The Links-Gould invariants as generalizations of the Alexander polynomial

Ben-Michael Kohli

Université de Bourgogne, U.F.R Sciences et Techniques
Institut de Mathématiques de Bourgogne
Ecole doctorale Carnot-Pasteur

THÈSE

pour l'obtention du grade de

Docteur de l'Université de Bourgogne

en Mathématiques

présentée et soutenue publiquement par

Ben-Michael Kohli

le 23 novembre 2016

Les invariants de Links-Gould comme généralisations du polynôme d'Alexander

Directeurs de thèse : **Peter Schauenburg, Emmanuel Wagner**

Jury composé de

Stéphane Baseilhac	Université de Montpellier	Rapporteur
Anna Beliakova	Universität Zürich	Examinateur
David Cimasoni	Université de Genève	Examinateur
Nathan Geer	Utah State University	Rapporteur
Peter Schauenburg	Université de Bougogne	Directeur
Emmanuel Wagner	Université de Bougogne	Directeur

Blondie - *You see, in this world there's two kinds of people, my friend ... those with loaded guns, and those who dig.*
You dig.
Tuco - *Where?*
Blondie - *Here.*

Remerciements

Je tiens avant tout à remercier Emmanuel Wagner pour m'avoir accompagné tout au long de ces trois années, et pour m'avoir orienté et guidé. Merci pour sa constance, son enthousiasme et ses conseils avisés. J'ai énormément appris et réfléchi au contact de ses idées toujours justes et pertinentes. Il m'a offert l'opportunité de travailler de manière autonome, tout en me donnant toujours l'impulsion nécessaire. Je remercie également Peter Schauenburg qui a accepté d'être mon directeur officiel. Je lui en suis reconnaissant.

Merci à Stéphane Baseilhac et Nathan Geer d'avoir accepté de rapporter ce manuscrit, pour l'intérêt qu'ils ont montré pour mon travail et le temps qu'ils y ont consacré. Merci aussi pour les remarques qu'ils m'ont faites dans leurs rapports qui ont permis de rendre le manuscrit meilleur. Merci à Anna Beliakova et David Cimasoni qui m'ont fait l'honneur de participer à mon jury de thèse.

Merci à l'ANR VasKho et à ses membres Benjamin Audoux, Paolo Bellingeri, Jean-Baptiste Meilhan et Emmanuel Wagner pour les nombreuses sessions de travail et les exposés auxquels j'ai eu la chance de pouvoir participer. Ils m'ont apporté tant d'ouverture d'esprit et d'inspiration pour mon travail. Les conférences *Winterbraids* et de *la Llagonne* ont également été des moments d'apprentissage et d'échange très précieux. Je souhaite sincèrement, pour tous les doctorants en topologie de basse dimension qui viendront après moi, qu'elles continuent encore longtemps d'exister. J'ai aussi eu la chance de travailler avec Peggy Cénac, Olivier Couture et Emmanuel Wagner lors des séances que Peggy a baptisées *MOB* - sur un mode très machiste ! Ce fut très intéressant pour moi d'adopter ce point de vue probabiliste sur les invariants quantiques, et cela a nourri d'idées mon travail de thèse. J'espère que nous allons bientôt pouvoir mener ce projet à bout. J'ai effectué deux séjours de recherche à Vannes où j'ai eu l'honneur de travailler avec Bertrand Patureau-Mirand. Notre travail a beaucoup bénéficié de ses idées et de sa rigueur, et j'ai pu profiter de son expérience et de ses conseils pour comprendre et approfondir de nombreux points. Enfin j'ai eu la chance d'effectuer un court mais instructif séjour à Genève où j'ai pu parler de mon travail avec David Cimasoni et Anthony Conway qui ont judicieusement suggéré que je m'intéresse aux propriétés de l'invariant *LG* sur les nœuds fibrés.

Le séminaire étudiants continue de bien fonctionner. C'est en grande partie grâce aux doctorants motivés qui se sont succédés ces derniers temps à son organisation : Sébastien Alvarez, Charlie Petitjean et Jérémy Rouot. Je souhaite de tout cœur que cela continue ainsi. C'est aussi grâce au soutien apporté par l'école doctorale Carnot-Pasteur et par son directeur, Hans Jauslin, aux différentes initiatives prises par les doctorants en mathématiques. Un grand merci également à Ibtissam Bourdadi et à Nadia Bader qui m'ont toujours aidé à accueillir au mieux les invités extérieurs. Enfin, tout ceci ne pourrait avoir lieu sans la bonne entente qui existe entre les doctorants au sein du laboratoire. Pour cela et tout le reste, je remercie les nouveaux comme les plus anciens.

J'ai au cours de ma thèse eu l'opportunité de discuter et d'échanger avec de nombreuses personnes au contact desquelles j'ai beaucoup réfléchi et appris. Merci à Benjamin Audoux, Paul Bakouche, Fathi Ben Aribi, Peggy Cénac, David Cimasoni, Bruno Cisneros, Anthony Conway, Olivier Couture, Adriana Da Luz, Céleste Damiani, Aziz Essalami, Simone Faldella, Thomas Fiedler, Catherine Gille, Hans Jauslin, Ahmed Jebrane, Catherine Labruère, Francis Léger, Jean-Mathieu Magot, Pavao Mardesic, Gaël Meigniez, Jean-Baptiste Meilhan, Michael Mignard, Filip Misev, Hugh Morton, Xavier Morvan, Lucy Moser-Jauslin, Vincent Nolot, Luis Paris, Jean-Philippe Rolin, Bruno Santiago, Martin Vogel et Jinhua Zhang. Merci en particulier à Catherine Labruère de m'avoir conseillé dans mon enseignement et de m'avoir confié un groupe de "ses" L1.

Merci à Mokhtar Beloual, Jean-Yves Bathelier, Stéphane Aulagnier et Michel Quercia de m'avoir donné envie de faire des mathématiques.

Un grand merci à Patrick Gabriel pour m'avoir mis le pied à l'étrier et pour ses nombreux et bienveillants conseils.

Merci à Papa et Maman pour toutes sortes de choses. Merci en particulier d'avoir préparé le pot de thèse ! Je ne m'en serais pas sorti seul . . .

Nora, reviens-nous vite. Pourvu que tes lunettes carrées se cassent entre temps.

Merci à Mathieu de bien vouloir collaborer avec moi. Peut-être, avec un peu de patience, la mouche finira-t-elle par trouver l'éclat dans le carreau.

Merci aux amis : Alexandre, Anthony, Antoine, AP, Augustin, Carole, Charlie, Emmanuelle, Guillaume, Hugues, Jean, Mathieu, Olivier, Pauline, Yann, et tous ceux que j'oublie car mon cerveau est mal organisé. Merci pour les bons moments, les souvenirs, et les méfaits accomplis !

Merci enfin à Sophie pour sa gentillesse et sa bonne humeur quotidienne. Merci de bien vouloir construire quelque chose avec moi. Merci aussi de me supporter, dans tous les sens du terme.

Un salut amical à celles et ceux qui, au cours de ces trois années, m'ont dispensé des leçons de morale. Désormais, je file droit.

Résumé

Les nœuds, tresses et autres objets noués sont des sources d'intérêt et de curiosité pour l'homme depuis de nombreuses années. Que ce soit à titre décoratif et architectural ou pour leur utilité dans l'art de la navigation, les nœuds sont et ont été présents dans de nombreuses sociétés humaines et nous entourent aujourd'hui encore, sans toujours que l'on s'en aperçoive. Sans que l'on puisse donner un sens précis à ses affirmations, le psychanalyste Jacques-Marie Lacan voit ainsi dans certains entrelacs des motifs d'interprétation de la subjectivité humaine [34].

Les premières traces de recherche en théorie des nœuds remontent au 18$^{\text{ème}}$ siècle avec les travaux du mathématicien français Alexandre-Théophile Vandermonde. Dans son papier *Remarques sur les problèmes de situation* [60] Vandermonde s'intéresse au problème du cavalier. La théorie des nœuds connut un regain d'intérêt grâce aux théories de Lord Kelvin qui postula que les atomes étaient des nœuds plongés dans l'éther. Celles-ci conduirent aux premières tables de nœuds réalisées par Peter Guthrie Tait. Il publia ses tables de nœuds à moins de 10 croisements ainsi que des conjectures désormais connues sous le nom de *conjectures de Tait* à la fin du 19$^{\text{ème}}$ siècle [57].

Le premier invariant de nœuds polynomial fut découvert par James Waddell Alexander [2] au cours des recherches qu'il a menées sur les groupes de nœuds et les propriétés homologiques des compléments de nœuds. C'est aussi à cette période de l'histoire mathématique que des progrès conséquents furent réalisés en matière de représentation des nœuds et entrelacs. Ce fut par exemple le cas du théorème de Reidemeister [50, 3] qui permet de rendre la représentation diagrammatique des entrelacs praticable. Des résultats sur le codage des entrelacs par les tresses sont aussi apparus :

Théorème (Alexander [1])**.** *Tout entrelacs est la clôture d'une certaine tresse.*

La contre partie du théorème de Reidemeister dans le cas des représentations en tresses

fut ensuite explicitée par Markov [38].

Alors que dans le même temps les techniques de géométrie hyperboliques furent introduites à la fin des années soixante-dix par William Thurston dans le champ de la théorie des nœuds, pendant un peu moins de soixante ans, le polynôme d'Alexander Δ demeura de manière *a posteriori* surprenante le seul invariant polynomial de nœuds connu des topologues. Cela jusqu'à l'introduction en 1984 par Vaughan Jones d'un nouvel invariant d'isotopie d'entrelacs défini de manière *combinatoire* grâce à son invariance par mouvements de Reidemeister : le polynôme de Jones [25]. Dès lors cet invariant a été largement étudié et il a été généralisé en ce qu'il est convenu d'appeler la théorie des *invariants quantiques*. Peu de temps après sa découverte, l'invariant de Jones fut utilisé avec succès afin de résoudre d'anciens problèmes ouverts, tels que les conjectures de Tait.

Malgré tout, la compréhension du polynôme en tant qu'objet demeure plus qu'incomplète précisément du fait de la construction combinatoire dont il dérive, qui ne permet pas de conception géométrique immédiate de son origine. A titre d'exemple, on ne sait toujours pas répondre à la question suivante malgré son énoncé élémentaire, et malgré la réponse clairement négative dans le cas du polynôme d'Alexander :

le polynôme de Jones distingue-t-il le nœud trivial ?

Plusieurs autes invariants furent construits sur le modèle du polynôme de Jones : les polynômes de Jones coloriés, l'invariant de Kontsevich qui peut être vu comme un invariant quantique universel, ... Même le polynôme d'Alexander peut être vu comme un invariant quantique [61]. Les même questions se posent pour tous ces invariants quantiques quant à la possibilité d'une interprétation classique homologique qui permettrait d'en extraire de l'information géométrique : information sur le genre du nœud, sur son caractère fibré, son caractère topologiquement bordant, etc. Ceci est d'un certain point de vue paradoxal puisque le théorème de Gordon-Luecke [21] a pour conséquence que tout invariant de nœud (premier) peut être extrait du groupe du nœud.

Le polynôme de HOMFLY-PT [16] a aussi été introduit à la suite de la découverte du polynôme de Jones. C'est une généralisation combinatoire commune aux invariants d'Alexander et de Jones. Le polynôme de HOMFLY-PT continent de l'information sur le genre canonique, mais pas sur le genre du nœud lui même [39].

Dans cette thèse nous concentrerons nos efforts sur une famille d'invariants quantiques d'entrelacs : les invariants de Links-Gould. Le premier invariant de Links-Gould $LG = LG^{2,1}$ a été défini en 1992 par Jon Links et Mark D. Gould [35] avant d'être étudié en détails par De Wit, Kauffman et Links en 1999 [11]. Les travaux d'Atsushi Ishii fournissent une collection simple de relations d'écheveaux pour LG [24] et posent la question de leur complétude. Par ailleurs, Ivan Marin et Emmanuel Wagner exhibent un ensemble de relations d'écheveaux complet mais difficile à manipuler pour cet invariant [37].

L'invariant de Links-Gould peut être généralisé dans au moins deux directions différentes. En 2001, De Wit étendit la construction de LG à une famille infinie d'invariants $LG^{n,m}$, $n, m \in \mathbb{N}^*$ [7]. Ces invariants furent alors généralisés à des invariants d'entrelacs à plusieurs variables par Nathan Geer et Bertrand Patureau-Mirand [20] tout comme on peut généraliser le polynôme d'Alexander à sa version multivariable. Dans ces perspectives, le polynôme d'Alexander devient un membre de la famille des invariants de Links-Gould [61]

$$\Delta = LG^{1,1}.$$

Ainsi, non seulement le polynôme d'Alexander devient un invariant quantique, mais les invariants $LG^{n,m}$ lui deviennent associés et construits suivant le même modèle que Δ. De fait, David De Wit, Atsushi Ishii et Jon Links conjecturent [10] que pour tout entrelacs L

$$LG^{n,m}(L; \tau, e^{\mathbf{i}\pi/m}) = \Delta_L(\tau^{2m})^n,$$

où Δ_L est le polynôme d'Alexander-Conway de L. Ils prouvent cette conjecture dans les cas $(n, m) = (1, m)$ complètement et $(n, m) = (2, 1)$ pour une famille restreinte de tresses. Une partie de ce travail sera consacrée à la preuve du cas $(n, 1)$ de la conjecture pour n quelconque et est issue de l'article *On the Links-Gould invariant and the square of the Alexander polynomial* [31] et du travail réalisé en collaboration avec Bertrand Patureau-Mirand *Other quantum relatives of the Alexander polynomial through the Links-Gould invariant* [33].

Théorème ([31, 33])**.** *Pour tout entrelacs L dans S^3,*

$$LG^{n,1}(L; \tau, -1) = \Delta_L(\tau^2)^n.$$

Les polynômes LG étant des généralisations du polynôme d'Alexander, ils contiennent davantage d'information sur l'entrelacs. En particulier, l'information homologique fournie par Δ est également présente dans LG. Le problème de démontrer le cas $(n, 1)$ de la conjecture de De Wit-Ishii-Links est plus complexe que celui de prouver le cas $(1, n)$ car on ne connait pas de collection complète de relations d'écheveaux pour Δ^n, $n \geqslant 2$.

On développera dans ce manuscrit deux stratégies différentes pour prouver le cas $(n, 1)$ de la conjecture. Pour n petit, on s'intéressera aux représentations des groupes de tresses dérivées de la R-matrice de la superalgèbre de Hopf $U_q\mathfrak{gl}(n|1)$ que l'on interprétera en termes de la représentation de Burau comme on l'a fait dans [31]. Si on note b_R^p la représentation de B_p induite par la R-matrice spécialisée à $q = -1$ de $LG^{2,1}$ ou $LG^{3,1}$ et Ψ_p l'extérieur de la somme directe de deux ou trois copies convenablement choisies de la représentation de Burau, ces représentations sont équivalentes :

Théorème ([31]). *Pour tout p, on peut construire I_p un automorphisme de $\mathbb{C}[B_p]$-modules. C'est à dire que I_p vérifie, pour toute tresse $b \in B_p$:*

$$\Psi_p(b) \circ I_p = I_p \circ b_R^p(b).$$

Malheureusement cette méthode requiert l'étude d'une R-matrice *explicite* pour chaque n, ce qui la rend difficilement praticable pour n quelconque. C'est pourquoi nous disséquerons ensuite directement la structure de la superalgèbre de Hopf $U_q\mathfrak{gl}(n|1)$ comme nous l'avons fait dans [33] afin d'obtenir le résultat pour tout n. Une étude précise de la superalgèbre de Hopf $U_q\mathfrak{gl}(n|1)$ conduit à obtenir une décomposition de cette algèbre quand q est évalué en -1 puis à une décomposition de la représentation de plus haut poids spécialisée de $U_q\mathfrak{gl}(n|1)$ en termes de la représentation de plus haut poids de $U_q\mathfrak{gl}(1|1)$. Avec les notations que nous introduirons dans le chapitre 3, cela donne :

Théorème ([33]). *Munie de l'action de $\otimes_i A_i$ induite par $\Theta : \otimes_i A_i \to \mathcal{A}_{-1}^{\sigma}/I$, la représentation de plus haut pois spécialisée $V_{-1}(0^n, \alpha)$ est isomorphe à la représentation irréductible $\otimes_i V^i$ où chaque V^i est un A_i-module isomorphe au U-module 2-dimensionnel $V_{q^{-\alpha}}$.*

Cependant, la première méthode demeure intéressante dans la perspective de comprendre les invariants de Links-Gould d'un point de vue classique. En effet, la connexion établie entre la R-matrice permettant de calculer $LG^{n,1}$ et la représentation de Burau donne bon espoir de pouvoir fournir une telle interprétation.

Ainsi il est on ne peut plus naturel de chercher à savoir si certaines propriétés de l'invariant d'Alexander s'étendent aux polynômes de Links-Gould, à commencer par le plus simple d'entre eux après Δ, $LG^{2,1}$. On conjecture ici que cela se produit dans au moins deux cas de figure.

Conjecture ([32], Conjecture 0.3). *Soit L un entrelacs dans S^3 à μ composantes.*

- **I** $span(LG^{2,1}(L; t_0, t_1)) \leqslant 2(2g(L) + \mu - 1)$,
- **II** *Si L est alterné, l'inégalité* **I** *est une égalité.*

Un résultat similaire et bien connu, énoncé dans la proposition 1.2.5, fait du polynôme d'Alexander un outil utile dans le calcul du genre des entrelacs. La démonstration de ce fait vient alors directement de la construction du polynôme Δ comme déterminant d'une matrice de Seifert, de la taille minimale d'une telle matrice et du degré de chaque coefficient.

Conjecture ([32], Conjecture 0.4). *Soit K un nœud dans S^3.*

- **I** *Si K est fibré alors $LG^{2,1}(K)$ est unitaire,*
- **II** *Si K est alterné, la réciproque est vraie.*

Ces conjectures furent initialement énoncées dans l'article *The Links-Gould invariant as a classical generalization of the Alexander polynomial ?* [32] et y sont vérifiées dans de nombreux cas.

Théorème ([32]). *La conjecture de genre est vérifiée pour tous les nœuds premiers possédant moins de 12 croisements et de nombreux nœuds premiers à 13 croisements, un double de Whitehead du nœud de trèfle, et plusieurs familles infinies d'entrelacs : les entrelacs à deux ponts, les nœuds twist et les nœuds pretzel. La conjecture sur les nœuds fibrés, elle, est vérifiée pour tous les nœuds premiers à moins de 12 croisements.*

Le double de Whitehead du nœud de trèfle pour lequel nous calculons le polynôme de Links-Gould afin de vérifier la conjecture de genre est un contre-exemple à une inégalité similaire dans le cas du polynôme de HOMFLT-PT dû à Hugh Morton [39].

De plus, étant données les évaluations connues pour $LG^{2,1}$,

$$LG^{2,1}(L;t_0,-t_0^{-1}) = \Delta_L(t_0^2)\ ,\ LG^{2,1}(L;t_0,t_0^{-1}) = \Delta_L(t_0)^2,$$

si ces deux conjectures s'avèrent exactes, ces résultats constitueraient une amélioration *systématique* des énoncés similaires pour le polynôme d'Alexander.

Rappelons également une conjecture due à Ishii qui constituerait un autre parallèle évident de comportement entre Δ et LG. Il énonce cette conjecture dans son papier *The Links-Gould polynomial as a generalization of the Alexander-Conway polynomial* [23] dans lequel il notait déjà la similitude de certaines propriétés élémentaires des polynômes de Links-Gould et d'Alexander. Ces propriétés sont les suivantes :

Théorème ([23]). *Le polynôme de Links-Gould vérifie :*

— $LG(\bigcirc) = 1$,

— *Si L^* est la réflexion de L, $LG(L^*;t_0,t_1) = LG(L;t_0^{-1},t_1^{-1})$,*

— *On a la symétrie suivante : $LG(L;t_0,t_1) = LG(L;t_1,t_0)$. En effet, LG ne détecte pas l'inversion,*

— *Si L et L' sont deux entrelacs, on désigne leur somme connexe par $L\#L'$. Alors : $LG(L\#L') = LG(L)LG(L')$,*

— *Si $L = L' \sqcup L''$ est l'union disjointe de L' et L'', alors $LG(L) = 0$,*

— $LG(L;t_0,1) = LG(L;1,t_1) = 1$ *si L est un nœud, 0 sinon.*

La conjecture affirme quant à elle :

Conjecture ([23])**.** *Le polynôme de Links-Gould* $LG(K; t_0, t_1) = \sum_{i,j} a_{ij} t_0^i t_1^j$ *d'un nœud alterné* K *est "alterné", cc'est à dire :* $a_{ij}a_{kl} \geqslant 0$ *si* $i + j + k + l$ *est pair, et* $a_{ij}a_{kl} \leqslant 0$ *sinon.*

Ce comportement vérifié expérimentalement est probablement la trace d'une construction classique sous-jacente pour les invariants de Links-Gould. Ainsi ces derniers semblent-ils être - le polynôme d'Alexander mis à part - le premier exemple d'invariants quantiques que l'on pourrait comprendre géométriquement. Qui plus est, il semble raisonnable de chercher à comprendre les invariants quantiques dans une perspective géométrique en commençant par l'étude des LG qui sont des cousins de Δ. La seule tactique que l'on puisse imaginer à ce point pour prouver les conjectures ci-dessus ou tout autre résultat significatif liant LG et la géométrie du nœud est d'exhiber une telle construction, ce à quoi nous nous attellerons à l'avenir.

Un motif d'espoir est que même dans le cas le plus étudié du polynôme de Jones, alors qu'aucune interprétation géométrique n'existe, une célèbre conjecture établit un lien entre les polynômes de Jones coloriés et la géométrie hyperbolique des complémentaires de nœuds. Si l'on définit l'invariant de Kashaev d'un nœud K par

$$\langle K \rangle_N = \lim_{q \to e^{2\pi i/N}} \frac{J_{K,N}(q)}{J_{O,N}(q)},$$

où $J_{K,N}(q)$ est le $N^{\text{ième}}$ polynôme de Jones colorié, alors la *conjecture du volume* prédit que :

Conjecture (Kashaev, [27])**.**

$$\lim_{N \to \infty} \frac{2\pi \log |\langle K \rangle_N|}{N} = \mathrm{vol}(K),$$

où $\mathrm{vol}(K)$ *désigne le volume hyperbolique du complémentaire du nœud* K *dans* S^3*.*

Le manuscrit de thèse qui suit est organisé en quatre parties. La première section présente les objets et introduit le contexte dans lequel cette recherche s'est effectuée, ainsi que les résultats fondamentaux qui sont à l'origine de ce travail. Après avoir détaillé plusieurs représentations possibles des entrelacs, on définit les polynômes d'Alexander et de Jones, avant d'exprimer chacun de ces invariants comme une trace quantique. On détaille ensuite la théorie de Reshetikhin-Turaev [52] qui permet de comprendre la genèse de ces expressions. On définit pour finir les invariants de Links-Gould qui sont l'objet de cette étude et on énonce la conjecture de De Wit-Ishii-Links dont la résolution était la cause finale de ce travail à son commencement. Les trois parties suivantes présentent les résultats obtenus. La deuxième et la troisième partie exposent les deux méthodes de résolution du cas $(n, 1)$ de la conjecture de De Wit-Ishii-Links que nous avons expliquées brièvement ci-dessus. Enfin, nous énonçons et étayons dans la dernière partie les conjectures citées plus haut.

Le reste de ce texte sera rédigé en anglais.

Abstract

Knots, braids and other knotted objects have sparked curiosity and interest among human societies since long ago. They are used still today for decoration in art and architectue, as well as in the science of navigation. On a perhaps more amusing note, French psychoanalyst Jacques-Marie Lacan sees in certain knots and links a symbolic interpretation of human subjectivity [34].

The scientific interest for knot theory starts late in the 18^{th} century with French mathematician Alexandre-Théophile Vandermonde and his paper *Remarques sur les problèmes de situation* [60] where he studies the knight's tour problem. The focus on the study of knots due to Lord Kelvin's theory that atoms were knots in the aether led to Peter Guthrie Tait's first knot tabulations. Tait published a table of knots with up to 10 crossings at the end of the 19^{th} century along with several conjectures that came to be known as *Tait conjectures* [57].

The first polynomial knot invariant was discovered in the effort conducted by James Waddell Alexander [2] to understand knot groups and homological properties of knot complements. At that same period in mathematical history, significant progress was made in the understanding of different ways in which one can represent knots and links. For example, Reidemeister's theorem [50, 3] allows us to use diagram representations of links effectively. Similar results were discovered for braid representations of links as well. First Alexander's theorem states:

Theorem (Alexander [1])**.** *Any link is the closure of a braid.*

Markov [38] then proved a result similar to the Reidemeister theorem in the case of braid representations.

While in that period of time William Thurston introduced hyperbolic geometry in the spectrum of techniques used to study knots at the end of the seventies, for practically sixty

years, the Alexander polynomial Δ was the only polynomial invariant known by topologists. But in 1984 Vaughan Jones introduced a new link isotopy invariant that is defined *combinatorially* using its invariance under Reidemeister moves: the Jones polynomial [25]. That invariant has since been studied in depth and much generalized to build the theory of *quantum invariants*. The Jones invariant has also been used successfully to solve old open problems such as the Tait conjectures soon after it was discovered.

However, the understanding of the polynomial itself remains quite incomplete precisely because the combinatorial construction it springs from does not offer a geometrical perspective for the object. For example, such an elementary question as the following, that is known to be untrue for the Alexander invariant, is still an open problem:

does the Jones polynomial detect the unknot?

Several other invariants were built on the model of the Jones polynomial: the colored Jones polynomials, the Kontsevich invariant that is in a sense a universal quantum invariant, ... Even the Alexander polynomial can be reinterpreted as a quantum invariant [61]. For all these quantum invariants the same questions arise regarding a possible classical homological intrepretation that would allow us to extract geometrical information from them: genus information, information on whether a knot is fibered or not, slice or not, etc. This is somewhat paradoxical since a consequence of the Gordon-Luecke theorem [21] is that the value of any knot invariant for a prime knot can be read in the group of the knot.

The HOMFLY-PT polynomial was also introduced [16] in the wake of the discovery of the Jones polynomial as a combinatorial generalization common to the Alexander and Jones invariants. It gives canonical genus information, but says nothing more about the minimal genus of a knot [39].

In this work we will focus on a family of quantum invariants: the Links-Gould invariants. The first Links-Gould invariant $LG = LG^{2,1}$ was introduced in 1992 by Jon Links and Mark D. Gould [35] and studied in detail by De Wit, Kauffman and Links in 1999 [11]. In his work, Atsushi Ishii gives a simple set of skein relations for LG [24] and wonders if that set is complete. On the other hand, Ivan Marin and Emmanuel Wagner give a complete but complicated set of skein relations for that invariant [37].

The Links-Gould invariant can be generalized in at least two distinct directions. In 2001, De Wit extended the construction of LG to an infinite family of invariants $LG^{n,m}$, $n, m \in \mathbb{N}^*$ [7]. These invariants were then generalized to multivariable link invariants by Nathan Geer and Bertrand Patureau-Mirand [20] in the same way that a multivariable version of the Alexander polynomial can be defined. These perspectives are quite interesting since the

Alexander polynomial becomes part of this family [61]

$$\Delta = LG^{1,1}.$$

Not only is the Alexander polynomial a quantum invariant, but the $LG^{n,m}$ invariants also turn out to be invariants associated to Δ and built on a same model. As a matter of fact, David De Wit, Atsushi Ishii and Jon Links conjectured [10] that for any link L

$$LG^{n,m}(L;\tau, e^{\mathbf{i}\pi/m}) = \Delta_L(\tau^{2m})^n,$$

where Δ_L is the Alexander-Conway polynomial of L. They proved the conjecture when $(n,m) = (1,m)$ and when $(n,m) = (2,1)$ for a particular class of braids. Part of our work here will be to prove the $(n,1)$ case of this conjecture for any n using two different methods, one of which was first detailed in the paper *On the Links-Gould invariant and the square of the Alexander polynomial* [31]. The second method was developed in the paper *Other quantum relatives of the Alexander polynomial through the Links-Gould invariant* [33] written with Bertrand Patureau-Mirand.

Theorem ([31, 33])**.** *For any link L in S^3,*

$$LG^{n,1}(L;\tau,-1) = \Delta_L(\tau^2)^n.$$

Since Links-Gould invariants are extensions of the Alexander polynomial, they contain more information on the link. In particular the classical information that can be recovered from Δ is enclosed in LG. Note that it is a harder problem to prove the $(n,1)$ case of the De Wit-Ishii-Links conjectures than to prove the $(1,n)$ case because there is no known complete set of skein relations for Δ^n, $n \geqslant 2$.

In this manuscript, we develop two distinct strategies to prove the $(n,1)$ case of the conjecture. For small values of n, we focus on the operator invariant derived from Hopf superalgebra $U_q\mathfrak{gl}(n|1)$ and give an interpretation of that operator in terms of the classical Burau representation of braid groups like we did in [31]. Denoting by b_R^p the representation of group B_p induced by the R-matrix for $LG^{2,1}$ or $LG^{3,1}$ specialized at $q=-1$, and calling Ψ_p the exterior representation of the direct sum of two or three well chosen copies of the Burau representation, these two representations are equivalent:

Theorem ([31])**.** *For any p, there exists a $\mathbb{C}[B_p]$-module automorphism I_p between the two representations we have just explicited. Which means that map I_p satisfies the following commutation relation for any braid $b \in B_p$:*

$$\Psi_p(b) \circ I_p = I_p \circ b_R^p(b).$$

The problem is that this method requires the study of an *explicit* R-matrix for each n, making it hard to implement as n grows. That is why we then focus on the structure of Hopf superalgebra $U_q\mathfrak{gl}(n|1)$ directly to prove the result for any n as we did in [33]. A precise study of Hopf superabgebra $U_q\mathfrak{gl}(n|1)$ allows us to obtain a specific decomposition of that algebra at $q=-1$. This splitting induces a decomposition of the specialized highest weight representation of superalgebra $U_q\mathfrak{gl}(n|1)$ in terms of the highest weight representation of $U_q\mathfrak{gl}(1|1)$. Using the notations we will introduce in Chapter 3, we can write:

Theorem ([33])**.** *Equipped with the action of $\otimes_i A_i$ induced by $\Theta : \otimes_i A_i \to \mathcal{A}^{\sigma}_{-1}/I$, the specialized highest weight representation $V_{-1}(0^n, \alpha)$ is isomorphic to the irreducible representation $\otimes_i V^i$ where each V^i is an A_i-module isomorphic to the 2-dimensional U-module $V_{q^{-\alpha}}$.*

However, the first method remains interesting in the prospect of giving a classical interpretation of Links-Gould invariants. Indeed, connecting the R-matrix representation we derive $LG^{n,1}$ from with the Burau representation offers hope and ideas in the pursuit of such an interpretation.

In fact it is natural to wonder at this point if properties of the Alexander invariant remain true for Links-Gould polynomials, and as a start for the first Links-Gould invariant after Δ, $LG^{2,1}$. We conjecture this happens in at least two cases.

Conjecture ([32], Conjecture 0.3)**.** *Set L a link in S^3 and μ the number of its components.*

- **I** $span(LG^{2,1}(L;t_0,t_1)) \leqslant 2(2g(L)+\mu-1)$,
- **II** *If L is alternating, then inequality* **I** *is an equality.*

A similar and well known result we recall in Proposition 1.2.5 shows that the Alexander polynomial is a very useful tool to compute the genus of knots and links. The proof of this fact is a direct consequence of the definition of polynomial Δ as the determinant of a Seifert matrix. By choosing such a matrix with minimal dimensions and considering the degree of each of its coefficients, we obtain the inequality.

Conjecture ([32], Conjecture 0.4)**.** *Set K a knot in S^3.*

- **I** *If K is fibered then $LG^{2,1}(K)$ is monic,*
- **II** *If K is alternating, the converse is true as well.*

These conjectures were first stated in the paper *The Links-Gould invariant as a classical generalization of the Alexander polynomial?* [32] where they are supported by evidence in several cases.

Theorem ([32]). *The genus conjecture is true for all prime knots with less than 12 crossings and a large selection of 13 crossing prime knots, a Whitehead double of the trefoil knot, and for different infinite families of links: 2-bridge links, twist knots and pretzel knots. On the other hand, the fiberedness conjecture has been successfully tested for prime knots with less than 12 crossings.*

The Whitehead double of the trefoil on which we tested the conjecture by computing its Links-Gould invariant is a counterexample to a similar inequality in the case of the HOMFLY-PT skein polynomial. This counterexample is due to Hugh Morton [39].

In addition, given the evaluations we have for $LG^{2,1}$,

$$LG^{2,1}(L; t_0, -t_0^{-1}) = \Delta_L(t_0^2)\ , LG^{2,1}(L; t_0, t_0^{-1}) = \Delta_L(t_0)^2,$$

if these two conjectures were to be true, the results would *systematically* improve the similar statements for the Alexander polynomial.

In the same vein, another conjecture due this time to Ishii would be one more trace of a similar behavior between LG and Δ. It can be found in his paper *The Links-Gould polynomial as a generalization of the Alexander-Conway polynomial* [23] where Ishii proves that the Links-Gould invariant $LG^{2,1}$ has all kinds of elementary Alexander-type features. In particular,

Theorem ([23]). *The Links-Gould polynomial satisfies the following properties:*

— $LG(\bigcirc) = 1$,
— *Denoting* L^* *the reflexion of* L, $LG(L^*; t_0, t_1) = LG(L; t_0^{-1}, t_1^{-1})$,
— *We have the following symmetry :* $LG(L; t_0, t_1) = LG(L; t_1, t_0)$*. Indeed* LG *does not detect inversion,*
— *For* L *and* L' *two links, denoting* $L\#L'$ *their connected sum :* $LG(L\#L') = LG(L)LG(L')$,
— *If* $L = L' \sqcup L''$ *is the split union of* L' *and* L''*, then* $LG(L) = 0$,
— $LG(L; t_0, 1) = LG(L; 1, t_1) = 1$ *if* L *is a knot,* 0 *otherwise.*

Moreover the Ishii conjecture states:

Conjecture ([23]). *The LG polynomial* $LG(K; t_0, t_1) = \sum_{i,j} a_{ij} t_0^i t_1^j$ *of an alternating knot* K *is "alternating", that is :* $a_{ij}a_{kl} \geqslant 0$ *if* $i + j + k + l$ *is even, and* $a_{ij}a_{kl} \leqslant 0$ *otherwise.*

This behavior we have tested experimentally is likely to be the trace of a classical construction for the LG invariants. So the LG invariants seem to be, apart from the Alexander invariant, the first example of quantum invariants we might understand from a geometrical point of view. Furthermore, it is reasonable to attempt to understand quantum invariants from a geometrical perspective by starting to study the LG invariants that are cousins of Δ. The only strategy we can imagine at this point to prove the two conjectures we stated and any other significative result linking LG with the geometry of the knot is to present such a construction. We intend to focus our future efforts on this goal.

A clear signal of hope is that even in the case of the Jones polynomial that has been the most thoroughly studied, although no topological meaning has been given up to now, a celebrated conjecture establishes a connection between the colored Jones polynomials and the hyperbolic geometry of knot complements. One can define Kashaev's invariant for a knot K as follows:

$$\langle K\rangle_N = \lim_{q\to e^{2\pi i/N}} \frac{J_{K,N}(q)}{J_{O,N}(q)},$$

where $J_{K,N}(q)$ stands for the N^{th} colored Jones polynomial. Then the *volume conjecture* predicts that

Conjecture (Kashaev, [27])**.**

$$\lim_{N\to\infty} \frac{2\pi \log|\langle K\rangle_N|}{N} = \mathrm{vol}(K),$$

where $\mathrm{vol}(K)$ *is the hyperbolic volume of the complement of knot* K *in* S^3*.*

The thesis manuscript that follows is divided into four distinct sections. The first section introduces the objects and explains the context in which we did this research, as well as the fundamental results that inspired and are at the origin of this work. We start by studiying in detail different representations that exist for knots and links. Then we introduce the Alexander and Jones polynomials, and we express each of these invariants as a quantum trace. We then explain what the Reshetikhin-Turaev theory is [52] and how it allows us to understand where these traces come from. Finally we give the definition of the objects we study here: the Links-Gould invariants of links. We state the De Wit-Ishii-Links conjecture that was the question that motivated this work at the start. The three last sections explain the results we achieved to prove. The second and third parts present the two methods we briefly exposed in this abstract to prove the $(n, 1)$ case of the De Wit-Ishii-Links conjecture. To conclude this work, we state the two conjectures we have formulated earlier and give evidence to support them.

Contents

Chapter 1

Introduction

1.1 Knots, links and invariants

1.1.1 Definitions

Knot theory is a subject for which the goal is quite easy to explain and somewhat natural, even for a non mathematician. If you consider two shoelaces, each of them can be entangled. Then one can glue the two ends of each shoelace together and ask:

Question 1.1.1. *Can you deform the first shoelace into the second one without ungluing or cutting it ?*

This is the question, quite simple in its terms, that people who study knot theory try to answer. The goal is to develop systematic and efficient methods to do so.

Definition 1.1.2. A link L of n components is a subset of $\mathbb{R}^3 \subset S^3$ consisting of n disjoint piecewise linear simple closed surves. When $n = 1$, we say L is a knot.

Remark 1.1.3. So we consider *oriented* links, even though we will often omit this precision in the following and in some figures.

Definition 1.1.4. Two links L_1 and L_2 are equivalent if there exists an orientation preserving piecewise linear homeomorphism $h : S^3 \longrightarrow S^3$ such that $h(L_1) = L_2$.

Soon, we will not say L_1 and L_2 are equivalent anymore, but simply that they are equal. The equivalence relation descried in Definition 1.1.4 is the mathematical translation of the deformation mentioned in Question 1.1.1.

One can grasp that the question of knowing if two links are equivalent is hard by taking a look at Figure 1.2. To measure this in another way, let us recall this surprising and non-trivial result:

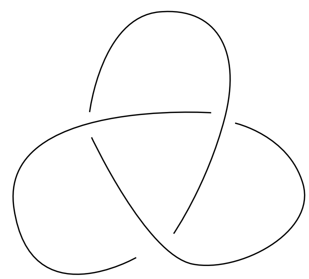


Figure 1.1 – The trefoil knot.

Theorem 1.1.5 (Gordon-Luecke theorem [21])**.** *Two* knots *are equivalent if and only if they have complements in S^3 that are orientation-preserving homeomorphic.*

So knot theory has direct and fundamental connections with the study of 3-manifolds. This alone can motivate the study of such objects. The main idea to be able to distinguish knots is one that is often used in algebraic topology. We try to compute equivalence invariants: quantities that are invariant on an equivalence class of links.

Definition 1.1.6 (link invariant)**.** A link invariant I is a map from the set of equivalence classes of links to a certain set E. Invariant I is said to be *complete* if I is an injection.

This definition is very general. Of course, the goal is to find interesting links invariants. An efficient invariant is one that is easy to compute and that at the same time distinguishes many links. In the following we will give several combinatorial descriptions of links that will allow us to build link invariants using different procedures.

1.1.2 Link diagrams

A link is a 1-dimensional object embedded in the 3-sphere. However, links can be represented in the 2-dimensional euclidean plane. These representations are called link diagrams.

