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ECOLE DOCTORALE DES SCIENCES FONDAMENTALES

THESE

présentée pour obtenir le grade de

DOCTEUR D'UNIVERSITE

Spécialité : Mathématiques appliquées

Par **Youssef KOSAD**

**Analyse spectrale et comportement asymptotique
des solutions de quelques modèles
d'équations de transport**

Soutenue publiquement le 19 décembre 2017 devant le jury composé de :

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Introduction Générale

1.1 Introduction

Le projet de thèse présenté dans ce manuscrit porte essentiellement sur la théorie spectrale de quelques classes d'opérateurs de transport et le comportement asymptotique (pour les temps grands) des solutions des problèmes de Cauchy gouvernés par ces derniers. Outre l'introduction, cette thèse est composée de cinq chapitres. L'objectif du deuxième chapitre est de collecter quelques résultats d'analyse fonctionnelle et les outils mathématiques utilisés tout au long de ce travail afin de rendre le texte plus lisible sans recourir systématiquement à la littérature. Le troisième chapitre est consacré aux propriétés spectrales des opérateurs d'advection et de transport des neutrons dans les espaces L^p , avec $1 \leq p < \infty$. L'objet de ce chapitre est d'étendre les résultats obtenus dans les travaux [62, 49] au cadre multidimensionnel avec des conditions aux limites générales. On donne entre autre une description fine du spectre ponctuel réel de l'opérateur de transport des neutrons (résultats d'existence ou d'inexistence des valeurs propres, finitude, infinitude, localisation des valeurs propres, estimation de leur nombre, etc.). On étudiera aussi l'irréductibilité du semi-groupe fortement continu engendré par l'opérateur de transport ce qui nous fournira une bonne description asymptotique ($t \rightarrow \infty$) de ce semi-groupe en norme d'opérateur. Le quatrième chapitre concerne les propriétés de régularité et le comportement, en temps long, de la solution du problème de Cauchy (1.3.1) gouverné par l'opérateur de transport considéré dans le chapitre 3 avec des conditions aux limites dissipatives et satisfaisant (1.3.4). Par souci d'exhaustivité, on étudie aussi le cas où l'opérateur frontière est multiplicatif compact. Le chapitre 5 porte sur le caractère bien posé et le comportement asymptotique de la solution d'une équation de transport des neutrons avec des sections efficaces non bornées. Contrairement au chapitre précédent, l'analyse de ce problème nécessite l'utilisation d'une théorie des perturbations non bornées (de semi-groupe) de type Miyadera. Dans le dernier chapitre, on s'intéresse à un problème linéaire issu d'un modèle introduit en 1974 par Lebowitz et Rubinow [51] décrivant la prolifération d'une population de cellules pour des longueurs des cycles infinies. Il s'agit d'une équation de transport à vitesse constante (égale à 1) avec des conditions aux limites générales. Ce chapitre est motivé par les travaux [48, 56, 1] et sa structure suit les mêmes cheminements que les

chapitres 3 et 4.

1.2 Description des résultats du Chapitre 3

Dans ce chapitre, on étudie les propriétés spectrales d'un opérateur de transport dans une géométrie bornée où les phénomènes aux bords sont modélisés par un opérateur frontière positif H reliant les flux rentrant et sortant. Plus précisément, on s'intéresse à la description du spectre asymptotique de l'opérateur intégral-différentiel suivant

$$\begin{aligned} A_H \psi(x, v) &= -v \cdot \nabla_x \psi(x, v) - \sigma(x, v) \psi(x, v) + \int_V \kappa(x, v, v') \psi(x, v') d\mu(v') \\ &= T_H \psi + K \psi \end{aligned}$$

où $(x, v) \in D \times V$. Ici D est un sous ensemble borné régulier de \mathbb{R}^n (espace physique où évoluent les particules), μ une mesure de Radon positive sur \mathbb{R}^n de support V , vérifiant $\mu(0) = 0$ et couvrant à priori les différents modèles (c'est à dire la mesure de Lebesgue sur \mathbb{R}^n ou sur le sphère ou même la combinaison des deux). Cette équation modélise le transport des particules (neutrons, photons, molécules de gaz, etc.) dans le domaine D . La fonction $\psi(x, v)$ représente la densité de probabilité des particules ayant la position x et la vitesse v . Les fonctions $\sigma(\cdot, \cdot)$ et $\kappa(\cdot, \cdot, \cdot)$, sont respectivement, la fréquence de collision et le noyau de l'opérateur K , appelé opérateur de collision. Enfin, l'opérateur non borné $T_H = A_H - K$ est appelé opérateur d'advection.

L'équation de transport a été considérée dans différents domaines de la physique mathématique pour décrire les processus de transport des particules. Ainsi, en cinétique des gaz où l'on doit décrire l'interaction des particules du gaz avec la paroi du mur limitant le domaine où le gaz évolue, le problème théorique consiste à lier la fonction distribution du flux des particules du gaz quittant le domaine à la fonction distribution du flux des particules du gaz entrant le domaine. D'un point de vu mathématique, l'interaction gaz/surface détermine les conditions aux bords. Cependant, les conditions aux limites décrivant cette interaction avec le bord du domaine sont très complexes et leur formulation mathématique précise est sujette à controverse (voir, par exemple, [15, 16, 26]). Néanmoins, un modèle souvent utilisé consiste à supposer qu'une part du flux sortant du domaine est réémise dans une direction déterministe (réflexion spéculaire), alors que l'autre part est rémise dans des directions aléatoires (réflexions diffuses). Dans notre contexte, les conditions aux limites sont modélisées par l'équation suivante

$$\psi^- = H\psi^+, \quad (1.2.1)$$

où ψ^- (resp. ψ^+) est la restriction de ψ à Γ_- (resp. Γ_+) avec Γ_- (resp. Γ_+) est la partie rentrante (resp. sortante) du domaine et H un opérateur linéaire défini sur des espaces des traces appropriés de l'espace de phase couvrant tous les modèles physiques bien connus.

Depuis les travaux de J. Lehner et G. M. Wing [52, 53], l'analyse spectrale joue un rôle central en théorie de transport. En effet, quand on considère le problème de Cauchy associé à l'opérateur

A_H (voir chapitre 3), on établit facilement, via le théorème de Hille-Yosida-Phillips (Théorème 2.3.1), que ce dernier est bien posé. Cette approche n'étant pas constructive, pour avoir plus d'informations sur la solution de ce problème, et en particulier sur son comportement en temps long, la connaissance du spectre asymptotique de A_H , $\sigma_{as}(A_H) := \sigma(A_H) \cap \{\lambda \in \mathbb{C} : \text{Re} \lambda > s(T_H)\}$, est fondamentalement importante. Ainsi, l'analyse spectrale est devenue maintenant un sujet classique en théorie de transport et une littérature abondante a été consacrée à l'étude spectrale des opérateurs de transport avec des conditions aux limites absorbantes, c'est à dire $H = 0$. Citons, par exemple, les contributions de J. Lehner et G. M. Wing [52], K. Jörgens [34], M.L. Demeru et B. Montagnini [24], S. Albertoni et B. Montagnini [2], E. W. Larsen et P. F. Zweifel [38], A. Palczewski [69], S. Ukai [80], I. Vidav [82], M. Mokhtar-Kharroubi [63, 65, 66] et les références qui s'y trouvent. On note que ces résultats sont basés sur des arguments de compacité (voir, par exemple, [65]). En présence d'un opérateur de frontière H non nul, excepté le cadre monodimensionnel où quelques progrès ont été réalisés (voir, par exemple, [19, 23, 41, 46]), on ne dispose que des résultats partiels. Citons, par exemple, les travaux de L. Arlotti [4], J. Chen et M. Z. Yang [21], G. Frosali, C. V. M. Van der Mee et Protopopescu [30]. En utilisant un résultat de compacité en dimension $N \geq 2$ établi dans [42], K. Latrach et B. Lods [44] ont fait une description fine du spectre de l'opérateur de transport A_H pour des conditions aux limites de type bounce-back. L'objet de ce chapitre est de généraliser ces résultats en donnant une analyse fine des spectres des opérateurs T_H et A_H pour des opérateurs frontières plus généraux lorsque l'espace des positions est un ouvert convexe et régulier de \mathbb{R}^N , avec $N \geq 2$. Notre analyse repose sur des arguments de compacité établis dans [42].

1.3 Description des résultats du chapitre 4

Ce chapitre est consacré à la régularité et le comportement asymptotique de la solution du problème de Cauchy suivant

$$\left\{ \begin{array}{l} \frac{\partial \psi}{\partial t}(x, v, t) = A_H \psi(x, v, t) := T_H \psi(x, v, t) + K \psi(x, v, t) \\ \qquad \qquad \qquad = -v \cdot \nabla_x \psi(x, v, t) - \sigma(v) \psi(x, v, t) + \int_{\mathbb{R}^n} \kappa(x, v, v') \psi(x, v', t) dv', \\ \psi(x, v, 0) = \psi_0(x, v), \end{array} \right. \quad (1.3.1)$$

où $(x, v) \in D \times \mathbb{R}^n$ et ψ_0 est la donnée initiale. Pour la signification des différents paramètres intervenant dans l'équation, on renvoie à la description du chapitre 3. Comme précédemment, les conditions aux limites satisfont l'équation (1.2.1).

Moyennant quelques hypothèses sur la norme de l'opérateur H , on montre que l'opérateur d'advection T_H engendre un semi-groupe fortement continu $(U_H(t))_{t \geq 0}$. En outre, comme l'opérateur de collision K est supposé borné, par le théorème classique des perturbations bornées de Phillips (voir, par exemple, [70, Theorem 1.1, p. 76]), A_H engendre un semi-groupe fortement

continu $(V_H(t))_{t \geq 0}$. Ce dernier est donné par la série de Dyson-Phillips

$$V_H(t) = \sum_{j=0}^{n-1} U_j(t) + R_n^H(t), \quad (1.3.2)$$

où $U_0(t) = U_H(t)$, $U_j(t) = \int_0^t U_H(s) K U_{j-1}(t-s) ds$ ($j \geq 1$) et le reste d'ordre n de la série (1.3.2) est donné par

$$R_n^H(t) = \sum_{j=n}^{\infty} U_j(t).$$

Si $\psi_0 \in D(A_H)$ (domaine de A_H), alors le problème de Cauchy (1.3.1) admet une solution unique donnée par $\psi(t) = V_H(t)\psi_0$. Cette procédure n'est pas constructive, et dans la mesure où on veut obtenir des informations sur cette solution, en particulier son comportement en temps long, la connaissance du spectre de A_H ou de $(V_H(t))_{t \geq 0}$ joue un rôle central. Ceci est lié directement à la compacité d'un reste d'ordre n , $R_n^H(t)$, de la série de Dyson-Phillips. En effet, si un certain reste $R_n^H(t)$ est compact, alors pour tout $\nu > 0$, $\sigma(V_H(t)) \cap \mathbb{C} \setminus B(0, r_{ess}(U_H(t)) + \nu)$ consiste au plus à un nombre fini des valeurs propres de multiplicité algébrique finie (voir Théorème 2.3.2). En d'autres termes,

$$\sigma(V_H(t)) \cap \mathbb{C} \setminus B(0, r_{ess}(U_H(t)) + \nu) = \{\lambda_1, \dots, \lambda_n\},$$

où $r_{ess}(U_H(t))$ est le rayon spectral essentiel de l'opérateur $U_H(t)$ (pour les définitions des spectres essentiels, voir Section 2.2). Ainsi, si on désigne par X_n le sous-espace spectral associé à $\{\lambda_1, \dots, \lambda_n\}$ et si $\psi_0 \in D(A_H)$, alors la solution du Problème (1.3.2) s'écrit sous la forme

$$\psi(t) = R(t) + \sum_{j=1}^n e^{\lambda_j t} e^{tD_j} P_j \psi_0, \quad (1.3.3)$$

où D_j (resp. P_j) désigne l'opérateur nilpotent (resp. la projection spectrale) associé à la valeur propre λ_j . La quantité $R(t)$ est bornée indépendamment du temps par $\varepsilon > 0$ (qui est très petit) et désigne la part de la solution dans un espace de dimension infini (supplémentaire de l'espace X_n), et donc le comportement asymptotique de la solution est donné par

$$\sum_{j=1}^n e^{\lambda_j t} e^{tD_j} P_j \psi_0$$

qui correspond à la part de la solution dans l'espace de dimension fini X_n .

Ces idées ont été tout d'abord introduites et développées par J. Lehner et G. M. Wing [52, 53], K. Jörgens [34] et I. Vidav [82, 83] pour des conditions aux limites absorbantes (c'est à dire $H = 0$). Ensuite, J. Voigt [84] a montré que le reste d'ordre deux de la série de Dyson Phillips, $R_2^0(t)$, est compact pour une classe d'opérateurs de collision assez générales. D'autres résultats

dans le même esprit ont été obtenus par M. Mokhtar-Kharroubi [65, chapitre 4] et L. Weis [91]. En présence d'un opérateur frontière abstrait ($H \neq 0$), le C_0 -semi-groupe $(U_H(t))_{t \geq 0}$ n'est pas en général explicite et donc l'étude de la compacité du reste de la série de Dyson-Phillips est très compliquée. En utilisant la formule de Dunford [70, Corollary 7.5, p. 29], une approche alternative souvent utilisée, appelée approche résolvente, consiste à faire une décomposition spectrale de la solution du Problème (1.3.3) pour ainsi obtenir le comportement asymptotique de cette dernière lorsque $\psi_0 \in D(A_H^2)$ [64] (voir aussi [40]). L'inconvénient de cette approche est que, contrairement à la première, elle nécessite que la donnée initiale ψ_0 soit dans $D(A_H^2)$ [44, 77]. Pour des conditions aux bords non absorbante ($H \neq 0$), cette approche est systématiquement utilisée excepté quelques cas particuliers, comme dans [45] où les auteurs ont montré, via une méthode d'interpolation, que le reste d'ordre un de la série de Dyson Phillips, $R_1^H(t)$, est compact dans les espaces L^p , $1 < p < \infty$, pour des conditions aux limites de type bounce-back boundary. L'objectif de ce chapitre est de poursuivre et améliorer les résultats obtenus dans [45]. On montre, via une méthode d'interpolation, que le reste d'ordre 1 de la série de Dyson-Phillips est compact dans les espaces L^p , avec $1 < p < \infty$, pour un opérateur frontière satisfaisant

$$H\psi(x, v) = \alpha I_1 \psi(x, v) + \beta \psi(x, -v), \quad \alpha, \beta \in [0, +\infty), \quad (1.3.4)$$

où I_1 est un opérateur compact et les constantes α et β sont choisies de sorte que $\|H\| < 1$. Ce qui nécessite uniquement l'hypothèse $\psi_0 \in D(A_H)$. Pour le cas $p = 1$, moyennant un résultat de compacité (cf. Proposition 4.1.1), on montre que tous les restes d'ordre n de la série de Dyson-Phillips sont compacts pour $n \geq 9$. On étudie également le cas où l'opérateur frontière est multiplicatif et compact.

1.4 Description des résultats du chapitre 5

Dans ce chapitre, on étudie le caractère bien posé et le comportement asymptotique (en temps), dans l'espace L^1 , d'une équation de transport singulier. Plus précisément, on s'intéresse à l'équation suivante

$$\left\{ \begin{array}{l} \frac{\partial \psi}{\partial t}(x, v, t) = A_H \psi(x, v, t) := T_H \psi(x, v, t) + K \psi(x, v, t) \\ \qquad \qquad \qquad = -v \cdot \nabla_x \psi(x, v, t) - \sigma(v) \psi(x, v, t) + \int_{\mathbb{R}^n} \kappa(x, v, v') \psi(x, v', t) dv', \\ \psi(x, v, 0) = \psi_0(x, v). \end{array} \right. \quad (1.4.1)$$

où $(x, v) \in D \times \mathbb{R}^n$ et ψ_0 est la donnée initiale. Pour la signification des différents paramètres intervenant dans l'équation, on renvoie à la description du chapitre 2. Les conditions aux limites sont modélisées par

$$(H\psi)(x, v, t) = \gamma \psi(x, -v, t), \quad (x, v) \in \Gamma_- \quad t > 0, \quad (1.4.2)$$

avec $\gamma \in (0, 1)$.

Par opérateur de transport singulier, on sous entend le fait que la fréquence de collision $\sigma(\cdot)$ et l'opérateur de collision

$$K\psi \rightarrow \int_{\mathbb{R}^n} \kappa(x, v, v')\psi(x, v', t)dv'$$

ne sont pas bornés.

Contrairement aux équations de transport classiques (comme, par exemple, le Problème (1.3.1)) l'étude de l'équation de transport singulier est plus délicate. En effet, l'opérateur A_H étant une perturbation non bornée de T_H , on ne peut pas faire appel à la théorie classique des perturbations pour en déduire l'existence d'une solution unique au sens du semi-groupe de l'équation (1.4.1). Dans ce cas l'analyse nécessite l'usage d'une théorie de perturbation non bornées de type Miyedera.

Depuis le papier de Chabi et Mokhtar-Kharroubi [65, Chapter 9], plusieurs travaux ont été consacrés à l'analyse de l'équation de transport singulier. L'existence et l'unicité ainsi que le comportement asymptotique de l'équation (1.4.1) ont été étudiés par Chabi et Mokhtar-Kharroubi [65, Chapter 9] et Lods [55] dans l'espace L^1 pour des conditions aux limites absorbantes (i.e. $H = 0$) dans une géométrie bornée. Dans [19] (resp. [20]), Chabi et Latrach ont étudié, dans le cadre monodimensionnel, le Problème (1.4.1) dans l'espace L^1 (resp. les espaces L^p , avec $1 < p < \infty$) pour des conditions aux bords non absorbantes (reflexives et periodiques). Notre but ici est d'étudier le Problème (1.4.1)-(1.4.2) dans le cadre multidimensionnel. Pour cela, on établit que ce dernier est bien posé et on montre ensuite que le reste d'ordre deux de la série de Dyson-Phillips est faiblement compacte ce qui implique, via des arguments classiques, le comportement asymptotique (pour les temps longs) de la solution.

1.5 Description des résultats du chapitre 6

Ce chapitre concerne l'analyse mathématique d'un modèle introduit par Lebowitz et Rubinow [51] décrivant la prolifération d'une population de cellules pour des longueurs des cycles infinies. On considère dans un premier temps le problème de Cauchy suivant

$$\begin{cases} \frac{\partial \psi}{\partial t}(a, l, t) = S_K \psi(a, l, t) := -\frac{\partial \psi}{\partial a}(a, l, t) - \mu(a, l)\psi(a, l, t) \\ \psi(a, l, 0) = \psi_0(a, l) \end{cases} \quad (1.5.1)$$

où

$$(a, l) \in \Omega := \{(a, l) : 0 \leq a \leq l, 0 \leq l_1 \leq l < l_2\}.$$

Cette équation est issue d'un modèle introduit en 1974 par Lebowitz et Rubinow [51] décrivant la prolifération d'une population de cellules. Selon leur point de vue, chaque cellule est distinguée par son âge a et sa longueur du cycle l . La longueur du cycle l d'une cellule est le temps entre la naissance et la division et c'est une propriété héréditaire. A la naissance, l'âge d'une cellule est

nul ($a = 0$) et à la division son âge est égal à la longueur du cycle l ($a = l$). La constante l_1 (resp. l_2) désigne la longueur minimale (resp. maximale) du cycle. La fonction $\psi(a, l, t)$ représente la densité de population ayant l'âge a et la longueur du cycle l à l'instant t . La fonction $\mu(\cdot, \cdot)$ est le taux de mortalité des cellules dû à des causes autres que la division.

La loi biologique reliant les cellules mères et les cellules filles est donnée par l'équation abstraite suivante

$$K\psi(l, l, t) = \psi(0, l, t), \quad (1.5.2)$$

où K est un opérateur linéaire, appelé opérateur de transition, défini sur des espaces des traces appropriés de l'espace de phase couvrant tous les modèles connus (voir [51, 89, 90, 48, 56] et les références qui s'y trouvent).

Lorsque la longueur maximale du cycle est finie ($l_2 < +\infty$), le problème (1.5.1)-(1.5.2) a été étudié par plusieurs auteurs pour des opérateurs de transition K particuliers. On note que l'analyse spectrale de l'opérateur S_K et le comportement asymptotique de la solution du problème (1.5.1)-(1.5.2) ont été étudiés par Webb [89, 90] dans l'espace des fonctions continues (qui n'est pas l'espace approprié pour ce problème). Dans [48], Latrach et Mokhtar-Kharroubi ont donné une description fine du spectre de S_K dans les espaces L^p , avec $p \in]1, \infty[$. Ils ont en outre montré, via un résultat de génération dû à Batty et Robinson [8], que l'opérateur S_K engendre un C_0 -semi-groupe dans l'espace L^1 pour des conditions au bords multiplicatives (i.e. $\|K\| \geq 1$). En supposant $l_1 > 0$, ce résultat a été généralisé ensuite par Boulanouar [11] aux espaces L^p , avec $p \in [1, \infty[$ et il a étudié par la même occasion le comportement asymptotique de la solution du problème (1.5.1)-(1.5.2). Moyennant un opérateur de transition K petit au voisinage de $l = 0$ quel que soit la taille de sa norme $\|K\|$. Lods et Mokhtar-Kharroubi [56] ont établi un résultat de génération pour le cas où $l_1 = 0$. A notre connaissance, excepté une version stationnaire du problème (1.5.1)-(1.5.2), ce problème n'a pas été étudié pour une longueur maximale du cycle infinie, c'est à dire $l_2 = +\infty$. Le but de ce chapitre est de combler cette lacune en considérant le cas $l_2 = +\infty$.

On commence notre travail par analyser le spectre de l'opérateur S_K . On montre que $\sigma(S_K) \cap \{\text{Re}\lambda > -\underline{\mu} + \varepsilon, \varepsilon > 0\}$, où $\underline{\mu}$ est défini dans la Section 6.2, consiste, au plus, à un nombre fini de valeurs propres isolées de multiplicité algébrique finie que l'on notera $\{\lambda_1, \dots, \lambda_n\}$. On établit ensuite un résultat de génération de semi-groupe. On termine cette première partie par donner une décomposition spectrale de la solution du Problème (1.5.1)-(1.5.2). En outre, on montre que pour toute donnée initiale, ψ_0 , appartenant à $D(S_K^2)$, cette solution satisfait

$$\|\psi(t) - \sum_{i=1}^n e^{\lambda_i t} e^{D_j t} P_j \psi_0\| \rightarrow 0, \text{ quand } t \rightarrow +\infty. \quad (1.5.3)$$

On note que l'inconvénient de ce résultat est dû au fait que l'estimation (1.5.3) n'est vraie que pour $\psi_0 \in D(S_K^2)$. Ce résultat ne peut pas être amélioré.

On considère ensuite le problème suivant qui est une perturbation du modèle (1.5.1)

$$\left\{ \begin{array}{l} \frac{\partial \psi}{\partial t}(a, l, t) = A_K \psi(a, l, t) := -\frac{\partial \psi}{\partial a}(a, l, t) - \mu(a, l) \psi(a, l, t) + (B\psi)(a, l, t), \\ \psi(a, l, 0) = \psi_0(a, l), \\ K(\psi') = \psi^0 \end{array} \right. = S_K \psi(a, l, t) + (B\psi)(a, l, t) \quad (1.5.4)$$

où

$$(B\psi)(a, l, t) := \int_{l_1}^{\infty} \kappa(a, l, l') \psi(a, l', t) \chi_{\Omega}(a, l') dl'.$$

Ici B est un opérateur linéaire qui décrit la transition des cellules de longueur de cycle l' en des cellules de longueur de cycle l et $\kappa(\cdot, \cdot, \cdot)$ est une fonction mesurable. Outre l'analyse spectrale de l'opérateur A_K , l'objectif de cette partie est de montrer que la solution du Problème (1.5.4) satisfait, et contrairement au Problème (1.5.1), l'estimation (1.5.3) pour toute donnée initiale, ψ_0 , appartenant à $D(S_K)$.

On termine cette introduction par quelques indications bibliographiques concernant ce travail.

- Le chapitre 3 a fait l'objet d'un article paru dans : *Journal of Mathematical Analysis and Applications*, 448, (2017), 506-537.
- Le chapitre 4 est soumis pour publication au journal : *Applicable Analysis*.
- Le chapitre 5 est soumis pour publication au journal : *Electronic Journal of Differential Equations*.
- Le chapitre 6 est soumis pour publication dans : *Journal of Mathematical Analysis and Applications*.

Résultats préliminaires

Dans ce chapitre, on rappelle quelques résultats d'analyse fonctionnelle et les différentes notions utilisées dans ce travail. Ceci permettra de le rendre plus autonome et sans avoir à recourir systématiquement à la littérature.

2.1 Éléments de la théorie spectrale

Soient X un espace de Banach complexe et $A : \mathcal{D}(A) \subseteq X \rightarrow X$ un opérateur linéaire fermé à domaine dense. L'ensemble résolvant de A est le sous-ensemble défini par

$$\rho(A) := \{\lambda \in \mathbb{C} : (\lambda - A) : \mathcal{D}(A) \subseteq X \rightarrow X \text{ est bijectif}\}.$$

Il s'agit d'un ouvert du complexe. Le complémentaire de $\rho(A)$ est appelé le spectre de l'opérateur A et est noté $\sigma(A) := \mathbb{C} \setminus \rho(A)$. Le spectre de A est par conséquent un ensemble fermé de \mathbb{C} .

Pour $\lambda \in \rho(A)$, on note $R(\lambda, A)$ la résolvante de A au point λ avec

$$R(\lambda, A) := (\lambda - A)^{-1}.$$

La fonction $\lambda \mapsto R(\lambda, A)$ est analytique sur $\rho(A)$ et à valeur dans $\mathcal{L}(X)$. Elle satisfait l'équation suivante (appelée identité de la résolvante)

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A) \quad \forall \lambda, \mu \in \rho(A).$$

On appelle borne spectrale de A , $s(A)$, le nombre réel suivant

$$s(A) := \sup \{\operatorname{Re} \lambda : \lambda \in \sigma(A)\}$$

avec la convention $s(A) = -\infty$ si $\sigma(A) = \emptyset$ et $s(A) = +\infty$ si $\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} \cap [0, \infty[$ est non borné. Si de plus A est un opérateur borné, alors $\sigma(A)$ est un ensemble compact non vide. Dans ce cas, on appelle rayon spectral de A , le réel défini par

$$r_\sigma(A) := \max \{ |\lambda| : \lambda \in \sigma(A) \}.$$

On va maintenant décrire quelques sous ensembles importants du spectre de l'opérateur A .

Définition 2.1.1 Soit A un opérateur fermé à domaine dense d'un espace de Banach X dans lui même.

- On appelle spectre ponctuel de A l'ensemble

$$\sigma_p(A) := \{ \lambda \in \mathbb{C} : (\lambda - A) : \mathcal{D}(A) \rightarrow X \text{ n'est pas injectif} \}.$$

- On appelle spectre résiduel de A l'ensemble

$$\sigma_r(A) := \{ \lambda \in \mathbb{C} : (\lambda - A) \text{ est inversible et } \overline{(\lambda - A)(X)} \neq X \}.$$

- On appelle spectre continu de A l'ensemble

$$\sigma_c(A) := \{ \lambda \in \mathbb{C} : (\lambda - A) \text{ est inversible et à image dense tel que } (\lambda - A)^{-1} \text{ est discontinu} \}.$$

De plus, les sous ensembles $\sigma_p(A)$, $\sigma_r(A)$ et $\sigma_c(A)$ forment une partition de $\sigma(A)$.

Les éléments de $\sigma_p(A)$ sont appelés valeurs propres de A . Si λ est une valeur propre de A , alors il existe un élément x non nul appartenant à $\mathcal{D}(A)$, appelé vecteur propre (ou fonction propre), tel que $(\lambda - A)x = 0$.

Définition 2.1.2 Soit A un opérateur fermé à domaine dense d'un espace de Banach X dans lui même. On appelle spectre approché l'ensemble

$$\sigma_{\text{app}}(A) := \{ \lambda \in \mathbb{C} : (\lambda - A) : \mathcal{D}(A) \rightarrow X \text{ n'est pas injectif ou à image non fermé} \}.$$

On rappelle que (cf. [68, p. 64])

$$\sigma(A) = \sigma_{\text{app}}(A) \cup \sigma_r(A). \quad (2.1.1)$$

On note que les sous ensembles $\sigma_{\text{app}}(A)$ et $\sigma_r(A)$ ne sont pas nécessairement disjoints.

Il résulte de l'équation (2.1.1) et de la Définition 2.1.1 que

$$\sigma_p(A) \cup \sigma_c(A) \subseteq \sigma_{\text{app}}(A).$$

Le résultat suivant donne une caractérisation du spectre approché.

Proposition 2.1.1 Soit $\lambda \in \sigma(A)$. Alors $\lambda \in \sigma_{\text{app}}(A)$ si, et seulement si, il existe une suite $(x_n)_{n \in \mathbb{N}}$ de $\mathcal{D}(A)$, appelée suite approximante, telle que

$$\|x_n\| = 1 \quad \text{et} \quad \|(\lambda - A)x_n\| \rightarrow 0 \quad \text{quand } n \rightarrow \infty.$$

Si λ est un point isolé du spectre de A , alors il existe un $\delta > 0$ tel que la fonction $\mu \rightarrow R(\mu, A)$ est développable en série de Laurent sur $D(\lambda, \delta) \setminus \{\lambda\}$ donnée par

$$R(\mu, A) = \sum_{n \in \mathbb{Z}} (\mu - \lambda)^n T_n,$$

où

$$T_n = \frac{1}{2i\pi} \int_{\gamma} \frac{R(\mu, A)}{(\mu - \lambda)^{n+1}} d\mu$$

avec γ est le cercle de centre λ et de rayon $\delta/2$ parcouru une fois dans le sens direct. Le résidu $P_\lambda := T_{-1} \in \mathcal{L}(X)$ est une projection appelée projection spectrale associée à λ . S'il existe un entier $k > 0$ tel que $T_{-k} \neq 0$ et $T_{-n} = 0$ pour tout entier $n > k$, alors λ est un pôle d'ordre k de la résolvante $R(\cdot, A)$. On dit alors que λ est une valeur propre isolée de A , la dimension de l'image de P_λ est appelée la multiplicité de λ (voir, par exemple, [35, p. 180]).

2.2 Spectre essentiel

Dans ce paragraphe, on rappelle quelques propriétés du spectre essentiel. Soient X un espace de Banach complexe et A un opérateur linéaire borné de X dans lui-même. On définit l'ensemble résolvant essentiel de A par

$$\rho_{\text{ess}}(A) := \rho(A) \cup \left\{ \lambda \in \sigma(A); \lambda \text{ est une valeur propre isolée de multiplicité algébrique finie} \right\}.$$

Le complémentaire de $\rho_{\text{ess}}(A)$ est le spectre essentiel de A , noté $\sigma_{\text{ess}}(A)$ [84]. On note qu'en général $\sigma(A) \setminus \sigma_{\text{ess}}(A) \subset \sigma_p(A)$ mais n'est pas réduit à des valeurs propres isolées de multiplicité algébrique finie. On définit le *rayon spectral essentiel* de A par

$$r_{\text{ess}}(A) := \sup \left\{ |\lambda|; \lambda \in \sigma_{\text{ess}}(A) \right\}.$$

Pour un opérateur fermé à domaine dense A , il existe plusieurs définitions non équivalentes du spectre essentiel. Toutefois, quel que soit la définition du spectre essentiel considérée le rayon spectral essentiel est toujours le même (cf. [28]). En connexion avec notre travail, on se limitera ici au spectre essentiel de Browder (pour la définition du spectre essentiel de Browder voir plus bas). En effet, si on note par $\sigma_{\text{eb}}(A)$ le spectre essentiel au sens de Browder de A , alors $\sigma(A) \setminus \sigma_{\text{eb}}(A)$ consiste, au plus, à des valeurs propres isolées de multiplicité algébrique finie. Par conséquent $\sigma_{\text{eb}}(A)$ coïncide avec le spectre essentiel des opérateurs bornés défini au début du paragraphe.

Soit A un opérateur linéaire fermé à domaine dense dans X . Les sous ensembles $R(A)$ et $N(A)$ sont appelés, respectivement, l'image et le noyau de l'opérateur A . On dit que A est un opérateur de Fredholm si:

(1) $\alpha(A) := \dim(N(A)) < \infty$;

(2) $R(A)$ est fermé;

(3) $\beta(A) := \text{codim}(R(A)) < \infty$

où $\text{codim}(R(A)) := \dim(X/\overline{R(A)})$.

Si A est un opérateur de Fredholm, alors on appelle indice de A , le nombre $i(A) := \alpha(A) - \beta(A)$. On définit les sous ensembles suivants

$$\rho_{esc}(A) := \{\lambda \in \mathbb{C}; \lambda - A \text{ est un opérateur de Fredholm sur } X \text{ et } i(A) = 0\}$$

et

$$\rho_{eb}(A) := \{\lambda \in \rho_{esc}(A); \text{ tel que tout scalaire proche de } \lambda \text{ appartient à } \rho(A)\}.$$

Les ensembles $\sigma_{esc}(A) := \mathbb{C} \setminus \rho_{esc}(A)$ et $\sigma_{eb}(A) := \mathbb{C} \setminus \rho_{eb}(A)$ sont appelés, respectivement, les spectres essentiels de Schechter et Browder. On a clairement $\sigma_{esc}(A) \subset \sigma_{eb}(A)$. Si A est un opérateur linéaire borné et auto-adjoint, alors $\sigma_{esc}(A)$ et $\sigma_{eb}(A)$ coïncident. Pour plus de détails concernant les opérateurs de Fredholm et les spectres essentiels on pourra, par exemple, consulter les ouvrages [61] et [74].

La définition ci-dessous donne une autre caractérisation du spectre essentiel de Schechter [73]:

Définition 2.2.1 *Si A est un opérateur fermé à domaine dense dans X , alors*

$$\sigma_{esc}(A) = \bigcap_{K \in \mathcal{K}(X)} \sigma(A + K)$$

où $\mathcal{K}(X) \subset \mathcal{L}(X)$ désigne l'ensemble des opérateurs compacts sur X .

On termine cette section par le théorème suivant, établi dans [43], dont on fera usage.

Théorème 2.2.1 *Soient (Ω, Σ, μ) un espace mesuré, A et B deux opérateurs fermés à domaines denses dans $L^1(\Omega, \Sigma, \mu)$. S'il existe $\lambda \in \rho(A) \cap \rho(B)$ tel que $(\lambda - A)^{-1} - (\lambda - B)^{-1}$ est faiblement compact, alors*

$$\sigma_{esc}(A) = \sigma_{esc}(B).$$

2.3 Théorie spectrale des semi-groupes

2.3.1 Semi-groupes fortement continus

Soit X un espace de Banach. On appelle semi-groupe fortement continu (ou encore C_0 -semi-groupe) toute famille d'opérateurs linéaires bornés $(G(t))_{t \geq 0}$ vérifiant

(i) $G(0) = I$, où I est l'application identique sur X .

(ii) $G(t + s) = G(t)G(s)$ pour tous $s, t \in \mathbb{R}^+$.

(iii) $\lim_{t \rightarrow 0^+} G(t)x = x$ pour tout $x \in X$.

On appelle générateur infinitésimal du C_0 -semi-groupe $(G(t))_{t \geq 0}$, l'opérateur A défini par

$$Ax = \lim_{h \rightarrow 0^+} \frac{G(h)x - x}{h} \quad \forall x \in \mathcal{D}(A).$$

Ainsi, $\mathcal{D}(A)$ (le domaine de l'opérateur A) est l'ensemble de tous les éléments de X tels que la limite ci-dessus existe. L'opérateur A , ainsi défini, est un opérateur fermé à domaine dense.

Si A est le générateur infinitésimal d'un C_0 -semi-groupe $(G(t))_{t \geq 0}$ (noté aussi $(e^{tA})_{t \geq 0}$), alors, pour tout $\psi_0 \in \mathcal{D}(A)$, le problème de Cauchy

$$\begin{cases} \frac{\partial \psi}{\partial t} = A\psi, \\ \psi(0) = \psi_0, \end{cases} \quad (2.3.1)$$

admet une solution unique donnée par $\psi(t) = G(t)\psi_0$.

Comme $(G(t))_{t \geq 0}$ satisfait les propriétés (ii) et (iii), par le théorème de Banach-Steinhaus, $(G(t))_{t \geq 0}$ est uniformément borné sur tout intervalle compact de \mathbb{R}^+ . Il existe donc deux constantes $M > 0$ et w tels que

$$\|G(t)\|_X \leq Me^{wt}, \quad \forall t \geq 0, \quad (2.3.2)$$

(voir, par exemple, [70, p. 4]).

On appelle le type du C_0 -semi-groupe $(G(t))_{t \geq 0}$, le réel $\omega(G(\cdot))$ défini par

$$\omega(G(\cdot)) := \inf \left\{ w \in \mathbb{R}, \exists M > 0 \text{ tel que l'équation (2.3.2) est satisfaite} \right\}.$$

Pour simplifier les notations, on désignera $\omega(G(\cdot))$ simplement par ω .

On note que ω satisfait les propriétés suivantes (voir, par exemple, [29, 70]):

(a) $\omega = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \|G(t)\|.$

(b) $r_\sigma(G(t)) = e^{\omega t} \quad \forall t \geq 0.$

(c) $s(A) \leq \omega$ et donc $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > \omega\} \subset \rho(A).$

(d) Pour tout nombre complexe λ tel que $\operatorname{Re} \lambda > \omega$, on a

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} G(t)x dt \quad \forall x \in X.$$

La propriété (d) exprime le fait que la résolvante de A est la transformée de Laplace du semi-groupe.

On est à présent en mesure d'énoncer, le théorème de Hille-Yosida, qui est un des résultats le plus important en théorie des semi-groupes linéaires. Ce résultat établit une correspondance biunivoque entre les C_0 -semi-groupes et leurs générateurs [33, 70, 29].

Théorème 2.3.1 *Un opérateur linéaire A sur un espace de Banach X est générateur d'un C_0 -semi-groupe $(G(t))_{t \geq 0}$ vérifiant $\|G(t)\| \leq Me^{\omega t}$ si et seulement si*

(a) A est fermé et $\overline{\mathcal{D}(A)} = X$;

(b) pour tout nombre complexe λ tel que $\operatorname{Re} \lambda > \omega$ et pour tout entier naturel n , on a

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\operatorname{Re} \lambda - \omega)^n}.$$

2.3.2 Type essentiel et comportement asymptotique

On rappelle ici l'incidence de la stabilité des types essentiels sur l'analyse du comportement asymptotique des solutions des problèmes de Cauchy. Dans [84] J. Voigt (et L. Weiss [91] via la mesure de non compacité) a montré l'existence d'un réel $\omega_{ess} \in [-\infty, \omega]$, appelé le type essentiel de $(G(t))_{t \geq 0}$, tel que

$$r_{ess}(G(t)) = e^{\omega_{ess} t}, \quad \forall t \geq 0.$$

Le comportement asymptotique de la solution du Problème (2.3.1) (lorsque t tend vers $+\infty$) peut être décrit en analysant directement le spectre du semi-groupe $(G(t))_{t \geq 0}$. Si de plus son générateur A est borné, alors on a

$$\sigma(G(t)) \setminus \{0\} = e^{t\sigma(A)}.$$

Lorsque l'égalité précédente est satisfaite, on dira que l'on a un théorème d'application spectrale. On note qu'en général

$$e^{t\sigma(A)} \subset \sigma(G(t)).$$

Le spectre approché (et en particulier le spectre continu) est responsable de l'absence d'un théorème d'application spectrale. Si $\omega_{ess} < \omega$, alors pour tout $w \in]\omega_{ess}, \omega[$ l'ensemble $\{\nu \in \sigma(G(t)) : |\nu| \geq e^{wt}\}$ est vide ou formé par des valeurs propres isolées de multiplicité algébrique finie. Comme un théorème de l'application spectral a lieu pour le spectre ponctuel (cf. [29, p. 277]), alors pour tout $\varepsilon > 0$ l'ensemble

$$\{\lambda \in \sigma(A) : \operatorname{Re} \lambda > w - \varepsilon\}$$

consiste, au plus, à un nombre fini des valeurs propres isolées de multiplicité algébrique finie que l'on notera $\lambda_1, \dots, \lambda_n$. Soient P_i et D_i , respectivement, la projection spectrale et l'opérateur

nilpotent associé à λ_i . On note par $P = P_1 + \dots + P_n$ la projection spectrale associée à l'ensemble $\{\lambda_1, \dots, \lambda_n\}$. Par le théorème de décomposition spectrale associée à l'ensemble compact $\{\lambda_1, \dots, \lambda_n\}$ (voir, par exemple, [35]), on a alors

$$G(t) = G(t)(I - P) + \sum_{i=1}^n e^{\lambda_i t} e^{D_i t} P_i.$$

De plus, si

$$\|G(t)(I - P)\| = o(e^{(\omega+\varepsilon)t}), \quad (2.3.3)$$

alors le comportement asymptotique de la solution du problème de Cauchy (2.3.1) est complètement déterminé par la part de $G(t)$ dans l'espace de dimension finie $P(X)$.

2.3.3 Perturbations bornées

Soient X un espace de Banach et A le générateur infinitésimal d'un C_0 -semi-groupe $(G(t))_{t \geq 0}$ sur X vérifiant $\|G(t)\| \leq M e^{\omega t}$. Si $B \in \mathcal{L}(X)$, alors par le théorème des perturbations bornées de Phillips (voir, par exemple, [70, Theorem 1.1, p. 76]), $A + B$ engendre un C_0 -semi-groupe $(S(t))_{t \geq 0}$ sur X vérifiant

$$\|S(t)\| \leq M e^{(\omega + \|B\|M)t} \text{ pour tout } t \geq 0.$$

De plus, pour tout $x \in X$ et pour tout $t \geq 0$, on a la formule de Duhamel

$$S(t)x = G(t)x + \int_0^t G(t-s)BS(s)x ds. \quad (2.3.4)$$

On note que le semi-groupe $(S(t))_{t \geq 0}$ est donné, en itérant l'équation (2.3.4), par

$$S(t) = \sum_{j=0}^{\infty} G_j(t) = \sum_{j=0}^{n-1} G_j(t) + R_n(t), \quad (2.3.5)$$

où $G_0(t) = G(t)$ et $G_j(t) = \int_0^t G(s)B G_j(t-s) ds$, $j = 1, 2, \dots$.

La série (2.3.5) est dite la série de Dyson-Phillips. Elle converge dans $\mathcal{L}(X)$ uniformément pour la topologie des opérateurs sur les intervalles bornés. En outre, le reste d'ordre n , $R_n(t)$, de cette série est donné par

$$R_n(t) = \sum_{j=n}^{\infty} G_j(t) = \int_{s_1 + \dots + s_n \leq t, s_i \geq 0} G(s_1)B \dots G(s_n)BS(t - \sum_{i=1}^n s_i) ds_1 \dots ds_n. \quad (2.3.6)$$

On rappelle maintenant le résultat suivant établi par I. Vidav [82] (voir aussi J. Voigt [84], L. Weis [91] ou M. Mohtar-Kharroubi [65]).

Théorème 2.3.2 Soit (S, Ω, μ) un espace mesuré. S'il existe un entier naturel n tel que le reste d'ordre n de la série de Dyson Phillips, $R_n(t)$, est compact sur $L^p(S, \Omega, \mu)$, avec $1 < p < +\infty$, (resp. faiblement compact sur $L^1(S, \Omega, \mu)$) pour tout $t \geq 0$, alors

$$r_{ess}(S(t)) = r_{ess}(G(t)).$$

Remarques 2.3.1 En particulier, si le reste d'ordre un de la série de Dyson Phillips, $R_1(t)$, est compact, alors, d'après le Théorème 2.2.1, on a

$$\sigma_{esc}(S(t)) = \sigma_{esc}(G(t)).$$

Ce résultat est certes intéressant néanmoins pour l'étude du comportement asymptotique des solutions des problèmes de Cauchy gouvernés par $S(t)$, la stabilité des rayons spectraux essentiels est amplement suffisante. \square

En pratique, sauf dans certains cas particuliers, il n'est pas facile d'établir un résultat de stabilité des rayons spectraux via la compacité du reste de la série de Dyson-Phillips. En effet, ceci est dû au fait que le semi-groupe $(G(t))_{t \geq 0}$ n'est en général pas explicite. En conséquence il est souvent difficile d'étudier la compacité d'un reste de la série de Dyson-Phillips. Néanmoins, l'analyse du spectre du générateur de $S(t)$ permet d'obtenir un résultat similaire à (2.3.3).

