \right). \end{aligned}$$

By (5.4.1) and (5.4.2),

$$\mathbb{E}_{i,y}^+ \left(\left| q_m^{-1} - q_\infty^{-1} \right| \right) \leq \mathbb{E}_{i,y}^+ \left(e^{-S_m} \right) + \mathbb{E}_{i,y}^+ \left(\sum_{k=0}^l e^{-S_k} |\eta_{k+1,m} - \eta_{k+1,\infty}| \right) + \eta \mathbb{E}_{i,y}^+ \left(\sum_{k=l+1}^{+\infty} e^{-S_k} \right).$$

Using Lemma 5.3.13 and the Lebesgue monotone convergence theorem,

$$\begin{aligned} & \mathbb{E}_{i,y}^+ \left(\left| q_m^{-1} - q_\infty^{-1} \right| \right) \\ & \leq \frac{c(1 + \max(y, 0)) e^y}{V(i, y)} \left(\frac{1}{m^{3/2}} + \eta \sum_{k=l+1}^{+\infty} \frac{1}{k^{3/2}} \right) + \mathbb{E}_{i,y}^+ \left(\sum_{k=0}^l e^{-S_k} |\eta_{k+1,m} - \eta_{k+1,\infty}| \right) \\ & \leq \frac{c(1 + \max(y, 0)) e^y}{V(i, y)} \left(\frac{1}{m^{3/2}} + \frac{\eta}{\sqrt{l}} \right) + \mathbb{E}_{i,y}^+ \left(\sum_{k=0}^l e^{-S_k} |\eta_{k+1,m} - \eta_{k+1,\infty}| \right). \end{aligned}$$

Moreover, by (5.4.1) and (5.4.2), we have $\sum_{k=0}^l e^{-S_k} |\eta_{k+1,m} - \eta_{k+1,\infty}| \leq \eta \sum_{k=0}^{+\infty} e^{-S_k}$ which is $\mathbb{P}_{i,y}^+$ -integrable by Lemma 5.3.13. Consequently, using the Lebesgue dominated convergence theorem and (5.4.2), when $m \rightarrow +\infty$, we obtain that for any $l \geq 1$,

$$\limsup_{m \rightarrow +\infty} \mathbb{E}_{i,y}^+ \left(\left| q_m^{-1} - q_\infty^{-1} \right| \right) \leq \frac{c\eta(1 + \max(y, 0)) e^y}{V(i, y)\sqrt{l}}.$$

Letting $l \rightarrow +\infty$ it proves (5.4.5).

Now, it follows easily from (5.3.4) that $q_\infty \leq 1$: for any $\varepsilon > 0$ and $m \geq 1$, we write that $\mathbb{P}_{i,y}^+ (q_\infty^{-1} < 1 - \varepsilon) \leq \mathbb{P}_{i,y}^+ (q_\infty^{-1} - q_m^{-1} < -\varepsilon)$. Since by (5.4.5), q_m^{-1} converges in $\mathbb{P}_{i,y}^+$ -probability to q_∞^{-1} , it follows that for any $\varepsilon > 0$, $\mathbb{P}_{i,y}^+ (q_\infty^{-1} < 1 - \varepsilon) = 0$ and so

$$q_\infty \leq 1 \quad \mathbb{P}_{i,y}^+ \text{-a.s.} \tag{5.4.7}$$

Consequently, $|q_m - q_\infty| = q_m q_\infty |q_m^{-1} - q_\infty^{-1}| \leq |q_m^{-1} - q_\infty^{-1}|$ and by (5.4.5), it proves (5.4.6). \square

Let U be a function defined on $\text{supp}(V)$ by

$$U(i, y) = \mathbb{E}_{i,y}^+ (q_\infty).$$

Note that for any $(i, y) \in \text{supp}(V)$, by (5.4.4), $q_\infty > 0$ $\mathbb{P}_{i,y}^+$ -a.s. and so

$$U(i, y) > 0. \tag{5.4.8}$$

By (5.4.7), we have also $U(i, y) \leq 1$.

Lemma 5.4.3. *Assume conditions of Theorem 5.2.1. For any $(i, y) \in \text{supp}(V)$ and $j \in \mathbb{X}$, we have*

$$\lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \mathbb{P}_i (Z_m > 0; X_n = j \mid \tau_y > n) = \nu(j)U(i, y).$$

Proof. By Lemma 5.4.1, for any $(i, y) \in \text{supp}(V)$, $j \in \mathbb{X}$ and $m \geq 1$, we have

$$\lim_{n \rightarrow +\infty} \mathbb{P}_i (Z_m > 0; X_n = j \mid \tau_y > n) = \nu(j)\mathbb{E}_{i,y}^+ (q_m).$$

By (5.4.6), we obtain the desired equality. \square

Lemma 5.4.4. *Assume conditions of Theorem 5.2.1. For any $(i, y) \in \text{supp}(V)$ and $\theta \in (0, 1)$,*

$$\lim_{m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \mathbb{P}_i \left(Z_m > 0, Z_{\lfloor \theta n \rfloor} = 0 \mid \tau_y > n \right) = 0.$$

Proof. Fix $(i, y) \in \text{supp}(V)$ and $\theta \in (0, 1)$. For any $m \geq 1$ and any $n \geq 1$ such that $\lfloor \theta n \rfloor \geq m + 1$ we define $\theta_n = \lfloor \theta n \rfloor$ and we write

$$\begin{aligned} I_0 &:= \mathbb{P}_i (Z_m > 0, Z_{\theta_n} = 0, \tau_y > n) \\ &= \mathbb{P}_i (Z_m > 0, \tau_y > n) - \mathbb{P}_i (Z_{\theta_n} > 0, \tau_y > n) \\ &= \mathbb{E}_i (\mathbb{P}_i (Z_m > 0 \mid X_1, \dots, X_m); \tau_y > n) - \mathbb{E}_i (\mathbb{P}_i (Z_{\theta_n} > 0 \mid X_1, \dots, X_{\theta_n}); \tau_y > n). \end{aligned}$$

By (5.3.2),

$$I_0 = \mathbb{E}_i (|q_m - q_{\theta_n}|; \tau_y > n).$$

We define $J_p(i, y) := \mathbb{P}_i (\tau_y > p)$ for any $(i, y) \in \mathbb{X} \times \mathbb{R}$ and $p \geq 0$ and consider

$$I_1 := \mathbb{P}_i (Z_m > 0, Z_{\theta_n} = 0 \mid \tau_y > n)$$

for any $(i, y) \in \text{supp}(V)$. By the Markov property, for any $(i, y) \in \text{supp}(V)$,

$$I_1 = \frac{I_0}{J_n(i, y)} = \mathbb{E}_i \left(|q_m - q_{\theta_n}| \frac{J_{n-\theta_n}(X_{\theta_n}, y + S_{\theta_n})}{J_n(i, y)}; \tau_y > \theta_n \right).$$

By the point 2 of Proposition 5.3.7,

$$I_1 \leq \frac{c}{\sqrt{(1-\theta)nJ_n(i, y)}} \mathbb{E}_i (|q_m - q_{\theta_n}| (1 + y + S_{\theta_n}); \tau_y > \theta_n).$$

Using also the point 3 of Proposition 5.3.6, we have

$$I_1 \leq \frac{c}{\sqrt{(1-\theta)nJ_n(i, y)}} \mathbb{E}_i (|q_m - q_{\theta_n}| (1 + V(X_{\theta_n}, y + S_{\theta_n})); \tau_y > \theta_n).$$

Using (5.3.4) and (5.3.24), we obtain that

$$I_1 \leq \frac{c}{\sqrt{(1-\theta)nJ_n(i, y)}} \left(\mathbb{P}_i (\tau_y > \theta_n) + V(i, y) \mathbb{E}_{i, y}^+ (|q_m - q_{\theta_n}|) \right).$$

Using the point 1 of Proposition 5.3.7, for any $(i, y) \in \text{supp}(V)$,

$$\frac{1}{\sqrt{(1-\theta)nJ_n(i, y)}} = \frac{1}{\sqrt{(1-\theta)n\mathbb{P}_i (\tau_y > n)}} \underset{n \rightarrow +\infty}{\sim} \frac{\sqrt{2\pi\sigma}}{2\sqrt{1-\theta}V(i, y)}.$$

Moreover using again the point 1 of Proposition 5.3.7 and using (5.4.6),

$$\mathbb{P}_i (\tau_y > \theta_n) + V(i, y) \mathbb{E}_{i, y}^+ (|q_m - q_{\theta_n}|) \xrightarrow{n \rightarrow +\infty} V(i, y) \mathbb{E}_{i, y}^+ (|q_m - q_\infty|).$$

Therefore, we obtain that, for any $m \geq 1$ and $\theta \in (0, 1)$,

$$\limsup_{n \rightarrow +\infty} I_1 \leq \frac{c}{\sqrt{1-\theta}} \mathbb{E}_{i, y}^+ (|q_m - q_\infty|).$$

Letting m go to $+\infty$ and using (5.4.6), we conclude that

$$\lim_{m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} I_1 = \lim_{m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \mathbb{P}_i (Z_m > 0, Z_{\theta_n} = 0 \mid \tau_y > n) = 0.$$

□

Lemma 5.4.5. *Assume conditions of Theorem 5.2.1. For any $(i, y) \in \text{supp}(V)$, $j \in \mathbb{X}$, and $\theta \in (0, 1)$,*

$$\lim_{n \rightarrow +\infty} \mathbb{P}_i \left(Z_{\lfloor \theta n \rfloor} > 0, X_n = j \mid \tau_y > n \right) = \nu(j)U(i, y).$$

In particular,

$$\lim_{n \rightarrow +\infty} \mathbb{P}_i \left(Z_{\lfloor \theta n \rfloor} > 0 \mid \tau_y > n \right) = U(i, y). \tag{5.4.9}$$

Proof. Fix $(i, y) \in \text{supp}(V)$ and $j \in \mathbb{X}$. Let $\theta_n := \lfloor \theta n \rfloor$ for any $\theta \in (0, 1)$ and $n \geq 1$. For any $m \geq 1$ and $n \geq 1$ such that $\theta_n \geq m + 1$, we write

$$\begin{aligned} & \mathbb{P}_i (Z_{\theta_n} > 0, X_n = j \mid \tau_y > n) \\ &= \mathbb{P}_i (Z_m > 0, Z_{\theta_n} > 0, X_n = j \mid \tau_y > n) \\ &= \mathbb{P}_i (Z_m > 0, X_n = j \mid \tau_y > n) - \mathbb{P}_i (Z_m > 0, Z_{\theta_n} = 0, X_n = j \mid \tau_y > n). \end{aligned}$$

By Lemma 5.4.4,

$$\begin{aligned} & \lim_{m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \mathbb{P}_i (Z_m > 0, Z_{\theta_n} = 0, X_n = j \mid \tau_y > n) \\ & \leq \lim_{m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \mathbb{P}_i (Z_m > 0, Z_{\theta_n} = 0 \mid \tau_y > n) = 0. \end{aligned}$$

Therefore, using Lemma 5.4.3, it follows that

$$\lim_{n \rightarrow +\infty} \mathbb{P}_i (Z_{\theta_n} > 0, X_n = j \mid \tau_y > n) = \nu(j)U(i, y).$$

□

Lemma 5.4.6. *Assume conditions of Theorem 5.2.1. For any $(i, y) \in \text{supp}(V)$,*

$$\lim_{p \rightarrow +\infty} \mathbb{P}_i (Z_p > 0 \mid \tau_y > p) = U(i, y).$$

Proof. Fix $(i, y) \in \text{supp}(V)$. For any $p \geq 1$ and $\theta \in (0, 1)$, we have

$$\mathbb{P}_i (Z_p > 0 \mid \tau_y > p) = \frac{\mathbb{P}_i \left(Z_p > 0, \tau_y > \frac{p}{\theta} + 1 \right) + \mathbb{P}_i \left(Z_p > 0, p < \tau_y \leq \frac{p}{\theta} + 1 \right)}{\mathbb{P}_i (\tau_y > p)}.$$

Let $n = \lfloor \frac{p}{\theta} \rfloor + 1$ and note that $\lfloor \theta n \rfloor = p$. So, by (5.4.9),

$$\lim_{p \rightarrow +\infty} \mathbb{P}_i (Z_p > 0 \mid \tau_y > p) = U(i, y) \lim_{p \rightarrow +\infty} \frac{\mathbb{P}_i (\tau_y > n)}{\mathbb{P}_i (\tau_y > p)} + \lim_{p \rightarrow +\infty} \frac{\mathbb{P}_i (Z_p > 0, p < \tau_y \leq n)}{\mathbb{P}_i (\tau_y > p)}.$$

By the point 1 of Proposition 5.3.7, we obtain that

$$\lim_{p \rightarrow +\infty} \mathbb{P}_i (Z_p > 0 \mid \tau_y > p) = U(i, y)\sqrt{\theta} + \lim_{p \rightarrow +\infty} \frac{\mathbb{P}_i (Z_p > 0, p < \tau_y \leq n)}{\mathbb{P}_i (\tau_y > p)}.$$

Moreover, using again the point 1 of Proposition 5.3.7, for any $\theta \in (0, 1)$,

$$\frac{\mathbb{P}_i (Z_p > 0, p < \tau_y \leq n)}{\mathbb{P}_i (\tau_y > p)} \leq \frac{\mathbb{P}_i (\tau_y > p) - \mathbb{P}_i (\tau_y > n)}{\mathbb{P}_i (\tau_y > p)} \xrightarrow{p \rightarrow +\infty} 1 - \sqrt{\theta}.$$

Letting $\theta \rightarrow 1$, we conclude that

$$\lim_{p \rightarrow +\infty} \mathbb{P}_i (Z_p > 0 \mid \tau_y > p) = U(i, y).$$

□

Lemma 5.4.7. *Assume conditions of Theorem 5.2.1. For any $(i, y) \in \text{supp}(V)$ and $\theta \in (0, 1)$,*

$$\lim_{n \rightarrow +\infty} \mathbb{P}_i \left(Z_{\lfloor \theta n \rfloor} > 0, Z_n = 0 \mid \tau_y > n \right) = 0.$$

Proof. For any $(i, y) \in \text{supp}(V)$, $\theta \in (0, 1)$ and $n \geq 1$,

$$\mathbb{P}_i \left(Z_{\lfloor \theta n \rfloor} > 0, Z_n = 0 \mid \tau_y > n \right) = \mathbb{P}_i \left(Z_{\lfloor \theta n \rfloor} > 0 \mid \tau_y > n \right) - \mathbb{P}_i \left(Z_n > 0 \mid \tau_y > n \right).$$

From (5.4.9) and Lemma 5.4.6, it follows

$$\mathbb{P}_i \left(Z_{\lfloor \theta n \rfloor} > 0, Z_n = 0 \mid \tau_y > n \right) \xrightarrow{n \rightarrow +\infty} U(i, y) - U(i, y) = 0.$$

