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## Branching processes and Erdős-Rényi graph

Pierre-Antoine Corre

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Université Pierre et  
Marie Curie



Laboratoire de Probabilités  
et Modèles Aléatoires

École doctorale de sciences mathématiques de Paris centre

# THÈSE DE DOCTORAT

Discipline : Mathématiques

présentée par

**Pierre-Antoine CORRE**

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## Processus de branchements et graphe d'Erdős–Rényi

---

dirigée par Julien BERESTYCKI

Soutenue le 29 novembre 2017 devant le jury composé de :

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*L'homme est en ce monde ainsi que l'oiseau sur  
la branche, la branche est attachée à l'arbre, qui  
s'attache à l'arbre suit de bons préceptes...*

Sganarelle  
dans *Dom Juan, ou le Festin de Pierre*  
de Molière.



# Résumé

Le fil conducteur de cette thèse, composée de trois parties, est la notion de branchement.

Le premier chapitre est consacré à l'arbre de Yule et à l'arbre binaire de recherche. Nous obtenons des résultats d'oscillations asymptotiques de l'espérance, de la variance et de la distribution de la hauteur de ces arbres, confirmant ainsi une conjecture de Drmota. Par ailleurs, l'arbre de Yule pouvant être vu comme une marche aléatoire branchante évoluant sur un réseau, nos résultats permettent de mieux comprendre ce genre de processus.

Dans le second chapitre, nous étudions le nombre de particules tuées en 0 d'un mouvement brownien branchant avec dérive surcritique conditionné à s'éteindre. Nous ferons enfin apparaître une nouvelle phase de transition pour la queue de distribution de ces variables.

L'objet du dernier chapitre est le graphe d'Erdős–Rényi dans le cas critique :  $G(n, 1/n)$ . En introduisant un couplage et un changement d'échelle, nous montrerons que, lorsque  $n$  augmente les composantes de ce graphe évoluent asymptotiquement selon un processus de coalescence-fragmentation qui agit sur des graphes réels. La partie coalescence sera de type multiplicatif et les fragmentations se produiront selon un processus ponctuel de Poisson sur ces objets.

## Mots-clés

Processus de Yule, arbres binaires de recherche, mouvement brownien branchant avec absorption, graphe d'Erdős–Rényi, coalescent multiplicatif.

# Branching processes and Erdős-Rényi graph.

## Abstract

This thesis is composed by three chapters and its main theme is branching processes.

The first chapter is devoted to the study of the Yule tree and the binary search tree. We obtain oscillation results on the expectation, the variance and the distribution of the height of these trees and confirm a Drmota's conjecture. Moreover, the Yule tree can be seen as a particular instance of lattice branching random walk, our results thus allow a better understanding of these processes.

In the second chapter, we study the number of particles killed at 0 for a Brownian motion with supercritical drift conditioned to extinction. We finally highlight a new phase transition in terms of the drift for the tail of the distributions of these variables.

The main object of the last chapter is the Erdős-Rényi graph in the critical case :  $G(n, 1/n)$ . By using coupling and scaling, we show that, when  $n$  grows, the scaling process is asymptotically a coalescence-fragmentation process which acts on real graphs. The coalescent part is of multiplicative type and the fragmentations happen according a certain Poisson point process.

## Keywords

Yule process, binary search tree, branching brownian motion with absorption, Erdős-Rényi graph, multiplicative coalescent

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# Introduction

## 0.1 Introduction aux processus de branchements

### 0.1.1 Une brève introduction historique

C'est par l'étude de la démographie que commence celle des processus de branchements. Au XVIII<sup>e</sup> siècle et dans la première moitié du XIX<sup>e</sup>, de nombreux mathématiciens cherchent à construire des modèles déterministes pour décrire l'évolution de populations. Ainsi Euler, en 1748, en s'appuyant sur des considérations empiriques et sur divers recensements propose le modèle suivant. Si  $P_n$  est le nombre d'individus à l'année  $n$  alors :

$$P_{n+1} = (1 + x)P_n, \forall n \in \mathbb{N} \quad (1)$$

avec  $x > 0$ . Quoique ce modèle de croissance géométrique illimitée soit contesté par Malthus et à sa suite par Verhulst (1838), compte tenu du fait que les subsistances ne s'accroissent pas géométriquement (simplement arithmétiquement pour Malthus), ce modèle semble une bonne approximation de la réalité pour beaucoup au XIX<sup>e</sup> siècle. Pour autant, malgré cette croissance supposée, de nombreux auteurs dont Thomas Doubleday et Alphonse de Candolle constatent que les noms des grandes familles tendent à disparaître. C'est pour comprendre ce phénomène que Bienaymé en 1845 et Galton et Watson en 1873 introduisent indépendamment deux modèles, dont l'un (celui de Bienaymé) peut être vu comme un cas particulier de l'autre. Ce modèle, qu'on appelle aujourd'hui le processus de Galton-Watson, était historiquement le suivant. Soient  $q \in \mathbb{N}^*$  et  $X$  une loi de probabilité sur  $\{1, \dots, q\}$ . On désigne par  $Z_n$  le nombre d'hommes adultes à la génération  $n$ . La suite  $(Z_n)$  est définie récursivement par  $Z_0 = N$ , où  $N \in \mathbb{N}^*$  et par :

$$Z_{n+1} = \sum_{i=1}^{Z_n} X_i^{(n)}, \quad (2)$$

où les  $X_i^{(n)}$  sont des copies indépendantes de  $X$ . Pour déterminer la probabilité qu'une famille soit éteinte au bout de  $r$  générations, Watson introduit la fonction génératrice de

$X$  (qui dans son cas est un polynôme) :

$$G(s) = \mathbb{E}(s^X). \quad (3)$$

En définissant  $G_n$  par  $G_0 = G$  et par la relation de récurrence :

$$G_{n+1}(s) = G(G_n(s)), \quad (4)$$

Watson constate que si  $N = 1$ ,  $\mathbb{P}(Z_n = k)$  est le  $k^e$  coefficient de  $G_n$ . Il en déduit que la probabilité  $q_n$  que la famille soit éteinte à la génération  $n$  satisfait  $q_0 = 0$  et :

$$q_{n+1} = G(q_n). \quad (5)$$

Si Watson comprend que  $q_\infty$ , la probabilité qu'a une famille de s'éteindre, doit satisfaire :

$$q_\infty = G(q_\infty), \quad (6)$$

il commet toutefois une erreur de calcul en traitant un cas particulier et conjecture que quelle que soit  $G$ , on a nécessairement  $q_\infty = 1$ , corroborant ainsi une idée assez répandue à l'époque selon laquelle l'extinction des grandes familles était inéluctable.

Après avoir été laissé de côté pendant plusieurs décennies, le modèle de Galton-Watson est réintroduit par Fisher (sans que l'on sache si Fisher eût connaissance des travaux de Galton et de Watson) en 1922, mais cette fois-ci dans le contexte de la théorie de l'évolution et de la génétique à la suite d'un demi-siècle de découvertes dans ce domaine (initiées notamment par la théorie de l'évolution exposée par Darwin dans *De l'origine des espèces* en 1859 et par les lois de l'hérédité établies par Mendel en 1865).

Ce sont ces mêmes disciplines qui motivent Yule à introduire en 1924, un modèle continu cherchant à décrire l'évolution du nombre d'espèces au sein d'un genre au cours du temps. Son modèle est le suivant. À chaque instant  $t$ , une espèce donne naissance à une autre espèce avec probabilité  $\lambda_e dt$  et un genre donne naissance à un autre genre avec probabilité  $\lambda_g dt$  et ce indépendamment des autres genres et espèces. Si l'on considère uniquement l'évolution du nombre d'espèces  $N_t$  au sein d'un genre en partant d'une seule espèce au temps 0, on obtient facilement à l'aide des équations de Kolmogorov, la probabilité  $p_n(t)$  d'avoir  $N_t = n$  espèces au temps  $t$  :

$$p_n(t) = e^{-\lambda_e t} \left(1 - e^{-\lambda_e t}\right)^{n-1} \quad (7)$$

et en conséquence :

$$\mathbb{E}(N_t) = e^{\lambda_e t}. \quad (8)$$

Ce processus porte le nom de processus de Yule ou encore de processus de naissance. Après avoir montré que, pour  $t$  asymptotiquement grand, la probabilité qu'un genre tiré uniformément existe depuis  $s$  unités de temps est de l'ordre de  $\lambda_g e^{-\lambda_g s} ds$  et en utilisant l'équation (7), Yule donne l'estimation asymptotique suivante de la probabilité  $q_n$  d'avoir  $n$  espèces au sein d'un genre choisi uniformément :

$$q_n = \frac{(n-1)! \left(\frac{\lambda_e}{\lambda_g}\right)^{n-1}}{\prod_{i=1}^n \left[1 + i \frac{\lambda_e}{\lambda_g}\right]}. \quad (9)$$

Yule compare ensuite ces résultats avec les données empiriques qu'il a recueillies et trouve une assez bonne concordance entre les deux.

Si les deux modèles précédents connaissent de nombreux développements tant du point de vue des mathématiques que de ses applications (chimie, physique nucléaire, épidémiologie...), ce n'est qu'en 1937 que la dimension spatiale est introduite indépendamment par Fisher d'une part et par Kolmogorov, Petrovsky, Piskounov d'autre part. Les deux travaux portent sur la diffusion spatiale d'un gène dominant et plus précisément sur l'équation suivante décrivant la proportion  $u(t, x)$  d'un gène au point  $x$  (nous sommes en dimension 1) au temps  $t$  :

$$\partial_t u = F(u) + d\partial_{xx}u, \quad (10)$$

où  $d > 0$  et  $F(u) = au(1-u)$ ,  $a > 0$  dans l'article de Fisher et vérifie les conditions suivantes plus générales chez Kolmogorov et al. :

- $F(0) = F(1) = 0$
- $F'(0) > 0$ ;
- $F(u) > 0, \forall u \in (0, 1)$ ;
- $F'(u) < F'(0), \forall u \in (0, 1]$ .

Leurs travaux mettent en évidence l'existence et l'unicité d'ondes voyageuses, c'est-à-dire de solutions de l'équation (10) de la forme  $u(t, x) = f(x - vt)$  et telles que  $f(x) \in (0, 1), \forall x \in \mathbb{R}$ ,  $f(x) \rightarrow 0$  lorsque  $x \rightarrow +\infty$  et  $f(x) \rightarrow 1$  lorsque  $x \rightarrow -\infty$ . Plus précisément, de telles solutions existent et sont uniques pour des vitesses  $v \geq v_c$  où  $v_c = 2\sqrt{F'(0)d}$  soit  $v_c = 2\sqrt{ad}$  dans le cas traité par Fisher. Par ailleurs, Kolmogorov et al. d'une part et Fisher d'autre part montrent que, sous une condition initiale de type Heaviside (c'est-à-dire  $u(0, x) = \mathbf{1}_{x>0}$ ), la solution de (10) se propage à vitesse  $v_c$  et que, recentrée en  $m_t = \inf\{x \in \mathbb{R}, u(t, x) > 1/2\}$ , cette solution converge vers l'onde voyageuse de vitesse  $v_c$ . Notons enfin que l'équation (10) est souvent appelée FKPP ou KPP.

Près de 40 ans plus tard, Mc Kean introduit un processus de branchements, qu'on appelle aujourd'hui le mouvement brownien branchant qui est fortement lié à l'équation KPP comme nous le verrons ultérieurement. En particulier, ce lien permet de donner une estimation extrêmement précise de  $m_t$ .

### 0.1.2 Définition de quelques modèles classiques

#### Arbres de Galton-Watson

Nous considérons ici les arbres enracinés et étiquetés selon les notations de Ulam-Harris. Dans ce cadre, l'ensemble des nœuds est représenté par :

$$\mathcal{U} = \bigcup_{n=0}^{+\infty} \mathbb{N}^{*n}, \quad (11)$$

où  $\mathbb{N}^{*0} = \emptyset$ . Un arbre  $\tau$  est alors un sous-ensemble de  $\mathcal{U}$  vérifiant les propriétés suivantes :

1.  $\emptyset \in \tau$
2. Si  $u_1 u_2 \dots u_n \in \tau$  alors  $u_1 u_2 \dots u_k \in \tau$  pour tout  $k \in \{1, \dots, n\}$ .
3. Si  $u_1 u_2 \dots u_n \in \tau$  alors  $u_1 u_2 \dots u_{n-1} j \in \tau$  pour tout  $j \in \{1, \dots, u_n\}$ .

Le nœud  $\emptyset$  correspond alors à la racine de l'arbre. Si  $u = u_1 \dots u_n \in \tau$ , on désigne par  $|u| := n$  la hauteur (ou la génération selon le contexte) d'un nœud, avec  $|\emptyset| = 0$ . Muni de ces notations, nous commençons par introduire un processus de branchements fondamental : l'arbre de Galton-Watson. Considérons un espace probabiliste  $(\Omega, \mathcal{F}, \mathbb{P})$ . Soit  $X$  une variable aléatoire à valeurs dans  $\mathbb{N}$  dont la loi  $\mu$  sera appelée loi de reproduction du processus de Galton-Watson. On construit par récurrence l'arbre de Galton-Watson et le processus de Galton-Watson sous-jacent de la manière suivante. On pose  $\tau_0 = \emptyset$  et  $Z_0 = 1$ . On suppose  $\tau_n$  construit. Posons  $G_n = \{u \in \tau_n, |u| = n\}$ . On considère  $(X_u)_{u \in G_n}$  une suite de variables indépendantes et de même loi que  $X$ . L'arbre  $\tau_{n+1}$  est alors défini par :

$$\tau_{n+1} = \tau_n \cup \{uj, u \in G_n, 1 \leq j \leq X_u\}. \quad (12)$$

L'arbre de Galton-Watson est alors :

$$\tau = \bigcup_{n=0}^{+\infty} \tau_n. \quad (13)$$

Le processus  $(Z_n)_{n \in \mathbb{N}}$  défini par  $Z_n = \text{Card}(G_n)$ , pour  $n \in \mathbb{N}^*$  et  $Z_0 = 1$  est alors appelé processus de Galton-Watson. Notons en passant qu'on peut s'affranchir de la structure d'arbre pour définir le processus de Galton-Watson par la relation de récurrence suivante (dont on peut vérifier facilement qu'elle est effectivement satisfaite par  $Z_n$  tel qu'on vient de le définir) :

$$\begin{cases} Z_0 = 1 \\ Z_{n+1} \stackrel{d}{=} \sum_{i=1}^{Z_n} X_{i,n} \end{cases}, \quad (14)$$

où les  $X_{i,n}$  sont des variables indépendantes et de même loi que  $X$ .

### Marches aléatoires branchantes

On peut ajouter une structure spatiale aux arbres de Galton-Watson et obtenir une marche aléatoire branchante définie de la manière suivante. Une particule se situe en 0 au temps 0. Cette particule meurt au temps 1 et donne naissance à un processus ponctuel  $\mathcal{L}$  (ici nous considérons uniquement le cas où le processus est sur  $\mathbb{R}$ ). Chaque nouvelle particule  $u$  meurt au temps 2 et donne naissance à une copie indépendante de  $\mathcal{L}$  recentrée en la position de  $u$ , notée  $V(u)$ . Le processus se reproduit indéfiniment.

Notons que dans de nombreux énoncés, le processus ponctuel  $\mathcal{L}$  est remplacé par  $N \in \mathbb{N}$  particules dont les déplacements sont des copies indépendantes de  $Y$ , où  $N$  et  $Y$  sont des variables aléatoires réelles et indépendantes.

### Mouvement brownien branchant

Avant de définir le mouvement brownien branchant, mentionnons que c'est un processus de Markov à temps continu qui décrit l'évolution spatiale de particules. Étant à temps continu, il est possible que le nombre de particules explose. Toutefois, cette situation ne se produit pas pour les lois de reproductions les plus souvent considérées (voir par exemple [20] pour une condition nécessaire et suffisante d'explosion), en particulier pour celles d'espérance finie. Dans le cas où l'explosion se produirait, la définition suivante ne s'appliquerait donc qu'avant le temps d'explosion.

Le mouvement brownien branchant est défini de la manière suivante. Considérons  $L$  une variable aléatoire à valeur dans  $\mathbb{N}$  telle que  $\mathbb{P}(L \leq 1) < 1$ . Une particule positionnée initialement en 0 suit un mouvement brownien pendant un temps déterminé par une variable aléatoire  $T_1$ , indépendante du mouvement et de  $L$ , de loi exponentielle de paramètre  $\beta$ . Au bout de ce temps, cette particule se divise en  $L$  particules. Chaque nouvelle particule  $u$  entame un mouvement brownien issu de la position de sa particule mère, indépendant du mouvement des autres particules, et ce pendant un temps  $T_u$  qui est une copie indépendante de  $T_1$ . Puis, cette particule va à son tour se diviser en  $L_u$  particules, où  $L_u$  est une copie indépendante de  $L$ . Le processus se reproduit ainsi indéfiniment.

Enfin, nous utiliserons les notations suivantes. L'ensemble des particules vivantes du mouvement brownien au temps  $t$  sera noté  $\mathcal{N}_t$ . Pour chaque  $u \in \mathcal{N}_t$ , on désignera par  $X_u(t)$ , la position de la particule  $u$  au temps  $t$ . On définira enfin la filtration  $(\mathcal{F}_t)$  par  $\mathcal{F}_t = \sigma\{X_u(s), u \in \mathcal{N}_s, 0 \leq s \leq t\}$ .

### Arbre binaire de recherche et arbre de Yule

L'arbre binaire de recherche  $(\mathcal{T}_n)_{n \in \mathbb{N}^*}$  est une chaîne de Markov à temps discret sur l'espace des arbres binaires qui peut être construit de la manière suivante :  $\mathcal{T}_1$  est un arbre



constitué d'une seule feuille. Étant donné  $\mathcal{T}_n$ , l'état suivant  $\mathcal{T}_{n+1}$  est obtenu en choisissant uniformément une feuille de  $\mathcal{T}_n$  qu'on remplace par un nœud avec deux feuilles attachées. L'arbre binaire de recherche est une structure naturelle pour tout ce qui concerne le stockage de données et est lié à l'algorithme de tri rapide.

On peut facilement transformer  $(\mathcal{T}_n)_{n \in \mathbb{N}^*}$  en un processus de Markov à temps continu par Poissonisation. Plus précisément, soit  $(N_t)_{t \in \mathbb{R}_+}$  un processus de naissance pur indépendant de  $(\mathcal{T}_n)_{n \in \mathbb{N}^*}$  qui saute de l'état  $n$  à l'état  $n + 1$  à taux  $n$ . Le processus

$$(\mathcal{T}_t^c)_{t \in \mathbb{R}_+} := (\mathcal{T}_{N_t})_{t \in \mathbb{R}_+} \quad (15)$$

est appelé processus d'arbre de Yule (Yule tree process). Nous pouvons, de manière équivalente, définir ce processus de la manière suivante. Au temps 0, l'arbre est réduit à une feuille. Chaque feuille vit pendant un temps aléatoire distribué exponentiellement de paramètre 1 et qui est indépendant des autres feuilles. Lorsqu'elle meurt, elle est remplacée par un nœud auquel on adjoint deux feuilles. Pour plus de détails nous recommandons [33, 86].

Notons que  $N_t$  est le cardinal de  $\mathcal{N}_t$ , l'ensemble des feuilles vivantes au temps  $t$ , et que si nous introduisons le temps d'arrêt suivant :

$$\tau_n = \inf\{t > 0, N_t = n\}, \quad (16)$$

l'équation (15) fournit :

$$(\mathcal{T}_n)_{n \in \mathbb{N}} = (\mathcal{T}_{\tau_n}^c)_{n \in \mathbb{N}}. \quad (17)$$

Ainsi, nous pouvons passer d'un modèle à l'autre.

À partir de l'arbre de Yule, nous pouvons définir une marche aléatoire branchante de la manière suivante. Nous désignons par  $X_u(t)$  la hauteur de la feuille  $u \in \mathcal{N}_t$ . Le processus à valeur mesure  $(X(t))_{t \in \mathbb{R}_+}$ , défini par

$$X(t) := \sum_{u \in \mathcal{N}_t} \delta_{X_u(t)}, \quad \forall t \in \mathbb{R}_+ \quad (18)$$

est une marche aléatoire en temps continu avec support en réseau (lattice support) qu'on appelle la marche aléatoire branchante de Yule. Chaque particule vit pendant un temps aléatoire distribué exponentiellement de paramètre 1 et, à la fin de ce temps, est remplacée par deux particules filles situées à une hauteur supérieure d'une unité à celle de leur mère. Fixons  $0 \leq t_0 \leq t_1 \leq t_2$ . Soit  $u \in \mathcal{N}_{t_1}$ , nous posons  $X_u(t_0) = X_v(t_0)$ , où  $v$  est l'ancêtre (qui est unique) de  $u$  dans  $\mathcal{N}_{t_0}$ . De façon cohérente, nous disons que  $X_u(t_2) = X_u(t_1)$ , même lorsque la particule  $u$  n'est plus vivante au temps  $t_2$ . La notation  $v < u$  indique que  $v$  est un ancêtre de  $u$ .

## 0.2 Particules extrêmes

Nous nous intéresserons dans le premier chapitre aux particules extrêmes de la marche aléatoire de Yule et aux feuilles de plus haut et de plus bas niveaux des arbres binaires de recherche. Dans cette section, nous présenterons dans un premier temps les résultats historiques concernant ces questions sur ces modèles afin d'introduire dans un second temps nos résultats. Enfin, dans un dernier temps, nous mettrons nos résultats en regard de ceux connus sur des processus de branchements plus généraux.

### 0.2.1 Arbre binaire de recherche et arbre de Yule

Étant donné que l'arbre binaire de recherche et l'arbre de Yule sont des modèles extrêmement proches, et que par ailleurs nous traiterons ultérieurement des marches aléatoires branchantes, nous n'évoquerons ici que les résultats concernant l'arbre binaire de recherche.

Introduisons dès à présent, pour  $\theta \in \mathbb{R}^*$ ,

$$c_\theta := \frac{(2e^\theta - 1)}{\theta}. \quad (19)$$

Nous appelons  $\theta^+$  (respectivement  $\theta^-$ ) la plus grande (respectivement la plus petite) solution de  $c_\theta = 2e^\theta$ . Par simplicité, nous noterons  $c^+ := c_{\theta^+}$  et  $c^- := c_{\theta^-}$ . Numériquement, [44] nous avons approximativement :

$$\begin{cases} c^+ = 4.311\dots \\ \theta^+ = 0.768\dots \end{cases} \quad \begin{cases} c^- = 0.373\dots \\ \theta^- = -1.678\dots \end{cases}.$$

La hauteur  $H_n$  de l'arbre binaire de recherche à  $n$  nœuds, sera la plus grande hauteur d'une feuille, le niveau de saturation  $l_n$ , sera la plus petite hauteur d'une feuille, plutôt que du minimum. Par ailleurs, les résultats sur la hauteur et sur le niveau de saturation étant similaires, nous ne mentionnerons que ceux sur la hauteur.

Initialement, Robson [88, 89] montre que  $(\mathbb{E}(H_n)/\log n)$  converge vers une constante entre 3.6 et  $c^+$  et en 1995, Devroye et Reed [42] prouvent que  $\text{Var}(H_n) = O_n((\log \log n)^2)$ . Ces estimations sont améliorées [40, 41, 43, 84] jusqu'à ce que Reed [86] et Drmota [44] démontrent indépendamment que :

$$\mathbb{E}(H_n) = a_{\log(n)} + O_n(1) \text{ et } \text{Var}(H_n) = O_n(1), \quad (20)$$

où  $a_t = c^+t - \frac{3 \log(t)}{2\theta^+}$ ,  $\forall t \geq 0$ . Introduisons maintenant, pour  $n \in \mathbb{N}$ ,  $\Upsilon_n$  la fonction

génératrice de  $(\mathbb{P}(H_k \leq n))_{k \in \mathbb{N}}$ , c'est-à-dire :

$$\Upsilon_n(x) := \sum_{k=0}^{+\infty} \mathbb{P}(H_k \leq n) x^k, \quad \forall x \in \mathbb{R}. \quad (21)$$

Drmotà [44] montre alors que la hauteur converge en distribution au sens suivant.

**Théorème** (Drmotà [44]). *Il existe une fonction décroissante  $\Psi : \mathbb{R}^+ \rightarrow (0, 1]$  satisfaisant  $\Psi(0) = 1$ ,  $\lim_{x \rightarrow +\infty} \Psi(x) = 0$  et l'équation*

$$y\Psi(y/e^{(1/c^+)}) = \int_0^y \Psi(z)\Psi(y-z)dz \quad (22)$$

telle que

$$\lim_{k \rightarrow +\infty} \sup_{n \in \mathbb{N}} \left| \mathbb{P}(H_k \leq n) - \Psi\left(\frac{k}{\Upsilon_n(1)}\right) \right| = 0. \quad (23)$$

Drmotà fournit en outre une bonne approximation de  $\Upsilon_n(1)$  en montrant que :

$$\Upsilon_n(1) = e^{\frac{n}{c^+} + \frac{3 \log n}{2(c^+-1)} + \kappa_n + o_n(1)}, \quad (24)$$

où  $(\kappa_n)$  est une suite bornée telle que  $\kappa_{n+1} - \kappa_n \rightarrow 0$ . Il montre par ailleurs que si  $(\kappa_n)$  converge alors il existe deux fonctions 1-périodiques  $R_1$  et  $R_2$  telles que :

$$\mathbb{E}(H_n) = a_{\log n} + R_1(a_{\log n}) + o_n(1), \quad (25)$$

$$\text{Var}(H_n) = R_2(a_{\log n}) + o_n(1). \quad (26)$$

Concluons cette section par des résultats portant sur  $F_n$  le nombre de feuilles à la plus haute position d'un arbre binaire de recherche à  $n$  feuilles. Drmotà montre, là encore, que si  $(\kappa_n)$  converge alors, il existe une fonction 1-périodique  $R_3$  telle que :

$$\mathbb{E}(F_n) = R_3(a_{\log n}) + o_n(1). \quad (27)$$

Il montre en outre que, sous cette hypothèse, les oscillations de  $(\mathbb{E}(F_n))$  autour de  $c^+$  sont au plus de l'ordre de  $10^{-4}$ . La question de savoir si  $(\mathbb{E}(F_n))$  converge se pose alors. Elle se pose d'autant plus que Drmotà montre que  $(\mathbb{E}(F_n))$  est croissante jusqu'à  $n = 100000$ . Évidemment, la croissance de  $(\mathbb{E}(F_n))$  sur  $\mathbb{N}$  prouverait la convergence de  $(\mathbb{E}(F_n))$ . Toutefois, il semble penser que  $(\mathbb{E}(F_n))$  ne converge pas. Toujours est-il que la question de la convergence de  $(\mathbb{E}(F_n))$  reste toujours ouverte, même si nos travaux permettront d'établir que  $(\kappa_n)$  converge et donc que les équations (25), (26), (27) sont valides.

Enfin mentionnons que Roberts transpose les résultats de [61] et montre dans [87] que :

$$\frac{1}{2\theta^+} = \liminf_{n \rightarrow +\infty} \frac{c^+ \log(n) - H_n}{\log \log(n)} < \limsup_{n \rightarrow +\infty} \frac{c^+ \log(n) - H_n}{\log \log(n)} = \frac{3}{2\theta^+}. \quad (28)$$

Il apporte aussi une information intéressante sur le comportement presque sûr de  $F_n$  :

$$\limsup_{n \rightarrow +\infty} F_n = +\infty. \quad (29)$$

### 0.2.2 Présentation des résultats du chapitre 1

Nous commencerons par présenter nos résultats sur la marche aléatoire branchante de Yule, car c'est à partir d'eux que nous obtiendrons ceux sur les arbres binaires de recherche. Nous conservons les notations de la Section 0.1.2.

Définissons  $\bar{X}(t) := \max_{u \in \mathcal{N}_t} X_u(t)$  et  $\underline{X}(t) := \min_{u \in \mathcal{N}_t} X_u(t)$ .

Pour  $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ , posons :

$$\begin{cases} \bar{h}(x, t) := \mathbb{P}(\bar{X}(t) \leq x) = \mathbb{P}(\bar{X}(t) \leq \lfloor x \rfloor) \\ \underline{h}(x, t) := \mathbb{P}(\underline{X}(t) \geq x) = \mathbb{P}(\underline{X}(t) \geq \lceil x \rceil) \end{cases}. \quad (30)$$

**Proposition 0.2.1.** *Les fonctions  $\bar{h}$  et  $\underline{h}$  sont solutions de l'équation :*

$$\partial_t h(x, t) = h^2(x - 1, t) - h(x, t), \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+. \quad (31)$$

On peut remarquer que le lien entre la marche aléatoire de Yule et l'équation (31) est de même nature que le lien qui unit le mouvement brownien branchant et l'équation F-KPP. De la même manière, nous disons que  $x \mapsto \phi(x)$  est une onde voyageuse solution de (31) de vitesse  $c \in \mathbb{R}$  si  $(x, t) \mapsto \phi(x - ct)$  est solution de (31). On peut facilement vérifier que  $\phi$  est une onde voyageuse de (31) de vitesse  $c \in \mathbb{R}$  si et seulement si elle vérifie l'équation :

$$c\phi'(x) = \phi(x) - \phi^2(x - 1), \quad x \in \mathbb{R}. \quad (32)$$

Contrairement au mouvement brownien branchant, dans le cas de la marche aléatoire de Yule, les fonctions  $\bar{h}$  et  $\underline{h}$  définies en (30) sont continues en  $t$  mais seulement continues par morceaux en  $x$ , ce qui implique que quel que soit le recentrage, nous ne pouvons espérer une convergence en distribution, à moins de nous restreindre à des sous-suites bien choisies. Cependant, le théorème suivant nous montre que les distributions asymptotiques oscillent autour des ondes voyageuses critiques.

**Théorème 0.2.2.** *Soit  $a_t = c^+t - \frac{3 \log(t)}{2\theta^+}$ ,  $b_t = c^-t - \frac{3 \log(t)}{2\theta^-}$  (nous rappelons que  $\theta^- < 0$ ) et  $\{x\} = x - \lfloor x \rfloor$ .*

*Il existe une onde voyageuse  $\bar{\phi}$  de vitesse  $c^+$  et une autre  $\underline{\phi}$  de vitesse  $c^-$  de (31) telles que :*

$$\lim_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}} |\mathbb{P}(\bar{X}(t) \leq a_t + x) - \bar{\phi}(x - \{a_t + x\})| = 0 \quad (33)$$

et :

$$\lim_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}} |\mathbb{P}(X(t) \geq b_t + x) - \underline{\phi}(x - \{b_t + x\})| = 0. \quad (34)$$

Notons que (33) est équivalent à :

$$\lim_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}} |\mathbb{P}(\bar{X}(t) \leq x) - \bar{\phi}(\lfloor x \rfloor - a_t)| = 0 \quad (35)$$

et que nous avons l'analogie pour (34). Nous allons maintenant étendre le théorème 0.2.2 à des fonctionnelles plus générales du maximum. Pour chaque fonction  $f$  continue par morceaux nous définissons  $G_f$ ,  $P_f$  et  $r_f$  par :

$$\begin{cases} G_f(t) := \mathbb{E} [f(\bar{X}(t) - a_t)], & t > 0, \\ P_f(s) := \sum_{k \in \mathbb{Z}} f(k - s) (\bar{\phi}(k - s) - \bar{\phi}(k - 1 - s)), & s \in \mathbb{R}, \\ r_f(t) := P_f(a_t), & t > 0, \end{cases} \quad (36)$$

lorsque ces fonctions sont bien définies. Notons que  $P_f$  est 1-périodique.

**Théorème 0.2.3.** *Il existe  $\delta > 0$  tel que pour toute fonction  $f$  continue par morceaux qui satisfait  $f(x) = o_{x \rightarrow \pm\infty}(e^{\delta|x|})$ , les fonctions  $G_f$ ,  $P_f$  et  $r_f$  sont bien définies et nous avons :*

$$\lim_{t \rightarrow +\infty} |G_f(t) - r_f(t)| = 0. \quad (37)$$

Par ailleurs, si chaque  $i^e$  dérivée de  $f \in \mathcal{C}^k(\mathbb{R})$ , pour  $0 \leq i \leq k$ , satisfait  $f^{(i)}(x) = o_{x \rightarrow \pm\infty}(e^{\delta|x|})$ , nous avons que  $G_f, P_f, r_f \in \mathcal{C}^k(\mathbb{R}^+)$  et :

$$\lim_{t \rightarrow +\infty} |G_f^{(k)}(t) - r_f^{(k)}(t)| = \lim_{t \rightarrow +\infty} |G_f^{(k)}(t) - (c^+)^k P_f^{(k)}(a_t)| = 0. \quad (38)$$

Nous pouvons établir les mêmes résultats pour le minimum. Observons qu'en prenant, pour  $x$  fixé,  $f$  définie par  $f(y) = \mathbf{1}_{\{y \leq x\}}$  dans le théorème 0.2.3 nous retrouvons (à l'unicité de la convergence près) le théorème 0.2.2.

Il est intéressant d'appliquer le théorème 0.2.3 à quelques cas particuliers. Posons

$$\tilde{F}_t := \text{Card}\{u \in \mathcal{N}_t, X_u(t) = \bar{X}(t)\}. \quad (39)$$

**Corollaire 0.2.4.** *Il existe trois fonctions lisses 1-périodiques  $Q_1, Q_2$  et  $Q_3$  telles que :*

$$\mathbb{E}(\bar{X}(t)) = a_t + Q_1(a_t) + o_t(1), \quad (40)$$

$$\text{Var}(\bar{X}(t)) = Q_2(a_t) + o_t(1), \quad (41)$$

$$\mathbb{E}(\tilde{F}_t) = Q_3(a_t) + o_t(1). \quad (42)$$

Nous pouvons déduire des théorèmes précédents des résultats analogues pour les arbres

binaires de recherche. Nous verrons qu'il existe deux fonctions  $\psi_K^+$  et  $\psi_K^-$  qui joueront un rôle analogue à celui de  $\bar{\phi}$  et de  $\underline{\phi}$ . Plus précisément, soient  $H_n$  la hauteur d'un arbre binaire de recherche à  $n$  nœuds et  $l_n$  son niveau de saturation.

**Théorème 0.2.5.**

$$\lim_{n \rightarrow +\infty} \sup_{x \in \mathbb{R}} |\mathbb{P}(H_n \leq \lfloor a_{\log(n)} + x \rfloor) - \psi_{K^+}^+(x - \{a_{\log(n)} + x\})| = 0, \quad (43)$$

et :

$$\lim_{n \rightarrow +\infty} \sup_{x \in \mathbb{R}} |\mathbb{P}(l_n \geq \lceil b_{\log(n)} + x \rceil) - \psi_{K^-}^-(x - \{b_{\log(n)} + x\})| = 0. \quad (44)$$

Le théorème 0.2.5 et certains résultats de Drmota que nous avons cités plus haut nous permettent d'obtenir un résultat analogue au corollaire 0.2.4. Soit  $F_n$  le nombre de feuilles à la plus haute position d'un arbre binaire de recherche à  $n$  feuilles.

**Corollaire 0.2.6.** *Il existe trois fonctions 1-périodiques  $R_1$ ,  $R_2$  et  $R_3$  telles que :*

$$\mathbb{E}(H_n) = a_{\log n} + R_1(a_{\log n}) + o_n(1), \quad (45)$$

$$\text{Var}(H_n) = R_2(a_{\log n}) + o_n(1), \quad (46)$$

$$\mathbb{E}(F_n) = R_3(a_{\log n}) + o_n(1). \quad (47)$$

### 0.2.3 Maximum du mouvement brownien branchant

Historiquement, les résultats les plus précis que nous allons présenter ont été d'abord démontrés pour le modèle du mouvement brownien branchant avant d'être généralisés. C'est pourquoi nous commençons par l'étude des particules extrémales dans ce cadre. On peut ajouter que par symétrie celle-ci se réduit à l'étude du maximum. Pour se conformer à la formulation originale des auteurs, on considère ici un mouvement brownien branchant binaire, c'est-à-dire que les particules se divisent à chaque fois en deux particules et on prend  $\beta = 1$ . Notons toutefois que des résultats récents permettent de les généraliser pour des lois de reproduction vérifiant  $\mathbb{E}(L \log^2 L) < \infty$ .

Mc Kean [79] constate dans son article, que si on appelle  $M_t$  la position de la particule la plus haute du mouvement brownien branchant au temps  $t$  alors  $u(t, x) = \mathbb{P}(M_t \leq x)$  est solution de l'équation KPP :

$$\partial_t u = \frac{1}{2} \partial_{xx}^2 u + u^2 - u, \quad (48)$$

avec condition initiale  $f(x) = \mathbf{1}_{x \geq 0}$ . À partir de ce lien, il parvient à redémontrer des résultats de Kolmogorov, Petrovsky et Piskounov. En particulier, si  $m_t$  est la médiane de  $u$  (c'est-à-dire si  $u(t, m_t) = 1/2$ ) alors :

$$\lim_{t \rightarrow +\infty} u(t, m_t + x) = \phi(x), \quad (49)$$

où  $\phi$  est une onde-voyageuse à vitesse  $\sqrt{2}$  de (48). Nous rappelons qu'une fonction  $f$  est dite onde voyageuse à vitesse  $c$ , si  $(t, x) \mapsto f(x-ct)$  est solution de (48). On peut facilement vérifier que  $f$  est une onde voyageuse si et seulement si  $f$  vérifie l'équation différentielle suivante :

$$\frac{1}{2}f'' + cf' + f^2 - f = 0. \quad (50)$$

En exploitant le lien entre mouvement brownien branchant et équation KPP, Mc Kean améliore l'estimation de  $m_t$ . Poursuivant dans cette voie, Bramson [27] donne l'estimation encore plus précise suivante :

$$m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + C + o_t(1), \quad (51)$$

où  $C \in \mathbb{R}$ .

Enfin, mentionnons que Lalley et Selke [70] donnent la représentation probabiliste de  $\phi$  (définie en (49)) suivante. On considère le processus  $(\partial W_t(\sqrt{2}))$  défini par :

$$\partial W_t(\sqrt{2}) = \sum_{u \in \mathcal{N}_t} (\sqrt{2}t - X_u(t)) e^{\sqrt{2}(X_u(t) - \sqrt{2}t)}, \forall t \geq 0.$$

Ce processus est appelé martingale dérivée, nous en reparlerons ultérieurement ce qui nous permettra de justifier à la fois son nom et sa notation. Le résultat de Lalley et Selke est le suivant.

**Théorème** (Lalley et Selke [70]). *La martingale dérivée converge presque sûrement vers une limite  $\partial W_\infty(\sqrt{2})$  strictement positive. Par ailleurs, il existe une constante  $C > 0$  telle que :*

$$\lim_{s \rightarrow +\infty} \lim_{t \rightarrow +\infty} \mathbb{P}(M_{t+s} \geq m_{t+s} + x | \mathcal{F}_s) = \exp(-C \partial W_\infty(\sqrt{2}) e^{-\sqrt{2}x}), \text{ p.s.}$$

Ceci implique :

$$\phi(x) = \mathbb{E}(\exp(-C \partial W_\infty(\sqrt{2}) e^{-\sqrt{2}x})).$$

#### 0.2.4 Minimum d'une marche aléatoire branchante

Afin d'homogénéiser la présentation des résultats, quitte à en modifier la formulation par rapport à celle de leurs auteurs, nous nous placerons dans le cas borné ("boundary case" en anglais) défini de la manière suivante :

$$\mathbb{E} \left( \sum_{|u|=1} 1 \right) > 1, \quad \mathbb{E} \left( \sum_{|u|=1} e^{-V(u)} \right) = 1, \quad \mathbb{E} \left( \sum_{|u|=1} V(u) e^{-V(u)} \right) = 0. \quad (52)$$

Ces hypothèses peuvent sembler restrictives, mais une simple renormalisation affine permet dans de très nombreux cas de se ramener à cette hypothèse.

Notons  $M_n$  le minimum d'une marche aléatoire branchante. La détermination de la vitesse linéaire date des années 70, voir par exemple Hammersley [55] qui montre que :

$$\lim_{n \rightarrow +\infty} \frac{M_n}{n} = 0, \text{ a.s.}$$

Les hypothèses pour démontrer ce résultat sont extrêmement faibles. Nous ne détaillerons pas les hypothèses des résultats qui vont suivre, mais mentionnerons simplement qu'elles sont généralement plus restrictives que celles imposées par le cas borné. En 2009, Addario-Berry et Reed [6] d'une part, Hu et Shi [61] d'autre part exhibent un terme de second ordre en  $\log n$ . Plus précisément, Addario-Berry et Reed montrent que  $M_n$  est exponentiellement tendu autour de  $3/2 \log n$  et Hu et Shi mettent en évidence des fluctuations asymptotiques de  $M_n$  au sens suivant :

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \frac{M_n}{\log n} &= \frac{3}{2}, \text{ a.s.} \\ \liminf_{n \rightarrow +\infty} \frac{M_n}{\log n} &= \frac{1}{2}, \text{ a.s.} \end{aligned}$$

Enfin, sous des conditions peu restrictives, Aïdékon [7] montre que  $M_n$  recentré en  $3/2 \log n$  converge en distribution. Afin d'énoncer son résultat, introduisons maintenant la martingale dérivée :

$$D_n := \sum_{|u|=n} V(u) e^{-V(u)}, \quad \forall n \in \mathbb{N}. \quad (53)$$

Sous les hypothèses d'Aïdékon, cette martingale converge presque sûrement vers une limite strictement positive  $D_\infty$ . Le résultat d'Aïdékon s'écrit alors :

**Théorème** (Aïdékon [7]). *Il existe  $C > 0$  telle que :*

$$\lim_{n \rightarrow +\infty} \mathbb{P} \left( M_n \geq \frac{3}{2} \log(n) + x \right) = \mathbb{E} \left( e^{-C e^x D_\infty} \right). \quad (54)$$

Là encore, nous ne détaillerons pas les hypothèses de son théorème. Toutefois, l'une d'entre elles est importante à relever dans notre contexte. Aïdékon considère en effet la situation où le support du processus ponctuel  $\mathcal{L}$  n'est pas un réseau (non-lattice case), c'est-à-dire un ensemble de la forme  $c\mathbb{Z} + d$ , où  $c, d \in \mathbb{R}$ . Or, précisément, la marche aléatoire branchante de Yule évolue sur un réseau. Ainsi, notre résultat permet de rendre manifeste l'importance de l'hypothèse posée par Aïdékon et de montrer comment le réseau vient perturber la convergence en distribution.



### 0.3 Mouvement brownien avec absorption

#### 0.3.1 Nombre de particules sur une barrière d'un mouvement brownien branchant

Posons  $m = \mathbb{E}(L)$  et  $\mu_0 = \sqrt{2\beta(m-1)}$ . Pour comprendre ce qui va suivre, revenons au maximum du mouvement brownien branchant. Nous avons donné plus haut des résultats extrêmement précis sur celui-ci dans le cas binaire et rappelé qu'ils pouvaient être étendus au cas  $\mathbb{E}(L \log^2 L) < \infty$ . Dans le cas plus général, où l'on suppose  $1 < m < +\infty$  les résultats suivants :

$$\lim_{t \rightarrow +\infty} \frac{M_t}{t} = \mu_0 \text{ a.s. et } \lim_{t \rightarrow +\infty} M_t - \mu_0 t = -\infty \text{ a.s.}$$

sont encore valides sous réserve de non-extinction.

On considère maintenant un mouvement brownien branchant avec dérive  $\mu$ , c'est-à-dire qu'au lieu de suivre un mouvement brownien simple, les particules suivent un mouvement brownien avec dérive  $\mu$ . Par ailleurs, on tue les particules en  $-x$  et on appelle  $Z_x$ , le nombre de particules ainsi tuées. On appelle alors ce processus, mouvement brownien branchant avec absorption.

Ainsi, toujours sous l'hypothèse  $1 < m < +\infty$ , trois régimes différents émergent selon la valeur de la dérive  $\mu$  par rapport à  $\mu_0$ .

1. Si  $\mu \leq -\mu_0$ , le processus s'éteint et  $Z_x < \infty$  presque sûrement.
2. Si  $|\mu| < \mu_0$ , la probabilité de survie est non-nulle et le nombre de particules absorbées est presque sûrement fini sur l'événement d'extinction et infini sinon.
3. Si  $\mu \geq \mu_0$ , la probabilité de survie est non-nulle et  $Z_x < \infty$  presque sûrement.

Lorsque  $m = +\infty$ , nous pouvons considérer que nous sommes dans le second cas. Le premier cas a sans doute été le plus étudié. En particulier, Neveu [81] montre que dans le cas où  $L \equiv 2$ , et  $\mu = -\mu_0$ , on a  $\mathbb{E}(Z_x) < \infty$  et  $\mathbb{E}(Z_x \log_+(Z_x)) = +\infty$ . Par ailleurs, dans le cadre des marches aléatoires branchantes et pour le cas analogue au premier cas mentionné, à la suite d'une conjecture d'Aldous, des estimations de plus en plus précises de  $T_x$ , le nombre total de particules ayant vécu, ont été apportées (voir [9], [2] et [10]). Si  $T_x$  et  $Z_x$  ne sont pas les mêmes variables, on peut constater facilement qu'on peut déduire l'une de l'autre lorsque  $L$  est une variable aléatoire constante. Pour en revenir au mouvement brownien branchant, plus récemment, le premier cas a été à nouveau étudié par Maillard [78] et le dernier par Berestycki, Brunet, Harris et Miłoś [21].

L'estimation donnée par Maillard du nombre de particules tuées sur la barrière  $-x$ ,  $x > 0$  lorsque  $\mu \leq -\mu_0$  est extrêmement fine. Plus précisément, il montre le résultat suivant. Posons  $\delta$  le pas de  $L$ , c'est-à-dire le plus grand entier tel que le support de  $L - 1$

est sur  $\delta\mathbb{Z}$ . Posons  $\lambda_1 := -\mu + \sqrt{\mu^2 - 2\beta}$ ,  $\lambda_2 := -\mu - \sqrt{\mu^2 - 2\beta}$  et  $d := \lambda_1/\lambda_2$ . Enfin introduisons  $G$  la fonction génératrice de  $L$  définie par :

$$G(s) = \sum_{i=0}^{\infty} p_i s^i, \quad \forall s \in D(0, 1),$$

où  $p_i = \mathbb{P}(L = i)$ .

**Théorème** (Maillard [78]). *Supposons que  $\mathbb{E}[L \log^2 L] < +\infty$ . Si  $\mu = -\mu_0$ , alors :*

$$\mathbb{P}(Z_x > n) \underset{n \rightarrow +\infty}{\sim} \frac{\mu_0 x e^{\mu_0 x}}{n(\log n)^2}. \quad (55)$$

Supposons maintenant que  $R_G$ , le rayon de convergence de  $G$  est strictement plus grand que 1, nous avons alors que :

— Si  $\mu = -\mu_0$  :

$$\mathbb{P}(Z_x = \delta n + 1) \underset{n \rightarrow +\infty}{\sim} \frac{\mu_0 x e^{\mu_0 x}}{\delta n^2 (\log n)^2}. \quad (56)$$

— Si  $\mu < -\mu_0$ , il existe  $K > 0$  tel que :

$$\mathbb{P}(Z_x = \delta n + 1) \underset{n \rightarrow +\infty}{\sim} K \frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{n^{d+1}}. \quad (57)$$

### 0.3.2 Présentation des résultats du chapitre 2

Notre objectif initial était d'étudier le second cas de la section précédente (i.e.  $|\mu| < \mu_0$ ), qui n'avait pas été traité. Dans les deux autres cas, un grand nombre d'informations sur  $Z_x$  sont recueillies grâce à sa fonction génératrice  $F_x$ , définie par :

$$F_x(s) = \mathbb{E}(s^{Z_x}),$$

pour tout  $s < R_F$ , où  $R_F$  est le rayon de convergence de la série entière associée naturellement à  $F_x$ .

Toutefois, dans notre cas, contrairement aux deux autres, le nombre de particules tuées est infini avec probabilité non nulle. Ainsi, lorsque  $R_F > 1$ ,  $F_x(s) = +\infty$  pour  $1 < s < R_F$  alors que la série entière associée à  $F_x$  converge en  $s$ . C'est pourquoi, il est intéressant d'introduire la fonction indicatrice  $\mathbf{1}_{\{Z_x < +\infty\}}$  dans l'espérance pour faire coïncider les deux objets. En rappelant que pour  $|\mu| < \mu_0$ ,  $\{Z_x < +\infty\} = \{\zeta_x < +\infty\}$ , où  $\zeta_x$  est le temps d'extinction du processus, cela revient à considérer  $f_x$  définie par :

$$f_x(s) := \mathbb{E}(s^{Z_x} \mathbf{1}_{\{\zeta_x < \infty\}}) = \sum_{i=0}^{\infty} q_i(x) s^i, \quad s \in \mathbb{R}^+, \quad (58)$$

où

$$q_i(x) = \mathbb{P}(Z_x = i, \zeta_x < \infty). \quad (59)$$

Notons que jusqu'à présent, le terme  $\{\zeta_x < \infty\}$  est superflu dans la définition de  $q_i(x)$ . Nous le conservons, car nous constaterons que l'étude de  $Z_x$  sur l'événement d'extinction pour  $|\mu| < \mu_0$  peut être vu comme un sous-cas naturel de cette même étude pour  $\mu > -\mu_0$ . Or pour  $\mu \geq \mu_0$ , l'étude de  $Z_x$  sur l'événement d'extinction n'est plus équivalente à celle de  $Z_x$  sans conditionnement, c'est pourquoi nous conserverons le terme  $\{\zeta_x < \infty\}$  dans la définition de  $q_i(x)$ .

Pour les raisons précédentes, nous travaillerons ici sous l'hypothèse :

$$\mu > -\mu_0 \text{ et } \zeta_x < +\infty. \quad (60)$$

Le rayon de convergence de  $f_x$ , qu'on notera  $R(\mu)$  (car nous verrons qu'il ne dépend pas de  $x$ ) donne une première information sur  $Z_x$ , en particulier à partir du théorème de Cauchy-Hadamard qui nous dit :

$$\limsup_{n \rightarrow +\infty} |q_n(x)|^{\frac{1}{n}} = \frac{1}{R(\mu)}. \quad (61)$$

Nous établirons donc dans un premier temps le résultat suivant.

**Théorème 0.3.1.** *La fonction  $\mu \mapsto R(\mu)$  est croissante et continue et vérifie :*

$$\lim_{\mu \rightarrow -\mu_0} R(\mu) = 1 \quad (62)$$

et

$$\lim_{\mu \rightarrow +\infty} R(\mu) = R_G. \quad (63)$$

Nous verrons que la distribution de  $Z_x$  sur l'événement d'extinction diffère qualitativement selon que  $R(\mu) = R_G$  ou  $R(\mu) < R_G$ . C'est pourquoi nous avons besoin de savoir si  $R_G$  est atteint ou pas. Posons donc

$$\mu_c = \inf\{\mu \in \mathbb{R}, R(\mu) = R_G\}. \quad (64)$$

Le théorème suivant nous donne une condition nécessaire et suffisante pour que  $R_G$  soit atteint.

**Théorème 0.3.2.**

$$\mu_c < \infty \Leftrightarrow \int_0^{R_G} G(s) ds < +\infty. \quad (65)$$

Dans le cas où  $R(\mu) < R_G$ , c'est-à-dire où  $-\mu_0 < \mu < \mu_c$ , nous pouvons en suivant la méthode de Maillard [78], qui s'appuie sur un lemme de transfert, donner une estimation très précise de  $q_i(x)$ . Afin de présenter notre théorème, nous aurons besoin d'un résultat auxiliaire. Posons  $Q(x) = \mathbb{P}(\zeta_x < \infty)$ . Nous verrons dans une prochaine section que  $Q$

vérifie l'équation des ondes voyageuses :

$$\frac{1}{2}y''(x) + \mu y'(x) + \beta (G(y(x)) - y(x)) = 0. \quad (66)$$

À partir de ce lien, nous établirons :

**Proposition 0.3.3.** *Si  $R_G > 1$ , il existe un intervalle  $I$  ouvert maximal contenant strictement  $\mathbb{R}^+$  tel que  $Q$  soit prolongeable en une fonction  $\tilde{Q}$  qui  $I$  vérifie (66). Par ailleurs, si nous notons  $J$  le sous-intervalle maximal de  $I$  tel que  $\tilde{Q}$  soit décroissante sur  $J$ , alors  $x_0 := \inf J$  vérifie :*

$$-\infty < x_0 < 0.$$

Par abus de notation, on posera  $Q = \tilde{Q}$ . Nous pouvons maintenant énoncer le résultat sur  $q_i(x)$ .

**Théorème 0.3.4.** *Lorsque  $-\mu_0 < \mu < \mu_c$ , nous avons pour  $x > 0$  :*

$$q_{\delta i+1}(x) \underset{i \rightarrow +\infty}{\sim} \frac{-Q'(x_0 + x)}{2R(\mu)^{\delta i + \frac{1}{2}} \sqrt{\delta \beta (G(R(\mu)) - R(\mu))} i^3 \pi}. \quad (67)$$

Les techniques utilisées pour démontrer le théorème précédent s'appuient sur des résultats d'analyse complexe présentés dans [53]. Pour appliquer ces méthodes, il faut que, pour  $x > 0$ ,  $s \mapsto f_x(s)$  puisse être prolongée analytiquement sur un domaine d'un certain type. Cette hypothèse est vérifiée lorsque  $\mu < \mu_c$ , mais n'est plus nécessairement valide pour des valeurs de  $\mu$  supérieure à  $\mu_c$ . C'est pourquoi, nous avons recours à des techniques moins efficaces, mais qui permettent tout de même de mettre en évidence des changements de comportement. Présentons deux de ces résultats. Le premier s'applique dans le cas où  $\mu = \mu_c$  et sous l'hypothèse sur  $G$  suivante :

$$G(s) - s \underset{s \rightarrow R_G}{\sim} C(R_G - s)^{-\alpha}, \quad (68)$$

où  $C > 0$ ,  $\alpha \in [0, 1)$  et  $s \in (q, R_G)$ .

**Proposition 0.3.5.** *Si la condition (68) est vérifiée et que  $\mu = \mu_c$ , alors*

$$\sum_{i=n}^{+\infty} q_i(x) R_G^i \underset{n \rightarrow +\infty}{\sim} \frac{-A Q'(x_0 + x)}{n^{\frac{1+\alpha}{2}}}, \quad (69)$$

$$\text{où } A := \frac{\sqrt{(1-\alpha)R_G^{1+\alpha}}}{(1+\alpha)\sqrt{\beta C}\Gamma(\frac{1-\alpha}{2})}.$$

Cette proposition montre que lorsque  $G(R_G) < \infty$  (ce qui correspond au cas  $\alpha = 0$ ), la distribution de  $Z_x$  a une queue de distribution dont le comportement qualitatif est assez

proche dans les cas  $\mu = \mu_c$  et  $\mu < \mu_c$ , mais que dans le cas inverse, ça n'est plus nécessairement le cas.

Nous présentons maintenant un résultat assez faible pour le cas  $\mu > \mu_c$ , mais qui a l'avantage de ne requérir aucune condition particulière sur  $G$ .

**Proposition 0.3.6.** *Soit  $k \in \mathbb{N}$ , si  $\mu > \mu_c$  nous avons :*

$$\sum_{i=0}^{\infty} q_i(x) R_G^i i^{k+2} < \infty \Leftrightarrow \sum_{i=0}^{\infty} p_i R_G^i i^k < \infty. \quad (70)$$

Ce résultat permet de percevoir que les liens sont beaucoup plus étroits entre la distribution de  $L$  et celle de  $Z_x$  lorsque  $\mu > \mu_c$  que lorsque  $\mu < \mu_c$ . Les méthodes employées ne nous permettent pas de donner une interprétation probabiliste à  $\mu_c$  et il semble difficile à l'heure actuelle d'expliquer pourquoi ce changement de phase intervient.

### 0.3.3 Quelques Martingales

Les résultats précédents sont établis en grande partie grâce à l'équation KPP. Il est donc intéressant de rappeler quelques liens entre le mouvement brownien branchant et cette équation. Dans cette perspective, nous allons introduire quelques martingales. Ici encore, nous conservons les notations de la section 0.1.2. Posons  $c_\lambda = \lambda/2 + \beta(m-1)/\lambda$ , où  $m = \mathbb{E}(L)$ .

1. Le processus  $(W_t(\lambda))$  défini par :

$$W_t(\lambda) = \sum_{u \in \mathcal{N}_t} e^{-\lambda(X_u(t) + c_\lambda t)} \quad (71)$$

est une  $(\mathcal{F}_t)$ -martingale appelée martingale additive.

2. De même, le processus  $(\partial W_t(\lambda))$  défini par :

$$\partial W_t(\lambda) = \sum_{u \in \mathcal{N}_t} (X_u(t) + \lambda t) e^{-\lambda(X_u(t) + c_\lambda t)} \quad (72)$$

est une  $(\mathcal{F}_t)$ -martingale appelée martingale dérivée.

On peut tout d'abord constater, qu'au signe près, la martingale dérivée s'obtient en dérivant la martingale additive par rapport à  $\lambda$ , ce qui justifie son nom. Ces deux martingales jouent un rôle considérable dans de nombreux développements en théorie du branchement. Nous avons déjà croisé à plusieurs reprises la martingale dérivée dans la section consacrée aux particules extrémales. Par ailleurs, ces vingt dernières années, les techniques de décomposition en épine dorsale qui s'appuient sur un changement de probabilité lié à ces

martingales ont permis d'établir un grand nombre de résultats. Plutôt que de donner une définition générale de cette technique, proposons en exemple le résultat de Chauvin et Rouault [34] qui fut d'ailleurs, le premier du genre. Pour simplifier, nous supposons ici que le processus ne peut pas s'éteindre. Considérons le changement de mesure donné par :

$$\frac{d\mathbb{Q}_\lambda}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = W_t(\lambda).$$

Nous avons alors :

**Théorème** (Chauvin, Rouault [34]). *Sous  $\mathbb{Q}_\lambda$ , le mouvement brownien branchant et l'épine peuvent se construire de la manière suivante :*

- *Au temps 0, une particule placée en 0 appartient à l'épine. Cette particule suit un mouvement brownien avec dérive  $\lambda$ .*
- *Au bout d'un temps exponentiel de paramètre  $m\beta$ , cette particule meurt et se divise en  $L'$  particules, où la loi de  $N'$  est donnée par :*

$$\mathbb{Q}_\lambda(L' = k) = \frac{k\mathbb{P}(L = k)}{m}.$$

- *Une des particules ainsi obtenue est choisie uniformément pour faire partie de l'épine et répétera le comportement de sa particule mère.*
- *Les autres particules donneront naissance à des  $\mathbb{P}$ -mouvements browniens branchants recentrés en la position de leur particule mère.*

À partir de cet exemple, nous pouvons comprendre la philosophie des techniques de décomposition en épine dorsale. Au lieu de considérer le mouvement brownien branchant dans son ensemble, la décomposition en épine dorsale permet de ramener certains problèmes à l'étude d'objets simples, comme ici à un mouvement brownien avec dérive, un processus de Poisson d'intensité  $m\beta$  et une loi de reproduction sur  $\mathbb{N}$ . Ces techniques ont permis de démontrer un grand nombre de résultats présentés dans cette introduction notamment ceux d'Hu et Shi [61], d'Aïdékon [8] ou encore les résultats qui vont suivre sur les ondes voyageuses de l'équation FKPP. Pour plus de détails sur ces techniques, nous renvoyons le lecteur à [56].

Notons enfin les résultats de convergence sur les martingales. Posons  $\lambda_0 = \sqrt{2\beta(m-1)}$ .

**Théorème** (Kyprianou [69], Yang, Ren [91]).  *$\lim_{t \rightarrow +\infty} W_t(\lambda) = W_\infty(\lambda)$  existe presque sûrement pour tout  $\lambda \in \mathbb{R}$ , De même,  $\lim_{t \rightarrow +\infty} \partial W_t(\lambda) = \partial W_\infty(\lambda)$  existe presque sûrement pour tout  $|\lambda| \geq \lambda_0$ . Par ailleurs, ces limites vérifient :*

1. *Si  $|\lambda| > \lambda_0$ , alors  $W_\infty(\lambda) = \partial W_\infty(\lambda) = 0$  presque sûrement.*
2. *Si  $|\lambda| = \lambda_0$ , alors  $W_\infty(\lambda) = 0$  presque sûrement. En outre,*
  - (a) *si  $\mathbb{E}(L \log^2 L) = +\infty$  alors  $\partial W_\infty(\lambda) = 0$  presque sûrement,*

- (b) sinon, avec probabilité  $q$ ,  $\partial W_\infty(\lambda) = 0$  et avec probabilité  $1 - q$ ,  $\partial W_\infty(\lambda) > 0$ .
3. Lorsque  $|\lambda| < \lambda_0$ ,
- (a) si  $\mathbb{E}(L \log L) = +\infty$ , alors  $W_\infty(\lambda) = 0$  presque sûrement,
- (b) sinon, avec probabilité  $q$ ,  $W_\infty(\lambda) = 0$  et avec probabilité  $1 - q$  :  $W_\infty(\lambda) > 0$ .

### 0.3.4 Ondes voyageuses de l'équation FKPP

Nous avons brièvement évoqué dans la section consacrée aux particules extrémales un lien entre une onde voyageuse de l'équation KPP et le mouvement brownien branchant. Nous allons dans cette section montrer que ces liens sont plus profonds. On conserve les notations de la section 0.1.2. On considère ici un mouvement brownien dans le cas général et non plus seulement un mouvement brownien binaire avec  $\beta = 1$ . Nous supposons que  $1 < m = \mathbb{E}(L) < \infty$  et l'on note  $q$  le plus petit point fixe de  $G$ , où  $G$  est la fonction génératrice de  $L$ . L'équation KPP s'écrira alors dans notre cadre :

$$\partial_t u = \frac{1}{2} \partial_{xx}^2 u + \beta(G(u) - u), \quad (73)$$

et plus généralement que dans le cas binaire (50),  $f$  sera une onde voyageuse de KPP si et seulement si elle vérifiera :

$$\frac{1}{2} f'' + c f' + \beta(G(f) - f) = 0. \quad (74)$$

Appelons  $E$  l'ensemble des ondes voyageuses  $\phi$  croissantes, définies sur  $\mathbb{R}$  à valeurs dans  $[q, 1]$  et telles que  $\lim_{x \rightarrow -\infty} \phi(x) = q$  et  $\lim_{x \rightarrow +\infty} \phi(x) = 1$ . Grâce à ces martingales, nous pouvons donner une représentation probabiliste aux ondes voyageuses de  $E$ . Posons  $c_0 = \lambda_0 = \sqrt{2\beta(m-1)}$ . Le résultat qui suit est obtenu à partir de [69] et [91].

**Théorème** (Kyprianou [69]). *Selon les vitesses, nous avons :*

1. Si  $|c| < c_0$ , alors  $E$  est vide.
2. Si  $|c| = c_0$  et que  $\mathbb{E}(L \log^2(L)) < \infty$ , alors  $E$  est non-vide. Par ailleurs, toute onde voyageuse de  $E$  est unique à une translation dans l'argument près et s'écrit :

$$\phi(x) = \mathbb{E} \left[ \exp \left( -e^{-\lambda_0 x} \partial W_\infty(\lambda_0) \right) \right].$$

3. Si  $|c| > c_0$  et que  $\mathbb{E}(L \log(L)) < \infty$ , alors  $E$  est non-vide. Posons  $\lambda$  tel que  $c_\lambda = c$ . Dans ce cas, toute onde voyageuse de  $E$  est unique à une translation dans l'argument près et s'écrit :

$$\phi(x) = \mathbb{E} \left[ \exp \left( -C e^{-\lambda x} W_\infty(\lambda) \right) \right].$$

Les ondes voyageuses définies sur  $\mathbb{R}^+$  et non plus sur  $\mathbb{R}$  tout entier peuvent aussi s'exprimer grâce au mouvement brownien branchant. Pour ce faire considérons un mouvement

brownien branchant avec dérive  $\mu$  et absorption. Nous avons évoqué précédemment que  $Q$ , la probabilité d'extinction, satisfaisait l'équation KPP. Plus précisément, à partir des arguments de [58] et [78] on obtient :

**Théorème** (Harris [58], Maillard [78]). *La fonction  $Q$  vérifie l'équation :*

$$\frac{1}{2}y''(x) + \mu y'(x) + \beta(G(y(x)) - y(x)) = 0, \quad y \in \mathcal{C}^2(\mathbb{R}^+, [0, 1]) \quad (75)$$

avec conditions aux bords :

$$y(0) = 1, y(\infty) = q. \quad (76)$$

Par ailleurs,

1. Si  $\mu > -\mu_0$ , alors  $Q$  est non-triviale et est l'unique solution de (75) satisfaisant les conditions aux bords (76).
2. Si  $\mu \leq -\mu_0$ , alors  $Q \equiv 1$  et (75) n'admet pas de solutions avec conditions aux bords décrites par (76).

## 0.4 Graphe d'Erdős–Rényi

### 0.4.1 Présentation du modèle

Nous commençons par définir le graphe d'Erdős–Rényi. Tout d'abord, notons que deux modèles légèrement différents portent ce nom. L'un d'eux, qui fut le premier introduit par Erdős et Rényi [50] est construit en munissant  $[n] = \{1, \dots, n\}$  d'une structure de graphe en ouvrant  $N$  arêtes distinctes uniformément parmi les  $n(n-1)/2$  arêtes possibles, où  $N$  est un entier fixé. Nous travaillerons pour notre part avec l'autre où chaque arête est ouverte avec probabilité  $p$  et on notera ce graphe aléatoire  $G(n, p)$ . On remarquera simplement qu'un grand nombre de résultats peuvent se transposer facilement d'un modèle à l'autre.

A priori, ce modèle semble éloigné des processus de branchements dont nous avons parlé jusqu'ici. Par ailleurs, à partir de sa seule définition, le graphe d'Erdős–Rényi peut paraître assez chaotique. Néanmoins, nous verrons que pour certaines valeurs de  $p$  la structure d'arbre émerge naturellement. Avant de discuter des arbres en question, rappelons quelques résultats classiques sur ce graphe.

On considère dorénavant que  $p$  dépend de  $n$  et on notera cette probabilité  $p_n$ . Une des premières questions que l'on peut se poser sur ce modèle est de déterminer de quel ordre de grandeur sont les tailles des composantes connexes lorsque  $n$  tend vers l'infini pour des  $p_n$  bien choisis. On pourra trouver dans [25] un grand nombre d'exemples dont certains seront mentionnés au chapitre 3. Afin d'illustrer notre propos, nous nous contenterons ici de parler des  $p_n$  de la forme  $p_n = c/n$ , avec  $c > 0$ , qui ont été parmi les premières à être



étudiées dans [51] (ou plus exactement leurs analogues pour la première version du graphe d'Erdős–Rényi citée plus haut).

Posons  $Z_i^n$  la taille de la  $i^e$  plus grande composante du graphe  $G(n, p_n)$ . Selon les valeurs de  $c$  nous avons :

1. Si  $c > 1$ , alors  $Z_1^n = \Theta(n)$  et  $Z_i^n = \Theta(\log(n))$  pour  $i \geq 2$ .
2. Si  $c = 1$ , alors  $Z_i^n = \Theta(n^{2/3})$ , pour tout  $i \in \mathbb{N}^*$ .
3. Si  $c < 1$ , alors  $Z_i^n = \Theta(\log(n))$ , pour tout  $i \in \mathbb{N}^*$ .

Cette disjonction de cas justifie qu'on appelle le cas  $p_n = 1/n$ , cas critique. Dans ce cas, et plus généralement pour des  $p_n$  de la forme  $p_n(\lambda) = 1/n + \lambda n^{-4/3}$ ,  $\lambda \in \mathbb{R}$ , Aldous [16] apporte des résultats beaucoup plus précis et fait apparaître un lien entre le graphe d'Erdős–Rényi et un objet appelé coalescent multiplicatif.

De manière informelle, les coalescents markoviens dont nous allons parler peuvent être définis de la manière suivante. On considère une suite finie ou dénombrable de réels positifs  $(m_i)$  qui peuvent être vues comme une suite de masses de particules. À chaque instant, chaque paire de particules de masse  $m_i$  et  $m_j$  fusionnent en une particule de masse  $m_i + m_j$  à taux  $\kappa(m_i, m_j)$ . Un coalescent est multiplicatif lorsque  $\kappa(x, y) = xy$ .

Nous aurons besoin à plusieurs reprises par la suite de considérer que les graphes  $G(n, p_n(\lambda))$  sont couplés. C'est pourquoi nous introduisons  $(U_{i,j})_{1 \leq i < j}$  une suite variables aléatoires indépendantes uniformément distribuées sur  $[0, 1]$ . Dans tout ce qui suit, pour  $1 \leq i < j \leq n$ , une arête  $ij$  sera ouverte dans  $G(n, p_n(\lambda))$  si et seulement si  $U_{i,j} < p_n(\lambda)$ .

Définissons maintenant quelques objets nécessaires à la présentation des résultats d'Aldous. On considère le mouvement brownien avec dérive parabolique  $(W_t(\lambda))$  défini par :

$$W_t(\lambda) = B_t + \lambda t - \frac{t^2}{2}, \quad \forall (t, \lambda) \in \mathbb{R}^+ \times \mathbb{R}, \quad (77)$$

où  $(B_t)$  est un mouvement brownien standard. À partir de  $(W_t(\lambda))$ , on définit le mouvement brownien avec dérive parabolique réfléchi  $(\beta_t(\lambda))$  par :

$$\beta_t(\lambda) = W_t(\lambda) - \inf_{0 \leq s \leq t} W_s(\lambda), \quad \forall (t, \lambda) \in \mathbb{R}^+ \times \mathbb{R}. \quad (78)$$

Aldous montre que les excursions de  $(\beta_t(\lambda))$  au-dessus de zéro peuvent être classées dans l'ordre décroissant de longueurs. Par ailleurs, en fixant  $(e_i(\lambda))_{i \geq 1}$  la suite des excursions ainsi classées et en posant  $(l_i(\lambda))_{i \geq 1}$  la suite de leurs longueurs, il montre que  $(l_i(\lambda)) \in (l^2)^\downarrow$ , où  $l^2 = \{u \in \mathbb{R}^{\mathbb{N}^*}, \sum_i u_i^2 < \infty\}$  et où la notation  $S^\downarrow$ , lorsque  $S$  est un ensemble de suites, désigne le sous-ensemble des suites décroissantes de  $S$ . Rappelons dès à présent que  $l^2$  est muni naturellement d'une structure d'espace métrique en considérant la distance  $d_2$  définie

par  $d_2(u, v) = \sqrt{\sum_i |u_i - v_i|^2}$ .