2.4 Espaces de Banach réticulés

On présente dans cette section quelques résultats liés à la positivité (au sens de l'ordre). On commence d'abord par définir la notion de la positivité dans les espaces de Banach. Un espace vectoriel X sur \mathbb{R} muni d'une relation d'ordre \leq est appelé un espace vectoriel ordonné, noté (X, \leq) , s'il satisfait les conditions suivantes:

$$f \leq g \Rightarrow f + h \leq g + h \text{ pour tous } f, g, h \in X; \quad (2.4.1)$$

$$f \leq g \Rightarrow \lambda f \leq \lambda g \text{ pour tous } f, g \in X \text{ et } \lambda \in \mathbb{R}^+.$$

L'invariance par translation (Eq. (2.4.1)) implique que pour tous $f, g \in X$, $f \leq g$ est équivalent à $g - f \geq 0$. Ainsi, la relation d'ordre sur X est complètement déterminée par la connaissance du cône positif de X donné par

$$X^+ := \{f \in X : f \geq 0\}.$$

Les éléments de X^+ sont appelés les éléments positifs de X et l'ensemble des éléments de X^+ qui sont strictement positifs est défini par

$$X_*^+ = \{f \in X : f > 0\}.$$

Le cône positif admet les propriétés suivantes:

-
- (a) pour tous $f, g \in X^+$ on $f + g \in X^+$;
 - (b) si $f \in X^+$, alors on $\lambda f \in X^+$ pour tout $\lambda \in \mathbb{R}^+$;
 - (c) si $f \in X^+$ et $-f \in X^+$, alors $f = 0$.

Un espace vectoriel ordonné (X, \leq) est dit réticulé si pour tout $(f, g) \in X^2$, on a

$$\sup(f, g) \in X \text{ et } \inf(f, g) \in X.$$

Soient (X, \leq) un espace vectoriel réticulé et $f \in X$. On pose

$$f^+ := \sup(f, 0) \quad \text{et} \quad f^- := -\inf(f, 0)$$

où f^+ (resp. f^-) est appelée la partie positive (resp. négative) de f . Il est clair que la valeur absolue de f , $|f|$, est donnée par $|f| = f^+ + f^-$.

Une norme $\|\cdot\|$ sur un espace vectoriel réticulé X est appelée une norme réticulée si, pour tous $f, g \in X$, on a

$$|f| \leq |g| \Rightarrow \|f\| \leq \|g\|.$$

Un espace de Banach X est dit réticulé s'il est muni d'une relation d'ordre \leq tel que $(X; \leq)$ est un espace vectoriel réticulé et si la norme sur cet espace est réticulée (l'espace X est évidemment complet pour la topologie de la norme).

Soient X et Y deux espaces de Banach réticulés et T un opérateur linéaire de X dans Y . On dit que T est un opérateur positif si

$$T(X^+) \subset Y^+.$$

L'opérateur T est appelé strictement positif si

$$T(X^+ \setminus \{0\}) \subset Y_*^+.$$

On termine ce paragraphe par rappeler quelques résultats concernant les opérateurs positifs dans le cadre des espaces L^p . Soit Ω un ouvert de \mathbb{R}^n , $n \geq 1$, et soit $Y_p := L^p(\Omega)$, avec $1 \leq p < \infty$. On commence par le théorème suivant dû à Dodds-Fremlin (voir, par exemple, [59, p. 223]). Bien que ce résultat soit valide pour des espaces de Banach réticulés généraux, on énonce une version de ce dernier adaptée aux espaces L^p .

Théorème 2.4.1 *Soient S et T deux opérateurs positifs de $\mathcal{L}(Y_p)$ tels que $S \leq T$.*

- (1) *Si $1 < p < \infty$ et T est compact, alors S est compact.*
- (2) *Si $p = 1$ et T est faiblement compact, alors S est faiblement compact.*

Définition 2.4.1 Un opérateur $T \in \mathcal{L}(Y_p)$ est dit irréductible s'il existe un entier naturel non nul n tel que T^n est strictement positif.

Soient A et B deux opérateurs positifs de $\mathcal{L}(Y_p)$. Il est bien connu que si $A \leq B$, alors $r_\sigma(A) \leq r_\sigma(B)$. Le résultat ci-dessous, dû à I. Marek [58, Theorem 4.4], donne une condition suffisante pour que l'inégalité précédente soit stricte.

Théorème 2.4.2 Soient A et B deux opérateurs positifs de $\mathcal{L}(Y_p)$ vérifiant $A \leq B$ et $A \neq B$. Si A n'est pas quasinilpotent (i.e. $r_\sigma(A) \neq 0$) et B est irréductible et admet une puissance compacte, alors $r_\sigma(A) < r_\sigma(B)$.

On rappelle aussi le théorème suivant établi dans [36, p.67], dont on aura besoin par la suite.

Théorème 2.4.3 Soit $A \in \mathcal{L}(Y_p)$ un opérateur positif compact vérifiant

$$\exists \varphi \geq 0, \quad \varphi \neq 0 \quad \text{et} \quad \alpha > 0 \quad \text{tels que} \quad A\varphi \geq \alpha\varphi.$$

Il existe alors une valeur propre λ_0 de A vérifiant $\lambda_0 \geq \alpha$ et la fonction propre associée à λ_0 est positive.

Comme conséquence, on a

Corollaire 2.4.1 Soit $A \in \mathcal{L}(Y_p)$ un opérateur positif, compact et non quasinilpotent. Alors $r_\sigma(A)$ est une valeur propre de A et la fonction propre associée à $r_\sigma(A)$ est positive.

Preuve. Soit $\lambda \in \mathbb{C}$ une valeur propre de A telle que $|\lambda| = r_\sigma(A)$. Il existe donc un $\varphi \neq 0$ tel que $A\varphi = \lambda\varphi$, ce qui implique que $|\lambda| |\varphi| \leq A |\varphi|$. D'après le Théorème 2.4.3 il existe un λ_0 tel que $\lambda_0 \geq |\lambda| = r_\sigma(A)$ et ceci achève la preuve. \square

2.5 Résultats supplémentaires

Dans cette section, on rappelle quelques résultats liés à la compacité dont on fera usage tout au long de ce travail. On commence par le théorème suivant appelé alternative de Gohberg-Shmulyan ou encore théorème de Fredholm analytique (voir, par exemple, [71, Theorem VI.14, p. 201]).

Théorème 2.5.1 Soient X un espace de Banach et U une partie connexe du plan complexe. Si $U \ni \lambda \mapsto \mathcal{F}(\lambda)$ est une famille d'opérateurs linéaires, compacts et analytiques dans X , alors soit

(1) $(1 - \mathcal{F}(\lambda))^{-1}$ n'existe pour aucun λ de U , ou

(2) $(1 - \mathcal{F}(\lambda))^{-1}$ existe pour tout $\lambda \in U \setminus S$ où S est un sous-ensemble discret de U . Dans ce cas $(1 - \mathcal{F}(\lambda))^{-1}$ est une fonction méromorphe sur U . Les éléments de S sont des pôles de $\mathcal{F}(\cdot)$ de parties principales dégénérées (i.e. les coefficients associés sont de rang fini) et si $\lambda \in S$, alors $\mathcal{F}(\lambda)\psi = \psi$ n'admet pas de solution dans X .

Remarque 2.5.1 On note que I. Vidav fut le premier à introduire cette alternative en neutronique (cf. [82]). □

On est en mesure maintenant d'établir un résultat de décroissance stricte du rayon spectral qui nous sera utile.

Lemme 2.5.1 Soient X un espace de Banach et U un ouvert connexe du plan complexe. Soit $\mathcal{F} : U \rightarrow \mathcal{L}(X)$ une application analytique telle que, pour tout $z \in U$, $\mathcal{F}(z)$ admet une puissance d'ordre N (N fixé) compact. Si l'application $U \ni z \mapsto r_\sigma(\mathcal{F}(z))$ est décroissante et $1 \notin \sigma(\mathcal{F}(z))$, alors elle est strictement décroissante.

Preuve. Il existe un entier $N > 1$ tel que, pour tout $z \in U$, $\mathcal{F}(z)^N$ soit compact. D'après le Théorème 2.5.1, $\sigma(\mathcal{F}(z)^N)$ est réduit à des valeurs propres isolées de multiplicité algébrique finie. Par le théorème d'application spectrale, $\sigma(\mathcal{F}(z))$ est aussi réduit à des valeurs propres isolées de multiplicité algébrique finie. En outre s'il existe $z_1, z_2 \in U$ tels que $z_1 > z_2$ et $r_\sigma(\mathcal{F}(z_1)) = r_\sigma(\mathcal{F}(z_2))$, alors $r_\sigma(\mathcal{F}(z)) = r_\sigma(\mathcal{F}(z_2))$ pour tout $z \in [z_1, z_2]$ et ceci contredit l'alternative du Théorème 2.5.1. □

On termine cette section par rappeler un résultat de (faible) compacité auquel on fera souvent appel (voir [27, Corollary 13 p. 510]).

Théorème 2.5.2 Soit (S, Σ, μ) un espace mesuré. Le produit (de composition) de deux opérateurs faiblement compacts dans $L^1(S, \Sigma, \mu)$ est compact. En particulier, si un opérateur A est faiblement compact dans $L^1(S, \Sigma, \mu)$, alors A^2 est compact.

Asymptotic spectrum of the linear Boltzmann equation with general boundary conditions in finite bodies

3.1 Introduction

This chapter is devoted to spectral properties of transport equation in finite bodies, when the behavior at the boundary is governed by a positive boundary operator H relating the incoming flux to the outgoing one. More precisely, we are concerned with the description of the asymptotic spectrum of the integro-differential operator

$$\begin{aligned} A_H\psi(x, v) &= -v \cdot \nabla_x \psi(x, v) - \sigma(x, v)\psi(x, v) + \int_V \kappa(x, v, v')\psi(x, v')d\mu(v') \\ &= T_H\psi + K\psi \end{aligned}$$

where $(x, v) \in D \times V$. Here D is a smooth open subset of \mathbb{R}^n , $\mu(\cdot)$ is a positive Radon measure on \mathbb{R}^n such that $\mu(0) = 0$ with support V . The functions $\sigma(\cdot, \cdot)$ and $\kappa(\cdot, \cdot, \cdot)$ are called, respectively, the collision frequency and the scattering kernel and the function $\psi(x, v)$ represents the number (or probability) density of gas particles having the position x and the velocity v . This operator describes the transport of particles (neutrons, photons, molecules of gas, etc.) in the domain D . The partial integral part of the operator A_H is denoted by K and called the collision operator while $T_H = A_H - K$ is called the advection operator (subjecte to the boundary operator H).

The transport equation was considered in different fields of mathematical physics to describe transport processes of particles. In kinetic theory of gas where we must describe the interaction of gas molecules with the walls bounding the region where the gas flows, the theoretical problem is to relate the distribution function of molecules leaving a solid surface to the distribution of the

molecules arriving at the same surface. The boundary conditions describing this interaction are very complex because the reaction of a gas molecules with a wall is so complicated. This is due, mainly, to our lack of knowledge of the structure of surface layers of solid bodies, and hence of the effective interaction of the gas molecules with the wall (see, for example, [15, 16, 26]). Nevertheless, a model which is often used consists in supposing that a part of the outgoing flux is re-emitted in a deterministic way (specular reflection), whereas the other part is re-emitted in random directions (diffuse reflections).

In our framework, the boundary conditions are modeled by

$$\psi_- = H(\psi_+), \quad (3.1.1)$$

where ψ_- (resp. ψ_+) is the restriction of ψ to Γ_- (resp. Γ_+) with Γ_- (resp. Γ_+) is the incoming (resp. outgoing) part of the boundary of the phase space $\mathbf{D} \times \mathbf{V}$ $\Gamma_{\pm} = \{(x, v) \in \partial\mathbf{D} \times \mathbf{V}, \pm v \cdot v_x \geq 0\}$, v_x being the outward unit normal vector at $x \in \partial\mathbf{D}$. The boundary conditions (3.1.1) means that the incoming flux $\psi|_{\Gamma_-}$ is related to the outgoing one flux $\psi|_{\Gamma_+}$ through a linear operator H that we shall assume to be bounded on some suitable trace spaces (see Section 3.2 for details). The known classical boundary conditions:

- perfect absorption (vacuum boundary): $H = 0$,
- periodic boundary conditions in a box,
- specular reflection: $H\varphi(x, v) = \varphi(x, v - 2(v \cdot v_x)v_x)$, where v_x stands for the outer unit normal vector at x ,
- reverse reflection: $H\varphi(x, v) = \varphi(x, -v)$,
- diffuse reflection: $H\varphi(x, v) = \int_{v' \cdot v_x > 0} h(x, v, v')\varphi(x, v')v' \cdot v_x d\mu(v')$,
- Maxwell-type boundary conditions which are a combination between a reflection and a diffuse reflection

are special examples of our framework setting.

The well posedness of the Cauchy problem

$$\begin{cases} \frac{\partial \psi}{\partial t} = A_H \psi, \\ \psi(0) = \psi_0, \end{cases} \quad (3.1.2)$$

for abstract boundary conditions was considered by many authors. In the case where the function $\sigma(\cdot, \cdot) = 0$, the first systematic treatment of generation results of T_H was given in [85]. Further, a comprehensive theory of the existence and uniqueness of solutions of the Problem (3.1.2) in an L^p -space setting has been developed in [9]. This theory can be also found in Chapter XI of [31]. It

should be noticed that the case of contractive boundary operators, $\|H\| < 1$, is well understood (in this case the fact that T_H (and so A_H) generates a C_0 -semigroup is an immediate consequence of Lumer-Phillips's theorem). The case of multiplying boundary conditions ($\|Hu\| > \|u\|$) has been investigated in [49, 12, 13, 54] and sufficient conditions on H guaranteeing that T_H generates a C_0 -semigroup were provided. Unfortunately, these sufficient conditions do not apply to conservative boundary operators H ($\|Hu\| = \|u\|$), in particular in the space $L^1(D \times V, dx \otimes d\mu)$. In fact T_H (and then A_H) is not necessary a generator of a C_0 -semigroup but there exists an extension B of T_H which generates a C_0 -semigroup of contractions. For more information in this direction, we refer to the works [5, 9, 31, 67, 85] and the references therein.

In the case where the Cauchy problem (3.1.2) is well posed, the determination of the time asymptotic of the solution (which is a cornerstone problem in linear transport theory) is intimately related to the spectrum of the operator A_H . More precisely, the knowledge of the asymptotic spectrum of A_H , $\sigma_{as}(A_H) := \sigma(A_H) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > s(T_H)\}$, is of fundamental importance. This is due to the fact that, when dealing with the Cauchy problem (3.1.2), we establish easily, via the Hille-Yosida-Phillips theorem, that it has a unique solution. Since this approach is not constructive, in order to get more information on the solution, in particular, on its behavior for large times, the knowledge of structure of the spectrum of the transport operator plays a central role. In fact, the location of $\sigma(A_H)$ gives information on the asymptotic behavior ($t \rightarrow \infty$) of the solution. Although the spectral analysis of transport operators is now a classical topic in the transport theory, most of the investigations was dedicated essentially to transport operators with vacuum boundary conditions i.e. $H = 0$. We quote, for example, the contributions by K. Jörgens [34], M.L. Demeru and B. Montagnini [24], S. Albertoni and B. Montagnini [2], E. W. Larsen and P. F. Zweifel [38], A. Palczewski [69], S. Ukai [80], I. Vidav [82], M. Mokhtar-Kharroubi [63, 65, 66] and the references therein. Other works dealing with the spectral theory of perturbed strongly continuous semigroups in L^p -spaces (and, more generally, in Banach lattices) are also available (see, for example, Chapters 2 and 3 in [65] and the references therein). The general theory is based on compactness arguments which are already present in the literature for particular models of neutron transport equations [2, 24, 38]. For vacuum boundary conditions (i.e. $H = 0$), it is well known [63, 76, 82] that, if some power of $K(\lambda - T_0)^{-1}$ or $(\lambda - T_0)^{-1}K$ is compact on $L^p(D \times V)$, then

$$\sigma(T_0 + K) \cap \{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda > s(T_0)\},$$

where $s(T_0) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(T_0)\}$, consists of, at most, isolated eigenvalues with finite algebraic multiplicity (here T_0 denotes the advection operator with vacuum boundary conditions). This hypothesis was verified by specific physical models [24, 38]. Despite this extensive work, when dealing with reentry boundary conditions such as periodic boundary conditions, specular reflections, diffuse reflections, generalized or mixed type boundary conditions, except the one dimensional case where many progress were made in the recent years (see, for example, [19, 23, 41, 42, 44, 46]), as far as we know, apart from the case of the reverse boundary conditions, [45], in the spaces $L^p(D \times \mathbb{R}^n, dx \otimes d\mu)$ with $1 < p < +\infty$, for higher dimensions, only few partial results for particular shapes of the spatial domain are available in the literature [4, 21, 30, 57, 69].

Note that, in the work [42], compactness properties of the operators $K(\lambda - T_H)^{-1}$ and $(\lambda - T_H)^{-1}K$ were discussed, here H is an abstract boundary operator. The main goal of this work is to make

use of these results in order to give a systematic analysis and a fine description of spectrum of the operators T_H and A_H for general boundary conditions. In particular, we discuss the asymptotic spectrum of the operators T_H and A_H (here we mean by asymptotic spectrum of T_H the part of the spectrum of T_H belonging to the half plane $\{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda > -\lambda^*\}$ where λ^* is defined in Section 3.2). In Section 3.2 we prepare the ground by introducing the functional setting of the problem and fixing the different notations and facts required in the sequel. The aim of Section 3.3 is to deal with the spectral theory of the streaming operator T_H involving both smooth (compact) and partly smooth boundary operators (cf. assumptions (A1) and (A2)). We give fine results concerning the asymptotic spectrum of T_H , $\sigma_{as}(T_H)$, we show, in particular, that the boundary operator H is a boundary perturbation of the streaming operator T_0 ($H = 0$) and behaves like an additive perturbation (in terms of the resolvents) and enters in play as a collision operator at the boundary. We give sufficient (some time necessary) condition in terms of H guaranteeing that $\sigma_{as}(T_H)$ is non empty or it consists of, at most, eigenvalues with finite algebraic multiplicities. We establish also that in the case where $\sigma_{as}(T_H) \neq \emptyset$, T_H admits a real leading eigenvalue and we give a sufficient condition on H ensuring that $\sigma_{as}(T_H) = \emptyset$ independently of the size of D . In Section 3.4, we present a fine description of the asymptotic spectrum of the operator A_H , $\sigma_{as}(A_H)$. We show, in particular, that, for this class of collision operators, $\sigma_{as}(A_H)$ consists of, at most, eigenvalues with finite algebraic multiplicities and, if $\sigma_{as}(A_H) \neq \emptyset$, then A_H admits a real leading eigenvalue. We give also a necessary and sufficient condition guaranteeing that $\sigma_{as}(A_H) \neq \emptyset$. Further, we show that there is a connection between the asymptotic spectrum of the operator A_H and that of its bounded part (denoted by \mathcal{B} in the text). Existence and nonexistence results of eigenvalues are given. The irreducibility of the C_0 -semigroup generated by A_H is discussed in Section 3.5. We give sufficient conditions in terms of H and K ensuring the irreducibility of the C_0 -semigroup generated by A_H which implies that the leading eigenvalue of A_H (if it exists) is strictly dominant with multiplicity 1 and the associated eigen-function is strictly positive. The problem concerning the strict monotonicity of the leading eigenvalue of the operator A_H with respect to the parameters of the equation is the purpose of Section 3.6. Further a compactness results (cf. Theorem 3.4.1), ours proofs use the comparison results of the spectral radius of positive operators (Theorem 2.4.2). We show that the leading eigenvalue (when it exists) increases strictly with respect to the boundary operator H and the collision operator K . Finally, in Section 3.7, we prove some technical lemmas required in this work.

3.2 Preliminaries

The goal of this section is to recall some basic definitions and results, for the usual neutron transport equation (without delayed neutrons), which we shall use in the sequel. Let D be a smooth open bounded convex subset of \mathbb{R}^n and let $\mu(\cdot)$ be a positive Radon measure on \mathbb{R}^n such that $\mu(0) = 0$. We denote by V its support which is called the velocities space. We note that the boundary of the phase space writes as $\partial D \times V := \Gamma_- \cup \Gamma_+ \cup \Gamma_0$ where

$$\Gamma_{\pm} = \{(x, v) \in \partial D \times V, \pm v \cdot \nu_x > 0\},$$

and $\Gamma_0 = \{(x, v) \in \partial D \times V, \pm v \cdot \nu_x = 0\}$ where ν_x stands for the outer unit normal vector at $x \in \partial D$. We will suppose throughout this work that Γ_0 is of zero measure with respect to $d\gamma_x d\mu(v)$, $d\gamma_x$ being the Lebesgue measure on ∂D . Let X_p , $1 \leq p < +\infty$, be the space

$$X_p := L^p(D \times V, dx \otimes d\mu(v)),$$

and define the partial Sobolev space $W_p = \{\psi \in X_p \text{ such that } v \cdot \nabla_x \psi \in X_p\}$. It is well known [17, 18, 31] that any function in W_p possesses traces ψ^\pm on Γ_\pm belonging to $L_{p,\text{loc}}^\pm(\Gamma_\pm, |v \cdot \nu_x| d\gamma_x d\mu(v))$. Note that, in applications, suitable L^p -spaces for the traces are

$$L_p^\pm := L^p(\Gamma_\pm, |v \cdot \nu_x| d\gamma_x d\mu(v)).$$

Accordingly, we define the set

$$\widetilde{W}_p = \{\psi \in W_p, \psi^- \in L_p^-\}.$$

It is well known that if $\psi \in W_p$, $1 \leq p < +\infty$, and $\psi^- \in L_p^-$, then $\psi^+ \in L_p^+$ and vice versa [17, 18, 31]. More precisely we have the identity

$$\widetilde{W}_p = \{\psi \in W_p, \psi^- \in L_p^-\} = \{\psi \in W_p, \psi^+ \in L_p^+\}.$$

Definition 3.2.1 Let $(x, v) \in \overline{D} \times V$. We set

$$\begin{aligned} t^\pm(x, v) &= \sup\{t > 0, x \pm sv \in D, 0 < s < t\} \\ &= \inf\{t > 0, x \pm tv \notin D\} \end{aligned}$$

and

$$\tau(x, v) := t^-(x, v) + t^+(x, v) \text{ for any } (x, v) \in \overline{\Omega} \times V.$$

Hence, for $(x, v) \in \Gamma_\pm$, one has $t^\pm(x, v) = 0$ and in all cases $x \mp t^\mp(x, v)v \in \Gamma_\mp$. The number $t^\pm(x, v)$ is the time required by a particle having the position $x \in D$ and the velocity $\pm v \in V$ to go out D .

Definition 3.2.2 We say that a linear operator $H \in \mathcal{L}(L_p^+, L_p^-)$ is a boundary operator if the following relation is satisfied:

$$H(\psi^+) = \psi^-.$$

Throughout this chapter, we assume that H is a positive operator (which is a natural hypothesis).

Let $H \in \mathcal{L}(L_p^+, L_p^-)$ be a boundary operator. Define the streaming operator T_H

$$\begin{cases} T_H : D(T_H) \subseteq X_p \longrightarrow X_p \\ \psi \longmapsto T_H \psi(x, v) = -v \cdot \nabla_x \psi(x, v) - \sigma(x, v) \psi(x, v) \end{cases}$$

with domain

$$D(T_H) = \{\psi \in \widetilde{W}_p \text{ such that } \psi^- = H(\psi^+)\}.$$

Here the function $\sigma(\cdot, \cdot)$ is assumed to be a non-negative measurable function belonging to $L^\infty(\mathbb{D} \times \mathbb{V}, dx \otimes d\mu(v))$.

Let $\varphi \in X_p$ and consider the resolvent equation for T_H

$$(\lambda - T_H)\psi = \varphi, \tag{3.2.1}$$

where λ is a complex number and the unknown ψ must be sought in $D(T_H)$. It is clear that if $\lambda \in \rho(T_H)$, then the solution of (3.2.1) is given by $\psi := (\lambda - T_H)^{-1}\varphi$. Let λ^* denote the real defined by

$$\lambda^* := \lim_{t \rightarrow \infty} \inf_{\{(x,v) \in \mathbb{D} \times \mathbb{V}; t \leq \tau(x,v)\}} \frac{1}{t} \int_0^t \sigma(x - sv, v) ds.$$

It is shown in [42] that, for $\text{Re} \lambda > -\lambda^*$, problem (3.2.1) can be solved formally to give

$$\begin{aligned} \psi(x, v) = \psi(x - t^-(x, v)v, v) e^{-\int_0^{t^-(x,v)} (\lambda + \sigma(x - sv, v)) ds} \\ + \int_0^{t^-(x,v)} e^{-\int_0^s (\lambda + \sigma(x - \tau v, v)) d\tau} \varphi(x - sv, v) ds. \end{aligned} \tag{3.2.2}$$

Moreover, for $(x, v) \in \Gamma_+$, (3.2.2) becomes

$$\begin{aligned} \psi(x, v) = \psi(x - t^-(x, v)v, v) e^{-\int_0^{\tau(x,v)} (\lambda + \sigma(x - sv, v)) ds} \\ + \int_0^{\tau(x,v)} e^{-\int_0^s (\lambda + \sigma(x - \tau v, v)) d\tau} \varphi(x - sv, v) ds. \end{aligned} \tag{3.2.3}$$

Hence, for $\operatorname{Re}\lambda > -\lambda^*$, the invertibility of $(\lambda - T_H)$, reduces to invertibility of the operator $\mathcal{P}_\lambda := I - M_\lambda H$ where

$$L_p^- \ni u \mapsto M_\lambda u = u(x - \tau(x, v)v, v)e^{-\int_0^{\tau(x, v)} (\lambda + \sigma(x - sv, v))ds} \in L_p^+$$

and the solution of (3.2.1) is given by

$$\psi = B_\lambda H \mathcal{P}_\lambda^{-1} G_\lambda \varphi + C_\lambda \varphi.$$

Accordingly,

$$(\lambda - T_H)^{-1} = B_\lambda H \mathcal{P}_\lambda^{-1} G_\lambda + C_\lambda. \quad (3.2.4)$$

Here

$$X_p \ni \varphi \mapsto C_\lambda \varphi = (\lambda - T_0)^{-1} \varphi = \int_0^{t^-(x, v)} e^{-\int_0^s (\lambda + \sigma(x - \tau v, v))d\tau} \varphi(x - sv, v) ds \in X_p,$$

$$X_p \ni \psi \mapsto G_\lambda \varphi := C_\lambda \varphi|_{\Gamma^+} = \int_0^{\tau(x, v)} e^{-\int_0^s (\lambda + \sigma(x - \tau v, v))d\tau} \varphi(x - sv, v) ds \in L_p^+$$

and

$$L_p^- \ni u \mapsto B_\lambda u := u(x - t^-(x, v)v, v)e^{-\int_0^{t^-(x, v)} (\lambda + \sigma(x - sv, v))ds} \in X_p.$$

These operators are bounded on their respective spaces. In fact, the norms of B_λ and C_λ are bounded above, respectively, by $[p(\operatorname{Re}\lambda + \lambda^*)]^{-\frac{1}{p}}$ and $(\operatorname{Re}\lambda + \lambda^*)^{-1}$ where q denotes the conjugate of p . Moreover, the operator $M_\lambda u = [B_\lambda u]|_{\Gamma^+}$ is a contraction and $G_\lambda \varphi = [C_\lambda \varphi]|_{\Gamma^+}$ is bounded with norm less than $[q(\operatorname{Re}\lambda + \lambda^*)]^{-\frac{1}{q}}$.

Remark 3.2.1 Note that, for any real $\alpha > -\lambda^*$ such that the operator $I - M_\alpha H$ is invertible, $\alpha \in \rho(T_H)$ and $(\alpha - T_H)^{-1}$ is given by (3.2.4). Clearly, if H is positive in the lattice sense, then the operator \mathcal{P}_α^{-1} is also positive. Since, for each real $\alpha > -\lambda^*$, the operators M_α , B_α and G_α are positive bounded linear operators, we infer that

$$\mathcal{H}(\alpha) = B_\alpha H \mathcal{P}_\alpha^{-1} G_\alpha \geq 0.$$

Since the operator C_α is nothing else but $(\alpha - T_0)^{-1}$, using the last estimate together with (3.2.4) we get

$$(\alpha - T_H)^{-1} \geq (\alpha - T_0)^{-1} \geq 0.$$

□

3.3 Asymptotic spectrum of T_H

Throughout this section we denote by Λ the subset of \mathbb{C} defined by

$$\Lambda := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > -\lambda^*\}.$$

Clearly, if $\|H\| < 1$, then using the fact that, for $\operatorname{Re} \lambda > -\lambda^*$, M_λ is a contraction, we get $\|M_\lambda H\| < 1$ and therefore \mathcal{P}_λ^{-1} exists. So, $(\lambda - T_H)^{-1}$ exists for any λ lying in Λ . Hence, since $\|H\| < 1$ we have

$$\sigma(T_H) \cap \Lambda = \emptyset.$$

However, the operator $(\lambda - T_H)^{-1}$ may exist if, for example, some power of $M_\lambda H$ is compact, say $(M_\lambda H)^k$ for some $k \in \mathbb{N} \setminus \{0, 1\}$. In this case the operator $(\lambda - T_H)^{-1}$ is well defined and it is a degenerate-meromorphic function on Λ (for details see the second part of the proof of Theorem 3.3.1(1)). Evidently, we have

$$\sigma(T_H) \cap \Lambda \neq \emptyset.$$

Let T_0 be the streaming operator with vacuum boundary conditions. Note that, for $\lambda \in \rho(T_0)$, we have $(\lambda - T_0)^{-1} = C_\lambda$. Unlike Section 3.4, where we are concerned with additive perturbations of the operator T_H , in this section H is a boundary perturbation of the streaming operator T_0 . Note that such a perturbation produces an additive perturbation of the resolvent of T_0 (see Eq. (3.2.4)) and it may produce also a change of the spectrum of T_0 (i.e. $\sigma(T_0) \neq \sigma(T_H)$). In fact, if $\|H\| < 1$, then $s(T_0) = s(T_H) = -\lambda^*$. However, if $\|H\| \geq 1$, then the spectrum of the operator H enters in play and H behaves like a collision operator at the boundary. In particular, we have $s(T_0) \leq s(T_H)$ and therefore $\sigma(T_H) \cap \Lambda$ may be not empty. The main goal here is to describe the set

$$\sigma(T_H) \cap \Lambda.$$

Note that $\sigma(T_H) \cap \Lambda$ may be viewed as the asymptotic spectrum of T_H , so, by analogy, we use the notation

$$\sigma_{as}(T_H) := \sigma(T_H) \cap \Lambda.$$

We shall now introduce the following assumption:

- (A1) there exists $N \in \mathbb{N} \setminus \{0, 1\}$ such that $(M_\lambda H)^N$ is compact if $p \in (1, \infty)$
or weakly compact for $p = 1$.

Remark 3.3.1 It is obvious that $\lambda \in \sigma_p(T_H)$ if and only if $1 \in \sigma_p(M_\lambda H)$. □

Throughout this section, d will denote the diameter of D , that is,

$$d := \sup \{ \|x - y\| : x, y \in D \}.$$

Theorem 3.3.1 *Let $p \in [1, \infty)$ and assume that (A1) holds true. Then we have*

- (1) $\sigma_{as}(T_H)$ consists of, at most isolated eigenvalues with finite algebraic multiplicities.
- (2) If $\sigma_{as}(T_H) \neq \emptyset$, then there exists a real leading eigenvalue λ_d , i.e. $\lambda_d \in \sigma_p(T_H)$ with $\lambda_d > -\lambda^*$ and $\sigma(T_H) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq \lambda_d\}$.
- (3) $\sigma_{as}(T_H) \neq \emptyset$ if and only if $\lim_{\lambda \rightarrow -\lambda^*} r_\sigma(M_\lambda H) > 1$.
- (4) If there exists $\lambda_0 \in \Lambda$ such that $r_\sigma(M_{\lambda_0} H) \leq 1$, then $\sigma_{as}(T_H) = \emptyset$ for each d .
- (5) If H is a finite rank operator and the phase space $D \times V$ is such that there exists $\varepsilon > 0$ such $\inf_{(x,v) \in \Gamma_+} \tau(x, v) > \varepsilon$, then $\sigma_{as}(T_H)$ is bounded and, for every $\omega > 0$, the set $\sigma_{as}(T_H) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -\lambda^* + \omega\}$ is finite.

Remark 3.3.2 This theorem is an extension to multidimensional case of some assertions established in Theorem 3.1 in [49] for slab geometry. Unfortunately, due to some technical difficulties, we were not able to prove the analogue of the assertions 5), 6), 8) and 9) of Theorem 3.1 in [49]. In fact, in a slab with thickness $2a$, the function $\psi(x, \xi)$ represents the number density of gas particles having the position x and the direction cosine of propagation ξ . (The variable ξ may be thought of as the cosine of the angle between the velocity of particles and the x -direction). Hence, in one-dimensional framework, the variable ξ plays the role of the velocity variable v . So, unlike our case, the operator M_λ is a strict contraction, that is, for any λ such that $\operatorname{Re}\lambda > -\lambda^*$, its norm is bounded above by $e^{-2a(\operatorname{Re}\lambda + \lambda^*)}$. In the multi-dimensional framework, we lost this explicit estimate which plays a central role in the proofs of Theorems 3.1 and 3.2 in [49]. \square

Proof of Theorem 3.3.1. 1) Note that, if, for all $\lambda \in \Lambda$, $r_\sigma(M_\lambda H) < 1$, then the operator $\mathcal{P}_\lambda = I - M_\lambda H$ is invertible. So, $\lambda \in \rho(T_H)$ and then $\sigma_{as}(T_H) = \emptyset$.

Assume that, for some $\lambda \in \Lambda$, $r_\sigma(M_\lambda H) > 1$. According to the hypothesis (A1) and Theorem 2.5.2, we infer that, for all $\lambda \in \Lambda$, $(M_\lambda H)^{2N}$ is compact. It is easy to see that $M_\lambda H \rightarrow 0$, in the strong operator topology, as $\operatorname{Re}\lambda \rightarrow \infty$. So, applying [35, Lemma 3.7, p. 151] we infer that $(M_\lambda H)^{2N+1} \rightarrow 0$, in the uniform operator topology, as $\operatorname{Re}\lambda \rightarrow \infty$. This yields

$$r_\sigma((M_\lambda H)^{2N+1}) \rightarrow 0 \text{ as } \operatorname{Re}\lambda \rightarrow \infty. \quad (3.3.1)$$

Using the analyticity of the operator valued function $\Lambda \ni \lambda \mapsto (M_\lambda H)^{2N+1}$ and Eq. (3.3.1) together with the Gohberg-Shmul'yan theorem (see Theorem 2.5.1) we infer that $(I - (M_\lambda H)^{2N+1})^{-1}$

is a degenerate-meromorphic operator function on Λ (i.e. $(I - (M_\lambda H)^{2N+1})^{-1}$ is holomorphic on Λ except a set S of isolated points where $(I - (M_\lambda H)^{k+1})^{-1}$ has poles and the coefficients of the principal part are degenerate). From the equation

$$I - (M_\lambda H)^{2N+1} = (I - M_\lambda H)(I + M_\lambda H + \dots + (M_\lambda H)^{2N}),$$

we conclude that

$$(I - M_\lambda H)^{-1} = (I + M_\lambda H + \dots + (M_\lambda H)^{2N})(I - (M_\lambda H)^{2N+1})^{-1}$$

which shows that $(I - M_\lambda H)^{-1}$ is also a degenerate-meromorphic function on Λ . Hence, according to Remark 3.3.1, if $\lambda \notin S$, $I - M_\lambda H$ is invertible and then $(\lambda - T_H)^{-1}$ exists and is a degenerate-meromorphic function on Λ . This implies that $\sigma_{as}(T_H) = S$ which ends the proof of (1).

2) Let $\lambda_0 \in \sigma_{as}(T_H)$. Using (1) and Remark 3.3.1, there exists $u \neq 0$ such that $M_{\lambda_0} H u = u$. Hence $(M_{\lambda_0} H)^{2N} u = u$ and then $|u| \leq |(M_{\beta_0} H)^{2N} u| \leq (M_{\beta_0} H)^{2N} |u|$ where $\beta_0 = \operatorname{Re} \lambda_0$. Therefore, applying Theorem 2.4.3, we infer that

$$r_\sigma((M_{\beta_0} H)^{2N}) \geq 1. \quad (3.3.2)$$

Further, according to Lemma 2.5.1, $r_\sigma((M_\beta H)^{2N})$ is a continuous strictly decreasing function of β on $(-\lambda^*, \infty)$. So, by the spectral mapping theorem (see, for example, [27, p. 563]), there exists $\alpha(\beta) \in \sigma(M_\beta H)$ such that $(\alpha(\beta))^{2N} = r_\sigma((M_\beta H)^{2N})$, i.e. $\alpha(\beta) = \sqrt[2N]{r_\sigma((M_\beta H)^{2N})}$. It is obvious that $\alpha(\beta)$ is also a continuous strictly decreasing function of β on $(-\lambda^*, \infty)$. On the other hand, Eq. (3.3.2) (resp. (3.3.1)) shows that $\alpha(\beta_0) \geq 1$ (resp. $\lim_{\beta \rightarrow \infty} \alpha(\beta) = 0$). Hence, there exists $\lambda \geq \beta_0$ such that $\alpha(\lambda) = 1$, i.e. $\lambda = \lambda_d$ (the leading eigenvalue of T_H).

3) By analyticity arguments, it suffices to restrict ourselves to $G_\sigma := \sigma(T_H) \cap (-\lambda^*, \infty)$. Arguing as in (2), we show that the leading eigenvalue λ_d is characterized by

$$r_\sigma(M_{\lambda_d} H) = 1. \quad (3.3.3)$$

If $G_\sigma \neq \emptyset$, then there exists a leading eigenvalue λ_d such that $r_\sigma(M_{\lambda_d} H) = 1$. Moreover, this shows that $G_\sigma \subset (-\lambda^*, \lambda_d]$. Since $r_\sigma(M_\lambda H)$ is a strictly decreasing function value in λ , we have $1 = r_\sigma(M_{\lambda_d} H) < r_\sigma(M_\lambda H)$ for all $\lambda \in G_\sigma \setminus \{\lambda_d\}$ and consequently $\lim_{\lambda \rightarrow -\lambda^*} r_\sigma(M_\lambda H) > 1$.

Conversely, if $\lim_{\lambda \rightarrow -\lambda^*} r_\sigma(M_\lambda H) > 1$, then the use of Eq. (3.3.1) gives $\lim_{\lambda \rightarrow \infty} r_\sigma(M_\lambda H) = 0$. Since $\lambda \mapsto r_\sigma(M_\lambda H)$ is a continuous strictly decreasing function (cf. Lemma 2.5.1) there exists $\lambda_1 > -\lambda^*$ such that $r_\sigma(M_{\lambda_1} H) = 1$. Hence λ_1 is the leading eigenvalue of T_H (cf. (3.3.3)), i.e. $\lambda_1 = \lambda_d$, and therefore $G_\sigma \neq \emptyset$.

4) It is an immediate consequence of the assertion (3).

5) Let $\lambda_2 > -\lambda^*$. If λ_2 is large enough, then, for all λ such that $\operatorname{Re} \lambda > \lambda_2$, we have $\|M_\lambda H\| < 1$ (indeed, using the compactness of H and reasoning as in the proof of the second part of 1) we

get $\lim_{\operatorname{Re}\lambda \rightarrow +\infty} \|M_\lambda H\| = 0$). So, we have $\sigma(T_H) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \lambda_2\} = \emptyset$. Thus, it suffices to restrict ourselves to the strip

$$G_{\lambda_2} := \{\lambda \in \mathbb{C} : -\lambda^* < \operatorname{Re}\lambda \leq \lambda_2\}.$$

Let $\lambda \in G_{\lambda_2}$. Since H is compact, according to Lemma 3.7.1, we have that $\lim_{|\operatorname{Im}\lambda| \rightarrow \infty} r_\sigma(M_\lambda H) = 0$. So, we conclude that there exists $M > 0$ such that, for all $\lambda \in G_{\lambda_2}$ satisfying $|\operatorname{Im}\lambda| > M$, we have $r_\sigma(M_\lambda H) < 1$. This proves the boundedness of $\sigma_{as}(T_H)$. Accordingly, for any $\omega > 0$ such that $-\lambda^* + \omega < \lambda_2$, $\sigma_{as}(T_H) \cap \{\lambda \in \mathbb{C} : -\lambda^* + \omega \leq \operatorname{Re}\lambda \leq \lambda_2\}$ is a discrete set confined in a compact subregion of G_{λ_2} and so it has no accumulation point. \square

We consider now the case of partly smooth boundary operators, that is

$$(A2) \quad \begin{cases} \mathcal{H} = H + \widetilde{H} \text{ with } 0 \leq \widetilde{H}, \\ M_\lambda \widetilde{H} \text{ is compact if } 1 < p < \infty \text{ and} \\ \text{weakly compact if } p = 1. \end{cases}$$

The boundary operator \mathcal{H} is a positive perturbation of H , i.e. $\widetilde{H} = \mathcal{H} - H$ is a positive operator on $\mathcal{L}(L_p^+, L_p^-)$.

Theorem 3.3.2 *Let $p \in [1, \infty)$ and suppose that and the condition (A2) is satisfied. If $\|H\| < 1$, then the following assertions hold true*

- (1) $\sigma(T_{\mathcal{H}}) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > s(T_{\mathcal{H}})\}$ consists of, at most, isolated eigenvalues with finite algebraic multiplicities.
- (2) If $\sigma(T_{\mathcal{H}}) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > s(T_{\mathcal{H}})\} \neq \emptyset$, then there exists a leading eigenvalue λ_d .
- (3) If $\lim_{\lambda \rightarrow s(T_{\mathcal{H}})} r_\sigma(M_\lambda \widetilde{H}) > 1$, then $\sigma(T_{\mathcal{H}}) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > s(T_{\mathcal{H}})\} \neq \emptyset$.

Remark 3.3.3 Theorem 3.3.2 is an generalization to multidimensional case of Theorem 3.2 in [49] established in slab geometry.

Proof of Theorem 3.3.2. 1) Let λ be such that $\operatorname{Re}\lambda > s(T_{\mathcal{H}})$ and let $\varphi \in X_p$. Consider the problem $(\lambda - T_{\mathcal{H}})\psi = \varphi$ where ψ is the unknown. The solution of this problem is given formally by (3.2.2) and, for $(x, v) \in \Gamma_+$, we get (cf. Eq. (3.2.3))

$$\psi_+ = M_\lambda H \psi_+ + M_\lambda \widetilde{H} \psi_+ + G_\lambda \varphi.$$

Since $\|M_\lambda H\| < 1$, the operator $I - M_\lambda H$ is invertible, and then the last equation can be written in the form

$$\psi_+ = S_\lambda \psi_+ + \Pi_\lambda \varphi$$

where

$$S_\lambda := (I - M_\lambda H)^{-1} M_\lambda \widetilde{H} \quad \text{and} \quad \Pi_\lambda := (I - M_\lambda H)^{-1} G_\lambda.$$

Let us first observe that under the condition $\|H\| < 1$, the quantity $\|(I - M_\lambda H)^{-1}\|$ is uniformly bounded independently of λ in the half plane $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > s(T_{\mathcal{H}})\}$. Indeed, since $\|M_\lambda H\| \leq \|H\| < 1$ (use the estimate $\|M_\lambda\| \leq 1$ (cf. Section 3.2)), we obtain

$$\|(I - M_\lambda H)^{-1}\| \leq \sum_{k \geq 0} \|H\|^k = \frac{1}{1 - \|H\|}. \quad (3.3.4)$$

In the remainder of the proof we treat separately the cases $p \in (1, \infty)$ and $p = 1$.

Let $p \in (1, \infty)$ and consider $\lambda \in \mathbb{C}$ such that $\operatorname{Re} \lambda > s(T_{\mathcal{H}})$. According to the hypothesis (A2), the operator $M_\lambda \widetilde{H}$ is compact and so the operator S_λ is compact too. Further, using the fact that $M_\lambda \widetilde{H} \rightarrow 0$ strongly as $\operatorname{Re} \lambda \rightarrow \infty$ and the compactness of S_λ we infer that $\|M_\lambda \widetilde{H} S_\lambda\| \rightarrow 0$ in the operator norm as $\operatorname{Re} \lambda$ goes to $+\infty$. Now, using the estimate (3.3.4), we get $\|(S_\lambda)^2\| \rightarrow 0$ as $\operatorname{Re} \lambda \rightarrow \infty$ and therefore $r_\sigma(S_\lambda) \rightarrow 0$ as $\operatorname{Re} \lambda \rightarrow \infty$.