□

Lemma 5.4.8. *Assume conditions of Theorem 5.2.1. For any $(i, y) \in \text{supp}(V)$ and $j \in \mathbb{X}$,*

$$\lim_{n \rightarrow +\infty} \mathbb{P}_i \left(Z_n > 0, X_n = j \mid \tau_y > n \right) = \nu(j)U(i, y).$$

Proof. For any $(i, y) \in \text{supp}(V)$, $j \in \mathbb{X}$, $\theta \in (0, 1)$ and $n \geq 1$,

$$\begin{aligned} \mathbb{P}_i \left(Z_n > 0, X_n = j \mid \tau_y > n \right) &= \mathbb{P}_i \left(Z_{\lfloor \theta n \rfloor} > 0, X_n = j \mid \tau_y > n \right) \\ &\quad - \mathbb{P}_i \left(Z_{\lfloor \theta n \rfloor} > 0, Z_n = 0, X_n = j \mid \tau_y > n \right) \end{aligned}$$

Using Lemmas 5.4.5 and 5.4.7, the result follows. □

Proof of Theorem 5.2.1. Fix $(i, j) \in \mathbb{X}^2$. For any $y \in \mathbb{R}$, we have

$$0 \leq \mathbb{P}_i \left(Z_n > 0, X_n = j \right) - \mathbb{P}_i \left(Z_n > 0, X_n = j, \tau_y > n \right) \leq \mathbb{P}_i \left(Z_n > 0, \tau_y \leq n \right). \quad (5.4.10)$$

Using (5.3.2),

$$\mathbb{P}_i \left(Z_n > 0, \tau_y \leq n \right) = \mathbb{E}_i \left(q_n ; \tau_y \leq n \right).$$

Moreover, by the definition of q_n in (5.3.1), for any $k \geq 1$,

$$q_k \leq f'_{X_k}(1) \times \cdots \times f'_{X_1}(1) = e^{S_k}.$$

Since $(q_k)_{k \geq 1}$ is non-increasing, we have $q_n = \min_{1 \leq k \leq n} q_k \leq e^{\min_{1 \leq k \leq n} S_k}$. Therefore

$$\begin{aligned} \mathbb{P}_i \left(Z_n > 0, \tau_y \leq n \right) &\leq \mathbb{E}_i \left(e^{\min_{1 \leq k \leq n} S_k} ; \tau_y \leq n \right) \\ &= e^{-y} \sum_{p=0}^{+\infty} \mathbb{E}_i \left(e^{\min_{1 \leq k \leq n} \{y + S_k\}} ; -(p+1) < \min_{1 \leq k \leq n} \{y + S_k\} \leq -p, \tau_y \leq n \right) \\ &\leq e^{-y} \sum_{p=0}^{+\infty} e^{-p} \mathbb{P}_i \left(\tau_{y+p+1} > n \right). \end{aligned} \quad (5.4.11)$$

By the point 2 of Proposition 5.3.7,

$$\mathbb{P}_i \left(Z_n > 0, \tau_y \leq n \right) = \frac{c e^{-y}}{\sqrt{n}} \sum_{p=0}^{+\infty} e^{-p} (1 + p + 1 + \max(y, 0)) \leq \frac{c e^{-y} (1 + \max(y, 0))}{\sqrt{n}}. \quad (5.4.12)$$

Note that from the point 3 of Proposition 5.3.6, it is clear that there exists $y_0 = y_0(i) < +\infty$ such that for any $y \geq y_0$, we have $V(i, y) > 0$ i.e. $(i, y) \in \text{supp}(V)$ (for more information on $\text{supp}(V)$ see [38]/Chapter 3). Using Lemma 5.4.8 and the point 1 of Proposition 5.3.7, for any $y \geq y_0$,

$$\sqrt{n}\mathbb{P}_i(Z_n > 0, X_n = j, \tau_y > n) \xrightarrow{n \rightarrow +\infty} \frac{2\nu(j)U(i, y)V(i, y)}{\sqrt{2\pi\sigma}}. \tag{5.4.13}$$

Let

$$I(i, j) = \liminf_{n \rightarrow +\infty} \sqrt{n}\mathbb{P}_i(Z_n > 0, X_n = j)$$

and

$$J(i, j) = \limsup_{n \rightarrow +\infty} \sqrt{n}\mathbb{P}_i(Z_n > 0, X_n = j).$$

Using (5.4.10), (5.4.12) and (5.4.13), we obtain that, for any $y \geq y_0(i)$,

$$\begin{aligned} \frac{2\nu(j)U(i, y)V(i, y)}{\sqrt{2\pi\sigma}} &\leq I(i, j) \\ &\leq J(i, j) \leq \frac{2\nu(j)U(i, y)V(i, y)}{\sqrt{2\pi\sigma}} + ce^{-y}(1 + \max(y, 0)) < +\infty. \end{aligned} \tag{5.4.14}$$

From (5.4.13), it is clear that $y \mapsto \frac{2U(i, y)V(i, y)}{\sqrt{2\pi\sigma}}$ is non-decreasing and from (5.4.14) the function is bounded by $I(i, j)/\nu(j) < +\infty$. Therefore

$$u(i) := \lim_{y \rightarrow +\infty} \frac{2U(i, y)V(i, y)}{\sqrt{2\pi\sigma}}$$

exists. Moreover by (5.4.8), for any $y \geq y_0(i)$,

$$u(i) \geq \frac{2U(i, y)V(i, y)}{\sqrt{2\pi\sigma}} > 0.$$

Taking the limit as $y \rightarrow +\infty$ in (5.4.14), we conclude that

$$\lim_{n \rightarrow +\infty} \sqrt{n}\mathbb{P}_i(Z_n > 0, X_n = j) = \nu(j)u(i),$$

which finishes the proof of Theorem 5.2.1.

5.5 Proofs in the strongly subcritical case

Assume the hypotheses of Theorem 5.2.2 that is Conditions 5.1-5.3 and $k'(1) < 0$. We fix $\lambda = 1$ and define the probability $\tilde{\mathbb{P}}_i$ and the corresponding expectation $\tilde{\mathbb{E}}_i$ by (5.3.37), such that, for any $n \geq 1$ and any $g: \mathbb{X}^n \rightarrow \mathbb{C}$,

$$\tilde{\mathbb{E}}_i(g(X_1, \dots, X_n)) = \frac{\mathbb{E}_i(e^{S_n} g(X_1, \dots, X_n)v_1(X_n))}{k(1)^n v_1(i)}. \tag{5.5.1}$$

By (5.3.2), we have, for any $(i, j) \in \mathbb{X}^2$ and $n \geq 1$,

$$\begin{aligned} \mathbb{P}_i(Z_{n+1} > 0, X_{n+1} = j) &= \mathbb{E}_i(q_{n+1}, X_{n+1} = j) \\ &= \tilde{\mathbb{E}}_i\left(\frac{e^{-S_{n+1}}}{v_1(X_{n+1})} q_{n+1}; X_{n+1} = j\right) k(1)^{n+1} v_1(i) \\ &= \tilde{\mathbb{E}}_i\left(e^{-S_n} q_n(f_j(0)); X_{n+1} = j\right) k(1)^{n+1} \frac{v_1(i) e^{-\rho(j)}}{v_1(j)}, \end{aligned}$$

where $q_n(s)$ is defined for any $s \in [0, 1]$ by (5.3.1). From Lemma 5.3.2, we write

$$\begin{aligned} e^{-S_n} q_n(f_j(0)) &= \left[\frac{1}{1 - f_j(0)} + \sum_{k=0}^{n-1} e^{S_n - S_k} \eta_{k+1, n}(f_j(0)) \right]^{-1} \\ &= \left[\frac{1}{1 - f_j(0)} + \sum_{k=1}^n e^{S_n - S_{n-k}} \eta_{n-k+1, n}(f_j(0)) \right]^{-1}. \end{aligned} \quad (5.5.2)$$

As in Section 5.3.2, we define the dual Markov chain $(X_n^*)_{n \geq 0}$, where the dual Markov kernel is given, for any $(i, j) \in \mathbb{X}^2$, by

$$\tilde{\mathbf{P}}_1^*(i, j) = \tilde{\mathbf{P}}_1(j, i) \frac{\tilde{\nu}_1(j)}{\tilde{\nu}_1(i)} = \mathbf{P}(j, i) \frac{e^{\rho(i)} \nu_1(j)}{k(1) \nu_1(i)}.$$

Let $(S_n^*)_{n \geq 0}$ be the associated Markov walk defined by (5.3.16) and

$$q_n^*(j) := \left[\frac{1}{1 - f_j(0)} + \sum_{k=1}^n e^{-S_k^*} \eta_k^*(j) \right]^{-1}, \quad (5.5.3)$$

where

$$\eta_k^*(j) := g_{X_k^*}(f_{X_{k-1}^*} \circ \cdots \circ f_{X_1^*} \circ f_j(0)) \quad \text{and} \quad \eta_1^*(j) := g_{X_1^*}(f_j(0)). \quad (5.5.4)$$

Following the proof of Lemma 5.3.2, we obtain

$$q_n^*(j) = e^{S_n^*} (1 - f_{X_n^*} \circ \cdots \circ f_{X_1^*} \circ f_j(0)). \quad (5.5.5)$$

We are going to apply duality Lemma 5.3.5. The following correspondences designed by the two-sided arrow \longleftrightarrow are included for the ease of the reader:

$$\begin{aligned} X_k^* &\longleftrightarrow X_{n-k+1}, \\ S_k^* &\longleftrightarrow S_{n-k} - S_n, \\ \eta_k^*(j) &\longleftrightarrow \eta_{n-k+1, n}(f_j(0)), \\ q_n^*(j) &\longleftrightarrow e^{-S_n} q_n(f_j(0)). \end{aligned}$$

Now Lemma 5.3.5 implies,

$$\mathbb{P}_i(Z_{n+1} > 0, X_{n+1} = j) = \tilde{\mathbb{E}}_j^*(q_n^*(j); X_{n+1}^* = i) k(1)^{n+1} \frac{\tilde{\nu}_1(j) v_1(i) e^{-\rho(j)}}{\tilde{\nu}_1(i) v_1(j)}, \quad (5.5.6)$$

where $\tilde{\mathbb{E}}_j^*$ is the expectation generated by the trajectories of the Markov chain $(X_n^*)_{n \geq 0}$ starting at $X_0^* = j$.

Note that, under Condition 5.2, by Lemma 5.3.3 we have, for any $j \in \mathbb{X}$ and $k \geq 1$,

$$0 \leq \eta_k^*(j) \leq \eta = \max_{i \in \mathbb{X}} \frac{f_i''(1)}{f_i'(1)^2} < +\infty \quad \tilde{\mathbb{P}}_j^* \text{-a.s.} \quad (5.5.7)$$

In particular, by (5.5.3),

$$q_n^*(j) \in (0, 1], \quad \forall n \geq 1.$$

For any $j \in \mathbb{X}$, consider the random variable

$$q_\infty^*(j) := \left[\frac{1}{1 - f_j(0)} + \sum_{k=1}^{\infty} e^{-S_k^*} \eta_k^*(j) \right]^{-1} \in [0, 1]. \quad (5.5.8)$$

Lemma 5.5.1. *Assume that the conditions of Theorem 5.2.2 are satisfied. For any $j \in \mathbb{X}$,*

$$\lim_{n \rightarrow +\infty} q_n^*(j) = q_\infty^*(j) \in (0, 1], \quad \tilde{\mathbb{P}}_j^* \text{-a.s.} \quad (5.5.9)$$

and

$$\lim_{n \rightarrow +\infty} \tilde{\mathbb{E}}_j^* (|q_n^*(j) - q_\infty^*(j)|) = 0. \quad (5.5.10)$$

Proof. Fix $j \in \mathbb{X}$. By the law of large numbers for finite Markov chains,

$$\frac{S_k^*}{k} \xrightarrow[k \rightarrow +\infty]{} \tilde{\nu}_1(-\rho), \quad \tilde{\mathbb{P}}_j^* \text{-a.s.}$$

This means that there exists a set N of null probability $\tilde{\mathbb{P}}_j^*(N) = 0$, such that for any $\omega \in \Omega \setminus N$ and any $\varepsilon > 0$, there exists $k_0(\omega, \varepsilon)$ such that for any $k \geq k_0(\omega, \varepsilon)$,

$$e^{-S_k^*(\omega)} \eta_k^*(j)(\omega) \leq e^{k\tilde{\nu}_1(\rho) + k\varepsilon} \eta,$$

where for the last inequality we used the bound (5.5.7). By Lemma 5.3.15, we have $\tilde{\nu}_1(\rho) = k'(1)/k(1) < 0$. Taking $\varepsilon = -\tilde{\nu}_1(\rho)/2$ we obtain that, for any $k \geq k_0(\omega)$,

$$0 \leq e^{-S_k^*(\omega)} \eta_k^*(j)(\omega) \leq e^{k\frac{\tilde{\nu}_1(\rho)}{2}} \eta.$$

Consequently, the series $(q_n^*(j))^{-1}$ converges a.s. to $(q_\infty^*(j))^{-1} \in [1, +\infty)$ which proves (5.5.9).

Now the sequence $(q_n^*(j))_{n \geq 1}$ belongs to $[0, 1]$ a.s. and so by the Lebesgue dominated convergence theorem,

$$\lim_{n \rightarrow +\infty} \tilde{\mathbb{E}}_j^* (|q_n^*(j) - q_\infty^*(j)|) = 0.$$

□

Lemma 5.5.2. *Assume that the conditions of Theorem 5.2.2 are satisfied. For any $(i, j) \in \mathbb{X}^2$,*

$$\lim_{n \rightarrow +\infty} \tilde{\mathbb{E}}_j^* (q_n^*(j); X_{n+1}^* = i) = \tilde{\nu}_1(i) \tilde{\mathbb{E}}_j^* (q_\infty^*(j)).$$

Proof. Let $m \geq 1$. For any $(i, j) \in \mathbb{X}^2$, and $n \geq m$,

$$\tilde{\mathbb{E}}_j^* (q_n^*(j); X_{n+1}^* = i) = \tilde{\mathbb{E}}_j^* (q_m^*(j); X_{n+1}^* = i) + \tilde{\mathbb{E}}_j^* (q_n^*(j) - q_m^*(j); X_{n+1}^* = i). \quad (5.5.11)$$

By the Markov property,

$$\tilde{\mathbb{E}}_j^* (q_m^*(j); X_{n+1}^* = i) = \tilde{\mathbb{E}}_j^* (q_m^*(j) (\tilde{\mathbf{P}}_1^*)^{n-m+1} (X_m^*, i)).$$

Using (5.3.18) (which holds also for $\tilde{\mathbf{P}}_1^*$ by Lemmas 5.3.14 and 5.3.4) and (5.5.10), we have

$$\lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \tilde{\mathbb{E}}_j^*(q_m^*(j); X_{n+1}^* = i) = \lim_{m \rightarrow +\infty} \tilde{\mathbb{E}}_j^*(q_m^*(j)) \tilde{\nu}_1(i) = \tilde{\mathbb{E}}_j^*(q_\infty^*(j)) \tilde{\nu}_1(i). \quad (5.5.12)$$

Moreover, again by (5.5.10),

$$\begin{aligned} \lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \left| \tilde{\mathbb{E}}_j^*(q_n^*(j) - q_m^*(j); X_{n+1}^* = i) \right| &\leq \lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \tilde{\mathbb{E}}_j^*(|q_n^*(j) - q_m^*(j)|) \\ &= \lim_{m \rightarrow +\infty} \tilde{\mathbb{E}}_j^*(|q_\infty^*(j) - q_m^*(j)|) \\ &= 0. \end{aligned}$$

Together with (5.5.11) and (5.5.12), this concludes the lemma. \square

Proof of Theorem 5.2.2. By (5.5.9), the function

$$u(j) = \frac{\tilde{\nu}_1(j) e^{-\rho(j)} \tilde{\mathbb{E}}_j^*(q_\infty^*(j))}{v_1(j)}$$

is positive. The result of the theorem follows from Lemma 5.5.2 and the identity (5.5.6).

