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# Géométrie $tt^*$ et applications pluriharmoniques

Lars Schäfer

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# Thèse

présentée pour l'obtention du titre de

Docteur de l'Université Henri Poincaré, Nancy-I

en Mathématiques

par

**Lars SCHÄFER**

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## Géométrie $tt^*$ et applications pluriharmoniques

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**and**  
**pluriharmonic maps**

**Dissertation**

zur

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# Contents

<b>1</b>	<b>Differential geometry on <math>\epsilon</math>complex manifolds</b>	<b>23</b>
1.1	$\epsilon$ complex manifolds . . . . .	23
1.2	$\epsilon$ Kähler manifolds . . . . .	28
1.3	Nearly $\epsilon$ Kähler manifolds . . . . .	29
1.4	Affine special $\epsilon$ complex and special $\epsilon$ Kähler manifolds . . . . .	30
1.5	The extrinsic construction of special $\epsilon$ Kähler manifolds . . . . .	35
1.5.1	The special Kähler case . . . . .	35
1.5.2	The special para-Kähler case . . . . .	36
1.6	Variations of $\epsilon$ Hodge structures . . . . .	37
1.6.1	$\epsilon$ Hodge structures and their variations . . . . .	37
1.6.2	$\epsilon$ VHS and special $\epsilon$ Kähler manifolds . . . . .	39
<b>2</b>	<b>Harmonic and <math>\epsilon</math>pluriharmonic maps</b>	<b>41</b>
2.1	Harmonic maps . . . . .	41
2.2	$\epsilon$ pluriharmonic maps from $\epsilon$ complex manifolds . . . . .	43
2.3	A generalization of $\epsilon$ pluriharmonic maps from almost $\epsilon$ complex manifolds into pseudo-Riemannian manifolds . . . . .	45
2.4	Special targets . . . . .	48
2.4.1	The space of pseudo-Riemannian metrics . . . . .	48
2.4.2	A remark on the space of Riemannian metrics . . . . .	52
2.4.3	The space of hermitian metrics . . . . .	53
2.4.4	The space of para-hermitian metrics . . . . .	56
2.5	The Lagrangian Grassmanians . . . . .	62
2.5.1	Definition and homogeneous model . . . . .	62
2.5.2	Holomorphic coordinates on the complex Lagrangian Grassmannian . . . . .	64

2.5.3	Para-holomorphic coordinates on the para-complex Lagrangian Grassmannian . . . . .	65
2.6	The space of compatible $\epsilon$ complex structures . . . . .	66
2.6.1	Differential geometry of the sets of compatible complex structures . . . . .	66
2.6.2	Differential geometry of the sets of compatible para-complex structures . . . . .	68
2.6.3	Lagrangian Grassmannians and $\epsilon$ complex structures . . . . .	70
2.7	Period domains of variations of $\epsilon$ Hodge structures . . . . .	71
<b>3</b>	<b><math>tt^*</math>-geometry and some of its solutions</b>	<b>75</b>
3.1	$tt^*$ -bundles . . . . .	75
3.2	Solutions on the tangent bundle of an almost $\epsilon$ complex manifold . . . . .	78
3.2.1	Solutions without metrics . . . . .	78
3.2.2	Solutions on almost $\epsilon$ hermitian manifolds . . . . .	82
3.2.3	Nearly $\epsilon$ Kähler manifolds . . . . .	85
3.2.4	Special $\epsilon$ complex and special $\epsilon$ Kähler manifolds . . . . .	86
3.3	Variations of $\epsilon$ Hodge structures . . . . .	88
3.4	Harmonic bundles . . . . .	89
<b>4</b>	<b><math>\epsilon tt^*</math>-geometry and <math>\epsilon</math>pluriharmonic maps</b>	<b>93</b>
4.1	The simply connected case . . . . .	93
4.2	The general case . . . . .	97
4.3	A rigidity result . . . . .	99
4.4	A special case of Lu's theorem . . . . .	99
<b>5</b>	<b>The <math>\epsilon</math>pluriharmonic maps associated to the above examples of <math>\epsilon tt^*</math>-bundles</b>	<b>101</b>
5.1	Solutions on the tangent bundle . . . . .	101
5.1.1	The classifying map of a flat nearly $\epsilon$ Kähler manifold . . . . .	101
5.2	The dual Gauß map of a special $\epsilon$ Kähler manifold with torsion . . . . .	103
5.3	The $\epsilon$ pluriharmonic map in the case of a special $\epsilon$ Kähler manifold . . . . .	104
5.3.1	The Gauß maps of a special Kähler manifold . . . . .	105
5.3.2	The local expression of the dual Gauß map . . . . .	106
5.3.3	The special Kähler metric in affine coordinates . . . . .	106
5.3.4	The Gauß maps of a special para-Kähler manifold . . . . .	109
5.3.5	The local expression of the dual Gauß map . . . . .	110

5.3.6	The special para-Kähler metric in an affine frame . . . . .	111
5.4	Variations of $\epsilon$ Hodge structures . . . . .	113
5.4.1	The period map of a variation of $\epsilon$ Hodge structures . . . . .	113
5.4.2	The period map of a variation of $\epsilon$ Hodge structures from the view- point of $\epsilon tt^*$ -geometry . . . . .	113
5.5	$\epsilon$ Harmonic bundles . . . . .	115





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# Introduction

Topological-antitopological fusion or  $tt^*$ -geometry is a topic of mathematical and physical interest. In about 1990 physicists studied topological-field-theories and their moduli spaces, in particular  $N = 2$  supersymmetric field-theories and found a special geometric structure called topological-antitopological fusion (see the works of Cecotti and Vafa [CV] and Dubrovin [D]). These geometries are realized on the tangent bundle of some manifold and part of their data is a Riemannian metric. One can replace the tangent bundle by an abstract vector bundle. This step allows to consider  $tt^*$ -bundles as a generalization of variations of Hodge structures, as it was done in Hertling's paper [Her].

The starting point of this thesis is a correspondence between  $tt^*$ -bundles and pluriharmonic maps into the space of positive definite metrics found by Dubrovin [D]. The aim is to find a version of this correspondence for  $tt^*$ -bundles on abstract vector bundles. The obvious interest of this correspondence is either to construct pluriharmonic maps or to construct  $tt^*$ -geometries. We first analyze pluriharmonic maps which are associated to solutions of  $tt^*$ -bundles coming from already known geometries. Famous examples of such solutions are harmonic bundles, variations of Hodge structures, special complex and special Kähler manifolds and flat nearly Kähler manifolds. In the last two cases indefinite metrics appear. This means one needs to understand the above-mentioned correspondence for  $tt^*$ -geometries with pseudo-Riemannian metrics.

Recently, special para-Kähler geometry was introduced in [CMMS]. It arises as one of the special geometries of Euclidean supersymmetry. This motivates us to search for para-complex versions of  $tt^*$ -geometries and for a correspondence to the para-complex analogue of pluriharmonic maps. In fact, we introduce the para-complex notion of  $tt^*$ -geometry, which we call para- $tt^*$ -geometry and establish a correspondence to the para-complex analogue of pluriharmonic maps. This result leads to the question, if there exist para-complex versions of the above-mentioned solutions of  $tt^*$ -geometry: harmonic bundles, variations of Hodge structures, special complex and special Kähler manifolds and flat nearly Kähler manifolds and if they are solutions of para- $tt^*$ -geometry. We answer positively to this question in this thesis, since we show, that one can generalize all these geometries to the para-complex category and that these generalizations supply solutions of para- $tt^*$ -geometry.

Let us describe the results of the work. It is a compilation of already published results and of newer unpublished ones. To compress this work we treat at places, where it does not impose too much confusion, the complex case and the para-complex case at the same time. The needed notions of para-complex geometry are introduced in chapter 1. For the rest of the introduction we may ask the reader who is not familiar with these notations

to skip the prefix 'para' if it disturbs him.

We explain the structure of a (para-) $tt^*$ -bundle: A (para-) $tt^*$ -bundle  $(E, D, S)$  consists of a (real) vector bundle  $E$  over a (para-)complex manifold  $(M, J)$  endowed with a connection  $D$  and a section  $S$  of  $T^*M \otimes \text{End}(E)$ , such that the family of connections

$$D^\theta := \begin{cases} D + \cos(\theta)S + \sin(\theta)S_J, \theta \in \mathbb{R}, & \text{for } M \text{ complex,} \\ D + \cosh(\theta)S + \sinh(\theta)S_J, \theta \in \mathbb{R}, & \text{for } M \text{ para-complex,} \end{cases}$$

is flat. A metric (para-) $tt^*$ -bundle  $(E, D, S, g)$  is a (para-) $tt^*$ -bundle  $(E, D, S)$  endowed with a  $D$ -parallel metric  $g$ , such that  $S$  is symmetric with respect to  $g$ .

First, we establish the correspondence between (para-) $tt^*$ -bundles on abstract vector bundles over simply connected manifolds and (para-)pluriharmonic maps, generalizing Dubrovin [D]. In fact we show the following result in theorem 4.1 and in theorem 4.2.:

### Theorem 1

- (i) *A metric (para-) $tt^*$ -bundle with a metric of signature  $(p, q)$  over a simply connected (para-)complex manifold  $(M, J)$  gives (after fixing a  $D^0$ -parallel frame) rise to an admissible (para-)pluriharmonic map  $f$  from  $M$  to  $GL(r, \mathbb{R})/O(p, q)$ .*
- (ii) *An admissible (para-)pluriharmonic map  $f$  from a simply connected (para-)complex manifold  $(M, J)$  to  $GL(r, \mathbb{R})/O(p, q)$  gives rise to a metric (para-) $tt^*$ -bundle  $(E = M \times \mathbb{R}^r, D, S)$ .*

For the definition of admissible (para-)pluriharmonic maps we refer to definition 2.9.

In other words we could roughly say, that our construction defines a bijection

$$\Phi : \left\{ \begin{array}{l} \text{framed metric (para-)}tt^*\text{-bundles} \\ \text{of rank } r \text{ and sign. } (p, q) \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{admissible (para-)pluriharmonic maps} \\ \text{into } GL(r, \mathbb{R})/O(p, q) \end{array} \right\}. \quad (0.0.1)$$

from the space of framed metric (para-) $tt^*$ -bundles of rank  $r$  over a simply connected (para-)complex manifold  $(M, J)$  to the space of (para-)pluriharmonic maps from  $(M, J)$  to  $GL(r, \mathbb{R})/O(p, q)$ . The case of a metric  $tt^*$ -bundle of rank  $r$  with metric of signature  $(r, 0)$  follows from this theorem, since in this case the pluriharmonic maps are shown to be admissible using a result of Sampson [Sam]. Our correspondence contains the classical correspondence shown by Dubrovin [D]. If the manifold  $M$  is not simply connected, one has to replace the (para-)pluriharmonic maps by twisted (para-)pluriharmonic maps. We also show a version for unimodular oriented metric  $tt^*$ -bundles. The target space of the (para-)pluriharmonic maps is for unimodular oriented metric  $tt^*$ -bundles the space  $SL(r, \mathbb{R})/SO(p, q)$ .

Adapting a rigidity result of Gordon [G] about harmonic maps to pluriharmonic maps we are able to prove a rigidity result for  $tt^*$ -bundles with a positive definite metric over a compact Kähler manifold (cf. theorem 4.6 and [Sch5]). Further we apply this to special Kähler manifolds and obtain a new proof Lu's theorem [Lu] in the case of a simply connected compact special Kähler manifold (cf. theorem 4.7 and [Sch5]).

We now shortly discuss the above-mentioned classes of  $tt^*$ -bundles:

From [Her] and [Sch1, Sch2] we knew, that harmonic bundles are objects, which are closely related to  $tt^*$ -bundles. A correspondence between these bundles and harmonic

maps from compact Kähler manifolds to  $GL(r, \mathbb{C})/U(r)$  was given in Simpson's paper [Sim]:

$$\Psi : \left\{ \begin{array}{l} \text{harmonic bundles} \\ \text{with pos. def. metric over} \\ \text{comp. Kähler manifolds } M \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{harmonic maps} \\ \text{from} \\ M \text{ to } GL(r, \mathbb{C})/U(r) \end{array} \right\}.$$

From Sampson's theorem [Sam] it follows that in this case the notion of harmonic map and pluriharmonic map coincide. In other words there exists a correspondence between harmonic bundles and pluriharmonic maps from compact Kähler manifolds to  $GL(r, \mathbb{C})/U(r)$ . This correspondence can be recovered from a more general result, discussed in this thesis, which is an application of our theorem 1. This is described briefly in the next paragraph and was published in [Sch4].

We generalize the notion of a harmonic bundle by admitting indefinite metrics. With this definition we construct metric and symplectic  $tt^*$ -bundles from harmonic bundles and we apply the correspondence of theorem 1 to prove that the target space of the admissible pluriharmonic maps can be restricted to the totally geodesic subspace  $GL(r, \mathbb{C})/U(p, q)$  of  $GL(2r, \mathbb{R})/O(2p, 2q)$ . This means, that the application of our construction roughly gives rise to a map:

$$\Psi : \left\{ \begin{array}{l} \text{framed harmonic bundles} \\ \text{over complex manifolds } M \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{admissible pluriharmonic maps} \\ \text{from } M \text{ to } GL(r, \mathbb{C})/U(p, q) \end{array} \right\}.$$

Simpson's result for positive definite signature is recovered, since for positive definite signature the above map  $\Phi$  (cf. equation (0.0.1)) is essentially bijective. Our result is a generalization of Simpson's work (for more information compare section 5.5), as arbitrary signature of the bundle metric is admitted and the compactness and the Kähler condition are not needed. We restrict to simply connected manifolds  $M$ , since the case with non-trivial fundamental group can be obtained by utilizing the corresponding theorems in chapter 4. The pluriharmonic maps are then replaced by twisted pluriharmonic maps.

Moreover, we introduce the notion of para-harmonic bundles, i.e. harmonic bundles in para-complex geometry (cf. [Sch9]). We use the same recipe as in complex geometry to relate these bundles to para-pluriharmonic maps into  $GL(r, \mathbb{C})/U^\pi(C^r)$ , where  $GL(r, \mathbb{C})$  is the para-complex analogue of the general complex linear group and  $U^\pi(C^r)$  is the para-complex version of the unitary group. Hence we extend the map  $\Psi$  to para-harmonic bundles:

$$\Psi : \left\{ \begin{array}{l} \text{framed para-harmonic bundles} \\ \text{over para-complex manifolds } M \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{admissible para-pluriharmonic maps} \\ \text{from } M \text{ to } GL(r, \mathbb{C})/U^\pi(C^r) \end{array} \right\}.$$

The next class of solutions are variations of Hodge structures (VHS). These are by Hertling's work [Her]  $tt^*$ -geometries. Locally VHS are described by their period map, i.e. a holomorphic map into the so-called period domain, which is an open set in a flag manifold. We weaken the second Riemannian bilinear relation. Then we relate the pluriharmonic map associated to a  $tt^*$ -bundle, which comes from a given VHS of odd weight, to the period map of this VHS. Likewise we introduce a para-complex version of VHS

and associate to this a kind of period map. The para-complex version of a VHS carries a metric para- $tt^*$ -bundle. For odd weight we express the para-pluriharmonic map associated to this para- $tt^*$ -bundle in terms of the para-complex period map.

In all these examples of (para-) $tt^*$ -geometries the (para-)complex structure of the base manifold  $(M, J)$  has been integrable. However, in the study of (para-) $tt^*$ -bundles  $(TM, D, S)$  on the tangent bundle  $TM$  it is reasonable to consider almost (para-)complex manifolds, since like this nearly (para-)Kähler manifolds with flat Levi-Civita connection arise as solutions of (para-) $tt^*$ -geometry. We give a constructive classification of Levi-Civita flat nearly (para-)Kähler manifolds in a common work with V. Cortés [CS2]. Let us explain the structure of this part of the thesis, which is also subject of [Sch7, Sch8]. Part of the  $tt^*$ -bundle  $(TM, D, S)$  is now a one-parameter family of flat connections  $D^\theta$  on the tangent bundle  $TM$ . Every almost (para-)complex manifold  $(M, J)$  endowed with a flat connection  $\nabla$  carries a natural one-parameter family of flat connections given by

$$\nabla^\theta = \exp(\theta J) \circ \nabla \circ \exp(-\theta J), \text{ with } \theta \in \mathbb{R}.$$

We study (para-) $tt^*$ -bundles for which the families  $D^\theta$  and  $\nabla^\theta$  are equivalent in the sense of the following:

**Definition 1** *Two one-parameter families of connections  $\nabla^\theta$  and  $D^\theta$  on some vector bundle  $E$  with  $\theta \in \mathbb{R}$  are called (linearly) equivalent with factor  $\alpha \in \mathbb{R}$  if they satisfy the equation  $\nabla^\theta = D^{\alpha\theta}$ .*

To consider such one-parameter families of connections is motivated by our previous study of special (para-)Kähler solutions of (para-) $tt^*$ -bundles. Like this we obtain a duality between Levi-Civita flat nearly (para-)Kähler manifolds and special (para-)Kähler manifolds, which are both of importance in mathematics and theoretical physics.

Afterwards we restrict to (para-) $tt^*$ -bundles  $(TM, D, S)$  as above such that the connection  $D$  is (para-)complex, i.e. satisfies  $DJ = 0$ . These are recovered uniquely from the (para-)complex structure  $J$  and the connection  $\nabla$ . In addition compatibility conditions on the pair  $(\nabla, J)$  are given and it is shown that for special (para-)complex and nearly (para-)Kähler manifolds these compatibility conditions on  $(\nabla, J)$  hold.

More precisely, we give a class of  $tt^*$ -bundles  $(TM, D, S)$ , which corresponds to special (para-)complex manifolds with torsion and non integrable almost (para-)complex structure  $J$  and a class of solutions which corresponds to flat almost (para-)complex manifolds satisfying the nearly Kähler condition (with torsion).

In the sequel we study whether the above (para-) $tt^*$ -bundles  $(TM, D, S)$  (over almost (para-)complex manifolds) provide *metric* and *symplectic* (para-) $tt^*$ -bundles, respectively. Solutions of the first type are, for example, given by special (para-)Kähler manifolds and solutions of the second kind arise on flat nearly (para-)Kähler manifolds. Otherwise, neither the nearly (para-)Kähler condition is compatible with metric (para-) $tt^*$ -bundles nor the condition to be special (para-)complex is compatible with symplectic (para-) $tt^*$ -bundles.

Finally it remains to analyze if one can transfer the relation between (para-)pluriharmonic maps and (para-) $tt^*$ -geometry to the case of (non-integrable) almost (para-)complex structure of the base  $(M, J)$ .

Since the (para-)complex structures are no longer integrable, we generalize the notion

of (para-)pluriharmonic maps to the case of a source manifold  $(M, J)$  with an (non-integrable) almost complex structure  $J$ : This is done by using the (para-)pluriharmonic map equation, where a nice connection (cf. definition 2.6) on  $M$  is chosen. Then we introduce  $\mathbb{S}_\epsilon^1$ -pluriharmonic maps which generalize the notion of associated families of pluriharmonic maps from complex manifolds (see for example [ET]) to maps from almost (para-)complex manifolds into pseudo-Riemannian manifolds. We give conditions for an  $\mathbb{S}_\epsilon^1$ -pluriharmonic map to be (para-)pluriharmonic and a result, which relates generalized (para-)pluriharmonic maps to harmonic maps. With these notions and results we associate pluriharmonic maps into  $\mathrm{Sp}(\mathbb{R}^{2n})/U(p, q)$ , respectively  $SO_0(2p, 2q)/U(p, q)$ , to the above metric and symplectic  $tt^*$ -bundles. Similarly we associate para-pluriharmonic maps into  $\mathrm{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$ , respectively into  $SO_0(n, n)/U^\pi(C^n)$ , to the above metric and symplectic para- $tt^*$ -bundles.

As already mentioned, special (para-)complex and special (para-)Kähler manifolds are an interesting class of  $\epsilon tt^*$ -bundles, respectively metric  $\epsilon tt^*$ -bundles. In the complex case this follows from the results of Hertling [Her] who associated a VHS of weight 1 to any special complex manifold. We give a direct differential geometric approach and a characterization of the tangent bundles of special complex and special Kähler manifolds as special  $tt^*$ -bundles (cf. [CS1]). The associated pluriharmonic map is expressed in terms of the dual Gauß map, which is a holomorphic map into the pseudo-Hermitian symmetric space  $\mathrm{Sp}(\mathbb{R}^{2n})/U(k, l)$ , where  $n = k + l$ . These results are generalized to para-complex geometry. This is done in [Sch3] and is detailed in this thesis. The approach, which uses a VHS of weight 1, has also been successfully transferred to (para-)complex geometry and is part of this thesis.





# Introduction

La fusion topologique-antitopologique est un sujet d'intérêt en physique comme en mathématique. Dans les années 1990, les physiciens ont étudié les espaces modules au sein des théories des champs topologiques et plus particulièrement dans celle des champs  $N=2$ -supersymétriques (cf. les travaux de Cecotti et Vafa [CV] et de Dubrovin [D]). Au cours de leur étude, ils ont découvert une structure géométrique, appelée fusion topologique-antitopologique, ou géométrie  $tt^*$ . À la base, cette géométrie était réalisée sur le fibré tangent à une variété et une métrique Riemannienne faisait partie des données géométriques. Mais on peut également remplacer le fibré tangent par un fibré vectoriel abstrait. Ce point de vue, que l'on trouve dans l'article de Hertling [Her], permet de considérer la géométrie  $tt^*$  comme une généralisation des variations de structures de Hodge.

Le point de départ de cette thèse est la correspondance existant entre les fibrés  $tt^*$  sur le fibré tangent et les applications pluriharmoniques dans l'espace des métriques définies positives. Cette correspondance a été découverte par Dubrovin [D]. Son intérêt est d'une part de construire des applications pluriharmoniques, d'autre part des fibrés  $tt^*$ . Nous analyserons les applications pluriharmoniques associées à des fibrés  $tt^*$  provenant de solutions déjà connues de la géométrie  $tt^*$ . Des exemples célèbres sont: les fibrés harmoniques, les variations de structures de Hodge, les variétés spéciales Kählériennes et les variétés approximativement Kählériennes plates. Dans les deux derniers cas les métriques peuvent être indéfinies. Il est donc nécessaire de généraliser la correspondance précédente pour le cas des fibrés  $tt^*$  avec des métriques indéfinies.

Plus récemment, la géométrie spéciale para-Kählérienne a été introduite par [CMMS] comme une des géométries spéciales de la supersymétrie Euclidienne. Notre motivation est d'étudier s'il existe des versions para-complexes des géométries  $tt^*$  et si l'on peut trouver une correspondance entre ces versions et les analogues para-complexes des applications pluriharmoniques. Nous répondons par l'affirmative à ces deux problèmes. Se pose alors la question de savoir s'il existe des versions para-complexes des solutions citées ci-dessus: des fibrés harmoniques, des variations de structures de Hodge, des variétés spéciales Kählériennes et des variétés approximativement Kählériennes plates, et de savoir si leurs généralisations sont des solutions de la géométrie para- $tt^*$ . Nous démontrons en effet que ces exemples peuvent être généralisés dans le cadre para-complexe et qu'ils sont des solutions de la géométrie para- $tt^*$ .

Décrivons les résultats de cette thèse. Ce travail rassemble des résultats publiés et des résultats plus récents. Pour comprimer le texte, nous avons, lorsque le risque de confusion n'est pas trop grand, traité les cas complexe et para-complexe en même temps. Les notions de base de géométrie para-complexe sont détaillées dans le premier chapitre.

Pour l'introduction, le lecteur peu habitué aux notions de la géométrie para-complexe, pourra supprimer le préfixe 'para'.

Expliquons la structure d'un fibré (para-) $tt^*$  : un fibré (para-) $tt^*$   $(E, D, S)$  est la donnée d'un fibré vectoriel sur une variété (para-)complexe  $(M, J)$ , muni d'une connexion  $D$ , et d'une section  $S$  dans  $T^*M \otimes \text{End}(E)$ , pour lesquels les connexions de la famille

$$D^\theta := \begin{cases} D + \cos(\theta)S + \sin(\theta)S_J, \theta \in \mathbb{R}, & \text{pour } M \text{ complexe,} \\ D + \cosh(\theta)S + \sinh(\theta)S_J, \theta \in \mathbb{R}, & \text{pour } M \text{ para-complexe,} \end{cases}$$

sont plates. Un fibré (para-) $tt^*$  métrique  $(E, D, S, g)$  est un fibré (para-) $tt^*$   $(E, D, S)$  muni d'une métrique parallèle pour  $D$  et pour laquelle la section  $S$  est  $g$ -symétrique.

Généralisant Dubrovin [D], nous établissons d'abord la correspondance entre les fibrés (para-) $tt^*$ , définis sur des fibrés vectoriels abstraits et des applications (para-)pluriharmoniques. En fait, nous démontrons dans les théorèmes 4.1 et 4.2 le résultat suivant :

### Théorème 1

- (i) *Un fibré (para-) $tt^*$  métrique  $(E, D, S, g)$  sur une variété (para-)complexe simplement connexe  $(M, J)$  induit (après avoir choisi un repère  $D^0$ -plat de  $E$ ) une application (para-)pluriharmonique admissible de la variété  $M$  dans  $GL(r, \mathbb{R})/O(p, q)$ .*
- (ii) *Une application (para-)pluriharmonique admissible d'une variété (para-)complexe simplement connexe  $(M, J)$  dans  $GL(r, \mathbb{R})/O(p, q)$  induit un fibré (para-) $tt^*$  métrique  $(E, D, S, g)$ .*

Pour la définition des applications (para-)pluriharmoniques admissibles, nous faisons référence à la définition 2.9.

Pour résumer, on pourrait dire que nous avons trouvé une bijection

$$\Phi : \left\{ \begin{array}{l} \text{fibrés (para-) } tt^* \text{ métriques} \\ \text{de rang } r \text{ et sign. } (p, q) \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{applications pluriharmoniques admissibles} \\ \text{dans } GL(r, \mathbb{R})/O(p, q) \end{array} \right\}. \quad (0.0.2)$$

entre l'espace des fibrés (para-) $tt^*$  métriques (avec repère fixé) de rang  $r$  et signature  $(p, q)$  sur une variété (para-)complexe  $(M, J)$  et l'espace des applications pluriharmoniques admissibles de  $(M, J)$  dans  $GL(r, \mathbb{R})/O(p, q)$ . Le cas d'un fibré (para-) $tt^*$  métrique avec une métrique de signature  $(r, 0)$  ou  $(0, r)$  est une conséquence de notre théorème, puisque dans ce cas, on peut montrer, en utilisant un théorème de Sampson [Sam], que les applications pluriharmoniques sont admissibles. Si la variété  $M$  n'est pas simplement connexe, il faut remplacer les applications (para-)pluriharmoniques par des applications (para-)pluriharmoniques twistées. Nous établissons également une version de ce résultat pour des fibrés (para-) $tt^*$  métriques orientés unimodulaires. Pour des fibrés (para-) $tt^*$  métriques orientés unimodulaires, l'espace cible des applications (para-)pluriharmoniques est donné par l'espace symétrique  $SL(r, \mathbb{R})/SO(p, q)$ .

En adaptant au cas des applications pluriharmoniques un résultat de rigidité de Gordon [G] concernant les applications harmoniques, nous sommes capables d'obtenir un résultat de rigidité pour des fibrés  $tt^*$  métriques sur des variétés Kählériennes compactes

(cf. théorème 4.6 et [Sch5]). Nous appliquons ensuite ce résultat au cas spécial Kählérien et nous obtenons une nouvelle preuve du théorème de Lu [Lu] dans le cas d'une variété spéciale Kählérienne compacte simplement connexe (cf. théorème 4.7 et [Sch5]).

Nous allons à présent examiner les classes de fibrés (para-) $tt^*$  citées ci-dessus :

Grâce au travail de Hertling [Her] et en utilisant [Sch1, Sch2], nous savions jusqu'alors que les fibrés harmoniques étaient des objets reliés à la géométrie  $tt^*$ . Une correspondance entre les fibrés harmoniques et les applications harmoniques des variétés Kählériennes compactes dans  $GL(r, \mathbb{C})/U(r)$  était donnée par Simpson [Sim]:

$$\Psi : \left\{ \begin{array}{l} \text{fibrés harmoniques} \\ \text{avec métrique pos. déf. sur} \\ \text{des var. Kähler. comp. } M \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{applications harmoniques} \\ \text{de } M \text{ dans} \\ GL(r, \mathbb{C})/U(r) \end{array} \right\}.$$

Le théorème de Sampson [Sam] implique que dans ce cas, les notions d'harmonicité et de pluriharmonicité coïncident. Ainsi, il existe une correspondance entre les fibrés harmoniques et les applications pluriharmoniques des variétés Kählériennes compactes dans  $GL(r, \mathbb{C})/U(r)$ . On peut également déduire cette correspondance d'un résultat plus général, qui est une application de notre théorème 1. Cette correspondance est explicitée brièvement dans le paragraphe suivant et a été publiée dans [Sch4].

Dans cette thèse, nous généralisons la notion de fibré harmonique en incluant le cas des métriques indéfinies. À partir de cette généralisation, nous construisons des fibrés  $tt^*$ . En appliquant alors notre correspondance donnée dans le théorème 1, nous démontrons que l'on peut restreindre les applications pluriharmoniques au sous-espace totalement géodésique  $GL(r, \mathbb{C})/U(p, q)$  de  $GL(r, \mathbb{R})/O(2p, 2q)$ . Ainsi, notre construction induit, sans détailler, une application

$$\Psi : \left\{ \begin{array}{l} \text{fibrés harmoniques} \\ \text{avec métrique de sign. } (p, q) \\ \text{sur des var. complexe } M \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{applications harmoniques admissibles} \\ \text{de } M \text{ dans} \\ GL(r, \mathbb{C})/U(p, q) \end{array} \right\}.$$

Notre résultat est une généralisation du travail de Simpson (plus d'informations se trouvent dans la section 5.5.) : en effet, d'une part, on peut retrouver son résultat avec une application  $\Phi$  essentiellement bijective, et d'autre part, nous admettons des métriques à signature arbitraire et nous n'avons besoin ni de la condition de compacité ni de la condition Kählérienne. Nous traitons le cas des variétés simplement connexes, le cas général pouvant être obtenu facilement en utilisant les théorèmes correspondants dans le chapitre 4 et en remplaçant les applications pluriharmoniques par des applications pluriharmoniques twistées.

Nous introduisons de plus la notion de fibré para-harmonique, c'est-à-dire de fibré harmonique en géométrie para-complexe (cf. [Sch9]). Nous utilisons par la suite une technique analogue afin d'obtenir une correspondance entre les fibrés para-harmoniques et des applications para-pluriharmoniques à valeurs dans  $GL(r, \mathbb{C})/U^\pi(C^r)$ . On désigne par  $GL(r, \mathbb{C})$  la version para-complexe du groupe linéaire général complexe et par  $U^\pi(C^r)$  l'analogie para-complexe du groupe unitaire. Nous généralisons ainsi l'application  $\Psi$  aux

fibrés para-harmoniques:

$$\Psi : \left\{ \begin{array}{l} \text{fibrés para-harmoniques} \\ \text{sur des var.} \\ \text{para-complexes } M \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{appl. para-pluriharm. admissibles} \\ \text{de } M \text{ dans} \\ GL(r, C)/U^\pi(C^r) \end{array} \right\}.$$

La prochaine classe de solutions est celle des variations de structures de Hodge (VHS). On sait d'après les travaux de Hertling [Her] qu'elles sont en effet des géométries  $tt^*$ . Localement, une VHS est décrite par son application de périodes, qui est une application holomorphe sur le domaine des périodes, sous-ensemble ouvert dans une variété de drapeaux. Nous affaiblissons la deuxième relation riemannienne bilinéaire. Nous donnons ensuite l'expression explicite de l'application pluriharmonique associée à la géométrie  $tt^*$  donnée par une VHS, en termes de l'application de périodes de cette VHS. De la même manière, nous introduisons une version para-complexe des variations de structures de Hodge, appelée les para-VHS. Nous associons une application de périodes à ces para-VHS. Les para-VHS sont des solutions de la géométrie para- $tt^*$ . L'application para-pluriharmonique associée à une géométrie para- $tt^*$ , qui provient d'une para-VHS, est exprimée à l'aide de l'application de périodes.

Dans tous les exemples des géométries (para-) $tt^*$  discutés ci-dessus, la structure (para-)complexe de la variété  $(M, J)$  était intégrable. Dans l'étude des fibrés (para-) $tt^*$   $(TM, D, S)$  sur le fibré tangent  $TM$  il était nécessaire d'analyser des variétés presque (para-)complexes, car des variétés approximativement Kählériennes plates apparaissaient alors comme solutions de la géométrie (para-) $tt^*$ . Une classification constructive des variétés approximativement Kählériennes plates est le sujet d'un travail en commun avec V. Cortés [CS2]. Expliquons la structure de cette partie de la thèse, dont le sujet est également développé dans [Sch7, Sch8]. La donnée d'un fibré (para-) $tt^*$  induit une famille à un paramètre de connexions plates  $D^\theta$ . D'autre part, chaque variété presque complexe  $(M, J)$  munie d'une connexion plate porte une famille naturelle à un paramètre de connexions défini par

$$\nabla^\theta = \exp(\theta J) \circ \nabla \circ \exp(-\theta J), \text{ avec } \theta \in \mathbb{R}.$$

Nous étudions les fibrés (para-) $tt^*$  pour lesquels les deux familles à un paramètre de connexions sont équivalentes dans le sens de la définition suivante:

**Définition 1** *Deux familles à un paramètre de connexions sont dites équivalentes linéaires avec facteur  $\alpha \in \mathbb{R}$ , si elles satisfont à l'équation  $\nabla^\theta = D^{\alpha\theta}$ .*

Nos études précédentes des solutions des fibrés (para-) $tt^*$  provenant des variétés spéciales (para-)Kählériennes ont motivé l'examen de ces familles à un paramètre de connexions. De cette manière, nous avons obtenu une dualité entre des variétés approximativement Kählériennes plates et des variétés spéciales (para-)Kählériennes. Dans les deux cas, il s'agit de géométries importantes en mathématique et en physique théorique.

Nous considérons ensuite comme ci-dessus la restriction du problème aux fibrés (para-) $tt^*$  du type  $(TM, D, S)$  pour lesquels la connexion  $D$  est (para-) complexe, c'est-à-dire vérifie  $DJ = 0$ . Ces fibrés sont donnés de façon unique par la structure (para-)complexe  $J$  et la connexion  $\nabla$ . De plus, on trouve des conditions de compatibilité pour  $(\nabla, J)$  et on

peut montrer que dans le cas des variétés spéciales (para-)complexes et approximativement (para-)Kählériennes ces conditions sont remplies. Plus précisément, nous donnons deux classes de solutions correspondant respectivement à des variétés spéciales (para-)complexes avec torsion d'une part, et à des variétés plates presque (para-)complexes satisfaisant la condition approximativement Kählérienne (avec torsion) d'autre part. Par la suite, nous étudions si ces fibrés (para-) $tt^*$   $(TM, D, S)$  peuvent donner des fibrés (para-) $tt^*$  métriques  $(TM, D, S, g)$  ou symplectiques  $(TM, D, S, \omega)$ . Les solutions du premier type proviennent par exemple des variétés spéciales (para-)Kählériennes et celles du second type de variétés approximativement (para-)Kählériennes. En effet, les variétés spéciales (para-)Kählériennes  $(M, J, \nabla, g)$  n'admettent pas de fibré  $tt^*$  symplectique  $(TM, D, S, \omega = g(J \cdot, \cdot))$ , de même que les variétés approximativement (para-)Kählériennes  $(M, J, g)$  n'admettent aucun fibré (para-) $tt^*$  métrique  $(TM, D, S, g)$ . Plus précisément, la condition pour une variété d'être approximativement (para-)Kählérienne n'est pas compatible avec des fibrés (para-) $tt^*$  métriques, et celle d'être spéciale (para-)complexe n'est pas compatible avec des fibrés symplectiques (para-) $tt^*$ .

En conclusion, il reste à analyser si l'on peut obtenir la même relation entre les applications (para-)pluriharmoniques et la géométrie (para-) $tt^*$  dans le cas où l'on a comme base une variété presque (para-)complexe  $(M, J)$ .

Comme les structures (para-)complexes ne sont alors plus intégrables, il faut généraliser la notion d'application pluriharmonique au cas d'une variété de départ  $(M, J)$  avec une structure presque complexe  $J$  : on y parvient en choisissant une connexion idoine (cf. définition 2.6) sur la variété  $M$  et en utilisant l'équation (para-)pluriharmonique. Nous introduisons ensuite la notion d'application  $S_\epsilon^1$ -pluriharmonique qui généralise la notion de famille associée à une application pluriharmonique (cf. [ET]) dans le cas des applications de variétés presque (para-)complexes vers des variétés pseudo-Riemanniennes. Nous donnons des conditions pour lesquelles une application  $S_\epsilon^1$ -pluriharmonique est (para-)pluriharmonique et nous trouvons des conditions d'harmonicité pour des applications (para-)pluriharmoniques. Ces notions nous permettent d'associer des applications pluriharmoniques vers  $\mathrm{Sp}(\mathbb{R}^{2n})/U(p, q)$  (respectivement vers  $SO_0(2p, 2q)/U(p, q)$ ) aux fibrés  $tt^*$  métriques (respectivement symplectiques) du dernier paragraphe. Nous associons également des applications pluriharmoniques vers  $\mathrm{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$  (respectivement vers  $SO_0(n, n)/U^\pi(C^n)$ ) aux fibrés para- $tt^*$  métriques (respectivement symplectiques) décrits ci-dessus.

Comme nous l'avons déjà remarqué, les variétés spéciales (para-)complexes et spéciales (para-)Kählériennes forment une classe intéressante de fibrés (para-) $tt^*$ , respectivement de fibrés (para-) $tt^*$  métriques. Dans le cas complexe, c'est une conséquence d'un travail de Hertling [Her], qui associe une VHS de poids 1 à chaque variété spéciale Kählérienne. Dans [CS1], nous donnons une approche utilisant la géométrie différentielle et une caractérisation des fibrés tangents des variétés spéciales complexes et spéciales Kählériennes comme des fibrés  $tt^*$ . L'application pluriharmonique associée peut être exprimée avec l'application de Gauß duale, qui est une application holomorphe dans l'espace symétrique pseudo-Hermitien  $\mathrm{Sp}(\mathbb{R}^{2n})/U(k, l)$  avec  $n = k + l$ . Ces résultats ont été généralisés à la géométrie para-complexe et publiés dans [Sch3]. Également, nous avons généralisé l'approche avec des VHS de poids 1. Plus précisément, on peut construire des para-VHS de poids 1 à partir d'une variété spéciale para-complexe. Les détails se trouvent dans cette thèse.



# Chapter 1

## Differential geometry on $\epsilon$ complex manifolds

In this chapter we recall some definitions and results of special  $\epsilon$ complex geometry given in [CMMS] and prove some results which are analogous to those proven for special Kähler manifolds in [ACD]. We give here just a sketch of the results needed in this thesis. The interested reader can find further information in [CMMS].

### 1.1 $\epsilon$ complex manifolds

#### Definition 1.1

- (i) A **complex structure** on a (real) finite dimensional vector space  $V$  is a linear map  $J \in \text{End}(V)$  satisfying  $J^2 = -\text{Id}_V$ . A **complex vector space**  $(V, J)$  is a vector space endowed with a complex structure  $J$ . A **complex subspace** of the complex vector space  $V$  is a subspace  $W$  of the real vector space  $V$ , such that the restriction of  $J$  to  $W$  is a complex structure, i.e.  $W$  is  $J$ -invariant.
- (ii) A **para-complex structure** on a (real) finite dimensional vector space  $V$  is a non-trivial involution  $\tau \in \text{End}(V)$ , i.e.  $\tau^2 = \text{Id}_V$  and  $\tau \neq \text{Id}_V$ , such that the two eigenspaces  $V^\pm := \ker(\text{Id} \mp \tau)$  of  $\tau$  have the same dimension. A **para-complex vector space**  $(V, \tau)$  is a vector space endowed with a para-complex structure  $\tau$ . A **para-complex subspace** of the para-complex vector space  $V$  is a subspace  $W$  of the real vector space  $V$ , such that the restriction of  $\tau$  to  $W$  is a para-complex structure.

**Remark 1.1** *It is well-known, that the eigenspaces of a complex structure have the same dimension. We remark, that for para-complex structures the condition on the eigenspaces to have the same dimension is not trivial. This condition can also be restated by requiring that the para-complex structure  $\tau$  is trace-free.*

In the rest of this work we want to enrich our language by the following  $\epsilon$ -notation: If a word has a prefix  $\epsilon$  with  $\epsilon \in \{\pm 1\}$ , i.e. is of the form  $\epsilon X$ , this expression is replaced



by

$$\epsilon X := \begin{cases} X, & \text{for } \epsilon = -1, \\ \text{para-}X, & \text{for } \epsilon = 1. \end{cases}$$

Using this construction we denote an  $\epsilon$ complex structure on the vector space  $V$  by the symbol  $J^\epsilon$ , where  $J^\epsilon$  satisfies  $J^{\epsilon^2} = \epsilon Id_V$ .

**Definition 1.2**

- (i) An almost  $\epsilon$ complex structure on a smooth manifold  $M$  is an endomorphism field  $J^\epsilon \in \Gamma(\text{End}(TM)), p \mapsto J_p^\epsilon$ , such that  $J_p^\epsilon$  is an  $\epsilon$ complex-structure for all  $p \in M$ .
- (ii) An almost para-complex structure is called integrable if the eigendistributions  $T^\pm M$  are both integrable.
- (iii) An integrable almost  $\epsilon$ complex structure is called  $\epsilon$ complex structure. A manifold with an  $\epsilon$ complex structure is called  $\epsilon$ complex manifold.

We remark, that the integrability of an almost  $\epsilon$ complex structure  $J^\epsilon$  is equivalent to the vanishing of the Nijenhuis<sup>1</sup> tensor of  $J^\epsilon$  defined by

$$N_{J^\epsilon}(X, Y) := [J^\epsilon X, J^\epsilon Y] + \epsilon[X, Y] - J^\epsilon[X, J^\epsilon Y] - J^\epsilon[J^\epsilon X, Y],$$

where  $X, Y \in \Gamma(TM)$ .

This is a well-known result in complex geometry. More information can be found in [KN] chapter IX. The para-complex case is done in [CMMS].

**Definition 1.3** A smooth map  $f : (M, J^\epsilon) \rightarrow (N, \tilde{J}^\epsilon)$  from an  $\epsilon$ complex manifold  $(M, J^\epsilon)$  to an  $\epsilon$ complex manifold  $(N, \tilde{J}^\epsilon)$  is called *eholomorphic* if  $df \circ J^\epsilon = \tilde{J}^\epsilon \circ df$  and *anti-eholomorphic* if  $df \circ J^\epsilon = -\tilde{J}^\epsilon \circ df$ .

To go further we introduce the algebra  $\mathbb{C}_\epsilon$  of  $\epsilon$ complex numbers. This is the real algebra generated by 1 and the symbol  $\hat{i}$  subject to the relation  $\hat{i}^2 = \epsilon$ . As one observes for  $\epsilon = -1$  this algebra coincides with the complex numbers  $\mathbb{C}$ . For  $\epsilon = 1$  the symbol  $\hat{i}$  is also denoted by  $e$ . We use the notation

$$\mathbb{C}_\epsilon = \begin{cases} \mathbb{C}, & \text{for } \epsilon = -1, \\ C, & \text{for } \epsilon = 1. \end{cases}$$

If one regards  $e$  as a unit vector in a one-dimensional  $\mathbb{R}$ -vector space with negative definite scalar product, then  $C$  is the (real) Clifford algebra  $Cl_{0,1} = \mathbb{R} \oplus \mathbb{R}$ . In the same manner we obtain  $Cl_{1,0} = \mathbb{C}$  by considering the complex unit  $i$  as a unit vector in a one-dimensional  $\mathbb{R}$ -vector space with positive definite scalar product (Here we used the sign convention of [LM].).

As for complex numbers we define the  $\epsilon$ complex conjugation by

$$\bar{\cdot} : \mathbb{C}_\epsilon \rightarrow \mathbb{C}_\epsilon, \quad x + \hat{i}y \mapsto x - \hat{i}y, \quad \text{for } x, y \in \mathbb{R}, \quad (1.1.1)$$

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<sup>1</sup>In [KN] the Nijenhuis tensor was defined with a factor 2.

which is a  $\mathbb{C}_\epsilon$ -anti-linear involution, i.e.  $\widehat{\hat{z}} = -\hat{z}$ .

Real and imaginary parts are defined as

$$x = \operatorname{Re} z := (z + \bar{z})/2 \text{ and } y = \operatorname{Im} z := \hat{\epsilon}i(z - \bar{z})/2. \quad (1.1.2)$$

One has  $z\bar{z} = x^2 - \epsilon y^2$  where  $z \in \mathbb{C}_\epsilon$ . Therefore the algebra  $C$  is sometimes called the **hypercomplex numbers**.

The circle

$$\mathbb{S}_{-1}^1 := \mathbb{S}^1 = \{z = x + iy \in \mathbb{C} \mid x^2 + y^2 = 1\}$$

is replaced by the four hyperbola

$$\{z = x + ey \in C \mid x^2 - y^2 = \pm 1\}.$$

We define  $\mathbb{S}_1^1$  to be the hyperbola given by the one parameter group  $z(\theta) = \cosh(\theta) + e \sinh(\theta)$ ,  $\theta \in \mathbb{R}$  :

$$\mathbb{S}_1^1 := \{z(\theta) = \cosh(\theta) + e \sinh(\theta) \mid \theta \in \mathbb{R}\}$$

and use the notation

$$\mathbb{S}_\epsilon^1 = \begin{cases} \mathbb{S}_{-1}^1 = \mathbb{S}^1, & \text{for } \epsilon = -1, \\ \mathbb{S}_1^1, & \text{for } \epsilon = 1. \end{cases}$$

In addition we define

$$\cos_\epsilon(x) := \begin{cases} \cos(x), & \text{for } \epsilon = -1, \\ \cosh(x), & \text{for } \epsilon = 1 \end{cases}$$

and

$$\sin_\epsilon(x) := \begin{cases} \sin(x), & \text{for } \epsilon = -1, \\ \sinh(x), & \text{for } \epsilon = 1 \end{cases}$$

to obtain with  $z_\epsilon(\theta) = \cos_\epsilon(\theta) + \hat{i} \sin_\epsilon(\theta)$  :

$$\mathbb{S}_\epsilon^1 = \begin{cases} z_\epsilon(\theta) \text{ with } \theta \in [0, 2\pi], & \text{for } \epsilon = -1, \\ z_\epsilon(\theta) \text{ with } \theta \in \mathbb{R}, & \text{for } \epsilon = 1. \end{cases}$$

Every  $\epsilon$ complex vector space  $V$  is isomorphic to a trivial free  $\mathbb{C}_\epsilon$ -module  $\mathbb{C}_\epsilon^k$  for some  $k$ . Obviously  $\epsilon$ complex subspaces  $W \subset V$  correspond to free submodules of  $V$ .

We regard further the  $\epsilon$ complexification

$$TM^{\mathbb{C}_\epsilon} = TM \otimes_{\mathbb{R}} \mathbb{C}_\epsilon$$

of the tangent bundle  $TM$  of an almost  $\epsilon$ complex manifold  $(M, J^\epsilon)$  and extend

$$J^\epsilon : TM \rightarrow TM$$

$\mathbb{C}_\epsilon$ -linearly to

$$J^\epsilon : TM^{\mathbb{C}_\epsilon} \rightarrow TM^{\mathbb{C}_\epsilon}.$$

Then for all  $p \in M$  the free  $\mathbb{C}_\epsilon$ -module  $T_p M^{\mathbb{C}_\epsilon}$  decomposes as  $\mathbb{C}_\epsilon$ -module into the direct sum of two free  $\mathbb{C}_\epsilon$ -modules

$$T_p M^{\mathbb{C}_\epsilon} = T_p^{1,0} M \oplus T_p^{0,1} M, \quad (1.1.3)$$

where

$$T_p^{1,0} M := \{X + \epsilon \hat{i} J^\epsilon X \mid X \in T_p M\} \text{ and } T_p^{0,1} M := \{X - \epsilon \hat{i} J^\epsilon X \mid X \in T_p M\}.$$

The subbundles  $T_p^{1,0} M$  and  $T_p^{0,1} M$  can be characterized as the  $\pm \hat{i}$ -eigenbundles of the linear map  $J^\epsilon : TM^{\mathbb{C}_\epsilon} \rightarrow TM^{\mathbb{C}_\epsilon}$ , i.e.  $J^\epsilon = \hat{i}$  on  $T^{1,0} M$  and  $J^\epsilon = -\hat{i}$  on  $T^{0,1} M$ .

