Ferromagnetic and ferroelectric thin structures

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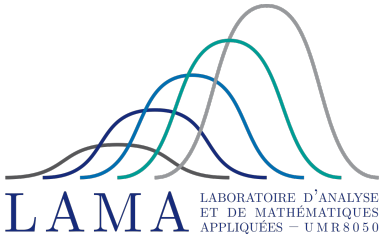

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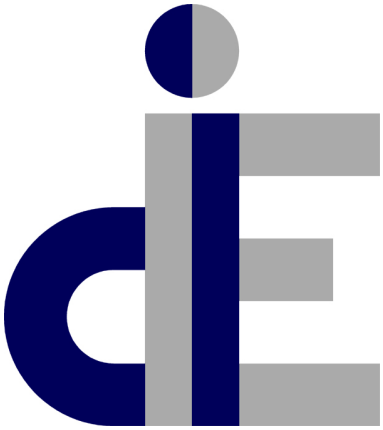

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Résumé

Cette thèse traite avec des équations aux dérivées partielles provenant de la physique mathématique. En particulier, à partir de modèles $3D$ ferromagnétisme et ferroélectricité, nous obtenons des modèles $1D$ et $2D$ par l'intermédiaire de processus asymptotiques basés sur des méthodes de réduction de dimension. Le modèle $3D$ ferromagnétisme a été proposé par W.F. Brown depuis les années 40 [9]. Il est également possible d'utiliser un modèle dynamique, décrivant l'aimantation au cours du temps, en utilisant un système introduit par L.D. Landau et E.M. Lifschitz en 1935 [50]. Pour le modèle ferroélectrique, nous nous référons aux papiers de P. Chandra et P.B. Littlewood [16], W. Zhang et K. Bhattacharya [65] et au livre de T. Mitsui, I. Taksuzaki et E. Nakamura [52].

Ma thèse est constituée de trois parties :

Au début, je considère l'énergie micromagnétique avec des coefficients dégénératifs dans un fil mince. Après avoir montrer l'existence de minimiseurs du problème, j'identifie l'énergie limite lorsque la section du fil tend vers zéro.

Dans la deuxième partie, j'étudie le comportement asymptotique des solutions dépendant du temps des problèmes micromagnétique dans une multi-structure constituée de la jonction de deux fils minces. En supposant que les volumes des deux fils tendent vers zéro avec la même vitesse. On obtient un problème limite couplé par une condition de jonction. Le problème limite reste non-convexe, mais devient complètement local.

Dans le dernier chapitre, à partir d'un modèle variationnel $3D$ non convexe et non-local pour la polarisation électrique dans un matériau ferroélectrique, et à l'aide d'un processus asymptotique basé sur la réduction de dimension, j'analyse des phénomènes de jonction pour deux films minces ferroélectriques joints orthogonaux. Selon la façon dont la réduction se passe, on obtient trois modèles différents de dimension 2. On remarque qu'un effet de mémoire du processus de réduction apparaît, ce dernier dépend de la compétition entre les épaisseurs des deux films: Le paramètre de guidage est la limite du rapport des épaisseurs des deux films.

Mots-clé:

Matériaux ferromagnétiques, matériaux ferroélectriques, film mince, fil mince, multi-structures, jonctions, analyse asymptotique.

Ferromagnetic and ferroelectric thin structures

Abstract

This thesis deals with partial differential equations coming from mathematical physics. Particularly, starting from $3D$ models for ferromagnetism and ferroelectricity, we derive $1D$ and $2D$ models via asymptotic processes based on dimensional reduction methods. The $3D$ model for ferromagnetism was proposed by W.F. Brown in the 40s [9] and it is based on a system introduced by L.D. Landau and E.M. Lifschitz in 1935 [50]. About the ferroelectric model, we refer to the papers of P. Chandra and P.B. Littlewood [16], W. Zhang and K. Bhattacharya [65] and to the book of T. Mitsui, I. Taksuzaki, and E. Nakamura [52].

This thesis based on three works:

At the beginning, we consider micromagnetic energy, with some degenerating coefficients, in a thin wire. After showing the existence of minimizers, we identify the limit energy as the section of the wire vanishes.

In the second part, we study the asymptotic behavior of the solutions of a time dependent micromagnetic problem in a multi-structure consisting of two joined thin wires. We assume that the volumes of the two wires vanish with same rate. We obtain two $1D$ limit problems coupled by a junction condition on the magnetization. The limit problem remains non-convex, but now it becomes completely local.

In the last chapter, starting from a non-convex and nonlocal $3D$ variational model for the electric polarization in a ferroelectric material, and using an asymptotic process based on dimensional reduction, we analyze junction phenomena for two orthogonal joined ferroelectric thin films. We obtain three different $2D$-variational models for joined thin films, depending on how the reduction happens. We note that, a memory effect of the reduction process appears, and it depends on the competition of the relative thickness of the two films: The guide parameter is the limit of the ratio between these two small thickness.

Keywords:

Ferromagnetic materials, ferroelectric materials, thin film, thin wire, multi-structures, junctions, asymptotic analysis.

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Introduction

A thin structure is a three-dimensional object with one (two) preponderant dimension(s), such as a wire, a thin film, a combination of wires, thin films, etc. In these structures, some physical phenomena take place that are generally described by variational problems. By starting from $3D$ models and using asymptotic mathematical methods, one tries to obtain $1D$ or $2D$ limit problems describing the physical phenomena in a thin structure. The reduced models are justified by reasons of simplicity and economy, by a numerical point of view, too. In this thesis we are interested in ferromagnetic problems and in ferroelectric problems in thin structures.

0.1 Ferromagnetic model

According to the classical theory of Weiss (1907), perfectioned by Landau and Lifshitz in 1935 (see [50] and, for a modern analysis, see [9]), on a microscopic scale a ferromagnetic body is magnetically saturated and is composed by uniformly magnetized regions separated by thin transition layers. The phenomena can be described by a magnetization field, defined on the domain in which the material is confined. The magnetization field on a microscopic scale has a fixed modulus and variable orientations. Then, the system can be studied through the functional representing its magnetic energy. It consists in several terms: the so-called exchange energy, which contains the space derivative of the magnetization field and is peculiar to ferromagnetic behavior, a term corresponding to magnetic anisotropy, and another one depending on the magnetic field, which is related to the magnetization via the magnetostatic equation. More precisely,

$$m : \Omega \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

denotes the magnetization and the body Ω is always locally magnetized to a saturation magnetization $|m(x)| = c(T) > 0$ unless the local temperature T is greater or equal to Curie temperature depending on the body. In the latter case $c(T) = 0$, and the material ceases to behave ferromagnetically. In the sequel, we suppose constant temperature lower than Curie temperature and,

without loss of generality, we assume that $c = 1$, that is $m(x) \in S^2$. The magnetic moment m and the magnetic field H are related by one of Maxwell's equations (no load, no current)

$$B = H + \overline{m}, \qquad in \quad \mathbb{R}^3$$

where we consider $\overline{m}$ as a field set in all $\mathbb{R}^3$, extending m by 0 on the exterior of Ω. The field B and $H(\overline{m})$ are defined in all $\mathbb{R}^3$.

0.1.1 Free energy of the classical micromagnetism model

The observed magnetic moments are (local) minimizers of the ferromagnetic energy

$$E(m) = \int_\Omega \left(\alpha |\nabla m|^2 + \varphi(m) + \frac{1}{2} \nabla \zeta m \right) dx. \tag{0.1.1}$$

Now, we will try to understand each term.

The exchange energy

The exchange energy is due to the existence of a responsible force to align the spins neighboring of two atoms. This contribution is local and it depends on the microscopic properties of materials. This energy is written

$$E_e(m) = \alpha \int_\Omega |\nabla m|^2 dx,$$

where α is the exchange coefficient.

The anisotropy energy

The magnetocrystalline anisotropy reflects the effects of anisotropy due to the crystal structure of the material. This is described by an even continuous function

$$\varphi : S^2 \longrightarrow \mathbb{R}^+.$$

Mainly, two cases are considered.

- *The uniaxial anisotropy:* this anisotropy is in the hexagonal crystals (Co). The expression of this energy is

  $$\varphi(m) = -K_1 m_z^2 + K_2 m_z^4,$$

  where m_z is the component along the third axis of the vector m, K_1 and K_2 are the coefficients of anisotropy depending on the temperature.

- *The cubic anisotropy:* in the case of cubic crystals (Fe, Ni), the axes x, y and z being chosen

along the crystal axes, the expression for the energy is

$$\varphi(m) = K_1(m_x^2 m_y^2 + m_y^2 m_z^2 + m_z^2 m_x^2) + K_2 m_x^2 m_y^2 m_z^2$$

where K_1 and K_2 are the coefficients of anisotropy, and m_x, m_y and m_z are the components of m.

The magnetostatic energy

The third term in (0.1.1) represents the magnetostatic energy. Starting from the Maxwell equations without charge or current, we find that the magnetic field is determined by the magnetization as a solution of the following problem

$$\begin{cases} \operatorname{div}(H(\overline{m}) + \overline{m}) = 0, & \text{in } \mathbb{R}^3, \\ \operatorname{curl} H(\overline{m}) = 0 & \text{in } \mathbb{R}^3. \end{cases}$$

Consequently, one obtains $H(\overline{m}) = -\nabla\zeta$ (ζ is the magnetic potential) and

$$E_{\text{mag}}(m) = \frac{1}{2}\int_{\Omega} \nabla\zeta m dx = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\zeta|^2 dx.$$

Note that the magnetostatic energy expresses a non-local interaction.

0.2 Asymptotic analysis for micromagnetics of nanowires of finite length governed by indefinite material coefficients

In a nanowire, we consider the micromagnetic free energy with some degenerating weights. In [41] G. Gioia and R. D. James (see also [13]) found that in the rescaled energy the exchange term is

$$\int_{\Omega} \frac{\alpha}{\delta} |\nabla m|^2 dy,$$

where α is the exchange constant. Thus, as $\delta \to 0$, the exchange term contributes an unacceptably large energy unless $|\nabla m| \sim 0$. In the limit $\delta \to \infty$, corresponds to phase theory see [28].

In [42] the authors studied the micromagnetism problem in the case of a thin plate with some degenerating weights. In this Chapter we will adapt the same argument of [42] to study the same problem in a nanowire of finite length. We identify the limit problem when the section of the wire tends to 0. Now, if we take into account the non degenerate case of the material coefficient α and $\alpha_* := \min_{x\in\overline{\Omega}} \alpha(x) > 0$, we can adapt the arguments to [41] and [13].

We consider a ferromagnetic nanowire occupying the $3D$ domain $\Omega^{(h)} =]-L, L[\times B_2(0,h)$, where $B_d(x,r)$ denotes the ball in $\mathbb{R}^d$ of radius r and center x. Let $\Omega =]-L, L[\times B_2(0,1)$. Moreover, let $\alpha : \overline{\Omega} \to [0,\infty)$ be a given continuous function, and set $A_0 = \alpha^{-1}(0)$. The aim is to study the minimization and the asymptotic behavior, as $h \to 0$, of the following non-convex and nonlocal problem:

$$\underline{\mathcal{E}}^{(h)}(\underline{m}^{(h)}) = \min_{m\in L^2(\Omega^{(h)},\mathbb{R}^3)} \underline{\mathcal{E}}^{(h)}(m), \tag{0.2.1}$$

where the functional $\underline{\mathcal{E}}^{(h)}$, defined in $L^2\left(\Omega^{(h)},\mathbb{R}^3\right)$, denotes the micromagnetism energy in $\Omega^{(h)}$ previously introduced:

$$\underline{\mathcal{E}}^{(h)}(m) := \begin{cases} \dfrac{1}{2L\pi h^2}\left(\displaystyle\int_{\Omega^{(h)}\setminus A_0} \alpha|\nabla m|^2 dx + \int_{\Omega^{(h)}} \varphi(m)dx + \frac{1}{2}\int_{\Omega^{(h)}} \nabla\zeta\cdot m dx\right), \\ \text{if } m \in H^1_{loc}\left(\Omega^{(h)}\setminus A_0,\mathbb{R}^3\right) \text{ and } \sqrt{\alpha}\nabla m \in L^2\left(\Omega^{(h)}\setminus A_0,\mathbb{R}^{3\times 3}\right), \\ \infty, \qquad \text{otherwise,} \end{cases} \tag{0.2.2}$$

subject to the constraints:

$$\operatorname{div}(-\nabla\zeta + \overline{m}) = 0, \text{ in } \mathbb{R}^3, \tag{0.2.3}$$

$$|m| = 1, \quad \text{a.e. in } \Omega^{(h)}, \tag{0.2.4}$$

where $\overline{m}$ denotes the zero-extension of m to $\mathbb{R}^3$.

The given continuous function $\alpha = \alpha(x)$ $(x \in \overline{\Omega})$ is the so-called material coefficient, and here, we suppose that it may degenerate somewhere on $\overline{\Omega}$. In order to work in a fixed domain, we introduce the following rescaling

$$x = (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3 \mapsto (x_1, hx_2, hx_3) \in \Omega^{(h)} \subset \mathbb{R}^3.$$

0.2.1 The main results

Let us set:

$$\alpha^\circ(x_1) := \alpha(x_1, 0, 0) \text{ for any } x_1 \in [-L, L], \text{ and } A_0^\circ := (\alpha^\circ)^{-1}(0).$$

We assume that

(a) $\mathcal{L}^3(A_0) = 0$,

(b) there exists a constant $C_\alpha \geq 1$, such that

$$\alpha^\circ(x_1) \leq \alpha(x) \leq C_\alpha \alpha^\circ(x_1), \text{ for all } x = (x_1, x_2, x_3) \in \overline{\Omega}.$$

We prove the following results.

Theorem 0.2.1. *Assume (a). Then, for $0 < h < 1$ problem (0.2.1) admits at least a minimizer solution $m^{(h)}$.*

Theorem 0.2.2. *Assume (a)-(b). Then, there exist a sequence $\{h_i \quad | \quad i = 1, 2, 3, \cdots\} \subset (0, 1)$ and a limiting function $m^\circ \in L^2(]-L, L[, \mathbb{R}^3)$, such that*

(i) $h_i \to 0$, $m^{(h_i)} \to m^\circ$ in $L^2(\Omega, \mathbb{R}^3)$, $\underline{\mathcal{E}}^{(h_i)}\left(m^{(h_i)}\right) \to \mathcal{E}^\circ(m^\circ)$, and

$$\begin{cases} \sqrt{\alpha^{(h_i)}}\partial_1 m^{(h_i)}(x_1, x_2, x_3) \to \sqrt{\alpha^\circ}\partial_1 m^\circ(x_1), \\ \\ \dfrac{\sqrt{\alpha^{(h_i)}}}{h_i}\partial_2 m^{(h_i)}(x_1, x_2, x_3) \to 0, \\ \\ \dfrac{\sqrt{\alpha^{(h_i)}}}{h_i}\partial_3 m^{(h_i)}(x_1, x_2, x_3) \to 0, \end{cases} \tag{0.2.5}$$

for a.e. $x_1 \in]-L, L[$ and a.e. $(x_2, x_3) \in B_2(0, 1)$, as $i \to \infty$,

(ii) the limit m° is a minimizer of $\mathcal{E}^\circ(m)$,

The functional $\mathcal{E}^\circ$ is defined on $L^2(]-L, L[, \mathbb{R}^3)$ by

$$\mathcal{E}^\circ(m) := \begin{cases} \Phi^\circ_\alpha(m) + \displaystyle\int_{-L}^{L} \varphi(m) dx_1 + \frac{1}{2}\int_{-L}^{L} \left(|m_2|^2 + |m_3|^2\right) dx_1, \\ \text{if } m \in L^2(]-L, L[, \mathbb{S}^2), \\ \\ \infty, \qquad\qquad \text{otherwise}, \end{cases} \tag{0.2.6}$$

where Φ°_α *iz the convex function on* $L^2(]-L,L[,\mathbb{R}^3)$*, defined as:*

$$\Phi^\circ_\alpha(m) := \begin{cases} \displaystyle\int_{]-L,L[\setminus A^\circ_0} \alpha^\circ |\nabla m|^2 dx_1, \ \textit{if} \, m \in H^1_{loc}(]-L,L[\setminus A^\circ_0,\mathbb{R}^3), \\ \infty, \qquad\qquad \textit{otherwise.} \end{cases} \tag{0.2.7}$$

0.3 Junction of quasi-stationary ferromagnetic thin multi-structures

In a joint work with L. Faella and C. Perugia, we study the asymptotic behavior of a system governed by the Landau-Lifshitz equation consisting of two joined roads with "vanishing" sections. So we attempt to simulate the behaviour of two joined nanowires. More precisely, let $\{h_n\}_{n\in\mathbb{N}} \subset]0,1[$ be a vanishing sequence. For every $n \in \mathbb{N}$, set

$$\Omega^a_n =]-h_n,0[^2 \times [0,1[, \ \ \Omega^{b,l}_n =]0,1[\times]-h_n,0[^2 , \ \ \Omega^{b,r}_n =]-h_n,0]^3 ,$$

$$\Omega_n = \Omega^a_n \cup \Omega^{b,l}_n \cup \Omega^{b,r}_n, \quad n \in \mathbb{N}.$$

Let us suppose that the body is homogeneous, isotropic and has uniform temperature. Let us introduce the magnetization M_n, the magnetic field H_{M_n} determined by M_n and the scalar potential U_{M_n} for this field (i.e.$H(M_n) \equiv -DU_{M_n}$). Let us denote by $\overline{M_n}$ the extension by zero of M_n outside Ω_n. Then, as previously described, the magnetic induction B_n and the magnetic field $H(M_n)$ are connected by the relations $B_n = -DU_{M_n} + \overline{M_n}$. Moreover, the static Maxwell equation and the magnetostatic equation (Faraday law) hold

$$\begin{cases} \nabla \times DU_{M_n} = 0, \\ div\left(-DU_{M_n} + \overline{M_n}\right) = div\,(B) = 0\,. \end{cases} \tag{0.3.1}$$

Fixed $M_{0_n} \in H^1(\Omega_n, S^2)$, ($U_{0_n}$ being the corresponding solution of Problem (0.3.1)), in [12], [24] and [63] it is proved that there exists at least a weak solution M_n of the following problem

$$
\begin{cases}
M_n \in L^\infty\left(0,T;H^1\left(\Omega_n,\mathbb{R}^3\right)\right)\cap C\left(\left[0,T\right];L^2\left(\Omega_n,\mathbb{R}^3\right)\right), \\
|M_n|=1 \ a.e. \text{ in } \ [0,T]\times\Omega_n, \quad \dfrac{\partial M_n}{\partial t}\in L^2\left(0,T;L^2(\Omega_n,\mathbb{R}^3)\right), \\
\forall\, \chi\in D(0,T), \text{ and } \ \psi\in H^1\left(\Omega_n,\mathbb{R}^3\right) \\
\displaystyle\int_0^T\int_{\Omega_n}\left(\frac{\partial M_n}{\partial t}+M_n\wedge\frac{\partial M_n}{\partial t}\right)\chi\psi dxdt=-2\int_0^T\int_{\Omega_n}\sum_{i=1}^{3}\left(M_n\wedge D_{x_i}M_n\right)\left(D_{x_i}\psi\right)\chi dxdt \\
\displaystyle -2\int_0^T\int_{\Omega_n}\left(M_n\wedge DU_{M_n}\right)\chi\psi dxdt, \\
M_n(0,x)=M_{0_n}(x), \text{ a.e. } x \quad \text{in} \quad \Omega_n, \\
U_{M_n} \text{ and } M_n \text{ linked by (0.3.1) for every } t\in[0,T].
\end{cases}
\tag{0.3.2}
$$

Moreover, it satisfies the following energy estimate

$$
E\left(M_n(t,\cdot)\right)+\int_0^t\left\|\frac{\partial M_n}{\partial t}\right\|^2_{\left(L^2(\Omega_n)\right)^3}ds\leq E\left(M_n(0,\cdot)\right)=E\left(M_{0_n}\right), \quad \text{for a.e. } t\in[0,T], \tag{0.3.3}
$$

where for every $t\in[0,T]$

$$
E\left(M_n(t,\cdot)\right)=\int_{\Omega_n}\left|DM_n\left(t,x\right)\right|^2dx+\frac{1}{2}\int_{\mathbb{R}^3}\left|DU_{M_n}\left(t,x\right)\right|^2dx,
$$

is the magnetic energy. Here, the terms $E_n^{exc}=\displaystyle\int_{\Omega_n}\left|DM_n(t,x)\right|^2dx$ is the exchange energy and $E_n^{mag}=\dfrac{1}{2}\displaystyle\int_{\mathbb{R}^3}\left|DU_{M_n}(t,x)\right|^2dx$ corresponds to the magnetostatic energy.

In what follows let us assume that

$$
E^{exc}\left(M_{0_n}\right)=O\left(h_n^2\right), \quad \forall n\in\mathbb{N}. \tag{0.3.4}
$$

0.3.1 The main result

We set

$$\Omega^a =]-1,0[^2\times[0,1[, \quad \Omega^{b,l} =]0,1[\times]-1,0[^2, \quad \Omega^{b,r} =]-1,0[^3,$$

and introduce the following space

$$\mathcal{W} = \left\{(\mu^a,\mu^{b,l}) \in H^1\left(]0,1[,\mathbb{R}^3\right)\times H^1\left(]0,1[,\mathbb{R}^3\right) \,:\, \mu^a(0)=\mu^{b,l}(0)\right\}. \tag{0.3.5}$$

Moreover, we set

$$\mathcal{M}=\mathcal{W}\cap\left\{\ H^1\left(\Omega^a,S^2\right)\times H^1\left(\Omega^{b,l},S^2\right)\ \right\}, \tag{0.3.6}$$

which explicitly takes into account the condition $|\mu|=1$.

Let us introduce the following problem

$$\begin{cases}
\mu=(\mu^a,\mu^{b,l})\in L^\infty\left(0,T;\mathcal{M}\right)\cap C\left([0,T];L^2\left(\Omega^a,\mathbb{R}^3\right)\times L^2\left(\Omega^{b,l},\mathbb{R}^3\right)\right),\\
\dfrac{\partial\mu}{\partial t}\in L^2\left(0,T;L^2\left(\Omega^a,\mathbb{R}^3\right)\times L^2\left(\Omega^{b,l},\mathbb{R}^3\right)\right),\\
\forall\,\chi\in D(0,T)\ \text{and}\,\psi=(\psi^a,\psi^{b,l})\in\mathcal{W},\\
\displaystyle\int_0^T\int_{\Omega^a}\left(\frac{\partial\mu^a}{\partial t}+\mu^a\wedge\frac{\partial\mu^a}{\partial t}\right)\chi\psi^a dxdt+\int_0^T\int_{\Omega^{b,l}}\left(\frac{\partial\mu^{b,l}}{\partial t}+\mu^{b,l}\wedge\frac{\partial\mu^{b,l}}{\partial t}\right)\chi\psi^{b,l}dxdt=\\
\displaystyle-2\int_0^T\int_{\Omega^a}\mu^a\wedge\frac{\partial\mu^a}{\partial x_3}\frac{\partial\psi^a}{\partial x_3}\chi dxdt-2\int_0^T\int_{\Omega^{b,l}}\mu^{b,l}\wedge\frac{\partial\mu^{b,l}}{\partial x_1}\frac{\partial\psi^{b,l}}{\partial x_1}\chi dxdt\\
\displaystyle-2\int_0^T\int_{\Omega^a}\mu^a\wedge\left((\mu^a,e_1)\Upsilon_1+(\mu^a,e_2)\Upsilon_2\right)\chi\psi^a dxdt\\
\displaystyle-2\int_0^T\int_{\Omega^{b,l}}\mu^{b,l}\wedge\left((\mu^{b,l},e_2)\Upsilon_1'+(\mu^{b,l},e_3)\Upsilon_2'\right)\chi\psi^{b,l}dxdt,\\
\mu^a(0,x)=\mu_0^a(x),\ \text{a.e. } x\text{ in }\Omega^a,\quad \mu^{b,l}(0,x)=\mu_0^{b,l}(x),\ \text{a.e. } x\text{ in }\Omega^{b,l},\quad \mu_0=(\mu_0^a,\mu_0^{b,l})\in\mathcal{M},
\end{cases} \tag{0.3.7}$$

where $\Upsilon_1=(\varepsilon_1,\varepsilon_2,0)$, $\Upsilon_2=(-\varepsilon_2,\varepsilon_1,0)$, $\Upsilon_1'=(0,\varepsilon_1,\varepsilon_2)$ and $\Upsilon_2'=(0,-\varepsilon_2,\varepsilon_1,)$, with ε_1, ε_2

constant depending on $]-1,0[^2$ (in the thesis we explicitly give the dependance of these constants on $]-1,0[^2$).

Then, we prove the following result.

Theorem 0.3.1. *Suppose that $M_{0_n} \in H^1(\Omega_n, S^2)$ and (0.3.4) holds, for every $n \in \mathbb{N}$. Let M_n be a solution of Problem (0.3.2). Then, there exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$, still denoted by $\{n\}$, $\mu_0 = \left(\mu_0^a, \mu_0^{b,l}\right) \in \mathcal{M}$, $\mu = (\mu^a, \mu^{b,l}) \in L^\infty(0,T;\mathcal{M})$, depending on the selected subsequence such that*

$$\begin{cases} \fint_{\Omega_n^a} |M_{0_n}(x_1,x_2,x_3) - \mu_0^a(x_3)|^2\, dx \to 0, \\ \fint_{\Omega_n^{b,l}} \left|M_{0_n}(x_1,x_2,x_3) - \mu_0^{b,l}(x_1)\right|^2 dx \to 0, \\ \fint_{\Omega_n^{b,r}} |M_{0_n}(x_1,x_2,x_3) - \mu_0^a(0)|^2\, dx \to 0, \end{cases}$$

as n diverges, for every $t \in [0,T]$

$$\begin{cases} \fint_{\Omega_n^a} |M_n(t,x_1,x_2,x_3) - \mu^a(t,x_3)|^2\, dxdt \to 0, \\ \fint_{\Omega_n^{b,l}} \left|M_n(t,x_1,x_2,x_3) - \mu^{b,l}(t,x_1)\right|^2 dxdt \to 0, \end{cases}$$

as n diverges, where $\mu = (\mu^a, \mu^{b,l})$ is a solution of Problem (0.3.7).

0.4 Ferroelectric model

Ferroelectricity is a property of some materials to have a spontaneous electrical polarization that can be reversed by the application of an external electric field. Hysteresis phenomena appear, so the behavior of these materials is very similar to the one of ferromagnetic materials. Analogously,

a Curie temperature T_C appears, too.

Let B be a $3D$ ferroelectric body. In B the electric displacement $\mathbf{D}$ is given by $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$, where $\varepsilon_0 > 0$ is the vacuum permeability, $\mathbf{E}$ is the applied external field, and $\mathbf{P}$ is the spontaneous electric polarization. Assume that $\mathbf{E}$ is the gradient of a potential ψ, i.e.

$$\mathbf{E} = D\psi, \tag{0.4.1}$$

and that the electric field generated by $\mathbf{P}$ derives from a potential $\varphi_{\mathbf{P}}$ satisfying the electrostatic equation

$$\mathrm{div}(-\varepsilon_0 D\varphi_{\mathbf{P}} + \mathbf{P}) = 0. \tag{0.4.2}$$

We limit ourselves to the case where no strong electric field has been applied on B, but only a very weak electric field acts on it (e.g. it is the case of iron in the ferromagnetism, before the magnetization, by analogy). Then, we can assume that there are not Weis domains (i.e. regions with different polarization separated by well defined interfaces), but only transition regions. In this framework we can assume that the polarization does not generate an electric field outside B. Consequently, equation (0.4.2) holds true in B, and the following boundary conditions on ∂B

$$\mathbf{P}\cdot\nu = 0, \quad D\varphi_{\mathbf{P}}\cdot\nu = 0 \text{ on } \partial B \tag{0.4.3}$$

can be added, where ν denotes the unit outer normal on ∂B.

One assumes that $\mathbf{P}$ minimizes the energy functional

$$\int_B \left(\beta|\mathrm{rot}\mathbf{P}|^2 + |\mathrm{div}\mathbf{P}|^2 + \alpha(|\mathbf{P}|^2 - 1)^2\right) dx + \int_{\mathbb{R}^3} |D\psi + D\varphi_{\mathbf{P}}|^2 dx, \tag{0.4.4}$$

where α and β are two positive constants independent of the external field and of the temperature. Here, $\int_B (\beta|\mathrm{rot}\mathbf{P}|^2 + |\mathrm{div}\mathbf{P}|^2)\, dx$ reduces to the classical energy $\int_B |D\mathbf{P}|^2 dx$ when $\beta = 1$ (see (0.5.6)), so roughly speaking this term penalizes the spatial variation of $\mathbf{P}$. The term $\alpha\int_B(|\mathbf{P}|^2 - 1)^2 dx$ obliges $|\mathbf{P}|$ to be near to 1, and it can induce a phase transition of $\mathbf{P}$. So the body is driven to have regions of uniform polarization separated by thin transition layers. The term $\int_{\mathbb{R}^3} |D\psi + D\varphi_{\mathbf{P}}|^2 dx$ is the electrostatic energy. As this last term is concerned, we have

$$\int_{\mathbb{R}^3} |D\psi + D\varphi_{\mathbf{P}}|^2 dx = \int_{\mathbb{R}^3} |\mathbf{E}|^2 dx + 2\int_B D\psi\cdot D\varphi_{\mathbf{P}} dx + \int_B |D\varphi_{\mathbf{P}}|^2 dx, \tag{0.4.5}$$

thanks to (0.4.1). On the other side, using (0.4.2) and (0.4.3) give

$$\int_B D\psi \cdot D\varphi_{\mathbf{P}} dx = \frac{1}{\varepsilon_0} \int_B D\psi \cdot \mathbf{P} dx. \tag{0.4.6}$$

Consequently, inserting (0.4.5) and (0.4.6) in (0.4.4), and remarking that $\int_{\mathbb{R}^3} |\mathbf{E}|^2 dx$ is constant with respect to $\mathbf{P}$, the energy functional minimized by $\mathbf{P}$ becomes

$$\int_B \left(\beta |\mathrm{rot}\mathbf{P}|^2 + |\mathrm{div}\mathbf{P}|^2 + \alpha(|\mathbf{P}|^2 - 1)^2\right) dx + \int_B |D\varphi_{\mathbf{P}}|^2 dx + \frac{2}{\varepsilon_0} \int_B \mathbf{E} \cdot \mathbf{P} dx, \tag{0.4.7}$$

where $\int_B |D\varphi_{\mathbf{P}}|^2 dx$ is the electrostatic energy induced by $\mathbf{P}$, and the external energy $\int_B \mathbf{E} \cdot \mathbf{P} dx$ favors the polarization parallel (but in opposite verse) to $\mathbf{E}$.

0.5 Fin junction of ferroelectric thin films

In a joint work with L. Carbone and A. Gaudiello, starting from problem (0.4.7) and using using an asymptotic process based on dimensional reduction, we obtain a simpler ferroelectric model, especially from a numerical point of view, for a multidomain composed of two orthogonal joined thin films.

Precisely, let $\{h_n^a\}_{n\in\mathbb{N}}$ and $\{h_n^b\}_{n\in\mathbb{N}} \subset]0,1[$ be two sequences such that

$$\lim_n h_n^a = 0 = \lim_n h_n^b, \quad \lim_n \frac{h_n^b}{h_n^a} = \ell \in [0, +\infty]. \tag{0.5.1}$$

For every $n \in \mathbb{N}$, set

$$\Omega_n^a = \left]-\tfrac{h_n^a}{2}, \tfrac{h_n^a}{2}\right[\times \left]-\tfrac{1}{2}, \tfrac{1}{2}\right[\times [0,1[, \quad \Omega_n^b = \left]-\tfrac{1}{2}, \tfrac{1}{2}\right[^2 \times \left]-h_n^b, 0\right[, \quad \Omega_n = \Omega_n^a \cup \Omega_n^b. \tag{0.5.2}$$

The multidomain Ω_n models a ferroelectric device consisting of two orthogonal joined thin films Ω_n^a and Ω_n^b with small thicknesses h_n^a and h_n^b, respectively. According to (0.4.7), we consider the following non-convex and nonlocal energy associated with Ω_n

$$\mathcal{E}_n : \mathbf{P} \in \mathcal{P}_n \longrightarrow \int_{\Omega_n} \left(\beta |\mathrm{rot}\mathbf{P}|^2 + |\mathrm{div}\mathbf{P}|^2 + \alpha(|\mathbf{P}|^2 - 1)^2 + |D\varphi_{\mathbf{P}}|^2 + (\mathbf{F}_n \cdot \mathbf{P})\right) dx, \tag{0.5.3}$$

where

$$\mathcal{P}_n = \left\{\mathbf{P} \in \left(H^1(\Omega_n)\right)^3 : \mathbf{P} \cdot \nu = 0 \text{ on } \partial\Omega_n\right\}, \tag{0.5.4}$$

α and β are two positive constants, $\mathbf{F}_n \in (L^2(\Omega_n))^3$, ν denotes the unit outer normal on $\partial\Omega_n$,

and $\varphi_{\mathbf{P}} \in H^1(\Omega_n)$ is the unique solution, up to an additive constant, of

$$\begin{cases} \operatorname{div}(-\varepsilon_0 D\varphi_{\mathbf{P}} + \mathbf{P}) = 0 \text{ in } \Omega_n, \\ \\ (-\varepsilon_0 D\varphi_{\mathbf{P}} + \mathbf{P}) \cdot \nu = 0 \text{ on } \partial\Omega_n. \end{cases} \tag{0.5.5}$$

Notice that $\mathbf{F}_n$ is a normalization of the external field. So to obtain (0.4.7), choose $\mathbf{F}_n = \frac{2}{\varepsilon_0}\mathbf{E}$.

Using (for instance compare [18] and Lemma 2.1 in [37])

$$\|D\mathbf{P}\|^2_{(L^2(\Omega_n))^9} = \|\mathrm{rot}\mathbf{P}\|^2_{(L^2(\Omega_n))^3} + \|\mathrm{div}\mathbf{P}\|^2_{L^2(\Omega_n)} \quad \forall \mathbf{P} \in \mathcal{P}_n, \quad \forall n \in \mathbb{N}, \tag{0.5.6}$$

and the direct method of Calculus of Variations give that problem

$$\min\{\mathcal{E}_n(\mathbf{P}) : \mathbf{P} \in \mathcal{P}_n\} \tag{0.5.7}$$

admits a solution. The aim of this chapter is to study the asymptotic behavior, as n diverges, of problems (0.5.7). As we shall show, its asymptotic behavior depends on ℓ given in (0.5.1).

In order to work on a fixed domain, we use the following maps

$$\begin{cases} x = (x_1, x_2, x_3) \in \Omega^a = \left]-\frac{1}{2}, \frac{1}{2}\right[\times \left]-\frac{1}{2}, \frac{1}{2}\right[\times]0, 1[\longrightarrow (h_n^a x_1, x_2, x_3) \in \mathrm{Int}(\Omega_n^a), \\ \\ x = (x_1, x_2, x_3) \in \Omega^b = \left]-\frac{1}{2}, \frac{1}{2}\right[\times \left]-\frac{1}{2}, \frac{1}{2}\right[\times]-1, 0[\longrightarrow (x_1, x_2, h_n^b x_3) \in \Omega_n^b, \end{cases} \tag{0.5.8}$$

where $\mathrm{Int}(\Omega_n^a)$ denotes the interior of Ω_n^a. Then, for every $n \in \mathbb{N}$ set

$$\begin{cases} f_n^a : x = (x_1, x_2, x_3) \in \Omega^a \longrightarrow \mathbf{F}_n(h_n^a x_1, x_2, x_3), \\ \\ f_n^b : x = (x_1, x_2, x_3) \in \Omega^b \longrightarrow \mathbf{F}_n(x_1, x_2, h_n^b x_3), \end{cases} \tag{0.5.9}$$

$$
\left\{
\begin{array}{l}
P_n = \Big\{ \quad (p^a, p^b) \in \left(H^1\left(\Omega^a\right)\right)^3 \times \left(H^1(\Omega^b)\right)^3 : \\
\qquad\qquad p^a \cdot \nu^a = 0 \text{ on } \partial\Omega^a \backslash \left(\left]-\frac{1}{2}, \frac{1}{2}\right[^2 \times \{0\} \right), \\
\qquad\qquad p^b \cdot \nu^b = 0 \text{ on } \partial\Omega^b \backslash \left(\left]-\frac{1}{2}, \frac{1}{2}\right[^2 \times \{0\} \right), \\
\qquad\qquad p_3^b = 0 \text{ on } \left(\left]-\frac{1}{2}, \frac{1}{2}\right[^2 \backslash \left(\left]-\frac{h_n^a}{2}, \frac{h_n^a}{2}\right[\times \left]-\frac{1}{2}, \frac{1}{2}\right[\right) \right) \times \{0\}, \\
\qquad\qquad p^a(x_1, x_2, 0) = p^b(h_n^a x_1, x_2, 0) \text{ in } \left]-\frac{1}{2}, \frac{1}{2}\right[^2 \Big\},
\end{array}
\right. \tag{0.5.10}
$$

where ν^a and ν^b denote the unit outer normals on $\partial\Omega^a$ and $\partial\Omega^b$, respectively, and

$$
U_n = \left\{ (\phi^a, \phi^b) \in H^1(\Omega^a) \times H^1(\Omega^b) \ : \ \phi^a(x_1, x_2, 0) = \phi^b(h_n^a x_1, x_2, 0) \text{ in } \left]-\frac{1}{2}, \frac{1}{2}\right[^2 \right\}. \tag{0.5.11}
$$

Then, $\mathcal{E}_n$ defined in (0.5.3) is rescaled by

$$
\left\{
\begin{array}{l}
E_n : \left(p^a, p^b\right) \in P_n \longrightarrow \\
\\
h_n^a \displaystyle\int_{\Omega^a} \left(\beta |\mathrm{rot}_n^a p^a|^2 + |\mathrm{div}_n^a p^a|^2 + \alpha(|p^a|^2 - 1)^2 + |D_n^a \phi^a_{(p^a, p^b)}|^2 + (f_n^a \cdot p^a) \right) dx \\
\\
+ h_n^b \displaystyle\int_{\Omega^b} \left(\beta |\mathrm{rot}_n^b p^b|^2 + |\mathrm{div}_n^b p^b|^2 + \alpha(|p^b|^2 - 1)^2 + |D_n^b \phi^b_{(p^a, p^b)}|^2 + (f_n^b \cdot p^b) \right) dx,
\end{array}
\right. \tag{0.5.12}
$$

where $\left(\phi^a_{(p^a,p^b)}, \phi^b_{(p^a,p^b)}\right)$ is the unique solution of

$$\begin{cases} \left(\phi^a_{(p^a,p^b)}, \phi^b_{(p^a,p^b)}\right) \in U_n, \quad \displaystyle\int_{\Omega^a} \phi^a_{(p^a,p^b)} dx = 0, \\ \\ h^a_n \displaystyle\int_{\Omega^a} ((-D^a_n \phi^a_{(p^a,p^b)} + p^a) \cdot D^a_n \phi^a) dx \\ \\ +h^b_n \displaystyle\int_{\Omega^b} \left(\left(-D^b_n \phi^b_{(p^a,p^b)} + p^b\right) \cdot D^b_n \phi^b\right) dx = 0 \quad \forall\, (\phi^a, \phi^b) \in U_n, \end{cases} \tag{0.5.13}$$

which rescales a weak formulation of (0.5.5), i.e.

$$\varphi_{\mathbf{P}} \in H^1(\Omega_n), \quad \int_{\Omega^a_n} \varphi_{\mathbf{P}}\, dx = 0, \quad \int_{\Omega_n} ((-\varepsilon_0 D\varphi_{\mathbf{P}} + \mathbf{P}) \cdot D\varphi)\ dx = 0 \quad \forall \varphi \in H^1(\Omega_n). \tag{0.5.14}$$

The Lax-Milgram Theorem provides that (0.5.14) admits solution and it is unique.

Note that if $\mathbf{P}_n$ solves (0.5.7), then (p^a_n, p^b_n) defined by

$$p^a_n(x_1, x_2, x_3) = \mathbf{P}_n(h^a_n x_1, x_2, x_3) \text{ in } \Omega^a, \quad p^b_n(x_1, x_2, x_3) = \mathbf{P}_n(x_1, x_2, h^b_n x_3) \text{ in } \Omega^b,$$

solves

$$\min\left\{E_n((p^a, p^b)) : (p^a, p^b) \in P_n\right\}. \tag{0.5.15}$$

Assume that

$$\begin{cases} f^a_n \rightharpoonup f^a = (f^a_1, f^a_2, f^a_3) \quad \text{weakly in } (L^2(\Omega^a))^3, \\ \\ f^b_n \rightharpoonup f^b = \left(f^b_1, f^b_2, f^b_3\right) \quad \text{weakly in } \left(L^2\left(\Omega^b\right)\right)^3. \end{cases} \tag{0.5.16}$$

0.5.1 The main results

The case $\ell \in]0, +\infty[$.

We state the limit result of problem (0.5.15) when in assumption (0.5.1) ℓ belongs to $]0,+\infty[$. Set

$$\left\{\begin{array}{l} P=\Big\{\quad \left(q^a,q^b\right)=\left(\left(q_2^a,q_3^a\right),\left(q_1^b,q_2^b\right)\right)\in \\ \\ \qquad \left(H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\times\left]0,1\right[\right)\right)^2\times\left(H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right)\right)^2: \\ \\ \qquad q^a\cdot\nu^a=0 \text{ on } \partial\left(\left]-\frac{1}{2},\frac{1}{2}\right[\times\left]0,1\right[\right), \qquad q^b\cdot\nu^b=0 \text{ on } \partial\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right), \\ \\ \qquad q_1^b(0,\cdot)=0, \quad q_2^a(\cdot,0)=q_2^b\left(0,\cdot\right) \text{ in } \left]-\frac{1}{2},\frac{1}{2}\right[\Big\}, \end{array}\right. \tag{0.5.17}$$

where ν^a and ν^b denote the unit outer normal on $\partial\left(\left]-\frac{1}{2},\frac{1}{2}\right[\times\left]0,1\right[\right)$ and $\partial\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right)$, respectively,

$$\left\{\begin{array}{l} U=\Big\{\quad \left(\psi^a,\psi^b\right)\in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\times\left]0,1\right[\right)\times H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right): \\ \qquad\qquad \psi^a(\cdot,0)=\psi^b(0,\cdot) \text{ in } \left]-\frac{1}{2},\frac{1}{2}\right[\Big\}, \end{array}\right. \tag{0.5.18}$$

and

$$\left\{\begin{array}{l} E:\left(q^a,q^b\right)\in P\longrightarrow \displaystyle\int_{\left]-\frac{1}{2},\frac{1}{2}\right[\times]0,1[}\Big(\beta|\mathrm{rot}q^a|^2+|\mathrm{div}q^a|^2+\alpha(|q^a|^2-1)^2 \\ \qquad\qquad +|D\psi^a_{(q^a,q^b)}|^2+\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_2^a,f_3^a)dx_1\cdot q^a\Big)dx_2dx_3 \\ \\ \qquad +\ell\displaystyle\int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2}\Big(\beta\left|\mathrm{rot}q^b\right|^2+\left|\mathrm{div}q^b\right|^2+\alpha\left(\left|q^b\right|^2-1\right)^2 \\ \qquad\qquad +\left|D\psi^b_{(q^a,q^b)}\right|^2+\displaystyle\int_{-1}^{0}\left(f_1^b,f_2^b\right)dx_3\cdot q^b\Big)\,dx_1dx_2, \end{array}\right. \tag{0.5.19}$$

where (f_2^a, f_3^a) and (f_1^b, f_2^b) are defined in (0.5.16), and $(\psi^a_{(q^a,q^b)}, \psi^b_{(q^a,q^b)})$ is the unique solution of

$$\begin{cases} \left(\psi^a_{(q^a,q^b)}, \psi^b_{(q^a,q^b)}\right) \in U, \quad \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,1[} \psi^a_{(q^a,q^b)} dx_2 dx_3 = 0, \\ \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,1[} \left(\left(-D_{(x_2,x_3)}\psi^a_{(q^a,q^b)} + q^a\right) \cdot D_{(x_2,x_3)}\psi^a\right) dx_2 dx_3 \\ \displaystyle +\ell \int_{]-\frac{1}{2},\frac{1}{2}[^2} \left(\left(-D_{(x_1,x_2)}\psi^b_{(q^a,q^b)} + q^b\right) \cdot D_{(x_1,x_2)}\psi^b\right) dx_1 dx_2 = 0 \quad \forall \left(\psi^a, \psi^b\right) \in U. \end{cases} \tag{0.5.20}$$

We prove the following result.

Theorem 0.5.1. *Assume (0.5.1) with $\ell \in]0, +\infty[$, and (0.5.16). For every $n \in \mathbb{N}$, let $\left(p_n^a, p_n^b\right)$ be a solution of (0.5.15), and $\left(\phi^a_{(p_n^a,p_n^b)}, \phi^b_{(p_n^a,p_n^b)}\right)$ be the unique solution of (0.5.13) with $\left(p^a, p^b\right) = \left(p_n^a, p_n^b\right)$. Moreover, let P and E be defined by (0.5.17) and (0.5.19)-(0.5.20), respectively. Then, there exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$ and (in possible dependence on the subsequence) $\left(\hat{p}^a, \hat{p}^b\right) = \left(\left(\hat{p}_2^a, \hat{p}_3^a\right), \left(\hat{p}_1^b, \hat{p}_2^b\right)\right) \in P$ such that*

$$\begin{cases} p^a_{n_i} \to (0, \hat{p}_2^a, \hat{p}_3^a) \text{ strongly in } \left(H^1(\Omega^a)\right)^3 \text{ and strongly in } \left(L^4(\Omega^a)\right)^3, \\ p^b_{n_i} \to \left(\hat{p}_1^b, \hat{p}_2^b, 0\right) \text{ strongly in } \left(H^1(\Omega^b)\right)^3 \text{ and strongly in } \left(L^4(\Omega^b)\right)^3, \end{cases} \tag{0.5.21}$$

$$\left(\frac{1}{h_n^a}\frac{\partial p_n^a}{\partial x_1}, \frac{1}{h_n^b}\frac{\partial p_n^b}{\partial x_3}\right) \to (0,0) \text{ strongly in } \left(L^2(\Omega^a)\right)^3 \times \left(L^2(\Omega^b)\right)^3, \tag{0.5.22}$$

$$\begin{cases} \left(\phi^a_{\left(p^a_{n_i},p^b_{n_i}\right)}, \phi^b_{\left(p^a_{n_i},p^b_{n_i}\right)}\right) \to \left(\psi^a_{(\hat{p}^a,\hat{p}^b)}, \psi^b_{(\hat{p}^a,\hat{p}^b)}\right) \text{ strongly in } H^1(\Omega^a) \times H^1(\Omega^b), \\ \left(\dfrac{1}{h_n^a}\dfrac{\partial \phi^a_{(p_n^a,p_n^b)}}{\partial x_1}, \dfrac{1}{h_n^b}\dfrac{\partial \phi^b_{(p_n^a,p_n^b)}}{\partial x_3}\right) \to (0,0) \text{ strongly in } L^2(\Omega^a) \times L^2\left(\Omega^b\right), \end{cases} \tag{0.5.23}$$

where $\left(\hat{p}^a, \hat{p}^b\right)$ solves

$$E\left(\left(\hat{p}^a, \hat{p}^b\right)\right) = \min\left\{E\left(\left(q^a, q^b\right)\right) : \left(q^a, q^b\right) \in P\right\}, \tag{0.5.24}$$

and $\left(\psi^a_{(\hat{p}^a,\hat{p}^b)}, \psi^b_{(\hat{p}^a,\hat{p}^b)}\right)$ *is the unique solution of* (0.5.20) *with* $(q^a, q^b) = \left(\hat{p}^a, \hat{p}^b\right)$. *Moreover,*

$$\lim_n \frac{E_n\left(\left(p_n^a, p_n^b\right)\right)}{h_n^a} = E\left(\left(\hat{p}^a, \hat{p}^b\right)\right). \tag{0.5.25}$$

The case $\ell = 0$.

We state the limit result of problem (0.5.15) when in (0.5.1) $\ell = 0$. Set

$$P_\circ = \left\{q^a = (q_2^a, q_3^a) \in \left(H^1\left(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[\times \left]0,1\right[\right)\right)^2 : q^a \cdot \nu^a = 0 \text{ on } \partial\left(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[\times \left]0,1\right[\right)\right\} \tag{0.5.26}$$

where ν^a denotes the unit outer normal on $\partial\left(\left]-\frac{1}{2}, \frac{1}{2}\right[\times \left]0,1\right[\right)$, and

$$\begin{cases} E_\circ : q^a \in \left(H^1\left(\left]-\frac{1}{2}, \frac{1}{2}\right[\times \left]0,1\right[\right)\right)^2 \longrightarrow \displaystyle\int_{\left]-\frac{1}{2},\frac{1}{2}\right[\times\left]0,1\right[} \Big(\beta|\mathrm{rot} q^a|^2 \\ \\ \qquad +|\mathrm{div} q^a|^2 + \alpha(|q^a|^2 - 1)^2 + |D\psi^a_{q^a}|^2 + \displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} (f_2^a, f_3^a) dx_1 \cdot q^a \Big) dx_2 dx_3, \end{cases} \tag{0.5.27}$$

where (f_2^a, f_3^a) is defined in (0.5.16), and $\psi^a_{q^a}$ is the unique solution of

$$\begin{cases} \psi^a_{q^a} \in H^1\left(\left]-\frac{1}{2}, \frac{1}{2}\right[\times \left]0,1\right[\right), \quad \displaystyle\int_{\left]-\frac{1}{2},\frac{1}{2}\right[\times\left]0,1\right[} \psi^a_{q^a} dx_2 dx_3 = 0, \\ \\ \displaystyle\int_{\left]-\frac{1}{2},\frac{1}{2}\right[\times\left]0,1\right[} \left(\left(-D_{(x_2,x_3)}\psi^a_{q^a} + q^a\right) \cdot D_{(x_2,x_3)}\psi^a\right) dx_2 dx_3 = 0 \\ \\ \forall \psi^a \in H^1\left(\left]-\frac{1}{2}, \frac{1}{2}\right[\times \left]0,1\right[\right), \end{cases} \tag{0.5.28}$$

We prove the following result.

Theorem 0.5.2. *Assume* (0.5.1) *with* $\ell = 0$, *and* (0.5.16). *For every* $n \in \mathbb{N}$, *let* $\left(p_n^a, p_n^b\right)$ *be a solution of* (0.5.15), *and* $\left(\phi^a_{(p_n^a,p_n^b)}, \phi^b_{(p_n^a,p_n^b)}\right)$ *be the unique solution of* (0.5.13) *with* $\left(p^a, p^b\right) = \left(p_n^a, p_n^b\right)$. *Moreover, let* $P_\circ$ *and* $E_\circ$ *be defined by* (0.5.26) *and* (0.5.27)-(0.5.28), *respectively. Then, there exist an increasing sequence of positive integer numbers* $\{n_i\}_{i\in\mathbb{N}}$ *and (in possible dependence on the*

subsequence) $\hat{p}^a = (\hat{p}_2^a, \hat{p}_3^a) \in P_\circ$ *such that*

$$\begin{cases} p_{n_i}^a \to (0, \hat{p}_2^a, \hat{p}_3^a) \text{ strongly in } \left(H^1(\Omega^a)\right)^3 \text{ and strongly in } \left(L^4(\Omega^a)\right)^3, \\ \\ \left(\dfrac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} p_n^b \to 0 \text{ strongly in } \left(H^1(\Omega^b)\right)^3 \text{ and strongly in } \left(L^4(\Omega^b)\right)^3, \end{cases} \tag{0.5.29}$$

$$\left(\frac{1}{h_n^a}\frac{\partial p_n^a}{\partial x_1}, \left(\frac{1}{h_n^a h_n^b}\right)^{\frac{1}{2}} \frac{\partial p_n^b}{\partial x_3}\right) \to (0,0) \text{ strongly in } \left(L^2(\Omega^a)\right)^3 \times \left(L^2\left(\Omega^b\right)\right)^3, \tag{0.5.30}$$

$$\begin{cases} \left(\phi^a_{\left(p^a_{n_i}, p^b_{n_i}\right)}, \left(\dfrac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} \phi^b_{(p_n^a, p_n^b)}\right) \to \left(\psi^a_{\hat{p}^a}, 0\right) \text{ strongly in } H^1(\Omega^a) \times H^1(\Omega^b), \\ \\ \left(\dfrac{1}{h_n^a}\dfrac{\partial \phi^a_{(p_n^a, p_n^b)}}{\partial x_1}, \left(\dfrac{1}{h_n^a h_n^b}\right)^{\frac{1}{2}} \dfrac{\partial \phi^b_{(p_n^a, p_n^b)}}{\partial x_3}\right) \to (0,0) \text{ strongly in } L^2(\Omega^a) \times L^2\left(\Omega^b\right) \end{cases} \tag{0.5.31}$$

where $\hat{p}^a$ *solves*

$$E_\circ\left(\hat{p}^a\right) = \min\left\{E_\circ\left(q^a\right) : q^a \in P_\circ\right\}, \tag{0.5.32}$$

and $\psi^a_{\hat{p}^a}$ *is the unique solution of* (0.5.28) *with* $q^a = \hat{p}^a$. *Moreover,*

$$\lim_n \frac{E_n\left(\left(p_n^a, p_n^b\right)\right)}{h_n^a} = E_\circ\left(\hat{p}^a\right). \tag{0.5.33}$$

The case $\ell = +\infty$.

