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Optimality conditions for optimal control problems and applications

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Conditions d'optimalité pour des problèmes en contrôle optimal et applications

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Abstract

The project of this thesis is twofold. The first concerns the extension of previous results on necessary optimality conditions for state constrained problems in optimal control and in calculus of variations. The second aim consists in working along two new research lines: derive viability results for a class of control systems with state constraints in which ‘standard inward pointing conditions’ are violated; and establish necessary optimality conditions for average cost minimization problems possibly perturbed by unknown parameters.

In the first part, we examine necessary optimality conditions which play an important role in finding candidates to be optimal solutions among all admissible solutions. However, in dynamic optimization problems with state constraints, some pathological situations might arise. For instance, it might occur that the multiplier associated with the objective function (to minimize) vanishes. In this case, the objective function to minimize does not intervene in first order necessary conditions: this is referred to as the *abnormal* case. A worse phenomenon, called the *degenerate* case shows that in some circumstances the set of admissible trajectories coincides with the set of candidates to be minimizers. Therefore the necessary conditions give no information on the possible minimizers. To overcome these difficulties, new additional hypotheses have to be imposed, known as *constraint qualifications*. We investigate these two issues (*normality* and *non-degeneracy*) for optimal control problems involving state constraints and dynamics expressed as a differential inclusion, when the minimizer has its left end-point in a region where the state constraint set is nonsmooth. We prove that under an additional information involving mainly the Clarke tangent cone, necessary conditions in the form of the Extended Euler-Lagrange condition are derived in the normal and non-degenerate form for two different classes of state constrained optimal control problems. Application of the normality result is shown also for the calculus of variations problem subject to a state constraint.

In the second part of the thesis, we consider first a class of state constrained control systems for which standard ‘first order’ constraint qualifications are not satisfied, but a higher (second) order constraint qualification is satisfied. We propose a new construction for feasible trajectories (a *viability* result) and we investigate examples (such as the Brockett nonholonomic integrator) providing in addition a non-linear estimate result. The other topic of the second part of the thesis concerns the study of a class of optimal control problems in which uncertainties appear in the data in terms of unknown parameters. Taking into consideration an average cost criterion, a crucial issue is clearly to be able to characterize optimal controls independently of the unknown parameter action: this allows to find a sort of ‘best compromise’ among all the possible realizations of the control system as the parameter varies. For this type of problems, we derive necessary optimality conditions in the form of Maximum Principle (possibly nonsmooth).

Résumé

Le projet de cette thèse est double. Le premier concerne l'extension des résultats précédents sur les conditions nécessaires d'optimalité pour des problèmes avec contraintes d'état, dans le cadre du contrôle optimal ainsi que dans le cadre de calcul des variations. Le deuxième objectif consiste à travailler sur deux nouveaux aspects de recherche: dériver des résultats de viabilité pour une classe de systèmes de contrôle avec des contraintes d'état dans lesquels les conditions dites 'standard inward pointing conditions' sont violées; et établir les conditions nécessaires d'optimalité pour des problèmes de minimisation de coût moyen éventuellement perturbés par des paramètres inconnus.

Dans la première partie, nous examinons les conditions nécessaires d'optimalité qui jouent un rôle important dans la recherche de candidats pour être des solutions optimales parmi toutes les solutions admissibles. Cependant, dans les problèmes d'optimisation dynamique avec contraintes d'état, certaines situations pathologiques pourraient survenir. Par exemple, il se peut que le multiplicateur associé à la fonction objective (à minimiser) disparaisse. Dans ce cas, la fonction objective à minimiser n'intervient pas dans les conditions nécessaires de premier ordre: il s'agit du cas dit *anormal*. Un phénomène pire, appelé le cas *dégénéré* montre que, dans certaines circonstances, l'ensemble des trajectoires admissibles coïncide avec l'ensemble des candidats minimiseurs. Par conséquent, les conditions nécessaires ne donnent aucune information sur les minimiseurs possibles. Pour surmonter ces difficultés, de nouvelles hypothèses supplémentaires doivent être imposées, appelées les *qualifications de la contrainte*. Nous étudions ces deux problèmes (*normalité* et *non dégénérescence*) pour des problèmes de contrôle optimal impliquant des contraintes dynamiques exprimées en termes d'inclusion différentielle, lorsque le minimiseur a son point de départ dans une région où la contrainte d'état est non lisse. Nous prouvons que sous une information supplémentaire impliquant principalement le cône tangent de Clarke, les conditions nécessaires sous la forme dite 'Extended Euler-Lagrange condition' sont satisfaites en forme normale et non dégénérée pour deux classes de problèmes de contrôle optimal avec contrainte d'état. Le résultat sur la normalité est également appliqué pour le problème de calcul des variations avec contrainte d'état.

Dans la deuxième partie de la thèse, nous considérons d'abord une classe de systèmes de contrôle avec contrainte d'état pour lesquels les qualifications de la contrainte standard du 'premier ordre' ne sont pas satisfaites, mais une qualification de la contrainte d'ordre supérieure (ordre 2) est satisfaite. Nous proposons une nouvelle construction des trajectoires admissibles (dit un résultat de *viabilité*) et nous étudions des exemples (tels que l'intégrateur non holonome de Brockett) fournissant en plus un résultat d'estimation non linéaire. L'autre sujet de la deuxième partie de la thèse concerne l'étude d'une classe de problèmes de contrôle optimal dans lesquels des incertitudes apparaissent dans les données en termes de paramètres inconnus. En tenant compte d'un critère de performance sous la forme de coût moyen, une question cruciale est clairement de pouvoir caractériser les contrôles optimaux indépendamment de l'action du paramètre inconnu: cela permet de trouver une sorte de 'meilleur compromis' parmi toutes les réalisations possibles du système de contrôle tant que le paramètre varie. Pour ce type de problèmes, nous obtenons des conditions nécessaires d'optimalité sous la forme du Principe du Maximum (éventuellement pour le cas non lisse).

Introduction

In control theory, a controller manipulates the inputs (known as *controls*) to a given system in order to obtain the desired effect on the output of the system. For instance, the driver controls the motion of the car by acting on the gas pedal and on the steering wheel, in order to achieve the desired behavior of steering the car into a parking spot. When we introduce a performance criterion (to minimize or to maximize) coupled with a control system, we step into optimal control theory domain: the objective of this theory is to provide tools for a choice among all possible strategies which accomplish the ‘best’ behavior. For instance, if $x(t)$ represents the number of tumor cells at time t , $u(t)$ the drug concentration, and we wish to minimize simultaneously the number of tumor cells at the end of the treatment period and the accumulated harmful effects of the drug on the body. This is an example of optimal control problem, where the control is the drug concentration and the objective functional to minimize would take into account the bad effects of both tumor cells and the drug.

Optimal control theory might be considered as a natural development of calculus of variations. The latter is a field of mathematics which possibly started in 1686 with J. Bernoulli who posed the ‘Brachistochrone problem’, which means *shortest time problem*: given two points A and B , the goal is to specify the curve along which the particle slides from A to B in minimal time. The gravitational attraction is the only force which affects the system. The challenge to solve this problem involved many mathematicians, for instance Euler, Lagrange, Laplace, etc, whose solutions marked the birth of new important mathematical tools. However, some recent works of Sussman and Willems [78], [79] show that looking at the Brachistochrone problem from the perspective of optimal control permits to give better and stronger results than the classical calculus of variations (regarding the nature, the existence and the uniqueness of optimal solutions to the problem).

An optimal control problem formulated in terms of controlled dynamics, can be written as follows:

$$(P) \quad \left\{ \begin{array}{l} \text{minimize } g(x(S), x(T)) \\ \text{over arcs } x(\cdot) \text{ and controls } u(\cdot) \text{ satisfying} \\ \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [S, T], \\ (x(S), x(T)) \in E_0 \times E_1, \\ u(t) \in U(t) \quad \text{a.e. } t \in [S, T], \end{array} \right.$$

the data for which comprise functions $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, and $f : [S, T] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$, closed sets $E_0, E_1 \subset \mathbb{R}^n$, and a multifunction $U(\cdot) : [S, T] \rightsquigarrow \mathbb{R}^m$. We call a local minimizer for (P) a process (\bar{u}, \bar{x}) satisfying the constraint of (P) and realizing the minimum cost in the neighborhood of any feasible process (u, x) .

Since optimal control problems deal with the issue of finding a control law for a given system such that an optimality criterion is achieved, many mathematicians have been interested in constructing conditions for which an optimal control can be derived. Considerable progress in the study of optimality conditions was made in the 1950s by the Russian mathematician

L. Pontryagin and his collaborators (cf. [66]), who formulated the celebrated Pontryagin's Maximum Principle (PMP), which provides a set of necessary conditions for problems like (P) which an optimum (if it exists) must satisfy. Based on the Pontryagin's Maximum Principle, optimal control strategies can be established for some real-life optimal control problems, for instance [55] and [8] (here, a hybrid version of the PMP is used). The alternative classical approach is based on the Dynamic Programming Principle (cf. [9]), which reduces the search of an optimal control function to the task of finding the solution to a partial differential equation known as the Hamilton-Jacobi-Bellman equation: this procedure gives a sufficient condition for minimizers. Many contributions concerning dynamic programming have been added to the literature, for instance [82], [70], and many others. This thesis is concerned with the necessary conditions.

The PMP states that, under mild hypotheses (we consider here regular data), and for a reference local minimizer (\bar{u}, \bar{x}) for (P), there exist an adjoint arc $p(\cdot) \in W^{1,1}([S, T], \mathbb{R}^n)$ and $\lambda \geq 0$ such that

- (i) $(p(\cdot), \lambda) \neq (0, 0)$;
- (ii) $-\dot{p}(t) = p(t)\nabla_x f(t, \bar{x}(t), \bar{u}(t)) \quad \text{a.e. } t$;
- (iii) $p(t) \cdot f(t, \bar{x}(t), \bar{u}(t)) = \max_{u \in U(t)} p(t) \cdot f(t, \bar{x}(t), u) \quad \text{a.e. } t$;
- (iv) $(p(S), -p(T)) \in \lambda \nabla_x g(\bar{x}(S), \bar{x}(T)) + N_{E_0 \times E_1}(\bar{x}(S), \bar{x}(T))$;

where $N_{E_0 \times E_1}(x, y)$ is the set of outward normals (in a sense that will be specified later) to $E_0 \times E_1$ at $(x, y) \in E_0 \times E_1$.

In optimal control problems, dynamics can be formulated also as a differential inclusion (see for instances [81], [47])

$$\dot{x}(t) \in F(t, x(t)) \quad \text{a.e. } t \in [S, T]. \quad (1)$$

Here, a solution is sometimes called an F -trajectory. We say that \bar{x} is a local minimizer for the differential inclusion optimal control problem, if it satisfies the constraint of the problem (i.e. feasible) and has minimum cost if compared with all feasible F -trajectories in a suitable neighborhood of \bar{x} . In this case, the necessary conditions can be expressed in different ways: for instance, well-known are the Extended Euler-Lagrange condition and the (Clarke's) Fully convexified Hamiltonian Inclusion. In this thesis, we deal with the Extended Euler-Lagrange condition, which is a generalization of Euler's equations of the classical calculus of variations theory, in the form of

$$\dot{p}(t) \in \text{co} \{ \eta \mid (\eta, p(t)) \in N_{\text{Gr } F(t, \cdot)}(\bar{x}(t), \dot{\bar{x}}(t)) \} \quad \text{a.e. } t \in [S, T].$$

(co X is the convex hull of the set X and $\text{Gr } F(t, \cdot)$ is the graph of $F(t, \cdot)$.) Indeed, an optimal control problem comprising a differential inclusion dynamics can be reformulated as

a variational problem: this is ensured if we add to the cost of the original problem a penalty term, taking account of the dynamic constraint:

$$g(x(S), x(T)) + \int_S^T L(t, x(t), \dot{x}(t)) dt$$

where $L(t, x, v) := \Psi_{G \cap F(t, \cdot)}(x, v)$. (Ψ_G denotes the indicator function of the set G .)

However, this reformulation brings new difficulties, since the penalty term has discontinuous derivatives. The issue is therefore to handle this discontinuity and to adapt traditional (smooth) necessary conditions to allow for nonsmooth cost integrands. In this framework, nonsmooth analysis emerges to solve this type of problems. It was initiated by F. Clarke [34] to generalize the concepts of differentiability in convex analysis to a larger class of problems. Different extensions of the usual differential was suggested also in the works of Frankowska [46], Ioffe [54], Mordukhovich [60], Rockafellar and Wets [72], and Warga [84]. Other significant developments for the PMP (employing many different techniques) can be found in [78] and [77] (and the references therein). In this dissertation, we make a large use of nonsmooth analysis tools providing a short overview in Chapter 1.

The thesis project can be divided into two main parts. The first one is to provide improvements of earlier results on nondegeneracy and normality properties of necessary optimality conditions (in the form of the Extended Euler-Lagrange condition: see Chapters 3 and 4). We consider as well a class of calculus of variations problems where slightly ameliorated results with respect to previous ones are established (cf. Chapter 5). The second part concerns the study of two challenging new class of problems: we provide the existence of feasible trajectories when the standard inward pointing condition is not satisfied (Chapter 6); we establish a maximum principle for optimal control problems with average cost (Chapter 7). The ideas are original to this thesis unless explicitly mentioned as adapted from earlier works.

The first class of problems addressed in the thesis is related to the study of *nondegeneracy* and *normality* of necessary conditions, in the form of the Extended Euler-Lagrange condition, for optimal control problems in which a pathwise state constraint is added to the problem formulation. Consider, for instance, the simpler case in which we have a scalar inequality constraint

$$h(x(t)) \leq 0 \quad \text{for all } t \in [S, T],$$

where $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuously differentiable function with locally Lipschitz derivatives. It is well known (cf. [81], [48]) that the presence of state constraints requires to consider an additional ‘multiplier’ in the form of a Borel measure μ which intervenes in the necessary conditions, by modifying slightly the adjoint arc $p(\cdot)$ as follows

$$p(t) + \int_{[S,t]} \nabla_x h(\bar{x}(s)) d\mu(s).$$

(Here, $\bar{x}(\cdot)$ is a reference local minimizer for the optimal control problem.)

In some circumstances, the optimality conditions convey no useful information about the minimizers (known as *degenerate case*), or apply in a form in which the cost multiplier vanishes (sometimes referred to as *abnormal case*).

Typical degeneracy phenomena might occur when the state constraint is active at the initial time. This is because, in such situations, the set of all admissible trajectories coincides with the set of candidates to be minimizers identified by the necessary optimality conditions. For instance, consider $E_0 = \{x_0\}$ and $h(x_0) = 0$, for some $x_0 \in \mathbb{R}^n$ such that $\nabla h(x_0) \neq 0$. In this case, it is straightforward to check that the necessary conditions are satisfied with the nonzero multipliers set

$$\lambda = 0 \quad \mu = \delta_S \quad \text{and} \quad p(\cdot) \equiv -\nabla_x h(x_0),$$

for any admissible process. Here, δ_S denotes the Dirac measure concentrated at S . Indeed, for such choice of multipliers, the expression $p(t) + \int_{[S,t]} \nabla_x h(\bar{x}(s)) d\mu(s)$, which features in the necessary conditions (for the state constrained case), vanishes for almost all times.

Another situation might occur when the necessary conditions are *abnormal*, meaning that the scalar multiplier associated with the objective function (denoted by λ) is equal to zero. In this case, the objective function term does not intervene in the necessary conditions.

Nondegenerate and normal forms of the necessary optimality conditions were studied along the years in order to facilitate the search of minimizers. In order to establish the nondegeneracy as well as the normality, conditions, known as *constraint qualifications*, are imposed. They permit to identify a class of problems for which strengthened forms of the necessary conditions hold.

The degeneracy of the necessary conditions for state constrained problems has already been developed in the literature even for nonsmooth data, see for example [37], [40], [39], [42], [57], in the framework of controlled dynamics; and [3], [69], [81], in the case of differential inclusion dynamics. These works, and many others, provide a variety of constraint qualifications. For the controlled dynamics case, a classical constraint qualification requires the existence of a control function pushing the state away from the boundary of the state constraint set (and inside the state constraint set) faster than the optimal control on a neighborhood of the initial time (cf. [39], [40]). Another constraint qualification used for instance in differential inclusion problems, requires the existence of a vector belonging to the set of velocities, which pushes the state inside the state constraint in a neighborhood of the initial time. More precisely, if the feasible trajectories are characterized by $h(x(t)) \leq 0$, where h is a continuously differentiable function with locally Lipschitz derivatives, the constraint qualification takes the following form: for a given minimizer $\bar{x}(\cdot)$ and for some $\delta > 0$ (see figure 0.1),

$$\min_{v \in F(t,x)} \langle v, \nabla h(x) \rangle < -\delta \quad \text{for each } t \text{ near } S \text{ and for each } x \text{ near } \bar{x}(S). \quad (2)$$

Such constraint qualification is considered for instance in [69], to establish nondegeneracy for differential inclusion problems. In this paper, the proof technique is to use an approach suggested in [68], and successively developed in later work: this requires exhibiting the concept of local existence of neighboring feasible trajectories satisfying some $W^{1,1}$ -linear estimates from the set of trajectories violating the state constraint. This concept is valid under the imposed inward pointing condition (2).

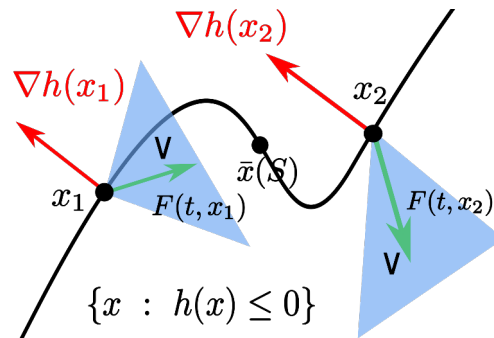


Figure 0.1 – Inward Pointing Condition

One natural question arises. What could be a possible inward pointing condition which can extend the nondegeneracy results to cover cases when the state constraint is not smooth? (For instance when $\bar{x}(S)$ is in a nonsmooth region of the state constraint). This question is investigated in Chapter 3 (whose results have been published in [15]), where we solve this issue for optimal control problems having dynamics formulated as in (1), and we allow the state constraint to be merely a closed set

$$x(t) \in A \quad \text{for all } t \in [S, T],$$

and the velocity set $F(., .)$ to be measurable w.r.t. the time variable. In earlier work either some regularity assumption w.r.t. the time variable is imposed to the velocity set (for instance Lipschitz continuous cf. [3], [81]; of bounded variation, see [62]), or the state constraint is represented by the intersection of regular sets, cf. [69]. Moreover, for the case in which a minimizer starts from a region where the state constraint is nonsmooth, we suggest a new constraint qualification in which no inward pointing condition involving the velocity set is required: the new condition invokes just hypertangent vectors of A at the initial state $\bar{x}(S)$ of a reference minimizer $\bar{x}(\cdot)$ and a distance property of trajectories (close to a reference minimizer w.r.t. the $W^{1,1}$ -norm) to the endpoint constraints. Under the imposed constraint qualification, and in order to establish the nondegeneracy of the necessary conditions in the nonsmooth case, we use a particular technique based on the construction, locally in time, of neighboring feasible trajectories verifying $W^{1,1}$ -linear estimates (cf. Chapter 2).

Normality of the necessary optimality conditions has been also studied for a large class of problems. In [43], [49], [42], [68], [13], [51], [50], and many other papers, it is shown, for controlled dynamics, that normality is guaranteed if an inward pointing condition is imposed on the neighborhood of each time for which the optimal trajectory touches the boundary of the state constraint. Normality was extended also to the case of differential inclusion dynamics for a state constraint expressed as a closed set (see for example [12]). In [12], the authors investigate a state constrained optimal control problem in which the cost to minimize comprises both integral and endpoint terms

$$g(x(S), x(T)) + \int_S^T L(t, x(t), \dot{x}(t)) dt,$$

where the Lagrangian $L(t, \cdot, \cdot)$ is a Lipschitz continuous function, measurable w.r.t. time, and the right endpoint set is $E_1 = \mathbb{R}^n$. Moreover, a time-independent velocity set is considered, namely

$$\dot{x}(t) \in F(x(t)) \quad \text{a.e. } t \in [S, T].$$

Normality (i.e. when $\lambda = 1$) is derived in [12] under the constraint qualification

$$\text{co } F(x) \cap \text{int } T_A(x) \neq \emptyset \quad \text{for all } x \in \partial A, \quad (3)$$

(here, $T_A(x)$ is the Clarke tangent cone to A at the point x) with a convexity assumption on A , if one of the two cases occurs: the first case is when $\bar{x}(S)$ belongs to a smooth region of the state constraint set, while the second case concerns $E_0 = \mathbb{R}^n$ (here $\bar{x}(S)$ is allowed to be a ‘corner’ of A). The proof technique, in both cases, is based on a global (on the time interval) construction of neighboring feasible trajectories with $W^{1,1}$ -linear estimates.

What if E_0 is allowed to be a possible strict subset of \mathbb{R}^n ? And under which new conditions normality can be still established when the initial state of the reference minimizer is located in a corner of the state constraint set?

An answer is given in Chapter 4 (results are published in [17]). In fact, in some situations, namely when the left endpoint constraint is merely a closed set, classical constraint qualifications in the sense of (3) cannot be used alone to guarantee normality of the necessary conditions. The key feature is therefore to prove that additional information involving tangent vectors to the left endpoint and the state constraint sets can be used to establish normality. The proof technique to establish normality when $\bar{x}(S)$ is at a corner of the state constraint is to construct a global result (in the spirit of [12]) on the existence of neighboring feasible trajectories with $W^{1,1}$ -linear estimates. The main novelty w.r.t. previous work on this topic is that the constraint qualification that we suggest is also less restrictive than earlier conditions for related cases.

A further problem addressed in the first part of the thesis concerns the study of the normality of necessary conditions for calculus of variations problems with state constraints, formulated as

$$(CV) \quad \left\{ \begin{array}{l} \text{minimize} \quad \int_S^T L(x(t), \dot{x}(t)) dt \\ \text{over arcs } x(\cdot) \in W^{1,1}([S, T], \mathbb{R}^n) \text{ satisfying} \\ \quad \quad \quad x(S) = x_0, \\ \quad \quad \quad x(t) \in A \quad \text{for all } t \in [S, T], \end{array} \right.$$

where $A \subset \mathbb{R}^n$ is a closed set.

Normality is derived in [39] for calculus of variations problems, studied for $W^{1,1}$ -local minimizers and with a state constraint expressed in terms of an inequality of a twice continuously differentiable function. This result has been extended in [43] to the nonsmooth case, for L^∞ -local minimizers, imposing a constraint qualification which makes use of some hybrid subgradients to cover situations in which the function, which defines the state constraint set, is not differentiable. In Chapter 5 (cf. [56]), we suggest a natural constraint qualification which assumes that if the interior of the Clarke tangent cone to the state constraint is nonempty, a

normality result for the necessary conditions is provided. This extends the normality theorem in [43] because our suggested constraint qualification covers cases in which the constraint qualification present in [43] is not satisfied. Moreover, one of our results is valid for $W^{1,1}$ -local minimizers not only for L^∞ -local minimizers.

The second part of the thesis explores two challenging problems in new directions. One topic is concerned with an innovative concept for constructing feasible trajectories for state constrained control systems, when the classical inward pointing condition like (2) is violated. To our knowledge, no results exist in this spirit and our approach to deal with this issue is completely new¹.

As already known, conditions in the form of (2) are crucial for the construction of feasible trajectories satisfying a $W^{1,1}$ -linear estimate, for state constrained control systems. However, in many cases of interest, for instance for the Brockett nonholonomic integrator, condition (2) is violated. Therefore, this type of construction is not possible.

How to overcome this difficulty? Can we propose a different methodology for the construction of feasible trajectories? Can we establish also some results on the estimate between a given reference trajectory possibly violating the constraint and the set of feasible trajectories?

These questions are examined in Chapter 6. In this framework, we consider first a (affine and smooth) control system in the form of

$$\dot{x} = f(x, u) = u_1 f_1(x) + u_2 f_2(x)$$

subject to a state constraint formulated as a scalar inequality function of class C^2 :

$$h(x(t)) \leq 0 \quad \text{for all } t \in [S, T].$$

We propose a particular construction of controls $v = (v_1, v_2)$, which will permit the corresponding vector field $f(x, v)$ to rotate in a suitable way and with sufficient intensity, in order to move in the interior of the state constraint: this is a viability result (cf. [6]). In this case, we prove that the constructed trajectory enters in the state constraint set ‘slower’ than when the classical inward pointing condition (2) holds true (cf. [69]: the rate with which the trajectory enters the state constraint is of order 1, while it is of order 3 for our analysis when (2) is violated).

This viability result constitutes the first step for neighboring feasible trajectories results, which will be examined for the Brockett nonholonomic integrator case. More precisely, we present two examples (always for the Brockett nonholonomic integrator case) where the global (in time) construction of neighboring feasible trajectories is straightforward, but leads to a nonlinear $W^{1,1}$ -estimate. (This requires to make use of the approach of the established viability result.)

The second topic concerns the study of a class of optimal control problems in which uncertainties appear in the data in terms of unknown parameters belonging to a given metric space. Though the state evolution is governed by a deterministic control system and the

¹This is an ongoing project initiated in collaboration with G. Colombo and F. Rampazzo. It started during my ‘Mobilité Sortante’, which allowed me to visit the University of Padova in Italy during my Ph.D.

initial datum is fixed (and well-known), the description of the dynamics depends on uncertain parameters which intervene also in the cost function and the right endpoint constraints. Taking into consideration an average cost criterion, a crucial issue is clearly to be able to characterize optimal controls independently of the unknown parameter action: this allows to find a sort of ‘best trade-off’ among all the possible realizations of the control system as the parameter varies. In this context, we provide in Chapter 7, under non restrictive assumptions, necessary optimality conditions, for an optimal control problem for which the *average cost* takes the form of:

$$\text{minimize } \int_{\Omega} g(x(T, \omega); \omega) d\mu(\omega),$$

where $g : \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}$ is a given function, T is the final time, and μ is a probability measure defined on Ω with a possibly non-finite support. The proof’s technique consists in approximating the measure μ by measures with finite support. This discretization yields properties about the approximate minimizers for an auxiliary problem, and making use of limit-taking results, necessary optimality conditions are established for the problem in the general case. An important feature of our results is that we allow the unknown parameters to belong to a mere complete separable metric space (not necessarily compact), which is crucial for applications in aerospace systems (cf. [73], [74]). The study of this new paradigm is, therefore, motivated not only by theoretical reasons but also by a recent growing interest in applications (such as aerospace engineering, heterogeneous systems, see for instance [73], [74], [80], [26]).

Organization of the manuscript

The dissertation is composed of 7 chapters. In Chapter 1, we provide some insights into nonsmooth analysis as well as optimal control theory, which will be crucial for the following chapters. We recall nonsmooth necessary optimality conditions for optimal control problems with and without state constraint, in the form of the Pontryagin’s Maximum Principle or the Extended Euler-Lagrange condition. Chapter 2 is devoted to establish new results concerning the existence of neighboring feasible trajectories with $W^{1,1}$ -linear estimates, to cover situations in which the initial data belongs to a nonsmooth region of the state constraint. The established results will be crucial for Chapters 3 and 4 to derive the nondegeneracy and the normality of the necessary optimality conditions in the form of the Extended Euler-Lagrange condition for state constrained optimal control problems. Examples at the end of each of Chapters 3 and 4 are provided to emphasize the results. Chapter 5 deals with calculus of variations problems with a state constraint formulated as a given closed set. We prove that under a certain constraint qualification, the necessary conditions apply in the normal form. In Chapter 6, we present a new approach to construct feasible trajectories when the classical inward pointing condition fails to hold true. Some examples are provided as well to extend the result to the construction of neighboring feasible trajectories with nonlinear $W^{1,1}$ -estimates. Chapter 7 is dedicated to present a theory for deriving necessary optimality conditions, in the form of PMP, for average cost minimization problems, involving uncertainties in the control system and in the right endpoint constraints.

Acknowledgments

If you are ambitious in challenging yourself, pushing yourself to new levels, achieving each day a new difficult goal and immersing yourself with no limited knowledge, then a doctorate may be indeed for you. You should know that a Ph.D. is a path, not an end. It needs accompanying people who will provide you with the light that you will use to uncover progressively your path. I was fortunate to have many people who step into my path to accomplish my dream of pursuing a Ph.D. in Mathematics. The most important is my advisor Professor Piernicola Bettiol. Your help, assist and guidance helped me during my Ph.D., as well as in my master thesis. I owe you my deep gratitude and I appreciate all your contributions of time and bright ideas to make my Ph.D. experience productive and stimulating. Your availability and wide knowledge were certainly prior for my own development. I thank you also for the helpful discussions when I asked for advice for my future plans and opportunities in research.

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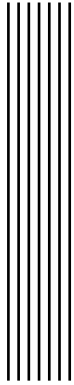
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To my strength and my weakness...

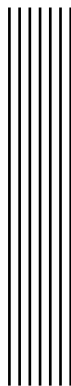
my family



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Notation

\mathbb{B}	Closed unit ball in the Euclidean space
$ x $	Euclidean norm of x
$d_C(x)$	Euclidean distance of x from C
$\text{diam } C$	Diameter of C
$\text{dom } f$	Domain of f
$\text{int } C$	Interior of C
∂C	Boundary of C
\bar{C}	Closure of C
$\text{co } C$	Convex hull of C
$N_C^P(x)$	Proximal normal cone to C at x
$N_C(x)$	Limiting normal cone to C at x
$\partial^P f(x)$	Proximal subdifferential of f at x
$\partial f(x)$	Limiting subdifferential of f at x
$\partial^> f(x)$	Hybrid subdifferential of f at x
$\nabla f(x)$	Gradient vector of f at x
$T_C(x)$	Clarke tangent cone to C at x
$T_C^B(x)$	Bouligand tangent cone to C at x
$\text{Gr } f$	Graph of f
$\text{epi } f$	Epigraph of f
$\pi_C(x)$	Projection of x on C
$x_i \xrightarrow{C} x$	$x_i \rightarrow x$ and $x_i \in C$ for all i
$x_i \xrightarrow{f} x$	$x_i \rightarrow x$ and $f(x_i) \rightarrow f(x)$ for all i
$\text{supp}(\mu)$	Support of a measure μ
$W^{1,1}(I, \mathbb{R}^n)$	Absolute continuous function $f : I \rightarrow \mathbb{R}^n$
\mathcal{L}	Lebesgue σ -algebra
\mathcal{B}^m	Borel σ -algebra of \mathbb{R}^m
$\mathcal{L} \times \mathcal{B}^m$	Product σ -algebra of \mathcal{L} and \mathcal{B}^m
δ_x	Dirac measure centered on x
$x \rightsquigarrow F(x)$	Set-valued map (or multifunction) F
Ψ_G	Indicator function of the set G



1

Preliminaries

The purpose of this chapter is to give a brief overview of nonsmooth analysis and optimal control problems with possibly a state constraint, and dynamic constraints expressed in terms of either a controlled differential equation or a differential inclusion. We also provide an existence result of minimizers and first order necessary conditions in the form of the Maximum Principle and the Extended Euler-Lagrange condition. The selected materials are crucial for the proofs of the main results of this thesis. The reader is referred to the monographs [81], [32], [5], [72], [34], [30], [60] for a deeper understanding of the tools and standard results exposed in this chapter.

Le but de ce chapitre est de donner un bref aperçu de l'analyse non lisse et des problèmes de contrôle optimal avec éventuellement une contrainte d'état et des contraintes dynamiques exprimées en termes d'équation différentielle contrôlée ou d'inclusion différentielle. Nous fournissons également un résultat d'existence des minimiseurs et des conditions nécessaires du premier ordre sous la forme du Principe du Maximum et des conditions dite 'Extended Euler-Lagrange condition'. Le matériel sélectionné est crucial pour les preuves des principaux résultats de cette thèse. Le lecteur est référé aux monographies [81], [32], [5], [72], [34], [30], [60] pour une compréhension plus approfondie des outils et des résultats standard exposés dans ce chapitre.

*“To understand the heart and mind
of a person, look not at what he
has already achieved, but at what
he aspires to.”*

— Gibran Khalil Gibran

1.1 Nonsmooth Analysis Tools

The term ‘nonsmooth’ refers to situations in which smoothness (differentiability) of the data is not necessarily postulated. In mathematics and optimization, there is an increasing interest in the possible occurrence of nonsmooth phenomena and there is a need to be able to deal with them. We are thus led to study how traditional smooth tools (normal vector to a smooth set, gradient of a differentiable function, etc.) can be generalized in a nonsmooth setting. This is the core of nonsmooth analysis which is well suited to this purpose. The interest and the utility of the tools and methods of nonsmooth analysis and optimization are not confined only to situations in which nonsmoothness is present. Sometimes in order to solve difficult smooth problems, we need to recall materials from the nonsmooth analysis in order to simplify the problem in hands. F. Clarke [32], [34] is one of the contributors to the nonsmooth analysis (a term that is due to him), and in particular for his main theory on generalized gradients. In this section, we are concerned with the local approximation of sets with nondifferentiable boundaries and of nondifferentiable functions. In other terms, we focus on the way with which we can adapt classical concepts of outward normals to subsets of vector spaces with smooth boundaries and of gradients of differentiable functions, to deal with situations in which the boundaries are nonsmooth and the functions are nondifferentiable. Since its inception in the early 1970s, there has been a fruitful interplay between nonsmooth analysis and optimal control (this interplay will be detailed in the next section). A familiarity with nonsmooth analysis is, therefore, essential for an in-depth understanding of present day research in optimal control. Emphasis is given to proximal normal vectors, proximal subgradients, and different types of limit of such vectors. The notion of tangency that does not require the set to be smooth or convex is presented also. The corresponding set is known as the Clarke tangent cone and it has a polarity relation with the normal cone.

In what follows, we take $C \subset \mathbb{R}^n$ (finite dimensional space) to be a closed set. A generalization of ‘outward normal vector’ to general closed sets is presented in the following definition:

Definition 1.1.1. *We say that a vector $\eta \in \mathbb{R}^n$ is a proximal normal vector to C at the point x if there exist a constant $M > 0$ such that*

$$\eta \cdot (y - x) \leq M|y - x|^2 \quad \text{for all } y \in C. \quad (1.1)$$

The cone of all proximal normal vectors to C at some point $x \in C$ is called the proximal normal cone, and denoted by $N_C^P(x)$.

Geometrically speaking, η is a proximal normal vector to C at x if there exists $r \geq 0$ such that

$$\eta = r(z - x),$$

where the point z has the point x as the unique closest point in C . Equivalently, the closed ball centered at z meets C only at x (see Figure 1.1 (a)).

We shall see in the coming sections that the proximal normal cone will not appear in the derivation of the necessary optimality conditions. This is because the derivation of many useful relations involving normal cones makes use of limit-taking with respect to the base-point x . If the cones in question are the proximal normal ones, the membership to such cones is not

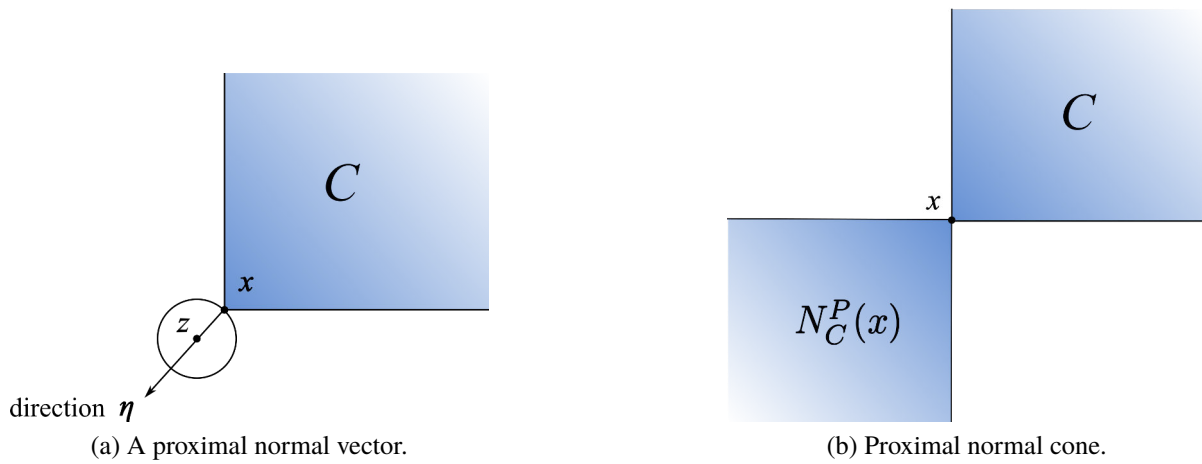


Figure 1.1 – Normal cones.

in general preserved under limit-taking. Therefore, a new tool, which is stable by limit-taking operation, is defined below and known as the limiting normal cone. It is constructed by adding extra normals which are limits of the proximal normal vectors at nearby base-points.

Definition 1.1.2. A vector $\eta \in \mathbb{R}^n$ is a limiting normal vector to C at $x \in C$ if there exist sequences $x_i \xrightarrow{C} x$ and $\eta_i \rightarrow \eta$ such that

$$\eta_i \in N_C^P(x_i) \text{ for all } i .$$

The cone of all limiting normal vectors to C at x is denoted by $N_C(x)$ and known as the limiting normal cone to C at x .

Figure 1.2 below gives a clear geometrical idea for the limiting normal cone.

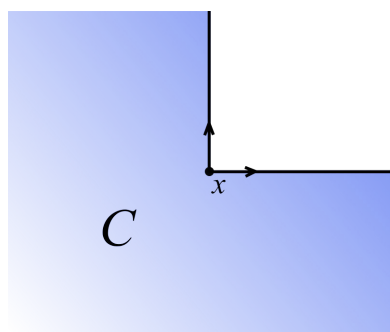


Figure 1.2 – Limiting normal cones comprising only two vectors

If we deal with convex closed sets C , we recover with Definitions 1.1.1 and 1.1.2 a familiar construction from convex analysis as follows:

Proposition 1.1.3. Let C be a closed and convex set and $x \in C$, then

$$N_C^P(x) = N_C(x) = \{ \eta \mid \eta \cdot (y - x) \leq 0 \text{ for all } y \in C \} .$$

We refer to [81] for a detailed proof of Proposition 1.1.3.

We list below some properties satisfied by the proximal and the limiting normal cones:

Proposition 1.1.4. *Take a closed subset C of \mathbb{R}^n and a point $x \in C$. Then, the proximal and limiting normal cones have the following properties:*

1) $N_C^P(x)$ and $N_C(x)$ are sets of \mathbb{R}^n containing $\{0\}$ such that

$$N_C^P(x) \subset N_C(x) . \quad (1.2)$$

2) If $x \in \text{int } C$, then $N_C^P(x) = N_C(x) = \{0\}$; and if $x \in \partial C$, $N_C(x)$ has nonzero elements.

3) $N_C^P(x)$ is convex (not necessarily closed), while $N_C(x)$ is closed.

Proof. From the analytical definitions of $N_C^P(x)$ and $N_C(x)$, we can easily deduce properties 1) and 3). An easy proof of property 2) can be deduced from the geometrical interpretation of the proximal normal cone. \square

Proposition 1.1.5. *Taking two closed subsets C_1 and C_2 of \mathbb{R}^n and a point $(x_1, x_2) \in C_1 \times C_2$. Then,*

$$N_{C_1 \times C_2}^P(x_1, x_2) = N_{C_1}^P(x_1) \times N_{C_2}^P(x_2) \quad \text{and} \quad N_{C_1 \times C_2}(x_1, x_2) = N_{C_1}(x_1) \times N_{C_2}(x_2) .$$

Proof. The proof is straightforward from the definition of proximal normal cones, and of normal cones as ‘limits’ of proximal normal cones. \square

In the next proposition, a ‘closure’ property of the limiting normal cone is given.

Proposition 1.1.6. *For a closed set C and $x \in C$, the set valued-map $x \rightsquigarrow N_C(x)$ has a closed graph; equivalently, for any sequence $x_i \xrightarrow{C} x$ and $\eta_i \rightarrow \eta$ such that $\eta_i \in N_C(x_i)$ for all i , we have $\eta \in N_C(x)$.*

Proximal and limiting normal cones intervene also in the definition of nonsmooth tools for functions (possibly extended-valued) $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ which are not differentiable. Definitions of the new tools require to introduce the epigraph of f , $\text{epi } f$, defined as

$$\text{epi } f := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \mid \alpha \geq f(x)\} .$$

Definition 1.1.7. *Take a lower semicontinuous function (possibly extended-valued) $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and a point $x \in \text{dom } f := \{y : f(y) < +\infty\}$.*

i) *The proximal subdifferential of f at x is the set*

$$\partial^P f(x) := \{\eta \mid (\eta, -1) \in N_{\text{epi } f}^P(x, f(x))\} .$$

Elements in $\partial^P f(x)$ are called proximal subgradients.

ii) The limiting subdifferential of f at x is the set

$$\partial f(x) := \{\eta \mid (\eta, -1) \in N_{\text{epi } f}(x, f(x))\}.$$

Elements in $\partial f(x)$ are called limiting subgradients.

We shall note that the class of lower semicontinuous functions is a natural choice to generalize the traditional notion of gradients. This is because for such functions, the epigraph is a closed set.

Proposition 1.1.8. *Take a lower semicontinuous function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$, which is in addition convex, and consider a point $x \in \text{dom } f$, then*

$$\partial^P f(x) = \partial f(x) = \{\eta \mid \eta \cdot (y - x) \leq f(y) - f(x) \text{ for all } y \in \mathbb{R}^n\}. \quad (1.3)$$

This coincides with the definition of subgradients in the convex analysis sense.

Proof. It suffices to notice that the set in (1.3) can be written in terms of the normal cone (in the convex analysis sense) to $\text{epi } f$ at $(x, f(x))$ and the result will follow from Proposition 1.1.3. \square

We enumerate below some properties of the proximal and limiting subdifferential cones.

Proposition 1.1.9. *Let f be a lower semicontinuous function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and consider a point $x \in \text{dom } f$. Then the following properties are satisfied:*

- 1) $\partial f(x)$ is a closed set and $\partial^P f(x) \subset \partial f(x)$.
- 2) If f is of class C^1 at x , then $\partial f(x) = \{\nabla f(x)\}$.

Proof. The proof follows from the definition of $\partial^P f(x)$ and $\partial f(x)$ in terms of the normal cones (cf. Definition 1.1.7) and from Proposition 1.1.4. \square

An alternative description of subgradients evokes a generalization of the familiar ‘difference quotient’ characterization of gradients.

Definition 1.1.10. *Let f be a lower semicontinuous function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and consider a point $x \in \text{dom } f$.*

i) A vector η belongs to $\partial^P f(x)$ if there exists $M > 0$ and $\epsilon > 0$ such that for all $y \in x + \epsilon\mathbb{B}$

$$\eta \cdot (y - x) \leq f(y) - f(x) + M|x - y|^2.$$

ii) A vector η belongs to $\partial f(x)$ if there exist sequences $x_i \xrightarrow{f} x$ and $\eta_i \rightarrow \eta$ such that

$$\eta_i \in \partial^P f(x_i) \text{ for all } i.$$

From a geometrical point of view, the proximal subgradient inequality in Definition 1.1.10 asserts that, locally, f is bounded below by the function

$$y \rightarrow f(x) + \eta \cdot (y - x) - M|x - y|^2.$$

The graph of this function is a parabola passing through $(x, f(x))$, and which has η as derivative at the point x . Therefore, proximal subgradients are the slopes at x of locally supporting parabolas to f as shown in the figure below.

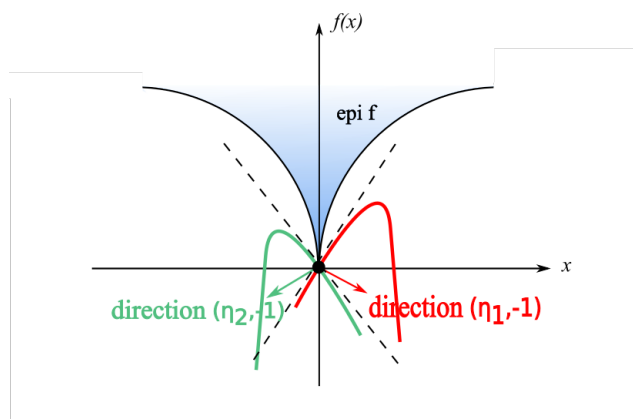


Figure 1.3 – Proximal subgradients η_1 and η_2

The following proposition ensures the ‘closure’ property of the graph of the set-valued map $x \rightsquigarrow \partial f(x)$. More precisely,

Proposition 1.1.11. *Take a lower semicontinuous function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and a point $x \in \mathbb{R}^n$. Then, for any sequences $x_i \xrightarrow{f} x$ and $\xi_i \rightarrow \xi$ such that $\xi_i \in \partial f(x_i)$ for all i , we have $\xi \in \partial f(x)$.*

Proof. This is straightforward from the definition of $\partial f(x)$ in terms of $N_{\text{epi } f}(x)$ and from Proposition 1.1.6. \square

Further properties of the proximal and limiting subdifferentials can be deduced if the function f is Lipschitz continuous.

Proposition 1.1.12. *Take a lower semicontinuous function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ and a point $x \in \mathbb{R}^n$. Assume that f is Lipschitz continuous on a neighborhood of x with Lipschitz constant K . Then,*

- 1) Characterizations i) and ii) in Definition 1.1.10 are satisfied globally, not merely in a neighborhood of x ;
- 2) $\partial f(x) \subset K\mathbb{B}$;
- 3) $\partial f(x)$ is a compact set (being closed and bounded).

4) (*Sum Rule*) $\partial(f + g)(x) \subset \partial f(x) + \partial g(x)$, for g lower semicontinuous extended-valued function which is Lipschitz on the neighborhood of $x \in \text{dom } f \cap \text{dom } g$.

From Proposition 1.1.6 and Proposition 1.1.11, we notice that the limiting normal cone and the limiting subdifferential have good convergence properties for limit-taking.

The next proposition provides a description of the limiting subdifferentials of the distance function when the base-point does not belong to the set:

Proposition 1.1.13. ([32] or [81, Lemma 4.8.3, Theorem 4.8.5]) Take a closed set $C \subset \mathbb{R}^n$, a point $x \notin C$, and a vector $\xi \in \partial d_C(x)$. Then x has a unique closest point \bar{x} in C and

$$\xi = \frac{x - \bar{x}}{|x - \bar{x}|}.$$

In particular, $\xi \in N_C(\bar{x})$.

We give now an alternative approach for subdifferentials of locally Lipschitz continuous functions, based on convex approximations. This concept, due to F. Clarke in the 1970s, permits to relate subdifferentials of nonsmooth functions and their counterparts in Convex Analysis. This approach provides a new representation of subdifferentials which intervene in applications.

Definition 1.1.14. Fix a point $x \in \mathbb{R}^n$ and take a Lipschitz continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ on a neighborhood of x . The generalized directional derivative of f in the direction v , denoted $f^0(x, v)$, is set to be

$$f^0(x, v) := \limsup_{y \rightarrow x, t \downarrow 0} \frac{f(y + tv) - f(y)}{t}.$$

It is straightforward to check that the function $v \mapsto f^0(x, v)$ is convex, Lipschitz continuous and positive homogeneous, in the sense that

$$f^0(x, \alpha v) = \alpha f^0(x, v) \quad \text{for all } v \in \mathbb{R}^n \text{ and } \alpha \geq 0.$$

We define the following set

$$\partial^C f(x) := \{\xi : f^0(x, v) \geq \xi \cdot v, \quad \text{for all } v \in \mathbb{R}^n\}$$

called the Clarke subdifferential. (The notation ∂^C refers to Clarke). This set is expressed as the subdifferential in the sense of Convex Analysis of the convex function $v \mapsto f^0(x, v)$ (which approximates f near x). It can be shown (cf. [81, Proposition 4.7.6, Theorem 4.7.7]) that

$$\begin{aligned} \partial^C f(x) &= \text{co } \partial f(x) \\ &= \text{co } \{\xi : \text{there exists } x_i \rightarrow x, x_i \notin E, \nabla f(x_i) \text{ exists and } \nabla f(x_i) \rightarrow \xi\}, \end{aligned} \quad (1.4)$$

where E is a zero-measure set in \mathbb{R}^n . We shall note that the right-hand side of the equality is well-defined owing to Rademacher's Theorem, which states that any locally Lipschitz continuous function is locally differentiable, i.e. its gradient ∇f exists a.e.

We define now a subset of the Clarke subdifferential evaluated only for points corresponding to $\{f > 0\}$.

Definition 1.1.15. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function and $x \in \mathbb{R}^n$. Assume that f is Lipschitz continuous on a neighborhood of x . Then, the hybrid subdifferential of f at x , denoted $\partial^\triangleright f(x)$, is the set

$$\partial^\triangleright f(x) := \text{co} \{ \eta \mid \text{there exists } x_i \xrightarrow{f} x \text{ such that } f(x_i) > 0 \text{ for all } i \text{ and } \nabla f(x_i) \rightarrow \eta \} . \quad (1.5)$$

If $f = f(x, y)$, then $\partial_x^\triangleright f(x, y)$ is the partial hybrid subdifferential with respect to the variable x , and is defined as

$$\partial_x^\triangleright f(x, y) := \text{co} \{ \eta \mid \text{there exists } (x_i, y_i) \xrightarrow{f} (x, y) \text{ such that } f(x_i, y_i) > 0 \text{ for all } i \text{ and } \nabla_x f(x_i, y_i) \rightarrow \eta \} . \quad (1.6)$$

Remark 1.1.16. The hybrid subdifferential is of interest because it will intervene in the necessary optimality conditions for state constrained optimal control problems.

For the particular case when $f = d_C$ where C is a closed set, we can establish the following:

Proposition 1.1.17. For a closed set $C \subset \mathbb{R}^n$ and $a \in \partial C$, we have

$$\partial^\triangleright d_C(a) \subset \text{co} (N_C(a) \cap \partial \mathbb{B}) .$$

A detailed proof is given in the *Appendix* of Chapter 5.

On the other hand, we make use of tangent cones which are a generalization of the notion of tangent space to a set with singularities (nonsmooth set). Roughly speaking, having a (closed) subset C of \mathbb{R}^n , we pick any direction $\eta \in \mathbb{R}^n$ and we start from x in the direction of η , ranging over the line $x + t\eta$ when $t > 0$. The tangent cone is the set of directions $\eta \in \mathbb{R}^n$ which, for small t , do not lead us far away from C . In nonlinear analysis, there exist many definitions for the tangent cone, including the Bouligand tangent cone, the Clarke tangent cone, etc, which play an important role in optimization and viability theory. The different definitions of tangent cones depend on the type of the limit involved. The Bouligand tangent cone (or *contingent*) (respectively, the Clarke tangent cone) to a closed set C at a point $x \in C$, denoted $T_C^B(x)$ (respectively $T_C(x)$), is defined as follows:

Definition 1.1.18. Let C be a closed subset of \mathbb{R}^n and $x \in C$.

1) The Bouligand tangent cone to C at x is the set

$$T_C^B(x) := \left\{ v : \liminf_{t \downarrow 0} \frac{d_C(x + tv)}{t} = 0 \right\} .$$

2) The Clarke tangent cone to C at x is the set

$$T_C(x) := \left\{ v : \lim_{t \downarrow 0, y \xrightarrow{C} x} \frac{d_C(y + tv)}{t} = 0 \right\} .$$

The Bouligand and Clarke tangent cone can be defined directly without recourse to the distance function, as shown in Proposition 1.1.19.

Proposition 1.1.19. *i) $T_C^B(x)$ comprises vectors η such that there exists a positive sequence $\{t_i\}$ converging to zero and $\{\eta_i\}$ converging to η , for which $x + t_i\eta_i \in C$ for all i ; equivalently*

$$T_C^B(x) := \limsup_{t \downarrow 0} \frac{C - x}{t} .$$

ii) $T_C(x)$ comprises vectors η such that for every positive sequence $\{t_i\}$ converging to zero and every sequence $\{x_i\}$ in C converging to x , there exists a sequence $\{\eta_i\}$ converging to η , such that $x_i + t_i\eta_i \in C$ for all i ; equivalently

$$T_C(x) := \liminf_{t \downarrow 0, y \xrightarrow{C} x} \frac{C - y}{t} .$$

Proposition 1.1.20. *For a closed set C and a point $x \in C$, the Bouligand and the Clarke tangent cone are closed cones containing the origin, such that $T_C(x) \subset T_C^B(x)$.*

Rockafellar [71] provides an equivalent characterization of the Clarke tangent cone of a closed set C at $x \in C$:

Proposition 1.1.21. *$\eta \in T_C(x)$, if and only if for every $\varepsilon > 0$, there exist $\delta > 0$ and $\lambda > 0$ such that*

$$C \cap [x' + t(\eta + \varepsilon\mathbb{B})] \neq \emptyset \quad \text{for all } x' \in C \cap (x + \delta\mathbb{B}), \quad t \in [0, \lambda] . \quad (1.7)$$

We introduce now another characterization of the Clarke tangent cone which uses the dual concept with normal cones. Let X be a subset of \mathbb{R}^n . The polar cone of X , denoted by X^* , is the set

$$X^* := \{\eta \mid \eta \cdot x \leq 0 \quad \text{for all } x \in X\} .$$

The Clarke tangent cone to a closed set C at some point $x \in C$ is related to the limiting normal cone according to:

Proposition 1.1.22. *$T_C(x) = N_C(x)^*$ for any closed set C and a point $x \in C$.*

This characterization simplifies, in some cases, the geometrical interpretation of the Clarke tangent cone, as shown in Figure 1.4.

Remark 1.1.23. *The Bouligand tangent cone for the set C in Figure 1.4 (a) is the whole space \mathbb{R}^2 . (This is because $N_C^P(x) = \{0\}$.)*

Surprisingly, $T_C(x)$ is a closed convex cone, without any convexity or smoothness assumptions on C . We refer the reader to [81, Proposition 4.10.3] for a possible proof. This is why the Clarke tangent cone is of interest in optimization fields, especially its interior, characterized by:

Theorem 1.1.24 (cf. [34], [71]). *$\eta \in \text{int } T_C(x)$ if and only if there exist $\varepsilon > 0$, $\delta > 0$, $\lambda > 0$ such that*

$$x' + t\eta' \in C \quad \text{for all } x' \in C \cap (x + \delta\mathbb{B}), \quad t \in [0, \lambda], \quad \eta' \in \eta + \varepsilon\mathbb{B} . \quad (1.8)$$



Figure 1.4 – Tangent cones.

Property (1.8) characterizes *hypertangent* vectors to the set C at the point $x \in C$.

Remark 1.1.25. *It is straightforward to notice that in the case of a set having cusps (for instance Figure 1.4), the interior of the Clarke tangent cone is empty.*

1.2 Optimal Control Problems

1.2.1 Control Systems and Differential Inclusions

Differential equations first came into existence with the invention of calculus by Newton and Leibniz in the 17th century. A differential equation to which we associate an initial condition (known as the Cauchy problem)

$$\begin{aligned} \dot{x}(t) &= f(t, x(t)) \\ x(t_0) &= x_0, \end{aligned}$$

is a relation between a state x and its rate of change $\dot{x} = \frac{dx}{dt}$. It models the evolution of a system and permits to predict its future evolution without changing its behavior. For instance, we can exactly predict time and locations of eclipses but we cannot modify them.

A control system is, however, a differential equation involving an external agent, called ‘controller’, who will affect the evolution of the system. This situation is modeled by the control system below. Namely,

$$\dot{x} = f(t, x, u), \quad u(\cdot) \in \mathcal{U} \tag{1.9}$$

where \mathcal{U} is a family of admissible control functions defined as

$$\mathcal{U} := \{u : \mathbb{R} \rightarrow \mathbb{R}^m; u(\cdot) \text{ measurable, } u(t) \in U(t) \text{ for a.e. } t\} \tag{1.10}$$

for a given nonempty multifunction $U(\cdot)$ such that $U(t) \subset \mathbb{R}^m$. In this case, the rate of change $\dot{x}(t)$ depends not only on the state x itself, but also on some external parameters, say $u = (u_1, \dots, u_m)$, which can also vary in time or space. The control function $u(\cdot)$, subject to some constraints, will be chosen by a controller in order to manage, command or regulate the behavior of the system and achieve certain predefined goals, for instance steer the system from one state to another, maximize the terminal value of one of the parameters, minimize or

maximize a certain cost functional, etc. We distinguish two types of controls: the time-variable control $t \rightarrow u(t)$ and the space-variable control $x \rightarrow u(x)$. The first is known as an *open loop control* while the second is a *closed loop control* or *feedback*. In an open loop control system, the control action from the controller is independent of the ‘process output’. A good example of this is a central heating boiler controlled only by a timer, so that heat is applied for a constant time, regardless of the temperature of the building. (The control action is the switching on/off of the boiler. The process output is the building temperature). In a closed loop control system, the control action from the controller depends on the process output. Considering the boiler, this would include a temperature thermostat to regulate the building temperature, and thereby feed back a signal to ensure that the controller maintains the temperature set on the thermostat. An open loop control is easier to implement since the only information needed is provided by a clock to measure time. In this work, we are interested in control systems involving time-variable controls.

The dynamics can also be represented as a differential inclusion which is a generalization of the concept of ordinary differential equation:

$$\dot{x} \in F(t, x) \quad (1.11)$$

where the set of velocities is given by

$$F(t, x) := \{y \mid y = f(t, x, u) \text{ for some } u \in U(t)\}$$

and F is a set-valued map, i.e. $F(t, x)$ is a set rather than a single point in \mathbb{R}^n .

It is clear that every trajectory for the control system (1.9) is also a solution for the differential inclusion (1.11). The converse is also true under some regularity assumptions on f .

Once these two types of dynamics are defined, we are ready to state optimal control problems which concern the properties of control functions that, when inserted into a differential equation, give solutions which minimize or maximize a certain ‘cost’ (for the case of control systems) and the properties of state trajectories and the set of velocities F achieving some minimum or maximum ‘cost’ (for the case of differential inclusions).

Let $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ be real-valued a cost function. We consider the optimal control problem involving a control system

$$(\text{CSP}) \quad \left\{ \begin{array}{l} \text{minimize } g(x(S), x(T)) \\ \text{over arcs } x(\cdot) \in W^{1,1} \text{ and measurable functions } u(\cdot) \text{ satisfying} \\ \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [S, T] \\ (x(S), x(T)) \in C \\ u(t) \in U(t) \quad \text{a.e. } t \in [S, T]. \end{array} \right.$$

The data for problem (CSP) involve a closed set C , a set-valued map $t \rightsquigarrow U(t) \subset \mathbb{R}^m$ and functions $f : [S, T] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$. We call *process* a couple $(x(\cdot), u(\cdot))$ such that $u(\cdot)$ is a Lebesgue measurable function satisfying $u(t) \in U(t)$ a.e. $t \in [S, T]$ and $x(\cdot)$ is the solution of the ordinary differential equation

$$\dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [S, T].$$

The process $(x(\cdot), u(\cdot))$ is called *feasible* if in addition $(x(S), x(T)) \in C$. We say that the process $(\bar{x}(\cdot), \bar{u}(\cdot))$ is a D -local minimizer for (CSP) if, for a given $\varepsilon > 0$

$$g(\bar{x}(S), \bar{x}(T)) \leq g(x(S), x(T))$$

for every feasible trajectory $(x(\cdot), u(\cdot))$ such that

$$\|x(\cdot) - \bar{x}(\cdot)\|_D \leq \varepsilon. \quad (1.12)$$

The process is called strong local minimizer when $D = L^\infty([S, T], \mathbb{R}^n)$, and weak local minimizer when $D = W^{1,1}([S, T], \mathbb{R}^n)$, which corresponds to the set of absolutely continuous functions. In some circumstances, we shall emphasize the dependence on the minimizer of ε and we would refer to it as a D local ε -minimizer. Since the set of absolutely continuous functions is larger than the set of L^∞ functions, the $W^{1,1}$ -norm is stronger than the L^∞ -norm. It follows that the $W^{1,1}$ -local minimizers would provide a sharper analysis on the local nature of the optimality conditions than would be the case with L^∞ -local minimizers.

An optimal control problem formulated in terms of a differential inclusion is defined as follow

$$(DIP) \quad \begin{cases} \text{minimize } g(x(S), x(T)) \\ \text{over arcs } x(\cdot) \in W^{1,1} \text{ satisfying} \\ \dot{x}(t) \in F(t, x(t)) \quad \text{a.e. } t \in [S, T] \\ (x(S), x(T)) \in C, \end{cases}$$

where $F : [S, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a set-valued map. A trajectory $x(\cdot)$ which solves the differential inclusion $\dot{x} \in F(t, x)$ is called an F -trajectory.

1.2.2 Existence of Solutions and Minimizers

The existence of minimizers for an optimal control problem requires first a theorem on the existence of solutions for the dynamics. Hence, we present first a theorem known as Filippov Existence Theorem which provides conditions for which a differential inclusion admits a solution.

Define the ε -tube around an arc y as follows:

$$T(y, \varepsilon) := \{(t, x) \in [S, T] \times \mathbb{R}^n \mid t \in [S, T], |x - y(t)| \leq \varepsilon\}.$$

Theorem 1.2.1 (Generalized Filippov Existence Theorem). *Take an open set Ω in $[S, T] \times \mathbb{R}^n$, a multifunction $F : \Omega \rightsquigarrow \mathbb{R}^n$, an arc $y \in W^{1,1}([S, T], \mathbb{R}^n)$, a point $\xi \in \mathbb{R}^n$ and $\varepsilon \in (0, \infty]$ such that $T(y, \varepsilon) \subset \Omega$. Assume that:*

- (i) $F(t, x')$ has nonempty values for all $(t, x') \in T(y, \varepsilon)$, F is $\mathcal{L} \times \mathcal{B}^m$ measurable and $\text{Gr } F(t, \cdot)$ is closed for each $t \in [S, T]$;
- (ii) there exists $k(\cdot) \in L^1$ such that

$$F(t, x') \subset F(t, x'') + k(t)|x' - x''|\mathbb{B} \quad (1.13)$$

for all $x', x'' \in y(t) + \varepsilon\mathbb{B}$ a.e. $t \in [S, T]$.

Assume further that

$$K \left(|\xi - y(S)| + \int_S^T \rho_F(t, y(t), \dot{y}(t)) dt \right) \leq \varepsilon$$

where $K := \exp\left(\int_S^T k(t) dt\right)$ and $\rho_F(t, y, v) := \inf\{|\eta - v| \mid \eta \in F(t, y)\}$ (extent to which y fails to be an F -trajectory).

Then, there exist an F -trajectory x satisfying $x(S) = \xi$ such that

$$\begin{aligned} \|x - y\|_{L^\infty} &\leq |x(S) - y(S)| + \int_S^T |\dot{x}(t) - \dot{y}(t)| dt \\ &\leq K \left(|\xi - y(S)| + \int_S^T \rho_F(t, y(t), \dot{y}(t)) dt \right). \end{aligned} \quad (1.14)$$

Proof. The detailed proof can be found in [81, Theorem 2.4.3]: the main idea is the generalization of the case of obtaining a solution to a differential equation via the well-known Picard iteration scheme. \square

Lemma 1.2.2 (Gronwall's Inequality). *For any absolutely continuous function $z : [S, T] \rightarrow \mathbb{R}^n$ and nonnegative integrable functions k and v verifying*

$$\left| \frac{d}{dt} z(t) \right| \leq k(t)|z(t)| + v(t) \quad \text{a.e. } t \in [S, T],$$

we have

$$|z(t)| \leq \exp\left(\int_S^t k(\sigma) d\sigma\right) \left[|z(S)| + \int_S^t \exp\left(-\int_S^\tau k(\sigma) d\sigma\right) v(\tau) d\tau \right].$$

The next corollary establishes the existence and uniqueness of solutions for an ordinary differential equation:

Corollary 1.2.3. *Take a function $g : [S, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, an arc $y \in W^{1,1}([S, T], \mathbb{R}^n)$, a point $\xi \in \mathbb{R}^n$ and $\varepsilon \in (0, \infty]$. Assume that:*

- (i) *For a fixed x , $g(\cdot, x)$ is \mathcal{L} -measurable;*
- (ii) *there exists an integrable function $k : [S, T] \rightarrow \mathbb{R}$ such that*

$$|g(t, x) - g(t, x')| \leq k(t)|x - x'| \quad (1.15)$$

for all $x, x' \in y(t) + \varepsilon\mathbb{B}$, a.e. $t \in [S, T]$.

Assume further that

$$K \left(|\xi - y(S)| + \int_S^T |\dot{y}(t) - g(t, y(t))| dt \right) \leq \varepsilon$$

where $K := \exp\left(\int_S^T k(t) dt\right)$.

Then, there exist a unique solution x to

$$\begin{cases} \dot{x}(t) = g(t, x(t)) & \text{a.e. } t \\ x(S) = \xi \end{cases}$$

satisfying

$$\begin{aligned} \|x - y\|_{L^\infty} &\leq |x(S) - y(S)| + \int_S^T |\dot{x}(t) - \dot{y}(t)| dt \\ &\leq K \left(|\xi - y(S)| + \int_S^T |\dot{y}(t) - g(t, y(t))| dt \right). \end{aligned} \quad (1.16)$$

Remark 1.2.4. Corollary 1.2.3 can be used to establish the existence and uniqueness of solutions for control systems of the form

$$\dot{x}(t) = f(t, x(t), u(t)) \quad x(S) = \xi. \quad (1.17)$$

Indeed, (1.17) is equivalent to

$$\dot{x}(t) = g(t, x(t)) \quad x(S) = \xi$$

where $g(t, x) := f(t, x, u(t))$. If we impose that $f(., x, .)$ is $\mathcal{L} \times \mathcal{B}^m$ -measurable for each $x \in \mathbb{R}^n$, $Gr U(., .)$ is measurable, and that $f(t, ., u)$ is locally Lipschitz, then the function $g(., .)$ verifies all the assumptions of Corollary 1.2.3 and therefore the control system (1.17) has a unique solution for a given control function $t \rightarrow u(t)$.