In the same manner we decompose

$$T^* M^{\mathbb{C}_\epsilon} = \Lambda^{1,0} T^* M \oplus \Lambda^{0,1} T^* M$$

into the  $\pm \hat{i}$ -eigenbundles of the dual  $\epsilon$ complex structure

$$(J^\epsilon)^* : T^* M^{\mathbb{C}_\epsilon} \rightarrow T^* M^{\mathbb{C}_\epsilon}.$$

This decomposition induces a bi-grading on the  $\mathbb{C}_\epsilon$ -valued exterior forms

$$\Lambda^k T^* M^{\mathbb{C}_\epsilon} = \bigoplus_{k=p+q} \Lambda^{p,q} T^* M.$$

We remark that the vector bundles  $\Lambda^{p,0} T^* M$  are  $\epsilon$ holomorphic vector bundles in the sense of the following definition (cf. [AK] for  $\epsilon = -1$  and [LS] for  $\epsilon = 1$ ):

**Definition 1.4**

(i) Let  $(M, J^\epsilon)$  be an  $\epsilon$ complex manifold. An  $\epsilon$ complex vector bundle of rank  $r$  is a smooth real vector bundle  $\pi : E \rightarrow M$  of rank  $2r$  where the total space  $E$  is endowed with a fiberwise  $\epsilon$ complex structure  $J^{\epsilon E} \in \Gamma(\text{End}(E))$ . We will denote it by  $(E, J^{\epsilon E})$ .

(ii) An  $\epsilon$ holomorphic vector bundle is an  $\epsilon$ complex vector bundle  $\pi : E \rightarrow M$  whose total space  $E$  is an  $\epsilon$ complex manifold, such that the projection  $\pi$  is an  $\epsilon$ holomorphic map and admits local  $\epsilon$ holomorphic trivializations.

An (local)  $\epsilon$ holomorphic section of an  $\epsilon$ holomorphic vector bundle  $\pi : E \rightarrow M$  is a (local) section of  $E$  which is an  $\epsilon$ holomorphic map. The set of  $\epsilon$ holomorphic sections of  $E$  will be denoted by  $\mathcal{O}(E)$ .

Finally we obtain a bi-grading on the  $\mathbb{C}_\epsilon$ -valued differential forms on  $M$

$$\Omega_{\mathbb{C}_\epsilon}^k(M) = \bigoplus_{k=p+q} \Omega^{p,q}(M).$$

In para-complex geometry there exists another bi-grading:

The decomposition of  $TM$  over a para-complex manifold  $M$  in  $T^+ M$  and  $T^- M$  induces a bi-grading on exterior forms

$$\Lambda^k T^* M = \bigoplus_{k=p+q} \Lambda^{p+,q-} T^* M. \quad (1.1.4)$$

We remark that for the cases  $(1, 1)$  and  $(1+, 1-)$  the two bi-gradings coincide in the sense that

$$\Lambda^{1,1} T^*M = (\Lambda^{1+,1-} T^*M) \otimes \mathbb{C}_\epsilon.$$

In complex geometry it is well-known, that every complex manifold admits a complex torsion-free connection (see for example [KN] chapter IX). We generalize this theorem to the  $\epsilon$ complex case, which was done in [Sch3]:

**Theorem 1.1** *Every almost  $\epsilon$ complex manifold  $(M, J^\epsilon)$  admits an almost  $\epsilon$ complex affine connection with torsion  $T$  satisfying*

$$N_{J^\epsilon} = -4\epsilon T,$$

where  $N_{J^\epsilon}$  is the Nijenhuis-tensor of the almost  $\epsilon$ complex structure  $J^\epsilon$ .

*Proof:* Let  $\nabla$  be a torsion-free connection on  $M$ . We define  $Q \in \Gamma((T^*M)^2 \otimes TM)$  as

$$4Q(X, Y) := [(\nabla_{J^\epsilon Y} J^\epsilon)X + J^\epsilon((\nabla_Y J^\epsilon)X) + 2J^\epsilon((\nabla_X J^\epsilon)Y)]$$

and further

$$\tilde{\nabla}_X Y = \nabla_X Y + \epsilon Q(X, Y).$$

Now we compute

$$\begin{aligned} (\tilde{\nabla}_X J^\epsilon)Y &= \tilde{\nabla}_X J^\epsilon Y - J^\epsilon \tilde{\nabla}_X Y = \nabla_X J^\epsilon Y + \epsilon Q(X, J^\epsilon Y) - J^\epsilon \nabla_X Y - \epsilon J^\epsilon Q(X, Y) \\ &= (\nabla_X J^\epsilon)Y + \underbrace{\epsilon(Q(X, J^\epsilon Y) - J^\epsilon Q(X, Y))}_{=: A(X, Y)}. \end{aligned}$$

Hence we have to show  $\epsilon A(X, Y) = -(\nabla_X J^\epsilon)Y$ . It is

$$\begin{aligned} 4Q(X, J^\epsilon Y) &= \epsilon(\nabla_Y J^\epsilon)X + J^\epsilon((\nabla_{J^\epsilon Y} J^\epsilon)X) + 2J^\epsilon((\nabla_X J^\epsilon)J^\epsilon Y), \\ 4J^\epsilon Q(X, Y) &= J^\epsilon(\nabla_{J^\epsilon Y} J^\epsilon)X + \epsilon((\nabla_Y J^\epsilon)X) + 2\epsilon((\nabla_X J^\epsilon)Y). \end{aligned}$$

With  $J^{\epsilon 2} = \epsilon \mathbb{1}$  we get  $J^\epsilon[(\nabla_X J^\epsilon)J^\epsilon Y] = -J^\epsilon[J^\epsilon(\nabla_X J^\epsilon)Y] = -\epsilon(\nabla_X J^\epsilon)Y$  and we obtain finally

$$4A = 4(Q(X, J^\epsilon Y) - J^\epsilon Q(X, Y)) = -4\epsilon(\nabla_X J^\epsilon)Y.$$

It remains to compute the torsion of  $\tilde{\nabla}$ :

$$T_{X,Y}^{\tilde{\nabla}} = T_{X,Y}^{\nabla} + \epsilon(Q(X, Y) - Q(Y, X)) = \epsilon(Q(X, Y) - Q(Y, X)).$$

With the definition of  $Q$  we find

$$\begin{aligned} 4\epsilon T_{X,Y}^{\tilde{\nabla}} &= (\nabla_{J^\epsilon Y} J^\epsilon)X + J^\epsilon((\nabla_Y J^\epsilon)X) + 2J^\epsilon((\nabla_X J^\epsilon)Y) \\ &\quad - ((\nabla_{J^\epsilon X} J^\epsilon)Y + J^\epsilon((\nabla_X J^\epsilon)Y) + 2J^\epsilon((\nabla_Y J^\epsilon)X)) \\ &= (\nabla_{J^\epsilon Y} J^\epsilon)X - (\nabla_{J^\epsilon X} J^\epsilon)Y + J^\epsilon((\nabla_X J^\epsilon)Y) - J^\epsilon((\nabla_Y J^\epsilon)X) \\ &= (\nabla_{J^\epsilon Y} J^\epsilon X) - (\nabla_{J^\epsilon X} J^\epsilon Y) - J^\epsilon(\nabla_{J^\epsilon Y} X - \nabla_{J^\epsilon X} Y) \\ &\quad + J^\epsilon[\nabla_X(J^\epsilon Y) - J^\epsilon \nabla_X Y] - J^\epsilon[\nabla_Y(J^\epsilon X) - J^\epsilon \nabla_Y X] \\ &= [J^\epsilon Y, J^\epsilon X] + \epsilon[Y, X] + J^\epsilon[\nabla_X(J^\epsilon Y) - \nabla_{J^\epsilon Y} X] + J^\epsilon[\nabla_{J^\epsilon X} Y - \nabla_Y J^\epsilon X] \\ &= [J^\epsilon Y, J^\epsilon X] + \epsilon[Y, X] - J^\epsilon[J^\epsilon Y, X] - J^\epsilon[Y, J^\epsilon X] = N_{J^\epsilon}(Y, X) = -N_{J^\epsilon}(X, Y). \end{aligned}$$

□

If the  $\epsilon$ complex structure is integrable we get a usefull corollary:

**Corollary 1.1** *Every  $\epsilon$ complex manifold  $(M, J^\epsilon)$  admits an  $\epsilon$ complex torsion-free affine connection.*

An important question in  $\epsilon$ complex geometry is which kind of connections on a given  $\epsilon$ complex vector bundle give rise to an  $\epsilon$ holomorphic structure. To answer to this question we first need the definition of **adapted connections**, which can be found in [AK] for  $\epsilon = -1$  and in [LS] for  $\epsilon = 1$ .

**Definition 1.5**

1. A connection  $\nabla$  on an  $\epsilon$ complex vector bundle  $(E, J^{\epsilon E})$  is called  **$\epsilon$ complex** if it commutes with the  $\epsilon$ complex structure on  $E$ , i.e.  $J^{\epsilon E}$  is  $\nabla$ -parallel. The set of all such connections will be denoted by  $\mathcal{P}(E, J^{\epsilon E})$ .
2. Let  $(E, J^{\epsilon E})$  be an  $\epsilon$ holomorphic vector bundle over an  $\epsilon$ complex manifold  $(M, J^\epsilon)$  and  $U \subset M$  be an arbitrary open set. Let  $\nabla$  be a connection on the vector bundle  $(E, J^{\epsilon E})$ . Then  $\nabla$  is called **adapted** if the following equation

$$\nabla_{J^\epsilon X} \xi = J^{\epsilon E} \nabla_X \xi \tag{1.1.5}$$

is satisfied for all  $X \in \Gamma(TM|_U)$ ,  $\xi \in \mathcal{O}(E|_U)$ .

Conversely, let  $(E, J^{\epsilon E})$  be an  $\epsilon$ holomorphic vector bundle over an  $\epsilon$ complex manifold  $(M, J^\epsilon)$  endowed with an adapted connection  $\nabla \in \mathcal{P}(E, J^{\epsilon E})$ , then a section  $\xi \in \Gamma(E|_U)$ , where  $U \subset M$  is an open set, is  $\epsilon$ holomorphic if and only if it satisfies equation (1.1.5) for all  $X \in \Gamma(TM|_U)$  (cf. Lemma 3 of [LS]).

The following proposition is well-known in complex geometry, compare for example the work of Atiyah, Hitchin and Singer [AHS] theorem 5.2 or proposition 3.7 in the book of Kobayashi [K]. The variety of its applications in complex geometry motivated us to search for a generalization.

For vector bundles over real surfaces this proposition was generalized to para-complex geometry by Erdem [E]. We gave in [LS] a different proof and more general result by adapting the methods of complex geometry to the para-complex setting.

**Proposition 1.1** *Let  $(E, J^{\epsilon E})$  be an  $\epsilon$ complex vector bundle over an  $\epsilon$ complex manifold  $(M, J^\epsilon)$  and  $\nabla$  be a connection in  $\mathcal{P}(E, J^{\epsilon E})$  with vanishing  $(0, 2)$ -curvature then there exists a unique  $\epsilon$ holomorphic vector bundle structure on  $(E, J^{\epsilon E})$  such that  $\nabla$  is adapted to this  $\epsilon$ holomorphic vector bundle structure.*

## 1.2 $\epsilon$ Kähler manifolds

The notion of a (pseudo-)Kähler manifold is classical and the notion of a para-Kähler manifold can be found in [CMMS].

**Definition 1.6** Let  $(V, J^\epsilon)$  be an  $\epsilon$ complex vector space. An  $\epsilon$ hermitian scalar product  $g$  on  $V$  is a pseudo-Euclidean scalar product for which  $J^\epsilon$  is an  $\epsilon$ -isometry, i.e.

$$(J^\epsilon)^*g = g(J^\epsilon \cdot, J^\epsilon \cdot) = -\epsilon g(\cdot, \cdot).$$

An  $\epsilon$ hermitian vector space  $(V, J^\epsilon, g)$  is an  $\epsilon$ complex vector space  $(V, J^\epsilon)$  endowed with an  $\epsilon$ hermitian scalar product  $g$ . The pair  $(J^\epsilon, g)$  is called  $\epsilon$ hermitian structure on the vector space  $V$ .

**Definition 1.7** Let  $(V, \tau, g)$  be a para-hermitian vector space. The para-unitary group of  $V$  is the automorphism group

$$U^\pi(V) := \text{Aut}(V, \tau, g) = \{L \in GL(V) \mid [L, \tau] = 0 \text{ and } L^*g = g\}.$$

Its Lie-algebra will be denoted by  $\mathfrak{u}^\pi(V)$ .

**Definition 1.8** An almost  $\epsilon$ hermitian manifold  $(M, J^\epsilon, g)$  is an almost  $\epsilon$ complex manifold  $(M, J^\epsilon)$  endowed with a pseudo-Riemannian metric  $g$  such that  $(J^\epsilon)^*g = -\epsilon g$ . If  $J^\epsilon$  is integrable, we call  $(M, J^\epsilon, g)$  an  $\epsilon$ hermitian manifold. The two-form  $\omega := g(J^\epsilon \cdot, \cdot)$  is called the fundamental two-form of the almost  $\epsilon$ hermitian manifold  $(M, J^\epsilon, g)$ .

**Definition 1.9** An  $\epsilon$ Kähler manifold  $(M, J^\epsilon, g)$  is an  $\epsilon$ hermitian manifold such that  $J^\epsilon$  is parallel with respect to the Levi-Civita-connection  $D$  of  $g$ , i.e.  $DJ^\epsilon = 0$ .

**Remark 1.2** The fundamental two-form  $\omega$  satisfies  $(J^\epsilon)^*\omega = -\epsilon\omega$  and hence is of type  $(1, 1)$  (considered as  $\mathbb{C}_\epsilon$ -valued two-form).

Since  $DJ^\epsilon = 0$  implies  $N_{J^\epsilon} = 0$  and  $d\omega = 0$ , any  $\epsilon$ Kählerian manifold is an  $\epsilon$ hermitian manifold with closed fundamental two-form.

On an  $\epsilon$ Kähler manifold the fundamental two-form  $\omega$  is called  $\epsilon$ Kähler-form. In fact,  $\epsilon$ Kähler manifolds are characterized to be  $\epsilon$ hermitian manifolds with closed fundamental two-form (compare [CMMS] for the para-complex case).

### 1.3 Nearly $\epsilon$ Kähler manifolds

In this section we introduce some notions and results of nearly  $\epsilon$ Kähler geometry. The complex case is due to Gray in his classical papers [G1, G2, G3]. Recent studies are the works Friedrich and Ivanov [FI] and Nagy [N1, N2]. The para-complex version is very recent and to our knowledge first appeared in the paper of Ivanov and Zamkovoy [IZ].

**Definition 1.10** An almost  $\epsilon$ hermitian manifold  $(M, J^\epsilon, g)$  is called nearly  $\epsilon$ Kähler manifold, if its Levi-Civita connection  $\nabla = \nabla^g$  satisfies the equation

$$(\nabla_X J^\epsilon)Y = -(\nabla_Y J^\epsilon)X, \quad \forall X, Y \in \Gamma(TM). \quad (1.3.1)$$

A nearly  $\epsilon$ Kähler manifold is called strict, if  $\nabla J^\epsilon \neq 0$ .

We recall that the tensor  $\nabla J^\epsilon$  defines two tensors  $A$  and  $B$  by

$$A(X, Y, Z) := g((\nabla_X J^\epsilon)Y, Z) \text{ and } B(X, Y, Z) := -\epsilon g((\nabla_X J^\epsilon)Y, J^\epsilon Z) \text{ with } X, Y, Z \in TM,$$

which are both (real) three-forms of type  $(3, 0) + (0, 3)$ .

A connection of particular importance in nearly  $\epsilon$ Kähler geometry is the connection  $\bar{\nabla}$  defined by

$$\bar{\nabla}_X Y := \nabla_X Y - \frac{1}{2}\epsilon(\nabla_X J^\epsilon)J^\epsilon Y, \text{ for all } X, Y \in \Gamma(TM). \quad (1.3.2)$$

The torsion of the connection  $\bar{\nabla}$  is given by

$$T^{\bar{\nabla}}(X, Y) = -\epsilon(\nabla_X J^\epsilon)J^\epsilon Y, \text{ for all } X, Y \in \Gamma(TM) \quad (1.3.3)$$

and it vanishes if and only if  $(M, J^\epsilon, g)$  is an  $\epsilon$ Kähler manifold.

We remark, that the connection  $\bar{\nabla}$  can be characterized to be the unique connection with totally skew-symmetric torsion (cf. Friedrich and Ivanov [FI] for case  $\epsilon = -1$  with a Riemannian metric.). In [CS2] we give a self-contained proof of this result using direct arguments for nearly pseudo-Kähler and nearly para-Kähler manifolds.

**Proposition 1.2** *Let  $(M, J^\epsilon, g)$  be a nearly  $\epsilon$ Kähler manifold. Then there exists a unique connection  $\bar{\nabla}$  with totally skew-symmetric torsion  $T^{\bar{\nabla}}$  satisfying  $\bar{\nabla}g = 0$  and  $\bar{\nabla}J^\epsilon = 0$ .*

*More precisely, it holds*

$$T^{\bar{\nabla}} = -2S \text{ with } S = -\frac{1}{2}\epsilon J^\epsilon \nabla^g J^\epsilon \quad (1.3.4)$$

*and  $\{S_X, J^\epsilon\} = 0$ , for all vector fields  $X$ .*

## 1.4 Affine special $\epsilon$ complex and special $\epsilon$ Kähler manifolds

**Definition 1.11** *An affine special  $\epsilon$ complex manifold  $(M, J^\epsilon, \nabla)$  is an  $\epsilon$ complex manifold  $(M, J^\epsilon)$  endowed with a torsion-free flat connection  $\nabla$  such that  $\nabla J^\epsilon$  is a symmetric  $(1, 2)$ -tensor field, i.e.  $(\nabla_X J^\epsilon)Y = (\nabla_Y J^\epsilon)X$  for all  $X, Y \in TM$ .*

*An affine special  $\epsilon$ Kähler manifold  $(M, J^\epsilon, g, \nabla)$  is an affine special  $\epsilon$ complex manifold  $(M, J^\epsilon, \nabla)$ , such that  $(M, J^\epsilon, g)$  is an  $\epsilon$ Kähler manifold and  $\nabla$  is symplectic, i.e.  $\nabla\omega = 0$ , where  $\omega$  is the  $\epsilon$ Kähler-form.*

Since projective special  $\epsilon$ complex and projective special  $\epsilon$ Kähler manifolds do not occur in this thesis, we omit the adjective affine. The definition of a special  $\epsilon$ Kähler manifold can be found in [ACD, F] for  $\epsilon = -1$ . Special para-Kähler manifolds were first considered in [CMMS] and special para-complex manifolds in [Sch3].

In the following part of this subsection we are going to generalize some results to  $\epsilon$ complex geometry, which are known from the affine special and the affine special Kähler case (see [ACD]). The para-complex results were published in [Sch3].

**Remark 1.3** Given a linear connection  $\nabla$  on the tangent bundle  $TM$  of a manifold  $M$  and an invertible endomorphism field  $A \in \Gamma(\text{End}(TM))$  we define the connection

$$\nabla^{(A)}X = A\nabla(A^{-1}X).$$

This connection is flat if and only if the connection  $\nabla$  is flat, since

$$\nabla X = 0 \Leftrightarrow \nabla^{(A)}(AX) = 0,$$

where  $X$  is a local vector field on  $M$ .

Again, given a linear flat connection on the real tangent bundle  $TM$  of an  $\epsilon$ complex manifold  $(M, J^\epsilon)$ , we define a one-parameter family of flat connections by

$$\nabla^\theta = \nabla^{(e^{\theta J^\epsilon})} = \nabla^{(\cos_\epsilon(\theta) \text{Id} + \sin_\epsilon(\theta) J^\epsilon)} \text{ for } \theta \in \mathbb{R}. \quad (1.4.1)$$

**Lemma 1.1** Let  $\nabla$  be a flat connection with torsion  $T$  on an  $\epsilon$ complex manifold  $(M, J^\epsilon)$ . Then it is

$$\nabla^\theta = \nabla + A^\theta, \text{ where } A^\theta = e^{\theta J^\epsilon} \nabla(e^{-\theta J^\epsilon}) = -\sin_\epsilon(\theta) e^{\theta J^\epsilon} \nabla J^\epsilon$$

and the torsion  $T^\theta$  of the connection  $\nabla^\theta$  is given by

$$T^\theta = T + \text{alt}(A^\theta) = T - \sin_\epsilon(\theta) e^{\theta J^\epsilon} d^\nabla J^\epsilon. \quad (1.4.2)$$

**Proposition 1.3** Let  $\nabla$  be a torsion-free flat connection on an  $\epsilon$ complex manifold  $(M, J^\epsilon)$ . Then the triple  $(M, J^\epsilon, \nabla)$  defines a special  $\epsilon$ complex manifold if and only if one of the following conditions holds:

a)  $d^\nabla J^\epsilon = 0$ .

b) The flat connection  $\nabla^\theta$  is torsion-free for some  $\theta$  with

$$\begin{cases} \theta \neq 0, & \text{for } \epsilon = 1, \\ \theta \not\equiv 0 \pmod{\pi}, & \text{for } \epsilon = -1. \end{cases}$$

b') The flat connection  $\nabla^\theta$  is torsion-free for all  $\theta$  with

$$\begin{cases} \theta \neq 0, & \text{for } \epsilon = 1, \\ \theta \not\equiv 0 \pmod{\pi}, & \text{for } \epsilon = -1. \end{cases}$$

c) There exists an element  $\theta$  with

$$\begin{cases} \theta \neq 0, & \text{for } \epsilon = 1, \\ \theta \not\equiv 0 \pmod{\pi}, & \text{for } \epsilon = -1, \end{cases}$$

such that  $[e^{\theta J^\epsilon} X, e^{\theta J^\epsilon} Y] = 0$  for all  $\nabla$ -parallel local vector fields  $X$  and  $Y$  on  $M$ .



c') It holds  $[e^{\theta J^\epsilon} X, e^{\theta J^\epsilon} Y] = 0$  for all  $\theta$  with

$$\begin{cases} \theta \neq 0, & \text{for } \epsilon = 1, \\ \theta \not\equiv 0 \pmod{\pi}, & \text{for } \epsilon = -1 \end{cases}$$

and for all  $\nabla$ -parallel local vector fields  $X$  and  $Y$  on  $M$ .

d) There exists an element  $\theta$  with

$$\begin{cases} \theta \neq 0, & \text{for } \epsilon = 1, \\ \theta \not\equiv 0 \pmod{\pi}, & \text{for } \epsilon = -1, \end{cases}$$

such that  $d(\eta \circ e^{-\theta J^\epsilon}) = 0$  for all  $\nabla$ -parallel local one-forms on  $M$ .

d') It holds  $d(\eta \circ e^{-\theta J^\epsilon}) = 0$  for all  $\theta$  with

$$\begin{cases} \theta \neq 0, & \text{for } \epsilon = 1, \\ \theta \not\equiv 0 \pmod{\pi}, & \text{for } \epsilon = -1 \end{cases}$$

and for all  $\nabla$ -parallel local one-forms on  $M$ .

*Proof:* The property a) defines special  $\epsilon$ -complex manifolds. As  $\nabla$  is torsion-free, the torsion of  $\nabla^\theta$  is by equation (1.4.2):

$$T^\theta = -\sin_\epsilon(\theta) e^{\theta J^\epsilon} d^\nabla J^\epsilon.$$

Since  $\sin_\epsilon(\theta) \neq 0$  for  $\theta$  with

$$\begin{cases} \theta \neq 0, & \text{for } \epsilon = 1, \\ \theta \not\equiv 0 \pmod{\pi}, & \text{for } \epsilon = -1, \end{cases}$$

we get the equivalence of a) and b) respectively b') and c').

Let  $X$  and  $Y$  be  $\nabla$ -parallel local vector fields. Then  $e^{\theta J^\epsilon} X$  and  $e^{\theta J^\epsilon} Y$  are  $\nabla^\theta$ -parallel, by the definition of  $\nabla^\theta$ . Therefore

$$T^\theta(e^{\theta J^\epsilon} X, e^{\theta J^\epsilon} Y) = [e^{\theta J^\epsilon} X, e^{\theta J^\epsilon} Y].$$

This gives  $b) \Leftrightarrow c)$  and  $b') \Leftrightarrow c')$ .

For a  $\nabla$ -parallel one-form  $\eta$  and  $X, Y$  as before we compute:

$$\begin{aligned} & d(\eta \circ e^{-\theta J^\epsilon})(e^{\theta J^\epsilon} X, e^{\theta J^\epsilon} Y) \\ &= e^{\theta J^\epsilon} X \eta(Y) - e^{\theta J^\epsilon} Y \eta(X) - \eta(e^{-\theta J^\epsilon} [e^{\theta J^\epsilon} X, e^{\theta J^\epsilon} Y]) \\ &= -\eta(e^{-\theta J^\epsilon} [e^{\theta J^\epsilon} X, e^{\theta J^\epsilon} Y]), \end{aligned}$$

as the functions  $\eta(X)$  and  $\eta(Y)$  are constant. This proves  $c) \Leftrightarrow d)$  and  $c') \Leftrightarrow d')$ .  $\square$

**Proposition 1.4** *If  $(M, J^\epsilon, \nabla)$  is a special  $\epsilon$ -complex manifold, then  $(M, J^\epsilon, \nabla^\theta)$  is a special  $\epsilon$ -complex manifold for any  $\theta$ .*

*If  $(M, J^\epsilon, g, \nabla)$  is a special  $\epsilon$ -Kähler manifold, then  $(M, J^\epsilon, g, \nabla^\theta)$  is a special  $\epsilon$ -Kähler manifold for any  $\theta$ .*

*Proof:* From above we know, that the connection  $\nabla^\theta$  is torsion-free and flat. In order to prove this proposition we compute  $\nabla^\theta J^\epsilon$  and  $\nabla^\theta \omega$ . Let  $X, Y, Z \in \Gamma(TM)$  :

$$\begin{aligned} (\nabla_X^\theta J^\epsilon)Y &= \nabla_X^\theta(J^\epsilon Y) - J^\epsilon \nabla_X^\theta Y = e^{\theta J^\epsilon} \nabla_X(e^{-\theta J^\epsilon} J^\epsilon Y) - J^\epsilon e^{\theta J^\epsilon} \nabla_X(e^{-\theta J^\epsilon} Y) \\ &= e^{\theta J^\epsilon} \nabla_X(J^\epsilon e^{-\theta J^\epsilon} Y) - e^{\theta J^\epsilon} J^\epsilon \nabla_X(e^{-\theta J^\epsilon} Y) \\ &= e^{\theta J^\epsilon} (\nabla_X J^\epsilon) e^{-\theta J^\epsilon} Y \stackrel{(*)}{=} e^{2\theta J^\epsilon} (\nabla_X J^\epsilon) Y. \end{aligned}$$

At (\*) we have used  $J^\epsilon(\nabla J^\epsilon) = -(\nabla J^\epsilon)J^\epsilon$ , which follows from  $J^{\epsilon 2} = \epsilon Id$ .

This shows  $d^{\nabla^\theta} J^\epsilon = e^{2\theta J^\epsilon} d^\nabla J^\epsilon = 0$ .

Further we find utilizing  $\omega(\cdot, e^{\theta J^\epsilon} \cdot) = \omega(e^{-\theta J^\epsilon} \cdot, \cdot)$ , which is a consequence of  $(J^\epsilon)^* \omega = -\epsilon \omega$  :

$$\begin{aligned} \nabla_Z^\theta \omega(X, Y) &= Z\omega(X, Y) - \omega(\nabla_Z^\theta X, Y) - \omega(X, \nabla_Z^\theta Y) \\ &= Z\omega(X, Y) - \omega(e^{\theta J^\epsilon} \nabla_Z e^{-\theta J^\epsilon} X, Y) - \omega(X, e^{\theta J^\epsilon} \nabla_Z e^{-\theta J^\epsilon} Y) \\ &= Z\omega(X, Y) - \omega(\nabla_Z e^{-\theta J^\epsilon} X, e^{-\theta J^\epsilon} Y) - \omega(e^{-\theta J^\epsilon} X, \nabla_Z e^{-\theta J^\epsilon} Y) \\ &= Z\omega(X, Y) - Z\omega(e^{-\theta J^\epsilon} X, e^{-\theta J^\epsilon} Y) = 0. \end{aligned}$$

□

Given an  $\epsilon$ complex manifold with a flat connection  $\nabla$ , we define the conjugate connection via

$$\nabla_X^c Y = \nabla_X^{(J^\epsilon)} Y = \epsilon J^\epsilon (\nabla_X J^\epsilon Y) = \nabla_X Y + \epsilon J^\epsilon (\nabla_X J^\epsilon) Y \text{ for } X, Y \in \Gamma(TM).$$

**Proposition 1.5** *Let  $(M, J^\epsilon)$  be an  $\epsilon$ complex manifold with a torsion-free flat connection  $\nabla$ . Then the following statements are equivalent:*

- a)  $(M, J^\epsilon, \nabla)$  is a special  $\epsilon$ complex manifold.
- b) The conjugate flat connection  $\nabla^c$  is torsion-free.

*Proof:* The torsion of the connection  $\nabla^c$  is

$$T^{\nabla^c} = T^\nabla + \epsilon \text{alt}(J^\epsilon(\nabla J^\epsilon)) = \epsilon J^\epsilon d^\nabla J^\epsilon.$$

Therefore  $\nabla^c$  is torsion-free if and only if  $d^\nabla J^\epsilon = 0$ . □

**Proposition 1.6** *Let  $(M, J^\epsilon, \nabla)$  be a special  $\epsilon$ complex manifold. Then  $D := \frac{1}{2}(\nabla + \nabla^c)$  defines a torsion-free  $\epsilon$ complex connection, i.e. a torsion-free connection such that  $DJ^\epsilon = 0$ .*

*Proof:* As it is a convex combination of torsion-free connections,  $D$  is a torsion-free connection. For any  $X \in \Gamma(TM)$  we compute:

$$D_X J^\epsilon = \nabla_X J^\epsilon + \frac{1}{2} \epsilon [J^\epsilon \nabla_X J^\epsilon, J^\epsilon] = \nabla_X J^\epsilon - \nabla_X J^\epsilon = 0.$$

□

**Proposition 1.7** *Let  $(M, J^\epsilon, g, \nabla)$  be a special  $\epsilon$ Kähler manifold and  $\nabla^g$  the Levi-Civita connection of  $g$ . Then the following hold:*

(i)  $\nabla^g = \frac{1}{2}(\nabla + \nabla^c) = D.$

(ii) *The conjugate connection  $\nabla^c$  is  $g$ -dual, i.e.:*

$$Xg(Y, Z) = g(\nabla_X^c Y, Z) + g(Y, \nabla_X Z).$$

*Equivalently*

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X^c Z)$$

*for all vector fields  $X, Y, Z \in \Gamma(TM)$ .*

(iii) *The tensor  $\nabla g$  is completely symmetric.*

*Proof:* (i) follows immediately from (ii) and proposition 1.6.

(ii) follows from a direct calculation which only uses the fact that  $\omega$  is  $\nabla$ -parallel and  $J^\epsilon$ - $\epsilon$ -anti-invariant: With  $X, Y, Z \in \Gamma(TM)$  one finds

$$\begin{aligned} Xg(Y, Z) &= X(\epsilon\omega(J^\epsilon Y, Z)) = \epsilon\omega(\nabla_X J^\epsilon Y, Z) + \epsilon\omega(J^\epsilon Y, \nabla_X Z) \\ &= -\omega(J^\epsilon \nabla_X J^\epsilon Y, J^\epsilon Z) + g(Y, \nabla_X Z) \\ &= \omega(J^\epsilon Z, J^\epsilon \nabla_X J^\epsilon Y) + g(Y, \nabla_X Z) \\ &= g(Z, \epsilon J^\epsilon \nabla_X (J^\epsilon Y)) + g(Y, \nabla_X Z) \\ &= g(\nabla_X^c Y, Z) + g(Y, \nabla_X Z). \end{aligned}$$

Finally we show (iii): From part (ii) it follows

$$\begin{aligned} (\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) &= Xg(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \\ &\quad - Yg(X, Z) + g(\nabla_Y X, Z) + g(X, \nabla_Y Z) \\ &\stackrel{(ii)}{=} -g(\nabla_X Y, Z) + g(\nabla_X^c Y, Z) \\ &\quad + g(\nabla_Y X, Z) - g(\nabla_Y^c X, Z) \\ &= g(-[X, Y] + [X, Y], Z) = 0. \end{aligned}$$

The symmetry of  $g$  finishes the proof. □

**Proposition 1.8** *Let  $(M, J^\epsilon, g, \nabla)$  be a special  $\epsilon$ Kähler manifold and  $D$  the Levi-Civita connection of  $g$ . Define the endomorphism field  $S$  as*

$$S := \nabla - D = \nabla - \frac{1}{2}(\nabla + \nabla^c) = \frac{1}{2}(\nabla - \nabla^c) = -\frac{1}{2}\epsilon J^\epsilon (\nabla J^\epsilon).$$

*Then  $S$  is*

(i) *symmetric, i.e.  $S_X Y = S_Y X; \forall X, Y \in \Gamma(TM)$ ,*

(ii)  *$\omega$ -skew-symmetric, i.e.  $\omega(S_X \cdot, \cdot) = -\omega(\cdot, S_X \cdot)$ ,*

(iii)  *$g$ -symmetric, i.e.  $g(S_X \cdot, \cdot) = g(\cdot, S_X \cdot)$  for all  $X \in \Gamma(TM)$  and*

(iv) *anti-commutes with  $J^\epsilon$ , i.e.*

$$\{S_X, J^\epsilon\} := S_X J^\epsilon + J^\epsilon S_X = 0 \text{ for all } X \in \Gamma(TM). \quad (1.4.3)$$

*Proof:* Let  $X, Y, Z \in \Gamma(TM)$ .

(i) For a special  $\epsilon$ complex manifold  $\nabla$  and  $\nabla^c$  are torsion-free (by definition and proposition 1.5), so  $\nabla - \nabla^c = -\epsilon J^\epsilon(\nabla J^\epsilon) = 2S$  is symmetric.

(ii) In fact  $Dg = 0$  (proposition 1.7) and  $DJ^\epsilon = 0$  (proposition 1.6) imply  $D\omega = 0$ . In addition  $\nabla\omega = 0$  yields

$$\omega(S_X Y, Z) + \omega(Y, S_X Z) = \omega((\nabla - D)_X Y, Z) + \omega(Y, (\nabla - D)_X Z) = (\nabla - D)_X \omega(Y, Z) = 0.$$

(iii) Using  $Xg(Y, Z) - g(\nabla_X Y, Z) = g(Y, \nabla_X^c Z)$  we prove the  $g$ -symmetry of  $S$

$$\begin{aligned} 2g(S_X Y, Z) &= g((\nabla - \nabla^c)_X Y, Z) = g(\nabla_X Y, Z) - g(\nabla_X^c Y, Z) \\ &= Xg(Y, Z) - g(Y, \nabla_X^c Z) - Xg(Y, Z) + g(Y, \nabla_X Z) \\ &= g(Y, (\nabla - \nabla^c)_X Z) = 2g(Y, S_X Z). \end{aligned}$$

(iv) Now we need only the  $\omega$ -skew-symmetry of  $S$ , the  $g$ -symmetry of  $S$  and  $\omega = g(J^\epsilon \cdot, \cdot) = -g(\cdot, J^\epsilon \cdot)$  to get for all  $X, Y, Z \in \Gamma(TM)$

$$g(S_X J^\epsilon Y, Z) = g(J^\epsilon Y, S_X Z) = \omega(Y, S_X Z) = -\omega(S_X Y, Z) = -g(J^\epsilon S_X Y, Z)$$

and consequently  $\{S_X, J^\epsilon\} = 0$ . □

## 1.5 The extrinsic construction of special $\epsilon$ Kähler manifolds

Now we shortly explain the extrinsic construction of special  $\epsilon$ Kähler manifolds given in [ACD, CMMS].

### 1.5.1 The special Kähler case

We consider the complex vector space  $V = T^*\mathbb{C}^n = \mathbb{C}^{2n}$  with canonical coordinates  $(z^1, \dots, z^n, w_1, \dots, w_n)$  endowed with the standard complex symplectic form

$$\Omega = \sum_{i=1}^n dz^i \wedge dw_i$$

and the standard real structure  $\kappa : V \rightarrow V$  with fixed points  $V^\kappa = T^*\mathbb{R}^n$ . These define a hermitian form  $\gamma := i\Omega(\cdot, \kappa \cdot)$ .

Let  $(M, J)$  be a complex manifold of complex dimension  $n$ . We call a holomorphic immersion  $\phi : M \rightarrow V$  **non-degenerate** (respectively **Lagrangian**) if  $\phi^*\gamma$  is non-degenerate (respectively, if  $\phi^*\Omega = 0$ ). If  $\phi$  is non-degenerate it defines a (possibly indefinite) Kähler metric  $g = \text{Re } \phi^*\gamma$  on the complex manifold  $(M, J)$  and the corresponding Kähler form  $g(J\cdot, \cdot)$  is a  $J$ -invariant symplectic form.

The following theorem gives a description of simply connected special Kähler manifolds in terms of the above data:

**Theorem 1.2** [ACD] *Let  $(M, J, g, \nabla)$  be a simply connected special Kähler manifold of complex dimension  $n$ , then there exists a holomorphic non-degenerate Lagrangian immersion  $\phi : M \rightarrow V = T^*\mathbb{C}^n$  inducing the Kähler metric  $g$ , the connection  $\nabla$  and the symplectic form  $\omega = g(J\cdot, \cdot) = 2\phi^*(\sum_{i=1}^n dx^i \wedge dy_i)$  on  $M$ . Moreover,  $\phi$  is unique up to an affine transformation of  $V$  preserving the complex symplectic form  $\Omega$  and the real structure  $\kappa$ . The flat connection  $\nabla$  is completely determined by the condition  $\nabla\phi^*dx^i = \nabla\phi^*dy_i = 0$ ,  $i = 1, \dots, n$ , where  $x^i = \operatorname{Re} z^i$  and  $y_i = \operatorname{Re} w_i$ .*

## 1.5.2 The special para-Kähler case

First we have to introduce a canonical non-degenerate exact  $C$ -valued two-form  $\Omega$  of type  $(2,0)$  on the cotangent bundle  $N = T^*M$  of an arbitrary para-complex manifold  $(M, \tau)$ , which is para-holomorphic, i.e. it is a para-holomorphic section of the para-holomorphic vector bundle  $\Lambda^{2,0}T^*N$ . Its explicit form is given by the following consideration:

We take local para-holomorphic coordinates  $(z^1, \dots, z^n)$  on an open subset  $U \subset M^n$ . Any point of  $T_p^*M \cong \operatorname{Hom}(T_p^*M, \mathbb{R}) \cong \operatorname{Hom}_C(T_p^*M, C)$ ,  $p \in U$ , where  $\operatorname{Hom}_C(T_p^*M, C)$  are the homomorphisms from the para-complex vector space  $(T_p^*M, \tau_p)$  to  $C$ , can be expressed as  $\sum w_i dz^i|_p$ . The coordinates  $z^i$  and  $w_i$  can be regarded as local para-holomorphic coordinates of the bundle  $T^*M|_U$ . The coordinates  $w_i$  induce linear para-holomorphic coordinates on each fiber  $T_p^*M$  for  $p \in U$ . In these coordinates the two form  $\Omega$  is given by

$$\Omega = \sum_{i=1}^n dz^i \wedge dw_i = -d \left( \sum_{i=1}^n w_i dz^i \right).$$

We observe, that  $\sum_{i=1}^n w_i dz^i$  does not depend on the choice of coordinates and hence  $\Omega$  does not depend on the choice of coordinates, too. The form  $\Omega$  will be called the symplectic form of  $T^*M$ .

In the following, we denote by  $V$  the para-holomorphic vector space  $T^*C^n = C^{2n}$ , endowed with its standard para-complex structure  $\tau_V$ , its symplectic form  $\Omega$  and the para-complex conjugation  $\bar{\cdot} : V \rightarrow V$ ,  $v \mapsto \bar{v}$  with fixed point set  $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$ . On this space we take a system of para-holomorphic linear coordinates  $(z^i, w_i)$  which are real-valued on  $T^*\mathbb{R}^n$ . The algebraic data  $(\Omega, \tau_V)$  defines a para-hermitian scalar product on  $V$  via

$$g_V(v, w) = \operatorname{Re} \gamma(v, w) = \operatorname{Re}(e\Omega(v, \bar{w})), \quad \forall v, w \in V \text{ with } \gamma(v, w) = e\Omega(v, \bar{w})$$

and  $(V, \tau_V, g_V)$  is a flat para-Kähler manifold, whose para-Kähler form is given by

$$\omega_V(v, w) := g_V(\tau_V v, w) = \operatorname{Im}(e\Omega(v, \bar{w})), \quad \forall v, w \in V.$$

Let  $(M, \tau)$  be a para-complex manifold. We call a para-holomorphic immersion  $\phi : M \rightarrow V$  **para-Kählerian** if  $g = \phi^*g_V$  is non-degenerate and **Lagrangian** if  $\phi^*\Omega = 0$ . Any para-Kählerian immersion  $\phi : M \rightarrow V$  induces on  $M$  the structure of a para-Kähler manifold  $(M, \tau, g)$  with para-Kähler form  $\omega(\cdot, \cdot) = g(\tau\cdot, \cdot) = \phi^*\omega_V$ . For a para-Kählerian Lagrangian immersion the para-Kähler form  $\omega = g(\tau\cdot, \cdot)$  of  $M$  is given by

$$\omega = 2 \sum_{i=1}^n d\tilde{x}^i \wedge d\tilde{y}_i,$$

where  $\tilde{x}^i = \operatorname{Re}(\phi^* z^i)$  and  $\tilde{y}^i = \operatorname{Re}(\phi^* w_i)$ . Additionally, a para-Kählerian Lagrangian immersion  $\phi : M \rightarrow V$  induces a canonical flat torsion-free connection  $\nabla$  on  $M$  which is characterized by the condition, that  $\nabla(\operatorname{Re} \phi^* df) = 0$  for all para-complex affine functions  $f$  on  $V$ .

With this information we now can give the extrinsic description of para-Kähler manifolds:

**Theorem 1.3** [CMMS] *Let  $\phi : M \rightarrow V$  be a para-Kählerian immersion with induced geometric data  $(\tau, g, \nabla)$ . Then  $(M, \tau, g, \nabla)$  is a special para-Kähler manifold. Conversely, any simply connected special para-Kähler manifold  $(M, \tau, g, \nabla)$  admits a para-Kählerian Lagrangian immersion inducing the special geometric data  $(\tau, g, \nabla)$  on  $M$ . The para-Kählerian Lagrangian immersion  $\phi$  is unique up to an affine linear transformation of  $V$  whose linear part belongs to the group  $\operatorname{Aut}(V, \Omega, \bar{\cdot}) = \operatorname{Aut}_{\mathbb{R}}(V, \tau_V, \Omega, \bar{\cdot}) = \operatorname{Sp}(\mathbb{R}^{2n})$ .*

## 1.6 Variations of $\epsilon$ Hodge structures

In this section we introduce the notion of variations of  $\epsilon$ Hodge structures in para-complex geometry and recall variations of Hodge structures which are classical objects in complex geometry. We follow the notations of [CMP] which is a reference for further study of variations of Hodge structures. The para-complex version seems to be new.

### 1.6.1 $\epsilon$ Hodge structures and their variations

#### Definition 1.12

- (a) *A real  $\epsilon$ Hodge structure of weight  $w \in \mathbb{N}$  is a real vector space  $H$  on the  $\epsilon$ complexification of which there is a decomposition into  $\epsilon$ complex vector spaces*

$$H^{\mathbb{C}_\epsilon} = \bigoplus_{w=p+q} H^{p,q} \quad \text{with } p, q \in \mathbb{N} \quad (1.6.1)$$

and where

$$\overline{H^{p,q}} = H^{q,p} \quad \text{with } p, q \in \mathbb{N}. \quad (1.6.2)$$

The  $\epsilon$ complex conjugation  $\bar{\cdot}$  is relative to the real structure on  $H^{\mathbb{C}_\epsilon} = H \otimes \mathbb{C}_\epsilon$ .

- (b) *Suppose, that an  $\epsilon$ Hodge structure of weight  $w$  carries a bilinear form  $b : H \times H \rightarrow \mathbb{R}$  which satisfies the following Riemannian bilinear relations*

- (i) *The  $\mathbb{C}_\epsilon$ -linear extension of the bilinear form  $b$ , also denoted by  $b$ , satisfies  $b(x, y) = 0$  if  $x \in H^{p,q}$  and  $y \in H^{r,s}$  for  $(r, s) \neq (w-p, w-q) = (q, p)$ ,*
- (ii) *The bilinear form  $b$  defines an hermitian sesquilinear scalar product (compare definition 2.10) on  $H^{p,q}$  by*

$$h(x, y) = (-1)^{w(w-1)/2} \hat{i}^{p-q} b(x, \bar{y}).$$

*Then we call this  $\epsilon$ Hodge structure weakly polarized.*

(c) Suppose, that a Hodge structure of weight  $w$  carries a bilinear form  $b : H \times H \rightarrow \mathbb{R}$  which satisfies the following Riemannian bilinear relations

- (i) The  $\mathbb{C}$ -linear extension of the bilinear form  $b$ , also denoted by  $b$ , satisfies  $b(x, y) = 0$  if  $x \in H^{p,q}$  and  $y \in H^{r,s}$  for  $(r, s) \neq (w-p, w-q) = (q, p)$ ,
- (ii) The bilinear form  $b$  defines a positive definite hermitian sesquilinear form on  $H^{p,q}$  by

$$h(x, y) = (-1)^{w(w-1)/2} i^{p-q} b(x, \bar{y}).$$

Then we call this Hodge structure **strongly polarized**.

(d) An  $\epsilon$ Hodge structure of weight  $w$  is called **polarized** if it is weakly polarized or strongly polarized.

Closely related to the  $\epsilon$ Hodge decomposition is the  $\epsilon$ Hodge filtration

$$F^p = \bigoplus_{a \geq p} H^{a,b}, \quad p = 0, \dots, w, \quad (1.6.3)$$

which satisfies for an  $\epsilon$ Hodge structure of weight  $w$  the relation

$$H^{\mathbb{C}\epsilon} = F^p \oplus \overline{F^{w-p+1}}, \quad p = 1, \dots, w. \quad (1.6.4)$$

Any filtration which obeys equation (1.6.4) is called an  $\epsilon$ Hodge filtration.

Such as an  $\epsilon$ Hodge decomposition induces an  $\epsilon$ Hodge filtration we obtain from an  $\epsilon$ Hodge filtration an  $\epsilon$ Hodge decomposition by

$$H^{p,q} = F^p \cap \overline{F^q}, \quad \text{with } p + q = w.$$

This  $\epsilon$ Hodge decomposition satisfies the relation (1.6.3).

We remark further, that the first Riemannian bilinear relation (cf. definition 1.12) is equivalent to

$$(F^p)^\perp = F^{w-p+1}, \quad p = 1, \dots, w,$$

where  $\perp$  is taken with respect to the bilinear form  $b$ .

Now we are going to consider deformations of these structures:

**Definition 1.13** A (real) variation of  $\epsilon$ Hodge structures ( $\epsilon$ VHS) is a triple  $(E, \nabla, F^p)$ , where  $E$  is a real vector bundle over an (connected)  $\epsilon$ complex base manifold  $(M, J^\epsilon)$ ,  $\nabla$  is a flat connection and  $F^p$  is a filtration of  $E^{\mathbb{C}\epsilon}$  by holomorphic subbundles of  $E^{\mathbb{C}\epsilon}$ , which is a point-wise  $\epsilon$ Hodge structure satisfying the infinitesimal period relation or the Griffiths transversality

$$\nabla_\chi F^p \subset F^{p-1}, \quad \forall \chi \in T^{1,0}M. \quad (1.6.5)$$

A polarization of a variation of  $\epsilon$ Hodge structures  $(E, \nabla, F^p)$  consists of a non-degenerate bilinear form

$$b \in \Gamma(E^* \otimes E^*) \quad (1.6.6)$$

having the following properties

- (i)  $b$  induces a polarization on each fiber obeying the first and the second bilinear relation.
- (ii)  $b$  is parallel with respect to  $\nabla$ .

## 1.6.2 $\epsilon$ VHS and special $\epsilon$ Kähler manifolds

Each fiber of the  $\epsilon$ complex tangent bundle

$$TM^{\mathbb{C}\epsilon} = T^{1,0}M \oplus T^{0,1}M$$

carries a natural  $\epsilon$ Hodge structure of weight 1 :

$$0 = F_x^2 \subset F_x^1 = T_x^{1,0}M \subset F_x^0 = T_x^{\mathbb{C}\epsilon}M. \quad (1.6.7)$$

The complex version of the next lemma and proposition was proven in [Her] and we generalize it to the para-complex case.