Consider now the case $p = 1$. Since the operator $M_\lambda \widetilde{H}$ is weakly compact, we have the weak compactness of S_λ . According to the Theorem 2.5.2, we get the compactness of $(S_\lambda)^2 = [(I - M_\lambda H)^{-1} M_\lambda \widetilde{H}]^2$. A similar reasoning as above shows that $\|M_\lambda \widetilde{H} (S_\lambda)^2\| \rightarrow 0$ in the operator norm as λ goes to $+\infty$ and therefore the use of (3.3.4) yields $\|(S_\lambda)^3\| \rightarrow 0$ as $\lambda \rightarrow \infty$. This shows that $r_\sigma(S_\lambda) \rightarrow 0$ as $\lambda \rightarrow +\infty$.

In summary, we have $\lim_{\lambda \rightarrow \infty} r_\sigma(S_\lambda) = 0$ for all $p \in [1, +\infty)$. This implies, in particular, that the operator $I - S_\lambda$ is invertible for all λ such that $\operatorname{Re} \lambda$ is large enough. Now, applying Theorem 2.5.1 and arguing as in the proof of Theorem 3.3.1 (1) we get the desired result.

2) The proof of (2) is similar to that of Theorem 3.3.1 (2), so it is omitted.

3) Let $\lambda > s(T_{\mathcal{H}})$. Since the operators M_λ , H and \widetilde{H} are positive, easy calculation show that

$$M_\lambda \widetilde{H} - M_\lambda H M_\lambda \widetilde{H} \leq M_\lambda \widetilde{H}.$$

Composing the two hand sides of the last equation by $(I - M_\lambda H)^{-1}$ we get

$$M_\lambda \widetilde{H} \leq S_\lambda$$

and therefore $r_\sigma(M_\lambda \widetilde{H}) \leq r_\sigma(S_\lambda)$. If $\lim_{\lambda \rightarrow s(T_{\mathcal{H}})} r_\sigma(M_\lambda \widetilde{H}) > 1$, then $\lim_{\lambda \rightarrow s(T_{\mathcal{H}})} r_\sigma(S_\lambda) > 1$. Moreover, since $(S_\lambda)^3$ is compact on L_p^+ (with $p \in [1, +\infty)$) and satisfies $\|(S_\lambda)^3\| \rightarrow 0$ as λ goes to $+\infty$, we conclude that $\lim_{\lambda \rightarrow \infty} r_\sigma(S_\lambda) = 0$. Now using the fact that $r_\sigma(S_\lambda)$ is continuous and strictly decreasing (see Lemma 2.5.1), we infer that there exists $\bar{\lambda} > s(T_{\mathcal{H}})$ such that $r_\sigma(S_{\bar{\lambda}}) = 1$. This proves that $\bar{\lambda}$ is the leading eigenvalue value of $T_{\mathcal{H}}$ which ends the proof. \square

3.4 Asymptotic spectrum of A_H

In this section we shall discuss the asymptotic spectrum of the transport operator A_H . To do so, we shall first introduce the class of regular collision operators and some of its properties on X_p , $1 \leq p < +\infty$. We define the collision operator

$$K : X_p \ni \varphi \mapsto K\varphi(x, v) = \int_{\mathbf{V}} \kappa(x, v, v') \varphi(x, v') d\mu(v'),$$

with a non-negative kernel. Note that K is local with respect to the space variable $x \in \mathbf{D}$. So, it may be regarded as an operator valued mapping from \mathbf{D} into $\mathcal{L}(L^p(\mathbf{V}, d\mu(v)))$, that is,

$$\mathbf{D} \ni x \mapsto K(x) \in \mathcal{L}(L^p(\mathbf{V}, d\mu(v))),$$

where

$$K(x) : L^p(\mathbf{V}) \ni \varphi \mapsto K(x)\varphi = \int_{\mathbf{V}} \kappa(x, v, v') \varphi(v') d\mu(v').$$

Let us first recall the definition of collectively compact (resp. collectively weakly compact) operators on Banach spaces.

Definition 3.4.1 *Let X and Y be two normed spaces and denote by \mathbf{B} the closed unit ball of X .*

(1) *A set \mathcal{C} of $\mathcal{L}(X, Y)$ is said to be collectively compact if, and only if, the set*

$$\mathcal{C}(\mathbf{B}) = \{C(x), C \in \mathcal{C}, x \in \mathbf{B}\} \text{ is relatively compact in } Y \text{ [3].}$$

(2) *A set \mathcal{W} of $\mathcal{L}(X, Y)$ is said to be collectively weakly compact if, and only if, the set*

$$\mathcal{W}(\mathbf{B}) = \{W(x), W \in \mathcal{W}, x \in \mathbf{B}\} \text{ is relatively weakly compact in } Y.$$

Now we are ready to state the definition of regular collisions operators [66] (see also [65]).

Definition 3.4.2 *Let $p \in (1, +\infty)$. A collision operator K is said to be regular on X_p if:*

(1) *$\{K(x) : x \in \mathbf{D}\}$ is a set of collectively compact operators on $L^p(\mathbf{V}, d\mu(v))$, i.e.*

$$\{K(x)\varphi, x \in \mathbf{D}, \|\varphi\| \leq 1\} \text{ is relatively compact in } L^p(\mathbf{V}, d\mu(v)).$$

(2) *For $\varphi' \in L^q(\mathbf{V}, d\mu(v))$,*

$$\{K'(x)\varphi', x \in \mathbf{D}, \|\varphi'\| \leq 1\} \text{ is relatively compact in } L^q(\mathbf{V}, d\mu(v)).$$

Here $K'(x)$ denotes the dual operator of $K(x)$ and $q = \frac{p}{p-1}$.

□

This class of operators enjoys the useful approximation property.

Lemma 3.4.1 [65, Proposition 4.1] *The class of regular collision operators is the closure in the operator norm of the class of collision operator with kernels of the form*

$$\kappa(x, v, v') = \sum_{i \in I} \alpha_i(x) f_i(v) g_i(v')$$

where I is finite, $\alpha(\cdot) \in L^\infty(\mathbf{D}; dx)$, $f_i(\cdot) \in L^p(\mathbf{V}, d\mu(v))$ and $g_i(\cdot) \in L^q(\mathbf{V}, d\mu(v))$ with $q = \frac{p}{p-1}$.

For $p = 1$, there is a more appropriate definition of regular collision operators [55, 79].

Definition 3.4.3 *We say that K is a regular collision operator on X_1 if, for almost all $x \in \mathbf{D}$, the operator*

$$\phi \in L^1(\mathbf{V}, d\mu(v)) \mapsto \int_{\mathbf{V}} \kappa(x, v, v') \phi(v') d\mu(v') \in L^1(\mathbf{V}, d\mu(v))$$

is weakly compact on $L^1(\mathbf{V}, d\mu(v))$ and the family of such operators on $L^1(\mathbf{V}, d\mu(v))$ indexed by $x \in \mathbf{D}$ is collectively weakly compact.

We now recall the following useful property of non-negative regular collision operators on X_1 .

Lemma 3.4.2 [55] *Let K be a regular non-negative collision operator. Then there exists a sequence $(K_n)_n$ of $\mathcal{L}(X_1)$ such that*

- (1) $0 \leq K_n \leq K$ for any $n \in \mathbb{N}$;
- (2) for any $n \in \mathbb{N}$, K_n is dominated by a rank-one operator in $\mathcal{L}(L^1(\mathbf{V}, d\mu(v)))$;
- (3) $\lim_{n \rightarrow \infty} \|K - K_n\| = 0$.

The Item (2) of the last lemma means that each operator K_n is dominated by an operator on X_1 which acts as follows

$$X_1 \ni \varphi \mapsto f_n(v) \int_{\mathbf{V}} \varphi(x, v') d\mu(v')$$

where f_n is a non-negative continuous bounded function with compact support on \mathbf{V} .

Remark 3.4.1 Note that, if K is a regular collision operator, then, according to Lemmas 3.4.1 and 3.4.2, it suffices to suppose that K has the following form

$$K : X_p \rightarrow X_p, \varphi \mapsto K\varphi(x, v) = \alpha(x) \int_V f(v)g(v')\varphi(x, v')d\mu(v'), \quad (3.4.1)$$

where $\alpha(\cdot) \in L^\infty(\mathbb{D}, dx)$, $f(\cdot) \in L^p(V, d\mu(v))$ and $g(\cdot) \in L^q(V, d\mu(v))$. By approximating f and g by continuous functions with compact support, we may suppose, that f and g are continuous with compact supports. Since $\alpha(\cdot) \in L^\infty(\mathbb{D}, dx)$, without loss of generality, we assume that the function $\alpha(\cdot)$ is constant equal to one. \square

From now on, we shall assume that the measure $\mu(\cdot)$ satisfies

$$(A3) \quad \begin{cases} \text{the hyperplanes have zero } \mu \text{-measure,} \\ \text{i.e. for each } e \in S^{n-1}, \mu\{v \in \mathbb{R}^n, v \cdot e = 0\} = 0 \\ \text{where } S^{n-1} \text{ denotes the unit sphere of } \mathbb{R}^n. \end{cases}$$

We introduce also the following geometrical property

$$(A4) \quad \int_{\alpha_1 \leq |x| \leq \alpha_2} d\mu(x) \int_0^{\alpha_3} \chi_A(tx) dt \rightarrow 0 \text{ as } |A| \rightarrow 0$$

for every $\alpha_1 < \alpha_2 < \infty$ and $\alpha_3 < \infty$, where $|A|$ is the Lebesgue measure of A and $\chi_A(\cdot)$ denotes the characteristic function of A .

Now, we establish the following compactness result which will play a fundamental role in the sequel.

Theorem 3.4.1 *Let K be a regular operator on X_p and let λ be such that $\operatorname{Re} \lambda > s(T_H)$.*

- (1) *If $p \in (1, \infty)$ and the condition (A3) is satisfied, then $(\lambda - T_H)^{-1}K$ and $K(\lambda - T_H)^{-1}$ are compact on X_p .*
- (2) *If $p = 1$ and the condition (A4) is satisfied, then $K(\lambda - T_H)^{-1}K$ is weakly compact on X_1 .*

Proof. The first item of the theorem was established in [42]. So it suffices to prove the second assertion.

Recall that

$$K(\lambda - T_H)^{-1}K = KB_\lambda H(I - M_\lambda)^{-1}G_\lambda K + KC_\lambda K.$$

Note however that C_λ is nothing else but the resolvent of the streaming operator with vacuum boundary condition T_0 . So, we know from [64] that, if the hypothesis (A4) holds true and K is a regular collision operator on X_1 , then the operator $KC_\lambda K$ is weakly compact operator on X_1 . Thus, in order to prove the weak compactness of $K(\lambda - T_H)^{-1}K$, it suffices to show that the operator KB_λ

is weakly compact. Since K is a regular collision operator, according to Lemma 3.4.2, it suffices to establish the result for a collision operator of the form

$$\varphi \in X_1 \rightarrow f(v) \int_{\mathbb{V}} \varphi(x, v') d\mu(v')$$

where $f(\cdot) \in L^1(\mathbb{V}, d\mu(v))$. For $\varphi \in L^{1,-}$, one can write

$$KB_\lambda \varphi(x, v) := f(v) \int_{\mathbb{V}} B_\lambda \varphi(x, v') d\mu(v'),$$

where $B_\lambda \varphi(x, v) = \varphi(x - t^-(x, v)v, v) e^{-t^-(x, v)(\lambda + \sigma(v))}$ and $\|B_\lambda\| \leq (\operatorname{Re} \lambda + \lambda^*)^{-1}$.

This yields that

$$\begin{aligned} \|KB_\lambda \varphi(x, v)\|_{X_1} &\leq \int_{\mathbb{V}} |f(v)| d\mu(v) \int_{\mathbb{D} \times \mathbb{V}} |B_\lambda \varphi(x, v')| dx d\mu(v') \\ &\leq \|f(\cdot)\|_{L^1(\mathbb{V}; d\mu(v))} \|B_\lambda \varphi(\cdot, \cdot)\|_{X_1} \\ &\leq (\operatorname{Re} \lambda + \lambda^*)^{-1} \|f(\cdot)\|_{L^1(\mathbb{V}; d\mu(v))} \|\varphi\|_{L^{1,-}}. \end{aligned}$$

So, we conclude that

$$\|KB_\lambda\| \leq (\operatorname{Re} \lambda + \lambda^*)^{-1} \|f(\cdot)\|_{L^1(\mathbb{V}; d\mu(v))}. \quad (3.4.2)$$

The estimate (3.4.2) shows that KB_λ depends continuously (for the uniform topology) on $f(\cdot) \in L^1(\mathbb{V}, d\mu(v))$. So, by approximating $f(\cdot)$ (in the L^1 -norm) by bounded functions, KB_λ is a limit (for operator topology) of integral operators with bounded kernel. Hence, KB_λ is weakly compact on X_1 (cf. [27, Corollary 11, p. 294]). \square

Remark 3.4.2 We point out that the assertion (2) of Theorem 3.4.1 is new and, as in slab geometry [39, Theorem 2.1], the weak compactness of $K(\lambda - T_H)^{-1}K$ does not depend on the boundary operator (see also Remarks 3.1 and 3.2 in [42]). \square

Before going further we first define the set $\sigma_{as}(A_H)$ (the asymptotic spectrum of the operator A_H) by

$$\sigma_{as}(A_H) := \sigma(A_H) \cap \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > s(T_H)\}.$$

Let $\lambda \in \sigma_{as}(A_H)$ and set $F(\lambda) := (\lambda - T_H)^{-1}K$.

Lemma 3.4.3 *Let K be a regular operator on X_p and assume that the conditions of Theorem 3.4.1 hold true. Then the map*

$$r_\sigma(F(\cdot)) : (s(T_H), +\infty) \rightarrow \mathbb{R}, \quad \alpha \mapsto r_\sigma(F(\alpha))$$

is continuous and strictly decreasing.

Proof. Let $\lambda_1, \lambda_2 \in (s(T_H), +\infty)$ such that $\lambda_1 > \lambda_2$. Using the resolvent identity, one sees that

$$F(\lambda_1) \leq F(\lambda_2),$$

and therefore

$$r_\sigma(F(\lambda_1)) \leq r_\sigma(F(\lambda_2)).$$

Further, according to Theorems 3.4.1 and 2.5.2, for all $\lambda \in (s(T_H), +\infty)$, the operator $F(\lambda)$ is power compact, so the desired result follows from Lemma 2.5.1. The continuity is a consequence of the analyticity of the resolvent of T_H . \square

Remark 3.4.3 In the proofs of the Theorems 3.4.2, 3.4.3 and 3.4.4, we use the fact that T_H is the generator of a C_0 -semigroup on X_p . As indicated in the introduction, they are sufficient conditions on the boundary operator H which guarantee that T_H generates a C_0 -semigroup on X_p even for multiplying boundary conditions. Let us here recall two facts which ensure that T_H is the generator of a C_0 -semigroup.

- (1) The first condition consists in assuming that the phase space $D \times V$ is such that there exists $\varepsilon > 0$ such that

$$\text{ess inf}_{(x,v) \in \Gamma_+} \tau(x, v) > \varepsilon. \quad (3.4.3)$$

In general, the sojourn time $\tau(x, v)$ may be arbitrary big as well as arbitrary small. Assumption (3.4.3) means that for all ψ in L_p^+ , $H\psi$ vanish on the set $\{(x, v) \in \Gamma_+ : \tau(x, v) \leq \varepsilon\}$ (this means that the tangential velocities are not taken into account by the boundary operator regardless of its norm). Under this hypothesis, for any boundary operator H , the streaming operator T_H generates a strongly continuous semigroup on X_p , $1 \leq p < +\infty$ independently of the size of $\|H\|$ [10, 12, 13, 49, 81] (see also [54] for more comment).

- (2) In the spirit of (1), it was established in [49] for $p = 1$ and in [54] for $1 < p < +\infty$ that T_H is the generator of a strongly continuous semigroup on X_p provided that the boundary operator H does not take too much into account the set $\{(x, v) \in \Gamma_+, \tau(x, v) = 0\}$. To be more precise, let $\varepsilon > 0$ and denote by χ_ε the multiplication operator in L_p^+ by the characteristic function of the set $\{(x, v) \in \Gamma_+, \tau(x, v) < \varepsilon\}$. The operator χ_ε is given by

$$\chi_\varepsilon \varphi(x, v) = \begin{cases} \varphi(x, v), & \text{if } \tau(x, v) \leq \varepsilon, \\ 0, & \text{else,} \end{cases}$$

for all $\varphi \in L_p^+$. Note that $\chi_\varepsilon \in \mathcal{L}(L_p^+)$. Let H be an arbitrary operator in $\mathcal{L}(L_p^+, L_p^-)$, it is established in [49, 54] that if

$$\limsup_{\varepsilon \rightarrow 0} \|H\chi_\varepsilon\|_{\mathcal{L}(L_p^+, L_p^-)} < 1,$$

then T_H generates a strongly continuous semigroup on X_p . \square

Hence, in Theorems 3.4.2, 3.4.3 and 3.4.4, the operator T_H is assumed to be the generator of a C_0 -semigroup on X_p (for simplicity, we shall make no assumptions on the operator H).

Theorem 3.4.2 *Let K be a regular collision operator on X_p and assume that $\sigma_{as}(A_H) \neq \emptyset$. If one of the following conditions is satisfied*

- (1) $p \in (1, \infty)$ and (A3) holds true,
- (2) $p = 1$ and (A4) holds true,

then $\sigma_{as}(A_H)$ consists of, at most, isolated eigenvalues with finite algebraic multiplicities. Moreover A_H admits a real leading eigenvalue $\bar{\lambda} > s(T_H)$.

Remark 3.4.4 Let us recall the following elementary facts: if λ is complex number such that $\operatorname{Re}\lambda > s(T_H)$, then

$$(\lambda - A_H) = (\lambda - T_H)[I - F(\lambda)].$$

If further $r_\sigma(F(\lambda)) < 1$, then

$$(\lambda - A_H)^{-1} = [I - F(\lambda)]^{-1}(\lambda - T_H)^{-1} = \sum_{k=0}^{+\infty} F(\lambda)^k (\lambda - T_H)^{-1}.$$

Since A_H is an additive perturbation of T_H , it is clear that if $\lambda \in \sigma_{as}(A_H)$, then λ is an eigenvalue of A_H if and only if 1 is an eigenvalue of $F(\lambda)$. □

Proof of Theorem 3.4.2. Let $\lambda \in G = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > s(T_H)\}$ and consider the operator valued function $F : G \rightarrow \mathcal{L}(X_p)$, $\lambda \rightarrow F(\lambda) = (\lambda - T_H)^{-1}K$. Since T_H generates a strongly continuous semigroup on X_p , we have $\|(\lambda - T_H)^{-1}K\| \rightarrow 0$ as $\operatorname{Re}\lambda \rightarrow \infty$ (cf. Theorem 2.3.1) and therefore

$$r_\sigma(F(\lambda)) \rightarrow 0 \text{ as } \operatorname{Re}\lambda \rightarrow \infty. \tag{3.4.4}$$

Hence, for λ large enough, the operator $I - F(\lambda)$ is invertible. Moreover, under the condition (1) or (2), the use of Theorems 3.4.1 and 2.5.2 implies that, for $n \geq 4$ and $\lambda \in G$, the operator $(F(\cdot))^n$ is compact on the connected subset G . So, Theorem 2.5.1 implies that $(I - F(\lambda))^{-1}$ exists on G except for a discrete set S of isolated points where $(I - F(\lambda))^{-1}$ has poles and the coefficients of the singular part of $(I - F(\lambda))^{-1}$ at each point of S have finite rank. Thus, $(\lambda - A_H)^{-1}$ is degenerate meromorphic on G (cf. Remark 3.4.4).

Let now $\lambda_0 \in \sigma_{as}(A_H)$ and set $\lambda_1 = \operatorname{Re}\lambda_0 > s(T_H)$. It follows from the first part of the proof (see also Remark 3.4.4) that there exists $\psi \neq 0$ such that $F(\lambda_0)\psi = \psi$ and then $|\psi| \leq F(\lambda_1)|\psi|$. Using the positivity of $F(\lambda_1)$ and Theorem 2.4.3 we get

$$r_\sigma(F(\lambda_1)) \geq 1. \tag{3.4.5}$$

By Lemma 3.4.3, the function $r_\sigma(F(\cdot))$ is continuous and strictly decreasing. So, using (3.4.4) and (3.4.5), we infer that there exists $\bar{\lambda} \in]s(T_H), +\infty[$ such that $r_\sigma(F(\bar{\lambda})) = 1$. Moreover, since for all $\lambda \in]s(T_H), +\infty[$, the operator $F(\lambda)$ is positive and power compact (cf. Theorems 3.4.1 and 2.5.2), so, using the Krein-Rutman theorem (cf. Theorem 4.1.4 in [59]), we infer that $r_\sigma(F(\lambda))$ is a eigenvalue of $F(\lambda)$. Hence, there exists $\bar{\psi} \neq 0$ such that $F(\bar{\lambda})\bar{\psi} = (\bar{\lambda} - T_H)^{-1}K\bar{\psi} = \bar{\psi}$ or else $A_H\bar{\psi} = \bar{\lambda}\bar{\psi}$. Since, for all $\lambda > \bar{\lambda}$, $r_\sigma(F(\lambda)) < 1$, we infer that $\bar{\lambda}$ is the leading eigenvalue of A_H . \square

Proposition 3.4.1 *If the assumptions of Theorem 3.4.2 hold true, then*

$$\sigma_{as}(A_H) \neq \emptyset \text{ if and only if } \lim_{\lambda \rightarrow -\lambda^*} r_\sigma(F(\lambda)) > 1.$$

Remark 3.4.5 It should be noticed that, as in Theorem 3.4.2, the result of Proposition 3.4.1 is independent of the size of $\|H\|$. \square

Proof of Proposition 3.4.1. By analyticity arguments, it suffices to restrict ourselves to the set $\sigma(A_H) \cap (s(T_H), +\infty)$. We know from the proof of the Theorem 3.4.2 that the leading eigenvalue $\bar{\lambda}$ is characterized by the following equation

$$r_\sigma(F(\bar{\lambda})) = 1. \tag{3.4.6}$$

If $\sigma(A_H) \cap (s(T_H), +\infty) \neq \emptyset$, then, by Theorem 3.4.2, there exists a leading eigenvalue $\bar{\lambda}$ such that $r_\sigma(F(\bar{\lambda})) = 1$ and so $\sigma(A_H) \cap (s(T_H), +\infty) \subset (s(T_H), \bar{\lambda}]$. Since $\lambda \mapsto r_\sigma(F(\lambda))$ is a strictly decreasing function, we have $1 = r_\sigma(F(\bar{\lambda})) < r_\sigma(F(\lambda))$ for all $\lambda \in \sigma(A_H) \cap (s(T_H), +\infty)$ and therefore $\lim_{\lambda \rightarrow -\lambda^*} r_\sigma(F(\lambda)) > 1$.

Conversely, if $\lim_{\lambda \rightarrow -\lambda^*} r_\sigma(F(\lambda)) > 1$, we know from Eq. (3.4.4) that $\lim_{\lambda \rightarrow \infty} r_\sigma(F(\lambda)) = 0$. Futher, since $\lambda \mapsto r_\sigma(F(\lambda))$ is a continuous and strictly decreasing function, there exists $\lambda_1 > -\lambda^*$ such that $r_\sigma(F(\lambda_1)) = 1$. Accordingly the use of Equation (3.4.6) shows that λ_1 is the leading eigenvalue of A_H and so $\sigma(A_H) \cap (s(T_H), +\infty) \neq \emptyset$. \square

Now we introduce the following assumption:

$$(A5) \quad \begin{cases} \mathcal{K} = K + \bar{K}, \\ \bar{K} \text{ is a regular collision operator on } X_p. \end{cases}$$

The collision operator \mathcal{K} is assumed to be a positive perturbation of the collision operator K and $\bar{K} = \mathcal{K} - K$ is a positive operator belonging to $\mathcal{L}(X_p)$. Before going further, let us first recall that by Λ we mean the set

$$\Lambda = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > -\lambda^*\}.$$

Theorem 3.4.3 *Let $p \in (1, +\infty)$ and let H be a boundary operator satisfying $\|H\| < 1$. If the conditions (A3) and (A5) are satisfied and $1 \notin \sigma((\lambda - T_H)^{-1}K)$, then $\sigma_{as}(A_H)$ consists of, at most, isolated eigenvalues with finite algebraic multiplicities.*

Remark 3.4.6 Theorem 3.4.3 is open for $p = 1$. □

Proof of Theorem 3.4.3. Let $\lambda \in \Lambda$ be such that $1 \notin \sigma((\lambda - T_H)^{-1}K)$ and consider the problem

$$\lambda\psi - A_H\psi = \varphi, \tag{3.4.7}$$

where φ is an assigned function in X_p and the unknown function ψ must be sought in $D(T_H)$. Equation (3.4.7) is equivalent to

$$\lambda\psi - T_H\psi - K\psi - \widetilde{K}\psi = \varphi.$$

The last equation may be written as

$$\psi - (\lambda - T_H)^{-1}K\psi - (\lambda - T_H)^{-1}\widetilde{K}\psi = (\lambda - T_H)^{-1}\varphi.$$

Since $1 \notin \sigma((\lambda - T_H)^{-1}K)$, the operator $I - (\lambda - T_H)^{-1}K$ is invertible and therefore

$$\psi - [I - (\lambda - T_H)^{-1}K]^{-1}(\lambda - T_H)^{-1}\widetilde{K}\psi = [I - (\lambda - T_H)^{-1}K]^{-1}(\lambda - T_H)^{-1}\varphi.$$

Set

$$F(\lambda) = [I - (\lambda - T_H)^{-1}K]^{-1}(\lambda - T_H)^{-1}\widetilde{K}.$$

Since \widetilde{K} is a regular collision operator on X_p , applying Theorem 3.4.1 (1), we infer that $(\lambda - T_H)^{-1}\widetilde{K}$ is compact and consequently $F(\lambda)$ is compact too (and depends analytically on λ).

Next, we know that T_H generates a strongly continuous semigroup on X_p (because $\|H\| < 1$), so by the Hille-Yosida theorem (Theorem 2.3.1), there exists a constant $M > 0$ such that, for all real $\lambda > s(T_H) = -\lambda^*$, we have

$$\|(\lambda - T_H)^{-1}K\| \leq \frac{M\|K\|}{\lambda + \lambda^*}. \tag{3.4.8}$$

It is clear that there exists a real $\lambda_0 > 0$ such that, for all $\lambda > \lambda_0$, we have $\frac{M\|K\|}{\lambda + \lambda^*} < 1$ and so, for such λ , we can write

$$[I - (\lambda - T_H)^{-1}K]^{-1} = \sum_{k=0}^{+\infty} ((\lambda - T_H)^{-1}K)^k$$

However, by (3.4.8), we conclude that there exists $\lambda_1 > \lambda_0$ and a real $\rho \in (0, 1)$ such that, for all $\lambda > \lambda_1$, we have $\|(\lambda - T_H)^{-1}K\| \leq \rho$. This implies $\|[I - (\lambda - T_H)^{-1}K]^{-1}\| \leq \frac{1}{1-\rho}$ and therefore we have (use (3.4.8))

$$\|F(\lambda)\| \leq \frac{1}{1-\rho} \|(\lambda - T_H)^{-1}\tilde{K}\| \rightarrow 0 \quad \text{as } \lambda \rightarrow +\infty.$$

Accordingly, for λ large enough, the operator $[I - F(\lambda)]$ is invertible. Now, applying Theorem 2.5.1, we infer that $(I - F(\lambda))^{-1}$ exists for all λ belonging to Λ except for a discrete set S contained in Λ consisting of poles with finite rank residues. This ends the proof. \square

Now we are able to formulate the following result.

Theorem 3.4.4 *Let $p \in [1, +\infty)$ and assume that the boundary operator $\mathcal{H} = H + \tilde{H}$ satisfies the condition (A2) and $\|M_\lambda H\| < 1$ for all λ such that $\text{Re}\lambda$ is large enough. Assume further that the hypothesis (A3) (resp. (A4)) holds true for $p \in (1, +\infty)$ (resp. $p = 1$). If K is a regular collision operator, then $\sigma_{as}(A_{\mathcal{H}})$ consists of, at most, isolated eigenvalues with finite algebraic multiplicities.*

Proof. Let λ be a complex number such that $\text{Re}\lambda$ is large enough, so by hypothesis, $r_\sigma(M_\lambda H) < 1$. We shall first prove the existence of $(\lambda - T_{\mathcal{H}})^{-1}$. As indicated in Section 3.2, the existence of $(\lambda - T_{\mathcal{H}})^{-1}$ reduces to the invertibility of the operator $(I - M_\lambda \mathcal{H})$ on L_p^+ . Since the operator $(I - M_\lambda H)^{-1}$ exists, Equation (3.2.3) may be written as

$$\psi^+ - \Delta_\lambda \psi^+ = (I - M_\lambda H)^{-1} G_\lambda \varphi$$

where

$$\Delta(\lambda) := (I - M_\lambda H)^{-1} M_\lambda \tilde{H}.$$

In order to obtain ψ^+ , it suffices to check that the operator $(I - \Delta(\lambda))$ is invertible. To this end, the use of the hypothesis (A3), shows that $[\Delta(\lambda)]^2$ is compact. (In fact, for $p \in (1, +\infty)$, $\Delta(\lambda)$ is compact and so $[\Delta(\lambda)]^2$ is compact too. Next, according to Theorem 2.5.2, we conclude that $[\Delta(\lambda)]^2$ is compact if $p = 1$.) As we already saw previously, we have $M_\lambda \tilde{H} \rightarrow 0$ strongly as $\text{Re}\lambda \rightarrow \infty$, using the compactness of $[\Delta(\lambda)]^2$, we deduce that $\|[\Delta(\lambda)]^3\| \rightarrow 0$ as $\text{Re}\lambda \rightarrow \infty$ (note that here we used also the fact that $\|(I - M_\lambda H)^{-1}\| \leq \frac{1}{1 - \|H\|}$, in other words, $\|(I - M_\lambda H)^{-1}\|$ is bounded independently of λ). This implies, in particular, that $r_\sigma(\Delta(\lambda)) < 1$ for λ large enough. Accordingly, for such λ , the operator $(I - \Delta(\lambda))$ is invertible and therefor $\lambda \in \rho(T_{\mathcal{H}})$.

On the other hand, for λ large enough, the problem

$$\lambda\psi - A_{\mathcal{H}}\psi = \varphi$$

may be written in the form

$$\psi - (\lambda - T_{\mathcal{H}})^{-1} K \psi = (\lambda - T_{\mathcal{H}})^{-1} \varphi.$$

Further, the regularity of the collision operator K together with Theorems 3.4.1 and 2.5.2 implies that the operator $[(\lambda - T_{\mathcal{H}})^{-1} K]^4$ is compact. Since T_H is the generator of a C_0 -semigroup on X_p , by the Hille-Yosida-Phillips theorem (Theorem 2.3.1), we get

$$\lim_{\operatorname{Re} \lambda \rightarrow +\infty} \|(\lambda - T_{\mathcal{H}})^{-1} K\| = 0.$$

Now, applying Theorem 2.5.1, we get the desired result. □

In the remainder of this section, we will assume that the collision frequency and the scattering kernel are *homogeneous*, that is,

$$\sigma(x, v) = \sigma(v) \quad \text{and} \quad \kappa(x, v, v') = \kappa(v, v').$$

Assume further that

$$\lim_{v \rightarrow 0} \sigma(v) = \lambda^*.$$

Now, we turn our attention to the bounded part of the transport operator A_H which we denote by \mathcal{B} . It is defined by

$$\begin{cases} \mathcal{B} : L^p(\mathcal{V}, d\mu(v)) \rightarrow L^p(\mathcal{V}, d\mu(v)); \\ \varphi \longrightarrow \mathcal{B}\varphi(v) = -\sigma(v)\varphi(v) + \int_{\mathcal{V}} \kappa(v, v')\varphi(v')d\mu(v'). \end{cases}$$

We shall discuss the relationship between the spectrum of A_H and that of \mathcal{B} . Let us first introduce the set

$$\sigma_{as}(\mathcal{B}) := \sigma(\mathcal{B}) \cap \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > -\lambda^*\}.$$

We begin with the following observation:

Proposition 3.4.2 [65, Proposition 5.5] *If K is a power compact operator on $L^p(\mathcal{V}, d\mu(v))$, then $\sigma_{as}(\mathcal{B})$ consists of, at most, isolated eigenvalues with finite algebraic multiplicities. Moreover, if $\sigma_{as}(\mathcal{B}) \neq \emptyset$, then there exists a leading eigenvalue λ_0 of \mathcal{B} .*

We are now in a position to prove

Theorem 3.4.5 *Let K be a regular collision operator on X_p and let H be a boundary operator satisfying $\|H\| < 1$. If $\sigma_{as}(A_H) \neq \emptyset$, then $\sigma_{as}(\mathcal{B}) \neq \emptyset$. Further, if $\sigma_{as}(A_H) \neq \emptyset$, then the leading eigenvalue of A_H is less than or equal to the leading eigenvalue of \mathcal{B} . Moreover, the latter is less than or equal to $-\lambda^* + r_{\sigma}(K)$ ($r_{\sigma}(K)$ denotes the spectral radius of K).*

Remark 3.4.7 (1) To our knowledge, the first results concerning the connection between the asymptotic spectrum of the transport operator and that of its bounded part are due to M. Mokhtar-Kharroubi [62].

(2) This result is an extension to abstract boundary conditions of a result established in [62, p. 941] proved vacuum boundary conditions ($H = 0$).

(3) For sufficient conditions in terms of collision operators ensuring that $\sigma_{as}(\mathcal{B}) = \emptyset$, we refer the reader to [62, p. 942].

Proof of Theorem 3.4.5. Suppose that $\sigma_{as}(A_H) \neq \emptyset$. It is clear, by Theorem 3.4.1, that A_H has a leading eigenvalue which we denote by $\bar{\lambda}$ and let $\bar{\psi} \neq 0$ denote its associated positive eigenvector. Hence, the equation $A_H \bar{\psi} = \bar{\lambda} \bar{\psi}$ may be written in the form

$$-v \cdot \nabla_x \bar{\psi}(x, v) - (\bar{\lambda} + \sigma(v)) \bar{\psi}(x, v) + \int_{\mathbb{V}} \kappa(v, v') \bar{\psi}(x, v') d\mu(v') = 0. \quad (3.4.9)$$

By integrating (3.4.9) with respect to x over D , we get

$$\int_D -v \cdot \nabla_x \bar{\psi}(x, v) dx - (\bar{\lambda} + \sigma(v)) \int_D \bar{\psi}(x, v) dx + \int_D \int_{\mathbb{V}} \kappa(v, v') \bar{\psi}(x, v') d\mu(v') dx = 0. \quad (3.4.10)$$

Set

$$\bar{\varphi}(v) = \int_D \bar{\psi}(x, v) dx.$$

It is clear that $\bar{\varphi} \geq 0$ and $\bar{\varphi} \neq 0$. Moreover, Eq. (3.4.10) becomes

$$\int_D -v \cdot \nabla_x \bar{\psi}(x, v) dx - (\bar{\lambda} + \sigma(v)) \bar{\varphi}(v) + \int_{\mathbb{V}} \kappa(v, v') \bar{\varphi}(v') d\mu(v') = 0. \quad (3.4.11)$$

We claim that

$$\int_D -v \cdot \nabla_x \bar{\psi}(x, v) dx \leq 0.$$

Indeed, let $v \in \mathbb{V}$, applying Green formula, we get

$$\begin{aligned} \int_D -v \cdot \nabla_x \bar{\psi}(x, v) dx &= - \int_{\partial D} \bar{\psi}(x, v) \frac{v}{\partial v_x} d\gamma_x \\ &= - \int_{\partial D} \bar{\psi}(x, v) v \cdot \nu_x d\gamma_x. \end{aligned}$$

Set

$$\partial D^\pm := \{x \in \partial D, \pm v \cdot \nu \geq 0\}.$$

It is obvious that

$$\partial D = \partial D^+ \cup \partial D^-.$$

So, we can write

$$-\int_{\partial D} \bar{\psi}(x, v) v \cdot \nu_x d\gamma_x = -\int_{\partial D^+} \bar{\psi}(x, v) v \cdot \nu_x d\gamma_x + \int_{\partial D^-} \bar{\psi}(x, v) v \cdot \nu_x d\gamma_x$$

which implies that

$$\int_D -v \cdot \nabla_x \bar{\psi}(x, v) dx \leq 0.$$

Evidently, the last equation is valid for each $v \in V$, so it follows from (3.4.11) that

$$-(\bar{\lambda} + \sigma(v))\bar{\varphi}(v) + \int_V \kappa(v, v')\bar{\varphi}(v')d\mu(v') \geq 0 \quad \text{for all } v \in V,$$

which can be written in the form

$$\int_V \frac{\kappa(v, v')}{\bar{\lambda} + \sigma(v)} \bar{\varphi}(v')d\mu(v') \geq \bar{\varphi}(v). \quad (3.4.12)$$

Let $\lambda > -\lambda^*$ and define the operator K_λ on $L_p(V; d\mu(v))$ by

$$\begin{cases} K_\lambda : L_p(V, d\mu(v)) \rightarrow L_p(V, d\mu(v)) \\ \varphi \longrightarrow K_\lambda \varphi(v) = \int_V \frac{\kappa(v, v')}{\lambda + \sigma(v)} \varphi(v')d\mu(v'). \end{cases}$$

It follows from the fact that K is regular on X_p that K_λ is a compact positive operator on $L_p(V, d\mu(v))$. Applying Krein-Rutman's theorem [59, Theorem 4.1.4], we infer that $r_\sigma(K_\lambda)$ is an eigenvalue of K_λ depending continuously on λ . On the other hand, Theorem 2.4.3 together with Eq. (3.4.12) implies that $r_\sigma(K_{\bar{\lambda}}) \geq 1$. Since $\lim_{\lambda \rightarrow \infty} r_\sigma(K_\lambda) = 0$, there exists $\lambda_0 \geq \bar{\lambda}$ such that $r_\sigma(K_{\lambda_0}) = 1$. This shows that there exists $\varphi_0 \neq 0$ in $L_p(V; d\mu(v))$ satisfying

$$K_{\lambda_0} \varphi_0 = \varphi_0. \quad (3.4.13)$$

This implies that $\mathcal{B}\varphi_0 = \lambda_0 \varphi_0$ which proves the first assertion.

To prove the last part of the theorem, we have only to observe that Eq. (3.4.13) implies

$$\int_V \kappa(v, v') |\varphi_0(v')| d\mu(v') = (\lambda_0 + \sigma(v)) |\varphi_0(v)| \geq (\lambda_0 + \lambda^*) |\varphi_0(v)|,$$

and so we conclude that $r_\sigma(K) \geq \lambda^* + \lambda_0$ or again

$$\lambda_0 \leq r_\sigma(K) - \lambda^*.$$

This ends the proof. □

As an immediate consequences of Theorem 3.4.5, we have

Corollary 3.4.1 *Assume that the hypotheses of Theorem 3.4.5 are satisfied, then the following assertions hold true.*

(1) *If K is quasinilpotent (i.e. $r_\sigma(K) = 0$), then $\sigma_{as}(A_H) = \emptyset$.*

(2) *If $p = 1$, $\sigma(v) > \lambda^*$ and*

$$\int_V \frac{\kappa(v, v')}{\sigma(v) - \lambda^*} d\mu(v') \leq 1, \quad (3.4.14)$$

then $\sigma_{as}(A_H) = \emptyset$.

(3) *If $p > 1$, $\sigma(v) > \lambda^*$ and*

$$\int_V \frac{1}{(\sigma(v) - \lambda^*)^p} d\mu(v) \left(\int_V \kappa(v, v')^q d\mu(v') \right)^{\frac{p}{q}} \leq 1, \quad (3.4.15)$$

then $\sigma_{as}(A_H) = \emptyset$.

(4) *If \mathcal{B} is subcritical (i.e. $\sigma_{as}(\mathcal{B}) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < 0\}$), then A_H is subcritical.*

Proof. 1) Assume the contrary, that is $\sigma_{as}(A_H) \neq \emptyset$. We know from Theorem 3.4.5 that $\sigma_{as}(\mathcal{B}) \neq \emptyset$ and then \mathcal{B} has a leading eigenvalue λ_0 . Using the expression of K_λ (see above), we get

$$K_{\lambda_0} \leq \frac{1}{\lambda_0 + \lambda^*} K.$$

Since $r_\sigma(K_{\lambda_0}) = 1$, we have necessary $r_\sigma(K) > 0$ which contradicts the fact that K is quasinilpotent.

2) By Theorem 3.4.5, it suffices to prove that (3.4.14) implies that $\sigma_{as}(\mathcal{B}) = \emptyset$. We argue by contradiction. Suppose that $\sigma_{as}(\mathcal{B}) \neq \emptyset$ and let λ_0 be its leading eigenvalue. Thus, we have

$$\frac{1}{\lambda_0 + \sigma(v)} \int_V \kappa(v, v') \varphi(v') d\mu(v') = \varphi(v), \quad \lambda_0 > -\lambda^*, \quad \varphi \in L_+^1(V, d\mu(v)),$$

where $\varphi \in L_+^1(V, d\mu(v))$ and $\varphi \neq 0$. Integrating over V , we get

$$\int_V \varphi(v') d\mu(v') \int_V \frac{\kappa(v, v')}{\lambda_0 + \sigma(v)} d\mu(v) = \|\varphi\|_{L^1(V)}$$

and then

$$\sup_{v' \in V} \int_V \frac{\kappa(v, v')}{\lambda_0 + \sigma(v)} d\mu(v) \|\varphi\|_{L^1(V)} \geq \|\varphi\|_{L^1(V)}.$$

Since $\lambda_0 > -\lambda^*$, the last equation implies

$$\sup_{v' \in V} \int_V \frac{\kappa(v, v')}{\sigma(v) - \lambda^*} d\mu(v) > 1$$

which contradicts (3.4.14).

3) As the previous item, we argue by contradiction. Suppose that $\sigma_{as}(\mathcal{B}) \neq \emptyset$ and let λ_0 be the leading eigenvalue of \mathcal{B} , that is,

$$\frac{1}{\lambda_0 + \sigma(v)} \int_V \kappa(v, v') \varphi(v') d\mu(v') = \varphi(v), \quad \lambda_0 > -\lambda^*, \quad \varphi \in L^p_+(V, d\mu(v)), \quad \varphi \neq 0.$$

Hölder's inequality leads to

$$\|\varphi\|_{L^p(V)}^p \frac{1}{(\lambda_0 + \sigma(v))^p} \left(\int_V \kappa(v, v')^q d\mu(v') \right)^{\frac{p}{q}} \geq |\varphi(v)|^p.$$

Integrating over V , we get

$$\|\varphi\|_{L^p(V)}^p \int_V \frac{1}{(\lambda_0 + \sigma(v))^p} \left(\int_V \kappa(v, v')^q d\mu(v') \right)^{\frac{p}{q}} d\mu(v) \geq \|\varphi\|_{L^p(V)}^p.$$

Due to the fact that $\lambda_0 > -\lambda^*$, we infer that

$$\int_V \frac{1}{(\sigma(v) - \lambda^*)^p} \left(\int_V \kappa(v, v')^q d\mu(v') \right)^{\frac{p}{q}} d\mu(v) > 1$$

which contradicts (3.4.15) and ends the proof.

4) We denote by λ_0 and $\bar{\lambda}$ the leading eigenvalues of \mathcal{B} and A_H , respectively. Note that, by Theorem 3.4.5, we have $\bar{\lambda} \leq \lambda_0$. If \mathcal{B} is subcritical, then $\lambda_0 < 0$ and therefore $\bar{\lambda} < 0$. \square

3.5 Irreducibility of the semigroup $(e^{tA_H})_{t \geq 0}$

In this section we shall discuss the irreducibility of the C_0 -semigroup $(e^{tA_H})_{t \geq 0}$ generated by the transport operator A_H . We recall that a positive C_0 -semigroup $(T(t))_{t \geq 0}$ on a Banach lattice X is called irreducible if there is no $(T(t))_{t \geq 0}$ -invariant closed ideal except $\{0\}$ and X (see, for example, [68] or [59] for the irreducibility of positive linear operators between Banach lattices). Note that the interest of the irreducibility lies in the fact that if $(e^{tA_H})_{t \geq 0}$ is irreducible, then the leading eigenvalue (if it exists) is strictly dominant with multiplicity 1 and the associated eigen-function is strictly positive. Hence, if this eigenvalue is strictly dominant, we obtain a simple description of the time asymptotic behavior ($t \rightarrow +\infty$) of the solution to the Cauchy problem

$$\begin{cases} \frac{d\psi(t)}{dt} = A_H \psi(t) \\ \psi(0) = u_0. \end{cases}$$

Throughout this section we assume that the operator T_H generates a strongly continuous semigroup on X_p and we impose no condition on the size of $\|H\|$.