We state the limit behavior of (0.5.15) when (0.5.1) is assumed with $\ell = +\infty$ and $h_n^b << \sqrt{h_n^a}$. Here we assume that the function $(\phi^a_{(p^a,p^b)}, \phi^b_{(p^a,p^b)})$ involved in (0.5.12) is the unique solution of

the following problem:

$$\begin{cases} \left(\phi^a_{(p^a,p^b)}, \phi^b_{(p^a,p^b)}\right) \in U_n, \quad \displaystyle\int_{\Omega^b} \phi^b_{(p^a,p^b)} dx = 0, \\ \\ h^a_n \displaystyle\int_{\Omega^a} ((-D^a_n \phi^a_{(p^a,p^b)} + p^a) \cdot D^a_n \phi^a) dx \\ \\ +h^b_n \displaystyle\int_{\Omega^b} \left(\left(-D^b_n \phi^b_{(p^a,p^b)} + p^b\right) \cdot D^b_n \phi^b\right) dx = 0 \quad \forall\, (\phi^a, \phi^b) \in U_n, \end{cases} \tag{0.5.34}$$

i.e. assumption $\displaystyle\int_{\Omega^a} \phi^a_{(p^a,p^b)} dx = 0$ is replaced with $\displaystyle\int_{\Omega^b} \phi^b_{(p^a,p^b)} dx = 0$, or equivalently, in (0.5.14), assumption $\displaystyle\int_{\Omega^a_n} \varphi_{\mathbf{P}} dx = 0$ is replaced with $\displaystyle\int_{\Omega^b_n} \varphi_{\mathbf{P}} dx = 0$. Obviously, $\mathcal{E}_n$ and E_n do not change. Set

$$\begin{cases} P_\infty = \Big\{ \quad q^b = (q^b_1, q^b_2) \in \left(H^1\left(\left]-\frac{1}{2}, \frac{1}{2}\right[^2\right)\right)^2 \,:\, q^b \cdot \nu^b = 0 \text{ on } \partial\left(\left]-\frac{1}{2}, \frac{1}{2}\right[^2\right), \\ \\ \qquad\qquad q^b_1(0,\cdot) = 0 \text{ in } \left]-\frac{1}{2}, \frac{1}{2}\right[\Big\}, \end{cases} \tag{0.5.35}$$

where ν^b denotes the unit outer normal on $\partial\left(\left]-\frac{1}{2}, \frac{1}{2}\right[^2\right)$, and

$$\begin{cases} E_\infty : q^b \in \left(H^1\left(\left]-\frac{1}{2}, \frac{1}{2}\right[\right)\right)^2 \longrightarrow \displaystyle\int_{\left]-\frac{1}{2}, \frac{1}{2}\right[^2} \Big(\beta |\mathrm{rot} q^b|^2 \\ \\ \qquad +|\mathrm{div} q^b|^2 + \alpha(|q^b|^2 - 1)^2 + |D\psi^b_{q^b}|^2 + \displaystyle\int_1^0 (f^b_1, f^b_2) dx_3 \cdot q^b \Big) dx_1 dx_2, \end{cases} \tag{0.5.36}$$

where (f_1^b, f_2^b) is defined in (0.5.16), and $\psi^b_{q^b}$ is the unique solution of

$$\begin{cases} \psi^b_{q^b} \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right), \quad \displaystyle\int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} \psi^b_{q^b} dx_1 dx_2 = 0, \\ \\ \displaystyle\int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} \left(\left(-D_{(x_1,x_2)}\psi^b_{q^b} + q^b\right)\cdot D_{(x_1,x_2)}\psi^b\right) dx_1 dx_2 = 0, \quad \forall \psi^b \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right). \end{cases} \tag{0.5.37}$$

We prove the following result.

Theorem 0.5.3. *Assume (0.5.1) with $\ell = +\infty$ and $h_n^b << \sqrt{h_n^a}$, and (0.5.16). For every $n \in \mathbb{N}$, let $\left(p_n^a, p_n^b\right)$ be a solution of (0.5.15), and $\left(\phi^a_{(p_n^a,p_n^b)}, \phi^b_{(p_n^a,p_n^b)}\right)$ be the unique solution of (0.5.34) with $\left(p^a, p^b\right) = \left(p_n^a, p_n^b\right)$. Moreover, let P_∞ and E_∞ be defined in (0.5.35) and (0.5.36)-(0.5.37), respectively. Then, there exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$ and (in possible dependence on the subsequence) $\hat{p}^b = \left(\hat{p}_1^b, \hat{p}_2^b\right) \in P_\infty$ such that*

$$\begin{cases} \left(\frac{h_n^a}{h_n^b}\right)^{\frac{1}{2}} p_n^a \to 0 \text{ strongly in } \left(H^1(\Omega^a)\right)^3 \text{ and strongly in } \left(L^4(\Omega^a)\right)^3, \\ \\ p^b_{n_i} \to \left(\hat{p}_1^b, \hat{p}_2^b, 0\right) \text{ strongly in } \left(H^1(\Omega^b)\right)^3 \text{ and strongly in } \left(L^4(\Omega^b)\right)^3 \end{cases} \tag{0.5.38}$$

$$\left(\left(\frac{1}{h_n^a h_n^b}\right)^{\frac{1}{2}} \frac{\partial p_n^a}{\partial x_1}, \frac{1}{h_n^b}\frac{\partial p_n^b}{\partial x_3}\right) \to 0 \text{ strongly in } \left(L^2(\Omega^a)\right)^3 \times \left(L^2(\Omega^b)\right)^3, \tag{0.5.39}$$

$$\begin{cases} \left(\left(\frac{h_n^a}{h_n^b}\right)^{\frac{1}{2}} \phi^a_{\left(p^a_{n_i},p^b_{n_i}\right)}, \phi^b_{\left(p^a_{n_i},p^b_{n_i}\right)}\right) \to \left(0, \psi^b_{\hat{p}^b}\right) \text{ strongly in } H^1(\Omega^a)\times H^1(\Omega^b), \\ \\ \left(\left(\frac{1}{h_n^a h_n^b}\right)^{\frac{1}{2}} \frac{\partial \phi^a_{(p_n^a,p_n^b)}}{\partial x_1}, \frac{1}{h_n^b}\frac{\partial \phi^b_{(p_n^a,p_n^b)}}{\partial x_3}\right) \to (0,0) \text{ strongly in } L^2(\Omega^a)\times L^2\left(\Omega^b\right), \end{cases} \tag{0.5.40}$$

where $\hat{p}^b$ solves

$$E_\infty\left(\hat{p}^b\right) = \min\left\{E_\infty\left(q^b\right) : q^b \in P_\infty\right\}, \tag{0.5.41}$$

and $\psi^b_{\hat{p}^b}$ is the unique solution of (0.5.37) with $q^b = \hat{p}^b$. Moreover, the convergence of the energies holds true, that is

$$\lim_n \frac{E_n\left(\left(p_n^a, p_n^b\right)\right)}{h_n^b} = E_\infty\left(\hat{p}^b\right). \tag{0.5.42}$$

Chapter 1

Asymptotic analysis for micromagnetics of nanowires of finite length governed by indefinite material coefficients

K. Chacouche. Asymptotic analysis for micromagnetics of nanowires of finite length governed by indefinite material coeficients, *to submit.*

Abstract. The objective of this paper, is dealt with a class of minimization problems, associated with the micromagnetics nanowires of finite length. Each minimization problem is characterized by the radius of the wire, denote by $0 < h < 1$, and it is considered under spatial indefinite and degenerative setting of the material coefficients. Based on the fundamental studies of the governing energy functionals, the existence of the minimizers, for every $0 < h < 1$, the $3D-1D$ asymptotic analysis for the studying minimization problems, as $h \to 0$, will be proved in Theorem 1.2.1 and Theorem 1.3.1, respectively.

Keywords: micromagnetics of nanowires, indefinite and degenerative material coefficient, $3D-1D$ asymptotic analysis.

2010 AMS subject classifications: 78A25, 49S05, 78M35

1.1 Introduction

In general, the theory of micromagnetism is used to model the particles ferromagnetic of small size (in the order of micrometer and less). This theory is proposed by W.F. Brown in the 40s, and aims to explain the nonlinear behavior of magnetic materials. The approach micromagnetic it is

first of all energetic, allowing to identify the main phenomena intervener in the configuration of the magnetization and associate their energy quantities. The equilibrium is achieved when the total energy is minimized.

In this paper, we study the energy released by a cylindrical ferromagnetic nanowires. The first main objective is to show, in the framework studied in [41–43, 63], that the corresponding minimization problem to the free total energy functional has at least one solution and we will perform it in the Theorem 1.2.1. The second objective by referring to the theories, that are studied in [13, 14, 33–35, 42, 43, 48, 59], is to rigorously derive a $1D$ reduced micromagnetic model for ferromagnetic nanowires and to find a definite association between the minimization problem in the three-dimensional domain and the limiting profile minimization in the one dimensional domain, which it will be proved in Theorem 1.3.1.

Let us assume that the finite $3D$ wire is a cylinder of length $2L$, and radius $0 < h < 1$. Consider a ferromagnetic nanowires occupying a bounded, possibly multi-connected domain $\Omega^{(h)} =]-L, L[\times B_2(0,h)$ of the Euclidean space $\mathbb{R}^3$ (see Figure 1.1) , where $]-L, L[$ is the bounded segment in $\mathbb{R}$, and $B_d(x,r)$ is the ball of radius r and center x in $\mathbb{R}^d$. Also, let us $\Omega :=]-L, L[\times B_2(0,1)$. Let $\alpha : \overline{\Omega} \to [0,\infty)$ be a given continuous function, and let $A_0 := \alpha^{-1}(0)$ be the set of zero-points of α on $\overline{\Omega}$.

The aim of this paper is to study the minimization and the asymptotic behavior, as $h \to 0$, of the following non-convex and nonlocal problem:

$$(P)^{(h)} = \begin{cases} \text{find a vectorial function } \underline{m}^{(h)} = (\underline{m}_1^{(h)}, \underline{m}_2^{(h)}, \underline{m}_3^{(h)}) \in L^2(\Omega^{(h)}, \mathbb{R}^3) \\ \text{of three variables, such that,} \\ \underline{\mathcal{E}}^{(h)}(\underline{m}^{(h)}) = \min\limits_{m \in L^2(\Omega^{(h)}, \mathbb{R}^3)} \underline{\mathcal{E}}^{(h)}(m), \end{cases}$$

where, the functional $\underline{\mathcal{E}}^{(h)}$ denotes the micromagnetism energy in $\Omega^{(h)}$ (see [9]), it is given by

$$\underline{\mathcal{E}}^{(h)}(m) := \begin{cases} \dfrac{1}{2L\pi h^2}\left(\displaystyle\int_{\Omega^{(h)}\setminus A_0} \alpha|\nabla m|^2 d\mathcal{L}^3 + \int_{\Omega^{(h)}} \varphi(m) d\mathcal{L}^3 + \frac{1}{2}\int_{\Omega^{(h)}} \nabla\zeta_{mag}\cdot m d\mathcal{L}^3\right), \\ \text{if } m \in H^1_{loc}\left(\Omega^{(h)}\setminus A_0, \mathbb{R}^3\right) \text{ and } \sqrt{\alpha}\nabla m \in L^2\left(\Omega^{(h)}\setminus A_0, \mathbb{R}^{3\times 3}\right), \\ \infty, \qquad \text{otherwise,} \end{cases} \tag{1.1.1}$$

$$\text{for any } m = (m_1, m_2, m_3) \in L^2\left(\Omega^{(h)}, \mathbb{R}^3\right),$$

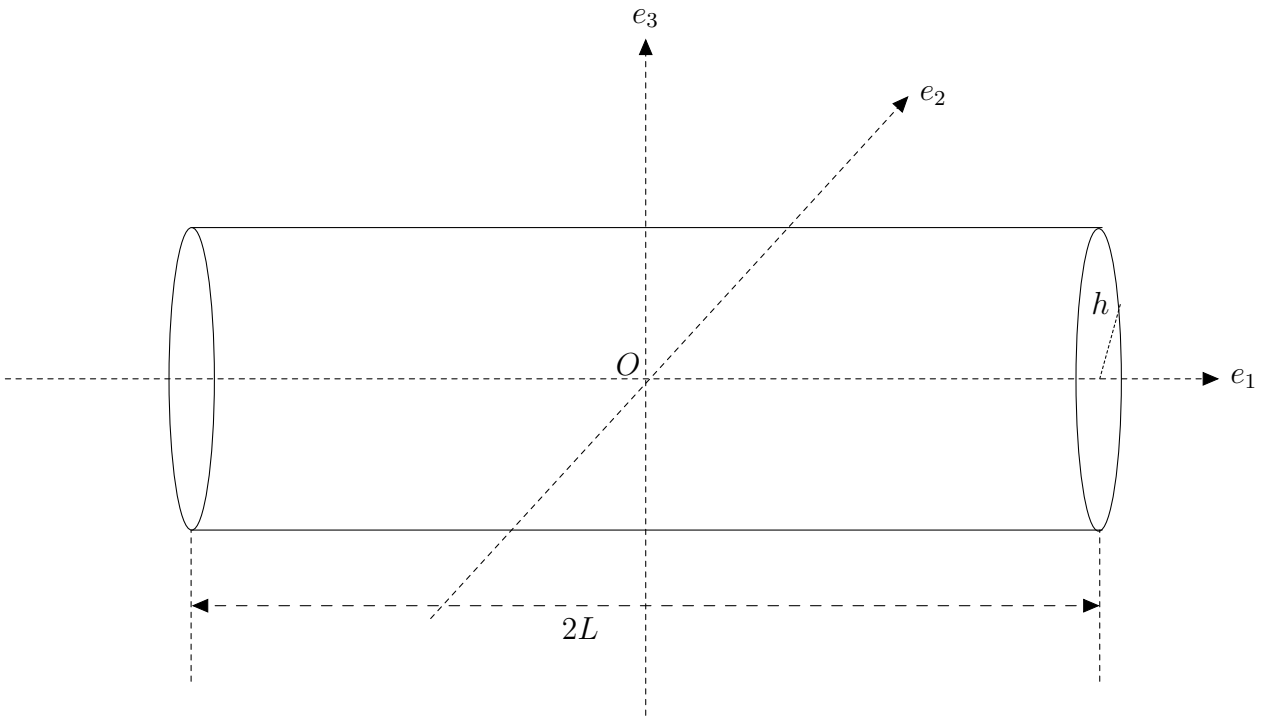

Figure 1.1: Domain $\Omega^{(h)}$.

subject to the constraints:

$$\operatorname{div}(-\nabla\zeta_{mag} + \overline{m}) = 0, \text{ in } \mathbb{R}^3, \tag{1.1.2}$$

$$|m| = m_s, \quad \mathcal{L}^3\text{-a.e. in } \Omega^{(h)}. \tag{1.1.3}$$

In (1.1.1), the funcional $\underline{\mathcal{E}}^{(h)}(m)$ is supposed to be the free energy, per unit volume, in a ferromgnetic nanowires (cf. Brown [9]). In the context, the index h and $\Omega^{(h)}$ denote the radius of the ball and the distribution region of the magnetic nanowires , respectively, and the unknown $m : \Omega^{(h)} \to \mathbb{R}^3$, $m = (m_1(x), m_2(x), m_3(x))$ $(x = (x_1, x_2, x_3) \in \Omega^{(h)})$ is a vectorial function of three variables, which describes the magnetization in $\Omega^{(h)}$. The given continuous function $\alpha = \alpha(x)$ $(x \in \overline{\Omega})$ is the so-called material coefficient, and here, it is supposed that is coefficient may degenerate somewhere on $\overline{\Omega}$.

The *exchange energy*

$$\int_{\Omega^{(h)}\setminus A_0} \alpha|\nabla m|^2 d\mathcal{L}^3.$$

The ferromagnetic behavior is essentially due to a quantistic force which tends to allign the molecular magnetic field. The most important contribution is due to the exchange energy.

The *anisotropy energy*

$$\int_{\Omega^{(h)}} \varphi(m) d\mathcal{L}^3,$$

models the existence of preferred directions of magnetization (*easy axes*), where $\varphi : \mathbb{R}^3 \to [0, \infty)$ is a given continuous, even function, exhibiting, whenever pertinent, crystallographic symmetry.

The scalar function $\zeta_{mag} : \mathbb{R}^3 \mapsto \mathbb{R}$ is supposed to be the potential of magnetic field $H \equiv -\nabla\zeta_{mag}$, whereby (1.1.2) $\nabla \times H = 0$ is automatically fulfilled, where equation (1.1.2) is a simplified version of the Maxwell equation, and hence its solution ζ_{mag}. Moreover, $\overline{m}$ denotes the zero-extension of m to $\mathbb{R}^3$.
The *energy due to the magnetic field* $H \equiv -\nabla\zeta_{mag}$ *(magnetostatic energy).*

$$\int_{\Omega^{(h)}} \nabla\zeta_{mag} \cdot m d\mathcal{L}^3 = \int_{\mathbb{R}^3} |\nabla\zeta_{mag}|^2 d\mathcal{L}^3.$$

Equation (1.1.3) embodies a fundamental constraint of micromagnetics, whereby a ferromagnetic body is always locally magnetized to a *saturation magnetization* $m_s(T)$, where T is the local temperature. Denoting the Curie temperature by T_c, $m_s > 0$ unless $T \geqslant T_c$, in the latter case $m_s = 0$ and the material ceases to behave ferromagnetically (for example the Curie temperature is 1043 K i.e. 770 $^\circ C$ for the iron). As a consequence of (1.1.3), a specimen at $T < T_c$ can achieve a demagnetized state only in an average sense. Throughout this work we suppose constant temperature in the ferromagnetic regime.

Reformulating the problem on a fixed domain through appropriate rescalings of the kind proposed by P. G. and P. Destuynder [17] and using the ideas were used by [42], it is proved the existence of a minimizers of problem $(P)^{(h)}$ (see Theorem 1.2.1). Also, by using the ideas as in [13, 14, 33–35, 41], we derive the limit problem (see Theorem 1.3.1). Specifically, we prove that

$$\lim_h (P)^{(h)} = \min\left\{\mathcal{E}^\circ(m) : \quad m = (m_1, m_2, m_3) \in L^2(]-L, L[, \mathbb{R}^3)\right\}.$$

where

$$\mathcal{E}^{\circ}(m) := \begin{cases} \Phi^{\circ}_{\alpha}(m) + \displaystyle\int_{]-L,L[} \varphi(m) d\mathcal{L}^1 + \frac{1}{2}\int_{]-L,L[} \left(|m_2|^2 + |m_3|^2\right) d\mathcal{L}^1, \\ \text{if } m \in L^2(]-L,L[,\mathbb{S}^2), \\ \\ \infty, \qquad\qquad \text{otherwise}, \end{cases}$$

for any $m = (m_1, m_2, m_3) \in L^2(]-L, L[, \mathbb{R}^3)$, and Φ°_{α} iz the convex function on $L^2(]-L, L[, \mathbb{R}^3)$, defined as:

$$\Phi^{\circ}_{\alpha}(m) := \begin{cases} \displaystyle\int_{]-L,L[\setminus A^{\circ}_0} \alpha^{\circ} |\nabla m|^2 d\mathcal{L}^1, \text{ if } m \in H^1_{loc}(]-L, L[\setminus A^{\circ}_0, \mathbb{R}^3), \\ \\ \infty, \qquad\qquad \text{otherwise}, \end{cases}$$

$$\text{for any } m = (m_1, m_2, m_3) \in L^2(]-L, L[, \mathbb{R}^3).$$

We obtain a infinite $1D$ reduced micromagnetic model for ferromagnetic nanowires. Moreover, the magnetostatic energy transforms into $\frac{1}{2}\displaystyle\int_{]-L,L[} \left(|m_2|^2 + |m_3|^2\right) d\mathcal{L}^1$, so that the limit problem is completely local and plays a role of additional anisotropy. In such wires, the additional anisotropy is uniaxial (directed along the wire). It easy to see that, if $\varphi = 0$, then the minimum in the limit problem is zero and it is attained by $(-1, 0, 0)$ or $(+1, 0, 0)$ (see Section 1.3).

The paper is organized as follows : In the next Section 1.2, we prove the existence of minimizers of problem $(P)^{(h)}$. The $3D - 1D$ asymptotic analysis for the observing minimization problem, it is obtained in Section 1.3.

1.2 The minimization problem

1.2.1 Nanowires scaling

As it is usual (see [17]), problem $(P)^{(h)}$ will be reformulated on a fixed domain.
From now on, for simplicity, let us set :

$$\mathcal{L}^3(\Omega) = 2L\pi = 1, \text{ and } m_s = 1;$$

and let us denote by $\psi^{(h)}$ the diffeomorphism, defined as:

$$\psi^{(h)} : x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mapsto (x_1, hx_2, hx_3) \in \mathbb{R}^3.$$

Also, let us put

$$\alpha^{(h)} := \alpha \circ \psi^{(h)} \in C(\overline{\Omega}) \text{ and } A_0^{(h)} := \left(\alpha^{(h)}\right)^{-1}(0).$$

Next, using the area formula in the fundamental calculations, it can be shown that the minimization problem $(P)^{(h)}$, for any $0 < h < 1$, has the below equivalent form, denoted by $(P')^{(h)}$. $(P')^{(h)}$ is for finding a vectorial function $m^{(h)} = (m_1^{(h)}, m_2^{(h)}, m_3^{(h)}) \in L^2(\Omega, \mathbb{R}^3)$ of three variables, which minimizes the following functional:

$$\mathcal{E}^{(h)}(m) := \begin{cases} \Phi_\alpha^{(h)}(m) + \displaystyle\int_\Omega \varphi(m)\, d\mathcal{L}^3 + \frac{1}{2}\int_\Omega \left(\partial_1\zeta\, m_1 + \tfrac{1}{h}\partial_2\zeta\, m_2 + \tfrac{1}{h}\partial_3\zeta\, m_3 \right) d\mathcal{L}^3, \\ \text{if } m \in L^2(\Omega, \mathbb{S}^2), \\ \infty, \qquad \text{otherwise}, \end{cases}$$

(1.2.1)

for any $m = (m_1, m_2, m_3) \in L^2(\Omega, \mathbb{R}^3)$,

subject to the constraints:

$$\partial_1\left(-\partial_1\zeta + \overline{m}_1\right) + \frac{1}{h}\left(\nabla' \cdot \left(-\frac{1}{h}\nabla'\zeta + \overline{m}'\right)\right) = 0, \text{ in } \mathbb{R}^3, \tag{1.2.2}$$

where

$$\overline{y}' = (\overline{y}_2, \overline{y}_3) \text{ for } y = (y_1, y_2, y_3) \in \mathbb{R}^3,$$

$$\nu' := (\nu_2, \nu_3) \in L^2(\Omega, \mathbb{R}^2), \text{ for } \nu = (\nu_1, \nu_2, \nu_3) \in L^2(\Omega, \mathbb{R}^3)$$

and the distributional gradient $\nabla' = (\partial_2, \partial_3)$ such that

$$\nabla'\nu = \begin{pmatrix} \partial_2\nu_1 & \partial_3\nu_1 \\ \partial_2\nu_2 & \partial_3\nu_2 \\ \partial_2\nu_3 & \partial_3\nu_3 \end{pmatrix}, \text{ for } \nu = (\nu_1, \nu_2, \nu_3) \in L^2(\Omega, \mathbb{R}^3),$$

and $\Phi_\alpha^{(h)}(m)$ is the convex function on $L^2(\Omega,\mathbb{R}^3)$, defined as:

$$\Phi_\alpha^{(h)}(m) := \begin{cases} \displaystyle\int_{\Omega\setminus A_0^{(h)}} \alpha^{(h)} \left(|\partial_1 m|^2 + \frac{1}{h^2}|\nabla' m|^2 \right) d\mathcal{L}^3, \\ \text{if } m \in H^1_{loc}\left(\Omega\setminus A_0^{(h)}, \mathbb{R}^3\right), \\ \\ \infty, \qquad\qquad\qquad \text{otherwise}, \end{cases} \tag{1.2.3}$$

$$\text{for any } m = (m_1, m_2, m_3) \in L^2(\Omega,\mathbb{R}^3).$$

Furthermore, for any $0 < h < 1$, the equality:

$$m^{(h)} = \underline{m}^{(h)} \circ \psi^{(h)} \text{ in } L^2(\Omega,\mathbb{R}^3), \tag{1.2.4}$$

holds between the minimizers $\underline{m}^{(h)}$ and $m^{(h)}$ of the respective problems $(P)^{(h)}$ and $(P')^{(h)}$.

In each case, the minimizer, described in (1.2.4), are supposed to represent the most probable profile of the magnetization in the studying ferromagnetic nanowires. However, when is the case of the very thin situation of the radius h, the problem $(P')^{(h)}$ / $(P)^{(h)}$ is usually reduced to another one (it will be discussed in Section 1.3).
For the detailed description of this matter, let us first set:

$$\alpha^\circ(x_1) := \alpha(x_1, 0, 0) \text{ for any } x_1 \in [-L, L], \text{ and } A_0^\circ := (\alpha^\circ)^{-1}(0).$$

Now, if we take into account the non degenerate case of the material coefficient α, referred to the case that:

$$A_0^{(h)} = A_0^\circ = \emptyset \text{ for } 0 < h < 1, \text{ and } \alpha_* := \min_{x\in\overline{\Omega}} \alpha(x) > 0,$$

then the convex part $\Phi_\alpha^{(h)}$ of the energy $\mathcal{E}^{(h)}$ satisfies the coercivity condition below:

$$\Phi_\alpha^{(h)}(m) \geq \alpha_* |\nabla m|^2_{L^2(\Omega,\mathbb{R}^{3\times 3})}, \text{ for all } m \in L^2(\Omega,\mathbb{R}^3). \tag{1.2.5}$$

1.2.2 The main results

The first goal of this study, is to impose the two conditions below for the material coefficient α:
(a) $\mathcal{L}^3(A_0) = 0$, and hence $\mathcal{L}^3(A_0^{(h)}) = 0$, for $0 < h < 1$.

(b) There exists a constant $C_\alpha \geq 1$, such that

$$\alpha^\circ(x_1) \leq \alpha(x) \leq C_\alpha \alpha^\circ(x_1), \text{ for all } x = (x_1, x_2, x_3) \in \overline{\Omega}$$

Consequently, we will show the following theorem:

Theorem 1.2.1. *Let us assume the condition (a). Then for $0 < h < 1$ the minimization problem $(P)^{(h)}$ admits at least one solution (minimizer) $m^{(h)}$, and hence the same holds for the problem $(P')^{(h)}$.*

1.2.3 Preliminaries

Notation. For any dimension $n \in \mathbb{N}$, the n-dimensional Lebesgue measure is denoted by $\mathcal{L}^n$, and for any Borel set $E \subset \mathbb{R}^n$, the characteristic function on E is denoted by χ_E.

For any abstract Banach space, the norm of X is denoted by $|\cdot|_X$. However, when X is an Euclidean space, the is simply denoted by $|\cdot|$. Also, we denote by $\mathrm{dist}_X(\xi, Y)$ the distance between any point $\xi \in X$ and any subset $Y \subset X$, that is defined as $\mathrm{dist}_X(\xi, Y) := \inf_{\delta \in Y} |\xi - \delta|_X$. Additionally, for any $\epsilon > 0$ and any functional $F : X \to [-\infty, \infty]$, we denote by $L(\epsilon, F)$ the sub-level set of F, more precisely:

$$L(\epsilon, F) := \{\xi \in X \quad | \quad F(\xi) \leq \epsilon\}.$$

For any abstract Hilbert space H, the inner product of H is denoted by $(\cdot, \cdot)_H$.
However, when H is an Euclidean space, the inner product between two vectors $\zeta, \delta \in H$ is simply denoted by $\zeta \cdot \delta$. Besides, for arbitrary $k, \ell \in \mathbb{N}$ and arbitrary k, ℓ-matrices $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{R}^{k \times \ell}$, the scalar product between these two matrices is denoted by $A : B := \sum_{i=1}^{k} \sum_{j=1}^{\ell} a_{ij} b_{ij}$.

To sum up the known-results, concerned the mathematical treatment of the coupled Maxwell equation (1.2.2), we recall what it is studied in [41, 47].

Remark 1.2.2. *(Summary of [47, Lemma 3.1]) Let us fix any constant $0 < h < 1$, and any $m = (m_1, m_2, m_3) \in L^2(\Omega, \mathbb{R}^3)$, taking into account that m vanishes outside Ω. Then, the solution of the equation (1.2.2) is defined as a function $\zeta \in V$ which solves the following variational identity:*

$$(\zeta, v) := \int_\Omega \left(m_1 \partial_1 v + \frac{1}{h} m' \cdot \nabla' v \right) d\mathcal{L}^3 = \int_{\mathbb{R}^3} \left(\partial_1 \zeta \partial_1 v + \frac{1}{h^2} \nabla' \zeta \cdot \nabla' v \right) d\mathcal{L}^3, \text{ for any } v \in V, \tag{1.2.6}$$

with $m' = (m_2, m_3)$ *and* $\nabla' = (\partial_2, \partial_3)$, *where* V, *the phase space for the Maxwell equation* (1.2.2) *is fixed as the following functional space, denoted by :*

$$V := \left\{ v \in H^1_{loc}(\mathbb{R}^3) \quad \middle| \quad \nabla v \in L^2(\mathbb{R}^3, \mathbb{R}^3) \text{ and } \int_{\tilde{B}_\Omega} v d\mathcal{L}^3 = 0 \right\},$$

and $\tilde{B}_\Omega$ *the (three-dimensional) open ball , which contains the cylindrical domain* $\Omega :=]-L, L[\times B_2(0,1)$. *The condition* $\int_{\tilde{B}_\Omega} v d\mathcal{L}^3 = 0$ *prevents trivial translations* $v \to v + c$. *As easily checked (cf. [5, Theorem 5.4.3]), this functional space is a Hilbert space, endowed with the inner product:*

$$(z, v) := \int_{\mathbb{R}^3} \left(\partial_1 z \partial_1 v + \frac{1}{h^2} \nabla' z \cdot \nabla' v \right) d\mathcal{L}^3, \text{ for all } z,\ v \in V,$$

which readily leads to the definition of the norm

$$|v|_V = (v, v)^{\frac{1}{2}}, \tag{1.2.7}$$

where $0 < h < 1$ *is the same constant as in* (1.2.2). *Furthermore, the Hilbert space* V *is compactly embedded into the space* $L^2(\tilde{B}_\Omega)$.

Remark 1.2.3. *(See [42, Proposition 1]) Let us fix any* $0 < h < 1$. *Then, for any* $m \in L^2(\Omega, \mathbb{R}^3)$, *the Maxwell equation* (1.2.2), *admits a unique solution* ζ. *Hence, the solution operator* $\mathcal{S}^{(h)} : L^2(\Omega, \mathbb{R}^3) \to V$, *that maps any* $m \in L^2(\Omega, \mathbb{R}^3)$ *to the solution* $\zeta \in V$ *of* (1.2.2), *is well-defined as a single-valued mapping. Moreover, the solution operator* $\mathcal{S}^{(h)}$ *is a bounded linear operator, such that:*

$$|\mathcal{S}^{(h)} m|_V \leq |m|_{L^2(\Omega, \mathbb{R}^3)}, \text{ for any } m \in L^2(\Omega, \mathbb{R}^3). \tag{1.2.8}$$

Remark 1.2.4. *(Sommary of [41]) For any* $0 < h < 1$, *the functional* $E^{(h)}_{mag}(m)$ *defined as*

$$E^{(h)}_{mag}(m) := \frac{1}{2} \int_\Omega \left(\partial_1 \zeta^{(h)} m_1 + \frac{1}{h} \nabla' \zeta^{(h)} \cdot m' \right) d\mathcal{L}^3, \tag{1.2.9}$$

$$\text{for any } m = (m_1, m_2, m_3) \in L^2(\Omega, \mathbb{R}^3),$$

links to the part of the free energy $\mathcal{E}^{(h)}$, *given in* (1.2.1), *that is involved in the coupled Maxwell equation* (1.2.2). *Moreover, in the light of Remark 1.2.2, setting* $v = \zeta$ *in* (1.2.6) *it follows that*

$$E^{(h)}_{mag}\left(m^{(h)}\right) = \frac{1}{2} \left|\zeta^{(h)}\right|^2_V \geq 0 \text{ for any } 0 < h < 1. \tag{1.2.10}$$

Next, let us use the Key-properties of the energy functionals.

We will start with a proposition concerned with a Hilbert space, associated with the effective domain of convex part of energy functional.

Proposition 1.2.5. *(See [42, Theorem 3.1]) Let us set:*

$$\begin{cases} A_0^\star := A_0^\circ \times B_2(0,1), \\ H_\alpha^\star := \left\{ m \in L^2(\Omega,\mathbb{R}^3) \quad \middle| \quad m \in H^1_{loc}(\Omega\backslash A_0^\star,\mathbb{R}^3), \quad \sqrt{\alpha^\circ}\nabla m \in L^2\left(\Omega\backslash A_0^\star,\mathbb{R}^{3\times 3}\right) \right\}. \end{cases} \tag{1.2.11}$$

Then, $H_\alpha^\star$ is a Hilbert space, endowed with the inner product:

$$(\xi,\Lambda)_{H_\alpha^\star} := \int_\Omega \xi\cdot\Lambda d\mathcal{L}^3 + \int_{\Omega\backslash A_0^\star} \alpha^\circ\nabla\xi : \nabla\Lambda d\mathcal{L}^3, \text{ for all } \xi,\ \Lambda \in H_\alpha^\star. \tag{1.2.12}$$

Hence, the functional $\Phi_\alpha^\star$ defined as:

$$\Phi_\alpha^\star(m) := \begin{cases} \displaystyle\int_{\Omega\backslash A_0^\star} \alpha^\circ|\nabla m|^2 d\mathcal{L}^3, & \text{if } m \in H_\alpha^\star, \\ \infty, & \text{otherwise}, \end{cases} \tag{1.2.13}$$

$$\text{for any } m=(m_1,m_2,m_3) \in L^2(\Omega,\mathbb{R}^3),$$

is proper l.s.c and convex on $L^2(\Omega,\mathbb{R}^3)$.

Just as in above proposition, we can prove the below Corollary.

Corollary 1.2.6. *Let us fix any $0<h<1$, and let us denote by $H_\alpha^{(h)}$ the effective domain of the convex function $\Phi_\alpha^{(h)}$, given in (1.2.3). Then, $H_\alpha^{(h)}$ is a Hilbert space, endowed with the inner product:*

$$(\xi,\Lambda)_{H_\alpha^{(h)}} := \int_\Omega \xi\cdot\Lambda d\mathcal{L}^3 + \int_{\Omega\backslash A_0^{(h)}} \alpha^{(h)}\left(\partial_1\xi\partial_1\Lambda + \frac{1}{h^2}\nabla'\xi\cdot\nabla'\Lambda\right) d\mathcal{L}^3,$$

$$\text{for all } \xi,\ \Lambda \in H_\alpha^{(h)}.$$

Hence, the convex function $\Phi_\alpha^{(h)}$ turns out to be proper and l.s.c on $L^2(\Omega,\mathbb{R}^3)$.

Lemma 1.2.7. *(Approximating open sets) For any $0<h<1$, there exists a sequence $\left\{\Omega_\lambda^{(h)} \quad | \quad \lambda=1,2,3,\cdots\right\} \subset \mathbb{R}^3$ of three-dimensional open sets, having Lipschitz boundaries,*

such that:

$$\emptyset \neq \Omega_1^{(h)} \subset\subset \Omega_2^{(h)} \subset\subset \Omega_3^{(h)} \subset\subset \cdots \subset\subset \Omega_\lambda^{(h)} \subset\subset \cdots \subset\subset \Omega \backslash A_0^{(h)} = \bigcup_{\lambda=1}^{\infty} \Omega_\lambda^{(h)}. \tag{1.2.14}$$

As a consequence, we infer that:

$$\begin{cases} a_\lambda^{(h)} := \min\limits_{x \in \overline{\Omega_\lambda^{(h)}}} \alpha^{(h)}(x) > 0, \quad \lambda = 1, 2, 3, \cdots, \text{ for any } 0 < h < 1. \\ \\ a_\lambda^{(h)} \to 0, \text{ as } \lambda \to \infty, \end{cases}$$

Proof. This Lemma is a direct consequence of the line of arguments, discussed in ([42], Lemma 4.1 and Remark 4]). □

Now, we generalize a result of [42, Theorem 3.2 page 7] for the three dimensional case.

Proposition 1.2.8. *(Compactness) Let us assume the condition (a), and let us take any $2 < p \leq \infty$. Then, for any $0 < h < 1$, any bounded sequence in $H_\alpha^{(h)} \cap L^p(\Omega, \mathbb{R}^3)$ is relatively compact in $L^p(\Omega, \mathbb{R}^3)$.*

Here is a corollary that is derived from the Proposition 1.2.8.

Corollary 1.2.9. *(I) Let us assume the condition (a), and let us take any $1 \leq p < 2$. Then, for any $0 < h < 1$, the Hilbert space $H_\alpha^{(h)}$ is compactly embedded into the Banach space $L^p(\Omega, \mathbb{R}^3)$.*

(II) Let us assume the condition (a), then for any $0 < h < 1$ and any $\epsilon > 0$, the sublevel set:

$$L\left(\epsilon, \mathcal{E}^{(h)}\right) := \left\{ m \in L^2(\Omega, \mathbb{R}^3) \quad | \quad \mathcal{E}^{(h)}(m) \leq \epsilon \right\},$$

is compact in $L^2(\Omega, \mathbb{R}^3)$.

Proof of Proposition 1.2.8. (The proof of the two dimensional case is made in [42, page 11])

Let us assume the condition (a), let us fix any $2 < q < p$, with $2 < p \leq \infty$, and let us set $\epsilon := \lim\limits_{\tilde{p} \to p} (\dfrac{\tilde{p}}{\tilde{p} - q})$. Besides, let us take any sequence $\left\{ u_*^{(i)} \quad | \quad i = 1, 2, 3, \cdots \right\} \subset H_\alpha^{(h)} \cap L^p(\Omega, \mathbb{R}^3)$, such that:

$$\sup_{i \in \mathbb{N}} \left| u_*^{(i)} \right|_{H_\alpha^{(h)}} \leq R_0 \text{ and } \sup_{i \in \mathbb{N}} \left| u_*^{(i)} \right|_{L^p(\Omega, \mathbb{R}^3)} \leq R_0, \tag{1.2.15}$$

for some constant R_0, independent of $i \in \mathbb{N}$. Then, noting that $\mathcal{L}^3(\Omega) = 1$, and:

$$\sup_{i\in\mathbb{N}} \left| \left|u_*^{(i)}\right|^2 \right|_{L^{\frac{q}{2}}(\Omega,\mathbb{R}^3)} = \sup_{i\in\mathbb{N}} \left|u_*^{(i)}\right|^2_{L^q(\Omega,\mathbb{R}^3)} \leq \sup_{i\in\mathbb{N}} \left|u_*^{(i)}\right|^2_{L^p(\Omega,\mathbb{R}^3)} \leq R_0^2,$$

we construct a sequence $\left\{ n_k^{(h)} \quad | \quad i = 1, 2, 3, \cdots \right\} \subset \mathbb{N}$, and functions $u_* \subset H_\alpha^{(h)}$ and $\gamma_* \in L^{\frac{q}{2}}(\Omega)$, such that:

$$\begin{cases} n_k^{(h)} \to \infty, \quad u_*^{(n_k^{(h)})} \to u_* \text{ weakly in } H_\alpha^{(h)}, \\ \text{and } \left|u_*^{(i)}\right|^2 \to \gamma_* \text{ weakly in } L^{\frac{q}{2}}(\Omega), \end{cases} \quad \text{as } k \to \infty. \tag{1.2.16}$$

The above convergence implies that:

$$\int_E \left|u_*^{(n_k^{(h)})}\right|^2 d\mathcal{L}^3 = \int_\Omega \left|u_*^{(n_k^{(h)})}\right|^2 \chi_E d\mathcal{L}^3 \to \int_\Omega \gamma_* \chi_E d\mathcal{L}^3 = \int_E \gamma_* d\mathcal{L}^3$$

$$\text{as } k \to \infty, \text{ for any Borel subset } E \subset \Omega.$$

So, applying the assumption (a) and Vitali-Hahn-Saks's theorem, we infer that

$$I_*^{(\lambda)} := \sup_{j\in\mathbb{N}} \int_{\Omega\setminus\Omega_\lambda^{(h)}} \left|u_*^{(n_j^{(h)})}\right|^2 d\mathcal{L}^3 \to 0, \text{ as } \lambda \to \infty. \tag{1.2.17}$$

Next, due to Lemma 1.2.7, the subsequence $\left\{ u_*^{(n_k^{(h)})} \quad | \quad k = 1, 2, 3, \cdots \right\} \subset \left\{ u_*^{(i)} \right\}$ turns out to be bounded in the space $H^1\left(\Omega_\lambda^{(h)}, \mathbb{R}^3\right)$, for any $\lambda \in \mathbb{N}$. Hence, Sobolev's embedding theorem enables to construct a decreasing family of subsequences:

$$\cdots \subset \left\{ n_k^{(\lambda)} \right\} \subset \cdots \subset \left\{ n_k^{(3)} \right\} \subset \left\{ n_k^{(2)} \right\} \subset \left\{ n_k^{(1)} \right\} \subset \left\{ n_k^{(h)} \right\},$$

to fulfill that:

$$\begin{cases} \bullet \text{ the subsequence } \left\{u_*^{(n_k^{(\lambda)})}\right\} \text{ admits a limit} \\ \eta_*^{(\lambda)} \in H^1\left(\Omega_\lambda^{(h)}, \mathbb{R}^3\right) \text{ in the strong topology} \\ \text{of } L^2\left(\Omega_\lambda^{(h)}, \mathbb{R}^3\right), \text{ as } k \to \infty, \\ \bullet \left|u_*^{(n_k^{(\lambda)})} - \eta_*^{(\lambda)}\right|^2_{L^2\left(\Omega_\lambda^{(h)}, \mathbb{R}^3\right)} \leq \frac{1}{\lambda}, \ k = 1, 2, 3, \cdots, \end{cases} \quad \text{for any } \lambda \in \mathbb{N}. \tag{1.2.18}$$

Now, let us set a function $\eta_* \in H^1_{loc}\left(\Omega \backslash A_0^{(h)}, \mathbb{R}^3\right)$, by putting:

$$\eta_*(x) := \eta_*^{(\lambda)}(x), \text{ if } x \in \Omega_\lambda^{(h)}, \text{ for } \mathcal{L}^3 - \text{a.e. } x \in \Omega.$$

Then, by virtue of (1.2.15) and the monotone convergence theorem,

$$\begin{aligned} \int_\Omega |\eta_*|^2 d\mathcal{L}^3 &= \lim_{\lambda\to\infty} \int_\Omega \chi_{\Omega_\lambda^{(h)}} |\eta_*|^2 d\mathcal{L}^3 \leq \sup_{\lambda\in\mathbb{N}} \int_{\Omega_\lambda^{(h)}} \left|\eta_*^{(\lambda)}\right|^2 d\mathcal{L}^3 \\ &= \sup_{\lambda\in\mathbb{N}} \left(\lim_{k\to\infty} \int_{\Omega_\lambda^{(h)}} \left|u_*^{(n_k^{(\lambda)})}\right|^2 d\mathcal{L}^3 \right) \leq \sup_{i\in\mathbb{N}} \left|u_*^{(i)}\right|^2_{H_\alpha^{(h)}} \leq R_0^2, \end{aligned} \tag{1.2.19}$$

therefore $\eta_* \in L^2(\Omega, \mathbb{R}^3)$.

Afterwards, let us set a subsequence $\left\{u_{**}^{(k)} \quad | \quad k = 1, 2, 3, \cdots\right\} \subset \left\{u_*^{(i)}\right\}$, by putting:

$$u_{**}^{(k)} = u_*^{(n_k^{(k)})} \text{ in } L^2(\Omega, \mathbb{R}^3) \text{ for } k = 1, 2, 3, \cdots.$$

Then, considering the assumption (a), and (1.2.17)-(1.2.19), we obtain that:

$$\left|u_{**}^{(k)}-\eta_*\right|^2_{L^2(\Omega,\mathbb{R}^3)}$$

$$\leq \left|u_*^{(n_k^{(k)})}-\eta_*\right|^2_{L^2\left(\Omega_k^{(h)},\mathbb{R}^3\right)}+2\left(\left|u_{**}^{(k)}\right|^2_{L^2\left(\Omega\setminus\Omega_k^{(h)},\mathbb{R}^3\right)}+\left|\eta_*\right|^2_{L^2\left(\Omega\setminus\Omega_k^{(h)},\mathbb{R}^3\right)}\right)$$

$$\leq \frac{1}{k}+2I_*^{(k)}+2\int_{\Omega\setminus\Omega_k^{(h)}}|\eta_*|^2d\mathcal{L}^3\to 0,\ \text{as } k\to\infty.$$

Thus, the subsequence $\left\{u_{**}^{(k)}\right\}$ is a convergent sequence in the topology of $L^2(\Omega,\mathbb{R}^3)$ and the limit η_* must coincide with the weak limit u_* as in (1.2.16). □

1.2.4 Proof of Theorem 1.2.1

The proof will be a slight modification of the argument, discussed in [42, Section 5.1]. In fact, under the condition (a), and under the fixed setting of $0<h<1$. Let us put $e^\star=(1,0,0)\in\mathbb{S}^2$. Then, by virtue of (1.2.1), (1.2.9) and (1.2.10),

$$0\leq E_*^{(h)}:=\inf_{m\in L^2(\Omega,\mathbb{R}^3)}\mathcal{E}^{(h)}(m)\leq\mathcal{E}^{(h)}(e^\star).$$

Therefore the infimum $E_*^{(h)}$ of $\mathcal{E}^{(h)}$ is finite in $L^2(\Omega,\mathbb{R}^3)$, we can find a *minimizing sequence* $\left\{m_*^{(i)}\ \ |\ \ i=1,2,3,\cdots\right\}\subset H_\alpha^{(h)}$, such that:

$$\mathcal{E}^{(h)}(m_*^{(i)})\searrow E_*^{(h)}\ \text{as } i\to\infty.$$

Here, on account of (II) in Corollary 1.2.9, and the constraint onto $L^2(\Omega,\mathbb{S}^2)$ as in (1.2.1), a convergence subsequence $\left\{m_*^{(i_k)}\ \ |\ \ k=1,2,3,\cdots\right\}\subset\left\{m_*^{(i)}\right\}$ will be found with the limit $m_*\in L^2(\Omega,\mathbb{R}^3)$, and such that:

$$\begin{cases} m_*^{(i_k)}\to m_*\ \text{in } L^2(\Omega,\mathbb{R}^3), \\ \varphi(m_*^{(i_k)})\to\varphi(m_*),\ \text{in } L^1(\Omega), \end{cases}\qquad \text{as } k\to\infty.$$

Let $\zeta_*^{(k)}$ the solution of the coupled Maxwell equation (1.2.2), when $m = m_*^{(i_k)}$, for any $k \in \mathbb{N}$, and also, let ζ_* the solution of (1.2.2), when $m = m_*$. Then, by Remark 1.2.3,

$$\zeta_*^{(k)} \to \zeta_* \text{ in } V, \quad \text{as } k \to \infty. \tag{1.2.20}$$

Thus by (1.2.10), and (1.2.20) we have,

$$E_{mag}^{(h)}\left(m_*^{(i_k)}\right) \to E_{mag}^{(h)}(m_*), \quad \text{as } k \to \infty. \tag{1.2.21}$$

Now, taking (1.2.21), and Proposition 1.2.5, we obtain that:

$$\begin{aligned} E_*^{(h)} &= \lim_{k\to\infty} \mathcal{E}^{(h)}\left(m_*^{(i_k)}\right) \\ &= \liminf_{k\to\infty} \Phi_\alpha^{(h)}\left(m_*^{(i_k)}\right) + \lim_{k\to\infty}\left(\left|\varphi\left(m_*^{(i_k)}\right)\right|_{L^1(\Omega)} + \frac{1}{2}\left|\zeta_*^{(k)}\right|_V^2\right) \\ &\geq \Phi_\alpha^{(h)}(m_*) + |\varphi(m_*)|_{L^1(\Omega)} + \frac{1}{2}|\zeta_*|_V^2 = \mathcal{E}^{(h)}(m_*) \geq E_*^{(h)}. \end{aligned}$$

Therefor, the limit m_* is the minimizer, that is denoted by $m^{(h)}$ in Theorem 1.2.1.

1.3 The $3D-1D$ asymptotic analysis for the observing minimization problem

1.3.1 The main results

Here, regarding the theories [13, 14, 33–35, 42, 43, 48, 59], to find a definite association between the limiting profile of $(P)^{(h)}$ as $h \to 0$, and the following minimization problem, denoted by $(P)^\circ$,

for the magnetization on the one dimensional domain $]-L, L[$.

$$(P)^\circ = \begin{cases} \text{find a vectorial function } m^\circ = (m_1^\circ, m_2^\circ, m_3^\circ) \in L^2(]-L, L[, \mathbb{R}^3) \\ \text{of three variables, such that,} \\ \mathcal{E}^\circ(m^\circ) = \min\limits_{m \in L^2(]-L,L[,\mathbb{R}^3)} \mathcal{E}^\circ(m), \end{cases}$$

where, the functional $\mathcal{E}^\circ$, it is given by

$$\mathcal{E}^\circ(m) := \begin{cases} \Phi_\alpha^\circ(m) + \int_{]-L,L[} \varphi(m) d\mathcal{L}^1 + \frac{1}{2}\int_{]-L,L[} \left(|m_2|^2 + |m_3|^2\right) d\mathcal{L}^1, \\ \text{if } m \in L^2(]-L, L[, \mathbb{S}^2), \\ \infty, \qquad\qquad \text{otherwise,} \end{cases} \tag{1.3.1}$$

$$\text{for any } m = (m_1, m_2, m_3) \in L^2(]-L, L[, \mathbb{R}^3),$$

where Φ_α° iz the convex function on $L^2(]-L, L[, \mathbb{R}^3)$, defined as:

$$\Phi_\alpha^\circ(m) := \begin{cases} \int_{]-L,L[\setminus A_0^\circ} \alpha^\circ |\nabla m|^2 d\mathcal{L}^1, \text{ if } m \in H^1_{loc}(]-L, L[\setminus A_0^\circ, \mathbb{R}^3), \\ \infty, \qquad\qquad \text{otherwise,} \end{cases} \tag{1.3.2}$$

$$\text{for any } m = (m_1, m_2, m_3) \in L^2(]-L, L[, \mathbb{R}^3).$$

Up to now, the above fact has been proved previously, by relying on the compactness of the sub level sets $\mathcal{E}^{(h)}$, that has been derived from the coercivity condition (1.2.5).

Now, let us consider the conditions (a) and (b) for the material coefficient α.

