Qualitative study of solutions of the system of Navier-Stokes equations with variable density

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Résumé

Dans cette thèse, on s'intéresse à deux problèmes provenant de l'étude mathématique des fluides incompressibles visqueux : la propagation de la régularité tangentielle et le mouvement d'une surface libre.

La première question concerne plus particulièrement l'étude qualitative de l'évolution de quantités thermodynamiques telles que la température dans l'équation de Boussinesq sans diffusion et la densité dans le système de Navier-Stokes non homogène. Typiquement, on suppose que ces deux quantités sont, à l'instant initial, discontinues le long d'une interface à régularité höldérienne. Comme conséquence de résultats de propagation de régularité tangentielle pour le champ de vitesses, on établit que la régularité des interfaces persiste pour tout temps aussi bien en dimension deux d'espace, qu'en dimension supérieure (avec condition de petitesse). Notre approche suit celle du travail de J.-Y. Chemin dans les années 90 pour le problème des poches de tourbillon dans les fluides incompressibles parfaits. Dans le cas présent, outre cette hypothèse de régularité tangentielle, nous n'avons besoin que d'une régularité critique sur le champ de vitesses. La démonstration repose sur le calcul para-différentiel et les espaces de multiplicateurs.

Dans la dernière partie de la thèse, on considère le problème à frontière libre pour le système de Navier-Stokes incompressible à deux phases. Ce système permet de décrire l'évolution d'un mélange de deux fluides non miscibles tels que l'huile et l'eau par exemple. Différents cas de figure sont étudiés : le cas d'un réservoir borné, d'une goutte ou d'une rivière à profondeur finie. On établit l'existence et l'unicité à temps petit pour ce problème. Notre démonstration repose fortement sur des propriétés de régularité maximale parabolique de type L_p-L_q.

Mots-clé:

Équations de Navier-Stokes incompressible à densité variable, équations de Boussinesq sans diffusion, régularité critique, régularité stratifiée, problème à frontière libre, écoulement diphasique.

Qualitative study of the solutions of density-dependent incompressible Navier-Stokes system

Abstract

This thesis is dedicated to two different problems in the mathematical study of the viscous incompressible fluids: the persistence of tangential regularity and the motion of a free surface.

The first problem concerns the study of the qualitative properties of some thermodynamical quantities in incompressible fluid models, such as the temperature for Boussinesq system with no diffusion and the density for the non-homogeneous Navier-Stokes system. Typically, we assume those two quantities to be initially piecewise constant along an interface with Hölder regularity. As a consequence of stability of certain directional smoothness of the velocity field, we establish that the regularity of the interfaces persist globally with respect to time both in the two dimensional and higher dimensional cases (under some smallness condition). Our strategy is borrowed from the pioneering works by J.-Y.Chemin in 1990s on the vortex patch problem for ideal fluids. Let us emphasize that, apart from the directional regularity, we only impose rough (critical) regularity on the velocity field. The proof requires tools from para-differential calculus and multiplier space theory.

In the last part of this thesis, we are concerned with the free boundary value problem for two-phase density-dependent Navier-Stokes system. This model is used to describe the motion of two immiscible liquids, like the oil and the water. Such mixture may occur in different situations, such as in a fixed bounded container, in a moving bounded droplet or in a river with finite depth. We establish the short time well-posedness for this problem. Our result strongly relies on the L_p-L_q maximal regularity theory for parabolic equations.

Keywords:

Inhomogeneous incompressible Navier-Stokes equations, Boussinesq equations with no diffusion, critical regularity, striated regularity, free boundary value problem, two-phase flow.

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Contents

Introduction

Le but de cette introduction est de replacer notre travail dans le contexte de la recherche mathématique actuelle portant sur la propagation des singularités dans des modèles de mécanique des fluides incompressibles et visqueux. Nous commençons par quelques rappels sur la dérivation formelle des modèles étudiés, puis exposons nos résultats princpaux ainsi que les techniques utilisées pour les démontrer. Des détails complémentaires sur ces techniques se trouvent en appendice.

0.1 Modélisation générale

Cette thèse est consacrée à plusieurs modèles *simples* décrivant le mouvement de fluides newtoniens. Tous ces modèles sont basés sur un principe expérimental, à savoir le *postulat d'état* (*postulat state* en anglais) stipulant que toutes les quantités thermodynamiques telles que la densité, l'énergie interne, la température, la pression et l'entropie, sont déterminées par des *équations d'état* mettant en jeu deux variables indépendantes. Bien sûr, les équations d'état utilisées dépendent de la situation physique considérée. Nous reviendrons sur ce point à la fin de cette section après une présentation des équations qui régissent le mouvement des fluides.

Pour dériver ces équations, il faut partir de plusieurs hypothèses physiques supplémentaires. La plus fondamentale est celle de *continuité*. Plus précisément, nous considérons le fluide comme un milieu continu, bien qu'un vrai gaz ou liquide soit composé de molécules (discrètes) à l'échelle microscopique. Pour nous, chaque *particule* de fluide sera infiniment petite par rapport à la propriété macroscopique qui nous intéresse, mais pourra néanmoins contenir un nombre très grand de molécules.

En partant de ce principe et de plusieurs lois de conservation, nous allons dériver des équations aux dérivées partielles régissant le mouvement des fluides. Nous suivrons la démarche de [4, 83] et, pour fixer les idées, nous nous concentrerons sur le cas physique de la dimension $N = 3$.

Conservation de la masse

Supposons que Ω soit un domaine simple (de classe $\mathcal{C}^1$) de $\mathbb{R}^3$ contenant un fluide. Soit $\boldsymbol{n}$ le vecteur normal extérieur à la frontière $\partial\Omega$ de Ω. Supposons que la densité ρ et le champ de vitesse $\boldsymbol{u}$ soient réguliers. S'il n'y a pas d'autre source de masse ou aucune réaction chimique à l'intérieur de Ω, la masse totale dans Ω vérifie

$$\frac{d}{dt}\int_{\Omega} \rho \, dx = -\int_{\partial\Omega} \rho \, \boldsymbol{u} \cdot \boldsymbol{n} \, dS. \tag{0.1}$$

L'équation (0.1) ci-dessus signifie que la variation de masse dans Ω est uniquement due à la perte (ou arrivée) de masse à travers la surface $\partial\Omega$. En physique, nous appelons $\rho\,\boldsymbol{u}$ *densité de moment*, et $\rho\,\boldsymbol{u} \cdot \boldsymbol{n}$ est le flux de masse passant à travers $\partial\Omega$ par unité de temps. Par *la théorème de flux-divergence*, (0.1) devient

$$\int_{\Omega} \partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) \, dx = \int_{\Omega} D_t \rho + \rho \operatorname{div} \boldsymbol{u} \, dx = 0, \tag{0.2}$$

où nous notons $D_t := \partial_t + \boldsymbol{u} \cdot \nabla$ la *dérivée particulaire*. La quantité $D_t \boldsymbol{u}$ représente l'accélération de la particule de fluide en mouvement.

Comme (0.2) est vraie pour tout domaine Ω de $\mathbb{R}^3$, l'intégrand est nul presque partout, et l'on a donc *l'équation de continuité* suivante :

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = D_t \rho + \rho \operatorname{div} \boldsymbol{u} = 0. \tag{0.3}$$

De toute évidence, si la densité du fluide est une constante (positive), alors (0.3) dégénère en une forme très simple :

$$\operatorname{div} \boldsymbol{u} = 0. \tag{0.4}$$

Par la suite, nous dirons que le fluide est *incompressible* si la condition (0.4) est satisfaite. Sinon, le fluide est dit *compressible*.

Conservation de la quantité de mouvement

En outre des définitions Ω, $\boldsymbol{n}$, ρ, et $\boldsymbol{u}$ ci-dessus, nous devons rappeler la définition de *tenseur de contrainte*, désigné par $\mathbb{T}$ ici, afin d'étudier la loi d'équilibre de la quantité de mouvement pour les fluides visqueux. En mécanique des milieux continus, le tenseur des contraintes décrit les forces internes entre des particules fluides adjacentes. Mathématiquement, cet effet de contrainte est représenté par un[1] tenseur d'ordre 2 qui peut être vu comme une matrice 3×3. Ce tenseur des contraintes permet de décrire les *forces de surface* (par opposition aux *forces volumiques* comme la gravité par exemple).

[1] Voir [24, le Chapitre 2] pour plus de discussions sur l'algèbre multilinéaire.

Supposons que Ω soit suffisamment petit. La variation de la quantité de mouvement dans Ω peut être causée par différents facteurs. Pour simplifier, considérons dans un premier temps un fluide *idéal* sans force interne et échange de chaleur. La force surfacique totale sur Ω se réduit alors à la pression $\mathfrak{p}$ et est donnée par

$$-\int_{\partial\Omega} \mathfrak{p}\, \boldsymbol{n}\, dS = -\int_{\Omega} \nabla \mathfrak{p}\, dx.$$

La formule ci-dessous permet donc d'exprimer la pression comme une force volumique, à savoir $-\nabla\mathfrak{p}$. Plus généralement, si le fluide est aussi soumis à une force volumique $\boldsymbol{f}$ (telle que la gravité ou la force de Coriolis), *la deuxième loi de motion de Newton* donne

$$\rho D_t \boldsymbol{u} = -\nabla\mathfrak{p} + \rho \boldsymbol{f}. \tag{0.5}$$

Dans le cas ρ constant, le système composé de (0.4) et (0.5) est appelé *système d'Euler incompressible (homogène).* En combinant (0.5) et (0.3), nous arrivons à une forme équivalente[2]

$$\partial_t(\rho\boldsymbol{u}) = -\mathrm{Div}\,\Pi + \rho\boldsymbol{f} \quad \text{avec} \quad \Pi := \mathfrak{p}\,\mathbb{I} + \rho\boldsymbol{u}\otimes\boldsymbol{u}. \tag{0.6}$$

Pour les fluides visqueux, c'est-à-dire pour lesquels il existe des forces de friction entre les particules de fluide adjacentes, il faut prendre en compte la dissipation d'énergie et modifier le tenseur Π dans (0.6) comme ci-dessous :

$$\mathbb{T} := \sigma - \mathfrak{p}\mathbb{I}.$$

La matrice de σ (de taille 3×3) est appelée *tenseur des contraintes visqueuses*, et dépend de la température θ, de la densité ρ et de la vitesse $\boldsymbol{u}$ du fluide.

Déterminons maintenant la forme de σ sous quelques hypothèses simplificatrices. Tout d'abord, comme le frottement intérieur résulte des mouvements relatifs entre les particules de fluide, on suppose que σ est une fonction *linéaire* de $\nabla\boldsymbol{u}$ (cette hypothèse est légitime si les variations de $\nabla\boldsymbol{u}$ ne sont pas trop importantes). Cela signifie en particulier que $\sigma\equiv 0$ pour un fluide isotrope dans les deux situations physiques suivantes:

1. le fluide se déplace à une vitesse constante $\boldsymbol{u}\equiv\boldsymbol{u}_0$;

2. le fluide tourne avec une vitesse angulaire constante $\boldsymbol{\Omega}_0$ (i.e. [3] $\boldsymbol{u}\equiv\boldsymbol{\Omega}_0\wedge x$).

D'autre part, par la conservation du *moment angulaire*, le tenseur des contraintes σ doit être *symétrique* (voir par exemple [4, le Chapitre 1]) et on peut donc l'écrire sous la forme :

[2]Pour toute matrice $\mathbb{A} = \big(A_k^j(x)\big)_{N\times N}$ définie sur $\mathbb{R}^N$, le symbole $\mathrm{Div}\,\mathbb{A}$ est le vecteur de $\mathbb{R}^N$ satisfaisant $(\mathrm{Div}\,\mathbb{A})^j := \sum_{k=1}^N \partial_k A_k^j$.

[3]Dans $\mathbb{R}^3$, nous notons $\boldsymbol{u}\wedge\boldsymbol{v} := (u^2v^3-u^3v^2, u^3v^1-u^1v^3, u^1v^2-u^2v^1)^{tr}$.

$$\sigma := 2\mu\, d + \lambda\, \mathrm{div}\, \boldsymbol{u}\, \mathbb{I},$$

où $d := \frac{1}{2}\mathbb{D}(\boldsymbol{u}) := \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{tr})$ désigne le *tenseur de déformation*. Les fonctions μ et λ (qui dépendent de (θ, ρ)) sont appelées respectivement coefficients de *cisaillement* et de *viscosité dynamique*. Sous toutes ces hypothèses, on dit que le fluide est dit *Newtonien*.

En utilisant l'expression ci-dessus pour σ, on obtient finalement la relation d'équilibre suivante pour la quantité de mouvement des fluides newtoniens :

$$\partial_t(\rho\boldsymbol{u}) = -\mathrm{Div}\,(\rho\boldsymbol{u}\otimes\boldsymbol{u}) + \mathrm{Div}\,\mathbb{T} + \rho\boldsymbol{f}. \tag{0.7}$$

Conservation de l'énergie

Bien que les relations sur l'équilibre de la masse et de la quantité de mouvement suffisent à décrire l'évolution des fluides considérés dans cette thèse, il est instructif de présenter la loi de conservation de l'énergie dans le cas général. Nous supposons ici que les contributions de l'énergie totale par unité de volume sont données par

$$\rho\frac{|\boldsymbol{u}|^2}{2} + \rho\, e,$$

où le premier terme est *l'énergie cinétique* et e représente *l'énergie interne* par unité de masse. En règle générale, l'énergie interne est causée par les mouvements des molécules du matériau. En outre, la variation de l'énergie totale dans Ω ici est réduite à la somme de l'énergie totale transportée à travers de la surface $\partial\Omega$, aux travaux exercés par les forces surfaciques et par la force externe, et par la quantité d'énergie fournie par l'extérieur de Ω. Nous avons donc

$$\begin{aligned}\frac{d}{d\,t}\int_\Omega \rho\Big(\frac{|\boldsymbol{u}|^2}{2} + e\Big)\, dx = -\int_{\partial\Omega}\rho\Big(\frac{|\boldsymbol{u}|^2}{2} + e\Big)\boldsymbol{u}\cdot\boldsymbol{n}\, dS + \int_{\partial\Omega}(\mathbb{T}\boldsymbol{n})\cdot\boldsymbol{u}\, dS \\ + \int_\Omega \rho\boldsymbol{f}\cdot\boldsymbol{u}\, dx - \int_{\partial\Omega}\boldsymbol{q}\cdot\boldsymbol{n}\, dS.\end{aligned} \tag{0.8}$$

La quantité $\boldsymbol{q}$ est appelée *vecteur de flux de chaleur* et satisfait (dans notre modélisation) *la loi de Fourier*

$$\boldsymbol{q} = -\kappa\nabla\theta,$$

avec θ et κ désignant respectivement *la température absolue* et *le coefficient de conductivité*. À partir de (0.8), compte tenu de la loi de Fourier, on obtient par le théorème de flux-divergence, (0.3) et (0.7),

$$\rho D_t e = \mathbb{T} : d + \mathrm{div}\,(\kappa\nabla\theta), \tag{0.9}$$

où $\mathbb{A} : \mathbb{B} := \sum_{j,k=1}^3 A_k^j B_k^j$ pour deux matrices $\mathbb{A}$ et $\mathbb{B}$.

Relations complémentaires

Pour préciser les conditions que doivent vérifier les coefficients, μ, λ et κ, rappelons *la relation de Gibbs*,

$$\theta d\, s = d\, e + \mathfrak{p}\, d(1/\rho) \tag{0.10}$$

avec s étant *l'entropie* par unité de masse. S'il n'y a pas d'autre source d'entropie, l'irréversibilité du processus de diffusion, c'est-à-dire *la deuxième loi de la thermodynamique*, se caractérise par *l'inégalité Clausius-Duhem* suivante :

$$\frac{d}{d\,t}\int_{\Omega} \rho s\, dx \geq -\int_{\partial\Omega}\Big(\rho s \boldsymbol{u}\cdot\boldsymbol{n} + \frac{\boldsymbol{q}\cdot\boldsymbol{n}}{\theta}\Big)\, dS.$$

Donc, par la loi de Fourier et le théorème de flux-divergence, nous avons

$$\partial_t(\rho s) + \operatorname{div}(\rho s \boldsymbol{u}) \geq \operatorname{div}\Big(\frac{\kappa\nabla\theta}{\theta}\Big)\cdot \tag{0.11}$$

Puis, grâce à (0.3) et (0.9), la relation de Gibbs (0.10) entraîne

$$\partial_t(\rho s) + \operatorname{div}(\rho s \boldsymbol{u}) = \frac{1}{\theta}\big(\sigma : d + \operatorname{div}(\kappa\nabla\theta)\big)\cdot \tag{0.12}$$

Maintenant, profitant de (0.11) et (0.12), nous obtenons

$$\frac{\sigma : d}{\theta} + \frac{\kappa|\nabla\theta|^2}{\theta^2} \geq 0. \tag{0.13}$$

Il n'est pas difficile de vérifier que

$$\mu \geq 0, \quad 2\mu + 3\lambda \geq 0, \quad \kappa \geq 0, \tag{0.14}$$

est une condition suffisante pour la validité de (0.13).

Maintenant notons que (0.3), (0.7) et (0.9) ne peuvent pas former un système fermé. Comme nous l'avons mentionné au tout début, pour que le système ne soit pas sous-déterminé, il faut lui adjoindre des *équations d'état* enfin de pouvoir exprimer toutes les quantités thermodynamiques à l'aide de deux inconnues indépendantes. Prenons par exemple ρ et s comme variables indépendantes et écrivons l'énergie interne comme

$$e = e(\rho, s). \tag{0.15}$$

On déduit de (0.10) que

$$\theta = \partial_s e \quad \text{et} \quad \mathfrak{p} = \rho^2 \partial_\rho e. \tag{0.16}$$

De cette manière, les équations (0.3), (0.7), (0.9), (0.15) et (0.16) forment un système fermé.

Il est également possible de prendre ρ et θ comme variables indépendantes, et les relations complémentaires

$$e = e(\rho, \theta) \;\text{ et }\; \mathfrak{p} = \mathfrak{p}(\rho, \theta), \tag{0.17}$$

ferment le système. La relation de Gibbs (0.10) et (0.17), donnent la condition de compatibilité (cachée) suivante :

$$\rho^2 \partial_\rho e + \theta \partial_\theta \mathfrak{p} = \mathfrak{p}, \tag{0.18}$$

tant que $s, e, \mathfrak{p}$ sont suffisamment lisses.

Un exemple classique d'équations d'état (0.17) est celui des gaz polytropiques visqueux parfaits. Un gaz est *parfait* si la pression $\mathfrak{p}$ est donnée par l'équation

$$\mathfrak{p} = R\rho\theta, \;\text{ pour une constante }\; R > 0. \tag{0.19}$$

En outre, si l'énergie interne satisfait

$$e = C_v\theta, \;\text{ pour une constante strictement positive }\; C_v, \tag{0.20}$$

alors on dit que le gaz est *polytropique*.

Quelques modèles asymptotiques

Dans cette thèse, nous étudions principalement des modèles simplifiés obtenus (formellement) à partir du système de Navier-Stokes complet présenté ci-dessus. L'heuristique est que dans les utilisations concrètes de ce système, certaines quantités physiques jouent un rôle négligeable, et peuvent être éliminées dans les équations. D'un point de vue mathématique, les modèles simplifiés avec moins de grandeurs physiques nous aident à mieux comprendre la structure.

L'approche standard de la simplification consiste à utiliser les arguments de mise à l'échelle et asymptotiques. Tout d'abord, nous pouvons choisir des grandeurs de référence pour les différentes quantités physiques et faire un changement d'échelle dans les équations. On obtient ainsi un système *adimensionnalisé*. Ensuite, nous pouvons faire tendre certains paramètres du système vers à 0 ou $\pm\infty$, et écrire les équations obtenues, censées être bien appropriés à la description de certains cas physiques extrêmes.

Précisons d'abord les notations. Si $\mathfrak{h}$ est une variable ou une inconnue scalaire, on désigne par $\mathfrak{h}_{ref}$ la constante de référence correspondante. Par exemple, t_{ref} représente le temps de référence, etc. D'autre part, nous introduisons $(L_{ref}, U_{ref}, F_{ref})$ la longueur de référence, la vitesse et la force externe, respectivement. On peut alors introduire les paramètres sans dimension suivants :

$$
\begin{aligned}
&(\textit{Nombre de Strouhal}) \quad Sr := \tfrac{L_{ref}}{\theta_{ref}U_{ref}}, \qquad (\textit{Nombre de Mach}) \quad Ma := \sqrt{\tfrac{\rho_{ref}}{\mathfrak{p}_{ref}}}U_{ref},\\
&(\textit{Nombre de Reynolds}) \quad Re := \tfrac{\rho_{ref}U_{ref}L_{ref}}{\mu_{ref}}, \quad (\textit{Nombre de Froude}) \quad Fr := \tfrac{U_{ref}}{\sqrt{L_{ref}F_{ref}}},\\
&(\textit{Nombre de Péclet}) \quad Pe := \tfrac{\mathfrak{p}_{ref}L_{ref}U_{ref}}{\theta_{ref}\kappa_{ref}}.
\end{aligned}
$$

Ensuite, on introduit les nouvelles variables (scalaires ou vectorielles) $\widehat{\mathfrak{h}} := \mathfrak{h}/\mathfrak{h}_{ref}$ avec $\mathfrak{h}$ appartenant à $\{t, x, \rho, \mathfrak{p}, \theta, e, \mu, \lambda, \kappa, \boldsymbol{u}, \boldsymbol{f}\}$. Il est donc naturel de faire le changmenet d'échelle :

$$
\widehat{\mathfrak{h}}(\widehat{x}, \widehat{t}) = \frac{1}{\mathfrak{h}_{ref}}\mathfrak{h}(L_{ref}\widehat{x}, t_{ref}\widehat{t}).
$$

En supposant $\mu_{ref} = \lambda_{ref}$ et $\mathfrak{p}_{ref} = \rho_{ref}e_{ref}$ (defini dans (0.18)), nous pouvons finalement écrire (0.3), (0.7) et (0.9) (en omettant les chapeaux par souci de lisibilité) comme suit :

$$
\begin{cases}
Sr\,\partial_t\rho + \mathrm{div}\,(\rho\boldsymbol{u}) = 0,\\
Sr\,\partial_t(\rho\boldsymbol{u}) + \mathrm{Div}\,(\rho\boldsymbol{u}\otimes\boldsymbol{u}) - \frac{1}{Re}\mathrm{Div}\,\sigma + \frac{1}{Ma^2}\nabla\mathfrak{p} = \frac{1}{Fr^2}\rho\boldsymbol{f},\\
Sr\,\partial_t(\rho e) + \mathrm{div}\,(\rho\boldsymbol{u}e) = \frac{Ma^2}{Re}\sigma : d - \mathfrak{p} : d + \frac{1}{Pe}\mathrm{div}\,(\kappa\nabla\theta).
\end{cases}
\tag{0.21}
$$

Le système (0.21) est *le système de Navier-Stokes complet* écrit sous forme adimensionnelle. Concentrons nous sur le cas

$$
Sr = Re = Pe = Ma/\varepsilon = Fr/\sqrt{\varepsilon} = 1,
$$

et sur l'asymptotique ε tend vers 0 (limite à faibles nombres de Mach et de Froude).

Dans le cas de gaz polytropiques parfaits avec $\mathfrak{p} = R\rho e/C_v$, nous obtenons formellement à la limite *les équations de Navier-Stokes inhomogènes* suivantes (voir les détails dans [89, Section 1.2]):

$$
\begin{cases}
\partial_t\rho + \mathrm{div}\,(\rho\boldsymbol{u}) = 0, \quad \mathrm{div}\,\boldsymbol{u} = 0,\\
\partial_t(\rho\boldsymbol{u}) + \mathrm{Div}\,(\rho\boldsymbol{u}\otimes\boldsymbol{u}) - \mathrm{Div}\,(2\mu\,d) + \nabla P = \rho\boldsymbol{f}.
\end{cases}
$$

Une autre simplification largement utilisée est *l'approximation d'Oberbeck-Boussinesq*. Maintenant, nous supposons formellement que

$$
\begin{aligned}
\rho &= \bar{\rho} + \varepsilon\rho^{(1)} + \varepsilon^2\rho^{(2)} + \ldots,\\
\boldsymbol{u} &= \boldsymbol{U} + \varepsilon\boldsymbol{u}^{(1)} + \varepsilon^2\boldsymbol{u}^{(2)} + \ldots,\\
\theta &= \bar{\theta} + \varepsilon\theta^{(1)} + \varepsilon^2\theta^{(2)} + \ldots,
\end{aligned}
$$

où le couple constant $(\bar{\rho}, \bar{\theta})$ est un équilibre thermodynamique. En injectant ces développements dans (0.21) et ignorant les termes d'ordre ε, nous arrivons à (voir [52, le Chapitre 4]) à :

$$
\begin{cases}
\bar\rho\partial_t \boldsymbol{U} + \bar\rho \mathrm{Div}\,(\boldsymbol{U}\otimes\boldsymbol{U}) - \mathrm{Div}\,(2\mu(\bar\theta)d)\nabla\Pi = r\nabla F, \quad \mathrm{div}\,\boldsymbol{U} = 0,\\
\bar\rho c_p(\bar\rho,\bar\theta)\big(\partial_t\Theta + \mathrm{div}\,(\Theta\boldsymbol{U})\big) - \mathrm{div}\,(G\boldsymbol{U}) - \mathrm{Div}\,\big(\kappa(\bar\theta)\nabla\Theta\big) = 0,
\end{cases}
$$

où F est un potentiel, $\Theta = \theta^{(1)}$, $G := \bar\rho\bar\theta\alpha(\bar\rho,\bar\theta)F$ et r vérifie la *la relation de Boussinesq*

$$
r + \bar\rho\alpha(\bar\rho,\bar\theta)\Theta = 0.
$$

Dans la suite de cette thèse, nous nous concentrerons sur l'étude du modèle de Navier-Stokes inhomogène et sur une approximation de type Oberbeck-Boussinesq avec $\kappa = 0$.

0.2 Principaux résultats

Dans cette section, nous exposerons les principaux résultats obtenus dans la thèse concernant les trois problèmes suivants :

(P_1) La persistance des "poches de température" pour le système de Boussinesq sans diffusion;

(P_2) L'évolution des "poches de densité pour le système de Navier-Stokes inhomogène;

(P_3) Un problème à frontière libre pour le système de Navier-Stokes inhomogène à deux phases.

Les résultats obtenus pour (P_1) et (P_2) sont les fruits d'une collaboration avec Raphaël Danchin et ont été publiés récemment dans [40, 41]. Le problème (P_3) est un travail en cours avec Hirokazu Saito et Yoshihiro Shibata.

Dans la suite de cette section, nous présenterons successivement (P_1)-(P_3). Cependant, l'objectif est seulement de décrire l'idée principale de nos résultats afin d'éviter les longues préparatifs pour des déclarations plus détaillées. Le lecteur intéressé peut trouver plus de références et des théorèmes plus précis dans les chapitres ultérieurs. Dans l'intervalle, nous aimerions également examiner d'autres contributions concernant les problèmes (P_1) et (P_2) après nos résultats [40, 41].

0.2.1 Les poches de température pour le système de Boussinesq

Le système de Boussinesq avec viscosité mais sans diffusion s'écrit

$$
\begin{cases}
\partial_t\theta + \mathrm{div}\,(\theta\boldsymbol{u}) = 0 & \text{dans} \quad \mathbb{R}^N\times]0,T[,\\
\partial_t\boldsymbol{u} + \mathrm{div}\,(\boldsymbol{u}\otimes\boldsymbol{u}) - \nu\Delta\boldsymbol{u} + \nabla P = \theta\boldsymbol{e}_{N} & \text{dans} \quad \mathbb{R}^N\times]0,T[,\\
\mathrm{div}\,\boldsymbol{u} = 0 & \text{dans} \quad \mathbb{R}^N\times]0,T[,\\
(\theta,\boldsymbol{u})|_{t=0} = (\theta_0,\boldsymbol{u}_0) & \text{dans} \quad \mathbb{R}^N.
\end{cases}
\tag{B}
$$

Les inconnues $(\theta, \boldsymbol{u}, P)$ désignent la température, la vitesse et la pression, respectivement, $\boldsymbol{e}_N := (0, ..., 1)^{tr}$ est le dernier vecteur de la base canonique de $\mathbb{R}^N$, et $\nu > 0$ est la viscosité cinématique, supposée constante pour simplifier. Le système ci-dessus est couramment utilisé comme modèle simplifié pour décrire l'évolution de fluides géophysiques. Mais une autre motivation pour son étude est que dans le cas non visqueux, ce système présente des analogies frappantes avec le système d'Euler incompressible en dimension 3, en ce qui concerne le phénomène d'étirement du tourbillon (*vortex stretching* en anglais), voir par exemple [71].

Notre but ici est d'étudier si, à l'instar du système d'Euler incompressible, le système de Boussinesq préserve la régularité stratifiée et, plus précisément, la régularité des fonctions caractéristiques pour la température (poche de température) pour tout temps (voir l'appendice A pour une présentation plus précise de la notion de régularité stratifiée pour le système d'Euler).

Rappelons que dans le cas de la dimension deux, l'existence globale et l'unicité de solutions à régularité Sobolev élevée pour (B) a été établie dans [16, 71]. Ces hypothèses de régularité ont été affaiblies dans [2, 38] où il est démontré que (B) admet une unique solution globale si les données vérifient les propriétés *de régularité critique* suivantes (notées $(\mathcal{H}_B)$ dans la suite) :

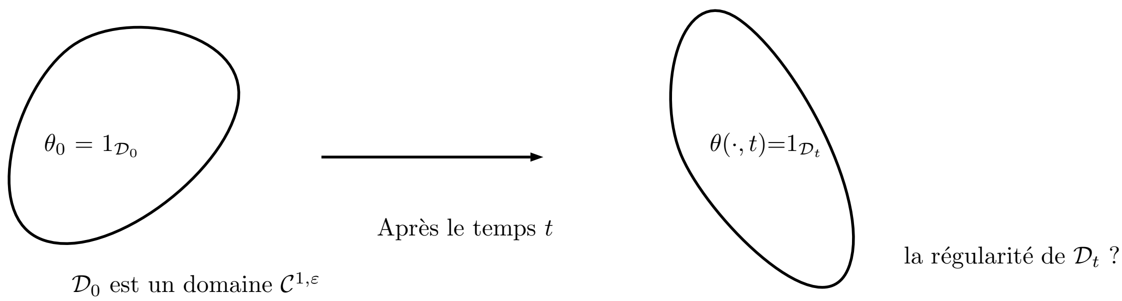
- $N = 2 : (\theta_0, \boldsymbol{u}_0) \in B^0_{2,1}(\mathbb{R}^2) \times \big(L_2(\mathbb{R}^2) \cap B^{-1}_{\infty,1}(\mathbb{R}^2)\big)^2$;
- $N \geq 3 : (\theta_0, \boldsymbol{u}_0) \in \big(L_{\frac{N}{3}}(\mathbb{R}^N) \cap \dot{B}^0_{N,1}(\mathbb{R}^N)\big) \times \big(L_{N,\infty}(\mathbb{R}^N) \cap \dot{B}^{\frac{N}{p}-1}_{p,1}(\mathbb{R}^N)\big)^N$ avec $p \in [N, \infty]$, et

$$\|\boldsymbol{u}_0\|_{L_{N,\infty}(\mathbb{R}^N)} + \nu^{-1}\|\theta_0\|_{L_{\frac{N}{3}}(\mathbb{R}^N)} \leq c\nu \quad \text{avec } \ c \text{ assez petit.}$$

La solution ainsi construite admet un flot ψ_u de classe $\mathcal{C}^1$. De ce fait, si l'on suppose que la température initiale est donnée par

$$\theta_0 = C_N \mathbb{1}_{\mathcal{D}_0} \tag{0.22}$$

avec $\mathcal{D}_0$ un domaine simple de $\mathbb{R}^N$, alors $\theta_t = C_N \mathbb{1}_{\mathcal{D}_t}$ avec $\mathcal{D}_t = \psi_u(t, \mathcal{D}_0)$, et la régularité $\mathcal{C}^1$ est préservée pour tout temps (pourvu que C_N soit assez petite si $N \geq 3$).



Supposons maintenant que $\mathcal{D}_0$ soit un domaine simple de classe $\mathcal{C}^{1,\varepsilon}$ $(0 < \varepsilon < 1)$. On se demande si cette régularité höldérienne supplémentaire est conservée au cours de l'évolution (voir la figure ci-dessus).

On ne peut pas espérer que l'hypothèse $(\mathcal{H}_B)$ assure plus que la régularité $\mathcal{C}^1$ pour $\boldsymbol{\psi}_u$. Mais nous allons démontrer que si θ_0 est donné par (0.22) et si u_0 est un plus régulier dans les directions tangentes à $\mathcal{D}_0$ alors il en est de même pour le flot $\boldsymbol{\psi}_u$. Pour mettre en valeur cette régularité de type anisotrope, on va faire appel à la stratégie employée par J.-Y. Chemin dans [19] concernant le système d'Euler incompressible.

Supposons que $\partial\mathcal{D}_0$ soit une courbe de Jordan de $\mathbb{R}^2$. Alors il existe une fonction f_0 de classe $\mathcal{C}^{1,\varepsilon}$ telle que $\partial\mathcal{D}_0 = f_0^{-1}(\{\mathbf{0}\})$. Pour mesurer la régularité tangentielle par rapport à la poche initiale d'une fonction g, il est donc naturel de considérer la dérivée de g par rapport au champ de vecteurs $\boldsymbol{X}_0 := \nabla^\perp f_0$, qui est bien tangent à $\partial\mathcal{D}_0$. En faisant évoluer $\boldsymbol{X}_0$ suivant le flot $\boldsymbol{\psi}_u$, c'est-à-dire en considérant le champ de vecteurs $\boldsymbol{X}$ dépendant du temps solution de l'équation[4]

$$D_t\boldsymbol{X} := (\partial_t + \boldsymbol{u}\cdot\nabla)\boldsymbol{X} = \partial_X\boldsymbol{u} := \sum_{k=1}^{N} X^k\partial_k\boldsymbol{u}, \quad X|_{t=0} = \boldsymbol{X}_0,$$

on constate que X_t coïncide avec $\nabla^\perp f_t$, où $f_t = f\circ\boldsymbol{\psi}_u(t,\cdot)$. Etant donné que $\mathcal{D}_t = \boldsymbol{\psi}_u(t,\mathcal{D}_0) = f_t^{-1}(\{\mathbf{0}\})$, pour démontrer la persistance de la régularité $\mathcal{C}^{1,\varepsilon}$ de la poche, il suffit donc de montrer la persistance de la régularité $\mathcal{C}^{0,\varepsilon}$ de $\boldsymbol{X}$. Vu l'équation ci-dessus, cette dernière est guarantie par le fait que $\partial_X\boldsymbol{u}$ soit dans $\mathcal{C}^{0,\varepsilon}(\mathbb{R}^2)^2$ ou encore que $\partial_X\omega$ soit la dérivée d'une fonction de $\mathcal{C}^{0,\varepsilon}(\mathbb{R}^2)$, où ω désigne le tourbillon (ou rotationnel) associé à u, c'est-à-dire $\omega := \partial_2u^1 - \partial_1u^2$.

En dimension $N \geq 3$, la méthode est similaire, mais il faut plusieurs champs de vecteurs tangents pour caractériser la régularité de $\partial\mathcal{D}_t$ (voir le Chapitre 1 pour plus de détails).

La difficulté principale est donc de propager la regularité de $\partial_X\boldsymbol{u}$ ou de $\partial_X\omega$ sachant que les seules propriétés de régularité globales vérifiées par $(\theta_0,\boldsymbol{u}_0)$ sont critiques (i.e. données en $(\mathcal{H}_B)$). Dans le cas de la dimension 2, il est plus aisé de travailler avec $\mathrm{div}\,(\boldsymbol{X}\omega)$ qui vérifie l'équation de type transport-diffusion suivante :

$$D_t\mathrm{div}\,(\boldsymbol{X}\omega) - \nu\Delta\mathrm{div}\,(\boldsymbol{X}\omega) = \text{terme source}.$$

En suivant l'approche utilisée pour les poches de tourbillon visqueuses (voir [6, Chapitre 7]), nous obtiendrons les résultats souhaités.

En dimension $N \geq 3$, propager la régularité stratifiée est plus délicat à cause du phénomène d'*étirement de la vorticité* (de ce fait, un terme d'ordre maximal supplémentaire apparaît dans l'équation de $\mathrm{div}\,(\boldsymbol{X}\omega)$). Pour cette raison, au lieu de chercher à démontrer directement que $\partial_X\boldsymbol{u}$ ou $\mathrm{div}\,(\boldsymbol{X}\omega)$ a la régularité souhaitée, on va travailler avec le parachamp T_X et s'intéresser à $\mathcal{T}_X\boldsymbol{u} := T_{X^k}\partial_k\boldsymbol{u}$ qui, dans notre contexte, peut être vu comme la partie principale de $\partial_X\boldsymbol{u}$. Rappelons que T est l'opérateur de paraproduit introduit par J.-M. Bony (pour plus de détails sur la définition et les propriétés du paraproduit, voir par exemple [6, Chapitre 2]). En appliquant le

[4] La première égalité est équivalente au fait que le commutateur $[D_t, \partial_X]$ s'annule.

parachamp $\mathcal{T}_X$ à l'équation du moment de (B), on obtient finalement le système de type Stokes suivant :

$$D_t \mathcal{T}_X \boldsymbol{u} - \nu \Delta \mathcal{T}_X \boldsymbol{u} + \nabla \mathcal{T}_X P = \text{terme source}, \quad \operatorname{div} \mathcal{T}_X \boldsymbol{u} = \operatorname{div}\,(\text{terme d'erreur}).$$

Grâce aux estimations classiques pour le système de Stokes et à des estimations de commutateur, on parvient à boucler les estimations pour tout temps. Pour plus de détails, le lecteur peut se référer au résultat général de persistance de la régularité stratifiée énoncé dans le théorème 1.1, ainsi qu'au théorème 1.3 du chapitre 1.

Donnons tout de suite les résultats obtenus dans le cadre strict des poches de température.

Corollaire 0.1 (poche de température 2-D). *Soit $(M_1, M_2) \in \mathbb{R}^2$. Supposons que $\theta_0 = M_1 \mathbb{1}_{\mathcal{D}_0}$ avec $\partial\mathcal{D}_0$ courbe de Jordan de classe $\mathcal{C}^{1,\varepsilon}$, et que le tourbillon ω_0 de $\boldsymbol{u}_0$ soit donné par*

$$\omega_0 = M_2 \mathbb{1}_{\mathcal{D}_0} - \widetilde{\omega}_0$$

avec $\widetilde{\omega}_0 \in L^r(\mathbb{R}^2)$ pour un $r > 1$, supporté dans $\overline{\mathcal{D}_0^\star}$ et tel que

$$\int_{\mathbb{R}^2} \widetilde{\omega}_0(x)\, dx = M_2\, |\mathcal{D}_0|.$$

Alors il existe une unique solution (θ, u) à (B). De plus, on a $\theta(t,\cdot) = M_1 \mathbb{1}_{\mathcal{D}_t}$ avec $\mathcal{D}(t) := \boldsymbol{\psi}_u(t, \mathcal{D}_0)$, et $\partial\mathcal{D}(t)$ demeure une courbe de Jordan de classe $\mathcal{C}^{1,\varepsilon}$ pour tout $t \geq 0$.

Corollaire 0.2 (poche de température N-D). *Soit $N \geq 3$ et m_1, m_2 constantes suffisamment petites. Supposons que $\theta_0 = m_1 \mathbb{1}_{\mathcal{D}_0}$ et que la i-ème composante de la vitesse initiale soit donnée par*

$$u_0^i := -\sum_{j=1}^{N} (-\Delta)^{-1} \partial_j (\Omega_0)_j^i$$

avec $\Omega_0 := m_2 \mathbb{1}_{J_0} \mathbb{A}_0$ où $\mathbb{A}_0$ est la matrice anti-symétrique définie par $(\mathbb{A}_0)_j^i = 1$ si $i < j$, et J_0 est un domaine simple borné de classe $\mathcal{C}^1$ tel que $\overline{\mathcal{D}_0} \subset J_0$.

Alors $\theta(t,\cdot) = m_1 \mathbb{1}_{\mathcal{D}_t}$ où $\mathcal{D}(t) := \boldsymbol{\psi}_u(t, \mathcal{D}_0)$, et $\mathcal{D}(t)$ demeure un domaine simple de classe $\mathcal{C}^{1,\varepsilon}$, pour tout $t \geq 0$.

A la suite de notre article [40], F. Gancedo et E. Garcia-Juarez [57] ont proposé une démonstration alternative et particulièrement simple pour la persistance de la régularité des poches de température dans le cas de la dimension 2. En supposant que la vitesse initiale est plus régulière : $\boldsymbol{u}_0 \in H^{\varepsilon+s}(\mathbb{R}^2)^2$ avec $s > 0$ et $\varepsilon + s < 1$ et que $\theta_0 := \mathbb{1}_{\mathcal{D}_0}$ comme ci-dessus, ils constatent que le flot $\boldsymbol{\psi}_u$ est $\mathcal{C}^{1,\varepsilon}$. De ce fait, la préservation de la régularité $\mathcal{C}^{1,\varepsilon}$ pour la poche devient triviale. Notons cependant que les hypothèses de [57] ne sont pas de même nature que les nôtres: dans leur article, il s'agit d'imposer suffisamment de régularité sur la vitesse pour avoir un flot globalement $\mathcal{C}^{1,\varepsilon}$ alors que nos hypothèses sont vraiment de nature locale et anisotrope, la seule régularité globale disponible étant critique.

0.2.2 Poches de densité pour les fluides incompressibles non homogènes

Notre second travail concerne le système de Navier-Stokes incompressible inhomogène:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\boldsymbol{u}) = 0 & \text{dans} \quad \mathbb{R}^N\times]0,T[, \\ \partial_t(\rho\boldsymbol{u}) + \operatorname{div}(\rho\boldsymbol{u}\otimes\boldsymbol{u}) - \mu\Delta\boldsymbol{u} + \nabla P = \boldsymbol{0} & \text{dans} \quad \mathbb{R}^N\times]0,T[, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{dans} \quad \mathbb{R}^N\times]0,T[, \\ (\rho,\boldsymbol{u})|_{t=0} = (\rho_0,\boldsymbol{u}_0) & \text{dans} \quad \mathbb{R}^N. \end{cases} \tag{INS}$$

La nouvelle inconnue ρ désigne la densité. Ce modèle peu être obtenu au moins formellement dans l'asymptotique à faible nombre de Mach à partir du système de Navier-Stokes compressible complet (voir par exemple [89, Chapitre 1]). Ce système a été abondamment étudié récemment. Nous renvoyons le lecteur au chapitre 2 pour plus de détails. Ici nous souhaitons uniquement présenter nos résultats principaux concernant la question (P_2).

Le problème des poches de densité pour (INS) a été proposé par P.-L. Lions dans son livre [89]. La théorie des solutions faibles exposée dans ce livre permet de construire une solution globale $(\rho,\boldsymbol{u})$ pour (INS) dès que la donnée initiale $(\rho_0,\boldsymbol{u}_0)$ vérifie

$$\sqrt{\rho_0}\boldsymbol{u}_0 \in L_2(\mathbb{R}^N)^N \quad \text{et} \quad \rho_0 := \mathbb{1}_{\mathcal{D}_0} \quad \text{pour un domaine simple} \quad \mathcal{D}_0 \subset \mathbb{R}^N.$$

De plus, à tout instant $t \geq 0$, on a $\rho(\cdot,t) = \mathbb{1}_{\mathcal{D}_t}$ avec $\mathcal{D}_t := \boldsymbol{\psi}_u(t,\mathcal{D}_0)$, et $\boldsymbol{\psi}_u$ flot généralisé de u. La question posée par P.-L. Lions concerne la régularité de la frontière du domaine. Quelles sont les hypothèses sur les données assurant sa conservation pour tout temps ? Si $\boldsymbol{u}_0$ est seulement L_2, le flot correspondant est très peu régulier (et sa régularité est susceptible de se dégrader au cours du temps, voir le Theorem A.2 de l'Appendice A). Pour répondre à la question de P.-L. Lions, il est donc naturel de se placer dans un cadre de régularité suffisant pour avoir non seulement l'existence, mais aussi l'unicité, ainsi que suffisamment de régularité pour le flot.

Schématiquement, si le fluide considéré ne comporte pas de vide à l'instant initial, alors les résultats d'existence et d'unicité pour (INS) sont similaires à ceux du cas à densité constante. Considérons par exemple une densité initiale

$$\rho_0 := \mathbb{1}_{\mathcal{D}_0} + \eta\mathbb{1}_{\mathcal{D}_0^c} \tag{0.23}$$

avec $\mathcal{D}_0$ domaine simple de $\mathbb{R}^N$ et $\eta > 0$ constant. Si l'on suppose de plus que $\eta - 1$ et que $\|\boldsymbol{u}_0\|_{\dot{B}^{N/p-1}_{p,1}(\mathbb{R}^N)}$ sont assez petits (avec $p \in]N-1, 2N[$), alors R. Danchin et P. Mucha ont établi dans [36] la persistance de la régularité $\mathcal{C}^1$ de $\mathcal{D}_t$, pour tout temps. Cependant, sous de telles hypothèses, le flot ψ_u n'est a priori que $\mathcal{C}^1$, et il n'est pas clair que l'on puisse propager plus de régularité sur $\mathcal{D}_t$.

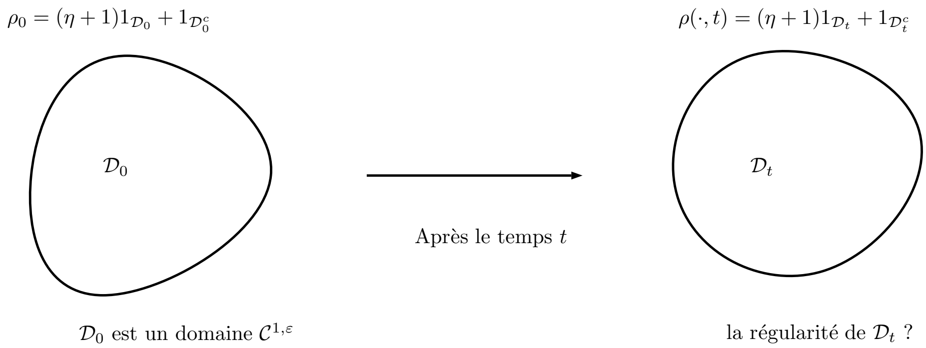
Dans [96], M. Paicu, P. Zhang et Z. Zhang ont réussi à supprimer l'hypothèse de petitesse sur $\eta - 1$ présente dans [36], en supposant que $\boldsymbol{u}_0 \in H^s(\mathbb{R}^2)^2$ pour un $s > 0$ (dans le cas de la dimension 2). Peu de temps après, X. Liao et P. Zhang dans [87, 88] ont utilisé les résutats de [96] pour traiter le problème des poches de densité associé à (INS). Plus précisément, ils supposent que la densité initiale est donnée par (0.23) avec $\mathcal{D}_0$ domaine borné de $\mathbb{R}^2$ de classe $W^{k+2,p}$ avec $k \geq 1$ et $p \in]2, \frac{2}{1-s}[$. Sous ces hypothèses, on peut trouver une fonction $f_0 \in W^{k+2,p}(\mathbb{R}^2; \mathbb{R})$ telle que

$$\partial\mathcal{D}_0 := f_0^{-1}(\{0\}) \quad \text{et} \quad \nabla f_0 \neq \mathbf{0} \ \text{ près de } \ \partial\mathcal{D}_0. \tag{0.24}$$

Si $\boldsymbol{X}_0 := \nabla^\perp f_0$ et la vitesse initiale u_0 vérifie

$$\boldsymbol{u}_0 \in B_{2,1}^{\varepsilon+s}(\mathbb{R}^2), \quad \partial_{X_0}^\ell \boldsymbol{u}_0 \in B_{2,1}^{s+\varepsilon\frac{k-\ell}{k}}(\mathbb{R}^2),$$

les articles [87, 88] assurent la persistance de la régularité $W^{k+2,p}$ ($\hookrightarrow C^{1+k,\varepsilon}$) de l'interface, sans condition de petitesse sur η.



Le but principal de notre travail est d'étudier la régularité intermédiaire entre les travaux de [36] (persistance de la régularité $\mathcal{C}^1$) et ceux de X. Liao et P. Zhang dans [87] (persistance de la régularité $W^{k,p}$ avec $k \geq 3$ et $p \in [2, 4]$). Notre approche repose sur la méthode utilisée pour le système de Boussinesq et exposée plus haut, et sur [36]. L'étape clef consiste à démontrer la propagation pour tout temps d'une régularité Besov adéquate pour $\partial_X \boldsymbol{u}$. Ensuite on utilise la quantité $\dot{\mathcal{T}}_X \boldsymbol{u} = \dot{T}_{X^k} \partial_k \boldsymbol{u}$ pour retrouver la régularité tangentielle de $\partial_X \boldsymbol{u}$ où $\dot{T}$ désigne l'opérateur de paraproduit homogène. A noter que le couplage entre ρ et $\boldsymbol{u}$ est quasi-linéaire dans (INS), alors que celui entre θ et $\boldsymbol{u}$ était linéaire dans (B). A cause de cette non-linéarité, appliquer directement $\dot{\mathcal{T}}_X$ à l'équation du moment de (INS) donne des termes problématiques comme $\dot{\mathcal{T}}_X(\rho\, \partial_t \boldsymbol{u})$. Dans le cas d'une densité constante par morceaux et d'un champ de vitesse à régularité seulement critique, ce terme empêche de boucler les estimations pour tout temps.

En utilisant le fait que $[D_t, \partial_X] \equiv 0$, on peut récrire l'équation de $\partial_X \boldsymbol{u}$ sous la forme

$$\rho D_t \dot{\mathcal{T}}_X \boldsymbol{u} - \mu \Delta \dot{\mathcal{T}}_X \boldsymbol{u} = \text{error terms.} \tag{0.25}$$

Ensuite, en s'inspirant de la technique d'espaces de multiplicateurs utilisée dans [34, 36], et, rappelée plus haut pour (B), on peut traiter (INS). Pour éviter de rentrer dans les détails techniques, donnons le résultat obtenu en dimension deux (le cas de la dimension $N \geq 3$ étant présenté dans le chapitre 2).

Corollaire 0.3. *Soit $\mathcal{D}_0$ un domaine simplement connexe de $\mathbb{R}^2$ et une fonction f_0 dans $\mathcal{C}^{1,\varepsilon}(\mathbb{R}^2;\mathbb{R})$ ($0<\varepsilon<1$) vérifiant (0.24). Il existe une constante η_0 ne dépendant que de $\mathcal{D}_0$ et telle que si*

$$\rho_0 := (1+\eta)\mathbb{1}_{\mathcal{D}_0} + \mathbb{1}_{\mathcal{D}_0^c} \quad \textit{avec} \quad \eta \in]-\eta_0, \eta_0[$$

et si le champ de vecteurs à divergence nulle $\boldsymbol{u}_0 \in \mathcal{S}'_h(\mathbb{R}^2)$ a un tourbillon $\omega_0 := \partial_1 u_0^2 - \partial_2 u_0^1$ vérifiant

$$\omega_0 = \overline{\omega}_0 + \widetilde{\omega}_0\, \mathbb{1}_{\mathcal{D}_0} \ \ \textit{avec} \ \ \mathrm{div}\,(\overline{\omega}_0 \nabla^\perp f_0) = 0 \quad \textit{et} \quad \int_{\mathbb{R}^2} \omega_0\, dx = 0$$

pour des fonctions suffisamment petites $(\overline{\omega}_0, \widetilde{\omega}_0)$ de $L_p(\mathbb{R}^2) \times \mathcal{C}^{\varepsilon'}(\mathbb{R}^2)$ avec $0<\varepsilon'<\varepsilon$ et $1<p<2/(2-\varepsilon)$, et à support compact, alors le système (INS) a une unique solution $(\rho, \boldsymbol{u}, \nabla P)$. De plus pour tout $t \geq 0$,

$$\rho(t,\cdot) := (1+\eta)\mathbb{1}_{\mathcal{D}_t} + \mathbb{1}_{\mathcal{D}_t^c} \ \ \textit{avec} \ \ \mathcal{D}_t := \boldsymbol{\psi}_u(t, \mathcal{D}_0),$$

et $\mathcal{D}_t$ demeure un domaine simplement connexe de classe $\mathcal{C}^{1,\varepsilon}$.

Pour clore cette section, signalons deux travaux très récents qui ont suivi [41]. Tout d'abord, dans [56], F. Gancedo et E. Garcia-Juarez ont considéré (INS) dans le cadre strict des poches de densité en dimension deux. En reprenant l'approche de [96], ils ont établi la persistance pour tout temps de la régularité $\mathcal{C}^{1,\varepsilon}$ pour toute constante $\eta > -1$. La clef de la démonstration est que le flot $\boldsymbol{\psi}_u$ est globalement dans $C^{1,\varepsilon}$. Une autre contribution très récente de R.Danchin et P.B.Mucha [35] concerne le cas avec vide : les auteurs établissent en particulier que si $\rho_0 := \mathbb{1}_{\mathcal{D}_0}$ avec $\mathcal{D}_0$ de classe $\mathcal{C}^{1,\varepsilon}$, et si $\boldsymbol{u}_0$ est H^1 alors le système admet une unique solution qui conserve la régularité $\mathcal{C}^{1,\varepsilon}$ de $\mathcal{D}_0$.

0.2.3 Un problème à frontière libre

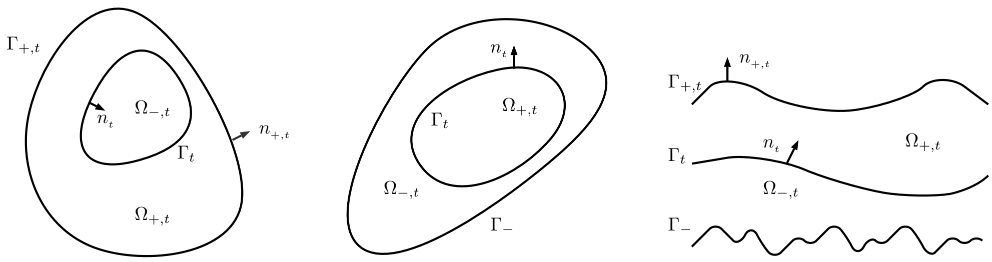
On peut voir le problème des poches de tourbillon comme un modèle de mélange pour deux fluides incompressibles visqueux et non miscibles. Dans le cas sans tension de surface, un modèle plus élaboré est le suivant :

$$
\begin{cases}
\partial_t(\rho \boldsymbol{v}) + \mathrm{Div}\,(\rho \boldsymbol{v} \otimes \boldsymbol{v}) - \mathrm{Div}\,\mathbb{T}(\boldsymbol{v}, \mathfrak{p}) = \rho \boldsymbol{f} & \text{dans} \quad \dot{\Omega}_t, \\
\partial_t \rho + \mathrm{div}\,(\rho \boldsymbol{v}) = 0, \;\; \mathrm{div}\, \boldsymbol{v} = 0 & \text{dans} \quad \dot{\Omega}_t, \\
[\![\mathbb{T}(\boldsymbol{v}, \mathfrak{p}) \boldsymbol{n}_t]\!] = \boldsymbol{0}, \;\; [\![\boldsymbol{v}]\!] = \boldsymbol{0} & \text{dans} \quad \Gamma_t, \\
\mathbb{T}(\boldsymbol{v}_+, \mathfrak{p}_+) \boldsymbol{n}_{+,t} = \boldsymbol{0} & \text{sur} \quad \Gamma_{+,t}, \\
\boldsymbol{v} = \boldsymbol{0} & \text{on} \quad \Gamma_-, \\
(\rho, \boldsymbol{v})|_{t=0} = (\rho_0, \boldsymbol{v}_0) & \text{sur} \quad \dot{\Omega}.
\end{cases}
\tag{$INS_\pm$}
$$

Nous nous intéressons ici au problème à deux phases séparées par une interface $\Gamma_t \neq \emptyset$. Le système ci-dessus permet de rendre compte de trois situations physiques différentes : l'évolution d'une goutte de fluide dans un autre fluide, un mélange dans un récipient borné avec parois rigides, et deux couches infinies (voir la figure ci-dessous). Pour simplifier la présentation, on suppose qu'il n'y a pas de frontière rigide : $\Gamma_- = \emptyset$. Dans ce cas, le modèle ($INS_\pm$) décrit le mouvement de deux liquides ne se mélangeant pas dans le domaine $\Omega_t := \dot{\Omega}_t \cup \Gamma_t := \Omega_{\pm,t} \cup \Gamma_t$ entouré par une surface mobile $\Gamma_{+,t}$ à l'instant t. Dans cette description, Γ_t est l'interface entre les deux phases qui remplissent $\Omega_{+,t}$ et $\Omega_{-,t}$, respectivement. On note $\boldsymbol{n}_t$ et $\boldsymbol{n}_{+,t}$ les vecteurs normaux extérieurs pour les surfaces Γ_t et $\Gamma_{+,t}$. Insistons sur le fait que le domaine de l'ensemble des fluides lui-même Ω_t dépend du temps : il est déterminé par

$$\Omega_{\pm,t} := \boldsymbol{\psi}_{\boldsymbol{v}}(t, \Omega_\pm)$$

avec $\dot{\Omega} := \Omega_\pm$ domaine à l'instant initial.



le cas: $\Gamma_- = \emptyset$ le cas: $\Gamma_{+,t} = \emptyset$ le cas: $\Gamma_-, \Gamma_{+,t} \neq \emptyset$

Le tenseur des contraintes $\mathbb{T}(\boldsymbol{v}, \mathfrak{q})$ est donné par

$$\mathbb{T}(\boldsymbol{v}, \mathfrak{q})(x,t) := \mu\big(\rho(x,t)\big)\mathbb{D}(\boldsymbol{v})(x,t) - \mathfrak{q}(x,t)\mathbb{I},$$

et le tenseur des déformations $\mathbb{D}(\boldsymbol{v})/2$ est défini par

$$\mathbb{D}(\boldsymbol{v}) = D_x \boldsymbol{v} + \nabla_x \boldsymbol{v}, \quad \text{avec} \quad (D_x \boldsymbol{v})^j_k \equiv (\nabla_x \boldsymbol{v})^k_j := \partial_{x_k} v^j \quad \text{pour} \quad j,k = 1, ..., N.$$

Dans le système ($INS_{\pm}$), on a utilisé les notations classiques rappelées ci-dessous : pour tout couple de vecteurs $\boldsymbol{u}$ et $\boldsymbol{v}$ de $\mathbb{R}^N$, le produit tensoriel $\boldsymbol{u}\otimes\boldsymbol{v}$ est la matrice $N\times N$ définie par $(\boldsymbol{u}\otimes\boldsymbol{v})^j_k := u^j v^k$ $(1\le j,k\le N)$. De plus, si $\mathbb{A} = \big(A^j_k(x)\big)_{N\times N}$ est une fonction définie sur $\mathbb{R}^N$ et à valeurs dans l'ensemble des matrices $N\times N$ alors $\operatorname{Div}\mathbb{A}$ désigne le champ de vecteurs sur $\mathbb{R}^N$ donné par $(\operatorname{Div}\mathbb{A})^j := \sum_{k=1}^N \partial_{x_k} A^j_k$. Finalement, le saut du vector $\boldsymbol{g}$ à travers la surface $\mathcal{S}$ est donné par la limite normale suivante :

$$[\![\boldsymbol{g}]\!](x_0) := \lim_{\delta\to 0+}\Big(\boldsymbol{g}\big(x_0+\delta\boldsymbol{\nu}(x_0)\big)-\boldsymbol{g}\big(x_0-\delta\boldsymbol{\nu}(x_0)\big)\Big) \quad \forall\ x_0\in\mathcal{S},$$

où $\boldsymbol{\nu}$ est le vecteur normal extérieur unitaire pour la surface $\mathcal{S}$.