Let $(T(t))_{t \geq 0}$ be a positive C_0 -semigroup. In the literature, there are various equivalent definitions of the irreducibility of $(T(t))_{t \geq 0}$ in L^p -spaces (or more generally in arbitrary Banach lattices) (see, for instance, [68, p. 306] or [86]).

We shall now recall the following proposition which we shall use throughout in this section. Although it is valid for general Banach lattices, here we restrict ourselves to the particular case of L^p -spaces.

Proposition 3.5.1 [65, Proposition 5.4] *Let $(T(t))_{t \geq 0}$ be a positive C_0 -semigroup on $L^p(\Omega)$ with generator S . Then the following assertions are equivalent:*

- (1) $(T(t))_{t \geq 0}$ is irreducible.
- (2) For every $f \in L^p_+(\Omega)$ (the positive cone of $L^p(\Omega)$) and $g \in L^q_+(\Omega)$ with $q = \frac{p}{p-1}$, there exists $t \geq 0$ such that $\langle T(t)f, g \rangle > 0$.
- (3) $(\lambda - S)^{-1}$ is strictly positive for some (every) $\lambda > s(S)$.
- (4) $(\lambda - S)^{-1}$ is irreducible for some (every) $\lambda > s(S)$.

For the definition of the (strictly) positive operator we refer to Section 2.4.

For transport operator with vacuum boundary conditions, i.e. $H = 0$, sufficient conditions of irreducibility of e^{tA_0} were proposed in the works [86, 32, 63]. In [41] the irreducibility of the transport operator with abstract boundary conditions was discussed only in slab geometry. The main goal of this section is to complete these works by extending some of their results concerning the irreducibility of the transport semigroup to abstract boundary conditions in bounded geometry. Our objective here is to discuss the influence of the boundary operators on the irreducibility of the semigroup generated by the transport operator A_H on X_p , $1 \leq p < \infty$.

We denote by $(V_{H,K}(t))_{t \geq 0}$ the semigroup generated by $A_H = T_H + K$. We know from Remark 3.2.1 that

$$(\lambda - T_H)^{-1} \geq (\lambda - T_0)^{-1} \quad \text{for all } \lambda > s(T_H).$$

Since K is a positive operator, we conclude that

$$((\lambda - T_H)^{-1}K)^n(\lambda - T_H)^{-1} \geq ((\lambda - T_0)^{-1}K)^n(\lambda - T_0)^{-1}, \quad \forall n \in \mathbb{N}. \quad (3.5.1)$$

Further, if $\lambda \in \rho(A_H) \cap \rho(A_0)$ and $r_\sigma((\lambda - T_H)^{-1}K) < 1$, then

$$(\lambda - A_H)^{-1} = (I - (\lambda - T_H)^{-1}K)^{-1}(\lambda - T_H)^{-1}$$

or

$$(\lambda - A_H)^{-1} = \sum_{n \geq 0} ((\lambda - T_H)^{-1}K)^n (\lambda - T_H)^{-1}. \quad (3.5.2)$$

Using (3.5.2) together with (3.5.1), we obtain

$$(\lambda - A_H)^{-1} \geq (\lambda - A_0)^{-1} \geq 0. \quad (3.5.3)$$

We fix $t > 0$, by the inequality (3.5.3), it is clear that, for all integer n such that $\frac{n}{t} > \max\{-\lambda^*, s(T_H)\}$, we have

$$\left(\frac{n}{t}\left(\frac{n}{t} - A_H\right)^{-1}\right)^n \psi \geq \left(\frac{n}{t}\left(\frac{n}{t} - A_0\right)^{-1}\right)^n \psi, \quad \forall \psi \geq 0.$$

Accordingly

$$\lim_{n \rightarrow \infty} \left(\frac{n}{t}\left(\frac{n}{t} - A_H\right)^{-1}\right)^n \psi \geq \lim_{n \rightarrow \infty} \left(\frac{n}{t}\left(\frac{n}{t} - A_0\right)^{-1}\right)^n \psi, \quad \forall \psi \geq 0. \quad (3.5.4)$$

Moreover, since A_H is the generator of C_0 -semigroup $V_{H,K}(t)$, using the exponential formula (see, for example, [35, p. 479]), we have

$$V_{H,K}(t) = \lim_{n \rightarrow \infty} \left(1 - \frac{t}{n} A_H\right)^{-n} = \lim_{n \rightarrow \infty} \left(\frac{n}{t}\left(\frac{n}{t} - A_H\right)^{-1}\right)^n.$$

Hence, according to (3.5.4), we have

$$V_{H,K}(t) \geq V_{0,K}(t), \quad \forall t \geq 0.$$

A similar reasoning gives

$$V_{H,K}(t) \geq V_{H,0}(t), \quad \forall t \geq 0. \quad (3.5.5)$$

Now we are in a position to state

Theorem 3.5.1 *If $(V_{0,K}(t))_{t \geq 0}$ or $(V_{H,0}(t))_{t \geq 0}$ is irreducible, then $(V_{H,K}(t))_{t \geq 0}$ is irreducible.*

Proof. If $(V_{0,K}(t))_{t \geq 0}$ is irreducible, then, by the Proposition 3.5.1, $(\lambda - A_0)^{-1}$ is strictly positive. Using (3.5.3), we infer that $(\lambda - A_H)^{-1}$ is strictly positive too and consequently $(V_{H,K}(t))_{t \geq 0}$ is irreducible. Assume now that $(V_{H,0}(t))_{t \geq 0}$ is irreducible. A similar reasoning as above using Proposition 3.5.1 and (3.5.5) gives the irreducibility of $(V_{H,K}(t))_{t \geq 0}$. \square

In the following result, we shall derive some sufficient conditions of irreducibility for $(V_{H,K}(t))_{t \geq 0}$, in terms of the boundary operator.

Corollary 3.5.1 *If $H\mathcal{P}(\lambda)^{-1}$ is strictly positive, then the semigroup $(V_{H,K}(t))_{t \geq 0}$ is irreducible.*

Proof. According to (3.2.4) and the positivity of C_λ we have the estimate

$$(\lambda - T_H)^{-1} \geq B_\lambda H\mathcal{P}(\lambda)^{-1} G_\lambda.$$

Note that G_λ is a strictly positive operator from X_p into L_p^+ and B_λ is a multiplication operator by a strictly positive function on $D \times V$. Hence, $(\lambda - T_H)^{-1}$ is a strictly positive operator on X_p . Consequently, the semigroup $(V_{H,0}(t))_{t \geq 0}$ is irreducible. The use of Theorem 3.5.1 gives the desired result. Q.E.D.

We have also the following corollary.

Corollary 3.5.2 *Assume that H satisfies $\|H\| < 1$. The semigroup $(V_{H,K}(t))_{t \geq 0}$ is irreducible if one of the following two conditions holds true:*

- (1) H is strictly positive,
- (2) there exists an integer $n_0 \in \mathbb{N}$ such that $(M_\lambda H)^{n_0}$ is strictly positive.

Proof. We know from Section 3.2 that $\|M_\lambda\| \leq 1$, so by hypothesis, we have $\|M_\lambda H\| < 1$. This implies that $M_\lambda H(I - M_\lambda H)^{-1} = \sum_{n \geq 1} (M_\lambda H)^n$. On the other hand, the operator M_λ is a multiplication operator by a strictly positive function, so the operator $H(I - M_\lambda H)^{-1}$ is strictly positive if and only if $M_\lambda H(I - M_\lambda H)^{-1}$ is strictly positive. But, due to the positivity of the operator $M_\lambda H$ we have

$$M_\lambda H(I - M_\lambda H)^{-1} \geq (M_\lambda H)^{n_0}.$$

So, if one of the conditions (i) and (ii) is satisfied, then $M_\lambda H(I - M_\lambda H)^{-1}$ is strictly positive. Accordingly, the result follows from Corollary 3.5.1. \square

Corollary 3.5.3 *Assume that the hypotheses of Theorem 3.4.5 hold true and $\sigma_{as}(A_H) \neq \emptyset$. If K is irreducible, then $\lambda_0 > \bar{\lambda}$ where λ_0 (resp. $\bar{\lambda}$) stands for the leading eigenvalue of \mathcal{B} (resp. A_H).*

Proof. If K is irreducible, then $K_{\bar{\lambda}}$ is also irreducible ($K_{\bar{\lambda}}$ is defined in the proof of Theorem 3.4.5). So, Eq. (3.4.12) together with Theorem 2.4.2 gives $r_\sigma(K_{\bar{\lambda}}) > 1$ which implies that $\lambda_0 > \bar{\lambda}$. This ends the proof. \square

3.6 Strict monotonicity of the leading eigenvalue

The aim of this section is to study the strict growth properties of the leading eigenvalue with respect to the parameters of the transport operator. Let us first note that, in [63], this problem was already considered for non-renty boundary conditions in bounded geometry ($H = 0$). Afterwards, a similar analysis was done for abstract boundary conditions in slab geometry in [41]. The objective of this section is to complete these works by extending some of their results to abstract boundary conditions in bounded geometry. We start our analysis by discussing the influence of the boundary operator on the monotonicity of the leading eigenvalue. To do so, let H_1 and H_2 be two boundary operators satisfying

$$(A6) \quad 0 \leq H_1 \leq H_2.$$

We denote by $\mathcal{P}_i(\lambda)$ the operator $(I - M_\lambda H_i)$, $i = 1, 2$. Under the hypothesis (A6), for any $\lambda > s(T_{H_i})$ ($s(T_{H_i})$ stands for the spectral bound of T_{H_i}), we have

$$(\lambda - T_{H_i})^{-1} = B_\lambda H_i [\mathcal{P}_i(\lambda)]^{-1} G_\lambda + C_\lambda, \quad i = 1, 2.$$

Further, for $\lambda > s(T_{H_i})$, $i = 1, 2$, the positivity of the operators H_1 , H_2 , K , M_λ , G_λ and C_λ , together with the formula

$$[\mathcal{P}_1(\lambda)]^{-1} - [\mathcal{P}_2(\lambda)]^{-1} = [\mathcal{P}_1(\lambda)]^{-1} M_\lambda (H_1 - H_2) [\mathcal{P}_2(\lambda)]^{-1}$$

and the hypothesis (A6) lead to

$$(\lambda - T_{H_1})^{-1} K \leq (\lambda - T_{H_2})^{-1} K. \quad (3.6.1)$$

In the following, we denote by $\lambda(H)$ the leading eigenvalue of the operator A_H (when it exists).

Theorem 3.6.1 *Let $p \in (1, +\infty)$ (resp. $p = 1$) and assume that the conditions (A3) (resp. (A4)) and (A6) hold true. Suppose further that K is a regular collision operator on X_p and $\lambda(H_1)$ exists. Then, $\lambda(H_2)$ exists and we have $\lambda(H_1) \leq \lambda(H_2)$. Furthermore, if $(\lambda(H_1) - T_{H_2})^{-1} K$ has a strictly positive power, then $\lambda(H_1) < \lambda(H_2)$.*

Proof. We first note that, since K is regular, Theorem 3.4.1 implies that $(\lambda(H_1) - T_{H_1})^{-1} K$ and $(\lambda(H_1) - T_{H_2})^{-1} K$ are power compact operators on X_p , $1 \leq p < +\infty$. Next, using Krein-Rutman's theorem [59, Theorem 4.1.4], we infer that $r_\sigma((\lambda(H_1) - T_{H_1})^{-1} K)$ is an eigenvalue of $(\lambda(H_1) - T_{H_1})^{-1} K$. Keeping in mind that $\lambda(H_1)$ is the leading eigenvalue of A_{H_1} , so (cf. (3.4.6)) we have

$$r_\sigma((\lambda(H_1) - T_{H_1})^{-1} K) = 1. \quad (3.6.2)$$

Since the operator $(\lambda(H_1) - T_{H_2})^{-1} K$ has a strictly positive power, using (3.6.1), (3.6.2) together with Theorem 2.4.2, we obtain

$$r_\sigma((\lambda(H_1) - T_{H_2})^{-1} K) > 1.$$

Next, combining (3.4.4), (3.6.2) and Lemma 3.4.3, we conclude that there exists a unique $\lambda_0 > \lambda(H_1)$ such that $r_\sigma((\lambda_0 - T_{H_2})^{-1} K) = 1$. This shows (cf. again (3.4.6)) that $\lambda_0 = \lambda(H_2)$ and completes the proof. \square

Remark 3.6.1 If one of the following condition is satisfied, then $(\lambda_{H_1} - T_{H_2})^{-1} K$ has a strictly positive power:

- (1) there exists an integer $n \geq 1$ such that $[C_{\lambda(H_1)} K]^n$ is strictly positive,
 - (2) there exists an integer $n \geq 1$ such that $[B_{\lambda(H_1)} H_2 (I - M_{\lambda(H_1)} H_2)^{-1} G_{\lambda(H_1)} K]^n$ is strictly positive.
- \square

Practical criteria of monotonicity of $\lambda(H)$ are given in the following corollary.

Corollary 3.6.1 *Assume that the hypotheses of Theorem 3.6.1 hold true and assume that one of the following conditions is satisfied*

- (1) H_2 is strictly positive and $\ker(K) \cap \{\varphi \in X_p, \varphi \geq 0\} = \{0\}$,
- (2) there exists $n_0 \in \mathbb{N}$ such that $(M_{\lambda(H_1)}H_2)^{n_0}$ is strictly positive and $\ker(K) \cap \{\varphi \in X_p, \varphi \geq 0\} = \{0\}$.

Then $\lambda(H_1) < \lambda(H_2)$.

Proof. Note that (3.6.1) and the fact that $B_{\lambda(H_1)}$ and $M_{\lambda(H_1)}$ are multiplication operators by a strictly positive function imply that $(\lambda(H_1) - T_{H_2})^{-1}K$ admits a strictly positive power if one of the conditions above is satisfied. Now, a similar reasoning as in the proof of Theorem 3.6.1 gives the desired result. \square

Now, we shall discuss the strict monotonicity of the leading eigenvalue of A_H with respect to the collision operator. To do so, let K_1 and K_2 be two non-negative regular collision operators satisfying $K_1 \leq K_2$ and $K_1 \neq K_2$. It is clear that

$$(\lambda - T_H)^{-1}K_1 \leq (\lambda - T_H)^{-1}K_2. \quad (3.6.3)$$

We denote by $\lambda(K)$ the leading eigenvalue of $A_H = T_H + K$, when it exists.

Theorem 3.6.2 *Let $p \in (1, \infty)$ (resp. $p = 1$) and assume that the condition (A3) (resp. (A4)) holds true. If $\lambda(K_1)$ exists, then $\lambda(K_2)$ exists and we have $\lambda(K_1) \leq \lambda(K_2)$. Furthermore, if $(\lambda(K_1) - T_H)^{-1}K_2$ has a strictly positive power, then $\lambda(K_1) < \lambda(K_2)$.*

Proof. We note first that, since K_1 and K_2 are regular collision operators, Theorems 2.5.2 and 3.4.1 imply that $(\lambda(K_1) - T_H)^{-1}K_1$ and $(\lambda(K_1) - T_H)^{-1}K_2$ are power compact on X_p , $1 \leq p < +\infty$. Next, using Krein-Rutman's theorem, infer that $\lambda(K_1)$ is an eigenvalue of $(\lambda(K_1) - T_H)^{-1}K_1$. Keeping in mind that $\lambda(K_1)$ is the leading eigenvalue of A_H , so (cf. (3.4.4)) we get

$$r_\sigma((\lambda(K_1) - T_H)^{-1}K_1) = 1. \quad (3.6.4)$$

Further, since $(\lambda(K_1) - T_H)^{-1}K_2$ has a strictly positive power, using (3.6.3), (3.6.4) together with Theorem 2.4.2, we have

$$r_\sigma((\lambda(K_1) - T_H)^{-1}K_2) > 1.$$

Finally, combining (3.4.6), (3.6.4) and Lemma 3.4.3, we conclude that there exists a unique $\lambda_0 > \lambda_{H_1}$ such that $r_\sigma((\lambda_0 - T_{H_2})^{-1}K) = 1$. Hence $\lambda_0 = \lambda(K_2)$ which ends the proof. \square

The following corollary is an immediate consequence of Theorem 3.6.2.

Corollary 3.6.2 *Assume that the hypotheses of Theorem 3.6.2 hold true and one of the following conditions is satisfied*

- (1) *H is strictly positive and $\ker(K_2) \cap \{\varphi \in X_p, \varphi \geq 0\} = \{0\}$,*
- (2) *there exists $n_0 \in \mathbb{N}$ such that $(M_{\lambda(K_1)}H_2)^{n_0}$ is strictly positive and $\ker(K_2) \cap \{\varphi \in X_p, \varphi \geq 0\} = \{0\}$.*

Then $\lambda(K_1) < \lambda(K_2)$.

The next result is concerned with the strict monotonicity of the leading eigenvalue of A_H with respect to both the collision operators and the boundary operators.

Consider two non-negative regular collision operators K_1 and K_2 satisfying $K_1 \leq K_2$ and $K_1 \neq K_2$ and let H_1 and H_2 be two non-negative boundary operators satisfying $H_1 \leq H_2$.

It is easy to check that

$$(\lambda - T_{H_1})^{-1}K_1 \leq (\lambda - T_{H_2})^{-1}K_2$$

We denote by $\lambda(H, K)$ the leading eigenvalue of the operator $A_H = T_H + K$, when it exists.

As an immediate consequence of Theorems 3.6.1 and 3.6.2, we have

Corollary 3.6.3 *Let $p \in (1, +\infty)$ (resp. $p = 1$) and assume that the conditions (A3), (resp. (A4)) and (A6) hold true. If K_1 and K_2 are two non-negative regular collision operators satisfying $K_1 \leq K_2$ and H_1 and H_2 are two non-negative boundary operators satisfying $H_1 \leq H_2$ and $\lambda(H_1, K_1)$ exists, then $\lambda(H_2, K_2)$ exists and $\lambda(H_1, K_1) \leq \lambda(H_2, K_2)$. Moreover, if $(\lambda(H_1, K_1) - T_{H_2})^{-1}K_2$ has a strictly positive power, then $\lambda(H_1, K_1) < \lambda(H_2, K_2)$.*

Remark 3.6.2 In [63, Theorem 4], it is established that the leading eigenvalue of the transport operator with vacuum boundary conditions ($H = 0$) increases strictly with the size of the domain. Unfortunately, in our framework, the monotonicity with respect to the size of the domain (even in the wide sense) remains an open problem. This problem is also open in slab geometry (cf. [41]).

3.7 Appendix

In this appendix, we shall give some Lemmas required in the proofs of some results discussed above.

Lemma 3.7.1 *If H is a compact operator and the phase space $D \times V$ is such that there exists $\varepsilon > 0$ such ess $\inf_{(x,v) \in \Gamma_+} \tau(x, v) > \varepsilon$, then*

$$\lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} r_\sigma(M_\lambda H) = 0.$$

Remark 3.7.1 If the boundary operator H is compact, then there exists a sequence of finite rank operator which converges, in the operator norm, to H . Hence, it suffices to establish the result for a finite rank operator, that is, $H = \sum_{k=1}^n H_k, H_k := \langle \cdot, \psi \rangle \varphi$ where $n \in \mathbb{N}, \psi \in L_q^+, \varphi \in L_p^-$ and q denotes the exponent of p . Thus, we restrict ourselves to a boundary operator of rank one which we denote also by H , namely, $Hu := \langle u, \psi \rangle \varphi$ where $\psi \in L_q^+$ and $\varphi \in L_p^-$. \square

Before proving Lemma 3.7.1, let us state two preliminary results. Let λ be a complex number such that $\operatorname{Re}\lambda > -\lambda^*$. The dual of the operator $M_\lambda H$ is given by $(M_\lambda H)' = H' \widetilde{M}_\lambda$, where

$$\widetilde{M}_\lambda : L_q^+ \rightarrow L_q^-, u \mapsto \widetilde{M}_\lambda u = u(x + \tau(x, v)v, v) e^{-\int_0^{\tau(x, v)} (\lambda + \sigma(x - sv, v)) ds}$$

and the operator H' is given by

$$H' : L_q^- \rightarrow L_q^+, u \mapsto H' u = \langle \varphi, u \rangle \psi \quad (3.7.1)$$

where φ and ψ are the functions appearing in the expression of H .

Lemma 3.7.2 Assume that the hypotheses of Lemma 3.7.1 hold true. Then, $(H' \widetilde{M}_\lambda)$ converges to zero for the strong operators topology as $|\operatorname{Im}\lambda| \rightarrow +\infty$.

Proof. Let $u \in L_q^+$. It follows from (3.7.1) that

$$H' \widetilde{M}_\lambda u = \langle \varphi, \widetilde{M}_\lambda u \rangle \psi$$

with

$$\langle \varphi, \widetilde{M}_\lambda u \rangle \psi = \psi(x', v') \int_{\Gamma^-} \varphi(x, v) u(x + \tau(x, v)v, v) e^{-\int_0^{\tau(x, v)} (\lambda + \sigma(x - sv, v)) ds} v \cdot \nu_x d\gamma_x d\mu(v).$$

Let λ be a complex numbers such that $\operatorname{Re}\lambda$ belongs to $(-\lambda^*, \lambda_0]$. For every $u \in L_q^+$, we have

$$|\langle \varphi, \widetilde{M}_\lambda u \rangle \psi| = |\psi(x', v') \int_{\Gamma^-} \varphi(x, v) u(x + \tau(x, v)v, v) e^{-\int_0^{\tau(x, v)} (\operatorname{Re}\lambda + i\operatorname{Im}\lambda + \sigma(x - sv, v)) ds} v \cdot \nu_x d\gamma_x d\mu(v)|.$$

On the other hand,

$$|\langle \varphi, \widetilde{M}_\lambda u \rangle \psi| = |\psi(x', v') \int_{\Gamma^-} e^{-i\operatorname{Im}\lambda\tau(x, v)} \varphi(x, v) \widetilde{M}_{\operatorname{Re}\lambda} u(x, v) v \cdot \nu_x d\gamma_x d\mu(v)|.$$

Since $\tau(x, v) > 0$ and applying the Riemann-Lebesgue theorem, we get

$$\lim_{|\operatorname{Im}\lambda| \rightarrow +\infty} |\psi(x', v') \int_{\Gamma^-} e^{-i\operatorname{Im}\lambda\tau(x,v)} \varphi(x, v) \widetilde{M}_{\operatorname{Re}\lambda} u(x, v) |v \cdot v_x| d\gamma_x d\mu(v)| = 0$$

and therefore, $\lim_{|\operatorname{Im}\lambda| \rightarrow +\infty} |\langle \varphi, \widetilde{M}_\lambda u \rangle_\psi| = 0$. Furthermore, we have

$$|\langle \varphi, \widetilde{M}_\lambda u \rangle_\psi| \leq |\psi(x', v')| \int_{\Gamma^-} |\varphi(x, v) \widetilde{M}_{\operatorname{Re}\lambda} u(x, v)| |v \cdot v_x| d\gamma_x d\mu(v)$$

which is an element of L_q^+ . Using the dominated convergence theorem of Lebesgue, we obtain

$$\lim_{|\operatorname{Im}\lambda| \rightarrow +\infty} \|\langle \varphi, \widetilde{M}_\lambda u \rangle_\psi\|_{L_q^+} = 0.$$

□

Lemma 3.7.3 *With the same notations and assumptions as in Lemma 3.7.2, the family $\{H' \widetilde{M}_\lambda, -\lambda^* < \operatorname{Re}\lambda \leq \lambda_0\}$ is collectively compact.*

Proof. Let B be the unit ball of the space L_q^+ and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in

$\cup_\lambda (H' \widetilde{M}_\lambda B)$, $\lambda \in \{\lambda : -\lambda^* < \operatorname{Re}\lambda \leq \lambda_0\}$. Then there exists a sequence $(q_n)_{n \in \mathbb{N}}$ in B such that $x_n = H' \widetilde{M}_\lambda q_n$, $n = 1, 2, \dots$. It is clear that the sequence $(y_n = \widetilde{M}_\lambda q_n)_{n \in \mathbb{N}}$ is bounded in L_q^- . So, it follows from the compactness of H' that $(x_n = H' y_n)_{n \in \mathbb{N}}$ has converging subsequence in $\cup_\lambda (H' \widetilde{M}_\lambda B)$ which achieves the proof. □

Proof of Lemma 3.7.1. In view of Lemmas 3.7.2 and 3.7.3 and [35, Lemma 3.7, p.151], we have

$$\lim_{|\operatorname{Im}\lambda| \rightarrow +\infty} \|(H' \widetilde{M}_\lambda)^2\| = 0 \text{ uniformly on } \{-\lambda^* < \operatorname{Re}\lambda \leq \lambda_0\}.$$

And therefore

$$\lim_{|\operatorname{Im}\lambda| \rightarrow +\infty} r_\sigma(H' \widetilde{M}_\lambda) = 0 \text{ uniformly on } \{-\lambda^* < \operatorname{Re}\lambda \leq \lambda_0\}.$$

Next, the use of the equality $r_\sigma(H' \widetilde{M}_\lambda) = r_\sigma(M_\lambda H)$ gives the result. □

Regularity of the solution to the linear Boltzmann equation in finite bodies

4.1 Introduction

The purpose of this chapter is to discuss the time asymptotic behavior of the solution of the following Cauchy problem governed by a transport operator

$$\left\{ \begin{array}{l} \frac{\partial \psi}{\partial t}(x, v, t) = A_H \psi(x, v, t) := T_H \psi(x, v, t) + K \psi(x, v, t) \\ \qquad \qquad \qquad = -v \nabla_x \psi(x, v, t) - \sigma(v) \psi(x, v, t) + \int_{\mathbb{R}^n} \kappa(x, v, v') \psi(x, v', t) dv', \\ \psi(x, v, 0) = \psi_0(x, v), \end{array} \right. \quad (4.1.1)$$

where $(x, v) \in D \times \mathbb{R}^n$ and K is the partial integral part of A_H and called the collisional operator. Here D is a bounded open subset of \mathbb{R}^n . The function $\psi(x, v, t)$ represents the number (or probability) density of gas particles having the position x and the velocity v at time t . The functions $\sigma(\cdot)$ and $\kappa(\cdot, \cdot, \cdot)$ are called, respectively, the collision frequency and the scattering kernel.

As in the previous chapter (cf. (3.1.1)), the boundary conditions are modeled by the following abstract operator

$$\psi_- = H(\psi_+). \quad (4.1.2)$$

Let A be the infinitesimal generator of the C_0 -semigroup $(U(t))_{t \geq 0}$ on a Banach space X and let $\mathcal{L}(X)$ denote the set of all bounded linear operators on X . If $B \in \mathcal{L}(X)$, then by the classical

perturbation theorem (cf. Section 2.3.3), $A+B$ generates a strongly continuous semigroup $(V(t))_{t \geq 0}$ given by the Dyson-Phillips expansion. Hence the Cauchy problem governed by the operator $A+B$ has a unique classical solution for any initial data belonging to $D(A)$. This procedure is not constructive, so in order to get more information on the solution, in particular, its behavior for large times, the knowledge of the spectrum of $A+B$ or $(V(t))_{t \geq 0}$ plays a central role.

According to Theorem 2.3.2 if some remainder term of the Dyson-Phillips expansion $R_n(t)$ (for the expression of $R_n(t)$ see (2.3.6)) is compact, then the operators $U(t)$ and $V(t)$ have the same essential type and therefore, for all $t \geq 0$ we have $r_{ess}(U(t)) = r_{ess}(V(t))$. Thus, for $\nu > 0$, $\sigma(V(t)) \cap \mathbb{C} \setminus B(0, r_{ess}(U(t)) + \nu)$ consists of at most isolated eigenvalues with finite algebraic multiplicities. Assuming the existence of such eigenvalues, the semigroup $(V(t))_{t \geq 0}$ can be decomposed into two parts: the first containing the time development of finitely many eigenmodes, the second being of faster decay. Using the spectral mapping theorem for the point spectrum, we infer that, for any $\eta > \omega$, $\sigma(A+B) \cap \{\text{Re} \lambda \geq \eta\}$ consists of finitely many isolated eigenvalues say $\{\lambda_1, \dots, \lambda_q\}$. Let $\beta_1 = \sup\{\text{Re} \lambda, \lambda \in \sigma(A+B), \text{Re} \lambda < \omega\}$, and $\beta_2 = \min\{\text{Re} \lambda_j, 1 \leq j \leq q\}$. The solution of the Cauchy problem satisfies

$$\left\| \psi(t) - \sum_{j=1}^q e^{\lambda_j t} e^{D_j t} P_j \psi_0 \right\| = o(e^{\beta^* t}) \text{ with } \beta_1 < \beta^* < \beta_2, \quad (4.1.3)$$

where $\psi_0 \in D(A)$, P_j and D_j denote, respectively, the spectral projection and the nilpotent operator associated with λ_j , $j = 1, 2, \dots, q$.

In the context of neutron transport theory, these ideas were initiated by I. Vidav [83] and developed afterwards by J. Voigt [84], L. Weis [91], M. Mokhtar-Kharroubi [64, 65, 66] and others (see the reference therein). The compactness of some order remainder term of the Dyson Phillips expansion in L^p -spaces, $1 \leq p < +\infty$, was established only for no-reentry boundary conditions (i.e. with zero incoming flux) [64, 65, 79, 84, 91]. It is clear that the success of this method is related to the possibility of computing some remainder order term of the Dyson-Phillips expansion and to the possibility of discussing their compactness properties. Unfortunately, when dealing with reentry boundary conditions, except the one-dimensional case with reflective or periodic boundary conditions [19, 23], the semigroup generated by the advection operator is not explicit (see, for example, [40, 42, 44, 46]) and therefore it is difficult to compute $R_n(t)$ because its expression involves the boundary operator. So, except in some simple cases, this approach does not work.

An alternative way to discuss the time structure of $\psi(t)$ is the so called resolvent approach. It is based on the following assumption:

$$\begin{cases} \exists m \in \mathbb{N} \text{ such that } [(\lambda - A)^{-1} B]^m \text{ is compact for all } \lambda \text{ such that } \text{Re} \lambda > \omega, \\ \lim_{|\text{Im} \lambda| \rightarrow +\infty} \| [(\lambda - A)^{-1} B]^m \| = 0 \text{ uniformly on } \{\text{Re} \lambda \geq \eta, \eta > \omega\}. \end{cases} \quad (4.1.4)$$

In [64, Theorem 1.1], it is proved that under the condition (4.1.4), the part of the spectrum of $A := S + B$ lying in the half plane $\{\text{Re} \lambda \geq \eta\}$ consists of, at most, a finite number of isolated

eigenvalues with finite algebraic multiplicity, say, $\{\lambda_1, \dots, \lambda_n\}$. Further, the solution of the Cauchy problem fulfills the equation (4.1.3) provided that $\psi_0 \in D(A^2)$.

Let us recall that the streaming operator T_H (where $\|H\| < 1$ or $\|H\| > 1$) generates a strongly continuous semigroup $(U_H(t))_{t \geq 0}$ on $L^p(\Omega \times \mathbb{R}^n, dx dv)$, $1 \leq p < +\infty$. Since the collision operator K is bounded, $A_H = T_H + K$ generates also a strongly continuous semigroup $(V_H(t))_{t \geq 0}$ on $L^p(\Omega \times \mathbb{R}^n, dx dv)$, $1 \leq p < +\infty$ given by

$$V_H(t) = \sum_{j=0}^{n-1} U_j^H(t) + R_n^H(t).$$

The resolvent approach was already applied to transport equations with vacuum boundary conditions ($H = 0$) in bounded geometry [64]. In [40], it is shown that this methods works also for one-dimensional transport equation for a large class of boundary conditions. The drawback of this method lies in the fact that, unlike (4.1.3), the quantity $\psi(t) - \sum_{i=1}^n e^{\lambda_i t} e^{D_i t} P_i \psi_0$ can be explicitied only for initial data ψ_0 belonging to $D(A^2)$. Note however that, since L^p -spaces, for $p \in (1, +\infty)$, are uniformly convex, using some arguments from functional analysis, the condition $\psi_0 \in D(A_H^2)$ was relaxed and, as in the semi-group approach, the estimate (4.1.3) is valid for all initial data ψ_0 belonging to $D(A_H)$ (we refer to [77] for no-reentry boundary conditions and to [44] for one dimensional transport equation with particular reentry boundary conditions). Unfortunately, except some particular boundary conditions in slab geometry [46], the results available in L^1 -spaces for reentry boundary conditions assume that ψ_0 belongs to $D(A_H^2)$.

The purpose of this chapter is twofold. First, we shall pursue the investigation started in the work [45] concerning the time asymptotic behavior of the solution to the Cauchy Problem (4.1.1)-(4.1.2) in L^1 -spaces for a slightly more general boundary operator than the bounce back one. Our main goal is to show that the estimate (4.1.3) is also valid for all initial data ψ_0 lying in $D(A_H)$. Our approach uses the following result established in [47]

Proposition 4.1.1 [47, Corollary 1.1] *Let T be the generator of a C_0 -semigroup $(U(t))_{t \geq 0}$ with type ω on a Banach space X and let B be a bounded linear operator on X . Assume that there exists $m \in \mathbb{N}$ and $\eta > \omega$ such that*

- (a) $(\lambda - T)^{-1}[B(\lambda - T)^{-1}]^m$ is compact for all λ such that $\text{Re} \lambda \geq \eta$,
- (b) $\lim_{|\text{Im} \lambda| \rightarrow \infty} |\text{Im} \lambda| \|(\lambda - T)^{-1}[B(\lambda - T)^{-1}]^m\| = 0$ for every λ such that $\text{Re} \lambda \geq \eta$.

Then $R_{2m+1}(t)$ is compact on X for each $t > 0$.

We shall establish that all order remainder terms $R_n^H(t)$, with $n \geq 9$, of the Dyson-Phillips expansion of the transport semigroup are compact on $L^1(\mathcal{D} \times \mathbb{R}^n, dx dv)$ and therefore, according the semigroup approach, we infer that the estimate (4.1.3) holds true for all initial data in $D(A_H)$. This shows that the condition $\psi_0 \in D(A_H^2)$ can be relaxed and replaced by $\psi_0 \in D(A_H)$.

Secondly, for $p \in (1, +\infty)$, using some arguments due to Brendle [14] (which were refined afterwards by Sbihi [72]) we shall prove that the first order remainder term of the Dyson-Phillips expansion, $R_1^H(t)$ is compact. In our opinion, regardless of its consequences, this result is interesting in itself. The latter implies that, for all $t \geq 0$, we have $\sigma_{ess}(V_H(t)) = \sigma_{ess}(U_H(t))$ and therefore $r_{ess}(V_H(t)) = r_{ess}(U_H(t))$. Accordingly, the estimate (4.1.3) holds true for all $\psi_0 \in D(A_H)$.

4.2 Preparatory results

The notations in this chapter are essentially the same that those introduced in Section 3.2. We have only to replace the velocity space V by \mathbb{R}^n , the Radon measure $\mu(\cdot)$ by the Lebesgue measure dv and $\sigma(x, v) := \sigma(v)$. So, the boundary operators M_λ , B_λ , G_λ and C_λ are now defined by

$$\begin{aligned} L_p^- \ni u &\mapsto M_\lambda u = u(x - \tau(x, v)v, v)e^{-\tau(x, v)(\lambda + \sigma(v))} \in L_p^+, \\ L_p^- \ni u &\mapsto B_\lambda u := u(x - t^-(x, v)v, v)e^{-t^-(x, v)(\lambda + \sigma(v))} \in X_p, \\ X_p \ni \varphi &\mapsto G_\lambda \varphi := C_\lambda \varphi|_{\Gamma^+} = \int_0^{\tau(x, v)} e^{-s(\lambda + \sigma(v))} \varphi(x - sv, v) ds \in L_p^+, \\ X_p \ni \varphi &\mapsto C_\lambda \varphi := (\lambda - T_0)^{-1} \varphi = \int_0^{t^-(x, v)} e^{-s(\lambda + \sigma(v))} \varphi(x - sv, v) ds \in X_p \end{aligned}$$

where $X_p := L^p(D \times \mathbb{R}^n, dx dv)$, $L_p^\pm := L^p(\Gamma_\pm : |v \cdot \nu_x| d\gamma_x dv)$ with $1 \leq p < +\infty$.

Our aim in this section is to establish some technical lemmas required in the sequel. We define the full transport operator A_H by $A_H = T_H + K$. Since the collision operator K is bounded, A_H is a bounded perturbation of T_H .

We suppose that the boundary operator H has the form

$$\begin{cases} H : L_p^+ \longrightarrow L_p^- \\ \psi^+ \longmapsto H\psi^+ = \alpha I_1 \psi^+ + \beta I_2 \psi^+, \quad \alpha, \beta \in [0, +\infty), \end{cases} \quad (4.2.1)$$

where I_1 is a compact operator and I_2 is defined by

$$I_2 u(x, v) = \alpha u(x, -v)$$

(I_2 is the so called bounce-back boundary operator). The constants α and β are chosen so that

$$\|H\| < 1. \quad (4.2.2)$$

For the sake of simplicity and in order to avoid some technical difficulties, we shall assume in this section that the collision frequency is an even function on \mathbb{R}^n , that is,

$$\sigma(-v) = \sigma(v), \quad \forall v \in \mathbb{R}^n.$$

Since $\|H\| < 1$, Equation (3.2.4) shows that, for all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda + \lambda^* > 0$, the resolvent of the operator T_H is given by

$$R(\lambda, T_H) = \sum_{n \geq 0} B_\lambda H (M_\lambda H)^n G_\lambda + C_\lambda.$$

This shows that

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -\lambda^*\} \subseteq \rho(T_H)$$

and, for any $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > -\lambda^*$, we have

$$\|KR(\lambda, T_H)K\| \leq \sum_{n \geq 0} \|KB_\lambda H (M_\lambda H)^n G_\lambda K\| + \|KC_\lambda K\|.$$

Let $\omega > 0$ and denote by Γ_ω the set

$$\Gamma_\omega = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq -\lambda^* + \omega\}.$$

Lemma 4.2.1 *If K is a regular collision operator, then, for all $r \in [0, 1]$, we have*

$$\lim_{|\operatorname{Im}\lambda| \rightarrow \infty} |\operatorname{Im}\lambda|^r \|KC_\lambda K\| = 0 \quad \text{uniformly on } \Gamma_\omega.$$

Proof. According to Remark 3.4.1, we may take K in the form (3.4.1). So, $KC_\lambda K$ writes in the form

$$(KC_\lambda K\varphi)(x, v) = \int_{\mathbb{R}^n} f(v)h(v')dv' \int_0^{t^-(x, v')} e^{-(\lambda + \sigma(v))t} dt \int_{\mathbb{R}^n} g(v'')\varphi(x - tv'', v'')dv'',$$

where $h(v') := g(v')f(v') \in L^1(\mathbb{R}^n)$. This yields the factorization $KC_\lambda K := A_3 A_2(\lambda) A_1$ with

$$A_1 : X_p \rightarrow L^p(\mathbb{D}), \quad \psi \mapsto \int_{\mathbb{R}^n} g(v)\psi(x, v)dv,$$

$$A_2(\lambda) : L^p(\mathbb{D}) \rightarrow L^p(\mathbb{D}), \quad \varphi \mapsto \int_{\mathbb{R}^n} h(v) \int_0^{t^-(x, v)} e^{-(\lambda + \sigma(v))t} \varphi(x - tv) dt dv$$

and

$$A_3 : L^p(\mathbb{D}) \rightarrow X_p, \quad \varphi \mapsto \varphi(x)f(v).$$

Since A_1 and A_3 are bounded operators and independent of the parameter λ , it suffices to establish the result for $A_2(\lambda)$. To this end, let $\varepsilon > 0$ and define the operator

$$\begin{cases} A_2^\varepsilon : L^p(\mathbb{D}) \longrightarrow L^p(\mathbb{D}) \\ \varphi \mapsto \int_{\mathbb{R}^n} h(v) \int_\varepsilon^{t^-(x, v)} e^{-(\lambda + \sigma(v))t} \varphi(x - tv) dt dv. \end{cases}$$

Since $\lim_{\varepsilon \rightarrow 0} \|A_2^\varepsilon(\lambda) - A_2(\lambda)\|_{\mathcal{L}(L^p(\mathbf{D}))} = 0$ uniformly on Γ_ω , so it suffices to establish the lemma for the operator $A_2^\varepsilon(\lambda)$. Now, using the convexity of \mathbf{D} , one sees that, for all $(x, v) \in \bar{\mathbf{D}} \times \mathbb{R}^n$,

$$t \in (0, t^-(x, v)) \Leftrightarrow y = x - tv \in \mathbf{D}.$$

Hence, using the change of variable $y = x - tv$, we get

$$(A_2^\varepsilon(\lambda)\varphi)(x) = \int_{\mathbf{D}} \varphi(y) dy \int_\varepsilon^\infty h\left(\frac{x-y}{t}\right) e^{-(\lambda + \sigma(\frac{x-y}{t}))t} \frac{dt}{t^n}.$$

Note that if $\lambda \in \Gamma_\omega$, then

$$\operatorname{Re}\lambda + \lambda^* - \frac{\omega}{2} \geq \frac{\omega}{2} > 0 \quad (4.2.3)$$

and then, for all $v \in \mathbb{R}^n$, we have

$$\sigma(v) - \lambda^* + \frac{\omega}{2} \geq \frac{\omega}{2} > 0. \quad (4.2.4)$$

Without loss of generality we may suppose that $h(\cdot)$ is a simple measurable function. For $x \in \mathbf{D}$, consider the function $\varphi_x(\cdot)$ defined by

$$(\varepsilon, +\infty) \ni t \mapsto \varphi_x(t) = \frac{1}{t^n} e^{-(\sigma(\frac{x}{t}) - \lambda^* + \frac{\omega}{2})t}.$$

It is clear that (4.2.4) implies $0 \leq \varphi_x(\cdot) \in L^1(\varepsilon, +\infty)$. We denote by $(l_{x,i}(\cdot))_{i \in \mathbb{N}}$ a sequence of non negative step functions which converges to $\varphi_x(\cdot)$ almost everywhere and satisfying

$$0 \leq l_{x,i}(\cdot) \leq \varphi_x(\cdot) \leq \frac{1}{\varepsilon} \quad \text{for all } i \in \mathbb{N}. \quad (4.2.5)$$

Let $A_{2,i}^\varepsilon(\lambda)$ be the sequence of operators defined, for all $i \in \mathbb{N}$, by

$$\begin{cases} A_{2,i}^\varepsilon(\lambda) : L^p(\mathbf{D}) \longrightarrow L^p(\mathbf{D}) \\ \varphi \mapsto \int_{\mathbf{D}} \varphi(y) dy \int_\varepsilon^\infty h\left(\frac{x-y}{t}\right) e^{-(\lambda + \lambda^* - \frac{\omega}{2})t} l_{x-y,i}(t) dt \end{cases}$$

We first claim that, for any $\varepsilon > 0$, $(A_{2,i}^\varepsilon(\lambda))_{i \in \mathbb{N}}$ converges uniformly on Γ_ω to $A_2^\varepsilon(\lambda)$ in $\mathcal{L}(L^p(\mathbf{D}))$. Indeed, for $\varphi \in L^p(\mathbf{D})$, we have

$$\|(A_{2,i}^\varepsilon(\lambda) - A_2^\varepsilon(\lambda))\varphi\|_{L^p(\mathbf{D})}^p = \int_{\mathbf{D}} dx \left| \int_{\mathbf{D}} \varphi(y) dy \int_\varepsilon^\infty h\left(\frac{x-y}{t}\right) e^{-(\lambda + \lambda^* - \frac{\omega}{2})t} \{l_{x-y,i}(t) - \varphi_{x-y}(t)\} dt \right|^p.$$

Applying the Hölder inequality, we get

$$\|(A_{2,i}^\varepsilon(\lambda) - A_2^\varepsilon(\lambda))\varphi\|_{L^p(\mathbf{D})}^p \leq |\mathbf{D}|^{\frac{p}{q}} \int_{\mathbf{D}} dx \int_{\mathbf{D}} dy \left| \int_\varepsilon^\infty \varphi(y) h\left(\frac{x-y}{t}\right) e^{-(\lambda + \lambda^* - \frac{\omega}{2})t} \{l_{x-y,i}(t) - \varphi_{x-y}(t)\} dt \right|^p.$$

It follows from Fubini's theorem and the change of variable $x' = x - y$, that

$$\|(A_{2,i}^\varepsilon(\lambda) - A_2^\varepsilon(\lambda))\varphi\|_{L^p(D)}^p \leq \sup |h(\cdot)|^p |\mathbb{D}|^{\frac{p}{q}} \int_{\mathbb{D}} dx' \left(\int_{\varepsilon}^{\infty} e^{-(\lambda+\lambda^*-\frac{\omega}{2})t} \{l_{x',i}(t) - \varphi_{x'}(t)\} dt \right)^p \|\varphi\|_{L^p(D)}^p.$$

Hence

$$\|A_{2,i}^\varepsilon(\lambda) - A_2^\varepsilon(\lambda)\|^p \leq \sup |h(\cdot)|^p |\mathbb{D}|^{\frac{p}{q}} \int_{\mathbb{D}} dx' \left(\int_{\varepsilon}^{\infty} e^{-(\operatorname{Re}\lambda+\lambda^*-\frac{\omega}{2})t} \{l_{x',i}(t) - \varphi_{x'}(t)\} dt \right)^p.$$

Now, using Equations (4.2.3)-(4.2.5), we get

$$e^{-(\operatorname{Re}\lambda+\lambda^*-\frac{\omega}{2})t} \{l_{x',i}(t) - \varphi_{x'}(t)\} \leq \frac{1}{\varepsilon} e^{-\frac{\omega}{2}t}.$$

By the continuity of the norm and the Lebesgue dominated convergence theorem, we conclude $\lim_{i \rightarrow \infty} \|A_{2,i}^\varepsilon(\lambda) - A_2^\varepsilon(\lambda)\| = 0$ uniformly on Γ_ω . This proves our claim.