The next theorem provides conditions for which an optimal control problem (in terms of differential inclusion dynamics) has a minimizer.

Proposition 1.2.5. [81, Proposition 2.6.2] Consider the problem (DIP). Assume that

- (a) the multifunction $F : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ has closed and nonempty values, $F(., .)$ is $\mathcal{L} \times \mathcal{B}^n$ and the graph of $F(t, .)$ is closed for a.e. $t \in [S, T]$;
- (b) there exist $\alpha \in L^1$ and $\beta \in L^1$ such that

$$F(t, x) \subset (\alpha(t)|x| + \beta(t))\mathbb{B} \quad \text{for all } (t, x);$$

- (c) C is closed and g is a given lower semicontinuous function;
- (d) one of these following sets is bounded:

$$C_0 := \{x_0 \in \mathbb{R}^n : (x_0, x_1) \in C, \text{ for some } x_1 \in \mathbb{R}^n\}$$

$$C_1 := \{x_1 \in \mathbb{R}^n : (x_0, x_1) \in C, \text{ for some } x_0 \in \mathbb{R}^n\};$$

- (e) the set of feasible F -trajectories $\{x : \dot{x}(t) \in F(t, x(t)) \text{ a.e. } t, \text{ and } (x(S), x(T)) \in C\}$ is nonempty;
- (f) $F(t, x)$ is convex for each (t, x) .

Then (DIP) has a minimizer.

The proof of Proposition 1.2.5 requires a crucial result on the compactness of trajectories (cf. [81, Theorem 2.5.3]):

Theorem 1.2.6 (Compactness of Trajectories Theorem). *Take an open set $\Omega \subseteq [S, T] \times \mathbb{R}^n$, a set-valued map $F : \Omega \rightsquigarrow \mathbb{R}^n$ and some closed set-valued map $X : [S, T] \rightsquigarrow \mathbb{R}^n$ such that $\text{Gr } X \subset \Omega$. Assume the following assumptions:*

- (i) F is a nonempty, closed and convex set-valued map;
- (ii) F is $\mathcal{L} \times \mathcal{B}^n$ measurable;
- (iii) for each $t \in [S, T]$, $\text{Gr } F(t, \cdot)$ restricted to $X(t)$ is closed.

Consider a sequence of $\{r_i(\cdot)\}$ in $L^1([S, T], \mathbb{R}^n)$ such that $\|r_i\|_{L^1} \rightarrow 0$ as $i \rightarrow \infty$, an absolutely continuous sequence of functions $\{x_i(\cdot)\}$, and a sequence $\{A_i\}$ of measurable subsets of $[S, T]$ such that $\text{meas } A_i \rightarrow |T - S|$ as $i \rightarrow \infty$. Suppose that

- (iv) $\text{Gr } x_i \subset \text{Gr } X$ for all i ;
- (v) $\{\dot{x}_i\}$ is a sequence of uniformly integrally bounded functions on $[S, T]$ such that $\{x_i(S)\}$ is bounded;
- (vi) there exists $c \in L^1$ such that

$$F(t, x_i(t)) \subset c(t)\mathbb{B} \quad \text{for a.e. } t \in A_i, \text{ and for } i = 1, 2, \dots$$

Suppose moreover that

$$\dot{x}_i(t) \in F(t, x_i(t)) + r_i(t)\mathbb{B} \quad \text{a.e. } t \in A_i.$$

Then, by subsequence extraction (without relabeling)

$$x_i \rightarrow x \quad \text{uniformly} \quad \text{and} \quad \dot{x}_i \rightarrow \dot{x} \quad \text{weakly in } L^1$$

for some $x \in W^{1,1}([S, T], \mathbb{R}^n)$ satisfying

$$\dot{x}(t) \in F(t, x(t)) \quad \text{a.e. } t \in [S, T].$$

Remark 1.2.7. 1) When F has no convex values,

$$\dot{x}(t) \in F(t, x(t)) \quad \text{a.e. } t \in [S, T]$$

will be replaced with

$$\dot{x}(t) \in \text{co } F(t, x(t)) \quad \text{a.e. } t \in [S, T]$$

in the statement of Theorem 1.2.6

- 2) Theorem 1.2.6 will play a crucial role in our future analysis regarding the limit-taking in the necessary conditions (and more precisely, the so-called ‘adjoint system’) for an auxiliary problem associated to the original optimal control problem.

1.2.3 Nonsmooth First Order Necessary Optimality Conditions

Since optimal control problems deal with the problem of finding a control law for a given system such that a certain optimality criterion (or ‘objective’) is achieved, many mathematicians were interested in constructing conditions for which an optimal control can be derived. The study of such conditions goes back to the 1950s with the work of L. Pontryagin and his famous Pontryagin’s Maximum Principle, which provides a set of necessary conditions which an optimum (if it exists) must satisfy; and the Dynamic Programming Principle, which simplifies the search of an optimal control function to the task of finding the solution to a partial differential equation known as the Hamilton-Jacobi-Bellman equation: this procedure gives a sufficient condition for an optimum. This thesis project treats only necessary optimality conditions in the form of the Maximum Principle for control systems (ordinary differential equation) and in the form of the Extended Euler-Lagrange condition (and the associated optimality conditions) for the differential inclusion dynamics.

Nonsmooth Pontryagin Maximum Principle

The PMP (Pontryagin Maximum Principle) was first proved, by Pontryagin et. al [66], for optimal control problems with ‘smooth’ dynamic constraints. The merge between nonsmooth analysis and optimal control problems motivated many mathematicians, like Clarke [29], Ioffe [54], Rockafellar [72], Vinter [81], Warga [84], among the others, in order to develop a nonsmooth version of the PMP. We present below a version of the PMP making use of the Clarke’s generalized gradient (in the sense of (1.4)). Recall the optimal control problem (CSP)

$$(CSP) \quad \left\{ \begin{array}{l} \text{minimize } g(x(S), x(T)) \\ \text{over arcs } x(\cdot) \in W^{1,1} \text{ and measurable functions } u(\cdot) \text{ satisfying} \\ \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [S, T] \\ (x(S), x(T)) \in C \\ u(t) \in U(t) \quad \text{a.e. } t \in [S, T]. \end{array} \right.$$

Theorem 1.2.8 (Nonsmooth Maximum Principle). (cf. [30], [33]) *Suppose that $(\bar{x}(\cdot), \bar{u}(\cdot))$ is a strong (L^∞ -)local minimizer for (CSP). Assume that, for some $\delta > 0$, the data for the problem satisfy the following assumptions:*

(CS.1) *$f(\cdot, x, \cdot)$ is $\mathcal{L} \times \mathcal{B}^m$ measurable, for fixed x . There exist a $\mathcal{L} \times \mathcal{B}^m$ measurable function $k(\cdot, \cdot) : [S, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that $t \rightarrow k(t, \bar{u}(t))$ is integrable and*

$$|f(t, x, u) - f(t, x', u)| \leq k(t, u)|x - x'|$$

for all $x, x' \in \bar{x}(t) + \delta\mathbb{B}$, $u \in U(t)$ a.e. $t \in [S, T]$;

(CS.2) *Gr U is $\mathcal{L} \times \mathcal{B}^m$ measurable set;*

(CS.3) *g is Lipschitz continuous on $(\bar{x}(S) + \delta\mathbb{B}) \times (\bar{x}(T) + \delta\mathbb{B})$.*

Then there exist an adjoint arc $p \in W^{1,1}([S, T], \mathbb{R}^n)$ and a cost multiplier $\lambda \geq 0$ such that:

- (i) $(\lambda, p) \neq (0, 0)$,
- (ii) $-\dot{p}(t) \in \text{co } \partial_x(p(t) \cdot f(t, \bar{x}(t), \bar{u}(t))) \quad \text{a.e. } t \in [S, T]$,
- (iii) $(p(S), -p(T)) \in \lambda \partial_x g(\bar{x}(S), \bar{x}(T)) + N_C(\bar{x}(S), \bar{x}(T))$,
- (iv) $p(t) \cdot f(t, \bar{x}(t), \bar{u}(t)) = \max_{u \in U(t)} p(t) \cdot f(t, \bar{x}(t), u) \quad \text{a.e. } t \in [S, T]$.

Remark 1.2.9. 1) The Nonsmooth Maximum Principle was extended to the case of weak $(W^{1,1}-)$ local minimizers by Vinter (cf. [81, Theorem 6.2.1]).

2) The nontriviality condition (i) can be equivalently expressed as a normalized condition

$$\lambda + \|p\|_{L^\infty} = 1. \quad (1.18)$$

This can be carried out by setting $\alpha := \lambda + \|p\|_{L^\infty}$, where $\alpha > 0$ (owing to (i)). Indeed, for $p_1(\cdot) := \frac{p(\cdot)}{\alpha}$ and $\lambda_1 := \frac{\lambda}{\alpha}$, we can easily check that

$$\lambda_1 + \max_{t \in [S, T]} |p_1(t)| = 1.$$

By observing that $(p_1(\cdot), \lambda_1)$ serves also as a set of multipliers (by positive homogeneity) and by rewriting p_1 and λ_1 as p and λ respectively, we can deduce the normalized condition (1.18).

Extended Euler-Lagrange Condition

After the establishment of the nonsmooth version of PMP for problems like (CSP), mathematicians were interested in deriving necessary optimality conditions for a broader framework by studying optimal control problems involving a dynamic constraint in the form of differential inclusion. The necessary optimality conditions of such problems are regarded as a generalization of conditions from the classical calculus of variations and are known as the Extended Euler-Lagrange condition.

Recall the state constrained optimal control problem (DIP) involving a dynamic constraint taking the form of a differential inclusion:

$$(DIP) \quad \left\{ \begin{array}{l} \text{minimize } g(x(S), x(T)) \\ \text{over arcs } x(\cdot) \in W^{1,1}([S, T]; \mathbb{R}^n) \text{ satisfying} \\ \quad \dot{x}(t) \in F(t, x(t)) \quad \text{a.e. } t \in [S, T] \\ \quad (x(S), x(T)) \in C. \end{array} \right.$$

We assume the following hypotheses for some $\delta > 0$, $c(\cdot) \in L^1$ and $k_F(\cdot) \in L^1$, where $\bar{x}(\cdot)$ is a given absolutely continuous arc:

- (DI.1) the multifunction $F : [S, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ has nonempty values, F is $\mathcal{L} \times \mathcal{B}^n$ measurable and $\text{Gr } F(t, \cdot)$ is closed for each $t \in [S, T]$;

(DI.2)

$$F(t, x) \subset c(t) \mathbb{B} \quad \text{for all } x \in \bar{x}(t) + \delta \mathbb{B}, \quad \text{a.e. } t \in [S, T];$$

(DI.3)

$$F(t, x) \subset F(t, y) + k_F(t)|x - y|\mathbb{B}, \quad \text{for all } x, y \in \bar{x}(t) + \delta \mathbb{B}, \quad \text{a.e. } t \in [S, T].$$

(DI.4) g is Lipschitz continuous on $(\bar{x}(S) + \delta \mathbb{B}) \times (\bar{x}(T) + \delta \mathbb{B})$.

Theorem 1.2.10. *Assume that for some local minimizer $\bar{x}(\cdot)$, assumptions (DI.1)-(DI.4) are satisfied. Then, there exist $p(\cdot) \in W^{1,1}([S, T]; \mathbb{R}^n)$ and $\lambda \geq 0$ such that:*

$$(i) \quad (\lambda, p) \neq (0, 0),$$

$$(ii) \quad \dot{p}(t) \in \text{co} \{ \eta \mid (\eta, p(t)) \in N_{\text{Gr } F(t, \cdot)}(\bar{x}(t), \dot{\bar{x}}(t)) \} \quad \text{a.e. } t \in [S, T],$$

$$(iii) \quad (p(S), -p(T)) \in \lambda \partial g(\bar{x}(S), \bar{x}(T)) + N_C(\bar{x}(S), \bar{x}(T)),$$

$$(iv) \quad p(t) \cdot \dot{\bar{x}}(t) = \max_{v \in F(t, \bar{x}(t))} p(t) \cdot v \quad \text{a.e. } t \in [S, T].$$

(For a proof, see [81, Theorem 7.4.1].)

Interpretation: Condition (ii) is called the Extended Euler-Lagrange condition because it is close to the classical Euler-Lagrange condition. Indeed, problem (DIP) can be regarded as a problem with a ‘penalty’ term in the cost:

$$(CVP) \quad \begin{cases} \text{minimize} & g(x(S), x(T)) + \int_S^T L(t, x(t), \dot{x}(t)) dt \\ \text{over arcs } x(\cdot) \in & W^{1,1}([S, T], \mathbb{R}^n) \text{ satisfying} \\ & (x(S), x(T)) \in C, \end{cases}$$

where

$$L(t, x, v) := \Psi_{\text{Gr } F(t, \cdot)}(x, v) = \begin{cases} 0 & \text{if } v \in F(t, x) \\ +\infty & \text{otherwise} \end{cases}$$

(Ψ_G denotes the indicator function of the set G). It is clear that the dynamic constraint has been eliminated with a penalty term in the cost. The related problem (CVP) is a problem in calculus of variations, with discontinuous data (an indicator function is indeed not smooth). Classical conditions including Euler-Lagrange condition will be therefore used to search for a minimizer to the problem (CVP):

$$\dot{p}(t) \in \text{co} \{ \eta \mid (\eta, p(t)) \in \partial_{x, \dot{x}} L(t, \bar{x}(t), \dot{\bar{x}}(t)) \}. \quad (1.19)$$

But since L is the indicator function of the closed set $\text{Gr } F(t, \cdot)$ and for $(x, v) \in \text{Gr } F(t, \cdot)$, we have

$$\partial_{x, v} L(t, x, v) = N_{\text{Gr } F(t, \cdot)}(x, v).$$

As a consequence, we can derive condition (ii) of Theorem 1.2.10.

1.3 State Constrained Optimal Control Problems

1.3.1 Description

This section introduces optimal control problems with pathwise constraints on the state trajectories. Several types of constraints exist, primarily equality constraints, inequality constraints and implicit constraints (expressed in terms of a closed set). A pathwise state constraint formulated as a scalar inequality constraint takes the form of

$$h(t, x(t)) \leq 0 \quad \text{for all } t \in [S, T]. \quad (1.20)$$

This type of constraint is the starting point to derive necessary optimality conditions for problems involving other types of state constraints, for instance multiple state constraints, implicit state constraint, etc. In this thesis, we study, most of the time, optimal control problems with an implicit state constraint

$$x(t) \in A \quad \text{for all } t \in [S, T]. \quad (1.21)$$

Remark 1.3.1. 1) *In the presence of state constraint, an F -trajectory $x(\cdot)$ is called feasible if it satisfies also condition (1.20) for problems with inequality state constraint, and condition (1.21) for those with implicit state constraint.*

2) *For optimal control problems with state constraints, the existence of minimizers is guaranteed by Proposition 1.2.5 if in addition to assumptions (a)-(c) and (f), either assumption (d) is satisfied or A is a bounded set, and (e) is replaced by*

(e') the set of feasible trajectories

$$\{x : \dot{x}(t) \in F(t, x(t)) \text{ a.e. } t, x(t) \in A, \text{ for all } t \text{ and } (x(S), x(T)) \in C\}$$

is nonempty.

More details can be found in [81, Proposition 2.6.2].

In what follows, we will provide first order necessary optimality conditions in the extended framework (i.e. a solution has to satisfy a pathwise constraint). We would expect that the necessary optimality conditions presented in the subsection 1.2.3 will change: a new multiplier, μ , which is a Borel measure, will be added to the set of multipliers. This is a consequence of the Riesz Representation Theorem. We refer the reader to [81, Section 9.1] for a deeper explanation regarding this point.

We state first an extremely useful technical proposition which is going to be exploited in the next chapters, dealing with state constrained optimal control problems.

1.3.2 Preliminary Result: Convergence of Measures

We consider closed subsets D and D_i , for $i = 1, 2, \dots$ of $[S, T] \times \mathbb{R}^K$. We denote by $D(\cdot)$, $D_i(\cdot) : [S, T] \rightsquigarrow \mathbb{R}^K$ the multifunctions defined as

$$D(t) := \{z \in \mathbb{R}^K : (t, z) \in D\} \quad \text{and}$$

$$D_i(t) := \{z \in \mathbb{R}^K : (t, z) \in D_i\} \quad \text{for all } i = 1, 2, \dots$$

Let $\{\mu_i\}$ be a convergent sequence of positive finite measures. Our aim now is to justify the limit-taking of sequences like

$$d\eta_i(t) = \gamma_i(t) d\mu_i(t) \quad i = 1, 2, \dots$$

in which $\{\gamma_i(t)\}$ is a sequence of Borel measurable functions satisfying

$$\gamma_i(t) \in D_i(t) \quad \mu_i - \text{a.e.}$$

The required convergence result is provided by Proposition 1.3.2 below, which can be also found in [81, Proposition 9.2.1] or [63].

Proposition 1.3.2. *Consider a sequence of positive finite measures $\{\mu_i\}$ such that $\mu_i \xrightarrow{*} \mu$ for some positive and finite measure μ , a sequence of uniformly bounded and closed sets $\{D_i \subset [S, T] \times \mathbb{R}^K\}$ such that*

$$\limsup_{i \rightarrow \infty} D_i \subset D, \tag{1.22}$$

for some closed and uniformly bounded set $D \subset [S, T] \times \mathbb{R}^K$, and a sequence of functions $\{\gamma_i : [S, T] \rightarrow \mathbb{R}^K\}$. Suppose that

- (i) $D(t)$ is convex for each $t \in \text{dom } D(\cdot)$;
- (ii) for $i = 1, 2, \dots$, γ_i is μ_i -measurable and

$$\gamma_i(t) \in D_i(t) \quad \mu_i - \text{a.e.} \quad \text{and} \quad \text{supp}(\mu_i) \subset \text{dom } D_i(\cdot) .$$

Define, for each i , the vector-valued measure $\eta_i := \gamma_i \mu_i$. Then, along a subsequence, we have

$$\eta_i \xrightarrow{*} \eta$$

where η is a vector-valued Borel measure (possibly not positive) on $[S, T]$ such that

$$d\eta(t) = \gamma(t) d\mu(t)$$

for some Borel measurable function $\gamma : [S, T] \rightarrow \mathbb{R}^K$ satisfying

$$\gamma(t) \in D(t) \quad \mu - \text{a.e. } t \in [S, T] .$$

Remark 1.3.3. 1) The ‘superior limit’ in (1.22) must be understood in the Kuratowski sense.

2) A sequence of measures $\{\mu_i\}$ converges weakly* to some measure μ (and we denote this convergence by $\mu_i \xrightarrow{*} \mu$) if $\int h d\mu_i \rightarrow \int h d\mu$ for all $h \in C([S, T], \mathbb{R})$.

1.3.3 Nonsmooth Maximum Principle

Consider the optimal control problem (with controlled dynamic) with explicit state constraint in the form of (1.20)

$$(\text{ExpCSP}) \quad \left\{ \begin{array}{l} \text{minimize } g(x(S), x(T)) \\ \text{over arcs } x(\cdot) \in W^{1,1}([S, T], \mathbb{R}^n) \text{ and measurable functions } u(\cdot) \text{ s.t.} \\ \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [S, T] \\ (x(S), x(T)) \in C \\ h(t, x(t)) \leq 0 \quad \text{for all } t \in [S, T] \\ u(t) \in U(t) \quad \text{a.e. } t \in [S, T]. \end{array} \right.$$

Theorem 1.3.4 (Constrained Nonsmooth Maximum Principle). (cf. [81, Theorem 9.3.1]) Suppose that $(\bar{x}(\cdot), \bar{u}(\cdot))$ is a $W^{1,1}$ -local minimizer for (ExpCSP). Assume that, for some $\delta > 0$, assumptions (CS.1)-(CS.3) are satisfied, in addition to the following assumption:

(CS.4) h is upper semicontinuous and there exists $K > 0$ such that

$$|h(t, x) - h(t, x')| \leq K|x - x'| \quad \text{for all } x, x' \in \bar{x}(t) + \delta\mathbb{B}, t \in [S, T].$$

Then, there exist an adjoint arc $p \in W^{1,1}([S, T], \mathbb{R}^n)$, $\lambda \geq 0$, a Borel measure $\mu(\cdot) : [S, T] \rightarrow \mathbb{R}$ and a μ -integrable function $\gamma(\cdot)$ such that

- (i) $(\lambda, p(\cdot), \mu(\cdot)) \neq (0, 0, 0)$,
- (ii) $-\dot{p}(t) \in \text{co } \partial_x(q(t) \cdot f(t, \bar{x}(t), \bar{u}(t))) \quad \text{a.e. } t \in [S, T]$,
- (iii) $(p(S), -q(T)) \in \lambda \partial_x g(\bar{x}(S), \bar{x}(T)) + N_C(\bar{x}(S), \bar{x}(T))$,
- (iv) $q(t) \cdot f(t, \bar{x}(t), \bar{u}(t)) = \max_{u \in U(t)} q(t) \cdot f(t, \bar{x}(t), u)$,
- (v) $\gamma(t) \in \partial_x^> h(t, \bar{x}(t)) \quad \text{and} \quad \text{supp}(\mu) \subset \{t \in [S, T] : h(t, \bar{x}(t)) = 0\}$,

where

$$q(t) = \begin{cases} p(S) & t = S \\ p(t) + \int_{[S,t]} \gamma(s) d\mu(s) & t \in (S, T]. \end{cases}$$

(Here, $\partial_x^> h(t, \cdot)$ is the partial hybrid subdifferential of h in the sense of (1.6).)

Remark 1.3.5. 1) If the reference minimizer $\bar{x}(\cdot)$ remains in the interior of the state constraint for all $t \in [S, T]$ (i.e. $h(\bar{x}(t)) < 0$ for all $t \in [S, T]$), the integral term $\int_{[S,t]} \gamma(s) d\mu(s)$ reduces to zero. Indeed, in this case, the set $\partial_x^> h(t, \bar{x}(t)) = \emptyset$ (which implies that $\text{supp}(\mu) = \emptyset$). Therefore, $q(t) = p(t)$ for all $t \in [S, T]$ and the ‘Constrained’ Maximum Principle coincides with the PMP established in the subsection 1.2.3 (cf. Theorem 1.2.8).

- 2) The necessary conditions for optimal control problems with state constraints (in the form of $h(t, x) \leq 0$) involve the partial hybrid subdifferential $\partial_x^> h(t, x)$ for its various properties, for instance its convexity and its closure. This permits to handle the limit-taking in $\int \gamma(s) d\mu(s)$ in the sense of Proposition 1.3.2, when the original problem is approached by an auxiliary one.

Consider now the same optimal control problem (with controlled dynamic) with an implicit state constraint in the sense of condition (1.21)

$$(\text{ImCSP}) \quad \left\{ \begin{array}{l} \text{minimize } g(x(S), x(T)) \\ \text{over arcs } x(\cdot) \in W^{1,1}([S, T], \mathbb{R}^n) \text{ and measurable functions } u(\cdot) \text{ satisfying} \\ \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [S, T] \\ (x(S), x(T)) \in C \\ x(t) \in A \quad \text{for all } t \in [S, T] \\ u(t) \in U(t) \quad \text{a.e. } t \in [S, T]. \end{array} \right.$$

In this case, assumption (CS.4) is replaced with

(CS.4') A is a closed set and $\text{co } N_A(\bar{x}(t))$ is pointed for all $t \in [S, T]$.

A convex cone $K \subset \mathbb{R}^n$ is said to be 'pointed' if for any nonzero elements $d_1, d_2 \in K$, $d_1 + d_2 \neq 0$. We will see in Remark 1.3.7 below why the pointedness assumption is important.

Theorem 1.3.6. *Suppose that for a given $W^{1,1}$ -local minimizer $(\bar{x}(\cdot), \bar{u}(\cdot))$ for (ImCSP) and for some $\delta > 0$, assumptions (CS.1)-(CS.3) and (CS.4') are satisfied. Then there exist an adjoint arc $p \in W^{1,1}([S, T], \mathbb{R}^n)$, $\lambda \geq 0$ and a function of bounded variation $v(\cdot) : [S, T] \rightarrow \mathbb{R}^n$ continuous from the right on (S, T) , such that: conditions (ii)-(iv) of Theorem 1.3.4 are satisfied with*

$$q(t) = \begin{cases} p(S) & t = S \\ p(t) + \int_{[S,t]} dv(s) & t \in (S, T], \end{cases}$$

in addition to

(i') $(\lambda, p(\cdot), v(\cdot)) \neq (0, 0, 0)$;

(v') $\int_{[S,T]} \xi(t) \cdot dv(t) \leq 0$ for all $\xi \in \{\xi' \in C([S, T], \mathbb{R}^n) : \xi'(t) \in T_A(\bar{x}(t)) \text{ } v - \text{a.e.}\}$ and $\text{supp}(v) \subset \{t \in [S, T] : \bar{x}(t) \in \partial A\}$.

Remark 1.3.7. *The passage from the optimality conditions of Theorem 1.3.4 to those of Theorem 1.3.6 is straightforward if in (1.21), we take the scalar function h to be*

$$h(x) := d_A(x),$$

where d_A denotes the Euclidean distance and we use the fact that

$$\partial^> d_A(\bar{x}(t)) \subset \text{co } N_A(\bar{x}(t)) \cap \{\xi : \xi \neq 0\}$$

owing to Proposition 1.1.17 and to the pointedness of $\text{co } N_A(\bar{x}(t))$ (we refer the reader to Proposition 5.3.4 for the detailed result on this inclusion). Therefore, writing $(p, \mu, \lambda, \gamma)$ as the

multipliers for the choice of h as d_A , the (vector-valued) function of bounded variations $v(\cdot)$ can be defined according to

$$dv(t) := \gamma(t)d\mu(t) \quad \text{where } \gamma(t) \in \partial_x^\circ d_A(\bar{x}(t)) \text{ } \mu - a.e.$$

The polarity relation between the Clarke tangent cone and the limiting normal cone (see Proposition 1.1.22) permits to retrieve condition (v').

We point out that the necessary conditions are of interest only when $0 \notin \partial_x^\circ d_A(\bar{x}(t))$ for all t such that $\bar{x}(t) \in \partial A$. This is because the necessary conditions are satisfied automatically along any arc \bar{x} such that

$$0 \in \partial_x^\circ d_A(\bar{x}(t')) \quad \text{and} \quad \bar{x}(t') \in \partial A$$

for some time t' , if we choose the set of multipliers to be

$$p(t) \equiv 0 \quad \lambda = 0 \quad \mu = \delta_{t'}.$$

The pointedness assumption permits, therefore, to exclude the existence of such arcs.

1.3.4 Extended Euler-Lagrange Condition

Consider now an implicit state constrained optimal control problem involving a dynamic constraint taking the form of a differential inclusion:

$$(ImDIP) \quad \left\{ \begin{array}{l} \text{minimize } g(x(S), x(T)) \\ \text{over arcs } x(\cdot) \in W^{1,1}([S, T]; \mathbb{R}^n) \text{ satisfying} \\ \dot{x}(t) \in F(t, x(t)) \quad \text{a.e. } t \in [S, T] \\ (x(S), x(T)) \in C \\ x(t) \in A \quad \text{for all } t \in [S, T]. \end{array} \right.$$

We assume that the previous stated hypotheses (DI.1)-(DI.4) are satisfied for some $\delta > 0$, $c(\cdot) \in L^1$ and $k_F(\cdot) \in L^1$, where $\bar{x}(\cdot)$ is a given absolutely continuous arc. We impose in addition that

(DI.5) the set A is closed and $\text{co } N_A(\bar{x}(t))$ is pointed for each $t \in [S, T]$.

Theorem 1.3.8. (cf. [81, Theorem 10.3.1]) Under assumptions (DI.1)-(DI.5) for some $W^{1,1}$ -local minimizer $\bar{x}(\cdot)$, there exist $p(\cdot) \in W^{1,1}([S, T]; \mathbb{R}^n)$, $\lambda \geq 0$, and a function of bounded variations $v(\cdot) : [S, T] \rightarrow \mathbb{R}^n$ continuous from the right such that

- (i) $(\lambda, p(\cdot), v(\cdot)) \neq (0, 0, 0)$,
- (ii) $\dot{p}(t) \in \text{co } \{\eta \mid (\eta, q(t)) \in N_{Gr F(t, \cdot)}(\bar{x}(t), \dot{\bar{x}}(t))\} \quad \text{a.e. } t \in [S, T]$,
- (iii) $(p(S), -q(T)) \in \lambda \partial g(\bar{x}(S), \bar{x}(T)) + N_C(\bar{x}(S), \bar{x}(T))$,
- (iv) $q(t) \cdot \dot{\bar{x}}(t) = \max_{v \in F(t, \bar{x}(t))} q(t) \cdot v \quad \text{a.e. } t \in [S, T]$,

$$(v) \int_{[S,T]} \xi(t) \cdot d\nu(t) \leq 0 \quad \text{for all } \xi \in \{\xi' \in C([S,T], \mathbb{R}^n) : \xi'(t) \in T_C(\bar{x}(t)) \nu - a.e.\},$$

where

$$q(t) = \begin{cases} p(S) & t = S \\ p(t) + \int_{[S,t]} d\nu(s) & t \in (S, T]. \end{cases}$$

The same theorem holds true when we consider a state constraint in the form of (1.20) and when the data satisfy assumptions (DI.1)-(DI.4), and (in place of (DI.5)) the following assumption:

(DI.5') h is upper semicontinuous and there exists a constant k_h such that

$$|h(t, x) - h(t, x')| \leq k_h |x - x'|$$

for all $x, x' \in \bar{x}(t) + \delta\mathbb{B}$, $t \in [S, T]$.

More precisely:

Theorem 1.3.9. (see [81, Theorem 10.3.1]) Under assumptions (DI.1)-(DI.4) and (DI.5') for some $W^{1,1}$ -local minimizer $\bar{x}(\cdot)$, there exist $p(\cdot) \in W^{1,1}([S, T]; \mathbb{R}^n)$, $\lambda \geq 0$, a Borel measure $\mu(\cdot) : [S, T] \rightarrow \mathbb{R}$ and a μ -integrable function γ such that

$$(i) (\lambda, p(\cdot), \mu(\cdot)) \neq (0, 0, 0),$$

$$(ii) \dot{p}(t) \in \text{co} \{ \eta \mid (\eta, q(t)) \in N_{Gr F(t, \cdot)}(\bar{x}(t), \dot{\bar{x}}(t)) \} \quad a.e. \ t \in [S, T],$$

$$(iii) (p(S), -q(T)) \in \lambda \partial g(\bar{x}(S), \bar{x}(T)) + N_C(\bar{x}(S), \bar{x}(T)),$$

$$(iv) q(t) \cdot \dot{\bar{x}}(t) = \max_{v \in F(t, \bar{x}(t))} q(t) \cdot v \quad a.e. \ t \in [S, T],$$

$$(v) \gamma(t) \in \partial_x^> h(t, \bar{x}(t)) \ \mu\text{-a.e.} \quad \text{and} \quad \text{supp}(\mu) \subset \{t : h(t, \bar{x}(t)) = 0\},$$

where

$$q(t) = \begin{cases} p(S) & t = S \\ p(t) + \int_{[S,t]} \gamma(s) d\mu(s) & t \in (S, T]. \end{cases}$$

Remark 1.3.10. We point out that we can always express the necessary conditions in an equivalent version when we consider the normalized version of the nontriviality condition (i), as follows

$$\lambda + \|p\|_{L^\infty} + \|\mu\|_{T.V.} = 1$$

by following the same reasoning of Remark 1.2.9 2). (Here, $\|\cdot\|_{T.V.}$ denotes the total variation.)

**2**

Neighboring Feasible Trajectories and $W^{1,1}$ -Linear Estimates

In this chapter, conditions, known as *inward pointing conditions*, are presented. We will see that such conditions guarantee, for a state constrained dynamics, the existence of a state trajectory satisfying a state constraint (this is a viability result) and a certain ($W^{1,1}$ -)estimate on the distance of the constructed feasible state trajectory from a specified state trajectory, in terms of the degree to which the specified state trajectory violates the state constraint. The results presented in this chapter cover situations in which the initial data belongs either to a smooth region of the state constraint or to a nonsmooth region of it which is the novelty according to this subject.

Dans ce chapitre, des conditions, connues sous le nom de '*inward pointing conditions*', sont présentées. Nous verrons que de telles conditions garantissent, pour une dynamique avec une contrainte d'état, l'existence d'une trajectoire d'état satisfaisant une contrainte d'état (c'est un résultat de viabilité) et une certaine ($W^{1,1}$ -)estimation sur la distance entre la trajectoire d'état admissible et une trajectoire de référence, en fonction du degré auquel la trajectoire de référence viole la contrainte d'état. Les résultats présentés dans ce chapitre couvrent des situations dans lesquelles les données initiales appartiennent soit à une région lisse de la contrainte d'état, soit à une région non lisse de celle-ci qui est la nouveauté selon ce sujet.

“When we go deep enough or high enough, we meet. It is only on the surface that we differ and sometimes clash. True, we do not always find our way to the depth or the height, or we do not take the trouble to do so.”

— Amin Rihani

2.1 Overview

Soner (cf. [75], [76]) was the pioneer in the derivation of conditions such that for a given reference F -trajectory $\hat{x}(\cdot)$ taking values in an ε -neighborhood of the state constraint, there exists a second (approximating) feasible trajectory $x(\cdot)$ which satisfies the L^∞ -linear estimate

$$\|x(\cdot) - \hat{x}(\cdot)\|_{L^\infty} \leq K\varepsilon$$

for a certain positive constant K . One important consequence of such estimate revealed by Soner, was the regularity property of the value function for state constrained dynamic optimization problems (more precisely, he established the bounded uniform continuity aspect of the value function). In his technique, Lipschitz continuity of the dynamics w.r.t. the time was required, in addition to the so-called ‘constraint qualification’, which requires the existence of a feedback control (i.e. depending on the state) for which the dynamic, expressed as a differential equation, points inside the state constraint set expressed as a closed set. Many other contributions to the result appeared by trying to relax the assumptions on the set of velocities (imposing only measurability w.r.t. the time) and to establish stronger estimates involving a $W^{1,1}$ -norm instead of the L^∞ -norm. We shall note that the results presented in this chapter differ from the kind of information given by viability theorems [6] because we provide an additional estimate from the set of feasible F -trajectories: viability theory gives only the existence of a feasible trajectory but not an estimate. Many applications of this result have been studied in the literature for optimal control problems involving a state constraint set, for instance deriving refined and unrestrictive conditions under which first order necessary conditions (the Maximum Principle or the Generalized Euler-Lagrange condition) are nondegenerate or normal, characterizing value functions as (possibly unique) solutions to the Hamilton Jacobi equations and establishing ‘sensitivity relations’ in which the costate trajectory and the Hamiltonian are interpreted in terms of generalized gradients of the value function. In what follows, we give an account of the theory. Rampazzo and Vinter [68] have considered the same problem as the one considered by Soner, but with an inequality state constraint, to which they adapted the constraint qualification. The contribution of their work was a refined regularity property of the value function associated to the state constrained problem: they proved that the value function is sub-Lipschitz. Later, a paper by Frankowska and Rampazzo [52] extended the neighboring feasible trajectory result to dynamics represented as differential inclusions and state constraints represented as a closed (possibly nonsmooth) set. They proved that, under a Soner-type hypothesis, adapted to set-valued maps, a $W^{1,1}$ -linear estimate can hold for the smooth state constraint, while a L^∞ -linear estimates holds for the nonsmooth data. In their paper, the results are concerned with the infinite horizon constrained problems. In the same year, Frankowska and Vinter [45] studied a class of problems involving a differential inclusion and multiple state constraints (it is a family of $C^{1,1}$ functional inequalities). The stated hypotheses do not require the set of velocities F to be continuous in time or convex valued, but merely measurable w.r.t. time. They supplied a tighter $W^{1,1}$ -estimate in place of the customary L^∞ -estimate. This is achieved under a new constraint qualification concerning the interaction of the state and dynamic constraints. This result was fundamental for Rampazzo and Vinter [69] in order to establish the nondegeneracy of the necessary optimality conditions (i.e. roughly speaking, when the set of admissible trajectories and arcs verifying the necessary conditions do

not coincide). In [10] and [11], Bettiol, Bressan and Vinter showed, by a counter-example, that linear $W^{1,1}$ - estimates are not in general valid for multiple state constraints: a state constraint set involving the intersection of two hyperplanes fails to guarantee a $W^{1,1}$ -estimate, when the initial data is at a ‘corner’ of the state constraint. They established moreover a ‘super-linear’ estimate w.r.t. the $W^{1,1}$ -norm for a constant set of velocities and a state constraint set represented as the intersection of two half spaces. A further improvement in this field was the work of Bressan and Facchi [25] where they generalized the analysis to problems with differential inclusion (depending merely on the state x) and an implicit state constraint (the state constraint set is a closed convex set). Their new constraint qualification involved an interaction between the set of velocities and the interior of the Clarke tangent cone of the state constraint set at every point belonging to the boundary of the constraint. The $W^{1,1}$ -estimate provided is in fact a ‘super-linear’ estimate, which is sharp if no additional assumption is imposed (by the counter-example of [10]). An early paper by Bettiol and Facchi [12] showed that the ‘super-linear’ estimate can be replaced by a linear one (w.r.t. the extent of the violation of the constraint), always for a convex state constraint set, if one of the following two cases occurs:

- The initial point $\hat{x}(S)$ of the reference trajectory ranges away from ‘corners’ of the state constraint set;
- the initial condition $x(S)$ of the approximate trajectory can be freely chosen.

Moreover, they applied their result to optimal control problems in order to establish normality of the necessary optimality conditions (i.e. when the multiplier λ associated to the cost is not zero). In [15], Bettiol and Khalil provide improvements on earlier work: we consider state-constrained differential inclusions, where the velocity set depends on both time and state variables (allowing the case where the velocity set is just measurable w.r.t. the time variable) and the state constraint is just a closed set (not necessarily locally smooth or convex). We propose a new constraint qualification involving just tangent vectors to the state constraint (no inward pointing condition involving the velocity set is required). This condition can be applied also when the starting point is in a region where the state constraint set is nonsmooth. Under this condition, we construct a local neighboring feasible trajectory which verifies a $W^{1,1}$ -linear estimate. In the same paper [15], we make use of the latter construction in order to formulate a theorem on the nondegeneracy of the necessary conditions in the case when a minimizer starts from a region where the state constraint is nonsmooth. A further refinement of earlier results is the work of Bettiol, Khalil and Vinter [17] where the local construction of feasible trajectories is extended to a global one, but here a convexity assumption on the state constraint set is a must. In this work, classical constraint qualifications are not enough to establish a global construction in the case when the starting point is at a ‘corner’ of the state constraint set. But an additional assumption concerning existence of hypertangent vectors at the starting point (without being in the velocity set) is imposed. In these circumstances, we prove that the additional information involving hypertangent vectors to the left endpoint and the state constraint sets can be used to establish normality of the necessary conditions.

We shall write A_0 the set of all points $y \in A$ where the state constraint A is locally regular: more precisely, $y \in A_0$ if and only if there exists a radius $r > 0$ and a function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ of class C^{1+} (i.e. everywhere differentiable with locally Lipschitz continuous derivatives) such that

$$\nabla\varphi(y) \neq 0, \quad A \cap (y + r \text{int } \mathbb{B}) = \{x : \varphi(x) \leq 0\} \cap (y + r \text{int } \mathbb{B}). \quad (2.1)$$

Observe that $A_0 \supset \text{int } A$.

2.2 Local Construction of Neighboring Feasible Trajectories with $W^{1,1}$ -Linear Estimates

Consider a system described by a differential inclusion and a state constraint set A

$$\begin{aligned} \dot{x}(t) &\in F(x(t)) \quad \text{a.e. } t \in [S, T] \\ x(t) &\in A \quad \text{for all } t \in [S, T], \end{aligned}$$

in which $[S, T]$ is a given time interval and $F(\cdot) : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a given multifunction and $A \subset \mathbb{R}^n$ is a non-empty set. Fix $r_0 > 0$ and assume that for some constants $c > 0$ and $k_F > 0$, and for $R_0 := e^{c(T-S)}(r_0 + 2)$, the following assumptions are satisfied:

(A.1) The set $A \subset \mathbb{R}^n$ is closed.

(A.2) (a) The multifunction F has non-empty values and $\text{Gr } F(\cdot)$ is closed ;

(b)

$$F(x) \subset c(1 + |x|) \mathbb{B} \quad \text{for all } x \in \mathbb{R}^n ;$$

(c)

$$F(x) \subset F(y) + k_F|x - y|\mathbb{B} \quad \text{for all } x, y \in R_0\mathbb{B} ;$$

(A.3)

$$\text{co } F(x) \cap \text{int } T_A(x) \neq \emptyset \quad \text{for all } x \in A \cap R_0\mathbb{B} .$$

($T_A(x)$ denotes the Clarke tangent cone to A at x in the sense of Proposition 1.1.21.)

Remark 2.2.1. 1) If the set of velocities $F(\cdot, \cdot)$ depends also on the time, then $F(\cdot, x)$ should be Lebesgue measurable for each $x \in \mathbb{R}^n$, $c(\cdot)$ and $k_F(\cdot)$ time-dependent functions and L^1 -integrable with values in \mathbb{R}^+ .

2) Hypothesis (A.2) is the standard requirement under which the existence of a solution to the differential inclusion is guaranteed (cf. Theorem 1.2.1).

3) Assumption (A.3) is a ‘classical’ (convexified) inward pointing condition: it states that for all points belonging to the boundary of the constraint set A , there exists a vector in the convex hull of the set of velocities and pointing inside the state constraint set A . This is the so-called ‘constraint qualification’. This type of condition appeared for the first time in the work of Soner [75] but for dynamics expressed as a differential equation. Such condition plays a crucial role in constructing a feasible state trajectory which stays ‘near’ a trajectory possibly violating the state constraint.

2.2.1 Smooth State Constraint

The following lemma asserts the existence of a trajectory which lies inside the state constraint for a small time interval and verifies a $W^{1,1}$ -linear estimate in the case where the initial data x_0 is away from ‘corners’ of the state constraint set.

Lemma 2.2.2. *Fix any $r_0 > 0$. Consider a multifunction $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ and a nonempty set A . Assume that (A.1)-(A.3) are satisfied (for some constants $c > 0$, $k_F > 0$ and $R_0 = e^{c(T-S)}(r_0 + 2)$). Take any $x_0 \in A_0 \cap r_0\mathbb{B}$. Then, we can find $\theta_0 \in (0, \frac{1}{4})$, $\tau_0 \in (0, T - S]$ and $K_0 > 0$ such that the following property is satisfied: given any F -trajectory $\hat{x}(\cdot) : [S, T] \rightarrow \mathbb{R}^n$ and $\varepsilon \geq 0$ such that*

$$\begin{cases} \max_{t \in [S, T]} d_A(\hat{x}(t)) \leq \varepsilon \\ \hat{x}(S) \in A \cap (x_0 + \theta_0\mathbb{B}), \end{cases}$$

we can find an F -trajectory $x(\cdot)$ on $[S, T]$ such that:

$$\begin{cases} x(S) = \hat{x}(S), \\ x(t) \in A \quad \text{for all } t \in [S, S + \tau_0], \\ \|x(\cdot) - \hat{x}(\cdot)\|_{W^{1,1}(S, T)} \leq K_0\varepsilon. \end{cases}$$

This lemma is a localized version of [10]. We provide more details of the proof in Section 2.4.

2.2.2 Nonsmooth State Constraint

The next lemma establishes the construction, locally in time, of a neighboring feasible trajectory verifying a $W^{1,1}$ -estimate in the case where the initial data x_0 belongs to a ‘corner’ of the state constraint (i.e. $x_0 \in A \setminus A_0$). Here, for the local construction, the existence of a vector belonging to the set of velocities and pointing inside the state constraint (i.e. assumption (A.3)) is not relevant.

Lemma 2.2.3. *(cf. [17]) Fix any $r_0 > 0$. Consider a multifunction $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ and a nonempty set $A \subset \mathbb{R}^n$. Suppose that only hypotheses (A.1) and (A.2) are satisfied (for some constants $c > 0$, $k_F > 0$ and $R_0 = e^{c(T-S)}(r_0 + 2)$). Take any $x_0 \in (A \setminus A_0) \cap r_0\mathbb{B}$, and any $w_0 \in \text{int } T_A(x_0)$. Then we can find $\theta_1 \in (0, \frac{1}{16})$, $\alpha \geq 0$, $\tau_1 \in (0, T - S]$ and $K_1 > 0$ such that the following property is satisfied: given any F -trajectory $\hat{x}(\cdot)$ on $[S, T]$ and $\varepsilon \geq 0$ such that*

$$\begin{cases} \max_{t \in [S, T]} d_A(\hat{x}(t)) \leq \varepsilon \\ \hat{x}(S) \in A \cap (x_0 + \theta_1\mathbb{B}), \end{cases} \quad (2.2)$$

we can find an F -trajectory $x(\cdot)$ on $[S, T]$ such that

$$\begin{cases} x(S) = \hat{x}(S) + \alpha\varepsilon w_0, \\ x(t) \in A \quad \text{for all } t \in [S, S + \tau_1], \\ \|x(\cdot) - \hat{x}(\cdot)\|_{W^{1,1}(S, T)} \leq K_1\varepsilon. \end{cases}$$

The proof of Lemma 2.2.3 is provided in Section 2.4.

2.3 Global Construction of Neighboring Feasible Trajectories with $W^{1,1}$ -Linear Estimates

This section is an extension of the local construction of neighboring feasible trajectories established in Section 2.2. The F -trajectory previously constructed has all the necessary properties of a feasible F -trajectory with one exception: it satisfies the state constraint only on a suitably small initial interval $[S, S + \tau]$ (for some $\tau \in (0, T - S]$). The idea behind the proof of extending the result to the whole time interval $[S, T]$ is to proceed with a recursive construction and to construct a finite sequence of sub-arcs, whose concatenation is an F -trajectory satisfying the state constraint on the whole time interval $[S, T]$ and which, at the same time, satisfies a linear, $W^{1,1}$ -estimate, with an increased constant of proportionality. By contrast, this construction requires a stronger assumption on the state constraint set. We shall therefore replace assumption (A.1) with

(A.1') The set $A \subset \mathbb{R}^n$ is closed, non-empty and convex.

The convexity of the state constraint set A will intervene in the iterative construction presented in the lemma below.

Lemma 2.3.1. *Fix $r_0 > 0$. Consider a multifunction $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ and a set $A \subset \mathbb{R}^n$ satisfying the assumptions (A.1'), (A.2) and (A.3) (for some constants $c > 0$, $k_F > 0$, and $R_0 = e^{c(T-S)}(r_0 + 2)$). Then there exists $\tau_2 \in (0, T - S]$ such that for each $0 < \tau \leq \tau_2$, we can find $\varepsilon_2 > 0$ and $K_2 > 0$ satisfying the following property: take any $t_0 \in [S, T]$, any F -trajectory $\hat{x}(\cdot)$ on $[t_0, T]$, and $\varepsilon \in [0, \varepsilon_2]$ satisfying:*

$$\begin{cases} \hat{x}(t_0) \in A \cap (e^{c(t_0-S)}(r_0 + 2) - 1)\mathbb{B}, \\ \hat{x}(t) \in A \quad \text{for all } t \in [t_0, t_0 + \tau], \\ d_A(\hat{x}(t)) \leq \varepsilon \quad \text{for all } t \in [t_0 + \tau, t_0 + 2\tau], \end{cases}$$

we can construct an F -trajectory $x(\cdot)$ on $[t_0, T]$ such that

$$\begin{cases} x(t_0) = \hat{x}(t_0), \\ x(t) \in A \quad \text{for all } t \in [t_0, t_0 + 2\tau], \\ \|x(\cdot) - \hat{x}(\cdot)\|_{W^{1,1}(t_0, T)} \leq K_2\varepsilon. \end{cases}$$

Proof. The proof follows the same analysis of [12, Lemma 1]. Indeed, fix any $r_0 > 0$ and $x_0 \in A \cap r_0\mathbb{B}$. Write $R_0 := e^{c(T-S)}(r_0 + 2)$ and $M_0 := c(1 + R_0)$. Observe that assumption (A.2) ensures the following a-priori bounds for the magnitude of all F -trajectories $x(\cdot)$ starting from $(r_0 + 1)\mathbb{B}$ and their velocities: if $x(\cdot)$ is an F -trajectory on $[S, T]$ with $x(S) \in (r_0 + 1)\mathbb{B}$, then

$$x(t) \in R_0\mathbb{B} \quad \text{for all } t \in [S, T]$$

and

$$\dot{x}(t) \in M_0\mathbb{B} \quad \text{a.e. } t \in [S, T].$$

Using the characterization of the interior of the Clarke tangent cone (in the sense of Theorem 1.1.24) and applying a standard compactness argument, we can find a finite number of points $y_i \in \partial A \cap R_0\mathbb{B}$ and numbers $\bar{\varepsilon} > 0$, $\delta_i \in (0, 1)$, $\rho_i \in (0, \frac{1}{4})$, for $i = 0, \dots, N$, such that

$$(\partial A + \bar{\varepsilon}\mathbb{B}) \cap R_0\mathbb{B} \subset \bigcup_{i=0}^N (y_i + \frac{\delta_i}{4} \text{int } \mathbb{B}),$$

and for all $i = 0, \dots, N$, there exists a vector $v_i \in \text{co } F(y_i) \cap \text{int } T_A(y_i)$ such that

$$y + [0, 2\delta_i](v_i + \rho_i\mathbb{B}) \subset A, \quad \text{for all } y \in (y_i + \delta_i\mathbb{B}) \cap A.$$

We highlight only the fact that the properties listed above (which are an immediate consequence of assumptions (A.2) and (A.3)) represent a localized version of the hypotheses invoked in [12, Lemma 1]. \square

We are now able to formulate two theorems (for the smooth and nonsmooth case) which extend the local construction of Section 2.2 to the whole time interval $[S, T]$.

Theorem 2.3.2 (Smooth case, [17]). *Assume that (A.1'), (A.2) and (A.3) are satisfied and let $x_0 \in A_0 \cap r_0\mathbb{B}$ be a given point. Then there exist $\theta \in (0, \frac{1}{4})$ and $K > 0$ such that for any F -trajectory $\hat{x}(\cdot)$ on $[S, T]$ and $\varepsilon \geq 0$ satisfying*

$$\begin{cases} \max_{t \in [S, T]} d_A(\hat{x}(t)) \leq \varepsilon \\ \hat{x}(S) \in A \cap (x_0 + \theta\mathbb{B}), \end{cases}$$

we can find an F -trajectory $x(\cdot)$ on $[S, T]$ satisfying

$$\begin{cases} x(S) = \hat{x}(S) \\ x(t) \in A \quad \text{for all } t \in [S, T] \\ \|x(\cdot) - \hat{x}(\cdot)\|_{W^{1,1}(S, T)} \leq K\varepsilon. \end{cases}$$

Comment: Our constraint qualification (A.3) is a ‘classical’ (convexified) inward pointing condition: we do not invoke the existence of regular feedback controls (cf. [68], [75]), or conditions requiring inward pointing vectors which are tangent to the velocity set (cf. [49]). But, known counter-examples (see [10] and [14]) clearly show that, in general, ‘classical’ inward pointing conditions are not enough to derive linear $W^{1,1}$ -estimates for nonsmooth state constraints. In a recent paper [12], Bettiol and Facchi proved that linear $W^{1,1}$ -estimates are still possible for convex (nonsmooth) domains and Lipschitz continuous velocity sets (time-independent) under the assumption (A.3), if the left endpoint for the approximating (feasible) trajectory is freely chosen. The next result (Theorem 2.3.3) presents an improvement on earlier demonstrated conditions when the starting point is located in a ‘corner’ of the state constraint set: the left endpoint for the approximating trajectory will be chosen along hypertangent directions of the state constraint A .

Theorem 2.3.3 (Nonsmooth case, [17]). *Under assumptions (A.1'), (A.2) and (A.3), consider $x_0 \in (A \setminus A_0) \cap r_0\mathbb{B}$ to be a given point. Then for any $w_0 \in \text{int } T_A(x_0)$ fixed, there exist $\theta \in (0, \frac{1}{4})$, $\alpha \geq 0$ and $K > 0$ such that for any F -trajectory $\hat{x}(\cdot)$ on $[S, T]$ and $\varepsilon \geq 0$ with*

$$\begin{cases} \max_{t \in [S, T]} d_A(\hat{x}(t)) \leq \varepsilon \\ \hat{x}(S) \in A \cap (x_0 + \theta\mathbb{B}), \end{cases} \quad (2.3)$$

we can find an F -trajectory $x(\cdot)$ on $[S, T]$ satisfying

$$\begin{cases} x(S) = \hat{x}(S) + \alpha\varepsilon w_0 \\ x(t) \in A \quad \text{for all } t \in [S, T] \\ \|x(\cdot) - \hat{x}(\cdot)\|_{W^{1,1}(S, T)} \leq K\varepsilon. \end{cases} \quad (2.4)$$

The proofs of Theorems 2.3.2 and 2.3.3 will be provided in Section 2.4, and can be found also in [17].

A consequence of Theorems 2.3.2 and 2.3.3 is the following result, in which the left endpoint $\hat{x}(S)$ of the possibly ‘violating’ F -trajectory $\hat{x}(\cdot)$ does not necessarily belong to the state constraint.

Corollary 2.3.4 (cf. [17]). *Suppose that all the assumptions (A.1'), (A.2) and (A.3) are satisfied (for some constants $c > 0$ and $k_F > 0$, and for $R_0 := e^{c(T-S)}(r_0 + 2)$). Let $x_0 \in A \cap r_0\mathbb{B}$ be a given point. Then, the following properties hold true.*

- (i) *If $x_0 \in A_0$, then there exist $\theta' > 0$ and $K' > 0$ such that for any F -trajectory $\hat{x}(\cdot)$ on $[S, T]$ and $\varepsilon \geq 0$ satisfying*

$$\begin{cases} \max_{t \in [S, T]} d_A(\hat{x}(t)) \leq \varepsilon \\ \hat{x}(S) \in x_0 + \theta'\mathbb{B}, \end{cases}$$

and for any given $y_0 \in A \cap (\hat{x}(S) + \varepsilon\mathbb{B})$, we can find an F -trajectory $x(\cdot)$ on $[S, T]$ satisfying

$$\begin{cases} x(S) = y_0 \\ x(t) \in A \quad \text{for all } t \in [S, T] \\ \|x(\cdot) - \hat{x}(\cdot)\|_{W^{1,1}(S, T)} \leq K'\varepsilon. \end{cases}$$

- (ii) *If $x_0 \in A \setminus A_0$, then, for any $w_0 \in \text{int } T_A(x_0)$ fixed, there exist $\alpha' \geq 0$, $\theta' > 0$ and $K' > 0$ such that for any F -trajectory $\hat{x}(\cdot)$ on $[S, T]$ and $\varepsilon \geq 0$ with*

$$\begin{cases} \max_{t \in [S, T]} d_A(\hat{x}(t)) \leq \varepsilon \\ \hat{x}(S) \in x_0 + \theta'\mathbb{B}, \end{cases} \quad (2.5)$$

and for any given $y_0 \in A \cap (\hat{x}(S) + \varepsilon\mathbb{B})$, we can find an F -trajectory $x(\cdot)$ on $[S, T]$ satisfying

$$\begin{cases} x(S) = y_0 + \alpha'\varepsilon w_0 \\ x(t) \in A \quad \text{for all } t \in [S, T] \\ \|x(\cdot) - \hat{x}(\cdot)\|_{W^{1,1}(S, T)} \leq K'\varepsilon. \end{cases} \quad (2.6)$$

Proof. We prove here just the case $x_0 \in A \setminus A_0$. The case $x_0 \in A_0$ is easier and can be similarly treated.

Fix any $x_0 \in A \setminus A_0$ and $w_0 \in \text{int } T_A(x_0)$. Consider the constants $\alpha \geq 0$, $\theta \in (0, \frac{1}{4})$ and $K > 0$ provided by Theorem 2.3.3. Define $\theta' := \frac{\theta}{4}$. Take any F -trajectory $\hat{x}(\cdot)$ and any number $\varepsilon \geq 0$ such that:

$$\begin{cases} \max_{t \in [S, T]} d_A(\hat{x}(t)) \leq \varepsilon \\ \hat{x}(S) \in x_0 + \theta' \mathbb{B} . \end{cases}$$

It is not restrictive to assume that $\varepsilon \leq \theta'$. Consider any $y_0 \in A \cap (\hat{x}(S) + \varepsilon \mathbb{B})$. Observe that $y_0 \in A \cap (x_0 + \theta \mathbb{B})$. It is obvious that $|y_0 - \hat{x}(S)| \leq \varepsilon$. Then invoking the Filippov Existence Theorem (cf. Theorem 1.2.1) to the reference trajectory $\hat{x}(\cdot)$ on $[S, T]$, we obtain an F -trajectory $y(\cdot)$ on $[S, T]$ such that $y(S) = y_0$ and

$$\|y(\cdot) - \hat{x}(\cdot)\|_{W^{1,1}(S, T)} \leq e^{k_F(T-S)} |y(S) - \hat{x}(S)| \leq e^{k_F(T-S)} \varepsilon . \quad (2.7)$$

Notice that

$$\max_{t \in [S, T]} d_A(y(t)) \leq \|y(\cdot) - \hat{x}(\cdot)\|_{L^\infty(S, T)} + \max_{t \in [S, T]} d_A(\hat{x}(t)) \leq (e^{k_F(T-S)} + 1) \varepsilon . \quad (2.8)$$

We write $\varepsilon' := (e^{k_F(T-S)} + 1) \varepsilon$. Theorem 2.3.3 is also applicable for the reference trajectory $y(\cdot)$ and number ε' : we obtain a feasible F -trajectory $x(\cdot)$ on $[S, T]$ such that

$$x(S) = y(S) + \alpha \varepsilon' w_0 = y_0 + \alpha \varepsilon' w_0 = y_0 + \alpha (e^{k_F(T-S)} + 1) \varepsilon w_0$$

and

$$\|x(\cdot) - y(\cdot)\|_{W^{1,1}(S, T)} \leq K \varepsilon' . \quad (2.9)$$

As a consequence, from (2.7) and (2.9), we have

$$\begin{aligned} \|x(\cdot) - \hat{x}(\cdot)\|_{W^{1,1}(S, T)} &\leq \|x(\cdot) - y(\cdot)\|_{W^{1,1}(S, T)} + \|y(\cdot) - \hat{x}(\cdot)\|_{W^{1,1}(S, T)} \\ &\leq K \varepsilon' + e^{k_F(T-S)} \varepsilon \\ &\leq (K(e^{k_F(T-S)} + 1) + e^{k_F(T-S)}) \varepsilon \end{aligned}$$

and the validity of (2.6) follows taking the constants $\theta' := \frac{\theta}{4}$, $\alpha' := \alpha(e^{k_F(T-S)} + 1)$ and $K' := K(e^{k_F(T-S)} + 1) + e^{k_F(T-S)}$. \square

We consider now another version of Theorems 2.3.2 and 2.3.3 in which assumptions (A.1'), (A.2) and (A.3) are replaced by assumptions (A.1')-(A.3') below which are satisfied only in a neighborhood of a reference arc $\bar{x}(\cdot)$. We assume that for some constants $\delta > 0$, $c > 0$ and $k_F > 0$ the following hypotheses are satisfied:

(A.1') The set $A \subset \mathbb{R}^n$ is closed, non-empty and convex.

(A.2') (a) The multifunction F has non-empty values, and $\text{Gr } F(\cdot)$ is closed;

(b)

$$F(x) \subset c \mathbb{B} \quad \text{for all } x \in \bar{x}(t) + \delta \mathbb{B}, \quad t \in [S, T] ;$$

(c)

$$F(x) \subset F(y) + k_F|x - y|\mathbb{B} \quad \text{for all } x, y \in \bar{x}(t) + \delta\mathbb{B}, \quad t \in [S, T];$$

(A.3')

$$\text{co } F(x) \cap \text{int } T_A(x) \neq \emptyset \quad \text{for all } x \in A \cap (\bar{x}(t) + \delta\mathbb{B}), \quad t \in [S, T].$$

Remark 2.3.5. When the set of velocities $F(., .)$ depends also on the time, then $F(., x)$ must be Lebesgue-measurable for each $x \in \mathbb{R}^n$, $c(., .)$ and $k_f(., .)$ will be time-dependent functions and L^1 -integrable from $[S, T]$ with values in \mathbb{R}^+ .

Theorem 2.3.6 (cf. [17]). Consider a set $A \subset \mathbb{R}^n$ and a multifunction $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ such that, for some constants $\delta > 0$, $c > 0$ and $k_F > 0$, and for a given arc $\bar{x}(.)$, hypotheses (A.1')-(A.3') are satisfied. Then, the following properties hold true.

(i) If $\bar{x}(S) \in A_0$, then there exist $\theta \in (0, \delta)$ and $K > 0$ such that for any F -trajectory $\hat{x}(.)$ on $[S, T]$ and $\varepsilon \geq 0$ with

$$\begin{cases} \max_{t \in [S, T]} d_A(\hat{x}(t)) \leq \varepsilon \\ \|\bar{x}(.) - \hat{x}(.)\|_{W^{1,1}(S, T)} \leq \theta, \end{cases}$$

and for any given $y_0 \in A \cap (\hat{x}(S) + \varepsilon\mathbb{B})$, we can find an F -trajectory $x(.)$ on $[S, T]$ satisfying

$$\begin{cases} x(S) = y_0 \\ x(t) \in A \quad \text{for all } t \in [S, T] \\ \|x(.) - \hat{x}(.)\|_{W^{1,1}(S, T)} \leq K\varepsilon. \end{cases}$$

(ii) If $\bar{x}(S) \in A \setminus A_0$, then, for any $w_0 \in \text{int } T_A(\bar{x}(S))$ fixed, there exist $\theta \in (0, \delta)$, $\alpha \geq 0$ and $K > 0$ such that for any F -trajectory $\hat{x}(.)$ on $[S, T]$ and $\varepsilon \geq 0$ with

$$\begin{cases} \max_{t \in [S, T]} d_A(\hat{x}(t)) \leq \varepsilon \\ \|\bar{x}(.) - \hat{x}(.)\|_{W^{1,1}(S, T)} \leq \theta, \end{cases}$$

and for any given $y_0 \in A \cap (\hat{x}(S) + \varepsilon\mathbb{B})$, we can find an F -trajectory $x(.)$ on $[S, T]$ satisfying

$$\begin{cases} x(S) = y_0 + \alpha\varepsilon w_0 \\ x(t) \in A \quad \text{for all } t \in [S, T] \\ \|x(.) - \hat{x}(.)\|_{W^{1,1}(S, T)} \leq K\varepsilon. \end{cases}$$

Observe that the current $W^{1,1}$ -estimate for the trajectory $\hat{x}(.)$ is much 'stronger' in the sense that we require that the F -trajectory $\hat{x}(.)$ is close w.r.t. the $W^{1,1}$ -norm to the reference arc $\bar{x}(.)$, rather than the mere left end-point $\hat{x}(S)$ is close to a given point x_0 .

2.4 Proof of the Results

Proof of Lemma 2.2.2. Since $x_0 \in A_0$ and owing to (2.1), there exist a C^{1+} function $h_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ and a constant $\epsilon_0 \in (0, 1)$ such that we have:

$$\begin{aligned} \nabla h_0(x_0) \neq 0, \quad A \cap (x_0 + \epsilon_0 \text{int } \mathbb{B}) &= \{x : h_0(x) \leq 0\} \cap (x_0 + \epsilon_0 \text{int } \mathbb{B}) \\ &\subset A_0. \end{aligned} \quad (2.10)$$

Take $\theta_0 := \frac{\epsilon_0}{4}$ and $\tau_0 := \frac{\epsilon_0}{2M_0}$, where $M_0 = c(1 + R_0)$ is an upper bound for the velocities of all F -trajectories with initial state in $(r_0 + 1)\mathbb{B}$. This ensures that any F -trajectory on $[S, T]$ starting from $x_0 + \theta_0\mathbb{B}$ remains in $x_0 + \frac{3}{4}\epsilon_0\mathbb{B}$ for all $t \in [S, S + \tau_0]$. Then, Theorem 2.2.2 represents a local version of known linear $W^{1,1}$ -estimates for smooth state constraints (cf. [10]). \square

Proof of Lemma 2.2.3. From the characterization of the interior of the Clarke tangent cone (cf. Theorem 1.1.24), since $w_0 \in \text{int } T_A(x_0)$, there exists $\epsilon_1 \in (0, \frac{1}{4})$ such that

$$y + [0, \epsilon_1](w_0 + 2\epsilon_1\mathbb{B}) \subset A \quad \text{for all } y \in (x_0 + \epsilon_1\mathbb{B}) \cap A. \quad (2.11)$$

Recall that the positive constant $M_0 = c(1 + R_0)$ represents an upper bound for the velocities of all F -trajectories with initial state in $(r_0 + 1)\mathbb{B}$.