**Lemma 1.2** *Let  $\nabla$  be a torsion-free flat connection on the  $\epsilon$ complex manifold  $(M, J^\epsilon)$ . Then  $F^1 = T^{1,0}M$  is an  $\epsilon$ holomorphic subbundle of  $F^0 = T^{\mathbb{C}\epsilon}M$  with respect to the  $\epsilon$ holomorphic structure defined by  $\nabla$  (compare proposition 1.1) if and only if  $(\nabla, J^\epsilon)$  is special (see definition 1.11).*

*Proof:* The condition of  $F^1$  to be  $\epsilon$ holomorphic is equivalent to

$$\nabla_{\bar{Y}}X = 0 \text{ for all } X, Y \in \mathcal{O}(T^{1,0}M)$$

and the condition of  $(\nabla, J^\epsilon)$  to be special is equivalent to

$$(\nabla_X J^\epsilon)(\bar{Y}) = (\nabla_{\bar{Y}} J^\epsilon)(X) \text{ for all } X, Y \in \mathcal{O}(T^{1,0}M),$$

due to the following short argument :

Let  $X, Y \in \Gamma(T^{1,0}M)$

$$(\nabla_X J^\epsilon)(Y) = \nabla_X J^\epsilon Y - J^\epsilon \nabla_X Y = \hat{i} \nabla_X Y - J^\epsilon \nabla_X Y,$$

which is symmetric as one sees by choosing vector fields  $X$  and  $Y$  such that  $[X, Y] = 0$ .

Let  $X, Y \in \Gamma(T^{0,1}M)$

$$(\nabla_X J^\epsilon)(Y) = \nabla_X J^\epsilon Y - J^\epsilon \nabla_X Y = -\hat{i} \nabla_X Y - J^\epsilon \nabla_X Y,$$

which is again symmetric as one sees by choosing vector fields  $X$  and  $Y$  such that  $[X, Y] = 0$ .

Let now  $X, Y \in \Gamma(T^{1,0}M)$  be  $\epsilon$ holomorphic vector fields, i.e.  $\mathcal{L}_X(J^\epsilon) = 0$  where  $\mathcal{L}$  is the Lie-derivative. Then it holds

$$\begin{aligned} 0 &= \mathcal{L}_X(J^\epsilon)\bar{Y} = [X, J^\epsilon\bar{Y}] - J^\epsilon[X, \bar{Y}] \\ &= \nabla_X J^\epsilon\bar{Y} - \nabla_{J^\epsilon\bar{Y}}X - J^\epsilon\nabla_X\bar{Y} + J^\epsilon\nabla_{\bar{Y}}X \\ &= (\nabla_X J^\epsilon)\bar{Y} - (\nabla_{\bar{Y}} J^\epsilon)X + \nabla_{\bar{Y}}J^\epsilon X - \nabla_{J^\epsilon\bar{Y}}X \\ &= [(\nabla_X J^\epsilon)\bar{Y} - (\nabla_{\bar{Y}} J^\epsilon)X] + 2\hat{i}\nabla_{\bar{Y}}X. \end{aligned}$$



This finishes the proof. □

From the lemma we obtain:

**Proposition 1.9** *Let  $(M, J^\epsilon)$  be an  $\epsilon$ complex manifold,  $\nabla$  be a torsion-free flat connection and  $F^\bullet$  defined as in equation (1.6.7).*

1. *Then  $(M, J^\epsilon, \nabla)$  is an affine special  $\epsilon$ complex manifold if and only if  $\nabla$  and  $F^\bullet$  give a variation of  $\epsilon$ Hodge structures of weight 1 on  $TM^{\mathbb{C}^\epsilon}$ .*
2. *Then  $(M, J^\epsilon, \nabla, g)$  is an affine special  $\epsilon$ Kähler manifold if and only if  $\nabla$ ,  $F^\bullet$  and  $\omega(\cdot, \cdot) = g(J^\epsilon \cdot, \cdot)$  give a variation of polarized  $\epsilon$ Hodge structures of weight 1 on  $TM^{\mathbb{C}^\epsilon}$ .*

# Chapter 2

## Harmonic and $\epsilon$ pluriharmonic maps

In this chapter we introduce the notion of harmonic maps and  $\epsilon$ pluriharmonic maps. We discuss the relation between them and we give a generalization of  $\epsilon$ pluriharmonic maps and of associated families of pluriharmonic maps to maps from almost  $\epsilon$ complex manifolds into pseudo-Riemannian manifolds. Afterwards we discuss the target spaces which are of importance in the context of  $\epsilon$ pluriharmonic maps associated to  $\epsilon$ tt\*-geometry.

### 2.1 Harmonic maps

First we recall the notion of a harmonic map.

**Definition 2.1** *Let  $(M, g)$  and  $(N, h)$  be pseudo-Riemannian manifolds and  $f : M \rightarrow N$  be a  $C^2$ -map.*

(i) *One defines the energy density of  $f$  by*

$$e(f) = \frac{1}{2}G(df, df), \quad (2.1.1)$$

*where  $df$  is seen as a section in  $T^*M \otimes f^*TN$  and  $G$  is the metric on  $T^*M \otimes f^*TN$  induced by the metrics  $g$  and  $h$ .*

(ii) *If the energy density  $e(f)$  is integrable we define the energy  $E(f)$  of  $f$  as*

$$E(f) = \int_M e(f) \text{vol}_g. \quad (2.1.2)$$

(iii) *The critical points of  $E(f)$  with respect to compact supported  $C^\infty$ -variations are called **harmonic maps** where the variation of  $E(f)$  with respect to the family of maps  $f_t$  with  $t \in (-\epsilon, \epsilon)$  is defined by*

$$\delta E(f) = \int_M \partial_t e(f_t) \text{vol}_g.$$

The following proposition states the well-known harmonic map equations, which are the Euler-Lagrange equations of the harmonic functional.

**Proposition 2.1** *Let  $(M, g)$  and  $(N, h)$  be pseudo-Riemannian manifolds and  $f : M \rightarrow N$  be a  $C^2$ -map. Denote by  $\nabla^g$  the Levi-Civita connection of  $g$ , by  $\nabla^h$  the Levi-Civita connection of  $h$  and by  $\nabla$  the connection induced by  $\nabla^g$  and  $\nabla^h$  on  $T^*M \otimes f^*TN$ . Then  $f$  is harmonic if and only if it satisfies the equation*

$$\operatorname{tr}_g \nabla df = 0. \quad (2.1.3)$$

First we recall a result about a special class of harmonic morphisms which is needed later:

**Proposition 2.2** *Let  $M, X$  and  $Y$  be pseudo-Riemannian manifolds and  $\Psi : X \rightarrow Y$  be a totally geodesic immersion. Then a map  $f : M \rightarrow X$  is harmonic if and only if  $\Psi \circ f : M \rightarrow Y$  is harmonic.*

*Proof:* Note  $\tau(f) = \operatorname{tr}_g \nabla df$  and let  $\Psi : X \rightarrow Y$  be an arbitrary map. Then we calculate

$$\operatorname{tr}(\nabla^X d(\Psi \circ f)) = \operatorname{tr}_g(\nabla^X d\Psi \circ df) = \operatorname{tr}_g(d\Psi(\nabla^Y df)) + \operatorname{tr}_g(II(df, df)),$$

where  $II$  is the second fundamental form of  $\Psi$ , which vanishes, if  $\Psi$  is totally geodesic. This shows

$$\tau(\Psi \circ f) = d\Psi \circ \tau(f).$$

The proof is finished, since  $\Psi$  is an immersion and therefore has maximal rank.  $\square$

We now restrict to compact source manifolds and to Riemannian metrics to obtain a theorem which is due to Gordon [G]. First we need a definition:

**Definition 2.2** *A subset  $U$  of a manifold  $Y$  is said to be **convex supporting** if and only if every compact subset of  $U$  has a  $Y$ -open neighborhood admitting a strictly convex  $C^2$ -function  $F$ . The function  $F$  is called **support function** and it is in general not globally defined.*

**Theorem 2.1** *(cf. [G] p. 434.) Let  $M$  and  $N$  be Riemannian manifolds with  $M$  compact and connected.*

- (A) *The image of any harmonic map  $f : M \rightarrow N$  cannot be contained in any convex supporting subset of  $N$  unless it is constant. Hence, any harmonic map from  $M$  to  $N$  is necessarily constant if  $N$  is convex supporting.*
- (B) *If  $\pi_1(M)$  is finite and  $N$  has a covering space which is convex supporting with respect to the lifted metric of  $N$ , then every harmonic map from  $M$  to  $N$  is necessarily constant.*

**Remark 2.1** *As also discussed in Example (a) in [G] (p. 434) a complete simply connected Riemannian manifold  $M$  with non-positive Riemannian sectional curvature is convex supporting. In fact, for a fixed  $p_0 \in M$  the squared geodesic distance from  $p_0$  to  $p$  is strictly convex and hence a support function. We are especially interested in the space  $GL(r, \mathbb{R})/O(r)$  and in the space  $SL(r, \mathbb{R})/SO(r)$ , which is a Riemannian symmetric space of non-compact type. As Riemannian symmetric spaces of non-compact type are non-positively curved, they are convex supporting (compare also [BR] p. 71). For  $GL(r, \mathbb{R})/O(r)$  we have the de Rham-decomposition  $GL(r, \mathbb{R})/O(r) = \mathbb{R} \times SL(r, \mathbb{R})/SO(r)$ , where  $\mathbb{R}$  corresponds to the connected central subgroup  $\mathbb{R}^{>0} = \{\lambda Id \mid \lambda > 0\} \subset GL(r, \mathbb{R})$ . Therefore  $GL(r, \mathbb{R})/O(r)$  is non-positively curved.*

## 2.2 $\epsilon$ pluriharmonic maps from $\epsilon$ complex manifolds

In this section we discuss general results about  $\epsilon$ pluriharmonic maps from  $\epsilon$ complex manifolds into pseudo-Riemannian manifolds.

**Definition 2.3** *An  $\epsilon$ complex curve or  $\epsilon$ Riemannian surface is an  $\epsilon$ complex manifold of  $\epsilon$ complex dimension one. An  $\epsilon$ complex curve in an  $\epsilon$ complex manifold  $M$  is an  $\epsilon$ complex curve  $\Sigma^\epsilon$  which is an  $\epsilon$ complex submanifold of  $M$ .*

**Definition 2.4** *A map  $f : (M, J^\epsilon) \rightarrow (N, h)$  from an  $\epsilon$ complex manifold  $(M, J^\epsilon)$  to a pseudo-Riemannian manifold  $(N, h)$  is  $\epsilon$ pluriharmonic if and only if the restriction of  $f$  to any  $\epsilon$ complex curve  $\Sigma^\epsilon$  in  $M$  is harmonic.*

**Remark 2.2** *Notice that the harmonicity of  $f$  restricted to  $\Sigma^\epsilon$  is independent of the choice of a (pseudo-)Riemannian metric in the conformal class induced by  $J^\epsilon$  on  $\Sigma^\epsilon$ , by conformal invariance of the harmonic map equation for (real) surfaces.*

The following notion was introduced in [AK] for holomorphic and in [LS] for paraholomorphic vector bundles.

**Definition 2.5** *Let  $(M, J^\epsilon)$  be an  $\epsilon$ complex manifold. A connection  $D$  on  $TM$  is called adapted if it satisfies*

$$D_{J^\epsilon Y} X = J^\epsilon D_Y X \tag{2.2.1}$$

*for all vector fields which satisfy  $\mathcal{L}_X J^\epsilon = 0$  (i.e. for which  $X + \hat{\epsilon}iJ^\epsilon X$  is  $\epsilon$ holomorphic).*

On every  $\epsilon$ complex manifold  $(M, J^\epsilon)$  there exists an  $\epsilon$ complex torsion-free connection, as we have shown in corollary 1.1. The following proposition ensures now the existence of an adapted connection.

**Proposition 2.3** *(cf. [CS1] for  $\epsilon = -1$ , [Sch3] for  $\epsilon = 1$ )*

- (i) *Every  $\epsilon$ complex torsion-free connection  $D$  on an  $\epsilon$ complex manifold  $(M, J^\epsilon)$  is adapted.*
- (ii) *On every  $\epsilon$ complex manifold there exists an adapted connection.*

*Proof:* (i) The conditions  $T^D = 0$  and  $DJ^\epsilon = 0$  yield

$$D_{J^\epsilon Y} X - J^\epsilon D_Y X = [J^\epsilon Y, X] + D_X(J^\epsilon Y) - J^\epsilon D_Y X = [J^\epsilon Y, X] - J^\epsilon [Y, X] = -(\mathcal{L}_X J^\epsilon)Y. \quad (2.2.2)$$

The right-hand side vanishes if  $\mathcal{L}_X J^\epsilon = 0$ .

(ii) The existence of an  $\epsilon$ complex torsion-free connection  $D$  on  $(M, J^\epsilon)$  follows from corollary 1.1. Part (i) implies now the statement (ii).  $\square$

**Proposition 2.4** (cf. [CS1] for  $\epsilon = -1$  and [Sch3] for  $\epsilon = 1$ ) *Let  $(M, J^\epsilon)$  be an  $\epsilon$ complex manifold and  $(N, h)$  be a pseudo-Riemannian manifold with Levi-Civita connection  $\nabla^h$ ,  $D$  an adapted connection on  $(M, J^\epsilon)$  and  $\nabla$  the connection on  $T^*M \otimes f^*TN$  which is induced by  $D$  and  $\nabla^h$ .*

*A map  $f : M \rightarrow N$  is  $\epsilon$ pluriharmonic if and only if it satisfies the equation*

$$\nabla'' \partial f = 0, \quad (2.2.3)$$

*where  $\partial f = df^{1,0} \in \Gamma(\wedge^{1,0} T^*M \otimes_{\mathbb{C}_\epsilon} (TN)^{\mathbb{C}_\epsilon})$  is the  $(1,0)$ -component of  $(df)^{\mathbb{C}_\epsilon}$  and  $\nabla''$  is the  $(0,1)$ -component of  $\nabla = \nabla' + \nabla''$ .*

*Equivalently one regards  $\alpha = \nabla df \in \Gamma(T^*M \otimes T^*M \otimes f^*TN)$ .*

*Then  $f$  is  $\epsilon$ pluriharmonic if and only if*

$$\alpha(X, Y) - \epsilon \alpha(J^\epsilon X, J^\epsilon Y) = 0$$

*for all  $X, Y \in TM$ . This can also be expressed as*

$$\alpha^{1,1} = 0.$$

*Moreover, the  $\epsilon$ pluriharmonic equation (2.2.3) is independent of the adapted connection chosen on  $M$ .*

We recall, that in the case  $(1,1)$  and  $(1+,1-)$  the two gradings defined for differential forms on para-complex manifolds in section 1.1 coincide in the sense that

$$\Lambda^{1,1} T^*M = (\Lambda^{1+,1-} T^*M) \otimes \mathbb{C}_\epsilon.$$

*Proof:* The fact that  $D$  is adapted implies  $D''Z = 0$  for all local  $\epsilon$ holomorphic vector fields  $Z$ , i.e.  $\Gamma_{\bar{\alpha}\beta}^\gamma = \Gamma_{\bar{\alpha}\beta}^{\bar{\gamma}} = 0$  in terms of Christoffel symbols of  $D$  with respect to  $\epsilon$ holomorphic coordinates  $z^\alpha$ . This implies that the Christoffel symbols of the connection  $D$  do not contribute to the  $\epsilon$ pluriharmonic equation (2.2.3). Therefore the  $\epsilon$ pluriharmonicity is independent of the adapted connection chosen on  $M$ . In the rest of the proof we suppose the connection  $D$  to be torsion-free (see proposition 2.3).

Let  $\Sigma^\epsilon \subset M$  be an  $\epsilon$ complex curve in  $(M, J^\epsilon)$ . On  $\Sigma^\epsilon$  an  $\epsilon$ hermitian metric  $g$  in the  $\epsilon$ conformal class of  $J^\epsilon$  is chosen. As  $g$  is  $\epsilon$ hermitian it is of type  $(1,1)$ . Hence the trace of  $\nabla df|_{\Sigma^\epsilon}$  with respect to  $g$  is zero if and only if  $\nabla'' \partial f|_{\Sigma^\epsilon} = 0$ , as  $\nabla df$  is symmetric. Since this holds for all curves  $\Sigma^\epsilon$  in  $M$  the proposition is proven.  $\square$

From the definition of  $\epsilon$ pluriharmonic maps and proposition 2.2 we obtain:

**Corollary 2.1** *Let  $(M, J^\epsilon)$  be an  $\epsilon$ complex manifold,  $X$  and  $Y$  be pseudo-Riemannian manifolds and  $\Psi : X \rightarrow Y$  a totally geodesic immersion. Then a map  $f : M \rightarrow X$  is  $\epsilon$ pluriharmonic if and only if  $\Psi \circ f : M \rightarrow Y$  is  $\epsilon$ pluriharmonic.*

Applying Theorem 2.1 to pluriharmonic maps we find:

**Corollary 2.2** *Let  $(M, J, g)$  be a connected compact Kähler manifold and  $N$  be a Riemannian manifold.*

- (i) *The image of any pluriharmonic map  $f : M \rightarrow N$  cannot be contained in any convex supporting subset of  $N$  unless it is constant. Hence, any pluriharmonic map from  $M$  to  $N$  is necessarily constant if  $N$  is convex supporting.*
- (ii) *If  $\pi_1(M)$  is finite and  $N$  has a covering space which is convex supporting with respect to the lifted metric of  $N$ , then every pluriharmonic map from  $M$  to  $N$  is necessarily constant.*

*Proof:* Since  $(M, J, g)$  is Kähler, the metric  $g$  is hermitian and the Levi-Civita connection  $D$  on  $M$  is adapted. Therefore we find

$$\operatorname{tr}_g \nabla df = \operatorname{tr}_g (\nabla df)^{1,1} = 0,$$

as  $(\nabla df)^{1,1}$  vanishes by the pluriharmonic map equation (2.2.3). □

## 2.3 A generalization of $\epsilon$ pluriharmonic maps from almost $\epsilon$ complex manifolds into pseudo-Riemannian manifolds

In this section, which is also subject of [Sch7, Sch8], we generalize the notion of an  $\epsilon$ pluriharmonic map to maps from almost  $\epsilon$ complex manifolds into pseudo-Riemannian manifolds. Afterwards we show that maps admitting a generalization of an associated family (compare the paper of Eschenburg and Tribuzy [ET]) give rise to an  $\epsilon$ pluriharmonic map and we give conditions under which an  $\epsilon$ pluriharmonic map is harmonic.

Let  $(M, J^\epsilon)$  be an almost  $\epsilon$ complex manifold of real dimension  $2n$ . From theorem 1.1 we know that on every almost  $\epsilon$ complex manifold there exists a connection with torsion  $T = -\frac{1}{4}\epsilon N_{J^\epsilon}$  where  $N_{J^\epsilon}$  is the Nijenhuis tensor of  $J^\epsilon$ .

**Definition 2.6** *Let  $(M, J^\epsilon)$  be an almost  $\epsilon$ complex manifold. A connection  $D$  on the tangent bundle of  $M$  is called nice if it is  $\epsilon$ complex and its torsion satisfies  $T = \lambda N_{J^\epsilon}$  for some function  $\lambda \in C^\infty(M, \mathbb{R})$ .*

We introduce the notion of an  $\epsilon$ pluriharmonic map from an almost  $\epsilon$ complex manifold:

**Definition 2.7** Let  $(M, J^\epsilon, D)$  be an almost  $\epsilon$ complex manifold endowed with a nice connection  $D$  on  $TM$  and  $N$  be a smooth manifold endowed with a connection  $\nabla^N$ . Denote by  $\nabla$  the connection on  $T^*M \otimes f^*TN$  which is induced by  $D$  and  $\nabla^N$ .

A smooth map  $f : M \rightarrow N$  is  $\epsilon$ pluriharmonic if and only if it satisfies the equation

$$(\nabla df)^{1,1} = 0. \quad (2.3.1)$$

We recall, that in the case (1,1) and (1+,1-) the two gradings defined for differential forms on para-complex manifolds in section 1.1 coincide in the sense that

$$\Lambda^{1,1} T^*M = (\Lambda^{1+,1-} T^*M) \otimes \mathbb{C}_\epsilon.$$

As preparation for associated families we recall an integrability condition satisfied by the differential of a smooth map. Let  $N$  be a smooth manifold with a connection  $\nabla^N$  on its tangent bundle having torsion tensor  $T^N$ . Given a second smooth manifold  $M$  and a smooth map  $f : M \rightarrow N$ , the differential  $F := df : TM \rightarrow f^*TN = E$  induces a vector bundle homomorphism between the tangent bundle of  $M$  and the pull-back of  $TN$  via  $f$ . The torsion tensor  $T^N$  of  $N$  induces a bundle homomorphism  $T^E : \Lambda^2 E \rightarrow E$  satisfying the identity

$$\nabla_V^E F(W) - \nabla_W^E F(V) - F([V, W]) = T^E(F(V), F(W)), \quad (2.3.2)$$

where  $\nabla^E = f^*\nabla^N$  denotes the pull-back connection, i.e. the connection which is induced on  $E$  by  $\nabla^N$  and where  $V, W \in \Gamma(TM)$ .

In the rest of the section we denote by  $D$  a nice connection on the almost  $\epsilon$ complex manifold  $(M, J^\epsilon)$ . Under this assumption we restate the condition (2.3.2)

$$\begin{aligned} T^E(F(V), F(W)) &= \nabla_V^E F(W) - \nabla_W^E F(V) - F([V, W]) \\ &= \nabla_V^E F(W) - \nabla_W^E F(V) \\ &\quad - F(D_V W) + F(D_W V) + F(T(V, W)) \\ &= \nabla_V^E F(W) - \nabla_W^E F(V) \\ &\quad - F(D_V W) + F(D_W V) + \lambda F(N_{J^\epsilon}(V, W)) \\ &= (\nabla_V F)W - (\nabla_W F)V + \lambda F(N_{J^\epsilon}(V, W)), \end{aligned} \quad (2.3.3)$$

where  $\nabla$  is the connection induced on  $T^*M \otimes E$  by  $D$  and  $\nabla^E$ .

Later in this work we consider the case where  $N$  is a pseudo-Riemannian symmetric space with its Levi-Civita connection  $\nabla^N$ .

Given an element  $\alpha \in \mathbb{R}$  we define  $\mathcal{R}_\alpha : TM \rightarrow TM$  as

$$\mathcal{R}_\alpha(X) = \cos_\epsilon(\alpha)X + \sin_\epsilon(\alpha)J^\epsilon X.$$

This defines a parallel endomorphism field on the tangent bundle  $TM$  of  $M$ . The eigenvalues of which are  $e^{\hat{i}\alpha}$  on  $T^{1,0}M$  and  $e^{-\hat{i}\alpha}$  on  $T^{0,1}M$ , as one sees easily.

An associated family for  $f$  is a family of maps  $f_\alpha : M \rightarrow N, \alpha \in \mathbb{R}$ , such that

$$\Phi_\alpha \circ df_\alpha = df \circ \mathcal{R}_\alpha, \quad \forall \alpha \in \mathbb{R}, \quad (2.3.4)$$

for some bundle isomorphism  $\Phi_\alpha : f_\alpha^*TN \rightarrow f^*TN, \alpha \in \mathbb{R}$ , which is parallel with respect to  $\nabla^N$  in the sense that

$$\Phi_\alpha \circ (f_\alpha^*\nabla^N) = (f^*\nabla^N) \circ \Phi_\alpha.$$

One observes, that each map  $f_\alpha$  of an associated family itself admits an associated family.

**Theorem 2.2** *Let  $(M, J^\epsilon)$  be an almost  $\epsilon$ complex manifold endowed with a nice connection  $D$ ,  $N$  a smooth manifold endowed with a torsion-free connection  $\nabla^N$  and let  $f : (M, D, J^\epsilon) \rightarrow (N, \nabla^N)$  be a smooth map admitting an associated family  $f_\alpha$ , then  $f$  is epluriharmonic. More precisely, each map of the associated family  $f_\alpha$  is epluriharmonic.*

*Proof:* As  $\Phi_\alpha$  is parallel with respect to  $\nabla^N$ ,  $\nabla^N$  is torsion free and  $D$  is nice, we can apply equation (2.3.3) to the family  $df_\alpha = F_\alpha = \Phi_\alpha^{-1} \circ df \circ \mathcal{R}_\alpha$  to obtain

$$(\nabla_V F_\alpha)W - (\nabla_W F_\alpha)V + \lambda F(N_{J^\epsilon}(V, W)) = 0.$$

Since  $\mathcal{R}_\alpha$  is  $D$ -parallel we obtain

$$(\nabla_X F_\alpha) = \Phi_\alpha^{-1} \circ (\nabla_X F) \circ \mathcal{R}_\alpha. \quad (2.3.5)$$

If  $Z = X - \hat{\epsilon}iJ^\epsilon X$  and  $W = Y + \hat{\epsilon}iJ^\epsilon Y$  have different type it holds  $N_{J^\epsilon}(Z, W) = 0$ , where we have extended the Nijenhuis tensor  $\epsilon$ complex linearly. This implies

$$(\nabla_Z F_\alpha)W = (\nabla_W F_\alpha)Z, \quad \forall \alpha \in \mathbb{R}$$

and using equation (2.3.5) we obtain

$$\begin{aligned} (\nabla_Z F_\alpha)W &= e^{\hat{i}\alpha} \Phi_\alpha^{-1}(\nabla_Z F)W \\ (\nabla_W F_\alpha)Z &= e^{-\hat{i}\alpha} \Phi_\alpha^{-1}(\nabla_W F)Z = e^{-\hat{i}\alpha} \Phi_\alpha^{-1}(\nabla_Z F)W \end{aligned}$$

for all  $\alpha \in \mathbb{R}$ . Since this should coincide, it follows  $(\nabla df)^{(1,1)} = 0$ , i.e.  $f : (M, D, J^\epsilon) \rightarrow (N, \nabla^N)$  is epluriharmonic. The rest follows, since each map of the associated family  $f_\alpha$  admits an associated family  $g_\beta = f_{(\alpha+\beta)}$ .  $\square$

This motivates the definition

**Definition 2.8** *Let  $(M, J^\epsilon)$  be an almost  $\epsilon$ complex manifold endowed with a nice connection  $D$  and  $N$  be a smooth manifold endowed with a torsion-free connection  $\nabla^N$ . A smooth map  $f : (M, D, J^\epsilon) \rightarrow (N, \nabla^N)$  is said to be  $\mathbb{S}_\epsilon^1$ -pluriharmonic if and only if it admits an associated family.*

Given an hermitian metric  $g$  on  $M$  then in general a nice connection  $D$  is not the Levi-Civita connection  $\nabla^g$  of  $g$ . Therefore the epluriharmonic equation (2.3.1) does not imply the harmonicity of  $f$ . But if the tensor  $D - \nabla^g$  is trace-free the epluriharmonic equation implies the harmonic equation. This is true in the case of a special  $\epsilon$ Kähler manifold  $(M, J^\epsilon, g, \nabla)$  and for a nearly  $\epsilon$ Kähler manifold  $(M, J^\epsilon, g)$ , where  $D = \bar{\nabla}$  and  $\bar{\nabla} - \nabla^g$  is skew-symmetric.

**Proposition 2.5** *Let  $(M, J^\epsilon, g)$  be an almost hermitian manifold endowed with a nice connection  $D$  and  $N$  be a pseudo-Riemannian manifold with its Levi-Civita connection  $\nabla^N$ . Suppose that the tensor  $S = \nabla^g - D$  is trace-free.*

*Then an epluriharmonic map  $f : (M, D, J^\epsilon) \rightarrow N$  is harmonic.*



*Proof:* We consider

$$\begin{aligned}
\mathrm{tr}_g(\nabla df) &= \sum_i g(e_i, e_i) [\nabla_{e_i}^E df(e_i) - df(D_{e_i} e_i)] \\
&= \sum_i g(e_i, e_i) [\nabla_{e_i}^E df(e_i) - df((\nabla^g - S)_{e_i} e_i)] \\
&= \sum_i g(e_i, e_i) [\nabla_{e_i}^E df(e_i) - df(\nabla_{e_i}^g e_i)] \\
&= \mathrm{tr}_g(\tilde{\nabla}^g df)
\end{aligned}$$

where  $\tilde{\nabla}^g$  is the connection induced on  $T^*M \otimes E$  by  $\nabla^g$  and  $\nabla^E$  and  $e_i$  is an orthogonal basis for  $g$  on  $TM$ . From the epluriharmonic equation and since  $g$  is hermitian we obtain

$$\mathrm{tr}_g(\nabla df) = \mathrm{tr}_g(\nabla df^{(1,1)}) = 0.$$

□

## 2.4 Special targets

In this subsection we discuss the manifolds, which are the target spaces of the epluriharmonic maps associated to  $\epsilon tt^*$ -bundles later in this work.

### 2.4.1 The space of pseudo-Riemannian metrics

To unify the results we use the notations

$$\begin{aligned}
G_0(r) &= GL(r, \mathbb{R}), \quad G_1(r) = SL(r, \mathbb{R}), \\
\mathfrak{g}_0 &= \mathfrak{gl}_{\mathbb{R}}(r), \quad \mathfrak{g}_1 = \mathfrak{sl}_{\mathbb{R}}(r), \\
K_0(p, q) &= O(p, q), \quad K_1(p, q) = SO(p, q), \\
\mathfrak{k}_0 &= \mathfrak{k}_1 = \mathfrak{so}(p, q), \\
S^0(p, q) &= S(p, q) = GL(r, \mathbb{R})/O(p, q), \quad S^1(p, q) = SL(r, \mathbb{R})/SO(p, q).
\end{aligned}$$

These objects are also written with an index  $i \in \{0; 1\}$ .

Let  $Sym_{p,q}^0(\mathbb{R}^r)$  be the symmetric  $r \times r$  matrices of symmetric signature  $(p, q)$  in  $G_0(r)$  and  $Sym_{p,q}^1(\mathbb{R}^r)$  the elements of  $Sym_{p,q}^0(\mathbb{R}^r)$  with determinant  $(-1)^q$ . These define pseudo-scalar products of same symmetric signature  $(p, q)$  by

$$\langle \cdot, \cdot \rangle_A = \langle A \cdot, \cdot \rangle_{\mathbb{R}^r},$$

where  $\langle \cdot, \cdot \rangle_{\mathbb{R}^r}$  is the Euclidean standard scalar product. The natural action of an element  $g \in G_i(r)$  is given by  $\langle g^{-1} \cdot, g^{-1} \cdot \rangle_A = \langle (g^{-1})^t A g^{-1} \cdot, \cdot \rangle_{\mathbb{R}^r}$ . This gives an action of  $G_i(r)$   $A \mapsto (g^{-1})^t A g^{-1}$  on  $Sym_{p,q}^i(\mathbb{R}^r)$  which we use to identify  $Sym_{p,q}^i(\mathbb{R}^r)$  with  $S^i(p, q)$  in the following proposition:

**Proposition 2.6** (cf. [Sch3, Sch6]) *Let  $\Psi^i$  be the canonical map*

$$\Psi^i : S^i(p, q) \xrightarrow{\sim} \text{Sym}_{p,q}^i(\mathbb{R}^r) \subset G_i(r)$$

where  $G_i(r)$  carries the pseudo-Riemannian metric induced by the Ad-invariant trace-form. Then  $\Psi^i$  is a totally-geodesic immersion and a map  $f$  from an  $\epsilon$ -complex manifold  $(M, J^\epsilon)$  to  $S^i(p, q)$  is epluriharmonic if and only if the map  $\Psi^i \circ f : M \rightarrow G_i(r)$  is epluriharmonic.

*Proof:* The proof is done by expressing the map  $\Psi^i$  in terms of the well-known Cartan immersion. For further information see for example [Hel], [CE], [GHL], [KN].

- 1) First we study the identification  $S^i(p, q) \xrightarrow{\sim} \text{Sym}_{p,q}^i(\mathbb{R}^r)$ .

The group  $G_i(r)$  operates on  $\text{Sym}_{p,q}^i(\mathbb{R}^r)$  via

$$G_i(r) \times \text{Sym}_{p,q}^i(\mathbb{R}^r) \rightarrow \text{Sym}_{p,q}^i(\mathbb{R}^r), \quad (g, B) \mapsto g \cdot B := (g^{-1})^t B g^{-1}.$$

The stabilizer of the point  $I_{p,q} = \text{diag}(\mathbb{1}_p, -\mathbb{1}_q)$  is  $K_i(p, q)$  and the above action is transitive by Sylvester's theorem. Therefore by the orbit-stabilizer theorem (compare the book of Gallot, Hulin, Lafontaine [GHL] 1.100) we obtain a diffeomorphism

$$\Psi^i : S^i(p, q) \xrightarrow{\sim} \text{Sym}_{p,q}^i(\mathbb{R}^r), \quad g K_i(p, q) \mapsto g \cdot I_{p,q} = (g^{-1})^t I_{p,q} g^{-1}.$$

- 2) We recall some results about symmetric spaces (For more information we refer to [CE] theorem 3.42 and [KN] volume II chapter X and XI and [Lo] to extend the proof of [CE] to non-compact groups  $G$ . A further reference is [ON].). Let  $G$  be a Lie-group and  $\sigma : G \rightarrow G$  a group-homomorphism with  $\sigma^2 = \text{Id}_G$ . Let  $K$  denote the subgroup  $K = G^\sigma = \{g \in G \mid \sigma(g) = g\}$ . The Lie-algebra  $\mathfrak{g}$  of  $G$  decomposes in  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$  with  $d\sigma_{\text{Id}_G}(\mathfrak{h}) = \mathfrak{h}$ ,  $d\sigma_{\text{Id}_G}(\mathfrak{p}) = -\mathfrak{p}$ . Moreover we have the following information: The map  $\phi : G/K \rightarrow G$  with  $\phi : [gK] \mapsto g\sigma(g^{-1})$  defines a totally geodesic immersion called the Cartan immersion.

We want to utilize this:

Therefore we define

$$\sigma : G_i(r) \rightarrow G_i(r), \quad g \mapsto (g^{-1})^\dagger$$

where  $g^\dagger = I_{p,q} g^t I_{p,q}$  is the adjoint with respect to the pseudo-scalar product  $\langle \cdot, \cdot \rangle_{I_{p,q}} = \langle \cdot, I_{p,q} \cdot \rangle_{\mathbb{R}^r}$ .

$\sigma$  is obviously a homomorphism and an involution with  $G_i(r)^\sigma = K_i(p, q)$ . By a direct calculation one gets  $d\sigma_{\text{Id}_{G_i}} = -h^\dagger$  and hence

$$\begin{aligned} \mathfrak{h} &= \{h \in \mathfrak{g}_i(r) \mid h^\dagger = -h\} = \mathfrak{o}(p, q) = \mathfrak{so}(p, q), \\ \mathfrak{p} &= \{h \in \mathfrak{g}_i(r) \mid h^\dagger = h\} =: \text{sym}^i(p, q). \end{aligned}$$

Thus we end up with

$$\phi^i : S^i(p, q) \rightarrow G_i(r), \tag{2.4.1}$$

$$g \mapsto g\sigma(g^{-1}) = gg^\dagger = (g I_{p,q} g^t) I_{p,q} = R_{I_{p,q}} \circ \Psi^i \circ \Lambda(g). \tag{2.4.2}$$

Here  $R_h$  is the right multiplication by  $h$  and  $\Lambda$  is the map induced by  $\Lambda : G_i \rightarrow G_i, h \mapsto (h^{-1})^t$  on  $G_i/K_i$ . Both maps are isometries of the invariant metrics. Hence  $\Psi^i$  is a totally-geodesic immersion.

3) Using point 1) and 2) corollary 2.1 finishes the proof.  $\square$

**Remark 2.3** (cf. [CS1, Sch3, Sch6])

Above we have identified  $G_i(r)/K_i(p, q)$  with  $\text{Sym}_{p,q}^i(\mathbb{R}^r)$  via  $\Psi^i$ .

Let us choose  $o = eK_i(p, q)$  as base point and suppose that  $\Psi^i$  is chosen to map  $o$  to  $I = I_{p,q}$ . By construction  $\Psi^i$  is  $G_i(r)$ -equivariant. We identify the tangent-space  $T_S \text{Sym}_{p,q}^i(\mathbb{R}^r)$  at  $S \in \text{Sym}_{p,q}^i(\mathbb{R}^r)$  with the (ambient) vector space of symmetric matrices:

$$T_S \text{Sym}_{p,q}^i(\mathbb{R}^r) = \text{Sym}^i(\mathbb{R}^r) := \{A \in \mathfrak{g}_i(r) \mid A^t = A\}. \quad (2.4.3)$$

For  $\Psi^i(\tilde{S}) = S$ , the tangent space  $T_{\tilde{S}} S^i(p, q)$  is canonically identified with the vector space of  $S$ -symmetric matrices:

$$T_{\tilde{S}} S^i(p, q) = \text{sym}^i(S) := \{A \in \mathfrak{g}_i(r) \mid A^t S = S A\}. \quad (2.4.4)$$

Note that  $\text{sym}^i(I_{p,q}) = \text{sym}^i(p, q)$ .

**Proposition 2.7** The differential of  $\varphi^i := (\Psi^i)^{-1}$  at  $S \in \text{Sym}_{p,q}^i(\mathbb{R}^r)$  is given by

$$\text{Sym}^i(\mathbb{R}^r) \ni X \mapsto -\frac{1}{2} S^{-1} X \in S^{-1} \text{Sym}^i(\mathbb{R}^r) = \text{sym}^i(S). \quad (2.4.5)$$

Using this proposition we relate now the differentials

$$df_x : T_x M \rightarrow \text{Sym}^i(\mathbb{R}^r) \quad (2.4.6)$$

of a map  $f : M \rightarrow \text{Sym}_{p,q}^i(\mathbb{R}^r)$  at  $x \in M$  and

$$d\tilde{f}_x : T_x M \rightarrow \text{sym}^i(f(x)) \quad (2.4.7)$$

of a map  $\tilde{f} = \varphi \circ f : M \rightarrow S^i(p, q)$ :  $d\tilde{f}_x = d\varphi df_x = -\frac{1}{2} f(x)^{-1} df_x$ .

One can interpret the one-form  $A := -2d\tilde{f} = f^{-1}df$  with values in  $\mathfrak{g}_i(r)$  as connection form on the vector bundle  $E = M \times \mathbb{R}^r$ . We note, that the definition of  $A$  is the pure gauge. This means, that  $A$  is gauge-equivalent to  $A' = 0$ , as for  $A' = 0$  one has  $A = f^{-1}A'f + f^{-1}df = f^{-1}df$ . The curvature vanishes, since it is independent of gauge. Thus we get:

**Proposition 2.8** Let  $f : M \rightarrow G_i(r)$  be a  $C^\infty$ -mapping and  $A := f^{-1}df : TM \rightarrow \mathfrak{g}_i(r)$ . Then the curvature of  $A$  vanishes, i.e. for  $X, Y \in \Gamma(TM)$  it holds

$$Y(A_X) - X(A_Y) = A_{[Y,X]} + [A_X, A_Y]. \quad (2.4.8)$$

In the next proposition we give the equations for  $\epsilon$ pluriharmonic maps from an  $\epsilon$ complex manifold to  $G_i(r)$ .

**Proposition 2.9** *Let  $(M, J^\epsilon)$  be an  $\epsilon$ complex manifold,  $f : M \rightarrow G_i(r)$  a  $C^\infty$ -map and  $A$  defined as in proposition 2.8.*

*The epluriharmonicity of  $f$  is equivalent to the equation*

$$Y(A_X) + \frac{1}{2}[A_Y, A_X] - \epsilon J^\epsilon Y(A_{J^\epsilon X}) - \epsilon \frac{1}{2}[A_{J^\epsilon Y}, A_{J^\epsilon X}] = 0, \quad (2.4.9)$$

for eholomorphic  $X, Y \in \Gamma(TM)$ .

*Proof:* Again the epluriharmonicity of the map  $f$  does not depend on the adapted connection chosen on  $M$ . This means, that we can take it torsion-free and  $\epsilon$ complex (compare proposition 2.3 and proposition 2.4). We calculate the tensor

$$\nabla df(X, Y) = \nabla_X^N(df(Y)) - df(D_X Y).$$

for (real parts of) eholomorphic vector fields  $X, Y$ . The contribution to the (1,1)-part of the second term vanishes for (real parts of) eholomorphic  $X, Y$ , since

$$D_X Y - \epsilon D_{J^\epsilon X} J^\epsilon Y = D_X Y - \epsilon J^\epsilon D_{J^\epsilon X} Y = D_X Y - \epsilon J^{\epsilon^2} D_X Y = 0.$$

Therefore we only have to regard the pulled back Levi-Civita connection  $\nabla$  on  $G_i(r)$ .

Let  $X, Y \in \Gamma(TM)$ . To find the epluriharmonic equations we write  $df(X)$  and  $df(Y)$  that are sections in  $f^* T G_i(r)$ , as linear combination of left invariant vector fields  $f^* \tilde{E}_{ij} = \tilde{E}_{ij} \circ f$ , with  $\tilde{E}_{ij}(g) = g E_{ij}$ ,  $\forall g \in G_i(r)$  and a basis  $E_{ij}, i, j = 1 \dots r$  of  $\mathfrak{g}_i(r)$ .

In this notation we have

$$df(X) = \sum_{ij} a_{ij} \tilde{E}_{ij} \circ f = \sum_{ij} a_{ij} f E_{ij} \text{ and } df(Y) = \sum_{ij} b_{ij} \tilde{E}_{ij} \circ f = \sum_{ij} b_{ij} f E_{ij},$$

with functions  $a_{ij}$  and  $b_{ij}$  on  $M$  and further

$$A_X = f^{-1} df(X) = \sum_{ij} a_{ij} E_{ij} \text{ and } A_Y = f^{-1} df(Y) = \sum_{ij} b_{ij} E_{ij}.$$

With this information we compute

$$\begin{aligned} (f^* \nabla)_Y df(X) &= (f^* \nabla)_Y \sum_{ij} a_{ij} \tilde{E}_{ij} \circ f \\ &= \sum_{ij} Y(a_{ij}) \tilde{E}_{ij} \circ f + \sum_{ij} a_{ij} (f^* \nabla)_Y \tilde{E}_{ij} \circ f \\ &= \sum_{ij} Y(a_{ij}) \tilde{E}_{ij} \circ f + \sum_{ij} a_{ij} \nabla_{df(Y)} \tilde{E}_{ij} \circ f \\ &= \sum_{ij} Y(a_{ij}) f E_{ij} + \sum_{abij} a_{ij} b_{ab} \underbrace{(\nabla_{\tilde{E}_{ab}} \tilde{E}_{ij}) \circ f}_{\frac{1}{2} f[E_{ab}, E_{ij}]} \\ &= f \left( Y(A_X) + \frac{1}{2}[A_Y, A_X] \right). \end{aligned}$$

Therefore the epluriharmonicity is equivalent to the equation

$$Y(A_X) + \frac{1}{2}[A_Y, A_X] - \epsilon J^\epsilon Y(A_{J^\epsilon X}) - \epsilon \frac{1}{2}[A_{J^\epsilon Y}, A_{J^\epsilon X}] = 0$$

for eholomorphic vector fields  $X, Y$ . □

## 2.4.2 A remark on the space of Riemannian metrics

In the complex case pluriharmonic maps into locally Riemannian symmetric spaces of non-compact type have a nice property.

Suppose that  $N$  is a locally Riemannian symmetric space with universal cover  $G/K$  with a non-compact semi-simple Lie group  $G$ , a maximal compact subgroup  $K$  and an associated Cartan decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ . In each point one identifies the tangent space of  $N$  with  $\mathfrak{p}$ . This is unique up to right action of  $K$  and left action of the fundamental group. All relevant structures are preserved by these actions. Therefore, given a map  $f : M \rightarrow N$ , we can regard  $df_x(T_x^{1,0}M)$ ,  $x \in M$  as a subspace of  $\mathfrak{p}^{\mathbb{C}}$ . For the ‘complexified’ sectional-curvature of  $N$  holds using the Killing-form  $b$

$$b(R(X, Y)\bar{Y}, \bar{X}) = -b([X, Y], [\bar{Y}, \bar{X}]) \leq 0. \quad (2.4.10)$$

It is a well-known result of Sampson [Sam], that a harmonic map of a compact complex manifold to a locally symmetric space of non-compact type is pluriharmonic and that its differential sends  $T^{1,0}M$  to an Abelian subspace of  $\mathfrak{p}^{\mathbb{C}}$ . The second claim, that the image of  $T^{1,0}M$  under the differential of a pluriharmonic map is Abelian is true on non-compact manifolds, too. We are going to prove, that the pluriharmonicity implies this property.

First we state a definition in a more general context, i.e. for  $\epsilon$ complex manifolds and locally pseudo-Riemannian symmetric spaces:

**Definition 2.9** *Let  $(M, J^\epsilon)$  be an  $\epsilon$ complex manifold and  $N$  a locally pseudo-Riemannian symmetric space with universal cover  $G/K$  and associated Cartan decomposition  $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ . A map  $f : (M, J^\epsilon) \rightarrow N$  is said to be **admissible**, if for all  $x \in M$  the  $\epsilon$ complex linear extension of its differential maps  $T_x^{1,0}M$  (equivalently  $T_x^{0,1}M$ ) to an Abelian subspace of  $\mathfrak{p}^{\mathbb{C}\epsilon}$ .*

**Theorem 2.3** *(compare [Sam]) Let  $(M, J)$  be a complex manifold and  $N$  be a locally Riemannian symmetric space with universal cover  $G/K$  and associated Cartan decomposition  $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ .*

*Then a pluriharmonic map  $f : M \rightarrow N$  is admissible.*

*The differential of a pluriharmonic map  $f : M \rightarrow N$  obeys the equation*

$$[df_x(X), df_x(Y)] = [df_x(JX), df_x(JY)]$$

*with  $X, Y \in T_xM$ ,  $x \in M$ .*

*Proof:* The strategy is to show the vanishing of the curvature.

Let  $X, Y, Z, W \in \Gamma(T^{1,0}M)$  be holomorphic

$$\begin{aligned} R^N(f_*X, f_*Y)f_*\bar{Z} &= R^{f^*\nabla^N}(X, Y)f_*\bar{Z} \\ &= (f^*\nabla^N)_X(f^*\nabla^N)_Y f_*\bar{Z} - (f^*\nabla^N)_Y(f^*\nabla^N)_X f_*\bar{Z} - (f^*\nabla^N)_{[X, Y]}f_*\bar{Z} \end{aligned}$$

We remark now, that the pluriharmonic equation for holomorphic vector fields does not depend on the adapted connection chosen on the manifold  $M$ . Hence it reduces to the equation  $(f^*\nabla^N)_X f_*\bar{Y} = 0$ , which implies  $R^N(f_*X, f_*Y)f_*\bar{Z} = 0$ . From equation (2.4.10)

we get  $b([f_*X, f_*Y], [f_*\bar{Z}, f_*\bar{W}]) = 0$  and in the end  $[f_*X, f_*Y] = 0$  for all  $X, Y$ . Let  $Z, W \in \Gamma(T^{1,0}M)$  be of the form  $Z = X - iJX$  and  $W = Y - iJY$  with  $X, Y \in \Gamma(TM)$  and compute  $[f_*Z, f_*W] = [f_*X, f_*Y] - [f_*JX, f_*JY] - i([f_*X, f_*JY] + [f_*JX, f_*Y])$ . Hence we conclude  $[df(X), df(Y)] = [df(JX), df(JY)]$ .  $\square$

**Corollary 2.3** *Let  $(M, J)$  be a complex manifold,  $f : M \rightarrow \text{Sym}_{r,0}^i(\mathbb{R}^r) \subset G_i(r)$  a pluriharmonic map induced by a pluriharmonic map to  $G_i(r)/K_i(r)$  and  $A$  defined as in proposition 2.8. If  $f$  is a pluriharmonic map, then the operators  $A$  satisfy for all  $X, Y \in T_xM$ , with  $x \in M$ , the equation  $[A_X, A_Y] = [A_{JX}, A_{JY}]$ .*

*Proof:* First, we apply theorem 2.3 to  $A = -2d\tilde{f}$  with a map  $\tilde{f} : M \rightarrow G_1/K_1$ . This yields the corollary for  $G_1 = SL(r, \mathbb{R})$ .

For  $S^0(r, 0) = S(r, 0)$  we have the de Rham decomposition  $S(r, 0) = \mathbb{R} \times S^1(r, 0)$ , where  $\mathbb{R}$  corresponds to the connected central subgroup  $\mathbb{R}^{>0} = \{\lambda Id \mid \lambda > 0\} \subset G_0 = GL(r, \mathbb{R})$ . Hence we have the decomposition of  $\mathfrak{gl}_{\mathbb{R}}(r) = \mathbb{R} \oplus \mathfrak{sl}_{\mathbb{R}}(r)$ , where the  $\mathbb{R}$ -factor is central. Therefore we are in the situation to apply the result for  $G_1$ .  $\square$

**Remark 2.4** *Since the trace-form on  $SL(r, \mathbb{R})$  is a multiple of the Killing-form and on  $GL(r, \mathbb{R})$  it corresponds to the metric on the decomposition  $S(r, 0) = \mathbb{R} \times S^1(r, 0)$ , we can choose the trace-form as metric and obtain the same result as in theorem 2.3 and corollary 2.3.*

### 2.4.3 The space of hermitian metrics

This subsection is published in [Sch4].