5.6 Proofs in the intermediate subcritical case

We assume the conditions of Theorem 5.2.3, that is Conditions 5.1-5.3 and $k'(1) = 0$. As in the critical case the proof is carried out through a series of lemmata.

The beginning of the reasoning is the same as in the strongly subcritical case. Keeping the same notation as in Section 5.5 (see (5.5.1)-(5.5.6)), we have

$$\mathbb{P}_i(Z_{n+1} > 0, X_{n+1} = j) = \tilde{\mathbb{E}}_j^*(q_n^*(j); X_{n+1}^* = i) k(1)^{n+1} \frac{\tilde{\nu}_1(j) v_1(i) e^{-\rho(j)}}{\tilde{\nu}_1(i) v_1(j)}. \quad (5.6.1)$$

Under the hypotheses of Theorem 5.2.3, the Markov walk $(S_n^*)_{n \geq 0}$ is centred under the probability $\tilde{\mathbb{P}}_j^*$ for any $j \in \mathbb{X}$: indeed $\tilde{\nu}_1(-\rho) = -k'(1)/k(1) = 0$ (see Lemma 5.3.15) and by Lemma 5.3.14, Conditions 5.1 and 5.3 hold for $\tilde{\mathbf{P}}_1$. In this case, by Lemma 5.3.4, Conditions 5.1 and 5.3 hold also for $\tilde{\mathbf{P}}_1^*$. Therefore all the results of Section 5.3.3 hold for the probability $\tilde{\mathbb{P}}^*$. Let τ_z^* be the exit time of the Markov walk $(z + S_n^*)_{n \geq 0}$:

$$\tau_z^* := \inf \{k \geq 1 : z + S_k^* \leq 0\}.$$

Denote by \tilde{V}_1^* the harmonic function defined by Proposition 5.3.6 with respect to the probability $\tilde{\mathbb{P}}^*$. As in (5.3.24), for any $(j, z) \in \text{supp}(\tilde{V}_1^*)$, define a new probability $\tilde{\mathbb{P}}_{j,z}^{*+}$ and its associated expectation $\tilde{\mathbb{E}}_{j,z}^{*+}$ on $\sigma(X_n^*, n \geq 1)$ by

$$\tilde{\mathbb{E}}_{j,z}^{*+}(g(X_1^*, \dots, X_n^*)) := \frac{1}{\tilde{V}_1^*(j, z)} \tilde{\mathbb{E}}_j^*(g(X_1^*, \dots, X_n^*) \tilde{V}_1^*(X_n^*, z + S_n^*); \tau_z^* > n),$$

for any $n \geq 1$ and any $g: \mathbb{X}^n \rightarrow \mathbb{C}$.

Lemma 5.6.1. *Assume that the conditions of Theorem 5.2.3 are satisfied. For any $m \geq 1$, $(j, z) \in \text{supp}(\tilde{V}_1^*)$, and $i \in \mathbb{X}$, we have*

$$\lim_{n \rightarrow +\infty} \tilde{\mathbb{E}}_j^*(q_m^*(j); X_{n+1}^* = i \mid \tau_z^* > n + 1) = \tilde{\mathbb{E}}_{j,z}^{*+}(q_m^*(j)) \tilde{\nu}_1(i).$$

Proof. The equation (5.5.5) gives an explicit formula for $q_m^*(j)$ in terms of (X_1^*, \dots, X_m^*) . Therefore, the assertion of the lemma is a straightforward consequence of Lemma 5.3.12. \square

As in Section 5.5, using Lemma 5.3.3 we have for any $(j, z) \in \text{supp}(\tilde{V}_1^*)$ and $k \geq 1$,

$$0 \leq \eta_k^*(j) \leq \eta = \max_{i \in \mathbb{X}} \frac{f_i''(1)}{f_i'(1)^2} < +\infty \quad \text{and} \quad q_n^*(j) \in (0, 1], \quad \tilde{\mathbb{P}}_{j,z}^{*+}\text{-a.s.} \quad (5.6.2)$$

Consider the random variable

$$q_\infty^*(j) := \left[\frac{1}{1 - f_j(0)} + \sum_{k=1}^{+\infty} e^{-S_k^*} \eta_k^*(j) \right]^{-1} \in [0, 1]. \quad (5.6.3)$$

Lemma 5.6.2. *Assume that the conditions of Theorem 5.2.3 are satisfied. For any $(j, z) \in \text{supp}(\tilde{V}_1^*)$*

$$\lim_{m \rightarrow +\infty} \tilde{\mathbb{E}}_{j,z}^{*+} \left(\left| (q_m^*(j))^{-1} - (q_\infty^*(j))^{-1} \right| \right) = 0, \quad (5.6.4)$$

and

$$\lim_{m \rightarrow +\infty} \tilde{\mathbb{E}}_{j,z}^{*+} (|q_m^*(j) - q_\infty^*(j)|) = 0. \quad (5.6.5)$$

Proof. Fix $(j, z) \in \text{supp}(\tilde{V}_1^*)$. By (5.5.3), (5.6.3) and (5.6.2), for any $m \geq 1$,

$$\tilde{\mathbb{E}}_{j,z}^{*+} \left(\left| (q_m^*(j))^{-1} - (q_\infty^*(j))^{-1} \right| \right) \leq \eta \tilde{\mathbb{E}}_{j,z}^{*+} \left(\sum_{k=m+1}^{+\infty} e^{-S_k^*} \right).$$

From this bound, by Lemma 5.3.13 and the dominated convergence theorem when $m \rightarrow +\infty$, we obtain (5.6.4).

Now by (5.6.2) and (5.6.3) we have for any $m \geq 1$,

$$\begin{aligned} \tilde{\mathbb{E}}_{j,z}^{*+} (|q_m^*(j) - q_\infty^*(j)|) &= \tilde{\mathbb{E}}_{j,z}^{*+} \left(|q_m^*(j)q_\infty^*(j)| \left| (q_m^*(j))^{-1} - (q_\infty^*(j))^{-1} \right| \right) \\ &\leq \tilde{\mathbb{E}}_{j,z}^{*+} \left(\left| (q_m^*(j))^{-1} - (q_\infty^*(j))^{-1} \right| \right), \end{aligned}$$

which proves (5.6.5). \square

Let U be the function defined on $\text{supp}(\tilde{V}_1^*)$ by

$$U^*(j, z) = \tilde{\mathbb{E}}_{j,z}^{*+} (q_\infty^*(j)).$$

Using (5.6.2) and Lemma 5.3.13, we have

$$\tilde{\mathbb{E}}_{j,z}^{*+} \left((q_\infty^*(j))^{-1} \right) \leq \frac{1}{1 - f_j(0)} + \eta \tilde{\mathbb{E}}_{j,z}^{*+} \left(\sum_{k=1}^{+\infty} e^{-S_k^*} \right) < +\infty. \quad (5.6.6)$$

Therefore $q_\infty^* > 0$ $\mathbb{P}_{i,y}^+$ -a.s. and so $U^*(j, z) > 0$. In addition, by (5.6.3), $U^*(j, z) \leq 1$. For any $(j, z) \in \text{supp}(\tilde{V}_1^*)$,

$$U^*(j, z) \in (0, 1]. \quad (5.6.7)$$

Lemma 5.6.3. *Assume that the conditions of Theorem 5.2.3 are satisfied. For any $(j, z) \in \text{supp}(\tilde{V}_1^*)$ and $i \in \mathbb{X}$, we have*

$$\lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \tilde{\mathbb{E}}_j^* \left(q_m^*(j); X_{n+1}^* = i \mid \tau_z^* > n + 1 \right) = U^*(j, z) \tilde{\nu}_1(i).$$

Proof. The assertion of the lemma is straightforward consequence of Lemmas 5.6.1 and 5.6.2. \square

Lemma 5.6.4. *Assume that the conditions of Theorem 5.2.3 are satisfied. For any $(j, z) \in \text{supp}(\tilde{V}_1^*)$ and $\theta \in (0, 1)$, we have*

$$\lim_{m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \tilde{\mathbb{E}}_j^* \left(\left| q_m^*(j) - q_{[\theta n]}^*(j) \right| \mid \tau_z^* > n + 1 \right) = 0.$$

Proof. Fix $(j, z) \in \text{supp}(\tilde{V}_1^*)$ and $\theta \in (0, 1)$. Let $m \geq 1$ and $n \geq 1$ be such that $\theta n \geq m + 1$. Set $\theta_n = \lfloor \theta n \rfloor$. Denote

$$I_0 := \tilde{\mathbb{E}}_j^* \left(\left| q_m^*(j) - q_{\theta_n}^*(j) \right| \mid \tau_z^* > n + 1 \right) \quad \text{and} \quad J_n(j, z) := \tilde{\mathbb{P}}_j^* (\tau_z^* > n).$$

Note that by the point 1 of Proposition 5.3.7, we have $J_n(j, z) > 0$ for any n large enough. By the Markov property and the point 2 of Proposition 5.3.7,

$$\begin{aligned} I_0 &= \frac{1}{J_{n+1}(j, z)} \tilde{\mathbb{E}}_j^* \left(\left| q_m^*(j) - q_{\theta_n}^*(j) \right| J_{n+1-\theta_n} \left(X_{\theta_n}^*, z + S_{\theta_n}^* \right) ; \tau_z^* > \theta_n \right) \\ &\leq \frac{c}{J_{n+1}(j, z) \sqrt{n+1-\theta_n}} \tilde{\mathbb{E}}_j^* \left(\left| q_m^*(j) - q_{\theta_n}^*(j) \right| \left(1 + z + S_{\theta_n}^* \right) ; \tau_z^* > \theta_n \right). \end{aligned}$$

Using the point 3 of Proposition 5.3.6 and (5.6.2),

$$\begin{aligned} I_0 &\leq \frac{c}{J_{n+1}(j, z) \sqrt{n(1-\theta)}} \tilde{\mathbb{E}}_j^* \left(\left| q_m^*(j) - q_{\theta_n}^*(j) \right| \left(1 + \tilde{V}_1^* \left(X_{\theta_n}^*, z + S_{\theta_n}^* \right) \right) ; \tau_z^* > \theta_n \right) \\ &\leq \frac{c}{J_{n+1}(j, z) \sqrt{n(1-\theta)}} \left(\tilde{\mathbb{P}}_j^* (\tau_z^* > \theta_n) + \tilde{V}_1(j, z) \tilde{\mathbb{E}}_{j,z}^{*+} \left(\left| q_m^*(j) - q_{\theta_n}^*(j) \right| \right) \right). \end{aligned}$$

By the point 1 of Proposition 5.3.7 and (5.6.5), we obtain that

$$\limsup_{n \rightarrow +\infty} I_0 \leq \limsup_{n \rightarrow +\infty} \frac{c\sqrt{n+1}}{\sqrt{n(1-\theta)}} \tilde{\mathbb{E}}_{j,z}^{*+} \left(\left| q_m^*(j) - q_{\theta_n}^*(j) \right| \right) = \frac{c}{\sqrt{(1-\theta)}} \tilde{\mathbb{E}}_{j,z}^{*+} \left(\left| q_m^*(j) - q_{\infty}^*(j) \right| \right).$$

Taking the limit as $m \rightarrow +\infty$ and using (5.6.5), we conclude that

$$\lim_{m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \tilde{\mathbb{E}}_j^* \left(\left| q_m^*(j) - q_{[\theta n]}^*(j) \right| \mid \tau_z^* > n + 1 \right) = 0.$$

\square

Lemma 5.6.5. *Assume that the conditions of Theorem 5.2.3 are satisfied. For any $(j, z) \in \text{supp}(\tilde{V}_1^*)$, $i \in \mathbb{X}$ and $\theta \in (0, 1)$, we have*

$$\lim_{n \rightarrow +\infty} \tilde{\mathbb{E}}_j^* \left(q_{[\theta n]}^*(j) ; X_{n+1}^* = i \mid \tau_z^* > n + 1 \right) = U^*(j, z) \tilde{\nu}_1(i).$$

Proof. For any $(j, z) \in \text{supp}(\tilde{V}_1^*)$, $i \in \mathbb{X}$, $\theta \in (0, 1)$, $m \geq 1$ and $n \geq m + 1$ such that $\lfloor \theta n \rfloor \geq m$, we have

$$\begin{aligned} I_0 &:= \tilde{\mathbb{E}}_j^* \left(q_{[\theta n]}^*(j) ; X_{n+1}^* = i \mid \tau_z^* > n + 1 \right) \\ &= \tilde{\mathbb{E}}_j^* \left(q_m^*(j) ; X_{n+1}^* = i \mid \tau_z^* > n + 1 \right) + \underbrace{\tilde{\mathbb{E}}_j^* \left(q_{[\theta n]}^*(j) - q_m^*(j) ; X_{n+1}^* = i \mid \tau_z^* > n + 1 \right)}_{=: I_1}. \end{aligned}$$

By Lemma 5.6.4,

$$\limsup_{m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} |I_1| \leq \lim_{m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \tilde{\mathbb{E}}_j^* \left(\left| q_{[\theta n]}^*(j) - q_m^*(j) \right| \mid \tau_z^* > n + 1 \right) = 0.$$

Consequently, using Lemma 5.6.3,

$$\lim_{n \rightarrow +\infty} I_0 = \lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \tilde{\mathbb{E}}_j^* \left(q_m^*(j); X_{n+1}^* = i \mid \tau_z^* > n + 1 \right) = U^*(j, z) \tilde{\nu}_1(i).$$

□

Lemma 5.6.6. *Assume that the conditions of Theorem 5.2.3 are satisfied. For any $(j, z) \in \text{supp}(\tilde{V}_1^*)$, we have*

$$\lim_{p \rightarrow +\infty} \tilde{\mathbb{E}}_j^* \left(q_p^*(j) \mid \tau_z^* > p + 1 \right) = U^*(j, z).$$