Define $\theta_1 := \frac{\epsilon_1}{4}$, $\alpha := \frac{1}{\epsilon_1}$ and $\tau_1 \in (0, \frac{\epsilon_1}{2M_0}]$ such that $\alpha|w_0|(e^{k_F\tau_1} - 1) \leq 1$. Take any F -trajectory $\hat{x}(\cdot)$ on $[S, T]$ and any $\epsilon \geq 0$ satisfying (2.2). It is not restrictive to assume that $\epsilon \leq \epsilon_1^2$. Consider the following arc $y(\cdot)$:

$$y(t) := \alpha\epsilon w_0 + \hat{x}(t), \quad \text{for all } t \in [S, T].$$

Observe that $y(t) = \hat{y}(t) + \alpha\epsilon w_0 + (\hat{x}(t) - \hat{y}(t))$, for all $t \in [S, T]$, where $\hat{y}(t) \in \pi_A(\hat{x}(t))$ (recall that $\pi_A(x)$ denotes the projection of x on the set A). Bearing in mind that $\hat{x}(\cdot)$ satisfies (2.2), we have

$$|\hat{x}(t) - \hat{y}(t)| \leq \epsilon, \quad \text{for all } t \in [S, T],$$

and therefore, for all $t \in [S, T]$,

$$\begin{aligned} y(t) + \epsilon\mathbb{B} &\subset \hat{y}(t) + \alpha\epsilon w_0 + 2\epsilon\mathbb{B} = \hat{y}(t) + \alpha\epsilon(w_0 + 2\epsilon_1\mathbb{B}) \\ &\subset \hat{y}(t) + \epsilon_1(w_0 + 2\epsilon_1\mathbb{B}), \end{aligned}$$

where in the last inclusion we have used the fact that $\epsilon \leq \epsilon_1^2$. As a consequence, as long as $\hat{y}(t) \in (x_0 + \epsilon_1\mathbb{B}) \cap A$ (details are shown in (2.12) below), from (2.11) we deduce that $y(t) + \epsilon\mathbb{B} \subset A$ for all $t \in [S, S + \tau_1]$. Indeed, this is guaranteed for all $t \in [S, S + \tau_1]$ owing to the choice of θ_1 and τ_1 ; more precisely, we have:

$$\begin{aligned} |x_0 - \hat{y}(t)| &\leq |x_0 - \hat{x}(S)| + |\hat{x}(S) - \hat{x}(t)| + |\hat{x}(t) - \hat{y}(t)| \\ &\leq \theta_1 + \tau_1 M_0 + \epsilon \leq \epsilon_1/4 + \epsilon_1/2 + \epsilon_1^2 \\ &< \epsilon_1. \end{aligned} \quad (2.12)$$

Consequently,

$$y(t) + \varepsilon \mathbb{B} \subset A, \quad \text{for all } t \in [S, S + \tau_1]. \quad (2.13)$$

Invoking the Filippov Existence Theorem (Theorem 1.2.1) to the reference trajectory $\hat{x}(\cdot)$, we obtain an F -trajectory $x(\cdot)$ on $[S, T]$ such that:

$$x(S) := y(S) = \hat{x}(S) + \alpha \varepsilon w_0$$

and for all $t \in (S, T]$:

$$\begin{aligned} \|x(\cdot) - \hat{x}(\cdot)\|_{L^\infty(S,t)} &\leq |x(S) - \hat{x}(S)| + \int_S^t |\dot{x}(s) - \dot{\hat{x}}(s)| ds \\ &\leq e^{k_F(t-S)} |x(S) - \hat{x}(S)|. \end{aligned} \quad (2.14)$$

Thus, for all $t \in (S, T]$, we have:

$$\|x(\cdot) - \hat{x}(\cdot)\|_{L^\infty(S,t)} \leq \alpha |w_0| e^{k_F(t-S)} \varepsilon$$

and

$$\|x(\cdot) - \hat{x}(\cdot)\|_{W^{1,1}(S,t)} \leq \alpha |w_0| (2e^{k_F(t-S)} - 1) \varepsilon, \quad (2.15)$$

(where $\|x\|_{W^{1,1}(S,T)} = \|x\|_{L^\infty(S,T)} + \|\dot{x}\|_{L^1(S,T)}$ for all $x \in W^{1,1}(S, T)$).

Since the arcs $x(\cdot)$ and $y(\cdot)$ have the same left end-point ($x(S) = y(S) = \hat{x}(S) + \alpha \varepsilon w_0$), and $\dot{y}(t) = \dot{\hat{x}}(t)$ for a.e. $t \in [S, T]$, from (2.14) we also obtain that

$$|x(t) - y(t)| \leq \int_S^t |\dot{x}(s) - \dot{\hat{x}}(s)| ds \leq \alpha |w_0| (e^{k_F(t-S)} - 1) \varepsilon, \quad \text{for all } t \in (S, T]$$

and then, from the choice of τ_1 ,

$$|x(t) - y(t)| \leq \varepsilon, \quad \text{for all } t \in [S, S + \tau_1]. \quad (2.16)$$

As a consequence, the statement of the lemma is confirmed by relations (2.13) and (2.16), which yield the (local) feasibility of the F -trajectory $x(\cdot)$ on $[S, S + \tau_1]$, and by (2.15), which provides the desired linear $W^{1,1}$ -estimate with constant $K_1 := \alpha |w_0| (2e^{k_F(T-S)} - 1)$. \square

Proof of Theorems 2.3.2 and 2.3.3. We build up the proof of both theorems in two steps. In the first step we recall the upper bounds for the magnitudes R_0 and M_0 of state trajectories and their velocities, respectively, having initial state on a bounded region. This allows us to use a standard compactness argument, and consequently apply a local analysis. In the second step, we treat separately the cases in which $x_0 \in A_0$ and $x_0 \in A \setminus A_0$. We shall make use of Lemmas 2.2.2 and 2.2.3 which give a local $W^{1,1}$ -estimate results for an initial subinterval, and we combine with the recursive argument of Lemma 2.3.1 ensuring the construction of the desired feasible F -trajectories.

Step 1. Fix any $r_0 > 0$ and $x_0 \in A \cap r_0 \mathbb{B}$. Write $R_0 := e^{c(T-S)}(r_0 + 2)$ and $M_0 := c(1 + R_0)$. The following a-priori bounds for the magnitude of all F -trajectories $x(\cdot)$ starting from $(r_0 + 1) \mathbb{B}$

and their velocities can be deduced from assumption (A.2): if $x(\cdot)$ is an F -trajectory on $[S, T]$ with $x(S) \in (r_0 + 1)\mathbb{B}$, then

$$x(t) \in R_0\mathbb{B} \quad \text{for all } t \in [S, T]$$

and

$$\dot{x}(t) \in M_0\mathbb{B} \quad \text{a.e. } t \in [S, T].$$

Using the characterization of the interior of the Clarke tangent cone (cf. property (1.8)) and applying a standard compactness argument, we can find a finite number of points $y_i \in \partial A \cap R_0\mathbb{B}$ and numbers $\bar{\varepsilon} > 0$, $\delta_i \in (0, 1)$, $\rho_i \in (0, \frac{1}{4})$, for $i = 0, \dots, N$, such that

$$(\partial A + \bar{\varepsilon}\mathbb{B}) \cap R_0\mathbb{B} \subset \bigcup_{i=0}^N (y_i + \frac{\delta_i}{4} \text{int } \mathbb{B}),$$

and for all $i = 0, \dots, N$, there exists a vector $v_i \in \text{co } F(y_i) \cap \text{int } T_A(y_i)$ such that

$$y + [0, 2\delta_i](v_i + \rho_i\mathbb{B}) \subset A, \quad \text{for all } y \in (y_i + \delta_i\mathbb{B}) \cap A.$$

Step 2. We treat separately the two cases ($x_0 \in A_0$) and ($x_0 \in A \setminus A_0$) using the local construction in Lemmas 2.2.2 and 2.2.3. After, we shall apply repeatedly the iterative construction technique based on Lemma 2.3.1. Assume first that $x_0 \in A_0$. Define

$$\theta := \theta_0, \quad \tau^* := \min\{\tau_0, \tau_2\}, \quad K^* := \max\{K_0, K_2\},$$

where θ_0 , τ_0 and K_0 are provided by Lemma 2.2.2, whereas τ_2 and K_2 are obtained using Lemma 2.3.1 (K_2 is related to time $\tau \leq \tau_2$). Write m the smaller integer such that $m\tau^* \geq T - S$, and consider the following standard iterative construction. We apply Lemma 2.2.2 to the reference F -trajectory $\hat{x}(\cdot)$ and $\varepsilon \geq 0$ satisfying $\max_{t \in [S, T]} d_A(\hat{x}(t)) \leq \varepsilon$ and $\hat{x}(S) \in (x_0 + \theta\mathbb{B}) \cap A$.

Therefore, we obtain an F -trajectory $z_0(\cdot)$ on $[S, T]$ such that

$$\begin{cases} z_0(t) \in A, & \text{for all } t \in [S, S + \tau^*] \\ \|z_0(\cdot) - \hat{x}(\cdot)\|_{W^{1,1}(S, T)} \leq K^* \varepsilon \\ \max_{t \in [S, T]} d_A(z_0(t)) \leq (K^* + 1)\varepsilon. \end{cases}$$

Observe that the F -trajectory $z_0(\cdot)$ verifies all the assumptions of Lemma 2.3.1 (for $t_0 = S$). As a result, we deduce the existence of an F -trajectory $z_1(\cdot)$ on $[S, T]$ satisfying the following relations:

$$\begin{cases} z_1(t) \in A, & \text{for all } t \in [S, S + 2\tau^*] \\ \|z_1(\cdot) - z_0(\cdot)\|_{W^{1,1}(S, T)} \leq K^*(K^* + 1)\varepsilon \\ \max_{t \in [S, T]} d_A(z_1(t)) \leq (K^* + 1)^2\varepsilon. \end{cases}$$

The above construction can be iteratively applied m -times, obtaining a finite sequence of F -trajectories $z_k(\cdot)$, for $k = 1, \dots, m$, such that:

$$\begin{cases} z_k(t) = z_{k-1}(t), & \text{for all } t \in [S, S + (k-1)\tau^*] \\ z_k(t) \in A, & \text{for all } t \in [S + (k-1)\tau^*, S + k\tau^*] \\ \|z_k(\cdot) - z_{k-1}(\cdot)\|_{W^{1,1}(S, T)} \leq K^*(K^* + 1)^k \varepsilon \\ \max_{t \in [S, T]} d_A(z_k(t)) \leq (K^* + 1)^{k+1} \varepsilon. \end{cases}$$

The F -trajectory $x(\cdot) := z_m(\cdot)$ satisfies the statement of Theorem 2.3.2, where for the $W^{1,1}$ -estimate we take the constant $K := (K^* + 1)^{m+1}$.

The proof of Theorem 2.3.3 in which $x_0 \in A \setminus A_0$ can be treated applying exactly the same argument, except the fact that we use Lemma 2.2.3 (nonsmooth case) instead of Lemma 2.2.2 in the local analysis. \square

Proof of Theorem 2.3.6. The proof is analogous to that one of Theorems 2.3.2 and 2.3.3 (and Corollary 2.3.4 whenever the initial state $\hat{x}(S)$ does not necessarily belong to the state constraint A). Indeed, we can use the same argument of step 1, which now is applied to the compact set

$$A \cap \{x : |x - \bar{x}(t)| \leq \delta, t \in [S, T]\}.$$

Estimates on an initial time interval (provided by Lemmas 2.2.2 and 2.2.3) are still valid if we replace the condition $\hat{x}(S) \in A \cap (x_0 + \theta\mathbb{B})$ with the condition

$$\|\hat{x}(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}(S,T)} \leq \theta,$$

for some $0 < \theta < \delta$. Observe that the latter is much ‘stronger’ in the sense that we require that the F -trajectory $\hat{x}(\cdot)$ is close w.r.t. the $W^{1,1}$ -norm to the reference arc $\bar{x}(\cdot)$, rather than the mere left end-point $\hat{x}(S)$ is close to a given point x_0 : we can take $0 < \theta < \delta$ small enough in such a manner that all the involved arcs and F -trajectories remain in the set

$$\{x : |x - \bar{x}(t)| \leq \delta/2, t \in [S, T]\},$$

in which assumptions (A.2) and (A.3) are always applicable. This guarantees the validity of all estimates of the iterative argument confirming all theorem assertions. \square

Remark 2.4.1. *The analysis in the proof of Theorem 2.3.6 justifies that for some closed non-empty set A , the statements of Lemmas 2.2.2 and 2.2.3 are satisfied in particular under assumptions (A.2’)-(A.3’) and (A.2’) respectively. Moreover, taking into account Remark 2.3.5, when $F(\cdot, \cdot)$ depends also in time, the statements remain valid for both lemmas. More precisely, for some integrable functions $c(\cdot)$ and $k_F(\cdot)$ with values in \mathbb{R}^+ , we consider the following two assumptions for a given reference arc $\bar{x}(\cdot)$ (these assumptions are slightly different according to the previous ones):*

(A.2’’) (a) *The multifunction $F(\cdot, \cdot)$ has non-empty values, $F(\cdot, x)$ is Lebesgue measurable for all $x \in \mathbb{R}^n$, and $\text{Gr } F(t, \cdot)$ is closed for all $t \in [S, T]$;*

(b) *$F(t, x) \subset c(t)\mathbb{B}$ for all $x \in \bar{x}(t) + \delta\mathbb{B}$, $t \in [S, T]$;*

(c) *$F(t, x) \subset F(t, y) + k_F(t)|x - y|\mathbb{B}$, for all $x, y \in \bar{x}(t) + \delta\mathbb{B}$, $t \in [S, T]$.*

(A.3’’) *$\bar{x}(S) \in A_0$ and for some $\tau \in (0, T - S]$, $c \geq 1$ and $\gamma > 0$ we have $\|c(\cdot)\|_{L^\infty} \leq c$, and*

$$\inf_{v \in F(t,x)} \nabla h(x) \cdot v \leq -\gamma \quad \text{for all } x \in (\bar{x}(t) + \delta\mathbb{B}) \cap A, \quad \text{a.e. } t \in [S, S + \tau].$$

(Here $h(\cdot)$ is the C^{1+} function with the property in (2.1) for $y = \bar{x}(S)$.)

The following two lemmas presented for the time-dependent differential inclusion case are a slightly different version of Lemmas 2.2.2 and 2.2.3.

Lemma 2.4.2 (see [15]). (Case where $\bar{x}(S) \in A_0$) Consider a (non-empty) closed set $A \subset \mathbb{R}^n$ and a multifunction $F : [S, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ such that for some constants $\delta > 0$, $c \geq 1$, $\gamma > 0$, for a function $k_F(\cdot) \in L^1([S, T], \mathbb{R}_+)$, and for a given F -trajectory $\bar{x}(\cdot)$, the assumptions (A.2'') and (A.3'') are satisfied. Then, there exist constants $\theta \in (0, \delta)$, $\tau' \in (0, \tau]$ and $K > 0$ such that: given any F -trajectory $\hat{x}(\cdot)$ on $[S, T]$ and $\varepsilon \geq 0$ satisfying

$$\begin{cases} \max_{t \in [S, T]} d_A(\hat{x}(t)) \leq \varepsilon \\ \|\hat{x}(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}(S, T)} \leq \theta, \end{cases}$$

we can find an F -trajectory $x(\cdot)$ on $[S, T]$ such that:

$$\begin{cases} x(S) \in \pi_A(\hat{x}(S)) \\ x(t) \in A \quad \text{for all } t \in [S, S + \tau'] \\ \|x(\cdot) - \hat{x}(\cdot)\|_{W^{1,1}(S, T)} \leq K\varepsilon. \end{cases}$$

Lemma 2.4.2 represents also a local version of known linear $W^{1,1}$ -estimates obtained for control systems with smooth state constraints (cf. [10]).

Lemma 2.4.3 (see [15]). (Case where $\bar{x}(S) \in A \setminus A_0$) Consider a (non-empty) closed set $A \subset \mathbb{R}^n$ and a multifunction $F : [S, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ such that for some constants $\delta > 0$, $c \geq 1$, for a function $k_F(\cdot) \in L^1([S, T], \mathbb{R}_+)$, and for a given F -trajectory $\bar{x}(\cdot)$, assumption (A.2'') is satisfied. Then, for any vector $w_0 \in \text{int } T_A(\bar{x}(S))$, there exist constants $\bar{\varepsilon} > 0$, $\alpha > 0$, $\theta \in (0, \delta)$, $\tau \in (0, T - S]$ and $K > 0$ such that: given any F -trajectory $\hat{x}(\cdot)$ on $[S, T]$ and $\varepsilon \in [0, \bar{\varepsilon}]$ satisfying

$$\begin{cases} \max_{t \in [S, S + \tau]} d_A(\hat{x}(t)) \leq \varepsilon \\ \|\hat{x}(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}(S, T)} \leq \theta, \end{cases} \quad (2.17)$$

and for any $y_0 \in \pi_A(\hat{x}(S))$, we can find an F -trajectory $x(\cdot)$ on $[S, T]$ such that:

$$\begin{cases} x(S) = y_0 + \alpha\varepsilon w_0 \\ x(t) \in A \quad \text{for all } t \in [S, S + \tau] \\ \|x(\cdot) - \hat{x}(\cdot)\|_{W^{1,1}(S, T)} \leq K\varepsilon. \end{cases} \quad (2.18)$$

The proof of Lemma 2.4.3 follows almost the same reasoning as the one used to prove Lemma 2.2.3.

The next two chapters will consider why the results on the neighboring feasible trajectories with associated $W^{1,1}$ -linear estimates are of big interest for providing additional information concerning the optimality condition for state-constrained optimal control problems. Indeed,

research into this subject is motivated by a broad implication in the dynamic optimization problems. More precisely, it intervenes in the derivation of non-degenerate necessary conditions as shown in Chapter 3. Moreover, it permits to show that $W^{1,1}$ -local minimizers satisfy a set of necessary optimality conditions in the normal form (i.e. the multiplier associated to the cost function is nonzero) as presented in Chapter 4.

**3**

Non-Degenerate Forms of the Generalized Euler-Lagrange Condition

This chapter holds an improvement on the necessary optimality conditions for optimal control problems with state constraints. We establish nondegeneracy of the generalized Euler-Lagrange conditions for state-constrained optimal control problems, in which the dynamic is represented in terms of a differential inclusion depending on both time and state variables (allowing the case where the velocity set is just measurable w.r.t. the time variable), the state and the end-point constraints are closed sets. We propose a new constraint qualification involving just tangent vectors to the state constraint (no inward pointing condition of the velocity set is required) and distance properties of trajectories to the end-point constraints; this condition can be applied also when the minimizer has left end-point in a region where the state constraint set is nonsmooth. The result is proved via a local construction of the neighboring feasible trajectories with $W^{1,1}$ -linear estimates. We also provide an example to emphasize the result.

Ce chapitre présente une amélioration des conditions nécessaires d'optimalité pour des problèmes de contrôle optimal avec des contraintes d'état. Nous établissons la non-dégénérescence des conditions d'Euler-Lagrange généralisées pour des problèmes de contrôle optimal avec contrainte d'état, dans lesquels la dynamique est représentée en termes d'inclusion différentielle en fonction des variables de temps et d'état (permettant au cas où l'ensemble de vitesse est juste mesurable par rapport au temps), la contrainte d'état ainsi que les contraintes initiale et finale sont des ensembles fermés. Nous proposons une nouvelle qualification de la contrainte impliquant seulement des vecteurs tangents à la contrainte d'état (aucune condition de pointage vers l'intérieur de l'ensemble de vitesse n'est requise) et des propriétés de distance des trajectoires aux contraintes initiale et finale. Cette condition peut également être appliquée lorsque le minimiseur a son point initial dans une région où la contrainte d'état n'est pas lisse. Le résultat est prouvé par une construction locale des trajectoires admissibles voisines avec des $W^{1,1}$ -estimations linéaires. Nous donnons aussi un exemple pour souligner le résultat.

“Exceptionally hard decisions can deplete your energy to the point at which you finally cave in. If you mentally crumble and degenerate into negative thinking, you’ll magnify the problem to the point where it can haunt you.”

— John C. Maxwell

3.1 Motivation

We shall recall the state-constrained optimal control problem (**ImDIP**) of Chapter 1 involving a dynamic constraint taking the form of a differential inclusion, and the end-point constraint takes the form of $E_0 \times E_1$, (E_0 and E_1 being two closed sets):

$$(P3) \quad \left\{ \begin{array}{l} \text{minimize } g(x(S), x(T)) \\ \text{over arcs } x(\cdot) \in W^{1,1}([S, T], \mathbb{R}^n) \text{ such that} \\ \dot{x}(t) \in F(t, x(t)) \quad \text{a.e. } t \in [S, T] \\ (x(S), x(T)) \in E_0 \times E_1 \\ x(t) \in A \quad \text{for all } t \in [S, T]. \end{array} \right.$$

3.1.1 Degeneracy Phenomenon

When a state constraint appears in an optimal control problem, it may happen that, using the necessary conditions for optimality (cf. Theorem 1.3.8), the Lagrange multiplier λ associated with the objective function (to minimize) takes the value zero (referred to as ‘abnormal’ case). In this case, the objective function does not appear in the necessary conditions, and as a consequence it does not provide useful information to find the minimizers. A worse phenomenon, called the ‘degenerate’ case might occur. It is well represented by classic examples (cf. [3] and [81]) which show that, in some circumstances, for instance, in the presence of pathwise state constraints active at the initial time, any feasible F -trajectory satisfies the necessary conditions, and therefore these give no information on the possible minimizers. In other words, the set of all feasible F -trajectories coincides with the set of candidates to the solution identified by the optimality conditions. Degeneracy may arise in other situations, for example when the pathwise state constraint is active at the final time or at an intermediate time (cf. [41]). Therefore, the non-degeneracy is a relevant issue in applying necessary optimality conditions. This explains the growing interest in the literature for non-degenerate forms (or even in the stronger version of normal forms) of necessary conditions for optimality (cf. [3], [40], [39], [57], [69], [19], and [68], [13], [50], [49], [43], and [12]). Along this chapter, we deal only with situations where the state constraint is active at the initial time $t = S$. For instance, assume that $E_0 = \{x_0\}$ where $x_0 \in \partial A$. Hence, one can consider the set of multipliers like

$$\lambda = 0, \quad \nu = \zeta \delta_S, \quad p(\cdot) \equiv -\zeta, \quad (3.1)$$

where $\zeta \in \partial^> d_A(x_0)$ and δ_S is the unit (Dirac) measure concentrated at S .

We easily notice that this set (called the trivial or degenerate set of multipliers) satisfies the conditions of Theorem 1.3.8 for any feasible trajectory x . Indeed, the quantity $p(t) + \int_{[S,t]} d\nu(s)$ for $t \in (S, T]$ vanishes almost everywhere. Therefore, all optimality condition (i)-(v) of Theorem 1.3.8 are satisfied independently of the minimizer \bar{x} . This degeneracy issue arises because of the incompleteness of the standard variants of the necessary optimality conditions for state-constrained problems. The idea to avoid the degeneracy phenomena consists in replacing the nontriviality condition (i) in Theorem 1.3.8 with a stronger condition which gets rid of the choice of degenerate multipliers in (3.1). However, this replacement cannot be carried out if all

local minimizers do not satisfy the strengthened form of optimality conditions. This difficulty is overcome by the introduction of additional hypotheses, known as *constraint qualifications*. They permit the identification of classes of problems under which strengthening forms of optimality conditions apply and are informative for all local minimizers. The need for the constraint qualification was remarked by Arutyunov and Aseev [3]: they noticed that for an example of minimization problem where the theorem due to Dubovitskii and Dubovitskii [37] applies, the degenerate multipliers are the only possible choice and cannot be eliminated:

$$\left\{ \begin{array}{ll} \text{minimize} & x_2(1) \\ \text{subject to} & \\ & (\dot{x}_1(t), \dot{x}_2(t)) = (tu(t), u(t)) \quad \text{a.e. } t \in [0, 1] \\ & (x_1(0), x_2(0)) = (0, 0) \\ & |u(t)| \leq 1 \quad \text{a.e. } t \in [0, 1] \\ & x_1(t) \geq 0 \quad \text{for all } t \in [0, 1]. \end{array} \right.$$

It is straightforward to notice that $(\bar{x}, \bar{u}) = (0, 0)$ is the unique solution of this problem. Applying Theorem 1.3.4 (in its smooth version), we show that the necessary conditions are only satisfied by the degenerate multipliers (up to a positive multiple)

$$\lambda = 0 \quad p(\cdot) = -h_x(0, 0) \quad \mu = \delta_0.$$

(Here, $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ is such that $h(x_1, x_2) = -x_1$).

3.1.2 Constraint Qualification

Our main contribution is to establish non-degenerate forms of the generalized Euler-Lagrange conditions for optimal control problems where the dynamic is represented in terms of a differential inclusion depending on both time and state variables (allowing the case where the velocity set is just measurable w.r.t. the time variable), the state constraint is merely a closed set, not necessarily locally smooth or convex, and we have both left and right end-point constraints which are just closed sets. We use an approach suggested in [68] and successively developed in earlier work: this requires exhibiting existence results of neighboring feasible trajectories (approximating a reference trajectory that possibly violates the state constraint) which allow to obtain linear estimates w.r.t. the $W^{1,1}$ -norm. Along this chapter, we make use of some of the results (mainly, the ‘local’ ones) on neighboring feasible trajectories with $W^{1,1}$ -linear estimates, established in Chapter 2.

More precisely, employing some linear $W^{1,1}$ -estimates which turn out to be locally valid, we can still prove that the generalized Euler-Lagrange condition for optimal control problems can be applied in the non-degenerate form (in the sense of (v) of Theorem 3.2.1 below) if one of the following constraint qualification is satisfied:

- (a) the left end-point of the reference minimizer $\bar{x}(\cdot)$ belongs to a region where the state constraint A is regular (i.e. the starting point is away from corners), and a classical inward pointing condition is satisfied;

- (b) there exist hypertangent vectors of the state constraint A ; moreover, for all F -trajectories $y(\cdot)$ which are feasible on an initial (small enough) time interval and close to the reference minimizer $\bar{x}(\cdot)$ (w.r.t. the $W^{1,1}$ -norm), the distance of the left end-point $y(S)$ to $E_0 \cap A$ is strictly smaller than the distance of the right end-point $y(T)$ to E_1 .

Thus, the novelties are the following ones:

- we allow the state constraint to be merely a closed set and the velocity set to be measurable w.r.t. the time variable, while in earlier work either some regularity assumption w.r.t. the time variable is imposed to the velocity set (for instance Lipschitz continuous cf. [3], [81]; of bounded variation, see [62], or the state constraint is represented by the intersection of regular sets, cf. [69];
- for the case in which a minimizer starts from a region where the state constraint is non-smooth, we suggest a new constraint qualification in which no inward pointing condition involving the velocity set is required: the new condition invokes just hypertangent vectors of A and a property of F -trajectories close to the reference minimizer $\bar{x}(\cdot)$ w.r.t. the $W^{1,1}$ -norm (see (CQ)(b) of Theorem 3.2.1 below).

3.2 Main Result

Theorem 3.2.1. *Let $\bar{x}(\cdot)$ be a $W^{1,1}$ δ -local minimizer for problem (P3) (in the sense of the definition (1.12)), in which we assume that for some functions $c_F(\cdot), k_F(\cdot) \in L^1([S, T], \mathbb{R}_+)$, the following hypotheses are satisfied:*

(H.1) *The subsets $A, E_0, E_1 \subset \mathbb{R}^n$ are closed.*

(H.2) (a) *The multifunction F has non-empty values, $F(\cdot, x)$ is Lebesgue measurable for each $x \in \mathbb{R}^n$, and $\text{Gr } F(t, \cdot)$ is closed for each $t \in [S, T]$;*

(b)

$$F(t, x) \subset c_F(t) \mathbb{B} \quad \text{for all } x \in \bar{x}(t) + \delta\mathbb{B}, \quad \text{a.e. } t \in [S, T];$$

(c)

$$F(t, x) \subset F(t, y) + k_F(t)|x - y|\mathbb{B} \quad \text{for all } x, y \in \bar{x}(t) + \delta\mathbb{B}, \quad \text{a.e. } t \in [S, T].$$

(H.3) *g is Lipschitz continuous on $(\bar{x}(S) + \delta\mathbb{B}) \times (\bar{x}(T) + \delta\mathbb{B})$.*

(CQ) *One of the following two conditions is satisfied:*

(a) *$\bar{x}(S) \in A_0$ and for some $\tau \in (0, T - S]$, $c \geq 1$ and $\gamma > 0$ we have $\|c_F(\cdot)\|_{L^\infty} \leq c$, and*

$$\inf_{v \in F(t, x)} \nabla h_0(x) \cdot v \leq -\gamma \quad \text{for all } x \in (\bar{x}(S) + \delta\mathbb{B}) \cap A, \quad \text{a.e. } t \in [S, S + \tau].$$

(Here $h_0(\cdot)$ is the C^{1+} function with the property in (2.1) for $y = \bar{x}(S)$.)

(b) $\text{int } T_A(\bar{x}(S)) \neq \emptyset$, and there exist numbers $\tau_0 \in (0, T - S]$ and $\rho_0 > 0$ such that for all arcs $y(\cdot) \neq \bar{x}(\cdot)$ verifying the following inclusion

$$y(\cdot) \in \{x(\cdot) \in W^{1,1}([S, T], \mathbb{R}^n) : \dot{x}(t) \in F(t, x(t)) \text{ a.e.}, \\ \|x(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}(S, T)} \leq \rho_0 \text{ and } x(t) \in A \text{ for all } t \in [S, S + \tau_0]\},$$

we have

$$d_{E_0 \cap A}(y(S)) < d_{E_1}(y(T)).$$

Then, there exist an adjoint arc $p(\cdot) \in W^{1,1}([S, T]; \mathbb{R}^n)$, $\lambda \geq 0$ and a function of bounded variation $v(\cdot) : [S, T] \rightarrow \mathbb{R}^n$, continuous from the right on (S, T) , such that

(i): for some positive Borel measure $\mu(\cdot)$ on $[S, T]$, whose support set satisfies

$$\text{supp}(\mu) \subset \{t \in [S, T] \mid \bar{x}(t) \in \partial A\},$$

and some Borel measurable selection

$$\gamma(t) \in \text{co } N_A(\bar{x}(t)) \cap \mathbb{B} \quad \mu - \text{a.e.} \quad t \in [S, T]$$

we have

$$v(t) = \int_{[S, t]} \gamma(s) d\mu(s) \quad \text{for all } t \in (S, T],$$

(ii): $\dot{p}(t) \in \text{co} \{\eta \mid (\eta, q(t)) \in N_{Gr F(t, \cdot)}(\bar{x}(t), \dot{\bar{x}}(t))\} \quad \text{a.e. } t \in [S, T],$

(iii): $(p(S), -q(T)) \in \lambda \partial g(\bar{x}(S), \bar{x}(T)) + N_{E_0 \cap A}(\bar{x}(S)) \times N_{E_1}(\bar{x}(T)),$

(iv): $q(t) \cdot \dot{\bar{x}}(t) = \max_{v \in F(t, \bar{x}(t))} q(t) \cdot v \quad \text{a.e. } t \in [S, T],$

(v):

$$\lambda + \int_{(S, T]} d\mu(s) + |p(S) + v(S)| \neq 0,$$

in which $q(\cdot) : [S, T] \rightarrow \mathbb{R}^n$ is the function

$$q(t) := \begin{cases} p(S) & \text{for } t = S, \\ p(t) + v(t) & \text{for } t \in (S, T]. \end{cases}$$

Remark 3.2.2. (a) We highlight the fact that the usual non-vanishing necessary condition:

$$\lambda + \|p\|_{L^\infty} + \int_{[S, T]} d\mu(s) \neq 0,$$

is here replaced by the non-degeneracy condition (v) in Theorem 3.2.1:

$$\lambda + |p(S) + v(S)| + \int_{(S, T]} d\mu(s) \neq 0. \quad (3.2)$$

This constitutes a relevant aspect in applying necessary optimality conditions: consider, for instance, the case in which $E_0 = \{x_0\}$ where $x_0 \in E_0$, and the particular choice of multipliers as in (3.1). Then condition (3.2) (where $\mu = \delta_S$) does not allow to consider such trivial set of multipliers and they are excluded from the possible set of multipliers.

(b) If, under the assumptions of Theorem 3.2.1, we suppose in addition that $F(., .)$ is convex valued, then (ii) implies also (cf. [81, Theorem 7.6.5])

(vi): $\dot{p}(t) \in \text{co}\{-\zeta \mid (\zeta, \dot{x}(t)) \in \partial H(t, \bar{x}(t), q(t))\}$ a.e. $t \in [S, T]$, where H is the Hamiltonian function:

$$H(t, x, q) := \max_{v \in F(t, x)} q \cdot v .$$

(c) Without loss of generality we may assume that the function $c_F(.)$ in (H.2) is actually essentially bounded, also when we consider the case represented by condition (CQ)(a), replacing $c_F(.)$ with a constant $c \geq 1$. This is a standard reduction procedure based on a change of time variables (cf. [29]). Indeed, let us consider a change of time scale via

$$s = \tau(t) \quad \text{where} \quad \tau(t) := \int_S^t c_F(r) dr . \quad (3.3)$$

Transformed arcs y corresponding to original ones x are given via

$$y(s) = x(t) = x(\tau^{-1}(s)) . \quad (3.4)$$

We shall define a new multifunction \hat{F} as follows:

$$\hat{F}(s, y) := \frac{1}{c_F(t)} F(t, y) , \quad t = \tau^{-1}(s) .$$

Moreover, x is an F -trajectory if and only if y is a trajectory of the multifunction \hat{F} defined above. Indeed, x is an F -trajectory means that $\dot{x}(t) \in F(t, x(t))$ a.e. $t \in [S, T]$. Dividing across by $c_F(t)$, we obtain

$$\frac{\dot{x}(t)}{c_F(t)} \in \frac{F(t, x(t))}{c_F(t)} = \hat{F}(s, x(\tau^{-1}(s))) = \hat{F}(s, y(s)) .$$

Notice that since $y(s) = x(t)$

$$\dot{y}(s) = \frac{dy}{ds} = \frac{dy}{dt} \cdot \frac{dt}{ds} = \dot{x}(t) \frac{dt}{ds} .$$

But $s = \tau(t)$. Then, $ds = \dot{\tau}(t)dt$. Therefore, owing to (3.3)

$$\frac{dt}{ds} = \frac{1}{\dot{\tau}(t)} = \frac{1}{c_F(t)} .$$

We deduce that

$$\dot{y}(s) = \frac{\dot{x}(t)}{c_F(t)} \in \hat{F}(s, y(s)) .$$

But the multifunction \hat{F} is bounded owing to the fact that

$$\hat{F}(s, y) = \frac{1}{c_F(t)} F(t, y) \subset \mathbb{B} .$$

Therefore we can suppose always that

$$F(t, x) \subset c \mathbb{B} \quad \text{for all } x \in \bar{x}(t) + \delta \mathbb{B}, \quad \text{a.e. } t \in [S, T], \quad (3.5)$$

where $c \geq 1$ a.e.

3.3 Example

We consider the optimal control problem with state constraints:

$$(Ex3) \quad \left\{ \begin{array}{l} \text{minimize } x_3(1) \\ \text{over arcs } x \in W^{1,1}([0, 1], \mathbb{R}^3) \text{ such that} \\ \dot{x}(t) \in F(t) \quad \text{a.e. } t \in [0, 1] \\ x(t) \in A \\ (x(0), x(1)) \in E_0 \times E_1, \end{array} \right.$$

where

$$\begin{aligned} A &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_2| \leq x_1 - x_3 \text{ and } x_3 \geq 0\}, \\ E_0 &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 = 0\} \text{ and } E_1 := \{(1, 1, 0)\}, \\ F(t) &:= \begin{cases} \{(1, u, 2 - u) : u \in \{-2, +2\}\} & \text{for } t = 0 \\ \{(1, u, 2 - u) : u \in \{-2, +2\}\} & \text{for } t \in (t_{2k+1}, t_{2k}] \\ \{(1, u, u + 2) : u \in \{-2, +2\}\} & \text{for } t \in (t_{2k}, t_{2k-1}]. \end{cases} \end{aligned}$$

in which $t_k = \frac{1}{3^k}$ for $k = 0, 1, 2, 3, \dots$

Consider the arc $z(\cdot) : [0, 1] \rightarrow \mathbb{R}$ defined as: $z(0) = 0$, $z(t_k) := (-1)^k t_k$, and $\dot{z}(t) = (-1)^k 2$ for $t \in (t_{k+1}, t_k]$. Then, the F -trajectory

$$\bar{x}(t) := (t, z(t), 0), \quad \text{for all } t \in [0, 1],$$

is a $W^{1,1}$ -minimizer for problem (Ex3), for $\bar{x}(0) = (0, 0, 0) \in E_0$, $\bar{x}(1) = (1, 1, 0) \in E_1$ and $\bar{x}_3(1) = 0$. We claim that condition (CQ)(b) is satisfied. Indeed, clearly $\text{int } T_A(\bar{x}(0)) \neq \emptyset$. Fix now any $\tau_0 \in (0, 1)$ and $\rho_0 > 0$. Consider the family of F -trajectories

$$\begin{aligned} \mathcal{W} &:= \{x(\cdot) \in W^{1,1}([0, 1], \mathbb{R}^3) : \dot{x}(t) \in F(t) \text{ a.e.}, \\ &\quad \|x(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}(0,1)} \leq \rho_0 \text{ and } x(t) \in A \text{ for all } t \in [0, \tau_0]\}. \end{aligned}$$

Observe that for all $y(\cdot) \in \mathcal{W}$ such that $y(\cdot) \neq \bar{x}(\cdot)$, we necessarily have $y(0) \in A$ and, so $y_1(0) > 0$.

Moreover, if $y_3(1) = 0$, then it follows that $y(1) = y(0) + \bar{x}(1)$, and we immediately have

$$d_{E_0 \cap A}(y(0)) = |y_2(0)| < |y(0)| = d_{E_1}(y(1)).$$

If $y_3(1) > 0$, then we obtain

$$d_{E_0 \cap A}(y(0)) = |y_2(0)| \leq y_1(0) < \sqrt{y_1^2(0) + y_3^2(0)} \leq d_{E_1}(y(1)).$$

We have proved the claim.

Therefore the necessary conditions apply in the non-degenerate form (in the sense of (v) of Theorem 3.2.1).

On the other hand, if we had a different choice for the left end-point condition in problem (Ex3), which does not satisfy the hypotheses of Theorem 3.2.1, for instance $E_0 = \{(0, 0, 0)\}$ (observe that in this particular case (CQ)(b) of Theorem 3.2.1 is not verified), then the necessary conditions for optimality would be applicable in the degenerate form, since they are compatible with the condition

$$\lambda + \int_{(0,1]} d\mu(s) + |p(0) + \nu(\{0\})| = 0. \quad (3.6)$$

Indeed, it is easy to verify that the necessary optimality conditions (i)-(iv) of Theorem 3.2.1 would be applicable in a degenerate form provided that $(p(\cdot) \equiv \text{const})$

$$p(0) \in N_{E_0 \cap A}(\bar{x}(0)) = \mathbb{R}^3 \quad (3.7)$$

and

$$p(0) \in -N_A(\bar{x}(0)). \quad (3.8)$$

Notice that (3.7) is just the left end-point transversality condition, whereas (3.8) follows immediately from the degenerate condition (3.6) and from the fact that $\nu(\{0\}) \in N_A(\bar{x}(0))$. Observe that for this particular choice of $E_0 (= \{(0, 0, 0)\})$, the necessary conditions apply (in the degenerate form) not only for the minimizer $\bar{x}(\cdot)$ but also for any feasible F -trajectory $x(\cdot)$ having $x(0) = (0, 0, 0)$ as left end-point. This is possible with the trivial choice of multipliers $\lambda = 0, \nu = \xi \delta_0, p(\cdot) = -\xi$, in which $\xi \in N_A((0, 0, 0)) \cap (\mathbb{B} \setminus \{0\})$, and $\mu = \delta_0$ is the unit measure concentrated at 0.

3.4 Proof of the Main Result (Theorem 3.2.1)

We shall start with the case when hypotheses (H.1)-(H.3) and (CQ)(b) are satisfied. The proof approach is along similar lines to that one of [69]. Nevertheless, since the nature of the state constraints (and in turn the related distance estimates provided by Lemma 2.4.3) differs from that one of [69], the analysis employed here requires some ideas, which are new with respect to earlier work.

To begin with, assumption (CQ)(b) of Theorem 3.2.1 guarantees the existence of a vector $\bar{v} \in \text{int } T_A(\bar{x}(S))$, which means that all the assertions of Lemma 2.4.3 are satisfied.

Consider the constants $\bar{\varepsilon} > 0, \alpha \geq 0, \theta \in (0, \delta), \tau \in (0, T - S]$, and $K > 0$ provided by Lemma 2.4.3. We can always suppose that $\tau \geq \tau_0$ and $\theta \leq \rho_0$ (where τ_0 and ρ_0 are the constant given by condition (CQ)(b)). Take any arbitrary sequence $\varepsilon_i \downarrow 0$ with $\varepsilon_i \leq \bar{\varepsilon}$ and, for each $i \in \mathbb{N}$, set the function $\phi_i : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$:

$$\phi_i(y_0, y_1, d) := \max\{g(y_0, y_1) - g(\bar{x}(S), \bar{x}(T)) + \varepsilon_i^2, d, d_{E_0 \cap A}(y_0), d_{E_1}(y_1)\}.$$

Define also the sets of functions

$$X := \{y(\cdot) \in W^{1,1}([S, T], \mathbb{R}^n) : \dot{y}(t) \in F(t, y(t)) \text{ a.e. and } \|y(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}(S, T)} \leq \theta\},$$

$$X_1 := \{y(\cdot) \in X : y(t) \in A \text{ for all } t \in [S, S + \tau]\}.$$

Consider now the space X with the metric induced by the $W^{1,1}$ -norm. Then, X turns out to be complete. Notice also that (w.r.t. the topology induced by the $W^{1,1}$ -norm) the set X_1 is closed. Moreover, the functional $\Phi_i : X \rightarrow \mathbb{R}$

$$y(\cdot) \rightarrow \Phi_i(y(\cdot)) := \phi_i(y(S), y(T), \max_{t \in [S+\tau, T]} d_A(y(t)))$$

is continuous on X , and so it is continuous also on X_1 . We notice that $\Phi_i(\bar{x}(\cdot)) = \varepsilon_i^2$. Then, $\bar{x}(\cdot) \in X_1$ is an ε_i^2 -minimizer for Φ_i on X_1 , for each $i \in \mathbb{N}$. Ekeland's Theorem (cf. [81, Theorem 3.3.1]) guarantees the existence of an arc $x_i(\cdot) \in X_1$ such that

$$\|x_i(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}(S,T)} \leq \varepsilon_i$$

and such that the functional

$$J_i(x(\cdot)) := \Phi_i(x(\cdot)) + \varepsilon_i \|x_i(\cdot) - x(\cdot)\|_{W^{1,1}(S,T)}$$

attains a (unique) minimum at $x_i(\cdot)$. As a consequence, by eventually a subsequence extraction (we do not relabel), we obtain that

$$\begin{cases} x_i(\cdot) \rightarrow \bar{x}(\cdot) & \text{w.r.t. the } W^{1,1}\text{-norm} \\ x_i(\cdot) & \text{minimizes the functional } J_i(\cdot) \text{ over } X_1. \end{cases}$$

By extracting subsequences (without relabeling) we can arrange that one of the following situations arises:

- (a) $x_i(\cdot) \neq \bar{x}(\cdot)$, for all $i = 1, 2, 3, \dots$ or
- (b) $x_i(\cdot) \equiv \bar{x}(\cdot)$, for all $i = 1, 2, 3, \dots$

Notice also that the functional $J_i(\cdot)$ is, in fact, uniformly Lipschitz continuous on X with respect to the $W^{1,1}$ -norm. Let $K_J > 0$ be a constant independent of i such that

$$|J_i(y(\cdot)) - J_i(x(\cdot))| \leq K_J \|y(\cdot) - x(\cdot)\|_{W^{1,1}(S,T)} \quad \text{for all } y(\cdot), x(\cdot) \in X.$$

Associated with any minimizer $x_i(\cdot)$ for $J_i(\cdot)$, we define the positive number

$$d_i := \max_{t \in [S+\tau, T]} d_A(x_i(t)). \quad (3.9)$$

The following lemma is an easy consequence of the Max-rule for subdifferentials, and the fact that either $x_i(\cdot) \neq \bar{x}(\cdot)$ or $x_i(\cdot) \equiv \bar{x}(\cdot)$ for all $i = 1, 2, 3, \dots$. Observe that, in the first case (i.e. when $x_i(\cdot) \not\equiv \bar{x}(\cdot)$), owing to assumption (CQ)(b), since each $x_i(\cdot) \in X_1$, we have that $d_{E_0 \cap A}(x_i(S)) < d_{E_1}(x_i(T))$.

Lemma 3.4.1. *Under (CQ)(b) take any subgradient*

$$\eta = (\eta_1, \eta_2, \eta_3) \in \partial \phi_i(x_i(S), x_i(T), d_i)$$

where $x_i(\cdot)$'s are the minimizers for $J_i(\cdot)$ and d_i 's are the corresponding numbers defined in (3.9). Then for all i large enough, we obtain either

Case (a) $x_i(\cdot) \neq \bar{x}(\cdot)$ (and then $d_{E_0 \cap A}(x_i(S)) < d_{E_1}(x_i(T))$). Then

$$\eta_3 \geq 0 \quad \text{and} \quad (\eta_1, \eta_2) \in a_1 \partial g(x_i(S), x_i(T)) + (0, \xi_1)$$

for some $a_1 \geq 0$ and $\xi_1 \in N_{E_1}(e_1)$, in which, $e_1 \in \pi_{E_1}(x_i(T))$ and

$$a_1 + \eta_3 + |\xi_1| = 1,$$

Case (b) or $x_i(\cdot) \equiv \bar{x}(\cdot)$ (and then $d_{E_0 \cap A}(x_i(S)) = d_{E_1}(x_i(T)) \equiv 0$). Then

$$\eta_3 = 0 \quad \text{and} \quad (\eta_1, \eta_2) \in \partial g(x_i(S), x_i(T)).$$

Proof. We shall observe first that for each i

$$\phi_i(x_i(S), x_i(T), d_i) > 0. \quad (3.10)$$

If not, $\phi_i = 0$ for some i , since ϕ_i s are nonnegative. But then,

$$x_i(t) \in A \quad \text{for all } t \in [S, T] \quad \text{where} \quad \|x_i(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}} \leq \theta < \delta$$

$$x_i(S) \in E_0 \cap A \quad \text{and} \quad x_i(T) \in E_1,$$

and

$$g(x_i(S), x_i(T)) < g(\bar{x}(S), \bar{x}(T)).$$

This violates the minimality of \bar{x} .

Case (a) If $d_{E_0 \cap A}(x_i(S)) < d_{E_1}(x_i(T))$, then

$$\phi_i(x_i(S), x_i(T), d_i) := \max\{g(x_i(S), x_i(T)) - g(\bar{x}(S), \bar{x}(T)) + \varepsilon_i^2, d_i, d_{E_1}(x_i(T))\}.$$

The max rule (cf. [81, Theorem 5.5.2]) asserts that there exist positive constants a_i , $i = 1, 2, 3$ such that $\sum_{i=1}^3 a_i = 1$ and

$$(\eta_1, \eta_2, \eta_3) \in a_1 \partial g(x_i(S), x_i(T)) \times \{0\} + a_2 \{0\} \times \{0\} \times \{1\} + a_3 \{0\} \times \partial d_{E_1}(x_i(T)) \times \{0\}.$$

Equivalently, $\eta_3 = a_2 \in [0, 1]$ and

$$(\eta_1, \eta_2) \in a_1 \partial g(x_i(S), x_i(T)) + (0, a_3 \xi'_1)$$

for some $\xi'_1 \in N_{E_1}(e_1)$ of unit norm, such that $e_1 \in \pi_{E_1}(x_i(T))$ (see Proposition 1.1.13). Writing $\xi_1 := a_3 \xi'_1$, we obtain that

$$(\eta_1, \eta_2) \in a_1 \partial g(x_i(S), x_i(T)) + (0, \xi_1) \quad (3.11)$$

and $a_1 + \eta_3 + |\xi_1| = 1$.

Case (b) If $d_{E_0 \cap A}(x_i(S)) = d_{E_1}(x_i(T)) \equiv 0$, in view of the strict inequality (3.10), we have that

$$\phi_i(y_0, y_1, d) = \max\{g(y_0, y_1) - g(x_i(S), x_i(T)) + \varepsilon_i^2, d\}$$

for all (y_0, y_1) near $(x_i(S), x_i(T))$. Consequently, (3.11) is valid for $\xi_1 = 0$ and $a_1 + \eta_3 = 1$.

□

Lemma 3.4.2. *Assume that $A \subset \mathbb{R}^n$ is a (nonempty) closed set and that hypothesis (H.2) holds true and $\text{int } T_A(\bar{x}(S)) \neq \emptyset$. Then there exists $i_0 \in \mathbb{N}$ such that, for all $i \geq i_0$ and for some $\delta' > 0$, $x_i(\cdot)$ is a $W^{1,1}$ δ' -local minimizer for the problem*

$$\begin{cases} \text{Minimize } \tilde{J}_i(y(\cdot)) := J_i(y(\cdot)) + K_J K \max_{t \in [S, S+\tau]} d_A(y(t)) \\ \text{over } y(\cdot) \in X. \end{cases}$$

(K and $\tau \in (0, T - S]$ being the constants provided by Lemma 2.4.3).

Proof. We shall first notice that assumptions (H.2) and $\text{int } T_A(\bar{x}(S)) \neq \emptyset$, are sufficient for the statement of Lemma 2.4.3 to hold when A is merely a closed set. The proof of this lemma is based on a standard penalization argument, which is applicable owing to Lemma 2.4.3. Consider the constants $\bar{\varepsilon}, \theta, \tau$ and K provided by Lemma 2.4.3. We write the proof here in detail to highlight the role of the $W^{1,1}$ -linear estimate provided by Lemma 2.4.3. Since $\varepsilon_i \downarrow 0$ with $\varepsilon_i \leq \bar{\varepsilon}$, there exists $i_0 \in \mathbb{N}$ such that we also have $\varepsilon_i \leq \frac{\theta}{2(K+1)}$, for all $i \geq i_0$. Suppose, by contradiction, that the statement of the lemma is false. Take $w_0 \in \text{int } T_A(\bar{x}(S))$. It follows that we can choose $\delta' \in (0, \min\{\bar{\varepsilon}, \frac{\theta}{2(K+1)}\})$ such that for all $j \in \mathbb{N}$, $j \geq i_0$, there exist $i \geq j$ and $\hat{y}(\cdot) \in X$ such that

$$\|\hat{y}(\cdot) - x_i(\cdot)\|_{W^{1,1}(S,T)} \leq \delta'$$

and

$$\tilde{J}_i(\hat{y}(\cdot)) < \tilde{J}_i(x_i(\cdot)).$$

Write $\varepsilon := \max_{t \in [S, S+\tau]} d_A(\hat{y}(t))$. Observe that the choice of δ' and the fact that the $x_i(\cdot)$ s belong to X_1 , guarantee that $\varepsilon \leq \delta' \leq \bar{\varepsilon}$. Then, owing to Lemma 2.4.3 (applied for the F -trajectory $\hat{y}(\cdot)$ and for the violation of the state constraint on $[S, S + \tau]$), there exist $\alpha \geq 0$ and an F -trajectory $y(\cdot)$ on $[S, T]$ such that

$$\begin{cases} \|y(\cdot) - \hat{y}(\cdot)\|_{W^{1,1}(S,T)} \leq K\varepsilon \\ y(t) \in A, \text{ for all } t \in [S, S + \tau] \\ y(S) = y_0 + \alpha\varepsilon w_0, \end{cases}$$

for any given $y_0 \in \pi_A(\hat{y}(S))$. Bearing in mind that also

$$\varepsilon \leq \max_{t \in [S, T]} d_A(\hat{y}(t)) \leq \|\hat{y}(\cdot) - \bar{x}(\cdot)\|_{L^\infty(S,T)} \leq \|\hat{y}(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}(S,T)}$$

(for $\bar{x}(\cdot)$ is feasible), it follows that

$$\begin{aligned} \|y(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}(S,T)} &\leq \|y(\cdot) - \hat{y}(\cdot)\|_{W^{1,1}(S,T)} + \|\hat{y}(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}(S,T)} \\ &\leq K\varepsilon + \|\hat{y}(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}(S,T)} \\ &\leq (K + 1)\|\hat{y}(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}(S,T)} \\ &\leq (K + 1)(\|\hat{y}(\cdot) - x_i(\cdot)\|_{W^{1,1}} + \|x_i(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}}) \\ &\leq \theta, \end{aligned}$$

and therefore $y(\cdot) \in X_1$. Moreover, we have

$$\begin{aligned} J_i(y(\cdot)) &\leq J_i(\hat{y}(\cdot)) + K_J \|y(\cdot) - \hat{y}(\cdot)\|_{W^{1,1}(S,T)} \leq J_i(\hat{y}(\cdot)) + K_J K \varepsilon \\ &\leq J_i(\hat{y}(\cdot)) + K_J K \varepsilon = J_i(\hat{y}(\cdot)) + K_J K \max_{t \in [S, S+\tau]} d_A(\hat{y}(t)) = \tilde{J}_i(\hat{y}(\cdot)) \\ &< \tilde{J}_i(x_i(\cdot)) = J_i(x_i(\cdot)), \end{aligned}$$

which contradicts the minimality of $x_i(\cdot)$ on X_1 . \square

A consequence of Lemma 3.4.2 is that, for $i \in \mathbb{N}$ with $i \geq i_0$, the state trajectory

$$\tilde{x}_i(\cdot) := \left(x_i(\cdot), y_i \equiv \max_{t \in [S+\tau, T]} d_A(x_i(t)), z_i \equiv 0, w_i \equiv 0 \right)$$

is a $W^{1,1}$ δ' -local minimizer for the state constrained optimal control problem

$$\text{(AuxP3)} \quad \begin{cases} \text{Minimize } \tilde{g}(\tilde{x}(S), \tilde{x}(T)) \\ \text{over } \tilde{x}(\cdot) = (x(\cdot), y(\cdot), z(\cdot), w(\cdot)) \in W^{1,1}([S, T]; \mathbb{R}^{n+3}) \text{ satisfying} \\ \dot{\tilde{x}}(t) = (\dot{x}(\cdot), \dot{y}(\cdot), \dot{z}(\cdot), \dot{w}(\cdot)) \in G(t, \tilde{x}(t)) \text{ a.e. } t \in [S, T], \\ \tilde{h}(t, \tilde{x}(t)) \leq 0 \quad \text{for all } t \in [S, T], \\ w(S) = 0, \end{cases}$$

in which the cost is

$$\begin{aligned} &\tilde{g}(\tilde{x}(S), \tilde{x}(T)) \\ &= \tilde{g}((x(S), y(S), z(S), w(S)), (x(T), y(T), z(T), w(T))) \\ &:= \phi_i(x(S), x(T), y(T)) + \varepsilon_i |x(S) - x_i(S)| + K_J K \max\{0, z(T)\} + w(T) \\ &= \max \{g(x(S), x(T)) - g(\bar{x}(S), \bar{x}(T)) + \varepsilon_i^2, y(T), d_{E_0 \cap A}(x(S)), d_{E_1}(x(T))\} \\ &\quad + \varepsilon_i |x(S) - x_i(S)| + K_J K \max\{0, z(T)\} + w(T), \end{aligned}$$

the velocity set is represented by the multivalued function

$$G(t, \tilde{x}) := \{(v, 0, 0, r) : v \in F(t, x), \quad r \geq \varepsilon_i |v - \dot{x}_i(t)|\},$$

and the function $\tilde{h} : [S, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ (which provides the state constraint by means of a functional inequality) is defined by:

$$\tilde{h}(t, x, y, z, w) := \begin{cases} d_A(x) - z & \text{if } t \in [S, S + \tau) \\ d_A(x) + \max\{-z, -y\} & \text{if } t = S + \tau \\ d_A(x) - y & \text{if } t \in (S + \tau, T]. \end{cases}$$

Indeed, suppose by contradiction, that there exists a state trajectory $X(\cdot) = (x(\cdot), y(\cdot), z(\cdot), w(\cdot))$ satisfying the state and dynamic constraints of problem (AuxP3), such that

$$\|X(\cdot) - \tilde{x}_i(\cdot)\|_{W^{1,1}(S,T)} \leq \delta', \quad \text{where } \delta' < \theta$$

and $\tilde{g}(X(S), X(T)) < \tilde{g}(\tilde{x}_i(S), \tilde{x}_i(T))$.

Observe that the state constraint condition $\tilde{h}(t, x(t), y(t), z(t), w(t)) \leq 0$ can be equivalently written as

$$\begin{cases} \max_{t \in [S, S+\tau]} d_A(x(t)) \leq z \\ d_A(x) + \max\{-z, -y\} \leq 0 \quad \text{when } t = S + \tau \\ \max_{t \in (S+\tau, T]} d_A(x(t)) \leq y. \end{cases}$$

Then we obtain:

$$\begin{aligned} \tilde{J}_i(x(\cdot)) &\leq \phi_i(x(S), x(T), y(T)) + \varepsilon_i \|x_i(\cdot) - x(\cdot)\|_{W^{1,1}} + K_J K \max_{[S, S+\tau]} d_A(x(t)) \\ &\leq \phi_i(x(S), x(T), y(T)) + \varepsilon_i \|x_i(\cdot) - x(\cdot)\|_{W^{1,1}} + K_J K z \\ &\leq \phi_i(x(S), x(T), y(T)) + \varepsilon_i \left(|x_i(S) - x(S)| + \int_S^T |\dot{x}_i(t) - \dot{x}(t)| dt \right) + K_J K \max\{0, z\} \\ &\leq \phi_i(x(S), x(T), y(T)) + \varepsilon_i |x_i(S) - x(S)| + w(T) + K_J K \max\{0, z\} \\ &= \tilde{g}(X(S), X(T)) < \tilde{g}(\tilde{x}_i(S), \tilde{x}_i(T)) = \tilde{J}_i(x_i(\cdot)). \end{aligned}$$

This contradicts the fact that $x_i(\cdot)$ is a $W^{1,1}$ -minimizer over X of $\tilde{J}_i(\cdot)$. (Along these inequalities, the $W^{1,1}$ -norm used is given by $\|x(\cdot)\|_{W^{1,1}(S,T)} = |x(S)| + \|\dot{x}(\cdot)\|_{L^1(S,T)}$ for any $x(\cdot) \in W^{1,1}(S, T)$.)

Assume now that we are in **case (a)**: $x_i(\cdot) \neq \bar{x}(\cdot)$, for all i .

Notice that the necessary conditions for the optimality referred to problem (AuxP3) (cf. Theorem 1.3.9) are applicable, and, consequently, for each i , we obtain a constant $\tilde{\lambda}_i \geq 0$, a function $\tilde{p}_i(\cdot) = (p_i(\cdot), \chi_i(\cdot), \zeta_i(\cdot), \psi_i(\cdot)) \in W^{1,1}([S, T]; \mathbb{R}^{n+3})$, which is the costate trajectory associated with the state $\tilde{x}_i(\cdot)$, a Borel measurable function $\mu_i(\cdot)$ on $[S, T]$ and a μ_i -integrable function $\tilde{\gamma}_i : [S, T] \rightarrow \mathbb{R}^{n+3}$, such that

- (i)': $\|p_i\|_{L^\infty} + \|\chi_i\|_{L^\infty} + \|\zeta_i\|_{L^\infty} + \|\psi_i\|_{L^\infty} + \tilde{\lambda}_i + \int_{[S,T]} d\mu_i(s) = 1$,
- (ii)': $\dot{\tilde{p}}_i(t) \in \text{co}\{\tilde{\eta} \in \mathbb{R}^{n+3} \mid (\tilde{\eta}, \tilde{q}_i(t)) \in N_{\text{Gr } G(t,\cdot)}(\tilde{x}_i(t), \dot{\tilde{x}}_i(t))\}$ a.e. $t \in [S, T]$,
- (iii)': $(\tilde{p}_i(S), -\tilde{q}_i(T)) \in \tilde{\lambda}_i \partial \tilde{g}(\tilde{x}_i(S), \tilde{x}_i(T)) + (\underbrace{\{0, \dots, 0\}}_{n+2}) \times \mathbb{R} \times (\underbrace{\{0, \dots, 0\}}_{n+3})$,
- (iv)': $\tilde{q}_i(t) \cdot \dot{\tilde{x}}_i(t) = \max_{\tilde{v} \in G(t, \tilde{x}_i(t))} \tilde{q}_i(t) \cdot \tilde{v}$ a.e. $t \in [S, T]$,
- (v)': $\tilde{\gamma}_i(t) \in \partial_{\tilde{x}}^{\geq} \tilde{h}(t, \tilde{x}_i(t))$ μ_i -a.e., and $\text{supp}(\mu_i) \subset \{t : \tilde{h}(t, \tilde{x}_i(t)) = 0\}$,

in which $\tilde{q}_i(\cdot) : [S, T] \rightarrow \mathbb{R}^{n+3}$ is the function

$$\tilde{q}_i(t) := \tilde{p}_i(t) + \int_{[S,t]} \tilde{\gamma}_i(s) d\mu_i(s) \quad \text{for } t \in (S, T].$$

Recall that $\text{supp}(\mu_i)$ denotes the support of the measure μ_i , and $\partial_{\tilde{x}}^{\geq} \tilde{h}(t, \tilde{x})$ is the set

$$\partial_{\tilde{x}}^{\geq} \tilde{h}(t, \tilde{x}) := \text{co} \{ \zeta \mid \text{there exist } \tilde{x}_j \rightarrow \tilde{x}, t_j \rightarrow t \text{ and } \zeta_j \rightarrow \zeta \text{ s. t.} \\ \nabla_{\tilde{x}} \tilde{h}(t_j, \tilde{x}_j) \text{ exists, } \zeta_j = \nabla_{\tilde{x}} \tilde{h}(t_j, \tilde{x}_j) \text{ and } \tilde{h}(t_j, \tilde{x}_j) > 0 \text{ for each } j \in \mathbb{N} \} .$$

We observe that the Euler-Lagrange inclusion (ii)', the transversality condition (iii)' and condition (v)', bearing in mind the information provided by Lemma 3.4.1, imply that there exist (for i large enough) $a_i, b_i \in [0, 1]$, $\beta_i^0, \beta_i^1 \in [0, 1]$, $\xi_i^1 \in \mathbb{R}^n$, such that

$$\text{(vi)'}: a_i + b_i + |\xi_i^1| = 1, \text{ and } \beta_i^0 + \beta_i^1 = 1$$

$$\text{(vii)'}: \xi_i^1 \in N_{E_1}(e_i^1), \text{ where } e_i^1 \in \pi_{E_1}(x_i(T))$$

and

$$(p_i(S), -q_i(T)) \in \tilde{\lambda}_i \left(a_i \partial g(x_i(S), x_i(T)) + (0, \xi_i^1) \right) + \tilde{\lambda}_i \varepsilon_i \mathbb{B} \times \{0\} \quad (3.12)$$

$$\chi_i = \dot{\chi}_i \equiv 0 \quad \text{and} \quad \int_{(S+\tau, T]} d\mu_i(s) + \beta_i^0 \mu_i(\{S + \tau\}) = \tilde{\lambda}_i b_i \quad (3.13)$$

$$\zeta_i = \dot{\zeta}_i \equiv 0 \quad \text{and} \quad \int_{[S, S+\tau)} d\mu_i(s) + \beta_i^1 \mu_i(\{S + \tau\}) \in [0, \tilde{\lambda}_i K_J K] \quad (3.14)$$

$$\dot{\psi}_i \equiv 0 \quad \text{and} \quad \psi_i(T) = -\tilde{\lambda}_i \quad (3.15)$$

$$\dot{p}_i(t) \in \text{co} \{ \eta \mid (\eta, q_i(t)) \in N_{\text{Gr } F(t, \cdot)}(x_i(t), \dot{x}_i(t)) + \{0\} \times \tilde{\lambda}_i \varepsilon_i \mathbb{B} \} \quad (3.16)$$

(The four adjoint systems corresponding to the adjoint arcs $p_i(\cdot)$, $\chi_i(\cdot)$, $\zeta_i(\cdot)$ and $\psi(\cdot)$ are obtained by expressing the graph of the set-valued map G in (ii)' as the epigraph of a certain map. A standard result of the nonsmooth analysis representing the limiting normal cone to the epigraph of a function by means of the subgradients of the function (cf. [81, Proposition 4.3.4]) is crucial to obtain the required adjoint systems in (3.13)-(3.16).)