Let  $\text{Herm}_{p,q}(\mathbb{C}^r)$  be the complex hermitian  $r \times r$  matrices with hermitian signature  $(p, q)$  and  $I = I_{p,q} = \text{diag}(\mathbb{1}_p, -\mathbb{1}_q)$ .

**Claim:**  $GL(r, \mathbb{C})$  operates on  $\text{Herm}_{p,q}(\mathbb{C}^r)$  via

$$\begin{aligned} GL(r, \mathbb{C}) \times \text{Herm}_{p,q}(\mathbb{C}^r) &\rightarrow \text{Herm}_{p,q}(\mathbb{C}^r), \\ (g, B) &\mapsto g \cdot B := (g^{-1})^H B g^{-1}, \end{aligned}$$

where  $g^H$  is the hermitian conjugate of  $g$ .

The stabilizer of  $I$  is

$$GL(r, \mathbb{C})_I = \{g \in GL(r, \mathbb{C}) \mid g \cdot I = (g^{-1})^H I g^{-1} = I\} = U(p, q)$$

and the action is transitive due to Sylvester's theorem. This yields, by identifying orbits and rest classes, a diffeomorphism

$$\begin{aligned} \Psi : GL(r, \mathbb{C})/U(p, q) &\xrightarrow{\sim} \text{Herm}_{p,q}(\mathbb{C}^r) \subset GL(r, \mathbb{C}), \\ gU(p, q) &\mapsto g \cdot I = (g^{-1})^H I g^{-1}. \end{aligned}$$

**Proposition 2.10** *The map  $\Psi : GL(r, \mathbb{C})/U(p, q) \xrightarrow{\sim} \text{Herm}_{p,q}(\mathbb{C}^r)$  is totally geodesic, where the target-space is carrying the (pseudo-)metric induced by the Ad-invariant trace-form (i.e.  $A, B \mapsto \text{tr}(AB)$ ) on  $\mathfrak{gl}(r, \mathbb{C})$ .*

Let  $(M, J)$  be a complex manifold. Then a map  $\phi : M \rightarrow H(p, q) := GL(r, \mathbb{C})/U(p, q)$  is pluriharmonic if and only if

$$\psi = \Psi \circ \phi : M \rightarrow GL(r, \mathbb{C})/U(p, q) \xrightarrow{\sim} \text{Herm}_{p,q}(\mathbb{C}^r) \subset GL(r, \mathbb{C})$$

is pluriharmonic.

*Proof:* Like in the last section the idea is to relate  $\Psi$  to the totally geodesic Cartan immersion. Therefore we define

$$\begin{aligned} \sigma & : GL(r, \mathbb{C}) \rightarrow GL(r, \mathbb{C}), \\ g & \mapsto (g^{-1})^\dagger. \end{aligned}$$

Here  $g^\dagger$  denotes the adjoint of  $g$  with respect to the hermitian scalar product defined by  $\langle \cdot, \cdot \rangle = \langle I_{p,q} \cdot, \cdot \rangle_{\mathbb{C}^r}$ , where  $\langle \cdot, \cdot \rangle_{\mathbb{C}^r}$  is the hermitian standard scalar product on  $\mathbb{C}^r$  and  $I = I_{p,q}$ . Explicitly it is  $g^\dagger = Ig^H I$ .

$\sigma$  is a homomorphism and an involution satisfying  $GL(r, \mathbb{C})^\sigma = U(p, q)$ .

Hence the Cartan immersion can be written as

$$\begin{aligned} i & : GL(r, \mathbb{C})/U(p, q) \rightarrow GL(r, \mathbb{C}), \\ g & \mapsto g\sigma(g^{-1}) = gg^\dagger = gIg^H I = R_I \circ \Psi \circ \Lambda(g), \end{aligned}$$

where  $R_h$  is the right-multiplication with  $h \in GL(r, \mathbb{C})$  and  $\Lambda$  the map induced on  $GL(r, \mathbb{C})/U(p, q)$  by  $\tilde{\Lambda} : GL(r, \mathbb{C}) \rightarrow GL(r, \mathbb{C})$ ,  $g \mapsto (g^{-1})^H$ . Both maps are isometries of the invariant metrics and therefore  $\Psi$  is totally geodesic. Corollary 2.1 finishes the proof.  $\square$

To be complete we mention the related symmetric decomposition:

$$\mathfrak{h} = \{h \in \mathfrak{gl}_r(\mathbb{C}) \mid h^\dagger = -h\} = \mathfrak{u}(p, q)$$

and

$$\mathfrak{p} = \{h \in \mathfrak{gl}_r(\mathbb{C}) \mid h^\dagger = h\} =: \text{herm}_{p,q}(\mathbb{C}^r). \quad (2.4.11)$$

Later in this work we need the relation between (pluriharmonic) maps coming from hermitian metrics and these coming from their real part. We are going to study their relation now:

In the rest of this subsection we identify  $\mathbb{C}^r$  with  $\mathbb{R}^r \oplus i\mathbb{R}^r = \mathbb{R}^{2r}$ . In this model the multiplication with  $i$  coincides with the automorphism  $j = \begin{pmatrix} 0 & \mathbb{1}_r \\ -\mathbb{1}_r & 0 \end{pmatrix}$  and  $GL(r, \mathbb{C})$  (respectively  $\mathfrak{gl}_r(\mathbb{C})$ ) consists of the elements in  $GL(2r, \mathbb{R})$  (respectively  $\mathfrak{gl}_{2r}(\mathbb{R})$ ), which commute with  $j$ .

An endomorphism  $C \in \text{End}(\mathbb{C}^r)$  decomposes in its real part  $A$  and its imaginary part  $B$ , i.e.  $C = A + iB$  with  $A, B \in \text{End}(\mathbb{R}^r)$ . In the above model  $C$  is identified with a real  $2r \times 2r$ -matrix. This identification we denote by  $\iota$ , i.e.

$$\iota(C) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

The complex conjugated of  $C$  is identified with

$$\iota(\bar{C}) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

the transpose  $C^t = A^t + iB^t$  yields

$$\iota(C^t) = \begin{pmatrix} A^t & -B^t \\ B^t & A^t \end{pmatrix}$$

and consequently the hermitian conjugated is identified with

$$\iota(\bar{C}^t) = \begin{pmatrix} A^t & B^t \\ -B^t & A^t \end{pmatrix}.$$

We observe, that  $\iota(\bar{C}^t) = \iota(C)^T$  where  $\cdot^T$  is the transpose in  $\text{End}(\mathbb{R}^{2r})$ .

The hermitian matrices  $\text{Herm}_{p,q}(\mathbb{C}^r)$  of signature  $(p, q)$  are identified with the subset of symmetric matrices  $H \in \text{Sym}_{2p,2q}(\mathbb{R}^{2r})$ , which commute with  $j$ , i.e.  $[H, j] = 0$ . Likewise,  $T_{I_{p,q}}\text{Herm}_{p,q}(\mathbb{C}^r)$  coincides with the symmetric matrices  $h \in \text{sym}(\mathbb{R}^{2r})$ , which commute with  $j$ , i.e. the hermitian matrices in  $\mathfrak{gl}_{2r}(\mathbb{R})$  which we denote by  $\text{herm}_{p,q}(\mathbb{C}^r)$ .

A hermitian scalar product  $h$  of signature  $(p, q)$  corresponds to a hermitian matrix  $H \in \text{Herm}_{p,q}(\mathbb{C}^r)$  of hermitian signature  $(p, q)$  defined by  $h(\cdot, \cdot) = (H\cdot, \cdot)_{\mathbb{C}^r}$ . The condition  $\bar{C}^t = C$ , i.e.  $C$  hermitian, means in our model, that  $C$  has the form

$$\iota(C) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

with  $A = A^t$  and  $B = -B^t$ .

Finally we find the explicit representation of the map  $\mathcal{R}$ , which corresponds to taking the real part of a hermitian metric  $h$ , i.e.  $\text{Re } h = (\mathcal{R}(H)\cdot, \cdot)_{\mathbb{R}^{2r}}$ :

$$\begin{aligned} \mathcal{R} &: \text{Herm}_{p,q}(\mathbb{C}^r) \rightarrow \text{Sym}_{2p,2q}(\mathbb{R}^{2r}), \\ H &\mapsto \iota\left(\frac{1}{2}(H + \bar{H}^t)\right) = \frac{1}{2}(\iota(H) + \iota(H)^T) = \iota(H). \end{aligned}$$

This map has maximal rank and is equivariant with respect to  $GL(r, \mathbb{C})$ .

Further we claim, that it is totally geodesic: The decomposition

$$\mathfrak{gl}_{2r}(\mathbb{R}) = \text{sym}_{2p,2q}(\mathbb{R}^{2r}) \oplus \mathfrak{o}(2p, 2q)$$

is a symmetric decomposition of the symmetric space  $GL(2r, \mathbb{R})/O(2p, 2q)$  and hence

$$[[\text{sym}_{2p,2q}(\mathbb{R}^{2r}), \text{sym}_{2p,2q}(\mathbb{R}^{2r})], \text{sym}_{2p,2q}(\mathbb{R}^{2r})] \subset \text{sym}_{2p,2q}(\mathbb{R}^{2r}).$$

From  $[A, j] = [B, j] = [C, j] = 0$ , we conclude with the Jacobi identity  $[[A, B], j] = 0$  and  $[[[A, B], C], j] = 0$ . Consequently  $T_{I_{p,q}}GL(r, \mathbb{C})/U(p, q) = \text{herm}_{p,q}(\mathbb{C}^r)$  is a Lie-triple-system<sup>1</sup> in  $T_{1_{2p,2q}}\text{Sym}_{2p,2q}(\mathbb{R}^{2r}) = \text{sym}_{2p,2q}(\mathbb{R}^{2r})$ , i.e.

$$[[\text{herm}_{p,q}(\mathbb{C}^r), \text{herm}_{p,q}(\mathbb{C}^r)], \text{herm}_{p,q}(\mathbb{C}^r)] \subset \text{herm}_{p,q}(\mathbb{C}^r).$$

<sup>1</sup>We refer to [Hel] Ch. IV.7, [KN] vol. 2, ch. XI.4 and [Lo] ch. III for more information on Lie-triple-systems and totally geodesic subspaces of symmetric spaces and [KN] vol. 2, ch. XI.2 for the (canonical) symmetric decomposition of a symmetric space.



Therefore  $GL(r, \mathbb{C})/U(p, q)$  is totally geodesic in  $GL(2r, \mathbb{R})/O(2p, 2q)$  and we have the commutative diagram:

$$\begin{array}{ccc}
 & \frac{GL(r, \mathbb{C})}{U(p, q)} & \xrightarrow{[i]} & \frac{GL(2r, \mathbb{R})}{O(2p, 2q)} & (2.4.12) \\
 & \uparrow \tilde{h} & & \downarrow \Psi^0 & \\
 M & & & & \\
 & \downarrow \Psi & & \downarrow & \\
 & \text{Herm}_{p, q}(\mathbb{C}^r) & \xrightarrow{\mathcal{R}} & \text{Sym}_{2p, 2q}(\mathbb{R}^{2r}), & \\
 & \uparrow h & & & 
 \end{array}$$

where  $[i]$  is induced by the inclusion  $i : GL(r, \mathbb{C}) \hookrightarrow GL(2r, \mathbb{R})$ . Since all other maps in the square of this diagram are totally geodesic, the map  $\mathcal{R} : \text{Herm}_{p, q}(\mathbb{C}^r) \rightarrow \text{Sym}_{2p, 2q}(\mathbb{R}^{2r})$  is a totally geodesic map. This gives the proposition:

**Proposition 2.11** *A map  $h : M \rightarrow \text{Herm}_{p, q}(\mathbb{C}^r)$  is pluriharmonic, if and only if  $g = \text{Re } h : M \rightarrow \text{Sym}_{2p, 2q}(\mathbb{R}^{2r})$  is pluriharmonic.*

*A map  $\tilde{h} : M \rightarrow H(p, q)$  is pluriharmonic, if and only if  $\tilde{g} = [i] \circ h : M \rightarrow S(2p, 2q)$  is pluriharmonic.*

*Proof:* As discussed above the map  $\mathcal{R} : \text{Herm}_{p, q}(\mathbb{C}^r) \rightarrow \text{Sym}_{2p, 2q}(\mathbb{R}^{2r})$  is totally geodesic and an immersion. This means that we are in the situation of corollary 2.1.

The second claim follows from the square commutative diagram (2.4.12) and the statements of proposition 2.10 and proposition 2.6, that the composition of a map  $f$  from  $M$  to  $\text{Herm}_{p, q}(\mathbb{C}^r)$  (respectively  $\text{Sym}_{2p, 2q}(\mathbb{R}^{2r})$ ) with  $\Psi^{-1}$  (respectively  $(\Psi^0)^{-1}$ ) is pluriharmonic, if and only if  $f$  is pluriharmonic.  $\square$

## 2.4.4 The space of para-hermitian metrics

In the following subsection we identify  $C^r$  with  $\mathbb{R}^r \oplus e\mathbb{R}^r = \mathbb{R}^{2r}$ . The multiplication with  $e$  equals the automorphism  $E = \begin{pmatrix} 0 & \mathbb{1}_r \\ \mathbb{1}_r & 0 \end{pmatrix}$  and  $GL(r, C)$  (respectively  $\mathfrak{gl}_r(C)$ ) consists of the elements in  $GL(2r, \mathbb{R})$  (respectively  $\mathfrak{gl}_{2r}(\mathbb{R})$ ) commuting with  $E$ .

First, we introduce the notion of para-hermitian sesquilinear scalar products on para-complex vector spaces

### Definition 2.10

1. A para-hermitian sesquilinear scalar product is a non-degenerate sesquilinear form  $h : C^r \times C^r \rightarrow C$ , i.e.

- (i)  $h$  is non-degenerate: Given  $w \in C^r$  such that for all  $v \in C^r$   $h(v, w) = 0$ , then it follows  $w = 0$ ,
- (ii)  $h(v, w) = \overline{h(w, v)}$ ,  $\forall v, w \in C^r$ ,
- (iii)  $h(\lambda v, w) = \lambda h(v, w)$ ,  $\forall \lambda \in C; v, w \in C^r$ .

2. Let  $z = (z^1, \dots, z^r)$  and  $w = (w^1, \dots, w^r)$  be two elements of  $C^r$ , then one defines the standard  $C$ -bilinear scalar product on  $C^r$  by

$$z \cdot w := \sum_{i=1}^r z^i w^i$$

and the standard para-hermitian sesquilinear scalar product by

$$(z, w)_{C^r} := z \cdot \bar{w}.$$

3. Given a matrix  $C$  of  $\text{End}(C^r) = \text{End}_C(C^r)$ , we define the para-hermitian conjugation by  $C \mapsto C^h = \bar{C}^t$ . We call  $C$  **para-hermitian** if and only if  $C^h = C$ . We denote by  $\text{herm}(C^r)$  the set of para-hermitian endomorphisms and by  $\text{Herm}(C^r) = \text{herm}(C^r) \cap \text{GL}(r, C)$ .

**Remark 2.5** We remark, that there is **no** notion of para-hermitian signature, since from  $h(v, v) = -1$  for a  $v \in C^r$  we obtain  $h(ev, ev) = 1$ .

**Proposition 2.12**

- (a) Given an element  $C$  of  $\text{End}(C^r)$  then it holds  $(Cz, w)_{C^r} = (z, C^h w)_{C^r}$ ,  $\forall z, w \in C^r$ .  
 (b) The set  $\text{herm}(C^r)$  is a real vector space.  
 (c) There is a bijective correspondence between  $\text{Herm}(C^r)$  and para-hermitian sesquilinear scalar products  $h$  on  $C^r$  given by

$$H \mapsto h(\cdot, \cdot) := (H\cdot, \cdot)_{C^r}.$$

An endomorphism  $C \in \text{End}(C^r)$  decomposes in its real part  $A$  and its imaginary part  $B$ , i.e.  $C = A + eB$  where  $A, B \in \text{End}(\mathbb{R}^r)$ . In the above identification the endomorphism  $C$  is identified via a map, which we denote by  $\iota$ , with the matrix

$$\iota(C) = \begin{pmatrix} A & B \\ B & A \end{pmatrix}.$$

The para-complex conjugation of  $C$ , i.e.  $\bar{C} = A - eB$ , corresponds to

$$\iota(\bar{C}) = \begin{pmatrix} A & -B \\ -B & A \end{pmatrix},$$

the transposition  $C^t = A^t + eB^t$  yields

$$\iota(C^t) = \begin{pmatrix} A^t & B^t \\ B^t & A^t \end{pmatrix} = \iota(C)^T$$

and the adjoint with respect to  $(\cdot, \cdot)_{C^r}$  is  $C^h = \bar{C}^t$  which corresponds to

$$\iota(C^h) = \begin{pmatrix} A^t & -B^t \\ -B^t & A^t \end{pmatrix} \stackrel{(*)}{=} \mathbb{1}_{r,r} \iota(C)^T \mathbb{1}_{r,r},$$

where  $\cdot^T$  is the transposition<sup>2</sup> in  $\text{End}(\mathbb{R}^{2r})$ . The equality in (\*) is due to the calculation:

$$\begin{aligned} \iota(C^h)\mathbb{1}_{r,r} &= \begin{pmatrix} A^t & -B^t \\ -B^t & A^t \end{pmatrix} \mathbb{1}_{r,r} = \begin{pmatrix} A^t & B^t \\ -B^t & -A^t \end{pmatrix} \\ &= \mathbb{1}_{r,r} \begin{pmatrix} A^t & B^t \\ B^t & A^t \end{pmatrix} = \mathbb{1}_{r,r} \iota(C)^T = \mathbb{1}_{r,r} \iota(C^t) \end{aligned} \quad (2.4.13)$$

with

$$\mathbb{1}_{r,r} = \begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & -\mathbb{1}_r \end{pmatrix}.$$

A para-hermitian sesquilinear scalar product  $h$  corresponds to a para-hermitian matrix  $H \in \text{Herm}(C^r)$  (compare with proposition 2.12) defined by  $h(\cdot, \cdot) = (H\cdot, \cdot)_{C^r}$ . The condition  $C^h = C$ , i.e.  $C$  para-hermitian, means in our model that  $C$  is of the form

$$\iota(C) = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

with  $A = A^t$  and  $B = -B^t$ .

Using this information we find the explicit representation of the map which corresponds to taking the real part  $\text{Re } h$  of  $h$ . This is the map  $\mathfrak{R}$  satisfying

$$\text{Re } h = (\mathfrak{R}(H)\cdot, \cdot)_{\mathbb{R}^{2r}},$$

where  $(\cdot, \cdot)_{\mathbb{R}^{2r}}$  is the Euclidean standard scalar product on  $\mathbb{R}^{2r}$ .

With  $z, w \in C^r$  we have

$$\beta(z, w) := \text{Re } (z, w)_{C^r} = \frac{1}{2} (z \cdot \bar{w} + \bar{z} \cdot w)$$

and

$$\begin{aligned} \text{Re } h(z, w) &= \text{Re } (Hz, w)_{C^r} \\ &= \frac{1}{2} [(Hz) \cdot \bar{w} + (\overline{Hz}) \cdot w] \\ &= \beta(Hz, w). \end{aligned}$$

Further we remark that  $\beta(\cdot, \cdot) = \text{Re } (\cdot, \cdot)_{C^r} = (\cdot, \cdot)_{\mathbb{R}^{r,r}}$ , where  $(\cdot, \cdot)_{\mathbb{R}^{r,r}} = (\mathbb{1}_{r,r}\cdot, \cdot)_{\mathbb{R}^{2r}}$  is the (pseudo-)Euclidean standard scalar product of signature  $(r, r)$  on  $\mathbb{R}^{2r}$ .

This yields

$$\text{Re } h(z, w) = (Hz, w)_{\mathbb{R}^{r,r}} = (\mathbb{1}_{r,r}Hz, w)_{\mathbb{R}^{2r}}$$

and for  $H = A + eB$  with  $A, B \in \text{End}(\mathbb{R}^r)$

$$\mathfrak{R}(H) = \mathbb{1}_{r,r} \iota(H) = \mathbb{1}_{r,r} \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}.$$

Since  $H$  is para-hermitian, we obtain  $\mathfrak{R}(H)^T = \mathfrak{R}(H)$ . The symmetric signature of the

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<sup>2</sup>To rest in the same notation as in the last section we use two symbols for the transposition, even if it here seems to be overkill.

symmetric matrix  $\mathcal{R}(H)$  is  $(r, r)$ , as it is the real part of a para-hermitian sesquilinear scalar product.

Summarizing we have

$$\begin{aligned} \mathcal{R} &: \text{Herm}(C^r) \rightarrow \text{Sym}_{r,r}(\mathbb{R}^{2r}), \\ H &\mapsto \mathcal{R}(H) = \mathbb{1}_{r,r} \iota(H). \end{aligned}$$

The map  $\mathcal{R}$  has maximal rank.

**Claim:**  $GL(r, C)$  operates on  $\text{Herm}(C^r)$  via

$$\begin{aligned} GL(r, C) \times \text{Herm}(C^r) &\rightarrow \text{Herm}(C^r), \\ (g, B) &\mapsto g \cdot B := (g^{-1})^h B g^{-1}, \end{aligned}$$

$g \cdot B$  is para-hermitian, since one has  $g \cdot B = (g \cdot B)^h$ .

We now show that  $\mathcal{R}$  is equivariant with respect to this  $GL(r, C)$ -action on  $\text{Herm}(C^r)$  and the  $GL(2r, \mathbb{R})$ -action on  $\text{Sym}_{r,r}(\mathbb{R}^{2r})$  given by

$$(g^{-1}, S) \mapsto g^{-1} \cdot S = g^T S g$$

with  $g \in GL(2r, \mathbb{R})$  and  $S \in \text{Sym}_{r,r}(\mathbb{R}^{2r})$ . In fact,

$$\begin{aligned} \mathcal{R}(g^{-1} \cdot H) &= \mathcal{R}(g^h H g) = \mathbb{1}_{r,r} \iota(g^h H g) \\ &= \mathbb{1}_{r,r} \iota(g^h) \iota(H) \iota(g) \stackrel{(2.4.13)}{=} \iota(g)^T \mathbb{1}_{r,r} \iota(H) \iota(g) \\ &= \iota(g)^T \mathcal{R}(H) \iota(g) = \iota(g)^{-1} \cdot \mathcal{R}(H). \end{aligned}$$

Our aim is to show, that this map is totally geodesic:

The decomposition

$$\mathfrak{gl}_{2r}(\mathbb{R}) = \text{sym}_{r,r}(\mathbb{R}^{2r}) \oplus \mathfrak{o}(r, r),$$

where  $\text{sym}_{r,r}(\mathbb{R}^{2r})$  are the symmetric matrices with respect to  $(\cdot, \cdot)_{\mathbb{R}^{r,r}}$ , is a symmetric decomposition associated to the symmetric space  $GL(2r, \mathbb{R})/O(r, r)$  and hence

$$[[\text{sym}_{r,r}(\mathbb{R}^{2r}), \text{sym}_{r,r}(\mathbb{R}^{2r})], \text{sym}_{r,r}(\mathbb{R}^{2r})] \subset \text{sym}_{r,r}(\mathbb{R}^{2r}).$$

Let  $A, B, C \in \text{herm}(C^r)$ . From  $[A, E] = [B, E] = [C, E] = 0$ , we conclude with the Jacobi identity  $[[A, B], E] = 0$  and  $[[[A, B], C], E] = 0$ . Hence

$$T_{\mathbb{1}_r} GL(r, C)/U^\pi(C^r) = \text{herm}(C^r)$$

is a Lie-triple-system in  $T_{\mathbb{1}_r, r} \text{Sym}_{r,r}(\mathbb{R}^{2r}) = \text{sym}_{r,r}(\mathbb{R}^{2r})$ , i.e.

$$[[\text{herm}(C^r), \text{herm}(C^r)], \text{herm}(C^r)] \subset \text{herm}(C^r)$$

and consequently  $GL(r, C)/U^\pi(C^r)$  is a totally geodesic submanifold of  $GL(2r, \mathbb{R})/O(r, r)$ .

The stabilizer of  $\mathbb{1}_r$  under the  $GL(r, C)$ -action on  $\text{Herm}(C^r)$  is

$$GL(r, C)_{\mathbb{1}_r} = \{g \in GL(r, C) \mid g \cdot \mathbb{1}_r = (g^{-1})^h \mathbb{1}_r g^{-1} = \mathbb{1}_r\} = U^\pi(C^r).$$

If the operation  $\cdot$  is transitive we obtain, by the orbit stabilizer theorem, a diffeomorphism

$$\begin{aligned} \Psi &: GL(r, C)/U^\pi(C^r) \xrightarrow{\sim} \text{Herm}(C^r) \subset GL(r, C), \\ gU^\pi(C^r) &\mapsto g \cdot \mathbb{1}_r = (g^{-1})^h \mathbb{1}_r g^{-1} = (g^{-1})^h g^{-1}. \end{aligned} \quad (2.4.14)$$

The transitivity is due to the following argument: Any para-hermitian sesquilinear scalar product is uniquely determined by its real part, which lies in  $\text{Sym}_{r,r}(\mathbb{R}^{2r})$ . On this space  $GL(2r, \mathbb{R})$  acts transitively.

**We claim:**  $h' = g \cdot h$  with some para-hermitian sesquilinear scalar product  $h$  and an element  $g \in GL(2r, \mathbb{R})$  is a para-hermitian sesquilinear scalar product if and only if  $g \in GL(r, C)$ .

*Proof:* This claim follows from a short calculation: Let  $v, w \in C^r$  and  $\lambda \in C$ :

On the one hand it holds

$$h'(\lambda v, w) = \lambda h'(v, w) = \lambda(g \cdot h)(v, w) = h(\lambda g^{-1}v, g^{-1}w)$$

and on the other hand

$$h'(\lambda v, w) = (g \cdot h)(\lambda v, w) = h(g^{-1}\lambda v, g^{-1}w).$$

Subtracting these two equations yields

$$h((g^{-1}\lambda - \lambda g^{-1})v, g^{-1}w) = 0.$$

Setting  $w = gw'$  with arbitrary  $w' \in C^r$  we obtain

$$h((g^{-1}\lambda - \lambda g^{-1})v, w') = 0.$$

Since  $g$  is invertible and  $h$  is non-degenerate we conclude  $g^{-1}\lambda v = \lambda g^{-1}v$ , which implies the  $C$ -linearity of  $g$ .  $\square$

We are now going to analyze para-pluriharmonic maps into these spaces

**Proposition 2.13** *Let  $(M, \tau)$  be a para-complex manifold and endow  $GL(r, C)/U^\pi(C^r)$  with the (pseudo-)metric induced by the trace-form on  $GL(r, C)$ . Then the map  $\Psi : GL(r, C)/U^\pi(C^r) \xrightarrow{\sim} \text{Herm}(C^r)$  defined in equation (2.4.14) is totally geodesic and a map  $\phi : M \rightarrow GL(r, C)/U^\pi(C^r)$  is para-pluriharmonic if and only if*

$$\psi = \Psi \circ \phi : M \rightarrow GL(r, C)/U^\pi(C^r) \xrightarrow{\sim} \text{Herm}(C^r) \subset GL(r, C)$$

*is para-pluriharmonic.*

*Proof:* To prove this we define

$$\begin{aligned} \sigma &: GL(r, C) \rightarrow GL(r, C), \\ g &\mapsto (g^{-1})^h. \end{aligned}$$

$\sigma$  is a homomorphism and an involution satisfying  $GL(r, C)^\sigma = U^\pi(C^r)$ .

Hence the Cartan immersion can be written as

$$\begin{aligned} i &: GL(r, C)/U^\pi(C^r) \rightarrow GL(r, C), \\ g &\mapsto g\sigma(g^{-1}) = gg^h = gg^h = \Psi \circ \Lambda(g), \end{aligned}$$

where  $\Lambda$  is the map induced on  $GL(r, C)/U^\pi(C^r)$  by  $\tilde{\Lambda} : GL(r, C) \rightarrow GL(r, C)$ ,  $g \mapsto (g^{-1})^h$  which is an isometry of the invariant metric, since  $g \mapsto g^h = \mathbb{1}_{r,r} g^T \mathbb{1}_{r,r}$  and  $g \mapsto g^{-1}$  are isometries of the invariant metric. Therefore  $\Psi$  is totally geodesic, since  $i$  is totally geodesic. Corollary 2.1 finishes the proof.  $\square$

To be complete we mention the related symmetric decomposition:

$$\mathfrak{h} = \{A \in \mathfrak{gl}_r(C) \mid A^h = -A\} = \mathfrak{u}^\pi(C^r)$$

and

$$\mathfrak{p} = \{A \in \mathfrak{gl}_r(C) \mid A^h = A\} = \text{herm}(C^r).$$

Summarizing our knowledge, we have the commutative diagram:

$$\begin{array}{ccc} & \frac{GL(r, C)}{U^\pi(C^r)} & \xrightarrow{[i]} & \frac{GL(2r, \mathbb{R})}{O(r, r)} & \\ & \uparrow \tilde{h} & & \downarrow \Psi^0 & \\ M & & & & \\ & \downarrow \Psi & & \downarrow & \\ & \text{Herm}(C^r) & \xrightarrow{\mathfrak{R}} & \text{Sym}_{r,r}(\mathbb{R}^{2r}), & \end{array} \quad (2.4.15)$$

where  $[i]$  is induced by the inclusion  $i : GL(r, C) \hookrightarrow GL(2r, \mathbb{R})$ . Since all other maps in the square of this diagram are totally geodesic, the map

$$\mathfrak{R} : \text{Herm}(C^r) \rightarrow \text{Sym}_{r,r}(\mathbb{R}^{2r}), H \mapsto \mathbb{1}_{r,r} \iota(H)$$

is a totally geodesic map.

Using the commutative diagram gives the proposition:

**Proposition 2.14** *A map  $h : M \rightarrow \text{Herm}(C^r)$  is para-pluriharmonic, if and only if  $g = \text{Re } h : M \rightarrow \text{Sym}_{r,r}(\mathbb{R}^{2r})$  is para-pluriharmonic.*

*A map  $\tilde{h} : M \rightarrow H(r) = GL(r, C)/U^\pi(C^r)$  is para-pluriharmonic, if and only if  $\tilde{g} = [i] \circ h : M \rightarrow S(r, r)$  is para-pluriharmonic.*

*Proof:* As discussed above in this section the map  $\mathfrak{R} : \text{Herm}(C^r) \rightarrow \text{Sym}_{r,r}(\mathbb{R}^{2r})$  is totally geodesic and an immersion. This means that we are in the situation of corollary 2.1.

The second claim follows from the square of the commutative diagram (2.4.15) and from the statements of proposition 2.13 and proposition 2.6, that the composition of a map  $f$  from  $M$  to  $\text{Herm}(C^r)$  (respectively  $\text{Sym}_{r,r}(\mathbb{R}^{2r})$ ) with  $\Psi^{-1}$  (respectively  $(\Psi^0)^{-1}$ ) is para-pluriharmonic, if and only if  $f$  is para-pluriharmonic.  $\square$

**Notation:** In the following work we use the notation  $H(r) = GL(r, C)/U^\pi(C^r)$  and

$$H^\epsilon(p, q) = \begin{cases} H(p, q), & \text{for } \epsilon = -1, \\ H(r), & \text{for } \epsilon = 1. \end{cases} \quad (2.4.16)$$

Further we introduce the notation for the unitary groups

$$U^\epsilon(p, q) = \begin{cases} U(p, q), & \text{for } \epsilon = -1, \\ U^\pi(C^r), & \text{for } \epsilon = 1. \end{cases} \quad (2.4.17)$$

## 2.5 The Lagrangian Grassmanians

### 2.5.1 Definition and homogeneous model

#### Complex version

Like in section 1.5.1 we consider the complex vector space  $V = T^*\mathbb{C}^n = \mathbb{C}^{2n}$  with canonical coordinates  $(z^1, \dots, z^n, w_1, \dots, w_n)$  endowed with the standard complex symplectic form  $\Omega = \sum_{i=1}^n dz^i \wedge dw_i$  and the standard real structure  $\kappa = \bar{\cdot} : V \rightarrow V$  with fixed points  $V^\kappa = T^*\mathbb{R}^n$  and the induced hermitian form  $\gamma := i\Omega(\cdot, \kappa\cdot)$ .

**Definition 2.11** *The subset of the Grassmannian of Lagrangian subspaces  $\mathcal{L}$  of the symplectic vector space  $(V, \Omega)$ , such that  $\gamma$  restricted to  $\mathcal{L}$  defines a hermitian metric of hermitian signature  $(k, l)$ , with  $n = k + l$  is called the hermitian Lagrangian Grassmannian of signature  $(k, l)$  and is denoted by  $Gr_0^{k, l}(\mathbb{C}^{2n})$ .*

We remark that  $Gr_0^{k, l}(\mathbb{C}^{2n})$  is an open subset of the Grassmannian of Lagrangian subspaces of  $(V, \Omega)$  and hence a complex submanifold of it.

**Proposition 2.15** *The real symplectic group  $Sp(\mathbb{R}^{2n})$  acts transitively on  $Gr_0^{k, l}(\mathbb{C}^{2n})$  and we have the following identification:*

$$Gr_0^{k, l}(\mathbb{C}^{2n}) = Sp(\mathbb{R}^{2n})/U(k, l). \quad (2.5.1)$$

Here  $U(k, l) \subset Sp(\mathbb{R}^{2n})$  is defined as the stabilizer of

$$W_o = \text{span}\left\{\frac{\partial}{\partial z^1} + i\frac{\partial}{\partial w_1}, \dots, \frac{\partial}{\partial z^k} + i\frac{\partial}{\partial w_k}, \frac{\partial}{\partial z^{k+1}} - i\frac{\partial}{\partial w_{k+1}}, \dots, \frac{\partial}{\partial z^n} - i\frac{\partial}{\partial w_n}\right\}. \quad (2.5.2)$$

The Grassmannian  $Gr_0^{k, l}(\mathbb{C}^{2n})$  is a pseudo-hermitian symmetric space and, in particular, a homogeneous pseudo-Kähler manifold.

*Proof:* Let  $\mathcal{L}, \mathcal{L}' \in Gr_0^{k, l}(\mathbb{C}^{2n})$ .

Since  $\gamma^\mathcal{L} = \gamma|_\mathcal{L}$  defines a hermitian sesquilinear form, we obtain from the definition of  $\gamma$  that

$$\mathcal{L} \cap \overline{\mathcal{L}} = \{0\}.$$

This means

$$V = \mathcal{L} \oplus \overline{\mathcal{L}}$$

defines an orthogonal decomposition with respect to  $\gamma$ .

The same applies to  $\mathcal{L}'$ .

Choosing a  $\gamma$ -hermitian base  $(f_i)_{i=1}^n$  (respectively  $(f'_i)_{i=1}^n$ ) of  $\mathcal{L}$  (respectively  $\mathcal{L}'$ ) and extending it to a base of  $V$  by  $(\sqrt{-1}f_i)_{i=1}^n$  (respectively  $(\sqrt{-1}f'_i)_{i=1}^n$ ) we construct two symplectic bases  $(f_i, \sqrt{-1}f_i)_{i=1}^n$  and  $(f'_i, \sqrt{-1}f'_i)_{i=1}^n$  of  $V$  and consider the base-change  $\beta$  from  $(f'_i, \sqrt{-1}f'_i)_{i=1}^n$  to  $(f_i, \sqrt{-1}f_i)_{i=1}^n$ .  $\beta$  respects  $\Omega$  and the real structure  $\kappa$ . This means it is an element of  $Sp(\mathbb{R}^{2n})$ . Hence the action of  $Sp(\mathbb{R}^{2n})$  on  $Gr_0^{k, l}(\mathbb{C}^{2n})$  is transitive.

If one considers the base point  $W_o$  one finds by a short calculation (see in the proof of theorem 5.5)

$$(\gamma|_{W_o})_{ij} = \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & -\mathbb{1}_l \end{pmatrix}.$$

□

## Para-complex version

In the para-complex setting (compare section 1.5.2) we denote by  $V$  the para-holomorphic vector space  $T^*C^n = C^{2n}$ , endowed with its standard para-complex structure  $\tau_V$ , its symplectic form  $\Omega$ , the para-complex conjugation  $\kappa = \bar{\cdot} : V \rightarrow V$ ,  $v \mapsto \bar{v}$  with fixed point set  $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$  and with the para-hermitian sesquilinear scalar product on  $V$  defined by  $\gamma(v, w) := e\Omega(v, \bar{w})$ . On this space we take a system of para-holomorphic linear coordinates  $(z^i, w_i)$  which are real valued on  $T^*\mathbb{R}^n$ .

**Definition 2.12** *The subset of the Grassmannian of Lagrangian subspaces  $\mathcal{L}$  of the symplectic vector space  $(V, \Omega)$ , such that  $\gamma$  restricted to  $\mathcal{L}$  defines a para-hermitian sesquilinear scalar product is called the para-hermitian Lagrangian Grassmannian and is denoted by  $Gr_0^n(C^{2n})$ .*

**Proposition 2.16** *The real symplectic group  $Sp(\mathbb{R}^{2n})$  acts transitively on  $Gr_0^n(C^{2n})$  and we have the following identification:*

$$Gr_0^n(C^{2n}) = Sp(\mathbb{R}^{2n})/U^\pi(C^n),$$

where  $U^\pi(C^n)$  is the stabilizer of

$$W_o = span_C \left\{ \frac{\partial}{\partial z^1} + e \frac{\partial}{\partial w_1}, \dots, \frac{\partial}{\partial z^n} + e \frac{\partial}{\partial w_n} \right\}. \quad (2.5.3)$$

*Proof:* Let  $\mathcal{L}, \mathcal{L}' \in Gr_0^n(C^{2n})$ .

Since  $\gamma^{\mathcal{L}} = \gamma|_{\mathcal{L}}$  defines a para-hermitian sesquilinear scalar product, we obtain from the definition of  $\gamma$

$$\mathcal{L} \cap \bar{\mathcal{L}} = \{0\}.$$

This means

$$V = \mathcal{L} \oplus \bar{\mathcal{L}} \quad (2.5.4)$$

defines an orthogonal decomposition with respect to  $\gamma$ .

The same applies to  $\mathcal{L}'$ .

The decomposition (2.5.4) and the fact that  $\mathcal{L}$  is Lagrangian implies  $\gamma(v, v) = e\Omega(v, \bar{v}) \neq 0$  for all  $0 \neq v \in \mathcal{L}$ . This allows us to choose a para-hermitian base  $(f_i)_{i=1}^n$  (respectively  $(f'_i)_{i=1}^n$ ) of  $\mathcal{L}$  (respectively  $\mathcal{L}'$ ). We extend this base to a base of  $V$  by  $(\bar{e}f_i)_{i=1}^n$  (respectively  $(\bar{e}f'_i)_{i=1}^n$ ) and obtain in this way two symplectic bases  $(f_i, \bar{e}f_i)_{i=1}^n$  and  $(f'_i, \bar{e}f'_i)_{i=1}^n$  of  $V$ . Further we consider the base-change  $\beta$  from  $(f'_i, \bar{e}f'_i)_{i=1}^n$  to  $(f_i, \bar{e}f_i)_{i=1}^n$ .  $\beta$  respects  $\Omega$  and the real structure  $\kappa$ . This means it is an element of  $Sp(\mathbb{R}^{2n})$ . Hence the action of  $Sp(\mathbb{R}^{2n})$  on  $Gr_0^n(C^{2n})$  is transitive.



If one considers elements  $\beta$  which leave the base point  $W_o$  invariant, one finds  $[\beta, \tau_{C^n}] = 0$  and  $\beta^* g_{C^n} = g_{C^n}$  with  $g_{C^n} = \text{Re}\gamma|_{W_o}$ .  $\square$

**Notation:** To unify the notation we introduce

$$Gr_0^{k,l}(\mathbb{C}_\epsilon^{2n}) := \begin{cases} Gr_0^{k,l}(\mathbb{C}^{2n}) & \text{for } \epsilon = -1, \\ Gr_0^n(\mathbb{C}^{2n}) & \text{for } \epsilon = 1. \end{cases}$$

## 2.5.2 Holomorphic coordinates on the complex Lagrangian Grassmannian

In this section (cf. [CS1]) we shall introduce a local model for the Grassmannian  $Gr_0^{k,l}(\mathbb{C}^{2n})$  and determine the corresponding local expression for the dual Gauß map. This model is a pseudo-Riemannian analog of the Siegel upper half-space

$$\text{Sym}^+(\mathbb{C}^n) := \{A \in \text{Mat}(n, \mathbb{C}) \mid A^t = A \text{ and } \text{Im } A \text{ is positive definite}\}. \quad (2.5.5)$$

Our aim is to construct holomorphic coordinates for the complex manifold  $Gr_0^{k,l}(\mathbb{C}^{2n})$  in a Zariski-open neighborhood of a point  $W_o$  of the Grassmannian represented by a Lagrangian subspace  $W_o \subset V$  of signature  $(k, l)$ . Using a transformation from  $\text{Sp}(\mathbb{R}^{2n})$  we can assume that  $W_o = W_\circ$ , see equation (2.5.2). Let  $U_o \subset Gr_0^{k,l}(\mathbb{C}^{2n})$  be the open subset consisting of  $W \in Gr_0^{k,l}(\mathbb{C}^{2n})$  such that the projection

$$\pi_{(z)} : V = T^*\mathbb{C}^n = \mathbb{C}^n \oplus (\mathbb{C}^n)^* \rightarrow \mathbb{C}^n \quad (2.5.6)$$

onto the first summand ( $z$ -space) induces an isomorphism

$$\pi_{(z)}|_W : W \xrightarrow{\sim} \mathbb{C}^n. \quad (2.5.7)$$

Notice that  $U_o \subset Gr_0^{k,l}(\mathbb{C}^{2n})$  is an open neighborhood of the base point  $W_o$ . For elements  $W \in U_o$  we can express  $w_i$  as a function of  $z = (z^1, \dots, z^n)$ . In fact,

$$w_i = \sum C_{ij} z^j, \quad (2.5.8)$$

where

$$(C_{ij}) \in \text{Sym}_{k,l}(\mathbb{C}^n) = \{A \in \text{Mat}(n, \mathbb{C}) \mid A^t = A \text{ and } \text{Im } A \text{ has hermitian signature } (k, l)\}. \quad (2.5.9)$$

**Proposition 2.17** *The map*

$$C : U_o \rightarrow \text{Sym}_{k,l}(\mathbb{C}^n), \quad W \mapsto C(W) := (C_{ij}) \quad (2.5.10)$$

*is a local holomorphic chart for the Grassmannian  $Gr_0^{k,l}(\mathbb{C}^{2n})$ .*

**Remark 2.6** *The open subset  $\text{Sym}_{k,l}(\mathbb{C}^n) \subset \text{Sym}(\mathbb{C}^n) = \{A \in \text{Mat}(n, \mathbb{C}) \mid A^t = A\}$  is a generalization of the famous Siegel upper half-space  $\text{Sym}_{n,0}(\mathbb{C}^n) = \text{Sym}^+(\mathbb{C}^n)$ , which is a Siegel domain of type I. In the latter case, we have  $U_o = \text{Sp}(\mathbb{R}^{2n})/\text{U}(n)$  and a global coordinate chart*

$$C : Gr_0^{n,0}(\mathbb{C}^{2n}) = \text{Sp}(\mathbb{R}^{2n})/\text{U}(n) \xrightarrow{\sim} \text{Sym}_{n,0}(\mathbb{C}^n). \quad (2.5.11)$$

### 2.5.3 Para-holomorphic coordinates on the para-complex Lagrangian Grassmannian

In this section (cf. [Sch3]) we introduce a local model of the Grassmannian  $\text{Gr}_0^n(C^{2n})$  of para-complex Lagrangian subspaces  $W \subset V$  of signature  $(n, n)$ , i.e. such that  $g_V = \text{Re } \gamma$  restricted to  $W$  has signature  $(n, n)$ .

This model is a para-complex pseudo-Riemannian analog of the Siegel upper half-space

$$\text{Sym}^+(\mathbb{C}^n) := \{A \in \text{Mat}(n, \mathbb{C}) \mid A^t = A \text{ and } \text{Im } A \text{ is positive definite}\}. \quad (2.5.12)$$

Given a point  $W \in \text{Gr}_0^n(C^{2n})$  we claim, that  $V = T^*C^n$  decomposes into the direct sum

$$V = W \oplus \bar{W} \cong W \oplus W^*. \quad (2.5.13)$$

Let  $\gamma^W = \gamma|_W$ ,  $\omega^W = (\omega_V)|_W$  and  $g^W = (g_V)|_W$ . Then the non-degeneracy of  $\gamma^W$ ,  $g^W$  and  $\omega^W$  are equivalent. One sees from the definition of  $\gamma^W$  that it is non-degenerate if and only if  $W \cap \bar{W} = \{0\}$ . Further it is  $\dim_{\mathbb{R}}(W) = \dim_{\mathbb{R}}(\bar{W}) = \frac{\dim_{\mathbb{R}}(V)}{2}$ , where the last equation follows since  $W$  is Lagrangian. This proves the claim.

One computes easily  $\gamma(\bar{v}, \bar{w}) = -\gamma(w, v)$ ,  $\forall v, w \in W$ . Hence  $g^{\bar{W}}$  has signature  $(n, n)$ , since  $g^W$  has signature  $(n, n)$ . Since  $\gamma = e\Omega(\cdot, \bar{\cdot})$  and  $W$  is Lagrangian, it follows that the decomposition (2.5.13) is  $\gamma$ -orthogonal. Using the isomorphism induced by the symplectic form  $\Omega$  on  $V = W \oplus \bar{W}$  yields an isomorphism of  $W^\perp = \bar{W} \cong W^*$  where  $\cdot^\perp$  is the orthogonal complement taken with respect to  $\gamma$ .

We now construct para-holomorphic coordinates for the para-complex Grassmannian  $\text{Gr}_0^n(C^{2n})$  in an open neighborhood of a point  $W_0$  of the Grassmannian represented by a Lagrangian subspace  $W_0 \subset V$  of signature  $(n, n)$ . Using the transitive action of the group  $Sp(\mathbb{R}^{2n})$  on  $\text{Gr}_0^n(C^{2n})$  we may assume  $W_0 = W_o$ , see equation (2.5.3). Let  $U_0 \subset \text{Gr}_0^n(C^{2n})$  be the open subset consisting of  $W \in \text{Gr}_0^n(C^{2n})$  such that the projection

$$\pi_{(z)} : V = T^*C^n = C^n \oplus (C^n)^* \rightarrow C^n \quad (2.5.14)$$

onto the first summand ( $z$ -space) induces an isomorphism

$$\pi_{(z)}|_W : W \xrightarrow{\sim} C^n. \quad (2.5.15)$$

Observe, that  $U_0 \subset \text{Gr}_0^n(C^{2n})$  is an open neighborhood of the base point  $W_o$ . For elements  $W \in U_0$  we can express  $w_i$  as a function of  $z = (z^1, \dots, z^n)$ . In fact,

$$w_i = \sum C_{ij} z^j \quad (2.5.16)$$

where

$$C_{ij} \in \text{Sym}_{n,n}(C^n) = \{A \in \text{Mat}(n, C) \mid A^t = A \text{ and } \text{Im}(A) \text{ has sym. signature } (n, n)\}. \quad (2.5.17)$$

**Proposition 2.18** *The map*

$$C : U_0 \rightarrow \text{Sym}_{n,n}(C^n), \quad W \mapsto C(W) := (C_{ij}) \quad (2.5.18)$$

*is a local para-holomorphic chart for the Grassmannian  $\text{Gr}_0^n(C^{2n})$ .*

## 2.6 The space of compatible $\epsilon$ complex structures

In this section (cf. [Sch7, Sch8]) we study the differential geometry of the spaces of  $\epsilon$ complex structures, which are compatible with a given metric or a given symplectic form. First we recall the definition of these spaces:

### Definition 2.13

(i) Let  $(V, \omega)$  be a real (finite dimensional) symplectic vector space. An  $\epsilon$ complex structure  $J^\epsilon$  is called **compatible** if and only if it satisfies

$$J^{\epsilon*}\omega = -\epsilon\omega. \quad (2.6.1)$$

The set of such  $\epsilon$ complex structures is denoted by  $\mathcal{J}^\epsilon(V, \omega)$ .