Proof. Fix $(j, z) \in \text{supp}(\tilde{V}_1^*)$. For any $p \geq 1$ and $\theta \in (0, 1)$ set $n = \lfloor p/\theta \rfloor + 1$. Note that $p = \lfloor \theta n \rfloor$. We write, for any $p \geq 1$,

$$\tilde{\mathbb{E}}_j^* \left(q_p^*(j) \mid \tau_z^* > p + 1 \right) = \frac{\tilde{\mathbb{E}}_j^* \left(q_p^*(j); \tau_z^* > n + 1 \right) + \tilde{\mathbb{E}}_j^* \left(q_p^*(j); p + 1 < \tau_z^* \leq n + 1 \right)}{\tilde{\mathbb{P}}_j^* \left(\tau_z^* > p + 1 \right)}.$$

By Lemma 5.6.5 and the point 1 of Proposition 5.3.7,

$$\begin{aligned} \frac{\tilde{\mathbb{E}}_j^* \left(q_p^*(j); \tau_z^* > n + 1 \right)}{\tilde{\mathbb{P}}_j^* \left(\tau_z^* > p + 1 \right)} &= \sum_{i \in \mathbb{X}} \tilde{\mathbb{E}}_j^* \left(q_p^*(j); X_{n+1}^* = i \mid \tau_z^* > n + 1 \right) \frac{\tilde{\mathbb{P}}_j^* \left(\tau_z^* > n + 1 \right)}{\tilde{\mathbb{P}}_j^* \left(\tau_z^* > p + 1 \right)} \\ &\xrightarrow{p \rightarrow +\infty} U^*(j, z) \sqrt{\theta}. \end{aligned}$$

Moreover, using (5.6.2) and the point 1 of Proposition 5.3.7,

$$\frac{\tilde{\mathbb{E}}_j^* \left(q_p^*(j); p + 1 < \tau_z^* \leq n + 1 \right)}{\tilde{\mathbb{P}}_j^* \left(\tau_z^* > p + 1 \right)} \leq 1 - \frac{\tilde{\mathbb{P}}_j^* \left(\tau_z^* > n + 1 \right)}{\tilde{\mathbb{P}}_j^* \left(\tau_z^* > p + 1 \right)} \xrightarrow{p \rightarrow +\infty} 1 - \sqrt{\theta}.$$

Therefore, for any $\theta \in (0, 1)$,

$$\left| \lim_{p \rightarrow +\infty} \tilde{\mathbb{E}}_j^* \left(q_p^*(j) \mid \tau_z^* > p + 1 \right) - U^*(j, z) \sqrt{\theta} \right| \leq 1 - \sqrt{\theta}.$$

Taking the limit as $\theta \rightarrow 1$ it concludes the proof. □

Lemma 5.6.7. *Assume that the conditions of Theorem 5.2.3 are satisfied. For any $(j, z) \in \text{supp}(\tilde{V}_1^*)$ and $\theta \in (0, 1)$, we have*

$$\lim_{n \rightarrow +\infty} \tilde{\mathbb{E}}_j^* \left(\left| q_{[\theta n]}^*(j) - q_n^*(j) \right| \mid \tau_z^* > n + 1 \right) = 0.$$

Proof. Using the fact that $\eta_k^*(j)$ are non-negative and the definition of $q_n^*(j)$ in (5.5.3), we see that $(q_n^*(j))_{n \geq 1}$ is non-increasing. Therefore, using Lemmas 5.6.5 and 5.6.6,

$$\begin{aligned} I_0 &:= \lim_{n \rightarrow +\infty} \tilde{\mathbb{E}}_j^* \left(\left| q_{[\theta n]}^*(j) - q_n^*(j) \right| \mid \tau_z^* > n + 1 \right) \\ &= \lim_{n \rightarrow +\infty} \sum_{i \in \mathbb{X}} \tilde{\mathbb{E}}_j^* \left(q_{[\theta n]}^*(j); X_{n+1}^* = i \mid \tau_z^* > n + 1 \right) - \lim_{n \rightarrow +\infty} \tilde{\mathbb{E}}_j^* \left(q_n^*(j) \mid \tau_z^* > n + 1 \right) \\ &= U^*(j, z) - U^*(j, z) = 0. \end{aligned}$$

□

Lemma 5.6.8. *Assume that the conditions of Theorem 5.2.3 are satisfied. For any $(j, z) \in \text{supp}(\tilde{V}_1^*)$ and $i \in \mathbb{X}$, we have*

$$\lim_{n \rightarrow +\infty} \tilde{\mathbb{E}}_j^* \left(q_n^*(j); X_{n+1}^* = i \mid \tau_z^* > n + 1 \right) = U^*(j, z) \tilde{\nu}_1(i).$$

Proof. By Lemmas 5.6.5 and 5.6.7, for any $(j, z) \in \text{supp}(\tilde{V}_1^*)$, $i \in \mathbb{X}$ and $\theta \in (0, 1)$,

$$\begin{aligned} I_0 &:= \lim_{n \rightarrow +\infty} \tilde{\mathbb{E}}_j^* \left(q_n^*(j); X_{n+1}^* = i \mid \tau_z^* > n + 1 \right) \\ &= \lim_{n \rightarrow +\infty} \tilde{\mathbb{E}}_j^* \left(q_{[\theta n]}^*(j); X_{n+1}^* = i \mid \tau_z^* > n + 1 \right) \\ &\quad + \lim_{n \rightarrow +\infty} \tilde{\mathbb{E}}_j^* \left(q_n^*(j) - q_{[\theta n]}^*(j); X_{n+1}^* = i \mid \tau_z^* > n + 1 \right) \\ &= U^*(j, z) \tilde{\nu}_1(i). \end{aligned}$$

□

Lemma 5.6.9. *Assume that the conditions of Theorem 5.2.3 are satisfied. There exists \tilde{u} a positive function on \mathbb{X} such that, for any $(i, j) \in \mathbb{X}^2$, we have*

$$\tilde{\mathbb{E}}_j^* \left(q_n^*(j); X_{n+1}^* = i \right) \underset{n \rightarrow +\infty}{\sim} \frac{\tilde{u}(j) \tilde{\nu}_1(i)}{\sqrt{n}}.$$

Proof. Fix $(i, j) \in \mathbb{X}^2$. For any $z \in \mathbb{R}$ and $n \geq 1$,

$$\begin{aligned} 0 \leq \tilde{\mathbb{E}}_j^* \left(q_n^*(j); X_{n+1}^* = i \right) - \tilde{\mathbb{E}}_j^* \left(q_n^*(j); X_{n+1}^* = i, \tau_z^* > n + 1 \right) \\ \leq \tilde{\mathbb{E}}_j^* \left(q_n^*(j); \tau_z^* \leq n + 1 \right). \end{aligned} \quad (5.6.8)$$

Since $q_n^*(j) \leq 1$ (see (5.6.2)), we have

$$\tilde{\mathbb{E}}_j^* \left(q_n^*(j); \tau_z^* \leq n + 1 \right) \leq \tilde{\mathbb{E}}_j^* \left(q_n^*(j); \tau_z^* \leq n \right) + \tilde{\mathbb{P}}_j \left(\tau_z^* = n + 1 \right). \quad (5.6.9)$$

By (5.5.5), $q_k^*(j) \leq e^{S_k^*}$, for any $k \leq n$. Since $(q_n^*(j))_{n \geq 1}$ is non-increasing, we have $q_n^*(j) = \min_{1 \leq k \leq n} q_k^*(j) \leq e^{\min_{1 \leq k \leq n} S_k^*}$. Consequently,

$$\begin{aligned} \tilde{\mathbb{E}}_j^* \left(q_n^*(j); \tau_z^* \leq n \right) &\leq e^{-z} \tilde{\mathbb{E}}_j^* \left(e^{\min_{1 \leq k \leq n} z + S_k^*}; \tau_z^* \leq n \right) \\ &\leq e^{-z} \sum_{p=0}^{+\infty} e^{-p} \tilde{\mathbb{P}}_j^* \left(-(p+1) < \min_{1 \leq k \leq n} z + S_k^* \leq -p, \tau_z^* \leq n \right) \\ &\leq e^{-z} \sum_{p=0}^{+\infty} e^{-p} \tilde{\mathbb{P}}_j^* \left(\tau_{z+p+1}^* > n \right). \end{aligned}$$

Using the point 2 of Proposition 5.3.7,

$$\tilde{\mathbb{E}}_j^* \left(q_n^*(j); \tau_z^* \leq n \right) \leq \frac{c e^{-z} (1 + \max(0, z))}{\sqrt{n}}. \quad (5.6.10)$$

By the point 3 of Proposition 5.3.6, there exists $z_0 \in \mathbb{R}$ such that for any $z \geq z_0$, $\tilde{V}_1^*(j, z) > 0$, which means that $(j, z) \in \text{supp}(\tilde{V}_1^*)$. Therefore, using the point 1 of Proposition 5.3.7, for any $z \geq z_0$,

$$\lim_{n \rightarrow +\infty} \sqrt{n} \tilde{\mathbb{P}}_j^* \left(\tau_z^* = n + 1 \right) = \lim_{n \rightarrow +\infty} \sqrt{n} \tilde{\mathbb{P}}_j^* \left(\tau_z^* > n \right) - \lim_{n \rightarrow +\infty} \sqrt{n} \tilde{\mathbb{P}}_j^* \left(\tau_z^* > n + 1 \right) = 0. \quad (5.6.11)$$

Putting together (5.6.9), (5.6.10) and (5.6.11), we obtain that, for any $z \geq z_0$,

$$\lim_{n \rightarrow +\infty} \sqrt{n} \tilde{\mathbb{E}}_j^*(q_n^*(j); \tau_z^* \leq n + 1) \leq c e^{-z} (1 + \max(0, z)). \tag{5.6.12}$$

Moreover, using Lemma 5.6.8 and the point 1 of Proposition 5.3.7,

$$\lim_{n \rightarrow +\infty} \sqrt{n} \tilde{\mathbb{E}}_j^*(q_n^*(j); X_{n+1}^* = i, \tau_z^* > n + 1) = \frac{2\tilde{V}_1^*(j, z)}{\sqrt{2\pi\tilde{\sigma}_1}} U^*(j, z) \tilde{\nu}_1(i), \tag{5.6.13}$$

where $\tilde{\sigma}_1$ is defined in (5.3.39). Denoting

$$I(i, j) = \liminf_{n \rightarrow +\infty} \sqrt{n} \tilde{\mathbb{E}}_j^*(q_n^*(j); X_{n+1}^* = i)$$

and

$$J(i, j) = \limsup_{n \rightarrow +\infty} \sqrt{n} \tilde{\mathbb{E}}_j^*(q_n^*(j); X_{n+1}^* = i)$$

and using (5.6.8), (5.6.12) and (5.6.13), we obtain that, for any $z \geq z_0$,

$$\begin{aligned} \frac{2\tilde{V}_1^*(j, z)}{\sqrt{2\pi\tilde{\sigma}_1}} U^*(j, z) \tilde{\nu}_1(i) &\leq I(i, j) \\ &\leq J(i, j) \leq \frac{2\tilde{V}_1^*(j, z)}{\sqrt{2\pi\tilde{\sigma}_1}} U^*(j, z) \tilde{\nu}_1(i) + c e^{-z} (1 + \max(0, z)). \end{aligned} \tag{5.6.14}$$

By (5.6.13), we observe that $z \mapsto \frac{2\tilde{V}_1^*(j, z) U^*(j, z)}{\sqrt{2\pi\tilde{\sigma}_1}}$ is non-decreasing and by (5.6.14), this function is bounded by $I(i, j)/\tilde{\nu}_1(i)$. Consequently the limit

$$\tilde{u}(j) := \lim_{z \rightarrow +\infty} \frac{2\tilde{V}_1^*(j, z) U^*(j, z)}{\sqrt{2\pi\tilde{\sigma}_1}}$$

exists and for any $z \geq z_0$, by (5.6.7),

$$\tilde{u}(j) \geq \frac{2\tilde{V}_1^*(j, z) U^*(j, z)}{\sqrt{2\pi\tilde{\sigma}_1}} > 0. \tag{5.6.15}$$

Taking the limit as $z \rightarrow +\infty$ in (5.6.14), we conclude that

$$I(i, j) = J(i, j) = \tilde{u}(j) \tilde{\nu}_1(i).$$

□

Proof of Theorem 5.2.3. By (5.6.15) the function

$$u(j) = \tilde{u}(j) \frac{\tilde{\nu}_1(j) e^{-\rho(j)}}{v_1(j)}, \quad \forall j \in \mathbb{X},$$

is positive on \mathbb{X} . The assertion of Theorem 5.2.3 is a consequence of (5.6.1) and Lemma 5.6.9.

5.7 Proofs in the weakly subcritical case

We assume the conditions of Theorem 5.2.4, that is Conditions 5.1-5.3 and $\nu(\rho) = k'(0) < 0, k'(1) > 0$. By Lemma 5.3.15, the function $\lambda \mapsto K'(\lambda)$ is non-decreasing (in fact increasing under Condition 5.3, see Lemma 10.3 in [37]/Lemma 4.10.3 of Chapter 4). Consequently, there exists $\lambda \in (0, 1)$ such that

$$K'(\lambda) = \frac{k'(\lambda)}{k(\lambda)} = \tilde{\nu}_\lambda(\rho) = 0. \tag{5.7.1}$$

For this λ and any $i \in \mathbb{X}$, define the changed probability measure $\tilde{\mathbb{P}}_i$ and the corresponding expectation $\tilde{\mathbb{E}}_i$ by (5.3.37), such that for any $n \geq 1$ and any $g: \mathbb{X}^n \rightarrow \mathbb{C}$,

$$\tilde{\mathbb{E}}_i(g(X_1, \dots, X_n)) = \frac{\mathbb{E}_i(e^{\lambda S_n} g(X_1, \dots, X_n) v_\lambda(X_n))}{k(\lambda)^n v_\lambda(i)}. \tag{5.7.2}$$

Our starting point is the following formula which is a consequence of (5.3.1): for any $(i, j) \in \mathbb{X}^2$ and $n \geq 1$,

$$\begin{aligned} &\mathbb{E}_i(q_{n+1}; X_{n+1} = j, \tau_y > n) \\ &= \tilde{\mathbb{E}}_i(e^{-\lambda S_n} q_n(f_j(0)); X_{n+1} = j, \tau_y > n) k(\lambda)^{n+1} \frac{v_\lambda(i)}{v_\lambda(j)} e^{-\lambda \rho(j)}. \end{aligned} \tag{5.7.3}$$

The transition probabilities of $(X_n)_{n \geq 0}$ under the changed measure are given by (5.3.34):

$$\tilde{\mathbf{P}}_\lambda(i, j) = \frac{e^{\lambda \rho(j)} v_\lambda(j)}{k(\lambda) v_\lambda(i)} \mathbf{P}(i, j).$$

By (5.7.1), the Markov walk $(S_n)_{n \geq 0}$ is centred under $\tilde{\mathbb{P}}_i$. Note that under the hypotheses of Theorem 5.2.4, by Lemma 5.3.14, Conditions 5.1 and 5.3 hold also for $\tilde{\mathbf{P}}_\lambda$. Therefore all the results of Section 5.3.3 hold for the Markov walk $(S_n)_{n \geq 0}$ under $\tilde{\mathbb{P}}_i$.