Ces objets ainsi définis, nous pouvons maintenant présenter les résultats principaux de [16]. Définissons comme précédemment  $Z_i^n(\lambda)$  la taille de la  $i^e$  composante connexe de  $G(n, p_n(\lambda))$  et  $S_i^n(\lambda)$  son surplus, c'est-à-dire le nombre d'arêtes à lui ôter pour obtenir un arbre. La suite des excursions étant donnée, on considère une suite de variables aléatoires indépendantes  $(P_i(\lambda))_{i \geq 1}$  telle que pour chaque  $i \geq 1$ ,  $P_i(\lambda)$  soit distribuée selon une loi de Poisson de paramètre  $\int_0^{l_i(\lambda)} e_i(s) ds$ . En premier lieu, Aldous établit le résultat suivant :

**Théorème** (Aldous [16]). *Lorsque  $n \rightarrow +\infty$ ,*

$$(n^{-2/3} Z_i^n(\lambda), S_i^n(\lambda))_{i \geq 1} \xrightarrow{d} (l_i(\lambda), P_i(\lambda))_{i \geq 1},$$

où la première convergence a lieu pour la distance  $d_2$  et la seconde pour la topologie produit.

Aldous s'intéresse par ailleurs au comportement lorsque  $n$  tend vers l'infini de  $\lambda \mapsto n^{-2/3} Z_i^n(\lambda)$ . Il montre que ce processus converge vers un coalescent multiplicatif  $Y$  qu'il appelle coalescent multiplicatif standard.

**Théorème** (Aldous [16]).

$$(n^{-2/3} Z_i^n(\lambda), \lambda \in \mathbb{R}) \xrightarrow{d} (Y(\lambda), \lambda \in \mathbb{R}),$$

où la convergence a lieu au sens de Skorokhod sur  $\mathbb{D}(\mathbb{R}, (l^2)^\downarrow)$ .

Les travaux d'Addario-Berry, Broutin et Goldschmidt [4] puis des mêmes auteurs et de Miermont [5] permettent d'approfondir la compréhension du lien qui unit le graphe d'Erdős-Rényi dans le cas critique et le mouvement brownien avec dérive parabolique en montrant que les composantes connexes de ce graphe, vues comme espaces métriques mesurés, convergent en distribution en un sens que l'on précisera vers des objets limites définis à partir du mouvement brownien avec dérive parabolique. Pour comprendre la nature des objets en question, nous aurons besoin d'introduire la notion d'arbre réel.

### 0.4.2 Arbres réels

S'il est très facile de concevoir des arbres tels que ceux de Galton-Watson, il est aussi intéressant de se demander comment s'étend la définition d'arbre pour des objets plus complexes et surtout à quoi ressemblent de tels objets. Nous donnons donc la définition suivante d'arbre réel.

**Définition.** *Un espace métrique  $(\mathcal{T}, d)$  est appelé un arbre réel (ou simplement un  $\mathbb{R}$ -arbre), si pour tout  $t_1, t_2 \in \mathcal{T}$ , les deux propriétés suivantes sont vérifiées :*

- Il existe une unique isométrie  $\phi$  entre  $[0, d(t_1, t_2)]$  et  $\mathcal{T}$  telle que  $\phi(0) = t_1$  et  $\phi(d(t_1, t_2)) = t_2$ .
- Si  $\psi : [0, 1] \rightarrow \mathcal{T}$  est une fonction continue et injective de  $[0, 1]$  dans  $\mathcal{T}$ , telle que  $\psi(0) = t_1$  et  $\psi(1) = t_2$ , alors :

$$\psi([0, 1]) = \phi([0, d(t_1, t_2)]).$$

Nous donnerons dans le chapitre 3 une définition équivalente à celle-là. Pour mieux décrire la structure des arbres, nous pouvons introduire les objets suivants. Soit  $\mathcal{T}$  un arbre réel et  $x$  un point de cet arbre. Le nombre de composantes connexes de  $\mathcal{T} \setminus \{x\}$  sera appelé degré de  $x$  et noté  $\text{deg}_{\mathcal{T}}(x)$ . Si  $\text{deg}_{\mathcal{T}}(x) = 1$ , alors  $x$  sera appelé feuille. L'ensemble des éléments de degré supérieur ou égal à 2 sera appelé squelette et sera noté  $\text{Skel}(\mathcal{T})$ . Si  $\text{deg}_{\mathcal{T}}(x) \geq 3$ , alors  $x$  sera appelé point de branchement.

Évidemment, si nous munissons les arbres de Galton-Watson de la distance usuelle de graphe et considérons que chaque arête entre deux nœuds est un segment de longueur 1, alors ce sont des arbres réels.

La construction d'arbres réels plus généraux qui va suivre, due à Duquesne et Le Gall [49], nous servira dans la présentation de nos résultats. Soit  $h$  une fonction réelle continue positive non-identiquement nulle de support compact  $[0, \sigma]$ , telle que  $h(0) = 0$ . La fonction  $d_h$  définie par :

$$d_h(x, y) = h(x) + h(y) - 2 \inf_{x \leq s \leq y} h(s) \quad (79)$$

est une semi-distance sur  $[0, \sigma]$ . En introduisant la relation d'équivalence  $\sim$  définie par  $x \sim y$  si  $d_h(x, y) = 0$ , nous obtenons un espace métrique :

$$\mathcal{T}_h = \mathbb{R}^+ / \sim, \quad (80)$$

qui est un arbre réel compact. Par ailleurs, la mesure image de la projection canonique de  $[0, \sigma]$  sur  $\mathcal{T}_h$  muni de sa tribu borélienne permet de définir un espace métrique mesuré.

### 0.4.3 Convergence des composantes du graphe d'Erdős-Rényi

Ces notions étant établies, nous allons pouvoir définir les objets limites vers lesquels les composantes d'Erdős-Rényi convergent. Nous avons vu qu'à partir d'une fonction positive  $h$  à support compact, nous pouvions définir un arbre réel  $\mathcal{T}_h$ . Par surcroît, nous pouvons à partir de  $\mathcal{T}_h$  construire de nouveaux espaces métriques mesurés en identifiant des points de  $\mathcal{T}_h$ . Plus précisément, donnons-nous un ensemble fini de points  $P$  inclus dans  $\mathcal{A}_h$  défini par :

$$\mathcal{A}_h := \{(x, y), x \in [0, \sigma], y \leq h(x)\}. \quad (81)$$

Définissons pour  $(x, y)$  dans  $P$ ,

$$r(x, y) := \inf\{x' \geq x, h(x') = y\}. \quad (82)$$

Soit  $\tau$  la projection canonique de  $[0, \sigma]$  sur  $\mathcal{T}_h$ . En identifiant  $\tau(x)$  et  $\tau(r(x, y))$  pour tout  $(x, y)$  dans  $P$  nous obtenons un nouvel espace métrique mesuré que l'on notera  $(g(h, P), d_{h, P}, \mu_{h, P})$ . Nous détaillerons dans le chapitre 3 comment construire rigoureusement à la fois ce procédé d'identification mais aussi la métrique sur  $g(h, P)$ . Pour l'instant, contentons-nous de dire que pour deux points  $t_1, t_2$  de  $g(h, P)$ , la distance  $d_{h, P}(t_1, t_2)$  est définie comme la longueur du plus court chemin sur le nouvel objet entre  $t_1$  et  $t_2$ . Le dessin ci-dessous illustre cette construction. La fonction en haut du schéma encode l'arbre en bas à gauche (les longueurs ne sont pas tout à fait respectées sur le schéma, mais la forme de l'arbre l'est). Pour simplifier, on considère que  $P$  est réduit à un point  $(x, y)$  représenté par la croix. La feuille  $\tau(x)$  est représentée par le triangle et la feuille  $\tau(r(x, y))$  par le carré. Le schéma en bas à droite représente le résultat du collage de  $\tau(x)$  et  $\tau(r(x, y))$ .

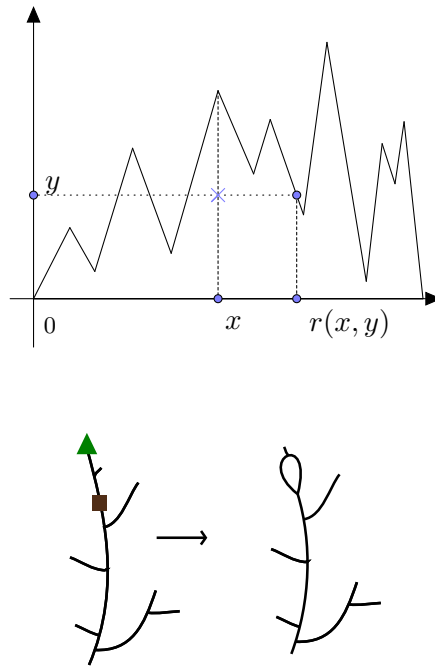


FIGURE 1 – Procédure de collage sur un arbre fini.

Considérons pour chaque  $i \in \mathbb{N}$  un processus ponctuel de Poisson sur  $\mathcal{A}_{e_i(\lambda)}$ , où nous rappelons que  $e_i(\lambda)$  est la  $i^e$  excursion de  $(\beta_t(\lambda))$ , ayant pour mesure d'intensité la mesure de Lebesgue. Par le procédé décrit plus haut, on peut définir pour chaque  $i$  un espace métrique mesuré  $(g(e_i(\lambda), P_i), d_{e_i(\lambda), P_i}, \mu_{e_i(\lambda), P_i})$  qu'on notera plus simplement  $(\mathcal{M}_i(\lambda), d_i, \mu_i)$  et qui sera l'objet limite dont nous avons parlé. En outre,  $\mathbb{M}(\lambda)$  désignera l'ensemble

$\sqcup_{i \geq 1} \mathcal{M}_i(\lambda)$  muni de la distance  $d_U$  définie par :

$$d_U(x, y) = \begin{cases} d_i(x, y) & \text{s'il existe } i \text{ tel que } x, y \in \mathcal{M}_i(\lambda), \\ \infty & \text{sinon} \end{cases} \quad (83)$$

et muni de la mesure borélienne  $\mu$  définie par :

$$\mu(A) = \sum_{i=1}^{+\infty} \mu_i(A \cap \mathcal{M}_i(\lambda)), \forall A \in \mathcal{B}(\mathbb{M}(\lambda)). \quad (84)$$

Nous appellerons enfin  $\hat{\mathbb{M}}(\lambda)$  la suite  $(\mathcal{M}_i(\lambda), d_i, \mu_i)_{i \geq 1}$ .

Notons dès à présent que, pour chaque  $i \in \mathbb{N}^*$ ,  $\mathcal{M}_i(\lambda)$  est un  $\mathbb{R}$ -graphe, c'est-à-dire que pour tout  $x \in \mathcal{M}_i(\lambda)$ , il existe  $\epsilon > 0$  tel que  $\mathcal{T}_{x, \epsilon} = \mathcal{M}_i(\lambda) \cap B(x, \epsilon)$  soit un arbre réel. On peut vérifier facilement que le degré de  $x$  dans  $\mathcal{T}_{x, \epsilon}$  ne dépend pas de l' $\epsilon$  choisi et par conséquent on notera  $\deg_{\mathcal{M}_i(\lambda)}(x)$  ce nombre. Tout comme pour les arbres réels, le squelette de  $\mathcal{M}_i(\lambda)$  qu'on notera  $\text{Skel}(\mathcal{M}_i(\lambda))$  sera l'ensemble des points  $x \in \mathcal{M}_i(\lambda)$  tels que  $\deg_{\mathcal{M}_i(\lambda)}(x) \geq 2$ . Et de même, on notera  $\text{Skel}(\mathbb{M}(\lambda)) = \cup_i \text{Skel}(\mathcal{M}_i(\lambda))$ .

Pour finir, rappelons qu'on peut munir  $G(n, p_n(\lambda))$  d'une structure d'espace métrique mesuré en en considérant  $\delta$  la distance de graphe et  $\nu$  la mesure de comptage. Pour obtenir des résultats de convergence, nous utiliserons des versions renormalisées de ces objets en posant  $\delta^n = n^{-1/3}\delta$  et  $\nu^n = n^{-2/3}\nu$ . Pour  $i \in \mathbb{N}^*$ , définissons  $\mathcal{C}_i^n(\lambda)$  la  $i^e$  composante connexe par ordre de taille de  $G(n, p_n(\lambda))$ . On munira pareillement  $\mathcal{C}_i^n(\lambda)$  d'une structure d'espace métrique mesuré en considérant  $\delta_i^n$  et  $\nu_i^n$ , les restrictions respectives de  $\delta^n$  et  $\nu^n$  à  $\mathcal{C}_i^n(\lambda)$ . La suite  $(\mathcal{C}_i^n(\lambda), \delta_i^n, \nu_i^n)_{i \geq 1}$  sera notée  $\hat{G}(n, p_n(\lambda))$ .

Enfin, l'espace dans lequel se produiront nos convergences sera  $(\mathbb{L}_4, \text{dist}_{GHP}^4)$  défini dans la section 0.4.5. Le résultat fondamental de [4] amélioré dans [5] s'écrit de la manière suivante.

**Théorème** (Addario-Berry, Broutin, Goldschmidt, Miermont [4], [5]). *Pour chaque  $\lambda \in \mathbb{R}$  fixé,*

$$\hat{G}(n, p_n(\lambda)) \xrightarrow{d} \hat{\mathbb{M}}(\lambda) \quad (85)$$

*pour la distance  $\text{dist}_{GHP}^4$  sur  $\mathbb{L}_4$ .*

#### 0.4.4 Présentation des résultats du chapitre 3

L'objectif du chapitre 3 est d'améliorer la compréhension qui nous est donnée par le théorème 0.4.3 du comportement asymptotique de  $\Upsilon(n) := \hat{G}(n, p_n(0))$  lorsque  $n$  tend vers l'infini (nous avons pris  $\lambda = 0$  uniquement pour simplifier les notations, mais le

cas général se traite exactement de la même manière). Pour ce faire, nous allons étudier  $\lambda \mapsto \Upsilon(n + \lfloor \lambda n^{2/3} \rfloor)$  lorsque  $n$  tend vers l'infini.

Notons tout d'abord qu'un simple développement limité nous donne :

$$G\left(n + \lfloor \lambda n^{2/3} \rfloor, \frac{1}{n + \lfloor \lambda n^{2/3} \rfloor}\right) = G\left(n + \lfloor \lambda n^{2/3} \rfloor, \frac{1}{n} - \frac{\lambda}{n^{4/3}} + \underset{n \rightarrow +\infty}{o}\left(\frac{1}{n^{4/3}}\right)\right).$$

Ce fait et les indépendances entre les variables  $U_{i,j}$  de notre modèle permettent de réduire notre problème à l'étude de  $F$  et  $C$  définis par :

$$F(n, \lambda) = \hat{G}\left(n, \frac{1}{n} - \frac{\lambda}{n^{4/3}}\right) \text{ et } C(n, \lambda) = \hat{G}\left(n + \lfloor \lambda n^{2/3} \rfloor, \frac{1}{n}\right), \quad (86)$$

pour tout  $n \geq 1$  et tout  $\lambda \in \mathbb{R}$ .

Commençons par l'étude de  $F$ . Fixons  $\lambda_0 \in \mathbb{R}$ . Posons  $N$  un processus ponctuel de Poisson sur  $\mathbb{R}_+ \times \text{Skel}(\mathbb{M}(\lambda))$ , avec mesure d'intensité  $d\lambda \otimes dl$ , où  $dl$  est la mesure de longueur sur  $\text{Skel}(\mathbb{M}(\lambda))$ . Nous définissons  $N_\lambda$  par  $N_\lambda(A) = N([0, \lambda] \times A)$ , pour tout borélien  $A \in \mathcal{B}(\text{Skel}(\mathbb{M}(\lambda)))$ . L'opérateur Frag sera défini précisément au chapitre 3. Grossièrement parlant, si  $X$  est un espace métrique et  $\Pi$  un processus ponctuel simple, alors  $\text{Frag}(X, \Pi)$  coupe  $X$  selon les points de  $\Pi$ .

**Théorème 0.4.1.** *Lorsque  $n \rightarrow +\infty$ , nous avons :*

$$(F(n, \lambda - \lambda_0))_{\lambda \geq 0} \xrightarrow{d} \text{Frag}(\mathbb{M}(\lambda_0), N_\lambda)_{\lambda \geq 0}, \quad (87)$$

au sens des distributions finies-dimensionnelles pour la distance  $\text{dist}_{GHP}^4$  sur  $\mathbb{L}_4$ .

Considérons maintenant ce processus limite lorsque nous renversons le temps et définissons  $\overleftarrow{F}(n, \lambda)$  par :

$$\overleftarrow{F}(n, \lambda) = F(n, -\lambda), \lambda \in \mathbb{R}. \quad (88)$$

Considérons  $P$  un processus ponctuel de Poisson sur  $\mathbb{R}_+ \times \mathbb{M}(\lambda_0) \times \mathbb{M}(\lambda_0)$  avec mesure intensité  $\frac{1}{2}d\lambda \otimes \mu \otimes \mu$  et fixons  $P_\lambda(A) := P([0, \lambda] \times A)$ , pour tout  $A \in \mathcal{B}(\mathbb{M}(\lambda_0) \times \mathbb{M}(\lambda_0))$ . L'opérateur Coal sera aussi défini plus précisément dans le chapitre 3. On peut le voir de la manière suivante. Si  $X$  est un espace métrique mesuré et  $\Pi_2$  un ensemble de couple de points de  $X$ , alors  $\text{Coal}(X, \Pi_2)$  colle  $x$  et  $y$  pour chaque  $(x, y) \in \Pi_2$ .

**Théorème 0.4.2.**

$$(\overleftarrow{F}(n, \lambda + \lambda_0))_{\lambda \geq 0} \xrightarrow{d} \text{Coal}(\mathbb{M}(\lambda_0), P_\lambda)_{\lambda \geq 0}, \quad (89)$$

au sens des distributions finies-dimensionnelles pour la distance  $\text{dist}_{GHP}^4$  sur  $\mathbb{L}_4$ .

Il nous reste à traiter le cas de  $C$ . Étant donné que son inversé en temps  $\overleftarrow{C}$  défini par :

$$\overleftarrow{C}(n, \lambda) = C(n, -\lambda), \lambda \geq -n^{1/3} \quad (90)$$

est plus simple à étudier, nous commencerons par l'étude de celui-ci. Le résultat suivant nous montre qu'asymptotiquement  $\overleftarrow{C}$  et  $F$  convergent en distribution vers le même processus.

**Théorème 0.4.3.**

$$(\overleftarrow{C}(n, \lambda - \lambda_0))_{\lambda \geq 0} \xrightarrow{d} \text{Frag}(\mathbb{M}(\lambda_0), N_\lambda)_{\lambda \geq 0}, \text{ as } n \rightarrow +\infty \quad (91)$$

au sens des distributions finies-dimensionnelles pour la distance  $\text{dist}_{GHP}^4$  sur  $\mathbb{L}_4$ .

Étant donné que  $\overleftarrow{C}$  et  $F$  convergent en distribution vers le même processus, il en est de même pour leurs retournés en temps, c'est-à-dire pour  $C$  et  $\overleftarrow{F}$ . Ainsi, nous avons :

**Théorème 0.4.4.**

$$(C(n, \lambda + \lambda_0))_{\lambda \geq 0} \xrightarrow{d} \text{Coal}(\mathbb{M}(\lambda_0), P_\lambda)_{\lambda \geq 0}, \quad (92)$$

au sens des distributions finies-dimensionnelles pour la distance  $\text{dist}_{GHP}^4$  sur  $\mathbb{L}_4$ .

On déduit facilement des théorèmes 0.4.1 et 0.4.4 le comportement asymptotique de  $\Upsilon(n + \lfloor \lambda n^{2/3} \rfloor)$ . L'opérateur FC sera, comme les précédents, défini au chapitre 3. On peut concevoir FC de la manière suivante. Étant donné  $X$  un espace métrique mesuré et  $\Pi$  et  $\Pi_2$  définis comme précédemment tels qu'aucun point de  $X$  n'appartient à un couple de  $\Pi_2$ ,  $\text{FC}(X, \Pi, \Pi_2)$  coupe simultanément  $X$  selon les points de  $\Pi$  et colle les points  $x$  et  $y$  tels que  $(x, y) \in \Pi_2$ . Nous avons alors le résultat suivant :

**Théorème 0.4.5.**

$$(\Upsilon(n + \lfloor \lambda n^{2/3} \rfloor))_{\lambda \geq 0} \xrightarrow{d} \text{FC}(\mathbb{M}(\lambda_0), P_\lambda, N_\lambda)_{\lambda \geq 0}, \quad (93)$$

au sens des distributions finies-dimensionnelles pour la distance  $\text{dist}_{GHP}^4$  sur  $\mathbb{L}_4$ .

### 0.4.5 Distance de Gromov-Hausdorff-Prokhorov

Les convergences que nous avons présentées font intervenir une distance sur l'ensemble des classes d'équivalence d'espaces métriques mesurés compacts, appelée distance de Gromov-Hausdorff-Prokhorov. Plus exactement, dans notre chapitre 3, nous en introduirons deux qui sont équivalentes (et même égales lorsque les mesures considérées sont des mesures de probabilités, voir Miermont [80]) et qui portent le même nom dans la littérature. Pour les besoins de cette introduction, nous n'en introduirons qu'une seule, qui

nous semble un peu plus intuitive.

Commençons par définir les distances de Hausdorff et de Prokhorov. Soit  $(M, \delta)$  un espace métrique polonais. Pour  $A \subset M$ , nous définissons  $A^\epsilon$ , l' $\epsilon$ -épaisi de  $A$  par :

$$A^\epsilon = \{x \in M, \exists y \in A, \delta(x, y) < \epsilon\}.$$

Soient  $K$  et  $K'$  deux sous-ensembles de  $M$  compacts. La distance de Hausdorff entre  $K$  et  $K'$  est définie par :

$$\delta_H(K, K') = \inf\{\epsilon > 0, K^\epsilon \supset K', (K')^\epsilon \supset K\}.$$

Posons maintenant  $\mathcal{M}_f(M)$  l'ensemble des mesures boréliennes finies sur  $M$ . Soient  $\mu$  et  $\nu$  deux mesures de  $\mathcal{M}_f(M)$ . La distance de Prokhorov entre  $\mu$  et  $\nu$  est définie par :

$$\delta_P(\mu, \nu) = \inf\{\epsilon > 0, \mu(A) \leq \nu(A_\epsilon) + \epsilon, \nu(A) \leq \mu(A_\epsilon) + \epsilon, \text{ pour tout fermé } A\}.$$

Pour comparer deux espaces métriques mesurés compacts  $(K, d, \mu)$  et  $(K', d', \mu')$ , en particulier lorsqu'ils ne sont pas définis sur le même espace, il est naturel de considérer leurs classes d'équivalences par isométrie  $[K, d, \mu]$  et  $[K', d', \mu']$ . Nous noterons  $\mathbb{K}$  l'ensemble des classes d'équivalences d'espaces métriques mesurés compacts. On peut munir  $\mathbb{K}$  d'une structure d'espace métrique en introduisant la distance de Gromov-Hausdorff-Prokhorov définie pour tout  $[K, d, \mu], [K', d', \mu'] \in \mathbb{K}$  par :

$$\delta_{GHP}([K, d, \mu], [K', d', \mu']) = \inf_{\phi, \phi'} (\delta_H(\phi(K), \phi'(K')) \vee \delta_P(\phi_*(\mu), \phi'_*(\mu'))), \quad (94)$$

où  $\phi$  (resp.  $\phi'$ ) parcourt l'ensemble des plongements isométriques de  $M$  (resp. de  $M'$ ) dans un espace métrique polonais  $(Z, d_Z)$ . Cette distance est particulièrement agréable, en particulier pour faire des probabilités, puisque l'espace  $(\mathbb{K}, \delta_{GHP})$  est polonais (voir [1]). Notons dès à présent que dès lors que cela ne posera pas de problème, on confondra une classe d'équivalence  $[K, d, \mu]$  avec un de ses représentants.

Définissons enfin à partir de cette distance l'espace dans lequel nous travaillerons. Posons  $Z_1$  la classe d'équivalence de  $(\{0\}, 0, 0)$  et  $Z = (Z_1, Z_1, Z_1, \dots)$ . Pour  $u, v \in \mathbb{K}^{\mathbb{N}^*}$ , on définit  $dist_{GHP}^4(u, v)$  par :

$$dist_{GHP}^4(u, v) = \left( \sum_{i=1}^{+\infty} d_{GHP}^4(u_i, v_i) \right)^{1/4}$$

et  $\mathbb{L}_4$  par :

$$\mathbb{L}_4 = \{u \in \mathbb{K}^{\mathbb{N}^*}, dist_{GHP}^4(u, Z) < \infty\}.$$



### 0.4.6 Encodage d'arbres finis

Il est quelquefois difficile de manipuler certains arbres directement, c'est pourquoi on a recours à des fonctions auxiliaires. Considérons donc un arbre  $\mathcal{T}$  ayant  $n$  nœuds, enraciné et dont les nœuds sont étiquetés selon les conventions d'Ulam-Harris. Pour simplifier, nous réétiquetons l'arbre en désignant chaque nœud par sa place selon l'ordre lexicographique en commençant par 0. Les fonctions suivantes encodent les arbres, dans la mesure où pour chacune d'entre elles correspond un unique arbre.

Une première fonction naturelle à associer à l'arbre est sa fonction de hauteur  $H$  qui à chaque entier  $i \in \{0, n-1\}$  associe la hauteur dans l'arbre du nœud étiqueté par  $i$ .

La fonction de contour est une autre fonction qui encode l'arbre et qui est assez proche de la fonction précédente. On peut la définir algorithmiquement en parcourant l'arbre de la manière suivante. A chaque étape  $k$ , on définit  $C(k)$  la fonction de contour à l'étape  $k$ ,  $L(k)$  la liste des nœuds parcourus à l'étape  $k$  et  $v_k$  le nœud sur lequel on se situe. On pose  $C(0) = 0$ ,  $v_0 = \emptyset$  et  $L_0 = \{v_0\}$ . Supposons que, pour  $k \in \mathbb{N}$ ,  $C$  soit construite sur  $\{0, \dots, k\}$  et que  $v_k$  et  $L_k$  soit définis. Si  $v_k$  est la racine et que tous les nœuds de l'arbre ont été parcourus alors on arrête l'algorithme. Sinon, dans le cas où  $v_k$  possède des enfants qui n'ont pas encore été parcourus, alors  $v_{k+1}$  est le premier d'entre eux dans l'ordre lexicographique et on adjoint  $v_{k+1}$  à  $L_k$ . Dans le cas contraire,  $v_{k+1}$  est la particule mère de  $v_k$  et  $L_k = L_{k+1}$ . Il est facile de voir que chaque nœud  $u$  est parcouru  $(N_u + 1)$  fois, où  $N_u$  est le nombre d'enfants de  $u$ . Sommant sur les  $n$  nœuds, on obtient que  $C$  est définie sur  $\{0, 2n-2\}$ .

Enfin, mentionnons la marche de Lukasiewicz. Elle est définie récursivement de la manière suivante sur  $\{0, \dots, n-1\}$  :

$$\begin{cases} W_0 = 0 \\ W_{k+1} = W_k + N_k - 1 \end{cases}, \quad (95)$$

où  $N_k$  est le nombre d'enfants du nœud étiqueté par  $k$ .

Ces fonctions sont très utiles pour décrire les arbres de Galton-Watson en raison de leurs liens avec des marches aléatoires. Nous allons mentionner quelques-uns de ces liens qui mettent en lumière l'intérêt de manipuler de telles fonctions. Considérons donc un arbre de Galton-Watson critique ou sous-critique de loi de reproduction  $\mu$ . On pose  $Z$  le nombre total de nœuds de l'arbre.

On définit la marche aléatoire  $(S_n)$  par  $S_0 = 0$ ,  $S_n = \sum_{i \in [n]} X_i$ , où la distribution de

$X_i$  est donnée par  $\mathbb{P}(X_i = k) = \mu(\{k + 1\}), \forall k \in \mathbb{N}$ . Fixons  $T_{-1}$  le temps d'arrêt pour la marche aléatoire  $(S_n)$  défini par :

$$T_{-1} = \inf\{n \geq 0, S_n = -1\}.$$

Nous avons alors :

**Théorème** (Pitman [82]). *Les marches  $(W_k)_{k \in \{0, \dots, Z\}}$  et  $(S_k)_{k \in \{0, \dots, T_{-1}\}}$  ont même lois. Par ailleurs, pour tout  $k \in \mathbb{N}$ , nous avons :*

$$\mathbb{P}(Z = k) = \mathbb{P}(T_{-1} = k) = \frac{1}{k} \mathbb{P}(S_k = k - 1).$$

Les fonctions de cette section jouent un grand rôle dès lors qu'on cherche à obtenir des résultats asymptotiques. En particulier le résultat d'Addario-Berry, Broutin et Goldschmidt [4] s'appuie sur ces fonctions.

#### 0.4.7 Arbres browniens et coalescent additif

##### Arbres browniens

L'arbre brownien, introduit par Aldous dans [12], a été l'un des premiers arbres réels non-triviaux à être défini. Il est lié à nos travaux et permet en outre de construire un coalescent, appelé coalescent additif, qui présente de nombreuses analogies avec le coalescent multiplicatif comme nous allons le voir. Nous donnons ici, trois manières de construire l'arbre brownien, qui sont équivalentes, et qui, pour chacune d'entre elles, apportent une intuition enrichissante de ce qu'est ce modèle.

La première donne une représentation de l'arbre brownien comme sous-espace de  $l^1$ . L'arbre brownien est alors construit de la manière suivante. Considérons un processus de Poisson d'intensité  $(N(t))_{t \in \mathbb{R}^+}$  d'intensité  $t$  sur la demi droite réelle. Définissons  $(J_n)_{n \in \mathbb{N}}$  la suite des instants de sauts de  $N$ , en prenant  $J_0 = 0$ . Définissons par ailleurs pour  $i \in \mathbb{N}^*$ , le vecteur  $e_i \in l^1$  par  $e_i = (\delta_{i,k})_{k \in \mathbb{N}^*}$ . On définit alors par récurrence sur  $\mathbb{N}$  la suite  $(X_n)$  de sous-espaces métriques de  $l^1$  de la manière suivante.

$$\begin{cases} X_0 = \{0\} \\ X_{n+1} = X_n \cup \{U_n + (t - J_n)e_{n+1}, J_n < t \leq J_{n+1}\} \end{cases}$$

où chaque  $U_n$  est un vecteur aléatoire tiré uniformément sur  $X_n$  (ainsi  $U_0 = 0$ ). En posant pour tout  $t \in \mathbb{R}^+$  :

$$\phi(t) = U_n + (t - J_n)e_{n+1}$$

et pour  $n \in \mathbb{N}^*$ ,  $\phi_n$  la restriction de  $\phi$  à  $[0, J_n]$ , on peut naturellement définir une mesure de probabilité  $\mu_n$  sur  $l^1$ , définie par  $\mu_n(A) = \mathcal{L}(\phi_n^{-1}(A))/J_n$  pour tout borélien  $A$ , où  $\mathcal{L}$  est

la mesure de Lebesgue sur  $\mathbb{R}$ . Posons enfin  $X_\infty = \cup_{n \in \mathbb{N}} X_n$ . Alors  $\overline{X}_\infty$ , l'adhérence de  $X_\infty$  est un arbre réel et Aldous [12] montre les résultats suivants.

**Théorème** (Aldous [12]). *Presque sûrement, nous avons :*

$$\lim_{n \rightarrow +\infty} \delta_H(X_n, \overline{X}_\infty) = 0. \quad (96)$$

*Il existe une mesure de probabilité  $\mu$  telle que :*

$$\lim_{n \rightarrow +\infty} d_P(\mu_n, \mu) = 0. \quad (97)$$

*Par ailleurs, nous avons les propriétés suivantes :*

1.  $\overline{X}_\infty$  est compact.
2.  $\overline{X}_\infty$  est le support de  $\mu$ .
3.  $\mu(X_\infty) = 0$ .

On peut facilement vérifier que  $X_\infty \setminus \cup_n \{\phi(J_n)\}$  est le squelette de l'arbre, que la mesure  $\mu$  est portée par les feuilles de  $\overline{X}_\infty$ , que tous les points de branchements sont de degré 3 et enfin que l'ensemble des points de branchement est dénombrable.

Mentionnons qu'une construction analogue pour les composantes  $\mathcal{M}_i(\lambda)$  est fournie dans [3].

On peut par ailleurs construire l'arbre brownien à partir d'une excursion brownienne standard (ce qui justifie son nom). En prenant  $h = e$ , où  $e$  est une excursion brownienne standard, dans la construction donnée par (80), on obtient l'arbre brownien. De plus, en considérant la projection  $p$  de  $[0, 1]$ , sur  $\mathcal{T}_e$ , on obtient la mesure  $\mu$  de la première construction en posant  $\mu = p_* \mathcal{L}$ , la mesure image de la mesure de Lebesgue  $\mathcal{L}$  par  $p$ . Là encore, on peut retrouver une analogie avec la construction de [4] des composantes  $\mathcal{M}_i(\lambda)$ .

On peut aussi concevoir l'arbre brownien à partir de limite d'arbres finis. Considérons par exemple, un arbre de Galton-Watson dont la loi de reproduction  $\nu$  est critique et de variance finie  $\sigma^2$ . On conditionne cet arbre à avoir  $n$  individus. On considère que chaque nœud a une masse  $1/n$  et que chaque arête a une longueur  $1/\sqrt{n}$ . Nous appelons  $H_n$  une variable aléatoire distribuée de cette manière. Enfin, nous désignerons pour un arbre réel  $T$  et une constante  $\alpha > 0$  par  $\alpha T$  l'arbre réel obtenu en multipliant par  $\alpha$  les longueurs dans  $T$ . Le résultat qui va suivre, est au sens strict celui de Haas et Miermont [54]. Cependant des versions extrêmement proches ont été démontrées antérieurement par plusieurs auteurs différents dont Aldous [12], Duquesne [48] et Le Gall [72].

**Théorème.** *Nous avons au sens de la topologie de Gromov-Hausdorff-Prokhorov :*

$$H_n \xrightarrow{d} \frac{2}{\sigma} \mathcal{T},$$

où  $\mathcal{T}$  est l'arbre brownien.

Le résultat de convergence des composantes du graphe d'Erdős-Rényi vers la suite  $(\mathcal{M}_i(\lambda))$  [3] peut une fois de plus être mis en regard de ce résultat.

### Coalescent additif

L'arbre brownien que nous avons défini est lié à un processus qu'on appelle le coalescent additif standard.

Le coalescent dont nous allons parler maintenant n'est pas directement lié à nos travaux, mais il permettra de les mettre en perspective. Considérons donc le cas où  $\kappa(x, y) = x + y$ . Un coalescent dont la dynamique est décrit par ce  $\kappa$  est appelé coalescent additif. Mentionnons deux manières de construire un coalescent additif particulièrement étudié, le coalescent additif standard. Considérons un arbre brownien comme nous venons de le définir. Pour simplifier, nous supposons que sa masse totale vaut 1. On définit par ailleurs sur son squelette un processus ponctuel de Poisson d'intensité  $dl \otimes dt$  où  $dl$  représente la mesure de longueur sur le squelette et  $dt$  la mesure de temps. Si l'on coupe le squelette selon ces points à l'instant  $t$ , on obtient une collection d'arbres réels dont la somme des masses vaut 1 (voir [16]). Considérons  $(M(t)) = (m_i(t), t \geq 0)$ , le processus qui décrit l'évolution des masses ordonnées de manière décroissante. Pour  $S$  un ensemble de suites réelles, on notera  $S^\downarrow$  l'ensemble des suites décroissantes de  $S$ . Le processus  $(M(t))$  est donc à valeur dans  $\Sigma_1^\downarrow$ , où  $\Sigma_1$  est défini par :

$$\Sigma_1 = \left\{ u \in \mathbb{R}_+^{\mathbb{N}^*}, \sum_{i=1}^{\infty} u_i = 1 \right\}.$$

On posant  $X(t) = M(e^{-t})$ , on obtient un coalescent additif défini sur  $\mathbb{R}$  tout entier appelé coalescent additif standard.

On peut noter que de la même manière que l'arbre brownien peut être vu comme limite d'arbres finis, le processus qu'on vient de construire peut être vu comme limite du retourné en temps d'un processus de masses obtenu en retirant des arêtes d'un arbre fini. Plus précisément, considérons un arbre  $\mathcal{T}_n$  distribué uniformément sur l'ensemble des arbres à  $n$  nœuds. On peut définir par récurrence à partir de  $\mathcal{T}_n$  le processus  $(\mathcal{F}^n(n-k))$  à valeur dans l'ensemble des forêts (c'est-à-dire des graphes acycliques) à  $n$  nœuds de la manière suivante. Étant un arbre,  $\mathcal{T}_n$  a  $n-1$  arêtes. On pose  $\mathcal{F}^n(n) = \mathcal{T}_n$ . On suppose que pour  $k \in [n-2]$ ,  $\mathcal{F}^n(n-k)$  est construit et on retire uniformément une des  $n-1-k$  arêtes restantes. La forêt ainsi obtenue est notée  $\mathcal{F}^n(n-k-1)$ .

La dynamique de  $(\mathcal{F}^n(m), 1 \leq m \leq n)$  obtenue en renversant le temps du précédent

processus peut être décrite de la manière suivante (voir [83]). À l'instant initial, on considère un graphe à  $n$  nœuds sans aucune arête (c'est-à-dire que nous avons une forêt de  $n$  arbres, où chaque arbre est constitué d'un seul nœud). Supposons pour  $m \in [n-1]$  que  $\mathcal{F}^n(m)$  est construit. Il y a alors  $m-1$  arêtes et  $k = n-m+1$  arbres. Appelons  $(t_1, \dots, t_k)$  les  $k$  arbres ainsi construits. Un couple d'arbre  $(t_i, t_j)$ ,  $1 \leq i < j \leq k$  est tiré avec probabilité  $(\text{Card}(t_i) + \text{Card}(t_j))/n(k-1)$ . Une fois ce couple donné on tire uniformément un nœud sur  $t_i$  et un nœud sur  $t_j$  et on ajoute une arête entre les deux.

À partir de  $\mathcal{F}^n$  on peut définir un coalescent additif de la manière suivante. Considérons une suite de variables aléatoires indépendantes  $(e_i)_{i \in [n-1]}$  telle que pour tout  $i \in [n-1]$ ,  $e_i$  soit distribuée comme une variable aléatoire exponentielle de paramètre  $n-i$ . On pose alors pour  $t \in \mathbb{R}^+$  :

$$\tilde{\mathcal{F}}^n(t) = \mathcal{F}^n(k(t) + 1),$$

où  $k(t) = \sup\{k \in [n-1], \sum_{1 \leq i \leq k} e_i \leq t\}$ . Autrement dit, on introduit entre chaque saut des variables exponentielles de paramètre  $k-1$ , où  $k$  est le nombre d'arbres présents. Définissons  $Z^n(t)$ , le vecteur des tailles des composantes de  $\tilde{\mathcal{F}}^n(t)$  réordonnées dans l'ordre décroissant. En posant pour  $t \geq -\log(n)$ ,  $X^n(t) = n^{-1}Z^n(t + \log n)$ , nous avons le résultat suivant :

**Théorème** (Aldous, Pitman [16]).  *$X^n$  est un coalescent additif sur  $[-\log n, +\infty)$ . Par ailleurs,*

$$X^n \xrightarrow{d} X,$$

au sens de la convergence de Skorokhod sur  $\mathbb{D}(\mathbb{R}, \Sigma_1^\downarrow)$ .

On peut alors observer une nouvelle analogie en comparant le théorème 0.4.1 que nous avons démontré et ces deux dernières constructions. Mentionnons pour finir, que le lecteur pourra trouver deux constructions semblables du coalescent additif et du coalescent multiplicatif dans l'article de Broutin et Marckert [29] proches de celle proposée par Bertoin pour le coalescent additif [22].

## 0.5 Problèmes ouverts et perspectives

Nous avons démontré dans le premier chapitre qu'un certain nombre de fonctions attachées à l'arbre binaire de recherche et à l'arbre de Yule oscillaient asymptotiquement, notamment  $n \mapsto \mathbb{E}(F_n)$  et son analogue continu  $t \mapsto \mathbb{E}(\tilde{F}_t)$ . La question de savoir si ces fonctions convergent ou non reste ouverte. Plusieurs chercheurs ont essayé de résoudre ce problème par différentes techniques (transformée de Fourier, approche combinatoire, martingales...), sans succès. Dans le cas où  $t \mapsto \mathbb{E}(\tilde{F}_t)$  ne convergerait pas, étant donné que nous avons une expression sous forme de série de la fonction 1-périodique limite  $Q_3$  impli-

quant  $\bar{\phi}$  et sa dérivée, nous pourrions éventuellement espérer qu'une approximation très fine de  $\bar{\phi}$  permette un jour de fournir une preuve de la non-convergence de  $t \mapsto \mathbb{E}(\tilde{F}_t)$ .

Dans le second chapitre, nous avons exhibé une nouvelle valeur critique  $\mu_c$ . Contrairement à  $\mu_0$ , la vitesse linéaire du maximum, il nous est difficile de donner une interprétation probabiliste simple de cette. Comme nous l'avons évoqué, l'analyse de  $D_x$ , le nombre de divisions du processus avant l'extinction, devrait fournir des éléments d'interprétation de  $\mu_c$ . Toutefois, contrairement à  $Z_x$ , la fonction génératrice de  $D_x$  ne satisfait pas d'équations différentielles simples liées à KPP, et donc d'autres méthodes que celles que nous avons employées doivent être envisagées.

Enfin, les résultats du dernier chapitre peuvent être améliorés. En particulier, les convergences établies sont au sens fini-dimensionnel et sont très probablement vraies au sens de Skorokhod. Quoique ce raffinement n'est pas immédiat, il semble plus simple que les deux précédents problèmes évoqués. En effet, une suite de fonctions aléatoires càdlàg à valeurs dans un espace polonais converge en distribution vers une fonction aléatoire càdlàg au sens de Skorokhod si elle converge au sens fini-dimensionnel et si elle est tendue. Il ne nous reste donc plus que la tension à démontrer, et pour ce faire, nous pouvons utiliser certains critères connus (voir par exemple Kallenberg [65]). Pour les appliquer, il nous faudra établir des estimations assez fines par exemple de celles du nombre d'arêtes  $e$  dans  $G(n, p_n(\lambda))$  qui vérifient la propriété suivante : si l'on ôte  $e$ , alors la composante connexe à laquelle  $e$  appartenait se sépare en deux composantes connexes dont l'une est de diamètre supérieur à  $a_n$  et l'autre à  $b_n$ , avec des suites  $a_n$  et  $b_n$  habilement choisies.



# Chapitre 1

## Oscillations in the height of the Yule tree and application to the binary search tree

This chapter is based on [38].

### 1.1 Introduction

We denote by  $\mathbb{N}$  the set  $\{0, 1, 2, \dots\}$  and by  $\mathbb{N}^*$  the set  $\mathbb{N} \setminus \{0\}$ . The binary search tree  $(\mathcal{T}_n)_{n \in \mathbb{N}^*}$  is a discrete time Markov process on the space of binary trees that can be constructed as follows :  $\mathcal{T}_1$  is a tree made of a single leaf. Given  $\mathcal{T}_n$ , the next state  $\mathcal{T}_{n+1}$  is obtained by uniformly choosing a leaf of  $\mathcal{T}_n$  which we change into an internal node with two leaves attached. The binary search tree is a natural structure to store data and is related with the Quicksort algorithm. For a general reference on the binary search tree, see for instance the monograph of Mahmoud [77].

One can easily transform  $(\mathcal{T}_n)_{n \in \mathbb{N}^*}$  into a continuous time Markov process simply by Poissonizing the jump times. More precisely, let  $(N_t)_{t \in \mathbb{R}_+}$  be a pure birth process independent of  $(\mathcal{T}_n)_{n \in \mathbb{N}^*}$  which jumps from state  $n$  to state  $n + 1$  at rate  $n$ . Then

$$(\mathcal{T}_t^c)_{t \in \mathbb{R}_+} := (\mathcal{T}_{N_t})_{t \in \mathbb{R}_+} \tag{1.1}$$

is called the Yule tree process. It is also a model of a random growing binary tree and is clearly a Markov process. Observe that we can give the following alternative description of the Yule tree process. At time 0, the tree is reduced to a leaf. Each leaf lives for a random time with exponential distribution with parameter 1 independent of the other leaves. When it dies, it is replaced by an internal node with two leaves. For more details and other constructions see e.g. [33, 86].



Note that  $N_t$  is the cardinal of  $\mathcal{N}_t$ , the set of leaves alive at time  $t$ , and that if we introduce the stopping time

$$\tau_n = \inf\{t > 0, N_t = n\}, \quad (1.2)$$

Equation (1.1) yields :

$$(\mathcal{T}_n)_{n \in \mathbb{N}} = (\mathcal{T}_{\tau_n}^c)_{n \in \mathbb{N}} \quad (1.3)$$

so one can go from one model to the other.

From the Yule tree process we can make a branching random walk in the following way : for a leaf  $u \in \mathcal{N}_t$ , let  $X_u(t)$  be its height in the tree (or its generation). Define  $(X(t))_{t \in \mathbb{R}_+}$ , the measure valued process by :

$$X(t) := \sum_{u \in \mathcal{N}_t} \delta_{X_u(t)}, \quad \forall t \in \mathbb{R}_+. \quad (1.4)$$

$X$  is then simply a branching random walk in continuous time with lattice-integer support which we call the Yule branching random walk (sometimes it is called the Yule-time process as in [35]). Each particle lives for an exponential(1) time and is then replaced by two daughter particles situated one unit of distance further. Fix  $0 \leq t_0 \leq t_1 \leq t_2$ . As usual, for  $u \in \mathcal{N}_{t_1}$ , we set  $X_u(t_0) = X_v(t_0)$ , where  $v$  is the unique ancestor of  $u$  in  $\mathcal{N}_{t_0}$ . Furthermore, for convenient we set  $X_u(t_2) = X_u(t_1)$ , even if  $u$  is not alive at time  $t_2$ . We will also write  $v < u$  when  $v$  is the an ancestor of  $u$ .

The main focus of the present work is the study of the asymptotic behaviour, of the height (i.e. the highest generation of a leaf) and of the saturation level (i.e. the maximal level  $l$  such that there are no leaves for all levels up to  $l$ ) of the Yule tree and the binary search tree. We point out that for the branching random walks we will talk about maximum and minimum rather than height and saturation level which are reserved to trees.

Our paper is organized as follows. In Section 1.2, we state the main results. In Section 1.3, we review the relevant literature and discuss our results. Section 1.4 is dedicated to the proofs of results concerning the Yule branching random walk and Section 1.5 to the application of these results to the binary search tree.

## 1.2 Main results

Let us define  $\overline{X}(t) := \max_{u \in \mathcal{N}_t} X_u(t)$  and  $\underline{X}(t) := \min_{u \in \mathcal{N}_t} X_u(t)$ . For  $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ , set :

$$\begin{cases} \overline{h}(x, t) := \mathbb{P}(\overline{X}(t) \leq x) = \mathbb{P}(\overline{X}(t) \leq \lfloor x \rfloor) \\ \underline{h}(x, t) := \mathbb{P}(\underline{X}(t) \geq x) = \mathbb{P}(\underline{X}(t) \geq \lceil x \rceil) \end{cases}. \quad (1.5)$$

**Proposition 1.2.1.**  $\bar{h}$  and  $\underline{h}$  solve the equation :

$$\partial_t h(x, t) = h^2(x - 1, t) - h(x, t), (x, t) \in \mathbb{R} \times \mathbb{R}^+. \quad (1.6)$$

The connection between the Yule branching random walk and Equation (1.6) is analogue to that between the branching Brownian motion and the F-KPP equation described in next section. We say that  $x \mapsto \phi(x)$  is a travelling-wave solution of (1.6) with speed  $c \in \mathbb{R}$  if  $(x, t) \mapsto \phi(x - ct)$  is a solution of (1.6). One can easily check that  $\phi$  is a travelling-wave solution of (1.6) with speed  $c \in \mathbb{R}$  if and only if it solves the delay differential equation :

$$c\phi'(x) = \phi(x) - \phi^2(x - 1), \quad x \in \mathbb{R}. \quad (1.7)$$

We will see in the next section that in the case of the branching Brownian motion and the F-KPP equation, the key result proved by Kolmogorov et al. in [68] is that  $x \mapsto \mathbb{P}(\bar{X}(t) \leq x + m_t)$  (where  $m_t$  is an appropriate centring term) converges uniformly in  $x$  to  $\omega^*$ , the critical traveling wave.

In the case of the Yule branching random walk, the functions  $\bar{h}$  and  $\underline{h}$  defined in (1.5) are continuous in  $t$  but piecewise constant in  $x$ , which implies that whatever the centring is, the distribution cannot converge. However, we see in the following theorem that the asymptotic distributions oscillate around the critical travelling-waves.

For  $\theta \in \mathbb{R}^*$ , introduce

$$c_\theta := \frac{(2e^\theta - 1)}{\theta}. \quad (1.8)$$

We denote by  $\theta^+$  (respectively by  $\theta^-$ ) the largest (resp. the smallest) solution of  $c_\theta = 2e^\theta$ . We also set  $c^+ := c_{\theta^+}$  and  $c^- := c_{\theta^-}$ . Numerically, [44] we have :

$$\begin{cases} c^+ = 4.311\dots \\ \theta^+ = 0.768\dots \end{cases} \quad \begin{cases} c^- = 0.373\dots \\ \theta^- = -1.678\dots \end{cases}.$$

**Theorem 1.2.2.** Let  $a_t = c^+t - \frac{3 \log(t)}{2\theta^+}$ ,  $b_t = c^-t - \frac{3 \log(t)}{2\theta^-}$  (recall that  $\theta^- < 0$ ) and  $\{x\} = x - \lfloor x \rfloor$ .

There exist a monotone travelling-wave solution  $\bar{\phi}$  at speed  $c^+$  and a monotone travelling-wave solution  $\underline{\phi}$  at speed  $c^-$  of (1.6) such that :

$$\lim_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}} |\mathbb{P}(\bar{X}(t) \leq a_t + x) - \bar{\phi}(x - \{a_t + x\})| = 0 \quad (1.9)$$

and :

$$\lim_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}} |\mathbb{P}(\underline{X}(t) \geq b_t + x) - \underline{\phi}(x - \{b_t + x\})| = 0. \quad (1.10)$$

Observe that (1.9) is equivalent to :

$$\lim_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}} |\mathbb{P}(\overline{X}(t) \leq x) - \overline{\phi}(\lfloor x \rfloor - a_t)| = 0 \quad (1.11)$$

and a similar formulation holds for (1.10). We will now extend Theorem 1.2.2. For each piecewise continuous function  $f$  we define  $G_f$ ,  $P_f$  and  $r_f$  by :

$$\begin{cases} G_f(t) := \mathbb{E} [f(\overline{X}(t) - a_t)], & t > 0, \\ P_f(s) := \sum_{k \in \mathbb{Z}} f(k - s) (\overline{\phi}(k - s) - \overline{\phi}(k - 1 - s)), & s \in \mathbb{R}, \\ r_f(t) := P_f(a_t), & t > 0, \end{cases} \quad (1.12)$$

when these functions are well-defined. Note that  $P_f$  is clearly 1-periodic. Before stating the theorem, let us give some notations. We denote by  $f^{(i)}$  the  $i$ th derivative of a function  $f \in \mathcal{C}^i(\mathbb{R})$ . For two real functions  $f$  and  $g$ , we write :

$$f(x) = o_x(g(x)),$$

when there exists a function  $\epsilon$  such that  $\lim_{x \rightarrow +\infty} \epsilon(x) = 0$  and such that  $f(x) = \epsilon(x)g(x)$  and

$$f(x) = o_{x \rightarrow \pm\infty}(g(x)),$$

when the function  $\epsilon$  also satisfies  $\lim_{x \rightarrow -\infty} \epsilon(x) = 0$ .

**Theorem 1.2.3.** *There exists  $\delta > 0$  such that for each piecewise continuous function  $f$  which satisfies  $f(x) = o_{x \rightarrow \pm\infty}(e^{\delta|x|})$   $G_f$ ,  $P_f$  and  $r_f$  are well-defined and we have :*

$$\lim_{t \rightarrow +\infty} |G_f(t) - r_f(t)| = 0. \quad (1.13)$$

Furthermore, if  $f \in \mathcal{C}^k(\mathbb{R})$  satisfies  $f^{(i)}(x) = o_{x \rightarrow \pm\infty}(e^{\delta|x|})$ , for  $0 \leq i \leq k$ , we have that  $G_f, P_f, r_f \in \mathcal{C}^k(\mathbb{R}^+)$  and :

$$\lim_{t \rightarrow +\infty} |G_f^{(k)}(t) - r_f^{(k)}(t)| = \lim_{t \rightarrow +\infty} |G_f^{(k)}(t) - (c^+)^k P_f^{(k)}(a_t)| = 0. \quad (1.14)$$

Similar results for the minimum also hold. Observe that by taking, for  $x$  fixed,  $f$  defined by  $f(y) = \mathbf{1}_{\{y \leq x\}}$  in Theorem 1.2.3 we find the result of Theorem 1.2.2 (if we omit the uniformity of the convergence).

It is interesting to see what Theorem 1.2.3 means in some particular cases. Set

$$\tilde{F}_t := \text{Card}\{u \in \mathcal{N}_t, X_u(t) = \overline{X}(t)\}. \quad (1.15)$$

$\tilde{F}_t$  is the so-called "fringe" which has been studied by Roberts in [87] and by Drmota in [44]

(see next section for more details). Applying Theorem 1.2.3 with power functions yields the following corollary.

**Corollary 1.2.4.** *There exist three 1-periodic smooth functions  $Q_1, Q_2$  and  $Q_3$  such that :*

$$\mathbb{E}(\bar{X}(t)) = a_t + Q_1(a_t) + o_t(1), \quad (1.16)$$

$$\text{Var}(\bar{X}(t)) = Q_2(a_t) + o_t(1), \quad (1.17)$$

$$\mathbb{E}(\tilde{F}_t) = Q_3(a_t) + o_t(1). \quad (1.18)$$

To prove Equation (1.18), we use the following lemma :

**Lemma 1.2.5.** *Let  $w(t) := \mathbb{E}(\bar{X}(t))$ ,  $\forall t \geq 0$ . The function  $w$  is a smooth function whose derivative is :*

$$w'(t) = \mathbb{E}(\tilde{F}_t), \forall t \geq 0. \quad (1.19)$$

Let us now turn to the binary search tree. Using (1.1) we show that Theorem 1.2.2 translates into an analogous oscillation result for the asymptotic distributions of the height (i.e. the highest generation of a leaf) and the saturation level (i.e. the maximal level  $l$  such that there are no leaves for all levels up to  $l$ ) of a binary search tree. We call  $\partial Z_\infty^+$  and  $\partial Z_\infty^-$  the limit of the derivative martingales of the binary search tree defined in (1.142), and for  $K > 0$  :

$$\begin{cases} \psi_K^+(x) := \mathbb{E} \left[ \exp \left( -K e^{-\theta^+ x} \partial Z_\infty^+ \right) \right] \\ \psi_K^-(x) := \mathbb{E} \left[ \exp \left( -K e^{-\theta^- x} \partial Z_\infty^- \right) \right] \end{cases}. \quad (1.20)$$

Let  $H_n$  be the height of a random binary search tree with  $n$  nodes and  $l_n$  its saturation level.

**Theorem 1.2.6.** *There exist  $K^+, K^- > 0$  such that :*

$$\lim_{n \rightarrow +\infty} \sup_{x \in \mathbb{R}} |\mathbb{P}(H_n \leq \lfloor a_{\log(n)} + x \rfloor) - \psi_{K^+}^+(x - \{a_{\log(n)} + x\})| = 0, \quad (1.21)$$

and :

$$\lim_{n \rightarrow +\infty} \sup_{x \in \mathbb{R}} |\mathbb{P}(l_n \geq \lceil b_{\log(n)} + x \rceil) - \psi_{K^-}^-(x - \{b_{\log(n)} + x\})| = 0. \quad (1.22)$$

We will further see in next section that Theorem 1.2.6 and results by Drmota allow us to obtain an analogue of Corollary 1.2.4. Let  $F_n$  be the number of particles at the highest position for a binary search tree with  $n$  leaves. Note that  $F_n$  is linked to  $\tilde{F}_t$  by the relation  $\tilde{F}_t = F_{N_t}$ .

**Corollary 1.2.7.** *There exist three 1-periodic functions  $R_1, R_2$  and  $R_3$  such that :*

$$\mathbb{E}(H_n) = a_{\log n} + R_1(a_{\log n}) + o_n(1), \quad (1.23)$$

$$\text{Var}(H_n) = R_2(a_{\log n}) + o_n(1), \quad (1.24)$$

$$\mathbb{E}(F_n) = R_3(a_{\log n}) + o_n(1). \quad (1.25)$$

## 1.3 Previous results and discussion

### 1.3.1 Extremal particles in a branching process

The position of the extremal particles in a branching process has been studied intensively. The case of the branching Brownian motion is the prototypical example. In that setting, it is then well known [79] that  $p(x, t) = \mathbb{P}(\bar{X}(t) \leq x)$ , where  $\bar{X}(t)$  is the position of the maximum at time  $t$ , solves the Fisher-Kolmogorov-Petrovskii-Piskunov (F-KPP) equation :

$$\partial_t p(x, t) = \frac{1}{2} \partial_{xx} p(x, t) + (p^2(x, t) - p(x, t)). \quad (1.26)$$

Kolmogorov, Petrovskii and Piskunov [68] show that, for a good centring term  $m_t$ ,

$$\lim_{t \rightarrow +\infty} p(x + m_t, t) = \omega^*(x), \quad (1.27)$$

where  $\omega^* \in \mathcal{C}^2(\mathbb{R})$  is the unique function (up to a shift) such that  $\tilde{p}(x, t) := \omega^*(x - \sqrt{2}t)$ ,  $\forall (x, t) \in \mathbb{R} \times \mathbb{R}_+$  is a solution of (1.26). One possible choice for  $m_t$  is the median  $m_t = \inf\{x \in \mathbb{R} : p(t, x) = 1/2\}$ . Bramson [27] later shows famously that any valid centring term must be of the form

$$m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + C + o(1). \quad (1.28)$$

The study of the minimum (or equivalently of the maximum) of a branching random walk also has a long story. Let us mention some remarkable results on  $M_n$ , the minimum of a branching random walk. For simplicity, we consider a branching random walk satisfying suitable assumptions and which survives almost surely. Hammersley [55] shows that  $(M_n/n)$  converges almost surely to a constant  $\gamma_1$ . The almost sure behaviour of the minimum is refined by Hu and Shi [61] who show that there exists a constant  $\gamma_2 > 0$  such that :

$$\liminf_{n \rightarrow +\infty} \frac{M_n - n\gamma_1}{\log n} = \frac{\gamma_2}{2} \text{ a.s.}, \quad \limsup_{n \rightarrow +\infty} \frac{M_n - n\gamma_1}{\log n} = \frac{3\gamma_2}{2} \text{ a.s.} \quad (1.29)$$

As far as the average of the minimum is concerned, Addario-Berry and Reed [6] show that :

$$\mathbb{E}(M_n) = \gamma_1 n + \frac{3\gamma_2}{2} \log n + O_n(1). \quad (1.30)$$

They also show that the minimum centred around its mean is tight and that its distribution has exponential tails, that is there exist  $C, \delta > 0$  such that :

$$\mathbb{P}(|M_n - \mathbb{E}(M_n)| \geq x) \leq C e^{-\delta x}, \quad \forall x \geq 0. \quad (1.31)$$

Aïdékon [7] proves a result similar to (1.27) for a large class of branching random walks in the non-lattice case. He shows that there exists  $K > 0$  such that for all  $x \in \mathbb{R}$  :

$$\lim_{n \rightarrow +\infty} \mathbb{P} \left( M_n \geq \gamma_1 n + \frac{3\gamma_2}{2} \log n + x \right) = \mathbb{E} \left( \exp \left( -K e^{-x/\gamma_2} D_\infty \right) \right), \quad (1.32)$$

where  $D_\infty$  is the limit of the derivative martingale whose definition is given in (1.150). The results of Addario-Berry and Reed (1.31) and of Aïdékon (1.32) will be useful for some proofs. They are therefore stated in more details in Appendix 1.B.

Bramson, Ding and Zeitouni [28] give an alternative proof of Aïdékon's Theorem in a slightly less general case. In particular, they assume that the displacements of each offspring of a particle are independent. Furthermore, the authors claim that their method works in the lattice case. Note that, even if we consider a discretized version of the Yule branching random walk, since the displacements of the particles after a split are not independent in our framework, we cannot expect to use directly the method of [28].

Let us now mention a work in the lattice case. Lifshits considers the following branching random walk in [73]. At time  $n = 0$ , a particle is at 0. At each time  $n \in \mathbb{N}$ , every particle produces two particles which are translated by 1 from their parents with probability  $0 < p < 1$  and by  $-1$  with probability  $1 - p$ . He proves that when  $p > \frac{1}{2}$ , the distribution of the centred maximum converges, but when  $p = \frac{1}{2}$ , oscillations as in (1.9) exist except that the centring term is not explicit (it involves the median).

In that context, Theorem 1.2.2 provides an example of oscillations of the centred distribution of the maximum of a lattice branching random walk around a function (the critical travelling-wave) with an explicit centring term. We have already mentioned Aïdékon's result which shows that this kind of oscillations does not appear in the non-lattice case. It is interesting to see in Theorem 1.2.3 how this phenomena of oscillations extends to a large set of functions applied to  $\bar{X}(t) - a_t$ . Indeed, Theorem 1.2.2 only yields (1.13) for very restrictive classes of functions, for instance the class of continuous functions with compact support.

Nevertheless, we point out that for a given function  $f$ , we cannot be sure, in general, that we have real oscillations in the sense that  $P_f$  defined in (1.12) can be constant. In some specific cases, we can determinate whether  $P_f$  is constant or not.

For instance, if  $f$  is defined by  $f(x) = \mathbf{1}_{x \leq y}$ , then  $P_f(x) = \bar{\phi}(y - \{y - x\})$ , and thus we have non-constant oscillations. Similarly, if we assume that  $f$  is a non-constant 1-periodic function, then we have  $P_f(x) = f(-x)$  and thus  $P_f$  is also non-constant. However, for  $f(x) = \bar{\phi}(x) + \bar{\phi}(x - 1)$ , we have that :

$$P_f(x) = \sum_{k \in \mathbb{Z}} \bar{\phi}^2(k - x) - \bar{\phi}^2(k - x - 1).$$

Since  $P_f$  is a telescoping sum, we have that  $P_f(x) = 1, \forall x \in \mathbb{R}$ .

These examples are quite anecdotal but show that there is no trivial general answer to that issue. The examples of Corollary 1.2.4 seem more interesting. Whether  $\mathbb{E}(\overline{X}(t) - a_t)$ ,  $\text{Var}(\overline{X}(t))$  and  $\mathbb{E}(\tilde{F}_t)$  converge or not to a constant is still an open problem. The proof of Corollary 1.2.4 shows that  $Q_3 = c^+ + c^+ Q'_1$  which implies that  $\mathbb{E}(\overline{X}(t) - a_t)$  converges to a constant if and only if  $\mathbb{E}(\tilde{F}_t)$  also converges to a constant. Indeed, since  $Q_1$  is 1-periodic,  $Q_1$  is constant if and only if  $Q'_1$  is constant.

Furthermore, the asymptotic behaviour of  $\mathbb{E}(\tilde{F}_t)$  is especially interesting because it is related to a more general question, namely the convergence of the extremal point process. In the case of the branching Brownian motion, it has been shown independently by Aïdékon, Berestycki, Brunet and Shi [8] and Arguin, Bovier and Kistler [18] that the extremal point process of branching Brownian motion converges and Madaule [76] proved the analogous result for branching random walks with non-lattice support. The lattice case has not been dealt with and the behaviour of  $\mathbb{E}(\tilde{F}_t)$  could shed a first light on this case.

Analogous questions for the binary search tree will also be discussed in a next section.

### 1.3.2 Yule generation process

One of the key step in studying the Yule branching random walk is to switch our point of view by swapping the role of space and time. We thus introduce the Yule-generation process defined as follows. Let  $\mathcal{M}_n$  be the set of the particles of the  $n$ th generation. For  $n \in \mathbb{N}$  and for  $u \in \mathcal{M}_n$  define

$$T_u(n) = \inf\{t > 0, X_u(t) = n\} \text{ and } T(n) = \sum_{u \in \mathcal{M}_n} \delta_{T_u(n)}. \quad (1.33)$$

Observe that  $T(n)$  is itself a branching random walk with discrete time and continuous spatial position with the following branching mechanism.

At time  $n = 0$ , a particle is at 0. At time  $n = 1$ , the particle dies and gives birth to the point process  $\xi := 2\delta_E$  where  $E$  is distributed as an exponential variable with parameter 1. We interpret  $\xi$  as two particles, both in position  $E$ . At each time  $n$ , the particles of the previous generation die and give birth to particles whose displacements from the parents are given by i.i.d. copies of  $\xi$ .

This connection was established by Chauvin and Rouault [35] and allowed them to prove the existence and uniqueness of travelling-waves.

The proof of Theorem 1.2.2 relies on the correspondence between the Yule branching random walk and the Yule generation process defined in (1.33). An advantage of such a change of point of view is that, as already mentioned, the non-lattice case is better understood. In particular, if we call  $\underline{T}(n) = \inf\{T_u(n), u \in \mathcal{M}_n\}$ , Aïdékon's result can be

transposed from the non-lattice case to our case thanks to the relation :

$$\mathbb{P}(\underline{T}(n) \leq t) = \mathbb{P}(\overline{X}(t) \geq n). \tag{1.34}$$

Indeed, the event  $\{\underline{T}(n) \leq t\}$  means that before  $t$  there exists a particle whose generation is  $n$ , which is equivalent to the fact that the maximal generation of a particle at time  $t$  is greater than  $n$ . To avoid confusions later on, we emphasize that if we take the complementaries of these events, (1.34) yields :

$$\mathbb{P}(\underline{T}(n+1) \geq t) = \mathbb{P}(\overline{X}(t) \leq n). \tag{1.35}$$

An illustration of this correspondence is given below. The generation of the particle  $u$  (the big circle) at time  $t = 4$  is 2 and the reaching time of the second generation for  $u$  is  $T_u(2)$ .

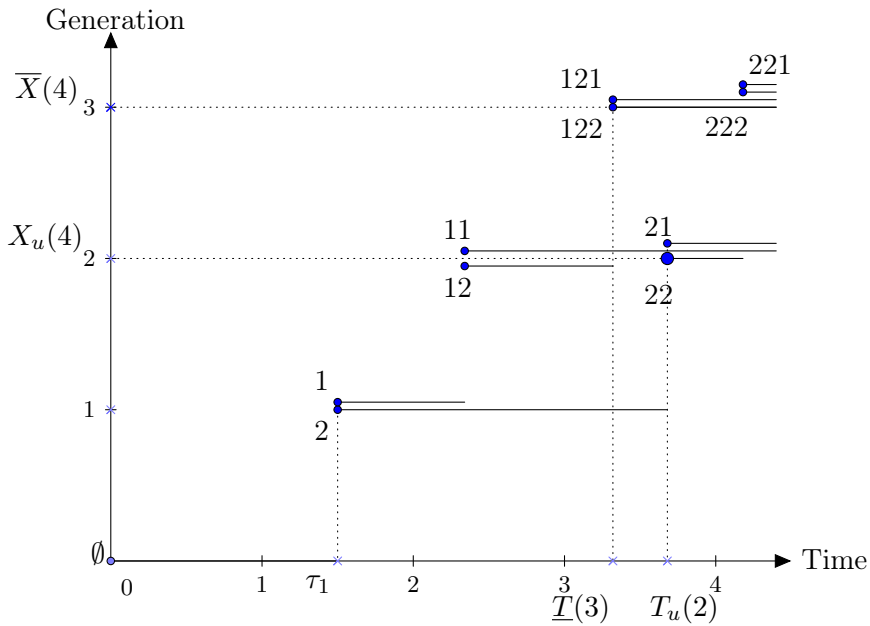


FIGURE 1.1 – Correspondence between the Yule branching random walk and the Yule generation process

### 1.3.3 Binary search tree

The average and the variance of the height and of the saturation level have been studied by many authors. Initially, Robson [88, 89] proved that  $(\mathbb{E}(H_n)/\log n)$  converges to a constant between 3.6 and  $c^+$  and in 1995, Devroye and Reed [42] showed that  $\text{Var}(H_n) = O_n((\log \log n)^2)$ . These asymptotics have been improved [40, 41, 43, 84] until Reed [86]



and Drmota [44] independently proved that

$$\mathbb{E}(H_n) = a_{\log(n)} + O_n(1) \text{ and } \text{Var}(H_n) = O_n(1), \quad (1.36)$$

where we recall that  $a_t = c^+t - \frac{3\log(t)}{2\theta^+}$ ,  $\forall t \geq 0$ . Moreover for  $n \in \mathbb{N}$ , let  $\Upsilon_n$  be the generating function of  $(\mathbb{P}(H_k \leq n))_{k \in \mathbb{N}}$ , that is :

$$\Upsilon_n(x) := \sum_{k=0}^{+\infty} \mathbb{P}(H_k \leq n)x^k, \quad \forall x \in \mathbb{R}. \quad (1.37)$$

Drmota proves the convergence of the distribution of the height in the following sense.

**Theorem (Drmota [44]).** *There exists a decreasing function  $\Psi : \mathbb{R}^+ \rightarrow (0, 1]$  with  $\Psi(0) = 1$  and  $\lim_{x \rightarrow +\infty} \Psi(x) = 0$  satisfying the integral equation*

$$y\Psi(y/e^{(1/c^+)}) = \int_0^y \Psi(z)\Psi(y-z)dz \quad (1.38)$$

such that

$$\lim_{k \rightarrow +\infty} \sup_{n \in \mathbb{N}} \left| \mathbb{P}(H_k \leq n) - \Psi\left(\frac{k}{\Upsilon_n(1)}\right) \right| = 0. \quad (1.39)$$

See also Drmota [46] and Chauvin and Drmota [32]. Moreover, Chauvin and Rouault [35] show that there exists a connection between  $\Psi$ , the function defined in Drmota's Theorem, and the derivative martingale of the binary search tree  $\partial Z_\infty^+$  defined in (1.142). Indeed, there exists  $K > 0$  such that

$$\Psi(x) = \mathbb{E} \left[ \exp \left( -Kx^{c^+-1} \partial Z_\infty^+ \right) \right]. \quad (1.40)$$

Observe that (1.40) tells us that that  $\psi_K^+(x) = \Psi(e^{-\frac{x}{c^+}})$ ,  $\psi_K^+$  being defined in (1.20).