En suivant la méthode classique pour le système de Navier-Stokes homogène avec conditions aux limites de Dirichlet (voir l'Appendice B), on cherche à résoudre ($INS_{\pm}$). Notre stratégie consiste à passer en coordonnées Lagrangiennes, en suivant V. A. Solonnikov dans [113]. Dans ces coordonnées, le système ($INS_{\pm}$) peut être considéré dans un domaine fixe $\dot{\Omega}$, avec des conditions aux limites non linéaires. En s'appuyant sur les travaux [94, 107] concernant les propriétés de régularité maximale $L_p - L_q$ du système linéarisé (ce qui nécessite de faire appel aux techniques de $\mathcal{R}$-bornitude rappelées dans l'appendice C), puis en utilisant le théorème de point fixe contractant dans les espaces métriques complets, on parvient alors à résoudre ($INS_{\pm}$) à temps petit. A titre d'exemple, on donne ci-dessous le résultat obtenu lorsque la densité de référence η ne prend que deux valeurs (voir les hypothèses $(\mathcal{H}1)-(\mathcal{H}3)$ du Chapitre 3).

Théorème 0.4. *Soit (p,q) dans $(I)\cup(II)$, avec*

$$(I) := \{(p,q)\in]2,\infty[\times]N,\infty[\} \quad \textit{et} \quad (II) := \Big\{(p,q)\in]1,2]\times]N,\infty[: \frac{1}{p}+\frac{N}{q}>1\Big\}.$$

Supposons que ρ_0 soit dans $W^1_q(\dot{\Omega})$ et vérifie

$$\|\eta-\rho_0\|_{L_\infty(\dot{\Omega})}\le c \quad \textit{pour une constante } c,$$

que $\boldsymbol{v}_0$ soit dans $\mathcal{D}^{2-2/p}_{q,p}(\dot{\Omega})$ [5] et que $\boldsymbol{f}$ appartienne à $L_p\big(0,2;W^1_\infty(\mathbb{R}^N)^N\big)$. Alors il existe un temps T et une constante C, ne dépendant que de p, q, $\boldsymbol{v}_0$ et $\boldsymbol{f}$, tels que le système ($INS_{\pm}$) admette une unique solution $(\rho,\boldsymbol{u},\mathfrak{q})$ telle que

$$\|\partial_t\boldsymbol{v}\|_{L_p(0,T;W^2_q(\dot{\Omega}_t))}+\|\nabla^2\boldsymbol{v}\|_{L_p(0,T;L_q(\dot{\Omega}_t))}+\|\nabla\mathfrak{p}\|_{L_p(0,T;L_q(\dot{\Omega}_t))}\le C.$$

De plus, ρ est borné dans $W^1_q(\dot{\Omega}_t)$.

Quelques commentaires sur le théorème 0.4: l'hypothèse (I) est naturelle, vu le cas monophasique traité dans [106] (voir aussi les références mentionnées dans cet article). Cependant, il

[5] Voir le chapitre 3 pour la définition de $\mathcal{D}^{2-2/p}_{q,p}(\dot{\Omega})$.

s'avère que (II) marche aussi pour ($INS_{\pm}$) grâce à des propriétés d'interpolation supplémentaires. En fait, il n'est pas clair que l'on puisse construire une solution, même locale pour tout couple $(p,q) \in]1,\infty[^2$.

Finalement, mentionnons que nous cherchons maintenant à étendre le théorème 0.4 (en renforçant éventuellement les hypothèses sur les données), pour tout temps.

Chapter 1

Temperature patch problems for Boussinesq system without diffusion

1.1 Introduction

In this chapter, we are concerned with the following *incompressible Boussinesq system with partial viscosity:*

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta = 0, \\ \partial_t u + u\cdot\nabla u - \nu\Delta u + \nabla\Pi = \theta e_N, \\ \operatorname{div} u = 0, \\ (\theta, u)|_{t=0} = (\theta_0, u_0). \end{cases} \tag{$B_{\nu,N}$}$$

Above, $e_N = (0, \cdots, 0, 1)$ stands for the unit vertical vector in $\mathbb{R}^N$ with $N \geq 2$. The unknowns are the scalar function θ (the so-called temperature), the velocity field $u = (u^1, u^2, ..., u^N)$ and the pressure Π, depending on the time variable $t \geq 0$ and on the space variable $x \in \mathbb{R}^N$. The viscosity ν is a positive constant.

The above Boussinesq system is a toy model for describing the convection phenomenon in geophysical flows (see e.g. [98]). It has been considered in a number of mathematical works in the last two decades, mainly in space dimension $N = 2$. In that setting, the existence of global strong (and unique) solutions has been established under different types of regularity assumptions, see e.g. [2, 16, 71, 80], and global weak solutions with finite energy (that is corresponding to initial data θ_0 and u_0 in $L^2(\mathbb{R}^2)$) have been constructed in [69]. Those 'weak' solutions have been proved to be unique shortly after, in [38], exactly as for the standard incompressible Navier-Stokes equations in $\mathbb{R}^2$ (see also [66] for a simpler proof). The reader may refer to [72] for an up-to-date review of results for System ($B_{\nu,2}$).

In contrast, only a few works have been dedicated to the higher dimensional case. As regards the strong solution theory, global existence and uniqueness statements for small data (in the sense of some critical norm) have been proved in [37, 38].

Our goal here is to investigate the propagation of a discontinuity of the temperature along an interface, both in the cases $N = 2$ and $N \geq 3$. We are interested in the stability of *temperature patches*, namely to initial temperatures θ_0 of the type $\mathbb{1}_{\mathcal{D}_0}$ with $\mathcal{D}_0$ being a bounded simply connected domain.

To better explain our approach, let us concentrate on the two-dimensional case for a while. Then, for any $\theta_0 = \mathbb{1}_{\mathcal{D}_0}$ and u_0 in $(L^2(\mathbb{R}^2))^2$, the work in [38] guarantees the existence of a unique solution $(\theta, u) \in \mathcal{C}\big(\mathbb{R}_+; L^2(\mathbb{R}^2)\big)^3$ of $(B_{\nu,2})$ satisfying for all $t \geq 0$:

$$\|\theta(t)\|_{L^2} = \|\theta_0\|_{L^2} \quad \text{and} \quad \|u(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u\|_{L^2}^2 \, dt' = \|u_0\|_{L^2}^2 + 2 \int_0^t \int_{\mathbb{R}^2} \theta u_2 \, dx \, dt'.$$

In [38], it has been proved that the velocity u is 'almost' in $\big(L^1_{loc}(\mathbb{R}_+; H^2)\big)^2$ (it belongs to the slightly larger space $\big(\widetilde{L}^1_{loc}(\mathbb{R}_+; H^2)\big)^2$ defined in (1.12)). Consequently, according to a result of J.-Y. Chemin and N. Lerner in [23], u has a unique flow ψ_u, solution of the following (integrated) ordinary differential equation:

$$\psi_u(t, x) := x + \int_0^t u(t', \psi_u(t', x)) \, dt'. \tag{1.1}$$

As θ satisfies a transport equation along u and $\theta_0 = \mathbb{1}_{\mathcal{D}_0}$, this guarantees that for all $t \geq 0$,

$$\theta(t, \cdot) = \mathbb{1}_{\mathcal{D}_t} \quad \text{where} \quad \mathcal{D}_t := \psi_u(t, \mathcal{D}_0).$$

The above flow being measure preserving (since $\operatorname{div} u = 0$) and continuous, the topology and measure of the patch $\mathcal{D}_t$ does not change through the time evolution. However, much less can be said concerning the regularity of the boundary of the patch, as the result of [23] just guarantees that ψ_u is in $\big(\mathcal{C}(\mathbb{R}_+; \mathcal{C}^{0,1-\eta})\big)^2$ for all $\eta > 0$. Therefore, it is not clear that the initial regularity of the patch does not deteriorate, if assuming only that u_0 is in $(L^2(\mathbb{R}^2))^2$. Now, if θ_0 and u_0 are slightly more regular like for instance in the Besov space [1] $B^0_{2,1}$ then, according to [2], all the entries of ∇u are in $L^1_{loc}(\mathbb{R}_+; \mathcal{C}_b)$, and the flow $\psi_u(t, \cdot)$ is thus $\mathcal{C}^1$. As a consequence, the $\mathcal{C}^1$ regularity of the temperature patch is preserved for all time.

Then a natural question arises: what if we assume that the patch has higher regularity ? Our concern has some similarity with the celebrated vortex patch problem for the 2-D incompressible Euler equations:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla \Pi = 0, \\ \operatorname{div} u = 0. \end{cases} \tag{E}$$

Indeed, the vorticity $\omega := \partial_1 u^2 - \partial_2 u^1$ associated to the velocity u of (E) satisfies the transport equation

[1] See the definition of Besov spaces in the next section.

$$\partial_t \omega + u \cdot \nabla \omega = 0,$$

so that, at least formally, if one assumes that $\omega_0 = \mathbb{1}_{\mathcal{D}_0}$ then we have $\omega(t) = \mathbb{1}_{\mathcal{D}_t}$ with $\mathcal{D}_t := \psi_u(t, \mathcal{D}_0)$. For (E), it has been proved (see e.g. [19, 20] and the references therein) that the $\mathcal{C}^{k,\varepsilon}$ Hölder regularity of the patch of vorticity persists for all time.

Proving that in our framework, too, the Hölder regularity of $\mathcal{D}_t$ is conserved is the main purpose of the present chapter. Just like for the vortex patch problem for Euler equations, our result will come up as a consequence of a much more general property of global-in-time persistence of regularity along a suitable family of vector fields that is tangent to the patch (this is the so-called *striated regularity*, a definition that originates from the work of J.-Y. Chemin in [17]).

In contrast with the Euler equations however, here we do not need a nondegenerate family of vector fields to ensure that the flow is Lipschitz, because we have that property for free whenever the initial data are in the Besov space $B^0_{2,1}$, see [2]. In fact, as the regularity result we want to achieve is *local in space*, it will be enough to consider vector fields that do not vanish *in a neighborhood of the patch.*

Before stating our main results, we need to introduce some notation, and to clarify what striated regularity is. Assume that[2] $X = X^k(x)\partial_k$ is some vector field acting on functions in $\mathcal{C}^1(\mathbb{R}^N;\mathbb{R})$. As usual, vector fields are identified with vector valued functions from $\mathbb{R}^N$ to $\mathbb{R}^N$, and $\partial_X f$ stands for the *directional derivative* of $f \in \mathcal{C}^1(\mathbb{R}^N;\mathbb{R})$ along the vector field X, namely

$$\partial_X f := X^k \partial_k f = X \cdot \nabla f.$$

The evolution $X_t(x) := X(t,x)$ of any continuous initial vector field X_0 along the flow of u is defined by:

$$X(t,x) := (\partial_{X_0}\psi_u)\big(\psi_u^{-1}(t,x)\big).$$

In the $\mathcal{C}^1$ case, combining the chain rule and the definition of the flow in (1.1) implies that X satisfies the following transport equation (omitting the index t in X_t for notational simplicity):

$$\begin{cases} \partial_t X + u \cdot \nabla X = \partial_X u, \\ X|_{t=0} = X_0. \end{cases} \tag{1.2}$$

Applying operator div to (1.2) and remembering that $\operatorname{div} u = 0$, we obtain in addition

$$\begin{cases} \partial_t \operatorname{div} X + u \cdot \nabla \operatorname{div} X = 0, \\ \operatorname{div} X|_{t=0} = \operatorname{div} X_0. \end{cases} \tag{1.3}$$

Therefore the divergence-free property is conserved through the evolution.

[2] We adopt Einstein summation convention in the whole text: summation is taken with respect to the repeated indices, whenever they occur both as a subscript and a superscript.

As explained into more details in Section 1.5, the temperature patch problem is closely related to the conservation of Hölder regularity $\mathcal{C}^{0,\varepsilon}$ for X. According to the classical theory of transport equations, if u is Lipschitz with respect to the space variable, then that property is equivalent to the fact that all the components of $\partial_X u$ have the regularity $\mathcal{C}^{0,\varepsilon}$ with respect to the space variable.

In the 2-D case, it is natural to recast the regularity of u along the vector field X in terms of the vorticity $\omega := \partial_1 u^2 - \partial_2 u^1$ as the simple transport-diffusion equation is fulfilled:

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega - \nu \Delta \omega = \partial_1 \theta, \\ \omega|_{t=0} = \omega_0, \end{cases} \tag{1.4}$$

and as it is known (see e.g. [6], Chap. 7) that[3]:

$$\|\partial_X u\|_{\mathscr{C}^\varepsilon} \lesssim \|\nabla u\|_{L^\infty} \|X\|_{\mathscr{C}^\varepsilon} + \|\mathrm{div}\,(X\omega)\|_{\mathscr{C}^{\varepsilon-1}}, \tag{1.5}$$

where for any real number s, we denote[4] $\mathscr{C}^s \equiv \mathscr{C}^s(\mathbb{R}^N) := B^s_{\infty,\infty}(\mathbb{R}^N)$.

Applying operators ∂_X and $\mathrm{div}\,(X\cdot)$ to the temperature and vorticity equations, respectively, and denoting

$$f := \nu \mathrm{div}\,\big(X\Delta\omega - \Delta(X\omega)\big) + \mathrm{div}\,(X\partial_1\theta),$$

we get the following system for $\big(\partial_X\theta, \mathrm{div}\,(X\omega)\big)$:

$$\begin{cases} \partial_t \partial_X \theta + u \cdot \nabla \partial_X \theta = 0, \\ \partial_t \mathrm{div}\,(X\omega) + u \cdot \nabla \mathrm{div}\,(X\omega) - \nu\Delta \mathrm{div}\,(X\omega) = f. \end{cases} \tag{1.6}$$

Let us recap. Knowing that ∇u is in $L^1_{loc}(\mathbb{R}_+; L^\infty)$, in order to propagate the $\mathscr{C}^\varepsilon$ regularity of X, it is enough to have $\partial_X u$ in $\big(L^1_{loc}(\mathbb{R}_+; \mathscr{C}^\varepsilon)\big)^2$, and this may be achieved thanks to (1.5), if $\mathrm{div}\,(X\omega)$ is in $L^1_{loc}(\mathbb{R}_+; \mathscr{C}^{\varepsilon-1})$. Now, assuming that $\mathrm{div}\,(X_0\omega_0)$ is in $\mathscr{C}^{\varepsilon-3}$ and taking advantage of smoothing properties of the heat flow, this latter information may be obtained if f is suitably bounded in the very negative space $L^1_{loc}(\mathbb{R}_+; \mathscr{C}^{\varepsilon-3})$. This requires to have $\partial_X\theta$ in $L^\infty_{loc}(\mathbb{R}_+; \mathscr{C}^{\varepsilon-2})$, and thus $\partial_{X_0}\theta_0 \in \mathscr{C}^{\varepsilon-2}$.

The following theorem states that it is indeed possible to propagate this type of regularity, globally in time, in the 2-D case.

Theorem 1.1. *Suppose that $(\varepsilon, q) \in]0,1[\times]1, 2/(2-\varepsilon)[$. Let θ_0 be in $B^{2/q-1}_{q,1}(\mathbb{R}^2)$ and u_0 be a divergence-free vector field in $\big(L^2(\mathbb{R}^2)\big)^2$, with vorticity $\omega_0 := \partial_1 u_0^2 - \partial_2 u_0^1$ in $B^{2/q-2}_{q,1}(\mathbb{R}^2)$. There exists a unique global solution (θ, u) of System $(B_{\nu,2})$, such that*

$$(\theta, u, \omega) \in \mathcal{C}(\mathbb{R}_+; B^{\frac{2}{q}-1}_{q,1}) \times \big(\mathcal{C}(\mathbb{R}_+; L^2)\big)^2 \times \big(\mathcal{C}(\mathbb{R}_+; B^{\frac{2}{q}-2}_{q,1}) \cap L^1_{loc}(\mathbb{R}_+; B^{\frac{2}{q}}_{q,1})\big). \tag{1.7}$$

[3] In Chapter 1, we agree that $A \lesssim B$ means $A \leq CB$ for some harmless constant C.

[4] Recall that $B^{k+\varepsilon}_{\infty,\infty}$ coincides with the standard Hölder space $\mathcal{C}^{k,\varepsilon}$ whenever $k \in \mathbb{N}$ and $\varepsilon \in]0,1[$, see [6].

Furthermore, for any vector field X_0 in $\big(\mathscr{C}^\varepsilon(\mathbb{R}^2)\big)^2$ such that

$$\partial_{X_0}\theta_0 \in \mathscr{C}^{\varepsilon-2}(\mathbb{R}^2) \quad \textit{and} \quad \operatorname{div}(X_0\omega_0) \in \mathscr{C}^{\varepsilon-3}(\mathbb{R}^2),$$

there exists a unique global solution[5] $X \in \big(\mathcal{C}_w(\mathbb{R}_+;\mathscr{C}^\varepsilon)\big)^2$ *to* (1.2) *and we have*[6]

$$\big(\partial_X\theta, \operatorname{div}(X\omega)\big) \in \mathcal{C}_w(\mathbb{R}_+;\mathscr{C}^{\varepsilon-2}) \times \big(\mathcal{C}_w(\mathbb{R}_+;\mathscr{C}^{\varepsilon-3}) \cap \widetilde{L}^1_{loc}(\mathbb{R}_+;\mathscr{C}^{\varepsilon-1})\big).$$

Additionally, there is a constant $C_{0,\nu}$ depending only on the initial data and viscosity constant such that for any $t \geq 0$,

$$\|X\|_{L^\infty_t(\mathscr{C}^\varepsilon)} \leq C_{0,\nu} \exp\Big(\exp\big(\exp(C_{0,\nu}t^4)\big)\Big).$$

A few comments are in order:

- The Besov space $B^{\frac{2}{q}-1}_{q,1}(\mathbb{R}^2)$ for θ_0 contains the characteristic function of any bounded $\mathcal{C}^1$ domain. Thanks to that, it will be possible to obtain the persistence of regularity for temperature patches as a consequence of the above theorem (see Corollary 1.2 below).

- It may be seen by means of elementary paradifferential calculus that if

$$(\varepsilon, q) \in]0,1[\times]1, \frac{2}{2-\varepsilon}[\,,$$

  then $\operatorname{div}(X_0\omega_0)$ and $\partial_{X_0}\theta_0$ are distributions of $\mathscr{C}^{-3}$ and $\mathscr{C}^{-2}$, respectively, and that the following (sharp) estimates are fulfilled:

$$\|\operatorname{div}(X_0\omega_0)\|_{\mathscr{C}^{-3}} \lesssim \|X_0\|_{\mathscr{C}^\varepsilon}\|\omega_0\|_{B^{\frac{2}{q}-2}_{q,1}} \quad \text{and} \quad \|\partial_{X_0}\theta_0\|_{\mathscr{C}^{-2}} \lesssim \|X_0\|_{\mathscr{C}^\varepsilon}\|\theta_0\|_{B^{\frac{2}{q}-1}_{q,1}}.$$

  Therefore, asking for $\operatorname{div}(X_0\omega_0) \in \mathscr{C}^{\varepsilon-3}$ and $\partial_{X_0}\theta_0 \in \mathscr{C}^{\varepsilon-2}$ is indeed additional hypothesis.

- The required level of regularity is much lower than for the 2-D vortex patch problem for (E) (there we need to have $\operatorname{div}(X_0\omega_0)$ in $\mathscr{C}^{\varepsilon-1}(\mathbb{R}^2)$). This is because the smoothing effect of the heat flow allows to gain two full derivatives compared to the data. Another difference is that $\nabla u \in L^1_{loc}(\mathbb{R}_+;L^\infty)$, by embedding. Therefore, it is not necessary to consider a family of vector fields that does not degenerate on the whole $\mathbb{R}^2$, to solve the temperature patch problem.

- Just like for the Euler equation, we expect our approach to be relevant for showing persistency of higher order striated regularity (thus leading to the propagation of $\mathcal{C}^{k,\varepsilon}$ regularity

[5] If E is a Banach space with predual E^* then $\mathcal{C}_w(\mathbb{R}_+;E)$ stands for the set of measurable functions $h:\mathbb{R}_+\to E$ such that for all $\phi\in E^*$, the function $t\mapsto\langle h(t),\phi\rangle_{E\times E^*}$ is continuous on $\mathbb{R}_+$.

[6] To fully benefit from the smoothing properties of the heat flow, one has to work in (close) superspaces of $L^1_{loc}(\mathbb{R}_+;\mathscr{C}^s)$ denoted by $\widetilde{L}^1_{loc}(\mathbb{R}_+;\mathscr{C}^s)$ and defined in (1.12).

for the temperature patches). We refrained from doing that here as it would make this chapter much more technical.

- A similar issue has been addressed recently by X. Liao and P. Zhang in [87, 88] for the inhomogeneous Navier-Stokes equations. For technical reasons (the system under consideration therein is much more nonlinear), only higher order regularities can be propagated, which requires to study the action of iterated vector fields on the solution. At the same, whether one can propagate just $\mathcal{C}^{1,\varepsilon}$ regularity for that more complicated system is our main task in next chapter.

Let us now go to the temperature patch problem in the 2-D case. So consider a $\mathcal{C}^{1,\varepsilon}$ simply connected bounded domain $\mathcal{D}_0$ of $\mathbb{R}^2$ (hence $\partial\mathcal{D}_0$ is a $\mathcal{C}^{1,\varepsilon}$ Jordan curve on $\mathbb{R}^2$). Let $\mathcal{D}_0^\star$ be a bounded domain of $\mathbb{R}^2$ such that

$$\overline{\mathcal{D}_0} \cap \overline{\mathcal{D}_0^\star} = \emptyset. \tag{1.8}$$

Then the following result holds true.

Corollary 1.2. *Let (M_1, M_2) be in $\mathbb{R}^2$. Assume that $\theta_0 = M_1 \mathbb{1}_{\mathcal{D}_0}$ and that the vorticity ω_0 of u_0 is given by*

$$\omega_0 = M_2 \mathbb{1}_{\mathcal{D}_0} - \widetilde{\omega}_0 \tag{1.9}$$

for some $\widetilde{\omega}_0 \in L^r(\mathbb{R}^2)$ with $r > 1$, supported in $\overline{\mathcal{D}_0^\star}$ and such that

$$\int_{\mathbb{R}^2} \widetilde{\omega}_0(x)\, dx = M_2\, |\mathcal{D}_0|. \tag{1.10}$$

Then there exists a unique solution (θ, u) to System $(B_{\nu,2})$, satisfying the properties listed in Theorem 1.1. Furthermore, we have $\theta(t,\cdot) = M_1 \mathbb{1}_{\mathcal{D}_t}$ where $\mathcal{D}(t) := \psi_u(t, \mathcal{D}_0)$, and $\partial\mathcal{D}(t)$ remains a $\mathcal{C}^{1,\varepsilon}$ Jordan curve of $\mathbb{R}^2$ for all $t \geq 0$.

Let us make some comments on that corollary.

- Hypothesis (1.10) ensures that the initial vorticity is mean free, which is needed to have $u_0 \in \big(L^2(\mathbb{R}^2)\big)^2$. The more natural assumption $\omega_0 = M_2 \mathbb{1}_{\mathcal{D}_0}$ would require our extending Theorem 1.1 to infinite energy velocity fields. As for (E), it is feasible, but introduces additional technicalities.

- Assumption (1.9) on the vorticity may seem somewhat artificial as it has no time persistency whatsoever, even in the asymptotics $\nu \to 0$ (in contrast with the slightly viscous vortex patch problem, see [27]). This is just to have a concrete example of initial velocity for which one can give a positive answer to the temperature patch problem. In fact, Corollary 1.2 remains true if the initial velocity $u_0 \in \big(L^2(\mathbb{R}^2)\big)^2$ is such that $\omega_0 \in B^{\frac{2}{q}-2}_{q,1}(\mathbb{R}^2)$ for some $1 < q < \frac{2}{2-\varepsilon}$ and satisfies $\operatorname{div}(X_0\omega_0) \in \mathscr{C}^{\varepsilon-3}(\mathbb{R}^2)$ for some vector field $X_0 \in \big(\mathscr{C}^\varepsilon(\mathbb{R}^2)\big)^2$ that does not vanish on $\partial\mathcal{D}_0$ and is tangent to $\partial\mathcal{D}_0$.

Let us finally consider the case $N \geq 3$. Then, compared to the 2-D case, the main difference is that the vorticity equation has an additional stretching term, and it is thus less natural to measure the striated regularity by means of $\operatorname{div}(X\Omega)$ with Ω denoting the matrix of $\operatorname{curl} u$ (even though our 2-D approach should be adaptable to the high-dimensional case, like in [55]). We shall thus look directly at $\partial_X u$. An additional (related) difficulty is that one cannot expect to prove global existence for general large initial data, since ($B_{\nu,N}$) contains the standard incompressible Navier-Stokes equations as a particular case. Therefore we shall prescribe some smallness condition on the data (the same one as in [38]) to achieve a global statement. This leads to the following theorem:

Theorem 1.3. *Suppose that $N \geq 3$ and that $(\varepsilon, p) \in]0,1[\times]N, N/(1-\varepsilon)[$. Assume that θ_0 is in $B^0_{N,1}(\mathbb{R}^N) \cap L^{\frac{N}{3}}(\mathbb{R}^N)$ and that the divergence-free vector field u_0 is in $B^{\frac{N}{p}-1}_{p,1}(\mathbb{R}^N)^N$ and in the weak Lebesgue space $L^{N,\infty}(\mathbb{R}^N)$. There exists a (small) positive constant c independent of p such that if*

$$\|u_0\|_{L^{N,\infty}} + \nu^{-1}\|\theta_0\|_{L^{\frac{N}{3}}} \leq c\nu,$$

then Boussinesq system ($B_{\nu,N}$) has a unique global solution $(\theta, u, \nabla\Pi)$ in the space

$$\mathcal{C}(\mathbb{R}_+; B^0_{N,1}) \times \big(\mathcal{C}(\mathbb{R}_+; B^{\frac{N}{p}-1}_{p,1}) \cap L^1_{loc}(\mathbb{R}_+; B^{\frac{N}{p}+1}_{p,1})\big)^N \times \big(L^1_{loc}(\mathbb{R}_+; B^{\frac{N}{p}-1}_{p,1})\big)^N.$$

Moreover, for all vector field X_0 is in the space $\big(\widetilde{\mathscr{C}}^\varepsilon(\mathbb{R}^N)\big)^N$ defined by

$$\big(\widetilde{\mathscr{C}}^\varepsilon(\mathbb{R}^N)\big)^N := \{Y \in \big(\mathscr{C}^\varepsilon(\mathbb{R}^N)\big)^N : \operatorname{div} Y \in \mathscr{C}^\varepsilon(\mathbb{R}^N)\},$$

and such that the components of $(\partial_{X_0}\theta_0, \partial_{X_0}u_0)$ are in $\mathscr{C}^{\varepsilon-2}(\mathbb{R}^N)$, System (1.2) has a unique solution X in $\mathcal{C}_w(\mathbb{R}_+; \widetilde{\mathscr{C}}^\varepsilon)$, and we have for some constant $C_{0,\nu}$ depending only on the initial data and on ν, and all $t \geq 0$,

$$\|X\|_{L^\infty_t(\widetilde{\mathscr{C}}^\varepsilon)} \leq C_{0,\nu} \exp\big(\exp(C_{0,\nu}t)\big).$$

Furthermore, the triplet $(\partial_X\theta, \partial_X u, \partial_X\nabla\Pi)$ belongs to

$$\mathcal{C}_w(\mathbb{R}_+; \mathscr{C}^{\varepsilon-2}) \times \big(\mathcal{C}_w(\mathbb{R}_+; \mathscr{C}^{\varepsilon-2}) \cap \widetilde{L}^1_{loc}(\mathbb{R}_+; \mathscr{C}^\varepsilon)\big)^N \times \big(\widetilde{L}^1_{loc}(\mathbb{R}_+; \mathscr{C}^{\varepsilon-2})\big)^N.$$

As in the 2-D case, the above result will enable us to solve the temperature patch problem. Indeed, fix some simply connected domain $\mathcal{D}_0$ of class $\mathcal{C}^{1,\varepsilon}$ (that is such that $\partial\mathcal{D}$ is a compact hypersurface of class $\mathcal{C}^{1,\varepsilon}$) and consider another $\mathcal{C}^1$ simply connected bounded domain J_0 such that $\overline{\mathcal{D}_0} \subset J_0$. Then we have the following statement[7]:

Corollary 1.4. *Let $N \geq 3$ and m_1, m_2 be sufficiently small constants. Assume that $\theta_0 = m_1\mathbb{1}_{\mathcal{D}_0}$ and that the ith component of initial velocity $u^i_0 := -\sum_{j=1}^N(-\Delta)^{-1}\partial_j(\Omega_0)^i_j$ satisfies $\Omega_0 := m_2\mathbb{1}_{J_0}\mathbb{A}_0$ where $\mathbb{A}_0$ stands for the anti-symmetric matrix defined by $(\mathbb{A}_0)^i_j = 1$ for $i < j$.*

[7]Like in the 2-D case, much more general initial velocities may be considered.

Then $\theta(t,\cdot) = m_1 \mathbb{1}_{\mathcal{D}_t}$ *where* $\mathcal{D}(t) := \psi_u(t, \mathcal{D}_0)$, *and* $\mathcal{D}(t)$ *remains a simply connected domain of class* $\mathcal{C}^{1,\varepsilon}$, *for any* $t \geq 0$.

The rest of this chapter unfolds as follows. In the next section, we shortly introduce Besov spaces and present some linear or nonlinear estimates which will be needed to achieve our results. Then the proofs of main theorems for the propagation of striated regularity will be revealed in Section 1.3 (for 2-D case) and Section 1.4 (for N-D case). Section 1.5 is devoted to proving Corollaries 1.2 and 1.4. Some technical commutator estimates are proved at last.

1.2 Basic notations and linear estimates

We here introduce definitions and notations that are used throughout the text, and recall some properties of Besov spaces and transport or transport-diffusion equations.

Let us begin with the definition of the *nonhomogeneous Littlewood-Paley decomposition* (for more details see [6], Chap. 2). Set

$$\mathfrak{B} := \{\xi \in \mathbb{R}^N : |\xi| \leq 4/3\} \ \text{ and } \ \mathfrak{C} := \{\xi \in \mathbb{R}^N : 3/4 \leq |\xi| \leq 8/3\}.$$

We fix two smooth radial functions χ and φ, supported in $\mathfrak{B}$ and $\mathfrak{C}$, respectively, and such that

$$\chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1, \ \ \forall\, \xi \in \mathbb{R}^N.$$

We then introduce the Fourier multipliers $\Delta_{-1} := \chi(D)$ and $\Delta_j := \varphi(2^{-j}D)$ with $j \geq 0$ (the so-called *nonhomogeneous dyadic blocks*) and the low frequency cut-off operator

$$S_j := \sum_{j' \leq j-1} \Delta_{j'}.$$

For $(s,p,r) \in \mathbb{R} \times [1,\infty]^2$, the nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^N)$ is defined by

$$B^s_{p,r}(\mathbb{R}^N) := \Big\{ u \in \mathcal{S}'(\mathbb{R}^N) : \|u\|_{B^s_{p,r}} := \big\| 2^{js} \|\Delta_j u\|_{L^p} \big\|_{\ell^r(\mathbb{N}\cup\{-1\})} < \infty \Big\}.$$

Let us next introduce the following *paraproduct* and *remainder* operators:

$$T_u v := \sum_{j\geq -1} S_{j-N_0} u \Delta_j v \quad \text{and} \quad R(u,v) \equiv \sum_{j\geq -1} \Delta_j u \widetilde{\Delta}_j v := \sum_{\substack{j\geq -1 \\ |j-k|\leq N_0}} \Delta_j u \Delta_k v,$$

where N_0 stands for some large enough (fixed) integer.

The following decomposition, first introduced by J.-M. Bony in [11]:

$$uv = T_u v + T_v u + R(u,v), \tag{1.11}$$

holds true whenever the product of the two tempered distribution u and v is defined. It will play a fundamental role in our study. Bilinear operators R and T possess continuity properties in a number of functional spaces (see e.g. Chap. 2 in [6]). We shall recall a few of them throughout the text, when needed.

When investigating evolutionary equations in Besov spaces and, in particular, parabolic type equations, it is natural to use the following *tilde Besov spaces* first introduced by J.-Y. Chemin in [21]: for any $t \in]0,\infty]$ and $(s,p,r,\rho) \in \mathbb{R}\times[1,\infty]^3$, we set

$$\widetilde{L}^\rho_t\big(B^s_{p,r}(\mathbb{R}^N)\big) := \Big\{u \in \mathcal{S}'(]0,t[\times\mathbb{R}^N) : \|u\|_{\widetilde{L}^\rho_t(B^s_{p,r})} := \big\|2^{js}\|\Delta_j u\|_{L^\rho(]0,t[;L^p)}\big\|_{\ell^r} < \infty\Big\}.$$

In the particular case where $p=r=2$ (resp. $p=r=\infty$), $B^s_{p,r}$ coincides with the Sobolev space H^s (resp. the generalized Hölder space $\mathscr{C}^s$), and we shall alternately denote

$$\widetilde{L}^\rho_t\big(H^s(\mathbb{R}^N)\big) := \widetilde{L}^\rho_t\big(B^s_{2,2}(\mathbb{R}^N)\big) \quad \text{and} \quad \widetilde{L}^\rho_t\big(\mathscr{C}^s(\mathbb{R}^N)\big) := \widetilde{L}^\rho_t\big(B^s_{\infty,\infty}(\mathbb{R}^N)\big). \tag{1.12}$$

The following a priori estimates for the transport and transport-diffusion equations in (nonhomogeneous) Besov spaces will be needed to establish our main results.

Proposition 1.5. *Let v be a divergence free vector field. Let $(p,p_1,r,\rho,\rho_1) \in [1,\infty]^5$ and $s\in\mathbb{R}$ satisfy*

$$-1-\min\Big(\frac{N}{p_1},\frac{N}{p'}\Big) < s < 1+\min\Big(\frac{N}{p},\frac{N}{p_1}\Big) \quad \text{and} \quad \rho_1 \le \rho.$$

Let f be a smooth solution of the following transport-diffusion equation with diffusion parameter $\nu \ge 0$:

$$\begin{cases} \partial_t f + \operatorname{div}(fv) - \nu\Delta f = g, \\ f|_{t=0} = f_0. \end{cases} \tag{TD_ν}$$

Then there exists a constant C depending on N, p, p_1 and s such that for all $t\ge 0$,

$$\begin{aligned} \nu^{\frac{1}{\rho}}\|f\|_{\widetilde{L}^\rho_t(B^{s+\frac{2}{\rho}}_{p,r})} \le Ce^{C(1+\nu t)^{\frac{1}{\rho}}V_{p_1}(t)}\Big((1+\nu t)^{\frac{1}{\rho}}\|f_0\|_{B^s_{p,r}} \\ + (1+\nu t)^{1+\frac{1}{\rho}-\frac{1}{\rho_1}}\nu^{\frac{1}{\rho_1}-1}\|g\|_{\widetilde{L}^{\rho_1}_t(B^{s-2+\frac{2}{\rho_1}}_{p,r})}\Big), \end{aligned} \tag{1.13}$$

where

$$V_{p_1}(t) := \int_0^t \|\nabla v\|_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty}\,dt'.$$

In the limit case $s=-1-\min\big(\frac{N}{p_1},\frac{N}{p'}\big)$, one needs to refine $\|\nabla v\|_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty}$ by $\|\nabla v\|_{B^{\frac{N}{p_1}}_{p_1,1}}$ in the definition of V_{p_1}.

Proof. The above statement has been proved in [32] in the case $p\le p_1$. That restriction came from the following commutator estimate [8]:

$$\left\|2^{js}\|[v\cdot\nabla,\Delta_j]f\|_{L^p}\right\|_{\ell^r}\lesssim\|\nabla v\|_{B^{\frac{N}{p_1}}_{p,1}}\|f\|_{B^s_{p,r}},$$

that has been proved only in the case $p\le p_1$ therein.

To handle the case $p>p_1$, let us set $\tilde v:=v-S_0v$ and write $[v\cdot\nabla,\Delta_j]f=\sum_{i=1}^6 R^i_j$ with

$$\begin{array}{rclrcl} R^1_j &:=& [T_{\tilde v^k},\Delta_j]\partial_k f, & R^2_j &:=& T_{\partial_k\Delta_j f}\tilde v^k,\\ R^3_j &:=& -\Delta_j T_{\partial_k f}\tilde v^k, & R^4_j &:=& \partial_k R(\tilde v^k,\Delta_j f),\\ R^5_j &:=& -\partial_k\Delta_j R(\tilde v^k,f), & R^6_j &:=& [S_0 v^k,\Delta_j]\partial_k f.\end{array}$$

In [32], Condition $p\le p_1$ is used only when bounding R^3_j. Now, if $p>p_1$ and $s<1+\frac{N}{p}$ then combining standard continuity results for the paraproduct with the embedding $B^{s-1}_{p,r}(\mathbb{R}^N)\hookrightarrow B^{s-1-\frac{N}{p}}_{\infty,r}(\mathbb{R}^N)$ implies that

$$\left\|2^{j(s+\frac{N}{p_1}-\frac{N}{p})}\|R^3_j\|_{L^p}\right\|_{\ell^r}\lesssim\|\nabla v\|_{B^{\frac{N}{p_1}}_{p_1,\infty}}\|\nabla f\|_{B^{s-1}_{p,r}},$$

whence the desired inequality. □

Remark 1.6. *We shall often use the above proposition in the particular case*

$$(\nu,p,r,\rho,\rho_1)=(0,\infty,\infty,\infty,1).$$

Then Inequality (1.13) *reduces to*

$$\|f\|_{L^\infty_t(\mathscr{C}^s)}\le Ce^{CV_{p_1}(t)}\big(\|f_0\|_{\mathscr{C}^s}+\|g\|_{\widetilde L^1_t(\mathscr{C}^s)}\big)\quad\textit{if}\quad -1-\frac{N}{p_1}<s<1.$$

We finally recall a refinement of Vishik's estimates for the transport equation [125] obtained by T. Hmidi and S. Keraani in [70], which is the key to the study of long-time behavior of the solution in critical spaces for Boussinesq system ($B_{\nu,N}$) (see [2, 38]).

Proposition 1.7. *Assume that v is divergence-free and that f satisfies the transport equation (TD_0). There exists a constant C such that for all $(p,r)\in[1,\infty]^2$ and $t>0$,*

$$\|f\|_{\widetilde L^\infty_t(B^0_{p,r})}\le C\big(\|f_0\|_{B^0_{p,r}}+\|g\|_{\widetilde L^1_t(B^0_{p,r})}\big)\left(1+\int_0^t\|\nabla v\|_{L^\infty}\,d\tau\right).$$

[8] We here adopt the usual notation $[A,B]$ for the commutator $AB-BA$.

1.3 Propagation of striated regularity in the 2D-case

This section is devoted to the proof of Theorem 1.1. To simplify the computations, we shall first make the change of unknowns

$$\widetilde{\theta}(t,x) = \nu^2\theta(\nu t, x), \quad \widetilde{u}(t,x) = \nu u(\nu t, x), \quad \widetilde{P}(t,x) = \nu^2 P(\nu t, x) \tag{1.14}$$

so as to reduce the study to the case $\nu = 1$.

Throughout this section, we always assume that

$$0 < \varepsilon < 1, \quad q > 1 \text{ and } \frac{\varepsilon}{2} + \frac{1}{q} > 1, \tag{1.15}$$

in accordance with the hypotheses of Theorem 1.1.

1.3.1 A priori estimates for the Lipschitz norm of the velocity field

Those estimates will be based on the following global existence theorem (see [2]).

Theorem 1.8. *Let θ_0 and the divergence-free vector field u_0 satisfy*

$$(\theta_0, u_0) \in B^0_{2,1}(\mathbb{R}^2) \times \big(L^2(\mathbb{R}^2) \cap B^{-1}_{\infty,1}(\mathbb{R}^2)\big)^2.$$

Then there exists a unique global solution $(u, \theta, \nabla\Pi)$ for System $(B_{1,2})$ such that

$$\begin{aligned} u \in \big(\mathcal{C}(\mathbb{R}_+; L^2 \cap B^{-1}_{\infty,1}) \cap L^2_{loc}(\mathbb{R}_+; H^1) \cap L^1_{loc}(\mathbb{R}_+; B^1_{\infty,1})\big)^2,\\ \theta \in \mathcal{C}_b(\mathbb{R}_+; B^0_{2,1}) \ \text{ and } \ \nabla\Pi \in \big(L^1_{loc}(\mathbb{R}_+; B^0_{2,1})\big)^2. \end{aligned} \tag{1.16}$$

Moreover, for any $t > 0$, there exists a constant C_0 depending only on the initial data such that

$$\|u\|_{L^1_t(B^1_{\infty,1})} \leq C_0 e^{C_0 t^4}.$$

We claim that the data (θ_0, u_0) of Theorem 1.1 fulfill the assumptions of the above theorem. Indeed, decompose $u_0 = \Delta_{-1}u_0 + (\mathrm{Id} - \Delta_{-1})u_0$ and apply the following *Biot-Savart law*

$$\nabla u = -\nabla(-\Delta)^{-1}\nabla^\perp\omega \quad \text{with} \quad \nabla^\perp := (-\partial_2, \partial_1).$$

Then thanks to the obvious embedding $B^{\frac{2}{q}-2}_{q,1}(\mathbb{R}^2) \hookrightarrow B^{-1}_{2,1}(\mathbb{R}^2)$, we obtain that

$$\begin{aligned} \|u_0\|_{B^{-1}_{\infty,1}} \lesssim \|u_0\|_{B^0_{2,1}} &\lesssim \|\Delta_{-1}u_0\|_{L^2} + \|(\mathrm{Id} - \Delta_{-1})\nabla u_0\|_{B^{-1}_{2,1}}\\ &\lesssim \|u_0\|_{L^2} + \|\omega_0\|_{B^{\frac{2}{q}-2}_{q,1}}. \end{aligned} \tag{1.17}$$

Besides, the obvious embedding $\theta_0 \in B^{\frac{2}{q}-1}_{q,1}(\mathbb{R}^2) \hookrightarrow B^0_{2,1}(\mathbb{R}^2)$ holds. Hence one may apply the above theorem to our data. The corresponding global solution (θ, u) fulfills (1.16) and the following inequality for all $t \geq 0$,

$$\|\nabla u\|_{L^1_t(L^\infty)} \lesssim \|u\|_{L^1_t(B^1_{\infty,1})} \leq C_0 e^{C_0 t^4}. \tag{1.18}$$

1.3.2 A priori estimates for θ and ω

We now want to prove that (θ, u) fulfills the additional property (1.7). To this end, the first observation is that θ satisfies a free transport equation. Hence, from the standard theory of transport equations (apply Proposition 1.5 with $\nu = 0$) and the bound (1.18), we deduce that θ is in $\mathcal{C}(\mathbb{R}_+; B^{\frac{2}{q}-1}_{q,1})$ and that

$$\|\theta\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}-1}_{q,1})} \leq e^{C\|\nabla u\|_{L^1_t(L^\infty)}} \|\theta_0\|_{B^{\frac{2}{q}-1}_{q,1}} \leq \|\theta_0\|_{B^{\frac{2}{q}-1}_{q,1}} \exp\left(C_0 \exp(C_0 t^4)\right) \quad \text{for all } \ t \in \mathbb{R}_+. \tag{1.19}$$

In order to bound ω, one may apply Proposition 1.5 to the vorticity equation, which yields for all $t \geq 0$,

$$\|\omega\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}-2}_{q,1})} + \|\omega\|_{L^1_t(B^{\frac{2}{q}}_{q,1})} \leq C(1+t)e^{C(1+t)\|\nabla u\|_{L^1_t(L^\infty)}} \left(\|\omega_0\|_{B^{\frac{2}{q}-2}_{q,1}} + \|\nabla\theta\|_{L^1_t(B^{\frac{2}{q}-2}_{q,1})}\right).$$

Hence, taking advantage of (1.18) and (1.19), we get

$$\|\omega\|_{L^\infty_t(B^{\frac{2}{q}-2}_{q,1})} + \|\omega\|_{L^1_t(B^{\frac{2}{q}}_{q,1})} \leq C \exp(\exp(C_0 t^4))\left(\|\omega_0\|_{B^{\frac{2}{q}-2}_{q,1}} + \|\theta_0\|_{B^{\frac{2}{q}-1}_{q,1}}\right). \tag{1.20}$$

Then Biot-Savart law allows to improve the regularity of the velocity u as follows:

$$U_q(t) := \|\nabla u\|_{L^1_t(B^{\frac{2}{q}}_{q,1})} \lesssim \|\omega\|_{L^1_t(B^{\frac{2}{q}}_{q,1})} \leq C_0 \exp\left(\exp(C_0 t^4)\right), \tag{1.21}$$

where C_0 depends only on the Lebesgue and Besov norms of the data in Theorem 1.1.

1.3.3 A priori estimates for the striated regularity

We shall need the following lemma which is a straightforward adaptation of Inequality (1.5) to time dependent functions.

Lemma 1.9. *For any $\varepsilon \in]0,1[$, there exists a constant C such that the following estimate holds true:*

$$\|\partial_X u\|_{\widetilde{L}^1_t(\mathscr{C}^\varepsilon)} \lesssim \int_0^t \|\nabla u\|_{L^\infty} \|X\|_{\mathscr{C}^\varepsilon}\, dt' + \|\mathrm{div}\,(X\omega)\|_{\widetilde{L}^1_t(\mathscr{C}^{\varepsilon-1})}. \tag{1.22}$$

Therefore, in order to propagate the Hölder regularity of X, it suffices to derive appropriate bounds for $\mathrm{div}\,(X\omega)$ in $\widetilde{L}^1_t(\mathscr{C}^{\varepsilon-1})$, for all $t \geq 0$. This will be based on the fact that $\mathrm{div}\,(X\omega)$ fulfills the transport-diffusion equation:

$$\partial_t \mathrm{div}\,(X\omega) + u \cdot \nabla \mathrm{div}\,(X\omega) - \Delta \mathrm{div}\,(X\omega) = f$$

with

$$f = \mathrm{div}\, F + \mathrm{div}\,(X\partial_1\theta) \quad \text{and} \quad F := X\Delta\omega - \Delta(X\omega).$$

Now, thanks to Proposition 1.5 and to (1.15), we have

$$\begin{aligned}\|\mathrm{div}\,(X\omega)\|_{L^\infty_t(\mathscr{C}^{\varepsilon-3})} + \|\mathrm{div}\,(X\omega)\|_{\widetilde{L}^1_t(\mathscr{C}^{\varepsilon-1})} \lesssim (1+t)\Big(&\|\mathrm{div}\,(X_0\omega_0)\|_{\mathscr{C}^{\varepsilon-3}} \\ &+ \int_0^t U_q' \,\|\mathrm{div}\,(X\omega)\|_{\mathscr{C}^{\varepsilon-3}}\, dt' + \|f\|_{\widetilde{L}^1_t(\mathscr{C}^{\varepsilon-3})}\Big), \end{aligned} \tag{1.23}$$

where U_q has been defined in (1.21).

Let us now bound the source term f in $\widetilde{L}^1_t(\mathscr{C}^{\varepsilon-3})$ of (1.23). It is not hard to estimate F via the following decomposition:

$$F = [T_X, \Delta]\omega + T_{\Delta\omega}X + R(X, \Delta\omega) - \Delta T_\omega X - \Delta R(\omega, X).$$

Using the commutator estimates of Lemma 1.11 and standard results of continuity for the paraproduct and remainder operators (see e.g. [6], Chap. 2), we get under Condition (1.15),

$$\|F\|_{\mathscr{C}^{\varepsilon-2}} \lesssim \|\omega\|_{B^{\frac{2}{q}}_{q,1}} \|X\|_{\mathscr{C}^\varepsilon}. \tag{1.24}$$

The last part of f may be decomposed into

$$\mathrm{div}\,(X\partial_1\theta) = \partial_1(\mathrm{div}\,(X\theta)) - \mathrm{div}\,(\theta\partial_1 X).$$

By Bony's decomposition (1.11),

$$\theta\partial_1 X = T_\theta \partial_1 X + T_{\partial_1 X}\theta + R(\theta, \partial_1 X).$$

As Condition (1.15) is fulfilled, we thus have

$$\|\theta\partial_1 X\|_{\mathscr{C}^{\varepsilon-2}} \lesssim \|\theta\|_{B^{\frac{2}{q}-1}_{q,1}} \|\partial_1 X\|_{\mathscr{C}^{\varepsilon-1}}. \tag{1.25}$$

Next, we see that

$$\mathrm{div}\,(X\theta) = \partial_X\theta + \theta\,\mathrm{div}\, X.$$

The last term may be bounded as in (1.25). Therefore, combining with Inequalities (1.24) and (1.25), integrating with respect to time, and using also $L^1_t(\mathscr{C}^{\varepsilon-3}) \hookrightarrow \widetilde{L}^1_t(\mathscr{C}^{\varepsilon-3})$, we end up with

$$\|f\|_{\widetilde{L}^1_t(\mathscr{C}^{\varepsilon-3})} \lesssim \int_0^t (\|\omega\|_{B^{\frac{2}{q}}_{q,1}} + \|\theta\|_{B^{\frac{2}{q}-1}_{q,1}})\|X\|_{\mathscr{C}^\varepsilon}\,dt' + \|\partial_X\theta\|_{L^1_t(\mathscr{C}^{\varepsilon-2})}. \tag{1.26}$$

Reverting to the transport equation (1.2) satisfied by X, combining with the last item of Proposition 1.5, and using also (1.22) and (1.23), we obtain

$$\begin{aligned}\|X\|_{L^\infty_t(\mathscr{C}^\varepsilon)} &\leq \|X_0\|_{\mathscr{C}^\varepsilon} + C\int_0^t \|\nabla u\|_{L^\infty}\|X\|_{\mathscr{C}^\varepsilon}\,dt' + C\|\partial_X u\|_{\widetilde{L}^1_t(\mathscr{C}^\varepsilon)}\\ &\leq \|X_0\|_{\mathscr{C}^\varepsilon} + C\int_0^t \|\nabla u\|_{L^\infty}\|X\|_{\mathscr{C}^\varepsilon}\,dt' + C\|\mathrm{div}\,(X\omega)\|_{\widetilde{L}^1_t(\mathscr{C}^{\varepsilon-1})}\\ &\leq \|X_0\|_{\mathscr{C}^\varepsilon} + C(1+t)\|\mathrm{div}\,(X_0\omega_0)\|_{\mathscr{C}^{\varepsilon-3}}\\ &+ C(1+t)\Big(\int_0^t U_q'\big(\|X\|_{\mathscr{C}^\varepsilon} + \|\mathrm{div}\,(X\omega)\|_{\mathscr{C}^{\varepsilon-3}}\big)\,dt' + \|f\|_{\widetilde{L}^1_t(\mathscr{C}^{\varepsilon-3})}\Big).\end{aligned} \tag{1.27}$$

Thus let us set

$$Z(t) := \|X\|_{L^\infty_t(\mathscr{C}^\varepsilon)} + \|\mathrm{div}\,(X\omega)\|_{L^\infty_t(\mathscr{C}^{\varepsilon-3})}$$

and

$$W(t) := U_q(t) + \int_0^t (\|\omega\|_{B^{\frac{2}{q}}_{q,1}} + \|\theta\|_{B^{\frac{2}{q}-1}_{q,1}})\,dt'.$$

Observe that the bounds (1.19), (1.20) and (1.21) imply that

$$W(t) \leq C_0\exp\big(\exp(C_0t^4)\big)\big(\|\omega_0\|_{B^{\frac{2}{q}-2}_{q,1}} + \|\theta_0\|_{B^{\frac{2}{q}-1}_{q,1}}\big). \tag{1.28}$$

Then putting together (1.23), (1.26) and (1.27) yields

$$Z(t) \lesssim (1+t)\bigg(Z(0) + \|\partial_X\theta\|_{L^1_t(\mathscr{C}^{\varepsilon-2})} + \int_0^t W'(t')Z(t')\,dt'\bigg).$$

Hence, by virtue of Gronwall lemma,

$$Z(t) \leq C(1+t)(Z(0) + \|\partial_X\theta\|_{L^1_t(\mathscr{C}^{\varepsilon-2})})e^{C(1+t)W(t)}. \tag{1.29}$$

Now, Proposition 1.5 and the fact that $\partial_X\theta$ satisfies a free transport equation imply that

$$\|\partial_X\theta\|_{L^\infty_t(\mathscr{C}^{\varepsilon-2})} \leq e^{CU_q(t)}\|\partial_{X_0}\theta_0\|_{\mathscr{C}^{\varepsilon-2}}. \tag{1.30}$$

Adding (1.28) and (1.30) to (1.29), we thus obtain

$$Z(t) \leq C_0\exp\Big(\exp\big(\exp(C_0t^4)\big)\Big). \tag{1.31}$$

Back to (1.23), one can eventually conclude that

$$\|\mathrm{div}\,(X\omega)\|_{\widetilde{L}^1_t(\mathscr{C}^{\varepsilon-1})} + \|\mathrm{div}\,(X\omega)\|_{L^\infty_t(\mathscr{C}^{\varepsilon-3})} \leq C_0 \exp\Big(\exp\big(\exp(C_0 t^4)\big)\Big). \tag{1.32}$$

1.3.4 Completing the proof of Theorem 1.1

Of course, the previous computations require the solutions to be smooth enough. To justify them in the context of Theorem 1.1, we first solve System $(B_{1,2})$ with smoothed out initial data

$$(\theta_0^n, u_0^n) := (S_n\theta_0, S_n u_0).$$

It is clear by embedding and (1.17) that the components of (θ_0, u_0) belong to all Sobolev spaces $H^s(\mathbb{R}^2)$. Thanks to the result in [16], we thus get a unique global smooth solution $(\theta^n, u^n, \nabla\Pi^n)$ having Sobolev regularity of any order. Furthermore, as θ_0^n belongs to all spaces $B^s_{q,1}$ and satisfies a linear transport equation with a smooth velocity field, we are guaranteed that $\theta^n \in \mathcal{C}(\mathbb{R}_+; B^s_{q,1})$ for all $s \in \mathbb{R}$. Then as ω^n is the solution of the transport-diffusion equation with $\omega_0^n \in B^s_{q,1}(\mathbb{R}^2)$ and source term $\partial_1\theta^n$ in $\mathcal{C}\big(\mathbb{R}_+; B^s_{q,1}(\mathbb{R}^2)\big)$ for all $s \in \mathbb{R}$, we conclude that (θ^n, ω^n) belongs to all sets E^s_q defined by

$$E^s_q := \Big\{(\vartheta, \sigma) : \vartheta \in L^\infty_{loc}(\mathbb{R}_+; B^{\frac{2}{q}-1+s}_{q,1}),\ \sigma \in \widetilde{L}^\infty_{loc}(\mathbb{R}_+; B^{\frac{2}{q}-2+s}_{q,1}) \cap L^1_{loc}(\mathbb{R}_+; B^{\frac{2}{q}+s}_{q,1})\Big\}.$$

Finally, regularizing X_0 into $X_0^n := S_n X_0$ and setting $X^n(t,x) := (\partial_{X_0^n}\psi_{u^n})(\psi_{u^n}^{-1}(t,x))$, we see that X^n belongs to Hölder spaces of any order, and satisfies (1.2) with velocity field u^n. The estimates that we proved so far are thus valid for (θ^n, u^n, X^n). We have to check whether one can bound the r.h.s. independently of n, though. As regards (1.19) and (1.28), this is obvious, for S_n maps L^p to itself for all $n \in \mathbb{N}$, with a norm independent of n, which guarantees that

$$\|\theta_0^n\|_{B^{\frac{2}{q}-1}_{q,1}} \lesssim \|\theta_0\|_{B^{\frac{2}{q}-1}_{q,1}}, \quad \|u_0^n\|_{L^2} \lesssim \|u_0\|_{L^2} \quad \text{and} \quad \|\omega_0^n\|_{B^{\frac{2}{q}-2}_{q,1}} \lesssim \|\omega_0\|_{B^{\frac{2}{q}-2}_{q,1}}. \tag{1.33}$$

Hence the sequence $(\theta^n, \omega^n)_{n\in\mathbb{N}}$ is bounded in E^0_q.