Let $(X_n^*)_{n \geq 0}$ be the dual Markov chain independent of $(X_n)_{n \geq 0}$, with transition probabilities $\tilde{\mathbf{P}}_\lambda^*$ defined by (cp. (5.3.15))

$$\tilde{\mathbf{P}}_\lambda^*(i, j) = \frac{\tilde{\nu}_\lambda(j)}{\tilde{\nu}_\lambda(i)} \tilde{\mathbf{P}}(j, i) = \frac{\nu_\lambda(j)}{\nu_\lambda(i)} \frac{e^{\rho(i)}}{k(\lambda)} \mathbf{P}(j, i). \tag{5.7.4}$$

As in Section 5.3.2, we define the dual Markov walk $(S_n^*)_{n \geq 0}$ by (5.3.16) and its exit time τ_z^* for any $z \in \mathbb{R}$ by (5.3.17). Let $\tilde{\mathbb{P}}_{i,j}$ be the probability on (Ω, \mathcal{F}) generated by the finite dimensional distributions of $(X_n, X_n^*)_{n \geq 0}$ starting at $(X_0, X_0^*) = (i, j)$. By (5.7.1), the Markov walk $(S_n^*)_{n \geq 1}$ is centred under $\tilde{\mathbb{P}}_{i,j}$:

$$\tilde{\nu}_\lambda(\rho) = \tilde{\nu}_\lambda(-\rho) = 0$$

and by Lemma 5.3.4, Conditions 5.1 and 5.3 hold for $\tilde{\mathbf{P}}_\lambda^*$. Let \tilde{V}_λ and \tilde{V}_λ^* be the harmonic functions of the Markov walks $(S_n)_{n \geq 0}$ and $(S_n^*)_{n \geq 0}$, respectively (see Proposition 5.3.6).

The idea of the proof is in the line with that of the previous sections: the positive trajectories (corresponding to the event $\{\tau_y > n\}$) affect the asymptotic behaviour of the survival probability. However, in the weakly subcritical case, the factor $e^{-\lambda S_n}$ in the expectation $\tilde{\mathbb{E}}_i(e^{-\lambda S_n} q_n(f_j(0)); X_{n+1} = j)$ contributes in such a way that, only the trajectories starting at $y \in \mathbb{R}$ conditioned to stay positive and to finish nearby 0, have an impact on the asymptotic of $\tilde{\mathbb{E}}_i(e^{-\lambda S_n} q_n(f_j(0)); X_{n+1} = j)$.

We start by some preliminary bounds. The following assertion is similar to Lemma 5.3.13.

Lemma 5.7.1. *Assume that the conditions of Theorem 5.2.4 are satisfied. For any $i \in \mathbb{X}$, $y \in \mathbb{R}$, $k \geq 1$ and $n \geq k + 1$, we have*

$$n^{3/2} \tilde{\mathbb{E}}_i \left(e^{-S_k} e^{-\lambda S_n}; \tau_y > n \right) \leq e^{(1+\lambda)y} (1 + \max(y, 0)) \frac{cn^{3/2}}{(n-k)^{3/2} k^{3/2}}.$$

Proof. Fix $i \in \mathbb{X}$, $y \in \mathbb{R}$, $k \geq 1$ and $n \geq k + 1$. By the Markov property,

$$\begin{aligned} I_0 &:= n^{3/2} \tilde{\mathbb{E}}_i \left(e^{-S_k} e^{-\lambda S_n}; \tau_y > n \right) \\ &\leq \sum_{p=0}^{+\infty} n^{3/2} e^{\lambda y} e^{-\lambda p} \tilde{\mathbb{E}}_i \left(e^{-S_k}; y + S_n \in [p, p + 1], \tau_y > n \right) \\ &= \sum_{p=0}^{+\infty} n^{3/2} e^{\lambda y} e^{-\lambda p} \tilde{\mathbb{E}}_i \left(e^{-S_k} J_{n-k}(X_k, y + S_k); \tau_y > k \right), \end{aligned}$$

where for any $i' \in \mathbb{X}$, $y' \in \mathbb{R}$ and $p \geq 1$

$$J_{n-k}(i', y') = \tilde{\mathbb{P}}_{i'}(y' + S_{n-k} \in [p, p + 1], \tau_{y'} > n - k).$$

By the point 2 of Proposition 5.3.9,

$$J_{n-k}(i', y') \leq \frac{c}{(n-k)^{3/2}} (1+p)(1 + \max(y', 0)).$$

Consequently,

$$\begin{aligned} I_0 &\leq e^{\lambda y} \frac{cn^{3/2}}{(n-k)^{3/2}} \tilde{\mathbb{E}}_i \left(e^{-S_k} (1 + y + S_k); \tau_y > k \right) \sum_{p=0}^{+\infty} e^{-\lambda p} (1+p) \\ &\leq e^{\lambda y} \frac{cn^{3/2}}{(n-k)^{3/2}} \tilde{\mathbb{E}}_i \left(e^{-S_k} (1 + y + S_k); \tau_y > k \right) \\ &\leq e^{(1+\lambda)y} \frac{cn^{3/2}}{(n-k)^{3/2}} \sum_{p=0}^{+\infty} e^{-p} (2+p) \tilde{\mathbb{P}}_i(y + S_k \in [p, p + 1]; \tau_y > k). \end{aligned}$$

Again by the point 2 of Proposition 5.3.9,

$$I_0 \leq e^{(1+\lambda)y} (1 + \max(y, 0)) \frac{cn^{3/2}}{(n-k)^{3/2} k^{3/2}} \sum_{p=0}^{+\infty} e^{-p} (2+p)(1+p).$$

This concludes the proof of the lemma. □

For any $l \geq 1$ and $n \geq l + 1$, set

$$q_{l,n}(f_j(0)) := 1 - f_{l+1,n}(f_j(0)) = 1 - f_{X_{l+1}} \circ \dots \circ f_{X_n} \circ f_j(0),$$

In the same way as in Lemma 5.3.2, we obtain:

$$q_{l,n}(f_j(0))^{-1} = \frac{e^{S_l - S_n}}{1 - f_j(0)} + \sum_{k=l}^{n-1} e^{S_l - S_k} \eta_{k+1,n}(f_j(0)), \tag{5.7.5}$$

where $\eta_{k+1,n}(s)$ are defined by (5.3.8). Moreover, similarly to (5.3.4), we have for any $n \geq l + 1 \geq 2$,

$$q_{l,n}(f_j(0)) \in (0, 1] \quad \tilde{\mathbb{P}}_i\text{-a.s.} \tag{5.7.6}$$

In addition, by Lemma 5.3.3, for any $k \leq n - 1$,

$$0 \leq \eta_{k+1,n}(f_j(0)) \leq \eta \quad \tilde{\mathbb{P}}_i\text{-a.s.} \tag{5.7.7}$$

Lemma 5.7.2. *Assume that the conditions of Theorem 5.2.4 are satisfied. For any $(i, j) \in \mathbb{X}^2$ and $y \in \mathbb{R}$, we have*

$$\lim_{l, m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} n^{3/2} \tilde{\mathbb{E}}_i \left(\left| e^{-S_{n-m}} q_{n-m, n} (f_j(0))^{-1} - e^{-S_l} q_{l, n} (f_j(0))^{-1} \right| e^{-\lambda S_n}; \tau_y > n \right) = 0.$$

Proof. Fix $(i, j) \in \mathbb{X}^2$ and $y \in \mathbb{R}$. For any $l \geq 1$, $m \geq 1$ and $n \geq l + m + 1$, we have

$$\begin{aligned} I_0 &:= n^{3/2} \tilde{\mathbb{E}}_i \left(\left| e^{-S_{n-m}} q_{n-m, n} (f_j(0))^{-1} - e^{-S_l} q_{l, n} (f_j(0))^{-1} \right| e^{-\lambda S_n}; \tau_y > n \right) \\ &= n^{3/2} \tilde{\mathbb{E}}_i \left(\sum_{k=l}^{n-m-1} e^{-S_k} \eta_{k+1, n} (f_j(0)) e^{-\lambda S_n}; \tau_y > n \right). \end{aligned}$$

Using (5.7.7) and Lemma 5.7.1,

$$I_0 \leq \eta \sum_{k=l}^{n-m-1} e^{(1+\lambda)y} (1 + \max(y, 0)) \frac{cn^{3/2}}{(n-k)^{3/2} k^{3/2}}.$$

Let $n_1 := \lfloor n/2 \rfloor$. We note that

$$\begin{aligned} \sum_{k=l}^{n-m-1} \frac{cn^{3/2}}{(n-k)^{3/2} k^{3/2}} &\leq \frac{cn^{3/2}}{(n-n_1)^{3/2}} \sum_{k=l}^{n_1} \frac{1}{k^{3/2}} + \frac{cn^{3/2}}{n_1^{3/2}} \sum_{k=n_1+1}^{n-m-1} \frac{1}{(n-k)^{3/2}} \\ &\leq c \sum_{k=l}^{+\infty} \frac{1}{k^{3/2}} + c \sum_{k=m}^{+\infty} \frac{1}{k^{3/2}}. \end{aligned}$$

Consequently,

$$\limsup_{n \rightarrow +\infty} I_0 \leq c\eta e^{(1+\lambda)y} (1 + \max(y, 0)) \left(\sum_{k=l}^{+\infty} \frac{1}{k^{3/2}} + \sum_{k=m}^{+\infty} \frac{1}{k^{3/2}} \right).$$

Taking the limits as $l \rightarrow +\infty$ and $m \rightarrow +\infty$, proves the lemma. \square

For any $l \geq 1$, $m \geq 1$ and $n \geq l + m + 1$, consider the random variables

$$\begin{aligned} r_n^{(l, m)}(j) &:= 1 - f_{l, l} \left(\left[1 - f'_{l+1, n-m}(1) (1 - f_{n-m+1, n}(f_j(0))) \right]^+ \right) \\ &= 1 - f_{X_1} \circ \dots \circ f_{X_l} \left(\left[1 - f'_{X_{l+1}}(1) \times \dots \right. \right. \\ &\quad \left. \left. \dots \times f'_{X_{n-m}}(1) (1 - f_{X_{n-m+1}} \circ \dots \circ f_{X_n} \circ f_j(0)) \right]^+ \right) \end{aligned}$$

where $[t]^+ = \max(t, 0)$ for any $t \in \mathbb{R}$. The random variable $r_n^{(l, m)}(j)$ approximates $q_n(f_j(0))$ in the following sense:

Lemma 5.7.3. *Assume that the conditions of Theorem 5.2.4 are satisfied. For any $(i, j) \in \mathbb{X}^2$ and $y \in \mathbb{R}$,*

$$\lim_{l, m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} n^{3/2} \tilde{\mathbb{E}}_i \left(\left| q_n(f_j(0)) - r_n^{(l, m)}(j) \right| e^{-\lambda S_n}; \tau_y > n \right) = 0.$$

Proof. Fix $(i, j) \in \mathbb{X}^2$ and $y \in \mathbb{R}$. Since for any $i' \in \mathbb{X}$, $f_{i'}$ is increasing and convex, the function $f_{l+1, n-m}$ is convex. So, for any $l \geq 1$, $m \geq 1$ and $n \geq l + m + 1$,

$$f_{l+1, n}(f_j(0)) = f_{l+1, n-m}(f_{n-m+1, n}(f_j(0))) \geq \left[1 - f'_{l+1, n-m}(1) (1 - f_{n-m+1, n}(f_j(0))) \right]^+.$$

Since $f_{1,l}$ is increasing,

$$q_n(f_j(0)) = 1 - f_{1,n}(f_j(0)) \leq r_n^{(l,m)}(j),$$

or equivalently

$$0 \leq r_n^{(l,m)}(j) - q_n(f_j(0)).$$

Moreover, by the convexity of $f_{1,l}$,

$$\begin{aligned} r_n^{(l,m)}(j) - q_n(f_j(0)) &= f_{1,l} \circ f_{l+1,n}(f_j(0)) - f_{1,l} \left(\left[1 - f'_{l+1,n-m}(1) (1 - f_{n-m+1,n}(f_j(0))) \right]^+ \right) \\ &\leq f'_{1,l}(1) \left(f_{l+1,n}(f_j(0)) - \left[1 - f'_{l+1,n-m}(1) (1 - f_{n-m+1,n}(f_j(0))) \right]^+ \right) \\ &\leq f'_{1,l}(1) \left(f'_{l+1,n-m}(1) q_{n-m,n}(f_j(0)) - q_{l,n}(f_j(0)) \right) \\ &= e^{S_{n-m}} q_{n-m,n}(f_j(0)) - e^{S_l} q_{l,n}(f_j(0)) \\ &= e^{S_{n-m}} q_{n-m,n}(f_j(0)) e^{S_l} q_{l,n}(f_j(0)) \\ &\quad \times \left(e^{-S_l} q_{l,n}(f_j(0))^{-1} - e^{-S_{n-m}} q_{n-m,n}(f_j(0))^{-1} \right). \end{aligned}$$

By (5.7.5), we have $q_{l,n}(f_j(0)) \leq e^{S_n - S_l}$ and so

$$r_n^{(l,m)}(j) - q_n(f_j(0)) \leq e^{2S_n} \left(e^{-S_l} q_{l,n}(f_j(0))^{-1} - e^{-S_{n-m}} q_{n-m,n}(f_j(0))^{-1} \right).$$

In addition, by the definition of $r_n^{(l,m)}(j)$ and $q_n(f_j(0))$, we have $r_n^{(l,m)}(j) - q_n(f_j(0)) \leq 1$. Therefore, $\tilde{\mathbb{P}}_i$ -a.s. it holds,

$$r_n^{(l,m)}(j) - q_n(f_j(0)) \leq \min \left(1, e^{2S_n} \left(e^{-S_l} q_{l,n}(f_j(0))^{-1} - e^{-S_{n-m}} q_{n-m,n}(f_j(0))^{-1} \right) \right).$$