Definition 1.1.7 (link diagram)**.** A link diagram is a 4-valent planar graph with over/under decorated vertices. Such a diagram represents a regular projection of the link on a properly chosen plane. Moreover, such a projection exists for any link type.

Figures 1.1 and 1.2 are examples of different diagrams of simple knots. This representation of links naturally gives rise to a first example of link invariant. For a link L, the *minimal number of crossings* of L is

$$n(L) := \min\{n(D), D \text{ diagram for } L\}$$

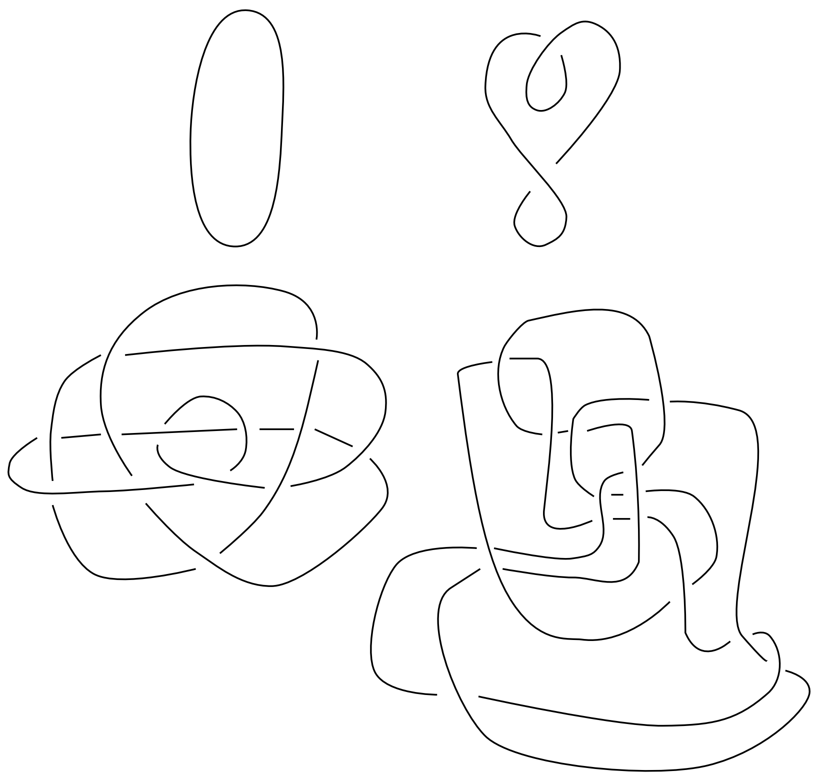


Figure 1.2 – Different "unknots".

where $n(D)$ is the number of crossings of a diagram D. Map $n : \{\text{links}\} \longrightarrow \mathbb{N}$ clearly is a link invariant. However, the value of n on a link is in general (very) hard to determine. That is why the minimal number of crossings is not an efficient link invariant. Nevertheless, some elegant results exist. For example:

Theorem 1.1.8 (First Tait conjecture [57, 28, 43, 44])**.** *Any reduced diagram of an alternating link has the fewest possible crossings.*

Considering two diagrams D_1 and D_2, one can wonder if they represent the same link. The Reidemeister theorem answers that question.

Theorem 1.1.9 (Reidemeister theorem [3, 50, 48])**.** *Set two links L and L'. If D is a diagram for L and D' is a diagram for L', then L and L' are equivalent if and only if D and D' are related by a finite sequence of isotopies of $\mathbb{R}^2$ and RIa, RIb, RII, RIII local moves shown in Figure 1.3.*

This result offers a new way to build a link invariant. A link invariant is a map $I : \{\text{link diagrams}\} \longrightarrow E$ such that the value of I is invariant under isotopies of $\mathbb{R}^2$ and Reidemeister moves.

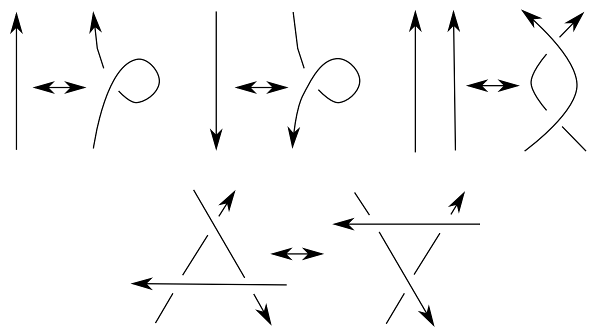


Figure 1.3 – Reidemeister moves RIa, RIb, RII and RIII.

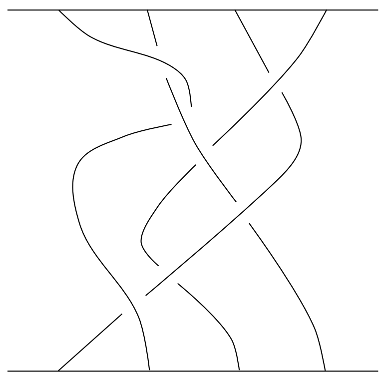


Figure 1.4 – Braid $\sigma_1\sigma_3^{-1}\sigma_1\sigma_2\sigma_3^{-1}\sigma_2^{-1}\sigma_1$.

1.1.3 Braid representations of links

Definition 1.1.10. A braid in n strands is a disjoint union of n strands embedded in $\mathbb{R}^2 \times [0, 1]$ such that no strand has a critical point with respect to the vertical coordinate. Two braids are equivalent if they are related by an isotopy of $\mathbb{R}^2 \times [0, 1]$ that preserves the vertical coordinate and its boundary. We consider braids up to equivalence.

An example of a braid in 4 strands can be found in Figure 1.4. We denote by B_n the set of braids in n strands up to isotopy. If $A, B \in B_n$ are two braids in n strands, we define the product of A and B denoted $A.B$ to be the braid obtained by putting A on top of B. That product is a group operation. The identity element is the braid where each strand goes straight down. The inverse of a given braid is its reflection. Group B_n is generated by the standard Artin generators σ_i for $1 \leqslant i \leqslant n-1$ and their inverses σ_i^{-1}, see Figure 1.5.

Braid group B_n equipped with that product and that set of generators can be *presented* as follows:

$$B_n =< \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_k\sigma_l = \sigma_l\sigma_k, |k-l| \geqslant 2, 1 \leqslant i \leqslant n-2 >$$

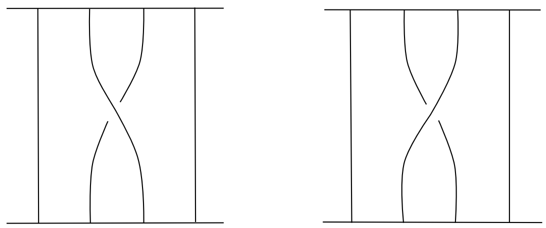


Figure 1.5 – Artin generator $\sigma_2 \in B_4$ and its inverse σ_2^{-1}.

Remark 1.1.11. The first relation in that presentation is called the *braid relation*.

Remark 1.1.12. Setting $\sigma_i^2 = 1$, we find that a quotient of B_n is the symmetric group S_n on a set of n symbols. This quotient corresponds to ignoring the over/under information at crossings.

This makes the topological objects braids happen to be quite pleasant to study since they are equivalently defined from that purely algebraic point of view. In particular, if V is a finite dimensional vector space over a field $\mathbb{K}$, we obtain a natural way of building a sequence $(\Psi_n)_{n\geq 2}$ of representations

$$\Psi_n : B_n \longrightarrow GL(V^{\otimes n}).$$

Indeed, setting $R \in GL(V \otimes V)$ one can define

$$\Psi_n(\sigma_i) := id_V^{\otimes i-1} \otimes R \otimes id_V^{\otimes n-i-1}.$$

Given the relations in B_n, representation Ψ_n is well defined if and only if

$$(R \otimes id_V) \circ (id_V \otimes R) \circ (R \otimes id_V) = (id_V \otimes R) \circ (R \otimes id_V) \circ (id_V \otimes R).$$

This equality is known as the *Yang-Baxter equation*. A solution R of that equation is called an *R-matrix*. Finding explicit R-matrices, especially as the dimension of V grows, is a hard problem and a motivation to the systematic study of quantum groups. We will explicit this in further detail in the following paragraphs.

Braids are interesting to us because they are after diagrams another efficient way to represent links. Indeed, one can build a link from any braid by closing it. See Figure 1.6. The surjectivity and kernel of that correspondence can be described explicitely. First let us recall this classical result:

Theorem 1.1.13 (Alexander's theorem [1])**.** *Any link is the closure of a certain braid.*

Note that although Alexander's paper does not give one, algorithms exist that transform any link diagram into a diagram that is the closure of a braid, see [62].

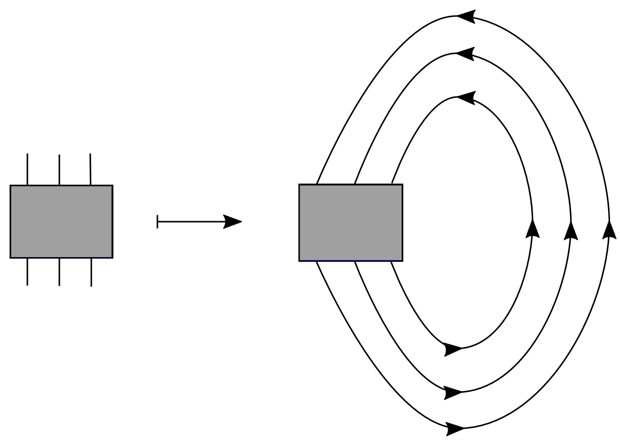


Figure 1.6 – The closure operation.

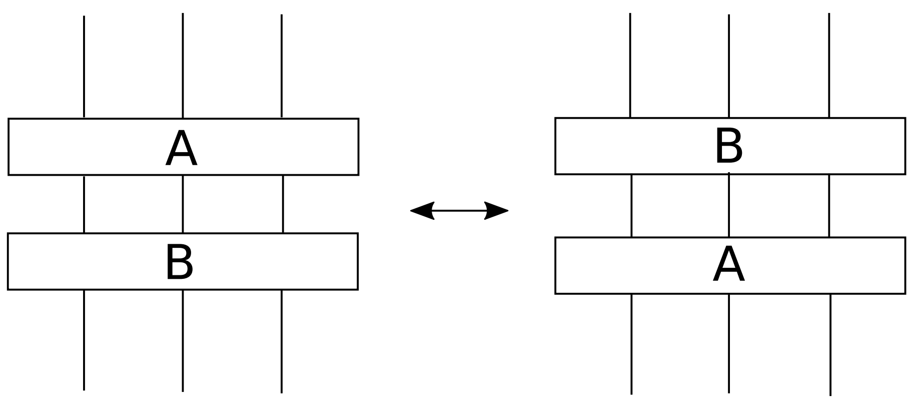


Figure 1.7 – Move MI.

However, the correspondence between braids and links is clearly not one-to-one. For example, the two braids $A.B$ and $B.A$ always have the same closure. The question of understanding how two braids with the same closure are related is addressed by Markov's theorem [38].

Theorem 1.1.14 (Markov's theorem)**.** *Two braids have the same closure if and only if they are related by a finite sequence of Markov moves MI, MII where:*

- **MI:** *for* $A, B \in B_n$, $A.B \longleftrightarrow B.A$;
- **MII:** *for* $A \in B_n$, $A \longleftrightarrow A.\sigma_n \in B_{n+1} \longleftrightarrow A.\sigma_n^{-1} \in B_{n+1}$.

Corollary 1.1.15. *A map* $I : \bigsqcup_{n\geqslant 1} B_n \longrightarrow E$ *that is invariant under MI and MII is a link invariant.*

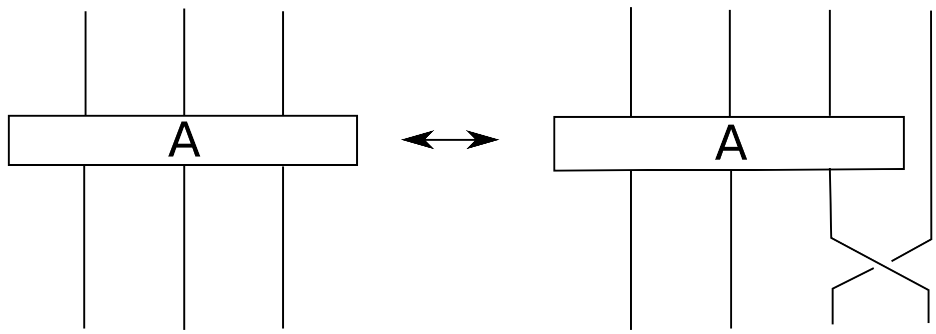


Figure 1.8 – Move MII.

1.2 The Alexander polynomial

The Alexander invariant is a polynomial link invariant that was introduced by James Waddell Alexander in 1923 [2]. It was the first polynomial invariant to be discovered and it remained the only one until the disclosure of the Jones polynomial in 1984. There are many different constructions of the Alexander polynomial, several of which we will explain and explore in this manuscript: derived from the Burau representation or Fox calculus, and even described as a quantum invariant, etc. For the moment we will define the Alexander invariant using a Seifert surface. We will then list some interesting properties of the invariant that are consequences of this construction.

1.2.1 Classical definition

Definition 1.2.1. (Seifert surface for a link) Set L a link in S^3. A Seifert surface for L is a compact, connected, orientable surface $\Sigma \subset S^3$ such that $\partial\Sigma = L$.

Such a surface exists for any link according to Seifert's algorithm [53]. An example of Seifert surface in the case of the trefoil knot is displayed in Figure 1.9. Note that because a Seifert surface is orientable, choosing an orientation, one can push the surface forward in the direction of a normal vector to the surface.

Remark 1.2.2. Any Seifert surface Σ being connected and orientable, one can define the genus $g(\Sigma)$ of Σ:

$$\chi(\Sigma) = 2 - 2g(\Sigma) - \mu$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ and μ is the number of components of link L.

Definition 1.2.3. (genus of a link) Let L be a link in S^3. The genus $g(L)$ of L is

$$g(L) := \min\{g(\Sigma), \Sigma \text{ Seifert surface for } L\}.$$

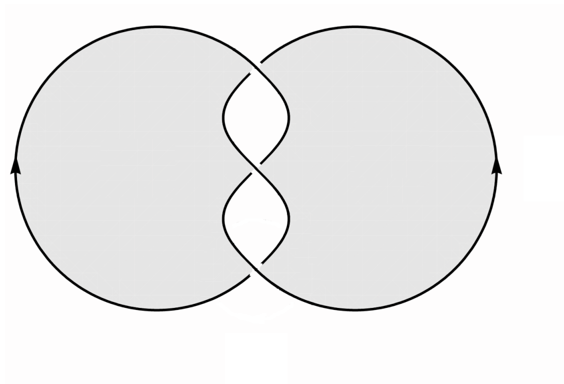


Figure 1.9 – A Seifert surface for the trefoil knot.

Like in the case of the minimal number of crossings, the genus is a link invariant that is quite easy to define but very hard to compute in practice. We will see the Alexander polynomial helps to do so since its degree is a lower bound for the genus. One aim of this work is to give evidence to support the assumption that we can systematically improve this lower bound.

Definition 1.2.4. (Alexander polynomial of a link) Set L a link in S^3 with μ components and choose Σ a Seifert surface for L. Then $H_1(\Sigma, \mathbb{Z})$ is a free abelian group of rank $1 - \chi(\Sigma) = 2g(\Sigma) + \mu - 1$. If v_{ij} is the linking number in S^3 of the i^{th} generator of $H_1(\Sigma, \mathbb{Z})$ with the pushoff of the j^{th} generator, then $V = (v_{ij})$ is called a Seifert matrix for L. The Alexander polynomial is computed from such a Seifert matrix setting

$$\Delta_L(t) = det(tV - {}^tV) \in \mathbb{Z}[t, t^{-1}].$$

With this definition, Δ_L is determined up to multiplication by $\pm t^n$, $n \in \mathbb{Z}$, that is up to an invertible element of $\mathbb{Z}[t, t^{-1}]$. The standard Alexander normalization consists in picking the representative with positive constant term. The Alexander-Conway normalization corresponds, at least in the case of a knot K, to choosing the symmetric Laurent polynomial with $\Delta_K(1) = 1$. In general, the Conway normalization is determined by its defining skein relations we recall in Theorem 1.4.12.

1.2.2 Consequences of this definition

Here we focus on some classical properties of the Alexander invariant. A very remarkable one is the following, that is a direct consequence of the previous definition of Δ:

Proposition 1.2.5. *For any link L, $deg(\Delta_L(t)) \leqslant 2g(L) + \mu - 1$.*

We now recall the definition of an alternating link.

Definition 1.2.6. A link is alternating when it has a link diagram with alternating underpasses and overpasses.

For alternating links, the inequality in Proposition 1.2.5 becomes an equality, see [6]. Moreover, the Alexander polynomial of an alternating link is "alternating", see [6, 41].

Proposition 1.2.7. *The Alexander polynomial $\Delta_L(t) = \sum_i a_i t^i$ of an alternating link L is alternating, that is: $(-1)^{i+j} a_i a_j \geqslant 0$ for any i, j.*

Another class of *knots* for which the Alexander invariant has special properties is the set of fibered knots.

Definition 1.2.8. A knot K in S^3 is said to be fibered if the two following conditions hold :

1. the complement of the knot is the total space of a locally trivial bundle over the base space S^1, i.e. there exists a map $p : S^3 \setminus K \longrightarrow S^1$ which is a locally trivial bundle.
2. there exists $V(K)$ a neighborhood of K and there exists a trivializing homeomorphism $\theta : V(K) \longrightarrow S^1 \times D^2$ such that $\pi \circ \theta(X) = p(X)$ for any $X \in V(K) \setminus K$, where $\pi(x, y) := \frac{y}{|y|}$.

The Alexander polynomial of a fibered knot is monic [45, 49, 55]. This means the coefficient of the highest degree term of the standard Alexander normalization of the polynomial is 1. For the Conway normalization, it means the leading coefficient is ± 1. The converse is not true in general. However, the condition is sufficient for prime knots with up to 10 crossings and alternating knots [42]. Also note that for fibered knots, the degree of the Alexander polynomial is exactly twice the genus of the knot, that is the genus of the corresponding fibre surface [49].

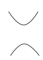
The last chapter of this manuscript is dedicated to pointing out evidence for possible improvements of all these results using Links-Gould invariants of links.

1.3 The Jones polynomial

Quite like the Alexander invariant, the Jones polynomial associates to every link a one variable polynomial. It was discovered by V. Jones in 1984 [25]. The computation is made by using a link diagram. So the invariance of the quantity we calculate under Reidemeister moves is the key point in the theory. The discovery of the Jones polynomial was the starting point of the blossoming new interset for knot theory that has never stopped since. That polynomial has been much generalized into what we now call *quantum link invariants*. Early after its disclosure, it was also used to prove conjectures that mathematicians had failed to solve without that powerful tool, for example Theorem 1.1.8. We will introduce the Jones polynomial using the Kauffman bracket [28].

1.3.1 The Kauffman bracket

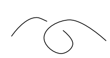
Definition 1.3.1. The Kauffman bracket is a function from *unoriented* link diagrams in the plane to Laurent polynomials with integer coefficients in one variable A. It maps a diagram D to its bracket $< D > \in \mathbb{Z}[A^{\pm 1}]$ and is defined in the following recursive way:

1. $< \bigcirc > = 1$,
2. $< \bigcirc \sqcup D > = (-A^2 - A^{-2}) < D >$,
3. $<$  $> = A <$  $> + A^{-1} <$  $>$.

The bracket polynomial of a diagram with m crossings can therefore be computed using **3** as a sum of 2^m diagrams with no crossings. Since the polynomial of a diagram with c components and no crossings is equal to $(-A^2 - A^{-2})^{c-1}$ by **1** and **2**, we obtain an expression for any link diagram in the plane. Note that the expression does not depend on the order in which crossings were resolved using **3**.

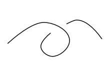
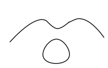
We now study the effect of a non oriented Reidemeister move on the value of the Kauffman bracket.

Proposition 1.3.2. **1.** *The effect of a type RI Reidemeister move on the bracket polynomial is the following:*

$<$  $> = -A^3 <$  $>, <$  $> = -A^{-3} <$  $>$.

2. *The Kauffman bracket is left unchanged by Reidemeister moves RII and RIII.*

Proof. For example, we study Reidemeister move RI.

$<$  $> = A <$  $> + A^{-1} <$  $>$

$= (A(-A^2 - A^{-2}) + A^{-1}) <$  $>$

$= -A^3 <$  $>$.

CQFD

We can modify the bracket slightly to obtain an invariant of *oriented* links.

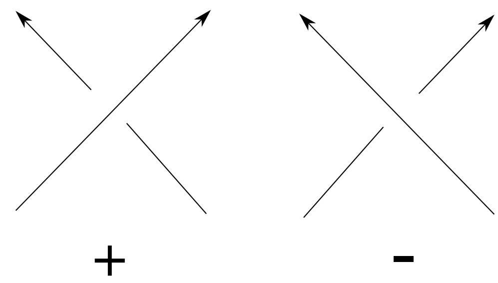


Figure 1.10 – A positive and a negative crossing.

1.3.2 Modifying the bracket to obtain a link invariant

Definition 1.3.3. The writhe $w(D)$ of a diagram of an *oriented* link is the sum of the signs of the crossings of D. See Figure 1.10.

Example 1.3.4. The writhe of the diagram shown in Figure 1.9 is -3.

It is easy to verify that $w(D)$ is invariant if D is changed under a type II or type III Reidemeister move, which is not the case for type I moves. This observation leads to the following result.

Theorem 1.3.5. *Let L be a link in S^3 and D be a diagram of L. The following quantity is an invariant of the oriented link L:*

$$(-A)^{-3w(D)} < D > .$$

Definition 1.3.6 (Jones polynomial)**.** The Jones polynomial of an oriented link L is the element of $\mathbb{Z}[t^{\pm 1/2}]$ defined by

$$V(L) = \left((-A)^{-3w(D)} < D > \right)_{t^{1/2}=A^{-2}}$$

where D is an oriented diagram for L.

Remark 1.3.7. For a link with an odd number of components, and in particular for a knot, $V(L) \in \mathbb{Z}[t^{\pm 1}]$.

Proposition 1.3.8. *The Jones invariant maps oriented links in S^3 to Laurent polynomials in the variable $t^{1/2}$. It satisfies the following skein relations:*

1. $V(\text{unknot}) = 1$,
2. *If L_+, L_- and L_0 are three links that are equal except in the neighborhood of a point where they are as drawn in Figure 1.11, then*

$$t^{-1}V(L_+) - tV(L_-) = (t^{1/2} - t^{-1/2})V(L_0).$$

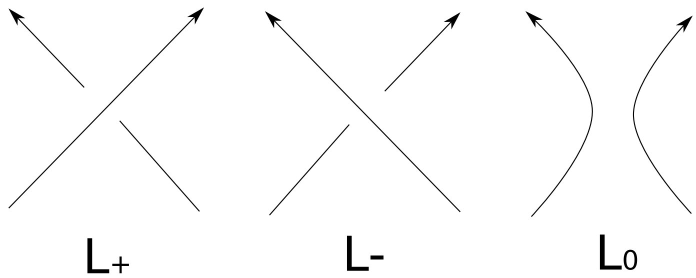


Figure 1.11 – Local changes in the skein relation.

Proof.

$$< \times > = A <)(> + A^{-1} < \asymp >,$$
$$< \times > = A^{-1} <)(> + A < \asymp > .$$

By multiplying the first equation by A and the second one by A^{-1}, and substracting these two equalities, we obtain

$$A < \times > - A^{-1} < \times > = (A^2 - A^{-2}) <)(> .$$

But $w(L_+) = w(L_0) + 1 = w(L_-) + 2$. So finally

$$-A^4 V(L_+) + A^{-4} V(L_-) = (A^2 - A^{-2}) V(L_0).$$

CQFD

1.4 The Jones and Alexander invariants expressed via modified traces

The two link invariants we have just studied can both be expressed in terms of R-matrix representations of braid groups we introduced in Subsection 1.1.3. This is what we will discuss in this section.

1.4.1 Trace and partial trace

Proposition 1.4.1 (Canonical identifications)**.** *Let V and W be two finite dimensional vector spaces over a field $\mathbb{K}$. Let $Hom(V, W)$ be the set of $\mathbb{K}$-linear maps $V \longrightarrow W$. Set also $End(V) := Hom(V, V)$ and $V^* := Hom(V, \mathbb{K})$. Then we have the following canonical identifications:*

$$Hom(V, W) = V^* \otimes W, End(V \otimes W) = End(V) \otimes End(W),$$

$$(V^*)^* = V, (V \otimes W)^* = V^* \otimes W^*.$$

Definition 1.4.2 (Trace)**.** The *trace* of an endomorphism is defined using the first identification of Proposition 1.4.1:

$$trace = \begin{cases} End(V) = V^* \otimes V & \longrightarrow & \mathbb{K} \\ f \otimes x & \longmapsto & f(x) \end{cases}.$$

Note that if an endomorphism f has a matrix A with respect to a fixed basis, then the trace of f is the sum of the diagonal entries of A.

Definition 1.4.3 (Partial trace)**.** The *partial trace* linear maps are defined for endomorphisms of a tensor product vector space $V \otimes W$ using the second identification of Proposition 1.4.1:

$$trace_1 = \begin{cases} End(V \otimes W) = End(V) \otimes End(W) & \longrightarrow & End(W) \\ f \otimes g & \longmapsto & trace(f)g \end{cases},$$

$$trace_2 = \begin{cases} End(V \otimes W) = End(V) \otimes End(W) & \longrightarrow & End(V) \\ f \otimes g & \longmapsto & trace(g)f \end{cases}.$$

This is not a complete definition in the sense that partial traces can be defined for endomorphisms of tensor products with more than two factors. When it is the case, the number of factors that are killed may vary as well. For example, for $f \otimes g \otimes h \in End(V \otimes W \otimes X) = End(V) \otimes End(W) \otimes End(X)$:

$$trace_2(f \otimes g \otimes h) = trace(g) f \otimes h \in End(V \otimes X),$$

$$trace_{2,3}(f \otimes g \otimes h) = trace(g) trace(h) f \in End(V).$$

Remark 1.4.4. Given these definitions, it is easy to see that for $f \otimes g \in End(V \otimes W) = End(V) \otimes End(W)$,

$$trace(trace_1(f \otimes g)) = trace(trace_2(f \otimes g)) = trace(f \otimes g).$$

Then, using linear combinations, this remains true if $f \otimes g$ is replaced by any $\varphi \in End(V \otimes W)$. So

$$trace \circ trace_1 = trace \circ trace_2 = trace.$$

This last observation will be useful to reduce expressions.

1.4.2 The Jones polynomial as a modified trace

Lemma 1.4.5. *Set V a finite dimensional vector space and fix $h \in GL(V)$ and $R \in GL(V \otimes V)$. If h and R satisfy*

1. $R \circ (h \otimes h) = (h \otimes h) \circ R$,
2. $trace_2\big((id_V \otimes h) \circ R^{\pm 1}\big) = id_V$,
3. $(R \otimes id_V) \circ (id_V \otimes R) \circ (R \otimes id_V) = (id_V \otimes R) \circ (R \otimes id_V) \circ (id_V \otimes R)$,

then R is a solution of the Yang-Baxter equation and as explained in Subsection 1.1.3 we can build a sequence of representations $\Psi_n : B_n \longrightarrow GL(V^{\otimes n})$ using R. Moreover, setting L a link in S^3 and $b \in B_n$ a braid with closure L,

$$trace\big(h^{\otimes n} \circ \Psi_n(b)\big)$$

only depends on L. Therefore, $L \mapsto trace\big(h^{\otimes n} \circ \Psi_n(b)\big)$ is a link invariant.

Proof. We will check that this quantity is invariant under Markov moves MI and MII. For MI we use **1**. Formula

$$R \circ (h \otimes h) = (h \otimes h) \circ R$$

implies that, for any $b \in B_n$, $\Psi_n(b)$ and $h^{\otimes n}$ commute. Therefore, considering $b, c \in B_n$:

$$\begin{aligned}
trace\big(h^{\otimes n} \circ \Psi_n(bc)\big) &= trace\big(h^{\otimes n} \circ \Psi_n(b) \circ \Psi_n(c)\big) \\
&= trace\big(\Psi_n(b) \circ h^{\otimes n} \circ \Psi_n(c)\big) \\
&= trace\Big(\Psi_n(b) \circ \big(h^{\otimes n} \circ \Psi_n(c)\big)\Big) \\
&= trace\Big(\big(h^{\otimes n} \circ \Psi_n(c)\big) \circ \Psi_n(b)\Big) \\
&= trace\big(h^{\otimes n} \circ \Psi_n(cb)\big).
\end{aligned}$$

The invariance under the MII move is a consequence of point **2**. For $b \in B_n$:

$$\begin{aligned}
trace\big(h^{\otimes n+1} \circ \Psi_{n+1}(\sigma_n^{\pm 1} b)\big) &= trace\big(h^{\otimes n+1} \circ \Psi_{n+1}(\sigma_n^{\pm 1}) \circ \Psi_{n+1}(b)\big) \\
&= trace\big(h^{\otimes n+1} \circ (id_V^{\otimes n-1} \otimes R^{\pm 1}) \circ (\Psi_n(b) \otimes id_V)\big) \\
&= trace\big((h^{\otimes n} \otimes id_V) \circ (id_V^{\otimes n} \otimes h) \circ (id_V^{\otimes n-1} \otimes R^{\pm 1}) \circ (\Psi_n(b) \otimes id_V)\big) \\
&= trace\Big(\big(h^{\otimes n} \otimes id_V\big) \circ \big(id_V^{\otimes n-1} \otimes ((id_V \otimes h) \circ R^{\pm 1})\big) \circ \big(\Psi_n(b) \otimes id_V\big)\Big).
\end{aligned}$$

But an elementary computation shows that

$$trace_2\big((f \otimes id_V) \circ g \circ (h \otimes id_V)\big) = f \circ trace_2(g) \circ h.$$

Applying this result:

$$\begin{aligned}
trace\big(h^{\otimes n+1} \circ \Psi_{n+1}(\sigma_n^{\pm 1} b)\big) &= trace\Big(h^{\otimes n} \circ trace_n\big(id_V^{\otimes n-1} \otimes ((id_V \otimes h) \circ R^{\pm 1})\big) \circ \Psi_n(b)\Big) \\
&= trace\Big(h^{\otimes n} \circ \big(id_V^{\otimes n-1} \otimes trace_2((id_V \otimes h) \circ R^{\pm 1})\big) \circ \Psi_n(b)\Big) \\
&= trace\big(h^{\otimes n} \circ (id_V^{\otimes n-1} \otimes id_V) \circ \Psi_n(b)\big) \\
&= trace\big(h^{\otimes n} \circ \Psi_n(b)\big).
\end{aligned}$$

CQFD

Let us consider a 2-dimensional vector space E with a basis (e_0, e_1). We define a linear map $R_0 \in GL(E \otimes E)$. It is represented in basis $(e_0 \otimes e_0, e_0 \otimes e_1, e_1 \otimes e_0, e_1 \otimes e_1)$ by the following matrix:

$$R_0 = \begin{pmatrix} -t^{1/2} & 0 & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & t & -t^{1/2} + t^{3/2} & 0 \\ 0 & 0 & 0 & -t^{1/2} \end{pmatrix}.$$

We also define

$$h_0 = \begin{pmatrix} -t^{-1/2} & 0 \\ 0 & -t^{1/2} \end{pmatrix} \in End(E).$$

Proposition 1.4.6. *Maps R_0 and h_0 satisfy the three relations of Lemma 1.4.5. So if we call the representations derived from the R-matrix (Φ_n),*

$$L \longmapsto trace(h_0^{\otimes n} \circ \Phi_n(b)),\ cl(b) = L$$

is an oriented link invariant.

Proof. Direct computation. CQFD

Theorem 1.4.7. *Let L be an oriented link and $b \in B_n$ a braid with closure L. The invariant defined above is equal up to a constant to the Jones polynomial $V(L)$ of L. To be more explicit*

$$trace(h_0^{\otimes n} \circ \Phi_n(b)) = (-t^{1/2} - t^{-1/2}) V(L)(t).$$

Proof. One can verify by hand that $t^{-1}R_0 - tR_0^{-1} = (t^{1/2} - t^{-1/2})id_{E\otimes E}$. So the two invariants satisfy the same skein relation. Therefore they are equal up to a constant, that can be determined by computing the value of each invariant on the unknot. CQFD

This is an alternative definition of the Jones polynomial in terms of representations of braid groups via R-matrices. In the next paragraph, we will try to obtain the Alexander polynomial with a similar construction, though we will see it is a little more tricky to do so in that second case.

1.4.3 The Alexander polynomial expressed as a partial trace

We introduce

$$R_1 = \begin{pmatrix} t^{-1/2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & t^{-1/2} - t^{1/2} & 0 \\ 0 & 0 & 0 & -t^{1/2} \end{pmatrix} \in End(E \otimes E)$$

and

$$h_1 = \begin{pmatrix} t^{1/2} & 0 \\ 0 & -t^{1/2} \end{pmatrix} \in End(E).$$

Lemma 1.4.8. *Maps R_1 and h_1 satisfy the three relations of Lemma 1.4.5. So if we call the representations derived from the R-matrix (Θ_n),*

$$L \longmapsto trace(h_0^{\otimes n} \circ \Theta_n(b)), cl(b) = L$$

is an oriented link invariant.

Unfortunately this invariant is not very interesting.

Proposition 1.4.9. *The invariant derived from R_1 and h_1 is equal to 0 for any oriented link.*

This is essentially due to the fact that the trace of h_1 is equal to zero, which was for example not the case for the Jones polynomial. For a detailed proof, see, for example, [46] Proposition 3.10.

To obtain a non trivial invariant from R_1 and h_1, we consider the following modification that, from a graphical point of view, consists in leaving the first strand of the braid open.

Theorem 1.4.10 (Alexander polynomial via a representation of B_n)**.** *Let L be an oriented link and $b \in B_n$ be any braid with closure L. Then:*

1. *There exists a scalar c such that* $trace_{2,3,\dots,n}((id_E \otimes h_1^{\otimes n-1}) \circ \Theta_n(b)) = c.id_E$,
2. *c is a link invariant and is equal to the Alexander polynomial of L,* $\Delta_L(t)$.

The existence and invariance of c under MI and MII will be proved by our study of operator invariants derived from ribbon Hopf algebras, see Section 1.5. For the proof of the equality of the invariant and the Alexander polynomial, we refer the reader to [46], Appendix C.

Corollary 1.4.11. *With the same notations, this formula follows from Theorem 1.4.10:*

$$\Delta_L(t) = \frac{1}{2}\, trace((id_E \otimes h_1^{\otimes n-1}) \circ \Theta_n(b)).$$

Proof. Applying the trace operator on each side of the formula that defines constant c, we obtain:

$$2c = trace(trace_{2,3,\dots,n}((id_E \otimes h_1^{\otimes n-1}) \circ \Theta_n(b))).$$

Since $trace \circ trace_{2,3,\dots,n} = trace$, this concludes. CQFD

Note that this definition of the Alexander polynomial, unlike the geometrical construction we explained before, does not contain any indeterminacy. The choice of the unit of $\mathbb{Z}[t^{\pm 1}]$ is intrinsic in our case. The Alexander polynomial, when it is chosen to have this privileged normalization, is often referred to as the *Alexander-Conway* polynomial.

Theorem 1.4.12. *The Alexander-Conway polynomial* Δ *satisfies the following skein relation:*

$$\Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{-1/2} - t^{1/2})\Delta_{L_0}(t)$$

where L_+, L_- *and* L_0 *are three links that are identical except in the neighborhood of a point where they are as drawn in Figure 1.11.*

Proof. Direct computation shows that $R_1 - R_1^{-1} = (t^{-1/2} - t^{1/2})id_{E\otimes E}$. CQFD

1.5 The Reshetikhin-Turaev theory

The logical question now is to try to understand *how* one can obtain couples of maps (R, h) that satisfy the three conditions of Lemma 1.4.5 and in particular R-matrices, especially when the dimension of the underlying vector space grows large. The Reshetikhin-Turaev theory [52] answers this question by introducing algebraic structures that, once represented, give birth to such maps. These algebraic structures are called *quantum groups*, or perhaps

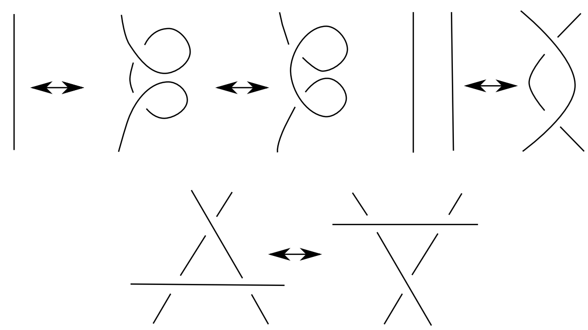


Figure 1.12 – Reidemeister moves RIc, RII and RIII for framed links.

more properly *ribbon Hopf algebras*. A first step for us will be to understand tangles, that are generalizations of braids, and their diagrams.

But as a preliminary let us introduce *framed links* that will be of interest to us in this section.

1.5.1 Framed objects

Definition 1.5.1. A *framed link* is the image of the embedding of a disjoint union of annuli $S^1 \times [0,1]$ into the 3-sphere.

The *underlying link* associated to a given framed link is the link obtained by restricting each annulus to $S^1 \times \{\frac{1}{2}\}$. The *framing* of a framed knot is the class of framed knots that have the same underlying knot as the framed knot itself. The framing of a framed knot can be determined by an integer: the linking number of the two boundary components of the annulus with orientations chosen so that the two components are oriented in the same direction. For a framed link, there is a framing for each annulus.

Finally, to a link diagram D in $\mathbb{R}^2$ we can associate a framed link by *blackboard framing*: the parallel always runs beside the link component in the 2-dimensional projection. Note that any framed link can be expressed by a link diagram by blackboard framing.

In the case of framed links, the Reidemeister theorem is slightly different from the classical one.

Theorem 1.5.2. *Let L and L' be two framed links. If D and D' are diagrams of these framed links by blackboard framing, then L and L' are isotopic if and only if D and D' are related by a finite sequence of isotopies of $\mathbb{R}^2$ and RIc, RII, RIII local moves shown in Figure 1.12.*

Corollary 1.5.3. *The Kauffman bracket is a framed link invariant.*

Proof. Consequence of Proposition 1.3.2. CQFD

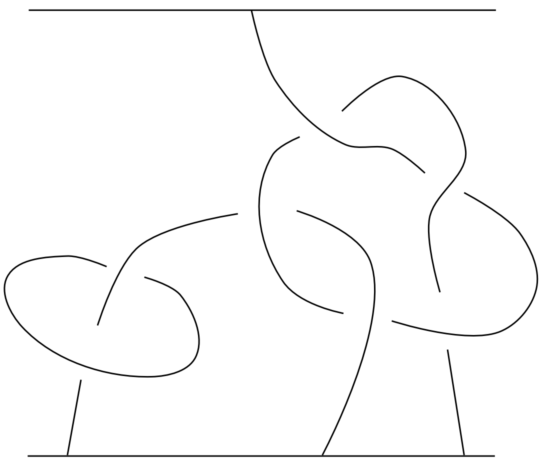


Figure 1.13 – A tangle.