The main difference between Drmota's Theorem and Theorem 1.2.6 is that  $\Upsilon_n(1)$  defined in (1.37) is implicit while  $a_{\log n}$  is explicit. However, Drmota provides a good approximation of  $\Upsilon_n(1)$  by showing that :

$$\Upsilon_n(1) = e^{\frac{n}{c^+} + \frac{3\log n}{2(c^+-1)} + \kappa_n + o_n(1)}, \quad (1.41)$$

where  $(\kappa_n)$  is a bounded sequence such that  $\kappa_{n+1} - \kappa_n \rightarrow 0$ . He also proves that if there exists  $\kappa_\infty \in \mathbb{R}$  such that the better asymptotic

$$\Upsilon_n(1) = e^{\frac{n}{c^+} + \frac{3\log n}{2(c^+-1)} + \kappa_\infty + o_n(1)}, \quad (1.42)$$

holds then Corollary 1.2.7 also holds. By comparing Drmota's Theorem and Theorem 1.2.2 we can give an equivalent to  $\Upsilon_n(1)$  and show that (1.42) holds thus yielding Corollary 1.2.7.

Drmotá also gives more details about  $F_n$ . He shows in [45] that the oscillations of  $(\mathbb{E}(F_n))$  around  $c^+$  are at most of order  $10^{-4}$ . Moreover, in the same paper, he tells us that  $(\mathbb{E}(F_n))$  is increasing until  $n = 100000$ . If  $(\mathbb{E}(F_n))$  were increasing on  $\mathbb{N}$  we could easily prove that  $(\mathbb{E}(F_n))$  converges. Whether or not  $(\mathbb{E}(F_n))$  is increasing and whether or not  $(\mathbb{E}(F_n))$  converges are still open questions.

The issue of the almost sure behaviour of the height and of the saturation level for the binary search tree has been dealt with by Roberts in [87], where he shows, relying on [61], that :

$$\frac{1}{2\theta^+} = \liminf_{n \rightarrow +\infty} \frac{c^+ \log(n) - H_n}{\log \log(n)} < \limsup_{n \rightarrow +\infty} \frac{c^+ \log(n) - H_n}{\log \log(n)} = \frac{3}{2\theta^+}, \quad (1.43)$$

and the analogous result for the saturation level. Roberts' result yields a similar behaviour for the maximum and minimum of the Yule branching random walk.

As far as the almost sure behaviour of  $F_n$  is concerned, Roberts shows [87] that :

$$\limsup_{n \rightarrow +\infty} F_n = +\infty. \quad (1.44)$$

## 1.4 Results on the Yule branching random walk

### 1.4.1 Proof of Proposition 1.2.1

This is a simple transposition of McKean's proof for the branching Brownian motion [79].

*Proof of Proposition 1.2.1.* We recall that the first division time  $\tau_2$  defined in (1.2) has an exponential law with parameter 1. Consider  $k \in \mathbb{Z}$  and  $t \geq 0$ .

- If  $k < 0$  then  $\bar{h}(k, t) = \mathbb{P}(\bar{X}(t) \leq k) = 0, \forall t \geq 0$  and thus Equation (1.6) is satisfied when  $k < 0$ .
- If  $k = 0$ ,  $\bar{h}(0, t) = \mathbb{P}(\bar{X}(t) \leq 0) = \mathbb{P}(\tau_2 \geq t) = e^{-t}$ . Therefore

$$\partial_t \bar{h}(0, t) = -e^{-t} = -\bar{h}(0, t) = \bar{h}^2(-1, t) - \bar{h}(0, t).$$

- If  $k > 0$ , we decompose the event  $\{\bar{X}(t) \leq k\}$  in two parts depending on whether  $\tau_2 \leq t$  or not. Since the event  $\{\tau_2 > t\}$  is included in  $\{\bar{X}(t) \leq k\}$ , we have :

$$\begin{aligned} \mathbb{P}(\bar{X}(t) \leq k) &= \mathbb{P}(\tau_2 > t) + \mathbb{P}(\{\bar{X}(t) \leq k\} \cap \{\tau_2 \leq t\}) \\ &= e^{-t} + \mathbb{P}(\{\bar{X}(t) \leq k\} \cap \{\tau_2 \leq t\}). \end{aligned}$$

By strong Markov property, the event  $\{\bar{X}(t) \leq k\} \cap \{\tau_2 \leq t\}$  is equal to  $\{\bar{X}^{(1)}(t - \tau_2) \leq k - 1\} \cap \{\bar{X}^{(2)}(t - \tau_2) \leq k - 1\} \cap \{\tau_2 \leq t\}$ , where  $\bar{X}^{(1)}$  and  $\bar{X}^{(2)}$  are two

independent copies of  $\bar{X}$  which are also independent of  $\tau_2$ . Therefore,

$$\begin{aligned} \mathbb{P}(\bar{X}(t) \leq k) &= e^{-t} + \int_0^t e^{-s} \mathbb{P}(\bar{X}(t-s) \leq k-1)^2 ds \\ &= e^{-t} + \int_0^t e^{-(t-s)} \mathbb{P}(\bar{X}(s) \leq k-1)^2 ds \\ \Rightarrow e^t \mathbb{P}(\bar{X}(t) \leq k) &= 1 + \int_0^t e^s \mathbb{P}(\bar{X}(s) \leq k-1)^2 ds. \end{aligned}$$

By differentiating with respect to  $t$ , we obtain :

$$\begin{aligned} e^t(\partial_t \bar{h}(k, t) + \bar{h}(k, t)) &= e^t \bar{h}^2(k-1, t) \\ \partial_t \bar{h}(k, t) &= \bar{h}^2(k-1, t) - \bar{h}(k, t), \end{aligned}$$

which is Equation (1.6).

We thus have proved that Equation (1.6) holds for  $k \in \mathbb{Z}$ . Since  $\bar{h}(x, t) = \bar{h}(\lfloor x \rfloor, t)$ , we have also proved that this equation holds for  $x \in \mathbb{R}$ .  $\square$

#### 1.4.2 Proof of Theorem 1.2.2

Since the proof of (1.9) is analogous to that of (1.10), we will just prove (1.9). We know that the critical travelling-waves are of the form  $\phi_{K, \theta^+}$  (see (1.134)), where  $K > 0$ . In order to simplify the notations, we will write  $\phi_K$  instead of  $\phi_{K, \theta^+}$  from now on. We thus want to prove that there exists  $K > 0$  such that :

$$\lim_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}} |\mathbb{P}(\bar{X}(t) \leq \lfloor a_t + x \rfloor) - \phi_K(\lfloor a_t + x \rfloor - a_t)| = 0, \quad (1.45)$$

where  $a_t = c^+ t - \frac{3 \log(t)}{2\theta^+}$ .

Furthermore, remember that the branching random walk  $(T(n))$  is obtained by switching time and space in the process  $(X(t))$  (see (1.33)). The advantage of such a change of point of view is that  $(T(n))$  is non-lattice and therefore, after some renormalizations, we can apply Aïdékon's Theorem (Appendix 1.B.2) and finally use the relation (1.34) to prove (1.45).

*Proof of Theorem 1.2.2.* Consider  $T'$  defined by :

$$T'(n) = \sum_{u \in \mathcal{M}_n} \delta_{(c^+ - 1)T_u(n) - \theta^+ n}, \quad \forall n \in \mathbb{N}. \quad (1.46)$$

The branching random walk  $T'$  can also be described as follows. At time  $n = 0$ , a particle is at 0. For every  $n \in \mathbb{N}$ , each particle alive at time  $n$  splits into two particles at time

$n + 1$ . The displacement of the two new particles with respect to their parent is given by two independent random variables which have the same law as  $(c^+ - 1)E - \theta^+$ , where  $E$  is an exponentially distributed random variable with parameter 1.

We will first ensure that  $T'$  satisfies Aïdékon's assumptions. Since the proofs of (1.148) and (1.149) are very close to that of Assumption (1.147), we will omit them. The first part of Assumption (1.147) is obvious since there is two particles at each division. Now consider  $E$  an exponentially distributed random variable with parameter 1. We recall that the Laplace transform of  $E$  is given by :

$$\mathbb{E}\left(e^{-\lambda E}\right) = \frac{1}{1 + \lambda}. \quad (1.47)$$

The term of the second part of Assumption (1.147) is :

$$\begin{aligned} \mathbb{E}\left(\sum_{u \in \mathcal{M}_1} e^{-T'_u(1)}\right) &= 2\mathbb{E}\left(e^{-[(c^+ - 1)E - \theta^+]}\right) \\ &= 2e^{\theta^+} \frac{1}{1 + (c^+ - 1)} = 2e^{\theta^+} \frac{1}{c^+}. \end{aligned}$$

By definition of  $c^+$  and  $\theta^+$ , we have  $c^+ = 2e^{\theta^+}$ . Therefore :

$$\mathbb{E}\left(\sum_{u \in \mathcal{M}_1} e^{-T'_u(1)}\right) = 1 \quad (1.48)$$

and thus the second part of (1.147) is proved. By differentiating (1.47) with respect to  $\lambda$ , we get :

$$\mathbb{E}\left(Ee^{-\lambda E}\right) = \frac{1}{(1 + \lambda)^2}. \quad (1.49)$$

This yields :

$$\begin{aligned} \mathbb{E}\left(\sum_{u \in \mathcal{M}_1} T'_u(1)e^{-T'_u(1)}\right) &= 2\mathbb{E}\left([(c^+ - 1)E - \theta^+] e^{-[(c^+ - 1)E - \theta^+]}\right) \\ &= 2e^{\theta^+} \frac{c^+ - 1}{(c^+)^2} - \theta^+. \end{aligned}$$

Using the fact that  $c^+ = 2e^{\theta^+}$  and  $c^+ = (2e^{\theta^+} - 1)/\theta^+$ , we get :

$$\begin{aligned} \mathbb{E}\left(\sum_{u \in \mathcal{M}_1} T'_u(1)e^{-T'_u(1)}\right) &= \frac{2e^{\theta^+} - 1}{c^+} - \theta^+ \\ &= 0. \end{aligned}$$

Hence,  $T'$  satisfies the assumptions of Aïdékon's Theorem. Now consider the derivative martingale  $(D_n)$  defined in (1.150) in the particular case of the branching random walk  $(T'(n))$  :

$$\begin{aligned} D_n &= \sum_{u \in \mathcal{M}_n} T'_u(n) e^{-T'_u(n)} \\ &= \sum_{u \in \mathcal{M}_n} ((c^+ - 1)T_u(n) - \theta^+ n) e^{-[(c^+ - 1)T_u(n) - \theta^+ n]} \\ &= \theta^+ \partial W_n^{GEN}(\theta^+), \end{aligned}$$

where  $\partial W_n^{GEN}(\theta^+)$  is defined in (1.136). Furthermore, the relation (1.138) yields :

$$D_\infty = \theta^+ \partial W_\infty(\theta^+). \quad (1.50)$$

For  $n \in \mathbb{N}$  and  $x \in \mathbb{R}$ , we define :

$$t_n = \frac{n}{c^+} + \frac{3 \log(n)}{2c^+\theta^+} \quad \text{and} \quad d_n(x) = \mathbb{P}\left(\underline{T}(n) \geq t_n - \frac{x}{c^+}\right). \quad (1.51)$$

After straightforward computations, we can show that

$$d_n(x) = \mathbb{P}\left(\underline{T}'(n) \geq \frac{3}{2} \log n + x\right),$$

where  $\underline{T}'(n) = \min\{T'_u(n), u \in \mathcal{M}_n\}$ . Aïdékon's result (1.151) and Equation (1.50) yield that there exists  $K > 0$  such that :

$$\lim_{n \rightarrow +\infty} d_n(x) = \phi_K(x), \forall x \in \mathbb{R}. \quad (1.52)$$

Moreover, for each  $n \in \mathbb{N}$ , the fact that  $d_n$  is a cumulative distribution function and the form of  $\phi_K$  (1.134) ensure that  $(d_n)$  and  $\phi_K$  satisfy the assumptions of Theorem 1.C.1. Therefore :

$$\lim_{n \rightarrow +\infty} \sup_{x \in \mathbb{R}} |d_n(x) - \phi_K(x)| = 0. \quad (1.53)$$

We will now use (1.35) to prove (1.45). We start by proving that there exists  $K > 0$  such that for all  $x \in \mathbb{R}$  :

$$\lim_{t \rightarrow +\infty} |\mathbb{P}(\overline{X}(t) \leq \lfloor a_t + x \rfloor) - \phi_K(\lfloor a_t + x \rfloor - a_t)| = 0. \quad (1.54)$$

We first rewrite  $\mathbb{P}(\overline{X}(t) \leq \lfloor a_t + x \rfloor)$  by using (1.35) :

$$\begin{aligned} \mathbb{P}(\overline{X}(t) \leq \lfloor a_t + x \rfloor) &= \mathbb{P}(\underline{T}(\lfloor a_t + x \rfloor + 1) \geq t) \\ &= \mathbb{P}(\underline{T}(\lfloor a_t + x \rfloor + 1) \geq t_{\lfloor a_t + x \rfloor + 1} + t - t_{\lfloor a_t + x \rfloor + 1}) \end{aligned}$$

$$\begin{aligned}
&= \mathbb{P} \left( \underline{T}(\lfloor a_t + x \rfloor + 1) \geq t_{\lfloor a_t + x \rfloor + 1} - \frac{c^+(t_{\lfloor a_t + x \rfloor + 1} - t)}{c^+} \right) \\
&= d_{\lfloor a_t + x \rfloor + 1} (c^+(t_{\lfloor a_t + x \rfloor + 1} - t)). \tag{1.55}
\end{aligned}$$

A straightforward computation gives :

$$c^+(t_{\lfloor a_t + x \rfloor + 1} - t) = \lfloor a_t + x \rfloor - a_t + 1 + \frac{3 \log(c^+)}{2\theta^+} + o_t(1),$$

where  $o_t(1) = \frac{3}{2c^+\theta^+} \log \left( \frac{\lfloor a_t + x \rfloor + 1}{a_t + x + 1} \right)$ . By the triangle inequality :

$$\begin{aligned}
&\left| d_{\lfloor a_t + x \rfloor + 1} (c^+(t_{\lfloor a_t + x \rfloor + 1} - t)) - \phi_K \left( \lfloor a_t + x \rfloor - a_t + 1 + \frac{3 \log(c^+)}{2\theta^+} \right) \right| \\
&\leq \left| d_{\lfloor a_t + x \rfloor + 1} (c^+(t_{\lfloor a_t + x \rfloor + 1} - t)) - \phi_K (c^+(t_{\lfloor a_t + x \rfloor + 1} - t)) \right| \\
&\quad + \left| \phi_K (c^+(t_{\lfloor a_t + x \rfloor + 1} - t)) - \phi_K \left( \lfloor a_t + x \rfloor - a_t + 1 + \frac{3 \log(c^+)}{2\theta^+} \right) \right|,
\end{aligned}$$

where  $K$  is fixed by (1.52). The first term on the right hand side goes to 0 by uniform convergence of  $(d_n)$  to  $\phi_K$ , see (1.53), and the second goes to 0 by uniform continuity of  $\phi_K$  (by (1.134) it is a continuous function with finite limits as  $x \rightarrow \pm\infty$ ).

We thus have proved that for all  $x \in \mathbb{R}$  :

$$\lim_{t \rightarrow +\infty} \left| \mathbb{P}(\overline{X}(t) \leq \lfloor a_t + x \rfloor) - \phi_K \left( \lfloor a_t + x \rfloor - a_t + 1 + \frac{3 \log(c^+)}{2\theta^+} \right) \right| = 0,$$

which is (1.54) (since the travelling-wave is defined up to an additive translation in the argument, we can incorporate  $1 + \frac{3 \log(c^+)}{2\theta^+}$  into  $K$ ).

With the help of (1.54), we will now prove (1.45). For that purpose, we start by proving that there exist  $t_0, x_0, x_1 \in \mathbb{R}$ ,  $x_0 < x_1$  such that :

$$\sup_{x \in ]-\infty, x_0] \cup [x_1, +\infty[} |\mathbb{P}(\overline{X}(t) \leq \lfloor a_t + x \rfloor) - \phi_K(\lfloor a_t + x \rfloor - a_t)| < \epsilon, \forall t > t_0. \tag{1.56}$$

First observe that :

$$x \leq \lfloor a_t + x \rfloor - a_t < x + 1, \forall x \in \mathbb{R}, \forall t > 0.$$

Furthermore,  $\phi_K$  is increasing and tends to 0 when  $x$  goes to  $-\infty$ . Therefore, for  $\epsilon > 0$ , there exists  $x_0 \in \mathbb{R}$  such that :

$$0 < \phi_K(\lfloor a_t + x \rfloor - a_t) < \epsilon/3, \forall t > 0, \forall x \leq x_0. \tag{1.57}$$

Equation (1.54) yields the existence of  $t_1 > 0$  such that :

$$|\mathbb{P}(\overline{X}(t) \leq \lfloor a_t + x_0 \rfloor) - \phi_K(\lfloor a_t + x_0 \rfloor - a_t)| < \epsilon/3, \forall t > t_1. \quad (1.58)$$

The functions  $x \mapsto \mathbb{P}(\overline{X}(t) \leq \lfloor a_t + x \rfloor)$  is non-decreasing and tends to 0 when  $x$  goes to  $-\infty$ . Therefore, Equations (1.57) and (1.58) yield

$$|\mathbb{P}(\overline{X}(t) \leq \lfloor a_t + x \rfloor)| < 2\epsilon/3, \forall x \leq x_0, \forall t > t_1. \quad (1.59)$$

Moreover, using the triangle inequality and with the help of (1.57) and (1.59), we get :

$$|\mathbb{P}(\overline{X}(t) \leq \lfloor a_t + x \rfloor) - \phi_K(\lfloor a_t + x \rfloor - a_t)| < \epsilon, \forall x \leq x_0, \forall t > t_1. \quad (1.60)$$

We can prove in a similar way that there exists  $t_2 > 0$  and  $x_1 > x_0$  such that :

$$|\mathbb{P}(\overline{X}(t) \leq \lfloor a_t + x \rfloor) - \phi_K(\lfloor a_t + x \rfloor - a_t)| < \epsilon, \forall x \geq x_1, \forall t > t_2. \quad (1.61)$$

By taking  $t_0 = \max\{t_1, t_2\}$ , we thus have (1.56). Moreover, since  $[x_0, x_1] \cap \mathbb{Z}$  is finite, we have that :

$$\lim_{t \rightarrow +\infty} \sup_{x \in [x_0, x_1] \cap \mathbb{Z}} |\mathbb{P}(\overline{X}(t) \leq \lfloor a_t + x \rfloor) - \phi_K(\lfloor a_t + x \rfloor - a_t)| = 0. \quad (1.62)$$

For fixed  $t$ ,  $x \mapsto |\mathbb{P}(\overline{X}(t) \leq \lfloor a_t + x \rfloor) - \phi_K(\lfloor a_t + x \rfloor - a_t)|$  is constant on each interval of the form :  $[\lfloor a_t \rfloor - a_t + k, \lfloor a_t \rfloor - a_t + k + 1[$ , and then for fixed  $t$

$$\begin{aligned} & \sup_{x \in [x_0, x_1] \cap \mathbb{Z}} |\mathbb{P}(\overline{X}(t) \leq \lfloor a_t + x \rfloor) - \phi_K(\lfloor a_t + x \rfloor - a_t)| \\ &= \sup_{x \in [x_0, x_1]} |\mathbb{P}(\overline{X}(t) \leq \lfloor a_t + x \rfloor) - \phi_K(\lfloor a_t + x \rfloor - a_t)|. \end{aligned} \quad (1.63)$$

Combining (1.56), (1.62) and (1.63) we obtain :

$$\lim_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}} |\mathbb{P}(\overline{X}(t) \leq \lfloor a_t + x \rfloor) - \phi_K(\lfloor a_t + x \rfloor - a_t)| = 0. \quad (1.64)$$

□

### 1.4.3 Proof of Theorem 1.2.3

To prove Theorem 1.2.3, we first need to establish estimates on the tails of the distribution of the maximum to determine a class of functions  $f$  sufficiently large for which  $G_f$  and  $P_f$  (1.12) are well defined. Again, we will work with the Yule generation process. More precisely, we rely on a result due to Addario-Berry and Reed [6] (see Appendix 1.B.1) which shows that the maximum of a branching random walk with i.i.d. displacements has

exponential tails.

*Proof of Theorem 1.2.3.* Although the process  $T$  does not have independent displacements, it is very easy to modify it to fit the hypothesis of Addario-Berry and Reed's Theorem (see Appendix 1.B.1). We can consider  $(T^{(2)}(n))$  defined by :

$$T^{(2)}(n) = \frac{1}{2} \sum_{u \in \mathcal{M}_{n+1}} \delta_{T_u(n+1) - \tau_2}, \quad \forall n \geq 0, \quad (1.65)$$

where  $\tau_2$  is defined in (1.2). Note that  $(T^{(2)}(n))$  is just a translation of  $(T(n))$  where we remove one element of each couple of particles. The branching random walk  $(T^{(2)}(n))$  can also be described as in Appendix 1.B.1 by taking  $B = 2$  and  $Y = E$ , where  $E$  is an exponential random variable with parameter 1. Furthermore,  $(T^{(2)}(n))$  satisfies the assumptions of Addario-Berry and Reed's Theorem. Indeed, Assumptions 1,2 and 3 are clearly satisfied. By taking  $\lambda_1 = -c^+\theta^+$ , we can also show, after straightforward computations, that Assumption 4 is satisfied.

Let  $\underline{T}^{(2)} = \min\{T_u^{(2)}(n), u \in \mathcal{M}_n\}$  and recall that  $t_n$  is defined by (1.51). Since the branching random walk  $T^{(2)}$  almost surely survives, Addario-Berry and Reed's Theorem yields that there exists a bounded sequence  $\rho$  such that :

$$\mathbb{E} \left( \underline{T}^{(2)}(n) \right) = t_n + \rho_n, \quad \forall n \geq 1 \quad (1.66)$$

and that there exist  $C, \alpha > 0$  such that for all  $n \geq 1$  and  $x \in \mathbb{R}^+$  :

$$\mathbb{P} \left( \left| \underline{T}^{(2)}(n) - t_n - \rho_n \right| \geq x \right) \leq C e^{-\alpha x}. \quad (1.67)$$

Since for each  $x \in \mathbb{R}^+$ ,  $\alpha \mapsto e^{-\alpha x}$  is non-increasing we can suppose that  $\alpha < 1$ . Furthermore, the relation (1.65) clearly implies :

$$\underline{T}(n) = \underline{T}^{(2)}(n-1) + \tau_2 \quad (1.68)$$

and that  $\tau_2$  is independent of  $T^{(2)}$ . Therefore, for  $n \geq 2$  and  $x \in \mathbb{R}$  we get by the triangular inequality :

$$\begin{aligned} & \mathbb{P} \left( \left| \underline{T}^{(2)}(n-1) - t_{n-1} - \rho_{n-1} \right| \geq x - \tau_2 \mid \tau_2 \right) \leq C e^{-\alpha(x-\tau_2)} \\ \Rightarrow & \mathbb{P} \left( \left| \underline{T}^{(2)}(n-1) - t_{n-1} - \rho_{n-1} \right| + \tau_2 \geq x \mid \tau_2 \right) \leq C e^{\alpha\tau_2} e^{-\alpha x} \\ \Rightarrow & \mathbb{P} \left( \left| \underline{T}^{(2)}(n-1) + \tau_2 - t_{n-1} - \rho_{n-1} \right| \geq x \mid \tau_2 \right) \leq C e^{\alpha\tau_2} e^{-\alpha x} \\ \Rightarrow & \mathbb{P} \left( \left| \underline{T}(n) - t_n + t_n - t_{n-1} - \rho_{n-1} \right| \geq x \mid \tau_2 \right) \leq C e^{\alpha\tau_2} e^{-\alpha x}. \end{aligned}$$



The sequence  $(|t_n - t_{n-1} - \rho_{n-1}|)$  is bounded, therefore there exists  $M > 0$  such that :

$$\begin{aligned} \mathbb{P}(|\underline{T}(n) - t_n| - M \geq x | \tau_2) &\leq C e^{\alpha \tau_2} e^{-\alpha x} \\ \Rightarrow \mathbb{P}(|\underline{T}(n) - t_n| \geq x | \tau_2) &\leq C e^{\alpha M} e^{\alpha \tau_2} e^{-\alpha x}. \end{aligned}$$

We have assumed that  $\alpha < 1$  and thus  $\mathbb{E}(e^{\alpha \tau_2}) < +\infty$ . This yields that there exists  $C_2 > 0$  such that :

$$\mathbb{P}(|\underline{T}(n) - t_n| \geq x) \leq C_2 e^{-\alpha x}. \quad (1.69)$$

Fix  $\eta > 0$ . Since  $a$  is not bounded in the neighbourhood of 0, we will now choose our times  $t$  in the set  $[\eta, +\infty)$ . With the help of (1.34) and (1.69), we now want to obtain that there exist  $C_3 > 0$  and  $\alpha' > 0$  such that for all  $t \geq \eta$  and all  $k \in \mathbb{N}$

$$\mathbb{P}(|\bar{X}(t) - \lfloor a_t \rfloor| \geq k) \leq C_3 e^{-\alpha' k}. \quad (1.70)$$

For that purpose, it suffices to prove that there exist  $C_4, C_5 > 0$  such that :

$$\mathbb{P}(\bar{X}(t) - \lfloor a_t \rfloor \geq k) \leq C_4 e^{-\alpha' k} \text{ and } \mathbb{P}(\bar{X}(t) - \lfloor a_t \rfloor \leq -k) \leq C_5 e^{-\alpha' k}. \quad (1.71)$$

Let us prove the first part of (1.71). By taking  $n = \lfloor a_t \rfloor + k$  in (1.69), we get :

$$\begin{aligned} \mathbb{P}(|\underline{T}(\lfloor a_t \rfloor + k) - t_{\lfloor a_t \rfloor + k}| \geq x) &\leq C_2 e^{-\alpha x} \\ \Rightarrow \mathbb{P}(\underline{T}(\lfloor a_t \rfloor + k) \leq t_{\lfloor a_t \rfloor + k} - x) &\leq C_2 e^{-\alpha x}. \end{aligned}$$

Equation (1.34) yields :

$$\mathbb{P}(\bar{X}(t_{\lfloor a_t \rfloor + k} - x) \geq \lfloor a_t \rfloor + k) \leq C_2 e^{-\alpha x}.$$

By taking  $x = t_{\lfloor a_t \rfloor + k} - t$ , we obtain :

$$\mathbb{P}(\bar{X}(t) - \lfloor a_t \rfloor \geq k) \leq C_2 e^{-\alpha(t_{\lfloor a_t \rfloor + k} - t)}.$$

A straightforward computation shows that :

$$t_{\lfloor a_t \rfloor + k} - t = \frac{1}{c^+} \left[ k + \lfloor a_t \rfloor - a_t + \frac{3}{2\theta^+} \log(c^+) \right] + o_t(1),$$

and thus the first part of (1.71) follows. Since the proofs of the two parts of (1.71) are similar, we omit the second. Equation (1.70) shows that the distribution of the centred maximum has exponentially bounded asymptotics. Moreover, we precisely know the asymptotics

of  $\bar{\phi}$  by [35] and (1.130) :

$$1 - \bar{\phi}(x) \underset{x \rightarrow +\infty}{\sim} x e^{-\theta x} \quad \text{and} \quad \bar{\phi}(x) \underset{x \rightarrow -\infty}{\sim} e^{\frac{x}{c^+}}. \quad (1.72)$$

Using these facts we will prove (1.13). Let  $\delta > 0$ , such that  $\delta < \min(\theta, \frac{1}{c^+}, \alpha')$ . Let  $f$  be a piecewise continuous function, such that  $f(x) = \underset{x \rightarrow \pm\infty}{o}(e^{\delta|x|})$  and for  $k \in \mathbb{Z}$ , let  $g_k$  and  $p_k$  be defined by :

$$\begin{aligned} g_k(t) &= f(k - a_t) \mathbb{P}(\bar{X}(t) = k) \\ &= f(k - a_t) [\bar{h}(k, t) - \bar{h}(k - 1, t)], \quad \forall t > 0 \end{aligned} \quad (1.73)$$

and

$$p_k(s) = f(k - s) [\bar{\phi}(k - s) - \bar{\phi}(k - 1 - s)], \quad \forall s \in \mathbb{R}. \quad (1.74)$$

We will first show that  $G_f$  and  $P_f$  are well-defined. For  $k \in \mathbb{Z}$  and  $t \geq \eta$ , we have :

$$\begin{aligned} \mathbb{P}(\bar{X}(t) = k + \lfloor a_t \rfloor) &\leq \mathbb{P}(|\bar{X}(t) - \lfloor a_t \rfloor| = |k|) \\ &\leq \mathbb{P}(|\bar{X}(t) - \lfloor a_t \rfloor| \geq |k|) \\ &\leq C_3 e^{-\alpha'|k|}, \end{aligned} \quad (1.75)$$

where (1.75) is a direct consequence of (1.70). Furthermore, the assumptions on  $f$  and (1.75) imply that there exists  $C_6 > 0$  such that :

$$g_{k+\lfloor a_t \rfloor}(t) \leq C_6 e^{-(\alpha' - \delta)|k|}, \quad \forall k \in \mathbb{Z}, \forall t \geq \eta. \quad (1.76)$$

Similarly the asymptotics in (1.72) and the assumptions on  $f$  yield that there exists  $(\delta', C_7) \in (\mathbb{R}^*)^2$  such that :

$$p_{k+\lfloor a_t \rfloor}(a_t) \leq C_7 e^{-\delta'|k|}, \quad \forall k \in \mathbb{Z}, \forall t \geq \eta. \quad (1.77)$$

We see that the sums  $P_f$  and  $G_f$  defined in (1.12) are invariant by translation by an integer. Thus :

$$G_f(t) = \sum_{k \in \mathbb{Z}} g_{k+\lfloor a_t \rfloor}(t) \quad \text{and} \quad r_f(t) = P_f(a_t) = \sum_{k \in \mathbb{Z}} p_{k+\lfloor a_t \rfloor}(a_t). \quad (1.78)$$

The upper bounds (1.76) and (1.77) ensure that  $G_f$  and  $r_f$  are well-defined on  $[\eta, +\infty)$ , for each  $\eta > 0$ . Therefore,  $G_f$  and  $r_f$  are well-defined on  $\mathbb{R}_+^*$ . We can also observe that if we further suppose that  $f$  is continuous on  $\mathbb{R}$ , since (1.76) and (1.77) yields the normal convergence of the series in (1.78),  $G_f$  and  $r_f$  are continuous and by a simple change of variables it is also the case for  $P_f$ . Let us now prove (1.13). The upper bounds (1.76) and

(1.77) yield that for  $\epsilon > 0$ , there exist  $k_0 \in \mathbb{Z}_-$  and  $k_1 \in \mathbb{N}$  such that for all  $t \geq \eta$  :

$$\sum_{k=-\infty}^{k_0} g_{k+[a_t]}(t) + \sum_{k=k_1}^{+\infty} g_{k+[a_t]}(t) < \epsilon, \quad (1.79)$$

and

$$\sum_{k=-\infty}^{k_0} p_{k+[a_t]}(a_t) + \sum_{k=k_1}^{+\infty} p_{k+[a_t]}(a_t) < \epsilon. \quad (1.80)$$

Furthermore, since there is a finite number of integers between  $k_0$  and  $k_1$ , Theorem 1.2.2 ensures that :

$$\lim_{t \rightarrow +\infty} \sum_{k=k_0}^{k=k_1} |g_{k+[a_t]}(t) - p_{k+[a_t]}(a_t)| = 0. \quad (1.81)$$

Equations (1.79), (1.80) and (1.81) thus yield that for all  $\epsilon > 0$ ; there exists  $t_0 > 0$  such that for all  $t > t_0$

$$\sum_{k=-\infty}^{+\infty} |g_{k+[a_t]}(t) - p_{k+[a_t]}(a_t)| < 3\epsilon, \quad (1.82)$$

which is equivalent to (1.13).

We have yet to deal with (1.14). For  $m \in \mathbb{N}$ , let  $f \in \mathcal{C}^m(\mathbb{R})$  with :

$$f^{(i)}(x) = o_{x \rightarrow \pm\infty}(e^{\delta|x|}), \quad \forall 0 \leq i \leq m. \quad (1.83)$$

The fact that  $f \in \mathcal{C}^m(\mathbb{R})$  obviously implies that  $t \mapsto p_k(a_t)$  and  $g_k$  belong to  $\mathcal{C}^m(\mathbb{R}_+^*)$ . We want to prove that  $G_f$  and  $P_f$  belong to  $\mathcal{C}^m(\mathbb{R}_+^*)$ . For that purpose, we will prove that, for each  $i \in \{1, \dots, m\}$ , the series  $\left(\sum g_k^{(i)}\right)$  and  $\left(\sum p_k^{(i)}\right)$  converge uniformly on every compact subset of  $\mathbb{R}_+^*$ .

For  $k \in \mathbb{Z}$ , define  $\tilde{f}_k$  by :

$$\tilde{f}_k(t) = f(k - a_t), \forall t > 0. \quad (1.84)$$

By an induction, we can show that for each  $i \in \{1, \dots, m\}$ , there exists a sequence of  $i - 1$  polynomial functions  $(R_{j,i})_{j \in \{1, \dots, i-1\}}$  with  $i$  variables such that :

$$\tilde{f}_k^{(i)}(t) = \sum_{j=1}^{i-1} R_{j,i}(a'_t, \dots, a_t^{(i)}) f^{(j)}(k - a_t) + (-a'_t)^i f^{(i)}(k - a_t) \quad (1.85)$$

and

$$R_{j,i}(a'_t, \dots, a_t^{(i)}) = o_t(1), \forall j \in \{1, \dots, i-1\}. \quad (1.86)$$

Equation (1.86) is simply a consequence of the fact that  $a'_t = c^+ + o_t(1)$  and  $a_t^{(j)} = o_t(1)$ ,

$\forall j \geq 2$ . We can also observe that the derivatives of  $a$  are bounded on  $[\eta, +\infty)$ . Therefore the hypothesis (1.83) on  $f$  and Equation (1.85) give us the existence of  $\tilde{C}_i > 0$  such that we have :

$$\left| \tilde{f}_k^{(i)}(t) \right| \leq \tilde{C}_i e^{\delta|k-a_t|}, \quad \forall t \geq \eta. \quad (1.87)$$

We now want to give by induction an upper bound to  $\partial_t^{(i)} \bar{h}(k, t)$ . First suppose that  $i = 1$  and that  $k \geq \lfloor a_t \rfloor$ . Since  $\partial_t \bar{h}(k, t)$  is negative Equation (1.6) yields for  $k \in \mathbb{Z}$  and  $t \geq \eta$  :

$$\begin{aligned} |\partial_t \bar{h}(k, t)| &= \bar{h}(k, t) - \bar{h}^2(k-1, t) \\ &\leq \bar{h}(k, t) \\ &\leq \mathbb{P}(\bar{X}(t) \leq (k - \lfloor a_t \rfloor) + \lfloor a_t \rfloor) \\ &\leq C_4 e^{-\alpha'(k - \lfloor a_t \rfloor)}, \end{aligned} \quad (1.88)$$

where the last inequality comes from (1.71). Similarly, if we now suppose that  $k \leq \lfloor a_t \rfloor$ , we get :

$$\begin{aligned} |\partial_t \bar{h}(k, t)| &= 1 - \mathbb{P}(\bar{X}(t) \geq k) - (1 - \mathbb{P}(\bar{X}(t) \geq k-1))^2 \\ &\leq 2\mathbb{P}(\bar{X}(t) \geq k-1) \\ &\leq 2e^{-\alpha'} C_5 e^{-\alpha'(\lfloor a_t \rfloor - k)}, \end{aligned} \quad (1.89)$$

where  $C_5$  is defined in (1.71). Therefore, Equations (1.88) and (1.89) and the fact that  $a_t - \lfloor a_t \rfloor \leq 1$  yield that there exists  $C_8 > 0$  such that :

$$|\partial_t \bar{h}(k, t)| \leq C_8 e^{-\alpha'|k-a_t|}. \quad (1.90)$$

Fix  $i \geq 2$  and suppose now that for each  $j \in \{1, \dots, i-1\}$ , there exists  $\tilde{C}'_j > 0$  such that for every  $k \in \mathbb{Z}$  and  $t \geq \eta$  :

$$\left| \partial_t^{(j)} \bar{h}(k, t) \right| \leq \tilde{C}'_j e^{-\alpha'|k-a_t|}. \quad (1.91)$$

If we differentiate  $i-1$  times with respect to  $t$  Equation (1.6) we get for  $k \in \mathbb{Z}$  and  $t > 0$  :

$$\partial_t^{(i)} \bar{h}(k, t) = \sum_{j=0}^{i-1} \binom{i-1}{j} \partial_t^{(j)} \bar{h}(k-1, t) \partial_t^{(i-j-1)} \bar{h}(k-1, t) - \partial_t^{(i-1)} \bar{h}(k, t). \quad (1.92)$$

Since  $|\bar{h}| \leq 1$  and using the upper bounds (1.91) into (1.92), we get that there exists  $\tilde{C}'_i > 0$  such that (1.91) holds with  $j = i$ . We thus have proved by induction that (1.91) holds for all  $j \in \{1, \dots, m\}$ .

Fix  $i \in \{1, \dots, m\}$ . By differentiating  $i$  times  $g_k$  defined in (1.73), we get :

$$g_k^{(i)}(t) = \sum_{j=0}^i \binom{i}{j} \tilde{f}_k^{(j)}(t) \left( \partial_t^{(i-j)} \bar{h}(k, t) - \partial_t^{(i-j)} \bar{h}(k-1, t) \right), \quad \forall t > 0. \quad (1.93)$$

By introducing the upper bounds (1.87) and (1.91) into (1.93), we obtain that there exists  $\tilde{C}_i^{(2)} > 0$  such that :

$$\left| g_k^{(i)}(t) \right| \leq \tilde{C}_i^{(2)} e^{-(\alpha' - \delta)|k - a_t|}, \quad \forall t \geq \eta. \quad (1.94)$$

Since  $a$  is smooth on  $\mathbb{R}_+^*$ , it is bounded on every compact subset of  $\mathbb{R}_+^*$ . Hence, (1.94) implies that the series  $\left( \sum g_k^{(i)} \right)$  converges uniformly on every compact subset of  $\mathbb{R}_+^*$  for each  $i \in \{1, \dots, m\}$ . We have already proved that it is also the case for  $i = 0$ . Therefore,  $G_f \in \mathcal{C}^m(\mathbb{R}_+^*)$ . With the same approach, we can show that  $P_f \in \mathcal{C}^m(\mathbb{R})$ .

Let us now prove (1.14). Recall the definition of  $r_f$  in (1.12). As for  $\tilde{f}$ , a induction yields for each  $i \in \{1, \dots, m\}$ , the existence of a sequence of  $i - 1$  polynomial functions  $(\tilde{R}_{j,i})_{j \in \{1, \dots, i-1\}}$  with  $i$  variables such that :

$$r_f^{(i)}(t) = a_t^{(i)} P_f^{(i)}(a_t) + \sum_{j=0}^{i-1} \tilde{R}_{j,i} \left( a_t', \dots, a_t^{(i-1)} \right) P_f^{(j)}(a_t) \quad (1.95)$$

and

$$\tilde{R}_{j,i} \left( a_t', \dots, a_t^{(i-1)} \right) = o_t(1), \quad \forall j \in \{1, \dots, i-1\}. \quad (1.96)$$

Since the derivatives of  $P_f$  are periodic, they are bounded. Therefore (1.95) and (1.96) yield :

$$r_f^{(i)}(t) = a_t^{(i)} P_f^{(i)}(a_t) + o_t(1) = (c^+)^i P_f^{(i)}(a_t) + o_t(1). \quad (1.97)$$

We recall that  $P_f$  and  $G_f$  are invariant by a translation by an integer. Therefore :

$$\begin{aligned} r_f^{(i)}(t) - G_f^{(i)}(t) &= \sum_{k=-\infty}^{+\infty} \left[ (c^+)^i p_k^{(i)}(a_t) - g_k^{(i)}(t) \right] + o_t(1) \\ &= \sum_{k=-\infty}^{+\infty} \left[ (c^+)^i p_{k+\lfloor a_t \rfloor}^{(i)}(a_t) - g_{k+\lfloor a_t \rfloor}^{(i)}(t) \right] + o_t(1). \end{aligned} \quad (1.98)$$

where

$$\begin{aligned} (c^+)^i p_{k+\lfloor a_t \rfloor}^{(i)}(a_t) &= \sum_{j=0}^i \left[ \binom{i}{j} f^{(j)}(k - \{a_t\}) (-c^+)^i \times \right. \\ &\quad \left. \left( \bar{\phi}^{(i-j)}(k - \{a_t\}) - \bar{\phi}^{(i-j)}(k - 1 - \{a_t\}) \right) \right] \end{aligned} \quad (1.99)$$

and

$$g_{k+[a_t]}^{(i)}(t) = \sum_{j=0}^i \left[ \binom{i}{j} f^{(j)}(k - \{a_t\}) (-c^+)^j \times \left( \partial_t^{(i-j)} \bar{h}(k + [a_t], t) - \partial_t^{(i-j)} \bar{h}(k + [a_t] - 1, t) \right) \right] + o_t(1). \quad (1.100)$$

To prove the final part of the Theorem, we can again cut the sum in (1.98) in three terms and proceed as for the proof of (1.13). We will simply show by induction that :

$$\lim_{t \rightarrow +\infty} \sup_{k \in \mathbb{Z}} \left| \partial_t^{(i)} \bar{h}(k + [a_t], t) - (-c^+)^i \bar{\phi}^{(i)}(k - \{a_t\}) \right| = 0. \quad (1.101)$$

The base case  $i = 0$  is Theorem 1.2.2. If we assume that the result holds for the rank  $i - 1$ , we can rewrite (1.92) :

$$\begin{aligned} \frac{\partial_t^{(i)} \bar{h}(k, t)}{(-c^+)^{i-1}} &= \sum_{j=0}^{i-1} \binom{i-1}{j} \bar{\phi}^{(j)}(k-1 - \{a_t\}) \bar{\phi}^{(j-i-1)}(k-1 - \{a_t\}) \\ &\quad - \bar{\phi}^{(i-1)}(k - \{a_t\}) + o_t(1), \end{aligned} \quad (1.102)$$

where  $o_t(1)$  is uniform in  $k$ . Differentiating  $i - 1$  times Equation (1.7) and combining the result with (1.102), we obtain (1.101).

The end of the proof is identical as for the proof of (1.13).  $\square$

For the minimum the proof is the same except that one of the asymptotics of  $\underline{\phi}$  (1.72) is different from those of  $\bar{\phi}$ . Indeed,  $\underline{\phi}$  is decreasing and thus (1.7) implies that for all  $x \in \mathbb{R}$ ,  $\underline{\phi}(x) \leq \underline{\phi}^2(x - 1)$ . Consequently, there is  $0 < A < 1$  such that :

$$\underline{\phi}(x) \leq A^{2^x}, \quad \forall x > 0.$$

#### 1.4.4 Proof of Corollary 1.2.4

We just prove Lemma 1.2.5. Indeed, if we suppose that Lemma 1.2.5 holds, and if we define  $f$  and  $g$  by  $f(x) = x$  and  $g(x) = x^2$ , Theorem 1.2.3 yields :

$$\begin{cases} \mathbb{E}(\bar{X}(t) - a_t) = G_f(t) \\ \text{Var}(\bar{X}(t)) = \text{Var}(\bar{X}(t) - a_t) = G_g(t) - G_f^2(t) \\ \mathbb{E}(\tilde{F}_t) = G'_f(t) + a'_t = G'_f(t) + c^+ + o_t(1) \end{cases}$$

and Corollary 1.2.4 immediately follows.

*Proof of Lemma 1.2.5.* Let  $t \geq 0$  and  $h > 0$ . We first recall that Theorem 1.2.3 applied to the identity ensures that  $w : t \mapsto \mathbb{E}(\bar{X}(t))$  is well-defined and differentiable. We will

show that  $w'(t) = \mathbb{E}(\tilde{F}_t)$ . For this purpose, we divide  $\bar{X}(t+h) - \bar{X}(t)$  into three parts depending on the number  $J_{t,h}$  of jumps of  $(N_t)$  between  $t$  and  $t+h$ . When a particle dies, it gives birth to 2 particles and thus  $J_{t,h} = N_{t+h} - N_t$ .

First consider  $\nu_0(t, h) := (\bar{X}(t+h) - \bar{X}(t)) \mathbf{1}_{\{N_{t+h} - N_t = 0\}}$ . If there is no division between  $t$  and  $t+h$ ,  $\bar{X}(t+h) = \bar{X}(t)$ . Therefore,  $\nu_0(t, h) = 0$ .

Now set  $\nu_1(t, h) := (\bar{X}(t+h) - \bar{X}(t)) \mathbf{1}_{\{N_{t+h} - N_t = 1\}}$ . By construction of the Yule tree process, when there is a division, the particle which splits is chosen uniformly and independently of the number of particles. Furthermore,  $\bar{X}$  increases by 1 after a split if and only if the split occurs for a particle situated at the maximal position. Therefore,

$$\mathbb{E}(\nu_1(t, h) | \mathcal{F}_t) = \frac{\tilde{F}_t}{N_t} \mathbb{P}(N_{t+h} - N_t = 1 | \mathcal{F}_t). \quad (1.103)$$

Since  $(N_t)$  is a pure birth process, we have :

$$\mathbb{P}(N_{t+h} - N_t = 1 | \mathcal{F}_t) = N_t h + o_h(h), \text{ uniformly for all } t. \quad (1.104)$$

Consequently,

$$\mathbb{E}(\nu_1(t, h)) = h \mathbb{E}(\tilde{F}_t) + o_h(h). \quad (1.105)$$

Finally, consider  $\nu_2(t, h) := (\bar{X}(t+h) - \bar{X}(t)) \mathbf{1}_{\{N_{t+h} - N_t \geq 2\}}$ . Since at each split  $(N_t)$  increases by one and  $\bar{X}$  increases at most by one, we have

$$\nu_2(t, h) \leq (N_{t+h} - N_t) \mathbf{1}_{\{N_{t+h} - N_t \geq 2\}}. \quad (1.106)$$

By replacing  $\mathbf{1}_{\{N_{t+h} - N_t \geq 2\}}$  by  $1 - \mathbf{1}_{\{N_{t+h} - N_t = 1\}} - \mathbf{1}_{\{N_{t+h} - N_t = 0\}}$ , we obtain

$$(N_{t+h} - N_t) \mathbf{1}_{\{N_{t+h} - N_t \geq 2\}} = N_{t+h} - N_t - \mathbf{1}_{\{N_{t+h} - N_t = 1\}}. \quad (1.107)$$

By taking the expectation of the terms of Equation (1.107), we get :

$$\begin{aligned} \mathbb{E}((N_{t+h} - N_t) \mathbf{1}_{\{N_{t+h} - N_t \geq 2\}}) &= e^{t+h} - e^t - h e^t + o_h(h) \\ &= o_h(h). \end{aligned}$$

Therefore,

$$\mathbb{E}(\nu_2(t, h)) = o_h(h). \quad (1.108)$$

By grouping  $\nu_0$ ,  $\nu_1$  and  $\nu_2$ , we get :

$$\mathbb{E}(\bar{X}(t+h) - \bar{X}(t)) = h \mathbb{E}(\tilde{F}_t) + o_h(h), \quad (1.109)$$

which concludes the proof.  $\square$

## 1.5 Application to the binary search tree

We recall that by (1.3) the binary search tree is embedded into the Yule tree. Consequently, if  $H_n$  is the height of a random binary search tree and if  $\tau_n$  is defined as in (1.2), we have :

$$H_n = \bar{X}(\tau_n). \quad (1.110)$$

Drawing our inspiration from Lalley and Selke [70] and with the help of Theorem 1.2.2, we will prove Theorem 1.2.6.

*Proof of Theorem 1.2.6.* For  $x \in \mathbb{R}$ , recall that  $\{x\} := x - \lfloor x \rfloor$ . Our proof is divided into two steps.

Step 1 : First, we will show that :

$$\exists K > 0 : \forall \epsilon > 0 \exists s_0 > 0 : \forall s > s_0 \exists t_s > 0 : \forall t > t_s :$$

$$\left| \mathbb{P}(\bar{X}(t+s-\log(W_s)) \leq \lfloor a_{t+s} + x \rfloor) - \mathbb{E} \left( \exp \left( -K e^{-\theta(\lfloor a_{t+s} + x \rfloor - a_{t+s})} \partial Z_\infty^+ \right) \right) \right| < \frac{\epsilon}{3}, \quad (1.111)$$

where  $W_s = W_s(0)$ . By Markov property, we have for  $t, s > 0$  and  $x \in \mathbb{R}$  that :

$$\begin{aligned} & \mathbb{P}(\bar{X}(t+s-\log(W_s)) \leq a_{t+s} + x | \mathcal{F}_s) \\ &= \prod_{u \in \mathcal{N}_s} \mathbb{P}(\bar{X}^u(t-\log(W_s)) \leq a_{t+s} + x - X_u(s)) \\ &= \prod_{u \in \mathcal{N}_s} \mathbb{P}(\bar{X}^u(t-\log(W_s)) \leq a_{t-\log W_s} + a_{t+s} + x - X_u(s) - a_{t-\log W_s}), \end{aligned} \quad (1.112)$$

where the processes  $\bar{X}^u$ , defined by  $\bar{X}^u(t) = \max\{X_v(t+s), v \in \mathcal{N}_{t+s}, u < v\}$ ,  $\forall t > 0$ ,  $\forall u \in \mathcal{N}_s$  are independent of  $\mathcal{F}_s$  and identically distributed. For a fixed  $s$ , we have by Theorem 1.2.2 that almost surely :

$$\begin{aligned} & \lim_{t \rightarrow +\infty} \left| \mathbb{P}(\bar{X}(t+s-\log(W_s)) \leq a_{t+s} + x | \mathcal{F}_s) \right. \\ & \quad \left. - \prod_{u \in \mathcal{N}_s} \phi_K(\lfloor a_{t+s} + x \rfloor - a_{t-\log W_s} - X_u(s)) \right| = 0, \end{aligned} \quad (1.113)$$

where  $\phi_K(x) = \mathbb{E} \left( \exp \left( -K e^{-\theta^+ x} \partial W_\infty(\theta^+) \right) \right) = \bar{\phi}(x)$ . Observe that the argument of  $\phi_K$  can be rewritten as :

$$\lfloor a_{t+s} + x \rfloor - a_{t-\log W_s} - X_u(s) = a_{t+s} + x - \{a_{t+s} + x\} - a_{t-\log(W_s)} - X_u(s)$$



$$= c^+ s + c^+ \log W_s + R_{x,s,t} + S_{s,t} - X_u(s),$$

where

$$R_{x,s,t} = x - \{a_{t+s} + x\} \quad \text{and} \quad S_{s,t} = \frac{3 \log\left(\frac{t - \log W_s}{t+s}\right)}{2\theta^+}.$$

Moreover,  $S_{s,t}$  goes to 0 when  $t$  goes to infinity.  $\phi_K$  is continuous and bounded by 1, and therefore by dominated convergence :

$$\begin{aligned} & \lim_{t \rightarrow +\infty} \left| \mathbb{P} \left( \bar{X}(t+s - \log(W_s)) \leq a_{t+s} + x \right) \right. \\ & \quad \left. - \mathbb{E} \left( \prod_{u \in \mathcal{N}_s} \phi_K(c^+ s + c^+ \log W_s + R_{x,s,t} - X_u(s)) \right) \right| = 0. \end{aligned} \quad (1.114)$$

We will now rewrite the product of the terms from (1.114) :

$$\prod_{u \in \mathcal{N}_s} \phi_K(c^+ s + c^+ \log W_s + R_{x,s,t} - X_u(s)) \quad (1.115)$$

$$= \prod_{u \in \mathcal{N}_s} \mathbb{E} \left( \exp \left( -K e^{-\theta(c^+ s + c^+ \log W_s + R_{x,s,t} - X_u(s))} \partial W_{\infty,u}(\theta^+) \right) \middle| \mathcal{F}_s \right) \quad (1.116)$$

$$= \mathbb{E} \left( \exp \left( -K e^{-\theta(c^+ \log W_s + R_{x,s,t})} \sum_{u \in \mathcal{N}_s} e^{\theta(X_u(s) - c^+ s)} \partial W_{\infty,u}(\theta^+) \right) \middle| \mathcal{F}_s \right), \quad (1.117)$$

where  $\partial W_{\infty,u}(\theta^+)$  are independent copies of  $\partial W_{\infty}(\theta^+)$  and independent of  $\mathcal{F}_s$ . Moreover we know ([35] and (1.143)) that :

$$\sum_{u \in \mathcal{N}_s} e^{\theta(X_u(s) - c^+ s)} \partial W_{\infty,u}(\theta^+) = \partial W_{\infty}(\theta^+) = \frac{W_{\infty}(0)^{c^++1}}{\Gamma(c^+)} \partial Z_{\infty}^+. \quad (1.118)$$

The definition of  $c^+$  means that  $c^+ \theta^+ = c^+ + 1$  and thus :

$$\begin{aligned} & \prod_{u \in \mathcal{N}_s} \phi_K(c^+ s - X_u(s) + c^+ \log W_s + R_{x,s,t}) \\ & = \mathbb{E} \left( \exp \left( -K e^{-\theta^+(c^+ \log W_s + R_{x,s,t})} \frac{W_{\infty}(0)^{c^++1}}{\Gamma(c^+)} \partial Z_{\infty}^+ \right) \middle| \mathcal{F}_s \right) \\ & = \mathbb{E} \left( \exp \left( -K' e^{-\theta^+ R_{x,s,t}} \left( \frac{W_{\infty}(0)}{W_s(0)} \right)^{c^++1} \partial Z_{\infty}^+ \right) \middle| \mathcal{F}_s \right), \end{aligned} \quad (1.119)$$

with  $K' = \frac{K}{\Gamma(c^+)}$ . Let  $\mu_{K'}(R, s) = \mathbb{E} \left( \exp \left( -K' e^{-\theta^+ R} \left( \frac{W_{\infty}(0)}{W_s(0)} \right)^{c^++1} \partial Z_{\infty}^+ \right) \right)$ . By dominated convergence :

$$\lim_{s \rightarrow +\infty} \mu_{K'}(R, s) = \psi(R).$$

Moreover for fixed  $s$ ,  $\mu(\cdot, s)$  is increasing and

$$\lim_{x \rightarrow +\infty} \mu(x, s) = 1 \quad \text{and} \quad \lim_{x \rightarrow -\infty} \mu(x, s) = 0.$$

Thus by Theorem 1.C.1 :

$$\lim_{s \rightarrow +\infty} \sup_{R \in \mathbb{R}} |\mu_{K'}(R, s) - \psi(R)| = 0.$$

We then have :

$$\lim_{s \rightarrow +\infty} \sup_{t > 0} \left| \mathbb{E} \left( \prod_{u \in \mathcal{N}_s} \phi_K(c^+ s - X_u(s) + c^+ \log W_s + R_{x,s,t}) \right) - \psi_{K'}(R_{x,s,t}) \right| = 0. \quad (1.120)$$

So from Equations (1.114) and (1.120) we deduce (1.111).

Step 2 : Noticing that  $\log(n) - \log(W_{\tau_n}(0)) = \tau_n$ , we have by (1.110) :

$$\begin{aligned} \mathbb{P}(H_n \leq \lfloor a_{\log(n)} + x \rfloor) &= \mathbb{P}(\bar{X}(\tau_n) \leq \lfloor a_{\log(n)} + x \rfloor) \\ &= \mathbb{P}(\bar{X}(\log(n) - \log(W_{\tau_n}(0))) \leq \lfloor a_{\log(n)} + x \rfloor). \end{aligned} \quad (1.121)$$

For every  $K > 0$ ,  $\psi_K$  is uniformly continuous and hence for  $\epsilon > 0$  there is  $\eta > 0$  such that for all  $(x, y) \in \mathbb{R}$  such that  $|x - y| < \eta$ ,

$$|\psi_K(x) - \psi_K(y)| < \frac{\epsilon}{3}. \quad (1.122)$$

For  $\eta' > 0$ , we may choose  $s_1 > s_0$  (where  $s_0$  is defined in (1.111)) such that for every  $s > s_1$  :

$$\mathbb{P} \left( |\log(W_s(0)) - \log(W_\infty(0))| > \frac{\eta'}{2} \right) < \frac{\epsilon}{6}. \quad (1.123)$$

by the almost sure convergence of  $W_s(0)$  to a positive random variable. In the same way, there exists  $n_1 \in \mathbb{N}$  such that :  $\log(n_1) > t_{s_1} + s_1$  and such that for all  $n \geq n_1$  :

$$\mathbb{P} \left( |\log(W_{\tau_n}(0)) - \log(W_\infty(0))| > \frac{\eta'}{2} \right) < \frac{\epsilon}{6}. \quad (1.124)$$

Introducing the approximations (1.123) and (1.124) in Equation (1.121) and using the monotonicity of  $\psi_{K'}$  (where  $K'$  is defined in (1.119)) we obtain for  $n \geq n_1$  :

$$\begin{aligned} \mathbb{P}(H_n \leq \lfloor a_{\log(n)} + x \rfloor) &\leq \mathbb{P}(\bar{X}(\log(n) - \log(W_{s_1}(0)) - \eta') \leq \lfloor a_{\log(n)} + x \rfloor) + \frac{\epsilon}{3} \\ &\leq \psi_{K'}(\lfloor a_{\log(n)} + x \rfloor - a_{\log(n) - \eta'}) + \frac{2\epsilon}{3}, \end{aligned} \quad (1.125)$$

where Equation (1.125) is a consequence of (1.111). So, taking  $\eta' > 0$  such that for all

$n \geq n_1$ ,  $|a_{\log(n)-\eta'} - a_{\log(n)}| < \eta$ , we obtain :

$$\begin{aligned} \mathbb{P}(H_n \leq \lfloor a_{\log(n)} + x \rfloor) &\leq \psi_{K'}(\lfloor a_{\log(n)} + x \rfloor - a_{\log(n)} + \eta) + \frac{2\epsilon}{3} \\ &\leq \psi_{K'}(\lfloor a_{\log(n)} + x \rfloor - a_{\log(n)}) + \epsilon, \end{aligned} \quad (1.126)$$

by (1.122). Similarly, we obtain :

$$\psi_{K'}(\lfloor a_{\log(n)} + x \rfloor - a_{\log(n)}) - \epsilon \leq \mathbb{P}(H_n \leq \lfloor a_{\log(n)} + x \rfloor), \quad (1.127)$$

and then for  $x \in \mathbb{R}$ ,

$$\lim_{n \rightarrow +\infty} |\mathbb{P}(H_n \leq \lfloor a_{\log(n)} + x \rfloor) - \psi_{K'}(\lfloor a_{\log(n)} + x \rfloor - a_{\log(n)})| = 0.$$

By the same arguments as in the proof of Theorem 1.2.2, we have, in fact, that :

$$\lim_{n \rightarrow +\infty} \sup_{x \in \mathbb{R}} |\mathbb{P}(H_n \leq \lfloor a_{\log(n)} + x \rfloor) - \psi_{K'}(\lfloor a_{\log(n)} + x \rfloor - a_{\log(n)})| = 0. \quad (1.128)$$

□

## 1.A Travelling-waves and martingales

### 1.A.1 Travelling-waves and martingales of the Yule branching random walk

It is well-known that travelling waves and some martingales play a key role in the study of the extremal particles of the branching random walk. In this appendix we define the relevant objects in our context and recall the pertinent results.

**Theorem (Chauvin, Rouault [35]).** *Equation (1.7) has monotone and bounded travelling-wave solutions at speed  $c$ , in  $\mathcal{C}^1(\mathbb{R})$  if and only if  $c \leq c^-$  or  $c \geq c^+$ . Moreover, uniqueness holds for each such  $c \neq 0$  (up to an additive constant in the argument). For  $c = 0$ , the whole set of solutions is :*

$$\{x \mapsto P(x)^{2^x} \mid P \text{ is 1-periodic and nonnegative}\}. \quad (1.129)$$

It is easy to see that we can find more than one monotone bounded solution in the set (1.129).

**Remark 1.** *Let us point out that this result is stated in a slightly different form in [35]. Indeed, Chauvin and Rouault proved uniqueness for this equation up to a decreasing change*

of variables and in the class of Laplace transforms. The key to their proof was an application of Liu [74, 75]. More recently, Alsmeyer, Biggins and Meiners [17] showed that there is uniqueness among decreasing functions in  $[0, 1]$ . We observe that the set of monotone bounded solutions of (1.7) is in fact the set of monotone solutions in  $[0, 1]$ . The proof that no travelling wave exist for  $c \in (c^-, c^+)$  can be lifted as is from Harris[59] for the branching Brownian motion.

Since it is useful for our proofs, let us mention that the method used in [35] also allows us to determine the left tail of the travelling-waves  $\phi$  with speed  $c \geq c^+$  of (1.7) :

$$\phi(x) \underset{x \rightarrow -\infty}{\sim} e^{\frac{x}{c}}. \quad (1.130)$$

We now define the derivative martingale and the additive martingale. For that purpose, let us consider the natural filtration  $(\mathcal{F}_t)_{t \geq 0}$  defined by  $\mathcal{F}_t := \sigma\{X_u(s), u \in \mathcal{T}_s^c, 0 \leq s \leq t\}$ ,  $\forall t \geq 0$ . We also fix  $\text{sgn}(x) = 1$ , when  $x \geq 0$  and  $\text{sgn}(x) = -1$  when  $x < 0$ .

**Theorem (Chauvin, Klein, Marckert and Rouault [33]).** *For  $\theta \in \mathbb{R}$ , the process  $(W_t(\theta))_{t \geq 0}$  defined by*

$$W_t(\theta) = \sum_{u \in \mathcal{N}_t} e^{\theta(X_u(t) - c_\theta t)} \quad (1.131)$$

is a  $(\mathcal{F}_t)$ -martingale called the additive martingale. Moreover, this martingale converges to an almost surely positive random variable  $W_\infty(\theta)$  when  $\theta \in (\theta^-, \theta^+)$ . In particular, the limit of the martingale is an exponential random variable with parameter 1 when  $\theta = 0$ .

Similarly, the process  $(\partial W_t(\theta))_{t \geq 0}$  defined by

$$\partial W_t(\theta) = \text{sgn}(\theta) \sum_{u \in \mathcal{N}_t} (2e^\theta t - X_u(t)) e^{\theta(X_u(t) - c_\theta t)} \quad (1.132)$$

is a  $(\mathcal{F}_t)$ -martingale called the derivative martingale. Moreover, for  $\theta \in \{\theta^-, \theta^+\}$ , the limit  $\partial W_\infty(\theta)$  of the derivative martingale exists and is almost surely positive.

Note that the derivative martingale of the Yule tree (1.132) is the derivative with respect to  $\theta$  (up to a change of sign for  $\theta < 0$ ) of the additive martingale. We now recall the link between travelling waves and these martingales.

**Theorem (Chauvin, and Drmota [32]).** *For  $\theta \in (\theta^-, \theta^+)$ , the travelling-wave at speed  $c_\theta$  has the following representation :*

$$\phi_{K,\theta}(x) = \mathbb{E} \left( \exp \left( -K e^{-\theta x} W_\infty(\theta) \right) \right), \quad (1.133)$$

where  $K > 0$  fixes the choice of the travelling wave.

For  $\theta \in \{\theta^-, \theta^+\}$ , the travelling-wave at speed  $c_\theta$  has the following representation :

$$\phi_{K,\theta}(x) = \mathbb{E} \left( \exp \left( -K e^{-\theta x} \partial W_\infty(\theta) \right) \right), \quad (1.134)$$

where  $K > 0$  fixes the choice of the travelling wave.

### 1.A.2 Martingales of the Yule generation process

We now define the additive and derivative martingales for the Yule generation process. First consider the filtration  $(\mathcal{F}_n^{GEN})$  defined by  $\mathcal{F}_n^{GEN} = \sigma\{T_u(k), u \in \mathcal{M}_k, k \leq n\}$ . For  $\theta \in \mathbb{R}$ , let  $(W_n^{GEN}(\theta))$  be defined by :

$$W_n^{GEN}(\theta) = e^{\theta n} \sum_{u \in \mathcal{M}_n} e^{-\theta c_\theta T_u(n)}, \quad \forall n \in \mathbb{N}. \quad (1.135)$$

Similarly, define  $(\partial W_n^{GEN}(\theta))$  by :

$$\partial W_n^{GEN}(\theta) = \text{sgn}(\theta) e^{\theta n} \sum_{u \in \mathcal{M}_n} (2e^\theta T_u(n) - n) e^{-\theta c_\theta T_u(n)}, \quad \forall n \in \mathbb{N}. \quad (1.136)$$

The following theorem groups together several results from [35]. We recall that  $W_\infty(\theta)$  and  $\partial W_\infty(\theta)$  are the limits of the additive and derivative martingales defined in (1.131) and in (1.132).

**Theorem (Chauvin, Rouault [35]).** *For  $\theta \in \mathbb{R}$ , the process  $(W_n^{GEN}(\theta))$  is a  $(\mathcal{F}_n^{GEN})$ -martingale called the additive martingale of the Yule generation process. Similarly,  $(\partial W_n^{GEN}(\theta))$  is a  $(\mathcal{F}_n^{GEN})$ -martingale called the derivative martingale of the Yule generation process. Furthermore, for  $\theta \in (\theta^-, \theta^+)$  :*

$$\lim_{n \rightarrow +\infty} W_n^{GEN}(\theta) = W_\infty(\theta) \text{ a.s.} \quad (1.137)$$

and for  $\theta \in \{\theta^-, \theta^+\}$  :

$$\lim_{n \rightarrow +\infty} \partial W_n^{GEN}(\theta) = \partial W_\infty(\theta) \text{ a.s.} \quad (1.138)$$

For the sake of brevity, we refrain from mentioning the idea of stopping lines in this article, we simply mention that the additive and the derivative martingales of the Yule generation process are the additive and the derivative martingales of the Yule branching random walk stopped on the sequence of stopping lines  $\mathcal{M}_n$  which partially explains Equations (1.137) and (1.138). For more references about stopping lines see for instance [31].

### 1.A.3 Martingales of the binary search tree

Let us first define the additive martingale and the derivative martingale for the binary search tree. Let  $\mathcal{L}_n$  be the set of leaves of  $\mathcal{T}_n$  and define  $\mathcal{F}_n^{BST} = \sigma(\{u \in \mathcal{T}_i\}, i \leq n, u \in \mathcal{T})$  where  $\mathcal{T}$  is the complete binary tree. For  $u \in \mathcal{L}_n$ , we denote by  $|u|$  the height of  $u$ . For

$z \in \mathbb{R} \setminus -\frac{\mathbb{N}}{2}$ , we fix :

$$U_0(z) = 1 \quad \text{and} \quad U_n(z) = \prod_{k=0}^{n-1} \frac{k+2z}{k+1}. \quad (1.139)$$

Then, the process  $(Z_n(z))$  introduced by Jabbour [63] and defined by

$$Z_n(z) = \frac{1}{U_n(z)} \sum_{u \in \mathcal{L}_n} z^{|u|}, \quad (1.140)$$

is a  $(\mathcal{F}_n^{BST})$ -martingale, which we will call the additive martingale of the binary search tree. The derivative martingale of the binary search tree is then simply defined by

$$\partial Z_n(z) = \frac{dZ_n(z)}{dz}. \quad (1.141)$$

The following theorem due to Chauvin and Rouault illuminates the connection between the Yule process and the derivative martingale (1.141).

**Theorem (Chauvin, Rouault [35]).** *The martingale  $(\partial Z_n(z))$  converges as  $n \rightarrow +\infty$  to a positive random variable when  $z = e^{\theta^+}$  or  $z = e^{\theta^-}$ . We denote these limits by :*

$$\lim_{n \rightarrow +\infty} \partial Z_n(e^{\theta^+}) = \partial Z_\infty^+ \quad \text{and} \quad \lim_{n \rightarrow +\infty} \partial Z_n(e^{\theta^-}) = \partial Z_\infty^- \quad \text{a.s.} \quad (1.142)$$

Moreover, by embedding the binary search tree into the Yule process, we have :

$$\partial W_\infty(\theta^+) = \frac{e^{-\theta^+} W_\infty(0)^{c^++1}}{\Gamma(c^+)} \partial Z_\infty^+ \quad (1.143)$$

and :

$$\partial W_\infty(\theta^-) = \frac{e^{-\theta^-} W_\infty(0)^{c^-+1}}{\Gamma(c^-)} \partial Z_\infty^-, \quad (1.144)$$

where the limit of the additive martingale  $W_\infty(0)$  has a random exponential 1 law and is independent of  $\partial Z_\infty^+$  and  $\partial Z_\infty^-$ .

## 1.B Results on the maximum of a branching random walk

Since Addario-Berry and Reed's result [6] and Aïdékon's result [7] are central to our argument, we here state them in more details than in Section 1.3.

### 1.B.1 Addario-Berry and Reed's result

Consider a branching random walk defined as follows. A particle is at 0 at time 0. This particle dies at time 1 and give birth to a random number of particles  $B \in \mathbb{N}$  whose displacements are independent copies of a random variable  $Y$ . Then, for each  $n \in \mathbb{N}$ , each

particle  $u$  of the  $n$ th generation gives birth to  $B_u$  particles, where  $B_u$  is an independent copy of  $B$ , and the displacement of each new particle with respect to its parent is a copy of  $Y$  independent of the others. We also define  $\Lambda$  by  $\Lambda(\lambda) := \log(\mathbb{E}(e^{\lambda Y}))$  and  $\overset{\circ}{D}_\Lambda$  the interior of the set of value for which  $\Lambda(\lambda)$  is finite. Finally, we fix  $M_n$  the minimum of the branching random walk and  $\mathcal{S}$  the survival event.