Since the multivalued map G is $k_F(\cdot)$ -Lipschitz w.r.t. the state variable around the G -trajectory $\tilde{x}(\cdot)$ (recall that, around the reference F -trajectory $\bar{x}(\cdot)$, $k_F(\cdot)$ is the function as defined in Theorem 3.2.1), we notice that (ii)' implies also that

$$|\dot{\tilde{p}}_i(t)| \leq k_F(t) |\tilde{q}_i(t)| . \quad (3.17)$$

Notice that the necessary conditions apply in the normal form. Indeed, assume that $\tilde{\lambda}_i = 0$, then from (3.12)-(3.15), we would obtain

$$\int_{[S, T]} d\mu_i(s) = 0 \quad \text{and} \quad \tilde{p}_i(S) = \tilde{q}_i(S) = 0 (= \tilde{p}_i(T)) ,$$

which, combined with (3.17), would imply $\tilde{p}_i \equiv 0$ (using Gronwall's Lemma). But this contradicts (i)'.

The above relations are valid for arbitrary $i \in \mathbb{N}$ sufficiently large. Standard convergence analysis (cf. [34] or [81]), following the extraction of subsequences, provides that $(\tilde{\lambda}_i, a_i, b_i, \beta_i^0, \beta_i^1) \rightarrow (\tilde{\lambda}, a, b, \beta^0, \beta^1)$ for some constants $\tilde{\lambda} \geq 0$ and $a, b, \beta^0, \beta^1 \in [0, 1]$, $\xi_i^1 \rightarrow \xi^1$ for some vector ξ^1

in $\mathbb{R}^n \cap \mathbb{B}$. Furthermore, $p_i(\cdot) \rightarrow p(\cdot)$ uniformly, $\dot{p}_i(\cdot) \rightarrow \dot{p}(\cdot)$ weakly in L^1 , $\mu_i(\cdot) \rightarrow \mu(\cdot)$ and $\gamma_i(\cdot)\mu_i(\cdot) \rightarrow \gamma(\cdot)\mu(\cdot)$ in the appropriate weak* topology (where $\gamma_i(\cdot)$, having its values in \mathbb{R}^n , is relative to the adjoint arc $p_i(\cdot)$ and it represents the first component of $\tilde{\gamma}_i(\cdot)$) and $q_i(\cdot) \rightarrow q(\cdot)$ a.e., for some absolutely continuous function $p(\cdot)$, function of bounded variation $q(\cdot)$, Borel measure $\mu(\cdot)$ and μ -integrable function $\gamma(\cdot)$. The limit of the relationships (i)', (iv)'-(vii)', (3.12)-(3.16) yield

$$(i)'': \|p\|_{L^\infty} + 2\tilde{\lambda} + \int_{[S,T]} d\mu(s) = 1,$$

$$(ii)'': \dot{p}(t) \in \text{co}\{\eta \in \mathbb{R}^n \mid (\eta, q(t)) \in N_{\text{Gr } F(t,\cdot)}(\bar{x}(t), \dot{\bar{x}}(t))\} \text{ a.e. } t \in [S, T],$$

$$(iii)'': (p(S), -q(T)) \in a\tilde{\lambda}\partial g(\bar{x}(S), \bar{x}(T)) + \tilde{\lambda}(0, \xi^1),$$

$$(iv)'': q(t) \cdot \dot{\bar{x}}(t) = \max_{v \in F(t, \bar{x}(t))} q(t) \cdot v \quad \text{a.e. } t \in [S, T],$$

$$(v)'': \int_{(S+\tau, T]} d\mu(s) + \beta^0 \mu(\{S + \tau\}) = \tilde{\lambda}b \quad \text{and} \quad \int_{[S, S+\tau)} d\mu(s) + \beta^1 \mu(\{S + \tau\}) \in [0, \tilde{\lambda}K_J K],$$

$$(vi)'': a + b + |\xi^1| = 1, \text{ and } \beta^0 + \beta^1 = 1,$$

$$(vii)'': \xi^1 \in N_{E_1}(\bar{x}(T)) \cap \mathbb{B}$$

in which $q(\cdot) : [S, T] \rightarrow \mathbb{R}^n$ is the function

$$q(t) := p(t) + \int_{[S,t]} \gamma(s) d\mu(s) \quad \text{for } t \in (S, T],$$

where $\gamma(\cdot) := (\tilde{\gamma}(\cdot))_p$, i.e. the first n -components of the μ -integrable function $\gamma(\cdot)$, corresponding to the p -variable. The state constraint in the optimal control problem (AuxP3) is formulated as a functional inequality constraint by means of the function \tilde{h} , which in turn involves the distance function to the set A , $d_A(\cdot)$. This translates in necessary optimality conditions for our reference problem (P3), in which we assume an implicit state constraints (i.e. the state constraint set is a general closed set). The simple analysis, involved in deriving the necessary conditions for the implicit constraints from the necessary conditions for the functional inequality constraints, is described in Remark 1.3.7 or [81] (see Remark (e) p. 332, and Remark (b) p. 370). This allows us to obtain property (i) of Theorem 3.2.1.

Note that condition (ii)'' implies that

$$|\dot{p}(t)| \leq k_F(t)|q(t)| \quad \text{a.e. } t \in [S, T]. \quad (3.18)$$

Observe that using the same argument as for the multipliers $\tilde{\lambda}_i$, we conclude that $\tilde{\lambda} \neq 0$. Define $\lambda := a\tilde{\lambda}$. Then, λ , $p(\cdot)$, $v(\cdot) := \int_{[S,\cdot]} \gamma(s) d\mu(s)$ satisfy assertions (ii)-(iv) of the theorem statement.

It remains to prove the non-degeneracy condition (assertion (v) in the statement of Theorem 3.2.1):

$$\lambda + \int_{(S,T]} d\mu(s) + |p(S) + v(S)| \neq 0, \quad (3.19)$$

Assume to the contrary that (3.19) is violated. Then $\lambda = 0$ (which, since $\tilde{\lambda} \neq 0$, implies $a = 0$), $|p(S) + \nu(S)| = 0$ and $\int_{(S,T]} d\mu(s) = 0$, that together with the first condition of (v)'' gives $b = 0$. As a consequence, from the first equality in (vi)'' we obtain that $|\xi^1| = 1$, and therefore, owing to (iii)'',

$$\left| p(T) + \int_{[S,T]} \gamma(s) d\mu(s) \right| = |q(T)| = |\tilde{\lambda} \xi^1| = \tilde{\lambda} > 0. \quad (3.20)$$

On the other hand, conditions $|p(S) + \nu(S)| = 0$ and $\int_{(S,T]} d\mu(s) = 0$ assumed above, with the help of (3.18), yield, owing to Gronwall's Lemma:

$$\left| p(t) + \int_{[S,t]} \gamma(s) d\mu(s) \right| = 0 \quad \text{for all } t \in (S, T]. \quad (3.21)$$

In particular for $t = T$. Therefore, the two relations (3.20) and (3.21) provide a contradiction. Thus, we have proved also the non-degeneracy condition (3.19).

Assume finally that we are in **case (b)**: $x_i(\cdot) = \bar{x}(\cdot)$, for all i .

Necessary optimality conditions for problem (AuxP3) still apply, providing the same conditions (i)''-(vii)'' above, for $\xi^1 \equiv 0$ and $b = 0$. This immediately implies that $a = 1$ and that the Lagrange multiplier associated with the cost function $\lambda = \tilde{\lambda} > 0$, and therefore, in this case the necessary conditions are non-degenerate (in fact they apply in the normal form), and this concludes the proof of Theorem 3.2.1 for the case when (CQ)(b) is satisfied.

On the other hand, when (CQ)(a) is assumed, the theorem can be proved applying a similar technique (invoking Lemma 2.4.2 instead of Lemma 2.4.3) and is easier to treat.

□

4

Normality of the Generalized Euler-Lagrange Condition: Differential Inclusions

Throughout this chapter, we consider state constrained optimal control problems in which the cost to minimize comprises both integral and end-point terms, establishing normality of the generalized Euler-Lagrange condition. Simple examples illustrate that the validity of the Euler-Lagrange condition (and related necessary conditions), in normal form, depends crucially on the interplay between velocity sets, the left end-point constraint set, and the state constraint set. We show that this is actually a common feature for general state constrained optimal control problems, in which the state constraint is represented by closed convex sets and the left end-point constraint is a closed set. In these circumstances, classical constraint qualifications involving the state constraints and the velocity sets cannot be used alone to guarantee normality of the necessary conditions. A key feature of this chapter is to prove that the additional information involving tangent vectors to the left end-point and the state constraint sets can be used to establish normality. The result uses techniques of a global construction of neighboring feasible trajectories with $W^{1,1}$ -linear estimates. An example is presented in order to emphasize the novelty of the result.

Tout au long de ce chapitre, nous considérons des problèmes de contrôle optimal avec contrainte d'état dans lesquels le coût à minimiser comprend à la fois des termes intégral et d'autres dépendent du point initial et du point final. Pour ce type de problème, on établit la normalité de la condition d'Euler-Lagrange généralisée. Des exemples simples illustrent que la validité de la condition d'Euler-Lagrange (et les conditions nécessaires complémentaires), dans la forme normale, dépend de manière cruciale de l'interaction entre les ensembles de vitesse, la contrainte initiale et la contrainte d'état. Nous montrons qu'il s'agit en fait d'une caractéristique commune pour les problèmes de contrôle optimale dans lesquels la contrainte d'état est représentée par des ensembles convexes fermés et la contrainte initiale est un ensemble fermé. Dans ces conditions, les qualifications de la contrainte classiques impliquant les contraintes

d'état et les ensembles de vitesse ne peuvent être utilisées seules pour garantir la normalité des conditions nécessaires. Le point clé de ce chapitre est de prouver que des informations supplémentaires impliquant des vecteurs tangents au point initial et à l'ensemble de contraintes d'état peuvent être utilisées pour établir la normalité. Le résultat utilise des techniques de construction globale de trajectoires admissibles voisines avec des $W^{1,1}$ -estimations linéaires. Un exemple est présenté afin de souligner la nouveauté du résultat.

“Satisfaction lies in the effort, not in the attainment, full effort is full victory.”

— Mahatma Gandhi

4.1 Introduction

Consider the state constrained optimal control problem involving time-independent differential inclusion dynamics:

$$(P4) \quad \left\{ \begin{array}{l} \text{minimize } J(x(\cdot)) := g(x(S), x(T)) + \int_S^T L(t, x(t), \dot{x}(t)) dt \\ \text{over arcs } x(\cdot) \in W^{1,1}([S, T]; \mathbb{R}^n) \text{ satisfying} \\ \quad \dot{x}(t) \in F(x(t)) \quad \text{a.e. } t \in [S, T] \\ \quad x(t) \in A \quad \text{for all } t \in [S, T] \\ \quad x(S) \in E_0, \end{array} \right.$$

in which $[S, T]$ is a given interval ($S < T$), $g(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ and $L(\cdot, \cdot, \cdot) : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ are given functions, $F(\cdot) : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a given multifunction with closed non-empty values, A and E_0 are given closed non-empty subsets of \mathbb{R}^n . An absolutely continuous arc $x(\cdot) : [a, b] \rightarrow \mathbb{R}^n$ (where $[a, b] \subset [S, T]$) which satisfies the differential inclusion $\dot{x}(t) \in F(x(t))$, a.e. $t \in [a, b]$, is called F -trajectory (on $[a, b]$). An F -trajectory $x(\cdot)$ for which $x(S) \in E_0$ and $x(t) \in A$ for all $t \in [a, b]$ is called *feasible*.

The infimum of the functional $J(x(\cdot))$ over all feasible state trajectories $x(\cdot)$ is the infimum cost for (P4). If no feasible state trajectories exist, a common interpretation of the infimum cost is to take the value $+\infty$. We say that an F -trajectory $\bar{x}(\cdot)$ is a $W^{1,1}$ -local minimizer for (P4), if there exists $\delta > 0$ such that

$$J(x(\cdot)) \geq J(\bar{x}(\cdot))$$

for all feasible F -trajectories $x(\cdot)$ satisfying $\|x(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}([S, T])} \leq \delta$. We shall also refer to $\bar{x}(\cdot)$ as a $W^{1,1}$ δ -local minimizer for (P4) when it is desirable to specify the positive number δ in the above definition.

Necessary conditions for optimal control problems in various forms have been known for many years (for example see the books [34], [81], and the references therein). But, for a significant class of problems these conditions can provide no useful information in finding minimizers (known as *degenerate* case, as already seen in Chapter 3), or apply in a form in which the cost multiplier vanishes (sometimes referred to as *abnormal* case). In the latter case the cost function would not intervene in detecting candidates to be minimizers. The importance of applying the necessary conditions for optimality with a non-zero cost multiplier, called the *normal* form, is confirmed also by the possibility to investigate regularity properties of minimizers (cf. [44], [53], [65]), second order necessary optimality conditions [22], the non-occurrence of gaps between local minimizers and minimizers for the convexified dynamics, the non-occurrence of the Lavrentieff phenomena, and other consequences of interest (cf. [81], [61], [50], [49]). A number of different techniques have been employed to obtain non-degeneracy (see for instance [3], [15], [19], [40], [39], [57] and [69]) or normality of necessary conditions (cf. [17], [13], [50], [49], [43], [68], [42], and [12]). In this chapter we use an approach suggested in [68] and successively developed in other papers: this is based on existence results of neighboring feasible

trajectories (approximating a reference trajectory that possibly violates the state constraint) with associated linear estimates. The class of problems, which we consider (comprising in the cost an integral term which depends on \dot{x}), would require exhibiting estimates which are linear w.r.t. the $W^{1,1}$ -norm for convex (in general non-smooth) state constraints. In addition, our constraint qualification involves a ‘classical’ (convexified) inward pointing condition: we do not invoke the existence of regular feedback controls (cf. [68]), or conditions requiring inward pointing vectors which are tangent to the velocity set (cf. [49]). But, known counter-examples (see [10] and [14]) clearly show that, in general, ‘classical’ inward pointing conditions are not enough to derive linear $W^{1,1}$ -estimates for non-smooth state constraints. However, a recent paper [12] shows that linear $W^{1,1}$ -estimates can be still considered for convex domains and Lipschitz continuous velocity sets, in either of the following two cases.

- (a) the left end-point of the reference trajectory belongs to a region where the state constraint A is regular (i.e. we ‘start away from corners’);
- (b) we are free to choose the left end-point for the approximating trajectory.

An application (shown in [12]) of this information is precisely the normality conditions for optimal control problems with convex state constraints when case (a) (whatever the left end-point constraint E_0 is), or alternatively (b) (that is $E_0 = \mathbb{R}^n$), takes place.

This chapter contributes to the normality literature. It provides, in particular, new conditions for normality, covering cases when the initial state is located in a ‘corner’ of intersection of the left end-point and state constraint sets. Indeed, the results of this chapter (cf. [17]) provide improvements on earlier demonstrated conditions for optimality, when the state constraint set A is non-smooth. In this respect: we allow the left end-point constraint set E_0 to be a possibly strict subset of \mathbb{R}^n , while in [12] E_0 was required to be the whole space \mathbb{R}^n . The conditions, suggested here, are also less restrictive than earlier conditions in [50], [49] for related cases, as is illustrated by an example in this chapter. Furthermore, normality is established in relation to the Extended Euler-Lagrange condition for optimal control problems for which the dynamic constraint is formulated as a differential inclusion, and not as a control-parameterized differential equation as in [50], [49].

Along this chapter, we shall employ the following norm on $W^{1,1}([a, b]; \mathbb{R}^n)$:

$$\|x(\cdot)\|_{W^{1,1}(a,b)} := \|x(\cdot)\|_{L^\infty(a,b)} + \|\dot{x}(\cdot)\|_{L^1(a,b)}, \quad \text{for all } x(\cdot) \in W^{1,1}([a, b]; \mathbb{R}^n),$$

which is equivalent to the classical $W^{1,1}$ -norm: $\|x(\cdot)\|_{W^{1,1}(a,b)} = \|x(\cdot)\|_{L^1(a,b)} + \|\dot{x}(\cdot)\|_{L^1(a,b)}$.

4.2 Main Result

We recall that the set denoted by A_0 is the set of all points $y \in A$ where the state constraint A is locally regular: more precisely, $y \in A_0$ if and only if there exists a radius $r > 0$ and a function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ of class C^{1+} (i.e. everywhere differentiable with locally Lipschitz continuous derivatives) such that

$$\nabla\varphi(y) \neq 0, \quad A \cap (y + r \text{ int } \mathbb{B}) = \{x : \varphi(x) \leq 0\} \cap (y + r \text{ int } \mathbb{B}).$$

(Observe that $A_0 \supset \text{int } A$.)

For some constants $\delta > 0$, $c > 0$, $k_F > 0$ and a reference arc $\bar{x}(\cdot)$, we recall assumptions (A.1')-(A.3') already established in Chapter 2:

(A.1') The set $A \subset \mathbb{R}^n$ is closed, non-empty and convex.

(A.2') (a) The multifunction F has non-empty values, and $\text{Gr } F(\cdot)$ is closed ;

(b)

$$F(x) \subset c \mathbb{B} \quad \text{for all } x \in \bar{x}(t) + \delta \mathbb{B} \quad t \in [S, T] ;$$

(c)

$$F(x) \subset F(y) + k_F |x - y| \mathbb{B} \quad \text{for all } x, y \in \bar{x}(t) + \delta \mathbb{B}, \quad t \in [S, T] ;$$

(A.3')

$$\text{co } F(x) \cap \text{int } T_A(x) \neq \emptyset \quad \text{for all } x \in A \cap (\bar{x}(t) + \delta \mathbb{B}), \quad t \in [S, T] .$$

Theorem 4.2.1. *Let $\bar{x}(\cdot)$ be a $W^{1,1}$ δ -local minimizer for problem (P4), in which we assume that for some constants $c > 0$ and $k_F > 0$, hypotheses (A.1')-(A.3') are satisfied. We assume further that the following two conditions are satisfied:*

(B.1) (a) g is Lipschitz continuous on $(\bar{x}(S) + \delta \mathbb{B}) \times (\bar{x}(T) + \delta \mathbb{B})$;

(b) $L(t, x, u)$ is measurable in t for fixed (x, u) , and for some constants $k_L > 0$ we have

$$|L(t, x, u) - L(t, x', u')| \leq k_L |(x, u) - (x', u')|$$

for all $(x, u), (x', u') \in (\bar{x}(t) + \delta \mathbb{B}) \times \mathbb{R}^n$, for all $t \in [S, T]$.

(B.2) One of the following two conditions is satisfied:

(a) $\bar{x}(S) \in A_0$;

(b) $\text{int } T_{E_0 \cap A}(\bar{x}(S)) \neq \emptyset$.

Then, there exist $p(\cdot) \in W^{1,1}([S, T]; \mathbb{R}^n)$ and a function of bounded variation $v(\cdot) : [S, T] \rightarrow \mathbb{R}^n$, continuous from the right on (S, T) such that

$$(i): \int_{[S, T]} \xi(t) \cdot d\nu(t) \leq 0 \quad \text{for all } \xi(\cdot) \in C([S, T]; \mathbb{R}^n) \text{ satisfying } \xi(t) \in T_A(\bar{x}(t)) ,$$

$$(ii): \dot{p}(t) \in \text{co} \{ \eta \mid (\eta, q(t)) \in \partial L(t, \bar{x}(t), \dot{\bar{x}}(t)) + N_{\text{Gr } F(\cdot)}(\bar{x}(t), \dot{\bar{x}}(t)) \} \quad \text{a.e. } t \in [S, T] ,$$

$$(iii): (p(S), -q(T)) \in \partial g(\bar{x}(S), \bar{x}(T)) + N_{E_0 \cap A}(\bar{x}(S)) \times \{0\} ,$$

$$(iv): q(t) \cdot \dot{\bar{x}}(t) - L(t, \bar{x}(t), \dot{\bar{x}}(t)) = \max_{v \in F(\bar{x}(t))} \{ q(t) \cdot v - L(t, \bar{x}(t), v) \} \quad \text{a.e. } t \in [S, T] ,$$

in which $q(\cdot) : [S, T] \rightarrow \mathbb{R}^n$ is the function

$$q(t) = \begin{cases} p(S) & t = S \\ p(t) + \int_{[S, t]} d\nu(s) & t \in (S, T]. \end{cases}$$

Remark 4.2.2. (a) Very similar proof techniques to those employed here yield an alternative version of Theorem 4.2.1, which provides necessary conditions in normal form in cases when the state constraint set A is expressed in terms of a functional inequality, that is

$$A = \{x \in \mathbb{R}^n : h(x) \leq 0\},$$

where $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is a Lipschitz function, and when the necessary conditions are expressed directly in terms of the inequality state constraint function $h(\cdot)$. More precisely, if assumptions (A.1')-(A.3') and (B.1)-(B.2) of Theorem 4.2.1 are satisfied, then the (normal) Extended Euler-Lagrange condition can be replaced by: there exist a costate arc $p(\cdot) \in W^{1,1}([S, T]; \mathbb{R}^n)$ associated with the state $\bar{x}(\cdot)$, a Borel measure $\mu(\cdot)$ on $[S, T]$ and a μ -integrable function $\gamma : [S, T] \rightarrow \mathbb{R}^n$, such that

- (ia): $\gamma(t) \in \partial_x^> h(\bar{x}(t))$ μ -a.e., and $\text{supp}(\mu) \subset \{t : h(\bar{x}(t)) = 0\}$,
- (iia): $\dot{p}(t) \in \text{co}\{\eta \mid (\eta, q(t)) \in \partial L(t, \bar{x}(t), \dot{\bar{x}}(t)) + N_{G_{F(\cdot)}}(\bar{x}(t), \dot{\bar{x}}(t))\}$ a.e. $t \in [S, T]$,
- (iiaa): $(p(S), -q(T)) \in \partial g(\bar{x}(S), \bar{x}(T)) + N_{E_0 \cap A}(\bar{x}(S)) \times \{0\}$,
- (iva): $q(t) \cdot \dot{\bar{x}}(t) - L(t, \bar{x}(t), \dot{\bar{x}}(t)) = \max_{v \in F(\bar{x}(t))} \{q(t) \cdot v - L(t, \bar{x}(t), v)\}$ a.e. $t \in [S, T]$,

in which $q(\cdot) : [S, T] \rightarrow \mathbb{R}^n$ is the function

$$q(t) = \begin{cases} p(S) & t = S \\ p(t) + \int_{[S,t]} \gamma(s) d\mu(s) & t \in (S, T]. \end{cases}$$

Here, $\text{supp}(\mu)$ denotes the support of the measure μ and $\partial_x^> h(x)$ is the hybrid subdifferential of h at x , which is defined as (in the sense of (1.5)):

$$\partial_x^> h(x) := \text{co}\{\eta \mid \text{there exist } x_i \rightarrow x \text{ and } \eta_i \rightarrow \eta \text{ s. t. } \nabla_x h(x_i) \text{ exists, } \eta_i = \nabla_x h(x_i) \text{ and } h(x_i) > 0 \text{ for each } i \in \mathbb{N}\}.$$

- (b) Besides [12], the special case in which $\bar{x}(S) \in \text{int } A$ (hypothesis (B.2)(a)) has been already treated in the literature: cf. [13] and [50] where normality of necessary conditions for optimality is proved for a generalized form of the maximum principle in the context of optimal control problems (involving time and control dependent differential equations). However, Theorem 4.2.1 deals with differential inclusion problems with state constraints and provides normality for the Extended Euler-Lagrange condition.
- (c) Assumption (B.2)(b) concerns the case in which $\bar{x}(S)$ belongs to a region where the state constraint A is not regular, and it involves tangent cones of the state and the left end-point constraints. An immediate consequence of (B.2)(b) is that the Generalized Euler-Lagrange conditions necessarily apply in a non-degenerate form (in the sense of (v) of Theorem 3.2.1 or [69]); however, here, we are interested in the stronger property of normality. Recent papers make use of similar conditions (comprising the intersection of tangent cones to the constraints) to deal with the case in which $\bar{x}(S)$ turns out to

be a ‘corner’ of A . In particular, [49] provides normality for a generalized version of the maximum principle, applicable for a wide class of optimal control problems, in which the state constraint is a closed set with non-empty interior tangent cones. Theorem 4.2.1 differs in the nature of the necessary conditions (here, exhibiting generalized Euler-Lagrange conditions), and also in the nature of the inward pointing condition. Indeed, the constraint qualification in [49] takes into account also inward vectors which are tangents to the velocity set, whereas here (A.3’) comprises inward pointing vectors belonging only to the velocity set. The following example illustrates a typical case in which (A.3’) holds true, (B.2) might be verified but the inward pointing condition of [49] is not satisfied.

4.3 Example

We consider the same example appearing in [12], employing a more general left end-point constraint E_0 to illustrate the crucial role played by hypothesis (B.2)(b) in providing normality of the necessary conditions of optimality. More precisely we look at the optimal control problem with state constraints:

$$(Ex4) \quad \left\{ \begin{array}{l} \text{minimize} \quad \int_0^1 L(t, x(t), \dot{x}(t)) dt \\ \text{over arcs } x(\cdot) \in W^{1,1}([0, 1]; \mathbb{R}^2) \text{ s.t.} \\ \dot{x}(t) \in F \quad \text{a.e. } t \in [0, 1] \\ x(t) \in A \quad \text{for all } t \in [0, 1] \\ x(0) \in E_0, \end{array} \right.$$

where

$$A := \{x = (x_1, x_2) \in \mathbb{R}^2 : h(x) \leq 0\} = \{(x_1, x_2) \in \mathbb{R}^2 : |x_2| \leq x_1\}.$$

$$F := \{(1, u) : u \in \{-2, 2\}\}, \quad L(t, u) := \begin{cases} 2 - u & \text{if } \dot{z}(t) = 2 \\ u + 2 & \text{if } \dot{z}(t) = -2. \end{cases}$$

The arc $z(\cdot) : [0, 1] \rightarrow \mathbb{R}$ is defined by: $z(0) = 0$, $z(t_k) := (-1)^k t_k$, and $\dot{z}(t) = (-1)^k 2$, for all $t \in (t_{k+1}, t_k]$, where $t_k := \frac{1}{3^k}$, for $k = 0, 1, 2, 3, \dots$. Notice that $L(t, u) = |\dot{z}(t) - u|$. Here, E_0 is any closed subset of \mathbb{R}^2 containing $\{(0, 0)\}$ and such that assumption (B.2)(b) is verified for $\bar{x}(S) = (0, 0)$ (not just $E_0 = \mathbb{R}^2$). The F -trajectory

$$\bar{x}(t) := (t, z(t)), \quad \text{for all } t \in [0, 1],$$

is a $W^{1,1}$ -minimizer for problem (Ex4), no matter the left end-point constraint is given, for we are assuming that $\bar{x}(0) = (0, 0) \in E_0$.

Applying the necessary conditions of optimality for the reference arc $\bar{x}(\cdot)$, if we suppose that the Lagrange multiplier (associated with the cost) vanishes, i.e. $\lambda = 0$, we obtain the existence of a co-state arc $p(\cdot) \equiv p(0) \neq 0$ such that (cf. [12] for the details)

$$p(0) \in N_{E_0 \cap A}(\bar{x}(0)) \tag{4.1}$$

and

$$p(0) \in -N_A(\bar{x}(0)) . \quad (4.2)$$

(We recall that (4.1) is just the left end-point transversality condition, whereas (4.2) follows from the Weierstrass and the right-end point transversality conditions.)

Now, assumption (B.2)(b) ensures the existence of a vector $w_0 \in \text{int } T_{E_0 \cap A}(\bar{x}(0))$. Notice that the convexity of A yields the inclusion $\text{int } T_{E_0 \cap A}(\bar{x}(0)) \subset \text{int } T_A(\bar{x}(0))$. Using the polar relation between the Clarke tangent cone and the (limiting) normal cone (see Proposition 1.1.22), since $w_0 \in \text{int } T_{E_0 \cap A}(\bar{x}(0))$, we have

$$w_0 \cdot \xi < 0, \quad \text{for all } \xi \in N_{E_0 \cap A}(\bar{x}(0)) \setminus \{0\} .$$

Taking $\xi = p(0)$ we obtain the following inequality:

$$w_0 \cdot p(0) < 0 . \quad (4.3)$$

And recalling also that $w_0 \in \text{int } T_A(\bar{x}(0))$:

$$w_0 \cdot \zeta < 0, \quad \text{for all } \zeta \in N_A(\bar{x}(0)) \setminus \{0\} .$$

Then, for $\zeta = -p(0)$, we have the inequality $w_0 \cdot (-p(0)) < 0$, which contradicts the previous relation (4.3). This confirms the normality of the necessary conditions once assumption (B.2)(b) is satisfied. The earlier work in [12] does not yield the information that there exists a normal minimizer in this case, since the left end-point constraint E_0 is not the whole space \mathbb{R}^2 .

Observe finally that if we had $E_0 = \{(0, 0)\}$, then hypothesis (B.2)(b) would not be satisfied, and the necessary conditions for optimality would apply in the degenerate form for the minimizer $\bar{x}(\cdot)$ (cf. [12]), confirming the relevance of assumption (B.2)(b).

4.4 Proof of the Main Result (Theorem 4.2.1)

We provide here a proof supposing that (A.1')-(A.3') and (B.1)-(B.2)(b) are satisfied. The proof when we assume (B.2)(a) instead of (B.2)(b) is simpler and has been already treated in [12] where we refer the reader for a proof.

Step 1. For some $\delta > 0$, let $\bar{x}(\cdot)$ be a $W^{1,1}$ δ -local minimizer for the optimal problem (P4). It is not restrictive to assume that $\delta \in (0, \frac{1}{2})$. Assumption (B.2)(b) guarantees the existence of a vector $w_0 \in \mathbb{R}^n$ such that

$$w_0 \in \text{int } T_{E_0 \cap A}(\bar{x}(S)).$$

Since A is convex, it follows that:

$$T_{E_0 \cap A}(\bar{x}(S)) \subset T_A(\bar{x}(S)).$$

Indeed,

$$T_{E_0 \cap A}(\bar{x}(S)) \subset T_{E_0 \cap A}^B(\bar{x}(S)) \subset T_{E_0}^B(\bar{x}(S)) \cap T_A^B(\bar{x}(S)) \subset T_A^B(\bar{x}(S)) \equiv T_A(\bar{x}(S)) \quad (4.4)$$

where $T_A^B(\bar{x}(S))$ denotes the Bouligand tangent cone (in the sense of Proposition 1.1.19), which coincides with the Clarke tangent cone for A convex (more details about the Bouligand tangent cone and its properties can be found in [7] or [81]). And therefore $w_0 \in \text{int } T_A(\bar{x}(S))$, which means that all the assumptions of case (ii) of Theorem 2.3.6 are satisfied. From the characterization of the interior of the Clarke tangent cone (cf. condition (1.8)), there exists $\bar{\delta} \in (0, 1)$ such that

$$y + [0, \bar{\delta}](w_0 + \bar{\delta}\mathbb{B}) \subset E_0 \cap A \quad \text{for all } y \in (\bar{x}(S) + 2\bar{\delta}\mathbb{B}) \cap (E_0 \cap A). \quad (4.5)$$

We take

$$\delta' := \min \left\{ \theta; \frac{\delta}{4(K+1)}; \frac{\bar{\delta}}{\alpha+1} \right\}, \quad (4.6)$$

where α , θ and K are the positive constants provided by Theorem 2.3.6.

Step 2. We shall define the following sets of functions

$$\mathfrak{R} := \{x(\cdot) \in W^{1,1}([S, T], \mathbb{R}^n) : \dot{x}(t) \in F(x(t)) \text{ a.e. } x(S) \in E_0 \cap A, \text{ and} \\ \|x(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}(S,T)} \leq \delta\},$$

$$\mathfrak{R}' := \{x(\cdot) \in \mathfrak{R} : \|x(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}(S,T)} \leq \delta'\}.$$

From assumption (B.1) it immediately follows that the functional $J(\cdot)$ is uniformly Lipschitz on \mathfrak{R} with respect to the $W^{1,1}$ -norm. Write $K_J > 0$ the Lipschitz constant of $J(\cdot)$:

$$|J(y(\cdot)) - J(x(\cdot))| \leq K_J \|y(\cdot) - x(\cdot)\|_{W^{1,1}(S,T)} \quad \text{for all } y(\cdot), x(\cdot) \in \mathfrak{R}. \quad (4.7)$$

Combining a penalization method with the linear estimate provided by Theorem 2.3.6, we obtain that $\bar{x}(\cdot)$ is also a $W^{1,1}$ -local minimizer for a new optimal control problem, in which the state constraint in problem (P4) is replaced by an extra penalty term in the cost.

Lemma 4.4.1. *Assume that all hypotheses of Theorem 4.2.1 are satisfied. Then, $\bar{x}(\cdot)$ is a $W^{1,1}$ -local minimizer for the problem*

$$\begin{cases} \text{minimize } \tilde{J}(y(\cdot)) = J(y(\cdot)) + K_J K \max_{t \in [S, T]} d_A(y(t)) \\ \text{over } y(\cdot) \in \mathfrak{R}'. \end{cases}$$

Proof. Take any $\hat{x}(\cdot) \in \mathfrak{R}'$. Notice that $\hat{x}(\cdot)$ is an F -trajectory such that $\hat{x}(S) \in A$ and $\|\hat{x}(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}(S,T)} \leq \theta$. Write $\varepsilon := \max_{t \in [S, T]} d_A(\hat{x}(t))$. Then invoking Theorem 2.3.6, we can find an F -trajectory $x(\cdot)$ such that

$$\begin{cases} x(t) \in A \quad \text{for all } t \in [S, T], \\ \|x(\cdot) - \hat{x}(\cdot)\|_{W^{1,1}(S,T)} \leq K\varepsilon \\ \text{and } x(S) = \hat{x}(S) + \alpha\varepsilon w_0. \end{cases} \quad (4.8)$$

Using the fact that

$$\varepsilon = \max_{t \in [S, T]} d_A(\hat{x}(t)) \leq \|\hat{x}(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}(S,T)} \leq \delta' \quad (4.9)$$

and the linear estimate (4.8) we obtain

$$\begin{aligned} \|x(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}(S,T)} &\leq \|x(\cdot) - \hat{x}(\cdot)\|_{W^{1,1}(S,T)} + \|\hat{x}(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}(S,T)} \\ &\leq K\varepsilon + \|\hat{x}(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}(S,T)} \leq (K+1) \|\hat{x}(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}(S,T)} \\ &\leq \delta. \end{aligned} \quad (4.10)$$

We prove that $x(S) \in E_0 \cap A$. Indeed, since $\hat{x}(\cdot) \in \mathfrak{R}'$, from the choice of δ' we also have $\hat{x}(S) \in (\bar{x}(S) + \delta\mathbb{B}) \cap (E_0 \cap A)$. On the other hand from (4.9) it follows that $\alpha\varepsilon \leq \alpha\delta' \leq \bar{\delta}$. Therefore, we can apply inclusion (4.5), obtaining that $x(S) = \hat{x}(S) + \alpha\varepsilon w_0 \in E_0 \cap A$. As a consequence $x(\cdot) \in \mathfrak{R}$.

Now, recalling that $K_J > 0$ is the Lipschitz constant of $J(\cdot)$ and that $\bar{x}(\cdot)$ is a $W^{1,1}$ -local minimizer for the reference problem (P4) (and in particular $\max_{t \in [S,T]} d_A(\bar{x}(t)) = 0$ for it is feasible), for any $\hat{x}(\cdot) \in \mathfrak{R}'$ we obtain

$$\begin{aligned} \tilde{J}(\bar{x}(\cdot)) &= J(\bar{x}(\cdot)) \leq J(x(\cdot)) \leq J(\hat{x}(\cdot)) + K_J \|x(\cdot) - \hat{x}(\cdot)\|_{W^{1,1}(S,T)} \\ &\leq J(\hat{x}(\cdot)) + K_J K \varepsilon = J(\hat{x}(\cdot)) + K_J K \max_{t \in [S,T]} d_A(\hat{x}(t)) = \tilde{J}(\hat{x}(\cdot)). \end{aligned}$$

This confirms the lemma statement. \square

Step 3. Consider the auxiliary state constrained optimal control problem

$$(AuxP4) \quad \left\{ \begin{array}{l} \text{minimize } \tilde{g}(X(S), X(T)) \\ \text{over } X(\cdot) = (x(\cdot), y(\cdot), z(\cdot)) \in W^{1,1}([S, T]; \mathbb{R}^{n+2}) \text{ satisfying} \\ \dot{X}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t)) \in G(t, X(t)) \text{ a.e. } t \in [S, T] \\ \tilde{h}(X(t)) \leq 0 \text{ for all } t \in [S, T] \\ (x(S), y(S), z(S)) \in (E_0 \cap A) \times \mathbb{R}_+ \times \{0\}. \end{array} \right.$$

The cost function \tilde{g} is defined by

$$\tilde{g}(X(S), X(T)) := g(x(S), x(T)) + K_J K \max\{0, y(T)\} + z(T).$$

The multivalued function is defined by

$$G(t, X) := \{(v, 0, r) : v \in F(x), r \geq L(t, x, v)\},$$

and the function $\tilde{h} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ (which provides the state constraint by means of a functional inequality) is defined by:

$$\tilde{h}(X) = \tilde{h}(x, y, z) := d_A(x) - y.$$

We added two extra state variables y and z , and applying known necessary conditions for optimality to a minimizer for problem (AuxP4), we shall derive normality for the reference

problem (P4).

Define the arc

$$\bar{z}(t) := \int_S^t L(s, \bar{x}(s), \dot{\bar{x}}(s)) ds, \quad \text{for all } t \in [S, T].$$

Observe that Lemma 4.4.1 implies that the G -trajectory

$$\bar{X}(\cdot) := (\bar{x}(\cdot), \bar{y} \equiv 0, \bar{z}(\cdot)),$$

is a $W^{1,1}$ δ' -local minimizer for problem (AuxP4).

Indeed, suppose by contradiction, that there exists a state trajectory $X(\cdot) = (x(\cdot), y(\cdot), z(\cdot))$ satisfying the state and dynamic constraints of (AuxP4), such that

$$\|(x(\cdot), y(\cdot), z(\cdot)) - (\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}(\cdot))\|_{W^{1,1}(S,T)} \leq \delta',$$

(δ' as defined in (4.6)) and $\tilde{g}(X(S), X(T)) < \tilde{g}(\bar{X}(S), \bar{X}(T))$.

Observe that $y(\cdot) \equiv y \geq 0$, and condition

$$\tilde{h}(x(t), y, z(t)) \leq 0 \quad \text{for all } t \in [S, T]$$

can be equivalently written as

$$\max_{t \in [S, T]} d_A(x(t)) \leq y.$$

Then we would obtain:

$$\begin{aligned} \tilde{J}(x(\cdot)) &\leq g(x(S), x(T)) + K_J K \max\{0, y\} + \int_S^T L(t, x(t), \dot{x}(t)) ds \\ &\leq \tilde{g}(X(S), X(T)) < \tilde{g}(\bar{X}(S), \bar{X}(T)) = \tilde{J}(\bar{x}(\cdot)). \end{aligned}$$

This contradicts the fact that $\bar{x}(\cdot)$ is a $W^{1,1}$ -minimizer over \mathfrak{R}' (by Lemma 4.4.1).

Step 4. Known necessary conditions for optimality are now applicable. In particular Theorem 1.3.9 (or eventually [81, Theorem 10.3.1]) implies that there exist a Lagrange multiplier $\lambda \geq 0$, a costate arc $P(\cdot) = (p(\cdot), \psi(\cdot), \phi(\cdot)) \in W^{1,1}([S, T], \mathbb{R}^{n+2})$ (associated with the minimizer $\bar{X}(\cdot) = (\bar{x}(\cdot), \bar{y} \equiv 0, \bar{z}(\cdot))$), a Borel measurable function $\mu(\cdot)$ on $[S, T]$ and a μ -integrable function $\gamma(\cdot) : [S, T] \rightarrow \mathbb{R}^{n+2}$ satisfying

- (a) $\|P\|_{L^\infty} + \lambda + \int_S^T d\mu(s) = 1$,
- (b) $\dot{P}(t) \in \text{co} \{w \in \mathbb{R}^{n+2} | (w, Q(t)) \in N_{G \circ G(t, \cdot)}((\bar{x}(t), \bar{y} \equiv 0, \bar{z}(t)), (\dot{\bar{x}}(t), 0, \dot{\bar{z}}(t)))\}$ a.e. $t \in [S, T]$,
- (c) $(P(S), -Q(T)) \in \lambda \partial \tilde{g}((\bar{x}(S), \bar{y} \equiv 0, \bar{z}(S)), (\bar{x}(T), \bar{y} \equiv 0, \bar{z}(T))) + N_{E_0 \cap A}(\bar{x}(S)) \times \mathbb{R}_- \times \mathbb{R} \times \{(0, 0, 0)\}$,
- (d) $Q(t) \cdot (\dot{\bar{x}}(t), 0, \dot{\bar{z}}(t)) = \max_{\tilde{v} \in G(t, (\bar{x}(t), \bar{y} \equiv 0, \bar{z}(t)))} Q(t) \cdot \tilde{v}$ a.e. $t \in [S, T]$,
- (e) $\gamma(t) \in \partial_{(x,y,z)}^> \tilde{h}(\bar{x}(t), \bar{y} \equiv 0, \bar{z}(t))$ μ -a.e., and $\text{supp}(\mu) \subset \{t : \tilde{h}(\bar{x}(t), \bar{y} \equiv 0, \bar{z}(t)) = 0\}$,

in which $Q(\cdot) : [S, T] \rightarrow \mathbb{R}^{n+2}$ is the function

$$Q(t) := P(t) + \int_{[S,t]} \gamma(s) d\mu(s) \text{ for } t \in (S, T].$$

Step 5. One can easily derive from relations (a)-(e) given above for the auxiliary problem (AuxP4), the desired necessary conditions for optimality for the reference problem (P4):

$$(p(S), -q(T)) \in \lambda \partial g(\bar{x}(S), \bar{x}(T)) + N_{E_0 \cap A}(\bar{x}(S)) \times \{0\},$$

$$\dot{\psi} \equiv 0, \quad \psi := \psi(S) \leq 0, \quad \text{and} \quad -\psi(T) + \int_{[S,T]} d\mu(s) \in [0, \lambda K_J K],$$

$$\dot{\phi} \equiv 0 \text{ and } \phi(T) = -\lambda,$$

$$\dot{p}(t) \in \text{co} \{ \eta \mid (\eta, q(t)) \in \lambda \partial L(t, \bar{x}(t), \dot{\bar{x}}(t)) + N_{G \circ F(\cdot)}(\bar{x}(t), \dot{\bar{x}}(t)) \} \text{ a.e. } t \in [S, T].$$

(The three adjoint systems corresponding to the adjoint arcs $p(\cdot)$, $\psi(\cdot)$ and $\phi(\cdot)$ are obtained by expressing the graph of the set-valued map G as the epigraph of a certain map. A standard result of the nonsmooth analysis representing the limiting normal cone to the epigraph of a function by means of the subgradients of the function (cf. [81, Proposition 4.3.4]) is crucial to obtain the required adjoint systems.)

Moreover, the necessary conditions apply in the normal form, for if $\lambda = 0$, then we would obtain

$$-\psi + \int_{[S,T]} d\mu(s) = 0, \quad p \equiv 0, \quad \psi \equiv 0 \quad \text{and} \quad \phi \equiv 0.$$

But this contradicts (a) of step 4. Therefore, relations (a)-(d) above yields conditions (ii)-(iv) of Theorem 4.2.1. Finally, since the function \tilde{h} is defined using the distance function to A , $d_A(\cdot)$, standard analysis (cf. Remark 1.3.7 or even [81] pages 332 and 370) allows to conclude the proof, confirming also property (i) of the theorem. \square

5

Calculus of Variations Problems: Applications to Normality

We consider autonomous variational problems in the calculus of variations with a state constraint represented by a given closed set. We prove that if the interior of the Clarke tangent cone of the state constraint set is non-empty (this is the constraint qualification that we suggest here), then the necessary optimality conditions apply in the normal form. We give the details of two possible techniques for the proof, both based on a *state augmentation* procedure: the first one uses the normality results for optimal control problems; the second technique is shorter (and simpler) and it employs a neighboring feasible trajectory result with L^∞ -linear estimates.

Nous considérons des problèmes variationnels autonomes dans le cadre de calcul des variations avec une contrainte d'état représentée par un ensemble fermé donné. Nous prouvons que si l'intérieur du cône tangent de Clarke de la contrainte d'état n'est pas vide (c'est la qualification de contrainte que nous suggérons ici), les conditions nécessaires d'optimalité s'appliquent sous la forme normale. Nous donnons les détails de deux techniques possibles pour la preuve, toutes les deux étant basées sur une procédure d'*augmentation de la variable d'état*: la première utilise les résultats de normalité pour des problèmes de contrôle optimal; la deuxième technique est plus courte (et plus simple) et elle emploie un résultat de trajectoire admissible voisine avec des L^∞ -estimations linéaires.

*“Tell them, that this their Queen of
theirs may have as much land as
she can cover with the hide of an
ox.”*

— From Alyssa, Queen of
Carthage

5.1 Introduction

Historically, variational calculus had possibly its beginnings in 1696 with J. Bernoulli with his famous *Brachistochrone Problem*, from the Ancient Greek words *brachistos* and *chronos*, meaning ‘shortest’ and ‘time’. This problem consists in finding a curve connecting two points a and b such that a mass point moves from a to b as fast as possible (i.e. minimal time) in a downward directed constant gravitational field, disregarding any friction. The solution to this problem is a cycloid curve. In the 18th century, the calculus of variations became an independent discipline of mathematics and much of the formulation of this field of mathematics was developed by Euler, Lagrange and Laplace. They established that solutions to minimization problems in the calculus of variations should verify the so-called Euler-Lagrange equations. Namely, each solution specified by the Euler-Lagrange equations is, thus, a candidate minimizer for the variational problem.

The Lipschitz continuity is a crucial regularity property that a reference minimizer must satisfy in order to derive the necessary optimality conditions for calculus of variations problems. This subject (that is the Lipschitz continuity of minimizers) has been studied in many papers for both cases of autonomous and nonautonomous Lagrangian. For instance, in the absence of state constraints, Clarke and Vinter [31] have established the Lipschitz continuity of the minimizers (for the nonautonomous case), if the Lagrangian, which is here $L := L(t, x, v)$ is convex w.r.t. v . This result has been extended by Dal Maso and Frankowska [36] to a non-convex Lagrangian (which is here autonomous), and under weaker assumptions on the regularity of the Lagrangian. Lipschitz regularity of minimizers is also known in the framework of state-constrained calculus of variations problems, see [30], [81] (for the autonomous case), and more recently [18] (for the nonautonomous case).

Another class of calculus of variations problems with boundary conditions was also studied in [23], [24] and [59] where some nice properties regarding the regularity of the minimizer are established. The existence of solutions for calculus of variations problems was also postulated in many papers, for instance in [28] (where some growth conditions are assumed, without a convexity assumption w.r.t. the dynamics) and [58] (where some boundary conditions exist and no growth assumptions on L are needed).

In this chapter, we present a normal form (i.e. when the Lagrangian multiplier associated to the cost function does not vanish) of the necessary optimality conditions for nonsmooth calculus of variations problems, concerning the minimization of an integral functional independent on time (autonomous case), over absolutely continuous arcs with fixed left end-point and with state constraints represented by a given closed set. We establish two normality results for $W^{1,1}$ -local minimizers and for $W^{1,1}$ -global minimizers. We identify therefore two different approaches in order to prove the normality of the necessary conditions, both based on a *state augmentation* technique, combined with:

- 1) either a construction of a suitable control and a normality result for optimal control problems (this is for the $W^{1,1}$ -local minimizers case);¹

¹The proof using approach 1) has been developed in [56] independently of the proof using approach 2).

- 2) or a ‘distance estimate’ result coupled with a standard maximum principle in optimal control (for the $W^{1,1}$ –global minimizers case).²

The establishment of the necessary optimality conditions in the normal form for calculus of variations problems, requires to impose new additional hypotheses, known as *constraint qualifications*, that permit to identify some class of problems for which normality is guaranteed. There has been a growing interest in the literature to ensure the normality of necessary optimality conditions for state-constrained calculus of variations problems, see for instance [39], [43]. In these papers, a useful technique employed to derive such optimality conditions consists in introducing an extra variable which reduces the reference calculus of variations problem to an optimal control problem with a terminal cost (this method is known as the ‘state augmentation’). In [39], Ferreira and Vinter use this approach to derive normality of the necessary conditions for calculus of variations problems, studied for $W^{1,1}$ –local minimizers and with a state constraint expressed in terms of an inequality of a twice continuously differentiable function. The constraint qualification referred to these smooth problems imposes that the gradient of the function representing the state constraint set is not zero at any point on the boundary of the state constraint set. The result in [39] has been extended by Fontes and Lopes in [43] to the nonsmooth case, for L^∞ –local minimizers, imposing a constraint qualification which makes use of some hybrid subgradients to cover situations in which the function, which defines the state constraint set, is not differentiable. More precisely the idea of the constraint qualification in [43] is the following: the angle between any couple of (hybrid) subgradients of the function that defines the state constraint set is ‘acute’.

In the present chapter, the state constraint set is given in the intrinsic form (i.e. it is a given closed set) and the constraint qualification we suggest is to assume that the interior of the Clarke tangent cone to the state constraint set is nonempty. We still reinterpret the reference calculus of variations problem as an optimal control problem with state constraints. In the first proof’s technique (which concerns $W^{1,1}$ –local minimizers), we invoke some stability properties of the interior of the Clarke tangent cone. This allows to construct a particular control which pushes the dynamic of the control system inside the state constraint more than the reference minimizer. In such circumstances, necessary conditions in optimal control apply in the normal form. This yields the desired normality property for our reference calculus of variations problem. The second proof’s technique (which concerns $W^{1,1}$ –global minimizers) deals with calculus of variations problems under a stronger (but still in the same spirit) constraint qualification than the one considered in the first technique. The proof in this case is much shorter and it employs a neighboring feasible trajectory result satisfying L^∞ –linear estimates (cf. [68] and [14]). This permits to find a global minimizer for an auxiliary optimal control problem to which we apply a standard constrained maximum principle, which allows to deduce the required normality form of the necessary conditions for the reference problem in calculus of variations.

Our result extends the normality theorem in [43] for the following reasons: first the constraint qualification (for both $W^{1,1}$ –local and global minimizers) present in this chapter covers cases in which the constraint qualification present in [43] is not satisfied (see Example 5.2.6 below). Moreover, our result for the first technique is valid for $W^{1,1}$ –local minimizers not only for L^∞ –local minimizers.

²This approach was suggested by a referee who reviewed the paper [56].

5.2 Main Results

This section provides the main results of this chapter: necessary optimality conditions in a normal form for autonomous problems in calculus of variations of the form:

$$(CV5) \quad \left\{ \begin{array}{l} \text{minimize} \quad \int_S^T L(x(t), \dot{x}(t)) dt \\ \text{over arcs } x \in W^{1,1}([S, T], \mathbb{R}^n) \text{ satisfying} \\ \quad x(S) = x_0, \\ \quad x(t) \in A \quad \text{for all } t \in [S, T], \end{array} \right.$$

where $A \subset \mathbb{R}^n$ is a closed set.

An absolutely continuous arc \bar{x} is said to be *admissible* if the left end-point and the state constraint of the problem (CV5) are satisfied. We say that an admissible arc \bar{x} is a $W^{1,1}$ -local minimizer if there exists $\epsilon > 0$ such that

$$\int_S^T L(\bar{x}(t), \dot{\bar{x}}(t)) dt \leq \int_S^T L(x(t), \dot{x}(t)) dt,$$

for all admissible arcs x satisfying

$$\|x(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}} \leq \epsilon,$$

where $\|x(\cdot)\|_{W^{1,1}} := |x(S)| + \|\dot{x}(\cdot)\|_{L^1}$ for all $x(\cdot) \in W^{1,1}([S, T], \mathbb{R}^n)$.

Assume that the following hypotheses are satisfied for a given reference arc $\bar{x} \in W^{1,1}([S, T], \mathbb{R}^n)$:

(CV.1) $L(\cdot, \cdot)$ is Borel measurable, bounded on bounded sets and there exist $\epsilon' > 0$, $K_L > 0$ such that

$$|L(x, v) - L(x', v)| \leq K_L |x - x'|, \quad \text{for all } x, x' \in \bar{x}(t) + \epsilon' \mathbb{B}, \quad t \in [S, T]$$

uniformly on $v \in \mathbb{R}^n$.

(CV.2) $v \mapsto L(x, v)$ is convex, for all $x \in \mathbb{R}^n$.

(CV.3) (Coercivity) There exists an increasing function $\theta : [0, \infty) \rightarrow [0, \infty)$ such that

$$\lim_{\alpha \rightarrow \infty} \frac{\theta(\alpha)}{\alpha} = +\infty,$$

and a constant β such that $L(x, v) > \theta(|v|) - \beta|v|$ for all $x \in \mathbb{R}^n, v \in \mathbb{R}^n$.

Remark 5.2.1. Hypotheses (CV.1) and (CV.2) imply that $v \mapsto L(x, v)$ is locally Lipschitz (cf. [34, Proposition 2.2.6]).

We are interested in providing normality conditions for problems in which the state constraint A is given in an implicit form: A is just a closed set. Using the distance function, the state constraint $x(t) \in A$ can be equivalently written as a pathwise functional inequality

$$d_A(x(t)) \leq 0 \quad \text{for all } t \in [S, T].$$

Theorem 5.2.2 (First Main Result). *Let \bar{x} be a $W^{1,1}$ -local minimizer for (CV5). Assume that hypotheses (CV.1)-(CV.3) are satisfied. Suppose also that*

(CQ)

$$\text{int } T_A(z) \neq \emptyset, \quad \text{for all } z \in \bar{x}([S, T]) \cap \partial A.$$

Then, there exist $p(\cdot) \in W^{1,1}([S, T], \mathbb{R}^n)$, a function of bounded variation $v(\cdot) : [S, T] \rightarrow \mathbb{R}^n$, continuous from the right on (S, T) , such that: for some positive Borel measure μ on $[S, T]$, whose support satisfies

$$\text{supp}(\mu) \subset \{t \in [S, T] : \bar{x}(t) \in \partial A\},$$

and some Borel measurable selection

$$\gamma(t) \in \partial_x^> d_A(\bar{x}(t)) \quad \mu - \text{a.e. } t \in [S, T]$$

we have

$$(a) \quad v(t) = \int_{[S, t]} \gamma(s) d\mu(s) \quad \text{for all } t \in (S, T];$$

$$(b) \quad \dot{p}(t) \in \text{co } \partial_x L(\bar{x}(t), \dot{\bar{x}}(t)) \quad \text{and} \quad q(t) \in \text{co } \partial_{\dot{x}} L(\bar{x}(t), \dot{\bar{x}}(t)) \quad \text{a.e. } t \in [S, T];$$

$$(c) \quad q(T) = 0;$$

where

$$q(t) = \begin{cases} p(S) & t = S \\ p(t) + \int_{[S, t]} \gamma(s) d\mu(s) & t \in (S, T]. \end{cases}$$

The following theorem concerns $W^{1,1}$ -global minimizers, establishing the same necessary optimality conditions (in the normal form) as Theorem 5.2.2. Here we invoke a slightly different assumption:

(CV.1') $L(\cdot, \cdot)$ is Borel measurable, bounded on bounded sets and for all $R_0 > 0$, there exists $K_L > 0$ such that for all $x, x' \in R_0\mathbb{B}$

$$|L(x, v) - L(x', v)| \leq K_L(t)|x - x'| \quad \text{uniformly in } v \in \mathbb{R}^n$$

Theorem 5.2.3 (Second Main Result). *Let \bar{x} be a $W^{1,1}$ -global minimizer for (CV5). Assume that hypotheses (CV.1'), (CV.2) and (CV.3) are satisfied. Suppose also that*

(CQ)

$$\text{int } T_A(z) \neq \emptyset, \quad \text{for all } z \in \partial A.$$

Then, there exist $p(\cdot) \in W^{1,1}([S, T], \mathbb{R}^n)$, a function of bounded variation $v(\cdot) : [S, T] \rightarrow \mathbb{R}^n$, continuous from the right on (S, T) , such that conditions (a)-(c) of Theorem 5.2.2 remain valid.

The two proof techniques of Theorem 5.2.2 and Theorem 5.2.3 are built up separately in section 5.3.

Remark 5.2.4. A crucial point due to assumptions (CV.1)-(CV.3) is the Lipschitz regularity of the minimizer $\bar{x}(\cdot)$ (as shown in Lemma 5.3.1). This property of Lipschitzianity will intervene in the proofs of Theorems 5.2.2 and 5.2.3. However, some recent works have showed that the Lipschitzianity of the minimizer can be still guaranteed if problem (CV5) is extended to the non-autonomous case; more precisely, we let the Lagrangian L to be time-dependent. Results in this context can be found in [29] and [62]. A part of an ongoing project is to establish Theorem 5.2.2 in the non-autonomous case, under weaker hypotheses than (CV.1)-(CV.3) (for example, to try to discard the convexity assumption of L w.r.t. the dynamics).

Discussion

We briefly discuss now in which sense our result improves previous literature. We recall that the problem of normality of necessary conditions in calculus of variations has been previously investigated in [39], for the case where the boundary of A is regular (of class C^2). This result has been successively extended to the case where the boundary of A is nonsmooth in [43]. More precisely, the state constraint A is expressed in the form of a functional inequality in the sense of the time-independent case of (1.20) (namely, $A := \{x : h(x) \leq 0\}$ where h is a Lipschitz continuous function); moreover the following constraint qualification is considered for a reference L^∞ -local minimizer \bar{x} for (CV5):

(CQ13') There exist positive constants c, ε such that for all $\tau \in \{\sigma \in [S, T] : h(\bar{x}(\sigma)) = 0\}$ and for all $x_1, x_2 \in \{\bar{x}(\sigma) : \sigma \in (\tau - \varepsilon, \tau] \cap [S, T]\}$

$$\gamma_1 \cdot \gamma_2 > c, \quad \text{for all } \gamma_1 \in \partial^> h(x_1), \quad \gamma_2 \in \partial^> h(x_2).$$

Furthermore, if $h(\bar{x}(S)) = 0$, then for all $x_1, x_2 \in \bar{x}(S) + \varepsilon\mathbb{B}$

$$\gamma_1 \cdot \gamma_2 > c, \quad \text{for all } \gamma_1 \in \partial^> h(x_1), \quad \gamma_2 \in \partial^> h(x_2).$$

(Here, $\partial^> h(\cdot)$ is the hybrid subdifferential in the sense of Definition 1.1.15.)

In [43], it was proved that if we assume that hypotheses (CV.1)-(CV.3) and (CQ13') are satisfied, then the necessary conditions apply in the normal form.

Consider now any point $y \in \{\bar{x}(\sigma) \in \partial A, \text{ for some } \sigma \in [S, T]\}$. Then, condition (CQ13') above implies the simpler property:

$$\gamma_1 \cdot \gamma_2 > c, \quad \text{for all } \gamma_1, \gamma_2 \in \partial^> h(y). \quad (5.1)$$

Theorem 5.2.2 extends the result in [43] in the following sense: first of all we merely consider $W^{1,1}$ -local minimizers (not the strong case of L^∞ -local minimizer). In addition, if $d_A(\cdot)$ satisfies (5.1) for some $c > 0$, then our condition (CQ) (and (\widetilde{CQ})) is verified. (This is proved by Proposition 5.2.5 below.) Therefore, given a $W^{1,1}$ -local (respectively global) minimizer for problem (CV5), the constraint qualification (CQ13') implies that the condition (CQ) is valid for $y \in \{\bar{x}(\sigma) \in \partial A, \text{ for some } \sigma \in [S, T]\}$ (respectively (\widetilde{CQ})) is valid for all $y \in \partial A$). Consequently, the optimality conditions in the normal form are provided by Theorem 5.2.2 (respectively Theorem 5.2.3).

Proposition 5.2.5. Consider a closed set $A \subset \mathbb{R}^n$ and assume that the distance function to A , $d_A(\cdot)$, satisfies the following condition: if for any given $y \in \partial A$ there exists $c > 0$ such that

$$\gamma_1 \cdot \gamma_2 > c, \quad \text{for all } \gamma_1, \gamma_2 \in \partial^\circ d_A(y),$$

then $\text{int } T_A(y) \neq \emptyset$.

Proof. Since the hypothesis (5.1) considered here (for the case where $h(\cdot) = d_A(\cdot)$) clearly implies that $0 \notin \partial^\circ d_A(y)$, then the proof uses the same ideas of [42, Proposition 2.1] which consists in showing that the cone $\mathbb{R}^+(\partial^\circ d_A(y))$ is closed and pointed (i.e. not containing zero) and that it is also equal to $\text{co } N_A(y)$. That is the convex hull of the limiting normal cone to A is pointed and, consequently, its polar, $T_A(y)$ has a nonempty interior. \square

We give below a simple and motivational example which shows that we can find a state constraint set A defined by a functional inequality (for some Lipschitz function h) such that (CQ) (and also $\widetilde{\text{CQ}}$) is always verified but (5.1) fails to hold true when a minimizer goes in a region where A is nonsmooth.

Example 5.2.6. Consider the set $A := \{(x, y) : h(x, y) \leq 0\}$, where $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$h(x, y) = |y| - x.$$

It is straightforward to check (owing to the definition of $\partial^\circ h(\cdot)$), that $\partial^\circ h(0, 0) = \text{co } \{\gamma_1, \gamma_2\}$, where $\gamma_1 := (-1, +1)$, and $\gamma_2 := (-1, -1)$ (cf. Figure 5.1 below). Therefore, at the point $(0, 0)$, condition (5.1) is violated when a minimizer \bar{x} is such that $(0, 0) \in \bar{x}([S, T])$. This is because we could find two vectors γ_1 and γ_2 such that $\gamma_1 \cdot \gamma_2 = 0$. However, $\text{int } T_A(0, 0) \neq \emptyset$, and more in general $\text{int } T_A(y) \neq \emptyset$ for all $y \in \partial A$. Then, (CQ) (and also $\widetilde{\text{CQ}}$) is always satisfied.

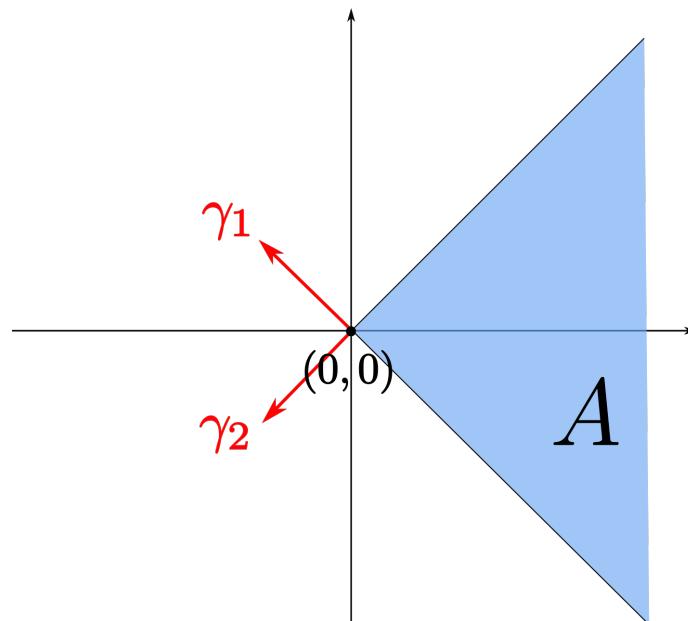


Figure 5.1 – Illustrative Example

Remark 5.2.7. Observe that if $\text{int } T_A(z) \neq \emptyset$ for all $z \in \partial A$ (that is (\widetilde{CQ})), then (CQ) is clearly satisfied: this provides a more general constraint qualification which does not involve the minimizer \bar{x} .

5.3 Two Proof Techniques for the Main Results

5.3.1 Proof of Theorem 5.2.2

Technique: Normality in Optimal Control Problems

Along this subsection, we prove Theorem 5.2.2 by introducing some technical lemmas and by making use of a normality result in optimal control for which a proof is established in the current subsection. More precisely, we identify a class of optimal control problems whose necessary optimality conditions apply in a normal form, under some constraint qualifications. The main purpose of introducing such problems is that calculus of variations problems can be regarded as an optimal control problem (owing to the ‘state augmentation’ procedure). Therefore, the results on optimal control problems will be used to establish normality of optimality conditions for the reference calculus of variations problem.

Technical Lemmas

Under assumptions (CV.1)-(CV.3), the Tonelli Existence Theorem (cf. [81, Theorem 11.1.1]) provides the existence of a minimizer for (CV5). The next lemma illustrates that the minimizers over the space of absolutely continuous functions are Lipschitz continuous (for the particular case of autonomous Lagrangian).

Lemma 5.3.1. (cf. [81, Theorem 11.5.1]) Let \bar{x} be a $W^{1,1}$ -local minimizer for (CV5). Assume that hypotheses (CV.1)-(CV.3) are satisfied. Then \bar{x} is a Lipschitz continuous function.

The next lemma says that one can select a particular bounded control $v(\cdot)$ which pulls the dynamics inward the state constraint set more than the reference minimizer.