Using the previous bound, it follows that, for any integer $N \geq 1$,

$$\begin{aligned} I_0 &:= n^{3/2} \tilde{\mathbb{E}}_i \left(\left| q_n(f_j(0)) - r_n^{(l,m)}(j) \right| e^{-\lambda S_n}; \tau_y > n \right) \\ &\leq e^{2(N-y)} n^{3/2} \tilde{\mathbb{E}}_i \left(\left| e^{-S_l} q_{l,n}(f_j(0))^{-1} - e^{-S_{n-m}} q_{n-m,n}(f_j(0))^{-1} \right| e^{-\lambda S_n}; \tau_y > n \right) \\ &\quad + n^{3/2} \tilde{\mathbb{E}}_i \left(e^{-\lambda S_n}; y + S_n > N, \tau_y > n \right). \end{aligned}$$

Moreover, using the point 2 of Proposition 5.3.9,

$$\begin{aligned} n^{3/2} \tilde{\mathbb{E}}_i \left(e^{-\lambda S_n}; y + S_n > N, \tau_y > n \right) &\leq \sum_{p=N}^{+\infty} e^{\lambda y} e^{-\lambda p} n^{3/2} \tilde{\mathbb{P}}_i \left(y + S_n \in [p, p+1], \tau_y > n \right) \\ &\leq c e^{\lambda y} (1 + \max(y, 0)) \sum_{p=N}^{+\infty} e^{-\lambda p} (1 + p). \end{aligned}$$

Consequently, using Lemma 5.7.2, we obtain that

$$\lim_{l,m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} I_0 \leq c e^{\lambda y} (1 + \max(y, 0)) \sum_{p=N}^{+\infty} e^{-\lambda p} (1 + p).$$

Taking the limit as $N \rightarrow +\infty$, proves the lemma. □

We now introduce the following random variable: for any $j \in \mathbb{X}$, $u \in \mathbb{R}$, $l \geq 1$ and $m \geq 1$

$$r_\infty^{(l,m)}(j, u) := 1 - f_{X_1} \circ \dots \circ f_{X_l} \left(\left[1 - e^{-S_l} e^u q_m^*(j) \right]^+ \right) \in [0, 1],$$

where, as in (5.5.3) and (5.5.5), for any $m \geq 1$,

$$q_m^*(j) := e^{S_m^*} \left(1 - f_{X_m^*} \circ \dots \circ f_{X_1^*} \circ f_j(0) \right) = \left[\frac{1}{1 - f_j(0)} + \sum_{k=1}^n e^{-S_k^*} \eta_k^*(j) \right]^{-1}$$

and as in (5.5.4), for any $k \geq 2$,

$$\eta_k^*(j) := g_{X_k^*} \left(f_{X_{k-1}^*} \circ \dots \circ f_{X_1^*} \circ f_j(0) \right) \quad \text{and} \quad \eta_1^* := g_{X_1^*} (f_j(0)).$$

For any $(i, y) \in \text{supp}(\tilde{V}_\lambda)$ and $(j, z) \in \text{supp}(\tilde{V}_\lambda^*)$, let $\tilde{\mathbb{P}}_{i,y,j,z}^+$ and $\tilde{\mathbb{E}}_{i,y,j,z}^+$ be, respectively, the probability and its associated expectation defined for any $n \geq 1$ and any function $g: \mathbb{X}^{l,m} \rightarrow \mathbb{C}$ by

$$\begin{aligned} \tilde{\mathbb{E}}_{i,y,j,z}^+ (g(X_1, \dots, X_l, X_m^*, \dots, X_1^*)) &= \tilde{\mathbb{E}}_{i,j} \left(g(X_1, \dots, X_l, X_m^*, \dots, X_1^*) \frac{\tilde{V}_\lambda(X_l, y + S_l)}{\tilde{V}_\lambda(i, y)} \times \right. \\ &\quad \left. \frac{\tilde{V}_\lambda^*(X_m^*, z + S_m^*)}{\tilde{V}_\lambda^*(j, z)}; \tau_y > l, \tau_z^* > m \right). \end{aligned} \tag{5.7.8}$$

For any $j \in \mathbb{X}$ let $z_0(j) \in \mathbb{R}$ be the unique real such that $(j, z) \in \text{supp}(\tilde{V}_\lambda^*)$ for any $z > z_0$ and $(j, z) \notin \text{supp}(\tilde{V}_\lambda^*)$ for any $z < z_0$ (see [38]/Chapter 3 for details on the domain of positivity of the harmonic function). Set $z_0(j)^+ = \max\{z_0(j), 0\}$.

Lemma 5.7.4. *Assume that the conditions of Theorem 5.2.4 are satisfied. For any $(i, y) \in \text{supp}(\tilde{V}_\lambda)$, $j \in \mathbb{X}$, $l \geq 1$ and $m \geq 1$,*

$$\begin{aligned} \lim_{n \rightarrow +\infty} n^{3/2} \tilde{\mathbb{E}}_i \left(r_n^{(l,m)}(j) e^{-\lambda S_n}; X_{n+1} = j, \tau_y > n \right) \\ = \frac{2}{\sqrt{2\pi\sigma^3}} e^{\lambda y} \int_{z_0(j)^+}^{+\infty} e^{-\lambda z} \tilde{\mathbb{E}}_{i,y,j,z}^+ \left(r_\infty^{(l,m)}(j, z - y) \tilde{V}_\lambda(i, y) \tilde{V}_\lambda^*(j, z) dz \tilde{\nu}_\lambda(j) \right). \end{aligned}$$

Proof. Fix $(i, y) \in \text{supp}(\tilde{V}_\lambda)$, $j \in \mathbb{X}$, $l \geq 1$ and $m \geq 1$ and let g be a function $\mathbb{X}^{l+m} \times \mathbb{R} \rightarrow \mathbb{R}_+$ defined by

$$\begin{aligned} g(i_1, \dots, i_l, i_{n-m+1}, \dots, i_n, z) &= e^{\lambda y} e^{-\lambda z} \mathbb{1}_{\{z \geq 0\}} \tilde{\mathbf{P}}_\lambda(i_n, j) \left[1 - f_{i_1} \circ \dots \right. \\ &\quad \left. \dots \circ f_{i_l} \left(\left[1 - e^{z-y-\rho(i_n)-\dots-\rho(i_{n-m+1})-\rho(i_l)-\dots-\rho(i_1)} \left(1 - f_{i_{n-m+1}} \circ \dots \circ f_{i_n} \circ f_j(0) \right) \right]^+ \right) \right] \end{aligned}$$

for all $(i_1, \dots, i_l, i_{n-m+1}, \dots, i_n, z) \in \mathbb{X}^{l+m} \times \mathbb{R}$ and note that on $\{\tau_y > n\}$,

$$g(X_1, \dots, X_l, X_{n-m+1}, \dots, X_n, y + S_n) = r_n^{(l,m)}(j) e^{-\lambda S_n} \tilde{\mathbf{P}}_\lambda(i_n, j).$$

Observe also that since $0 \leq g(i_1, \dots, i_l, i_{n-m+1}, \dots, i_n, z) \leq e^{\lambda y} e^{-\lambda z} \mathbb{1}_{\{z \geq 0\}}$, the function g belongs to the set, say $\mathcal{C}^+(\mathbb{X}^{l+m} \times \mathbb{R}_+)$, of non-negative function $g: \mathbb{X}^{l+m} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying the following properties:

- for any $(i_1, \dots, i_{l+m}) \in \mathbb{X}^{l+m}$, the function $z \mapsto g(i_1, \dots, i_{l+m}, z)$ is continuous,
- there exists $\varepsilon > 0$ such that $\max_{i_1, \dots, i_{l+m} \in \mathbb{X}} \sup_{z \geq 0} g(i_1, \dots, i_{l+m}, z)(1+z)^{2+\varepsilon} < +\infty$.

Therefore, by the Markov property and Proposition 5.3.10, we obtain that

$$\begin{aligned} I_0 &:= \lim_{n \rightarrow +\infty} n^{3/2} \tilde{\mathbb{E}}_i \left(r_n^{(l,m)}(j) e^{-\lambda S_n} ; X_{n+1} = j, \tau_y > n \right) \\ &= \lim_{n \rightarrow +\infty} n^{3/2} \tilde{\mathbb{E}}_i \left(g(X_1, \dots, X_l, X_{n-m+1}, \dots, X_n, y + S_n) ; \tau_y > n \right) \\ &= \frac{2}{\sqrt{2\pi\sigma^3}} \int_0^{+\infty} e^{-\lambda(z-y)} \sum_{j' \in \mathbb{X}} \tilde{\mathbb{E}}_{i,j'} \left(r_\infty^{(l,m)}(j, z-y) \tilde{\mathbf{P}}_\lambda(X_1^*, j) \tilde{V}_\lambda(X_l, y + S_l) \right. \\ &\quad \left. \times \tilde{V}_\lambda^*(X_m^*, z + S_m^*) ; \tau_y > l, \tau_z^* > m \right) \tilde{\nu}_\lambda(j') dz. \end{aligned}$$

Since $\tilde{\nu}_\lambda$ is $\tilde{\mathbf{P}}_\lambda^*$ -invariant, we write

$$\begin{aligned} I_0 &= \frac{2}{\sqrt{2\pi\sigma^3}} \int_0^{+\infty} e^{-\lambda(z-y)} \sum_{j_1 \in \mathbb{X}} \tilde{\mathbf{P}}_\lambda(j_1, j) \tilde{\nu}_\lambda(j_1) \tilde{\mathbb{E}}_i \left(r_\infty^{(l,m)}(j, z-y) \tilde{V}_\lambda(X_l, y + S_l) \right. \\ &\quad \left. \times \tilde{V}_\lambda^*(X_m^*, z + S_m^*) ; \tau_y > l, \tau_z^* > m \mid X_1^* = j_1 \right) dz. \end{aligned}$$

Using the definition of $\tilde{\mathbf{P}}_\lambda^*$ in (5.7.4), we have

$$\begin{aligned} I_0 &= \frac{2}{\sqrt{2\pi\sigma^3}} \int_0^{+\infty} e^{-\lambda(z-y)} \tilde{\nu}_\lambda(j) \tilde{\mathbb{E}}_{i,j} \left(r_\infty^{(l,m)}(j, z-y) \tilde{V}_\lambda(X_l, y + S_l) \right. \\ &\quad \left. \times \tilde{V}_\lambda^*(X_m^*, z + S_m^*) ; \tau_y > l, \tau_z^* > m \right) dz. \end{aligned}$$

Now, note that when $(j, z) \notin \text{supp}(\tilde{V}_\lambda^*)$, using the point 1 of Proposition 5.3.6,

$$\begin{aligned} &\tilde{\mathbb{E}}_{i,j} \left(r_\infty^{(l,m)}(j, z-y) \tilde{V}_\lambda(X_l, y + S_l) \tilde{V}_\lambda^*(X_m^*, z + S_m^*) ; \tau_y > l, \tau_z^* > m \right) \\ &\leq \tilde{\mathbb{E}}_i \left(\tilde{V}_\lambda(X_l, y + S_l) ; \tau_y > l \right) \tilde{\mathbb{E}}_j^* \left(\tilde{V}_\lambda^*(X_m^*, z + S_m^*) ; \tau_z^* > m \right) = \tilde{V}_\lambda(i, y) \tilde{V}_\lambda^*(j, z) = 0. \end{aligned}$$

Together with (5.7.8), it proves the lemma. \square

Consider for any $l \geq 1, j \in \mathbb{X}$ and $u \in \mathbb{R}$,

$$r_\infty^{(l,\infty)}(j, u) = 1 - f_{X_1} \circ \dots \circ f_{X_l} \left(\left[1 - e^{-S_l} e^u q_\infty^*(j) \right]^+ \right) \in [0, 1], \quad (5.7.9)$$

where as in (5.5.8),

$$q_\infty^*(j) = \left[\frac{1}{1 - f_j(0)} + \sum_{k=1}^{\infty} e^{-S_k^*} \eta_k^*(j) \right]^{-1}.$$

Lemma 5.7.5. *Assume that the conditions of Theorem 5.2.4 are satisfied. For any $(i, y) \in \text{supp}(\tilde{V}_\lambda)$, $(j, z) \in \text{supp}(\tilde{V}_\lambda^*)$, $l \geq 1$ and $u \in \mathbb{R}$,*

$$\lim_{m \rightarrow +\infty} \tilde{\mathbb{E}}_{i,y,j,z}^+ \left(\left| r_\infty^{(l,m)}(j, u) - r_\infty^{(l,\infty)}(j, u) \right| \right) = 0.$$

Proof. Fix $(i, y) \in \text{supp}(\tilde{V}_\lambda)$, $(j, z) \in \text{supp}(\tilde{V}_\lambda^*)$, $l \geq 1$ and $u \in \mathbb{R}$. By the convexity of $f_{1,l}$, for any $m \geq 1$, we have $\tilde{\mathbb{P}}_{i,y,j,z}^+$ a.s.,

$$\begin{aligned} &\left| r_\infty^{(l,m)}(j, u) - r_\infty^{(l,\infty)}(j, u) \right| \\ &\leq (f_{X_1} \circ \dots \circ f_{X_l})'(1) \left| \left[1 - e^{-S_l} e^u q_m^*(j) \right]^+ - \left[1 - e^{-S_l} e^u q_\infty^*(j) \right]^+ \right| \\ &\leq e^{S_l} \left| e^{-S_l} e^u q_m^*(j) - e^{-S_l} e^u q_\infty^*(j) \right| \\ &= e^u \left| q_m^*(j) q_\infty^*(j) \right| \left| (q_\infty^*(j))^{-1} - (q_m^*(j))^{-1} \right|. \end{aligned}$$

Moreover, for any $m \geq 1$,

$$q_m^*(j) = \left[\frac{1}{1 - f_j(0)} + \sum_{k=1}^m e^{-S_k^*} \eta_k^*(j) \right]^{-1} \in (0, 1],$$

$$q_\infty^*(j) = \left[\frac{1}{1 - f_j(0)} + \sum_{k=1}^{\infty} e^{-S_k^*} \eta_k^*(j) \right]^{-1} \in [0, 1]$$

and by Lemma 5.3.3, for any $k \geq 1$,

$$0 \leq \eta_k^*(j) \leq \eta. \quad (5.7.10)$$

Therefore,

$$\left| r_\infty^{(l,m)}(j, u) - r_\infty^{(l,\infty)}(j, u) \right| \leq e^u \eta \sum_{k=m+1}^{+\infty} e^{-S_k^*}.$$

Using Lemma 5.3.13 and the Lebesgue dominated convergence theorem,

$$\tilde{\mathbb{E}}_{i,y,j,z}^+ \left(\left| r_\infty^{(l,m)}(j, u) - r_\infty^{(l,\infty)}(j, u) \right| \right) \leq e^u \eta \sum_{k=m+1}^{+\infty} \tilde{\mathbb{E}}_{i,y,j,z}^+ \left(e^{-S_k^*} \right).$$

By Lemma 5.3.13, we conclude that

$$\lim_{m \rightarrow +\infty} \tilde{\mathbb{E}}_{i,y,j,z}^+ \left(\left| r_\infty^{(l,m)}(j, u) - r_\infty^{(l,\infty)}(j, u) \right| \right) = 0.$$