Now we have only to prove the lemma for $A_{2,i}^\varepsilon(\lambda)$. Note that easy calculations show that

$$\|A_{2,i}^\varepsilon\|^p \leq |\mathbb{D}|^{\frac{p}{q}} \int_{\mathbb{D}} dx \left| \int_{\varepsilon}^{\infty} e^{-(\lambda+\lambda^*-\frac{\omega}{2})t} h\left(\frac{x}{t}\right) l_{x,i}(t) dt \right|^p.$$

Let x be a fixed real in \mathbb{D} . Clearly, the map $G_x(\cdot) : (\varepsilon, \infty) \rightarrow \mathbb{R}^n$, $t \mapsto e^{-\lambda^4 t} h\left(\frac{x}{t}\right) l_{x,i}(t)$ is a simple function. Let $(t_j)_{1 \leq j \leq m}$ denote a subdivision of its support such that

$$\forall j \in \{1, \dots, m-1\}, \quad G_x(t) = G_x(t_j) \quad \forall t \in [t_j, t_{j+1}[.$$

Hence

$$\begin{aligned} \int_{\varepsilon}^{\infty} e^{-(\lambda^4+\lambda+\lambda^*-\frac{\omega}{2})t} G_x(t) dt &= \sum_{j=1}^{m-1} G_x(t_j) \int_{t_j}^{t_{j+1}} e^{-(\lambda^4+\lambda+\lambda^*-\frac{\omega}{2})t} dt \\ &= \frac{1}{-\lambda^4 + \lambda + \lambda^* - \frac{\omega}{2}} \sum_{j=1}^{m-1} G_x(t_j) \left(e^{-(\lambda^4+\lambda+\lambda^*-\frac{\omega}{2})t_j} - e^{-(\lambda^4+\lambda+\lambda^*-\frac{\omega}{2})t_{j+1}} \right) \end{aligned}$$

and consequently

$$\left| \int_{\varepsilon}^{\infty} e^{-(\lambda^4+\lambda+\lambda^*-\frac{\omega}{2})t} G_x(t) dt \right|^p \leq \frac{e^{-p\operatorname{Im}\lambda^4(1+\frac{\operatorname{Re}\lambda}{\operatorname{Im}\lambda}-\frac{6\operatorname{Re}\lambda^2}{\operatorname{Im}\lambda^2})} \sup |h(\cdot)|^p (2(m-1))^p}{|\operatorname{Im}\lambda|^{4p} \varepsilon^{np}}.$$

Therefore

$$|\operatorname{Im}\lambda|^r \|A_{2,i}^\varepsilon\| \leq \frac{2(m-1)|\mathcal{D}|^p e^{-\operatorname{Im}\lambda^4(1+\frac{\operatorname{Re}\lambda}{\operatorname{Im}\lambda}-\frac{6\operatorname{Re}\lambda^2}{\operatorname{Im}\lambda^2})} \sup|h(\cdot)|}{|\operatorname{Im}\lambda|^{4-r} \varepsilon^n}.$$

This ends the proof because, for any $r \in [0, 1]$, we have

$$\lim_{|\operatorname{Im}\lambda| \rightarrow \infty} \frac{2(m-1)|\mathcal{D}|^p e^{-\operatorname{Im}\lambda^4(1+\frac{\operatorname{Re}\lambda}{\operatorname{Im}\lambda}-\frac{6\operatorname{Re}\lambda^2}{\operatorname{Im}\lambda^2})} \sup|h(\cdot)|}{|\operatorname{Im}\lambda|^{4-r} \varepsilon^n} = 0.$$

□

Our next task is to estimate the quantity $|\operatorname{Im}\lambda|^r \|KB_\lambda H(I - M_\lambda H)^{-1} G_\lambda K\|$ as $|\operatorname{Im}\lambda|$ goes to $+\infty$ for all $r \in [0, 1]$. Set $H(\lambda) := M_\lambda H$. Since $\|H\| < 1$, the use of Eq. (3.2.4) imply that

$$\|KB_\lambda H(I - H(\lambda))^{-1} G_\lambda K\| \leq \sum_{n \geq 0} \|KB_\lambda H(H(\lambda))^n G_\lambda K\|.$$

Further, according to (4.2.1), one sees that the operator $H(\lambda)$ may be expressed as

$$H(\lambda) = H^1(\lambda) + H^2(\lambda),$$

where $H^1(\lambda) := \alpha M_\lambda I_1$ and $H^2(\lambda) := \beta M_\lambda I_2$. Since the operators $H^i(\lambda)$, $i = 1, 2$, do not commute, the operator $(H(\lambda))^n = (H^1(\lambda) + H^2(\lambda))^n$ is a sum of 2^n different terms, that is, $(H(\lambda))^n = \sum_{j=1}^{2^n} P_j$

where each P_j is a product of n factors formed from the operators $H^i(\lambda)$, $i = 1, 2$. These 2^n terms P_j may be divided into two classes C_1 and C_2 . The class C_i consists of those P_j , $1 \leq j \leq (2)^n$, for which the last factor (on the right hand side) is $H^i(\lambda)$, $i = 1, 2$. It follows that, for each integer n , we have

$$\|KB_\lambda H(H(\lambda))^n G_\lambda K\| \leq \sum_{j=1}^{2^n} \|KB_\lambda H P_j G_\lambda K\|,$$

where j is an integer belonging to the set $\{1, 2, \dots, 2^n\}$.

Lemma 4.2.2 *Let K be a regular collision operator. If the boundary operator H is in the form (4.2.1) and satisfies (4.2.2), then*

$$\lim_{|\operatorname{Im}\lambda| \rightarrow \infty} |\operatorname{Im}\lambda|^r \|KB_\lambda H(I - H(\lambda))^{-1} G_\lambda K\| = 0, \quad \text{uniformly on } \Gamma_\omega,$$

for all $r \in [0, 1]$.

Before proving this lemma, let us establish two preliminary results.

Lemma 4.2.3 *If the hypotheses of the Lemma 4.2.2 hold true, then, for all $r \in [0, 1]$, we have*

$$\lim_{|\operatorname{Im}\lambda| \rightarrow \infty} |\operatorname{Im}\lambda|^r \|I_1 G_\lambda K\| = 0 \quad \text{uniformly on } \Gamma_\omega.$$

Proof. Since I_1 is a compact operator, as in Remark 3.4.1, we can establish the result for an operator of rank one, that is

$$I_1 : L_p^+ \rightarrow L_p^-, \varphi \mapsto I_1 \varphi(x, v) = \theta_1(x, v) \int_{\Gamma_+} \theta_2(x', v') \varphi(x', v') |v' \cdot \nu_{x'}| d\gamma_{x'} dv', \quad \forall (x, v) \in \Gamma_-,$$

where $\theta_1 \in L_p^-$ and $\theta_2 \in L_q^+$. So, the operator $I_1 G_\lambda K$ writes in the form

$$(I_1 G_\lambda K \varphi)(x, v) = \theta_1(x, v) \int_{\Gamma_+} \theta_2(x', v') |v' \cdot \nu_{x'}| d\gamma_{x'} dv' \int_0^{\tau(x', v')} e^{-(\lambda + \sigma(v))t} f(v') dt \int_{\mathbb{R}^n} g(v'') \varphi(x' - tv'', v'') dv''.$$

Hence, it can be factorized as

$$I_1 G_\lambda K = A_3 A_2(\lambda) A_1,$$

where

$$A_1 : X_p \ni \psi \mapsto \int_{\mathbb{R}^n} g(v) \psi(x, v) dv \in L_p(\mathbb{D}),$$

$$A_2(\lambda) : L_p(\mathbb{D}) \ni \varphi \mapsto \int_{\Gamma_+} \theta_2(x, v) f(v) |v \cdot \nu_x| d\gamma_x dv \int_0^{\tau(x, v)} e^{-(\sigma(v) + \lambda)t} \varphi(x - tv) dt \in \mathbb{R}$$

and

$$A_3 : \mathbb{R} \ni \gamma \mapsto \gamma \theta_1(x, v) \in L_p^-.$$

Since A_1 and A_3 are bounded linear operators independent of the parameter λ , it suffices to establish the lemma for $A_2(\lambda)$. Let $\varepsilon > 0$ and define the operator $A_2^\varepsilon(\lambda)$ by

$$\begin{cases} A_2^\varepsilon(\lambda) : L^p(\mathbb{D}) \longrightarrow \mathbb{R} \\ \varphi \mapsto \int_{\Gamma_+} \theta_2(x, v) f(v) |v \cdot \nu_x| d\gamma_x dv \int_\varepsilon^{\tau(x, v)} e^{-(\sigma(v) + \lambda)t} \varphi(x - tv) dt. \end{cases}$$

Since $A_2^\varepsilon(\lambda) \rightarrow A_2(\lambda)$, in the operator norm, as $\varepsilon \rightarrow 0$, it suffices to show the lemma for the operator $A_2^\varepsilon(\lambda)$.

Let $(x, v) \in \Gamma_+$. We know from Section 3.2 that $\tau(x, v) = t^-(x, v)$, so using the convexity of \mathbb{D} , we get the following equivalence

$$0 < t < t^-(x, v) \iff y = x - tv \in \mathbb{D}$$

and therefore the change of variable $y = x - tv$ leads to

$$A_2^\varepsilon(\lambda)\varphi = \int_{\mathbb{D}} \varphi(y)F_{\lambda,\varepsilon}(y)dy,$$

where

$$F_{\lambda,\varepsilon}(y) = \int_\varepsilon^\infty \frac{dt}{t^n} \int_{\partial\mathbb{D}^+} \theta_2(x, \frac{x-y}{t}) f(\frac{x-y}{t}) |\frac{x-y}{t} \cdot \nu_x| e^{-(\lambda+\sigma(\frac{x-y}{t}))t} d\gamma_x.$$

Now arguing as in the last part of the proof of Lemma 4.2.1 we reach the desired result. □

Lemma 4.2.4 *If the hypotheses of the Lemma 4.2.2 hold true, then, for all $r \in [0, 1]$, we have*

$$\lim_{|\operatorname{Im}\lambda| \rightarrow \infty} |\operatorname{Im}\lambda|^r \|I_2 G_\lambda K\| = 0 \quad \text{uniformly on } \Gamma_\omega.$$

Proof. The proof of this result is similar to that of Lemma 4.2.3. □

Proof of Lemma 4.2.2. According to the definition of C_i , $i = 1, 2$, implies that $P_j = P'_j H^i(\lambda)$. So, one can write

$$\begin{aligned} \|KB_\lambda H(I - H(\lambda))^{-1} G_\lambda K\| &\leq \sum_{n \geq 0} \sum_{j=1}^{2^n} \|KB_\lambda H P'_j G_\lambda K\| \\ &\leq \sum_{n \geq 0} \sum_{j=1}^{2^n} \|KB_\lambda H P'_j\| \|H^i(\lambda) G_\lambda K\|. \end{aligned}$$

The operator $H^i(\lambda) G_\lambda K$ can be expressed, according to the values of i , as

$$H^i(\lambda) G_\lambda K = \begin{cases} M_\lambda I_1 G_\lambda K & \text{if } i = 1, \\ M_\lambda I_2 G_\lambda K & \text{if } i = 2. \end{cases}$$

By virtue of the uniform boundedness of the operators $KB_\lambda H P'_j$ and M_λ on Γ_ω , it is sufficient to establish the result for the operators $I_1 G_\lambda K$ and $I_2 G_\lambda K$. This follows from Lemmas 4.2.3 and 4.2.4. □

4.3 Main results

4.3.1 Dissipative boundary conditions

As in the previous section, the collision frequency $\sigma(\cdot)$ is also assumed to be an even on \mathbb{R}^N in this subsection, that is,

$$\sigma(v) = \sigma(-v), \text{ for all } v \in \mathbb{R}^n.$$

Remark 4.3.1 Note however that, the assumptions (A3) and (A4) are fulfilled by the usual Lebesgue measure. So, the result of Theorem 3.4.1 is also true for Lebesgue measure on the velocity space.

Now we are ready to prove the following result.

Theorem 4.3.1 *Let K be a regular collision operator and let H be a boundary operator satisfying (4.2.1) and (4.2.2). If the hypotheses of Theorem 3.4.1 hold true, then $[(\lambda - T_H)^{-1}K]^n$ is compact on X_p ($1 \leq p < \infty$) for $n \geq 4$. Further, for all $r \in [0, 1]$, we have*

$$\lim_{|\operatorname{Im}\lambda| \rightarrow \infty} |\operatorname{Im}\lambda|^r \|K(\lambda - T_H)^{-1}K\| = 0 \text{ uniformly on } \Gamma_\omega.$$

Proof. If $p \in (1, +\infty)$, then, according to Theorem 3.4.1(1), the operator $(\lambda - T_H)^{-1}K$ is compact on X_p . If $p = 1$, then, by Theorem 3.4.1(2), we infer that $K(\lambda - T_H)^{-1}K$ weakly compact on X_1 and therefore $(\lambda - T_H)^{-1}K(\lambda - T_H)^{-1}K$ is weakly compact on X_1 . Next, this together with Theorem 2.5.2, implies that $[(\lambda - T_H)^{-1}K(\lambda - T_H)^{-1}K]^2 = [(\lambda - T_H)^{-1}K]^4$ is compact on X_1 which proves the first assertion. The second statement follows immediately from Lemmas 4.2.1 and 4.2.2. \square

The remainder of the subsection is devoted to give a spectral decomposition of the solution to Problem (4.1.1). Before going further, we first recall that the set $\sigma_{as}(A_H)$ (the asymptotic spectrum of the operator A_H) is defined in Section 3.4 by

$$\sigma_{as}(A_H) := \sigma(A_H) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -\lambda^*\}.$$

Lemma 4.3.1 *If the hypotheses of Theorem 4.3.1 hold true, then*

- (1) $\sigma_{as}(A_H)$ consists of at most isolated eigenvalues with finite algebraic multiplicities;
- (2) if $\varepsilon > 0$ then $\sigma(A_H) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -\lambda^* + \varepsilon\}$ is finite;
- (3) if $\varepsilon > 0$ then $\|(\lambda - A_H)^{-1}\|$ is uniformly bounded in $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -\lambda^* + \varepsilon\}$ for large $|\operatorname{Im}\lambda|$.

Proof. Let λ be such that $\operatorname{Re}\lambda > -\lambda^*$. Since K is a regular collision operator, according to Theorem 4.3.1 with $n = 4$, the operator $[(\lambda - T_H)^{-1}K]^4$ is compact on X_p , $1 \leq p \leq \infty$. Next, applying again Theorem 4.3.1 with $r = 0$, we get

$$\lim_{|\operatorname{Im}\lambda| \rightarrow \infty} \|[(\lambda - T_H)^{-1}K]^4\| = 0 \quad \text{uniformly in } \Gamma_\omega.$$

Now, the result follows from [64, Lemma 1.1]. □

Assume that K is a regular collision operator on X_p and H satisfies the hypotheses (4.2.1) and (4.2.2). Then, by Lemma 4.3.1, the spectrum of A_H in the half plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -\lambda^* + \varepsilon\}$ consists of, a finite number of isolated eigenvalues with finite algebraic multiplicity which we denote $\lambda_1, \lambda_2, \dots, \lambda_n$. Let P_i and D_i denote, respectively, the spectral projection and the nilpotent operator associated with $\lambda_i, i = 1, 2, \dots, n$. Then $P := P_1 + \dots + P_n$ is the spectral projection of the compact set $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$. According to the spectral decomposition theorem corresponding to the compact set $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ (see, for example, [35]), we may write

$$V_H(t) = \tilde{V}_H(t) + \sum_{i=1}^n e^{\lambda_i t} e^{D_i t} P_i$$

where $\tilde{V}_H(t) := V_H(t)(I - P)$ is the C_0 -semigroup on the Banach space $(I - P)X_p$ with generator $\tilde{A}_H := A_H(I - P)$ (\tilde{A}_H is the part of A_H on the closed subspace $(I - P)X_p$).

Now, we are in a position to give the main result in this chapter.

Theorem 4.3.2 *Let $p = 1$ and K be a regular collision operator. Assume that the boundary operator satisfies (4.2.1) and (4.2.2). Then, there exists $\varepsilon > 0$ small enough and $M > 0$ such that*

$$\left\| V_H(t)\psi_0 - \sum_{i=1}^n e^{(\lambda_i + D_i)t} P_i \psi_0 \right\|_{X_1} \leq M e^{(-\lambda^* + \varepsilon)t}, \quad \forall t > 0,$$

where $\psi_0 \in D(A_H)$.

Proof. Let λ be a complex number satisfying $\operatorname{Re}\lambda > -\lambda^*$. We know from Theorem 4.3.1 that if K is a regular collision operator, then $(\lambda - T_H)^{-1}[K(\lambda - T_H)^{-1}]^4$ is compact on X_1 . Next, a simple calculation shows that

$$\|(\lambda - T_H)^{-1}[K(\lambda - T_H)^{-1}]^4\| \leq \|(\lambda - T_H)^{-1}\|^3 \|K(\lambda - T_H)^{-1}K\|^2.$$

So, applying Theorem 4.3.1 with $r = 1$, we get

$$\lim_{|\operatorname{Im}\lambda| \rightarrow \infty} |\operatorname{Im}\lambda| \|(\lambda - T_H)^{-1}[K(\lambda - T_H)^{-1}]^4\| = 0.$$

Now, according to Proposition 4.1.1, we get the compactness of $R_n^H(t)$ for $n \geq 9$. Then by Theorem 2.3.2, the C_0 -Semigroups $U_H(t)$ and $V_H(t)$ have the same essential type, that is, for all $t \geq 0$ we have

$$r_e(U_H(t)) = r_e(V_H(t)) \leq e^{-\lambda^* t}.$$

This implies that, outside the spectral disc $|\mu| \leq e^{(\varepsilon - \lambda^*)t}$, the spectrum of $V_H(t)$ consists of at most a finite number of isolated eigenvalues with finite algebraic multiplicity. On the other hand, all points μ' satisfying $|\mu'| \geq e^{(\varepsilon - \lambda^*)t}$ belong to the resolvent set of $\tilde{V}_H(t)$ and consequently,

$$\|\tilde{V}_H(t)\| < e^{(\varepsilon - \lambda^*)t}.$$

Hence, $\|\tilde{V}_H(t)\| = o(e^{(-\lambda^* + \varepsilon)t})$ as $t \rightarrow \infty$. This completes the proof. \square

In the remainder of this subsection, our aim is to establish a similar result to that of Theorem 4.3.2 for $p \in (1, +\infty)$. In fact, we have the following result.

Theorem 4.3.3 *Let $p \in (1, \infty)$ and let K be a regular collision operator. Assume that the boundary operator satisfies (4.2.1) and (4.2.2). Then, there exists $\varepsilon > 0$ small enough and $M > 0$ such that*

$$\left\| V_H(t)\psi_0 - \sum_{i=1}^n e^{(\lambda_i + D_i)t} P_i \psi_0 \right\|_{X_p} \leq M e^{(-\lambda^* + \varepsilon)t}, \quad \forall t > 0,$$

where $\psi_0 \in D(A_H)$.

The proof of the Theorem 4.3.3 is based on the following result.

Proposition 4.3.1 *Assume that the hypotheses of Theorem 4.3.3 hold true, then the first order remainder term of the Dyson-Phillips expansion, $R_1^H(t)$, is compact on X_p .*

As an immediate consequence of Proposition 4.3.1 we have

Corollary 4.3.1 *Assume that the hypotheses of Theorem 4.3.3 hold true, then*

$$\sigma_{ess}(V_H(t)) = \sigma_{ess}(U_H(t)) \text{ for any } t > 0.$$

Proof of Proposition 4.3.1. Making use Equation (2.3.6), we have

$$R_1^H(t) = \int_0^t U_H(s) K V_H(t-s) ds.$$

Thus, it is obvious that $R_1^H(t)$ depends linearly and continuously, in the norm operator topology, on the collision operator K . On the other hand, according to Remark 3.4.1 K has the following form

$$K : X_p \rightarrow X_p, \varphi \mapsto K\varphi(x, v) = \alpha(x) \int_{\mathbb{R}^n} f(v)g(v')\varphi(x, v')dv', \quad p \in (1, \infty)$$

where f and g are continuous functions with compact supports. Hence, $R_1^H(t)$ maps X_q into itself for all $q \in (1, \infty)$. Taking into account of Krasnoselskii interpolation [37, Theorem 3.10, p. 57], we may restrict ourselves to the Hilbert space X_2 . On the other hand, let λ be a complex number such that $\operatorname{Re} \lambda > -\lambda^*$. According to Theorem 3.4.1(1), we obtain the compactness of $(\lambda - T_H)^{-1}K$. Furthermore, using Theorem 4.3.1 with $r = 0$, we get

$$\lim_{|\operatorname{Im} \lambda| \rightarrow \infty} \|K(\lambda - T_H)^{-1}K\| = 0 \quad \text{uniformly in } \Gamma_\omega.$$

Finally, the dissipativity of T_H (because H satisfies the condition (4.2.2)) together with [72, Theorem 2.2, Corollary 2.1] give the desired result. \square

Proof of Theorem 4.3.3. It is clear that, according to Corollary 4.3.1, we have

$$r_{ess}(V_H(t)) = r_{ess}(U_H(t)), \quad \text{for all } t > 0.$$

Now, arguing as in the last part of the proof of the Theorem 4.3.2, we obtain the desired result. \square

Corollary 4.3.2 *Let $p \in [1, \infty)$ and let K be a regular collision operator. Assume that the boundary operator satisfies (4.2.1) and (4.2.2). If $(V_H(t))_{t \geq 0}$ is irreducible, then there exists $\varepsilon > 0$ small enough and $M > 0$ such that*

$$\|V_H(t)\psi_0 - e^{t\bar{\lambda}} \langle \psi_0, \psi_1 \rangle \varphi_1\|_{X_p} \leq M e^{(-\lambda^* + \varepsilon)t}, \quad \forall t > 0 \quad \forall \psi_0 \in D(A_H),$$

where $\bar{\lambda}$ is the leading eigenvalue of A_H and φ_1 (resp. ψ_1) is a (strictly positive) eigen-function of A_H (resp. of A_H^* , the adjoint of A_H) corresponding to $\bar{\lambda}$.

Proof. For all $\varepsilon' > 0$ small enough, we have

$$\sigma(A_H) := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq \bar{\lambda} - \varepsilon'\} = \{\bar{\lambda}\}.$$

Further, if the semigroup $(V_H(t))_{t \geq 0}$ is irreducible, then

$$V_H(t) = \tilde{V}_H(t) + e^{t\bar{\lambda}} P_{\bar{\lambda}}$$

where $\dim(P_{\bar{\lambda}}) = 1$. Next, according to [22, P. 224 and Theorem 8.17] there exists two strictly positive functions $\varphi_1 \in X_p$ and $\psi_1 \in X_q$ satisfying $\langle \varphi_1, \psi_1 \rangle = 1$ such that

$$P_{\bar{\lambda}} \varphi_0 = \langle \psi_0, \psi_1 \rangle \varphi_1.$$

Now, the rest of the proof follows from Theorems 4.3.2 and 4.3.3. \square

Remark 4.3.2 It should be noticed that, by using the so called resolvent approach, one can give a different proof of Theorems 4.3.2 and 4.3.3 based on the Riesz-Thorin interpolation theorem. One of the main steps in the proofs consists in showing that the asymptotic spectrum of the operator A_H remains unchanged in all X_p for $p \in [1, \infty)$. Unlike our proofs, this approach uses many complex computations (cf. [46, Section 5]). \square

4.3.2 Multiplying compact boundary conditions

The aim of this subsection is to extend the results of the last subsection to the case of multiplying compact boundary conditions.

Remark 4.3.3 It should be noticed that the streaming operator T_H with conservative boundary conditions does not generate a strongly continuous semigroup but it possesses an extension \widetilde{T}_H which generates a strongly continuous semigroup. So, in this work we consider only dissipative boundary conditions $\|H\| < 1$ (cf. Subsection 4.3.1) and multiplying boundary conditions $\|H\| > 1$.

Throughout this subsection, we shall assume that following condition is satisfied:

let D be a smooth open subset of \mathbb{R}^n for which there exists $\delta > 0$ such that, $\text{ess inf}_{(x,v) \in \Gamma_+} \tau(x, v) > \delta$.

Note that, in general, the sojourn time $\tau(x, v)$ may be arbitrary big as well as arbitrary small. Our assumption means that the boundary operator H vanishes on the set $\{(x, v) \in \Gamma_+ : \tau(x, v) \leq \delta\}$ (the tangential velocities are not taken into account by the boundary operator regardless of its norm). We suppose further that the boundary operator satisfies

$$\|H\| > 1. \quad (4.3.1)$$

Set

$$\lambda_0 = -\lambda^* + \frac{1}{\delta} \ln(\|H\|).$$

As in Section 3.2, in order to derive the expression of $(\lambda - T_H)^{-1}$, we consider the equation $(\lambda - T_H)\psi = \varphi$, where φ is a given function in X_p , λ is a complex number. The unknown ψ must be sought in $D(T_H)$. For $\text{Re}\lambda > -\lambda^*$, the invertibility of $(\lambda - T_H)$, reduces to the invertibility of the operator $\mathcal{P}_\lambda := I - M_\lambda H$. In particular, if λ is such that $\text{Re}\lambda > \lambda_0$, then $\|M_\lambda H\| < 1$. Let Γ_{λ_0} be the set defined by

$$\Gamma_{\lambda_0} := \{\lambda \in \mathbb{C} \text{ such that } \text{Re}\lambda > \lambda_0\}.$$

Clearly, if $\lambda \in \Gamma_{\lambda_0}$, the operator \mathcal{P}_λ is invertible and

$$(\mathcal{P}_\lambda)^{-1} = \sum_{n \geq 0} (M_\lambda H)^n. \quad (4.3.2)$$

Next, substituting (4.3.2) in the Equation (3.2.4), we get

$$(\lambda - T_H)^{-1} = \sum_{n \geq 0} B_\lambda H (M_\lambda H)^n G_\lambda + C_\lambda. \quad (4.3.3)$$

Unlike in Subsection 4.3.1, here the strip $\{\lambda \in \mathbb{C} \text{ such that } -\lambda^* < \text{Re}\lambda \leq \lambda_0\}$ is not necessary contained in $\rho(T_H)$. However, with additional hypotheses (for example, the compactness of H), we can obtain more information about its structure.

As before, we denote by G the set $\{\lambda \in \mathbb{C} \text{ such that } \text{Re}\lambda > -\lambda^*\}$.

Lemma 4.3.2 *Let $\lambda \in G$. If H is a compact operator satisfying (4.3.1), then there exists a discrete subset S of G such that $G \setminus S \subset \rho(T_H)$.*

Proof. Let $\lambda \in G$. It is easy to see that $M_\lambda \rightarrow 0$, as $\operatorname{Re} \lambda \rightarrow \infty$, in the strong operator topology. Using the compactness of H together with [35, Lemma 3.7, p. 151], we infer that $M_\lambda H \rightarrow 0$, as $\operatorname{Re} \lambda \rightarrow \infty$, in the operator topology. Applying Gohberg-Smul'yan's theorem (Theorem 2.5.1) we deduce that $(I - M_\lambda H)$ is invertible for all $\lambda \in G \setminus S$ where S is a discrete subset of G . \square

Lemma 4.3.3 *Let H be a boundary operator satisfying (4.3.1). Then the following holds*

$$\lim_{\operatorname{Re} \lambda \rightarrow +\infty} \|(\lambda - T_H)^{-1}\| = 0.$$

Proof. This follows from the fact that T_H is the generator of a C_0 -semigroup. \square

Lemma 4.3.4 *If H is a compact boundary operator satisfying (4.3.1), then $(I - M_\lambda H)^{-1}$ exists for $\lambda \in G$ such that $|\operatorname{Im} \lambda|$ is large enough.*

Proof. The result is clear if $\lambda \in \Gamma_{\lambda_0}$. Next, let $\lambda \in G \setminus \Gamma_{\lambda_0}$. According to Lemma 3.7.1, there exists $M > 0$ such that, for $|\operatorname{Im} \lambda| > M$, we have $r_\sigma(M_\lambda H) < 1$ which conclude the proof. \square

Now, we are ready to state one of the main result of this section. Before going further, we first recall that the set Γ_ω is defined by

$$\Gamma_\omega = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq -\lambda^* + \omega\}.$$

where $\omega > 0$ is small enough.

Proposition 4.3.2 *Let K be a regular collision operator. If H is a compact boundary operator satisfying (4.3.1), then*

$$\lim_{|\operatorname{Im} \lambda| \rightarrow \infty} |\operatorname{Im} \lambda|^r \|K(\lambda - T_H)^{-1} K\| = 0 \text{ uniformly on } \Gamma_\omega,$$

where $r \in [0, 1]$.

Proof. According to Remark 3.4.1, it suffices to establish the result for a one rank collision operator which we define by

$$K : X_p \rightarrow X_p, \varphi \mapsto K\varphi(x, v) = \int_{\mathbb{R}^n} f(v)g(v')\varphi(x, v')dv',$$

where $f(\cdot) \in L_p(\mathbb{R}^n)$ et $g(\cdot) \in L_q(\mathbb{R}^n)$.

Since, $K(\lambda - T_H)^{-1}K = KB_\lambda H(I - M_\lambda H)^{-1}G_\lambda K + KC_\lambda K$ (use Equation (4.3.3)), it follows from Lemma 4.2.1, that $\lim_{|\text{Im}\lambda| \rightarrow \infty} |\text{Im}\lambda|^r \|KC_\lambda K\| = 0$ uniformly on Γ_ω . Hence, it suffices to establish the result for the operator $KB_\lambda H(I - M_\lambda H)^{-1}G_\lambda K$. To do so, we shall proceed in two steps.

(i) We will first prove that the family of operators $\{H(I - M_\lambda H)^{-1}G_\lambda K, -\lambda^* + \omega \leq \text{Re}\lambda \leq \lambda_0\}$ is collectively compact. Let B be the unit ball of X_p and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\cup_\lambda H(I - M_\lambda H)^{-1}G_\lambda KB$, $\lambda \in \{\lambda, -\lambda^* + \omega \leq \text{Re}\lambda \leq \lambda_0\}$. Then there exists $(q_n)_{n \in \mathbb{N}}$ in B such that $x_n = H(I - M_\lambda H)^{-1}G_\lambda K q_n$ $n = 1, 2, \dots$. It is clear that the sequence $(y_n = (I - M_\lambda H)^{-1}G_\lambda K q_n)_{n \in \mathbb{N}}$ is bounded in L_p^+ . So, it follows from the compactness of H that $(x_n = Hy_n)_{n \in \mathbb{N}}$ has converging subsequence in $\cup_\lambda H(I - M_\lambda H)^{-1}G_\lambda KB$.

(ii) Now we shall establish that $\{|\text{Im}\lambda|^r KB_\lambda, -\lambda^* + \omega \leq \text{Re}\lambda \leq \lambda_0\}$ converges strongly to zero as $|\text{Im}\lambda|$ goes to infinity. For $\varphi \in L_p^-$ we have

$$(KB_\lambda \varphi)(x, v) = f(v) \int_{\mathbb{R}^n} g(v') e^{-i\text{Im}\lambda t^-(x, v')} B_{\text{Re}\lambda} \varphi(x, v') dv'.$$

So, it can be decomposed as $KB_\lambda := A_2(\text{Im}\lambda)A_1(\text{Re}\lambda)$, where

$$A_1(\text{Re}\lambda) : L_p^+ \rightarrow X_p, \psi \mapsto (A_1(\text{Re}\lambda)\psi)(x, v) = B_{\text{Re}\lambda} \psi(x, v)$$

and

$$A_2(\text{Im}\lambda) : X_p \rightarrow X_p, \varphi \mapsto (A_2(\text{Im}\lambda)\varphi)(x, v) = f(v) \int_{\mathbb{R}^n} g(v') e^{-i\text{Im}\lambda t^-(x, v')} \varphi(x, v') dv'.$$

Note that $A_1(\text{Re}\lambda)$ is independent of $\text{Im}\lambda$ and $t^-(x, v') \in (\delta, \mathbf{d})$ (\mathbf{d} stands for the diameter of D). Now, arguing as in the last part of the proof of the Lemma 4.2.1, we get

$$\lim_{|\text{Im}\lambda| \rightarrow \infty} |\text{Im}\lambda|^r \|A_2(\text{Im}\lambda)\varphi\|_{X_p} = 0.$$

Now, according to (i), (ii) and [35, Lemma 3.7, p.151], we get the desired result. \square

Let $\bar{\lambda}$ be an element of S with the greater real part (S is the subset obtained in Lemma 4.3.2).

Lemma 4.3.5 *Let K be a regular collision operator and let H be compact boundary operator satisfying (4.3.1). Then*

- (1) $\sigma(A_H) \cap \{\lambda \in \mathbb{C} : \text{Re}\lambda > \text{Re}\bar{\lambda}\}$ consists of, at most, isolated eigenvalues with finite algebraic multiplicities,
- (2) if $\varepsilon > 0$ then $\sigma(A_H) \cap \{\lambda \in \mathbb{C} : \text{Re}\lambda > \text{Re}\bar{\lambda} + \varepsilon\}$ is finite,
- (3) if $\varepsilon > 0$ then $\|(\lambda - A_H)^{-1}\|$ is uniformly bounded in $\{\lambda \in \mathbb{C} : \text{Re}\lambda > \text{Re}\bar{\lambda} + \varepsilon\}$ for large $|\text{Im}\lambda|$.

Proof. The proof is similar to that of Lemma 4.3.1 and so it is omitted. □

Assume that K is a regular collision operator on X_p and H is an arbitrary compact boundary operator satisfying (4.3.1). Then, by Lemma 4.3.3, the spectrum of A_H in the half plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \operatorname{Re}\lambda + \varepsilon\}$ consists of, a finite number of isolated eigenvalues with finite algebraic multiplicity which we denote $\lambda_1, \lambda_2, \dots, \lambda_n$. Let P_i and D_i denote, respectively, the spectral projection and the nilpotent operator associated with $\lambda_i, i = 1, 2, \dots, n$. Then $P := P_1 + \dots + P_n$ is the spectral projection of the compact set $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$. According to the spectral decomposition theorem corresponding to the compact set $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ (see, for example, [35]), we may write

$$V_H(t) = \tilde{V}_H(t) + \sum_{i=1}^n e^{\lambda_i t} e^{D_i t} P_i$$

where $\tilde{V}_H(t) := V_H(t)(I - P)$ is the C_0 -semigroup on the Banach space $(I - P)X_p$ with generator $\tilde{A}_H := A_H(I - P)$ (\tilde{A}_H is the part of A_H on the closed subspace $(I - P)X_p$).

We are in a position to state the main result of this subsection.

Theorem 4.3.4 *Assume that $\lambda^* = \inf_{v \in V} \sigma(v) > 0$ and $\delta \geq \frac{\ln(\|H\|)}{\lambda^*}$. If the hypotheses of Lemma 4.3.3 hold true, then, there exists $\varepsilon > 0$ small enough and $M > 0$ such that*

$$\left\| V_H(t)\psi_0 - \sum_{i=1}^n e^{(\lambda_i + D_i)t} P_i \psi_0 \right\|_{X_p} \leq M e^{(\varepsilon + \lambda_0)t} \quad \forall t > 0$$

where $\psi_0 \in D(A_H)$ and $p \in [1, \infty)$.

Proof. According to Theorems 3.4.1 and 2.5.2, $(\lambda - T_H)^{-1}K$ is power compact. Further, using Proposition 4.3.2, for all $r \in [0, 1]$, we have $\|\operatorname{Im}\lambda\|^r \|K(\lambda - T_H)^{-1}K\| \rightarrow 0$ as $|\operatorname{Im}\lambda| \rightarrow \infty$. Now the rest of the proof may modeled in a similar way to those of Theorems 4.3.2 and 4.3.3. □

Remark 4.3.4 We close this section by noticing that it is not difficult to prove that the first order remainder term of the Dyson-Phillips, $R_1^H(t)$, is compact on X_p for $p \in (1, +\infty)$. The proof may be derived in the same way as that of Proposition 4.3.1. However, the weak compactness of $R_1^H(t)$ on X_1 , even for dissipative boundary conditions, is still an open problem.

On a singular neutron transport equation with bounce-back boundary

5.1 Introduction

In this chapter, we discuss the well-posedness of a Cauchy problem governed by a singular transport operator in bounded geometry. To be more precise, we are concerned with the following equation

$$\left\{ \begin{array}{l} \frac{\partial \psi}{\partial t}(x, v, t) = A_H \psi(x, v, t) := T_H \psi(x, v, t) + K \psi(x, v, t) \\ \qquad \qquad \qquad = -v \cdot \nabla_x \psi(x, v, t) - \sigma(v) \psi(x, v, t) + \int_{\mathbb{R}^n} \kappa(x, v, v') \psi(x, v', t) dv', \\ \psi(x, v, 0) = \psi_0(x, v), \end{array} \right. \quad (5.1.1)$$

where $(x, v) \in D \times \mathbb{R}^n$ and K is the partial integral part of A_H , it is called the collision operator. Here D is a bounded open subset of \mathbb{R}^n . The function $\psi(x, v, t)$ represents the number (or probability) density of gas particles having the position x and the velocity v at the time t . The functions $\sigma(\cdot)$ and $\kappa(\cdot, \cdot, \cdot)$ are called, respectively, the collision frequency and the scattering kernel.

Problem (5.1.1) is supplemented by bounce-back boundary conditions, that is

$$(H\psi)(x, v, t) = \gamma \psi(x, -v, t), \quad (x, v) \in \Gamma_-, \quad t > 0, \quad (5.1.2)$$

where $\gamma \in (0, 1)$ and Γ_- represents the incoming part of the boundary of the phase space (see Section 5.2 for more details).

By singular transport operator, we mean that both the collision frequency $\sigma(\cdot)$ and the collision operator

$$\psi \mapsto K\psi = \int_{\mathbb{R}^n} \kappa(x, v, v')\psi(x, v', t)dv'$$

are not bounded in $L^1(D \times \mathbb{R}^n, dx dv)$.

Let us now introduce the following assumptions

- (i) there exists $\sigma_0 > 0$ and a closed subset Λ of \mathbb{R}^n with zero Lebesgue measure such that

$$\sigma(\cdot) \in L_{\text{loc}}^\infty(\mathbb{R}^n \setminus \Lambda), \quad \sigma(v) \geq \sigma_0 \quad \text{a.e. on } \mathbb{R}^n$$

and

$$\sigma(v) = \sigma(-v) \quad \forall v \in \mathbb{R}^n,$$

- (ii) the collision operator K viewed as an operator from $L^1(\mathbb{R}^n, \sigma(v)dv)$ into $L^1(\mathbb{R}^n, dv)$ is non negative and weakly compact.

In the context of singular transport equations, these hypotheses were considered by many authors (see, for example, [65, Chapter 9] and [19, 20, 55]) and were motivated by free gas models [24, 78]. The first part of Assumption (i) means that the collision frequency is locally bounded on $\mathbb{R}^n \setminus \Lambda$.

Since the earlier work of Chabi and Mokhtar-Kharroubi [65, Chapter 9], the singular transport equation was investigated by several authors. Their main objective was to study the well-posedness and to discuss also the time asymptotic behavior of the solution of Problem (5.1.1) in L^p spaces, with $1 \leq p < \infty$. Unlike [64, 40, 45] and Chapter 3, when dealing with singular transport equations (i.e. unbounded collision frequencies and unbounded collision operators), we can not appeal to the standard perturbation theory to deduce that Cauchy Problem (5.1.1) has a unique classical solution in the sense of semigroup theory. We point that, in the litterature, the study of Problem (5.1.1) were done separately for $p = 1$ and $p > 1$. In all cases, the general theory is based on a non standard perturbation theory of strongly continuous semigroups for a class of unbounded perturbations. This class was introduced by Miyadera [60] refined and developed afterwards by Voigt, Desh, Arlotti, Mokhtar-Kharroubi, Banasiak and others in the framework of positive semigroups. For more information in this direction, we refer to the works [4, 6, 25, 65, 87, 88] and the references therein. We quote the contributions by Chabi and Mokhtar-Kharroubi [65, Chapter 9] and Lods [55] in L^1 spaces with vacuum boundary conditions (i.e. $H = 0$) in bounded geometry. In [19] (resp. [20]), Chabi and Latrach studied Problem (5.1.1) in slab geometry for reentry boundary conditions (both specular reflection and periodic boundary conditions) in L^1 spaces (resp. L^p spaces, with $1 < p < \infty$).

Let X be a Banach space and let (Σ, μ) be a measure space. An operator defined on $L^1(\Sigma, d\mu)$ is said to be resolvent positive if there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and $(\lambda - A)^{-1}$ is positive

(in the lattice sense) for all $\lambda > \omega$. Now, we recall an abstract generation theorem, in the context of L^1 spaces, concerning Miyadera perturbation of strongly continuous semigroups due to Desch [25] (see also [7, Section 5.2] or [65, Theorem 8.1]) which will play a crucial role in our analysis.

Theorem 5.1.1 *Let (Σ, μ) be a measure space and let A be the generator of a positive C_0 -semigroup $(U(t))_{t \geq 0}$ in $L^1(\Sigma, \mu)$. Assume that $B \in \mathcal{L}(D(A), L^1(\Sigma, \mu))$ is a positive operator. If there exists $\lambda \geq s(A)$ such that*

$$r_\sigma(B(\lambda - A)^{-1}) < 1,$$

then $A + B$ generates a positive C_0 -semigroup $(V(t))_{t \geq 0}$ given by the Dyson-Phillips expansion.

According to Theorem 2.3.2, if some remainder term of the Dyson-Phillips expansion $R_n(t)$ is weakly compact, then $\omega_e(V(t)) = \omega_e(U(t))$. As already explained in the introduction of Chapter 4, the semigroup $(V(t))_{t \geq 0}$ can be decomposed into two parts, the first one containing the time development of finitely many eigenmodes and the second one being of faster decay.

The main goal of this chapter is to study Problem (5.1.1)-(5.1.2) and to extend, to high dimension, the results obtained by Chabi and Latrach in [19]. We shall establish, under the assumptions (i) and (ii), the well-posedness of Problem (5.1.1)-(5.1.2) and show afterwards the weak compactness of the second-order remainder term of the Dyson-Phillips expansion. This implies that the essential types of the streaming semigroup and the transport semigroup coincide. Hence, we derive easily, via standard arguments, the time asymptotic behavior of the solution.

5.2 Preliminary results

In this section we introduce only the notions which do not defined in the two previous chapters.