Lemma 5.3.2. Let $\bar{x}(\cdot)$ be a $W^{1,1}$ -local minimizer for (CV5). Assume that hypotheses (CV.1)-(CV.3) and (CQ) are satisfied. Then, there exist positive constants ε , ρ , β , C , C_1 and a measurable function $v(\cdot)$ such that:

$$(i) \quad \|v - \dot{\bar{x}}\|_{L^\infty} \leq C \text{ and } \|v\|_{L^\infty} \leq C_1.$$

$$(ii) \quad \text{For all } \tau \in \{\sigma \in [S, T] : \bar{x}(\sigma) \in \partial A\},$$

$$\sup_{\eta \in \text{co}(N_A(\bar{x}(s)) \cap \partial \mathbb{B})} (v(t) - \dot{\bar{x}}(t)) \cdot \eta < -\beta, \quad \text{a.e. } s, t \in (\tau - \varepsilon, \tau].$$

$$(iii) \quad \text{If } x_0 = \bar{x}(S) \in \partial A, \text{ then}$$

$$\sup_{\substack{\eta \in \text{co}(N_A(x) \cap \partial \mathbb{B}) \\ x \in (x_0 + \rho \mathbb{B}) \cap \partial A}} (v(t) - \dot{\bar{x}}(t)) \cdot \eta < -\beta, \quad \text{a.e. } t \in [S, S + \varepsilon].$$

Normality in Optimal Control Problems

We recall that ‘normality’ means that the Lagrangian multiplier associated with the objective function – here written λ – is different from zero (it can be taken equal to 1).

Consider the fixed left end-point optimal control problem (P5) with a state constraint set $A \subset \mathbb{R}^n$ which is merely a closed set:

$$(P5) \quad \left\{ \begin{array}{l} \text{minimize } g(x(T)) \\ \text{over } x \in W^{1,1}([S, T], \mathbb{R}^n) \text{ and measurable functions } u \text{ satisfying} \\ \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [S, T] \\ x(S) = x_0 \\ x(t) \in A \quad \text{for all } t \in [S, T] \\ u(t) \in U(t) \quad \text{a.e. } t \in [S, T]. \end{array} \right.$$

The data for this problem comprise functions $g : \mathbb{R}^n \rightarrow \mathbb{R}$, $f : [S, T] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$, an initial state $x_0 \in \mathbb{R}^n$, and a multifunction $U(\cdot) : [S, T] \rightsquigarrow \mathbb{R}^m$. The set of control functions for (P5), denoted by \mathcal{U} , is the set of all measurable functions $u : [S, T] \rightarrow \mathbb{R}^m$ such that $u(t) \in U(t)$ a.e. $t \in [S, T]$.

We say that an admissible process (\bar{x}, \bar{u}) is a $W^{1,1}$ -local minimizer if there exists $\epsilon > 0$ such that

$$g(\bar{x}(T)) \leq g(x(T)),$$

for all admissible processes (x, u) satisfying

$$\|x(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}} \leq \epsilon.$$

There follows a ‘normal’ version of the maximum principle for state constrained problems. For a $W^{1,1}$ -local minimizer (\bar{x}, \bar{u}) of interest, slightly different hypotheses than the ones invoked in Chapter 1 (namely, (CS.1)-(CS.3)) are presented below. There exists a positive scalar δ such that:

(H.1) The function $(t, u) \mapsto f(t, x, u)$ is $\mathcal{L} \times \mathcal{B}^m$ measurable for each $x \in \mathbb{R}^n$. There exists a $\mathcal{L} \times \mathcal{B}^m$ measurable function $k(t, u)$ such that $t \mapsto k(t, \bar{u}(t))$ is integrable and

$$|f(t, x, u) - f(t, x', u)| \leq k(t, u)|x - x'|$$

for $x, x' \in \bar{x}(t) + \delta\mathbb{B}$, $u \in U(t)$, a.e. $t \in [S, T]$. Furthermore there exist scalars $K_f > 0$ and $\epsilon' > 0$ such that

$$|f(t, x, u) - f(t, x', u)| \leq K_f|x - x'|$$

for $x, x' \in \bar{x}(S) + \delta\mathbb{B}$, $u \in U(t)$, a.e. $t \in [S, S + \epsilon']$.

(H.2) $\text{Gr } U(\cdot)$ is $\mathcal{L} \times \mathcal{B}^m$ measurable.

(H.3) The function g is Lipschitz continuous on $\bar{x}(T) + \delta\mathbb{B}$.

Reference is also made to the following constraint qualifications. There exist positive constants $K, \tilde{\varepsilon}, \tilde{\beta}, \tilde{\rho}$ and a control $\hat{u} \in \mathcal{U}$ such that

(CQ1)

$$|f(t, \bar{x}(t), \bar{u}(t)) - f(t, \bar{x}(t), \hat{u}(t))| \leq K, \quad \text{for a.e. } t \in (\tau - \tilde{\varepsilon}, \tau] \cap [S, T]$$

and

$$\eta \cdot [f(t, \bar{x}(t), \hat{u}(t)) - f(t, \bar{x}(t), \bar{u}(t))] < -\tilde{\beta},$$

for all $\eta \in \partial_x^> d_A(\bar{x}(s))$, a.e. $s, t \in (\tau - \tilde{\varepsilon}, \tau] \cap [S, T]$ and for all $\tau \in \{\sigma \in [S, T] : \bar{x}(\sigma) \in \partial A\}$.

(CQ2) If $x_0 \in \partial A$, then for a.e. $t \in [S, S + \tilde{\varepsilon})$

$$|f(t, x_0, \hat{u}(t))| \leq K, \quad |f(t, x_0, \bar{u}(t))| \leq K,$$

and

$$\eta \cdot [f(t, x_0, \hat{u}(t)) - f(t, x_0, \bar{u}(t))] < -\tilde{\beta}$$

for all $\eta \in \partial_x^> d_A(x)$, $x \in (x_0 + \tilde{\rho}\mathbb{B}) \cap \partial A$.

(CQ3)

$$\text{co } N_A(\bar{x}(t)) \text{ is pointed for each } t \in [S, T].$$

We recall that $\text{co } N_A(\bar{x}(t))$ is ‘pointed’ if for any nonzero elements $d_1, d_2 \in \text{co } N_A(\bar{x}(t))$,

$$d_1 + d_2 \neq 0.$$

Theorem 5.3.3. *Let (\bar{x}, \bar{u}) be a $W^{1,1}$ -local minimizer for (P5). Assume that hypotheses (H.1)-(H.3) and the constraint qualifications (CQ1)-(CQ3) hold. Then, there exist $p(\cdot) \in W^{1,1}([S, T], \mathbb{R}^n)$, a Borel measure $\mu(\cdot)$ and a μ -integrable function $\gamma(\cdot)$ such that*

$$(i) \quad -\dot{p}(t) \in \text{co } \partial_x(q(t) \cdot f(t, \bar{x}(t), \bar{u}(t))) \quad \text{a.e. } t \in [S, T],$$

$$(ii) \quad -q(T) \in \partial g(\bar{x}(T)),$$

$$(iii) \quad q(t) \cdot f(t, \bar{x}(t), \bar{u}(t)) = \max_{u \in U(t)} q(t) \cdot f(t, \bar{x}(t), u),$$

$$(iv) \quad \gamma(t) \in \partial^> d_A(\bar{x}(t)) \text{ and } \text{supp}(\mu) \subset \{t \in [S, T] : \bar{x}(t) \in \partial A\},$$

where

$$q(t) = \begin{cases} p(S) & t = S \\ p(t) + \int_{[S,t]} \gamma(s) d\mu(s) & t \in (S, T]. \end{cases}$$

This result was proved in [43] for L^∞ -local minimizers. We will show shortly that it remains valid also for the weaker case of $W^{1,1}$ -local minimizers.

Proof of Theorem 5.2.2. We first employ a standard argument, the so-called ‘state augmentation’, which allows to write the problem of calculus of variations (CV5) as an optimal control problem of type (P5).

Indeed, it is enough to add an extra absolutely continuous state variable

$$z(t) = \int_S^t L(x(s), \dot{x}(s)) ds$$

and consider the dynamics $\dot{x} = u$. We notice that $z(S) = 0$ and $U(t) = \mathbb{R}^n$.

Then, the problem (CV5) can be written as the optimal control problem (P5’):

$$(P5') \quad \left\{ \begin{array}{l} \text{minimize } z(T) \\ \text{over } W^{1,1} - \text{arcs } (x(\cdot), z(\cdot)) \text{ and measurable functions } u(\cdot) \in \mathbb{R}^n \text{ satisfying} \\ \quad (\dot{x}(t), \dot{z}(t)) = (u(t), L(x(t), u(t))) \quad \text{a.e. } t \in [S, T] \\ \quad (x(S), z(S)) = (x_0, 0) \\ \quad x(t) \in A \quad \text{for all } t \in [S, T] \\ \quad u(t) \in U(t) = \mathbb{R}^n \quad \text{a.e. } t \in [S, T]. \end{array} \right.$$

We set $\varpi(t) := \begin{pmatrix} x(t) \\ z(t) \end{pmatrix}$ and $\tilde{f}(\varpi(t), u(t)) := \begin{pmatrix} u(t) \\ L(x(t), u(t)) \end{pmatrix}$. Here the set of controls \mathcal{U} is the set of dynamics $\dot{x}(t) \in \mathbb{R}^n$ for a.e. $t \in [S, T]$.

It is easy to prove that if \bar{x} is a $W^{1,1}$ -local minimizer for the reference calculus of variations problem (CV5), then $\left(\bar{\varpi}(t) = \begin{pmatrix} \bar{x}(t) \\ \bar{z}(t) \end{pmatrix}, \bar{u} \right)$ is a $W^{1,1}$ -local minimizer for (P5’) where $\dot{\bar{z}}(t) = L(\bar{x}(t), \bar{u}(t))$ and $\bar{u} = \dot{\bar{x}}$.

The proof of Theorem 5.2.2 is now given in three steps. The first step is devoted to show that the constraint qualifications (CQ1)-(CQ3) of Theorem 5.3.3 are implied by (CQ) (of Theorem 5.2.2). In step 2, we verify that hypotheses (H.1)-(H.3) of Theorem 5.3.3 can be deduced from hypotheses (CV.1)-(CV.3) of Theorem 5.2.2. In step 3, we apply Theorem 5.3.3 to (P5’) and then obtain the assertions of Theorem 5.2.2.

We recall first the following result (cf. Proposition 1.1.17), for which a short proof is provided in Section 5.4 at the end of the present chapter.

Proposition 5.3.4. (i) $\partial^> d_A(a) \subset \text{co}(N_A(a) \cap \partial \mathbb{B})$ for a closed set $A \subset \mathbb{R}^n$ and $a \in \partial A$.

(ii) If in addition $\text{co } N_A(a)$ is pointed, then $0 \notin \text{co}(N_A(a) \cap \partial \mathbb{B})$.

Step 1. Prove that the constraint qualifications (CQ1)-(CQ3) of Theorem 5.3.3 are mainly implied by (CQ) (of Theorem 5.2.2).

The constraint qualifications (CQ1) and (CQ2) for the optimal problem (P5’) become as follows: there exist positive constants $K, \tilde{\varepsilon}, \tilde{\beta}, \tilde{\rho}$ and a control $\hat{u} \in \mathcal{U}$ such that

(CQ1)'

$$|\tilde{f}((\bar{x}(t), \bar{z}(t)), \bar{u}(t)) - \tilde{f}((\bar{x}(t), \bar{z}(t)), \hat{u}(t)))| \leq K, \text{ a.e. } t \in (\tau - \tilde{\varepsilon}, \tau] \cap [S, T] \quad (5.2)$$

and

$$\tilde{\eta} \cdot [\tilde{f}((\bar{x}(t), \bar{z}(t)), \hat{u}(t)) - \tilde{f}((\bar{x}(t), \bar{z}(t)), \bar{u}(t))] < -\tilde{\beta}, \quad (5.3)$$

for all $\tilde{\eta} \in \partial_{\omega}^{\geq} d_A(\bar{x}(s))$ a.e. $s, t \in (\tau - \tilde{\varepsilon}, \tau] \cap [S, T]$ and for all $\tau \in \{\sigma \in [S, T] : \bar{x}(\sigma) \in \partial A\}$. Here $\partial_{\omega}^{\geq} d_A(\bar{x}(s)) := \text{co}\{(a, b) : \text{there exists } s_i \rightarrow s \text{ such that } d_A(\bar{x}(s_i)) > 0 \text{ for all } i, d_A(\bar{x}(s_i)) \rightarrow d_A(\bar{x}(s)) \text{ and } \nabla_{\omega} d_A(\bar{x}(s_i)) \rightarrow (a, b)\}$.

(CQ2)' If $x_0 \in \partial A$ then for a.e. $t \in [S, S + \tilde{\varepsilon}]$

$$|\tilde{f}((x_0, 0), \hat{u}(t))| \leq K \quad \text{and} \quad |\tilde{f}((x_0, 0), \bar{u}(t))| \leq K, \quad (5.4)$$

and

$$\tilde{\eta} \cdot [\tilde{f}((x_0, 0), \hat{u}(t)) - \tilde{f}((x_0, 0), \bar{u}(t))] < -\tilde{\beta}, \quad (5.5)$$

for all $\tilde{\eta} \in \partial_{\omega}^{\geq} d_A(x)$, $x \in (x_0 + \tilde{\rho}\mathbb{B}) \cap \partial A$. Here, $\partial_{\omega}^{\geq} d_A(x) = \text{co}\{(a, b) : \text{there exists } x_i \xrightarrow{d_A} x \text{ such that } d_A(x_i) > 0 \text{ for all } i \text{ and } \nabla_{\omega} d_A(x_i) \rightarrow (a, b)\}$.

1. We start with the proof of condition (5.4) of (CQ2)'

Since \bar{x} is a minimizer for the problem of calculus of variations (CV5), Lemma 5.3.1 ensures that \bar{x} is a Lipschitz continuous function. Take $R := |x_0| + \|\dot{\bar{x}}\|_{L^{\infty}}(1 + T - S) > 0$. Then $\bar{x}([S, T]) \subset R\mathbb{B}$ and $|\dot{\bar{x}}(t)| \leq R$ for a.e. $t \in [S, T]$. Since the function $L(\cdot, \cdot)$ is bounded on bounded sets (owing to (CV.1)), we obtain

$$|\tilde{f}((x_0, 0), \bar{u}(t))| = \left| \begin{pmatrix} \bar{u}(t) = \dot{\bar{x}}(t) \\ L(x_0, \bar{u}(t)) \end{pmatrix} \right| \leq K \quad \text{a.e. } t \in [S, T], \text{ for some } K > 0.$$

Moreover, under (CQ), Lemma 5.3.2 ensures the existence of a measurable function $v(\cdot)$ and positive constants C and C_1 such that $\|v\|_{L^{\infty}} \leq C_1$ and $\|v - \dot{\bar{x}}\|_{L^{\infty}} \leq C$. Therefore, by choosing a control \hat{u} such that $\hat{u}(t) = v(t)$ a.e. t , we deduce easily that

$$|\tilde{f}((x_0, 0), \hat{u}(t))| \leq K \quad \text{a.e. } t \in [S, T] \quad \text{for some } K > 0.$$

Condition (5.4) is therefore satisfied.

2. Condition (5.2) follows also from the particular choice of $\hat{u}(t)$ to be the same bounded control $v(\cdot)$ of Lemma 5.3.2, and from the boundedness of $L(x, \cdot)$ on bounded sets.

3. To prove (5.5), we suppose that $x_0 \in \partial A$. Now we invoke Lemma 5.3.2 and we consider the control function $\hat{u} : [S, S + \varepsilon] \rightarrow \mathbb{R}^n$ defined by $\hat{u}(t) := v(t)$, where $v(t)$ is a measurable function which satisfies the properties of Lemma 5.3.2. Condition (iii) of Lemma 5.3.2 implies that there exist positive constants ε, ρ, β such that

$$\eta \cdot (\hat{u}(t) - \dot{\bar{x}}(t)) < -\beta \quad \text{for a.e. } t \in [S, S + \varepsilon] \quad (5.6)$$

for all $\eta \in \text{co}(N_A(x) \cap \partial\mathbb{B})$ and for all $x \in (x_0 + \rho\mathbb{B}) \cap \partial A$. In particular for all $\eta \in \partial^> d_A(x)$ (owing to Proposition 5.3.4(i)) and by choosing $0 < \tilde{\varepsilon} \leq \varepsilon$, $0 < \tilde{\rho} \leq \rho$ and $\tilde{\beta} = \beta$. Take any $\tilde{\eta} \in \partial_{\tilde{\omega}}^{\geq} d_A(x)$ where $x \in (x_0 + \tilde{\rho}\mathbb{B}) \cap \partial A$. By definition of $\partial_{\tilde{\omega}}^{\geq} d_A(x)$

$$\tilde{\eta} \in \left(\text{co} \{ a : \text{there exists } x_i \xrightarrow{d_A} x \text{ such that } d_A(x_i) > 0 \text{ for all } i, \text{ and } \nabla_x d_A(x_i) \rightarrow a \}, 0 \right).$$

This is because $\nabla_z d_A(x_i) = 0$. We conclude that $\tilde{\eta}$ can be written as:

$$\tilde{\eta} := (\eta, 0) \quad \text{where } \eta \in \partial^> d_A(x).$$

It follows that, owing to (5.6):

$$(\eta, 0) \cdot [\tilde{f}((x_0, 0), \hat{u}(t)) - \tilde{f}((x_0, 0), \bar{u}(t))] = \eta \cdot (\hat{u}(t) - \bar{u}(t)) < -\tilde{\beta},$$

for all $\eta \in \partial^> d_A(x)$ a.e. $t \in [S, S + \tilde{\varepsilon})$ for all $x \in (x_0 + \tilde{\rho}\mathbb{B}) \cap \partial A$. Therefore, condition (5.5) is confirmed.

4. For the proof of (5.3), we take any $\tau \in \{\sigma \in [S, T] : \bar{x}(\sigma) \in \partial A\}$. Consider again the positive constants ε and β and the selection $v(\cdot)$ provided by Lemma 5.3.2. Property (ii) of the latter lemma implies that for a.e. $s, t \in (\tau - \varepsilon, \tau]$

$$\eta \cdot (v(t) - \dot{\bar{x}}(t)) < -\beta \tag{5.7}$$

for all $\eta \in \text{co}(N_A(\bar{x}(s)) \cap \partial\mathbb{B})$. In particular for all $\eta \in \partial_x^> d_A(\bar{x}(s))$ (owing to Proposition 5.3.4(i)) and for $0 < \tilde{\varepsilon} \leq \varepsilon$ and $\tilde{\beta} = \beta$ and by considering the control function $\hat{u}(t) := v(t)$. Now we take an element $\tilde{\eta}$ in $\partial_{\tilde{\omega}}^{\geq} d_A(\bar{x}(s))$ for $s \in (\tau - \tilde{\varepsilon}, \tau] \cap [S, T]$. It follows that,

$$\tilde{\eta} \in \left(\text{co} \{ a : \text{there exists } t_i \rightarrow s \text{ s.t. } d_A(\bar{x}(s_i)) > 0 \text{ for all } i, d_A(\bar{x}(s_i)) \rightarrow d_A(\bar{x}(s)) \text{ and } \nabla_x d_A(\bar{x}(s_i)) \rightarrow a \}, 0 \right)$$

which is equivalent to write $\tilde{\eta}$ as

$$\tilde{\eta} := (\eta, 0) \quad \text{where } \eta \in \partial^> d_A(\bar{x}(s))$$

and $s \in (\tau - \tilde{\varepsilon}, \tau] \cap [S, T]$. Making use of (5.7), it follows that

$$(\eta, 0) \cdot [\tilde{f}((\bar{x}(t), \bar{z}(t)), \hat{u}(t)) - \tilde{f}((\bar{x}(t), \bar{z}(t)), \bar{u}(t))] = \eta \cdot (\hat{u}(t) - \bar{u}(t)) < -\tilde{\beta},$$

for all $\eta \in \partial^> d_A(\bar{x}(s))$ for a.e. $s, t \in (\tau - \tilde{\varepsilon}, \tau] \cap [S, T]$ and for all $\tau \in \{\sigma \in [S, T] : \bar{x}(\sigma) \in \partial A\}$. We conclude that condition (5.3) is satisfied.

5. Finally, it is clear that (CQ) implies that (CQ3) is satisfied.

Step 2. This step is devoted to prove that hypotheses (H.1)-(H.3) adapted to the optimal control problem (P5') are satisfied.

- Owing to (CV.1), it is straightforward to see that the function $u \mapsto \tilde{f}((x, z), u) = \begin{pmatrix} u \\ L(x, u) \end{pmatrix}$ is a Borel-measurable function, for each $x \in \mathbb{R}^n$. Moreover, since $L(\cdot, u)$ is uniformly Lipschitz on $u \in \mathbb{R}^n$ (cf. (CV.1)), we deduce that

$$|\tilde{f}((x, z), u) - \tilde{f}((x', z'), u)| = \left| \begin{pmatrix} 0 \\ L(x, u) - L(x', u) \end{pmatrix} \right| \leq K_L |x - x'|,$$

for all $x, x' \in \bar{x}(t) + \epsilon' \mathbb{B}$, for all $t \in [S, T]$ uniformly on $u \in \mathbb{R}^n$. The Lipschitz continuity w.r.t. z is obvious since $(x, z) \mapsto \tilde{f}((x, z), u)$ is independent on z . Hypothesis (H.1) is therefore satisfied by choosing $\delta := \epsilon'$.

- Since $U(t) = \mathbb{R}^n$, we have that $\text{Gr } U(\cdot) = [S, T] \times \mathbb{R}^n$ which is $\mathcal{L} \times \mathcal{B}^n$ measurable. Hypothesis (H.2) is therefore satisfied.
- Hypothesis (H.3) is verified since the cost function of problem (P5'), $g(x, z) := z$, is Lipschitz.

Step 3. Apply Theorem 5.3.3 to (P5') and then obtain the assertions of Theorem 5.2.2.

We recall that if \bar{x} is a $W^{1,1}$ -local minimizer for the reference calculus of variations problem (CV5), then $((\bar{x}, \bar{z}), \dot{\bar{x}} = \bar{u})$ is a $W^{1,1}$ -local minimizer for the optimal control problem (P5'). Therefore, Theorem 5.3.3 can be applied to the problem (P5'). Namely, there exist a couple of absolutely continuous functions $(p_1, p_2) \in W^{1,1}([S, T], \mathbb{R}^n) \times W^{1,1}([S, T], \mathbb{R})$, a Borel measure $\mu(\cdot)$ and a μ -integrable function $\gamma(\cdot)$, such that:

- (i)' $-(\dot{p}_1(t), \dot{p}_2(t)) \in \text{co } \partial_{(x,z)}((q_1(t), q_2(t)) \cdot \tilde{f}((\bar{x}(t), \bar{z}(t)), \bar{u}(t)))$ a.e. $t \in [S, T]$,
- (ii)' $-(q_1(T), q_2(T)) \in \partial_{(x,z)} g(\bar{x}(T), \bar{z}(T))$,
- (iii)' $(q_1(t), q_2(t)) \cdot \tilde{f}((\bar{x}(t), \bar{z}(t)), \bar{u}(t)) = \max_{u \in U(t)} (q_1(t), q_2(t)) \cdot \tilde{f}((\bar{x}(t), \bar{z}(t)), u)$,
- (iv)' $\gamma(t) \in \partial^> d_A(\bar{x}(t))$ and $\text{supp}(\mu) \subset \{t \in [S, T] : \bar{x}(t) \in \partial A\}$,

where

$$q_1(t) = \begin{cases} p_1(S) & t = S \\ p_1(t) + \int_{[S,t]} \gamma(s) d\mu(s), & t \in (S, T] \end{cases}$$

and

$$q_2(t) = p_2(t), \quad \text{for } t \in [S, T].$$

The transversality condition (ii)' ensures that $q_1(T) = 0$. This implies that condition (c) of Theorem 5.2.2 is satisfied. Furthermore, $q_2(T) = -1 = p_2(T)$.

The adjoint system (i)' ensures, by expanding it and applying a well-known nonsmooth calculus rule, that:

$$-(\dot{p}_1(t), \dot{p}_2(t)) \in \text{co } \partial_{(x,z)} (q_1(t) \cdot \bar{u}(t) + q_2(t) L(\bar{x}(t), \bar{u}(t))).$$

Since $L(\cdot, u)$ is Lipschitz continuous on a neighborhood of $\bar{x}(t)$ and owing to Proposition 1.1.12 4), we obtain

$$-(\dot{p}_1(t), \dot{p}_2(t)) \in q_2(t) \text{ co } \partial_x L(\bar{x}(t), \bar{u}(t)) \times \{0\}.$$

Therefore, $\dot{p}_2(t) = 0$ and $-\dot{p}_1(t) \in q_2(t) \text{ co } \partial_x L(\bar{x}(t), \bar{u}(t))$. We deduce that $p_2(t) = q_2(t) = -1$ and $\dot{p}_1(t) \in \text{co } \partial_x L(\bar{x}(t), \bar{u}(t))$ a.e. t .

The maximization condition (iii)' is equivalent to:

$$q_1(t) \cdot \bar{u}(t) - L(\bar{x}(t), \bar{u}(t)) = \max_{u \in \mathbb{R}^n} \{q_1(t) \cdot u - L(\bar{x}(t), u)\}.$$

Deriving the expression above in terms of u , and evaluating it at $u = \dot{\bar{x}}$, we obtain

$$0 \in \text{co } \partial_{\dot{x}} \{q_1(t) \cdot \bar{u}(t) - L(\bar{x}(t), \bar{u}(t))\}.$$

This yields that

$$0 \in \text{co } \{\partial_{\dot{x}}(q_1(t) \cdot \dot{\bar{x}}(t)) + \partial_{\dot{x}}(-L(\bar{x}(t), \bar{u}(t)))\} \quad \text{a.e.}$$

since $u \mapsto L(x, u)$ is locally Lipschitz (cf. Remark 5.2.1). Moreover, since $\text{co } \partial(-f) = -\text{co } \partial(f)$, and making use of Proposition 1.1.12 4), we deduce that

$$0 \in q_1(t) - \text{co } \partial_{\dot{x}}(L(\bar{x}(t), \bar{u}(t))).$$

Equivalently,

$$q_1(t) \in \text{co } \partial_{\dot{x}} L(\bar{x}(t), \bar{u}(t)) \quad \text{a.e. } t.$$

Condition (b) of Theorem 5.2.2 is therefore satisfied. Condition (a) is straightforward owing to (iv)'. By consequence, the proof of Theorem 5.2.2 is complete. \square

Proof of Normality for Optimal Problems

Proof of Theorem 5.3.3. The proof follows similar reasoning as in [43] and [40]. It is divided into three steps. In the first step, we apply the maximum principle of Theorem 1.3.6. Step 2 is dedicated to consider a sequence of auxiliary problems which approximate the original problem and to apply the necessary optimality conditions in the form of Theorem 1.3.4. We also establish a nondegeneracy result under (CQ2). In the last step, we ensure that the necessary conditions apply in a normal form under (CQ1).

Step 1. Since hypotheses (H.1)-(H.3) and (CQ3) are satisfied, we can apply the maximum principle for state constraints using Theorem 1.3.6.

Step 2. In this step, we intend to strengthen the nontriviality condition of the maximum principle by

$$\int_{(S,T]} d\mu(s) + \|q\|_{L^\infty} + \lambda \neq 0. \quad (5.8)$$

As in [40], we consider a sequence of approximating problems differing from (P5) in the dynamics near the left end-point.

Choose $\alpha \in (0, T - S]$. Consider any measurable control function v , and any absolutely continuous function x that satisfies:

$$(W) \quad \begin{cases} \dot{x}(t) = f(t, x(t), \bar{u}(t)) + v(t) \cdot \Delta f(t, x(t)) & t \in [S, S + \alpha] \\ x(S) = x_0, \\ v(t) \in \{0\} \cup \{1\} & \text{a.e. } t \in [S, S + \alpha] \end{cases}$$

where $\Delta f(t, x) := f(t, x, \hat{u}(t)) - f(t, x, \bar{u}(t))$ and \hat{u} is the control function featuring in the constraint qualification (CQ2).

By reducing the size of α , we have that for any trajectory x solving system (W)

$$d_A(x(t)) \leq 0 \quad \text{for all } t \in [S, S + \alpha].$$

Now, take a decreasing sequence $\{\alpha_i\}$ on $(0, \alpha)$, converging to zero. Associate with each α_i the following problem (P_i) , where the state constraint is imposed only on the subinterval $[S + \alpha_i, T]$:

$$(P_i) \quad \begin{cases} \text{Minimize} & g(x(T)) \\ \text{subject to} & \dot{x}(t) = f(t, x(t), \bar{u}(t)) + v(t) \cdot \Delta f(t, x(t)) \\ & \text{a.e. } t \in [S, S + \alpha_i] \\ & \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [S + \alpha_i, T] \\ & v(t) \in \{0\} \cup \{1\} \quad \text{a.e. } t \in [S, S + \alpha_i] \\ & x(S) = x_0 \\ & u(t) \in U(t) \quad \text{a.e. } t \in [S + \alpha_i, T] \\ & \tilde{d}(t, x(t)) \leq 0 \quad \text{for all } t \in [S, T]. \end{cases}$$

where

$$\tilde{d}(t, x) := \begin{cases} 0 & t \in [S, S + \alpha_i] \\ d_A(x) & t \in [S + \alpha_i, T]. \end{cases}$$

Note that we can write the first dynamic equation as

$$\dot{x}(t) = f(t, x(t), \tilde{u}(t))$$

where

$$\tilde{u}(t) =: \begin{cases} \bar{u}(t), & \text{if } v(t) = 0 \\ \hat{u}(t), & \text{if } v(t) = 1 \end{cases} \quad \text{a.e. } t \in [S, S + \alpha_i].$$

The function \tilde{u} is a measurable function and $\tilde{u} \in U(t)$. These facts combine with $d_A(x(t)) \leq 0$ for all $t \in [S, S + \alpha_i]$ to ensure that all admissible state trajectories x for (P_i) are contained in the set of admissible solutions for (P5). Moreover the process $(\bar{x}, (\bar{u}, 0))$ for (P_i) has a cost identical to that of (P5). Therefore, the trajectory $(x \equiv \bar{x})$ and the controls $((u, v) \equiv (\bar{u}, 0))$ are $W^{1,1}$ -local optimal for all problems (P_i) .

The Weierstrass condition from the maximum principle for state constraints (cf. Theorem 1.3.4) to these problems is written as: for almost every $t \in [S, S + \alpha_i]$,

$$0 = \max_{v \in \{0\} \cup \{1\}} v p_i(t) \cdot \left(f(t, \bar{x}(t), \hat{u}(t)) - f(t, \bar{x}(t), \bar{u}(t)) \right). \quad (5.9)$$

Passing to the limit, as $i \rightarrow \infty$, in the others conditions, we obtain the necessary conditions of optimality for the original problem (P5).

Suppose to the contrary that

$$\int_{(S,T]} d\mu(s) + \|q\|_{L^\infty} + \lambda = 0.$$

Due to the non-vanishing condition and considering the case when $\bar{x}(S) \in \partial A$, we must have

$$\lambda = 0, \quad \mu = \delta_S, \quad 0 \neq p(t) = \text{const} = -\gamma \quad \text{for some } \gamma \in \partial^> d_A(\bar{x}(S)).$$

By using the constraint qualification (CQ2), we arrive at a contradiction of the Weierstrass condition (5.9).

Step 3. In this step, we prove the normality using the same techniques of [43].

Expanding the internal product and applying a well-known nonsmooth calculus rule (see [34, Proposition 2.3.3]) to the adjoint inclusion in Theorem 1.3.4, we obtain

$$-\dot{p}(t) \in \text{co } \partial_x \left(\sum_{i=1}^n q_i(t) f_i(t, \bar{x}(t), \bar{u}(t)) \right) \subset \sum_{i=1}^n q_i(t) \text{co } \partial_x f_i(t, \bar{x}(t), \bar{u}(t))$$

a.e. $t \in [S, T]$.

Define the matrix $\xi(t) = \begin{pmatrix} \xi_1(t) \\ \cdots \\ \xi_n(t) \end{pmatrix}$ for some $\xi_i(t) \in \text{co } \partial_x f_i(t, \bar{x}(t), \bar{u}(t))$ conveniently selected such that

$$-\dot{p}(t) = q(t) \cdot \xi(t) \quad \text{a.e. } t \in [S, T].$$

It follows that

$$p(t) = p(T) + \int_t^T q(s) \xi(s) ds$$

or equivalently

$$q(t) = q(T) + \int_t^T q(s) \xi(s) ds - \int_{[t,T]} \gamma(s) d\mu(s).$$

Now suppose in contradiction that the multiplier λ is equal to zero. Then, $q(T) = 0$ and

$$q(t) = \int_t^T q(s) \xi(s) ds - \int_{[t,T]} \gamma(s) d\mu(s).$$

Let $\tau = \inf\{t \in [S, T] : \int_{[t,T]} d\mu(s) = 0\}$. (τ is the first time the reference minimizer \bar{x} leaves the boundary of the state constraint set.)

If $\tau = S$, then $\int_{(S,T]} d\mu(s) = 0$. This implies that $q(t) = 0$ for all $t \in [S, T]$. Hence $\int_{(S,T]} d\mu(s) + \|q\|_{L^\infty} + \lambda = 0$ and we arrive at a contradiction with the non degeneracy condition (5.8).

It remains to consider the case when $\tau > S$. We show that when $\lambda = 0$ and (CQ1) is verified, the maximization condition of Theorem 5.3.3 cannot be satisfied.

Defining $\Phi(t, s)$ as the transition matrix for the linear system $\dot{z}(t) = \xi(t)z(t)$, the function q can be written as

$$q(t) = - \int_{[t, T]} \gamma(s) \Phi(s, t) d\mu(s).$$

For $t \in (\tau - \tilde{\varepsilon}, \tau]$, where $\tilde{\varepsilon}$ is the positive constant in (CQ1), we set $\Delta f(t, \bar{x}(t)) = f(t, \bar{x}(t), \hat{u}(t)) - f(t, \bar{x}(t), \bar{u}(t))$, where \hat{u} is the control function chosen in (CQ1) for $t \in (\tau - \tilde{\varepsilon}, \tau] \cap [S, T]$ and is equal to \bar{u} a.e. on $[\tau, T]$. We have for the same positive $\tilde{\beta}$ of (CQ1):

$$\begin{aligned} q(t) \cdot \Delta f(t, \bar{x}(t)) &= - \int_{[t, T]} \gamma(s) \Phi(s, t) \Delta f(t, \bar{x}(t)) d\mu(s) \\ &= - \int_{[t, \tau]} \gamma(s) \Phi(s, t) \Delta f(t, \bar{x}(t)) d\mu(s) \\ &= - \int_{[t, \tau]} \gamma(s) \Delta f(t, \bar{x}(t)) d\mu(s) \\ &\quad - \int_{[t, \tau]} \gamma(s) [\Phi(s, t) - \Phi(\tau, \tau)] \Delta f(t, \bar{x}(t)) d\mu(s) \\ &> \tilde{\beta} \mu\{[t, \tau]\} - \int_{[t, \tau]} \gamma(s) [\Phi(s, t) - \Phi(\tau, \tau)] \Delta f(t, \bar{x}(t)) d\mu(s). \end{aligned}$$

As Φ is continuous we can ensure the existence of a positive scalar δ_1 such that $|\Phi(s, t) - \Phi(\tau, \tau)| < \frac{\tilde{\beta}}{2K}$ for all (s, t) satisfying $|(s, t) - (\tau, \tau)| < \delta_1$. (K being the constant of condition (CQ1).) Hence, for a.e. $t \in (\tau - \tilde{\varepsilon}, \tau] \cap (\tau - \delta_1, \tau]$ we have

$$q(t) \cdot \Delta f(t, \bar{x}(t)) > \tilde{\beta} \mu\{[t, \tau]\} - \frac{\tilde{\beta}}{2} \mu\{[t, \tau]\} > 0$$

contradicting the maximization condition (iii)' of Theorem 5.3.3. Therefore, $\lambda = 1$, and the necessary conditions of Theorem 5.2.2 apply in the normal form. \square

5.3.2 Proof of Theorem 5.2.3

Technique: Neighboring Feasible Trajectories with L^∞ -Linear Estimates

In this subsection we give details of a shorter proof based on a simple technique using the neighboring feasible trajectory result with linear estimate (initially introduced in [68]), while regarding the calculus of variations problems as an optimal control problem with final cost. The result we aim to prove is valid for global $W^{1,1}$ -minimizers under the stronger constraint qualification

(\widetilde{CQ})

$$\text{int } T_A(z) \neq \emptyset, \quad \text{for all } z \in \partial A.$$

Proof of Theorem 5.2.2. Let $\bar{x}(\cdot)$ be a global $W^{1,1}$ -minimizer for the calculus of variations problem (CV5).

We employ the same standard argument (known as *state augmentation*) previously stated in the first proof technique. This allows to write the problem of calculus of variations (CV5) as an optimal control problem. Indeed, by adding an extra absolutely continuous state variable

$$w(t) = \int_S^t L(x(s), \dot{x}(s)) ds$$

and by considering the dynamics $\dot{x} = u$, the problem (CV5) can be written as the optimal control problem ($\widetilde{P5}$):

$$(\widetilde{P5}) \quad \left\{ \begin{array}{ll} \text{minimize} & w(T) \\ \text{over } W^{1,1} \text{ arcs } & (x(\cdot), w(\cdot)) \text{ satisfying} \\ & (\dot{x}(t), \dot{w}(t)) \in F(t, x) \quad \text{a.e } t \in [S, T] \\ & (x(S), w(S)) = (x_0, 0) \\ & (x(t), w(t)) \in A \times \mathbb{R} \quad \text{for all } t \in [S, T]. \end{array} \right.$$

where

$$F(t, x) := \begin{cases} \{(u, L(x, u)) : u \in \bar{u}(t) + M\mathbb{B}\} & \text{if } \dot{\bar{x}}(t) \text{ exists and } |\dot{\bar{x}}(t)| \leq \|\dot{\bar{x}}(\cdot)\|_{L^\infty} \\ \{(u, L(x, u)) : u \in M\mathbb{B}\} & \text{otherwise,} \end{cases} \quad (5.10)$$

in which $\bar{u}(\cdot) = \dot{\bar{x}}(\cdot)$ and $M > 0$ large enough (for instance $M \geq 2\|\dot{\bar{x}}\|_{L^\infty}$).

It is easy to prove that if $\bar{x}(\cdot)$ is a $W^{1,1}$ -global minimizer for (CV5), then $(\bar{x}(\cdot), \bar{w}(\cdot))$ is a $W^{1,1}$ -global minimizer for ($\widetilde{P5}$).

The proof of Theorem 5.2.3 is now given in three steps. In Step 1, we show that the neighboring feasible trajectory theorem in [14, Theorem 2.3] holds true for our velocity set $F(\cdot, \cdot)$ and our constraint qualification $A \times \mathbb{R}$. In Step 2, we combine a penalization method with the L^∞ -linear estimate provided by [14, Theorem 2.3] which will permit to derive a minimizer for a state constrained-free problem involving a penalty term in the cost. Step 3 is devoted to apply a standard Maximum Principle to an auxiliary problem and deduce the assertions of Theorem 5.2.3 in their normal form.

Step 1. In this step, we prove that $(F(\cdot, \cdot), A \times \mathbb{R})$ satisfies the hypothesis of [14, Theorem 2.3]. Indeed, the set of velocities is nonempty, of closed values owing to the Lipschitz continuity of $u \rightarrow L(x, u)$, and $F(\cdot, x)$ is \mathcal{L} -measurable for all $x \in \mathbb{R}^n$. Moreover, the set $F(t, x)$ is bounded on bounded sets, since hypothesis (CV.1') assumes that $L(\cdot, \cdot)$ is bounded on bounded set, and Lemma 5.3.1 ensures the boundedness of $\bar{u}(t)$ a.e. $t \in [S, T]$. The Lipschitz continuity of $F(t, \cdot)$ is a direct consequence of the Lipschitz continuity of $x \rightarrow L(x, u)$ provided by assumption (CV.1'). Finally, we claim that the constraint qualification

$$F(t, x) \cap \text{int } T_{A \times \mathbb{R}}(x, w) \neq \emptyset \quad \text{for each } (t, (x, w)) \in [S, T] \times (R_1\mathbb{B} \cap \partial A) \times \mathbb{R}$$

is well verified, for some $R_1 > 0$. Indeed, for our case,

$$F(t, x) \cap \text{int } T_{A \times \mathbb{R}}(x, w) = \begin{cases} \left((\bar{u}(t) + M\mathbb{B}) \cap \text{int } T_A(x) \right) \times \mathbb{R} & \text{if } \dot{\hat{x}}(t) \text{ exists and } |\dot{\hat{x}}(t)| \leq \|\dot{\hat{x}}(\cdot)\|_{L^\infty} \\ \left(M\mathbb{B} \cap \text{int } T_A(x) \right) \times \mathbb{R} & \text{otherwise} \end{cases}$$

The constraint qualification (\widetilde{CQ}) that we suggest

$$\text{int } T_A(x) \neq \emptyset \quad \text{for all } x \in \partial A,$$

and the boundedness of $\|\bar{u}\|_{L^\infty} = \|\dot{\hat{x}}\|_{L^\infty}$ guarantee that the claim is confirmed for $M > 0$ chosen large enough. Therefore, [14, Theorem 2.3] is applicable.

Step 2. In this step, we combine a penalization technique with the linear estimate given by [14, Theorem 2.3]. We obtain that (\bar{x}, \bar{w}) is also a global minimizer for a new optimal control problem, in which the state constraint in problem ($\widetilde{P5}$) is replaced with an extra penalty term in the cost:

Lemma 5.3.5. *Assume that all hypotheses of Theorem 5.2.3 are satisfied. Then, (\bar{x}, \bar{w}) is a global minimizer for the problem:*

$$(\widetilde{CV5}) \quad \begin{cases} \text{minimize} & w(T) + K \max_{t \in [S, T]} d_{A \times \mathbb{R}}(x(t), z(t)) =: J(x(\cdot), w(\cdot)) \\ \text{over arcs} & (x, w) \in W^{1,1}([S, T], \mathbb{R}^n \times \mathbb{R}) \text{ satisfying} \\ & (\dot{x}(t), \dot{w}(t)) \in F(t, x(t)) \quad \text{a.e. } t \in [S, T] \\ & (x(S), w(S)) = (x_0, 0) \quad x(S) \in A. \end{cases}$$

Here, K is the constant provided by [14, Theorem 2.3].

Proof. Suppose that there exists a global minimizer $(\hat{x}(\cdot), \hat{w}(\cdot))$ for ($\widetilde{CV5}$) such that

$$J(\hat{x}(\cdot), \hat{w}(\cdot)) < J(\bar{x}(\cdot), \bar{w}(\cdot)).$$

Denote by $\hat{\varepsilon} := \max_{t \in [S, T]} d_{A \times \mathbb{R}}(\hat{x}(t), \hat{w}(t))$, the extent to which the reference trajectory $(\hat{x}(\cdot), \hat{w}(\cdot))$ violates the state constraint $A \times \mathbb{R}$. By the neighboring feasible trajectory (with L^∞ -estimates) [14, Theorem 2.3], there exists an F -trajectory $(x(\cdot), w(\cdot))$ and $K > 0$ such that

$$\begin{cases} (x(S), w(S)) = (x_0, 0) \\ (x(t), w(t)) \in A \times \mathbb{R} \quad \text{for all } t \in [S, T] \\ \|(x(\cdot), w(\cdot)) - (\hat{x}(\cdot), \hat{w}(\cdot))\|_{L^\infty(S, T)} \leq K\hat{\varepsilon}. \end{cases}$$

In particular

$$|w(T) - \hat{w}(T)| \leq K\hat{\varepsilon}.$$

Therefore,

$$w(T) \leq \hat{w}(T) + K\hat{\varepsilon} = J(\hat{x}(\cdot), \hat{w}(\cdot)) < J(\bar{x}(\cdot), \bar{w}(\cdot)) = \bar{w}(T).$$

But this contradicts the minimality of (\bar{x}, \bar{w}) for ($\widetilde{P5}$). The proof is therefore complete. \square

A consequence of Lemma 5.3.5 is the following:

Lemma 5.3.6. $\bar{X}(\cdot) := (\bar{x}(\cdot), \bar{w}(\cdot) = \int_S^T L(\bar{x}(t), \dot{\bar{x}}(t)) dt, \bar{z}(\cdot) \equiv 0)$ is a global minimizer for

$$\left\{ \begin{array}{l} \text{minimize } g(X(T)) \\ \text{over } X(\cdot) = (x(\cdot), w(\cdot), z(\cdot)) \in W^{1,1}([S, T], \mathbb{R}^{n+2}) \text{ satisfying} \\ \dot{X}(t) = (\dot{x}(t), \dot{w}(t), \dot{z}(t)) \in G(t, X(t)) \quad \text{a.e. } t \in [S, T] \\ h(X(t)) \leq 0 \quad \text{for all } t \in [S, T] \\ (x(S), w(S), z(S)) \in \{x_0\} \times \{0\} \times \mathbb{R}^+ . \end{array} \right.$$

The cost function g is defined by

$$g(X(T)) := w(T) + Kz(T).$$

The multivalued function is defined by

$$G(t, X(t)) := \begin{cases} \{(u, L(x, u), 0) : u \in \bar{u}(t) + M\mathbb{B}\} & \text{if } \dot{\bar{x}}(t) \text{ exists and } |\dot{\bar{x}}(t)| \leq \|\dot{\bar{x}}(\cdot)\|_{L^\infty} \\ \{(u, L(x, u), 0) : u \in M\mathbb{B}\} & \text{otherwise,} \end{cases}$$

and the function $h : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ (which provides the state constraint in terms of a functional inequality) is given by:

$$h(X) = h(x, w, z) := d_A(x) - z$$

Proof. By contradiction, suppose that there exists a state trajectory $X(\cdot) := (x(\cdot), w(\cdot), z(\cdot))$ satisfying the state and dynamic constraints of the problem, such that

$$g(X(T)) < g(\bar{X}(T)).$$

Observe that $z(\cdot) \equiv z \geq 0$, and the state constraint condition is equivalent to

$$\max_{t \in [S, T]} d_A(x(t)) \leq z.$$

Then, we would obtain

$$\begin{aligned} J(x(\cdot), w(\cdot)) &= w(T) + K \max_{t \in [S, T]} d_A(x(t)) \\ &\leq w(T) + Kz \\ &= g(X(T)) < g(\bar{X}(T)) = J(\bar{x}(\cdot), \bar{w}(\cdot)). \end{aligned}$$

This contradicts the fact that $(\bar{x}(\cdot), \bar{w}(\cdot))$ is a global minimizer for (CV5). \square

Step 3. In this step we apply known necessary optimality conditions (cf. Theorem 1.3.4). There exist costate arcs $P(\cdot) = (p_1(\cdot), p_2(\cdot), p_3(\cdot)) \in W^{1,1}([S, T], \mathbb{R}^{n+2})$ associated with the minimizer $(\bar{x}(\cdot), \bar{w}(\cdot), \bar{z} \equiv 0)$, a Lagrange multiplier $\lambda \geq 0$, a Borel measure $\mu(\cdot) : [S, T] \rightarrow \mathbb{R}$ and a μ -integrable function $\gamma(\cdot) = (\gamma_1(\cdot), \gamma_2(\cdot), \gamma_3(\cdot))$ such that:

- (i) $(\lambda, P(\cdot), \mu(\cdot)) \neq (0, 0, 0)$,
- (ii) $-\dot{P}(t) \in \text{co } \partial_{(x,w,z)} \left(Q(t) \cdot (\dot{x}(t), L(\bar{x}(t), \dot{x}(t)), 0) \right)$,
- (iii) $(P(S), -Q(T)) \in \lambda \partial g(\bar{x}(T), \bar{w}(T), \bar{z} \equiv 0) + (\mathbb{R} \times \mathbb{R} \times \mathbb{R}^-) \times (\{0\} \times \{0\} \times \{0\})$,
- (iv) $Q(t) \cdot (\dot{x}(t), L(\bar{x}(t), \dot{x}(t)), 0)$
 $= \begin{cases} \max_{u \in \bar{u}(t) + M\mathbb{B}} Q(t) \cdot (u, L(\bar{x}(t), u), 0) & \text{if } \dot{x}(t) \text{ exists and } |\dot{x}(t)| \leq \|\dot{x}(\cdot)\|_{L^\infty} \\ \max_{u \in M\mathbb{B}} Q(t) \cdot (u, L(\bar{x}(t), u), 0) & \text{otherwise,} \end{cases}$
- (v) $\gamma(t) \in \partial_{(x,w,z)}^> h(\bar{x}(t), \bar{w}(t), \bar{z} \equiv 0) \quad \mu\text{-a.e.} \quad \text{and} \quad \text{supp}(\mu) \subset \{t : h(\bar{x}(t), \bar{w}(t), \bar{z} \equiv 0) = 0\}$.

Here, $Q(\cdot) : [S, T] \rightarrow \mathbb{R}^{n+2}$ is the function

$$Q(t) := P(t) + \int_{[S,t]} \gamma(s) d\mu(s) \quad \text{for } t \in (S, T],$$

where $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t)) \in \partial^> d_A(\bar{x}(t)) \times \{0\} \times \{-1\} \quad \mu\text{-a.e.}$ (owing to condition (v)). From the conditions above, we derive the following:

- (i') $(\lambda, p_1(\cdot), p_2(\cdot), p_3(\cdot), \mu(\cdot)) \neq (0, 0, 0, 0, 0)$,
- (ii') $-\dot{p}_1(t) \in p_2(t) \text{co } \partial_x L(\bar{x}(t), \dot{x}(t))$, a.e. $t \in [S, T]$ and $\dot{p}_2 = \dot{p}_3 \equiv 0$,
- (iii') $-p_1(T) + \int_S^T \gamma_1(s) d\mu(s) = 0$, $p_3(S) = p_3 \leq 0$, $-(p_3(T) - \int_{[S,T]} d\mu(s)) = \lambda K$ and $p_2(T) = p_2 = -\lambda$,
- (iv') $(p_1(t) + \int_S^t \gamma_1(s) d\mu(s)) \cdot \dot{x}(t) - \lambda L(\bar{x}(t), \dot{x}(t))$
 $= \begin{cases} \max_{u \in \bar{u}(t) + M\mathbb{B}} (p_1(t) + \int_S^t \gamma_1(s) d\mu(s)) \cdot u - \lambda L(\bar{x}(t), u) & \text{if } \dot{x}(t) \text{ exists and } |\dot{x}(t)| \leq \|\dot{x}(\cdot)\|_{L^\infty} \\ \max_{u \in M\mathbb{B}} (p_1(t) + \int_S^t \gamma_1(s) d\mu(s)) \cdot u - \lambda L(\bar{x}(t), u) & \text{otherwise} \end{cases}$
- (v') $\gamma_1(t) \in \partial_x^> d_A(\bar{x}(t)) \quad \mu\text{-a.e.} \quad \text{and} \quad \text{supp}(\mu) \subset \{t : h(\bar{x}(t), \bar{w}(t), \bar{z} \equiv 0) = 0\}$

We prove that conditions (i')-(v') apply in the normal form (i.e. with $\lambda = 1$). Indeed, suppose that $\lambda = 0$, then

$$-p_3 + \int_{[S,T]} d\mu(s) = 0 \quad p_1 = 0 \quad p_2 = 0 \quad p_3 = 0.$$

But this contradicts the nontriviality condition (i'). Therefore, the relations (i')-(v') apply in the normal form. Moreover, notice that the convexity of $L(x, \cdot)$ yields that the maximality condition (iv') is verified globally (i.e. for all $u \in \mathbb{R}^n$), and by deriving it w.r.t. $u = \dot{x}$, and making use of the max rule ([81, Theorem 5.5.2]) we deduce that

$$p_1(t) + \int_{[S,t]} \gamma_1(s) d\mu(s) \in \text{co } \partial_{\dot{x}} L(\bar{x}(t), \dot{x}(t)) \quad \text{a.e. } t \in [S, T].$$

This permits to conclude the necessary optimality conditions in the normal form of Theorem 5.2.3. □

5.4 Proofs of Proposition 5.3.4 and Lemma 5.3.2

Proof of Proposition 5.3.4. (i). Let $\zeta \in \partial^> d_A(a)$ for $a \in \partial A$. Set

$$X := \{\xi \in \mathbb{R}^n : \text{there exists } a_i \xrightarrow{d_A} a, d_A(a_i) > 0, \text{ for all } i \text{ and } \nabla d_A(a_i) \rightarrow \xi\}.$$

We know that $\partial^> d_A(a) = \text{co } X$. Therefore, we can write that

$$\zeta \in \text{co } X.$$

From the Caratheodory theorem: $\zeta = \sum_{j=0}^n \lambda_j \zeta_j$ where $\sum_{j=0}^n \lambda_j = 1$ for $\lambda_j \in [0, 1]$, and $\zeta_j \in X$,

for all $j = 0, \dots, n$. Therefore, owing to the definition of X , for each j , there exists $a_{j_i} \xrightarrow{d_A} a$ such that $d_A(a_{j_i}) > 0$ and $\zeta_j = \lim_{i \rightarrow +\infty} \nabla d_A(a_{j_i})$. Therefore, Proposition 1.1.13 ensures that

$$\nabla d_A(a_{j_i}) = \frac{a_{j_i} - b_{j_i}}{|a_{j_i} - b_{j_i}|} \text{ where } b_{j_i} := \pi_A(a_{j_i}), \text{ for all } j = 0, \dots, n.$$

Set $\eta_{j_i} := \nabla d_A(a_{j_i})$. It is clear that $|\eta_{j_i}| = 1$. Moreover, $|b_{j_i} - a| \leq |b_{j_i} - a_{j_i}| + |a_{j_i} - a| \leq |c - a_{j_i}| + |a_{j_i} - a|$ for all $c \in A$, which implies that $|b_{j_i} - a| \leq 2|a - a_{j_i}| \xrightarrow{i \rightarrow +\infty} 0$ for the particular choice of $c = a \in A$. Therefore, $\zeta_j \in N_A(a)$, such that $|\zeta_j| = 1$, for all $j = 0, \dots, n$. We conclude that $\zeta = \sum_{j=0}^n \lambda_j \zeta_j \in \text{co}(N_A(a) \cap \partial \mathbb{B})$ for $\lambda_j \in [0, 1]$, $j = 0, \dots, n$ and $\sum_{j=0}^n \lambda_j = 1$. Proposition 5.3.4(i) is confirmed.

(ii). We proceed by contradiction by supposing that

$$0 \in \text{co}(N_A(a) \cap \partial \mathbb{B}).$$

Owing to the Caratheodory theorem, $0 = \sum_{j=0}^n \lambda_j \zeta_j$ where $\sum_{j=0}^n \lambda_j = 1$ for $\lambda_j \in [0, 1]$, and where $\zeta_j \in N_A(a) \cap \partial \mathbb{B}$ for all $j = 0, \dots, n$. Therefore, there exists at least one λ_j which is not zero. Assume that $\lambda_0 \neq 0$ and define $\bar{\lambda} := \sum_{j=1}^n \lambda_j$. It is clear that $\bar{\lambda} \neq 0$, otherwise $\lambda_0 \zeta_0 = 0$ which contradicts that $\lambda_0 \neq 0$ and $\zeta_0 \neq 0$. We deduce that

$$\frac{\lambda_0}{\bar{\lambda}} \zeta_0 + \sum_{j=1}^n \frac{\lambda_j}{\bar{\lambda}} \zeta_j = 0.$$

This is a contradiction with the pointedness of $\text{co } N_A(a)$, by setting $d_1 := \frac{\lambda_0}{\bar{\lambda}} \zeta_0 \in N_A(a) \subset \text{co } N_A(a)$ and $d_2 := \sum_{j=1}^n \frac{\lambda_j}{\bar{\lambda}} \zeta_j \in \text{co } N_A(a)$ (since $\sum_{j=1}^n \frac{\lambda_j}{\bar{\lambda}} = 1$). Indeed, since $d_1 \neq 0$ and $d_1 + d_2 = 0$ then $d_2 \neq 0$.

□

For the proof of Lemma 5.3.2, we shall invoke the following preliminary lemma:

Lemma 5.4.1. Fix $R > 0$. Assume that $\text{int } T_A(z) \neq \emptyset$ for all $z \in \partial A \cap (R+1)\mathbb{B}$. Then we can find positive numbers $\beta, \epsilon_0, \epsilon_1, \dots, \epsilon_k$, points $z_1, \dots, z_k \in \partial A \cap (R+1)\mathbb{B}$, and vectors $\zeta_j \in \text{int } T_A(z_j)$, for $j = 1, \dots, k$, such that

$$(i) \quad \bigcup_{j=1}^k (z_j + \frac{\epsilon_j}{2} \text{int } \mathbb{B}) \supset (\partial A + \epsilon_0 \mathbb{B}) \cap (R+1)\mathbb{B},$$

(ii)

$$\sup_{\substack{\eta \in \text{co}(N_A(z) \cap \partial \mathbb{B}) \\ z \in (z_j + \epsilon_j \mathbb{B}) \cap \partial A}} \zeta_j \cdot \eta < -\beta, \quad \text{for all } j = 1, \dots, k.$$

Proof of Lemma 5.4.1. (i). A is a closed set, then ∂A is also closed, which implies that $\partial A \cap (R+1)\mathbb{B}$ is a compact set. For all $z \in \partial A \cap (R+1)\mathbb{B}$, from the Rockafellar's Characterization of the hypertangent vectors (in the sense of Theorem 1.1.24), since $\text{int } T_A(z) \neq \emptyset$, there exist $\zeta_z \in \text{int } T_A(z)$ and $\epsilon_z > 0$ such that:

$$z' + [0, 2\epsilon_z](\zeta_z + 2\epsilon_z \mathbb{B}) \subset A, \quad \text{for all } z' \in A \cap (z + 2\epsilon_z \mathbb{B}). \quad (5.11)$$

Take the following covering:

$$\partial A \cap (R+1)\mathbb{B} \subset \bigcup_{z \in \partial A \cap (R+1)\mathbb{B}} (z + \frac{\epsilon_z}{2} \text{int } \mathbb{B}).$$

Due to a compactness argument, there exist finite number of points $z_1, \dots, z_k \in \partial A \cap (R+1)\mathbb{B}$ and numbers $\epsilon_1, \dots, \epsilon_k > 0$ (we denote $\epsilon_j := \epsilon_{z_j}$), such that

$$\partial A \cap (R+1)\mathbb{B} \subset \bigcup_{j=1}^k (z_j + \frac{\epsilon_j}{2} \text{int } \mathbb{B}). \quad (5.12)$$

And more precisely, for some $\epsilon_0 > 0$, we have (i).

(ii). Consider now vectors $\zeta_j \in \text{int } T_A(z_j)$ for $j = 1, \dots, k$ associated with points $z_1, \dots, z_k \in \partial A \cap (R+1)\mathbb{B}$ verifying (5.12). Define $\bar{\epsilon} := \min_{j=1, \dots, k} \epsilon_j$. From Rockafellar's Characterization (1.1.24), we obtain in particular

$$z' + [0, 2\bar{\epsilon}](\zeta_j + 2\bar{\epsilon} \mathbb{B}) \subset A, \quad \text{for all } z' \in (z_j + 2\epsilon_j \mathbb{B}) \cap A. \quad (5.13)$$

However, for any $z \in (z_j + \epsilon_j \mathbb{B}) \cap \partial A$, we have

$$(z + \epsilon_j \mathbb{B}) \cap \partial A \subset (z_j + 2\epsilon_j \mathbb{B}) \cap \partial A.$$

Therefore, condition (5.13) is verified for any $x \in (z + \epsilon_j \mathbb{B}) \cap \partial A$.

Consequently

$$x + [0, \bar{\epsilon}](\zeta_j + \bar{\epsilon} \mathbb{B}) \subset A, \quad \text{for all } x \in (z + \epsilon_j \mathbb{B}) \cap A.$$

This implies

$$\zeta_j \in \text{int } T_A(z), \quad \text{for all } z \in (z_j + \epsilon_j \mathbb{B}) \cap \partial A \quad (5.14)$$

We claim that for each $j \in \{1, \dots, k\}$ there exists $\beta_j > 0$ such that

$$\zeta_j \cdot \eta \leq -\beta_j, \quad (5.15)$$

for all $\eta \in \text{co}(N_A(z) \cap \partial \mathbb{B})$, and for all $z \in (z_j + \epsilon_j \mathbb{B}) \cap \partial A$.

Suppose by contradiction that (5.15) is not satisfied. So, there exist a subsequence of points $z_{j_\ell} \in (z_j + \epsilon_j \mathbb{B}) \cap \partial A$ and a subsequence of associated vectors η_{j_ℓ} such that $\eta_{j_\ell} \in \text{co}(N_A(z_{j_\ell}) \cap \partial \mathbb{B})$ and

$$\zeta_j \cdot \eta_{j_\ell} \downarrow 0 \quad \text{as } \ell \rightarrow \infty. \quad (5.16)$$

By compactness, we know that there exist $\bar{\eta}_j \in \mathbb{R}^n$ and $\bar{z}_j \in \mathbb{R}^n$ for each $j = 1, \dots, n$, such that by subsequence extraction (we do not relabel): $\eta_{j_\ell} \rightarrow \bar{\eta}_j$ and $z_{j_\ell} \rightarrow \bar{z}_j$ as $\ell \rightarrow \infty$. Owing to the closure of the graph of $N_A(\cdot)$ (cf. Proposition 1.1.6), we obtain that

$$\bar{\eta}_j \in \text{co}(N_A(\bar{z}_j) \cap \partial \mathbb{B}).$$

where $\bar{z}_j \in (z_j + \epsilon_j \mathbb{B}) \cap \partial A$. Therefore, by (5.16), as $\ell \rightarrow \infty$,

$$\zeta_j \cdot \bar{\eta}_j = 0, \quad \text{for each } j = 1, \dots, k.$$

This is a contradiction because, owing to (5.14), and since $\bar{z}_j \in (z_j + \epsilon_j \mathbb{B}) \cap \partial A$, we have that $\zeta_j \in \text{int } T_A(\bar{z}_j)$. The claim is therefore confirmed.

Taking now $\beta = \min_{j=1, \dots, k} \beta_j$, we conclude that:

$$\sup_{\substack{\eta \in \text{co}(N_A(z) \cap \partial \mathbb{B}) \\ z \in (z_j + \epsilon_j \mathbb{B}) \cap \partial A}} \zeta_j \cdot \eta < -\beta, \quad \text{for all } j = 1, \dots, k.$$

This ends the proof. \square

Proof of Lemma 5.3.2. Since $\bar{x}(\cdot)$ is a minimizer, Lemma 5.3.1 states that $\bar{x}(\cdot)$ is a Lipschitz continuous function. Therefore, there exists $R := |x_0| + \|\dot{\bar{x}}\|_{L^\infty}(1 + T - S)$, such that the following a-priori bound is satisfied:

$$|\bar{x}(t)| \leq R \quad \text{for all } t \in [S, T];$$

namely, $\bar{x}([S, T]) \subset R\mathbb{B} \subset (R + 1)\mathbb{B}$ and

$$|\dot{\bar{x}}(t)| \leq R \quad \text{a.e. } t \in [S, T],$$

where R is the chosen to be the same constant as in Lemma 5.4.1. Therefore, the assertions of Lemma 5.4.1 are valid for all $z \in \partial A \cap \bar{x}([S, T])$ (i.e. when (CQ) is assumed). Therefore, there exist positive numbers $\beta, \epsilon_0, \epsilon_1, \dots, \epsilon_k$, points $z_1, \dots, z_k \in \partial A \cap \bar{x}([S, T])$ and vectors $\zeta_j \in \text{int } T_A(z_j)$, for $j = 1, \dots, k$, such that conditions (i) and (ii) of Lemma 5.4.1 are satisfied. We can always arrange $\epsilon_0, \epsilon_1, \dots, \epsilon_k$ in such a way that $\epsilon_0 < \frac{1}{2} \min_{j=1, \dots, k} \frac{\epsilon_j}{2}$. Set $\varepsilon := \frac{\epsilon_0}{R}$ and

$$C := k \min_{j=1, \dots, k} |\zeta_j|.$$

Notice that $\dot{\bar{x}}(t)$ might be non defined on a zero-measure set, so we consider:

$$[\dot{\bar{x}}(t)] := \begin{cases} \dot{\bar{x}}(t) & \text{where } \dot{\bar{x}}(t) \text{ is defined} \\ 0 & \text{elsewhere.} \end{cases}$$

To simplify notation, we write $\dot{\bar{x}}(t)$ for the function $[\dot{\bar{x}}(t)]$. Define the function $v(\cdot) : [S, T] \rightarrow \mathbb{R}^n$ as follows

$$v(t) := \dot{\bar{x}}(t) + \sum_{j=1}^k \chi_{z_j + \frac{\epsilon_j}{2} \text{int } \mathbb{B}}(\bar{x}(t)) \zeta_j, \quad t \in [S, T]$$

where we denote by χ_Y the characteristic function of the subset $Y \subset \mathbb{R}^n$, defined by

$$\chi_Y(x) := \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $v(\cdot)$ is a measurable function (being the sum of two measurable functions).

We start by proving (i):

$$|v(t) - \dot{\bar{x}}(t)| = \left| \sum_{j=1}^k \chi_{z_j + \frac{\epsilon_j}{2} \text{int } \mathbb{B}}(\bar{x}(t)) \zeta_j \right| \leq k \max_{j=1, \dots, k} |\zeta_j| =: C, \text{ for a.e. } t \in [S, T].$$

Furthermore, $|v(t)| \leq |v(t) - \dot{\bar{x}}(t)| + |\dot{\bar{x}}(t)| \leq C + R =: C_1$, for a.e. $t \in [S, T]$, which implies (i).