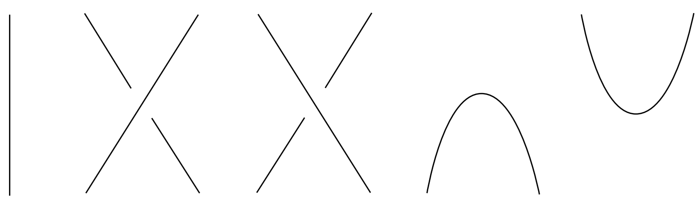


Figure 1.14 – The elementary tangle diagrams.

1.5.2 Tangles and their diagrams

A *tangle* is a compact 1-dimensional manifold embedded in $\mathbb{R}^2 \times [0,1]$ in a way that the boundary of the tangle is a finite set of isolated points in $\{0\} \times \mathbb{R} \times \{0,1\}$. Two tangles are *equivalent*, or *isotopic*, when they are related by an isotopy of the ambient space $\mathbb{R}^2 \times [0,1]$ that fixes the boundary points. A *framed tangle* is a tangle for which each component has a framing. If each component is oriented, we obtain an *oriented (framed) tangle*.

Note that an (oriented) (framed) link is an (oriented) (framed) tangle with no boundary points.

A *tangle diagram* is a diagram of a tangle in $\mathbb{R} \times [0,1]$. See Figure 1.13. A tangle diagram is made of crossings, neighborhoods of critical points with respect to the height function $\mathbb{R} \times [0,1] \to [0,1]$ and vertical paths, that is embedded curves such that the tangent line at each point of the curve is not horizontal. In other words, any tangle diagram can be expressed, up to isotopy, as a *composition* of *tensor products* of copies of *elementary tangle diagrams* described in Figure 1.14. The composition of two tangles T and S is obtained by putting T on top of S and is denoted TS, while the tensor product $T \otimes S$ is defined by putting T next to S.

We consider a particular class of tangle diagrams we call *sliced diagrams*. Sliced tangle

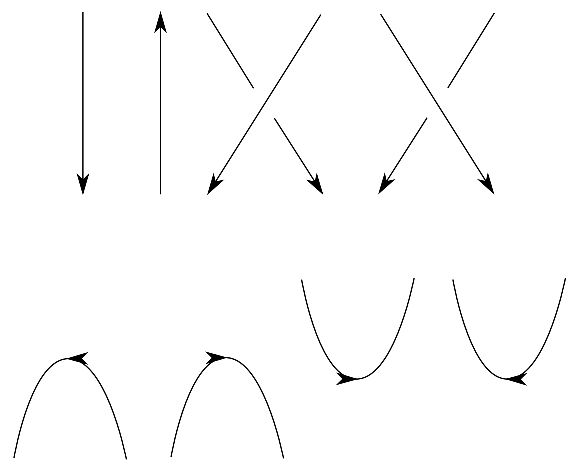


Figure 1.15 – The elementary oriented tangle diagrams.

diagrams are tangle diagrams sliced by horizontal lines such that each element of the product these lines define is the tensor product of at most one crossing or critical point with as many vertical lines as needed. Each tangle can be represented by a sliced diagram, and we will only consider such representations from now on.

Oriented tangles can be represented by *oriented tangle diagrams* as well. They are tangle diagrams where each component is oriented. Any oriented tangle can be represented by an *oriented sliced tangle diagram* where the *elementary oriented tangle diagrams* are shown in Figure 1.15.

Remark 1.5.4. Figure 1.15 shows that all elementary oriented crossings are oriented downwards. Hence one has to rotate all crossings oriented differently before slicing the oriented tangle into elementary diagrams. See Figure 1.16.

Work by Turaev [58, 59] and Freed and Yetter [15] explicits how two (framed) (oriented) sliced tangle diagrams are related whenever they represent the same (framed) (oriented) tangle.

Theorem 1.5.5. *Two sliced (framed) tangle diagrams represent the same (framed) tangle if and only if they are related by a finite sequence of Turaev moves:*

1. $T \times$ *trivial tangle diagram* $\leftrightarrow T \leftrightarrow$ *trivial tangle diagram* $\times T$*, where a trivial tangle diagram is a tensor product of vertical lines,*
2. $(T \otimes$ *trivial tangle diagram*$) \times ($*trivial tangle diagram* $\otimes S) \leftrightarrow ($*trivial tangle diagram* $\otimes S) \times (T \otimes$ *trivial tangle diagram*$)$,

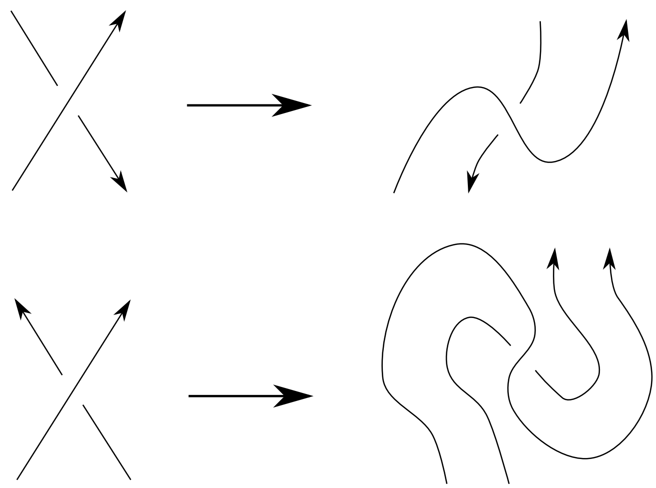
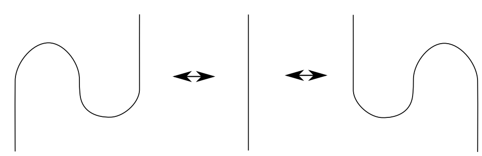
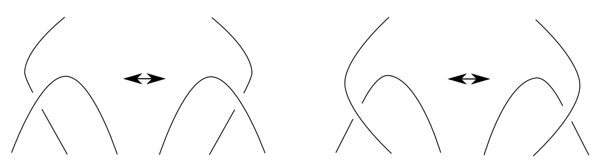


Figure 1.16 – We rotate crossings that are not oriented downwards.

3.



4.

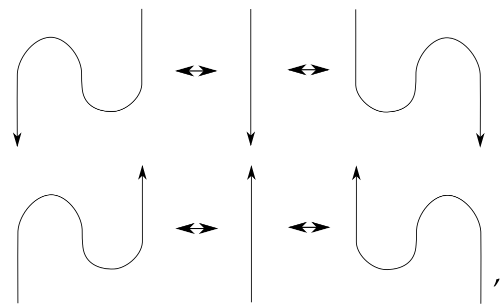


5. *Unoriented Reidemeister moves, see Figure 1.3 and forget the orientations for tangles. For diagrams of* framed *tangles, consider Figure 1.12.*

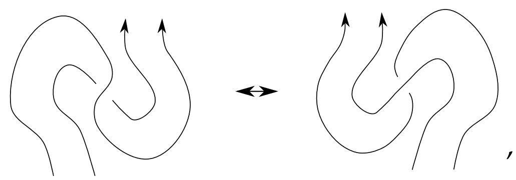
Theorem 1.5.6. *Two* oriented *sliced (framed) tangle diagrams represent the same oriented (framed) tangle if and only if they are related by a finite sequence of Turaev moves:*

1. $T \times$ *trivial tangle diagram* $\leftrightarrow$ T $\leftrightarrow$ *trivial tangle diagram* $\times$ T*, where a trivial tangle diagram is a tensor product of vertical lines,*
2. $(T \otimes$ *trivial tangle diagram*$) \times ($*trivial tangle diagram* $\otimes S) \leftrightarrow ($*trivial tangle diagram* $\otimes S) \times$ $(T \otimes$ *trivial tangle diagram*$)$,

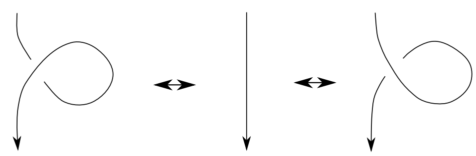
3.



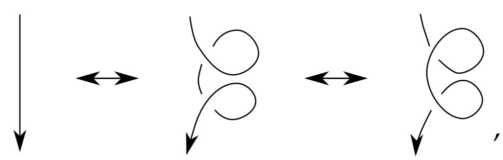
4.



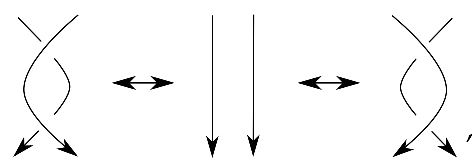
5.



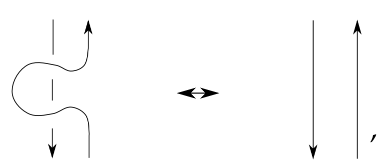
or in the case of framed *tangles:*



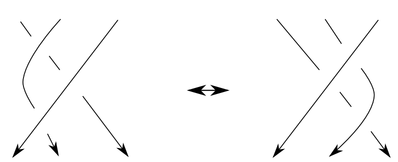
6.



7.



8.



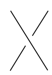
These Turaev theorems play the same role in the construction of *operator invariants* of tangles than the Markov theorem in the case of R-matrix representations of braids, since they give conditions for the operators representing the topological objects to be isotopy invariants.

1.5.3 Operator invariants

The unoriented case

Set V a finite dimensional $\mathbb{C}$-vector space. We introduce the Reshetikhin-Turaev functor that sends a sliced tangle diagram with a lower ends and b upper ends to an element of $Hom(V^{\otimes a}, V^{\otimes b})$. Then, we study under what conditions this functor is topologically invariant.

To explicit that functor, we need only to define the image of an elementary tangle diagram and that of the product and tensor product of two diagrams. The image of the product of two sliced diagrams is the composition of the two image maps, while the image of the tensor product of two diagrams is the tensor product map of the two images. The elementary tangle diagrams are mapped as follows:

$$\Big| \mapsto id_V \ ;$$  $$\mapsto R \in GL(V \otimes V) \ ;$$  $$\mapsto R^{-1} \in GL(V \otimes V) \ ;$$

 $$\mapsto n \in (V \otimes V)^* \ ;$$  $$\mapsto u \in Hom(\mathbb{C}, V \otimes V).$$

That way we map any sliced diagram D to a linear map $[D]$ we will call the *bracket* of D. In particular, the bracket of a sliced link diagram belongs to $End(\mathbb{C}) = \mathbb{C}$. We now focus on what conditions to impose on R, n, u for the functor to be topological, meaninng that the image of a sliced diagram D only depends on the tangle represented by D (or on the framed tangle represented by blackboard framing by D in the framed case). We need to make sure $[D]$ is invariant under Turaev moves.

Lemma 1.5.7. *Suppose that the bracket is a (framed) tangle invariant. Then u is uniquely determined by n. Moreover, n is a non degenerate bilinear form.*

Proof. Let $(e_1, \ldots, e_n)$ be a basis of V. Since the bracket is invariant under Turaev moves, translating Theorem 1.5.5 *3.*, we get the following condition on operators:

$$(n \otimes id_V) \circ (id_V \otimes u) = (id_V \otimes n) \circ (u \otimes id_V) = id_V.$$

We write n and u in terms of the basis:

$$n(e_i \otimes e_j) = n_{i,j} \ , \ u(x) = x \sum_{i,j} u_{i,j} e_i \otimes e_j.$$

Define $N = (n_{i,j})$ and $U = (u_{i,j})$. These are two square matrices. Using the condition on n, u we just explicited:

$$(n \otimes id_V) \circ (id_V \otimes u)(e_k) = \sum_{i,j} n_{k,i} u_{i,j} e_j = e_k \ ,$$

$$(id_V \otimes n) \circ (u \otimes id_V)(e_k) = \sum_{i,j} u_{i,j} n_{j,k} e_i = e_k \ .$$

This exactly means that $N.U = U.N = I$. So U and N are invertible and $U = N^{-1}$. Hence u is uniquely determined by n. Moreover, $n \in (V \otimes V)^*$ is a non degenerate bilinear form. CQFD

Definition 1.5.8. Setting R an invertible element of $End(V \otimes V)$ and $n \in (V \otimes V)^*$ a non degenerate bilinear form, and defining $u \in Hom(\mathbb{C}, V \otimes V)$ from n like above, the bracket $[\ .\ ]$ obtained like mentioned before is called the bracket associated with R and n.

Theorem 1.5.9 ([58, 59, 15])**.** *Set T a (framed) tangle and D a sliced diagram of T (representing T by blackboard framing in the case where T is a framed tangle). If an invertible element $R \in End(V \otimes V)$ and a non degenerate bilinear form $n \in (V \otimes V)^*$ satisfy the following relations:*

1. $(id_V \otimes n) \circ (R^{\pm 1} \otimes id_V) = (n \otimes id_V) \circ (id_V \otimes R^{\mp 1})$,
2. $(R \otimes id_V) \circ (id_V \otimes R) \circ (R \otimes id_V) = (id_V \otimes R) \circ (R \otimes id_V) \circ (id_V \otimes R)$,
3. *$n \circ R = n$, or in the case of framed objects $n \circ R = c.n$ for some non zero scalar c,*

then the bracket $[D]$ associated with R and n is an isotopy invariant of T. That invariant is denoted by $[T]$ and is called the operator invariant *of a (framed) tangle T associated with R and n.*

A particular case is for a (framed) link L. In that case the operator invariant is an element of $End(\mathbb{C}) = \mathbb{C}$, where the identification is given by $f \mapsto f(1)$.

Proof. The proof consists in showing the bracket is preserved by Turaev moves. We will not go into further detail. CQFD

Remark 1.5.10. The Kauffman bracket can be generalized to framed tangles and can then be seen as an operator invariant, which is coherent with Corollary 1.5.3. See for example [46], Chapter 3, where the proper R-matrix and bilinear form are explicited.

We now investigate the case of oriented (framed) tangles.

The oriented case

To define the Reshetikhin-Turaev functor in the case of sliced oriented tangle diagrams, we connsider once more a finite dimensional $\mathbb{C}$-vector space V. We also consider $Hom(V, \mathbb{C}) = V^*$ the dual vector space. The fuctor will send a sliced oriented tangle diagram to a linear map between two tensor products of copies of V and V^*. The elementary oriented tangle diagrams are mapped the following way:

 $\mapsto id_V$; $\mapsto id_{V^*}$;  $\mapsto R \in GL(V \otimes V)$;  $\mapsto R^{-1} \in GL(V \otimes V)$;

 $\mapsto n \in (V \otimes V^*)^*$;  $\mapsto n' \in (V^* \otimes V)^*$;

 $\mapsto u \in Hom(\mathbb{C}, V^* \otimes V)$;  $\mapsto u' \in Hom(\mathbb{C}, V \otimes V^*)$.

We set $B = (e_1, \ldots, e_n)$ a basis of V. Then $(e_1^*, \ldots, e_n^*)$ stands for the dual basis to B. Morphism R is an invertible element of $End(V \otimes V)$ and we derive n, n', u, u' from an invertible element $h \in End(V)$ as follows:

$$n(v \otimes f) = f(h(v)) \ , \ n'(v \otimes f) = f(v),$$

$$u(z) = z \sum_i e_i^* \otimes h(e_i) \ , \ u'(z) = z \sum_i e_i \otimes e_i^*.$$

The bracket of an oriented sliced diagram D is then defined in the same way as it was built in the unoriented case, and is denoted $[D]$ once more.

Identifying vector spaces as explained in Proposition 1.4.1, we regard an endomorphism $f \in End(V \otimes W)$ as an element of the following tensor product:

$$f \in End(V \otimes W) = (V \otimes W)^* \otimes (V \otimes W) = W^* \otimes V^* \otimes V \otimes W.$$

Using Ohtsuki's notations [46], we define $f^{\circlearrowright}$ and $f^{\circlearrowleft}$ the two elementary cyclic permutations of the entries of the tensor product defining f in $W^* \otimes V^* \otimes V \otimes W$.

$$f^{\circlearrowright} \in W \otimes W^* \otimes V^* \otimes V = (W \otimes W^*)^* \otimes (V^* \otimes V) = Hom(W \otimes W^*, V^* \otimes V),$$

$$f^{\circlearrowleft} \in V^* \otimes V \otimes W \otimes W^* = (V^* \otimes V)^* \otimes (W \otimes W^*) = Hom(V^* \otimes V, W \otimes W^*).$$

Theorem 1.5.11 ([58, 59, 15])**.** *Set T an oriented (framed) tangle and D a sliced diagram of T (representing T by blackboard framing in the case where T is framed). If an invertible element $R \in End(V \otimes V)$ and an invertible map $h \in End(V)$ satisfy the following relations:*

1. $R \circ (h \otimes h) = (h \otimes h) \circ R$,
2. $(R \otimes id_V) \circ (id_V \otimes R) \circ (R \otimes id_V) = (id_V \otimes R) \circ (R \otimes id_V) \circ (id_V \otimes R)$,
3. $trace_2\big((id_V \otimes h) \circ R^{\pm 1}\big) = id_V$*, or in the case of framed objects* $trace_2\big((id_V \otimes h) \circ R^{\pm 1}\big) = c^{\pm 1}.id_V$ *for some non zero scalar* c,
4. $(R^{-1})^{\circlearrowleft} \circ \big((id_V \otimes h) \circ R \circ (h^{-1} \otimes id_V)\big)^{\circlearrowright} = id_{V \otimes V}$,

then the bracket $[D]$ associated with R and h is an isotopy invariant of T. That invariant is denoted by $[T]$ and is called the operator invariant *of an oriented (framed) tangle T associated with R and h.*

Proof. Translation of Turaev moves. CQFD

Remark 1.5.12. The element c that is introduced in the case of framed oriented tangles is called the *twist* of the operator invariant. Comparing the framed and the unframed case, observe that an operator invariant of framed oriented tangles becomes an invariant of unframed oriented tangles once the R-matrix is divided by the twist. Also note that such a modification was not possible in the unoriented case.

Remark 1.5.13. Comparing this result with Lemma 1.4.5, we see that we need one more condition to be able to generalize a link invariant to one that is defined on oriented tangles. Indeed,

$$trace\big(h^{\otimes n} \circ \Psi_n(b)\big)$$

happpens to be exactly the operator invariant of the oriented link L when that link is represented (as an oriented tangle with no ends) as the closure of braid b.

The Jones and Alexander polynomials as operator invariants

In Subsections 1.4.2 and 1.4.3, we recovered the Jones and Alexander polynomials as modified traces of representations of the braid groups derived from R-matrices. Here we try to understand these expressions we recall in terms of operator invariants:

$$(-t^{1/2} - t^{-1/2})V(L)(t) = trace(h_0^{\otimes n} \circ \Phi_n(b)),$$

$$\Delta_L(t).id_E = trace_{2,3,\ldots,n}((id_E \otimes h_1^{\otimes n-1}) \circ \Theta_n(b)),$$

where $b \in B_n$ is a braid with closure L, Φ_n is the representation of B_n built from R_0, and Θ_n is the representation built using R_1.

Lemma 1.5.14. *The two pairs of maps (R_0, h_0) and (R_1, h_1) introduced in 1.4.2 and 1.4.3 satisfy points 1., 2., 3., and 4. of Theorem 1.5.11. Hence they both induce an operator invariant of oriented tangles.*

Remark 1.5.13 immediately shows that the operator invariant associated with R_0 and h_0 is a generalization of the Jones polynomial to oriented tangles.

In the case of the Alexander polynomial, note that the fact that the partial trace is a scalar multiple of the identity has yet to be justified. This will be achieved in the next section.

However, we can still try to understand what the meaning of the partial trace is in terms of the operator invariant associated with R_1 and h_1. In fact, the partial trace

$$trace_{2,3,\dots,n}((id_E \otimes h_1^{\otimes n-1}) \circ \Theta_n(b))$$

is the value of the bracket of the 1-1-tangle obtained from b by closing all strands except the first one. This is a way of building a non trivial link invariant from the operator invariant even though its value on links in always zero as we already mentioned.

1.5.4 Ribbon Hopf algebras and invariants of links

However a question still remains to be answered: how can one find pairs of linear maps (R, h) associated with an operator invariant systematically ? Where, for example, do (R_0, h_0) and (R_1, h_1) come from ? The ribbon Hopf algebra structure achieves this goal quite elegantly. The idea is that given a ribbon Hopf algebra A, special elements of the algebra will be represented on a finite dimensional vector space by maps associated to an operator invariant of oriented tangles. The results about Hopf algebras we present here are originally due to [13]. Most statements will not be proved in this paragraph. We follow the way ideas are explained in [46], Chapter 4.

Ribbon Hopf algebras

Setting A an algebra over $\mathbb{C}$ with a unit element 1, we denote by $m : A \otimes A \longrightarrow A$ the product map. We also define $i : \mathbb{C} \longrightarrow A, z \longmapsto z.1$.

Definition 1.5.15 (Hopf algebra)**.** We say A is a Hopf algebra when it is equipped with a *comultiplication* homomorphism $\Delta : A \longrightarrow A \otimes A$, an anti-homomorphism $S : A \longrightarrow A$ called *antipode* and a *counit* homomorphism $\varepsilon : A \longrightarrow \mathbb{C}$ subject to the following relations:

1. $(\Delta \otimes id_A) \circ \Delta = (id_A \otimes \Delta) \circ \Delta$,
2. $(\varepsilon \otimes id_A) \circ \Delta = id_A$,
3. $(id_A \otimes \varepsilon) \circ \Delta = id_A$,
4. $m \circ (S \otimes id_A) \circ \Delta = i \circ \varepsilon$,
5. $m \circ (id_A \otimes S) \circ \Delta = i \circ \varepsilon$.

Definition 1.5.16 (Quasi-triangular Hopf algebra)**.** A quasi-triangular Hopf algebra is a pair $(A, \mathcal{R})$ where A is a Hopf algebra and $\mathcal{R} \in A \otimes A$ is an invertible element subject to:

1. $(\Pi \circ \Delta)(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}$, for any $a \in A$,
2. $(\Delta \otimes id_A)(\mathcal{R}) = \mathcal{R}_{1,3}\mathcal{R}_{2,3}$,
3. $(id_A \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{1,3}\mathcal{R}_{1,2}$,

where $\Pi(a \otimes b) = b \otimes a$ for $a, b \in A$, $\mathcal{R}_{1,2} = \mathcal{R} \otimes 1$, $\mathcal{R}_{2,3} = 1 \otimes \mathcal{R}$, $\mathcal{R}_{1,3} = \sum_i a_i \otimes 1 \otimes b_i$ where $\mathcal{R} = \sum_i a_i \otimes b_i$. When all these conditions are fulfilled, $\mathcal{R}$ is called a *universal R-matrix*.

If $(A, \mathcal{R})$ is a quasi-triangular Hopf algebra, still writing $\mathcal{R} = \sum_i a_i \otimes b_i$, we define an element $u \in A$, sometimes called *pivotal element*, by setting

$$u = \sum_i S(b_i) a_i.$$

Proposition 1.5.17 ([13, 46]). *The pivotal element $u \in A$ has the following properties:*

$$S^2(a) = uau^{-1} \text{ for any } a \in A,$$

$$u^{-1} = \sum_j S^{-1}(d_j) c_j,$$

where we write $\mathcal{R}^{-1} = \sum_j c_j \otimes d_j$.

Proposition 1.5.18 (Some useful properties of the universal R-matrix [13, 46]). *In a quasi-triangular Hopf algebra $(A, \mathcal{R})$, the universal R-matrix satisfies*

1. $\mathcal{R}_{1,2}\mathcal{R}_{1,3}\mathcal{R}_{2,3} = \mathcal{R}_{2,3}\mathcal{R}_{1,3}\mathcal{R}_{1,2}$,
2. $(\varepsilon \otimes id_A)(\mathcal{R}) = 1 = (id_A \otimes \varepsilon)(\mathcal{R})$,
3. $(S \otimes id_A)(\mathcal{R}) = \mathcal{R}^{-1} = (id_A \otimes S^{-1})(\mathcal{R})$,
4. $(S \otimes S)(\mathcal{R}) = \mathcal{R}$,
5. $(u \otimes u).\mathcal{R} = \mathcal{R}.(u \otimes u)$.

Proposition 1.5.19 (Properties of u [13, 46]). *The coproduct and counit of u can be expressed as follows:*

$$\Delta(u) = (u \otimes u).(\Pi(\mathcal{R})\mathcal{R})^{-1},$$

$$\Delta(S(u)) = (S(u) \otimes S(u)).(\Pi(\mathcal{R})\mathcal{R})^{-1},$$

$$\varepsilon(u) = 1.$$

To derive operator invariants from some quasi-triangular Hopf algebras, we need to restrict the class of Hopf algebras we consider one last time. Before we introduce ribbon Hopf algebras, and in order to understand what motivates this new definition, we study the properties of $w = S(u)u \in A$.

Proposition 1.5.20 ([13, 46]). *Set $w = S(u)u \in A$. Then w is central in A and*

$$\Delta(w) = (w \otimes w).(\Pi(\mathcal{R})\mathcal{R})^{-2},$$

$$S(w) = w,$$

$$\varepsilon(w) = 1.$$

Definition 1.5.21 (Ribbon Hopf algebra [52])**.** A quasi-triangular Hopf algebra $(A, \mathcal{R})$ is called a *ribbon Hopf algebra* if there is an element $v \in A$ such that

1. v is a central element,
2. $v^2 = S(u)u$,
3. $\Delta(v) = (v \otimes v).(\Pi(\mathcal{R})\mathcal{R})^{-1}$,
4. $S(v) = v$,
5. $\varepsilon(v) = 1$.

So a ribbon Hopf algebra $(A, \mathcal{R}, v)$ can be obtained from a quasi-triangular Hopf algebra by fixing a square root of $w = S(u)u$.

Operator invariants defined from ribbon Hopf algebras

Setting a representation of a ribbon Hopf algebra, we can derive an operator invariant of framed tangles. This operator is compatible with the module structure induced by the representation.

Let $(A, \mathcal{R}, v)$ be a ribbon Hopf algebra and V be a finite dimensional $\mathbb{C}$-vector space. Set $\rho : A \longrightarrow End(V)$ a representation of A on V. This representation defines a A-module structure on V.

Theorem 1.5.22 (Schur's lemma)**.** *If V is an* irreducible *A-module and $f : V \longrightarrow V$ is an A-linear map, then f is a scalar multiple of the identity. In particular, any central element of A acts by a scalar multiple of the identity.*

For an irreducible representation ρ of A on V, we consider the two following maps:

$$R = P \circ \big((\rho \otimes \rho)(\mathcal{R})\big) \in End(V \otimes V),$$

$$h = \rho(uv^{-1}) \in End(V),$$

where $P : V \otimes V \to V \otimes V, x \otimes y \mapsto y \otimes x$.

Theorem 1.5.23. *The two previous maps R and h satisfy the relations of Theorem 1.5.11 for some non-zero scalar c. Therefore, they define an operator invariant of framed oriented tangles and an invariant of unframed oriented tangles once R is divided by the twist.*

Proof. This is a consequence of Propositions 1.5.17, 1.5.18 and 1.5.19 and Definition 1.5.21.
CQFD

If $\rho_V : A \to End(V)$ and $\rho_W : A \to End(W)$ are two representations of A, we define two representations of A on $V \otimes W$ and on V^*.

$$\rho_{V\otimes W}(a) = (\rho_V \otimes \rho_W)(\Delta(a)) \in End(V) \otimes End(W) = End(V \otimes W),$$

$$\rho_{V^*}(a) \in End(V^*) \text{ defined by } \rho_{V^*}(a)(f) = f \circ \rho_V(S(a)).$$

This defines a A-module structure inductively on any tensor power of V and V^* where V is a fixed A-module. Moreover, the A-module structure we choose on $\mathbb{C}$ is derived from the *unit representation* $\rho_{\mathbb{C}} = \varepsilon : A \to \mathbb{C}$.

Let us recall that an A-module homomorphism $f : V \to W$ is a linear map that commutes with the action of any element of A:

$$f \circ (\rho_V(a)) = (\rho_W(a)) \circ f$$

for any $a \in A$.

Theorem 1.5.24. *Denote the operator invariant of an oriented (framed) tangle T derived from a representation of a ribbon Hopf algebra A on V by $[T]_{A,V}$. Then $[T]_{A,V}$ is a A-module homomorphism with respect to the A-module structures we introduced.*

Corollary 1.5.25. *If T is a (framed) oriented $1,1$-tangle, that is a tangle with one uper end and one lower end, then its operator invariant $[T]_{A,V}$ derived from ribbon Hopf algebra A and representation V is a scalar multiple of the identity.*

Proof. Apply Schur's lemma (Theorem 1.5.22) to $[T]_{A,V}$. CQFD

1.5.5 The Jones and Alexander polynomials derived from ribbon Hopf algebras

The operator invariant that we recovered the Jones polynomial from in the previous section is derived from the 2-dimensional irreducible family of representations of quantum group $U_q\mathfrak{sl}(2)$ at a generic q, once the R-matrix is divided by the twist.

On the other hand, the operator invariant that generalizes the Alexander-Conway invariant to oriented tangles can be recovered in two different ways: either by considering the quasi triangular Hopf algebra associated to $U_q\mathfrak{sl}(2)$ at fourth roots of unity, or by considering the super Hopf algebra $U_q\mathfrak{gl}(1|1)$ [61]. We will study these two interpretations in detail in Chapter 3. Moreover, knowing this, it becomes clear using Corollary 1.5.25 that the expression

$$trace_{2,3,\dots,n}((id_E \otimes h_1^{\otimes n-1}) \circ \Theta_n(b))$$

is a scalar multiple of the identity.

We shall now introduce a family of quantum link invariants that we will study in the rest of this manuscript: the Links-Gould invariants of links.

1.6 A family of quantum invariants: the Links-Gould invariants

1.6.1 The Links-Gould invariants of links $LG^{n,m}$

The first Links-Gould invariant was introduced by Links and Gould in 1992 [35], and studied in further detail by De Wit, Kauffman and Links in [11]. It was generalized to a family of invariants $LG^{n,m}$, $n, m \in \mathbb{N}^*$ by De Wit [7]. The first Links-Gould invariant is $LG^{2,1}$ in this family, and is sometimes simply denoted LG when it is not ambiguous to do so. The Links-Gould invariants are two variable polynomial invariants. They are derived from *super* Hopf Algebras $U_q\mathfrak{gl}(n|m)$. The *super* means that the algebra is equipped with an additional structure: a $\mathbb{Z}/2\mathbb{Z}$ grading that modifies the axioms slightly compared to those of a standard Hopf algebra. However, this will not be important for us since there is a procedure, called *bosonization* and due to Majid [36], that transforms a super Hopf algebra into an ordinary one. This trick will be explained in Chapter 3. We will present super Hopf algebra $U_q\mathfrak{gl}(n|m)$ and its characteristic elements when it is necessary for our study, that is in the case of $U_q\mathfrak{gl}(n|1)$, in the course of Chapter 3. For a completely general definition of $U_q\mathfrak{gl}(n|m)$, we refer the reader to [7, 8]. Whenever it is possible, we choose to stay at the level of operator invariants.

The work of Viro [61] shows that the simplest Links-Gould invariant $LG^{1,1}$ is non other than the Alexander-Conway polynomial. However and except for the $(1,1)$ case, unlike for most polynomial link invariants, properties that are considered to be at the base of the understanding of an invariant are not known for LG. For example, if Marin and Wagner give a complete set of skein relations for $LG^{2,1}$ [37], the cubic relation is barely practicable, and a simple set of relations has still to be found. On the other hand, the investigations we went through and that are explained in Chapter 4 suggest that the Links-Gould invariants are much closer to the Alexander polynomial in terms of a geometrical interpretation that any other known quantum invariant. These paradoxical facts are underlined by the questions we raise and answers we provide in the course of this text.

The simplest Links-Gould invariant $LG^{2,1}$ can be defined as derived from an operator invariant as follows.

Definition 1.6.1. Set $\mathbb{K} := \mathbb{C}(t_0^{\pm\frac{1}{2}}, t_1^{\pm\frac{1}{2}})$. Let $W = < e_1, \ldots, e_4 >$ be a four-dimensional $\mathbb{K}$-vector space. The following linear map R, expressed in basis $(e_1 \otimes e_1, e_1 \otimes e_2, e_1 \otimes e_3, e_1 \otimes e_4, e_2 \otimes e_1, e_2 \otimes e_2, e_2 \otimes e_3, \ldots)$, is an automorphism of $W \otimes W$ and an R-matrix ([11], p.186) :

$$\begin{pmatrix}
t_0 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & t_0^{1/2} & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & t_0^{1/2} & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . \\
. & t_0^{1/2} & . & . & t_0-1 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & -1 & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & t_0t_1-1 & . & . & -t_0^{1/2}t_1^{1/2} & . & . & -t_0^{1/2}t_1^{1/2}Y & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & t_1^{1/2} & . & . \\
. & . & t_0^{1/2} & . & . & . & . & . & t_0-1 & . & . & . & . & . & . & . \\
. & . & . & . & . & . & -t_0^{1/2}t_1^{1/2} & . & . & . & . & . & Y & . & . & . \\
. & . & . & . & . & . & . & . & . & . & -1 & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & t_1^{1/2} & . \\
. & . & . & 1 & . & . & -t_0^{1/2}t_1^{1/2}Y & . & . & Y & . & . & Y^2 & . & . & . \\
. & . & . & . & . & . & . & t_1^{1/2} & . & . & . & . & . & t_1-1 & . & . \\
. & . & . & . & . & . & . & . & . & . & . & t_1^{1/2} & . & . & t_1-1 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & t_1
\end{pmatrix}$$

where $Y = ((t_0-1)(1-t_1))^{1/2}$.
We denote by b_R^n the representation of braid group B_n derived from this R-matrix, given by the standard formula

$$b_R^n(\sigma_i) = id_W^{\otimes i-1} \otimes R \otimes id_W^{\otimes n-i-1}\ , i = 1, \ldots, n-1.$$

Theorem 1.6.2. *Let L be an unframed oriented link, and $b \in B_n$ a braid with closure L. Set μ the linear map defined by*

$$\mu = \begin{pmatrix} t_0^{-1} & . & . & . \\ . & -t_1 & . & . \\ . & . & -t_0^{-1} & . \\ . & . & . & t_1 \end{pmatrix} \in End(W).$$

Then :
1) there exists an element $c \in \mathbb{K}$ such that $trace_{2,3,\ldots,n}((id_W \otimes \mu^{\otimes n-1}) \circ b_R^n(b)) = c.id_W$,
2) c is an oriented link invariant called Links-Gould invariant of L. We will denote it by $LG^{2,1}(L;t_0,t_1) = LG(L;t_0,t_1)$.

Remark 1.6.3. In fact $LG(L;t_0,t_1) \in \mathbb{Z}[t_0^{\pm1}, t_1^{\pm1}]$ [23].

Remark 1.6.4. With the notations we use, $LG(L;q^{-2\alpha}, q^{2\alpha+2})$ is the Links-Gould invariant introduced in [11], using a one parameter family of representations of quantum superalgebra $U_q(\mathfrak{gl}(2|1))$.

Remark 1.6.5. For practical reasons, we will use the *opposite* R-matrix in Chapter 2, but we will use this one in Chapter 4. Moreover, this is the R-matrix used the most often in papers on the subject, and in particular in Ishii's paper comparing properties of $LG^{2,1}$ and Δ [23] that relates to Chapter 4.

Corollary 1.6.6. *As in Corollary 1.4.11, we have the following formula for LG:*

$$LG(L; t_0, t_1) = \frac{1}{4}\, trace((id_W \otimes \mu^{\otimes n-1}) \circ b_R^n(b)).$$

1.6.2 Different sets of variables

There are several sets of variables used in papers studying LG invariants. Four of them appear regularly: (t_0, t_1), (τ, q), (p, q) and (q^α, q). Each set can be expressed in terms of the others using the following defining relations:

$$t_0 = q^{-2\alpha}, t_1 = q^{2\alpha+2},$$

$$\tau = t_0^{1/2} = q^{-\alpha}, p^a q^b = t_0^{-\frac{a}{4}+\frac{b}{2}} t_1^{\frac{a}{4}+\frac{b}{2}}.$$

In the case of $LG^{2,1}$, variables (t_0, t_1) nicely lead to a symmetric Laurent polynomial.

1.7 The De Wit-Ishii-Links conjecture

David De Wit, Atsushi Ishii and Jon Links conjectured [10] that for any link L

$$LG^{n,m}(L; \tau, e^{\mathbf{i}\pi/m}) = \Delta_L(\tau^{2m})^n,$$

where Δ_L is the Alexander-Conway polynomial of L. They proved the conjecture when $(n, m) = (1, m)$ and when $(n, m) = (2, 1)$ for a particular class of braids. A complete proof of the $(n, 1)$ case for $n = 2, 3$ is given in Chapter 2. This is achieved by studying the invariants at hand at the level of *representations*, which requires computation of an explicit R-matrix for each $LG^{n,1}$. This makes that method hard to implement as n grows.

In Chapter 3 we prove the $(n, 1)$ case of the conjecture for any n:

$$LG^{n,1}(L; \tau, -1) = \Delta_L(\tau^2)^n.$$

To do so we study the structure of the *universal* objects directly, and in particular the (super) Hopf algebras and universal R-matrices that are involved. However, the method that consists in understanding the specialized R-matix representation in terms of the Burau representation remains interesting since it relates that representation to the topology of the complement of the link.

These results were the first pieces of evidence we had for the conjectures we state and support in Chapter 4. Indeed, once LG is known to be a generallization of Δ, one can wonder if Δ's properties remain true for the Links-Gould polynomials. The very surprising fact is that most of Δ's homological properties seem to propagate to LG. In that spirit, let us recall that a conjecture by Ishii [23] states:

Conjecture 1.7.1. *The LG polynomial* $LG(K; t_0, t_1) = \sum_{i,j} a_{ij} t_0^i t_1^j$ *of an alternating knot K is "alternating", that is :* $a_{ij}a_{kl} \geqslant 0$ *if* $i + j + k + l$ *is even, and* $a_{ij}a_{kl} \leqslant 0$ *otherwise.*

We give evidence for more results that extend the well known properties of the Alexander invariant we recalled in Subsection 1.2.2. We conjecture that the span of the LG invariant is a lower bound for the genus of a link. We also conjecture that for fibered knots, there are conditions on the leading coefficients of the LG polynomial. We believe that this behavior is the trace of a classical construction for the Links-Gould invariants, and that the general proof of these results will have to use this construction.

However, note that still now, the strong version of the De Wit-Ishii-Links conjecture remains open.

Chapter 2

Understanding the evaluations at the level of representations

In this chapter [1], we prove the following identity we can express using different sets of variables, namely (τ, q) and (t_0, t_1):

$$LG^{n,1}(L;\tau,-1) = \Delta_L(\tau^2)^n,$$

$$LG^{n,1}(L;t_0,t_0^{-1}) = \Delta_L(t_0)^n$$

when $n = 2, 3$. Our strategy will be to take advantage of the robustness of the braid structure to encode links. We use the expression of the Alexander-Conway polynomial as a quantum trace we gave in Theorem 1.4.10. Then we prove the R-matrix representation of braid group B_n used to define reduced Links-Gould invariant $LG^{2,1}$ (resp. $LG^{3,1}$) is isomorphic to the exterior power of a direct sum of Burau representations. That way, the specialized Links-Gould invariants can be written as products of terms, each of which can be identified with the Alexander polynomial of our link.