Let

$$X := L^1(D \times \mathbb{R}^n, dx dv),$$

where D denotes a smooth bounded open convex subset of \mathbb{R}^n and

$$X^\sigma := L^1(D \times \mathbb{R}^n, \sigma(v) dx dv), \quad L_1 := L^1(\mathbb{R}^n, dv) \quad \text{and} \quad L_1^\sigma := L^1(\mathbb{R}^n, \sigma(v) dv).$$

We first define the streaming operator T_H by

$$\begin{cases} T_H : D(T_H) \subseteq X^\sigma \longrightarrow X \\ \psi \longmapsto T_H \psi(x, v) = -v \cdot \nabla_x \psi(x, v) - \sigma(v) \psi(x, v) \\ D(T_H) = \{ \psi \in \widetilde{W} \text{ such that } \psi \in X^\sigma \text{ and } H\psi(x, v) = \gamma \psi(x, -v) \}, \end{cases}$$

where $\gamma \in (0, 1)$.

Let us now introduce the following assumptions:

(A7) there exists $\sigma_0 > 0$ and a closed subset Λ of \mathbb{R}^n with zero Lebesgue measure such that

$$\sigma(\cdot) \in L_{\text{loc}}^\infty(\mathbb{R}^n \setminus \Lambda); \quad \sigma(v) \geq \sigma_0 \text{ a.e. } v \in \mathbb{R}^n \text{ and } \sigma(v) = \sigma(-v) \quad \forall v \in \mathbb{R}^n,$$

and

(A8) the collision operator K viewed as an operator from X^σ into X is a weakly compact and positive operator.

Before going further, let us introduce the function $m_\lambda(\cdot, \cdot)$ defined by

$$\begin{cases} m_\lambda : D \times \mathbb{R}^n \longrightarrow \mathbb{R}^+ \\ (x, v) \longmapsto m_\lambda(x, v) := \gamma e^{-\tau(x, v)(\lambda + \sigma(v))}, \end{cases}$$

for any complex number λ .

Lemma 5.2.1 *For any complex numbers λ satisfying $\text{Re}\lambda + \sigma_0 > 0$, there exists a real number $C_{\lambda, \gamma}$ belonging to $(0, 1)$ such that*

$$|m_\lambda(x, v)| \leq C_{\lambda, \gamma} \quad \text{a.e. } (x, v) \in D \times \mathbb{R}^n.$$

Remark 5.2.1 Let us denote by D_v^0 and D_v^1 , the subsets of $\bar{D} \times \mathbb{R}^n$, defined by

$$D_v^0 := \{(x, v) \in \bar{D} \times \mathbb{R}^n \text{ such that } \tau(x, v) = 0\}$$

and

$$D_v^1 := \{(x, v) \in \bar{D} \times \mathbb{R}^n \text{ such that } \tau(x, v) > 0\}.$$

It is clear that

$$\bar{D} \times \mathbb{R}^n = D_v^0 \cup D_v^1 \quad \text{and} \quad D_v^0 \cap D_v^1 = \emptyset.$$

Proof of Lemma 5.2.1. If $(x, v) \in D_v^0$, then $C_{\lambda, \gamma} = \gamma$ and so we have the desired result.

Now, if $(x, v) \in D_v^1$, then there exists a real $\tau_0 > 0$ such that $\tau(x, v) \geq \tau_0$. And therefore, we have

$$|m_\lambda(x, v)| \leq e^{-\tau_0(\sigma_0 + \text{Re}\lambda)}.$$

In summary, we have

$$C_{\lambda, \gamma} = \begin{cases} \gamma & \text{if } \tau(x, v) = 0; \\ e^{-\tau_0(\sigma_0 + \text{Re}\lambda)} & \text{else.} \end{cases}$$

□

Let $\varphi \in X$ and consider the problem

$$(\lambda - T_H)\psi = \varphi, \quad (5.2.1)$$

where λ is a complex number and the unknown ψ must be sought in $D(T_H)$. For $\operatorname{Re}\lambda + \sigma_0 > 0$, the solution of (5.2.1) is given by

$$\psi(x, v) = e^{-(\lambda + \sigma(v))t^-(x, v)}\psi(x - t^-(x, v)v, v) + \int_0^{t^-(x, v)} e^{-t(\lambda + \sigma(v))}\varphi(x - tv, v)dt.$$

Using the last equation and the fact that the boundary operator H satisfies (5.1.2), a similar reasoning to that in the proof of Proposition 3.1 in [45] yields that the resolvent of T_H is given by

$$(\lambda - T_H)^{-1} := \mathcal{P}_\lambda + \mathcal{S}_\lambda + \mathcal{Q}_\lambda, \quad (5.2.2)$$

where

$$\left\{ \begin{array}{l} \mathcal{P}_\lambda : X \longrightarrow X \\ \varphi \mapsto (\mathcal{P}_\lambda\varphi)(x, v) := \left(1 - m_\lambda^2(x, v)\right)^{-1} \\ \quad \int_0^{\tau(x, v)} e^{-t(\lambda + \sigma(v))}\varphi(x + (t - t^-(x, v))v, -v)dt, \end{array} \right.$$

$$\left\{ \begin{array}{l} \mathcal{S}_\lambda : X \longrightarrow X \\ \varphi \mapsto (\mathcal{S}_\lambda\varphi)(x, v) := \left(1 - m_\lambda^2(x, v)\right)^{-1} e^{-(t^-(x, v) + \tau(x, v))(\lambda + \sigma(v))} \\ \quad \int_0^{\tau(x, v)} e^{-t(\lambda + \sigma(v))}\varphi(x + (\tau(x, v) - t^-(x, v) - t)v, v)dt \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \mathcal{Q}_\lambda : X \longrightarrow X \\ \varphi \mapsto (\mathcal{Q}_\lambda\varphi)(x, v) := \int_0^{t^-(x, v)} e^{-t(\lambda + \sigma(v))}\varphi(x - tv, v)dt. \end{array} \right.$$

Remark 5.2.2 Note that, for all real number such that $\lambda > -\sigma_0$, these operators are positive (in the lattice sense) on X . So, from (5.2.2) the operator $(\lambda - T_H)^{-1}$ is also positive on X for any $\lambda > -\sigma_0$.

Since the boundary operator H satisfies Hypothesis (5.1.2), then by the Lumer-Phillips theorem, the streaming operator T_H generates a C_0 -semigroup $(U_H(t))_{t \geq 0}$. Next, according to the hypothesis (A7), Remark 5.2.2 and [45, Theorem 3.3], we infer that, for all $\varphi \in X$, $(U_H(t))_{t \geq 0}$ is given explicitly by

$$(U_H(t)\varphi)(x, v) = (U_0(t)\varphi)(x, v) + \sum_{n \geq 0} (I_n(t) + J_n(t))\varphi(x, v), \quad (5.2.3)$$

where $(U_0(t)\varphi)(x, v) := e^{-\sigma(v)t}\varphi(x - tv, v)\chi_{(0, t^-(x, v))}(t)$ and

$$\begin{cases} (I_n(t)\varphi)(x, v) := \varphi(x + tv, -v)i_n(t, x, v), \\ (J_n(t)\varphi)(x, v) := \varphi(x - tv, v)j_n(t, x, v) \end{cases}$$

with

$$\begin{cases} i_n(t, x, v) = \gamma^{2n+1} e^{-\sigma(v)(t+2t^-(x, v)+2n\tau(x, v))} \chi_{(-t^-(x, v), t^+(x, v))}(t), \\ j_n(t, x, v) = \gamma^{2n+2} e^{-\sigma(v)(t+(2n+2)\tau(x, v))} \chi_{(-t^-(x, v), t^+(x, v))}(t). \end{cases}$$

Remark 5.2.3 Note however that, $(U_0(t))_{t \geq 0}$ is nothing else but the C_0 -semigroup generated by the streaming operator with vacuum boundary condition T_0 (see [65, Chapter 9]). \square

5.3 Generation results

In this section, we shall discuss the well-posedness of Cauchy Problem (5.1.1)-(5.1.2). To do so, it suffices to establish that operator A_H generates a C_0 -semigroup.

Lemma 5.3.1 $D(T_H)$ is continuously embedded in X^σ .

Proof. Let λ be a complex number such that $\operatorname{Re}\lambda + \sigma_0 > 0$. It follows from Section 5.2 that $\lambda \in \rho(T_H)$ and

$$(\lambda - T_H)^{-1} := \mathcal{P}_\lambda + \mathcal{S}_\lambda + \mathcal{Q}_\lambda.$$

So, to prove the lemma, it suffices to show that subsets $\mathcal{P}_\lambda(X)$, $\mathcal{S}_\lambda(X)$ and $\mathcal{Q}_\lambda(X)$ are contained in X^σ . Note that, \mathcal{Q}_λ is nothing else but $(\lambda - T_0)^{-1}$ (the resolvent of the streaming operator with vacuum boundary condition), applying Proposition 9.2 in [65], we conclude that $\mathcal{Q}_\lambda(X) \subseteq X^\sigma$ and

$$\|\mathcal{Q}_\lambda\varphi\|_{\mathcal{L}(X, X^\sigma)} \leq \sup_{v \in \mathbb{R}^d} \frac{\sigma(v)}{\operatorname{Re}\lambda + \sigma(v)}.$$

Let $\varphi \in X$, it is easy to see that

$$|\mathcal{P}_\lambda\varphi(x, v)| \leq C_{1, \lambda} \int_0^{\tau(x, v)} |\varphi(x + (t - t^-(x, v))v, -v)| e^{-t(\operatorname{Re}\lambda + \sigma(v))} dt,$$

where $C_{1,\lambda} := \frac{1}{1 - C_{\lambda,\gamma}^2}$ with $C_{\lambda,\gamma}$ is the real introduced in proof of Lemma 5.2.1. Hence

$$\|\mathcal{P}_\lambda \varphi(x, v)\|_{X^\sigma} \leq C_{1,\lambda} \int_{\mathbb{D} \times \mathbb{R}^n} \sigma(v) dx dv \int_0^{\tau(x,v)} |\varphi(x + (t - t^-(x, v))v, -v)| e^{-t(\operatorname{Re}\lambda + \sigma(v))} dt.$$

Let $(x, v) \in \bar{\mathbb{D}} \times \mathbb{R}^n$. Using the convexity of \mathbb{D} , one sees that

$$t - t^-(x, v) \in (-t^-(x, v), t^+(x, v)) \Leftrightarrow y = x + (t - t^-(x, v))v \in \mathbb{D}.$$

The change of variable $y = x + (t - t^-(x, v))v$ together with the fact that $\tau(y - (t - t^-(x, v))v, v) = \tau(y, v)$ leads to

$$\begin{aligned} \|\mathcal{P}_\lambda \varphi(x, v)\|_{X^\sigma} &\leq C_{1,\lambda} \int_{\mathbb{D} \times \mathbb{R}^n} \sigma(v) |\varphi(y, -v)| dy dv \int_0^{\tau(y,v)} e^{-t(\operatorname{Re}\lambda + \sigma(v))} dt \\ &\leq C_{1,\lambda} \int_{\mathbb{D} \times \mathbb{R}^n} \frac{\sigma(v)}{\operatorname{Re}\lambda + \sigma(v)} |\varphi(y, -v)| dy dv \\ &\leq C_{1,\lambda} \sup_{v \in \mathbb{R}^n} \frac{\sigma(v)}{\operatorname{Re}\lambda + \sigma(v)} \|\varphi\|_X. \end{aligned}$$

Since \mathcal{P}_λ and \mathcal{S}_λ have the same structure, similar calculations as above give the following estimate

$$\|\mathcal{S}_\lambda\|_{\mathcal{L}(X, X^\sigma)} \leq C_{1,\lambda} \sup_{v \in \mathbb{R}^n} \frac{\sigma(v)}{\operatorname{Re}\lambda + \sigma(v)}$$

and therefore

$$\|(\lambda - T_H)^{-1}\|_{\mathcal{L}(X, X^\sigma)} \leq C_{2,\lambda} \sup_{v \in \mathbb{R}^n} \frac{\sigma(v)}{\operatorname{Re}\lambda + \sigma(v)}, \quad (5.3.1)$$

where $C_{2,\lambda} := (1 + 2C_{1,\lambda})$. □

Remark 5.3.1 Since $\sigma(\cdot)$ is unbounded, it follows from Eq. (5.3.1) that

$$\|(\lambda - T_H)^{-1}\|_{\mathcal{L}(X, X^\sigma)} \leq C_{2,\lambda}.$$

It clear that if $\|K\|_{\mathcal{L}(X^\sigma, X)} < \frac{1}{C_{2,\lambda}}$, then

$$\|K(\lambda - T_H)^{-1}\|_{\mathcal{L}(X)} < 1. \quad (5.3.2)$$

So, since K is positive, the use of Eq. (5.3.2) together with Theorem 5.1.1 implies that $T_H + K$ generates a C_0 -semigroup. This result is restrictive because it depends on the size of $\|K\|_{\mathcal{L}(X^\sigma, X)}$.

We are in a position to state a more general generation result.

Theorem 5.3.1 *If hypotheses (A7) and (A8) hold true, then $T_H + K$ generates a positive C_0 -semigroup $(V_H(t))_{t \geq 0}$ given by the Dyson-Phillipps expansion, that, is*

$$V_H(t) = \sum_{j=0}^{n-1} U_j(t) + R_n^H(t),$$

where $U_0(t) = U_H(t)$, $U_j(t) = \int_0^t U_H(s) K U_{j-1}(t-s) ds$.

The n^{th} remainder term $R_n^H(t)$ is given by

$$R_n^H(t) = \sum_{j=n}^{\infty} U_j(t) = \int_{s_1+s_2+\dots+s_n \leq t, s_i \geq 0} U_H(s_1) K \cdots U_H(s_n) K V_H(t - \sum_{i=1}^n s_i) ds_1 \cdots ds_n.$$

As already mentioned above, the expression of $U_H(t)$ is given by (5.2.3). To establish this theorem we need to prove the following result.

Lemma 5.3.2 *If hypotheses (A7) and (A8) are satisfied, then*

$$\lim_{\text{Re } \lambda \rightarrow +\infty} \left\| (K(\lambda - T_H)^{-1})^2 \right\|_{\mathcal{L}(X)} = 0.$$

Proof. According to Lemma 5.3.1, $K(\lambda - T_H)^{-1}$ depends continuously on $K \in \mathcal{L}(L_1^\sigma, L_1)$. Using Assumption (A8), one sees that there exists a sequence of finite rank operators which converges, in the operator norm, to K . Without loss of generality, we may restrict ourselves to a collision operator of rank one which we denote also by K , that is,

$$(K\varphi)(x, v) = f(v) \int_{\mathbb{R}^n} g(v') \varphi(x, v') dv',$$

where $f \in L_1$ and $g/\sigma \in L^\infty(\mathbb{R}^n, dv)$. Moreover, since $(\lambda - T_H)^{-1}$ is a bounded operator, it suffices to prove the lemma for $K(\lambda - T_H)^{-1}K$. So, using Equation (5.2.2), we have

$$\left\| K(\lambda - T_H)^{-1}K \right\| := \|K\mathcal{P}_\lambda K\| + \|KS_\lambda K\| + \|KQ_\lambda K\|.$$

Using the fact that the operators $K\mathcal{P}_\lambda K$, $KS_\lambda K$ and $KQ_\lambda K$ have the same structure, we restrict ourselves to the operator $K\mathcal{P}_\lambda K$. It is easy to see that $K\mathcal{P}_\lambda K$ can be factorized as

$$K\mathcal{P}_\lambda K := A_3 A_2(\lambda) A_1,$$

where

$$X^\sigma \ni \psi \mapsto A_1 \psi = \int_{\mathbb{R}^n} g(v) \psi(x, -v) dv \in L^1(D, dx),$$

$$L^1(\mathbb{D}, dx) \ni \varphi \mapsto A_2(\lambda)\varphi = \int_{\mathbb{R}^n} \frac{g(v)f(-v)}{1 - m_\lambda^2(x, v)} dv \\ \int_0^{\tau(x, v)} e^{-t(\lambda + \sigma(v))} \varphi(x + (t - t^-(x, v))v) dt \in L^1(\mathbb{D}, dx),$$

and

$$L^1(\mathbb{D}, dx) \ni \varphi \mapsto A_3\varphi = \varphi(x)f(v) \in X.$$

Since A_1 and A_3 are bounded linear operators independent of the parameter λ , it suffices to prove the lemma for $A_2(\lambda)$. Indeed, let us first observe that

$$\|A_2(\lambda)\varphi(x)\|_{L^1(\mathbb{D}, dx)} \leq \frac{1}{1 - C_{\lambda, \gamma}^2} \int_{\mathbb{D} \times \mathbb{R}^n} |f(-v)| |g(v)| dx dv \\ \int_0^{\tau(x, v)} e^{-t(\operatorname{Re}\lambda + \sigma(v))} |\varphi(x + (t - t^-(x, v))v)| dt$$

where $C_{\lambda, \gamma}$ is the constant defined in the proof of Lemma 5.2.1. Now, arguing as in the proof of Lemma 5.3.1, the change of variable $y = x + (t - t^-(x, v))v$, gives

$$\|A_2(\lambda)\varphi(x)\|_{L^1(\mathbb{D}, dx)} \leq \frac{1}{1 - C_{\lambda, \gamma}^2} \int_{\mathbb{D} \times \mathbb{R}^n} |f(v)| |g(-v)| |\varphi(y)| dy dv \int_0^{\tau(y, v)} e^{-t(\operatorname{Re}\lambda + \sigma(v))} dt \\ \leq \frac{1}{(1 - C_{\lambda, \gamma}^2)(\operatorname{Re}\lambda + \sigma_0)} \int_{\mathbb{D} \times \mathbb{R}^n} \sigma(v) |f(v)| \frac{|g(-v)|}{\sigma(v)} |\varphi(y)| dy dv \\ \leq \frac{C_3}{(1 - C_{\lambda, \gamma}^2)(\operatorname{Re}\lambda + \sigma_0)} \|\varphi(\cdot)\|_{L^1(\mathbb{D}, dx)}.$$

where $C_3 := \left\| \frac{g(\cdot)}{\sigma(\cdot)} \right\|_{L^\infty(\mathbb{R}^n, dv)} \|\sigma(\cdot)\|_{L^\infty(\mathbb{R}^n \setminus \Lambda)} \|f(\cdot)\|_{L^1(\mathbb{R}^n, dv)}$ which end the proof. \square

Proof of Theorem 5.3.1. According to Lemma 5.3.2, for all complex number λ such that $\operatorname{Re}\lambda$ is large enough, we have

$$r_\sigma(K(\lambda - T_H)^{-1}) < 1.$$

Using the positivity of K together with Theorem 5.1.1 gives the desired result. \square

Remark 5.3.2 It is not difficult to see that if $K = K_1 + K_2$ where K_1 satisfies the assumption (A_8) and K_2 is a positive bounded linear operator on X , then $T_H + K$ generates a positive C_0 -semigroup on X . This is an immediate consequence of Theorem 5.3.1 and the classical perturbation theorem for bounded operators. \square

5.4 Asymptotic behavior of $(V_H(t))_{t \geq 0}$

The purpose of this section is to discuss the weak compactness of the second order remainder term of the Dyson-phillips expansion on the space \mathbf{X} . In fact, as a consequence of this result, the essential type of the C_0 -semigroup $(V_H(t))_{t \geq 0}$ is less than or equal to that of $(U_H(t))_{t \geq 0}$ (see, for example, [88, Theorem 1] or [65, Theorem 7.2]). This allows us to derive the time asymptotic behavior of the solution to the Cauchy problem (5.1.1)-(5.1.2) for large times. Let us first recall that the operator $R_2^H(t)$ is defined by

$$R_2^H(t) = \int_{s_1+s_2 \leq t, s_1, s_2 \geq 0} U_H(s_1)K U_H(s_2)K V_H(t-s_1-s_2) ds_1 ds_2.$$

The difficulty here is twofold. First, the operator K is not defined on the whole space \mathbf{X} . Secondly, $R_2^H(t)$ involves strong time integrals of operators which are not locally bounded in time. So, in order to overcome those difficulties, we shall use Proposition 5.4.2 below.

Proposition 5.4.1 *For all $t \in (0, \infty)$ there exists an integer $n(t)$ such that*

$$U_H(t)\varphi = U_0(t)\varphi + \sum_{n=0}^{n(t)} (I_n(t) + J_n(t))\varphi.$$

Proof. Let us first recall that

$$\mathbf{D} \times \mathbb{R}^n = \mathbf{D}_v^0 \cup \mathbf{D}_v^1 \quad \text{and} \quad \mathbf{D}_v^0 \cap \mathbf{D}_v^1 = \emptyset,$$

where \mathbf{D}_v^0 and \mathbf{D}_v^1 are the subsets defined in Remark 5.2.1. Note that, if $(x, v) \in \mathbf{D}_v^0$, then $\tau(x, v) = 0$ and consequently $t = 0$ which contradicts the hypothesis.

Let $(x, v) \in \mathbf{D}_v^1$. Then, there exists a real number $\tau_0 > 0$ such that $\tau(x, v) \geq \tau_0$. Set $n(t) = \left[\frac{t}{2\tau_0} + \frac{1}{2} \right] + 1$ where $\left[\frac{t}{2\tau_0} + \frac{1}{2} \right]$ is the integer part of $t/2\tau_0 + 1/2$. So, for any integer $n \geq n(t)$ and $\varphi \in \mathbf{X}$, we have

$$I_n\varphi = J_n\varphi = 0$$

which ends the proof. □

Now, we are able to establish the following result.

Proposition 5.4.2 *Let $t \in (0, \infty)$ and let $\varphi \in \mathbf{X}$. If hypotheses (A7) and (A8) are satisfied, then*

$$(a) \quad U_H(t)\varphi \in \mathbf{X}^\sigma \quad \text{and} \quad \int_0^t \|U_H(s)\varphi\|_{\mathbf{X}^\sigma} ds < \infty,$$

$$(b) \int_0^t \|V_H(s)\varphi\|_{X^\sigma} ds \leq M(t) \|\varphi\|_X,$$

where $M(t)$ is a constant depending on t .

Proof. As seen in the proof of the previous proof, if $(x, v) \in D_v^0$, then $t = 0$ which contradicts the fact that $t \in (0, \infty)$. We suppose now that $(x, v) \in D_v^1$.

(a) It is not difficult to check that

$$|i_n(t, x, v)| \leq e^{-\sigma(v)t} \quad \text{and} \quad |j_n(t, x, v)| \leq e^{-\sigma(v)t} \quad \forall t > 0.$$

Moreover, using the above inequality and the fact that, $ue^{-u} \leq e^{-1}$, for all $u \geq 0$, we have

$$\|U_0(t)\varphi\|_{X^\sigma} \leq \frac{1}{te} \|\varphi\|_X, \quad \|I_n(t)\varphi\|_{X^\sigma} \leq \frac{1}{te} \|\varphi\|_X \quad \text{and} \quad \|J_n(t)\varphi\|_{X^\sigma} \leq \frac{1}{te} \|\varphi\|_X$$

for $n = 0, 1, \dots, n(t)$, where $n(t)$ is the integer introduced in Proposition 5.4.1.

Now, using the last estimates and the expression of U_H (cf. (5.2.3)), we get

$$\|U_H(t)\varphi\|_{X^\sigma} \leq \frac{2n(t) + 3}{te} \|\varphi\|_X < \infty.$$

This proves the first part of assertion (a).

Next, let $t \in (0, \infty)$ and let $\varphi \in X$. According to the expression of $i_n(t, x, v)$ and after an easy change of variable, we get

$$\begin{aligned} \int_0^t \|I_n(s)\varphi\|_{X^\sigma} ds &\leq \int_0^t \int_{D \times \mathbb{R}^n} |\varphi(x + sv, -v)| \sigma(v) e^{-\sigma(v)s} \chi_{(-t-(x,v), t+(x,v))}(s) dx dv ds \\ &\leq \int_0^d \int_{D \times \mathbb{R}^n} |\varphi(y, -v)| \sigma(v) e^{-\sigma(v)s} ds dx dv, \end{aligned}$$

where d stands for the diameter of D . Now, the use of the Fubini theorem gives

$$\int_0^t \|I_n(s)\varphi\|_{X^\sigma} ds \leq \|\varphi\|_X. \quad (5.4.1)$$

Using the same computations as above, yield

$$\int_0^t \|U_0(s)\varphi\|_{X^\sigma} ds \leq \|\varphi\|_X \quad \text{and} \quad \int_0^t \|J_n(s)\varphi\|_{X^\sigma} ds \leq \|\varphi\|_X. \quad (5.4.2)$$

Note that for all $s \in (0, t)$, $n(s) = [s/2\tau_0 + 1/2] + 1$ is less than or equal to $n(t) = [t/2\tau_0 + 1/2] + 1$. According to Equations (5.4.1) and (5.4.2), we have

$$\begin{aligned}
 \int_0^t \|U_H(s)\varphi\|_{\mathcal{X}^\sigma} ds &\leq \int_0^t \|U_0(s)\varphi\|_{\mathcal{X}^\sigma} ds + \int_0^t \sum_{n=0}^{n(s)} \{\|I_n(s)\varphi\|_{\mathcal{X}^\sigma} + \|J_n(s)\varphi\|_{\mathcal{X}^\sigma}\} ds \\
 &\leq \|\varphi\|_{\mathcal{X}} + \int_0^t \sum_{n=0}^{n(t)} \{\|I_n(s)\varphi\|_{\mathcal{X}^\sigma} + \|J_n(s)\varphi\|_{\mathcal{X}^\sigma}\} ds \\
 &\leq \|\varphi\|_{\mathcal{X}} + \sum_{n=0}^{n(t)} \int_0^t \{\|I_n(s)\varphi\|_{\mathcal{X}^\sigma} + \|J_n(s)\varphi\|_{\mathcal{X}^\sigma}\} ds \\
 &\leq (2n(t) + 3) \|\varphi\|_{\mathcal{X}}
 \end{aligned}$$

which prove the second part of assertion (a).

(b) The proof of second statement is based on (5.4.1) and (5.4.2) and follows from [65, Proposition 4.9]. □.

Now, we are in a position to state and to prove the main result of this section.

Theorem 5.4.1 *If Assumptions (A7) and (A8) are satisfied, then the second order remainder term of the Dyson-Phillips expansion, $R_2^H(t)$, is weakly compact on \mathcal{X} .*

Proof. We first note that, according to [65, Theorem 2.6], $R_2^H(t)$ is weakly compact if and only if

$$U_2(t) = \int_0^t ds \int_0^{t-s} U_H(s) K U_H(r) K U_H(t-s-r) dr$$

is a weakly compact operator. Further, since $U_H(t)$ is bounded and the set of weakly compact operator has the convex weakly compactness property [75], then it suffices to show the weakly compactness of $K U_H(t) K$. By Proposition 5.4.2, the latter depends continuously on $K \in \mathcal{L}(L_1^\sigma, L_1)$. Moreover, according to the Assumption (A8), there exists a sequence of finite rank operators which converges, in the operator norm, to K . So, without loss of generality, we can restrict ourselves to a collision operator of rank one which we denote also K , that is

$$(K\varphi)(x, v) = f(v) \int_{\mathbb{R}^n} g(v') \varphi(x, v') dv',$$

where $f \in L_1$ and $g/\sigma \in L^\infty(\mathbb{R}^n, dv)$.

According to the expression of U_H (cf. Equation (5.2.3)), we have

$$K U_H K = K U_0(t) K + \sum_{n=0}^{n(t)} (K I_n K + K J_n K).$$

Since $KU_0(t)K$, KI_nK and KJ_nK have the same structure, we shall restrict ourselves to KI_nK . Note that KI_nK can be written as

$$KI_n(t)K = A_3A_{2,n}(t)A_1$$

with

$$A_1 : X^\sigma \ni \psi \mapsto \int_{\mathbb{R}^n} g(v)\psi(x, -v)dv \in L^1(\mathbf{D}, dx),$$

$$A_{2,n} : L^1(\mathbf{D}, dx) \ni \varphi \mapsto \int_{\mathbb{R}^n} g(v)f(-v)i_n(t, x, v)\varphi(x + tv)dv \in L^1(\mathbf{D}, dx),$$

and

$$A_3 : L^1(\mathbf{D}, dx) \ni \varphi \mapsto f(v)\varphi(x) \in X.$$

Since the operators A_1 and A_3 are bounded, it suffices to establish the weak compactness of $A_{2,n}$. Note that the convexity of \mathbf{D} implies that

$$-t^-(x, v) < t < t^+(x, v) \Leftrightarrow y = x + tv \in \mathbf{D},$$

and therefore, using the change of the variable $y = x + tv$, we infer

$$(A_{2,n}\varphi)(x) = \int_{\mathbf{D}} \varphi(y)K_n(x, y)dy,$$

where

$$K_n(x, y) := \frac{\gamma^{2n+1}}{t^N} g\left(\frac{y-x}{t}\right) f\left(\frac{x-y}{t}\right) \exp\left(-\sigma\left(\frac{y-x}{t}\right)(t + 2t^-(x, \frac{y-x}{t}) + 2n\tau(x, \frac{y-x}{t}))\right).$$

We point out that $A_{2,n}$ is not a convolution operator because of the dependance in x of $\tau(\cdot, \cdot)$. So, $K(x, y)$ may be factorized as $K(x, y) := N(x-y)M_n(x, x-y)$ where

$$N(z) := \frac{\gamma^{2n+1}}{t^N} g\left(\frac{z}{t}\right) f\left(-\frac{z}{t}\right) \exp\left(-\sigma\left(\frac{z}{t}\right)t\right)$$

and

$$M_n(x, z) := \exp\left(-2\sigma\left(\frac{z}{t}\right)(t^-(x, \frac{z}{t}) + n\tau(x, \frac{z}{t}))\right).$$

It is clear that $|K_n(x, y)| \leq |N(x-y)|$, so

$$\|A_{2,n}\varphi\|_{L^1(\mathbf{D}, dx)} \leq \int_{\mathbf{D}} |\varphi(y)| dy \int_{\mathbf{D}} \sup_{y \in \mathbf{D}} |N(x-y)| dx.$$

Moreover, we have the following estimate

$$\begin{aligned} \int_{\mathbb{D}} |N(z)| dz &\leq \int_{\mathbb{D}} \left| g\left(\frac{z}{t}\right) \right| \left| f\left(-\frac{z}{t}\right) \right| \exp\left(-\sigma\left(\frac{y-x}{t}\right)t\right) \frac{dz}{t^n} \\ &\leq \left\| \frac{g}{\sigma(\cdot)} \right\|_{L^\infty(\mathbb{R}^n, d\nu)} e^{-1} t \|f(\cdot)\|_{L^1}. \end{aligned}$$

And so,

$$\|KI_n(t)K\| \leq t \|A_1\| \|A_2\| \left\| \frac{g}{\sigma(\cdot)} \right\|_{L^\infty(\mathbb{R}^n, d\nu)} e^{-1} \|f(\cdot)\|_{L^1}, \quad (5.4.3)$$

Inequality (5.4.3) means that $KI_n(t)K$ depends continuously (for the uniform topology) on $f(\cdot) \in L^1$. So, by approximating $f(\cdot)$ (in the L^1 -norm) by bounded functions, $KI_n(t)K$ is a limit (for operator topology) of integral operators with bounded kernel. Hence, $KI_n(t)K$ is weakly compact on X (cf., [27, Corollary 11, p. 294]). \square

As an immediate consequence of Theorem 5.4.1, we have

Corollaire 5.4.1 *If the hypotheses of Theorem 5.4.1 hold true, then*

$$r_e(V_H(t)) \leq r_e(U_H(t)) \quad \text{for all } t > 0.$$

Spectral analysis and time asymptotic behavior of the solution to a model of age-cycle length structured cell population with inherited properties

6.1 Introduction

In 1974, Lebowitz and Rubinow [51] introduced a mathematical model of age-cycle length structured cell population modeled by the following equation

$$\begin{cases} \frac{\partial \psi}{\partial t}(a, l, t) = S_K \psi(a, l, t) := -\frac{\partial \psi}{\partial a}(a, l, t) - \mu(a, l) \psi(a, l, t) \\ \psi(a, l, 0) = \psi_0(a, l) \end{cases} \quad (6.1.1)$$

where

$$(a, l) \in \{(a, l) : 0 \leq a \leq l, 0 \leq l_1 \leq l \leq l_2\}.$$

According to their viewpoint the cell cycle length l of individual cells is an inherent characteristic determined at birth. The variable l is a heritable and unchanging property of individual cells. The variable a is the age of the individual cells. It is defined so that cells born at $a = 0$ (daughter cells) and divide at $a = l$ (mother cells). The constant l_1 (resp. l_2) denotes the minimum cycle length (resp. maximum cycle length) of cells. The function $\psi(a, l, t)$ is the density of the population with respect to age a and cell cycle length l at time t while $\mu(\cdot, \cdot)$ is the rate of cells mortality or loss due to causes other than division.

The boundary conditions are given by the following abstract equation

$$\psi^0 = K(\psi^l) \tag{6.1.2}$$

where $\psi^0 := \psi(0, l)$, $\psi^l := \psi(l, l)$ and K is a linear bounded operator called the transition operator. Equation (6.1.2) is a reproduction law describing the transition from mother cells of cycle length l to daughter cells of cycle length l which stands for the boundary conditions and it covers, in particular, the usual biological models (cf. [51, 89, 90, 48, 56] and the references therein).

For particular transitions operators K and the hypothesis that maximum cycle length is finite ($l_2 < \infty$), Problem (6.1.1)-(6.1.2) was considered by many authors. We first note that the spectral analysis of operator S_K and the time asymptotic behavior of the Problem (6.1.1)-(6.1.2) were discussed by Webb [89, 90] in the space of continuous functions (which is not the adequate space of the problem). In [48], Latrach and Mokhtar-Kharroubi gave a detailed spectral analysis of S_K in L^p -spaces ($p \in]1, \infty[$) for a compact positive transition operator K . Their analysis use the same technical as that used by the authors in the context of neutrons transport equations [65, 48]. They proved also, via a generation result owing to Batty and Robinson [8], that S_K generates a C_0 -semigroup in L^1 -space for multiplying boundary conditions (i.e. $\|K\| \geq 1$) because, for biological models, at mitosis cells divide. Assuming that $l_1 > 0$, this result was generalized afterwards by Boulanouar [11] to all L^p -spaces ($p \in [1, \infty[$) also discussed the time asymptotic behavior of the solution to the Problem (6.1.1)-(6.1.2). In order to avoid the restriction $l_1 > 0$, Lods and Mokhtar-Kharroubi [56] established a generation result for the case $l_1 = 0$ under the condition that the operator K is small in the neighborhood of $l = 0$ regardless the size of its norm $\|K\|$ as it was done in the context of transport theory [65]. To our knowledge, except a nonlinear stationary version of Problem (6.1.1) (see [1]), it seems that this problem is not yet investigated in the case where the maximum cycle length is infinite, that is $l_2 = \infty$. The purpose of this article is to fill this gap and to allow to the maximum cycle length, l_2 , to be infinite.

The goal of this chapter is twofold: Firstly, we consider the original model of Lebowitz and Rubinow (6.1.1)-(6.1.2) in the case where $l_2 = +\infty$, that is,

$$\begin{cases} \frac{\partial \psi}{\partial t}(a, l, t) = S_K \psi(a, l, t) := -\frac{\partial \psi}{\partial a}(a, l, t) - \mu(a, l)\psi(a, l, t), \\ \psi(a, l, 0) = \psi_0(a, l), \\ K(\psi^l) = \psi^0 \end{cases} \tag{6.1.3}$$

where

$$(a, l) \in \Omega := \{(a, l) : 0 \leq a \leq l, 0 < l_1 \leq l < \infty\}.$$

We start our analysis by investigating the spectrum of S_K for both smooth and partly smooth transition operators (cf. (A9) and (A10)). We prove that $\sigma(S_K) \cap \{\text{Re} \lambda > -\underline{\mu} + \varepsilon, \varepsilon > 0\}$, where $\underline{\mu}$ is defined in Section 6.2, consists of finitely many isolated eigenvalues, say $\{\bar{\lambda}_1, \dots, \bar{\lambda}_n\}$. Next, using a result due to Batty and Robinson [8], we establish a generation result in the space $L^1(\Omega, dadl)$. Note however that this result is open in the spaces $L^p(\Omega, dadl)$ for $p \in (1, +\infty)$. (In fact, for infinite maximum cycle length and $p \in (1, +\infty)$, we did not succeed to establish that functions f

belonging to $L^p(\Omega, dadl)$ such that $\frac{\partial f}{\partial a}(a, l) \in L^p(\Omega, dadl)$ possess traces on the boundaries of the phase space as in the case where $l_2 < +\infty$ (cf. [48, Proposition 2.1].) Finally, we give a spectral decomposition of the solution to Cauchy Problem (6.1.3) for weakly compact transition operator. Further, we show that, for all $\psi_0 \in D(S_K^2)$, this solution satisfies the following estimate

$$\|\psi(t) - \sum_{i=1}^n e^{\lambda_i t} e^{D_j t} P_j \psi_0\| \rightarrow 0, \text{ as } t \rightarrow +\infty. \quad (6.1.4)$$

It should be noticed that the weakness of this result lies in the fact that the estimate above (which gives the time asymptotic behavior for large times) holds true only for initial data belonging to $D(S_K^2)$. It seems that this results can not be improved.

Secondly, we consider the following perturbation of problem (6.1.3)

$$\begin{cases} \frac{\partial \psi}{\partial t}(a, l, t) = A_K \psi(a, l, t) := -\frac{\partial \psi}{\partial a}(a, l, t) - \mu(a, l) \psi(a, l, t) + (B\psi)(a, l, t), \\ \psi(a, l, 0) = \psi_0(a, l), \\ K(\psi') = \psi^0 \end{cases} \quad (6.1.5)$$

where

$$(B\psi)(a, l, t) := \int_{l_1}^{\infty} \kappa(a, l, l') \psi(a, l', t) \chi_{\Omega}(a, l') dl'.$$

Here B denotes a linear operator which describes the transition of cells of cycle length l' to cells of cycle length l and $\kappa(\cdot, \cdot, \cdot)$ is a measurable function from $\mathbb{R}^+ \times [l_1, +\infty) \times [l_1, +\infty)$ into \mathbb{R} . (We recall that a nonlinear boundary value problem derived from (6.1.5) was discussed for finite maximum cycle length in [50] and for infinite maximum cycle length in [1].) As above, we have $(a, l) \in \Omega$ and S_K denotes the operator appearing in Problem (6.1.1)-(6.1.2). Besides a fine spectral analysis of the operator A_K and the weak compactness result for the operator $(\lambda - S_K)^{-1} B$ for smooth operators B (cf. Section 6.5) and $\text{Re} \lambda > -\underline{\mu}$, our main objective here is to discuss the regularity of the solution to Problem (6.1.5) in $L^1(\Omega, dadl)$ (when it exists). By regularity, we mean the possibility that this solution satisfies an estimate like (6.1.4) for all initial data belonging to $D(A_K)$. To do so, we shall adopt the same strategy as in the context of neutrons transport theory in the previous chapter. It consists in showing that the strongly continuous semigroups generated by S_K and A_K possess the same essential type. Making use of Proposition 4.1.1, we prove that some remainder terms of the Dyson-Phillips expansion is compact on $L^1(\Omega, dadl)$. So, according to a standard argument and in contrast to model (6.1.3), we infer that the estimate (6.1.4) is valid for all initial data in $D(A_K)$.

Let ζ be a real number. Throughout this chapter we shall use the notation

$$\mathbb{C}_{\zeta} := \{\lambda \in \mathbb{C} : \text{Re} \lambda > \zeta\}.$$

6.2 Functional framework and preliminary

The aim of this section is to introduce the functional setting of the problem and to draw some preliminary facts related to the problem. Let X be the following Banach space

$$X := L^1(\Omega, dadl),$$

where $\Omega := \{(a, l) : 0 \leq a \leq l, 0 < l_1 \leq l < +\infty\}$ and denote by

$$L^1(\Gamma_i, dl) \quad i = 1, 2,$$

the boundary spaces where

$$\Gamma_1 := \{(0, l) : l \in [l_1, +\infty)\} \quad \text{and} \quad \Gamma_2 := \{(l, l) : l \in [l_1, +\infty)\}.$$

We define the partial Sobolev space W by

$$W = \left\{ \psi \in X \text{ such that } \frac{\partial \psi}{\partial a} \in X \right\}.$$

Let us now recall the following trace lemma established in [1].

Lemma 6.2.1 *Let $\psi \in W$. If $\psi|_{\Gamma_1} \in L^1(\Gamma_1, dl)$, then $\psi|_{\Gamma_2} \in L^1(\Gamma_2, dl)$ and vice versa.*

It follows from the above lemma that we can define the space

$$\widetilde{W} := \left\{ \psi \in W \text{ such that } \psi|_{\Gamma_1} \in L^1(\Gamma_1, dl) \right\} = \left\{ \psi \in W \text{ such that } \psi|_{\Gamma_2} \in L^1(\Gamma_2, dl) \right\}.$$

So, each function belonging to \widetilde{W} possesses traces in the boundary spaces $L^1(\Gamma_1, dl)$ and $L^1(\Gamma_2, dl)$.

For the sake of simplicity, we shall identify $L^1(\Gamma_1, dl)$ and $L^1(\Gamma_2, dl)$ with the space

$$Y := L^1([l_1, +\infty); dl).$$

Definition 6.2.1 *We say that a linear operator $K \in \mathcal{L}(Y)$ is a transition operator if, for each $\psi \in \widetilde{W}$, we have*

$$K(\psi^l) = \psi^0,$$

where $\psi^l(l) := \psi(l, l)$ and $\psi^0(l) := \psi(0, l)$. □

Note that, in applications, the transition operator is positive (in the lattice sense). So, throughout this chapter, K will be assumed to be positive without any condition on the size of its norm.

Let $K \in \mathcal{L}(Y)$ be a transition operator. We define the operator S_K by

$$\begin{cases} S_K : D(S_K) \subseteq X \longrightarrow X \\ \psi \longmapsto S_K \psi(a, l, t) = -\frac{\partial \psi}{\partial a}(a, l, t) - \mu(a, l) \psi(a, l, t) \end{cases}$$

with domain

$$D(S_K) = \{\psi \in \widetilde{W} \text{ such that } \psi^0 = K(\psi^l)\},$$

where $\mu(\cdot, \cdot)$ is a nonnegative function belonging to $L^\infty(\Omega)$. We denote by $\underline{\mu}$ (resp. $\bar{\mu}$) the real number

$$\underline{\mu} := \text{ess - inf}_{(a,l) \in \Omega} \mu(a, l) \quad (\text{resp. } \bar{\mu} := \text{ess - sup}_{(a,l) \in \Omega} \mu(a, l)).$$

Remark 6.2.1 It is not difficult to check that the operator S_0 (corresponding to $K = 0$) is a closed densely defined linear operator on X and, for $\lambda \in \mathbb{C}_{-\underline{\mu}}$, its resolvent is given by

$$(\lambda - S_0)^{-1} \varphi(a, l) := \int_0^a e^{-\int_s^a (\lambda + \mu(\tau, l)) d\tau} \varphi(s, l) ds.$$

Hence $\mathbb{C}_{-\underline{\mu}} \subseteq \rho(S_0)$. □

Let $\varphi \in X$ and consider the resolvent equation of the operator S_K ,

$$(\lambda - S_K)\psi = \varphi \tag{6.2.1}$$

where λ is a complex number and the unknown ψ must be sought in $D(S_K)$.

Let $\lambda \in \mathbb{C}_{-\underline{\mu}}$. The solution of Equation (6.2.1) is formally given by

$$\psi(a, l) = \psi(0, l) e^{-\int_0^a (\lambda + \mu(s, l)) ds} + \int_0^a e^{-\int_s^a (\lambda + \mu(\tau, l)) d\tau} \varphi(s, l) ds. \tag{6.2.2}$$

For $a = l$, we get

$$\psi(l, l) = \psi(0, l) e^{-\int_0^l (\lambda + \mu(s, l)) ds} + \int_0^l e^{-\int_s^l (\lambda + \mu(\tau, l)) d\tau} \varphi(s, l) ds. \tag{6.2.3}$$

To compute the resolvent of A_K , we introduce the operators Θ_λ , Λ_λ and Ξ_λ where

$$Y \ni u \mapsto (\Theta_\lambda u)(l, l) := u(0, l) e^{-\int_0^l (\lambda + \mu(s, l)) ds} \in Y,$$

$$Y \ni u \mapsto (\Lambda_\lambda u)(a, l) := u(0, l) e^{-\int_0^a (\lambda + \mu(s, l)) ds} \in X,$$

and

$$X \ni \varphi \mapsto (\Xi_\lambda \varphi)(l, l) := \int_0^l e^{-\int_s^l (\lambda + \mu(\tau, l)) d\tau} \varphi(s, l) ds \in Y.$$

Remark 6.2.2 (a) For all $\lambda \in \mathbb{C}_{-\underline{\mu}}$, one checks readily that these operators are bounded on their respective spaces. Further, the norms of Λ_λ and $(\lambda - S_0)^{-1}$ are bounded above by $(\text{Re} \lambda + \underline{\mu})^{-1}$ while the norms of the operators Θ_λ and Ξ_λ are bounded above by 1.