To justify the uniform boundedness of $(X^n, \partial_{X^n}\theta^n, \mathrm{div}\,(X^n\omega^n))_{n\in\mathbb{N}}$ in

$$\big(L^\infty_{loc}(\mathbb{R}_+; \mathscr{C}^\varepsilon)\big)^2 \times L^\infty_{loc}(\mathbb{R}_+; \mathscr{C}^{\varepsilon-2}) \times \big(L^\infty_{loc}(\mathbb{R}_+; \mathscr{C}^{\varepsilon-3}) \cap \widetilde{L}^1_{loc}(\mathbb{R}_+; \mathscr{C}^{\varepsilon-1})\big),$$

it is enough to check that the 'constant' C_0^n in the r.h.s. of (1.30) and (1.31) may be bounded independently of n. In fact, besides the norms appearing in (1.33), C_0^n depends only (and continuously) on

$$\|X_0^n\|_{\mathscr{C}^\varepsilon}, \quad \|\partial_{X_0^n}\theta_0^n\|_{\mathscr{C}^{\varepsilon-2}} \quad \text{and} \quad \|\mathrm{div}\,(X_0^n\omega_0^n)\|_{\mathscr{C}^{\varepsilon-3}}.$$

Arguing as in (1.33), we see that $\|X_0^n\|_{\mathscr{C}^\varepsilon}$ can be uniformly controlled by $\|X_0\|_{\mathscr{C}^\varepsilon}$. Furthermore, combining Lemma 1.13 and Lemma 2.97 in [6], we get

$$\|\partial_{X_0^n}\theta_0^n\|_{\mathscr{C}^{\varepsilon-2}} \lesssim \|\partial_{X_0}\theta_0\|_{\mathscr{C}^{\varepsilon-2}} + \|\theta_0\|_{B^{\frac{2}{q}-1}_{q,1}}\|X_0\|_{\mathscr{C}^{\varepsilon}}.$$

Finally, we claim that

$$\|\mathrm{div}\,(X_0^n\omega_0^n)\|_{\mathscr{C}^{\varepsilon-3}} \lesssim \|\mathrm{div}\,(X_0\omega_0)\|_{\mathscr{C}^{\varepsilon-3}} + \|\omega_0\|_{B^{\frac{2}{q}-2}_{q,1}}\|X_0\|_{\mathscr{C}^{\varepsilon}}.$$

This is a consequence of the decomposition

$$\mathrm{div}\,(Yf) - \mathcal{T}_Y f = \mathrm{div}\,(T_f Y + R(f,Y)) + [\partial_k, T_{Y^k}]f,$$

applied to $(Y,f) = (X_0^n, \omega_0^n)$ or (X_0, ω_0), and of the fact that

$$\mathcal{T}_{X_0^n}\omega_0^n = S_n\mathcal{T}_{X_0}\omega_0 + [T_{(X_0^n)^k}, S_n]\partial_k\omega_0 + \mathcal{T}_{(X_0^n - X_0)}\omega_0^n,$$

where the last term $\mathcal{T}_{(X_0^n-X_0)}\omega_0^n$ vanishes if N_0 in (1.11) is taken larger than 1.

Let us now establish that $(\theta^n, u^n, \nabla\Pi^n)$ converges (strongly) to some solution $(\theta, u, \nabla\Pi)$ of $(B_{1,2})$ belonging to the space $F_2^0(\mathbb{R}^2)$ where, for all $s\in\mathbb{R}$, $p\in[1,\infty]$ and $N\geq 2$, we set

$$\begin{aligned}F_p^s(\mathbb{R}^N) := \Big\{(\vartheta, v, \nabla P) : \vartheta \in \widetilde{L}^\infty_{loc}(\mathbb{R}_+; B^{\frac{N}{p}-1+s}_{p,1}(\mathbb{R}^N)), \nabla P \in \Big(L^1_{loc}\big(\mathbb{R}_+; B^{\frac{N}{p}-1+s}_{p,1}(\mathbb{R}^N)\big)\Big)^N \\ \text{and} \quad v \in \Big(\widetilde{L}^\infty_{loc}\big(\mathbb{R}_+; B^{\frac{N}{p}-1+s}_{p,1}(\mathbb{R}^N)\big) \cap L^1_{loc}\big(\mathbb{R}_+; B^{\frac{N}{p}+1+s}_{p,1}(\mathbb{R}^N)\big)\Big)^N\Big\}.\end{aligned}$$

Let us first prove that $(\theta^n, u^n, \nabla\Pi^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $F_2^{-1}(\mathbb{R}^2)$. Indeed, if setting

$$(\delta\theta_n^m, \delta u_n^m, \delta\Pi_n^m) := (\theta^m - \theta^n, u^m - u^n, \Pi^m - \Pi^n),$$

then we get from $(B_{1,N})$ that

$$\begin{cases}\partial_t\delta\theta_n^m + \mathrm{div}\,(u^m\delta\theta_n^m) = -\mathrm{div}\,(\delta u_n^m\,\theta^n),\\ \partial_t\delta u_n^m + \mathrm{div}\,(u^m\otimes\delta u_n^m) - \Delta\delta u_n^m + \nabla\delta\Pi_n^m = \delta\theta_n^m\, e_N - \mathrm{div}\,(\delta u_n^m\otimes u^n),\\ \mathrm{div}\,\delta u_n^m = 0.\end{cases} \tag{$B^{m,n}_{1,N}$}$$

We infer from Bony decomposition (1.11) and continuity results for the paraproduct and remainder, that

$$\|\mathrm{div}\,(\delta u_n^m\theta^n)\|_{B^{-1}_{2,1}} \lesssim \|\delta u_n^m\|_{B^1_{2,1}}\|\theta^n\|_{B^0_{2,1}} \quad\text{and}\quad \|\mathrm{div}\,(\delta u_n^m\otimes u^n)\|_{B^{-1}_{2,1}} \lesssim \|\delta u_n^m\|_{B^{-1}_{2,1}}\|u^n\|_{B^2_{2,1}}.$$

Using the above two inequalities and the estimates for the transport and transport-diffusion equations stated in Proposition 1.5, we end up with

$$
\begin{aligned}
&\|\delta\theta_n^m\|_{\widetilde{L}_t^\infty(B_{2,1}^{-1})} \leq e^{C\|\nabla u^m\|_{L_t^1(B_{2,1}^1)}}\left(\|\delta\theta_n^m(0)\|_{B_{2,1}^{-1}} + \int_0^t \|\delta u_n^m\|_{B_{2,1}^1}\|\theta^n\|_{B_{2,1}^0}\,dt'\right),\\
&\mathscr{E}_n^m(t) \lesssim (1+t^2)e^{C(1+t)\|\nabla u^m\|_{L_t^1(B_{2,1}^1)}}\bigg(\|\delta u_n^m(0)\|_{B_{2,1}^{-1}} + \|\delta\theta_n^m(0)\|_{B_{2,1}^{-1}}\\
&\qquad\qquad + \int_0^t \left(\|u^n\|_{B_{2,1}^1} + \|\theta^n\|_{B_{2,1}^0}\right)\mathscr{E}_n^m(t')\,dt'\bigg),
\end{aligned}
$$

where $\mathscr{E}_n^m(t) := \|\delta u_n^m\|_{\widetilde{L}_t^\infty(B_{2,1}^{-1})} + \|\delta u_n^m\|_{L_t^1(B_{2,1}^1)}$.

Combining Gronwall lemma with the bounds before, we deduce that $(\theta^n, u^n, \nabla\Pi^n)_{n\in\mathbb{N}}$ strongly converges in the norm of $\|\cdot\|_{F_2^{-1}(\mathbb{R}^2)}$. Then, interpolating, we discover that strong convergence holds true in $F_2^{-\eta}(\mathbb{R}^2)$, too, for all $\eta > 0$, which allows to justify that $(\theta, u, \nabla\Pi)$ satisfies $(B_{1,2})$. Now, from Fatou property of Besov spaces, we gather that the limit $(\theta, u, \nabla\Pi)$ belongs to $F_2^0(\mathbb{R}^2)$. In addition, as the sequence (θ^n, ω^n) is bounded in E_q^0, Fatou property also implies that (θ, ω) belongs to E_q^0. Finally, using (complex) interpolation, we obtain that for any $0 < \eta, \delta < 1$, sequence $(\theta^n, \omega^n)_{n\in\mathbb{N}}$ converges to (θ, ω) in the space $E_{q_\delta}^{-\delta\eta}$ with $q_\delta := \frac{\delta}{2} + \frac{1-\delta}{q}$.

To complete the proof of Theorem 1.1, we have to check that the announced proprieties of striated regularity are fulfilled. In fact, taking advantage of the (uniform) Lipschitz-continuity of u^n, we may obtain that for all $\eta > 0$ (see [20]),

$$
X = \lim_{n\to\infty} X^n \text{ in } L_{loc}^\infty(\mathbb{R}_+; \mathscr{C}^{\varepsilon-\eta}), \tag{1.34}
$$

which allows to justify that X satisfies (1.2). Moreover, we also have that for all $\eta' > 0$, Sequence $\big(\partial_{X^n}\theta^n, \operatorname{div}(X^n\omega^n)\big)_{n\in\mathbb{N}}$ converges in the space

$$
L_{loc}^\infty(\mathbb{R}_+; \mathscr{C}^{\varepsilon-2-\eta'}) \times L_{loc}^1(\mathbb{R}_+; \mathscr{C}^{\varepsilon-1-\eta'}).
$$

As we know in addition that (1.30) and (1.32) are fulfilled by $\partial_{X^n}\theta^n$ and $\operatorname{div}(X^n\omega^n)$ for all $n \in \mathbb{N}$, one can conclude that

$$
\partial_X\theta \in L_{loc}^\infty(\mathbb{R}_+; \mathscr{C}^{\varepsilon-2}) \quad \text{and} \quad \operatorname{div}(X\omega) \in L_{loc}^\infty(\mathbb{R}_+; \mathscr{C}^{\varepsilon-3}) \times \widetilde{L}_{loc}^1(\mathbb{R}_+; \mathscr{C}^{\varepsilon-1}),
$$

and that both (1.30) and (1.32) hold true.

1.4 Propagation of striated regularity in the general case $N \geq 3$

This section is dedicated to the proof of Theorem 1.3. The first step is to extend the global existence result of R. Danchin and M. Paicu in [38], to nonhomogeneous Besov spaces. The second step is to propagate striated regularity. As pointed out in the introduction, it suffices to bound

$\partial_X u$ in the space $\widetilde{L}^1_t(\mathscr{C}^\varepsilon)$, which requires our using the smoothing properties of the heat flow. By applying the directional derivative ∂_X to the velocity equation, we discover that $\partial_X u$ may be seen as the solution to some evolutionary Stokes system with a *nonhomogeneous* divergence condition, namely (using the fact that $\operatorname{div} u = 0$),

$$\operatorname{div}(\partial_X u) = \operatorname{div}\big(\partial_u X - u \operatorname{div} X\big).$$

In order to reduce our study to that of a solution to the standard Stokes system, it is thus natural to decompose $\partial_X u$ into

$$\partial_X u = z + \partial_u X - u \operatorname{div} X.$$

Now, z is indeed divergence free but the above decomposition introduces additional source terms in the Stokes system satisfied by z. One of those terms is $\Delta \partial_u X$, that we do not know how to handle without losing regularity. Moreover, this also leads to the failure of global estimates technically.

To overcome this difficulty, we here propose to replace the directional derivative ∂_X by the *para-vector field* $\mathcal{T}_X$ that is defined by

$$\mathcal{T}_X u := T_{X^k} \partial_k u.$$

The gain is that, as we shall see below, the aforementioned loss of regularity does not occur anymore when estimating $\mathcal{T}_X u$, and that $\mathcal{T}_X u$ may be seen of the leading order part of $\partial_X u$, as shown by the following two inequalities (that hold true whenever $\frac{N}{p} + \varepsilon - 1 \geq 0$, see Lemma 1.13):

$$\|\partial_X u - \mathcal{T}_X u\|_{L^\infty_t(\mathscr{C}^{\varepsilon-2})} \lesssim \|X\|_{L^\infty_t(\widetilde{\mathscr{C}}^\varepsilon)} \|u\|_{L^\infty_t(B^{\frac{N}{p}-1}_{p,1})}, \tag{1.35}$$

$$\|\partial_X u - \mathcal{T}_X u\|_{L^1_t(\mathscr{C}^\varepsilon)} \lesssim \|X\|_{L^\infty_t(\mathscr{C}^\varepsilon)} \|u\|_{L^1_t(B^{\frac{N}{p}+1}_{p,1})}. \tag{1.36}$$

The second step of the proof of Theorem 1.3 will be mainly devoted to proving a priori estimates for $\mathcal{T}_X u$ in $\big(L^\infty_t(\mathscr{C}^{\varepsilon-2}) \cap \widetilde{L}^1_t(\mathscr{C}^\varepsilon)\big)^N$.

Finally, in the last step of the proof, we shall smooth out the data (as in the 2-D case) so as to justify that the a priori estimates of the first two steps are indeed satisfied.

1.4.1 Global existence in nonhomogeneous Besov spaces

The following result is an obvious modification of the work by R.Danchin and M. Paicu in [38].

Theorem 1.10. *Let $N \geq 3$ and $p \in]N, \infty[$. Assume that $\theta_0 \in B^0_{N,1}(\mathbb{R}^N) \cap L^{\frac{N}{3}}(\mathbb{R}^N)$ and that the initial divergence-free velocity field u_0 is in $(B^{\frac{N}{p}-1}_{p,1}(\mathbb{R}^N) \cap L^{N,\infty}(\mathbb{R}^N))^N$. There exists a (small) positive constant c independent of p such that if*

$$\|u_0\|_{L^{N,\infty}} + \nu^{-1}\|\theta_0\|_{L^{\frac{N}{3}}} \leq c\nu, \tag{1.37}$$

then the Boussinesq system ($B_{\nu,N}$) *has a unique global solution* $(\theta, u, \nabla\Pi)$ *belonging to the space*

$$\mathcal{C}(\mathbb{R}_+; B^0_{N,1}) \cap \big(\mathcal{C}(\mathbb{R}_+; B^{\frac{N}{p}-1}_{p,1}) \cap L^1_{loc}(\mathbb{R}_+; B^{\frac{N}{p}+1}_{p,1})\big)^N \cap \big(L^1_{loc}(\mathbb{R}_+; B^{\frac{N}{p}-1}_{p,1})\big)^N.$$

Moreover, there exists some positive constant C *independent on* N *such that for any* $t \geq 0$*, we have*

$$\|\theta(t)\|_{L^{\frac{N}{3}}} = \|\theta_0\|_{L^{\frac{N}{3}}}, \quad \|u(t)\|_{L^{N,\infty}} \leq C(\|u_0\|_{L^{N,\infty}} + \nu^{-1}\|\theta_0\|_{L^{\frac{N}{3}}}), \tag{1.38}$$

$$U_p(t) \leq A(t), \tag{1.39}$$

$$\|\theta\|_{\widetilde{L}^\infty_t(B^0_{N,1})} \leq C\|\theta_0\|_{B^0_{N,1}}(1 + \nu^{-1}A(t)), \tag{1.40}$$

$$\|(\partial_t u, \nabla\Pi)\|_{L^1_t(B^{\frac{N}{p}-1}_{p,1})} \leq C\nu^{-1}A^2(t) + Ct\|\theta_0\|_{B^0_{N,1}}\big(1 + A(t)\big), \tag{1.41}$$

where

$$U_p(t) := \|u\|_{\widetilde{L}^\infty_t(B^{\frac{N}{p}-1}_{p,1})} + \nu\|u\|_{L^1_t(B^{\frac{N}{p}+1}_{p,1})} \quad \textit{and}$$

$$A(t) := C(\|u_0\|_{B^{\frac{N}{p}-1}_{p,1}} + \nu)e^{C\nu^{-1}t\|\theta_0\|_{B^0_{N,1}}} + C\nu^2\Big(\frac{\|\theta_0\|_{B^0_{N,1}} + \nu^2}{\|\theta_0\|^2_{B^0_{N,1}}}\Big)\Big(e^{C\nu^{-1}t\|\theta_0\|_{B^0_{N,1}}} - 1\Big).$$

The above nonhomogeneous estimates are based on a very simple observation allowing to control the low frequency part of the velocity field : for r satisfying $1 + \frac{1}{p} = \frac{1}{N} + \frac{1}{r}$, we have as a consequence of refined Young's inequalities (see [6]):

$$\|\Delta_{-1}u(t)\|_{L^p} \lesssim \|\mathcal{F}^{-1}\chi\|_{L^r}\|u(t)\|_{L^{N,\infty}}, \tag{1.42}$$

whose r.h.s. is bounded according to (1.38) thanks to Theorem 1.4 in [38].

For the reader convenience, let us outline the proof of (1.40) and (1.41) for a smooth solution of ($B_{\nu,N}$). Let $\mathbb{P} := \mathrm{Id} - \nabla\Delta^{-1}\mathrm{div}$ be the Leray projector onto divergence-free vector fields. On one hand, applying $\mathbb{P}\Delta_j$ to the momentum equation in ($B_{\nu,N}$) and noting that Lemma 4.4 in [38] still holds for nonhomogeneous Besov space, we get some constant C and sequence $(c_j)_{j\in\mathbb{N}}$ where $\|(c_j)\|_{\ell^1} \leq 1$ such that for any $j \geq 0$ and $t \geq 0$,

$$\begin{aligned}\|\Delta_j u(t)\|_{L^p} \leq e^{-C\nu t 2^{2j}}\|\Delta_j u_0\|_{L^p} + \int_0^t e^{-C\nu(t-t')2^{2j}}\big(\|\Delta_j\theta\|_{L^p} \\ + C2^{j(\frac{N}{p}-1)}c_j\|u\|_{\mathscr{C}^{-1}}\|u\|_{B^{\frac{N}{p}+1}_{p,1}}\big)\, dt'.\end{aligned} \tag{1.43}$$

On the other hand, the low frequency part of u could be controlled according to (1.42), and we get assuming the smallness condition (1.37) on initial data,

$$\begin{aligned}\|\Delta_{-1}u\|_{L^\infty_t(L^p)}+\nu\|\Delta_{-1}u\|_{L^1_t(L^p)}&\lesssim(1+\nu t)\|u\|_{L^\infty_t(L^{N,\infty})}\\&\lesssim(1+\nu t)\big(\|u_0\|_{L^{N,\infty}}+\nu^{-1}\|\theta_0\|_{L^{\frac N3}}\big).\end{aligned}\tag{1.44}$$

Putting (1.43) and (1.44) together thus implies that

$$U_p(t)\lesssim\|u_0\|_{B^{\frac Np-1}_{p,1}}+\|\theta\|_{L^1_t(B^{\frac Np-1}_{p,1})}+\nu^{-1}\|u\|_{L^\infty_t(\mathscr{C}^{-1})}U_p(t)+(1+\nu t)\big(\|u_0\|_{L^{N,\infty}}+\nu^{-1}\|\theta_0\|_{L^{\frac N3}}\big).$$

As $L^{N,\infty}\hookrightarrow\mathscr{C}^{-1}$ and (1.38) is fulfilled, the third term of the r.h.s. above may be absorbed by the l.h.s. if c is small enough in (1.37).

As regards θ, thanks to Proposition 1.7 and to the embedding $B^0_{N,1}\hookrightarrow B^{\frac Np-1}_{p,1}$ for $p\geq N$, we have

$$\|\theta(t)\|_{B^{\frac Np-1}_{p,1}}\lesssim\|\theta\|_{\widetilde L^\infty_t(B^0_{N,1})}\lesssim\|\theta_0\|_{B^0_{N,1}}\big(1+\|\nabla u\|_{L^1_t(L^\infty)}\big)\quad\text{for all }\ t\geq0.\tag{1.45}$$

Adding up the above estimates to the bound of $U_p(t)$ yields

$$U_p(t)\leq C(\|u_0\|_{B^{\frac Np-1}_{p,1}}+c\nu)+C(\|\theta_0\|_{B^0_{N,1}}+c\nu^2)t+C\nu^{-1}\|\theta_0\|_{B^0_{N,1}}\int_0^t U_p\,dt'.$$

The fact that $\int_0^t t'e^{B(t-t')}\,dt'\leq\frac{1}{B^2}(e^{Bt}-1)$ for any positive constant B, allows to bound the term $U_p(t)$ by $A(t)$ according to Gronwall lemma.

The estimates for $\nabla\Pi$ can be deduced from the equation, as

$$\nabla\Pi=-\mathbb{Q}(u\cdot\nabla u+\theta e_N)\quad\text{with}\quad\mathbb{Q}:=\mathrm{Id}-\mathbb{P}.$$

Finally, the bound for $\partial_t u$ may be determined from the momentum equation.

1.4.2 A priori estimates for striated regularity

In this subsection, we assume that $\nu=1$ for simplicity (which is not restrictive, owing to (1.14)).

As explained at the beginning of this section, we shall concentrate on the proof of estimates for $\mathcal{T}_X u$ in $L^\infty_t(\mathscr{C}^{\varepsilon-2})\cap\widetilde L^1_t(\mathscr{C}^\varepsilon)$, for all $t\geq0$, and this will be (mainly) based on the smoothing properties of the heat flow. More precisely, applying the para-vector field $\mathcal{T}_X$ to the velocity equation of $(B_{1,N})$, we discover that

$$\partial_t\mathcal{T}_X u+u\cdot\nabla\mathcal{T}_X u-\Delta\mathcal{T}_X u+\nabla\mathcal{T}_X\Pi=g,$$

with
$$g := -[\mathcal{T}_X, \partial_t + u\cdot\nabla]u + [\mathcal{T}_X, \Delta]u - [\mathcal{T}_X, \nabla]\Pi + (\mathcal{T}_X\theta)e_N. \tag{1.46}$$

In general, the divergence-free property is not satisfied by $\mathcal{T}_X u$ as
$$\operatorname{div}\mathcal{T}_X u = \operatorname{div}(T_{\partial_k X}u^k - T_{\operatorname{div}X}u).$$

In order to enter into the standard theory for the Stokes system, we set
$$v := \mathcal{T}_X u - w \quad \text{with} \quad w := T_{\partial_k X}u^k - T_{\operatorname{div}X}u.$$

Now, v satisfies:
$$\begin{cases} \partial_t v - \Delta v + \nabla\mathcal{T}_X\Pi = \widetilde{g}, \\ \operatorname{div} v = 0, \\ v|_{t=0} = v_0, \end{cases} \tag{S}$$
with $\widetilde{g} := g - u\cdot\nabla\mathcal{T}_X u - (\partial_t w - \Delta w)$ and g defined in (1.46).

Based on that observation, proving a priori estimates for striated regularity involves three steps. The first step is dedicated to bounding $\widetilde{g}$ (which mainly requires the commutator estimates of the appendix). In the second step, we take advantage of the smoothing effect of the heat flow so as to estimate v. In the third step, we revert to $\mathcal{T}_X u$ and also bound X so as to complete the proof of our study of striated regularity.

First step: bounds of $\widetilde{g}$

To start with, let us bound g in $\big(\widetilde{L}^1_t(\mathscr{C}^{\varepsilon-2})\big)^N$ for any positive t. According to Proposition 1.15 and to the remark that follows, the first commutator of g satisfies

$$\begin{aligned}\|[\mathcal{T}_X, \partial_t + u\cdot\nabla]u\|_{\widetilde{L}^1_t(\mathscr{C}^{\varepsilon-2})} &\lesssim \|u\|_{L^\infty_t(\mathscr{C}^{-1})}\|\mathcal{T}_X u\|_{\widetilde{L}^1_t(\mathscr{C}^{\varepsilon})} \\ &+ \int_0^t \|u\|_{B^{\frac{N}{p}+1}_{p,1}}\|\mathcal{T}_X u\|_{\mathscr{C}^{\varepsilon-2}}\,dt' + \int_0^t \|X\|_{\widetilde{\mathscr{C}}^{\varepsilon}}\|u\|_{B^{\frac{N}{p}+1}_{p,1}}\|u\|_{B^{\frac{N}{p}-1}_{p,1}}\,dt'.\end{aligned} \tag{1.47}$$

Next, thanks to the commutator estimates (1.64) we have the following two bounds:
$$\|[\mathcal{T}_X, \Delta]u\|_{L^1_t(\mathscr{C}^{\varepsilon-2})} \lesssim \int_0^t \|\nabla X\|_{\mathscr{C}^{\varepsilon-1}}\|\nabla u\|_{\mathscr{C}^0}\,dt', \tag{1.48}$$

$$\|[\mathcal{T}_X, \nabla]\Pi\|_{L^1_t(\mathscr{C}^{\varepsilon-2})} \lesssim \int_0^t \|\nabla X\|_{\mathscr{C}^{\varepsilon-1}}\|\nabla\Pi\|_{\mathscr{C}^{-1}}\,dt'. \tag{1.49}$$

Taking Lemma 1.13 into account and integrating from 0 to t gives

$$\|\mathcal{T}_X\theta\|_{L^1_t(\mathscr{C}^{\varepsilon-2})} \lesssim \|\partial_X\theta\|_{L^1_t(\mathscr{C}^{\varepsilon-2})} + \int_0^t \|X\|_{\widetilde{\mathscr{C}}^\varepsilon}\|\theta\|_{B^{\frac{N}{p}-1}_{p,1}}\,dt'. \tag{1.50}$$

Putting together (1.47) to (1.50) thus yields

$$\begin{aligned}\|g\|_{\widetilde{L}^1_t(\mathscr{C}^{\varepsilon-2})} &\lesssim \|u\|_{L^\infty_t(\mathscr{C}^{-1})}\|\mathcal{T}_X u\|_{\widetilde{L}^1_t(\mathscr{C}^\varepsilon)} + \int_0^t \|u\|_{B^{\frac{N}{p}+1}_{p,1}}\|\mathcal{T}_X u\|_{\mathscr{C}^{\varepsilon-2}}\,dt' \\ &+ \int_0^t \Big(\|u\|_{B^{\frac{N}{p}+1}_{p,1}}(\|u\|_{B^{\frac{N}{p}-1}_{p,1}}+1) + \|\nabla\Pi\|_{\mathscr{C}^{-1}} + \|\theta\|_{B^{\frac{N}{p}-1}_{p,1}}\Big)\|X\|_{\widetilde{\mathscr{C}}^\varepsilon}\,dt' + \|\partial_X\theta\|_{L^1_t(\mathscr{C}^{\varepsilon-2})}.\end{aligned}$$

Bounding the second term of $\widetilde{g}$ is obvious : taking advantage of Bony's decomposition, and remembering that $\frac{N}{p}+\varepsilon>1$ and that $\operatorname{div} u=0$, we get

$$\|u\cdot\nabla\mathcal{T}_X u\|_{\widetilde{L}^1_t(\mathscr{C}^{\varepsilon-2})} \lesssim \|u\|_{L^\infty_t(\mathscr{C}^{-1})}\|\mathcal{T}_X u\|_{\widetilde{L}^1_t(\mathscr{C}^\varepsilon)} + \int_0^t \|u\|_{B^{\frac{N}{p}+1}_{p,1}}\|\mathcal{T}_X u\|_{\mathscr{C}^{\varepsilon-2}}\,dt'.$$

Finally, to bound the term $\partial_t w-\Delta w$ in $\widetilde{L}^1_t(\mathscr{C}^{\varepsilon-2})$, we use the decomposition

$$\partial_t w-\Delta w=\sum_{\alpha=1}^{3}W_\alpha,$$

with

$$W_1 := T_{\partial_k X}\partial_t u^k - T_{\operatorname{div} X}\partial_t u,\quad W_2 := T_{\partial_k\partial_t X}u^k - T_{\operatorname{div}\partial_t X}u,\quad W_3 := \Delta\big(T_{\operatorname{div} X}u - T_{\partial_k X}u^k\big).$$

By embedding, it suffices to bound the components of W_α in $L^1_t(B^{\frac{N}{p}+\varepsilon-2}_{p,1})$ with $\alpha=1,2,3$.

Now, standard continuity results for the paraproduct ensure that

$$\begin{aligned}\|W_1\|_{L^1_t(B^{\frac{N}{p}+\varepsilon-2}_{p,1})} &\lesssim \int_0^t \|\nabla X\|_{\mathscr{C}^{\varepsilon-1}}\|\partial_t u\|_{B^{\frac{N}{p}-1}_{p,1}}\,dt', \\ \|W_2\|_{L^1_t(B^{\frac{N}{p}+\varepsilon-2}_{p,1})} &\lesssim \int_0^t \|\partial_t X\|_{\mathscr{C}^{\varepsilon-2}}\|u\|_{B^{\frac{N}{p}+1}_{p,1}}\,dt', \\ \|W_3\|_{L^1_t(B^{\frac{N}{p}+\varepsilon-2}_{p,1})} &\lesssim \int_0^t \|\nabla X\|_{\mathscr{C}^{\varepsilon-1}}\|u\|_{B^{\frac{N}{p}+1}_{p,1}}\,dt'.\end{aligned}$$

To estimate $\partial_t X$ in the bound of W_2, we use the fact that

$$\partial_t X = -u\cdot\nabla X + \partial_X u.$$

The convection term $u \cdot \nabla X = \operatorname{div}(u \otimes X)$ may be easily bounded (use Bony's decomposition and continuity results for the remainder and paraproduct operators) as follows:

$$\|u \cdot \nabla X\|_{\mathscr{C}^{\varepsilon-2}} \lesssim \|u\|_{B^{\frac{N}{p}-1}_{p,1}} \|X\|_{\mathscr{C}^{\varepsilon}}.$$

Bounding $\partial_X u$ according to Lemma 1.13 yields

$$\|W_2\|_{L^1_t(B^{\frac{N}{p}+\varepsilon-2}_{p,1})} \lesssim \int_0^t (\|X\|_{\widetilde{\mathscr{C}}^{\varepsilon}} \|u\|_{B^{\frac{N}{p}-1}_{p,1}} + \|\mathcal{T}_X u\|_{\mathscr{C}^{\varepsilon-2}}) \|u\|_{B^{\frac{N}{p}+1}_{p,1}} \, dt'.$$

Putting together all the above estimates and using (slightly abusively) the notation $U_p'(\tau) := \|u(\tau)\|_{B^{\frac{N}{p}+1}_{p,1}}$ for any $\tau \geq 0$, we eventually get the following estimate for $\widetilde{g}$:

$$\|\widetilde{g}\|_{\widetilde{L}^1_t(\mathscr{C}^{\varepsilon-2})} \lesssim \|u\|_{L^\infty_t(\mathscr{C}^{-1})} \|\mathcal{T}_X u\|_{\widetilde{L}^1_t(\mathscr{C}^{\varepsilon})} + \|\partial_X \theta\|_{L^1_t(\mathscr{C}^{\varepsilon-2})} + \int_0^t U_p' \|\mathcal{T}_X u\|_{\mathscr{C}^{\varepsilon-2}} \, dt'$$
$$+ \int_0^t \Big(U_p'(\|u\|_{B^{\frac{N}{p}-1}_{p,1}} + 1) + \|(\partial_t u, \nabla \Pi, \theta)\|_{B^{\frac{N}{p}-1}_{p,1}}\Big) \|X\|_{\widetilde{\mathscr{C}}^{\varepsilon}} \, dt'. \quad (1.51)$$

Second step: Bounds for v

The second step is devoted to bounding v in $\big(L^\infty_t(\mathscr{C}^{\varepsilon-2}) \cap \widetilde{L}^1_t(\mathscr{C}^{\varepsilon})\big)^N$. Granted with (1.51), this will essentially follow from the smoothing properties of the heat flow after spectral localization. More precisely, applying $\mathbb{P}\Delta_j$ to the equation (S) yields for all $j \geq -1$,

$$\begin{cases} \partial_t \Delta_j v - \Delta \Delta_j v = \mathbb{P}\Delta_j \widetilde{g} \\ \Delta_j v|_{t=0} = \Delta_j v_0. \end{cases}$$

Now, Lemma 2.1 in [21] implies that if $j \geq 0$,

$$\|\Delta_j v(t)\|_{L^\infty} \leq C\bigg(e^{-ct2^{2j}} \|\Delta_j v_0\|_{L^\infty} + \int_0^t e^{-c(t-t')2^{2j}} \|\Delta_j \widetilde{g}(t')\|_{L^\infty} \, dt'\bigg).$$

Therefore, taking the supremum over $j \geq 0$, we find that the high frequency part of v satisfies

$$\sup_{j\geq 0} 2^{j(\varepsilon-2)} \|\Delta_j v\|_{L^\infty_t(L^\infty)} + \sup_{j \geq 0} 2^{j\varepsilon} \|\Delta_j v\|_{L^1_t(L^\infty)} \lesssim \|v_0\|_{\mathscr{C}^{\varepsilon-2}} + \|\widetilde{g}\|_{\widetilde{L}^1_t(\mathscr{C}^{\varepsilon-2})}.$$

Bounding the norm $\|v_0\|_{\mathscr{C}^{\varepsilon-2}}$ stems from the definition of v_0 and from Lemma 1.13: we get

$$\|v_0\|_{\mathscr{C}^{\varepsilon-2}} \lesssim \|\mathcal{T}_{X_0} u_0\|_{\mathscr{C}^{\varepsilon-2}} + \|u_0\|_{\mathscr{C}^{-1}} \|X_0\|_{\mathscr{C}^{\varepsilon}} \lesssim \|\partial_{X_0} u_0\|_{\mathscr{C}^{\varepsilon-2}} + \|u_0\|_{B^{\frac{N}{p}-1}_{p,1}} \|X_0\|_{\widetilde{\mathscr{C}}^{\varepsilon}}. \quad (1.52)$$

To handle the low frequencies of v, we first observe from the definition of $\|\cdot\|_{\mathscr{C}^s}$ that

$$\|\Delta_{-1}v\|_{L_t^\infty(L^\infty)} + \|\Delta_{-1}v\|_{L_t^1(L^\infty)} \lesssim \|v\|_{L_t^\infty(\mathscr{C}^{-2})} + \|v\|_{L_t^1(L^\infty)}.$$

Then we recall the definition $v = \mathcal{T}_X u - T_{\partial_k X} u^k + T_{\operatorname{div} X} u$, and hence continuity results for paraproduct yield

$$\|v\|_{L_t^\infty(\mathscr{C}^{-2})} + \|v\|_{L_t^1(L^\infty)} \lesssim \|X\|_{L_t^\infty(L^\infty)}(\|u\|_{\widetilde{L}_t^\infty(B_{p,1}^{\frac{N}{p}-1})} + \|u\|_{L_t^1(B_{p,1}^{\frac{N}{p}+1})}).$$

Moreover, the fact that X satisfies the transport equation (1.2) ensures

$$\|X\|_{L_t^\infty(L^\infty)} \leq \|X_0\|_{L^\infty} e^{\|\nabla u\|_{L_t^1(L^\infty)}}.$$

Hence, using the notation U_p of Theorem 1.10, we control the low frequency part of v via

$$\|\Delta_{-1}v\|_{L_t^\infty(L^\infty)} + \|\Delta_{-1}v\|_{L_t^1(L^\infty)} \lesssim \|X_0\|_{L^\infty} U_p(t) e^{CU_p(t)}.$$

Putting the estimates for low and high frequency parts of v together with (1.52), we end up with

$$\widetilde{\mathscr{H}}(t) \lesssim \|\partial_{X_0} u_0\|_{\mathscr{C}^{\varepsilon-2}} + \|u_0\|_{B_{p,1}^{\frac{N}{p}-1}} \|X_0\|_{\widetilde{\mathscr{C}}^\varepsilon} + \|\widetilde{g}\|_{\widetilde{L}_t^1(\mathscr{C}^{\varepsilon-2})} + \|X_0\|_{L^\infty} U_p(t) e^{CU_p(t)}, \tag{1.53}$$

where we denoted

$$\widetilde{\mathscr{H}}(t) := \|v\|_{L_t^\infty(\mathscr{C}^{\varepsilon-2})} + \|v\|_{\widetilde{L}_t^1(\mathscr{C}^\varepsilon)}.$$

Third step: bounds for striated regularity

Keeping (1.35) and (1.36) in mind, the core of the proof consists in deriving appropriate bounds for $\mathcal{T}_X u$ and X. Now, remembering that

$$\mathcal{T}_X u = v + w \quad \text{with} \quad w = T_{\partial_k X} u^k - T_{\operatorname{div} X} u,$$

it is easy to bound the following quantity:

$$\mathscr{H}(t) := \|\mathcal{T}_X u\|_{L_t^\infty(\mathscr{C}^{\varepsilon-2})} + \|\mathcal{T}_X u\|_{\widetilde{L}_t^1(\mathscr{C}^\varepsilon)}.$$

Indeed, continuity results for paraproduct operators guarantee that

$$\|w\|_{L_t^\infty(\mathscr{C}^{\varepsilon-2})} + \|w\|_{\widetilde{L}_t^1(\mathscr{C}^\varepsilon)} \lesssim \|u\|_{L_t^\infty(\mathscr{C}^{-1})} \|X\|_{L_t^\infty(\mathscr{C}^\varepsilon)} + \int_0^t \|u\|_{\mathscr{C}^1} \|\nabla X\|_{\mathscr{C}^{\varepsilon-1}} \, dt'. \tag{1.54}$$

Let us now bound X in $L_t^\infty(\widetilde{\mathscr{C}}^\varepsilon)$. As X and $\operatorname{div} X$ satisfy (1.2) and (1.3), respectively, Hölder estimates for transport equations and Lemma 1.13 imply that

$$
\begin{aligned}
\|X\|_{L_t^\infty(\widetilde{\mathscr{C}}^\varepsilon)} &\leq \|X_0\|_{\widetilde{\mathscr{C}}^\varepsilon} + \int_0^t \|\nabla u\|_{L^\infty} \|X\|_{\widetilde{\mathscr{C}}^\varepsilon} \, dt' + \|\partial_X u\|_{\widetilde{L}_t^1(\mathscr{C}^\varepsilon)} \\
&\leq \|X_0\|_{\widetilde{\mathscr{C}}^\varepsilon} + \int_0^t U_p' \|X\|_{\widetilde{\mathscr{C}}^\varepsilon} \, dt' + \|\mathcal{T}_X u\|_{\widetilde{L}_t^1(\mathscr{C}^\varepsilon)}.
\end{aligned} \tag{1.55}
$$

Inserting this in (1.54), one gets

$$
\|w\|_{L_t^\infty(\mathscr{C}^{\varepsilon-2})} + \|w\|_{\widetilde{L}_t^1(\mathscr{C}^\varepsilon)} \lesssim \|u\|_{L_t^\infty(\mathscr{C}^{-1})} \|\mathcal{T}_X u\|_{\widetilde{L}_t^1(\mathscr{C}^\varepsilon)} + \|X_0\|_{\widetilde{\mathscr{C}}^\varepsilon} + \int_0^t U_p' \|X\|_{\widetilde{\mathscr{C}}^\varepsilon} \, dt'. \tag{1.56}
$$

By the embedding $L^{N,\infty}(\mathbb{R}^N) \hookrightarrow \mathscr{C}^{-1}(\mathbb{R}^N)$ and smallness condition (1.37), we know that $\|u\|_{L_t^\infty(\mathscr{C}^{-1})}$ is small. Therefore, from (1.51), (1.53) and (1.56), we infer that

$$
\begin{aligned}
\mathscr{H}(t) \lesssim \|\partial_{X_0} u_0\|_{\mathscr{C}^{\varepsilon-2}} &+ \|X_0\|_{\widetilde{\mathscr{C}}^\varepsilon} (\|u_0\|_{B_{p,1}^{\frac{N}{p}-1}} + 1) + \|X_0\|_{L^\infty} U_p(t) e^{U_p(t)} + \|\partial_X \theta\|_{L_t^1(\mathscr{C}^{\varepsilon-2})} \\
&+ \int_0^t U_p' \mathscr{H} \, dt' + \int_0^t \big(U_p'(U_p + 1) + \|(\partial_t u, \nabla \Pi, \theta)\|_{B_{p,1}^{\frac{N}{p}-1}} \big) \|X\|_{\widetilde{\mathscr{C}}^\varepsilon} \, dt'.
\end{aligned} \tag{1.57}
$$

Denoting

$$
\mathscr{K}(t) := \mathscr{H}(t) + \|X\|_{L_t^\infty(\widetilde{\mathscr{C}}^\varepsilon)}
$$

and using (1.55) and (1.57), we obtain that

$$
\begin{aligned}
\mathscr{K}(t) \lesssim \big(\|\partial_{X_0} u_0\|_{\mathscr{C}^{\varepsilon-2}} + \|X_0\|_{\widetilde{\mathscr{C}}^\varepsilon} (\|u_0\|_{B_{p,1}^{\frac{N}{p}-1}} + 1) + \|X_0\|_{L^\infty} U_p(t) e^{CU_p(t)} + \|\partial_X \theta\|_{L_t^1(\mathscr{C}^{\varepsilon-2})} \big) \\
+ \int_0^t \Big(U_p'(U_p + 1) + \|(\partial_t u, \nabla \Pi, \theta)\|_{B_{p,1}^{\frac{N}{p}-1}} \Big) \mathscr{K}(t') \, dt'.
\end{aligned}
$$

In order to bound $\partial_X \theta$, it suffices to remember that

$$
\partial_t \partial_X \theta + u \cdot \nabla \partial_X \theta = 0.
$$

As $\frac{N}{p} + \varepsilon > 1$, applying Proposition 1.5 gives

$$
\|\partial_X \theta\|_{L_t^\infty(\mathscr{C}^{\varepsilon-2})} \lesssim \|\partial_{X_0} \theta_0\|_{\mathscr{C}^{\varepsilon-2}} e^{CU_p(t)}.
$$

Furthermore, using embedding and Vishik type estimate (1.45) for θ yields

$$
\|\theta\|_{L_t^\infty(B_{p,1}^{\frac{N}{p}-1})} \lesssim \|\theta\|_{\widetilde{L}_t^\infty(B_{N,1}^0)} \lesssim \|\theta_0\|_{B_{N,1}^0} (1 + U_p(t)).
$$

Applying Gronwall lemma to $\mathscr{K}(t)$ and mingling with the above bounds for $(\partial_X \theta, \theta)$ and with the bounds of Theorem 1.10 for U_p and $(\partial_t u, \nabla \Pi)$ yields

$$\mathscr{K}(t) \leq C\big(\|\partial_{X_0} u_0\|_{\mathscr{C}^{\varepsilon-2}} + \|X_0\|_{\widetilde{\mathscr{C}}^{\varepsilon}}(\|u_0\|_{B^{\frac{N}{p}-1}_{p,1}} + 1) + \|X_0\|_{L^\infty} + \|\partial_{X_0}\theta_0\|_{\mathscr{C}^{\varepsilon-2}}\big)$$
$$\exp\Big(C\big((A^2(t)+1)(t+1)(\|\theta_0\|_{B^0_{N,1}}+1)\big)\Big) \leq C_0 \exp\big(\exp(C_0 t)\big),$$

where $A(t)$ has been defined in Theorem 1.10. Then we deduce from (1.35) and (1.36) that

$$\|\partial_X u\|_{L^\infty_t(\mathscr{C}^{\varepsilon-2})} + \|\partial_X u\|_{\widetilde{L}^1_t(\mathscr{C}^{\varepsilon})} \lesssim (1 + A(t))\mathscr{K}(t).$$

Finally, in order to bound $\partial_X \nabla\Pi$, we use the identity

$$\partial_X \nabla\Pi - \nabla\mathcal{T}_X\Pi = \operatorname{div} R(X, \nabla\Pi) - R(\operatorname{div} X, \nabla\Pi) + T_{\nabla^2\Pi}\cdot X - \mathcal{T}_{\nabla X}\Pi,$$

from which we deduce that

$$\|\partial_X \nabla\Pi - \nabla\mathcal{T}_X\Pi\|_{\widetilde{L}^1_t(\mathscr{C}^{\varepsilon-2})} \lesssim \|X\|_{L^\infty_t(\widetilde{\mathscr{C}}^{\varepsilon})}\|\nabla\Pi\|_{L^1_t(B^{\frac{N}{p}-1}_{p,1})}.$$

Now, the fact that $\nabla\mathcal{T}_X\Pi$ may be estimated in $\big(\widetilde{L}^1_{loc}(\mathbb{R}_+;\mathscr{C}^{\varepsilon-2})\big)^N$ is a consequence of

$$\nabla\mathcal{T}_X\Pi = \mathbb{Q}\widetilde{g}$$

and of (1.51) once noticed that the projector $\mathbb{Q}$ on gradient-like vector fields is a 0-th order Fourier multiplier and that the Fourier transform of $\nabla\mathcal{T}_X\Pi$ is supported away from 0. This completes the proof of a priori estimates for striated regularity in Theorem 1.3.

1.4.3 End of the proof of Theorem 1.3

Like in the 2-D case, in order to prove the global existence in the desired spaces, we first smooth out the initial data:

$$(\theta_0^n, u_0^n) := (S_n\theta_0, S_n u_0).$$

Since S_n is a (uniformly) bounded operator on all Besov spaces $B^s_{q,r}$, Lebesgue spaces L^q and weak Lebesgue spaces $L^{q,\infty}$ with $1 < q < \infty$, we can apply Theorem 1.10 and obtain for all $n \in \mathbb{N}$ a unique global solution $(\theta^n, u^n, \nabla\Pi^n)$ which fulfills Inequalities (1.38) to (1.41) with right-hand sides independent of n. In particular, sequence $(\theta^n, u^n, \nabla\Pi^n)_{n\in\mathbb{N}}$ is bounded in the space $F^0_p(\mathbb{R}^N)$ of Subsection 1.3.4.

If we consider as in the 2-D case, System ($B^{m,n}_{1,N}$) satisfied by

$$(\delta\theta^m_n, \delta u^m_n, \delta\Pi^m_n) := (\theta^m - \theta^n, u^m - u^n, \Pi^m - \Pi^n),$$

and combine estimates for transport and transport-diffusion equations, and product laws like e.g.

$$\begin{aligned} &\|\operatorname{div}(\delta u_n^m \theta^n)\|_{B^{\frac{N}{p}-2}_{p,1}} \lesssim \|\delta u_n^m\|_{B^{\frac{N}{p}}_{p,1}} \|\theta^n\|_{B^{\frac{N}{p}-1}_{p,1}} \\ \text{and} \quad &\|\operatorname{div}(\delta u_n^m \otimes u^n)\|_{B^{\frac{N}{p}-2}_{p,1}} \lesssim \|\delta u_n^m\|_{B^{\frac{N}{p}-2}_{p,1}} \|u^n\|_{B^{\frac{N}{p}+1}_{P,1}}, \end{aligned}$$

then we can easily prove that the sequence $(\theta^n, u^n, \nabla\Pi^n)_{n\in\mathbb{N}}$ strongly converges in $F_p^{-1}(\mathbb{R}^N)$ to some limit $(\theta, u, \nabla\Pi)$. By uniform estimates and interpolation, we see that convergence holds true in $F_p^{-\eta}(\mathbb{R}^N)$ for any $\eta > 0$, and functional analysis arguments similar to those of Subsection 1.3.4 allow to show that $(\theta, u, \nabla\Pi)$ is indeed a solution to $(B_{1,N})$ that belongs to F_p^0. Moreover, $\theta \in \widetilde{L}^\infty_{loc}(\mathbb{R}_+; B^0_{N,1})$, owing to (1.40).

Next, to establish conservation of striated regularity, we smooth out X_0 into $X_0^n := S_n X_0$ and define $X^n(t,x) := (\partial_{X_0^n}\psi_{u^n})(\psi_{u^n}^{-1}(t,x))$. By adapting the arguments of the 2-D case, it is easy to make all the bounds obtained in the previous step independent of n, and to conclude to the full statement of Theorem 1.3. The details are left to the reader.

1.5 The temperature patch problem with Hölder regularity

This section is devoted to solving the temperature patch problem, through Corollaries 1.2 and 1.4. We start with general considerations that will be useful both in the 2-D and in the N-D cases.

Because $\mathcal{D}_0$ is a simply connected bounded $\mathcal{C}^{1,\varepsilon}$ domain of $\mathbb{R}^N$ (see Definition above Corollary 1.4), it is orientable (as $\partial\mathcal{D}_0$ is compact), and one can adopt the so-called level-set characterization (see Chap. 2 of [47] and references therein). More precisely, for any (bounded) open neighborhood V_0 of $\overline{\mathcal{D}}_0$, there exists some open set W_0 with $\overline{\mathcal{D}}_0 \subset W_0 \subset V_0$ and a function $f_0 \in \mathcal{C}^{1,\varepsilon}(\mathbb{R}^N;\mathbb{R})$ such that $\partial\mathcal{D}_0 = f_0^{-1}(\{0\}) \cap W_0$, ∇f_0 is supported in $\overline{W}_0$ and does not vanish on $\partial\mathcal{D}_0$. The key to that global characterization is the *tubular neighborhood theorem.* In the case $N = 2$, we fix V_0 so that in addition $V_0 \cap \overline{\mathcal{D}_0^\star} = \emptyset$ (see Corollary 1.2) while if $N \geq 3$, we assume that $\overline{V_0} \cap (\mathbb{R}^N \setminus J_0) = \emptyset$. Finally, we also fix some smooth cut-off function χ_0 with compact support in V_0, and value 1 on W_0.

As for the classical vortex patch problem for the incompressible Euler equations, the result will come up as a consequence of persistence of striated regularity. Indeed, at time t the boundary of $\mathcal{D}_t = \psi_u(t, \mathcal{D}_0)$ is the level set of the function $f(t,\cdot) := f_0\big(\psi_u^{-1}(t,\cdot)\big)$ satisfying

$$\begin{cases} \partial_t f + u\cdot\nabla f = 0, \\ f|_{t=0} = f_0. \end{cases} \tag{1.58}$$

In order to control the regularity in $\mathcal{C}^{1,\varepsilon}$ of $\partial\mathcal{D}_t$, it is thus sufficient to show that the components of ∇f remain in $\mathcal{C}^{0,\varepsilon}$ for all time. In the 2-D case, the proof is rather direct because it will be possible to apply directly Theorem 1.1 with the (divergence free) vector field $X_0 := \nabla^\perp f_0$

provided u_0 satisfies suitable striated regularity properties with respect to X_0. In the N-D case, more (not necessarily divergence-free) vector fields will be needed.

Before going to the proof of our corollaries, we would like to point out two important properties. The first one is that the characteristic function of any bounded $\mathcal{C}^1$ bounded simply connected domain $\mathcal{D}$ of $\mathbb{R}^N$ for $N \geq 2$ satisfies (see [36] and references therein):

$$\mathbb{1}_{\mathcal{D}} \in B^{\frac{1}{q}}_{q,\infty}(\mathbb{R}^N) \hookrightarrow B^{\frac{N}{p}-1}_{p,1}(\mathbb{R}^N) \quad \text{whenever} \ \ (N-1) < q \leq p \leq \infty. \tag{1.59}$$

The second observation concerns L^N boundedness of velocity field u_0. In the 2-D case, having the vorticity compactly supported, in L^r for some $r > 1$, and *mean free* ensures that $u_0 \in \big(L^2(\mathbb{R}^2)\big)^2$ (see e.g. Chap.3 in [92]).

In dimension $N \geq 3$, that u_0 is in L^N is an easy consequence of Biot-Savart law and Hardy-Littlewood-Sobolev inequality. Indeed, we have

$$\|u_0\|_{L^N(\mathbb{R}^N)} \lesssim \| |\cdot|^{1-N} \star \Omega_0\|_{L^N(\mathbb{R}^N)} \lesssim \|\Omega_0\|_{L^{\frac{N}{2}}(\mathbb{R}^N)}. \tag{1.60}$$

In what follows, whenever Y is a (tangential) vector field on $\mathbb{R}^N$, we shall denote by $\widetilde{Y}_t$ its restriction along the surface $\partial\mathcal{D}_t$.

1.5.1 The 2-D case

This short paragraph is devoted to proving Corollary 1.2. Of course, one can assume with no loss of generality that the exponent r is in $]1, \frac{2}{2-\varepsilon}[$ so that we have $\omega_0 \in B^{\frac{2}{q}-2}_{q,1}(\mathbb{R}^2)$ for some $1 < q < \frac{2}{2-\varepsilon}$. Thus, remembering that u_0 is in $\big(L^2(\mathbb{R}^2)\big)^2$ (as explained above), and using (1.59), we see that both θ_0 and u_0 fulfill the regularity assumptions of Theorem 1.1.

Next, we observe that the boundary $\partial\mathcal{D}_0$ of the initial patch is some Jordan curve of the plane. Setting $X_0 := \nabla^\perp f_0$ provides us with a parametrization γ_0 of the curve $\partial\mathcal{D}_0$ through the following ODE:

$$\begin{cases} \partial_\sigma\gamma_0(\sigma) = \widetilde{X}_0\big(\gamma_0(\sigma)\big), \ \ \forall\sigma \in \mathbb{S}^1, \\ \gamma_0(\sigma_0) = x_0 \in \partial\mathcal{D}_0. \end{cases}$$

It is obvious that $\partial_{X_0}\theta_0 = \operatorname{div}(X_0\theta_0) \equiv 0$ in the sense of distributions. Similarly, because $X_0 \equiv 0$ on some neighborhood of $\overline{\mathcal{D}_0^\star}$, we have $\operatorname{div}(X_0\omega_0) = \partial_{X_0}\omega_0 \equiv 0$. Hence Theorem 1.1 applies.

In order to conclude the proof of Corollary 1.2, we observe that a parametrization for $\partial\mathcal{D}_t$ is given by $\gamma_t(\sigma) := \psi_u(t, \gamma_0(\sigma))$ and that, obviously,

$$\begin{cases} \partial_\sigma \gamma_t(\sigma) = \widetilde{X}_t\big(\gamma_t(\sigma)\big), \ \ \forall \sigma \in \mathbb{S}^1, \\ \gamma_t(\sigma_0) = \psi_u(t, x_0) \in \partial \mathcal{D}_t, \end{cases} \quad \text{with } \ X_t(x) := (\partial_{X_0}\psi_u)\big(\psi_u^{-1}(t,x)\big).$$

As pointed out before, X_t satisfies Equation (1.2), from which we can deduce that X_t is in $\big(\mathcal{C}^{0,\varepsilon}(\mathbb{R}^2)\big)^2$ for all $t \geq 0$, and thus $\gamma_t \in \mathcal{C}^{1,\varepsilon}(\mathbb{S}^1; \mathbb{R}^2)$.

1.5.2 The N-D case

If we take data θ_0 and u_0 fulfilling the hypotheses of Corollary 1.4 then (1.59) and standard embedding ensure that

$$\theta_0 \in B^0_{N,1}(\mathbb{R}^N) \quad \text{and} \quad (\Omega_0)^i_j \in B^{\frac{1}{p}}_{p,\infty}(\mathbb{R}^N) \hookrightarrow B^{\frac{N}{p}-2}_{p,1}(\mathbb{R}^N) \ \text{ as long as } \ p > \frac{N-1}{2}.$$

Since in addition $u^i_0 \in L^N(\mathbb{R}^N)$ (see (1.60)), we have $u^i_0 \in B^{\frac{N}{p}-1}_{p,1}(\mathbb{R}^N)$ as may be seen through the following decomposition

$$u^i_0 = \Delta_{-1} u^i_0 - \sum_j (\mathrm{Id} - \Delta_{-1})(-\Delta)^{-1}\partial_j (\Omega_0)^i_j,$$

and the fact that $(\mathrm{Id} - \Delta_{-1})(-\Delta)^{-1}\partial_j$ is a homogeneous multiplier of degree -1 away from a neighborhood of the origin. Indeed, we have for all $p > \frac{N-1}{2}$,

$$\|u_0\|_{B^{\frac{N}{p}-1}_{p,1}} \lesssim \|\Omega_0\|_{L^{\frac{N}{2}}} + \|\Omega_0\|_{B^{\frac{1}{p}}_{p,\infty}}. \tag{1.61}$$

Therefore, if m_1 and m_2 in Corollary 1.4 are sufficiently small, then one may apply the first part of Theorem 1.3 and get a unique global solution (θ, u) for System ($B_{\nu,N}$), satisfying (1.38), (1.39), (1.40) and (1.41).