(ii) Let  $(V, \langle \cdot, \cdot \rangle)$  be a real (finite dimensional) pseudo-Euclidean vector space. An  $\epsilon$ complex structure  $J^\epsilon$  is called **compatible** if and only if it satisfies

$$J^{\epsilon*}\langle \cdot, \cdot \rangle = -\epsilon\langle \cdot, \cdot \rangle. \quad (2.6.2)$$

The set of such  $\epsilon$ complex structures is denoted by  $\mathcal{J}^\epsilon(V, \langle \cdot, \cdot \rangle)$ .

We use with  $g = \langle \cdot, \cdot \rangle$  the following notations

$$\mathcal{J}^\epsilon(V, \omega) = \begin{cases} \mathcal{J}(V, \omega), & \text{for } \epsilon = -1, \\ \mathcal{P}(V, \omega), & \text{for } \epsilon = 1 \end{cases}$$

and

$$\mathcal{J}^\epsilon(V, g) = \begin{cases} \mathcal{J}(V, g), & \text{for } \epsilon = -1, \\ \mathcal{P}(V, g), & \text{for } \epsilon = 1. \end{cases}$$

One easily shows the next proposition.

**Proposition 2.19** Let  $\omega_{J^\epsilon} := g(J^\epsilon \cdot, \cdot)$  and  $g_{J^\epsilon} := \epsilon\omega(J^\epsilon \cdot, \cdot)$ . Then it holds:

- (a) Given  $J^\epsilon \in \mathcal{J}^\epsilon(V, \omega)$  then it is  $J^\epsilon \in \mathcal{J}^\epsilon(V, g_{J^\epsilon})$ .
- (b) Given  $J^\epsilon \in \mathcal{J}^\epsilon(V, g)$  then it is  $J^\epsilon \in \mathcal{J}^\epsilon(V, \omega_{J^\epsilon})$ .

### 2.6.1 Differential geometry of the sets of compatible complex structures

#### The metric case

One can consider  $\mathcal{J}(V, \langle \cdot, \cdot \rangle) = \mathcal{J}^{-1}(V, \langle \cdot, \cdot \rangle)$ , where  $V = \mathbb{C}^n = (\mathbb{R}^{2n}, j_0)$  is endowed with its standard complex structure  $j_0$  and its standard scalar product  $\langle \cdot, \cdot \rangle$  of hermitian signature  $(p, q)$ , as a subset in the vector space  $\mathfrak{so}(2p, 2q) = \mathfrak{so}(V) \subset \text{Mat}(\mathbb{R}^{2n})$  characterized by the equations

$$f(j) = -\mathbb{1}_{2n}, \quad (2.6.3)$$

where  $f : \text{Mat}(\mathbb{R}^{2n}) \rightarrow \text{Mat}(\mathbb{R}^{2n})$  is given by  $f : A \mapsto A^2$ . The differential of this map is  $df_A(H) = \{A, H\}$  for  $A, H \in \text{Mat}(\mathbb{R}^{2n})$ . In addition,  $df$  has constant rank in points  $j$  satisfying equation (2.6.3), since one sees

$$\begin{aligned} \ker df_j &= \{A \in \mathfrak{so}(V) \mid \{j, A\} = 0\}, \\ \text{im } df_j &\cong \{A \in \mathfrak{so}(V) \mid [j, A] = 0\} \cong \mathfrak{u}(p, q). \end{aligned}$$

Applying the regular value theorem  $\mathcal{J}(V, \langle \cdot, \cdot \rangle)$  is shown to be a submanifold of  $\mathfrak{so}(V)$ . Its tangent space at  $j \in \mathcal{J}(V, \langle \cdot, \cdot \rangle)$  is

$$T_j \mathcal{J}(V, \langle \cdot, \cdot \rangle) = \ker df_j = \{A \in \mathfrak{so}(V) \mid \{j, A\} = 0\}. \quad (2.6.4)$$

Moreover,  $\mathcal{J}(V, \langle \cdot, \cdot \rangle)$  can be identified with the pseudo-Riemannian symmetric space  $SO_0(2p, 2q)/U(p, q)$ , where  $SO_0(2p, 2q)$  is the identity component of the special pseudo-orthogonal group  $SO(2p, 2q)$  and  $U(p, q)$  is the unitary group of signature  $(p, q)$ , by the map

$$\begin{aligned} \Phi &: SO_0(2p, 2q)/U(p, q) \rightarrow \mathcal{J}(V, \langle \cdot, \cdot \rangle), \\ gK &\mapsto g j_0 g^{-1}, \end{aligned}$$

which maps the canonical base point  $o = eK$  to  $j_0$ .

Any  $j \in \mathcal{J}(V, \langle \cdot, \cdot \rangle)$  defines a symmetric decomposition of  $\mathfrak{so}(V)$  by

$$\begin{aligned} \mathfrak{p}(j) &= \{A \in \mathfrak{so}(V) \mid \{j, A\} = 0\}, \\ \mathfrak{k}(j) &= \{A \in \mathfrak{so}(V) \mid [j, A] = 0\} \cong \mathfrak{u}(p, q). \end{aligned}$$

In particular  $\mathfrak{k}(j_0) = \mathfrak{u}(p, q)$ . Moreover, one observes  $T_j \mathcal{J}(V, \langle \cdot, \cdot \rangle) = \mathfrak{p}(j)$ .

Let  $\tilde{j} \in SO_0(2p, 2q)/U(p, q)$  and  $j = \Phi(\tilde{j})$ , then  $T_{\tilde{j}} SO_0(2p, 2q)/U(p, q)$  is canonically identified with  $\mathfrak{p}(j)$ . We determine now the differential of the above identification:

**Proposition 2.20** *Let  $\Psi = \Phi^{-1} : \mathcal{J}(V, \langle \cdot, \cdot \rangle) \rightarrow SO_0(2p, 2q)/U(p, q)$ . Then it holds at  $j \in \mathcal{J}(V, \langle \cdot, \cdot \rangle)$*

$$d\Psi : T_j \mathcal{J}(V, \langle \cdot, \cdot \rangle) \ni X \mapsto -\frac{1}{2}j^{-1}X \in \mathfrak{p}(j). \quad (2.6.5)$$

## The symplectic case

Now we discuss the differential geometry of  $\mathcal{J}(V, \omega_0) = \mathcal{J}^{-1}(V, \omega_0)$ , where  $\omega_0$  is the standard symplectic form on  $V = \mathbb{C}^n = (\mathbb{R}^{2n}, j_0)$ .

First, we consider  $\mathcal{J}(V, \omega_0)$  as a subset of the vector space  $\mathfrak{sp}(\mathbb{R}^{2n}) \subset \text{Mat}(\mathbb{R}^{2n})$  characterized by the set of equations

$$f(j) = -\mathbb{1}_{2n}, \quad (2.6.6)$$

where  $f : \text{Mat}(\mathbb{R}^{2n}) \rightarrow \text{Mat}(\mathbb{R}^{2n})$  is given as above. Again,  $df$  has constant rank in points  $j$  satisfying equation (2.6.6), since one sees

$$\begin{aligned} \ker df_j &= \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \mid \{j, A\} = 0\}, \\ \text{im } df_j &\cong \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \mid [j, A] = 0\} \cong \mathfrak{u}(p, q). \end{aligned}$$

Applying the regular value theorem we obtain that  $\mathcal{J}(V, \omega_0)$  is a submanifold of  $\mathfrak{sp}(\mathbb{R}^{2n})$ . Its tangent space at  $j \in \mathcal{J}(V, \omega_0)$  is

$$T_j \mathcal{J}(V, \omega_0) = \ker df_j = \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \mid \{j, A\} = 0\}. \quad (2.6.7)$$

In addition the manifold  $\mathcal{J}(V, \omega_0)$  can be identified with the pseudo-Riemannian symmetric space  $\mathrm{Sp}(\mathbb{R}^{2n})/U(p, q)$ , where  $(p, q)$  is the hermitian signature of the hermitian metric  $g(\cdot, \cdot) = \omega(J\cdot, \cdot)$ , by the map

$$\begin{aligned} \Phi : \mathrm{Sp}(\mathbb{R}^{2n})/U(p, q) &\rightarrow \mathcal{J}(V, \omega_0), \\ gK &\mapsto g j_0 g^{-1}, \end{aligned}$$

which maps the canonical base point  $o = eK$  to  $j_0$ .

Any  $j \in \mathcal{J}(V, \omega_0)$  defines a symmetric decomposition of  $\mathfrak{sp}(\mathbb{R}^{2n})$  by

$$\begin{aligned} \mathfrak{p}(j) &= \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \mid \{j, A\} = 0\}, \\ \mathfrak{k}(j) &= \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \mid [j, A] = 0\} \cong \mathfrak{u}(p, q). \end{aligned}$$

In particular  $\mathfrak{k}(j_0) = \mathfrak{u}(p, q)$ . Moreover, one observes  $T_j \mathcal{J}(V, \omega_0) = \mathfrak{p}(j)$ .

Let  $\tilde{j} \in \mathrm{Sp}(\mathbb{R}^{2n})/U(p, q)$  and  $j = \Phi(\tilde{j})$ , then  $T_{\tilde{j}} \mathrm{Sp}(\mathbb{R}^{2n})/U(p, q)$  is canonically identified with  $\mathfrak{p}(j)$  and for the differential of the identification one obtains:

**Proposition 2.21** *Let  $\Psi = \Phi^{-1} : \mathcal{J}(V, \omega_0) \rightarrow \mathrm{Sp}(\mathbb{R}^{2n})/U(p, q)$ . Then it holds at  $j \in \mathcal{J}(V, \omega_0)$*

$$d\Psi : T_j \mathcal{J}(V, \omega_0) \ni X \mapsto -\frac{1}{2} j^{-1} X \in \mathfrak{p}(j). \quad (2.6.8)$$

## 2.6.2 Differential geometry of the sets of compatible para-complex structures

### The metric case

One can consider  $\mathcal{P}(V, \langle \cdot, \cdot \rangle) = \mathcal{J}^1(V, \langle \cdot, \cdot \rangle)$ , where  $V = C^n = \mathbb{R}^n \oplus e\mathbb{R}^n = (\mathbb{R}^{2n}, j_0)$  is endowed with its standard para-complex structure  $j_0$  and its standard scalar product  $\langle \cdot, \cdot \rangle$ , as a subset in the vector space  $\mathfrak{so}(n, n) = \mathfrak{so}(V) \subset \mathrm{Mat}(\mathbb{R}^{2n})$  characterized by the equations

$$f(j) = \mathbb{1}_{2n}, \quad (2.6.9)$$

where  $f : \mathrm{Mat}(\mathbb{R}^{2n}) \rightarrow \mathrm{Mat}(\mathbb{R}^{2n})$  is given in the last subsection. We remark, that elements satisfying equation (2.6.9) define automatically para-complex structures, since they are trace-free and hence their eigenspaces to the eigenvalues  $\pm 1$  have the same dimension. As before  $df$  has constant rank in points  $j$  satisfying equation (2.6.9), since one sees

$$\begin{aligned} \ker df_j &= \{A \in \mathfrak{so}(V) \mid \{j, A\} = 0\}, \\ \mathrm{im} df_j &\cong \{A \in \mathfrak{so}(V) \mid [j, A] = 0\} \cong \mathfrak{u}^\pi(C^n). \end{aligned}$$

Applying the regular value theorem  $\mathcal{P}(V, \langle \cdot, \cdot \rangle)$  is shown to be a submanifold of  $\mathfrak{so}(V)$ . Its tangent space at  $j \in \mathcal{P}(V, \langle \cdot, \cdot \rangle)$  is

$$T_j \mathcal{P}(V, \langle \cdot, \cdot \rangle) = \ker df_j = \{A \in \mathfrak{so}(V) \mid \{j, A\} = 0\}. \quad (2.6.10)$$

Moreover,  $\mathcal{P}(V, \langle \cdot, \cdot \rangle)$  can be identified with the pseudo-Riemannian symmetric space  $SO_0(n, n)/U^\pi(C^n)$ , where  $SO_0(n, n)$  is the identity component of the special pseudo-orthogonal group  $SO(n, n)$  and  $U^\pi(C^n)$  is the para-unitary group, by the map

$$\begin{aligned} \Phi & : SO_0(n, n)/U^\pi(C^n) \rightarrow \mathcal{P}(V, \langle \cdot, \cdot \rangle), \\ gK & \mapsto g j_0 g^{-1}, \end{aligned}$$

which maps the canonical base point  $o = eK$  to  $j_0$ .

Any  $j \in \mathcal{P}(V, \langle \cdot, \cdot \rangle)$  defines a symmetric decomposition of  $\mathfrak{so}(V)$  by

$$\begin{aligned} \mathfrak{p}(j) & = \{A \in \mathfrak{so}(V) \mid \{j, A\} = 0\}, \\ \mathfrak{k}(j) & = \{A \in \mathfrak{so}(V) \mid [j, A] = 0\} \cong \mathfrak{u}^\pi(C^n). \end{aligned}$$

In particular  $\mathfrak{k}(j_0) = \mathfrak{u}^\pi(C^n)$ . Moreover, one observes  $T_j \mathcal{P}(V, \langle \cdot, \cdot \rangle) = \mathfrak{p}(j)$ .

Let  $\tilde{j} \in SO_0(n, n)/U^\pi(C^n)$  and  $j = \Phi(\tilde{j})$ , then  $T_{\tilde{j}}SO_0(n, n)/U^\pi(C^n)$  is canonically identified with  $\mathfrak{p}(j)$ . We determine now the differential of the above identification:

**Proposition 2.22** *Let  $\Psi = \Phi^{-1} : \mathcal{P}(V, \langle \cdot, \cdot \rangle) \rightarrow SO_0(n, n)/U^\pi(C^n)$ . Then it holds at  $j \in \mathcal{P}(V, \langle \cdot, \cdot \rangle)$*

$$d\Psi : T_j \mathcal{P}(V, \langle \cdot, \cdot \rangle) \ni X \mapsto -\frac{1}{2}j^{-1}X \in \mathfrak{p}(j). \quad (2.6.11)$$

## The symplectic case

Now we discuss the differential geometry of  $\mathcal{P}(V, \omega_0) = \mathcal{J}^1(V, \omega_0)$ , where  $\omega_0$  is the standard symplectic form on  $V = C^n = (\mathbb{R}^{2n}, j_0)$ .

First, we consider  $\mathcal{P}(V, \omega_0)$  as a subset of the vector space  $\mathfrak{sp}(\mathbb{R}^{2n}) \subset \text{Mat}(\mathbb{R}^{2n})$  characterized by the set of equations

$$f(j) = \mathbb{1}_{2n}, \quad (2.6.12)$$

where  $f : \text{Mat}(\mathbb{R}^{2n}) \rightarrow \text{Mat}(\mathbb{R}^{2n})$  is given as above. Again,  $df$  has constant rank in points  $j$  satisfying equation (2.6.12), since one sees

$$\begin{aligned} \ker df_j & = \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \mid \{j, A\} = 0\}, \\ \text{im } df_j & \cong \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \mid [j, A] = 0\} \cong \mathfrak{u}^\pi(C^n). \end{aligned}$$

Applying the regular value theorem we obtain that  $\mathcal{P}(V, \omega_0)$  is a submanifold of  $\mathfrak{sp}(\mathbb{R}^{2n})$ . Its tangent space at  $j \in \mathcal{P}(V, \omega_0)$  is

$$T_j \mathcal{P}(V, \omega_0) = \ker df_j = \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \mid \{j, A\} = 0\}. \quad (2.6.13)$$

In addition the manifold  $\mathcal{P}(V, \omega_0)$  can be identified with the pseudo-Riemannian symmetric space  $\text{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$  by the map

$$\begin{aligned} \Phi & : \text{Sp}(\mathbb{R}^{2n})/U^\pi(C^n) \rightarrow \mathcal{P}(V, \omega_0), \\ gK & \mapsto g j_0 g^{-1}, \end{aligned}$$

which maps the canonical base point  $o = eK$  to  $j_0$ .

Any  $j \in \mathcal{P}(V, \omega_0)$  defines a symmetric decomposition of  $\mathfrak{sp}(\mathbb{R}^{2n})$  by

$$\begin{aligned}\mathfrak{p}(j) &= \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \mid \{j, A\} = 0\}, \\ \mathfrak{k}(j) &= \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \mid [j, A] = 0\} \cong \mathfrak{u}^\pi(C^n).\end{aligned}$$

In particular  $\mathfrak{k}(j_0) = \mathfrak{u}^\pi(C^n)$ . Moreover, one observes  $T_j \mathcal{P}(V, \omega_0) = \mathfrak{p}(j)$ .

Let  $\tilde{j} \in \mathrm{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$  and  $j = \Phi(\tilde{j})$ , then  $T_{\tilde{j}}\mathrm{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$  is canonically identified with  $\mathfrak{p}(j)$  and for the differential of the identification one obtains:

**Proposition 2.23** *Let  $\Psi = \Phi^{-1} : \mathcal{P}(V, \omega_0) \rightarrow \mathrm{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$ . Then it holds at  $j \in \mathcal{P}(V, \omega_0)$*

$$d\Psi : T_j \mathcal{P}(V, \omega_0) \ni X \mapsto -\frac{1}{2}j^{-1}X \in \mathfrak{p}(j). \quad (2.6.14)$$

### 2.6.3 Lagrangian Grassmannians and $\epsilon$ complex structures

We are now going to identify the spaces of compatible  $\epsilon$ complex structures  $\mathcal{J}^\epsilon(V, \omega)$  on  $V = \mathbb{C}_\epsilon^n = (\mathbb{R}^{2n}, j_0^\epsilon)$  with the above discussed Lagrangian Grassmannians.

Given an element  $J^\epsilon \in \mathcal{J}^\epsilon(V, \omega)$  we define  $g(\cdot, \cdot) = \epsilon\omega(J^\epsilon \cdot, \cdot)$ .

The data  $(g, \omega)$  defines on  $V$  an  $\epsilon$ hermitian sesquilinear scalar product by

$$h = g + \hat{i}\omega. \quad (2.6.15)$$

In the complex case, i.e. for  $\epsilon = -1$ , the hermitian signature  $(k, l)$  of which is determined by the symmetric signature  $(2k, 2l)$  of  $g$ .

**Proposition 2.24** *(cf. Woodhouse [W] ch. 5 for  $\epsilon = -1$ ) Let  $V^{\mathbb{C}_\epsilon} = V \otimes \mathbb{C}_\epsilon$  be the  $\epsilon$ complexification of  $V$ . Then there is a bijective correspondence between Lagrangian subspaces  $\mathcal{L} \in \mathrm{Gr}_0^{k,l}(\mathbb{C}_\epsilon^{2n})$  and compatible  $\epsilon$ complex structures in  $\mathcal{J}^\epsilon(V, \omega)$ .*

*Proof:* First, let a compatible  $\epsilon$ complex structure  $J^\epsilon$  be given.

The map

$$\begin{aligned}\Gamma &: V \rightarrow V^{\mathbb{C}_\epsilon} \\ X &\mapsto \frac{1}{2}(X + \hat{\epsilon}iJ^\epsilon X)\end{aligned}$$

identifies the  $\epsilon$ complex vector space  $(V, J^\epsilon)$  with the Lagrangian subspace

$$\mathcal{L}_{J^\epsilon} = \{X + \hat{\epsilon}iJ^\epsilon X \mid X \in V\} \subset V^{\mathbb{C}_\epsilon},$$

i.e. the maximal subspace  $W$  satisfying  $J^\epsilon|_W = \hat{i}$ , in such a way, that  $h$  coincides with the product

$$\langle Z, Z' \rangle = -2\hat{i}\omega(Z, \bar{Z}').$$

Conversely, we start with a Lagrangian subspace  $\mathcal{L} \subset V^{\mathbb{C}_\epsilon}$  such that  $h|_{\mathcal{L}}$  is non degenerate. This is equivalent to the condition  $\mathcal{L} \cap \bar{\mathcal{L}} = \{0\}$ .

**Claim:**  $\mathcal{L}$  defines a unique  $J^\epsilon$  such that  $\mathcal{L} = \mathcal{L}_{J^\epsilon}$ .

From  $\mathcal{L} \cap \bar{\mathcal{L}} = \{0\}$  we get  $V^{\mathbb{C}_\epsilon} = \mathcal{L} \oplus \bar{\mathcal{L}}$ . Therefore  $J^\epsilon X$  is uniquely given by expressing  $X$  as  $X = Z + \bar{Z}$  with  $Z \in \mathcal{L}$  and defining  $J^\epsilon(X) = \hat{i}(Z - \bar{Z})$ .  $\square$

## 2.7 Period domains of variations of $\epsilon$ Hodge structures

We recall some information about period domains of variations of  $\epsilon$ Hodge structures and have a closer look at the description of these either as homogeneous spaces or as flag manifolds, since this is crucial to understand our later results. A reference for the complex case is the book [CMP]. Again the complex case is classical and the para-complex case is new.

We introduce the period domain parameterizing the set of polarized  $\epsilon$ Hodge structures on a fixed real vector space  $H$  having a fixed weight  $w$  and fixed  $\epsilon$ Hodge numbers  $h^{p,q}$ . Such an  $\epsilon$ Hodge structure is determined by specifying a flag  $F^w \subset F^{w-1} \subset \dots \subset F^0$  of fixed type satisfying the two bilinear relations. The set of such flags satisfying the first bilinear relation is usually called  $\tilde{D}$  and can be described in a homogeneous model  $G_{\mathbb{C}_\epsilon}/B$  where  $G_{\mathbb{C}_\epsilon}$  is the group of automorphisms of  $H^{\mathbb{C}_\epsilon}$  fixing the polarization  $b$  and  $B$  is the stabilizer of some given reference structure  $F_o^\bullet$ .

**Proposition 2.25** *The set  $\tilde{D}$  classifying  $\epsilon$ Hodge decompositions of weight  $w$  with fixed  $\epsilon$ Hodge numbers  $h^{p,q}$  which obey the first bilinear relation is a flag manifold of type  $(f_w, \dots, f_v)$ ,  $f_p = \dim F^p$ ,  $v = \lfloor \frac{w+1}{2} \rfloor$ , such that*

- (i) *in the case of even weight  $w = 2v$  each  $F^p$ , for  $p = w, \dots, v+1$ , is isotropic with respect to the bilinear form  $b$ .*
- (ii) *in the case of odd weight  $w = 2v - 1$  each  $F^p$ , for  $p = w, \dots, v$ , is isotropic with respect to the bilinear form  $b$ .*

*It can also be identified with the homogeneous manifold  $G_{\mathbb{C}_\epsilon}/B$ .*

*Proof:*

- (i) In the case of even weight we recover the spaces  $F^p$ , for  $p = 0, \dots, (w-v+1) = v+1$ , from  $F^p$ , for  $p = w, \dots, v$ , by using the decomposition

$$H^{\mathbb{C}_\epsilon} = F^p \oplus_{\perp} \overline{F^{w-p+1}},$$

where  $\perp$  is taken with respect to the non-degenerate  $\epsilon$ hermitian sesquilinear form  $b(\cdot, \bar{\cdot})$ . The condition on  $F^p$ , for  $p = w, \dots, v+1$ , to be isotropic is the first Riemannian bilinear relation.

- (ii) In fact, for odd weight, one can recover the whole flag from  $F^p$  for  $p = w, \dots, v$ , by using the decomposition

$$H^{\mathbb{C}_\epsilon} = F^p \oplus_{\perp} \overline{F^{w-p+1}},$$

where  $\perp$  is taken with respect to the non-degenerate  $\epsilon$ hermitian sesquilinear form  $b(\cdot, \bar{\cdot})$ . The condition on  $F^p$ , for  $p = w, \dots, v$ , to be isotropic is in the case of odd weight  $w$  inherited from the first Riemannian bilinear relation.  $\square$



In the complex case  $B$  is a parabolic subgroup. There seems to be no equivalent paracompact notion in the literature.

The subset of  $\tilde{D}$  classifying  $\epsilon$ Hodge structures which also satisfy the second bilinear relation is called  $D$ . As a non-degeneracy or a positivity condition the second bilinear relation defines an open subset of  $\tilde{D}$ .

**Proposition 2.26** *The period domain  $D$  classifying  $\epsilon$ Hodge filtrations  $F^\bullet$  of fixed dimension  $f^p = \dim F^p$  satisfying both bilinear relations is an open subset of  $\tilde{D}$  and it is a homogeneous manifold*

$$D = G/V,$$

where  $G$  is the group of linear automorphisms of  $H$  preserving  $b$  and  $V = G \cap B$ .

We consider the case of Hodge structures which are strongly polarized. Given the space  $G/V$ , we call  $G/K$  where  $K$  is the maximal compact subgroup of  $G$  the ‘associated symmetric space’ and denote the canonical map by

$$\pi : G/V \rightarrow G/K.$$

## The case of odd weight

We now have a glance at the groups  $G, V$  and  $K$  and the associated flag manifolds for Hodge structures of odd weight. Using this we describe for strongly polarized variations of Hodge structures the map  $\pi$  at the level of flag manifolds. This description is needed later to relate the (classical) period map to the  $\epsilon$ pluriharmonic maps appearing in  $\epsilon$ tt\*-geometry.

In the case of odd weight  $w = 2l + 1$  for  $l = v - 1$  the form  $b$  is anti-symmetric due to the first Riemannian bilinear relation. In particular the real dimension of  $H$  is even. Hence the group  $G$  is the symplectic group  $Sp(H, b) \cong Sp(\mathbb{R}^r)$  with  $r = \dim_{\mathbb{R}} H \in 2\mathbb{N}$ . The maximal compact subgroup of  $Sp(\mathbb{R}^r)$  is  $K = U(r)$ .

We define the  $b$ -isotropic  $\epsilon$ complex vector space

$$\mathcal{L} = \bigoplus_{p=0}^l H^{w-p,p} = F^{w-l} = F^v.$$

One sees by equation (1.6.4)

$$H^{C_\epsilon} = \mathcal{L} \oplus \overline{\mathcal{L}}. \quad (2.7.1)$$

Since they have the same dimension,  $\mathcal{L}$  and  $\overline{\mathcal{L}}$  are, by the first bilinear relation, Lagrangian subspaces.

We further fix a reference structure  $F_o^\bullet$ .

Taking successively  $\epsilon$ unitary bases<sup>3</sup>

$$\{f^i\}_{i=1}^{\dim(\mathcal{L})}$$

---

<sup>3</sup>This means a basis with  $h(f_i, f_j) = \pm \delta_{ij}$ .

and

$$\{f_o^i\}_{i=1}^{\dim(\mathcal{L}_o)} \tag{2.7.2}$$

with respect to the hermitian sesquilinear scalar product

$$h(\cdot, \cdot) = (-1)^{w(w-1)/2} \hat{i}^{p-q} \bar{b}(\cdot, \bar{\cdot})$$

of the flags

$$H^{w,0} \subset H^{w,0} \oplus H^{w-1,1} \subset \dots \subset \mathcal{L}$$

and

$$H_o^{w,0} \subset H_o^{w,0} \oplus H_o^{w-1,1} \subset \dots \subset \mathcal{L}_o$$

and extending these with  $\{\bar{f}_o^i\}_{i=1}^{\dim(\mathcal{L}_o)}$  and  $\{\bar{f}_o^i\}_{i=1}^{\dim(\mathcal{L}_o)}$  to symplectic bases of  $\bar{\mathcal{L}}$  and  $\bar{\mathcal{L}}_o$  one sees that  $Sp(\mathbb{R}^r)$  acts transitively by change of the basis from  $\{f_o^i\}_{i=1}^{\dim(\mathcal{L}_o)}$  to  $\{\bar{f}_o^i\}_{i=1}^{\dim(\mathcal{L}_o)}$ .

(i) First we discuss the complex case.

If we have a strongly polarized variation of Hodge structures, then the stabilizer of  $F_o^\bullet$  is the group  $V = \prod_{p=0}^l U(h^{w-p,p})$ . The map  $\pi : G/V \rightarrow G/K$  is at this level nothing else than the forgetful map from the flag  $H^{w,0} \subset H^{w,0} \oplus H^{w-1,1} \subset \dots \subset \mathcal{L}$  to the subspace  $\mathcal{L}$ . We remark, that the stabilizer of  $\mathcal{L}_o$  is contained in the group  $U(r)$ , if we assume the variation of Hodge structures to be strongly polarized.

If we consider a weakly polarized variation of Hodge structures, then the stabilizer of  $F_o^\bullet$  is the group  $V = \prod_{p=0}^l U(k_p, l_p)$ , where  $(k_p, l_p)$ , with  $h^{p,q} = k_p + l_p$ , is the hermitian signature of  $h$  restricted to  $H^{w-p,p}$  with  $q = w - p$ .

The stabilizer of  $\mathcal{L}_o$  is in this case an element of the group  $U(k, l)$ , where  $r = 2(k+l)$  and  $(k, l)$  is the hermitian signature of  $h$  on  $\mathcal{L}_o$ , i.e.  $k = \sum k_p$  and  $l = \sum l_p$ .

Given a variation of Hodge structures of odd weight over the complex base manifold  $(M, J)$  we denote by  $L$  the (holomorphic) map

$$L : M \rightarrow Sp(\mathbb{R}^r)/U(k, l), \tag{2.7.3}$$

$$x \mapsto \mathcal{L}_x. \tag{2.7.4}$$

The Grassmannian of Lagrangian subspaces, on which  $h$  has signature  $(k, l)$  will be denoted by  $Gr_0^{k,l}(\mathbb{C}^r)$  and on which  $h$  is positive definite will be denoted by  $Gr_0(\mathbb{C}^r) = Gr_0^{r,0}(\mathbb{C}^r)$ .

(ii) In the para-complex case the stabilizer of  $\mathcal{L}_o$  is the group  $U^\pi(C^n)$ , with  $r = 2n$ , compare definition 1.7. As before given a variation of para-Hodge structures of odd weight  $w$  over the para-complex base manifold  $(M, \tau)$  we denote by  $L$  the (para-holomorphic) map

$$L : M \rightarrow Sp(\mathbb{R}^r)/U^\pi(C^n), \tag{2.7.5}$$

$$x \mapsto \mathcal{L}_x. \tag{2.7.6}$$

The associated Grassmannian of Lagrangian subspaces will be denoted by  $Gr_0^n(C^{2n})$  with  $r = 2n$ .



# Chapter 3

## $tt^*$ -geometry and some of its solutions

In the first part of this chapter we introduce  $\epsilon tt^*$ -bundles and characterize these in terms of explicit geometric data and equations on this data. In the second section we study  $\epsilon tt^*$ -bundles on the tangent bundle  $TM$  of a given almost  $\epsilon$ complex manifold  $(M, J^\epsilon)$ . In particular special  $\epsilon$ complex, special  $\epsilon$ Kähler and Levi-Civita flat nearly  $\epsilon$ Kähler manifolds are solutions of  $tt^*$ -geometry on the tangent bundle  $TM$ . These three classes of solutions are discussed separately. The last two sections of this chapter deal with variations of  $\epsilon$ Hodge structures and  $\epsilon$ harmonic bundles as solutions of  $tt^*$ -geometry.

### 3.1 $tt^*$ -bundles

In this section we introduce the real differential geometric definition of an  $\epsilon tt^*$ -bundle. For integrable  $\epsilon$ complex structures the complex geometric version was given in [CS1, Sch6] and the para-complex geometric version was introduced in [Sch4]. The non-integrable case and the symplectic version were first considered in complex geometry in [Sch7] and in para-complex geometry in [Sch8].

**Definition 3.1** *An  $\epsilon tt^*$ -bundle  $(E, D, S)$  over an almost  $\epsilon$ complex manifold  $(M, J^\epsilon)$  is a real vector bundle  $E \rightarrow M$  endowed with a connection  $D$  and a section  $S \in \Gamma(T^*M \otimes \text{End } E)$  satisfying the  $\epsilon tt^*$ -equation*

$$R^\theta = 0 \quad \text{for all } \theta \in \mathbb{R}, \quad (3.1.1)$$

where  $R^\theta$  is the curvature tensor of the connection  $D^\theta$  defined by

$$D_X^\theta := D_X + \cos_\epsilon(\theta)S_X + \sin_\epsilon(\theta)S_{J^\epsilon X} \quad \text{for all } X \in TM. \quad (3.1.2)$$

A **metric  $\epsilon tt^*$ -bundle**  $(E, D, S, g)$  is an  $\epsilon tt^*$ -bundle  $(E, D, S)$  endowed with a possibly indefinite  $D$ -parallel fiber metric  $g$  such that  $S$  is  $g$ -symmetric, i.e. for all  $p \in M$

$$g(S_X Y, Z) = g(Y, S_X Z) \quad \text{for all } X, Y, Z \in T_p M. \quad (3.1.3)$$

A unimodular metric  $\epsilon tt^*$ -bundle  $(E, D, S, g)$  is a metric  $\epsilon tt^*$ -bundle  $(E, D, S, g)$  such that  $\text{tr } S_X = 0$  for all  $X \in TM$ .

An oriented unimodular metric  $\epsilon tt^*$ -bundle  $(E, D, S, g, \text{or})$  is a unimodular metric  $\epsilon tt^*$ -bundle endowed with an orientation  $\text{or}$  of the bundle  $E$ .

A symplectic  $\epsilon tt^*$ -bundle  $(E, D, S, \omega)$  is an  $\epsilon tt^*$ -bundle  $(E, D, S)$  endowed with the structure of a symplectic vector bundle<sup>1</sup>  $(E, \omega)$ , such that  $\omega$  is  $D$ -parallel and  $S$  is  $\omega$ -symmetric, i.e. for all  $p \in M$

$$\omega(S_X Y, Z) = \omega(Y, S_X Z) \quad \text{for all } X, Y, Z \in T_p M. \quad (3.1.4)$$

In the case of moduli spaces of topological quantum field theories [CV, D] and the moduli spaces of singularities [Her], the complexified  $tt^*$ -bundle  $E^{\mathbb{C}}$  (This means we consider  $\epsilon = -1$ .) is identified with  $T^{1,0}M$  and the metric  $g$  is positive definite. The case  $E = TM$ , and hence  $E^{\mathbb{C}} = T^{1,0}M + T^{0,1}M$  includes special complex and special Kähler manifolds, as we have proven in [CS1] and follows from [Her] in the complex situation. This was shown in [Sch4] in the para-complex framework. We discuss this later in more details.

### Remark 3.1

1) If  $(E, D, S)$  is an  $\epsilon tt^*$ -bundle then  $(E, D, S^\theta)$  is an  $\epsilon tt^*$ -bundle for all  $\theta \in \mathbb{R}$ , where

$$S^\theta := D^\theta - D = \cos_\epsilon(\theta)S + \sin_\epsilon(\theta)S_{J^\epsilon}. \quad (3.1.5)$$

The same remark applies to metric and symplectic  $\epsilon tt^*$ -bundles.

2) Notice that an oriented unimodular metric  $\epsilon tt^*$ -bundle  $(E, D, S, g, \text{or})$  carries a canonical metric volume element  $\nu \in \Gamma(\wedge^r E^*)$ ,  $r = \text{rk } E$ , determined by  $g$  and  $\text{or}$ , which is  $D^\theta$ -parallel for all  $\theta \in \mathbb{R}$ .

Further, a symplectic  $\epsilon tt^*$ -bundle  $(E, D, S, \omega)$  of rank  $2r$  carries a  $D$ -parallel volume given by  $\underbrace{\omega \wedge \dots \wedge \omega}_{r \text{ times}}$ .

The following proposition characterizes  $\epsilon tt^*$ -bundles  $(E, D, S)$  in form of explicit equations for  $D$  and  $S$ . These equations are important in the later calculations.

**Proposition 3.1** *Let  $E$  be a real vector bundle over an (almost)  $\epsilon$ -complex manifold  $(M, J^\epsilon)$  endowed with a connection  $D$  and a section  $S \in \Gamma(T^*M \otimes \text{End } E)$ .*

*Then  $(E, D, S)$  is an  $\epsilon tt^*$ -bundle if and only if  $D$  and  $S$  satisfy the following equations:*

$$R^D + S \wedge S = 0, \quad (3.1.6)$$

$$S \wedge S \text{ is of type } (1,1), \quad (3.1.7)$$

$$[D_X, S_Y] - [D_Y, S_X] - S_{[X,Y]} = 0, \quad \forall X, Y \in \Gamma(TM), \quad (3.1.8)$$

$$[D_X, S_{J^\epsilon Y}] - [D_Y, S_{J^\epsilon X}] - S_{J^\epsilon[X,Y]} = 0, \quad \forall X, Y \in \Gamma(TM). \quad (3.1.9)$$

Fixing a torsion-free connection on  $(M, J^\epsilon)$  the last two equations are equivalent to

$$d^D S = 0 \quad \text{and} \quad d^D S_{J^\epsilon} = 0. \quad (3.1.10)$$

<sup>1</sup>see D. Mc Duff and D. Salamon [McDS]

*Proof:* As the attentive reader observes, it is easier to show this proposition after  $\epsilon$ complexifying  $TM$ . But since one idea of this work was to formulate these results in terms of real differential geometry, we give a real version of the proof.

To prove the proposition, we have to compute the curvature tensor of  $D^\theta$ .

Let  $X, Y \in \Gamma(TM)$  be arbitrary:

$$\begin{aligned}
 R_{X,Y}^\theta &= R_{X,Y}^D \\
 &+ [D_X, \cos_\epsilon(\theta)S_Y + \sin_\epsilon(\theta)S_{J^\epsilon Y}] \\
 &+ [\cos_\epsilon(\theta)S_X + \sin_\epsilon(\theta)S_{J^\epsilon X}, D_Y] \\
 &+ [\cos_\epsilon(\theta)S_X + \sin_\epsilon(\theta)S_{J^\epsilon X}, \cos_\epsilon(\theta)S_Y + \sin_\epsilon(\theta)S_{J^\epsilon Y}] \\
 &- \cos_\epsilon(\theta)S_{[X,Y]} - \sin_\epsilon(\theta)S_{J^\epsilon[X,Y]} \\
 &= R_{X,Y}^D \\
 &+ \sin_\epsilon^2(\theta)[S_{J^\epsilon X}, S_{J^\epsilon Y}] \\
 &+ \cos_\epsilon^2(\theta)[S_X, S_Y] \\
 &+ \cos_\epsilon(\theta)\sin_\epsilon(\theta) ([S_X, S_{J^\epsilon Y}] + [S_{J^\epsilon X}, S_Y]) \\
 &+ \cos_\epsilon(\theta) ([D_X, S_Y] + [S_X, D_Y] - S_{[X,Y]}) \\
 &+ \sin_\epsilon(\theta) ([S_{J^\epsilon X}, D_Y] + [D_X, S_{J^\epsilon Y}] - S_{J^\epsilon[X,Y]}).
 \end{aligned}$$

We recall the theorems of addition

$$\begin{aligned}
 \cos_\epsilon(\theta)\sin_\epsilon(\theta) &= \frac{1}{2}\sin_\epsilon(2\theta), \\
 \cos_\epsilon^2(\theta) &= \frac{1}{2}(1 + \cos_\epsilon(2\theta)) \text{ and} \\
 \sin_\epsilon^2(\theta) &= \frac{1}{2}\epsilon(\cos_\epsilon(2\theta) - 1)
 \end{aligned} \tag{3.1.11}$$

to find

$$\begin{aligned}
 R_{X,Y}^\theta &= R_{X,Y}^D + \frac{1}{2} ([S_X, S_Y] - \epsilon[S_{J^\epsilon X}, S_{J^\epsilon Y}]) \\
 &+ \cos_\epsilon(\theta) ([D_X, S_Y] + [S_X, D_Y] - S_{[X,Y]}) \\
 &+ \sin_\epsilon(\theta) ([S_{J^\epsilon X}, D_Y] + [D_X, S_{J^\epsilon Y}] - S_{J^\epsilon[X,Y]}) \\
 &+ \frac{1}{2}\cos_\epsilon(2\theta) ([S_X, S_Y] + \epsilon[S_{J^\epsilon X}, S_{J^\epsilon Y}]) \\
 &+ \frac{1}{2}\sin_\epsilon(2\theta) ([S_X, S_{J^\epsilon Y}] + [S_{J^\epsilon X}, S_Y]).
 \end{aligned}$$

Taking 'Fourier-coefficients' yields

$$\begin{aligned}
 R_{X,Y}^D + \frac{1}{2} ([S_X, S_Y] - \epsilon[S_{J^\epsilon X}, S_{J^\epsilon Y}]) &= 0, \\
 [S_X, S_Y] + \epsilon[S_{J^\epsilon X}, S_{J^\epsilon Y}] &= 0, \\
 [S_X, S_{J^\epsilon Y}] + [S_{J^\epsilon X}, S_Y] &= 0, \\
 [D_X, S_Y] + [S_X, D_Y] - S_{[X,Y]} &= 0, \\
 [S_{J^\epsilon X}, D_Y] + [D_X, S_{J^\epsilon Y}] - S_{J^\epsilon[X,Y]} &= 0.
 \end{aligned}$$

The first three equations give

$$R_{X,Y}^D + [S_X, S_Y] = 0, \quad S \wedge S(X, Y) = [S_X, S_Y] = -\epsilon[S_{J^\epsilon X}, S_{J^\epsilon Y}].$$

Choosing a torsion-free connection on  $M$  the last two equations yield

$$d^D S = 0 \text{ and } d^D S_{J^\epsilon} = 0.$$

□

## 3.2 Solutions on the tangent bundle of an almost $\epsilon$ complex manifold

The following sections are contained in [Sch7, Sch8].

### 3.2.1 Solutions without metrics

Given an almost  $\epsilon$ complex manifold  $(M, J^\epsilon)$  with a flat connection  $\nabla$  it is natural to consider the one-parameter family of connections  $\nabla^\theta$ , which is defined by

$$\nabla_X^\theta Y = \exp(\theta J^\epsilon) \nabla_X (\exp(-\theta J^\epsilon) Y) \text{ for } X, Y \in \Gamma(TM), \quad (3.2.1)$$

where  $\exp(\theta J^\epsilon) = \cos_\epsilon(\theta) Id + \sin_\epsilon(\theta) J^\epsilon$ .

Recall, that the flatness of  $\nabla$  implies the flatness of the family of connections  $\nabla^\theta$  (compare remark 1.3).

Let us recall a definition

**Definition 3.2** *Two one-parameter families of connections  $\nabla^\theta$  and  $D^\theta$  on some vector bundle  $E$  with  $\theta \in \mathbb{R}$  are called (linearly) equivalent with factor  $\alpha \in \mathbb{R}$  if they satisfy the equation  $\nabla^\theta = D^{\alpha\theta}$ .*

We are now going to analyze the form of  $\epsilon tt^*$ -bundles  $(TM, D, S)$  on the tangent bundle  $TM$  of  $M$  for which the family of connections  $D^\theta$  defined in equation (3.1.2) is linearly equivalent to the family of connections  $\nabla^\theta$  defined in equation (3.2.1).

**Proposition 3.2** *Given an almost  $\epsilon$ complex manifold  $(M, J^\epsilon)$  with a flat connection  $\nabla$  and a decomposition of  $\nabla = D + S$  in a connection  $D$  and a section  $S$  in  $T^*M \otimes \text{End}(TM)$ . Then  $(TM, D, S)$  defines an  $\epsilon tt^*$ -bundle, such that the family of connections  $D^\theta$  is linearly equivalent to the family of connections  $\nabla^\theta$  with factor  $\alpha = \pm 2$  if and only if  $S$  and  $D$  satisfy*

$$S_{J^\epsilon X} = \pm J^\epsilon S_X Y$$

and

$$-(D_X J^\epsilon) Y = J^\epsilon S_X Y + S_X J^\epsilon Y = \{S_X, J^\epsilon\} Y$$

for all  $X, Y \in \Gamma(TM)$ .

*Proof:* First one has to analyze the family of connections  $\nabla^\theta$  for  $X, Y \in \Gamma(TM)$

$$\begin{aligned}\nabla_X^\theta Y &= \exp(\theta J^\epsilon)(D_X + S_X)[(\cos_\epsilon(\theta)Id - \sin_\epsilon(\theta)J^\epsilon)Y] \\ &= D_X Y - \exp(\theta J^\epsilon)\sin_\epsilon(\theta)(D_X J^\epsilon)Y \\ &\quad + (\cos_\epsilon(\theta)Id + \sin_\epsilon(\theta)J^\epsilon)S_X(\cos_\epsilon(\theta)Id - \sin_\epsilon(\theta)J^\epsilon)Y \\ &= D_X Y - (\cos_\epsilon(\theta)\sin_\epsilon(\theta) + \sin_\epsilon^2(\theta)J^\epsilon)(D_X J^\epsilon)Y + \cos_\epsilon^2(\theta)S_X Y \\ &\quad - \sin_\epsilon^2(\theta)J^\epsilon S_X J^\epsilon Y - \cos_\epsilon(\theta)\sin_\epsilon(\theta)[S_X, J^\epsilon]Y,\end{aligned}$$

which yields with the theorems of addition (see equation (3.1.11)), the identity

$$\begin{aligned}\nabla_X^\theta Y &= D_X Y - \frac{1}{2}\sin_\epsilon(2\theta)(D_X J^\epsilon)Y - \frac{1}{2}\epsilon(\cos_\epsilon(2\theta) - 1)J^\epsilon(D_X J^\epsilon)Y \\ &\quad + \frac{1}{2}(1 + \cos_\epsilon(2\theta))S_X Y - \frac{1}{2}\epsilon(\cos_\epsilon(2\theta) - 1)J^\epsilon S_X J^\epsilon Y - \frac{1}{2}\sin_\epsilon(2\theta)[S_X, J^\epsilon]Y \\ &= D_X Y + \frac{1}{2}[S_X + \epsilon J^\epsilon S_X J^\epsilon + \epsilon J^\epsilon D_X J^\epsilon]Y \\ &\quad + \frac{1}{2}\sin_\epsilon(2\theta)[[J^\epsilon, S_X] - D_X J^\epsilon]Y \\ &\quad + \frac{1}{2}\cos_\epsilon(2\theta)[S_X - \epsilon J^\epsilon S_X J^\epsilon - \epsilon J^\epsilon D_X J^\epsilon]Y \\ &\stackrel{!}{=} D_X Y + \cos_\epsilon(\vartheta)T_X Y + \sin_\epsilon(\vartheta)T_{J^\epsilon X} Y \text{ with } \vartheta = \pm 2\theta,\end{aligned}$$

where we have to determine  $T \in \Gamma(T^*M \otimes \text{End}(TM))$ .

Comparing coefficients of  $1, \cos_\epsilon(n\vartheta), \sin_\epsilon(n\vartheta)$  with  $n = 1, 2$  yields

$$-\epsilon J^\epsilon(D_X J^\epsilon)Y = S_X Y + \epsilon J^\epsilon S_X J^\epsilon Y, \text{ or equivalently} \quad (3.2.2)$$

$$-(D_X J^\epsilon)Y = J^\epsilon S_X Y + S_X J^\epsilon Y = \{S_X, J^\epsilon\}Y,$$

$$T_X Y = \frac{1}{2}(S_X Y - \epsilon J^\epsilon S_X J^\epsilon Y - \epsilon J^\epsilon(D_X J^\epsilon)Y) \stackrel{(3.2.2)}{=} S_X Y, \quad (3.2.3)$$

$$\begin{aligned}T_{J^\epsilon X} Y &= \pm \frac{1}{2}([J^\epsilon, S_X]Y - (D_X J^\epsilon)Y) \\ &\stackrel{(3.2.2)}{=} \pm \frac{1}{2}(J^\epsilon S_X Y - S_X J^\epsilon Y + J^\epsilon S_X Y + S_X J^\epsilon Y) \\ &= \pm J^\epsilon S_X Y.\end{aligned} \quad (3.2.4)$$

The last two equations yield the constraint on  $S$

$$S_{J^\epsilon X} = \pm J^\epsilon S_X Y$$

and the first equation the one on  $D$  and  $S$ . □

Now we suppose the connection  $D$  to be  $\epsilon$ complex. Such a connection exists on every almost  $\epsilon$ complex manifold, as we have shown in theorem 1.1.