(b) It is clear that if $\lambda \in \mathbb{R}$ such that $\lambda > -\underline{\mu}$, these operators are positive with respect to the natural positive cones of their respective spaces. □

Using the operators above and the fact that ψ must satisfy the boundary conditions, Equation (6.2.3) may be written abstractly in the form

$$\psi^l = \Theta_\lambda K \psi^l + \Xi \varphi. \quad (6.2.4)$$

Similarly, Equation (6.2.2) becomes

$$\psi = \Lambda_\lambda K \psi^l + (\lambda - S_0)^{-1} \varphi. \quad (6.2.5)$$

As already mentioned in [48, Theorem 4.1], solving Equation (6.2.1) is equivalent to solve the boundary Fredholm Equation (6.2.4). Further, a sufficient condition for solving (6.2.4) is $r_\sigma(\Theta_\lambda K) < 1$. Hence, we have the following result.

Proposition 6.2.1 *Assume that there exists a real η such that*

$$r_\sigma(\Theta_\lambda K) < 1 \quad \text{for all } \lambda \in \mathbb{C}_\eta. \quad (6.2.6)$$

Then $\mathbb{C}_\eta \subseteq \rho(S_K)$ and, for all $\lambda \in \mathbb{C}_\eta$, we have

$$(\lambda - S_K)^{-1} = \sum_{n \geq 0} \Lambda_\lambda K (\Theta_\lambda K)^n \Xi_\lambda + (\lambda - S_0)^{-1}. \quad (6.2.7)$$

Proof. It is clear that, for all $\lambda \in \mathbb{C}_\eta$, the operator $(I - \Theta_\lambda K)$ is invertible and therefore the solution of (6.2.4) is given by $\psi^l = \sum_{n \geq 0} (\Theta_\lambda K)^n \Xi_\lambda \varphi$. Now, substituting ψ^l into (6.2.5) we get (6.2.7). \square

Note that, condition (6.2.6) is optimal for S_K to have a positive resolvent. We point out that (6.2.6) is independent of the size of l_1 . However, if $l_1 = 0$, then condition (6.2.6) is satisfied for any operator K "small in the neighborhood" of 0 (see [56, Section 3]). When $l_1 > 0$, any bounded operator K satisfies (6.2.6). Indeed, if $l_1 > 0$, then, for all real number λ satisfying $\lambda > -\underline{\mu}$, we have $\Theta_\lambda \leq e^{-(\lambda + \underline{\mu}) l_1} Id$ (where Id denotes the identity operator of the space Y). Using the positivity of K , for all $n \geq 1$, we get $(\Theta_\lambda K)^n \leq e^{-n(\lambda + \underline{\mu}) l_1} K^n$ and therefore

$$\|(\Theta_\lambda K)^n\|^{\frac{1}{n}} \leq e^{-(\lambda + \underline{\mu}) l_1} \|K^n\|^{\frac{1}{n}}.$$

This yields that

$$r_\sigma(\Theta_\lambda K) \leq e^{-(\lambda + \underline{\mu}) l_1} r_\sigma(K). \quad (6.2.8)$$

So, a sufficient condition guaranteeing that $r_\sigma(\Theta_\lambda K) < 1$ is that $e^{-(\lambda + \underline{\mu}) l_1} r_\sigma(K) < 1$. Let λ_0 be the real number defined by

$$\lambda_0 = \begin{cases} -\underline{\mu} & \text{if } r_\sigma(K) \leq 1, \\ -\underline{\mu} + \frac{1}{l_1} \ln(r_\sigma(K)) & \text{if } r_\sigma(K) > 1. \end{cases} \quad (6.2.9)$$

Remark 6.2.3 It is clear from the discussion above that, for all λ in \mathbb{C}_{λ_0} , condition (6.2.6) is satisfied and, for such λ , $(\lambda - S_K)^{-1}$ is given by (6.2.7). So, we have the inclusion $\mathbb{C}_{\lambda_0} \subset \rho(S_K)$. \square

6.3 Spectral analysis of S_K

Our objective here is to describe the so-called asymptotic spectrum of the operator S_K , that is, the set defined by

$$\sigma_{as}(S_K) := \sigma(S_K) \cap \mathbb{C}_{-\underline{\mu}}.$$

The motivation of this section relies on the fact that, as we will see below, if $r_\sigma(K) > 1$, then K acts as an additive perturbation. This produces some changes in the picture of the spectrum of S_K .

Remarque 6.3.1 It should be noticed that, if $r_\sigma(K) \leq 1$, then the spectral bound of the operator S_K is $-\underline{\mu}$. So, the asymptotic spectrum of S_K is empty. \square

6.3.1 Smooth transition operators

Before going further, we shall establish the following lemma required below.

Lemma 6.3.1 *Let $\lambda \in \mathbb{C}_{-\underline{\mu}}$. If K is a weakly compact transition operator, then*

$$\lim_{|\operatorname{Im}\lambda| \rightarrow \infty} r_\sigma(\Theta_\lambda K) = 0.$$

Proof. This result is clear if $\operatorname{Re}\lambda > \lambda_0$.

Assume that λ belongs to the strip $\Lambda_0 = \{\lambda \in \mathbb{C} \text{ such that } -\underline{\mu} \leq \operatorname{Re}\lambda \leq \lambda_0\}$. Since the space Y has the Dunford-Pettis property, it follows from Theorem 2.5.2 that $\mathcal{H}_\lambda := K\Theta_\lambda K$ is compact on Y . Now, arguing as in the proof of Lemma 6.2 in [48] where the operator K is replaced by \mathcal{H}_λ (in fact, the cycle length does not play any role), we derive that

$$\lim_{|\operatorname{Im}\lambda| \rightarrow +\infty} \|(\Theta_\lambda \mathcal{H}_\lambda)^2\| = \lim_{|\operatorname{Im}\lambda| \rightarrow +\infty} \|(\Theta_\lambda K)^4\| = 0 \quad \text{uniformly on } \Lambda_0. \quad (6.3.1)$$

Next, using (6.3.1) together with the estimate $r_\sigma(\Theta_\lambda K) \leq \|(\Theta_\lambda K)^n\|^{\frac{1}{n}}$, $n = 1, 2, \dots$, we get the desired result. \square

Remark 6.3.1 This lemma was already established in [48, Lemma 6.2] in L^p -spaces, $1 \leq p < +\infty$, under the hypotheses that K is compact and the maximum cycle length l_2 is finite. \square

Let us now introduce the following assumption

$$(A9) \quad \begin{cases} K \text{ is positive (in the lattice sense)} \\ \text{and some power of } K \text{ is compact.} \end{cases}$$

Theorem 6.3.1 *Assume that the hypothesis (A9) holds true. Then the following assertions are satisfied:*

- (1) $\sigma_{as}(S_K)$ consists of, at most, isolated eigenvalues with finite algebraic multiplicities.
- (2) If $\lim_{\lambda \rightarrow -\underline{\mu}} r_\sigma(\Theta_\lambda K) > 1$, then there exists a real leading eigenvalue $\bar{\lambda}$ of S_K .
- (3) If $\sigma_{as}(S_K) \neq \emptyset$, then $r_\sigma(K) > 1$ and

$$-\underline{\mu} \leq \bar{\lambda} \leq -\underline{\mu} + \frac{1}{l_1} \ln(r_\sigma(K)).$$

- (4) If K is weakly compact, then $\sigma_{as}(S_K)$ is bounded. For every $\varepsilon > 0$, the set

$$\sigma_{as}(S_K) \cap \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > -\underline{\mu} + \varepsilon\} \text{ is finite.}$$

Remark 6.3.2 It is obvious that $\lambda \in \sigma_p(S_K)$ if and only if $1 \in \sigma_p(\Theta_\lambda K)$. □

Proof of Theorem 6.3.1 (1) Let us first note that, for every real number λ satisfying $\lambda > -\underline{\mu}$, we have $\Theta_\lambda \leq e^{-(\lambda+\underline{\mu})l_1} Id$ and therefore

$$\Theta_\lambda K \leq e^{-(\lambda+\underline{\mu})l_1} K, \quad \forall \lambda > -\underline{\mu}. \tag{6.3.2}$$

Note that an immediate consequence of the estimate (6.3.2), is that $\Theta_\lambda K \rightarrow 0$, in the strong operator topology, as $\lambda \rightarrow \infty$. Moreover, by the hypothesis (A9) there exists an integer $N \in \mathbb{N} \setminus \{0, 1\}$ such that K^N is compact. Now, applying Theorem 2.4.1 together with the fact that $(\Theta_\lambda K)^N \leq K^N$ (cf. Eq.(6.3.2)), we have the compactness of $(\Theta_\lambda K)^N$ for all $\lambda > -\underline{\mu}$. Hence, according to [35, Lemma 3.7, p. 151], we infer that $(\Theta_\lambda K)^{N+1} \rightarrow 0$ in the uniform operator topology as $\lambda \rightarrow \infty$. Now arguing as in the last two line of the proof of Lemma 6.3.1, we infer that

$$r_\sigma((\Theta_\lambda K)^{N+1}) \rightarrow 0 \quad \lambda \rightarrow \infty. \tag{6.3.3}$$

This implies that $1 \notin \sigma((\Theta_\lambda K)^{N+1})$. On the other hand, $\mathbb{C}_{-\underline{\mu}}$ is a connected subset of \mathbb{C} and the mapping $\mathbb{C}_{-\underline{\mu}} \ni \lambda \mapsto (\Theta_\lambda K)^{N+1}$ is a holomorphic compact operator valued function. Now the use of (6.3.3) together with Theorem 2.5.1 implies that $(I - (\Theta_\lambda K)^{N+1})^{-1}$ is a holomorphic function on $\mathbb{C}_{-\underline{\mu}}$ except for a discrete set S where, for each point of S , $(I - (\Theta_\lambda K)^{N+1})^{-1}$ has a poles and the coefficients of their principal parts have finite rank. It follows from the following equality

$$I - (\Theta_\lambda K)^{N+1} = (I - \Theta_\lambda K)(I + \Theta_\lambda K + \dots + (\Theta_\lambda K)^N)$$

that

$$(I - \Theta_\lambda K)^{-1} = (I + \Theta_\lambda K + \dots + (\Theta_\lambda K)^N)(I - (\Theta_\lambda K)^{N+1})^{-1}.$$

Further, the operator valued function $\mathbb{C}_{-\underline{\mu}} \ni \lambda \mapsto (I - \Theta_\lambda K)^{-1}$ is holomorphic on $\mathbb{C}_{-\underline{\mu}}$ except for the discrete set S . Hence, if $\lambda \notin S$, then $I - \Theta_\lambda K$ is invertible and so $(\lambda - S_K)^{-1}$ exists. In fact, the operator valued function $\mathbb{C}_{-\underline{\mu}} \ni \lambda \mapsto (\lambda - S_K)^{-1}$ is a degenerate-meromorphic function on $\mathbb{C}_{-\underline{\mu}}$.

(2) To prove the second item, we shall proceed in two steps.

(i) We first give a characterization of the leading eigenvalue (when it exists) of S_K . To this end, let $\lambda_0 \in \sigma_{as}(S_K)$. Using (1) and Remark 6.3.2, there exists $u \neq 0$ such that $\Theta_{\lambda_0} K u = u$. Hence $(\Theta_{\lambda_0} K)^N u = u$ and so $|u| \leq |(\Theta_{\lambda_0} K)^N u| \leq (\Theta_{\beta_0} K)^N |u|$ where $\beta_0 = \operatorname{Re} \lambda_0$. So, we infer that

$$r_\sigma((\Theta_{\beta_0} K)^N) \geq 1. \quad (6.3.4)$$

Further, according to Lemma 2.5.1, $r_\sigma((\Theta_\beta K)^N)$ is a continuous strictly decreasing function of β on $(-\underline{\mu}, \infty)$. So, by the spectral mapping theorem (see, for example, [27, p. 563]), there exists $\alpha(\beta) \in \sigma(\Theta_\beta K)$ such that $(\alpha(\beta))^N = r_\sigma((\Theta_\beta K)^N)$. It is obvious that $\alpha(\beta)$ is also a continuous strictly decreasing function of β on $(-\underline{\mu}, \infty)$. On the other hand, Eq. (6.3.4) (resp. (6.3.3)) shows that $\alpha(\beta_0) \geq 1$ (resp. $\lim_{\beta \rightarrow \infty} \alpha(\beta) = 0$). Hence, there exists $\lambda \geq \beta_0$ such that $\alpha(\lambda) = 1$, i.e. $\lambda = \bar{\lambda}$ (the leading eigenvalue of S_K). Accordingly, if $\sigma_{as}(S_K) \neq \emptyset$, then S_K admits a leading eigenvalue $\bar{\lambda}$ characterized by

$$r_\sigma(\Theta_{\bar{\lambda}} K) = 1. \quad (6.3.5)$$

(ii) Next, if $\lim_{\lambda \rightarrow -\underline{\mu}} r_\sigma(\Theta_\lambda K) > 1$, then, using (6.3.3) and the characterization of the leading eigenvalue of S_K established in (i), we get the desired result.

(3) For $\lambda > -\underline{\mu}$, Equation (6.3.2) shows that

$$r_\sigma(\Theta_\lambda K) \leq e^{-(\lambda + \underline{\mu})l_1} r_\sigma(K).$$

Suppose now that $\bar{\lambda}$ exists. Using the last equation together with (6.3.5) we obtain $1 \leq e^{-(\bar{\lambda} + \underline{\mu})l_1} r_\sigma(K)$. This shows that $r_\sigma(K) > 1$ and

$$-\underline{\mu} \leq \bar{\lambda} \leq -\underline{\mu} + \frac{1}{l_1} \ln(r_\sigma(K)). \quad (6.3.6)$$

(4) By definition of the real λ_0 , we know that $\sigma(S_K) \cap \mathbb{C}_{\lambda_0} = \emptyset$. Thus, it suffices to restrict ourselves to the strip

$$\Lambda_0 := \{\lambda \in \mathbb{C} : -\underline{\mu} < \operatorname{Re} \lambda \leq \lambda_0\}.$$

Let $\lambda \in \Lambda_0$. Since K is weakly compact, according to Lemma 6.3.1, we have that $\lim_{|\operatorname{Im} \lambda| \rightarrow \infty} r_\sigma(\Theta_\lambda K) = 0$. Therefore, there exists $M > 0$ such that, for all $\lambda \in \Lambda_0$ satisfying $|\operatorname{Im} \lambda| > M$,

we have $r_\sigma(\Theta_\lambda K) < 1$. This shows that $\sigma_{as}(S_K)$ is a bounded subset of $\mathbb{C}_{-\underline{\mu}}$. So for any $\varepsilon > 0$ satisfying $-\underline{\mu} + \varepsilon < \lambda_0$, $\sigma(S_K) \cap \{\lambda \in \mathbb{C} : -\underline{\mu} + \varepsilon \leq \operatorname{Re}\lambda \leq \lambda_0\}$ is confined in a compact subset of Λ_0 . Accordingly $\sigma(S_K) \cap \{\lambda \in \mathbb{C} : -\underline{\mu} + \varepsilon \leq \operatorname{Re}\lambda \leq \lambda_0\}$ is at most a finite subset of $\mathbb{C}_{-\underline{\mu}}$. \square

6.3.2 Partly smooth transition operators

Now, we turn our attention to the case of partly smooth transition operators. So, instead of (A9) and shall use the hypothesis

$$(A10) \quad \begin{cases} \mathcal{K} = K + \widetilde{K} \text{ with } 0 \leq \widetilde{K}, \\ \text{and } \widetilde{K} \text{ is weakly compact.} \end{cases}$$

The transition operator \mathcal{K} is a positive perturbation of K , i.e. $\widetilde{K} = \mathcal{K} - K$ is a positive operator of $\mathcal{L}(Y)$. According to Equation (6.2.9), there exists a real number $\lambda_0 < +\infty$ depending on $r_\sigma(K)$, such that

$$r_\sigma(\Theta_\lambda K) < 1 \quad \text{for all } \lambda > \lambda_0. \quad (6.3.7)$$

Theorem 6.3.2 *Assume that the condition (A10) holds true. Then the following assertions are satisfied:*

- (1) $\sigma(S_{\mathcal{K}}) \cap \mathbb{C}_{\lambda_0}$ consists of, at most, isolated eigenvalues with finite algebraic multiplicities.
- (2) If $\sigma(S_{\mathcal{K}}) \cap \mathbb{C}_{\lambda_0} \neq \emptyset$, then there exists a leading eigenvalue.
- (3) If $r_\sigma(\Theta_\lambda \widetilde{K}) > 1$, then $\sigma(S_{\mathcal{K}}) \cap \mathbb{C}_{\lambda_0} \neq \emptyset$.

Proof. (1) Let λ be a real number such that $\lambda > \lambda_0$ and let $\varphi \in X$. Consider the problem $(\lambda - S_{\mathcal{K}})\psi = \varphi$, where the unknown function ψ must be sought in the domain of $S_{\mathcal{K}}$. The solution of this problem is given formally by (6.2.2) and, for $a = l$, we get (cf. Eq. (6.2.3))

$$\psi|_{\Gamma_2} = \Theta_\lambda K \psi|_{\Gamma_2} + \Theta_\lambda \widetilde{K} \psi|_{\Gamma_2} + \Xi_\lambda \varphi.$$

According to (6.3.7), the last equation writes as

$$\psi|_{\Gamma_2} = \mathcal{S}_\lambda \psi|_{\Gamma_2} + L_\lambda \varphi$$

where $\mathcal{S}_\lambda := (I - \Theta_\lambda K)^{-1} \Theta_\lambda \widetilde{K}$ and $L_\lambda = (I - \Theta_\lambda K)^{-1} \Xi_\lambda$.

Let $\lambda_1 > \lambda_0$. Since, for all $\lambda > \lambda_1$, the estimate $(I - \Theta_\lambda K)^{-1} \leq (I - \Theta_{\lambda_1} K)^{-1}$ holds true, we infer that

$$(\mathcal{S}_\lambda)^3 = (I - \Theta_\lambda K)^{-1} \Theta_\lambda \widetilde{K} (\mathcal{S}_\lambda)^2 \leq (I - \Theta_{\lambda_1} K)^{-1} \Theta_\lambda \widetilde{K} (\mathcal{S}_\lambda)^2 \quad \forall \lambda > \lambda_1.$$

By the assumption (A10), the operator \widetilde{K} is weakly compact and therefore \mathcal{S}_λ is weakly compact too. So, using Theorem 2.4.1, we deduce that the operator $(\mathcal{S}_\lambda)^2$ is compact. Moreover, it is clear that $\Theta_\lambda \widetilde{K} \rightarrow 0$ strongly as $\lambda \rightarrow \infty$. Now, according to [35, Lemma 3.7, p. 151] we conclude that

$$\|(\mathcal{S}_\lambda)^3\| \leq \|(I - \Theta_{\lambda_1} K)^{-1}\| \|\Theta_\lambda \widetilde{K} (\mathcal{S}_\lambda)^2\| \rightarrow 0 \quad \text{as } \lambda \rightarrow \infty,$$

that is $(\mathcal{S}_\lambda)^3 \rightarrow 0$ in the uniform operator topology. Proceeding as in the proof of the first assertion of Theorem 6.3.1 we deduce that $\lim_{\lambda \rightarrow \infty} r_\sigma(\mathcal{S}_\lambda) = 0$. Now, the remainder of the proof is similar to that of the last part of the proof of Theorem 6.3.1 (1) and so it is omitted.

(2) The proof of (2) is similar to that of Theorem 6.3.1 (2) (Step (ii)), so it is omitted.

(3) Let λ be a real number such that $\lambda > \lambda_0$. According to (6.3.7) and the positivity of the operators K , \widetilde{K} and Θ_λ , one can write

$$\mathcal{S}_\lambda = \sum_{n \geq 0} (\Theta_\lambda K)^n \Theta_\lambda \widetilde{K} \geq \Theta_\lambda \widetilde{K}$$

and therefore

$$r_\sigma(\mathcal{S}_\lambda) \geq r_\sigma(\Theta_\lambda \widetilde{K}). \quad (6.3.8)$$

Next, using Lemma 2.5.1 and (6.3.8) together with the hypothesis (A10), we conclude that

$$\lim_{\lambda \rightarrow \lambda_0} r_\sigma(\mathcal{S}_\lambda) \geq r_\sigma(\Theta_\lambda \widetilde{K}) > 1.$$

Accordingly, there exists $\bar{\lambda} > \lambda_0$ such that $r_\sigma(\mathcal{S}_{\bar{\lambda}}) = 1$. This proves that $\bar{\lambda}$ is the leading eigenvalue of $S_{\mathcal{K}}$ and completes the proof. \square

6.4 Cauchy problem (6.1.3)

The objective of this section is to establish a generation result and to discuss the properties of the solution to the Cauchy problem governed by the unbounded operator S_K . It is clear that S_0 generates a C_0 -semigroup, $(T_0(t))_{t \geq 0}$ given by

$$(T_0(t)\varphi)(a, l) = \begin{cases} \exp\left(-\int_{a-t}^a \mu(\tau, l) d\tau\right) \varphi(a-t, l) & \text{if } t < a, \\ 0 & \text{otherwise.} \end{cases}$$

The case of dissipative transition operator (i.e. $\|K\| \leq 1$) follows from a direct application of the Lumer-Phillips theorem. So, we consider only the case of multiplying transition operators, that is,

$$\|Ku\| \geq \|u\| \quad \forall u \in Y. \quad (6.4.1)$$

Here we will proceed exactly as in [48]. In fact, we use a generation result due to Batty and Robinson [8] for positive resolvent operators (see, for example, [48, Theorem 5.1]). Now, we recall that a linear operator A on a Banach lattice \mathcal{Z} is called a resolvent positive operator, if there exists $\tau \in \mathbb{R}$ such that $(\tau, +\infty) \subset \rho(A)$ and, for all $\lambda > \tau$, $(\lambda - A)^{-1}$ is a positive operator in the lattice sense.

Remark 6.4.1 If K is a positive transition operator, then S_K is a resolvent positive operator on X . Indeed, according to Proposition 6.2.1 and Remark 6.2.3, for all λ such that $\lambda > \lambda_0$, we have

$$(\lambda - S_K)^{-1} = \Lambda_\lambda K(I - \Theta_\lambda K)^{-1} \Xi_\lambda + (\lambda - S_0)^{-1}.$$

Next, using Remark 6.2.2(b), we infer that

$$(\lambda - S_K)^{-1} \geq (\lambda - S_0)^{-1} \geq 0.$$

□

Theorem 6.4.1 *Let K be a positive transition operator satisfying (6.4.1). Then S_K generates a positive strongly continuous semigroup $(T_K(t))_{t \geq 0}$ on X .*

Theorem 6.4.1 was already established for finite cycle length ($l_2 < +\infty$) in [48].

Proof. The proof is similar to that of Theorem 5.2 in [48]. In fact, when we replace l_2 by $+\infty$, the proof works again. So, it is omitted. □

The rest of the section is devoted to discuss the properties of the solution of Problem (6.1.3).

Lemma 6.4.1 *Suppose that the transition operator K is weakly compact. Then the following assertions hold true*

- (1) $\|(\lambda - S_K)^{-1}\|$ converges to zero as $\operatorname{Re} \lambda \rightarrow \infty$,
- (2) if $\eta > -\underline{\mu}$, then $\|(\lambda - S_K)^{-1}\|$ is uniformly bounded in \mathbb{C}_η for large $|\operatorname{Im} \lambda|$.

Proof. Since S_K generates a strongly continuous semigroup on X (cf. Theorem 6.4.1), the first item is a consequence of the semigroup theory. Further, according to Lemma 6.3.1 the operator $(I - \Theta_\lambda K)^{-1}$ exists for all $\lambda \in \mathbb{C}_{-\underline{\mu}}$ with $|\operatorname{Im} \lambda|$ sufficiently large. Thus, arguing as in the proof of Lemma 6.1(3) in [48], we get the desired result. □

Now we are in a position to establish the following result.

Theorem 6.4.2 *Let $\eta > 0$. Assume that the transition operator K is weakly compact and $\sigma_{as}(S_K)$ is nonempty. According to Theorem 6.3.1, $\sigma(A_K) \cap \{\operatorname{Re} \lambda \geq -\underline{\mu} + \eta\}$ consists of, at most, a finite number of isolated eigenvalues with finite algebraic multiplicities, say $\{\lambda_1, \dots, \lambda_n\}$. Let*

$$\delta_1 = \sup\{\operatorname{Re} \lambda, \lambda \in \sigma(S_K) \text{ and } \operatorname{Re} \lambda < -\underline{\mu} + \eta\} \quad \text{and} \quad \delta_2 = \min\{\operatorname{Re} \lambda_j, 1 \leq j \leq n\}.$$

Obviously, we have $\delta_1 < \delta_2$. Let δ^* be a real number satisfying $\delta_1 < \delta^* < \delta_2$. Then for $\psi_0 \in D(S_K)$

$$R(t) = \lim_{\gamma \rightarrow +\infty} \frac{1}{2i\pi} \int_{\delta^* - i\gamma}^{\delta^* + i\gamma} e^{\lambda t} (\lambda - S_K)^{-1} \psi_0 d\lambda \quad \text{exists}$$

and the solution $\psi(t)$ of the Equation (6.1.3) is given by

$$\psi(t) = R(t) + \sum_{j=1}^n e^{\lambda_j t} e^{tD_j} P_j \psi_0,$$

where P_j and D_j are, respectively, the spectral projection and the nilpotent operator associated with the eigenvalue λ_j . Further, if $\psi_0 \in D(S_K^2)$ and $\delta^* \neq 0$, then

$$R(t) = \frac{e^{\delta^* t}}{2\pi} \int_{\mathbb{R}} \frac{(\delta^* t + iv - S_K)^{-1} S_K^2 \psi_0}{(\delta^* + iv)^2} e^{ivt} dv + \alpha(\delta^*)(\psi_0 + tS_K \psi_0),$$

where

$$\alpha(\delta^*) = \begin{cases} 1 & \text{if } \delta^* > 0, \\ 0 & \text{if } \delta^* < 0. \end{cases}$$

In particular, if $\delta_2 < 0$, then we have

$$\|R(t)\| \leq \tau(\delta^*) \|S_K^2 \psi_0\| e^{\delta^* t} \quad (\delta_1 < \delta^* < 0), \quad (6.4.2)$$

where

$$\tau(\delta^*) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\|(\delta^* t + iv - S_K)^{-1}\|}{|(\delta^* + iv)|^2} dv.$$

Proof. The proof of this theorem is similar to that of Theorem 6.1 in [48], so it is omitted. \square

Remark 6.4.2 We point out that the drawback of this results lies in the fact that the estimate (6.4.2) is valid only for initial data ψ_0 belonging to $D(S_K^2)$. Unfortunately, it seems that it can not be improved. \square

6.5 A perturbation of the model (6.1.1)

The remainder of the chapter deals with the spectral analysis and the regularity properties of the solution to the Cauchy problem governed by the following operator

$$\begin{aligned} A_K \psi(a, l) &= \frac{\partial \psi}{\partial a}(a, l, t) + \mu(a, l) \psi(a, l, t) + \int_{l_1}^{+\infty} \kappa(a, l, l') \psi(a, l', t) \chi_{\Omega}(a, l') dl' \\ &= S_K \psi + B\psi \end{aligned} \quad (6.5.1)$$

where B is the partial integral part of A_K , $\kappa(\cdot, \cdot, \cdot)$ is a measurable function on $\mathbb{R}^+ \times [l_1, \infty) \times [l_1, \infty)$ and $\chi_\Omega(\cdot, \cdot)$ denotes the characteristic function of the set Ω .

Let \widetilde{X} be the Banach space defined by

$$\widetilde{X} := L^1([0, \infty) \times [l_1, \infty), dadl).$$

Definition 6.5.1 We say that $\kappa(\cdot, \cdot, \cdot)$ is an admissible kernel if the linear operator \mathbf{B} defined by

$$\begin{cases} \mathbf{B} : \widetilde{X} \rightarrow \widetilde{X}, \\ \varphi \mapsto (\mathbf{B}\varphi)(a, l) = \int_{l_1}^{\infty} \kappa(a, l, l')\varphi(a, l')dl' \end{cases}$$

is bounded. □

A simple example of admissible kernel is given by

$$\kappa(a, l, l') = \theta(a)\tau(l)v(l')$$

where $\theta \in L^\infty([0, \infty), da)$, $\tau \in L^1([l_1, \infty), dl)$ and $v \in L^\infty([l_1, \infty), dl)$. Indeed, let $\psi \in \widetilde{X}$, easy calculations using Fubini's theorem show that

$$\|\mathbf{B}\psi\|_{\widetilde{X}} \leq \|\theta\|_\infty \|\tau\|_{L^1} \|v\|_\infty \|\psi\|_{\widetilde{X}}.$$

This proves that the operator \mathbf{B} is bounded and

$$\|\mathbf{B}\|_{\mathcal{L}(\widetilde{X})} \leq \|\theta\|_\infty \|\tau\|_{L^1} \|v\|_\infty.$$

Remark 6.5.1 In the remainder of this paper the kernel $\kappa(\cdot, \cdot, \cdot)$ will be always assumed to be admissible. □

Note that \mathbf{B} is local with respect to the space variable $a \in [0, +\infty)$. So, it may be regarded as an operator valued mapping from $[0, +\infty)$ into $\mathcal{L}(Y)$, that is,

$$a \in [0, +\infty) \mapsto \mathbf{B}(a) \in \mathcal{L}(Y),$$

where

$$\mathbf{B}(a) : Y \ni \varphi \mapsto \mathbf{B}(a)\varphi = \int_{l_1}^{+\infty} \kappa(a, l, l')\varphi(l')dl'.$$

We assume that the function $\mathbf{B}(\cdot)$ is strongly measurable, i.e. for every $u \in Y$, the function

$$[0, \infty) \ni a \mapsto \mathbf{B}(a)u \in \mathcal{L}(Y)$$

is measurable and the map $[0, \infty) \ni a \rightarrow \|\mathbf{B}(a)\|_{\mathcal{L}(Y)}$ is essentially bounded.

Definition 6.5.2 We say that B is a smooth operator if, for almost all $a \in [0, +\infty)$, the operator

$$Y \ni \varphi \mapsto (\mathbf{B}(a)\varphi)(l) = \int_{l_1}^{+\infty} \kappa(a, l, l')\varphi(l')dl' \in Y$$

is weakly compact and the family of such operators on Y indexed by $a \in [0, +\infty)$ is collectively weakly compact. \square

The class of smooth operators enjoys the following useful approximation property.

Lemma 6.5.1 *If B is a smooth operator, then it can be approximated, in the uniform operator topology, by a sequence $(B_n)_{n \in \mathbb{N}}$ of linear operators such that, for each $n \in \mathbb{N}$, B_n is given by*

$$(B_n\varphi)(a, l) := \int_{l_1}^{\infty} \left(\sum_{n \in I} f_n(l) \right) \varphi(a, l') \chi_{\Omega}(a, l') dl'$$

where $f_n(\cdot) \in L^1([l_1, \infty), dl)$ and I is finite.

Proof. Note that according to [55, Theorem 2.4] (2), \mathbf{B} can be approximated, in the uniform operator topology, by a sequence $(\mathbf{B}_n)_n$ of linear operator with kernels of the form $\sum_{n \in I} f_n(l)$, where $f_n(\cdot) \in L^1([l_1, \infty), dl)$ and I is finite.

Let $\varphi \in \tilde{X}$. We get

$$\begin{aligned} \|(B - B_n)\varphi_{\chi_{\Omega}}\|_X &= \|(\mathbf{B} - \mathbf{B}_n)\varphi_{\chi_{\Omega}}\|_{\tilde{X}} \\ &\leq \|(\mathbf{B} - \mathbf{B}_n)\varphi\|_{\tilde{X}}. \end{aligned}$$

And therefore

$$\|(B - B_n)\|_X \leq \|\mathbf{B} - \mathbf{B}_n\|_{\tilde{X}}.$$

The desired result follows by letting $n \rightarrow \infty$. \square

We close this section by establishing the following results which shall play fundamental role in the sequel.

Theorem 6.5.1 *Let B be a smooth operator on X . Then, for all $\lambda \in \mathbb{C}_{\lambda_0}$, $(\lambda - S_K)^{-1}B$ is weakly compact on X .*

Proof. Let $\lambda \in \mathbb{C}_{\lambda_0}$. According to Equation (6.2.7), $(\lambda - S_K)^{-1}B$ is given by

$$(\lambda - S_K)^{-1}B = \Lambda_{\lambda}K(I - \Theta_{\lambda}K)^{-1}\Xi_{\lambda}B + (\lambda - S_0)^{-1}B.$$

To prove the weak compactness of $(\lambda - S_K)^{-1}B$ on X , it suffices to establish that the operators Ξ_λ and $(\lambda - S_0)^{-1}B$ are weakly compact on X . By Lemma 6.5.1, it suffices to prove the theorem for an operator B in the form

$$(B\varphi)(a, l) := \int_{l_1}^{\infty} f(l)\varphi(a, l')\chi_{\Omega}(a, l')dl',$$

where $f \in L^1([l_1, \infty), dl)$.

Let $\varphi \in X$. We have

$$(\Xi_\lambda B\varphi)(l, l) = f(l) \int_0^a e^{-\int_s^a (\lambda + \mu(\tau, l))d\tau} ds \int_{l_1}^{\infty} \varphi(s, l')dl'.$$

This yields the factorization $\Xi_\lambda B := A_2(\lambda)A_1$ where

$$A_1 : X \ni \varphi \mapsto \int_{l_1}^{\infty} \varphi(a, l)dl \in L^1([0, l], da)$$

and

$$A_2(\lambda) : L^1([0, l], da) \ni \psi \mapsto f(l) \int_0^a e^{-\int_s^a (\lambda + \mu(\tau, l))d\tau} \psi(s)ds \in Y.$$

Since A_1 is bounded, it suffices to prove that the operator $A_2(\lambda)$ is weakly compact. Next, let $\psi \in X$, one can write

$$\begin{aligned} \|A_2(\lambda)\psi\|_Y &\leq \int_{l_1}^{\infty} |f(l)| dl \int_0^a e^{-(a-s)(\operatorname{Re}\lambda + \underline{\mu})} |\psi(s)| ds \\ &\leq \|f(\cdot)\|_{L^1([l_1, \infty), dl)} \|\psi(\cdot)\|_{L^1([0, l], da)} \end{aligned}$$

and therefore

$$\|A_2(\lambda)\| \leq \|f(\cdot)\|_{L^1([l_1, \infty), dl)}.$$

This last estimate shows that $A_2(\lambda)$ depends continuously (in the uniform topology) on $f \in L^1([l_1, \infty), dl)$. So, by approximating f (in the L^1 -norm) by bounded functions, $A_2(\lambda)$ is a limit (for uniform topology) of integral operators of bounded kernel. Hence, $A_2(\lambda)$ is weakly compact (cf. [27, Corollary 11, p. 294]).

A similar reasoning allows us to reach the same result for the operator $(\lambda - S_0)^{-1}B$. □

Remark 6.5.2 We note that A_K is a bounded perturbation of S_K ($A_K - S_K = B \in \mathcal{L}(X)$). Since S_K generates the strongly continuous semigroup $(T_K(t))_{t \geq 0}$ on X , it follows from Theorem 6.4.1 and the classical perturbation theory (see, for example, [35]) that A_K generates also a strongly continuous semigroup $(W_K(t))_{t \geq 0}$ on X given by the Dyson-Phyllips expansion

$$W_K(t) = \sum_{j=0}^{n-1} T^j(t) + R_n(t), \quad (6.5.2)$$

where $T^0(t) = T_K(t)$, $T^j(t) = \int_0^t T_K(s)BT^{j-1}(t-s)ds$ ($j \geq 1$) and the series (6.5.2) converges uniformly in $\mathcal{L}(X)$ in bounded time intervals. The n^{th} remainder term $R_n(t)$ is given by

$$R_n(t) = \sum_{j=n}^{\infty} T^j(t) = \int_{s_1+s_2+\dots+s_n \leq t, s_i \geq 0} T_K(s_1)B\dots T_K(s_n)BW_K(t - \sum_{i=1}^n s_i)ds_1 \cdots ds_n.$$

□

6.6 Asymptotic spectrum of A_K

This section is devoted to the asymptotic spectrum of the operator A_K . Let λ_0 be the real defined by (6.2.9) and define $\sigma_{as}(A_K)$ (the asymptotic spectrum of A_K) by

$$\sigma_{as}(A_K) := \sigma(A_K) \cap \mathbb{C}_{\lambda_0}.$$

Let $F : \mathbb{C}_{\lambda_0} \rightarrow \mathcal{L}(X)$ be the function defined by

$$\mathbb{C}_{\lambda_0} \ni \lambda \mapsto F(\lambda) := (\lambda - S_K)^{-1}B.$$

Theorem 6.6.1 *Let $\lambda \in \mathbb{C}_{\lambda_0}$ and assume that B is a smooth operator on X . Then $\sigma_{as}(A_K)$ consists of, at most, isolated eigenvalues with finite algebraic multiplicities. Moreover, if $\sigma_{as}(A_K) \neq \emptyset$, then A_K admits a real leading eigenvalue $\underline{\lambda} > \lambda_0$.*

Remark 6.6.1 Let $\lambda \in \mathbb{C}_{\lambda_0}$. Since A_K is an additive perturbation of S_K , it is clear that λ is an eigenvalue of A_K if and only if 1 is an eigenvalue of $F(\lambda)$. Indeed, if λ is an eigenvalue of A_K , then there exists $\psi \neq 0$ such that $A_K\psi = S_K\psi + B\psi = \lambda\psi$. This implies that $F(\lambda)\psi = (\lambda - S_K)^{-1}B\psi = \psi$ which proves that 1 is an eigenvalue of $F(\lambda)$. The converse is immediate. □

Proof of Theorem 6.6.1. Let $\lambda \in \mathbb{C}_{\lambda_0}$. Since S_K generates a strongly continuous semigroup on X , we have $\|(\lambda - S_K)^{-1}B\| \rightarrow 0$ as $\text{Re}\lambda \rightarrow \infty$ and therefore

$$r_\sigma(F(\lambda)) \rightarrow 0 \text{ as } \text{Re}\lambda \rightarrow \infty. \quad (6.6.1)$$

This shows that $1 \notin \sigma(F(\lambda))$. Applying the spectral mapping theorem, we infer that $1 \notin \sigma([F(\lambda)]^2)$. Moreover, using the smoothness of B we deduce that, via Theorem 6.5.1, $F(\lambda)$ is weakly compact and therefore $[F(\lambda)]^2$ is compact (use Theorem 2.5.2). Next, the use of Theorem 2.5.1 implies that $(I - [F(\lambda)]^2)^{-1}$ exists on \mathbb{C}_{λ_0} except for a discrete set S where, for each point of S , $(I - [F(\lambda)]^2)^{-1}$ has a pole of finite order. Making use of the relation $I - [F(\lambda)]^2 = (I - F(\lambda))(I + F(\lambda))$ we conclude that

$$(I - F(\lambda))^{-1} = (I + F(\lambda))(I - [F(\lambda)]^2)^{-1}$$

and the operator valued function $\lambda \mapsto (I - F(\lambda))^{-1}$ is holomorphic on $\mathbb{C}_{\lambda_0} \setminus S$. So, according to Remark 6.6.1, for all $\lambda \in \mathbb{C}_{\lambda_0} \setminus S$, the operator $(\lambda - A_K)$ is invertible and consequently $\sigma_{as}(A_K) = S$.

Suppose now that $\sigma_{as}(A_K) \neq \emptyset$. Let $\lambda_1 \in \sigma_{as}(A_K)$ and set $\lambda_2 = \operatorname{Re} \lambda_1 > \lambda_0$. It follows from the first part of the proof that there exists $\psi \neq 0$ such that $F(\lambda_1)\psi = \psi$ and then $|\psi| \leq F(\lambda_2)|\psi|$. So, using the positivity of $F(\lambda_2)$ together Theorem 2.4.3, we get

$$r_\sigma(F(\lambda_2)) \geq 1. \tag{6.6.2}$$

According to Lemma 2.5.1, the function $] \lambda_0, +\infty[\ni \lambda \mapsto r_\sigma(F(\lambda))$ is continuous and strictly decreasing. So, by (6.6.1) and (6.6.2), we infer that there exists $\underline{\lambda} \in] \lambda_0, +\infty[$ such that $r_\sigma(F(\underline{\lambda})) = 1$. Moreover, since for all $\lambda \in] \lambda_0, +\infty[$, the operator $F(\lambda)$ is positive and power compact (cf. Theorem 6.5.1), by the Krein-Rutman theorem (cf. Theorem 4.1.4 in [59]), we conclude that $r_\sigma(F(\lambda))$ is a eigenvalue of $F(\lambda)$. Hence, there exists $\psi_0 \neq 0$ such that $F(\underline{\lambda})\psi_0 = (\underline{\lambda} - S_K)^{-1}B\psi_0 = \psi_0$ or else $A_K\psi_0 = \underline{\lambda}\psi_0$. Now remembering the fact that, for all $\lambda > \underline{\lambda}$, $r_\sigma(F(\lambda)) < 1$, we conclude that $\underline{\lambda}$ is the leading eigenvalue of A_K . \square

Proposition 6.6.1 *If the hypotheses of Theorem 6.6.1 hold true, then*

$$\sigma_{as}(A_K) \neq \emptyset \text{ if and only if } \lim_{\lambda \rightarrow \lambda_0} r_\sigma(F(\lambda)) > 1.$$

Proof. By analyticity arguments, it suffices to restrict ourselves to the set $\sigma(A_K) \cap (\lambda_0, +\infty)$. In view of the proof of the Theorem 6.6.1, the leading eigenvalue $\underline{\lambda}$ is characterized by the following equation

$$r_\sigma(F(\underline{\lambda})) = 1. \tag{6.6.3}$$

If $\sigma(A_K) \cap (\lambda_0, +\infty) \neq \emptyset$, then, according to the Theorem 6.6.1, there exists a leading eigenvalue $\underline{\lambda}$ such that $r_\sigma(F(\underline{\lambda})) = 1$ and so $\sigma(A_K) \cap (\lambda_0, +\infty) \subset (\lambda_0, \underline{\lambda}]$. Since the function $] \lambda_0, +\infty[\ni \lambda \mapsto r_\sigma(F(\lambda))$ is continuous and strictly decreasing (see Lemma 2.5.1), we have $1 = r_\sigma(F(\underline{\lambda})) < r_\sigma(F(\lambda))$ for all $\lambda \in (\lambda_0, \underline{\lambda})$ and therefore $\lim_{\lambda \rightarrow \lambda_0} r_\sigma(F(\lambda)) > 1$.

Conversely, assume that $\lim_{\lambda \rightarrow \lambda_0} r_\sigma(F(\lambda)) > 1$. We know from equation (6.6.1) that $\lim_{\lambda \rightarrow \infty} r_\sigma(F(\lambda)) = 0$. Invoking again Lemma 2.5.1, we conclude that there exists $\lambda_1 > \lambda_0$ such that $r_\sigma(F(\lambda_1)) = 1$. Hence, the use of (6.6.3) shows that λ_1 is the leading eigenvalue of S_K and consequently $\sigma_{as}(A_K) \neq \emptyset$. \square

Now, we introduce the following assumption

$$(A11) \quad \begin{cases} \mathcal{B} = B + \widetilde{B}, \\ \widetilde{B} \text{ is a smooth operator.} \end{cases}$$

Theorem 6.6.2 *Let λ be a complex number in \mathbb{C}_{λ_0} and assume that the condition (A11) holds true. Then $\sigma_{as}(A_K)$ consists of, at most, isolated eigenvalue with finite algebraic multiplicity.*

Proof. Let $\lambda \in \mathbb{C}_{\lambda_0}$ and consider the problem

$$\varphi = \lambda\psi - A_K\psi = \lambda\psi - S_K\psi - B\psi - \widetilde{B}\psi \quad (6.6.4)$$

where φ is an assigned function in X and the unknown function ψ must be sought in $D(S_K)$. Since $\lambda \in \rho(T_K)$, the last equation can be written as

$$\psi - (\lambda - S_K)^{-1}B\psi - (\lambda - S_K)^{-1}\widetilde{B}\psi = (\lambda - S_K)^{-1}\varphi.$$

As we already seen in the first part of the proof of Theorem 6.6.1, the operator $(I - (\lambda - S_K)^{-1}B)$ is invertible for all λ such that $\operatorname{Re}\lambda$ is large enough. Therefore, for such λ , we get

$$\psi - F(\lambda)\psi = [I - (\lambda - S_K)^{-1}B]^{-1}(\lambda - S_K)^{-1}\varphi,$$

where

$$F(\lambda) := (I - (\lambda - S_K)^{-1}B)^{-1}(\lambda - S_K)^{-1}\widetilde{B}.$$

Now, using (A11) together with Theorem 6.5.1, we infer that $(\lambda - S_K)^{-1}\widetilde{B}$ is weakly compact and consequently $[F(\lambda)]^2$ is compact (apply Theorem 2.5.2).