To prove (ii) we take any $\tau \in \{\sigma \in [S, T] : \bar{x}(\sigma) \in \partial A\}$. Then, from the compactness argument of Lemma 5.4.1, there exists $j_0 \in \{1, \dots, k\}$ such that $\bar{x}(\tau) \in (z_{j_0} + \frac{\epsilon_{j_0}}{2} \text{int } \mathbb{B}) \cap \partial A$ for a certain $z_{j_0} \in \partial A \cap \bar{x}([S, T])$.

We claim that $\bar{x}(s) \in z_{j_0} + \epsilon_{j_0} \mathbb{B}$, for a.e. $s \in (\tau - \varepsilon, \tau]$. Indeed, owing to the Lipschitz continuity of $\bar{x}(\cdot)$, we obtain

$$|\bar{x}(\tau) - \bar{x}(s)| \leq R|s - \tau| \leq R\varepsilon = \epsilon_0.$$

Therefore, from the choice of ϵ_0 such that $\epsilon_0 < \frac{1}{2} \min_{j=1, \dots, k} \frac{\epsilon_j}{2}$, we deduce that

$$|\bar{x}(\tau) - \bar{x}(s)| < \frac{\epsilon_{j_0}}{2},$$

which implies that $\bar{x}(s) \in z_{j_0} + \epsilon_{j_0} \mathbb{B}$. Therefore, the claim is confirmed. Similarly, $\bar{x}(t) \in z_{j_0} + \epsilon_{j_0} \mathbb{B}$, for a.e. $t \in (\tau - \varepsilon, \tau]$.

Moreover, if $\bar{x}(t) \in z_j + \frac{\epsilon_j}{2} \text{int } \mathbb{B}$ for a certain $j \in \{1, \dots, k\}$, then since $|t - s| < 2\varepsilon$, it is straightforward to see that $\bar{x}(s) \in z_j + \epsilon_j \mathbb{B}$, owing to the particular choice of ϵ_0 . (Bearing in mind that we shall consider only points $\bar{x}(s)$ belonging to ∂A , otherwise $\text{co}(N_A(\bar{x}(s)) \cap \partial \mathbb{B})$ is an empty set.) As a consequence, making use of Lemma 5.4.1 (ii), we have that

$$\begin{aligned} & \sup_{\eta \in \text{co}(N_A(\bar{x}(s)) \cap \partial \mathbb{B})} (v(t) - \dot{\bar{x}}(t)) \cdot \eta \leq \\ & \sum_{j=1}^k \sup_{\eta \in \text{co}(N_A(\bar{x}(s)) \cap \partial \mathbb{B})} \chi_{z_j + \frac{\epsilon_j}{2} \text{int } \mathbb{B}}(\bar{x}(t)) \zeta_j \cdot \eta < -\beta. \end{aligned}$$

To prove (iii), we take $x_0 \in \partial A \cap (R+1)\mathbb{B}$. The compactness argument of Lemma 5.4.1 ensures the existence of $j_0 \in \{1, \dots, k\}$ such that $x_0 \in (z_{j_0} + \frac{\epsilon_{j_0}}{2} \text{int } \mathbb{B}) \cap \partial A$ for $z_{j_0} \in \partial A \cap \bar{x}([S, T])$. Choose $\rho := \epsilon_0$ of Lemma 5.4.1. We show that, if $x \in (x_0 + \rho\mathbb{B}) \cap \partial A$, then $x \in (z_{j_0} + \epsilon_{j_0}\mathbb{B}) \cap \partial A$. Indeed, $|x - z_{j_0}| \leq |x - x_0| + |x_0 - z_{j_0}| \leq \rho + \frac{\epsilon_{j_0}}{2}$. Then, $|x - z_{j_0}| \leq \epsilon_{j_0}$. Furthermore, it is easy to check that $\bar{x}(t) \in z_{j_0} + \epsilon_{j_0}\mathbb{B}$, for $t \in [S, S + \varepsilon)$. Moreover, if $\bar{x}(t) \in z_j + \frac{\epsilon_j}{2} \text{int } \mathbb{B}$ for some $j \in \{1, \dots, k\}$, then

$$|\bar{x}(t) - x| \leq |\bar{x}(t) - x_0| + |x_0 - x| \leq \varepsilon R + \rho \leq 2\varepsilon_0 < \frac{\epsilon_j}{2}.$$

It follows that $x \in (z_j + \epsilon_j\mathbb{B}) \cap \partial A$. Then, for $t \in [S, S + \varepsilon)$ and $\bar{x}(S) = x_0 \in \partial A$, Lemma 5.4.1 (ii) implies that

$$\sup_{\substack{\eta \in \text{co}(N_A(x) \cap \partial\mathbb{B}) \\ x \in (x_0 + \rho\mathbb{B}) \cap \partial A}} (v(t) - \dot{\bar{x}}(t)) \cdot \eta \leq \sum_{j=1}^k \sup_{\substack{\eta \in \text{co}(N_A(x) \cap \partial\mathbb{B}) \\ x \in (x_0 + \rho\mathbb{B}) \cap \partial A}} \chi_{z_j + \frac{\epsilon_j}{2} \text{int } \mathbb{B}}(\bar{x}(t)) \zeta_j \cdot \eta < -\beta.$$

□

6

Construction of Feasible Trajectories When the Classical Inward Pointing Condition Is Violated

In this chapter, we present first a viability result for the construction of feasible trajectories for a control system $\dot{x} = f(x, u)$ subject to a state constraint $h(x) \leq 0$, in the case when the *classical inner pointing vector field condition* fails to hold true. More precisely, we propose a new approach, based on a particular construction of controls, which permits the corresponding vector field to rotate in a suitable sense and with sufficient intensity in order to point inside the state constraint. This viability result is the first step for a stronger result: neighboring feasible trajectories satisfying some $W^{1,1}$ -estimates. The particular case of the *Brockett nonholonomic integrator* is treated, and two examples are provided to show that, even a construction of neighboring feasible trajectories is still possible (globally in time), but with a nonlinear $W^{1,1}$ -estimate.

Dans ce chapitre, nous présentons d'abord un résultat de viabilité pour la construction de trajectoires admissibles pour un système de contrôle $\dot{x} = f(x, u)$ auquel est associée une contrainte d'état $h(x) \leq 0$, dans le cas où la condition dite *classical inner pointing vector field condition* est violée. Plus précisément, nous proposons une nouvelle approche, basée sur une construction particulière de contrôles, qui permet au champ vectoriel correspondant de faire une rotation dans un sens approprié et avec une intensité suffisante pour pointer à l'intérieur de la contrainte d'état. Ce résultat de viabilité est la première étape pour un résultat plus fort: des trajectoires admissibles voisines satisfaisant des $W^{1,1}$ -estimations. Le cas particulier de l'intégrateur non holonomique de Brockett est traité et deux exemples sont fournis pour montrer que, même une construction de trajectoires admissibles voisines (globale par rapport au temps) est encore possible, mais avec une $W^{1,1}$ -estimation non linéaire.

*“Il dubbio è padre dell’invenzione.”
“Doubt is the father of invention.”*

— Galileo Galilei

6.1 Motivation

For a control system $\dot{x} = f(x, u)$ to which we impose a state constraint formulated as a scalar inequality function of class C^2 ($h(x) \leq 0$), the so-called *inner pointing vector field condition* states the existence of a constant $\delta > 0$ so that, at each point x of the constraint's boundary (i.e. $h(x) = 0$), one can choose a control u verifying

$$\langle \nabla h(x), f(x, u) \rangle < -\delta, \quad (6.1)$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathbb{R}^n . This type of condition, as already discussed in Chapter 2, is sufficient for the existence of neighboring feasible trajectories verifying $W^{1,1}$ -linear estimates. In turn, this is a crucial hypothesis for some nice properties valid in the unconstrained system to hold also in the presence of an actual constraint. Instances of such properties are: semicontinuity of the solution map, uniqueness of a solution for associated Hamilton-Jacobi boundary value problems, nondegeneracy issues in the Pontryagin Maximum Principle and in the optimality conditions in the Extended Euler-Lagrange form. The first key feature of this chapter is to deal with state constrained differential equations parametrized by a control, and to propose a new construction of controls, which is still sufficient for the feasible trajectories to exist (this is a viability result), in cases when the (uniform) inward pointing condition (6.1) is violated. This construction constitutes the primary step for a neighboring feasible trajectory result, which we are able to establish for two particular examples (cf. Section 6.4). The main difference between our construction of viable trajectories and earlier works (cf. [69]), is that we are able to locally construct a trajectory which enters inside the state constraint set, when (6.1) is violated, but 'slower' than in the case when (6.1) is satisfied. This innovative topic to deal with viability results, in the first place, and consequently with neighboring feasible trajectories satisfying some estimates with respect to the $W^{1,1}$ -norm, initiated in 2015-2016 with an idea raised by G. Colombo¹ and F. Rampazzo². This is motivated by two simple examples presented below:

• **Motivational example 1: Brockett Nonholonomic Integrator:** Consider the (affine) control system in \mathbb{R}^3

$$\dot{x} = f_1(x)u_1 + f_2(x)u_2 = f(x, u), \quad u = (u_1, u_2) \text{ is a control s.t. } |u| \leq 1$$

where f_1 and f_2 are two vector fields given by

$$f_1 = \begin{pmatrix} 1 \\ 0 \\ -x_2 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 \\ 1 \\ x_1 \end{pmatrix}.$$

We can easily see that

$$[f_1, f_2] = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix},$$

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where $[f_1, f_2]$ is another vector field which denotes the Lie Bracket of f_1 and f_2 and is defined as

$$[f_1, f_2] := Df_2f_1 - Df_1f_2.$$

Impose the flat state constraint

$$x \in A \quad \text{where} \quad A := \{x = (x_1, x_2, x_3) : h(x) \leq 0\},$$

and $h : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that $h(x) = x_3 - 1$. Let $F(x) := f(x, U)$ be the set of velocities and U the set of controls (i.e. $U := \{u : |u| \leq 1\}$). It is clear (cf. Figure 6.1 below) that the vector fields involved in this example fail to satisfy the inward pointing condition (6.1) at a point $x_0 = (0, 0, 1) \in \partial A$. Indeed, observe that at the point $x_0 = (0, 0, 1) \in \partial A$, we have

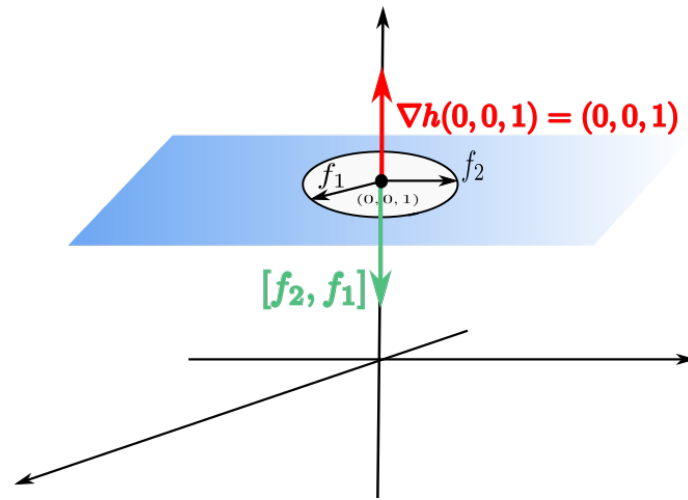


Figure 6.1 – Violated Inward Pointing Condition (6.1)

$$F(x_0) = D^1 \times \{0\}, \quad \text{where } D^1 \text{ denotes the unit disk in } \mathbb{R}^2.$$

By consequence, for any u such that $|u| \leq 1$

$$\langle f(x_0, u), \nabla h(x_0) \rangle = 0.$$

(The same situation occurs at any point belonging to the x_3 -axis.) On the contrast, while $[f_2, f_1] \notin F(x_0)$, one has

$$\langle [f_2, f_1](x), \nabla h(x) \rangle < 0,$$

for all $x \in \mathbb{R}^3$, in particular for $x = x_0$. This interplay between the dynamics (through Lie brackets) and the boundary target ∂A will be crucial for the construction of a feasible trajectory as will be shown in Theorem 6.2.1 below.

• **Motivational example 2:** Consider the vector field $f(x_1, x_2) = (1, 0)$ in \mathbb{R}^2 , and consider the dynamics given by

$$(\dot{x}_1, \dot{x}_2) = f(x_1, x_2)u, \quad u \in [-1, 1].$$

Let $A := \{(x_1, x_2) : x_2 \leq x_1^2\}$ be a state constraint set in \mathbb{R}^2 . Consider the family of trajectories

$$\gamma_\varepsilon(t) := (-\varepsilon, \varepsilon^2) + (t, 0), \quad \varepsilon \geq 0.$$

For $\varepsilon > 0$, the optimal neighboring feasible trajectory is the constant $(-\varepsilon, \varepsilon^2)$. Observe, however, that for $\varepsilon = 0$,

$$\gamma_0(t) = (t, 0)$$

and the trajectory $\gamma_0(\cdot)$ is feasible. But, no construction of neighboring feasible trajectories with a reasonable estimate with the deviation can be expected. This is because the first (uniform) inward pointing condition, but also some controllability are missing: there is no way to control the x_1 -coordinate.

6.2 A Local Viability Result

In this section, we deal with state constrained (affine) control systems in \mathbb{R}^n , and we propose a viability result when the inward pointing condition (6.1) fails to hold true. This is based on a particular construction of controls, and on a set of assumptions depending on the geometrical structure of the problem. Consider the following (affine) control system in \mathbb{R}^n :

$$\dot{x} = f(x, u) = f_1(x)u_1 + f_2(x)u_2, \quad x(0) = x_0 \quad (6.2)$$

where the vector fields f_1, f_2 and the controls u_1, u_2 satisfy the following assumptions:

(V.1) $f_1, f_2 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are of class C^2 , bounded and with common Lipschitz constant k (to be determined later) and Df_1, Df_2 are Lipschitz;

(V.2) $u_1, u_2 \in \{u = (u_1, u_2) \in \mathbb{R}^2 : u_1^2 + u_2^2 \leq 1\}$.

We impose a pathwise state constraint ($x \in A$) formulated as a scalar inequality constraint

$$A := \{x \in \mathbb{R}^n : h(x) \leq 0\}$$

where $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is of class C^2 and $D^2h(x)$ is bounded for all $x \in \mathbb{R}^n$. The set A being the closure of an open set with a $C^{1,1}$ boundary (i.e. $\nabla h(x) \neq 0$ at all x such that $h(x) = 0$).

We denote by \mathcal{S} the set of points for which the inward pointing condition (6.1) is not verified; namely

$$\mathcal{S} := \{x \in \mathbb{R}^n : \min_{|u| \leq 1} \langle \nabla h(x), f(x, u) \rangle = 0\}, \quad (6.3)$$

The set \mathcal{S} will be called the ‘singular set’.

For simplicity, we will establish the statement of the viability result for the case of a flat constraint, namely when

$$h(x) := \zeta \cdot x + \zeta_0 \quad \text{with } \zeta \neq 0. \quad (6.4)$$

Theorem 6.2.1 (Viability result [35]). *Assume that f_1, f_2 and u_1, u_2 satisfy hypotheses (V.1) and (V.2) and consider the flat state constraint in the form of (6.4). Let $x_0 \in \mathcal{S} \cap \partial A$ be such that $f_1(x_0) \neq 0$ and $f_2(x_0) \neq 0$. Assume moreover that $\zeta \cdot [f_2, f_1](x_0) > 0$. Then there exist $K > 0$, $\tau > 0$, $\omega = \omega(\tau)$, and $\varphi \in [0, 2\pi)$, such that if the following assumptions are verified:*

(C.1) φ is chosen such that $f_1(x_0) \cos \varphi + f_2(x_0) \sin \varphi$ is orthogonal to ζ ;

(C.2) τ small enough so that for all $t \in [0, \tau]$

$$\zeta \cdot \left(\cos\left(\varphi + \frac{\omega t}{2}\right) f_1(y(t)) + \sin\left(\varphi + \frac{\omega t}{2}\right) f_2(y(t)) \right) \leq 0;$$

(C.3) $\zeta \cdot \left(Df_1(x_0)f_2(x_0) + Df_2(x_0)f_1(x_0) \right) \sin(2\varphi + \omega t) \geq 0$ for all $t \in [0, \tau]$;

(C.4) $\zeta \cdot Df_1(x_0)f_1(x_0) \geq 0$ and $\zeta \cdot Df_2(x_0)f_2(x_0) \geq 0$;

then we can find a trajectory $y(\cdot)$ solution of (6.2) corresponding to particular controls $u_i^{\varphi, \omega}(t)$, $i = 1, 2$, such that

$$h(y(t)) \leq -\frac{Kt^3}{\tau} \quad \text{for all } t \in [0, \tau], \quad (6.5)$$

Proof. Set for $\ell, m = 1, 2$

$$A_\ell(x) := \nabla h(x) \cdot f_\ell(x) \quad (6.6)$$

$$A_{\ell m}(x) := \frac{1}{2} \nabla h(x) \cdot [f_m, f_\ell](x), \quad (6.7)$$

$$S_{\ell m}(x) := \frac{1}{2} \nabla h(x) \cdot \left(Df_m(x)f_\ell(x) + Df_\ell(x)f_m(x) \right). \quad (6.8)$$

Observe that for all $\ell, m = 1, 2$,

$$S_{\ell m}(x) \text{ is symmetric, while } A_{\ell m}(x) \text{ is antisymmetric }^3, \text{ and} \quad (6.9)$$

$$S_{\ell m}(x) + A_{\ell m}(x) = \nabla h(x) \cdot Df_\ell(x)f_m(x). \quad (6.10)$$

We fix now $\tau > 0$ and set

$$\omega := \frac{\pi}{k\tau}$$

where $k \in \mathbb{Z}$ is the common Lipschitz constant of f_1 and f_2 to be determined later. Let $\varphi \in [0, 2\pi)$, to be also determined later. Define the controls $u_1(t) = \cos t$ and $u_2(t) = \sin t$, and for $i = 1, 2$,

$$u_i^{\varphi, \omega}(t) := u_i(\varphi + \omega t) \quad \text{and} \quad U_i^{\varphi, \omega}(t) := \int_0^t u_i^{\varphi, \omega}(s) ds.$$

³This is owing to the antisymmetry of the Lie Bracket operator, i.e. $[X, Y] = -[Y, X]$ for two vector fields X and Y of class C^1 .

Denote by $y(\cdot)$ the solution at the time interval $[0, \tau]$ of the control system (6.2), corresponding to the controls $u_i^{\varphi, \omega}$, $i = 1, 2$. Then, for all $t \in [0, \tau]$,

$$h(y(t)) - h(x_0) = \int_0^t \sum_{\ell=1}^2 \nabla h(y(s)) \cdot f_\ell(y(s)) u_\ell^{\varphi, \omega}(s) ds.$$

An integration by parts implies that

$$h(y(t)) - h(x_0) = \sum_{\ell=1}^2 \nabla h(y(t)) \cdot f_\ell(y(t)) U_\ell^{\varphi, \omega}(t) - \int_0^t \sum_{\ell=1}^2 U_\ell^{\varphi, \omega}(s) \frac{d}{ds} (\nabla h(y(s)) \cdot f_\ell(y(s))) ds.$$

Expliciting the derivative term and bearing in mind that $D^2h(y) \equiv 0$ (for the flat constraint case), we obtain

$$\begin{aligned} h(y(t)) - h(x_0) &= \sum_{\ell=1}^2 \nabla h(y(t)) \cdot f_\ell(y(t)) U_\ell^{\varphi, \omega}(t) \\ &\quad - \int_0^t \sum_{\ell=1}^2 U_\ell^{\varphi, \omega}(s) \left[\nabla h(y(s)) \cdot \sum_{m=1}^2 Df_\ell(y(s)) f_m(y(s)) u_m^{\varphi, \omega}(s) \right] ds. \end{aligned}$$

Therefore, recalling (6.10), and computing $U_\ell^{\varphi, \omega}(t)$ for $\ell = 1, 2$, we have

$$\begin{aligned} h(y(t)) - h(x_0) &= \frac{\sin(\varphi + \omega t) - \sin \varphi}{\omega} \nabla h(y(t)) \cdot f_1(y(t)) \\ &\quad + \frac{\cos \varphi - \cos(\varphi + \omega t)}{\omega} \nabla h(y(t)) \cdot f_2(y(t)) \\ &\quad - \sum_{\ell, m=1}^2 \int_0^t U_\ell^{\varphi, \omega}(s) u_m^{\varphi, \omega}(s) [S_{\ell m}(y(s)) + A_{\ell m}(y(s))] ds. \end{aligned}$$

Via some trigonometric formulas, we can write

$$\begin{aligned} &h(y(t)) - h(x_0) \\ &= \frac{2 \cos\left(\varphi + \frac{\omega t}{2}\right) \sin \frac{\omega t}{2}}{\omega} \nabla h(y(t)) \cdot f_1(y(t)) + \frac{2 \sin\left(\varphi + \frac{\omega t}{2}\right) \sin \frac{\omega t}{2}}{\omega} \nabla h(y(t)) \cdot f_2(y(t)) \\ &\quad - \sum_{\ell, m=1}^2 (S_{\ell m}(x_0) + A_{\ell m}(x_0)) \int_0^t U_\ell^{\varphi, \omega}(s) u_m^{\varphi, \omega}(s) ds \\ &\quad - \sum_{\ell, m=1}^2 \int_0^t [S_{\ell m}(y(s)) - S_{\ell m}(x_0) + A_{\ell m}(y(s)) - A_{\ell m}(x_0)] U_\ell^{\varphi, \omega}(s) u_m^{\varphi, \omega}(s) ds. \end{aligned}$$

For simplicity, we shall write $h(y(t)) - h(x_0)$ as the sum of three terms (I),(II) and (III), where

$$\begin{aligned} \text{(I)} &:= \frac{2 \cos\left(\varphi + \frac{\omega t}{2}\right) \sin \frac{\omega t}{2}}{\omega} \nabla h(y(t)) \cdot f_1(y(t)) + \frac{2 \sin\left(\varphi + \frac{\omega t}{2}\right) \sin \frac{\omega t}{2}}{\omega} \nabla h(y(t)) \cdot f_2(y(t)) \\ \text{(II)} &:= - \sum_{\ell, m=1}^2 (S_{\ell m}(x_0) + A_{\ell m}(x_0)) \int_0^t U_{\ell}^{\varphi, \omega}(s) u_m^{\varphi, \omega}(s) ds \\ \text{(III)} &:= - \sum_{\ell, m=1}^2 \int_0^t \left[S_{\ell m}(y(s)) - S_{\ell m}(x_0) + A_{\ell m}(y(s)) - A_{\ell m}(x_0) \right] U_{\ell}^{\varphi, \omega}(s) u_m^{\varphi, \omega}(s) ds. \end{aligned}$$

We now explicit each term apart. We start with (II). Owing to a simple computation and making use of property (6.9), we obtain

$$\begin{aligned} (S_{11}(x_0) + A_{11}(x_0)) \int_0^t U_1^{\varphi, \omega}(s) u_1^{\varphi, \omega}(s) ds &= \zeta \cdot Df_1(x_0) f_1(x_0) \frac{2 \cos^2\left(\varphi + \frac{\omega t}{2}\right) \sin^2 \frac{\omega t}{2}}{\omega^2}; \\ (S_{22}(x_0) + A_{22}(x_0)) \int_0^t U_2^{\varphi, \omega}(s) u_2^{\varphi, \omega}(s) ds &= \zeta \cdot Df_2(x_0) f_2(x_0) \frac{2 \sin^2\left(\varphi + \frac{\omega t}{2}\right) \sin^2 \frac{\omega t}{2}}{\omega^2}; \\ (S_{12}(x_0) + A_{12}(x_0)) \int_0^t U_1^{\varphi, \omega}(s) u_2^{\varphi, \omega}(s) ds + (S_{21}(x_0) + A_{21}(x_0)) \int_0^t U_2^{\varphi, \omega}(s) u_1^{\varphi, \omega}(s) ds \\ &= \frac{1}{2} \zeta \cdot [f_2, f_1](x_0) \left(\frac{\omega t - \sin \omega t}{\omega^2} \right) + \frac{1}{2} \zeta \cdot \left(Df_2(x_0) f_1(x_0) + Df_1(x_0) f_2(x_0) \right) \frac{2 \sin^2 \frac{\omega t}{2} \sin(2\varphi + \omega t)}{\omega^2}. \end{aligned}$$

Denote by Δ_0 the Lipschitz constant of $S_{\ell m}(y(\cdot)) + A_{\ell m}(y(\cdot))$, $\ell, m = 1, 2$, which can be estimated from the data, independently of the trajectory $y(\cdot)$ (using also the boundedness of f_1, f_2, Df_1 and Df_2). Then, owing to assumption (V.1), we obtain

$$|\text{(III)}| \leq \Delta_0 \left| \sum_{\ell, m=1}^2 \int_0^t s u_m^{\varphi, \omega}(s) U_{\ell}^{\varphi, \omega}(s) \right| ds.$$

Referring the reader to the computations in the *Appendix* of this chapter (made by the Mathematica Software), we obtain

$$|\text{(III)}| \leq \Delta_1 t^3,$$

where Δ_1 can be computed explicitly, and, for $t \in [0, \tau]$, it is independent of φ and ω . From all these computations, we can deduce that

$$\begin{aligned} &h(y(t)) - h(x_0) \\ &\leq \frac{2 \sin \frac{\omega t}{2}}{\omega} \zeta \cdot \left(\cos\left(\varphi + \frac{\omega t}{2}\right) f_1(y(t)) + \sin\left(\varphi + \frac{\omega t}{2}\right) f_2(y(t)) \right) \\ &\quad - \frac{2 \sin^2 \frac{\omega t}{2}}{\omega^2} \zeta \cdot \left(Df_1(x_0) f_1(x_0) \cos^2\left(\varphi + \frac{\omega t}{2}\right) + Df_2(x_0) f_2(x_0) \sin^2\left(\varphi + \frac{\omega t}{2}\right) \right) \\ &\quad - \zeta \cdot \left(Df_2(x_0) f_1(x_0) + Df_1(x_0) f_2(x_0) \right) \frac{\sin^2 \frac{\omega t}{2} \sin(2\varphi + \omega t)}{\omega^2} \\ &\quad + \frac{1}{2} \zeta \cdot [f_2, f_1](x_0) \frac{\sin \omega t - \omega t}{\omega^2} + \Delta_1 t^3. \end{aligned}$$

Recalling the choice of $\omega := \frac{\pi}{k\tau}$ and taking into account assumptions (C.3)-(C.4), we obtain

$$h(y(t)) - h(x_0) \leq t \zeta \cdot \left(\cos\left(\varphi + \frac{\pi t}{2k\tau}\right) f_1(y(t)) + \sin\left(\varphi + \frac{\pi t}{2k\tau}\right) f_2(y(t)) \right) - \Delta_2 \zeta \cdot [f_2, f_1](x_0) \frac{t^3}{\tau} + \Delta_1 t^3,$$

where $\Delta_2 > 0$ is a constant independent of φ and ω for $t \in [0, \tau]$. Our aim is to provide sufficient conditions under which we can guarantee that

$$h(y(t)) \leq 0 \quad \text{for all } t \in [0, \tau].$$

If we choose $k = 1$ and taking into account assumptions (C.1)-(C.2) and the fact that

$$\zeta \cdot [f_2, f_1](x_0) > 0,$$

we deduce that

$$h(y(t)) \leq -K \frac{t^3}{\tau} \quad \text{for all } t \in [0, \tau], \quad (6.11)$$

for some constant $K > 0$. This concludes the proof. \square

Remark 6.2.2. 1) Inequality (6.11) is weaker than the one established in [69]. The main difference is the rate with which the constructed trajectory goes inside the state constraint set: indeed, our construction yields a trajectory which enters the constraint ‘slower’ than the one constructed in [69] (where the rate is t instead of t^3). Moreover, it is made of interior points provided ω (called the angular velocity or the spin) is positive and sufficiently large. That is, the vector field $f(x, u(t))$ has to rotate (in a suitable verse: this is the meaning of the positivity of ω) and with a sufficient intensity.

2) For the general state constraint set expressed as

$$A := \{x \in \mathbb{R}^n : h(x) \leq 0\},$$

where h is of class C^2 and $D^2h(x)$ is bounded for all $x \in \mathbb{R}^n$, the same theorem holds true, but more sophisticated conditions on the data should be imposed in order to ensure that inequality (6.5) holds true. In this case, we shall set, for $\ell, m = 1, 2$

$$S_{\ell m}(x) := f_\ell(x) \cdot D^2h(x) f_m(x) + \frac{1}{2} \nabla h(x) \cdot (Df_m(x) f_\ell(x) + Df_\ell(x) f_m(x)).$$

Instance for such constraint in \mathbb{R}^3 is the paraboloid

$$x \rightarrow h_\alpha(x) := x_3 + c|x_1^2 + x_2^2|^\alpha,$$

for $\alpha \geq 1$ and $c > 0$. Observe that the constant c , that measures the curvature of the graph of the state constraint set A , may be allowed to be arbitrarily large: of course, the larger c , the larger is the spin ω needed to enter inside the state constraint A .

The next example shows that the Brockett nonholonomic integrator, subject to a flat state constraint, verifies the assumptions imposed in Theorem 6.2.1, and consequently a viability result holds in this case. But here, a clockwise rotation (rather than counter-clockwise as established in Theorem 6.2.1) is required in order to construct the feasible trajectory.

Example 6.2.3. We recall the Brockett nonholonomic integrator in \mathbb{R}^3 :

$$\dot{x} = f_1(x)u_1 + f_2(x)u_2, \quad u = (u_1, u_2) \text{ is a control s.t. } |u| \leq 1$$

where f_1 and f_2 are two vector fields given by

$$f_1 = \begin{pmatrix} 1 \\ 0 \\ -x_2 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 \\ 1 \\ x_1 \end{pmatrix}.$$

Impose the flat state constraint

$$x \in A \quad \text{where } A := \{x = (x_1, x_2, x_3) : h(x) = x_3 \leq 0\}.$$

Following the same procedure as in the proof of Theorem 6.2.1, we construct two controls

$$u_1^{\varphi, \omega}(t) = -\cos(\varphi + \omega t) \quad \text{and} \quad u_2^{\varphi, \omega}(t) = \sin(\varphi + \omega t).$$

The minus sign, which appears in the expression of $u_1^{\varphi, \omega}(t)$, will be crucial in order to construct the feasible trajectory. We will see how shortly. If we denote by $y(\cdot)$ the trajectory corresponding to the controls $u_1^{\varphi, \omega}(t)$ and $u_2^{\varphi, \omega}(t)$ (such that $y(0) = (0, 0, 0)$), we can write, owing to a simple computation that:

$$y_1(t) = \frac{\sin \varphi - \sin(\varphi + \omega t)}{\omega}$$

$$y_2(t) = \frac{\cos \varphi - \cos(\varphi + \omega t)}{\omega}.$$

Therefore, it is easy to check that (I) ≤ 0 . Here, we use the fact that

$$\nabla h(y(t)) \cdot \left(\cos\left(\varphi + \frac{\omega t}{2}\right) f_1(y(t)) + \sin\left(\varphi + \frac{\omega t}{2}\right) f_2(y(t)) \right) \leq 0$$

(and consequently, assumption (C.2) is verified (for all $t \in [0, \frac{\pi}{2\omega}]$ and $\varphi \in [0, \frac{\pi}{4}]$). Clearly

assumption (C.1) is verified. Moreover, since $Df_1 f_1 = Df_2 f_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (and consequently

assumption (C.4) is verified), and $Df_2 f_1 + Df_1 f_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (and consequently assumption (C.3)

is verified), the expression of (II) simplifies to:

$$(II) = -\frac{1}{2} \nabla h(x_0) \cdot [f_2, f_1](x_0) \left(\frac{-\omega t + \sin \omega t}{\omega^2} \right).$$

By Taylor expansion, we obtain that

$$(II) = \nabla h(x_0) \cdot [f_2, f_1](x_0) \frac{\omega}{12} t^3.$$

Since the chosen controls corresponds to a clockwise rotation, the condition that $\nabla h \cdot [f_2, f_1](x_0)$ must verify, should be reversed, i.e. $\nabla h(x_0) \cdot [f_2, f_1](x_0) < 0$. This is true for the Brockett Nonholonomic Integrator because $\nabla h(x_0) \cdot [f_2, f_1](x_0) = -1$. All the assumptions of Theorem 6.2.1 are therefore satisfied and the constructed trajectory $y(\cdot)$ turns out to be feasible for all $t \in [0, \tau]$ (where $\tau := \frac{\pi}{2\omega}$).

6.3 Brockett Nonholonomic Integrator with Flat Constraint: Local Construction

In this section, we will present the construction of feasible trajectories for a particular control system, the so-called *Brockett nonholonomic integrator* in the presence of a given trajectory possibly violating the state constraint. The approach is based on the viability result established in Theorem 6.2.1. More precisely, we will consider a reference trajectory $x(\cdot)$ which violates the state constraint at some time τ_1 , where the classical inward pointing condition is not satisfied (here $x(\tau_1)$ plays the role of x_0 in the statement of Theorem 6.2.1). And we will use similar construction of controls (i.e. the corresponding vector field has to rotate), in order to construct a trajectory with interior points, which enters the constraint set with a rate $(t - \tau_1)^3$ (like in the statement of Theorem 6.2.1), and stays inside for a small time (i.e. locally feasible). The time for which the constructed trajectory is feasible will depend on the extent the reference trajectory $x(\cdot)$ violates the state constraint.

Fix $T > 0$. Consider the (affine) control system in \mathbb{R}^3 already presented in section 6.1:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) = f_1(x(t))u_1(t) + f_2(x(t))u_2(t), & \text{a.e. } t \in [0, T] \\ x(0) = (x_1^0, x_2^0, x_3^0) = x_0 \\ u(t) = (u_1(t), u_2(t)), |u(t)| \leq 1 \end{cases}$$

where

$$f_1(x) = \begin{pmatrix} 1 \\ 0 \\ -x_2 \end{pmatrix}, \quad f_2(x) = \begin{pmatrix} 0 \\ 1 \\ x_1 \end{pmatrix}.$$

Equivalently, for a.e. $t \in [0, T]$

$$(BRO) \quad \begin{cases} \dot{x}_1(t) = u_1(t) & x_1(0) = x_1^0 \\ \dot{x}_2(t) = u_2(t) & x_2(0) = x_2^0 \\ \dot{x}_3(t) = -x_2(t)u_1(t) + x_1(t)u_2(t) & x_3(0) = x_3^0. \end{cases}$$

We impose a flat state constraint given by $x(t) \in A$ for all $t \in [0, T]$, where

$$A := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \leq 0\}.$$

It is clear that in this case the singular set (where the classical inward pointing condition (6.1) is violated) coincides with the x_3 -axis, namely:

$$\mathcal{S} = \{x \in \mathbb{R}^3 : x_1 = x_2 = 0\}.$$

Let $T > 0$. Define, for a trajectory $x(\cdot) = (x_1(\cdot), x_2(\cdot), x_3(\cdot))$ of (BRO), the extent of violation of the state constraint A

$$\varepsilon := \max_{t \in [0, T]} |x_3(t)|.$$

Proposition 6.3.1 (Brockett Nonholonomic Integrator [35]). *Let $x(\cdot)$ be a reference trajectory (solution of (BRO)) corresponding to the controls $(u_1(\cdot), u_2(\cdot))$ such that $x(0) \in A$ (i.e. $x_3^0 \leq 0$). Let $\tau_1 := \inf\{t \in [0, T] : x_3(t) > 0\}$. Then there exist a constant $K_1 > 0$, a time $\tau(\varepsilon) > 0$ (with $\tau(0+) = \tau(0) = 0$) and two particular controls $v_1(\cdot), v_2(\cdot)$ such that the corresponding trajectory $y(\cdot)$ satisfies the following properties:*

$$\begin{cases} y(t) = x(t) & \text{for all } t \in [0, \tau_1] \\ y_3(t) < 0 & \text{for all } t \in (\tau_1, \tau_1 + \tau(\varepsilon)] \\ y_3(\tau_1 + \tau(\varepsilon)) = -K_1 \tau^2(\varepsilon). \end{cases} \quad (6.12)$$

Proof. Let $x(\cdot)$ be a solution of (BRO). Of course, for all $t \in [0, T]$, we have

$$\begin{aligned} x_1(t) &= x_1^0 + \int_0^t u_1(s) ds, & x_2(t) &= x_2^0 + \int_0^t u_2(s) ds \\ x_3(t) &= x_3^0 + \int_0^t -x_2(s)u_1(s) + x_1(s)u_2(s) ds, & x_3^0 &\leq 0. \end{aligned}$$

We define $\tau_1 := \inf\{t \in [0, T] : x_3(t) > 0\}$, i.e. τ_1 corresponds to the first time the reference trajectory $x(\cdot)$ leaves the state constraint A . For some $\eta > 0$, two possible situations may occur:

Case (a): $x_1^2(\tau_1) + x_2^2(\tau_1) \geq \eta^2$;

Case (b): $x_1^2(\tau_1) + x_2^2(\tau_1) < \eta^2$.

The first case is equivalent to say that the point $(x_1(\tau_1), x_2(\tau_1), x_3(\tau_1))$ is far from the x_3 -axis (i.e. far from the singular set). Here, we can repeat the same construction as in [69], because the classical inward pointing condition (6.1) is satisfied. The relevant and innovative case to study is the second one, when the point $(x_1(\tau_1), x_2(\tau_1), x_3(\tau_1))$ is close to the x_3 -axis. We suppose $\varepsilon > 0$, for it $\varepsilon = 0$, the reference trajectory $x(\cdot)$ remains always in the state constraint A . Set

$$\tau(\varepsilon) := \frac{\sqrt{3}\pi}{\sqrt{\pi-2}} \sqrt{\varepsilon} \quad (6.13)$$

and

$$\omega := \frac{\pi}{2\tau(\varepsilon)}.$$

ω is called the ‘angular velocity’ or the ‘spin’. We shall notice that the rotation (or the spiraling) is faster as $\tau(\varepsilon)$ decreases.

Construct, for $t \in [0, \tau_1 + \tau(\varepsilon)]$ and for some φ (called ‘phase’ to be determined later), two particular controls $v_1(\cdot)$ and $v_2(\cdot)$ as follows

$$v_1(t) := u_1(t)\chi_{[0,\tau_1]}(t) - \cos(\omega(t - \tau_1) + \varphi)\chi_{[\tau_1,\tau_1+\tau(\varepsilon)]}(t) \quad (6.14)$$

$$v_2(t) := u_2(t)\chi_{[0,\tau_1]}(t) + \sin(\omega(t - \tau_1) + \varphi)\chi_{[\tau_1,\tau_1+\tau(\varepsilon)]}(t), \quad (6.15)$$

where

$$\chi_{[a,b]}(t) := \begin{cases} 0 & \text{if } t \notin [a, b] \\ 1 & \text{otherwise.} \end{cases}$$

Namely, the vector field $f(x, v(t))$, corresponding to the control $v = (v_1, v_2)$, has to rotate with a certain intensity (and in a specific verse depending on the sign of ω) in order to enter in the state constraint A .

Denote now by $y(\cdot)$ the trajectory corresponding to the choice of the controls v_1 and v_2 . It is straightforward that, when $\tau_1 > 0$,

$$y(t) = x(t) \quad \text{for all } t \in [0, \tau_1].$$

On the time interval $[\tau_1, \tau_1 + \tau(\varepsilon)]$, the trajectory $y(\cdot)$ verifies

$$y_1(t) = x_1(\tau_1) + \int_{\tau_1}^t -\cos(\omega(s - \tau_1) + \varphi) ds$$

$$y_2(t) = x_2(\tau_1) + \int_{\tau_1}^t \sin(\omega(s - \tau_1) + \varphi) ds$$

$$y_3(t) = x_3(\tau_1) + \int_{\tau_1}^t y_2(s) \cos(\omega(s - \tau_1) + \varphi) + y_1(s) \sin(\omega(s - \tau_1) + \varphi) ds.$$

Via some computations and making use of some trigonometric identities, we obtain for all $t \in [\tau_1, \tau_1 + \tau(\varepsilon)]$

$$y_1(t) = x_1(\tau_1) + \frac{\sin \varphi - \sin(\omega(t - \tau_1) + \varphi)}{\omega}$$

$$y_2(t) = x_2(\tau_1) + \frac{\cos \varphi - \cos(\omega(t - \tau_1) + \varphi)}{\omega}$$

$$y_3(t) = x_3(\tau_1) + \frac{\sin(\omega(t - \tau_1)) - \omega(t - \tau_1)}{\omega^2} + \frac{2 \sin \frac{\omega(t - \tau_1)}{2}}{\omega} \left[\sin \left(\varphi + \frac{\omega(t - \tau_1)}{2} \right) x_1(\tau_1) + \cos \left(\varphi + \frac{\omega(t - \tau_1)}{2} \right) x_2(\tau_1) \right].$$

In particular, owing to the choices of ω and $\tau(\varepsilon)$, we obtain that

$$y_3(\tau_1 + \tau(\varepsilon)) = x_3(\tau_1) - 6\varepsilon + \frac{2\sqrt{6}}{\sqrt{\pi - 2}} \sqrt{\varepsilon} \left[\sin \left(\varphi + \frac{\pi}{4} \right) x_1(\tau_1) + \cos \left(\varphi + \frac{\pi}{4} \right) x_2(\tau_1) \right],$$

where $x_3(\tau_1) = 0$. We may distinguish two sub-cases of the case (b) (that is, when $x_1^2(\tau_1) + x_2^2(\tau_1) < \eta^2$):

Case (b.1): $x_1(\tau_1) = x_2(\tau_1) = 0$;

Case (b.2): $(x_1(\tau_1), x_2(\tau_1)) \neq (0, 0)$ (but still near to the singular set).

For the case (b.1), $y_3(t)$ simplifies to

$$\begin{aligned} y_3(t) &= x_3(\tau_1) + \frac{\sin(\omega(t - \tau_1)) - \omega(t - \tau_1)}{\omega^2} \\ &= \frac{4\tau^2(\varepsilon)}{\pi^2} \left(\sin\left(\frac{\pi(t - \tau_1)}{2\tau(\varepsilon)}\right) - \frac{\pi(t - \tau_1)}{2\tau(\varepsilon)} \right). \end{aligned}$$

For all $t \in (\tau_1, \tau_1 + \tau(\varepsilon)]$, we have that

$$y_3(t) < 0.$$

This is compatible with the statement of Theorem 6.2.1 because $y_3(t)$ is of rate $(t - \tau_1)^3$ on $[\tau_1, \tau_1 + \tau(\varepsilon)]$. In particular

$$y_3(\tau_1 + \tau(\varepsilon)) = \frac{2(2 - \pi)}{\pi^2} \tau^2(\varepsilon) = -6\varepsilon.$$

Thus, the statement (6.12) of Proposition 6.3.1 is verified for the case (b.1) in the interval $(\tau_1, \tau_1 + \tau(\varepsilon)]$ with $K_1 := \frac{2(\pi-2)}{\pi^2}$.

On the other hand, for the case (b.2), namely when $(x_1(\tau_1), x_2(\tau_1)) \neq (0, 0)$, we choose the phase φ such that the vector

$$\left(\sin\left(\varphi + \frac{\omega\tau(\varepsilon)}{2}\right), \cos\left(\varphi + \frac{\omega\tau(\varepsilon)}{2}\right) \right)$$

is parallel and opposite to $(x_1(\tau_1), x_2(\tau_1))$. Then, for all $t \in (\tau_1, \tau_1 + \tau(\varepsilon)]$

$$\left(\sin\left(\varphi + \frac{\omega(t - \tau_1)}{2}\right), \cos\left(\varphi + \frac{\omega(t - \tau_1)}{2}\right) \right) \cdot (x_1(\tau_1), x_2(\tau_1)) < 0. \quad (6.16)$$

Indeed, in such interval, the variation of the argument is at most $\frac{\pi}{4}$. Therefore, in this case too, we have that

$$y_3(t) < 0 \quad \text{for all } t \in [0, \tau_1 + \tau(\varepsilon)].$$

In particular,

$$y_3(\tau_1 + \tau(\varepsilon)) \leq -6\varepsilon,$$

and the statement of Proposition 6.3.1 is verified for a certain $K_1 > 0$. This concludes the proof. \square

6.4 Neighboring Feasible Trajectories with Nonlinear $W^{1,1}$ -estimate

Throughout all the analysis in the previous sections of this chapter (mainly in the statement of Proposition 6.3.1), we did not give any global construction of a feasible trajectory, neither an estimate between the reference trajectory violating the state constraint and the constructed (approximating) feasible trajectory. This is because the theory behind is still ‘under construction’. However, in this section, we will invoke first a ‘wished’ theorem merely for the Brockett nonholonomic integrator case subject to a flat state constraint. We provide after two examples where the global construction of neighboring feasible trajectories is straightforward, but, leads to a nonlinear $W^{1,1}$ -estimate.

6.4.1 Conjecture

Consider the same control system as (BRO) subject to the flat state constraint

$$A := \{x \in \mathbb{R}^3 : x_3 \leq 0\}.$$

Let $T > 0$. Define, for a trajectory $x(\cdot) = (x_1, x_2, x_3)(\cdot)$ of (BRO), the extent of violation of the state constraint:

$$\varepsilon := \max_{t \in [0, T]} |x_3(t)|.$$

Proposition 6.4.1 (In progress). *There exist constants $\varepsilon_0 > 0$, $K > 0$, $\alpha \in (0, 1]$ such that, for any trajectory $x(\cdot)$ of (BRO) such that $x(0) \in A$ (i.e. $x_3^0 \leq 0$) and $\varepsilon \leq \varepsilon_0$, there exists a trajectory $y(\cdot)$ of (BRO) with $y(0) = x(0)$ such that*

$$\begin{cases} y(t) \in A & \text{for all } t \in [0, T] \\ \|y - x\|_{W^{1,1}(0, T)} \leq K\varepsilon^\alpha. \end{cases}$$

Here for any arc $x(\cdot) \in W^{1,1}([0, T], \mathbb{R}^3)$, the corresponding norm is

$$\|x(\cdot)\|_{W^{1,1}(0, T)} = |x(0)| + \|\dot{x}(\cdot)\|_{L^1(0, T)}.$$

Proof. The complete proof of the proposition is omitted because it is not ready yet. But the idea behind is to construct particular controls on the whole set $[0, T]$ and not merely on $[0, \tau_1 + \tau(\varepsilon)]$ (where τ_1 and $\tau(\varepsilon)$ are the quantities defined in Proposition 6.3.1):

$$\begin{aligned} v_1(t) &:= u_1(t)\chi_{[0, \tau_1]}(t) - \cos(\omega(t - \tau_1) + \varphi)\chi_{[\tau_1, \tau_1 + \tau(\varepsilon)]}(t) + u_1(t - \tau(\varepsilon))\chi_{[\tau_1 + \tau(\varepsilon), T]}(t) \\ v_2(t) &:= u_2(t)\chi_{[0, \tau_1]}(t) + \sin(\omega(t - \tau_1) + \varphi)\chi_{[\tau_1, \tau_1 + \tau(\varepsilon)]}(t) + u_2(t - \tau(\varepsilon))\chi_{[\tau_1 + \tau(\varepsilon), T]}(t) \end{aligned} \quad (6.17)$$

This is equivalent to say that on the time interval $[0, \tau_1 + \tau(\varepsilon)]$, we follow the same construction as in the proof of Proposition 6.3.1. And, starting from the time $\tau_1 + \tau(\varepsilon)$, we shall construct the arc $y(\cdot)$ by using the same controls $u_1(\cdot)$ and $u_2(\cdot)$ as the reference trajectory $x(\cdot)$, but with

a delay in time. Therefore, on the time interval $[\tau_1 + \tau(\varepsilon), T]$, the corresponding arc can be expressed as follows owing to a standard computation:

$$\begin{aligned}
 y_1(t) &= x_1(t - \tau(\varepsilon)) - \frac{2}{\omega} \sin\left(\frac{\omega\tau(\varepsilon)}{2}\right) \cos\left(\varphi + \frac{\omega\tau(\varepsilon)}{2}\right) \\
 y_2(t) &= x_2(t - \tau(\varepsilon)) + \frac{2}{\omega} \sin\left(\frac{\omega\tau(\varepsilon)}{2}\right) \sin\left(\varphi + \frac{\omega\tau(\varepsilon)}{2}\right) \\
 y_3(t) &= x_3(t - \tau(\varepsilon)) + \frac{\sin \omega\tau(\varepsilon) - \omega\tau(\varepsilon)}{\omega^2} \\
 &\quad + \frac{2}{\omega} \sin\left(\frac{\omega\tau(\varepsilon)}{2}\right) \left[\sin\left(\varphi + \frac{\omega\tau(\varepsilon)}{2}\right) \left(x_1(\tau_1) - \int_{\tau_1}^{t-\tau(\varepsilon)} u_1(s) ds\right) \right. \\
 &\quad \left. + \cos\left(\varphi + \frac{\omega\tau(\varepsilon)}{2}\right) \left(x_2(\tau_1) - \int_{\tau_1}^{t-\tau(\varepsilon)} u_2(s) ds\right) \right]. \quad (6.18)
 \end{aligned}$$

The future step is to prove that this piece of arc has also interior points in the state constraint set A for $t \in [\tau_1 + \tau(\varepsilon), T]$, and to establish after the $W^{1,1}$ -estimate from the set of reference (possibly violating) trajectories. \square

6.4.2 Examples: Brockett Nonholonomic Integrator

The following two examples assert that the statement of Proposition 6.4.1 is in fact true. More precisely, we provide two motivational examples (always for the Brockett nonholonomic integrator subject to a flat state constraint), where we construct two particular controls (in the sense of (6.17)), for which a feasible trajectory exists globally and verifies a nonlinear $W^{1,1}$ -estimate from the set of reference trajectories violating the state constraint.

Example 6.4.2. Consider the control system which corresponds to the Brockett nonholonomic integrator in \mathbb{R}^3 (cf. (BRO)) with the flat state constraint $A := \{x : x_3 \leq 0\}$. Let $a > 0$, $\bar{t} > 0$ and consider the controls

$$(u_1(t), u_2(t)) = \begin{cases} (a, a) & t \leq \bar{t} \\ (-a, a) & \bar{t} < t \leq 2\bar{t} \\ (-a, -a) & 2\bar{t} < t \leq 3\bar{t} \\ (a, -a) & 3\bar{t} < t \leq 4\bar{t}. \end{cases} \quad (6.19)$$

Via some standard computations, we can deduce that a solution of the control system is a trajectory $x(\cdot)$ starting at $(0, 0, 0)$, and

$$\varepsilon = 4a^2\bar{t}^2 \quad \text{and} \quad \tau_1 = \bar{t}.$$

We recall that ε is the extent of violation of the state constraint, while τ_1 is the first time the reference trajectory $x(\cdot)$ leaves the state constraint set. Denote by $y : [0, 4\bar{t}] \rightarrow \mathbb{R}^3$ the trajectory constructed on the time interval $[0, \bar{t} + \tau(\varepsilon)]$, according to the procedure in the proof

of Proposition 6.3.1. We assume further that the constant a is so small so that $\tau(\varepsilon) < \bar{t}$. The construction in Proposition 6.3.1 gives

$$\tau(\varepsilon) = \frac{2\sqrt{3}\pi}{\sqrt{\pi-2}}a\bar{t} \quad \text{and} \quad \omega = \frac{\sqrt{\pi-2}}{4\sqrt{3}a\bar{t}}.$$

Moreover, since $(x_1(\tau_1), x_2(\tau_1)) \neq (0, 0)$, the same construction tells that

$$y_3(t) < 0 \quad \text{for all } t \in [0, \tau_1 + \tau(\varepsilon)],$$

if the phase φ is chosen such that the vector

$$\left(\sin\left(\varphi + \frac{\pi}{4}\right), \cos\left(\varphi + \frac{\pi}{4}\right) \right)$$

is parallel and opposite to $(x_1(\tau_1), x_2(\tau_1)) = (a\bar{t}, a\bar{t})$. This gives that $\frac{\pi}{4} + \varphi = \frac{5\pi}{4}$. Therefore, $\varphi = \pi$, and the expression of $y_3(t)$, as computed in (6.18), gives the following on $[\bar{t} + \tau(\varepsilon), 4\bar{t}]$:

$$y_3(t) \leq -5\varepsilon - \frac{2\sqrt{3}}{\sqrt{\pi-2}}\sqrt{\varepsilon}\left(4a\bar{t} - x_1(t - \tau(\varepsilon)) - x_2(t - \tau(\varepsilon))\right).$$

We can easily check that $y_3(t) \leq 0$ for all $t \in (\tau_1 + \tau(\varepsilon), 4\bar{t}]$. Therefore, we have

$$y(t) \in A \quad \text{for all } t \in [0, 4\bar{t}].$$

We wish now to estimate the quantity $\|y - x\|_{W^{1,1}(0,4\bar{t})}$. Explicitly,

$$\|y - x\|_{W^{1,1}(0,4\bar{t})} = |y(0) - x(0)| + \int_0^{4\bar{t}} |\dot{y}(t) - \dot{x}(t)| dt.$$

But $y(t) \equiv x(t)$ for all $t \in [0, \tau_1]$. It follows that

$$\|y - x\|_{W^{1,1}(0,4\bar{t})} = \int_{\tau_1}^{\tau_1 + \tau(\varepsilon)} |\dot{y}(t) - \dot{x}(t)| dt + \int_{\tau_1 + \tau(\varepsilon)}^{4\bar{t}} |\dot{y}(t) - \dot{x}(t)| dt.$$

Approximating each quantity apart, we obtain that

$$\int_{\tau_1}^{\tau_1 + \tau(\varepsilon)} |\dot{y}(t) - \dot{x}(t)| dt \leq 2c\tau(\varepsilon)$$

where c denotes the uniform bound on the vector fields f_1 and f_2 . On the other hand, for all $t \in (\tau_1 + \tau(\varepsilon), 4\bar{t}]$, from the Lipschitz continuity of the vector fields f_1 and f_2 (because $y(\cdot)$ and $x(\cdot)$ have the same controls on $(\tau_1 + \tau(\varepsilon), 4\bar{t}]$), we have:

$$|\dot{y}(t) - \dot{x}(t)| \leq k|y(t) - x(t)|$$

where k is the common Lipschitz constant to f_1 and f_2 . Invoking the Gronwall's Lemma (cf. Lemma 1.2.2) on $(\tau_1 + \tau(\varepsilon), 4\bar{t}]$, it follows that

$$|y(t) - x(t)| \leq \exp\left(k(t - (\tau_1 + \tau(\varepsilon)))\right)|y(\tau_1 + \tau(\varepsilon)) - x(\tau_1 + \tau(\varepsilon))|.$$

Therefore,

$$|y(t) - x(t)| \leq \exp\left(k(t - (\tau_1 + \tau(\varepsilon)))\right) 2c\tau(\varepsilon).$$

We deduce that for all $t \in (\tau_1 + \tau(\varepsilon), 4\bar{t}]$

$$|\dot{y}(t) - \dot{x}(t)| \leq k \exp\left(k(t - (\tau_1 + \tau(\varepsilon)))\right) 2c\tau(\varepsilon).$$

This gives that

$$\|y - x\|_{W^{1,1}(0,4\bar{t})} \leq 2c\tau(\varepsilon) + 2c\tau(\varepsilon) \int_{\tau_1 + \tau(\varepsilon)}^{4\bar{t}} k \exp\left(k(t - (\tau_1 + \tau(\varepsilon)))\right) dt.$$

Therefore,

$$\|y - x\|_{W^{1,1}(0,4\bar{t})} \leq K\sqrt{\varepsilon},$$

and Proposition 6.4.1 is verified for $\alpha := \frac{1}{2}$ and $K := \frac{2c\sqrt{3}\pi}{\sqrt{\pi-2}} \exp\left(k(3\bar{t} - \tau(\varepsilon))\right)$.

Example 6.4.3. Let $m \in \mathbb{N}$ and $T = \pi$, and consider the controls

$$(u_1(t), u_2(t)) = (\cos 2mt, \sin 2mt) \quad \text{on } t \in [0, \pi].$$

The corresponding solution for the Brockett Nonholonomic Integrator, starting from a point $(0, 0, 0)$, is

$$\begin{cases} x_1(t) = \frac{\sin 2mt}{2m} \\ x_2(t) = \frac{1 - \cos 2mt}{2m} \\ x_3(t) = \frac{2mt - \sin 2mt}{4m^2}. \end{cases}$$

In this case, it is easy to check that:

$$\tau_1 = 0 \quad \text{and} \quad \varepsilon := \max_{t \in [0, \pi]} |x_3(t)| = \frac{\pi}{2m}.$$

The construction in Proposition 6.3.1 yields

$$\tau(\varepsilon) = \frac{\sqrt{3}\pi}{\sqrt{\pi-2}} \frac{\sqrt{\pi}}{\sqrt{2m}} \quad \text{and} \quad \omega = \sqrt{\frac{\pi-2}{6\pi}} \sqrt{m}.$$

Since $(x_1(0), x_2(0)) = (0, 0)$, the phase φ is irrelevant: we take $\varphi = 0$. The computation in (6.18) implies that, for all $t \in (\tau(\varepsilon), \pi]$,

$$y_3(t) \leq 0.$$

The constraint is therefore satisfied for all $t \in [0, \pi]$ (it is satisfied on $[0, \tau(\varepsilon)]$ owing to Proposition 6.3.1). The $W^{1,1}$ -estimate is straightforward and gives

$$\|y - x\|_{W^{1,1}(0, \pi)} \leq K\sqrt{\varepsilon}.$$

Therefore, Proposition 6.4.1 is satisfied for $\alpha = \frac{1}{2}$ and some $K > 0$ which can be determined explicitly.

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Appendix

```

In[1]:= u1[t_] = Cos[om t + phi]
U1[t_] = Integrate[u1[s], {s, 0, t}]
u2[t_] = Sin[om t + phi]
U2[t_] = Integrate[u2[s], {s, 0, t}]

Out[1]:= Cos[phi + om t]
Out[2]:= 
$$\frac{-\text{Sin}[\text{phi}] + \text{Sin}[\text{phi} + \text{om } t]}{\text{om}}$$

Out[3]:= Sin[phi + om t]
Out[4]:= 
$$\frac{\text{Cos}[\text{phi}] - \text{Cos}[\text{phi} + \text{om } t]}{\text{om}}$$


In[5]:= UU11[t_] = Simplify[Integrate[u1[s] U1[s], {s, 0, t}]]
Out[5]:= 
$$\frac{(\text{Sin}[\text{phi}] - \text{Sin}[\text{phi} + \text{om } t])^2}{2 \text{om}^2}$$


In[6]:= UU12[t_] = Simplify[Integrate[u1[s] U2[s], {s, 0, t}]]
Out[6]:= 
$$-\frac{2 \text{om } t + \text{Sin}[2 \text{phi}] - 2 \text{Sin}[\text{om } t] + \text{Sin}[2 (\text{phi} + \text{om } t)] - 2 \text{Sin}[2 \text{phi} + \text{om } t]}{4 \text{om}^2}$$


In[7]:= UU21[t_] = Simplify[Integrate[u2[s] U1[s], {s, 0, t}]]
Out[7]:= 
$$\frac{2 \text{om } t - 4 \text{Cos}[\text{phi}] \text{Sin}[\text{phi}] + 4 \text{Cos}[\text{phi} + \text{om } t] \text{Sin}[\text{phi}] + \text{Sin}[2 \text{phi}] - \text{Sin}[2 (\text{phi} + \text{om } t)]}{4 \text{om}^2}$$


In[8]:= UU22[t_] = Simplify[Integrate[u2[s] U2[s], {s, 0, t}]]
Out[8]:= 
$$\frac{(\text{Cos}[\text{phi}] - \text{Cos}[\text{phi} + \text{om } t])^2}{2 \text{om}^2}$$


In[9]:= good = Simplify[UU12[t] - UU21[t]]
Out[9]:= 
$$\frac{-\text{om } t + \text{Sin}[\text{om } t]}{\text{om}^2}$$


In[10]:= Simplify[UU12[t] + UU21[t]]
Out[10]:= 
$$\frac{2 \text{Sin}[\frac{\text{om } t}{2}]^2 \text{Sin}[2 \text{phi} + \text{om } t]}{\text{om}^2}$$


In[11]:= bad = Simplify[
  Integrate[s (u1[s] U1[s] + u1[s] U2[s] + u2[s] U1[s] + u2[s] U2[s]), {s, 0, t}]]
Out[11]:= 
$$-\frac{1}{4 \text{om}^3} (3 \text{Cos}[2 \text{phi}] + \text{Cos}[2 (\text{phi} + \text{om } t)]) - 4 (\text{Cos}[2 \text{phi} + \text{om } t] + \text{Sin}[\text{om } t]) +$$


$$2 \text{om } t (2 \text{Cos}[\text{om } t] + \text{Sin}[2 (\text{phi} + \text{om } t)] - 2 \text{Sin}[2 \text{phi} + \text{om } t])$$


In[14]:= Series[bad, {t, 0, 7}]
Out[14]:= 
$$\frac{1}{3} (1 + \text{Sin}[2 \text{phi}]) t^3 + \frac{3}{8} \text{om} \text{Cos}[2 \text{phi}] t^4 + \left( -\frac{\text{om}^2}{30} - \frac{7}{30} \text{om}^2 \text{Sin}[2 \text{phi}] \right) t^5 -$$


$$\frac{5}{48} (\text{om}^3 \text{Cos}[2 \text{phi}]) t^6 + \frac{1}{840} (\text{om}^4 + 31 \text{om}^4 \text{Sin}[2 \text{phi}]) t^7 + O[t]^8$$


In[15]:= Series[good, {t, 0, 7}]
Out[15]:= 
$$-\frac{\text{om } t^3}{6} + \frac{\text{om}^3 t^5}{120} - \frac{\text{om}^5 t^7}{5040} + O[t]^8$$


```


Necessary Optimality Conditions For Average Cost Minimization Problems

Control systems involving unknown parameters appear a natural framework for applications in which the model design has to take into account various uncertainties. In these circumstances the performance criterion can be given in terms of an *average cost*, providing a paradigm which differs from the traditional minimax or robust optimization criteria. In this chapter, we provide necessary optimality conditions for a nonrestrictive class of optimal control problems in which unknown parameters intervene in the dynamics, the cost function, and the right end-point constraint. An important feature of our results is that we allow the unknown parameters belonging to a mere complete separable metric space (not necessarily compact).

Les systèmes de contrôle impliquant des paramètres inconnus apparaissent comme un cadre naturel pour les applications dans lesquelles la conception du modèle doit prendre en compte diverses incertitudes. Dans ces conditions, le critère de performance peut être donné en termes de *coût moyen*, fournissant un paradigme qui diffère du minimax traditionnel ou des critères d'optimisation robustes. Dans ce chapitre, nous fournissons des conditions nécessaires d'optimalité pour une classe non restrictive de problèmes de contrôle optimal dans lesquels des paramètres inconnus interviennent dans la dynamique, la fonction de coût et la contrainte finale. Une caractéristique importante de nos résultats est que nous autorisons les paramètres inconnus à appartenir à un espace métrique séparable complet (pas nécessairement compact).

*“Pleasure in the job puts perfection
in the work.”*

— Aristotelis

7.1 Introduction

In this chapter we consider a class of optimal control problems in which uncertainties appear in the data in terms of unknown parameters belonging to a given metric space. Though the state evolution is governed by a deterministic control system and the initial datum is fixed (and well-known), the description of the dynamics depends on uncertain parameters which intervene also in the cost function and the right end-point constraints. Taking into consideration an *average cost* criterion, a crucial issue is clearly to be able to characterize optimal controls independently of the unknown parameter action: this allows to find a sort of ‘best trade-off’ among all the possible realizations of the control system as the parameter varies. In this context we provide, under non restrictive assumptions, necessary optimality conditions. More precisely, we consider the following average cost minimization problem:

$$(P7) \quad \left\{ \begin{array}{l} \text{minimize } J_{\Omega}((u(\cdot), \{x(\cdot, \omega)\})) := \int_{\Omega} g(x(T, \omega); \omega) d\mu(\omega) \\ \text{over measurable functions } u : [0, T] \rightarrow \mathbb{R}^m \text{ and } W^{1,1} \text{ arcs } \{x(\cdot, \omega) : [0, T] \rightarrow \mathbb{R}^n \mid \omega \in \Omega\} \\ \text{such that } u(t) \in U(t) \quad \text{a.e. } t \in [0, T] \\ \text{and, for each } \omega \in \Omega, \\ \quad \dot{x}(t, \omega) = f(t, x(t, \omega), u(t), \omega) \quad \text{a.e. } t \in [0, T], \\ \quad x(0, \omega) = x_0 \quad \text{and } x(T, \omega) \in C(\omega). \end{array} \right.$$

The data for this problem comprise a time interval $[0, T]$, a probability measure μ defined on a metric space Ω , functions $g : \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}$ and $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \Omega \rightarrow \mathbb{R}^n$, a nonempty multifunction $U : [0, T] \rightsquigarrow \mathbb{R}^m$, and a family of closed sets $\{C(\omega) \subset \mathbb{R}^n \mid \omega \in \Omega\}$. A measurable function $u : [0, T] \rightarrow \mathbb{R}^m$ that satisfies

$$u(t) \in U(t) \quad \text{a.e. } t \in [0, T]$$

is called a *control function*. The set of all control functions is written \mathcal{U} . A *process* $(u, \{x(\cdot, \omega) : \omega \in \Omega\})$ is a control function u coupled with a family of arcs $\{x(\cdot, \omega) \in W^{1,1}([0, T], \mathbb{R}^n) : \omega \in \Omega\}$, satisfying, for each $\omega \in \Omega$, the dynamic constraint:

$$\dot{x}(t, \omega) = f(t, x(t, \omega), u(t), \omega) \quad \text{a.e. } t \in [0, T], \quad x(0, \omega) = x_0. \quad (7.1)$$

A process is said to be *feasible* if, in addition, the arcs $x(\cdot, \omega)$ ’s satisfy the right end-point constraint

$$x(T, \omega) \in C(\omega) \quad \text{for all } \omega \in \text{supp}(\mu). \quad (7.2)$$

For a given control function $u(\cdot)$, the ‘set-valued’ states for the system given by (7.1)-(7.2) is illustrated in Figure 7.1, where each trajectory corresponds to a unknown parameter ω .