In order to propagate the regularity of the temperature patch, we shall borrow the ideas of [28, 55] pertaining to the standard vortex patch problem. Firstly, we claim that if X_0 is in $\big(\widetilde{\mathscr{C}}^\varepsilon(\mathbb{R}^N)\big)^N$ then having $\partial_{X_0}\Omega_0$ in $\mathscr{C}^{\varepsilon-3}$ implies that $\partial_{X_0} u_0$ is in $\big(\mathscr{C}^{\varepsilon-2}(\mathbb{R}^N)\big)^N$. Indeed, Lemma 1.13 yields for $\frac{N}{p} + \varepsilon > 1$

$$\|\partial_{X_0} u_0\|_{\mathscr{C}^{\varepsilon-2}} \lesssim \|\mathcal{T}_{X_0} u_0\|_{\mathscr{C}^{\varepsilon-2}} + \|X_0\|_{\widetilde{\mathscr{C}}^\varepsilon} \|u_0\|_{B^{\frac{N}{p}-1}_{p,1}}.$$

Then by Biot-Savart Law, we have the following identity

$$\begin{aligned} \mathcal{T}_{X_0} u^i_0 &= \sum_j T_{X^k_0} \partial_k \Delta^{-1} \partial_j (\Omega_0)^i_j \\ &= \sum_j \big(\Delta^{-1}\partial_j \psi_{N_0}(D)\big)\mathcal{T}_{X_0}(\Omega_0)^i_j + [T_{X^k_0}, \Delta^{-1}\partial_j \psi_{N_0}(D)]\partial_k (\Omega_0)^i_j, \end{aligned}$$

where ψ_{N_0} is some suitable (smooth) cut-off function away from 0. From the above identity and

Lemma 1.11, we deduce that

$$\|\mathcal{T}_{X_0}u_0\|_{\mathscr{C}^{\varepsilon-2}} \lesssim \|\mathcal{T}_{X_0}\Omega_0\|_{\mathscr{C}^{\varepsilon-3}} + \|X_0\|_{\widetilde{\mathscr{C}}^{\varepsilon}}\|\Omega_0\|_{B^{\frac{1}{q}}_{q,\infty}} \quad \text{for } q \geq N-1,$$

whence, thanks to Lemma 1.13 and (1.61),

$$\|\partial_{X_0}u_0\|_{\mathscr{C}^{\varepsilon-2}} \lesssim \|X_0\|_{\widetilde{\mathscr{C}}^{\varepsilon}}(\|\Omega_0\|_{L^{\frac{N}{2}}} + \|\Omega_0\|_{B^{\frac{1}{q}}_{q,\infty}}) + \|\partial_{X_0}\Omega_0\|_{\mathscr{C}^{\varepsilon-3}}. \tag{1.62}$$

Now, let us construct a first family of vector fields $\{E_j\}_{j=1}^N$ according to (1.62) and Theorem 1.3. Let $\{e_1, \cdots, e_N\}$ be the canonical basis of $\mathbb{R}^N$ and assume (with no loss of generality) that the cut-off function χ_0 defined at the beginning of this section is supported in the domain J_0. Let E_j be the solution to

$$\begin{cases} \partial_t E_j + u\cdot\nabla E_j = \partial_{E_j}u, \\ E_j|_{t=0} = E_{j,0} := \chi_0\, e_j. \end{cases}$$

Taking $X_0 = E_{j,0}$ in (1.62), the last term $\partial_{E_{j,0}}\Omega_0$ on the r.h.s. vanishes by the definition of J_0. Hence Theorem 1.3 applies and one can conclude that $E_j \in \big(L^\infty_{loc}(\mathbb{R}_+;\widetilde{\mathscr{C}}^{\varepsilon})\big)^N$ for all $j = 1, \cdots, N$. Furthermore, because $\nabla\theta_0 \in \big(B^{\frac{1}{q}-1}_{q,\infty}(\mathbb{R}^N)\big)^N$ for all $q \in [1, +\infty]$, we have $\nabla\theta_0 \in \big(\mathscr{C}^{\varepsilon-2}(\mathbb{R}^N)\big)^N$ by embedding (just take $q \geq \frac{N-1}{1-\varepsilon}$). Of course, as $E_{j,0}$ is smooth and compactly supported, and as any nonhomogeneous Besov space is stable by multiplication by functions in $\mathcal{C}^\infty_c$, one can conclude that $\partial_{E_{j,0}}\theta_0 \in \mathscr{C}^{\varepsilon-2}(\mathbb{R}^N)$.

Before constructing a second family of vector fields, let us introduce one more notation. For any family $\mathcal{P} = (P_1, ..., P_{N-1})$ of $N-1$ vectors of $\mathbb{R}^N$, we define $\wedge\mathcal{P} \equiv P_1 \wedge \cdots \wedge P_{N-1}$ to be the unique vector of $\mathbb{R}^N$ satisfying

$$\wedge\mathcal{P}\cdot Q = \det(P_1, ..., P_{N-1}, Q), \quad \forall Q \in \mathbb{R}^N.$$

Assume that $\mathcal{X} = (X_1, \cdots, X_{N-1})$ where the time dependent vector fields X_j satisfy System (1.2). Thanks to $\operatorname{div} u = 0$, the vector field $\wedge\mathcal{X}$ satisfies the equation,

$$\partial_t \wedge\mathcal{X} + u\cdot\nabla\wedge\mathcal{X} = -(\nabla u)^{tr}\cdot(\wedge\mathcal{X}). \tag{1.63}$$

Set $\Lambda = \{\lambda = (\lambda_1, ..., \lambda_{N-1}) : 1 \leq \lambda_1 < \cdots < \lambda_{N-1} \leq N\}$. For any $\lambda \in \Lambda$, we define Y_λ to be the solution of

$$\begin{cases} \partial_t Y_\lambda + u\cdot\nabla Y_\lambda = \partial_{Y_\lambda}u, \\ Y_\lambda|_{t=0} = e_{\lambda_1}\wedge\cdots\wedge e_{\lambda_{N-2}}\wedge\nabla f_0. \end{cases}$$

Extend λ to a permutation $\bar\lambda := (\lambda_1, \cdots, \lambda_{N-1}, \lambda_N)$ of $\{1, \cdots, N\}$, with signature $\tau(\bar\lambda)$. By definition of $\bar\lambda$, we have

$$Y_{\lambda,0} \equiv Y_\lambda|_{t=0} = \tau(\bar\lambda)(\partial_{\lambda_{N-1}}f_0\, e_N - \partial_{\lambda_N}f_0\, e_{N-1}),$$

which is divergence free in $\big(\mathscr{C}^\varepsilon(\mathbb{R}^N)\big)^N$. Because $\widetilde{Y}_{\lambda,0}$ is a (non-degenerate) tangential vector field along $\partial\mathcal{D}_0$, the function $\partial_{Y_{\lambda,0}}\theta_0$ vanishes. In addition, from the fact that

$$\mathrm{Supp}\nabla(\Omega_0)^i_j \cap \mathrm{Supp}\chi_0 = \emptyset \text{ and } \nabla f_0 = \chi_0\nabla f_0,$$

we gather that $\partial_{Y_{\lambda,0}}\Omega_0$ vanishes for any λ. Applying (1.62) with $X_0 = Y_{\lambda,0}$ and Theorem 1.3, we get $Y_\lambda \in \big(L^\infty_{loc}(\mathbb{R}_+;\mathscr{C}^\varepsilon)\big)^N$. Note that in fact $Y_\lambda \in \big(L^\infty_{loc}(\mathbb{R}_+;\widetilde{\mathscr{C}}^\varepsilon)\big)^N$ as all those vector fields are divergence free.

Finally, putting together what we proved hitherto, we deduce that the vector fields W_λ defined by

$$W_\lambda := E_{\lambda_1} \wedge \cdots \wedge E_{\lambda_{N-2}} \wedge Y_\lambda,$$

are in $\big(L^\infty_{loc}(\mathbb{R}_+;\mathscr{C}^\varepsilon)\big)^N$. Furthermore, according to the definition of E_j and Y_λ, they satisfy Equation (1.63). The expression of $Y_{\lambda,0}$ and the fact that $\nabla f_0 = \chi_0\nabla f_0$ imply that

$$W_{\lambda,0} \equiv W_\lambda|_{t=0} = -(\partial_{\lambda_{N-1}} f_0 e_{N-1} + \partial_{\lambda_N} f_0 e_N),$$

and summing up over $\lambda \in \Lambda$ thus yields

$$\sum_{\lambda\in\Lambda} W_{\lambda,0} = -(N-1)\nabla f_0.$$

Therefore

$$W := -\frac{1}{N-1}\sum_{\lambda\in\Lambda} W_\lambda \in \big(L^\infty_{loc}(\mathbb{R}_+;\mathscr{C}^\varepsilon)\big)^N,$$

coincides with ∇f thanks to (1.63) and to the uniqueness of solutions for the equation satisfied by ∇f, namely,

$$\begin{cases} \partial_t\nabla f + u\cdot\nabla(\nabla f) = -(\nabla u)^{tr}\nabla f, \\ \nabla f|_{t=0} = \nabla f_0. \end{cases}$$

This completes the proof of the conservation of $\mathcal{C}^{1,\varepsilon}$ regularity for domain $\mathcal{D}_t$ in dimension $N \geq 3$.

1.6 Commutator Estimates

We here prove some commutator estimates that were needed in the previous sections. They strongly rely on continuity results in Besov spaces for the paraproduct and remainder operators, and on the following classical result (see e.g. [6], Chap. 2).

Lemma 1.11. *Let $A : \mathbb{R}^N \to \mathbb{R}$ be a smooth function, homogeneous of degree m away from a neighborhood of 0. Let $(\varepsilon, s, p, r, p_1, p_2) \in]0,1[\times\mathbb{R}\times[1,\infty]^4$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}\cdot$ Then there exists* *a constant C, depending only on s, ε, N and A such that,*

$$\|[T_g, A(D)]u\|_{B^{s-m+\varepsilon}_{p,r}} \leq C\|\nabla g\|_{B^{\varepsilon-1}_{p_1,\infty}}\|u\|_{B^s_{p_2,r}}.$$

Remark 1.12. *A similar inequality holds for time-dependent distributions, as may be seen by following the proof of Lemma 1.11, treating the time as a parameter, and applying (time) Hölder inequality when appropriate. For example, for any $(\rho, \rho_1, \rho_2) \in [1,\infty]^3$ with $\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$, and $(\varepsilon, s, p, r, p_1, p_2)$ as above, we have*

$$\|[T_g, A(D)]u\|_{\widetilde{L}^\rho_T(B^{s-m+\varepsilon}_{p,r})} \leq C\|\nabla g\|_{\widetilde{L}^{\rho_1}_T(B^{\varepsilon-1}_{p_1,\infty})}\|u\|_{\widetilde{L}^{\rho_2}_T(B^s_{p_2,r})}. \tag{1.64}$$

Lemma 1.13. *Let the vector field X be in $\left(\mathscr{C}^\varepsilon(\mathbb{R}^N)\right)^N$ for some $\varepsilon \in]0,1[$, and f be in $B^s_{p,r}$ with $(s,p,r) \in]-\infty, 1+\frac{N}{p}[\times[1,\infty]^2$. Then we have:*

1. *If in addition $s+\varepsilon > 1$ or $\{s+\varepsilon = 1, r = 1\}$, then*

$$\|\mathcal{T}_X f - \partial_X f\|_{\mathscr{C}^{s+\varepsilon-\frac{N}{p}-1}} \lesssim \|X\|_{\mathscr{C}^\varepsilon}\|\nabla f\|_{B^{s-1}_{p,r}}.$$

The previous estimate holds in the case $s = 1+\frac{N}{p}$, if replacing $\|\nabla f\|_{B^{\frac{N}{p}}_{p,r}}$ by $\|\nabla f\|_{B^{\frac{N}{p}}_{p,\infty}\cap L^\infty}$.

2. *If in addition $\operatorname{div} X \in \mathscr{C}^\varepsilon(\mathbb{R}^N)$ then for $s+\varepsilon > 0$ or $\{s+\varepsilon = 0, r = 1\}$, we have*

$$\|\mathcal{T}_X f - \partial_X f\|_{\mathscr{C}^{s+\varepsilon-\frac{N}{p}-1}} \lesssim \|X\|_{\widetilde{\mathscr{C}}^\varepsilon}\|f\|_{B^s_{p,r}}.$$

Proof. One may start from the Bony decomposition as follows:

$$\begin{aligned}\partial_X f &= \mathcal{T}_X f + T_{\partial_k f}X^k + R(X^k, \partial_k f)\\ &= \mathcal{T}_X f + T_{\partial_k f}X^k + \partial_k R(X^k, f) - R(\operatorname{div} X, f).\end{aligned}$$

Then applying standard continuity results for paraproduct and remainder operators (see e.g. [6], Chap. 2) yields the desired inequalities. □

If the integer N_0 in the definition of Bony's paraproduct and remainder is large enough (for instance $N_0 = 4$ does), then the following fundamental lemma holds.

Lemma 1.14. *Let $(\varepsilon, s, s_k, p, p_k, r, r_k) \in]0,1[\times\mathbb{R}^2\times[1,\infty]^4$ for $k = 1,2$ satisfying*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \text{ and } \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}\cdot$$

1. *If $s_2 < 0$ then we have*

$$\|\mathcal{T}_X T_g f - T_g \mathcal{T}_X f - T_{\mathcal{T}_X g} f\|_{B^{s_1+s_2+\varepsilon-1}_{p,r}} \leq C\|X\|_{\mathscr{C}^\varepsilon}\|f\|_{B^{s_1}_{p_1,r}}\|g\|_{B^{s_2}_{p_2,\infty}}.$$

The above inequality still holds in the case $(s_2, r_2) = (0, \infty)$, *if one replaces* $\|g\|_{B^0_{\infty,\infty}}$ *by* $\|g\|_{L^\infty}$.

2. *If* $s_1 + s_2 + \varepsilon - 1 > 0$ *then we have*

$$\|\mathcal{T}_X R(f,g) - R(\mathcal{T}_X f, g) - R(f, \mathcal{T}_X g)\|_{B^{s_1+s_2+\varepsilon-1}_{p,r}} \leq C\|X\|_{\mathscr{C}^\varepsilon}\|f\|_{B^{s_1}_{p_1,r_1}}\|g\|_{B^{s_2}_{p_2,r_2}}.$$

The above inequality still holds in the limit case $s_1+s_2+\varepsilon-1 = 0$, $r = \infty$ *and* $\frac{1}{r_1} + \frac{1}{r_2} = 1$.

Proof. The key to our proof is a generalized Leibniz formula for the para-vector field operators which was derived by J.-Y. Chemin in [17]. Define the following Fourier multipliers for $k \in 1, \ldots, N$

$$\Delta_{k,j} := \begin{cases} \varphi_k(2^{-j}D) & j \geq 0, \quad \text{with } \varphi_k(\xi) := i\xi_k\varphi(\xi), \\ \chi_k(D) & j = -1, \quad \text{with } \chi_k(\xi) := i\xi_k\chi(\xi), \\ 0 & j \leq -2. \end{cases}$$

Then we have

$$\begin{aligned} \mathcal{T}_X T_g f &= \sum_{j\geq -1}(S_{j-N_0}g\mathcal{T}_X\Delta_j f + \Delta_j f \mathcal{T}_X S_{j-N_0}g) + \sum_{j\geq -1}(T_{1,j} + T_{2,j}) \\ &= T_g\mathcal{T}_X f + T_{\mathcal{T}_X g}f + \sum_{\substack{j\geq -1 \\ \alpha=1,\ldots,4}} T_{\alpha,j}, \end{aligned}$$

where

$$\begin{aligned} T_{1,j} &:= \sum_{\substack{j\leq j'\leq j+1 \\ j-N_0-1\leq j''\leq j'-N_0-1}} 2^{j'}\Delta_{j''}X^k\big(\Delta_{k,j'}(\Delta_j f S_{j-N_0}g) - \Delta_{k,j'}\Delta_j f S_{j-N_0}g\big), \\ T_{2,j} &:= \sum_{\substack{j'\leq j-2 \\ j'-N_0\leq j''\leq j-N_0-2}} 2^{j'}\Delta_{j''}X^k(\Delta_j f)\Delta_{k,j'}S_{j-N_0}g, \\ T_{3,j} &:= S_{j-N_0}g[T_{X^k}, \Delta_j]\partial_k f, \\ T_{4,j} &:= \Delta_j f[T_{X^k}, S_{j-N_0}]\partial_k g. \end{aligned}$$

Bounding $T_{1,j}$ and $T_{2,j}$ is straightforward : just use the definition of Besov norms. Lemma 2.100 of [6] allows to bound $T_{3,j}$ and $T_{4,j}$ provided $\varepsilon < 1$. We end up with the desired inequality.

In order to prove the second item, let us set $A_{j,j'} := [j - N_0 - 1, j' - N_0 - 1] \cup [j' - N_0, j - N_0 - 2] \cap \mathbb{Z}$. We thus have

$$\begin{aligned}\mathcal{T}_X R(f,g) &= \sum_{j\geq -1}(\widetilde{\Delta}_j g \mathcal{T}_X \Delta_j f + \Delta_j f \mathcal{T}_X \widetilde{\Delta}_j g) + \sum_{j\geq -1}(R_{1,j}+R_{2,j})\\ &= R(\mathcal{T}_X f, g) + R(f, \mathcal{T}_X g) + \sum_{\substack{j\geq -1\\ \alpha=1,\dots,4}} R_{\alpha,j},\end{aligned}$$

where, denoting $\widetilde{\Delta}_j := \Delta_{j-N_0} + \cdots + \Delta_{j+N_0}$,

$$\begin{aligned}R_{1,j} &:= \sum_{\substack{|j'-j|\leq N_0+1\\ j''\in A_{j,j'}}} \operatorname{sgn}(j'-j+1)2^{j'}\Delta_{j''}X^k\big(\Delta_{k,j'}(\Delta_j f\widetilde{\Delta}_j g) - \Delta_j f \Delta_{k,j'}\widetilde{\Delta}_j g\big)\\ &\qquad + \sum_{\substack{j-1\leq j'\leq j\\ j'-N_0\leq j''\leq j-N_0}} 2^{j'}\Delta_{j''}X^k(\Delta_{k,j'}\Delta_j f)\widetilde{\Delta}_j g,\\ R_{2,j} &:= \sum_{\substack{j'\leq j-N_0-2\\ j'-N_0\leq j''\leq j-N_0-2}} 2^{j'}\Delta_{j''}X^k\Delta_{k,j'}(\Delta_j f\widetilde{\Delta}_j g),\\ R_{3,j} &:= \widetilde{\Delta}_j g[T_{X^k},\Delta_j]\partial_k f,\\ R_{4,j} &:= \Delta_j f[T_{X^k},\widetilde{\Delta}_j]\partial_k g.\end{aligned}$$

Here again, bounding $R_{1,j}$ and $R_{2,j}$ follows from the definition of Besov norms, while Lemma 2.100 of [6] allows to estimate $R_{3,j}$ and $R_{4,j}$. □

We are in position to prove our key commutator estimate that involves the convective derivative and some para-vector field evolving according to (1.2).

Proposition 1.15. *Suppose that* $(\varepsilon,p)\in(0,1)\times[1,\infty]$ *with* $\frac{N}{p}+\varepsilon\geq 1$. *Consider a couple of vector fields* (X,v) *such that* $\operatorname{div} v = 0$ *and*

$$(X,v)\in\big(L^\infty_{loc}(\mathbb{R}_+;\widetilde{\mathscr{C}^\varepsilon})\big)^N\times\big(L^\infty_{loc}(\mathbb{R}_+;B^{\frac{N}{p}-1}_{p,1})\cap L^1_{loc}(\mathbb{R}_+;B^{\frac{N}{p}+1}_{p,1})\big)^N,$$

satisfying the following equation

$$\begin{cases}(\partial_t + v\cdot\nabla)X = \partial_X v,\\ X|_{t=0} = X_0.\end{cases}\tag{1.65}$$

Then there is a constant C *such that:*

$$\begin{aligned}\|[\mathcal{T}_X,\partial_t+v\cdot\nabla]v\|_{\mathscr{C}^{\varepsilon-2}} \leq C(&\|X\|_{\widetilde{\mathscr{C}^\varepsilon}}\|v\|_{B^{\frac{N}{p}+1}_{p,1}}\|v\|_{B^{\frac{N}{p}-1}_{p,1}}\\ &+\|v\|_{\mathscr{C}^{-1}}\|\mathcal{T}_X v\|_{\mathscr{C}^\varepsilon} + \|v\|_{B^{\frac{N}{p}+1}_{p,1}}\|\mathcal{T}_X v\|_{\mathscr{C}^{\varepsilon-2}}).\end{aligned}\tag{1.66}$$

Proof. A particular case has been considered in [27], as it was needed to study the viscous vortex patch problem. For the reader's convenience, let us here recall the main arguments. Because $\operatorname{div} v = 0$, we may write

$$[\mathcal{T}_X, \partial_t + v^\ell \partial_\ell]v = -v^\ell \partial_\ell T_{X^k} \partial_k v - T_{\partial_t X^k} \partial_k v + T_{X^k} \partial_k (v^\ell \partial_\ell v)$$
$$= -T_{\partial_t X^k} \partial_k v + \partial_\ell \mathcal{T}_X (v^\ell v) - \mathcal{T}_{\partial_\ell X}(v^\ell v) - v^\ell \partial_\ell \mathcal{T}_X v.$$

Hence, decomposing $v^\ell v$ according to Bony's decomposition, we discover that

$$[\mathcal{T}_X, \partial_t + v^\ell \partial_\ell]v = \sum_{\alpha=1}^{\alpha=5} R_\alpha$$

with

$$R_1 := -T_{\partial_t X^k} \partial_k v, \qquad R_2 := \partial_\ell (\mathcal{T}_X T_{v^\ell} v + \mathcal{T}_X T_v v^\ell),$$
$$R_3 := \partial_\ell \mathcal{T}_X R(v^\ell, v), \qquad R_4 := -\mathcal{T}_{\partial_\ell X}(v^\ell v),$$
$$R_5 := -v^\ell \partial_\ell \mathcal{T}_X v.$$

To complete the proof, it suffices to check that all the terms R_α may be bounded by the r.h.s. of (1.66).

• **Bound of R_1:**

From the equation (1.65), we have

$$R_1 = T_{v \cdot \nabla X^k} \partial_k v - T_{\partial_X v^k} \partial_k v.$$

Hence using standard continuity results for the paraproduct, we deduce that

$$\|R_1\|_{\mathscr{C}^{\varepsilon-2}} \lesssim \|\nabla v\|_{\mathscr{C}^0} \big(\|v \cdot \nabla X\|_{\mathscr{C}^{\varepsilon-2}} + \|\partial_X v\|_{\mathscr{C}^{\varepsilon-2}} \big).$$

The last term may be bounded according to Lemma 1.13. As for the first term, we use $\operatorname{div} v = 0$ and the following decomposition

$$v \cdot \nabla X = \mathcal{T}_v X + T_{\partial_\ell X} v^\ell + \partial_\ell R(v^\ell, X),$$

which allows to get, since $\varepsilon + \frac{N}{p} \geq 1$,

$$\|R_1\|_{\mathscr{C}^{\varepsilon-2}} \lesssim \|\nabla v\|_{L^\infty} (\|v\|_{B^{\frac{N}{p}-1}_{p,1}} \|X\|_{\widetilde{\mathscr{C}}^\varepsilon} + \|\mathcal{T}_X v\|_{\mathscr{C}^{\varepsilon-2}}).$$

• **Bound of R_2:**

Due to Lemma 1.14 (i) and continuity of paraproduct operator, we have

$$\|R_2\|_{\mathscr{C}^{\varepsilon-2}} \lesssim \|X\|_{\mathscr{C}^\varepsilon}\|v\|_{\mathscr{C}^1}\|v\|_{\mathscr{C}^{-1}} + \|v\|_{\mathscr{C}^{-1}}\|\mathcal{T}_X v\|_{\mathscr{C}^\varepsilon} + \|v\|_{\mathscr{C}^1}\|\mathcal{T}_X v\|_{\mathscr{C}^{\varepsilon-2}}.$$

• **Bound of R_3:**

Applying Lemma 1.14 (ii) and continuity of remainder operator under the assumption $\frac{N}{p}+\varepsilon \geq 1$ yields

$$\|R_3\|_{\mathscr{C}^{\varepsilon-2}} \lesssim \|X\|_{\mathscr{C}^\varepsilon}\|v\|_{B^{\frac{N}{p}+1}_{p,1}}\|v\|_{\mathscr{C}^{-1}} + \|v\|_{B^{\frac{N}{p}+1}_{p,1}}\|\mathcal{T}_X v\|_{\mathscr{C}^{\varepsilon-2}}.$$

• **Bound of R_4:**

From Bony decomposition, it is easy to get

$$\|v^l v\|_{B^{\frac{N}{p}}_{p,1}} \lesssim \|v\|_{\mathscr{C}^{-1}}\|v\|_{B^{\frac{N}{p}+1}_{p,1}}.$$

Hence

$$\|R_4\|_{B^{\frac{N}{p}+\varepsilon-2}_{p,1}} \lesssim \|\nabla X\|_{\mathscr{C}^{\varepsilon-1}}\|v\|_{\mathscr{C}^{-1}}\|v\|_{B^{\frac{N}{p}+1}_{p,1}}.$$

• **Bound of R_5:**

Applying Bony decomposition and using that $\operatorname{div} v = 0$ and $\frac{N}{p}+\varepsilon \geq 1$ give

$$\|R_5\|_{\mathscr{C}^{\varepsilon-2}} \lesssim \|v\|_{\mathscr{C}^{-1}}\|\mathcal{T}_X v\|_{\mathscr{C}^\varepsilon} + \|v\|_{B^{\frac{N}{p}+1}_{p,1}}\|\mathcal{T}_X v\|_{\mathscr{C}^{\varepsilon-2}}.$$

Combining the above estimates for all R_α, with $\alpha = 1, \ldots, 5$ yields (1.66). □

Remark 1.16. *Slight modifications of the above proof allow to handle time dependent functions with tilde type Besov norms. In particular, one may prove the following estimate for all $t > 0$:*

$$\|[\mathcal{T}_X, \partial_t + v\cdot\nabla]v\|_{\widetilde{L}^1_t(\mathscr{C}^{\varepsilon-2})} \lesssim \|v\|_{L^\infty_t(\mathscr{C}^{-1})}\|\mathcal{T}_X v\|_{\widetilde{L}^1_t(\mathscr{C}^\varepsilon)}$$
$$+ \int_0^t \|v\|_{B^{\frac{N}{p}+1}_{p,1}}\|\mathcal{T}_X v\|_{\mathscr{C}^{\varepsilon-2}}\,dt' + \int_0^t \|X\|_{\widetilde{\mathscr{C}}^\varepsilon}\|v\|_{B^{\frac{N}{p}+1}_{p,1}}\|v\|_{B^{\frac{N}{p}-1}_{p,1}}\,dt'.$$

Chapter 2

Density patch problem for inhomogeneous Navier-Stokes equations

2.1 Introduction

We are concerned with the following *inhomogeneous incompressible Navier-Stokes equations* in the whole space $\mathbb{R}^N$ with $N \geq 2$:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho(\partial_t u + u \cdot \nabla u) - \mu \Delta u + \nabla P = 0, \\ \operatorname{div} u = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0). \end{cases} \tag{INS}$$

Above, the unknowns $(\rho, u, P) \in \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}$ stand for the density, velocity vector field and pressure, respectively, and the so-called viscosity coefficient μ is a positive constant.

A number of recent works have been dedicated to the mathematical analysis of System (INS). In particular, it is well-known that if ρ_0 is positive and bounded, and $\sqrt{\rho_0}u_0$ is in $L^2(\mathbb{R}^N)$, then System (INS) admits a global weak solution with finite energy (see [4] and the references therein). That result has been extended by J. Simon in [109] to the case $\rho_0 \geq 0$, and by P.-L. Lions in [89] to viscosity coefficients depending on the density.

In [89], P.-L. Lions raised the so-called *density patch problem*. It may be stated as follows: assume that $\rho_0 = \mathbb{1}_{\mathcal{D}_0}$ for some domain $\mathcal{D}_0$. Can we find conditions on u_0 that ensure that

$$\rho(t) = \mathbb{1}_{\mathcal{D}_t} \quad \text{for all} \ \ t \geq 0 \tag{2.1}$$

for some domain $\mathcal{D}_t$ *with the same regularity as the initial one* ?

Whenever $\sqrt{\rho_0}u_0$ is in $L^2(\mathbb{R}^N)$, the renormalized solutions theory in [46] by R. Di Perna and P.-L. Lions for transport equations ensures that the global weak solutions mentioned above have a volume preserving generalized flow ψ and that we do have (2.1) with $\mathcal{D}_t$ being the image of $\mathcal{D}_0$ by $\psi(t,\cdot)$. However, without assuming more on u_0, it is very unlikely that one can get information on the persistence of regularity of $\mathcal{D}_t$ for positive times.

The present chapter aims at making one more step toward solving P.-L.Lions' question, by considering the case where

$$\rho_0 = \eta_1 \mathbb{1}_{\mathcal{D}_0} + \eta_2 \mathbb{1}_{\mathcal{D}_0^c}, \tag{2.2}$$

for some simply connected bounded domain $\mathcal{D}_0$ of class $\mathcal{C}^{1,\varepsilon}$.

Our goal is to find as general as possible conditions on u_0, that guarantee that for all time $t \geq 0$, the domain $\mathcal{D}_t := \psi(t, \mathcal{D}_0)$ remains $\mathcal{C}^{1,\varepsilon}$, and the density reads

$$\rho(t,\cdot) = \eta_1 \mathbb{1}_{\mathcal{D}_t} + \eta_2 \mathbb{1}_{\mathcal{D}_t^c}. \tag{2.3}$$

Several recent works give a partial answer to that issue if $|\eta_1 - \eta_2|$ is small enough. Indeed, the paper [36] by R.Danchin and P.B. Mucha ensures that if ρ_0 is given by (2.2) and u_0 belongs to the critical Besov space $\dot{B}^{\frac{N}{p}-1}_{p,1}(\mathbb{R}^N)$ (see the definition below in (2.11)), then (2.3) is fulfilled for small time (and for all time if u_0 is small) and the $\mathcal{C}^1$ regularity is preserved. Likewise, according to the work [73] by J. Huang, M. Paicu and P. Zhang (see also [39]), one may solve (INS) if the initial density is close (for the L^∞ norm) to some positive constant and u_0 belongs to $\dot{B}^{\frac{N}{p}-1}_{p,r}(\mathbb{R}^N) \cap \dot{B}^{\frac{N}{p}+\delta-1}_{p,r}(\mathbb{R}^N)$ for some $1 < r < \infty$. As the flow of the corresponding solution is $\mathcal{C}^{1,\delta'}$ for all $\delta' < \delta$, one can deduce that if ρ_0 is given by (2.2) then the $\mathcal{C}^{1,\varepsilon}$ regularity of the boundary is preserved *provided that* $\varepsilon < \delta$ (as the flow need not be in $\mathcal{C}^{1,\delta}$).

Finally, as noticed in [33] then improved by M. Paicu, P. Zhang and Z. Zhang [96], in the 2D case, if working *within the energy framework*, then one may avoid the smallness condition on the density and solve (INS) globally if ρ_0 and u_0 just satisfy

$$0 < \eta_1 \leq \rho_0 \leq \eta_2, \quad u_0 \in H^s(\mathbb{R}^2) \quad \text{for some } \; s > 0. \tag{2.4}$$

As the constructed velocity field therein admits a $\mathcal{C}^1$ flow, one can readily deduce that, if ρ_0 is given by (2.2) with $\mathcal{D}_0 \subset \mathbb{R}^2$ then the $\mathcal{C}^1$ regularity of the boundary is preserved.

The common point between the above works is that the hypotheses on u_0 do not take into account the non-isotropic structure of ρ_0. Consequently, the maximal regularity that can be propagated for the patch is limited by the overall regularity of the initial velocity. In two recent papers [88],[87] devoted to the 2D case (see also [86] for the 3-D case), X. Liao and P. Zhang pointed out that *only tangential regularity along the boundary of* $\mathcal{D}_0$ was needed to propagate high Sobolev regularity of the patch. They followed J.-Y. Chemin's approach in his work [19] dedicated to the vortex patches problem for the 2-D incompressible Euler equations, and characterized the regularity of the boundary of the domain by means of one (or several) tangent vector fields that evolve according to the flow of the velocity field.

More precisely, assume with no loss of generality that $\partial\mathcal{D}_0$ coincides with the level set $f_0^{-1}(\{0\})$ of some (at least $\mathcal{C}^1$) function $f_0 : \mathbb{R}^N \to \mathbb{R}$ that does not degenerate in a neighborhood of $\partial\mathcal{D}_0$, namely there exists some open neighborhood V_0 of $\mathcal{D}_0$ such that

$$\mathcal{D}_0 = f_0^{-1}(\{0\}) \cap V_0 \quad \text{and} \quad \nabla f_0 \ \text{ does not vanish on } \ V_0. \tag{2.5}$$

Then $\mathcal{D}_t$ coincides with $f_t^{-1}(\{0\})$ where $f_t \equiv f(t,\cdot) := f_0 \circ \psi_t^{-1}$ with $\psi_t := \psi(t,\cdot)$ and ψ being the solution of the (integrated) ordinary differential equation:

$$\psi(t,x) = x + \int_0^t u\big(\tau, \psi(\tau,x)\big)\, d\tau. \tag{2.6}$$

Now, the tangent vector field $X_t := \nabla^\perp f_t$ coincides with the evolution of $X_0 := \nabla^\perp f_0$ along the flow of u, namely[1] :

$$X(t,\cdot) := (\partial_{X_0}\psi_t) \circ \psi_t^{-1}, \tag{2.7}$$

and thus satisfies the transport equation

$$\begin{cases} \partial_t X + u \cdot \nabla X = \partial_X u, \\ X|_{t=0} = X_0. \end{cases} \tag{2.8}$$

Consequently, the problem of persistence of regularity for the patch reduces to that of the vector field X solution to (2.8).

In their first paper [88], X. Liao and P. Zhang justified that heuristics in the case where the jump $|\eta_1 - \eta_2|$ is small enough, and $u_0 \in \big(W^{1,p}(\mathbb{R}^2)\big)^2$ for $2 < p < 4$. Their proof was essentially based on weighted $L_p - L_q$ estimates for the velocity and allowed to propagate Sobolev regularity $W^{k,p}$ of the boundary, with k large enough (in particular the boundary is at least $\mathcal{C}^{2,\varepsilon}$ for some $\varepsilon > 0$). In a second paper [87], after revisiting the approach of [96] (that is Sobolev spaces H^s with $s > 0$ and thus finite energy framework), X. Liao and P. Zhang succeeded in proving a similar result for *general* positive η_1 and η_2 in (2.2). The corresponding level set function f_0 has to be in $W^{2+k,p}(\mathbb{R}^2)$ for some integer number $k \geq 1$ and $p \in]2,4[$, hence $\mathcal{D}_0$ is still at least $\mathcal{C}^{2,\varepsilon}$. As regards the initial velocity field u_0, it has to satisfy the following *striated regularity* property along the vector field $X_0 := \nabla^\perp f_0$:

$$\big(\partial_{X_0}^\ell u_0 \in B_{2,1}^{s+\varepsilon\frac{k-\ell}{k}}(\mathbb{R}^2)\big)^2 \text{ for all } \ \ell \in \{0, \cdots, k\}$$

[1]For any vector field $Y = Y^k(x)\partial_k$ and function f in $\mathcal{C}^1(\mathbb{R}^N;\mathbb{R})$, we denote by $\partial_Y f$ the *directional derivative* of f along Y, that is, with the Einstein summation convention, $\partial_Y f := Y^k\partial_k f = Y \cdot \nabla f$.

with $0 < s < 1-\varepsilon$ and (s,p) in $]0,1[\times]2,\min\{4, 2/(1-s)\}[$.

In this chapter, we propose a simpler approach that allows to propagate just $\mathcal{C}^{1,\varepsilon}$ Hölder regularity (for all $\varepsilon\in]0,1[$), within a *critical* regularity framework. By critical, we mean that the solution space that we shall consider has the same scaling invariance by time and space dilations as (INS) itself, namely:

$$(\rho, u, P)(t,x) \to (\rho, \lambda u, \lambda^2 P)(\lambda^2 t, \lambda x) \quad \text{and} \quad (\rho_0, u_0)(x) \to (\rho_0, \lambda u_0)(\lambda x). \tag{2.9}$$

That framework is by now classical for the homogeneous Navier-Stokes equations (that is ρ is a positive constant in (INS)) in the whole space $\mathbb{R}^N$ (see e.g. [6, 84] and the references therein). As observed by R.Danchin in [30] (see also H. Abidi in [1] and H. Abidi and M. Paicu in [3]), working in a suitable critical functional framework is still relevant in the inhomogeneous situation.

2.2 Results

Before stating our main result, we need to introduce a few notations. First, we recall the definition of Besov spaces : following [6, Section 2.2], we consider two smooth radial functions χ and φ supported in $\{\xi\in\mathbb{R}^N : |\xi|\leq 4/3\}$ and $\{\xi\in\mathbb{R}^N : 3/4\leq|\xi|\leq 8/3\}$, respectively, and satisfying

$$\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi) = 1, \ \ \forall\ \xi\in\mathbb{R}^N\backslash\{0\}, \quad \chi(\xi)+\sum_{j\geq 0}\varphi(2^{-j}\xi) = 1, \ \ \forall\xi\in\mathbb{R}^N. \tag{2.10}$$

Then we define Fourier truncation operators as follows:

$$\dot\Delta_j := \varphi(2^{-j}D), \ \ \dot S_j := \chi(2^{-j}D), \ \ \forall j\in\mathbb{Z}; \qquad \Delta_j := \varphi(2^{-j}D), \ \ \forall j\geq 0, \quad \Delta_{-1} := \chi(D).$$

For all triplet $(s,p,r)\in\mathbb{R}\times[1,\infty]^2$, the homogeneous Besov space $\dot B^s_{p,r}(\mathbb{R}^N)$ (just denoted by $\dot B^s_{p,r}$ if the value of the dimension is clear from the context) is defined by

$$\dot B^s_{p,r}(\mathbb{R}^N) := \big\{\, u\in\mathcal{S}'_h(\mathbb{R}^N) : \|u\|_{\dot B^s_{p,r}} := \big\|2^{js}\|\dot\Delta_j u\|_{L^p}\big\|_{\ell^r(\mathbb{Z})} < \infty\,\big\}, \tag{2.11}$$

where $\mathcal{S}'_h(\mathbb{R}^N)$ is the subspace of tempered distributions $\mathcal{S}'(\mathbb{R}^N)$ defined by

$$\mathcal{S}'_h(\mathbb{R}^N) := \big\{\, u\in\mathcal{S}'(\mathbb{R}^N) : \lim_{j\to-\infty}\dot S_j u = 0\,\big\}.$$

We shall also use sometimes the following inhomogeneous Besov spaces:

$$B^s_{p,r}(\mathbb{R}^N) := \big\{\, u\in\mathcal{S}'(\mathbb{R}^N) : \|u\|_{B^s_{p,r}} := \big\|2^{js}\|\Delta_j u\|_{L^p}\big\|_{\ell^r(\mathbb{N}\cup\{-1\})} < \infty\,\big\}. \tag{2.12}$$

Throughout, we adopt the common notation $b^s_{p,r}(\mathbb{R}^N)$ to designate $B^s_{p,r}(\mathbb{R}^N)$ or $\dot B^s_{p,r}(\mathbb{R}^N)$.

It is well-known that Sobolev or Hölder spaces belong to the Besov spaces hierarchy. For instance $\dot B^s_{2,2}(\mathbb{R}^N)$ coincides with the homogeneous Sobolev space $\dot H^s(\mathbb{R}^N)$, and we have

$$B^s_{\infty,\infty}(\mathbb{R}^N)=\mathcal{C}^{0,s}(\mathbb{R}^N)=L^\infty(\mathbb{R}^N)\cap\dot B^s_{\infty,\infty}(\mathbb{R}^N)\quad\text{if}\quad s\in]0,1[. \tag{2.13}$$

To emphasize that connection, we shall often use the notation $\dot{\mathscr{C}}^s:=\dot B^s_{\infty,\infty}$ (or $\mathscr{C}^s:=B^s_{\infty,\infty}$) for *any* $s\in\mathbb{R}$.

When investigating evolutionary equations in critical Besov spaces, it is wise to use the following *tilde homogeneous Besov spaces* first introduced by J.-Y. Chemin in [21]: for any $T\in]0,+\infty]$ and $(s,p,r,\gamma)\in\mathbb{R}\times[1,+\infty]^3$, we set[2]

$$\widetilde L^\gamma_T(\dot B^s_{p,r}):=\Big\{u\in\mathcal{S}'(]0,T[\times\mathbb{R}^N):\lim_{j\to-\infty}\dot S_ju=0\text{ in }L^\gamma_T(L^\infty)\text{ and }\|u\|_{\widetilde L^\gamma_T(\dot B^s_{p,r})}<\infty\Big\},$$

where

$$\|u\|_{\widetilde L^\gamma_T(\dot B^s_{p,r})}:=\big\|2^{js}\|\dot\Delta_ju\|_{L^\gamma_T(L^p)}\big\|_{\ell^r(\mathbb{Z})}<\infty.$$

The index T will be omitted if equal to $+\infty$, and we shall denote

$$\widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot B^s_{p,r}):=\widetilde L^\infty(\mathbb{R}_+;\dot B^s_{p,r})\cap\mathcal{C}(\mathbb{R}_+;\dot B^s_{p,r}).$$

We also need to introduce the following spaces for $(\sigma,p,T)\in\mathbb{R}\times[1,\infty]\times]0,\infty]$:

$$\dot E^\sigma_p(T):=\big\{(v,\nabla Q):v\in\widetilde{\mathcal{C}}_b\big([0,T[;\dot B^{\frac Np-1+\sigma}_{p,1}\big),\ (\partial_tv,\nabla^2v,\nabla Q)\in L^1_T\big(\dot B^{\frac Np-1+\sigma}_{p,1}\big)\big\},$$

endowed with the norm

$$\|(v,\nabla Q)\|_{\dot E^\sigma_p(T)}:=\|v\|_{\widetilde L^\infty_T\big(\dot B^{\frac Np+\sigma-1}_{p,1}\big)}+\|(\partial_tv,\nabla^2v,\nabla Q)\|_{L^1_T\big(\dot B^{\frac Np+\sigma-1}_{p,1}\big)}.$$

For notational simplicity, we shall omit σ or T in the notation $\dot E^\sigma_p(T)$ whenever σ is zero or $T=\infty$. For instance, $\dot E_p:=\dot E^0_p(\infty)$.

Finally, we shall make use of *multiplier spaces* associated to couples (E,F) of Banach spaces included in the set of tempered distributions. The definition goes as follows:

Definition. *Let E and F be two Banach spaces embedded in $\mathcal{S}'(\mathbb{R}^N)$. The* multiplier space *$\mathcal{M}(E\to F)$ (simply denoted by $\mathcal{M}(E)$ if $E=F$) is the set of those functions φ satisfying $\varphi u\in F$ for all u in E and, additionally,*

[2]For $T\in]0,+\infty[$, $p\in[1,+\infty]$ and E a Banach space, the notation $L^p_T(E)$ designates the space of L^p functions on $]0,T[$ with values in E, and $L^p(\mathbb{R}_+;E)$ corresponds to the case $T=+\infty$. We keep the same notation for vector or matrix-valued functions.

$$\|\varphi\|_{\mathcal{M}(E\to F)} := \sup_{\substack{u\in E\\ \|u\|_E\leq 1}} \|\varphi u\|_F < \infty. \tag{2.14}$$

It goes without saying that $\|\cdot\|_{\mathcal{M}(E\to F)}$ is a norm on $\mathcal{M}(E\to F)$ and that one may restrict the supremum in (2.14) to any *dense* subset of E.

The following result that has been proved in [36] is the starting point of our analysis[3]:

Theorem 2.1. *Let $p\in[1,2N[$ and u_0 be a divergence-free vector field with coefficients in $\dot B^{\frac{N}{p}-1}_{p,1}$. Assume that ρ_0 belongs to the multiplier space $\mathcal{M}(\dot B^{\frac{N}{p}-1}_{p,1})$. There exist two constants c and C depending only on p and on N such that if*

$$\|\rho_0-1\|_{\mathcal{M}\left(\dot B^{\frac{N}{p}-1}_{p,1}\right)} + \|u_0\|_{\dot B^{\frac{N}{p}-1}_{p,1}} \leq c$$

then System (INS) in $\mathbb{R}^N$ with $N\geq 2$ has a unique solution $(\rho,u,\nabla P)$ satisfying

$$\rho\in L^\infty\Big(\mathbb{R}_+;\mathcal{M}\big(\dot B^{\frac{N}{p}-1}_{p,1}\big)\Big) \quad \textit{and} \quad (u,\nabla P)\in \dot E_p.$$

Furthermore, the following inequality is fulfilled:

$$\|u\|_{\widetilde L^\infty\left(\mathbb{R}+;\dot B^{\frac{N}{p}-1}_{p,1}\right)} + \|\partial_t u,\nabla^2 u,\nabla P\|_{L^1\left(\mathbb{R}+;\dot B^{\frac{N}{p}-1}_{p,1}\right)} \leq C\|u_0\|_{\dot B^{\frac{N}{p}-1}_{p,1}}. \tag{2.15}$$

By classical embedding, having $\nabla^2 u$ in $L^1(\mathbb{R}_+;\dot B^{\frac{N}{p}-1}_{p,1})$ implies that ∇u is in $L^1(\mathbb{R}_+;\mathcal{C}_b)$. Therefore the flow ψ of u defined by (2.6) is in $\mathcal{C}^1$. Now, it has been observed in [36] that for any uniformly $\mathcal{C}^1$ bounded domain $\mathcal{D}_0$, the function $\mathbb{1}_{\mathcal{D}_0}$ belongs to $\mathcal{M}(\dot B^s_{p,1})$ whenever $-1+\frac{1}{p}<s<\frac{1}{p}$. Therefore, one may deduce from Theorem 2.1 that if ρ_0 is given by (2.2), if u_0 is in $\dot B^{\frac{N}{p}-1}_{p,1}$ for some $N-1<p<2N$ and if

$$\|u_0\|_{\dot B^{\frac{N}{p}-1}_{p,1}} + |\eta_2-\eta_1| \quad \text{is small enough}$$

then System (INS) admits a unique global solution $(\rho,u,\nabla P)$ with $(u,\nabla P)$ in $\dot E_p$ and ρ given by (2.3) with $\mathcal{D}_t$ in $\mathcal{C}^1$ for all time $t\geq 0$.

The (parabolic type) gain of regularity for u pointed out in Theorem 2.1 is optimal, as well as the embedding of $\dot B^{\frac{N}{p}}_{p,1}$ in the set of continuous bounded functions. Therefore, one cannot expect the flow of u given by Theorem 2.1 to be in any Hölder space $\mathcal{C}^{1,\alpha}$ for some $\alpha>0$, which prevents our propagating more than $\mathcal{C}^1$ regularity. Following [87, 88], it is natural to make an additional

[3] As the viscosity coefficient μ will be fixed once and for all, we shall set it to 1 for notational simplicity. Likewise, we shall assume the reference density at infinity to be 1.

tangential regularity assumption on u_0. This motivates the following general result of persistence of geometric structures for (INS).

Theorem 2.2. *Let ε be in $]0,1[$ and p satisfy*

$$\frac{N}{2} < p < \min\Big\{\frac{N}{1-\varepsilon}, 2N\Big\}. \tag{2.16}$$

Let u_0 be a divergence-free vector field with coefficients in $\dot{B}^{\frac{N}{p}-1}_{p,1}$. Assume that the initial density ρ_0 is bounded and belongs to the multiplier space $\mathcal{M}\big(\dot{B}^{\frac{N}{p}-1}_{p,1}\big) \cap \mathcal{M}\big(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\big)$. There exists a constant c depending only on p and N such that if

$$\|\rho_0 - 1\|_{\mathcal{M}\big(\dot{B}^{\frac{N}{p}-1}_{p,1}\big)\cap\mathcal{M}\big(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\big)\cap L^\infty} + \|u_0\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}} \leq c, \tag{2.17}$$

then System (INS) in $\mathbb{R}^N$ has a unique global solution $(\rho, u, \nabla P)$ with

$$\rho \in L^\infty\Big(\mathbb{R}_+; L^\infty \cap \mathcal{M}\big(\dot{B}^{\frac{N}{p}-1}_{p,1}\big) \cap \mathcal{M}\big(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\big)\Big) \quad \textit{and} \quad (u, \nabla P) \in \dot{E}_p.$$

Moreover, for any vector field X_0 with $\mathcal{C}^{0,\varepsilon}$ regularity (assuming in addition that $\varepsilon > 2 - \frac{N}{p}$ if $\operatorname{div} X_0 \not\equiv 0$), if the following conditions are fulfilled

$$\partial_{X_0}\rho_0 \in \mathcal{M}\big(\dot{B}^{\frac{N}{p}-1}_{p,1} \to \dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\big) \quad \textit{and} \quad \partial_{X_0} u_0 \in \dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1},$$

then System (2.8) in $\mathbb{R}^N$ has a unique global solution $X \in C_w(\mathbb{R}_+; \mathcal{C}^{0,\varepsilon})$, and we have

$$\partial_X \rho \in L^\infty\Big(\mathbb{R}_+; \mathcal{M}\big(\dot{B}^{\frac{N}{p}-1}_{p,1} \to \dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\big)\Big) \quad \textit{and} \quad (\partial_X u, \partial_X \nabla P) \in \dot{E}^{\varepsilon-1}_p.$$

Some comments are in order:

- The divergence-free property on X_0 is conserved during the evolution because if one takes the divergence of (2.8), then we get, remembering that $\operatorname{div} u = 0$,

$$\begin{cases} \partial_t \operatorname{div} X + u \cdot \nabla \operatorname{div} X = 0, \\ \operatorname{div} X|_{t=0} = \operatorname{div} X_0. \end{cases}$$

- In the case $\operatorname{div} X_0 \neq 0$, the additional constraint on (ε, p) is due to the fact that the product of a general $\mathcal{C}^{0,\varepsilon}$ function with a $\dot{B}^{\frac{N}{p}-2}_{p,1}$ distribution need not be defined if the sum of regularity coefficients, namely $\varepsilon + \frac{N}{p} - 2$, is negative.

- The vector field X given by (2.8) has the Finite Propagation Speed Property. Indeed, from the definitions of the flow and of the space $\dot{E}_p$, and from the embedding of $\dot{B}^{\frac{N}{p}}_{p,1}(\mathbb{R}^N)$ in

$\mathcal{C}_b(\mathbb{R}^N)$, we readily get[4] for all $t \geq 0$ and $x \in \mathbb{R}^N$,

$$\big|\psi(t,x) - x\big| \lesssim \sqrt{t}\,\|u\|_{L^2_t(\dot B^{\frac{N}{p}}_{p,1})} \leq C\sqrt{t}\,\|u_0\|_{\dot B^{\frac{N}{p}-1}_{p,1}}.$$

Therefore, if the initial vector field X_0 is supported in the set K_0 then $X(t)$ is supported in some set K_t such that

$$\mathrm{diam}(K_t) \leq \mathrm{diam}(K_0) + C\sqrt{t}\,\|u_0\|_{\dot B^{\frac{N}{p}-1}_{p,1}}.$$

- One can prove a similar result (only local in time) for *large* u_0 in $\dot B^{\frac{N}{p}-1}_{p,1}$. Moreover, we expect our method to be appropriate for handling Hölder regularity $\mathcal{C}^{k,\varepsilon}$ if making suitable assumptions on $\partial^j_{X_0}\rho_0$ and $\partial^j_{X_0}u_0$ for $j = 0, \cdots, k$. We refrained from writing out here this generalization to keep the presentation as short as possible.

In the density patch situation (that is if ρ_0 is given by (2.2)) the condition on $\partial_{X_0}\rho_0$ is trivially satisfied as the derivative of the density along any continuous vector field that is tangent to $\partial\mathcal{D}_0$, vanishes. This implies the following statement of propagation of Hölder regularity of density patches for (INS) in the plane:

Theorem 2.3. *Let $\mathcal{D}_0$ be a simply connected bounded domain of $\mathbb{R}^2$ satisfying (2.5) for some function $f_0 \in \mathcal{C}^{1,\varepsilon}(\mathbb{R}^2;\mathbb{R})$ with ε in $]0,1[$. There exists a constant η_0 depending only on $\mathcal{D}_0$ and such that if*

$$\rho_0 := (1+\eta)\mathbb{1}_{\mathcal{D}_0} + \mathbb{1}_{\mathcal{D}_0^c} \quad \textit{with} \quad \eta \in]-\eta_0,\eta_0[\tag{2.18}$$

and if the divergence free vector-field $u_0 \in \mathcal{S}'_h(\mathbb{R}^2)$ has vorticity $\omega_0 := \partial_1 u_0^2 - \partial_2 u_0^1$ given by

$$\omega_0 = \overline{\omega}_0 + \widetilde{\omega}_0\,\mathbb{1}_{\mathcal{D}_0} \;\; \textit{with} \;\; \mathrm{div}\,(\overline{\omega}_0\nabla^\perp f_0) = 0 \quad \textit{and} \quad \int_{\mathbb{R}^2}\omega_0\,dx = 0 \tag{2.19}$$

for some small enough compactly supported functions $(\overline{\omega}_0,\widetilde{\omega}_0)$ in $L^p(\mathbb{R}^2)\times\mathcal{C}^{\varepsilon'}(\mathbb{R}^2)$ with $0<\varepsilon'<\varepsilon$ and $1<p<2/(2-\varepsilon)$, then System (INS) has a unique solution $(\rho,u,\nabla P)$ with the properties listed in Theorem 2.1. Moreover, if we denote by $\psi(t,\cdot)$ the flow of u then for all $t \geq 0$, we have

$$\rho(t,\cdot) := (1+\eta)\mathbb{1}_{\mathcal{D}_t} + \mathbb{1}_{\mathcal{D}_t^c} \;\; \textit{with} \;\; \mathcal{D}_t := \psi(t,\mathcal{D}_0), \tag{2.20}$$

and $\mathcal{D}_t$ remains a simply connected bounded domain of class $\mathcal{C}^{1,\varepsilon}$.

Remark 2.4. *Of course, one can take $\overline{\omega}_0 \equiv 0$ or $\widetilde{\omega}_0 \equiv 0$ in the above statement.*

The zero average condition guarantees that u_0 belongs to some homogeneous Besov space $\dot B^{\frac{2}{p}-1}_{p,1}$ (so as to apply Theorem 2.2). It is not needed if we have constant vortex pattern in $\mathbb{R}^2$ (see Theorem 2.9)

[4]In this chapter as before, we agree that $A \lesssim B$ means $A \leq CB$ for some harmless "constant" C, the meaning of which may be guessed from the context.

or if the dimension $N = 3$ (see Theorem 2.10).

Remark 2.5. *We imposed the particular structure of the vorticity in Theorem 2.3 just to give an explicit example for which (2.3) with regularity $\mathcal{C}^{1,\varepsilon}$ holds true. It goes without saying that one can consider a much more general class of initial velocities : according to Theorem 2.2, it suffices that u_0 satisfies the smallness condition of Theorem 2.1 and that $\operatorname{div}(\nabla^\perp f_0 \otimes u_0)$ is in $\dot{B}^{\frac{2}{p}+\varepsilon-2}_{p,1}$ for some $1 < p < \min\{4, 2/(1-\varepsilon)\}$. In other words, we just need "u_0 to have ε more regularity in the direction that is tangential to the patch of density." This is of course satisfied if u_0 vanishes in a neighborhood of $\mathcal{D}_0$. However, one may consider much more singular examples like the case where u_0 is compactly supported and behaves locally near some $x_0 \notin \partial\mathcal{D}_0$, like the function $|x - x_0|^{-1}(-\log|x-x_0|)^{-(1+\delta)}$ with $\delta > 0$.*

Remark 2.6. *Similar results, only local in time, hold true for large u_0 with critical regularity.*

We end this section with a short presentation of the main ideas of the proof of Theorem 2.2. From Theorem 2.1, we have a global solution $(\rho, u, \nabla P)$ such that $\rho \in L^\infty\Big(\mathbb{R}_+; \mathcal{M}\big(\dot{B}^{\frac{N}{p}-1}_{p,1}\big)\Big)$ and $(u, \nabla P) \in \dot{E}_p$. As already explained, our main task is to prove that $X(t,\cdot)$ remains in $\mathcal{C}^{0,\varepsilon}$ for all time. Now, in light of (2.8), we have

$$X(t,x) = X_0\big(\psi_t^{-1}(x)\big) + \int_0^t \partial_X u\Big(t', \psi_{t'}\big(\psi_t^{-1}(x)\big)\Big) dt'.$$

Therefore, because ψ_t is a $\mathcal{C}^1$ diffeomorphism, it suffices to show that $\partial_X u \in L^1_{loc}(\mathbb{R}_+; \mathcal{C}^{0,\varepsilon})$. To this end, it is natural to look for a suitable evolution equation for $\partial_X u$. Since (2.8) means that $[D_t, \partial_X] = 0$, where $D_t := \partial_t + u \cdot \nabla$ stands for the material derivative associated to u, differentiating the momentum equation of (INS) along X yields

$$\rho D_t \partial_X u + \partial_X \rho D_t u - \partial_X \Delta u + \partial_X \nabla P = 0. \tag{2.21}$$

Even though (2.21) has some similarities with the Stokes system, it is not clear that it does have the same smoothing properties, as its coefficients have very low regularity. One of the difficulties lies in the product of the discontinuous function ρ with $D_t \partial_X u$, as having only $\partial_X u$ in $\mathcal{C}^{0,\varepsilon}$ suggests that $D_t \partial_X u$ has *negative* regularity. At the same time, the term with $\partial_X \rho$ is harmless as, owing to $[D_t, \partial_X] = 0$ and to the mass equation, we have

$$D_t \partial_X \rho = 0. \tag{2.22}$$

Hence any (reasonable) regularity assumption for $\partial_{X_0} \rho_0$ persists through the evolution.