**Theorem (Addario-Berry and Reed [6]).** *Consider a branching random walk satisfying the following assumptions :*

1.  $\mathbb{E}(B) > 1$ ,
2. *there exists  $d \geq 2$ , such that  $\mathbb{P}(B \leq d) = 1$ ,*
3. *there exists  $\lambda_0 > 0$ , such that  $\mathbb{E}(e^{\lambda_0 Y}) < +\infty$ ,*
4. *there exists  $\lambda_1 \in \overset{\circ}{D}_\Lambda \cap \mathbb{R}_-^*$ , such that  $\lambda_1 \Lambda'(\lambda_1) - \Lambda(\lambda_1) = \log(\mathbb{E}(B))$ .*

Then

$$\mathbb{E}(M_n | \mathcal{S}) = \Lambda(\lambda_1)n - \frac{3}{2\lambda_1} \log n + O_n(1). \quad (1.145)$$

Furthermore, there exist  $C, \delta > 0$  such that :

$$\mathbb{P}(|M_n - \mathbb{E}(M_n | \mathcal{S})| \geq x | \mathcal{S}) \leq Ce^{-\delta x}, \quad \forall x \geq 0. \quad (1.146)$$

### 1.B.2 Aïdékon's result

The class of branching random walks in Aïdékon's article is slightly different. As before, a particle is at 0 at time 0. This particle dies at time 1 and give birth this time to a non-lattice point process  $\mathcal{L}$ . Then, for each  $n \in \mathbb{N}$ , the particles of generation  $n$  give birth to independent copies of  $\mathcal{L}$ , translated to their position. We call  $\mathbb{T}$  the genealogical tree of the process and for each  $u \in \mathbb{T}$ , we denote by  $|u|$  its generation and by  $V(u)$  its position on the real line. We conserve the other notations of Addario-Berry and Reed's result.

Aïdékon considers the boundary case, which is quite general after some renormalizations, and which corresponds to the following assumptions :

$$\mathbb{E} \left( \sum_{|u|=1} 1 \right) > 1, \quad \mathbb{E} \left( \sum_{|u|=1} e^{-V(u)} \right) = 1, \quad \mathbb{E} \left( \sum_{|u|=1} V(u)e^{-V(u)} \right) = 0. \quad (1.147)$$

Furthermore, if we set  $E_1 := \sum_{|u|=1} e^{-V(u)}$  and  $E_2 := \sum_{|u|=1} V(u)_+ e^{-V(u)}$ , where  $y_+ = \max(0, y)$ , the suppositions that :

$$\mathbb{E} \left( \sum_{|u|=1} V(u)^2 e^{-V(u)} \right) < +\infty \quad (1.148)$$

and

$$\mathbb{E} \left( E_1 (\log_+ E_1)^2 \right) < \infty, \quad \mathbb{E} (E_2 \log_+ E_2) < \infty \quad (1.149)$$

are made. We now introduce the derivative martingale :

$$D_n := \sum_{|u|=n} V(u)e^{-V(u)}, \quad \forall n \in \mathbb{N}. \quad (1.150)$$

It is known [24] that  $(D_n)$  converges to a positive limit  $D_\infty$ . Finally, still writing  $M_n$  for the minimum of the branching random walk, Aïdékon shows under Assumptions (1.147), (1.148) and (1.149) the following theorem :

**Theorem (Aïdékon [7]).** *There exists  $C > 0$  such that :*

$$\lim_{n \rightarrow +\infty} \mathbb{P} \left( M_n \geq \frac{3}{2} \log(n) + x \right) = \mathbb{E} \left( e^{-Ce^x D_\infty} \right). \quad (1.151)$$

## 1.C A theorem of uniform convergence

**Theorem 1.C.1.** *Let  $(f_n)$  be a sequence of non-decreasing functions from  $\mathbb{R}$  to  $[0, 1]$ , such that :*

$$\lim_{x \rightarrow -\infty} f_n(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow +\infty} f_n(x) = 1.$$

*If  $(f_n)$  converges pointwise to a continuous function  $f$  such that for all  $n \in \mathbb{N}$  :*

$$\lim_{x \rightarrow -\infty} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow +\infty} f(x) = 1,$$

*then the convergence is uniform.*

This theorem is a simple extension of the following result (Problem 127 of [85]).

**Theorem 1.C.2.** *Fix  $a, b \in \mathbb{R}$  such that  $a < b$ . Let  $(f_n)$  be a sequence of non-decreasing functions from  $[a, b]$  to  $\mathbb{R}$ . If  $(f_n)$  converges pointwise to a continuous function  $f$ , then the convergence is uniform.*

*Proof of Theorem 1.C.1.* Let  $(f_n)$  and  $f$  be defined as in the statement of Theorem 1.C.1. Fix  $\epsilon > 0$ . By hypothesis, there exists  $x_0 \in \mathbb{R}$  such that  $0 \leq f(x_0) < \epsilon/6$ . By pointwise convergence of  $(f_n)$  to  $f$  there exists  $n_0$  such that for all  $n \geq n_0$  :

$$|f_n(x_0) - f(x_0)| < \epsilon/6. \quad (1.152)$$

The choice of  $x_0$  and Equation (1.152) imply that for all  $n \geq n_0$ ,

$$0 \leq f_n(x_0) < \epsilon/3. \quad (1.153)$$

Since  $(f_n)$  is a sequence of non-decreasing functions and since the limit of such a sequence is itself non-decreasing, we get that for all  $n \geq n_0$  and  $x \leq x_0$  :

$$0 \leq f_n(x) < \epsilon/3 \quad \text{and} \quad 0 \leq f(x) \leq \epsilon/6. \quad (1.154)$$



This yields :

$$|f_n(x) - f(x)| < \epsilon, \forall x \leq x_0, \forall n \geq n_0. \quad (1.155)$$

Similarly, we can show that there exists  $x_1 > x_0$  and  $n_1 \in \mathbb{N}$  such that :

$$|f_n(x) - f(x)| < \epsilon, \forall x \geq x_1, \forall n \geq n_1. \quad (1.156)$$

We can directly apply Theorem 1.C.1 to the sequence  $(f_n)$  restricted to  $[x_0, x_1]$ , which gives the existence of  $n_2 \in \mathbb{N}$ , such that :

$$|f_n(x) - f(x)| < \epsilon, \forall x \in [x_0, x_1], \forall n \geq n_2. \quad (1.157)$$

By taking  $n_3 = \max\{n_0, n_1, n_2\}$ , and combining (1.155), (1.156) (1.157) we get :

$$|f_n(x) - f(x)| < \epsilon, \forall x \in \mathbb{R}, \forall n \geq n_3, \quad (1.158)$$

which concludes the proof. □

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## Chapitre 2

# Number of particles absorbed in a BBM on the extinction event

This chapter is based on [37].

### 2.1 Introduction and main results

We consider a branching Brownian motion which starts from 0 with drift  $\mu \in \mathbb{R}$ , branching rate  $\beta > 0$ , and reproduction law  $L$ . Let us recall the definition of such a process : a particle starts from 0, lives during an exponential  $\beta$  random time and moves as a Brownian motion with drift  $\mu$ .  $\mathbb{N}$  denotes as usual the set  $\{0, 1, 2, \dots\}$ . When a particle dies, it gives birth to a random number  $L \in \mathbb{N}$  of independent branching Brownian motions started at the position where it dies. We denote by  $G$  the generating function of  $L$ , that is

$$G(s) = \mathbb{E}(s^L) = \sum_{i=0}^{\infty} p_i s^i, \quad (2.1)$$

where  $p_i = \mathbb{P}(L = i)$  and we call  $R_G$  the radius of convergence of  $G$ . In this article, we will assume, unless otherwise stated, that :

$$m = \mathbb{E}(L) \in (1, +\infty) \text{ and } \mu > -\mu_0 \text{ or } m = +\infty. \quad (2.2)$$

where  $\mu_0 = \sqrt{2\beta(m-1)}$ , which corresponds, when  $m < \infty$ , to the speed of the maximum  $M_t$  of a branching Brownian motion without drift in the sense that  $M_t/t \xrightarrow{+\infty} \mu_0$  almost surely on the survival event.

In our model, we kill the particles when they first hit the position  $-x$ ,  $x \geq 0$ . We call  $\zeta_x$  the extinction time of the process, that is the first time when all particles have been

killed. Let us define the extinction probability

$$Q(x, \mu) := \mathbb{P}(\zeta_x < +\infty) \quad (2.3)$$

(or often simply  $Q(x)$  when no confusion can arise). We will, throughout this paper, use the classical notation  $(\mathcal{T}, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$  to denote the filtered probability space on which the Branching Brownian motion evolves, see for instance [57] for more details.

Our purpose is to study the number  $Z_x$  of particles killed at  $-x$  for a branching Brownian motion on the extinction event. We can distinguish 3 different cases according to the drift value when  $1 < m < +\infty$ .

1. If  $\mu \leq -\mu_0$  there is extinction almost-surely and  $Z_x < \infty$  a.s.
2. If  $|\mu| < \mu_0$  then the survival probability is non-zero. The number of particles is almost-surely finite if extinction occurs and is almost-surely infinite otherwise.
3. If  $\mu \geq \mu_0$  then the survival probability is non-zero. The number of absorbed particles is almost-surely finite, whether extinction occurs or not.

When  $m = +\infty$ , we can consider that we are in the second case. The first case has been studied by Maillard [78] and the third case by Berestycki et al. [21]. Here, we consider both case 2 and 3 (that is  $\mu > -\mu_0$ ), on the extinction event. We can point out that since  $Z_x = \infty$  a.s. on the survival event in case 2, the restriction to the extinction event in this provides a whole description of  $Z_x$ .

Initially, in the context of branching random walk with absorption on a barrier, the issue of the total number of particles  $Y$  that have lived before extinction on a barrier had been studied by Aldous [11]. He conjectured that there exist  $K, b > 1$  such that in the critical case (which is the analogue of  $\mu = -\mu_0$  for the branching Brownian motion) we have that  $\mathbb{P}(Y > n) \sim_{n \rightarrow +\infty} K/n^b$  and that in the sub-critical case (which is the analogue of  $\mu < -\mu_0$ ) we have that  $\mathbb{E}(Y) < +\infty$  and  $\mathbb{E}(Y \log Y) = +\infty$ . This problem has been solved by Addario-Berry et al. in [2] and Aïdékon et al. in [10] refined their results. As far as  $Z_x$  is concerned, Neveu [81] showed that  $\mathbb{E}(Z_x) < +\infty$  and  $\mathbb{E}(Z_x \log_+(Z_x)) = +\infty$  in the binary case, when  $\mu = -\mu_0$ . More recently, Maillard has given a very precise description of  $Z_x$  when  $\mu \leq -\mu_0$ . More precisely, he showed the following result. Fix  $\delta$  the span of  $L$ , that is the greatest positive integer such that the support of  $L - 1$  is on  $\delta\mathbb{Z}$ . Define  $\lambda_1 := -\mu + \sqrt{\mu^2 - 2\beta}$ ,  $\lambda_2 := -\mu - \sqrt{\mu^2 - 2\beta}$ , and  $d := \lambda_1/\lambda_2$ . The following theorem is the combinaison of Theorem 1.1 and Remark 1.6 of [78].

**Theorem (Maillard [78]).** *Assume that  $\mathbb{E}[L \log^2 L] < +\infty$ . If  $\mu = -\mu_0$ , then*

$$\mathbb{P}(Z_x > n) \underset{n \rightarrow +\infty}{\sim} \frac{\mu_0 x e^{\mu_0 x}}{n(\log n)^2}. \quad (2.4)$$

*Assume now that  $R_G$ , the radius of convergence of the generating function of  $L$ , is strictly greater than 1. We then have that :*

— If  $\mu = -\mu_0$  :

$$\mathbb{P}(Z_x = \delta n + 1) \underset{n \rightarrow +\infty}{\sim} \frac{\mu_0 x e^{\mu_0 x}}{\delta n^2 (\log n)^2}. \quad (2.5)$$

— If  $\mu < -\mu_0$ , there exists  $K > 0$  such that :

$$\mathbb{P}(Z_x = \delta n + 1) \underset{n \rightarrow +\infty}{\sim} K \frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{n^{d+1}}. \quad (2.6)$$

To prove this theorem Maillard introduces the generating function of  $Z_x$  defined for  $s \in \mathbb{R}^+$  by :

$$F_x(s) = \mathbb{E}(s^{Z_x}). \quad (2.7)$$

Since we want to work on the extinction event we will rather work with :

$$f_x(s) := \mathbb{E}(s^{Z_x} \mathbf{1}_{\{\zeta_x < \infty\}}) = \sum_{i=0}^{\infty} q_i(x) s^i, \quad s \in \mathbb{R}^+, \quad (2.8)$$

where

$$q_i(x) = \mathbb{P}(Z_x = i, \zeta_x < \infty), \quad (2.9)$$

to prove an analogous theorem. Note that  $F_x(s)$  and  $f_x(s)$  coincide in two cases. The first one, dealt with by Maillard, happens when  $m < \infty$  and  $\mu \leq -\mu_0$  because the process becomes extinct almost surely. The second case happens for  $s \in [0, 1)$  when  $m = +\infty$  or when  $m < \infty$  and  $|\mu| < \mu_0$ , since for this range of  $\mu$  the event  $\{\zeta_x = \infty\}$  is almost surely equal to  $\{Z_x = \infty\}$ . The reason for which we chose to consider  $f_x$  instead of  $F_x$  is that, for  $s > 1$ , and  $|\mu| < \mu_0$ ,  $F_x(s)$  is infinite (because  $\{Z_x = \infty\}$  happens with non-zero probability). Even in the case  $\mu \geq \mu_0$ , our situation is clearly different from that in [21], since we restrict to extinction and only the binary branching mechanism is considered in [21].

As a first step we will focus on the radius of convergence of  $f_x$  denoted by  $R(\mu)$  and we will show that it depends on  $\mu$  but not on  $x$ , which justifies the notation  $R(\mu)$ . The quantity  $R(\mu)$  gives us a first information on  $Z_x$ , in particular via the Cauchy-Hadamard Theorem (see for instance [71]) which tells us that :

$$\limsup_{n \rightarrow +\infty} |q_n(x)|^{\frac{1}{n}} = \frac{1}{R(\mu)}. \quad (2.10)$$

A key tool in the present work is  $Q$ . It satisfies the KPP travelling wave equation and is its unique solution under some boundary conditions. This result, which is stated in [58] in the binary case ( $L \equiv 2$ ), is given in the following theorem. Let  $q$  be the probability of extinction without killing on the barrier or equivalently the smallest non-negative fixed point of  $G$  (defined in (2.1)).

**Theorem (Harris et al. [58]).** *The function  $Q$  is the unique solution in  $\mathcal{C}(\mathbb{R}^+, [0, 1])$  of*

the equation :

$$\frac{1}{2}y''(x) + \mu y'(x) + \beta (G(y(x)) - y(x)) = 0, \quad (2.11)$$

with boundary conditions :

$$y(0) = 1, y(\infty) = q, \quad (2.12)$$

when  $m = +\infty$  or  $\mu > -\mu_0$ . There is no such solutions when  $m < +\infty$  and  $\mu \leq -\mu_0$ .

The arguments presented in [58] work without modification in the general case except one. Indeed, the non-triviality of  $Q$  is proved when  $\mu > -\mu_0$  by using the convergence of the additive martingale to a non-trivial limit (see for instance [57]). But this convergence requires the condition  $\mathbb{E}(L \log L) < \infty$ . Maillard gives a proof of the non-triviality of  $Q$  in the supercritical case, without assumptions on  $L$ , which proves that this theorem is always true. Note finally that  $x \mapsto F_x(s)$  and  $x \mapsto f_x(s)$  also satisfy (2.11), but only  $x \mapsto Q(x) = f_x(1)$  satisfies the boundary conditions (2.12).

As a solution of (2.11), we can extend  $Q$  to an open interval containing  $\mathbb{R}_+^*$  and also to a complex domain. The following result is a reformulation in our setting of two classical theorems (Theorem 3.1 of Chapter II of [36] and Section 12.1 of [62]) applied to (2.24), which gives such extensions. We define a neighbourhood of a point by a simply connected open set which contains this point.

**Proposition 2.1.1.** *There exists a maximal open interval  $I$  such that we can extend  $Q$  on  $I$  as a solution of (2.11) and such that  $Q(x) \in (-R_G, R_G), \forall x \in I$ . This extension is unique. Let us define  $x_l = \inf I$ , we further have that if  $x_l > -\infty$ , then :*

$$\lim_{x \rightarrow x_l^+} |Q(x)| = R_G \text{ or } \limsup_{x \rightarrow x_l^+} |Q'(x)| = +\infty. \quad (2.13)$$

Moreover for each  $x \in I$ ,  $Q$  admits an analytic continuation on a neighbourhood of  $x$  (in the complex sense).

Since the extension described in Proposition 2.1.1 is unique, we will make a slight abuse of notation and write  $Q$  to denote this extension. If  $x_l > -\infty$  either  $Q$  cannot be extended analytically left of  $x_l$  or such an extension would exit  $(-R_G, R_G)$ .

Finally, we give the connection between  $f_x$  and  $Q$ . The branching property yields :

$$f_{x+y}(s) = f_y(f_x(s)), \forall (x, y, s) \in (\mathbb{R}^+)^2 \times \mathbb{R}^+, \quad (2.14)$$

where the two sides can possibly be equal to  $+\infty$ , see Maillard [78] for the analogous property for  $F$ . Now consider  $J$  the maximal open interval included in  $I$  which contains  $(0, +\infty)$  such that  $Q$  is decreasing on  $J$ . The function  $Q$  is thus invertible on  $J$ . We define  $x_0(\mu)$  as :

$$x_0(\mu) := \inf J. \quad (2.15)$$

By another slight abuse of notation, we define  $Q(x_0(\mu))$  as the right-limit of  $Q$  when  $x$  goes to  $x_0$ . Note that this limit exists because  $Q$  is decreasing and bounded on  $J$ . For  $x \in \mathbb{R}^+$ , since  $f_x(1) = Q(x)$ , we can derive from (2.14) that :

$$f_x(s) = Q(Q^{-1}(s) + x), \forall q < s \leq R(\mu) \wedge Q(x_0(\mu)). \quad (2.16)$$

We choose in the previous equation  $q < s \leq R(\mu) \wedge Q(x_0(\mu))$  to ensure that the two terms of the equality are well-defined. Actually, the following description of  $R(\mu)$  shows us that we can chose  $s \in (q, R(\mu)]$ .

**Theorem 2.1.2.** *Let  $\mu > -\mu_0$ . The radius of convergence  $R(\mu)$  can be described in terms of  $x_0(\mu)$  and  $Q$  by the relation :*

$$R(\mu) = Q(x_0(\mu), \mu). \quad (2.17)$$

We will write  $x_0$  rather than  $x_0(\mu)$  when no confusions can arise. With the help of Theorem 2.1.2, we can state the behaviour of  $R(\mu)$  with respect to  $\mu$ .

**Theorem 2.1.3.** *The radius of convergence  $R(\mu)$  is a non-decreasing continuous function of  $\mu \in \mathbb{R}$  such that :*

$$\lim_{\mu \rightarrow -\mu_0} R(\mu) = 1 \quad (2.18)$$

and

$$\lim_{\mu \rightarrow +\infty} R(\mu) = R_G. \quad (2.19)$$

Furthermore, if  $R_G > 1$ ,  $R$  is increasing on  $R^{-1}[(1, R_G)]$ .

Even if we are essentially interested by the case  $\mu > -\mu_0$ , we know (see [78]) that for  $\mu \leq -\mu_0$ , we have  $R(\mu) = 1$ , which allow us to state the previous theorem for  $\mu \in \mathbb{R}$ . Furthermore, we can already mention that if  $R_G = 1$  then  $R(\mu) = 1, \forall \mu \in \mathbb{R}$ .

We want now to have a more accurate result than (2.10) concerning the asymptotic behaviour of  $q_n(x)$  when  $n$  tends to infinity. For this, we will first find a good domain (we will say what good means later) on which  $f_x$  is analytic. We will next study the behaviour of  $f_x$  near  $R(\mu)$  (in the real or complex sense, as appropriate). More precisely, the goal is to obtain a classical function equivalent of  $f_x$  in a neighbourhood of  $R(\mu)$ . If these two conditions are satisfied, we can give an exact equivalent of  $q_n(x)$  when  $n$  tends to infinity thanks to analytic methods. Fortunately, this is the case when  $R(\mu) < R_G$ . A natural question is then to know whether  $R(\mu)$  reaches  $R_G$  for a finite  $\mu$ . Let us define  $\mu_c$  as :

$$\mu_c = \inf\{\mu \in \mathbb{R}, R(\mu) = R_G\}. \quad (2.20)$$

In virtue of Theorem 2.1.3, we can distinguish three cases :

1.  $\mu_c = -\infty$  if and only if  $R(\mu) = R_G = 1, \forall \mu \in \mathbb{R}$ ,

2.  $\mu_c = +\infty$  if and only if  $R(\mu) < R_G, \forall \mu \in \mathbb{R}$ ,
3.  $\mu_c \in (-\mu_0, +\infty)$  else.

The following figure shows what happens when  $-\mu_0 < \mu < \mu_c$  or when  $\mu > \mu_c$ .

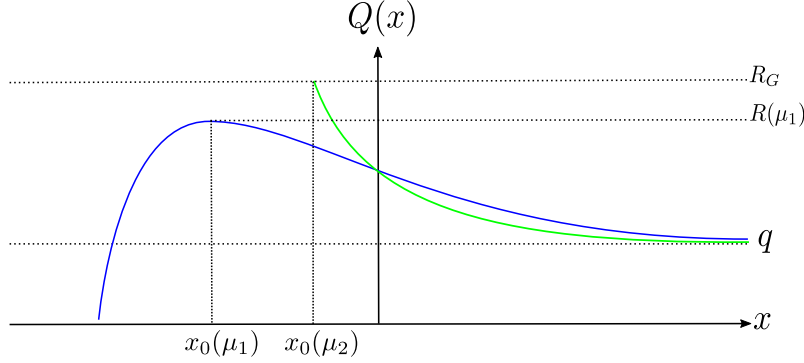


FIGURE 2.1 – Let  $(\mu_1, \mu_2) \in \mathbb{R}^2$  such that  $-\mu_0 < \mu_1 < \mu_c < \mu_2$ . We represent in blue  $Q(\cdot, \mu_1)$  and in green  $Q(\cdot, \mu_2)$ .

The next theorem gives us a criterion to know whether  $R_G$  is reached or not by  $R(\mu)$  for a finite  $\mu$ .

**Theorem 2.1.4.** *The radius of convergence  $R_G$  is reached if and only if  $\mu_c < +\infty$ , which is equivalent to :*

$$\mu_c < +\infty \Leftrightarrow \int_0^{R_G} G(s) ds < +\infty. \quad (2.21)$$

Note that the case  $R_G = +\infty$  is included in the case  $\int_0^{R_G} G(s) ds = +\infty$  of Theorem 2.1.4. We can now give an asymptotic equivalent to  $q_n(x)$  when  $n$  tends to infinity for  $\mu < \mu_c$ . We recall that  $\delta$  is the span of  $G$  and  $x_0$  is defined above as the position of the local maximum of  $Q$  the nearest to 0.

**Theorem 2.1.5.** *When  $-\mu_0 < \mu < \mu_c$ , for  $x > 0$ , we have the following asymptotic estimation of  $q_{\delta i+1}$  :*

$$q_{\delta i+1}(x) \underset{i \rightarrow +\infty}{\sim} \frac{-Q'(x_0(\mu) + x)}{2R(\mu)^{\delta i + \frac{1}{2}} \sqrt{\delta \beta(G(R(\mu)) - R(\mu)) i^3 \pi}}. \quad (2.22)$$

When  $\mu \geq \mu_c > -\mu_0$  we cannot use the same techniques. We will explain why in the last section, but roughly speaking, the reason is that in the general case, we cannot extend  $f_x$  on a complex domain big enough called a  $\Delta$ -domain to apply the transfer Lemma (see [53] and Section 2.4), which is the key to Theorem 2.1.5. Nevertheless, we can obtain some results for not too restrictive hypothesis by applying a classical Tauberian theorem. We

present some particular examples with assumptions like

$$G^{(m)}(s) \underset{s \rightarrow R_G}{\sim} \frac{C}{(R_G - s)^\alpha},$$

where  $\alpha \in (0, 1)$  and  $C > 0$  in the last section. By determining asymptotic equivalent for  $\sum_{i=n}^{+\infty} q_i(x) R_G^i$ , we highlight a change of regime when  $\mu = \mu_c$  and when  $\mu > \mu_c$ .

The paper is organised as follows. Section 2 concerns the extinction probability. Some important results are recalled. In Section 3, we will give the main properties of  $R$  and prove Theorems 2.1.3 and 2.1.4. Section 4 is devoted to the proof with analytic methods of Theorem 2.1.5. Finally, in the last section, we will consider the case  $\mu \geq \mu_c$ .

## 2.2 First results on the extinction probability

In this section, we give the main properties on  $Q$  which will allow us to determine the radius of convergence of  $f_x$  in the next section. Since we want to use phase portraits techniques, we consider :

$$X(x, \mu) := (Q(x, \mu), Q'(x, \mu)) \in \mathbb{R}^2. \quad (2.23)$$

Once again, we will often write  $X(x)$  instead of  $X(x, \mu)$ . We can rewrite the KPP travelling wave equation satisfied by  $Q$  as :

$$X' = \Gamma(X, \mu), \text{ where } \Gamma((x, y), \mu) = (y, -2\mu y - 2\beta(G(x) - x)). \quad (2.24)$$

We will need the precise behaviour of  $Q$  and  $Q'$  in the neighbourhood of  $+\infty$ . In [58], Harris et al. give the asymptotic equivalent of  $Q$  in the binary case, we will state here a more precise version of this result in the general case.

**Theorem 2.2.1.** *If  $m = +\infty$  or  $m < +\infty$  and  $\mu > -\mu_0$ , there exists  $k > 0$  such that :*

$$Q(x) = q + ke^{-\tilde{\lambda}x} + o_{x \rightarrow +\infty}(e^{-\tilde{\lambda}x}) \quad (2.25)$$

and

$$Q'(x) \underset{x \rightarrow +\infty}{\sim} -k\tilde{\lambda}e^{-\tilde{\lambda}x}, \quad (2.26)$$

where  $\tilde{\lambda} = \sqrt{2\beta(1 - G'(q)) + \mu^2} + \mu$ .

If  $q = 0$ ,  $G'(q) = p_1 = 0$  and thus  $\lambda := \sqrt{2\beta + \mu^2} + \mu = \tilde{\lambda}$ , which is exactly the result in [58]. Note that this result could be refined by showing that  $Q$  is a Dirichlet series as it



is done for another travelling-wave in [21].

The following lemma reformulates the KPP equation in two ways. The first one is obtained by stopping the process at the first branching time. The second one is obtained by multiplying all terms of the KPP travelling-wave equation by  $e^{2\mu x}$ , and by integrating this equation from 0 to  $x$ .

**Lemma 2.2.2.** *Let  $\alpha = \sqrt{2\beta + \mu^2}$  and  $\lambda = \alpha + \mu$ . For  $x \geq 0$ , we have :*

$$Q(x) = e^{-\lambda x} + \frac{\beta}{\alpha} \int_0^{+\infty} e^{-\mu y} e^{\mu y} G(Q(y)) \left[ e^{-\alpha|y-x|} - e^{-\alpha(y+x)} \right] dy. \quad (2.27)$$

Moreover, for  $x > x_l$ , we have :

$$Q'(x) = \left( Q'(0) - 2\beta \int_0^x e^{2\mu y} (G(Q(y)) - Q(y)) dy \right) e^{-2\mu x}. \quad (2.28)$$

*Proof.* We prove (2.27) only. Let  $x \geq 0$ . We decompose the event  $\{\zeta_x < +\infty\}$  on two sub-events :

$$\begin{aligned} Q(x) &= \mathbb{E} \left( \mathbf{1}_{\{\zeta_x < +\infty\}} \mathbf{1}_{\{T_1 \geq \zeta_x\}} \right) + \mathbb{E} \left( \mathbf{1}_{\{\zeta_x < +\infty\}} \mathbf{1}_{\{T_1 < \zeta_x\}} \right) \\ &= \mathbb{P}(T_1 \geq \zeta_x) + \mathbb{E} \left( \mathbf{1}_{\{\zeta_x < +\infty\}} \mathbf{1}_{\{T_1 < \zeta_x\}} \right), \end{aligned} \quad (2.29)$$

where  $T_1$  is the time of first split. The first term in the right-hand side of (2.29) is the probability that a Brownian motion with drift  $\mu$  starting from  $x$  reaches 0 before a exponential time with parameter  $\beta$ . By formula 1.1.2 p.250 of [26], we thus have :

$$\mathbb{P}(T_1 \geq \zeta_x) = e^{-\lambda x}. \quad (2.30)$$

We now look the second term. It is the probability that a Brownian motion with drift  $\mu$  splits before reaching 0 and that each process starting from its children becomes extinct. Consider  $(K_s^x)$  a Brownian motion with drift  $\mu$  starting from  $x$  and killed at 0. Using the Markov property and the independence between the Brownian motions, the first split time and the number of children, we have that :

$$\begin{aligned} \mathbb{E} \left( \mathbf{1}_{\{\zeta_x < +\infty\}} \mathbf{1}_{\{T_1 < \zeta_x\}} \right) &= \mathbb{E} \left( \mathbf{1}_{\{K_{T_1}^x > 0\}} Q(K_{T_1}^x)^L \right) \\ &= \mathbb{E} \left( \mathbf{1}_{\{K_{T_1}^x > 0\}} G(Q(K_{T_1}^x)) \right) \\ &= \int_0^{+\infty} G(Q(y)) \mathbb{P}(K_{T_1}^x \in dy). \end{aligned} \quad (2.31)$$

We can derive from 1.0.5 and 1.1.6 p.250-251 of [26] that :

$$\mathbb{P}(K_{T_1}^x \in dy) = \frac{\beta}{\alpha} e^{-\mu x} e^{\mu y} \left[ e^{-\alpha|y-x|} - e^{-\alpha(y+x)} \right] dy \quad (2.32)$$

and thus by plugging (2.32) into (2.31) we get the desired result.  $\square$

We will see in Proposition 2.2.5 that  $x_0 := \inf J$  is finite. Therefore, Equation (2.28) provides the existence in  $\mathbb{R} \cup \{-\infty\}$  of the right-limit  $Q'$  as  $x$  tends to  $x_0$ . Actually, this limit is in  $\mathbb{R}$  (see the remark just after Lemma 2.3.4). As above for  $Q$ , we denote by  $Q'(x_0)$  this limit, when it is finite.

To prove Theorem 2.2.1, we now give a bound of  $Q$  in the following lemma. Although stated for the binary case in Lemma 15 of [58], the result holds more generally when we just suppose  $G(0) = 0$  (which is equivalent to  $p_0 = 0$  or  $q = 0$ ).

**Lemma 2.2.3.** *If  $G(0) = 0$ ,  $\mu > -\mu_0$  then for all  $0 < y < x$  :*

$$Q(x) \leq (Q(y)e^{\rho y})e^{-\rho x}, \quad (2.33)$$

where  $\rho = \sqrt{\mu^2 + 2\beta(1 - Q(y))} + \mu$ .

In particular, this lemma tells us that for any  $\epsilon > 0$  there exist  $x_1, k > 0$  such that for any  $x > x_1$  :

$$Q(x) \leq ke^{-(\lambda - \epsilon)x}, \quad (2.34)$$

where  $\lambda = \sqrt{\mu^2 + 2\beta} + \mu$ . The proof of Lemma 2.2.3 is identical to that of [58] except that we are not in the binary case. Therefore, if  $Y_t$  is a Brownian motion with drift  $\mu$  starting from 0 and  $\tau_z := \inf\{t : Y_t = -z\}$ , the process  $(M_t)$  defined by :

$$M_t := Q(Y_{t \wedge \tau_x}) \exp \left( \beta \int_0^{\tau_x} \left( \frac{G(Q(Y_s))}{Q(Y_s)} - 1 \right) ds \right)$$

replaces the process  $(M_t)$  in [58]. The rest of the proof of Theorem 2.2.1 is from now on different from [58]. We begin by proving the case where  $q = 0$ .

**Lemma 2.2.4.** *Suppose  $q = 0$  and that  $m = \infty$  or  $\mu > -\mu_0$ . Then there exists  $C > 0$  such that :*

$$Q(x) = Ce^{-\lambda x} + o_{x \rightarrow +\infty}(e^{-\lambda x}) \quad (2.35)$$

and

$$Q'(x) \underset{x \rightarrow +\infty}{\sim} -C\lambda e^{-\lambda x}, \quad (2.36)$$

where  $\lambda = \mu + \sqrt{\mu^2 + 2\beta} > 0$ .

*Proof.* We have supposed that  $p_0 = p_1 = 0$ , which implies that  $G(s) \leq s^2, \forall s \in [0, 1]$ . Hence, for  $0 < \epsilon < \frac{\lambda}{2}$  and  $y > 0$ , we have :

$$\begin{aligned} e^{\lambda y} G(Q(y)) &\leq e^{\lambda y} Q^2(y) \\ &\leq C_1 e^{\lambda y} e^{-2(\lambda-\epsilon)y} \\ &\leq C_1 e^{-(\lambda-2\epsilon)y}, \end{aligned} \quad (2.37)$$

where  $C_1 > 0$ . The second inequality in (2.37) is a consequence of Lemma 2.2.3. Let  $x \geq 0$ , we rewrite (2.27) :

$$Q(x)e^{\lambda x} = 1 + \frac{\beta}{\alpha} \left[ \int_0^x e^{\lambda y} G(Q(y)) dy \right. \quad (2.38)$$

$$\left. + e^{2\alpha x} \int_x^{+\infty} e^{-(\alpha-\mu)y} G(Q(y)) dy \right. \quad (2.39)$$

$$\left. - \int_0^{+\infty} e^{-(\alpha-\mu)y} G(Q(y)) dy \right]. \quad (2.40)$$

The inequality (2.37) implies that the integral terms in (2.39) and in (2.40) are well defined and that the integral term in (2.38) is convergent in  $+\infty$ . In the same way, the term in (2.39) converges to 0 when  $x$  goes to infinity, and the term in (2.40) does not depend on  $x$ . Hence, we have that :

$$\lim_{x \rightarrow +\infty} Q(x)e^{\lambda x} = 1 + \frac{\beta}{\alpha} \int_0^{+\infty} [e^{\lambda y} - e^{(\alpha-\mu)y}] G(Q(y)) dy, \quad (2.41)$$

which is (2.35). Now we will establish (2.36) by differentiating  $x \mapsto e^{\lambda x} Q(x)$  :

$$Q'(x)e^{\lambda x} + \lambda Q(x)e^{\lambda x} = \frac{\beta e^{\lambda x} G(Q(x))}{\alpha} \quad (2.42)$$

$$+ 2\beta e^{2\alpha x} \int_x^{+\infty} e^{-(\alpha-\mu)y} G(Q(y)) dy \quad (2.43)$$

$$- \frac{\beta e^{\lambda x} G(Q(x))}{\alpha}. \quad (2.44)$$

The right-hand side of (2.42) cancels the term in (2.44). Moreover, (2.43) can be bounded by using (2.37). We then get that for  $\epsilon > 0$  small enough, there exists  $C_2 > 0$  such that :

$$2\beta e^{2\alpha x} \int_x^{+\infty} e^{-(\alpha-\mu)y} G(Q(y)) dy \leq C_2 e^{-(\lambda-2\epsilon)x}. \quad (2.45)$$

Therefore,

$$\lim_{x \rightarrow +\infty} 2\beta e^{2\alpha x} \int_x^{+\infty} e^{-(\alpha-\mu)y} G(Q(y)) dy = \lim_{x \rightarrow +\infty} Q'(x)e^{\lambda x} + \lambda Q(x)e^{\lambda x} = 0 \quad (2.46)$$

and thus :

$$\lim_{x \rightarrow +\infty} Q'(x)e^{\lambda x} = - \lim_{x \rightarrow +\infty} \lambda Q(x)e^{\lambda x}. \quad (2.47)$$

Equations (2.41) and (2.47) yield (2.36).  $\square$

Now, relying on Lemma 2.2.4 we prove Theorem 2.2.1 dropping the hypothesis  $q = 0$ .

*Proof.* We consider for  $x > 0$  and  $s \in [0, 1]$  :

$$\tilde{Q}(x) := \frac{Q(x) - q}{1 - q} \text{ and } \tilde{G}(s) := \frac{G((1 - q)s + q) - q}{1 - q}. \quad (2.48)$$

It is easy to show that there exists  $(\tilde{p}_i) \in (\mathbb{R}_+)^{\mathbb{N}}$  such that  $\tilde{p}_0 = 0$ ,  $\sum_{i=0}^{\infty} \tilde{p}_i = 1$  and  $\tilde{G}(s) = \sum_{i=0}^{\infty} \tilde{p}_i s^i$ . Besides,  $\tilde{Q}(0) = 1$ ,  $\tilde{Q}(+\infty) = 0$  and  $\tilde{Q}$  solves the equation :

$$\frac{1}{2}y''(x) + \mu y'(x) + \beta \left( \tilde{G}(y(x)) - y(x) \right) = 0, \quad \forall x > 0, \quad (2.49)$$

which means that  $\tilde{Q}$  is the extinction probability of a branching Brownian motion with reproduction law  $\tilde{L}$  with generating function  $\tilde{G}$ . The random variable  $\tilde{L}$  has the following probabilistic interpretation, which can be found more precisely in [20]. Consider a supercritical Galton-Watson tree with reproduction law  $L$  and generating function  $G$ . If we condition the tree to survive and if we keep only the prolific individuals (that is these which give birth to an infinite tree) we obtain a Galton-Watson tree with reproduction law  $\tilde{L}$ . In the same way, the branching Brownian motion with reproduction law  $\tilde{L}$  without killing on a barrier is the branching Brownian motion with reproduction law  $L$  without killing on a barrier conditioned to survive where we keep only the prolific individuals.

The assumptions of Lemma 2.2.4 are almost satisfied. Furthermore, we can simply ignore reproduction events corresponding to  $\tilde{p}_1$  and replace our branching Brownian motion with rate  $\beta$  and reproduction law described by  $\tilde{G}$  by one where the branching rate is  $(1 - \tilde{p}_1)\beta$  and the reproduction law is described by the generating function :

$$G_2(s) := \frac{\tilde{G}(s) - \tilde{p}_1 s}{1 - \tilde{p}_1}. \quad (2.50)$$

We now have that :  $G_2(0) = G_2'(0) = 0$  and we thus can apply Lemma 2.2.4 :

$$\frac{Q(x) - q}{1 - q} = \tilde{Q}(x) \underset{x \rightarrow +\infty}{\sim} k e^{-\tilde{\lambda} x}, \quad (2.51)$$

where  $k > 0$  and  $\tilde{\lambda} = \sqrt{2\beta(1 - \tilde{p}_1) + \mu^2} + \mu = \sqrt{2\beta(1 - G'(q)) + \mu^2} + \mu$ .  $\square$

We now consider  $J$ , the maximal open subinterval of  $I$  (defined in Proposition 2.1.1) such that  $(0, \infty) \subset J$  and on which  $Q$  is decreasing.

**Proposition 2.2.5.** *Fix  $\mu \in (-\mu_0, +\infty)$ . Let  $J$  be defined as above. We have  $x_0(\mu) := \inf J > -\infty$  and either  $Q'(x_0(\mu), \mu) = 0$  or  $Q(x_0(\mu), \mu) = R_G$ .*

*Proof.* We begin by proving that  $x_0(\mu)$  is finite. First, suppose that  $R_G = 1$ . By definition of  $I$ , we have in this case  $I = (0, +\infty)$  and thus  $J = I$  which implies  $x_0(\mu) := \inf J = 0$ . Furthermore,  $Q(x_0(\mu), \mu) = R_G = 1$ . Suppose, now that  $R_G > 1$ . By Proposition 2.1.1, this implies that  $I$  is strictly bigger than  $(0, +\infty)$ . Moreover,  $Q$  is decreasing on  $(0, +\infty)$  and  $Q'(0) \neq 0$  because  $Q'(0) = 0$  would imply that  $Q \equiv 1$  by Cauchy-Lipschitz Theorem. Therefore  $Q'(0) < 0$ . This implies that  $Q$  is decreasing on a interval of the form  $[y, 0)$ , with  $y < 0$ . Suppose that the lower bound of  $J$  is  $-\infty$  (or equivalently that  $J = I = \mathbb{R}$ ). Since on  $I$ ,  $Q(x) < R_G$ , and since  $Q$  is decreasing on  $\mathbb{R}$ , there exists  $l \in (1, R_G]$  such that :

$$\lim_{x \rightarrow -\infty} Q(x) = l. \quad (2.52)$$

Let  $\epsilon \in (0, l - 1)$ . By the Intermediate Value Theorem, there exists  $x_\epsilon \in \mathbb{R}^-$  such that  $Q(x_\epsilon) = l - \epsilon > 1$ . Moreover, by integrating (2.11), we get that there exists  $C \in \mathbb{R}$ , such that :

$$\frac{1}{2}Q'(x) + \mu Q(x) + \int_{x_\epsilon}^x \beta(G(Q(x)) - Q(x)) dx = C, \quad \forall x \in \mathbb{R}. \quad (2.53)$$

Since for  $x < x_\epsilon$ ,  $1 < Q(x) \leq R_G$  and since  $u \mapsto G(u) - u$  is increasing and positive on  $(1, R_G)$ , Equation (2.53) yields :

$$\begin{aligned} Q'(x) &\geq -2 \int_{x_\epsilon}^x \beta(G(Q(x)) - Q(x)) dx + 2C - 2|\mu|R_G \\ &\geq -2(x - x_\epsilon) \beta(G(l - \epsilon) - l - \epsilon) + 2C - 2|\mu|R_G. \end{aligned} \quad (2.54)$$

Consequently,  $\lim_{x \rightarrow -\infty} Q'(x) = +\infty$  which is in contradiction with the fact that  $Q$  is decreasing on  $\mathbb{R}$ . Therefore,  $x_0(\mu)$  is finite.

Let us prove the last part of the proposition. Suppose that  $Q(x_0(\mu), \mu) \neq R_G$ . In this case  $G(Q(y)) < +\infty$ ,  $\forall y \in [x_0(\mu), 0]$ . Hence, (2.28) yields that :

$$\limsup_{x \downarrow x_0(\mu)} |Q'(x)| = \lim_{x \downarrow x_0(\mu)} |Q'(x)| < +\infty.$$

Thus, by Proposition 2.1.1,  $x_0(\mu) \neq x_l$ . Therefore, by definition of  $x_0(\mu)$ , there exists  $\epsilon > 0$  such that  $Q$  is defined and smooth on an interval  $J_\epsilon = (x_0(\mu) - \epsilon, x_0(\mu) + \epsilon)$ , not decreasing on  $(x_0(\mu) - \epsilon, x_0(\mu))$  and decreasing on  $(x_0(\mu), x_0(\mu) + \epsilon)$ . We then have that  $Q'(x_0(\mu), \mu) = 0$ . Hence, either  $Q'(x_0(\mu), \mu) = 0$  or  $Q(x_0(\mu), \mu) = R_G$ .  $\square$

As we said before, we will often write  $x_0$  instead of  $x_0(\mu)$  in what follows. By definition of  $J$ ,  $Q$  is decreasing on  $J$ . This implies that  $Q'(x) \leq 0$  for all  $x \in J$ . More precisely, we

show that  $Q'(x) < 0$  for all  $x \in J$ .

**Proposition 2.2.6.** *For all  $x \in J$ ,  $Q'(x) < 0$ .*

*Proof.* We recall that  $J$  is an open interval and thus  $x_0$  is not included in  $J$ . Consider now  $x_1 \in J$  and suppose that  $Q'(x_1) = 0$ . The KPP equation (2.11) yields that :

$$Q''(x_1) = -2\beta(G(Q(x_1)) - Q(x_1)). \quad (2.55)$$

If  $x_1 \in (x_0, 0)$ , then  $Q''(x_1) < 0$  since  $Q(x) > Q(0) = 1, \forall x \in (x_0, 0)$  and since  $G(s) > s, \forall s > 1$ . Therefore  $x_1$  is a local maximum, which is in contradiction with the fact that on  $J$ ,  $Q$  is decreasing.

Similarly, if  $x_1 \in (0, +\infty)$ , then  $Q''(x_1) > 0$  which also contradicts the decrease of  $Q$  on  $J$ .

We have already proved that  $Q'(0) \neq 0$ . Therefore for all  $x \in J$ ,  $Q'(x) < 0$ .  $\square$

## 2.3 Radius of convergence

In this section, we will focus on  $R(\mu)$  the radius of convergence of  $f_x$ . We will show that it is a function of  $\mu$  which does not depend of  $x > 0$  and determine how this radius evolves with respect to  $\mu$ . Furthermore, since  $R(\mu) = 1$  and  $Q(x, \mu) = 1$  for  $\mu \leq -\mu_0$ , we will often state the results in this section for  $\mu \in \mathbb{R}$  and not only for  $\mu > -\mu_0$ . As a first step, we bound  $R(\mu)$ .

**Proposition 2.3.1.** *For any  $\mu \in \mathbb{R}$ , we have :*

$$1 \leq R(\mu) \leq R_G.$$

Observe that the case where  $\mu \leq -\mu_0$  is trivial. Indeed,  $R_G$  is always greater or equal to 1 (since  $G$  is a generating function) and  $R(\mu) = 1$  for this range of  $\mu$  (see [78]).

*Proof.* The fact that  $R(\mu) \geq 1$  is obvious since  $f_x(1) = Q(x) < +\infty, \forall x \geq 0$ . Let  $D_x$  be defined as the total number of birth event (which include the case  $L = 0$ ) before  $\zeta_x$  and fix  $k \in \mathbb{N} \setminus \{1\}$ . To prove that  $R(\mu) \leq R_G$ , we will calculate  $\tilde{q}_k(x) = \mathbb{P}(D_x = 1, Z_x = k, \zeta_x < +\infty)$ . Since these computations are very similar to those of Lemma 2.2.2, we will skip some details. Like in Lemma 2.2.2, we denote by  $(K_s^x)$  a Brownian motion with drift  $\mu$  starting from  $x$  and killed at 0 and by  $T_1$  an exponential random variable with parameter  $\beta$ . We also recall that  $p_k = \mathbb{P}(L = k)$ ,  $\alpha = \sqrt{2\beta + \mu^2}$  and  $\lambda = \alpha + \mu$ . Using Equations (2.30) and

(2.32), we get :

$$\begin{aligned}
\mathbb{P}(D_x = 1, Z_x = k, \zeta_x < +\infty) &= \int_0^{+\infty} \mathbb{P}(K_{T_1}^x \in dy) p_k \mathbb{P}(T_1 \geq \zeta_y)^k \\
&= \frac{\beta p_k}{\alpha} \int_0^{+\infty} e^{-\mu x} e^{\mu y} \left[ e^{-\alpha|y-x|} - e^{-\alpha(y+x)} \right] e^{-\lambda k y} dy \\
&= 2\beta p_k \left[ \frac{e^{-\lambda x} - e^{-k\lambda x}}{\lambda(k-1)(\alpha - \mu + \lambda k)} \right]. \tag{2.56}
\end{aligned}$$

We know that multiplying the coefficients of a power series by a rational function does not change its radius of convergence. Furthermore, the term  $e^{-\lambda x} - e^{-k\lambda x}$  is equivalent to  $e^{-\lambda x}$  when  $k$  goes to infinity and  $x > 0$ . Therefore, the radius of convergence of the power series whose coefficients are the left-hand side of (2.56), is  $R_G$ . Since

$$\mathbb{P}(D_x = 1, Z_x = k, \zeta_x < +\infty) \leq q_k(x) = \mathbb{P}(Z_x = k, \zeta_x < +\infty), \tag{2.57}$$

we can easily show, for instance with Cauchy-Hadamard Theorem (2.10), that  $R(\mu) \leq R_G$ .  $\square$

This proposition proves in particular that  $R_G = 1$  implies that  $R(\mu) = 1$  for any  $\mu \in \mathbb{R}$ . That is why we can suppose that  $R_G > 1$  (which implies that  $m < +\infty$ ) throughout this section. We now prove Theorem 2.1.2.

*Proof.* Fix  $s_0 = Q(x_0)$ . By Proposition 2.2.5,  $Q'(x) < 0$ ,  $\forall x \in (x_0, +\infty)$  and  $Q$  is continuous on  $(x_0, +\infty)$  and right-continuous at  $x_0$ . Therefore,  $Q^{-1}$  is well-defined on  $(q, s_0)$ . By right-continuity of  $Q$  we even have that  $Q^{-1}(s_0) = x_0$ . Furthermore, we recall that for  $x > 0$  :

$$f_x(s) = Q(Q^{-1}(s) + x), \forall s \leq R(\mu) \wedge s_0. \tag{2.16}$$

Let us show that  $s_0 \leq R(\mu)$ . Suppose not. Let us define  $x_1 = Q^{-1}(R(\mu))$ . By Proposition 2.1.1 there is a complex neighbourhood  $V_1$  of  $x_1$  such that  $Q$  admits an analytic continuation on  $V_1$ . Furthermore,  $Q'(x_1) \neq 0$ , which yields, by Theorem 10.30 of [90], the existence of  $V_2$ , a complex neighbourhood of  $x_1$  included in  $V_1$  such that  $Q$  admits a complex analytic inverse on  $V_2$ . Let us call  $\phi$  the analytic continuation of  $Q^{-1}$  on  $Q(V_2)$ . We now fix  $x > 0$ . Similarly, there exists  $V_3$  a neighbourhood of  $x_1$  such that  $Q$  admits an analytic continuation on the open  $V_3 + x$ . Vivanti-Pringsheim's Theorem (see Theorem 5.7.1 of [60]) ensures that if  $R(\mu)$  is the radius of convergence of  $f_x(s)$  then  $s \mapsto f_x(s)$  cannot have an analytic extension around  $R(\mu)$ . But  $s \mapsto Q(\phi(s) + x)$  is precisely such an extension on  $Q(V_2 \cap V_3)$ . This is in contradiction with the assumption that  $R(\mu) < s_0$  and therefore we have :

$$s_0 \leq R(\mu). \tag{2.58}$$

We will now prove that  $s_0 = R(\mu)$ . By Proposition 2.2.5, we have  $Q(x_0) = R_G$  or  $Q'(x_0) =$

0.

Suppose that  $s_0 = Q(x_0) = R_G$ , we have by (2.58) that  $s_0 = R_G \leq R(\mu)$ . Proposition 2.3.1 tells us that  $R(\mu) \leq R_G$ , and thus  $R(\mu) = R_G = Q(x_0)$  in this case.

Suppose now that  $Q'(x_0) = 0$ . On  $(q, s_0)$  we have that :

$$f'_x(s) = (Q^{-1})'(s)Q'(Q^{-1}(s) + x), \quad (2.59)$$

where  $f'_x$  is the derivative of  $f_x$  with respect to  $s$ . Let us prove by contradiction that  $s_0 \geq R(\mu)$ . Suppose that  $s_0 < R(\mu)$ . Since the radius of convergence of  $f_x$  is strictly greater than  $s_0$ , the left-limit in  $s_0$  of the left-hand side of (2.59) tends to a finite limit. However, since we suppose that  $Q'(x_0) = 0$  we have that

$$\lim_{s \uparrow s_0} (Q^{-1})'(s) = -\infty$$

and since  $Q'(Q^{-1}(s_0) + x) < 0$ , the left-limit of the right-hand side of (2.59) is not finite, which is a contradiction. Therefore,  $s_0 \geq R(\mu)$ . This fact and (2.58) yield :

$$Q(x_0) = R(\mu).$$

□

We thus have proved that the radius of convergence is  $Q(x_0, \mu)$ . We want now to focus on the variation of  $R(\mu)$  with respect to  $\mu$ . For this purpose, inspired by Maillard's approach [78], we introduce a new object  $a$ , defined by :

$$a(s, \mu) := \partial_x f_0(s) = Q'(Q^{-1}(s)), \forall s \in (q, Q(x_0(\mu))), \quad (2.60)$$

where the second equality is given by (2.16) and justifies the existence of  $a$ . Although in this article we will only see  $a$  as a mean to simplify some proofs, there are deeper reasons for its use. We know, see for instance Neveu [81], that  $(Z_x)_{x \geq 0}$  is a Galton-Watson process. Let us consider its infinitesimal generator defined by  $b(s) := \partial_x F_0(s)$  (this is  $a$  in [78]), where  $F$  is defined as in (2.7). As we will use  $f$  instead of  $F$ ,  $a$  will be a slightly different object, which will be nevertheless identical to  $b$  when  $\mu < \mu_0$  and  $s \in (q, 1)$ , and which will satisfy the same properties. The function  $a$  is also a power series with radius of convergence  $R(\mu)$ . Since we do not use this fact in the present work, we will not prove it. Besides, we can notice that the definition of  $a$  implies that the trajectory of  $X$  (defined in (2.23)) for  $x \in (x_0, +\infty)$  is the same of the one of  $s \mapsto (s, a(s))$  for  $s \in (q, Q(x_0))$ . We can therefore work with either of them, depending on the situation. Finally, the travelling wave equation (2.11) and the definition of  $a$  yield :

$$a'(s)a(s) = -2\mu a(s) - 2\beta(G(s) - s) \forall s \in (q, Q(x_0)), \quad (2.61)$$



where  $x_0$  is defined in Proposition 2.2.5. Furthermore, the definition of  $a$  (2.60) and Proposition 2.2.6 implies that :

$$a(s) < 0, \forall s \in (q, Q(x_0)). \quad (2.62)$$

Therefore, the application of the results in Section 12.1 of [62] to (2.61) yields that for every  $s \in (q, Q(x_0))$ , we can analytically extend  $a$  to a complex neighbourhood of  $s$ .

We can now use  $a$  to determine the variation of  $R$  with respect to  $\mu$ .

**Proposition 2.3.2.** *Fix  $\mu_1$  and  $\mu_2$  such that  $-\mu_0 < \mu_1 < \mu_2$ . Set  $r = R(\mu_1) \wedge R(\mu_2)$  and define  $\phi(s) := a(s, \mu_1) - a(s, \mu_2)$  for  $s \in [q, r)$ . The function  $\phi$  is positive on  $(q, r)$  and increasing on  $(1, r)$ . Therefore,  $R$  is a non-decreasing function on  $\mathbb{R}$  and more specifically an increasing function at each  $\mu$  such that  $R(\mu) \in (1, R_G)$ .*

*Proof.* We will prove by contradiction that  $\phi$  is positive. Let us define

$$H := \{s \in (q, r), a(s, \mu_1) \leq a(s, \mu_2)\} \quad (2.63)$$

and suppose that  $H$  is non-empty. We can then define  $h := \inf H < +\infty$ . The functions  $a(\cdot, \mu_1)$  and  $a(\cdot, \mu_2)$  are continuous on  $(q, r)$  and

$$\lim_{s \rightarrow q^+} a(s, \mu_1) = \lim_{s \rightarrow q^+} a(s, \mu_2) = 0,$$

by definition of  $a$  (2.60). Hence,  $h \in H \cup \{q\}$ . We now need an equivalent of  $a$  when  $s$  goes to  $q$ . For  $s > q$ , fix  $x = Q^{-1}(s)$ . We recall that for  $\mu \in \mathbb{R}$ , we define  $\tilde{\lambda}$  by  $\tilde{\lambda} = \sqrt{2\beta(1 - G'(q)) + \mu^2} + \mu$ . With the help of (2.25), we get :

$$\begin{aligned} Q(x) &= s \\ q + Ce^{-\tilde{\lambda}x} + \underset{x \rightarrow +\infty}{o} \left( e^{-\tilde{\lambda}x} \right) &= s \\ -\tilde{\lambda}x + \log(C) + \log \left( 1 + \underset{x \rightarrow +\infty}{o} (1) \right) &= \log(s - q) \\ Q^{-1}(s) = x &= -\frac{\log(s - q)}{\tilde{\lambda}} + \frac{\log(C)}{\tilde{\lambda}} + \underset{x \rightarrow +\infty}{o} (1). \end{aligned}$$

Similarly, using (2.26), we finally obtain :

$$a(s) = Q'(Q^{-1}(s)) \sim -\tilde{\lambda}(s - q) \text{ as } s \downarrow q. \quad (2.64)$$

Furthermore,  $\tilde{\lambda}$  is increasing with respect to  $\mu$  and thus there exists a neighbourhood  $V$  of  $q$  such that :

$$a(s, \mu_1) > a(s, \mu_2), \forall s \in V \cap (q, 1), \quad (2.65)$$

and thus  $h > q$ . This fact and the fact that  $a(\cdot, \mu_1)$  and  $a(\cdot, \mu_2)$  are continuous imply in

particular :

$$a(h, \mu_1) = a(h, \mu_2). \quad (2.66)$$

By recalling (2.62), we know that  $a(s, \mu_1) \neq 0$  and  $a(s, \mu_2) \neq 0$ ,  $\forall s \in (q, r)$ . Equation (2.61) thus implies that for  $s \in (q, r)$  :

$$a'(s, \mu_1) - a'(s, \mu_2) = 2(\mu_2 - \mu_1) - 2\beta \left( \frac{1}{a(s, \mu_1)} - \frac{1}{a(s, \mu_2)} \right) (G(s) - s). \quad (2.67)$$

In particular taking  $s = h$  in (2.67), we have by using (2.66) :

$$a'(h, \mu_1) - a'(h, \mu_2) = 2(\mu_2 - \mu_1) > 0. \quad (2.68)$$

Equations (2.66) and (2.68) and the fact that  $h > q$  yield that there exists  $\epsilon > 0$  such that  $a(s, \mu_1) \leq a(s, \mu_2)$ ,  $\forall s \in (h - \epsilon, h)$ , which contradicts the definitions of  $H$  and  $h$ . Hence  $H$  is empty and :

$$a(s, \mu_1) > a(s, \mu_2), \forall s \in (q, r). \quad (2.69)$$

We have thus proved that  $\phi$  is positive on  $(q, r)$ . Furthermore, since  $G(s) > s$ ,  $\forall s \in (1, R_G)$ , Equation (2.67) yields that  $\phi' > 0$  on  $(1, r)$ .

Let us now focus on  $\mu \mapsto R(\mu)$ . If  $R(\mu_2) = R_G$ , we know by Proposition 2.3.1 that  $R(\mu_1) \leq R(\mu_2)$ .

Now, suppose that  $R(\mu_2) < R_G$ . Proposition 2.2.5 yields  $Q'(x_0(\mu_2), \mu_2) = a(R(\mu_2), \mu_2) = 0$ . Let us first prove by contradiction that  $R(\mu_2) \geq R(\mu_1)$ . If  $R(\mu_1) > R(\mu_2)$ , then  $a(R(\mu_2), \mu_1)$  is well-defined. Furthermore, by taking the left-limit when  $s$  goes to  $r = R(\mu_2)$  in Equation (2.69), we get that

$$a(R(\mu_2), \mu_1) \geq a(R(\mu_2), \mu_2) = 0. \quad (2.70)$$

Since  $R(\mu_1) > R(\mu_2)$ , Equation (2.70) is in contradiction with (2.62). Hence,  $R(\mu_1) \leq R(\mu_2) < R_G$ .

Let us now prove that  $R(\mu_1) < R(\mu_2)$ . Since  $R(\mu_1) < R_G$ , Proposition 2.2.5 yields  $a(R(\mu_1), \mu_1) = 0$ . Moreover, the increase of  $\phi$  implies :

$$-a(R(\mu_1), \mu_2) = a(R(\mu_1), \mu_1) - a(R(\mu_1), \mu_2) > a(1, \mu_1) - a(1, \mu_2) > 0. \quad (2.71)$$

Consequently,  $R(\mu_1)$  cannot be equal to  $R(\mu_2)$  and thus  $R(\mu_1) < R(\mu_2)$ .  $\square$

We will now prove the continuity of  $R$ . As a first step, we will prove the continuity of  $X(x, \cdot) = (Q(x, \cdot), Q'(x, \cdot)) \in \mathbb{R}^2$  for fixed  $x \in \mathbb{R}^+$  by probabilistic methods. In fact, for our purpose, it would be sufficient to show the continuity of  $\mu \mapsto Q'(0, \mu)$ . But if we have the continuity of  $Q$ , it is simple to establish the continuity of  $Q'$  with respect to  $\mu$ . Next,

using the fact that  $Q$  is solution of KPP, we will extend the continuity of  $Q$  to negative half-line and deduce from it the continuity of  $R$ .

**Proposition 2.3.3.** *For any  $x \geq 0$ , the functions  $\mu \mapsto Q(x, \mu)$  and  $\mu \mapsto Q'(x, \mu)$  (where the derivative is with respect to  $x$ ) are continuous on  $\mathbb{R}$ .*

We recall we suppose throughout this section that  $R_G > 1$  (which implies in particular that  $m < \infty$  and that the process cannot explode in finite time). Although Proposition 2.3.3 holds in general, this assumption allows us to avoid unnecessary technical complications.

*Proof.* Fix  $x_1 > 0$ . In order to prove the continuity of  $Q$  and  $Q'$  with respect to  $\mu$ , it is easier to consider a branching Brownian motion starting from 0 without drift killed on the barrier :

$$\gamma_{x_1, \mu} = \{(t, x) \in \mathbb{R}^2, x = -\mu t - x_1\}. \quad (2.72)$$

rather than a branching Brownian motion with drift  $\mu$  and killed at  $-x_1$ . We can thus move the barrier by changing  $\mu$  for a fixed  $\omega \in \mathcal{T}$ .

Let us fix some notations. We call  $\mathcal{N}_t$  the set of particles alive at time  $t$  without killing and for  $\mu \in \mathbb{R}$ , we call  $\mathcal{S}_t^\mu$  the set of particles stopped on  $\gamma_{x_1, \mu}$  at time  $t$ ,  $\mathcal{A}_t^\mu$  the set of particles alive for the branching Brownian motion with killing on  $\gamma_{x_1, \mu}$  and  $Z_{x_1, \mu} := |\mathcal{S}_t^\mu|$ , that is the number of particles killed on  $\gamma_{x_1, \mu}$ . For  $u \in \mathcal{N}_t$  and  $s \leq t$ , we call  $X_u(s)$  the position of the ancestor of  $u$  alive at time  $s$ . We denote by  $K_\mu$  the event  $\{\zeta_{x_1, \mu} < +\infty\}$  which means "All particles are killed on  $\gamma_{x_1, \mu}$ ". We thus have  $Q(x_1, \mu) = \mathbb{P}(K_\mu)$ . The function  $\mu \mapsto Q(x_1, \mu)$  is non-increasing because if  $\mu_1 \leq \mu_2$  then  $K_{\mu_2} \subset K_{\mu_1}$ . Therefore,  $\mu \mapsto Q(x_1, \mu)$  has a left-limit and right-limit at every point.

We temporarily suppose that  $p_0 = G(0) = 0$ . To prove the continuity of  $\mu \mapsto Q(x_1, \mu)$  let us start by proving its left-continuity. The left-continuity for  $\mu \leq -\mu_0$  is obvious, since for this range of  $\mu$ ,  $Q(x, \mu) = 1, \forall x \in \mathbb{R}^+$ . Suppose that  $\mu \mapsto Q(x_1, \mu)$  is not left-continuous for a  $\mu_1 > -\mu_0$ , which is equivalent to the fact that :

$$\mathcal{L}_{\mu_1} := \bigcap_{\mu < \mu_1} K_\mu \cap K_{\mu_1}^c = \bigcap_{\mu < \mu_1, \mu \in \mathbb{Q}} K_\mu \cap K_{\mu_1}^c. \quad (2.73)$$

happens with non-zero probability (the second equality ensures that  $\mathcal{L}_{\mu_1}$  is measurable). Fix  $\omega \in \mathcal{L}_{\mu_1}$ . Since we have supposed that  $G(0) = 0$ , the function  $\mu \mapsto Z_{x_1, \mu}(\omega)$  is non-decreasing on  $(-\infty, \mu_1)$ . It thus has a left-limit when  $\mu$  goes to  $\mu_1$ ,  $l(\omega) \in \mathbb{R} \cup \{+\infty\}$ . We will first prove that this limit is infinite. Fix  $M \in \mathbb{N}$ . On  $K_{\mu_1}^c$  (and consequently on  $\mathcal{L}_{\mu_1}$ ) we know that almost surely the number of particles in  $\mathcal{A}_t^{\mu_1}$  increases to infinity (see for instance [67]) as  $t$  tends to infinity. Therefore, for almost every  $\omega \in \mathcal{L}_{\mu_1}$  there exists  $t(\omega) > 0$  such that  $N_{x_1, \mu_1}(t(\omega))$ , the number of particles alive at time  $t(\omega)$ , is larger than

*M.* Now, fix :

$$\epsilon := \inf_{u \in \mathcal{A}_t^{\mu_1}} \{X_u(s) + \mu_1 s + x_1, s \leq t(\omega)\} \text{ and } \mu_2(\omega) := \mu_1 - \frac{\epsilon}{2t(\omega)}. \quad (2.74)$$

$\epsilon$  is the infimum of a finite number of strictly positive continuous functions and thus is strictly positive.  $\mu_2(\omega)$  has been chosen such that  $\mu_2(\omega) < \mu_1$  and such that every particle which has not been killed on  $\gamma_{x_1, \mu_1}$  before  $t(\omega)$  is not killed on  $\gamma_{x_1, \mu_2(\omega)}$  before  $t(\omega)$  either. Since  $G(0) = 0$  and  $\omega \in \mathcal{L}_{\mu_1} \subset K_{\mu_2(\omega)}$ , we will necessarily have  $Z_{x_1, \mu_2(\omega)}(\omega) \geq N_{x_1, \mu_1}(t(\omega)) \geq M$  and thus  $l(\omega) \geq M$ . As  $M$  is arbitrary, we have for almost every  $\omega \in \mathcal{L}_{\mu_1}$  :

$$\lim_{\mu \rightarrow \mu_1^-} Z_{x_1, \mu}(\omega) = +\infty. \quad (2.75)$$

The fact that  $\mathbb{P}(\mathcal{L}_{\mu_1}) > 0$  and (2.75) imply that :

$$\mathbb{E} \left[ \liminf_{\mu \rightarrow \mu_1^-} \mathbf{1}_{\mathcal{L}_{\mu_1}} Z_{x_1, \mu} \right] = +\infty. \quad (2.76)$$

Furthermore, by Fatou's lemma we get :

$$\mathbb{E} \left[ \liminf_{\mu \rightarrow \mu_1^-} \mathbf{1}_{\mathcal{L}_{\mu_1}} Z_{x_1, \mu} \right] \leq \liminf_{\mu \rightarrow \mu_1^-} \mathbb{E} \left[ \mathbf{1}_{\mathcal{L}_{\mu_1}} Z_{x_1, \mu} \right] \quad (2.77)$$

$$\leq \liminf_{\mu \rightarrow \mu_1^-} \mathbb{E} \left[ \mathbf{1}_{\{\zeta_{x_1, \mu} < +\infty\}} Z_{x_1, \mu} \right] \quad (2.78)$$

$$\leq \liminf_{\mu \rightarrow \mu_1^-} \frac{Q'(x_1, \mu)}{Q'(0, \mu)}. \quad (2.79)$$

Inequality (2.78) comes from the definition of  $\mathcal{L}_{\mu_1}$  which implies that for any  $\mu < \mu_1$ ,  $\mathcal{L}_{\mu_1} \subset \{\zeta_{x, \mu} < +\infty\}$  and (2.79) is obtained by the differentiation of  $f_{x_1}$  with respect to  $s$  at 1 and from (2.16). We recall that for  $x \geq 0$ ,  $q \leq Q(x) \leq 1$  and for  $s \in (q, 1)$ ,  $G(s) \leq s$ . Therefore, for any  $\mu > -\mu_0$ , (2.28) yields :

$$\frac{Q'(x_1, \mu)}{Q'(0, \mu)} \leq e^{-2\mu x_1}. \quad (2.80)$$

The left-hand of (2.77) is thus bounded by  $e^{-2\mu_1 x_1}$ , which contradicts (2.76). Hence,  $\mathbb{P}(\mathcal{L}_{\mu_1}) = 0$  and consequently  $\mu \mapsto Q(x_1, \mu)$  is left-continuous.

Suppose now that  $\mu \mapsto Q(x_1, \mu)$  is not right-continuous at  $\mu_1 \geq -\mu_0$ . That implies that the event  $\mathcal{R}_{\mu_1}$ , defined by :

$$\mathcal{R}_{\mu_1} := \bigcap_{\mu > \mu_1} K_\mu^c \cap K_{\mu_1} = \bigcap_{\mu > \mu_1, \mu \in \mathbb{Q}} K_\mu^c \cap K_{\mu_1} \quad (2.81)$$

happens with positive probability. We define for  $u \in \mathcal{N}_t : Y_u(t) = X_u(t) + \mu_1 t + x_1$  and  $\tau_u = \inf\{s \leq t, Y_u(s) = 0\}$ . Furthermore, we call  $H$  the event :

$$H := \bigcap_{n \in \mathbb{N}^*} \bigcap_{u \in \mathcal{S}_n^{\mu_1}} \bigcup_{s \in (0, n - \tau_u) \cap \mathbb{Q}} \{Y_u(\tau_u + s) < 0\}. \quad (2.82)$$

Let us briefly show that  $\mathbb{P}(H^c) = 0$ . Let  $B$  be a Brownian motion and  $\tau = \inf\{s \in \mathbb{R}^+, B_s = -\mu_1 s - x_1\}$ . A Brownian motion cannot stay above a barrier after reaching this barrier and the many-to-one lemma (see for instance Theorem 8.5 of [56]) will ensure that none particle of the branching Brownian motion can do it. More formally, we have :

$$\begin{aligned} \mathbb{P}(H^c) &= \mathbb{P} \left( \bigcup_{n \in \mathbb{N}^*} \bigcup_{u \in \mathcal{S}_n^{\mu_1}} \bigcap_{s \in (0, n - \tau_u) \cap \mathbb{Q}} \{Y_u(\tau_u + s) \geq 0\} \right) \\ &\leq \sum_{n=1}^{+\infty} \mathbb{E} \left( \sum_{u \in \mathcal{S}_n^{\mu_1}} \mathbf{1}_{\{Y_u(\tau_u + s) \geq 0, \forall s \in (0, n - \tau_u) \cap \mathbb{Q}\}} \right) \\ &\leq \sum_{n=1}^{+\infty} e^{\beta(m-1)n} \mathbb{P}(\tau < n; Y_{\tau+s} \geq 0, \forall s \in (0, n - \tau) \cap \mathbb{Q}) \end{aligned} \quad (2.83)$$

$$\leq 0, \quad (2.84)$$

where  $Y$  is a Brownian motion with drift  $\mu_1$  starting from  $x_1$ . Inequality (2.83) is just many-to-one lemma and inequality (2.84) comes from the strong Markov property.