**Corollary 3.1** *Given an almost  $\epsilon$ complex manifold  $(M, J^\epsilon)$  with a flat connection  $\nabla$  and a decomposition of  $\nabla = D + S$  in a connection  $D$  and a section  $S$  in  $T^*M \otimes \text{End}(TM)$ , such that  $J^\epsilon$  is  $D$ -parallel, i.e.  $DJ^\epsilon = 0$ . Then  $(TM, D, S)$  defines an  $ett^*$ -bundle, such*



that the family of connections  $D^\theta$  is linearly equivalent to the family of connections  $\nabla^\theta$  with factor  $\alpha = \pm 2$  if and only if  $S$  satisfies

$$S_{J^\epsilon X} = \pm J^\epsilon S_X \text{ and } \{S_X, J^\epsilon\} = 0.$$

*Proof:* The second constraint in proposition 3.2 is for  $DJ^\epsilon = 0$  the condition  $\{S_X, J^\epsilon\} = 0$ . The first constraint of proposition 3.2 is exactly  $S_{J^\epsilon X} = \pm J^\epsilon S_X \stackrel{\{S_X, J^\epsilon\}=0}{=} \mp S_X J^\epsilon$ .  $\square$

We are going to show some uniqueness result. Therefore we prove the

**Lemma 3.1** *Let  $(M, J^\epsilon)$  be an almost  $\epsilon$ complex manifold. Given a connection  $\nabla$  on  $M$  which decomposes as  $\nabla = D + S$ , where  $D$  is a connection on  $M$  and  $S$  is a section in  $T^*M \otimes \text{End}(TM)$ , such that  $J^\epsilon$  is  $D$ -parallel, i.e.  $DJ^\epsilon = 0$  and  $S$  anti-commutes with  $J^\epsilon$ , i.e.  $\{S_X, J^\epsilon\} = 0$  for all  $X \in \Gamma(TM)$ . Then  $S$  and  $D$  are uniquely given by*

$$S_X Y = -\frac{1}{2}\epsilon J^\epsilon (\nabla_X J^\epsilon) Y \text{ and } D_X Y = \nabla_X Y - S_X Y \text{ for } X, Y \in \Gamma(TM). \quad (3.2.5)$$

*Otherwise, given a connection  $\nabla$  and define  $D$  and  $S$  by equation (3.2.5), then  $D$  and  $S$  satisfy  $DJ^\epsilon = 0$  and  $\{S_X, J^\epsilon\} = 0$ .*

*Proof:* First we observe  $\nabla = D + S$  and

$$S_X J^\epsilon Y = -\frac{1}{2}\epsilon J^\epsilon (\nabla_X J^\epsilon) J^\epsilon Y = \frac{1}{2}\epsilon J^{\epsilon^2} (\nabla_X J^\epsilon) Y = -J^\epsilon S_X Y,$$

where the second equality follows from deriving  $J^{\epsilon^2} = \epsilon Id$ . Further it is

$$(D_X J^\epsilon) Y = (\nabla_X J^\epsilon) Y - [S_X, J^\epsilon] Y \stackrel{\{S_X, J^\epsilon\}=0}{=} (\nabla_X J^\epsilon) Y + 2J^\epsilon S_X Y = 0.$$

Now we prove the uniqueness: Suppose there exist  $D'$  and  $S'$  with the same properties. Thus we get

$$0 = (D'_X J^\epsilon) Y = (\nabla_X J^\epsilon) Y - [S'_X, J^\epsilon] Y = (\nabla_X J^\epsilon) Y + 2J^\epsilon S'_X Y$$

and consequently

$$S'_X Y = -\frac{1}{2}\epsilon J^\epsilon (\nabla_X J^\epsilon) Y = S_X Y \text{ and } D'_X Y = \nabla_X Y - S'_X Y = \nabla_X Y - S_X Y = D_X Y.$$

$\square$

Summarizing corollary 3.1 and lemma 3.1 we find the following uniqueness result:

**Theorem 3.1** *Given an almost  $\epsilon$ complex manifold  $(M, J^\epsilon)$  with a flat connection  $\nabla$  and a decomposition of  $\nabla = D + S$  in a connection  $D$  and a section  $S$  in  $T^*M \otimes \text{End}(TM)$ , such that  $J^\epsilon$  is  $D$ -parallel, i.e.  $DJ^\epsilon = 0$ . If  $(TM, D, S)$  defines an  $ett^*$ -bundle, such that*

the family of connections  $D^\theta$  is linearly equivalent to the family of connections  $\nabla^\theta$  with factor  $\alpha = \pm 2$ , then  $D$  and  $S$  are uniquely determined by the equations

$$S = -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon) \quad (3.2.6)$$

and

$$D = \nabla - S. \quad (3.2.7)$$

Moreover,  $(TM, D, S)$  as given by equation (3.2.6) and (3.2.7) defines an  $ett^*$ -bundle, such that the family of connections  $D^\theta$  is linearly equivalent to the family of connections  $\nabla^\theta$  with factor  $\alpha = \pm 2$ , if and only if  $J^\epsilon$  satisfies  $(\nabla_{J^\epsilon X} J^\epsilon) = \pm J^\epsilon(\nabla_X J^\epsilon)$  and  $D$  and  $S$  are given by  $S = -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon)$  and  $D = \nabla - S$ .

In the following propositions we are going to give some classes of examples which satisfy the condition  $S_{J^\epsilon X} = \pm J^\epsilon S_X$ .

**Proposition 3.3** *Given an almost  $\epsilon$ complex manifold  $(M, J^\epsilon)$  with a connection  $\nabla$  and let  $S$  be the section in  $T^*M \otimes \text{End}(TM)$  defined by*

$$S := -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon). \quad (3.2.8)$$

If the pair  $(\nabla, J^\epsilon)$  satisfies one of the following conditions

- (i)  $(\nabla, J^\epsilon)$  is special, i.e.  $(\nabla_X J^\epsilon)Y = (\nabla_Y J^\epsilon)X$  for all  $X, Y \in \Gamma(TM)$ ,
- (ii)  $(\nabla, J^\epsilon)$  satisfies the nearly  $\epsilon$ Kähler condition, i.e.  $(\nabla_X J^\epsilon)Y = -(\nabla_Y J^\epsilon)X$  for all  $X, Y \in \Gamma(TM)$ ,

then it holds  $S_{J^\epsilon X}Y = -J^\epsilon S_X Y$ .

*Proof:* If the condition (i) or (ii) holds, we obtain the identity

$$\begin{aligned} (\nabla_{J^\epsilon X} J^\epsilon)Y &= \pm(\nabla_Y J^\epsilon)J^\epsilon X = \pm[\epsilon \nabla_Y X - J^\epsilon \nabla_Y(J^\epsilon X)] \\ &= \mp J^\epsilon[\nabla_Y(J^\epsilon X) - J^\epsilon \nabla_Y X] = \mp J^\epsilon(\nabla_Y J^\epsilon)X = -J^\epsilon(\nabla_X J^\epsilon)Y. \end{aligned}$$

The following calculation finishes the proof

$$S_{J^\epsilon X}Y = -\frac{1}{2}\epsilon J^\epsilon(\nabla_{J^\epsilon X} J^\epsilon)Y = \frac{1}{2}\epsilon J^{\epsilon^2}(\nabla_X J^\epsilon)Y = -J^\epsilon S_X Y.$$

□

**Proposition 3.4** *Given an  $\epsilon$ complex manifold  $(M, J^\epsilon)$  with a connection  $\nabla$  and let  $S$  be the section in  $T^*M \otimes \text{End}(TM)$  defined by*

$$S := -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon). \quad (3.2.9)$$

If  $\nabla$  is (anti-)adapted, i.e.  $\nabla_{J^\epsilon X} Y = \pm J^\epsilon \nabla_X Y$  for all  $\epsilon$ holomorphic vector fields  $X, Y$ , then it holds  $S_{J^\epsilon X}Y = \pm J^\epsilon S_X Y$ .

*Proof:* Since  $\nabla$  is (anti-)adapted, we obtain for all  $\epsilon$ holomorphic vector fields  $X, Y$

$$(\nabla_{J^\epsilon X} J^\epsilon)Y = \pm J^\epsilon(\nabla_X J^\epsilon)Y.$$

The following computation gives the proof

$$S_{J^\epsilon X}Y = -\frac{1}{2}\epsilon J^\epsilon(\nabla_{J^\epsilon X} J^\epsilon)Y = \mp \frac{1}{2}\epsilon J^{\epsilon^2}(\nabla_X J^\epsilon)Y = \pm J^\epsilon S_X Y.$$

□

### Remark 3.2

*One sees easily that condition (i) of proposition 3.3 is the symmetry of  $S_X Y$  and condition (ii) is its anti-symmetry. We recall that if the connection  $\nabla$  is torsion-free, flat and special then  $(M, J^\epsilon, \nabla)$  is a special  $\epsilon$ complex manifold, see subsection 1.4.  $\epsilon$ tt\*-bundles coming from special  $\epsilon$ complex manifolds and special  $\epsilon$ Kähler manifolds were studied in [CS1, Sch3] and are discussed later in subsection 3.2.4.*

*Further we want to remark that the second condition arises in nearly  $\epsilon$ Kählerian geometry and therefore is quite natural. These geometries as solutions of  $tt^*$ -geometry are discussed in subsection 3.2.3.*

*Finally, the notion of adapted connections appeared in the study of decompositions on ( $\epsilon$ holomorphic) vector bundles over  $\epsilon$ complex manifolds, compare the paper of Abe and Kurosu [AK] for the complex and a common paper with M.-A. Lawn-Paillusseau [LS] for the para-complex case.*

## 3.2.2 Solutions on almost $\epsilon$ hermitian manifolds

In this section we consider almost  $\epsilon$ complex manifolds  $(M, J^\epsilon)$  endowed with a flat connection  $\nabla$  such that  $(\nabla, J^\epsilon)$  is special or satisfies the nearly  $\epsilon$ Kähler condition and analyze under which assumptions these define symplectic or metric  $\epsilon$ tt\*-bundles.

First, we recall a lemma from tensor-algebra:

**Lemma 3.2** *Let  $V$  be a vector space  $\alpha \in T^3(V^*)$  an element in the third tensorial power of  $V^*$ , the dual space of  $V$ . Suppose that  $\alpha(X, Y, Z)$  is symmetric (resp. anti-symmetric) in  $X, Y$  and  $Y, Z$  and  $\alpha(X, Y, Z)$  is anti-symmetric (resp. symmetric) in  $X, Z$  then  $\alpha = 0$ .*

*Proof:* It is

$$\alpha(X, Y, Z) = \sigma\alpha(Y, X, Z) = \sigma\alpha(X, Z, Y)$$

with  $\sigma \in \{\pm 1\}$  which implies

$$\alpha(X, Y, Z) = \sigma\alpha(Y, X, Z) = \sigma^2\alpha(Y, Z, X) = \sigma^3\alpha(Z, Y, X).$$

But further it holds

$$\alpha(X, Y, Z) = -\sigma\alpha(Z, Y, X)$$

and consequently

$$-\alpha(Z, Y, X) = \sigma^2\alpha(Z, Y, X) = \alpha(Z, Y, X).$$

This shows  $\alpha = 0$ . □

The subsequent proposition shows that the condition to be special is not compatible with symplectic  $ett^*$ -bundles:

**Proposition 3.5** *Given an almost  $\epsilon$ hermitian manifold  $(M, J^\epsilon, g)$  with a flat connection  $\nabla$ , such that  $(\nabla, J^\epsilon)$  is special. Define  $S$ , a section in  $T^*M \otimes \text{End}(TM)$ , by*

$$S := -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon), \quad (3.2.10)$$

*then  $(TM, D = \nabla - S, S)$  defines an  $ett^*$ -bundle. Suppose, that  $(TM, D, S, \omega = g(J^\epsilon \cdot, \cdot))$  is a symplectic  $ett^*$ -bundle, then it is trivial, i.e.  $S = 0$ .*

*Proof:* In fact we know from theorem 3.1 and proposition 3.3, that  $(TM, D, S)$  is an  $ett^*$ -bundle.

Suppose, that  $(TM, D, S, \omega = g(J^\epsilon \cdot, \cdot))$  is a symplectic  $ett^*$ -bundle. To finish the proof, we define the tensor

$$\alpha(X, Y, Z) := \omega(S_X Y, Z) = g(J^\epsilon S_X Y, Z), \text{ with } X, Y, Z \in T_p M.$$

$\alpha(X, Y, Z)$  is symmetric in  $X, Y$ , since  $\nabla J^\epsilon$  is special, i.e. is symmetric in  $X, Y$ .

Further it holds

$$\begin{aligned} \alpha(X, Y, Z) &= \omega(S_X Y, Z) = -\omega(Z, S_X Y) \\ &= -\omega(Z, S_Y X) = -\omega(S_Y Z, X) = -\omega(S_Z Y, X) = -\alpha(Z, Y, X), \end{aligned}$$

which is the anti-symmetry of  $\alpha(X, Y, Z)$  in  $X, Z$ . Finally

$$\begin{aligned} \alpha(X, Y, Z) &= \omega(S_X Y, Z) = \omega(Y, S_X Z) \\ &= \omega(Y, S_Z X) = -\omega(S_Z X, Y) = -\alpha(Z, X, Y) = -\alpha(X, Z, Y), \end{aligned}$$

i.e. the anti-symmetry of  $\alpha(X, Y, Z)$  in  $Y, Z$ .

Hence  $\alpha$  vanishes and consequently  $S$ . □

Otherwise, the nearly  $\epsilon$ Kähler condition is not compatible with metric  $ett^*$ -bundles:

**Proposition 3.6** *Given an almost  $\epsilon$ hermitian manifold  $(M, J^\epsilon, g)$  with a flat connection  $\nabla$ , such that  $(\nabla, J^\epsilon)$  satisfies the nearly  $\epsilon$ Kähler condition. Define  $S$ , a section in  $T^*M \otimes \text{End}(TM)$ , by*

$$S := -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon), \quad (3.2.11)$$

*then  $(TM, D = \nabla - S, S)$  defines an  $ett^*$ -bundle. Suppose, that  $(TM, D, S, g)$  is a metric  $ett^*$ -bundle, then it is trivial, i.e.  $S = 0$ .*

*Proof:* In fact we know from theorem 3.1 and proposition 3.3, that  $(TM, D, S)$  is an  $ett^*$ -bundle.

Suppose, that it is a metric  $ett^*$ -bundle. To finish the proof, we define the tensor

$$\alpha(X, Y, Z) := g(S_X Y, Z), \text{ with } X, Y, Z \in T_p M.$$

$\alpha(X, Y, Z)$  is anti-symmetric in  $X, Y$ , since  $\nabla J^\epsilon$  is anti-symmetric in  $X, Y$  by the nearly  $\epsilon$ Kähler condition.

Further it holds

$$\begin{aligned}\alpha(X, Y, Z) &= g(S_X Y, Z) = g(Z, S_X Y) \\ &= -g(Z, S_Y X) = -g(S_Y Z, X) = g(S_Z Y, X) = \alpha(Z, Y, X)\end{aligned}$$

which is the symmetry of  $\alpha(X, Y, Z)$  in  $X, Z$ . Finally

$$\begin{aligned}\alpha(X, Y, Z) &= g(S_X Y, Z) = g(Y, S_X Z) \\ &= -g(Y, S_Z X) = -g(S_Z X, Y) = -\alpha(Z, X, Y) = \alpha(X, Z, Y),\end{aligned}$$

i.e. the symmetry of  $\alpha(X, Y, Z)$  in  $Y, Z$ .

Hence  $\alpha$  vanishes by the above lemma and so does  $S$ .  $\square$

The following theorem gives solutions of symplectic  $\epsilon tt^*$ -bundles on the tangent bundle, which are more general than the later discussed nearly  $\epsilon$ Kähler manifolds in the sense, that we admit the connection  $\nabla$  to have torsion, but more special in the sense, that our connection  $\nabla$  has to be flat.

**Theorem 3.2** *Given an almost hermitian manifold  $(M, J^\epsilon, g)$  with a flat metric connection  $\nabla$ , such that  $(\nabla, J^\epsilon)$  satisfies the nearly  $\epsilon$ Kähler condition. Define  $S$ , a section in  $T^*M \otimes \text{End}(TM)$ , by*

$$S := -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon), \quad (3.2.12)$$

*then  $(TM, D = \nabla - S, S, \omega = g(J^\epsilon \cdot, \cdot))$  defines a symplectic  $\epsilon tt^*$ -bundle. In addition, it is  $DJ^\epsilon = 0$ . Moreover, the torsion  $T^D$  of  $D$  and the torsion  $T^\nabla$  of  $\nabla$  are related by  $T^D = T^\nabla - 2S$ .*

*Proof:* In fact we know from theorem 3.1 and proposition 3.3, that  $(TM, D, S)$  is an  $\epsilon tt^*$ -bundle.

It remains to check  $D\omega = 0$  and that  $S$  is  $\omega$ -symmetric.

First we remark, that, since  $g$  is hermitian and  $\nabla g = 0$ ,  $\nabla_X J^\epsilon$  is skew-symmetric with respect to  $g$ . Using this we show by the following calculation, that  $S$  is skew-symmetric with respect to  $g$ :

$$\begin{aligned}-2\epsilon g(S_X Y, Z) &= g(J^\epsilon(\nabla_X J^\epsilon)Y, Z) = -g((\nabla_X J^\epsilon)Y, J^\epsilon Z) \\ &= g(Y, (\nabla_X J^\epsilon)J^\epsilon Z) = -g(Y, J^\epsilon(\nabla_X J^\epsilon)Z) = 2\epsilon g(Y, S_X Z).\end{aligned}$$

The definition of  $\omega = g(J^\epsilon \cdot, \cdot)$  and  $\{S_X, J^\epsilon\} = 0$  yield the  $\omega$ -symmetry of  $S_X$ . Further it holds  $D = \nabla + \frac{1}{2}\epsilon J^\epsilon \nabla J^\epsilon$ , which implies

$$DJ^\epsilon = \nabla J^\epsilon + \frac{1}{2}\epsilon [J^\epsilon \nabla J^\epsilon, J^\epsilon] = 0.$$

Hence we see, that  $D\omega = 0$  if and only if  $Dg = 0$ . But  $\nabla g = 0$  and  $S$  is skew-symmetric with respect to  $g$ , so  $g$  is parallel for  $D = \nabla - S$ .

This shows, that  $(TM, D = \nabla - S, S, \omega)$  is a symplectic  $ett^*$ -bundle.

Calculating the torsion we find  $T^D(X, Y) = T^\nabla(X, Y) - S_X Y + S_Y X = T^\nabla(X, Y) - 2S_X Y$ .  
 $\square$

The next theorem gives solutions of metric  $ett^*$ -bundles on the tangent bundle, which are more general than special  $\epsilon$ Kähler manifolds in the sense, that we admit connections  $\nabla$  with torsion.

**Theorem 3.3** *Given an almost  $\epsilon$ hermitian manifold  $(M, J^\epsilon, g)$  with a flat connection  $\nabla$ , such that  $(\nabla, J^\epsilon)$  is special and the fundamental two-form  $\omega = g(J^\epsilon \cdot, \cdot)$  is  $\nabla$ -parallel. Define  $S$ , a section in  $T^*M \otimes \text{End}(TM)$ , by*

$$S := -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon), \quad (3.2.13)$$

*then  $(TM, D = \nabla - S, S, g)$  defines a metric  $ett^*$ -bundle. In addition, it is  $DJ^\epsilon = 0$  and the torsion  $T^D$  of  $D$  equals the torsion  $T^\nabla$  of  $\nabla$ .*

*Suppose that  $\nabla$  is torsion-free, then  $D$  is the Levi-Civita connection of  $g$ ,  $(M, J^\epsilon, g)$  is an  $\epsilon$ Kähler manifold and  $(M, J^\epsilon, g, \nabla)$  is a special  $\epsilon$ Kähler manifold.*

*Proof:* In fact we know from theorem 3.1 and proposition 3.3, that  $(TM, D, S)$  is an  $ett^*$ -bundle.

It remains to check  $Dg = 0$  and that  $S$  is  $g$ -symmetric.

First we remark that  $\omega(J^\epsilon X, Y) = -\omega(X, J^\epsilon Y)$  as  $g$  is  $\epsilon$ hermitian. This yields using  $\nabla\omega = 0$  the  $\omega$ -skew-symmetry of  $\nabla_X J^\epsilon$ , which implies that  $S_X = -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon)$  is  $\omega$ -skew-symmetric, since  $J^\epsilon(\nabla_X J^\epsilon) = -(\nabla_X J^\epsilon)J^\epsilon$ . Finally  $\{S_X, J^\epsilon\} = 0$  shows the  $g$ -symmetry of  $S_X$ .

Further it is

$$DJ^\epsilon = \nabla J^\epsilon + \frac{1}{2}\epsilon[J^\epsilon \nabla J^\epsilon, J^\epsilon] = 0$$

and consequently  $Dg = 0$  is equivalent to  $D\omega = 0$ .

From  $\nabla\omega = 0$  and the  $\omega$ -skew-symmetry of  $S$  it follows  $D\omega = (\nabla - S)\omega = 0$ .

The symmetry of  $\nabla J^\epsilon$ , i.e.  $(\nabla_X J^\epsilon)Y = (\nabla_Y J^\epsilon)X$  for all  $X, Y \in TM$  implies  $S_X Y = S_Y X$ . This shows using  $D = \nabla - S$  that  $T^D = T^\nabla$ .

Suppose now that  $\nabla$  is torsion-free, then  $D$  is torsion-free and consequently the Levi-Civita connection of  $g$ . Therefore  $DJ^\epsilon = 0$  implies the vanishing of the Nijenhuis tensor. Further the equation  $\nabla\omega = 0$  implies  $d\omega = 0$  since  $\nabla$  is torsion-free. Hence  $(M, J^\epsilon, g)$  is  $\epsilon$ Kähler. In addition  $(M, J^\epsilon, \nabla)$  is special  $\epsilon$ complex by the conditions on  $\nabla$  and  $J^\epsilon$ . As it holds  $\nabla\omega = 0$ ,  $(M, J^\epsilon, \nabla, g)$  is special  $\epsilon$ Kähler.  $\square$

In [CS1, Sch3] we studied special  $\epsilon$ Kähler solutions of  $ett^*$ -geometry in more details. The results are discussed in subsection 3.2.4.

### 3.2.3 Nearly $\epsilon$ Kähler manifolds

In this section we want to apply the above results to nearly  $\epsilon$ Kähler manifolds and we use the notation of subsection 1.3.

**Corollary 3.2** *Given a nearly  $\epsilon$ Kähler manifold  $(M, J^\epsilon, g)$  such that its Levi-Civita connection  $\nabla = \nabla^g$  is flat and let  $S$  be the section in  $T^*M \otimes \text{End}(TM)$  defined by*

$$S := -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon), \quad (3.2.14)$$

*then  $(TM, \bar{\nabla}, S)$  defines an  $\epsilon tt^*$ -bundle. Suppose, that  $(TM, \bar{\nabla}, S, g)$  is a metric  $\epsilon tt^*$ -bundle, then it is trivial, i.e.  $S = 0$  and consequently  $(M, J^\epsilon, g)$  is  $\epsilon$ Kähler.*

*Proof:* By setting  $D = \bar{\nabla}$  we are in the situation of proposition 3.6. □

**Theorem 3.4** *Given a nearly  $\epsilon$ Kähler manifold  $(M, J^\epsilon, g)$  such that its Levi-Civita connection  $\nabla$  is flat. Let  $S$  be the section in  $T^*M \otimes \text{End}(TM)$  defined by*

$$S := -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon), \quad (3.2.15)$$

*then  $(TM, \bar{\nabla}, S, \omega := g(J^\epsilon \cdot, \cdot))$  is a symplectic  $\epsilon tt^*$ -bundle. Further it holds*

$$B(X, Y, Z) = -2g(S_X Y, Z) \text{ and } \bar{\nabla} J^\epsilon = 0. \quad (3.2.16)$$

*Proof:* By setting  $D = \bar{\nabla}$  we are in the situation of theorem 3.2. In addition it holds

$$2g(S_X Y, Z) = -\epsilon g(J^\epsilon(\nabla_X J^\epsilon)Y, Z) = \epsilon g((\nabla_X J^\epsilon)Y, J^\epsilon Z) = -B(X, Y, Z).$$

□

### Remark 3.3

*Nearly Kähler manifolds  $(M, J, g)$  such that their Levi-Civita connection  $\nabla^g$  is flat were characterized in common work with V. Cortés [CS2]. More precisely, a constructive classification of nearly Kähler manifolds with flat Levi-Civita connection was given. We further recall that a Levi-Civita flat nearly Kähler cannot be strict. This means that the more interesting examples appear for non definite signature.*

## 3.2.4 Special $\epsilon$ complex and special $\epsilon$ Kähler manifolds

In this subsection we consider another time  $\epsilon tt^*$ -bundles on the tangent-bundle  $TM$  of an  $\epsilon$ complex manifold  $(M, J^\epsilon)$  and discuss the results which were published in [CS1] for  $\epsilon = -1$  and [Sch3] for  $\epsilon = 1$ . More precisely, we analyze solutions coming from special  $\epsilon$ complex and special  $\epsilon$ Kähler manifolds. In this context it is natural to restrict to  $\epsilon tt^*$ -bundles, such that the family of connections  $D^\theta$  is torsion-free.

**Definition 3.3** *An  $\epsilon tt^*$ -bundle  $(TM, D, S)$  over an  $\epsilon$ complex manifold  $(M, J^\epsilon)$  is called special if  $D^\theta$  is torsion-free and special, i.e.  $D^\theta J^\epsilon$  is symmetric, for all  $\theta \in \mathbb{R}$ .*

**Proposition 3.7** *An  $\epsilon tt^*$ -bundle  $(TM, D, S)$  is special if and only if  $D$  is torsion-free and  $DJ^\epsilon$ ,  $S$  and  $S_{J^\epsilon}$  are symmetric.*

*Proof:* The torsion  $T^\theta$  of  $D^\theta$  equals

$$T^\theta(X, Y) = T(X, Y) + \cos_\epsilon(\theta)(S_X Y - S_Y X) + \sin_\epsilon(\theta)(S_{J^\epsilon X} Y - S_{J^\epsilon Y} X) \quad (3.2.17)$$

where  $T$  is the torsion-tensor of  $D$ . This implies, that  $T^\theta = 0$  for all  $\theta \in \mathbb{R}$  if and only if  $T = 0$  and  $S$  and  $S_{J^\epsilon}$  are symmetric.

The equation

$$\begin{aligned} (D_X^\theta J^\epsilon)Y &= (D_X J^\epsilon)Y + \cos_\epsilon(\theta)[S_X, J^\epsilon]Y + \sin_\epsilon(\theta)[S_{J^\epsilon X}, J^\epsilon]Y \quad (3.2.18) \\ \stackrel{\{S_X, J^\epsilon\}=0}{=} & (D_X J^\epsilon)Y - 2\cos_\epsilon(\theta) J^\epsilon S_X Y - 2\sin_\epsilon(\theta) J^\epsilon S_{J^\epsilon X} Y \end{aligned}$$

shows that  $D^\theta J^\epsilon$  is symmetric if and only if  $DJ^\epsilon$ ,  $S$  and  $S_{J^\epsilon}$  are symmetric.

Conversely, let  $T^\theta = 0$  and  $D^\theta J^\epsilon$  be symmetric: Then the first part of the proof yields, that  $S$  and  $S_{J^\epsilon}$  are symmetric and  $T = 0$ . Equation (3.2.18) implies finally the symmetry of  $DJ^\epsilon$ .  $\square$

### Theorem 3.5

(i) Let  $(M, J^\epsilon, \nabla)$  be a special  $\epsilon$ complex manifold. Put  $S := -\frac{1}{2}\epsilon J^\epsilon \nabla J^\epsilon$  and  $D := \nabla - S$ . Then  $(TM, D, S)$  is a special  $ett^*$ -bundle with the following additional properties:

- a)  $S_X J^\epsilon = -J^\epsilon S_X$  for all  $X \in TM$  and
- b)  $DJ^\epsilon = 0$ .

This defines a map  $\Phi$  from special  $\epsilon$ complex manifolds to special  $ett^*$ -bundles.

- (ii) Let  $(TM, D, S)$  be a special  $ett^*$ -bundle over an  $\epsilon$ complex manifold  $(M, J^\epsilon)$ . Then  $(M, J^\epsilon, \nabla := D + S)$  is a special  $\epsilon$ complex manifold. This defines a map  $\Psi$  from special  $ett^*$ -bundles to special  $\epsilon$ complex manifolds such that  $\Psi \circ \Phi = Id$ . If  $(TM, D, S)$  is a special  $ett^*$ -bundle satisfying the conditions a) and b) of (i), then  $\Phi(\Psi(TM, D, S)) = (TM, D, S)$ .
- (iii) Let  $(M, J^\epsilon, g, \nabla)$  be a special  $\epsilon$ Kähler manifold with  $S$  and  $D$  as in (i). Then  $(TM, D, S, g)$  defines a special metric  $ett^*$ -bundle satisfying a) and b) of (i). This defines a map, also called  $\Phi$ , from special  $\epsilon$ Kähler manifolds to special metric  $ett^*$ -bundles.
- (iv) Let  $(TM, D, S, g)$  be a special metric  $ett^*$ -bundle over an hermitian manifold  $(M, J^\epsilon, g)$  satisfying the conditions a) and b) in (i). Then  $(M, J^\epsilon, g, \nabla := D + S)$  is a special  $\epsilon$ Kähler manifold. In particular, we have a map  $\Psi$  from special metric  $ett^*$ -bundles over hermitian manifolds  $(M, J^\epsilon, g)$  satisfying the conditions a) and b) in (i) to special  $\epsilon$ Kähler manifolds. Moreover  $\Psi$  is a bijection and  $\Psi^{-1} = \Phi$ .
- (v) Let  $(TM, D, S, g)$  be a metric  $ett^*$ -bundle over an hermitian manifold  $(M, J^\epsilon, g)$  satisfying the conditions a) and b) in (i) and such that  $D$  is torsion-free. Then it is special if and only if  $(M, J^\epsilon, g, \nabla := D + S)$  is a special  $\epsilon$ Kähler manifold.



*Proof:*

(i) From theorem 3.1 and proposition 3.3 we know, that  $(TM, D, S)$  is an  $\epsilon tt^*$ -bundle. This  $\epsilon tt^*$ -bundle is special, since the family of connections  $D^\theta$  and the family of connections  $\nabla^\theta$  are linearly equivalent and since by proposition 1.4  $(M, J^\epsilon, \nabla^\theta)$  is a special  $\epsilon$ Kähler manifold. The additional properties hold, as  $(M, J^\epsilon, \nabla)$  is a special  $\epsilon$ complex manifold (compare proposition 1.6 and 1.8).

(ii) In order to prove the second statement, let  $(TM, D, S)$  be a special  $\epsilon tt^*$ -bundle, i.e.  $D^\theta$  is flat, torsion-free and special. In particular,  $\nabla = D + S = D^0$  is flat, torsion-free and special. Hence  $(M, J^\epsilon, \nabla)$  is a special  $\epsilon$ complex manifold. Obviously we have  $\Psi \circ \Phi = Id$ . Conversely, let  $(TM, D, S)$  be a special  $\epsilon tt^*$ -bundle satisfying  $DJ^\epsilon = 0$  and  $S_X J^\epsilon = -J^\epsilon S_X$  for all  $X \in T_p M$ . Then we use lemma 3.1 to recover  $D$  and  $S$  uniquely from  $\nabla = D + S$  by the formulas  $S = -\frac{1}{2}\epsilon J^\epsilon \nabla J^\epsilon$  and  $D = \nabla - S$ .

(iii) Let  $(M, J^\epsilon, g, \nabla)$  be a special  $\epsilon$ Kähler manifold with  $D$  and  $S$  defined as in (i). Then  $(TM, D, S)$  is a special  $\epsilon tt^*$ -bundle satisfying a) and b), due to (i). Proposition 1.7 implies, that  $Dg = 0$  and proposition 1.8 implies, that  $S$  is  $g$ -symmetric and hence that  $(TM, D, S, g)$  is a special metric  $\epsilon tt^*$ -bundle.

(iv) Let  $(TM, D, S, g)$  be a special metric  $\epsilon tt^*$ -bundle over an  $\epsilon$ hermitian manifold  $(M, J^\epsilon, g)$  satisfying a) and b) in (i). By (ii), we know already, that  $(M, J^\epsilon, \nabla := D + S)$  is a special  $\epsilon$ complex manifold. Therefore it remains to prove  $\nabla\omega = 0$ . This implies  $d\omega = 0$ , as  $\nabla$  is torsion-free. We have  $Dg = 0$  and  $DJ^\epsilon = 0$  (property b) in (i)) and consequently  $D\omega = 0$ . As  $D\omega = 0$ ,  $\nabla\omega = 0$  is equivalent to the  $\omega$ -skew-symmetry of  $S$  and finally to the  $g$ -symmetry of  $S$ , since  $\{J^\epsilon, S_X\} = 0$ . But by the definition of a metric  $\epsilon tt^*$ -bundle  $S$  is  $g$ -symmetric. Therefore  $(M, J^\epsilon, \nabla, g)$  is a special  $\epsilon$ Kähler manifold. The rest of part (iv) follows from part (ii).

(v) It remains to show the direction which does not follow from (iv). Let  $(TM, D, S, g)$  be a metric  $\epsilon tt^*$ -bundle over an  $\epsilon$ hermitian manifold  $(M, J^\epsilon, g)$ , such that  $(M, J^\epsilon, g, \nabla = D + S) = \Psi(TM, D, S, g)$  is a special  $\epsilon$ Kähler manifold. If  $D$  is torsion-free, then it is the Levi-Civita connection of  $g$ , and therefore  $D = \nabla + \frac{1}{2}\epsilon J^\epsilon \nabla J^\epsilon$ , see proposition 1.7. This shows, that  $\Phi(M, J^\epsilon, g, \nabla) = (TM, D, S, g)$  and that  $(TM, D, S, g)$  is a special metric  $\epsilon tt^*$ -bundle.  $\square$

**Corollary 3.3** *A special metric  $\epsilon tt^*$ -bundle  $(TM, D, S, g)$  over an  $\epsilon$ hermitian manifold  $(M, J^\epsilon, g)$  which satisfies a) and b) in theorem 3.5 is oriented and unimodular.*

*Proof:* By theorem 3.5,  $(M, J^\epsilon, g, \nabla = D + S)$  is a special  $\epsilon$ Kähler manifold. Hence we can orient it by  $\omega \wedge \dots \wedge \omega$ , where  $\omega$  is its  $\epsilon$ Kähler-form. Its  $\epsilon$ Kähler-form is parallel with respect to the connections  $D$  and  $\nabla$  and therefore invariant under  $S_X = \nabla_X - D_X$ . This shows  $\text{tr } S_X = 0$ .  $\square$

### 3.3 Variations of $\epsilon$ Hodge structures

In this section we recall the result of Hertling [Her] that variations of Hodge structures give solutions of metric  $tt^*$ -bundles and generalize it to para-complex geometry and symplectic  $\epsilon tt^*$ -bundles. Our presentation differs from that of [Her], since we give this result in the language of real differential geometry. Again, the para-complex version seems to

be new.

Let  $(E, \nabla, F^p)$  be a (real) variation of  $\epsilon$ Hodge structures of weight  $w$ . The  $\epsilon$ complexified connection of  $\nabla$  on  $E^{\mathbb{C}\epsilon} = E \otimes \mathbb{C}_\epsilon$  will be denoted by  $\nabla^c$ . Griffiths transversality and the holomorphicity of the subbundles  $F^p$  gives

$$\nabla^c : \Gamma(F^p) \rightarrow \Lambda^{1,0}(F^{p-1}) + \Lambda^{0,1}(F^p) \quad (3.3.1)$$

and  $\epsilon$ complex conjugation yields

$$\nabla^c : \Gamma(\overline{F^p}) \rightarrow \Lambda^{0,1}(\overline{F^{p-1}}) + \Lambda^{1,0}(\overline{F^p}). \quad (3.3.2)$$

Summarizing one obtains with  $H^{p,w-p} = F^p \cap \overline{F}^{w-p}$

$$\nabla^c : \Gamma(H^{p,w-p}) \rightarrow \underbrace{\Lambda^{1,0}(H^{p,w-p}) + \Lambda^{0,1}(H^{p,w-p})}_D + \underbrace{\Lambda^{1,0}(H^{p-1,w+1-p}) + \Lambda^{0,1}(H^{p+1,w-1-p})}_S. \quad (3.3.3)$$

Using the decomposition induced by the  $\epsilon$ Hodge structure and by the bi-degree of differential forms, one can find, that the curvature of  $\nabla^c$  vanishes if and only if  $(E^c, D, S)$  defines an  $\epsilon tt^*$ -bundle. In addition the  $\epsilon$ complex conjugation  $\kappa = \bar{\cdot}$  respects the  $\epsilon$ Hodge decomposition and it is  $\nabla^c \kappa = 0$ . Again the decomposition induced by the  $\epsilon$ Hodge structure and by the bi-degree of differential forms implies that  $D\kappa = 0$ , i.e.  $D$  leaves  $E$  invariant and that  $S\kappa = \kappa S$ , i.e.  $S$  leaves  $E$  invariant, too.

If  $b$  is a polarization of the above variation of  $\epsilon$ Hodge structures  $(E, \nabla, F^p)$ , then  $\nabla b = 0$  and  $\nabla^c \kappa = 0$  yield after decomposing with respect to  $\epsilon$ Hodge structure the equations  $Dg = 0$  and  $g(S\cdot, \cdot) = g(\cdot, S\cdot)$  with  $g = \text{Re } h$ . Concluding we obtain the proposition

**Proposition 3.8** *Let  $(E, \nabla, F^p)$  be a (real) variation of  $\epsilon$ Hodge structures of weight  $w$  with a polarization  $b$ , then  $(E, D, S, g = \text{Re } h)$  with  $D$  and  $S$  as defined in equation (3.3.3) is a metric  $\epsilon tt^*$ -bundle.*

The above consideration holds for  $\Omega = \text{Im } h$ , too. This implies  $D\Omega = 0$  and  $\Omega(S\cdot, \cdot) = \Omega(\cdot, S\cdot)$ . Hence we have proven

**Proposition 3.9** *Let  $(E, \nabla, F^p)$  be a (real) variation of  $\epsilon$ Hodge structures of weight  $w$  with a polarization  $b$ , then  $(E, D, S, \Omega = \text{Im } h)$  with  $D$  and  $S$  as defined in equation (3.3.3) is a symplectic  $\epsilon tt^*$ -bundle.*

## 3.4 Harmonic bundles

In this section (cf. [Sch4] for the complex case, i.e.  $\epsilon = -1$ ) we introduce the notion of an  $\epsilon$ harmonic bundle and show that every such bundle gives two solutions of the  $\epsilon tt^*$ -equations. The first is a metric and the second is a symplectic  $\epsilon tt^*$ -bundle.

To introduce the notion of an  $\epsilon$ harmonic bundle we need a definition:

**Definition 3.4** An  $\epsilon$ hermitian sesquilinear metric  $h$  on an  $\epsilon$ complex vector bundle  $E$  over an  $\epsilon$ complex manifold  $(M, J^\epsilon)$  is a smooth fiberwise  $\epsilon$ hermitian sesquilinear product.

**Definition 3.5** An  $\epsilon$ harmonic bundle  $(E \rightarrow M, D, C, \bar{C}, h)$  consists of the following data:

An  $\epsilon$ complex vector bundle  $E$  over an  $\epsilon$ complex manifold  $(M, J^\epsilon)$ , an  $\epsilon$ hermitian sesquilinear metric  $h$  on  $E$ , a metric connection  $D$  with respect to  $h$  and two  $C^\infty$ -linear maps  $C : \Gamma(E) \rightarrow \Gamma(\Lambda^{1,0}T^*M \otimes E)$  and  $\bar{C} : \Gamma(E) \rightarrow \Gamma(\Lambda^{0,1}T^*M \otimes E)$ , such that the connection

$$D^{(\lambda)} = D + \lambda C + \lambda^{-1} \bar{C}$$

is flat for all  $\lambda \in \mathbb{S}_\epsilon^1$  and  $h(C_Z a, b) = h(a, \bar{C}_{\bar{Z}} b)$  for all  $a, b \in \Gamma(E)$  and  $Z \in \Gamma(T^{1,0}M)$ .

**Remark 3.4**

In the case  $\epsilon = -1$  and positive definite metric  $h$ , this definition is equivalent to the definition of a harmonic bundle given in Simpson's paper [Sim]. Equivalent structures with metrics of arbitrary signature have been also considered in [Her].

**Theorem 3.6** Let  $(E \rightarrow M, D, C, \bar{C}, h)$  be an  $\epsilon$ harmonic bundle over the  $\epsilon$ complex manifold  $(M, J^\epsilon)$ , then  $(E, D, S, g = \operatorname{Re} h [\omega = \operatorname{Im} h])$  with  $S_X := C_Z + \bar{C}_{\bar{Z}}$  for  $X = Z + \bar{Z} \in TM$  and  $Z \in T^{1,0}M$  is a metric [symplectic]  $\epsilon$ tt\*-bundle.

*Proof:* For  $\lambda = \cos_\epsilon(\alpha) + \hat{i} \sin_\epsilon(\alpha) \in \mathbb{S}_\epsilon^1$  we have a look at  $D^{(\lambda)}$  :

$$\begin{aligned} D_X^{(\lambda)} &= D_X + \lambda C_Z + \bar{\lambda} \bar{C}_{\bar{Z}} = D_X + \cos_\epsilon(\alpha)(C_Z + C_{\bar{Z}}) + \sin_\epsilon(\alpha)(\hat{i}C_Z - \hat{i}C_{\bar{Z}}) \\ &= D_X + \cos_\epsilon(\alpha)S_X + \sin_\epsilon(\alpha)(C_{J^\epsilon Z} + C_{J^\epsilon \bar{Z}}) \\ &= D_X + \cos_\epsilon(\alpha)S_X + \sin_\epsilon(\alpha)S_{J^\epsilon X} = D_X^\alpha. \end{aligned}$$

Hence we have

$$D^\alpha = D^{(\lambda)} \tag{3.4.1}$$

and  $D^\alpha$  is flat if and only if  $D^{(\lambda)}$  is flat.

Further we claim, that  $S$  is  $g$ -symmetric [ $\omega$ -symmetric]. With  $X = Z + \bar{Z}$  for  $Z \in T^{1,0}M$  one finds

$$h(S_X \cdot, \cdot) = h(C_Z + C_{\bar{Z}} \cdot, \cdot) = h(\cdot, C_Z + C_{\bar{Z}} \cdot) = h(\cdot, S_X \cdot)$$

and consequently the symmetry of  $S$  with respect to

$$g = \operatorname{Re} h$$

and

$$\omega = \operatorname{Im} h.$$

Finally we show  $Dg = 0$  and  $D\omega = 0$

$$\begin{aligned} X(h(e, f) \pm h(f, e)) &= (Z + \bar{Z})(h(e, f) \pm h(f, e)) \\ &= h(D_Z e, f) + h(e, D_{\bar{Z}} f) + h(D_{\bar{Z}} e, f) + h(e, D_Z f) \\ &\pm [h(D_Z f, e) + h(f, D_{\bar{Z}} e) + h(D_{\bar{Z}} f, e) + h(f, D_Z e)] \\ &= h((D_Z + D_{\bar{Z}})e, f) + h(e, (D_{\bar{Z}} + D_Z)f) \\ &\pm [h((D_Z + D_{\bar{Z}})f, e) + h(f, (D_{\bar{Z}} + D_Z)e)] \\ &= h(D_X e, f) + h(e, D_X f) \pm h(D_X f, e) \pm h(f, D_X e). \end{aligned}$$

Summarizing we obtain

$$Xg(e, f) = g(D_X e, f) + g(e, D_X f)$$

and

$$X\omega(e, f) = \omega(D_X e, f) + \omega(e, D_X f).$$

This proves, that  $(E, D, S, g = \operatorname{Re} h [\omega = \operatorname{Im} h])$  is a metric [symplectic]  $\epsilon tt^*$ -bundle.  $\square$



# Chapter 4

## $\epsilon tt^*$ -geometry and $\epsilon$ pluriharmonic maps

In this section we are going to state and prove the central results which give the correspondence between  $\epsilon$ pluriharmonic maps and  $\epsilon tt^*$ -bundles. In the first section we consider  $\epsilon tt^*$ -bundles over simply connected manifolds. The case of non trivial fundamental group is dicussed in the second section. These results are part of [Sch6, Sch3]. The third section deals with a kind of rigidity result for  $tt^*$ -bundles over compact Kähler manifolds with finite fundamental group. Applying this rigidity result to simply connected compact special Kähler manifolds in the fourth section we obtain a special case of Lu's theorem for simply connected compact special Kähler manifolds.

### 4.1 The simply connected case

Let  $(M, J^\epsilon)$  be an  $\epsilon$ complex manifold and let  $f$  be a map  $f : M \rightarrow G_i(r)$ . Like in section 2.4.1 one regards the mapping  $A = f^{-1}df = -2d\tilde{f}$  as a flat connection  $A : TM \rightarrow \mathfrak{g}_i(r)$  on the bundle  $E = M \times \mathbb{R}^r$ .

**Theorem 4.1** (cf. [Sch6, Sch3]) *Let  $(M, J^\epsilon)$  be a simply connected  $\epsilon$ complex manifold. Let  $(E, D, S, g [, or])$  be a metric [an oriented unimodular metric]  $\epsilon tt^*$ -bundle where  $E$  has rank  $r$  and  $M$  dimension  $n$ .*

*Then the matrix representing the metric  $g$  in a  $D^\theta$ -flat frame of  $E$   $f : M \rightarrow \text{Sym}_{p,q}^i(\mathbb{R}^r)$  induces an admissible  $\epsilon$ pluriharmonic map  $\tilde{f} : M \xrightarrow{f} \text{Sym}_{p,q}^i(\mathbb{R}^r) \xrightarrow{\sim} S^i(p, q)$ , where  $S^i(p, q)$  carries the metric induced by the bi-invariant pseudo-Riemannian trace-form on  $\mathfrak{g}_i(r)$ .*

*Let  $s'$  be another  $D^\theta$ -flat frame. Then  $s' = s \cdot U$  for a constant matrix and the  $\epsilon$ pluriharmonic map associated to  $S'$  is  $f' = U^t f U$ .*

**Remark 4.1** (cf. [CS1, Sch6, Sch3])

*Before proving the theorem we make some remarks on the condition that the map  $\tilde{f}$  is admissible. Let  $x \in M$  and  $\tilde{f}(x) = uo$ . If  $d\tilde{f}(T_x^{1,0}M)$  consist of commuting matrices, then*

$dL_u^{-1}d\tilde{f}(T_x^{1,0}M)$  is commutative, too. This follows from the fact, that

$$dL_u : T_oS^i(p, q) \rightarrow T_{u_o}S^i(p, q) = T_{\tilde{f}(x)}S^i(p, q)$$

equals

$$Ad_u : sym^i(p, q) = sym^i(I_{p,q}) \rightarrow sym^i(u \cdot I_{p,q}) = sym^i(\tilde{f}(x)),$$

which preserves the Lie-bracket.

*Proof:* Using remark 3.1.1) it suffices to prove the case  $\theta = \pi$  for  $\epsilon = -1$  or  $\theta = 0$  for  $\epsilon = 1$ .

We first consider a metric  $\epsilon tt^*$ -bundle  $(E, D, S, g)$ .

Let  $s := (s_1, \dots, s_r)$  be a  $D^\theta$ -flat frame of  $E$  (i.e.  $Ds = -\epsilon Ss$ ),  $f$  the matrix  $g(s_k, s_l)$  and further  $S^s$  the matrix-valued one-form representing  $S$  in the frame  $s$ . For  $X \in \Gamma(TM)$  we get:

$$\begin{aligned} X(f) &= Xg(s, s) = g(D_X s, s) + g(s, D_X s) & (4.1.1) \\ &= -\epsilon(g(S_X s, s) + g(s, S_X s)) \\ &= -2\epsilon g(S_X s, s) = -2\epsilon f \cdot S^s(X) = -2\epsilon f \cdot S_X^s. \end{aligned}$$

Consequently  $A_X = -2\epsilon S_X^s$ . We now prove the  $\epsilon$ pluriharmonicity using

$$d^D S(X, Y) = D_X(S_Y) - D_Y(S_X) - S_{[X, Y]} = 0, \quad (4.1.2)$$

$$d^D S_{J^\epsilon}(X, Y) = D_X(S_{J^\epsilon Y}) - D_Y(S_{J^\epsilon X}) - S_{J^\epsilon[X, Y]} = 0. \quad (4.1.3)$$

The equation (4.1.3) implies

$$\begin{aligned} 0 = d^D S_{J^\epsilon}(J^\epsilon X, Y) &= D_{J^\epsilon X}(S_{J^\epsilon Y}) - \underbrace{\epsilon D_Y(S_X)}_{\stackrel{(4.1.2)}{=} \epsilon(D_X(S_Y) - S_{[X, Y]})} - S_{J^\epsilon[J^\epsilon X, Y]} \\ &= D_{J^\epsilon X}(S_{J^\epsilon Y}) - \epsilon D_X(S_Y) + \epsilon S_{[X, Y]} - S_{J^\epsilon[J^\epsilon X, Y]}. \end{aligned}$$

In local holomorphic coordinate fields  $X, Y$  on  $M$  we get in the frame  $s$

$$J^\epsilon X(S_{J^\epsilon Y}^s) - \epsilon X(S_Y^s) + [S_X^s, S_Y^s] - \epsilon[S_{J^\epsilon X}^s, S_{J^\epsilon Y}^s] = 0.$$

Now  $A = -2\epsilon S^s$  gives equation (2.4.9) and proves the  $\epsilon$ pluriharmonicity of  $f$ .

Using  $A_X = -2\epsilon S_X^s = -2d\tilde{f}(X)$ , we find the property of the differential, as  $S \wedge S$  is of type (1,1) by the  $\epsilon tt^*$ -equations, see proposition 3.1.