Our strategy is to assume that ρ belongs to the multiplier space corresponding to the space to which $D_t \partial_X u$ is expected to belong. As the flow is $\mathcal{C}^1$, propagating multiplier informations is straightforward (see Lemma 2.14). This new viewpoint spares us the tricky energy estimates and iterated differentiation along vector fields (requiring higher regularity of the patch) that were the cornerstone of the work by X. Liao and P. Zhang. In fact, *under the smallness assumption* (2.17) which, unfortunately, forces the fluid to have small density variations, we succeed in closing the estimates using *only one* differentiation along X. This makes the proof rather elementary and allows us to propagate low Hölder regularity.

Whether one can differentiate terms like Δu or ∇P along X within our critical regularity framework is not totally clear, though. Therefore, as in previous chapter dedicated to the incompressible Boussinesq system, we shall replace differentiation along vector-fields by *para-differentiation.*

Let us briefly recall how it works. Fix some suitably large integer N_0 and introduce the following *paraproduct* and *remainder* operators (after J.-M. Bony in [11]):

$$\dot{T}_u v := \sum_{j\in\mathbb{Z}} \dot{S}_{j-N_0} u \dot{\Delta}_j v \quad \text{and} \quad \dot{R}(u,v) \equiv \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \widetilde{\dot{\Delta}}_j v := \sum_{\substack{j\in\mathbb{Z}\\ |j-k|\leq N_0}} \dot{\Delta}_j u \dot{\Delta}_k v.$$

Then any product may be formally decomposed as follows:

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v). \tag{2.23}$$

To overcome the problem with the definition of $\partial_X \Delta u$ and $\partial_X \nabla P$, we shall change the vector field X to the para-vector field operator $\dot{\mathcal{T}}_X \cdot := \dot{T}_{X^k} \partial_k \cdot$ which, in our regularity framework, will turn out to be *the principal part* of operator ∂_X. Indeed, for any couple (X, f), Decomposition (2.23) ensures that

$$(\dot{\mathcal{T}}_X - \partial_X) f = \dot{T}_{\partial_k f} X^k + \partial_k \dot{R}(f, X^k) - \dot{R}(f, \operatorname{div} X).$$

Therefore, taking advantage of classical continuity results for operators $\dot{T}$ and $\dot{R}$ (see [6]), we discover that

$$\|(\dot{\mathcal{T}}_X - \partial_X) f\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} \lesssim \|f\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}} \|X\|_{\mathscr{C}^\varepsilon} \tag{2.24}$$

whenever $(\varepsilon, p) \in]0,1[\times[1,+\infty]$ fulfills:

$$\frac{N}{p} \in]\,1-\varepsilon,\, 2\,[\;\text{ if }\; \operatorname{div} X = 0, \quad \text{and} \quad \frac{N}{p} \in]\,2-\varepsilon,\, 2\,[\;\text{ otherwise.} \tag{2.25}$$

In our situation, we will apply (2.24) with $f = \nabla P$ or Δu, which are in $L^1\big(\mathbb{R}_+; \dot{B}^{\frac{N}{p}-1}_{p,1}(\mathbb{R}^N)\big)$.

Now, incising the term $\partial_X u$ by the scalpel $\dot{\mathcal{T}}_X$ in (2.21) and applying $\dot{\mathcal{T}}_X$ to the third equation of (INS) yield

$$\begin{cases} \rho D_t \dot{\mathcal{T}}_X u - \Delta \dot{\mathcal{T}}_X u + \nabla \dot{\mathcal{T}}_X P = g, \\ \operatorname{div} \dot{\mathcal{T}}_X u = \operatorname{div} (\dot{T}_{\partial_k X} u^k - \dot{T}_{\operatorname{div} X} u), \\ \dot{\mathcal{T}}_X u|_{t=0} = \dot{\mathcal{T}}_{X_0} u_0 \end{cases} \tag{2.26}$$

with

$$g := -\rho[\dot{\mathcal{T}}_X, D_t]u + [\dot{\mathcal{T}}_X, \Delta]u - [\dot{\mathcal{T}}_X, \nabla]P + (\partial_X - \dot{\mathcal{T}}_X)(\Delta u - \nabla P) \\ - \partial_X \rho D_t u + \rho(\dot{\mathcal{T}}_X - \partial_X) D_t u. \tag{2.27}$$

This surgery leading to (2.26) is effective for three reasons. First, all the commutator terms in (2.27) are under control (see the last section of this chapter). Second, $D_t \partial_X u$ and $D_t \dot{\mathcal{T}}_X u$ are in the same Besov space, and the multiplier type regularity on the density that was pointed out before is thus appropriate. Last, Condition (2.17) ensures that $(\dot{\mathcal{T}}_X - \partial_X)u$ is a remainder term. Of course, the divergence free condition need not be satisfied by $\dot{\mathcal{T}}_X u$, but one can further modify (2.26) so as to enter in the standard maximal regularity theory. Then, under the smallness condition (2.17), one can close the estimates involving $\partial_X u$ or $\partial_X \rho$, globally in time.

The rest of this chapter unfolds as follows. In the next section, we show that Theorem 2.2 entails a general (but not so explicit) result of persistence of Hölder regularity for patches of density in any dimension. We shall then obtain Theorem 2.3, and an analogous result in dimension $N = 3$. Section 2.4 is devoted to the proof of our general result of all-time persistence of striated regularity (Theorem 2.2). Some technical results pertaining to commutators and multiplier spaces are postponed in last two sections.

2.3 The density patch problem

This section is devoted to the proof of results of persistence of regularity for patches of constant densities, taking Theorem 2.2 for granted. Throughout this section we shall use repeatedly the fact (proved in e.g. [36, Lemma A.7]) that for any (not necessarily bounded) domain $\mathcal{D}$ of $\mathbb{R}^N$ with uniform $\mathcal{C}^1$ boundary, we have

$$\mathbb{1}_{\mathcal{D}} \in \mathcal{M}\big(\dot{B}^s_{p,r}(\mathbb{R}^N)\big) \quad \text{whenever} \quad (s, p, r) \in]\frac{1}{p} - 1, \frac{1}{p}[\times]1, \infty[\times[1, \infty].$$

From that property, we deduce that if $(\varepsilon, p) \in]0, 1[\times]N - 1, \frac{N-1}{1-\varepsilon}[$, then the density ρ_0 given by (2.18) belongs to $\mathcal{M}\big(\dot{B}^{\frac{N}{p}-1}_{p,r}(\mathbb{R}^N)\big) \cap \mathcal{M}\big(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,r}(\mathbb{R}^N)\big)$.

As a start, let us give a result of persistence of regularity, under rather general hypotheses.

Proposition 2.7. *Assume that ρ_0 is given by (2.18) with small enough η and some $\mathcal{C}^{1,\varepsilon}$ domain $\mathcal{D}_0$ of $\mathbb{R}^N$ satisfying (2.5). Let u_0 be a small enough divergence free vector field with coefficients in*

$\dot{B}^{\frac{N}{p}-1}_{p,1}$ *for some*

$$N-1<p<\min\Big\{\frac{N-1}{1-\varepsilon},2N\Big\}\cdot \tag{2.28}$$

Consider a family $(X_{\lambda,0})_{\lambda\in\Lambda}$ *of* $\mathcal{C}^{0,\varepsilon}$ *divergence free vector fields tangent to* $\mathcal{D}_0$ *and such that* $\partial_{X_{\lambda,0}}u_0\in\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}$ *for all* $\lambda\in\Lambda$.

Then the unique solution $(\rho,u,\nabla P)$ *of* (INS) *given by Theorem 2.1 satisfies the following additional properties:*

- $\rho(t,\cdot)$ *is given by* (2.20),
- *all the time-dependent vector fields* X_λ *solutions to* (2.8) *with initial data* $X_{\lambda,0}$ *belong to* $L^\infty_{loc}(\mathbb{R}_+;\mathcal{C}^{0,\varepsilon})$ *and remain tangent to the patch for all time.*

Proof. Assumptions (2.18) and (2.28) guarantee that ρ_0 is in $\mathcal{M}(\dot{B}^{\frac{N}{p}-1}_{p,1})\cap\big(\dot{B}^{\frac{N}{p}-2+\varepsilon}_{p,1}\big)$, and that (2.17) is fulfilled if η and u_0 are small enough. Of course, $\partial_{X_{\lambda,0}}\rho_0\equiv 0$ for all $\lambda\in\Lambda$ because the vector fields $X_{\lambda,0}$ are tangent to the boundary. Therefore, applying Theorem 2.2 ensures that all the vector fields X_λ are in $L^\infty_{loc}(\mathbb{R}_+;\mathcal{C}^{0,\varepsilon})$. Now, if we consider a level set function f_0 in $\mathcal{C}^{1,\varepsilon}$ associated to $\mathcal{D}_0$ as in (2.5), then $f_t:=f_0\circ\psi_t$ is associated to the transported domain $\mathcal{D}_t=\psi_t(\mathcal{D}_0)$, and we have

$$D_t\nabla f=-\nabla u\cdot\nabla f\quad\text{with}\;\;(\nabla u)_{ij}=\partial_i u^j. \tag{2.29}$$

Therefore, as X_λ satisfies (2.8), we have

$$D_t(X_\lambda\cdot\nabla f)=(D_tX_\lambda)\cdot\nabla f+X_\lambda\cdot(D_t\nabla f)=0,$$

which ensures that X_λ remains tangent to the patch for all time. □

Example. As a consequence of Bony decomposition and of div $X_{\lambda,0}\equiv 0$, we have

$$\partial_{X_{\lambda,0}}u_0=\dot{\mathcal{T}}_{X_{\lambda,0}}u_0+\dot{T}_{\partial_k u_0}X^k_{\lambda,0}+\operatorname{div}\dot{R}(u_0,X_{\lambda,0}).$$

Hence, if $u_0\in\dot{B}^{\frac{N}{p}-1}_{p,1}\cap\dot{B}^{\frac{N}{p}+\varepsilon-1}_{p,1}$ with p satisfying (2.28), then the conditions of Proposition 2.7 are fulfilled. In fact, the additional regularity $\dot{B}^{\frac{N}{p}+\varepsilon-1}_{p,1}$ of u_0 implies that the flow ψ_t is in $\mathcal{C}^{1,\varepsilon}$, because the solution $(u,\nabla P)$ lies in $\dot{E}^\varepsilon_q$ for some $q>(N-1)/(1-\varepsilon)$. This is a consequence of the following result that may be obtained along the lines of the proof of Theorem 2.2.

Proposition 2.8. *If the initial data* (ρ_0,u_0) *are as in Theorem 2.1 and if in addition* u_0 *is in* $\dot{B}^{\frac{N}{q}+\varepsilon-1}_{q,1}$ *and*

$$\|\rho_0-1\|_{\mathcal{M}\big(\dot{B}^{\frac{N}{q}+\varepsilon-1}_{q,1}\big)}\leq c\quad\textit{for small constant }c,\;\;0<\varepsilon<1\;\;\textit{and}\;\;\frac{N-1}{1-\varepsilon}<q\leq\infty,$$

then, beside the properties listed in Theorem 2.1, the unique global solution $(\rho, u, \nabla P)$ *of System* (INS) *satisfies*

$$\rho \in L^\infty\Big(\mathbb{R}_+; \mathcal{M}\big(\dot{B}^{\frac{N}{q}+\varepsilon-1}_{q,1}\big)\Big) \quad \textit{and} \quad (u, \nabla P) \in \dot{E}^\varepsilon_q.$$

2.3.1 The two-dimensional case

Here we prove Theorem 2.3. As a start, we have to show that if the vorticity ω_0 is given by (2.19) then u_0 is in $\dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2)$. This will be achieved by using the fact that u_0 can be computed from ω_0 by means of the following *Biot-Savart law*:

$$u_0 = K_2 * \omega_0, \quad \text{with} \quad K_2(x) := \frac{1}{2\pi|x|^2}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}. \tag{2.30}$$

Recall that ω_0 is in $L^p(\mathbb{R}^2)$ and is supported in some ball $B(0,R)$. Now, on the one hand, one may write for all x in $B(0,2R)$,

$$|u_0(x)| \le \frac{1}{2\pi}\int_{B(0,3R)} |\omega_0(x-y)|\,\frac{dy}{|y|},$$

which, by convolution inequalities and our choice of p, implies that $|u_0|\mathbb{1}_{B(0,2R)}$ is in $L^r(\mathbb{R}^2)$ for any r satisfying

$$r \in \Big[1, \frac{2p}{2-p}\Big[\subset \Big[1, \frac{2}{1-\varepsilon}\Big[. \tag{2.31}$$

On the other hand, for $|x| \ge 2R$, owing to the zero average condition for ω_0, we have

$$u_0(x) = \frac{1}{2\pi}\int_{|y|\le R} \big(K_2(x-y) - K_2(x)\big)\omega_0(y)\,dy.$$

Therefore, by computing $K_2(x-y) - K_2(x)$, it is not difficult to see that we have for some constant C_R depending only on R,

$$|u_0(x)| \le \frac{C_R}{|x|^2}\|\omega_0\|_{L^1} \quad \text{for all } \ x \ \text{ such that } \ |x| \ge 2R.$$

Then putting the two informations together, we get u_0 in L^r for all r given by (2.31).

Next, let us write that

$$u_0 = \dot{S}_0 u_0 + (\mathrm{Id} - \dot{S}_0)u_0.$$

To handle the first term, we infer from the embedding of L^r in $\dot{B}^{\frac{2}{p}-\frac{2}{r}}_{p,\infty}$ for all $1 < r < p < 2$,

$$\|\dot{S}_0 u_0\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} \lesssim \|\dot{S}_0 u_0\|_{\dot{B}^{\frac{2}{p}-\frac{2}{r}}_{p,\infty}} \lesssim \|\dot{S}_0 u_0\|_{L^r} \lesssim \|u_0\|_{L^r} \lesssim \|\omega_0\|_{L^p}.$$

As regards the high frequency part of u_0, the Fourier multiplier $(\mathrm{Id} - \dot{S}_0)\nabla^\perp(-\Delta)^{-1}$ is homogeneous of degree -1 away from a neighborhood of 0, which yields

$$\begin{aligned}\|(\mathrm{Id}-\dot{S}_0)u_0\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} &= \|(\mathrm{Id}-\dot{S}_0)\nabla^\perp(-\Delta)^{-1}\omega_0\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} \\ &\lesssim \|(\mathrm{Id}-\dot{S}_0)\omega_0\|_{\dot{B}^{\frac{2}{p}-2}_{p,1}} \lesssim \|\omega_0\|_{L^p}.\end{aligned} \tag{2.32}$$

Next, consider the divergence free vector field $X_0=\nabla^\perp f_0$ where f_0 is given by (2.5) and is (with no loss of generality) compactly supported. If it is true that

$$\partial_{X_0}u_0 \in \dot{B}^{\frac{2}{p}-2+\varepsilon}_{p,1}, \tag{2.33}$$

then one can apply Proposition 2.7 which ensures that the transported vector field X_t remains in $\mathcal{C}^{0,\varepsilon}$ for all $t\geq 0$. Now, it is classical that we have $X_t=(\nabla f_t)^\perp$ with $f_t=f_0\circ\psi_t^{-1}$. Hence $\mathcal{D}_t$ has a $\mathcal{C}^{1,\varepsilon}$ boundary.

Let us establish (2.33). First note that

$$X_0\in\mathscr{C}^\varepsilon_c \hookrightarrow b^\varepsilon_{p,\infty}\cap b^\alpha_{p,r} \quad \text{provided} \quad 1\leq r\leq\infty \text{ and } -\frac{2}{p'}<\alpha<\varepsilon \tag{2.34}$$

due to Proposition 2.13 and Proposition 2.12. Now, (2.24) ensures that

$$\|\dot{\mathcal{T}}_{X_0}u_0-\partial_{X_0}u_0\|_{\dot{B}^{\frac{2}{p}+\varepsilon-2}_{p,1}} \lesssim \|u_0\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}\|X_0\|_{\dot{\mathscr{C}}^\varepsilon} \quad \text{for any } 1<p<\frac{2}{1-\varepsilon}\cdot \tag{2.35}$$

Then thanks to (2.30), we obtain

$$\dot{\mathcal{T}}_{X_0}u_0=\dot{\mathcal{T}}_{X_0}(-\Delta)^{-1}\nabla^\perp\omega_0=(-\Delta)^{-1}\nabla^\perp\dot{\mathcal{T}}_{X_0}\omega_0+[\dot{\mathcal{T}}_{X_0},(-\Delta)^{-1}\nabla^\perp]\omega_0,$$

whence using Lemma 2.15 and (2.32),

$$\|\dot{\mathcal{T}}_{X_0}u_0-(-\Delta)^{-1}\nabla^\perp\dot{\mathcal{T}}_{X_0}\omega_0\|_{\dot{B}^{\frac{2}{p}+\varepsilon-2}_{p,1}} \lesssim \|X_0\|_{\dot{\mathscr{C}}^\varepsilon}\|\omega_0\|_{L^p}. \tag{2.36}$$

Next, we notice that

$$\dot{\mathcal{T}}_{X_0}\omega_0-\mathrm{div}\,(X_0\omega_0)=-\mathrm{div}\,\big(\dot{T}_{\omega_0}X_0+\dot{R}(\omega_0,X_0)\big).$$

Therefore, taking advantage of standard continuity results for $\dot{T}$ and $\dot{R}$, we have

$$\|\dot{\mathcal{T}}_{X_0}\omega_0-\mathrm{div}\,(X_0\omega_0)\|_{\dot{B}^{\frac{2}{p}+\varepsilon-3}_{p,1}} \lesssim \|\omega_0\|_{L^p}\|X_0\|_{\dot{\mathscr{C}}^\varepsilon}, \quad \text{as } 1<p<\frac{2}{2-\varepsilon}\cdot \tag{2.37}$$

Since $\mathrm{div}\,(X_0\,\overline{\omega}_0)\equiv 0$ by assumption, it is sufficient to study $\mathrm{div}\,(X_0\,\widetilde{\omega}_0\,\mathbb{1}_{\mathcal{D}_0})$. Recall that $\widetilde{\omega}_0\in\mathscr{C}^{\varepsilon'}_c$ for some $0<\varepsilon'<\varepsilon$, and that $\mathrm{div}\,X_0=0$. As $\partial_{X_0}\mathbb{1}_{\mathcal{D}_0}=0$, Corollary 2.19 implies that

$$\mathrm{div}\,(X_0\,\widetilde{\omega}_0\,\mathbb{1}_{\mathcal{D}_0}) \in \dot{B}^{\frac{2}{p}+\varepsilon-3}_{p,1}.$$

Putting (2.35), (2.36) and (2.37) together, we conclude that (2.33) is fulfilled, which completes the proof of Theorem 2.3. □

If dropping off the zero average condition for the function ω_0 in Theorem 2.3, then the corresponding initial velocity field u_0 cannot be in $L^r(\mathbb{R}^2)$ for any $r \in]1,2]$. Still, one can get a similar statement in the particular case where $(\overline{\omega}_0, \widetilde{\omega}_0) \equiv (0, \eta')$ for some small enough η'. Indeed, from (2.30) and Hardy-Littlewood-Sobolev inequality, we deduce that u_0 belongs to all spaces $L^r(\mathbb{R}^2)$ with $r \in]2,\infty[$. Repeating the first part of the proof of Theorem 2.3 thus yields $u_0 \in \dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2)$ for any $2 < p < \infty$. Now, as ω_0 is bounded and compactly supported, it is in $\dot{B}^{\frac{2}{q}+\varepsilon-2}_{q,1}(\mathbb{R}^2)$ for any $0 < \varepsilon < 1$ and $1 < q < \infty$, which implies that $u_0 \in \dot{B}^{\frac{2}{q}+\varepsilon-1}_{q,1}$. Hence, applying Proposition 2.8, and using the fact that the flow of the solution constructed therein is in $\mathcal{C}^{1,\varepsilon}$, we conclude to the following generalization of of [88, Remark 1.1].

Theorem 2.9. *Let $\mathcal{D}_0$ satisfy (2.5) for some ε in $]0,1[$. There exists a constant η_0 depending only on $\mathcal{D}_0$ so that for all $\eta, \eta' \in]-\eta_0, \eta_0[$ if*

$$\rho_0 := (1+\eta)\mathbb{1}_{\mathcal{D}_0} + \mathbb{1}_{\mathcal{D}_0^c},$$

and if the divergence free vector-field u_0 in $W^{1,p}(\mathbb{R}^2)$ for some $p > 2$ is given by

$$u_0 := (-\Delta)^{-1}\nabla^\perp(\eta'\mathbb{1}_{\mathcal{D}_0}),$$

then System (INS) has a unique solution $(\rho, u, \nabla P)$ with the properties listed in Proposition 2.8 for some suitable In addition, (2.3) is fulfilled for all $t \geq 0$, and $\mathcal{D}_t$ remains a simply connected bounded domain of class $\mathcal{C}^{1,\varepsilon}$.

2.3.2 The three-dimensional case

As another application of Proposition 2.7, one can generalize Theorem 2.3 to the three dimensional case. Our result reads as follows.

Theorem 2.10. *Let $\mathcal{D}_0$ be a $\mathcal{C}^{1,\varepsilon}$ simply connected bounded domain of $\mathbb{R}^3$ with $\varepsilon \in]0,1[$. Let ρ_0 be given by (2.18) with small enough η. Assume that the initial velocity u_0 has coefficients in $\mathcal{S}'_h(\mathbb{R}^3)$ and vorticity*[5]

$$\Omega_0 := \nabla \wedge u_0 = \widetilde{\Omega}_0 \mathbb{1}_{\mathcal{D}_0},$$

for some small enough $\widetilde{\Omega}_0$ in $\mathcal{C}^{0,\delta}(\mathbb{R}^3;\mathbb{R}^3)$ ($\delta \in]0,\varepsilon[$) with $\mathrm{div}\,\widetilde{\Omega}_0 = 0$ and $\widetilde{\Omega}_0 \cdot \boldsymbol{n}_{\mathcal{D}_0}|_{\partial\mathcal{D}_0} \equiv 0$ (here $\boldsymbol{n}_{\mathcal{D}_0}$ denotes the outwards unit normal of the domain $\mathcal{D}_0$).

[5]For any point $Y \in \mathbb{R}^3$, we set $X \wedge Y := (X^2Y^3 - X^3Y^2,\, X^3Y^1 - X^1Y^3,\, X^1Y^2 - X^2Y^1)$ where X stands for an element of $\mathbb{R}^3$ or for the ∇ operator.

There exists a unique solution $(\rho, u, \nabla P)$ to System (INS) with the properties listed in Theorem 2.1 for some suitable p satisfying

$$2 < p < \min\left\{\frac{2}{1-\varepsilon}, 6\right\}. \tag{2.38}$$

Furthermore, for all $t \geq 0$, we have (2.20) and $\mathcal{D}_t$ remains a simply connected bounded domain of class $\mathcal{C}^{1,\varepsilon}$.

Proof. With no loss of generality, one may assume that $\widetilde{\Omega}_0$ is compactly supported. Like in the 2D case, we first have to check that u_0 fulfills the assumptions of Proposition 2.7. As it is divergence free and decays at infinity (recall that $u_0 \in \mathcal{S}'_h$), it is given by the Biot-Savart law:

$$u_0 = (-\Delta)^{-1}\nabla \wedge \Omega_0, \quad \text{with } \ \Omega_0 = \widetilde{\Omega}_0 \, \mathbb{1}_{\mathcal{D}_0}. \tag{2.39}$$

We claim that u_0 belongs to $\dot{B}^{\frac{3}{p}-1}_{p,1}$ for some p satisfying (2.38). Indeed, the characteristic function of any bounded domain with $\mathcal{C}^1$ regularity belongs to all Besov spaces $B^{\frac{1}{q}}_{q,\infty}$ with $1 \leq q \leq \infty$ (see e.g. [123]). Hence combining Proposition 2.12 and the embedding (2.70) gives

$$\mathbb{1}_{\mathcal{D}_0} \in \ \mathcal{E}' \cap B^{\frac{1}{q}}_{q,\infty} \hookrightarrow \dot{B}^{\frac{3}{q}-2}_{q,1} \quad \text{for any } \ q \in]1, \infty[. \tag{2.40}$$

Now, using Bony's decomposition and standard continuity results for operators $\dot{R}$ and $\dot{T}$, we discover that

$$\widetilde{\Omega}_0 \in \mathscr{C}^\delta_c \hookrightarrow \mathcal{M}\big(\dot{B}^{\frac{3}{q}-2}_{q,1}\big) \quad \text{for any } \ q \in \left]\frac{3}{2}, \frac{3}{2-\delta}\right[.$$

Hence the definition of Multiplier space and (2.40) yield

$$\Omega_0 = \widetilde{\Omega}_0 \, \mathbb{1}_{\mathcal{D}_0} \in \dot{B}^{\frac{3}{q}-2}_{q,1} \quad \text{for any } \ q \in \left]\frac{3}{2}, \frac{3}{2-\delta}\right[. \tag{2.41}$$

As u_0 is in $\mathcal{S}'_h$ and $(-\Delta^{-1})^{-1}\nabla\wedge$ in (2.39) is a homogeneous multiplier of degree -1, one can conclude that

$$u_0 \in \dot{B}^{\frac{3}{q}-1}_{q,1} \hookrightarrow \dot{B}^{\frac{3}{p}-1}_{p,1}, \quad \text{for any } p \geq q.$$

Note that for any δ in $]0, 1[$, one can find some p satisfying the above conditions and (2.38) altogether.

Next, consider some (compactly supported) level set function f_0 associated to $\partial\mathcal{D}_0$, and the three $\mathcal{C}^{0,\varepsilon}$ vector-fields $X_{k,0} := e_k \wedge \nabla f_0$ with (e_1, e_2, e_3) being the canonical basis of $\mathbb{R}^3$. It is clear that those vector-fields are divergence free and tangent to $\partial\mathcal{D}_0$. Let us check that we have $\partial_{X_{k,0}} u_0 \in \dot{B}^{\frac{3}{p}-2+\varepsilon}_{p,1}$ for some p satisfying (2.38). As in the two-dimensional case, this will follow from Biot-Savart law and the special structure of Ω_0. Indeed, from (2.24) and div $X_{k,0} = 0$, we have

$$\|\dot{\mathcal{T}}_{X_{k,0}} u_0 - \partial_{X_{k,0}} u_0\|_{\dot{B}^{\frac{3}{p}+\varepsilon-2}_{p,1}} \lesssim \|u_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \|X_0\|_{\dot{\mathscr{C}}^\varepsilon}, \quad \forall p \in \left]\frac{3}{2}, \frac{3}{1-\varepsilon}\right[.$$

Then (2.39) yields

$$\dot{\mathcal{T}}_{X_{k,0}} u_0 = \dot{\mathcal{T}}_{X_{k,0}} (-\Delta)^{-1} \nabla \wedge \Omega_0 = (-\Delta)^{-1} \nabla \wedge \dot{\mathcal{T}}_{X_{k,0}} \Omega_0 + [\dot{\mathcal{T}}_{X_{k,0}}, (-\Delta)^{-1} \nabla \wedge]\, \Omega_0.$$

Thanks to Lemma 2.15 and homogeneity of $(-\Delta^{-1})^{-1} \nabla \wedge$, it is thus sufficient to verify that $\dot{\mathcal{T}}_{X_{k,0}} \Omega_0$ belongs to $\dot{B}^{\frac{3}{p}+\varepsilon-3}_{p,1}$ for some p satisfying (2.38). In fact, from the decomposition

$$\dot{\mathcal{T}}_{X_{k,0}} \Omega_0 - \mathrm{div}\,(X_{k,0} \Omega_0) = -\mathrm{div}\,\big(\dot{T}_{\Omega_0} X_{k,0} + \dot{R}(\Omega_0, X_{k,0})\big),$$

and continuity results for $\dot{R}$ and $\dot{T}$, we get

$$\|\dot{\mathcal{T}}_{X_{k,0}} \Omega_0 - \mathrm{div}\,(X_{k,0} \Omega_0)\|_{\dot{B}^{\frac{3}{q}+\varepsilon-3}_{q,1}} \lesssim \|\Omega_0\|_{\dot{B}^{\frac{3}{q}-2}_{q,1}} \|X_{k,0}\|_{\mathscr{C}^\varepsilon}, \quad \text{for any } q \in \Big]\frac{3}{2}, \frac{3}{2-\varepsilon}\Big[.$$

Thus, remembering (2.41) and $0 < \delta < \varepsilon$, we have to choose some p satisfying (2.38), such that the following standard embedding holds

$$\dot{B}^{\frac{3}{q}+\varepsilon-3}_{q,1} \hookrightarrow \dot{B}^{\frac{3}{p}+\varepsilon-3}_{p,1} \quad \text{for some } q \in \Big]\frac{3}{2}, \frac{3}{2-\delta}\Big[\text{ with } q \le p. \tag{2.42}$$

Now, because $\partial_{X_{k,0}} \mathbb{1}_{\mathcal{D}_0} \equiv 0$ and $\widetilde{\Omega}_0$ is in $\mathcal{C}^\delta$, Corollary 2.19 yields for all $0 < \delta_\star < \delta$,

$$\partial_{X_{k,0}} \Omega_0 = \mathrm{div}\,(X_{k,0} \otimes \Omega_0) = \mathrm{div}\,(X_{k,0} \otimes \widetilde{\Omega}_0\, \mathbb{1}_{\mathcal{D}_0}) \in \dot{B}^{\delta_\star - 1}_{q,1} \quad \text{for all } q \ge 1.$$

One can thus conclude that $\partial_{X_{k,0}} u_0 \in \dot{B}^{\frac{3}{p}-2+\varepsilon}_{p,1}$ for any index p satisfying $p \ge q$ with q satisfying Condition (2.42) and $\frac{3}{q} + \varepsilon - 2 = \delta^* \in]0, \delta[$.

As one can require in addition the index p to fulfill (2.38), Proposition 2.7 applies with the family $(X_{k,0})_{1 \le k \le 3}$. Denoting by $(X_k)_{1 \le k \le 3}$ the corresponding family of divergence free vector fields in $\mathcal{C}^{0,\varepsilon}$ given by (2.8) with initial data $X_{0,k}$, and introducing $Y_1 := X_3 \wedge X_1$, $Y_2 := X_3 \wedge X_1$ and $Y_3 = X_1 \wedge X_2$, we discover that for $\alpha = 1, 2, 3$,

$$\begin{cases} \partial_t Y_\alpha + u \cdot \nabla Y_\alpha = -\nabla u \cdot Y_\alpha, \\ (Y_\alpha)|_{t=0} = \partial_\alpha f_0 \, \nabla f_0. \end{cases} \tag{2.43}$$

By (2.29), it is clear that the time-dependent vector field $\big(\partial_\alpha f_0(\psi_t^{-1})\big)\, \nabla f_t$ also satisfies (2.43), hence we have, by uniqueness, $Y_\alpha(t, \cdot) = \big((\partial_\alpha f_0)(\psi_t^{-1})\big) \nabla f_t$. So finally,

$$\big|\nabla f_0 \circ \psi_t^{-1}\big|^2 \, \nabla f_t = \sum_{\alpha=1}^{3} Y_\alpha(t, \cdot)\, \partial_\alpha f_0 \circ \psi_t^{-1}.$$

As ψ_t^{-1} is $\mathcal{C}^1$ and as both Y_α and ∇f_0 are in $\mathcal{C}^{0,\varepsilon}$, one can conclude that ∇f_t is $\mathcal{C}^{0,\varepsilon}$ in some neighborhood of $\partial \mathcal{D}_0$. Therefore $\mathcal{D}_t$ remains of class $\mathcal{C}^{1,\varepsilon}$ for all time. $\square$

Remark 2.11. *In contrast with the 2D case, one cannot consider constant vortex patterns for the condition* $\widetilde{\Omega}_0 \cdot \boldsymbol{n}_{\mathcal{D}_0}|_{\partial\mathcal{D}_0} \equiv 0$ *is not fulfilled. One can define directly* u_0 *through* $u_0 = (-\Delta)^{-1}\nabla \wedge e$ *where* e *is a constant vector of* $\mathbb{R}^3$ *(as we did for the Boussinesq system in previous chapter), but then,* $\nabla \wedge u_0$ *does not coincides with* e.

2.4 The proof of persistence of striated regularity

That section is devoted to the proof of Theorem 2.2. The first step is to apply Theorem 2.1. From it, we get a unique global solution $(\rho, u, \nabla P)$ with $\rho \in \mathcal{C}_b\big(\mathbb{R}_+; \mathcal{M}(\dot{B}^{\frac{N}{p}-1}_{p,1})\big)$ and $(u, \nabla P) \in \dot{E}_p$, satisfying (2.15). Because the product of functions maps $\dot{B}^{\frac{N}{p}-1}_{p,1} \times \dot{B}^{\frac{N}{p}}_{p,1}$ to $\dot{B}^{\frac{N}{p}-1}_{p,1}$, we deduce that $D_t u = \partial_t u + u \cdot \nabla u$ is also bounded by the right-hand side of (2.15). So finally,

$$\|(u, \nabla P)\|_{\dot{E}_p} + \|D_t u\|_{L^1_t\big(\dot{B}^{\frac{N}{p}-1}_{p,1}\big)} \le C\|u_0\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}}. \tag{2.44}$$

In order to complete the proof of the theorem, it is only a matter of showing that the additional multiplier and striated regularity properties are conserved for all positive times. We shall mainly concentrate on the proof of a priori estimates for the corresponding norms, just explaining at the end how a suitable regularization process allows to make it rigorous.

2.4.1 Bounds involving multiplier norms

As already pointed out in the introduction, because ∇u is in $L^1(\mathbb{R}_+; \dot{B}^{\frac{N}{p}}_{p,1})$ and $\dot{B}^{\frac{N}{p}}_{p,1}$ is embedded in $\mathcal{C}_b$, the flow ψ of u is $\mathcal{C}^1$ and we have for all $t \ge 0$, owing to (2.15),

$$\|\nabla\psi_t^{\pm 1}\|_{L^\infty} \le \exp\left(\int_0^t \|\nabla u\|_{L^\infty}\, d\tau\right) \le C \tag{2.45}$$

for a suitably large universal constant C.

Now, from the mass conservation equation and (2.22), we gather that

$$\rho(t,\cdot) = \rho_0 \circ \psi_t^{-1} \quad \text{and} \quad (\partial_X \rho)(t,\cdot) = (\partial_{X_0}\rho_0) \circ \psi_t^{-1}.$$

Hence $\|\rho(t,\cdot)\|_{L^\infty}$ is time independent, and Lemma 2.14 (keeping in mind Condition (2.16)) guarantees that for all $t \in \mathbb{R}_+$,

$$\|\rho(t) - 1\|_{\mathcal{M}\big(\dot{B}^{\frac{N}{p}-1}_{p,1}\big)} \le C\|\rho_0 - 1\|_{\mathcal{M}\big(\dot{B}^{\frac{N}{p}-1}_{p,1}\big)}, \tag{2.46}$$

$$\|\rho(t) - 1\|_{\mathcal{M}\big(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\big)} \le C\|\rho_0 - 1\|_{\mathcal{M}\big(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\big)}, \tag{2.47}$$

$$\|(\partial_X \rho)(t)\|_{\mathcal{M}\big(\dot{B}^{\frac{N}{p}-1}_{p,1} \to \dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\big)} \le C\|\partial_{X_0}\rho_0\|_{\mathcal{M}\big(\dot{B}^{\frac{N}{p}-1}_{p,1} \to \dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\big)}. \tag{2.48}$$

2.4.2 Estimates for the striated regularity

Recall that $\dot{\mathcal{T}}_X u$ satisfies the Stokes-like system (2.26). As $\dot{\mathcal{T}}_X u$ need not be divergence free, to enter into the standard theory, we set

$$v := \dot{\mathcal{T}}_X u - w \quad \text{with} \quad w := \dot{T}_{\partial_k X} u^k - \dot{T}_{\operatorname{div} X} u.$$

Denoting $\widetilde{g} := g - \rho u \cdot \nabla \dot{\mathcal{T}}_X u - (\rho \partial_t w - \Delta w)$ with g defined in (2.27), we see that v satisfies:

$$\begin{cases} \rho \partial_t v - \Delta v + \nabla \dot{\mathcal{T}}_X P = \widetilde{g}, \\ \operatorname{div} v = 0, \\ v|_{t=0} = v_0. \end{cases} \tag{S}$$

We shall decompose the proof of a priori estimates for striated regularity into three steps. The first one is dedicated to bounding $\widetilde{g}$ (which mainly requires the commutator estimates of the appendix). In the second step, we take advantage of the smoothing effect of the heat flow so as to estimate v. In the third step, we revert to $\dot{\mathcal{T}}_X u$ and eventually bound X.

First step: bounds of $\widetilde{g}$

Recall that $\widetilde{g} := g - \rho u \cdot \nabla \dot{\mathcal{T}}_X u - (\rho \partial_t w - \Delta w)$ with

$$g = -\rho[\dot{\mathcal{T}}_X, D_t]u + [\dot{\mathcal{T}}_X, \Delta]u - [\dot{\mathcal{T}}_X, \nabla]P + (\partial_X - \dot{\mathcal{T}}_X)(\Delta u - \nabla P) - \partial_X \rho D_t u + \rho(\dot{\mathcal{T}}_X - \partial_X) D_t u.$$

The first term of g may be bounded according to Proposition 2.17 and to the definition of multiplier spaces. We get, under assumption (2.25),

$$\begin{aligned} \|\rho[\dot{\mathcal{T}}_X, D_t]u\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} \lesssim \|\rho\|_{\mathcal{M}(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1})} \Big(& \|u\|_{\dot{\mathscr{C}}^{-1}} \|\dot{\mathcal{T}}_X u\|_{\dot{B}^{\frac{N}{p}+\varepsilon}_{p,1}} \\ & + \|u\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}} \|\dot{\mathcal{T}}_X u\|_{\dot{\mathscr{C}}^{\varepsilon-2}} + \|u\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}} \|u\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}} \|X\|_{\dot{\mathscr{C}}^{\varepsilon}} \Big). \end{aligned} \tag{2.49}$$

Next, thanks to the commutator estimates in Lemma 2.15, we have

$$\|[\dot{\mathcal{T}}_X, \Delta]u\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} \lesssim \|\nabla X\|_{\dot{\mathscr{C}}^{\varepsilon-1}} \|\nabla u\|_{\dot{B}^{\frac{N}{p}}_{p,1}}, \tag{2.50}$$

$$\|[\dot{\mathcal{T}}_X, \nabla]P\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} \lesssim \|\nabla X\|_{\dot{\mathscr{C}}^{\varepsilon-1}} \|\nabla P\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}}. \tag{2.51}$$

Bounding the fourth term of g stems from (2.24): we have

$$\|(\dot{\mathcal{T}}_X - \partial_X)(\Delta u - \nabla P)\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} \lesssim \|(\Delta u, \nabla P)\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}} \|X\|_{\dot{\mathscr{C}}^{\varepsilon}}. \tag{2.52}$$

Then the definition of multiplier spaces yields

$$\|\partial_X \rho D_t u\|_{\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}} \lesssim \|\partial_X \rho\|_{\mathcal{M}\left(\dot B^{\frac{N}{p}-1}_{p,1}\to \dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} \|D_t u\|_{\dot B^{\frac{N}{p}-1}_{p,1}}. \tag{2.53}$$

Finally, using again (2.24) and the definition of multiplier spaces, we may write

$$\|\rho(\dot{\mathcal{T}}_X - \partial_X) D_t u\|_{\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}} \lesssim \|\rho\|_{\mathcal{M}\left(\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} \|X\|_{\dot{\mathscr{C}}^{\varepsilon}} \|D_t u\|_{\dot B^{\frac{N}{p}-1}_{p,1}}. \tag{2.54}$$

Putting together (2.49) – (2.54) and integrating with respect to time, we end up with

$$\begin{aligned}\|g\|_{L^1_t\left(\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} \lesssim \int_0^t \|\rho\|_{\mathcal{M}\left(\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} \big(\|u\|_{\dot{\mathscr{C}}^{-1}} \|\dot{\mathcal{T}}_X u\|_{\dot B^{\frac{N}{p}+\varepsilon}_{p,1}} + \|\nabla u\|_{\dot B^{\frac{N}{p}}_{p,1}} \|\dot{\mathcal{T}}_X u\|_{\dot{\mathscr{C}}^{\varepsilon-2}}\big)\, dt' \\ + \int_0^t \|X\|_{\dot{\mathscr{C}}^{\varepsilon}} \Big(\big(\|\nabla u\|_{\dot B^{\frac{N}{p}}_{p,1}} \|u\|_{\dot B^{\frac{N}{p}-1}_{p,1}} + \|D_t u\|_{\dot B^{\frac{N}{p}-1}_{p,1}}\big) \|\rho\|_{\mathcal{M}\left(\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} + \|(\nabla^2 u, \nabla P)\|_{\dot B^{\frac{N}{p}-1}_{p,1}}\Big) dt' \\ + \int_0^t \|\partial_X \rho\|_{\mathcal{M}\left(\dot B^{\frac{N}{p}-1}_{p,1}\to \dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} \|D_t u\|_{\dot B^{\frac{N}{p}-1}_{p,1}}\, dt'.\end{aligned} \tag{2.55}$$

Bounding the second term of $\widetilde{g}$ is obvious : taking advantage of Bony's decomposition (2.23) and remembering that $\frac{N}{p} + \varepsilon > 1$ and that $\operatorname{div} u = 0$, we get

$$\begin{aligned}\|\rho u \cdot \nabla \dot{\mathcal{T}}_X u\|_{L^1_t\left(\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} \lesssim \int_0^t \|\rho\|_{\mathcal{M}\left(\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} \big(\|u\|_{\dot{\mathscr{C}}^{-1}} \|\dot{\mathcal{T}}_X u\|_{\dot B^{\frac{N}{p}+\varepsilon}_{p,1}} \\ + \|u\|_{\dot B^{\frac{N}{p}+1}_{p,1}} \|\dot{\mathcal{T}}_X u\|_{\dot{\mathscr{C}}^{\varepsilon-2}}\big)\, dt'.\end{aligned} \tag{2.56}$$

To bound the last term of $\widetilde{g}$, we use the decomposition

$$\rho \partial_t w - \Delta w = \rho(W_1 + W_2) + W_3,$$

with

$$W_1 := \dot T_{\partial_k X} \partial_t u^k - \dot T_{\operatorname{div} X} \partial_t u, \quad W_2 := \dot T_{\partial_k \partial_t X} u^k - \dot T_{\operatorname{div} \partial_t X} u, \quad W_3 := \Delta\big(\dot T_{\operatorname{div} X} u - \dot T_{\partial_k X} u^k\big).$$

Continuity results for the paraproduct and the definition of $\mathcal{M}\big(\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}\big)$ ensure that

$$\|\rho W_1\|_{L^1_t\left(\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} \int_0^t \|\rho\|_{\mathcal{M}\left(\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} \|\nabla X\|_{\dot{\mathscr{C}}^{\varepsilon-1}} \|\partial_t u\|_{\dot B^{\frac{N}{p}-1}_{p,1}}\, dt', \tag{2.57}$$

$$\|\rho W_2\|_{L^1_t\left(\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} \lesssim \int_0^t \|\rho\|_{\mathcal{M}\left(\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} \|\partial_t X\|_{\dot{\mathscr{C}}^{\varepsilon-2}} \|u\|_{\dot B^{\frac{N}{p}+1}_{p,1}}\,dt', \tag{2.58}$$

$$\|W_3\|_{L^1_t\left(\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} \lesssim \int_0^t \|\nabla X\|_{\dot{\mathscr{C}}^{\varepsilon-1}} \|u\|_{\dot B^{\frac{N}{p}+1}_{p,1}}\,dt'. \tag{2.59}$$

To estimate $\partial_t X$ in (2.58), we use the fact that

$$\partial_t X = -u\cdot\nabla X + \partial_X u = -\mathrm{div}\,(u\otimes X) + \partial_X u.$$

Hence using (2.23), and continuity results for the remainder and paraproduct operators, we get under Condition (2.25),

$$\|\partial_t X\|_{\dot{\mathscr{C}}^{\varepsilon-2}} \lesssim \|u\|_{\dot B^{\frac{N}{p}-1}_{p,1}} \|X\|_{\dot{\mathscr{C}}^{\varepsilon}} + \|\partial_X u\|_{\dot{\mathscr{C}}^{\varepsilon-2}}.$$

Therefore, taking advantage of (2.24) yields

$$\|\rho W_2\|_{L^1_t\left(\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} \lesssim \int_0^t \|\rho\|_{\mathcal{M}\left(\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} (\|X\|_{\dot{\mathscr{C}}^{\varepsilon}} \|u\|_{\dot B^{\frac{N}{p}-1}_{p,1}} + \|\dot{\mathcal{T}}_X u\|_{\dot{\mathscr{C}}^{\varepsilon-2}}) \|\nabla u\|_{\dot B^{\frac{N}{p}}_{p,1}}\,dt'. \tag{2.60}$$

Combining (2.57), (2.58) and (2.60), we eventually obtain

$$\begin{aligned}\|\rho\partial_t w - \Delta w\|_{L^1_t\left(\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} &\lesssim \int_0^t \|\dot{\mathcal{T}}_X u\|_{\dot{\mathscr{C}}^{\varepsilon-2}} \|\nabla u\|_{\dot B^{\frac{N}{p}}_{p,1}} \|\rho\|_{\mathcal{M}\left(\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)}\,dt' \\ &+ \int_0^t \|X\|_{\dot{\mathscr{C}}^{\varepsilon}} \Big(\big(\|\rho\|_{\mathcal{M}\left(\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} \|u\|_{\dot B^{\frac{N}{p}-1}_{p,1}} + 1\big) \|\nabla u\|_{\dot B^{\frac{N}{p}}_{p,1}} + \|\rho\|_{\mathcal{M}\left(\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} \|\partial_t u\|_{\dot B^{\frac{N}{p}-1}_{p,1}} \Big)\,dt',\end{aligned} \tag{2.61}$$

whence, putting together estimate (2.55), (2.56) and (2.61),

$$\begin{aligned}\|\widetilde{g}\|_{L^1_t\left(\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} \lesssim &\int_0^t \|\rho\|_{\mathcal{M}\left(\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} \big(\|u\|_{\dot{\mathscr{C}}^{-1}} \|\dot{\mathcal{T}}_X u\|_{\dot B^{\frac{N}{p}+\varepsilon}_{p,1}} + \|\nabla u\|_{\dot B^{\frac{N}{p}}_{p,1}} \|\dot{\mathcal{T}}_X u\|_{\dot{\mathscr{C}}^{\varepsilon-2}}\big)\,dt' \\ &+ \int_0^t \|X\|_{\dot{\mathscr{C}}^{\varepsilon}} \big(\|\nabla u\|_{\dot B^{\frac{N}{p}}_{p,1}} \|u\|_{\dot B^{\frac{N}{p}-1}_{p,1}} + \|(\partial_t u, D_t u)\|_{\dot B^{\frac{N}{p}-1}_{p,1}}\big) \|\rho\|_{\mathcal{M}\left(\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)}\,dt' \\ &+ \int_0^t \|X\|_{\dot{\mathscr{C}}^{\varepsilon}} \|(\nabla^2 u, \nabla P)\|_{\dot B^{\frac{N}{p}-1}_{p,1}}\,dt' + \int_0^t \|\partial_X \rho\|_{\mathcal{M}\left(\dot B^{\frac{N}{p}-1}_{p,1} \to \dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} \|D_t u\|_{\dot B^{\frac{N}{p}-1}_{p,1}}\,dt'.\end{aligned} \tag{2.62}$$

Second step: bounds of v

We now want to bound v in $\widetilde{L}^\infty_t\big(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\big) \cap L^1_t\big(\dot{B}^{\frac{N}{p}+\varepsilon}_{p,1}\big)$, knowing (2.62). This will follow from the smoothing properties of the heat flow. More precisely, introduce the projector $\mathbb{P}$ over divergence-free vector fields, and apply $\mathbb{P}\dot{\Delta}_j$ (with $j \in \mathbb{Z}$) to the equation (S). We get

$$\begin{cases} \partial_t \dot{\Delta}_j v - \Delta \dot{\Delta}_j v = \mathbb{P}\dot{\Delta}_j(\widetilde{g} + (1-\rho)\partial_t v) \\ \dot{\Delta}_j v|_{t=0} = \dot{\Delta}_j v_0. \end{cases}$$

Lemma 2.1 in [21] implies that if $p \in [1,\infty]$,

$$\|\dot{\Delta}_j v(t)\|_{L^p} \leq e^{-ct2^{2j}}\|\dot{\Delta}_j v_0\|_{L^p} + C\int_0^t e^{-c(t-t')2^{2j}}\|\dot{\Delta}_j(\widetilde{g} + (1-\rho)\partial_t v)(t')\|_{L^p}\,dt'.$$

Therefore, taking the supremum over $j \in \mathbb{Z}$, using the fact that

$$\partial_t v = \Delta v + \mathbb{P}\big(\widetilde{g} + (1-\rho)\partial_t v\big)$$

and that $\mathbb{P} : \dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1} \to \dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}$, we find that

$$\begin{aligned} \|v\|_{\widetilde{L}^\infty_t\big(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\big)} + \|v\|_{L^1_t\big(\dot{B}^{\frac{N}{p}+\varepsilon}_{p,1}\big)} + \|\partial_t v\|_{L^1_t\big(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\big)} \\ \lesssim \|v_0\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} + \|\widetilde{g}\|_{L^1_t\big(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\big)} + \|(1-\rho)\partial_t v\|_{L^1_t\big(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\big)}. \end{aligned} \tag{2.63}$$

The smallness condition (2.17) combined with Inequality (2.47) ensure that the last term of (2.63) may be absorbed by the left-hand side, and we thus end up with

$$\|v\|_{\widetilde{L}^\infty_t\big(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\big)\cap L^1_t\big(\dot{B}^{\frac{N}{p}+\varepsilon}_{p,1}\big)} + \|\partial_t v\|_{L^1_t\big(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\big)} \lesssim \|v_0\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} + \|\widetilde{g}\|_{L^1_t\big(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\big)}.$$

Next, we use the fact that by definition of v_0,

$$\begin{aligned} v_0 &= \dot{\mathcal{T}}_{X_0} u_0 - \dot{T}_{\partial_k X_0} u_0^k + \dot{T}_{\operatorname{div} X_0} u_0 \\ &= \partial_{X_0} u_0 - \dot{T}_{\partial_k u_0} X_0^k - \partial_k \dot{R}(X_0^k, u_0) + \dot{R}(\operatorname{div} X_0, u_0) - \dot{T}_{\partial_k X_0} u_0^k + \dot{T}_{\operatorname{div} X_0} u_0. \end{aligned}$$

Hence continuity results for the paraproduct yield, under Condition (2.25),

$$\|v_0\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} \lesssim \|\partial_{X_0} u_0\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} + \|X_0\|_{\mathscr{C}^\varepsilon}\|u_0\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}}.$$

Thus

$$\|v\|_{\widetilde{L}_t^\infty\big(\dot{B}_{p,1}^{\frac{N}{p}+\varepsilon-2}\big)\cap L_t^1\big(\dot{B}_{p,1}^{\frac{N}{p}+\varepsilon}\big)}+\|\partial_t v\|_{L_t^1\big(\dot{B}_{p,1}^{\frac{N}{p}+\varepsilon-2}\big)}$$
$$\lesssim \|\partial_{X_0}u_0\|_{\dot{B}_{p,1}^{\frac{N}{p}+\varepsilon-2}}+\|X_0\|_{\dot{\mathscr{C}}^\varepsilon}\|u_0\|_{\dot{B}_{p,1}^{\frac{N}{p}-1}}+\|\widetilde{g}\|_{L_t^1\big(\dot{B}_{p,1}^{\frac{N}{p}+\varepsilon-2}\big)}. \tag{2.64}$$

Third step: bounds for striated regularity

Remembering that

$$\dot{\mathcal{T}}_X u=v+w \quad \text{with} \quad w=\dot{T}_{\partial_k X}u^k-\dot{T}_{\operatorname{div}X}u,$$

it is now easy to bound the following quantity:

$$\mathscr{H}(t):=\|\dot{\mathcal{T}}_X u\|_{\widetilde{L}_t^\infty\big(\dot{B}_{p,1}^{\frac{N}{p}+\varepsilon-2}\big)}+\|\dot{\mathcal{T}}_X u\|_{L_t^1\big(\dot{B}_{p,1}^{\frac{N}{p}+\varepsilon}\big)}+\|\nabla\dot{\mathcal{T}}_X P\|_{L_t^1\big(\dot{B}_{p,1}^{\frac{N}{p}+\varepsilon-2}\big)}.$$

Indeed, we have

$$\nabla\dot{\mathcal{T}}_X P=(\mathrm{Id}-\mathbb{P})(\widetilde{g}-\rho\partial_t v), \tag{2.65}$$

and thus $\|\nabla\dot{\mathcal{T}}_X P\|_{L_t^1\big(\dot{B}_{p,1}^{\frac{N}{p}+\varepsilon-2}\big)}$ may be bounded by the right-hand side of (2.64). Note also that continuity results for paraproduct operators guarantee that

$$\|w\|_{\widetilde{L}_t^\infty\big(\dot{B}_{p,1}^{\frac{N}{p}+\varepsilon-2}\big)}\lesssim\|u\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{N}{p}-1})}\|X\|_{L_t^\infty(\dot{\mathscr{C}}^\varepsilon)},$$
$$\|w\|_{L_t^1\big(\dot{B}_{p,1}^{\frac{N}{p}+\varepsilon}\big)}\lesssim\int_0^t\|u\|_{\dot{B}_{p,1}^{\frac{N}{p}+1}}\|\nabla X\|_{\dot{\mathscr{C}}^{\varepsilon-1}}\,dt'.$$

Hence we have

$$\mathscr{H}(t)\lesssim\|\partial_{X_0}u_0\|_{\dot{B}_{p,1}^{\frac{N}{p}+\varepsilon-2}}+\|X_0\|_{\dot{\mathscr{C}}^\varepsilon}\|u_0\|_{\dot{B}_{p,1}^{\frac{N}{p}-1}}+\|\widetilde{g}\|_{L_t^1\big(\dot{B}_{p,1}^{\frac{N}{p}+\varepsilon-2}\big)}$$
$$+\|u\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{N}{p}-1})\cap L_t^1(\dot{B}_{p,1}^{\frac{N}{p}+1})}\|X\|_{L_t^\infty(\dot{\mathscr{C}}^\varepsilon)}. \tag{2.66}$$

Because X satisfies (2.8), standard Hölder estimates for transport equations imply that

$$\|X\|_{L_t^\infty(\dot{\mathscr{C}}^\varepsilon)}\le\|X_0\|_{\dot{\mathscr{C}}^\varepsilon}+\int_0^t\|\nabla u\|_{L^\infty}\|X\|_{\dot{\mathscr{C}}^\varepsilon}\,dt'+\int_0^t\|\partial_X u\|_{\dot{\mathscr{C}}^\varepsilon}\,dt'.$$

Now, recall that

$$\partial_X u-\dot{\mathcal{T}}_X u=\dot{T}_{\partial_k u}X^k+\dot{R}(\partial_k u,X^k).$$

Hence, using standard continuity results for operators $\dot{T}$ and $\dot{R}$, and embedding,

$$\|\dot{\mathcal{T}}_X u-\partial_X u\|_{\dot{\mathscr{C}}^\varepsilon}\lesssim\|\dot{\mathcal{T}}_X u-\partial_X u\|_{\dot{B}_{p,1}^{\frac{N}{p}+\varepsilon}}\lesssim\|\nabla u\|_{\dot{B}_{p,1}^{\frac{N}{p}}}\|X\|_{\dot{\mathscr{C}}^\varepsilon}. \tag{2.67}$$

Therefore we have

$$\|X\|_{L_t^\infty(\dot{\mathscr{C}}^\varepsilon)} \leq \|X_0\|_{\dot{\mathscr{C}}^\varepsilon} + \int_0^t \|\nabla u\|_{\dot{B}^{\frac{N}{p}}_{p,1}} \|X\|_{\dot{\mathscr{C}}^\varepsilon}\, dt' + \|\dot{\mathcal{T}}_X u\|_{L_t^1\left(\dot{B}^{\frac{N}{p}+\varepsilon}_{p,1}\right)}. \tag{2.68}$$

Then, using (2.44) and plugging the above inequality in (2.66), we get

$$\begin{aligned}\mathscr{H}(t) \lesssim \|\partial_{X_0} u_0\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} + \|X_0\|_{\dot{\mathscr{C}}^\varepsilon}\|u_0\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}} + \|\widetilde{g}\|_{L_t^1\left(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} \\ + \|u_0\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}}\left(\|\dot{\mathcal{T}}_X u\|_{L_t^1\left(\dot{B}^{\frac{N}{p}+\varepsilon}_{p,1}\right)} + \int_0^t \|\nabla u\|_{\dot{B}^{\frac{N}{p}}_{p,1}} \|X\|_{\dot{\mathscr{C}}^\varepsilon}\, dt'\right).\end{aligned}$$

Choosing c small enough in (2.17), we see that the first term of the second line may be absorbed by the left-hand side. Therefore, setting

$$\mathscr{K}(t) := \mathscr{H}(t) + \|X\|_{L_t^\infty(\dot{\mathscr{C}}^\varepsilon)}$$

and using again (2.68) and the smallness of u_0,

$$\mathscr{K}(t) \lesssim \|\partial_{X_0} u_0\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} + \|X_0\|_{\dot{\mathscr{C}}^\varepsilon} + \|\widetilde{g}\|_{L_t^1\left(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} + \int_0^t \|\nabla u\|_{\dot{B}^{\frac{N}{p}}_{p,1}} \|X\|_{\dot{\mathscr{C}}^\varepsilon}\, dt'.$$

In order to close the estimates, it suffices to bound $\widetilde{g}$ by means of (2.62). Then the above inequality becomes, after using (2.47) and (2.48) (and the fact that $\|\rho_0 - 1\|_{\mathcal{M}(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1})}$ is small implies that $\|\rho_0\|_{\mathcal{M}(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1})}$ is of order one),

$$\begin{aligned}\mathscr{K}(t) \lesssim \|\partial_{X_0} u_0\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} + \|X_0\|_{\dot{\mathscr{C}}^\varepsilon} + \int_0^t \left(\|u\|_{\dot{\mathscr{C}}^{-1}}\|\dot{\mathcal{T}}_X u\|_{\dot{B}^{\frac{N}{p}+\varepsilon}_{p,1}} + \|\nabla u\|_{\dot{B}^{\frac{N}{p}}_{p,1}}\|\dot{\mathcal{T}}_X u\|_{\dot{\mathscr{C}}^{\varepsilon-2}}\right) dt' \\ + \int_0^t \|X\|_{\dot{\mathscr{C}}^\varepsilon}\left(\|\nabla u\|_{\dot{B}^{\frac{N}{p}}_{p,1}}\|u\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}} + \|(\partial_t u, D_t u, \nabla^2 u, \nabla P)\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}}\right) dt' \\ + \|\partial_{X_0}\rho_0\|_{\mathcal{M}\left(\dot{B}^{\frac{N}{p}-1}_{p,1} \to \dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} \int_0^t \|D_t u\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}}\, dt'.\end{aligned}$$

The smallness of u_0 and (2.15) imply that all the terms of the right-hand side (except for the ones pertaining to the data), may be absorbed by the left-hand side. Therefore using the bounds for $D_t u$ in (2.44), we eventually get

$$\mathscr{K}(t) \lesssim \|\partial_{X_0} u_0\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} + \|X_0\|_{\dot{\mathscr{C}}^\varepsilon} + \|\partial_{X_0}\rho_0\|_{\mathcal{M}\left(\dot{B}^{\frac{N}{p}-1}_{p,1} \to \dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)}\|u_0\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}}. \tag{2.69}$$

From (2.67), we gather that $\partial_X u$ is bounded by the right-hand side of (2.69). Next, in order to control the whole nonhomogeneous Hölder norm of X, it suffices to remember that

$$\|X\|_{\mathcal{C}^{0,\varepsilon}} = \|X\|_{L^\infty} + \|X\|_{\dot{\mathscr{C}}^\varepsilon}$$

and that Relation (2.7) together with (2.45) directly yield

$$\|X_t\|_{L^\infty} \leq \|\partial_{X_0}\psi_t\|_{L^\infty} \leq C\|X_0\|_{L^\infty}.$$

Finally, to estimate $\partial_X \nabla P$, we use Inequality (2.24) and get

$$\|\partial_X \nabla P - \nabla \dot{\mathcal{T}}_X P\|_{L^1_t\left(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)} \lesssim \|X\|_{L^\infty_t(\dot{\mathscr{C}}^\varepsilon)}\|\nabla P\|_{L^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}.$$

Therefore $\|\partial_X \nabla P\|_{L^1_t\left(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\right)}$ may be bounded like $\mathscr{K}(t)$.