We now fix  $n = \lfloor \zeta_{x_1, \mu_1} \rfloor + 1$ . Since  $n \geq \zeta_{x_1, \mu_1}$  and  $\mathbb{P}(H) = 1$ , we have on  $\mathcal{R}_{\mu_1}$  that for all  $u \in \mathcal{N}_n$  there exists  $s_u \in (0, n)$  such that  $Y_u(s_u) < 0$ . The fact that  $\mathcal{N}_n$  is finite implies there exists  $\epsilon > 0$  such that  $Y_u(s_u) \leq -\epsilon, \forall u \in \mathcal{N}_n$ . If we take  $\mu = \mu_1 + \frac{x_1}{n} + \frac{\epsilon}{2n}$  then each particle of  $\mathcal{N}_n$  reaches  $\gamma_{x_1, \mu}$  before  $n$ , which means that the process dies on  $\gamma_{x_1, \mu}$ . This is in contradiction with the definition of  $\mathcal{R}_{\mu_1}$ . Therefore,

$$\mathbb{P}(\mathcal{R}_{\mu_1}) = 0. \quad (2.85)$$

We have proved that  $\mu \mapsto Q(x_1, \mu)$  is also right-continuous and thus continuous in the case where  $G(0) = 0$ . The argument of the proof of Theorem 2.2.1, which consists of looking the tree of prolific individuals can again be applied to prove the result in the general case.

Let us now prove the continuity of  $\mu \mapsto Q'(x, \mu)$  for any  $x > 0$ . We can deduce from (2.27) that :

$$Q'(0, \mu) = 2\beta \int_0^{+\infty} e^{-(\alpha-\mu)y} G(Q(y, \mu)) dy - \lambda, \quad (2.86)$$

where we recall that  $\alpha = \sqrt{\mu^2 + 2\beta}$  and  $\lambda = \mu + \alpha$ . Let  $K$  be a compact subset of  $\mathbb{R}$ . For

any  $\mu \in K$  and for any  $y \geq 0$ , we have :

$$e^{-(\alpha-\mu)y}G(Q(y, \mu)) \leq e^{-Cy}, \quad (2.87)$$

where  $C = \min_{\mu \in K} \{\alpha - \mu\} > 0$ . Moreover, we know that  $\mu \mapsto Q(x, \mu)$  is continuous and, as a composition of continuous functions,  $\mu \mapsto e^{-(\alpha-\mu)y}G(Q(y, \mu))$  is also continuous. Therefore,  $\mu \mapsto Q'(0, \mu)$  is continuous. Similarly, with the help of (2.28), we can easily prove that  $\mu \mapsto Q'(x, \mu)$  is continuous. □

The continuity of  $Q$  with respect to  $\mu$  will be useful to prove the continuity of  $R$ . Before proving this continuity, we just prove the right-continuity of  $R$  in  $-\mu_0$ .

**Lemma 2.3.4.** *The function  $R$  is continuous at 1, which is equivalent to :*

$$\lim_{\mu \rightarrow -\mu_0} R(\mu) = 1. \quad (2.88)$$

*Proof.* Suppose first that  $R_G = 1$ . In that case, Proposition 2.3.1 yields  $R(\mu) = 1$ ,  $\forall \mu \in \mathbb{R}$  and thus the lemma is proved. Now suppose that  $R_G > 1$  (which implies that  $m < \infty$ ). Let  $-\mu_0 < \mu < 0$ . We recall that for  $s \in (q, R(\mu))$  :

$$a'(s)a(s) = -2\mu a(s) - 2\beta(G(s) - s). \quad (2.61)$$

For  $1 \leq s < R(\mu)$ , we have  $G(s) \geq s$ ,  $a(s) < 0$  and thus  $a'(s) \geq -2\mu$ . By integrating the previous equation, we obtain :

$$a(s) \geq -2\mu(s - 1) + a(1). \quad (2.89)$$

Knowing that  $a(1) = Q'(0, \mu) \leq 0$ , we have that  $s_1(\mu) := \frac{Q'(0, \mu)}{2\mu} + 1$  cancels the right hand side of (2.89). By the Intermediate Value Theorem, there is  $s_2(\mu) \leq s_1(\mu)$  such that  $a(s_2(\mu)) = 0$  and therefore

$$1 \leq R(\mu) \leq s_1(\mu). \quad (2.90)$$

We have by Proposition 2.3.3 that :

$$\lim_{\mu \rightarrow -\mu_0} Q'(0, \mu) = 0, \quad (2.91)$$

which means that :

$$\lim_{\mu \rightarrow -\mu_0} s_1(\mu) = 1. \quad (2.92)$$

Equations (2.90) and (2.92) finally provide :

$$\lim_{\mu \rightarrow -\mu_0} R(\mu) = 1. \quad (2.93)$$

□

Note that (2.89) implies that  $Q'(x_0(\mu), \mu) > -\infty$ . We now can more generally prove that the radius of convergence  $R(\mu)$  is continuous on  $R^{-1}([1, R_G])$ .

**Lemma 2.3.5.**  $\mu \mapsto R(\mu)$  is continuous on  $R^{-1}([1, R_G])$ .

Essentially, the key to the proof of Lemma 2.3.5 is Proposition 2.3.3 and the continuity of the flow. We give this proof in Appendix.

We will now tackle the last point of this section. As we mentioned in the introduction, whether  $R(\mu) = R_G$  or  $R(\mu) < R_G$  will be decisive to determine precisely the asymptotic behaviour of  $q_n(x)$ . We know that  $R$  is non-decreasing and bounded by  $R_G$  (we recall that  $R_G$  can be infinite) and thus has a limit (not necessary finite) smaller or equal to  $R_G$ . We first show this limit is precisely  $R_G$ . After that, we will distinguish two cases which will allow us to determine whether there exists  $\mu$  such that  $R(\mu) = R_G$  or not.

**Proposition 2.3.6.** Let  $r \in [1, R_G]$ , if  $\int_0^r G(x)dx < +\infty$  then there exists  $\mu_r$  such that  $R(\mu_r) \geq r$ .

Actually, the condition  $\int_0^r G(x)dx < +\infty$  is always satisfied for  $r < R_G$ , but we choose to formulate Proposition 2.3.6 in these terms to avoid repetitions.

*Proof.* Fix  $r \in [1, R_G]$  and assume that  $\int_0^r G(x)dx < +\infty$ . Furthermore, we suppose that for all  $\mu \in \mathbb{R}$ ,  $R(\mu) < r$ . Let  $\mu \geq 0$  and  $s < R(\mu)$ . By integrating (2.61), we get :

$$\begin{aligned} \frac{1}{2} (a^2(s, \mu) - a^2(1, \mu)) &= -2\mu \int_1^s a(u, \mu) du - 2\beta \int_1^s (G(u) - u) du \\ a^2(s, \mu) &= (Q')^2(0, \mu) - 4\mu \int_1^s a(u, \mu) du - 4\beta \int_1^s (G(u) - u) du \\ a^2(s, \mu) &\geq (Q')^2(0, \mu) - 4\beta \int_1^r (G(u) - u) du. \end{aligned} \quad (2.94)$$

Fix  $\mu_1, y_0 > 0$ . Since  $Q$  is decreasing as  $y$  or  $\mu$  increases we have for any  $y \geq 0$  and  $\mu \geq \mu_1$  :

$$Q(y, \mu) \leq \mathbf{1}_{\{y \leq y_0\}} + \mathbf{1}_{\{y > y_0\}} Q(y_0, \mu_1).$$

Introducing the previous inequality into (2.86), we get :

$$Q'(0, \mu) \leq 2\beta \left[ \frac{1 + (G(Q(y_0, \mu_1)) - 1)e^{-(\alpha - \mu)y_0}}{\alpha - \mu} \right] - \lambda.$$

A simple Taylor development yields :

$$Q'(0, \mu) \leq 2\mu(G(Q(y_0, \mu_1)) - 1) + o(1),$$

which provides that :  $\lim_{\mu \rightarrow +\infty} (Q')^2(0, \mu) = +\infty$  and therefore for  $\mu$  large enough there is  $M > 0$  such that :  $a^2(s, \mu) > M$  for all  $s < R(\mu)$ . We cannot then have that  $a^2(R(\mu), \mu) = 0$ , which is in contradiction with the fact that  $R(\mu) < r \leq R_G$ .  $\square$

From Proposition 2.3.6, we can obviously derive the following corollary.

**Corollary 2.3.7.** *The asymptotic behaviour of  $R$  is given by :*

$$\lim_{\mu \rightarrow +\infty} R(\mu) = R_G.$$

We now give the proof of Theorem 2.1.4 which is a criterion to know whether  $R_G$  is reached by  $R(\mu)$  or not. In Proposition 2.3.6 we have proved the implication :

$$\int_0^{R_G} G(s) ds < +\infty \implies \mu_c < \infty, \quad (2.95)$$

where  $\mu_c$  is defined in (2.20). We prove in the following proposition the reciprocal implication.

**Proposition 2.3.8.** *If  $\int_0^{R_G} G(s) ds = +\infty$  then for all  $\mu \in \mathbb{R}$ ,  $R(\mu) < R_G$ .*

Note that if  $R_G = +\infty$ , we have  $\int_0^{R_G} G(s) ds = +\infty$ .

*Proof.* We suppose by contradiction that there exists  $\mu > -\mu_0$  such that  $R(\mu) = R_G$ . Therefore, in this case by Theorem 2.1.2, and by definition of  $x_0$  in Proposition 2.2.5, we have that  $x_0 < 0$ ,  $Q$  is decreasing on  $(x_0, 0)$  and  $Q(x_0) = R_G$ . Let  $x \in (x_0, 0)$ , we have by change of variable :

$$\int_0^x e^{2\mu y} (G(Q(y)) - Q(y)) dy = \int_1^{Q(x)} e^{2\mu Q^{-1}(s)} \frac{(G(s) - s)}{Q'(Q^{-1}(s))} ds. \quad (2.96)$$

Moreover, (2.28) implies that :

$$Q'(Q^{-1}(s)) \geq Q'(0) e^{-2\mu Q^{-1}(s)}. \quad (2.97)$$

By introducing (2.97) into (2.96) we obtain :

$$\int_0^x e^{2\mu y} (G(Q(y)) - Q(y)) dy \leq \int_1^{Q(x)} e^{4\mu Q^{-1}(s)} \frac{(G(s) - s)}{Q'(0)} ds. \quad (2.98)$$

We now suppose that  $\mu \geq 0$ . Using (2.28), (2.98) and the fact that  $Q^{-1}(s) \geq x$  we obtain :

$$Q'(x) \geq \left( Q'(0) - \frac{2\beta e^{4\mu x}}{Q'(0)} \int_1^{Q(x)} (G(s) - s) ds \right) e^{-2\mu x}. \quad (2.99)$$



We then have by comparison theorem :

$$\lim_{x \rightarrow x_0} Q'(x) = +\infty, \quad (2.100)$$

which is in contradiction with the assumption that  $Q$  is decreasing on  $(x_0, 0)$ . We have supposed that  $\mu \geq 0$  but since  $R(\mu)$  is an non-decreasing function this result also holds for  $\mu \in (-\mu_0, 0]$ .  $\square$

By gathering Proposition 2.3.8 and (2.95), we obtain Theorem 2.1.4. We finish this section by giving an exhaustive description of  $(Q(x_0), Q'(x_0))$ .

**Proposition 2.3.9.** *Let  $\mu \in (-\mu_0, +\infty)$ .*

1. *If  $\mu < \mu_c$  then  $Q'(x_0) = 0$  and  $Q(x_0) < R_G$  ;*
2. *if  $\mu = \mu_c$  then  $Q'(x_0) = 0$  and  $Q(x_0) = R_G$  ;*
3. *if  $\mu > \mu_c$ ,  $Q'(x_0) < 0$  and  $Q(x_0) = R_G$ .*

Furthermore,  $\mu \mapsto R(\mu)$  is continuous on  $\mathbb{R}$ .

The proof of this proposition concludes the proof of Theorem 2.1.3. Note that if  $R_G = 1$  we are always in the third case of this proposition, and if  $\int_0^{R_G} G(x)dx = +\infty$  we are always in the first case. The continuity of  $R$  have already been seen on  $R^{-1}[1, R_G)$  and the first point is already known. We can reformulate what it remains to prove with the following lemma.

**Lemma 2.3.10.** *If  $R_G > 1$  and  $\int_0^{R_G} G(x)dx < +\infty$ ,  $\mu_c$  is the unique  $\mu$  such that  $Q'(x_0(\mu), \mu) = 0$ ,  $Q(x_0(\mu), \mu) = R_G$ . Moreover,  $R$  is continuous at  $\mu_c$ .*

As for Lemma 2.3.5, we give the proof of this lemma in Annexes.

## 2.4 Case $R(\mu) < R_G$

We consider the case where  $-\mu_0 < \mu < \mu_c$ , which is equivalent to  $1 < R_\mu < R_G$ . We chose to dedicate a section to this case because in this situation we can give an exact equivalent to  $q_n(x)$  when  $n$  tends to  $+\infty$  by using complex analytic methods. The general idea to use complex analysis and more specifically the singularity analysis in this context is due to Maillard. Since the behaviour of  $f_x$  near its singularities is different from that of  $F_x$  in [78], we nevertheless need to do some adjustments. We start with some notations and known results. The next lemma is Lemma 6.1 of [78].

**Lemma 2.4.1.** *The span of  $Z_x$  and  $\delta$  (the span of  $G$ ) are equal.*

We also give an adaptation in our context of Lemma 6.2 of [78]. Let  $x_0$  be defined as in Proposition 2.2.5 and  $s_0 = Q(x_0) = R(\mu)$ . For  $z \in \mathbb{C}$  and  $r > 0$ ,  $D(z, r)$  will denote the

open disc of center  $z$  and radius  $r$  and we fix  $D = D(0, s_0)$  and  $D_\delta = D(0, s_0^\delta)$ . As usual, the frontier of a set  $S$  is denoted by  $\partial S$ .

**Lemma 2.4.2.** *Fix  $x > 0$ . If  $\delta = 1$ , then  $f_x$  is analytic at every  $s \in \partial D \setminus \{s_0\}$ . If  $\delta \geq 2$ , then there exists an analytic function on  $D_\delta : h_x$ , such that :*

$$f_x(s) = sh_x(s^\delta), \quad \forall s \in D. \quad (2.101)$$

Moreover,  $h_x$  is analytic at every  $s \in \partial D_\delta \setminus \{s_0^\delta\}$ .

The proof of the previous result can be adapted from [78] to our case with one exception. Indeed, we need to have  $f_x(s_0) < \infty$ , whose analogue is always satisfied in Maillard's case (since in his situation  $s_0 = 1$  and  $F_x(1) = 1$ ) but which is not obvious in ours. However, (2.16) and the fact that  $R(\mu) = Q(x_0)$  yield  $\lim_{s \rightarrow s_0} f_x(s) = Q(x_0 + x) < \infty$ . Moreover, the coefficients of  $f_x$  as a power series are non-negative. Therefore, by applying for instance the monotone convergence theorem we see that  $f_x(s_0) = Q(x_0 + x) < \infty$ .

Finally, we state a reformulation in our framework of Corollary VI.1 of [53]. For  $z \in \mathbb{C}$ ,  $\arg(z)$  is chosen in  $(-\pi, \pi]$ . We call a  $\Delta$ -domain, as in [53] and in [78], a set defined for  $\varphi \in (0, \pi/2)$ ,  $s > 0$  and  $r > 0$  by :

$$\Delta(\varphi, r, s) := \{z \in D(0, s+r) \setminus \{s\} : |\arg(z-s)| > \varphi\}. \quad (2.102)$$

The following result is called the transfer Lemma.

**Theorem (Flajolet, Corollary VI.1 of [53]).** *Let  $\varphi \in (0, \pi/2)$ ,  $s > 0$  and  $r > 0$ . Let  $\mathcal{H}(z) := \sum_{n=0}^{+\infty} H_n z^n$  be an analytic function on  $\Delta(\varphi, r, s)$ . If there exists  $\alpha \in \mathbb{R} \setminus \mathbb{Z}_-$  such that :*

$$\mathcal{H}(z) \underset{z \rightarrow r}{\sim} \frac{1}{(r-z)^\alpha}, \quad z \in \Delta(\varphi, r, s)$$

then

$$H_n \underset{n \rightarrow +\infty}{\sim} \frac{n^{\alpha-1}}{r^{n+\alpha} \Gamma(\alpha)}.$$

To apply this theorem, we need the behaviour of  $f_x$  when  $\delta = 1$  (resp.  $h_x$ , when  $\delta \geq 2$ ) near its singularity  $s_0$  (resp.  $s_0^\delta$ ).

Let us introduce the complex logarithm defined for  $z \in \mathbb{C} \setminus \mathbb{R}^-$  by

$$\log(z) = \log|z| + \mathbf{i} \arg(z)$$

and the complex square root defined on the same set by

$$\sqrt{z} = e^{\frac{\log(z)}{2}}.$$

**Lemma 2.4.3.** *For each  $x > 0$ , there exists  $r_{1,x} > 0$  such that  $s \mapsto f_x(s)$  is analytic on  $D(s_0, r_{1,x}) \setminus (s_0, +\infty)$ , and for  $s$  in this set we have :*

$$f'_x(s) \underset{s \rightarrow s_0}{\sim} \frac{-Q'(x_0 + x)}{2\sqrt{\beta(s_0 - s)(G(s_0) - s_0)}}. \quad (2.103)$$

*Similarly, when  $\delta \geq 2$ , for each  $x > 0$ , there exists  $r_{\delta,x} > 0$  such that  $s \mapsto h_x(s)$  is analytic on  $D(s_0^\delta, r_{\delta,x}) \setminus (s_0^\delta, +\infty)$ , and for  $s$  in this set we have :*

$$h'_x(s) \underset{s \rightarrow s_0^\delta}{\sim} -\frac{Q'(x_0 + x)}{2\sqrt{\beta\delta s_0^{\delta+1}(s_0^\delta - s)(G(s_0) - s_0)}}. \quad (2.104)$$

*Proof.* To prove the analyticity of  $f_x$ , we will use and extend in a complex sense Equation (2.16). In this equation, the inverse function  $Q^{-1}$  is only defined on  $Q(J)$  (defined in Proposition 2.2.5), that is why we will find an analytic function defined near (in a sense we will precise below)  $s_0$  which coincides with  $Q^{-1}$  on  $Q(J)$ .

By Proposition 2.3.9, when  $\mu < \mu_c$ ,  $Q'(x_0) = 0$ . Moreover, Equation (2.11) implies that  $Q''(x_0) < 0$ . Since  $Q(x_0) < R_G$ ,  $Q$  admits an analytic extension near  $x_0$  by Proposition 2.1.1. Thus in the complex plane near  $x_0$  we have :

$$Q(z) = Q(x_0) + (z - x_0)^2 \frac{Q''(x_0)}{2} + \underset{z \rightarrow x_0}{o} \left( (z - x_0)^2 \right). \quad (2.105)$$

The function  $Q$  is analytic on an neighbourhood of  $x_0$ , which is a zero of order 2 of  $Q(z) - Q(x_0)$ . Theorem 10.32 of [90] thus ensures that there exists  $r_1 > 0$  such that on  $D(x_0, r_1)$ , there exists an analytic invertible function  $\psi : D(x_0, r_1) \rightarrow \psi(D(x_0, r_1)) \subset \mathbb{C}$  such that :

$$Q(z) = Q(x_0) + \frac{Q''(x_0)}{2} \psi(z)^2 \quad (2.106)$$

and such that  $\psi'(z) \neq 0$ , for all  $z \in D(x_0, r_1)$ , which ensures in particular that  $(\psi^{-1})'$  is well defined on  $\psi(D(x_0, r_1))$ . Note that Equation (2.106) implies that for  $z \in J \cap D(x_0, r_1)$  we have  $\psi(z) \in \mathbb{R} \cup i\mathbb{R}$ . More precisely,  $\psi(z) \in \mathbb{R}$ , because if  $\psi(z) \in i\mathbb{R}$ , we would have  $Q(z) > Q(x_0)$ , which is in contradiction with the definition of  $J$ . Furthermore, the Intermediate Value Theorem and the fact that  $Q(z) < Q(x_0), \forall z \in J$  implies that if there exists  $z_0 \in J \cap D(x_0, r_1)$  such that  $\psi(z_0) > 0$  then for all  $z \in J \cap D(x_0, r_1)$ ,  $\psi(z) > 0$ . Finally, since we can substitute  $-\psi$  for  $\psi$  in (2.106), we can chose  $\psi$  such that  $\psi > 0$  on  $J \cap D(x_0, r_1)$ .

We now fix  $s \in [0, s_0) \cap Q(D(x_0, r_1))$  (note that  $Q$  is analytic, and thus it is an open application, which implies that this intersection is not empty). By choosing  $z = Q^{-1}(s)$  in

(2.106) and using (2.11), we get :

$$\psi^{-1} \left( \sqrt{\frac{s_0 - s}{\beta(G(s_0) - s_0)}} \right) = Q^{-1}(s). \quad (2.107)$$

By considering the complex square root, we can define

$$\eta(s) := \psi^{-1} \left( \sqrt{\frac{s_0 - s}{\beta(G(s_0) - s_0)}} \right), \quad \forall s \in Q(D(x_0, r_1)) \setminus (s_0, +\infty). \quad (2.108)$$

We now recall that for  $s \in [0, s_0)$  and  $x > 0$  :

$$f_x(s) = Q(Q^{-1}(s) + x). \quad (2.16)$$

By Proposition 2.1.1, there exists  $r_2 > 0$ , such that  $Q$  is analytic on  $D(x_0 + x, r_2)$ . Furthermore, Equations (2.16), (2.107) and (2.108) yield  $r_3 > 0$  small enough such that  $f_x(s)$  and  $Q(\eta(s) + x)$  exist for  $s \in [0, s_0) \cap D(s_0, r_3)$  and coincide. Formally, we choose  $0 < r_3 \leq r_1$  such that :

$$\begin{cases} D(s_0, r_3) \subset Q(D(x_0, r_1)) \\ Q^{-1}(D(s_0, r_3) \cap [0, s_0)) + x \subset D(x_0 + x, r_2). \end{cases} \quad (2.109)$$

Moreover,  $D(s_0, r_3) \setminus (s_0, +\infty)$  is an open connected set and  $[0, s_0) \cap D(s_0, r_3)$  is a subset of it with an accumulation point. Therefore,  $f_x(s)$  admits a unique analytic extension on  $D(s_0, r_3) \setminus (s_0, +\infty)$  which is  $Q(\eta(s) + x)$ . Thus, we have that, for  $s \in D(s_0, r_3) \setminus (s_0, +\infty)$  :

$$f'_x(s) = -\frac{(\psi^{-1})' \left( \sqrt{\frac{s_0 - s}{\beta(G(s_0) - s_0)}} \right)}{2\sqrt{\beta(s_0 - s)(G(s_0) - s_0)}} Q' \left( \psi^{-1} \left( \sqrt{\frac{s_0 - s}{\beta(G(s_0) - s_0)}} \right) + x \right), \quad (2.110)$$

which implies that :

$$f'_x(s) \underset{s \rightarrow s_0}{\sim} -\frac{(\psi^{-1})'(0)}{2\sqrt{\beta(s_0 - s)(G(s_0) - s_0)}} Q'(x_0 + x). \quad (2.111)$$

To get the value of  $(\psi^{-1})'(0)$  we first differentiate (2.106) with respect to  $z$  :

$$Q'(z) = Q''(x_0)\psi'(z)\psi(z). \quad (2.112)$$

Furthermore, Taylor's formula yields :

$$\begin{cases} Q'(z) = Q''(x_0)(z - x_0) + o(z - x_0) \\ \psi(z) = \psi'(x_0)(z - x_0) + o(z - x_0). \end{cases} \quad (2.113)$$

Equations (2.112) and (2.113) provide  $(\psi')^2(x_0) = 1$ . We have chosen  $\psi$  such that  $\psi > 0$

on  $(x_0, x_0 + r_1)$  and thus  $\psi'(x_0) = 1$ . As a consequence,  $(\psi^{-1})'(0) = 1$ , which yields (2.103).

We can derive from the results on  $f_x$  the analogous results on  $h_x$ . Let us define on  $\mathbb{C} \setminus \mathbb{R}^-$  the function  $z \mapsto \sqrt[\delta]{z} := e^{\frac{\log(z)}{\delta}}$ . Lemma 2.4.2 yields :

$$h_x(s) = \frac{f_x(\sqrt[\delta]{s})}{\sqrt[\delta]{s}}, \forall s \in D(0, s_0^\delta) \setminus \mathbb{R}^-. \quad (2.114)$$

Besides, since there exists  $0 < r_{1,x} < s_0$  such that  $f_x$  is analytic on  $D(s_0, r_{1,x}) \setminus (s_0, +\infty)$ , we can show after some change of variable that there exists  $0 < r_{\delta,x} < s_0^\delta$  such that the right term of (2.114) is analytic on  $D(s_0^\delta, r_{\delta,x}) \setminus (s_0^\delta, +\infty)$ . The function  $h_x$  and the right term of (2.114) coincide on a open subset of the connected set  $D(s_0^\delta, r_{\delta,x}) \setminus (s_0^\delta, +\infty)$  and thus  $h_x$  has an analytic extension on  $D(s_0^\delta, r_{\delta,x}) \setminus (s_0^\delta, +\infty)$ .

Let us turn to the behaviour of  $h_x$  near  $s_0^\delta$ . By differentiating (2.114) with respect to  $s$ , we get :

$$h'_x(s) = \frac{f'_x(\sqrt[\delta]{s}) \sqrt[\delta]{s} - f_x(\sqrt[\delta]{s})}{\delta s \sqrt[\delta]{s}}. \quad (2.115)$$

Equation (2.111) yields :

$$\begin{aligned} f'_x(\sqrt[\delta]{s}) &\underset{s \rightarrow s_0^\delta}{\sim} - \frac{Q'(x_0 + x)}{2\sqrt{\beta \left( \sqrt[\delta]{s_0^\delta} - \sqrt[\delta]{s} \right) (G(s_0) - s_0)}} \\ &\underset{s \rightarrow s_0^\delta}{\sim} - \frac{Q'(x_0 + x) \sqrt{\delta}}{2\sqrt{\beta (s_0^\delta - s) (s_0^{1-\delta}) (G(s_0) - s_0)}}, \end{aligned} \quad (2.116)$$

since :

$$\lim_{s \rightarrow s_0^\delta} \frac{\sqrt[\delta]{s_0^\delta} - \sqrt[\delta]{s}}{s_0^\delta - s} = \frac{1}{\delta} (s_0^\delta)^{\frac{1}{\delta} - 1}.$$

Moreover, we have seen that  $f_x(s_0) < +\infty$ . Therefore, by introducing (2.116) into (2.115), we get (2.104).  $\square$

The end of the proof of Theorem 2.1.5 is almost identical to that from Theorem 1.2 in [78] up to the fact that the asymptotic is not the same.

*Proof of Theorem 2.1.5.* Let us just give the principal steps when  $\delta = 1$ . Let us take  $\varphi_0 \in [0, \pi/2)$  small enough such that  $s_0 e^{i\varphi_0} \in D(s_0, r_x)$ . By Lemma 2.4.2, for each  $s \in S := \{s_0 e^{i\varphi}, \varphi \in [\varphi_0, 2\pi - \varphi_0]\}$ , there exists  $\mathcal{R}_s > 0$  such that  $f_x$  admit an analytic extension on  $D(s, \mathcal{R}_s)$ . Furthermore,  $S$  is compact. Hence, there exist  $k \in \mathbb{N}$  and  $(z_i) \in S^k$  such that :

$$S \subset T := \bigcup_{i \in \{1, \dots, k\}} D(z_i, \mathcal{R}_{z_i}).$$

Since  $f_x$  is analytic on  $T \cup D(0, s_0)$  and by Lemma 2.4.3 on  $D(s_0, r_x) \setminus (s_0, +\infty)$ , we can

find  $\epsilon > 0$  such that  $f_x$  is analytic on  $\mathcal{D} := D(0, s_0 + \epsilon) \setminus (s_0, +\infty)$ . We now can apply Corollary VI.1 of [53]. Indeed,  $\mathcal{D}$  contains a  $\Delta$ -domain which satisfies the assumptions of this Corollary and we precisely know the asymptotic of  $f'_x$  near  $s_0$  by Lemma 2.4.3. Since the  $i$ th coefficient of  $f'_x$  as a power series is  $(i+1)q_{i+1}(x)$ , we get :

$$(i+1)q_{i+1}(x) \underset{i \rightarrow +\infty}{\sim} \frac{-Q'(x_0 + x)}{2\sqrt{\beta(G(R(\mu)) - R(\mu))}} \times \frac{1}{R(\mu)^{i+\frac{1}{2}}\sqrt{i}\Gamma(\frac{1}{2})},$$

which is Theorem 2.1.5 when  $\delta = 1$ .

When  $\delta \geq 2$ , we can also find a good  $\Delta$ -domain on which  $h_x$  is analytic. Furthermore, by definition of  $h_x$  (2.101), the  $i$ th coefficient of  $h'_x$  is  $(i+1)q_{\delta(i+1)+1}$ . Therefore Corollary VI.1 of [53] and Lemma 2.4.3 similarly provide (2.22). □

## 2.5 Case where $R(\mu) = R_G$

Let  $\mu > -\mu_0$ . In the previous section, we handled the case  $R(\mu) < R_G$  which includes the case  $\mu_c = +\infty$ . We now consider the case where  $\mu_c < +\infty$  ( $\mu_c$  can be equal to  $-\infty$ ) and  $\mu \geq \mu_c$ . This is equivalent to the case  $\int_0^{R_G} G(s)ds < +\infty$  and  $R(\mu) = R_G$ . The asymptotic behaviour of  $q_k(x)$  when  $k$  tends to  $+\infty$  was obtained when  $R(\mu) < R_G$  by studying  $f_x$  near its radius of convergence. Since  $R(\mu) = R_G$ , it is not possible anymore to extend  $f_x$  to a  $\Delta$ -domain analytically as in the previous section. However, in some case, the behaviour of  $G$  as a real function near  $R_G$  gives us weaker results.

Suppose first that  $\mu = \mu_c$ . As a first step, we will give the behaviour of  $f_x(s)$ , when  $s \rightarrow R_G^-$ ,  $s \in \mathbb{R}$ .

**Lemma 2.5.1.** *If  $\mu = \mu_c$ , we have :*

$$f'_x(s) \underset{s \rightarrow R_G}{\sim} -\frac{Q'(x+x_0)}{2\sqrt{\beta \int_s^{R_G} (G(u) - u)du}}, \quad s \in (q, R_G). \quad (2.117)$$

We cannot use the exact same arguments as in the proof of Lemma 2.4.3 since  $Q$  is not analytic at  $x_0$  anymore.

*Proof.* Suppose that  $\mu = \mu_c$  and fix  $s < R_G$ . As a consequence of Proposition 2.3.9, when  $s$  tends to  $R_G$ ,  $-2\mu_c a(s)$  tends to 0. Moreover, if  $R_G = 1$ , then  $\forall \mu \in \mathbb{R}, R(\mu) = R_G$  and therefore, by definition of  $\mu_c$ ,  $\mu_c = -\infty$ . Here, since we have supposed  $\mu_c = \mu \in \mathbb{R}$ ,  $\mu_c > -\infty$  and thus  $R_G > 1$ . Hence, we have that  $2\beta(G(s) - s) \geq C$ , where  $C > 0$ , for  $s$  in

a neighbourhood of  $R_G$ . Therefore, (2.61) yields :

$$a'(s)a(s) \underset{s \rightarrow R_G}{\sim} -2\beta(G(s) - s). \quad (2.118)$$

We have that  $\int_0^{R_G} G(u)du < \infty$ , and therefore by integration :

$$-\frac{1}{2}a^2(s) \underset{s \rightarrow R_G}{\sim} -\int_s^{R_G} 2\beta(G(u) - u) du. \quad (2.119)$$

Since  $a \leq 0$ , (2.119) implies that :

$$a(s) \underset{s \rightarrow R_G}{\sim} -2\sqrt{\beta \int_s^{R_G} (G(u) - u) du}. \quad (2.120)$$

On the other hand, by differentiating (2.16), we obtain :

$$f'_x(s) = \frac{Q'(Q^{-1}(s) + x)}{a(s)} \underset{s \rightarrow R_G}{\sim} \frac{Q'(x_0 + x)}{a(s)}. \quad (2.121)$$

Combining (2.120) and (2.121) we get the result.  $\square$

Observe that if  $G(R_G) < \infty$ ,

$$f'_x(s) \underset{s \rightarrow R_G}{\sim} -\frac{Q'(x_0 + x)}{2\sqrt{\beta(R_G - s)(G(R_G) - R_G)}}. \quad (2.122)$$

In this case,  $f'_x$  has the same kind of asymptotic near its radius of convergence as in the previous section and thus it is likely we will have the same kind of asymptotic for  $q_i(x)$ . However, the asymptotic of  $f_x$  is, this time, only in the real sense. Nevertheless, a Tauberian theorem can be used to obtain a rougher description of the large  $i$  behaviour of  $q_i(x)$ . In what follows, we consider a case a slightly more general than  $G(R_G) < \infty$  by supposing that :

$$G(s) - s \underset{s \rightarrow R_G}{\sim} C(R_G - s)^{-\alpha}, \quad (2.123)$$

where  $C > 0$ ,  $\alpha \in [0, 1)$  and  $s \in (q, R_G)$ . Since we are in the case  $\mu_c < \infty$ , we necessarily have that  $\int_0^{R_G} G(s)ds < +\infty$ , which explains why  $\alpha$  must be in  $[0, 1)$ . Note that if  $\alpha = 0$ , Equation (2.123) is equivalent to  $G(R_G) < \infty$  and in this case  $C = G(R_G) - R_G$ , whereas if  $\alpha > 0$  we can replace  $G(s) - s$  by  $G(s)$  in (2.123).

**Proposition 2.5.2.** *When  $\mu = \mu_c$ , if Equation (2.123) holds then :*

$$\sum_{i=n}^{+\infty} q_i(x)R_G^i \underset{n \rightarrow +\infty}{\sim} \frac{-AQ'(x_0 + x)}{n^{\frac{1+\alpha}{2}}}, \quad (2.124)$$

where  $A := \frac{\sqrt{(1-\alpha)R_G^{1+\alpha}}}{(1+\alpha)\sqrt{\beta C}\Gamma(\frac{1-\alpha}{2})}$ .

We can observe that if we give an equivalent of  $\sum_{i=n}^{+\infty} q_i(x)R_G^i$  when  $\mu < \mu_c$  with the help of Theorem 2.1.5, we get the same result as in Proposition 2.5.2 when  $\alpha = 0$ .

*Proof.* Combining Equation (2.117) and (2.123) we get :

$$f'_x(s) \underset{s \rightarrow R_G}{\sim} -\frac{Q'(x+x_0)\sqrt{1-\alpha}}{2\sqrt{\beta C}(R_G-s)^{1-\alpha}}. \quad (2.125)$$

Let us rescale  $f'_x$  by defining  $b_x(s) = f'_x(sR_G)$ . Since the radius of convergence of  $f_x$  is  $R_G$ , the radius of convergence of  $b_x$  is 1. By rescaling (2.125) we obtain :

$$b_x(s) \underset{s \rightarrow R_G}{\sim} -\frac{Q'(x+x_0)\sqrt{1-\alpha}}{2\sqrt{\beta C}R_G^{1-\alpha}(1-s)^{1-\alpha}}. \quad (2.126)$$

Using Theorem 5. Chapitre XIII Section 5 of [52] we get :

$$V(n) := \sum_{i=0}^{n-1} (i+1)q_{i+1}(x)R_G^i \underset{n \rightarrow +\infty}{\sim} -\frac{n^{\frac{1-\alpha}{2}}Q'(x+x_0)}{\sqrt{1-\alpha}\Gamma(\frac{1-\alpha}{2})\sqrt{\beta C}R_G^{1-\alpha}}. \quad (2.127)$$

The definition of  $V$  yields :

$$q_i(x)R_G^i = \frac{(V(i) - V(i-1))R_G}{i}. \quad (2.128)$$

We recall that  $f_x(R(\mu)) = f_x(R_G) < +\infty$ . Therefore, by using (2.127) and (2.128) and the fact that the terms of each of the series which follow are all positive we get :

$$\begin{aligned} \sum_{i=n}^{\infty} q_i(x)R_G^i &= \sum_{i=n}^{\infty} \frac{(V(i) - V(i-1))R_G}{i} \\ &= \sum_{i=n}^{\infty} \frac{V(i)R_G}{i(i+1)} - \frac{V(n-1)R_G}{n} \end{aligned} \quad (2.129)$$

$$\begin{aligned} &\underset{n \rightarrow +\infty}{\sim} -\frac{Q'(x+x_0)R_G}{\sqrt{1-\alpha}\Gamma(\frac{1-\alpha}{2})\sqrt{\beta C}R_G^{1-\alpha}} \left[ \sum_{i=n}^{\infty} \frac{1}{i^{\frac{3+\alpha}{2}}} - \frac{1}{n^{\frac{1+\alpha}{2}}} \right] \\ &\underset{n \rightarrow +\infty}{\sim} -\frac{AQ'(x+x_0)}{n^{\frac{1+\alpha}{2}}}, \end{aligned} \quad (2.130)$$

where  $A := \frac{\sqrt{(1-\alpha)R_G^{1+\alpha}}}{(1+\alpha)\sqrt{\beta C}\Gamma(\frac{1-\alpha}{2})}$ . Equation (2.129) is obtained by integration by parts and Equation (2.130) by a classical comparison between sums and integrals.  $\square$

The influence of  $G$  on the  $q_i(x)$  seems difficult to understand when  $\mu = \mu_c$ . We recall



that in this section  $\mu > -\mu_0$ . We will now see that in the case where  $\mu > \mu_c$ , there exists a stronger link between the  $p_i$  and the  $q_i(x)$ . As a first step we give a result which does not require any specific knowledge on  $G$ .

**Proposition 2.5.3.** *Let  $k \in \mathbb{N}$ , if  $\mu > \mu_c$  we have :*

$$\sum_{i=0}^{\infty} q_i(x) R_G^i i^{k+2} < \infty \Leftrightarrow \sum_{i=0}^{\infty} p_i R_G^i i^k < \infty, \forall x > 0. \quad (2.131)$$

*Proof.* We begin by prove this result for  $k = 0$ . First suppose that

$$G(R_G) = \sum_{i=0}^{\infty} p_i R_G^i < +\infty. \quad (2.132)$$

We recall from (2.61) that :

$$a'(s)a(s) = -2\mu a(s) - 2\beta (G(s) - s) \forall s \in (q, Q(x_0)).$$

Since  $\lim_{s \rightarrow R_G} a(s) \in \mathbb{R}^*$ , Equations (2.61) and (2.132) imply that  $\lim_{s \rightarrow R_G} a'(s) < \infty$ . As above, by using Kolmogorov equations we get :

$$a(s)f'_x(s) = a(f_x(s)) \quad (2.133)$$

$$a'(s)f'_x(s) + a(s)f''_x(s) = f'_x(s)a'(f_x(s)). \quad (2.134)$$

Equation (2.133) implies that  $\lim_{s \rightarrow R_G} f'_x(s) < \infty$  and (2.134) similarly implies that  $\lim_{s \rightarrow R_G} f''_x(s) < \infty$ . The coefficients of the power series  $f''_x(s)$  are positive, therefore  $f''_x(R_G) < \infty$ , which is equivalent to

$$\sum_{i=0}^{\infty} q_i(x) R_G^i i^2 < \infty. \quad (2.135)$$

If we now suppose that  $k \in \mathbb{N}$ ,  $\sum_{i=0}^{\infty} p_i R_G^i i = \infty$ , we can similarly prove that

$$\sum_{i=0}^{\infty} q_i(x) R_G^i i^2 = \infty. \quad (2.136)$$

The general case can be proved by induction. The proof is almost identical to the case  $k = 0$ , we differentiate (2.133)  $k + 1$  times and (2.61)  $k$  times, and use the induction hypothesis to determinate what is finite or not.  $\square$

This result is pretty weak, but informally it shows that a link exists between  $p_i$  and  $q_i(x)i^2$ . Once again, with more specific assumptions on  $G$  we can give a more accurate result on  $q_i(x)$  which confirms this link.

**Proposition 2.5.4.** *Let  $m \in \mathbb{N}$  and suppose that  $G^{(m)}(s) \underset{s \rightarrow R_G}{\sim} C(R_G - s)^{-\alpha}$ , with  $C > 0$ ,  $\alpha \in (0, 1)$  and  $s \in (q, R_G)$ . Suppose that  $\mu > \mu_c$ , then for  $t \in \mathbb{R}$  such that  $t < m + 2 - \alpha$ , we have :*

$$\begin{aligned} \sum_{i=n}^{+\infty} q_i(x) R_G^i i^t &\underset{n \rightarrow +\infty}{\sim} \frac{K_1 Q'(x + x_0)}{n^{m+2-t-\alpha}}, \\ &\underset{n \rightarrow +\infty}{\sim} K_2 Q'(x + x_0) \sum_{i=n}^{+\infty} p_i R_G^i i^{t-2}, \end{aligned} \quad (2.137)$$

where  $K_1 = \frac{2\beta C R_G^{m+2-\alpha}}{\Gamma(\alpha)(m+2-t-\alpha)(Q')^3(x_0)}$  and  $K_2 = \frac{2\beta R_G^2}{(Q')^3(x_0)}$ .

We recall that  $\int_0^{R_G} G(s) ds < +\infty$  in this case.

*Proof.* Since the proof of this proposition is very close to that of Proposition 2.5.2 we will skip some details. We will first determine an asymptotic equivalent for  $\sum_{i=n}^{+\infty} p_i R_G^i i^{t-2}$ . Define  $\phi(s) := G^{(m)}(s R_G)$ . We have by assumption :

$$\phi(s) = \sum_{i=0}^{+\infty} p_{i+m} \frac{i!}{(i-m)!} R_G^i s^i \underset{s \rightarrow 1}{\sim} \frac{C}{R_G^\alpha (1-s)^\alpha}.$$

Applying Theorem 5. Chapitre XIII Section 5 of [52], we obtain :

$$W(n) := \sum_{i=0}^{n-1} p_{i+m} \frac{(i+m)!}{i!} R_G^i \underset{n \rightarrow +\infty}{\sim} \frac{C n^\alpha}{R_G^\alpha \Gamma(\alpha + 1)}. \quad (2.138)$$

Therefore :

$$\begin{aligned} p_{n-1+m} R_G^{n-1} &= \frac{(n-1)!}{(n-1+m)!} [W(n) - W(n-1)]. \\ p_n R_G^n n^{t-2} &= R_G^m \frac{n^{t-2} (n-m)!}{n!} [W(n-m+1) - W(n-m)]. \\ \sum_{i=n}^{+\infty} p_i R_G^i i^{t-2} &= R_G^m \sum_{i=n}^{+\infty} \left[ W(i-m+1) \frac{(i-m)!}{i!} i^{t-2} \left( 1 - \left( 1 + \frac{1}{i} \right)^{t-2} \frac{(i+1-m)}{i+1} \right) \right] \\ &\quad - R_G^m n^{t-2} \frac{(n-m)!}{n!} W(n-m). \end{aligned} \quad (2.139)$$

By equation (2.138) and classical Taylor expansion, the term in the sum of (2.139) has the following equivalent :

$$W(i-m+1) \frac{(i-m)!}{i!} i^{t-2} \left( 1 - \left( 1 + \frac{1}{i} \right)^{t-2} \frac{(i+1-m)}{i+1} \right) \underset{i \rightarrow +\infty}{\sim} \frac{C i^{\alpha+t-3-m} (m-t+2)}{R_G^\alpha \Gamma(\alpha + 1)}.$$

After classical comparison between sums and integrals, we get :

$$\begin{aligned} \sum_{i=n}^{+\infty} p_i R_G^i i^{t-2} &\underset{i \rightarrow +\infty}{\sim} \frac{C R_G^{m-\alpha} (m-t+2)}{\Gamma(\alpha+1)(m+2-t-\alpha)} n^{\alpha+t-2-m} - \frac{C R_G^{m-\alpha} n^{t-2-m+\alpha}}{\Gamma(\alpha+1)} \\ &\underset{i \rightarrow +\infty}{\sim} \frac{C R_G^{m-\alpha} n^{\alpha+t-2-m}}{(m+2-t-\alpha)\Gamma(\alpha)}. \end{aligned} \quad (2.140)$$

We will now determine an asymptotic equivalent for  $\sum_{i=n}^{+\infty} q_i(x) R_G^i i^t$ . For  $0 \leq k \leq m$  and  $s \in (q, R_G)$  we get by differentiating  $k$  times (2.61) that :

$$\sum_{i=0}^k \binom{k}{i} a^{(i+1)}(s) a^{(k-i)}(s) = -2\mu a^{(k)}(s) - 2\beta \left( G^{(k)}(s) - \phi^{(k)}(s) \right), \quad (2.141)$$

where  $\phi(x) = x$ . Thanks to (2.141), we can show by induction that for all  $k \leq m$ ,  $\lim_{s \rightarrow R_G} a^{(k)}(s) < +\infty$  and thus that :

$$a^{(m+1)}(s) Q'(x_0) \underset{s \rightarrow R_G}{\sim} -\frac{2C\beta}{(R_G - s)^\alpha}. \quad (2.142)$$

By differentiating 2 times Equation (2.16) we get for  $q < s < R_G$  :

$$f_x''(s) = \frac{Q''(Q^{-1}(s) + x) - a'(s)Q'(Q^{-1}(s) + x)}{a^2(s)}. \quad (2.143)$$

If we differentiate again  $m$  times Equation (2.143), we can show that

$$f_x^{(m+2)}(s) \underset{s \rightarrow R_G}{\sim} \frac{-a^{(m+1)}(s)Q'(x_0 + x)}{(Q')^2(x_0)}, \quad (2.144)$$

since the others terms involve at most the  $m$ -th derivative of  $a$  and thus are finite near  $R_G$ . Combining (2.142) and (2.144), we obtain :

$$f_x^{(m+2)}(s) \underset{s \rightarrow R_G}{\sim} \frac{2C\beta Q'(x_0 + x)}{(Q')^3(x_0)(R_G - s)^\alpha}. \quad (2.145)$$

With this equivalent, we can proceed as before with the equivalent of  $G$  and by replacing  $m$  by  $(m+2)$ ,  $t-2$  by  $t$  and  $C$  by  $(2C\beta Q'(x_0 + x))(Q')^{-3}(x_0)$  in (2.140) we get the desired result. □

We recall that the probability to have  $i$  particles on the barrier at the extinction time and only one splits before the extinction calculated in (2.56) is of order  $p_i/i^2$ . This fact and the two previous propositions lead us to think that when  $\mu > \mu_c$ , there is also a change of regime for the number of divisions  $D_x$  before extinction and we can conjecture, for instance, that the radius of convergence of the generating function (on the extinction event) of  $D_x$

is infinite. In any case, the law of  $D_x$  should cast light on the change of behaviour of  $Z_x$  when  $\mu > \mu_c$  or when  $\mu = \mu_c$ . Unfortunately, the study of  $D_x$  seems much more difficult than that of  $Z_x$ , since its generating function does not solve simple equations.

## 2.A Proof of Lemma 2.3.5

*Proof.* We have to prove that  $R$  is continuous on  $R^{-1}([1, R_G])$ . We recall that when  $R_G > 1$ , we have  $R(\mu) > 1, \forall \mu > -\mu_0$ , and  $R(\mu) = 1$  else. This fact and Lemma 2.3.4 imply that it remains only to prove the continuity on  $R^{-1}((1, R_G))$ .

Let  $\mu_1 \in \mathbb{R}$  such that  $R(\mu_1) \in (1, R_G)$ . Roughly speaking, to prove the continuity of  $R$  in  $\mu_1$  we will consider a domain around the trajectory of  $X(\cdot, \mu_1)$  (defined in (2.23)) narrow enough such that for  $\mu$  close enough to  $\mu_1$ , the trajectory of  $X(\cdot, \mu)$  is in this domain and cuts the  $x$  axis near  $Q(x_0, \mu_1)$ .

We have proved that there exists  $x_0 \in \mathbb{R}^-$  such that  $X(x_0, \mu_1) = (R(\mu_1), 0)$  and  $Q$  is decreasing on  $(x_0, +\infty)$ . Furthermore, if we define  $x_l$  as in Proposition 2.1.1 and if  $R(\mu_1) < R_G$ , we necessarily have that  $x_0 > x_l$ . Indeed, since in this case  $Q'(x_0) = 0$  and  $Q(x_0) < R_G$ , Equation (2.13) implies that  $x_0 \neq x_l$ .

Let  $0 < \epsilon < (R_G - R(\mu_1))/2$ , there exists  $0 < \eta_0 < x_l - x_0$  such that :

$$|Q(x, \mu_1) - R(\mu_1)| + |Q'(x, \mu_1)| < \epsilon, \forall x \in (x_0 - \eta_0, x_0 + \eta_0) \quad (2.146)$$

and

$$Q'(x, \mu_1) \neq 0, \forall x \in (x_0 - \eta_0, x_0 + \eta_0) \setminus \{x_0\}. \quad (2.147)$$

Equation (2.146) comes from the continuity of  $X$  with respect to  $x$  and (2.147) is a consequence of the fact that  $Q''(x_0, \mu_1) < 0$ . Let  $(x_1, x_2) \in (x_0 - \eta_0, x_0) \times (x_0, x_0 + \eta_0)$ . Since the function  $Q'$  is negative and continuous on  $[x_2, 0]$ , there exists  $\epsilon_2 > 0$  such that  $Q'(x) < -\epsilon_2, \forall x \in [x_2, 0]$ . We now define  $D$  the domain of  $\mathbb{R}^2$  :

$$D = \bigcup_{x \in [0, x_1]} B(X(x, \mu_1), \min\{\epsilon, \epsilon_2\}), \quad (2.148)$$

where  $B(x, r)$  is the open ball for the norm  $\|\cdot\|_1$  of radius  $r$  centered at  $x$ . Let  $I_0$  be an open bounded interval such that  $\mu_1 \in I_0 \subset (-\mu_0, +\infty)$  and for  $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ ,  $\psi(\cdot, \xi, \mu)$  is defined as the maximal solution of (2.24) such that  $\psi(0, \xi, \mu) = \xi$ . The function  $\Gamma$ , defined in (2.24), is continuous and uniformly Lipschitz with respect to  $((x, y), \mu)$  on  $D \times I_0$ . Therefore by Theorem 7.4 Section 7 Chapter 1 of [36], there exists  $\delta > 0$  such that if  $(\xi, \mu) \in V$ , where  $V$  is defined by :

$$V = \{(\xi = (\xi_1, \xi_2), \mu), |\xi_1 - Q(0, \mu_1)| + |\xi_2 - Q'(0, \mu_1)| + |\mu - \mu_1| < \delta\} \quad (2.149)$$

then for all  $x \in [0, x_1]$ ,  $\psi(x, \xi, \mu)$  is defined and  $\psi(x, \xi, \mu) \in D$ . Moreover,  $\psi$  is continuous in  $[0, x_1] \times V$ . We have shown in Proposition 2.3.3 that  $\mu \mapsto (Q(0, \mu), Q'(0, \mu))$  is continuous. Therefore, there exists  $\eta_1 > 0$  such that if  $\mu \in I_0$  satisfies  $|\mu - \mu_1| < \eta_1$  then  $(Q(0, \mu), Q'(0, \mu), \mu) \in V$ . Since  $(Q(x, \mu), Q'(x, \mu)) = \psi(x, (Q(0, \mu), Q'(0, \mu)), \mu)$ , we have that  $X$  is continuous on  $V_2 := [0, x_1] \times (\mu_1 - \eta_1, \mu_1 + \eta_1)$  and  $X(V_2) \subset D$ .

As an easy consequence of the fact that  $Q''(x_0, \mu_1) < 0$  and of (2.147) we have  $Q'(x_1, \mu_1) > 0$ . Therefore, by continuity of  $X$ , there exists  $\eta_2 < \eta_1$  such that if  $|\mu - \mu_1| < \eta_2$  then  $Q'(x_1, \mu) > 0$  and  $Q'(x_2, \mu) < 0$ . Hence, by the Intermediate Value Theorem, there exists  $x_3 \in (x_1, x_2)$  such that  $Q'(x_3, \mu) = 0$ . Therefore,  $R(\mu) \leq Q(x_3, \mu)$ . Furthermore, consider  $(q, q') \in D$ , such that  $q \geq 1$  and  $q' = 0$ . By definition of  $D$ , there exists  $x \in [0, x_1]$  such that :

$$|q - Q(x, \mu_1)| + |Q'(x, \mu_1)| < \min\{\epsilon, \epsilon_2\}. \quad (2.150)$$

If by contradiction  $x \in (0, x_2)$ , then by definition of  $\epsilon_2$ , we have :

$$|q - Q(x, \mu_1)| < \min\{\epsilon, \epsilon_2\} - \epsilon_2 < 0.$$

Therefore  $x \in [x_2, x_1] \subset (x_0 - \eta_0, x_0 + \eta_0)$ . By Equations (2.146) and (2.150), we have :

$$|R(\mu_1) - q| < 2\epsilon. \quad (2.151)$$

Moreover the choice of  $\epsilon$  implies that  $q < R_G$  and since  $X(V_2) \subset D$ , we have  $Q(x_3, \mu) < R_G$ . Furthermore, the fact that  $X(V_2) \subset D$  and (2.151) implies that we cannot have a  $x \in [0, x_1]$  such that  $Q(x, \mu) < R(\mu_1) - 2\epsilon$  and  $Q'(x, \mu) = 0$ . Therefore,  $R(\mu) \geq R(\mu_1) - 2\epsilon$ . Similarly,  $(Q(x_3, \mu), 0) \in D$  and  $R(\mu) \leq Q(x_3, \mu)$  and thus  $R(\mu) < R(\mu_1) + 2\epsilon$ . Therefore,  $R$  is continuous on  $R^{-1}[(1, R_G)]$ .  $\square$

## 2.B Proof of Lemma 2.3.10

*Proof.* The uniqueness comes from the increase of  $\phi$  described in Proposition 2.3.2. Furthermore, we cannot have  $R(\mu_c) < R_G$ . Indeed, the proof of Lemma 2.3.5, implies that if  $R(\mu_c) < R_G$  there exists  $\mu_1$  such that  $R(\mu_c) < R(\mu_1) < R_G$ , which is in contradiction with the definition of  $\mu_c$ .

We want now to prove that  $Q'(x_0, \mu_c) = 0$  and that  $R$  is continuous at  $\mu_c$ . Since the proof of these two points are very similar to that from Lemma 2.3.5, we will skip some details. However, we cannot as in this lemma choose a domain which contains  $(R_G, Q'(x_0, \mu_c))$ . Indeed, on such a domain, we would not have necessarily the Lipschitz condition because of the singularity of  $G$  in  $R_G$ . We will thus take a slightly different one. Let  $\epsilon_1 > 0$ . We

can take  $x_1 > x_0$  such that :  $\|X(x_1, \mu_c) - X(x_0, \mu_c)\|_1 < \epsilon_1$  and define  $D$  by :

$$D = \bigcup_{x \in [0, x_1]} B(X(x, \mu_c), \epsilon_2), \quad (2.152)$$

where  $0 < \epsilon_2 < \epsilon_1$  is small enough to have that  $d(D, \mathbb{R} \times \{0\} \cup \{R_G\} \times \mathbb{R}) > 0$ . There exists  $\eta > 0$ , such that for all  $\mu$  satisfying  $|\mu - \mu_c| < \eta$ , we have  $\|X(x_1, \mu_c) - X(x_1, \mu)\|_1 < \epsilon_2$ , and  $X(x, \mu) \in D, \forall x \in [0, x_1]$ . This implies in particular that for  $|\mu - \mu_c| < \eta$ ,  $R(\mu) \geq R_G - \epsilon_1 - \epsilon_2$ . Furthermore, we know that  $R(\mu) \leq R_G$  and thus  $R$  is continuous at  $\mu_c$ .

Suppose now that we have  $Q'(x_0, \mu_c) < 0$  and  $\epsilon_1 < |Q'(x_0, \mu_c)|/4$ . Let  $s_1 = Q(x_1, \mu_c)$  and choose  $\mu$  and  $s$  such that  $|\mu_c - \mu| < \eta \wedge |\mu_c|$  and  $s_1 < s < R(\mu)$ . Equation (2.61) yields :

$$\begin{aligned} a'(s, \mu)a(s, \mu) &= -2\mu a(s, \mu) - 2\beta(G(s) - s) \\ \frac{1}{2}(a^2(s, \mu) - a^2(s_1, \mu)) &= -2\mu \int_{s_1}^s a(u, \mu) du - 2\beta \int_{s_1}^s (G(u) - u) du \\ |a(s, \mu) - a(s_1, \mu)| &\leq \frac{4}{|a(s_1, \mu)|} \left[ |\mu|(R_G - s_1) \sup_{x \in [x_0, 0]} |Q'(x, \mu_c)| \right. \\ &\quad \left. + \beta \int_{s_1}^{R_G} (G(u) - u) du \right]. \\ |a(s, \mu) - a(s_1, \mu)| &\leq \frac{8}{|Q'(x_0, \mu_c)|} \left[ 2|\mu_c|(R_G - s_1) \sup_{x \in [x_0, 0]} |Q'(x, \mu_c)| \right. \\ &\quad \left. + \beta \int_{s_1}^{R_G} (G(u) - u) du \right]. \end{aligned} \quad (2.153)$$

Let  $0 < \epsilon < |Q'(x_0, \mu_c)|/2$ , by choosing  $\epsilon_1$  small enough we have for all  $\mu$  such that  $|\mu_c - \mu| < \eta \wedge |\mu_c|$ ,  $|a(s_1, \mu) - a(R_G, \mu_c)| < \epsilon/2$  and thanks to (2.153) that  $|a(s, \mu) - a(s_1, \mu)| < \epsilon/2$  which implies that :  $|a(s, \mu)| > |Q'(x_0, \mu_c)|/2$  for all  $s_1 < s < R(\mu)$  and thus  $R(\mu) = R_G$ . If  $\mu < \mu_c$  we then get a contradiction with the definition of  $\mu_c$ . Therefore  $Q'(x_0, \mu_c) = 0$ .  $\square$

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## Chapitre 3

# Coalescence and fragmentation on the Erdős–Rényi graph

This is a work in progress. Some proofs have to be simplified. We also have to provide some additional developments, for instance in the last section and to give some more rigorous justifications at some points.

### 3.1 Introduction

#### 3.1.1 Model and main results

We consider the Erdős–Rényi graph  $G(n, p)$  on  $[n] = \{1, \dots, n\}$ , with  $p \in [0, 1]$ , introduced in [50] and defined as follows. An integer  $i$  is called a vertex and an unordered pair  $\{i, j\}$  (which we will often simply write  $ij$ ) is called an edge. Let us fix  $(U_{ij})_{1 \leq i < j}$  a sequence of independent uniform variables on  $[0, 1]$  defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . An edge  $ij$  is open if and only if  $U_{ij} < p$ . Throughout this article, the graphs  $G(n, p)$  are defined on the same space, which means that for different values of  $p$  and  $n$  we keep the same  $(U_{ij})$ . The graph  $G(n, p)$  can be endowed with the structure of a measured metric graph defined as in Section 3.B.3 in the following way. Each edge is an open interval with length 1 which connects two vertices. The distance  $\rho_0$  between two points (vertices or points on an edge) is the length of a shortest path between these two points. The measure  $\nu_0$  is given by : for any Borelian  $A$ ,  $\nu_0(A)$  is the number of vertices in  $A$ .

We here focus on the critical case, that is when  $p = p_n(\lambda) = 1/n + \lambda n^{-4/3}$ . As we will see below, this case justifies that we consider rescaled distance  $\rho_n = n^{-1/3} \rho_0$  and measure  $\nu_n = n^{-2/3} \nu_0$ . Since most of the time we will work with these rescaled distances and measures, when no confusion can arise, we will implicitly consider that  $G(n, p_n(\lambda))$  is endowed with them. If however, we need to work with the unscaled distance and measure, we will precise it. Furthermore, for any Borelian  $A$ ,  $\nu_0(A)$  will be called the size of  $A$  and  $\nu_n(A)$



will be called the mass of  $A$ . Finally, in order to state convergence results, we will need to consider the sequence of connected components of  $G(n, p_n(\lambda))$  in non-increasing order of sizes instead of  $G(n, p_n(\lambda))$  itself. We will denote by  $\hat{G}(n, p_n(\lambda)) := (\mathcal{C}_i^n(\lambda), \rho_{n,i}, \nu_{n,i})_i$  this sequence, where  $\nu_{n,i}$  and  $\rho_{n,i}$  are the restrictions of  $\rho_n$  and  $\nu_n$  to the  $i^{\text{th}}$  component  $\mathcal{C}_i^n(\lambda)$ .

It is known (see [3] and Subsection 3.1.2), that  $\hat{G}(n, p_n(\lambda))$  converges in distribution as  $n \rightarrow +\infty$  to a certain sequence of measured metric spaces  $\hat{\mathbb{M}}(\lambda)$  defined in (3.16). Roughly speaking, each component of  $\hat{\mathbb{M}}(\lambda)$  is a Continuum real tree (defined in Subsection 3.1.2) with some additional cycles. Our goal is to describe more precisely the asymptotic behaviour of  $\Upsilon(n) := \hat{G}(n, p_n(0))$  (we choose  $\lambda = 0$  in order to simplify then otations, but the general case is identical). For that purpose, we will show that the random function  $\lambda \mapsto \Upsilon(n + \lfloor \lambda n^{2/3} \rfloor)$  converges in distribution, as  $n \rightarrow +\infty$  to a certain coalescence-fragmentation process.

A simple Taylor expansion shows that :

$$\hat{G}\left(n + \lfloor \lambda n^{2/3} \rfloor, \frac{1}{n + \lfloor \lambda n^{2/3} \rfloor}\right) = \hat{G}\left(n + \lfloor \lambda n^{2/3} \rfloor, \frac{1}{n} - \frac{\lambda}{n^{4/3}} + \underset{n \rightarrow +\infty}{o}\left(\frac{1}{n^{4/3}}\right)\right),$$

Motivated by this fact, we will see that the asymptotic study of  $\Upsilon(n + \lfloor \lambda n^{2/3} \rfloor)$  reduces to the studies of  $F$  and  $C$  defined by :

$$F(n, \lambda) = \hat{G}\left(n, \frac{1}{n} - \frac{\lambda}{n^{4/3}}\right) \text{ and } C(n, \lambda) = \hat{G}\left(n + \lfloor \lambda n^{2/3} \rfloor, \frac{1}{n}\right). \quad (3.1)$$

Let us start by the study of  $F$ . Fix  $\lambda_0 \in \mathbb{R}$ . Consider the measured metric space  $\mathbb{M}(\lambda)$  defined in (3.17) and its skeleton  $\text{Skel}(\mathbb{M}(\lambda))$  defined in Subsection 3.1.2. Set  $N$  a Poisson point process on  $\mathbb{R}_+ \times \text{Skel}(\mathbb{M}(\lambda))$ , with intensity measure  $d\lambda \otimes dl$ , where  $l$  is the Lebesgue measure on the skeleton. We define  $N_\lambda$  by  $N_\lambda(A) = N([0, \lambda] \times A)$ , for all  $A \in \mathcal{B}(\text{Skel}(\mathbb{M}(\lambda)))$ . The operator  $\text{Frag}$  is defined in Subsection 3.2.2. Roughly speaking, given a measured metric space  $X$  and a simple point process  $\Pi$ ,  $\text{Frag}(X, \Pi)$  cuts  $X$  along the set of points of  $\Pi$ . We finally refer the reader to Appendix 3.A for the definitions of  $\text{dist}_{GHP}^4$  and  $\mathbb{L}_4$ .

**Theorem 3.1.1.**

$$(F(n, \lambda - \lambda_0))_{\lambda \geq 0} \xrightarrow{d} \text{Frag}(\mathbb{M}(\lambda_0), N_\lambda)_{\lambda \geq 0}, \text{ as } n \rightarrow +\infty \quad (3.2)$$

in the sense of finite dimensional distributions for the metric  $\text{dist}_{GHP}^4$  on  $\mathbb{L}_4$ .

We now consider this limit process when we reverse time. Define

$$\overleftarrow{F}(n, \lambda) = F(n, -\lambda), \lambda \in \mathbb{R}. \quad (3.3)$$

Fix  $\lambda_0 \in \mathbb{R}$ . Consider  $P$  a Poisson point process on  $\mathbb{R}_+ \times \mathbb{M}(\lambda_0) \times \mathbb{M}(\lambda_0)$  with intensity measure  $\frac{1}{2}d\lambda \otimes \mu \otimes \mu$  and fix  $P_\lambda(A) := P([0, \lambda] \times A)$ , for all  $A \in \mathcal{B}(\mathbb{M}(\lambda_0) \times \mathbb{M}(\lambda_0))$ . The operator  $\text{Coal}$  is defined in Subsection 3.2.2. Roughly speaking, if  $X$  is a metric space and  $\Pi_2$  a set of couple of points in  $X$ , then  $\text{Coal}(X, \Pi_2)$  glues  $X$  along  $\Pi_2$ .

**Theorem 3.1.2.**

$$(\overleftarrow{F}(n, \lambda + \lambda_0))_{\lambda \geq 0} \xrightarrow{d} \text{Coal}(\mathbb{M}(\lambda_0), P_\lambda)_{\lambda \geq 0}, \quad (3.4)$$

in the sense of finite dimensional distributions for the metric  $\text{dist}_{GHP}^4$  on  $\mathbb{L}_4$ .

We now define the time reversal  $\overleftarrow{C}$  of  $C$  by

$$\overleftarrow{C}(n, \lambda) = C(n, -\lambda), \lambda \geq -n^{1/3}. \quad (3.5)$$

The operator  $\lambda \mapsto \overleftarrow{C}(n, \lambda)$  is a "fragmentation" operator since we remove vertices. The following result shows that asymptotically  $\overleftarrow{C}$  and  $F$  have the same behaviour.

**Theorem 3.1.3.**

$$(\overleftarrow{C}(n, \lambda - \lambda_0))_{\lambda \geq 0} \xrightarrow{d} \text{Frag}(\mathbb{M}(\lambda_0), N_\lambda)_{\lambda \geq 0}, \text{ as } n \rightarrow +\infty \quad (3.6)$$

in the sense of finite dimensional distributions for the metric  $\text{dist}_{GHP}^4$  on  $\mathbb{L}_4$ .

Since  $\overleftarrow{C}$  and  $F$  converge in distribution to the same process, it is also the case for their time-reversal, which implies the following.

**Theorem 3.1.4.**

$$(C(n, \lambda + \lambda_0))_{\lambda \geq 0} \xrightarrow{d} \text{Coal}(\mathbb{M}(\lambda_0), P_\lambda)_{\lambda \geq 0}, \quad (3.7)$$

in the sense of finite dimensional distributions for the metric  $\text{dist}_{GHP}^4$  on  $\mathbb{L}_4$ .

Theorems 3.1.1 and 3.1.4 yield the asymptotic behaviour of  $\Upsilon(n + \lfloor \lambda n^{2/3} \rfloor)$ . Consider  $\text{FC}$  the operator defined in Section 3.2.2. In order to describe  $\text{FC}$  consider the set  $\Pi$  and  $\Pi_2$  defined as above, with the restriction that none point of  $\Pi$  is a point of a couple of points in  $\Pi_2$ . Given a metric space  $X$ ,  $\text{FC}(X, \Pi, \Pi_2)$  is the metric space obtained by cutting  $X$  along  $\Pi$  and simultaneously gluing each point of each couple in  $\Pi_2$ . The previous theorems and the independence between the  $U_{i,j}$  provides the following.

**Theorem 3.1.5.**

$$(\Upsilon(n + \lfloor \lambda n^{2/3} \rfloor))_{\lambda \geq 0} \xrightarrow{d} \text{FC}(\mathbb{M}(0), P_\lambda, N_\lambda)_{\lambda \geq 0}, \quad (3.8)$$

in the sense of finite dimensional distributions for the metric  $\text{dist}_{GHP}^4$  on  $\mathbb{L}_4$ .

### 3.1.2 Previous results

#### $\mathbb{R}$ -graphs

Following [3], [4] and [5], we provide some standard definitions.

**Définition 3.1.6.** Fix  $(X, d)$  a metric space and  $x, y \in X$ .

- A geodesic between  $x$  and  $y$  is an isometric embedding  $\phi : [0, d(x, y)] \rightarrow X$  such that  $\phi(0) = x$  and  $\phi(d(x, y)) = y$ .
- $(X, d)$  is called geodesic if for each  $x, y \in X$  there exists a geodesic between  $x$  and  $y$ .
- A cycle of  $X$  is the image of a continuous injective function  $\phi : \mathbb{S}^1 \rightarrow X$ , where  $\mathbb{S}^1 = \{z \in \mathbb{C}, |z| = 1\}$ .
- $X$  is said to be acyclic when it has no cycle.

More specifically, we work on the two objects defined as follows.

**Définition 3.1.7.** A geodesic acyclic metric space  $(\mathcal{T}, d)$  is called a real tree (or a  $\mathbb{R}$ -tree).

**Définition 3.1.8.** A compact geodesic metric space  $(X, d)$  is called a  $\mathbb{R}$ -graph if for each  $x \in X$ , there exists  $r_0 > 0$  such that  $B(0, r_0) \cap X$  is a real tree.

Each element of a  $\mathbb{R}$ -graph is called a vertex. The degree of a vertex  $x \in X$  is the number of connected components in  $B(x, r_0) \setminus \{x\}$ , where  $r_0 > 0$  is chosen small enough for having that  $B(x, r_0) \cap X$  is a tree. The skeleton of  $X$  is the set of vertices with degree greater or equal to 2 and is denoted by  $\text{Skel}(X)$ . More generally, we conserve these notations and definitions when  $X = \sqcup_i X_i$ , where  $X_i$  is a  $\mathbb{R}$ -graph.