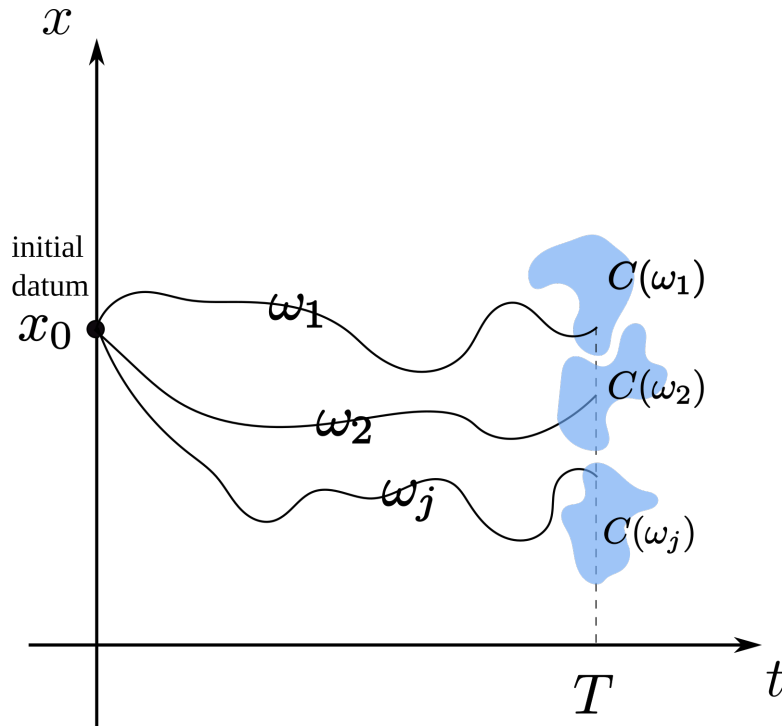


Figure 7.1 – The evolution of a state trajectory for any given control

If the integral term in (P7) does not exist for a feasible process $(u, \{x(\cdot, \omega) : \omega \in \Omega\})$, then we set $J_\Omega(u(\cdot), \{x(\cdot, \omega)\}) = +\infty$. To underline the dependence on a given control $u(\cdot) \in \mathcal{U}$, sometimes we shall employ the notation $x(\cdot, u, \omega)$ for the feasible arc belonging to the family of trajectories $\{x(\cdot, \omega) : \omega \in \Omega\}$ associated with the control $u(\cdot)$ and the element $\omega \in \Omega$. A feasible process $(\bar{u}, \{\bar{x}(\cdot, \omega) : \omega \in \Omega\})$ is said to be a $W^{1,1}$ -local minimizer for (P7) if there exists $\epsilon > 0$ such that

$$\int_\Omega g(\bar{x}(T, \omega); \omega) d\mu(\omega) \leq \int_\Omega g(x(T, \omega); \omega) d\mu(\omega)$$

for all feasible processes $(u, \{x(\cdot, \omega) : \omega \in \Omega\})$ such that

$$\|\bar{x}(\cdot, \omega) - x(\cdot, \omega)\|_{W^{1,1}} \leq \epsilon \quad \text{for all } \omega \in \text{supp}(\mu). \tag{7.3}$$

Control systems involving unknown parameters have been well-studied in literature finding widespread applications particularly from the point of view of the robust (worst-case) control, see for instance the monographs [1] and [21] (and the references therein), and the paper [83] on minimax optimal control. A rising interest has recently emerged in considering an ‘averaged’ (or ‘expected’ with respect to an given measure) approach, exploring various issues, directions and applications: see for instance a recent series of papers on aerospace systems [73], [74], [26], and the articles [2] and [86] on averaged controllability (from different viewpoints); see also [80] for results on heterogeneous systems. Therefore, motivated not only by theoretical reasons but also by a recent growing interest in applications (such as aerospace engineering, see in particular [73] and [74]), in our chapter we consider the ‘new average cost’ paradigm rather

than the more ‘classical’ criteria employed in the minimax/robust optimization framework. For the general (nonsmooth) case we derive necessary optimality conditions ensuring the existence of a *costate function* $p(., .) : [0, T] \times \Omega \rightarrow \mathbb{R}^n$ which satisfies a (standard) non-triviality condition and an averaged (on Ω) maximality condition. Moreover, the costate arcs $p(., \omega)$ ’s satisfy also the somewhat expected adjoint system and transversality condition, when ω belongs at least to a countable dense subset $\tilde{\Omega}$ of $\text{supp}(\mu)$. We also show that these last two necessary conditions extend to the whole metric space Ω for free right end-point problem, if we impose (suitable) regularity assumptions on the dynamics and the cost function.

This chapter is organized as follows. We first study the simpler case in which the measure μ has a finite support (Section 2), which constitutes a discretization model for the general case of an arbitrary measure on a complete separable metric space (which is investigated successively). The main results are displayed in Section 3, and their proofs are given in Section 5. Section 4 is devoted to recall some fundamental theorems in measure theory and provide a limit-taking lemma which play a crucial role in our analysis. The approach that we suggest in our chapter consists in approximating the measure μ by measures with finite support (convex combination of Dirac measures). Owing to Ekeland’s variational principle, we construct a suitable family of auxiliary optimal control problems whose solutions approximate the reference problem (P7). Invoking the maximum principle (applicable in a more traditional version) for the approximating minimizers, we obtain properties which, taking the limit (in a suitable sense), allow us to derive the desired necessary conditions. An important source of inspiration for the techniques here employed is represented by Vinter’s paper [83] (which is devoted to minimax optimal control but, in fact, contains flexible and effective analytical tools that can be extended or adapted to our case). As one may expect, the necessary conditions that we obtain differ from those ones in the minimax context (in particular for the general nonsmooth case), for the nature of the minimization criterion is different. We highlight that an important feature of our chapter is the unrestrictive nature of our assumptions: indeed, we allow not only nonsmooth data (on the dynamics, the cost function and the right end-point constraint), but we also provide results for unknown parameters belonging to a mere complete separable metric space Ω . This aspect is particularly relevant for some applications (cf. [73]) where Ω (and the support of the reference measure μ) need not to be compact.

Along this chapter, we denote by (Ω, ρ_Ω) a metric space, and by \mathcal{B}_Ω the σ -algebra of Borel sets in Ω . A probability measure μ on the measurable space $(\Omega, \mathcal{B}_\Omega)$ verifies the σ -additivity property and is such that $\mu(\Omega) = 1$. The family of all probability measures on $(\Omega, \mathcal{B}_\Omega)$ is denoted by $\mathcal{M}(\Omega)$. Recall that a sequence $\{\mu_i\}$ of measures in $\mathcal{M}(\Omega)$ is said to converge weakly* to a measure $\mu \in \mathcal{M}(\Omega)$ (in symbol $\mu_i \xrightarrow{*} \mu$), if $\int_\Omega h d\mu_i \rightarrow \int_\Omega h d\mu$ for every bounded continuous function h on Ω .

7.2 Average on Measures with Finite Support

We start considering the particular and simple case of optimal control problems of the form (P7), where the probability measure μ of the integral functional has a finite support: it is a convex combination of unit Dirac measures. This constitutes also a preliminary step to derive

necessary conditions for the general case.

The following assumptions will be needed throughout this section. For a given $W^{1,1}$ -local minimizer $(\bar{u}, \{\bar{x}(\cdot, \omega) : \omega \in \Omega\})$ and for some $\delta > 0$, we shall suppose:

- (H.1) (i) The function $f(\cdot, x, \cdot, \omega)$ is $\mathcal{L} \times \mathcal{B}^m$ measurable for each $(x, \omega) \in \mathbb{R}^n \times \Omega$.
 (ii) The multifunction $t \rightsquigarrow U(t)$ has nonempty values, and $\text{Gr } U(\cdot)$ is a $\mathcal{L} \times \mathcal{B}^m$ measurable set.
- (H.2) There exists a $\mathcal{L} \times \mathcal{B}^m$ measurable function $k_f : [0, T] \times \mathbb{R}^m \rightarrow \mathbb{R}$ such that $t \rightarrow k_f(t, \bar{u}(t))$ is integrable, and for each $\omega \in \Omega$,

$$|f(t, x, u, \omega) - f(t, x', u, \omega)| \leq k_f(t, u)|x - x'|$$

for all $x, x' \in \bar{x}(t, \omega) + \delta\mathbb{B}$, $u \in U(t)$, a.e. $t \in [0, T]$.

- (H.3) The function $g(\cdot, \omega)$ is Lipschitz continuous on $\bar{x}(T, \omega) + \delta\mathbb{B}$ for all $\omega \in \text{supp}(\mu)$.

Proposition 7.2.1. *Let $(\bar{u}, \{\bar{x}(\cdot, \omega) : \omega \in \Omega\})$ be a $W^{1,1}$ -local minimizer for (P7). Assume that μ is a given probability measure with finite support and that for some $\delta > 0$, hypotheses (H.1)-(H.3) are satisfied. Then, there exist a family of arcs $\{p(\cdot, \omega) \in W^{1,1}([0, T], \mathbb{R}^n) : \omega \in \Omega\}$ and a number $\lambda \in [0, 1]$ such that*

$$(a) \quad \lambda + \int_{\Omega} \max_{t \in [0, T]} |p(t, \omega)| \, d\mu(\omega) = 1 ;$$

$$(b) \quad \int_{\Omega} p(t, \omega) \cdot f(t, \bar{x}(t, \omega), \bar{u}(t), \omega) \, d\mu(\omega) = \max_{u \in U(t)} \int_{\Omega} p(t, \omega) \cdot f(t, \bar{x}(t, \omega), u, \omega) \, d\mu(\omega) \quad \text{a.e. } t ;$$

$$(c) \quad -\dot{p}(t, \omega) \in \text{co } \partial_x [p(t, \omega) \cdot f(t, \bar{x}(t, \omega), \bar{u}(t), \omega)] \quad \text{a.e. } t \in [0, T] \text{ and for } \mu - \text{a.e. } \omega \in \Omega ;$$

$$(d) \quad -p(T, \omega) \in \lambda \partial_x g(\bar{x}(T, \omega); \omega) + N_{C(\omega)}(\bar{x}(T, \omega)) \quad \text{for } \mu - \text{a.e. } \omega \in \Omega .$$

Proof. The measure μ can be written as a convex combination of Dirac measures at points $\omega_j \in \Omega$, for $j = 1, \dots, N$, where N is a suitable integer, as follows:

$$\mu = \sum_{j=1}^N \alpha_j \delta_{\omega_j}, \quad \sum_{j=1}^N \alpha_j = 1, \quad \alpha_j \in (0, 1]. \quad (7.4)$$

As a consequence the integral functional to minimize in (P7) reduces to the following finite sum:

$$\sum_{j=1}^N \alpha_j g(x(T, \omega_j); \omega_j),$$

and, the minimization problem (P7) turns out to be easily treated, for it can be equivalently written as a standard optimal control problem:

$$(P7)_N \quad \left\{ \begin{array}{l} \text{minimize} \quad \sum_{j=1}^N \alpha_j g(x(T, \omega_j); \omega_j) \\ \text{over controls } u(\cdot) \text{ such that } u(t) \in U(t) \text{ a.e. } t \in [0, T] \\ \text{and arcs } x(\cdot, \omega_j) \text{ such that for each } j = 1, \dots, N \\ \quad \dot{x}(t, \omega_j) = f(t, x(t, \omega_j), u(t), \omega_j) \quad \text{a.e. } t \in [0, T] \\ \quad x(0, \omega_j) = x_0 \quad \text{and} \quad x(T, \omega_j) \in C(\omega_j) . \end{array} \right.$$

Observe that in writing (P7)_N, we can restrict attention only to elements ω belonging to the $\text{supp}(\mu) = \{\omega_1, \dots, \omega_N\}$. Clearly, since $(\bar{u}, \{\bar{x}(\cdot, \omega) : \omega \in \Omega\})$ is a $W^{1,1}$ -local minimizer, $(\bar{u}, \{\bar{x}(\cdot, \omega_j) : j = 1, \dots, N\})$ is a $W^{1,1}$ -local minimizer for (P7)_N. Under the stated assumptions (H.1)-(H.3) and using the sum rule (cf. [81, Theorem 5.4.1]), the necessary conditions for (P7)_N can be derived from the nonsmooth maximum principle (cf. Theorem 1.2.8 or [81, Theorem 6.2.1] which guarantees the existence of a multiplier $\lambda \geq 0$ and arcs $\tilde{p}(\cdot, \omega_j) \in W^{1,1}([0, T], \mathbb{R}^n)$, for $j = 1, \dots, N$ such that

- (i) $(\lambda, \tilde{p}(\cdot, \omega_1), \dots, \tilde{p}(\cdot, \omega_N)) \neq (0, \dots, 0)$;
- (ii) $-\dot{\tilde{p}}(t, \omega_j) \in \text{co } \partial_x [\tilde{p}(t, \omega_j) \cdot f(t, \bar{x}(t, \omega_j), \bar{u}(t), \omega_j)] \quad \text{a.e. } t \text{ and for all } j = 1, \dots, N$;
- (iii) $-\tilde{p}(T, \omega_j) \in \lambda \alpha_j \partial_x g(\bar{x}(T, \omega_j); \omega_j) + N_{C(\omega_j)}(\bar{x}(T, \omega_j)) \quad \text{for all } j = 1, \dots, N$;
- (iv) $\sum_{j=1}^N \tilde{p}(t, \omega_j) \cdot f(t, \bar{x}(t, \omega_j), \bar{u}(t), \omega_j) = \max_{u \in U(t)} \sum_{j=1}^N \tilde{p}(t, \omega_j) \cdot f(t, \bar{x}(t, \omega_j), u, \omega_j) \quad \text{a.e. } t \in [0, T]$.

For each j , we set

$$p(\cdot, \omega_j) := \frac{\tilde{p}(\cdot, \omega_j)}{\alpha_j} .$$

From the homogeneity property of subdifferentials and taking into account that the maximality condition (iv) can be expressed as an integral form owing to (7.4), we obtain

- (i)' $(\lambda, p(\cdot, \omega_1), \dots, p(\cdot, \omega_N)) \neq (0, \dots, 0)$;
- (ii)' $-\dot{p}(t, \omega_j) \in \text{co } \partial_x [p(t, \omega_j) \cdot f(t, \bar{x}(t, \omega_j), \bar{u}(t), \omega_j)]$;
- (iii)' $-p(T, \omega_j) \in \lambda \partial_x g(\bar{x}(T, \omega_j); \omega_j) + N_{C(\omega_j)}(\bar{x}(T, \omega_j))$;
- (iv)' $\int_{\Omega} p(t, \omega) \cdot f(t, \bar{x}(t, \omega), \bar{u}(t), \omega) d\mu(\omega) = \max_{u \in U(t)} \int_{\Omega} p(t, \omega) \cdot f(t, \bar{x}(t, \omega), u, \omega) d\mu(\omega)$.

Now let $d := \lambda + \int_{\Omega} \max_{t \in [0, T]} |p(t, \omega)| d\mu(\omega)$. From the nontriviality condition (i)', we have $d > 0$, and eventually, dividing across the family of arcs $\{p(\cdot, \omega)\}$ and the multiplier λ by the constant d (we do not relabel), we deduce (a)-(d) of the proposition statement. This concludes the proof. \square

7.3 Main Results

We take now a probability space $(\Omega, \mathcal{B}_\Omega, \mu)$ where μ is a (general) probability measure. For a given $W^{1,1}$ -local minimizer $(\bar{u}, \{\bar{x}(\cdot, \omega) : \omega \in \Omega\})$ and for some $\delta > 0$, we shall suppose:

(A.1) (Ω, ρ_Ω) is a complete separable metric space.

(A.2) (i) The function $f(\cdot, x, \cdot, \cdot)$ is $\mathcal{L} \times \mathcal{B}^m \times \mathcal{B}_\Omega$ measurable for each $x \in \mathbb{R}^n$.

(ii) The multifunction $t \rightsquigarrow U(t)$ has nonempty values and $\text{Gr } U(\cdot)$ is a $\mathcal{L} \times \mathcal{B}^m$ measurable set.

(iii) The set $f(t, x, U(t), \omega)$ is closed for all $x \in \bar{x}(t, \omega) + \delta\mathbb{B}$, and $(t, \omega) \in [0, T] \times \Omega$.

(A.3) There exist a constant $c > 0$ and an integrable function $k_f : [0, T] \rightarrow \mathbb{R}$ such that

$$|f(t, x, u, \omega) - f(t, x', u, \omega)| \leq k_f(t)|x - x'| \quad \text{and} \quad |f(t, x, u, \omega)| \leq c$$

for all $x, x' \in \bar{x}(t, \omega) + \delta\mathbb{B}$, $u \in U(t)$, $\omega \in \Omega$ a.e. $t \in [0, T]$.

(A.4) (i) The function g is $\mathcal{B}^n \times \mathcal{B}_\Omega$ measurable.

(ii) There exist positive constants k_g and M_g such that for all $\omega \in \Omega$ we have

$$|g(x, \omega)| \leq M_g \quad \text{for all } x \in \bar{x}(T, \omega) + \delta\mathbb{B},$$

$$|g(x, \omega) - g(x', \omega)| \leq k_g|x - x'| \quad \text{for all } x, x' \in \bar{x}(T, \omega) + \delta\mathbb{B}.$$

(iii) There exists a modulus of continuity $\theta_g(\cdot)$ such that for all $\omega \in \Omega$ and $x \in \bar{x}(T, \omega) + \delta\mathbb{B}$ we have

$$|g(x, \omega_1) - g(x, \omega_2)| \leq \theta_g(\rho_\Omega(\omega_1, \omega_2)) \quad \text{for all } \omega_1, \omega_2 \in \Omega.$$

(A.5) There exists a modulus of continuity $\theta_f(\cdot)$ such that for all $\omega, \omega_1, \omega_2 \in \Omega$,

$$\int_0^T \sup_{x \in \bar{x}(t, \omega) + \delta\mathbb{B}, u \in U(t)} |f(t, x, u, \omega_1) - f(t, x, u, \omega_2)| dt \leq \theta_f(\rho_\Omega(\omega_1, \omega_2)).$$

(We say that $\theta : [0, \infty) \rightarrow [0, \infty)$ is a modulus of continuity if $\theta(s)$ is increasing and $\lim_{s \downarrow 0} \theta(s) = 0$.)

The first result provides necessary optimality conditions for the general nonsmooth case.

Theorem 7.3.1. *Let $(\bar{u}, \{\bar{x}(\cdot, \omega) : \omega \in \Omega\})$ be a $W^{1,1}$ -local minimizer for (P7) in which $\mu \in \mathcal{M}(\Omega)$ is given. Assume that, for some $\delta > 0$, hypotheses (A.1)-(A.5) are satisfied. Then, there exist $\lambda \geq 0$, a $\mathcal{L} \times \mathcal{B}_\Omega$ measurable function $p(\cdot, \cdot) : [0, T] \times \Omega \rightarrow \mathbb{R}^n$ and a countable dense subset $\widehat{\Omega}$ of $\text{supp}(\mu)$ such that*

(i) $p(\cdot, \omega) \in W^{1,1}([0, T], \mathbb{R}^n)$ for all $\omega \in \widehat{\Omega}$;

(ii) $\int_\Omega p(t, \omega) \cdot f(t, \bar{x}(t, \omega), \bar{u}(t), \omega) d\mu(\omega) = \max_{u \in U(t)} \int_\Omega p(t, \omega) \cdot f(t, \bar{x}(t, \omega), u, \omega) d\mu(\omega)$ a.e. t ;

(iii) $p(\cdot, \omega) \in \text{co } \mathcal{P}(\omega)$ for all $\omega \in \widehat{\Omega}$ where

$$\mathcal{P}(\omega) := \left\{ q(\cdot, \omega) \in W^{1,1}([0, T], \mathbb{R}^n) : \|q(\cdot, \cdot)\|_{L^\infty} \leq 1, \lambda + \sum_{\omega \in \widehat{\Omega}} \max_{t \in [0, T]} |q(t, \omega)| = 1, \right. \\ \left. - \dot{q}(t, \omega) \in \text{co } \partial_x [q(t, \omega) \cdot f(t, \bar{x}(t, \omega), \bar{u}(t, \omega))] \quad \text{a.e. } t \in [0, T], \right. \\ \left. \text{and } -q(T, \omega) \in \lambda \partial_x g(\bar{x}(T, \omega); \omega) + N_{C(\omega)}(\bar{x}(T, \omega)) \right\}.$$

Remark 7.3.2. Notice that (iii) of Theorem 7.3.1 guarantees a non-triviality condition for the pair $(\lambda, p(\cdot, \cdot))$. Observe also that, if there is no right end-point constraint ($C(\omega) \equiv \mathbb{R}^n$), then condition (iii) of Theorem 7.3.1 immediately implies that the necessary conditions apply in the normal form (i.e. $\lambda > 0$). If, in addition, we impose regularity assumptions on the dynamics and the terminal cost function, properties (i) and (iii) of Theorem 7.3.1 extend to the whole parameter set Ω , as stated in Theorem 7.3.3 below.

Theorem 7.3.3 (Smooth case). Let $(\bar{u}, \{\bar{x}(\cdot, \omega) : \omega \in \Omega\})$ be a $W^{1,1}$ -local minimizer for (P7) where $\mu \in \mathcal{M}(\Omega)$ is given. Suppose that, for some $\delta > 0$, hypotheses (A.1)-(A.3), (A.4)(i) and (A.5) are satisfied. In addition, assume that

(C.1) $g(\cdot, \omega)$ is differentiable for each $\omega \in \Omega$, and $\nabla_x g(\cdot, \cdot)$ is continuous;

(C.2) $f(t, \cdot, u, \omega)$ is continuously differentiable on $\bar{x}(t, \omega) + \delta \mathbb{B}$ for all $u \in U(t)$ and $\omega \in \Omega$ a.e. $t \in [0, T]$, and $\omega \rightarrow \nabla_x f(t, x, u, \omega)$ is uniformly continuous with respect to $(t, x, u) \in \{(t', x', u') \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \mid u' \in U(t')\}$;

(C.3) $C(\omega) := \mathbb{R}^n$.

Then, there exists a $\mathcal{L} \times \mathcal{B}_\Omega$ measurable function $p(\cdot, \cdot) : [0, T] \times \Omega \rightarrow \mathbb{R}^n$ such that

(i)' $p(\cdot, \omega) \in W^{1,1}([0, T], \mathbb{R}^n)$ for all $\omega \in \Omega$;

(ii)' $\int_\Omega p(t, \omega) \cdot f(t, \bar{x}(t, \omega), \bar{u}(t, \omega)) d\mu(\omega) = \max_{u \in U(t)} \int_\Omega p(t, \omega) \cdot f(t, \bar{x}(t, \omega), u, \omega) d\mu(\omega)$ a.e. t ;

(iii)' $-\dot{p}(t, \omega) = [\nabla_x f(t, \bar{x}(t, \omega), \bar{u}(t, \omega))]^T p(t, \omega)$ a.e. $t \in [0, T]$, for all $\omega \in \Omega$;

(iv)' $-p(T, \omega) = \nabla_x g(\bar{x}(T, \omega); \omega)$ for all $\omega \in \Omega$.

7.4 Preliminary Results in Measure Theory

This section is devoted to display results which will be relevant for the proofs of Theorems 7.3.1 and 7.3.3. We shall make repeatedly use of the following theorem (also referred to as Portmanteau Theorem, cf. [4, Theorem 4.5.1] or [64, Theorem 6.1. pp. 39]) which provides conditions characterizing the weak* convergence of probability measures on a metric space (Ω, ρ_Ω) .

Theorem 7.4.1. *Let (Ω, ρ_Ω) be a metric space. Take a sequence of measures $\{\mu_i\}$ in $\mathcal{M}(\Omega)$ and a measure $\mu \in \mathcal{M}(\Omega)$. The following conditions are equivalent:*

- (a) $\int_\Omega h d\mu_i \rightarrow \int_\Omega h d\mu$ for any bounded continuous function h on Ω (i.e. $\mu_i \xrightarrow{*} \mu$);
- (b) $\int_\Omega h d\mu_i \rightarrow \int_\Omega h d\mu$ for any bounded uniformly continuous function h on Ω ;
- (c) $\lim \mu_i(B) = \mu(B)$ for every Borel set B whose boundary has μ -measure zero. (Such sets are also referred to as μ -continuity sets);
- (d) $\limsup \mu_i(C) \leq \mu(C)$ for every closed set C in Ω ;
- (e) $\liminf \mu_i(E) \geq \mu(E)$ for every open set E in Ω .

We recall that $\mu \in \mathcal{M}(\Omega)$ is said to be *tight* if for each $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset \Omega$ such that $\mu(\Omega \setminus K_\varepsilon) < \varepsilon$. A very well-known result asserts that when (Ω, ρ_Ω) is a complete separable metric space, then every $\mu \in \mathcal{M}(\Omega)$ is tight (cf. [64, Theorem 3.2. pp. 29]). We shall invoke also Prokhorov Theorem [67] (see below) which provides a useful characterization of the relatively compact subsets of $\mathcal{M}(\Omega)$, when Ω is a complete separable metric space. This result will be crucial to derive measure convergence properties (see Lemma 7.4.3 below).

Theorem 7.4.2 (Prokhorov Theorem). *Let (Ω, ρ_Ω) be a complete separable metric space and consider a subset $\Upsilon \subset \mathcal{M}(\Omega)$. Then, Υ is relatively compact in $\mathcal{M}(\Omega)$ if and only if Υ is uniformly tight; in particular a sequence of measures $\{\mu_i\}$ in $\mathcal{M}(\Omega)$ admits a weakly* convergent subsequence in $\mathcal{M}(\Omega)$ if and only if the sequence $\{\mu_i\}$ is uniformly tight.*

We consider now subsets D and D_i , for $i = 1, 2, \dots$, of $\Omega \times \mathbb{R}^K$. We denote respectively by $D(\cdot)$, $D_i(\cdot) : \Omega \rightsquigarrow \mathbb{R}^K$ the multifunctions defined as

$$D(\omega) := \{z \in \mathbb{R}^K : (\omega, z) \in D\} \quad \text{and} \quad D_i(\omega) := \{z \in \mathbb{R}^K : (\omega, z) \in D_i\} \quad \text{for all } i = 1, 2, \dots$$

Let $\{\mu_i\}$ be a weak* convergent sequence of measures in $\mathcal{M}(\Omega)$. Our aim is to justify the limit-taking of sequences like

$$d\eta_i(\omega) = \gamma_i(\omega) d\mu_i(\omega) \quad i = 1, 2, \dots$$

in which $\{\gamma_i(\omega)\}$ is a sequence of Borel measurable functions satisfying

$$\gamma_i(\omega) \in D_i(\omega) \quad \mu_i - \text{a.e.}$$

The required convergence result is provided by Lemma 7.4.3 below, which represents an extension of [81, Proposition 9.2.1] and [83, Proposition 6.1] (presented also in Proposition 1.3.2) to the case in which Ω is an arbitrary complete separable metric space (not necessarily compact).

Lemma 7.4.3. *Let Ω be a complete separable metric space. Consider a sequence of measures $\{\mu_i\}$ in $\mathcal{M}(\Omega)$ such that $\mu_i \xrightarrow{*} \mu$ for some $\mu \in \mathcal{M}(\Omega)$, a sequence of sets $\{D_i \subset \Omega \times \mathbb{R}^K\}$ such that*

$$\limsup_{i \rightarrow \infty} D_i \subset D, \quad (7.5)$$

for some closed set $D \subset \Omega \times \mathbb{R}^K$, and a sequence $\{\gamma_i : \Omega \rightarrow \mathbb{R}^K\}$ of Borel functions. Suppose that

- (i) $D(\omega)$ is convex for each $\omega \in \text{dom } D(\cdot)$;
- (ii) the multifunctions $\omega \rightsquigarrow D(\omega)$ and $\omega \rightsquigarrow D_i(\omega)$, for all i , are uniformly bounded;
- (iii) for each $i = 1, 2, \dots$, $\gamma_i(\cdot)$ is measurable, $\gamma_i(\omega) \in D_i(\omega)$ μ_i -a.e. and $\text{supp}(\mu_i) \subset \text{dom } D_i(\cdot)$.

Define, for each i , the vector-valued measure $\eta_i := \gamma_i \mu_i$. Then, along a subsequence, we have

$$\eta_i \xrightarrow{*} \eta$$

where η is a vector-valued Borel measure on Ω such that

$$d\eta(\omega) = \gamma(\omega) d\mu(\omega)$$

for some Borel measurable function $\gamma : \Omega \rightarrow \mathbb{R}^K$ satisfying

$$\gamma(\omega) \in D(\omega) \quad \mu\text{-a.e. } \omega \in \Omega.$$

(The upper limit in (7.5) above must be understood in the Kuratowski sense, cf. [7] or [81].) More precisely,

$$\limsup_{i \rightarrow \infty} D_i = \left\{ (\omega, y) \in \Omega \times \mathbb{R}^K \mid \begin{array}{l} \text{for all open neighborhoods } V \text{ of } (\omega, y), \\ V \cap D_i \neq \emptyset \text{ for infinitely many } i \end{array} \right\}.$$

Proof. Since Ω is a complete separable metric space, the sequence $\{\mu_i\}$ turns out to be uniformly tight as result of Theorem 7.4.2. We also know that $\gamma_i(\omega) \in D_i(\omega)$ μ_i -a.e. and $D_i(\omega)$ is uniformly bounded for all i . It follows that there exists a constant $M > 0$ such that

$$|\gamma_i(\omega)| \leq M \quad \mu_i\text{-a.e.} \quad (7.6)$$

For each i , the vector-valued measure $\eta_i = \gamma_i \mu_i$ can be expressed as $\eta_i = (\eta_{i,1}, \dots, \eta_{i,K})$. From the tightness of $\{\mu_i\}$ and (7.6), it immediately follows that, for all $k \in \{1, \dots, K\}$, $\{\eta_{i,k}\}$ is a family of uniformly tight, eventually signed measures. Therefore according to Theorem 7.4.2, for each $k \in \{1, \dots, K\}$ one can extract a subsequence $\{\eta_{i,k}\}$ (we do not relabel) which converges weakly* to some η_k . We show that $\eta := (\eta_1, \dots, \eta_K)$ is absolutely continuous with respect to μ . Let $\eta_{i,k} = \eta_{i,k}^+ - \eta_{i,k}^-$ and $\eta_k = \eta_k^+ - \eta_k^-$ be the Jordan decompositions of $\eta_{i,k}$ and η_k , where η_k^+ and η_k^- are respectively the weak* limits of $\eta_{i,k}^+$ and $\eta_{i,k}^-$. Let $B_{\eta, \mu}$ be the common

family of continuity sets (in the sense of (c) of Theorem 7.4.1) for the measures $\eta_1^+, \dots, \eta_K^+$, $\eta_1^-, \dots, \eta_K^-$ and μ . Take any Borel set B in $B_{\eta, \mu}$, we have

$$\left| \int_B d\eta \right| = \lim_i \left| \int_B d\eta_i \right| = \lim_i \left| \int_B \gamma_i(\omega) d\mu_i(\omega) \right| \leq M \lim_i \int_B d\mu_i(\omega) = M \int_B d\mu(\omega) .$$

But since $B_{\eta, \mu}$ generates all the Borel sets of Ω , (details are given in Theorem A.7, it follows that η is absolutely continuous with respect to μ . Therefore, by the Radon-Nikodym Theorem, there exists a \mathbb{R}^K -valued, Borel measurable and μ -integrable function γ on Ω such that for any Borel subset B of Ω we have

$$\eta(B) = \int_B d\eta(\omega) = \int_B \gamma(\omega) d\mu(\omega) ;$$

equivalently,

$$d\eta(\omega) = \gamma(\omega) d\mu(\omega) .$$

It remains to show that $\gamma(\omega) \in D(\omega)$ μ -a.e. $\omega \in \Omega$. For all $j \in \mathbb{N}$ fixed, following the approach suggested in [81, Proposition 9.2.1], we define $D^j(\omega) := D(\omega) + \frac{1}{j}\mathbb{B} \subset \mathbb{R}^K$. We fix $q \in \mathbb{R}^K$. Since $D(\omega)$ is uniformly bounded and D is closed, the multifunction $D^j(\cdot)$ is upper semicontinuous. Then, for $\bar{R} > 0$ large enough, the marginal function defined by

$$\sigma_q(\omega) := \begin{cases} \max\{q \cdot d : d \in D^j(\omega)\} & \text{if } D^j(\omega) \neq \emptyset \\ \bar{R} & \text{otherwise} \end{cases}$$

turns out to be upper semicontinuous and bounded on Ω , owing to the Maximum Theorem (cf. [7, Theorem 1.4.16]). From standard results on semicontinuous maps (cf. [4, A6.6]), there exists a sequence of bounded continuous functions $\{\psi_q^\ell : \Omega \rightarrow \mathbb{R}, \ell = 1, 2, \dots\}$ such that:

$$\lim_{\ell \rightarrow \infty} \psi_q^\ell(\omega) = \sigma_q(\omega) \quad \text{and} \quad \sigma_q(\omega) \leq \psi_q^\ell(\omega) \quad \text{for all } \ell = 1, 2, \dots \quad (7.7)$$

Recalling that the sets $D(\omega)$ and $D_i(\omega)$ for $i = 1, 2, \dots$, are uniformly bounded, and owing to (1.22), we have that, for all $j \in \mathbb{N}$, there exists i_j such that for all $i \geq i_j$, $D_i(\omega) \subset D^j(\omega)$. Then for $q \in \mathbb{R}^K$ and for any Borel subset B of Ω , for all $i \geq i_j$, we have

$$\begin{aligned} q \cdot \int_B d\eta_i(\omega) &= q \cdot \int_B \gamma_i(\omega) d\mu_i(\omega) = q \cdot \int_{B \cap \text{dom } D^j(\cdot)} \gamma_i(\omega) d\mu_i(\omega) \\ &\leq \int_B \sigma_q(\omega) d\mu_i(\omega) \leq \int_B \psi_q^\ell(\omega) d\mu_i(\omega) . \end{aligned} \quad (7.8)$$

The last inequality is a consequence of (7.7). Before passing to the limit, we observe that

$$\text{supp}(\eta) \subset \text{dom } D^j(\cdot) . \quad (7.9)$$

Indeed, take any open set $E \subset \Omega \setminus \text{dom } D^j(\cdot)$. Since $\text{supp}(\eta_i) \subset \text{dom } D^j(\cdot)$ for i sufficiently large, from (e) of Theorem 7.4.1, we have

$$0 \leq \int_E d\eta_k^+(\omega) \leq \liminf_i \int_E d\eta_{i,k}^+(\omega) \leq 0 .$$

We deduce that $\eta_k^+(E) = 0$ for all $k = 1, \dots, K$. Following the same reasoning, one can conclude that $\eta_k^-(E) = 0$ for all $k = 1, \dots, K$. Hence, $\eta(E) = 0$ for all open subsets $E \subset \Omega \setminus \text{dom } D^j(\cdot)$ and $\text{supp}(\eta) \subset \text{dom } D^j(\cdot)$. The inclusion (7.9) is therefore proved. By passing to the limit in (7.8) as $i \rightarrow \infty$, since $\psi_q^\ell(\cdot)$ is bounded continuous on Ω , we obtain for any Borel set $B \in \mathcal{B}_{\eta, \mu}$

$$q \cdot \int_B d\eta(\omega) \leq \int_B \psi_q^\ell(\omega) d\mu(\omega).$$

As $\int_B d\eta(\omega) = \int_B \gamma(\omega) d\mu(\omega)$, for any $B \in \mathcal{B}_{\eta, \mu}$, we have

$$q \cdot \int_B \gamma(\omega) d\mu(\omega) \leq \int_B \psi_q^\ell(\omega) d\mu(\omega). \quad (7.10)$$

Recalling that $\mathcal{B}_{\eta, \mu}$ generates the Borel σ -algebra \mathcal{B}_Ω (cf. Theorem A.7), we deduce that (7.10) is actually valid for all Borel subsets of Ω . As a consequence, $q \cdot \gamma(\omega) \leq \psi_q^\ell(\omega) \quad \mu - \text{a.e.}$, and letting $\ell \rightarrow \infty$, we obtain

$$q \cdot \gamma(\omega) \leq \sigma_q(\omega) \quad \mu - \text{a.e.} \quad (7.11)$$

We prove now that inequality (7.11) holds for all $q \in \mathbb{R}^K$ with $|q| = 1$. From the continuity of the map

$$q \mapsto \max\{q \cdot d : d \in D^j(\omega)\},$$

it is enough to establish inequality (7.11) for all $q \in \mathbb{Q}^K$. The analysis above allows to associate with each $q \in \mathbb{Q}^K$ a set $E_q \subset \Omega$ of full measure for μ such that

$$\max_{d \in D^j(\omega)} q \cdot d - q \cdot \gamma(\omega) \geq 0 \quad \text{for all } \omega \in E_q. \quad (7.12)$$

Taking $E := \bigcap_{q \in \mathbb{Q}^K} E_q$, we obtain a set of full measure for μ such that for all $q \in \mathbb{Q}^K$

$$\max_{d \in D^j(\omega)} q \cdot d - q \cdot \gamma(\omega) \geq 0 \quad \text{for all } \omega \in E. \quad (7.13)$$

Since $D^j(\omega)$ is a closed and convex set, for each $\omega \in \text{dom } D(\cdot)$, invoking the Hahn-Banach separation theorem, we obtain that

$$\gamma(\omega) \in D^j(\omega) \quad \mu - \text{a.e.}$$

Taking the limit as $j \rightarrow \infty$, we deduce that $\gamma(\omega) \in \bigcap_{j \in \mathbb{N}} D^j(\omega) = D(\omega) \quad \mu - \text{a.e.}$ which concludes the proof. \square

7.5 Proofs of Theorem 7.3.1 and Theorem 7.3.3

We first employ a standard hypotheses reduction argument establishing that we can, without loss of generality, replace assumptions (A.3)-(A.5) by the stronger conditions in which $\delta = +\infty$ (i.e. the conditions are satisfied globally).

(A.3)' There exist a constant $c > 0$ and an integrable function $k_f : [0, T] \rightarrow \mathbb{R}$ such that

$$|f(t, x, u, \omega) - f(t, x', u, \omega)| \leq k_f(t)|x - x'| \quad \text{and} \quad |f(t, x, u, \omega)| \leq c$$

for all $x, x' \in \mathbb{R}^n$, $u \in U(t)$, $\omega \in \Omega$, a.e. $t \in [0, T]$.

(A.4)' (i) The function g is $\mathcal{B}^n \times \mathcal{B}_\Omega$ measurable.

(ii) There exist positive constants k_g and M_g such that for all $\omega \in \Omega$
 $|g(x, \omega)| \leq M_g$ for all $x \in \mathbb{R}^n$,

$$|g(x, \omega) - g(x', \omega)| \leq k_g|x - x'| \text{ for all } x, x' \in \mathbb{R}^n .$$

(iii) There exists a modulus of continuity $\theta_g(\cdot)$ such that we have

$$|g(x, \omega_1) - g(x, \omega_2)| \leq \theta_g(\rho_\Omega(\omega_1, \omega_2)) \quad \text{for all } \omega_1, \omega_2 \in \Omega \text{ and } x \in \mathbb{R}^n .$$

(A.5)' There exists a modulus of continuity $\theta_f(\cdot)$ such that for all $\omega_1, \omega_2 \in \Omega$,

$$\int_0^T \sup_{u \in U(t), x \in \mathbb{R}^n} |f(t, x, u, \omega_1) - f(t, x, u, \omega_2)| dt \leq \theta_f(\rho_\Omega(\omega_1, \omega_2)).$$

This is possible if we consider the ‘truncation’ function $tr_{y,\delta} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, defined to be

$$tr_{y,\delta}(x) := \begin{cases} x & \text{if } |x - y| < \delta \\ y + \delta \frac{x-y}{|x-y|} & \text{if } |x - y| \geq \delta, \end{cases}$$

and we replace f and g by their local expression \tilde{f} and \tilde{g} defined as follows

$$\tilde{f}(t, x, u, \omega) := f(t, tr_{\bar{x}(t,\omega),\delta}(x), u, \omega) \quad \text{and} \quad \tilde{g}(x, \omega) := g(tr_{\bar{x}(T,\omega),\delta}(x); \omega) .$$

Indeed, the problems involving the functions (f, g) and (\tilde{f}, \tilde{g}) do coincide in a neighborhood of the $W^{1,1}$ -local minimizer $(\bar{u}, \{\bar{x}(\cdot, \omega) \mid \omega \in \Omega\})$ for (P7). Therefore, $(\bar{u}, \{\bar{x}(\cdot, \omega) \mid \omega \in \Omega\})$ does remain a $W^{1,1}$ -local minimizer for the problem (P7) when we substitute the pair (f, g) with (\tilde{f}, \tilde{g}) .

We provide two technical lemmas which will be employed in the approximation techniques used in the theorems proof. These preliminary results establish the uniform continuity of trajectories with respect to ω and the existence of a sequence of suitable finite support measures approximating the reference measure μ . Throughout this section, $d_{\mathcal{G}}(\cdot, \cdot)$ denotes the Ekeland metric defined on the control set \mathcal{U} as

$$d_{\mathcal{G}}(u_1, u_2) := \text{meas} \{t \in [0, T] \mid u_1(t) \neq u_2(t)\}.$$

We recall that, given a control $u(\cdot)$, we shall also employ the alternative notation $x(\cdot, u, \omega)$ for the feasible arc belonging to the family of trajectories $\{x(\cdot, \omega) : \omega \in \Omega\}$ associated with the control $u(\cdot)$.

Lemma 7.5.1. *Let (Ω, ρ_Ω) be a metric space. Suppose that assumptions (A.2)(i)-(ii), (A.3)' and (A.5)' are satisfied. Then,*

(i) *we can find $\beta > 0$ such that, for all $\varepsilon > 0$,*

$$\sup_{\omega \in \Omega} \{ \|x(\cdot, u, \omega) - x(\cdot, u', \omega)\|_{L^\infty} \mid u(\cdot), u'(\cdot) \in \mathcal{U}, d_g(u, u') \leq \beta\varepsilon \} \leq \varepsilon. \quad (7.14)$$

(ii) *for all $\tilde{\varepsilon} > 0$, we can find $\tilde{r} > 0$, such that for any given $u(\cdot) \in \mathcal{U}$,*

$$\|x(\cdot, u, \omega) - x(\cdot, u, \omega')\|_{L^\infty} < \tilde{\varepsilon} \text{ for all } \omega, \omega' \in \Omega \text{ such that } \rho_\Omega(\omega, \omega') < \tilde{r}.$$

Proof. (i). Write

$$\beta := \frac{1}{2c \exp\left(\int_0^T k_f(s) ds\right)}.$$

Fix any $\varepsilon > 0$. Take any $u(\cdot), u'(\cdot) \in \mathcal{U}$ such that $d_g(u, u') < \beta\varepsilon$. Owing to Filippov Existence Theorem (cf. Theorem 1.2.1 or [81, Theorem 2.4.3]) (recall that we have the same initial datum x_0), for each $\omega \in \Omega$, we obtain

$$\begin{aligned} & \|x(\cdot, u, \omega) - x(\cdot, u', \omega)\|_{L^\infty} \\ & \leq \exp\left(\int_0^T k_f(s) ds\right) \int_0^T |f(t, x(t, u', \omega), u'(t), \omega) - f(t, x(t, u', \omega), u(t), \omega)| dt \\ & \leq 2c \exp\left(\int_0^T k_f(s) ds\right) d_g(u, u'). \end{aligned}$$

The last inequality is a consequence of the bound on the dynamic (assumption (A.3)'). The definition of Ekeland metric and the choice of β imply that for any $\varepsilon > 0$

$$\|x(\cdot, u, \omega) - x(\cdot, u', \omega)\|_{L^\infty} < \varepsilon \quad \text{for each } \omega \in \Omega.$$

(ii). Fix now any $\tilde{\varepsilon} > 0$. Take a control $u(\cdot) \in \mathcal{U}$. Choose $\tilde{r} > 0$ such that

$$\theta_f(r') \leq \frac{\tilde{\varepsilon}}{\exp\left(\int_0^T k_f(s) ds\right)} \quad \text{for all } 0 < r' \leq \tilde{r}.$$

Take $\omega, \omega' \in \Omega$ such that $\rho_\Omega(\omega, \omega') < \tilde{r}$. Taking two different trajectories $x(\cdot, u, \omega)$ and $x(\cdot, u, \omega')$ with the same initial point x_0 and the same control $u(\cdot)$, for all $t \in [0, T]$ we have,

$$\begin{aligned} & |x(t, u(t), \omega) - x(t, u(t), \omega')| \\ & \leq \int_0^t |f(s, x(s, \omega), u(s), \omega) - f(s, x(s, \omega'), u(s), \omega')| ds \\ & \leq \int_0^t |f(s, x(s, \omega), u(s), \omega) - f(s, x(s, \omega'), u(s), \omega)| ds \\ & \quad + \int_0^t |f(s, x(s, \omega'), u(s), \omega) - f(s, x(s, \omega'), u(s), \omega')| ds. \end{aligned} \quad (7.15)$$

Taking into account assumptions (A.3)' and (A.5)', we conclude that

$$|x(t, u(t), \omega) - x(t, u(t), \omega')| \leq \int_0^t k_f(s) |x(s, \omega) - x(s, \omega')| ds + \theta_f(\rho_\Omega(\omega, \omega')) .$$

Applying Gronwall Lemma (cf. Lemma 1.2.2), for all $t \in [0, T]$, we deduce

$$|x(t, u(t), \omega) - x(t, u(t), \omega')| \leq \theta_f(\rho_\Omega(\omega, \omega')) \exp \left(\int_0^t k_f(s) ds \right) .$$

The particular choice of \tilde{r} and assumption (A.5)' ($\lim_{s \downarrow 0} \theta_f(s) = 0$) allow to conclude the proof. \square

Lemma 7.5.2. *Suppose that conditions (A.1), (A.2)(i)-(ii), (A.3)'-(A.5)' are satisfied, and $\mu \in \mathcal{M}(\Omega)$. Then, there exist a sequence of finite subsets of Ω , $\{\Omega^\ell := \{\omega_j^\ell : j = 0, \dots, N_\ell\}\}_{\ell \geq 1}$ and a sequence of convex combinations of Dirac measures $\{\mu_\ell\}_{\ell \geq 1}$, such that the following properties are satisfied.*

(i) $\Omega^\ell \subset \Omega^{\ell+1}$ for all integer $\ell \geq 1$, and $\widehat{\Omega} := \bigcup_{\ell \geq 1} \Omega^\ell$ is a countable dense subset of $\text{supp}(\mu)$;

(ii) $\mu_\ell = \sum_{j=0}^{N_\ell} \alpha_j^\ell \delta_{\omega_j^\ell}$, where $\alpha_j^\ell \in (0, 1]$ and $\sum_{j=0}^{N_\ell} \alpha_j^\ell = 1$, and $\mu_\ell \xrightarrow{*} \mu$;

(iii) for each $\varepsilon > 0$, we can find $\ell_\varepsilon \in \mathbb{N}$ such that for all $\ell \geq \ell_\varepsilon$,

$$\left| \int_{\Omega} g(x(T, u, \omega); \omega) d\mu_\ell - \int_{\Omega} g(x(T, u, \omega); \omega) d\mu \right| \leq \varepsilon \quad \text{for all } u(\cdot) \in \mathcal{U} .$$

Proof. (i). Since Ω is a complete separable metric space, the measure μ is tight. As a consequence, for all integer $\ell \geq 1$, there exists a compact set $K_\ell \subset \Omega$ such that $\mu(\Omega \setminus K_\ell) < \frac{1}{\ell}$. Write $\Omega_0^\ell := (\Omega \setminus K_\ell) \cap \text{supp}(\mu)$.

Since $K_\ell \cap \text{supp}(\mu)$ is a compact metric space, then for all $\omega \in K_\ell \cap \text{supp}(\mu)$, there exists a certain radius r_ω and open balls $\text{int } B(\omega, \frac{1}{r_\omega})$ such that

$$K_\ell \cap \text{supp}(\mu) \subset \bigcup_{\omega \in \Omega} \text{int } B\left(\omega, \frac{1}{r_\omega}\right) .$$

By a standard compactness argument, there exists $N_\ell \in \mathbb{N}$ and points ω_j^ℓ , for $j = 1, \dots, N_\ell$ such that

$$K_\ell \cap \text{supp}(\mu) \subset \bigcup_{j=1}^{N_\ell} \text{int } B\left(\omega_j^\ell, \frac{1}{2\ell}\right) .$$

For a fixed $\ell \in \mathbb{N}$, we denote by $\Omega^\ell := \{\omega_1^\ell, \dots, \omega_{N_\ell}^\ell\}$ the finite subset of Ω . Set $\omega_j^{\ell+1} := \omega_j^\ell$ for all $j = 1, \dots, N_\ell$. It is easy to notice that

$$\widehat{\Omega} := \left(K_\ell \cap \text{supp}(\mu) \right) \setminus \bigcup_{j=1}^{N_\ell} \text{int } B\left(\omega_j^{\ell+1}, \frac{1}{2(\ell+1)}\right)$$

is compact. It can be covered by open balls $\text{int } B(\omega, \frac{1}{r'_\omega})$ where $\omega \in \hat{\Omega}$ and r'_ω is some positive radius. By standard compactness argument, there exists $\hat{N}_\ell \in \mathbb{N}$ and points $\hat{\omega}_1, \dots, \hat{\omega}_{\hat{N}_\ell}$ in $\hat{\Omega}$ such that

$$\hat{\Omega} \subset \bigcup_{k=1}^{\hat{N}_\ell} \text{int } B\left(\hat{\omega}_k, \frac{1}{2(\ell+1)}\right).$$

We can now complete the family $\{\omega_j^{\ell+1} := \omega_j^\ell : j = 1, \dots, N_\ell\}$ (which coincides with the set Ω_ℓ) with

$$\{\omega_j^{\ell+1} := \hat{\omega}_{j-N_\ell} \quad \text{for } j = N_\ell + 1, \dots, N_\ell + \hat{N}_\ell =: N_{\ell+1}\}.$$

The union of the two sets is written $\Omega^{\ell+1}$. Thus, we can construct an increasing sequence of finite sets $\Omega^\ell \subset \Omega^{\ell+1} \subset \dots$, and we denote by $\hat{\Omega}$ the union of these sets:

$$\hat{\Omega} := \bigcup_{\ell \geq 1} \Omega^\ell,$$

equivalently, there exists a countable sequence $\{\omega_k\}$ such that $\hat{\Omega} := \{\omega_k\}_{k \geq 1}$ and $\hat{\Omega}$ is dense in $\text{supp}(\mu)$. Denote now

$$\begin{aligned} \Omega_1^\ell &:= B\left(\omega_1^\ell, \frac{1}{2^\ell}\right) \quad \text{and} \\ \Omega_j^\ell &:= B\left(\omega_j^\ell, \frac{1}{2^\ell}\right) \setminus \bigcup_{k=1}^{j-1} \Omega_k^\ell \quad \text{for all } j = 2, \dots, N_\ell. \end{aligned}$$

Since $K_\ell \cap \text{supp}(\mu)$ is totally bounded, we can write

$$K_\ell \cap \text{supp}(\mu) = \bigcup_{j=1}^{N_\ell} \Omega_j^\ell$$

such that $\Omega_j^\ell \cap \Omega_k^\ell = \emptyset$, $j \neq k$, $\Omega_j^\ell \in \mathcal{B}_\Omega$ and $\text{diam}(\Omega_j^\ell) \leq \frac{1}{2^\ell}$, for all $j = 1, \dots, N_\ell$. Here $\text{diam}(\Omega_j^\ell) = \sup_{a, b \in \Omega_j^\ell} \rho_\Omega(a, b)$.

Therefore, employing this iterative argument, a suitable choice of the compact set K_ℓ allows to obtain, for each $\ell \geq 1$, a family of disjoint Borel subsets $\{\Omega_j^\ell\}_{j=0, \dots, N_\ell}$, for some $N_\ell \in \mathbb{N}$, such that the following properties are satisfied:

- (a) $\text{supp}(\mu) = \bigcup_{j=0}^{N_\ell} \Omega_j^\ell$ (this is because $\text{supp}(\mu) = \Omega_0^\ell \cup (K_\ell \cap \text{supp}(\mu))$);
- (b) for each $j \in \{1, \dots, N_\ell\}$, $\overline{\Omega_j^\ell} \subset K_\ell$ (and so $\overline{\Omega_j^\ell}$ is compact whenever $j \neq 0$) and $\text{diam}(\Omega_j^\ell) \leq \frac{1}{2^\ell}$.
- (c) $\mu(\Omega_0^\ell) < \frac{1}{2}$ and $\Omega_0^\ell \supset \Omega_0^{\ell+1}$.

We can also choose elements $\omega_j^\ell \in \Omega_j^\ell$, for all $j = 0, 1, \dots, N_\ell$, in such a manner that we have $\{\omega_j^\ell\}_{j=0, \dots, N_\ell} \subset \{\omega_j^{\ell+1}\}_{j=0, \dots, N_{\ell+1}}$. If $\text{supp}(\mu)$ is compact, then we can always assume that $\Omega_0^\ell = \emptyset$ for all integer $\ell \geq 1$. In this case, we can relabel the elements chosen in the Borel sets Ω_j^ℓ 's, taking

$$\omega_j^\ell \in \Omega_{j+1}^\ell, \quad \text{for all } j = 0, 1, \dots, N_\ell - 1$$

and we replace N_ℓ with $\tilde{N}_\ell := N_\ell - 1$. In any case, we obtain, for each $\ell \geq 1$, a finite set $\Omega^\ell := \{\omega_j^\ell\}_j$ such that $\Omega^\ell \subset \Omega^{\ell+1}$.

(ii). We assume here that $\text{supp}(\mu)$ is not compact (the compact case can be treated in a similar and easier way). Consider, for each $\ell \geq 1$, the family of Borel disjoint subsets of Ω , $\{\Omega_j^\ell\}_{j=0, \dots, N_\ell}$ and the finite sequence of elements $\{\omega_j^\ell\}_{j=0, \dots, N_\ell}$, with $\omega_j^\ell \in \Omega_j^\ell$, provided in the proof of (i). We define the measure μ_ℓ

$$\mu_\ell := \sum_{j=0}^{N_\ell} \mu(\Omega_j^\ell) \delta_{\omega_j^\ell}.$$

Owing to Theorem 7.4.1, we can check the weak* convergence of the sequence $\{\mu_\ell\}$ on the set of bounded real valued uniformly continuous functions on (Ω, ρ_Ω) (instead of the set of bounded continuous functions). Take any bounded uniformly continuous function $h : \Omega \rightarrow \mathbb{R}$. Write $M := \sup_{\omega \in \Omega} |h(\omega)|$. Fix any $\varepsilon > 0$. Then, there exists $r_\varepsilon > 0$ such that

$$|h(\omega_1) - h(\omega_2)| \leq \frac{\varepsilon}{6} \quad \text{for all } \omega_1, \omega_2 \in \Omega \quad \text{with } \rho_\Omega(\omega_1, \omega_2) \leq r_\varepsilon.$$

Let $\ell_\varepsilon \in \mathbb{N}$ such that $\frac{1}{\ell_\varepsilon} \leq \min\{r_\varepsilon; \frac{\varepsilon}{4M}\}$. Then for all $\ell \geq \ell_\varepsilon$, we have

$$\int_{\Omega} h \, d\mu_\ell - \int_{\Omega} h \, d\mu = \sum_{j=0}^{N_\ell} \mu(\Omega_j^\ell) h(\omega_j^\ell) - \sum_{j=0}^{N_\ell} \int_{\Omega_j^\ell} h(\omega) \, d\mu(\omega) = \sum_{j=0}^{N_\ell} \int_{\Omega_j^\ell} h(\omega_j^\ell) - h(\omega) \, d\mu(\omega). \quad (7.16)$$

For each $j \in \{1, \dots, N_\ell\}$, we define

$$\beta_j^\ell := \inf_{\omega \in \Omega_j^\ell} h(\omega) \quad \text{and} \quad \gamma_j^\ell := \sup_{\omega \in \Omega_j^\ell} h(\omega),$$

and, recall that, from the construction employed in the proof of (i), each $\overline{\Omega_j^\ell}$ is compact (we are considering $j \neq 0$), we can find $y_j^\ell, z_j^\ell \in \Omega_j^\ell$ such that

$$h(y_j^\ell) \leq \beta_j^\ell + \frac{\varepsilon}{6} \quad \text{and} \quad h(z_j^\ell) \geq \gamma_j^\ell - \frac{\varepsilon}{6}.$$

Then for all $\ell \geq \ell_\varepsilon$,

$$\gamma_j^\ell - \beta_j^\ell \leq h(z_j^\ell) - h(y_j^\ell) + \frac{2}{6}\varepsilon \leq \frac{\varepsilon}{2}, \quad \text{for all } j = 1, \dots, N_\ell. \quad (7.17)$$

As a consequence, for all $\ell \geq \ell_\varepsilon$, using (7.16) we deduce

$$\begin{aligned} & \left| \int_{\Omega} h d\mu_\ell - \int_{\Omega} h d\mu \right| \\ & \leq \sum_{j=0}^{N_\ell} \int_{\Omega_j^\ell} |h(\omega_j^\ell) - h(\omega)| d\mu(\omega) \\ & \leq \sum_{j=1}^{N_\ell} \int_{\Omega_j^\ell} \left(\sup_{\omega' \in \Omega_j^\ell} h(\omega') - \inf_{\omega'' \in \Omega_j^\ell} h(\omega'') \right) d\mu(\omega) + \int_{\Omega_0^\ell} |h(\omega_0^\ell) - h(\omega)| d\mu(\omega). \end{aligned}$$

Then, from inequality (7.17) and the choice of ℓ_ε , for all $\ell \geq \ell_\varepsilon$, we obtain

$$\left| \int_{\Omega} h d\mu_\ell - \int_{\Omega} h d\mu \right| \leq \sum_{j=1}^{N_\ell} \int_{\Omega_j^\ell} (\gamma_j^\ell - \beta_j^\ell) d\mu + 2M\mu(\Omega_0^\ell) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon.$$

Setting $\alpha_j^\ell := \mu(\Omega_j^\ell) > 0$, for $j = 0, \dots, N_\ell$, we conclude the proof of (ii).

(iii). Fix any $\varepsilon > 0$. Choose $r_0 > 0$ such that

$$\theta_g(r) \leq \frac{\varepsilon}{4} \quad \text{for all } 0 < r \leq r_0.$$

Take any $\omega_1, \omega_2 \in \Omega$ such that $\rho_\Omega(\omega_1, \omega_2) < r_0$. Then, from assumption (A4)'(iii)

$$|g(x, \omega_1) - g(x, \omega_2)| < \frac{\varepsilon}{4} \quad \text{for all } x \in \mathbb{R}^n.$$

Take any $u(\cdot) \in \mathcal{U}$. From Lemma 7.5.1(ii), there exists $\tilde{r} > 0$ such that for all $\omega_1, \omega_2 \in \Omega$ verifying $\rho_\Omega(\omega_1, \omega_2) < \tilde{r}$, we have

$$|x(t, u, \omega_1) - x(t, u, \omega_2)| \leq \frac{\varepsilon}{4k_g} \quad \text{for all } t \in [0, T].$$

Write $r_\varepsilon := \min\{\tilde{r}, r_0\}$. For all $\omega_1, \omega_2 \in \Omega$ verifying $\rho_\Omega(\omega_1, \omega_2) \leq r_\varepsilon$, from assumption (A4)'(ii), we deduce

$$\begin{aligned} & |g(x(T, u, \omega_1); \omega_1) - g(x(T, u, \omega_2); \omega_2)| \\ & \leq |g(x(T, u, \omega_1); \omega_1) - g(x(T, u, \omega_1); \omega_2)| + |g(x(T, u, \omega_1); \omega_2) - g(x(T, u, \omega_2); \omega_2)| \\ & \leq \frac{\varepsilon}{4} + k_g |x(T, u, \omega_1) - x(T, u, \omega_2)| = \frac{\varepsilon}{2}. \end{aligned}$$

Therefore, for each $u(\cdot) \in \mathcal{U}$, the map $\omega \mapsto g(x(T, u, \omega); \omega)$ is uniformly continuous, and from (A.4)' (uniformly) bounded by the constant M_g (observe that M_g and r_ε above do not depend on $u(\cdot)$). Invoking the same argument employed in the proof of (ii) we conclude that, whenever we fix $\varepsilon > 0$, we can find $\ell_\varepsilon \in \mathbb{N}$ such that for all $\ell \geq \ell_\varepsilon$, we have

$$\left| \int_{\Omega} g(x(T, u, \omega); \omega) d\mu_\ell(\omega) - \int_{\Omega} g(x(T, u, \omega); \omega) d\mu(\omega) \right| \leq \varepsilon.$$

This confirms property (iii). □

Proof of Theorem 7.3.1. The proof is build up in four parts. The first part consists in approximating the reference problem with a given probability measure by an auxiliary problem which involves measures with finite support. This is possible invoking the result on the weak* convergence established in Lemma 7.5.2 and the Ekeland variational Principle. In the second part, we apply necessary optimality conditions (cf. Proposition 7.2.1 previously obtained) for the auxiliary problem. In the third part, we pass to the limit a first time to obtain optimality conditions on a countable dense subset of $\text{supp}(\mu)$. The last part of the proof is devoted to derive, via a second limit-taking process, all the desired necessary conditions of the theorem statement.