Consequently, we can conclude the Theorem 1.3.1, that is shown as following:

Theorem 1.3.1. *Under the conditions (a)-(b), there exist a sequence* $\{h_i \quad | \quad i = 1, 2, 3, \cdots\} \subset (0, 1)$ *and limiting function* $m^\circ \in L^2(]-L, L[, \mathbb{R}^3)$ *of one variable, such that:*

(i) $h_i \to 0$, $m^{(h_i)} \to m^\circ$ *in* $L^2(\Omega, \mathbb{R}^3)$, $\mathcal{E}^{(h_i)}\left(m^{(h_i)}\right) \to \mathcal{E}^\circ(m^\circ)$, *and*

$$\begin{cases} \sqrt{\alpha^{(h_i)}}\partial_1 m^{(h_i)}(x_1,x_2,x_3) \to \sqrt{\alpha^\circ}\partial_1 m^\circ(x_1) \ \left(= \sqrt{\alpha^\circ}\nabla m^\circ(x_1)\right), \\ \dfrac{\sqrt{\alpha^{(h_i)}}}{h_i}\partial_2 m^{(h_i)}(x_1,x_2,x_3) \to 0, \\ \dfrac{\sqrt{\alpha^{(h_i)}}}{h_i}\partial_3 m^{(h_i)}(x_1,x_2,x_3) \to 0, \end{cases} \tag{1.3.3}$$

for $\mathcal{L}^1$-a.e. $x_1 \in]-L, L[$ and $\mathcal{L}^2$-a.e. $(x_2,x_3) \in B_2(0,1)$, as $i \to \infty$,

(ii) the limit m° solves the problem $(P)^\circ$,
where $\left\{m^{(h)} \ \middle| \ 0<h<1\right\}$ is the sequence of minimizer $m^{(h)}$, $0<h<1$, obtained in Theorem 1.2.1.

Additionally, let us look toward the limiting observation for (1.2.2), as $h \to 0$. As a innovative work for this theme, we can refer to [14, Theorem 2.1], or [59, Sections 3], stated as follows.

Proposition 1.3.2. *(Summary of [14, Theorem 2.1]) Let $\left\{\tilde{m}^{(h)} \ \middle| \ 0<h<1\right\} \subset L^2(\Omega,\mathbb{R}^3)$ be a fixed sequence, such that $\tilde{m}^{(h)} \to \tilde{m}$ in $L^2(\Omega,\mathbb{R}^3)$ as $h \to 0$, for some $\tilde{m} = (\tilde{m}_1,\tilde{m}_2,\tilde{m}_3) \in L^2(\Omega,\mathbb{R}^3)$. For any $0<h<1$, let $\zeta^{(h)}$ be the solution of the Maxwell equation (1.2.2) when $m = \tilde{m}^{(h)}$. Let $E^{(h)}_{mag}$ the functional defined in (1.2.9), and let E°_{mag} be functionals on $L^2(\Omega,\mathbb{R}^3)$, which is defined as:*

$$E^\circ_{mag}(m) := \frac{1}{2}\int_\Omega \left(|m_2|^2+|m_3|^2\right) d\mathcal{L}^3, \tag{1.3.4}$$
$$\text{for any } m=(m_1,m_2,m_3) \in L^2(\Omega,\mathbb{R}^3).$$

Then,

$$\begin{cases} \nabla\zeta^{(h)} \to 0 \ \ in \ \ V, \\ \dfrac{1}{h}\partial_2\zeta^{(h)} \to \tilde{m}_2, \ \ \dfrac{1}{h}\partial_3\zeta^{(h)} \to \tilde{m}_3 \ \ in \ \ L^2(\Omega,\mathbb{R}^3), \end{cases} \quad as\ h \to 0, \tag{1.3.5}$$

and hence

$$E^{(h)}_{mag}\left(\tilde{m}^{(h)}\right) \to E^\circ_{mag}(\tilde{m}), \ as\ h \to 0.$$

Lemma 1.3.3. *There exists a sequence $\{I_\lambda \ | \ \lambda = 1,2,3,\cdots\} \subset \mathbb{R}$ of one-dimensional open interval, such that:*

$$\emptyset \neq I_1 \subset\subset I_2 \subset\subset I_3 \subset\subset \cdots \subset\subset I_\lambda \subset\subset \cdots \subset\subset]-L,L[\setminus A^\circ_0 = \bigcup_{\lambda=1}^{\infty} I_\lambda.$$

As a consequence, we infer that:

$$\begin{cases} a_\lambda^\circ := \min\limits_{x\in\overline{I_\lambda}} \alpha^\circ(x) > 0, \quad \lambda = 1, 2, 3, \cdots, \\ a_\lambda^\circ \to 0, \ as\ \lambda \to \infty; \end{cases}$$

with $\overline{I_\lambda}$ is a closed interval in $\mathbb{R}$.

Proof of Lemma 1.3.3. This Lemma is directly similar of Lemma 1.2.7, made for the one-dimensional situation. □

Remark 1.3.4. *(See [42, Section 3 and Section 4])*

(I) Similarly as in the Corollary 1.2.6, let us denote by H_α° the effective domain of the convex function Φ_α°, given in (1.3.2). Then, H_α° is a Hilbert space, endowed with the inner product:

$$(\xi, \Lambda)_{H_\alpha^\circ} := \int_{]-L,L[} \xi \cdot \Lambda d\mathcal{L}^1 + \int_{]-L,L[\setminus A_0^\circ} \alpha^\circ \nabla\xi : \nabla\Lambda d\mathcal{L}^1, \ \textit{for all}\ \xi,\ \Lambda \in H_\alpha^\circ.$$

Hence, the convex function Φ_α° turns out to be proper and l.s.c on $L^2(]-L, L[, \mathbb{R}^3)$.

(II) As it is easily checked, the two convex functions $\Phi_\alpha^\star$ and Φ_α°, as in above assumption (I), Proposition 1.2.5, and Corollary 1.2.6, coincide with as functionals on $L^2(]-L, L[, \mathbb{R}^3)$, namely:

$$\Phi_\alpha^\star(m) = \Phi_\alpha^\circ(m), \ \textit{if}\ m \in L^2(]-L, L[, \mathbb{R}^3).$$

(III) If we assume that $\mathcal{L}^1(A_0^\circ) = 0$ and let us take any $2 < p \leq \infty$, we have

(i) Any bounded sequence in $H_\alpha^\circ \cap L^p(]-L, L[, \mathbb{R}^3)$ is relatively compact in $L^p(]-L, L[, \mathbb{R}^3)$.

(ii) For any $\epsilon > 0$, the sublevel set:

$$L(\epsilon, \mathcal{E}^\circ) := \left\{ m \in L^2(\Omega, \mathbb{R}^3) \quad | \quad \mathcal{E}^\circ(m) \leq \epsilon \right\},$$

is compact in $L^2(]-L, L[, \mathbb{R}^3)$.

Remark 1.3.5. *Let us assume the condition (b), then the sequences $\left\{\Omega_\lambda^{(h)}\right\}$, $0 < h < 1$, can be taking independently of h. In fact, since the condition (b) implies that:*

$(\star)$ $A_0^{(h)} = A_0^\star$, *for any* $0 < h < 1$,
it easily to checked that for all of open sets, given as:

$$\Omega_\lambda^\star := I_\lambda \times B_2(0, 1), \ \lambda = 1, 2, 3, \cdots,$$

have Lipschitz boundaries, and the sequence $\{\Omega_\lambda^\star \quad | \quad \lambda = 1, 2, 3, \cdots\}$.

$(\star\star)$ *Let us take account of assumptions (a)-(b), Proposition 1.2.5, and assertion* $(\star)$ *we can show that:* $\mathcal{L}^3\left(A_0^{(h)}\right) = \mathcal{L}^3(A_0^\star) = \mathcal{L}^1(A_0^\circ) = 0$, $\Phi_\alpha^{(h)} \geq \Phi_\alpha^\star$ *on* $L^2(\Omega, \mathbb{R}^3)$ *and hence* $H_\alpha^{(h)} \subset H_\alpha^\star$, *for any* $0 < h < 1$.

1.3.2 Proof of Theorem 1.3.1

The proof of this theorem will be a modification of the argument, discussed in [42] Section 4 and Section 5.2.

We will do this proof in several steps. The first two steps will show that we call Γ-*convergence* when $h \to 0$ of the functional $\Phi_\alpha^{(h)}$. In the third step we will show that when the radius of the ball goes to zero the magnetization function converge to a limit function which has a single variable and solves the problem $(P)^\circ$. Finally, in the last step, all we have to do is to show in which meaning the pointwise convergence has asserted in (1.3.3).

At the beginning, in the Step 1 and Step 2, we deal with the Γ-convergence from $\Phi_\alpha^{(h)}$ to Φ_α° as $h \to 0$. We will show that under the conditions (a)-(b), the sequence $\left\{\Phi_\alpha^{(h)} \quad | \quad 0 < h < 1\right\}$ of convex function Γ-converge to the convex function Φ_α°, on $L^2(\Omega, \mathbb{R}^3)$, as $h \to 0$. More precisely, by referring to [1, 20] this is equivalent to show that:

$(\gamma 1)$ $\liminf_{h\to 0} \Phi_\alpha^{(h)}\left(\mu^{(h)}\right) \geq \Phi_\alpha^{(h)}(\mu)$, if $\left\{\mu^{(h)} \quad | \quad 0 < h < 1\right\} \subset L^2(\Omega, \mathbb{R}^3)$, $\mu \in L^2(\Omega, \mathbb{R}^3)$, and $\mu^{(h)} \to \mu$ in $L^2(\Omega, \mathbb{R}^3)$ as $h \to 0$,

$(\gamma 2)$ for any $\nu \in H_\alpha^\circ$ $(\subset L^2(]-L, L[, \mathbb{R}^3))$, there exists a sequence $\left\{\mu_\nu^{(h)} \quad | \quad 0 < h < 1\right\} \subset L^2(\Omega, \mathbb{R}^3)$, such that $\mu_\nu^{(h)} \to \nu$ in $L^2(\Omega, \mathbb{R}^3)$ and $\Phi_\alpha^{(h)}\left(\mu_\nu^{(h)}\right) \to \Phi_\alpha^\circ(\nu)$, as $h \to 0$.

Indeed, firstly let us assume the assumption $(\star\star)$ in Remark 1.3.5.

Step 1. *Verification of* $\gamma 1$. Let us take any sequence $\left\{\mu^{(h)} \quad | \quad 0 < h < 1\right\} \subset L^2(\Omega, \mathbb{R}^3)$ and any $\mu \in L^2(\Omega, \mathbb{R}^3)$, such that:

$$\mu^{(h)} \to \mu \text{ weakly in } L^2(\Omega, \mathbb{R}^3), \text{ as } h \to 0. \tag{1.3.6}$$

Then, it is enough to consider only the case when $\liminf_{h\to 0} \Phi_\alpha^{(h)}\left(\mu^{(h)}\right) < \infty$, since another case is trivial. In this case we construct a sequence $\left\{\hat{h}_i \quad | \quad i = 1, 2, 3, \cdots\right\} \subset (0, 1)$ and a constant R_1,

independent of the index $i \in \mathbb{N}$, such that:

$$\begin{cases} \hat{h}_{i+1}^2 < \hat{h}_i^2 < \dfrac{1}{2^i}, \\ \dfrac{1}{\hat{h}_i^2}\displaystyle\int_{\Omega\setminus A_0^\star} \alpha^{(\hat{h}_i)}\left(\left|\partial_2\mu^{(\hat{h}_i)}\right|^2 + \left|\partial_3\mu^{(\hat{h}_i)}\right|^2\right) d\mathcal{L}^3 \leq \Phi_\alpha^{(\hat{h}_i)}\left(\mu^{(\hat{h}_i)}\right) \leq R_1, \end{cases} \quad \text{for } i = 1, 2, 3, \cdots,$$

and

$$\lim_{i\to\infty} \Phi_\alpha^{(\hat{h}_i)}\left(\mu^{(\hat{h}_i)}\right) = \liminf_{h\to 0} \Phi_\alpha^{(h)}\left(\mu^{(h)}\right) < \infty. \tag{1.3.7}$$

Here, taking into account (1.3.6)-(1.3.7) and Lemma 1.2.7,

$$\begin{cases} \left|\partial_2\mu^{(\hat{h}_i)}\right|^2_{L^2(\Omega_\lambda^\star,\mathbb{R}^3)} \leq \dfrac{\Phi_\alpha^{(\hat{h}_i)}\left(\mu^{(\hat{h}_i)}\right)}{a_\lambda^{(\hat{h}_i)}}\hat{h}_i^2 \leq \dfrac{R_1}{a_\lambda^{(\hat{h}_i)}}\hat{h}_i^2 \leq \dfrac{R_1}{a_\lambda^{(1/2^i)}}\hat{h}_i^2 \to 0, \text{ as } i \to \infty, \\ \partial_2\mu = 0 \text{ in } L^2(\Omega_\lambda^\star, \mathbb{R}^3), \\ \left|\partial_3\mu^{(\hat{h}_i)}\right|^2_{L^2(\Omega_\lambda^\star,\mathbb{R}^3)} \leq \dfrac{\Phi_\alpha^{(\hat{h}_i)}\left(\mu^{(\hat{h}_i)}\right)}{a_\lambda^{(\hat{h}_i)}}\hat{h}_i^2 \leq \dfrac{R_1}{a_\lambda^{(\hat{h}_i)}}\hat{h}_i^2 \leq \dfrac{R_1}{a_\lambda^{(1/2^i)}}\hat{h}_i^2 \to 0, \text{ as } i \to \infty, \\ \partial_3\mu = 0 \text{ in } L^2(\Omega_\lambda^\star, \mathbb{R}^3), \end{cases} \quad \text{for } \lambda = 1, 2, 3, \cdots.$$

Therefore, it is possible to write μ as a function of one-variable $\hat{\mu}^{(\lambda)} \in L^2(I_\lambda, \mathbb{R}^3)$, for any $\lambda \in \mathbb{N}^*$, such that:

$$\mu(x_1, x_2, x_3) = \hat{\mu}^{(\lambda)}(x_1), \text{ for } \mathcal{L}^1\text{-a.e. } x_1 \in I_\lambda \text{ and } \mathcal{L}^2\text{-a.e. } (x_2, x_3) \in B_2(0, 1).$$

Thereafter, let us set:

$$\hat{\mu}(x_1) := \begin{cases} \hat{\mu}^{(\lambda)}(x_1), \text{ if } \lambda \in \mathbb{N} \text{ and } x_1 \in I_\lambda, \\ 0, \qquad \text{otherwise}, \end{cases}$$

$$\text{for } \mathcal{L}^1\text{-a.e. } x_1 \in]-L, L[.$$

Then with helps from (a)-(b) Fubini's theorem and monotone convergence theorem, it is deduced

that:

$$\infty > \tfrac{1}{\pi}|\mu|^2_{L^2(\Omega,\mathbb{R}^3)} \geq \tfrac{1}{\pi}\lim_{\lambda\to\infty}\int_{\Omega^\star_\lambda}|\mu|^2 d\mathcal{L}^3 = \frac{1}{\pi}\lim_{\lambda\to\infty}\int_{B_2(0,1)}\int_{I_\lambda}|\hat{\mu}|^2 d\mathcal{L}^1 d\mathcal{L}^2$$
$$= \frac{1}{\pi}\lim_{\lambda\to\infty}\int_{B_2(0,1)}\int_{]-L,L[\setminus A_0^\circ}\chi_{I_\lambda}|\hat{\mu}|^2 d\mathcal{L}^1 d\mathcal{L}^2 = \frac{1}{\pi}\int_{B_2(0,1)}\int_{]-L,L[\setminus A_0^\circ}|\hat{\mu}|^2 d\mathcal{L}^1 d\mathcal{L}^2 =$$
$$\int_{]-L,L[}|\hat{\mu}|^2 d\mathcal{L}^1,$$

and

$$\int_\Omega |\mu-\hat{\mu}|^2 d\mathcal{L}^3 = \lim_{\lambda\to\infty}\int_{B_2(0,1)}\int_{]-L,L[}\chi_{I_\lambda}|\mu-\hat{\mu}|^2 d\mathcal{L}^1 d\mathcal{L}^2$$
$$= \lim_{\lambda\to\infty}\int_{B_2(0,1)}\int_{I_\lambda}|\mu-\hat{\mu}|^2 d\mathcal{L}^1 d\mathcal{L}^2 = 0.$$

Hence, the limit μ can be regarded as the functional $\hat{\mu}\in L^2(]-L,L[,\mathbb{R}^3)$ of one-variable.

Now, taking into account of the assumptions $(\star\star)$ in Remark 1.3.5, Proposition 1.2.5 and (II) in Remark 1.3.4, we conclude that:

$$\liminf_{h\to 0}\Phi^{(h)}_\alpha\left(\mu^{(h)}\right) \geq \liminf_{h\to 0}\Phi^\star_\alpha\left(\mu^{(h)}\right) \geq \Phi^\star_\alpha(\mu) = \Phi^\circ_\alpha(\mu)$$

Step 2. *Verification of* $\gamma 2$. Let $\nu\in H^\circ_\alpha$. Then, under (a)-(b), by constructing the required sequence $\overline{\left\{\mu^{(h)}_\nu \quad | \quad 0<h<1\right\}}$, we define:

$$\mu^{(h)}_\nu = \nu \in H^\star_\alpha \,(= H^{(h)}_\alpha) \text{ for any } 0<h<1. \tag{1.3.8}$$

Here, noting that:

$$\begin{cases} \alpha^{(h)} \to \alpha^\circ \text{ in } C(\overline{\Omega}), \text{ as } h\to 0, \\ \left|\alpha^{(h)}\right| \,\left(=\alpha^{(h)}\right) \leq C_\alpha\alpha^\circ \text{ on } \overline{\Omega}, \text{ for any } 0<h<1, \end{cases}$$

Consequently, by applying dominated convergence theorem we obtain that:

$$\Phi_{\alpha}^{(h)}\left(\mu_{\nu}^{(h)}\right) = \int_{\Omega\setminus A_0^{\star}} \alpha^{(h)}|\nabla\nu|^2 d\mathcal{L}^3 \to \int_{\Omega\setminus A_0^{\star}} \alpha^{\circ}|\nabla\nu|^2 d\mathcal{L}^3 \tag{1.3.9}$$

$$= \int_{]-L,L[\setminus A_0^{\circ}} \alpha^{\circ}|\nabla\nu|^2 d\mathcal{L}^1 = \Phi_{\alpha}^{\circ}(\nu) \text{ as } h \to 0.$$

Remark 1.3.6. *(See [42, Corollary 3 page 8] and according to [1, Lemma 2.3]) Under same assumption as in Step 1 and Step 2, the sequence $\left\{\mathcal{E}^{(h)} \quad | \quad 0<h<1\right\}$ of free energy functionals converges to the limiting one functional $\mathcal{E}^{\circ}$, on $L^2(\Omega,\mathbb{R}^3)$, in the sense of Γ-convergence, as $h\to 0$ if and only if:*

(i) $\liminf_{h\to 0}\mathcal{E}^{(h)}\left(\mu^{(h)}\right) \geq \mathcal{E}^{(h)}(\mu)$, *if* $\left\{\mu^{(h)} \quad | \quad 0<h<1\right\} \subset L^2(\Omega,\mathbb{R}^3)$, $\mu \in L^2(\Omega,\mathbb{R}^3)$, *and* $\mu^{(h)} \to \mu$ *(strong) in* $L^2(\Omega,\mathbb{R}^3)$, *as* $h\to 0$,

(ii) for any $\nu \in H_{\alpha}^{\circ}$ $(\subset L^2(]-L,L[,\mathbb{R}^3))$, *there exists a sequence* $\left\{\mu_{\nu}^{(h)} \quad | \quad 0<h<1\right\} \subset L^2(\Omega,\mathbb{R}^3)$, *such that* $\mu_{\nu}^{(h)} \to \nu$ *in* $L^2(\Omega,\mathbb{R}^3)$ *and* $\mathcal{E}^{(h)}\left(\mu_{\nu}^{(h)}\right) \to \mathcal{E}^{\circ}(\nu)$, *as* $h\to 0$.

Step 3. Let us assume the conditions (a)-(b), and let us take a sequence $\left\{m^{(h)} \quad | \quad 0<h<1\right\}$ of minimizes of $\mathcal{E}^{(h)}$, $0<h<1$. Means that:

$$\mathcal{E}^{(h)}\left(m^{(h)}\right) \leq \mathcal{E}^{(h)}(m), \text{ for all } m\in L^2(\Omega,\mathbb{R}^3), \text{ and all } 0<h<1. \tag{1.3.10}$$

For any $0<h<1$, let $\zeta_{e^{\star}}^{(h)}$ be the solution of the coupled Maxwell equation (1.2.2), when $m\equiv e^{\star}$, $\mathcal{L}^3$-a.e in Ω. Then, by (1.2.6) and (1.2.10) yields that $E_{mag}^{(h)}(e^{\star})<1$, for any $0<h<1$ (for details see [42, Section 5.2]). In light of this, it is furthermore considered that:

$$\Phi_{\alpha}^{(h)}\left(m^{(h)}\right) \leq \mathcal{E}^{(h)}\left(m^{(h)}\right) \leq \mathcal{E}^{(h)}(e^{\star}) = \Phi_{\alpha}^{(h)}(e^{\star}) + |\varphi(e^{\star})|_{L^1(\Omega)} + E_{mag}^{(h)}(e^{\star})$$

$$\leq \varphi(e^{\star}) + 1, \text{ for all } 0<h<1. \tag{1.3.11}$$

Due to (1.3.11), [42, Theorem 3.4 page 8] and the constraint onto $L^2(\Omega,\mathbb{S}^2)$ as in (1.2.1), we find a sequence $\left\{\check{h}_i \quad | \quad k=1,2,3,\cdots\right\}\subset(0,1)$ and a limiting function $m^{\circ}\in L^2(\Omega,\mathbb{S}^2)$, such that:

$$\begin{cases} \check{h}_i \to 0, \quad m^{(\check{h}_i)} \to m^{\circ} \text{ in } L^2(\Omega,\mathbb{R}^3), \\ \varphi\left(m^{(\check{h}_i)}\right) \to \varphi(m^{\circ}), \text{ in } L^1(\Omega), \end{cases} \text{ as } i\to\infty. \tag{1.3.12}$$

Equally, considering (1.3.11) and Remark 1.3.6, it will be noted that:

$$\Phi^{\circ}_{\alpha}(m^{\circ}) \leq \mathcal{E}^{\circ}(m^{\circ}) \leq \liminf_{i\to\infty} \mathcal{E}^{(\check{h}_i)}\left(m^{(\check{h}_i)}\right) \leq \varphi(e^{\star}) + 1$$

and hence $m^{\circ} \in H^{\circ}_{\alpha} \cap L^2(]-L, L[, \mathbb{S}^2)$. Moreover, by (1.3.8)-(1.3.9), we obtain that:

$$\mathcal{E}^{\circ}(m^{\circ}) \leq \limsup_{i\to\infty} \mathcal{E}^{(\check{h}_i)}\left(m^{(\check{h}_i)}\right) \leq \lim_{i\to\infty} \mathcal{E}^{(\check{h}_i)}(m) = \mathcal{E}^{\circ}(m),$$

$$\text{for any } m \in H^{\circ}_{\alpha} \cap L^2(]-L, L[, \mathbb{S}^2),$$

and

$$\mathcal{E}^{\circ}(m^{\circ}) \leq \liminf_{i\to\infty} \mathcal{E}^{(\check{h}_i)}\left(m^{(\check{h}_i)}\right) \leq \limsup_{i\to\infty} \mathcal{E}^{(\check{h}_i)}(m) \leq \mathcal{E}^{\circ}(m^{\circ}).$$

It implies that m° solves the limiting problem $(P)^{\circ}$ and it is deduced that:

$$\mathcal{E}^{(h)}\left(m^{(h)}\right) \to \mathcal{E}^{\circ}(m^{\circ}) \text{ as } h \to 0.$$

Step 4. Now, we will to show that the pointwise convergence, asserted in (1.3.3). For this matter we will used the argument discussed in the proof of [41, Theorem 4.1] and [42, Section 5.2]. Indeed, by (1.3.11) we have that $\mathcal{E}^{(h)}\left(m^{(h)}\right) \leq \varphi(e^{\star}) + 1$, for all $0 < h < 1$, where $m^{(h)}$ is the energy minimizer and $e^{\star} = (1, 0, 0) \in \mathbb{S}^2$, Afterward

$$\left|\sqrt{\alpha^{(h)}}\partial_1 m^{(h)}\right|^2_{L^2(\mathbb{R}^3)} \leq D_1, \quad \frac{1}{h^2}|\sqrt{\alpha^{(h)}}\nabla' m^{(h)}|^2_{L^2(\mathbb{R}^3)} \leq D_2, \tag{1.3.13}$$

where D_1 and D_2 are two fixed constants. It follows from (1.3.13) the existence of a subsequence $\left\{\check{\check{h}}_i \quad | \quad i = 1, 2, 3, \cdots\right\} \subset \{\check{h}_i\}$ with a limiting function $\sqrt{\alpha^{\circ}}\partial_1 m^{\circ} \in L^2(\Omega \backslash A^{\star}_0)$, such that:

$$\sqrt{\alpha^{(\check{\check{h}}_i)}}\nabla m^{(\check{\check{h}}_i)} \rightharpoonup \sqrt{\alpha^{\circ}}\nabla m^{\circ}, \quad \sqrt{\alpha^{(\check{\check{h}}_i)}}\nabla' m^{(\check{\check{h}}_i)} \to 0, \tag{1.3.14}$$

in $L^2(\Omega \backslash A^{\star}_0)$, as $i \to \infty$. Furthermore, by the condition (a) and (1.3.14) we obtain that $\partial_2 m^{\circ} = 0$, $\partial_3 m^{\circ} = 0$ in $L^2(\Omega \backslash A^{\star}_0)$. To show that $\sqrt{\alpha^{(\check{\check{h}}_i)}}\partial_1 m^{(\check{\check{h}}_i)}$ converge strongly to $\sqrt{\alpha^{\circ}}\partial_1 m^{\circ}$ in $L^2(\Omega \backslash A^{\star}_0)$, we now compare $\mathcal{E}^{(\check{\check{h}}_i)}(m^{(\check{\check{h}}_i)})$ to $\mathcal{E}^{(\check{\check{h}}_i)}(m^{\circ})$:

$$\begin{aligned}
&\int_{\Omega \backslash A^{\star}_0} \alpha^{(\check{\check{h}}_i)} \left(\left|\partial_1 m^{(\check{\check{h}}_i)}\right|^2 + \frac{1}{\check{\check{h}}_i^2} \left|\nabla' m^{(\check{\check{h}}_i)}\right|^2 \right) d\mathcal{L}^3 + \int_{\Omega} \varphi(m^{(\check{\check{h}}_i)}) d\mathcal{L}^3 + E^{(\check{\check{h}}_i)}_{mag}\left(m^{(\check{\check{h}}_i)}\right) \\
&\leq \int_{\Omega \backslash A^{\star}_0} \alpha^{(\check{\check{h}}_i)} |\partial_1 m^{\circ}|^2 d\mathcal{L}^3 + \int_{\Omega} \varphi(m^{\circ}) d\mathcal{L}^3 + E^{(\check{\check{h}}_i)}_{mag}(m^{\circ}).
\end{aligned}$$

By Proposition 1.3.2 (for $\tilde{m} = m^\circ$) and (1.3.12), simplifies to

$$\int_{\Omega\backslash A_0^\star} \alpha^{(\overset{\circ}{h}_i)} \left(\left| \partial_1 m^{(\overset{\circ}{h}_i)} \right|^2 + \frac{1}{\overset{\circ}{h}_i^2} \left| \nabla' m^{(\overset{\circ}{h}_i)} \right|^2 \right) d\mathcal{L}^3 \leq \int_{\Omega\backslash A_0^\star} \alpha^{(\overset{\circ}{h}_i)} |\partial_1 m^\circ|^2 d\mathcal{L}^3 + o(1). \tag{1.3.15}$$

From the first part of (1.3.14), we can write $\sqrt{\alpha^{(\overset{\circ}{h}_i)}}\partial_1 m^{(\overset{\circ}{h}_i)} = \sqrt{\alpha^\circ}\partial_1 m^\circ + g^{(\overset{\circ}{h}_i)}$ with $g^{(\overset{\circ}{h}_i)} \rightharpoonup 0$ in $L^2(\Omega)$. Substituting this expression into the left-hand side of (1.3.15), it follows that

$$\begin{aligned} &\int_{\Omega\backslash A_0^\star} \alpha^\circ |\partial_1 m^\circ|^2 + 2\sqrt{\alpha^\circ}\partial_1 m^\circ . g^{(\overset{\circ}{h}_i)} + |g^{(\overset{\circ}{h}_i)}|^2 + \frac{\alpha^{(\overset{\circ}{h}_i)}}{\overset{\circ}{h}_i^2} \left| \nabla' m^{(\overset{\circ}{h}_i)} \right|^2 d\mathcal{L}^3 \\ &\leq \int_{\Omega\backslash A_0^\star} \alpha^{(\overset{\circ}{h}_i)} |\partial_1 m^\circ|^2 d\mathcal{L}^3 + o(1), \end{aligned} \tag{1.3.16}$$

this is equivalent to write (1.3.16) as:

$$\int_{\Omega\backslash A_0^\star} \left(\alpha^\circ - \alpha^{(\overset{\circ}{h}_i)} \right) |\partial_1 m^\circ|^2 + 2\sqrt{\alpha^\circ}\partial_1 m^\circ . g^{(\overset{\circ}{h}_i)} + |g^{(\overset{\circ}{h}_i)}|^2 + \frac{\alpha^{(\overset{\circ}{h}_i)}}{\overset{\circ}{h}_i^2} \left| \nabla' m^{(\overset{\circ}{h}_i)} \right|^2 d\mathcal{L}^3 \leq o(1). \tag{1.3.17}$$

Here, we can easily checked by applying Lebesgue's dominated convergence theorem that the first term in (1.3.17) tends to zero, for the situation that:

$$\begin{cases} \alpha^{(\overset{\circ}{h}_i)} |\partial_1 m^\circ|^2 \to \alpha^\circ |\partial_1 m^\circ|^2, \\ \alpha^{(\overset{\circ}{h}_i)} |\partial_1 m^\circ|^2 \leq C_\alpha \alpha^\circ |\nabla m^\circ|^2, \end{cases} \quad \mathcal{L}^3\text{-a.e. in } \Omega\backslash A_0^\star,$$

since $g^{(\overset{\circ}{h}_i)} \rightharpoonup 0$, the second term in (1.3.17) tends to zero and, therefore,

$$\int_{\Omega\backslash A_0^\star} \left| g^{(\overset{\circ}{h}_i)} \right|^2 d\mathcal{L}^3 \to 0, \quad \frac{1}{\overset{\circ}{h}_i^2} \int_{\Omega\backslash A_0^\star} \alpha^{(\overset{\circ}{h}_i)} \left| \nabla' m^{(\overset{\circ}{h}_i)} \right|^2 d\mathcal{L}^3 \to 0, \tag{1.3.18}$$

thus (1.3.18) with (a) and $\mathcal{L}^1(A_0^\circ) = 0$, implies

$$\sqrt{\alpha^{(\overset{\circ}{h}_i)}}\partial_1 m^{(\overset{\circ}{h}_i)} \to \sqrt{\alpha^\circ}\partial_1 m^\circ, \quad \frac{1}{\overset{\circ}{h}_i}\sqrt{\alpha^{(\overset{\circ}{h}_i)}}\partial_2 m^{(\overset{\circ}{h}_i)} \to 0 \text{ and } \frac{1}{\overset{\circ}{h}_i}\sqrt{\alpha^{(\overset{\circ}{h}_i)}}\partial_3 m^{(\overset{\circ}{h}_i)} \to 0, \tag{1.3.19}$$

$$\text{in } L^2(\Omega\backslash A_0^\star), \text{ as } i \to \infty.$$

On account of (a)-(b) and $\mathcal{L}^1(A_0^\circ) = 0$, the above convergence (1.3.19) implies the existence of a subsequence $\{h_i \quad | \quad i = 1, 2, 3, \cdots\} \subset \left\{ \overset{\circ}{h}_i \right\} \subset \{\check{h}_i\}$, satisfying (1.3.3). □

Chapter 2

Junction of quasi-stationary ferromagnetic thin multi-structures

K. Chacouche, L. Faella, C. Perugia. Junction of quasi-stationary ferromagnetic thin multi-structures, *to submit.*

Abstract. In this Chapter we study the asymptotic behavior of the solutions of time dependent micromagnetism problem in a multi-structure consisting of two joined nano-wires. We assume that the volumes of the two parts composing the multi-structure vanish with same rate. We obtain two $1D$ limit problems coupled by a junction condition on the magnetization. The limit problem remains non-convex, but now it becomes completely local.

Keywords: Micromagnetics, variational problem, thin films, Landau-Lifschitz equation.

AMS subject classifications. 78A25, 74K35, 78M35

2.1 Introduction

According to the classical theory of Weiss (1907), perfectioned by Landau and Lifshitz in 1935 (see [50] and for a modern analysis see [9]), on a microscopic scale a ferromagnetic body is magnetically saturated and is composed by uniformly magnetized regions separated by thin transition layers. The phenomena can be described by a magnetization field, defined on the domain in which the material is confined. The magnetization field on a microscopic scale has a fixed modulus and variable orientations. Then, the system can be studied through the functional representing its magnetic energy. This energy consists in several terms: the so called exchange energy, which contains the space derivative of the magnetization field and is peculiar to ferromagnetic behav-

ior, a term corresponding to magnetic anisotropy, and another one depending on the magnetic field, which is related to the magnetization via the equations of magnetostatic. Precisely, let

$$\Omega_n^a =]-h_n, 0[^2 \times [0,1[, \quad \Omega_n^{b,l} =]0,1[\times]-h_n, 0[^2, \quad \Omega_n^{b,r} =]-h_n, 0]^3 \quad n \in \mathbb{N},$$

$$\Omega_n = \Omega_n^a \cup \Omega_n^{b,l} \cup \Omega_n^{b,r}, \quad n \in \mathbb{N}, \tag{2.1.1}$$

be a $3D$ ferromagnetic multidomain consisting of two orthogonal joined nano-wires (see Fig.2.1), with $\{h_n\}_{n\in\mathbb{N}} \subset]0,1[$ be a vanishing sequence of positive numbers tending to zero, as n diverges. Let us suppose that the body is homogeneous, isotropic and has uniform temperature. As usual

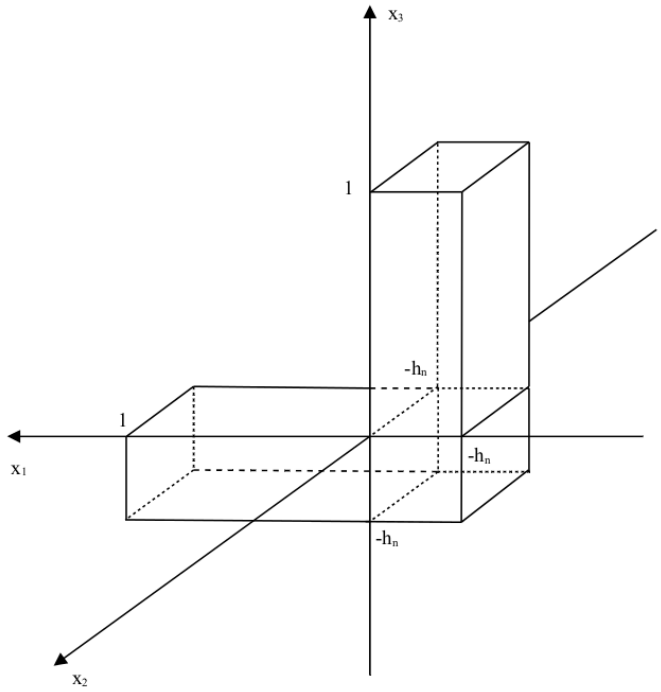

Figure 2.1: Ω_n

let us introduce the magnetization M_n, the magnetic field H_{M_n} determined by M_n and the scalar potential U_{M_n} for this field (i.e.$H(M_n) \equiv -DU_{M_n}$). Let us denote by $\overline{M_n}$ the extension by zero of M_n outside Ω_n. Then the magnetic induction B_n and the magnetic field $H(M_n)$ are connected by the relations $B_n = -DU_{M_n} + \overline{M_n}$. Moreover, the static Maxwell equation and the magnetostatic

equation (Faraday law) hold:

$$\begin{cases} \nabla \times DU_{M_n} = 0, \\ \\ div\left(-DU_{M_n} + \overline{M_n}\right) = div\left(B_n\right) = 0\ . \end{cases} \tag{2.1.2}$$

In the quasi-stationary case, the system is governed by Landau-Lifshitz equation (see [12] and [63])

$$\begin{cases} |M_n(x)| = 1 & \text{in } \Omega_n, \\ \\ \dfrac{\partial M_n}{\partial t} + M_n \wedge \dfrac{\partial M_n}{\partial t} = 2M_n \wedge (\Delta M_n - DU_{M_n}) & \text{in } \Omega_n \times]0,T[, \\ M_n(0,x) = M_{0_n}(x) & \text{in } \Omega_n, \\ \\ U_{M_n} \text{ and } M_n \text{ linked by (2.1.2) for every } t, \end{cases} \tag{2.1.3}$$

where $|M_{0_n}(x)| = 1$ in Ω_n.
In this work we study the asymptotic behavior of a system governed by the Landau-Lifshitz equation consisting of two joined roads when the thicknesse h_n converges to zero. So we attempt to simulate the behaviour of two joined nanowires. The existence result for this problem is proved, in a more general case, in [63, Theorem 2] and in [12, Section 3 and Section 5]. We observe (see [12, 63]) that the corresponding configuration satisfies an energy estimate. After having reformulated on a fixed domain

$$\begin{aligned} &\Omega^a =]-1,0[^2 \times [0,1[\,, \quad \Omega^{b,l} =]0,1[\times]-1,0[^2\,, \quad \Omega^{b,r} =]-1,0[^3 \\ &\Omega = \Omega^a \cup \Omega^{b,l} \cup \Omega^{b,r}, \end{aligned} \tag{2.1.4}$$

through appropriate rescalings of the kind proposed by Ciarlet and Destuynder [17], we derive the limit problem. Precisely, in Theorem 2.3.1, assuming that the initial energy is an $O(h_n^2)$, we prove that the solutions of (2.1.3) converge in mean square, for every t, up to a subsequence, to

solutions of the following limit problem

$$\begin{cases} \dfrac{\partial \mu^a}{\partial t} + \left(\mu^a \wedge \dfrac{\partial \mu^a}{\partial t}\right) = 2\mu^a \wedge \left(\dfrac{\partial^2 \mu^a}{\partial x_3^2} - ((\mu^a, e_1)\Upsilon_1 + (\mu^a, e_2)\Upsilon_2)\right) \quad \text{in } \ ]0,T[\times]0,1[, \\ \dfrac{\partial \mu^{b,l}}{\partial t} + \left(\mu^{b,l} \wedge \dfrac{\partial \mu^{b,l}}{\partial t}\right) = 2\mu^{b,l} \wedge \left(\dfrac{\partial^2 \mu^{b,l}}{\partial x_1^2} - \left((\mu^{b,l}, e_2)\Upsilon_1' + (\mu^{b,l}, e_3)\Upsilon_2'\right)\right) \quad \text{in } \ ]0,T[\times]0,1[, \\ \mu^a(0,x_3) = \mu_0^a(x), \ \text{ in } \ ]0,1[, \\ \mu^{b,l}(0,x_1) = \mu_0^{b,l}(x) \quad \text{in } \ ]0,1[, \\ (\mu^a)'(0) = \left(\mu^{b,l}\right)'(0), \\ |\mu^a| = 1 \quad \text{for } x \text{ in } \ ]0,1[, \ \left|\mu^{b,l}\right| = 1 \quad \text{for } x \text{ in } \ ]0,1[, \\ \mu^a(0) = \mu^{b,l}(0), \end{cases} \tag{2.1.5}$$

where $e_1 = (1,0)$, $e_2 = (0,1)$, $\Upsilon_1(\varepsilon_1, \varepsilon_2, 0)$, $\Upsilon_2(-\varepsilon_2, \varepsilon_1, 0)$, $\Upsilon_1'(0, \varepsilon_1, \varepsilon_2)$ and $\Upsilon_2'(0, -\varepsilon_2, \varepsilon_1)$, with ε_1, ε_2 constant depending on the geometry of the problem.

We obtain, two $1D$ limit problems coupled by junction condition on the magnetization $\mu^a(0) = \mu^{b,l}(0)$. The paper is organize as follows: in Section 2.2, we recall the definition and some properties of the Beppo Levi space on $\mathbb{R}^2$; in Section 2.3 we give the main result; in Section 2.4 we identify the limit of the magnetostatic energy; in Section 2.5 we gives the case of single wire. While it is quite classical in the thin film, where only the component of the magnetization orthogonal to the film appears (see [24]), it become more complicated in the wire where the following combination of the two components of the magnetization with coefficient involving solutions of PDE in Beppo Levi space on $\mathbb{R}^2$ intervene. These coefficient depend on the geometry of the cross section of the wire. To this aim we have to use different rescaling and symmetry arguments which take into account the geometry and that the limit problem will be coupled (see [35]).
As the case wire-wire is concerned, where a singular point appears due to a strong variation of its derivatives, we remark that its behavior reduces to the behavior of a single wire.

Problems of dimension reduction in magnetostatic were treated by several authors. A pioneering work is the paper of Stoner and Wohlfarth (1948). A rigorous treatment in this case was given by De Simone [27]. Carbou treated the case of magnetic wire in ([14], [15]) and the case of thin films again in [13], see also [42] and [48]. Other regimes are considered in [25] and [26] in the case of the films. In [34] and [35] Gaudiello and Hadiji studies the behavior of minimizers of free energy in a

multidomain. In what concerns the study of a ferroelectric materials see also ([36],[37]). See [11], [33], [38], [39], for junction $3D-1D$, and [32] for junction $1D-1D$. For other recent problems with thin multistructures, see also [40], [30] and [31]. As Gioia and James [41] in the stationary case, Carbou in [13] studies the limit behavior of the isotropic ferromagnetic films when the thicknesses goes to zero, in the quasy stationary case. Other similar problems are studied by Ammari et al. [4]. The homogenization of the Landau-Lifschitz equation in periodically perforated domain was studied in [57]. In [23] and in [24], the authors study the asymptotic behavior of the solutions of time dependent micromagnetism problem in a multi-domain consisting of two joined ferromagnetic thin films, different regimes depending on the limit of the ratio between the small thickness of the two films were considered.

2.2 Preliminaries

Let

$$W^1(\mathbb{R}^2)=\left\{\phi\in L^2_{loc}(\mathbb{R}^2):D\phi\in\left(L^2(\mathbb{R}^2)\right)^2\right\}/\mathbb{R}$$

equipped with the inner product

$$(\phi_1,\phi_2)\in W^1(\mathbb{R}^2)\times W^1(\mathbb{R}^2)\to\int\limits_{\mathbb{R}^2}D\phi_1D\phi_2dydz, \tag{2.2.1}$$

where (y,z) denote the coordinates in $\mathbb{R}^2$. It is well known that $W^1(\mathbb{R}^2)$ is a Hilbert space (see [22], Corol. 1.1) and it is separable. Consequently, if $S\subset\mathbb{R}^2$ is a bounded open set , every one of the following problems

$$\begin{cases} p\in W^1(\mathbb{R}^2),\\ \displaystyle\int\limits_{\mathbb{R}^2}DpD\phi\,dydz=\int\limits_S D_y\phi\,dydz,\quad\forall\phi\in W^1(\mathbb{R}^2),\end{cases} \tag{2.2.2}$$

$$\begin{cases} q\in W^1(\mathbb{R}^2),\\ \displaystyle\int\limits_{\mathbb{R}^2}DqD\phi\,dydz=\int\limits_S D_z\phi\,dydz,\quad\forall\phi\in W^1(\mathbb{R}^2),\end{cases} \tag{2.2.3}$$

$$\begin{cases} p_k \in W^1(\mathbb{R}^2), \\ \displaystyle\int_{\mathbb{R}^2} Dp_k D\phi \, dydz = \int_S kD\phi \, dydz, \quad \forall \phi \in W^1(\mathbb{R}^2), \end{cases} \tag{2.2.4}$$

with $k = (k_1, k_2) \in \mathbb{R}^2$ admits a unique solution which obviously depends on S. Then, we set

$$\alpha(S) = \int_{\mathbb{R}^2} |Dp|^2 dydz, \quad \beta(S) = \int_{\mathbb{R}^2} |Dq|^2 dydz, \quad \gamma(S) = 2\int_{\mathbb{R}^2} DpDqdydz, \tag{2.2.5}$$

where (y, z) denotes the coordinates in $\mathbb{R}^2$.

We remark that if S is sufficiently smooth, problems (2.2.2), (2.2.3) and definitions (2.2.5) are equivalent respectively to

$$\begin{cases} p \in W^1(\mathbb{R}^2), \\ \Delta p = 0 \text{ in } S, \quad \Delta p = 0 \text{ in } \mathbb{R}^2 \setminus S, \\ \left[\dfrac{\partial p}{\partial \nu}\right] = \nu e_1 \text{ on } \partial S, \end{cases} \qquad \begin{cases} q \in W^1(\mathbb{R}^2), \\ \Delta q = 0 \text{ in } S, \quad \Delta q = 0 \text{ in } \mathbb{R}^2 \setminus S, \\ \left[\dfrac{\partial q}{\partial \nu}\right] = \nu e_2 \text{ on } \partial S, \end{cases} \tag{2.2.6}$$

where ν the exterior unit normal to ∂S, $\left[\frac{\partial \cdot}{\partial \nu}\right]$ the jump of $\frac{\partial \cdot}{\partial \nu}$ on ∂S, and $e_1 = (1, 0)$, $e_2 = (0, 1)$, and

$$\begin{cases} \alpha(S) = \displaystyle\int_{\partial S} p\nu e_1 ds, \quad \beta(S) = \int_{\partial S} q\nu e_2 ds, \\ \gamma(S) = 2\displaystyle\int_{\partial S} q\nu e_1 ds = 2\int_{\partial S} p\nu e_2 ds = \int_{\partial S} q\nu e_1 ds + \int_{\partial S} p\nu e_2 ds \\ \varepsilon(S) = \displaystyle\int_S Dpdydz, \quad \delta(S) = \int_S Dqdydz. \end{cases} \tag{2.2.7}$$

For instance, if $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$, p and q can be explicitly computed (a formula can be found in [[56], p. 177], it result that $\alpha(S) = \beta(S) = \dfrac{\pi}{2}$ and $\gamma(S) = 0$.

In the sequel, we shall use the following results.

Lemma 2.2.1. *Let p and q be the unique solutions of (2.2.2) and (2.2.3), respectively. Then, i) for every $k = (k_1, k_2) \in \mathbb{R}^2$, the unique solution p_k of (2.2.4) is given by:*

$$p_k = k_1 p + k_2 q;$$

ii)Dp and Dq are linearly independent;
iii) for every $k = (k_1, k_2) \in \mathbb{R}^2$, it holds

$$\int_S |k_1 Dp + k_2 Dq|^2 \, dydz = \int_S (k_1 Dp + k_2 Dq)(k_1, k_2) dydz = k_1 \alpha(S) + k_2 \beta(S) + 2k_1 k_2 \gamma(S). \tag{2.2.8}$$

Proof. Let $k_1, k_2 \in \mathbb{R}$ be such that

$$k_1 Dp + k_2 Dq = 0, \text{ a.e. in } \mathbb{R}^2.$$

Then, comparing (2.2.2) with (2.2.3), one obtains that

$$\int_S (k_1 D_y \phi + k_2 D_z \phi) \, dydz = 0, \quad \forall \phi \in W^1(\mathbb{R}^2),$$

which provides $k_1 = k_2 = 0$. □

Lemma 2.2.2. *Let be $S =]-1, 0[^2$, then*

$$\alpha(S) = \beta(S), \quad \gamma(S) = 0,$$

and

$$\varepsilon(S) = (\varepsilon_1(S), \varepsilon_2(S)), \quad \delta(S) = (\delta_1(S), \delta_2(S))),$$

with

$$\delta_1(S) = -\varepsilon_2(S), \quad \delta_2(S) = \varepsilon_1(S) \tag{2.2.9}$$

Proof. Let us observe that the functions p and q, solutions of the problems (2.2.6) for $S =]-1, 0[^2$, are obtained by traslation of the solutions of the problems (2.2.6) for $S =]-\frac{1}{2}, \frac{1}{2}[^2$. Moreover, the solutions p and q related to this set are a rotated the other (to fix the idea q is the rotated function of p) with respect $\theta = \frac{\pi}{2}$. Then, $q = p(x_2, -x_1)$ and $Dq = (-\frac{\partial p}{\partial x_2}, \frac{\partial p}{\partial x_1})$. Hence, $\alpha(S) = \beta(S)$, $\gamma(S) = 0$, $\varepsilon(S) = (\varepsilon_1(S), \varepsilon_2(S))$ and $\delta(S) = (-\varepsilon_2(S), \varepsilon_1(S))$. □

We recall the Poincaré Lemma (which is well known if the domain is bounded).

Lemma 2.2.3. *Let $\xi \in \left(L^2(\mathbb{R}^2)\right)^2$ such that rot $\xi = 0$. Then, there exists a unique $w \in W^1(\mathbb{R}^2)$ such that $\xi = Dw$.*

Proof. The fact that rot $\xi = 0$ provides the existence of $T \in \mathcal{D}'(\mathbb{R}^2)$ such that $\xi = DT$, and T is unique up to a constant (see [58, Ch. II, Th. VI, page 59]). On the other hand, since $\xi \in \left(L^2(\mathbb{R}^2)\right)^2$, Kryloff Theorem assures that $T \in L^2_{loc}(\mathbb{R}^2)$ (see [58, Ch. VI, Th. XV, page 181]). □

The following result was suggested by F. Murat [53].

Proposition 2.2.1. *Let $u \in L^2_{loc}(\mathbb{R}^2)$ be such that $Du \in \left(L^2(\mathbb{R}^2)\right)^2$. Then, there exist a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^2)$ such that $D\varphi_n \to Du$ strongly in $\left(L^2(\mathbb{R}^2)\right)^2$.*

2.3 Statement of the problem and main results

In the sequel, $x = (x_1, x_2, x_3)$ denotes the generic point of $\mathbb{R}^3$. If η_1, η_2, $\eta_3 \in \mathbb{R}^3$, then $(\eta_1|\eta_2|\eta_3)$ denotes the 3×3 real matrix having η_1^T as first column, η_2^T as second column, and η_3^T as third column. In according to this notation, if $v : A \subset \mathbb{R}^3 \to \mathbb{R}^3$, then Dv denotes the 3×3 real matrix $(D_{x_1}v|D_{x_2}v|D_{x_3}v)$, where $D_{x_i}v \in \mathbb{R}^3$, i=1,2,3, stands for the derivative of v with respect to x_i. More precisely, let $\{h_n\}_{n\in\mathbb{N}} \subset]0,1[$ be sequence. For every $n \in \mathbb{N}$, set

$$\Omega_n^a =]-h_n, 0[^2 \times [0,1[,\ \Omega_n^{b,l} =]0,1[\times]-h_n,0[^2,\ \Omega_n^{b,r} =]-h_n,0]^3 \text{ and } \Omega_n = \Omega_n^a \cup \Omega_n^{b,l} \cup \Omega_n^{b,r},$$

which approximates two joined wires as in Figure 2.1.

Let $B =]-2,2[^3$, and set

$$\mathcal{U} = \left\{ U \in L^1_{loc}\left(\mathbb{R}^3\right) : U \in L^2\left(B\right),\ DU \in \left(L^2\left(\mathbb{R}^3\right)\right)^3,\ \int_B U dx = 0 \right\}. \tag{2.3.1}$$

It is easy to prove that $\mathcal{U}$ is contained in $L^2_{loc}\left(\mathbb{R}^3\right)$ and it is an Hilbert space with the inner product $(U,V) = \int_{\mathbb{R}^3} DUDV dx + \int_B UV dx$. Moreover, from Poincaré-Wirtinger inequality it follows that a norm on $\mathcal{U}$ equivalent to $(U,U)^{\frac{1}{2}}$ is given by $\left(\int_{\mathbb{R}^3} |DU|^2\, dx\right)^{\frac{1}{2}}$.

Let $M \in L^2(\Omega_n, \mathbb{R}^3)$ then the following problem

$$U_M \in \mathcal{U}, \quad \int_{\mathbb{R}^3} DU_M DU = \int_{\Omega_n} MDUdx \quad \forall U \in \mathcal{U}, \tag{2.3.2}$$

admits a unique solution $U_M \in \mathcal{U}$. This solution is characterized as the unique minimizer of the following problem:

$$\min \left\{ \frac{1}{2} \int_{\mathbb{R}^3} \left| DU - \overline{M} \right|^2 dx : U \in \mathcal{U} \right\}, \tag{2.3.3}$$

where as usual $\overline{M}$ denotes the zero extension of M in $\mathbb{R}^3 \backslash \Omega_n$. Moreover $U_M \in H^1(\mathbb{R}^3)$ up to an additive constant, see [47].