The last statement is obvious.

In the case of an oriented unimodular metric  $\epsilon tt^*$ -bundle  $(E, D, S, g, or)$  we can take the frame  $s$  to be oriented and of volume 1, with respect to the canonical  $D^\theta$ -parallel-metric volume  $\nu$ . Therefore the map  $f$  takes values in  $Sym_{p,q}^1(\mathbb{R}^r)$  and the above arguments show the rest.  $\square$

**Theorem 4.2** (cf. [Sch6, Sch3]) *Let  $(M, J^\epsilon)$  be a simply connected  $\epsilon$ complex manifold and put  $E = M \times \mathbb{R}^r$ .*

Then an  $\epsilon$ pluriharmonic map  $\tilde{f} : M \rightarrow S^i(p, q)$  gives rise to an  $\epsilon$ pluriharmonic map  $f : M \xrightarrow{\tilde{f}} S^i(p, q) \xrightarrow{\sim} \text{Sym}_{p,q}^i(\mathbb{R}^r) \subset G_i(r)$ .

If the map  $\tilde{f}$  is admissible, then the map  $f$  induces a metric  $\epsilon tt^*$ -bundle [an oriented unimodular metric  $\epsilon tt^*$ -bundle]  $(E, D = \partial - \epsilon S, S = \epsilon d\tilde{f}, g = \langle f \cdot, \cdot \rangle_{\mathbb{R}^r}$  [, or]) on  $M$  where  $\partial$  is the canonical flat connection on  $E$  and or is the canonical orientation on  $E$ .

**Remark 4.2** We observe, that for  $\epsilon$ Riemannian surfaces  $M = \Sigma$  the condition on the differential holds, since  $T^{1,0}\Sigma$  is one-dimensional.

*Proof:*

Let  $\tilde{f} : M \rightarrow S^i(p, q)$  be an  $\epsilon$ pluriharmonic map. Then by proposition 2.6 we know, that  $f : M \xrightarrow{\sim} \text{Sym}_{p,q}^i(\mathbb{R}^r) \subset G_i(r)$  is  $\epsilon$ pluriharmonic.

Since  $E = M \times \mathbb{R}^r$ , we can regard sections of  $E$  as  $r$ -tuples of  $C^\infty(M, \mathbb{R})$ -functions.

In the spirit of section 2.4.1 we regard the one-form  $A = -2d\tilde{f} = f^{-1}df = -2\epsilon S$  with values in  $\mathfrak{g}_i(r)$  as a connection on  $E$ . We remind, that the curvature of this connection vanishes (proposition 2.8).

a) First, we check the conditions on the metric:

**Lemma 4.1** *The connection  $D$  is compatible with the metric  $g$  and  $S$  is symmetric with respect to  $g$ .*

*Proof:* This is a direct computation with  $X \in \Gamma(TM)$  and  $v, w \in \Gamma(E)$  using the relations (\*)  $S = -\frac{1}{2}\epsilon f^{-1}df$ , (\*\*)  $df_x : T_x M \rightarrow T_{f(x)}\text{Sym}_{p,q}^i(\mathbb{R}^r) = \text{Sym}^i(\mathbb{R}^r)$  (compare remark 2.3) and  $g = \langle f \cdot, \cdot \rangle_{\mathbb{R}^r} = \langle \cdot, f \cdot \rangle_{\mathbb{R}^r}$  which follows from  $f : M \rightarrow \text{Sym}_{p,q}^i(\mathbb{R}^r)$  :

$$\begin{aligned} X(g(v, w)) &= X(\langle f v, w \rangle_{\mathbb{R}^r}) = \langle X(f)v, w \rangle_{\mathbb{R}^r} + \langle f(\partial_X v), w \rangle_{\mathbb{R}^r} + \langle f v, \partial_X w \rangle_{\mathbb{R}^r} \\ &\stackrel{(**)}{=} \frac{1}{2} \langle X(f)v, w \rangle_{\mathbb{R}^r} + \frac{1}{2} \langle v, X(f)w \rangle_{\mathbb{R}^r} + \langle f(\partial_X v), w \rangle_{\mathbb{R}^r} + \langle f v, \partial_X w \rangle_{\mathbb{R}^r} \\ &= \frac{1}{2} \langle f \cdot f^{-1}(X(f))v, w \rangle_{\mathbb{R}^r} + \frac{1}{2} \langle v, f \cdot f^{-1}(X(f))w \rangle_{\mathbb{R}^r} \\ &\quad + \langle f \partial_X v, w \rangle_{\mathbb{R}^r} + \langle f v, \partial_X w \rangle_{\mathbb{R}^r} \\ &\stackrel{(*),(**)}{=} g(X.v - \epsilon S_X v, w) + g(v, X.w - \epsilon S_X w) \\ &= g(D_X v, w) + g(v, D_X w). \end{aligned}$$

For  $x \in M$   $d\tilde{f}_x$  takes by remark 2.3 values in  $\text{sym}^i(f(x))$ . This shows that  $S = \epsilon d\tilde{f}$  is symmetric with respect to  $g = \langle f \cdot, \cdot \rangle_{\mathbb{R}^r}$ .  $\square$

To finish the proof, we have to check the  $\epsilon tt^*$ -equations. The second  $\epsilon tt^*$ -equation

$$-\epsilon[S_X, S_Y] = [S_{J^\epsilon X}, S_{J^\epsilon Y}] \tag{4.1.4}$$

for  $S$  follows from the assumption that the image of  $T^{1,0}M$  under  $(d\tilde{f})^{\mathbb{C}^\epsilon}$  is Abelian. In fact, this is equivalent to  $[d\tilde{f}(J^\epsilon X), d\tilde{f}(J^\epsilon Y)] = -\epsilon[d\tilde{f}(X), d\tilde{f}(Y)] \forall X, Y \in TM$ .

$$\begin{aligned} d^D S(X, Y) &= [D_X, S_Y] - [D_Y, S_X] - S_{[X, Y]} \\ &= \partial_X(S_Y) - \partial_Y(S_X) - 2\epsilon[S_X, S_Y] - S_{[X, Y]} = 0 \end{aligned}$$



is equivalent to the vanishing of the curvature of  $A = -2\epsilon S$  interpreted as a connection on  $E$  (see proposition 2.8).

Finally one has for eholomorphic coordinate fields  $X, Y \in \Gamma(TM)$

$$\begin{aligned}
d^D S_{J^\epsilon}(J^\epsilon X, Y) &= [D_{J^\epsilon X}, S_{J^\epsilon Y}] - \epsilon [D_Y, S_X] \\
&= [\partial_{J^\epsilon X} - \epsilon S_{J^\epsilon X}, S_{J^\epsilon Y}] - \epsilon [\partial_Y - \epsilon S_Y, S_X] \\
&= \partial_{J^\epsilon X}(S_{J^\epsilon Y}) - \epsilon \partial_Y(S_X) - \epsilon [S_{J^\epsilon X}, S_{J^\epsilon Y}] - [S_X, S_Y] \\
&\stackrel{(4.1.4)}{=} -\frac{1}{2}\epsilon (\partial_{J^\epsilon X}(A_{J^\epsilon Y}) - \epsilon \partial_Y(A_X)) \\
&\stackrel{(2.4.8)}{=} -\frac{1}{2}\epsilon (\partial_{J^\epsilon X}(A_{J^\epsilon Y}) - \epsilon \partial_X(A_Y) - \epsilon [A_X, A_Y]) \\
&\stackrel{(4.1.4)}{=} -\frac{1}{2}\epsilon \left( \partial_{J^\epsilon X}(A_{J^\epsilon Y}) - \epsilon \partial_X(A_Y) - \frac{1}{2}\epsilon [A_X, A_Y] + \frac{1}{2}[A_{J^\epsilon X}, A_{J^\epsilon Y}] \right) \\
&\stackrel{(2.4.9)}{=} 0.
\end{aligned}$$

This shows the vanishing of the tensor  $d^D S_{J^\epsilon}$ .

It remains to show the curvature equation for  $D$ . We observe, that  $D + \epsilon S = \partial - \epsilon S + \epsilon S = \partial$  and that the connection  $\partial$  is flat, to find

$$0 = R_{X,Y}^{D+\epsilon S} = R_{X,Y}^D + \epsilon d^D S(X, Y) + [S_X, S_Y] \stackrel{d^D S=0}{=} R_{X,Y}^D + [S_X, S_Y].$$

- b) With the same proof as in part a) we get a metric  $ett^*$ -bundle. The orientation is given by the orientation of  $E = M \times \mathbb{R}^r$ .

It remains to check the condition on the trace of  $S$ . This property is clear, since in this case  $d\tilde{f}_x$  takes values in  $\text{sym}^1(f(x))$  for all  $x \in M$ .  $\square$

We want to emphasize the last result in case of metric  $tt^*$ -bundles with positive definite metric over a complex manifold  $(M, J)$ .

**Theorem 4.3** *Let  $(M, J)$  be a simply connected complex manifold and put  $E = M \times \mathbb{R}^r$ . Then a pluriharmonic map  $\tilde{f} : M \rightarrow S^i(r, 0)$  is admissible. Moreover, it induces a second pluriharmonic map  $f : M \xrightarrow{\tilde{f}} S^i(r, 0) \xrightarrow{\sim} \text{Sym}_{r,0}^i(\mathbb{R}^r) \subset G_i(r)$  and a metric  $ett^*$ -bundle [an oriented unimodular metric  $ett^*$ -bundle]  $(E, D = \partial + S, S = -d\tilde{f}, g = \langle f \cdot, \cdot \rangle_{\mathbb{R}^r} [, or])$  on  $M$  where  $\partial$  is the canonical flat connection on  $E$  and  $or$  is the canonical orientation of  $E$ .*

*Proof:* In the case of signature  $(r, 0)$  corollary 2.3 implies that any pluriharmonic map  $\tilde{f} : M \rightarrow S^i(r, 0)$  is admissible as required in theorem 4.2.  $\square$

In the situation of theorem 4.2 the two constructions are inverse in the following sense:

**Proposition 4.1**

1. *Let  $(E, D, S, g [, or])$  be a metric [an oriented unimodular metric]  $ett^*$ -bundle on an  $\epsilon$ -complex manifold  $(M, J^\epsilon)$  and let  $\tilde{f}$  be the associated epluriharmonic map constructed to a  $D^\theta$ -flat frame  $s$  in theorem 4.1. Then  $\tilde{f}$  is admissible and the metric*

[oriented unimodular metric]  $\epsilon tt^*$ -bundle  $(M \times \mathbb{R}^r, \tilde{D} = \partial - \epsilon \tilde{S}, \tilde{S}, \tilde{g}, [or])$  associated to  $\tilde{f}$  in theorem 4.2 is the representation of  $(E, D, S, g [or])$  in the frame  $s$ .

2. Given an  $\epsilon$ pluriharmonic map  $\tilde{f}$  from an  $\epsilon$ complex manifold  $(M, J^\epsilon)$  to  $S^i(p, q)$ , then one obtains via theorem 4.2 a metric [an oriented unimodular metric]  $\epsilon tt^*$ -bundle  $(M \times \mathbb{R}^r, D, S, g [or])$ . The  $\epsilon$ pluriharmonic map associated to this metric  $\epsilon tt^*$ -bundle is conjugated to the map  $\tilde{f}$  by a constant matrix in  $G_i(r)$ .

*Proof:* Using again remark 3.1.1) we can set  $\theta = \pi$  for  $\epsilon = -1$  or  $\theta = 0$  for  $\epsilon = 1$ .

1. The maps  $f, \tilde{f}$  and the metric  $\tilde{g} = \langle f \cdot, \cdot \rangle_{\mathbb{R}^r}$  express the metric  $g$  in the frame  $s$ . In the computations of theorem 4.1 and with theorem 4.2 one finds  $2\tilde{S} = -\epsilon A = -\epsilon f^{-1}df = 2S^s$ . From  $0 = D^\theta s = Ds + \epsilon Ss$  we obtain that the connection  $D$  in the frame  $s$  is just  $\partial - \epsilon S^s = \partial + \frac{A}{2} = \partial - \epsilon \tilde{S} = \tilde{D}$ .
2. To find the  $\epsilon$ pluriharmonic map associated to  $(M \times \mathbb{R}^r, D, S, g [or])$  we have to express the metric  $g$  in a  $D^\theta$ -flat frame  $s$ . But  $D^\theta = \partial - \epsilon S + \epsilon S = \partial$ . Hence we can take  $s$  as the standard-basis of  $\mathbb{R}^r$  and we get  $f$ . Every other basis gives a conjugated result.  $\square$

## 4.2 The general case

In this section we are going to transfer the results in the simply connected case to manifolds with non-trivial fundamental group.

**Definition 4.1** Let  $p : \tilde{M} \rightarrow M$  be the universal cover of an  $\epsilon$ complex manifold  $(M, J^\epsilon)$  with the pulled back  $\epsilon$ complex structure.

Let  $(E, D, S)$  be an  $\epsilon tt^*$ -bundle, then we define the pulled back  $\epsilon tt^*$ -bundle of  $(E, D, S)$  to be given by  $(p^*E, p^*D, p^*S)$ .

Let  $(E, D, S, g)$  be a metric  $\epsilon tt^*$ -bundle, then we define the pulled back metric  $\epsilon tt^*$ -bundle of  $(E, D, S, g)$  to be given by  $(p^*E, p^*D, p^*S, p^*g)$ .

Finally, let  $(E, D, S, g, or)$  be an oriented unimodular metric  $\epsilon tt^*$ -bundle, then we define the pulled back oriented unimodular metric  $\epsilon tt^*$ -bundle of  $(E, D, S, g, or)$  to be given by  $(p^*E, p^*D, p^*S, p^*g, p^*or)$ .

**Remark 4.3** The pulled back  $\epsilon tt^*$ -bundles, metric  $\epsilon tt^*$ -bundles and oriented unimodular metric  $\epsilon tt^*$ -bundles are  $\epsilon tt^*$ -bundles, metric  $\epsilon tt^*$ -bundles and oriented unimodular metric  $\epsilon tt^*$ -bundles respectively, as one checks easily. This motivates the above definition.

**Theorem 4.4** Let  $(M, J^\epsilon)$  be an  $\epsilon$ complex manifold.

Let  $(E, D, S, g [or])$  be a [an oriented unimodular] metric  $\epsilon tt^*$ -bundle where  $E$  has rank  $r$  and  $M$  dimension  $n$  and  $(p^*E, p^*D, p^*S, p^*g [p^*or])$  the corresponding pulled-back [oriented unimodular] metric  $\epsilon tt^*$ -bundle on the universal cover  $\tilde{M}$  of  $M$ .

Denote by  $f^* : \tilde{M} \rightarrow S^i(p, q)$  the  $\epsilon$ pluriharmonic map obtained from theorem 4.1 in the  $p^*D^\theta$ -flat frame  $p^*s$ , where  $s$  is a  $D^\theta$ -flat frame and  $f : M \rightarrow S^i(p, q)$  the map obtained

from the representation of  $g$  in the frame  $s$ . Then  $f^*$  is a  $\pi_1(M)$ -equivariant map (Here equivariant means by the left-action on  $M$  and via the holonomy on  $S^i(p, q)$ .) and the lift  $p^*f$  of  $f$ . In other words  $f$  is a twisted epluriharmonic map.

*Proof:* The equivariance follows, since we have pulled back all structures. If  $s$  is  $D^\theta$ -flat,  $p^*s$  is  $p^*D^\theta$ -flat, too.

The map  $f^*$  at  $\tilde{x} \in \tilde{M}$  with  $p(\tilde{x}) = x$  is given by

$$f^*(\tilde{x}) = p^*g(p^*s, p^*s)(x) = g_{p(\tilde{x})}(s \circ p(\tilde{x}), s \circ p(\tilde{x})) = f(x) = f \circ p(\tilde{x}) = p^*f(x).$$

□

**Theorem 4.5** *Let  $(M, J^\epsilon)$  be an  $\epsilon$ complex manifold,  $p : \tilde{M} \rightarrow M$  its universal covering with the pulled back  $\epsilon$ complex structure, also called  $J^\epsilon$ . Set  $E = \tilde{M} \times \mathbb{R}^r$ .*

*Let  $\tilde{f}^* : \tilde{M} \rightarrow S^i(p, q)$  be an admissible epluriharmonic map, which is equivariant with respect to a representation  $\rho : \pi_1(M) \rightarrow G_i(r)$  and  $f^* : \tilde{M} \rightarrow \text{Sym}_{p,q}^i(\mathbb{R}^r)$  the corresponding map. Then  $\tilde{f}^*$  induces by theorem 4.2 a [an unimodular oriented] metric  $ett^*$ -bundle  $(E, D = \partial - \epsilon S, S = \epsilon d\tilde{f}^*, g = \langle f^* \cdot, \cdot \rangle_{\mathbb{R}^r})$  on  $\tilde{M}$  where  $\partial$  is the canonical flat connection on  $E$ . This [oriented unimodular] metric  $ett^*$ -bundle induces a [an oriented unimodular] metric  $ett^*$ -bundle  $(F, D = \partial - \epsilon T, T, h)$  on  $M$ , such that the [unimodular oriented] metric  $ett^*$ -bundle  $(E, D = \partial - \epsilon S, S = \epsilon d\tilde{f}^*, g = \langle f^* \cdot, \cdot \rangle_{\mathbb{R}^r})$  is its pull back.*

*Proof:*

a) We want to regard the action of  $\pi_1(M)$  on  $E$ , given by

$$(\gamma, m, v) \in \pi_1(M) \times E \mapsto (\gamma.m, \rho(\gamma)v) =: \gamma.(m, v) \in E \quad (4.2.1)$$

which induces the action

$$(\gamma, m, A) \in \pi_1(M) \times \text{End}(E) \mapsto (\gamma.m, \rho(\gamma)A\rho(\gamma)^{-1}) =: \gamma.(m, A) \in \text{End}(E) \quad (4.2.2)$$

of  $\pi_1(M)$  on  $\text{End}(E)$ . The quotient of  $E$  by the action of  $\pi_1(E)$  gives a vector bundle  $F \rightarrow M$  over  $M$ .

The equivariance of the map  $\tilde{f}^* : \tilde{M} \rightarrow S(p, q)$  means for  $m \in \tilde{M}$  :

$$\tilde{f}^*(\gamma.m) = \rho(\gamma)\tilde{f}^*(m)\rho(\gamma)^{-1}, \quad (4.2.3)$$

which implies for  $X \in T_m\tilde{M}$ ,  $m \in \tilde{M}$

$$d\tilde{f}_{\gamma.m}^*(d\gamma X) = \rho(\gamma)d\tilde{f}_m^*(X)\rho(\gamma)^{-1}. \quad (4.2.4)$$

Equation (4.2.3) is the equivariance of  $g$  and equation (4.2.4) is the equivariance of  $S$ . Hence they descend to a metric  $h$  on  $F$  and an endomorphism field  $T$  on  $F$ , which is  $h$ -symmetric. Since  $\partial$  is  $\pi_1(M)$ -invariant, it defines connection on  $F$  and since  $S$  is equivariant  $D = \partial - \epsilon T$  defines connection on  $F$  which preserves  $h$ . With the same argument the family  $D^\theta = D + \cos_\epsilon(\theta)T + \sin_\epsilon(\theta)T_{J^\epsilon}$  defines a family of connections on  $F$  which is flat. Hence  $(F, D = \partial - \epsilon T, T, h)$  is a metric  $ett^*$ -bundle on  $F$  over  $M$ .

b) One gets the data  $(F, D = \partial - \epsilon T, T, h)$  as in part a). The orientation is given by the orientation of  $E = \tilde{M} \times \mathbb{R}^r$ , since  $\rho$  takes values in  $SL(r)$ . □

### 4.3 A rigidity result

In [Sch5] we showed a rigidity result, which will be used later to obtain a new proof of Lu's theorem [Lu] in the case of simply connected compact special Kähler manifolds.

**Theorem 4.6** *Let  $M$  be a compact Kähler manifold of dimension  $n$  with finite fundamental group  $\pi_1(M)$  (i.e., the universal cover of  $M$  is compact). Let  $(E, D, S, g)$  be a metric  $tt^*$ -bundle, where  $E$  has rank  $r$ , with positive definite metric  $g$ . Then  $(E, D, S, g)$  is trivial, i.e.  $S = 0$ ,  $D$  is flat and  $g$   $D^\theta$ -parallel.*

*Proof:* Pulling back all structures to the universal cover of  $M$  we suppose that  $M$  is simply connected.  $S = 0$  if and only if the same holds for its pull back. Let  $s$  be a  $D^\pi$ -flat frame of  $E$ . The associated pluriharmonic map  $\tilde{f} : M \rightarrow GL(r, \mathbb{R})/O(r)$  obtained from theorem 4.1 is constant by corollary 2.2. Hence, the representing matrix  $G^s$  of  $g$  in the frame  $s$  is constant. We recall the relation between the representation  $S^s$  of  $S$  in the frame  $s$  with  $G^s$  which we found in equation (4.1.1):

$$X(G^s) = 2G^s \cdot S_X^s.$$

This shows  $S^s = 0$  and consequently  $S = 0$  and  $D^\theta = D$  for all  $\theta \in \mathbb{R}$ . Hence  $D$  is flat and  $D^\theta g = 0$ . □

### 4.4 A special case of Lu's theorem

As a corollary of our rigidity result, theorem 4.6, we obtain Lu's theorem [Lu] for simply connected compact manifolds. Another proof of Lu's theorem was given in [BC1]. The authors immersed any simply connected special Kähler manifold  $M^n$  as a parabolic affine hypersphere into  $\mathbb{R}^{n+1}$  and obtained Lu's theorem from a result of Calabi and Pogorelov.

**Theorem 4.7** *Let  $(M, J, g, \nabla)$  be a simply connected compact special Kähler manifold of dimension  $n$ . Then  $M$  is a point.*

*Proof:* Using theorem 3.5 the data  $(TM, D = \nabla - S, S = \frac{1}{2}J\nabla J, g)$  defines a metric  $tt^*$ -bundle. Then theorem 4.6 yields  $S = 0$  and hence  $D = \nabla$ . From  $Dg = 0$  and the torsion-freeness of  $\nabla$  it follows that  $D$  is the Levi-Civita connection. Therefore  $M$  is Levi-Civita flat, compact and simply connected, i.e.  $M$  is a point. □



# Chapter 5

## The $\epsilon$ pluriharmonic maps associated to the above examples of $\epsilon tt^*$ -bundles

In this chapter we analyze and apply the correspondence between  $\epsilon tt^*$ -bundles and  $\epsilon$ pluriharmonic maps for the classes of solutions which were discussed in chapter 3. In addition we associate generalized  $\epsilon$ pluriharmonic maps to the geometries with non integrable  $\epsilon$ complex structures.

### 5.1 Solutions on the tangent bundle

This subsection is also subject of [Sch7, Sch8].

#### 5.1.1 The classifying map of a flat nearly $\epsilon$ Kähler manifold

In this section we consider simply connected almost  $\epsilon$ hermitian manifolds  $(M, J^\epsilon, g)$  endowed with a flat metric connection  $\nabla$  such that  $(\nabla, J^\epsilon)$  satisfies the nearly  $\epsilon$ Kähler condition.

In particular, simply connected flat nearly  $\epsilon$ Kähler manifolds  $(M^{2n}, J^\epsilon, g)$ , i.e. nearly  $\epsilon$ Kähler manifolds  $(M, J^\epsilon, g)$  with flat Levi-Civita connection  $\nabla^g$  are of this type.

Since  $(M, g, \nabla)$  is simply connected and flat, we may identify by fixing a  $\nabla$ -parallel frame  $s_0$  its tangent bundle  $TM$  with  $(M \times V, \langle \cdot, \cdot \rangle)$ , where  $V = \mathbb{C}_\epsilon^n = (\mathbb{R}^{2n}, j_0^\epsilon)$  is endowed with the standard scalar product  $\langle \cdot, \cdot \rangle$  of the same hermitian signature  $(p, q)$  as the hermitian metric  $g$  for  $\epsilon = -1$  and of symmetric signature  $(n, n)$  for  $\epsilon = 1$ .

The compatible  $\epsilon$ complex structure  $J^\epsilon$  defines via this identification a map

$$J^\epsilon : M \rightarrow \mathcal{J}^\epsilon(V, \langle \cdot, \cdot \rangle),$$

where  $\mathcal{J}^\epsilon(V, \langle \cdot, \cdot \rangle)$  is the set of  $\epsilon$ complex structures on  $V$  which are compatible with  $\langle \cdot, \cdot \rangle$  and the orientation of  $V = \mathbb{R}^{2n}$ . The differential geometry of this set was discussed in section 2.5.

**Theorem 5.1** *Let  $(M, J^\epsilon, g)$  be a simply connected almost  $\epsilon$ hermitian manifold endowed with a flat metric connection  $\nabla$  such that  $(\nabla, J^\epsilon)$  satisfies the nearly  $\epsilon$ Kähler condition,*

then  $(TM, D = \nabla - S, S = -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon), \omega = g(J^\epsilon, \cdot))$  defines a symplectic  $\epsilon$ tt\*-bundle and the matrix of  $J^\epsilon$  in a  $D^\theta$ -flat frame  $s^\theta = (s_i^\theta)$  defines an  $\mathbb{S}_\epsilon^1$ -pluriharmonic map  $\tilde{J}^{\epsilon^\theta} : M \rightarrow \mathcal{J}^\epsilon(V, \langle \cdot, \cdot \rangle) \rightarrow SO_0(2p, 2q)/U^\epsilon(p, q)$ .

In particular, given a nice connection  $D$  on  $M$  the map

$$\tilde{J}^{\epsilon^\theta} : (M, J^\epsilon, D) \rightarrow SO_0(2p, 2q)/U^\epsilon(p, q)$$

is  $\epsilon$ pluriharmonic.

*Proof:* We observe  $D^\theta g = 0$  since  $\nabla g = 0$ ,  $D^0 = \nabla$  and  $S_X^\theta := \cos_\epsilon(\theta)S_X + \sin_\epsilon(\theta)S_{J^\epsilon X}$  is skew-symmetric with respect to  $g$ . Therefore we can choose for each  $\theta \in \mathbb{R}$  the  $D^\theta$ -flat frame  $s^\theta$  orthonormal, such that  $s^{\theta=0} = s_0$ . This yields using  $DJ^\epsilon = 0$  (compare theorem 3.1 and lemma 3.1)

$$Xg(J^\epsilon s_i^\theta, s_j^\theta) = g(D_X^\theta(J^\epsilon s_i^\theta), s_j^\theta) = g((D_X^\theta J^\epsilon)s_i^\theta, s_j^\theta) = g([S_X^\theta, J^\epsilon]s_i^\theta, s_j^\theta) = -2g(J^\epsilon S_X^\theta s_i^\theta, s_j^\theta).$$

Let  $S^{s^\theta}$  and  $J^{\epsilon s^\theta}$  be the representation of  $S$  and  $J^\epsilon$  in the frame  $s^\theta$ , then

$$(J^{\epsilon s^\theta})^{-1}X(J^{\epsilon s^\theta}) = -2S^{s^\theta}$$

or

$$d\tilde{J}^{\epsilon^\theta} = (s^\theta)^{-1} \circ S^\theta \circ s^\theta,$$

where the frame  $s^\theta$  is seen as a map  $s^\theta : M \times V \rightarrow TM$ . This shows for  $X \in \Gamma(TM)$

$$\begin{aligned} d\tilde{J}^{\epsilon^\theta}(X) &= (s^\theta)^{-1} \circ S_X^\theta \circ (s^\theta) = (s^\theta)^{-1} \circ S_{\mathcal{R}_\theta X} \circ (s^\theta) \\ &= ((s^\theta)^{-1} s^0) \circ d\tilde{J}^\epsilon(\mathcal{R}_\theta X) \circ ((s^0)^{-1} s^\theta) \\ &= Ad_{\alpha_\theta}^{-1} \circ d\tilde{J}^\epsilon(\mathcal{R}_\theta X) = \Phi_\theta^{-1} \circ d\tilde{J}^\epsilon(\mathcal{R}_\theta X), \end{aligned}$$

where  $\alpha_\theta = (s^\theta)^{-1} s^0$  is the frame change from  $s^0$  to  $s^\theta$  and  $\Phi_\theta = Ad_{\alpha_\theta}$  which is parallel with respect to the Levi-Civita connection on  $SO_0(2p, 2q)/U^\epsilon(p, q)$ . This shows, that  $\tilde{J}^{\epsilon^\theta}$  is  $\mathbb{S}_\epsilon^1$ -pluriharmonic. Given a nice connection  $D$  on  $M$  theorem 2.2 shows that  $\tilde{J}^{\epsilon^\theta}$  is  $\epsilon$ pluriharmonic.  $\square$

We emphasize the nearly  $\epsilon$ Kähler setting:

**Corollary 5.1** *Let  $(M, J^\epsilon, g)$  be a flat nearly  $\epsilon$ Kähler manifold and  $(TM, \bar{\nabla} = \nabla^g - S, S = -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon), \omega(\cdot, \cdot) = g(J^\epsilon, \cdot))$  the associated symplectic  $\epsilon$ tt\*-bundle, then the matrix of  $J^\epsilon$  in a  $D^\theta$ -flat frame  $s^\theta = (s_i^\theta)$  defines an  $\mathbb{S}_\epsilon^1$ -pluriharmonic map  $\tilde{J}^{\epsilon^\theta} : M \rightarrow \mathcal{J}^\epsilon(V, \langle \cdot, \cdot \rangle) \rightarrow SO_0(2p, 2q)/U^\epsilon(p, q)$ .*

For nearly  $\epsilon$ Kähler manifolds we have more precise informations about the map  $\tilde{J}^{\epsilon^\theta}$ :

**Theorem 5.2** *Let  $(M, J^\epsilon, g)$  be a flat nearly  $\epsilon$ Kähler manifold and  $(TM, \bar{\nabla} = \nabla^g - S, S = -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon), \omega(\cdot, \cdot) = g(J^\epsilon, \cdot))$  the associated symplectic  $\epsilon$ tt\*-bundle. Then the connection  $\bar{\nabla}$  is nice and the matrix of  $J^\epsilon$  in a  $D^\theta$ -flat frame  $s^\theta = (s_i^\theta)$  defines an  $\epsilon$ pluriharmonic map  $\tilde{J}^{\epsilon^\theta} : (M, J^\epsilon, \bar{\nabla}) \rightarrow \mathcal{J}^\epsilon(V, \langle \cdot, \cdot \rangle) \rightarrow SO_0(2p, 2q)/U^\epsilon(p, q)$ . Moreover, the map  $\tilde{J}^{\epsilon^\theta}$  is harmonic.*

*Proof:* First we show, that  $\bar{\nabla}$  is nice. Therefore we rewrite the Nijenhuis tensor

$$\begin{aligned} N_{J^\epsilon}(X, Y) &= (\nabla_{J^\epsilon X} J^\epsilon)Y - (\nabla_{J^\epsilon Y} J^\epsilon)X - J^\epsilon(\nabla_X J^\epsilon)Y + J^\epsilon(\nabla_Y J^\epsilon)X \\ &= -4J^\epsilon(\nabla_X J^\epsilon)Y, \end{aligned}$$

where the second equality follows from the nearly  $\epsilon$ Kähler condition and by

$$(\nabla_{J^\epsilon X} J^\epsilon)Y = -(\nabla_Y J^\epsilon)J^\epsilon X = J^\epsilon(\nabla_Y J^\epsilon)X = -J^\epsilon(\nabla_X J^\epsilon)Y.$$

But the torsion of  $\bar{\nabla}$  is by equation (1.3.3)

$$T^{\bar{\nabla}}(X, Y) = \epsilon J^\epsilon(\nabla_X J^\epsilon)Y.$$

This shows that  $\bar{\nabla}$  is nice.

By corollary 5.1 the map  $\tilde{J}^{\epsilon^\theta}$  is  $\mathbb{S}_\epsilon^1$ -pluriharmonic. Since  $\bar{\nabla}$  is nice, theorem 2.2 implies that  $\tilde{J}^{\epsilon^\theta}$  is  $\epsilon$ pluriharmonic. From the skew-symmetry of  $S$  and proposition 2.5 we obtain that  $\tilde{J}^{\epsilon^\theta}$  is harmonic.  $\square$

## 5.2 The dual Gauß map of a special $\epsilon$ Kähler manifold with torsion

In this subsection we consider a simply connected almost  $\epsilon$ hermitian manifold  $(M, J^\epsilon, g)$  with a flat connection  $\nabla$ , such that  $(\nabla, J^\epsilon)$  is special and the two-form  $\omega = g(J^\epsilon \cdot, \cdot)$  is  $\nabla$ -parallel.

Using the flat connection  $\nabla$  we identify by fixing a  $\nabla$ -parallel symplectic frame  $s_0$  the tangent space  $(TM, \omega)$  with  $(M \times V, \omega_0)$  where  $V = \mathbb{R}^{2n}$  and  $\omega_0$  is its standard symplectic form.

The compatible  $\epsilon$ complex structure  $J^\epsilon$  is seen as a map

$$J^\epsilon : M \rightarrow \mathcal{J}^\epsilon(V, \omega_0),$$

where  $\mathcal{J}^\epsilon(V, \omega_0)$  is the set of  $\epsilon$ complex structures on  $V$  which are compatible with  $\omega_0$ .

The differential geometry of this set was discussed in section 2.5.

Recall, that under the above assumptions  $(TM, D = \nabla - S, S = -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon), g)$  defines a metric  $\epsilon tt^*$ -bundle. Analogous to the last section we obtain:

**Theorem 5.3** *Let  $(M, J^\epsilon, g)$  be a simply connected almost  $\epsilon$ hermitian manifold with a flat connection  $\nabla$ , such that  $(\nabla, J^\epsilon)$  is special and the two-form  $\omega = g(J^\epsilon \cdot, \cdot)$  is  $\nabla$ -parallel and let  $(TM, D = \nabla - S, S = -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon), g)$  be the associated metric  $\epsilon tt^*$ -bundle. Then the matrix of  $J^\epsilon$  in a  $D^\theta$ -flat frame  $s^\theta = (s_i^\theta)$  defines an  $\mathbb{S}_\epsilon^1$ -pluriharmonic map  $\tilde{J}^{\epsilon^\theta} : M \rightarrow \mathcal{J}^\epsilon(V, \omega_0) \rightarrow \text{Sp}(\mathbb{R}^{2n})/U^\epsilon(p, q)$ .*

*In particular, given a nice connection  $D$  on  $(M, J^\epsilon)$  then the map  $\tilde{J}^{\epsilon^\theta} : (M, J^\epsilon, D) \rightarrow \text{Sp}(\mathbb{R}^{2n})/U^\epsilon(p, q)$  is  $\epsilon$ pluriharmonic.*



*Proof:* Since  $D^0\omega = \nabla\omega = (D + S)\omega = 0$  and  $S_X^\theta := \cos_\epsilon(\theta)S_X + \sin_\epsilon(\theta)S_{J^\epsilon X}$  is skew-symmetric with respect to  $\omega$ , we obtain  $D\omega = 0$  and  $D^\theta\omega = 0$ . Therefore we can choose for each  $\theta \in \mathbb{R}$  the  $D^\theta$ -parallel frame  $s^\theta$  as a symplectic frame, such that  $s^{\theta=0} = s_0$ . This yields using  $DJ^\epsilon = 0$  (compare theorem 3.1 and lemma 3.1)

$$X\omega(J^\epsilon s_i^\theta, s_j^\theta) = \omega(D_X^\theta(J^\epsilon s_i^\theta), s_j^\theta) = \omega((D_X^\theta J^\epsilon) s_i^\theta, s_j^\theta) = \omega([S_X^\theta, J^\epsilon] s_i^\theta, s_j^\theta) = -2\omega(J^\epsilon S_X^\theta s_i^\theta, s_j^\theta).$$

Let  $S^{s^\theta}$  and  $J^{\epsilon s^\theta}$  be the representation of  $S$  and  $J^\epsilon$  in the frame  $s^\theta$ , then

$$(J^{\epsilon s^\theta})^{-1}X(J^{\epsilon s^\theta}) = -2S^{s^\theta}$$

or

$$d\tilde{J}^{\epsilon^\theta} = (s^\theta)^{-1} \circ S^\theta \circ s^\theta,$$

where the frame  $s^\theta$  is seen as a map  $s^\theta : M \times V \rightarrow TM$ . This shows for  $X \in \Gamma(TM)$

$$\begin{aligned} d\tilde{J}^{\epsilon^\theta}(X) &= (s^\theta)^{-1} \circ S_X^\theta \circ (s^\theta) = (s^\theta)^{-1} \circ S_{\mathcal{R}_\theta X} \circ (s^\theta) \\ &= ((s^\theta)^{-1} s^0) \circ d\tilde{J}^{\epsilon^\theta}(\mathcal{R}_\theta X) \circ ((s^0)^{-1} s^\theta) \\ &= Ad_{\alpha_\theta}^{-1} \circ d\tilde{J}^{\epsilon^\theta}(\mathcal{R}_\theta X) = \Phi_\theta^{-1} \circ d\tilde{J}^{\epsilon^\theta}(\mathcal{R}_\theta X), \end{aligned}$$

where  $\alpha_\theta = (s^\theta)^{-1} s^0$  is the frame change from  $s^0$  to  $s^\theta$  and  $\Phi_\theta = Ad_{\alpha_\theta}$  which is parallel with respect to the Levi-Civita connection on  $\mathrm{Sp}(\mathbb{R}^{2n})/U^\epsilon(p, q)$ . In other words we have found an associated family. Given a nice connection  $D$  on  $(M, J^\epsilon)$  theorem 2.2 shows that  $\tilde{J}^{\epsilon^\theta}$  is  $\epsilon$ pluriharmonic.  $\square$

If the above  $\epsilon tt^*$ -bundle comes from a special  $\epsilon$ Kähler manifold we have the

**Theorem 5.4** *Let  $(M, J^\epsilon, g, \nabla)$  be a special  $\epsilon$ Kähler manifold and  $(TM, D = \nabla - S, S = -\frac{1}{2}\epsilon J^\epsilon \nabla J^\epsilon, g)$  the associated metric  $\epsilon tt^*$ -bundle, then the matrix of  $J^\epsilon$  in a  $D^\theta$ -flat frame  $s^\theta = (s_i^\theta)$  defines an  $\epsilon$ pluriharmonic map  $\tilde{J}^{\epsilon^\theta} : (M, J^\epsilon, D) \rightarrow \mathrm{Sp}(\mathbb{R}^{2n})/U^\epsilon(p, q)$ . Moreover,  $\tilde{J}^{\epsilon^\theta}$  is harmonic.*

*Proof:* By theorem 5.3 the map  $\tilde{J}^{\epsilon^\theta}$  is  $\mathbb{S}_\epsilon^1$ -pluriharmonic. In the special  $\epsilon$ Kähler case we know that  $D$  is the Levi-Civita connection and hence torsion-free. The  $\epsilon$ complex structure  $J^\epsilon$  is integrable and so  $N_{J^\epsilon} = 0$ . This means, that  $D$  is nice and theorem 2.2 shows that  $\tilde{J}^{\epsilon^\theta}$  is  $\epsilon$ pluriharmonic. Since  $S$  is trace-free we get from proposition 2.5 that  $\tilde{J}^{\epsilon^\theta}$  is harmonic.  $\square$

We remark, that the last result can also be obtained by observing, that the map  $\tilde{J}^{\epsilon^\theta}$  is  $\epsilon$ pluriharmonic and that the manifold  $M$  is  $\epsilon$ Kähler, as  $\epsilon$ pluriharmonic maps from  $\epsilon$ Kähler manifolds are harmonic.

### 5.3 The $\epsilon$ pluriharmonic map in the case of a special $\epsilon$ Kähler manifold

The results of this subsection were published in [CS1] for  $\epsilon = -1$  and in [Sch3] for  $\epsilon = 1$ .

### 5.3.1 The Gauß maps of a special Kähler manifold

Let  $(M, J, g, \nabla)$  be a special Kähler manifold of complex dimension  $n = k + l$  and of hermitian signature  $(k, l)$ , i.e.  $g$  has symmetric signature  $(2k, 2l)$ . Let  $(\tilde{M}, J, g, \nabla)$  be its universal covering with the pullback special Kähler structure, which is again denoted by  $(J, g, \nabla)$ . According to Theorem 1.2, there exists a (holomorphic) Kählerian Lagrangian immersion  $\phi : \tilde{M} \rightarrow V = T^*\mathbb{C}^n = \mathbb{C}^{2n}$ , which is unique up to a complex affine transformation of  $V$  with linear part in  $\mathrm{Sp}(\mathbb{R}^{2n})$ . We consider the *dual Gauß map* of  $\phi$

$$L : \tilde{M} \rightarrow \mathrm{Gr}_0^{k,l}(\mathbb{C}^{2n}), \quad p \mapsto L(p) := T_{\phi(p)}\tilde{M} := d\phi_p T_p\tilde{M} \subset V \quad (5.3.1)$$

into the Grassmannian of complex Lagrangian subspaces  $W \subset V$  of *signature*  $(k, l)$ , i.e. such that the restriction of  $\gamma$  to  $W$  is a hermitian form of signature  $(k, l)$ . The map  $L : \tilde{M} \rightarrow \mathrm{Gr}_0^{k,l}(\mathbb{C}^{2n})$  is in fact the dual of the *Gauß map*

$$L^\perp : \tilde{M} \rightarrow \mathrm{Gr}_0^{l,k}(\mathbb{C}^{2n}), \quad p \mapsto L(p)^\perp = \overline{L(p)} \cong L(p)^*. \quad (5.3.2)$$

Here  $L(p)^\perp$  stands for the  $\gamma$ -orthogonal complement of  $L(p)$  and the isomorphism  $\overline{L(p)} \cong L(p)^*$  is induced by the symplectic form  $\Omega$  on  $V = L(p) \oplus \overline{L(p)}$ .

#### Proposition 5.1

(i) The dual Gauß map  $L : \tilde{M} \rightarrow \mathrm{Gr}_0^{k,l}(\mathbb{C}^{2n})$  is holomorphic.

(ii) The Gauß map  $L^\perp : \tilde{M} \rightarrow \mathrm{Gr}_0^{l,k}(\mathbb{C}^{2n})$  is anti-holomorphic.

*Proof:* The holomorphicity of  $L$  follows from that of  $\phi$ . Part (ii) follows from (i), since  $L^\perp = \overline{L} : p \mapsto \overline{L(p)}$ .  $\square$

The Gauß maps  $L$  and  $L^\perp$  induce Gauß maps

$$L_M : M \rightarrow \Gamma \backslash \mathrm{Gr}_0^{k,l}(\mathbb{C}^{2n}) \quad (5.3.3)$$

$$L_M^\perp : M \rightarrow \Gamma \backslash \mathrm{Gr}_0^{l,k}(\mathbb{C}^{2n}) \quad (5.3.4)$$

into the quotient of the Grassmannian by the holonomy group  $\Gamma = \mathrm{Hol}(\nabla) \subset \mathrm{Sp}(\mathbb{R}^{2n})$  of the flat symplectic connection  $\nabla$ .

#### Corollary 5.2

(i) The dual Gauß map  $L_M : M \rightarrow \Gamma \backslash \mathrm{Gr}_0^{k,l}(\mathbb{C}^{2n})$  of  $M$  is holomorphic.

(ii) The Gauß map  $L_M^\perp : M \rightarrow \Gamma \backslash \mathrm{Gr}_0^{l,k}(\mathbb{C}^{2n})$  is anti-holomorphic.

If  $\Gamma \subset \mathrm{Sp}(\mathbb{R}^{2n})$  acts properly discontinuously on  $\mathrm{Gr}_0^{k,l}(\mathbb{C}^{2n})$  then  $\Gamma \backslash \mathrm{Gr}_0^{k,l}(\mathbb{C}^{2n})$  is a locally symmetric space of pseudo-hermitian type.

### 5.3.2 The local expression of the dual Gauß map

We shall now describe the dual Gauß map  $L$  in local holomorphic coordinates in neighborhoods of  $p_0 \in \tilde{M}$  and  $L(p_0) \in Gr_0^{k,l}(\mathbb{C}^{2n})$ . Applying a transformation from  $\mathrm{Sp}(\mathbb{R}^{2n})$ , if necessary, we can assume that  $L(p_0) \in U_0$ , where  $U_0$  was defined in section 2.5.2. We put  $U := L^{-1}(U_0)$ . The open subset  $U \subset \tilde{M}$  is a neighborhood of  $p_0$ .

Let  $\phi : \tilde{M} \rightarrow T^*\mathbb{C}^n$  be the Kählerian Lagrangian immersion. It defines a system of local (special) holomorphic coordinates

$$\varphi := \pi_{(z)} \circ \phi|_U : U \xrightarrow{\sim} U' \subset \mathbb{C}^n, \quad p \mapsto (z^1(\phi(p)), \dots, z^n(\phi(p))) \quad (5.3.5)$$

with  $\pi_{(z)}$  as defined in section 2.5.2.

This yields the following commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{L} & U_0 \\ \varphi \downarrow & & \downarrow C \\ U' & \xrightarrow{L_U} & \mathrm{Sym}_{k,l}(\mathbb{C}^n), \end{array} \quad (5.3.6)$$

where the vertical arrows are holomorphic diffeomorphisms and  $L_U$  at  $z = (z^1, \dots, z^n)$  is given by

$$L_U(z) = (F_{ij}(z)) := \left( \frac{\partial^2 F(z)}{\partial z^i \partial z^j} \right). \quad (5.3.7)$$

Here  $F = F(z)$  is a holomorphic function on  $U' \subset \mathbb{C}^n$ , called the prepotential, determined, up to a constant, by the equations

$$w_j(\phi(p)) = \left. \frac{\partial F}{\partial z^j} \right|_{z(\phi(p))}. \quad (5.3.8)$$

Summarizing, we obtain the following proposition:

**Proposition 5.2** *The dual Gauß map  $L$  has the following coordinate expression*

$$L_U = C \circ L \circ \varphi^{-1} = (F_{ij}), \quad (5.3.9)$$

where  $\varphi : U \rightarrow \mathbb{C}^n$  is the (special) holomorphic chart of  $\tilde{M}$  associated to the Kählerian Lagrangian immersion  $\phi$ , see equation (5.3.5), and  $C : U_0 \rightarrow \mathrm{Sym}(\mathbb{C}^n)$  is the holomorphic chart of  $Gr_0^{k,l}(\mathbb{C}^{2n})$  constructed in equation (2.5.10).

### 5.3.3 The special Kähler metric in affine coordinates

As before, let  $(M, J, g, \nabla)$  be a special Kähler manifold of hermitian signature  $(k, l)$ ,  $k + l = n = \dim_{\mathbb{C}} M$ , and  $(\tilde{M}, J, g, \nabla)$  its universal covering. As in chapter 4, we shall now consider the metric  $g$  in a  $\nabla$ -parallel frame. Such a frame is provided by the Kählerian Lagrangian immersion  $\phi : \tilde{M} \rightarrow V$ . In fact, any point  $p \in \tilde{M}$  has a neighborhood in which the functions  $\tilde{x}^i := \mathrm{Re} z^i \circ \phi$ ,  $\tilde{y}_i := \mathrm{Re} w_i \circ \phi$ ,  $i = 1, \dots, n$ , form a system of local  $\nabla$ -affine coordinates. We recall that the  $\nabla$ -parallel Kähler form is given by  $\omega = 2 \sum d\tilde{x}^i \wedge d\tilde{y}_i$ .

This implies that the globally defined one-forms  $\sqrt{2}d\tilde{x}^i$ ,  $\sqrt{2}d\tilde{y}_i$  constitute a  $\nabla$ -parallel unimodular frame

$$(e^a)_{a=1,\dots,2n} = (e^1, \dots, e^{2n}) := (\sqrt{2}d\tilde{x}^1, \dots, \sqrt{2}d\tilde{x}^n, \sqrt{2}d\tilde{y}_1, \dots, \sqrt{2}d\tilde{y}_n) \quad (5.3.10)$$

of  $T^*\tilde{M}$  with respect to the metric volume form  $\nu = (-1)^{n+1} \frac{\omega^n}{n!} = 2^n d\tilde{x}^1 \wedge \dots \wedge d\tilde{y}_n$ . The dual frame  $(e_a)$  of  $T\tilde{M}$  is also  $\nabla$ -parallel and unimodular. The metric defines a smooth map

$$G : \tilde{M} \rightarrow \text{Sym}_{2k,2l}^1(\mathbb{R}^{2n}) = \{A \in \text{Mat}(2n, \mathbb{R}) \mid A^t = A, \det A = 1 \text{ has signature } (2k, 2l)\} \quad (5.3.11)$$

by

$$p \mapsto G(p) := (g_{ab}(p)) := (g_p(e_a, e_b)). \quad (5.3.12)$$

We will call  $G = (g_{ab})$  the *fundamental matrix* of  $\phi$ . As before, we identify

$$\text{Sym}_{2k,2l}^1(\mathbb{R}^{2n}) = \text{SL}(2n, \mathbb{R})/\text{SO}(2k, 2l). \quad (5.3.13)$$

This is a pseudo-Riemannian symmetric space. For conventional reasons, in this section,  $\text{SO}(2k, 2l) \subset \text{SL}(2n, \mathbb{R})$  is defined as the stabilizer of the symmetric matrix

$$E_o^{k,l} := \text{diag}(\mathbb{1}_k, -\mathbb{1}_l, \mathbb{1}_k, -\mathbb{1}_l). \quad (5.3.14)$$

The fundamental matrix induces a map

$$G_M : M \rightarrow \Gamma \backslash \text{Sym}_{2k,2l}^1(\mathbb{R}^{2n}) \quad (5.3.15)$$

into the quotient of  $\text{Sym}_{2k,2l}^1(\mathbb{R}^{2n})$  by the action of the holonomy group  $\Gamma = \text{Hol}(\nabla) \subset \text{Sp}(\mathbb{R}^{2n}) \subset \text{SL}(2n, \mathbb{R})$ . The target  $\Gamma \backslash \text{Sym}_{2k,2l}^1(\mathbb{R}^{2n})$  is a pseudo-Riemannian locally symmetric space, provided that  $\Gamma$  acts properly discontinuously.