□

For any $l \geq 1$, $j \in \mathbb{X}$ and $u \in \mathbb{R}$, set

$$s_l(j, u) = \left[1 - e^{-S_l} e^u q_\infty^*(j) \right]^+. \quad (5.7.11)$$

Note that, by Lemma 5.3.13, $(q_\infty^*(j))^{-1}$ is integrable and so finite a.s. (see (5.6.6)). Therefore $s_l(j, u) \in [0, 1]$. In addition, by the convexity of $f_{X_{l+1}}$, we have for any $j \in \mathbb{X}$, $u \in \mathbb{R}$ and $l \geq 1$,

$$\begin{aligned} f_{X_{l+1}}(s_{l+1}(j, u)) &\geq 1 - f'_{X_{l+1}}(1) (1 - s_{l+1}(j, u)) \\ &\geq 1 - e^{\rho(X_{l+1})} e^{-S_{l+1}} e^u q_\infty^*(j) = 1 - e^{-S_l} e^u q_\infty^*(j). \end{aligned}$$

Since $f_{X_{l+1}}$ is non-negative on $[0, 1]$, we see that $f_{X_{l+1}}(s_{l+1}(j, u)) \geq s_l(j, u)$ and so for any $k \geq 1$, $(f_{k+1,l}(s_l(j, u)))_{l \geq k}$ is non-decreasing and bounded by 1. Using the continuity of g_{X_k} and (5.3.14), we deduce that $(\eta_{k,l}(s_l(j, u)))_{l \geq k}$ converges and we denote for any $k \geq 1$,

$$\eta_{k,\infty}(j, u) := \lim_{l \rightarrow +\infty} \eta_{k,l}(s_l(j, u)). \quad (5.7.12)$$

Moreover, by Lemma 5.3.3, we have for any $k \geq 1$, $l \geq k$ and $u \in \mathbb{R}$,

$$0 \leq \eta_{k,l}(s_l(j, u)) \leq \eta \quad \text{and} \quad 0 \leq \eta_{k,\infty}(j, u) \leq \eta. \quad (5.7.13)$$

For any $j \in \mathbb{X}$ and $u \in \mathbb{R}$, set

$$r_\infty(j, u) := \left[\frac{e^{-u}}{q_\infty^*(j)} + \sum_{k=0}^{+\infty} e^{-S_k} \eta_{k+1,\infty}(j, u) \right]^{-1}.$$

Lemma 5.7.6. *Assume that the conditions of Theorem 5.2.4 are satisfied. For any $(i, y) \in \text{supp}(\tilde{V}_\lambda)$, $(j, z) \in \text{supp}(\tilde{V}_\lambda^*)$ and $u \in \mathbb{R}$,*

$$\lim_{l \rightarrow +\infty} \tilde{\mathbb{E}}_{i,y,j,z}^+ \left(\left| r_\infty^{(l,\infty)}(j, u) - r_\infty(j, u) \right| \right) = 0.$$

Proof. Fix $(i, y) \in \text{supp}(\tilde{V}_\lambda)$, $(j, z) \in \text{supp}(\tilde{V}_\lambda^*)$ and $u \in \mathbb{R}$. By (5.7.9), Lemma 5.3.2 and (5.7.11), we have

$$\left(r_\infty^{(l,\infty)}(j, u) \right)^{-1} = \frac{e^{-S_l}}{1 - s_l(j, u)} + \sum_{k=0}^{l-1} e^{-S_k} \eta_{k+1,l}(s_l(j, u)).$$

So, for any $p \geq 1$ and $l \geq p$, using (5.7.13),

$$\begin{aligned} \left| \left(r_\infty^{(l,\infty)}(j, u) \right)^{-1} - r_\infty(j, u)^{-1} \right| &\leq \sum_{k=0}^p e^{-S_k} \left| \eta_{k+1,l}(s_l(j, u)) - \eta_{k+1,\infty}(j, u) \right| \\ &\quad + \left| \frac{e^{-u}}{q_\infty^*(j)} - \frac{e^{-S_l}}{1 - s_l(j, u)} \right| + 2\eta \sum_{k=p+1}^{+\infty} e^{-S_k}. \end{aligned}$$

Therefore,

$$\begin{aligned} I_0 &:= \tilde{\mathbb{E}}_{i,y,j,z}^+ \left(\left| \left(r_\infty^{(l,\infty)}(j, u) \right)^{-1} - r_\infty(j, u)^{-1} \right| \right) \\ &\leq \sum_{k=0}^p \tilde{\mathbb{E}}_{i,y,j,z}^+ \left(e^{-S_k} \left| \eta_{k+1,l}(s_l(j, u)) - \eta_{k+1,\infty}(j, u) \right| \right) \\ &\quad + \tilde{\mathbb{E}}_{i,y,j,z}^+ \left(\left| \frac{e^{-u}}{q_\infty^*(j)} - e^{-S_l} \right| ; e^{-S_l} > \frac{e^{-u}}{q_\infty^*(j)} \right) + 2\eta \tilde{\mathbb{E}}_{i,y}^+ \left(\sum_{k=p+1}^{+\infty} e^{-S_k} \right), \end{aligned}$$

where $\tilde{\mathbb{P}}_{i,y}^+$ is the marginal law of $\tilde{\mathbb{P}}_{i,y,j,z}^+$ on $\sigma(X_n, n \geq 1)$. Using Lemma 5.3.13 and the Lebesgue dominated convergence theorem,

$$\begin{aligned} I_0 &\leq \tilde{\mathbb{E}}_{i,y}^+ \left(e^{-S_l} \right) + \sum_{k=0}^p \tilde{\mathbb{E}}_{i,y,j,z}^+ \left(e^{-S_k} \left| \eta_{k+1,l}(s_l(j, u)) - \eta_{k+1,\infty}(j, u) \right| \right) + 2\eta \sum_{k=p+1}^{+\infty} \tilde{\mathbb{E}}_{i,y}^+ \left(e^{-S_k} \right) \\ &\leq \frac{c(1 + \max(y, 0)) e^y}{V(i, y)} \left(\frac{1}{l^{3/2}} + \sum_{k=p+1}^{+\infty} \frac{\eta}{k^{3/2}} \right) \\ &\quad + \sum_{k=0}^p \tilde{\mathbb{E}}_{i,y,j,z}^+ \left(e^{-S_k} \left| \eta_{k+1,l}(s_l(j, u)) - \eta_{k+1,\infty}(j, u) \right| \right). \end{aligned}$$

Since $|\eta_{k+1,l}(s_l(j, u)) - \eta_{k+1,\infty}(j, u)| \leq 2\eta$, by the Lebesgue dominated convergence theorem and (5.7.12)

$$\limsup_{l \rightarrow +\infty} I_0 \leq \frac{c(1 + \max(y, 0)) e^y}{V(i, y)} \sum_{k=p+1}^{+\infty} \frac{\eta}{k^{3/2}}.$$

Letting $p \rightarrow +\infty$, we obtain that $\lim_{l \rightarrow +\infty} I_0 = 0$. Moreover, by (5.7.9) for any $l \geq 1$, $r_\infty^{(l,\infty)}(j, u) \in [0, 1]$. In the same manner as we proved (5.4.7), we have also

$$r_\infty(j, u) \leq 1.$$

Consequently,

$$\lim_{l \rightarrow +\infty} \tilde{\mathbb{E}}_{i,y,j,z}^+ \left(\left| r_\infty^{(l,\infty)}(j, u) - r_\infty(j, u) \right| \right) \leq \lim_{l \rightarrow +\infty} I_0 = 0.$$

□

We now consider the function

$$U(i, y, j) := \frac{2e^{\lambda(y-\rho(j))} v_\lambda(i)}{\sqrt{2\pi\sigma^3} v_\lambda(j)} \int_{z_0(j)^+}^{+\infty} e^{-\lambda z} \tilde{\mathbb{E}}_{i,y,j,z}^+ (r_\infty(j, z-y)) \tilde{V}_\lambda(i, y) \tilde{V}_\lambda^*(j, z) dz \tilde{\nu}_\lambda(j).$$

Using (5.7.10), (5.7.13) and Lemma 5.3.13, for any $(i, y) \in \text{supp}(\tilde{V}_\lambda)$, $(j, z) \in \text{supp}(\tilde{V}_\lambda^*)$ and $u \in \mathbb{R}$,

$$\tilde{\mathbb{E}}_{i,y,j,z}^+ (r_\infty(j, u)^{-1}) \leq e^{-u} \left(\frac{1}{1-f_j(0)} + \eta \tilde{\mathbb{E}}_{i,y,j,z}^+ \left(\sum_{k=1}^{+\infty} e^{-S_k^*} \right) \right) + \eta \tilde{\mathbb{E}}_{i,y,j,z}^+ \left(\sum_{k=1}^{+\infty} e^{-S_k} \right) < +\infty.$$

So $r_\infty(j, u) > 0$ $\tilde{\mathbb{P}}_{i,y,j,z}^+$ -a.s. and therefore, for any $(i, y) \in \text{supp}(\tilde{V}_\lambda)$, $j \in \mathbb{X}$,

$$U(i, y, j) > 0. \quad (5.7.14)$$

Lemma 5.7.7. *Assume that the conditions of Theorem 5.2.4 are satisfied. For any $(i, y) \in \text{supp}(\tilde{V}_\lambda)$ and $j \in \mathbb{X}$, we have*

$$\mathbb{E}_i(q_{n+1}; X_{n+1} = i, \tau_y > n) \underset{n \rightarrow +\infty}{\sim} \frac{U(i, y, j) k(\lambda)^{n+1}}{(n+1)^{3/2}}.$$

Proof. Fix $(i, y) \in \text{supp}(\tilde{V}_\lambda)$ and $j \in \mathbb{X}$. By (5.7.3), for any $n \geq 1$,

$$\begin{aligned} I_0 &:= \frac{(n+1)^{3/2}}{k(\lambda)^{n+1}} \mathbb{E}_i(q_{n+1}; X_{n+1} = i, \tau_y > n) \\ &= \frac{v_\lambda(i) e^{-\lambda\rho(j)}}{v_\lambda(j)} (n+1)^{3/2} \tilde{\mathbb{E}}_i(e^{-\lambda S_n} q_{n+1}; X_{n+1} = j, \tau_y > n). \end{aligned}$$

Using Lemmas 5.7.3 and 5.7.4,

$$\begin{aligned} \lim_{n \rightarrow +\infty} I_0 &= \lim_{(l,m) \rightarrow +\infty} \lim_{n \rightarrow +\infty} \frac{v_\lambda(i) e^{-\lambda\rho(j)}}{v_\lambda(j)} (n+1)^{3/2} \tilde{\mathbb{E}}_i(r_n^{(l,m)}(j) e^{-\lambda S_n}; X_{n+1} = j, \tau_y > n) \\ &= \lim_{(l,m) \rightarrow +\infty} \frac{2v_\lambda(i)}{\sqrt{2\pi\sigma^3} v_\lambda(j)} e^{\lambda(y-\rho(j))} \int_{z_0(j)^+}^{+\infty} e^{-\lambda z} \tilde{\mathbb{E}}_{i,y,j,z}^+ (r_\infty^{(l,m)}(j, z-y)) \\ &\quad \times \tilde{V}_\lambda(i, y) \tilde{V}_\lambda^*(j, z) dz \tilde{\nu}_\lambda(j). \end{aligned}$$

Since for any $l \geq 1$, $m \geq 1$ and $u \in \mathbb{R}$, $r_\infty^{(l,m)}(j, u) \leq 1$, by the Lebesgue dominated convergence theorem and Lemmas 5.7.5 and 5.7.6,

$$\begin{aligned} \lim_{n \rightarrow +\infty} I_0 &= \frac{2v_\lambda(i)}{\sqrt{2\pi\sigma^3} v_\lambda(j)} e^{\lambda(y-\rho(j))} \int_{z_0(j)^+}^{+\infty} e^{-\lambda z} \lim_{l \rightarrow +\infty} \tilde{\mathbb{E}}_{i,y,j,z}^+ (r_\infty^{(l,\infty)}(j, z-y)) \\ &\quad \times \tilde{V}_\lambda(i, y) \tilde{V}_\lambda^*(j, z) dz \tilde{\nu}_\lambda(j) \\ &= \frac{2v_\lambda(i)}{\sqrt{2\pi\sigma^3} v_\lambda(j)} e^{\lambda(y-\rho(j))} \int_{z_0(j)^+}^{+\infty} e^{-\lambda z} \tilde{\mathbb{E}}_{i,y,j,z}^+ (r_\infty(j, z-y)) \tilde{V}_\lambda(i, y) \tilde{V}_\lambda^*(j, z) dz \tilde{\nu}_\lambda(j) \\ &= U(i, y, j). \end{aligned}$$

□

Proof of Theorem 5.2.4. We use arguments similar to those of the proof of Lemma 5.6.9. Fix $(i, j) \in \mathbb{X}^2$. For any $y \in \mathbb{R}$ and $n \geq 1$, let

$$I_0 := \frac{(n+1)^{3/2}}{k(\lambda)^{n+1}} \mathbb{E}_i(q_{n+1}; X_{n+1} = j)$$

and

$$\begin{aligned} I_1 &:= I_0 - \frac{(n+1)^{3/2}}{k(\lambda)^{n+1}} \mathbb{E}_i(q_{n+1}; X_{n+1} = j, \tau_y > n) \\ &= \frac{(n+1)^{3/2}}{k(\lambda)^{n+1}} \mathbb{E}_i(q_n(f_j(0)); X_{n+1} = j, \tau_y \leq n). \end{aligned} \quad (5.7.15)$$

Since $f_j(0) \geq 0$, it is easy to see that $q_n(f_j(0)) \leq q_n(0)$. Using the fact that $(q_k(0))_{k \geq 1}$ is non-increasing and Lemma 5.3.2, it holds $q_n(f_j(0)) \leq \min_{1 \leq k \leq n} q_k(0) \leq e^{\min_{1 \leq k \leq n} S_k}$. Therefore, as in (5.4.11),

$$\begin{aligned} I_1 &\leq \frac{(n+1)^{3/2}}{k(\lambda)^{n+1}} \mathbb{E}_i\left(e^{\min_{1 \leq k \leq n} S_k}; X_{n+1} = j, \tau_y \leq n\right) \\ &\leq \frac{(n+1)^{3/2}}{k(\lambda)^{n+1}} e^{-y} \sum_{p=0}^{+\infty} e^{-p} \mathbb{P}_i(X_{n+1} = j, \tau_{y+p+1} > n). \end{aligned}$$