Next, we know (cf. Theorem 6.4.1) that S_K generates a strongly continuous semigroup on X . So, by Hille-Yosida-Phillips's theorem, there exists a constant M such that $\|(\lambda - S_K)^{-1}B\| \leq \frac{M\|B\|}{\operatorname{Re}\lambda - \lambda_0}$. So, for λ such that $\operatorname{Re}\lambda$ is large enough, a simple calculation shows that

$$\left\| (I - (\lambda - S_K)^{-1}B)^{-1} \right\| \leq \left(1 - \frac{M\|B\|}{\operatorname{Re}\lambda - \lambda_0} \right)^{-1}$$

and therefore

$$\lim_{\operatorname{Re}\lambda \rightarrow +\infty} \|F(\lambda)\| = 0.$$

Now, arguing as in the proof of the Theorem 6.6.1, we get the desired result. \square

In the remainder of this section we shall discuss the relationship between the asymptotic spectrum of the A_K and that of its bounded part which we denote by \mathcal{A} . Here we suppose that the rate of cell mortality and the kernel are independent of the age a , that is,

$$\mu(a, l) = \mu(l) \quad \text{and} \quad \kappa(a, l, l') = \kappa(l, l').$$

Hence the operator \mathcal{A} can be written as

$$\begin{cases} \mathcal{A} : L^1((l_1, \infty), dl) \rightarrow L^1((l_1, \infty), dl); \\ \varphi \mapsto \mathcal{A}\varphi(l) = -\mu(l)\varphi(l) + \int_{l_1}^{\infty} \kappa(l, l')\varphi(l')dl'. \end{cases}$$

We begin with the following result.

Proposition 6.6.2 *If B is a power compact operator on $L^1((l_1, \infty), dl)$, then $\sigma_{as}(\mathcal{A}) := \sigma(\mathcal{A}) \cap \mathbb{C}_{\lambda_0}$ consists of, at most, isolated eigenvalues with finite algebraic multiplicities. Further, if $\sigma_{as}(\mathcal{A}) \neq \emptyset$, then there exists a real leading eigenvalue $\bar{\lambda}$ of \mathcal{A} such that $\bar{\lambda} > \lambda_0$.*

Proof. The resolvent of the following multiplication operator

$$L^1((l_1, \infty), dl) \ni \varphi \mapsto -\mu(l)\varphi(l)$$

is given by

$$L^1((l_1, \infty), dl) \ni \varphi \mapsto (\lambda + \mu(l))^{-1}\varphi(l), \quad \text{Re}\lambda > \lambda_0.$$

Furthermore, for all real λ such that $\lambda > \lambda_0$, the operator $(\lambda + \mu(l))^{-1}B$ is dominated by $(\lambda + \mu)^{-1}B$ which is a power compact operator. Now, using Theorem 2.5.1 and arguing as in the proof of Theorem 6.6.1, we obtain the desired results. \square

In the remainder of this section, we suppose that the transition operator satisfies the estimate

$$K \leq Id \tag{6.6.5}$$

where Id is the identity operator of the space Y .

Theorem 6.6.3 *Let B be a power compact operator in $L^1((l_1, \infty), dl)$. If $\sigma_{as}(A_K) \neq \emptyset$, then $\sigma_{as}(\mathcal{A}) \neq \emptyset$ and the leading eigenvalue of A_K is less than or equal to that of \mathcal{A} . Moreover, the latter is less than or equal to $-\underline{\mu} + r_{\sigma}(B)$.*

Proof. Suppose that $\sigma_{as}(A_K) \neq \emptyset$. According to Theorem 6.6.1, A_K has a real leading eigenvalue denoted by $\underline{\lambda}$ and let $\psi_0 \neq 0$ be an associated positive eigenvector. The equation $A_K\psi_0 = \underline{\lambda}\psi_0$ writes as

$$-\frac{\partial\psi_0}{\partial a}(a, l) - (\underline{\lambda} + \mu(l))\psi_0(a, l) + \int_{l_1}^{\infty} \kappa(l, l')\psi_0(a, l')dl' = 0. \tag{6.6.6}$$

Integrating Equation (6.6.6) with respect to the variable a , we get

$$(K - I)\psi_0(l, l) - (\underline{\lambda} + \mu(l)) \int_0^l \psi_0(a, l)da + \int_0^l \int_{l_1}^{\infty} \kappa(l, l')\psi_0(a, l')dl'da = 0. \tag{6.6.7}$$

Set

$$\varphi_0(l) = \int_0^l \psi_0(a, l) da.$$

It is clear that $\varphi_0 \geq 0$ and $\varphi_0 \neq 0$. Now, Equations (6.6.5) and (6.6.7) lead to

$$-(\underline{\lambda} + \mu(l))\varphi_0(l) + \int_{l_1}^{\infty} \kappa(l, l')\varphi_0(l') dl \geq 0$$

and therefore

$$\int_{l_1}^{\infty} \frac{\kappa(l, l')}{\underline{\lambda} + \mu(l)} \varphi_0(l') dl' \geq \varphi_0(l). \quad (6.6.8)$$

Let λ be a real number such that $\lambda > \lambda_0$ and define the operator B_λ by

$$\begin{cases} B_\lambda : L^1((l_1, \infty), dl) \rightarrow L^1((l_1, \infty), dl) \\ \varphi \mapsto B_\lambda \varphi(l) = \int_{l_1}^{\infty} \frac{\kappa(l, l')}{\lambda + \mu(l)} \varphi(l') dl'. \end{cases}$$

Since B_λ is a power compact operator on $L^1((l_1, \infty), dl)$, it follows from the Krein-Rutman theorem [59, Theorem 4.14] that $r_\sigma(B_\lambda)$ is an eigenvalue of B_λ depending continuously on λ . Moreover, Equation (6.6.8) implies that $r_\sigma(B_\lambda) \geq 1$. Now, using the fact that $r_\sigma(B_\lambda)$ is a continuous strictly decreasing function of λ (cf. Lemma 2.5.1) together with $\lim_{\lambda \rightarrow \infty} r_\sigma(B_\lambda) = 0$, we infer that there exists $\tilde{\lambda} \geq \underline{\lambda}$ such that $r_\sigma(B_{\tilde{\lambda}}) = 1$. This shows that there exists $\bar{\varphi} \in L^1((l_1, \infty), dl)$ such that

$$B_{\tilde{\lambda}} \bar{\varphi} = \bar{\varphi}. \quad (6.6.9)$$

This implies that $\mathcal{A}\bar{\varphi} = \tilde{\lambda}\bar{\varphi}$ and proves the first part of the theorem.

On the other hand, the use of Equation (6.6.8), gives

$$\int_{l_1}^{\infty} \kappa(l, l') |\bar{\varphi}(l')| dl' \geq (\tilde{\lambda} + \mu(l)) |\bar{\varphi}(l)| \geq (\tilde{\lambda} + \underline{\mu}) |\bar{\varphi}(l)|,$$

which implies that $r_\sigma(B) \geq \tilde{\lambda} + \underline{\mu}$ and therefore $\tilde{\lambda} \leq r_\sigma(B) - \underline{\mu}$. \square

Corollary 6.6.1 *If the hypotheses of Theorem 6.6.3 are satisfied, then the following assertions hold true.*

- (1) *If B is quasinilpotent (i.e., $r_\sigma(B) = 0$), then $\sigma_{as}(A_K) = \emptyset$.*
- (2) *If \mathcal{A} is subcritical (i.e., $\sigma_{as}(\mathcal{A}) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < 0\}$), then A_K is subcritical.*

Proof. (1) Assume that $\sigma_{as}(A_K) \neq \emptyset$. The use of the expression of B_λ implies that

$$B_{\tilde{\lambda}} \leq \frac{1}{\tilde{\lambda} + \underline{\mu}} B,$$

where $\tilde{\lambda}$ is the leading eigenvalue of \mathcal{A} . Moreover, since $r_\sigma(B_{\tilde{\lambda}}) = 1$, we deduce that $r_\sigma(B) > 0$ which contradicts the fact that B is quasinilpotent.

(2) According to Theorem 6.6.3, we have $\underline{\lambda} \leq \tilde{\lambda}$, where $\underline{\lambda}$ and $\tilde{\lambda}$ are the leading eigenvalues of A_K and \mathcal{A} , respectively. Now, if \mathcal{A} is subcritical, then $\tilde{\lambda} < 0$ and so $\underline{\lambda} < 0$. \square

6.7 Regularity of the solution to the Cauchy problem (6.1.5)

The main goal of this section is to discuss the time asymptotic behavior (for large times) of the solution to Cauchy Problem (6.1.5).

Lemma 6.7.1 *Let $\lambda \in \mathbb{C}_{\lambda_0}$. If B is a smooth operator, then, for all $r \in [0, 1]$, we have*

$$\lim_{|\operatorname{Im}\lambda| \rightarrow +\infty} |\operatorname{Im}\lambda|^r \|(\lambda - S_K)^{-1} B\| = 0.$$

Proof. The use of Eq. (6.2.7) allows us to write

$$(\lambda - S_K)^{-1} B = \Lambda_\lambda K (I - \Theta_\lambda K)^{-1} \Xi_\lambda B + (\lambda - S_0)^{-1} B.$$

Since Λ_λ , K and $(I - \Theta_\lambda K)^{-1}$ are uniformly bounded on \mathbb{C}_{λ_0} , it suffices to establish the lemma for the operators $\Xi_\lambda B$ and $(\lambda - S_0)^{-1} B$. It follows from Lemma 6.5.1 that we have only to establish the result in the case where the operator B is in the form

$$(B\varphi)(a, l) := \int_{l_1}^{+\infty} f(l)\varphi(a, l')\chi_\Omega(a, l')dl',$$

where $f \in L^1([l_1, \infty), dl)$. So, for all $\varphi \in X$, we have

$$(\Xi_\lambda B\varphi)(l, l) = f(l) \int_0^a e^{-\int_s^a (\lambda + \mu(\tau, l))d\tau} ds \int_{l_1}^{+\infty} \varphi(s, l')dl'.$$

Hence, the operator $\Xi_\lambda B$ can be factorized as $\Xi_\lambda B := A_2(\lambda)A_1$ where

$$A_1 : X \ni \varphi \mapsto \int_{l_1}^{+\infty} \varphi(a, l)dl \in L^1([0, l], da)$$

and

$$A_2(\lambda) : L^1([0, l], da) \ni \psi \mapsto f(l) \int_0^a e^{-\int_s^a (\lambda + \mu(\tau, l))d\tau} \psi(s)ds \in Y.$$

Since A_1 is a bounded operator independent of λ , it suffices to establish the result for $A_2(\lambda)$. Arguing as in the proof of Lemma 4.2.1 we have the desired result.

Using the same arguments, we prove also that $\lim_{|\operatorname{Im}\lambda| \rightarrow +\infty} \|\operatorname{Im}\lambda^r(\lambda - S_0)^{-1}B\| = 0$. This completes the proof. \square

Lemma 6.7.2 *Let $\lambda \in \mathbb{C}_{\lambda_0}$ and let B be a smooth operator. Then following statements are satisfied.*

- (1) $\sigma(A_K) \cap \mathbb{C}_{\lambda_0}$ consists of, at most, isolated eigenvalues with finite algebraic multiplicities.
- (2) If $\varepsilon > 0$, then $\sigma(A_K) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \lambda_0 + \varepsilon\}$ is, at most, finite.
- (3) If $\varepsilon > 0$, then $\|(\lambda - A_K)^{-1}\|$ is uniformly bounded in $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq \lambda_0 + \varepsilon\}$ for $|\operatorname{Im}\lambda|$ large enough.

Proof. Let $\lambda \in \mathbb{C}_{\lambda_0}$. It follows from Theorems 6.5.1 and 2.5.2 that the operator $[(\lambda - S_K)^{-1}B]^2$ is compact on X . Applying Lemma 6.7.1 with $r = 0$, we get

$$\lim_{|\operatorname{Im}\lambda| \rightarrow \infty} \|(\lambda - S_K)^{-1}B\| = 0.$$

Now the rest of the proof is similar to that of Lemma 1.1 in [64]. \square

It follows from Lemma 6.7.2 that $\sigma(A_K) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \lambda_0 + \varepsilon\}$ consists of, at most, a finite number of isolated eigenvalues with finite algebraic multiplicities, say $\lambda_1, \lambda_2, \dots, \lambda_n$. Let P_i and D_i denote, respectively, the spectral projection and the nilpotent operator associated to $\lambda_i, i = 1, 2, \dots, n$. Then $P = P_1 + \dots + P_n$ is the spectral projection associated to the compact set $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$. It follows from the spectral decomposition theorem corresponding to sets $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ and $\sigma(A_K) \setminus \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ (see, for example, [35, Theorem 6.17, p. 178]), that

$$W_K(t) = \widetilde{W}_K(t) + \sum_{i=1}^n e^{\lambda_i t} e^{D_i t} P_i,$$

where $\widetilde{W}_K(t) := W_K(t)(I - P)$. $(\widetilde{W}_K(t))_{t \geq 0}$ is a C_0 -semigroup on the Banach space $(I - P)X$ with generator $\widetilde{A}_K := A_K(I - P)$ (\widetilde{A}_K is the part of A_K on the closed subspace $(I - P)X$).

Now, we are in a position to state the main result of this section.

Theorem 6.7.1 *Let $\lambda \in \mathbb{C}_{\lambda_0}$ and let B be a smooth operator on X . Then, for each $\varepsilon > 0$, there exists $M > 0$ such that*

$$\|W_K(t)\psi_0 - \sum_{i=1}^n e^{(\lambda_i + D_i)t} P_i \psi_0\| \leq M e^{(\lambda_0 + \varepsilon)t}, \quad \forall t > 0, \quad (6.7.1)$$

where $\psi_0 \in D(A_K)$. Further, if $r_\sigma(K) < e^{I\mu}$, then

$$\|W_K(t)\psi_0 - \sum_{i=1}^n e^{(\lambda_i + D_i)t} P_i \psi_0\| \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Remark 6.7.1 Unlike to the estimate (6.4.2) (cf. Remark 6.4.2), the estimate (6.7.1) is valid for all initial data ψ_0 belonging to $D(A_K)$. □

Proof of Theorem 6.7.1. Let $\lambda \in \mathbb{C}_{\lambda_0}$. It follows from Theorems 6.5.1 and 2.5.2 that the operator $(\lambda - S_K)^{-1}[B(\lambda - S_K)^{-1}]^3$ is compact on X . Since

$$\|(\lambda - S_K)^{-1}[B(\lambda - S_K)^{-1}]^3\| \leq \|(\lambda - S_K)^{-1}\| \|(\lambda - S_K)^{-1}B\|^3,$$

applying Lemma 6.7.1 with $r = 1$, we conclude that

$$\lim_{|\operatorname{Im}\lambda| \rightarrow +\infty} |\operatorname{Im}\lambda| \|(\lambda - S_K)^{-1}[B(\lambda - S_K)^{-1}]^3\| = 0.$$

Now, the use of Proposition 4.1.1 implies that, for $n \geq 7$, $R_n(t)$ is compact. Applying Proposition 2.3.2 we infer that the strongly continuous semigroups $(T_K(t))_{t \geq 0}$ and $(W_K(t))_{t \geq 0}$ have the same essential type and so, for all $t \geq 0$, we have

$$r_{ess}(T_K(t)) = r_{ess}(W_K(t)) \leq e^{\lambda_0 t}.$$

Thus, outside the spectral disc $|\mu| \leq e^{(\lambda_0 + \varepsilon)t}$, the spectrum of $(W_K(t))_{t \geq 0}$ consists of, at most, a finite number of isolated eigenvalues with finite algebraic multiplicities. On other hand, each point μ' satisfying $|\mu'| \geq e^{(\lambda_0 + \varepsilon)t}$ belongs to the resolvent set of $\widetilde{W}_K(t)$ and consequently,

$$r_\sigma(\widetilde{W}_K(t)) \leq e^{(\lambda_0 + \varepsilon)t}$$

Hence, $\|\widetilde{W}_K(t)\| = o(e^{(\lambda_0 + \varepsilon)t})$ as $t \rightarrow \infty$. This completes the proof. □

6.8 Further results

We close this paper by some results in connection with the asymptotic spectrum of the operator A_K such as the behavior of the leading eigenvalue of A_K with respect to the parameters of the equation, the irreducibility of the strongly continuous semigroup generated by A_K and the description of some essential spectra of A_K .

6.8.1 Strict monotonicity of the leading eigenvalue of A_K

The goal of this subsection is to discuss the strict growth of the leading eigenvalue (when it exists) of the operator A_K with respect to the parameters of the equation.

We first consider the strict monotonicity of the leading eigenvalue with respect to the transition operator. Let K_1 and K_2 be two transition operators satisfying $K_1 \leq K_2$ and $K_1 \neq K_2$. We denote by λ_K the leading eigenvalue of the operator A_K , when it exists.

Let λ be a real number such that $\lambda > -\underline{\mu} + \frac{1}{l_1} \max(\ln(r_\sigma(K_1)), \ln(r_\sigma(K_2)))$. It is clear that, for $i \in \{1, 2\}$, we have $\lambda \in \rho(S_{K_i})$ and

$$(\lambda - S_{K_i})^{-1} = \Lambda_\lambda K_i (I - \Theta_\lambda K_i)^{-1} \Xi_\lambda + (\lambda - S_0)^{-1}. \quad (6.8.1)$$

Further, the use of the positivity of the operators K_1 , K_2 , B , Λ_λ , Θ_λ , Ξ_λ and Π_λ , together with the formula

$$(I - \Theta_\lambda K_1)^{-1} - (I - \Theta_\lambda K_2)^{-1} = (I - \Theta_\lambda K_1)^{-1} \Theta_\lambda (K_1 - K_2) (I - \Theta_\lambda K_2)^{-1}$$

leads to the estimate

$$(\lambda - S_{K_1})^{-1} B \leq (\lambda - S_{K_2})^{-1} B. \quad (6.8.2)$$

Theorem 6.8.1 *Let B be a smooth operator on X . If λ_{K_1} exists, then λ_{K_2} exists and $\lambda_{K_1} \leq \lambda_{K_2}$. Further, if $(\lambda_{K_1} - S_{K_2})^{-1} B$ has a strictly positive power, then we have $\lambda_{K_1} < \lambda_{K_2}$.*

Proof. It follows from Theorem 6.5.1 that the operators $(\lambda_{K_1} - S_{K_1})^{-1} B$ and $(\lambda_{K_1} - S_{K_2})^{-1} B$ are power compact on X . Applying Krein-Rutman's theorem (cf. [59, Theorem 4.1.4]), we infer that $r_\sigma((\lambda_{K_1} - S_{K_1})^{-1} B)$ is an eigenvalue of $(\lambda_{K_1} - S_{K_1})^{-1} B$. Keeping mind that λ_{K_1} is the leading eigenvalue of A_{K_1} , we conclude that (cf. (6.6.3))

$$r_\sigma((\lambda_{K_1} - S_{K_1})^{-1} B) = 1. \quad (6.8.3)$$

Since the operator $(\lambda_{K_1} - S_{K_2})^{-1} B$ has a strictly positive power, the use of (6.8.2) and (6.8.3) together with Theorem 2.4.2 implies that

$$r_\sigma((\lambda_{K_1} - S_{K_2})^{-1} B) > 1.$$

Now, combining (6.6.1), (6.8.3) and the fact that the map $]\lambda_0, +\infty[\ni \lambda \mapsto r_\sigma((\lambda - S_K)^{-1} B)$ is continuous and strictly decreasing, we conclude that there exists a unique $\lambda_1 > \lambda_{K_1}$ such that $r_\sigma((\lambda_1 - S_{K_2})^{-1} B) = 1$. This shows (cf. again (6.6.3)) that $\lambda_1 = \lambda_{K_2}$ and completes the proof. \square

Remark 6.8.1 If one of the following conditions holds true, then $(\lambda_{K_1} - S_{K_2})^{-1} B$ has a strictly positive power:

- (a) there exists an integer $n \geq 1$ such that $(\Pi_{\lambda_{K_1}} B)^n$ is strictly positive,
- (b) there exists an integer $n \geq 1$ such that $[\Lambda_{\lambda_{K_1}} K_2 (I - \Theta_{\lambda_{K_1}} K_2)^{-1} \Xi_{\lambda_{K_1}} B]^n$ is strictly positive.

\square

Practical criteria of monotonicity of $\lambda(K)$ are given in the following corollary.

Corollary 6.8.1 *Assume that the hypotheses of Theorem 6.8.1 are satisfied. Then, if λ_{K_1} exists, then λ_{K_2} exists and $\lambda_{K_1} \leq \lambda_{K_2}$. If, further, there exists $n_0 \in \mathbb{N}$ such that $(\Theta_{\lambda_{K_1}} K_2)^{n_0}$ is strictly positive and $\ker(B) \cap \{\varphi \in X, \varphi \geq 0\} = \{0\}$, then $\lambda_{K_1} < \lambda_{K_2}$.*

Proof. It follows from (6.8.2) together with the fact that $\Lambda_{\lambda_{K_1}}$ and $\Xi_{\lambda_{K_1}}$ are a multiplication operators by strictly positive functions, the operator $(\lambda_{K_1} - T_{K_2})^{-1}B$ admits a strictly positive power if one of the conditions (a) and (b) holds true. Now, a similar reasoning as in the proof of Theorem 6.8.1 gives the desired result. \square

We shall discuss now the strict monotonicity of the leading eigenvalue of A_K with respect to the operator B . To do so, let B_1 and B_2 be two smooth operators on X satisfying $B_1 \leq B_2$ and $B_1 \neq B_2$. It is obvious that

$$(\lambda - S_K)^{-1}B_1 \leq (\lambda - S_K)^{-1}B_2. \quad (6.8.4)$$

We denote by λ_B the leading eigenvalue of the operator $S_K + B$ (when it exists).

Theorem 6.8.2 *Let B_1 and B_2 be two smooth operators on X . If λ_{B_1} exists, then λ_{B_2} exists and $\lambda_{B_1} \leq \lambda_{B_2}$. Further, if $(\lambda_{B_1} - S_K)^{-1}B_2$ has a strictly positive power, then $\lambda_{B_1} < \lambda_{B_2}$.*

Proof. According to Theorem 6.5.1, $(\lambda_{B_1} - S_K)^{-1}B_1$ and $(\lambda_{B_1} - S_K)^{-1}B_2$ are power compact operators on X . By the Krein-Rutman theorem, we infer that λ_{B_1} is an eigenvalue of $(\lambda_{B_1} - S_K)^{-1}B_1$. Using the fact that λ_{B_1} is the leading eigenvalue of $S_K + B_1$, we get

$$r_\sigma((\lambda_{B_1} - S_K)^{-1}B_1) = 1. \quad (6.8.5)$$

Further, since the operator $(\lambda_{B_1} - S_K)^{-1}B_2$ has a strictly positive power, using (6.8.4) and (6.8.5) together with Theorem 2.4.2, we conclude that

$$r_\sigma((\lambda_{B_1} - S_K)^{-1}B_2) > 1.$$

Next, combining (6.6.1), (6.8.5) and the fact that the map $] \lambda_0, +\infty[\ni \lambda \mapsto r_\sigma((\lambda - S_K)^{-1}B)$ is continuous and strictly decreasing, we infer that there exists a unique $\lambda_1 > \lambda_{B_1}$ such that $r_\sigma((\lambda_1 - S_K)^{-1}B_2) = 1$. This shows (cf. again (6.6.3)) that $\lambda_1 = \lambda_{B_2}$ and completes the proof. \square

The following corollary is an immediate consequence of Theorem 6.8.2.

Corollary 6.8.2 *Assume that the hypotheses of Theorem 6.8.2 are satisfied. If λ_{B_1} exists, then λ_{B_2} exists and $\lambda_{B_1} \leq \lambda_{B_2}$. Further, if there exists $n_0 \in \mathbb{N}$ such that $(\Theta_{\lambda_{B_1}} K)^{n_0}$ is strictly positive and $\ker(B_2) \cap \{\varphi \in X, \varphi \geq 0\} = \{0\}$, then $\lambda_{B_1} < \lambda_{B_2}$.*

We close this subsection by considering the strict monotonicity of the leading eigenvalue of A_K with respect to both K and B . Assume that B_1 and B_2 are two smooth operators on X such that $B_1 \leq B_2$ and $B_1 \neq B_2$ and let K_1 and K_2 be two transition operators satisfying $K_1 \leq K_2$ and $K_1 \neq K_2$. Arguing as above, it is not difficult to check that for all $l \in \mathbb{R}$ such that $\lambda > -\underline{\mu} + \frac{1}{l_1} \max(\ln(r_\sigma(K_1)), \ln(r_\sigma(K_2)))$, we have

$$(\lambda - S_{K_1})^{-1} B_1 \leq (\lambda - S_{K_2})^{-1} B_2.$$

We denote by $\lambda_{K,B}$ the leading eigenvalue of the operator $A_K = S_K + B$ (when it exists). As consequence of Theorems 6.8.1 and 6.8.2, we have

Corollary 6.8.3 *Under the hypotheses introduced just above, if λ_{K_1, B_1} exists, then λ_{K_2, B_2} exists and $\lambda_{K_1, B_1} \leq \lambda_{K_2, B_2}$. Further, if $(\lambda_{K_1, B_1} - S_{K_2})^{-1} B_2$ has a strictly positive power, then $\lambda_{K_1, B_1} < \lambda_{K_2, B_2}$.*

6.8.2 Irreducibility of the semigroup e^{tA_K}

In this subsection, we shall discuss the irreducibility of the strongly continuous semigroup $(W_K(t))_{t \geq 0}$ generated by the operator A_K . The interest of the irreducibility lies in the fact that if $(U(t))_{t \geq 0}$ is irreducible, then the leading eigenvalue (if it exists) is strictly dominant with multiplicity 1 and the associated eigen-function is strictly positive. This provides a simple description of the time asymptotic behavior ($t \rightarrow +\infty$) of the solution to Cauchy problem. Our goal here is to discuss the influence of the transition operator on the irreducibility of the C_0 -semigroup $(W_{K,B}(t))_{t \geq 0}$, generated by the operator $A_{K,B} = S_K + B$ on X . Here B is assumed to be a positive operator.

Let λ be a real number such that $\lambda > \lambda_0$. Since K and Θ_λ are positive in the lattice sense, then the operator $(I - \Theta_\lambda K)^{-1}$ is also positive. So, according to the positivity of Λ_λ and Ξ_λ together with (6.2.7), we get

$$(\lambda - S_K)^{-1} \geq (\lambda - S_0)^{-1} \quad \forall \lambda > \lambda_0.$$

Next, using the positivity of B , we conclude that

$$((\lambda - S_K)^{-1} B)^n (\lambda - S_K)^{-1} \geq ((\lambda - S_0)^{-1} B)^n (\lambda - S_0)^{-1} \quad \forall n \in \mathbb{N}. \quad (6.8.6)$$

Further, if $\lambda \in \rho(A_K) \cap \rho(A_0)$ and $r_\sigma((\lambda - S_K)^{-1} B) < 1$, then

$$(\lambda - A_K)^{-1} = \sum_{n \geq 0} ((\lambda - S_K)^{-1} B)^n (\lambda - S_K)^{-1}. \quad (6.8.7)$$

Now, using (6.8.6) together with (6.8.7), we obtain

$$(\lambda - A_K)^{-1} \geq (\lambda - A_0)^{-1} \geq 0. \quad (6.8.8)$$

We fix $t > 0$. It follows from the estimate (6.8.8) that, for all integer n such that $\frac{n}{t} > \lambda_0$, we have

$$\left(\frac{n}{t}\left(\frac{n}{t} - A_K\right)^{-1}\right)^n \psi \geq \left(\frac{n}{t}\left(\frac{n}{t} - A_0\right)^{-1}\right)^n \psi \quad \forall \psi \geq 0.$$

Next, using the exponential formula (see, for example, [35, p. 479]), we get

$$W_{K,B}(t) = \lim_{n \rightarrow \infty} \left(\frac{n}{t}\left(\frac{n}{t} - A_K\right)^{-1}\right)^n \geq \lim_{n \rightarrow \infty} \left(\frac{n}{t}\left(\frac{n}{t} - A_0\right)^{-1}\right)^n = W_{0,B}(t)$$

and therefore

$$W_{K,B}(t) \geq W_{0,B}(t) \quad \forall t \geq 0.$$

A similar reasoning gives.

$$W_{K,B}(t) \geq W_{K,0}(t), \quad \forall t \geq 0. \tag{6.8.9}$$

Now we are in a position to state the following result.

Theorem 6.8.3 *Let B be a smooth positive operator on X . If $(W_{0,B}(t))_{t \geq 0}$ or $(W_{K,0}(t))_{t \geq 0}$ is irreducible, then $(W_{K,B}(t))_{t \geq 0}$ is irreducible.*

Proof. If $(W_{0,B}(t))_{t \geq 0}$ is irreducible, then, by the Proposition 3.5.1, $(\lambda - A_0)^{-1}$ is strictly positive. Using (6.8.8), we infer that $(\lambda - A_K)^{-1}$ is strictly positive too and consequently $(W_{K,B}(t))_{t \geq 0}$ is irreducible. Next, if $(W_{K,0}(t))_{t \geq 0}$ is irreducible, then a similar reasoning as above using Proposition 3.5.1 and (6.8.9) gives the irreducibility of $(W_{K,B}(t))_{t \geq 0}$. \square

In the following result, we shall derive some sufficient conditions of irreducibility of $(W_{K,B}(t))_{t \geq 0}$, in terms of transitions operators.

Corollary 6.8.4 *Let B be a smooth positive operator on X . If for some (every) $\lambda \in \mathbb{C}_{\lambda_0}$, $K(I - \Theta_\lambda K)^{-1}$ is strictly positive, then $(W_{K,B}(t))_{t \geq 0}$ is irreducible.*

Proof. According to (6.2.7) and the positivity of $(\lambda - S_0)^{-1}$ we have the estimate

$$(\lambda - S_K)^{-1} \geq \Lambda_\lambda K(I - \Theta_\lambda K)^{-1} \Xi_\lambda. \tag{6.8.10}$$

On the other hand, since Ξ_λ is a strictly positive operator from X into Y and Λ_λ is a multiplication operator by a strictly positive function on Ω , we infer that for all $\varphi \in X^+$ with $\varphi \neq 0$, $\Lambda_\lambda K(I - \Theta_\lambda K)^{-1} \Xi_\lambda \varphi > 0$ a.e. on Ω . Now the result follows from (6.8.10), Proposition 3.5.1 and Theorem 6.8.3. \square

We have also the following corollary

Corollary 6.8.5 *Let B be a smooth positive operator on X and let $\lambda \in \mathbb{C}_{\lambda_0}$. The strongly continuous positive semigroup $W_{K,B}(t)_{t \geq 0}$ is irreducible if one of the following two conditions holds true:*

- (1) K is strictly positive,
- (2) there exists an integer $n_0 \in \mathbb{N}$ such that $(\Theta_\lambda K)^{n_0}$ is strictly positive.

Proof. Since Θ_λ is a multiplication operator by a strictly positive function, one checks readily that $K(I - \Theta_\lambda K)^{-1}$ is strictly positive if and only if $\Theta_\lambda K(I - \Theta_\lambda K)^{-1}$ is strictly positive. Since the operator $\Theta_\lambda K$, is positive, for all $n_0 \in \mathbb{N} \setminus \{0\}$, one obtains

$$\Theta_\lambda K(I - \Theta_\lambda K)^{-1} \geq (\Theta_\lambda K)^{n_0}.$$

Hence, if one of the conditions (1) and (2) is satisfied, then $\Theta_\lambda K(I - \Theta_\lambda K)^{-1}$ is strictly positive. Now the result follows from Corollary 6.8.4. \square

6.8.3 Essential spectra of A_K

In this subsection we shall describe the Browder essential spectrum of the operator A_K which has some connections with its asymptotic spectrum.

In the following we denote by S_0 the operator defined by

$$\left\{ \begin{array}{l} S_0 : D(S_0) \subseteq X \longrightarrow X \\ \psi \longrightarrow S_0 \psi(a, l, t) = -\frac{\partial \psi}{\partial a}(a, l, t) - \mu(a, l) \psi(a, l, t) \\ D(S_0) = \{ \psi \in \tilde{W} \text{ such that } \psi(0, l) = 0 \}. \end{array} \right.$$

For each $n \in \mathbb{N}$, we define the set

$$\Omega_n := (0, l) \times (nl_1, (n+1)l_1).$$

Let us now introduce the following technical assumption.

$$(A4) \quad \text{For each } \varepsilon \in (0, \frac{1}{l}), \text{ we have } \quad \mu(a, l) \leq \underline{\mu} + \varepsilon \quad \text{for all } (a, l) \in \Omega_n.$$

Lemma 6.8.1 *If the assumption (A4) holds true, then*

$$\{ \lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq -\underline{\mu} \} \subset \sigma_{app}(S_0).$$

Proof. Let us first prove that

$$\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < -\underline{\mu}\} \subset \sigma_{app}(S_0).$$

We have clearly $\Omega_n \subset \Omega$.

Set $g_n(a, l) := \chi_{\Omega_n}(a, l)e^{-ia\operatorname{Im} \lambda}$. It is clear that

$$\|g_n(a, l)\|_X = \|\chi_{\Omega_n}(a, l)\| = \frac{(2n+1)l_1^2}{2}.$$

For all $(a, l) \in \Omega$, set

$$f_n(a, l) := (\lambda - S_0)^{-1}g_n(a, l) = e^{-ia\operatorname{Im} \lambda} \int_0^a e^{-\int_s^a (\operatorname{Re} \lambda + \mu(\tau, l))d\tau} \chi_{\Omega_n}(s, l)ds.$$

We note that $f_n \in D(S_0)$. It follows from the previous equation that $e^{ia\operatorname{Im} \lambda} f_n(a, l)$ is a nonnegative real. Hence, easy calculations using (A4) yield that

$$e^{ia\operatorname{Im} \lambda} f_n(a, l) \geq e^{-1} \int_0^a \chi_{\Omega_n}(s, l)ds$$

It follows that

$$\begin{aligned} \|f_n\|_X &= \|e^{ia\operatorname{Im} \lambda} f_n(a, l)\|_X \geq e^{-1} \int_{nl_1}^{(n+1)l_1} dl \int_0^l da \int_0^a ds \\ &= e^{-1} \frac{3n^2 l_1^3 + 3nl_1^2 + 1}{6} \\ &> 0. \end{aligned}$$

And finally, we get

$$\begin{aligned} \frac{\|(\lambda - S_0)f_n\|_X}{\|f_n\|_X} &= \frac{\|g_n\|_X}{\|f_n\|_X} \\ &\leq \frac{2nl_1^2 + l_1^2}{2} \frac{6e}{3n^2 l_1^3 + 3nl_1^2 + 1} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

This proves that the half plane $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < -\underline{\mu}\}$ is contained in the approximate point spectrum. Next, using the fact that the spectrum is closed, we conclude the desired result. \square

Note that, by Remark 6.2.1, we have $\sigma(S_0) \subseteq \{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re} \lambda \leq -\underline{\mu}\}$. Now, using the previous lemma we conclude that

$$\sigma(S_0) = \{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re} \lambda \leq -\underline{\mu}\}.$$

Lemma 6.8.2 (a) *The point spectrum of the operator S_0 is empty.*

(b) *The residual spectrum of the operator S_0 is empty.*

Proof. (a) Let $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda \leq -\underline{\mu}$. The solution of the equation $(\lambda - S_0)\psi = 0$ is given by

$$\psi(a, l) = g(l) \exp\left(-\int_0^a (\lambda + \mu(\tau, l))d\tau\right)$$

where $g(l) \in Y$. Since ψ satisfies the boundary conditions, we obtain $0 = \psi(0, l) = g(l)$ and therefore $\psi \equiv 0$ which concludes the proof of the first item.

(b) Let S_0^* be the dual operator of S_0 . It is given by

$$\begin{cases} S_0^* : D(S_0^*) \subseteq L^\infty(\Omega, dadl) \longrightarrow L^\infty(\Omega, dadl) \\ \psi \longrightarrow S_0^*\psi(a, l, t) = \frac{\partial\psi}{\partial a}(a, l, t) - \mu(a, l)\psi(a, l, t) \\ D(S_0^*) = \{\psi \in W_\infty \text{ such that } \psi^l = 0\}, \end{cases}$$

where $W_\infty = \{\psi \in L^\infty(\Omega, dadl) \text{ such that } \frac{\partial\psi}{\partial a} \in L^\infty(\Omega, dadl)\}$.

It is well know that $\sigma_r(S_0) \subset \sigma_p(S_0^*)$. In order to get information on the residual spectrum of S_0 , we shall consider the eigenvalue problem $(\lambda - S_0^*)\psi = 0$ with $\operatorname{Re}\lambda \leq -\underline{\mu}$. The solution of $(\lambda - S_0^*)\psi = 0$ is given by

$$\psi(a, l) = g(l) \exp\left(\int_0^a (\lambda + \mu(\tau, l))d\tau\right)$$

where $g(l) \in Y$. As above, the solution ψ must belong to $D(S_0^*)$, so

$$0 = \psi(l, l) = g(l) \exp\left(\int_0^l (\lambda + \mu(\tau, l))d\tau\right).$$

It is clear that, for all $l \in [l_1, +\infty)$, we have $\exp\left(\int_0^l (\lambda + \mu(\tau, l))d\tau\right) \neq 0$. Accordingly, one has $g(\cdot) \equiv 0$ and then $\psi = 0$. This proves that $\sigma_p(S_0^*) = \emptyset$ which implies that $\sigma_r(S_0) = \emptyset$. \square

It follows from Lemma 6.8.2 that $\sigma(S_0) = \sigma_{\text{app}}(S_0) = \sigma_c(S_0)$. Next using the fact that the Schechter and the Browder essential spectra are enlargement of the continuous spectrum, we conclude that

$$\sigma_{\text{eb}}(S_0) = \sigma_{\text{esc}}(S_0) = \{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda \leq -\underline{\mu}\}. \quad (6.8.11)$$

Theorem 6.8.4 *Let B be a smooth operator on X and assume that (A4) holds true. Then*

$$\sigma_{\text{esc}}(A_K) = \sigma_{\text{esc}}(S_K).$$

Moreover, if K is a weakly compact operator, then

$$\sigma_{\text{esc}}(A_K) = \{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda \leq -\underline{\mu}\}.$$

Proof. Let $\lambda \in \mathbb{C}_{\lambda_0}$ be such that $r_\sigma[(\lambda - S_K)^{-1}B] < 1$. Then $\lambda \in \rho(S_K + B)$ and

$$(\lambda - A_K)^{-1} - (\lambda - S_K)^{-1} = \sum_{n \geq 1} [(\lambda - S_K)^{-1}B]^n (\lambda - S_K)^{-1}.$$

It follows from Theorem 6.5.1 that $(\lambda - A_K)^{-1} - (\lambda - S_K)^{-1}$ is weakly compact on X . It follows from Theorem 2.2.1, that $\sigma_{esc}(A_K) = \sigma_{esc}(S_K)$.

Further, if K is weakly compact, then

$$(\lambda - A_K)^{-1} - (\lambda - S_0)^{-1} = \Lambda_\lambda K(I - \Theta_\lambda K)^{-1} \Xi_\lambda + \sum_{n \geq 1} [(\lambda - S_K)^{-1}B]^n (\lambda - S_K)^{-1}$$

is also weakly compact. Now, arguing as above and using (6.8.11) we conclude the desired result. \square

Now, we are in a position to establish the main result of this subsection.

Theorem 6.8.5 *Let B be a smooth operator on X and let K be a weakly compact such that $\|K\| \leq 1$. If the condition (A4) is satisfied, then*

$$\sigma_{eb}(A_K) = \{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re} \lambda \leq -\underline{\mu}\}.$$

Proof. Let $\lambda \in \mathbb{C}_{\lambda_0}$. It follows from Theorem 6.6.1 that either λ belongs to $\rho(A_K)$ or λ is an isolated eigenvalue with finite algebraic multiplicity. So, according to the definition of $\rho_{eb}(A_K)$ (see above) we infer that

$$\sigma_{eb}(A_K) \subset \{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re} \lambda \leq \lambda_0\}.$$

Further, if $\|K\| \leq 1$, then $\lambda_0 = -\underline{\mu}$ and therefore

$$\sigma_{eb}(A_K) \subset \{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re} \lambda \leq -\underline{\mu}\}.$$

Next, using the fact that K is weakly compact, the use of the inclusion $\sigma_{es}(A_K) \subseteq \sigma_{eb}(A_K)$ together with the second assertion of Theorem 6.8.4 gives the desired result. \square

Remark 6.8.2 As a consequence of this theorem is that $\sigma_{as}(A_K) = \Pi(A_K) = \sigma(A_K) \setminus \sigma_{eb}(A_K)$ is contained in the half plane $\mathbb{C}_{-\underline{\mu}}$. It seems that this result is more restrictive than Theorem 6.6.1 (cf. the hypotheses on K), but the novelty here is that the set $\{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re} \lambda \leq -\underline{\mu}\}$ is contained in $\sigma(A_K)$. In fact, this set coincide with the Browder essential spectrum of A_K , $\sigma_{eb}(A_K)$, while the result of Theorem 6.6.1 cannot exclude the existence of holes in $\{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re} \lambda \leq -\underline{\mu}\}$ belonging to the resolvent set of A_K .

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Résumé. Cette thèse est consacrée à la théorie spectrale de quelques opérateurs de transport et le comportement asymptotique (pour les temps grands) des solutions des problèmes de Cauchy gouvernés par ces derniers. Dans la première partie, on s'est intéressé aux propriétés spectrales des opérateurs d'advection et de transport des neutrons dans le cadre multidimensionnel pour des conditions aux limites générales. Après avoir établi un résultat de compacité de type lemmes de moyenne indispensable dans notre analyse, on a donné entre autre une description fine du spectre asymptotique de l'opérateur de transport. Ce travail a été complété par l'étude des propriétés de régularité et le comportement asymptotique de la solution du problème de Cauchy gouverné par l'opérateur de transport étudié précédemment pour des conditions aux limites de type bounce-back plus un opérateur compact dans l'espace L^1 . Ensuite, on a étudié le caractère bien posé et le comportement asymptotique de la solution d'une équation de transport des neutrons avec des sections efficaces non bornées. Contrairement à la première partie, l'analyse de ce problème nécessite l'usage d'une théorie de perturbation de Miyadera-Voigt pour les opérateurs non bornés. La dernière partie de ce travail porte sur un problème linéaire issu d'un modèle introduit en 1974 par Lebowitz et Rubinow décrivant la prolifération d'une population de cellules structurée par l'âge et la longueur du cycle. Notre analyse a porté sur le cas où la longueur du cycle maximale est infinie.

Mots-clés : Problème de Cauchy, opérateur de transport, C_0 -semi-groupe, série de Dyson-Phillips, spectre asymptotique, valeur propre isolée, positivité au sens de l'ordre.

Abstract. This thesis is devoted to the spectral theory and the time asymptotic behavior of the solution to Cauchy problems governed by various transport operators. In the first part, we discussed the spectral properties of streaming and transport operators in finite bodies with general boundary conditions. After establishing a compactness result essential to our analysis, we gave a fine description of the asymptotic spectrum of the transport operator. We also derive the regularity and the asymptotic behavior of the solution to Cauchy problem governed by the transport operator supplemented by bounce-back boundary conditions plus a compact operator in the space L^1 . In the second part, we discussed the well-posedness and the asymptotic behavior of the solution to Cauchy problem governed by a singular transport operator. Unlike the first part, the analysis of this problem requires the use of Miyadera-Voigt perturbation theory for unbounded operators. In the last part of this work, a Cauchy problem governed by a linear operator introduced by Lebowitz and Rubinow describing a proliferating cell population structured by age and the cycle length was considered. Here our analysis was devoted to the case where the maximum cycle length is infinite.

Keywords : Cauchy problem, transport operator, C_0 -semigroup, Dyson-Phillips expansion, asymptotic spectrum, isolated eigenvalue, positivity in the lattice sense.