We can construct real trees in a simple way with a function as it is shown in [49]. Consider a continuous function  $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  with compact support  $[0, \sigma]$ , not identically zero and such that  $h(0) = 0$ . Then :

$$d_h(x, y) = h(x) + h(y) - 2 \inf_{x \leq s \leq y} h(s) \quad (3.9)$$

defines a semi-distance on  $[0, \sigma]$ . By introducing the equivalence relation  $\sim$  defined by  $x \sim y$  if  $d_h(x, y) = 0$ , we get a metric space :

$$\mathcal{T}_h = \mathbb{R}^+ / \sim, \quad (3.10)$$

which is a real tree. Note in particular that if we take  $h = e^1$ , where  $e^1$  is a standard Brownian excursion, we get the Brownian tree, sometimes called the Continuum real tree (CRT) see for instance [12], [13], [14] or [72]

With such trees, we can also define new metric spaces by identifying some points. More precisely, consider a finite set of points  $P$  on

$$\mathcal{A}_h := \{(x, y), x \in [0, \sigma], y \leq h(x)\}. \quad (3.11)$$

For  $(x, y)$  in  $P$ , define

$$r(x, y) := \inf\{x' \geq x, h(x') = y\}. \quad (3.12)$$

Let  $\tau$  be the canonical projection from  $[0, \sigma]$  onto  $\mathcal{T}_h$ . For each  $(x, y)$  in  $P$ , by identifying  $\tau(x)$  with  $\tau(r(x, y))$  (see Appendix 3.B) we get a new metric space  $(g(h, P), d_{h,P})$ , where  $d_{h,P}$  and  $d_h$  are respectively the  $d_{\mathcal{R}}$  and the  $d$  in (3.141). Finally, by considering the Borelian measures  $\tau_*\mathcal{L}^1$  and  $\tilde{\tau}_*\mathcal{L}^1$  (see Section 3.2.1), where  $\tilde{\tau}$  is the projection from  $[0, \sigma]$  to  $g(h, P)$  and  $\mathcal{L}^1$  the Lebesgue measure on  $\mathbb{R}$ , we can endow  $\mathcal{T}_h$  and  $g(h, P)$  with a structure of measured metric space.

Following [4], we construct the limit measured metric spaces  $(\mathcal{M}_i(\lambda), d_i, \mu_i)$ . Consider then the Brownian motion with parabolic drift  $(W_t^\lambda)$  defined by :

$$W_t^\lambda = B_t + \lambda t - \frac{t^2}{2}, \forall t \geq 0 \quad (3.13)$$

and :

$$\beta_t^\lambda = W_t^\lambda - \inf_{0 \leq s \leq t} W_s^\lambda, \forall t \geq 0. \quad (3.14)$$

We will denote by  $(\tilde{e}_i(\lambda))$  the sequence of excursions above 0 of  $\beta_t^\lambda$  in decreasing order of lengths and by  $\tilde{e}^{(\sigma)}$  such an excursion conditioned to have length  $\sigma > 0$ . Note that by Section 5 of [4], the distribution of  $\tilde{e}^{(\sigma)}$  depends neither on the starting point of the excursion nor on  $\lambda$ . We can characterise the distribution of  $\tilde{e}^{(\sigma)}$  as follows. Let us denote by Exc the space of excursions, that is :

$$\text{Exc} := \{f \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+), \exists \sigma > 0, f(x) > 0 \Leftrightarrow x \in (0, \sigma)\}.$$

We endow Exc with the topology induced by the supremum norm,  $\|f\|_\infty = \sup_{s \geq 0} |f(s)|$ . Let us denote by  $e^{(\sigma)}$  an excursion of the standard Brownian motion with length  $\sigma > 0$ . Then we have (see [4]) :

$$\mathbb{P}(\tilde{e}^{(\sigma)} \in B) = \frac{\mathbb{E}\left(\mathbf{1}_{\{e^{(\sigma)} \in B\}} \exp\left(\int_0^\sigma e_u^{(\sigma)} du\right)\right)}{\mathbb{E}\left(\exp\left(\int_0^\sigma e_u^{(\sigma)} du\right)\right)}, \quad (3.15)$$

for all  $B \in \mathcal{B}(\text{Exc})$ . Obviously, the excursion  $\tilde{e}^{(\sigma)}$  encodes a real tree by (3.10).

For each  $i \in \mathbb{N}^*$ , let  $\mathcal{P}_i$  be a Poisson point process on  $\mathcal{A}_{2\tilde{e}_i(\lambda)}$  with intensity measure

$(1/2)\mathcal{L}^2$ , where  $\mathcal{L}^2$  is the Lebesgue measure on  $\mathbb{R}^2$ . We can thus define a sequence of compact measured metric spaces  $\hat{\mathbb{M}}(\lambda) := (\mathcal{M}_i(\lambda), d_i, \mu_i)_{i \geq 1}$  defined by :

$$\mathcal{M}_i(\lambda) := g(2\tilde{e}_i(\lambda), \mathcal{P}_i), \forall i \geq 1, \quad (3.16)$$

with distances and measures defined as above.

Sometimes, it is more convenient to see the components  $\mathcal{M}_i(\lambda)$  as subsets of a same space. Therefore, let  $J$  be a subset of  $\mathbb{N}^*$ , we define for  $\lambda \in \mathbb{R}$  :

$$\mathbb{M}_J(\lambda) = \bigcup_{i \in J} \mathcal{M}_i(\lambda). \quad (3.17)$$

We endow  $\mathbb{M}_J(\lambda)$  with the metric  $d_U$ , defined in (3.142), and with its Borelian  $\sigma$ -algebra  $\mathcal{B}(\mathbb{M}_J(\lambda))$ . The measure  $\mu_J$  on  $\mathbb{M}_J(\lambda)$  is then defined by :

$$\mu_J(A) = \sum_{i \in J} \mu_i(A \cap \mathcal{M}_i(\lambda)), \forall A \in \mathcal{B}(\mathbb{M}_J(\lambda)). \quad (3.18)$$

If  $J = \mathbb{N}^*$ , we simply write  $\mathbb{M}_J(\lambda) = \mathbb{M}(\lambda)$  and  $\mu = \mu_J$ . We finally mention (see Section 2.1 of [3]) that if  $\tilde{e}^{(\sigma)}$  is an excursion with law given in (3.15),  $\mathcal{P}$  a Poisson point process on  $\mathcal{A}_{2\tilde{e}^{(\sigma)}}$  with intensity measure  $(1/2)\mathcal{L}^2$  and  $e$  a standard Brownian excursion then conditionally on  $\text{Card}(\mathcal{P}) = s \geq 0$ , we have for any bounded continuous function  $f$  of  $\text{Exc}$  :

$$\mathbb{E} \left( f(\tilde{e}^{(\sigma)}) \mid \text{Card}(\mathcal{P}) = s \right) = \frac{\mathbb{E} \left( f(\sqrt{\sigma}e(\frac{\cdot}{\sigma})) \left( \int_0^1 e_u d_u \right)^s \right)}{\mathbb{E} \left( \left( \int_0^\sigma e_u d_u \right)^s \right)}, \quad (3.19)$$

which means that, unlike in (3.15), on the event  $\text{Card}(\mathcal{P}) = s$ , we can derive the results in distribution on  $g(2\tilde{e}^{(\sigma)}, \mathcal{P})$  from those on  $g(2\tilde{e}^{(1)}, \mathcal{P})$  by a simple scaling.

### Erdős-Renyi graph

One of the most natural question about this model is to know the sizes and the structures of the different connected components, when  $p = p_n$  depends on  $n$ , for  $n$  large enough. In order to get a global picture as far as the sizes of these components are concerned, we give a non-exhaustive list of results and refer the reader to [25] and [64] for more explanations. We denote by  $Z_i^{(n)}$  the size of the  $i^{\text{th}}$  biggest connected component of the graph  $G(n, p)$  and we refer the reader to Section 3.2.1 for the definition of the Landau notations.

1. Suppose that  $p_n = \frac{\log n}{n} + \frac{c}{n}$ , where  $c \in \mathbb{R}$  and fix  $A_{n,k}$  the event : "there exist one connected component with  $n - k$  elements and  $k$  isolated points". In this case, we have :

$$\lim_{n \rightarrow +\infty} \mathbb{P}(A_{n,k}) = \frac{e^{-ck}}{k!} e^{e^{-c}}.$$

2. Suppose now that  $p_n = \frac{c}{n}$ , where  $c > 1$ . In this case, we have :

$$Z_1^{(n)} \underset{n \rightarrow +\infty}{\sim} yn \text{ and } Z_i^{(n)} = \underset{n \rightarrow +\infty}{\Theta}(\log n), \forall i \geq 2,$$

in probability, where  $y$  satisfy the equation  $e^{-cy} = 1 - y$ .

3. Suppose now that  $p_n = \frac{1}{n} + \frac{\lambda}{n^{\frac{4}{3}}}$ , where  $\lambda \in \mathbb{R}$ . In this case, we have :

$$Z_i^{(n)} = \underset{n \rightarrow +\infty}{\Theta}(n^{\frac{2}{3}}), \forall i \geq 1,$$

in probability.

4. Suppose that  $p_n = \frac{c}{n}$ , where  $0 < c < 1$ . In this case, we have :

$$Z_i^{(n)} = \underset{n \rightarrow +\infty}{\Theta}(\log(n)), \forall i \geq 1,$$

in probability.

5. Finally, consider the event  $B_n$  defined by : "all connected components are isolated points". The function  $n^{-2}$  is a threshold function for this property which means that :

$$\begin{cases} \lim_{n \rightarrow +\infty} \mathbb{P}(B_n) = 1 & \text{when } p(n) = \underset{n \rightarrow +\infty}{o}\left(\frac{1}{n^2}\right) \\ \lim_{n \rightarrow +\infty} \mathbb{P}(B_n) = 0 & \text{when } \frac{1}{n^2} = \underset{n \rightarrow +\infty}{o}(p(n)). \end{cases}$$

We mention that the case 1 is historically the first to have been studied in the paper of Erdős and Rényi [50]. Furthermore, cases 1 and 5 provide the boundaries of the non-trivial cases. Cases 2, 3 and 4 show the existence of a phase transition at  $p(n) = 1/n$ . For more on this, see [25].

In this article, we focus on case 3, that is  $p = p_n(\lambda) = n^{-1} + \lambda n^{-4/3}, \lambda \in \mathbb{R}$ . The following result due to Aldous [15] explains the meaning of parameter  $\lambda$ . Let  $S_i^{(n)}(\lambda)$  be the surplus of the  $i^{th}$  component of  $G(n, p_n(\lambda))$ , that is the minimal number of edges that need to be removed in order to obtain a tree. Finally, call  $Z_i(\lambda)$  the length of  $\tilde{e}_i(\lambda)$  defined in (3.14) and  $S_i(\lambda)$  a Poisson random variable with intensity  $\int_0^{Z_i(\lambda)} \tilde{e}_i(\lambda)(s) ds$ .

**Theorem (Aldous [15]).** *At  $\lambda \in \mathbb{R}$  fixed,*

$$(n^{-2/3} Z_i^{(n)}(\lambda), S_i^{(n)}(\lambda))_{i \geq 1} \xrightarrow{d} (Z_i(\lambda), S_i(\lambda))_{i \geq 1}, \text{ as } n \text{ goes to } +\infty \quad (3.20)$$

where the convergence of the first coordinate holds for the topology induced by  $\|\cdot\|_2$  on  $l_{\downarrow}^2$  and the convergence of the second holds for the product topology.

Furthermore, we have the following convergence in distribution for the Skorokhod topology

on  $\mathbb{D}(\mathbb{R}, l_{\downarrow}^2)$  :

$$\left( (n^{-2/3} Z_i^{(n)}(\lambda))_{i \geq 1}, \lambda \in \mathbb{R} \right) \xrightarrow{d} (\xi(\lambda), \lambda \in \mathbb{R}) \text{ as } n \text{ goes to } +\infty \quad (3.21)$$

where  $\xi$  is a random process on  $\mathbb{D}(\mathbb{R}, l_{\downarrow}^2)$  called the standard multiplicative coalescent.

A construction of the multiplicative coalescents is given in [15]. Informally, the coalescents we consider are continuous-time Markov processes on a certain subset  $S$  of the set of real non-increasing sequences. Each of its states  $X(t) \in S$  can be seen as the sequence in non-increasing order of sizes of clusters. The dynamics are given by : each pair of clusters of size  $(x, y)$  merges at rate  $\kappa(x, y)$  into a cluster of size  $x + y$ . When,  $S = l_{\downarrow}^2$  and  $\kappa(x, y) = xy$ , it is called the multiplicative coalescent. The standard multiplicative coalescent is the multiplicative coalescent whose marginals are given by the  $(Z_i(\lambda))$  of Equation (3.20). Note that Aldous' result can be extended to the study of the dynamic of  $(Z_i(\lambda), S_i(\lambda))$  which is called the augmented multiplicative coalescent, see [23].

Later, Armendariz [19] (unpublished) and Broutin and Marckert [29] have shown by different methods that the multiplicative coalescent can be constructed with a unique Brownian motion with parabolic drift. More precisely,

**Theorem (Broutin and Marckert [29]).** *If for each  $\lambda \in \mathbb{R}$ , the  $W_t^\lambda$  in (3.13) are defined for the same  $B_t$ , then :*

$$((Z_i(\lambda))_i, \lambda \in \mathbb{R}) \text{ is a standard multiplicative coalescent.}$$

Addario-Berry, Broutin, Goldschmidt in [3] and the same authors and Miermont [5] show that there is not only a link between the sizes of the components of the Erdős-Rényi graph and the lengths of the excursions of a Brownian motion with parabolic drift but also that at  $\lambda$  fixed, the sequence of connected components  $\hat{G}(n, p_n(\lambda))$  converges in distribution to the sequence of components  $\hat{M}(\lambda)$  defined in (3.16). Consider the metric space  $(\mathbb{L}_4, d_{GHP}^4)$  defined in Appendix 3.A.

**Theorem (Addario-Berry et al. [5]).** *For any  $\lambda \in \mathbb{R}$ ,*

$$\hat{G}(n, p_n(\lambda)) \xrightarrow{d} \hat{M}(\lambda) \quad (3.22)$$

for the topology induced by  $\text{dist}_{GHP}^4$  on  $\mathbb{L}_4$ .

### 3.1.3 Discussion and perspectives

Theorems 3.1.1 and 3.1.2 complete the picture by showing how asymptotically the components  $\mathcal{C}_i^n(\lambda)$  evolve when  $\lambda$  increases or decreases. We also mention that Theorem 3.1.1

has its intrinsic interest and has to be put into perspective with the standard additive coalescent. If, keeping the same notations as above for the multiplicative coalescent,  $S = l_{\downarrow}^1$  and  $\kappa(x, y) = x + y$ , then the coalescent is called additive coalescent. Aldous and Pitman [16] show that if  $Y(t)$  is the sequence of sizes in decreasing order at time  $t$  of the components obtained from a CRT after cutting it along a Poisson point process on its skeleton with intensity  $dl \otimes dt$ , then  $Y(e^{-t})$  is an additive coalescent called the standard additive coalescent. In our framework, the components are not CRT and the change of time is not the same as in the case of the standard additive coalescent, which explains why we get a different process.

A natural extension of our work is to prove that our convergences hold for the Skorokhod topology on  $\mathbb{D}(\mathbb{R}, \mathbb{L}^4)$ . The main difficulty for proving this extension is to deal with the geometry of the components. A lot is known about the asymptotic sizes of components and less is known about their geometry. Furthermore, this last aspect is more complicated. For instance, if we split a component  $X$  in two other components  $Y$  and  $Z$ , we do not have natural inequalities between  $\text{Diam}^4(X)$  and  $\text{Diam}^4(Y) + \text{Diam}^4(Z)$  while we have  $\text{Mass}(X) = \text{Mass}(Y) + \text{Mass}(Z)$  and  $\text{Mass}^2(X) \geq \text{Mass}^2(Y) + \text{Mass}^2(Z)$ , where we denote by  $\text{Mass}$  and by  $\text{Diam}$  the mass and the diameter of a measured metric space. Therefore, in order to solve this problem, we need more specific knowledge on the geometrical repartition of the edges in the  $G(n, p)$ .

A second extension is to study more precisely the process of the masses of the two members of Theorem 3.1.5. Even if we already have a convergence in the sense of the finite dimensional distributions, we would prefer a convergence for the Skorokhod topology  $\mathbb{D}(\mathbb{R}, l^2)$ . Obviously, we can derive from the first extension this one. But it is also possible to make a more simple proof relying on already existing results. It is interesting to note that the process  $(\text{Mass}(\Upsilon(n + \lfloor \lambda n^{2/3} \rfloor)))_{\lambda \geq 0}$  is not Markovian with respect to its canonical filtration and it seems that FCM defined by :

$$\text{FCM} : \lambda \mapsto (\text{Mass}(\text{FC}(\mathbb{M}(0), P_{\lambda}, N_{\lambda})))_{\lambda \geq \lambda_0} \quad (3.23)$$

is not Markovian either. We can give the following heuristic and non-rigorous argument to explain why. Suppose that two components distributed as two independent  $g(2\tilde{e}^{(\sigma)}, \mathcal{P})$  coalesce at a time  $t$ , then the expected surplus of the new component is the sum of the expected surplus of the two previous. But this sum is not equal to the expected surplus of a  $g(2\tilde{e}^{2\sigma}, \mathcal{P})$ . In other terms, there are in expectation less cycles in the new component than in a component of mass  $2\sigma$  without information on its past. Hence, even if we can prove that CM defined by :

$$\text{CM} : \lambda \mapsto \text{Mass}(\text{Coal}(\mathbb{M}(0), P_{\lambda})) \quad (3.24)$$



and FM defined by :

$$\text{FM} : \lambda \mapsto \text{Mass}(\text{Frag}(\mathbb{M}(0), N_\lambda)), \quad (3.25)$$

are Markovian, it is not the case for FCM. Indeed, CM is the standard multiplicative coalescent and FM its reversal-time. We can heuristically understand why are these two processes Markovian in the following way. The process CM is Markovian because only the masses have an effect on the dynamic of this process and the geometry of the components does not matter. For the process FM, the geometry is important, but Lemma 3.3.12 tells us (for the discrete version) that when we fragment a component of mass  $m$  into a sequence of masses  $(m_i)_i$ , the fact that a component of size  $m_i$  comes from a component of size  $m$  has no influence on its distribution as a measured metric space. Therefore, when we combine fragmentation and coalescence, the geometrical structure of the coalesced elements, which had no importance when we considered only the coalescence, is the reason of the non-Markovianity.

## 3.2 Basic Notations and main tools

### 3.2.1 Basic Notations

In this subsection, we set some basic notations and definitions.

- For a metric space  $(M, d)$ , we write  $\mathcal{B}(M)$  for the set of its Borelians. If  $(M, d, \mu)$  and  $(M', d', \mu')$  are two Borelian measurable metric spaces and  $\phi : M \rightarrow M'$  is a measurable function, then  $\phi_*\mu$  is the measure on  $(M', d')$  defined by :  $\phi_*\mu(A) = \mu(\phi^{-1}(A))$ , for all  $A \in \mathcal{B}(M')$ .
- The isometry class of a metric space  $(M, \delta)$  is denoted by  $[M, \delta]$ . Fix  $(M, \delta, \mu)$  and  $(M', \delta', \mu')$  two Borelian measurable metric spaces and  $\phi : M \rightarrow M'$  a measurable function. If  $\phi$  is a bijective isometry and that  $\mu' = \phi_*\mu$ , then  $(M, \delta, \mu)$  and  $(M', \delta', \mu')$  are isometric-equivalent measured metric spaces and we denote their class of equivalence by  $[M, \delta, \mu]$ . Finally, when we distinguish  $k$  points  $\mathbf{m} = (m_1, \dots, m_k)$  in  $M$  and  $k$  points  $\mathbf{m}' = (m'_1, \dots, m'_k)$  in  $M'$  and when the previous function  $\phi$  satisfies  $\phi(m_i) = m'_i$  for each  $i \in [k]$ , we say that  $(M, d, \mu, \mathbf{m})$  and  $(M', d', \mu', \mathbf{m}')$  are isometric-equivalent pointed measured metric spaces and we denote their class of equivalence by  $[M, d, \mu, \mathbf{m}]$ . When there is non ambiguity, we will sometimes write by a slight abuse of notation  $M$  instead of  $[M, \delta]$ ,  $[M, \delta, \mu]$  or  $[M, \delta, \mu, \mathbf{m}]$ .
- For  $k \in \mathbb{N}^*$ , the space  $l^k$  is the set of real sequences  $u$  satisfying :

$$\|u\|_k^k = \sum_{i=0}^{+\infty} u_i^k < +\infty,$$

and is endowed with the norm  $\|\cdot\|_k$ . The space  $l_{\downarrow}^k$  is the subset of non-increasing sequences of  $l^k$ .

- For a collection of measured metric spaces  $\{(X_i, d_i, \mu_i)\}$ , we define, when such an object exists :

$$\{(X_i, d_i, \mu_i)\}^{\downarrow} \quad (3.26)$$

as the sequence in non-increasing order of mass (that is the  $\mu_i(X_i)$ ) of the components of  $\{(X_i, d_i, \mu_i)\}$ .

- For a complete separable metric space  $X$  we denote by  $\mathfrak{M}_{f,c}(X)$  the set of simple finite counting measure, that the measures of the form :

$$m = \sum_{i=1}^N \delta_{x_i},$$

where  $N \in \mathbb{N}$ ,  $x_i \in X$ ,  $\forall i \in [N]$  and  $x_i \neq x_j$  when  $i \neq j$ .

- For two sequences  $(a_n)$  and  $(b_n)$ , we use the following Landau notations :

1.

$$a_n = \underset{n \rightarrow +\infty}{o}(b_n),$$

when there exist a sequence  $(\epsilon_n)$  and  $n_0 \in \mathbb{N}$ , such that  $\epsilon_n \rightarrow 0$  when  $n$  tends to  $+\infty$  and such that  $a_n = \epsilon_n b_n, \forall n \geq n_0$ ;

2.

$$a_n \underset{n \rightarrow +\infty}{\sim} b_n,$$

when there exist a sequence  $(c_n)$  and  $n_0 \in \mathbb{N}$ , such that  $c_n \rightarrow 1$  when  $n$  tends to  $+\infty$  and such that  $a_n = c_n b_n, \forall n \geq n_0$ ;

3.

$$a_n = \underset{n \rightarrow +\infty}{O}(b_n),$$

when there exist  $C > 0$  and  $n_0 \in \mathbb{N}$ , such that  $a_n \leq C b_n, \forall n \geq n_0$ ;

4.

$$a_n = \underset{n \rightarrow +\infty}{\Theta}(b_n),$$

when there exist  $C_1, C_2 > 0$  and  $n_0 \in \mathbb{N}$ , such that  $C_1 b_n \leq a_n \leq C_2 b_n, \forall n \geq n_0$ .

If now  $(X_n)$  is a sequence of random variables, we say that :

$$X_n = \underset{n \rightarrow +\infty}{\Theta}(b_n), \text{ in probability,}$$

when for every  $\epsilon > 0$ , there exist  $C_1, C_2 > 0$  and  $n_0 \in \mathbb{N}$ , such that :

$$\mathbb{P}(C_1 b_n \leq X_n \leq C_2 b_n) \geq 1 - \epsilon, \forall n \geq n_0.$$

Analogous definitions hold for the three other previous cases.

- For a set  $S$ , we denote by  $\text{Int}(S)$  and by  $\text{Adh}(S)$  its interior and its adherence. Given another set  $S_2$ , we fix :

$$\mathcal{P}_2(S, S_2) = \mathcal{P}_2(S_2, S) = \{\{s, s_2\}, s \in S, s_2 \in S_2\}.$$

If  $S = S_2$ , we simply write  $\mathcal{P}_2(S, S) = \mathcal{P}_2(S)$ .

- By convention, if  $(X_i, d_i, \mu_i)_{i \in [n]}$  is a finite sequence of measured metric spaces, we fix for  $j > n$ ,  $(X_j, d_j, \mu_j) = (\{0\}, 0, 0)$ , in order to obtain the whole sequence  $(X_i, d_i, \mu_i)_{i \geq 1}$ .

### 3.2.2 Operators on measured metric spaces

We define in this section the main operators we will use. Consider  $(X_i, d_i, \mu_i)_i$  a sequence of disjoint measured connected compact  $\mathbb{R}$ -graph endowed with their Borelian  $\sigma$ -algebra. We further assume that each component has a finite number of cycles and that the maximal degree of a vertex of each component is also finite. We recall that we can see  $X = \sqcup_i X_i$  as a measured metric space by endowing it with the metric  $d_U$  defined in (3.141) and with measure  $\mu$  defined by  $\mu(A) = \sum_i \mu_i(A \cap X_i), \forall A \in \mathcal{B}(X)$ .

We first introduce the operator  $\text{Frag}$  defined in the following way. Fix  $N = \sum_{x \in \mathcal{N}} \delta_x$ , where  $\mathcal{N}$  is a countable subset of  $X$ . For each  $i \in \mathbb{N}$ , let us define by  $(X_{i,j})_j$  the collection of connected subsets of  $X_i \setminus \mathcal{N}$ . We also define  $d_{i,j}$  as the restriction to  $X_{i,j}$  of the metric  $d_C$  defined in (3.146) and  $(\bar{X}_{i,j}, \bar{d}_{i,j})$  as the completion of  $(X_{i,j}, d_{i,j})$ . We endow  $\bar{X}_{i,j}$ , with the measure  $\mu_{i,j}$  defined by  $\bar{\mu}_{i,j}(A) = \mu_i(A \cap X_{i,j}), \forall A \in \mathcal{B}(\bar{X}_{i,j})$  (see Theorem 3.C.1 in Appendix for the justification of the well-definition of  $\bar{\mu}_{i,j}$ ). Finally, we define :

$$\text{Frag}(X, N) = \{(\bar{X}_{i,j}, \bar{d}_{i,j}, \bar{\mu}_{i,j})\}^\downarrow, \quad (3.27)$$

where  $\{\cdot\}^\downarrow$  is defined as in (3.26).

Let us define the function  $\text{Coal}$  in the following way. Consider that  $J = \sum_{(x,y) \in \mathcal{J}} \delta_{(x,y)}$ , where  $\mathcal{J}$  is a countable subset of  $X \times X$ . In order to obtain an equivalence relation, we consider  $\tilde{\mathcal{J}}_0$  defined as the set of  $\{x, y\}$  such that  $(x, y) \in J$  or  $(y, x) \in J$  and  $\tilde{\mathcal{J}}$  defined as the set of  $\{x, y\}$  satisfying the property :

$$\exists k \in \mathbb{N}, \exists (z_i) \in X^k, \{z_i, z_{i+1}\} \in \mathcal{J}_0, \forall i \in [k-1], z_1 = x, z_k = y.$$

Hence,  $\tilde{\mathcal{J}}$  contains  $\tilde{\mathcal{J}}_0$  and we can define the following equivalence relation on  $X$  by :

$$x \mathcal{R} y \Leftrightarrow x = y \text{ or } \{x, y\} \in \tilde{\mathcal{J}}.$$

We can now glue our spaces by the process described in Appendix 3.B and consider  $Y$  the length space hence obtained. We endow  $Y$  with the Borelian measure  $\nu = p_*\mu$ , where  $p$  is the projection from  $X$  to  $Y$  (which is continuous in virtue of (3.143)). As before, we can define by completion  $(\bar{Y}, \bar{d}, \bar{\nu})$ . By calling  $(\bar{Y}_i, \bar{d}_i, \bar{\nu}_i)$  the collection of connected components endowed with their respective restrictions of  $\bar{d}$  and  $\bar{\nu}$ , we finally define :

$$\text{Coal}(X, J) := \{(\bar{Y}_i, \bar{d}_i, \bar{\mu}_i)\}^\downarrow. \quad (3.28)$$

We can finally define an operator FC which takes in account cuts and gluing. Consider  $N, \mathcal{N}, J$  and  $\mathcal{J}$  defined as above. We further define  $\mathcal{J}_1$  as the set of elements  $x \in X$ , such that there exists  $y \in X$  such that  $\{x, y\} \in \mathcal{J}$ . We suppose here that  $\mathcal{J}_1 \cap \mathcal{N} = \emptyset$ . The function FC is then defined by :

$$\text{FC}(X, N, J) := \text{Coal}(\text{Frag}(X, N), J). \quad (3.29)$$

Note that the sequence  $\{(X_i, d_i, \mu_i)\}^\downarrow$  can be not defined. It is the case, when for instance, the set of masses of the  $(X_i, d_i, \mu_i)$  is  $\mathbb{Q} \cap [0, 1]$ . However in our framework, we will never be in such a case, since the masses will always be in  $l^2$ . Furthermore, it is possible to have several components with same mass  $m$ . Actually, it will not be a real problem in our framework. Indeed, we will see that almost surely this never happens for the asymptotic components and since our results are asymptotic, we can give any arbitrary order for the Erdős–Rényi components with same sizes.

We finally mention that if  $\phi$  is an isometry of measured metric space between  $(X, d, \mu)$  and  $(X', d', \mu')$  and  $N$  (resp.  $N'$ ) is an atomic measure of  $X$  (resp.  $X'$ ) such that  $\phi_*N = N'$  then  $\text{Frag}((X, d, \mu), N)$  and  $\text{Frag}((X', d', \mu'), N')$  are isometric-equivalent measured metric spaces. Similarly, if we apply one of the two other operators to two isometric-equivalent measured metric spaces we get two isometric-equivalent measured metric spaces. Therefore, by a slight abuse of notation we can consider these operators either on measured metric space or on its class of equivalence. More generally, since the properties we will study are isometric-equivalent, we will often confound the measured metric spaces and their equivalence classes.

### 3.2.3 Ordered Depth-first search

We recall some facts about the ordered depth-first search and its applications to Erdős–Rényi graph present in [4], which will be useful in what follows.

Let  $G = (V, E)$  be a graph (possibly with multiple edges and loops), with  $V = [n]$ . The ordered depth-first search ( $\text{oDFS}(G)$ ) is an algorithm for searching a forest by always

exploring the smallest-labeled possible vertex. More precisely, we will define by induction, for  $i \in [n]$ , the set of vertices already explored  $\mathcal{A}_i$  and the stack  $\mathcal{O}_i$  of open vertices at time  $i$  (an open vertex is a vertex at (unscaled) distance 1 of the vertices already explored). A vertex  $u$  has been seen at time  $i$  if  $u$  is in  $\mathcal{O}_i \cup \mathcal{A}_i$ . Let  $c_i$  be the number of components already explored. The algorithm is then defined by the following procedure :

1. Initialisation : Fix  $\mathcal{O}_0 = (1)$ ,  $\mathcal{A}_0 = \emptyset$  and  $c_0 = 1$ .
2. Heredity : Suppose that for a fixed  $i \in \{0, \dots, n-1\}$ ,  $\mathcal{O}_i$ ,  $\mathcal{A}_i$  and  $c_i$  are known. Let  $v_i$  be the first vertex of  $\mathcal{O}_i$  and  $\mathcal{N}_i$  the set of neighbours of  $v_i$  in  $[n] \setminus (\mathcal{A}_i \cup \mathcal{O}_i)$ . Put  $\mathcal{A}_{i+1} = \mathcal{A}_i \cup \{v_i\}$ . We construct  $\mathcal{O}_{i+1}$  from  $\mathcal{O}_i$  by first deleting  $v_i$  and next by concatenating the elements of  $\mathcal{N}_i$  in increasing order to the top of the new stack. When  $\mathcal{O}_{i+1} \neq \emptyset$ , set  $c_{i+1} = c_i$ . Otherwise, if  $[n] \setminus \mathcal{A}_{i+1}$  is non-empty, fix  $c_{i+1} = c_i + 1$  and add to  $\mathcal{O}_{i+1}$  the vertex with lowest label of  $[n] \setminus \mathcal{A}_{i+1}$ , else stop the algorithm.

We will now restrain ourselves to the case of a connected graph on  $[n]$ . In this case, this procedure provides a rooted (the root is the first label) spanning tree  $T(G)$  which we call a depth-first tree. Some processes naturally encode this tree and we mention two of them. The first one is called the depth-first walk and is defined by  $\mathcal{X}(i) = |\mathcal{O}_i \setminus \{v_i\}|$ ,  $\forall 0 \leq i < n$ . We further fix  $a(T) = \sum_{1 \leq i \leq n-1} \mathcal{X}(i)$ . The second one is called the height process, and associate to each  $i$  the height  $H(i)$  of the vertex  $v_i$  with respect to the root.

Consider now  $G_Z^p$  a connected component of  $G(n, p)$  conditioned to have size  $Z$ . In order to simplify, we here consider that the labelling is given by the depth-first order algorithm starting from 0. We first introduce a graph exactly distributed as  $G_Z^p$ . Pick a labeled tree  $\tilde{T}_Z^p$  in  $\{0, \dots, Z-1\}$  such that  $\mathbb{P}(\tilde{T}_Z^p = T) \propto (1-p)^{-a(T)}$ . Then  $\tilde{T}_Z^p$  is distributed as  $T(G_Z^p)$  and the process  $\tilde{G}_Z^p$  constructed from  $\tilde{T}_Z^p$  by adding for each couple  $(i, j)$ , where  $j \in \mathcal{O}_i$ , an edge  $ij$  with probability  $p$ , is also distributed as  $G_Z^p$ . Therefore, if we fix a binomial set of points  $Q_p$  with intensity  $p$  on  $\mathbb{N} \times \mathbb{N}$  independent of  $\mathcal{X}$ , and we define  $G^{\mathcal{X}}(\tilde{T}_Z^p, Q_p)$  as the graph obtained from  $\tilde{T}_Z^p$  by adding for each point  $(i, j) \in Q_p$  under the curve of  $\mathcal{X}$ , the edge  $ij_1$ , where  $j_1$  is the  $j^{\text{th}}$  vertex starting from the bottom of  $\mathcal{O}_i$ , we obtain a graph with same distribution as  $G_Z^p$ .

If now, we consider the graph obtained from  $\tilde{T}_Z^p$  by adding for each point  $(i, j) \in Q_p$  under the curve of  $2H$ , the edge  $ij_2$ , where  $j_2$  is the vertex on the path from 0 (the root) to  $i$  at distance  $2j-1$  from the root, we get another graph  $G^H(\tilde{T}_Z^p, Q_p)$ . This object has not exactly the same distribution as  $G_Z^p$ , but Lemma 16 and 20 of [4] show us that when  $p = 1/n + \lambda n^{-4/3}$  and  $Z = \lfloor \sigma n^{2/3} \rfloor$ , for a good probability space chosen, the Gromov-Hausdorff distance between these two objects tends to 0 as  $n$  goes to infinity. The graph  $G^H(\tilde{T}_Z^p, Q_p)$  is essentially introduced in [4] because it is easier to manipulate than  $G^{\mathcal{X}}(\tilde{T}_Z^p, Q_p)$ .

### 3.3 The fragmentation process

In this section, we will prove Theorem 3.1.1. We start this section by defining the main objects we will use.

#### 3.3.1 Main notations of this section

Fix  $\lambda_0 \in \mathbb{R}$  and  $\sigma > 0$ . In order to simplify our notations we will fix in this section for each  $(i, n, \lambda) \in \mathbb{N}^2 \times \mathbb{R}$ ,  $q_n(\lambda) = p_n(-\lambda) = n^{-1} - \lambda n^{-4/3}$  and  $\mathcal{N}_i(\lambda) = \mathcal{M}_i(-\lambda)$ , where  $\mathcal{M}_i(-\lambda)$  is defined as in (3.16). We recall that each edge of  $G(n, q_n(\lambda))$  has length  $n^{-1/3}$  and each of its vertices has mass  $n^{-2/3}$ . We denote by :

- $(M_n(\lambda_0), \rho_n, \nu_n)$  a connected component of  $G(n, q_n(\lambda_0))$  conditioned to have size  $Z = \lfloor \sigma n^{2/3} \rfloor$  seen as measured metric graph.
- $(M_{n,i}(\lambda_0, \lambda), \rho_{n,i}, \nu_{n,i})$  the  $i^{\text{th}}$  biggest connected component of  $G(n, q_n(\lambda))$  whose vertices belong to  $M_n(\lambda_0)$ . The sequence  $(M_{n,i}(\lambda_0, \lambda), \rho_{n,i}, \nu_{n,i})_{i \geq 1}$  is simply written  $M_n(\lambda_0, \lambda)$ ;

Fix  $\tilde{e}^{(\sigma)}$  an excursion with distribution given by (3.15) and  $\mathcal{P}$  a Poisson point process of intensity  $(1/2)\mathcal{L}^2$  on  $\mathcal{A}_{2\tilde{e}^{(\sigma)}}$  defined in (3.11), where  $\mathcal{L}^2$  is the Lebesgue measure on  $\mathbb{R}^2$ . We denote by :

- $(T_\infty(\lambda_0), d_T, \mu_T)$  the real tree associated to  $2\tilde{e}^{(\sigma)}$  as in (3.10);
- $(M_\infty(\lambda_0), d, \mu)$  the measured metric space  $g(2\tilde{e}^{(\sigma)}, \mathcal{P})$  defined as in introduction;
- $N$  a Poisson point process on  $[\lambda_0, +\infty) \times \text{Skel}(M_\infty(\lambda_0))$ , with intensity measure  $d\lambda \otimes dl$ , where  $l$  is the Lebesgue measure on the skeleton. For  $\lambda \geq \lambda_0$ , we define  $N_\lambda$  by  $N_\lambda(A) = N([\lambda_0, \lambda] \times A)$ , for all  $A \in \mathcal{B}(\text{Skel}(M_\infty(\lambda_0)))$ ;
- $M_\infty(\lambda_0, \lambda)$  the vector  $\text{Frag}(M_\infty(\lambda_0), N_\lambda)$ ;
- $(M_{\infty,i}(\lambda_0, \lambda), d_i, \mu_i)$  the  $i^{\text{th}}$  component of  $M_\infty(\lambda_0, \lambda)$ .

We recall that the distribution of  $T_\infty(\lambda_0)$  and of  $M_\infty(\lambda_0)$  does not depend on  $\lambda_0$ . Nevertheless, we choose to put  $\lambda_0$  into our notations because on the one hand, the distribution of the whole sequence of components  $(\mathcal{N}_i(\lambda_0))$  depends on  $\lambda_0$  and on the other hand  $\lambda_0$  can be seen as a starting time and our purpose is to see how evolve these components when the time  $\lambda$  varies.

We now introduce some concepts used in [3]. Consider a connected labeled graph  $G = (V, E)$  and define  $s(G) = \text{Card}(E) - \text{Card}(V) + 1$  its surplus, that is the minimal number of edges to delete if we want to obtain a tree. The core of  $G$  is the maximal subset of  $G$  whose vertices has minimum degree two. The kernel of  $G$  is the multigraph obtained from the core by replacing with an edge all paths whose internal vertices all have degree two in the core and whose endpoints have degree at least three in the core.

We now define the analogous objects for the asymptotic components. We define the surplus of  $M_\infty(\lambda_0)$  as the cardinal of  $\mathcal{P}$  and we denote it by  $s(M_\infty(\lambda_0))$ . We recall that  $\mathcal{P}_x :=$

$\{x, (x, y) \in \mathcal{P}\}$  and  $\mathcal{P}_r := \{r(x, y), (x, y) \in \mathcal{P}\}$ , where  $r(x, y) = \inf\{x' \geq x, 2\tilde{e}^{(\sigma)}(x') = y\}$ . The precore  $T_\infty^0(\lambda_0)$  of  $M_\infty(\lambda_0)$  is the union of all shortest paths in  $T_\infty(\lambda_0)$  between the points in  $\tau(\mathcal{P}_x) \cup \tau(\mathcal{P}_r) \cup \{\tau(0)\}$ , where  $\tau$  is the projection from  $[0, \sigma]$  onto  $T_\infty(\lambda_0)$ . It is thus a rooted tree with root  $\tau(0)$ . We define the core  $C_\infty(\lambda_0)$  as the metric space obtained by gluing the points  $\tau(r(x, y))$  and  $\tau(x)$  in  $T_\infty^0(\lambda_0)$ .

Let  $U_1, \dots, U_k$  be  $k$  i.i.d. points of  $M_\infty(\lambda_0)$  sampled according to  $\mu$  and consider that  $U_0 = \tau(0)$  is the root of  $M_\infty(\lambda_0)$ . Similarly, for  $n \in \mathbb{N}^*$ , let  $U_1^n, \dots, U_k^n$  be  $k$  i.i.d vertices of  $M_n(\lambda_0)$  sampled according to  $\nu_n$  and consider that the smallest-labelled vertex  $U_0^n$  is the root of  $M_n(\lambda_0)$ . We denote by :

- $(M_\infty^k(\lambda_0), d, \mu^k)$  the measured metric space defined as the union of  $C_\infty(\lambda_0)$  and all paths from the  $U_i$ 's to the core with measure  $\mu^k = \sum_{i \in [k]} \frac{\sigma}{k} \delta_{U_i}$  (we recall that  $[k] = \{1, \dots, k\}$  and thus  $U_0$  does not have mass);
- $M_\infty^k(\lambda_0, \lambda)$  the vector  $\text{Frag}(M_\infty^k(\lambda_0), N_\lambda)$ ;
- $(M_{\infty, i}^k(\lambda_0, \lambda), d_i^k, \mu_i^k)$  the  $i^{\text{th}}$  component of  $M_\infty^k(\lambda_0, \lambda)$ ;
- $(M_n^k(\lambda_0), \rho_n, \nu_n^k)$  the measured metric graph defined as the union of the core of  $M_n(\lambda_0)$  and all paths from the  $U_i^n$  to the core, with measure  $\nu_n^k = \sum_{i \in [k]} \frac{\sigma}{k} \delta_{U_i^n}$
- $(M_{n, i}^k(\lambda_0, \lambda), \rho_{n, i}^k, \nu_{n, i}^k)$  the  $i^{\text{th}}$  biggest component of  $G(n, q_n(\lambda))$  whose vertices belong to  $M_n^k(\lambda_0)$ .
- $M_n^k(\lambda_0, \lambda)$  the sequence  $(M_{n, i}^k(\lambda_0, \lambda), \rho_{n, i}^k, \nu_{n, i}^k)_{i \geq 1}$ .

### 3.3.2 Strategy of the proof

Theorem 3.1.1 tells us that the asymptotic evolution of the components of  $\hat{G}(n, q_n(\lambda))$  starting at time  $\lambda_0$  is described by a fragmentation along a Poisson point process on the skeleton. We start by proving this result for one connected component at time  $\lambda_0$ . More precisely, fix  $\lambda_0, \lambda \in \mathbb{R}, \lambda \geq \lambda_0$ .

#### Proposition 3.3.1.

$$M_n(\lambda_0, \lambda) \xrightarrow{d} M_\infty(\lambda_0, \lambda) \text{ as } n \rightarrow +\infty, \quad (3.30)$$

for the product topology induced by  $\delta_{GHP}$ .

Define the process  $(N_\lambda^n)_{\lambda \geq \lambda_0}$  as follows. Consider  $E^n(\lambda)$  the set of opened edges in  $G(n, q_n(\lambda))$  and  $\bar{E}^n(\lambda)$  the set of closed edges. For two vertices  $i, j$  linked by an edge  $e = ij$ , we denote by  $\text{mid}(e)$ , the midpoint of  $e$ , that is a point such that  $d(\text{mid}(e), i) = d(\text{mid}(e), j) = d(j, i)/2$ . Then  $(N_\lambda^n)_{\lambda \geq \lambda_0}$  is defined by :

$$N_\lambda^n = \sum_{e \in E^n(\lambda_0)} \delta_{\text{mid}(e)} \mathbf{1}_{e \in \bar{E}^n(\lambda)}, \forall \lambda \geq \lambda_0. \quad (3.31)$$

Since  $\text{Frag}(M_n(\lambda_0), N_\lambda^n)$  is obtained by cutting edges at their midpoints and  $M_n(\lambda_0, \lambda)$  is

obtained by deleting the same edges, it is straightforward to show that :

$$d_{GHP}(\text{Frag}(M_n(\lambda_0), N_\lambda^n), M_n(\lambda_0, \lambda)) \leq \frac{1}{2n^{1/3}}. \quad (3.32)$$

The same inequality holds if we replace  $M_n(\lambda_0)$  with  $M_n^k(\lambda_0)$  and  $M_{n,i}(\lambda_0, \lambda)$  with  $M_{n,i}^k(\lambda_0, \lambda)$ . Therefore, we can approximate  $M_n(\lambda_0, \lambda)$  (resp.  $M_n^k(\lambda_0, \lambda)$ ) by  $\text{Frag}(M_n(\lambda_0), N_\lambda^n)$  (resp.  $\text{Frag}(M_n^k(\lambda_0), N_\lambda^n)$ ). Furthermore, we can heuristically see that in a certain sense :

$$N^n \xrightarrow{d} N. \quad (3.33)$$

A Borel measure  $\mu$  on  $X$  is said to be boundedly finite, if for each  $A \in \mathcal{B}(X)$ , such that  $A$  is bounded, we have  $\mu(A) < +\infty$ . The classical theorems of convergence of point processes (see for instance [39]) assume that the considered measures are boundedly finite, which is not the case of the length measure of the skeleton. The skeleton can indeed be constructed from  $\mathbb{R}^+$  (see [3]) and is included in a ball of finite radius. Furthermore, we need to find a good topology for which the convergence in (3.33) implies the convergence of  $\text{Frag}(M_n(\lambda_0), N_\lambda^n)$  to  $\text{Frag}(M_\infty(\lambda_0), N_\lambda)$ . It seems thus difficult to make a direct proof. That is why, our strategy is to proceed by successive approximations. We start by proving that the asymptotic components can be approximated by the finite graphs obtained by fragmentation of  $M_\infty^k(\lambda_0)$ .

**Proposition 3.3.2.** *For any  $\lambda \geq \lambda_0$ , we have :*

$$\text{Frag}(M_\infty^k(\lambda_0), N_\lambda) \rightarrow \text{Frag}(M_\infty(\lambda_0), N_\lambda) \text{ as } k \rightarrow +\infty \text{ a.s.} \quad (3.34)$$

for the product topology induced by  $\delta_{GHP}$ .

Note that the case  $\lambda = \lambda_0$  corresponds to :

$$M_\infty^k(\lambda_0) \rightarrow M_\infty(\lambda_0) \text{ as } k \rightarrow +\infty \text{ a.s.} \quad (3.35)$$

We next prove the following convergence.

**Proposition 3.3.3.** *For any  $\lambda \geq \lambda_0$ , we have :*

$$\text{Frag}(M_n^k(\lambda_0), N_\lambda^n) \xrightarrow{d} \text{Frag}(M_\infty^k(\lambda_0), N_\lambda) \text{ as } n \rightarrow +\infty, \quad (3.36)$$

for the product topology induced by  $\delta_{GHP}$ .

We next approximate  $M_{n,i}(\lambda_0, \lambda)$  by  $M_{n,i}^k(\lambda_0, \lambda)$ .

**Proposition 3.3.4.** *For any  $\lambda \geq \lambda_0$ , for any  $i \in \mathbb{N}^*$  and  $\epsilon > 0$ , for  $n$  and  $k$  large enough, we have :*

$$\mathbb{P}(d_{GHP}(M_{n,i}^k(\lambda_0, \lambda), M_{n,i}(\lambda_0, \lambda)) \geq \epsilon) \leq \epsilon. \quad (3.37)$$



Combining the three previous propositions we get Proposition 3.3.1. By using convergence results in [4], we deduce that this convergence holds for the whole process and in the sense of the topology induced by  $\text{dist}_{GHP}^4$ . By Markov property, we finally have that the convergence in Theorem 3.1.1 is in the sense of finite dimensional distribution.

### 3.3.3 Proof of Proposition 3.3.2

We start this section with the proof of two lemmas.

**Lemma 3.3.5.** *For any  $x \in M_\infty(\lambda_0)$  and  $\epsilon > 0$  we have  $\mu(B(x, \epsilon)) > 0$ .*

*Proof.* Recall that  $\tilde{\tau}$  is the projection from  $[0, \sigma]$  onto  $M_\infty(\lambda_0)$ . Let  $x \in [0, \sigma]$ . By continuity of  $2\tilde{e}^{(\sigma)}$  there is  $\eta > 0$  such that for each  $y \in [0, \sigma]$  :

$$|x - y| < \eta \Rightarrow |2\tilde{e}^{(\sigma)}(x) - 2\tilde{e}^{(\sigma)}(y)| < \epsilon/2.$$

Since the distance between two points in the tree induced by  $2\tilde{e}^{(\sigma)}$  is larger than the distance between the same points in  $M_\infty(\lambda_0)$ , this implies that for any  $y \in (x - \eta, x + \eta)$  we have  $d(\tilde{\tau}(x), \tilde{\tau}(y)) < \epsilon$ . Therefore, we have :

$$\{\tilde{\tau}(z) : x < z < \eta + x\} \subset B(\tilde{\tau}(x), \epsilon). \quad (3.38)$$

And since  $\mu(\{\tilde{\tau}(z) : x < z < \eta + x\}) = \eta$ , we have the desired result by monotonicity of the measure.  $\square$

**Lemma 3.3.6.** *For any  $x \in \text{Skel}(M_\infty(\lambda_0))$ , there exists  $k > 0$  such that  $x \in M_\infty^k(\lambda_0)$ .*

*Proof.* By definition of the skeleton, there exist  $y, z$  such that  $x$  is on a path  $\gamma_1$  between  $y$  and  $z$  and such that  $x, y$  and  $z$  are distinct.

Fix  $r = [d(x, y)/3] \wedge [d(y, z)/3] \wedge [d(x, z)/3]$ . The measures of the balls  $B(y, r)$  and  $B(z, r)$  are non-zero by Lemma 3.3.5. Therefore, almost surely there exist  $i, j \in \mathbb{N}^*$  such that  $U_i \in B(y, r)$  and  $U_j \in B(z, r)$ . This yields a path  $\gamma_2$  between  $U_i$  and  $U_j$  such that  $x \in \gamma_2$  and by construction of  $M_\infty^k(\lambda_0)$ , for  $k = i \vee j$ , we have  $M_\infty^k(\lambda_0) \supset \gamma_2$ , and thus  $x \in M_\infty^k(\lambda_0)$ .  $\square$

We now give some properties of the masses of  $M_\infty(\lambda_0, \lambda)$ .

**Lemma 3.3.7.** *For any  $\lambda > \lambda_0$ , the masses of  $M_\infty(\lambda_0, \lambda)$  satisfy :*

1.  $\sum_i \mu(M_{\infty,i}(\lambda_0, \lambda)) = \sigma$ ,
2.  $\forall i \in \mathbb{N}, \mu(M_{\infty,i}(\lambda_0, \lambda)) > 0$ .
3.  $\forall i, j \in \mathbb{N}, i \neq j \Rightarrow \mu(M_{\infty,i}(\lambda_0, \lambda)) \neq \mu(M_{\infty,j}(\lambda_0, \lambda))$ .

*Proof.* The first point can be proved by adapting Lemma 10 of [16], which tells us that a Poisson point process of cuts on the skeleton of a CRT keeps the total mass (we can easily extend this lemma for a CRT with mass different from 1 by a simple scaling). The only change is that we must consider  $M_\infty^k(\lambda_0)$  instead of  $\mathcal{R}(k)$  as defined in [16] and that the density of the total length of  $M_\infty^k(\lambda_0)$ , conditionally on  $s(M_\infty(\lambda_0)) = S$  is equal to :

$$f(L) = \frac{L^d}{(d-1)!} \exp\left(-\frac{1}{2}L^2\right), \quad (3.39)$$

where  $d = (3S - 1) \wedge 0 + k$ . Equation (3.39) can be deduced from Lemma 14 of [3].

Another way to prove the first point, and which will be useful for the two others, is to recall the Procedure 1 in [3]. Conditionally on  $s(M_\infty(\lambda_0)) = S$ , the measured metric space  $M_\infty(\lambda_0)$  is a CRT when  $S = 0$  and can be constructed by gluing  $I(S)$  couples of points of  $J(S)$  CRT  $(\mathcal{T}_1, \dots, \mathcal{T}_{J(S)})$  whose vector of masses is equal to  $\sigma \times \text{Dirichlet}(\frac{1}{2}, \dots, \frac{1}{2})$ , where  $I(S) = 2(S-1)\mathbf{1}_{S \geq 2} + 2\mathbf{1}_{S=1}$  and  $J(S) = 3(S-1)\mathbf{1}_{S \geq 2} + 2\mathbf{1}_{S=1}$ . Therefore, by applying directly Lemma 10 of [16] on each of these CRT, we see that the mass is conserved by the Poisson point process of cuts. Furthermore, a finite number of gluing does not affect the total mass.

Similarly, the second point derives from Procedure 1 in [3]. Remark that this second point is equivalent to the assertion "there exists an infinite sequence of components of  $M_\infty(\lambda_0, \lambda)$  with mass non-zero". Let us take  $\mathcal{T}_1$  one of the CRT described above. The components of  $\text{Frag}(\mathcal{T}_1, N_\lambda)$ , except for a finite number of them (those with a gluing point), will be in  $M_\infty(\lambda_0, \lambda)$ . By Theorem 4 of [16] these components are almost surely non-zero, and thus the second point is proved.

For the last point, we just give the main steps of the proof. The key is that if we take two independent real random variables  $X$  and  $Y$  whose laws have no atoms, then  $\mathbb{P}(X = Y) = 0$ . Furthermore, conditionally on  $s(M_\infty(\lambda_0)) = S$ , the set of masses of  $M_\infty(\lambda_0, \lambda)$  is included in the set  $\Sigma$  :

$$\Sigma = \left\{ \sum_{i=1}^k \pi_i, 1 \leq k \leq I(S), (\pi_i)_{i \in [k]} \in \Gamma^k \right\},$$

where  $\Gamma$  is the set of non-zero masses of trees obtained by the fragmentation of the  $(\mathcal{T}_i)$ . If we want to isolate some independent random variables it is more simple to consider the distribution of a size-biased random permutation of the sequence of masses of a  $\mathcal{T}_i$  rather than its distribution in decreasing order. It is given (after a scaling) by Corollary 5 of [16]. Then by considering  $j + j'$  distinct couples of integers  $(i_1, k_1), \dots, (i_{j+j'}, k_{j+j'})$  and by defining  $m_{i,k}$  as the mass of the  $k^{\text{th}}$  element of the size-biased random permutation of  $\text{Frag}(\mathcal{T}_i, N_\lambda)$ , we can show, for instance by isolating a Gaussian random variable given by

Corollary 5 of [16], that :

$$\mathbb{P}\left(m_{i_1, k_1} + \dots + m_{i_j, k_j} = m_{i_{j+1}, k_{j+1}} + \dots + m_{i_{j+j'}, k_{j+j'}}\right) = 0.$$

The universal bound  $\mathbb{P}(\cup A_i) \leq \sum \mathbb{P}(A_i)$  concludes the proof.  $\square$

Using the previous lemmas, we can now prove Proposition 3.3.2.

*Proof of Proposition 3.3.2.* Suppose that  $(X, d, \nu)$  and  $(Y, d, \mu)$  are two measured metric spaces such that  $X \subset Y$ . In order to compare the equivalence classes of  $X$  and of  $Y$  in the Gromov-Hausdorff-Prokhorov sense, it is often more convenient to compare the particular representatives  $X$  and  $Y$  in the Hausdorff sense and  $\nu$  (more exactly its natural extension to  $Y$ ) and  $\mu$  in the Prokhorov sense and next apply the inequality (3.126). We will proceed accordingly.

In this proof we have to consider the components before and after completion (for the metric  $d_C$  defined as in (3.146)). Therefore, for this proof and only for this proof,  $M_{\infty, i}(\lambda_0, \lambda)$  and  $M_{\infty, i}^k(\lambda_0, \lambda)$  will be the components before completion and  $\overline{M}_{\infty, i}(\lambda_0, \lambda)$  and  $\overline{M}_{\infty, i}^k(\lambda_0, \lambda)$  will be the same components after completion.

Fix  $i \in \mathbb{N}^*$ . Since  $\mu(M_{\infty, i}(\lambda_0, \lambda)) > 0$ , there exists  $k \in \mathbb{N}^*$  large enough, such that one of the  $(U_j)_{j \leq k}$  sampled according to  $\mu$  is in  $M_{\infty, i}(\lambda_0, \lambda)$ . Fix such a  $k$ . We consider  $\theta_k$  a random permutation of  $\mathbb{N}^*$  such that :

$$M_{\infty, \theta_k(i)}^k(\lambda_0, \lambda) \subset M_{\infty, i}(\lambda_0, \lambda).$$

We first show that :

$$\overline{M}_{\infty, \theta_k(i)}^k(\lambda_0, \lambda) \rightarrow \overline{M}_{\infty, i}(\lambda_0, \lambda) \text{ as } k \rightarrow +\infty \text{ a.s.} \quad (3.40)$$

First remark that by construction of  $M^k(\lambda_0)$ , if  $x, y \in M_{\infty, \theta_k(i)}^k(\lambda_0, \lambda)$  and if a path  $\gamma$  between  $x$  and  $y$  lies in  $M_{\infty, i}(\lambda_0, \lambda)$  then  $\gamma$  lies in  $M_{\infty, \theta_k(i)}^k(\lambda_0, \lambda)$ . Therefore, the metric on  $M_{\infty, \theta_k(i)}^k(\lambda_0, \lambda)$  is the restriction of the metric of  $M_{\infty, i}(\lambda_0, \lambda)$ . Implicitly, we work here on  $M_{\infty, i}(\lambda_0, \lambda)$  with the restriction of the metric  $d_C$  (defined in (3.146)) to  $M_{\infty, i}(\lambda_0, \lambda)$ .

Fix  $\epsilon > 0$ . Since  $\mu$  is finite and diffuse, it is also the case for  $\mu_i$  its restriction to  $M_{\infty, i}(\lambda_0, \lambda)$ . Therefore, for each  $x \in M_{\infty, i}(\lambda_0, \lambda)$ , there exists  $0 < \eta_x < \epsilon/2$  such that  $\mu_i(B(x, \eta_x)) < \epsilon/2$ . Furthermore, by Theorem 3.C.1,  $\overline{M}_{\infty, i}(\lambda_0, \lambda)$  is compact, and thus there exists a finite number  $\kappa$  of  $x_j \in M_{\infty, i}(\lambda_0, \lambda)$  such that  $M_{\infty, i}(\lambda_0, \lambda)$  is included in  $\cup_j B(x_j, \eta_{x_j})$ .

Fix  $j \in [\kappa]$  and  $y \in B(x_j, \eta_{x_j})$ . Since  $\mu(M_{\infty, i}(\lambda_0, \lambda)) \neq 0$  there exists another point  $z \in M_{\infty, i}(\lambda_0, \lambda)$  and by definition of  $M_{\infty, i}(\lambda_0, \lambda)$  there is a path  $]y, z[[$  in  $M_{\infty, i}(\lambda_0, \lambda)$ . Take  $a \in ]y, z[[$  such that  $d_C(a, y) < \eta_{x_j} - d_C(y, x_j)$ . By triangular inequality  $d_C(a, x_j) < \eta_{x_j}$ .

Furthermore,  $a$  is the skeleton. Lemma 3.3.6 yields that there exists  $k_0 \in \mathbb{N}$  such that  $a \in M_\infty^{k_0}(\lambda_0)$ . Therefore  $a \in M_{\infty, \theta_{k_0}(i)}^{k_0}(\lambda_0, \lambda) \cap B(x_j, \eta_{x_j})$ . Since  $\kappa$  is finite, there exists  $k_1 \in \mathbb{N}^*$  such that for each  $k \geq k_1$ ,

$$\delta_H(M_{\infty, i}(\lambda_0, \lambda), M_{\infty, \theta_{k_1}(i)}^{k_1}(\lambda_0, \lambda)) < \epsilon.$$

By considering that  $M_{\infty, i}(\lambda_0, \lambda)$  and  $M_{\infty, \theta_{k_1}(i)}^{k_1}(\lambda_0, \lambda)$  are embedded into a Polish space, their adherence are complete and since for two bounded sets  $A$  and  $B$ ,  $\delta_H(A, B) = \delta_H(\text{Adh}(A), \text{Adh}(B))$ , we have :

$$\delta_H(\overline{M}_{\infty, i}(\lambda_0, \lambda), \overline{M}_{\infty, \theta_{k_1}(i)}^{k_1}(\lambda_0, \lambda)) < \epsilon.$$

Consider now the partition  $(A_j)_{j \in [\kappa]}$  of  $M_{\infty, i}(\lambda_0, \lambda)$  defined recursively by  $A_1 = B(x_1, \eta_{x_1})$  and for  $j \in \{2, \dots, \kappa\}$ ,

$$A_j = [B(x_j, \eta_{x_j})] \setminus \cup_{k \in [j-1]} A_k.$$

For  $k \in \mathbb{N}^*$ , we denote by  $\mu_i^{k_2}$  the natural extension of the measure on  $M_{\infty, \theta_k(i)}^k(\lambda_0, \lambda)$  to  $M_{\infty, i}(\lambda_0, \lambda)$ . By the strong law of large numbers there exists  $k_2 > k_1$  such that for each  $j \in [\kappa]$ ,

$$|\mu_i^{k_2}(A_j) - \mu_i(A_j)| < \epsilon/\kappa.$$

Fix  $F$  a closed subset of  $M_{\infty, i}(\lambda_0, \lambda)$ . By definition of  $\eta_{x_j}$ , if  $F \cap B(x_j, \eta_{x_j}) \neq \emptyset$  then  $F^\epsilon \supset B(x_j, \eta_{x_j})$ , where  $F^\epsilon$  is defined as in (3.122). This yields  $F^\epsilon \supset A_j$ . Therefore,

$$\begin{aligned} \mu_i^{k_2}(F) &\leq \sum_{j=1}^{\kappa} \mu_i^{k_2}(A_j) \mathbf{1}_{\{F \cap B(x_j, \eta_{x_j}) \neq \emptyset\}} \\ &\leq \sum_{j=1}^{\kappa} [\mu_i(A_j) \mathbf{1}_{\{F \cap B(x_j, \eta_{x_j}) \neq \emptyset\}}] + \kappa \times \frac{\epsilon}{\kappa} \\ &\leq \mu_i(F^\epsilon) + \epsilon. \end{aligned}$$

The other inequality in (3.124) is obtained similarly, and thus

$$\delta_P(\mu_i^{k_2}, \mu_i) < \epsilon.$$

By considering the same embedding as above, we have the same inequality for the measures on the completed spaces, which concludes the proof of (3.40). We will now show that for any  $i \in \mathbb{N}^*$ , there exists  $k \in \mathbb{N}$  such that for any  $j \in [i]$ ,

$$M_{\infty, \theta_k(j)}^k(\lambda_0, \lambda) = M_{\infty, j}^k(\lambda_0, \lambda). \quad (3.41)$$

For  $i \in \mathbb{N}$ , fix  $m_i = \mu(M_{\infty,i}(\lambda_0, \lambda))$  and  $m_i^k = \mu^k(M_{\infty,\theta_k(i)}^k(\lambda_0, \lambda))$ . Fix  $i_0 \in \mathbb{N}$ . There exists  $i_1 \in \mathbb{N}$  such that :

$$\sum_{l=i_1}^{+\infty} m_l < \frac{m_{i_0}}{3}.$$

Fix  $h = \inf\{m_j - m_{j+1}, j \in [i_1]\}$ . By Lemma 3.3.7, we have  $h > 0$ . Furthermore, by strong law of large numbers, there exists  $k \in \mathbb{N}$  such that for any  $j \in [i_1]$ ,  $|m_j^k - m_j| < h/3$  and

$$\left| \sum_{l \geq i_1} [m_l^k - m_l] \right| < h/3.$$

This yields firstly that  $m_j^k > m_l^k$  if  $j, l \in [i_1]$  and  $j < l$ , and secondly that for any  $j \in [i_0]$  and  $l \geq i_1$ ,  $m_j^k > m_l^k$ . Therefore, the masses  $(m_j^k)_{j \in [i_0]}$  are the  $i_0$  first masses in decreasing order of the components of  $M_{\infty}^k(\lambda_0, \lambda)$ . Combining this result with (3.40), we have the desired result. □

### 3.3.4 Proof of Proposition 3.3.3

We now consider the set  $\mathfrak{g}$  of measured metric graph. A rigorous construction of such graphs is given in Appendix 3.B. Intuitively, a measured metric graph is just a finite graph where the edges has lengths and endowed with a structure of measured metric spaces. We recall (see Appendix 3.B) that we call edges the interior of the segments which connects two vertices. The distance between two points is then the length of a shortest path between these two points in the graph. The objects in  $\mathfrak{g}$  can be compared with the Gromov-Hausdorff-Prokhorov distance. Unfortunately, the function  $\text{Frag}$  is not continuous in its first variable for this distance. Consider for instance, the following two subsets of  $\mathbb{R}^2$  :

$$I_1 = [0, 1] \times \{0\} \text{ and } I_2(n) = ([0, 1] \times \{0\}) \cup B(1/2, 1/n). \quad (3.42)$$

The equivalence class of the metric space  $I_2(n)$  converges in the Gromov-Hausdorff-Prokhorov sense to  $I_1$ . But  $\text{Frag}(I_2(n), \delta_{1/2})$  does not converge to  $\text{Frag}(I_1, \delta_{1/2})$ . That is why we introduce a topology more appropriate in our framework. In order to simplify our proof, we will define the sets  $\mathcal{H}_0$  and  $\mathcal{H}$  by :

$$\mathcal{H}_0 = \{(G, d, \mu) \in \mathfrak{g}, \text{supp}(\mu) \subset l(G)\}, \quad (3.43)$$

$$\mathcal{H} = \{[(G, d, \mu), N], (G, d, \mu) \in \mathcal{H}_0, N \in \mathfrak{M}_{f,c}(E(G))\}, \quad (3.44)$$

where we denote by  $l(G)$  the set of leaves of  $G$ , that is the set of vertices which are connected to the others by at most one edge, and by  $E(G)$  the set of opened edges of  $G$ .

Let  $(G, d, \mu)$  and  $(G', d', \mu')$  be two elements of  $\mathcal{H}_0$ . We say that  $G$  and  $G'$  have the

same shape if :

1. They share the same set of vertices, let us say  $[n]$ ,
2. For each couple  $(i, j)$  such that  $1 \leq i \leq j \leq n$ , the number  $K(i, j)$  of opened edges between  $i$  and  $j$  in  $G$  is the same as in  $G'$ .

We further say that  $G$  and  $G'$  have the same masses if  $\mu(\{i\}) = \mu'(\{i\}), \forall i \in [n]$ .

We define for each  $G \in \mathcal{H}_0$ ,  $E_{i,j,k}(G)$  as the  $k^{th}$  opened edge in  $G$  between  $i$  and  $j$ , where  $1 \leq i \leq j \leq n$ ,  $k \in K(i, j)$ . We also denote by  $l_{i,j,k}(G)$ , the length of  $E_{i,j,k}(G)$ . We finally define for each  $x \in E_{i,j,k}(G) \cup \{i\}$ ,  $\varrho^{(1)}(x)$  (we delete the superscript when non confusions can arise) as the length of the segment  $[[i, x]]$  included in  $\text{Adh}(E_{i,j,k}(G))$  and  $\varrho^{(2)}(x) = l_{i,j,k}(G) - \varrho^{(1)}(x)$ .

We now define a distance  $\Delta$  on  $\mathcal{H}$ . Let us fix  $[(G, d, \mu), N]$  and  $[(G', d', \mu'), N']$ , two elements of  $\mathcal{H}$ . If  $G$  and  $G'$ , do not have the same shape or the same masses, then :

$$\Delta([(G, d, \mu), N], [(G', d', \mu'), N']) = +\infty.$$

Suppose now that  $[(G, d, \mu), N]$  and  $[(G', d', \mu'), N']$  have same shape and same masses. Fix  $(i, j, k) \in [n]^3$ , such that  $1 \leq i \leq j \leq n$  and  $1 \leq k \leq K(i, j)$ . In order to simplify the notations, we here put  $e(G) = E_{i,j,k}(G)$ ,  $e(G') = E_{i,j,k}(G')$ ,  $l(G) = l_{i,j,k}(G)$  and  $l(G') = l_{i,j,k}(G')$ .

— If  $i \neq j$ , define  $\Delta_{i,j,k}(G, G')$  by :

$$\Delta_{i,j,k}(G, G') = |l(G) - l(G')| \vee d_P(\varrho_* N|_{e(G)}, \varrho_* N'|_{e(G')}), \quad (3.45)$$

and for  $x \in e(G)$  and  $x' \in e(G')$  define :

$$d_{i,j,k}(x, x') = |\varrho(x) - \varrho(x')|. \quad (3.46)$$

— If now  $i = j$ , define  $\Delta_{i,i,k}(G, G')$  by :

$$\Delta_{i,i,k}(G, G') = |l(G) - l(G')| \vee \inf_{m, m' \in \{1, 2\}} d_P(\varrho_*^{(m)} N|_{e(G)}, \varrho_*^{(m')} N'|_{e(G')}) \quad (3.47)$$

and for  $x \in e(G)$  and  $x' \in e(G')$  define :

$$d_{i,i,k}(x, x') = \inf_{m, m' \in \{1, 2\}} |\varrho^{(m)}(x) - \varrho^{(m')}(x')|. \quad (3.48)$$

We can now define the distance  $\Delta$  by :

$$\Delta([(G, d, \mu), N], [(G', d', \mu'), N']) = \sup_{i,j,k} \Delta_{i,j,k}(G, G'). \quad (3.49)$$

It is straightforward to show that  $\Delta$  is well a distance. Furthermore, we can naturally define a distance  $\Delta_0$  between elements of  $\mathcal{H}_0$  by taking :

$$\Delta_0((G, d, \mu), (G', d', \mu')) = \Delta([(G, d, \mu), 0], [(G', d', \mu'), 0]),$$

where 0 is the null measure.

**Proposition 3.3.8.** *The convergence in the sense of  $\Delta_0$  implies the convergence in the sense of the Gromov-Hausdorff-Prokhorov topology.*

*Proof.* Suppose that  $\Delta_0(G, G') < \epsilon$ . This implies that  $G$  and  $G'$  have same shape and same masses. In the following, for  $(x, x') \in G \times G'$ , we will say that  $x$  and  $x'$  are on the same edge  $E_{i,j,k}$  (or on the same loop) if there are on the edges indexed by the same integers  $i, j, k$ . Hence, define the correspondence  $C$  between  $G$  and  $G'$ , by  $(x, x') \in G \times G'$  is in  $C$  if and only if one of the following condition is satisfied :

1.  $x$  and  $x'$  are the same vertex ;
2.  $x$  and  $x'$  are on the same edge (or the same loop)  $E_{i,j,k}$  and  $d_{i,j,k}(x, x') < \epsilon$ , where  $d_{i,j,k}$  is defined as in (3.46) and (3.48).