In Section 2.1, we recall the definitions we will need in this chapter, and in particular the definition of the Links-Gould invariant of oriented links we will follow here. In Section 2.2 we show that the specialized Links-Gould invariant $LG^{2,1}$ can be written as a product by proving two representations of the braid group are isomorphic. We then identify in Section 2.3 each part of the product with the Alexander-Conway invariant. Section 2.4 is dedicated to extending the proof to the next Links-Gould invariant $LG^{3,1}$.

1. This chapter is based on the paper *On the Links-Gould invariant and the square of the Alexander polynomial* published in the Journal of Knot Theory and Its Ramifications [31].

2.1 Definitions and main result

Here we define precisely what characteristic elements we consider for the quantum version of the Alexander polynomial Δ and for $LG^{2,1}$.

2.1.1 The Alexander polynomial

Definition 2.1.1. (*Reduced and non-reduced Burau representations of a braid*)
Set $\mathbb{K} := \mathbb{C}(t^{\pm\frac{1}{2}})$. Let $W_n =< f_1, \dots, f_n >$ be a n-dimensional $\mathbb{K}$-vector space, and B_n be the braid group on n strands. We denote by $\sigma_1, \dots, \sigma_{n-1}$ the standard Artin generators of the group. The non-reduced Burau representation $\Psi_{W_n} : B_n \longrightarrow GL(W_n)$ is given by :

$$\Psi_{W_n}(\sigma_i)(f_j) = \begin{cases} (1-t)f_i + t^{1/2} f_{i+1} & \text{if } j = i \,, \\ t^{1/2} f_i & \text{if } j = i+1 \,, \\ f_j & \text{otherwise.} \end{cases}$$

Denote by $\delta_n := t^{-(n-1)/2} f_1 + t^{-(n-2)/2} f_2 + \dots + t^{-1/2} f_{n-1} + f_n$. One can verify that for any $b \in B_n$, $\Psi_{W_n}(b)(\delta_n) = \delta_n$. Hence the reduced Burau representation $\Psi_{\widehat{W_n}} : B_n \longrightarrow GL(\widehat{W_n})$ is given by :

$$\Psi_{\widehat{W_n}}(b)(\overline{x}) = \overline{\Psi_{W_n}(b)(x)}$$

where $\widehat{W_n} := W_n / < \delta_n >$.

Recall that Alexander's theorem states that any link can be obtained as the closure of a given braid. Moreover, Markov's theorem allows us to define link invariants through braids with closure the link. A possible definition of the classical Alexander link invariant, different from that using a Seifert surface, exploits that procedure.

Definition 2.1.2. (*Alexander polynomial of a link through the Burau representation*)
The Alexander polynomial of an oriented link L is defined as :

$$\Delta_L(t) \doteq \frac{1-t}{1-t^n}\, det(I - \Psi_{\widehat{W_n}}(b))$$

where b is any braid in B_n with closure L, and the notation $\doteq$ means equality up to multiplication by a unit of $\mathbb{C}[t^{\pm 1}]$.

But it is not this definition of the Alexander polynomial that will be useful to us in the following. We exploit the definition of Δ as a partial trace to stay closer to the definition of $LG^{2,1}$ we gave. We multiply the R-matrix we used in Theorem 1.4.10 by a factor $t^{1/2}$, therefore renouncing to use the Conway normalization of the polynomial. However this is not a major problem since the result we prove is an equality up to $\pm t^{n/2}, n \in \mathbb{Z}$.

Definition 2.1.3. Let V be a 2-dimensional $\mathbb{K}$-vector space, and (e_0, e_1) be a basis of V. We define a representation $\Psi_{V^{\otimes n}} : B_n \longrightarrow GL(V^{\otimes n})$ of B_n :

$$\Psi_{V^{\otimes n}}(\sigma_i) = id_V^{\otimes i-1} \otimes R_1 \otimes id_V^{\otimes n-i-1}$$

where

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & t^{1/2} & 0 \\ 0 & t^{1/2} & 1-t & 0 \\ 0 & 0 & 0 & -t \end{pmatrix} \in End(V \otimes V)$$

is an R-matrix, that is a solution of the Yang-Baxter equation.

Theorem 2.1.4. *Let L be an oriented link and $b \in B_n$ be any braid with closure L. We define*

$$h = \begin{pmatrix} t^{1/2} & 0 \\ 0 & -t^{1/2} \end{pmatrix} \in End(V).$$

Then :
1) There exists a scalar $c \in \mathbb{K}$ such that $trace_{2,3,\dots,n}((id_V \otimes h^{\otimes n-1}) \circ \Psi_{V^{\otimes n}}(b)) = c.id_V$,
2) c is a link invariant and is equal to the Alexander polynomial of L, $\Delta_L(t)$.

Keep in mind Corollary 1.4.11 that transforms the partial trace into something easier to manipulate:

$$\Delta_L(t) = \frac{1}{2}\, trace((id_V \otimes h^{\otimes n-1}) \circ \Psi_{V^{\otimes n}}(b)).$$

2.1.2 The Links-Gould invariant $LG^{2,1}$ of links

In the whole chapter, we consider the *opposite* R-matrix to define $LG^{2,1}$ compared to Definition 1.6.1:

$$R = \begin{pmatrix}
-t_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -t_0^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -t_0^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\
\cdot & -t_0^{1/2} & \cdot & \cdot & 1-t_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1-t_0t_1 & \cdot & \cdot & t_0^{1/2}t_1^{1/2} & \cdot & \cdot & t_0^{1/2}t_1^{1/2}Y & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -t_1^{1/2} & \cdot & \cdot \\
\cdot & \cdot & -t_0^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & 1-t_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_0^{1/2}t_1^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & -Y & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -t_1^{1/2} & \cdot \\
\cdot & \cdot & \cdot & -1 & \cdot & \cdot & t_0^{1/2}t_1^{1/2}Y & \cdot & \cdot & -Y & \cdot & \cdot & -Y^2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -t_1^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & 1-t_1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -t_1^{1/2} & \cdot & \cdot & 1-t_1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -t_1
\end{pmatrix}$$

where $Y = ((t_0 - 1)(1 - t_1))^{1/2}$.

This matrix still computes the same polynomial up to $\pm t^{n/2}, n \in \mathbb{Z}$. Using notations that were introduced in Definition 1.6.1, the invariant is given by the formula

$$LG(L; t_0, t_1) = \frac{1}{4}\, trace((id_W \otimes \mu^{\otimes n-1}) \circ b_R^n(b)).$$

The main result of the chapter states:

Theorem 2.1.5. *For any oriented link L, $LG^{2,1}(L; t_0, t_0^{-1}) \doteq \Delta_L(t_0)^2$, where $\doteq$ stands for equality up to $\pm t^{n/2}, n \in \mathbb{Z}$.*

For a right choice of characteristic elements, the $\doteq$ can be transformed into an equality with the Conway normalization for the Alexander invariant, as we will see in the next chapter.

2.2 The reduced Links-Gould invariant expressed as a product

We derive a representation of the braid group B_n from the Burau representation. We identify it with a specialization of the R-matrix representation given in Subsection 2.1.2. Then we use this identification to express the specialized Links-Gould invariant as a product.

2.2.1 A representation of B_n isomorphic to $b_R^n(t_0, t_0^{-1})$

Denote by F the following Burau representation of B_n on vector space $W_n =< f_1, \dots, f_n >$ where we replace t_0 by t_0^{-1}:

$$F(\sigma_i)(f_j) = \begin{cases} (1-t_0^{-1})f_i + t_0^{-1/2} f_{i+1} & \text{if } j = i\,, \\ t_0^{-1/2} f_i & \text{if } j = i+1\,, \\ f_j & \text{otherwise.} \end{cases}$$

In a similar way, let G be the representation of B_n on n-dimensional vector space $W_n =< g_1, \dots, g_n >$ given by:

$$G(\sigma_i)(g_j) = \begin{cases} -t_0^{1/2} g_{i+1} & \text{if } j = i\,, \\ -t_0^{1/2} g_i + (1-t_0) g_{i+1} & \text{if } j = i+1\,, \\ g_j & \text{otherwise.} \end{cases}$$

Proposition 2.2.1. *Representation G is isomorphic to the Burau representation of B_n.*

Proof. One can verify that for $i = 1, 2, \dots, n-1 : J_n \circ \Psi_{W_n}(\sigma_i) = G(\sigma_i) \circ J_n$ where J_n can be defined inductively: $J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and

$$J_n = \left(\begin{array}{ccc|c} & & & (-1)^2 t^{(n-2)/2} \\ & & & \vdots \\ & J_{n-1} & & (-1)^{n-1} t^{1/2} \\ & & & (-1)^n t^{0/2} \\ \hline (-1)^{n+1} t^{-(n-2)/2} & \dots\ (-1)^{n+1} t^{-1/2} & (-1)^{n+1} t^{-0/2} & 0 \end{array}\right).$$

Moreover, evaluating the determinant of J_n, we deduce that J_n is an automorphism. Indeed, $detJ_{n+1} = (-1)^{n+1}(t^{1/2} + t^{-1/2})detJ_n + detJ_{n-1}$. So $detJ_n \in \mathbb{Z}[t^{\pm 1/2}]$ is invertible in $\mathbb{Q}(t^{\pm 1/2})$ since it has degree $n-2$ in both variables $t^{1/2}$ and $t^{-1/2}$. CQFD

Definition 2.2.2. If $F \oplus G$ is the representation of the n-strand braid group on $W_n \oplus W_n$ built from F and G, we consider the exterior representation $\Psi_n := \bigwedge(F \oplus G)$ on exterior algebra $\bigwedge(W_n \oplus W_n)$.

Remark 2.2.3. Note that $b_R^n(t_0, t_0^{-1})$ and Ψ_n both are $4^n = 2^{2n}-$dimensional representations.

We are going to show these two representations are isomorphic. For that we study first the case where $n = 2$. Since $t_1 = t_0^{-1}$, we have a simpler R-matrix R:

$$R=\begin{pmatrix}
-t_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -t_0^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -t_0^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\
\cdot & -t_0^{1/2} & \cdot & \cdot & 1-t_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & -Y & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_0^{-1/2} & \cdot & \cdot \\
\cdot & \cdot & -t_0^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & 1-t_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & -Y & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_0^{-1/2} & \cdot \\
\cdot & \cdot & \cdot & -1 & \cdot & \cdot & -Y & \cdot & \cdot & -Y & \cdot & \cdot & -Y^2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_0^{-1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & 1-t_0^{-1} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_0^{-1/2} & \cdot & \cdot & 1-t_0^{-1} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -t_0^{-1}
\end{pmatrix}$$

where $Y = t_0^{1/2} - t_0^{-1/2}$.

R-matrix R can be rewritten in basis $\mathcal{B} = (|e_1 \otimes e_1|, |e_4 \otimes e_4|, |e_2 \otimes e_2|, |e_3 \otimes e_3|, |e_1 \otimes e_2, e_2 \otimes e_1|, |e_1 \otimes e_3, e_3 \otimes e_1|, |e_3 \otimes e_4, e_4 \otimes e_3|, |e_2 \otimes e_4, e_4 \otimes e_2|, |e_1 \otimes e_4, e_2 \otimes e_3, e_3 \otimes e_2, e_4 \otimes e_1|)$ as follows:

$$\begin{pmatrix}
-t_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & -t_0^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 0 & -t_0^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -t_0^{1/2} & 1-t_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & -t_0^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -t_0^{1/2} & 1-t_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & t_0^{-1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_0^{-1/2} & 1-t_0^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & t_0^{-1/2} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_0^{-1/2} & 1-t_0^{-1} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & -1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & -1 & -Y \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & -1 & 0 & -Y \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -Y & -Y & -Y^2
\end{pmatrix}$$

Family (f_1, f_2, g_1, g_2) is a basis for $W_2 \oplus W_2$. Since $B_2 =< \sigma_1 >$, we are looking for a linear automorphism $I : W^{\otimes 2} \longrightarrow \bigwedge(W_2 \oplus W_2)$ such that $\Psi_2(\sigma_1) \circ I = I \circ \underbrace{b_R^2(\sigma_1)}_{R}$.

In basis (f_1, f_2, g_1, g_2),

$$(F \oplus G)(\sigma_1) = \begin{pmatrix} 1-t_0^{-1} & t_0^{-1/2} & \cdot & \cdot \\ t_0^{-1/2} & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & -t_0^{1/2} \\ \cdot & \cdot & -t_0^{1/2} & 1-t_0 \end{pmatrix}$$

Therefore, computation of $\Psi_2(\sigma_1)$ shows that in basis $\mathcal{C} = (|g_1 \wedge g_2|, |f_1 \wedge f_2|, |1|, |f_1 \wedge f_2 \wedge g_1 \wedge g_2|, |g_1, g_2|, |f_2 \wedge g_1 \wedge g_2, f_1 \wedge g_1 \wedge g_2|, |f_1 \wedge f_2 \wedge g_1, f_1 \wedge f_2 \wedge g_2|, |f_2, f_1|, |f_2 \wedge g_1, f_2 \wedge g_2, f_1 \wedge g_1, f_1 \wedge g_2|)$ we obtain the same matrix:

$$\left(\begin{array}{cccccccccccccccc}
-t_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & -t_0^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 0 & -t_0^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -t_0^{1/2} & 1-t_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & -t_0^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -t_0^{1/2} & 1-t_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & t_0^{-1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_0^{-1/2} & 1-t_0^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & t_0^{-1/2} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_0^{-1/2} & 1-t_0^{-1} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & -1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & -1 & -Y \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & -1 & 0 & -Y \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -Y & -Y & -Y^2
\end{array}\right)$$

Setting $I : W^{\otimes 2} \longrightarrow \bigwedge(W_2 \oplus W_2)$ the linear map that transforms $\mathcal{B}$ into $\mathcal{C}$, we obtain an automorphism that preserves the $\mathbb{C}[B_2]$-module structure:

$$\Psi_2(\sigma_1) \circ I = I \circ R.$$

The idea is to generalize that construction for n larger than 2. We choose the following reference basis for $\bigwedge(W_n \oplus W_n)$:

$$(f_{i_1} \wedge \ldots \wedge f_{i_p} \wedge g_{j_1} \wedge \ldots \wedge g_{j_m})_{1 \leq i_1 < \ldots < i_p \leq n\ ,\ 1 \leq j_1 < \ldots < j_m \leq n}.$$

When we refer to $Reord(u_{i_1} \wedge \ldots \wedge u_{i_r})$, where the u_{i_k} are distinct elements of $\{f_1, \ldots, f_n, g_1, \ldots, g_n\}$, we mean that we rewrite the element so that it becomes part of the reference basis we just mentioned.

We set $I_1 = \begin{cases} W & \longrightarrow & \bigwedge(W_1 \oplus W_1) \\ e_1 & \longmapsto & g_1 \\ e_2 & \longmapsto & 1 \\ e_3 & \longmapsto & f_1 \wedge g_1 \\ e_4 & \longmapsto & f_1 \end{cases}$ and $I_2 = \begin{cases} W^{\otimes 2} & \longrightarrow & \bigwedge(W_2 \oplus W_2) \\ e_i \otimes e_1 & \longmapsto & I_1(e_i) \wedge g_2 \\ e_i \otimes e_2 & \longmapsto & I_1(e_i) \\ e_i \otimes e_3 & \longmapsto & Reord(I_1(e_i) \wedge f_2 \wedge g_2) \\ e_i \otimes e_4 & \longmapsto & Reord(I_1(e_i) \wedge f_2) \end{cases}$.

An elementary calculation shows that $I_2 = I$. We can extend these maps by induction setting:

$$I_n = \begin{cases} W^{\otimes n} & \longrightarrow & \bigwedge(W_n \oplus W_n) \\ e_{i_1} \otimes \ldots \otimes e_{i_{n-1}} \otimes e_1 & \longmapsto & I_{n-1}(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}}) \wedge g_n \\ e_{i_1} \otimes \ldots \otimes e_{i_{n-1}} \otimes e_2 & \longmapsto & I_{n-1}(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}}) \\ e_{i_1} \otimes \ldots \otimes e_{i_{n-1}} \otimes e_3 & \longmapsto & Reord(I_{n-1}(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}}) \wedge f_n \wedge g_n) \\ e_{i_1} \otimes \ldots \otimes e_{i_{n-1}} \otimes e_4 & \longmapsto & Reord(I_{n-1}(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}}) \wedge f_n) \end{cases}.$$

It is easy to see that map I_n sends the natural basis of $W^{\otimes n}$ derived from (e_1, e_2, e_3, e_4) on our reference basis of $\bigwedge(W_n \oplus W_n)$. In particular, I_n is a linear automorphism. Note that map I_n can also be written directly:

$$I_n(e_{i_1} \otimes \ldots \otimes e_{i_n}) = \Big(\bigwedge_{k\,:\,i_k=3,4} f_k \Big) \wedge \Big(\bigwedge_{k\,:\,i_k=1,3} g_k \Big)$$

Proposition 2.2.4. *Map I_n is a $\mathbb{C}[B_n]$-module automorphism. That is, for any $b \in B_n$:*

$$\Psi_n(b) \circ I_n = I_n \circ b_R^n(b).$$

Proof. We show the commutation by induction on n, the number of strands in the braid group we consider. Note that it has already been verified when $n = 1, 2$. Let us now suppose the equality holds for $n-1$, $n \geqslant 3$. We only need to prove the result for $b = \sigma_k$, $k = 1, \ldots, n-1$.

<u>For σ_k, $k \leq n-2$</u>:

$I_n(b_R^n(\sigma_k)(e_{i_1} \otimes \ldots \otimes e_{i_n}))$
$= I_n(e_{i_1} \otimes \ldots \otimes R(e_{i_k} \otimes e_{i_{k+1}}) \otimes \ldots \otimes e_{i_n})$
$= I_n(b_R^{n-1}(\sigma_k)(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}}) \otimes e_{i_n})$

$$= \begin{cases} I_{n-1}(b_R^{n-1}(\sigma_k)(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}})) \wedge g_n & \text{if } i_n = 1 \\ I_{n-1}(b_R^{n-1}(\sigma_k)(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}})) & \text{if } i_n = 2 \\ Reord(I_{n-1}(b_R^{n-1}(\sigma_k)(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}})) \wedge f_n \wedge g_n) & \text{if } i_n = 3 \\ Reord(I_{n-1}(b_R^{n-1}(\sigma_k)(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}})) \wedge f_n) & \text{if } i_n = 4 \end{cases}$$

$$= \begin{cases} \Psi_{n-1}(\sigma_k)(I_{n-1}(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}})) \wedge g_n & \text{if } i_n = 1 \\ \Psi_{n-1}(\sigma_k)(I_{n-1}(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}})) & \text{if } i_n = 2 \\ Reord(I_{n-1}(b_R^{n-1}(\sigma_k)(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}})) \wedge f_n \wedge g_n) & \text{if } i_n = 3 \\ Reord(I_{n-1}(b_R^{n-1}(\sigma_k)(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}})) \wedge f_n) & \text{if } i_n = 4 \end{cases} \quad \text{(inductive hypothesis)}$$

On the other hand:

$$\Psi_n(\sigma_k)(I_n(e_{i_1} \otimes \ldots \otimes e_{i_n}))$$
$$= \begin{cases} \Psi_n(\sigma_k)(I_{n-1}(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}}) \wedge g_n) & \text{if } i_n = 1 \\ \Psi_n(\sigma_k)(I_{n-1}(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}})) & \text{if } i_n = 2 \\ \Psi_n(\sigma_k)(Reord(I_{n-1}(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}}) \wedge f_n \wedge g_n)) & \text{if } i_n = 3 \\ \Psi_n(\sigma_k)(Reord(I_{n-1}(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}}) \wedge f_n)) & \text{if } i_n = 4 \end{cases}$$

$$= \begin{cases} \Psi_n(\sigma_k)(I_{n-1}(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}})) \wedge \Psi_n(\sigma_k)(g_n) & \text{if } i_n = 1 \\ \Psi_n(\sigma_k)(I_{n-1}(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}})) = \Psi_{n-1}(\sigma_k)(I_{n-1}(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}})) & \text{if } i_n = 2 \\ \Psi_n(\sigma_k)(Reord(I_{n-1}(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}}) \wedge f_n \wedge g_n)) & \text{if } i_n = 3 \\ \Psi_n(\sigma_k)(Reord(I_{n-1}(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}}) \wedge f_n)) & \text{if } i_n = 4 \end{cases}$$

For $i_n = 1$, since $\Psi_n(\sigma_k)(g_n) = g_n$, we obtain the result. We also observe the equality holds when $i_n = 2$. We now study the two remaining cases.

Let $\mu(e_{i_1} \otimes \ldots \otimes e_{i_n})$ be the total of the number of e_1 and the number of e_3 in that elementary tensor. Given the expression of I_n and our reference basis of $\bigwedge(W_n \oplus W_n)$, it is obvious that:

$$Reord(I_{n-1}(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}}) \wedge f_n) = (-1)^{\mu(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}})} I_{n-1}(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}}) \wedge f_n$$

and

$$Reord(I_{n-1}(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}}) \wedge f_n \wedge g_n) = (-1)^{\mu(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}})} I_{n-1}(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}}) \wedge f_n \wedge g_n.$$

Therefore:

$$\Psi_n(\sigma_k)(Reord(I_{n-1}(e_{i_1}\otimes\ldots\otimes e_{i_{n-1}})\wedge f_n\wedge g_n))$$
$$=(-1)^{\mu(e_{i_1}\otimes\ldots\otimes e_{i_{n-1}})}\Psi_n(\sigma_k)(I_{n-1}(e_{i_1}\otimes\ldots\otimes e_{i_{n-1}})\wedge f_n\wedge g_n)$$
$$=(-1)^{\mu(e_{i_1}\otimes\ldots\otimes e_{i_{n-1}})}\Psi_{n-1}(\sigma_k)(I_{n-1}(e_{i_1}\otimes\ldots\otimes e_{i_{n-1}}))\wedge f_n\wedge g_n$$

Moreover, given the specific form of matrix R, every term that appears in $b_R^{n-1}(\sigma_k)(e_{i_1}\otimes\ldots\otimes e_{i_{n-1}})$ has the same total number of e_1 and e_3 as $e_{i_1}\otimes\ldots\otimes e_{i_{n-1}}$. Hence:

$$Reord(I_{n-1}(b_R^{n-1}(\sigma_k)(e_{i_1}\otimes\ldots\otimes e_{i_{n-1}}))\wedge f_n\wedge g_n)$$
$$=(-1)^{\mu(e_{i_1}\otimes\ldots\otimes e_{i_{n-1}})}I_{n-1}\circ b_R^{n-1}(\sigma_k)(e_{i_1}\otimes\ldots\otimes e_{i_{n-1}})\wedge f_n\wedge g_n$$
$$=(-1)^{\mu(e_{i_1}\otimes\ldots\otimes e_{i_{n-1}})}\Psi_{n-1}(\sigma_k)\circ I_{n-1}(e_{i_1}\otimes\ldots\otimes e_{i_{n-1}})\wedge f_n\wedge g_n$$

Thus we obtain the identity in case $i_n=3$. Similar calculations show it is also true when $i_n=4$. The only remaining question is for the last generator of B_n.

<u>For σ_{n-1}</u> : We show that $\Psi_n(\sigma_{n-1})\circ I_n(e_{i_1}\otimes\ldots\otimes e_{i_n})=I_n\circ b_R^n(\sigma_{n-1})(e_{i_1}\otimes\ldots\otimes e_{i_n})$ for each of the 16 possible ordered pairs (i_{n-1},i_n):

$\boxed{(1,1)}$:

$$\Psi_n(\sigma_{n-1})(I_n(e_{i_1}\otimes\ldots\otimes e_1\otimes e_1))$$
$$=\Psi_n(\sigma_{n-1})(I_{n-2}(e_{i_1}\otimes\ldots\otimes e_{i_{n-2}})\wedge g_{n-1}\wedge g_n)$$
$$=I_{n-2}(e_{i_1}\otimes\ldots\otimes e_{i_{n-2}})\wedge(-t_0^{1/2}g_n)\wedge(-t_0^{1/2}g_{n-1}+(1-t_0)g_n)$$
$$=-t_0I_{n-2}(e_{i_1}\otimes\ldots\otimes e_{i_{n-2}})\wedge g_{n-1}\wedge g_n$$

$$I_n\circ(1\otimes 1\otimes\ldots\otimes R)(e_{i_1}\otimes\ldots\otimes e_1\otimes e_1)$$
$$=I_n(e_{i_1}\otimes\ldots\otimes e_{i_{n-2}}\otimes -t_0e_1\otimes e_1)$$
$$=-t_0I_{n-2}(e_{i_1}\otimes\ldots\otimes e_{i_{n-2}})\wedge g_{n-1}\wedge g_n$$

Now that we have explicited one case, we give the results for the remaining ones.

$\boxed{(4,4)}$: $\Psi_n(\sigma_{n-1})(I_n(e_{i_1}\otimes\ldots\otimes e_4\otimes e_4))=I_n\circ(1\otimes 1\otimes\ldots\otimes R)(e_{i_1}\otimes\ldots\otimes e_4\otimes e_4)$
$=-t_0^{-1}I_{n-2}(e_{i_1}\otimes\ldots\otimes e_{i_{n-2}})\wedge f_{n-1}\wedge f_n$

$\boxed{(2,2)}$: $\Psi_n(\sigma_{n-1})(I_n(e_{i_1}\otimes\ldots\otimes e_2\otimes e_2))=I_n\circ(1\otimes 1\otimes\ldots\otimes R)(e_{i_1}\otimes\ldots\otimes e_2\otimes e_2)$
$=I_{n-2}(e_{i_1}\otimes\ldots\otimes e_{i_{n-2}})$

$\boxed{(3,3)}: \Psi_n(\sigma_{n-1})(I_n(e_{i_1} \otimes \ldots \otimes e_3 \otimes e_3)) = I_n \circ (1 \otimes 1 \otimes \ldots \otimes R)(e_{i_1} \otimes \ldots \otimes e_3 \otimes e_3)$
$= I_{n-2}(e_{i_1} \otimes \ldots \otimes e_{i_{n-2}}) \wedge f_{n-1} \wedge f_n \wedge g_{n-1} \wedge g_n$

$\boxed{(1,2)}: \Psi_n(\sigma_{n-1})(I_n(e_{i_1} \otimes \ldots \otimes e_1 \otimes e_2)) = I_n \circ (1 \otimes 1 \otimes \ldots \otimes R)(e_{i_1} \otimes \ldots \otimes e_1 \otimes e_2)$
$= -t_0^{1/2} I_{n-2}(e_{i_1} \otimes \ldots \otimes e_{i_{n-2}}) \wedge g_n$

$\boxed{(2,1)}: \Psi_n(\sigma_{n-1})(I_n(e_{i_1} \otimes \ldots \otimes e_2 \otimes e_1)) = I_n \circ (1 \otimes 1 \otimes \ldots \otimes R)(e_{i_1} \otimes \ldots \otimes e_2 \otimes e_1)$
$= I_{n-2}(e_{i_1} \otimes \ldots \otimes e_{i_{n-2}}) \wedge (-t_0^{1/2} g_{n-1} + (1-t_0) g_n)$

$\boxed{(1,3)}: \Psi_n(\sigma_{n-1})(I_n(e_{i_1} \otimes \ldots \otimes e_1 \otimes e_3)) = I_n \circ (1 \otimes 1 \otimes \ldots \otimes R)(e_{i_1} \otimes \ldots \otimes e_1 \otimes e_3)$
$= (-1)^{\mu(e_{i_1} \otimes \ldots \otimes e_{i_{n-2}} \otimes e_1)} t_0^{1/2} I_{n-2}(e_{i_1} \otimes \ldots \otimes e_{i_{n-2}}) \wedge f_{n-1} \wedge g_{n-1} \wedge g_n$

$\boxed{(3,1)}: \Psi_n(\sigma_{n-1})(I_n(e_{i_1} \otimes \ldots \otimes e_3 \otimes e_1)) = I_n \circ (1 \otimes 1 \otimes \ldots \otimes R)(e_{i_1} \otimes \ldots \otimes e_3 \otimes e_1)$
$= (-1)^{\mu(e_{i_1} \otimes \ldots \otimes e_{i_{n-2}})} I_{n-2}(e_{i_1} \otimes \ldots \otimes e_{i_{n-2}}) \wedge ((1-t_0) f_{n-1} \wedge g_{n-1} \wedge g_n - t_0^{1/2} f_n \wedge g_{n-1} \wedge g_n)$

$\boxed{(3,4)}: \Psi_n(\sigma_{n-1})(I_n(e_{i_1} \otimes \ldots \otimes e_3 \otimes e_4)) = I_n \circ (1 \otimes 1 \otimes \ldots \otimes R)(e_{i_1} \otimes \ldots \otimes e_3 \otimes e_4)$
$= t_0^{-1/2} I_{n-2}(e_{i_1} \otimes \ldots \otimes e_{i_{n-2}}) \wedge f_{n-1} \wedge f_n \wedge g_n$

$\boxed{(4,3)}: \Psi_n(\sigma_{n-1})(I_n(e_{i_1} \otimes \ldots \otimes e_4 \otimes e_3)) = I_n \circ (1 \otimes 1 \otimes \ldots \otimes R)(e_{i_1} \otimes \ldots \otimes e_4 \otimes e_3)$
$= I_{n-2}(e_{i_1} \otimes \ldots \otimes e_{i_{n-2}}) \wedge (t_0^{-1/2} f_{n-1} \wedge f_n \wedge g_{n-1} + (1-t_0^{-1}) f_{n-1} \wedge f_n \wedge g_n)$

$\boxed{(2,4)}: \Psi_n(\sigma_{n-1})(I_n(e_{i_1} \otimes \ldots \otimes e_2 \otimes e_4)) = I_n \circ (1 \otimes 1 \otimes \ldots \otimes R)(e_{i_1} \otimes \ldots \otimes e_2 \otimes e_4)$
$= t_0^{-1/2} (-1)^{\mu(e_{i_1} \otimes \ldots \otimes e_{i_{n-2}})} I_{n-2}(e_{i_1} \otimes \ldots \otimes e_{i_{n-2}}) \wedge f_{n-1}$

$\boxed{(4,2)}: \Psi_n(\sigma_{n-1})(I_n(e_{i_1} \otimes \ldots \otimes e_4 \otimes e_2)) = I_n \circ (1 \otimes 1 \otimes \ldots \otimes R)(e_{i_1} \otimes \ldots \otimes e_4 \otimes e_2)$
$= (-1)^{\mu(e_{i_1} \otimes \ldots \otimes e_{i_{n-2}})} I_{n-2}(e_{i_1} \otimes \ldots \otimes e_{i_{n-2}}) \wedge ((1-t_0^{-1}) f_{n-1} + t_0^{-1/2} f_n)$

$\boxed{(1,4)}: \Psi_n(\sigma_{n-1})(I_n(e_{i_1} \otimes \ldots \otimes e_1 \otimes e_4)) = I_n \circ (1 \otimes 1 \otimes \ldots \otimes R)(e_{i_1} \otimes \ldots \otimes e_1 \otimes e_4)$
$= -(-1)^{\mu(e_{i_1} \otimes \ldots \otimes e_{i_{n-2}})} I_{n-2}(e_{i_1} \otimes \ldots \otimes e_{i_{n-2}}) \wedge f_{n-1} \wedge g_n$

$\boxed{(2,3)}: \Psi_n(\sigma_{n-1})(I_n(e_{i_1} \otimes \ldots \otimes e_2 \otimes e_3)) = I_n \circ (1 \otimes 1 \otimes \ldots \otimes R)(e_{i_1} \otimes \ldots \otimes e_2 \otimes e_3)$
$= (-1)^{\mu(e_{i_1} \otimes \ldots \otimes e_{i_{n-2}})} I_{n-2}(e_{i_1} \otimes \ldots \otimes e_{i_{n-2}}) \wedge (-f_{n-1} \wedge g_{n-1} - Y f_{n-1} \wedge g_n)$

$\boxed{(3,2)}: \Psi_n(\sigma_{n-1})(I_n(e_{i_1} \otimes \ldots \otimes e_3 \otimes e_2)) = I_n \circ (1 \otimes 1 \otimes \ldots \otimes R)(e_{i_1} \otimes \ldots \otimes e_3 \otimes e_2)$

$= (-1)^{\mu(e_{i_1}\otimes\ldots\otimes e_{i_{n-2}})} I_{n-2}(e_{i_1}\otimes\ldots\otimes e_{i_{n-2}}) \wedge (-Yf_{n-1}\wedge g_n - f_n\wedge g_n)$

$\boxed{(4{,}1)}: \Psi_n(\sigma_{n-1})(I_n(e_{i_1}\otimes\ldots\otimes e_4\otimes e_1)) = I_n\circ(1\otimes 1\otimes\ldots\otimes R)(e_{i_1}\otimes\ldots\otimes e_4\otimes e_1)$
$= (-1)^{\mu(e_{i_1}\otimes\ldots\otimes e_{i_{n-2}})} I_{n-2}(e_{i_1}\otimes\ldots\otimes e_{i_{n-2}}) \wedge (-Yf_{n-1}\wedge g_{n-1} - f_n\wedge g_{n-1} - Y^2 f_{n-1}\wedge g_n - Yf_n\wedge g_n)$

Which ends the proof. CQFD

2.2.2 A convenient expression for $LG^{2,1}$

Now we have built an exterior representation that is isomorphic to $b_R^n(t_0, t_0^{-1})$, we use it to write the reduction of the Links-Gould polynomial as the product of two quantities we will then identify.

Using proposition 2.2.4, we can write:

$$LG(L; t_0, t_0^{-1}) = \frac{1}{4}\, trace(\underbrace{I_n\circ(id_W\otimes\mu^{\otimes n-1})\circ I_n^{-1}}_{\tilde{\mu}}\circ\Psi_n(b)).$$

We wish to explicit $\tilde{\mu}$.

Lemma 2.2.5. *Map $\tilde{\mu}$ can be expressed on the reference basis of $\bigwedge(W_n\oplus W_n)$:*

$$\tilde{\mu}(f_{i_1}\wedge\ldots\wedge f_{i_p}\wedge g_{j_1}\wedge\ldots\wedge g_{j_m}) = t_0^{-(n-1)}(-1)^{n-1}(-1)^{\#\{k\in\{2,\ldots,n\}|f_k \text{ appears}\}}$$
$$(-1)^{\#\{k\in\{2,\ldots,n\}|g_k \text{ appears}\}} f_{i_1}\wedge\ldots\wedge f_{i_p}\wedge g_{j_1}\wedge\ldots\wedge g_{j_m}.$$

Proof. If $f_{i_1}\wedge\ldots\wedge f_{i_p}\wedge g_{j_1}\wedge\ldots\wedge g_{j_m}$ is an element of the basis of $\bigwedge(W_n\oplus W_n)$, we denote by $e_{l_1}\otimes\ldots\otimes e_{l_n}$ its image under I_n^{-1}. That way, $I_n(e_{l_1}\otimes\ldots\otimes e_{l_n}) = f_{i_1}\wedge\ldots\wedge f_{i_p}\wedge g_{j_1}\wedge\ldots\wedge g_{j_m}$.

$$\begin{aligned}\tilde{\mu}(f_{i_1}\wedge\ldots\wedge f_{i_p}\wedge g_{j_1}\wedge\ldots\wedge g_{j_m}) &= I_n\circ(id_W\otimes\mu^{\otimes n-1})(e_{l_1}\otimes\ldots\otimes e_{l_n})\\ &= t_0^{-(n-1)}(-1)^{\#\{k\in\{2,\ldots,n\}|l_k=2\}}(-1)^{\#\{k\in\{2,\ldots,n\}|l_k=3\}}\\ &\quad I_n(e_{l_1}\otimes\ldots\otimes e_{l_n})\\ &= t_0^{-(n-1)}(-1)^{\#\{k\in\{2,\ldots,n\}|l_k=2\}}\\ &\quad(-1)^{\#\{k\in\{2,\ldots,n\}|l_k=3\}} f_{i_1}\wedge\ldots\wedge f_{i_p}\wedge g_{j_1}\wedge\ldots\wedge g_{j_m}\end{aligned}$$

But:
$(-1)^{\#\{k\in\{2,\ldots,n\}|l_k=2\}}(-1)^{\#\{k\in\{2,\ldots,n\}|l_k=3\}}$
$= (-1)^{n-1}(-1)^{\#\{k\in\{2,\ldots,n\}|l_k=1\}}(-1)^{\#\{k\in\{2,\ldots,n\}|l_k=4\}}$
$= (-1)^{n-1}(-1)^{\#\{k\in\{2,\ldots,n\}|l_k=1\}}(-1)^{\#\{k\in\{2,\ldots,n\}|l_k=4\}}\left((-1)^{\#\{k\in\{2,\ldots,n\}|l_k=3\}}\right)^2$
$= (-1)^{n-1}(-1)^{\#\{k\in\{2,\ldots,n\}|l_k=3 \text{ or } l_k=4\}}(-1)^{\#\{k\in\{2,\ldots,n\}|l_k=1 \text{ or } l_k=3\}}$
$= (-1)^{n-1}(-1)^{\#\{k\in\{2,\ldots,n\}|f_k \text{ appears}\}}(-1)^{\#\{k\in\{2,\ldots,n\}|g_k \text{ appears}\}}$

This provides the result.

CQFD

Given the expression for $\tilde{\mu}$ we just obtained, and the special form of representation Ψ_n we have:

Proposition 2.2.6. *Invariant $LG(L;t_0,t_0^{-1})$ can be written as a product, with each term depending only on one of the copies of the Burau representation.*

Proof. Recall $LG(L;t_0,t_0^{-1}) = \frac{1}{4}\, trace((id_W \otimes \mu^{\otimes n-1}) \circ b_R^n(b))$, where

$$\mu = \begin{pmatrix} t_0^{-1} & . & . & . \\ . & -t_0^{-1} & . & . \\ . & . & -t_0^{-1} & . \\ . & . & . & t_0^{-1} \end{pmatrix}.$$

Using that we can write

$$\begin{aligned} LG(L;t_0,t_0^{-1}) &= \frac{1}{4}\, trace(\tilde{\mu} \circ \Psi_n(b)) \\ &= \frac{1}{4} \sum_{\substack{1\le i_1<\ldots<i_p\le n \\ 1\le j_1<\ldots<j_m\le n}} (f_{i_1}\wedge\ldots\wedge g_{j_m})^* \left(\tilde{\mu}\circ\Psi_n(b)(f_{i_1}\wedge\ldots\wedge g_{j_m})\right) \end{aligned}$$

where $(f_{i_1}\wedge\ldots\wedge g_{j_m})^*$ indicates a vector of the dual basis of the reference basis. But given

Lemma 2.2.5,

$$(f_{i_1} \wedge \ldots \wedge g_{j_m})^* \left(\tilde{\mu} \circ \Psi_n(b)(f_{i_1} \wedge \ldots \wedge g_{j_m})\right) = (-t_0)^{-(n-1)} (-1)^{\#\{k \in \{2,\ldots,n\} | f_k \text{ appears}\}} (-1)^{\#\{k \in \{2,\ldots,n\} | g_k \text{ appears}\}} (f_{i_1} \wedge \ldots \wedge g_{j_m})^* \left(\Psi_n(b)(f_{i_1} \wedge \ldots \wedge g_{j_m})\right).$$

Also, $(f_{i_1} \wedge \ldots \wedge f_{i_p} \wedge g_{j_1} \wedge \ldots \wedge g_{j_m})^* (\underbrace{\Psi_n(b)(f_{i_1} \wedge \ldots \wedge f_{i_p} \wedge g_{j_1} \wedge \ldots \wedge g_{j_m})}_{\bigwedge F(b)(f_{i_1} \wedge \ldots \wedge f_{i_p}) \wedge \bigwedge G(b)(g_{j_1} \wedge \ldots \wedge g_{j_m})})$

$$= (f_{i_1} \wedge \ldots \wedge f_{i_p})^* (\bigwedge F(b)(f_{i_1} \wedge \ldots \wedge f_{i_p}))\, (g_{j_1} \wedge \ldots \wedge g_{j_m})^* (\bigwedge G(b)(g_{j_1} \wedge \ldots \wedge g_{j_m})).$$

That way we have the following expression for $LG(L; t_0, t_0^{-1})$:

$$\frac{1}{4}\,(-t_0)^{-(n-1)} \sum_{1 \le i_1 < \ldots < i_p \le n} \left((-1)^{\#\{k \in \{2,\ldots,n\} | f_k \text{ appears}\}} (f_{i_1} \wedge \ldots \wedge f_{i_p})^* (\bigwedge F(b)(f_{i_1} \wedge \ldots \wedge f_{i_p})) \right)$$
$$* \sum_{1 \le j_1 < \ldots < j_m \le n} \left((-1)^{\#\{k \in \{2,\ldots,n\} | g_k \text{ appears}\}} (g_{j_1} \wedge \ldots \wedge g_{j_m})^* (\bigwedge G(b)(g_{j_1} \wedge \ldots \wedge g_{j_m})) \right).$$

CQFD

Now we wish to show that each of these two sums is equal to $\Delta_L(t_0)$ up to multiplication by a unit of $\mathbb{C}[t_0^{\pm 1/2}]$, that is up to multiplication by $\pm t_0^{n/2}$, $n \in \mathbb{Z}$.