**Theorem 5.5** *The fundamental matrix*

$$G : \tilde{M} \rightarrow \text{Sym}_{2k,2l}^1(\mathbb{R}^{2n}) = \text{SL}(2n, \mathbb{R})/\text{SO}(2k, 2l) \quad (5.3.16)$$

takes values in the totally geodesic submanifold

$$i : Gr_0^{k,l}(\mathbb{C}^{2n}) = \text{Sp}(\mathbb{R}^{2n})/\text{U}(k, l) \hookrightarrow \text{SL}(2n, \mathbb{R})/\text{SO}(2k, 2l) \quad (5.3.17)$$

and coincides with the dual Gauß map  $L : \tilde{M} \rightarrow Gr_0^{k,l}(\mathbb{C}^{2n})$  in the sense that

$$G = i \circ L.$$

*Proof:* The proof will follow from a geometric description of the inclusion  $i$ . To any Lagrangian subspace  $W \in Gr_0^{k,l}(\mathbb{C}^{2n})$  we can associate the scalar product

$$g^W := \text{Re } \gamma|_W$$

of signature  $(2k, 2l)$  on  $W \subset V$ . The projection onto the real points

$$\text{Re} : V = T^*\mathbb{C}^n \rightarrow T^*\mathbb{R}^n = \mathbb{R}^{2n}, \quad v \mapsto \text{Re } v = \frac{1}{2}(v + \bar{v}) \quad (5.3.18)$$

induces an isomorphism of real vector spaces  $W \xrightarrow{\sim} \mathbb{R}^{2n}$  the inverse of which we denote by  $\psi = \psi_W$ .

We claim that

$$i(W) = \psi^* g^W =: (g_{ab}^W) =: G^W. \quad (5.3.19)$$

To check this, it is sufficient to prove that the map

$$Gr_0^{k,l}(\mathbb{C}^{2n}) \ni W \mapsto G^W \in \text{Sym}_{2k,2l}^1(\mathbb{R}^{2n}) \quad (5.3.20)$$

is  $\text{Sp}(\mathbb{R}^{2n})$ -equivariant and maps the base point  $W_o$  with stabilizer  $U(k, l)$ , see (2.5.2), to the base point  $E_o^{k,l}$  with stabilizer  $\text{SO}(2k, 2l)$ , see (5.3.14). Let us verify that indeed  $G^{W_o} = E_o^{k,l}$ .

Using the definition of  $\gamma$ , one finds that in the basis of  $V$  given by

$$(e_j^\pm) := \left( \frac{\partial}{\partial z^j} \pm i \frac{\partial}{\partial w_j} \right) \quad (5.3.21)$$

the only non-vanishing components of  $\gamma$  are  $\gamma(e_j^\pm, e_j^\pm) = \pm 2$ . This shows that  $g^{W_o} = \text{Re } \gamma|_{W_o}$  is represented by the matrix  $2E_o^{k,l}$  with respect to the basis

$$(e_1^+, \dots, e_k^+, e_1^-, \dots, e_l^-, ie_1^+, \dots, ie_k^+, ie_1^-, \dots, ie_l^-). \quad (5.3.22)$$

In order to calculate  $G^{W_o} = (g_{ab}^{W_o}) = (g(\psi e_a, \psi e_b))$ , we need to pass from the real basis (5.3.22) of  $W_o$  to the real basis  $(\psi e_a)$ .

Recall that the real structure  $\kappa$  is complex conjugation with respect to the coordinates  $(z^j, w_i)$ . This implies that

$$\psi^{-1}(e_j^+) = \frac{\partial}{\partial x^j} = \sqrt{2}e_j, \quad \psi^{-1}(ie_j^+) = -\frac{\partial}{\partial y_j} = -\sqrt{2}e_{n+j}, \quad j = 1, \dots, k, \quad (5.3.23)$$

$$\psi^{-1}(e_j^-) = \frac{\partial}{\partial x^j} = \sqrt{2}e_j, \quad \psi^{-1}(ie_j^-) = \frac{\partial}{\partial y_j} = \sqrt{2}e_{n+j}, \quad j = 1, \dots, l \quad (5.3.24)$$

and shows that  $G^{W_o} = E_o^{k,l}$ .

It remains to check the equivariance of  $W \mapsto G^W = \psi_W^* g$ . Using the definition of the map  $\psi = \psi_W : \mathbb{R}^{2n} \rightarrow W$ , one easily checks that, under the action of  $\Lambda \in \text{Sp}(\mathbb{R}^{2n})$ ,  $\psi$  transforms as

$$\psi_{\Lambda W} = \Lambda \circ \psi_W \circ \Lambda^{-1}|_{\mathbb{R}^{2n}}. \quad (5.3.25)$$

From this we deduce the transformation law of  $G^W$ :

$$G^{\Lambda W} = \psi_{\Lambda W}^* g^{\Lambda W} = (\Lambda^{-1})^* \psi_W^* \Lambda^* g^{\Lambda W} = (\Lambda^{-1})^* \psi_W^* g^W = (\Lambda^{-1})^* G^W = \Lambda \cdot G^W. \quad (5.3.26)$$

The above claim (5.3.19), together with the fact that

$$g^{L(p)} = g_p \quad \text{and} \quad G^{L(p)} = G(p) \quad (5.3.27)$$

for all  $p \in \tilde{M}$ , implies that

$$i(L(p)) = G^{L(p)} = G(p). \quad (5.3.28)$$

□

**Corollary 5.3** *The fundamental matrix*

$$G : \tilde{M} \rightarrow \text{Sym}_{2k,2l}^1(\mathbb{R}^{2n}) \quad (5.3.29)$$

*is pluriharmonic.*

*Proof:* The map  $G = i \circ L$  is the composition of the holomorphic map  $L : \tilde{M} \rightarrow Gr_0^{k,l}(\mathbb{C}^{2n})$  with the totally geodesic inclusion  $Gr_0^{k,l}(\mathbb{C}^{2n}) \subset \text{Sym}_{2k,2l}^1(\mathbb{R}^{2n})$ . The composition of a holomorphic map with a totally geodesic map is pluriharmonic.  $\square$

### 5.3.4 The Gauß maps of a special para-Kähler manifold

Now we are going to introduce the Gauß maps of a special para-Kähler manifold, which are the para-complex analogue of the Gauß maps introduced in section 5.3.2 and were introduced in [Sch3].

Let  $(M, \tau, g, \nabla)$  be a special para-Kähler manifold of para-complex dimension  $n$ . Consequently the metric  $g$  has signature  $(n, n)$ . Let  $(\tilde{M}, \tau, g, \nabla)$  be the universal cover of  $M$  with the pull-back special para-Kähler structure, which we denote again by  $(\tau, g, \nabla)$ . According to Theorem 1.3, there exists a (para-holomorphic) Kählerian Lagrangian immersion  $\Phi : \tilde{M} \rightarrow V = C^{2n} = T^*C^n$ , which is unique up to an affine transformation of  $V$  with linear part in  $\text{Aut}(V, \Omega, \cdot) = \text{Sp}(\mathbb{R}^{2n})$ .

We consider the **dual Gauß map** of  $\phi$ , i.e.

$$L : \tilde{M} \rightarrow \text{Gr}_0^n(C^{2n}), \quad p \mapsto L(p) := T_{\phi(p)}\tilde{M} := d\phi_p T_p\tilde{M} \subset V$$

into the Grassmannian  $\text{Gr}_0^n(C^{2n})$  of para-complex Lagrangian subspaces  $W \subset V$  of signature  $(n, n)$ , i.e.  $g_V = \text{Re } \gamma$  restricted to  $W$  has signature  $(n, n)$ . The map  $L : \tilde{M} \rightarrow \text{Gr}_0^n(C^{2n})$  is in fact the dual of the **Gauß map**

$$L^\perp : \tilde{M} \rightarrow \text{Gr}_0^n(C^{2n}), \quad p \mapsto L(p)^\perp = \bar{L}(p) \cong L(p)^*.$$

With  $L(p)^\perp$  we mean the  $\gamma$ -orthogonal complement of  $L(p)$  and the isomorphism  $L(p)^\perp \cong L(p)^*$  is induced by the symplectic form  $\Omega$  on  $V = L(p) \oplus \bar{L}(p)$ . The structure of a para-complex manifold on  $\text{Gr}_0^n(C^{2n})$  was introduced in section 2.5.3.

#### Proposition 5.3

(i) *The dual Gauß map  $L : \tilde{M} \rightarrow \text{Gr}_0^n(C^{2n})$  is para-holomorphic.*

(ii) *The Gauß map  $L^\perp : \tilde{M} \rightarrow \text{Gr}_0^n(C^{2n})$  is anti-para-holomorphic.*

*Proof:* The para-holomorphicity of  $L$  follows from that of  $\phi$  and part (ii) follows from  $L^\perp = \bar{L} : p \mapsto \bar{L}(p)$ .  $\square$

The Gauß maps  $L$  and  $L^\perp$  induce Gauß maps

$$\begin{aligned} L : M &\rightarrow \Gamma \backslash \text{Gr}_0^n(C^{2n}), \\ L^\perp : M &\rightarrow \Gamma \backslash \text{Gr}_0^n(C^{2n}) \end{aligned}$$

into the quotient of the Grassmannian by the holonomy group  $\Gamma \subset \text{Hol}(\nabla) \subset \text{Sp}(\mathbb{R}^{2n})$  of the flat symplectic connection  $\nabla$ . This yields the

**Corollary 5.4**

- (i) The dual Gauß map  $L_M : M \rightarrow \Gamma \backslash Gr_0^n(C^{2n})$  is para-holomorphic.
- (ii) The Gauß map  $L_M^\perp : M \rightarrow \Gamma \backslash Gr_0^n(C^{2n})$  is anti-para-holomorphic.

If  $\Gamma \subset Sp(\mathbb{R}^{2n})$  acts properly discontinuously on  $Gr_0^n(C^{2n})$  then  $\Gamma \backslash Gr_0^n(C^{2n})$  is a locally symmetric space and a para-Kähler manifold.

**5.3.5 The local expression of the dual Gauß map**

We now describe the dual Gauß map  $L$  in local para-holomorphic coordinates of  $p_0 \in \tilde{M}$  and  $L(p_0) \in Gr_0^n(C^{2n})$ . Utilizing a transformation of  $Sp(\mathbb{R}^{2n})$ , if necessary, we can assume  $L(p_0) \in U_0$ . For the definition of  $U_0$  we refer to section 2.5.3. We put  $U := L^{-1}(U_0)$ . The set  $U \subset \tilde{M}$  is an open neighborhood of  $p_0$ .

Let  $\phi : \tilde{M} \rightarrow T^*C^n$  be the para-Kählerian Lagrangian immersion. It defines a system of local (special) para-holomorphic coordinates

$$\varphi := \pi_{(z)} \circ \phi|_U : U \xrightarrow{\sim} U' \subset C^n, \quad p \mapsto (z^1(\phi(p)), \dots, z^n(\phi(p))), \quad (5.3.30)$$

where  $\pi_{(z)}$  was introduced in section 2.5.3.

This means that we have the following commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{L} & U_0 \\ \varphi \downarrow & & \downarrow C \\ U' & \xrightarrow{L_U} & \text{Sym}_{n,n}(C^n), \end{array} \quad (5.3.31)$$

where the vertical arrows are para-holomorphic diffeomorphisms and  $L_U$  at  $z = (z^1, \dots, z^n)$  is given by

$$L_U(z) = (F_{ij}(z)) := \left( \frac{\partial^2 F(z)}{\partial z^i \partial z^j} \right). \quad (5.3.32)$$

Here  $F(z)$  is a para-holomorphic function on  $U' \subset C^n$ , called prepotential (see [CMMS]), which is up to a constant determined by the equations

$$w_j((\phi(p))) = \frac{\partial F}{\partial z^j} \Big|_{z(\phi(p))}. \quad (5.3.33)$$

Summarizing, we obtain the proposition:

**Proposition 5.4** *The dual Gauß map  $L$  has the following coordinate expression*

$$L_U = C \circ L \circ \varphi^{-1} = (F_{ij}), \quad (5.3.34)$$

where  $\varphi : U \rightarrow C^n$  is the (special) para-holomorphic chart of  $\tilde{M}$  associated to the para-Kählerian Lagrangian immersion  $\phi$ , see equation (5.3.30), and  $C : U_0 \rightarrow \text{Sym}_{n,n}(C^n)$  is the para-holomorphic chart of  $Gr_0^n(C^{2n})$  constructed in equation (2.5.18).

### 5.3.6 The special para-Kähler metric in an affine frame

In this section we show that the para-pluriharmonic map associated to a para-Kähler manifold coincides with the dual Gauß map.

As above, let  $(M, \tau, g, \nabla)$  be a special para-Kähler manifold of dimension  $n = \dim_{\mathbb{C}} M$  and  $(\tilde{M}, \tau, g, \nabla)$  be its universal covering. Like in chapter 4 we now consider the metric  $g$  in a  $\nabla$ -parallel frame. Such a frame is provided by the para-Kählerian Lagrangian immersion  $\phi : \tilde{M} \rightarrow V$ . In fact, an arbitrary point  $p \in \tilde{M}$  has a neighborhood in which the functions  $\tilde{x}^i := \operatorname{Re} z^i \circ \phi$  and  $\tilde{y}_i := \operatorname{Re} w_i \circ \phi$ ,  $i = 1, \dots, n$  form a system of local  $\nabla$ -affine coordinates. We recall that the  $\nabla$ -parallel Kähler form is given by  $\omega = 2 \sum d\tilde{x}^i \wedge d\tilde{y}_i$ . Therefore the globally defined one-forms  $\sqrt{2}d\tilde{x}^i$  and  $\sqrt{2}d\tilde{y}_i$  constitute a  $\nabla$ -parallel unimodular frame

$$(e^a)_{a=1, \dots, 2n} := (\sqrt{2}d\tilde{x}^1, \dots, \sqrt{2}d\tilde{x}^n, \sqrt{2}d\tilde{y}_1, \dots, \sqrt{2}d\tilde{y}_n) \quad (5.3.35)$$

of  $T^*\tilde{M}$  with respect to the metric volume form  $\nu = (-1)^{n+1} \omega^n / n! = 2^n d\tilde{x}^1 \wedge \dots \wedge d\tilde{y}_n$ . The dual frame  $e_a$  of  $T\tilde{M}$  is also  $\nabla$ -parallel and unimodular. The metric  $g$  defines a smooth map

$$G : \tilde{M} \rightarrow \operatorname{Sym}_{n,n}^1(\mathbb{R}^{2n}) = \{A \in \operatorname{Mat}(2n, \mathbb{R}) \mid A^t = A, \det(A) = (-1)^n \text{ of signature } (n, n)\}$$

by

$$p \mapsto G(p) = (g_{ab}(p)) := (g_p(e_a, e_b)). \quad (5.3.36)$$

We call  $G = (g_{ab})$  the **fundamental matrix** of  $\phi$ . As before, we have the identification

$$\operatorname{Sym}_{n,n}^1(\mathbb{R}^{2n}) = SL(2n, \mathbb{R}) / SO(n, n)$$

of  $\operatorname{Sym}_{n,n}^1(\mathbb{R}^{2n})$  with a pseudo-Riemannian symmetric space.

The group  $SO(n, n) \subset SL(2n, \mathbb{R})$  is in this section considered as the stabilizer of the symmetric matrix

$$E_0^n = \operatorname{diag}(-\mathbb{1}_n, \mathbb{1}_n). \quad (5.3.37)$$

The fundamental matrix induces a map

$$G_M : M \rightarrow \Gamma \backslash \operatorname{Sym}_{n,n}^1(\mathbb{R}^{2n})$$

into the quotient of  $\operatorname{Sym}_{n,n}^1(\mathbb{R}^{2n})$  by the action of the holonomy group  $\Gamma = \operatorname{Hol}(\nabla) \subset \operatorname{Sp}(\mathbb{R}^{2n}) \subset SL(2n, \mathbb{R})$ . The target  $\Gamma \backslash \operatorname{Sym}_{n,n}^1(\mathbb{R}^{2n})$  is a pseudo-Riemannian locally symmetric space, provided that  $\Gamma$  acts properly discontinuously.

**Theorem 5.6** *The fundamental matrix*

$$G : \tilde{M} \rightarrow \operatorname{Sym}_{n,n}^1(\mathbb{R}^{2n}) = SL(2n, \mathbb{R}) / SO(n, n) \quad (5.3.38)$$

*takes values in the totally geodesic submanifold*

$$i : Gr_0^n(C^{2n}) = Sp(\mathbb{R}^{2n}) / U^\pi(C^n) \hookrightarrow SL(2n, \mathbb{R}) / SO(n, n)$$

*and coincides with the dual Gauß map  $L : \tilde{M} \rightarrow Gr_0^n(C^{2n})$  in the sense that*

$$G = i \circ L.$$



*Proof:* The proof follows from a geometric interpretation of the inclusion  $i$ . To any Lagrangian subspace  $W \in \text{Gr}_0^n(C^{2n})$  we associate the scalar product

$$g^W := \text{Re } \gamma|_W$$

of signature  $(n, n)$  on  $W \subset V$ . The projection onto the real points

$$\text{Re} : V = T^*C^n \mapsto T^*\mathbb{R}^n = \mathbb{R}^{2n}, v \mapsto \text{Re } v = \frac{1}{2}(v + \bar{v}) \quad (5.3.39)$$

induces an isomorphism of real vector spaces  $W \xrightarrow{\sim} \mathbb{R}^{2n}$  with inverse  $\psi = \psi_W$ .

We claim that

$$i(W) = \psi_W^* g =: (g_{ab}^W) =: G^W. \quad (5.3.40)$$

To check the claim, we have to show the  $\text{Sp}(\mathbb{R}^{2n})$ -equivariance of the map

$$\text{Gr}_0^n(C^{2n}) \ni W \mapsto G^W \in \text{Sym}_{n,n}^1(\mathbb{R}^{2n})$$

and that it maps the base point  $W_o$ , see equation (2.5.3), to  $E_0^n$  (equation (5.3.37)).

By the definition of  $\gamma$  we find for the basis

$$e_j^\pm := \frac{\partial}{\partial z^j} \pm e \frac{\partial}{\partial w_j} \quad (5.3.41)$$

of  $V$  that the only non-vanishing components of  $\gamma$  are  $\gamma(e_j^\pm, e_j^\pm) = \mp 2$ . This shows that  $g^{W_o}$  is represented by the matrix  $2E_0^n$  with respect to the real basis

$$(e_1^+, \dots, e_n^+, e e_1^+, \dots, e e_n^+). \quad (5.3.42)$$

In order to calculate  $G^{W_o} = (g_{ab}^{W_o}) = (g(\psi e_a, \psi e_b))$ , we need to pass from the real basis (5.3.42) to the real basis  $(\psi e_a)$  of  $W_o$ .

Recall that the real structure is the para-complex conjugation with respect to the coordinates  $(z^i, w_i)$ . This implies that

$$\psi^{-1}(e_j^+) = \frac{\partial}{\partial x^j} = \sqrt{2}e_j, \quad \psi^{-1}(e e_j^+) = \frac{\partial}{\partial y^j} = \sqrt{2}e_{n+j}, \quad j = 1, \dots, n, \quad (5.3.43)$$

$$\psi^{-1}(e_j^-) = \frac{\partial}{\partial x^j} = \sqrt{2}e_j, \quad \psi^{-1}(e e_j^-) = -\frac{\partial}{\partial y^j} = -\sqrt{2}e_{n+j}, \quad j = 1, \dots, n. \quad (5.3.44)$$

This shows that  $G^{W_o} = E_0^n$ .

It remains to show the equivariance of  $W \mapsto G^W = \psi_W^* g$ . Using the definition of the map  $\psi = \psi^W : \mathbb{R}^{2n} \rightarrow W$ , one easily checks that, under the action of  $\Lambda \in \text{Sp}(\mathbb{R}^{2n})$ ,  $\psi$  transforms as

$$\psi_{\Lambda W} = \Lambda \circ \Psi_W \circ \Lambda_{\mathbb{R}^{2n}}^{-1}. \quad (5.3.45)$$

This implies the transformation law for  $G^W$ :

$$G^{\Lambda W} = \psi_{\Lambda W}^* g^{\Lambda W} = (\Lambda^{-1})^* \psi_W^* \Lambda^* g^{\Lambda W} = (\Lambda^{-1})^* \psi_W^* g^W = (\Lambda^{-1})^* G^W = \Lambda \cdot G^W. \quad (5.3.46)$$

The above claim (5.3.40) and the fact

$$g^{L(p)} = g_p \text{ and } G^{L(p)} = G(p) \quad (5.3.47)$$

for all  $p \in \tilde{M}$  imply

$$i(L(p)) = G^{L(p)} = G(p). \quad (5.3.48)$$

□

**Corollary 5.5** *The fundamental matrix  $G : \tilde{M} \rightarrow \text{Sym}_{n,n}^1(\mathbb{R}^{2n})$  is para-pluriharmonic.*

*Proof:* In fact,  $G = i \circ L$  is the composition of a para-holomorphic map  $L : \tilde{M} \rightarrow \text{Gr}_0^n(C^{2n})$  with the totally geodesic inclusion  $\text{Gr}_0^n(C^{2n}) \subset \text{Sym}_{n,n}^1(\mathbb{R}^{2n})$ . The composition of a para-holomorphic map with a totally geodesic one is para-pluriharmonic.  $\square$

## 5.4 Variations of $\epsilon$ Hodge structures

### 5.4.1 The period map of a variation of $\epsilon$ Hodge structures

Like period domains describe  $\epsilon$ Hodge structures,  $\epsilon$ holomorphic maps into period domains describe variations of  $\epsilon$ Hodge structures, in the sense of the following proposition which is in the complex case due to Griffiths. We only consider the simply connected case:

**Proposition 5.5** *Let  $(M, J^\epsilon)$  be a simply connected  $\epsilon$ complex manifold and  $G/V$  the period domain classifying polarized  $\epsilon$ Hodge structures of given weight and  $\epsilon$ Hodge numbers, then giving a variation of  $\epsilon$ Hodge structures is equivalent to giving an  $\epsilon$ holomorphic map from  $M$  to  $G/V$  which satisfies the Griffiths transversality condition. Such maps are called period maps.*

The following result is known for strongly polarized complex variations of Hodge structures and will be generalized for variations of  $\epsilon$ Hodge structures of odd weight later in this work.

**Theorem 5.7** *(cf. [CMP] Theorem 14.4.1) Let  $f : M \rightarrow G/V$  be a period mapping and  $\pi : G/V \rightarrow G/K$ , as defined in section 2.7, the canonical map to the associated locally symmetric space. The  $\pi \circ f$  is pluriharmonic.*

### 5.4.2 The period map of a variation of $\epsilon$ Hodge structures from the viewpoint of $\epsilon tt^*$ -geometry

Let  $(E, \nabla, F^p)$  be a variation of  $\epsilon$ Hodge structures of odd weight  $w$  over the  $\epsilon$ complex base manifold  $(M, J^\epsilon)$  endowed with a polarization  $b$  where  $E$  has rank  $r$  and where  $f_p = \dim F_p$ . Denote by  $(E, D, S, g)$  the corresponding  $\epsilon tt^*$ -bundle constructed in proposition 3.8. We suppose, that  $M$  is simply connected.

Like in chapter 4 we examine the metric  $g$  in a  $D^0 = \nabla$ -parallel frame  $s$  of  $E$ . The metric  $g$  defines a smooth map

$$G : M \rightarrow \text{Sym}_{p,q}(\mathbb{R}^r) = \{A \in \text{Mat}(\mathbb{R}^r) \mid A = A^t \text{ and } A \text{ has signature } (p, q)\}. \quad (5.4.1)$$

In the complex case  $(p, q) = (2k, 2l)$  is the symmetric signature of  $g$ . We remark that for a variation of para-Hodge structures the metric  $g$  is forced to have split signature  $(p, q) = (n, n)$  with  $n = \frac{1}{2} \dim_{\mathbb{R}} H$ .

The map  $G$  will be called the **fundamental matrix** of the variation of  $\epsilon$ Hodge structures  $(E, \nabla, F^p)$  and as above  $\text{Sym}_{p,q}(\mathbb{R}^r)$  is identified with the pseudo-Riemannian symmetric space  $GL(r, \mathbb{R})/O(p, q)$ .

We recall that for odd weight each fibre of  $E$  has the structure of a symplectic vector space and consequently it holds  $\text{rk}_{\mathbb{R}} E = r = 2n \in 2\mathbb{N}$ .

**Theorem 5.8** *Let  $(E, \nabla, F^p)$  be a polarized variation of  $\epsilon$ Hodge structures of odd weight  $w$  with polarization  $b$  over the  $\epsilon$ complex base manifold  $(M, J^\epsilon)$ . Let  $r = 2n$  be the real rank of  $E$ .*

*Then the fundamental matrix  $G$  takes values in the totally geodesic submanifold*

$$i : Gr_0^{k,l}(\mathbb{C}^{2n}) = Sp(\mathbb{R}^{2n})/U(k, l) \rightarrow GL(r, \mathbb{R})/O(2k, 2l), \text{ for } \epsilon = -1, \quad (5.4.2)$$

$$i : Gr_0^n(\mathbb{C}^{2n}) = Sp(\mathbb{R}^{2n})/U^\pi(C^n) \rightarrow GL(r, \mathbb{R})/O(n, n), \text{ for } \epsilon = 1 \quad (5.4.3)$$

*and coincides with the map  $L$ , i.e.  $G = i \circ L : M \rightarrow GL(r, \mathbb{R})/O(p, q)$ .*

*Proof:* Given a point  $x \in M$  we put  $V = H_x^{\mathbb{C}^\epsilon}$  and  $V^{\mathbb{R}} = H_x \cong \mathbb{R}^r$ . To any polarized  $\epsilon$ Hodge structure  $F^p$  of odd weight  $w$  with polarization  $b$  the map  $L$  associated a Lagrangian subspace  $\mathcal{L} \in Gr_0^{k,l}(V)$  in the complex and a Lagrangian subspace  $\mathcal{L} \in Gr_0^n(V)$  in the para-complex case (see section 2.7). We define a scalar product  $g^{\mathcal{L}} = \text{Re } h|_{\mathcal{L}}$  on  $\mathcal{L} \subset V$ . The projection onto the real points

$$\text{Re} : V \rightarrow V^{\mathbb{R}} \quad (5.4.4)$$

induces an isomorphism  $\mathcal{L} \cong V^{\mathbb{R}}$ . Its inverse we call  $\Phi = \Phi_{\mathcal{L}} : V^{\mathbb{R}} \rightarrow \mathcal{L}$ .

We are going to prove

$$i(\mathcal{L}) = \Phi_{\mathcal{L}}^* g^{\mathcal{L}} =: G^{\mathcal{L}}. \quad (5.4.5)$$

We first show the  $Sp(\mathbb{R}^r)$  equivariance of the map

$$\mathcal{L} \mapsto G^{\mathcal{L}}. \quad (5.4.6)$$

From the definition of  $\Phi_{\mathcal{L}}$  we obtain with  $\Lambda \in Sp(\mathbb{R}^r)$  :

$$\Phi_{\Lambda\mathcal{L}} = \Lambda \circ \Phi_{\mathcal{L}} \circ \Lambda_{|\mathbb{R}^r}^{-1} \quad (5.4.7)$$

and from this the transformation law of  $G^{\mathcal{L}}$

$$G^{\Lambda\mathcal{L}} = \Phi_{\Lambda\mathcal{L}}^* g^{\Lambda\mathcal{L}} = (\Lambda^{-1})^* \Phi_{\mathcal{L}}^* \Lambda^* g^{\Lambda\mathcal{L}} = (\Lambda^{-1})^* \Phi_{\mathcal{L}}^* g^{\mathcal{L}} = (\Lambda^{-1})^* G^{\mathcal{L}} = \Lambda \cdot G^{\mathcal{L}}. \quad (5.4.8)$$

Let  $F_o^p$  be the reference flag of  $V_o^{\mathbb{C}^\epsilon} = H_o^{\mathbb{C}^\epsilon}$  with  $\dim F_o^p = f_p$ . We calculate  $G^{\mathcal{L}_o}$  in the basis  $\{f_o^i\}_{i=1}^{\dim(\mathcal{L}_o)}$  constructed in equation (2.7.2)

$$(G^{\mathcal{L}_o}(\text{Re } f_i, \text{Re } f_j)) = \mathbb{1}_{p,q}, \text{ after permutation.} \quad (5.4.9)$$

This yields

$$\Phi_{\mathcal{L}_o}^* g^{\mathcal{L}_o} = \mathbb{1}_{p,q}. \quad (5.4.10)$$

The proof is finished, since  $G(x) = G^{L(x)} = i(L(x))$ .  $\square$

**Corollary 5.6** *Let  $(E, \nabla, F^p)$  be a polarized variation of  $\epsilon$ Hodge structures of odd weight  $w$  with polarization  $b$  over the  $\epsilon$ complex base manifold  $(M, J^\epsilon)$ . Then the map  $L : M \rightarrow Gr_0^{k,l}(\mathbb{C}^r) = Sp(\mathbb{R}^r)/U^\epsilon(k, l)$  is  $\epsilon$ pluriharmonic.*

*Proof:* This follows from the  $\epsilon$ pluriharmonicity of the fundamental matrix  $G : M \rightarrow GL(r, \mathbb{R})/O(p, q)$ , since  $G = i \circ L$ , where  $i$  is a totally geodesic immersion and consequently, by corollary 2.1, the  $\epsilon$ pluriharmonicity of  $L$  is equivalent to that of  $G$ .  $\square$

The last theorem and the last corollary can be specialized for variations of Hodge structures (This means  $\epsilon = -1$ .), which are strongly polarized:

**Theorem 5.9** *Let  $(E, \nabla, F^p)$  be a strongly polarized variation of Hodge structures of odd weight  $w$  with polarization  $b$  over the complex base manifold  $(M, J)$ . Then the fundamental matrix  $G$  takes values in the totally geodesic submanifold*

$$i : Gr_0(\mathbb{C}^r) = Gr_0^{r,0}(\mathbb{C}^r) = Sp(\mathbb{R}^r)/U(r) \rightarrow GL(r, \mathbb{R})/O(r) \quad (5.4.11)$$

and coincides with the map  $L = \pi \circ \mathcal{P} : M \rightarrow G/K$ , i.e.  $G = i \circ L : M \rightarrow GL(r, \mathbb{R})/O(r)$ .

With the same argument as before, we obtain the

**Corollary 5.7** *Let  $(E, \nabla, F^p)$  be a strongly polarized variation of Hodge structures of odd weight  $w$  with polarization  $b$  over the complex base manifold  $(M, J)$ . Then the map  $L : M \rightarrow Gr_0(\mathbb{C}^r) = Gr_0^{r,0}(\mathbb{C}^r) = Sp(\mathbb{R}^r)/U(r)$  is pluriharmonic.*

## 5.5 $\epsilon$ Harmonic bundles

The complex version of this chapter was published in [Sch4].

Collecting our knowledge from the previous chapters we obtain the corollary:

**Corollary 5.8** *Let  $(E \rightarrow M, D, C, \bar{C}, h)$  be an  $\epsilon$ harmonic bundle of  $\epsilon$ complex rank  $r$  over the simply connected  $\epsilon$ complex manifold  $(M, J^\epsilon)$ , then the representation of  $g = \text{Re } h$  in a  $D^{(\lambda)}$ -flat frame defines an  $\epsilon$ pluriharmonic map  $\Phi_g : M \rightarrow S^\epsilon(2p, 2q)$  where we define*

$$S^\epsilon(2p, 2q) := \begin{cases} S(2p, 2q) & \text{for } \epsilon = -1, \\ S(r, r) & \text{for } \epsilon = 1, \end{cases}$$

where  $(p, q)$  with  $r = p + q$  is the hermitian signature of  $h$  for  $\epsilon = -1$ .

*Proof:* By theorem 3.6 the  $\epsilon$ harmonic bundle  $(E \rightarrow M, D, C, \bar{C}, h)$  induces a metric  $\epsilon tt^*$ -bundle  $(E, D, S, g = \text{Re } h)$  with  $S_X := C_Z + \bar{C}_{\bar{Z}}$  for  $X = Z + \bar{Z} \in TM$  and  $Z \in T^{1,0}M$ . The identity (3.4.1), i.e.  $D_X^{(\lambda)} = D_X^\alpha$  for  $\lambda = \cos_\epsilon(\alpha) + i\sin_\epsilon(\alpha) \in \mathbb{S}_\epsilon^1$  and theorem 4.1 prove the corollary.  $\square$

With our considerations about  $\epsilon$ pluriharmonic maps we are going to show the next theorem. First we introduce a notion:

$$\text{Herm}_{p,q}^\epsilon(\mathbb{C}^r) := \begin{cases} \text{Herm}_{p,q}(\mathbb{C}^r), & \text{for } \epsilon = -1, \\ \text{Herm}(\mathbb{C}^r), & \text{for } \epsilon = 1. \end{cases}$$

**Theorem 5.10** *Let  $(E \rightarrow M, D, C, \bar{C}, h)$  be an  $\epsilon$ harmonic bundle over the simply connected  $\epsilon$ complex manifold  $(M, J^\epsilon)$ . Then the representation of  $h$  in a  $D^{(\lambda)}$ -flat frame defines an  $\epsilon$ pluriharmonic map  $\phi_h : M \rightarrow \text{Herm}_{p,q}^\epsilon(\mathbb{C}^r)$ , which itself induces an admissible  $\epsilon$ pluriharmonic map  $\tilde{\phi}_h : M \rightarrow H^\epsilon(p, q)$  (The space  $H^\epsilon(p, q)$  was defined in equation (2.4.16)).*

*Proof:* The  $\epsilon$ pluriharmonicity of the map  $\phi_h$  follows from corollary 5.8 and propositions 2.11 and 2.14. For the second part we observe, that the differential of  $\mathcal{R} : \mathfrak{gl}_r(\mathbb{C}^\epsilon) \rightarrow \mathfrak{gl}_{2r}(\mathbb{R})$  is a homomorphism of Lie-algebras and therefore preserves the vanishing of the Lie-bracket.  $\square$

The following theorem gives the converse statement:

**Theorem 5.11** *Let  $(M, J^\epsilon)$  be a simply connected  $\epsilon$ complex manifold and  $E = M \times \mathbb{C}^r$ . An admissible  $\epsilon$ pluriharmonic map  $\tilde{\phi}_h : M \rightarrow H^\epsilon(p, q)$  induces an admissible  $\epsilon$ pluriharmonic map  $\tilde{\phi}_g = [i] \circ \tilde{\phi}_h : M \rightarrow S^\epsilon(2p, 2q)$  and an  $\epsilon$ harmonic bundle  $(E, D = \partial - \epsilon(C + \bar{C}), C = \epsilon(d\tilde{\phi}_h)^{1,0}, h = (\phi_h \cdot, \cdot)_{\mathbb{C}^r})$ , where  $\partial$  is the  $\epsilon$ complex linear extension on  $TM^{\mathbb{C}^\epsilon}$  of the flat connection on  $E = M \times \mathbb{C}^r$ .*

*If  $M = \Sigma$  is an  $\epsilon$ Riemannian surface, then every  $\epsilon$ pluriharmonic map  $\tilde{\phi}_h$  is admissible. If  $(M, J)$  is a complex manifold and the signature is  $(r, 0)$  or  $(0, r)$ , then every pluriharmonic map  $\tilde{\phi}_h$  is admissible.*

*Proof:* Due to proposition 2.11 and 2.14 the map  $\tilde{\phi}_g$  is  $\epsilon$ pluriharmonic. Hence one obtains from theorem 4.2 an  $\epsilon$ tt\*-bundle  $(E = M \times \mathbb{R}^{2r}, D = \partial - \epsilon S, S = \epsilon d\tilde{\phi}_g, g = \langle \phi_g \cdot, \cdot \rangle_{\mathbb{R}^{2r}})$ , since the condition on  $d\tilde{\phi}_g|_x$  is obtained as in theorem 5.10. We are now going to use the additional information, we have from the fact, that the map  $\tilde{\phi}_g$  comes from  $\tilde{\phi}_h$ , to show that  $(E, D = \partial - \epsilon(C + \bar{C}), C = \epsilon(d\tilde{\phi}_h)^{1,0}, h = (\phi_h \cdot, \cdot)_{\mathbb{C}^r})$  is an  $\epsilon$ harmonic bundle.

The  $\epsilon$ hermitian sesquilinear metric  $h$  is given by

$$h = g + \hat{i} \omega$$

with  $\omega = g(j^\epsilon \cdot, \cdot)$ . This is the standard relation between  $\epsilon$ hermitian metrics on  $\epsilon$ complex vector spaces and the  $\epsilon$ hermitian metrics on the underlying real vector spaces.

We observe  $Dj^\epsilon = [\partial - \epsilon S, j^\epsilon] = -\epsilon[S_X, j^\epsilon] = 0$ , because  $S$  is the derivation of a map from  $M$  to  $GL(r, \mathbb{C}^\epsilon)$  and hence commutes with  $j^\epsilon$ . Therefore  $D\omega = 0$  follows from  $Dg = 0$  and  $Dh = 0$  from  $D\omega = 0$  and  $Dg = 0$ .

From the definition of  $S$  and  $S_{J^\epsilon}$  in theorem 3.6, i.e.

$$\begin{aligned} S_X &= C_Z + \bar{C}_{\bar{Z}}, \\ S_{J^\epsilon X} &= C_{J^\epsilon Z} + \bar{C}_{J^\epsilon \bar{Z}} \end{aligned}$$

for  $X = Z + \bar{Z}$  and  $Z \in T^{1,0}M$  we obtain the definition of

$$2C_Z = S_X + \epsilon j^\epsilon S_{J^\epsilon X},$$

$$2\bar{C}_{\bar{Z}} = S_X - \epsilon j^\epsilon S_{J^\epsilon X}.$$

In addition we have the identity  $D_X^{(\lambda)} = D_X^\alpha$  for  $\lambda = \cos_\epsilon(\alpha) + \hat{i} \sin_\epsilon(\alpha) \in \mathbb{S}_\epsilon^1$  which again gives the equivalence between the flatness of  $D^{(\lambda)}$  and  $D^\alpha$ .

It remains to show

$$h(C_Z \cdot, \cdot) = h(\cdot, \bar{C}_{\bar{Z}} \cdot).$$

We recall the relations  $j^{\epsilon*}g = -\epsilon g$  and  $(*)g(j^\epsilon \cdot, \cdot) = -g(\cdot, j^\epsilon \cdot)$ , which implies the anti-symmetry of  $\omega = g(j^\epsilon \cdot, \cdot)$  and  $(*)'\omega(j^\epsilon \cdot, \cdot) = -\omega(\cdot, j^\epsilon \cdot)$ . Further we use the identities  $(**) [S, j^\epsilon] = [S_{J^\epsilon}, j^\epsilon] = 0$  and that  $(***)S, S_{J^\epsilon}$  are  $g$ -symmetric. Due to  $(**)$  and  $(***)$  we get  $(***')S, S_{J^\epsilon}$   $\omega$ -symmetric. These identities imply

$$\begin{aligned} 2h(C_Z \cdot, \cdot) &= g(S_X + \epsilon j^\epsilon S_{J^\epsilon X \cdot}, \cdot) + \hat{i}\omega(S_X + \epsilon j^\epsilon S_{J^\epsilon X \cdot}, \cdot) \\ &\stackrel{(*), (**), (***)}{=} g(\cdot, S_X - \epsilon j^\epsilon S_{J^\epsilon X \cdot}) + \hat{i}\omega(S_X + \epsilon j^\epsilon S_{J^\epsilon X \cdot}, \cdot) \\ &\stackrel{(*'), (**), (***)}{=} g(\cdot, S_X - \epsilon j^\epsilon S_{J^\epsilon X \cdot}) + \hat{i}\omega(\cdot, S_X - \epsilon j^\epsilon S_{J^\epsilon X \cdot}) \\ &= 2h(\cdot, \bar{C}_{\bar{Z}} \cdot). \end{aligned}$$

Using  $S = \epsilon d\tilde{\phi}_g = \epsilon d([i] \circ \tilde{\phi}_h) = \epsilon d\tilde{\phi}_h$  we find extending  $S$  on  $TM^{C_\epsilon}$  to  $S^{C_\epsilon}$  for  $Z \in T^{1,0}M$  the equations  $C_Z = S_Z^{C_\epsilon} = \epsilon d\tilde{\phi}_h(Z)$  and  $\bar{C}_{\bar{Z}} = \epsilon d\tilde{\phi}_h(\bar{Z})$ .  $\square$

In [Sim] section 1 Simpson studied Higgs-bundles with harmonic positive definite metrics, i.e. harmonic bundles, over a compact Kähler-manifold  $M^n$  and related these to harmonic maps from  $M$  in  $GL(n, \mathbb{C})/U(n)$ . From his results one can find, that a given flat bundle with a harmonic metric induces a harmonic map from  $M$  in  $GL(n, \mathbb{C})/U(n)$ . Conversely, a harmonic map from  $M$  in  $GL(n, \mathbb{C})/U(n)$  and a flat bundle give rise to a harmonic bundle. From Sampson's theorem [Sam] one obtains, that in the above case the notion of harmonic and pluriharmonic coincide.

Simpson's result follows from the theorems 5.10 and 5.11, since the condition on the differential of  $\tilde{\phi}_h$  is satisfied in the case of signature  $(r, 0)$  and  $(0, r)$ . We remark, that the theorems 5.10 and 5.11 are in fact more general, since the compactness of  $M$  and the Kähler condition are not needed. Simpson uses Kähler-identities for vector bundles over compact Kähler manifolds in his proof. Further he needs the compactness, since he uses arguments from harmonic map theory, which are developed from Sius Bochner formula for harmonic maps to obtain the vanishing of the object which he calls pseudo-curvature and which is the integrability constraint for a flat bundle to define a Higgs bundle. Dubrovin's work [D] and this thesis deal with pluriharmonic maps. The results are proven by direct calculations using the pluriharmonic and the  $tt^*$ -equations, respectively. In the case of signature  $(r, 0)$  and  $(0, r)$  we needed only the second statement of Sampson's theorem [Sam] and therefore compactness is not needed.

The next theorem gives a rigidity result for harmonic bundles:

**Theorem 5.12** *Let  $(M, J)$  be a compact Kähler manifold of dimension  $n$  with finite fundamental group  $\pi_1(M)$  (i.e., the universal cover of  $M$  is compact). Let  $(E \rightarrow M, D, C, \bar{C}, h)$  be a harmonic bundle over  $(M, J)$  with positive definite hermitian metric  $h$ . Then  $(E \rightarrow M, D, C, \bar{C}, h)$  is trivial, i.e.  $C = \bar{C} = 0$ ,  $D^{(\lambda)} = D$  for all  $\lambda \in \mathbb{S}^1$ ,  $D$  is flat and  $h$  is  $D^{(\lambda)}$ -parallel.*

*Proof:* Pulling back all structures to the universal cover of  $M$  we suppose that  $M$  is simply connected.  $C = \bar{C} = 0$  if and only if the same holds for its pull back.

Let  $s$  be a  $D^{(1)}$ -flat frame of  $E$ . The associated pluriharmonic map  $\tilde{f} : M \rightarrow GL(r, \mathbb{C})/U(r)$  obtained from theorem 5.10 is constant by corollary 2.2. We consider again the representation  $H^s$  of  $h$  in the frame  $s$  to compute the representations  $C^s$  and  $\bar{C}^s$  of  $C$  and  $\bar{C}$  in the frame  $s$  for  $Z \in \Gamma(T^{1,0}M)$  :

$$\begin{aligned} Z(H^s) &= h(D_Z s, s) + h(s, D_{\bar{Z}} s) \\ &= -h(C_Z s, s) - h(s, \bar{C}_{\bar{Z}} s) \\ &= -2h(C_Z s, s) = -2H^s \cdot C_Z^s, \\ \bar{Z}(H^s) &= h(D_{\bar{Z}} s, s) + h(s, D_Z s) \\ &= -h(\bar{C}_{\bar{Z}} s, s) - h(s, C_Z s) \\ &= -2h(\bar{C}_{\bar{Z}} s, s) = -2H^s \cdot \bar{C}_{\bar{Z}}^s. \end{aligned}$$

This yields  $C^s = \bar{C}^s = 0$ . It follows  $C = \bar{C} = 0$  and  $D^{(\lambda)} = D$ . □

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In this work we introduce the real differential geometric notion of a  $tt^*$ -bundle  $(E, D, S)$ , a metric  $tt^*$ -bundle  $(E, D, S, g)$  and a symplectic  $tt^*$ -bundle  $(E, D, S, \omega)$  on an abstract vector bundle  $E$  over an almost complex manifold  $(M, J)$ . With this notion we construct, generalizing Dubrovin [D], a correspondence between metric  $tt^*$ -bundles over complex manifolds  $(M, J)$  and *admissible* pluriharmonic maps from  $(M, J)$  into the pseudo-Riemannian symmetric space  $GL(r, \mathbb{R})/O(p, q)$  where  $(p, q)$  is the signature of the metric  $g$ . Moreover, we show a rigidity result for  $tt^*$ -bundles over compact Kähler manifolds and we obtain as application a special case of Lu's theorem.

In addition we study solutions of  $tt^*$ -bundles  $(TM, D, S)$  on the tangent bundle  $TM$  of  $(M, J)$  and characterize an interesting class of these solutions which contains special complex manifolds and flat nearly Kähler manifolds. We analyze which elements of this class admit metric or symplectic  $tt^*$ -bundles. Further we consider solutions coming from variations of Hodge structures (VHS) and harmonic bundles.

Applying our correspondence to harmonic bundles we generalize a correspondence given by Simpson. Analyzing the associated pluriharmonic maps we obtain roughly speaking for special Kähler manifolds the dual Gauß map and for VHS of odd weight the period map. In the case of non-integrable complex structures, we need to generalize the notions of pluriharmonic maps and some results.

Apart from the rigidity result we generalize all above results to para-complex geometry.

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Dans cette thèse nous introduisons la notion de fibré  $tt^*$   $(E, D, S)$ , de fibré  $tt^*$  métrique  $(E, D, S, g)$  et de fibré  $tt^*$  symplectique  $(E, D, S, \omega)$  sur un fibré vectoriel  $E$  au-dessus d'une variété complexe, dans le langage de la géométrie différentielle réelle. Grâce à cette notion on obtient une correspondance entre des fibrés  $tt^*$  métriques et des applications pluriharmoniques *admissibles* de  $(M, J)$  dans l'espace symétrique pseudo-Riemannien  $GL(r, \mathbb{R})/O(p, q)$ , avec  $(p, q)$  la signature de la métrique  $g$ . En utilisant ce résultat on obtient dans le cas où  $M$  est compact Kählérienne, un résultat de rigidité, puis un cas particulier du théorème de Lu.

De plus nous étudions des fibrés  $tt^*$  sur le fibré tangent  $TM$  et caractérisons une classe spéciale qui contient les variétés spéciales complexes et les variétés nearly Kählériennes plates, et la sous-classe qui admet un fibré  $tt^*$  métrique ou symplectique. En outre on analyse les fibrés  $tt^*$  qui proviennent de variations de structures de Hodge (VHS) et de fibrés harmoniques. Pour les fibrés harmoniques, la correspondance permet de généraliser un résultat de Simpson. L'application pluriharmonique associée à une variété spécialement Kählérienne est reliée à l'application de Gauß duale, et celle associée à une VHS de poids impair est l'application de périodes. Si la structure complexe n'est pas intégrable, on doit généraliser la notion de pluriharmonicité.

Hors la rigidité ces résultats sont généralisés au cas para-complexe.

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**Discipline:** Mathématiques

**Mots clés:** géométrie  $tt^*$ , applications pluriharmoniques, fibrés harmoniques, géométrie spéciale complexe et Kählérienne, nearly Kählériennes, espaces symétriques pseudo-Riemanniennes