By (5.7.2),

$$\begin{aligned} I_1 &\leq \frac{(n+1)^{3/2}}{n^{3/2}} \frac{v_\lambda(i)}{v_\lambda(j)} e^{-y-\lambda\rho(j)} \sum_{p=0}^{+\infty} e^{-p} \tilde{\mathbb{E}}_i\left(e^{-\lambda S_n}; \tau_{y+p+1} > n\right) \\ &\leq c \frac{v_\lambda(i)}{v_\lambda(j)} e^{-y-\lambda\rho(j)} \sum_{p=0}^{+\infty} e^{-p} \sum_{l=0}^{+\infty} e^{\lambda(y+p+1)} e^{-\lambda l} \\ &\quad \times n^{3/2} \tilde{\mathbb{P}}_i(y+p+1+S_n \in [l, l+1]; \tau_{y+p+1} > n). \end{aligned}$$

Using the point 2 of Proposition 5.3.9,

$$\begin{aligned} I_1 &\leq c \frac{v_\lambda(i)}{v_\lambda(j)} e^{-\lambda\rho(j)} e^{-(1-\lambda)y} \sum_{p=0}^{+\infty} e^{-(1-\lambda)p} \sum_{l=0}^{+\infty} e^{-\lambda l} (1 + \max(y+p+1, 0))(1+l) \\ &\leq c \frac{v_\lambda(i)}{v_\lambda(j)} e^{-\lambda\rho(j)} e^{-(1-\lambda)y} (1 + \max(y, 0)). \end{aligned}$$

Moreover, there exists $y_0(i) \in \mathbb{R}$ such that, for any $y \geq y_0(i)$ it holds $(i, y) \in \text{supp}(\tilde{V}_\lambda)$. Using (5.7.15) and Lemma 5.7.7, we obtain that, for any $y \geq y_0(i)$,

$$\begin{aligned} U(i, y, j) &\leq \liminf_{n \rightarrow +\infty} I_0 \leq \limsup_{n \rightarrow +\infty} I_0 \leq U(i, y, j) \\ &\quad + c \frac{v_\lambda(i)}{v_\lambda(j)} e^{-\lambda\rho(j)} e^{-(1-\lambda)y} (1 + \max(y, 0)). \end{aligned} \quad (5.7.16)$$

This proves that $\limsup_{n \rightarrow +\infty} I_0$ is a finite real which does not depend on y and so $y \mapsto U(i, y, j)$ is a bounded function. Moreover, by Lemma 5.7.7,

$$U(i, y, j) = \lim_{n \rightarrow \infty} \frac{(n+1)^{3/2}}{k(\lambda)^{n+1}} \mathbb{E}_i(q_{n+1}; X_{n+1} = j, \tau_y > n)$$

and so $y \mapsto U(i, y, j)$ is non-decreasing. Let u be its limit:

$$u(i, j) := \lim_{y \rightarrow +\infty} U(i, y, j) \in \mathbb{R}.$$

By (5.7.14), for any $y \geq y_0(i)$,

$$u(i, j) \geq U(i, y, j) > 0.$$

Taking the limit as $y \rightarrow +\infty$ in (5.7.16),

$$\lim_{n \rightarrow +\infty} I_0 = u(i, j).$$

Finally, by (5.3.2),

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{(n+1)^{3/2}}{k(\lambda)^{n+1}} \mathbb{P}_i(Z_{n+1} > 0, X_{n+1} = j) &= \lim_{n \rightarrow +\infty} \frac{(n+1)^{3/2}}{k(\lambda)^{n+1}} \mathbb{E}_i(q_{n+1}; X_{n+1} = j) \\ &= u(i, j). \end{aligned}$$

Bibliography

- [1] V. I. Afanasyev. Limit theorems for a moderately subcritical branching process in a random environment. *Discrete Mathematics and Applications*, 8(1):35–52, 2009.
- [2] A. Agresti. Bounds on the extinction time distribution of a branching process. *Advances in Applied Probability*, 6(2):322–335, 1974.
- [3] K. B. Athreya and S. Karlin. Branching processes with random environments II: Limit theorems. *The Annals of Mathematical Statistics*, 42(6):1843–1858, 1971.
- [4] K. B. Athreya and S. Karlin. On branching processes with random environments I: Extinction probabilities. *The Annals of Mathematical Statistics*, 42(5):1499–1520, 1971.
- [5] K. B. Athreya and P. E. Ney. *Branching Processes*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1972. DOI: 10.1007/978-3-642-65371-1.
- [6] J. Bertoin and R. A. Doney. On conditioning a random walk to stay nonnegative. *The Annals of Probability*, 22(4):2152–2167, 1994.
- [7] N. H. Bingham. Limit theorems in fluctuation theory. *Advances in Applied Probability*, 5(03):554–569, 1973.
- [8] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular variation*, volume 27. Cambridge university press, 1989.
- [9] E. Bolthausen. On a functional central limit theorem for random walks conditioned to stay positive. *The Annals of Probability*, 4(3):480–485, 1976.
- [10] A. A. Borovkov. On the asymptotic behavior of distributions of first-passage times, I. *Mathematical Notes*, 75(1-2):23–37, 2004.
- [11] A. A. Borovkov. On the asymptotic behavior of distributions of first-passage times, II. *Mathematical Notes*, 75(3-4):322–330, 2004.
- [12] D. Buraczewski, E. Damek, and Y. Guivarc’h. Convergence to stable laws for a class of multidimensional stochastic recursions. *Probability Theory and Related Fields*, 148(3-4):333–402, 2009.
- [13] F. Caravenna. A local limit theorem for random walks conditioned to stay positive. *Probability Theory and Related Fields*, 133(4):508–530, 2005.
- [14] F. M. Dekking. On the survival probability of a branching process in a finite state i.i.d. environment. *Stochastic Processes and their Applications*, 27:151–157, 1987.
- [15] A. Dembo, J. Ding, and F. Gao. Persistence of iterated partial sums. *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, 49(3):873–884, 2013.
- [16] D. Denisov, D. Korshunov, and V. Wachtel. Harmonic functions and stationary distributions for asymptotically homogeneous transition kernels on \mathbb{Z}^+ . *arXiv:1312.2201 [math]*, 2013. arXiv: 1312.2201.

- [17] D. Denisov, V. Vatutin, and V. Wachtel. Local probabilities for random walks with negative drift conditioned to stay nonnegative. *Electronic Journal of Probability*, 19:1–17, 2014.
- [18] D. Denisov and V. Wachtel. Conditional limit theorems for ordered random walks. *Electronic Journal of Probability*, 15:292–322, 2010.
- [19] D. Denisov and V. Wachtel. Exit times for integrated random walks. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 51(1):167–193, 2015.
- [20] D. Denisov and V. Wachtel. Random walks in cones. *The Annals of Probability*, 43(3):992–1044, 2015.
- [21] R. A. Doney. Conditional limit theorems for asymptotically stable random walks. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 70(3):351–360, 1985.
- [22] R. A. Doney. On the asymptotic behaviour of first passage times for transient random walk. *Probability Theory and Related Fields*, 81(2):239–246, 1989.
- [23] J. C. D'souza and B. M. Hambly. On the survival probability of a branching process in a random environment. *Advances in Applied Probability*, 29(01):38–55, 1997.
- [24] N. Dunford and J. T. Schwartz. *Linear operators*. Interscience publishers, 1971.
- [25] J. Duraj. Random walks in cones: The case of nonzero drift. *Stochastic Processes and their Applications*, 124(4):1503–1518, 2014.
- [26] J. Duraj and V. Wachtel. Invariance principles for random walks in cones. *arXiv:1508.07966 [math]*, 2015. arXiv: 1508.07966.
- [27] P. Eichelsbacher and W. König. Ordered random walks. *Electronic Journal of Probability*, 13:1307–1336, 2008.
- [28] D. J. Emery. Limiting behaviour of the distributions of the maxima of partial sums of certain random walks. *Journal of Applied Probability*, 9(03):572–579, 1972.
- [29] W. Feller. *An Introduction to Probability Theory and Its Applications, Vol. 2, 2nd Edition*. John Wiley & Sons, Inc., New York, 2nd edition edition, 1971.
- [30] Z. Gao, Y. Guivarc'h, and É. Le Page. Stable laws and spectral gap properties for affine random walks. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 51(1):319–348, 2015.
- [31] R. Garbit. A central limit theorem for two-dimensional random walks in a cone. *Bulletin de la SMF*, 139(2):271–286, 2011.
- [32] J. Geiger and G. Kersting. The survival probability of a critical branching process in a random environment. *Theory of Probability & Its Applications*, 45(3):517–525, 2001.
- [33] J. Geiger, G. Kersting, and V. A. Vatutin. Limit theorems for subcritical branching processes in random environment. *Annales de l'institut Henri Poincaré (B) Probabilités et Statistiques*, 39:593–620, 2003.
- [34] B. V. Gnedenko. On a local limit theorem of the theory of probability. *Uspekhi Matematicheskikh Nauk*, 3(3):187–194, 1948.
- [35] M. I. Gordin. Central limit theorem for stationary processes. *Doklady Akademii Nauk SSSR*, 188(4):739–741, 1969.

- [36] I. Grama, R. Lauvergnat, and É. Le Page. Limit theorems for affine Markov walks conditioned to stay positive. *Annales de l'institut Henri Poincaré (B) Probabilités et Statistiques*, in press, 2016. arXiv:1601.02991 [math].
- [37] I. Grama, R. Lauvergnat, and É. Le Page. Conditioned local limit theorems for random walks defined on finite Markov chains. *arXiv:1707.06129 [math]*, 2017.
- [38] I. Grama, R. Lauvergnat, and É. Le Page. Limit theorems for Markov walks conditioned to stay positive under a spectral gap assumption. *Annals of Probability*, in press, 2017. arXiv:1607.07757 [math].
- [39] I. Grama and É. Le Page. Bounds in the local limit theorem for a random walk conditioned to stay positive. In *Modern Problems of Stochastic Analysis and Statistics*, pages 1–21, 2017.
- [40] I. Grama, É. Le Page, and M. Peigné. On the rate of convergence in the weak invariance principle for dependent random variables with application to Markov chains. *Colloquium Mathematicum*, 134(1):1–55, 2014.
- [41] I. Grama, É. Le Page, and M. Peigné. Conditioned limit theorems for products of random matrices. *Probability Theory and Related Fields*, pages 1–39, 2016.
- [42] Y. Guivarc'h and J. Hardy. Théorèmes limites pour une classe de chaînes de Markov et applications aux difféomorphismes d'Anosov. *Annales de l'IHP Probabilités et statistiques*, 24:73–98, 1988.
- [43] Y. Guivarc'h and É. Le Page. On spectral properties of a family of transfer operators and convergence to stable laws for affine random walks. *Ergodic Theory and Dynamical Systems*, 28(02):423–446, 2008.
- [44] Y. Guivarc'h and Q. Liu. Propriétés asymptotiques des processus de branchement en environnement aléatoire. *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, 332(4):339–344, 2001.
- [45] G. H. Hardy. *Divergent series*. Clarendon Press, 1949.
- [46] T. E. Harris. *The theory of branching processes*. Springer-Verlag Berlin Heidelberg, 1963.
- [47] D. L. Iglehart. Functional central limit theorems for random walks conditioned to stay positive. *The Annals of Probability*, 2(4):608–619, 1974.
- [48] D. L. Iglehart. Random walks with negative drift conditioned to stay positive. *Journal of Applied Probability*, 11(4):742–751, 1974.
- [49] C. T. Ionescu Tulcea and G. Marinescu. Théorie ergodique pour des classes d'opérations non complètement continues. *Annals of Mathematics*, 52(1):140–147, 1950.
- [50] T. Kato. *Perturbation Theory for Linear Operators*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1976.
- [51] J. Komlós, P. Major, and G. Tusnády. An approximation of partial sums of independent RV's, and the sample DF. I. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 32(1-2):111–131, 1975.
- [52] J. Komlós, P. Major, and G. Tusnády. An approximation of partial sums of independent RV's, and the sample DF. II. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 34(1):33–58, 1976.

- [53] M. Kozlov. On the asymptotic behavior of the probability of non-extinction for critical branching processes in a random environment. *Theory of Probability & Its Applications*, 21(4):791–804, 1977.
- [54] É. Le Page. Théorèmes limites pour les produits de matrices aléatoires. In *Probability Measures on Groups*, pages 258–303. Springer, Berlin, Heidelberg, 1982. DOI: 10.1007/BFb0093229.
- [55] É. Le Page and M. Peigné. A local limit theorem on the semi-direct product of \mathbb{R}^{*+} and \mathbb{R}^d . *Annales de l'Institut Henri Poincaré (B) Probability and Statistics*, 33:223–252, 1997.
- [56] É. Le Page and Y. Ye. The survival probability of a critical branching process in a Markovian random environment. *Comptes Rendus Mathématique*, 348(5-6):301–304, 2010.
- [57] P. Lévy. *Processus stochastiques et mouvement brownien*. Gauthier-Villars, Paris, 1965.
- [58] Q. Liu. On the survival probability of a branching process in a random environment. *Annales de l'I.H.P. Probabilités et statistiques*, 32(1):1–10, 1996.
- [59] M. F. Norman. *Markov processes and learning models*, volume 84. 1972.
- [60] E. Presman. Boundary problems for sums of lattice random variables, defined on a finite regular markov chain. *Theory of Probability and Its Applications*, 12(2):323–328, 1967.
- [61] E. Presman. Methods of factorization and a boundary problems for sums of random variables defined on a markov chain. *Izvestija Akademii Nauk SSSR*, 33:861–990, 1969.
- [62] W. L. Smith and W. E. Wilkinson. On Branching Processes in Random Environments. *The Annals of Mathematical Statistics*, 40(3):814–827, 1969.
- [63] W. L. Smith and W. E. Wilkinson. Branching processes in Markovian environments. *Duke Mathematical Journal*, 38(4):749–763, 1970.
- [64] F. Spitzer. A combinatorial lemma and its application to probability theory. *Transactions of the American Mathematical Society*, 82(2):323–339, 1956.
- [65] F. Spitzer. A Tauberian theorem and its probability interpretation. *Transactions of the American Mathematical Society*, 94(1):150–169, 1960.
- [66] F. Spitzer. *Principles of random walk*. The University Series in Higher Mathematics. D. Van Nostrand Co., Inc., Princeton, NJ-Toronto-London, 1964.
- [67] C. Stone. A Local Limit Theorem for Nonlattice Multi-Dimensional Distribution Functions. *The Annals of Mathematical Statistics*, 36(2):546–551, 1965.
- [68] N. Th. Varopoulos. Potential theory in conical domains. *Mathematical Proceedings of the Cambridge Philosophical Society*, 125(2):335–384, 1999.
- [69] N. Th. Varopoulos. Potential theory in conical domains. II. *Mathematical Proceedings of the Cambridge Philosophical Society*, 129(2):301–320, 2000.
- [70] N. Th. Varopoulos. Potential Theory in Lipschitz Domains. *Canadian Journal of Mathematics*, 53(5):1057–1120, 2001.
- [71] V. A. Vatutin and V. Wachtel. Local probabilities for random walks conditioned to stay positive. *Probability Theory and Related Fields*, 143(1-2):177–217, 2008.