2.3 Proof of the main theorem

Since we consider equality up to a unit of $\mathbb{C}[t_0^{\pm 1/2}]$, we can add a coefficient in front of the trace in the expression of the Alexander-Conway polynomial:

$$\Delta_L(t_0) \doteq t_0^{-(n-1)/2}\, \frac{1}{2}\, trace((id_V \otimes h^{\otimes n-1}) \circ \Psi_{V^{\otimes n}}(b)).$$

So we can write more simply

$$\Delta_L(t_0) \doteq \frac{1}{2}\, trace((id_V \otimes \tilde{h}^{\otimes n-1}) \circ \Psi_{V^{\otimes n}}(b)), \text{ where } \tilde{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Proposition 2.3.1. *Let* $J_n : V^{\otimes n} \longrightarrow \bigwedge W_n$, $e_{i_1} \otimes \ldots \otimes e_{i_n} \longmapsto \bigwedge_{k\,:\, i_k = 1} f_k$. *Then* J_n *is a* $\mathbb{C}[B_n]$*-module* *automorphism:*

$$\bigwedge \Psi_{W_n}(b) \circ J_n = J_n \circ \Psi_{V^{\otimes n}}(b), \forall b \in B_n.$$

The proof is quite similar to the one we did in the previous section. It is detailed in [46], appendix C, where what we just called J_n is denoted by I_n, and is introduced by induction.

Applying J_n, we can express the Alexander polynomial differently:

$$\Delta_L(t_0) \doteq \frac{1}{2}\, trace\big(\underbrace{J_n \circ (id_V \otimes \tilde{h}^{\otimes n-1}) \circ J_n^{-1}}_{\mu_1} \circ \bigwedge \Psi_{W_n}(b)\big).$$

Where

$$\begin{aligned}\mu_1(f_{i_1} \wedge \ldots \wedge f_{i_p}) &= J_n \circ (id_V \otimes \tilde{h}^{\otimes n-1})(e_0 \otimes \ldots \otimes e_0 \otimes \underbrace{e_1}_{i_1^{th} \text{ position}} \otimes e_0 \otimes \ldots \otimes e_0 \otimes \underbrace{e_1}_{i_2^{th} \text{ position}} \otimes \ldots) \\ &= (-1)^{\#\{k\in\{2,\ldots,n\} | i_k = 1\}} f_{i_1} \wedge \ldots \wedge f_{i_p} \\ &= (-1)^{\#\{k\in\{2,\ldots,n\} | f_k \text{ appears}\}} f_{i_1} \wedge \ldots \wedge f_{i_p}.\end{aligned}$$

Therefore,

$$\Delta_L(t_0) \doteq \frac{1}{2} \sum_{1\leq i_1<\ldots<i_p\leq n} \Big((-1)^{\#\{k\in\{2,\ldots,n\} | f_k \text{ appears}\}} (f_{i_1} \wedge \ldots \wedge f_{i_p})^* (\bigwedge \Psi_{W_n}(b)(f_{i_1} \wedge \ldots \wedge f_{i_p})) \Big).$$

But F and Ψ_{W_n} are identical once you change t_0 into t_0^{-1}. That way we can identify the first factor of our product with $\Delta_L(t_0^{-1})$. But the Alexander polynomial is symmetric: $\Delta_L(t_0) \doteq \Delta_L(t_0^{-1})$ [14]. So the only remaining problem is to identify the second sum with the Alexander invariant to be able to conclude. To do that we have to modify the representation of $V^{\otimes n}$ we used up to now to define $\Delta_L(t_0)$, and especially R-matrix R_1 we introduced at the beginning.

Lemma 2.3.2. *We can slightly modify R-matrix R_1 so that the new representations $\rho_{V^{\otimes n}}$ of the braid groups we obtain that way still verify*

$$\Delta_L(t_0) \doteq \frac{1}{2}\, trace((id_V \otimes \tilde{h}^{\otimes n-1}) \circ \rho_{V^{\otimes n}}(b)).$$

Proof. For the moment, we can write: $\Delta_L(t_0) \doteq \frac{1}{2}\, trace((id_V \otimes \tilde{h}^{\otimes n-1}) \circ \Psi_{V^{\otimes n}}(b))$, where $\Psi_{V^{\otimes n}}$

is the representation associated to R-matrix $R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & t_0^{1/2} & 0 \\ 0 & t_0^{1/2} & 1-t_0 & 0 \\ 0 & 0 & 0 & -t_0 \end{pmatrix}$.

We can replace R_1 by $R_2 = -t_0^{-1} R_1 = \begin{pmatrix} -t_0^{-1} & 0 & 0 & 0 \\ 0 & 0 & -t_0^{-1/2} & 0 \\ 0 & -t_0^{-1/2} & 1-t_0^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ in the definition of $\Psi_{V^{\otimes n}}$, and we will still have $\Delta_L(t_0) \overset{\bullet}{=} \frac{1}{2}\, trace((id_V \otimes \tilde{h}^{\otimes n-1}) \circ \Psi_{V^{\otimes n}}(b))$. At last, we replace t_0 by t_0^{-1} in R_2 to obtain $R_3 = \begin{pmatrix} -t_0 & 0 & 0 & 0 \\ 0 & 0 & -t_0^{1/2} & 0 \\ 0 & -t_0^{1/2} & 1-t_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. We define the representation of B_n associated with R_3:

$$\rho_{V^{\otimes n}}(\sigma_i) = id_V^{\otimes i-1} \otimes R_3 \otimes id_V^{\otimes n-i-1}.$$

Since the Alexander polynomial is symmetric, we have the following expression for $\Delta_L(t_0)$, that will help us to conclude

$$\Delta_L(t_0) \overset{\bullet}{=} \frac{1}{2}\, trace((id_V \otimes \tilde{h}^{\otimes n-1}) \circ \rho_{V^{\otimes n}}(b)).$$

CQFD

Using the same strategy as previously, we wish to find $K_n : V^{\otimes n} \longrightarrow \bigwedge W_n$ such that for any $b \in B_n : \bigwedge G(b) \circ K_n = K_n \circ \rho_{V^{\otimes n}}(b)$.

Proposition 2.3.3. *We set* $K_1 = \left\{ \begin{array}{lll} V & \longrightarrow & \bigwedge W_1 \\ e_1 & \longmapsto & 1 \\ e_0 & \longmapsto & g_1 \end{array} \right.$ *and, for* $n \geqslant 2$,

$$K_n = \left\{ \begin{array}{ccl} V^{\otimes n} & \longrightarrow & \bigwedge W_n \\ e_{i_1} \otimes \ldots \otimes e_{i_{n-1}} \otimes e_0 & \longmapsto & K_{n-1}(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}}) \wedge g_n \\ e_{i_1} \otimes \ldots \otimes e_{i_{n-1}} \otimes e_1 & \longmapsto & K_{n-1}(e_{i_1} \otimes \ldots \otimes e_{i_{n-1}}) \end{array} \right. .$$

Then, for any $b \in B_n$*:* $\bigwedge G(b) \circ K_n = K_n \circ \rho_{V^{\otimes n}}(b)$.

Proof. We leave it to the reader to verify that a proof by induction resembling the one we did with I_n concludes. CQFD

That way,

$$\begin{aligned}\Delta_L(t_0) &\doteq \frac{1}{2}\, trace((id_V \otimes \tilde{h}^{\otimes n-1}) \circ \rho_{V^{\otimes n}}(b)) \\ &= \frac{1}{2}\, trace(\underbrace{K_n \circ (id_V \otimes \tilde{h}^{\otimes n-1}) \circ K_n^{-1}}_{\nu} \circ \bigwedge G(b)).\end{aligned}$$

We can set $K_n(e_{i_1} \otimes \ldots \otimes e_{i_n}) = g_{j_1} \wedge \ldots \wedge g_{j_m}$. That allows us to explicit the values of ν on the natural basis of $\bigwedge W_n$.

$$\begin{aligned}\nu(g_{j_1} \wedge \ldots \wedge g_{j_m}) &= K_n \circ id_V \otimes \tilde{h}^{\otimes n-1}(e_{i_1} \otimes \ldots \otimes e_{i_n}) \\ &= (-1)^{\#\{k\in\{2,\ldots,n\}|i_k=1\}} K_n(e_{i_1} \otimes \ldots \otimes e_{i_n}) \\ &= (-1)^{n-1}(-1)^{\#\{k\in\{2,\ldots,n\}|i_k=0\}} g_{j_1} \wedge \ldots \wedge g_{j_m} \\ &= (-1)^{n-1}(-1)^{\#\{k\in\{2,\ldots,n\}|g_k \text{ appears}\}} g_{j_1} \wedge \ldots \wedge g_{j_m}.\end{aligned}$$

So

$$\begin{aligned}\Delta_L(t_0) &= \frac{1}{2}\, trace(\nu \circ \bigwedge G(b)) \\ &\doteq \frac{1}{2} \sum_{1\leq j_1<\ldots<j_m\leq n} \left((-1)^{\#\{k\in\{2,\ldots,n\}|g_k \text{ appears}\}} (g_{j_1} \wedge \ldots \wedge g_{j_m})^* (\bigwedge G(b)(g_{j_1} \wedge \ldots \wedge g_{j_m})) \right).\end{aligned}$$

And finally $LG(L; t_0, t_0^{-1}) \doteq \Delta_L(t_0)^2$ for any link L.

2.4 Generalizing the proof

2.4.1 Writing the conjecture using variables (t_0, t_1) and other considerations

The completely general conjecture states, using variables (τ, q):

$$LG^{m,n}(L; \tau, e^{i\pi/n}) = \Delta_L(\tau^{2n})^m \text{ , for any link } L.$$

We can rewrite it using variables (t_0, t_1). Indeed, since $q = e^{i\pi/n}$, the variables are related by $t_1^{1/2} = \tau^{-1} e^{i\pi/n}$ and $t_0^{1/2} = \tau$. Therefore, the conjecture can be expressed the following way:

$$LG^{m,n}(L; t_0, e^{2i\pi/n} t_0^{-1}) = \Delta_L(t_0^n)^m \text{ , for any link } L.$$

We can now explore in which cases it seems reasonable to attempt to generalize the strategy we used to evaluate the reduction of $LG^{2,1}$. An obvious obstruction to that concerns the dimension of both representations we built and showed they were isomorphic. Let's calculate the dimensions of the natural generalizations of these representations in case (m,n).

The vector space corresponding to what we denoted W is the highest weight $U_q(\mathfrak{gl}(m|n))$-module used to define $LG^{m,n}$. It is 2^{nm}-dimensional. So the representation of braid group B_p defined thanks to the corresponding R-matrix is 2^{nmp}-dimensional. On the other hand, the representation of B_p we want to define to produce $\Delta_L(t_0^n)^m$ is

$$\bigwedge \underbrace{(W_p \oplus \ldots \oplus W_p)}_{m \text{ times}},$$

where each W_p is a $\mathbb{C}[B_p]$-module isomorphic to a version of Ψ_{W_p} where t_0 is replaced by $t_0^{\pm n}$. Such a representation is 2^{mp}-dimensional. These two representations can not be isomorphic if $n > 1$.

That is why a straightforward use of our method can only be applied to prove cases $(m,1)$.

2.4.2 Proof of case $(3,1)$

We give the essential steps to prove the result that interests us in the case $(m,n) = (3,1)$. We follow the same ideas we used to study $LG^{2,1}$.

Theorem 2.4.1. *For any oriented link L, $LG^{3,1}(L;t_0,t_0^{-1}) \doteq \Delta_L(t_0)^3$.*

For an explicit definition of $LG^{3,1}$, see [7], p.17. The author uses variables (τ,q), but denotes $\tau = q^{-\alpha}$. We will only use the reduced version of $LG^{3,1}$. It is obtained by setting $q = -1$ and $q^{-\alpha} = t_0^{1/2}$.

Remark 2.4.2. Since we are going to set $q = -1$ in the R-matrix of [7], we have to chose precisely what the roots that are written formally are. We have chosen : $[\alpha+1]^{1/2} = q^{-1/2}[\alpha]^{1/2}$ and $[\alpha+2]^{1/2} = -[\alpha]^{1/2}$.

Definition 2.4.3. (*R-matrix S*)
Set $\mathbb{F} := \mathbb{C}(t_0^{\pm 1/2})$. Let $W =< e_1,\ldots,e_8 >$ be a 8-dimensional $\mathbb{F}$-vector space. We define S an automorphism of $W \otimes W$ as the direct sum of the following automorphisms (S is globally

multiplied by $t_0^{-3/2}$ in comparison with the R-matrix explicited in [7]):

$$\begin{pmatrix}
1 & . & . & . & . & . & . & . \\
. & -t_0^{-1} & . & . & . & . & . & . \\
. & . & -t_0^{-1} & . & . & . & . & . \\
. & . & . & -t_0^{-1} & . & . & . & . \\
. & . & . & . & t_0^{-2} & . & . & . \\
. & . & . & . & . & t_0^{-2} & . & . \\
. & . & . & . & . & . & t_0^{-2} & . \\
. & . & . & . & . & . & . & -t_0^{-3}
\end{pmatrix}$$

in basis $(e_1 \otimes e_1, e_2 \otimes e_2, e_3 \otimes e_3, e_4 \otimes e_4, e_5 \otimes e_5, e_6 \otimes e_6, e_7 \otimes e_7, e_8 \otimes e_8)$;

several copies of

$$\begin{pmatrix} 0 & t_0^{-1/2} \\ t_0^{-1/2} & 1 - t_0^{-1} \end{pmatrix}$$

in bases $(e_1 \otimes e_2, e_2 \otimes e_1)$, $(e_1 \otimes e_3, e_3 \otimes e_1)$ and $(e_1 \otimes e_4, e_4 \otimes e_1)$;

several copies of

$$t_0^{-2} \begin{pmatrix} 0 & t_0^{-1/2} \\ t_0^{-1/2} & 1 - t_0^{-1} \end{pmatrix}$$

in bases $(e_7 \otimes e_8, e_8 \otimes e_7)$, $(e_6 \otimes e_8, e_8 \otimes e_6)$ and $(e_5 \otimes e_8, e_8 \otimes e_5)$;

several copies of

$$-t_0^{-1} \begin{pmatrix} 0 & t_0^{-1/2} \\ t_0^{-1/2} & 1 - t_0^{-1} \end{pmatrix}$$

in bases $(e_2 \otimes e_5, e_5 \otimes e_2)$, $(e_3 \otimes e_5, e_5 \otimes e_3)$, $(e_2 \otimes e_6, e_6 \otimes e_2)$, $(e_4 \otimes e_6, e_6 \otimes e_4)$, $(e_3 \otimes e_7, e_7 \otimes e_3)$ and $(e_4 \otimes e_7, e_7 \otimes e_4)$;

several copies of

$$t_0^{-1} \begin{pmatrix}
. & . & . & 1 \\
. & . & 1 & t_0^{1/2} - t_0^{-1/2} \\
. & 1 & . & t_0^{1/2} - t_0^{-1/2} \\
1 & t_0^{1/2} - t_0^{-1/2} & t_0^{1/2} - t_0^{-1/2} & (t_0^{1/2} - t_0^{-1/2})^2
\end{pmatrix}$$

in bases $(e_1 \otimes e_5, e_2 \otimes e_3, e_3 \otimes e_2, e_5 \otimes e_1)$, $(e_1 \otimes e_6, e_2 \otimes e_4, e_4 \otimes e_2, e_6 \otimes e_1)$ and $(e_1 \otimes e_7, e_3 \otimes e_4, e_4 \otimes e_3, e_7 \otimes e_1)$;

several copies of

$$-t_0^{-2}\begin{pmatrix} . & . & . & 1 \\ . & . & 1 & t_0^{1/2}-t_0^{-1/2} \\ . & 1 & . & t_0^{1/2}-t_0^{-1/2} \\ 1 & t_0^{1/2}-t_0^{-1/2} & t_0^{1/2}-t_0^{-1/2} & (t_0^{1/2}-t_0^{-1/2})^2 \end{pmatrix}$$

in bases $(e_4 \otimes e_8, e_6 \otimes e_7, e_7 \otimes e_6, e_8 \otimes e_4)$, $(e_3 \otimes e_8, e_5 \otimes e_7, e_7 \otimes e_5, e_8 \otimes e_3)$ and $(e_2 \otimes e_8, e_5 \otimes e_6, e_6 \otimes e_5, e_8 \otimes e_2)$;

$$t_0^{-3/2}\begin{pmatrix} . & . & . & . & . & . & . & 1 \\ . & . & . & . & . & . & 1 & t_0^{1/2}-t_0^{-1/2} \\ . & . & . & . & . & 1 & . & t_0^{1/2}-t_0^{-1/2} \\ . & . & . & . & 1 & . & . & t_0^{1/2}-t_0^{-1/2} \\ . & . & . & 1 & . & t_0^{1/2}-t_0^{-1/2} & t_0^{1/2}-t_0^{-1/2} & (t_0^{1/2}-t_0^{-1/2})^2 \\ . & . & 1 & . & t_0^{1/2}-t_0^{-1/2} & . & t_0^{1/2}-t_0^{-1/2} & (t_0^{1/2}-t_0^{-1/2})^2 \\ . & 1 & . & . & t_0^{1/2}-t_0^{-1/2} & t_0^{1/2}-t_0^{-1/2} & . & (t_0^{1/2}-t_0^{-1/2})^2 \\ 1 & t_0^{1/2}-t_0^{-1/2} & t_0^{1/2}-t_0^{-1/2} & t_0^{1/2}-t_0^{-1/2} & (t_0^{1/2}-t_0^{-1/2})^2 & (t_0^{1/2}-t_0^{-1/2})^2 & (t_0^{1/2}-t_0^{-1/2})^2 & (t_0^{1/2}-t_0^{-1/2})^3 \end{pmatrix}$$

in basis $(e_1 \otimes e_8, e_4 \otimes e_5, e_3 \otimes e_6, e_2 \otimes e_7, e_7 \otimes e_2, e_6 \otimes e_3, e_5 \otimes e_4, e_8 \otimes e_1)$.

Then S is an R-matrix. So we can denote by b_S^n the representation of braid group B_n derived from S. It is given by the usual expression

$$b_S^n(\sigma_i) = id_W^{\otimes i-1} \otimes S \otimes id_W^{\otimes n-i-1}\ , i = 1, \dots, n-1.$$

Definition 2.4.4 (Reduced Links-Gould invariant $LG^{3,1}$)**.** Let L be any oriented link, and $b \in B_n$ be a braid with closure L. The reduced version of Links-Gould invariant $LG^{3,1}$ is given by the following formula:

$$LG^{3,1}(L; t_0, t_0^{-1}) = \frac{1}{8}\, trace((id_W \otimes \mu^{\otimes n-1}) \circ b_S^n(b))$$

where

$$\mu = t_0^{3/2} \begin{pmatrix} 1 & . & . & . & . & . & . & . \\ . & -1 & . & . & . & . & . & . \\ . & . & -1 & . & . & . & . & . \\ . & . & . & -1 & . & . & . & . \\ . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & 1 & . & . \\ . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & -1 \end{pmatrix} \in End(W).$$

We set three n-dimensional vector spaces $< f_1, \dots, f_n >, < g_1, \dots, g_n >$ and $< h_1, \dots, h_n >$ that will be all refered to as W_n. On each of them, we define a representation isomorphic to the Burau representation:

$$F(\sigma_i)(f_j) = \begin{cases} t_0^{-1/2} f_{i+1} & \text{if } j = i \ , \\ t_0^{-1/2} f_i + (1 - t_0^{-1}) f_{i+1} & \text{if } j = i+1 \ , \\ f_j & \text{otherwise.} \end{cases}$$

We designate by G and H representations on $< g_1, \dots, g_n >$ and $< h_1, \dots, h_n >$ defined by the exact same formula. Then we set Φ_n the representation of B_n on $\bigwedge(W_n \oplus W_n \oplus W_n)$ given by

$$\Phi_n := \bigwedge(F \oplus G \oplus H).$$

When $n = 2$, one can compute $\Phi_2(\sigma_1)$ and notice that its matrix is equal to S in a well chosen basis. A precise look at this basis gave us the idea to define the following map by induction. Note that retrospectively one can recover this basis simply by computing the image by our map of the basis we used to express S when $n = 2$.

Theorem 2.4.5. *We set* $I_1 = \begin{cases} V & \longrightarrow & \bigwedge W_1 \\ e_1 & \longmapsto & 1 \\ e_2 & \longmapsto & f_1 \\ e_3 & \longmapsto & g_1 \\ e_4 & \longmapsto & h_1 \\ e_5 & \longmapsto & f_1 \wedge g_1 \\ e_6 & \longmapsto & f_1 \wedge h_1 \\ e_7 & \longmapsto & g_1 \wedge h_1 \\ e_8 & \longmapsto & f_1 \wedge g_1 \wedge h_1 \end{cases}$ *and, for* $n \geqslant 2$*,*

$$I_n = \begin{cases} V^{\otimes n} & \longrightarrow & \bigwedge W_n \\ e_1 \otimes e_{i_{n-1}} \otimes \ldots \otimes e_{i_1} & \longmapsto & I_{n-1}(e_{i_{n-1}} \otimes \ldots \otimes e_{i_1}) \\ e_2 \otimes e_{i_{n-1}} \otimes \ldots \otimes e_{i_1} & \longmapsto & Reord(I_{n-1}(e_{i_{n-1}} \otimes \ldots \otimes e_{i_1}) \wedge f_n) \\ e_3 \otimes e_{i_{n-1}} \otimes \ldots \otimes e_{i_1} & \longmapsto & Reord(I_{n-1}(e_{i_{n-1}} \otimes \ldots \otimes e_{i_1}) \wedge g_n) \\ e_4 \otimes e_{i_{n-1}} \otimes \ldots \otimes e_{i_1} & \longmapsto & I_{n-1}(e_{i_{n-1}} \otimes \ldots \otimes e_{i_1}) \wedge h_n \\ e_5 \otimes e_{i_{n-1}} \otimes \ldots \otimes e_{i_1} & \longmapsto & Reord(I_{n-1}(e_{i_{n-1}} \otimes \ldots \otimes e_{i_1}) \wedge f_n \wedge g_n) \\ e_6 \otimes e_{i_{n-1}} \otimes \ldots \otimes e_{i_1} & \longmapsto & Reord(I_{n-1}(e_{i_{n-1}} \otimes \ldots \otimes e_{i_1}) \wedge f_n \wedge h_n) \\ e_7 \otimes e_{i_{n-1}} \otimes \ldots \otimes e_{i_1} & \longmapsto & Reord(I_{n-1}(e_{i_{n-1}} \otimes \ldots \otimes e_{i_1}) \wedge g_n \wedge h_n) \\ e_8 \otimes e_{i_{n-1}} \otimes \ldots \otimes e_{i_1} & \longmapsto & Reord(I_{n-1}(e_{i_{n-1}} \otimes \ldots \otimes e_{i_1}) \wedge f_n \wedge g_n \wedge h_n) \end{cases}.$$

Then the following identity holds for $n \geqslant 1$ *and* $b \in B_n$*:*

$$\Phi_n(b) \circ I_n = I_n \circ b^n_S(\hat{b}).$$

Remark 2.4.6. As in the previous sections, *Reord* refers to a reference basis of $\bigwedge(W_n \oplus W_n \oplus W_n)$ that is

$$(f_{i_1} \wedge \ldots \wedge f_{i_p} \wedge g_{j_1} \wedge \ldots \wedge g_{j_m} \wedge h_{k_1} \wedge \ldots \wedge h_{k_q})_{1 \le i_1 < \ldots < i_p \le n,\, 1 \le j_1 < \ldots < j_m \le n,\, 1 \le k_1 < \ldots < k_q \le n}.$$

Remark 2.4.7. For $b = \sigma_{i_1}^{\varepsilon_1} \ldots \sigma_{i_p}^{\varepsilon_p} \in B_n$, we define $\hat{b} := \sigma_{n-i_1}^{\varepsilon_1} \ldots \sigma_{n-i_p}^{\varepsilon_p}$. $\hat{b}$ is braid b "looked at from the other side". That way we have elementary properties : $closure(b) = closure(\hat{b})$; $\hat{\sigma_k} = \sigma_{n-k}$; for any $\sigma, \tau \in B_n$: $\widehat{\sigma\tau} = \hat{\sigma}\hat{\tau}$.

We can use I_n to express $LG^{3,1}$ differently.

$$\begin{aligned}LG^{3,1}(L;t_0,t_0^{-1}) &= \frac{1}{8}\, trace((id_W\otimes\mu^{\otimes n-1})\circ b_S^n(b))\\ &= \frac{1}{8}\, trace(\underbrace{I_n\circ(id_W\otimes\mu^{\otimes n-1})\circ I_n^{-1}}_{\tilde{\mu}}\circ\Phi_n(\hat{b})).\end{aligned}$$

Denoting as we already did several times $I_n(e_{i_n}\otimes\ldots\otimes e_{i_1}) = f_{i_1}\wedge\ldots\wedge h_{k_q}$, we can compute $\tilde{\mu}$:

$$\begin{aligned}\tilde{\mu}(f_{i_1}\wedge\ldots\wedge h_{k_q}) &= I_n\circ(id_W\otimes\mu^{\otimes n-1})(e_{i_n}\otimes\ldots\otimes e_{i_1})\\ &= t_0^{3(n-1)/2}(-1)^{\#\{k\in\{1,\ldots,n-1\}|i_k\in\{2,3,4,8\}\}}I_n(e_{i_n}\otimes\ldots\otimes e_{i_1})\\ &= t_0^{3(n-1)/2}(-1)^{\#\{k\in\{1,\ldots,n-1\}|\text{ an odd number of the following appear : }\{f_k,g_k,h_k\}\}}\\ &\quad f_{i_1}\wedge\ldots\wedge h_{k_q}\\ &= t_0^{3(n-1)/2}(-1)^{\#\{k\in\{1,\ldots,n-1\}|f_k\text{ appears}\}}(-1)^{\#\{k\in\{1,\ldots,n-1\}|g_k\text{ appears}\}}\\ &\quad(-1)^{\#\{k\in\{1,\ldots,n-1\}|h_k\text{ appears}\}}f_{i_1}\wedge\ldots\wedge h_{k_q}.\end{aligned}$$

So

$$\begin{aligned}LG^{3,1}(L;t_0,t_0^{-1}) &= \frac{1}{8}\, trace(\tilde{\mu}\circ\Phi_n(\hat{b}))\\ &= \frac{1}{8}\sum_{\substack{1\le i_1<\ldots<i_p\le n\\ 1\le j_1<\ldots<j_m\le n\\ 1\le k_1<\ldots<k_q\le n}}(f_{i_1}\wedge\ldots\wedge h_{k_q})^*(\tilde{\mu}\circ\Phi_n(\hat{b})(f_{i_1}\wedge\ldots\wedge h_{k_q})).\end{aligned}$$

But

$$\begin{aligned}&(f_{i_1}\wedge\ldots\wedge h_{k_q})^*(\tilde{\mu}\circ\Phi_n(\hat{b})(f_{i_1}\wedge\ldots\wedge h_{k_q}))\\ &= t_0^{3(n-1)/2}(-1)^{\#\{k\in\{1,\ldots,n-1\}|f_k\text{ appears}\}}(-1)^{\#\{k\in\{1,\ldots,n-1\}|g_k\text{ appears}\}}\\ &\quad(-1)^{\#\{k\in\{1,\ldots,n-1\}|h_k\text{ appears}\}}(f_{i_1}\wedge\ldots\wedge h_{k_q})^*(\Phi_n(\hat{b})(f_{i_1}\wedge\ldots\wedge h_{k_q}))\\ &= t_0^{3(n-1)/2}(-1)^{\cdots}(f_{i_1}\wedge\ldots\wedge f_{i_p})^*(\bigwedge F(\hat{b})(f_{i_1}\wedge\ldots\wedge f_{i_p}))\\ &\quad(g_{j_1}\wedge\ldots\wedge g_{j_m})^*(\bigwedge G(\hat{b})(g_{j_1}\wedge\ldots\wedge g_{j_m}))\;(h_{k_1}\wedge\ldots\wedge h_{k_q})^*(\bigwedge H(\hat{b})(h_{k_1}\wedge\ldots\wedge h_{k_q})).\end{aligned}$$

So finally

$$LG^{3,1}(L;t_0,t_0^{-1}) = t_0^{3(n-1)/2}\left(\frac{1}{2}\sum_{1\leq i_1<\ldots<i_p\leq n}(-1)^{\#\{k\in\{1,\ldots,n-1\}|f_k \text{ appears}\}}\right.$$
$$\left.(f_{i_1}\wedge\ldots\wedge f_{i_p})^*(\bigwedge F(\hat{b})(f_{i_1}\wedge\ldots\wedge f_{i_p}))\right)*\ldots$$

The only thing that remains to be shown is that each of the three terms in the product is equal to $\Delta_{\hat{b}}(t) = \Delta_b(t)$. The proof is similar to the one we did for $LG^{2,1}$. The main point is to find R-matrices associated to their representations on $V^{\otimes n}$ such that

$$\Delta_L(t) = \frac{1}{2}\,trace((id_V\otimes h^{\otimes n-1})\circ\Psi_{V^{\otimes n}}(b))$$

and that up to conjugation the trace is one of the three sums. We will not detail this argument.

2.4.3 A remark around case $(n,1)$

To prove the identity

$$LG^{n,1}(L;t_0,t_0^{-1}) \doteq \Delta_L(t_0)^n$$

when $n = 2$, 3, we have used the crucial fact that we know an explicit formula for the R-matrix and the left handle (the maps we called μ) in these two cases. Solving the conjecture for any n using the same ideas therefore requires the R-matrix to be computed in all cases. In [7], the calculations are explicit but up to $n = 4$ only. This is why it seems hard to solve the problem using these techniques when n grows larger. In the next chapter, we study the problem at the level of (super) Hopf algebras to prove all cases at the same time.

Chapter 3

Understanding the evaluations at the level of universal objects

Oleg Viro studied two interpretations of the (multivariable) Alexander polynomial as a quantum link invariant in [61]: either by considering the quasi triangular Hopf algebra associated to $U_q\mathfrak{sl}(2)$ at fourth roots of unity, or by considering super Hopf algebra $U_q\mathfrak{gl}(1|1)$. In this chapter [1], we show these Hopf algebras share properties with the -1 specialization of $U_q\mathfrak{gl}(n|1)$ leading to a complete proof of the $(n,1)$ case of the De Wit-Ishii-Links conjecture.

3.1 Hopf algebras for the Alexander polynomial

We first define a Hopf algebra U which is an essential ingredient in the study of quantum relatives of the Alexander polynomial. Unfortunately this algebra is only braided in a weak sense. Then we recall the definition of two quantum groups which can be seen as central extensions of U. One was first used by Murakami [40], both were studied by Viro in [61]. Finally we compare the braidings of these two Hopf algebras.

3.1.1 A braided Hopf algebra U

The following Hopf algebra U is a version of quantum $\mathfrak{sl}(2)$ when the quantum parameter q is chosen to be a fourth root $\mathbf{i}$ of 1. The complex algebra U is finitely presented by generators $k^{\pm 1}, e, f$ and relations

$$ke + ek = kf + fk = e^2 = f^2 = 0 \quad \text{and} \quad ef - fe = k - k^{-1}.$$

1. This chapter is based on the paper *Other quantum relatives of the Alexander polynomial through the Links-Gould invariants* written with Bertrand Patureau-Mirand [33].

The coproduct, counity and antipode of U are given by

$$\begin{aligned}
\Delta(e) &= 1\otimes e + e\otimes k, & \varepsilon(e) &= 0, & S(e) &= -ek^{-1},\\
\Delta(f) &= k^{-1}\otimes f + f\otimes 1, & \varepsilon(f) &= 0, & S(f) &= -kf,\\
\Delta(k) &= k\otimes k, & \varepsilon(k) &= 1, & S(k) &= k^{-1}.
\end{aligned}$$

This Hopf algebra can be seen in some sense as a a "double" of Bodo Pareigis' Hopf algebra [47] that would be $< k, f >$ with our notations. A pivotal structure is a group like element ϕ that induces a conjugation map equal to the square of the antipode. There is non obviously a better choice that can be made by setting $\phi = k^{-1}$.

Let $\tau : x\otimes y \mapsto y\otimes x$ be the switch of factors. Hopf algebra U is not quasi-triangular but it is braided in the sense of [51]: there exists an (outer) algebra automorphism $\mathscr{R} : U\otimes U \to U\otimes U$ different from τ that satisfies

$$\mathscr{R} \circ \Delta = \tau \circ \Delta, \tag{3.1}$$

$$\Delta_1 \circ \mathscr{R} = \mathscr{R}_{13}\mathscr{R}_{23}, \tag{3.2}$$

$$\Delta_2 \circ \mathscr{R} = \mathscr{R}_{13}\mathscr{R}_{12}. \tag{3.3}$$

Automorphism $\mathscr{R}$ admits a regular splitting (see [51]) $\mathscr{R} = \mathscr{D} \circ \mathrm{Ad}_{\check{R}}$ where $\mathrm{Ad}_{\check{R}}$ is the conjugation by the invertible element

$$\check{R} = 1 + e\otimes f$$

and $\mathscr{D}$ is an outer automorphism satisfying equations similar to (3.2) and (3.3) and defined by:

$$\mathscr{D} \circ \tau = \tau \circ \mathscr{D}\,, \quad \mathscr{D}(e\otimes 1) = e\otimes k\,, \quad \mathscr{D}(f\otimes 1) = f\otimes k^{-1} \quad \text{and} \quad \mathscr{D}(k\otimes 1) = k\otimes 1.$$

Elements $k^{\pm 2}$ generate a central Hopf subalgebra and for any $g \in \mathbb{C} \setminus \{0, 1\}$, the quotient $U/(k^2 - g)$ is a 8-dimensional semi-simple Hopf algebra with two isomorphism classes of irreducible representations $V_{\pm a}$ where $a^2 = g$. The representation V_a is 2-dimensional and can be written as follows in a certain basis (e_0, e_1):

$$k = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ a - \frac{1}{a} & 0 \end{pmatrix}. \tag{3.4}$$

Then the central element $ef + fe$ acts by $(a - a^{-1})I_2$.

3.1.2 The $\mathfrak{sl}(2)$ model and the Alexander polynomial

From [61, 5] the $\mathfrak{sl}(2)$ model is the unrolled version of quantum $\mathfrak{sl}(2)$ at $q = \mathbf{i} = \exp(\mathbf{i}\pi/2)$. It is an algebra $U_{\mathbf{i}}^H\mathfrak{sl}(2)$ generated by $K^{\pm1}, E, F, H$. Its presentation is obtained from that of U ($U_0 = \langle K^{\pm1}, E, F\rangle \simeq U$) by adding a generator H and the following relations:

$$[H, K] = 0\,, \quad [H, E] = 2E\,, \quad [H, F] = -2F.$$

We will consider the category $\mathscr{C}$ of weight modules, that is finite dimensional vector spaces where element H acts diagonally and

$$K = \mathbf{i}^H = \exp(\mathbf{i}\pi H/2). \tag{3.5}$$

The pivotal Hopf algebra structure U is extended to $U_{\mathbf{i}}^H\mathfrak{sl}(2)$ using the following relations: [2]

$$\Delta(H) = 1\otimes H + H\otimes 1 \qquad \varepsilon(H) = 0, \qquad S(H) = -H.$$

Like in the case of U, the pivotal element is $\Phi = K^{-1}$ and therefore $S^2(\cdot) = \Phi \cdot \Phi^{-1}$. With this pivotal structure, category $\mathscr{C}$ is ribbon with braiding given by the switch $\tau : x\otimes y \mapsto y\otimes x$ composed with the action of the universal R-matrix:

$$R^H = \mathbf{i}^{H\otimes H/2}(1 + E\otimes F).$$

Lemma 3.1.1. *For any two representations $V, W \in \mathscr{C}$, the conjugation by $D^H := \mathbf{i}^{H\otimes H/2}$ in $V\otimes W$ induces an automorphism $\mathscr{D}^H$ of $\mathrm{End}_{\mathbb{C}}(V\otimes W)$ that satisfies*

$$\rho_{V\otimes W} \circ \mathscr{D} = \mathscr{D}^H \circ \rho_{V\otimes W} : U\otimes U \to \mathrm{End}_{\mathbb{C}}(V\otimes W).$$

Proof. This is an easy consequence of Equation (3.5). More generally, if $x, y \in U$ satisfy $[H, x] = 2mx$ and $[H, y] = 2m'y$, then $H\otimes H.x\otimes y = x\otimes y.(H + 2m)\otimes(H + 2m')$ so

$$\begin{aligned}
\mathbf{i}^{\rho_{V\otimes W}(H\otimes H/2)}\rho_{V\otimes W}(x\otimes y) &= \rho_{V\otimes W}(x\otimes y)\mathbf{i}^{\rho_{V\otimes W}((H+2m)\otimes(H+2m')/2)}\\
&= \rho_{V\otimes W}((x\otimes K^m)(K^{m'}\otimes y))\mathbf{i}^{\rho_{V\otimes W}(H\otimes H/2)}\\
&= \rho_{V\otimes W}(\mathscr{D}(x\otimes y))\mathbf{i}^{\rho_{V\otimes W}(H\otimes H/2)}.
\end{aligned}$$

CQFD

For any complex number α that is not an odd integer, $U_{\mathbf{i}}^H\mathfrak{sl}(2)$ possesses up to isomorphism a unique two dimensional irreducible representation V_α with $\mathrm{Spec}(H) = \{\alpha + 1, \alpha - 1\}$. When restricted to U, this is representation V_a where $a = \mathbf{i}^{\alpha+1}$ and the action of H is

2. Compared to Viro, we use the opposite coproduct here.

given by $H = \begin{pmatrix} \alpha+1 & 0 \\ 0 & \alpha-1 \end{pmatrix}$.