1. Take a $W^{1,1}$ -local minimizer $(\bar{u}, \{\bar{x}(\cdot, \omega) : \omega \in \Omega\})$ for problem (P7). It is not restrictive to assume that $\text{supp}(\mu) = \Omega$. Then there exists $\bar{\varepsilon} > 0$ such that

$$\int_{\Omega} g(\bar{x}(T, \omega); \omega) d\mu(\omega) \leq \int_{\Omega} g(x(T, \omega); \omega) d\mu(\omega)$$

for all feasible processes $(u, \{x(\cdot, \omega) : \omega \in \Omega\})$ such that

$$\|\bar{x}(\cdot, \omega) - x(\cdot, \omega)\|_{W^{1,1}} \leq \bar{\varepsilon} \quad \text{for all } \omega \in \Omega (= \text{supp}(\mu)).$$

Consider the sequence of convex combinations of Dirac measures $\{\mu_{\ell}\}$ provided by Lemma 7.5.2. Recall, in particular, that $\mu_{\ell} \xrightarrow{*} \mu$ and

$$\mu_{\ell} = \sum_{j=0}^{N_{\ell}} \alpha_j^{\ell} \delta_{\omega_j^{\ell}}$$

where $\alpha_j^{\ell} \in (0, 1]$, for all $j = 0, \dots, N_{\ell}$ and $\sum_{j=0}^{N_{\ell}} \alpha_j^{\ell} = 1$. Define the set

$$W := \{(u(\cdot), \{x(\cdot, \omega) : \omega \in \Omega\}) \text{ feasible processes for the control system in (P7)} \\ \text{s.t. } \|x(\cdot, \omega) - \bar{x}(\cdot, \omega)\|_{W^{1,1}} \leq \bar{\varepsilon} \text{ for all } \omega \in \Omega\}.$$

Observe that $(W, d_{\mathcal{G}})$ is a complete metric space. Take a sequence $\epsilon_i \downarrow 0$. For each $i \geq 1$, owing to Lemma 7.5.2(iii), there exists $\ell_i \in \mathbb{N}$ such that for all $\ell \geq \ell_i$, we have

$$\left| \int_{\Omega} g(x(T, u, \omega); \omega) d\mu_{\ell} - \int_{\Omega} g(x(T, u, \omega); \omega) d\mu \right| \leq \frac{\epsilon_i^2}{2}$$

for all $(u(\cdot), \{x(\cdot, \omega) : \omega \in \Omega\}) \in W$. Write $\mu_i := \mu_{\ell_i}$ the corresponding convex combination of $(N_i + 1 = N_{\ell_i} + 1)$ Dirac measures which approximate μ , and $\omega_j^i := \omega_j^{\ell_i}$, $j = 0, 1, \dots, N_i$, and $\Omega^i := \{\omega_j^i\}_{j=0}^{N_i}$. For each i , we define the functional $J_i : (W, d_{\mathcal{G}}) \rightarrow \mathbb{R}$ as follows:

$$J_i((u, \{x(\cdot, \omega)\})) := \int_{\Omega} g(x(T, u, \omega); \omega) d\mu_i(\omega) - \int_{\Omega} g(\bar{x}(T, \omega); \omega) d\mu(\omega) + \frac{\epsilon_i^2}{2}.$$

It is clear that $J_i((u, \{x(\cdot, \omega)\})) \geq 0$, for all processes $(u(\cdot), \{x(\cdot, \omega) : \omega \in \Omega\}) \in W$, and

$$J_i((\bar{u}, \{\bar{x}(\cdot, \omega)\})) \leq \inf_{(u, \{x(\cdot, \omega)\} : \omega \in \Omega)} J_i((u, \{x(\cdot, \omega)\})) + \epsilon_i^2.$$

Therefore, $(\bar{u}, \{\bar{x}(\cdot, \omega) : \omega \in \Omega\})$ is an ϵ_i^2 -minimizer for J_i . Then, since J_i is a continuous function on $(W, d_{\mathcal{G}})$, we deduce from Ekeland Theorem (cf. [81, Theorem 3.3.1]) that there exists $(u_i, \{x_i(\cdot, \omega)\}) \in W$ such that

$$d_{\mathcal{G}}(u_i, \bar{u}) \leq \epsilon_i \quad \text{and} \quad (7.18)$$

$$J_i((u_i, \{x_i(\cdot, \omega)\})) + \epsilon_i d_{\mathcal{G}}(u_i, u_i) = \min_{(u, \{x(\cdot, \omega)\}) \in W} \{J_i((u, \{x(\cdot, \omega)\})) + \epsilon_i d_{\mathcal{G}}(u, u_i)\}. \quad (7.19)$$

Now we introduce the $\mathcal{L} \times \mathcal{B}^m$ -measurable function

$$m_i(t, u) := \begin{cases} 0 & \text{if } u = u_i(t) \\ 1 & \text{otherwise.} \end{cases}$$

Then we deduce:

$$d_{\mathcal{G}}(u, u_i) = \int_0^T m_i(t, u(t)) dt.$$

Write $\tilde{J}_i : (W, d_{\mathcal{G}}) \rightarrow \mathbb{R}$ the functional

$$\tilde{J}_i((u, \{x(\cdot, \omega)\})) := J_i((u, \{x(\cdot, \omega)\})) + \epsilon_i d_{\mathcal{G}}(u, u_i) \left(= J_i((u, \{x(\cdot, \omega)\})) + \epsilon_i \int_0^T m_i(t, u(t)) dt \right).$$

The minimizing property (7.19) can be expressed in terms of the following auxiliary optimal control problem

$$(P_i) \left\{ \begin{array}{l} \text{minimize } \tilde{J}_i((u, \{x(\cdot, \omega)\})) = J_i((u, \{x(\cdot, \omega)\})) + \epsilon_i \int_0^T m_i(t, u(t)) dt \\ \text{over controls } u(\cdot) \in \mathcal{U} \text{ such that } u(t) \in U(t) \quad \text{a.e. } t \in [0, T] \\ \text{and family of } W^{1,1} \text{-arcs } \{x(\cdot, \omega)\} \text{ s.t. for all } \omega \in \Omega^i \\ \dot{x}(t, \omega) = f(t, x(t, \omega), u, \omega) \quad \text{a.e. } t \in [0, T] \\ \dot{\gamma}(t) = m_i(t, u(t)) \quad \text{a.e. } t \in [0, T] \\ x(0, \omega) = x_0 \quad \text{and} \quad x(T, \omega) \in C(\omega) \\ \gamma(0) = 0, \end{array} \right.$$

whose minimizer is the family $(u_i, (\gamma_i \equiv 0, \{x_i(\cdot, \omega_j^i) : j = 0, 1, \dots, N_i\}))$ verifying, as $i \rightarrow \infty$, $d_{\mathcal{G}}(u_i, \bar{u}) \rightarrow 0$ and

$$\sup_{\omega \in \Omega} \|\bar{x}(\cdot, \omega) - x(\cdot, u_i, \omega)\|_{L^\infty} \rightarrow 0. \quad (7.20)$$

Observe that (7.20) is a consequence of Lemma 7.5.1 (i).

2. The second step of the proof consists in applying necessary optimality conditions (cf. Proposition 7.2.1) to problem (P_i) for each i sufficiently large: for all $\omega \in \Omega^i$ (that is for μ_i -a.e. $\omega \in \Omega$), there exist $W^{1,1}$ -arcs $p_i(\cdot, \omega)$ (associated with the state variable x), $q_i(\cdot)$ (associated with the variable γ) and a scalar $\lambda_i \in [0, 1]$ such that

$$\lambda_i + \int_{\Omega} \max_{t \in [0, T]} |p_i(t, \omega)| d\mu_i(\omega) + \max_{t \in [0, T]} |q_i(t)| = 1. \quad (7.21)$$

The transversality condition leads to

$$-p_i(T, \omega) \in \lambda_i \partial_x g(x_i(T, \omega); \omega) + N_{C(\omega)}(x_i(T, \omega)) \quad \text{and} \quad -q_i(T) = \lambda_i \epsilon_i. \quad (7.22)$$

The adjoint system gives $-\dot{q}_i(t) \equiv 0$, which implies that $q_i(t) \equiv -\lambda_i \epsilon_i$, and

$$-\dot{p}_i(t, \omega) \in \text{co } \partial_x [p_i(t, \omega) \cdot f(t, x_i(t, \omega), u_i(t, \omega))] \quad \text{a.e. } t \in [0, T]. \quad (7.23)$$

Expliciting the maximality condition, we obtain

$$\begin{aligned} & \int_{\Omega} p_i(t, \omega) \cdot f(t, x_i(t, \omega), u_i(t, \omega)) d\mu_i(\omega) \\ &= \max_{u \in U(t)} \left(\int_{\Omega} p_i(t, \omega) \cdot f(t, x_i(t, \omega), u, \omega) d\mu_i(\omega) - \lambda_i \epsilon_i m_i(t, u) \right) \quad \text{for a.e. } t \in [0, T]. \end{aligned}$$

This implies that for a.e. $t \in [0, T]$ and for every $u \in U(t)$

$$\int_{\Omega} p_i(t, \omega) \cdot [f(t, x_i(t, \omega), u, \omega) - f(t, x_i(t, \omega), u_i(t, \omega))] d\mu_i(\omega) \leq \lambda_i \epsilon_i. \quad (7.24)$$

Property (7.18) yields

$$u_i = \bar{u}(t) \quad \text{on a set } A_{\epsilon_i} \subset [0, T] \text{ such that } \text{meas}([0, T] \setminus A_{\epsilon_i}) \leq \epsilon_i.$$

Therefore, for each i large enough and μ_i -a.e. $\omega \in \Omega$, from the optimality conditions (7.21)-(7.24), we have

- (a1) $\lambda_i + \sum_{\omega \in \Omega^i} \max_{t \in [0, T]} |p_i(t, \omega)| + \lambda_i \epsilon_i = 1$.
- (a2) $-\dot{p}_i(t, \omega) \in \text{co } \partial_x [p_i(t, \omega) \cdot f(t, x_i(t, \omega), \bar{u}(t), \omega)]$ for all $t \in A_{\epsilon_i}$;
- (a3) $-p_i(T, \omega) \in \lambda_i \partial_x g(x_i(T, \omega); \omega) + N_{C(\omega)}(x_i(T, \omega))$;
- (a4) $\int_{\Omega} p_i(t, \omega) \cdot [f(t, x_i(t, \omega), u, \omega) - f(t, x_i(t, \omega), \bar{u}(t), \omega)] d\mu_i(\omega) \leq \lambda_i \epsilon_i$ for all $t \in A_{\epsilon_i}$ and for any $u \in U(t)$.

Observe that from (7.18) and Lemma 7.5.1 (i), we can also deduce that there exists a sequence $\epsilon'_i \downarrow 0$ ($\epsilon'_i := \frac{\epsilon_i}{\beta}$) such that:

$$x_i(t, \omega) \in \bar{x}(t, \omega) + \epsilon'_i \mathbb{B} \quad \text{for all } \omega \in \Omega, \text{ and for all } t \in [0, T].$$

3. We derive now consequences of the limit-taking for conditions (a1)-(a3) of the previous step. Recall that from Lemma 7.5.2, we have a countable dense subset $\widehat{\Omega}$ of $\text{supp}(\mu) \subset \Omega$, such that $\widehat{\Omega} = \bigcup_{i \geq 1} \Omega^i$, where $\Omega^i = \{\omega_j^i : j = 0, \dots, N_i\}$ provides an increasing sequence of finite subsets of Ω : $\Omega^1 \subset \dots \subset \Omega^i \subset \Omega^{i+1} \subset \dots$. Since $\widehat{\Omega}$ is a countable set, we can write it as the collection of the elements of a sequence $\{\omega_k\}_{k \geq 1}$ such that

$$\widehat{\Omega} = \{\omega_k\}_{k \geq 1}.$$

Fix $i \in \mathbb{N}$. When we take $\omega_k \in \widehat{\Omega}$, two possible cases may occur: either $\omega_k \in \Omega^i$ for the fixed $i \in \mathbb{N}$; or $\omega_k \in \widehat{\Omega} \setminus \Omega^i$. In the first case, it means that there exists $j \in \{0, \dots, N_i\}$ such that $\omega_k = \omega_j^i$ and the corresponding adjoint arc $p_i(\cdot, \omega_j^i)$ satisfies conditions (a1)–(a4). So, we can define the arc $p_i(\cdot, \omega_k)$ as follows:

$$p_i(\cdot, \omega_k) := \begin{cases} p_i(\cdot, \omega_j^i) & \text{if } \omega_k \in \Omega^i \text{ (and } \omega_j^i = \omega_k) \\ 0 & \text{if } \omega_k \in \widehat{\Omega} \setminus \Omega^i. \end{cases}$$

Therefore, by iterating on i , associated to each $\omega_k \in \widehat{\Omega}$, we can construct a sequence of families of arcs $\{p_i(\cdot, \omega_k) : \omega_k \in \widehat{\Omega}\}_{i \geq 1}$. Observe that there exists always $i_k \in \mathbb{N}$ such that, for all $i \geq i_k$, $p_i(\cdot, \omega_k)$ is an adjoint arc for which (a1)–(a4) hold true. Condition (a1) implies that, for each fixed k , the sequence $\{p_i(\cdot, \omega_k)\}_i$ is bounded, and $\{\dot{p}_i(\cdot, \omega_k)\}$ is integrally bounded. The hypotheses are satisfied under which the Compactness Theorem (see Theorem 1.2.6 or [81, Theorem 2.5.3]) is applicable to

$$-\dot{p}_i(t, \omega_k) \in \text{co } \partial_x [p_i(t, \omega_k) \cdot f(t, x_i(t, \omega_k), \bar{u}(t), \omega_k)] \quad \text{for all } t \in A_{\varepsilon_i}.$$

We conclude that, along some subsequence (we do not relabel),

$$p_i(\cdot, \omega_k) \xrightarrow{i} \widehat{p}(\cdot, \omega_k) \quad \text{and} \quad \dot{p}_i(t, \omega_k) \rightharpoonup \widehat{\dot{p}}(t, \omega_k) \quad \text{weakly in } L^1 \quad (7.25)$$

for some $\widehat{p}(\cdot, \omega_k) \in W^{1,1}$ which satisfies (for the fixed k)

$$-\widehat{\dot{p}}(t, \omega_k) \in \text{co } \partial_x [\widehat{p}(t, \omega_k) \cdot f(t, \bar{x}(t, \omega_k), \bar{u}(t), \omega_k)] \quad \text{a.e. } t \in [0, T].$$

We can also take the subsequence in such a manner that $\{\lambda_i\}$ converges to some $\lambda \in [0, 1]$. Moreover, from the closure of the graph of the limiting subdifferential and the normal cone (seen as multifunctions), we have that

$$-\widehat{p}(T, \omega_k) \in \lambda \partial_x g(\bar{x}(T, \omega_k); \omega_k) + N_{C(\omega_k)}(\bar{x}(T, \omega_k)).$$

But $\widehat{\Omega} = \{\omega_k\}_k$ is a countable set. Then, we can repeat the similar analysis for each $\omega_k \in \widehat{\Omega}$, taking into account the subsequence obtained for the previous element ω_{k-1} . As a consequence, we have a collection of subsequences $\{\widehat{p}_i(\cdot, \omega)\}$ verifying the convergence properties (7.25) to a collection of adjoint arcs $\{\widehat{p}(\cdot, \omega)\}$ which satisfies, for all $\omega \in \widehat{\Omega}$

$$-\widehat{\dot{p}}(t, \omega) \in \text{co } \partial_x [\widehat{p}(t, \omega) \cdot f(t, \bar{x}(t, \omega), \bar{u}(t), \omega)] \quad \text{a.e. } t \in [0, T] \quad (7.26)$$

and

$$-\widehat{p}(T, \omega) \in \lambda \partial_x g(\bar{x}(T, \omega); \omega) + N_{C(\omega)}(\bar{x}(T, \omega)). \quad (7.27)$$

Furthermore, since for all i , $\widehat{p}_i(\cdot, \cdot)$ is $\mathcal{L} \times \mathcal{B}_{\widehat{\Omega}}$ measurable, we obtain that its limit $\widehat{p}(\cdot, \cdot)$ is also $\mathcal{L} \times \mathcal{B}_{\widehat{\Omega}}$ measurable. From [38, Theorem 4.2.5], we can extend $\widehat{p}(\cdot, \cdot)$ to a $\mathcal{L} \times \mathcal{B}_{\Omega}$ measurable function $p(\cdot, \cdot)$ on $[0, T] \times \Omega$ which satisfies conditions (7.26) and (7.27) for all $\omega \in \widehat{\Omega}$, and such that $\|p(\cdot, \cdot)\|_{L^\infty} \leq 1$. Finally, passing to the limit as $i \rightarrow \infty$ in condition (a1) gives the nontriviality condition

$$\lambda + \sum_{\omega \in \widehat{\Omega}} \max_{t \in [0, T]} |p(t, \omega)| = 1 \quad (7.28)$$

4. In the last part of the proof, we want to use also the information contained in the maximality condition (a4) (or in its alternative version (7.24)) as $i \rightarrow \infty$. This task requires to use Castaing Representation Theorem (cf. [27, Theorem III.7], the Aumann's Measurable Selection Theorem (cf. [27, Theorem III.22]), and Lemma 7.4.3 which has a central role for the limit-taking of all the necessary conditions obtained in step 2 at the same time. Write

$$F(t, \omega) := f(t, \bar{x}(t, \omega), U(t), \omega) .$$

Owing to assumption (A.2) and the Lipschitz continuity of $f(t, \cdot, u, \omega)$, we obtain that $(t, \omega) \rightsquigarrow F(t, \omega)$ is a $\mathcal{L} \times \mathcal{B}_\Omega$ measurable with closed values (cf. Appendix B for the proof). Using Castaing Representation Theorem, we know that there exists a countable family of $\mathcal{L} \times \mathcal{B}_\Omega$ measurable functions $\{f_j(t, \omega)\}_{j \geq 0}$, such that

$$F(t, \omega) = \overline{\bigcup_{j \geq 0} \{f_j(t, \omega)\}} \quad \text{for all } (t, \omega) \in E ,$$

in which $E \subset [0, T] \times \Omega$ is a set of full-measure. We can also assume that $f_0(t, \omega) = f(t, \bar{x}(t, \omega), \bar{u}(t), \omega)$. For all $j \geq 1$, define the multifunction

$$\tilde{U}_j(t, \omega) := \begin{cases} \bar{u}(t) & \text{if } (t, \omega) \notin E \\ \{u \in U(t) : f_j(t, \omega) = f(t, \bar{x}(t, \omega), u, \omega)\} & \text{if } (t, \omega) \in E . \end{cases}$$

The graph of $\tilde{U}_j(\cdot, \cdot)$ is a $\mathcal{L} \times \mathcal{B}_\Omega \times \mathcal{B}^m$ measurable set. Indeed, we have

$$\begin{aligned} \text{Gr } \tilde{U}_j(\cdot, \cdot) &= \{((t, \omega), u) : u \in U(t), (t, \omega) \in E \text{ and } f(t, \bar{x}(t, \omega), u, \omega) - f_j(t, \omega) = 0\} \\ &\quad \bigcup \{((t, \omega), u) : (t, \omega) \notin E, u = \bar{u}(t)\} , \end{aligned}$$

which is the union of two $\mathcal{L} \times \mathcal{B}_\Omega \times \mathcal{B}^m$ measurable sets. Now invoking Aumann's Measurable Selection Theorem, $\tilde{U}_j(\cdot, \cdot)$ has a measurable selection $v_j(t, \omega) \in \tilde{U}_j(t, \omega)$.

Let now \mathcal{D} be a countable and dense subset of $[0, T]$. Consider the sequence of intervals $\{[s_i, t_i]\}_{i \geq 1}$ having extrema in $\mathcal{D} : \bigcup_{i \geq 1} [s_i, t_i] = \mathcal{D}$. We construct now a further countable family of controls $\{\tilde{v}_{j,i}(t, \omega)\}_{j \geq 1, i \geq 1}$ as follows

$$\tilde{v}_{j,i}(t, \omega) := \begin{cases} v_j(t, \omega) & \text{on } [s_i, t_i] \times \Omega \\ \bar{u}(t) & \text{on } ([0, T] \setminus [s_i, t_i]) \times \Omega . \end{cases} \quad (7.29)$$

Writing $\{\tilde{u}_k(t, \omega)\}_{k \geq 0} = \{\tilde{v}_{j,i}(t, \omega)\}_{j \geq 1, i \geq 1} \cup \{\bar{u}(\cdot)\}$, in such a manner that (up to a reordering) $\tilde{u}_0(\cdot, \omega) = \bar{u}(\cdot)$, we obtain

$$F(t, \omega) = \overline{\bigcup_{k \geq 0} \{f(t, \bar{x}(t, \omega), \tilde{u}_k(t, \omega), \omega)\}} \quad \text{for all } (t, \omega) \in E . \quad (7.30)$$

Following an effective technique proposed by Vinter [83], for a fixed integer K , we introduce the operators $\Psi_k(\cdot, \cdot)$ and $\Psi_k^i(\cdot, \cdot)$ on $W^{1,1}([0, T], \mathbb{R}^n) \times \Omega$ (linear with respect to their first variable): for $k = 1, \dots, K$, we set

$$\Psi_k(p(\cdot), \omega) := \int_0^T p(t) \cdot [f(t, \bar{x}(t, \omega), \tilde{u}_k(t, \omega), \omega) - f(t, \bar{x}(t, \omega), \bar{u}(t), \omega)] dt ,$$

and, for all integers $i \geq 1$,

$$\Psi_k^i(p(\cdot), \omega) := \int_0^T p(t) \cdot [f(t, x_i(t, \omega), \tilde{u}_k(t, \omega), \omega) - f(t, x_i(t, \omega), u_i(t, \omega))] dt .$$

Define also the subsets D_i , for all $i \geq 1$, and D of $\Omega \times \mathbb{R}^K$ as follows:

$D_i := \{(\omega, \xi) \in \Omega \times \mathbb{R}^K \mid \omega \in \Omega \text{ and } \xi = (\Psi_k^i(p(\cdot, \omega), \omega))_{k=1, \dots, K} \text{ for some } \mathcal{L} \times \mathcal{B}_\Omega \text{ measurable function } p : [0, T] \times \Omega \rightarrow \mathbb{R}^n \text{ such that } p(\cdot, \omega) \in \mathcal{P}_i(\omega) \text{ for all } \omega \in \Omega^i\}$, where $\{\Omega^i\}$ is the increasing sequence of (finite) subsets introduced in step 3 (cf. Lemma 7.5.2) and

$\mathcal{P}_i(\omega)$

$$:= \left\{ q(\cdot, \omega) \in W^{1,1} : \|q(\cdot, \cdot)\|_{L^\infty} \leq 1, -\dot{q}(t, \omega) \in \text{co } \partial_x [q(t, \omega) \cdot f(t, \bar{x}(t, \omega) + \epsilon'_i \mathbb{B}, \bar{u}(t), \omega)] \right.$$

on a set A_i such that $\text{meas}([0, T] \setminus A_i) \leq \epsilon_i$, and there exists $\lambda_i \in [0, 1]$ such that

$$\left. \lambda_i + \sum_{\omega \in \Omega^i} \max_{t \in [0, T]} |q(t, \omega)| + \lambda_i \epsilon_i = 1 \text{ and } -q(T, \omega) \in \bigcup_{x \in \bar{x}(T, \omega) + \epsilon'_i \mathbb{B}} \lambda_i \partial_x g(x, \omega) + N_{C(\omega)}(x) \right\} .$$

The set D is written

$D := \{(\omega, \xi) \in \Omega \times \mathbb{R}^K \mid \omega \in \Omega \text{ and } \xi = (\Psi_k(p(\cdot, \omega), \omega))_{k=1, \dots, K} \text{ for some } \mathcal{L} \times \mathcal{B}_\Omega \text{ measurable function } p : [0, T] \times \Omega \rightarrow \mathbb{R}^n \text{ such that } p(\cdot, \omega) \in \text{co } \mathcal{P}(\omega) \text{ for all } \omega \in \widehat{\Omega}\}$

where $\widehat{\Omega}$ is the countable dense subset of $\text{supp}(\mu)$ provided by Lemma 7.5.2 and

$$\mathcal{P}(\omega) := \left\{ q(\cdot, \omega) \in W^{1,1}([0, T], \mathbb{R}^n) : \text{for some } \lambda \in [0, 1], \text{ we have } \|q(\cdot, \cdot)\|_{L^\infty} \leq 1, \right. \\ \left. \lambda + \sum_{\omega \in \widehat{\Omega}} \max_{t \in [0, T]} |q(t, \omega)| = 1, -\dot{q}(t, \omega) \in \text{co } \partial_x [q(t, \omega) \cdot f(t, \bar{x}(t, \omega), \bar{u}(t), \omega)] \text{ a.e. } t, \right. \\ \left. -q(T, \omega) \in \lambda \partial_x g(\bar{x}(T, \omega); \omega) + N_{C(\omega)}(\bar{x}(T, \omega)) \right\} .$$

Now, we define the multifunctions $D_i(\cdot)$, for $i = 1, 2, \dots$, and $D(\cdot)$ on Ω , taking values in the subsets of \mathbb{R}^K as follow:

$$D_i(\omega) := \{(\xi_1, \dots, \xi_K) \in \mathbb{R}^K : (\omega, \xi) \in D_i\} \text{ and } D(\omega) := \{(\xi_1, \dots, \xi_K) \in \mathbb{R}^K : (\omega, \xi) \in D\} .$$

The multifunctions $\omega \rightsquigarrow D(\omega)$ and $\omega \rightsquigarrow D_i(\omega)$, for all i , are uniformly bounded. The necessary optimality conditions (a1)-(a3) corresponding to the auxiliary problem (P_i) of step 2 guarantee that the set $D_i(\omega)$ is non-empty : indeed there exist $\mathcal{L} \times \mathcal{B}_\Omega$ measurable functions $p_i : [0, T] \times \Omega \rightarrow \mathbb{R}^n$ such that $p_i(\cdot, \omega) \in \mathcal{P}_i(\omega)$ μ_i -a.e. $\omega \in \Omega$ and so

$$(\Psi_k^i(p_i(\cdot, \omega), \omega))_{k=1, \dots, K} \in D_i(\omega) \quad \mu_i - \text{a.e. } \omega \in \Omega .$$

Moreover, the linearity of the operator Ψ_k with respect to the first variable p and the convexity of the set $\text{co } \mathcal{P}(\omega)$ guarantee the convexity of the set $D(\omega)$ for each $\omega \in \text{dom } D(\cdot)$. It follows that hypotheses (i)-(iii) of Lemma 7.4.3 are satisfied. We claim that

$$\limsup_{i \rightarrow \infty} D_i \subset D .$$

Indeed, take any $(\omega, \xi) \in \limsup_{i \rightarrow \infty} D_i$. From the definition of the limsup in the Kuratowski sense, there exists a subsequence $i_h \rightarrow \infty$ and $(\omega_{i_h}, \xi_{i_h}) \in D_{i_h}$ such that

$$\lim_{i_h \rightarrow \infty} (\omega_{i_h}, \xi_{i_h}) = (\omega, \xi)$$

We shall show that $(\omega, \xi) \in D$. Since $(\omega_{i_h}, \xi_{i_h}) \in D_{i_h}$, there exists a sequence of $\mathcal{L} \times \mathcal{B}_\Omega$ measurable functions $p_{i_h} : [0, T] \times \Omega \rightarrow \mathbb{R}^n$ such that $p_{i_h}(\cdot, \omega) \in \mathcal{P}_{i_h}(\omega)$ for all $\omega \in \Omega^{i_h}$. From the analysis of step 3, we have established the existence of a map p on $[0, T] \times \Omega$ which is $\mathcal{L} \times \mathcal{B}_\Omega$ measurable, verifying conditions (7.26)-(7.28) for all $\omega \in \widehat{\Omega}$, and such that $\|p(\cdot, \cdot)\|_{L^\infty} \leq 1$. Moreover, the uniform convergence of $\{p_{i_h}(\cdot, \omega) : \omega \in \widehat{\Omega}\}$, Lemma 7.5.1 and assumption (A.3)' guarantee that, for $k = 1, \dots, K$ and for all $\omega \in \widehat{\Omega}$,

$$\int_0^T p_{i_h}(t, \omega) \cdot [f(t, x_{i_h}(t, \omega), \widetilde{u}_k(t, \omega), \omega) - f(t, x_{i_h}(t, \omega), u_i(t), \omega)] dt$$

converges, as $i_h \rightarrow \infty$, to

$$\int_0^T p(t, \omega) \cdot [f(t, \bar{x}(t, \omega), \widetilde{u}_k(t, \omega), \omega) - f(t, \bar{x}(t, \omega), \bar{u}(t), \omega)] dt .$$

Therefore, $(\omega, \xi) \in D$ and the claim is confirmed. Consequently, all required hypotheses of Lemma 7.4.3 are satisfied for $\gamma_i(\omega) = (\gamma_{i,1}(\omega), \dots, \gamma_{i,K}(\omega))$ where for $k = 1, \dots, K$,

$$\gamma_{i,k}(\omega) = \int_0^T p_i(t, \omega) \cdot [f(t, x_i(t, \omega), \widetilde{u}_k(t, \omega), \omega) - f(t, x_i(t, \omega), u_i(t), \omega)] dt$$

which is μ_i -measurable. Defining, for each i , the vector-valued measure $\eta_i := \gamma_i \mu_i$, and applying Lemma 7.4.3, we obtain, along a subsequence (we do not relabel) $\eta_i \xrightarrow{*} \eta$ where η is a vector-valued Borel measure on Ω such that $d\eta(\omega) = \gamma(\omega) d\mu(\omega)$, for some Borel measurable function $\gamma : \Omega \rightarrow \mathbb{R}^K$ satisfying

$$\gamma(\omega) \in D(\omega) \quad \mu - \text{a.e. } \omega \in \Omega .$$

In addition, from the definition of the set D (associated with each $K \in \mathbb{N}$), there exists a $\mathcal{L} \times \mathcal{B}_\Omega$ measurable function $p_K : [0, T] \times \Omega \rightarrow \mathbb{R}^n$ such that $\|p_K(\cdot, \cdot)\|_{L^\infty} \leq 1$, $p_K(\cdot, \omega) \in \text{co } \mathcal{P}(\omega)$ for all $\omega \in \widehat{\Omega}$, and $\gamma(\omega) := (\Psi_k(p_K(\cdot, \omega), \omega))_{k=1, \dots, K}$ verifying

$$\int_\Omega \gamma_i(\omega) d\mu_i(\omega) \xrightarrow{i \rightarrow \infty} \int_\Omega \gamma(\omega) d\mu(\omega) .$$

In other terms, for each $k = 1, \dots, K$

$$\begin{aligned} & \int_{\Omega} \int_0^T p_i(t, \omega) \cdot [f(t, x_i(t, \omega), \tilde{u}_k(t, \omega), \omega) - f(t, x_i(t, \omega), u_i(t, \omega))] dt d\mu_i(\omega) \\ & \xrightarrow{i \rightarrow \infty} \int_{\Omega} \int_0^T p_K(t, \omega) \cdot [f(t, \bar{x}(t, \omega), \tilde{u}_k(t, \omega), \omega) - f(t, \bar{x}(t, \omega), \bar{u}(t, \omega))] dt d\mu(\omega) . \end{aligned} \quad (7.31)$$

The maximality condition (7.24), after inserting $u = \tilde{u}_k(t, \omega)$, gives

$$\int_{\Omega} p_i(t, \omega) \cdot [f(t, x_i(t, \omega), \tilde{u}_k(t, \omega), \omega) - f(t, x_i(t, \omega), u_i(t, \omega))] d\mu_i(\omega) \leq \lambda_i \epsilon_i \text{ a.e. } t \in [0, T]. \quad (7.32)$$

Since the integrand function in (7.32) is $\mathcal{L} \times \mathcal{B}_{\Omega}$ -measurable, and the integral function in (7.32) is \mathcal{L} -measurable, making use of Fubini-Tonelli, we obtain

$$\int_{\Omega} \int_0^T p_i(t, \omega) \cdot [f(t, x_i(t, \omega), \tilde{u}_k(t, \omega), \omega) - f(t, x_i(t, \omega), u_i(t, \omega))] dt d\mu_i(\omega) \leq \lambda_i \epsilon_i T .$$

Therefore, letting $i \rightarrow \infty$ and invoking (7.31), we have that

$$\int_{\Omega} \int_0^T p_K(t, \omega) \cdot [f(t, \bar{x}(t, \omega), \tilde{u}_k(t, \omega), \omega) - f(t, \bar{x}(t, \omega), \bar{u}(t, \omega))] dt d\mu(\omega) \leq 0 . \quad (7.33)$$

For each $K \in \mathbb{N}$, the map $\omega \rightarrow p_K(\cdot, \omega)$ can be interpreted as a \mathcal{B}_{Ω} -measurable element of the μ -a.e. equivalence class in the Hilbert space

$$\mathcal{H} := L^2_{\mu}(\Omega, L^2([0, T]; \mathbb{R}^n))$$

endowed with the inner product

$$\langle p, p' \rangle_{\mu} := \int_{\Omega} \int_0^T p(t, \omega) \cdot p'(t, \omega) dt d\mu(\omega) .$$

Now consider $\widehat{\mathcal{P}}$ to be the set of $\mathcal{L} \times \mathcal{B}_{\Omega}$ measurable functions \widehat{q} of \mathcal{H} defined on $[0, T] \times \Omega$ such that $\widehat{q}(\cdot, \omega) \in \text{co } \mathcal{P}(\omega)$ for all $\omega \in \widehat{\Omega}$:

$$\widehat{\mathcal{P}} := \{\widehat{q} \in \mathcal{H} \mid \widehat{q}(\cdot, \omega) \in \text{co } \mathcal{P}(\omega) \text{ for all } \omega \in \widehat{\Omega}\} .$$

Note that $\widehat{\mathcal{P}}$ is nonempty since $p_K(\cdot, \omega) \in \text{co } \mathcal{P}(\omega)$ for all $\omega \in \widehat{\Omega}$. Moreover, it is a straightforward task to prove that $\widehat{\mathcal{P}}$ is a closed and convex subset in \mathcal{H} (owing to the convexity and the closure of the set $\text{co } \mathcal{P}(\omega)$ for all $\omega \in \widehat{\Omega}$). Therefore, $\widehat{\mathcal{P}}$ is weakly closed, as well. Moreover, the sequence $\{\omega \rightarrow p_K(\cdot, \omega)\}_{K=1}^{\infty}$ is (uniformly) bounded, w.r.t. the norm induced by $\langle \cdot, \cdot \rangle_{\mu}$ because it belongs to the bounded set $\text{co } \mathcal{P}(\omega)$ for all $\omega \in \widehat{\Omega}$. By subsequence extraction (without relabeling), there exists a weakly convergent subsequence to $\{\omega \rightarrow p(\cdot, \omega)\}$ for some

$p \in \widehat{\mathcal{P}}$. The weak convergence $p_K \rightarrow p$ in the Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_\mu)$, employed in inequality (7.33), implies that

$$\int_{\Omega} \int_0^T p(t, \omega) \cdot [f(t, \bar{x}(t, \omega), \tilde{u}_k(t, \omega), \omega) - f(t, \bar{x}(t, \omega), \bar{u}(t, \omega))] dt d\mu(\omega) \leq 0. \quad (7.34)$$

We observe that condition (7.30) yields the following inclusion for all $t \in \mathcal{S}$

$$\int_{\Omega} p(t, \omega) \cdot f(t, \bar{x}(t, \omega), U(t), \omega) d\mu(\omega) \subset \overline{\bigcup_{k \geq 0} \int_{\Omega} p(t, \omega) \cdot f(t, \bar{x}(t, \omega), \tilde{u}_k(t, \omega), \omega) d\mu(\omega)}, \quad (7.35)$$

where \mathcal{S} is a set of full measure in $[0, T]$. Define now the set $\mathcal{S}' \subset \mathcal{S}$, still of full measure in $[0, T]$, containing the Lebesgue points for the map $\Gamma : [0, T] \rightarrow \mathbb{R}$ defined as

$$s \mapsto \Gamma(s) := \int_{\Omega} p(s, \omega) \cdot [f(s, \bar{x}(s, \omega), \tilde{u}_k(s, \omega), \omega) - f(s, \bar{x}(s, \omega), \bar{u}(s, \omega))] d\mu(\omega)$$

for all k . Take any $t \in \mathcal{S}'$ and $u \in U(t)$. Owing to (7.35), there exists a subsequence $\{k_\ell\}_\ell$ such that

$$\int_{\Omega} p(t, \omega) \cdot f(t, \bar{x}(t, \omega), u, \omega) d\mu(\omega) = \lim_{\ell} \int_{\Omega} p(t, \omega) \cdot f(t, \bar{x}(t, \omega), \tilde{u}_{k_\ell}(t, \omega), \omega) d\mu(\omega).$$

In other words, for a sequence $\beta_\ell \downarrow 0$ (eventually taking a subsequence of \tilde{u}_{k_ℓ}), we have

$$\left| \int_{\Omega} p(t, \omega) \cdot f(t, \bar{x}(t, \omega), u, \omega) d\mu(\omega) - \int_{\Omega} p(t, \omega) \cdot f(t, \bar{x}(t, \omega), \tilde{u}_{k_\ell}(t, \omega), \omega) d\mu(\omega) \right| \leq \beta_\ell. \quad (7.36)$$

For the Lebesgue point $t \in \mathcal{S}'$, we can also consider a sequence of intervals $\{[s_i, t_i]\}_{i \geq 1}$, having extrema in a countable dense set \mathcal{D} of $[0, T]$ (in the sense of (7.29)) and such that $s_i \uparrow t$ and $t_i \downarrow t$. Recalling the definition (7.29) of $\tilde{v}_{j,i}$ and replacing in (7.34) \tilde{u}_k by v_j on $[s_i, t_i] \times \Omega$, and by $\bar{u}(t)$ on $([0, T] \setminus [s_i, t_i]) \times \Omega$, using Fubini-Tonelli (since the integrand is $\mathcal{L} \times \mathcal{B}_\Omega$ -measurable) and dividing across by $|t_i - s_i|$, we obtain

$$\frac{1}{|t_i - s_i|} \int_{s_i}^{t_i} \int_{\Omega} p(s, \omega) \cdot [f(s, \bar{x}(s, \omega), v_j(s, \omega), \omega) - f(s, \bar{x}(s, \omega), \bar{u}(s, \omega))] d\mu(\omega) ds \leq 0. \quad (7.37)$$

Since t is a Lebesgue point for the map Γ , we deduce

$$\begin{aligned} & \int_{\Omega} p(t, \omega) \cdot [f(t, \bar{x}(t, \omega), \tilde{u}_k(t, \omega), \omega) - f(t, \bar{x}(t, \omega), \bar{u}(t, \omega))] d\mu(\omega) \\ &= \lim_i \frac{1}{|t_i - s_i|} \int_{s_i}^{t_i} \int_{\Omega} p(s, \omega) \cdot [f(s, \bar{x}(s, \omega), v_j(s, \omega), \omega) - f(s, \bar{x}(s, \omega), \bar{u}(s, \omega))] d\mu(\omega) ds. \end{aligned} \quad (7.38)$$

Therefore, owing to (7.36)-(7.38), we have

$$\int_{\Omega} p(t, \omega) \cdot [f(t, \bar{x}(t, \omega), u, \omega) - f(t, \bar{x}(t, \omega), \bar{u}(t), \omega)] d\mu(\omega) \leq \beta_\ell + 0,$$

for any $\beta_\ell \downarrow 0$ and any $u \in U(t)$. We conclude that

$$\int_{\Omega} p(t, \omega) \cdot [f(t, \bar{x}(t, \omega), u, \omega) - f(t, \bar{x}(t, \omega), \bar{u}(t), \omega)] d\mu(\omega) \leq 0$$

for any $u \in U(t)$ and for all $t \in S'$, a set of full measure in $[0, T]$. Therefore, now all the assertions stated in Theorem 7.3.1 are confirmed (included the maximality condition (ii)), which completes the proof. \square

Proof of Theorem 7.3.3. A scrutiny of Theorem 7.3.1 proof reveals that steps 1, 2 and 3 are applicable providing a simplified result. Indeed, taking into account hypotheses (C.1)-(C.2) on $f(t, \cdot, u, \omega)$ and $g(\cdot, \omega)$, we obtain a family of costate arcs $\tilde{p}(\cdot, \omega)$, for $\omega \in \widehat{\Omega}$ ($\widehat{\Omega}$ is a countable dense subset of Ω), satisfying the properties listed at the end of the step 3 of the proof of Theorem 7.3.1, where (7.26) and (7.27) read now as

$$-\dot{\tilde{p}}(t, \omega) = [\nabla_x f(t, \bar{x}(t, \omega), \bar{u}(t), \omega)]^T \tilde{p}(t, \omega) \quad \text{a.e. } t \in [0, T], \quad (7.39)$$

and

$$-\tilde{p}(T, \omega) = \nabla_x g(\bar{x}(T, \omega), \omega) \quad (7.40)$$

for all $\omega \in \widehat{\Omega}$. Notice, that the multiplier λ cannot take the value 0, for otherwise we would obtain a contradiction with the nontriviality condition (7.28). Then, normalizing we can take $\lambda = 1$.

We claim now that we can extend in a unique way the family of arcs $\tilde{p}(\cdot, \omega)$, for $\omega \in \widehat{\Omega}$, to a $\mathcal{L} \times \mathcal{B}_\Omega$ measurable function $p(\cdot, \cdot) : [0, T] \times \Omega \rightarrow \mathbb{R}^n$ such that for all $\omega \in \Omega$ we have:

$$(i)'' \quad p(\cdot, \omega) \in W^{1,1}([0, T], \mathbb{R}^n);$$

$$(ii)'' \quad -\dot{p}(t, \omega) = [\nabla_x f(t, \bar{x}(t, \omega), \bar{u}(t), \omega)]^T p(t, \omega) \quad \text{a.e. } t \in [0, T];$$

$$(iii)'' \quad -p(T, \omega) = \nabla_x g(\bar{x}(T, \omega); \omega).$$

Indeed, take any $\omega \in \Omega \setminus \widehat{\Omega}$. Then, since $\widehat{\Omega}$ is dense in Ω , there exists a sequence $\{\widehat{\omega}_i\} \subset \widehat{\Omega}$ converging to ω . Assumptions (A.3)' and (A.4)' guarantee that $|\nabla_x f(t, \bar{x}(t, \widehat{\omega}_i), \bar{u}(t), \widehat{\omega}_i)| \leq k_f(t)$ a.e. $t \in [0, T]$ and $|\nabla_x g| \leq k_g$. From (7.39) we deduce that $\{\tilde{p}(\cdot, \widehat{\omega}_i)\}$ is uniformly integrally bounded, and (7.40) guarantees that $|\tilde{p}(T, \widehat{\omega}_i)| \leq k_g$. Then, by a standard compactness argument, taking a subsequence (we do not relabel), there exists $p(\cdot, \omega) \in W^{1,1}([0, T], \mathbb{R}^n)$ such that

$$\begin{aligned} \tilde{p}(t, \widehat{\omega}_i) &\rightarrow p(t, \omega) && \text{uniformly on } [0, T] \text{ as } i \rightarrow \infty \\ \dot{\tilde{p}}(t, \widehat{\omega}_i) &\rightharpoonup \dot{p}(t, \omega) && \text{weakly in } L^1 \end{aligned}$$

and

$$-\dot{p}(t, \omega) = [\nabla_x f(t, \bar{x}(t, \omega), \bar{u}(t), \omega)]^T p(t, \omega) \quad \text{a.e. } t \in [0, T], \quad (7.41)$$

$$-p(T, \omega) = \nabla_x g(\bar{x}(T, \omega); \omega) . \quad (7.42)$$

(The last two equalities are a consequence of Lemma 7.5.1 (ii).)

This, being true for any sequence $\{\widehat{\omega}_i\} \subset \widehat{\Omega}$ converging to $\omega \in \Omega \setminus \widehat{\Omega}$, since the limit arc satisfies the same conditions (7.41)-(7.42), we conclude that we can extend the family of arcs $\widetilde{p}(\cdot, \omega)$ simply taking the limit:

$$p(\cdot, \omega) := \lim_{\rho_{\Omega}(\omega, \widehat{\omega}) \rightarrow 0, \widehat{\omega} \in \widehat{\Omega}} \widetilde{p}(\cdot, \widehat{\omega}) , \quad (7.43)$$

confirming the claim above. It remains to prove the Weierstrass condition (ii)'. We follow exactly the same analysis of step 4 of Theorem 7.3.1 proof, taking now the simplified version of the definition of the set D in which we take into account the regularity of functions f and g , the fact that $\lambda = 1$ and we do not have end-point constraints:

$$D := \{(\omega, \xi) \in \Omega \times \mathbb{R}^K \mid \omega \in \Omega \text{ and } \xi = (\Psi_k(p(\cdot, \omega), \omega))_{k=1, \dots, K} \text{ for some } \mathcal{L} \times \mathcal{B}_{\Omega} \text{ measurable function } p : [0, T] \times \Omega \rightarrow \mathbb{R}^n \text{ such that } p(\cdot, \omega) \in \mathcal{P}_S(\omega) \text{ for all } \omega \in \Omega\}$$

where now, for a suitable constant $M_q > 0$ (which depends only on the data of the problem), we set

$$\mathcal{P}_S(\omega) := \left\{ q(\cdot, \omega) \in W^{1,1}([0, T], \mathbb{R}^n) : \|q(\cdot, \cdot)\|_{L^\infty} \leq M_q, -q(T, \omega) = \nabla_x g(\bar{x}(T, \omega); \omega) \right. \\ \left. - \dot{q}(t, \omega) = [\nabla_x f(t, \bar{x}(t, \omega), \bar{u}(t), \omega)]^T q(t, \omega) \text{ a.e. } t \in [0, T] \right\} .$$

The uniqueness of solutions to systems appearing in $\mathcal{P}_S(\omega)$ allows to conclude.

Appendix

A Continuity Sets

For any set Ω and \mathcal{A} a collection of subsets of Ω (usually written $\mathcal{A} \subset \mathcal{P}(\Omega)$ where $\mathcal{P}(\Omega)$ is the power set of Ω), we shall denote by $\Sigma(\mathcal{A})$ the σ -algebra on \mathcal{A} . We recall that if $\mathcal{A} \subset \mathcal{B} \subset \Sigma(\mathcal{A})$, then $\Sigma(\mathcal{A}) = \Sigma(\mathcal{B})$. Indeed, $\mathcal{A} \subset \mathcal{B} \subset \Sigma(\mathcal{B}) \subset \Sigma(\mathcal{A})$. The last inclusion follows from $\mathcal{B} \subset \Sigma(\mathcal{A})$ and the definition of a σ -algebra as the smallest algebra on the set. Moreover, $\mathcal{A} \subset \Sigma(\mathcal{B})$ implies that $\Sigma(\mathcal{A}) \subset \Sigma(\mathcal{B})$.

We shall summarize in this section some topological relevant results which will permit to derive that the family of continuity sets (i.e. Borel sets for which the measure of the boundary is zero) for a certain measure generates the Borel σ -algebra denoted by \mathcal{B}_Ω .

Definition A.1. Let (Ω, \mathcal{T}) be a topological space. Then the σ -algebra on \mathcal{T} is the Borel σ -algebra; equivalently

$$\Sigma(\mathcal{T}) = \mathcal{B}_\Omega .$$

Definition A.2. A topological space (Ω, \mathcal{T}) is called second-countable if there exists a countable base \mathcal{C} of the topology \mathcal{T} of Ω .

Lemma A.3. *Let (Ω, \mathcal{T}) be a second-countable topological space. Consider a countable base \mathcal{C} for the topology \mathcal{T} and take any $\mathcal{B} \subset \mathcal{B}_\Omega$ such that $\mathcal{C} \subset \mathcal{B}$. Then*

$$\Sigma(\mathcal{B}) = \mathcal{B}_\Omega .$$

Proof. We claim that $\Sigma(\mathcal{C}) = \mathcal{B}_\Omega$. Indeed, $\mathcal{C} \subset \mathcal{T} \subset \Sigma(\mathcal{T}) \equiv \mathcal{B}_\Omega$. It follows that $\Sigma(\mathcal{C}) \subset \mathcal{B}_\Omega$. It remains to prove that $\Sigma(\mathcal{T}) \subset \Sigma(\mathcal{C})$. It suffices to prove that $\mathcal{T} \subset \Sigma(\mathcal{C})$. Indeed, since \mathcal{C} is a countable base of the topology \mathcal{T} , then for all $T \in \mathcal{T}$

$$T = \bigcup_{\substack{C \in \mathcal{C} \\ \text{countable}}} C .$$

But $\bigcup_{\substack{C \in \mathcal{C} \\ \text{countable}}} C \in \Sigma(\mathcal{C})$ (by definition of σ -algebra). Since this is true for all $T \in \mathcal{T}$, it follows that $\mathcal{T} \subset \Sigma(\mathcal{C})$. The claim is therefore confirmed. To finish the proof, notice that

$$\mathcal{C} \subset \mathcal{B} \subset \mathcal{B}_\Omega = \Sigma(\mathcal{C}) .$$

We conclude that $\Sigma(\mathcal{B}) = \mathcal{B}_\Omega$. □

Let μ be now a measure defined on the Borel σ -algebra \mathcal{B}_Ω . Define B_μ to be the set of μ -continuity sets:

$$B_\mu := \{B : B \in \mathcal{B}_\Omega \text{ s.t. } \mu(\partial B) = 0\} .$$

Proposition A.4. ([20, Proposition 8.2.8]) *If Ω is completely regular, then B_μ contains a base of the topology of Ω . In particular, for Ω a metric space.*

Lemma A.5. ([4, A5], [85, 16B.2]) *Let (Ω, \mathcal{T}) be a second-countable topological space. For any base \mathcal{C} of the topology \mathcal{T} , there exists a countable base $\widehat{\mathcal{C}}$ such that $\widehat{\mathcal{C}} \subset \mathcal{C}$.*

Proposition A.6. *A metrizable topological space is separable if and only if it is second-countable. In particular, any separable metric space is second-countable.*

All the results above will be useful for the establishment of the following theorem:

Theorem A.7. *Let (Ω, ρ_Ω) be a separable metric space and \mathcal{T} the topology induced from ρ_Ω . Then, B_μ generates the Borel σ -algebra \mathcal{B}_Ω .*

Proof. We shall prove that \mathcal{B}_Ω is generated by the μ -continuity sets. Indeed, since (Ω, ρ_Ω) is a metric space, then from Proposition A.4, B_μ contains a base \mathcal{C} of the topology of Ω . According to Lemma A.5 and Proposition A.6, one can construct a countable base $\widehat{\mathcal{C}}$ such that $\widehat{\mathcal{C}} \subset \mathcal{C}$. Therefore,

$$\widehat{\mathcal{C}} \subset B_\mu \quad \text{and} \quad B_\mu \subset \mathcal{B}_\Omega .$$

As a consequence of Lemma A.3 (and Proposition A.6)

$$\Sigma(B_\mu) = \mathcal{B}_\Omega .$$

□

B Measurability of $F(t, \omega)$

Recall that

$$F(t, \omega) := f(t, \bar{x}(t, \omega), U(t), \omega) .$$

We claim that the set-valued function

$$(t, \omega) \rightsquigarrow f(t, \bar{x}(t, \omega), U(t), \omega) \quad \text{is } \mathcal{L} \times \mathcal{B}_\Omega \text{-measurable.}$$

Indeed, take a \mathcal{L} -measurable set A , a \mathcal{B}_Ω -measurable set B and an open set C of \mathbb{R}^m . Define the set

$$\{(t, \omega) \in [0, T] \times \Omega : ((t, U(t)), \omega) \cap (A \times C) \times B \neq \emptyset\} = (A \cap U^{-1}(C)) \times B .$$

This is a $\mathcal{L} \times \mathcal{B}_\Omega$ -measurable set. Define the family of sets $\Sigma \subset [0, T] \times \mathbb{R}^m \times \Omega$ as follows:

$$\Sigma := \left\{ G \subset [0, T] \times \mathbb{R}^m \times \Omega : \{(t, \omega) \in [0, T] \times \Omega : ((t, U(t)), \omega) \cap G \neq \emptyset\} \text{ is } \mathcal{L} \times \mathcal{B}_\Omega \text{-measurable} \right\} .$$

Σ is a σ -algebra, such that it contains $(A \times C) \times B$, where A is \mathcal{L} -measurable, C is an open set of \mathbb{R}^m (i.e. Borel set), B is \mathcal{B}_Ω -measurable. Hence, $(\mathcal{L} \times \mathcal{B}^m) \times \mathcal{B}_\Omega \subset \Sigma$ (by definition of σ -algebra as the smallest algebra defined on the set). Now take any open set $O \subset \mathbb{R}^n$. Since $f(., x, ., .)$ is $\mathcal{L} \times \mathcal{B}^m \times \mathcal{B}_\Omega$ -measurable function, then $f^{-1}(., x, ., .)(O)$ is $\mathcal{L} \times \mathcal{B}^m \times \mathcal{B}_\Omega$ -measurable. Hence, $f^{-1}(., x, ., .)(O) \in \Sigma$. This implies that

$$\left\{ (t, \omega) \in [0, T] \times \Omega : ((t, U(t)), \omega) \cap f^{-1}(., x, ., .)(O) \neq \emptyset \right\} \quad \text{is } \mathcal{L} \times \mathcal{B}_\Omega \text{-measurable.}$$

Equivalently,

$$\left\{ (t, \omega) \in [0, T] \times \Omega : f(t, x, U(t), \omega) \cap O \neq \emptyset \right\} \quad \text{is } \mathcal{L} \times \mathcal{B}_\Omega \text{-measurable.} \quad (7.44)$$

Take now a dense sequence $\{x_i\} \subset \mathbb{R}^n$. For each $k \in \mathbb{N}$, define the set-valued map

$$\phi_k(t, \omega) := f(t, x_i, U(t), \omega)$$

in which $x_i \in \mathbb{R}^n$ is the first term such that

$$|\bar{x}(t, \omega) - x_i| \leq \frac{1}{k} .$$

Then,

$$\phi_k(t, \omega) \xrightarrow[k \rightarrow \infty]{} f(t, \bar{x}(t, \omega), U(t), \omega)$$

for all (t, ω) fixed. Since, from (7.44), each ϕ_k is a $\mathcal{L} \times \mathcal{B}_\Omega$ -measurable multifunction, we deduce that the set-valued map defined as

$$(t, \omega) \rightsquigarrow f(t, \bar{x}(t, \omega), U(t), \omega) \quad (\text{called the pointwise limit})$$

is also $\mathcal{L} \times \mathcal{B}_\Omega$ -measurable. This is a consequence of [7, Theorem 8.2.5] since the set $f(t, x_i, U(t), \omega)$ is assumed to have closed values. The claim is therefore confirmed. \square



Conclusions and Perspectives

This thesis has established improvements (nondegeneracy and normality) on the necessary optimality conditions in the form of the Extended Euler-Lagrange condition for state constrained optimal control problems. A contribution on calculus of variations problems with state constraint has been also derived from optimal control results. We have investigated problems concerning viability results for a new class of state constrained control systems, and necessary optimality conditions for optimal control problems involving uncertain parameters in the data.

Chapter 2 has added contributions on neighbouring feasible trajectories results with $W^{1,1}$ -linear estimates for state constrained differential inclusions of the form:

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) & \text{a.e. } t \in [S, T] \\ x(t) \in A & \text{for all } t \in [S, T], \end{cases}$$

in which $[S, T]$ is a given time interval and $F(., .) : [S, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a given multifunction and $A \subset \mathbb{R}^n$ is a non-empty closed set. It was proved in [69], that under an inward pointing condition asserting the existence of a vector belonging to the velocity set and pointing inside the state constraint, a neighbouring feasible trajectories result (locally in time) with $W^{1,1}$ -estimates holds true. By a counter-example in [10], it was proved that the $W^{1,1}$ -linear estimate is not well satisfied if the initial datum is situated in a ‘corner’ of the state constraint set. As we have showed in Chapter 2 (cf. [15], [17]), the same result is satisfied when the initial datum belongs to a nonsmooth region of the state constraint set represented as merely a closed set, and under a weaker constraint qualification: there exists a vector which belongs to the interior of the Clarke tangent cone, while in previous works, the vector must belong also to the set of velocities.

The extension of neighbouring feasible trajectories results to a global (in time) was treated for instance in [12] for time-independent velocity sets and for a convex state constraint set A . In this paper, it was proved that under a classical inward pointing condition of the type

$$\text{co } F(x) \cap \text{int } T_A(x) \neq \emptyset \quad \text{for all } x \in A \cap R\mathbb{B},$$

(for some positive constant R), one can construct a neighbouring feasible trajectory (globally in time) verifying a linear $W^{1,1}$ -estimate in two cases: the first case is when the initial datum is away from corners of the state constraint set, and the second case is when we are free to choose the initial condition of the approximating trajectory. We have studied in Chapter 2 (cf. [17]) the same state constraint differential inclusion as the one considered in [12] and we have established that with an additional assumption concerning existence of hypertangent vectors (without being in the velocity set) at the starting point which belongs to a corner of the state constraint set A , the neighbouring feasible trajectory holds globally in time with a $W^{1,1}$ -linear estimate.

As already shown in literature, neighbouring feasible trajectories theorems providing linear $W^{1,1}$ -estimates play a crucial role in establishing regularity properties of the value function, providing ‘sensitivity relations’ in which the adjoint arc and the Hamiltonian are interpreted in terms of generalized gradients of the value function, deriving the necessary optimality conditions (in the form of the Maximum Principle or the Extended Euler-Lagrange) in the nondegenerate and the normal form. Chapter 3 (see [15]) (respectively Chapter 4 see [17]) has focused on the nondegeneracy (respectively normality) of the necessary conditions in the form of the Extended Euler-Lagrange condition when the minimizer has left end-point in a region where the state constraint set is nonsmooth. We have used an approach based on the local (respectively global) neighbouring feasible trajectories results involving $W^{1,1}$ -linear estimates.

In Chapter 5 (cf. [56]) we have studied a class of autonomous calculus of variations problems with a state constraint, already studied in the literature [43], for a state constraint formulated as a scalar inequality which is not differentiable. We have extended this result to a state constraint represented as a closed set and we have proved that if the interior of the Clarke tangent cone is nonempty, then the necessary optimality conditions apply in the normal form. Two techniques are given to prove this result. The first one concerns problems with $W^{1,1}$ -local minimizers and employs normality result for optimal control problems, while the second provides normality for $W^{1,1}$ -global minimizers and makes use of neighbouring feasible trajectories results with L^∞ -linear estimates.

In this thesis, we have also explored two new research directions. The first problem (cf. Chapter 6 and [35]) studied a state constrained (affine) control system of the form

$$\begin{cases} \dot{x}(t) = f_1(x(t))u_1(t) + f_2(x(t))u_2(t) & \text{for } u_1^2(t) + u_2^2(t) \leq 1 \quad \text{a.e. } t \in [0, T] \\ x(t) \in A & \text{for all } t \in [0, T], \end{cases} \quad (1)$$

where the state constraint set A is formulated as a scalar (smooth) inequality constraint

$$A := \{x : h(x) \leq 0\},$$

but standard (first order) inward pointing conditions are not satisfied.

We have showed that for some classes of problems, for instance for the Brockett nonholonomic integrator, the classical inward pointing condition fails to hold true. By consequence, the construction of a feasible trajectory cannot be made. We have developed, therefore, a local viability result for state constrained control systems of the form (1) following a particular construction of controls which allows the corresponding velocity vector to rotate in order to goes inside the state constraint set. Examples have been given to extend the analysis to the

construction of neighbouring feasible trajectories with $W^{1,1}$ -estimates. We have considered as a reference model the Brockett nonholonomic integrator and we have proved that, given a reference trajectory which possibly violates the constraint, we can construct particular controls, for which a feasible trajectory exists globally in time and verifies a nonlinear $W^{1,1}$ -estimate.

One can naturally ask if we can construct a neighbouring feasible trajectory result, with nonlinear $W^{1,1}$ -estimates, not merely for the Brockett nonholonomic integrator problem but for any control systems in the form of (1). (This is an ongoing research project.) Certainly, some preliminary steps to establish first a general result for the Brockett nonholonomic integrator have been made. However, the difficulties we have encountered show that the problem is not trivial, and do not allow straightforward extensions of previous results which involve the presence of (standard) inward pointing conditions.

Finally in Chapter 7 (see [16]), we have studied optimal control problems of the form:

$$(P) \quad \left\{ \begin{array}{l} \text{minimize} \quad \int_{\Omega} g(x(T, \omega); \omega) d\mu(\omega) \\ \text{over measurable functions } u(\cdot) \text{ and } W^{1,1} \text{ arcs } x(\cdot, \omega) \\ \text{such that} \quad u(t) \in U(t) \quad \text{a.e. } t \in [0, T] \\ \text{and, for each } \omega \in \Omega, \\ \quad \dot{x}(t, \omega) = f(t, x(t, \omega), u(t), \omega) \quad \text{a.e. } t \in [0, T], \\ \quad x(0, \omega) = x_0 \quad \text{and} \quad x(T, \omega) \in C(\omega). \end{array} \right.$$

Here, the performance criterion is given in terms of an average cost, since the control system and the right endpoint constraint involve unknown parameters $\omega \in \Omega$. For this class of problems, we have derived necessary optimality conditions in the form of Maximum Principle when the unknown parameters belong to a mere complete separable metric space (not necessarily compact).

A natural subsequent step is to consider the same optimal control problem as (P), imposing a state constraint condition, and to establish the necessary optimality conditions in this context. This is also a part of an ongoing research project.

Dissemination

Peer-reviewed Journals

P. Bettiol, **N. Khalil**, R. B. Vinter. “Normality of generalized Euler-Lagrange conditions for state constrained optimal control problems”. *J. Convex Anal* 23.1 (2016), pp. 291–311.

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P. Bettiol, **N. Khalil**. “Non-degenerate forms of the generalized Euler-Lagrange condition for state-constrained optimal control problems: In Imaging and Geometric Control”. *Variational Methods*. 18 vols. Radon Series on Computational and Applied Mathematics. De Gruyter, 2016, pp. 297–315.

Submitted paper

P. Bettiol, **N. Khalil**. “Necessary optimality conditions for average cost minimization problems”.

Paper in preparation

G. Colombo, **N. Khalil**, F. Rampazzo. “Construction of feasible trajectories when the classical inward pointing condition is violated”.



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Résumé des Travaux de Thèse

Nathalie T. Khalil

Conditions d'optimalité pour des problèmes en contrôle optimal et applications

Le projet de cette thèse est double. Le premier concerne l'extension des résultats précédents sur les conditions nécessaires d'optimalité pour des problèmes avec contraintes d'état, dans le cadre du contrôle optimal ainsi que dans le cadre de calcul des variations. Le deuxième objectif consiste à travailler sur deux nouveaux aspects de recherche: dériver des résultats de viabilité pour une classe de systèmes de contrôle avec des contraintes d'état dans lesquels les conditions dites 'standard inward pointing conditions' sont violées; et établir les conditions nécessaires d'optimalité pour des problèmes de minimisation de coût moyen éventuellement perturbés par des paramètres inconnus.

Dans la première partie, nous examinons les conditions nécessaires d'optimalité qui jouent un rôle important dans la recherche de candidats pour être des solutions optimales parmi toutes les solutions admissibles. Cependant, dans les problèmes d'optimisation dynamique avec contraintes d'état, certaines situations pathologiques pourraient survenir. Par exemple, il se peut que le multiplicateur associé à la fonction objective (à minimiser) disparaisse. Dans ce cas, la fonction objective à minimiser n'intervient pas dans les conditions nécessaires de premier ordre: il s'agit du cas dit *anormal*. Un phénomène pire, appelé le cas *dégénéré* montre que, dans certaines circonstances, l'ensemble des trajectoires admissibles coïncide avec l'ensemble des candidats minimiseurs. Par conséquent, les conditions nécessaires ne donnent aucune information sur les minimiseurs possibles. Pour surmonter ces difficultés, de nouvelles hypothèses supplémentaires doivent être imposées, appelées les *qualifications de la contrainte*. Nous étudions ces deux problèmes (*normalité* et *non dégénérescence*) pour des problèmes de contrôle optimal impliquant des contraintes dynamiques exprimées en termes d'inclusion différentielle, lorsque le minimiseur a son point de départ dans une région où la contrainte d'état est non lisse. Nous prouvons que sous une information supplémentaire impliquant principalement le cône tangent de Clarke, les conditions nécessaires sous la forme dite 'Extended Euler-Lagrange condition' sont satisfaites en forme normale et non dégénérée pour deux classes de problèmes de contrôle optimal avec contrainte d'état. Le résultat sur la normalité est également appliqué pour le problème de calcul des variations avec contrainte d'état.

Dans la deuxième partie de la thèse, nous considérons d'abord une classe de systèmes de contrôle avec contrainte d'état pour lesquels les qualifications de la contrainte standard du 'premier ordre' ne sont pas satisfaites, mais une qualification de la contrainte d'ordre supérieure (ordre 2) est satisfaite. Nous proposons une nouvelle construction des trajectoires admissibles (dit un résultat de *viabilité*) et nous étudions des exemples (tels que l'intégrateur non holonomique de Brockett) fournissant en plus un résultat d'estimation non linéaire. L'autre sujet de la deuxième partie de la thèse concerne l'étude d'une classe de problèmes de contrôle optimal dans lesquels des incertitudes apparaissent dans les données en termes de paramètres inconnus. En tenant compte d'un critère de performance sous la forme de coût moyen, une question cruciale est clairement de pouvoir caractériser les contrôles optimaux indépendamment de l'action du paramètre inconnu: cela permet de trouver une sorte de 'meilleur compromis' parmi toutes les réalisations possibles du système de contrôle tant que le paramètre varie. Pour ce type de problèmes, nous obtenons des conditions nécessaires d'optimalité sous la forme du Principe du Maximum (éventuellement pour le cas non lisse).

Mots clefs: Contrôle Optimal, Théorie de Contrôle, Conditions Nécessaires d'Optimalité, Normalité, Non-Dégénérescence, Analyse Non Lisse, Calcul des Variations

Optimality conditions for optimal control problems and applications

The project of this thesis is twofold. The first concerns the extension of previous results on necessary optimality conditions for state constrained problems in optimal control and in calculus of variations. The second aim consists in working along two new research lines: derive viability results for a class of control systems with state constraints in which ‘standard inward pointing conditions’ are violated; and establish necessary optimality conditions for average cost minimization problems possibly perturbed by unknown parameters.

In the first part, we examine necessary optimality conditions which play an important role in finding candidates to be optimal solutions among all admissible solutions. However, in dynamic optimization problems with state constraints, some pathological situations might arise. For instance, it might occur that the multiplier associated with the objective function (to minimize) vanishes. In this case, the objective function to minimize does not intervene in first order necessary conditions: this is referred to as the *abnormal* case. A worse phenomenon, called the *degenerate* case shows that in some circumstances the set of admissible trajectories coincides with the set of candidates to be minimizers. Therefore the necessary conditions give no information on the possible minimizers. To overcome these difficulties, new additional hypotheses have to be imposed, known as *constraint qualifications*. We investigate these two issues (*normality* and *non-degeneracy*) for optimal control problems involving state constraints and dynamics expressed as a differential inclusion, when the minimizer has its left end-point in a region where the state constraint set is nonsmooth. We prove that under an additional information involving mainly the Clarke tangent cone, necessary conditions in the form of the Extended Euler-Lagrange condition are derived in the normal and non-degenerate form for two different classes of state constrained optimal control problems. Application of the normality result is shown also for the calculus of variations problem subject to a state constraint.

In the second part of the thesis, we consider first a class of state constrained control systems for which standard ‘first order’ constraint qualifications are not satisfied, but a higher (second) order constraint qualification is satisfied. We propose a new construction for feasible trajectories (a *viability* result) and we investigate examples (such as the Brockett nonholonomic integrator) providing in addition a non-linear estimate result. The other topic of the second part of the thesis concerns the study of a class of optimal control problems in which uncertainties appear in the data in terms of unknown parameters. Taking into consideration an average cost criterion, a crucial issue is clearly to be able to characterize optimal controls independently of the unknown parameter action: this allows to find a sort of ‘best compromise’ among all the possible realizations of the control system as the parameter varies. For this type of problems, we derive necessary optimality conditions in the form of Maximum Principle (possibly nonsmooth).

Keywords: Optimal Control, Control Theory, Necessary Optimality Conditions, Normality, Non-Degeneracy, NonSmooth Analysis, Calculus of Variations