Fixed $M_{0_n} \in H^1(\Omega_n, S^2)$, ($U_{0_n} \in \mathcal{U}$ being the corresponding solution of Problem (2.3.2)), in [12], [24] and [63] it is proved that there exists at least a weak solution M_n of the following problem

$$\begin{cases} M_n \in L^\infty(0,T; H^1(\Omega_n, \mathbb{R}^3)) \cap C([0,T]; L^2(\Omega_n, \mathbb{R}^3)), \\ |M_n| = 1 \ a.e. \text{ in } [0,T] \times \Omega_n, \quad \dfrac{\partial M_n}{\partial t} \in L^2(0,T; L^2(\Omega_n, \mathbb{R}^3)), \\ \forall\, \chi \in D(0,T), \text{ and } \psi \in H^1(\Omega_n, \mathbb{R}^3) \\ \displaystyle\int_0^T \int_{\Omega_n} \left(\frac{\partial M_n}{\partial t} + M_n \wedge \frac{\partial M_n}{\partial t} \right) \chi \psi dxdt = -2 \int_0^T \int_{\Omega_n} \sum_{i=1}^{3} (M_n \wedge D_{x_i} M_n)(D_{x_i} \psi) \chi dxdt \\ \displaystyle -2 \int_0^T \int_{\Omega_n} (M_n \wedge DU_{M_n}) \chi \psi dxdt, \\ M_n(0,x) = M_{0_n}(x), \text{ a.e. } x \quad \text{in} \quad \Omega_n, \\ U_{M_n} \text{ and } M_n \text{ linked by (2.3.2) for every } t \in [0,T]. \end{cases} \tag{2.3.4}$$

Moreover, it satisfies the following energy estimate:

$$E(M_n(t,\cdot)) + \int_0^t \left\| \frac{\partial M_n}{\partial t} \right\|^2_{(L^2(\Omega_n))^3} ds \leq E(M_n(0,\cdot)) = E(M_{0_n}), \quad \text{for a.e. } t \in [0,T], \tag{2.3.5}$$

where for every $t \in [0,T]$

$$E\left(M_n(t,\cdot)\right) = \int\limits_{\Omega_n} \left|DM_n\left(t,x\right)\right|^2 dx + \frac{1}{2}\int\limits_{\mathbb{R}^3} \left|DU_{M_n}\left(t,x\right)\right|^2 dx,$$

is the magnetic energy. Here, the terms $E_n^{exc} = \displaystyle\int\limits_{\Omega_n} \left|DM_n(t,x)\right|^2 dx$ is the exchange energy and $E_n^{mag} = \dfrac{1}{2}\displaystyle\int\limits_{\mathbb{R}^3} \left|DU_{M_n}(t,x)\right|^2 dx$ corresponds to the magnetostatic energy.

In what follows let us assume that

$$E^{exc}\left(M_{0_n}\right) = O\left(h_n^2\right), \quad \forall n \in \mathbb{N}. \tag{2.3.6}$$

Remark 2.3.1. *By density argument (for instance, see [46, Lemma 1.9 pag. 39] and also [29]), Problem (2.3.4) is equivalent to that obtained by choosing as test function* $\Phi \in D\left(]0,T[\times \overline{\Omega}\right)$.

Namely, setting

$$\Omega^a =]-1,0[^2 \times [0,1[, \quad \Omega^{b,l} =]0,1[\times]-1,0[^2, \quad \Omega^{b,r} =]-1,0[^3,$$

let us introduce the following space

$$\mathcal{W} = \Big\{(\mu^a,\mu^{b,l}) \in H^1\left(\Omega^a,\mathbb{R}^3\right) \times H^1(\Omega^{b,l},\mathbb{R}^3) \ : \ \mu^a \text{ is independent of } (x_1,x_2),$$

$$\mu^{b,l} \text{ is independent of } (x_2,x_3), \quad \mu^a(0) = \mu^{b,l}(0)\Big\} \simeq \tag{2.3.7}$$

$$\left\{(\mu^a,\mu^{b,l}) \in H^1\left(]0,1[,\mathbb{R}^3\right) \times H^1\left(]0,1[,\mathbb{R}^3\right) \ : \ \mu^a(0) = \mu^{b,l}(0)\right\}.$$

Moreover, we can pose

$$\mathcal{M} = \mathcal{W} \cap \left\{ \ H^1\left(\Omega^a,S^2\right) \times H^1\left(\Omega^{b,l},S^2\right) \ \right\}, \tag{2.3.8}$$

which explicitely takes into account the condition $|\mu| = 1$. Then, the equivalent $3D$ variational formulation of the Problem (2.1.5) is the following one:

$$
\begin{cases}
\mu = (\mu^a, \mu^{b,l}) \in L^\infty(0,T;\mathcal{M}) \cap C\left([0,T]; L^2(\Omega^a,\mathbb{R}^3) \times L^2\left(\Omega^{b,l},\mathbb{R}^3\right)\right), \\
\dfrac{\partial \mu}{\partial t} \in L^2\left(0,T; L^2(\Omega^a,\mathbb{R}^3) \times L^2\left(\Omega^{b,l},\mathbb{R}^3\right)\right), \\
\forall\, \chi \in D(0,T) \text{ and } \psi = (\psi^a, \psi^{b,l}) \in \mathcal{W}, \\
\displaystyle\int_0^T \int_{\Omega^a} \left(\frac{\partial \mu^a}{\partial t} + \mu^a \wedge \frac{\partial \mu^a}{\partial t}\right) \chi \psi^a dx dt + \int_0^T \int_{\Omega^{b,l}} \left(\frac{\partial \mu^{b,l}}{\partial t} + \mu^{b,l} \wedge \frac{\partial \mu^{b,l}}{\partial t}\right) \chi \psi^{b,l} dx dt = \\
\displaystyle -2\int_0^T \int_{\Omega^a} \mu^a \wedge \frac{\partial \mu^a}{\partial x_3} \frac{\partial \psi^a}{\partial x_3} \chi dx dt - 2\int_0^T \int_{\Omega^{b,l}} \mu^{b,l} \wedge \frac{\partial \mu^{b,l}}{\partial x_1} \frac{\partial \psi^{b,l}}{\partial x_1} \chi dx dt \\
\displaystyle -2\int_0^T \int_{\Omega^a} \mu^a \wedge \left((\mu^a, e_1)\Upsilon_1 + (\mu^a, e_2)\Upsilon_2\right) \chi \psi^a dx dt - 2\int_0^T \int_{\Omega^{b,l}} \mu^{b,l} \wedge \left((\mu^{b,l}, e_2)\Upsilon_1' + (\mu^{b,l}, e_3)\Upsilon_2'\right) \chi \psi^{b,l} dx dt, \\
\mu^a(0,x) = \mu_0^a(x), \text{ a.e. } x \text{ in } \Omega^a, \quad \mu^{b,l}(0,x) = \mu_0^{b,l}(x), \text{ a.e. } x \text{ in } \Omega^{b,l}, \quad \mu_0 = (\mu_0^a, \mu_0^{b,l}) \in \mathcal{M},
\end{cases}
\tag{2.3.9}
$$

where $\Upsilon_1 = (\varepsilon_1, \varepsilon_2, 0)$, $\Upsilon_2 = (-\varepsilon_2, \varepsilon_1, 0)$, $\Upsilon_1' = (0, \varepsilon_1, \varepsilon_2)$ and $\Upsilon_2' = (0, -\varepsilon_2, \varepsilon_1,)$, with ε_1, ε_2 constant depending on $S =]-1,0[^2$. To Problem (2.3.9), for a.e. $t \in [0,T]$, the following energy will be associated,

$$
E(t) + \int_0^t \left\| \frac{\partial \mu^a}{\partial t} \right\|^2_{(L^2(]0,1[))^3} ds + \int_0^t \left\| \frac{\partial \mu^{b,l}}{\partial t} \right\|^2_{(L^2(]0,1[))^3} ds\,, \tag{2.3.10}
$$

where

$$
\begin{aligned}
E(t) = &\int_0^1 \left|\frac{\partial \mu^a}{\partial x_3}\right|^2 dx_3 + \int_0^1 \left|\frac{\partial \mu^{b,l}}{\partial x_1}\right|^2 dx_1 + \\
&\frac{1}{2}\alpha\left(]-1,0[^2\right) \left[\int_0^1 |\mu_1^a|^2 + |\mu_2^a|^2 dx_3 + \frac{1}{2}\int_0^1 |\mu_2^{b,l}|^2 + |\mu_3^{b,l}|^2 dx_1\right],
\end{aligned}
\tag{2.3.11}
$$

$\alpha\left(]-1,0[^2\right)$ is defined by (2.2.5) with $S = \left(]-1,0[^2\right)$. Here, the term

$$
E^{exc}(t) = \int_0^1 \left|\frac{\partial \mu^a}{\partial x_3}\right|^2 dx_3 + \int_0^1 \left|\frac{\partial \mu^{b,l}}{\partial x_1}\right|^2 dx_1, \tag{2.3.12}
$$

can be considered an exchange energy and the term

$$E^{mag}(t) = \frac{1}{2}\alpha\left(]-1,0[^2\right)\left[\int_0^1 |\mu_1^a|^2 + |\mu_2^a|^2 dx_3 + \frac{1}{2}\int_0^1 |\mu_2^{b,l}|^2 + |\mu_3^{b,l}|^2 dx_1\right], \tag{2.3.13}$$

can be considered the equivalent of a magnetostatic energy.

Theorem 2.3.1. *Suppose that $M_{0_n} \in H^1(\Omega_n, S^2)$ and (2.3.6) holds, for every $n \in \mathbb{N}$. Let M_n be a solution of Problem (2.3.4). Then, there exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$, still denoted by $\{n\}$, $\mu_0 = \left(\mu_0^a, \mu_0^{b,l}\right) \in \mathcal{M}$, $\mu = (\mu^a, \mu^{b,l}) \in L^\infty(0,T;\mathcal{M})$, depending on the selected subsequence such that:*

$$\begin{cases} \fint_{\Omega_n^a} |M_{0_n}(x_1,x_2,x_3) - \mu_0^a(x_3)|^2\, dx \to 0, \\ \fint_{\Omega_n^{b,l}} \left|M_{0_n}(x_1,x_2,x_3) - \mu_0^{b,l}(x_1)\right|^2 dx \to 0, \\ \fint_{\Omega_n^{b,r}} |M_{0_n}(x_1,x_2,x_3) - \mu_0^a(0)|^2\, dx \to 0, \end{cases} \tag{2.3.14}$$

as n diverges, for every $t \in [0,T]$

$$\begin{cases} \fint_{\Omega_n^a} |M_n(t,x_1,x_2,x_3) - \mu^a(t,x_3)|^2\, dxdt \to 0, \\ \fint_{\Omega_n^{b,l}} \left|M_n(t,x_1,x_2,x_3) - \mu^{b,l}(t,x_1)\right|^2 dxdt \to 0, \end{cases} \tag{2.3.15}$$

as n diverges, where $\mu = (\mu^a, \mu^{b,l})$ is a solution of Problem (2.3.9).

2.4 The rescaled problem

By setting

$$\begin{cases} \mathbb{R}^3_a = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}, \\ \mathbb{R}^3_{b,l} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < 0,\ x_1 > 0\}, \\ \mathbb{R}^3_{b,r} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < 0,\ x_1 < 0\}, \end{cases}$$

for every $n \in \mathbb{N}$, problem (2.3.4) is reformulated on a fixed domain through the following rescaling

$$T_n : (x_1, x_2, x_3) \in \mathbb{R}^3 \to T_n(x_1, x_2, x_3) = \begin{cases} (h_n x_1, h_n x_2, x_3), \text{ if } (x_1, x_2, x_3) \in \mathbb{R}^3_a, \\ (x_1, h_n x_2, h_n x_3), \text{ if } (x_1, x_2, x_3) \in \mathbb{R}^3_{b,l}, \\ (h_n x_1, h_n x_2, h_n x_3), \text{ if } (x_1, x_2, x_3) \in \mathbb{R}^3_{b,r}. \end{cases} \tag{2.4.1}$$

Namely, setting

$$B^a_n = \left]-\frac{2}{h_n}, \frac{2}{h_n}\right[^2 \times]0, 2[, \quad B^{b,l}_n =]0, 2[\times \left]-\frac{2}{h_n}, 0\right[^2, \quad B^{b,r}_n = \left]-\frac{2}{h_n}, 0\right[^3, \quad n \in \mathbb{N},$$

the space $\mathcal{U}$ defined in (2.3.1) is rescaled in the following

$$\begin{aligned}
\mathcal{U}_n = \Big\{ \quad & (u^a, u^{b,l}, u^{b,r}) \in L^1_{loc}(\overline{\mathbb{R}^3_a}) \times L^1_{loc}(\overline{\mathbb{R}^3_{b,l}}) \times L^1_{loc}(\overline{\mathbb{R}^3_{b,r}}) \ : \\
& (u^a_{|_{B^a_n}}, u^{b,l}_{|_{B^{b,l}_n}}, u^{b,r}_{|_{B^{b,r}_n}}) \in L^2(B^a_n) \times L^2(B^{b,l}_n) \times L^2(B^{b,r}_n), \\
& (Du^a, Du^{b,l}, Du^{b,r}) \in (L^2(\mathbb{R}^3_a))^3 \times (L^2(\mathbb{R}^3_{b,l}))^3 \times (L^2(\mathbb{R}^3_{b,r}))^3, \\
& \int_{B^a_n} u^a dx + \int_{B^{b,l}_n} u^{b,l} dx + h_n \int_{B^{b,r}_n} u^{b,r} dx = 0, \\
& u^a(x_1, x_2, 0) = u^{b,l}(h_n x_1, x_2, 0), \text{ for } (x_1, x_2) \text{ a.e. in }]0, +\infty[\times\mathbb{R}, \\
& u^a(x_1, x_2, 0) = u^{b,r}(x_1, x_2, 0), \text{ for } (x_1, x_2) \text{ a.e. in }]-\infty, 0[\times\mathbb{R}, \\
& u^{b,l}(0, x_2, x_3) = u^{b,r}(0, x_2, x_3), \text{ for } (x_2, x_3) \text{ a.e. in } \mathbb{R}\times]-\infty, 0[\Big\}.
\end{aligned} \tag{2.4.2}$$

Then, for every $(m^a, m^{b,l}, m^{b,r}) \in L^2(\Omega^a, \mathbb{R}^3) \times L^2(\Omega^{b,l}, \mathbb{R}^3) \times L^2(\Omega^{b,r}, \mathbb{R}^3)$, the following equation

$$\begin{cases}
(u^a, u^{b,l}, u^{b,r}) \in \mathcal{U}_n, \\
\displaystyle\int_{\mathbb{R}^3_a} \left(\frac{1}{h_n} D_{x_1} u^a, \frac{1}{h_n} D_{x_2} u^a, D_{x_3} u^a\right) \left(\frac{1}{h_n} D_{x_1} v^a, \frac{1}{h_n} D_{x_2} v^a, D_{x_3} v^a\right) dx + \\
\displaystyle\int_{\mathbb{R}^3_{b,l}} \left(D_{x_1} u^{b,l}, \frac{1}{h_n} D_{x_2} u^{b,l}, \frac{1}{h_n} D u^{b,l}\right) \left(D_{x_1} v^{b,l}, \frac{1}{h_n} D_{x_2} v^{b,l}, \frac{1}{h_n} D_{x_3} v^{b,l}\right) dx + \\
\displaystyle\frac{1}{h_n} \int_{\mathbb{R}^3_{b,r}} \left(D_{x_1} u^{b,r}, D_{x_2} u^{b,r}, D_{x_3} u^{b,r}\right) \left(D_{x_1} v^{b,r}, D_{x_2} v^{b,r}, D_{x_3} v^{b,r}\right) dx = \\
\displaystyle\int_{\Omega^a} \left(\frac{1}{h_n} D_{x_1} v^a, \frac{1}{h_n} D_{x_2} v^a, D_{x_3} v^a\right) m^a dx + \\
\displaystyle\int_{\Omega^{b,l}} \left(D_{x_1} v^{b,l}, \frac{1}{h_n} D_{x_2} v^{b,l}, \frac{1}{h_n} D_{x_3} v^{b,l}\right) m^{b,l} dx + \\
\displaystyle\int_{\Omega^{b,r}} \left(D_{x_1} v^{b,r}, D_{x_2} v^{b,r}, D_{x_3} v^{b,r}\right) m^{b,r} dx, \quad \forall (v^a, v^{b,l}, v^{b,r}) \in \mathcal{U}_n,
\end{cases} \tag{2.4.3}$$

which rescales equation (2.3.2), admits a unique solution. We note that $(u^a, u^{b,l}, u^{b,r})$ belongs to $H^1(\mathbb{R}^3_a) \times H^1(\mathbb{R}^3_{b,l}) \times H^1(\mathbb{R}^3_{b,r})$ up to an additive constant.

For every $n \in \mathbb{N}$, let us consider the following space

$$\mathcal{W}_n = \Big\{ \quad (m^a, m^{b,l}, m^{b,r}) \in H^1(\Omega^a, \mathbb{R}^3) \times H^1(\Omega^{b,l}, \mathbb{R}^3) \times H^1(\Omega^{b,r}, \mathbb{R}^3) \ : \\ m^a(x_1, x_2, 0) = m^{b,r}(x_1, x_2, 0), \text{ for } (x_1, x_2) \text{ a.e. in }]-1, 0[^2, \\ m^{b,l}(0, x_2, x_3) = m^{b,r}(0, x_2, x_3), \text{ for } (x_2, x_3) \text{ a.e. in }]-1, 0[^2 \Big\}, \tag{2.4.4}$$

For simplicity of notation, let us introduce the space

$$\mathcal{M}_n = \mathcal{W}_n \cap \left\{ H^1\left(\Omega^a, S^2\right) \times H^1\left(\Omega^{b,l}, S^2\right) \times H^1\left(\Omega^{b,r}, S^2\right) \right\}, \tag{2.4.5}$$

which explicitly takes into account the condition $|m| = 1$. Let $m_{0_n} = \left(m^a_{0_n}, m^{b,l}_{0_n}, m^{b,r}_{0_n}\right) \in \mathcal{M}_n$, then, there exists at least a solution m_n of the following rescaled problem

$$
\begin{cases}
m_n = \left(m_n^a, m_n^{b,l}, m_n^{b,r}\right) \in L^\infty\left(0,T;\mathcal{M}_n\right) \cap C\left([0,T]; L^2\left(\Omega^a,\mathbb{R}^3\right) \times L^2\left(\Omega^{b,l},\mathbb{R}^3\right) \times L^2\left(\Omega^{b,r},\mathbb{R}^3\right)\right), \\
\dfrac{\partial m_n}{\partial t} = \left(\dfrac{\partial m_n^a}{\partial t}, \dfrac{\partial m_n^b}{\partial t}\right) \in L^2\left(0,T; L^2\left(\Omega^a,\mathbb{R}^3\right) \times L^2\left(\Omega^{b,l},\mathbb{R}^3\right) \times L^2\left(\Omega^{b,r},\mathbb{R}^3\right)\right), \\
\forall\, \chi \in D(0,T) \text{ and } \psi = \left(\psi^a, \psi^{b,l}, \psi^{b,r}\right) \in \mathcal{W}_n, \\
\displaystyle\int_0^T \int_{\Omega^a} \left(\frac{\partial m_n^a}{\partial t} + m_n^a \wedge \frac{\partial m_n^a}{\partial t}\right) \chi\psi^a dxdt + \int_0^T \int_{\Omega^{b,l}} \left(\frac{\partial m_n^{b,l}}{\partial t} + m_n^{b,l} \wedge \frac{\partial m_n^{b,l}}{\partial t}\right) \chi\psi^{b,l} dxdt \\
+h_n \displaystyle\int_0^T \int_{\Omega^{b,r}} \left(\frac{\partial m_n^{b,r}}{\partial t} + m_n^{b,r} \wedge \frac{\partial m_n^{b,r}}{\partial t}\right) \chi\psi^{b,r} dxdt = \\
-2\displaystyle\int_0^T \int_{\Omega^a} \left[\sum_{i=1}^{2} m_n^a \wedge \frac{1}{h_n} D_{x_i} m^a \frac{1}{h_n} \left(D_{x_i}\psi^a\right)\chi + m_n^a \wedge D_{x_3} m_n^a \left(D_{x_3}\psi^a\right)\chi\right] dxdt \\
-2\displaystyle\int_0^T \int_{\Omega^a} m_n^a \wedge \left(\frac{1}{h_n} D_{x_1} u_{m_n}^a, \frac{1}{h_n} D_{x_2} u_{m_n}^a, D_{x_3} u_{m_n}^a\right) \chi\psi^a dxdt \\
-2\displaystyle\int_0^T \int_{\Omega^{b,l}} \left[m_n^{b,l} \wedge D_{x_1} m^b \left(D_{x_1}\psi^{b,l}\right)\chi + \sum_{i=2}^{3} m_n^{b,l} \wedge \frac{1}{h_n} D_{x_i} m_n^{b,l} \frac{1}{h_n} \left(D_{x_i}\psi^{b,l}\right)\chi\right] dxdt \\
-2h_n \displaystyle\int_0^T \int_{\Omega^{b,r}} \left[\sum_{i=1}^{3} m_n^{b,r} \wedge \frac{1}{h_n} D_{x_i} m_n^{b,r} \frac{1}{h_n} \left(D_{x_i}\psi^{b,r}\right)\chi\right] dxdt \\
-2\displaystyle\int_0^T \int_{\Omega^{b,l}} m_n^{b,l} \wedge \left(D_{x_1} u_{m_n}^{b,l}, \frac{1}{h_n} D_{x_2} u_{m_n}^{b,l}, \frac{1}{h_n} D_{x_3} u_{m_n}^{b,l}\right) \chi\psi^{b,l} dxdt \\
-2h_n \displaystyle\int_0^T \int_{\Omega^{b,r}} m_n^{b,r} \wedge \left(\frac{1}{h_n} D_{x_1} u_{m_n}^{b,r}, \frac{1}{h_n} D_{x_2} u_{m_n}^{b,r}, \frac{1}{h_n} D_{x_3} u_{m_n}^{b,r}\right) \chi\psi^{b,r} dxdt, \\
m_n^a(0,x) = m_{0_n}^a(x), \text{ a.e. } x \text{ in } \quad \Omega^a, \quad m_n^{b,l}(0,x) = m_{0_n}^{b,l}(x), \text{ a.e. } x \text{ in } \Omega^{b,l}, \\
m_n^{b,r}(0,x) = m_{0_n}^{b,r}(x), \text{ a.e. } x \text{ in } \Omega^{b,r}, \\
u_{m_n} \text{ and } m_n \text{ are linked by (2.4.3) for every } t \in [0,T].
\end{cases}
\tag{2.4.6}
$$

Moreover it satisfies the following energy estimate:

$$E_n(t) + \int_0^t \left\| \frac{\partial m_n^a}{\partial t} \right\|^2_{(L^2(\Omega^a))^3} ds + \int_0^t \left\| \frac{\partial m_n^{b,l}}{\partial t} \right\|^2_{(L^2(\Omega^{b,l}))^3} ds + h_n \int_0^t \left\| \frac{\partial m_n^{b,r}}{\partial t} \right\|^2_{(L^2(\Omega^{b,r}))^3} ds$$

$$\leq E_n(0) = E(m_{0_n}), \text{ for a.e. } t \in [0,T],$$

(2.4.7)

where, for $a.e.\, t \in [0,T]$:

$$\begin{aligned}
E_n(t) = &\int_{\Omega^a} \left| \left(\frac{1}{h_n} D_{x_1} m_n^a, \frac{1}{h_n} D_{x_2} m_n^a, D_{x_3} m_n^a \right) \right|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3_a} \left| \left(\frac{1}{h_n} D_{x_1} u^a_{m_n}, \frac{1}{h_n} D_{x_2} u^a_{m_n}, D_{x_3} u^a_{m_n} \right) \right|^2 dx \\
&+ \int_{\Omega^{b,l}} \left| \left(D_{x_1} m_n^{b,l}, \frac{1}{h_n} D_{x_2} m_n^{b,l}, \frac{1}{h_n} D_{x_3} m_n^{b,l} \right) \right|^2 dx \\
&+ \frac{1}{2} \int_{\mathbb{R}^3_{b,l}} \left| \left(D_{x_1} u^{b,l}_{m_n}, \frac{1}{h_n} D_{x_2} u^{b,l}_{m_n}, \frac{1}{h_n} D_{x_3} u^{b,l}_{m_n} \right) \right|^2 dx \\
&+ h_n \int_{\Omega^{b,r}} \left| \left(\frac{1}{h_n} D_{x_1} m_n^{b,r}, \frac{1}{h_n} D_{x_2} m_n^{b,r}, \frac{1}{h_n} D_{x_3} m_n^{b,r} \right) \right|^2 dx \\
&+ \frac{1}{2} \frac{1}{h_n} \int_{\mathbb{R}^3_{b,r}} \left| \left(D_{x_1} u^{b,r}_{m_n}, D_{x_2} u^{b,r}_{m_n}, D_{x_3} u^{b,r}_{m_n} \right) \right|^2 dx.
\end{aligned}$$

(2.4.8)

Then, we will denote

$$E_n(0) = E_n(m_{0_n}) \tag{2.4.9}$$

In the sequel we denote, for every $n \in \mathbb{N}$ and for a.e. $t \in [0,T]$

$$\begin{aligned}
E_n^{exc}(t) = &\int_{\Omega^a} \left| \left(\frac{1}{h_n} D_{x_1} m_n^a, \frac{1}{h_n} D_{x_2} m_n^a, D_{x_3} m_n^a \right) \right|^2 dx \\
&+ \int_{\Omega^{b,l}} \left| \left(D_{x_1} m_n^{b,l}, \frac{1}{h_n} D_{x_2} m_n^{b,l}, \frac{1}{h_n} D_{x_3} m_n^{b,l} \right) \right|^2 dx \\
&+ h_n \int_{\Omega^{b,r}} \left| \left(\frac{1}{h_n} D_{x_1} m_n^{b,r}, \frac{1}{h_n} D_{x_2} m_n^{b,r}, \frac{1}{h_n} D_{x_3} m_n^{b,r} \right) \right|^2 dx
\end{aligned} \tag{2.4.10}$$

and

$$E_n^{mag}(t) = \frac{1}{2}\int_{\mathbb{R}^3_a} \left|\left(\frac{1}{h_n}D_{x_1}u^a_{m_n}, \frac{1}{h_n}D_{x_2}u^a_{m_n}, D_{x_3}u^a_{m_n}\right)\right|^2 dx$$
$$+\frac{1}{2}\int_{\mathbb{R}^3_{b,l}} \left|\left(D_{x_1}u^{b,l}_{m_n}, \frac{1}{h_n}D_{x_2}u^{b,l}_{m_n}, \frac{1}{h_n}D_{x_3}u^{b,l}_{m_n}\right)\right|^2 dx \tag{2.4.11}$$
$$+\frac{1}{2}\frac{1}{h_n}\int_{\mathbb{R}^3_{b,r}} \left|\left(D_{x_1}u^{b,r}_{m_n}, D_{x_2}u^{b,r}_{m_n}, D_{x_3}u^{b,r}_{m_n}\right)\right|^2 dx.$$

So, by virtue of (2.4) , $E_n(t)$ can be rewritten as

$$E_n(t) = E_n^{exc}(t) + E_n^{mag}(t), \tag{2.4.12}$$

the sum of the exchange and magnetostatic energies.
Indeed, we can observe that, for every $t \in [0,T]$, the function defined by

$$\begin{cases} M_n(t, h_n x_1, h_n x_2, x_3) \text{ for a.e. in } \Omega^a, \\ M_n(t, x_1, h_n x_2, h_n x_3) \text{ for a.e. in } \Omega^{b,l}, \\ M_n(t, h_n x_1, h_n x_2, h_n x_3) \text{ for a.e. in } \Omega^{b,r}, \end{cases} \tag{2.4.13}$$

with M_n solution of the Problem (2.3.4), is a solution of Problem (2.4.6) with the following initial data:

$$\begin{cases} m^a_{0_n}(x_1,x_2,x_3) = M_{0_n}(h_n x_1, h_n x_2, x_3), \text{ for a.e. in } \Omega^a, \\ m^{b,l}_{0_n}(x_1,x_2,x_3) = M_{0_n}(x_1, h_n x_2, h_n x_3) \text{ for a.e. in } \Omega^{b,l}, \\ m^{b,r}_{0_n}(x_1,x_2,x_3) = M_{0_n}(h_n x_1, h_n x_2, h_n x_3) \text{ for a.e. in } \Omega^{b,r}. \end{cases} \tag{2.4.14}$$

Then, we will denote

$$E_n^{exc}(0) = E_n^{exc}(m_{0_n}) \tag{2.4.15}$$

Also, consider the hypothesis

$$\exists\, C \in]0,+\infty[: \quad E_n^{exc}(m_{0_n}) \le C, \quad \forall n \in \mathbb{N}. \tag{2.4.16}$$

Let us denotes that we can reformulate Theorem 2.3.1 in the rescaled form:

Theorem 2.4.1. *Suppose that* $m_{0_n} = \left(m^a_{0_n}, m^{b,l}_{0_n}, m^{b,r}_{0_n}\right) \in \mathcal{M}_n$ *and* (2.4.16) *holds, for every* $n \in \mathbb{N}$, *let* $m_n = \left(m^a_n, m^b_n\right)$ *be the solution of Problem* (2.4.6). *Then, there exist an increasing sequence of positive integer numbers* $\{n_i\}_{i\in\mathbb{N}}$, *still denoted by* $\{n\}$, $\mu_0 = \left(\mu^a_0, \mu^{b,l}_0\right) \in \mathcal{M}$, $\mu = \left(\mu^a, \mu^{b,l}\right) \in L^\infty\left(0,T;\mathcal{M}\right)$, *depending on the selected subsequence such that:*

$$\begin{cases} m^a_{0_n} \rightharpoonup \mu^a_0 \ weakly \ in \ H^1\left(\Omega^a, \mathbb{R}^3\right), \\ m^{b,l}_{0_n} \rightharpoonup \mu^{b,l}_0 \ weakly \ in \ H^1\left(\Omega^{b,l}, \mathbb{R}^3\right), \\ m^{b,r}_{0_n} \rightharpoonup \mu^a_0(0) = \mu^{b,l}_0(0) \ weakly \ in \ H^1\left(\Omega^{b,r}, \mathbb{R}^3\right), \end{cases} \tag{2.4.17}$$

$$\begin{cases} m^a_n \rightharpoonup \mu^a \ weakly \ * \ in \ L^\infty\left(0,T;H^1\left(\Omega^a,\mathbb{R}^3\right)\right), \\ m^{b,l}_n \rightharpoonup \mu^{b,l} \ weakly \ * \ in \ L^\infty\left(0,T;H^1\left(\Omega^{b,l},\mathbb{R}^3\right)\right), \\ m^a_n \to \mu^a \ in \ C\left(0,T;L^2\left(\Omega^a,\mathbb{R}^3\right)\right), \\ m^{b,l}_n \to \mu^{b,l} \ in \ C\left(0,T;L^2\left(\Omega^{b,l},\mathbb{R}^3\right)\right), \\ m^{b,r}_n \rightharpoonup \mu^a(t,0) = \mu^{b,l}(t,0) \ weakly \ * \ in \ L^\infty\left(0,T;H^1\left(\Omega^{b,r},\mathbb{R}^3\right)\right), \end{cases} \tag{2.4.18}$$

as n *diverges, where* $\mu = (\mu^a, \mu^{b,l})$ *is a solution of Problem* (2.3.9).

Theorem 2.3.1, is immediate consequence of Theorem 2.4.1. Indeed, we have to observe that (2.3.6) is equivalent to (2.4.16). So, we can apply Theorem 2.4.1. Then, we can use the observed equivalence between Problem (2.3.4) and Problem (2.4.6). So a change of variables and convergences (2.4.17) give (2.3.14) and the third and fourth convergences in (2.4.18) give the convergences (2.3.15).

2.4.1 Compactness like results

Let us obtain a priori estimates for the sequence of the solutions of the Problem (2.4.6). Let us introduce the following compactness like results

Proposition 2.4.1. *Assume (2.4.16). For every $n \in \mathbb{N}$, let $m_{0_n} = \left(m_{0_n}^a, m_{0_n}^{b,l}, m_{0_n}^{b,r}\right) \in \mathcal{M}_n$ and let $m_n = \left(m_n^a, m_n^{b,l}, m_n^{b,r}\right)$ be the solution of problem (2.4.6). Then, there exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$, still denoted by $\{n\}$, $\mu_0 = \left(\mu_0^a, \mu_0^{b,l}\right) \in \mathcal{M}$, $\mu = (\mu^a, \mu^{b,l}) \in L^\infty(0,T;\mathcal{M})$, depending on the subsequence, such that*

$$\begin{cases} m_{0_n}^a \rightharpoonup \mu_0^a \text{ weakly in } H^1\left(\Omega^a, \mathbb{R}^3\right), \\ m_{0_n}^{b,l} \rightharpoonup \mu_0^{b,l} \text{ weakly in } H^1\left(\Omega^{b,l}, \mathbb{R}^3\right), \\ m_{0_n}^{b,r} \rightharpoonup \mu_0^a(0) = \mu_0^{b,l}(0) \text{ weakly in } H^1\left(\Omega^{b,r}, \mathbb{R}^3\right), \end{cases} \tag{2.4.19}$$

$$\begin{cases} i)\, m_n^a \rightharpoonup \mu^a \text{ weakly} * \text{ in } L^\infty(0,T;H^1\left(\Omega^a,\mathbb{R}^3\right)), \\ ii)\, m_n^{b,l} \rightharpoonup \mu^{b,l} \text{ weakly} * \text{ in } L^\infty\left(0,T;H^1\left(\Omega^{b,l},\mathbb{R}^3\right)\right), \\ iii)\, m_n^{b,r} \rightharpoonup \mu^a(t,0) = \mu^{b,l}(t,0) \text{ weakly} * \text{ in } L^\infty\left(0,T;H^1\left(\Omega^{b,r},\mathbb{R}^3\right)\right), \\ iv)\, m_n^a \to \mu^a \text{ in } C\left(0,T;L^2\left(\Omega^a,\mathbb{R}^3\right)\right), \\ v)\, m_n^{b,l} \to \mu^{b,l} \text{ in } C\left(0,T;L^2\left(\Omega^{b,l},\mathbb{R}^3\right)\right), \\ vi)\, \dfrac{\partial m_n^a}{\partial t} \rightharpoonup \dfrac{\partial \mu^a}{\partial t} \text{ weakly in } L^2\left(0,T;L^2\left(\Omega^a,\mathbb{R}^3\right)\right), \\ vii)\, \dfrac{\partial m_n^{b,l}}{\partial t} \rightharpoonup \dfrac{\partial \mu^{b,l}}{\partial t} \text{ weakly in } L^2\left(0,T;L^2\left(\Omega^{b,l},\mathbb{R}^3\right)\right), \\ viii)\, h_n \dfrac{\partial m_n^{b,r}}{\partial t} \rightharpoonup 0 \text{ weakly in } L^2\left(0,T;L^2\left(\Omega^{b,r},\mathbb{R}^3\right)\right), \end{cases} \tag{2.4.20}$$

as n diverges. Moreover

$$\mu^a(0,x) = \mu_0^a(x), \quad \text{a.e. } x \text{ in } \Omega^a, \quad \mu^{b,l}(0,x) = \mu_0^{b,l}(x), \quad \text{a.e. } x \text{ in } \Omega^{b,l}. \tag{2.4.21}$$

Proof. Observe that by (2.4.15) and (2.4.16), we have

$$\begin{cases} \left\|D_{x_1} m_{0_n}^a\right\|_{(L^2(\Omega^a))^3} \leq Ch_n, \quad \left\|D_{x_2} m_{0_n}^a\right\|_{(L^2(\Omega^a))^3} \leq Ch_n, \quad \left\|D_{x_3} m_{0_n}^a\right\|_{(L^2(\Omega^a))^3} \leq C, \\ \left\|D_{x_1} m_{0_n}^{b,l}\right\|_{(L^2(\Omega^{b,l}))^3} \leq C, \quad \left\|D_{x_2} m_{0_n}^{b,l}\right\|_{(L^2(\Omega^{b,l}))^3} \leq Ch_n, \quad \left\|D_{x_3} m_{0_n}^{b,l}\right\|_{(L^2(\Omega^{b,l}))^3} \leq Ch_n, \\ \left\|D_{x_1} m_{0_n}^{b,r}\right\|_{(L^2(\Omega^{b,r}))^3} \leq C\sqrt{h_n}, \quad \left\|D_{x_2} m_{0_n}^{b,r}\right\|_{(L^2(\Omega^{b,r}))^3} \leq C\sqrt{h_n}, \\ \left\|D_{x_3} m_{0_n}^{b,r}\right\|_{(L^2(\Omega^{b,r}))^3} \leq C\sqrt{h_n}, \end{cases} \tag{2.4.22}$$

for every $n \in \mathbb{N}$, where C is a constant independent on n. Then there exist a subsequence (not rellebelled) $\mu_0^a \in H^1(\Omega^a, \mathbb{R}^3)$ and $\mu_0^{b,l} \in H^1\left(\Omega^{b,l}, \mathbb{R}^3\right)$ such that the following convergences are verified

$$\begin{cases} m_{0_n}^a \rightharpoonup \mu_0^a \; weakly \text{ in } H^1(\Omega^a, \mathbb{R}^3), \\ m_{0_n}^{b,l} \rightharpoonup \mu_0^{b,l} \; weakly \text{ in } H^1\left(\Omega^{b,l}, \mathbb{R}^3\right), \\ m_{0_n}^{b,r} \rightharpoonup c \; weakly \text{ in } H^1\left(\Omega^{b,r}, \mathbb{R}^3\right). \end{cases} \tag{2.4.23}$$

Since $\left|m_{0_n}^a(x)\right| = 1$ x a.e. in Ω^a, by (2.4.23), $|\mu_0^a(x)| = 1$ for x $a.e.$ in Ω^a (respectively $\left|\mu_0^{b,l}(x)\right| = 1$ for x $a.e.$ in $\Omega^{b,l}$). Moreover, by (2.4.22) we obtain that μ_0^a is independent of (x_1, x_2) and $\mu_0^{b,l}$ is independent of (x_2, x_3). Actually, since one has that

$$\begin{cases} m_{0_n}^a(x_1, x_2, 0) = m_{0_n}^{b,r}(x_1, x_2, 0), \text{ for } (x_1, x_2) \text{ a.e. in }]-1, 0[^2, \\ m_{0_n}^{b,l}(0, x_2, x_3) = m_{0_n}^{b,r}(0, x_2, x_3), \text{ for } (x_2, x_3) \text{ a.e. in }]-1, 0[^2, \end{cases}$$

for every $n \in \mathbb{N}$, by trace convergence properties we obtain $\mu_0^a(0) = c = \mu_0^{b,l}(0)$, that is $\mu_0 = (\mu_0^a, \mu_0^{b,l}) \in \mathcal{M}$, and so (2.4.19).

Moreover, by (2.4.7) and hypotheses (2.4.16), the following estimates are satisfied:

$$
\begin{cases}
i)\ \|D_{x_1} m_n^a\|_{L^\infty\left(0,T;(L^2(\Omega^a))^3\right)} \leq Ch_n, \quad \|D_{x_2} m_n^a\|_{L^\infty\left(0,T;(L^2(\Omega^a))^3\right)} \leq Ch_n, \\
\quad \|D_{x_3} m_n^a\|_{L^\infty\left(0,T;(L^2(\Omega^a))^3\right)} \leq C, \\
ii)\ \left\|D_{x_1} m_n^{b,l}\right\|_{L^\infty\left(0,T;(L^2(\Omega^{b,l}))^3\right)} \leq C, \quad \left\|D_{x_2} m_n^{b,l}\right\|_{L^\infty\left(0,T;(L^2(\Omega^{b,l}))^3\right)} \leq Ch_n, \\
\quad \left\|D_{x_3} m_n^{b,l}\right\|_{L^\infty\left(0,T;(L^2(\Omega^{b,l}))^3\right)} \leq Ch_n, \\
iii)\ \left\|D_{x_1} m_n^{b,r}\right\|_{L^\infty\left(0,T;(L^2(\Omega^{b,r}))^3\right)} \leq C\sqrt{h_n}, \quad \left\|D_{x_2} m_n^{b,r}\right\|_{L^\infty\left(0,T;(L^2(\Omega^{b,r}))^3\right)} \leq C\sqrt{h_n}, \\
\quad \left\|D_{x_3} m_n^{b,r}\right\|_{L^\infty\left(0,T;(L^2(\Omega^{b,r}))^3\right)} \leq C\sqrt{h_n}, \\
iv)\ \left\|\dfrac{\partial m_n^a}{\partial t}\right\|_{L^2\left((0,T;(L^2(\Omega^a))^3\right)} \leq C, \quad \left\|\dfrac{\partial m_n^{b,l}}{\partial t}\right\|_{L^2\left(0,T;(L^2(\Omega^{b,l}))^3\right)} \leq C, \\
v)\ \sqrt{h_n}\left\|\dfrac{\partial m_n^{b,r}}{\partial t}\right\|_{L^2\left(0,T;(L^2(\Omega^{b,r}))^3\right)} \leq C,
\end{cases}
\tag{2.4.24}
$$

for every $n \in \mathbb{N}$ where C is a constant independent on n. By (2.4.24)$i)$ and (2.4.24)$ii)$ it follows that there exist a subsequence (not rellebelled) $\mu^a \in L^\infty\left(0,T;H^1\left(\Omega^a,\mathbb{R}^3\right)\right)$ and $\mu^{b,l} \in L^\infty\left(0,T;H^1\left(\Omega^{b,l},\mathbb{R}^3\right)\right)$ such that

$$
\begin{cases}
i) \quad m_n^a \rightharpoonup \mu^a \ weakly* \ \text{in } L^\infty(0,T;H^1\left(\Omega^a,\mathbb{R}^3\right)), \\
ii) \quad m_n^{b,l} \rightharpoonup \mu^{b,l} \ weakly* \ \text{in } L^\infty\left(0,T;H^1\left(\Omega^{b,l},\mathbb{R}^3\right)\right),
\end{cases}
\tag{2.4.25}
$$

as n diverges.

By equiboundness of $\{m_n^a\}_{n\in\mathbb{N}}$, $\{m_n^{b,l}\}_{n\in\mathbb{N}}$, and by (2.4.24)$iv)$ arguing similar as in [24, Proposition 4.1], we obtain, up to a subsequence, convergences (2.4.20)$iv)$ and (2.4.20)$v)$.

About initial conditions, we observe that

$$
m_n^a\left(0,\cdot\right) = m_{0_n}^a \quad \text{and} \quad m_n^{b,l}\left(0,\cdot\right) = m_{0_n}^{b,l}.
$$

Then, by $(2.4.20)iv)$ and $(2.4.20)v)$, it follows

$$\begin{cases} m_n^a(0,\cdot) \to \mu^a(0,\cdot) \quad \text{in } L^2(\Omega^a, \mathbb{R}^3), \\ m_n^{b,l}(0,\cdot) \to \mu^{b,l}(0,\cdot) \quad \text{in } L^2\left(\Omega^{b,l}, \mathbb{R}^3\right). \end{cases}$$

Then, by the first two convergences in (2.4.19), we get (2.4.21).
Furthermore, let us point out that, by first and second estimates in $(2.4.24)i)$ and second and third estimate in $(2.4.24)ii)$, the functions μ^a and $\mu^{b,l}$ do not depend on (x_1, x_2) and (x_2, x_3) respectively.

Indeed by $(2.4.20)i)$ we get that

$$m_n^a \rightharpoonup \mu^a \text{ weakly in } L^2\left(0,T; H^1\left(\Omega^a, \mathbb{R}^3\right)\right).$$

Consequently, by lower semicontinuity theorem for a convex functional, we obtain

$$\int_0^T \|D_{x_1}\mu^a\|^2_{(L^2(\Omega^a))^3} \leq \liminf_n \int_0^T \|D_{x_1} m_n^a\|^2_{(L^2(\Omega^a))^3}.$$

Then, by $(2.4.24)i)$, for $a.e.\, t \in [0,T]$ we obtain

$$\|D_{x_1} m_n^a\|^2_{(L^2(\Omega^a))^3} \leq C h_n.$$

So, by (2.4.24), since m_n^a is bounded in $L^\infty(0,T; H^1(\Omega^a, \mathbb{R}^3))$ and h_n goes to zero as n diverges, we obtain, for a.e. $t \in [0,T]$, that

$$\|D_{x_1}\mu^a\|_{(L^2(\Omega^a))^3} = 0.$$

Then for a.e. $t \in [0,T]$ we get

$$D_{x_1}\mu^a = 0, \qquad \text{a.e. } x \text{ in } \Omega^a.$$

Similarly

$$D_{x_2}\mu^a = 0, \text{ a.e. } x \text{ in } \Omega^a, \ D_{x_2}\mu^{b,l} = 0 \text{ and } D_{x_3}\mu^{b,l} = 0, \text{ a.e. } x \text{ in } \Omega^{b,l}.$$

Let us observe that $|m_n^a(t,x)| = 1$ for every $t \in [0,T]$ and x a.e. in Ω^a. So, by $(2.4.20)\, iv)$, $|\mu^a(t,x)| = 1$ for every $t \in [0,T]$ and x $a.e.$ in Ω^a (respectively $\left|\mu^{b,l}(t,x)\right| = 1$ for every $t \in [0,T]$ and x $a.e.$ in $\Omega^{b,l}$).

By $(2.4.24)iv)$ and by definition of distributional derivative (see [64, Chapter 23]),arguing as in [24, Proposition 4.1], ones prove $(2.4.20)vi)$ and $(2.4.20)vii)$.

Moreover, from $(2.4.24)iii)$ there exist a subsequence (not rellebelled) and $\zeta^{b,r} \in L^\infty\left(0,T; H^1\left(\Omega^{b,r}, \mathbb{R}^3\right)\right)$ such that

$$m_n^{b,r} \rightharpoonup \zeta^{b,r} \ weakly* \text{ in } L^\infty\left(0,T; H^1\left(\Omega^{b,r}, \mathbb{R}^3\right)\right), \tag{2.4.26}$$

as n diverges.

In particular we obtain

$$m_n^{b,r} \rightharpoonup \zeta^{b,r} \ weakly \text{ in } L^2\left(0,T; H^1\left(\Omega^{b,r}, \mathbb{R}^3\right)\right).$$

Now, let us identify $\zeta^{b,r}$.

By lower semicontinuity theorem for a convex functional, we obtain

$$\int_0^T \left\|D\zeta^{b,r}\right\|^2_{(L^2(\Omega^{b,r}))^3} \leq \liminf_n \int_0^T \|Dm_n^{b,r}\|^2_{(L^2(\Omega^{b,r}))^3}.$$

By $(2.4.24)iii)$ we get $\left\|D\zeta^{b,r}\right\|^2_{(L^2(\Omega^{b,r}))^3} = 0$ a.e. $t \in [0,T]$, then $\zeta^{b,r} = c(t)$ for a.e. t.

Since, $\left|m_n^{b,r}(t,x)\right| = 1$ for every $t \in [0,T]$ and x a.e. in $\Omega^{b,r}$ we obtain $c(t) \in S^2$ for every $t \in [0,T]$.

In order to verify that $\mu = (\mu^a, \mu^{b,l}) \in L^\infty(0,T;\mathcal{M})$ it remains to prove that $\mu^a(t,0) = \mu^{b,l}(t,0)$ almost everywhere in $]0,T[$. At first let us observe that for a.e. $t \in [0,T]$,

$$\begin{cases} m_n^a(t,x_1,x_2,0) = m_n^{b,r}(t,x_1,x_2,0), \text{ for } (x_1,x_2) \text{ a.e. in }]-1,0[^2, \\ \\ m_n^{b,l}(t,0,x_2,x_3) = m_n^{b,r}(t,0,x_2,x_3), \text{ for } (x_2,x_3) \text{ a.e. in }]-1,0[^2, \end{cases}$$

consequently

$$\int_0^T \int_{]-1,0[^2\times\{0\}} m_n^a\left(t,x_1,x_2,0\right)\varphi dx_1dx_2dt = \int_0^T \int_{]-1,0[^2\times\{0\}} m_n^{b,r}\left(t,x_1,x_2,0\right)\varphi dx_1dx_2dt \quad (2.4.27)$$

for every n and for every $\varphi \in C_0^\infty\left((0,T)\times\Omega^a\right)$, and

$$\int_0^T \int_{]-1,0[^2\times\{0\}} m_n^{b,l}\left(t,x_2,x_3,0\right)\varphi dx_2dx_3dt = \int_0^T \int_{]-1,0[^2\times\{0\}} m_n^{b,r}\left(t,x_2,x_3,0\right)\varphi dx_2dx_3dt \quad (2.4.28)$$

for every n and for every $\varphi \in C_0^\infty\left((0,T)\times\Omega^{b,l}\right)$.
Now, by 2.4.20i) (respectively 2.4.20iii)) , $\int_0^T m_n^a\varphi dt$ (respectively $\int_0^T m_n^{b,r}\varphi dt$) is bounded in $H^1(\Omega^a)$ (respectively in $H^1(\Omega^{b,r})$) and converges weakly to $\int_0^T \mu^a\varphi dt$ (respectively to $\int_0^T c(t)\varphi dt$). Thus their trace converges strongly in $L^2(]-1,0[^2\times 0$. Then

$$\lim_n \int_0^T \int_{]-1,0[^2\times\{0\}} m_n^a\left(t,x_1,x_2,0\right)\varphi dx_1dx_2dt = \int_0^T \int_{]-1,0[^2\times\{0\}} \mu^a\left(t,x_1,x_2,0\right)\varphi dx_1dx_2dt \quad (2.4.29)$$

for every $\varphi \in C_0^\infty\left((0,T)\times\Omega^a\right)$ and

$$\lim_n \int_0^T \int_{]-1,0[^2\times\{0\}} m_n^{b,r}\left(t,x_1,x_2,0\right)\varphi dx_1dx_2dt = \int_0^T \int_{]-1,0[^2\times\{0\}} c(t)\varphi dx_1dx_2dt \quad (2.4.30)$$

for every $\varphi \in C_0^\infty\left((0,T)\times\Omega^a\right)$. By combining (2.4.27), (2.4.29) and (2.4.30) and remembering that μ^a is independent of (x_1,x_2), we get $\mu^a(t,0) = c(t)$ almost everywhere in $]0,T[$. In the same way, by (2.4.28), we prove that $\mu^{b,l}(t,0) = c(t)$ almost everywhere in $]0,T[$. So we can conclude that $\mu \in L^\infty(0,T;\mathcal{M})$.
The convergence (2.4.20) $viii$) is easily obtained by (2.4.24)v) and by definition of distributional derivative.

□

2.4.2 A convergence result for the magnetostatic energy

In what follows $S =]-1,0[^2$, so we will omit the explicit dependence of coefficients $\alpha, \beta, \gamma, \delta, \varepsilon$, defined in (2.2.7), by S. Moreover, let us pose

$$\begin{cases} P(x_1,x_2,x_3) = p(x_1,x_2) & \text{for every } (x_1,x_2) \in]-1,0[^2 \\ Q(x_1,x_2,x_3) = q(x_1,x_2) & \text{for every } (x_1,x_2) \in]-1,0[^2 \\ P'(x_1,x_2,x_3) = p(x_2,x_3) & \text{for every } (x_2,x_3) \in]-1,0[^2 \\ Q'(x_1,x_2,x_3) = q(x_2,x_3) & \text{for every } (x_2,x_3) \in]-1,0[^2 \end{cases} \tag{2.4.31}$$

with p (resp. q) the unique solution of (2.2.2) (resp. (2.2.3)).