Once it is written in representation $V_\alpha \otimes V_\beta$ with respect to basis $(e_0 \otimes e_0, e_0 \otimes e_1, e_1 \otimes e_0, e_1 \otimes e_1)$, the braiding is:

$$\mathbf{i}^{\frac{\alpha\beta-1}{2}} \begin{pmatrix} \mathbf{i}^{\frac{\alpha+\beta+2}{2}} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{i}^{\frac{\alpha-\beta}{2}} & 0 \\ 0 & \mathbf{i}^{\frac{-\alpha+\beta}{2}} & \mathbf{i}^{\frac{-\alpha+\beta}{2}}(\mathbf{i}^{\beta+1} - \mathbf{i}^{-\beta-1}) & 0 \\ 0 & 0 & 0 & \mathbf{i}^{\frac{-\alpha-\beta+2}{2}} \end{pmatrix}.$$

In the case where $\alpha = \beta$, the R-matrix then takes the particular form

$$\tau R^H = \mathbf{i}^{\frac{\alpha^2-1}{2}} \begin{pmatrix} t^{-1/2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & (t^{-1/2} - t^{1/2}) & 0 \\ 0 & 0 & 0 & -t^{1/2} \end{pmatrix}$$

where we set $t^{1/2} = \mathbf{i}^{-\alpha-1}$.

The ribbon category we consider here allows us to apply the Reshetikhin-Turaev theory [52] to construct a framed link isotopy invariant in S^3. It becomes an unframed link isotopy invariant if one divides the above R-matrix on $V_\alpha \otimes V_\alpha$ by the value of the twist $\theta_\alpha = \mathbf{i}^{\frac{\alpha^2-1}{2}}$ to obtain matrix R_1 of Subsection 1.4.3. In this particular case, the invariant we find is the Conway normalization of the classical Alexander polynomial, see [61] and Subsection 1.4.3. Recall the Reshetikhin-Turaev functor gives representations of braid groups B_ℓ

$$\begin{array}{rcl} \Psi_{V_\alpha^{\otimes \ell}} : B_\ell & \longrightarrow & GL(V_\alpha^{\otimes \ell}) \\ \sigma_i & \mapsto & \mathrm{Id}_{V_\alpha}^{\otimes i-1} \otimes \theta_\alpha^{-1} \tau R^H \otimes \mathrm{Id}_{V_\alpha}^{\otimes \ell-i-1}, \end{array}$$

where σ_i is the i^{th} standard Artin generator of braid group B_ℓ.

As we already mentioned, setting L an oriented link in S^3 obtained as closure of a braid in ℓ strands $b \in B_\ell$, we obtain the following result:

1) There exists a scalar c such that $\mathrm{trace}_{2,3,\ldots,\ell}((\mathrm{Id}_{V_\alpha} \otimes (K^{-1})^{\otimes \ell-1}) \circ \Psi_{V_\alpha^{\otimes \ell}}(b)) = c.\,\mathrm{Id}_{V_\alpha}$,

2) $L \mapsto c$ is a link invariant and is equal to the Alexander polynomial of L, $\Delta_L(t)$.

3.1.3 An example of bosonization: the $\mathfrak{gl}(1|1)$ model

Bosonization

Here we recall Majid's trick [36] to transform a super Hopf algebra into an ordinary one.

Let H be a pivotal super Hopf algebra and $\mathscr{C}$ be its even monoidal category of representations (morphisms are formed by even H-linear maps). Let H^σ be the bosonization of H: as an algebra, H^σ is the semi-direct product of H with $\mathbb{Z}/2\mathbb{Z} = \{1, \sigma\}$ where the action of σ or equivalently the commutation relations in H^σ are given by

$$\forall x \in H,\ \sigma x = (-1)^{|x|} x\sigma.$$

The coproduct Δ^σ on H^σ is given by $\Delta^\sigma \sigma = \sigma \otimes \sigma$ and

$$\forall x \in H,\ \Delta^\sigma(x) = \sum_i x_i \sigma^{|x_i'|} \otimes x_i' \text{ where } \Delta(x) = \sum_i x_i \otimes x_i'.$$

If $R = \sum_i R_i^{(1)} \otimes R_i^{(2)}$ is the universal R-matrix in H, then the following formula defines a universal R-matrix in H^σ:

$$R^\sigma = R_1 \sum_i R_i^{(1)} \sigma^{|R_i^{(2)}|} \otimes R_i^{(2)}, \text{ where } R_1 = \frac{1}{2}(1 \otimes 1 + \sigma \otimes 1 + 1 \otimes \sigma - \sigma \otimes \sigma).$$

Given a super representation $V = V_{\bar{0}} \oplus V_{\bar{1}}$ of H we get a representation of H^σ by setting $\sigma_{|V} = \mathrm{Id}_{V_{\bar{0}}} - \mathrm{Id}_{V_{\bar{1}}}$. On the other hand, since $\sigma^2 = 1$, every H^σ-module inherits a natural $\mathbb{Z}/2Z$ grading: W splits into $W = W_{\bar{0}} \oplus W_{\bar{1}}$ where we define $W_{\bar{0}} = \ker(\sigma - 1)$ and $W_{\bar{1}} = \ker(\sigma + 1)$.

Theorem 3.1.2 ([36] Theorem 4.2)**.** *The even category of super H-modules can be identified with the category of H^σ-modules.*

Note that the antipode of H^σ is given by $x \mapsto \sigma^{|x|} S(x)$. Also, if H has a pivot ϕ then one can choose $\phi^\sigma = \sigma\phi$ as a pivot in H^σ.

The $\mathfrak{gl}(1|1)$ model

Using the same notations as Viro: $U_q\mathfrak{gl}(1|1)$ is the pivotal super Hopf algebra generated by two odd generators X, Y and two even generators I, G satisfying the relations

$$XY + YX = \frac{C - C^{-1}}{q - q^{-1}}, \quad X^2 = Y^2 = 0,$$

$$[I, X] = [I, Y] = [I, G] = 0,$$

$$[G, X] = X, \quad [G, Y] = -Y,$$

where $C = q^I$, with coproduct

$$\Delta(I) = 1\otimes I + I\otimes 1\,, \quad \Delta(G) = 1\otimes G + G\otimes 1,$$

$$\Delta(X) = X\otimes C^{-1} + 1\otimes X\,, \quad \Delta(Y) = Y\otimes 1 + C\otimes Y,$$

counit

$$\varepsilon(X) = \varepsilon(Y) = \varepsilon(I) = \varepsilon(G) = 0,$$

antipode

$$S(I) = -I, S(G) = -G, S(X) = -XC, S(Y) = -YC^{-1},$$

pivot

$$\phi = K$$

and universal R-matrix

$$R = (1 + (q - q^{-1})(X\otimes Y)(C\otimes C^{-1}))q^{-I\otimes G - G\otimes I}.$$

Its bosonization $U_q\mathfrak{gl}(1|1)^\sigma$ contains a Hopf subalgebra U_1 isomorphic to U defined by setting

$$e = (q - q^{-1})X\sigma\,, \quad f = Y \quad \text{and} \quad k = C^{-1}\sigma.$$

Indeed, these elements satisfy the following:

$$ef - fe = (q - q^{-1})(X\sigma Y - YX\sigma) = (q - q^{-1})(-XY - YX)\sigma = k - k^{-1},$$

$$ke + ek = kf + fk = 0,$$

$$\Delta^\sigma(e) = (q - q^{-1})\Delta^\sigma(X\sigma) = (q - q^{-1})(X\otimes C^{-1} + \sigma\otimes X)(\sigma\otimes\sigma) = e\otimes k + 1\otimes e,$$

$$\Delta^\sigma(f) = \Delta^\sigma(Y) = Y\otimes 1 + C\sigma\otimes Y = f\otimes 1 + k^{-1}\otimes f,$$

$$\Delta^\sigma(k) = k\otimes k.$$

In the bosonization, the universal R-matrix is

$$R^\sigma = R_1 q^{-(I\otimes G + G\otimes I)}(1 + e\otimes f), \text{where } R_1 = \frac{1}{2}(1\otimes 1 + \sigma\otimes 1 + 1\otimes\sigma - \sigma\otimes\sigma).$$

Lemma 3.1.3. *Denoting $D' = q^{-I\otimes G - G\otimes I}$ and $D^\sigma = R_1 D'$ we have that for any $x, y \in U = U_1$*

$$R_1(x\otimes y)R_1^{-1} = \sigma^{|y|}x\otimes y\sigma^{|x|}\,, \quad D'(x\otimes y)(D')^{-1} = xC^{-d_G(y)}\otimes yC^{-d_G(x)},$$

$$D^\sigma(x \otimes y)(D^\sigma)^{-1} = (C^{-1}\sigma)^{d_G(y)} x \otimes y (C^{-1}\sigma)^{d_G(x)} = \mathscr{D}(x \otimes y),$$

where $d_G(x) \in \mathbb{Z}$ is defined by $[G, x] = d_G(x)x$.

Remark 3.1.4. For a homogeneous $a \in U_0$, $|a| = d_G(a)$ modulo 2.

Let us recall the expression of a family of 2-dimensional $U_q\mathfrak{gl}(1|1)^\sigma$-modules. This family is parametrized by two complex numbers (j, J) and $\varepsilon \in \{0, 1\}$, see [61]. It extends the representation V_a of U_1 where $a = (-1)^\varepsilon q^{-2j}$. Written in matrix form,

$$I = \begin{pmatrix} 2j & 0 \\ 0 & 2j \end{pmatrix}, G = \begin{pmatrix} \frac{J+1}{2} & 0 \\ 0 & \frac{J-1}{2} \end{pmatrix},$$

$$X = \begin{pmatrix} 0 & \frac{q^{2j}-q^{-2j}}{q-q^{-1}} \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \sigma = \begin{pmatrix} (-1)^\varepsilon & 0 \\ 0 & -(-1)^\varepsilon \end{pmatrix}.$$

3.1.4 Comparing the actions of R^σ and R^H

$U_0 \subset U_{\mathbf{i}}^H\mathfrak{sl}(2)$ and $U_1 \subset U_q\mathfrak{gl}(1|1)^\sigma$ are two isomorphic Hopf algebras. The goal of this paragraph is to show the action of

$$R^H = \mathbf{i}^{H \otimes H/2}(1 + E \otimes F) \in U_{\mathbf{i}}^H\mathfrak{sl}(2) \otimes U_{\mathbf{i}}^H\mathfrak{sl}(2)$$

and that of

$$R^\sigma = R_1 q^{-(I \otimes G + G \otimes I)}(1 + e \otimes f) \in U_q\mathfrak{gl}(1|1)^\sigma \otimes U_q\mathfrak{gl}(1|1)^\sigma$$

on two representations $V_1^H \otimes V_2^H$ of $U_{\mathbf{i}}^H\mathfrak{sl}(2)$ and $V_1^\sigma \otimes V_2^\sigma$ of $U_q\mathfrak{gl}(1|1)^\sigma$ are identical up to a scalar multiple of the identity, when V_i^H and V_i^σ have the same underlying $U_0 = U_1$-module structure.

We recall conjugations by $D^H = \mathbf{i}^{H \otimes H/2}$ in $V_1^H \otimes V_2^H$ and D^σ in $V_1^\sigma \otimes V_2^\sigma$ induce the same automorphism $\mathscr{D}$ of $U \otimes U$.

Proposition 3.1.5. *For $i = 1, 2$ let V_i^H be a representation of $U_{\mathbf{i}}^H\mathfrak{sl}(2)$ and V_i^σ be a representation of $U_q\mathfrak{gl}(1|1)^\sigma$ that both restrict to the same irreducible representation of $U = U_0 = U_1$. Then $D^H(D^\sigma)^{-1} \in \mathrm{End}_\mathbb{C}(V_1 \otimes V_2)$ is a scalar multiple of the identity.*

Proof. The density theorem states that if V is a finite dimensional irreducible representation of an algebra A over an algebraically closed field, then $A \twoheadrightarrow \mathrm{End}(V)$ is surjective. Denote the representations at hand $\rho_{V_i^H}, \rho_{V_i^\sigma}$ for $i = 1, 2$. We supposed

$$\rho_{V_i^H}|_U = \rho_{V_i^\sigma}|_U.$$

So if $\rho_H = \rho_{V_1^H} \otimes \rho_{V_2^H}$ and $\rho_\sigma = \rho_{V_1^\sigma} \otimes \rho_{V_2^\sigma}$ we define $\rho := \rho_{H\,|U\otimes U} = \rho_{\sigma\,|U\otimes U}$. Using Lemma 3.1.1 and Lemma 3.1.3, for any $x, y \in U$:

$$\rho_H\big(D^H\big)\rho(x\otimes y)\rho_H\big((D^H)^{-1}\big) = \rho\big(\mathscr{D}(x\otimes y)\big) = \rho_\sigma\big(D^\sigma\big)\rho(x\otimes y)\rho_\sigma\big((D^\sigma)^{-1}\big).$$

Which means

$$\rho_H\big(D^H\big)^{-1}\rho_\sigma\big(D^\sigma\big)\rho(x\otimes y) = \rho(x\otimes y)\rho_H\big(D^H\big)^{-1}\rho_\sigma\big(D^\sigma\big).$$

Using the density theorem, $\rho_H\big(D^H\big)^{-1}\rho_\sigma\big(D^\sigma\big)$ commutes with any element in $\mathrm{End}_{\mathbb{C}}(V_1)\otimes \mathrm{End}_{\mathbb{C}}(V_2) = \mathrm{End}_{\mathbb{C}}(V_1\otimes V_2)$. So this linear map is a scalar multiple of the identity.

CQFD

From now on, we consider Hopf algebra $A = U_{\mathbf{i}}^H\mathfrak{sl}(2) \otimes_U U_q\mathfrak{gl}(1|1)^\sigma$. That algebra contains both algebras $U_{\mathbf{i}}^H\mathfrak{sl}(2)$ and $U_q\mathfrak{gl}(1|1)^\sigma$.

Formally, setting $q = e^h$, $q^T := e^{hT}$ and $\mathbf{i}^\alpha = e^{\mathbf{i}\frac{\pi}{2}\alpha}$, we also consider that

$$\mathbf{i}^H = k = q^{-I}\sigma$$

which means that we will only study representations of A that satisfy this relation. Recall from Equations (3.4) the representation of U with parameter a. We can look for the representations of A that simultaneously extend to the representations of $U_{\mathbf{i}}^H\mathfrak{sl}(2)$ and $U_q\mathfrak{gl}(1|1)^\sigma$ we already described. If $\varepsilon \in \{0,1\}$ is the degree of the first vector e_0 of the basis (e_0, e_1) we choose, direct computation of such a representation $V(\alpha, a, 2j, \varepsilon, J)$ shows it is well defined if and only if:

$$\begin{cases} (-1)^\varepsilon q^{-2j} = a \\ a = e^{\mathbf{i}\frac{\pi}{2}(\alpha+1)} = \mathbf{i}^{\alpha+1} \end{cases} \tag{3.6}$$

Setting $s = q^j\mathbf{i}^{\frac{\alpha-3-2\varepsilon}{2}} = \pm 1$, we can compute the coefficient $R^H/R^\sigma = D^H/D^\sigma$ given by Proposition 3.1.5 in our case.

Proposition 3.1.6. $R^H/R^\sigma = D^H/D^\sigma = ss'(-1)^{\varepsilon\varepsilon'}\mathbf{i}^{\varepsilon+\varepsilon'}\mathbf{i}^{\frac{\alpha\alpha'-1}{2}}q^{jJ'+j'J}$.

Proof. Using representation $V\otimes V' = V(\alpha, a, 2j, \varepsilon, J)\otimes V(\alpha', a', 2j', \varepsilon', J')$ in basis $(e_0\otimes e_0, e_0\otimes e_1, e_1\otimes e_0, e_1\otimes e_1)$, we can write:

$$D^H = \mathbf{i}^{\alpha\alpha'/2}\begin{pmatrix} \mathbf{i}^{\frac{\alpha+\alpha'+1}{2}} & 0 & 0 & 0 \\ 0 & \mathbf{i}^{\frac{-\alpha+\alpha'-1}{2}} & 0 & 0 \\ 0 & 0 & \mathbf{i}^{\frac{\alpha-\alpha'-1}{2}} & 0 \\ 0 & 0 & 0 & \mathbf{i}^{\frac{-\alpha-\alpha'+1}{2}} \end{pmatrix}.$$

Moreover, $D^\sigma = R_1 D'$ and

$$R_1 = (-1)^{\varepsilon\varepsilon'} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (-1)^{\varepsilon} & 0 & 0 \\ 0 & 0 & (-1)^{\varepsilon'} & 0 \\ 0 & 0 & 0 & (-1)^{\varepsilon+\varepsilon'+1} \end{pmatrix},$$

$$D' = q^{-jJ'-j'J} \begin{pmatrix} q^{-j-j'} & 0 & 0 & 0 \\ 0 & q^{j-j'} & 0 & 0 \\ 0 & 0 & q^{-j+j'} & 0 \\ 0 & 0 & 0 & q^{j+j'} \end{pmatrix}.$$

Since $a = \mathbf{i}^\alpha = (-1)^{\varepsilon+1}\mathbf{i}q^{-2j}$, the formulas make two square roots of a appear:

$$\sqrt[\alpha]{a} = \mathbf{i}^{\alpha/2} \text{ and } \sqrt[j]{a} = \mathbf{i}^{\varepsilon+\frac{3}{2}}q^{-j} = s\sqrt[\alpha]{a}.$$

That way, one can compute any of the diagonal coefficients of $D^H(D^\sigma)^{-1}$ to end the proof.
CQFD

3.2 An integral form of $U_q\mathfrak{gl}(n|1)$ and its specialization

3.2.1 Quasitriangular Hopf superalgebra $U_q\mathfrak{gl}(n|1)$

Here we define the h-adic quasitriangular Hopf superalgebra $U_q\mathfrak{gl}(n|1)$ that we will use to construct the Links-Gould invariant $LG^{n,1}$. The conventions we use for generators and relations are those chosen by Zhang and De Wit in [63, 8]. $\mathcal{I} = \{1, 2, \ldots, n+1\}$ will be the set of indices. We introduce a grading $[a] \in \mathbb{Z}/2\mathbb{Z}$ for any $a \in \mathcal{I}$ by setting

$$[a] = 0 \text{ if } a \leqslant n \text{ and } [a] = 1 \text{ when } a = n+1.$$

The superalgebra has $(n+1)^2$ generators divided into three families. There are $n+1$ even Cartan generators E_a^a. There are $\frac{1}{2}n(n+1)$ lowering generators E_a^b parametrized by $a < b$. Finally there are $\frac{1}{2}n(n+1)$ raising generators E_b^a, with $a < b$. The degree of E_a^b is given by $[a] + [b]$.

For $a \in \mathcal{I}$, $a \neq n+1$, set $K_a = q^{E_a^a}$, and set $K_{n+1} = q^{-E_{n+1}^{n+1}}$. In the following $[X, Y]$ denotes the super commutator $[X, Y] = XY - (-1)^{[X][Y]}YX$.

Now let us present the relations there are between elements of $U_q\mathfrak{gl}(n|1)$.

For any $a, b \in \mathcal{I}$ with $|a-b| \geq 2$ and for any c in the interval between a and b,

$$E_b^a = E_c^a E_b^c - q^{\mathrm{sign}(a-b)} E_b^c E_c^a.$$

For any $a, b \in \mathcal{I}$,

$$E_a^a E_b^b = E_b^b E_a^a,\ E_a^a E_{b\pm 1}^b = E_{b\pm 1}^b \left(E_a^a + \delta_b^a - \delta_{b\pm 1}^a\right)$$

$$[E_{a+1}^a, E_b^{b+1}] = \delta_b^a \frac{K_a K_{a+1}^{-1} - K_a^{-1} K_{a+1}}{q-q^{-1}}$$

$$\text{which generalizes for } a<b \text{ to } [E_b^a, E_a^b] = \frac{K_a K_b^{-1} - K_a^{-1} K_b}{q-q^{-1}},$$

$$(E_{n+1}^n)^2 = (E_n^{n+1})^2 = 0, \text{ which implies } (E_{n+1}^i)^2 = (E_i^{n+1})^2 = 0 \text{ for } i < n+1.$$

The Serre relations: for any $a, b \in \mathcal{I}$ with $|a-b| \geq 2$,

$$E_a^{a+1} E_b^{b+1} = E_b^{b+1} E_a^{a+1},\ E_{a+1}^a E_{b+1}^b = E_{b+1}^b E_{a+1}^a,$$

and for $a \leq n-1$,

$$E_{a+1}^a E_{a+2}^a = q E_{a+2}^a E_{a+1}^a\,, \quad E_a^{a+1} E_a^{a+2} = q E_a^{a+2} E_a^{a+1}\,,$$

$$E_{a+2}^a E_{a+2}^{a+1} = q E_{a+2}^{a+1} E_{a+2}^a \quad \text{and} \quad E_a^{a+2} E_{a+1}^{a+2} = q E_{a+1}^{a+2} E_a^{a+2}.$$

These relations can be completed into a set of "quasi-commutation" relations indexed by pairs of root vectors (see [8, Lemma 1] where a reordering algorithm gives a constructive proof of the Poincaré-Birkhoff-Witt theorem) but these relations are redundant over the field $\mathbb{C}(q)$.

We consider the Hopf algebra structure given by the coproduct

$$\Delta(E_{a+1}^a) = E_{a+1}^a \otimes K_a K_{a+1}^{-1} + 1 \otimes E_{a+1}^a\,, \quad \Delta(E_a^{a+1}) = K_a^{-1} K_{a+1} \otimes E_a^{a+1} + E_a^{a+1} \otimes 1$$

$$\Delta(K_a) = K_a \otimes K_a \quad \text{and} \quad \Delta(E_a^a) = E_a^a \otimes 1 + 1 \otimes E_a^a$$

which admits [3] the universal R-matrix $R^{\mathfrak{gl}} = D^{\mathfrak{gl}} \check{R}^{\mathfrak{gl}}$ with $D^{\mathfrak{gl}} = q^{\sum_{i\leq n} E_i^i \otimes E_i^i - E_{n+1}^{n+1} \otimes E_{n+1}^{n+1}}$ and

$$\check{R}^{\mathfrak{gl}} = \prod_{i=1}^{n} \left(\prod_{j=i+1}^{n} \mathrm{e}_q((q-q^{-1}) E_j^i \otimes E_i^j) \right) \mathrm{e}_q'(E_{n+1}^i \otimes E_i^{n+1}),$$

3. we use here the coproduct and R-matrix of [29] conjugated by $D^{\mathfrak{gl}}$.

where $\mathrm{e}'_q(x) = (1 - (q - q^{-1})x)$, $\mathrm{e}_q(x) = \sum_{k=0}^{+\infty} \frac{x^k}{(k)_q!}$, $(k)_q = \frac{1-q^k}{1-q}$ and $(k)_q! = (1)_q(2)_q \dots (k)_q$. Do note that the order in which the factors are written in $\check{R}^{\mathfrak{gl}}$ matters.

3.2.2 Integral form and interesting subalgebras

We now give an integral form of $U_q\mathfrak{gl}(n|1)$ that supports evaluation at $q = -1$. Let $\mathcal{A}_q$ be the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_q\mathfrak{gl}(n|1)$ generated by elements K_a, $\mathcal{E}^a_b := \left(q - q^{-1}\right) E^a_b$ when $a < b$ and $\mathcal{E}^a_b := E^a_b$ when $a > b$. The relations of $U_q\mathfrak{gl}(n|1)$

$$[E^a_b, E^b_a] = \frac{K_a K_b^{-1} - K_a^{-1} K_b}{q - q^{-1}}$$

for $a < b$, are replaced in algebra $\mathcal{A}_q$ by

$$[\mathcal{E}^a_b, \mathcal{E}^b_a] = K_a K_b^{-1} - K_a^{-1} K_b.$$

Still, $\mathcal{A}_q$ admits a presentations similar to that of $U_q\mathfrak{gl}(n|1)$. No additional relations are needed because the analog of the above commutation relations are enough to express any element in the Poincaré-Birkhoff-Witt basis.

In the bosonization $\mathcal{A}_q^\sigma$ of $\mathcal{A}_q$, define for $i = 1, \dots, n$ the algebra

$$A_i = \left\langle e_i = -\mathcal{E}^i_{n+1}\sigma,\ f_i = \mathcal{E}^{n+1}_i,\ k_i = K_i K_{n+1}^{-1}\sigma \right\rangle \subset \mathcal{A}_q^\sigma.$$

Proposition 3.2.1. *Algebra A_i is isomorphic to U. Indeed:*

$$e_i f_i - f_i e_i = k_i - k_i^{-1},$$

$$k_i e_i + e_i k_i = k_i f_i + f_i k_i = 0.$$

Proof. Direct computations from the defining relations of $\mathcal{A}_q$ and Lemma 1 of [8]. In particular, $e_i f_i - f_i e_i = -\mathcal{E}^i_{n+1}\sigma\mathcal{E}^{n+1}_i + \mathcal{E}^{n+1}_i\mathcal{E}^i_{n+1}\sigma = [\mathcal{E}^i_{n+1}, \mathcal{E}^{n+1}_i]\sigma = k_i - k_i^{-1}$. CQFD

Remark 3.2.2. However, A_i is not isomorphic to U as a Hopf algebra (except for A_n), which can be seen by looking at the coproduct of elements of A_i in $\mathcal{A}_q$. This will not be a problem for us.

Set $1 \leqslant i \neq j \leqslant n$. Using [8] Lemma 1 once again, we want to see at what conditions any $x \in A_i$ and $y \in A_j$ commute.

Lemma 3.2.3. *We have the following commutations:*

$$e_ie_j = -q^{-1}e_je_i,\ f_if_j = -q^{-1}f_jf_i,\ k_ik_j = k_jk_i,$$

$$\text{if } i<j,\ e_if_j - f_je_i = \sigma K_jK_{n+1}^{-1}\mathcal{E}^i_j,\ \text{otherwise } e_if_j - f_je_i = \sigma\left(q-q^{-1}\right)\mathcal{E}^i_jK_{n+1}K_i^{-1},$$

$$k_je_i = -q^{-1}e_ik_j,\ k_jf_i = -qf_ik_j.$$

Proof. The first two equalities correspond to [8, Eq. (38) and (39)] and the two brackets $[e_i, f_j]$ correspond to [8, Eq. (36) (c) and (d)]. CQFD

Corollary 3.2.4. *Setting $q=-1$, in any quotient of $\mathcal{A}^\sigma_{-1}$ where for any $1\leqslant i<j\leqslant n$, $\mathcal{E}^i_j = 0$, the elements of two distinct A_i commute.*

3.2.3 Highest weight representation $V(0^n,\alpha)$

Let $V(0^n,\alpha)$ be the highest weight irreducible 2^n-dimensional representation of $U_q\mathfrak{gl}(n|1)$ of weight $(0^n,\alpha)$, with $\alpha\notin\mathbb{Z}$. So E^i_i is represented by 0, except for E^{n+1}_{n+1} that is represented by α. Set v_0 a highest weight vector in $V(0^n,\alpha)$ and let $V_q(0^n,\alpha) = \mathcal{A}_qv_0$. The Poincaré-Birkhoff-Witt theorem proves that

$$\left(\prod_{i=1}^n f_i^{m_i}v_0\right)_{m_i\in\{0,1\}}$$

is a basis for vector space $V(0^n,\alpha)$ and for the free $\mathbb{Z}[q,q^{-1}]$-module $V_q(0^n,\alpha)$. Set $\mathcal{A}^\sigma_{-1} = \mathcal{A}^\sigma_q\otimes_{q=-1}\mathbb{C}$ and $V_{-1}(0^n,\alpha) = V_q(0^n,\alpha)\otimes_{q=-1}\mathbb{C}$

Proposition 3.2.5. *In the representation $V_{-1}(0^n,\alpha)$, $\mathcal{E}^i_j = 0$ for any $1\leqslant i<j\leqslant n$. So $\mathcal{E}^i_j$ belongs to the kernel I of the representation $\mathcal{A}^\sigma_{-1}\longrightarrow \mathrm{End}(V_{-1}(0^n,\alpha))$. As a consequence, the following map is well defined:*

$$\begin{array}{rccc}\Theta: & \bigotimes_{i=1}^n A_i & \longrightarrow & \mathcal{A}^\sigma_{-1}/I \\ & \otimes_i x_i & \mapsto & \prod_i x_i\end{array}.$$

Proof. We want to show that for any basis vector $v\in V_{-1}(0^n,\alpha)$ and for $1\leqslant i<j\leqslant n$, $\mathcal{E}^i_jv = 0$. We can write $v = f_1^{i_1}\dots f_n^{i_n}v_0$ where $i_k = 0,1$.
Using [8] Lemma 1 once more, if

$c<i$ then $[E^i_j, E^{n+1}_c] = 0$ by [8, Eq. (40)]
$c=i$ then $[E^i_j, E^{n+1}_c] = -K_iK_j^{-1}E^{n+1}_j$ by [8, Eq. (36)(a)]
$i<c<j$ then $[E^i_j, E^{n+1}_c] = -(q-q^{-1})K_cK_j^{-1}E^i_cE^{n+1}_j$ by [8, Eq. (43)(a)]
$j\leq c$ then $[E^i_j, E^{n+1}_c] = 0$ by [8, Eq. (37),(40)].

In all cases, $[\mathcal{E}^i_j, f_c] = [\mathcal{E}^i_j, \mathcal{E}^{n+1}_c] = (q-q^{-1})[E^i_j, E^{n+1}_c] = 0$ in $\mathcal{A}^\sigma_{-1}$. So $\mathcal{E}^i_jv = f_1^{i_1}\dots f_n^{i_n}(\mathcal{E}^i_jv_0)$.

But $\mathcal{E}^i_j$ is a raising generator, so $\mathcal{E}^i_j v_0 = 0$. Using Corollary 3.2.4, for $i \neq j$ A_i and A_j commute in that representation. CQFD

3.2.4 $\check{R}^{\mathfrak{gl}}$ makes sense when $q = -1$

Here we intend to show that the non diagonal part $\check{R}^{\mathfrak{gl}}$ of the universal R-matrix of $U_q\mathfrak{gl}(n|1)$ supports evaluation at $q = -1$, which is not obvious given the formula defining $\check{R}^{\mathfrak{gl}}$. In the bosonization $U_q\mathfrak{gl}(n|1)^\sigma$, the universal R-matrix is given by

$$(R^{\mathfrak{gl}})^\sigma = D^{\mathfrak{gl}}(\check{R}^{\mathfrak{gl}})^\sigma = D^{\mathfrak{gl}}\prod_{i=1}^{n}\left(\prod_{j=i+1}^{n} \mathrm{e}_q(\mathcal{E}^i_j \otimes \mathcal{E}^j_i)\right)(1 + e_i \otimes f_i).$$

Proposition 3.2.6. *For any* $1 \leqslant i < j \leqslant n$*,*

$$\left(\mathrm{e}_q(\mathcal{E}^i_j \otimes \mathcal{E}^j_i) - 1\right) V_q(0^n, \alpha) \otimes V_q(0^n, \alpha) \subset (q+1)\mathbb{Z}[q, q^{-1}]_{loc} V_q(0^n, \alpha) \otimes V_q(0^n, \alpha)$$

where $\mathbb{Z}[q, q^{-1}]_{loc}$ *is the localization of* $\mathbb{Z}[q, q^{-1}]$ *at* $(q+1)$*. Hence* $(R^{\mathfrak{gl}})^\sigma$ *induces a well defined automorphism of* $V_{-1}(0^n, \alpha) \otimes V_{-1}(0^n, \alpha)$ *where the action of* $(\check{R}^{\mathfrak{gl}})^\sigma$ *is given by*

$$(\check{R}^{\mathfrak{gl}})^\sigma = \prod_{i=1}^{n}(1 + e_i \otimes f_i).$$

Proof. Define $V = \mathbb{Z}[q, q^{-1}]_{loc} V_q(0^n, \alpha) \subset V(0^n, \alpha)$ so that $V_{-1}(0^n, \alpha) \cong V \otimes_{q=-1} \mathbb{C}$. We wish to prove that for $1 \leqslant i < j \leqslant n$, in the representation $V \otimes V$, $\mathrm{e}_q(\mathcal{E}^i_j \otimes \mathcal{E}^j_i) = 1 \bmod (q+1)$. Set $1 \leqslant i < j \leqslant n$. We show by induction on $k \geqslant 1$, that

$$\frac{(\mathcal{E}^i_j)^k}{(k)_q!} V \subset (q+1)V.$$

For $k = 1$, this follows from $\mathcal{E}^i_j \in I$ (see Proposition 3.2.5). Now let us suppose that the result holds for any $l \in \{1, \ldots, k-1\}$. Since $\frac{(\mathcal{E}^i_j)^k}{(k)_q!} = \frac{(\mathcal{E}^i_j)^{k-1}}{(k-1)_q!}\frac{\mathcal{E}^i_j}{(k)_q}$ it is enough to show that $\frac{\mathcal{E}^i_j}{(k)_q} V \subset V$.

We know that $\mathcal{E}^i_j V \subset (q+1)V$, so $\frac{\mathcal{E}^i_j}{(k)_q} V \subset \frac{q+1}{(k)_q} V$.

If k is even, $(k)_q = (q+1)(\frac{k}{2})_{q^2}$ with $(\frac{k}{2})_{q^2} = \frac{k}{2} \bmod (q+1)$ so $\dfrac{\mathcal{E}^i_j}{(k)_q} V \subset \dfrac{1}{(\frac{k}{2})_{q^2}} V = V$. If k is odd, $(k)_q = 1 \bmod (q+1)$ and therefore $\dfrac{\mathcal{E}^i_j}{(k)_q} V \subset (q+1)V$. This concludes the proof. CQFD

3.3 Links-Gould invariants and the conjecture

3.3.1 Links-Gould invariants $LG^{n,1}$

The Links-Gould invariants $LG^{n,1}$ are the framed link invariants obtained by applying the modified (one has to use a modified trace, see [19]) Reshetikhin-Turaev construction to the ribbon Hopf algebras $U_q\mathfrak{gl}(n|1)^\sigma$ we just studied. Like in the Alexander case, the R-matrix can be divided by the value of the twist so that $LG^{n,1}$ becomes an *unframed* link invariant. Note that this definition and Viro's work [61] show that the first LG invariant $LG^{1,1}$ coincides with the Alexander-Conway polynomial Δ.

Here we are interested in what happens to $LG^{n,1}$ when you evaluate q at -1, or in other words when you set $t_0t_1 = 1$.

3.3.2 Proof of the conjecture

Our study of ribbon Hopf algebra $U_q\mathfrak{gl}(n|1)^\sigma$ allows us to prove the $(n,1)$ case of the De Wit-Ishii-Links conjecture:

Theorem 3.3.1. *For any link L in S^3, $LG^{n,1}(L;\tau,-1) = \Delta_L(\tau^2)^n$. This can be translated in variables (t_0,t_1):*

$$LG^{n,1}(L;t_0,t_0^{-1}) = \Delta_L(t_0)^n.$$

Remark 3.3.2. Here we prove an equality, and not only an equality up to an ivertible element.

The rest of the section is devoted to proving this identity. First we identify $V_{-1}(0^n,\alpha)$ as a $\otimes_i A_i$-module:

Proposition 3.3.3. *Let us denote by V^i an A_i-module isomorphic to the 2-dimensional U-module $V_{q^{-\alpha}}$. Equipped with the action of $\otimes_i A_i$ induced by $\Theta : \otimes_i A_i \to \mathcal{A}^\sigma_{-1}/I$, representation $V_{-1}(0^n,\alpha)$ is isomorphic to the irreducible representation $\otimes_i V^i$.*

Proof. By $\otimes_i V^i$, we mean the representation

$$\otimes_i\rho_i : \otimes_i A_i \to \otimes_i \operatorname{End}_{\mathbb{C}}(V^i) \cong \operatorname{End}_{\mathbb{C}}(\otimes_i V^i) \text{ where } \rho_i : A_i \to \operatorname{End}_{\mathbb{C}}(V^i).$$

Set $a = q^{-\alpha}$. For each i, k_i^2 acts by a^2 on $V_{-1}(0^n,\alpha)$. Thus $V_{-1}(0^n,\alpha)$ is a representation of the 8^n-dimensional semi-simple algebra $\bigotimes_{i=1}^j \left(A_i/(k_i^2-a^2)\right)$. But for each A_i, v_0 is a highest weight vector of weight a. So it belongs to a summand of the $\bigotimes_{i=1}^j \left(A_i/(k_i^2-a^2)\right)$-module $V_{-1}(0^n,\alpha)$ of the form $\otimes_i V^i$. Comparing the dimensions which are equal to 2^n for both vector spaces, we have that $V_{-1}(0^n,\alpha) \simeq \otimes_i V^i$. CQFD

Now we study the action of the pivotal element of $\mathcal{A}_q^\sigma$ in the representation at $q = -1$.

Proposition 3.3.4. *If $K_{2\rho}^\sigma$ is the pivotal element of $\mathcal{A}_q^\sigma$, in the representation $V_{-1}(0^n, \alpha)$,*

$$K_{2\rho}^\sigma = \Theta(\otimes_i \phi_i)$$

where $\phi_i = k_i^{-1} \in A_i$.

Proof. The antipode of $U_q\mathfrak{gl}(n|1)$ satisfies $S(E_{i+1}^i) = -E_{i+1}^i K_{i+1} K_i^{-1}$ and $S^2(E_{i+1}^i) = K_i K_{i+1}^{-1} E_{i+1}^i K_{i+1} K_i^{-1} = K_{2\rho} E_{i+1}^i K_{2\rho}^{-1}$. We can write $K_{2\rho}$ in terms of Cartan generators:

$$K_{2\rho} = K_{n+1}^n \prod_{i=1}^n K_i^{n-2i}.$$

Denoting $\langle a|b\rangle := \sum_{i=1}^n a_i b_i - a_{n+1} b_{n+1}$, and ρ the graded half sum of all positive roots, we find:

$$2\rho = \sum_{i=1}^n (n-2i)\varepsilon_i + n\varepsilon_{n+1},$$

where ε_i is the i^{th} basis vector of $\mathbb{C}^{n+1}$ and we write any vector $x = \sum_{i=1}^{n+1} x_i \varepsilon_i$ in this basis. $K_{2\rho}$ conjugates element $e_i \in A_i$ as follows:

$$\begin{aligned} K_{2\rho} e_i K_{2\rho}^{-1} &= q^{\langle 2\rho|\varepsilon_i - \varepsilon_{n+1}\rangle} e_i \\ &= q^{(n-2i+n)} e_i \\ &= q^{2n-2i} e_i. \end{aligned}$$

So if $q = -1$,

$$\begin{aligned} \sigma K_{2\rho} e_i K_{2\rho}^{-1} \sigma &= -e_i \\ &= \phi_i e_i \phi_i^{-1} \\ &= \Theta(\otimes_j \phi_j) e_i \Theta(\otimes_j \phi_j^{-1}). \end{aligned}$$

Similarly to Proposition 3.1.5, we therefore can say that in the irreducible $\otimes_i A_i$-module $V_{-1}(0^n, \alpha)$, $K_{2\rho}^\sigma$ is a scalar multiple of $\Theta(\otimes_j \phi_j)$. We call this element λ. Since the two maps both act by $q^{n\alpha}$ on the highest weight vector, we find that $\lambda = 1$. CQFD

Proposition 3.3.5. *For any $x \in A_i \otimes A_i \subset \mathcal{A}_q \otimes \mathcal{A}_q$, we have*

$$D^{\mathfrak{gl}} x (D^{\mathfrak{gl}})^{-1} = \mathscr{D}(x)$$

where we identified $A_i \otimes A_i \cong U \otimes U$.