2.4.3 The regularization process

In all the above computations, we implicitly assumed that X and $\partial_X u$ were in $L^\infty_{loc}(\mathbb{R}_+;\mathcal{C}^{0,\varepsilon})$ and $L^1_{loc}(\mathbb{R}_+;\mathcal{C}^{0,\varepsilon})$, respectively. However, Theorem 2.1 just ensures continuity of those vector-fields, not Hölder regularity.

To overcome that difficulty, one may smooth out the initial velocity (not the density, not to destroy the multiplier hypotheses) by setting for example $u_0^n := \dot{S}_n u_0$. Then Condition (2.17) is satisfied by (ρ_0, u_0^n) and, as in addition u_0^n belongs to all Besov spaces $\dot{B}^{\frac{N}{p}-1}_{\widetilde{p},r}$ with $\widetilde{p} \geq p$ and $r \geq 1$, one can apply[6] [39, Theorem 1.1] for solving (INS) with initial data (ρ_0, u_0^n). This provides us with a unique global solution $(\rho^n, u^n, \nabla P^n)$ which, among others, satisfies

$$\nabla u^n \in L^r(\mathbb{R}_+;\dot{B}^{\frac{N}{p}}_{\widetilde{p},r}) \quad \text{for all } \ r \in]1,\infty[\ \text{ and } \ \max\left(p, \frac{Nr}{3r-2}\right) \leq \widetilde{p} \leq \frac{Nr}{r-1}.$$

By taking r sufficiently close to 1 and using embedding, we see that this implies that ∇u^n is in $L^1_{loc}(\mathbb{R}_+;\dot{\mathscr{C}}^{0,\delta})$ for all $0<\delta<1$ and thus the corresponding flow ψ^n is (in particular) in $\mathcal{C}^{1,\varepsilon}$. This ensures, thanks to (2.7), that X^n is in $L^\infty_{loc}(\mathbb{R}_+;\mathcal{C}^{0,\varepsilon})$ and thus that $\partial_{X^n} u^n$ is in $L^1_{loc}(\mathbb{R}_+;\mathcal{C}^{0,\varepsilon})$.

From the previous steps and the fact that the data (ρ_0, u_0^n) satisfy (2.17) uniformly, we get uniform bounds for ρ^n, u^n, ∇P^n and X^n, and standard arguments thus allow to show that u^n tends to u in $L^1_{loc}(\mathbb{R}_+;L^\infty)$ and thus $(\psi^n - \psi) \to 0$ in $L^\infty_{loc}(\mathbb{R}_+;L^\infty)$. Interpolating with the uniform bounds and using standard functional analysis arguments, one can conclude that $X^n \to X$ in $L^\infty_{loc}(\mathbb{R}_+;\mathcal{C}^{0,\varepsilon'})$ for all $\varepsilon' < \varepsilon$ (and similar results for $(u^n)_{n\in\mathbb{N}}$) and that all the estimates of the previous steps are satisfied. The details are left to the reader. □

[6]That paper concerns the half-space; having the same result in the whole space setting is much easier.

2.5 Multiplier spaces

The following relationship between the nonhomogeneous Besov spaces $B^s_{p,r}(\mathbb{R}^N)$ and the homogeneous Besov spaces $\dot{B}^s_{p,r}(\mathbb{R}^N)$ for *compactly supported* functions or distributions has been established in [34, Section 2.1].

Proposition 2.12. *Let $(p,r) \in [1,\infty]^2$ and $s > -\frac{N}{p'} := -N(1-\frac{1}{p})$ (or just $s \geq -\frac{N}{p'}$ if $r = \infty$). For any u in the set $\mathcal{E}'(\mathbb{R}^N)$ of compactly supported distributions on $\mathbb{R}^N$, we have*

$$u \in B^s_{p,r}(\mathbb{R}^N) \Longleftrightarrow u \in \dot{B}^s_{p,r}(\mathbb{R}^N).$$

Moreover, there exists a constant $C = C(s,p,r,N,\operatorname{Supp} u)$ such that

$$C^{-1}\|u\|_{\dot{B}^s_{p,r}} \leq \|u\|_{B^s_{p,r}} \leq C\|u\|_{\dot{B}^s_{p,r}}.$$

A simple consequence of Proposition 2.12 and of standard embeddings for nonhomogeneous Besov spaces is that for any (s,p,r) as above, we have

$$\mathcal{E}'(\mathbb{R}^N) \cap \dot{B}^{s+\delta}_{p,r}(\mathbb{R}^N) \hookrightarrow \mathcal{E}'(\mathbb{R}^N) \cap \dot{B}^s_{p,r}(\mathbb{R}^N) \quad \text{for any} \quad \delta > 0. \tag{2.70}$$

We also used the following statement:

Proposition 2.13. *Let (s,p,r) be arbitrary in $\mathbb{R} \times [1,\infty]^2$. Then for all $u \in B^s_{\infty,r}(\mathbb{R}^N) \cap \mathcal{E}'(\mathbb{R}^N)$, we have $u \in B^s_{p,r}(\mathbb{R}^N)$ and there exists $C = C(s,p,\operatorname{Supp} u)$ such that*

$$\|u\|_{B^s_{p,r}} \leq C\|u\|_{B^s_{\infty,r}}.$$

Proof. Let u be in $B^s_{\infty,r}(\mathbb{R}^N)$ with compact support. Fix some smooth cut-off function ϕ so that $\phi \equiv 1$ on $\operatorname{Supp} u$. Being compactly supported and smooth, ϕ belongs to any nonhomogeneous Besov space. Then, using (the nonhomogeneous version of) the decomposition (2.23) and that $u = \phi u$, we get

$$u = T_\phi u + T_u\phi + R(u,\phi).$$

Because ϕ is in L^p and u in $B^s_{\infty,r}$, standard continuity results for the paraproduct ensure that $T_\phi u$ is in $B^s_{p,r}$. For the second term, we just use that u is in $\mathscr{C}^{-|s|-1}$ and ϕ in $B^{|s|+1+s}_{p,r}$ hence $T_u\phi$ is in $B^s_{p,r}$. For the remainder term, we use for instance the fact that ϕ is in $\mathscr{C}^{|s|+1}$. Putting all those informations together completes the proof. □

The following result was the key to bounding the density terms in our study of (INS).

Lemma 2.14. *Let $(s,s_k,p,p_k,r,r_k) \in]-1,1[^2 \times [1,\infty]^4$ with $k = 1,2$, and $Z : \mathbb{R}^N \to \mathbb{R}^N$ be a C^1 measure preserving diffeomorphism such that DZ and DZ^{-1} are bounded. When we consider the homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^N)$ or $\dot{B}^{s_k}_{p_k,r_k}(\mathbb{R}^N)$, we assume in addition that $s \in]-\frac{N}{p'},\frac{N}{p}[$ and $s_k \in]-\frac{N}{p'_k},\frac{N}{p_k}[$ for $k = 1,2$. Then we have:*

1. *If* $b^s_{p,r}(\mathbb{R}^N)$ *stands for* $B^s_{p,r}(\mathbb{R}^N)$ *or* $\dot{B}^s_{p,r}(\mathbb{R}^N)$*, then the mapping* $u \mapsto u \circ Z$ *is continuous on* $b^s_{p,r}(\mathbb{R}^N)$*: there is a positive constant* $C_{Z,s,p,r}$ *such that*

$$\|u \circ Z\|_{b^s_{p,r}} \leq C_{Z,s,p,r}\|u\|_{b^s_{p,r}}. \tag{2.71}$$

2. *If* $b^{s_k}_{p_k,r_k}$ *with* $k = 1, 2$*, denote the same type of Besov spaces, then the mapping* $\varphi \mapsto \varphi \circ Z$ *is continuous on* $\mathcal{M}\big(b^{s_1}_{p_1,r_1}(\mathbb{R}^N) \to b^{s_2}_{p_2,r_2}(\mathbb{R}^N)\big)$*, that is*

$$\|\varphi \circ Z\|_{\mathcal{M}\left(b^{s_1}_{p_1,r_1} \to b^{s_2}_{p_2,r_2}\right)} \leq C_{Z^{-1},s_1,p_1,r_1} C_{Z,s_2,p_2,r_2} \|\varphi\|_{\mathcal{M}\left(b^{s_1}_{p_1,r_1} \to b^{s_2}_{p_2,r_2}\right)}.$$

3. *We have the following equivalence for any* $\varphi \in \mathcal{E}'(\mathbb{R}^N)$*,*

$$\varphi \in \mathcal{M}\big(B^{s_1}_{p_1,r_1}(\mathbb{R}^N) \to B^{s_2}_{p_2,r_2}(\mathbb{R}^N)\big) \Longleftrightarrow \varphi \in \mathcal{M}\big(b^{s_1}_{p_1,r_1}(\mathbb{R}^N) \to b^{s_2}_{p_2,r_2}(\mathbb{R}^N)\big).$$

Here $b^{s_1}_{p_1,r_1}$ *and* $b^{s_2}_{p_2,r_2}$ *can be different type of Besov spaces but obey our convention on the index* s_k *for homogeneous Besov space.*

Proof. Item (i) in the case $b = \dot{B}$ has been proved in [34, Lemma 2.1.1]. One may easily modify the proof to handle nonhomogeneous Besov spaces: use the finite difference characterization of [123, Page 98] if $s > 0$, argue by duality if $s < 0$ and interpolate for the case $s = 0$. We get $C_{Z,s,p,r} \approx 1 + \|DZ\|^{s+\frac{N}{r}}_{L^\infty}$ if $s > 0$, and $C_{Z,s,p,r} \approx 1 + \|DZ^{-1}\|^{-s+\frac{N}{r'}}_{L^\infty}$ if $s < 0$.

Part (ii) is immediate according to (2.14) and (2.71). Indeed we may write:

$$\begin{aligned}
\|\varphi \circ Z\|_{\mathcal{M}\left(b^{s_1}_{p_1,r_1} \to b^{s_2}_{p_2,r_2}\right)} &= \sup_{\|u\|_{b^{s_1}_{p_1,r_1}} \leq 1} \|(\varphi \circ Z)\, u\|_{b^{s_2}_{p_2,r_2}} \\
&= \sup_{\|u\|_{b^{s_1}_{p_1,r_1}} \leq 1} \|(\varphi\,(u \circ Z^{-1})) \circ Z\|_{b^{s_2}_{p_2,r_2}} \\
&\leq C_{Z,s_2,p_2,r_2} \sup_{\|u\|_{b^{s_1}_{p_1,r_1}} \leq 1} \|\varphi\,(u \circ Z^{-1})\|_{b^{s_2}_{p_2,r_2}} \\
&\leq C_{Z,s_2,p_2,r_2} \|\varphi\|_{\mathcal{M}(b^{s_1}_{p_1,r_1} \to b^{s_2}_{p_2,r_2})} \sup_{\|u\|_{b^{s_1}_{p_1,r_1}} \leq 1} \|u \circ Z^{-1}\|_{b^{s_1}_{p_1,r_2}} \\
&\leq C_{Z^{-1},s_1,p_1,r_1} C_{Z,s_2,p_2,r_2} \|\varphi\|_{\mathcal{M}(b^{s_1}_{p_1,r_1} \to b^{s_2}_{p_2,r_2})}.
\end{aligned}$$

To prove the last item, it suffices to check that if φ belongs to $\mathcal{E}' \cap \mathcal{M}\big(B^{s_1}_{p_1,r_1} \to B^{s_2}_{p_2,r_2}\big)$, then φ is also in the multiplier space between the general type Besov spaces. Take $u \in b^{s_1}_{p_1,r_1}$ with compact support, and some smooth and compactly supported nonnegative cut-off function ψ satisfying $\psi \equiv 1$ on $\operatorname{Supp}\varphi$. Then from Proposition 2.12 and (2.14), we have

$$
\begin{aligned}
\|\varphi u\|_{b^{s_2}_{p_2,r_2}} &= \|\varphi\psi u\|_{b^{s_2}_{p_2,r_2}} \lesssim \|\varphi\psi u\|_{B^{s_2}_{p_2,r_2}} \\
&\lesssim \|\varphi\|_{\mathcal{M}\left(B^{s_1}_{p_1,r_1}\to B^{s_2}_{p_2,r_2}\right)} \|\psi u\|_{B^{s_1}_{p_1,r_1}} \\
&\lesssim \|\varphi\|_{\mathcal{M}\left(B^{s_1}_{p_1,r_1}\to B^{s_2}_{p_2,r_2}\right)} \|\psi u\|_{b^{s_1}_{p_1,r_1}} \\
&\lesssim \|\varphi\|_{\mathcal{M}\left(B^{s_1}_{p_1,r_1}\to B^{s_2}_{p_2,r_2}\right)} \|\psi\|_{\mathcal{M}\left(b^{s_1}_{p_1,r_1}\right)} \|u\|_{b^{s_1}_{p_1,r_1}}.
\end{aligned}
$$

For the last inequality, we used $\mathcal{C}^\infty_c \hookrightarrow \mathcal{M}\big(b^{s_1}_{p_1,r_1}\big)$ (see [34, Corollary 2.1.1]). □

2.6 Commutator Estimates

We here recall and prove some commutator estimates that were crucial in this Chapter. All of them strongly rely on continuity results in Besov spaces for the paraproduct and remainder operators, and on the following classical result (see e.g. [6, Section 2.10]).

Lemma 2.15. *Let $A : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ be a smooth function, homogeneous of degree m. Let $(\varepsilon, s, p, r, r_1, r_2, p_1, p_2) \in]0,1[\times\mathbb{R}\times[1,\infty]^6$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ and*

$$
s - m + \varepsilon < \frac{N}{p} \quad \textit{or} \quad \Big\{ s - m + \varepsilon < \frac{N}{p} \ \textit{ and } \ r = 1 \Big\}\cdot
$$

There exists a constant C depending only on s, ε, N and A such that,

$$
\|[\dot{T}_g, A(D)]u\|_{\dot{B}^{s-m+\varepsilon}_{p,r}} \leq C\|\nabla g\|_{\dot{B}^{\varepsilon-1}_{p_1,r_1}} \|u\|_{\dot{B}^{s}_{p_2,r_2}}.
$$

If the integer N_0 in the definition of Bony's paraproduct and remainder is large enough (for instance $N_0 = 4$ does), then the following fundamental lemma holds.

Lemma 2.16 (Chemin-Leibniz Formula). *Let $(\varepsilon, s_k, p, p_k, p_3, r, r_k) \in]0,1[\times\mathbb{R}\times[1,\infty]^5$ for $k = 1, 2$ satisfying*

$$
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \quad \textit{and} \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}\cdot
$$

1. *If $s_2 < 0$ and $s_1 + s_2 + \varepsilon - 1 < \frac{N}{p}$ or $\{s_1 + s_2 + \varepsilon - 1 = \frac{N}{p}$ and $r = 1\}$, then we have*

$$
\|\dot{\mathcal{T}}_X \dot{T}_g f - \dot{T}_g \dot{\mathcal{T}}_X f - \dot{T}_{\dot{\mathcal{T}}_X g} f\|_{\dot{B}^{s_1+s_2+\varepsilon-1}_{p,r}} \leq C\|X\|_{\dot{B}^{\varepsilon}_{p_3,\infty}} \|g\|_{\dot{B}^{s_2}_{p_2,\infty}} \|f\|_{\dot{B}^{s_1}_{p_1,r}}.
$$

   The above inequality still holds for $s_2 = 0$, if one replaces $\|g\|_{\dot{B}^0_{p_2,\infty}}$ by $\|g\|_{L^{p_2}}$.

2. *If $s_1 + s_2 + \varepsilon - 1 \in]0, \frac{N}{p}[$ or $\{s_1 + s_2 + \varepsilon - 1 = \frac{N}{p}$ and $r = 1\}$, then we have*

$$
\|\dot{\mathcal{T}}_X \dot{R}(f, g) - \dot{R}(\dot{\mathcal{T}}_X f, g) - \dot{R}(f, \dot{\mathcal{T}}_X g)\|_{\dot{B}^{s_1+s_2+\varepsilon-1}_{p,r}} \leq C\|X\|_{\dot{B}^{\varepsilon}_{p_3,\infty}} \|f\|_{\dot{B}^{s_1}_{p_1,r_1}} \|g\|_{\dot{B}^{s_2}_{p_2,r_2}}.
$$

The above inequality still holds for $s_1+s_2+\varepsilon-1=0$, $r=\infty$ *and* $\frac{1}{r_1}+\frac{1}{r_2}=1$.

Proof. This is a mere adaptation of Lemma 1.14 to the homogeneous framework. The proof is based on a generalized Leibniz formula for para-vector field operators which was derived by J.-Y. Chemin in [17]. More precisely, define the following Fourier multipliers

$$\dot{\Delta}_{k,j} := \varphi_k(2^{-j}D) \qquad \text{with } \varphi_k(\xi) := i\xi_k\varphi(\xi) \ \text{ for } \ k\in\{1,\cdots,N\} \ \text{ and } \ j\in\mathbb{Z}.$$

Then we have

$$\begin{aligned}\dot{\mathcal{T}}_X\dot{T}_g f &= \sum_{j\in\mathbb{Z}}(\dot{S}_{j-N_0}g\dot{\mathcal{T}}_X\dot{\Delta}_j f+\dot{\Delta}_j f\dot{\mathcal{T}}_X\dot{S}_{j-N_0}g)+\sum_{j\in\mathbb{Z}}(\dot{T}_{1,j}+\dot{T}_{2,j})\\ &= \dot{T}_g\dot{\mathcal{T}}_X f+\dot{T}_{\dot{\mathcal{T}}_X g}f+\sum_{\substack{j\in\mathbb{Z}\\ \alpha=1,\dots,4}}\dot{T}_{\alpha,j},\end{aligned}$$

where

$$\begin{aligned}\dot{T}_{1,j} &:= \sum_{\substack{j\le j'\le j+1\\ j-N_0-1\le j''\le j'-N_0-1}} 2^{j'}\dot{\Delta}_{j''}X^k\big(\dot{\Delta}_{k,j'}(\dot{\Delta}_j f\dot{S}_{j-N_0}g)-\dot{\Delta}_{k,j'}\dot{\Delta}_j f\dot{S}_{j-N_0}g\big),\\ \dot{T}_{2,j} &:= \sum_{\substack{j'\le j-2\\ j'-N_0\le j''\le j-N_0-2}} 2^{j'}\dot{\Delta}_{j''}X^k(\dot{\Delta}_j f)\dot{\Delta}_{k,j'}\dot{S}_{j-N_0}g,\\ \dot{T}_{3,j} &:= \dot{S}_{j-N_0}g[\dot{T}_{X^k},\dot{\Delta}_j]\partial_k f,\\ \dot{T}_{4,j} &:= \dot{\Delta}_j f[\dot{T}_{X^k},\dot{S}_{j-N_0}]\partial_k g.\end{aligned}$$

Bounding $\dot{T}_{1,j}$ and $\dot{T}_{2,j}$ stems from the definition of Besov norms, and Lemmas 2.99, 2.100 of [6] allow to bound $\dot{T}_{3,j}$ and $\dot{T}_{4,j}$ provided $\varepsilon<1$.

In order to prove the second item, let us set

$$A_{j,j'} := \big\{j-N_0-1,\cdots,j'-N_0-1\big\}\cup\big\{j'-N_0,\cdots,j-N_0-2\big\}\cdot$$

We have

$$\begin{aligned}\dot{\mathcal{T}}_X\dot{R}(f,g) &= \sum_{j\in\mathbb{Z}}(\widetilde{\dot{\Delta}}_j g\dot{\mathcal{T}}_X\dot{\Delta}_j f+\dot{\Delta}_j f\dot{\mathcal{T}}_X\widetilde{\dot{\Delta}}_j g)+\sum_{j\in\mathbb{Z}}(\dot{R}_{1,j}+\dot{R}_{2,j})\\ &= \dot{R}(\dot{\mathcal{T}}_X f,g)+\dot{R}(f,\dot{\mathcal{T}}_X g)+\sum_{\substack{j\in\mathbb{Z}\\ \alpha=1,\dots,4}}\dot{R}_{\alpha,j},\end{aligned}$$

where, denoting $\widetilde{\dot{\Delta}}_j := \dot{\Delta}_{j-N_0}+\cdots+\dot{\Delta}_{j+N_0}$,

$$
\begin{aligned}
\dot R_{1,j} &:= \sum_{\substack{|j'-j|\leq N_0+1\\ j''\in A_{j,j'}}} \operatorname{sgn}(j'-j+1)2^{j'}\dot\Delta_{j''}X^k\big(\dot\Delta_{k,j'}(\dot\Delta_j f\widetilde{\dot\Delta}_j g)-\dot\Delta_j f\dot\Delta_{k,j'}\widetilde{\dot\Delta}_j g\big)\\
&\qquad\qquad\qquad\qquad + \sum_{\substack{j-1\leq j'\leq j\\ j'-N_0\leq j''\leq j-N_0}} 2^{j'}\dot\Delta_{j''}X^k(\dot\Delta_{k,j'}\dot\Delta_j f)\widetilde{\dot\Delta}_j g,\\
\dot R_{2,j} &:= \sum_{\substack{j'\leq j-N_0-2\\ j'-N_0\leq j''\leq j-N_0-2}} 2^{j'}\dot\Delta_{j''}X^k\dot\Delta_{k,j'}(\dot\Delta_j f\widetilde{\dot\Delta}_j g),\\
\dot R_{3,j} &:= \widetilde{\dot\Delta}_j g[\dot T_{X^k},\dot\Delta_j]\partial_k f,\\
\dot R_{4,j} &:= \dot\Delta_j f[\dot T_{X^k},\widetilde{\dot\Delta}_j]\partial_k g.
\end{aligned}
$$

Here again, bounding $\dot R_{1,j}$ and $\dot R_{2,j}$ follows from the definition of Besov norms, while Lemma 2.100 of [6] allows to bound $\dot R_{3,j}$ and $\dot R_{4,j}$. □

Proposition 2.17. *Let* (ε,p) *be in* $]0,1[\times[1,\infty]$. *Consider a couple of vector fields* (X,v) *in*

$$
\big(L^\infty_{loc}(\mathbb{R}_+;\dot{\mathscr{C}}^\varepsilon)\big)^N\times\big(L^\infty_{loc}(\mathbb{R}_+;\dot B^{\frac{N}{p}-1}_{p,1})\cap L^1_{loc}(\mathbb{R}_+;\dot B^{\frac{N}{p}+1}_{p,1})\big)^N,
$$

satisfying $\operatorname{div} v=0$ *and the transport equation*

$$
\begin{cases}
(\partial_t+v\cdot\nabla)X=\partial_X v,\\
X|_{t=0}=X_0.
\end{cases}\tag{2.72}
$$

If in addition

$$
\frac{N}{p}>2-\varepsilon,\quad \text{or}\quad \frac{N}{p}>1-\varepsilon\ \text{ and }\ \operatorname{div}X\equiv 0,\tag{2.73}
$$

then there exists a constant C *such that:*

$$
\begin{aligned}
\|[\dot{\mathcal{T}}_X,\partial_t+v\cdot\nabla]v\|_{\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}} &\leq C(\|X\|_{\dot{\mathscr{C}}^\varepsilon}\|v\|_{\dot B^{\frac{N}{p}+1}_{p,1}}\|v\|_{\dot B^{\frac{N}{p}-1}_{p,1}}\\
&\qquad + \|v\|_{\dot{\mathscr{C}}^{-1}}\|\dot{\mathcal{T}}_X v\|_{\dot B^{\frac{N}{p}+\varepsilon}_{p,1}} + \|v\|_{\dot B^{\frac{N}{p}+1}_{p,1}}\|\dot{\mathcal{T}}_X v\|_{\dot{\mathscr{C}}^{\varepsilon-2}}).
\end{aligned}\tag{2.74}
$$

Proof. This is essentially the proof of [40, Proposition A.5]. For the reader convenience, we here give a sketch of it. Because $\operatorname{div} v=0$, we may write

$$
\begin{aligned}
[\dot{\mathcal{T}}_X,\partial_t+v^\ell\partial_\ell]v &= -v^\ell\partial_\ell\dot T_{X^k}\partial_k v-\dot T_{\partial_t X^k}\partial_k v+\dot T_{X^k}\partial_k(v^\ell\partial_\ell v)\\
&= -\dot T_{\partial_t X^k}\partial_k v+\partial_\ell\dot{\mathcal{T}}_X(v^\ell v)-\dot{\mathcal{T}}_{\partial_\ell X}(v^\ell v)-v^\ell\partial_\ell\dot{\mathcal{T}}_X v.
\end{aligned}
$$

Hence, decomposing $v^\ell v$ according to Bony's decomposition, we discover that

$$[\dot{\mathcal{T}}_X, \partial_t + v^\ell \partial_\ell]v = \sum_{\alpha=1}^{\alpha=5} \dot{R}_\alpha$$

with

$$\begin{aligned}
&\dot{R}_1 := -\dot{T}_{\partial_t X^k} \partial_k v, && \dot{R}_2 := \partial_\ell(\dot{\mathcal{T}}_X \dot{T}_{v^\ell} v + \dot{\mathcal{T}}_X \dot{T}_v v^\ell),\\
&\dot{R}_3 := \partial_\ell \dot{\mathcal{T}}_X \dot{R}(v^\ell, v), && \dot{R}_4 := -\dot{\mathcal{T}}_{\partial_\ell X}(v^\ell v),\\
&\dot{R}_5 := -v^\ell \partial_\ell \dot{\mathcal{T}}_X v.
\end{aligned}$$

It suffices to check that all the terms $\dot{R}_\alpha$ may be bounded by the right-hand side of (2.74).

• **<u>Bound of $\dot{R}_1$</u>:**

From the equation (2.72), we have

$$\dot{R}_1 = \dot{T}_{v\cdot\nabla X^k} \partial_k v - \dot{T}_{\partial_X v^k} \partial_k v.$$

Hence using standard continuity results for the paraproduct, we deduce that

$$\|\dot{R}_1\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} \lesssim \|\nabla v\|_{\dot{B}^{\frac{N}{p}}_{p,1}} \big(\|v\cdot\nabla X\|_{\dot{\mathscr{C}}^{\varepsilon-2}} + \|\partial_X v\|_{\dot{\mathscr{C}}^{\varepsilon-2}}\big).$$

Keeping in mind (2.73), the last term may be bounded according to (2.24), after using the embedding $\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}(\mathbb{R}^N) \hookrightarrow \dot{\mathscr{C}}^{\varepsilon-2}(\mathbb{R}^N)$. We get

$$\|\partial_X v - \dot{\mathcal{T}}_X v\|_{\dot{\mathscr{C}}^{\varepsilon-2}} \lesssim \|\nabla v\|_{\dot{B}^{\frac{N}{p}-2}_{p,1}} \|X\|_{\dot{\mathscr{C}}^{\varepsilon}}.$$

As for the first term, we use the fact $\operatorname{div} v = 0$ and the following decomposition

$$v\cdot\nabla X = \dot{\mathcal{T}}_v X + \dot{T}_{\partial_\ell X} v^\ell + \partial_\ell \dot{R}(v^\ell, X),$$

which allow to get, as long as (2.73) holds

$$\|\dot{R}_1\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} \lesssim \|\nabla v\|_{\dot{B}^{\frac{N}{p}}_{p,1}} \big(\|v\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}} \|X\|_{\dot{\mathscr{C}}^{\varepsilon}} + \|\dot{\mathcal{T}}_X v\|_{\dot{\mathscr{C}}^{\varepsilon-2}}\big).$$

• **<u>Bound of $\dot{R}_2$</u>:**

Due to Lemma 2.16 (i) and continuity of paraproduct operator, we have

$$\|\dot{R}_2\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} \lesssim \|X\|_{\dot{\mathscr{C}}^{\varepsilon}} \|v\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}} \|v\|_{\dot{\mathscr{C}}^{-1}} + \|v\|_{\dot{\mathscr{C}}^{-1}} \|\dot{\mathcal{T}}_X v\|_{\dot{B}^{\frac{N}{p}+\varepsilon}_{p,1}} + \|v\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}} \|\dot{\mathcal{T}}_X v\|_{\dot{\mathscr{C}}^{\varepsilon-2}}.$$

• **Bound of $\dot{R}_3$:**

Applying Lemma 2.16 (ii) and continuity of remainder operator under the condition $\frac{N}{p}+\varepsilon-1>0$ yields

$$\|\dot{R}_3\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} \lesssim \|X\|_{\dot{\mathscr{C}}^{\varepsilon}}\|v\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}}\|v\|_{\dot{\mathscr{C}}^{-1}} + \|v\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}}\|\dot{\mathcal{T}}_X v\|_{\dot{\mathscr{C}}^{\varepsilon-2}}.$$

• **Bound of $\dot{R}_4$:**

From Bony decomposition (2.23), it is easy to get

$$\|v^l v\|_{\dot{B}^{\frac{N}{p}}_{p,1}} \lesssim \|v\|_{\dot{\mathscr{C}}^{-1}}\|v\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}}.$$

Hence

$$\|\dot{R}_4\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} \lesssim \|\nabla X\|_{\dot{\mathscr{C}}^{\varepsilon-1}}\|v\|_{\dot{\mathscr{C}}^{-1}}\|v\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}}.$$

• **Bound of $\dot{R}_5$:**

Applying Bony decomposition and using that $\operatorname{div} v=0$ and $\frac{N}{p}+\varepsilon>1$ give

$$\|\dot{R}_5\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} \lesssim \|v\|_{\dot{\mathscr{C}}^{-1}}\|\dot{\mathcal{T}}_X v\|_{\dot{B}^{\frac{N}{p}+\varepsilon}_{p,1}} + \|v\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}}\|\dot{\mathcal{T}}_X v\|_{\dot{\mathscr{C}}^{\varepsilon-2}}.$$

Combining the above estimates for all $\dot{R}_\alpha$, with $\alpha=1,\dots,5$ yields (2.74). □

Another consequence of Lemma 2.16 is the following estimate of $\operatorname{div}(Xfg)$:

Proposition 2.18. *Let (s,p,r) be in $]0,1[\times[1,\infty]^2$ and η in $]0,1-s[$. Consider a bounded vector field X and two bounded functions f,g satisfying*

$$X\in\big(\dot{B}^s_{p,r}(\mathbb{R}^N)\cap\mathscr{C}^{s+\eta}\big)^N,\ \ (f,g)\in\dot{B}^s_{p,r}(\mathbb{R}^N)\times\dot{B}^{-\eta}_{p,r}(\mathbb{R}^N)\ \ \textit{and}\ \ \partial_X g\in\dot{B}^{s-1}_{p,r}(\mathbb{R}^N).$$

If in addition $\operatorname{div} X$ *belongs to* $\mathcal{M}\big(\dot{B}^s_{p,r}(\mathbb{R}^N)\to\dot{B}^{s-1}_{p,r}(\mathbb{R}^N)\big)$, *and there exists some* $q\in[1,p[$ *such that*

$$\operatorname{div} X\in\dot{B}^{s_{p,q}}_{q,r}(\mathbb{R}^N)\ \ \textit{with}\ \ s_{p,q}:=s-1+N\Big(\frac{1}{q}-\frac{1}{p}\Big)>0, \tag{2.75}$$

then we have $\operatorname{div}(Xfg)\in\dot{B}^{s-1}_{p,r}(\mathbb{R}^N)$, *and the following estimate holds true:*

$$\begin{aligned}\|\operatorname{div}(Xfg)\|_{\dot{B}^{s-1}_{p,r}} &\lesssim \|X\|_{\dot{B}^s_{p,r}\cap\mathscr{C}^{s+\eta}}\|f\|_{L^\infty\cap\dot{B}^s_{p,r}}\|g\|_{L^\infty\cap\dot{B}^{-\eta}_{p,r}} + \|f\|_{L^\infty}\|\partial_X g\|_{\dot{B}^{s-1}_{p,r}}\\ &\quad + \|\operatorname{div} X\|_{\dot{B}^{s_{p,q}}_{q,r}\cap\mathcal{M}(\dot{B}^s_{p,r}\to\dot{B}^{s-1}_{p,r})}\|g\|_{L^\infty}\|f\|_{\dot{B}^s_{p,r}\cap L^\infty}.\end{aligned}$$

Proof. In light of Bony's decomposition (2.23), and denoting $\dot{T}'_g f:=\dot{T}_g f+\dot{R}(f,g)$, we can decompose $\operatorname{div}(Xfg)$ into

$$\operatorname{div}(Xfg) = \operatorname{div}\big(\dot{T}'_{fg}X + \dot{T}_X(fg)\big) = \sum_{\alpha=1}^{4} \dot{F}_\alpha,$$

where

$$\dot{F}_1 := \operatorname{div}(\dot{T}'_{fg}X), \qquad \dot{F}_2 := \dot{T}_{\operatorname{div} X}(fg),$$
$$\dot{F}_3 := \dot{\mathcal{T}}_X \dot{T}'_g f, \qquad \dot{F}_4 := \dot{\mathcal{T}}_X \dot{T}_f g.$$

• **Bound of $\dot{F}_1$:**

As $s > 0$, standard continuity results for $\dot{T}$ and $\dot{R}$ yield

$$\|\dot{F}_1\|_{\dot{B}^{s-1}_{p,r}} \lesssim \|\dot{T}'_{fg}X\|_{\dot{B}^{s}_{p,r}} \lesssim \|f\|_{L^\infty}\|g\|_{L^\infty}\|X\|_{\dot{B}^{s}_{p,r}}.$$

• **Bound of $\dot{F}_2$:**

Thanks to continuity results for $\dot{T}$, we have for $s < 1$,

$$\|\dot{F}_2\|_{\dot{B}^{s-1}_{p,r}} \lesssim \|\operatorname{div} X\|_{\dot{B}^{s-1}_{p,r}}\|f\|_{L^\infty}\|g\|_{L^\infty}.$$

• **Bound of $\dot{F}_3$:**

Because X and g are bounded and $s > 0$, we readily have

$$\|\dot{F}_3\|_{\dot{B}^{s-1}_{p,r}} \lesssim \|X\|_{L^\infty}\|\dot{T}'_g f\|_{\dot{B}^{s}_{p,r}} \lesssim \|X\|_{L^\infty}\|g\|_{L^\infty}\|f\|_{\dot{B}^{s}_{p,r}}.$$

• **Bound of $\dot{F}_4$:**

Because $0 < s < s+\eta < 1$, Lemma 2.16 and continuity results for the paraproduct $\dot{T}$ imply that

$$\begin{aligned}
\|\dot{\mathcal{T}}_X \dot{T}_f g\|_{\dot{B}^{s-1}_{p,r}} &\lesssim \|X\|_{\dot{\mathscr{C}}^{s+\eta}}\|f\|_{L^\infty}\|g\|_{\dot{B}^{-\eta}_{p,r}} + \|\dot{T}_f \dot{\mathcal{T}}_X g\|_{\dot{B}^{s-1}_{p,r}} + \|\dot{T}_{\dot{\mathcal{T}}_X f} g\|_{\dot{B}^{s-1}_{p,r}}\\
&\lesssim \|X\|_{\dot{\mathscr{C}}^{s+\eta}}\|f\|_{L^\infty}\|g\|_{\dot{B}^{-\eta}_{p,r}} + \|f\|_{L^\infty}\|\dot{\mathcal{T}}_X g\|_{\dot{B}^{s-1}_{p,r}} + \|g\|_{L^\infty}\|\dot{\mathcal{T}}_X f\|_{\dot{B}^{s-1}_{p,r}}.
\end{aligned}$$

To bound the last term, one may use the decomposition

$$\dot{\mathcal{T}}_X f = \operatorname{div}(\dot{T}_X f) - f \operatorname{div} X + \dot{T}_f \operatorname{div} X + \dot{R}(f, \operatorname{div} X).$$

Hence using continuity results for $\dot{R}$ and $\dot{T}$ and the fact that $(s_{p,q}, q)$ satisfies (2.75),

$$\|\dot{\mathcal{T}}_X f\|_{\dot{B}^{s-1}_{p,r}} \lesssim \|f\|_{\dot{B}^{s}_{p,r}}\big(\|X\|_{L^\infty} + \|\operatorname{div} X\|_{\mathcal{M}(\dot{B}^{s}_{p,r}\to\dot{B}^{s-1}_{p,r})}\big) + \|f\|_{L^\infty}\|\operatorname{div} X\|_{\dot{B}^{s_{p,q}}_{q,r}}.$$

Finally, to bound the term with $\dot{\mathcal{T}}_X g$, we use the fact that

$$\partial_X g - \dot{\mathcal{T}}_X g = \dot{T}_{\nabla g} \cdot X + \operatorname{div} \dot{R}(X, g) - \dot{R}(\operatorname{div} X, g),$$

whence

$$\|\partial_X g - \dot{\mathcal{T}}_X g\|_{\dot{B}^{s-1}_{p,r}} \lesssim \|g\|_{L^\infty} \big(\|X\|_{\dot{B}^s_{p,r}} + \|\operatorname{div} X\|_{\dot{B}^{s_{p,q}}_{q,r}}\big). \tag{2.76}$$

This completes the proof of the proposition. □

Proposition 2.18 above reveals that the bounded function g may behave like some element in $\mathcal{M}(\dot{B}^{s-1}_{p,\infty})$ under a suitable additional structure assumption. If in addition g has compact support, then one can relax a bit the regularity of X and f to study $\partial_X(fg)$, and get the following generalization of [28, Lemma A.6].

Corollary 2.19. *Consider a divergence-free vector field X with coefficients in $\mathscr{C}^\varepsilon$, and some function f in $\mathscr{C}^{\varepsilon'}$ with $0 < \varepsilon, \varepsilon' < 1$. Let $g \in L^\infty$ be compactly supported and satisfy $\partial_X g \in \dot{B}^{\alpha-1}_{p,r}$ for some $(p, r) \in [1, \infty]^2$ and $\alpha \in]0, \min\{\varepsilon, \varepsilon'\}[$. Then $\operatorname{div}(Xfg) = \partial_X(fg) \in \dot{B}^{\alpha-1}_{p,r}$.*

Proof. Let $\psi \in \mathcal{C}^\infty_c$ be a cut-off function such that $\psi \equiv 1$ near $\operatorname{Supp} g$. Denote $(\widetilde{X}, \widetilde{f}) := (\psi X, \psi f)$. From Proposition 2.12 and the proof of Proposition 2.13, we know that

$$(\widetilde{X}, \widetilde{f}, g) \in (B^\varepsilon_{q,\infty})^N \times B^{\varepsilon'}_{q,\infty} \times B^{-\eta}_{q,r} \hookrightarrow (\dot{B}^\alpha_{q,1} \cap L^\infty)^{N+1} \times \dot{B}^{-\eta}_{q,r},$$

for any $q \in [1, \infty]$ and some $\eta \in]0, \min\{N/q', \varepsilon - \alpha\}[$. It is also clear that $\partial_X(fg) = \operatorname{div}(\widetilde{X}\widetilde{f}g)$ and $\partial_{\widetilde{X}} g = \partial_X g$. Hence applying Proposition 2.18 gives the result. □

Chapter 3

Two-phase inhomogeneous flow

3.1 Introduction

3.1.1 Description of the problem

In this chapter, we consider the following Cauchy problem in $\mathbb{R}^N$ $(N \geq 2)$,

$$\begin{cases} \partial_t(\rho \boldsymbol{v}) + \mathrm{Div}\,(\rho \boldsymbol{v} \otimes \boldsymbol{v}) - \mathrm{Div}\,\mathbb{T}(\boldsymbol{v}, \mathfrak{p}) = \rho \boldsymbol{f} & \text{in} \quad \dot{\Omega}_t, \\ \partial_t \rho + \mathrm{div}\,(\rho \boldsymbol{v}) = 0, \quad \mathrm{div}\, \boldsymbol{v} = 0 & \text{in} \quad \dot{\Omega}_t, \\ [\![\mathbb{T}(\boldsymbol{v}, \mathfrak{p}) \boldsymbol{n}_t]\!] = \boldsymbol{0}, \quad [\![\boldsymbol{v}]\!] = \boldsymbol{0} & \text{in} \quad \Gamma_t, \\ \mathbb{T}(\boldsymbol{v}_+, \mathfrak{p}_+) \boldsymbol{n}_{+,t} = \boldsymbol{0} & \text{on} \quad \Gamma_{+,t}, \\ \boldsymbol{v}_- = \boldsymbol{0} & \text{on} \quad \Gamma_-, \\ (\rho, \boldsymbol{v})|_{t=0} = (\rho_0, \boldsymbol{v}_0) & \text{on} \quad \dot{\Omega}, \end{cases} \tag{$INS_\pm$}$$

which describes the motion of two immiscible viscous incompressible liquids at time instant t in some domain $\Omega_t := \dot{\Omega}_t \cup \Gamma_t := \Omega_{+,t} \cup \Omega_{-,t} \cup \Gamma_t$ surrounded by time-dependent surface $\Gamma_{+,t}$ and fixed boundary Γ_- without taking surface tension into account. For simplicity, we use the notations $\Omega := \dot{\Omega} \cup \Gamma := \Omega_\pm \cup \Gamma \subset \mathbb{R}^N$ for the initial domain with boundaries $\Gamma_\pm$ by dropping off the subscript $t = 0$.

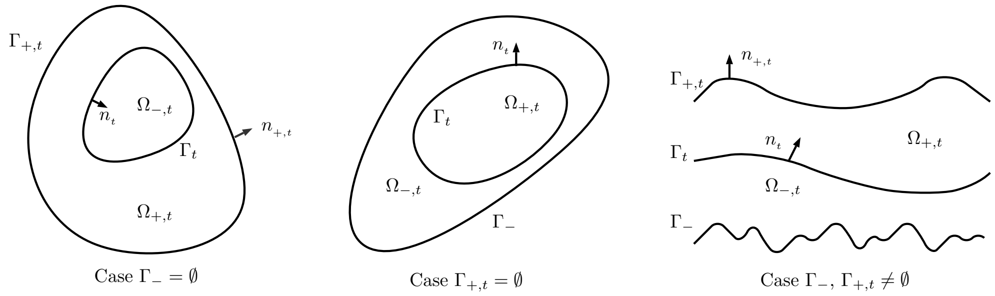
In fact there are three typical physical situations characterized by ($INS_\pm$) (see the figure below) but we only concentrate on (Ω_1):

(Ω_1): Ω_t is some bounded domain surrounded by the free surface $\Gamma_{+,t}$ with $\Gamma_- \equiv \emptyset$;

(Ω_2): Ω_t is some bounded container with solid compact boundary Γ_- by assuming $\Gamma_{+,t} \equiv \emptyset$;

(Ω_3): $\Omega_\pm$ are two infinite layers with some rigid bottom Γ_-.

Besides, $\boldsymbol{n}_t$ and $\boldsymbol{n}_{+,t}$ are outwards unit normals subject to the moving interface Γ_t between two bulks $\Omega_{\pm,t}$ and the free surface $\Gamma_{+,t}$ respectively at time instant t.



With above settings on domains $\dot{\Omega}_t$, our aim is to study the solvability of ($INS_\pm$), that is to determine the unknowns $(\rho, \boldsymbol{v}, \mathfrak{p}, \dot{\Omega}_t)$: the density, the velocity field, the pressure and the domain with free surfaces, whenever the external force $\boldsymbol{f}$ and initial states $(\rho_0, \boldsymbol{v}_0)$ are given. In addition, the standard stress tensor $\mathbb{T}(\boldsymbol{v}, \mathfrak{q})$ is defined by

$$\mathbb{T}(\boldsymbol{v}, \mathfrak{q})(x,t) := \mu\big(\rho(x,t)\big)\mathbb{D}(\boldsymbol{v})(x,t) - \mathfrak{q}(x,t)\mathbb{I},$$

and the deformation tensor $\mathbb{D}(\boldsymbol{v})/2$ is given by

$$\mathbb{D}(\boldsymbol{v}) = D_x\boldsymbol{v} + \nabla_x\boldsymbol{v}, \quad \text{with} \quad (D_x\boldsymbol{v})^j_k \equiv (\nabla_x\boldsymbol{v})^k_j := \partial_{x_k} v^j \quad \text{for} \quad j,k = 1, ..., N.$$

In System ($INS_\pm$), we also adopt the following standard notations. For any two vectors $\boldsymbol{u}, \boldsymbol{v}$ in $\mathbb{R}^N$, the tensor product $\boldsymbol{u} \otimes \boldsymbol{v}$ stands for a $N \times N$ matrix with the (j,k)-entry $(\boldsymbol{u} \otimes \boldsymbol{v})^j_k := u^j v^k$ $(1 \leq j,k \leq N)$. Additionally, if $\mathbb{A} = \big(A^j_k(x)\big)_{N\times N}$ is any $N \times N$ matrix defined in $\mathbb{R}^N$, then $\operatorname{Div}\mathbb{A}$ means a vector in $\mathbb{R}^N$ satisfying $(\operatorname{Div}\mathbb{A})^j := \sum_{k=1}^N \partial_{x_k} A^j_k$. Lastly, the jump of the vector $\boldsymbol{g}$ across some surface $\mathcal{S}$ is given by the non-tangential limit

$$[\![\boldsymbol{g}]\!](x_0) := \lim_{\delta\to 0+} \Big(\boldsymbol{g}\big(x_0 + \delta\boldsymbol{\nu}(x_0)\big) - \boldsymbol{g}\big(x_0 - \delta\boldsymbol{\nu}(x_0)\big)\Big) \quad \forall\ x_0 \in \mathcal{S},$$

where $\boldsymbol{\nu}$ is the unit outwards normal along the surface $\mathcal{S}$.

Our main aim here is to investigate the solution of the two-phase model ($INS_\pm$) within $L_p - L_q$ maximal regularity, but one may modify our proof for two-phase problem to handle several classical one-phase cases. More precisely, assuming $\Gamma_t \equiv \emptyset$ and $\rho_0 \equiv 1$, let Ω satisfy either of the following physical settings:

(Ω_4): Ω is some bounded container with assuming $\Gamma_- = \emptyset$;

(Ω_5): Ω is some infinite layer with some rigid bottom Γ_-.

Before describing the main results in this context, let us recall the history of free boundary problem on the motion of viscous liquid for the cases (Ω_2)- (Ω_5). The first breakthrough is [113] by V.A. Solonnikov for the case (Ω_4), where the author first came up with Lagrangian coordinates approach and studied classical solutions with Hölder continuity. Indeed, V.A. Solonnikov in [113] succeeded in establishing the short time existence of a unique solution in some bounded domain Ω_t with free surface $\Gamma_{+,t}$, as long as the given data satisfy

$$\boldsymbol{v}_0 \in \mathcal{C}^{2,\varepsilon}(\Omega),\ \Gamma_+ \in \mathcal{C}^{2,\varepsilon} \text{ and } \boldsymbol{f}, \nabla \boldsymbol{f} \in (\mathcal{C}_t^{\varepsilon/2} \cap \mathcal{C}_x^{\varepsilon})(\mathbb{R}^3 \times]0,T[)$$

for some $0 < \varepsilon < 1$ and some $T > 0$. Later, V.A. Solonnikov in [115] investigated the global solvability in the Sobolev space framework where the author assumed that Γ_+ is $W_p^{2-\frac{1}{p}}$ regular for some $p > N$. Compared with [113, 115], V.A. Solonnikov studied the role of of the surface tension in [114, 116]. Undoubtedly, these early works [113–116] by V.A.Solonnikov on free boundary problems of the viscous flow have a profound impact on the further improvements.

Later, J.T.Beale considered the unbounded layer (Ω_5) filled with one-phase fluid, and he studied the (local and global) wellposedness issues in [7] without taking surface tension into account, and in [8] with surface tension involved. The author of [7, 8] used the L^2 framework, where the initial state $\boldsymbol{v}_0$ lies in $W_2^s(\Omega)$ for some $s \in]2, 5/2[$ and satisfies some suitable compatibility boundary conditions. Of course, the global result in [8] is about the stability near the equilibrium state. Furthermore, A.Tani in [121] and A.Tani and N.Tanaka in [122] formulated the problem for (Ω_5) in fractional Sobolev-Slobodetskiĭ spaces and treated more carefully the regularity of moving surface $\Gamma_{+,t}$. More recently, Y.Guo and I.Tice gave a series of works [63–65] about the wellposedness issues and decay property of (Ω_5) based on a new energy method.

Besides, for the bounded domain occupied by two-phase inhomogeneous immiscible liquids in case of (Ω_2), N.Tanaka in [119, 120] proved that the equilibrium state is stable in L_2 framework with including surface tension. Inspired by [119, 120], L.Xu and Z.Zhang in [127] studied the double-layer case (Ω_3) with gravity additionally involved. However, for the piecewise constant density, the work [42] by I.V.Denisova and her further result [44] with V.A.Solonnikov implied the global solvability in Hölder space with or without surface for the domain (Ω_2).

Apart from the L_2 or Hölder framework, let us mention recent contributions [78, 100–103] to the L_p approach for two-phase problems by J. Prüss and his collaborators, especially for the case with surface tension. For instance, the authors in [102] showed that the interface between two immiscible liquids becomes instantaneously real analytic whenever the initial data lie in some Sobolev spaces W_p^s for large enough p. The L_p-approach by J. Prüss et al. is no longer based on the Lagrangian coordinate from V.A. Solonnikov. Instead, the so-called *Direct Mapping Method* via a *Hanzawa* transform plays a vital role.

Whether we can solve the two-phase problem within $L_p - L_q$ maximal regularity is the main task of this chapter, which will rely on the recent contributions [94, 107] to the linearized model problem. In [94, 107], the authors employed the Multiplier Theorem characterized by $\mathcal{R}$-*boundedness theory* in [126] to study the corresponding resolvent problems. For $\mathcal{R}$-boundedness theory, one may see [45, 78] for more discussions.

3.1.2 Main results

To state our main results in this context concerning the solution of $(INS_{\pm}^{\Omega})$ with $L_p - L_q$ maximal regularity, we need to specify our assumptions on the domain Ω and on the viscosity μ as in [94]. First, the smoothness of domain Ω we need here reads as follows.

Definition 3.1. *We say that a connected open subset Ω in $\mathbb{R}^N$ ($N \geq 2$) is of class $W_r^{2-1/r}$ for some $1 < r < \infty$, if and only if for any point $x_0 \in \partial\Omega$, one can choose a Cartesian coordinate system with origin x_0 (up to some translation and rotation) and coordinates $y = (y', y_N) := (y_1, ..., y_{N-1}, y_N)$, as well as positive constants α, β, K and some $W_r^{2-1/r}$ function h satisfying $\|h\|_{W_r^{2-1/r}} \leq K$ such that the neighborhood of x_0*

$$U_{\alpha,\beta,h}(x_0) := \{(y', y_N) : h(y') - \beta < y_N < h(y') + \beta, |y'| < \alpha\}$$

satisfies

$$U^-_{\alpha,\beta,h}(x_0) := \{(y', y_N) : h(y') - \beta < y_N < h(y'), |y'| < \alpha\} = \Omega \cap U_{\alpha,\beta,h}(x_0),$$

and

$$\partial\Omega \cap U_{\alpha,\beta,h}(x_0) = \{(y', y_N) : y_N = h(y'), |y'| < \alpha\}.$$

Above α, β, K, h may vary with respect to the different location on the boundary. Whenever the choices of α, β, K are independent of the location of x_0, Ω is called uniform $W_r^{2-1/r}$ domain. Note that if the boundary $\partial\Omega$ is compact, then the uniformness is satisfied automatically. Sometimes Ω is just called $W_r^{2-1/r}$ regular for simplicity.