Proposition 2.4.2. *Let* $\left\{m_n = \left(m_n^a, m_n^{b,l}, m_n^{b,r}\right)\right\}_{n\in\mathbb{N}}$ *and let* $\left(\mu^a, \mu^{b,l}\right) = \left(\left(\mu_1^a, \mu_2^a, \mu_3^a\right), \left(\mu_1^{b,l}, \mu_2^{b,l}, \mu_3^{b,l}\right)\right) \in L^\infty(0,T;\mathcal{M})$ *such that,*

$$\begin{cases} m_n^a \to \mu^a \quad \textit{in } C\left(0,T;L^2\left(\Omega^a,\mathbb{R}^3\right)\right), \\ m_n^{b,l} \to \mu^{b,l} \quad \textit{in } C\left(0,T;L^2\left(\Omega^{b,l},\mathbb{R}^3\right)\right), \\ m_n^{b,r} \rightharpoonup \mu^a(t,0) = \mu^{b,l}(t,0) \textit{ weakly}* \textit{ in } L^\infty\left(0,T;H^1\left(\Omega^{b,r},\mathbb{R}^3\right)\right), \end{cases} \tag{2.4.32}$$

as n diverges. Moreover, for every $n \in \mathbb{N}$, *let* $u_{m_n} = \left(u_{m_n}^a, u_{m_n}^{b,l}, u_{m_n}^{b,r}\right)$ *be the unique solution of* (2.4.3) *corresponding to* m_n *and let* E_n^{mag} *be defined by* (2.4.11).
Then it result that

$$\begin{cases} \dfrac{1}{h_n} D_{x_1} u_n^a \rightharpoonup \xi_1^a, \quad \dfrac{1}{h_n} D_{x_2} u_n^a \rightharpoonup \xi_2^a, \quad D_{x_3} u_n^a \rightharpoonup 0 \quad \textit{weakly } * \textit{ in } L^\infty\left(0,T;L^2\left(\mathbb{R}_a^3\right)\right), \\ D_{x_1} u_n^{b,l} \rightharpoonup 0, \quad \dfrac{1}{h_n} D_{x_2} u_n^{b,l} \rightharpoonup \xi_2^{b,l}, \quad \dfrac{1}{h_n} D_{x_3} u_n^{b,l} \rightharpoonup \xi_3^{b,l} \quad \textit{weakly } * \textit{ in } L^\infty\left(0,T;L^2\left(\mathbb{R}_{b,l}^3\right)\right), \\ D u_n^{b,r} \rightharpoonup 0 \quad \textit{weakly}* \textit{ in } L^\infty\left(0,T;\left(L^2\left(\mathbb{R}_{b,r}^3\right)\right)^3\right), \end{cases} \tag{2.4.33}$$

as n diverges, where

$$(\xi_1^a,\xi_2^a,\xi_3^a)(x_1,x_2,x_3)=\begin{cases}(0,0,0),\ \textit{a.e. in}\ \mathbb{R}^2\times]1,+\infty[,\\ \\ \mu_1^a(x_3)DP(x_1,x_2,x_3)+\mu_2^a(x_3)DQ(x_1,x_2,x_3),\ \textit{a.e. in}\ \mathbb{R}^2\times]0,1[,\end{cases}\tag{2.4.34}$$

$$(\xi_1^{b,l},\xi_2^{b,l},\xi_3^{b,l})(x_1,x_2,x_3)=\begin{cases}(0,0,0),\ \textit{a.e. in}\ ]1,+\infty[\times\mathbb{R}\times]-\infty,0[,\\ \\ \mu_2^{b,l}(x_1)DP'(x_1,x_2,x_3)+\mu_3^{b,l}(x_1)DQ'(x_1,x_2,x_3),\ \textit{a.e. in}\ ]0,1[\times\mathbb{R}\times]-\infty,0[,\end{cases}\tag{2.4.35}$$

with P, P', Q and Q' defined in (2.4.31). Furthermore, one has that

$$\begin{aligned}&\lim_n\Bigg[\int_{\Omega^a}\left(\frac{1}{h_n}D_{x_1}u_n^a,\frac{1}{h_n}D_{x_2}u_n^a,D_{x_3}u_n^a\right)m_n^a dx+\\ &\int_{\Omega^{b,l}}\left(D_{x_1}u_n^{b,l},\frac{1}{h_n}D_{x_2}u_n^{b,l},\frac{1}{h_n}D_{x_3}u_n^{b,l}\right)m_n^{b,l}dx+\\ &\int_{\Omega^{b,r}}\left(D_{x_1}u_n^{b,r},D_{x_2}u_n^{b,r},D_{x_3}u_n^{b,r}\right)m_n^{b,r}dx\Bigg]=\\ &\alpha\left(\int_0^1|\mu_1^a|^2dx_3+\int_0^1|\mu_2^a|^2dx_3+\int_0^1|\mu_2^{b,l}|^2dx_1+\int_0^1|\mu_3^{b,l}|^2dx_1\right),\end{aligned}\tag{2.4.36}$$

where α,is defined by (2.2.5).

Proof. Choose $(u_n^a,u_n^{b,l},u_n^{b,r})$ as test function in (2.4.3), take into account that for every t $\{(m_n^a,m_n^{b,l},m_n^{b,r})\}_{n\in\mathbb{N}}\subset L^2(\Omega^a,S^2)\times L^2(\Omega^{b,l},S^2)\times L^2(\Omega^{b,r},S^2)$ and that by (2.4.32) the norms in these spaces are equi-bounded, then there exists $C\in]0,+\infty[$ such that

$$\begin{cases}\left\|\left(\frac{1}{h_n}D_{x_1}u_n^a,\frac{1}{h_n}D_{x_2}u_n^a,D_{x_3}u_n^a\right)\right\|_{(L^2(R_a^3))^3}\le C,\\ \\ \left\|\left(D_{x_1}u_n^{b,l},\frac{1}{h_n}D_{x_2}u_n^{b,l},\frac{1}{h_n}D_{x_3}u_n^{b,l}\right)\right\|_{(L^2(R_{b,l}^3))^3}\le C,\\ \\ \frac{1}{\sqrt{h_n}}\left\|\left(D_{x_1}u_n^{b,r},D_{x_2}u_n^{b,r},D_{x_3}u_n^{b,r}\right)\right\|_{(L^2(R_{b,r}^3))^3}\le C,\end{cases}\tag{2.4.37}$$

for every $n\in\mathbb{N}$.

We observe that the last estimate in (2.4.37) gives the last limit in (2.4.33).

By arguing as in the first part of the proof of [34, Proposition 5.1], from the first two estimates in (2.4.37) one derives the third and the fourth limit in (2.4.33), up to a subsequence. From the first estimate in (2.4.37) one obtains the first two limits in (2.4.33) and from the second estimate in (2.4.37) one obtains the fifth and sixth limits in (2.4.33), up to a subsequence.

Next step is devoted to identify $\xi^a = (\xi_1^a, \xi_2^a)$.

Let as fix $t \in [0,T]$, in equation (2.4.3) with $(m^a, m^{b,l}, m^{b,r}) = (m_n^a, m_n^{b,l}, m_n^{b,r})$ choose $v^a = \varphi + c_n$, $v^{b,l} = c_n$ and $v^{b,r} = c_n$, with $\varphi \in C_0^\infty(\mathbb{R}^3_a)$ and $c_n = -\left(|B_n^a| + |B_n^{b,l}| + |B_n^{b,r}|\right)^{-1} \int_{B_n^a} \varphi dx$ (such that $(v^a, v^{b,l}, v^{b,r}) \in \mathcal{U}_n$). By multiplying this equation by h_n, for all $t \in [0,T]$ one has

$$\begin{cases} \displaystyle\int_{\mathbb{R}^3_a} \left(\frac{1}{h_n} D_{x_1} u_n^a, \frac{1}{h_n} D_{x_2} u_n^a, D_{x_3} u_n^a\right) (D_{x_1}\varphi, D_{x_2}\varphi, h_n D_{x_3}\varphi)\, dx = \\ \displaystyle\int_{\Omega^a} (D_{x_1}\varphi, D_{x_2}\varphi, h_n D_{x_3}\varphi)\, m_n^a dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^3_a). \end{cases} \tag{2.4.38}$$

Then, passing to the limit, as n diverges, in (2.4.38), for every $t \in [0,T]$ convergences (2.4.32) and (2.4.33) give that

$$\int_{\mathbb{R}^3_a} (\xi_1^a, \xi_2^a)(D_{x_1}\varphi, D_{x_2}\varphi) dx = \int_0^1 \left((\mu_1^a, \mu_2^a) \int_{]-1,0[^2} (D_{x_1}\varphi, D_{x_2}\varphi) dx_1 dx_2 \right) dx_3, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^3_a).$$

Then, observe that:

$$D_{x_2}\left(\frac{1}{h_n} D_{x_1} u_n^a\right) = D_{x_1}\left(\frac{1}{h_n} D_{x_2} u_n^a\right) \text{ in } \mathcal{D}'(\mathbb{R}^3_a), \quad \forall n \in \mathbb{N},$$

and using the first two limits in (2.4.33) , one obtains that

$$\int_{\mathbb{R}^3_a} \xi_1^a D_{x_2}\varphi dx = \int_{\mathbb{R}^3_a} \xi_2^a D_{x_1}\varphi dx, \quad \forall \varphi \in H_0^1(\mathbb{R}^3_a). \tag{2.4.39}$$

By taking $\varphi(x) = \phi(x_1, x_2)\chi(x_3)$ with $\phi \in H^1(\mathbb{R}^2)$ and $\chi \in C_0^\infty(]0, +\infty[)$ and recalling that $H^1(\mathbb{R}^2)$ is separable, it follows from (2.4.39) that

$$\begin{cases} \text{for } x_3 \text{ a.e. in }]0,+\infty[, \quad \displaystyle\int_{\mathbb{R}^2} \xi_1^a(x_1, x_2, x_3) D_{x_2}\phi(x_1, x_2) dx_1 dx_2 = \\ \displaystyle\int_{\mathbb{R}^2} \xi_2^a(x_1, x_2, x_3) D_{x_1}\phi(x_1, x_2) dx_1 dx_2, \quad \forall \phi \in H^1(\mathbb{R}^2). \end{cases}$$

Consequently, by virtue of the Poincaré Lemma (see Section 2.2), it results that

$$\begin{cases} \text{for } x_3 \text{ a.e. in }]0,+\infty[, \quad \exists! w(\cdot,\cdot,x_3) \in W^1(\mathbb{R}^2) : \\ \\ \xi_1^a(\cdot,\cdot,x_3) = D_{x_1} w(\cdot,\cdot,x_3), \quad \xi_2^a(\cdot,\cdot,x_3) = D_{x_2} w(\cdot,\cdot,x_3), \quad \text{a.e. in } \mathbb{R}^2. \end{cases} \tag{2.4.40}$$

Consequently, arguing as above, taking into account that $W^1(\mathbb{R}^2)$ is separable, and using Proposition 2.2.1 and (2.4.40), it follows that, for x_3 a.e. in $]1,+\infty[$, $w(\cdot,\cdot,x_3)$ solves the following problem:

$$\begin{cases} w(\cdot,\cdot,x_3) \in W^1(\mathbb{R}^2), \\ \\ \displaystyle\int_{\mathbb{R}^2} (D_{x_1} w(x_1,x_2,x_3), D_{x_2} w(x_1,x_2,x_3))(D_{x_1}\phi(x_1,x_2), D_{x_2}\phi(x_1,x_2)) dx_1 dx_2 = 0, \quad \forall \phi \in W^1(\mathbb{R}^2), \end{cases}$$

while, for x_3 a.e. in $]0,1[$, $w(\cdot,\cdot,x_3)$ solves the following one:

$$\begin{cases} w(\cdot,\cdot,x_3) \in W^1(\mathbb{R}^2), \\ \\ \displaystyle\int_{\mathbb{R}^2} (D_{x_1} w(x_1,x_2,x_3), D_{x_2} w(x_1,x_2,x_3))(D_{x_1}\phi(x_1,x_2), D_{x_2}\phi(x_1,x_2)) dx_1 dx_2 = \\ \\ (\mu_1^a(x_3), \mu_2^a(x_3)) \displaystyle\int_{]-1,0[^2} (D_{x_1}\phi(x_1,x_2), D_{x_2}\phi(x_1,x_2)) dx_1 dx_2, \quad \forall \phi \in W^1(\mathbb{R}^2). \end{cases} \tag{2.4.41}$$

Then, by virtue of Lemma 2.2.1, it results that, for x_3 a.e. in $]0,+\infty[$,

$$w(\cdot,\cdot,x_3) = \begin{cases} 0, \text{ a.e. in } \mathbb{R}^2, \text{ if } x_3 > 1, \\ \\ \mu_1^a(x_3) p(\cdot,\cdot) + \mu_2^a(x_3) q(\cdot,\cdot), \text{ a.e. in } \mathbb{R}^2, \text{ if } x_3 < 1, \end{cases} \tag{2.4.42}$$

with p (resp. q) the unique solution of (2.2.2) (resp. (2.2.3)).

Moreover, since Tonelli theorem assures that ξ^a and $\mu_1^a Dp + \mu_2^a Dq$ belong to $(L^2(\mathbb{R}^3_a))^2 \subset$

$(L^1_{\text{loc}}(\mathbb{R}^3_a))^2$, using Fubini theorem with (2.4.40) and (2.4.42) one entails that

$$\int_{\mathbb{R}^3_a} \xi^a \varphi dx = \int_0^{+\infty} \left(\int_{\mathbb{R}^2} \xi^a \varphi dx_1 dx_2 \right) dx_3 = \int_0^1 \left(\int_{\mathbb{R}^2} \left(\mu_1^a Dp + \mu_2^a Dq \right) \varphi dx_1 dx_2 \right) dx_3 =$$

$$\int_{\mathbb{R}^2 \times]0,1[} \left(\mu_1^a Dp + \mu_2^a Dq \right) \varphi dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^3_a),$$

that is

$$\xi^a(x_1, x_2, x_3) = \begin{cases} (0,0), \text{ a.e. in } \mathbb{R}^2 \times]1, +\infty[, \\ \\ \mu_1^a(x_3) Dp(x_1, x_2) + \mu_2^a(x_3) Dq(x_1, x_2), \text{ a.e. in } \mathbb{R}^2 \times]0, 1[, \end{cases} \tag{2.4.43}$$

with p (resp. q) the unique solution of (2.2.2) (resp. (2.2.3)). Consequently, the first two limits in (2.4.33) and (2.4.34) hold true for the whole sequence.

For every $t \in [0, T]$ using the first limit in (2.4.32) and (2.4.34) we obtain that

$$\begin{aligned} &\lim_n \int_{\Omega^a} \left(\frac{1}{h_n} D_{x_1} u_n^a, \frac{1}{h_n} D_{x_2} u_n^a, D_{x_3} u_n^a \right) m_n^a dx = \\ &\lim_n \int_{\Omega^a} \left(\mu_1^a DP + \mu_2^a DQ \right) (\mu_1^a, \mu_2^a) = \end{aligned} \tag{2.4.44}$$

By $iii)$ of Lemma 2.2.1 with $k = (\mu_1^a, \mu_2^a)$ we get

$$\begin{aligned} &\lim_n \int_{\Omega^a} \left(\frac{1}{h_n} D_{x_1} u_n^a, \frac{1}{h_n} D_{x_2} u_n^a, D_{x_3} u_n^a \right) m_n^a dx = \\ &\alpha \left(\int_0^1 |\mu_1^a|^2 dx_3 + \int_0^1 |\mu_2^a|^2 dx_3 \right), \end{aligned} \tag{2.4.45}$$

Using the third limits in (2.4.32) and the third inequality in (2.4.37) we obtain, for every t,

$$\lim_n \int_{\Omega^{b,r}} \left(D_{x_1} u_n^{b,r}, D_{x_2} u_n^{b,r}, D_{x_3} u_n^{b,r} \right) m_n^{b,r} dx = 0. \tag{2.4.46}$$

Next step is devoted to identify $\xi^{b,l} = (\xi_2^{b,l}, \xi_3^{b,l})$, we introduce other rescalings. Specifically, by setting

$$\mathbb{R}^3_{a,r} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0,\ x_1 < 0\}, \quad \mathbb{R}^3_l = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0\},$$

for every $n \in \mathbb{N}$, problem (2.3.2) will be reformulated on a fixed domain through the following rescaling:

$$\mathcal{T}_n : (x_1, x_2, x_3) \in \mathbb{R}^3 \to \mathcal{T}_n(x_1, x_2, x_3) = \begin{cases} (h_n x_1, h_n x_2, x_3), \text{ if } (x_1, x_2, x_3) \in \mathbb{R}^3_{a,r}, \\ (x_1, h_n x_2, h_n x_3), \text{ if } (x_1, x_2, x_3) \in \mathbb{R}^3_{l}, \\ (h_n x_1, h_n x_2, h_n x_3), \text{ if } (x_1, x_2, x_3) \in \mathbb{R}^3_{b,r} \end{cases} \tag{2.4.47}$$

(note that $T_{n|\mathbb{R}^3_{b,r}} = \mathcal{T}_{n|\mathbb{R}^3_{b,r}}$, and $T_n(\Omega^a) = \mathcal{T}_n(\Omega^a) = \Omega^a_n$, $T_n(\Omega^{b,l}) = \mathcal{T}_n(\Omega^{b,l}) = \Omega^{b,l}_n$). Namely, setting

$$B^{a,r}_n = \left]-\frac{2}{h_n}, 0\right[^2 \times]0, 2[, \quad B^l_n =]0, 2[\times \left]-\frac{2}{h_n}, \frac{2}{h_n}\right[^2, \quad B^{b,r}_n = \left]-\frac{2}{h_n}, 0\right[^3, \quad n \in \mathbb{N},$$

space $\mathcal{U}$ defined in (2.3.1) is rescaled in the following

$$\mathcal{V}_n = \Bigg\{ \begin{array}{l} (v^{a,r}, v^l, v^{b,r}) \in L^1_{loc}(\overline{\mathbb{R}^3_{a,r}}) \times L^1_{loc}(\overline{\mathbb{R}^3_l}) \times L^1_{loc}(\overline{\mathbb{R}^3_{b,r}}) \ : \\ (v^{a,r}_{|_{B^{a,r}_n}}, v^l_{|_{B^l_n}}, v^{b,r}_{|_{B^{b,r}_n}}) \in L^2(B^{a,r}_n) \times L^2(B^l_n) \times L^2(B^{b,r}_n), \\ (Dv^{a,r}, Dv^l, Dv^{b,r}) \in (L^2(\mathbb{R}^3_{a,r}))^3 \times (L^2(\mathbb{R}^3_l))^3 \times (L^2(\mathbb{R}^3_{b,r}))^3, \\ \displaystyle\int_{B^{a,r}_n} v^{a,r} dx + \int_{B^l_n} v^l dx + h_n \int_{B^{b,r}_n} v^{b,r} dx = 0, \\ v^l(0, x_2, x_3) = v^{a,r}(0, x_2, h_n x_3), \text{ for } (x_2, x_3) \text{ a.e. in } \mathbb{R}\times]0, +\infty[, \\ v^l(0, x_2, x_3) = v^{b,r}(0, x_2, x_3), \text{ for } (x_2, x_3) \text{ a.e. in } \mathbb{R}\times]-\infty, 0[, \\ v^{a,r}(x_1, x_2, 0) = v^{b,r}(x_1, x_2, 0), \text{ for } (x_1, x_2) \text{ a.e. in }]-\infty, 0[\times\mathbb{R} \Bigg\}. \end{array} \tag{2.4.48}$$

Then, for every $(m^a, m^{b,l}, m^{b,r}) \in L^2(\Omega^a, \mathbb{R}^3) \times L^2(\Omega^{b,l}, \mathbb{R}^3) \times L^2(\Omega^{b,r}, \mathbb{R}^3)$, the following equa-

tion:

$$\begin{cases} (v^{a,r}, v^l, v^{b,r}) \in \mathcal{V}_n, \\ \displaystyle\int_{\mathbb{R}^3_{a,r}} \left(\frac{1}{h_n} D_{x_1} v^{a,r}, \frac{1}{h_n} D_{x_2} v^{a,r}, D_{x_3} v^{a,r}\right) \left(\frac{1}{h_n} D_{x_1} w^{a,r}, \frac{1}{h_n} D_{x_2} w^{a,r}, D_{x_3} w^{a,r}\right) dx + \\ \displaystyle\int_{\mathbb{R}^3_l} \left(D_{x_1} v^l, \frac{1}{h_n} D_{x_2} v^l, \frac{1}{h_n} D_{x_3} v^l\right) \left(D_{x_1} w^l, \frac{1}{h_n} D_{x_2} w^l, \frac{1}{h_n} D_{x_3} w^l\right) dx + \\ \displaystyle\frac{1}{h_n} \int_{\mathbb{R}^3_{b,r}} \left(D_{x_1} v^{b,r}, D_{x_2} v^{b,r}, D_{x_3} v^{b,r}\right) \left(D_{x_1} w^{b,r}, D_{x_2} w^{b,r}, D_{x_3} w^{b,r}\right) dx = \\ \displaystyle\int_{\Omega^a} \left(\frac{1}{h_n} D_{x_1} w^{a,r}, \frac{1}{h_n} D_{x_2} w^{a,r}, D_{x_3} w^{a,r}\right) m^a dx + \\ \displaystyle\int_{\Omega^{b,l}} \left(D_{x_1} w^l, \frac{1}{h_n} D_{x_2} w^l, \frac{1}{h_n} D_{x_3} w^l\right) m^{b,l} dx + \\ \displaystyle\int_{\Omega^{b,r}} \left(D_{x_1} w^{b,r}, D_{x_2} w^{b,r}, D_{x_3} w^{b,r}\right) m^{b,r} dx, \quad \forall (w^{a,r}, w^l, w^{b,r}) \in \mathcal{V}_n \end{cases} \tag{2.4.49}$$

admits a unique solution. This equation rescales equation (2.3.2) by rescaling (2.4.47),

For every $n \in \mathbb{N}$, let $(v_n^{a,r}, v_n^l, v_n^{b,r})$ be the unique solution of (2.4.49) corresponding to $(m_n^a, m_n^{b,l}, m_n^{b,r})$. Arguing as in the first part of this proof, for a symmetric argument, one can easily prove that

$$D_{x_1} v_n^l \rightharpoonup 0, \quad \frac{1}{h_n} D_{x_2} v_n^l \rightharpoonup \xi_2^l, \quad \frac{1}{h_n} D_{x_3} v_n^l \rightharpoonup \xi_3^l \text{ weakly in } L^2(\mathbb{R}^3_l), \tag{2.4.50}$$

as n diverges, where

$$(\xi_2^l, \xi_3^l)(x_1, x_2, x_3) = \begin{cases} (0,0), \text{ a.e. in }]1, +\infty[\times \mathbb{R}^2, \\ \\ \mu_2^b(x_1) Dp(x_2, x_3) + \mu_3^b(x_1) Dq(x_2, x_3), \text{ a.e. in }]0, 1[\times \mathbb{R}^2, \end{cases}$$

with p (resp. q) the unique solution of (2.2.2) (resp. (2.2.3)). Furthermore, arguing as before by

$iii)$ of Lemma 2.2.1 with $k = (\mu_2^{b,l}, \mu_3^{b,l})$ one has that

$$\lim_n \int_{\Omega^{b,l}} \left(D_{x_1} v_n^l, \frac{1}{h_n} D_{x_2} v_n^l, \frac{1}{h_n} D_{x_3} v_n^l, \right) m_n^b dx = \alpha \left(\int_0^1 |\mu_2^b|^2 dx_1 + \int_0^1 |\mu_3^b|^2 dx_1 \right), \tag{2.4.51}$$

where α is defined by (2.2.5).

Now, to conclude it is enough to note that

$$\mathcal{T}_n^{-1}(T_n(x)) = x, \quad \forall x \in \mathbb{R}^3 \setminus \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \geq 0,\ x_3 \geq 0\}, \quad \forall n \in \mathbb{N},$$

$$v_n \left(\mathcal{T}_n^{-1}(T_n(x)) \right) = u_n(x), \quad \forall x \in \mathbb{R}^3, \quad \forall n \in \mathbb{N},$$

where $u_n = (u_n^a, u_n^{b,l}, u_n^{b,r})$ and $v_n = (v_{m,n}^{a,r}, v_{m,n}^l, v_{m,n}^{b,r})$. Consequently, it results that

$$v_n(x) = u_n(x), \quad \forall x \in \mathbb{R}^3 \setminus \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \geq 0,\ x_3 \geq 0\}, \quad \forall n \in \mathbb{N}. \tag{2.4.52}$$

Then, for every $t \in [0, T]$ combining (2.4.50) and (2.4.51) with (2.4.52), one obtains the fifth and the sixth limit in (2.4.33) and consequently (2.4.35) hold. Moreover,

$$\lim_n \int_{\Omega^{b,l}} \left(D_{x_1} u_n^l, \frac{1}{h_n} D_{x_2} u_n^l, \frac{1}{h_n} D_{x_3} u_n^l \right) m_n^b dx = \alpha \left(\int_0^1 |\mu_2^b|^2 dx_1 + \int_0^1 |\mu_3^b|^2 dx_1 \right). \tag{2.4.53}$$

So, for every $t \in [0, T]$ combining (2.4.45) with (2.4.53), also limit (2.4.36) holds true. □

2.4.3 Proof of Theorem 2.4.1

In this subsection, our aim is to study the asymptotic behavior, as n diverges, of Problem (2.4.6). If μ is the limit given in (2.4.20), we want to identify μ as solution of Problem (2.3.9).

Let us pose $S = \{0\} \times \{0\} \times [0, 1] \cup [0, 1] \times \{0\} \times \{0\}$. We denote the space of the Lipschitz continuous functions on S, taking values in $\mathbb{R}^3$, by $\mathcal{L}$. In the following with slight abuse of notation, we will continue to denote with $\mathcal{L}$ the space of functions $\psi = \left(\psi^a, \psi^{b,l}\right)$ on $\Omega^a \cup \Omega^{b,l}$ such that $\psi = \left(\psi^a, \psi^{b,l}\right)$ restricted to S is in $\mathcal{L}$, ψ is constant in (x_1, x_2) in Ω^a and is constant in (x_2, x_3) in $\Omega^{b,l}$.

Remark 2.4.1. *Let $\mathcal{W}$ be the space defined in (2.3.7). Then, $\mathcal{L}$ is dense in $\mathcal{W}$ (see [51] and [24,*

Proposition 2.2]).

Now, let us build a suitable couple of test functions. To this aim, for $\left(\psi^a, \psi^{b,l}\right) \in \mathcal{L}$, let us set

$$v = \begin{cases} \psi^a, \text{ in } \Omega^a, \\ \\ \psi^{b,l}, \text{ in } \Omega^{b,l}, \\ \\ \psi^a(0) = \psi^{b,l}(0), \text{ in } \Omega^{b,r}. \end{cases}$$

Obviously, $v \in \mathcal{W}_n$, for every $n \in N$.

Now, let us choose, v as test function in (2.4.6). So, we want to pass to the limit as n diverges in (2.4.6) term by term.
By $(2.4.20)vi)$, $|m_n^a| = 1$ and $(2.4.20)iv)$ we have

$$\int_0^T \int_{\Omega^a} \left(\frac{\partial m_n^a}{\partial t} + m_n^a \wedge \frac{\partial m_n^a}{\partial t} \right) \chi v dx dt \to \int_0^T \int_{\Omega^a} \left(\frac{\partial \mu^a}{\partial t} + \mu^a \wedge \frac{\partial \mu^a}{\partial t} \right) \chi \psi^a dx dt,$$

$$\forall \chi \in D(0,T), \ \ \forall \left(\psi^a, \psi^{b,l}\right) \in \mathcal{L}.$$

By $(2.4.20)vii)$, $|m_n^{b,l}| = 1$ and $(2.4.20)v)$ we get

$$\int_0^T \int_{\Omega^{b,l}} \left(\frac{\partial m_n^{b,l}}{\partial t} + m_n^{b,l} \wedge \frac{\partial m_n^{b,l}}{\partial t} \right) \chi v dx dt \to \int_0^T \int_{\Omega^{b,l}} \left(\frac{\partial \mu^{b,l}}{\partial t} + \mu^{b,l} \wedge \frac{\partial \mu^{b,l}}{\partial t} \right) \chi \psi^{b,l} dx dt,$$

$$\forall \chi \in D(0,T), \ \ \forall \left(\psi^a, \psi^{b,l}\right) \in \mathcal{L}.$$

By $(2.4.20)viii)$ and $|m_n^{b,r}| = 1$ we get

$$h_n \int_0^T \int_{\Omega^{b,r}} \left(\frac{\partial m_n^{b,r}}{\partial t} + m_n^{b,r} \wedge \frac{\partial m_n^{b,r}}{\partial t} \right) \chi v dx dt \to 0, \quad \forall \chi \in D(0,T), \ \ \forall \left(\psi^a, \psi^{b,l}\right) \in \mathcal{L}.$$

By $(2.4.20)iv)$ and $|m_n^a| = 1$, remembering that $|D_{x_i} v| = |D_{x_i} \psi^a| = 0$ on Ω^a for $i = 1, 2$, we

obtain

$$2\int_0^T \int_{\Omega^a} \left[\sum_{i=1}^{2} m_n^a \wedge \frac{1}{h_n} D_{x_i} m_n^a \frac{1}{h_n} \left(D_{x_i} v\right) \chi + m_n^a \wedge D_{x_3} m_n^a \left(D_{x_3} v\right) \chi\right] dxdt \to$$
$$2\int_0^T \int_{\Omega^a} \mu^a \wedge \frac{\partial \mu^a}{\partial x_3} \frac{\partial \psi^a}{\partial x_3} \chi, \quad \forall \chi \in D(0,T), \ \ \forall \left(\psi^a, \psi^{b,l}\right) \in \mathcal{L}.$$

By $(2.4.20)v)$ and $|m_n^{b,l}| = 1$ remembering that $|D_{x_i} v| = \left|D_{x_i} \psi^{b,l}\right| = 0$ on $\Omega^{b,l}$ for $i = 2, 3$, one has

$$2\int_0^T \int_{\Omega^{b,l}} \left[m_n^{b,l} \wedge D_{x_1} m_n^{b,l} \left(D_{x_1} v\right) \chi + \sum_{i=2}^{3} m_n^{b,l} \wedge \frac{1}{h_n} D_{x_i} m_n^{b,l} \frac{1}{h_n} \left(D_{x_i} v\right) \chi\right] dxdt \to$$
$$2\int_0^T \int_{\Omega^{b,l}} \mu^{b,l} \wedge \frac{\partial \mu^{b,l}}{\partial x_1} \frac{\partial \psi^{b,l}}{\partial x_1} \chi, \quad \forall \chi \in D(0,T), \ \ \forall \left(\psi^a, \psi^{b,l}\right) \in \mathcal{L}.$$

By $(2.4.20)iii)$ one has

$$2h_n \int_0^T \int_{\Omega^{b,r}} \left[\sum_{i=1}^{3} m_n^{b,r} \wedge \frac{1}{h_n} D_{x_i} m_n^{b,r} \frac{1}{h_n} \left(D_{x_i} v\right) \chi\right] dxdt = 0, \quad \forall \chi \in D(0,T), \ \ \forall \left(\psi^a, \psi^{b,l}\right) \in \mathcal{L}.$$

By $(2.4.20)iv)$, $|m_n^a| = 1$, first three convergences in (2.4.33) and (2.4.34), gives

$$\begin{aligned} &2\int_0^T \int_{\Omega^a} m_n^a \wedge \left(\frac{1}{h_n} D_{x_1} u_{m_n}^a, \frac{1}{h_n} D_{x_2} u_{m_n}^a, D_{x_3} u_{m_n}^a\right) \chi v dxdt \to \\ &2\int_0^T \int_{\Omega^a} \mu^a \wedge \left(\mu_1^a DP + \mu_2^a DQ\right) \chi \psi^a dxdt, \quad \forall \chi \in D(0,T), \ \ \forall \left(\psi^a, \psi^{b,l}\right) \in \mathcal{L}, \end{aligned} \tag{2.4.54}$$

with p (resp. q) the unique solution of (2.2.2) (resp.(2.2.3)). By taking into account $(2.4.20)v)$,

$|m_n^{b,l}| = 1$, fourth , fifth and sixth convergences in (2.4.33) and (2.4.35), we get

$$\begin{aligned} &2\int_0^T \int_{\Omega^{b,l}} m_n^{b,l} \wedge \left(D_{x_1} u_{m_n}^{b,l}, \frac{1}{h_n} D_{x_2} u_{m_n}^{b,l}, \frac{1}{h_n} D_{x_3} u_{m_n}^{b,l} \right) \chi v dx dt \to \\ &2\int_0^T \int_{\Omega^{b,l}} \mu^{b,l} \wedge \left(\mu_2^{b,l} DP' + \mu_3^{b,l} DQ' \right) \chi \psi^{b,l} dx dt, \quad \forall \chi \in D(0,T), \ \forall \left(\psi^a, \psi^{b,l} \right) \in \mathcal{L}, \end{aligned} \tag{2.4.55}$$

with p (resp. q) the unique solution of (2.2.2) (resp.(2.2.3)).
By taking into account (2.4.20)iii), $|m_n^{b,r}| = 1$ and last convergence in (2.4.33), we get

$$2h_n \int_0^T \int_{\Omega^{b,r}} m_n^{b,r} \wedge \left(\tfrac{1}{h_n} D_{x_1} u_{m_n}^{b,r}, \tfrac{1}{h_n} D_{x_2} u_{m_n}^{b,l}, \tfrac{1}{h_n} D_{x_3} u_{m_n}^{b,r} \right) \chi v dx dt \to 0, \quad \forall \chi \in D(0,T), \ \forall \left(\psi^a, \psi^{b,l} \right) \in \mathcal{L}.$$

Let us observe that $(\psi^a, \psi^{b,l})$ can be any arbitrarily element of $\mathcal{L}$. Being $\mathcal{L}$ dense in $\mathcal{W}$, we obtain that the above convergences hold true for every $\psi = (\psi^a, \psi^{b,l}) \in \mathcal{W}$. By using Lemma 2.2.2, we get (2.3.9).

2.5 A single wire

Let

$$\Omega_n =]-h_n, 0[^2 \times [0,1[, \quad \Omega =]-1, 0[^2 \times [0,1[,$$

let us introduce the following space

$$\mathcal{W} = \left\{ (\mu \in H^1(\Omega, \mathbb{R}^3) \ : \ \mu \text{ is independent of } (x_1, x_2) \right\} \simeq \left\{ (\mu \in H^1(]0,1[, \mathbb{R}^3) \right\}. \tag{2.5.1}$$

Moreover, we can pose

$$\mathcal{M} = \mathcal{W} \cap \left\{ \ H^1(\Omega, S^2) \ \right\}, \tag{2.5.2}$$

which explicitely takes into account the condition $|\mu| = 1$.

Theorem 2.5.1. *Suppose that $M_{0_n} \in H^1(\Omega_n, S^2)$ and (2.3.6) holds, for every $n \in \mathbb{N}$. Let M_n be a solution of Problem (2.3.4) . Then, there exist an increasing sequence of positive integer numbers $\{n_i\}_{i \in \mathbb{N}}$, still denoted by $\{n\}$, $\mu_0 \in \mathcal{M}$, $\mu \in L^\infty(0,T; \mathcal{M})$, depending on the selected subsequence*

such that:

$$\left\{ \fint_{\Omega_n} |M_{0_n}(x_1,x_2,x_3)-\mu_0(x_3)|^2\,dx \to 0, \right. \tag{2.5.3}$$

as n *diverges, for every* $t\in[0,T]$

$$\left\{ \fint_{\Omega_n} |M_n(t,x_1,x_2,x_3)-\mu(t,x_3)|^2\,dxdt \to 0, \right. \tag{2.5.4}$$

as n *diverges, where* μ *is a solution of the following problem*

$$\left\{ \begin{array}{l} \mu\in L^\infty(0,T;\mathcal{M})\cap C\left([0,T];L^2(\Omega,\mathbb{R}^3)\right),\ \dfrac{\partial\mu}{\partial t}\in L^2\left(0,T;L^2(\Omega,\mathbb{R}^3)\right), \\ \\ \forall\,\chi\in D(0,T)\ \textit{and}\,\psi\in\mathcal{W}, \\ \\ \displaystyle\int_0^T\int_\Omega \left(\frac{\partial\mu}{\partial t}+\mu\wedge\frac{\partial\mu}{\partial t}\right)\chi\psi dxdt = \\ \\ \displaystyle -2\int_0^T\int_\Omega \mu\wedge\frac{\partial\mu}{\partial x_3}\frac{\partial\psi}{\partial x_3}\chi dxdt-2\int_\Omega \mu\wedge\left((\mu,e_1)\Upsilon_1+(\mu,e_2)\Upsilon_2\right)\chi\psi dxdt, \\ \\ \mu(0,x)=\mu_0(x),\ \textit{a.e. } x\ \textit{in}\,\Omega, \end{array} \right. \tag{2.5.5}$$

where $\Upsilon_1=(\varepsilon_1,\varepsilon_2,0)$, $\Upsilon_2=(-\varepsilon_2,\varepsilon_1,0)$, *with* ε_1, ε_2 *constant depending on* $S=]-1,0[^2$.

To Problem (2.5.5), for a.e. $t\in[0,T]$, the following energy will be associated,

$$E(t)+\int_0^t \left\|\frac{\partial\mu}{\partial t}\right\|^2_{(L^2(]0,1[))^3} ds \tag{2.5.6}$$

where

$$E(t)=\int_0^1 \left|\frac{\partial\mu}{\partial x_3}\right|^2 dx_3+\frac{1}{2}\alpha\left(]-1,0[^2\right)\left[\int_0^1 |\mu_1|^2+|\mu_2|^2 dx_3\right], \tag{2.5.7}$$

$\alpha\left(]-1,0[^2\right)$ is defined by (2.2.5) with $S=\left(]-1,0[^2\right)$. Here, the term $E^{exc}(t)=\displaystyle\int_0^1 \left|\frac{\partial\mu^a}{\partial x_3}\right|^2 dx_3$,

can be considered an exchange energy and the term $E^{mag}(t) = \frac{1}{2}\alpha\left(]-1,0[^2\right)\left[\int_0^1 |\mu_1^a|^2 + |\mu_2^a|^2 dx_3\right]$,can be considered the equivalent of a magnetostatic energy.

Chapter 3

Fin junction of ferroelectric thin films

L. Carbone, K. Chacouche, A. Gaudiello. Fin junction ferroelectric thin films, *submitted*.

Abstract. In this paper, starting from a non-convex and nonlocal 3D variational model for the electric polarization in a ferroelectric material, and using an asymptotic process based on dimensional reduction, we analyze junction phenomena for two orthogonal joined ferroelectric thin films. We obtain three different 2D-variational models for joined thin films, depending on how the reduction happens. Indeed, a memory effect of the reduction process appears, and it depends on the competition of the relative thickness of the two films. The guide parameter is the limit of the ratio between these two small thickness.

Keywords: Electric polarization; Nonlocal problems; Thin film; Junctions.

2010 AMS subject classifications: 35Q61, 78A25

3.1 Introduction.

Ferroelectricity is a property of some materials to have a spontaneous electrical polarization that can be reversed by the application of an external electric field. Hysteresis phenomena appear, so the behavior of these materials is very similar to the one of ferromagnetic materials. Analogously, a Curie temperature T_C appears, too.

The idea of existence of materials which can have stable electric polarization is as old as the study of electrical phenomena. The quest was perhaps opened by S. Gray in the middle of eighteenth century. O. Heaveside is quoted as the creator of term "electret" for this kind of materials in 1885, borrowing the name from magnet, by analogy. T. Iguchi obtained the first electret at the beginning of Twenties of the last century by mixing and heating some organic natural materials. In

the Twenties again, J. Valasek discovered the presence of a hysteresis cycle (and so the first ferroelectric material) in Rochelle salt, a common salt but chemically and crystallographically complex enough. Immediately later, another ferroelectric salt was discovered (KH_2PO_4). Then the study of ferroelectric phenomena became easier and some theoretical model were proposed. In the Forties, the family of ferroelectric material enlarged, e.g. ferroelectric properties were demonstrated in barium titanate $(BaTiO_3)$ and lead titanate $(PbTiO_3)$. These simple materials opened the way for industrial use of materials with ferroelectric properties and also the modeling of these materials was more intensively studied. Properties of ferroelectric materials are now applied in a wide variety of contests. In particular, due to switching effect of hysteresis cycle, thin ferroelectric materials are used in electronic circuits with miniaturized and integrated forms in memory and storage devices as, for instance, radio frequency identification cards (RFID). Moreover, also the ferroelectric tunnel junction (FTJ) seems to offer great opportunities. We refer to [7], [16], and [19], about the history and applications of ferroelectric material. Recently, the mathematical modeling (in the static case) of thin structures of ferroelectric materials was studied starting from a non-convex and nonlocal 3-variational model for the electric polarization. Via an asymptotic process based on dimensional reduction, 2D-variational models for thin films were obtained in [37], and 1D-variational models for thin wires were obtained in [37].

Now, we summarize the essential features of the model that we consider (see also [7], [16], [19], [52], [54], [62], and [65]). We do not take into account any deformation of the ferroelectric material. The electric displacement $\mathbf{D}$ is given by $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$, where $\varepsilon_0 > 0$ is the vacuum permeability, $\mathbf{E}$ is the applied external field, and $\mathbf{P}$ is the spontaneous electric polarization in a ferroelectric body B. Assume that $\mathbf{E}$ is the gradient of a potential ψ, i.e.

$$\mathbf{E} = D\psi, \tag{3.1.1}$$

and that the electric field generated by $\mathbf{P}$ derives from a potential $\varphi_{\mathbf{P}}$ satisfying the electrostatic equation

$$\operatorname{div}(-\varepsilon_0 D\varphi_{\mathbf{P}} + \mathbf{P}) = 0. \tag{3.1.2}$$

We limit ourselves to the case where no strong electric field has been applied on B, but only a very weak electric field acts on it (e.g. it is the case of iron in the ferromagnetism, before the magnetization, by analogy). Then, we can assume that there are not Weis domains (i.e. regions with different polarization separated by well defined interfaces), but only transition regions. In this framework we can assume that the polarization does not generate an electric field outside B. Consequently, equation (3.1.2) holds true in B, and the following boundary conditions on ∂B

$$\mathbf{P} \cdot \nu = 0, \quad D\varphi_{\mathbf{P}} \cdot \nu = 0 \text{ on } \partial B \tag{3.1.3}$$

can be added, where ν denotes the unit outer normal on ∂B.

One assumes that $\mathbf{P}$ minimizes the energy functional

$$\int_B \left(\beta|\mathrm{rot}\mathbf{P}|^2 + |\mathrm{div}\mathbf{P}|^2 + \alpha(|\mathbf{P}|^2-1)^2\right) dx + \int_{\mathbb{R}^3} |D\psi + D\varphi_{\mathbf{P}}|^2 dx, \tag{3.1.4}$$

where α and β are two positive constants independent of the external field and of the temperature. Here, $\int_B (\beta|\mathrm{rot}\mathbf{P}|^2 + |\mathrm{div}\mathbf{P}|^2)\, dx$ reduces to the classical energy $\int_B |D\mathbf{P}|^2 dx$ when $\beta = 1$ (see (3.1.13)), so roughly speaking this term penalizes the spatial variation of $\mathbf{P}$. The term $\alpha \int_B (|\mathbf{P}|^2 - 1)^2 dx$ obliges $|\mathbf{P}|$ to be near to 1, and it can induce a phase transition of $\mathbf{P}$. So the body is driven to have regions of uniform polarization separated by thin transition layers. The term $\int_{\mathbb{R}^3} |D\psi + D\varphi_{\mathbf{P}}|^2 dx$ is the electrostatic energy. As this last term is concerned, we have

$$\int_{\mathbb{R}^3} |D\psi + D\varphi_{\mathbf{P}}|^2 dx = \int_{\mathbb{R}^3} |\mathbf{E}|^2 dx + 2\int_B D\psi \cdot D\varphi_{\mathbf{P}} dx + \int_B |D\varphi_{\mathbf{P}}|^2 dx, \tag{3.1.5}$$

thanks to (3.1.1). On the other side, using (3.1.2) and (3.1.3) give

$$\int_B D\psi \cdot D\varphi_{\mathbf{P}} dx = \frac{1}{\varepsilon_0} \int_B D\psi \cdot \mathbf{P} dx. \tag{3.1.6}$$

Consequently, inserting (3.1.5) and (3.1.6) in (3.1.4), and remarking that $\int_{\mathbb{R}^3} |\mathbf{E}|^2 dx$ is constant with respect to $\mathbf{P}$, the energy functional minimized by $\mathbf{P}$ becomes

$$\int_B \left(\beta|\mathrm{rot}\mathbf{P}|^2 + |\mathrm{div}\mathbf{P}|^2 + \alpha(|\mathbf{P}|^2-1)^2\right) dx + \int_B |D\varphi_{\mathbf{P}}|^2 dx + \frac{2}{\varepsilon_0} \int_B \mathbf{E} \cdot \mathbf{P} dx, \tag{3.1.7}$$

where $\int_B |D\varphi_{\mathbf{P}}|^2 dx$ is the electrostatic energy induced by $\mathbf{P}$, and the external energy $\int_B \mathbf{E} \cdot \mathbf{P} dx$ favors the polarization parallel (but in opposite verse) to $\mathbf{E}$.

In this paper, starting from this 3D-variational model and using an asymptotic process based on dimensional reduction, we analyze junction phenomena in a fin-like shaped structure composed of two orthogonal joined ferroelectric thin films (see Fig.3.1). Such a structure appears in some types of non-planar transistor used in the design of modern processors, the so called Fin Field Effect Transistor (FinFET). We obtain three different 2D-variational models for joined thin films, depending on how the reduction happens. Indeed, a memory effect of the reduction process appears, and it depends on the competition of the relative thickness of the two films. The guide parameter is the limit of the ratio between these two small thickness. The reduced models are justified by reasons of simplicity and economy, by a numerical point of view, too.

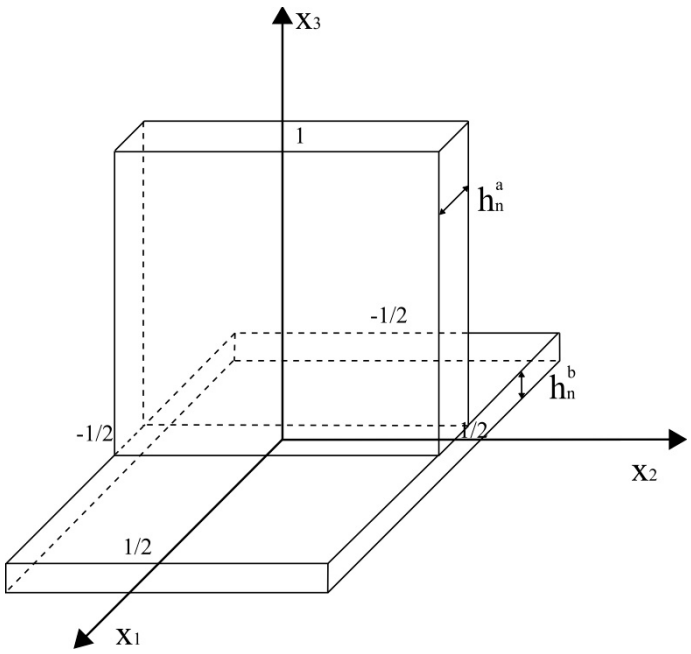

Figure 3.1: Ω_n

Let $\{h_n^a\}_{n\in\mathbb{N}}$ and $\left\{h_n^b\right\}_{n\in\mathbb{N}} \subset]0,1[$ be two sequences such that

$$\lim_n h_n^a = 0 = \lim_n h_n^b, \quad \lim_n \frac{h_n^b}{h_n^a} = \ell \in [0,+\infty]. \tag{3.1.8}$$

For every $n \in \mathbb{N}$, set (see Fig.3.1)

$$\Omega_n^a = \left]-\tfrac{h_n^a}{2}, \tfrac{h_n^a}{2}\right[\times \left]-\tfrac{1}{2},\tfrac{1}{2}\right[\times [0,1[, \quad \Omega_n^b = \left]-\tfrac{1}{2},\tfrac{1}{2}\right[^2 \times \left]-h_n^b, 0\right[, \quad \Omega_n = \Omega_n^a \cup \Omega_n^b. \tag{3.1.9}$$

The multidomain Ω_n models a ferroelectric device consisting of two orthogonal joined thin films Ω_n^a and Ω_n^b with small thicknesses h_n^a and h_n^b, respectively. According to (3.1.7), we consider the following non-convex and nonlocal energy associated with Ω_n

$$\mathcal{E}_n : \mathbf{P} \in \mathcal{P}_n \longrightarrow \int_{\Omega_n} \left(\beta|\mathrm{rot}\mathbf{P}|^2 + |\mathrm{div}\mathbf{P}|^2 + \alpha(|\mathbf{P}|^2-1)^2 + |D\varphi_{\mathbf{P}}|^2 + (\mathbf{F}_n \cdot \mathbf{P})\right) dx, \tag{3.1.10}$$

where

$$\mathcal{P}_n = \left\{\mathbf{P} \in \left(H^1(\Omega_n)\right)^3 : \mathbf{P}\cdot\nu = 0 \text{ on } \partial\Omega_n\right\}, \tag{3.1.11}$$

α and β are two positive constants, $\mathbf{F}_n \in (L^2(\Omega_n))^3$, ν denotes the unit outer normal on $\partial\Omega_n$,

and $\varphi_{\mathbf{P}} \in H^1(\Omega_n)$ is the unique solution, up to an additive constant, of

$$\begin{cases} \operatorname{div}(-\varepsilon_0 D\varphi_{\mathbf{P}} + \mathbf{P}) = 0 \text{ in } \Omega_n, \\ \\ (-\varepsilon_0 D\varphi_{\mathbf{P}} + \mathbf{P}) \cdot \nu = 0 \text{ on } \partial\Omega_n. \end{cases} \tag{3.1.12}$$

Notice that $\mathbf{F}_n$ is a normalization of the external field. So to obtain (3.1.7), choose $\mathbf{F}_n = \frac{2}{\varepsilon_0}\mathbf{E}$.

Using

$$\|D\mathbf{P}\|^2_{(L^2(\Omega_n))^9} = \|\mathrm{rot}\mathbf{P}\|^2_{(L^2(\Omega_n))^3} + \|\mathrm{div}\mathbf{P}\|^2_{L^2(\Omega_n)} \quad \forall \mathbf{P} \in \mathcal{P}_n, \quad \forall n \in \mathbb{N}, \tag{3.1.13}$$

(this formula is true due to the special geometry of the domain, for instance compare [18] and Lemma 2.1 in [37]) and the direct method of Calculus of Variations give that problem

$$\min\{\mathcal{E}_n(\mathbf{P}) : \mathbf{P} \in \mathcal{P}_n\} \tag{3.1.14}$$

admits a solution. Note that problem (3.1.14) is an optimal control problem.

We rescale $\mathbf{F}_n$ in $\Omega^a = \left]-\frac{1}{2},\frac{1}{2}\right[\times \left]-\frac{1}{2},\frac{1}{2}\right[\times]0,1[$ and $\Omega^b = \left]-\frac{1}{2},\frac{1}{2}\right[\times \left]-\frac{1}{2},\frac{1}{2}\right[\times]-1,0[$ (see (3.2.2)), and we impose suitable convergence assumptions on these rescaled fields (see (3.2.9)). The goal of this paper is to study the asymptotic behavior, as n diverges, of (3.1.14).