Note that  $\Delta_0(G, G') < \epsilon$ , and in particular the fact that the difference between the lengths of the edges with same index are less than  $\epsilon$  implies that  $C$  is a correspondence. Fix also  $\Pi$  the measure on  $G \times G'$  defined by :

$$\Pi = \sum_{i \in [n]} m_i \delta_{\{i\} \times \{i\}},$$

where  $m_i$  is the mass on  $i$  for  $G$  and  $G'$  (since they have the same shape). Fix now  $(x, x'), (y, y') \in C$ . Consider in  $G$  a shortest path between  $x$  and  $y$  and  $i_1, \dots, i_l$  the vertices in order of passage in this path. We have thus  $d(x, y) = d(x, i_1) + \dots + d(i_l, y)$ . The fact that  $G$  and  $G'$  have the same shape and the fact that  $x'$  and  $y'$  are in correspondence with  $x$  and  $y$  implies that there exists a path (not necessarily the shortest) between  $x'$  and  $y'$  whose vertices reached in order of passage are  $i_1, \dots, i_l$ . We thus have :  $d(x', y') \leq d(x', i_1) + \dots + d(i_l, y')$ . The fact that  $\Delta_0(G, G') < \epsilon$  and the definition of the correspondence implies :

$$d(x', y') - d(x, y) \leq (l + 1)\epsilon \leq (|E(G)| + 1)\epsilon$$

where  $|E(G)|$  is the number of edges in  $G$  and  $G'$ . Proceeding similarly by reversing the roles of  $(x, y)$  and  $(x', y')$  we get that  $\text{Dis}(C) \leq (|E(G)| + 1)\epsilon$ . We further have  $\text{Dis}(\Pi) = 0$  and  $\Pi(C^c) = 0$  and thus :

$$d_{GHP}(G, G') < \frac{(|E(G)| + 1)\epsilon}{2},$$

which concludes the proof. □

We are now interested by the operation of cutting some points. In order to prove the convergence in distribution of Proposition 3.3.3, we prove the continuity of Frag for our topology.

**Lemma 3.3.9.** *Frag :  $(\mathcal{H}, \Delta) \rightarrow (\mathcal{H}_0, \Delta_0)$  is continuous.*

*Proof.* We recall that in the definition of  $\mathcal{H}$  we choose  $N$  such that  $N$  is simple and such that its atoms are on the edges. It is simple to see that without these assumptions the function Frag is not continuous for the topologies induced by  $\Delta$  and  $\Delta_0$ . Fix  $x \in e_0 = E_{i,j,k}(G)$ ,  $1 \leq i \leq j \leq n$  and  $k \in [K(i, j)]$ . We consider  $G_x$  the graph obtained from  $G$  by deleting the edge  $e_1 = E_{i,j,k}(G)$ , by adding two new vertices  $v_1$  and  $v_2$ , two new edges  $E_{i,v_1,1}(G_x)$  and  $E_{j,v_2,1}(G_x)$  with respective lengths  $\varrho^{(1)}(x)$  and  $\varrho^{(2)}(x)$ . From  $G$  we derive naturally a distance  $d_x$  and a measure  $\mu_x$  on  $G_x$  and we have in the Gromov-Hausdorff-Prokhorov sense :

$$(G_x, d_x, \mu_x) = \text{Frag}((G, d, \mu), \delta_x).$$

Suppose now that  $[(G_n, d_n, \mu_n), N_n]$  converges to  $[(G, d, \mu), \delta_x] \in \mathcal{H}$  for the metric  $\Delta$ . This first implies that  $G_n$  and  $G$  have the same shape and the same masses for  $n$  large enough. The fact that  $N_n$  is a counting measure and the Prokhorov convergence of  $\varrho_* N_n$  to  $\varrho_* \delta_x$  implies that for  $n$  large enough  $N_n = \delta_{x_n}$ , where  $x_n \in G$  and  $x_n \rightarrow x$ , when  $n \rightarrow +\infty$ . Furthermore, for each  $(i', j', k') \neq (i, j, k)$  the fragmentation has no effects on the edges  $E_{i',j',k'}$ , and thus on the term  $\Delta_{i',j',k'}(G, G_n)$ . As an easy consequence,  $\text{Frag}((G_n, d_n, \mu_n), N_n)$  converges to  $\text{Frag}((G, d, \mu), \delta_x)$ . The general case derives from a simple induction using the fact that the counting measures are simple.  $\square$

We finally consider the operation of gluing two vertices. Let  $(G, d, \mu) \in \mathcal{H}_0$ . Gluing  $i$  and  $j$ ,  $i \leq j$  is equivalent to consider the new graph  $G_{i,j}$  obtained in the following way. First delete  $j$ . Next, for each  $k \in [n]$  replace each edge of  $G$  between  $k$  and  $j$  by an edge between  $k$  and  $i$  (if  $k = j$  replace each loop on  $j$  by a loop on  $i$ ). For the new graph the masses are unchanged for all vertices except for  $i$ , which has now the mass  $\mu(\{i, j\})$  Call  $(G_{i,j}, d_{i,j}, \mu_{i,j})$  the new measured metric graph. Consider  $(G', d', \mu') \in \mathcal{H}_0$  another graph and  $(G'_{i,j}, \mu'_{i,j}, d'_{i,j})$  define as for  $G$ , the following lemma is immediate :

**Lemma 3.3.10.** *If  $G$  and  $G'$  have same shape and same masses, then :*

$$\Delta_0((G_{i,j}, \mu_{i,j}, d_{i,j}), (G'_{i,j}, \mu'_{i,j}, d'_{i,j})) = \Delta_0((G, d, \mu), (G', d', \mu')).$$

We can now prove the Proposition 3.3.3.

*Proof of Proposition 3.3.3.* Let us begin by proving that :

$$M_n^k(\lambda_0) \xrightarrow{d} M_\infty^k(\lambda_0), \text{ as } n \rightarrow +\infty, \tag{3.50}$$



for the metric  $\Delta_0$ . Actually, although formulated differently, the main arguments of the convergence in (3.50) are in [4]. However, for self-completeness, we give the principal reasons of this convergence.

We recall that in Section 3.2.3, we have seen that  $M_n(\lambda_0)$  ( $= G_Z^p$  in Section 3.2.3) is distributed as  $G^{\mathcal{X}}(\tilde{T}_Z^p, Q_p)$ , where  $p = q_n(\lambda_0)$  and  $Z = \lfloor \sigma n^{2/3} \rfloor$  and  $\tilde{T}_Z^p$  and  $Q_p$  are defined as in Section 3.2.3. For simplicity, we will from now just write  $G^{\mathcal{X}}$  instead of  $G^{\mathcal{X}}(\tilde{T}_Z^p, Q_p)$  and similarly for  $G^H$  defined in Section 3.2.3).

Fix  $U_0^n = 1$  be the smallest-labelled vertex in  $\tilde{T}_Z^p$ . If we take uniformly and independently  $k$  vertices  $U_1^n, \dots, U_k^n$  in  $G^{\mathcal{X}}$  and if we consider the union of the core of  $G^{\mathcal{X}}$  and of the paths from the  $U_i^n$  to the core of  $G^{\mathcal{X}}$ , and if we put a mass  $\sigma/k$  to each  $U_i^n$ ,  $i \in [k]$ , we get a metric graph  $M_n^{\mathcal{X},k}(\lambda_0)$  distributed as  $M_n^k(\lambda_0)$ , where the vertices are the leaves, the  $U_i^n$  and the branching points. If now, we take the same  $U_0^n, \dots, U_k^n$  (and the same masses) but consider the union of the core of  $G^H$  and of the paths from the  $U_i^n$  to the core of  $G^H$ , we get another metric graph  $M_n^{H,k}(\lambda_0)$ .

The proof of the convergence in distribution of  $M_n^{H,k}(\lambda_0)$  to  $M_\infty^k(\lambda_0)$  for the metric  $\Delta_0$  is close to the proof of the convergence of  $M_n(\lambda_0)$  to  $M_\infty(\lambda_0)$  in Theorem 4.1 of [5]) for the Gromov-Hausdorff-Prokhorov metric. Therefore, we just give the main steps. Fix  $S^n$  the surplus of  $M_n(\lambda_0)$  and  $Q_p = \{(V_i^n, W_i^n), i \in [S^n]\}$ . By the proof of Theorem 22 in [4] and Theorem 4.1 of [5]), we have jointly :

$$S^n \xrightarrow{d} \text{Card}(\mathcal{P}) \quad (3.51)$$

$$(n^{-1/3} H^n(\lfloor n^{2/3} t \rfloor), 0 \leq t < \Sigma^n) \xrightarrow{d} (2\tilde{e}^{(\sigma)}(t), 0 \leq t < \sigma) \quad (3.52)$$

$$(n^{-2/3} V_i^n, n^{-1/3} W_i^n, i \in [S^n]) \xrightarrow{d} \mathcal{P} \quad (3.53)$$

$$n^{-2/3}(U_0^n, U_1^n, \dots, U_k^n) \xrightarrow{d} (0, U_1, \dots, U_k). \quad (3.54)$$

By Skorokhod representation theorem, we may work on a probability space where these convergences hold almost surely. Consider  $T_k^n(\lambda_0)$  the sub-graph of  $\tilde{T}_Z^p$  defined as the union of all paths connecting the root, the vertices  $U_i^n$  and the vertices  $V_i^n$ . As before, we see  $T_k^n(\lambda_0)$  as a metric graph whose vertices are the root, the vertices  $U_i^n$ , the vertices  $V_i^n$ , the vertices  $A_i^n$  at height  $2W_i^n - 1$  on  $]U_0^n, V_i^n[[$  and the branching points. Define  $T_k(\lambda_0)$  similarly for  $M_n(\lambda_0)$ . The above convergences and the fact that almost surely all the vertices are disjoint in  $T_k(\lambda_0)$  and of degree at most 3 imply straightforwardly the convergence of  $T_k^n(\lambda_0)$  to  $T_k(\lambda_0)$  for the metric  $\Delta_0$ . The number of edges added to obtain  $M_n^{H,k}(\lambda_0)$  is bounded and the length of edges is  $n^{-1/3}$ , therefore the fact to glue the vertices  $A_i^n$  and  $V_i^n$  or to add an edges between them has asymptotically no importance. Hence, Lemma

3.3.10 implies the convergence of  $M_n^{H,k}(\lambda_0)$  to  $M_\infty^k(\lambda_0)$  for the metric  $\Delta_0$ .

Finally, in order to prove (3.50), we only have to prove that :

$$\Delta_0(M_n^{H,k}(\lambda_0), M_n^{\mathcal{X},k}(\lambda_0)) \xrightarrow{d} 0. \quad (3.55)$$

For  $i \in [S^n]$ , consider  $A_i^n$  defined as above and  $B_i^n$  the  $(W_i^n)^{th}$  vertex in  $\mathcal{O}_{V_i^n}$  starting from the end as defined in Section 3.2.3. We recall that we add the edges  $V_i^n A_i^n$ ,  $i \in [S^n]$  to obtain  $M_n^{H,k}(\lambda_0)$  and the edges  $V_i^n B_i^n$ ,  $i \in [S^n]$  to obtain  $M_n^{\mathcal{X},k}(\lambda_0)$ . Let  $d_T(A_i^n, B_i^n)$  be the unscaled distance between  $A_i^n$  and  $B_i^n$  in  $\tilde{T}_Z^n$ . The proof of Lemma 20 in [4] shows us that :

$$d_T(A_i^n, B_i^n) \leq \|H - 2\mathcal{X}\| + 4. \quad (3.56)$$

Furthermore, Lemma 16 of the same article tells us there exists  $K > 0$  such that for  $n$  large enough,

$$\mathbb{P}(\|H - 2\mathcal{X}\| \geq n^{1/4}) \leq Kn^{-1/24}. \quad (3.57)$$

Equations (3.56) and (3.57) implies that :

$$d_T(A_i^n, B_i^n) \xrightarrow{d} 0 \text{ as } n \rightarrow +\infty. \quad (3.58)$$

Combining (3.58) and (3.52), we can straightforwardly show that the probability that there exists a branching point with a  $U_j^n$  or another  $V_j^n$  or another  $A_j^n$  or  $B_j^n$  on the path  $[[A_i^n, B_i^n]]$  tends to 0. This yields that the probability that the shapes of  $M_n^{H,k}(\lambda_0)$  and  $M_n^{\mathcal{X},k}(\lambda_0)$  are identical tends to 1. Furthermore, (3.58) also implies the convergence of the lengths and thus we have (3.55) which itself yields (3.50).

Finally let us look the cuts. We first recall that  $N_\lambda^n$  is simple and has no atoms on the vertices by construction. Furthermore, we consider only its restriction to  $M_n^k(\lambda_0)$  seen as a metric graph as before. Fix  $\tilde{N}_\lambda^n = \varrho_* N_\lambda^n$ , where  $\varrho$  is defined as in the beginning of this section. Consider  $e$  an edge (in the sense of measured metric graph) with length  $ln^{1/3}$  (in order to simplify the notations we delete the  $[\cdot]$  which have no impact asymptotically). Fix  $0 \leq a < b \leq l$ , such that  $b - a = c$ . The probability that there is  $k$  cuts at time  $\lambda$  knowing that this path is open at time 0 is :

$$\mathbb{P}\left(\tilde{N}_\lambda^n([an^{1/3}, bn^{1/3}]) = k\right) = \frac{\binom{cn^{1/3}}{k} \left(\frac{1}{n} - \frac{\lambda}{n^{4/3}}\right)^{cn^{1/3}-k} \frac{\lambda^k}{n^{4k/3}}}{\frac{1}{n^{cn^{1/3}}}}$$

and using Stirling's Formula, we get :

$$\mathbb{P}\left(\tilde{N}_\lambda^n([an^{1/3}, bn^{1/3}]) = k\right) \underset{n \rightarrow +\infty}{\sim} \frac{c^k \lambda^k}{k!} e^{-\lambda c}. \quad (3.59)$$

Since the  $U_{i,j}$  are independent, we have trivially the convergence in finite dimensional distributions, which is equivalent in this case to the weak convergence, which itself is equivalent to the convergence in the sense of Prokhorov, (see [39]). Therefore  $(M_n^k(\lambda_0), N_\lambda^n)$  converges in the sense of  $\Delta$  to  $(M_\infty^k(\lambda_0), N_\lambda)$ . Lemma 3.3.9 and Proposition 3.3.8 conclude the proof.  $\square$

### 3.3.5 Proof of Proposition 3.3.4

The global idea of the proof is the following. We will show that the mass of each component  $M_{n,i}(\lambda_0, \lambda)$  converges in distribution to the mass of  $M_{\infty,i}(\lambda_0, \lambda)$  and that conditionally on its mass  $m$ , a component  $K_n(\lambda)$  of  $M_n(\lambda_0, \lambda)$  is distributed as a  $M_n(\lambda)$  (where  $\sigma n^{2/3} = Z$ ). By proving that each component of  $M_n^k(\lambda_0, \lambda)$  is close to a space distributed as a  $M_n^k(\lambda)$  and that  $M_n^k(\lambda)$  is asymptotically close to  $M_n(\lambda)$ , we get the desired result.

Let us begin by proving that  $M_n^k(\lambda_0)$  is close to  $M_n(\lambda_0)$ . We recall that  $\nu_n$  is the measure on  $M_n(\lambda_0)$  and  $\nu_n^k$  the measure on  $M_n^k(\lambda_0)$ .

**Lemma 3.3.11.**  $\forall \epsilon > 0, \exists n_0, k_0 \in \mathbb{N}, \forall n \geq n_0, \forall k \geq k_0 :$

$$\mathbb{P}(\delta_H(M_n(\lambda_0), M_n^k(\lambda_0)) \geq \epsilon) \leq \epsilon. \quad (3.60)$$

and

$$\mathbb{P}(\delta_P(\nu_n^k, \nu_n) \geq \epsilon) \leq \epsilon. \quad (3.61)$$

Naturally, by Equation (3.126), we can provide the equivalent result for the Gromov-Hausdorff-Prokhorov distance. However, it will be later useful to use this stronger result.

*Proof.* We could make a direct proof, but we will rather use what we have already done. Furthermore, since the proof of (3.61) is identical to the proof of (3.60), we skip it. We recall that using Theorem 4.1 of [5]) and Skorokhod's representation theorem, we get in the proof of Proposition 3.3.3 the existence of a space such that

$$(M_n^k(\lambda_0), M_n(\lambda_0)) \rightarrow (M_\infty^k(\lambda_0), M_\infty(\lambda_0)), \text{ as } n \rightarrow +\infty \text{ a.s.} \quad (3.62)$$

in the Gromov-Hausdorff-Prokhorov sense. We now take particular representatives such that  $M_\infty^k(\lambda_0) \subset M_\infty(\lambda_0) \subset Z_\infty$  and  $M_n^k(\lambda_0) \subset M_n(\lambda_0) \subset Z_n$ , where for each  $n \in \mathbb{N}^* \cup \{\infty\}$ ,  $Z_n$  is a Polish space. Equation (3.62) provides for each  $n \in \mathbb{N}$  the existence of an isometry  $\phi_n$  from  $Z_n$  into  $Z_\infty$  such that :

$$(\phi_n(M_n^k(\lambda_0)), \phi_n(M_n(\lambda_0))) \rightarrow (M_\infty^k(\lambda_0), M_\infty(\lambda_0)), \text{ as } n \rightarrow +\infty \text{ a.s.} \quad (3.63)$$

in the Hausdorff sense. Furthermore, we have proved in Proposition 3.3.2 the almost sure

convergence :

$$M_\infty^k(\lambda_0) \rightarrow M_\infty(\lambda_0), \text{ as } k \rightarrow +\infty. \quad (3.64)$$

in the Hausdorff sense. The triangular inequality provides :

$$\begin{aligned} \delta_H(\phi_n(M_n^k(\lambda_0)), \phi_n(M_n(\lambda_0))) &\leq \delta_H(\phi_n(M_n^k(\lambda_0)), M_\infty^k(\lambda_0)) \\ &\quad + \delta_H(M_\infty^k(\lambda_0), M_\infty(\lambda_0)) \\ &\quad + \delta_H(M_\infty(\lambda_0), \phi_n(M_n(\lambda_0))) \end{aligned} \quad (3.65)$$

Since  $\delta_H(\phi_n(M_n^k(\lambda_0)), \phi_n(M_n(\lambda_0))) = \delta_H(M_n^k(\lambda_0), M_n(\lambda_0))$ , combining Equations (3.62), (3.64) and (3.65) we get for a realisation  $\omega$  on the probability space given by the Skorokhod representation theorem, the existence of  $N(\omega), K(\omega) \in \mathbb{N}$  such that for all  $n(\omega) \geq N(\omega)$ ,  $k(\omega) \geq K(\omega)$  we have :

$$\delta_H(M_{n(\omega)}(\lambda_0), M_{n(\omega)}^{k(\omega)}(\lambda_0)) \leq \epsilon.$$

By taking  $n_0$  and  $k_0$  large enough such that  $A = \{N(\omega) \geq n_0, K(\omega) \geq k_0\}$  satisfies  $\mathbb{P}(A) \leq \epsilon$ , we get the desired result.  $\square$

We now show that, conditionally to their sizes, the connected components of  $M_n(\lambda_0, \lambda)$  have the same structure as a  $M_n(\lambda)$  in the following sense. As before  $\lambda \geq \lambda_0$ . Consider  $K_n(\lambda)$  a connected component of  $G(n, q_n(\lambda))$  and  $J_n(\lambda)$  its size. Fix  $K_n(\lambda_0)$  the connected component of  $G(n, q_n(\lambda_0))$  which contains the vertices of  $K_n(\lambda)$ , and call  $J_n(\lambda_0)$  its size. Then we have :

**Lemma 3.3.12.** *For  $m, m' \in [n]$ , such that  $m \leq m'$ , we have :*

$$\mathbb{P}(K_n(\lambda) = G | J_n(\lambda) = m, J_n(\lambda_0) = m') = \mathbb{P}(K_n(\lambda) = G | J_n(\lambda) = m). \quad (3.66)$$

*Proof.* Fix  $P = \mathbb{P}(K_n(\lambda) = G | J_n(\lambda) = m, J_n(\lambda_0) = m')$ .

$$\begin{aligned} P &= \frac{\mathbb{P}(K_n(\lambda) = G, J_n(\lambda) = m, J_n(\lambda_0) = m')}{\mathbb{P}(J_n(\lambda) = m, J_n(\lambda_0) = m')} \\ &= \frac{\mathbb{P}(J_n(\lambda_0) = m' | K_n(\lambda) = G, J_n(\lambda) = m) \times \mathbb{P}(K_n(\lambda) = G, J_n(\lambda) = m)}{\mathbb{P}(J_n(\lambda_0) = m' | J_n(\lambda) = m) \times \mathbb{P}(J_n(\lambda) = m)} \end{aligned}$$

The law of the number of vertices in  $K_n(\lambda_0)$  knowing  $K_n(\lambda)$  depends uniquely on the number of vertices of  $K_n(\lambda)$ , therefore :

$$\begin{aligned} P &= \frac{\mathbb{P}(J_n(\lambda_0) = m' | J_n(\lambda) = m) \times \mathbb{P}(K_n(\lambda) = G, J_n(\lambda) = m)}{\mathbb{P}(J_n(\lambda_0) = m' | J_n(\lambda) = m) \times \mathbb{P}(J_n(\lambda) = m)} \\ &= \mathbb{P}(K_n(\lambda) = G | J_n(\lambda) = m). \end{aligned}$$

$\square$

Let  $(m_{n,i}(\lambda))_i$ ,  $(m_{\infty,i}(\lambda))_i$ ,  $(m_{n,i}^k(\lambda))_i$  and  $(m_{\infty,i}^k(\lambda))_i$  be the sequence of masses of  $M_n(\lambda_0, \lambda)$ ,  $M_\infty(\lambda_0, \lambda)$ ,  $M_n^k(\lambda_0, \lambda)$  and  $M_\infty^k(\lambda_0, \lambda)$  each ordered in non-increasing order of size.

**Lemma 3.3.13.** *We have in the sense of the product topology :*

$$(m_{n,i}(\lambda), i \in \mathbb{N}^*) \xrightarrow{d} (m_{\infty,i}(\lambda), i \in \mathbb{N}^*) \quad (3.67)$$

*Proof.* Define  $p_{n,i}(\lambda)$  and  $\tilde{p}_{n,i}^k(\lambda)$  as

$$p_{n,i}(\lambda) = \frac{m_{n,i}(\lambda)}{\sum_j m_{n,j}(\lambda)} \text{ and } \tilde{p}_{n,i}^k(\lambda) = \frac{\tilde{m}_{n,i}^k(\lambda)}{\sum_j \tilde{m}_{n,j}^k(\lambda)},$$

where

$$\tilde{m}_{n,i}^k(\lambda) = \frac{\sigma}{k} \text{Card}(\{U_j^n \in M_{n,i}(\lambda_0, \lambda), j \in [k]\}),$$

which corresponds to the mass of the component of  $M_n^k(\lambda_0, \lambda)$  included in  $M_{n,i}(\lambda_0, \lambda)$ . Conditionally to  $p_{n,i}(\lambda)$ , we have that :

$$\tilde{p}_{n,i}^k(\lambda) = \frac{\sum_{j=1}^k X_j}{k},$$

where the  $X_j$  are i.i.d. Bernoulli random variables of parameter  $p_{n,i}(\lambda)$ . Therefore, using Tchebychev inequality we get :

$$\begin{aligned} \mathbb{P}(|\tilde{p}_{n,i}^k(\lambda) - p_{n,i}(\lambda)| \geq x | p_{n,i}(\lambda)) &\leq \frac{p_{n,i}(\lambda)(1 - p_{n,i}(\lambda))}{x^2 k} \\ \mathbb{P}(|\tilde{p}_{n,i}^k(\lambda) - p_{n,i}(\lambda)| \geq x) &\leq \frac{\mathbb{E}(p_{n,i}(\lambda))}{x^2 k} \\ \mathbb{P}(\exists i \in \mathbb{N}^*, |\tilde{p}_{n,i}^k(\lambda) - p_{n,i}(\lambda)| \geq x) &\leq \frac{1}{x^2 k}. \end{aligned} \quad (3.68)$$

Consider now  $\underline{m}_{n,i}^k(\lambda)$  the mass of the component of  $M_n(\lambda_0, \lambda)$  which contains the  $i^{\text{th}}$  component of  $M_n^k(\lambda_0, \lambda)$ , and :

$$\underline{p}_{n,i}^k(\lambda) = \frac{\underline{m}_{n,i}^k(\lambda)}{\sum_j \underline{m}_{n,j}^k(\lambda)} \text{ and } p_{n,i}^k(\lambda) = \frac{m_{n,i}^k(\lambda)}{\sum_j m_{n,j}^k(\lambda)}.$$

By a simple reordering and using Equation (3.68), we get :

$$\mathbb{P}(\exists i \in \mathbb{N}^*, |\underline{p}_{n,i}^k(\lambda) - p_{n,i}^k(\lambda)| \geq x) \leq \frac{1}{x^2 k}. \quad (3.69)$$

Proposition 3.3.3 implies :

$$(p_{n,i}^k(\lambda), i \in \mathbb{N}^*) \xrightarrow{d} (p_{\infty,i}^k(\lambda), i \in \mathbb{N}^*) \text{ as } n \rightarrow +\infty. \quad (3.70)$$

Furthermore, Proposition 3.3.2 yields :

$$(p_{\infty,i}^k(\lambda), i \in \mathbb{N}^*) \rightarrow (p_{\infty,i}(\lambda), i \in \mathbb{N}^*) \text{ as } k \rightarrow +\infty. \quad (3.71)$$

Combining Equations (3.69), (3.70) and (3.71), we get easily :

$$(\underline{p}_{n,i}^k(\lambda), i \in \mathbb{N}^*) \xrightarrow{d} (p_{\infty,i}(\lambda), i \in \mathbb{N}^*) \text{ as } k, n \rightarrow +\infty. \quad (3.72)$$

Proceeding as for the proof of Proposition 3.3.2, we get that for any  $i_0$  and for  $n$  and  $k$  large enough :

$$\mathbb{P}(\exists i \in [i_0], \underline{p}_{n,i}^k(\lambda) \neq p_{n,i}(\lambda)) \leq \epsilon. \quad (3.73)$$

Equations (3.72) and (3.73) yield the convergence for  $(p_{n,i}(\lambda))$  and by rescaling  $(p_{n,i}(\lambda))$  by  $\lfloor n^{2/3}\sigma \rfloor n^{-2/3}$ , we conclude the proof.  $\square$

We can now prove the Proposition 3.3.4.

*Proof of Proposition 3.3.4.* Fix  $i \in \mathbb{N}^*$  and  $\tilde{M}_{n,i}^k(\lambda_0, \lambda)$  the component of  $M_n^k(\lambda_0, \lambda)$  included in  $M_{n,i}(\lambda_0, \lambda)$ . Consider also  $\tilde{\mathbf{M}}_{n,i}^k(\lambda_0, \lambda)$  the measured metric space obtained from  $\tilde{M}_{n,i}^k(\lambda_0, \lambda)$  by replacing each mass  $\sigma/k$ , by a mass  $\tilde{\mathbf{m}}_{n,i}^k(\lambda) = m_{n,i}(\lambda)/k'$ , where  $k' = kp_{n,i}^k(\lambda)$ . Lemma 3.3.12 tells us that conditionally to its mass  $m_{i,n}(\lambda)$ , the distribution of  $M_{n,i}(\lambda_0, \lambda)$  is the same as the distribution of a  $M_n(\lambda)$  (conditioned to have the same mass). Moreover, let  $R_{n,i}^k(\lambda_0, \lambda)$  be the measured metric space defined as the union of all the paths in  $M_{n,i}(\lambda_0, \lambda)$  connecting the  $U_j^n$  in  $M_{n,i}(\lambda_0, \lambda)$  to the core of  $M_{n,i}(\lambda_0, \lambda)$  and with mass  $\tilde{\mathbf{m}}_{n,i}^k(\lambda)$  on each of such a  $U_j^n$ . Hence,  $R_{n,i}^k(\lambda_0, \lambda)$  is distributed as a  $M_n^k(\lambda)$ . Furthermore,  $R_{n,i}^k(\lambda_0, \lambda) \subset \tilde{\mathbf{M}}_{n,i}^k(\lambda_0, \lambda)$  and the (extension of the) measure on  $R_{n,i}^k(\lambda_0, \lambda)$  is equal to the measure  $\eta_{n,i}^k$  on  $\tilde{\mathbf{M}}_{n,i}^k(\lambda_0, \lambda)$ . It is straightforward to show that if  $A \subset B \subset C$ , then :

$$\delta_H(B, C) \leq \delta_H(A, C).$$

Furthermore, we know thanks to Lemma 3.3.13 and Lemma 3.3.7 that  $m_{i,n}(\lambda)$  converges in distribution to a positive random variable. Therefore, by Lemma 3.3.11, for  $k$  and  $n$  large enough, we get :

$$\mathbb{P}(\delta_P(\eta_{n,i}^k, \nu_{n,i}) \geq \epsilon) \leq \epsilon, \quad (3.74)$$

where  $\nu_{n,i}$  is the measure on  $M_{n,i}(\lambda_0, \lambda)$  and :

$$\mathbb{P}(\delta_H(\tilde{M}_{n,i}^k(\lambda_0, \lambda), M_{n,i}(\lambda_0, \lambda)) \geq \epsilon) \leq \epsilon, \quad (3.75)$$

Moreover, Lemma 3.3.13 implies that for  $n$  and  $k$  large enough :

$$\mathbb{P}(\delta_P(\eta_{n,i}^k, \tilde{\nu}_{n,i}^k) \geq \epsilon) \leq \epsilon, \quad (3.76)$$

where  $\tilde{\nu}_{n,i}^k$  is the measure on  $\tilde{M}_{n,i}^k(\lambda_0, \lambda)$ . Finally, proceeding as for the proof of Proposition 3.3.2, we can show that for  $n$  and  $k$  large enough :

$$\mathbb{P}(\tilde{M}_{n,i}^k(\lambda_0, \lambda) \neq M_{n,i}^k(\lambda_0, \lambda)) \leq \epsilon. \quad (3.77)$$

Combining (3.74), (3.75), (3.76) and (3.77), we get the desired result.  $\square$

### 3.3.6 Proof of Theorem 3.1.1

The proof of Proposition 3.3.1 is immediate combining Propositions 3.3.2, 3.3.3 and 3.3.4. Furthermore, the theorem of convergence in [5] (see (3.22)) yields that (non-jointly) :

$$F(n, -\lambda_0) \xrightarrow{d} \hat{\mathbb{M}}(\lambda_0), \text{ as } n \rightarrow +\infty \quad (3.78)$$

and

$$F(n, \lambda - \lambda_0) \xrightarrow{d} \hat{\mathbb{M}}(\lambda_0 - \lambda) \text{ as } n \rightarrow +\infty, \quad (3.79)$$

for the topology induced by  $\text{dist}_{GHP}^4$  on  $\mathbb{L}_4$ . For  $i \in \mathbb{N}^*$ , call  $F_i(n, \lambda)$  the  $i^{\text{th}}$  component of  $F(n, \lambda)$  and call  $z$  the measured metric space  $(\{0\}, 0, 0)$ . Fix also  $\epsilon > 0$ . Equations (3.78) and (3.79) imply that there exists  $i_0 \in \mathbb{N}^*$  such that with probability greater than  $1 - \epsilon$ , we jointly have :

$$\sum_{i=i_0}^{+\infty} d_{GHP}^4(F_i(n, -\lambda_0), z) < \epsilon \text{ and } \sum_{i=i_0}^{+\infty} d_{GHP}^4(F_i(n, \lambda - \lambda_0), z) < \epsilon. \quad (3.80)$$

Define  $F^k(n, -\lambda_0)$  as the sequence of the  $k$  first components in non-increasing order of size of  $F(n, -\lambda_0)$  and  $\tilde{F}^k(n, \lambda - \lambda_0)$  the sequence (reordered in non-increasing order) of components of  $F(n, \lambda - \lambda_0)$  whose vertices belong to the components of  $F^k(n, -\lambda_0)$ . We write for simplicity  $\mathcal{M}_i(\lambda_0)$  instead of  $(\mathcal{M}_i(\lambda_0), d_i, \mu_i)$ . By Proposition 3.3.1, we have as  $n \rightarrow +\infty$  :

$$(F^k(n, -\lambda_0), \tilde{F}^k(n, \lambda - \lambda_0)) \xrightarrow{d} ((\mathcal{M}_i(\lambda_0))_{i \in [k]}, \text{Frag}(\mathbb{M}_{[k]}(\lambda_0), N_\lambda)). \quad (3.81)$$

It is straightforward to show that there exists  $k$  large enough such that  $\tilde{F}^k(n, \lambda - \lambda_0)$  contains all elements of  $F_{i_0}^k(n, \lambda - \lambda_0)$  with probability greater than  $1 - \epsilon$  and we conclude easily with Equations (3.80) and (3.81) that :

$$(F(n, -\lambda_0), F(n, \lambda - \lambda_0)) \xrightarrow{d} (\hat{\mathbb{M}}(\lambda_0), \text{Frag}(\mathbb{M}(\lambda_0), N_\lambda)). \quad (3.82)$$

Finally, the fact that  $(F(n, \lambda - \lambda_0))_{\lambda \geq 0}$  and  $(\text{Frag}((\mathcal{M}_i(\lambda_0))_{i \in \mathbb{N}^*}, N_\lambda))_{\lambda \geq 0}$  are Markov processes yields Theorem 3.1.1 in the sense of finite-dimensional distributions.

### 3.4 The coalescent process

We will prove in this section Theorem 3.1.2. We fix the notations for this section. Let  $\lambda \in \mathbb{R}$ . As before  $p_n(\lambda) = 1/n + \lambda n^{-4/3}$ . We recall that we denote by  $m_i^n(\lambda)$  the mass of  $\mathcal{C}_i^n(\lambda)$  (that is  $n^{-2/3}Z_i^{(n)}(\lambda)$ ) and by  $m_{\infty,i}(\lambda)$  the mass of  $\mathcal{M}_i(\lambda)$ . As in introduction, we fix  $\mathbb{M}(\lambda) = \cup_{i \geq 1} \mathcal{M}_i(\lambda)$ . We also set :

$$\mathcal{C}_i^n(\lambda) = \bigcup_{1 \leq k \leq i} \mathcal{C}_k^n(\lambda) \text{ and } \mathbb{M}_i(\lambda) = \bigcup_{1 \leq k \leq i} \mathcal{M}_k(\lambda). \quad (3.83)$$

Consider  $E^n(\lambda)$  the set of opened edges in  $G(n, p_n(\lambda))$  and  $\bar{E}^n(\lambda)$  the set of closed edges of  $G(n, p_n(\lambda))$ . For  $\lambda_0 \leq \lambda$ , we define  $P_{\lambda_0, \lambda}^n$  by :

$$P_{\lambda_0, \lambda}^n = \sum_{uv \in \bar{E}^n(\lambda_0), u \leq v} [B_{uv} \delta_{(u,v)} + (1 - B_{uv}) \delta_{(v,u)}] \mathbf{1}_{uv \in E^n(\lambda)}, \quad (3.84)$$

where the  $B_{uv}$  are i.i.d Bernoulli random variables with parameter  $1/2$ .

In what follows, if  $Q$  is defined on  $Y$  and that  $X \subset Y$ , we will consider that  $\text{Coal}(X, Q) = \text{Coal}(X, \tilde{Q})$ , where  $\tilde{Q}$  is the restriction of  $Q$  to  $X$ .

Fix  $\mathcal{U}_i^n(\lambda_0, \lambda) = \text{Coal}(\mathcal{C}_i^n(\lambda_0), P_{\lambda_0, \lambda}^n)$ . Set  $\mathcal{U}_{i,k}^n(\lambda_0, \lambda)$  the  $k^{\text{th}}$  component of  $\mathcal{U}_i^n(\lambda_0, \lambda)$  and  $\mathbb{U}_{i,k}^n(\lambda_0, \lambda) = \cup_{1 \leq j \leq k} \mathcal{U}_{i,j}^n(\lambda_0, \lambda)$ .

Consider now  $P$  a Poisson process on  $\mathbb{R}^2 \times \mathbb{M}(\lambda_0) \times \mathbb{M}(\lambda_0)$  with intensity  $(1/2)\mathcal{L}^2 \otimes \mu \otimes \mu$ . We define  $P_{\lambda_0, \lambda}$  by  $P_{\lambda_0, \lambda}(A) = P([\lambda_0, \lambda] \times A)$ , for all  $A \in \mathcal{B}(\mathbb{M}(\lambda_0) \times \mathbb{M}(\lambda_0))$ .

Fix  $\mathcal{U}_i(\lambda_0, \lambda) = \text{Coal}(\mathbb{M}_i(\lambda_0), P_{\lambda_0, \lambda})$ . Set  $\mathcal{U}_{i,k}(\lambda_0, \lambda)$  the  $k^{\text{th}}$  component of  $\mathcal{U}_i(\lambda_0, \lambda)$  and  $\mathbb{U}_{i,k}(\lambda_0, \lambda) = \cup_{1 \leq j \leq k} \mathcal{U}_{i,j}(\lambda_0, \lambda)$ . Finally, define  $\mathcal{H}(\lambda_0, \lambda) = \text{Coal}(\mathbb{M}(\lambda_0), P_{\lambda_0, \lambda})$ .

Like in the previous Section, since the results of convergence are easier to prove for finite sets, we divide this proof in three parts. As a first step, we show the following proposition.

**Proposition 3.4.1.** *As  $n \rightarrow +\infty$ ,*

$$\mathcal{U}_i^n(\lambda_0, \lambda) \xrightarrow{d} \mathcal{U}_i(\lambda_0, \lambda). \quad (3.85)$$

Next, we will prove the following.

**Proposition 3.4.2.**

$$\lim_{i \rightarrow +\infty} d_{GHP}(\mathcal{H}_k(\lambda_0, \lambda), \mathcal{U}_{i,k}(\lambda_0, \lambda)) = 0. \quad (3.86)$$

Finally, we state the following result.



**Proposition 3.4.3.** *For any  $k \in \mathbb{N}^*$  and  $\epsilon > 0$ , there exist  $i_0, n_0 \in \mathbb{N}^*$  such that for all  $i \geq i_0$ ,  $n \geq n_0$  :*

$$\mathbb{P} \left( d_{GHP} \left( \mathcal{C}_k^n(\lambda), \mathcal{U}_{i,k}^n(\lambda_0, \lambda) \right) > \epsilon \right) < \epsilon. \quad (3.87)$$

### 3.4.1 Proof of Proposition 3.4.1

Fix  $i \in \mathbb{N}^*$ . Let  $(X_j, \mu_j, d_j)_{1 \leq j \leq i}$  and  $(X_j^n, \mu_j^n, d_j^n)_{1 \leq j \leq i, n \in \mathbb{N}}$  be compact measured metric length spaces. In order to avoid useless complications, we assume that each sums of masses of  $(X_j, \mu_j, d_j)_{1 \leq j \leq i}$  are distinct, that is for  $A, B \in \mathcal{P}([i])$ ,

$$\sum_{j \in A} \mu_j(X_j) = \sum_{j \in B} \mu_j(X_j) \Rightarrow A = B.$$

Fix also, for  $k, n \in \mathbb{N}^*$ ,  $(x_l, y_l)_{1 \leq l \leq k}$  (resp.  $(x_j^n, y_j^n)_{1 \leq j \leq k}$ ) be couples of distinct points of  $\cup X_j$  (resp.  $\cup X_j^n$ ). For  $j \in [i]$ , we denote by  $z_j$  (resp.  $z_j^n$ ) the vector of points of the form  $x_l$  or  $y_l$  which belong to  $X_j$  (resp.  $X_j^n$ ). We finally call  $(\hat{X}_j, \hat{\mu}_j, \hat{d}_j)$  (resp.  $(\hat{X}_j^n, \hat{\mu}_j^n, \hat{d}_j^n)$ ) the vector  $\text{Coal}((X_j, \mu_j, d_j)_{j \in [i]}, \sum_{l \in [k]} \delta_{(x_l, y_l)})$  (resp.  $\text{Coal}((X_j^n, \mu_j^n, d_j^n)_{j \in [i]}, \sum_{l \in [k]} \delta_{(x_l^n, y_l^n)})$ ).

**Lemma 3.4.4.** *If for each  $j \in [i]$ ,*

$$d_{pGHP}[(X_j, \mu_j, d_j, z_j), (X_j^n, \mu_j^n, d_j^n, z_j^n)] \rightarrow 0, \text{ as } n \rightarrow +\infty, \quad (3.88)$$

*then :*

$$(\hat{X}_j^n, \hat{\mu}_j^n, \hat{d}_j^n) \rightarrow (\hat{X}_j, \hat{\mu}_j, \hat{d}_j), \text{ as } n \rightarrow +\infty$$

*in the sense that each component converges for the topology induced by  $d_{GHP}$ .*

*Proof.* This proof is close to the proof of Lemma 4.2. in [5]. We start by the case  $k = 1$ . Suppose first that  $x_1$  and  $y_1$  are in the same component  $X_{j_0}$ . For each  $j \in [i]$  such that  $j \neq j_0$ , Equation (3.88) implies obviously the convergence of  $\hat{X}_j^n$  to  $\hat{X}_j$ . Suppose now that  $j = j_0$ . Fix  $\epsilon > 0$  and take  $n$  large enough such that :

$$d_{pGHP}[(X_j, \mu_j, d_j, (x_1, y_1)), (X_j^n, \mu_j^n, d_j^n, (x_1^n, y_1^n))] < \epsilon. \quad (3.89)$$

There exists by definition a correspondence  $C$  between  $X_j$  and  $X_j^n$  and a measure  $\pi$  on  $X_j \times X_j^n$  such that :  $\text{Dis}(C) < \epsilon$ ,  $\pi(C^c) < \epsilon$ ,  $D(\pi, \mu_j, \mu_j^n) < \epsilon$  and  $(x_1, x_1^n), (y_1, y_1^n) \in C$ . Consider  $p$  (resp.  $p^n$ ) the natural projection from  $(X_j)$  (resp.  $(X_j^n)$ ) to  $(\hat{X}_j)$  (resp.  $(\hat{X}_j^n)$ ). The subset  $\hat{C}$  of  $\hat{X}_j \times \hat{X}_j^n$  defined by :

$$\hat{C} := \{(p(x), p^n(y)), (x, y) \in C\}$$

is a correspondence between  $\hat{X}_j$  and  $\hat{X}_j^n$ . Define further the measure  $\hat{\pi} := (p, p^n)_* \pi$ . By

definition of  $\hat{C}$ , we have :

$$\begin{aligned}\hat{C}^c &= \{(\hat{x}, \hat{y}) \in \hat{X}_j \times \hat{X}_j^n, (\hat{x}, \hat{y}) \text{ has no antecedent in } C\} \\ &\subset \{(p(x), p^n(y)), (x, y) \in C^c\}.\end{aligned}$$

Therefore,  $\hat{\pi}(\hat{C}^c) < \epsilon$ . Similarly, let  $k_0$  be an integer,  $(A_l)_{l \in [k_0]}$  be a measurable partition of  $\hat{X}_j$  and  $q$  (resp.  $\hat{q}$ ) be the projection from  $X_j \times X_j^n$  (resp.  $\hat{X}_j \times \hat{X}_j^n$ ) to  $X_j$  (resp.  $\hat{X}_j$ ).

We have :

$$\begin{aligned}\sum_{l=1}^{k_0} |\hat{q}_* \hat{\pi}(A_l) - \hat{\mu}_j(A_l)| &= \sum_{l=1}^{k_0} |q_* \pi(p^{-1}(A_l)) - \mu_j(p^{-1}(A_l))| \\ &\leq \|q_* \pi - \mu_j\|\end{aligned}$$

which implies  $\|\hat{q}_* \hat{\pi} - \hat{\mu}_j\| \leq \|q_* \pi - \mu_j\|$  and proceeding similarly for the other part in the definition of the discrepancy, we get

$$D(\hat{\pi}, \hat{\mu}_j, \hat{\mu}_j^n) \leq D(\pi, \mu_j, \mu_j^n) < \epsilon.$$

Let  $a_1, b_1 \in \hat{X}_j$  and  $a_2, b_2 \in \hat{X}_j^n$  such that  $(a_1, a_2), (b_1, b_2) \in \hat{C}$ . By symmetry of the problem, we can also suppose that :

$$|\hat{d}(a_1, b_1) - \hat{d}^n(a_2, b_2)| = \hat{d}(a_1, b_1) - \hat{d}^n(a_2, b_2).$$

If  $a_1, b_1 \neq p(x_1)$  and  $a_2, b_2 \neq p^n(x_1^n)$  then  $a_1$  and  $b_1$  (resp.  $a_2$  and  $b_2$ ) each have an only antecedent by  $p$  (resp.  $p^n$ ), which we call  $a'_1$  and  $b'_1$  (resp.  $a'_2$  and  $b'_2$ ). By definition of  $\hat{d}$ , we have :  $\hat{d}(a_1, b_1) = d(a'_1, b'_1)$  or  $\hat{d}(a_1, b_1) = d(a'_1, x_1) + d(y_1, b'_1)$  or  $\hat{d}(a_1, b_1) = d(a'_1, y_1) + d(x_1, b'_1)$  and so does for  $\hat{d}^n(a_2, b_2)$ . By symmetries, we can reduce the problem to five cases :

1. If  $\hat{d}(a_1, b_1) = d(a'_1, b'_1)$  and  $\hat{d}^n(a_2, b_2) = d^n(a'_2, b'_2)$ , then :

$$\hat{d}(a_1, b_1) - \hat{d}^n(a_2, b_2) = d(a'_1, b'_1) - d^n(a'_2, b'_2) < \epsilon.$$

2. If  $\hat{d}(a_1, b_1) = d(a'_1, x_1) + d(y_1, b'_1)$  and  $\hat{d}^n(a_2, b_2) = d^n(a'_2, b'_2)$ , then :

$$\hat{d}(a_1, b_1) - \hat{d}^n(a_2, b_2) \leq d(a'_1, b'_1) - d^n(a'_2, b'_2) < \epsilon.$$

3. If  $\hat{d}(a_1, b_1) = d(a'_1, b'_1)$  and  $\hat{d}^n(a_2, b_2) = d^n(a'_2, x_1^n) + d^n(y_1^n, b'_2)$ , then :

$$\hat{d}(a_1, b_1) - \hat{d}^n(a_2, b_2) \leq d(a'_1, x_1) + d(y_1, b'_1) - [d^n(a'_2, x_1^n) + d^n(y_1^n, b'_2)] < 2\epsilon.$$

4. If  $\hat{d}(a_1, b_1) = d(a'_1, x_1) + d(y_1, b'_1)$  and  $\hat{d}^n(a_2, b_2) = d^n(a'_2, x_1^n) + d^n(y_1^n, b'_2)$ , then :

$$\hat{d}(a_1, b_1) - \hat{d}^n(a_2, b_2) = d(a'_1, x_1) + d(y_1, b'_1) - [d^n(a'_2, x_1^n) + d^n(y_1^n, b'_2)] < 2\epsilon.$$

5. If  $\hat{d}(a_1, b_1) = d(a'_1, x_1) + d(y_1, b'_1)$  and  $\hat{d}^n(a_2, b_2) = d^n(a'_2, y_1^n) + d^n(x_1^n, b'_2)$ , then :

$$\hat{d}(a_1, b_1) - \hat{d}^n(a_2, b_2) \leq d(a'_1, x_1) + d(y_1, b'_1) - [d^n(a'_2, x_1^n) + d^n(y_1^n, b'_2)] < 2\epsilon.$$

The cases where  $a_1 = p(x_1)$  or  $b_1 = p(x_1)$  or  $a_2 = p^n(x_1^n)$  or  $b_2 = p^n(x_1^n)$  are similar and thus we have  $\text{Dis}(\hat{C}) < 2\epsilon$ , which concludes the proof in the case where  $x_1$  and  $y_1$  are in the same component.

Suppose now that  $x_1 \in X_{j_1}$  and  $y_1 \in X_{j_2}$  with  $j_1 \neq j_2$ . The proof is quite similar and thus we skip some details. We call  $\hat{X}_{j_3}$  the component obtained by gluing  $X_{j_1}$  and  $X_{j_2}$ . As before, for  $j \neq j_1, j_2$  the components  $X_j$  are unchanged (but their order can change). For  $n$  large enough, for  $j = j_1$  and  $j = j_2$ , Equation (3.89) is satisfied. For  $l = 1, 2$ , fix  $C_l$  a correspondence between  $X_{j_l}$  and  $X_{j_l}^n$  and a measure  $\pi_l$  on  $X_{j_l} \times X_{j_l}^n$  such that :  $\text{Dis}(C_l) < \epsilon$ ,  $\pi(C_l^c) < \epsilon$ ,  $D(\pi_l, \mu_{j_l}, \mu_{j_l}^n) < \epsilon$  and  $(x_1, x_1^n) \in C_1$  and  $(y_1, y_1^n) \in C_2$ . Fix also  $p_l$  (resp  $p_l^n$ ) the projection from  $X_{j_l}$  to  $\hat{X}_{j_3}$  (resp. from  $X_{j_l}^n$  to  $\hat{X}_{j_3}^n$ ). We define the correspondence  $\hat{C}_3$  between  $\hat{X}_{j_3}$  and  $\hat{X}_{j_3}^n$  by :

$$\hat{C}_3 = \{(p_1(x), p_1^n(y)), (x, y) \in C_1\} \cup \{(p_2(x), p_2^n(y)), (x, y) \in C_2\}$$

and the measure  $\hat{\pi}_3$  on  $\hat{X}_{j_3} \times \hat{X}_{j_3}^n$  by :

$$\hat{\pi}_3 = (p_1, p_1^n)_* \pi_1 + (p_2, p_2^n)_* \pi_2.$$

Proceeding as before, we get  $\text{Dis}(\hat{C}_3) < 2\epsilon$ ,  $\hat{\pi}_3(\hat{C}_3^c) < 2\epsilon$  and  $D(\hat{\pi}_3, \hat{\mu}_{j_3}, \hat{\mu}_{j_3}^n) < 2\epsilon$ .

We thus proved the result for  $k = 1$ . The general case follows by immediate induction using the fact that :

$$\text{Coal}((X_j, \mu_j, d_j)_{j \in [i]}, \sum_{l \in [k]} \delta_{(x_l, y_l)}) = \text{Coal}((\tilde{X}_j, \tilde{\mu}_j, \tilde{d}_j)_{j \in [i]}, \sum_{2 \leq l \leq k} \delta_{(\hat{x}_l, \hat{y}_l)}),$$

where  $(\tilde{X}_j, \tilde{\mu}_j, \tilde{d}_j)$  is the sequence of spaces obtained by gluing  $x_1$  and  $y_1$ . □

We can now prove the Proposition 3.4.1.

*Proof of Proposition 3.4.1.* We first relabel the vertices of  $G(n, p_n(\lambda_0))$ . We take the greatest component and give to it the labelling associated to the ordered depth-first search. We next take the second greatest component and start the labelling associated to the ordered

depth-first search by  $Z_1^{(n)}(\lambda_0) + 1$ . We proceed similarly for the others components and define  $H$  the height function for this new ordering. As in introduction, we consider the sequence of excursions of the reflected Brownian motion with parabolic drift  $(\tilde{e}_i(\lambda_0))_{i \geq 1}$  and the sequence of their length  $(m_{\infty,i}(\lambda_0))_{i \geq 1}$ . Define  $h$  in the following way :

$$h(s) = \tilde{e}_{I(s)}(\lambda_0) \left( s - \sum_{j=1}^{I(s)-1} m_{\infty,j}(\lambda_0) \right),$$

where  $I(s) = \inf\{i \in \mathbb{N}, \sum_{j=1}^i m_{\infty,j}(\lambda_0) > s\}$  and  $\sum_{j=1}^0 m_{\infty,j}(\lambda_0) = 0$ . Consider also  $\mathcal{P}$  a Poisson point process on  $\mathbb{R}^2$  with intensity  $\mathcal{L}^2$ . We recall that by the proof of Theorem 22 in [4] and Theorem 4.1 of [5]), we have jointly :

$$(n^{-1/3} H^n(\lfloor n^{2/3} t \rfloor)) \xrightarrow{d} h(t) \tag{3.90}$$

and the convergence of the points which provide the surplus. By Skorokhod representation theorem, we may work on a probability space where these convergences hold almost surely.

Fix  $H_{n,i}$  and  $H_i$  defined by :

$$H_{n,i} := \left[ 0, \sum_{k=1}^i m_{n,k}(\lambda_0) \right]^2 \quad \text{and} \quad H_i := \left[ 0, \sum_{k=1}^i m_{\infty,k}(\lambda_0) \right]^2.$$

Consider  $\tilde{P}_{\lambda_0, \lambda}^{n,i}$  the point process on  $H_{n,i}$  defined by :

$$\tilde{P}_{\lambda_0, \lambda}^{n,i} = \sum_{uv \in \bar{E}^n(\lambda_0), u \leq v} [B_{uv} \delta_{(n^{-2/3}u, n^{-2/3}v)} + (1 - B_{uv}) \delta_{(n^{-2/3}v, n^{-2/3}u)}] \mathbf{1}_{uv \in E^n(\lambda)}, \tag{3.91}$$

where the  $B_{uv}$  is Bernouilli i.i.d. random variables with parameter 1/2. Consider also  $\tilde{P}_{\lambda_0, \lambda}^i$  a Poisson point process on  $H_i$  with intensity  $1/2(\lambda - \lambda_0)\mathcal{L}^2$ , where  $\mathcal{L}^2$  is the Lebesgue measure on  $\mathbb{R}^2$ . We know (see (3.20)) that :

$$\lim_{n \rightarrow +\infty} \sum_{k=1}^i m_{n,k}(\lambda_0) = \sum_{k=1}^i m_{\infty,k}(\lambda_0). \tag{3.92}$$

The probability that an edge  $ij$  is opened at time  $\lambda$  knowing that it is closed at time  $\lambda_0$  is :

$$p_{ij}(\lambda_0, \lambda) := \frac{\mathbb{P}(\frac{1}{n} + \frac{\lambda_0}{n^{4/3}} \leq U_{i,j} < \frac{1}{n} + \frac{\lambda}{n^{4/3}})}{\mathbb{P}(\frac{1}{n} + \frac{\lambda_0}{n^{4/3}} \leq U_{i,j})} \underset{n \rightarrow +\infty}{\sim} \frac{\lambda - \lambda_0}{n^{4/3}}.$$

Furthermore, consider  $A = (a, b) \times (c, d)$  a subset of  $H_i$ , it is straightforward to show that :

$$\lim_{n \rightarrow +\infty} \mathbb{P}(\tilde{P}_{\lambda_0, \lambda}^n(A) = k) = \frac{((1/2)(\lambda - \lambda_0)(b - a)(d - c))^k}{k!} e^{-(1/2)(\lambda - \lambda_0)(b - a)(d - c)}. \tag{3.93}$$

We can easily extend this result to union and intersection of squares and then get by the independence of the  $U_{i,j}$  the convergence of  $\tilde{P}_{\lambda_0,\lambda}^{n,i}$  to  $\tilde{P}_{\lambda_0,\lambda}^i$  as  $n \rightarrow +\infty$  in the sense of finite-dimensional distributions. Furthermore,  $\mathcal{L}^2(H_i)$  is finite and thus the convergence holds in the weak sense (see [39]). Consider  $p$  the canonical projection from  $\mathbb{R} \setminus \{\sum_{i=1}^j m_{\infty,i}(\lambda_0), j \in \mathbb{N}^*\}$  to  $\mathbb{M}(\lambda_0)$ . Since  $\mu(\text{Skel}(\mathbb{M}_i)) = 0$ , it is simple to see that the projection of  $\tilde{P}_{\lambda_0,\lambda}^i$  by  $p$  is a Poisson point process on  $\mathbb{M}(\lambda_0) \times \mathbb{M}(\lambda_0)$  with intensity  $(1/2)(\lambda - \lambda_0)[\mu \otimes \mu]$ . Furthermore, for  $j \in [i]$ , let  $S_j$  be the vector of elements  $p(x)$ , such that  $x \in [\sum_{k=1}^{j-1} m_{\infty,k}(\lambda_0), \sum_{k=1}^j m_{\infty,k}(\lambda_0))$  and such that there exists  $y \in \mathbb{R}^+$  such that  $(x, y)$  or  $(y, x)$  is an atom of  $\tilde{P}_{\lambda_0,\lambda}^i$  and  $S_j^n$  defined similarly for  $\tilde{P}_{\lambda_0,\lambda}^{n,i}$ . The previous convergences yield for each  $j \in [i]$  :

$$d_{pGHP}((C_j^n(\lambda_0), S_j^n), (\mathcal{M}_j(\lambda_0), S_j)) \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

And we conclude by using Lemma 3.4.4. □

### 3.4.2 Proof of Proposition 3.4.2

We will in this section show that  $\mathcal{U}_i(\lambda_0, \lambda)$  is close to  $\mathcal{H}(\lambda_0, \lambda)$  for  $i$  large enough in the sense of finite dimensional distribution.

Set  $I$  a subset of  $\mathbb{N}^*$ ,  $\mathbb{M}_I(\lambda_0) = (\mathcal{M}_j(\lambda_0))_{j \in I}$  and  $\mathbf{m}_I(\lambda_0) = (m_j(\lambda_0))_{j \in I}$ . Consider the graph  $\mathcal{W}(\mathbf{m}_I(\lambda_0), \lambda - \lambda_0)$  on  $\mathbb{N}^*$  defined in the following way. Fix :

$$A_{j_1, j_2} = \{(u, v), u \in \mathcal{M}_{j_1}(\lambda_0), v \in \mathcal{M}_{j_2}(\lambda_0) \text{ or } v \in \mathcal{M}_{j_1}(\lambda_0), u \in \mathcal{M}_{j_2}(\lambda_0)\}.$$

For each  $j_1, j_2 \in I$  such that  $j_1 \neq j_2$ , the edge between  $j_1$  and  $j_2$  is open if and only  $P_{\lambda_0,\lambda}(A_{j_1, j_2}) \neq 0$ . In other terms, knowing  $\mathbf{m}_I(\lambda_0)$  the edge between  $j_1$  and  $j_2$  is open at time  $\lambda - \lambda_0$  with probability  $1 - e^{-(\lambda - \lambda_0)m_{j_1}(\lambda_0)m_{j_2}(\lambda_0)}$ . Notice that the graph  $\mathcal{W}(\mathbf{m}_{\mathbb{N}^*}(\lambda_0), \lambda - \lambda_0)$  describes which components of  $\mathbb{M}(\lambda_0)$  are connected in  $\mathcal{H}(\lambda_0, \lambda)$ . Furthermore, let us call  $X(\mathbf{m}_I(\lambda_0), \lambda - \lambda_0)$  the vector of masses (in non-increasing order) of the different connected components of  $\mathcal{W}(\mathbf{m}_I(\lambda_0), \lambda - \lambda_0)$ . We know (see Aldous [15]) that  $(X(\mathbf{m}_I(\lambda_0), \lambda - \lambda_0, \lambda \geq \lambda_0))$  is the standard multiplicative coalescent. Hence, we know that the distribution of masses of  $\mathcal{H}(\lambda_0, \lambda)$  is in  $l_2^1$  and that :

$$X(\mathbf{m}_{[i]}(\lambda_0), \lambda - \lambda_0) \rightarrow X(\mathbf{m}_{\mathbb{N}^*}(\lambda_0), \lambda - \lambda_0), \text{ as } i \rightarrow +\infty, \quad (3.94)$$

for the metric  $\|\cdot\|_2$ . Hence, we essentially have to prove the Hausdorff part of the convergence in (3.86). As a first step let us show that each component of  $\mathcal{H}(\lambda_0, \lambda)$  is compact.

**Proposition 3.4.5.** *Each component of  $\mathcal{H}(\lambda_0, \lambda)$  is compact.*

It is straightforward to show that gluing a finite number of compact spaces along a finite

number of couple of points yields a sequence of compact spaces. Indeed, the operator  $\text{Coal}$  provides by definition the completeness. Moreover, the precompactness of each component before gluing yields the precompactness of each component after gluing. In the general case, gluing a infinite number of compact spaces does not yield the compactness of the components after gluing. For instance, the sequence of segments  $[n, n + 1]$ ,  $n \in \mathbb{N}$ , glued at their endpoints in the obvious way yields  $\mathbb{R}$ . Fortunately, in our case, the particular geometry of the problem will give us the compactness of the different components.

Our strategy is to construct a sequence of vector of metric measured spaces  $[K_l^i(n)]_{l \in i, n \in \mathbb{N}}$  from  $\mathbb{M}_i(\lambda_0)$  by gluing successively some spaces to them. This sequence will converge to a vector of compact metric measured space  $[K_l^i(\infty)]_{l \in N(i)}$ . Finally, we will prove that a finite number of gluing between the  $[K_l^i(\infty)]_{l \in [i]}$  yields the first components of  $\mathcal{H}(\lambda_0, \lambda)$ .

We define by induction on  $(n, l)$  in the lexicographical order  $(\leq_l)$  on  $\mathbb{N} \times [i]$  a sequence of vector of subsets of  $\mathbb{N}^*$ ,  $[J_l^i(n)]_{l \in [i], n \in \mathbb{N}^*}$ , in the following way. Put  $[J_l^i(0)]_{l \in [i]} = (\{j\})_{j \in [i]}$ . Suppose that for all  $(n, l) \leq_l (n_0, l_0)$ , where  $(n_0, l_0) \geq_l (0, i)$ ,  $J_l^i(n)$  is defined and call  $(n_1, l_1)$  the immediate successor of  $(n_0, l_0)$ . We define  $J_{l_1}^i(n_1)$  by

$$J_{l_1}^i(n_1) = \{j \in \mathbb{N} \setminus \bigcup_{(n,l) <_l (n_1, l_1)} J_l^i(n), P_{\lambda_0, \lambda}(A_{j,k}) \neq 0, k \in J_{l_1}^i(n_1 - 1)\}.$$

The sequence  $[J_l^i(n)]_{l \in i, n \in \mathbb{N}^*}$  hence defined, we define  $K_l^i(n)$  for each  $(n, l) \in \mathbb{N} \times [i]$  by :

$$K_l^i(n) = \text{Coal} \left( \mathbb{M}_{L_l^i(n)}, P_{\lambda_0, \lambda} \right),$$

where  $L_l^i(n) = \cup_{0 \leq n \leq n_1} J_{l_1}^i(n)$ .

Roughly speaking,  $K_l^i(n)$  has been constructed after  $n$  generations of gluing.  $L_l^i(n)$  is the set of indices of components  $\mathcal{M}_k(\lambda_0)$  which have been glued in order to produce  $K_l^i(n)$  and  $J_l^i(k)$  for  $k \leq n$  can be seen as the set of indices of such components glued at the generation  $k$ . We thus construct  $K_l^i(n + 1)$  from  $K_l^i(n)$  by gluing all components which are connected to the  $n^{\text{th}}$  generation of  $K_l^i(n)$  and which have not been already glued. By commodity, we will often say that a component  $A$  contains a component  $B$  when  $A$  is obtained by gluing several components to  $B$ . However, this must not be considered as an inclusion in the mathematical sense.

**Lemma 3.4.6.** *For each  $(l, n) \in [i] \times \mathbb{N}$ ,  $K_l^i(n)$  is compact.*

*Proof.* Gluing a finite number of points in a compact space yields a compact space and thus  $K_l^i(0)$  is compact. Suppose now that  $K_l^i(n)$  is compact. The space  $K_l^i(n + 1)$  is complete by definition of the operator  $\text{Coal}$ . Furthermore, take  $\eta > 0$ . Since  $K_l^i(n)$  is compact there

exists a finite number  $n_1$  of points  $x_k \in K_l^i(n)$  such that  $K_l^i(n) \subset \cup_{k \in [n_1]} B(x_k, \eta)$ . Define  $Pr$  as the projection from  $K_l^i(n)$  to  $K_l^i(n+1)$  and for  $k \in \mathbb{N}$ ,  $d_k$  the distance on  $K_l^i(k)$ . By definition of the new distance in a glued space (see (3.141)), we have :

$$d_n(x, y) \geq d_{n+1}(Pr(x), Pr(y)), \forall x, y \in K_l^i(n). \quad (3.95)$$

Therefore,  $Pr(K_l^i(n)) \subset \cup_{k \in [n_1]} B(Pr(x_k), \eta)$ . It is straightforward to show that this implies  $[Pr(K_l^i(n))]^\eta \subset \cup_{k \in [n_1]} B(Pr(x_k), 2\eta)$ , where  $[K_l^i(n)]^\eta$  is defined in (3.122). Moreover, the number of components  $\mathcal{M}_k(\lambda_0)$  with diameter greater than  $\eta$  is finite. Therefore, for each component added to  $K_l^i(n)$  with diameter greater than  $\eta$  we can put a ball with radius  $2\eta$ . Using again (3.95), we get that  $K_l^i(n+1)$  is recovered by a finite number of balls with radius  $2\eta$ , for each  $\eta > 0$  and thus is precompact, which concludes the proof for the compactness of  $K_l^i(n+1)$ .  $\square$

We define the  $\sigma$ -algebra  $\mathcal{G} := \sigma(m_j(\lambda_0), j \in \mathbb{N}^*)$  and the filtration  $(\mathcal{F}^{n,l})$  by

$$\mathcal{F}^{n,l} = \mathcal{G} \vee \sigma(J_{l'}^i(n'), (n', l') \leq_l (n, l)).$$

Throughout this section, we fix  $0 < \epsilon < 1$  and take  $i$  large enough such that  $(\lambda - \lambda_0) \sum_{l \geq i} m_l^2(\lambda_0) < \epsilon$  and  $m_i(\lambda_0) < 1$ . Call  $\pi_{n,l}$  the mass of the generation  $n$  in  $K_{n,l}^i$  defined by :

$$\pi_{n,l} = \sum_{j \in J_l^i(n)} m_j(\lambda_0).$$

**Lemma 3.4.7.** *Fix  $\epsilon_1 > 0$  such that  $\epsilon < \epsilon_1 < 1$ . We have :*

$$\mathbb{P} \left( \limsup_{n \rightarrow +\infty} \{\pi_{n,l} \geq \epsilon_1^n\} \mid \mathcal{G} \right) = 0.$$

*Proof.* The components of  $K_l^i(n+1)$  of the  $(n+1)^{th}$  generation are necessarily connected to the components of the  $n^{th}$  and not to the components of the previous generations, else, by construction, they would be already in  $K_{n,l}^i$ . Moreover, these new components are chosen among the components with index greater than  $i$ . Therefore,

$$\begin{aligned} \mathbb{E}(\pi_{n+1,l} \mid \mathcal{F}^{n,l}) &\leq \sum_{k \geq i+1} \sum_{j \in J_l^i(n)} m_k(\lambda_0) \left( 1 - e^{-(\lambda - \lambda_0)m_k(\lambda_0)m_j(\lambda_0)} \right) \\ &\leq \sum_{k \geq i+1} (\lambda - \lambda_0) m_k^2(\lambda_0) \pi_{n,l} \\ &\leq \epsilon \pi_{n,l} \end{aligned}$$

This naturally yields :

$$\mathbb{E}(\pi_{n,l} \mid \mathcal{G}) \leq \epsilon^n m_l(\lambda_0). \quad (3.96)$$

By Markov inequality, we have :

$$\mathbb{P}(\pi_{n,l} \geq \epsilon_1^n | \mathcal{G}) \leq \left( \frac{\epsilon}{\epsilon_1} \right)^n m_l(\lambda_0). \quad (3.97)$$

And Borel-Cantelli Lemma concludes the proof.  $\square$

Let us call  $S_{n,l}^i$  the surplus of  $K_l^i(n)$  and let us show that the sequence  $(S_{n,l}^i)$  is bounded.

**Lemma 3.4.8.** *The sequence  $(S_{n,l}^i)$  is almost surely bounded.*

*Proof.* At generation 0,  $S_{0,l}^i$  is equal to the sum of the surplus of  $\mathcal{M}_l(\lambda_0)$  and of  $P_{\lambda_0,\lambda}(\mathcal{M}_l(\lambda_0) \times \mathcal{M}_l(\lambda_0))$ . We already know that the first term is finite, and the second is also finite since the mass of  $\mathcal{M}_l(\lambda_0)$  is finite. Now, let us look  $S_{n+1,l}^i - S_{n,l}^i$ . There are three way to have new cycles in  $K_l^i(n+1)$  with respect to  $K_l^i(n)$ . The first one is to have cycles inside an added component. The second one is to have a component of the  $(n+1)^{th}$  generation which have at least two connections with the components of the  $n^{th}$  and the third is to have components of the  $(n+1)^{th}$  generation which are directly connected. For  $k \in \{1, 2, 3\}$ , let us call  $A_k(n)$  the number of new cycles of type  $k$ . Let us start by  $A_1(n)$ .

$$\begin{aligned} A_1(n) &= \sum_{j \in J_l^i(n+1)} [S(\mathcal{M}_j(\lambda_0)) + P_{\lambda_0,\lambda}(\mathcal{M}_j(\lambda_0) \times \mathcal{M}_j(\lambda_0))] \\ \mathbb{E}(A_1(n) | \mathcal{F}^{n,l}) &= \sum_{j \in J_l^i(n+1)} \left[ \mathbb{E} \left[ \int_0^{m_j(\lambda_0)} \tilde{e}^{m_j(\lambda_0)}(s) ds \right] + (\lambda - \lambda_0) \frac{m_j^2(\lambda_0)}{2} \right]. \end{aligned} \quad (3.98)$$

The first term in the sum is equal to :

$$\begin{aligned} \mathbb{E} \left[ \int_0^{m_j(\lambda_0)} \tilde{e}^{m_j(\lambda_0)}(s) ds \right] &= \frac{\mathbb{E} \left[ m_j^{3/2}(\lambda_0) \int_0^1 e(s) ds \exp \left( m_j^{3/2}(\lambda_0) \int_0^1 e(s) ds \right) \right]}{\mathbb{E} \left[ \exp \left( m_j^{3/2}(\lambda_0) \int_0^1 e(s) ds \right) \right]} \\ &\leq m_j(\lambda_0) \mathbb{E} \left[ \int_0^1 e(s) ds \exp \left( \int_0^1 e(s) ds \right) \right], \end{aligned}$$

where as in introduction  $e$  denotes a standard Brownian excursion. The previous inequality comes from  $m_j \leq m_i < 1$ . Therefore, we have :

$$\begin{aligned} \sum_{n=0}^{+\infty} \mathbb{E}(A_1(n) | \mathcal{F}^{0,l}) &\leq \sum_{n=0}^{+\infty} \mathbb{E}(\pi_{n,l} | \mathcal{F}^{0,l}) \mathbb{E} \left[ \int_0^1 e(s) ds \exp \left( \int_0^1 e(s) ds \right) \right] + \frac{\epsilon}{2} \\ &\leq \frac{1}{1-\epsilon} m_l(\lambda_0) \mathbb{E} \left[ \int_0^1 e(s) ds \exp \left( \int_0^1 e(s) ds \right) \right] + \frac{\epsilon}{2} < \infty. \end{aligned}$$

Hence, the total number of cycles of first type is finite.