Apart from the $W_r^{2-1/r}$ regularity on domain Ω, we need the unique solvability of the so-called *weak elliptic transmission problem.* To be more precise, some notations are introduced below. Here, consider some domain $\Omega := \dot{\Omega} \cup \Gamma := \Omega_{\pm} \cup \Gamma$ in $\mathbb{R}^N$ surrounded by two disjoint surfaces $\Gamma_{\pm}$, and suppose that $\Gamma_+ \neq \emptyset$. Then the Banach space $X^1_{q,\Gamma_+}(\Omega)$ for any $1 < q < \infty$ with the word $X \in \{W, \widehat{W}\}$ is defined as below [1],

$$X^1_{q,\Gamma_+}(\Omega) := \{f \in X^1_q(\Omega) : f \equiv 0 \text{ on } \Gamma_+\} \quad \text{and} \quad \|f\|_{X^1_{q,\Gamma_+}(\Omega)} := \|f\|_{X^1_q(\Omega)}.$$

Above the nonhomogeneous and homogeneous Sobolev spaces $W^1_q(\Omega)$ and $\widehat{W}^1_q(\Omega)$ are standard:

$$\begin{aligned} W^1_q(\Omega) &:= \{f \in L_q(\Omega) : \|f\|_{W^1_q(\Omega)} := \|f\|_{L_q(\Omega)} + \|\nabla f\|_{L_q(\Omega)} < \infty\}, \\ \widehat{W}^1_q(\Omega) &:= \{f \in L_{q,loc}(\Omega) : \|f\|_{\widehat{W}^1_q(\Omega)} := \|\nabla f\|_{L_q(\Omega)} < \infty\}. \end{aligned}$$

Definition 3.2. *Consider some domain Ω as above with $\Gamma_+ \neq \emptyset$. Assume that $\mathcal{W}^1_q(\Omega)$ is a closed subspace $\widehat{W}^1_{q,\Gamma_+}(\Omega)$ for some $1 < q < \infty$ and that $W^1_{q,\Gamma_+}(\Omega)$ is dense in $\mathcal{W}^1_q(\Omega)$. Suppose that the*

[1] If $\Gamma_+ = \emptyset$, then set $X^1_{q,\Gamma_+}(\Omega) := X^1_q(\Omega)$.

step function $\eta := \eta_+\mathbb{1}_{\Omega_+} + \eta_-\mathbb{1}_{\Omega_-}$ for some strictly positive constants $\eta_\pm$. Then we say that the weak elliptic transmission problem is uniquely solvable on $\mathcal{W}^1_q(\Omega)$ for $\eta_\pm$ if the following assertions hold true: For any $\boldsymbol{f} \in L_q(\Omega)^N$, there is a unique $\theta \in \mathcal{W}^1_q(\Omega)$ satisfying variational equations [2] *as below,*

$$(\eta^{-1}\nabla\theta, \nabla\varphi)_{\dot{\Omega}} = (\boldsymbol{f}, \nabla\varphi)_\Omega, \quad \text{for all } \varphi \in \mathcal{W}^1_{q'}(\Omega).$$

Moreover, there exists a constant C independent on the choices of θ, φ and $\boldsymbol{f}$ such that

$$\|\nabla\theta\|_{L_q(\Omega)} \leq C\|\boldsymbol{f}\|_{L_q(\Omega)}.$$

Remark 3.3. *For some $1 < q < \infty$, we write*

$$W^1_q(\dot{\Omega}) + \mathcal{W}^1_q(\Omega) := \{\theta_1 + \theta_2 : \theta_1 \in W^1_q(\dot{\Omega}) \text{ and } \theta_2 \in \mathcal{W}^1_q(\Omega)\}.$$

Suppose that the week elliptic transmission problem is uniquely solvable on $\mathcal{W}^1_q(\Omega)$ for $\eta_\pm$ and Ω as in Definition 3.2. Then for any $(\boldsymbol{\alpha}, \beta, \gamma) \in L_q(\dot{\Omega})^N \times W^{1-1/q}_q(\Gamma) \times W^{1-1/q}_q(\Gamma_+)$ and any test function $\varphi \in \mathcal{W}^1_{q'}(\Omega)$, there exists a unique $\theta \in W^1_q(\dot{\Omega}) + \mathcal{W}^1_q(\Omega)$ satisfying

$$(\eta^{-1}\nabla\theta, \nabla\varphi)_{\dot{\Omega}} = (\boldsymbol{\alpha}, \nabla\varphi)_{\dot{\Omega}}, \quad [\![\theta]\!] = \beta \text{ on } \Gamma \quad \text{and} \quad \theta = \gamma \text{ on } \Gamma_+.$$

In addition, there is a positive constant C independent on the choices of $\boldsymbol{\alpha}$, β, γ and φ such that,

$$\|\nabla\theta\|_{L_q(\dot{\Omega})} \leq C\Big(\|\boldsymbol{\alpha}\|_{L_q(\dot{\Omega})} + \|\beta\|_{W_q^{1-\frac{1}{q}}(\Gamma)} + \|\gamma\|_{W_q^{1-\frac{1}{q}}(\Gamma_+)}\Big).$$

For brevity, we write $\theta := \mathcal{K}(\boldsymbol{\alpha}, \beta, \gamma)$, which satisfies above properties.

With above definitions, we will make the following three hypotheses to investigate System ($INS_\pm$):

($\mathcal{H}1$) The domain $\dot{\Omega}$ is uniformly $W^{2-1/r}_r$ regular for some $r > N$, i.e. $\Omega_\pm$ are uniform $W^{2-1/r}_r$ domains;

($\mathcal{H}2$) The weak Elliptic transmission problem is uniquely solvable on $\mathcal{W}^1_q(\Omega)$ and $\mathcal{W}^1_{q'}(\Omega)$ for $\eta = \eta_+\mathbb{1}_{\Omega_+} + \eta_-\mathbb{1}_{\Omega_-}$ and $q \in]1, \infty[$.

($\mathcal{H}3$) $\mu\big(\rho_0(x)\big)$ is a strictly positive function on $\dot{\Omega}$ satisfying

$$\widetilde{\mu}_+\mathbb{1}_{\Omega_+} + \widetilde{\mu}_-\mathbb{1}_{\Omega_-} \leq \mu\big(\rho_0(\cdot)\big) \leq \bar{\mu}_+\mathbb{1}_{\Omega_+} + \bar{\mu}_-\mathbb{1}_{\Omega_-},$$

where $\widetilde{\mu}_\pm$ and $\bar{\mu}_\pm$ are all strictly positive constants. In addition, we assume that $\mu(\cdot) \in W^1_{r,loc}(\mathbb{R}_+)$ and r is given as in ($\mathcal{H}1$).

[2] Here for any vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ defined in $G \subset \mathbb{R}^N$, $(\boldsymbol{u}, \boldsymbol{v})_G := \int_G \boldsymbol{u} \cdot \boldsymbol{v}\, dx = \sum_{j=1}^N \int_G u^j v^j\, dx$.

Motivated by the pioneering work [113] by Solonnikov, we shall take advantage of the so-called Lagrangian coordinates to study System ($INS_{\pm}$) under the previous fundamental assumptions $(\mathcal{H}_1)$-$(\mathcal{H}_3)$. Thus the fact that the surfaces Γ_t, $\Gamma_{+,t}$ and Γ_- consist of the exactly same fluid particles at all time instants t, is taken for granted. Indeed, if we denote $\boldsymbol{u}(\xi,t) := \boldsymbol{v}\big(\boldsymbol{X}_u(\xi,t),t\big)$ and consider the following transformation

$$\boldsymbol{X}_u(\xi,t) := \xi + \int_0^t \boldsymbol{u}(\xi,\tau)\,d\tau \quad \text{for all } \ \xi \in \Omega \cup \Gamma_-, \tag{3.1}$$

then $(\Gamma_t,\Gamma_{+,t},\Gamma_-) \equiv \boldsymbol{X}_u\big((\Gamma,\Gamma_+,\Gamma_-),t\big)$. Of course, the unknown regions $\Omega_{\pm,t}$ at time instant t are also the image of $\Omega_{\pm}$ respectively under $\boldsymbol{X}_u(\cdot,t)$.

To rewrite System ($INS_{\pm}$) by (3.1), we adopt the similar notations as in [113] here and subsequently:

- For any $\mathcal{C}^1$ vector $\boldsymbol{Y}(\xi)$ defined in $\dot{\Omega}$, we write $D_\xi \boldsymbol{Y}$ for the Jacobian matrix of $\boldsymbol{Y}$, i.e. $(D_\xi \boldsymbol{Y})^j_k := \partial_{\xi_k} Y^j$ with $1 \le j,k \le N$, and $\nabla_\xi \boldsymbol{Y} := (D_\xi \boldsymbol{Y})^{tr}$.
- For simplicity, $\mathscr{A}_u$ stands for the cofactor matrix of $D_\xi \boldsymbol{X}_u$. Moreover, we use the following notations for the derivatives and stress tensors related to (3.1),

$$\nabla_u := \mathscr{A}_u \nabla_\xi, \ \ \mathrm{div}_{\,u} = \mathrm{Div}_{\,u} := \nabla_u \cdot \ \text{ and } \ \mathbb{T}_u(\boldsymbol{w},\mathfrak{q}) := \mu\big(\rho_0(\xi)\big)\mathbb{D}_u(\boldsymbol{w}) - \mathfrak{q}\mathbb{I},$$

  where the entry of the matrix $\mathbb{D}_u(\boldsymbol{u})$

$$\mathbb{D}_u(\boldsymbol{w})^j_k := (D_\xi \boldsymbol{w} \cdot \mathscr{A}_u^{tr} + \mathscr{A}_u \cdot \nabla_\xi \boldsymbol{w})^j_k = \sum_{\ell=1}^N \Big((\mathscr{A}_u)^k_\ell \frac{\partial w^j}{\partial \xi_\ell} + (\mathscr{A}_u)^j_\ell \frac{\partial w^k}{\partial \xi_\ell} \Big).$$

- Suppose that $\boldsymbol{n}$ and $\boldsymbol{n}_+$ are the unit normal for the interface Γ and the boundary Γ_+ respectively.

$$(\overline{\boldsymbol{n}},\overline{\boldsymbol{n}}_+)(\xi,t) := (\boldsymbol{n}_t,\boldsymbol{n}_{+,t})\big(\boldsymbol{X}_u(\xi,t)\big) = \Big(\frac{\mathscr{A}_u \boldsymbol{n}}{|\mathscr{A}_u \boldsymbol{n}|}, \frac{\mathscr{A}_{u_+}\boldsymbol{n}_+}{|\mathscr{A}_{u_+}\boldsymbol{n}_+|}\Big)(\xi,t), \quad \forall \xi \in \Gamma \cup \Gamma_+.$$

- For any vector $\boldsymbol{\nu}$ and $\boldsymbol{h}$ defined along some surface S, we define the operator

$$\mathcal{T}_{\boldsymbol{\nu}} \boldsymbol{h} := \boldsymbol{h} - (\boldsymbol{h}\cdot\boldsymbol{\nu})\boldsymbol{\nu},$$

  which is a projection into the hypersurface orthogonal to $\boldsymbol{\nu}$.

Now, note that the mass conservation law in ($INS_{\pm}$) is reduced to $\rho\big(\boldsymbol{X}_u(\xi,t),t\big) \equiv \rho_0(\xi)$. Then introduce $\mathfrak{q}(\xi,t) := \mathfrak{p}\big(\boldsymbol{X}_u(\xi,t),t\big)$. Thanks to ($INS_{\pm}$), it is not hard to verify that $(\boldsymbol{u},\mathfrak{q})$ satisfies the following equations

$$
\begin{cases}
\rho_0\partial_t \boldsymbol{u} - \mathrm{Div}_{\,u}\mathbb{T}_u(\boldsymbol{u},\mathfrak{q}) = \rho_0 \boldsymbol{f}\big(\boldsymbol{X}_u(\xi,t),t\big) & \text{in} \quad \dot{\Omega}\times]0,T[,\\
\mathrm{div}_{\,u}\boldsymbol{u} = 0 & \text{in} \quad \dot{\Omega}\times]0,T[,\\
[\![\mathbb{T}_u(\boldsymbol{u},\mathfrak{q})\overline{\boldsymbol{n}}]\!] = [\![\boldsymbol{u}]\!] = \boldsymbol{0} & \text{in} \quad \Gamma\times]0,T[,\\
\mathbb{T}_{u_+}(\boldsymbol{u}_+,\mathfrak{q}_+)\overline{\boldsymbol{n}}_+ = \boldsymbol{0} & \text{in} \quad \Gamma_+\times]0,T[,\\
\boldsymbol{u}_- = \boldsymbol{0} & \text{on} \quad \Gamma_-,\\
\boldsymbol{u}|_{t=0} = \boldsymbol{v}_0 & \text{on} \quad \dot{\Omega}.
\end{cases}
\qquad (INS^{\mathfrak{L}}_{\pm})
$$

For $(INS^{\mathfrak{L}}_{\pm})_2$, we remark that $\mathscr{A}_u = (\nabla_\xi \boldsymbol{X}_u)^{-1}$ due to the fact $\det(D_\xi \boldsymbol{X}_u) \equiv 1$ and *Liouville Theorem.*

To give a suitable space for the initial data $\boldsymbol{v}_0$, let us recall the linear mapping $\mathcal{K}$ in Remark 3.3. For any $1 < q < \infty$ and any vector $\boldsymbol{u} \in W^2_q(\dot{\Omega})^N$, we consider

$$
\begin{aligned}
\boldsymbol{\alpha}_u &:= \eta^{-1}\mathrm{Div}\,\big(\mu\mathbb{D}(\boldsymbol{u})\big) - \nabla \mathrm{div}\,\boldsymbol{u},\\
\beta_u &:= [\![\mu\mathbb{D}(\boldsymbol{u})\boldsymbol{n}]\!]\boldsymbol{n} - [\![\mathrm{div}\,\boldsymbol{u}]\!],\\
\gamma_u &:= \big(\mu\mathbb{D}(\boldsymbol{u})\boldsymbol{n}_+\big)\boldsymbol{n}_+ - \mathrm{div}\,\boldsymbol{u},
\end{aligned}
$$

and write $K(\boldsymbol{u}) := \mathcal{K}(\boldsymbol{\alpha}_u, \beta_u, \gamma_u) \in W^1_q(\dot{\Omega}) + \mathcal{W}^1_q(\Omega)$ for short. This is the key step to handle the two-phase reduced Stokes system (for more details see [94]).

Next, keeping $K(\boldsymbol{u})$ above in mind, we denote Stokes operator for two phase problem by $\mathcal{A}_q\boldsymbol{u} := \eta^{-1}\mathrm{Div}\,\mathbb{T}\big(\boldsymbol{u}, K(\boldsymbol{u})\big)$, whose domain $\mathcal{D}(\mathcal{A}_q)$ is given by

$$
\begin{aligned}
\mathcal{D}(\mathcal{A}_q) := \{\boldsymbol{u} \in W^2_q(\dot{\Omega})^N \cap J_q(\dot{\Omega}) : [\![\boldsymbol{u}]\!]|_\Gamma = [\![\mathcal{T}_{\boldsymbol{n}}\big(\mu\mathbb{D}(\boldsymbol{u})\boldsymbol{n}\big)]\!]|_\Gamma = \boldsymbol{0},&\\
\mathcal{T}_{\boldsymbol{n}_+}\big(\mu\mathbb{D}(\boldsymbol{u})\boldsymbol{n}_+\big)|_{\Gamma_+} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{u}|_{\Gamma_-} = 0\}.&
\end{aligned}
$$

Besides, recall the real interpolation functor (see next section) and the spaces

$$
J_q(\dot{\Omega}) := \{\boldsymbol{f} \in L_q(\Omega)^N : (\boldsymbol{f}, \nabla\varphi)_{\dot{\Omega}} = 0 \quad \forall \varphi \in \mathcal{W}^1_{q'}(\Omega)\}.
$$

Then the spaces for $\boldsymbol{v}_0$ and for the solution $\boldsymbol{u}$ in some time interval $I \subset \mathbb{R}$ are defined by

$$
\mathcal{D}^{2-\frac{2}{p}}_{q,p}(\dot{\Omega}) := \big(J_q(\dot{\Omega}), \mathcal{D}(\mathcal{A}_q)\big)_{1-\frac{1}{p},p} \;\text{ and }\; W^{2,1}_{q,p}(\dot{\Omega}\times I) := L_p\big(I; W^2_q(\dot{\Omega})\big) \cap W^1_p\big(I; L_q(\dot{\Omega})\big).
$$

One may see more discussions on $\mathcal{D}^{2-2/p}_{q,p}(\dot{\Omega})$ later. Now we can state our main result upon the local solvability of System $(INS^{\mathfrak{L}}_{\pm})$ for the cases (Ω_1)-(Ω_3):

Theorem 3.4. *Let (p,q) be in $(I) \cup (II)$ with the sets (I) and (II) given by*

$$
(I) := \{(p,q) \in]2,\infty[\times]N,\infty[\} \quad \textit{and} \quad (II) := \{(p,q) \in]1,2]\times]N,\infty[: \frac{1}{p} + \frac{N}{q} > 1\}.
$$

Additionally, the Hypothesis $(\mathcal{H}1)-(\mathcal{H}3)$ *on the domain* $\dot{\Omega}$ *and the viscosity* μ *are fulfilled. Assume that* ρ_0 *lies in* $W^1_q(\dot{\Omega})$ *satisfying*

$$\|\eta-\rho_0\|_{L_\infty(\dot{\Omega})}\leq c \quad \textit{for some (small enough) constant } c,$$

$\boldsymbol{v}_0$ *is in* $\mathcal{D}^{2(1-\frac{1}{p})}_{q,p}(\dot{\Omega})$ *and* $\boldsymbol{f}$ *belongs to* $L_p\big(0,2;W^1_\infty(\mathbb{R}^N)^N\big)$*. There are some time instant* $T(<1)$ *and constant* C*, only depending on* p*,* q*,* $\boldsymbol{v}_0$ *and* $\boldsymbol{f}$*, such that the System* ($INS^{\mathfrak{L}}_{\pm}$) *admits a unique solution* $(\boldsymbol{u},\mathfrak{q})$ *satisfying*

$$\|\boldsymbol{u}\|_{W^{2,1}_{q,p}(\dot{\Omega}\times]0,T[)}+\|\nabla\mathfrak{q}\|_{L_p(0,T;L_q(\dot{\Omega}))}\leq C. \tag{3.2}$$

In addition, if μ *is piecewise constant, we can relax the condition* $\rho_0\in W^1_q(\dot{\Omega})$ *to* $\rho_0\in L_\infty(\Omega)$*.*

As the case where μ is piecewise constant is simpler, we only prove more general situation with μ being density-dependent in following sections. Now let us indicate that Theorem 3.4 yields some local existence result for the system ($INS_{\pm}$).

By (3.2), we first note that $\boldsymbol{X}_u$ in (3.1) is well defined as long as T is small enough. Moreover, $\boldsymbol{X}_u$ is a $\mathcal{C}^1$ diffeomorphism from $\dot{\Omega}$ onto $\dot{\Omega}_t$ and measure preserving. Denote $\boldsymbol{X}_u$ the inverse mapping of $\boldsymbol{X}_u$. For any smooth function h over $\dot{\Omega}$ and $r\in[1,\infty]$, we have

$$\|h\cdot\boldsymbol{X}_u^{-1}\|_{L_r(\dot{\Omega}_t)}\lesssim\|h\|_{L_r(\dot{\Omega})}.$$

Now recall the relation

$$(\rho,\boldsymbol{v},\mathfrak{p})=(\rho_0,\boldsymbol{u},\mathfrak{q})\circ\boldsymbol{X}_u^{-1},$$

and the definition $\mathscr{A}_u:=(D_\xi X_u)^{-1}$. By Lemma 3.7, (3.2) and

$$\nabla_x(\boldsymbol{v},\mathfrak{p})=\big(\mathscr{A}_u\nabla_\xi(\boldsymbol{u},\mathfrak{q})\big)\circ\boldsymbol{X}_u^{-1},$$

we obtain that the components of $\nabla_x(\boldsymbol{v},\mathfrak{p})$ belong to $L_p\big(0,T;L_q(\dot{\Omega}_t)\big)$.

On the other hand, it is not hard to see

$$\partial_t\boldsymbol{v}=(\partial_t\boldsymbol{u})\circ\boldsymbol{X}_u^{-1}+(\boldsymbol{u}\circ\boldsymbol{X}_u^{-1})\cdot\nabla_x\boldsymbol{v}.$$

Then combining the above estimates yields that $\partial_t\boldsymbol{v}\in L_p\big(0,T;L_q(\dot{\Omega}_t)\big)^N$. Finally, taking advantage of Lemma 3.7 and the equality

$$\nabla^2\boldsymbol{v}=\mathrm{div}\,(\mathscr{A}_u^{tr}\mathscr{A}_u\nabla\boldsymbol{u})\circ\boldsymbol{X}_u^{-1},$$

one can verify that the second order derivatives of $\boldsymbol{v}$ are bounded in $L_p\big(0,T;L_q(\dot{\Omega}_t)\big)^N$. Therefore combining above discussions, we conclude the following result for system ($INS_{\pm}$).

Theorem 3.5. *Under the assumptions of Theorem 3.4, there are some time instant $T(<1)$ and constant C, only depending on p, q, $\boldsymbol{v}_0$ and $\boldsymbol{f}$, such that the System ($INS_\pm$) admits a unique solution $(\rho, \boldsymbol{u}, \mathfrak{q})$ satisfying*

$$\|\partial_t \boldsymbol{v}\|_{L_p(0,T;W^2_q(\dot\Omega_t))} + \|\nabla^2 \boldsymbol{v}\|_{L_p(0,T;L_q(\dot\Omega_t))} + \|\nabla \mathfrak{p}\|_{L_p(0,T;L_q(\dot\Omega_t))} \leq C,$$

and ρ bounded in $W^1_q(\dot\Omega)$. In addition, if μ is piecewise constant, we can relax the condition $\rho_0 \in W^1_q(\dot\Omega)$ to $\rho_0 \in L_\infty(\Omega)$.

3.2 Functional spaces and some linear estimates

3.2.1 Functional spaces

In order to prove our main results precisely, let us firstly introduce some notations for functional spaces setting, which will be used throughout this paper. For any domain $G \subset \mathbb{R}^m$ $(1 \leq m \in \mathbb{N})$ and some Banach space E, the notations $W^k_p(G;E)$ with $p \in [1,\infty]$ and $k \in \mathbb{N}$ means standard $E-$valued Sobolev spaces. Whenever E coincides with $\mathbb{R}$ or $\mathbb{C}$, we will just write $W^k_p(G)$ for the usual scalar valued Sobolev functions. Moreover, the $L_p(G)$ $(L_p(G;E))$ stands for usual ($E-$valued) Lebegue space in G. Of course, we will admit similar conventions for spaces $W^k_{p,loc}(G;E)$ and the usual homogeneous spaces $\widehat{W}^k_p(G;E)$. For some open (time) interval I in $\mathbb{R}$ and some postive constant γ, we define the following exponentially weighted Lebesgue and Sobolev spaces,

$$\begin{aligned}
L_{p,\gamma}(I;E) &:= \{f : I \to E : e^{-\gamma t} f \in L_p(I;E)\}, \\
L_{p,\gamma,0}(\mathbb{R};E) &:= \{f \in L_{p,\gamma}(\mathbb{R};E) : f(\cdot,t) = 0 \text{ for } t<0\}, \\
W^m_{p,\gamma}(I;E) &:= \{f \in L_{p,\gamma}(I;E) : \partial^j_t f(\cdot,t) \in L_{p,\gamma}(I;E),\ 1 \leq j \leq m\}, \\
W^m_{p,\gamma,0}(\mathbb{R};E) &:= W^m_{p,\gamma}(\mathbb{R};E) \cap L_{p,\gamma,0}(\mathbb{R};E).
\end{aligned}$$

Moreover, the norm of $W^m_{p,0,\gamma}(\mathbb{R};E)$ with $m \geq 0$ is given by

$$\|f\|_{W^m_{p,0,\gamma}(\mathbb{R};E)} := \sum_{0 \leq j \leq m} \|e^{-\gamma t} \partial^j_t f(\cdot,t)\|_{L_p(\mathbb{R};E)}.$$

Recall the notion of *Japanese* bracket $\langle y \rangle := (1+|y|^2)^{\frac12}$ for any point $y \in \mathbb{R}^m$ and $\langle D_y \rangle (\equiv (I-\Delta_y)^{\frac12})$ denotes the Fourier Multiplier whose symbol is $\langle y \rangle$. With such kind of multiplier, the standard and also exponentially weighted Bessel potential spaces are defined for $s \geq 0$,

$$\begin{aligned}
H^s_p(\mathbb{R};E) :=& \{f \in L_p(\mathbb{R};E) : (\langle D_t \rangle^s f)(\cdot,t) \in L_p(\mathbb{R};E)\}, \\
H^s_{p,0,\gamma}(\mathbb{R};E) :=& \{f \in L_{p,0,\gamma}(\mathbb{R};E) : e^{-\gamma t}(\langle D_t \rangle^s f)(\cdot,t) \in L_p(\mathbb{R};E)\}.
\end{aligned}$$

Above the norm of the space $H^s_p(\mathbb{R};E)$ is given by

$$\|f\|_{H^s_p(\mathbb{R};E)} := |\langle D_t\rangle f|_{H^s_p(\mathbb{R};E)} + \|f\|_{L_p(\mathbb{R};E)} := \|\langle D_t\rangle f\|_{L_p(\mathbb{R};E)} + \|f\|_{L_p(\mathbb{R};E)}.$$

and the norm $\|\cdot\|_{H^s_{p,0,\gamma}(\mathbb{R};E)}$ is defined similarly. On the other hand, we can also define some "multiplier" operator via Laplace transform for smooth enough function f. We denote the complex parameter $\lambda = \gamma + i\tau$ in the following formulas for Laplace transform $\mathcal{L}$ and its inverse $\mathcal{L}^{-1}$,

$$\mathcal{L}f(\lambda) := \int_{-\infty}^{\infty} e^{-\lambda t} f(t)\,dt, \quad \mathcal{L}^{-1}g(t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\lambda\tau} g(\lambda)\,d\tau.$$

Then we define $(\Lambda^s_\gamma)f := \mathcal{L}^{-1}\big(\lambda^s\mathcal{L}f(\cdot)\big)(t)$ for nonnegative real number s. Another type Bessel potential spaces related to Λ^s_γ can be defined as follows,

$$\begin{aligned}\mathcal{H}^s_{p,\gamma}(\mathbb{R};E) :=&\{f \in L_{p,\gamma}(\mathbb{R};E) : e^{-\gamma t}(\Lambda^s_\gamma f)(\cdot,t) \in L_p(\mathbb{R};E)\},\\ \mathcal{H}^s_{p,0,\gamma}(\mathbb{R};E) :=&\mathcal{H}^s_{p,\gamma}(\mathbb{R};E) \cap L_{p,0,\gamma}(\mathbb{R};E),\end{aligned}$$

One can verify that $H^s_{p,0,\gamma}(\mathbb{R};E)$ coincides with $\mathcal{H}^s_{p,0,\gamma}(\mathbb{R};E)$ for γ large enough.

In addition, if I stands for some open (time) interval in $\mathbb{R}$ and q belongs to $[1,\infty]$, then we set the following mixed derivative spaces for simplicity,

$$\begin{aligned}W^{2,1}_{q,p}(G\times I) :=&L_p\big(I;W^2_q(G)\big) \cap W^1_p\big(I;L_q(G)\big),\\ W^{2,1}_{q,p,0,\gamma}(G\times\mathbb{R}) :=&L_{p,0,\gamma}\big(\mathbb{R};W^2_q(G)\big) \cap W^1_{p,0,\gamma}\big(\mathbb{R};L_q(G)\big),\\ H^{1,1/2}_{q,p}(G\times\mathbb{R}) :=&L_p\big(\mathbb{R};W^1_q(G)\big) \cap H^{1/2}_p\big(\mathbb{R};L_q(G)\big),\\ H^{1,1/2}_{q,p,0,\gamma}(G\times\mathbb{R}) :=&L_{p,0,\gamma}\big(\mathbb{R};W^1_q(G)\big) \cap H^{1/2}_{p,0,\gamma}\big(\mathbb{R};L_q(G)\big).\end{aligned}$$

Here for any two Banach spaces E_0 and E_1, each of which is embedded in tempered distribution $\mathcal{S}'(\mathbb{R}^m)$, the norm of Banach space $E_0\cap E_1$ is understood by $\|\cdot\|_{E_0\cap E_1} := \|\cdot\|_{E_0}+\|\cdot\|_{E_1}$. Besides, $\mathcal{L}(E_0,E_1)$ stands for the space of all bounded linear mapping from E_0 to E_1. Moreover, we also need to employ the notations $(E_0,E_1)_{\theta,p}$ and $(E_0,E_1)_{[\theta]}$ for the real and complex interpolation functors between some interpolation couple E_0 and E_1 respectively for some $\theta \in]0,1[$ and $p \in]1,\infty[$. Here we avoid discussing the endpoint case $\theta = 0$ or 1 (for more details see [9]).

3.2.2 Linear estimates

Our main results will based on the results upon the following linear two phase Stokes equtions in fixed domain Ω,

$$\begin{cases} \eta\, \partial_t \boldsymbol{u} - \operatorname{Div} \mathbb{T}(\boldsymbol{u}, \mathfrak{q}) = \boldsymbol{f} & \text{in} \quad \dot{\Omega} \times]0, T[, \\ \operatorname{div} \boldsymbol{u} = g = \operatorname{div} \boldsymbol{R} & \text{in} \quad \dot{\Omega} \times]0, T[, \\ [\![\mathbb{T}(\boldsymbol{u}, \mathfrak{q}) \boldsymbol{n}]\!] = [\![\boldsymbol{h}]\!], \ \ [\![\boldsymbol{u}]\!] = \boldsymbol{0} & \text{in} \quad \Gamma \times]0, T[, \\ \mathbb{T}_+(\boldsymbol{u}_+, \mathfrak{q}_+) \boldsymbol{n}_+ = \boldsymbol{k} & \text{on} \quad \Gamma_+ \times]0, T[, \\ \boldsymbol{u}_- = \boldsymbol{0} & \text{on} \quad \Gamma_- \times]0, T[, \\ \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0 & \text{on} \quad \dot{\Omega}. \end{cases} \tag{$S_\pm$}$$

where η is a piecewise constant in $\dot{\Omega}$, i.e. $\eta = \eta_+ \mathbb{1}_{\Omega_+} + \eta_- \mathbb{1}_{\Omega_-}$ and $\eta_\pm \in]0, \infty[$. The following theorem proved in [94, Theorem 2.8] treated ($S_\pm$) above with $T = \infty$. However, we firstly need to refine the hypothesis upon the viscosity function as follows.

($\mathcal{H}3'$) $\mu(x)$ is a strictly positive functions on $\dot{\Omega}$ satisfying

$$\widetilde{\mu}_+ \mathbb{1}_{\Omega_+} + \widetilde{\mu}_- \mathbb{1}_{\Omega_-} \leq \mu(\cdot) \leq \bar{\mu}_+ \mathbb{1}_{\Omega_+} + \bar{\mu}_- \mathbb{1}_{\Omega_-},$$

where $\widetilde{\mu}_\pm$ and $\bar{\mu}_\pm$ are all strictly positive constants. In addition, we assume that $\mu(\cdot) \in W^1_r(\dot{\Omega})$ and r as given in ($\mathcal{H}1$).

On the other hand, we denote the dual space of $\mathcal{W}^1_{q'}(\Omega)$ by $\mathcal{W}^{-1}_q(\Omega)$ for some $1 < q < \infty$ and set $\boldsymbol{W}^{-1}_q(\Omega) := L_q(\Omega) \cap \mathcal{W}^{-1}_q(\Omega)$. Thus the linear result in [94] reads:

Theorem 3.6. *Let $(p, q) \in]1, \infty[^2$ and $r \geq \max\{q, q'\}$ with standard notation for conjugate index $q' := q/(q-1)$. Assume that η are two strictly positive constants and Hypothesis ($\mathcal{H}1$), ($\mathcal{H}2$) and ($\mathcal{H}3'$) hold for $\dot{\Omega}$ and η. Then there exists $\gamma_0 \geq 1$ such that the following assertions hold true with taking $T = \infty$ in System ($S_\pm$) :*

1. *For any initial data $\boldsymbol{u}_0 \in \mathcal{D}^{2-2/p}_{q,p}(\dot{\Omega})$ and $(\boldsymbol{f}, \boldsymbol{R}, g, \boldsymbol{h}, \boldsymbol{k}) \equiv \boldsymbol{0}$, the System ($S_\pm$) admits a unique solution*

$$(\boldsymbol{u}, \mathfrak{q}) \in W^{2,1}_{q,p,\gamma_0}(\dot{\Omega} \times \mathbb{R}_+) \times L_{p,\gamma_0}\big(\mathbb{R}_+; W^1_q(\dot{\Omega}) + \mathcal{W}^1_q(\Omega)\big),$$

which satisfies the following inequalities for some constant C_{p,q,γ_0}

$$\|e^{-\gamma_0 t}(\partial_t \boldsymbol{u}, \boldsymbol{u}, \nabla \boldsymbol{u}, \nabla^2 \boldsymbol{u}, \nabla \mathfrak{q})\|_{L_p(\mathbb{R}_+; L_q(\dot{\Omega}))} \leq C_{p,q,\gamma_0} \|\boldsymbol{u}_0\|_{\mathcal{D}^{2-2/p}_{q,p}(\dot{\Omega})}$$

2. *Assume $\boldsymbol{u}_0 = \boldsymbol{0}$ and $(\boldsymbol{f}, \boldsymbol{R}, g, \boldsymbol{h}, \boldsymbol{k}) \in \mathcal{Y}_{p,q,\gamma_0}$. In other words, $(\boldsymbol{f}, \boldsymbol{R}, g, \boldsymbol{h}, \boldsymbol{k})$ satisfying*

$$\boldsymbol{f} \in L_{p,0,\gamma_0}\big(\mathbb{R}; L_q(\dot{\Omega})^N\big), \quad \boldsymbol{R} \in W^1_{p,0,\gamma_0}\big(\mathbb{R}; L_q(\dot{\Omega})^N\big),$$
$$g \in L_{p,0,\gamma_0}\big(\mathbb{R}; W^1_q(\dot{\Omega}) \cap \boldsymbol{W}^{-1}_q(\Omega)\big) \cap H^{\frac{1}{2}}_{p,0,\gamma_0}\big(\mathbb{R}; L_q(\dot{\Omega})\big),$$

$$\begin{aligned}\boldsymbol{h} &\in L_{p,0,\gamma_0}\big(\mathbb{R}; W^1_q(\dot\Omega)^N\big) \cap H^{\frac12}_{p,0,\gamma_0}\big(\mathbb{R}; L_q(\dot\Omega)^N\big),\\ \boldsymbol{k} &\in L_{p,0,\gamma_0}\big(\mathbb{R}; W^1_q(\Omega_+)^N\big) \cap H^{\frac12}_{p,0,\gamma_0}\big(\mathbb{R}; L_q(\Omega_+)^N\big),\end{aligned}$$

where $\boldsymbol{R}$ is some representative of $\mathcal{G}(g)$. Then the System ($S_\pm$) admits a unique solution

$$(\boldsymbol{u}, \mathfrak{q}) \in W^{2,1}_{q,p,0,\gamma_0}\big(\dot\Omega \times \mathbb{R}\big) \times L_{p,0,\gamma_0}\big(\mathbb{R}; W^1_q(\Omega) + \mathcal{W}^1_q(\Omega)\big)$$

possessing the following estimates for some C_{p,q,γ_0} independent on t

$$\begin{aligned}\|e^{-\gamma_0 t}(\partial_t \boldsymbol{u}, \langle D_t\rangle^{\frac12}\nabla \boldsymbol{u}, \nabla^2 \boldsymbol{u}, \nabla \mathfrak{q})\|_{L_p(\mathbb{R}; L_q(\dot\Omega))} + \|e^{-\gamma_0 t}\boldsymbol{u}\|_{L_p(\mathbb{R}; L_q(\dot\Omega))}&\\ \leq C_{p,q,\gamma_0}\|(\boldsymbol{f}, \boldsymbol{R}, g, \boldsymbol{h}, \boldsymbol{k})\|_{\mathcal{Y}_{p,q,\gamma_0}},&\end{aligned}$$

with the corresponding norm $\|\cdot\|_{\mathcal{Y}_{p,q,\gamma_0}}$ defined by

$$\begin{aligned}\|(\boldsymbol{f}, \boldsymbol{R}, g, \boldsymbol{h}, \boldsymbol{k})\|_{\mathcal{Y}_{p,q,\gamma_0}} := \|e^{-\gamma_0 t}(\boldsymbol{f}, \partial_t \boldsymbol{R})\|_{L_p(\mathbb{R}; L_q(\dot\Omega))} + \|e^{-\gamma_0 t}(g, \boldsymbol{h})\|_{L_p(\mathbb{R}; W^1_q(\dot\Omega))}&\\ + \|e^{-\gamma_0 t}\boldsymbol{k}\|_{L_p(\mathbb{R}; W^1_q(\Omega_+))} + \|e^{-\gamma_0 t}\langle D_t\rangle^{\frac12}(g, \boldsymbol{h})\|_{L_p(\mathbb{R}; L_q(\dot\Omega))}&\\ + \|e^{-\gamma_0 t}\langle D_t\rangle^{\frac12}\boldsymbol{k}\|_{L_p(\mathbb{R}; L_q(\Omega_+))}.&\end{aligned}$$

3.3 Short time existence

As we mentioned above, our approach for construction of short time solution of ($INS^{\mathfrak{L}}_\pm$) will be based on Theorem 3.6 and Fixed Point Theorem. For convenience, we rewrite ($INS^{\mathfrak{L}}_\pm$) into "Stokes-like" form as follows:

$$\begin{cases} \eta\,\partial_t \boldsymbol{u} - \operatorname{Div}_\xi \mathbb{T}(\boldsymbol{u}, \mathfrak{q}) = \boldsymbol{f}_{u,\mathfrak{q}} & \text{in} \quad \dot\Omega \times]0, T[,\\ \operatorname{div}_\xi \boldsymbol{u} = g_u = \operatorname{div}_\xi \boldsymbol{R}_u & \text{in} \quad \dot\Omega \times]0, T[,\\ [\![\mathbb{T}(\boldsymbol{u}, \mathfrak{q})\boldsymbol{n}]\!] = [\![\boldsymbol{h}_{u,\mathfrak{q}}]\!], \quad [\![\boldsymbol{u}]\!] = \boldsymbol{0} & \text{in} \quad \Gamma \times]0, T[,\\ \mathbb{T}(\boldsymbol{u}_+, \mathfrak{q}_+)\boldsymbol{n}_+ = \boldsymbol{k}_{u_+,\mathfrak{q}_+} & \text{in} \quad \Gamma_+ \times]0, T[,\\ \boldsymbol{u}_- = \boldsymbol{0} & \text{on} \quad \Gamma_- \times]0, T[,\\ \boldsymbol{u}|_{t=0} = \boldsymbol{v}_0 & \text{on} \quad \dot\Omega, \end{cases} \tag{3.3}$$

where the nonlinear terms above $(\boldsymbol{f}_{u,\mathfrak{q}}, g_u, \boldsymbol{R}_u, \boldsymbol{h}_{u,\mathfrak{q}}, \boldsymbol{k}_{u_+\mathfrak{q}_+})$ are defined by

$$\begin{aligned}\boldsymbol{f}_{u,\mathfrak{q}} &:= \rho_0 \boldsymbol{f}\big(\boldsymbol{X}_u(\xi, t), t\big) + (\eta - \rho_0)\partial_t \boldsymbol{u} - \big(\operatorname{Div}_\xi \mathbb{T}(\boldsymbol{u}, \mathfrak{q}) - \operatorname{Div}_u \mathbb{T}_u(\boldsymbol{u}, \mathfrak{q})\big),\\ g_u &:= D_\xi \boldsymbol{u} : (\mathbb{I} - \mathscr{A}^{tr}_u), \quad \boldsymbol{R}_u := (\mathbb{I} - \mathscr{A}^{tr}_u)\boldsymbol{u},\end{aligned}$$

$$\boldsymbol{h}_{u,\mathfrak{q}} := \mathbb{T}(\boldsymbol{u},\mathfrak{q})\boldsymbol{n} - \mathbb{T}_u(\boldsymbol{u},\mathfrak{q})\overline{\boldsymbol{n}},$$
$$\boldsymbol{k}_{u_+,\mathfrak{q}_+} := \mathbb{T}(\boldsymbol{u}_+,\mathfrak{q}_+)\boldsymbol{n}_+ - \mathbb{T}_{u_+}(\boldsymbol{u}_+,\mathfrak{q}_+)\overline{\boldsymbol{n}}_+.$$

As a start point, let us consider the following linear system with non-zero initial state,

$$\begin{cases} \eta\,\partial_t\boldsymbol{u}_L - \operatorname{Div}_\xi\mathbb{T}(\boldsymbol{u}_L,\mathfrak{q}_L) = \boldsymbol{0} & \text{in} \quad \dot\Omega\times\mathbb{R}_+, \\ \operatorname{div}_\xi\boldsymbol{u}_L = 0 & \text{in} \quad \dot\Omega\times\mathbb{R}_+, \\ [\![\mathbb{T}(\boldsymbol{u}_L,\mathfrak{q}_L)\boldsymbol{n}]\!] = [\![\boldsymbol{u}_L]\!] = \boldsymbol{0} & \text{in} \quad \Gamma\times\mathbb{R}_+, \\ \mathbb{T}_+(\boldsymbol{u}_{L,+},\mathfrak{q}_{L,+})\boldsymbol{n}_+ = \boldsymbol{0} & \text{on} \quad \Gamma_+\times\mathbb{R}_+, \\ \boldsymbol{u}_{L,-} = \boldsymbol{0} & \text{on} \quad \Gamma_-\times\mathbb{R}_+, \\ \boldsymbol{u}_L|_{t=0} = \boldsymbol{v}_0 & \text{on} \quad \dot\Omega. \end{cases} \tag{3.4}$$

Thanks to Theorem 3.6, we obtain a (unique) global solution $(\boldsymbol{u}_L,\mathfrak{q}_L)$ of (3.4) such that

$$(\boldsymbol{u}_L,\mathfrak{q}_L) \in W^{2,1}_{q,p,\gamma_0}(\dot\Omega\times\mathbb{R}_+)\times L_{p,\gamma_0}\big(\mathbb{R}_+;W^1_q(\dot\Omega)+\mathcal{W}^1_q(\Omega)\big)$$

for some $\gamma_0\geq 1$. Moreover, there exists a positive constant C_{p,q,γ_0} such that

$$\|e^{-\gamma_0 t}(\partial_t\boldsymbol{u}_L,\boldsymbol{u}_L,\nabla\boldsymbol{u}_L,\nabla^2\boldsymbol{u}_L,\nabla\mathfrak{q}_L)\|_{L_p(\mathbb{R}_+;L_q(\dot\Omega))} \leq C_{p,q,\gamma_0}\|\boldsymbol{v}_0\|_{\mathcal{D}^{2-2/p}_{q,p}(\dot\Omega)}. \tag{3.5}$$

By the definition of $\mathcal{D}^{2-2/p}_{q,p}(\dot\Omega)$ and the classical embedding

$$L_p(I;E_1)\cap W^1_p(I;E_0)\hookrightarrow \mathcal{BUC}\big(I;(E_0,E_1)_{1-1/p,p}\big) \quad \forall\, p\in]1,\infty[,$$

we have the following inequality

$$\|e^{-\gamma_0 t}\boldsymbol{u}_L\|_{L_\infty(\mathbb{R}_+;\mathcal{D}^{2-2/p}_{q,p}(\dot\Omega))} \lesssim \|e^{-\gamma_0 t}\partial_t\boldsymbol{u}_L\|_{L_p(\mathbb{R}_+;L_q(\dot\Omega))} + \|e^{-\gamma_0 t}\boldsymbol{u}_L\|_{L_p(\mathbb{R}_+;W^2_q(\dot\Omega))}.$$

Thus above inequality and (3.5) yield

$$\|\boldsymbol{u}_L\|_{L_\infty(0,T;\mathcal{D}^{2-2/p}_{q,p}(\dot\Omega))} \leq C_{p,q,\gamma_0}e^{\gamma_0 T}\|\boldsymbol{v}_0\|_{\mathcal{D}^{2-2/p}_{q,p}(\dot\Omega)}, \quad \text{for any finite } T>0. \tag{3.6}$$

Then combining (3.5) and (3.6), we arrive for any finite $T>0$,

$$\begin{aligned}\|\boldsymbol{u}_L\|_{L_\infty(0,T;\mathcal{D}^{2-2/p}_{q,p}(\dot\Omega))} + \|(\partial_t\boldsymbol{u}_L,\boldsymbol{u}_L,\nabla\boldsymbol{u}_L,\nabla^2\boldsymbol{u}_L,\nabla\mathfrak{q}_L)\|_{L_p(0,T;L_q(\dot\Omega))}& \\ \leq C_{p,q,\gamma_0}e^{\gamma_0 T}\|\boldsymbol{v}_0\|_{\mathcal{D}^{2-2/p}_{q,p}(\dot\Omega)}.&\end{aligned} \tag{3.7}$$

With $(\boldsymbol{u}_L, \mathfrak{q}_L)$ defined as above, our main aim is to find some solution $(\boldsymbol{u}, \mathfrak{q})$ of (3.3) which coincides with $(\boldsymbol{u}_L + \boldsymbol{U}, \mathfrak{q}_L + Q)$ in some short time interval. Here the equations of $(\boldsymbol{U}, Q)$ are easily deduced from (3.3) and (3.4),

$$\begin{cases} \eta\, \partial_t \boldsymbol{U} - \operatorname{Div}_\xi \mathbb{T}(\boldsymbol{U}, Q) = \overline{\boldsymbol{f}}_{U,Q} & \text{in} \quad \dot{\Omega}\times]0,T[, \\ \operatorname{div}_\xi \boldsymbol{U} = \overline{g}_U = \operatorname{div}_\xi \overline{\boldsymbol{R}}_U & \text{in} \quad \dot{\Omega}\times]0,T[, \\ [\![\mathbb{T}(\boldsymbol{U}, Q)\boldsymbol{n}]\!] = [\![\overline{\boldsymbol{h}}_{U,Q}]\!], \ [\![\boldsymbol{U}]\!] = \boldsymbol{0} & \text{in} \quad \Gamma\times]0,T[, \\ \mathbb{T}_+(\boldsymbol{U}_+, Q_+)\boldsymbol{n}_+ = \overline{\boldsymbol{k}}_{U_+,Q_+} & \text{on} \quad \Gamma_+\times]0,T[, \\ \boldsymbol{U}_- = \boldsymbol{0} & \text{on} \quad \Gamma_-\times]0,T[, \\ \boldsymbol{U}|_{t=0} = \boldsymbol{0} & \text{on} \quad \dot{\Omega}, \end{cases} \tag{3.8}$$

where $(\overline{\boldsymbol{f}}_{U,Q}, \overline{g}_U, \overline{\boldsymbol{R}}_U, \overline{\boldsymbol{h}}_{U,Q}, \overline{\boldsymbol{k}}_{U_+,Q_+}) \equiv (\boldsymbol{f}_{u,\mathfrak{q}}, g_u, \boldsymbol{R}_u, \boldsymbol{h}_{u,\mathfrak{q}}, \boldsymbol{k}_{u_+\mathfrak{q}_+})$ as we defined before (below (3.3)). Then the local solvability of System (3.3) is reduced to the existence of solution $(\boldsymbol{U}, Q)$ of (3.8) in some short time interval $]0,T[$. Hence we assume $T < 1$ hereafter for convenience.

Furthermore, let us introduce the solution space for System (3.8). For any given $(p, q) \in]1,\infty[^2$ and $\gamma_0 > 0$,

$$\mathscr{E}(T) := W^{2,1}_{q,p}(\dot{\Omega}\times]0,T[) \times L_p\big(0,T; W^1_q(\Omega) + \mathcal{W}^1_q(\Omega)\big) \times H^{1,\frac{1}{2}}_{q,p,0,\gamma_0}(\dot{\Omega}\times\mathbb{R}) \times H^{1,\frac{1}{2}}_{q,p,0,\gamma_0}(\dot{\Omega}_+\times\mathbb{R})$$

with the norm $\|\cdot\|_{\mathscr{E}(T)}$ given by

$$\begin{aligned} \|(\boldsymbol{w}, \nabla P, \Pi, \Pi_+)\|_{\mathscr{E}(T)} := {} & \|\boldsymbol{w}\|_{W^{2,1}_{q,p}(\dot{\Omega}\times]0,T[)} + \|\nabla P\|_{L_p(0,T;L_q(\dot{\Omega}))} \\ & + \|\Pi\|_{H^{1,\frac{1}{2}}_{q,p,0,\gamma_0}(\dot{\Omega}\times\mathbb{R})} + \|\Pi_+\|_{H^{1,\frac{1}{2}}_{q,p,0,\gamma_0}(\dot{\Omega}_+\times\mathbb{R})}. \end{aligned}$$

We say $(\boldsymbol{w}, P, \Pi, \Pi_+)$ belongs to $\mathscr{E}_L(T)$ for some $L > 0$, if and only if the following assertions hold true:

1. $(\boldsymbol{w}, P, \Pi, \Pi_+)$ belongs to $\mathscr{E}(T)$ satisfying

$$\begin{aligned} \boldsymbol{w}|_{\Gamma_-} = \boldsymbol{0} \ \text{ on } \Gamma_-, \quad & [\![(P-\Pi)\boldsymbol{n}]\!] = [\![\boldsymbol{w}]\!] = \boldsymbol{0} \ \text{ on } \Gamma\times]0,T[, \\ \text{and} \quad & P_+\boldsymbol{n}_+ = \Pi_+\boldsymbol{n}_+ \ \text{ on } \Gamma_+\times]0,T[; \end{aligned}$$

2. The norm of $(\boldsymbol{w}, \nabla P, \Pi, \Pi_+)$ is bounded by L, i.e. $\|(\boldsymbol{w}, \nabla P, \Pi, \Pi_+)\|_{\mathscr{E}(T)} \leq L$.

Now assume $(\boldsymbol{w}, P, \Pi, \Pi_+)$ belongs to $\mathscr{E}_L(T)$ for some $q \in]N,\infty[$ and choose some parameter L large enough such that for some small constant c

$$\|\boldsymbol{u}_L\|_{L_\infty(0,2;\mathcal{D}^{2-2/p}_{q,p}(\dot{\Omega}))} + \|\boldsymbol{u}_L\|_{W^{2,1}_{q,p}(\dot{\Omega}\times]0,2[)} + \|\boldsymbol{f}, \nabla \mathfrak{q}_L\|_{L_p(0,2;L_q(\dot{\Omega}))} \leq L,$$

$$L^{-1}\|\boldsymbol{f}\|_{L_p(0,2;L_q(\mathbb{R}^N))} + \|\eta - \rho_0\|_{L_\infty(\dot\Omega)} \leq c.$$

Obviously, the choice of L also depends on the quantity $\|\boldsymbol{v}_0\|_{\mathcal{D}^{2-2/p}_{q,p}(\dot\Omega)}$ from (3.7). For such $(\boldsymbol{w}, P, \Pi, \Pi_+)$ in $\mathscr{E}_L(T)$, consider the following *linearized* Stokes equations from (3.8),

$$\begin{cases} \eta\,\partial_t \boldsymbol{U} - \operatorname{Div}_\xi \mathbb{T}(\boldsymbol{U}, Q) = \overline{\boldsymbol{f}}_{w,P} & \text{in} \quad \dot\Omega\times]0,T[, \\ \operatorname{div}_\xi \boldsymbol{U} = \overline{g}_w = \operatorname{div}_\xi \overline{\boldsymbol{R}}_w & \text{in} \quad \dot\Omega\times]0,T[, \\ [\![\mathbb{T}(\boldsymbol{U}, Q)\boldsymbol{n}]\!] = [\![\overline{\boldsymbol{h}}_{w,P}]\!], \quad [\![\boldsymbol{U}]\!] = \boldsymbol{0} & \text{in} \quad \Gamma\times]0,T[, \\ \mathbb{T}_+(\boldsymbol{U}_+, Q_+)\boldsymbol{n}_+ = \overline{\boldsymbol{k}}_{w_+,P_+} & \text{on} \quad \Gamma_+\times]0,T[, \\ \boldsymbol{U}_- = \boldsymbol{0} & \text{on} \quad \Gamma_-\times]0,T[, \\ \boldsymbol{U}|_{t=0} = \boldsymbol{0} & \text{on} \quad \dot\Omega, \end{cases} \tag{3.9}$$

where $(\overline{\boldsymbol{f}}_{w,P}, \overline{g}_w, \overline{\boldsymbol{R}}_w, \overline{\boldsymbol{h}}_{w,P}, \overline{\boldsymbol{k}}_{w_+,P_+})$ are defined like $(\overline{\boldsymbol{f}}_{U,Q}, \overline{g}_U, \overline{\boldsymbol{R}}_U, \overline{\boldsymbol{h}}_{U,Q}, \overline{\boldsymbol{k}}_{U_+,Q_+})$ with replacing (U, Q) by (w, P).

Let us first sketch the strategy. For any point $(\boldsymbol{w}, P, \Pi, \Pi_+)$ in $\mathscr{E}_L(T)$, our goal is to find some

$$(\boldsymbol{U}, Q, \Xi, \Xi_+) := \Phi(\boldsymbol{w}, P, \Pi, \Pi_+) \in \mathscr{E}_L(T)$$

with $(\boldsymbol{U}, Q)$ solving (3.9) in $]0,T[$. The construction of such mapping Φ will be a consequence of Theorem 3.6. Hence we need to find suitable extensions $(\widetilde{\boldsymbol{f}}_{w,P}, \widetilde{g}_w, \widetilde{\boldsymbol{R}}_w, \widetilde{\boldsymbol{h}}_{w,P}, \widetilde{\boldsymbol{k}}_{w_+,P_+})$ over time interval $\mathbb{R}$ such that

$$(\widetilde{\boldsymbol{f}}_{w,P}, \widetilde{g}_w, \widetilde{\boldsymbol{R}}_w, \widetilde{\boldsymbol{h}}_{w,P}, \widetilde{\boldsymbol{k}}_{w_+,P_+})|_{]0,T[} = (\overline{\boldsymbol{f}}_{w,P}, \overline{g}_w, \overline{\boldsymbol{R}}_w, \overline{\boldsymbol{h}}_{w,P}, \overline{\boldsymbol{k}}_{w_+,P_+})$$

More importantly, for some $\gamma_0 > 1$ and $q \in]N, \infty[$, we have

$$\|(\widetilde{\boldsymbol{f}}_{w,P}, \widetilde{g}_w, \widetilde{\boldsymbol{R}}_w, \widetilde{\boldsymbol{h}}_{w,P}, \widetilde{\boldsymbol{k}}_{w_+,P_+})\|_{\mathcal{Y}_{p,q,\gamma_0}} < \infty \tag{3.10}$$

Thanks to (3.10), the solvability of the following equations is guaranteed by Theorem 3.6,

$$\begin{cases} \eta\,\partial_t \widetilde{\boldsymbol{U}} - \operatorname{Div}_\xi \mathbb{T}(\widetilde{\boldsymbol{U}}, \widetilde{Q}) = \widetilde{\boldsymbol{f}}_{w,P} & \text{in} \quad \dot\Omega\times\mathbb{R}_+, \\ \operatorname{div}_\xi \widetilde{\boldsymbol{U}} = \widetilde{g}_w = \operatorname{div}_\xi \widetilde{\boldsymbol{R}}_w & \text{in} \quad \dot\Omega\times\mathbb{R}_+, \\ [\![\mathbb{T}(\widetilde{\boldsymbol{U}}, \widetilde{Q})\boldsymbol{n}]\!] = [\![\widetilde{\boldsymbol{h}}_{w,P}]\!], \quad [\![\widetilde{\boldsymbol{U}}]\!] = \boldsymbol{0} & \text{in} \quad \Gamma\times\mathbb{R}_+, \\ \mathbb{T}_+(\widetilde{\boldsymbol{U}}_+, \widetilde{Q}_+)\boldsymbol{n}_+ = \widetilde{\boldsymbol{k}}_{w_+,P_+} & \text{on} \quad \Gamma_+\times\mathbb{R}_+, \\ \widetilde{\boldsymbol{U}}_- = \boldsymbol{0} & \text{on} \quad \Gamma_-\times\mathbb{R}_+, \\ \widetilde{\boldsymbol{U}}|_{t=0} = \boldsymbol{0} & \text{on} \quad \dot\Omega, \end{cases} \tag{3.11}$$

In other words, we can obtain some global solution $(\widetilde{\boldsymbol{U}}, \widetilde{Q})$ of System (3.11). Moreover, $(U, Q) := (\widetilde{U}, \widetilde{Q})|_{[0,T[}$ for some small enough T is exactly the local solution of System (3.9). Next we define

$$\Xi := \big(\mu(\rho_0)\mathbb{D}(\widetilde{\boldsymbol{U}})\boldsymbol{n}\big) \cdot \boldsymbol{n} - \widetilde{\boldsymbol{h}}_{w,P} \cdot \boldsymbol{n} \tag{3.12}$$

$$\Xi_+ := \big(\mu(\rho_0)\mathbb{D}(\widetilde{\boldsymbol{U}}_+)\boldsymbol{n}_+\big) \cdot \boldsymbol{n}_+ - \widetilde{\boldsymbol{k}}_{w_+,P_+} \cdot \boldsymbol{n}_+. \tag{3.13}$$

With above definitions, we can set $\Phi(\boldsymbol{w}, P, \Pi, \Pi_+) := (\boldsymbol{U}, Q, \Xi, \Xi_+)$. Furthermore, by smallness of T, we can find that $(\boldsymbol{U}, Q, \Xi, \Xi_+)$ belongs to $\mathscr{E}_L(T)$ and that Φ is a contracting mapping. This will complete our proof for local solvability of System (3.8) (as well as System (3.3)) by standard fixed point arguments.

The main goal of this section is to find some solution mapping Φ from from $\mathscr{E}_L(T)$ into itself and the contracting property of Φ will be discussed in the next section.