Now, we describe the three different limit regimes depending on ratio ℓ given in (3.1.8), according

to $0<\ell<+\infty$, $\ell=0$, or $\ell=+\infty$. Precisely, when $0<\ell<+\infty$, i.e. $h_n^b \simeq h_n^a$, we prove that

$$
\left\{
\begin{array}{l}
\lim\limits_n \min\left\{\dfrac{1}{|\Omega_n|}\mathcal{E}_n(\mathbf{P}) : \mathbf{P}\in \left(H^1\left(\Omega_n\right)\right)^3 : \mathbf{P}\cdot\nu=0 \text{ on } \partial\Omega_n\right\} = \\
\\
\min\left\{\dfrac{1}{1+\ell}\displaystyle\int_{]-\frac12,\frac12[\times]0,1[}\Big(\beta|\mathrm{rot} q^a|^2+|\mathrm{div} q^a|^2+\alpha(|q^a|^2-1)^2\right. \\
\\
\qquad\qquad \left.+|D\psi^a_{(q^a,q^b)}|^2+\displaystyle\int_{-\frac12}^{\frac12}(f_2^a,f_3^a)dx_1\cdot q^a\Big)dx_2dx_3\right. \\
\\
+\dfrac{\ell}{1+\ell}\displaystyle\int_{]-\frac12,\frac12[^2}\Big(\beta\left|\mathrm{rot} q^b\right|^2+\left|\mathrm{div} q^b\right|^2+\alpha\left(\left|q^b\right|^2-1\right)^2 \\
\\
\qquad\qquad +\left|D\psi^b_{(q^a,q^b)}\right|^2+\displaystyle\int_{-1}^{0}\left(f_1^b,f_2^b\right)dx_3\cdot q^b\Big)\,dx_1dx_2 : \\
\\
\left(q^a,q^b\right)=\left(\left(q_2^a,q_3^a\right),\left(q_1^b,q_2^b\right)\right)\in\left(H^1\left(\left]-\frac12,\frac12\right[\times]0,1[\right)\right)^2\times\left(H^1\left(\left]-\frac12,\frac12\right[^2\right)\right)^2, \\
\\
q^a\cdot\nu^a=0 \text{ on } \partial\left(\left]-\frac12,\frac12\right[\times]0,1[\right), \qquad q^b\cdot\nu^b=0 \text{ on } \partial\left(\left]-\frac12,\frac12\right[^2\right), \\
\\
q_1^b(0,\cdot)=0, \quad q_2^a(\cdot,0)=q_2^b\left(0,\cdot\right) \text{ in } \left]-\frac12,\frac12\right[\Big\},
\end{array}
\right. \tag{3.1.15}
$$

where ν^a and ν^b denote the unit outer normals on $\partial\left(\left]-\frac12,\frac12\right[\times]0,1[\right)$ and $\partial\left(\left]-\frac12,\frac12\right[^2\right)$, respectively, $f^a=(f_1^a,f_2^a,f_3^a)$ and $f^b=(f_1^b,f_2^b,f_3^b)$ denote the L^2-weak limits of the rescaled fields of $\mathbf{F}_n$ on Ω^a and Ω^b, respectively, q^a and $\psi^a_{(q^a,q^b)}$ depend on (x_2,x_3) only, q^b and $\psi^b_{(q^a,q^b)}$ depend on (x_1,x_2) only, and $\left(\psi^a_{(q^a,q^b)},\psi^b_{(q^a,q^b)}\right)\in H^1\left(\left]-\frac12,\frac12\right[\times]0,1[\right)\times H^1\left(\left]-\frac12,\frac12\right[^2\right)$ is the unique

weak solution, up to an additive constant, of

$$\begin{cases} \operatorname{div}(-\varepsilon_0 D\psi^a_{(q^a,q^b)} + q^a) = 0 \text{ in } \left]-\frac{1}{2},\frac{1}{2}\right[\times]0,1[\,, \\ \operatorname{div}(-\varepsilon_0 D\psi^b_{(q^a,q^b)} + q^b) = 0 \text{ in } \left]-\frac{1}{2},0\right[\times \left]-\frac{1}{2},\frac{1}{2}\right[\,, \\ \operatorname{div}(-\varepsilon_0 D\psi^b_{(q^a,q^b)} + q^b) = 0 \text{ in } \left]0,\frac{1}{2}\right[\times \left]-\frac{1}{2},\frac{1}{2}\right[\,, \\ (-\varepsilon_0 D\psi^a_{(q^a,q^b)} + q^a)\cdot\nu^a = 0 \text{ on } \partial\left(\left]-\frac{1}{2},\frac{1}{2}\right[\times]0,1[\right) \setminus \left(\{0\} \times \left]-\frac{1}{2},\frac{1}{2}\right[\right), \\ (-\varepsilon_0 D\psi^b_{(q^a,q^b)} + q^b)\cdot\nu^b = 0 \text{ on } \partial\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right), \\ \psi^a_{(q^a,q^b)}(\cdot,0) = \psi^b_{(q^a,q^b)}(0,\cdot) \text{ in } \left]-\frac{1}{2},\frac{1}{2}\right[\,, \\ \partial_{x_3}\psi^a_{(q^a,q^b)}(\cdot,0) = \ell[\partial_{x_1}\psi^b_{(q^a,q^b)}](0,\cdot) \text{ in } \left]-\frac{1}{2},\frac{1}{2}\right[\,, \end{cases} \tag{3.1.16}$$

with $[\partial_{x_1}\psi^b_{(q^a,q^b)}]$ denoting the jump of $\partial_{x_1}\psi^b_{(q^a,q^b)}$ on $\{0\} \times \left]-\frac{1}{2},\frac{1}{2}\right[$.

More precisely, in Theorem 3.3.2 we obtain a limit polarization $(0,\hat{p}^a_2,\hat{p}^a_3)$ in the vertical thin film and a limit polarization $\left(\hat{p}^b_1,\hat{p}^b_2,0\right)$ in the horizontal thin film, and the couple $\left((\hat{p}^a_2,\hat{p}^a_3),\left(\hat{p}^b_1,\hat{p}^b_2\right)\right)$ solves problem (3.1.15)-(3.1.16). Roughly speaking, $(\hat{p}^a_2,\hat{p}^a_3)$ is independent of x_1 and solves a 2D-problem in the vertical thin film "similar" to the original problem; while $\left(\hat{p}^b_1,\hat{p}^b_2\right)$ is independent of x_3 and solves a 2D-problem in the horizontal thin film "similar" to the original problem. Moreover, a memory effect of the original 3D-problem appears. Indeed, $(\hat{p}^a_2,\hat{p}^a_3)$ and $\left(\hat{p}^b_1,\hat{p}^b_2\right)$ are subjected to a transmission condition on the intersection of the thin films (see the two last lines in (3.1.15) and in (3.1.16)). We also study the limit behavior of the rescaled potential of electric field associated with the polarization.

In the case $\ell = 0$, i.e. $h_n^b \ll h_n^a$, we prove that

$$\begin{cases} \lim\limits_n \min\left\{ \dfrac{1}{|\Omega_n|}\mathcal{E}_n(\mathbf{P}) : \mathbf{P} \in \left(H^1(\Omega_n)\right)^3 : \mathbf{P}\cdot\nu = 0 \text{ on } \partial\Omega_n \right\} = \\ \min\left\{ \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,1[} \Big(\beta|\mathrm{rot} q^a|^2 + |\mathrm{div} q^a|^2 + \alpha(|q^a|^2-1)^2 \right. \\ \left. \qquad + |D\psi^a_{q^a}|^2 + \displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} (f_2^a, f_3^a) dx_1 \cdot q^a \Big) dx_2 dx_3 : \right. \\ \left. q^a = (q_2^a, q_3^a) \in \left(H^1\left(\left]-\tfrac{1}{2},\tfrac{1}{2}\right[\times]0,1[\right)\right)^2, \quad q^a\cdot\nu^a = 0 \text{ on } \partial\left(\left]-\tfrac{1}{2},\tfrac{1}{2}\right[\times]0,1[\right) \right\}, \end{cases} \tag{3.1.17}$$

where $\psi^a_{q^a}$ is the unique weak solution, up to an additive constant, of

$$\begin{cases} \mathrm{div}(-\varepsilon_0 D\psi^a_{q^a} + q^a) = 0 \text{ in } \left]-\tfrac{1}{2},\tfrac{1}{2}\right[\times]0,1[, \\ (-\varepsilon_0 D\psi^a_{q^a} + q^a)\cdot\nu^a = 0 \text{ on } \partial\left(\left]-\tfrac{1}{2},\tfrac{1}{2}\right[\times]0,1[\right). \end{cases} \tag{3.1.18}$$

Roughly speaking, in the limit process the energy, renormalized by $\frac{1}{|\Omega_n|}$, reduces to the energy of a 2D-problem in the vertical thin film. Also in this case a memory effect of the original 3D-problem appears. Indeed, a solution p^a of problem (3.1.17) satisfies the boundary condition $p_3^a = 0$ and $\partial_{x_3}\psi^a_{p^a} = 0$ on $\left]-\frac{1}{2},\frac{1}{2}\right[\times\{0\}$ (see the last lines in (3.1.17) and (3.1.18), respectively). See Theorem 3.3.4 about the limit behavior of the rescaled polarization and of the rescaled potential of electric field associated with it.

In the case $\ell = +\infty$, i.e. $h_n^a \ll h_n^b$, we are able to study the asymptotic behavior of problem (3.1.14) under the additional assumption $h_n^b \ll \sqrt{h_n^a}$ (see Remark 3.6.5). In this case we prove

that

$$\begin{cases} \lim\limits_{n} \min\left\{ \dfrac{1}{|\Omega_n|}\mathcal{E}_n(\mathbf{P}) : \mathbf{P} \in \left(H^1\left(\Omega_n\right)\right)^3 : \mathbf{P}\cdot\nu = 0 \text{ on } \partial\Omega_n \right\} = \\ \min\left\{ \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[^2} \left(\beta \left|\mathrm{rot} q^b\right|^2 + \left|\mathrm{div} q^b\right|^2 + \alpha\left(\left|q^b\right|^2 - 1\right)^2 \right.\right. \\ \qquad \left.\left. + \left|D\psi^b_{q^b}\right|^2 + \displaystyle\int_{-1}^{0} \left(f_1^b, f_2^b\right) dx_3 \cdot q^b \right) dx_1 dx_2 : \right. \\ q^b = \left(q_1^b, q_2^b\right) \in \left(H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right)\right)^2, \qquad q^b\cdot\nu^b = 0 \text{ on } \partial\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right), \\ \left. q_1^b(0,\cdot) = 0 \text{ in } \left]-\frac{1}{2},\frac{1}{2}\right[\right\}, \end{cases} \tag{3.1.19}$$

where $\psi^b_{q^b}$ is the unique weak solution, up to an additive constant, of

$$\begin{cases} \mathrm{div}(-\varepsilon_0 D\psi^b_{q^b} + q^b) = 0 \text{ in } \left(\left]-\frac{1}{2},\frac{1}{2}\right[\right)^2, \\ (-\varepsilon_0 D\psi^b_{q^b} + q^b)\cdot\nu^b = 0 \text{ on } \partial\left(\left]-\frac{1}{2},\frac{1}{2}\right[\right)^2. \end{cases} \tag{3.1.20}$$

Roughly speaking, in the limit process the energy, renormalized by $\frac{1}{|\Omega_n|}$, reduces to the energy of a 2D-problem in the horizontal thin film. Also in this case a memory effect of the original 3D-problem appears. Indeed, a solution p^b of problem (3.1.19) satisfies the boundary condition $p_1^b = 0$ and $[\partial_{x_1}\psi^b_{q^b}] = 0$ on $\{0\}\times\left]-\frac{1}{2},\frac{1}{2}\right[$ (see the last lines in (3.1.19) and the first line in (3.1.20), respectively). See Theorem 3.3.7 about the limit behavior of the rescaled polarization and of the rescaled potential of electric field associated with it.

In what follows, we assume $\varepsilon_0 = 1$.

In all the three cases, we reformulate the problem on a fixed domain through appropriate rescalings of the kind proposed in [17] and impose suitable convergence assumptions on the rescaled

fields of $\mathbf{F}_n$. Then, we use accurate *a priori* estimates which provide a first characterization of the $(H^1(\Omega^a))^3 \times (H^1(\Omega^b))^3$-weak limit of the rescaled polarization. For obtaining the transmission conditions (the memory effect) on the limit polarization in the case $0 < \ell < +\infty$, we refer to [34]; while the proofs of the memory effects in the case $\ell = 0$ and $\ell = +\infty$ are completely new and more intricate, mainly when $\ell = +\infty$. The properties of the limit polarization drive us in the study of the asymptotic behavior of the electrostatic energy associated with the polarization. First, we obtain *a priori* estimates of the electrostatic potential which provide a limit electrostatic potential. Then, using the main ideas of the Γ-convergence method introduced in [21] (see [8], [10], and [20], too), we identify the limit electrostatic equation and the problem satisfied by the limit polarization. In this last two steps, the main difficulties arise in the construction of the "recovery sequence". For instance, as the polarization is concerned (the same holds for the electrostatic potential), the main difficulties depend on the fact that the test functions (p^a, p^b) (of the rescaled problem of (3.1.14)) are not H^1 of all the rescaled domain, but $p^a \in (H^1(\Omega^a))^3$, $p^b \in \left(H^1(\Omega^b)\right)^3$, and they satisfy the junction condition $p^a(x_1, x_2, 0) = p^b(h_n^a x_1, x_2, 0)$ on the interface separating Ω^a and Ω^b, and also some boundary conditions. We build the recovery sequence for enough regular test functions, so some density results are needed to be proved.

The 3D model of ferromagnetic micro devices is close to our model. For the limit behavior of a ferromagnetic thin structures we refer to [34] for joined ferromagnetic thin films in the stationary case, and [23] and [24] in the quasi-stationary case. For the study of ferromagnetic thin films, we refer to [2], [3], [4], [6], [13], [25], [35], [41], [42], [48], and [49]. For problems of junctions, we refer to [51].

3.2 The rescaled problem

As it is usual (see [17]), (3.1.14) is reformulated on a fixed domain through the maps

$$\begin{cases} x = (x_1, x_2, x_3) \in \Omega^a = \left]-\frac{1}{2}, \frac{1}{2}\right[\times \left]-\frac{1}{2}, \frac{1}{2}\right[\times]0, 1[\longrightarrow (h_n^a x_1, x_2, x_3) \in \text{Int}(\Omega_n^a), \\ \\ x = (x_1, x_2, x_3) \in \Omega^b = \left]-\frac{1}{2}, \frac{1}{2}\right[\times \left]-\frac{1}{2}, \frac{1}{2}\right[\times]-1, 0[\longrightarrow (x_1, x_2, h_n^b x_3) \in \Omega_n^b, \end{cases} \tag{3.2.1}$$

where $\text{Int}(\Omega_n^a)$ denotes the interior of Ω_n^a. Precisely, for every $n \in \mathbb{N}$ set

$$D_n^a : p^a \in \left(H^1(\Omega^a)\right)^k \longrightarrow \left(\frac{1}{h_n^a}\frac{\partial p^a}{\partial x_1}, \frac{\partial p^a}{\partial x_2}, \frac{\partial p^a}{\partial x_3}\right) \in (L^2(\Omega^a))^{3k}, \quad k \in \{1, 3\},$$

$$D_n^b : p^b \in \left(H^1(\Omega^b)\right)^k \longrightarrow \left(\frac{\partial p^b}{\partial x_1}, \frac{\partial p^b}{\partial x_2}, \frac{1}{h_n^b}\frac{\partial p^b}{\partial x_3}\right) \in \left(L^2\left(\Omega^b\right)\right)^{3k}, \quad k \in \{1,3\},$$

$$\text{div}_n^a : p^a = (p_1^a, p_2^a, p_3^a) \in \left(H^1(\Omega^a)\right)^3 \longrightarrow \frac{1}{h_n^a}\frac{\partial p_1^a}{\partial x_1} + \frac{\partial p_2^a}{\partial x_2} + \frac{\partial p_3^a}{\partial x_3} \in L^2(\Omega^a),$$

$$\text{div}_n^b : p^b = \left(p_1^b, p_2^b, p_3^b\right) \in \left(H^1(\Omega^b)\right)^3 \longrightarrow \frac{\partial p_1^b}{\partial x_1} + \frac{\partial p_2^b}{\partial x_2} + \frac{1}{h_n^b}\frac{\partial p_3^b}{\partial x_3} \in L^2(\Omega^b),$$

$$\text{rot}_n^a : p^a = (p_1^a, p_2^a, p_3^a) \in \left(H^1\left(\Omega^a\right)\right)^3 \longrightarrow \left(\frac{\partial p_3^a}{\partial x_2} - \frac{\partial p_2^a}{\partial x_3}, \frac{\partial p_1^a}{\partial x_3} - \frac{1}{h_n^a}\frac{\partial p_3^a}{\partial x_1}, \frac{1}{h_n^a}\frac{\partial p_2^a}{\partial x_1} - \frac{\partial p_1^a}{\partial x_2}\right) \in (L^2(\Omega^a))^3,$$

$$\text{rot}_n^b : p^b = \left(p_1^b, p_2^b, p_3^b\right) \in \left(H^1\left(\Omega^b\right)\right)^3 \longrightarrow \left(\frac{\partial p_3^b}{\partial x_2} - \frac{1}{h_n^b}\frac{\partial p_2^b}{\partial x_3}, \frac{1}{h_n^b}\frac{\partial p_1^b}{\partial x_3} - \frac{\partial p_3^b}{\partial x_1}, \frac{\partial p_2^b}{\partial x_1} - \frac{\partial p_1^b}{\partial x_2}\right) \in (L^2(\Omega^b))^3,$$

$$\begin{cases} f_n^a : x = (x_1, x_2, x_3) \in \Omega^a \longrightarrow \mathbf{F}_n(h_n^a x_1, x_2, x_3), \\ \\ f_n^b : x = (x_1, x_2, x_3) \in \Omega^b \longrightarrow \mathbf{F}_n(x_1, x_2, h_n^b x_3), \end{cases} \tag{3.2.2}$$

$$\begin{cases} P_n = \Big\{ \ (p^a, p^b) \in \left(H^1\left(\Omega^a\right)\right)^3 \times \left(H^1(\Omega^b)\right)^3 : \\ \\ p^a \cdot \nu^a = 0 \text{ on } \partial\Omega^a \backslash \left(\left]-\frac{1}{2}, \frac{1}{2}\right[^2 \times \{0\}\right), \\ \\ p^b \cdot \nu^b = 0 \text{ on } \partial\Omega^b \backslash \left(\left]-\frac{1}{2}, \frac{1}{2}\right[^2 \times \{0\}\right), \\ \\ p_3^b = 0 \text{ on } \left(\left]-\frac{1}{2}, \frac{1}{2}\right[^2 \backslash \left(\left]-\frac{h_n^a}{2}, \frac{h_n^a}{2}\right[\times \left]-\frac{1}{2}, \frac{1}{2}\right[\right)\right) \times \{0\}, \\ \\ p^a(x_1, x_2, 0) = p^b(h_n^a x_1, x_2, 0) \text{ in } \left]-\frac{1}{2}, \frac{1}{2}\right[^2 \Big\}, \end{cases} \tag{3.2.3}$$

where ν^a and ν^b denote the unit outer normals on $\partial\Omega^a$ and $\partial\Omega^b$, respectively, and

$$U_n = \left\{ (\phi^a, \phi^b) \in H^1(\Omega^a) \times H^1(\Omega^b) \ : \ \phi^a(x_1, x_2, 0) = \phi^b(h_n^a x_1, x_2, 0) \text{ in } \left]-\tfrac{1}{2}, \tfrac{1}{2}\right[^2 \right\}. \tag{3.2.4}$$

Then, $\mathcal{E}_n$ defined in (3.1.10) is rescaled by

$$\left\{ \begin{array}{l} E_n : \left(p^a, p^b\right) \in P_n \longrightarrow \\ \\ h_n^a \displaystyle\int_{\Omega^a} \left(\beta |\mathrm{rot}_n^a p^a|^2 + |\mathrm{div}_n^a p^a|^2 + \alpha(|p^a|^2 - 1)^2 + |D_n^a \phi^a_{(p^a,p^b)}|^2 + (f_n^a \cdot p^a) \right) dx \\ \\ + h_n^b \displaystyle\int_{\Omega^b} \left(\beta |\mathrm{rot}_n^b p^b|^2 + |\mathrm{div}_n^b p^b|^2 + \alpha(|p^b|^2 - 1)^2 + |D_n^b \phi^b_{(p^a,p^b)}|^2 + (f_n^b \cdot p^b) \right) dx, \end{array} \right. \tag{3.2.5}$$

where $\left(\phi^a_{(p^a,p^b)}, \phi^b_{(p^a,p^b)}\right)$ is the unique solution of

$$\left\{ \begin{array}{l} \left(\phi^a_{(p^a,p^b)}, \phi^b_{(p^a,p^b)}\right) \in U_n, \quad \displaystyle\int_{\Omega^a} \phi^a_{(p^a,p^b)} dx = 0, \\ \\ h_n^a \displaystyle\int_{\Omega^a} ((-D_n^a \phi^a_{(p^a,p^b)} + p^a) \cdot D_n^a \phi^a) dx \\ \\ + h_n^b \displaystyle\int_{\Omega^b} \left(\left(-D_n^b \phi^b_{(p^a,p^b)} + p^b \right) \cdot D_n^b \phi^b \right) dx = 0 \quad \forall\, (\phi^a, \phi^b) \in U_n, \end{array} \right. \tag{3.2.6}$$

which rescales a weak formulation of (3.1.12), i.e.

$$\varphi_{\mathbf{P}} \in H^1(\Omega_n), \quad \int_{\Omega_n^a} \varphi_{\mathbf{P}} \, dx = 0, \quad \int_{\Omega_n} ((-\varepsilon_0 D\varphi_{\mathbf{P}} + \mathbf{P}) \cdot D\varphi) \ dx = 0 \quad \forall \varphi \in H^1(\Omega_n). \tag{3.2.7}$$

The Lax-Milgram Theorem provides that (3.2.7) admits solution and it is unique.

Note that if $\mathbf{P}_n$ solves (3.1.14), then (p_n^a, p_n^b) defined by

$$p_n^a(x_1, x_2, x_3) = \mathbf{P}_n(h_n^a x_1, x_2, x_3) \text{ in } \Omega^a, \quad p_n^b(x_1, x_2, x_3) = \mathbf{P}_n(x_1, x_2, h_n^b x_3) \text{ in } \Omega^b,$$

solves

$$\min\left\{E_n((p^a,p^b)) : (p^a,p^b)\in P_n\right\}. \tag{3.2.8}$$

Conversely, if (p_n^a, p_n^b) solves (3.2.8), then $\mathbf{P}_n$ defined by

$$\mathbf{P}_n(x_1,x_2,x_3) = p_n^a\left(\frac{x_1}{h_n^a},x_2,x_3\right) \text{ in } \Omega_n^a, \quad \mathbf{P}_n(x_1,x_2,x_3) = p_n^b\left(x_1,x_2,\frac{x_3}{h_n^b}\right) \text{ in } \Omega_n^b,$$

solves (3.1.14). Therefore, the goal of this paper becomes to study the asymptotic behavior, as n diverges, of (3.2.8). To this aim, we assume

$$\begin{cases} f_n^a \rightharpoonup f^a = (f_1^a, f_2^a, f_3^a) \quad \text{weakly in } (L^2(\Omega^a))^3, \\ \\ f_n^b \rightharpoonup f^b = \left(f_1^b, f_2^b, f_3^b\right) \quad \text{weakly in } \left(L^2\left(\Omega^b\right)\right)^3. \end{cases} \tag{3.2.9}$$

We conclude this section noting that rescalings in (3.2.1) transform (3.1.13) into

$$\begin{cases} h_n^a \left\|D_n^a p^a\right\|^2_{(L^2(\Omega^a))^9} + h_n^b \left\|D_n^b p^b\right\|^2_{(L^2(\Omega^b))^9} = h_n^a \|\mathrm{rot}_n^a p^a\|^2_{(L^2(\Omega^a))^3} + h_n^a \|\mathrm{div}_n^a p^a\|^2_{L^2(\Omega^a)} \\ \\ + h_n^b \left\|\mathrm{rot}_n^b p^b\right\|^2_{(L^2(\Omega^b))^3} + h_n^b \left\|\mathrm{div}_n^b p^b\right\|^2_{L^2(\Omega^b)} \quad \forall (p^a,p^b)\in P_n, \quad \forall n\in\mathbb{N}. \end{cases} \tag{3.2.10}$$

3.3 The main results

3.3.1 The case $\ell \in]0,+\infty[$

This subsection is devoted to state Theorem 3.3.2 describing the limit behavior of (3.2.8) when (3.1.8) is assumed with $\ell \in]0,+\infty[$. Theorem 3.3.2 will be proved in Section 3.4.

Set

$$\left\{\begin{array}{l} P=\Big\{\quad \left(q^a,q^b\right)=\left(\left(q_2^a,q_3^a\right),\left(q_1^b,q_2^b\right)\right)\in \\ \\ \qquad \left(H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\times\left]0,1\right[\right)\right)^2\times\left(H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right)\right)^2: \\ \\ \qquad q^a\cdot\nu^a=0 \text{ on } \partial\left(\left]-\frac{1}{2},\frac{1}{2}\right[\times\left]0,1\right[\right), \qquad q^b\cdot\nu^b=0 \text{ on } \partial\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right), \\ \\ \qquad q_1^b(0,\cdot)=0, \quad q_2^a(\cdot,0)=q_2^b\left(0,\cdot\right) \text{ in } \left]-\frac{1}{2},\frac{1}{2}\right[\Big\}, \end{array}\right. \tag{3.3.1}$$

where ν^a and ν^b denote the unit outer normal on $\partial\left(\left]-\frac{1}{2},\frac{1}{2}\right[\times\left]0,1\right[\right)$ and $\partial\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right)$, respectively,

$$\left\{\begin{array}{l} U=\Big\{\quad \left(\psi^a,\psi^b\right)\in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\times\left]0,1\right[\right)\times H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right): \\ \qquad\qquad \psi^a(\cdot,0)=\psi^b(0,\cdot) \text{ in } \left]-\frac{1}{2},\frac{1}{2}\right[\Big\}, \end{array}\right. \tag{3.3.2}$$

and

$$\left\{\begin{array}{l} E:\left(q^a,q^b\right)\in P\longrightarrow \displaystyle\int_{\left]-\frac{1}{2},\frac{1}{2}\right[\times]0,1[}\Big(\beta|\mathrm{rot}q^a|^2+|\mathrm{div}q^a|^2+\alpha(|q^a|^2-1)^2 \\ \qquad\qquad +|D\psi^a_{(q^a,q^b)}|^2+\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_2^a,f_3^a)dx_1\cdot q^a\Big)dx_2dx_3 \\ \qquad +\ell\displaystyle\int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2}\Big(\beta\left|\mathrm{rot}q^b\right|^2+\left|\mathrm{div}q^b\right|^2+\alpha\left(\left|q^b\right|^2-1\right)^2 \\ \qquad\qquad +\left|D\psi^b_{(q^a,q^b)}\right|^2+\displaystyle\int_{-1}^{0}\left(f_1^b,f_2^b\right)dx_3\cdot q^b\Big)\,dx_1dx_2, \end{array}\right. \tag{3.3.3}$$

where (f_2^a, f_3^a) and (f_1^b, f_2^b) are defined in (3.2.9), and $(\psi^a_{(q^a,q^b)}, \psi^b_{(q^a,q^b)})$ is the unique solution of

$$\begin{cases} \left(\psi^a_{(q^a,q^b)}, \psi^b_{(q^a,q^b)}\right) \in U, \quad \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,1[} \psi^a_{(q^a,q^b)} dx_2 dx_3 = 0, \\ \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,1[} \left(\left(-D_{(x_2,x_3)}\psi^a_{(q^a,q^b)} + q^a\right) \cdot D_{(x_2,x_3)}\psi^a\right) dx_2 dx_3 \\ \displaystyle +\ell \int_{]-\frac{1}{2},\frac{1}{2}[^2} \left(\left(-D_{(x_1,x_2)}\psi^b_{(q^a,q^b)} + q^b\right) \cdot D_{(x_1,x_2)}\psi^b\right) dx_1 dx_2 = 0 \quad \forall \left(\psi^a, \psi^b\right) \in U. \end{cases} \tag{3.3.4}$$

Note that (3.3.4) admits a solution and it is unique since the set

$$\left\{ \left(\psi^a, \psi^b\right) \in U : \int_{]-\frac{1}{2},\frac{1}{2}[\times]0,1[} \psi^a \, dx_2 dx_3 = 0 \right\}$$

is a Hilbert space with the inner product

$$< \left(\psi^a, \psi^b\right), \left(\varphi^a, \varphi^b\right) >= \int_{]-\frac{1}{2},\frac{1}{2}[\times]0,1[} D\psi^a D\varphi^a dx_2 dx_3 + \ell \int_{]-\frac{1}{2},\frac{1}{2}[^2} D\psi^b D\varphi^b dx_1 dx_2.$$

Remark 3.3.1. *In what follows, any element of U (resp. P) is assumed to be extended to the element of $H^1(\Omega^a) \times H^1(\Omega^b)$ (resp. $\left(H^1(\Omega^a)\right)^2 \times \left(H^1(\Omega^b)\right)^2$) with the first (resp. first two) component(s) independent of x_1 and the second (resp. last two) component(s) independent of x_3.*

Theorem 3.3.2. *Assume (3.1.8) with $\ell \in]0, +\infty[$, and (3.2.9). For every $n \in \mathbb{N}$, let $\left(p_n^a, p_n^b\right)$ be a solution of (3.2.8), and $\left(\phi^a_{(p_n^a,p_n^b)}, \phi^b_{(p_n^a,p_n^b)}\right)$ be the unique solution of (3.2.6) with $\left(p^a, p^b\right) = \left(p_n^a, p_n^b\right)$. Moreover, let P and E be defined by (3.3.1) and (3.3.3)-(3.3.4), respectively. Then, there exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$ and (in possible dependence on the subsequence) $\left(\hat{p}^a, \hat{p}^b\right) = \left(\left(\hat{p}_2^a, \hat{p}_3^a\right), \left(\hat{p}_1^b, \hat{p}_2^b\right)\right) \in P$ such that*

$$\begin{cases} p^a_{n_i} \to (0, \hat{p}_2^a, \hat{p}_3^a) \text{ strongly in } \left(H^1(\Omega^a)\right)^3 \text{ and strongly in } \left(L^4(\Omega^a)\right)^3, \\ p^b_{n_i} \to \left(\hat{p}_1^b, \hat{p}_2^b, 0\right) \text{ strongly in } \left(H^1(\Omega^b)\right)^3 \text{ and strongly in } \left(L^4(\Omega^b)\right)^3, \end{cases} \tag{3.3.5}$$

$$\left(\frac{1}{h_n^a}\frac{\partial p_n^a}{\partial x_1}, \frac{1}{h_n^b}\frac{\partial p_n^b}{\partial x_3}\right) \to (0,0) \text{ strongly in } \left(L^2(\Omega^a)\right)^3 \times \left(L^2(\Omega^b)\right)^3, \tag{3.3.6}$$

$$\begin{cases} \left(\phi^a_{\left(p^a_{n_i}, p^b_{n_i}\right)}, \phi^b_{\left(p^a_{n_i}, p^b_{n_i}\right)}\right) \to \left(\psi^a_{(\hat{p}^a,\hat{p}^b)}, \psi^b_{(\hat{p}^a,\hat{p}^b)}\right) \text{ strongly in } H^1(\Omega^a) \times H^1(\Omega^b), \\ \left(\dfrac{1}{h_n^a}\dfrac{\partial \phi^a_{(p_n^a,p_n^b)}}{\partial x_1}, \dfrac{1}{h_n^b}\dfrac{\partial \phi^b_{(p_n^a,p_n^b)}}{\partial x_3}\right) \to (0,0) \text{ strongly in } L^2(\Omega^a) \times L^2\left(\Omega^b\right), \end{cases} \tag{3.3.7}$$

where $\left(\hat{p}^a, \hat{p}^b\right)$ *solves*

$$E\left(\left(\hat{p}^a, \hat{p}^b\right)\right) = \min\left\{E\left(\left(q^a, q^b\right)\right) : \left(q^a, q^b\right) \in P\right\}, \tag{3.3.8}$$

and $\left(\psi^a_{(\hat{p}^a,\hat{p}^b)}, \psi^b_{(\hat{p}^a,\hat{p}^b)}\right)$ *is the unique solution of* (3.3.4) *with* $(q^a, q^b) = \left(\hat{p}^a, \hat{p}^b\right)$. *Moreover,*

$$\lim_n \frac{E_n\left(\left(p_n^a, p_n^b\right)\right)}{h_n^a} = E\left(\left(\hat{p}^a, \hat{p}^b\right)\right). \tag{3.3.9}$$

3.3.2 The case $\ell = 0$

This subsection is devoted to state Theorem 3.3.4 describing the limit behavior of (3.2.8) when (3.1.8) is assumed with $\ell = 0$. Theorem 3.3.4 will be proved in Section 3.5.

Set

$$P_\circ = \left\{q^a = (q_2^a, q_3^a) \in \left(H^1\left(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[\times \left]0,1\right[\right)\right)^2 : q^a \cdot \nu^a = 0 \text{ on } \partial\left(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[\times \left]0,1\right[\right)\right\} \tag{3.3.10}$$

where ν^a denotes the unit outer normal on $\partial\left(\left]-\frac{1}{2}, \frac{1}{2}\right[\times \left]0,1\right[\right)$, and

$$\begin{cases} E_\circ : q^a \in \left(H^1\left(\left]-\frac{1}{2}, \frac{1}{2}\right[\times \left]0,1\right[\right)\right)^2 \longrightarrow \displaystyle\int_{\left]-\frac{1}{2},\frac{1}{2}\right[\times]0,1[} \Big(\beta|\mathrm{rot} q^a|^2 \\ \qquad + |\mathrm{div} q^a|^2 + \alpha(|q^a|^2 - 1)^2 + |D\psi^a_{q^a}|^2 + \displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} (f_2^a, f_3^a) dx_1 \cdot q^a\Big) dx_2 dx_3, \end{cases} \tag{3.3.11}$$

where (f_2^a, f_3^a) is defined in (3.2.9), and $\psi_{q^a}^a$ is the unique solution of

$$\begin{cases} \psi_{q^a}^a \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\times\left]0,1\right[\right), \quad \displaystyle\int_{\left]-\frac{1}{2},\frac{1}{2}\right[\times]0,1[} \psi_{q^a}^a dx_2 dx_3 = 0, \\ \displaystyle\int_{\left]-\frac{1}{2},\frac{1}{2}\right[\times]0,1[} \left(\left(-D_{(x_2,x_3)}\psi_{q^a}^a + q^a\right)\cdot D_{(x_2,x_3)}\psi^a\right) dx_2 dx_3 = 0 \\ \forall \psi^a \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\times\left]0,1\right[\right), \end{cases} \tag{3.3.12}$$

Remark 3.3.3. *In what follows, any element of $H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\times\left]0,1\right[\right)$ (resp. $P_\circ$) is assumed to be extended to the element of $H^1(\Omega^a)$ (resp. $\left(H^1(\Omega^a)\right)^2$) independent of x_1.*

Theorem 3.3.4. *Assume (3.1.8) with $\ell = 0$, and (3.2.9). For every $n \in \mathbb{N}$, let $\left(p_n^a, p_n^b\right)$ be a solution of (3.2.8), and $\left(\phi^a_{(p_n^a,p_n^b)}, \phi^b_{(p_n^a,p_n^b)}\right)$ be the unique solution of (3.2.6) with $\left(p^a,p^b\right) = \left(p_n^a,p_n^b\right)$. Moreover, let $P_\circ$ and $E_\circ$ be defined by (3.3.10) and (3.3.11)-(3.3.12), respectively. Then, there exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$ and (in possible dependence on the subsequence) $\hat{p}^a = (\hat{p}_2^a, \hat{p}_3^a) \in P_\circ$ such that*

$$\begin{cases} p_{n_i}^a \to (0,\hat{p}_2^a,\hat{p}_3^a) \text{ strongly in } \left(H^1(\Omega^a)\right)^3 \text{ and strongly in } \left(L^4(\Omega^a)\right)^3, \\ \left(\dfrac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} p_n^b \to 0 \text{ strongly in } \left(H^1(\Omega^b)\right)^3 \text{ and strongly in } \left(L^4(\Omega^b)\right)^3, \end{cases} \tag{3.3.13}$$

$$\left(\frac{1}{h_n^a}\frac{\partial p_n^a}{\partial x_1}, \left(\frac{1}{h_n^a h_n^b}\right)^{\frac{1}{2}}\frac{\partial p_n^b}{\partial x_3}\right) \to (0,0) \text{ strongly in } \left(L^2(\Omega^a)\right)^3 \times \left(L^2\left(\Omega^b\right)\right)^3, \tag{3.3.14}$$

$$\begin{cases} \left(\phi^a_{\left(p^a_{n_i},p^b_{n_i}\right)}, \left(\dfrac{h_n^b}{h_n^a}\right)^{\frac{1}{2}}\phi^b_{(p_n^a,p_n^b)}\right) \to \left(\psi^a_{\hat{p}^a}, 0\right) \text{ strongly in } H^1(\Omega^a)\times H^1(\Omega^b), \\ \left(\dfrac{1}{h_n^a}\dfrac{\partial \phi^a_{(p_n^a,p_n^b)}}{\partial x_1}, \left(\dfrac{1}{h_n^a h_n^b}\right)^{\frac{1}{2}}\dfrac{\partial \phi^b_{(p_n^a,p_n^b)}}{\partial x_3}\right) \to (0,0) \text{ strongly in } L^2(\Omega^a)\times L^2\left(\Omega^b\right) \end{cases} \tag{3.3.15}$$

where $\hat{p}^a$ solves

$$E_\circ\left(\hat{p}^a\right) = \min\left\{E_\circ\left(q^a\right) : q^a \in P_\circ\right\}, \tag{3.3.16}$$

and $\psi^a_{\hat{p}^a}$ *is the unique solution of* (3.3.12) *with* $q^a = \hat{p}^a$. *Moreover,*

$$\lim_n \frac{E_n\left(\left(p^a_n, p^b_n\right)\right)}{h^a_n} = E_\circ\left(\hat{p}^a\right). \tag{3.3.17}$$

Remark 3.3.5. *Note that* (3.3.17) *combined with* (3.3.13)-(3.3.15) *improves the last convergence in* (3.3.13). *Precisely, also one obtains*

$$\left(\frac{h^b_n}{h^a_n}\right)^{\frac{1}{4}} p^b_n \to 0 \text{ strongly in } \left(L^4(\Omega^b)\right)^3.$$

3.3.3 The case $\ell = +\infty$

This subsection is devoted to state Theorem 3.3.7 describing the limit behavior of (3.2.8) when (3.1.8) is assumed with $\ell = +\infty$ and $h^b_n << \sqrt{h^a_n}$. Theorem 3.3.7 will be proved in Section 3.6.

Here we assume that the function $(\phi^a_{(p^a,p^b)}, \phi^b_{(p^a,p^b)})$ involved in (3.2.5) is the unique solution of the following problem:

$$\begin{cases} \left(\phi^a_{(p^a,p^b)}, \phi^b_{(p^a,p^b)}\right) \in U_n, \quad \displaystyle\int_{\Omega^b} \phi^b_{(p^a,p^b)} dx = 0, \\ \\ h^a_n \displaystyle\int_{\Omega^a} ((-D^a_n \phi^a_{(p^a,p^b)} + p^a) \cdot D^a_n \phi^a) dx \\ \\ +h^b_n \displaystyle\int_{\Omega^b} \left(\left(-D^b_n \phi^b_{(p^a,p^b)} + p^b\right) \cdot D^b_n \phi^b\right) dx = 0 \quad \forall\, (\phi^a, \phi^b) \in U_n, \end{cases} \tag{3.3.18}$$

i.e. assumption $\displaystyle\int_{\Omega^a} \phi^a_{(p^a,p^b)} dx = 0$ is replaced with $\displaystyle\int_{\Omega^b} \phi^b_{(p^a,p^b)} dx = 0$, or equivalently, in (3.2.7), assumption $\displaystyle\int_{\Omega^a_n} \varphi_{\mathbf{P}} dx = 0$ is replaced with $\displaystyle\int_{\Omega^b_n} \varphi_{\mathbf{P}} dx = 0$. Obviously, $\mathcal{E}_n$ and E_n do not change.

Set

$$\left\{ P_\infty = \left\{ \begin{array}{l} q^b = (q_1^b, q_2^b) \in \left(H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right)\right)^2 \ : \ q^b \cdot \nu^b = 0 \text{ on } \partial\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right), \\ \\ q_1^b(0,\cdot) = 0 \text{ in } \left]-\frac{1}{2},\frac{1}{2}\right[\end{array} \right\}, \right. \tag{3.3.19}$$

where ν^b denotes the unit outer normal on $\partial\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right)$, and

$$\left\{ E_\infty : q^b \in \left(H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\right)\right)^2 \longrightarrow \int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} \left(\begin{array}{l} \beta|\mathrm{rot} q^b|^2 \\ \\ +|\mathrm{div} q^b|^2 + \alpha(|q^b|^2-1)^2 + |D\psi_{q^b}^b|^2 + \int_1^0 (f_1^b, f_2^b) dx_3 \cdot q^b \end{array} \right) dx_1 dx_2, \right. \tag{3.3.20}$$

where (f_1^b, f_2^b) is defined in (3.2.9), and $\psi_{q^b}^b$ is the unique solution of

$$\left\{ \begin{array}{l} \psi_{q^b}^b \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right), \quad \int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} \psi_{q^b}^b dx_1 dx_2 = 0, \\ \\ \int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} \left(\left(-D_{(x_1,x_2)}\psi_{q^b}^b + q^b\right) \cdot D_{(x_1,x_2)}\psi^b \right) dx_1 dx_2 = 0, \quad \forall \psi^b \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right). \end{array} \right. \tag{3.3.21}$$

Remark 3.3.6. *In what follows, any element of* $H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right)$ *(resp.* P_∞*) is assumed to be extended to the element of* $H^1(\Omega^b)$ *(resp.* $\left(H^1(\Omega^b)\right)^2$*) independent of* x_3*.*

Theorem 3.3.7. *Assume* (3.1.8) *with* $\ell = +\infty$ *and* $h_n^b << \sqrt{h_n^a}$*, and* (3.2.9)*. For every* $n \in \mathbb{N}$*, let* $\left(p_n^a, p_n^b\right)$ *be a solution of* (3.2.8)*, and* $\left(\phi^a_{(p_n^a,p_n^b)}, \phi^b_{(p_n^a,p_n^b)}\right)$ *be the unique solution of* (3.3.18) *with* $\left(p^a, p^b\right) = \left(p_n^a, p_n^b\right)$*. Moreover, let* P_∞ *and* E_∞ *be defined in* (3.3.19) *and* (3.3.20)-(3.3.21)*, respectively. Then, there exist an increasing sequence of positive integer numbers* $\{n_i\}_{i\in\mathbb{N}}$ *and (in*

possible dependence on the subsequence) $\hat{p}^b = \left(\hat{p}_1^b, \hat{p}_2^b\right) \in P_\infty$ *such that*

$$\begin{cases} \left(\dfrac{h_n^a}{h_n^b}\right)^{\frac{1}{2}} p_n^a \to 0 \text{ strongly in } \left(H^1(\Omega^a)\right)^3 \text{ and strongly in } \left(L^4(\Omega^a)\right)^3, \\ \\ p_{n_i}^b \to \left(\hat{p}_1^b, \hat{p}_2^b, 0\right) \text{ strongly in } \left(H^1(\Omega^b)\right)^3 \text{ and strongly in } \left(L^4(\Omega^b)\right)^3 \end{cases} \tag{3.3.22}$$

$$\left(\left(\frac{1}{h_n^a h_n^b}\right)^{\frac{1}{2}} \frac{\partial p_n^a}{\partial x_1}, \frac{1}{h_n^b}\frac{\partial p_n^b}{\partial x_3}\right) \to 0 \text{ strongly in } \left(L^2(\Omega^a)\right)^3 \times \left(L^2(\Omega^b)\right)^3, \tag{3.3.23}$$

$$\begin{cases} \left(\left(\dfrac{h_n^a}{h_n^b}\right)^{\frac{1}{2}} \phi^a_{\left(p^a_{n_i}, p^b_{n_i}\right)}, \phi^b_{\left(p^a_{n_i}, p^b_{n_i}\right)}\right) \to \left(0, \psi^b_{\hat{p}^b}\right) \text{ strongly in } H^1(\Omega^a) \times H^1(\Omega^b), \\ \\ \left(\left(\dfrac{1}{h_n^a h_n^b}\right)^{\frac{1}{2}} \dfrac{\partial \phi^a_{\left(p_n^a, p_n^b\right)}}{\partial x_1}, \dfrac{1}{h_n^b}\dfrac{\partial \phi^b_{\left(p_n^a, p_n^b\right)}}{\partial x_3}\right) \to (0,0) \text{ strongly in } L^2(\Omega^a) \times L^2\left(\Omega^b\right), \end{cases} \tag{3.3.24}$$

where $\hat{p}^b$ *solves*

$$E_\infty\left(\hat{p}^b\right) = \min\left\{E_\infty\left(q^b\right) : q^b \in P_\infty\right\}, \tag{3.3.25}$$

and $\psi^b_{\hat{p}^b}$ *is the unique solution of* (3.3.21) *with* $q^b = \hat{p}^b$. *Moreover, the convergence of the energies holds true, that is*

$$\lim_n \frac{E_n\left(\left(p_n^a, p_n^b\right)\right)}{h_n^b} = E_\infty\left(\hat{p}^b\right). \tag{3.3.26}$$

Remark 3.3.8. *Note* (3.3.26) *combined with* (3.3.22)-(3.3.24) *improves the second convergence in* (3.3.22). *Precisely, one obtains also*

$$\left(\frac{h_n^a}{h_n^b}\right)^{\frac{1}{4}} p_n^a \to 0 \text{ strongly in } \left(L^4(\Omega^a)\right)^3.$$

3.4 The proofs in the case $\ell \in]0, +\infty[$

This section is devoted to proving Theorem 3.3.2.

3.4.1 *A priori* estimates on the polarization

Proposition 3.4.1. *Assume* (3.1.8) *with* $\ell \in]0,+\infty[$, *and* (3.2.9). *For every* $n \in \mathbb{N}$, *let* $\left(p_n^a, p_n^b\right)$ *be a solution of* (3.2.8). *Then, there exists a constant* c *such that*

$$\|p_n^a\|_{(L^4(\Omega^a))^3} \leq c, \quad \left\|p_n^b\right\|_{(L^4(\Omega^b))^3} \leq c \quad \forall n \in \mathbb{N}, \tag{3.4.1}$$

and

$$\|D_n^a p_n^a\|_{(L^2(\Omega^a))^9} \leq c, \quad \left\|D_n^b p_n^b\right\|_{(L^2(\Omega^b))^9} \leq c \quad \forall n \in \mathbb{N}. \tag{3.4.2}$$

Proof. Function 0 belonging to P_n gives

$$\left\{\begin{array}{l} \displaystyle\int_{\Omega^a} \left(\beta \left|\mathrm{rot}_n^a p_n^a\right|^2 + \left|\mathrm{div}_n^a p_n^a\right|^2 + \alpha\left(\left|p_n^a\right|^4 - 2\left|p_n^a\right|^2\right) + \left|D_n^a \phi^a_{(p_n^a, p_n^b)}\right|^2\right) dx \\ \\ \displaystyle + \frac{h_n^b}{h_n^a} \int_{\Omega^b} \left(\beta \left|\mathrm{rot}_n^b p_n^b\right|^2 + \left|\mathrm{div}_n^b p_n^b\right|^2 + \alpha\left(\left|p_n^b\right|^4 - 2\left|p_n^b\right|^2\right) + \left|D_n^b \phi^b_{(p_n^a, p_n^b)}\right|^2\right) dx \\ \\ \displaystyle \leq \frac{1}{2}\int_{\Omega^a} \left(|f_n^a|^2 + |p_n^a|^2\right) dx + \frac{h_n^b}{h_n^a}\frac{1}{2}\int_{\Omega^b} \left(|f_n^b|^2 + |p_n^b|^2\right) dx \quad \forall n \in \mathbb{N}. \end{array}\right. \tag{3.4.3}$$

Estimate (3.4.3) implies

$$\left\{\begin{array}{l} \displaystyle\int_{\Omega^a} \alpha\left(\left|p_n^a\right|^4 - \left(2 + \frac{1}{2\alpha}\right)\left|p_n^a\right|^2\right) dx + \frac{h_n^b}{h_n^a}\int_{\Omega^b} \alpha\left(\left|p_n^b\right|^4 - \left(2 + \frac{1}{2\alpha}\right)\left|p_n^b\right|^2\right) dx \\ \\ \displaystyle \leq \frac{1}{2}\int_{\Omega^a} \left|f_n^a\right|^2 dx + \frac{h_n^b}{h_n^a}\frac{1}{2}\int_{\Omega^b} \left|f_n^b\right|^2 dx \quad \forall n \in \mathbb{N}, \end{array}\right.$$

which gives

$$\left\{\begin{array}{l} \displaystyle\int_{\Omega^a} \alpha\left(\left|p_n^a\right|^2 - \left(1 + \frac{1}{4\alpha}\right)\right)^2 dx + \frac{h_n^b}{h_n^a}\int_{\Omega^b} \alpha\left(\left|p_n^b\right|^2 - \left(1 + \frac{1}{4\alpha}\right)\right)^2 dx \\ \\ \displaystyle \leq \alpha\left(1 + \frac{1}{4\alpha}\right)^2 \left(|\Omega^a| + \frac{h_n^b}{h_n^a}|\Omega^b|\right) + \frac{1}{2}\int_{\Omega^a} |f_n^a|^2 dx + \frac{h_n^b}{h_n^a}\frac{1}{2}\int_{\Omega^b} \left|f_n^b\right|^2 dx \quad \forall n \in \mathbb{N}. \end{array}\right. \tag{3.4.4}$$

Then, estimates in (3.4.1) follow from (3.4.4), (3.1.8) with $\ell \in]0,+\infty[$, and (3.2.9). Estimates in (3.4.2) follow from (3.4.3), (3.1.8) with $\ell \in]0,+\infty[$, (3.2.9), (3.4.1), the continuous embedding of L^4 into L^2, and (3.2.10). □

Corollary 3.4.2. *Assume* (3.1.8) *with* $\ell \in]0,+\infty[$, *and* (3.2.9). *For every* $n \in \mathbb{N}$, *let* $\left(p_n^a, p_n^b\right)$ *be a solution of* (3.2.8). *Let* P *be defined in* (3.3.1). *Then, there exist a subsequence of* $\mathbb{N}$, *still denoted by* $\{n\}$, *and (in possible dependence on the subsequence)* $(\hat{p}^a, \hat{p}^b) = \left(\left(\hat{p}_2^a, \hat{p}_3^a\right), \left(\hat{p}_1^b, \hat{p}_2^b\right)\right) \in P$ *such that*

$$\begin{cases} p_n^a \rightharpoonup \left(0, \hat{p}_2^a, \hat{p}_3^a\right) \text{ weakly in } \left(H^1(\Omega^a)\right)^3 \text{ and strongly in } \left(L^4(\Omega^a)\right)^3, \\ \\ p_n^b \rightharpoonup \left(\hat{p}_1^b, \hat{p}_2^b, 0\right) \text{ weakly in } \left(H^1(\Omega^b)\right)^3 \text{ and strongly in } \left(L^4(\Omega^b)\right)^3. \end{cases} \tag{3.4.5}$$

Proof. Proposition 3.4.1 ensures that there exist a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence on the subsequence) $\left(\hat{p}_1^a, \hat{p}_2^a, \hat{p}_3^a\right) \in \left(H^1\left(\Omega^a\right)\right)^3$ independent of x_1, and $\left(\hat{p}_1^b, \hat{p}_2^b, \hat{p}_3^b\right) \in \left(H^1\left(\Omega^b\right)\right)^3$ independent of x_3 such that

$$\begin{cases} p_n^a \rightharpoonup \left(\hat{p}_1^a, \hat{p}_2^a, \hat{p}_3^a\right) \text{ weakly in } \left(H^1(\Omega^a)\right)^3 \text{ and strongly in } \left(L^4(\Omega^a)\right)^3, \\ \\ p_n^b \rightharpoonup \left(\hat{p}_1^b, \hat{p}_2^b, \hat{p}_3^b\right) \text{ weakly in } \left(H^1(\Omega^b)\right)^3 \text{ and strongly in } \left(L^4(\Omega^b)\right)^3, \end{cases}$$

and

$$\left(\hat{p}_1^a, \hat{p}_2^a, \hat{p}_3^a\right) \cdot \nu^a = 0 \text{ on } \partial\Omega^a \backslash \left(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[^2 \times \{0\}\right), \ \left(\hat{p}_1^b, \hat{p}_2^b, \hat{p}_3^b\right) \cdot \nu^b = 0 \text{ on } \partial\Omega^b \backslash \left(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[^2 \times \{0\}\right).$$

In particular, this implies

$$\hat{p}_1^a = 0 \text{ in } \Omega^a, \quad \hat{p}_3^b = 0 \text{ in } \Omega^b.$$

Moreover, as proved in Proposition 5.4 in [34], one has

$$\left(\hat{p}_1^a, \hat{p}_2^a, \hat{p}_3^a\right)(\cdot, 0) = \left(\hat{p}_1^b, \hat{p}_2^b, \hat{p}_3^b\right)(0, \cdot) \text{ in } \left]-\tfrac{1}{2}, \tfrac{1}{2}\right[.$$

Consequently, one obtains

$$\hat{p}_3^a(\cdot,0) = 0 = \hat{p}_1^b(0,\cdot) \text{ in } \left]-\tfrac{1}{2},\tfrac{1}{2}\right[, \tag{3.4.6}$$

which completes the proof. □

3.4.2 A convergence result for problem (3.2.6)

Let

$$U_{\text{reg}} = \left\{ \begin{array}{l} \left(\psi^a,\psi^b\right) \in C^1\left(\left[-\tfrac{1}{2},\tfrac{1}{2}\right]\times[0,1]\right) \times C\left(\left[-\tfrac{1}{2},\tfrac{1}{2}\right]^2\right) : \\ \\ \psi^b_{\left|_{\left[-\frac{1}{2},0\right]\times\left[-\frac{1}{2},\frac{1}{2}\right]}\right.} \in C^1\left(\left[-\tfrac{1}{2},0\right]\times\left[-\tfrac{1}{2},\tfrac{1}{2}\right]\right), \\ \\ \psi^b_{\left|_{\left[0,\frac{1}{2}\right]\times\left[-\frac{1}{2},\frac{1}{2}\right]}\right.} \in C^1\left(\left[0,\tfrac{1}{2}\right]\times\left[-\tfrac{1}{2},\tfrac{1}{2}\right]\right), \\ \\ \psi^a(\cdot,0) = \psi^b(0,\cdot) \text{ in } \left[-\tfrac{1}{2},\tfrac{1}{2}\right] \end{array} \right\}. \tag{3.4.7}$$

Proposition 3.4.3. *Let U and U_{reg} be defined in (3.3.2) and (3.4.7), respectively. Then, U_{reg} is dense in U.*

Proof. Let $\left(\psi^a,\psi^b\right) \in U$. The goal is to find a sequence $\left\{\left(\psi_n^a,\psi_n^b\right)\right\}_{n\in\mathbb{N}} \subset U_{\text{reg}}$ such that

$$\left(\psi_n^a,\psi_n^b\right) \to \left(\psi^a,\psi^b\right) \text{ strongly in } H^1\left(\left]-\tfrac{1}{2},\tfrac{1}{2}\right[\times\left]0,1\right[\right) \times H^1\left(\left]-\tfrac{1}{2},\tfrac{1}{2}\right[^2\right). \tag{3.4.8}$$

To this aim, split $\psi^b = \psi^e + \psi^o$ in the even part and in the odd part with respect to x_1. Note that $\psi^e, \psi^o \in H^1\left(\left]-\tfrac{1}{2},\tfrac{1}{2}\right[^2\right)$, and

$$\psi^e(0,\cdot) = \psi^b(0,\cdot) = \psi^a(\cdot,0), \quad \psi^o(0,\cdot) = 0 \text{ in } \left]-\tfrac{1}{2},\tfrac{1}{2}\right[.$$

Consequently, a convolution argument allows us to build three sequences (see [51], and also compare the first part of the proof of Proposition 5.5 in [34])) $\{\zeta_n^a\}_{n\in\mathbb{N}} \subset C^\infty\left(\left[-\tfrac{1}{2},\tfrac{1}{2}\right]\times[0,1]\right)$,

$\{\zeta_n^e\}_{n\in\mathbb{N}} \subset C\left(\left[-\frac{1}{2},\frac{1}{2}\right]^2\right)$, and $\{\zeta_n^o\}_{n\in\mathbb{N}} \subset C^\infty\left(\left[-\frac{1}{2},\frac{1}{2}\right]^2\right)$ such that

$$\begin{cases} \left\{\zeta_{n|_{\left[-\frac{1}{2},0\right]\times\left[-\frac{1}{2},\frac{1}{2}\right]}}^e\right\}_{n\in\mathbb{N}} \subset C^\infty\left(\left[-\frac{1}{2},0\right]\times\left[-\frac{1}{2},\frac{1}{2}\right]\right), \\ \left\{\zeta_{n|_{\left[0,\frac{1}{2}\right]\times\left[-\frac{1}{2},\frac{1}{2}\right]}}^e\right\}_{n\in\mathbb{N}} \subset C^\infty\left(\left[0,\frac{1}{2}\right]\times\left[-\frac{1}{2},\frac{1}{2}\right]\right), \\ \zeta_n^a \to \psi^a \text{ strongly in } H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\times\left]0,1\right[\right), \\ \zeta_n^e \to \psi^e \text{ strongly in } H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right), \quad \zeta_n^o \to \psi^o \text{ strongly in } H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right), \\ \zeta_n^a(\cdot,0) = \zeta_n^e(0,\cdot), \quad \zeta_n^o(0,\cdot) = 0 \text{ in } \left]-\frac{1}{2},\frac{1}{2}\right[\quad \forall n\in\mathbb{N}. \end{cases}$$

This implies (3.4.8), setting $\psi_n^a = \zeta_n^a$ and $\psi_n^b = \zeta_n^e + \zeta_n^o$. □

Proposition 3.4.4. *Assume (3.1.8) with $\ell \in]0,+\infty[$. Let $\left\{\left(q_n^a, q_n^b\right)\right\}_{n\in\mathbb{N}} \subset (L^2(\Omega^a))^3 \times (L^2(\Omega^b))^3$, and $\left(q^a,q^b\right) = \left(\left(q_2^a,q_3^a\right),\left(q_1^b,q_2^b\right)\right) \in (L^2(\Omega^a))^2 \times (L^2(\Omega^b))^2$ be such that q^a is independent of x_1, q^b is independent of x_3, and*

$$\left(q_n^a,q_n^b\right) \to \left(\left(0,q_2^a,q_3^a\right),\left(q_1^b,q_2^b,0\right)\right) \text{ strongly in } (L^2(\Omega^a))^3 \times (L^2(\Omega^b))^3. \tag{3.4.9}$$

Moreover, for $n\in\mathbb{N}$ let $\left(\phi^a_{(q_n^a,q_n^b)}, \phi^b_{(q_n^a,q_n^b)}\right)$ be the unique solution of

$$\begin{cases} \left(\phi^a_{(q_n^a,q_n^b)}, \phi^b_{(q_n^a,q_n^b)}\right) \in U_n, \quad \displaystyle\int_{\Omega^a} \phi^a_{(q_n^a,q_n^b)} dx = 0, \\ \displaystyle\int_{\Omega^a} \left(\left(-D_n^a \phi^a_{(q_n^a,q_n^b)} + q_n^a\right)\cdot D_n^a\phi^a\right) dx \\ \displaystyle + \frac{h_n^b}{h_n^a}\int_{\Omega^b}\left(\left(-D_n^b\phi^b_{(q_n^a,q_n^b)} + q_n^b\right)\cdot D_n^b\phi^b\right) dx = 0 \quad \forall\,(\phi^a,\phi^b)\in U_n, \end{cases} \tag{3.4.10}$$

where U_n is defined in (3.2.4). Furthermore, let $\left(\psi^a_{(q^a,q^b)}, \psi^b_{(q^a,q^b)}\right)$ be the unique solution of (3.3.4). Then, one has

$$\left(\phi^a_{(q^a_n,q^b_n)}, \phi^b_{(q^a_n,q^b_n)}\right) \to \left(\psi^a_{(q^a,q^b)}, \psi^b_{(q^a,q^b)}\right) \text{ strongly in } H^1(\Omega^a) \times H^1(\Omega^b), \tag{3.4.11}$$

$$\left(\frac{1}{h^a_n}\frac{\partial \phi^a_{(q^a_n,q^b_n)}}{\partial x_1}, \frac{1}{h^b_n}\frac{\partial \phi^b_{(q^a_n,q^b_n)}}{\partial x_3}\right) \to (0,0) \text{ strongly in } L^2(\Omega^a) \times L^2(\Omega^b), \tag{3.4.12}$$

Proof. In this proof, c denotes any positive constant independent of $n \in \mathbb{N}$.