Proof. By a direct computation,

$$D^{\mathfrak{gl}} E^j_{n+1} \otimes 1 = E^j_{n+1} \otimes 1 q^{\sum_{i\leq n}((E^i_i+\delta^i_j)\otimes E^i_i-(E^{n+1}_{n+1}-1)\otimes E^{n+1}_{n+1}} = E^j_{n+1} \otimes K_j K^{-1}_{n+1} D^{\mathfrak{gl}}$$

Thus $D^{\mathfrak{gl}} e_j \otimes 1 (D^{\mathfrak{gl}})^{-1} = e_j \otimes k_j$. Similarly $D^{\mathfrak{gl}} f_j \otimes 1 (D^{\mathfrak{gl}})^{-1} = f_j \otimes k_j^{-1}$. Finally $k_i \otimes 1$ clearly commutes with $D^{\mathfrak{gl}}$ and we can conclude using $\tau \circ D^{\mathfrak{gl}} = D^{\mathfrak{gl}} \circ \tau$. CQFD

Proof of Theorem 3.3.1. Let us sum up what we proved up to now to obtain 3.3.1. Let $V_{-1}(0^n, \alpha) \simeq \bigotimes_{i=1}^n V^i$ be the isomorphic representations of Proposition 3.3.3. In the following we fix such an isomorphism. Let V^i_H be a $U^H_{\mathbf{i}}\mathfrak{sl}(2)$-module structure on V^i extending the representation of A_i. We therefore obtain n commuting R-matrices $R^i = D^i \check{R}^i$ in $\mathrm{End}_{\mathbb{C}}(V^i \otimes V^i) \hookrightarrow \mathrm{End}_{\mathbb{C}}(V_{-1}(0^n, \alpha) \otimes V_{-1}(0^n, \alpha))$, where the explicit inclusion maps are given by $\iota_i : v \otimes w \mapsto (id^{\otimes i-1} \otimes v \otimes id^{\otimes n-i}) \otimes (id^{\otimes i-1} \otimes w \otimes id^{\otimes n-i})$. By Proposition 3.2.6,

$$\check{R}^{\mathfrak{gl}}_{|q=-1} = \prod_i \iota_i(\check{R}^i) \in \mathrm{End}_{\mathbb{C}}(V_{-1}(0^n, \alpha) \otimes V_{-1}(0^n, \alpha)).$$

Moreover, using Lemma 3.1.1, Proposition 3.3.5, and the density Lemma, the conjugation by $\prod_i \iota_i(D^i)$ is equal to the conjugation by $D^{\mathfrak{gl}}$ in $\mathrm{End}_{\mathbb{C}}(V_{-1}(0^n, \alpha) \otimes V_{-1}(0^n, \alpha))$. Hence the braidings on $(\bigotimes_{i=1}^n V^i_H) \otimes (\bigotimes_{i=1}^n V^i_H)$ and on $V_{-1}(0^n, \alpha) \otimes V_{-1}(0^n, \alpha)$ are proportional. Now in the process of computing both the Links-Gould invariant and the Alexander polynomial, the R-matrices are rescaled by the inverse of their twist θ^{-1} so that the invariants become framing independent:

$$\mathrm{trace}_2(\theta^{-1}(\mathrm{Id} \otimes \phi)\tau R) = \mathrm{Id}_{V_{-1}(0^n,\alpha)}$$

(here ϕ denotes any of the pivotal structures which are equal by Proposition 3.3.4). Hence the rescaled R-matrices $R^{\mathfrak{gl}}_{|q=-1} = \prod_i \iota_i(R^H_{V^i \otimes V^i})$ and $\bigotimes_i R^H_{V^i \otimes V^i}$ are equal up to reordering factors. Finally, for any braid $\beta \in B_\ell$, the associated operators by the Reshetikhin-Turaev construction correspond up to reordering as well:

$$\Psi^{\mathfrak{gl}}_{V_{-1}(0^n,\alpha)^{\otimes \ell}}(\beta) = \left(\Psi^{U^H_{\mathbf{i}}\mathfrak{sl}(2)}_{V^{\otimes \ell}_{-\alpha}}(\beta) \right)^{\otimes n}.$$

At the end, if $\mathrm{trace}_{2,3,\ldots,\ell}\left((\mathrm{Id}_{V_{-1}(0^n,\alpha)} \otimes \phi^{\otimes \ell-1}) \circ \Psi^{\mathfrak{gl}}_{V_{-1}(0^n,\alpha)^{\otimes \ell}}(\beta) \right) = d.\,\mathrm{Id}_{V_{-1}(0^n,\alpha)}$ when $\mathrm{trace}_{2,3,\ldots,\ell}\left((\mathrm{Id}_{V_\alpha} \otimes \phi_U^{\otimes \ell-1}) \circ \Psi^{U^H_{\mathbf{i}}\mathfrak{sl}(2)}_{V^{\otimes \ell}_\alpha}(\beta) \right) = c.\,\mathrm{Id}_{V_\alpha}$, we obtain

$$d = c^n$$

by considering the trace of these two maps. Indeed, the trace is blind to reordering factors.
CQFD

Remark 3.3.6. In [19], the LG invariant is extended to a multivariable link invariant $M(L; q, q_1, \ldots, q_c)$ for links with $c \geq 2$ ordered components, taking its values in Laurent polynomials $\mathbb{Z}[q^{\pm}, q_1^{\pm}, \ldots, q_c^{\pm}]$. It is shown in [20] that

$$LG^{n,1}(\tau, q) = \left(\prod_{i=0}^{n-1} \frac{q^i}{\tau} - \frac{\tau}{q^i} \right) M(L; q, \tau^{-1}, \ldots, \tau^{-1}).$$

The proof in this chapter should adapt to show that

$$M(L; -1, q_1, \ldots, q_c) = \nabla(q_1, \ldots, q_c)^n$$

where ∇ is the Conway potential function, a version of the multivariable Alexander polynomial.

Chapter 4

A classical generalization of the Alexander invariant ?

At this point, the Alexander-Conway polynomial of a link Δ_L can be recovered from LG in at least two ways. David de Wit, Atsushi Ishii and Jon Links showed [10]

$$LG(L; t_0, -t_0^{-1}) = \Delta_L(t_0^2).$$

On the other hand, we just proved that the square of the Alexander polynomial can also be obtained evaluating LG:

$$LG(L; t_0, t_0^{-1}) = \Delta_L(t_0)^2.$$

Knowing this, it is natural to wonder :

Question 4.0.1. *Are there properties of Δ that extend to LG ?*

Conjecture 1.7.1 is a first attempt to give a positive answer to this question. In this final chapter [1] we give evidence for more positive answers to Question 4.0.1. We conjecture that the span of the LG invariant is a lower bound for the genus of a link.

Conjecture 4.0.2. *Set L a link in S^3 and μ the number of its components.*

- **I** $span(LG(L; t_0, t_1)) \leqslant 2(2g(L) + \mu - 1)$,
- **II** *If L is alternating, then inequality* **I** *is an equality.*

We also conjecture that for fibered knots, there are conditions on the leading coefficients of the LG polynomial.

1. This last chapter was first summed up in the paper *The Links-Gould invariant as a classical generalization of the Alexander polynomial ?* [32] that has been accepted for publication in Experimental Mathematics.

Conjecture 4.0.3. *Set K a knot in S^3.*

- **I** *If K is fibered then $LG(K)$ is monic,*
- **II** *If K is alternating, the converse is true as well.*

We base these conjectures on computations for the first prime knots and on partial skein relations for LG that allow its evaluation on various infinite families of links. Notice that if the genus conjecture were true, LG would systematically give a better lower bound for the genus of a link than the one given by the Alexander invariant. Also, the criterion we conjecture for fibered knots would refine the well known similar statement for Δ.

A proof of these two statements would show *quantum* invariant LG can be used to find information on the geometry of links.

The R-matrix for $LG^{2,1}$ we consider in this chapter in the one introduced in Definition 1.6.1.

4.1 The Links-Gould invariant and the genus of links

We believe that Proposition 1.2.5 can be extended to the similar statement expressed in Conjecture 4.0.2. We will explain how and why it would be an extension of 1.2.5. The goal of Section 4.2 is to give a range of evidence to support that conjecture.

Definition 4.1.1. Set $P \in \mathbb{Z}[t_0^{\pm 1}, t_1^{\pm 1}]$. For $(n, m) \in \mathbb{Z}^2$, we define

$$deg(t_0^n t_1^m) := n - m.$$

For a general $P = \sum_{i,j \in \mathbb{Z}} a_{ij} t_0^i t_1^j$, we can extend that definition to introduce the span of P :

$$span(P) := \max_{a_{ij} \neq 0} \{deg(t_0^i t_1^j)\} - \min_{a_{ij} \neq 0} \{deg(t_0^i t_1^j)\}.$$

Remark 4.1.2. The span satisfies the usual elementary degree properties :

$$span(PQ) = span(P) + span(Q),$$

$span(P + Q) \leqslant max\{span(P), span(Q)\}$ **if** P and Q are symmetric Laurent polynomials.

Conjecture 4.0.2 generalizes Proposition 1.2.5 since this well known result shows Conjecture 4.0.2 is true when $t_0t_1 = 1$ and $t_0t_1 = -1$ via the evaluations we already mentionned

$$LG(L; t_0, -t_0^{-1}) = \Delta_L(t_0^2)\ ;\ LG(L; t_0, t_0^{-1}) = \Delta_L(t_0)^2.$$

These evaluations also explain why our definition for the span was natural to try and push the lower bound a little further.

Proposition 4.1.3.

1 *In Conjecture 4.0.2,* **I** *implies* **II***,*

2 *If Conjecture 4.0.2 is true, it systematically improves the lower bound for the genus provided by* Δ *:*

$$\text{for any } L \text{ link, } 2deg(\Delta_L(t)) \leqslant span(LG(L; t_0, t_1)).$$

Moreover, there are links where

$$2deg(\Delta_L(t)) < span(LG(L; t_0, t_1)).$$

Proof. Since $LG(L; t_0, -t_0^{-1}) = \Delta_L(t_0^2)$, $2deg(\Delta_L(t)) = deg(\Delta_L(t^2)) = span(LG(L; t, -t^{-1}))$. So to prove **2**, we wish to show

$$span(LG(L; t, -t^{-1})) \leqslant span(LG(L; t_0, t_1)).$$

If we denote $LG(L; t_0, t_1) = \sum\limits_{\substack{i,j \in \mathbb{Z}/\\ a_{ij} \neq 0}} a_{ij} t_0^i t_1^j$, then

$$LG(L; t, -t^{-1}) = \sum_{k \in \mathbb{Z}} \Big(\sum_{\substack{i,j \in \mathbb{Z}/\\ a_{ij} \neq 0 \\ i-j=k}} a_{ij} (-1)^j \Big) t^k.$$

This clearly shows that if the coefficient in front of t^k in $LG(L; t, -t^{-1})$ is non zero, then there is at least one non zero coefficient in front of a monomial of degree k in the expression of $LG(L; t_0, t_1)$, which yields **2**. Moreover, some examples where the equality does not hold are given in Proposition 4.2.1.

Now suppose **I** holds for any link and set L an alternating link. Then [6], Theorem 3.5, states

$$deg(\Delta_L(t)) = 2g(L) + \mu - 1.$$

So we have the following inequality chain :

$$\begin{aligned} 2.(2g(L)+\mu-1) &\geqslant span(LG(L;t_0,t_1)) && \text{(Conjecture 4.0.2)}\\ &\geqslant 2deg(\Delta_L(t)) && \text{(point } \mathbf{2}\text{)}\\ &= 2.(2g(L)+\mu-1) && \text{(reference [6])} \end{aligned}$$

CQFD

4.2 Evidence supporting the genus conjecture

We wish to give evidence of the likeliness of Conjecture 4.0.2. In particular we verify the bound for small prime knots, prove it for several infinite families of knots and links and verify that the genus conjecture holds on an untwisted Whitehead double of the trefoil knot, which is a counter example due to Hugh Morton in a ressembling situation we will explain.

4.2.1 Less than 13 crossing prime knots

When we consider knots, μ is equal to 1 and the inequality becomes

$$span(LG(K;t_0,t_1)) \leqslant 4g(K).$$

We tested that inequality on all prime knots with less than 12 crossings, and on a large selection of non alternating prime knots with 13 crossings. To do that, we used the computations of LG for prime knots one can access via David de Wit's LINKS-GOULD EXPLORER [9]. To find genus information up to 12 crossings, we used Cha and Livingston's KNOTINFO [4]. For non alternating 13 crossing prime knots, data is obtained from Stoimenow's website KNOT DATA TABLES [56]. Knots are listed with respect to the HTW ordering for tables of prime knots of up to 16 crossings [22].

The reason why we did not test all non alternating 13 crossing prime knots is explained in [12]: The LINKS-GOULD EXPLORER's database contains evaluations only for LG of knots with string index at most 5, and from time to time 6 or 7. Indeed, the memory required increases dramatically with braid width. This still provides values for LG for 2096 non alternating prime knots with 13 crossings among the 5110 which exist.

Proposition 4.2.1.

1 *Conjecture 4.0.2 holds for every knot tested. In particular, for all alternating knots tested, the equality holds.*

2 *For all prime knots with less than 10 crossings, the span of LG exactly is 4 times the genus of the knot.*

3 *The list of prime knots with 11 or 12 crossings where there is no equality is the following :* 11^N_{34}, 11^N_{42}, 11^N_{45}, 11^N_{67}, 11^N_{73}, 11^N_{97}, 11^N_{152}, 12^N_{28}, 12^N_{31}, 12^N_{51}, 12^N_{56}, 12^N_{63}, 12^N_{87}, 12^N_{129}, 12^N_{132}, 12^N_{221}, 12^N_{231}, 12^N_{256}, 12^N_{257}, 12^N_{264}, 12^N_{267}, 12^N_{268}, 12^N_{313}, 12^N_{430}, 12^N_{665}, 12^N_{808}, 12^N_{812}.

4 *Among these knots we get a more precise lower bound with LG than with* Δ *for several of them :* 11^N_{34}, 11^N_{42}, 11^N_{67}, 11^N_{97}, 12^N_{31}, 12^N_{51}, 12^N_{129}, 12^N_{256}, 12^N_{257}, 12^N_{264}, 12^N_{267}, 12^N_{268}, 12^N_{313}, 12^N_{430}, 12^N_{665}, 12^N_{812}.

5 *Some of the spans are "half integers" in the sense that they are multiples of 2, as they necessarily are, but not multiples of 4 :* 11^N_{34}, 11^N_{42}, 11^N_{67}, 11^N_{97}, 12^N_{51}, 12^N_{256}, 12^N_{257}, 12^N_{264}, 12^N_{267}, 12^N_{268}, 12^N_{313}, 12^N_{430}, 12^N_{665}, 12^N_{812}.

Remark 4.2.2. If true, the span inequality is meaningful when $2deg(\Delta_L(t)) < span(LG(L;t_0,t_1))$. However, as long as it remains a conjecture, the hard case for the inequality is when $deg(\Delta_L(t)) = 2g(L) + \mu - 1$ precisely because there is no choice on the value of the span of LG for it not to be a counter-example.

4.2.2 The untwisted Whitehead double of the trefoil knot

Here we compute the Links-Gould polynomial of the untwisted double of the trefoil knot that is drawn is Figure 1. This is an interesting knot to study since it is a counterexample to a genus type bound for another generalization of the Alexander-Conway polynomial : the HOMFLY-PT polynomial. Precisely, Hugh Morton shows in [39], theorem 2, that the monomial of highest degree with respect to the Alexander variable in the HOMFLY-PT invariant gives a lower bound for $2\tilde{g}(L) + \mu - 1$, where $\tilde{g}(L)$ is the *canonical genus* of link L. The double of the trefoil is the example Morton gives to show that degree is *not* in general a lower bound for $2g(L) + \mu - 1$. There is no such problem here :

Proposition 4.2.3. *The untwisted Whitehead double of the right handed trefoil with positive clasp* K_0 *has the following properties :*

- $g(K_0) = 1$,
- $\Delta_{K_0}(t) = 1$,
- *A braid presentation of* K_0 *in braid group* B_6 *as a word in the standard Artin generators* σ_1, $\ldots$, σ_5 *is :*

$$\sigma_4\sigma_3\sigma_2\sigma_3^{-1}\sigma_4^{-1}\sigma_5\sigma_3\sigma_2\sigma_1\sigma_2^{-1}\sigma_3^{-1}\sigma_5\sigma_4\sigma_3\sigma_4^{-1}\sigma_5^{-1}\sigma_3\sigma_2\sigma_3^{-1}\sigma_2\sigma_1^2\sigma_2^{-1},$$

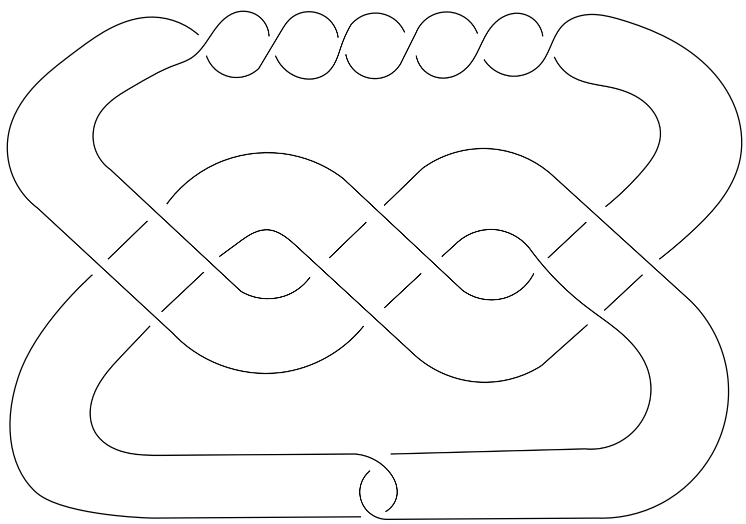


Figure 4.1 – The untwisted Whitehead double of the right handed trefoil with a positive clasp.

— $LG(K_0; t_0, t_1) = 3 - 4t_1 + 2t_1^2 - 4t_0 + 6t_1t_0 - 2t_1^2t_0 + 2t_0^2 - 2t_1t_0^2 - 2t_1^2t_0^2 + 4t_1^3t_0^2 - 2t_1^4t_0^2 + 4t_1^2t_0^3 - 10t_1^3t_0^3 + 8t_1^4t_0^3 - 2t_1^5t_0^3 - 2t_1^2t_0^4 + 8t_1^3t_0^4 - 8t_1^4t_0^4 + 2t_1^5t_0^4 - 2t_1^3t_0^5 + 2t_1^4t_0^5 + 4t_1^5t_0^5 - 6t_1^6t_0^5 + 2t_1^7t_0^5 - 6t_1^5t_0^6 + 8t_1^6t_0^6 - 2t_1^7t_0^6 + 2t_1^5t_0^7 - 2t_1^6t_0^7.$

So the span of $LG(K_0; t_0, t_1)$ is 4 and the span inequality is verified in this case.

Remark 4.2.4. The value of $LG(K_0; t_0, t_1)$ was obtained by direct computation of the formula given by Theorem 1.6.2 with the R-matrix in Definition 1.6.1 using MATHEMATICA 10.

4.2.3 Infinite families of knots using partial skein relations

Here we verify the genus bound on several infinite families of knots or links. To do that, we will use basic Alexander-type properties of LG we will recall, and partial skein relations that will make the computations practicable.

Some properties of LG and useful skein relations

To compute LG for infinite families of knots, we need to have a more efficient way to evaluate it than simply using the formula in 1.6.2. We first recall some general facts about the LG polynomial.

Proposition 4.2.5. *The Links-Gould polynomial satisfies the following properties :*

— $LG(\bigcirc) = 1$,

— *Denoting L^* the reflexion of L,* $LG(L^*; t_0, t_1) = LG(L; t_0^{-1}, t_1^{-1})$,

— *We have the following symmetry :* $LG(L; t_0, t_1) = LG(L; t_1, t_0)$*. Indeed* LG *does not detect inversion,*
— *For* L *and* L' *two links, denoting* $L\#L'$ *their connected sum :* $LG(L\#L') = LG(L)LG(L')$*,*
— *If* $L = L' \sqcup L''$ *is the split union of* L' *and* L''*, then* $LG(L) = 0$*.*

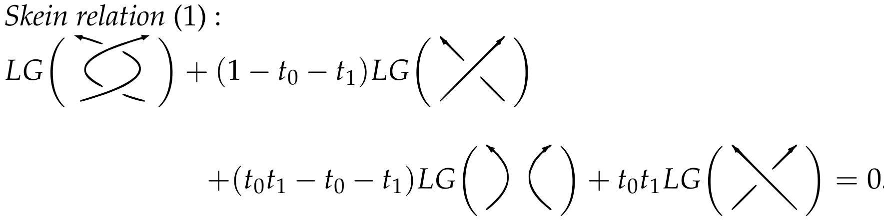
Proof. For proofs of these facts, we refer the reader to [24, 11, 7]. CQFD

Remark 4.2.6. The last two points show that LG and Δ behave similarily concerning sums and disjoint unions.

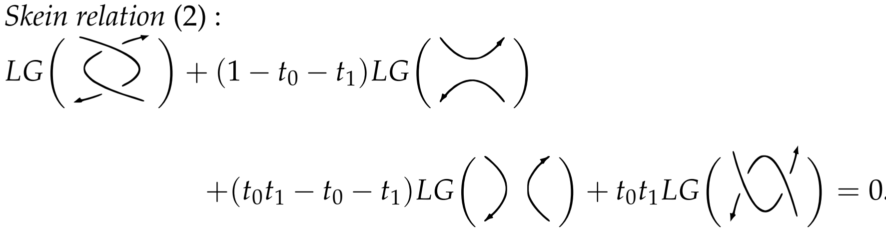
Let us also cite a list of skein relations that are known to be true for LG. Whether the associated skein module is generated by the unknot or not is a problem pointed out by Ishii [24]. It is to the best of our knowledge an open question.

Proposition 4.2.7. *LG verifies the following skein relations.*

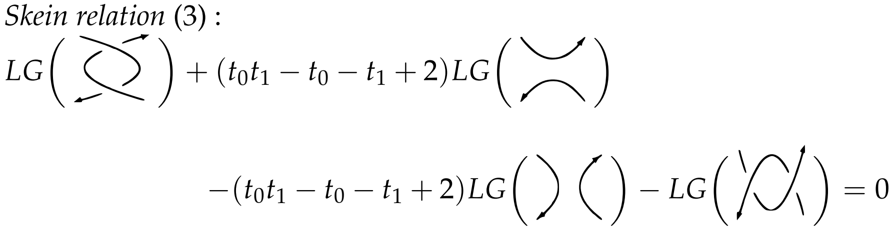
Skein relation (1) :



Skein relation (2) :

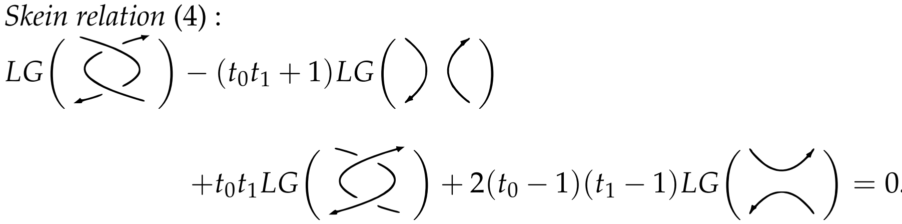


Skein relation (3) :



.

Skein relation (4) :



Proof. See [24, 11, 35]. CQFD

Remark 4.2.8. (1) and (2) are equivalent, by adding each time a well chosen tangle from the left.

Remark 4.2.9. As explained in [24], (4) is a consequence of (2) and (3).

Remark 4.2.10. Set V the 4-dimensional irreducible $U_q(\mathfrak{gl}(2|1))$-module that gives rise to the Links-Gould invariant. Then the tensor product of two copies of V decomposes with respect to the $U_q(\mathfrak{gl}(2|1))$-module structure.

$$V \otimes V = V_1 \oplus V_2 \oplus W, \text{ with } dimV_1 = dimV_2 = 4 \text{ and } dimW = 8.$$

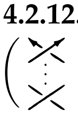
Morevover, V_1, V_2 and W are non isomorphic irreducible $U_q(\mathfrak{gl}(2|1))$-modules. For details, see [18, 11]. Using this and denoting $A := U_q(\mathfrak{gl}(2|1))$, we have the following identification :

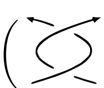
$$End_A(V \otimes V) \simeq End_A(V_1) \oplus End_A(V_2) \oplus End_A(W) \simeq \mathbb{C}(t_0^{\pm 1}, t_1^{\pm 1})^3.$$

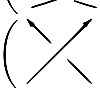
In particular, for *any* three $(2,2)$-tangles such that the associated maps in $End_A(V \otimes V)$ are linearly independent, *any* other can be expressed as a linear combination of the first three. This potentially generates a great variety of skein relations for LG.

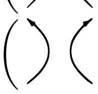
Remark 4.2.11. Using points 2 and 3 of Proposition 4.2.5, we can modify the previous skein relations : orientation of the strands, signs of the crossings. We will use these modified relations, though we will not write them down here.

Corollary 4.2.12. *Using notations in [24], LG satisfies the following skein relations :*

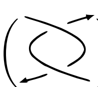
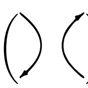
— $LG\Big($  $\Big\}$ *n half twists* $\Big)$

$= \left(\frac{(-1)^n}{(t_0+1)(t_1+1)} + \frac{t_0^n}{(t_0+1)(t_0-t_1)} + \frac{t_1^n}{(t_1+1)(t_1-t_0)} \right) LG\Big($  $\Big)$

$- \left(\frac{(-1)^n(t_0+t_1)}{(t_0+1)(t_1+1)} + \frac{t_0^n(t_1-1)}{(t_0+1)(t_0-t_1)} + \frac{t_1^n(t_0-1)}{(t_1+1)(t_1-t_0)} \right) LG\Big($  $\Big)$

$+ \left(\frac{(-1)^n t_0 t_1}{(t_0+1)(t_1+1)} - \frac{t_0^n t_1}{(t_0+1)(t_0-t_1)} - \frac{t_1^n t_0}{(t_1+1)(t_1-t_0)} \right) LG\Big($  $\Big)$.

— $LG\Big($  $\Big\}$ *n full twists* $\Big)$

$= \frac{t_0^n t_1^n - 1}{t_0 t_1 - 1} LG\Big($  $\Big) + \left(1 - \frac{t_0^n t_1^n - 1}{t_0 t_1 - 1}\right) LG\Big($  $\Big)$

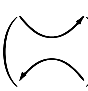
$+ \frac{2(t_0-1)(t_1-1)}{t_0 t_1 - 1} \left(n - \frac{t_0^n t_1^n - 1}{t_0 t_1 - 1} \right) LG\Big($  $\Big)$.

Figure 4.2 – Generators S_1 and S_2 of braid group B_3.

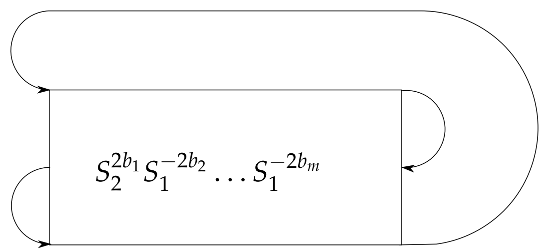


Figure 4.3 – $D(b_1, \dots, b_m)$ when m is even.

— *LG*(} *n full twists*)
$= a_1(n) LG(\,) + a_2(n) LG(\,) + a_3(n) LG(\,)$,

where :

$$a_1(n) = \frac{t_0^n t_1^n - 1}{t_0 t_1 - 1},$$
$$a_2(n) = \frac{2n(t_0-1)(t_1-1)}{t_0 t_1 - 1} - a_1(n)\left(\frac{(t_0 t_1 + 1)(t_0 - 1)(t_1 - 1)}{t_0 t_1 - 1} + 1\right),$$
$$a_3(n) = (t_0 - 1)(t_1 - 1) a_1(n) + 1.$$

We will now use all these properties to compute LG, or at least its span, on some infinite families of links.

2-bridge links

A 2-bridge link is a link with bridge number 2. As explained in [26], an oriented 2-bridge link can always be written in terms of the two generators S_1 and S_2 of 3-string braid group B_3. We use the notations one can find in [26, 24], setting $D(b_1, b_2, \dots, b_m)$ the oriented 2-bridge link drawn in Figures 3 and 4. We can suppose $b_1, \dots, b_m > 0$, thereby choosing an alternating diagram to represent any 2-bridge link. In particular, any 2-bridge link is alternating, so the inequality in Conjecture 4.0.2 should be an equality.

Remark 4.2.13. If m is even, $D(b_1, \dots, b_m)$ is a knot. It is a link with two components when m is odd.

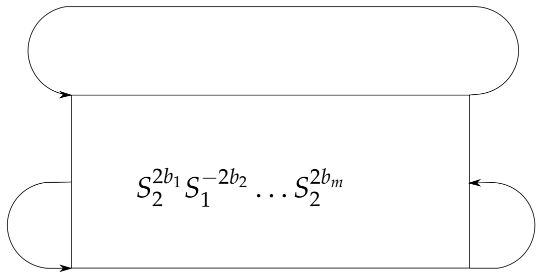


Figure 4.4 – $D(b_1, \dots, b_m)$ when m is odd.

Proposition 4.2.14. *If $b_i \neq 0$ for any i, then*

$$g(D(b_1, \dots, b_m)) = \frac{m - \mu + 1}{2}$$

where μ is the number of components [54].

Therefore, Conjecture 4.0.2 can be rephrased

$$span(LG(D(b_1, \dots, b_m))) = 2(2g(D(b_1, \dots, b_m)) + \mu - 1) = 2m.$$

Theorem 4.2.15. *For any $b_1, b_2, \dots, b_m > 0$,*

$$span(LG(D(b_1, \dots, b_m))) = 2m.$$

Proof. First we note that $a_1(n), a_2(n), a_3(n)$ are symmetric *polynomials* with respect to variables t_0 and t_1. We can compute the span in each case.

$$span(a_1(n)) = 0, span(a_2(n)) = 2, span(a_3(n)) = 2.$$

We will indicate by $\tilde{a_i}(n)$ the quantity $a_i(n)(t_0^{-1}, t_1^{-1})$. Let's prove the span equality by induction on m.

$\boxed{m = 1}$ Using the mirror of skein relation 3 of Corollary 4.2.12 :

$$LG(D(b_1)) = \tilde{a_1}(b_1)LG(\bigcirc) + \tilde{a_2}(b_1)LG(\bigcirc) + \tilde{a_3}(b_1)LG(\bigcirc\bigcirc) = \tilde{a_1}(b_1) + \tilde{a_2}(b_1).$$

So the span of $LG(D(b_1))$ is 2.

$\boxed{m = 2}$ Still using the same skein relation, we can compute $LG(D(b_1, b_2))$.

$$LG(D(b_1, b_2)) = a_1(b_2)LG(D(b_1 - 1)) + a_2(b_2)LG(D(b_1)) + a_3(b_2)LG(\bigcirc).$$

The second part of the sum has span $2+2=4$. The third part has span $2+0=2$. More care has to be taken with the first term, and in particular with $LG(D(b_1-1))$. If $b_1-1>0$, $LG(D(b_1-1))$ has span 2. In the other case, $D(0)=\bigcirc\bigcirc$ so $LG(D(0))=0$. So in any case the span of the sum is 4.

Let us now set $m \geqslant 3$ and suppose the equality stands for any $D(b_1,\ldots,b_k)$ with $k \leqslant m-1$. For $D(b_1,\ldots,b_m)$ we can apply skein relation 3 of 4.2.12 or its mirror image to the crossings that correspond to $S_1^{-2b_m}$ or $S_2^{2b_m}$ depending on whether m is odd or even. Say m is even.

$$\begin{aligned} LG(D(b_1,\ldots,b_m)) =& a_1(b_m)LG(D(b_1,\ldots,b_{m-1}-1)) \\ &+ a_2(b_m)LG(D(b_1,\ldots,b_{m-1})) \\ &+ a_3(b_m)LG(D(b_1,\ldots,b_{m-2})). \end{aligned}$$

Since $LG(D(b_1,\ldots,b_{m-1},0)) = LG(D(b_1,\ldots,b_{m-2}))$ the first element in the sum has a span smaller than $0+2(m-1)=2m-2$. The second part has span $2+2(m-1)=2m$ and the third $2+2(m-2)=2m-2$. In the end

$$span(LG(D(b_1,\ldots,b_m))) = 2m.$$

CQFD

Twist knots

Definition 4.2.16. A twist knot is a Whitehead double of the unknot. We will denote by K_n the twist knot shown in Figure 5 when $2n-1$ is positive. If $2n-1$ is negative, there are $1-2n$ crossings of the opposite sort.

For instance K_0 is the unknot, K_1 is the trefoil knot and K_2 is knot 5_2.

Proposition 4.2.17. *For $n \neq 0$, $g(K_n)=1$.*

Proposition 4.2.18.

— $LG(K_0) = LG(\bigcirc) = 1.$

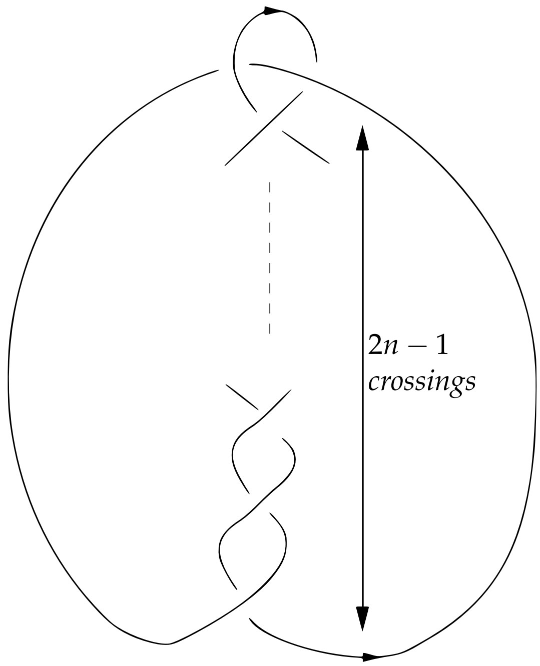


Figure 4.5 – Twist knot K_n.

— *For $n \geqslant 1$,*

$$LG(K_n) =$$
$$(-t_0^{-2}t_1^{-1} - t_0^{-1}t_1^{-2} + t_0^{-2} + 2t_0^{-1}t_1^{-1} + t_1^{-2} - t_0^{-1} - t_1^{-1} + 1)(t_0^{-1}t_1^{-1}\tilde{a}_1(n-1) + 1)$$
$$+ (t_0^{-1} - 1)^2(t_1^{-1} - 1)^2 \frac{(t_0^{-1}t_1^{-1} + 1)\tilde{a}_1(n-1) - 2(n-1)}{t_0^{-1}t_1^{-1} - 1}$$
$$+ \tilde{a}_1(n-1)(t_0^{-1} - 1)^2(t_1^{-1} - 1)^2 - t_0^{-1}t_1^{-1}\tilde{a}_1(n-1).$$

It has span 4.

— *For $n \leqslant -1$,*

$$LG(K_n) =$$
$$(t_0^{-1} - 1)(t_1^{-1} - 1)\left(a_1(-n) + (t_0 - 1)(t_1 - 1)\frac{2n + a_1(-n)(t_0t_1 + 1)}{t_0t_1 - 1}\right)$$
$$+ (t_0 - 1)(t_1 - 1)a_1(-n) + 1.$$

It has span 4 as well.

Proof. When $n \geqslant 1$, we first write the third skein relation of 4.2.12 for $n-1$ full twists. On

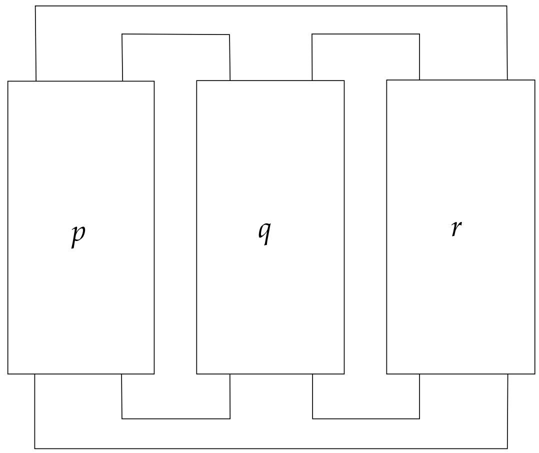


Figure 4.6 – $L(p,q,r)$.

each of the three links that appear, we use the first point of 4.2.12. We find the formula written in the theorem. A close look at that expression shows that

$$span(LG(K_n)) \leqslant 4.$$

To see it is equal to 4, we can for example evaluate $t_1 = -t_0^{-1}$. We know we will find (and can verify)

$$LG(K_n)(t_0, -t_0^{-1}) = \Delta_{K_n}(t_0^2) = nt_0^2 - (2n-1) + nt_0^{-2}.$$

So $span(LG(K_n)) \geqslant 4$. Similar computations can be made when $n \leqslant -1$. CQFD

Pretzel knots

Definition 4.2.19. Set $p,q,r \in \mathbb{Z}$. The (p,q,r)-pretzel link $L(p,q,r)$ is a union of three pairs of strands half-twisted p,q,r times and attached along the tops and bottoms as shown in Figure 6. The half-twists are oriented according to whether the integer is positive or negative.

For example, pretzel knot $L(-2,3,7)$ is represented in Figure 7.

Proposition 4.2.20.

$$L(p,q,r) \textit{ is a knot } \iff \big(\textit{at most one of the three integers } p,q \textit{ and } r \textit{ is even}\big).$$

In that case pretzel knot $L(p,q,r)$ is denoted by $K(p,q,r)$.

In [30], Kim and Lee explicit the genus for all pretzel knots. Verifying the genus conjecture on this family of knots is quite interesting since the genus does not behave the same way

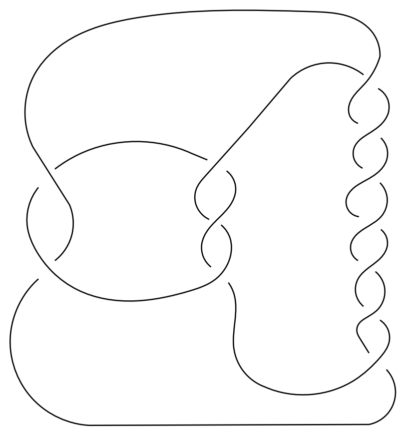


Figure 4.7 – $L(-2,3,7)$.

as a function of parameters (p,q,r) in all cases. The next theorem is proved in [30].