We now deal with  $A_2(n)$ . We work in what follows conditionally to  $\mathcal{F}^{\infty,l} = \bigvee_{n \in \mathbb{N}} \mathcal{F}^{n,l}$ . Take  $k \in J_l^i(n+1)$ . The number of edges  $E_{k,n,l}$  between  $K_l^i(n)$  and the component indexed by  $k$  follows a Poisson law with intensity  $(\lambda - \lambda_0)\pi_{n,l}m_k(\lambda_0)$  conditioned to be greater than 1. Therefore, we have :

$$\begin{aligned} \mathbb{E}(A_2(n)|\mathcal{F}^{\infty,l}) &= \sum_{k \in J_l^i(n+1)} \mathbb{E}(E_{k,n,l} - 1 | E_{k,n,l} \geq 1) \\ &= \sum_{k \in J_l^i(n+1)} e^{(\lambda - \lambda_0)m_k(\lambda_0)\pi_{n,l}} - 1 \end{aligned}$$

Since the sequences  $(\pi_{n,l})$  and  $(m_k(\lambda_0))$  are bounded, there exists  $C > 0$  such that :

$$\begin{aligned} \mathbb{E}\left(A_2(n)|\mathcal{F}^{\infty,l}\right) &\leq \sum_{k \in J_l^i(n+1)} C(\lambda - \lambda_0)m_k(\lambda_0)\pi_{n,l} \\ \mathbb{E}\left(\sum_{n=0}^{+\infty} A_2(n)|\mathcal{F}^{\infty,l}\right) &\leq \sum_{n=0}^{+\infty} Ce(\lambda - \lambda_0)\pi_{n+1,l}\pi_{n,l}. \end{aligned}$$

By Cauchy-Schwarz inequality and the fact that  $l^1 \subset l^2$ , we have straightforwardly that the right-hand side of the last inequality is finite. Therefore  $\sum_{n=0}^{+\infty} A_2(n) < +\infty$ . The treatment of  $A_3(n)$  is very similar. Let us call  $E_{k_1,k_2,n,l}$  the edges between the components indexed by  $k_1$  and  $k_2$ , when  $k_1, k_2 \in J_l^i(n+1)$ .

$$\begin{aligned} \mathbb{E}(A_3(n)|\mathcal{F}^{\infty,l}) &= \sum_{k_1 \in J_l^i(n+1)} \sum_{k_2 \in J_l^i(n+1), k_2 \neq k_1} \mathbb{E}(E_{k_1,k_2,n,l}) \\ &= \sum_{k_1 \in J_l^i(n+1)} \sum_{k_2 \in J_l^i(n+1), k_2 \neq k_1} m_{k_1}(\lambda_0)m_{k_2}(\lambda_0)(\lambda - \lambda_0) \\ &\leq \pi_{n+1,l}^2(\lambda - \lambda_0), \end{aligned}$$

which similarly implies that  $\sum_{n=0}^{+\infty} A_3(n) < +\infty$ . By combining the previous results, we thus have that  $(S_{n,l}^i)_n$  is bounded.  $\square$

Call  $K_l^i(\infty) = \text{Coal}\left(\mathbb{M}_{L_l^i}, P_{\lambda_0,\lambda}\right)$ , where  $L_l^i = \bigcup_{n \in \mathbb{N}} J_l^i(n)$ .

**Lemma 3.4.9.** *The measured metric space  $K_l^i(\infty)$  is compact and :*

$$\lim_{n \rightarrow +\infty} d_{GHP}(K_l^i(n), K_l^i(\infty)) = 0. \quad (3.99)$$

*Proof.* Fix  $\mathcal{H} = \mathcal{F}^{\infty,l} \vee \sigma(S_{n,l}^i, n \in \mathbb{N}, l \in i)$ . Take  $n_0$  large enough such that there is no more cycles and such that for any  $n \geq n_0$ ,  $\pi_{n,l} \geq \epsilon_1^n$ , with  $\epsilon_1$  taken as in Lemma 3.4.7. For  $n \geq n_0$  and  $k \in J_l^i(n)$ , set  $D_k$  as the diameter of  $\mathcal{M}_k(\lambda_0)$ . Since there is no surplus in  $\mathcal{M}_k(\lambda_0)$ ,  $\mathcal{M}_k(\lambda_0)$  can be encoded by an excursion with length  $m_k(\lambda_0)$  of a standard Brownian motion. If  $e_*^{m_k(\lambda_0)}$  is the maximum of this excursion, we know that

$D_k \leq 2e_*^{m_k(\lambda_0)}$ . Consider now  $e_*$  the maximum of a standard excursion and fix  $\epsilon_2 > 0$  such that  $\epsilon_1 < \epsilon_2 < 1$ . We have :

$$\begin{aligned} \mathbb{P}\left(D_k \geq (\epsilon_2)^{n/2} | \mathcal{H}\right) &\leq \mathbb{P}\left(2e_*^{m_k(\lambda_0)} \geq (\epsilon_2)^{n/2} | \mathcal{H}\right) \\ &\leq \mathbb{P}\left(e_* \geq \frac{(\epsilon_2)^{n/2}}{2\sqrt{m_k(\lambda_0)}} \middle| \mathcal{H}\right). \end{aligned} \quad (3.100)$$

Theorem 1 of [66] yields the exact distribution of  $e_*$  :

$$\mathbb{P}(e_* \geq x) = 2 \sum_{j=1}^{+\infty} (4x^2 j^2 - 1) e^{-2x^2 j^2}. \quad (3.101)$$

A simple calculation yields the existence of  $y_0 > 0$ , such that  $2(4y^2 - 1) \exp(-2y^2) \leq y^{-2}$  for any  $y \geq y_0$ . Furthermore,

$$\sqrt{m_k(\lambda_0)} \leq \sqrt{\pi_{n,l}} \leq (\epsilon_1)^{n/2}.$$

Therefore, there exists  $n_1 \geq n_0$  such that, for any  $n \geq n_1$  and  $k \in J_l^i(n)$ , we have :

$$\frac{(\epsilon_2)^{n/2}}{2\sqrt{m_k(\lambda_0)}} \geq y_0.$$

This implies :

$$\begin{aligned} \mathbb{P}\left(D_k \geq (\epsilon_2)^{n/2} | \mathcal{H}\right) &\leq \frac{4m_k(\lambda_0)}{\epsilon_2^n} \sum_{j=1}^{+\infty} \frac{1}{j^2} \\ \sum_{k \in J_l^i(n)} \mathbb{P}\left(D_k \geq (\epsilon_2)^{n/2} | \mathcal{H}\right) &\leq \frac{2\pi^2 \epsilon_1^n}{3\epsilon_2^n} \\ \sum_{n \geq n_1} \sum_{k \in J_l^i(n)} \mathbb{P}\left(D_k \geq (\epsilon_2)^{n/2} | \mathcal{H}\right) &< \infty. \end{aligned}$$

Hence, by Borel-Cantelli Lemma there exists  $n_2 \geq n_1$  such that for any  $n \geq n_2$ ,  $k \in J_l^i(n)$  :

$$D_k \leq (\epsilon_2)^{n/2}. \quad (3.102)$$

Fix now  $\eta > 0$  and take  $n_3 > n_2$  such that :

$$\frac{\epsilon_2^{n_3/2}}{1 - \epsilon_2} < \eta.$$

Since  $K_l^i(n_3)$  is compact we can take a finite number  $k$  of points  $(x_j)_{j \in [k]}$  such that :  $K_l^i(n_3) \subset_{j \in [k]} B(x_j, \eta)$ . Since there is no more surplus in  $K_l^i(\infty)$  than in  $K_l^i(n_3)$ ,  $K_l^i(n_3)$

can be isometrically injected into  $K_l^i(\infty)$  (see Proposition 3.B.3). The inequality (3.102) and the definition of  $n_3$  yield that  $K_l^i(\infty)$  is recovered by the balls  $B(x_j, \eta)$  and thus is precompact. Since  $K_l^i(\infty)$  is also complete, it is compact. Furthermore, since for any  $\epsilon > 0$  there exists  $n_4$  such that for any  $n \geq n_4$ , we have :

$$\sum_{k \geq n} D_k < \epsilon \text{ and } \sum_{k \geq n} \pi_{k,l} < \epsilon$$

we have :  $\delta_{GHP}(K_l^i(\infty), K_l^i(n)) < \epsilon$  and thus Equation (3.99) is proved.  $\square$

We need now to make the gluing between the  $K_l^i(\infty)$  in order to obtain the components of  $\mathcal{H}(\lambda_0, \lambda) = \text{Coal}(\mathbb{M}(\lambda_0), P_{\lambda_0, \lambda})$  which contain the  $i$  first components  $\mathcal{M}_j(\lambda_0)$ . Set  $L^i = \cup_{l=1}^i L_l^i$ , let  $p$  be the projection from  $\mathbb{M}_{L^i}(\lambda_0)$  to  $\cup_{l \in [i]} K_l^i(\infty)$  and fix  $\tilde{P}_{\lambda_0, \lambda} = \sum \delta_{(p(u), p(v))}$ , such that the couples  $(u, v)$  are the atoms of  $P_{\lambda_0, \lambda}$  whose elements are in  $\mathbb{M}_{L^i}(\lambda_0)$ . Fix  $(C_k^i)_{k \in [k_0]} := \text{Coal}(K_l^i(\infty)_{l \in [i]}, \tilde{P}_{\lambda_0, \lambda})$ . The  $C_k^i$  are the components of  $\text{Coal}(\mathbb{M}(\lambda_0), P_{\lambda_0, \lambda})$  which contain the  $i$  first components  $\mathcal{M}_j(\lambda_0)$ . Furthermore, they are compact as proved in the following lemma.

**Lemma 3.4.10.** *For each  $k \in [k_0]$ ,  $C_k^i$  is compact.*

*Proof.* By construction of the  $K_l^i(\infty)$ , we only have to add the edges between the  $K_l^i(\infty)$ . By the properties of independence of the Poisson point process, and conditionally to the masses  $x_{i,l}$  of the components  $K_l^i(\infty)$ , the expected number  $N$  of edges between the  $K_l^i(\infty)$  is :

$$N = (\lambda_0 - \lambda) \sum_{1 \leq j < k \leq k_0} x_{i,j} x_{i,k}.$$

Since this number is finite, the sequence  $(C_k^i)$  is obtained by gluing a finite number of compact spaces along a finite number of points and thus is a sequence of compact spaces.  $\square$

Since each component of  $\mathcal{H}(\lambda_0, \lambda)$  can be seen as a  $C_k^i$  by taking  $i$  large enough, we thus have proved Proposition 3.4.5. Let us prove Proposition 3.4.2. Call  $\mathbb{U}_{i,k}(\lambda_0, \lambda) = \cup_{1 \leq j \leq k} \mathcal{U}_{i,j}(\lambda_0, \lambda)$  and  $\mathbb{U}_k(\lambda_0, \lambda) = \cup_{1 \leq j \leq k} \mathcal{H}_j(\lambda_0, \lambda)$

*Proof of Proposition 3.4.2.* Fix  $\epsilon > 0$ . Since each  $\mathcal{H}_j(\lambda_0, \lambda)$  is compact, we can take a finite number  $N$  of  $x_l \in \mathbb{U}_k(\lambda_0, \lambda)$  such that the balls  $B(x_l, \epsilon)$  recover  $\mathbb{U}_k(\lambda_0, \lambda)$ . The paths between the  $x_l$  cross a finite number of components and so do the cycles (which are in finite number). Put  $I$  the set of indices of these components and set  $i_0 = \max I$ . Fix  $(m_j^i(\lambda_0, \lambda))_{j \in [i]}$  (resp.  $(m_j(\lambda_0, \lambda))_j$ ) the vector of masses of the components of  $\mathcal{U}_i(\lambda_0, \lambda)$  (resp.  $\mathcal{H}(\lambda_0, \lambda)$ ). Since we know that  $(m_j^i(\lambda_0, \lambda))_{j \in [i]}$  converges to  $(m_j(\lambda_0, \lambda))_j$  (see (3.94)) for the metric  $\|\cdot\|_2$ , there exists  $i_1 \geq i_0$  such that for each  $i \geq i_1$  and  $j \in [k]$ ,

$$|m_j^i(\lambda_0, \lambda) - m_j(\lambda_0, \lambda)| < \epsilon. \quad (3.103)$$

Since all the cycles in  $\mathbb{U}_k(\lambda_0, \lambda)$  are in  $\mathbb{U}_{i_1, k}(\lambda_0, \lambda)$ , we can isometrically inject  $\mathbb{U}_{i_1, k}(\lambda_0, \lambda)$  into  $\mathbb{U}_k(\lambda_0, \lambda)$ . The fact that the  $x_j$  are in  $\mathbb{U}_{i_1, k}(\lambda_0, \lambda)$  and Equation (3.103) imply for each  $j \in [k]$  :

$$\delta_{GHP}(\mathcal{H}_j(\lambda_0, \lambda), \mathcal{U}_{i_1, j}(\lambda_0, \lambda)) < \epsilon,$$

which concludes the proof.  $\square$

### 3.4.3 Proof of Proposition 3.4.3

Proposition 3.4.1 yields that the number of edges added between the vertices of the  $i$  first components of  $G(n, p_n(\lambda_0))$  between the times  $\lambda_0$  and  $\lambda$  is  $\mathcal{O}_n(1)$  in probability. Since the lengths of the edges are  $n^{-1/3}$ , we thus have for any  $\epsilon > 0$  and  $k, i \in \mathbb{N}^*$ ,  $k \leq i$  :

$$\lim_{n \rightarrow +\infty} \mathbb{P} \left( d_{GHP} \left( \tilde{\mathcal{U}}_{i, k}^n(\lambda_0, \lambda), \mathcal{U}_{i, k}^n(\lambda_0, \lambda) \right) > \epsilon \right) = 0, \quad (3.104)$$

where  $\tilde{\mathcal{U}}_i^n(\lambda_0, \lambda)$  is the vector of connected components obtained from  $\mathbb{C}_i^n(\lambda_0)$  by adding the edges  $jl$ , where  $j, l$  are vertices of  $\mathbb{C}_i^n(\lambda_0)$ , such that

$$n^{-1} + \lambda_0 n^{-4/3} \leq U_{jl} \leq n^{-1} + \lambda n^{-4/3}.$$

In other terms, it is asymptotically equivalent to glue the vertices of  $\mathbb{C}_i^n(\lambda_0)$  or to add an edge between them.

*Proof.* In order to prove that :

$$\mathbb{P} \left( d_{GHP} \left( \mathbb{C}_k^n(\lambda), \mathcal{U}_{i, k}^n(\lambda_0, \lambda) \right) > \epsilon \right) < \epsilon, \quad (3.105)$$

we will rely on what we have already done in the previous section. We keep the same notations as in the beginning of the previous section (except for the roles of  $\lambda$  and  $\lambda_0$  which are reversed). Roughly speaking, our approach is first to show that there exists  $k_0 \in \mathbb{N}$  such that a component  $M_n(\lambda)$  is close to the object obtained by gluing the  $k_0$  first components obtained after fragmentation of  $M_n(\lambda)$ . We have seen that for  $k_1$  large enough we have :

$$d_{GHP} \left( M_\infty^{k_1}(\lambda), M_\infty(\lambda) \right) < \epsilon. \quad (3.106)$$

Since the total length of  $M_\infty^{k_1}(\lambda)$  is finite, the number of cuts of the Poisson point process between time  $\lambda$  and  $\lambda_0$  on it is also finite and thus the number of components of  $M_\infty^{k_1}(\lambda, \lambda_0)$  is also finite. Take  $i_0$  the biggest index of a component of  $M_\infty(\lambda, \lambda_0)$  which contains a component of  $M_\infty^{k_1}(\lambda, \lambda_0)$ . For any  $\epsilon > 0$ , we can take  $k_2 \geq k_1$  such that for each  $i \in [i_0]$ ,  $M_{\infty, i}^{k_2}(\lambda, \lambda_0) \subset M_{\infty, i}(\lambda, \lambda_0)$ ,  $d_{GHP}(M_{\infty, i}^{k_2}(\lambda, \lambda_0), M_{\infty, i}(\lambda, \lambda_0)) < \epsilon/2^{K+i_0}$ , where  $K$  is the number of cuts on  $M_\infty^{k_1}(\lambda)$ , and the mass  $m_{\infty, i}^{k_2}(\lambda_0, \lambda)$  is distinct from the others masses. We have further seen that we can take a probability space such that almost surely we

simultaneously have :

$$\left( M_n^{k_2}(\lambda), N_{\lambda_0-\lambda}^n \right) \rightarrow \left( M_\infty^{k_2}(\lambda), N_{\lambda_0-\lambda} \right) \quad (3.107)$$

$$M_n(\lambda) \rightarrow M_\infty(\lambda), \quad (3.108)$$

$$M_n(\lambda, \lambda_0) \rightarrow M_\infty(\lambda, \lambda_0) \quad (3.109)$$

where the first convergence is in the sense of  $\Delta$ , the second for the *GHP* metric and the third for the product topology associated to the *GHP* metric. These convergences and the choice of  $k_2$  yield that for  $n$  large enough, for each  $i \in [i_0]$ , we have  $M_{n,i}^{k_2}(\lambda, \lambda_0) \subset M_{n,i}(\lambda, \lambda_0)$ . We have seen that a cut on a metric graph provides a new metric graph with two new vertices. Furthermore, Equation (3.107) yield that after a certain  $n_0$ , the set of such couples of vertices  $C$  is identical in  $M_n^{k_2}(\lambda, \lambda_0)$  and in  $M_\infty^{k_2}(\lambda, \lambda_0)$ , if we see these spaces as measured metric graphs. Moreover, it is straightforward to show that the convergence of  $M_n^{k_2}(\lambda, \lambda_0)$  to  $M_\infty^{k_2}(\lambda, \lambda_0)$  in the sense of  $\Delta_0$  implies the convergence for the same metric of its restriction  $\cup_{i \in [i_0]} M_{n,i}^{k_2}(\lambda, \lambda_0)$  to  $\cup_{i \in [i_0]} M_{\infty,i}^{k_2}(\lambda, \lambda_0)$ . Furthermore, Proposition 3.3.8 and Lemma 3.3.10 provide the convergence in the Gromov-Hausdorff-Prokhorov sense :

$$\text{Coal} \left( (M_{n,i}^{k_2}(\lambda, \lambda_0))_{i \in [i_0]}, \sum_{l,j \in \tilde{C}} \delta_{x_l^n x_j^n} \right) \rightarrow \text{Coal} \left( (M_{\infty,i}^{k_2}(\lambda, \lambda_0))_{i \in [i_0]}, \sum_{l,j \in \tilde{C}} \delta_{x_l x_j} \right), \quad (3.110)$$

where  $\tilde{C}$  is the restriction of  $C$  to the couple of vertices which belong to the measured metric graph  $\cup_{i \in [i_0]} M_{n,i}^{k_2}(\lambda, \lambda_0)$  and  $\cup_{i \in [i_0]} M_{\infty,i}^{k_2}(\lambda, \lambda_0)$  and the  $x_l$  and  $x_l^n$  are the points associated to the label  $l$  in  $(M_{n,i}^{k_2}(\lambda, \lambda_0))_{i \in [i_0]}$  and  $(M_{\infty,i}^{k_2}(\lambda, \lambda_0))_{i \in [i_0]}$  without the structure of graphs. Since the right-hand side of (3.110) contains  $M^{k_1}(\lambda)$ , Equations (3.106) and (3.108) imply that there exists  $n_0$  such that for any  $n \geq n_0$  the left-hand side, which we call  $H_{n,i_0}^{k_2}$  is at distance at most  $3\epsilon$  of  $M_n(\lambda)$ . Furthermore, since by construction all the cycles are in  $M^{k_1}(\lambda)$ , the above convergences in the sense of  $\Delta$  imply that all cycles of  $M_n(\lambda)$  are in  $H_{n,i_0}^{k_2}$ . Therefore, for  $n$  large enough if a vertex belongs to  $M_{n,i}(\lambda, \lambda_0)$ ,  $i \in [i_0]$  and is the endpoint of an edge of  $M_n(\lambda)$  which becomes close between time  $\lambda$  and  $\lambda_0$ , then it belongs to  $M_{n,i}^{k_2}(\lambda, \lambda_0)$ . Hence,  $H_{n,i_0}^{k_2}$  can be seen as a subset of  $H_{n,i_0}$  defined by :

$$H_{n,i_0}^{k_2} := \text{Coal} \left( (M_{n,i}(\lambda, \lambda_0))_{i \in [i_0]}, \sum_{l,j \in \tilde{C}} \delta_{x_l^n x_j^n} \right).$$

Moreover, the convergences (3.107) and (3.109) and the choice of  $k_2$  implies that for  $n$  large enough for any  $i \in [i_0]$ ,  $d_{GHP}(M_{n,i}^{k_2}(\lambda, \lambda_0), M_{n,i}(\lambda, \lambda_0)) < \epsilon/2^{K+i_0}$ . By using the inequalities in Lemma 3.4.4 and by bounding the number of edges between the  $M_{n,i}(\lambda, \lambda_0)$  by  $K + i_0$ ,

by a simple induction we have that :

$$d_{GHP}(H_{n,i_0}^{k_2}, H_{n,i_0}) < \epsilon.$$

This inequality could be improved but is sufficient for our purpose. We thus deduce that there exists  $n_1 \geq n_0$  such that for any  $n \geq n_1$  :

$$d_{GHP}(M_n(\lambda), H_{n,i_0}) < 4\epsilon.$$

It is then straightforward to extend this result to the  $k$  first components of  $G(n, p_n(\lambda))$  and to deduce the desired result.  $\square$

We can now conclude this section by the proof of Theorem 3.1.2.

*Proof of Theorem 3.1.2.* The proof is now immediate. Propositions 3.4.1, 3.4.2 and 3.4.3 provide the convergence for the product topology induced by  $d_{GHP}$  :

$$\overleftarrow{F}(n, \lambda + \lambda_0) \xrightarrow{d} \text{Coal}(\mathbb{M}(\lambda_0), P_\lambda). \quad (3.111)$$

Furthermore, since  $G(n, p_n(\lambda))$  converges for the topology induced by  $\text{dist}_{GHP}^4$  in  $\mathbb{L}^4$ , it is straightforward to extend the convergence (3.111) for the metric induced by  $\text{dist}_{GHP}^4$ . The fact that  $(\overleftarrow{F}(n, \lambda + \lambda_0))_{\lambda \geq 0}$  and  $\text{Coal}(\mathbb{M}(\lambda_0), P_\lambda)_{\lambda \geq 0}$  are Markovian finally yield the convergence in the sense of finite-dimensional distribution.  $\square$

### 3.5 Extension of the results to the other processes

We start by the proof of Theorem 3.1.3. Fix  $\phi_\lambda(n) = n + \lfloor -\lambda^- n^{2/3} \rfloor$ . Contrary to previous sections, it is important to work on a filtration which does not take account on the initial labelling. We thus define the filtration  $(\mathcal{H}_\lambda^n)$  by :

$$\mathcal{H}_\lambda^n = \sigma\{B_{\psi,n,A}, \psi \in \mathfrak{S}_{\phi_\lambda(n)}, A \subset \mathcal{P}_2([\phi_\lambda(n)])\},$$

where :

$$B_{\psi,n,A} = \bigcap_{\{i,j\} \in A} \{U_{\psi(i)\psi(j)} \leq 1/n\} \bigcap_{\{i,j\} \in A^c} \{U_{\psi(i)\psi(j)} > 1/n\}.$$

Throughout this section, we will work on this filtration. Hence, if we consider  $\lambda_0 \leq \lambda$ , we obtain  $G(\phi_\lambda(n), 1/n)$  from  $G(\phi_{\lambda_0}(n), 1/n)$  by deleting uniformly  $\phi_{\lambda_0}(n) - \phi_\lambda(n)$  vertices on  $G(\phi_{\lambda_0}(n), 1/n)$ . Fix  $n_1 = \phi_{\lambda_0}(n)$ . A simple Taylor development shows :

$$G\left(\phi_{\lambda_0}(n), \frac{1}{n}\right) = G\left(n_1, \frac{1}{n_1} - \frac{\lambda_0}{n_1^{4/3}} + \underset{n \rightarrow +\infty}{o}\left(\frac{1}{n_1^{4/3}}\right)\right).$$

The results of convergence in [5] or [3] are stated for  $p_n(\lambda) = n^{-1} + \lambda n^{-4/3}$ . However, these results can be adapted without difficulties for  $p_n(\lambda) = n^{-1} + \lambda n^{-4/3} + o_n(n^{-4/3})$ . Proposition 4 of [15] shows indeed that the term  $o_n(n^{-4/3})$  does not affect the convergence in distribution of the vector of masses. Furthermore, Theorem 22 of [4] (or Lemma 4.3 of [5] for its adaptation for the metric  $d_{GHP}$ ) shows that if we consider two sequences  $p(n)$  and  $m(n)$  such that  $p(n)n \rightarrow 1$  and  $m(n)n^{-2/3} \rightarrow \sigma$  as  $n \rightarrow +\infty$ , then a rescaled connected component of  $G(n, p(n))$  conditioned on having size  $m(n)$  converges in distribution to  $g(2\tilde{e}^{(\sigma)}, \mathcal{P})$  as defined in introduction. This means that with such a conditioning, the limit distribution does not depend on the term  $\lambda n^{-4/3} + o_n(n^{-4/3})$ . Hence, the proof of Theorem 25 in [5] and Theorem 4.1 in [3] are identical with a term  $o_n(n^{-4/3})$  in addition.

Therefore,  $G(\phi_{\lambda_0}(n), 1/n)$  seen as a sequence of measured metric spaces converges in distribution to  $(\mathcal{M}_i(-\lambda_0), d_i, \mu_i)_i$  for the metric  $\text{dist}_{GHP}^4$ . The strategy of the proof of Theorem 3.1.3 is identical to that of Theorem 3.1.1. We will thus skip a lot of details. Recall the notations of Section 3.3.1 and fix  $\tilde{M}_n(\lambda_0)$  a component of  $G(\phi_{\lambda_0}(n), 1/n)$  conditioned on having size  $\lfloor \sigma n^{2/3} \rfloor$  and  $\tilde{M}_n^k(\lambda_0)$  defined as  $M_n^k(\lambda_0)$  for  $M_n(\lambda_0)$ . We also set a labelling of  $G(\phi_{\lambda_0}(n), 1/n)$  which is given by a uniform random permutation of the initial labelling. We call  $\tilde{M}_n(\lambda_0, \lambda)$  the sequence of connected components of  $G(\phi_\lambda(n), 1/n)$ ,  $\lambda \geq \lambda_0$  whose vertices belong to  $\tilde{M}_n(\lambda_0)$  and define similarly  $\tilde{M}_n^k(\lambda_0, \lambda)$ . Since  $\tilde{M}_n(\lambda) = M_{n_1}(\lambda_0 + o(1))$ , we have the convergences as  $n \rightarrow +\infty$  :

$$\tilde{M}_n(\lambda) \xrightarrow{d} M_\infty(\lambda_0) \text{ and } \tilde{M}_n^k(\lambda) \xrightarrow{d} M_\infty^k(\lambda_0), \tag{3.112}$$

where the first convergence holds for the  $GHP$  metric and the second for the  $\Delta_0$  metric. Since this time we remove vertices instead of edges, we need to do a slight adaptation for proving that the process of cuts on  $\tilde{M}_n^k(\lambda_0)$  converges to a Poisson point process. Call  $V(\lambda)$  the set of vertices of  $G(\phi_{\lambda_0}(n), 1/n)$  which belong to  $\tilde{M}_n^k(\lambda_0)$  and which are deleted between time  $\lambda_0$  and  $\lambda$  and define  $L_\lambda^n$  by :

$$L_\lambda^n = \sum_{v \in V(\lambda)} \delta_v.$$

**Lemma 3.5.1.**

$$(\tilde{M}_n^k(\lambda_0), L_\lambda^n) \xrightarrow{d} (M_\infty^k(\lambda_0), N_\lambda),$$

for the metric  $\Delta$ .

*Proof.* We already know that the first term converges in distribution for the metric  $\Delta_0$ . We recall that by construction,  $L_\lambda^n$  must have no atoms on a branching point or a leaf of  $\tilde{M}_n^k(\lambda_0)$ . Call  $K_n$  the number of branching points and leaves of  $\tilde{M}_n^k(\lambda_0)$ . Since we remove  $\kappa_n = \phi_{\lambda_0}(n) - \phi_\lambda(n) = (\lambda_0 - \lambda)n^{2/3} + O_n(1)$  vertices between time  $\lambda_0$  and  $\lambda$ , it is straight-

forward to bound the probability knowing  $\mathcal{H}_{\lambda_0}^n$  to have a cut on a branching point or a leaf by :  $K_n(\lambda_0 - \lambda)n^{-1/3}$ . Furthermore, since  $K_n$  converges in distribution to a finite random variable, the probability that there is a cut on a branching point or a leaf of  $\tilde{M}_n^k(\lambda_0)$  tends to 0. Take now  $A_1^n, \dots, A_k^n$ ,  $k$  disjoint paths on  $\tilde{M}_n^k(\lambda_0)$  with unscaled respective lengths  $l_k^n$ , such that  $n^{-1/3}l_k^n \rightarrow l_k$  as  $n \rightarrow +\infty$ .

$$\begin{aligned} \mathbb{P}(L_\lambda^n(A_1^n) = i_1, \dots, L_\lambda^n(A_k^n) = i_k) &= \binom{n - \sum_{j=1}^k l_j^n}{\kappa_n - \sum_{j=1}^k i_j} \times \frac{\prod_{j=1}^k \binom{l_j^n}{i_j}}{\binom{n}{\kappa_n}}, \\ &\underset{+\infty}{\sim} e^{-[\lambda - \lambda_0] \sum_{j=1}^k l_j} \prod_{j=1}^k \frac{l_j^{i_j}}{i_j!}, \end{aligned}$$

which provides the desired convergence. □

Lemma 3.5.1 yields the equivalent of Proposition 3.3.3.

**Proposition 3.5.2.** *For any  $\lambda \geq \lambda_0$ , we have :*

$$\text{Frag}(\tilde{M}_n^k(\lambda_0), L_\lambda^n) \xrightarrow{d} \text{Frag}(M_\infty^k(\lambda_0), N_\lambda) \text{ as } n \rightarrow +\infty, \tag{3.113}$$

*in the sense of finite dimensional distributions.*

We can also state an equivalent of Proposition 3.3.4.

**Proposition 3.5.3.** *For any  $\lambda \geq \lambda_0$ , for any  $i \in \mathbb{N}^*$  and  $\epsilon > 0$ , for  $n$  and  $k$  large enough, such that  $k \leq n^{1/3-\epsilon}$ , we have :*

$$\mathbb{P}(d_{GHP}(\tilde{M}_{n,i}^k(\lambda_0, \lambda), \tilde{M}_{n,i}(\lambda_0, \lambda)) \geq \epsilon) \leq \epsilon. \tag{3.114}$$

We choose to assume  $k \leq n^{1/3-\epsilon}$  in order to have a probability to remove a mass of  $\tilde{M}_n^k(\lambda_0)$  which tends to 0 and hence to have a similar proof. However, this assumption is unnecessary. The proofs of this proposition and of Theorem 3.1.3 being similar we omit them.

The proof of Theorem 3.1.4 is now immediate. Since  $F$  and  $\overleftarrow{C}$  converge to the same process, their time reversals converge also to the same process. Theorem 3.1.5 is also a straightforward consequence of Theorem 3.1.1, Theorem 3.1.4 and of the independence of the  $U_{i,j}$ .

## 3.A Gromov-Hausdorff-Prokhorov

### 3.A.1 A first distance

We refer the reader to Section 2. of [5] for a more complete description of the tools presented in this section.



Fix  $(X, d)$  and  $(X', d')$  two metrics space. Consider  $C$  a subset of  $X \times X'$ . The distortion of  $C$  is defined by :

$$\text{Dis}(C) := \sup\{|d(x, y) - d'(x', y')|, (x, x'), (y, y') \in C\}. \quad (3.115)$$

Let  $\pi_1$  (resp.  $\pi_2$ ) be the natural projection from  $C$  to  $X$  (resp. to  $X'$ ). We say that  $C$  is a correspondence if  $\pi_1(C) = X$  and  $\pi_2(C) = X'$  and we denote by  $C(X, X')$  the set of correspondence between  $X$  and  $X'$ . We define the Gromov-Hausdorff distance between  $[X, d]$  and  $[X', d']$  by :

$$d_{GH}(X, X') = \frac{1}{2} \inf_{C \in C(X, X')} \text{Dis}(C). \quad (3.116)$$

Fix  $\mathcal{M}_f(M)$  the set of all finite Borel measure on  $M$ . The discrepancy of a measure  $\pi \in \mathcal{M}_f(X \times X')$  with respect to  $\mu_1 \in \mathcal{M}_f(X)$  and  $\mu_2 \in \mathcal{M}_f(X')$  is defined by :

$$D(\pi, \mu_1, \mu_2) := \|\mu_1 - p_{1*}\pi\| + \|\mu_2 - p_{2*}\pi\|, \quad (3.117)$$

where  $\|\nu\|$  denotes the total variation of the signed measure  $\nu$ . We can now define the Gromov-Hausdorff-Prokhorov distance between  $[X, d, \mu]$  and  $[X', d', \mu']$  by :

$$d_{GHP}(X, X') := \inf \left\{ \frac{1}{2} \text{Dis}(C) \vee D(\pi, \mu_1, \mu_2) \vee \pi(C^c) \right\}, \quad (3.118)$$

where the infimum is taken over all  $C \in C(X, X')$  and  $\pi \in B(X \times X')$ . Fix  $\mathbf{x} = (x_1, \dots, x_k)$  and  $\mathbf{x}' = (x'_1, \dots, x'_k)$ . We also can define a pointed Gromov Hausdorff Prokhorov distance for classes of equivalence of pointed measured metric space  $(X, d, \mu, \mathbf{x})$  and  $(X', d', \mu', \mathbf{x}')$  by :

$$d_{pGHP}(X, X') := \inf \left\{ \frac{1}{2} \text{Dis}(C) \vee D(\pi, \mu_1, \mu_2) \vee \pi(C^c) \right\}, \quad (3.119)$$

where the infimum is taken over all  $C \in C(X, X')$  such that  $(x_i, x'_i) \in C$  for all  $i \in [k]$  and  $\pi \in \mathcal{M}_f(X \times X')$ . Consider now  $\mathbb{K}$  the set of isometry-equivalence classes of measured compact metric spaces. Let  $Z$  be the equivalence class of  $(\{x\}, 0, 0)$  and  $Y = (Z, Z, Z, \dots)$ . For  $A, B \in \mathbb{K}^{\mathbb{N}^*}$ , we define  $\text{dist}_{GHP}^p(A, B)$  by :

$$\text{dist}_{GHP}^p(A, B) = \left( \sum_{i \geq 1} d_{GHP}(A_i, B_i)^p \right)^{\frac{1}{p}} \quad (3.120)$$

We further define  $\mathbb{L}_p$  by :

$$\mathbb{L}_p = \left\{ X \in \mathbb{K}^{\mathbb{N}^*}, \text{dist}_{GHP}^p(X, Y) < \infty \right\}. \quad (3.121)$$

### 3.A.2 A second distance

We now present a second approach, which is essentially in [1].

We begin by the definitions of the Hausdorff and the Prokhorov distances. Consider  $(M, \delta)$  a Polish metric space. For  $A \subset M$ , we denote by  $A^\epsilon$  the set :

$$A^\epsilon = \{x \in M, \exists y \in A, \delta(x, y) < \epsilon\}. \quad (3.122)$$

Now consider  $K$  and  $K'$  two compact subsets of  $M$ . The Hausdorff distance between  $K$  and  $K'$  is :

$$\delta_H(K, K') = \inf\{\epsilon > 0, K^\epsilon \supset K', (K')^\epsilon \supset K\}. \quad (3.123)$$

Let  $\mu, \nu \in \mathcal{M}_f(M)$ . The Prokhorov distance between  $\mu$  and  $\nu$  is :

$$\delta_P(\mu, \nu) = \inf\{\epsilon > 0, \mu(A) \leq \nu(A_\epsilon) + \epsilon, \nu(A) \leq \mu(A_\epsilon) + \epsilon, \text{ for all closed } A\}. \quad (3.124)$$

We can define a Gromov-Hausdorff Prokhorov distance between  $[M, d, \mu]$  and  $[M', d', \mu']$  by :

$$\delta_{GHP}([M, d, \mu], [M', d', \mu']) = \inf_{\phi, \phi'} (\delta_H(\phi(M), \phi'(M')) \vee \delta_P(\phi_*(\mu), \phi'_*(\mu'))), \quad (3.125)$$

where  $\phi$  (resp.  $\phi'$ ) is an isometric embeddings from  $M$  (resp.  $M'$ ) into some Polish space  $(Z, d_Z)$ . Note that some authors, as in [1], change  $\vee$  by  $+$ , which yields an equivalent distance. Furthermore, we can remark that if  $M$  and  $M'$  are subsets of the same metric space, then, by definition of  $\delta_{GHP}$ , we have :

$$\delta_{GHP}([M, d, \mu], [M', d, \mu']) \leq \delta_H(M, M') \vee \delta_P(\mu, \mu'). \quad (3.126)$$

Fix  $k$  points  $\mathbf{m} = (m_1, \dots, m_k)$  in  $M$  and  $k$  points  $\mathbf{m}' = (m'_1, \dots, m'_k)$  in  $M'$ . As for the previous distance, we can also define a Gromov Hausdorff Prokhorov distance for pointed measured metric spaces by :

$$\delta_{pGHP}(M, M') = \inf_{\phi, \phi'} \left\{ \delta_H(\phi(M), \phi'(M')) \vee \delta_P(\phi_*(\mu), \phi'_*(\mu')) \bigvee_{i=1}^k d_Z(\phi(m_i), \phi'(m'_i)) \right\}, \quad (3.127)$$

where  $\phi$  (resp.  $\phi'$ ) is an isometric embeddings from  $M$  (resp.  $M'$ ) into some Polish space  $(Z, d_Z)$ . We can also remark that Proposition 6 of [80] shows that  $\delta_{GHP} = d_{GHP}$  when the considered measures are probability measures. By adapting the proof of this Proposition, we can show that :

**Theorem 3.A.1.**

$$\frac{1}{2} \delta_{GHP} \leq d_{GHP} \leq \delta_{GHP}. \quad (3.128)$$

These bounds are maybe not optimal but largely sufficient for our purpose. Since we do not find references for the proof of (3.128) when the measures are not probability measures, we give here a sketch of proof inspired by Proposition 6 of [80]. We refer the reader to this proposition for more details.

*Proof.* Fix  $[M, d, \mu]$  and  $[M', d', \mu']$ . We begin by proving the right inequality. Suppose first that  $\delta_{GHP}(M, M') < r' < r$ . Remark 7.3.12 of [30] yields the existence of  $\phi$  and  $\phi'$  two embeddings of  $M$  and  $M'$  into a separable space  $(Z, d_Z)$  such that  $\delta_H(\phi(M), \phi'(M')) < r'$  and  $\delta_P(\phi_*(\mu), \phi'_*(\mu')) < r'$ . Consider the correspondence

$$\mathcal{R} = \{(m, m') \in M \times M' : d_Z(\phi(m), \phi(m')) \leq r'\}. \quad (3.129)$$

As in [80], we get

$$\text{Dis}(\mathcal{R}) \leq 2r'. \quad (3.130)$$

We work with finite measures, and thus  $\mu(M) < \infty$  and  $\mu'(M') < \infty$ . If  $\mu(M) = 0$  (and similarly if  $\mu'(M') = 0$ ), by taking  $\pi$  the null measure on  $M \times M'$ , we can easily see that :

$$D(\pi, \mu, \mu') = \mu'(M') = \delta_P(\phi_*(\mu), \phi'_*(\mu')), \quad (3.131)$$

where  $D$  is defined in (3.117). Furthermore,  $\pi(\mathcal{R}^c) = 0$  and thus in this case the right inequality is proved. Suppose then that  $\mu(M)$  and  $\mu'(M')$  are strictly positive. The measures  $\tilde{\mu} := \frac{\phi_*(\mu)}{\mu(M)}$  and  $\tilde{\mu}' := \frac{\phi'_*(\mu')}{\mu'(M')}$  are thus probability measures. Suppose that  $\mu(M) \leq \mu'(M')$  (the opposite case is identical) and fix  $F$  a closed subset of  $Z$ . The definition of the Prokhorov distance implies :

$$\phi_*\mu(F) \leq \phi'_*(\mu')(F^{r'}) + r' \quad (3.132)$$

$$\begin{aligned} \frac{\phi_*\mu(F)}{\mu(M)} &\leq \frac{\phi'_*(\mu')(F^{r'})}{\mu(M)} + \frac{r'}{\mu(M)} \\ \frac{\phi_*\mu(F)}{\mu(M)} &\leq \frac{\phi'_*(\mu')(F^{r'})}{\mu'(M')} + \frac{r'}{\mu(M)} \\ \tilde{\mu}(F) &\leq \tilde{\mu}'(F^{r'}) + \frac{r'}{\mu(M)} \end{aligned} \quad (3.133)$$

Theorem 11.6.2 of [47] and Equation (3.133) yield the existence of a coupling  $\nu_1$  on  $Z \times Z$  between  $\tilde{\mu}$  and  $\tilde{\mu}'$  such that :

$$\nu_1(\{(x, y), d_Z(x, y) > r\}) \leq \frac{r}{\mu(M)} \quad (3.134)$$

By a change of measures as in [80], we get a coupling  $\nu_2$  between  $\frac{\mu}{\mu(M)}$  and  $\frac{\mu'}{\mu'(M')}$  such

that  $\nu_2(\mathcal{R}^c) \leq \frac{r}{\mu(M)}$ . Consider thus  $\nu_3 := \mu(M)\nu_2$ . We obviously have :

$$\nu_3(\mathcal{R}^c) \leq r. \quad (3.135)$$

Furthermore if  $p_1$  (resp.  $p_2$ ) is the canonical projection from  $M \times M'$  to  $M$  (resp. to  $M'$ ), the discrepancy of  $\nu_3$  with respect to  $\mu$  and  $\mu'$  is :

$$\begin{aligned} D(\nu_3, \mu, \mu') &= \|\mu - p_{1*}\nu_3\| + \|\mu' - p_{2*}\nu_3\| \\ &= \|\mu - \mu(M)\frac{\mu}{\mu(M)}\| + \|\mu' - \mu(M)\frac{\mu'}{\mu'(M')}\| \\ &= \mu'(M')\left|1 - \frac{\mu(M)}{\mu'(M')}\right| \\ &= |\mu'(M') - \mu(M)| < r. \end{aligned} \quad (3.136)$$

Hence Equations (3.130), (3.135) and (3.136) yield the right inequality.

We now prove the left inequality. Suppose that  $d_{GHP}(M, M') < r$ . There exists a correspondence  $\mathcal{R}$  between  $M$  and  $M'$  and a finite measure  $\nu$  on  $M \times M'$  such that :  $\nu(\mathcal{R}^c) \leq r$  and  $D(\nu, \mu, \mu') \leq r$ . Consider the space  $Z := M \sqcup M'$  endowed with the distance  $d_Z$  defined such that  $d_Z = d$  on  $M^2$ ,  $d_Z = d'$  on  $M'^2$  and :

$$d_Z(m, m') = \inf\{d(m, x) + r + d'(y, m'), (x, y) \in \mathcal{R}\}, \quad (3.137)$$

when  $(m, m') \in M \times M'$ . By a slight abuse of notations we will write  $\mu$  instead of the measure image by the canonical injection from  $M$  to  $Z$ . We do not make the proof that  $d_Z$  is a distance which is in part in [80], but we mention that the fact that the condition  $1/2 \text{Dis}(\mathcal{R}) \leq r$  is necessary to have the triangular inequality. If  $m \in M$ , by definition of a correspondence there exists  $m' \in M'$  such that  $(m, m') \in \mathcal{R}$  and thus  $d_Z(m, m') = r$ , which proves that there exist an embedding of  $M$  and  $M'$  such that the Hausdorff distance between  $M$  and  $M'$  is  $r$ . Furthermore, if  $d_Z(m, m') = r$ , the definition of  $d_Z$  implies that  $(m, m') \in \bar{\mathcal{R}}$ . Consider now  $F$  a closed subset of  $M$ . We denote by  $\mathcal{R}^F$  the set  $\{y \in M', (x, y) \in \mathcal{R}, x \in F\}$ . Since  $D(\nu, \mu, \mu') \leq r$ , we have :

$$\begin{aligned} &|\mu(F) - p_{1*}\nu(F)| + |\mu'(F^r \cap M') - p_{2*}\nu(F^r \cap M')| \leq r \\ &\mu(F) \leq r + \mu'(F^r) + \nu(F \times M') - \nu(M \times F^r \cap M') \\ &\mu(F) \leq r + \mu'(F^r) + \nu(F \times \mathcal{R}^F) + \nu(F \times (\mathcal{R}^F)^c) - \nu(F \times \bar{\mathcal{R}}^F) - \nu(F^c \times \bar{\mathcal{R}}^F) \\ &\mu(F) \leq r + \mu'(F^r) + \nu(\mathcal{R}^c) \\ &\mu(F) \leq 2r + \mu'(F^{2r}). \end{aligned} \quad (3.138)$$

If now  $F$  is a closed subset of  $Z$ , since  $\mu(F \cap M') = 0$ , Equation (3.138) holds. Since we

can proceed in the same way by switching  $\mu$  and  $\mu'$ , the left inequality is proved.  $\square$

We have the same inequality for the Gromov Hausdorff distances between pointed measured metric spaces.

**Theorem 3.A.2.**

$$\frac{1}{2}\delta_{pGHP} \leq d_{pGHP} \leq \delta_{pGHP}. \quad (3.139)$$

*Proof.* The proof of this point is almost identical to the previous proof and thus we just give the changes. Let us distinguish  $k$  points  $\mathbf{m} = (m_1, \dots, m_k)$  in  $M$  and  $k$  points  $\mathbf{m}' = (m'_1, \dots, m'_k)$  in  $M'$ , where  $M$  and  $M'$  are measured metric spaces.

Suppose  $\delta_{pGHP}(M, M') < r$ , and consider the embeddings at the beginning of the previous proof and the correspondence (3.129). Since  $\delta_{pGHP}(M, M') < r$ , we have  $\bigvee_{i=1}^k d_Z(\phi(m_i), \phi'(m'_i)) < r$ , which implies that for any  $i \in [k]$ ,  $(m_i, m'_i)$  is in the correspondence (3.129). Hence, if we proceed as before, we obtain the right inequality.

Suppose now that  $d_{pGHP}(M, M') < r$ . Let us introduce  $Z, \mathcal{R}, \nu, d_Z$  as in Equation (3.137) and just above. The assumption implies that for any  $i \in [k]$ ,  $(m_i, m'_i)$  is in the correspondence  $\mathcal{R}$ . But we have seen in the previous proof that if  $(x, x')$  is in  $\mathcal{R}$  then  $d_Z(x, x') = r$ . Therefore, we have  $\bigvee_{i=1}^k d_Z(\phi(m_i), \phi'(m'_i)) \leq r$ . The rest of the proof is identical.  $\square$

## 3.B Gluing spaces

### 3.B.1 Generalities

We refer the reader to [30] for more details. We first need to define what is a length space. Fix  $X$  a topological space. A continuous path is a continuous function  $\gamma : I \mapsto X$ , where  $I$  is an interval of  $\mathbb{R}$ . Let  $A$  be a subset of the continuous paths called the set of admissible paths, and  $L : A \mapsto \mathbb{R}_+ \cup \{\infty\}$ . The couple  $(A, L)$  is called length structure on  $X$  if it satisfies the followings which follow. In this case,  $A$  is called the set of admissible paths and  $L$  the length of the paths.

1.  $A$  is closed under restrictions. If  $\gamma : [a, b] \rightarrow X$  is in  $A$  then for all  $a \leq c \leq d \leq b$ ,  $\gamma|_{[c, d]} \in A$ .
2.  $A$  is closed under concatenation. Let  $\gamma_1 : [a, b] \rightarrow X$  and  $\gamma_2 : [b, c] \rightarrow X$ , such that  $\gamma_1, \gamma_2 \in A$  and  $\gamma_1(b) = \gamma_2(b)$ . Then the function  $\gamma = \gamma_1 \cdot \gamma_2$  defined on  $[a, c]$  by :  $\gamma_1 \cdot \gamma_2 = \gamma_1$  on  $[a, b]$  and  $\gamma_1 \cdot \gamma_2 = \gamma_2$  on  $[b, c]$  is in  $A$ .
3.  $A$  is closed under linear reparametrisations. Let  $\gamma \in A$  defined on  $[a, b]$  and  $\phi : t \mapsto \alpha t + \beta$ , with  $\alpha \neq 0$ . Then  $\gamma \circ \phi$  restricted to  $\phi^{-1}([a, b])$  is in  $A$ .

4. Let  $a \leq b \leq c$  and  $\gamma : [a, c] \rightarrow X$ . Then  $L(\gamma) = L(\gamma|_{[a,b]}) + L(\gamma|_{[b,c]})$ .
5. Let  $\gamma \in A$  defined on  $[a, b]$ . The map  $t \in [a, b] \mapsto L(\gamma|_{[a,t]})$  is continuous.
6.  $L(\gamma \circ \phi) = L(\gamma)$  for  $\phi$  any linear reparametrisation.
7. Let  $A_{x,y}$  the set of map  $\gamma$  defined on an interval  $[a, b]$  satisfying  $\gamma(a) = x$ ,  $\gamma(b) = y$  and  $\gamma \in A$ .

$$\inf\{L(\gamma), \gamma \in A_{x,y}\} > 0, \forall (x, y) \in X^2, x \neq y.$$

The length structure provides a natural distance  $d_L$  on  $X$ , defined by :

$$d_L(x, y) = \inf\{L(\gamma), \gamma \in A_{x,y}\}. \quad (3.140)$$

A metric space  $(X, d)$  such that  $d = d_L$  is called a length space.

We can now explain how gluing spaces. Let  $(X, d)$  be a metric space and  $\mathcal{R}$  an equivalence relation on  $X$ . Consider the quantity  $d_{\mathcal{R}}$  defined by :

$$d_{\mathcal{R}}(x, y) = \inf_{k \in \mathbb{N}^*} \left\{ \sum_{i=1}^k d(p_i, q_i), p_0 = x, q_k = y, q_i \mathcal{R} p_{i+1}, \forall i \in \{0, \dots, k-1\} \right\}. \quad (3.141)$$

The quantity  $d_{\mathcal{R}}$  is a semi-distance on  $X$  and is a distance on  $X/d_{\mathcal{R}}$ , where  $X/d_{\mathcal{R}}$  is the set of classes of equivalence for the relation  $\sim_{d_{\mathcal{R}}}$  defined by  $x \sim_{d_{\mathcal{R}}} y$ , if  $d_{\mathcal{R}}(x, y) = 0$ . We called the resulting space the quotient metric space and we say that it is obtained by gluing  $X$  along  $\mathcal{R}$ . Note that gluing a length metric space provides a length metric space.

We can gluing different length spaces by considering a collection of length as an only length space in the following way. Consider  $(X_i, L_i)$  a collection of length spaces. We consider the disjoint union  $\cup_i X_i$ . We define  $d_U$  in the following way :

$$d_U(x, y) = \begin{cases} d_i(x, y) & \text{if there exists } i \text{ such that } x, y \in X_i, \\ \infty & \text{else.} \end{cases} \quad (3.142)$$

The quantity  $d_U$  is then a length metric on  $\cup_i X_i$ . Therefore if we have a equivalence relation on  $\cup_i X_i$ , we can construct a glued space, which is a length space, by the previous method. Finally, we mention that if  $p$  is the projection from  $X$  to  $X/d_{\mathcal{R}}$  then we have :

$$d_{\mathcal{R}}(p(x), p(y)) \leq d(x, y). \quad (3.143)$$

### 3.B.2 Some properties of gluing

We here provide some properties which theoretically justify our construction in Section 3.4. Consider  $(X_i)_{i \in \mathbb{N}}$  a sequence of compact connected and disjoint length spaces. Fix  $J \subset K \subset \mathbb{N}$ ,  $A = \cup_{i \in J} X_i$  and  $B = \cup_{i \in K} X_i$ . Fix  $\mathcal{R}_B$  an equivalence relation on  $B$ . We will always suppose in what follows that each class of equivalence of  $\mathcal{R}_B$  contains at most 2

elements (which is an assumption almost surely satisfied in our framework). Fix also  $\mathcal{R}_A$  the restriction of  $\mathcal{R}_B$  to  $A$  (which is thus also an equivalence relation on  $B$ ). Fix  $p_B$  (resp.  $p_A$ ) the projection from  $B$  into  $B/d_{\mathcal{R}_B}$  (resp.  $B/d_{\mathcal{R}_A}$ ), where  $d_{\mathcal{R}_B}$  is defined as in (3.141). Finally, say for  $x, y \in B/d_{\mathcal{R}_A}$  that  $x\mathcal{R}^q y$  if  $x$  and  $y$  have each an antecedent by  $p_A$ ,  $x'$  and  $y'$  such that  $x'\mathcal{R}_B y'$ . It is straightforward to show that  $\mathcal{R}^q$  is an equivalence relation on  $B/d_{\mathcal{R}_A}$ . Fix  $q$  the projection from  $B/d_{\mathcal{R}_A}$  into  $[B/d_{\mathcal{R}_A}]/d_{\mathcal{R}^q}$ . The following proposition justifies that we can successively glue the metric spaces.

**Proposition 3.B.1.** *The map :*

$$\phi : \begin{array}{ccc} B/d_{\mathcal{R}_B} & \longrightarrow & [B/d_{\mathcal{R}_A}]/d_{\mathcal{R}^q} \\ \bar{b} & \longmapsto & q \circ p_A(b), \end{array}$$

where  $b$  is an antecedent of  $\bar{b}$  by  $p_B$  is well-defined and is a bijective isometry.

*Proof.* Let  $b_1, b_2 \in B$ . The distance  $d_{\mathcal{R}^q}(q \circ p_A(b_1), q \circ p_A(b_2))$  is obtained by taking the infimum of the sums of the form :

$$\sum_{i=0}^n d_{\mathcal{R}_A}(x_i, y_i)$$

such that  $x_0 = p_A(b_1)$ ,  $y_n = p_A(b_2)$  and  $x_i\mathcal{R}^q y_{i+1}$  for all  $0 \leq i \leq n-1$ . The last point is equivalent to the fact there exists  $x'_i$  and  $y'_i$  such that  $x_i = p_A(x'_i)$ ,  $y_i = p_A(y'_i)$  and  $x'_i\mathcal{R}_B y'_i$ . By definition,  $d_{\mathcal{R}_A}(p_A(x'_i), p_A(y'_i)) = d_{\mathcal{R}_A}(x'_i, y'_i)$ . Therefore,  $d_{\mathcal{R}^q}(q \circ p_A(b_1), q \circ p_A(b_2))$  is obtained by taking the infimum of the sums of the form :

$$\sum_{i=0}^n \sum_{j=0}^m d(x_{i,j}, y_{i,j}), \quad (3.144)$$

where  $x_{i,0} = x'_i$ ,  $y_{i,m} = y'_i$ ,  $x_{0,0} = b_1$ ,  $y_{n,m} = b_2$ ,  $x'_i\mathcal{R}_B y'_i$  and  $x_{i,j}\mathcal{R}_A y_{i,j+1}$ . Notice that by changing  $x_{i,j}\mathcal{R}_A y_{i,j+1}$  by  $x_{i,j}\mathcal{R}_B y_{i,j+1}$ , we get a reformulation of the definition of  $d_{\mathcal{R}_B}$ . Furthermore,  $\mathcal{R}_A \subset \mathcal{R}_B$ , and thus :

$$d_{\mathcal{R}^q}(q \circ p_A(b_1), q \circ p_A(b_2)) \geq d_{\mathcal{R}_B}(b_1, b_2).$$

Now, if we take a sum  $S$  of the form :

$$S = \sum_{i=0}^n d(x_i, y_i),$$

where  $x_0 = b_1$ ,  $y_n = b_2$  and  $x_i\mathcal{R}_B y_{i+1}$ ,  $S$  can be rewritten in the form (3.144) by putting

$x_{i,j} = 0$  and  $y_{i,j} = 0$  for all  $x_{i,j}$  and  $y_{i,j}$  different from  $x_{i,0}$  and  $y_{i,m}$ . Therefore,

$$d_{\mathcal{R}^q}(q \circ p_A(b_1), q \circ p_A(b_2)) = d_{\mathcal{R}_B}(b_1, b_2).$$

This equality yields that  $\phi$  is well-defined and isometric. The surjectivity is obvious.  $\square$

Note that in our framework, we work with the completions of the glued spaces. However, it is easy to show that if two metric spaces are isometric, their completions are isometric too. Similarly, the following proposition justifies that to glue two components along a set of points is equivalent to glue all points in a component, next in another component and finally between the two components. Set  $C = B \setminus A$  and consider  $p'_A$  the projection from  $A$  into  $A/d_{\mathcal{R}_A}$  and define similarly the projection  $p'_C$ . We say that  $x\mathcal{R}_{A,C}y$  if there exists  $\{x', y'\} \in \mathcal{P}_2(A, C)$  such that  $p'_A(x') = x$ ,  $p'_C(y') = y$  and  $x'\mathcal{R}_B y'$ . Fix  $Y = A/d_{\mathcal{R}_A} \sqcup C/d_{\mathcal{R}_C}$ . Since the proof of the following Proposition is equivalent to the previous, we omit it.

**Proposition 3.B.2.** *The spaces  $Y/d_{\mathcal{R}_{A,C}}$  and  $B/d_{\mathcal{R}_B}$  are isometric.*

Consider now the graph (possibly infinite)  $G$  such that a node is labelled  $J$  (we recall that  $J$  is the set of indices which define  $A$ ) and the others are labelled by the  $k \in K \setminus J$ . There is an edge between the node  $J$  and a node  $k \in K \setminus J$  if there exists  $\{x, y\} \in \mathcal{P}_2(A, X_k)$  such that  $x\mathcal{R}_B y$  and a node between  $k$  and  $k'$  if there exists  $\{x, y\} \in \mathcal{P}_2(X_k, X_{k'})$  such that  $x\mathcal{R}_B y$ .

**Proposition 3.B.3.** *The map :*

$$\psi : \begin{array}{ccc} A/d_{\mathcal{R}_A} & \longrightarrow & B/d_{\mathcal{R}_B} \\ \bar{a} & \longmapsto & p_B(a), \end{array}$$

where  $a$  is an antecedent of  $\bar{a}$  by  $p'_A$  is well-defined. Moreover, we have :

$$d_{\mathcal{R}_A}(a_1, a_2) \geq d_{\mathcal{R}_B}(\psi(a_1), \psi(a_2)), \forall a_1, a_2 \in A/d_{\mathcal{R}_A}.$$

If further the graph  $G$  defined above is a tree, then  $\psi$  is an isometry.

*Proof.* The fact that  $\psi$  is well-defined and the inequality are obvious. Suppose now that  $G$  is a tree and take  $a_1, a_2 \in A/d_{\mathcal{R}_A}$ . Let  $a'_1$  and  $a'_2$  be their antecedents by  $p'_A$  and consider a sum of the form :

$$\sum_{i=0}^n d(x_i, y_i)$$

such that  $x_0 = a'_1$ ,  $y_n = a'_2$  and  $y_i\mathcal{R}_B x_{i+1}$ . Since  $G$  is a tree, we can restrict ourselves to elements of  $A$  and we recover the definition of the distance  $d_{\mathcal{R}_A}$ . Therefore,  $\psi$  is an isometry.  $\square$



### 3.B.3 Measured metric graphs

We give here a rigorous construction of measured metric finite graphs. Fix  $n \in \mathbb{N}$ . Each element of  $[n]$  is called a vertex. For each couple  $(i, j)$  such that  $1 \leq i \leq j \leq n$ , we fix  $K(i, j) \in \mathbb{N}$ , called the number of opened edges between  $i$  and  $j$  when  $i < j$  and the number of loops on  $i$  when  $i = j$ . If  $K(i, j) \neq 0$ , we put a collection of  $K(i, j)$  segments  $E_{i,j,k}$  of length  $l_{i,j,k} > 0$ , where  $k \in [K(i, j)]$  endowed with the usual metric (which is obviously a length metric in the sense of Appendix 3.B) and with a Borelian measure  $\mu_{i,j,k}$ . To avoid confusions, even if it is an unnecessary assumption, we assume that for each endpoint  $e$  of  $E_{i,j,k}$  we have  $\mu_{i,j,k}(\{e\}) = 0$ . We also suppose that each  $\{i\}$  has a measure  $\mu_i = a_i \delta_i$ , where  $a_i \geq 0$ . Let us define  $(X, d_U, \mu)$  the measured metric space :

$$X = \bigsqcup_{1 \leq k \leq K(i,j)} E_{i,j,k} \bigsqcup_{l \in [n]} \{l\}, \quad (3.145)$$

with distance  $d_U$  defined as in the end of the previous subsection and where :

$$\mu = \sum_{0 \leq i < j \leq n} \mu_{i,j,k} + \sum_{1 \leq i \leq n} \mu_i.$$

For each  $i \in [n]$ , we define the set  $R_i$  as the union of  $\{i\}$  and the left endpoints of the  $E_{i,j,k}$ , when  $i \leq j \leq n$  and  $1 \leq k \leq K(i, j)$  and the right endpoints of the  $E_{h,i,k}$ , when  $1 \leq h < i$  and  $1 \leq k \leq N(h, i)$ . We say that  $x \mathcal{R} y$  when there exists  $i \in [n]$  such that  $x \in R_i$  and  $y \in R_i$ . Define the semi-distance  $d_R$  as in (3.141). Fix  $G = (X/d_R)$ . The space  $(G, d_R)$  is hence a finite metric graph. If we further consider  $p$  the canonical projection  $p : X \rightarrow G$ , we have naturally a measured metric finite graph  $(G, d_R, p_*\mu)$ . The set  $p(\text{Int}(E_{i,j,k}))$  will be called edge and  $\{p(i)\}$  will be called a vertex. We finally denote by  $E(G)$  (resp.  $V(G)$ ) the set of edges (resp. vertices) of  $G$ .

## 3.C Fragmentation of length spaces

Consider  $X$  a length space,  $A$  its set of admissible paths and  $L$  the length of the paths. If  $N$  is a set of points of  $X$ , we can endow  $X \setminus N$  with the metric  $d_C$  defined by :

$$d_C(x, y) = \inf\{L(\gamma), \gamma \in A_{x,y}^N\}, \quad (3.146)$$

where  $A_{x,y}^N$  is the set of  $\gamma \in A$  defined on an interval  $[a, b]$  such that  $\gamma(a) = x$ ,  $\gamma(b) = y$  and for all  $z \in N$ ,  $z \notin \text{Im}(\gamma)$ . As for the gluing, we can extend this definition for a disjoint collection of length spaces. We can further show, as we have done for the gluing, that we can make successively the cuts and that we can permutate gluing and cuts, when there is

no cuts on a gluing point. We finally give the reverse inequality of (3.143) :

$$d_C(x, y) \geq d(x, y). \quad (3.147)$$

We will now show that the fragmentation of the objects we manipulate conserves some interesting properties. In order to simplify the following proofs, we consider stronger assumptions than previously. Fix  $(X, d, \mu)$  a compact connected  $\mathbb{R}$ -graph such that  $\mu$  is a Borelian measure and such that the number of cycles embedded in  $X$  and the maximal degree of a point of  $X$  are finite. Consider  $N$  a countable subset of the skeleton of  $X$ . Let  $Y$  be a connected component of  $X \setminus N$ . Fix  $N_Y = \{x \in N, d(x, Y) = 0\}$ . We denote by  $\text{Adh}_d$  the adherence of  $Y$  in  $X$  for the metric  $d$ . Let  $(\overline{Y}, \overline{d_C})$  be the completion of  $(Y, d_C)$ .

**Theorem 3.C.1.** *We have the following properties :*

1.  $\text{Adh}_d(Y) = Y \cup N_Y$ . Therefore,  $Y \in \mathcal{B}((X, d))$ .
2.  $\overline{Y} \setminus Y$  is countable.
3. The restriction of the distances  $d$  and  $d_C$  to  $Y$  induce the same topology on  $Y$  and thus the same Borelian sets.
4.  $(\overline{Y}, \overline{d_C})$  is compact.

*Proof.* Let us prove the first point. Let  $x \in X$  such that  $d(x, Y) = 0$ . Fix  $y \in Y$ . The number  $n_1$  of paths  $(\gamma)_{[n_1]}$  in  $X$  between  $x$  and  $y$  is finite since the number of cycles embedded in  $X$  is finite. Suppose that for each  $i \in [n_1]$ , there exists a point  $x_i$  of  $N$  on the interior of  $\gamma_i$ .

Suppose by contradiction that there exists a point  $y_2$  in  $Y$  such that  $d(y_2, x) < \min_{i \in [n_1]} d(x, x_i)$ . Since  $Y$  is connected, there is a path  $\gamma_{n_1+1}$  between  $y_2$  and  $y$  without points of  $N$ . By compactness, there exists  $z \in \gamma_{n_1+1}$  which minimizes the distance  $d(x, \gamma_{n_1+1})$ . By definition of the  $\mathbb{R}$ -graph, there exists also a path  $\gamma_{n_1+2}$  between  $x$  and  $z$  with length  $d(x, z)$ . By definition of  $z$ , the concatenation of  $\gamma_{n_1+2}$  and  $\gamma_{n_1+1}$  is a path  $\gamma_{n_1+3}$ . Furthermore, for any  $i \in [n_2]$ , there exists no point  $x' \in N \cap \gamma_{n_1+3}$  such that  $d(x', x) = d(x, x_i)$ , and thus  $\gamma_{n_1+3} \neq \gamma_i$ . We thus have a contradiction with the fact that there is only  $n_1$  paths between  $y$  and  $x$ . Hence, when for each  $i \in [n_1]$ , there exists a point  $x_i$  of  $N$  on the interior of  $\gamma_i$ ,  $x$  is not in  $\text{Adh}_d(Y)$ .

If now, there exists a path  $\gamma$  between  $x$  and  $y$  without point of  $N$  in its interior, either  $x$  is a point of  $N$  either  $x$  is connected to  $y$  and thus is in  $Y$  by definition.

Let us deal with the second point. Consider  $(y_n)$  a Cauchy sequence of  $Y$  for  $d_C$ . By (3.147),  $(y_n)$  is also a Cauchy sequence for  $d$  and thus converges to a point of  $Y \cup N_Y$  by the previous point. Suppose that  $(y_n)$  converges to  $x \in N_Y$  for  $d$ . Take  $\epsilon > 0$  such that  $B_d(x, \epsilon) \cap X$  is a  $\mathbb{R}$ -tree. By assumption, the number  $n_2$  of connected components of  $[B_d(x, \epsilon/3) \cap X] \setminus \{x\}$  is finite. Let us call  $(X_i)_{i \in [n_2]}$  these components. If for  $1 \leq i < j \leq n_2$ ,

$\gamma$  is a path in  $X$  between  $x_i \in X_i$  and  $x_j \in X_j$  such that  $L(\gamma) \leq 2/3\epsilon$  then  $x \in \gamma$ . Indeed, in the contrary case,  $B_d(x, \epsilon) \cap X$  would not be a  $\mathbb{R}$ -tree. Furthermore, there exists  $n_0 \in \mathbb{N}$ , such that for any  $n, m \geq n_0$ ,  $y_n \in B_d(x, \epsilon/3) \cap X \setminus \{x\}$  and  $d_C(y_n, y_m) < 2/3\epsilon$ . Therefore, there exists  $i \in [n_2]$ , such that for any  $n \geq n_0$ ,  $y_n \in X_i$ .

Consider now another Cauchy sequence  $(z_n)$  for  $d_C$ . Suppose first that  $(z_n)$  does not converge to  $x$  for  $d$ . In virtue of (3.147), we cannot have  $d_C(z_n, y_n) \rightarrow 0$  as  $n \rightarrow +\infty$ . Suppose now that  $(z_n)$  converges to  $x$  for  $d$ , but is not always in  $X_i$  for  $n$  large enough. By the same arguments as above, we would have  $d_C(y_n, z_n) \geq 2/3\epsilon$  for  $n$  large enough. Suppose finally that  $(z_n)$  converges to  $x$  for  $d$  and is always in  $X_i$  for  $n$  large enough. The fact that there exists an only path between two components of  $B_d(x, \epsilon/3) \cap X \setminus \{x\}$  with length smaller than  $2/3\epsilon$  implies that the restrictions of  $d$  and  $d_C$  to  $X_i$  are equal. Therefore  $d_C(y_n, z_n) \rightarrow 0$ .

Consider finally  $S$  the set of equivalence classes on the set of Cauchy sequences of  $Y$  for the relation  $u \sim v$  if and only if  $d_C(u_n, v_n) \rightarrow 0$ . Fix  $\tilde{Y}$  the equivalence classes of the constant sequences of  $Y$ . We have thus proved that  $S \setminus \tilde{Y}$  can be indexed by a subset of  $N_Y \times [D]$ , where  $D$  is the maximal degree of a point of  $X$ . By the classical construction of the completion, this implies the second point.

We now consider the third point. Fix  $y \in Y$  and  $(y_n) \in Y^{\mathbb{N}}$ . If  $d_C(y, y_n) \rightarrow 0$  then  $d(y, y_n) \rightarrow 0$  by (3.147). If now  $d(y, y_n) \rightarrow 0$ , by using the same arguments as above, we can find  $\epsilon > 0$  such that on  $B_d(y, \epsilon)$ ,  $d$  and  $d_C$  coincide and thus  $d_C(y, y_n) \rightarrow 0$ .

Finally, let  $(y_n)$  be a sequence of  $\overline{Y}$ . We can define a sequence  $(\tilde{y}_n)$  of  $Y \cup N_Y$ , such that for any  $n$ ,  $\tilde{y}_n = y_n$  if  $y_n \in Y$  and else  $\tilde{y}_n = x$ , where  $x$  is the element of  $N_Y$  naturally associated to  $y_n$  as in the proof of the second point.  $Y \cup N_Y$  is compact for the topology induced by  $d$  as a closed subset of a compact. Therefore, there exists a subsequence  $(\tilde{y}_{\sigma(n)})$  of  $(\tilde{y}_n)$  such that  $(\tilde{y}_{\sigma(n)})$  converges to  $y \in Y \cup N_Y$  for the metric  $d$ . If  $y \in Y$ , then by the same arguments as above  $(y_{\sigma(n)})$  converges to  $y$ . If now,  $y \in N_Y$ , since  $D$  the maximal degree of a point of  $X$  is finite, there exists a subsequence  $(\tilde{y}_{\sigma(\psi(n))})$  which is always in a component  $X_i$  defined as above, and by using the same arguments,  $(y_{\sigma(\psi(n))})$  converges for  $d_C$  to a point of  $(\overline{Y}, \overline{d_C})$ . Therefore,  $(\overline{Y}, \overline{d_C})$  is compact. □

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