Choosing $\left(\phi^a, \phi^b\right) = \left(\phi^a_{(q^a_n,q^b_n)}, \phi^b_{(q^a_n,q^b_n)}\right)$ as test function in (3.4.10), applying the Young inequality, and using (3.1.8) with $\ell \in]0,+\infty[$ and (3.4.9) give

$$\left\| D^a_n \phi^a_{(q^a_n,q^b_n)} \right\|_{(L^2(\Omega^a))^3} \leq c, \quad \left\| D^b_n \phi^b_{(q^a_n,q^b_n)} \right\|_{(L^2(\Omega^b))^3} \leq c \quad \forall n \in \mathbb{N}. \tag{3.4.13}$$

The first estimate in (3.4.13) implies

$$\left\| \phi^a_{(q^a_n,q^b_n)} \right\|_{H^1(\Omega^a)} \leq c \quad \forall n \in \mathbb{N}, \tag{3.4.14}$$

since $\displaystyle\int_{\Omega^a} \phi^a_{(q^a_n,q^b_n)} dx = 0$.

The next step is devoted to proving

$$\left\| \phi^b_{(q^a_n,q^b_n)} \right\|_{H^1(\Omega^b)} \leq c \quad \forall n \in \mathbb{N}. \tag{3.4.15}$$

The junction condition in (3.2.4) gives

$$\begin{cases} \displaystyle\int_{]-\frac{h^a_n}{2},\frac{h^a_n}{2}[\times]-\frac{1}{2},\frac{1}{2}[} \left|\phi^b_{(q^a_n,q^b_n)}(x_1,x_2,0)\right|^2 dx_1 dx_2 \\ \\ \displaystyle = h^a_n \int_{]-\frac{1}{2},\frac{1}{2}[^2} \left|\phi^b_{(q^a_n,q^b_n)}(h^a_n x_1,x_2,0)\right|^2 dx_1 dx_2 \\ \\ \displaystyle = h^a_n \int_{]-\frac{1}{2},\frac{1}{2}[^2} \left|\phi^a_{(q^a_n,q^b_n)}(x_1,x_2,0)\right|^2 dx_1 dx_2 \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.4.16}$$

Then, (3.4.16) and (3.4.14) provide

$$\left\|\phi^b_{(q_n^a,q_n^b)}\right\|_{L^2\left(\right]-\frac{h_n^a}{2},\frac{h_n^a}{2}\left[\times\right]-\frac{1}{2},\frac{1}{2}\left[\times\{0\}\right)} \leq c\sqrt{h_n^a} \quad \forall n \in \mathbb{N},$$

which implies

$$\left\|\phi^b_{(q_n^a,q_n^b)}\right\|_{L^2\left(\right]-\frac{h_n^a}{2},\frac{h_n^a}{2}\left[\times\right]-\frac{1}{2},\frac{1}{2}\left[\times\right]-1,0\left[\right)} \leq c\sqrt{h_n^a} \quad \forall n \in \mathbb{N}, \tag{3.4.17}$$

by virtue of the second estimate in (3.4.13) and (3.1.8) with $\ell \in]0,+\infty[$. Combining (3.4.17) with the second estimate in (3.4.13) ensures

$$\left\|\phi^b_{q_n}\right\|_{L^2(\{0\}\times\left]-\frac{1}{2},\frac{1}{2}\right[\times]-1,0[)} \leq c \quad \forall n \in \mathbb{N}$$

which combined again with the second estimate in (3.4.13) proves (3.4.15).

Estimates (3.4.13), (3.4.14), and (3.4.15) ensure the existence of a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence on the subsequence) $\left(\tau^a,\tau^b\right) \in U$ and $(\xi^a,\xi^b) \in L^2(\Omega^a) \times L^2\left(\Omega^b\right)$ such that

$$\left(\phi^a_{(q_n^a,q_n^b)},\phi^b_{(q_n^a,q_n^b)}\right) \rightharpoonup \left(\tau^a,\tau^b\right) \text{ weakly in } H^1(\Omega^a) \times H^1\left(\Omega^b\right), \tag{3.4.18}$$

$$\left(\frac{1}{h_n^a}\frac{\partial\phi^a_{(q_n^a,q_n^b)}}{\partial x_1},\frac{1}{h_n^b}\frac{\partial\phi^b_{(q_n^a,q_n^b)}}{\partial x_3}\right) \rightharpoonup \left(\xi^a,\xi^b\right) \text{ weakly in } L^2(\Omega^a) \times L^2\left(\Omega^b\right), \tag{3.4.19}$$

$$\int_{\left]-\frac{1}{2},\frac{1}{2}\right[\times]0,1[} \tau^a \, dx_2dx_3 = 0. \tag{3.4.20}$$

Note that junction condition $\tau^a(\cdot,0) = \tau^b(0,\cdot)$ in $\left]-\frac{1}{2},\frac{1}{2}\right[$ can be obtained arguing as in the proof of Proposition 5.4 in [34].

The next step is devoted to proving that $\left(\tau^a,\tau^b\right)$ solves (3.3.4). To this aim, for every couple $\left(\psi^a,\psi^b\right) \in U_{\text{reg}}$, where U_{reg} is defined in (3.4.7), consider a sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset H^1\left(\Omega^a\right)$

(depending on $\left(\psi^a, \psi^b\right)$) such that

$$\begin{cases} (\mu_n^a, \psi^b) \in U_n \quad \forall n \in \mathbb{N}, \\ \mu_n \to \psi^a \text{ strongly in } L^2\left(\Omega^a\right), \\ \left(\dfrac{1}{h_n^a} D_{x_1}\mu_n, D_{x_2}\mu_n, D_{x_3}\mu_n\right) \to \left(0, D_{x_2}\psi^a, D_{x_3}\psi^a\right) \text{ strongly in } \left(L^2\left(\Omega^a\right)\right)^3. \end{cases} \tag{3.4.21}$$

For instance, setting

$$\mu_n(x) = \begin{cases} \psi^a(x_2, x_3) \quad \text{if } x = (x_1, x_2, x_3) \in \left]-\frac{1}{2}, \frac{1}{2}\right[^2 \times]h_n^a, 1[, \\ \psi^a\left(x_2, h_n^a\right)\dfrac{x_3}{h_n^a} + \psi^b\left(h_n^a x_1, x_2\right)\dfrac{h_n^a - x_3}{h_n^a} \quad \text{if } x = (x_1, x_2, x_3) \in \left]-\frac{1}{2}, \frac{1}{2}\right[^2 \times [0, h_n^a], \end{cases}$$

the first two proprieties in (3.4.21) can be immediately verified by the properties of U_{reg}, while the last one follows from

$$\int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2 \times]0, h_n^a[} \left|\frac{1}{h_n^a} D_{x_1}\mu_n\right|^2 dx \le \left\|\psi^b\right\|^2_{W^{1,\infty}\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right)} h_n^a \quad \forall n \in \mathbb{N},$$

$$\int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2 \times]0, h_n^a[} \left|D_{x_2}\mu_n\right|^2 dx \le 2\Big(\left\|\psi^a\right\|^2_{W^{1,\infty}\left(\left]-\frac{1}{2},\frac{1}{2}\right[\times]0,1[\right)} + \left\|\psi^b\right\|^2_{W^{1,\infty}\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right)} \Big) h_n^a \quad \forall n \in \mathbb{N},$$

and

$$\int_{]-\frac{1}{2},\frac{1}{2}[^2\times]0,h_n^a[} |D_{x_3}\mu_n|^2\,dx = \int_{]-\frac{1}{2},\frac{1}{2}[^2} \frac{1}{h_n^a}\left|\psi^a\left(x_2,h_n^a\right)-\psi^b\left(h_n^a x_1,x_2\right)\right|^2 dx_1dx_2$$

$$= \int_{]-\frac{1}{2},\frac{1}{2}[^2} \frac{1}{h_n^a}\left|\psi^a\left(x_2,h_n^a\right)-\psi^a\left(x_2,0\right)+\psi^b\left(0,x_2\right)-\psi^b\left(h_n^a x_1,x_2\right)\right|^2 dx_1dx_2$$

$$\leq 2\left(\left\|\psi^a\right\|^2_{W^{1,\infty}(]-\frac{1}{2},\frac{1}{2}[\times]0,1[)}+\left\|\psi^b\right\|^2_{W^{1,\infty}\left(]-\frac{1}{2},\frac{1}{2}[^2\right)}\right)h_n^a \quad \forall n\in\mathbb{N},$$

where again the properties of U_{reg} played a crucial role.

Now, fixing $\left(\psi^a,\psi^b\right)\in U_{\text{reg}}$, choosing $\left(\mu_n,\psi^b\right)$ as test function in (3.4.10) with $\{\mu_n\}$ satisfying (3.4.21) , passing to the limt as n diverges, and using (3.1.8) with $\ell\in]0,+\infty[$, (3.4.9), (3.4.18), (3.4.19), and (3.4.21), one obtains

$$\begin{cases} \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,1[}\left(\left(-D_{(x_2,x_3)}\tau^a+q^a\right)\cdot D_{(x_2,x_3)}\psi^a\right)dx_2dx_3 \\ \\ \displaystyle+\ell\int_{]-\frac{1}{2},\frac{1}{2}[^2}\left(\left(-D_{(x_1,x_2)}\tau^b+q^b\right)\cdot D_{(x_1,x_2)}\psi^b\right)dx_1dx_2=0. \end{cases} \tag{3.4.22}$$

By virtue of Proposition 3.4.3, equation (3.4.22) holds true also with any test function in U. Consequently, thanks to (3.4.20), $\left(\tau^a,\tau^b\right)$ is the unique solution of (3.3.4), i.e.

$$\left(\tau^a,\tau^b\right)=\left(\psi^a_{(q^a,q^b)},\psi^b_{(q^a,q^b)}\right) \text{ a.e. in } \Omega^a\times\Omega^b. \tag{3.4.23}$$

Finally, using (3.4.10), (3.1.8) with $\ell\in]0,+\infty[$, (3.4.9), (3.4.18), (3.4.23), (3.4.19), (3.3.4), and a l.s.c.

argument, one has

$$\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,1[} \left| D_{(x_2,x_3)} \psi^a_{(q^a,q^b)} \right|^2 \, dx_2 dx_3 + \int_{\Omega^a} |\xi^a|^2 dx$$

$$+\ell \int_{]-\frac{1}{2},\frac{1}{2}[^2} \left| D_{(x_1,x_2)} \psi^b_{(q^a,q^b)} \right|^2 \, dx_1 dx_2 + \ell \int_{\Omega^b} |\xi^b|^2 dx$$

$$\leq \lim_n \left(\int_{\Omega^a} \left| D^a_n \phi^a_{(q^a_n,q^b_n)} \right|^2 \, dx + \frac{h^b_n}{h^a_n} \int_{\Omega^b} \left| D^b_n \phi^b_{(q^a_n,q^b_n)} \right|^2 \, dx \right)$$

$$= \lim_n \left(\int_{\Omega^a} D^a_n \phi^a_{(q^a_n,q^b_n)} \cdot q^a_n dx + \frac{h^b_n}{h^a_n} \int_{\Omega^b} D^b_n \phi^b_{(q^a_n,q^b_n)} \cdot q^b_n dx \right)$$

$$= \int_{]-\frac{1}{2},\frac{1}{2}[\times]0,1[} D_{(x_2,x_3)} \psi^a_{(q^a,q^b)} \cdot q^a dx_2 dx_3 + \ell \int_{]-\frac{1}{2},\frac{1}{2}[^2} D_{(x_1,x_2)} \psi^b_{(q^a,q^b)} \cdot q^b dx_1 dx_2$$

$$= \int_{]-\frac{1}{2},\frac{1}{2}[\times]0,1[} \left| D_{(x_2,x_3)} \psi^a_{(q^a,q^b)} \right|^2 dx_2 dx_3 + \ell \int_{]-\frac{1}{2},\frac{1}{2}[^2} \left| D_{(x_1,x_2)} \psi^b_{(q^a,q^b)} \right|^2 dx_1 dx_2,$$

which implies that $\xi^a = 0, \xi^b = 0$, and that convergences (3.4.18) and (3.4.19) are strong. Note that previous convergences hold true for the whole sequence, since the limits are uniquely identified. □

3.4.3 The proof of Theorem 3.3.2

Let

$$
\begin{cases}
P_{\text{reg}} = \left\{ \left(q^a, q^b\right) = \left(\left(q_2^a, q_3^a\right), \left(q_1^b, q_2^b\right)\right) \in \left(C^1\left(\left[-\frac{1}{2}, \frac{1}{2}\right] \times [0,1]\right)\right)^2 \times \left(C\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^2\right)\right)^2 : \right. \\
q^b_{\left|_{\left[-\frac{1}{2},0\right]\times\left[-\frac{1}{2},\frac{1}{2}\right]}\right.} \in \left(C^1\left(\left[-\frac{1}{2}, 0\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right]\right)\right)^2,\ q^b_{\left|_{\left[0,\frac{1}{2}\right]\times\left[-\frac{1}{2},\frac{1}{2}\right]}\right.} \in \left(C^1\left(\left[0, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right]\right)\right)^2, \\
q_2^a(\cdot, 0) = q_2^b(0, \cdot) \text{ in } \left]-\frac{1}{2}, \frac{1}{2}\right[, \quad \text{and for some } \delta > 0 \text{ (depending on } \left(q^a, q^b\right)) \\
q_2^a = 0 \text{ in } \left(\left[-\frac{1}{2}, -\frac{1}{2} + \delta\right] \cup \left[\frac{1}{2} - \delta, \frac{1}{2}\right]\right) \times [0,1], \\
q_3^a = 0 \text{ in } \left[-\frac{1}{2}, \frac{1}{2}\right] \times ([0, \delta] \cup [1 - \delta, 1]), \\
q_1^b = 0 \text{ in } \left(\left[-\frac{1}{2}, -\frac{1}{2} + \delta\right] \cup [-\delta, \delta] \cup \left[\frac{1}{2} - \delta, \frac{1}{2}\right]\right) \times \left[-\frac{1}{2}, \frac{1}{2}\right], \\
\left. q_2^b = 0 \text{ in } \left[-\frac{1}{2}, \frac{1}{2}\right] \times \left(\left[-\frac{1}{2}, -\frac{1}{2} + \delta\right] \cup \left[\frac{1}{2} - \delta, \frac{1}{2}\right]\right) \right\}.
\end{cases}
\tag{3.4.24}
$$

Proposition 3.4.5. *Let P and P_{reg} be defined in (3.3.1) and (3.4.24), respectively. Then, P_{reg} is dense in P.*

Proof. Fix $\left(q^a, q^b\right) = \left(\left(q_2^a, q_3^a\right), \left(q_1^b, q_2^b\right)\right) \in P$. The goal is to find a sequence $\left\{\left(\theta_n^a, \theta_n^b\right)\right\}_{n\in\mathbb{N}} \subset P_{\text{reg}}$ such that

$$
\left(\theta_n^a, \theta_n^b\right) \to \left(q^a, q^b\right) \text{ strongly in } \left(H^1\left(\left]-\frac{1}{2}, \frac{1}{2}\right[\times \left]0,1\right[\right)\right)^2 \times \left(H^1\left(\left]-\frac{1}{2}, \frac{1}{2}\right[^2\right)\right)^2. \tag{3.4.25}
$$

At first, note that there exist two sequences $\{\eta_n^a\}_{n\in\mathbb{N}} \subset C^\infty\left(\left[-\frac{1}{2}, \frac{1}{2}\right] \times [0,1]\right)$ and $\left\{\eta_n^b\right\}_{n\in\mathbb{N}} \subset$

$C^\infty\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^2\right)$ such that

$$\begin{cases} \eta_n^a = 0 \text{ in } \left]-\frac{1}{2}, \frac{1}{2}\right[\times \left(\left]0, \frac{1}{n}\right[\cup \left]1-\frac{1}{n}, 1\right[\right) \quad \forall n \in \mathbb{N}, \\ \eta_n^b = 0 \text{ in } \left(\left]-\frac{1}{2}, -\frac{1}{2}+\frac{1}{n}\right[\cup \left]-\frac{1}{n}, \frac{1}{n}\right[\cup \left]\frac{1}{2}-\frac{1}{n}, \frac{1}{2}\right[\right) \times \left]-\frac{1}{2}, \frac{1}{2}\right[\quad \forall n \in \mathbb{N}, \\ \eta_n^a \to q_3^a \text{ strongly in } H^1\left(\left[-\frac{1}{2}, \frac{1}{2}\right] \times [0,1]\right), \quad \eta_n^b \to q_1^b \text{ strongly in } H^1\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^2\right). \end{cases}$$

Now, split $q_2^b = q^e + q^o$ in the even part and in the odd part with respect to x_1 (compare the proof of Proposition 3.4.3). By convolution, one can build three sequences $\{\zeta_n^a\}_{n\in\mathbb{N}} \subset C^\infty\left(\left[-\frac{1}{2}, \frac{1}{2}\right] \times [0,1]\right)$,

$\{\zeta_n^e\}_{n\in\mathbb{N}} \subset C\left(\left[-\frac{1}{2},\frac{1}{2}\right]^2\right)$ and $\{\zeta_n^o\}_{n\in\mathbb{N}} \subset C^\infty\left(\left[-\frac{1}{2},\frac{1}{2}\right]^2\right)$ such that

$$
\begin{cases}
\left\{\zeta_{n|_{\left[-\frac{1}{2},0\right]\times\left[-\frac{1}{2},\frac{1}{2}\right]}}^e\right\}_{n\in\mathbb{N}} \subset C^\infty\left(\left[-\frac{1}{2},0\right]\times\left[-\frac{1}{2},\frac{1}{2}\right]\right), \\
\left\{\zeta_{n|_{\left[0,\frac{1}{2}\right]\times\left[-\frac{1}{2},\frac{1}{2}\right]}}^e\right\}_{n\in\mathbb{N}} \subset C^\infty\left(\left[0,\frac{1}{2}\right]\times\left[-\frac{1}{2},\frac{1}{2}\right]\right), \\
\zeta_n^a \to q_2^a \text{ strongly in } H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\times\left]0,1\right[\right), \\
\zeta_n^e \to q^e \text{ strongly in } H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right), \quad \zeta_n^o \to q^o \text{ strongly in } H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right), \\
\zeta_n^a(\cdot,0) = \zeta_n^e(0,\cdot), \quad \zeta_n^o(0,\cdot) = 0 \text{ in } \left]-\frac{1}{2},\frac{1}{2}\right[\quad \forall n\in\mathbb{N}, \\
\zeta_n^a = 0 \text{ in } \left(\left]-\frac{1}{2},-\frac{1}{2}+\frac{1}{n}\right[\cup\left]\frac{1}{2}-\frac{1}{n},\frac{1}{2}\right[\right)\times\left]0,1\right[\quad \forall n\in\mathbb{N}, \\
\zeta_n^e = 0 = \zeta_n^o \text{ in } \left]-\frac{1}{2},\frac{1}{2}\right[\times\left(\left]-\frac{1}{2},-\frac{1}{2}+\frac{1}{n}\right[\cup\left]\frac{1}{2}-\frac{1}{n},\frac{1}{2}\right[\right) \quad \forall n\in\mathbb{N}.
\end{cases}
$$

Finally, (3.4.25) is achieved by setting $\theta_n^a = (\zeta_n^a, \eta_n^a)$, $\theta_n^b = \left(\eta_n^b, \zeta_n^e + \zeta_n^0\right)$. □

Proof of Theorem 3.3.2 In what follows, $p_{n,i}^a$ (resp. $p_{n,i}^b$) denotes the ith-component, $i = 1,2,3$, of p_n^a, (resp. p_n^b).

Corollary 3.4.2 asserts that there exist a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence on the subsequence) $\left(\hat{p}^a,\hat{p}^b\right) = \left(\left(\hat{p}_2^a,\hat{p}_3^a\right),\left(\hat{p}_1^b,\hat{p}_2^b\right)\right) \in P$ satisfying (3.4.5).

The next step is devoted to proving the existence of a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence on the subsequence) of

$$
z^a \in L^2\left(H_0^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\right),\left]-\frac{1}{2},\frac{1}{2}\right[\times\left]0,1\right[\right)\times\left(L^2\left(H_m^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\right),\left]-\frac{1}{2},\frac{1}{2}\right[\times\left]0,1\right[\right)\right)^2
$$

$$z^b \in \left(L^2\left(\left]-\tfrac{1}{2},\tfrac{1}{2}\right[^2, H^1_m\left(]-1,0[\right)\right)\right)^2 \times L^2\left(\left]-\tfrac{1}{2},\tfrac{1}{2}\right[^2, H^1\left(]-1,0[\right)\right)$$

with $z^b(\cdot,\cdot,-1)=0$ in $\left]-\tfrac{1}{2},\tfrac{1}{2}\right[^2$ such that such that

$$\frac{1}{h^a_n}\frac{\partial p^a_n}{\partial x_1} \rightharpoonup \frac{\partial z^a}{\partial x_1} \text{ weakly in } \left(L^2(\Omega^a)\right)^3, \quad \frac{1}{h^b_n}\frac{\partial p^b_n}{\partial x_3} \rightharpoonup \frac{\partial z^b}{\partial x_3} \text{ weakly in } \left(L^2(\Omega^b)\right)^3. \tag{3.4.26}$$

Indeed, the boundary condition on p^a_n and the Poincaré inequality give

$$\begin{cases} \left\| \dfrac{1}{h^a_n} p^a_{n,1}(\cdot,x_2,x_3) \right\|_{H^1_0\left(]-\frac{1}{2},\frac{1}{2}[\right)} \leq \dfrac{c}{h^a_n} \left\| \dfrac{\partial p^a_{n,1}(\cdot,x_2,x_3)}{\partial x_1} \right\|_{L^2\left(]-\frac{1}{2},\frac{1}{2}[\right)} \\ \\ (x_2,x_3) \text{ a.e. in } \left]-\tfrac{1}{2},\tfrac{1}{2}\right[\times]0,1[, \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.4.27}$$

where c is the Poincaré constant in $H^1_0\left(]-\frac{1}{2},\frac{1}{2}[\right)$, while, for $i=2,3$, the Poincaré-Wirtinger inequality gives

$$\begin{cases} \left\| \dfrac{1}{h^a_n} \left(p^a_{n,i}(\cdot,x_2,x_3) - \displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} p^a_{n,i}(x_1,x_2,x_3)\,dx_1 \right) \right\|_{H^1_m\left(]-\frac{1}{2},\frac{1}{2}[\right)} \\ \\ \leq \dfrac{c'}{h^a_n} \left\| \dfrac{\partial p^a_{n,i}(\cdot,x_2,x_3)}{\partial x_1} \right\|_{L^2\left(]-\frac{1}{2},\frac{1}{2}[\right)} \quad (x_2,x_3) \text{ a.e. in } \left]-\tfrac{1}{2},\tfrac{1}{2}\right[\times]0,1[, \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.4.28}$$

where the subscript ``m" means zero average, and c' is the Poincaré-Wirtinger constant in $H^1_m\left(]-\frac{1}{2},\frac{1}{2}[\right)$. Integrating (3.4.27) and (3.4.28) over $(x_2,x_3) \in \left]-\tfrac{1}{2},\tfrac{1}{2}\right[\times]0,1[$ and using the first estimate in (3.4.2) imply the first convergence in (3.4.26). Similarly, one proves the second convergence in (3.4.26). More precisely,

$$z^b_3 \in L^2\left(\left]-\tfrac{1}{2},\tfrac{1}{2}\right[^2, H^1_0\left(]-1,0[\right)\right),$$

since $p^b_{n,3}(\cdot,\cdot,0)=0$ on $\left]-\tfrac{1}{2},\tfrac{1}{2}\right[^2 \setminus \left(\left]-\frac{h^a_n}{2},\frac{h^a_n}{2}\right[\times \left]-\tfrac{1}{2},\tfrac{1}{2}\right[\right)$, for every $n \in \mathbb{N}$.

The next step is devoted to identifying $(\hat{p}^a_2,\hat{p}^a_3)$, $(\hat{p}^b_1,\hat{p}^b_2)$, z^a, and z^b. To this aim, for every $(q^a,q^b) = \left((q^a_2,q^a_3),(q^b_1,q^b_2)\right) \in P_{\text{reg}}$, where P_{reg} is defined in (3.4.24), consider a sequence

$\{v_n\}_{n\in\mathbb{N}} \subset \left(H^1\left(\Omega^a\right)\right)^3$ (depending on (q^a, q^b)) such that

$$\begin{cases} (v_n, (q_1^b, q_2^b, 0)) \in P_n \quad \forall n \in \mathbb{N}, \\ v_n \to (0, q_2^a, q_3^a) \text{ strongly in } \left(L^4\left(\Omega^a\right)\right)^3, \\ \left(\dfrac{1}{h_n^a} D_{x_1} v_n, D_{x_2} v_n, D_{x_3} v_n\right) \to \left(0, \left(0, D_{x_2} q^a\right), \left(0, D_{x_3} q^a\right)\right) \text{ strongly in } \left(L^2\left(\Omega^a\right)\right)^9. \end{cases} \tag{3.4.29}$$

For instance, setting

$$v_n(x) = \begin{cases} \left(0, q_2^a, q_3^a\right)(x_2, x_3), & \text{if } x = (x_1, x_2, x_3) \in \left]-\frac{1}{2}, \frac{1}{2}\right[^2 \times \left]h_n^a, 1\right[, \\ \left(q_1^b\left(h_n^a x_1, x_2\right)\dfrac{h_n^a - x_3}{h_n^a}, q_2^a\left(x_2, h_n^a\right)\dfrac{x_3}{h_n^a} + q_2^b\left(h_n^a x_1, x_2\right)\dfrac{h_n^a - x_3}{h_n^a}, q_3^a\left(x_2, x_3\right)\right), & \\ & \text{if } x = (x_1, x_2, x_3) \in \left]-\frac{1}{2}, \frac{1}{2}\right[^2 \times [0, h_n^a], \end{cases}$$

the first two proprieties in (3.4.29) can be immediately verified by virtue of the properties of P_{reg}, while the last one follows from

$$\int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2 \times]0, h_n^a[} \left|\frac{1}{h_n^a} D_{x_1} v_n\right|^2 dx \le \left(\left\|q_1^b\right\|^2_{W^{1,\infty}\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right)} + \left\|q_2^b\right\|^2_{W^{1,\infty}\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right)}\right) h_n^a \quad \forall n \in \mathbb{N},$$

$$\begin{aligned} \int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2 \times]0, h_n^a[} \left|D_{x_2} v_n\right|^2 dx \le & \left(\left\|q_1^b\right\|^2_{W^{1,\infty}\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right)} + 2\left\|q_2^a\right\|^2_{W^{1,\infty}\left(\left]-\frac{1}{2},\frac{1}{2}\right[\times]0,1[\right)} \right. \\ & \left. + 2\left\|q_2^b\right\|^2_{W^{1,\infty}\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right)} + \left\|q_3^a\right\|^2_{W^{1,\infty}\left(\left]-\frac{1}{2},\frac{1}{2}\right[\times]0,1[\right)} \right) h_n^a \quad \forall n \in \mathbb{N}, \end{aligned}$$

and

$$\int_{]-\frac{1}{2},\frac{1}{2}[^2\times]0,h_n^a[} |D_{x_3} v_n|^2 \, dx \leq \int_{]-\frac{1}{2},\frac{1}{2}[^2} \frac{1}{h_n^a} \left| q_1^b \left(h_n^a x_1, x_2 \right) \right|^2 dx_1 dx_2$$

$$+ \int_{]-\frac{1}{2},\frac{1}{2}[^2} \frac{1}{h_n^a} \left| q_2^a \left(x_2, h_n^a \right) - q_2^b \left(h_n^a x_1, x_2 \right) \right|^2 dx_1 dx_2 + \| q_3^a \|^2_{W^{1,\infty}(]-\frac{1}{2},\frac{1}{2}[\times]0,1[)} \, h_n^a$$

$$= \int_{]-\frac{1}{2},\frac{1}{2}[^2} \frac{1}{h_n^a} \left| q_1^b \left(h_n^a x_1, x_2 \right) - q_1^b \left(0, x_2 \right) \right|^2 dx_1 dx_2$$

$$+ \int_{]-\frac{1}{2},\frac{1}{2}[^2} \frac{1}{h_n^a} \left| q_2^a \left(x_2, h_n^a \right) - q_2^a \left(x_2, 0 \right) + q_2^b \left(0, x_2 \right) - q_2^b \left(h_n^a x_1, x_2 \right) \right|^2 dx_1 dx_2$$

$$+ \| q_3^a \|^2_{W^{1,\infty}(]-\frac{1}{2},\frac{1}{2}[\times]0,1[)} \, h_n^a \leq \Big(\left\| q_1^b \right\|^2_{W^{1,\infty}\left(]-\frac{1}{2},\frac{1}{2}[^2\right)} + 2 \left\| q_2^a \right\|^2_{W^{1,\infty}(]-\frac{1}{2},\frac{1}{2}[\times]0,1[)}$$

$$+2 \left\| q_2^b \right\|^2_{W^{1,\infty}\left(]-\frac{1}{2},\frac{1}{2}[^2\right)} + \| q_3^a \|^2_{W^{1,\infty}(]-\frac{1}{2},\frac{1}{2}[\times]0,1[)} \Big) h_n^a \quad \forall n \in \mathbb{N},$$

where again the properties of P_{reg} played a crucial role.

Now, fixing $\left(q^a, q^b\right) = \left(\left(q_2^a, q_3^a\right), \left(q_1^b, q_2^b\right)\right) \in P_{\text{reg}}$, and choosing $\left(v_n, \left(q_1^b, q_2^b, 0\right)\right)$ as test function in (3.2.8) with $\{v_n\}_{n\in\mathbb{N}}$ satisfying (3.4.29) give

$$\frac{1}{h_n^a} E_n \left(\left(p_n^a, p_n^b \right) \right) \leq \frac{1}{h_n^a} E_n \left(\left(v_n, \left(q_1^b, q_2^b, 0 \right) \right) \right) \quad \forall n \in \mathbb{N}. \tag{3.4.30}$$

Then, passing to the limit in (3.4.30), as n diverges, and using (3.1.8) with $\ell \in]0,+\infty[$, (3.2.9),

(3.4.5), (3.4.26), (3.4.29), Proposition 3.4.4, and a l.s.c. argument imply

$$
\begin{cases}
\beta \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,1[} \left|\mathrm{rot}\hat{p}^a\right|^2 dx_2dx_3 \\
\\
+ \displaystyle\int_{\Omega^a} \left(\beta \left(\left| \frac{\partial z_3^a}{\partial x_1} \right|^2 + \left| \frac{\partial z_2^a}{\partial x_1} \right|^2 \right) + \left| \mathrm{div}\,(\hat{p}^a) + \frac{\partial z_1^a}{\partial x_1} \right|^2 \right) dx \\
\\
+ \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,1[} \left(\alpha \left(|\hat{p}^a|^2 - 1 \right)^2 + \left| D\psi^a_{(\hat{p}^a,\hat{p}^b)} \right|^2 + \int_{-\frac{1}{2}}^{\frac{1}{2}} (f_2^a, f_3^a) dx_1 \cdot \hat{p}^a \right) dx_2dx_3 \\
\\
+ \ell \Bigg(\beta \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[^2} \left|\mathrm{rot}\hat{p}^b\right|^2 dx_1dx_2 \\
\\
+ \displaystyle\int_{\Omega^b} \left(\beta \left(\left| \frac{\partial z_2^b}{\partial x_3} \right|^2 + \left| \frac{\partial z_1^b}{\partial x_3} \right|^2 \right) + \left| \mathrm{div}\,(\hat{p}^b) + \frac{\partial z_3^b}{\partial x_3} \right|^2 \right) dx \\
\\
+ \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[^2} \left(\alpha \left(\left|\hat{p}^b\right|^2 - 1 \right)^2 + \left| D\psi^b_{(\hat{p}^a,\hat{p}^b)} \right|^2 + \int_{-1}^{0} \left(f_1^b, f_2^b \right) dx_3 \cdot \hat{p}^b \right) dx_1dx_2 \Bigg) \\
\\
\leq \liminf_n \dfrac{E_n\left(\left(p_n^a, p_n^b\right)\right)}{h_n^a} \leq \limsup_n \dfrac{E_n\left(\left(p_n^a, p_n^b\right)\right)}{h_n^a} \leq \lim_n \dfrac{E_n\left(\left(v_n, \left(q_1^b, q_2^b, 0\right)\right)\right)}{h_n^a} \\
\\
= E\left(\left(q^a, q^b\right)\right).
\end{cases}
\tag{3.4.31}
$$

On the other hand, the properties of $\hat{p}^a$, z_1^a, $\hat{p}^b$, and z_3^b give

$$\begin{cases} \displaystyle\int_{\Omega^a} \left| \operatorname{div}\left(\hat{p}^a\right) + \frac{\partial z_1^a}{\partial x_1} \right|^2 dx = \int_{]-\frac{1}{2},\frac{1}{2}[\times]0,1[} \left|\operatorname{div}\hat{p}^a\right|^2 dx_2 dx_3 + \int_{\Omega^a} \left| \frac{\partial z_1^a}{\partial x_1} \right|^2 dx, \\ \displaystyle\int_{\Omega^b} \left| \operatorname{div}\left(\hat{p}^b\right) + \frac{\partial z_3^b}{\partial x_3} \right|^2 dx = \int_{]-\frac{1}{2},\frac{1}{2}[^2} \left|\operatorname{div}\hat{p}^b\right|^2 dx_1 dx_2 + \int_{\Omega^b} \left| \frac{\partial z_3^b}{\partial x_3} \right|^2 dx. \end{cases} \tag{3.4.32}$$

Hence, inserting (3.4.32) in (3.4.31) and taking into account Proposition 3.4.5, one has

$$\begin{cases} \min\{1,\beta,\ell,\beta\ell\} \left(\displaystyle\int_{\Omega^a} \left| \frac{\partial z^a}{\partial x_1} \right|^2 dx + \int_{\Omega^b} \left| \frac{\partial z^b}{\partial x_3} \right|^2 dx \right) + E\left(\left(\hat{p}^a,\hat{p}^b\right)\right) \\ \leq \liminf_n \dfrac{E_n\left(\left(p_n^a,p_n^b\right)\right)}{h_n^a} \leq \limsup_n \dfrac{E_n\left(\left(p_n^a,p_n^b\right)\right)}{h_n^a} \leq E\left(\left(q^a,q^b\right)\right) \quad \forall \left(q^a,q^b\right) \in P, \end{cases} \tag{3.4.33}$$

from which, thanks to the properties of z^a and z^b,

$$z^a = 0 \text{ in } \Omega^a \text{ and } z^b = 0 \text{ in } \Omega^b \tag{3.4.34}$$

follow. Consequently, inserting (3.4.34) in (3.4.33), one obtains that $\left(\hat{p}^a,\hat{p}^b\right)$ solves (3.3.8) and convergence (3.3.9) holds true. Note that convergences in (3.3.9) and in (3.4.26) hold true for the whole sequence since the limits are uniquely identified. Moreover, (3.3.7) follows from (3.4.5) and Proposition 3.4.4. Now, it remains to prove that convergences in (3.4.5) and (3.4.26) are strong. To this aim, (3.3.9), (3.1.8) with $\ell \in]0,\infty[$, (3.2.9), (3.3.7), and (3.4.5) imply

$$\lim_n \left(\int_{\Omega^a} \left(\beta \left|\operatorname{rot}_n^a p_n^a\right|^2 + \left|\operatorname{div}_n^a p_n^a\right|^2\right) dx + \frac{h_n^b}{h_n^a} \int_{\Omega^b} \left(\beta \left|\operatorname{rot}_n^b p_n^b\right|^2 + \left|\operatorname{div}_n^b p_n^b\right|^2\right) dx \right)$$

$$= \int_{\Omega^a} \left(\beta \left|\operatorname{rot}\left(0,\hat{p}_2^a,\hat{p}_3^a\right)\right|^2 + \left|\operatorname{div}\left(0,\hat{p}_2^a,\hat{p}_3^a\right)\right|^2\right) dx + \ell \int_{\Omega^b} \left(\beta \left|\operatorname{rot}\left(\hat{p}_1^b,\hat{p}_2^b,0\right)\right|^2 + \left|\operatorname{div}\left(\hat{p}_1^b,\hat{p}_2^b,0\right)\right|^2\right) dx,$$

from which, using (3.1.8) with $\ell \in]0,\infty[$, (3.4.5), (3.4.26), and (3.4.34), one deduces

$$\begin{cases} \left(\mathrm{rot}_n^a p_n^a, \mathrm{rot}_n^b p_n^b\right) \to \left(\mathrm{rot}\left(0,\hat{p}_2^a,\hat{p}_3^a\right), \mathrm{rot}\left(\hat{p}_1^b,\hat{p}_1^b,0\right)\right) \text{ strongly in } \left(L^2\left(\Omega^a\right)\right)^3 \times \left(L^2\left(\Omega^b\right)\right)^3, \\ \left(\mathrm{div}_n^a p_n^a, \mathrm{div}_n^b p_n^b\right) \to \left(\mathrm{div}\left(0,\hat{p}_2^a,\hat{p}_3^a\right), \mathrm{div}\left(\hat{p}_1^b,\hat{p}_1^b,0\right)\right) \text{ strongly in } L^2\left(\Omega^a\right) \times L^2\left(\Omega^b\right). \end{cases}$$

Consequently, recalling (3.2.10) and (3.1.8) with $\ell \in]0,\infty[$, one has

$$\begin{cases} \|D_n^a p_n^a\|^2_{(L^2(\Omega^a))^9} + \dfrac{h_n^b}{h_n^a}\left\|D_n^b p_n^b\right\|^2_{(L^2(\Omega^b))^9} \\ = \|\mathrm{rot}_n^a p_n^a\|^2_{(L^2(\Omega^a))^3} + \|\mathrm{div}_n^a p_n^a\|^2_{L^2(\Omega^a)} + \dfrac{h_n^b}{h_n^a}\left\|\mathrm{rot}_n^b p_n^b\right\|^2_{(L^2(\Omega^b))^3} + \dfrac{h_n^b}{h_n^a}\left\|\mathrm{div}_n^b p_n^b\right\|^2_{L^2(\Omega^b)} \\ \longrightarrow \|\mathrm{rot}\left(0,\hat{p}_2^a,\hat{p}_3^a\right)\|^2_{(L^2(\Omega^a))^3} + \|\mathrm{div}\left(0,\hat{p}_2^a,\hat{p}_3^a\right)\|^2_{L^2(\Omega^a)} \\ +\ell\left\|\mathrm{rot}\left(\hat{p}_1^b,\hat{p}_2^b,0\right)\right\|^2_{(L^2(\Omega^b))^3} + \ell\left\|\mathrm{div}\left(\hat{p}_1^b,\hat{p}_2^b,0\right)\right\|^2_{L^2(\Omega^b)} \\ = \|D\left(0,\hat{p}_2^a,\hat{p}_3^a\right)\|^2_{(L^2(\Omega^a))^9} + \ell\left\|D\left(\hat{p}_1^b,\hat{p}_2^b,0\right)\right\|^2_{(L^2(\Omega^b))^9}, \text{ as } n \to +\infty, \end{cases} \tag{3.4.35}$$

where the boundary conditions of $\left(\hat{p}^a,\hat{p}^b\right)$ play a crucial role in the last equality. Finally, combining (3.4.35) with (3.4.5), (3.4.26), (3.4.34) and, (3.1.8) with $\ell \in]0,\infty[$ again, one obtains that convergences in (3.4.5) and (3.4.26) are strong.

3.5 The proofs in the case $\ell = 0$

This section is devoted to proving Theorem 3.3.4.