Theorem 4.2.21. *Let p,q,r be integers. The genus of $K(p,q,r)$ is as follows :*

1 $K(p,\pm 1,\mp 1)$, $K(\pm 2,\mp 1,\pm 3)$ *have genus 0 for any p,*

2 $K(p,q,r)$ *has genus 1 if p,q,r are odd and we are not in case* **1**,

3 $K(\pm 2,\mp 1,\pm r)$ *has genus* $\frac{|r-2|-1}{2}$,

4 $K(2l,q,r)$ *has genus* $\frac{|q|+|r|}{2}$ *if q,r have the same sign and we are not in any of the previous cases,*

5 $K(2l,q,r)$ *has genus* $\frac{|q|+|r|-2}{2}$ *if q,r have different signs and we are not in cases* **1**, **2** *or* **3**.

We rewrite that theorem so that different cases exclude each other. Doing this makes computations more specific and somewhat easier in each case. Moreover, since $K(p,q,r)^* = K(-p,-q,-r)$, we will consider $p \geqslant 0$. Also, $K(p,q,r) = K(q,r,p) = K(r,p,q)$. So we can restrict our study to the cases where q,r are odd.

Corollary 4.2.22. *Given the restrictions mentioned, setting $p \geqslant 0$ an integer and q,r two odd integers, the genus g of $K(p,q,r)$ is as follows :*

1 $g = 0$ *for $K(p,\pm 1,\mp 1)$ and $K(2,-1,3)$, that is when $K(p,q,r)$ is the unknot,*

2 $g = 1$ *when p,q,r are odd and $K(p,q,r)$ is not the unknot,*

3 $g = \frac{|r-2|-1}{2}$ *for $K(2,-1,r)$,*

4 $g = \frac{|q|+|r|}{2}$ *if :*

— *p is even and q,r are positive,*

— *p is even and different from 2, $q=-1$ and r is negative,*

— *p is even, q is negative and different from -1, r is negative,*

5 $g = \frac{|q|+|r|-2}{2}$ *if :*

— *p is even and different from 2, $q = -1$ and $r \geqslant 3$,*

— *p is even, $q \geqslant 0$, $r \leqslant 0$ and $(p,q,r) \neq (p,1,-1)$,*

— *p is even, $q \leqslant -3$ and $r \geqslant 0$.*

Theorem 4.2.23. *For all pretzel knots,*

$$span(LG(K(p,q,r))) \leqslant 4g(K(p,q,r)).$$

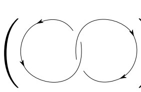
Proof. We compute the span of $LG(K(p,q,r))$ in each case of Corollary 4.2.22.

$\boxed{1}$ $K(p,\pm 1,\mp 1)$ and $K(2,-1,3)$ are different representations of the trivial knot that is part of the small cases we already checked.

$\boxed{2}$ Using the fact that $K(p,q,r)^* = K(-p,-q,-r)$ and $K(p,q,r) = K(q,r,p) = K(r,p,q)$, we have only two cases to consider : when p,q,r have the same sign and when two out of the three have the same sign. For example we can choose the following configurations : $p,q,r \geqslant 0$ and $p \geqslant 0, q, r \leqslant 0$. In each case, using skein relation 2 of Corollary 4.2.12 on the three pairs of strands, we find a sum of 27 terms, each of which is symmetric of span smaller than 4.

$\boxed{3}$ For $r = 1,3$, $K(2,-1,r)$ is the unknot. If $r \geqslant 5$, $K(2,-1,r)$ is drawn in Figure 8 once it is simplified. The same kind of isotopy can be operated on $K(2,-1,r)$ when $r \leqslant -1$ and the result is shown in Figure 9.

For example if $r \geqslant 5$ we can apply skein relation 1 of Corollary 4.2.12 to the $r-2$ half twists.

$$LG(K(2,-1,r)) = \left(\frac{-1}{(t_0+1)(t_1+1)} + \frac{t_0^{r-2}}{(t_0+1)(t_0-t_1)} + \frac{t_1^{r-2}}{(t_1+1)(t_1-t_0)}\right) LG\Bigg($$

$$\Bigg)$$

$$\begin{aligned} &- \left(\frac{-t_0-t_1}{(t_0+1)(t_1+1)} + \frac{t_0^{r-2}(t_1-1)}{(t_0+1)(t_0-t_1)} + \frac{t_1^{r-2}(t_0-1)}{(t_1+1)(t_1-t_0)}\right) LG(\bigcirc) \\ &+ \left(\frac{-t_0t_1}{(t_0+1)(t_1+1)} - \frac{t_0^{r-2}t_1}{(t_0+1)(t_0-t_1)} - \frac{t_1^{r-2}t_0}{(t_1+1)(t_1-t_0)}\right) LG(\bigcirc\bigcirc) \\ &= \frac{(t_0-t_1)(t_0t_1+1) - t_0^{r-1}(t_1-1)(t_1+1) + t_1^{r-1}(t_0-1)(t_0+1)}{(t_0+1)(t_1+1)(t_0-t_1)} \end{aligned}$$

The numerator has span $2r-2$ and the denominator has span 4. So

$$span(LG(K(2,-1,r))) = (2r-2) - 4 = 2r - 6 = 4\left(\frac{r-3}{2}\right) = 4\left(\frac{|r-2|-1}{2}\right).$$

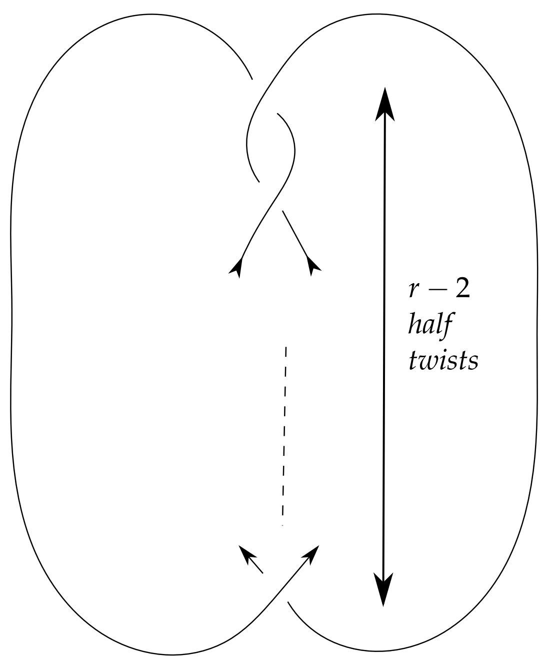


Figure 4.8 – $K(2,-1,r)$ when $r \geqslant 3$.

Computations can be led in a similar way in the other case.

$\boxed{4}$ *If p is positive and even, and q, r are positive and odd.* Choosing an orientation for $K(p,q,r)$, we can apply skein relation 2 of 4.2.12 on the p half twists and skein relation 1 of 4.2.12 on the q and r half twists. As in a previous case, we get a sum of 27 terms, each of which can be computed easily. All these terms have a span smaller than $2q + 2r = 4g$. Therefore we have the inequality in this case.

If p is positive, even and different from 2, $q = -1$, and r is odd and negative. Choosing an orientation here again, we can use skein relation 2 on the p half twists and skein relation 1 on the r half twists. Each of the 9 parts of the sum such obtained has a span smaller than $2 - 2r$. So once again

$$span(LG(K(p,-1,r))) \leqslant 2 - 2r = 4\left(\frac{1-r}{2}\right) = 4g.$$

If p is even and positive, q is odd negative and different from -1, and r is odd and negative. An extended computation similar to the two previous ones proves the bound in this case as well.

$\boxed{5}$ *If p is even, positive and different from 2, $q = -1$, and r is positive, odd and different from 1.* Applying skein 2 of 4.2.12 on the $p/2$ full twists, we find three links, each of which is the

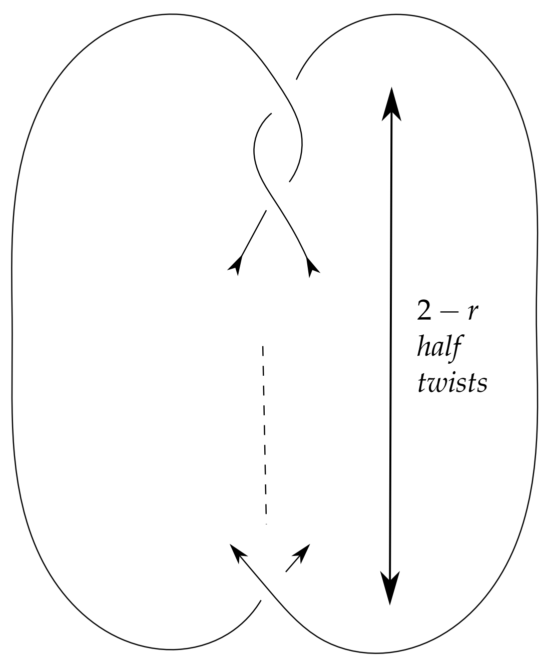


Figure 4.9 – $K(2,-1,r)$ when $r \leqslant -1$.

closure of a power of generator σ_1 of B_2. We can then use skein relation 1 of 4.2.12 on each of these links to find that these three terms have a span smaller than $2r-2 = 4\left(\frac{r-1}{2}\right)$.

If p is even and positive, q is odd and positive, r is odd and negative, and $(q,r) \neq (1,-1)$. This is the most tricky case. Indeed, if we compute LG naively using skein relations 1 and 2 as we did for the moment, some parts of the sum we obtain have a span larger than $4g = 2q-2r-4$. We therefore have to look at these particular terms to see that what goes past the bound we hope actually compensates. This is achieved in the appendix (Section 4.4).

If p is positive and even, q is odd and $q \leqslant -3$, and r is odd and positive. $K(p,q,r)$ can be isotoped as follows :

$$K(p,q,r) = K(r,q,p) = K(p,r,q).$$

The last form $K(p,r,q)$ shows that this case is a consequence of the two previous cases of point **5**. CQFD

4.2.4 A generalization of Conjecture 4.0.2 to $LG^{n,1}$

Though we lack computations for $LG^{m,n}$ when $(m,n) \neq (1,1),(2,1)$, the $(m,1)$ and $(1,m)$ cases of the De Wit-Ishii-Links conjecture, that are known to be true, extend the evaluations we have for $LG^{2,1}$, that motivated Conjecture 4.0.2. So a potential homological interpretation for $LG^{2,1}$ should extend to $LG^{n,1}$ as well.

Question 4.2.24. *Set L a link in S^3 and $n \geqslant 3$. Do we have, as it seems to be the case when $n=2$:*

- **I** $span(LG^{n,1}(L;t_0,t_1)) \leqslant n(2g(L)+\mu-1)$,
- **II** *If L is alternating, then inequality* **I** *is an equality ?*

Remark 4.2.25. For example, the equality holds for all prime knots with less than 10 crossings when $n=3$.

Remark 4.2.26. As a consequence, one could ask, as n tends to infinity, if

$$span(LG^{n,1}(L;t_0,t_1)) \underset{+\infty}{\sim} n(2g(L)+\mu-1)\ ?$$

However this *cannot* be true. Indeed, there are pairs of mutant knots with different genera, and neither of the $LG^{n,1}$ detects mutation.

4.3 The Links-Gould polynomial and fiberedness

In addition to genus information, the Links-Gould polynomial seems to contain signs of whether a knot is fibered or not. This is another well known feature of the Alexander invariant. Conjecture 4.0.3, if it were to be true, would refine the standard Alexander polynomial criterion. This is the object of this section.

Let us recall that a knot K in S^3 is said to be *fibered* when the two following conditions hold:

1 the complement of the knot is the total space of a locally trivial bundle over the base space S^1, i.e. there exists a map $p : S^3 \setminus K \longrightarrow S^1$ which is a locally trivial bundle.

2 there exists $V(K)$ a neighborhood of K and there exists a trivializing homeomorphism $\theta : V(K) \longrightarrow S^1 \times D^2$ such that $\pi \circ \theta(X) = p(X)$ for any $X \in V(K) \setminus K$, where $\pi(x, y) := \frac{y}{|y|}$.

Proposition 4.3.1. *Set K a fibered knot. If Conjecture 4.0.2 is true, then*

$$span(LG(K; t_0, t_1)) = 4g(K).$$

Proof. To have such a result we can more generally consider a set of links E such that, for any $L \in E$, $deg(\Delta_L(t)) = 2g(L) + \mu - 1$. This is the case here, and it was also the case in Proposition 4.1.3 where it is proved completely. CQFD

As we already said, we can express the Links-Gould polynomial with different sets of variables : for example with variables (t_0, t_1) as we did up to now, but also (p, q) where

$$p^a q^b = t_0^{-\frac{a}{4}+\frac{b}{2}} t_1^{\frac{a}{4}+\frac{b}{2}}.$$

These are the variables used in the LINKS-GOULD EXPLORER as well as in de Wit's papers on the subject. He sometimes uses $P = p^2$. In variables (p, q) the LG polynomial of a link L can be written

$$LG(L; p, q) = a_0 + \sum_{k \in \mathbb{N}^*} P_k(q)(p^{2k} + p^{-2k})$$

where $a_0 \in \mathbb{Z}$ and $P_k(q) \in \mathbb{Z}[q^{\pm 1}]$. Note that if $P_l(q) \neq 0$ and $P_k(q) = 0$ for any $k > l$, then $span(LG(L; p, q)) = 2l$.

Definition 4.3.2. Set K a knot. We say $LG(K)$ is *monic* when the term in $LG(K)$ of highest and lowest degrees can be written $q^{2m}(p^{4l} + p^{-4l})$ with $l \in \mathbb{N}$ and $m \in \mathbb{Z}$. In terms of variables (t_0, t_1), this condition is expressed by saying the terms of highest and lowest degrees are monic monomials of the form $t_0^{\alpha} t_1^{\beta}$ with $\alpha + \beta$ even.

Proposition 4.3.3. *Set K a knot. If $LG(K; t_0, t_1)$ is monic, then $\Delta_K(t)$ is monic as well.*

Proof. Consequence of $LG(K; t, -t^{-1}) = \Delta_K(t^2)$. CQFD

Remark 4.3.4. Point **I** in Conjecture 4.0.3 implies point **II**. This is a consequence of Proposition 4.3.3 and of the fact that when the Alexander polynomial is monic for an alternating knot, the knot is fibered.

Remark 4.3.5. Given Proposition 4.3.3, criterion 4.0.3 would be an improvement of the criterion provided by the Alexander invariant. In addition, we will see in the following that there are examples of knots where Δ is monic but LG is not.

Proposition 4.3.6.

1 *Conjecture 4.0.3 holds for every prime knot up to 12 crossings. In particular, for all alternating knots tested, fiberedness and having monic LG are equivalent.*

2 *For a prime knot K with at most 11 crossings, K is fibered if and only if $LG(K)$ is monic.*

By work of Friedl and Kim [17], there are 13 non-fibered 12-crossing prime knots which have monic Alexander polynomials such that $deg(\Delta_K(t)) = 2g(K)$: 12^N_{57}, 12^N_{210}, 12^N_{214}, 12^N_{258}, 12^N_{279}, 12^N_{382}, 12^N_{394}, 12^N_{464}, 12^N_{483}, 12^N_{535}, 12^N_{650}, 12^N_{801}, 12^N_{815}. Among these, LG manages to "detect" non-fiberedness of some, but not all.

Proposition 4.3.7. *The following knots have monic LG :* 12^N_{57}, 12^N_{258}, 12^N_{279}, 12^N_{464}, 12^N_{483}, 12^N_{650}, 12^N_{815}. *So there are knots that are non-fibered but that have monic LG.*

Proposition 4.3.8. *The following knots have non-monic LG :* 12^N_{210}, 12^N_{214}, 12^N_{382}, 12^N_{394}, 12^N_{535}, 12^N_{801}. *So Δ sometimes is monic when LG is not.*

Verifications were done using de Wit's LINKS-GOULD EXPLORER [9] and Cha and Livingston's KNOTINFO [4].

4.4 Appendix : proof of the harder case in Theorem 4.2.23

Here we prove the remaining case of Theorem 4.2.23. Consider pretzel knot $K(p, q, r)$ when p is positive and even, q is positive and odd, r is negative and odd, and $(q, r) \neq (1, -1)$. We want to show that

$$span(LG(K(p, q, r))) \leqslant 4g = 2q - 2r - 4.$$

We first consider $r = -1$. Then $4g = 2q - 2$. In that precise configuration, using skein relation 2 of 4.2.12 on the $p/2$ full twists and skein relation 1 of the same corollary on the q half twists, you find a sum of terms, each of which has a span smaller than $2q - 2$.

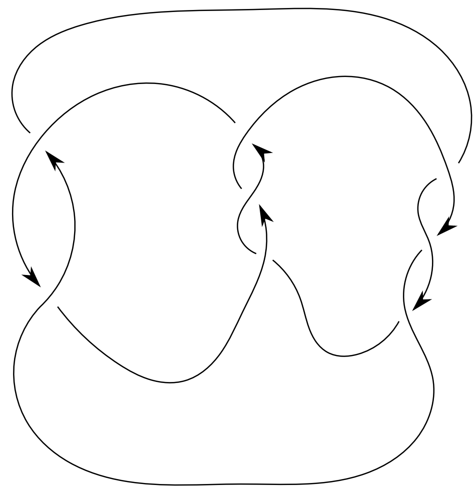


Figure 4.10 – $K(2,3,-3)$.

In general, the computation is not that easy. We show knot $K(2,3,-3)$ in Figure 10 to fix the orientation chosen here. Let us introduce some notations :

$$x(n) = \frac{(-1)^n}{(t_0+1)(t_1+1)} + \frac{t_0^n}{(t_0+1)(t_0-t_1)} + \frac{t_1^n}{(t_1+1)(t_1-t_0)},$$

$$y(n) = \frac{(-1)^n(t_0+t_1)}{(t_0+1)(t_1+1)} + \frac{t_0^n(t_1-1)}{(t_0+1)(t_0-t_1)} + \frac{t_1^n(t_0-1)}{(t_1+1)(t_1-t_0)},$$

$$z(n) = \frac{(-1)^n t_0 t_1}{(t_0+1)(t_1+1)} - \frac{t_0^n t_1}{(t_0+1)(t_0-t_1)} - \frac{t_1^n t_0}{(t_1+1)(t_1-t_0)}.$$

We transform the $-r$ half twists in $K(p,q,r)$ using skein relation 1 of 4.2.12.

$$\begin{aligned} LG(K(p,q,r)) &= x(-r)(t_0^{-1},t_1^{-1})LG(K(p,q,-2)) \\ &\quad - y(-r)(t_0^{-1},t_1^{-1})LG(K(p,q,-1)) \\ &\quad + z(-r)(t_0^{-1},t_1^{-1})LG(K(p,q,0)). \end{aligned}$$

The span of $LG(K(p,q,-1))$ is $2q-2$ with the previous case. Also $span(y(-r)(t_0^{-1},t_1^{-1})) = -2r-4$. So the second term of the sum above has span $2q-2r-6$ and we need only to consider the first and third terms in the rest of the proof.

Also, using skein relation 1 of Proposition 4.2.7,

$$\begin{aligned}x(-r)(t_0^{-1},t_1^{-1})LG(K(p,q,-2)) &= x(-r)(t_0^{-1},t_1^{-1})(t_0^{-1}+t_1^{-1}-1)LG(K(p,q,-1))\\ &\quad + x(-r)(t_0^{-1},t_1^{-1})(t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1})LG(K(p,q,0))\\ &\quad - x(-r)(t_0^{-1},t_1^{-1})t_0^{-1}t_1^{-1}LG(K(p,q,1)).\end{aligned}$$

Again, the first term of this sum has span $(-2r-4)+2+(2q-2)=2q-2r-4$ so we can ignore it as well and we are interested in the following sum:

$$\begin{aligned}&\Big(x(-r)(t_0^{-1},t_1^{-1})(t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1})+z(-r)(t_0^{-1},t_1^{-1})\Big)LG(K(p,q,0))\\ &\qquad -\Big(x(-r)(t_0^{-1},t_1^{-1})t_0^{-1}t_1^{-1}\Big)LG(K(p,q,1)).\end{aligned}$$

However, $span(z(-r)(t_0^{-1},t_1^{-1}))=-2r-6$ and $span(x(-r)(t_0^{-1},t_1^{-1})(t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1}))=(-2r-4)+2=-2r-2$. So we can reduce our concerns to

$$\begin{aligned}&\Big(x(-r)(t_0^{-1},t_1^{-1})(t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1})\Big)LG(K(p,q,0))\\ &\qquad -\Big(x(-r)(t_0^{-1},t_1^{-1})t_0^{-1}t_1^{-1}\Big)LG(K(p,q,1)).\end{aligned}$$

Using the usual skein relations on $LG(K(p,q,0)$, we get the following value modulo terms with a small span:

$$\begin{aligned}LG(K(p,q,0)) =&\left(\frac{(t_0^{-1}t_1^{-1})^{p/2}-1}{t_0^{-1}t_1^{-1}-1}\big(-(t_0^{-1}-1)(t_1^{-1}-1)\big)x(q)(t_0+t_1-1-t_0t_1)\right)\\ &+\left(\frac{2(t_0^{-1}-1)(t_1^{-1}-1)}{t_0^{-1}t_1^{-1}-1}\right)\left(\frac{p}{2}-\frac{(t_0^{-1}t_1^{-1})^{p/2}-1}{t_0^{-1}t_1^{-1}-1}\right)x(q)\big(-(t_0-1)(t_1-1)\big).\end{aligned}$$

The two pieces of the sum have span $2q$. Similarily, ignoring non extremal span terms:

$$\begin{aligned}&LG(K(p,q,1)) =\\ &\quad\left(\frac{(t_0^{-1}t_1^{-1})^{p/2}-1}{t_0^{-1}t_1^{-1}-1}x(q)(t_0+t_1-1)(t_0^{-1}+t_1^{-1}-2-t_0^{-1}t_1^{-1})\big(-(t_0-1)(t_1-1)\big)\right)\\ &\quad+\left(\frac{2(t_0^{-1}-1)(t_1^{-1}-1)}{t_0^{-1}t_1^{-1}-1}\right)\left(\frac{p}{2}-\frac{(t_0^{-1}t_1^{-1})^{p/2}-1}{t_0^{-1}t_1^{-1}-1}\right)x(q+1)\big(-(t_0-1)(t_1-1)\big).\end{aligned}$$

So the quantity we are interested in is

$$\begin{aligned}
&x(-r)(t_0^{-1},t_1^{-1})(t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1})\Big[\frac{(t_0^{-1}t_1^{-1})^{p/2}-1}{t_0^{-1}t_1^{-1}-1}\big(-(t_0^{-1}-1)(t_1^{-1}-1)\big)x(q)(t_0+t_1)\\
&\qquad+\frac{2(t_0^{-1}-1)(t_1^{-1}-1)}{t_0^{-1}t_1^{-1}-1}\left(\frac{p}{2}-\frac{(t_0^{-1}t_1^{-1})^{p/2}-1}{t_0^{-1}t_1^{-1}-1}\right)x(q)(t_0+t_1)\Big]\\
&\qquad+x(-r)(t_0^{-1},t_1^{-1})(-t_0^{-1}t_1^{-1})\Big[\frac{(t_0^{-1}t_1^{-1})^{p/2}-1}{t_0^{-1}t_1^{-1}-1}x(q)(t_0+t_1)(t_0^{-1}+t_1^{-1})(t_0+t_1)\\
&\qquad+\frac{2(t_0^{-1}-1)(t_1^{-1}-1)}{t_0^{-1}t_1^{-1}-1}\left(\frac{p}{2}-\frac{(t_0^{-1}t_1^{-1})^{p/2}-1}{t_0^{-1}t_1^{-1}-1}\right)x(q+1)(t_0+t_1)\Big]\\
&=x(-r)(t_0^{-1},t_1{}^{-1})(t_0+t_1)\Big[(t_0^{-1}+t_1^{-1})^2\frac{(t_0^{-1}t_1^{-1})^{p/2}-1}{t_0^{-1}t_1^{-1}-1}x(q)\\
&\qquad+(t_0^{-1}+t_1^{-1})\frac{2(t_0^{-1}-1)(t_1^{-1}-1)}{t_0^{-1}t_1^{-1}-1}\left(\frac{p}{2}-\frac{(t_0^{-1}t_1^{-1})^{p/2}-1}{t_0^{-1}t_1^{-1}-1}\right)x(q)\\
&\qquad+\frac{(t_0^{-1}t_1^{-1})^{p/2}-1}{t_0^{-1}t_1^{-1}-1}x(q)(t_0+t_1)(t_0^{-1}+t_1^{-1})(-t_0^{-1}t_1^{-1})\\
&\qquad+(-t_0^{-1}t_1^{-1})\frac{2(t_0^{-1}-1)(t_1^{-1}-1)}{t_0^{-1}t_1^{-1}-1}\left(\frac{p}{2}-\frac{(t_0^{-1}t_1^{-1})^{p/2}-1}{t_0^{-1}t_1^{-1}-1}\right)x(q+1)\Big]\\
&=x(-r)(t_0^{-1},t_1{}^{-1})(t_0+t_1)\Big[x(q)\frac{(t_0^{-1}t_1^{-1})^{p/2}-1}{t_0^{-1}t_1^{-1}-1}(t_0^{-2}+t_1^{-2}-t_0^{-1}t_1^{-1}(t_0t_1^{-1}+t_1t_0^{-1}))\\
&\qquad+\frac{2(t_0^{-1}-1)(t_1^{-1}-1)}{t_0^{-1}t_1^{-1}-1}\left(\frac{p}{2}-\frac{(t_0^{-1}t_1^{-1})^{p/2}-1}{t_0^{-1}t_1^{-1}-1}\right)\big((t_0^{-1}+t_1^{-1})x(q)-t_0^{-1}t_1^{-1}x(q+1)\big)\Big].
\end{aligned}$$

But $t_0^{-2}+t_1^{-2}-t_0^{-1}t_1^{-1}(t_0t_1^{-1}+t_1t_0^{-1})=0$. So to show the two terms of highest and lowest degree disappear in that polynomial we show that modulo lower degree terms, $\alpha = (t_0^{-1}+t_1^{-1})x(q)-t_0^{-1}t_1^{-1}x(q+1)=0$. Let's look at $x(q)$ first of all:

$$x(q)=\frac{t_0^q-t_1^q+\text{ other terms}}{(t_0+1)(t_1+1)(t_0-t_1)}=M+m+\text{ other terms},$$

where M is the term of highest degree in $x(q)$, and m the one of smallest degree. That way

$$t_0^q-t_1^q+\text{ other terms }=(t_0+1)(t_1+1)(t_0-t_1)(M+m+\text{ other terms}).$$

And identifying the highest and lowest degree terms on each side we find

$$M=t_0^{q-2}\text{ and }m=t_1^{q-2}.$$

Finally, modulo lower degree terms,

$$\begin{aligned}\alpha &= (t_0^{-1}+t_1^{-1})(t_0^{q-2}+t_1^{q-2}) - t_0^{-1}t_1^{-1}(t_0^{q-1}+t_1^{q-1}) \\ &= t_1^{-1}t_0^{q-2} - t_0^{-1}t_1^{-1}t_0^{q-1} + t_0^{-1}t_1^{q-2} - t_0^{-1}t_1^{-1}t_1^{q-1} \\ &= 0.\end{aligned}$$

In conclusion,

$$span(LG(K(p,q,r))) \leqslant (2q-2r-2) - \underline{2} = 2q-2r-4 = 4\left(\frac{q-r-2}{2}\right).$$

List of Figures

Bibliography

[1] J. W. Alexander. *A lemma on a system of knotted curves*. Proc. Natl. Acad. Sci. USA, 9: 93–95.

[2] J. W. Alexander. *Topological Invariants of Knots and Links*. Trans. Amer. Math. Soc., 30, 275-306, 1928.

[3] J. W. Alexander, G. B. Briggs. *On types of knotted curves*. Ann. of Math., 28 (1926): 562–586.

[4] J. C. Cha, C. Livingston. *KnotInfo: Table of Knot Invariants*. `http://www.indiana.edu/~knotinfo`

[5] F. Costantino, N. Geer, B. Patureau-Mirand. *Quantum invariants of 3-manifolds via link surgery presentations and non-semi-simple categories*, J. Topology (2014) 7 (4) 1005–1053.

[6] R. H. Crowell. *Genus of Alternating Link Types*. Ann. of Math., (2) Vol. 69 (1959), 258-275.

[7] D. De Wit. *An infinite suite of Links-Gould invariants*. J. Knot Theory Ramifications. 10 (2001), no.1, 37-62. MR 2002e:57015

[8] D. De Wit - *Automatic construction of explicit R matrices for the one-parameter families of irreducible typical highest weight representations of* $U_q[\mathfrak{gl}(m|n)]$, Comput. Phys. Commun. 145:2, 205-255.

[9] D. De Wit. *Links-Gould Explorer*. `http://www.maths.uq.edu.au/cmp/Links--Gould%20Explorer.html`

[10] D. De Wit, A. Ishii, J. Links - *Infinitely many two-variable generalisations of the Alexander–Conway polynomial*, Algebraic & Geometric Topology 5 (2005) 405–418.

[11] D. De Wit, L.H. Kauffman, J. Links. *On the Links-Gould invariant of links*. J. Knot Theory Ramifications. 8 (1999), no.2, 165-199. MR 2000j:57020

[12] D. De Wit, J. Links. *Where the Links-Gould invariant first fails to distinguish nonmutant prime knots*. J. Knot Theory Ramifications. 16, 1021 (2007). DOI: 10.1142/S0218216507005658.

[13] V. G. Drinfel'd. *On almost cocommutative Hopf algebras*. Algebra i Analiz, 1 (1989), 30-47.

[14] R.H. Fox, G. Torres. *Dual presentations of the group of a knot*. The Annals of Mathematics. Second Series, vol.59, no.2 (Mar., 1954), 211-218.

[15] P. Freed, D. Yetter. *Braided compact closed categories with applications to low dimensional topology*. Adv. in Math., 77(2) (1989), 156-182.

[16] P. Freed, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, A. Ocneanu. *Braided compact closed categories with applications to low dimensional topology*. Bulletin of the American Mathematical Society, 12(2) (1985), 239-246.

[17] S. Friedl, T. Kim. *The Thurston norm, fibered manifolds and twisted Alexander polynomials*. Topology, 45, 929-953 (2006).

[18] N. Geer, B. Patureau-Mirand. *Multivariable link invariants arising from sl(2|1) and the Alexander polynomial*. J. Pure Appl. Algebr., Volume 210, Issue 1, July 2007, Pages 283–298.

[19] N. Geer, B. Patureau-Mirand. *Multivariable link invariants arising from Lie superalgebras of type I.* J. Knot Theory Ramifications 19, Issue 1 (2010) 93–115.

[20] N. Geer, B. Patureau-Mirand. *On the Colored HOMFLY-PT, Multivariable and Kashaev Link Invariants.* Commun. Contemp. Math. 10 (2008), no. 1 supp, 993-1011.

[21] C. M. Gordon, J. Luecke. *Knots are determined by their complements*. J. Amer. Math. Soc., 2 (1989) 371-415.

[22] J. Hoste, M. Thistlethwaite, J. Weeks. *The first 1,701,936 knots*. The Mathematical Intelligencer. 20(4):33–48, 1998.

[23] A. Ishii. *The Links-Gould polynomial as a generalization of the Alexander-Conway polynomial*. Pacific J. Math., 225 (2006) 273–285.

[24] A. Ishii. *Algebraic links and skein relations of the Links-Gould invariant*. Proc. Amer. Math. Soc., 132 (2004), pp. 3741-3749.

[25] V. F. R. Jones. *A polynomial invariant for knots via von Neumann algebras*. Bull. Amer. Math. Soc., 12 (1985) 103-111.

[26] T. Kanenobu. *Genus and Kauffman polynomial of a 2-bridge knot*. Osaka J. Math., Volume 29, Number 3 (1992), 635-651.

[27] R. M. Kashaev. *The hyperbolic volume of knots from the quantum dilogarithm*. Letters in Mathematical Physics, 39 (3) (1997), 269-275.

[28] L. H. Kauffman. *State Models and the Jones Polynomial*. Topology, 26, 395-407, 1987.

[29] S.M. Khoroshkin, V.N. Tolstoy. *Universal R-matrix for quantized (super)algebras.* Comm. Math. Phys. **141** (1991), no. 3, 599–617.

[30] D. Kim, J. Lee. *Some invariants of pretzel links*. Bull. Austral. Math. Soc., Vol. 75 (2007), 253-271.

[31] B.-M. Kohli. *On the Links-Gould invariant and the square of the Alexander polynomial*. J. Knot Theory Ramifications, Vol. 25, No. 02, 1650006 (2016).

[32] B.-M. Kohli. *The Links-Gould invariant as a classical generalization of the Alexander polynomial ?* Preprint, 2016, `https://hal.archives-ouvertes.fr/hal-01342901`, accepted for publication in Exper. Math.

[33] B.-M. Kohli, B. Patureau-Mirand. *Other quantum relatives of the Alexander polynomial through the Links-Gould invariant*. Preprint, 2016, `http://arxiv.org/abs/1606.02538`.

[34] J. Lacan. *Écrits: A Selection*. Trans. Alan Sheridan, London: Tavistock Publications, 1977.

[35] J. Links, M.D. Gould. *Two variable link polynomials from quantum supergroups*. Letters in Mathematical Physics, 26(3):187-198, November 1992.

[36] S. Majid. *Cross products by braided groups and bosonization*, Journal of Algebra 163 (1), 165-190.

[37] I. Marin, E. Wagner. *A cubic defining algebra for the Links-Gould polynomial*. Advances in Math., 248 (2013), 1332-1365.

[38] A. A. Markov. *Über die freie Äquivalenz der geschlossenen Zöpfe*. Mat. Sbornik, 43, 73-78, 1936.

[39] H. R. Morton. *Seifert circles and knot polynomials*. Proc. Camb. Phil. Soc., 99 (1986), 107-109.

[40] J. Murakami. *The multi-variable Alexander polynomial and a one-parameter family of representations of* $U_q(\mathfrak{sl}(2,\mathbb{C}))$ *at* $q^2 = -1$. Quantum groups (Leningrad, 1990), 350–353, Lecture Notes in Math., 1510, Springer, Berlin, 1992.

[41] K. Murasugi. *On the genus of the alternating knot II*. J. Math. Soc. Japan, Volume 10, Number 3 (1958), 235-248.

[42] K. Murasugi. *On a certain subgroup of the group of an alternating link*. Amer. J. Math., 85 (1963), 544-550.

[43] K. Murasugi. *The Jones Polynomial and Classical Conjectures in Knot Theory*. Topology, 26, 187-194, 1987.

[44] K. Murasugi. *Jones Polynomials and Classical Conjectures in Knot Theory II*. Math. Proc. Cambridge Philos. Soc., 102, 317-318, 1987.

[45] L. Neuwirth. *Knot Groups*. Ann. of Math. Stud. 56, Princeton University Press, Princeton, N.J., 1965.

[46] T. Ohtsuki. *Quantum invariants. A study of knots, 3–manifolds, and their sets.* Series on Knots and Everything, 29. World Scientific Publishing Co., Inc., River Edge, NJ, (2002).

[47] B. Pareigis. *A non-commutative non-cocommutative Hopf algebra in "nature"*. Journal of Algebra, Volume 70, Issue 2, June 1981, Pages 356-374.

[48] M. Polyak. *Minimal generating sets of Reidemeister moves*. Quantum Topol., 1 (2010), 399–411.

[49] E. Rapaport. *On the commutator subgroup of a knot group*. Ann. of Math., (2) 71 (1960), 157-162.

[50] K. Reidemeister. *Elementare Begründung der Knotentheorie*. Abh. Math. Sem. Univ. Hamburg, 5 (1): 24–32, 1927.

[51] N. Reshetikhin. *Quasitriangularity of Quantum Groups at Roots of 1.* Comm. Math. Phys. 170 (1995), 79–99.

[52] N. Reshetikhin, V.G. Turaev. *Ribbon graphs and their invariants derived from quantum groups.* Comm. Math. Phys. 127 (1990), no. 1, 1–26.

[53] H. Seifert. *Über das Geschlecht von Knoten*. Math. Annalen, 110 (1): 571–592, 1934.

[54] L. Siebenmann. *Exercices sur les nœuds rationnels*. unpublished notes from the Département de Mathématiques d'Orsay, 1975.

[55] J. Stallings. *On fibering certain 3-manifolds*. In Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961), Prentice-Hall, Englewood Cliffs, N.J., 1962, 95-100.

[56] A. Stoimenow. *Knot Data Tables*. `http://stoimenov.net/stoimeno/homepage/ptab/13n.gen`

[57] P. G. Tait. *On Knots I, II, III*. Scientific Papers, Vol. 1. London: Cambridge University Press, pp. 273-347, 1900.

[58] V. G. Turaev. *The Yang-Baxter equation and invariants of links*. Invent. Math., 92 (1988), 527-553.

[59] V. G. Turaev. *Operator invariants of tangles, and R-matrices*. Izv. Akad. Nauk SSSR Ser. Math, 53 (1989).

[60] A. T. Vandermonde. *Remarques sur les problèmes de situation.* 1771.

[61] O. Viro. *Quantum relatives of the Alexander polynomial.* Algebra i Analiz 18 (2006), no. 3, 63-157; translation in St. Petersburg Math. J. 18 (2007), no. 3, 391-457.

[62] P. Vogel. *Representation of links by braids. A new algorithm*. Comm. Math. Helv., 65 (1990), pp 104-113.

[63] R.B. Zhang. *Universal L operator and invariants of the quantum supergroup* $U_q(\mathfrak{gl}(m|n))$, J. Math. Phys. 33 (1992), 1970–1979.

Résumé

On s'intéresse dans cette thèse aux rapports qui existent entre deux invariants d'entrelacs. D'une part l'invariant d'Alexander Δ qui est l'invariant de nœuds le plus classique, et le plus étudié avec le polynôme de Jones, et d'autre part la famille des invariants de Links-Gould $LG^{n,m}$ qui sont des invariants quantiques dérivés des super algèbres de Hopf $U_q\mathfrak{gl}(n|m)$. On démontre en particulier un cas de la conjecture de De Wit-Ishii-Links : certaines spécialisations des polynômes de Links-Gould fournissent des puissances du polynôme d'Alexander. Les polynômes LG sont donc des généralisations du polynôme d'Alexander. On conjecture de plus que ces invariants conservent certaines propriétés homologiques bien connues de Δ permettant d'évaluer le genre des entrelacs et de tester le caractère fibré des nœuds.

Mots clés nœud, entrelacs, polynôme d'Alexander, invariants de Links-Gould, algèbre de Hopf, R-matrice, genre, nœud fibré.

Abstract

In this thesis we focus on the connections that exist between two link invariants: first the Alexander-Conway invariant Δ that was the first polynomial link invariant to be discovered, and one of the most thoroughly studied since alongside with the Jones polynomial, and on the other hand the family of Links-Gould invariants $LG^{n,m}$ that are quantum link invariants derived from super Hopf algebras $U_q\mathfrak{gl}(n|m)$. We prove a case of the De Wit-Ishii-Links conjecture: in some cases we can recover powers of the Alexander polynomial as evaluations of the Links-Gould invariants. So the LG polynomials are generalizations of the Alexander invariant. Moreover we give evidence that these invariants should still have some of the most remarkable properties of the Alexander polynomial: they seem to offer a lower bound for the genus of links and a criterion for fiberedness of knots.

Keywords knot, link, Alexander polynomial, Links-Gould invariants, Hopf algebra, R-matrix, genus, fiberedness.