3.5.1 *A priori* estimates on the polarization

The same arguments used to proving Proposition 3.4.1 give

Proposition 3.5.1. *Assume* (3.1.8) *with* $\ell = 0$, *and* (3.2.9). *For every* $n \in \mathbb{N}$, *let* $\left(p_n^a, p_n^b\right)$ *be a solution of* (3.2.8). *Then, there exists a constant* c *such that*

$$\|p_n^a\|_{(L^4(\Omega^a))^3} \leq c, \quad \left\| \left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{4}} p_n^b \right\|_{(L^4(\Omega^b))^3} \leq c \quad \forall n \in \mathbb{N}, \tag{3.5.1}$$

and

$$\|D_n^a p_n^a\|_{(L^2(\Omega^a))^9} \leq c, \quad \left\| \left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} D_n^b p_n^b \right\|_{(L^2(\Omega^b))^9} \leq c \quad \forall n \in \mathbb{N}. \tag{3.5.2}$$

Corollary 3.5.2. *Assume* (3.1.8) *with* $\ell = 0$, *and* (3.2.9). *For every* $n \in \mathbb{N}$, *let* $\left(p_n^a, p_n^b\right)$ *be a solution of* (3.2.8). *Let* $P_\circ$ *be defined in* (3.3.10). *Then,*

$$\left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} p_n^b \rightharpoonup 0 \text{ weakly in } \left(H^1(\Omega^b)\right)^3 \text{ and strongly in } \left(L^4\left(\Omega^b\right)\right)^3. \tag{3.5.3}$$

Moreover, there exist an increasing sequence of positive integer numbers, still denoted by $\{n\}$, *and (in possible dependence on the subsequence)* $\hat{p}^a = (\hat{p}_2^a, \hat{p}_3^a) \in P_\circ$ *such that*

$$p_n^a \rightharpoonup (0, \hat{p}_2^a, \hat{p}_3^a) \text{ weakly in } \left(H^1(\Omega^a)\right)^3 \text{ and strongly in } \left(L^4(\Omega^a)\right)^3. \tag{3.5.4}$$

Proof. Proposition 3.5.1 implies (3.5.3) and the existence of a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence on the subsequence) of $(\hat{p}_1^a, \hat{p}_2^a, \hat{p}_3^a) \in \left(H^1\left(\Omega^a\right)\right)^3$ independent of x_1, such that

$$(\hat{p}_1^a, \hat{p}_2^a, \hat{p}_3^a) \cdot \nu^a = 0 \text{ on } \partial\Omega^a \backslash \left(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[^2 \times \{0\} \right)$$

and

$$p_n^a \rightharpoonup (\hat{p}_1^a, \hat{p}_2^a, \hat{p}_3^a) \text{ weakly in } \left(H^1(\Omega^a)\right)^3 \text{ and strongly in } \left(L^4(\Omega^a)\right)^3. \tag{3.5.5}$$

In particular, this ensures

$$\hat{p}_1^a = 0 \text{ in } \Omega^a.$$

To complete the proof, it remains to verify

$$\hat{p}_3^a(\cdot, 0) = 0 \text{ in } \left]-\tfrac{1}{2}, \tfrac{1}{2}\right[. \tag{3.5.6}$$

The trace of $p^b_{n,3}$ vanishing on $\left]-\frac{1}{2},\frac{1}{2}\right[^2\times\{-1\}$ implies

$$\left|\frac{1}{(h^a_n)^{\frac{1}{2}}}p^b_{n,3}\left(x_1,x_2,0\right)\right|^2\leq\int_{-1}^{0}\left|\frac{1}{(h^a_n)^{\frac{1}{2}}}\partial_t p^b_{n,3}\left(x_1,x_2,t\right)\right|^2dt \text{ in } \left]-\tfrac{1}{2},\tfrac{1}{2}\right[^2,\quad \forall n\in\mathbb{N}.$$

Integrating this inequality over $\left]-\frac{1}{2},\frac{1}{2}\right[^2$ and using the second estimate in (3.5.2) gives

$$\frac{1}{(h^a_n)^{\frac{1}{2}}}p^b_{n,3}(\cdot,\cdot,0)\to 0 \text{ strongly in } L^2\left(\left]-\tfrac{1}{2},\tfrac{1}{2}\right[^2\right). \tag{3.5.7}$$

Finally, (3.5.6) follows from (3.5.5), the junction condition in (3.2.3), and (3.5.7). Indeed,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\hat{p}^a_3(x_2,0)\right|^2dx_2=\lim_n\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|p^a_{n,3}(x_1,x_2,0)\right|^2dx_1dx_2$$

$$=\lim_n\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|p^b_{n,3}\left(h^a_nx_1,x_2,0\right)\right|^2dx_1dx_2=\lim_n\int_{-\frac{h^a_n}{2}}^{\frac{h^a_n}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{1}{(h^a_n)^{\frac{1}{2}}}p^b_{n,3}(x_1,x_2,0)\right|^2dx_1dx_2=0.$$

□

3.5.2 A convergence result for problem (3.2.6)

Proposition 3.5.3. *Assume (3.1.8) with $\ell=0$. Let $\{(q^a_n,q^b_n)\}_{n\in\mathbb{N}}\subset\left(L^2\left(\Omega^a\right)\right)^3\times\left(L^2\left(\Omega^b\right)\right)^3$ and $q^a=(q^a_2,q^a_3)\in(L^2(\Omega^a))^2$ be such that q^a is independent of x_1, and*

$$\left(q^a_n,\left(\frac{h^b_n}{h^a_n}\right)^{\frac{1}{2}}q^b_n\right)\to\left((0,q^a),0\right) \text{ strongly in } (L^2(\Omega^a))^3\times\left(L^2\left(\Omega^b\right)\right)^3. \tag{3.5.8}$$

Moreover, for $n \in \mathbb{N}$ *let* $\left(\phi^a_{(q^a_n,q^b_n)}, \phi^b_{(q^a_n,q^b_n)}\right)$ *be the unique solution of*

$$\begin{cases} \left(\phi^a_{(q^a_n,q^b_n)}, \phi^b_{(q^a_n,q^b_n)}\right) \in U_n, \quad \displaystyle\int_{\Omega^a} \phi^a_{(q^a_n,q^b_n)} dx = 0, \\ \\ h^a_n \displaystyle\int_{\Omega^a} ((-D^a_n \phi^a_{(q^a_n,q^b_n)} + q^a_n) \cdot D^a_n \phi^a) dx \\ \\ +h^b_n \displaystyle\int_{\Omega^b} \left(\left(-D^b_n \phi^b_{(q^a_n,q^b_n)} + q^b_n\right) \cdot D^b_n \phi^b\right) dx = 0 \quad \forall\, (\phi^a, \phi^b) \in U_n, \end{cases} \tag{3.5.9}$$

where U_n *is defined in* (3.2.4). *Furthermore, let* $\psi^a_{q^a}$ *be the unique solution of* (3.3.12). *Then, one has*

$$\left(\phi^a_{(q^a_n,q^b_n)}, \left(\frac{h^b_n}{h^a_n}\right)^{\frac{1}{2}} \phi^b_{(q^a_n,q^b_n)}\right) \to \left(\psi^a_{q^a}, 0\right) \text{ strongly in } H^1(\Omega^a) \times H^1\left(\Omega^b\right), \tag{3.5.10}$$

$$\left(\frac{1}{h^a_n} \frac{\partial \phi^a_{(q^a_n,q^b_n)}}{\partial x_1}, \frac{1}{h^b_n} \frac{\partial \left(\left(\frac{h^b_n}{h^a_n}\right)^{\frac{1}{2}} \phi^b_{(q^a_n,q^b_n)}\right)}{\partial x_3}\right) \to (0,0) \text{ strongly in } L^2(\Omega^a) \times L^2\left(\Omega^b\right). \tag{3.5.11}$$

Proof. Arguing as in the proof of Proposition 3.4.4, one can prove the existence of a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence on the subsequence) of $\left(\tau^a, \tau^b\right) \in H^1\left(\Omega^a\right) \times H^1\left(\Omega^b\right)$ with τ^a independent of x_1 and τ^b independent of x_3, and of $\left(\xi^a, \xi^b\right) \in L^2(\Omega^a) \times L^2\left(\Omega^b\right)$ such that

$$\left(\phi^a_{(q^a_n,q^b_n)}, \left(\frac{h^b_n}{h^a_n}\right)^{\frac{1}{2}} \phi^b_{(q^a_n,q^b_n)}\right) \rightharpoonup (\tau^a, \tau^b) \text{ weakly in } H^1(\Omega^a) \times H^1(\Omega^b), \tag{3.5.12}$$

$$\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,1[} \tau^a dx_2 dx_3 = 0, \tag{3.5.13}$$

$$\left(\frac{1}{h^a_n} \frac{\partial \phi^a_{(q^a_n,q^b_n)}}{\partial x_1}, \frac{1}{h^b_n} \frac{\partial \left(\left(\frac{h^b_n}{h^a_n}\right)^{\frac{1}{2}} \phi^b_{(q^a_n,q^b_n)}\right)}{\partial x_3}\right) \rightharpoonup (\xi^a, \xi^b) \text{ weakly in } L^2(\Omega^a) \times L^2\left(\Omega^b\right). \tag{3.5.14}$$

The next step is devoted to proving

$$\tau^b(0,\cdot)=0 \text{ in } \left]-\tfrac{1}{2},\tfrac{1}{2}\right[. \tag{3.5.15}$$

Indeed, the junction condition in (3.2.4) gives

$$\begin{cases} \displaystyle\int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2}\left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}}\phi^a_{(q_n^a,q_n^b)}(x_1,x_2,0)\varphi(x_2)dx_1dx_2 \\ \\ \displaystyle=\int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2}\left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}}\phi^b_{(q_n^a,q_n^b)}(h_n^a x_1,x_2,0)\varphi(x_2)dx_1dx_2 \quad \forall\varphi\in C_0^\infty\left(\left]-\frac{1}{2},\frac{1}{2}\right[\right). \end{cases} \tag{3.5.16}$$

Moreover, (3.1.8) with $\ell=0$ and the convergence of the first term in (3.5.12) imply

$$\left(\tfrac{h_n^b}{h_n^a}\right)^{\frac{1}{2}}\phi^a_{(q_n^a,q_n^b)}(\cdot,\cdot,0)\to 0 \text{ strongly in } L^2\left(\left]-\tfrac{1}{2},\tfrac{1}{2}\right[^2\right), \tag{3.5.17}$$

while, using the second convergences in (3.5.12) and (3.5.14), again (3.1.8) with $\ell=0$, and arguing as in the proof of Proposition 5.4 in [34], one can prove

$$\begin{cases} \displaystyle\lim_n\int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2}\left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}}\phi^b_{(q_n^a,q_n^b)}(h_n^a x_1,x_2,0)\varphi(x_2)dx_1dx_2=\int_{-\frac{1}{2}}^{\frac{1}{2}}\tau^b(x_2,0)\varphi(x_2)dx_2 \\ \\ \forall\varphi\in C_0^\infty\left(\left]-\tfrac{1}{2},\tfrac{1}{2}\right[\right). \end{cases} \tag{3.5.18}$$

Then, (3.5.15) follows from (3.5.16), (3.5.17), and (3.5.18).

To identity $\tau^a=\psi^a_{q^a}$, it is enough to pass to the limit, as n diverges, in the equation in (3.5.9) with a test functions (ϕ^a,ϕ^b) such that

$$\phi^a(x)=\frac{1}{h_n^a}\psi^a(x_2,x_3) \text{ if } x=(x_1,x_2,x_3)\in\Omega^a, \quad \phi^b(x)=\frac{1}{h_n^a}\psi^a(x_2,0) \text{ if } x=(x_1,x_2,x_3)\in\Omega^b,$$

with $\psi^a\in C^\infty\left(\left[-\tfrac{1}{2},\tfrac{1}{2}\right]\times[0,1]\right)$, to use (3.1.8) with $\ell=0$, previous convergences, the density of $C^\infty\left(\left[-\tfrac{1}{2},\tfrac{1}{2}\right]\times[0,1]\right)$ in $H^1\left(\left]-\tfrac{1}{2},\tfrac{1}{2}\right[\times]0,1[\right)$, and (3.5.13).

To identify $\tau^b=0$, it is enough to pass to the limit, as n diverges, in the equation in (3.5.9) with

test functions (ϕ^a, ϕ^b) such that

$$\phi^a(x) = 0 \text{ if } x = (x_1, x_2, x_3) \in \Omega^a, \quad \phi^b(x) = \left(\frac{1}{h_n^a h_n^b}\right)^{\frac{1}{2}} \psi^b(x_1, x_2) \text{ if } x = (x_1, x_2, x_3) \in \Omega^b$$

with

$$\psi^b \in A = \left\{ v \in C^\infty\left(\left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^2\right) : v = 0 \text{ in }]-\delta, \delta[\times]-\tfrac{1}{2}, \tfrac{1}{2}[\text{, for some } \delta > 0 \text{ (depending on } v) \right\},$$

to use previous convergences, the density of A in the space of functions in $H^1\left(]-\frac{1}{2}, \frac{1}{2}[^2\right)$ with zero trace on $\{0\} \times]-\frac{1}{2}, \frac{1}{2}[$, and to take into account (3.5.15).

To identify ξ^a, ξ^b and to prove that all the previous convergences are strong one can argue as in the last part of the proof of Proposition 3.4.4. □

3.5.3 Proof of Theorem 3.3.4

Proof. Corollary 3.5.2 asserts that (3.5.3) holds true and that there exist a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence on the subsequence) of $\hat{p}^a = (\hat{p}_2^a, \hat{p}_3^a) \in P_\circ$ satisfying (3.5.4). Moreover, one can prove the existence of a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence on the subsequence) of

$$z^a \in L^2\left(H_0^1\left(]-\tfrac{1}{2}, \tfrac{1}{2}[\right),]-\tfrac{1}{2}, \tfrac{1}{2}[\times]0,1[\right) \times \left(L^2\left(H_m^1\left(]-\tfrac{1}{2}, \tfrac{1}{2}[\right),]-\tfrac{1}{2}, \tfrac{1}{2}[\times]0,1[\right)\right)^2$$

such that (the proof of Theorem 3.3.2)

$$\left(\frac{1}{h_n^a}\frac{\partial p_n^a}{\partial x_1}, \frac{1}{h_n^b}\frac{\partial\left(\left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} p_n^b\right)}{\partial x_3}\right) \rightharpoonup \left(\frac{\partial z^a}{\partial x_1}, \zeta\right) \text{ weakly in } \left(L^2(\Omega^a)\right)^3 \times \left(L^2(\Omega^b)\right)^3. \tag{3.5.19}$$

The next step is to identify $(\hat{p}_2^a, \hat{p}_3^a)$, z^a, and ζ. Let

$$P_\circ^{\text{reg}} = \left\{ q^a = (q_2^a, q_3^a) \in \left(C^1\left(\left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \times [0,1]\right)\right)^2 : \text{ for some } \delta > 0 \text{ (depending on } q^a) \right.$$

$$\left. q_2^a = 0 \text{ in } \left(\left[-\tfrac{1}{2}, -\tfrac{1}{2} + \delta\right] \cup \left[\tfrac{1}{2} - \delta, \tfrac{1}{2}\right]\right) \times [0,1], \; q_3^a = 0 \text{ in } \left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \times ([0,\delta] \cup [1-\delta, 1]) \right\}.$$

Now, for $q^a = (q_2^a, q_3^a) \in P_\circ^{\rm reg}$, the couple $((0, q_2^a, q_3^a), (0, q_2^a(\cdot, 0), 0))$ belongs to P_n. Consequently,

$$\frac{1}{h_n^a} E_n\left(\left(p_n^a, p_n^b\right)\right) \leq \frac{1}{h_n^a} E_n\left(((0, q_2^a, q_3^a), (0, q_2^a(\cdot, 0), 0))\right) \quad \forall\, (q_2^a, q_3^a) \in P_\circ^{\rm reg}, \quad \forall n \in \mathbb{N}. \tag{3.5.20}$$

Then, passing to the limit in (3.5.20), as n diverges, and using (3.1.8) with $\ell = 0$, (3.2.9), (3.5.3), (3.5.4), (3.5.19), Proposition 3.5.3, and a l.s.c. argument imply

$$\beta \int_{]-\frac{1}{2},\frac{1}{2}[\times]0,1[} |{\rm rot}\hat{p}^a|^2\, dx_2 dx_3 + \int_{\Omega^a} \left(\beta \left(\left| \frac{\partial z_3^a}{\partial x_1} \right|^2 + \left| \frac{\partial z_2^a}{\partial x_1} \right|^2 \right) + \left| {\rm div}\,(\hat{p}^a) + \frac{\partial z_1^a}{\partial x_1} \right|^2 \right) dx$$

$$+ \int_{]-\frac{1}{2},\frac{1}{2}[\times]0,1[} \left(\alpha \left(|\hat{p}^a|^2 - 1 \right)^2 + \left| D\psi_{\hat{p}^a}^a \right|^2 + \int_{-\frac{1}{2}}^{\frac{1}{2}} (f_2^a, f_3^a) dx_1 \cdot \hat{p}^a \right) dx_2 dx_3$$

$$+ \beta \int_{\Omega^b} \left(|\zeta_2|^2 + |\zeta_1|^2 \right) dx + \int_{\Omega^b} |\zeta_3|^2\, dx \leq \liminf_n \frac{E_n\left(\left(p_n^a, p_n^b\right)\right)}{h_n^a} \leq \limsup_n \frac{E_n\left(\left(p_n^a, p_n^b\right)\right)}{h_n^a}$$

$$\leq \lim_n \frac{E_n\left(((0, q_2^a, q_3^a), (0, q_2^a(\cdot, 0), 0))\right)}{h_n^a} = E_\circ\left(q^a\right) \quad \forall q^a = (q_2^a, q_3^a) \in P_\circ^{\rm reg}.$$

This inequalities hold true also for any $q^a \in P_\circ$, since $P_\circ^{\rm reg}$ is dense in $P_\circ$. One can conclude the proof arguing as in the last part of the proof of Theorem 3.3.2. □

3.6 The proofs in the case $\ell = +\infty$

This section is devoted to proving Theorem 3.3.7.

3.6.1 *A priori* estimates on the polarization

The same arguments used to proving Proposition 3.4.1 give

Proposition 3.6.1. *Assume* (3.1.8) *with* $\ell = +\infty$, *and* (3.2.9). *For every* $n \in \mathbb{N}$, *let* $\left(p_n^a, p_n^b\right)$ *be a*

solution of (3.2.8). *Then, there exists a constant* c *such that*

$$\left\| \left(\frac{h_n^a}{h_n^b} \right)^{\frac{1}{4}} p_n^a \right\|_{(L^4(\Omega^a))^3} \leq c, \quad \left\| p_n^b \right\|_{(L^4(\Omega^b))^3} \leq c \quad \forall n \in \mathbb{N}, \tag{3.6.1}$$

and

$$\left\| \left(\frac{h_n^a}{h_n^b} \right)^{\frac{1}{2}} D_n^a p_n^a \right\|_{(L^2(\Omega^a))^9} \leq c, \quad \left\| D_n^b p_n^b \right\|_{(L^2(\Omega^b))^9} \leq c \quad \forall n \in \mathbb{N}. \tag{3.6.2}$$

Corollary 3.6.2. *Assume* (3.1.8) *with* $\ell = +\infty$, *and* (3.2.9). *For every* $n \in \mathbb{N}$, *let* $\left(p_n^a, p_n^b\right)$ *be a solution of* (3.2.8). *Let* P_∞ *be defined by* (3.3.19). *Then*

$$\left(\frac{h_n^a}{h_n^b} \right)^{\frac{1}{2}} p_n^a \rightharpoonup 0 \text{ weakly in } \left(H^1(\Omega^a)\right)^3 \text{ and strongly in } \left(L^4(\Omega^a)\right)^3. \tag{3.6.3}$$

Moreover, there exist an increasing sequence of positive integer numbers, still denoted by $\{n\}$, *and (in possible dependence on the subsequence)* $\hat{p}^b = \left(\hat{p}_1^b, \hat{p}_2^b\right) \in \left(H^1\left(\left]-\frac{1}{2}, \frac{1}{2}\right[^2\right)\right)^2$ *with* $p^b \cdot \nu^b = 0$ *on* $\partial\left(\left]-\frac{1}{2}, \frac{1}{2}\right[^2\right)$ *such that*

$$p_n^b \rightharpoonup \left(\hat{p}_1^b, \hat{p}_2^b, 0\right) \text{ weakly in } \left(H^1(\Omega^b)\right)^3 \text{ and strongly in } \left(L^4(\Omega^b)\right)^3. \tag{3.6.4}$$

Furthermore, if and $h_n^b << \sqrt{h_n^a}$, *then*

$$p_1^b(0, \cdot) = 0 \text{ in } \left]-\tfrac{1}{2}, \tfrac{1}{2}\right[. \tag{3.6.5}$$

Proof. Proposition 3.6.1 implies (3.6.3) and the existence of a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence on the subsequence) $\left(\hat{p}_1^b, \hat{p}_2^b, \hat{p}_3^b\right) \in \left(H^1\left(\Omega^b\right)\right)^3$ independent of x_3, such that

$$\left(\hat{p}_1^b, \hat{p}_2^b, \hat{p}_3^b\right) \cdot \nu^b = 0 \text{ on } \partial\Omega^b \backslash \left(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[^2 \times \{0\}\right)$$

and

$$p_n^b \rightharpoonup \left(\hat{p}_1^b, \hat{p}_2^b, \hat{p}_3^b\right) \text{ weakly in } \left(H^1(\Omega^b)\right)^3 \text{ and strongly in } \left(L^4\left(\Omega^b\right)\right)^3.$$

In particular, this ensures

$$\hat{p}_3^b = 0 \text{ a.e. in } \Omega^b.$$

Let us prove (3.6.5) under assumption $h_n^b << \sqrt{h_n^a}$.

In what follows c denotes a constant satisfying Proposition 3.6.1. Let $\varphi \in C_0^\infty\left(\left]-\frac{1}{2}, \frac{1}{2}\right[\right)$.

Convergence (3.6.4) and the independence of p^b of x_3 is imply

$$\lim_n \int_{]-\frac{1}{2},\frac{1}{2}[\times]-1,0[} p^b_{n,1}(0,x_2,x_3)\varphi(x_2)dx_2dx_3 = \int_{-\frac{1}{2}}^{\frac{1}{2}} p^b_1(0,x_2)\varphi(x_2)dx_2. \tag{3.6.6}$$

On the side, the transmission condition in the last line of (3.2.3) allows us to split the left-hand side of (3.6.6) as follows

$$\begin{cases} \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[\times]-1,0[} p^b_{n,1}(0,x_2,x_3)\varphi(x_2)dx_2dx_3 = \int_{\Omega^b} p^b_{n,1}(0,x_2,x_3)\varphi(x_2)dx = \\ \displaystyle\int_{\Omega^b} \left(p^b_{n,1}(0,x_2,x_3) - (p^b_{n,1}(h^a_n x_1,x_2,x_3)\right)\varphi(x_2)dx + \\ \displaystyle\int_{\Omega^b} \left(p^b_{n,1}(h^a_n x_1,x_2,x_3) - p^b_{n,1}(h^a_n x_1,x_2,0)\right)\varphi(x_2)dx + \\ \displaystyle\frac{1}{\alpha_n}\int_{\Omega^{a\star}_n} \left(p^a_{n,1}(x_1,x_2,0) - p^a_{n,1}(x_1,x_2,x_3)\right)\varphi(x_2)dx + \\ \displaystyle\frac{1}{\alpha_n}\int_{\Omega^{a\star}_n} p^a_{n,1}(x_1,x_2,x_3)\varphi(x_2)dx \quad \forall n\in\mathbb{N}, \end{cases} \tag{3.6.7}$$

where$\{\alpha_n\}_{n\in\mathbb{N}} \subset]0,1[$ is a vanishing sequence of positive numbers which will be defined later, and $\Omega^{a\star}_n = \left]-\frac{1}{2},\frac{1}{2}\right[\times \left]-\frac{1}{2},\frac{1}{2}\right[\times]0,\alpha_n[$.

As far as the second line and the third line in (3.6.7) are concerned, the Hölder inequality and the

second estimate in (3.6.2) get

$$\begin{cases} \left| \int_{\Omega^b} \left(p^b_{n,1}(0,x_2,x_3) - (p^b_{n,1}(h^a_n x_1, x_2, x_3) \right) \varphi(x_2) dx \right| \leq \\ \\ \|\varphi\|_{L^\infty(]-\frac{1}{2},\frac{1}{2}[)} \left(\int_{\Omega^b} \left| p^b_{n,1}(0,x_2,x_3) - (p^b_{n,1}(h^a_n x_1, x_2, x_3) \right|^2 dx \right)^{\frac{1}{2}} = \\ \\ \|\varphi\|_{L^\infty(]-\frac{1}{2},\frac{1}{2}[)} \left(\int_{\Omega^b} \left| \int_{h^a_n x_1}^{0} \partial_t p^b_{n,1}(t,x_2,x_3) dt \right|^2 dx \right)^{\frac{1}{2}} \leq \\ \\ \|\varphi\|_{L^\infty(]-\frac{1}{2},\frac{1}{2}[)} \sqrt{h^a_n} \left(\int_{\Omega^b} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |\partial_t p^b_{n,1}(t,x_2,x_3)|^2 dt \right) dx \right)^{\frac{1}{2}} = \\ \\ \|\varphi\|_{L^\infty(]-\frac{1}{2},\frac{1}{2}[)} \sqrt{h^a_n} \|\partial_{x_1} p^b_{n,1}\|_{L^2(\Omega^b)} \leq \|\varphi\|_{L^\infty(]-\frac{1}{2},\frac{1}{2}[)} \sqrt{h^a_n}\, c \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.6.8}$$

$$
\begin{cases}
\left| \int_{\Omega^b} \left(p^b_{n,1}(h^a_n x_1, x_2, x_3) - p^b_{n,1}(h^a_n x_1, x_2, 0) \right) \varphi(x_2) dx \right| \leq \\
\\
\|\varphi\|_{L^\infty(]-\frac{1}{2},\frac{1}{2}[)} \int_{\Omega^b} \left| \int_0^{x_3} \partial_t p^b_{n,1}(h^a_n x_1, x_2, t) dt \right| dx \leq \\
\\
\|\varphi\|_{L^\infty(]-\frac{1}{2},\frac{1}{2}[)} \int_{\Omega^b} \left(\int_{-1}^{0} \left| \partial_t p^b_{n,1}(h^a_n x_1, x_2, t) \right| dt \right) dx = \\
\\
\|\varphi\|_{L^\infty(]-\frac{1}{2},\frac{1}{2}[)} \frac{1}{h^a_n} \int_{]-\frac{h^a_n}{2},\frac{h^a_n}{2}[\times]-\frac{1}{2},\frac{1}{2}[\times]-1,0[} |\partial_{x_3} p^b_{n,1}(x)| dx \leq \\
\\
\|\varphi\|_{L^\infty(]-\frac{1}{2},\frac{1}{2}[)} \frac{1}{h^a_n} \sqrt{h^a_n} \|\partial_{x_3} p^b_{n,1}\|_{L^2(\Omega^b)} \leq \\
\\
\|\varphi\|_{L^\infty(]-\frac{1}{2},\frac{1}{2}[)} \frac{1}{\sqrt{h^a_n}} h^b_n c \quad \forall n \in \mathbb{N}.
\end{cases}
\tag{3.6.9}
$$

Note that the last line of (3.6.9) vanishes when $h^b_n << \sqrt{h^a_n}$.

As far as the fourth line and the fifth line in (3.6.7) are concerned, the Hölder inequality, the first

estimate in (3.6.2) and the boundary condition on $p^a_{n,1}$ get

$$\begin{cases} \dfrac{1}{\alpha_n}\left|\displaystyle\int_{\Omega_n^{a\star}} \left(p^a_{n,1}(x_1,x_2,0) - p^a_{n,1}(x_1,x_2,x_3)\right)\varphi(x_2)dx\right| \leq \\ \\ \|\varphi\|_{L^\infty(]-\frac{1}{2},\frac{1}{2}[)}\dfrac{1}{\alpha_n}\displaystyle\int_{\Omega_n^{a\star}}\left|\int_{x_3}^{0}\partial_t p^a_{n,1}(x_1,x_2,t)dt\right|dx \leq \\ \\ \|\varphi\|_{L^\infty(]-\frac{1}{2},\frac{1}{2}[)}\displaystyle\int_{\Omega_n^{a\star}}|\partial_{x_3}p^a_{n,1}(x)|dx \leq \|\varphi\|_{L^\infty(]-\frac{1}{2},\frac{1}{2}[)}\sqrt{\alpha_n}\|\partial_{x_3}p^a_{n,1}\|_{L^2(\Omega^a)} \leq \\ \\ \|\varphi\|_{L^\infty(]-\frac{1}{2},\frac{1}{2}[)}\sqrt{\alpha_n}c\sqrt{\dfrac{h^b_n}{h^a_n}} \quad \forall n\in\mathbb{N}, \end{cases} \tag{3.6.10}$$

$$\begin{cases} \dfrac{1}{\alpha_n}\left|\displaystyle\int_{\Omega_n^{a\star}} p^a_{n,1}(x_1,x_2,x_3)\varphi(x_2)dx\right| \leq \\ \\ \|\varphi\|_{L^\infty(]-\frac{1}{2},\frac{1}{2}[)}\dfrac{1}{\alpha_n}\displaystyle\int_{\Omega_n^{a\star}}\left|\int_{-\frac{1}{2}}^{x_1}\partial_t p^a_{n,1}(t,x_2,x_3)dt\right|dx \leq \\ \\ \|\varphi\|_{L^\infty(]-\frac{1}{2},\frac{1}{2}[)}\dfrac{1}{\alpha_n}\displaystyle\int_{\Omega_n^{a\star}}|\partial_{x_1}p^a_{n,1}(x)|dx \leq \|\varphi\|_{L^\infty(]-\frac{1}{2},\frac{1}{2}[)}\dfrac{1}{\alpha_n}\sqrt{\alpha_n}\|\partial_{x_1}p^a_{n,1}\|_{L^2(\Omega^a)} \leq \\ \\ \|\varphi\|_{L^\infty(]-\frac{1}{2},\frac{1}{2}[)}\dfrac{1}{\sqrt{\alpha_n}}ch^a_n\sqrt{\dfrac{h^b_n}{h^a_n}} \quad \forall n\in\mathbb{N}. \end{cases} \tag{3.6.11}$$

Note that choosing $\alpha_n = h^a_n\sqrt{h^b_n}$ provides that the last line in (3.6.10) and the last line in (3.6.11) vanish as n diverges.

Finally, choosing $\alpha_n = h^a_n\sqrt{h^b_n}$, passing to the limit, as n diverges, in (3.6.7) and taking into account (3.6.8), (3.6.9), (3.6.10), and (3.6.11) gives

$$\lim_n \int_{]-\frac{1}{2},\frac{1}{2}[\times]-1,0[} p^b_{n,1}(0,x_2,x_3)\varphi(x_2)dx_2dx_3 = 0, \tag{3.6.12}$$

which implies (3.6.5) thanks to (3.6.6) and the fact that φ is a generic function in $C_0^\infty\left(\left]-\frac{1}{2},\frac{1}{2}\right[\right)$.

□

3.6.2 A convergence result for problem (3.3.18)

Lemma 3.6.3. *Assume (3.1.8) with $\ell = +\infty$. For every $n \in \mathbb{N}$, let $\left(\phi_n^a, \phi_n^b\right) \in U_n$, with U_n defined in (3.2.4), be such that*

$$\left\|\phi_n^b\right\|_{L^2(\Omega^b)} \leq c \quad \left\|D_n^b \phi_n^b\right\|_{(L^2(\Omega^b))^3} \leq c, \quad \forall n \in \mathbb{N}, \tag{3.6.13}$$

where c is a positive constant independent of n. Then,

$$\lim_n \left\| \left(\frac{h_n^a}{h_n^b}\right)^{\frac{1}{2}} \phi_n^a \right\|_{L^2\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2 \times \{0\}\right)} = 0. \tag{3.6.14}$$

Proof. Estimates (3.6.13) imply the existence of an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$, and (in possible dependence on the subsequence) $\psi^b \in H^1\left(\Omega^b\right)$ with ψ^b independent of x_3 such that

$$\phi_{n_i}^b \rightharpoonup \psi^b \text{ weakly in } H^1\left(\Omega^b\right). \tag{3.6.15}$$

By virtue of the transmission condition in (3.2.4), to obtain (3.6.14) it is enough to prove

$$\lim_i \int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} \frac{h_{n_i}^a}{h_{n_i}^b} |\phi_{n_i}^b(h_{n_i}^a x_1, x_2, 0)|^2 dx_1 dx_2 = 0. \tag{3.6.16}$$

At first, we prove the existence of $\bar{x}_3 \in \left]-1,0\right[$, and of an increasing sequence of positive integer numbers $\{i_k\}_{k\in\mathbb{N}}$ such that

$$\phi_{n_{i_k}}^b(\cdot,\cdot,\bar{x}_3) \rightharpoonup \psi^b \text{ weakly in } H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right). \tag{3.6.17}$$

Indeed,

$$\forall i \in \mathbb{N}\ \exists A_i \subseteq \left]-1,0\right[: |A_i| = 0,\ \phi_{n_i}^b(\cdot,\cdot,x_3) \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2 \times \{x_3\}\right),\ \forall x_3 \in \left]-1,0\right[\setminus A_i.$$

For every $i \in \mathbb{N}$ let

$$\rho_i : x_3 \in]-1,0[\setminus \bigcup_{j=1}^{+\infty} A_j \to$$

$$\int_{]-\frac{1}{2},\frac{1}{2}[^2} \left(|D_{x_1}\phi^b_{n_i}(x_1,x_2,x_3)|^2 + |D_{x_2}\phi^b_{n_i}(x_1,x_2,x_3)|^2 + |\phi^b_{n_i}(x_1,x_2,x_3)|^2 \right) dx_1 dx_2.$$

Fatou's Lemma and (3.6.13) imply

$$\int_{-1}^{0} \liminf_i \rho_i(x_3) dx_3 \leq \liminf_i \int_{-1}^{0} \rho_i(x_3) dx_3 < +\infty.$$

Consequently,

$$\exists c \in]0,+\infty[, \quad \exists \bar{x}_3 \in]-1,0[\setminus \bigcup_{j=1}^{+\infty} A_j, \quad \exists \{i_k\}_{k\in\mathbb{N}} \subset \mathbb{N} \ : \ \rho_{i_k}(\bar{x}_3) < c \quad \forall k \in \mathbb{N},$$

which gives (3.6.17) thanks to (3.6.15).

Now, for proving (3.6.16), it is enough to show that each term of the right-hand side of the following splitting

$$\begin{cases} \dfrac{h^a_{n_{i_k}}}{h^b_{n_{i_k}}} \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[^2} \left| \phi^b_{n_{i_k}} \left(h^a_{n_{i_k}} x_1, x_2, 0 \right) \right|^2 dx_1 dx_2 \\ \leq 2 \dfrac{h^a_{n_{i_k}}}{h^b_{n_{i_k}}} \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[^2} \left| \phi^b_{n_{i_k}} \left(h^a_{n_{i_k}} x_1, x_2, 0 \right) - \phi^b_{n_{i_k}} \left(h^a_{n_{i_k}} x_1, x_2, \bar{x}_3 \right) \right|^2 dx_1 dx_2 \\ + 4 \dfrac{h^a_{n_{i_k}}}{h^b_{n_{i_k}}} \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[^2} \left| \phi^b_{n_{i_k}} \left(h^a_{n_{i_k}} x_1, x_2, \bar{x}_3 \right) - \phi^b_{n_{i_k}} \left(0, x_2, \bar{x}_3 \right) \right|^2 dx_1 dx_2 \\ + 4 \dfrac{h^a_{n_{i_k}}}{h^b_{n_{i_k}}} \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[} \left| \phi^b_{n_{i_k}} \left(0, x_2, \bar{x}_3 \right) \right|^2 dx_2, \quad \forall k \in \mathbb{N}, \end{cases} \tag{3.6.18}$$

vanishes as k diverges.

By virtue of the second estimate in (3.6.13), and (3.1.8) with $\ell = +\infty$, one has

$$\left\{\begin{array}{l} \limsup_k \left(\dfrac{h^a_{n_{i_k}}}{h^b_{n_{i_k}}} \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[^2} \left| \phi^b_{n_{i_k}} \left(h^a_{n_{i_k}} x_1, x_2, 0 \right) - \phi^b_{n_{i_k}} \left(h^a_{n_{i_k}} x_1, x_2, \bar{x}_3 \right) \right|^2 dx_1 dx_2 \right) \\ = \limsup_k \left(\dfrac{h^a_{n_{i_k}}}{h^b_{n_{i_k}}} \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[^2} \left| \int_{\bar{x}_3}^{0} D_{x_3} \phi^b_{n_{i_k}} \left(h^a_{n_{i_k}} x_1, x_2, x_3 \right) dx_3 \right|^2 dx_1 dx_2 \right) \\ \leq \limsup_k \left(\dfrac{h^a_{n_{i_k}}}{h^b_{n_{i_k}}} \displaystyle\int_{\Omega^b} \left| D_{x_3} \phi^b_{n_{i_k}} \left(h^a_{n_{i_k}} x_1, x_2, x_3 \right) \right|^2 dx_1 dx_2 dx_3 \right) \\ \leq \limsup_k \left(\dfrac{1}{h^b_{n_{i_k}}} \displaystyle\int_{\Omega^b} \left| D_{x_3} \phi^b_{n_{i_k}} (x) \right|^2 dx \right) = 0. \end{array}\right. \tag{3.6.19}$$

By virtue of (3.6.17), and (3.1.8) with $\ell = +\infty$, one has

$$\left\{\begin{array}{l} \limsup_k \left(\dfrac{h^a_{n_{i_k}}}{h^b_{n_{i_k}}} \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[^2} \left| \phi^b_{n_{i_k}} \left(h^a_{n_{i_k}} x_1, x_2, \bar{x}_3 \right) - \phi^b_{n_{i_k}} \left(0, x_2, \bar{x}_3 \right) \right|^2 dx_1 dx_2 \right) \\ = \limsup_k \left(\dfrac{h^a_{n_{i_k}}}{h^b_{n_{i_k}}} \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[^2} \left| \int_{0}^{h^a_{n_{i_k}} x_1} D_t \phi^b_{n_{i_k}} \left(t, x_2, \bar{x}_3 \right) dt \right|^2 dx_1 dx_2 \right) \\ \leq \limsup_k \left(\dfrac{h^a_{n_{i_k}}}{h^b_{n_{i_k}}} \dfrac{h^a_{n_{i_k}}}{2} \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[^2} \left| D_{x_1} \phi^b_{n_{i_k}} (x_1, x_2, \bar{x}_3) \right|^2 dx_1 dx_2 \right) = 0. \end{array}\right. \tag{3.6.20}$$

As far as the last term of the right-hand side in (3.6.18) is concerned, (3.6.17) provides

$$\phi^b_{n_{i_k}} \left(0, \cdot, \bar{x}_3 \right) \to \psi^b \left(0, \cdot \right) \text{ strongly in } L^2 \left(\left] -\tfrac{1}{2}, \tfrac{1}{2} \right[\right),$$

which implies

$$\lim_k \left(\frac{h^a_{n_{i_k}}}{h^b_{n_{i_k}}} \int_{]-\frac{1}{2},\frac{1}{2}[} \left| \phi^b_{n_{i_k}} \left(0, x_2, \bar{x}_3 \right) \right|^2 dx_2 \right) = 0, \tag{3.6.21}$$

thanks to (3.1.8) with $\ell = +\infty$.

Finally, (3.6.16) is obtained by passing to the limit in (3.6.18), as $k \to +\infty$, and by taking into account (3.6.19), (3.6.20), and (3.6.21). □

Proposition 3.6.4. *Assume (3.1.8) with $\ell = +\infty$. Let $\{(q_n^a, q_n^b)\}_{n\in\mathbb{N}} \subset \left(L^2\left(\Omega^a\right)\right)^3 \times \left(L^2\left(\Omega^b\right)\right)^3$ and $q^b = \left(q_1^b, q_2^b\right) \in \left(L^2\left(\Omega^b\right)\right)^2$ be such that q^b is independent of x_3, and*

$$\left(\left(\frac{h_n^a}{h_n^b}\right)^{\frac{1}{2}} q_n^a, q_n^b\right) \to \left(0, \left(q^b, 0\right)\right) \text{ strongly in } \left(L^2\left(\Omega^a\right)\right)^3 \times \left(L^2\left(\Omega^b\right)\right)^3. \tag{3.6.22}$$

Moreover, for $n \in \mathbb{N}$ let $\left(\phi^a_{(q_n^a,q_n^b)}, \phi^b_{(q_n^a,q_n^b)}\right)$ be the unique solution of

$$\begin{cases} \left(\phi^a_{(q_n^a,q_n^b)}, \phi^b_{(q_n^a,q_n^b)}\right) \in U_n, \quad \displaystyle\int_{\Omega^b} \phi^b_{(q_n^a,q_n^b)} dx = 0, \\ \\ h_n^a \displaystyle\int_{\Omega^a} ((-D_n^a \phi^a_{(q_n^a,q_n^b)} + q_n^a) \cdot D_n^a \phi^a) dx \\ \\ +h_n^b \displaystyle\int_{\Omega^b} \left(\left(-D_n^b \phi^b_{(q_n^a,q_n^b)} + q_n^b\right) \cdot D_n^b \phi^b\right) dx = 0 \quad \forall\, (\phi^a, \phi^b) \in U_n, \end{cases} \tag{3.6.23}$$

where U_n is defined in (3.2.4). Furthermore, let $\psi^b_{q^b}$ the unique solution of (3.3.21). Then, one has

$$\left(\left(\frac{h_n^a}{h_n^b}\right)^{\frac{1}{2}} \phi^a_{(q_n^a,q_n^b)}, \phi^b_{(q_n^a,q_n^b)}\right) \to \left(0, \psi^b_{q^b}\right) \text{ strongly in } H^1\left(\Omega^a\right) \times H^1(\Omega^b), \tag{3.6.24}$$

$$\left(\frac{1}{h_n^a} \frac{\partial\left(\left(\frac{h_n^a}{h_n^b}\right)^{\frac{1}{2}} \phi^a_{(q_n^a,q_n^b)}\right)}{\partial x_1}, \frac{1}{h_n^b} \frac{\partial \phi^b_{(q_n^a,q_n^b)}}{\partial x_3}\right) \to (0,0) \text{ strongly in } L^2(\Omega^a) \times L^2\left(\Omega^b\right). \tag{3.6.25}$$

Proof. In this proof, c denotes any positive constant independent of $n \in \mathbb{N}$.

Choosing $\left(\phi^a, \phi^b\right) = \left(\frac{1}{h_n^b}\phi^a_{(q_n^a,q_n^b)}, \frac{1}{h_n^b}\phi^b_{(q_n^a,q_n^b)}\right)$ as test function in (3.6.23), applying the Young inequality, and using (3.6.22) and (3.1.8) with $\ell = +\infty$ give

$$\left\| \left(\frac{h_n^a}{h_n^b}\right)^{\frac{1}{2}} D_n^a \phi^a_{(q_n^a,q_n^b)} \right\|_{(L^2(\Omega^a))^3} \le c, \quad \left\| D_n^b \phi^b_{(q_n^a,q_n^b)} \right\|_{(L^2(\Omega^b))^3} \le c \quad \forall n \in \mathbb{N}. \tag{3.6.26}$$

The second estimate in (3.6.26) implies

$$\left\|\phi^b_{(q^a_n,q^b_n)}\right\|_{H^1(\Omega^b)} \leq c \quad \forall n \in \mathbb{N}, \tag{3.6.27}$$

since $\displaystyle\int_{\Omega^b} \phi^b_{(q^a_n,q^b_n)} dx = 0.$

On the other side, the second estimate in (3.6.26), and (3.6.27) provides (see Lemma 3.6.3)

$$\lim_n \left\| \left(\frac{h^a_n}{h^b_n}\right)^{\frac{1}{2}} \phi^a_{(q^a_n,q^b_n)} \right\|_{L^2\left(]-\frac{1}{2},\frac{1}{2}[^2\times\{0\}\right)} = 0, \tag{3.6.28}$$

which combined with the first estimate in (3.6.26) implies

$$\left\| \left(\frac{h^a_n}{h^b_n}\right)^{\frac{1}{2}} \phi^a_{(q^a_n,q^b_n)} \right\|_{H^1(\Omega^a)} \leq c \quad \forall n \in \mathbb{N}. \tag{3.6.29}$$

Estimates (3.6.26), (3.6.27) and (3.6.29), and convergence (3.6.28) ensure the existence of a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence on the subsequence) of $\left(\tau^a, \tau^b\right) \in H^1\left(\Omega^a\right) \times H^1\left(\Omega^b\right)$ with τ^a independent of x_1 and τ^b independent of x_3, and of $\left(\xi^a, \xi^b\right) \in L^2(\Omega^a) \times L^2\left(\Omega^b\right)$ such that

$$\left(\left(\frac{h^a_n}{h^b_n}\right)^{\frac{1}{2}} \phi^a_{(q^a_n,q^b_n)}, \phi^b_{(q^a_n,q^b_n)} \right) \rightharpoonup (\tau^a, \tau^b) \text{ weakly in } H^1(\Omega^a) \times H^1(\Omega^b),$$

$$\tau^a = 0 \text{ a.e. on }]-\tfrac{1}{2},\tfrac{1}{2}[\times \{0\}, \tag{3.6.30}$$

$$\int_{]-\frac{1}{2},\frac{1}{2}[^2} \tau^b dx_1 dx_2 = 0, \tag{3.6.31}$$

$$\left(\frac{1}{h^a_n} \frac{\partial \left(\left(\frac{h^a_n}{h^b_n}\right)^{\frac{1}{2}} \phi^a_{(q^a_n,q^b_n)} \right)}{\partial x_1}, \frac{1}{h^b_n} \frac{\partial \phi^b_{(q^a_n,q^b_n)}}{\partial x_3} \right) \rightharpoonup (\xi^a, \xi^b) \text{ weakly in } L^2(\Omega^a) \times L^2\left(\Omega^b\right).$$

To identity $\tau^b = \psi^b_{q^b}$, it is enough to pass to the limit, as n diverges, in the equation in (3.6.23) with test functions (ϕ^a, ϕ^b) such that

$$\phi^a(x) = \frac{1}{h^b_n} \psi^b\left(0, x_2\right) \text{ if } x = (x_1, x_2, x_3) \in \Omega^a,$$

$$\phi^b(x) = \begin{cases} \dfrac{1}{h_n^b}\psi^b\left(0, x_2\right) \text{ if } x = (x_1, x_2, x_3) \in \Omega^b \text{ and } |x_1| \leq \dfrac{h_n^a}{2}, \\ \\ \dfrac{1}{h_n^b}\psi^b\left(x_1 - \text{sign}(x_1)\dfrac{h_n^a}{2}, x_2\right) \text{ if } x = (x_1, x_2, x_3) \in \Omega^b \text{ and } |x_1| > \dfrac{h_n^a}{2}, \end{cases} \tag{3.6.32}$$

with $\psi^b \in C^\infty\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^2\right)$, to use (3.1.8) with $l = +\infty$, previous convergences, the density of $C^\infty\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^2\right)$ in $H^1\left(\left]-\frac{1}{2}, \frac{1}{2}\right[^2\right)$, and (3.6.31).

To identity $\tau^a = 0$, it is enough to pass to the limit, as n diverges, in the equation in (3.6.23) with test functions (ϕ^a, ϕ^b) such that

$$\phi^a(x) = \left(\frac{1}{h_n^a h_n^b}\right)^{\frac{1}{2}} \psi^a\left(x_2, x_3\right) \text{ if } x = (x_1, x_2, x_3) \in \Omega^a, \quad \phi^b(x) = 0 \text{ if } x = (x_1, x_2, x_3) \in \Omega^b,$$

with

$$\psi^a \in \left\{v \in H^1\left(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[\times]0, 1[\right) : v = 0 \text{ on } \left]-\tfrac{1}{2}, \tfrac{1}{2}\right[\times \{0\}\right\},$$

to use previous convergences, and (3.6.30).

To identify ξ^a, ξ^b, and to prove that all the previous convergences are strong one can argue as in the last part of the proof of Proposition 3.4.4. □

3.6.3 Proof of Theorem 3.3.7

Proof. Corollary 3.6.2 asserts that (3.6.3) holds true and that there exist a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence on the subsequence) of $\hat{p}^b = \left(\hat{p}_1^b, \hat{p}_2^b\right) \in P_\infty$ satisfying (3.6.4). Moreover, one can prove the existence of a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence on the subsequence) of

$$\zeta \in \left(L^2(\Omega^a)\right)^3, \quad z^b \in \left(L^2\left(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[^2, H_m^1\left(]-1, 0[\right)\right)\right)^2 \times L^2\left(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[^2, H_0^1\left(]-1, 0[\right)\right)$$

such that (compare the proof of Theorem 3.3.2)

$$\left(\frac{1}{h_n^a}\frac{\partial\left(\left(\frac{h_n^a}{h_n^b}\right)^{\frac{1}{2}} p_n^a\right)}{\partial x_1}, \frac{1}{h_n^b}\frac{\partial p_n^b}{\partial x_3}\right) \rightharpoonup \left(\zeta, \frac{\partial z^b}{\partial x_3}\right) \text{ weakly in } \left(L^2(\Omega^a)\right)^3 \times \left(L^2(\Omega^b)\right)^3. \tag{3.6.33}$$

The next step is to identify $\hat{p}^b = \left(\hat{p}_1^b, \hat{p}_2^b\right)$, z^b, and ζ. To this aim, for every $(q_1^b, q_2^b) \in P_\infty^{\text{reg}}$, where

$$\begin{cases} P_\infty^{\text{reg}} = & \left\{q^b = \left(q_1^b, q_2^b\right) \in \left(C^1\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right)\right)^2 : \text{ for some } \delta > 0 \,(\text{depending on } q^b)\right. \\ & \\ & q_1^b = 0 \text{ in } \left(\left[-\frac{1}{2}, -\frac{1}{2} + \delta\right] \cup \right] - \delta, \delta[\cup \left[\frac{1}{2} - \delta, \frac{1}{2}\right]\right) \times \left]-\frac{1}{2}, \frac{1}{2}\right[, \\ & \\ & \left. q_2^b = 0 \text{ in } \left]-\frac{1}{2}, \frac{1}{2}\right[\times \left(\left[-\frac{1}{2}, -\frac{1}{2} + \delta\right] \cup \left[\frac{1}{2} - \delta, \frac{1}{2}\right]\right) \right\}, \end{cases} \tag{3.6.34}$$

consider a sequence $\{v_n\}_{n\in\mathbb{N}} \subset \left(H^1(\Omega^a)\right)^3$ (depending on q^b) such that

$$\left(v_n, (q_1^b, q_2^b, 0)\right) \in P_n, \quad \|v_n\|_{(L^4(\Omega^a)^3} \leq c, \quad \|D_n^a v_n\|_{(L^2(\Omega^a)^9} \leq c \quad \forall n \in \mathbb{N}, \tag{3.6.35}$$

where c is a positive constant independent of n. For instance, (3.6.35) is satisfied setting

$$v_n(x) = \left(q_1^b(h_n^a x_1, x_2), q_2^b(h_n^a x_1, x_2), 0\right) \text{ if } x = (x_1, x_2, x_3) \in \Omega^a. \tag{3.6.36}$$

Now, fixing $(q_1^b, q_2^b) \in P_\infty^{\text{reg}}$, and choosing $\left(v_n, (q_1^b, q_2^b, 0)\right)$ as test function in (3.2.8) with $\{v_n\}_{n\in\mathbb{N}}$ satisfying (3.6.35) give

$$\frac{1}{h_n^b} E_n\left(\left(p_n^a, p_n^b\right)\right) \leq \frac{1}{h_n^b} E_n\left(\left(v_n, \left(q_1^b, q_2^b, 0\right)\right)\right) \quad \forall n \in \mathbb{N}. \tag{3.6.37}$$

Then, passing to the limit in (3.6.37), as n diverges, and using (3.1.8) (with $\ell = \infty$), (3.2.9), (3.6.3),

(3.6.4), (3.6.33), (3.6.35), Proposition 3.6.4, and a l.s.c. argument imply

$$
\begin{cases}
\displaystyle\int_{\Omega^a} \beta\left(|\zeta_3|^2 + |\zeta_2|^2\right) dx + \int_{\Omega^a} |\zeta_1|^2 \, dx + \beta \int_{]-\frac{1}{2},\frac{1}{2}[^2} \left|\mathrm{rot}(\hat{p}^b)\right|^2 dx_1 dx_2 \\
\displaystyle + \int_{\Omega^b} \left(\beta \left(\left| \frac{\partial z_2^b}{\partial x_3} \right|^2 + \left| \frac{\partial z_1^b}{\partial x_3} \right|^2 \right) + \left| \mathrm{div}\left(\hat{p}^b\right) + \frac{\partial z_3^b}{\partial x_3} \right|^2 \right) dx \\
\displaystyle + \int_{]-\frac{1}{2},\frac{1}{2}[^2} \left(\alpha \left(\left|\hat{p}^b\right|^2 - 1 \right)^2 + \left| D\psi^b_{\hat{p}^b} \right|^2 + \int_{-1}^{0} \left(f_1^b, f_2^b\right) dx_3 \cdot \hat{p}^b \right) dx_1 dx_2 \leq \\
\displaystyle \leq \liminf_n \frac{E_n\left(\left(p_n^a, p_n^b\right)\right)}{h_n^b} \leq \limsup_n \frac{E_n\left(\left(p_n^a, p_n^b\right)\right)}{h_n^b} \\
\displaystyle \leq \lim_n \left(\frac{E_n\left(v_n, (q_1^b, q_2^b, 0)\right)}{h_n^b} \right) = E_\infty\left(q^b\right), \quad \forall q^b = \left(q_1^b, q_2^b\right) \in P_\infty^{\mathrm{reg}}.
\end{cases}
\tag{3.6.38}
$$

This inequalities hold true also for any $q^b \in P_\infty$, since P_∞^{reg} is dense in P_∞. One can conclude the proof arguing as in the last part of the proof of Theorem 3.3.2. □

Remark 3.6.5. *At first we point out that in this section we are treating the case $h_n^a << h_n^b$. Let $\{v_n\}_{n\in\mathbb{N}}$ be the sequence defined by (3.6.36). Note that $\left(v_n, (q_1^b, q_2^b, 0)\right) \in P_n$, and in particular $v_n\left(\pm\frac{1}{2}, \cdot\right) = 0$ in $\left]-\frac{1}{2}, \frac{1}{2}\right[$. This property is true since $(q_1^b, q_2^b) \in P_\infty^{reg}$, and in particular $q_1^b = 0$ in $]-\delta, \delta[\times\left]-\frac{1}{2}, \frac{1}{2}\right[$. The closure of P_∞^{reg} is P_∞. Consequently any elements $(q_1^b, q_2^b) \in P_\infty$ satisfies $q_1^b(0, \cdot) = 0$ in $\left]-\frac{1}{2}, \frac{1}{2}\right[$. We are able to prove this property for the limit function only when $h_n^b << \sqrt{h_n^a}$. The case $h_n^b \cong \sqrt{h_n^a}$ or $\sqrt{h_n^a} << h_n^b$ is an open problem.*

Perspectives

1) Let $\Omega^{(h)}$ be defined as in of Chapter 1, we would like to study the asymptotic behavior of the following problem

$$\min\left\{\frac{1}{|\Omega^{(h)}|}\int_{\Omega^{(h)}}\left(\underline{\alpha}_h(x)|\nabla\underline{m}|^2+\varphi(\underline{m})+\frac{1}{2}\underline{\beta}_h(x)\nabla\underline{\zeta}\underline{m}\right)dx,\underline{m}\in H^1(\Omega^{(h)},S^{n-1})\right\} \tag{3.6.39}$$

and is subjected to the following constraints

$$div\left(-\underline{\gamma}_h(x)D\underline{\zeta}+\underline{\beta}_h(x)\underline{m}\right)=0 \quad \text{in } \mathbb{R}^n, \tag{3.6.40}$$

where

$$\underline{\alpha}_h\in L^2(\Omega^{(h)}),\quad \underline{\beta}_h\in L^2(\Omega^{(h)}),\quad \underline{\gamma}_h\in L^2(\mathbb{R}^n), \tag{3.6.41}$$

$$0\leq\underline{\beta}_h(x),\quad 0\leq\underline{\alpha}_h(x),\quad 0\leq\underline{\gamma}_h(x)\quad x \text{ a.e. in } \Omega^{(h)},\quad \forall h. \tag{3.6.42}$$

There is partial results studied for this problem in [61].

2) At first we point out that in Section (3.6) of Chapter 3, we are treating the case $h_n^a << h_n^b$. Let $\{v_n\}_{n\in\mathbb{N}}$ be the sequence defined by (3.6.36). Note that $\left(v_n,(q_1^b,q_2^b,0)\right)\in P_n$, and in particular $v_n\left(\pm\frac{1}{2},\cdot\right)=0$ in $\left]-\frac{1}{2},\frac{1}{2}\right[$. This property is true since $(q_1^b,q_2^b)\in P_\infty^{\text{reg}}$, and in particular $q_1^b=0$ in $]-\delta,\delta[\times\left]-\frac{1}{2},\frac{1}{2}\right[$. The closure of P_∞^{reg} is P_∞. Consequently any elements $(q_1^b,q_2^b)\in P_\infty$ satisfies $q_1^b(0,\cdot)=0$ in $\left]-\frac{1}{2},\frac{1}{2}\right[$. We are able to prove this property for the limit function only when $h_n^b<<\sqrt{h_n^a}$. The case $h_n^b\cong\sqrt{h_n^a}$ or $\sqrt{h_n^a}<<h_n^b$ is an open problem.

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