To define our solution operator Φ, it is convenient to introduce some standard extension operator $E_{(t)}$ such as in [106, Theorem 3.2]. For any (scalar- or vector-valued) mapping $\mathfrak{h}$ defined on $]0,T[$ and any fixed parameter $t \in]0,T]$,

$$E_{(t)}\mathfrak{h}(\cdot, s) := \begin{cases} \mathfrak{h}(\cdot, s) & \text{if} \quad s \in]0,t[, \\ \mathfrak{h}(\cdot, 2t - s) & \text{if} \quad s \in]t, 2t[, \\ 0 & \text{otherwise.} \end{cases}$$

For any Banach space E and $(p, \gamma) \in]1,\infty[\times]0,\infty[$, $E_{(t)} \in \mathcal{L}\big(L_p(0,T;E), L_{p,0,\gamma_0}(\mathbb{R};E)\big)$. Indeed, we have

$$\|e^{-\gamma s} E_{(t)}\mathfrak{h}(\cdot, s)\|_{L_p(\mathbb{R};E)} \le 2\|\mathfrak{h}(\cdot, s)\|_{L_p(0,T;E)}, \quad \text{for any } \ \gamma \ge 0 \ \text{ and } \ t \in]0,T]. \tag{3.14}$$

If $\mathfrak{h}(\cdot, 0) = 0$, then clearly

$$\partial_s E_{(t)}\mathfrak{h}(\cdot, s) = \begin{cases} \partial_s\mathfrak{h}(\cdot, s) & \text{if} \quad s \in]0,t[, \\ -(\partial_s\mathfrak{h})(\cdot, 2t - s) & \text{if} \quad s \in]t, 2t[, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $E_{(t)} \in \mathcal{L}\big(\widehat{W}^1_{p,0}(0,T;E), \widehat{W}^1_{p,0,\gamma_0}(\mathbb{R};E)\big)$. In fact, we have

$$\|e^{-\gamma s}\partial_s\big(E_{(t)}\mathfrak{h}(\cdot, s)\big)\|_{L_p(\mathbb{R};E)} \le 2\|(\partial_s\mathfrak{h})(\cdot, s)\|_{L_p(0,T;E)}, \ \ \forall \ \gamma \ge 0 \ \text{ and } \ t \in]0,T]. \tag{3.15}$$

For simplicity, we also introduce the notations $(\boldsymbol{W}, \Theta) := (\boldsymbol{u}_{_L} + \boldsymbol{w}, \mathfrak{p}_{_L} + P)$. Thanks to (3.93) and previous conventions on (T, L), the following important decay properties holds

$$\|\nabla \boldsymbol{W}\|_{L_1(0,T;L_\infty(\dot\Omega))} \le C_N T^{\frac{1}{p'}} \|(\nabla \boldsymbol{w}, \nabla \boldsymbol{u}_{_L})\|_{L_p(0,T;L_\infty(\dot\Omega))} \le C_N T^{\frac{1}{p'}+\sigma_{p,q}} L, \tag{3.16}$$

where, for any $1 < p < \infty$ and $N < q < \infty$, the non-negative index $\sigma_{p,q}$ is given by

$$\sigma_{p,q} := \begin{cases} \frac{1}{2}\big(1-\frac{N}{q}\big), & \text{for} \quad \frac{2}{p}+\frac{N}{q}>1; \\ 0, & \text{for} \quad \frac{2}{p}+\frac{N}{q}\leq 1. \end{cases}$$

Thus X_W, $\mathscr{A}_W$ and $\overline{\boldsymbol{n}}_W := (\mathscr{A}_W\boldsymbol{n})/|\mathscr{A}_W\boldsymbol{n}|$ are well defined so long as T small enough satisfying

$$C_N T^{\frac{1}{p'}+\sigma_{p,q}} L \leq 1/2. \tag{3.17}$$

Combining this decay property and Condition (3.17), we infer from Lemma 3.7 and Lemma 3.9 that

$$\begin{aligned} \|(\mathscr{A}_W-\mathbb{I},\overline{\boldsymbol{n}}_W-\boldsymbol{n})\|_{L_\infty(\dot{\Omega}\times]0,T[)} &\lesssim T^{\frac{1}{p'}+\sigma_{p,q}}L, \\ \|\nabla_\xi(\mathscr{A}_W-\mathbb{I},\overline{\boldsymbol{n}}_W-\boldsymbol{n})\|_{L_\infty(0,T;L_q(\dot{\Omega}))} &\lesssim T^{\frac{1}{p'}}L, \\ \|\partial_t(\mathscr{A}_W-\mathbb{I},\overline{\boldsymbol{n}}_W-\boldsymbol{n})\|_{L_p(0,T;L_\infty(\dot{\Omega}))} &\lesssim T^{\sigma_{p,q}}L, \\ \|\partial_t\nabla_\xi(\mathscr{A}_W-\mathbb{I},\overline{\boldsymbol{n}}_W-\boldsymbol{n})\|_{L_p(0,T;L_q(\dot{\Omega}))} &\lesssim L. \end{aligned} \tag{3.18}$$

In the rest of this subsection, we devote ourselves to verify the bound (3.10) with keeping bounds (3.14) and (3.15) in mind. The fact that $W^1_q(\dot{\Omega})$ is a Banach algebra for $N<q<\infty$ will be also constantly used without mention.

Bounds for $\widetilde{\boldsymbol{f}}_{w,P}$

Now, we recall the definition of $\overline{\boldsymbol{f}}_{w,P}$ as follows,

$$\begin{aligned} \overline{\boldsymbol{f}}_{w,P} = {}& \rho_0\boldsymbol{f}\big(\boldsymbol{X}_W(\xi,t),t\big) + (\eta-\rho_0)\partial_t\boldsymbol{W} - \operatorname{Div}_\xi\Big(\mu(\rho_0)\big(\mathbb{H}_W + (\mathbb{I}-\mathscr{A}_W^{tr})\mathbb{D}(W)\big)\Big) \\ & + \operatorname{Div}_\xi\big(\mu(\rho_0)(\mathbb{I}-\mathscr{A}_W^{tr})\mathbb{H}_W\big) + \operatorname{Div}_\xi\big(\Theta(\mathbb{I}-\mathscr{A}_W^{tr})\big), \end{aligned}$$

where we adopt the notation

$$\mathbb{H}_W := D_\xi\boldsymbol{W}(\mathbb{I}-\mathscr{A}_W^{tr}) + (\mathbb{I}-\mathscr{A}_W)\nabla_\xi\boldsymbol{W} \equiv \mu(\rho_0)^{-1}\big(\mathbb{T}(\boldsymbol{W},\Theta)-\mathbb{T}_W(\boldsymbol{W},\Theta)\big).$$

The main aim of this step is to seek some suitable extension $\widetilde{\boldsymbol{f}}_{w,P} \in L_{p,0,\gamma_0}\big(\mathbb{R};L_q(\dot{\Omega})\big)$ of the source term $\overline{\boldsymbol{f}}_{w,P}$. To this end, let us introduce some notations which are also very useful in the rest of this context. Assume that $\chi\in\mathcal{C}^\infty(\mathbb{R})$ is some cut-off function satisfying $\chi(t)\equiv 1$ for $|t|\leq 1$ and $\chi(t)\equiv 0$ for $|t|\geq 2$. Thanks to (3.7), we set

$$\widetilde{\boldsymbol{u}}_L(\cdot,t) := \chi(t)e^{-|t|\mathcal{A}_q}\boldsymbol{v}_0(\cdot) \quad \text{for} \ \ t\in\mathbb{R}.$$

From the definition of $\widetilde{\boldsymbol{u}}_L$, it is not hard to see that $\widetilde{\boldsymbol{u}}_L(\cdot,t)=\boldsymbol{u}_L(\cdot,t)$ for any $t\in[0,1]$. Moreover, the mixed derivative theorem implies that

$$\|\widetilde{\boldsymbol{u}}_L\|_{H_p^{\frac{1}{2}}(\mathbb{R};W_q^1(\dot{\Omega}))} \lesssim \|\widetilde{\boldsymbol{u}}_L\|_{W_{q,p}^{2,1}(\dot{\Omega}\times\mathbb{R})} \lesssim \|\boldsymbol{u}_L\|_{W_{q,p}^{2,1}(\dot{\Omega}\times]0,2[)} \lesssim L. \tag{3.19}$$

Next with such $\widetilde{\boldsymbol{u}}_L$, consider $\widetilde{\boldsymbol{W}} := E_{(T)}\boldsymbol{w} + \widetilde{\boldsymbol{u}}_L$ which satisfies

$$\widetilde{\boldsymbol{W}}(\cdot,t) \equiv \boldsymbol{W}(\cdot,t) \text{ for any } t \in [0,T[\quad \text{and} \quad \widetilde{\boldsymbol{W}}(\cdot,t) \equiv \boldsymbol{0} \text{ for any } t \notin]-2,2[.$$

Furthermore, (3.19) and the mixed derivative theorem yield

$$\|\widetilde{\boldsymbol{W}}\|_{H_p^{\frac{1}{2}}(\mathbb{R};W_q^1(\dot{\Omega}))} \lesssim \|\widetilde{\boldsymbol{W}}\|_{W_{q,p}^{2,1}(\dot{\Omega}\times\mathbb{R})} \lesssim \|E_{(T)}\boldsymbol{w}\|_{W_{q,p}^{2,1}(\dot{\Omega}\times]0,2T[)} + \|\widetilde{\boldsymbol{u}}_L\|_{W_{q,p}^{2,1}(\dot{\Omega}\times\mathbb{R})} \lesssim L. \tag{3.20}$$

Recalling (3.16), $\widetilde{\boldsymbol{W}}$ has the similar decay property due to (3.20)

$$\|\nabla_\xi \widetilde{\boldsymbol{W}}\|_{L_1(0,2T;L_\infty(G))} \lesssim T^{\frac{1}{p'}} \|\nabla_\xi \widetilde{\boldsymbol{W}}\|_{L_p(0,2T;L_\infty(G))} \lesssim T^{\frac{1}{p'}+\sigma_{p,q}} L. \tag{3.21}$$

Now, introduce the following two matrices to simplify our writing,

$$\begin{aligned}
\widetilde{\mathbb{H}}_W &:= D_\xi \widetilde{\boldsymbol{W}} \cdot E_{(T)}(\mathbb{I} - \mathscr{A}_W^{tr}) + E_{(T)}(\mathbb{I} - \mathscr{A}_W) \cdot \nabla_\xi \widetilde{\boldsymbol{W}}, \\
\widetilde{\mathbb{D}}_W &:= E_{(T)}(\mathbb{I} - \mathscr{A}_W^{tr}) \cdot \mathbb{D}(\widetilde{\boldsymbol{W}}).
\end{aligned}$$

Then the following vector will be one desired extension,

$$\begin{aligned}
\widetilde{\boldsymbol{f}}_{w,P} :={}& \rho_0 E_{(T)} \boldsymbol{f}\big(\boldsymbol{X}_W(\xi,t),t\big) + (\eta - \rho_0) E_{(T)} \partial_t \boldsymbol{W} - \operatorname{Div}_\xi\big(\mu(\rho_0)(\widetilde{\mathbb{H}}_W + \widetilde{\mathbb{D}}_W)\big) \\
&+ \operatorname{Div}_\xi\big(\mu(\rho_0) E_{(T)}(\mathbb{I} - \mathscr{A}_W^{tr})\widetilde{\mathbb{H}}_W\big) + \operatorname{Div}_\xi\big((E_{(T)}\Theta) E_{(T)}(\mathbb{I} - \mathscr{A}_W^{tr})\big).
\end{aligned} \tag{3.22}$$

Obviously, $\widetilde{\boldsymbol{f}}_{w,P}|_{t\in]0,T[} = \overline{\boldsymbol{f}}_{w,P}$ is fulfilled. More importantly, we will demonstrate

$$\|\widetilde{\boldsymbol{f}}_{w,P}\|_{L_{p,0,\gamma_0}(\mathbb{R};L_q(\dot{\Omega}))} \lesssim cL + T^{\frac{1}{p'}+\sigma_{p,q}} L^2 \big(\|\mu\|_{L_\infty(\dot{\Omega})} + T^{\sigma_{p,q}} \|\nabla\mu\|_{L_q(\dot{\Omega})}\big) + T^{\frac{1}{p'}} L^2. \tag{3.23}$$

To verify (3.23), first note that Condition (3.17) (up to the choice of C_N) yields,

$$\Big\| \det\big(\mathbb{I} + \int_0^t \nabla \boldsymbol{W}(\cdot,\tau)d\tau\big)\Big\|_{L_\infty(0,T;L_\infty(\dot{\Omega}))} \leq \frac{1}{2}.$$

Thus we have for some $\gamma_0 > 0$,

$$\big\|E_{(T)}\boldsymbol{f}\big(\boldsymbol{X}_W(\xi,t),t\big)\big\|_{L_{p,0,\gamma_0}(\mathbb{R};L_q(\dot{\Omega}))} \lesssim \|\boldsymbol{f}\|_{L_p(0,T;L_q(\dot{\Omega}))} \lesssim cL. \tag{3.24}$$

Next, the second term on the right hand side of (3.22) is easy bounded by

$$\|(\eta - \rho_0) E_{(T)} \partial_t \boldsymbol{W}\|_{L_{p,0,\gamma_0}(\mathbb{R};L_q(\dot{\Omega}))} \lesssim \|\eta - \rho_0\|_{L_\infty(\dot{\Omega}\times]0,T[)} L \lesssim cL. \tag{3.25}$$

To study the nonlinear terms in (3.22), we know from (3.18) and (3.21) that for any $1 \leq j, k, \ell, m \leq N$,

$$\big\|\big(E_{(T)}(\mathbb{I} - \mathscr{A}_W)\big)^j_k (\nabla_\xi \widetilde{\boldsymbol{W}})^\ell_m\big\|_{L_p(0,2T;W^1_q(\dot{\Omega}))} \lesssim T^{\frac{1}{p'}+\sigma_{p,q}} L^2, \tag{3.26}$$

$$\big\|\big(E_{(T)}(\mathbb{I} - \mathscr{A}_W)\big)^j_k (\nabla_\xi \widetilde{\boldsymbol{W}})^\ell_m\big\|_{L_p(0,2T;L_\infty(\dot{\Omega}))} \lesssim T^{\frac{1}{p'}+2\sigma_{p,q}} L^2. \tag{3.27}$$

Then combining the bounds (3.26) and (3.27), we obtain that for any $\gamma_0 > 0$,

$$\|\mu(\rho_0)(\widetilde{\mathbb{H}}_W, \widetilde{\mathbb{D}}_W)\|_{L_{p,0,\gamma_0}(\mathbb{R};W^1_q(\dot{\Omega}))} \lesssim T^{\frac{1}{p'}+\sigma_{p,q}} L^2\big(\|\mu\|_{L_\infty(\dot{\Omega})} + T^{\sigma_{p,q}}\|\nabla\mu\|_{L_q(\dot{\Omega})}\big). \tag{3.28}$$

Moreover, compared with (3.28), $\mu(\rho_0)\widetilde{\mathbb{H}}_W \cdot E_{(T)}(\mathbb{I} - \mathscr{A}_W)$ is higher order term with respect to time,

$$\|\mu(\rho_0)\widetilde{\mathbb{H}}_W \cdot E_{(T)}(\mathbb{I} - \mathscr{A}_W)\|_{L_{p,0,\gamma_0}(\mathbb{R};W^1_q(\dot{\Omega}))} \lesssim T^{\frac{2}{p'}+2\sigma_{p,q}} L^3\big(\|\mu\|_{L_\infty(\dot{\Omega})} + T^{\sigma_{p,q}}\|\nabla\mu\|_{L_q(\dot{\Omega})}\big). \tag{3.29}$$

Therefore, under Condition (3.17), (3.28) and (3.29) yield

$$\begin{aligned}\|\mathrm{Div}_\xi\big(\mu(\rho_0)(\widetilde{\mathbb{H}}_W + \widetilde{\mathbb{D}}_W)\big) - \mathrm{Div}_\xi\big(\mu(\rho_0)E_{(T)}(\mathbb{I} - \mathscr{A}^{tr}_W)\widetilde{\mathbb{H}}_W\big)\|_{L_{p,0,\gamma_0}(\mathbb{R};L_q(\dot{\Omega}))}&\\ \lesssim T^{\frac{1}{p'}+\sigma_{p,q}} L^2\big(\|\mu\|_{L_\infty(\dot{\Omega})} + T^{\sigma_{p,q}}\|\nabla\mu\|_{L_q(\dot{\Omega})}\big).&\end{aligned} \tag{3.30}$$

The bound of last term of $\widetilde{\boldsymbol{f}}_{w,P}$ is immediate from (3.18),

$$\|\mathrm{Div}_\xi\big((E_{(T)}\Theta)E_{(T)}(\mathbb{I} - \mathscr{A}^{tr}_W)\big)\|_{L_{p,0,\gamma_0}(\mathbb{R};L_q(\dot{\Omega}))} \lesssim T^{\frac{1}{p'}} L^2. \tag{3.31}$$

At last, putting together the bounds (3.24), (3.25), (3.30) and (3.31) completes the proof of our claim (3.23).

Bound of $\widetilde{g}_w$

Based on the expression

$$\overline{g}_w = D_\xi \boldsymbol{W} : (\mathbb{I} - \mathscr{A}^{tr}_W) = \mathrm{div}_\xi \overline{\boldsymbol{R}}_w = \mathrm{div}_\xi\big((\mathbb{I} - \mathscr{A}^{tr}_W)\boldsymbol{W}\big),$$

let us consider the following extension

$$\widetilde{g}_w := E_{(T)}\big(D_\xi \widetilde{\boldsymbol{W}} : (\mathbb{I} - \mathscr{A}^{tr}_W)\big) = \mathrm{div}\,\Big(E_{(T)}\big((\mathbb{I} - \mathscr{A}^{tr}_W)\widetilde{\boldsymbol{W}}\big)\Big) =: \mathrm{div}\,\widetilde{\boldsymbol{R}}_w. \tag{3.32}$$

By (3.32), we immediately know that $\widetilde{g}_w(\cdot,t) = \overline{g}_w(\cdot,t)$ for $t \in [0,T[$ and vanishes for $t \notin]0,2T[$. To prove such $\widetilde{g}_w$ is an admissible extension, it is sufficient to verify for some $\gamma_0 > 1$,

$$\widetilde{g}_w \in L_{p,0,\gamma_0}\big(\mathbb{R}; W^1_q(\dot{\Omega}) \cap \boldsymbol{W}^{-1}_q(\Omega)\big) \cap H^{\frac{1}{2}}_{p,0,\gamma_0}\big(\mathbb{R}; L_q(\dot{\Omega})\big). \tag{3.33}$$

In fact, the first part of (3.33) is guaranteed by our previous discussion (3.26),

$$\|\widetilde{g}_w\|_{L_{p,0,\gamma_0}(\mathbb{R};W^1_q(\dot\Omega))} \lesssim \|\widetilde{g}_w\|_{L_p(0,2T;W^1_q(\dot\Omega))} \lesssim T^{\frac{1}{p'}+\sigma_{p,q}}L^2. \tag{3.34}$$

Next, to prove the assertion $\widetilde{g}_w \in L_{p,0,\gamma_0}\big(\mathbb{R};\boldsymbol{W}^{-1}_q(\Omega)\big)$, note the fact that $\widetilde{\boldsymbol{W}}|_{\Gamma_-} = \mathbf{0}$. Thus we obtain

$$\widetilde{\boldsymbol{R}}_w = \big(E_{(T)}(\mathbb{I}-\mathscr{A}^{tr}_W)\big)\widetilde{\boldsymbol{W}} = \mathbf{0} \ \text{ on } \ \Gamma_-.$$

On the other hand, we claim the following property

$$[\![\widetilde{\boldsymbol{R}}_w\cdot\boldsymbol{n}]\!] = [\![\big(E_{(T)}(\mathbb{I}-\mathscr{A}^{tr}_W)\big)\widetilde{\boldsymbol{W}}\cdot\boldsymbol{n}]\!] = 0 \ \text{ on } \ \Gamma, \tag{3.35}$$

holds true, which yields our desired result concerning $\boldsymbol{W}^{-1}_q(\Omega)$ norm for $\widetilde{g}_w$ (for more discussions on $\boldsymbol{W}^{-1}_q(\Omega)$, see [94, Section 1.2]).

Now, we assume $t\in]0,T]$ and the proof of the case $t\in]0,2T[$ follows along the similar lines. As $[\![\widetilde{\boldsymbol{W}}]\!]$ is continuous across Γ, it is sufficient to check the fact

$$[\![\mathscr{A}^{tr}_W\boldsymbol{W}\cdot\boldsymbol{n}]\!] = \boldsymbol{W}\cdot[\![\mathscr{A}_W\boldsymbol{n}]\!] = 0 \ \text{ on } \ \Gamma, \quad \forall\, t\in]0,T].$$

This is true due to [43, Remark 3.1], which completes the proof for (3.35).

Finally, it remains to check $\widetilde{g}_w \in H^{\frac12}_{p,0,\gamma_0}\big(\mathbb{R}_+;L_q(\dot\Omega)\big)$. However, for technical reason, we here divide the proof into two cases:

$$(I) := \{(p,q)\in]2,\infty[\times]N,\infty[\} \quad \text{and} \quad (II) := \{(p,q)\in]1,2]\times]N,\infty[: 1/p+N/q>1\}.$$

For the case (I), Lemma 3.11, (3.18) and (3.20) imply

$$\begin{aligned}\|\widetilde{g}_w\|_{H^{\frac12}_{p,0,\gamma_0}(\mathbb{R};L_q(\dot\Omega))} &\lesssim \|\mathbb{I}-\mathscr{A}^{tr}_W\|^{\frac12}_{L_\infty(\dot\Omega\times]0,T[)}\Big(\|\nabla_\xi\mathscr{A}^{tr}_W\|_{L_\infty(0,T;L_q(\dot\Omega))} + \|\mathbb{I}-\mathscr{A}^{tr}_W\|_{L_\infty(\dot\Omega\times]0,T[)} \\ &\quad + T^{\frac{q-N}{pq}}\|\partial_t\mathscr{A}^{tr}_W\|^{1-\frac{N}{2q}}_{L_\infty(0,T;L_q(\dot\Omega))}\|\partial_t\mathscr{A}^{tr}_W\|^{\frac{N}{2q}}_{L_p(0,T;H^1_q(\dot\Omega))}\Big)^{\frac12}\|D_\xi\widetilde{\boldsymbol{W}}\|_{H^{\frac12,\frac12}_{q,p}(\dot\Omega\times\mathbb{R})} \\ &\lesssim T^{\frac{1}{2p'}+\frac{\sigma_{p,q}}{2}}L^{\frac32}\Big(T^{\frac{1}{p'}}L + T^{\frac{q-N}{pq}}\|\boldsymbol{W}\|^{1-\frac{N}{2q}}_{L_\infty(0,T;W^1_q(\dot\Omega))}L^{\frac{N}{2q}}\Big)^{\frac12}.\end{aligned}$$

As $\boldsymbol{W}\in L_\infty\big(0,T;\mathcal{D}^{2-2/p}_{q,p}(\dot\Omega)\big)\hookrightarrow L_\infty\big(0,T;W^1_q(\dot\Omega)^N\big)$ for $p>2$, the above inequalities yield

$$\|\widetilde{g}_w\|_{H^{\frac12}_{p,0,\gamma_0}(\mathbb{R};L_q(\dot\Omega))} \lesssim T^{\frac12\left(\frac{pq-N}{pq}+\sigma_{p,q}\right)}L^2. \tag{3.36}$$

On the other hand, if $p\in]1,2]$ such that $\frac1p+\frac Nq>1$, then we infer from Lemma 3.12 and (3.18) that

$$\begin{aligned}\|\widetilde{g}_w\|_{H^{\frac{1}{2}}_{p,0,\gamma_0}(\mathbb{R};L_q(\dot\Omega))} &\lesssim \|\mathbb{I}-\mathscr{A}_W^{tr}\|^{\frac{1}{2}}_{L_\infty(\dot\Omega\times]0,T[)}\Big(\|\nabla_\xi\mathscr{A}_W^{tr}\|_{L_\infty(0,T;L_q(\dot\Omega))}+\|\mathbb{I}-\mathscr{A}_W^{tr}\|_{L_\infty(\dot\Omega\times]0,T[)}\\ &\quad+T^{\frac{3}{2}-\frac{1}{p}-\frac{N}{2q}-\frac{N}{2\beta}}\|\partial_t(\mathbb{I}-\mathscr{A}_W^{tr})\|_{L_{p/\theta}(0,T;L_\beta(\dot\Omega))}\Big)^{\frac{1}{2}}\|D_\xi\widetilde{\boldsymbol{W}}\|_{H^{\frac{1}{2},\frac{1}{2}}_{q,p}(\dot\Omega\times\mathbb{R})}\\ &\lesssim T^{\frac{1}{2p'}+\frac{\sigma_{p,q}}{2}}L^{\frac{3}{2}}\Big(T^{\frac{1}{p'}}L+T^{\frac{3}{2}-\frac{1}{p}-\frac{N}{2q}-\frac{N}{2\beta}}\|\partial_t\mathscr{A}_W\|_{L_{p/\theta}(0,T;L_\beta(\dot\Omega))}\Big)^{\frac{1}{2}}\end{aligned}$$

for some θ and β such that $1-\frac{2(1-\theta)}{p}=\frac{N}{q}-\frac{N}{\beta}$ and $\frac{N}{q}+\frac{N}{\beta}\le 1$. Moreover, apply Lemma 3.7 and the comments below Proposition 3.10,

$$\|\partial_t\mathscr{A}_W\|_{L_{p/\theta}(0,T;L_\beta(\dot\Omega))}\lesssim\|\nabla_\xi\boldsymbol{W}\|_{L_{p/\theta}(0,T;L_\beta(\dot\Omega))}\lesssim L. \tag{3.37}$$

Thus we conclude

$$\|\widetilde{g}_w\|_{H^{\frac{1}{2}}_{p,0,\gamma_0}(\mathbb{R};L_q(\dot\Omega))}\lesssim T^{\frac{1}{p'}+\frac{\sigma_{p,q}}{2}}L^2. \tag{3.38}$$

Finally, combining the estimates (3.34), (3.36) and (3.38) yields

$$\|\widetilde{g}_w\|_{H^{1,\frac{1}{2}}_{q,p,0,\gamma_0}(\dot\Omega\times\mathbb{R})}\lesssim(T^{\frac{1}{p'}+\sigma_{p,q}}+T^{s_{p,q}})L^2\lesssim T^{s_{p,q}}L^2, \tag{3.39}$$

where the index $s_{p,q}$ is defined by

$$s_{p,q}:=\begin{cases}\frac{1}{2}\big(\frac{pq-N}{pq}+\sigma_{p,q}\big), & \text{for}\quad (p,q)\in(I);\\ \frac{1}{p'}+\frac{\sigma_{p,q}}{2}, & \text{for}\quad (p,q)\in(II).\end{cases}$$

Bound of $\widetilde{\boldsymbol{R}}_w$

Recall the definition $\widetilde{\boldsymbol{R}}_w=\big(E_{(T)}(\mathbb{I}-\mathscr{A}_W^{tr})\big)\widetilde{\boldsymbol{W}}$ in (3.32). Obviously $\widetilde{\boldsymbol{R}}_w(\cdot,t)\equiv\overline{\boldsymbol{R}}_w(\cdot,t)$ for $t\in]0,T[$. Besides, employing Lemma 3.7 and (3.18) implies that

$$\|\partial_t\widetilde{\boldsymbol{R}}_W\|_{L_{p,0,\gamma_0}(\mathbb{R};L_q(\dot\Omega))}\lesssim\|\widetilde{W}^j\partial_{\xi_\ell}E_{(T)}W^k\|_{L_p(0,2T;L_q(\dot\Omega))}+T^{\frac{1}{p'}+\sigma_{p,q}}L^2. \tag{3.40}$$

So to handle the nonlinear term $\widetilde{W}^j\partial_{\xi_\ell}E_{(T)}W^k$ on the right hand side of (3.40), let us first consider the case where $N/q+2/p\le 1$. In this situation, we have the embeddings $\mathcal{D}^{2-2/p}_{q,p}(\dot\Omega)\hookrightarrow W^1_q(\dot\Omega)\hookrightarrow L_\infty(\dot\Omega)$ and thus

$$\|\widetilde{W}^j\partial_{\xi_\ell}E_{(T)}W^k\|_{L_p(0,2T;L_q(\dot\Omega))}\lesssim T^{\frac{1}{p}}\|\widetilde{\boldsymbol{W}}\|_{L_\infty(\dot\Omega\times]0,2T[)}\|\nabla_\xi\boldsymbol{W}\|_{L_\infty(0,T;L_q(\dot\Omega))}\lesssim T^{\frac{1}{p}}L^2. \tag{3.41}$$

On the other hand, assuming $N/q+2/p>1$, we infer from (3.16),

$$\|\widetilde{W}^j\partial_{\xi_\ell}E_{(T)}W^k\|_{L_p(0,T;L_q(\dot\Omega))}\lesssim\|\nabla_\xi\boldsymbol{W}\|_{L_p(0,T;L_\infty(\dot\Omega))}\|\widetilde{\boldsymbol{W}}\|_{L_\infty(0,2T;L_q(\dot\Omega))}\lesssim T^{\sigma_{p,q}}L^2. \tag{3.42}$$

At last, adding (3.41) and (3.42) into (3.40) yields for all $(p,q)\in]1,\infty[\times]N,\infty[$,

$$\|\partial_t \widetilde{\boldsymbol{R}}_W\|_{L_{p,0,\gamma_0}(\mathbb{R};L_q(\dot{\Omega}))} \lesssim T^{\widetilde{\sigma}_{p,q}} L^2 \quad \text{with } \widetilde{\sigma}_{p,q} := \min\Big\{\frac{1}{p}, \frac{1}{2}-\frac{N}{2q}\Big\} > 0. \tag{3.43}$$

Bound of $\widetilde{\boldsymbol{h}}_{w,P}$

Recall the symbol $\mathbb{H}_W = \mathbb{D}(\boldsymbol{W}) - \mathbb{D}_W(\boldsymbol{W})$ and write out

$$\overline{\boldsymbol{h}}_{w,P} = \mu(\rho_0)\mathbb{H}_W \boldsymbol{n} + \mu(\rho_0)\mathbb{H}_W(\overline{\boldsymbol{n}}_W - \boldsymbol{n}) - \mu(\rho_0)\mathbb{D}(\boldsymbol{W})(\overline{\boldsymbol{n}}_W - \boldsymbol{n}) + \Theta\,(\overline{\boldsymbol{n}}_W - \boldsymbol{n})$$

In fact, this motivates us to introduce

$$\widetilde{\boldsymbol{h}}_{w,P} := \mu(\rho_0)\widetilde{\mathbb{H}}_W \boldsymbol{n} + \mu(\rho_0)\widetilde{\mathbb{H}}_W E_{(T)}(\overline{\boldsymbol{n}}_W - \boldsymbol{n}) - \mu(\rho_0)\mathbb{D}(\widetilde{\boldsymbol{W}})E_{(T)}(\overline{\boldsymbol{n}}_W - \boldsymbol{n}) + \Pi\, E_{(T)}(\overline{\boldsymbol{n}}_W - \boldsymbol{n}).$$

Clearly $[\![\widetilde{\boldsymbol{h}}_{w,P}(\cdot,t)]\!] = [\![\overline{\boldsymbol{h}}_{w,P}(\cdot,t)]\!]$ on Γ for any $t\in]0,T[$. If the following property holds true,

$$\widetilde{\boldsymbol{h}}_{w,P} \in L_{p,0,\gamma_0}\big(\mathbb{R}; W^1_q(\dot{\Omega})^N\big) \cap H^{\frac{1}{2}}_{p,0,\gamma_0}\big(\mathbb{R}; L_q(\dot{\Omega})^N\big) \quad \text{for some } \gamma_0 > 1, \tag{3.44}$$

then $\widetilde{\boldsymbol{h}}_{w,P}$ is exactly one desired extension. In the rest of this step, our main task is to check (3.44).

Firstly, the inequalities (3.26), (3.27) and Lemma 3.9 imply that

$$\|\mu(\rho_0)(\widetilde{\mathbb{H}}_W, \widetilde{\mathbb{H}}_W \boldsymbol{n})\|_{L_{p,0,\gamma_0}(\mathbb{R};W^1_q(\dot{\Omega}))} \lesssim T^{\frac{1}{p'}+\sigma_{p,q}} L^2\big(\|\mu\|_{L_\infty(\dot{\Omega})} + T^{\sigma_{p,q}}\|\nabla\mu\|_{L_q(\dot{\Omega})}\big). \tag{3.45}$$

Now thanks to (3.18), (3.45) and (3.21) yield respectively,

$$\|\mu(\rho_0)\widetilde{\mathbb{H}}_W E_{(T)}(\overline{\boldsymbol{n}}_W - \boldsymbol{n})\|_{L_{p,0,\gamma_0}(\mathbb{R};W^1_q(\dot{\Omega}))} \lesssim T^{\frac{2}{p'}+2\sigma_{p,q}} L^3\big(\|\mu\|_{L_\infty(\dot{\Omega})} + T^{\sigma_{p,q}}\|\nabla\mu\|_{L_q(\dot{\Omega})}\big), \tag{3.46}$$

$$\|\mu(\rho_0)\mathbb{D}(\widetilde{\boldsymbol{W}})E_{(T)}(\overline{\boldsymbol{n}}_W - \boldsymbol{n})\|_{L_{p,0,\gamma_0}(\mathbb{R};W^1_q(\dot{\Omega}))} \lesssim T^{\frac{1}{p'}+\sigma_{p,q}} L^2\big(\|\mu\|_{L_\infty(\dot{\Omega})} + T^{\sigma_{p,q}}\|\nabla\mu\|_{L_q(\dot{\Omega})}\big). \tag{3.47}$$

The estimate of the last term in $\widetilde{\boldsymbol{h}}_{w,P}$ is immediate from (3.18),

$$\|\Pi\, E_{(T)}(\overline{\boldsymbol{n}}_W - \boldsymbol{n})\|_{L_{p,0,\gamma_0}(\mathbb{R};W^1_q(\dot{\Omega}))} \lesssim T^{\frac{1}{p'}} L^2. \tag{3.48}$$

Therefore, combining the bounds (3.45)-(3.48) and Condition (3.17), we arrive

$$\|\widetilde{\boldsymbol{h}}_{w,P}\|_{L_{p,0,\gamma_0}(\mathbb{R};W^1_q(\dot{\Omega}))} \lesssim T^{\frac{1}{p'}+\sigma_{p,q}} L^2\big(\|\mu\|_{L_\infty(\dot{\Omega})} + T^{\sigma_{p,q}}\|\nabla\mu\|_{L_q(\dot{\Omega})}\big) + T^{\frac{1}{p'}} L^2, \tag{3.49}$$

which completes the first part proof of the claim (3.44).

To verify the rest property in (3.44), we first recall the proof of (3.39) and obtain that

$$\|\widetilde{\mathbb{H}}_W\|_{H^{1,\frac{1}{2}}_{q,p,0,\gamma_0}(\dot{\Omega}\times\mathbb{R})} \lesssim T^{s_{p,q}}L^2. \tag{3.50}$$

Thus the bound (3.27) and Lemma 3.9 imply that

$$\|\mu(\rho_0)(\widetilde{\mathbb{H}}_W, \widetilde{\mathbb{H}}_W\boldsymbol{n})\|_{H^{1,\frac{1}{2}}_{q,p,0,\gamma_0}(\dot{\Omega}\times\mathbb{R})} \lesssim \|\mu\|_{L_\infty(\dot{\Omega})}T^{s_{p,q}}L^2 + \|\nabla\mu(\rho_0)\|_{L_q(\dot{\Omega})}T^{\frac{1}{p'}+2\sigma_{p,q}}L^2. \tag{3.51}$$

Now, note that Lemma 3.9 and (3.37) yield

$$\|\partial_t(\overline{\boldsymbol{n}}_W - \boldsymbol{n})\|_{L_{p/\theta}(0,T;L_\beta(\dot{\Omega}))} \lesssim \|\partial_t\mathscr{A}_W\|_{L_{p/\theta}(0,T;L_\beta(\dot{\Omega}))} \lesssim L.$$

Thus by means of (3.18), we can produce similar arguments as (3.36) and (3.38) with replacing $\mathscr{A}_W - \mathbb{I}$ by $\overline{\boldsymbol{n}}_W - \boldsymbol{n}$. Indeed, we can conclude

$$\begin{aligned} \|\mu(\rho_0)\widetilde{\mathbb{H}}_W E_{(T)}(\overline{\boldsymbol{n}}_W - \boldsymbol{n})\|_{H^{\frac{1}{2}}_{p,0,\gamma_0}(\mathbb{R};L_q(\dot{\Omega}))} &\lesssim \|\mu\|_{L_\infty(\dot{\Omega})}T^{2s_{p,q}}L^3 \\ &\quad + \|\nabla\mu(\rho_0)\|_{L_q(\dot{\Omega})}T^{\frac{1}{p'}+2\sigma_{p,q}+s_{p,q}}L^3. \end{aligned} \tag{3.52}$$

$$\|\mu(\rho_0)\mathbb{D}(\widetilde{\boldsymbol{W}})E_{(T)}(\overline{\boldsymbol{n}}_W - \boldsymbol{n})\|_{H^{\frac{1}{2}}_{p,0,\gamma_0}(\mathbb{R};L_q(\dot{\Omega}))} \lesssim \|\mu\|_{L_\infty(\dot{\Omega})}T^{s_{p,q}}L^2. \tag{3.53}$$

$$\|\Pi\, E_{(T)}(\overline{\boldsymbol{n}}_W - \boldsymbol{n})\|_{H^{\frac{1}{2}}_{p,0,\gamma_0}(\mathbb{R};L_q(\dot{\Omega}))} \lesssim T^{s_{p,q}}L^2. \tag{3.54}$$

Lastly, keeping the following convention in mind

$$CT^{s_{p,q}}L \leq 1 \quad \text{for some constant } C, \tag{3.55}$$

we infer from (3.51)-(3.54)

$$\|\widetilde{\boldsymbol{h}}_{w,P}\|_{H^{\frac{1}{2}}_{p,0,\gamma_0}(\mathbb{R}_+;L_q(\dot{\Omega}))} \lesssim (\|\mu\|_{L_\infty(\dot{\Omega})}+1)T^{s_{p,q}}L^2 + \|\nabla\mu(\rho_0)\|_{L_q(\dot{\Omega})}T^{\frac{1}{p'}+2\sigma_{p,q}}L^2. \tag{3.56}$$

Hence combining the estimates (3.49) and (3.56), we end up with

$$\|\widetilde{\boldsymbol{h}}_{w,P}\|_{H^{1,\frac{1}{2}}_{q,p,0,\gamma_0}(\dot{\Omega}\times\mathbb{R})} \lesssim \|\mu\|_{L_\infty(\dot{\Omega})}T^{s_{p,q}}L^2 + \|\nabla\mu(\rho_0)\|_{L_q(\dot{\Omega})}T^{\frac{1}{p'}+2\sigma_{p,q}}L^2 + T^{\min\{s_{p,q},\frac{1}{p'}\}}L^2. \tag{3.57}$$

Bound of $\widetilde{\boldsymbol{k}}_{w_+,P_+}$

Inspired by previous step, we introduce the following notations,

$$\begin{aligned}\widetilde{\mathbb{K}}_{W_+} &:= D_\xi \widetilde{\boldsymbol{W}}_+ E_{(T)}(\mathbb{I} - \mathscr{A}^{tr}_{W_+}) + E_{(T)}(\mathbb{I} - \mathscr{A}_{W_+})\nabla_\xi \widetilde{\boldsymbol{W}}_+,\\ \widetilde{\boldsymbol{k}}_{w_+,P_+} &:= \mu(\rho_0)\widetilde{\mathbb{K}}_{W_+}\boldsymbol{n}_+ + \mu(\rho_0)\widetilde{\mathbb{K}}_{W_+}E_{(T)}(\overline{\boldsymbol{n}}_{W_+} - \boldsymbol{n}_+)\\ &\quad - \mu(\rho_0)\mathbb{D}(\widetilde{\boldsymbol{W}}_+)E_{(T)}(\overline{\boldsymbol{n}}_{W_+} - \boldsymbol{n}) + \Pi_+ E_{(T)}(\overline{\boldsymbol{n}}_{W_+} - \boldsymbol{n}_+).\end{aligned}$$

In fact, $\widetilde{\boldsymbol{k}}_{w_+,P_+}$ defined above is one desired extension. Indeed, taking advantage of similar arguments for $\widetilde{\boldsymbol{h}}_{w,P}$, we can find

$$\|\widetilde{\boldsymbol{k}}_{w_+,P_+}\|_{H^{1,\frac12}_{q,p,0,\gamma_0}(\dot\Omega_+\times\mathbb{R})} \lesssim \|\mu\|_{L_\infty(\dot\Omega)}T^{s_{p,q}}L^2 + \|\nabla\mu(\rho_0)\|_{L_q(\dot\Omega)}T^{\frac{1}{p'}+2\sigma_{p,q}}L^2 + T^{\min\{s_{p,q},\frac{1}{p'}\}}L^2. \tag{3.58}$$

Finally, putting the bounds (3.23), (3.39), (3.43), (3.57) and (3.58) together yields

$$\begin{aligned}\|(\widetilde{\boldsymbol{f}}_{w,P}, \widetilde{g}_w, \widetilde{\boldsymbol{R}}_w, \widetilde{\boldsymbol{h}}_{w,P}, \widetilde{\boldsymbol{k}}_{w_+,P_+})\|_{\mathcal{Y}_{p,q,\gamma_0}} &\lesssim \|\boldsymbol{f}\|_{L_p(0,T;L_q(\dot\Omega))} + \|\eta - \rho_0\|_{L_\infty(\dot\Omega\times]0,T[)}L\\ &+ \|\mu\|_{L_\infty(\dot\Omega)}T^{s_{p,q}}L^2 + \|\nabla\mu(\rho_0)\|_{L_q(\dot\Omega)}T^{\frac{1}{p'}+2\sigma_{p,q}}L^2 + T^{\min\{\widetilde\sigma_{p,q},s_{p,q},\frac{1}{p'}\}}L^2.\end{aligned}$$

Thus taking the difference $\eta - \rho_0$ and T small enough, we find that

$$\|\widetilde{\boldsymbol{U}}\|_{W^{2,1}_{q,p,0,\gamma_0}(\dot\Omega\times\mathbb{R})} + \|\nabla_\xi\widetilde{Q}\|_{L_{p,0,\gamma_0}(\mathbb{R};L_q(\dot\Omega))} \le cL + C_{p,q,\gamma_0}T^{\min\{\widetilde\sigma_{p,q},s_{p,q},\frac{1}{p'}\}}L^2. \tag{3.59}$$

Recalling the definitions (3.12) and (3.13), we infer from above inequality that

$$\|\Xi\|_{H^{1,\frac12}_{q,p,0,\gamma_0}(\dot\Omega\times\mathbb{R})} + \|\Xi_+\|_{H^{1,\frac12}_{q,p,0,\gamma_0}(\dot\Omega_+\times\mathbb{R})} \le cL + C_{p,q,\gamma_0}T^{\min\{\widetilde\sigma_{p,q},s_{p,q},\frac{1}{p'}\}}L^2. \tag{3.60}$$

Combining (3.59) and (3.60), we know that for small enough T and for any $(\boldsymbol{w}, P, \Pi, \Pi_+)$ in $\mathscr{E}_L(T)$,

$$(\boldsymbol{U}, Q, \Xi, \Xi_+) := (\widetilde{\boldsymbol{U}}|_{]0,T[}, \widetilde{Q}|_{]0,T[}, \Xi, \Xi_+) \in \mathscr{E}_L(T).$$

3.4 Stability

In this section, we shall show the existence of a unique fixed point of $\Phi(\cdot)$ in $\mathscr{E}_L(T)$, as long as T small enough. Here our proof is still based on extension arguments. So let us introduce some notations for convenience. Assume $(\boldsymbol{w}_k, P_k, \Pi_k, \Pi_{+,k})$ with $k = 1, 2$, are any two distinct points in $\mathscr{E}_L(T)$ and $(\widetilde{\boldsymbol{U}}_k, \widetilde{Q}_k)$ are corresponding solution (global in time) of System (3.9) provided with the extensions $(\widetilde{\boldsymbol{f}}_{w_k,P_k}, \widetilde{g}_{w_k}, \widetilde{\boldsymbol{R}}_{w_k}, \widetilde{\boldsymbol{h}}_{w_k,P_k}, \widetilde{\boldsymbol{k}}_{w_{k,+},P_{k,+}})$ as before. For $k = 1, 2$, set

$$
\begin{aligned}
(\delta\widetilde{\boldsymbol{U}}, \delta\widetilde{Q}, \delta\Pi, \delta\Pi_+, \delta\overline{\boldsymbol{n}}) &:= (\widetilde{\boldsymbol{U}}_2 - \widetilde{\boldsymbol{U}}_1, \widetilde{Q}_2 - \widetilde{Q}_1, \Pi_2 - \Pi_1, \Pi_{+,2} - \Pi_{+,1}, \overline{\boldsymbol{n}}_{W_2} - \overline{\boldsymbol{n}}_{W_1}),\\
(\boldsymbol{W}_k, \Theta_k, \widetilde{\boldsymbol{W}}_k, \delta\mathscr{A}_W) &:= (\boldsymbol{u}_L + \boldsymbol{w}_k, \mathfrak{p}_L + P_k, E_{(T)}\boldsymbol{w}_k + \widetilde{\boldsymbol{u}}_L, \mathscr{A}_{W_2} - \mathscr{A}_{W_1}),\\
(\delta\boldsymbol{w}, \delta P, \delta\widetilde{\boldsymbol{W}}) &:= (\boldsymbol{w}_2 - \boldsymbol{w}_1, P_2 - P_1, \widetilde{\boldsymbol{W}}_2 - \widetilde{\boldsymbol{W}}_1)\\
&\equiv (\boldsymbol{W}_2 - \boldsymbol{W}_1, \Theta_2 - \Theta_1, E_{(T)}\delta\boldsymbol{w}).
\end{aligned}
$$

Moreover, we know from Equations (3.9) that $(\delta\widetilde{\boldsymbol{U}}, \delta\widetilde{Q})$ solves

$$
\begin{cases}
\eta\,\partial_t\delta\widetilde{\boldsymbol{U}} - \operatorname{Div}_\xi\mathbb{T}(\delta\widetilde{\boldsymbol{U}}, \delta\widetilde{Q}) = \delta\widetilde{\boldsymbol{f}} & \text{in} \quad \dot{\Omega}\times\mathbb{R}_+,\\
\operatorname{div}_\xi\delta\widetilde{\boldsymbol{U}} = \delta\widetilde{g} = \operatorname{div}_\xi\delta\widetilde{\boldsymbol{R}} & \text{in} \quad \dot{\Omega}\times\mathbb{R}_+,\\
[\![\mathbb{T}(\delta\widetilde{\boldsymbol{U}}, \delta\widetilde{Q})\boldsymbol{n}]\!] = [\![\delta\widetilde{\boldsymbol{h}}]\!], \;\; [\![\delta\widetilde{\boldsymbol{U}}]\!] = \boldsymbol{0} & \text{in} \quad \Gamma\times\mathbb{R}_+,\\
\mathbb{T}_+(\delta\widetilde{\boldsymbol{U}}_+, \delta\widetilde{Q}_+)\boldsymbol{n}_+ = \delta\widetilde{\boldsymbol{k}} & \text{on} \quad \Gamma_+\times\mathbb{R}_+,\\
\delta\widetilde{\boldsymbol{U}}_- = \boldsymbol{0} & \text{on} \quad \Gamma_-\times\mathbb{R}_+,\\
\delta\widetilde{\boldsymbol{U}}|_{t=0} = \boldsymbol{0} & \text{on} \quad \dot{\Omega},
\end{cases}
\tag{3.61}
$$

where $(\delta\widetilde{\boldsymbol{f}}, \delta\widetilde{g}, \delta\widetilde{\boldsymbol{R}}, \delta\widetilde{\boldsymbol{h}}, \delta\widetilde{\boldsymbol{k}})$ are given by

$$
\begin{aligned}
(\delta\widetilde{\boldsymbol{f}}, \delta\widetilde{\boldsymbol{h}}, \delta\widetilde{\boldsymbol{k}}) &:= (\widetilde{\boldsymbol{f}}_{w_2,P_2} - \widetilde{\boldsymbol{f}}_{w_1,P_1}, \widetilde{\boldsymbol{h}}_{w_2,P_2} - \widetilde{\boldsymbol{h}}_{w_1,P_1}, \widetilde{\boldsymbol{k}}_{w_{2,+},P_{2,+}} - \widetilde{\boldsymbol{k}}_{w_{1,+},P_{1,+}}),\\
\delta\widetilde{g} &:= D_\xi E_{(T)}\delta\boldsymbol{w} : E_{(T)}(\mathbb{I} - \mathscr{A}^{tr}_{W_2}) - D_\xi\widetilde{\boldsymbol{W}}_1 : E_{(T)}\delta\mathscr{A}^{tr}_W,\\
\delta\widetilde{\boldsymbol{R}} &:= E_{(T)}(\mathbb{I} - \mathscr{A}^{tr}_{W_2})\,E_{(T)}\delta\boldsymbol{w} - E_{(T)}\delta\mathscr{A}^{tr}_W\,\widetilde{\boldsymbol{W}}_1.
\end{aligned}
$$

Therefore, to apply Theorem 3.6 to System (3.61), we need $(\delta\widetilde{\boldsymbol{f}}, \delta\widetilde{g}, \delta\widetilde{\boldsymbol{R}}, \delta\widetilde{\boldsymbol{h}}, \delta\widetilde{\boldsymbol{k}}) \in \mathcal{Y}_{p,q,\gamma_0}$ for some $\gamma_0 > 0$. More importantly, setting

$$
\mathfrak{L} := \|(\delta\widetilde{\boldsymbol{U}}|_{]0,T[}, \delta\widetilde{Q}|_{]0,T[}, \delta\Pi, \delta\Pi_+)\|_{\mathscr{E}(T)}
$$

and taking T small enough, we can prove

$$
\|\Phi(\delta\boldsymbol{w}, \delta P, \delta\Pi, \delta\Pi_+)\|_{\mathscr{E}(T)} \leq \frac{1}{2}\mathfrak{L}. \tag{3.62}
$$

which yields the mapping Φ is contracting. Therefore,Φ has a unique fixed point $(\boldsymbol{U}, Q, \Xi, \Xi_+)$ in $\mathscr{E}_L(T)$ such that $(\boldsymbol{U}, Q)$ solves the nonlinear System (3.8).

To prove (3.62), we will constantly use the decay estimate in the rest of this section,

$$
\|\nabla_\xi\delta\boldsymbol{w}\|_{L_1(0,T;L_\infty(G))} \lesssim T^{\frac{1}{p'}}\|\nabla_\xi\delta\boldsymbol{w}\|_{L_p(0,T;L_\infty(G))} \lesssim T^{\frac{1}{p'}+\sigma_{p,q}}\mathfrak{L}. \tag{3.63}
$$

Combining this decay property and Condition (3.17), we infer from Lemma 3.8 and Lemma 3.9 that

$$\|(\delta\mathscr{A}_W, \delta\overline{\boldsymbol{n}})\|_{L_\infty(\dot\Omega\times]0,T[)} \lesssim T^{\frac{1}{p'}+\sigma_{p,q}}\mathfrak{L}, \qquad \|\nabla_\xi(\delta\mathscr{A}_W, \delta\overline{\boldsymbol{n}})\|_{L_\infty(0,T;L_q(\dot\Omega))} \lesssim T^{\frac{1}{p'}}\mathfrak{L}, \tag{3.64}$$
$$\|\partial_t(\delta\mathscr{A}_W, \delta\overline{\boldsymbol{n}})\|_{L_p(0,T;L_\infty(\dot\Omega))} \lesssim T^{\sigma_{p,q}}\mathfrak{L}, \qquad \|\partial_t\nabla_\xi(\delta\mathscr{A}_W, \delta\overline{\boldsymbol{n}})\|_{L_p(0,T;L_q(\dot\Omega))} \lesssim \mathfrak{L}.$$

Bound of $\delta\widetilde{\boldsymbol{f}}$

The proof here is similar to the bound of $\widetilde{\boldsymbol{f}}_{w,P}$ before. So it is convenient to use the following notations for $k = 1, 2$,

$$\begin{aligned}
\delta\widetilde{\mathbb{H}} := \widetilde{\mathbb{H}}_{W_2} - \widetilde{\mathbb{H}}_{W_1} &= D_\xi E_{(T)}\delta\boldsymbol{w}\cdot E_{(T)}(\mathbb{I} - \mathscr{A}^{tr}_{W_2}) - D_\xi\widetilde{\boldsymbol{W}}_1\cdot E_{(T)}\delta\mathscr{A}^{tr}_W \\
&\quad + E_{(T)}(\mathbb{I} - \mathscr{A}_{W_2})\cdot\nabla_\xi E_{(T)}\delta\boldsymbol{w} - E_{(T)}\delta\mathscr{A}_W\cdot\nabla_\xi\widetilde{\boldsymbol{W}}_1, \\
\delta\widetilde{\mathbb{D}} := \widetilde{\mathbb{D}}_{W_2} - \widetilde{\mathbb{D}}_{W_1} &= E_{(T)}(\mathbb{I} - \mathscr{A}^{tr}_{W_2})\cdot\mathbb{D}(E_{(T)}\delta\boldsymbol{w}) - E_{(T)}\delta\mathscr{A}^{tr}_W\cdot\mathbb{D}(\widetilde{\boldsymbol{W}}_1).
\end{aligned}$$

With above conventions in mind, we may write $\delta\widetilde{\boldsymbol{f}}$ as below,

$$\begin{aligned}
\delta\widetilde{\boldsymbol{f}} := \rho_0 E_{(T)}&\Big(\boldsymbol{f}\big(\boldsymbol{X}_{W_2}(\xi,t),t\big) - \boldsymbol{f}\big(\boldsymbol{X}_{W_1}(\xi,t),t\big)\Big) \\
&+ (\eta - \rho_0)E_{(T)}\partial_t\boldsymbol{w} - \mathrm{Div}_\xi\big(\mu(\rho_0)(\delta\widetilde{\mathbb{H}} + \delta\widetilde{\mathbb{D}})\big) \\
&+ \mathrm{Div}_\xi\Big(\mu(\rho_0)\big(E_{(T)}(\mathbb{I} - \mathscr{A}^{tr}_{W_2})\cdot\delta\widetilde{\mathbb{H}} - E_{(T)}\delta\mathscr{A}^{tr}_W\cdot\widetilde{\mathbb{H}}_{W_1}\big)\Big) \\
&+ \mathrm{Div}_\xi\big((E_{(T)}\delta P)E_{(T)}(\mathbb{I} - \mathscr{A}^{tr}_{W_2}) - (E_{(T)}\Theta_1)E_{(T)}\delta\mathscr{A}^{tr}_W\big).
\end{aligned}$$

On one hand, it is clear from our conventions on $\boldsymbol{f}$ that

$$\big\|\boldsymbol{f}\big(\boldsymbol{X}_{W_2}(\xi,t),t\big) - \boldsymbol{f}\big(\boldsymbol{X}_{W_1}(\xi,t),t\big)\big\|_{L_p(0,T;L_q(\dot\Omega))} \lesssim T\|\nabla_\xi\boldsymbol{f}\|_{L_p(0,T;L_\infty(\mathbb{R}^N))}\|\delta\boldsymbol{w}\|_{L_\infty(0,T;L_q(\dot\Omega))}.$$

On the other hand, it is not hard to see from (3.64) that for $1 \leq j, k, \ell, m \leq N$,

$$\big\|\big(D_\xi\widetilde{\boldsymbol{W}}_1\big)^j_k\big(E_{(T)}\delta\mathscr{A}_W\big)^\ell_m\big\|_{L_{p,0,\gamma_0}(\mathbb{R};W^1_q(\dot\Omega))} \lesssim T^{\frac{1}{p'}+\sigma_{p,q}}L\mathfrak{L}, \tag{3.65}$$
$$\big\|\big(D_\xi\widetilde{\boldsymbol{W}}_1\big)^j_k\big(E_{(T)}\delta\mathscr{A}_W\big)^\ell_m\big\|_{L_{p,0,\gamma_0}(\mathbb{R};L_\infty(\dot\Omega))} \lesssim T^{\frac{1}{p'}+2\sigma_{p,q}}L\mathfrak{L}. \tag{3.66}$$

Putting together (3.65) and (3.66) implies

$$\|\mu(\rho_0)(\delta\widetilde{\mathbb{H}}, \delta\widetilde{\mathbb{D}})\|_{L_{p,0,\gamma_0}(\mathbb{R};W^1_q(\dot\Omega))} \lesssim T^{\frac{1}{p'}+\sigma_{p,q}}L\mathfrak{L}\big(\|\mu\|_{L_\infty(\dot\Omega)} + T^{\sigma_{p,q}}\|\nabla\mu\|_{L_q(\dot\Omega)}\big), \tag{3.67}$$

Therefore, applying the similar computations as $\widetilde{\boldsymbol{f}}_{w,P}$, we conclude

$$\|\delta\widetilde{\boldsymbol{f}}\|_{L_{p,0,\gamma_0}(\mathbb{R};L_q(\dot\Omega))} \lesssim c\mathfrak{L} + T^{\frac{1}{p'}+\sigma_{p,q}}L\mathfrak{L}\big(\|\mu\|_{L_\infty(\dot\Omega)} + T^{\sigma_{p,q}}\|\nabla\mu\|_{L_q(\dot\Omega)}\big) + T^{\frac{1}{p'}}L\mathfrak{L}. \tag{3.68}$$

Bound of $\delta\widetilde{g}$

To verify the estimates of $\delta\widetilde{g}$, we immediately know from the bound (3.65) that

$$\|\delta\widetilde{g}\|_{L_{p,0,\gamma_0}(\mathbb{R};W^1_q(\dot\Omega))} \lesssim T^{\frac{1}{p'}+\sigma_{p,q}} L\mathfrak{L}. \tag{3.69}$$

Next, remembering Lemma 3.8 and (3.63), we obtain for $1 \leq j,k,\ell,m \leq N$,

$$\big\|\big(D_\xi \widetilde{\boldsymbol{W}}_1\big)^j_k \big(E_{(T)}\delta\mathscr{A}_W\big)^\ell_m\big\|_{H^{\frac12}_{p,0,\gamma_0}(\mathbb{R};L_q(\dot\Omega))} \lesssim T^{s_{p,q}} L\mathfrak{L}. \tag{3.70}$$

On the other hand, thanks to Lemma 3.8,(3.63) and (3.17), we have the similar assertion as (3.37),

$$\|\partial_t \delta\mathscr{A}_W\|_{L_{p/\theta}(0,T;L_\beta(\dot\Omega))} \lesssim \mathfrak{L}. \tag{3.71}$$

Thus the arguments for the bounds (3.36) and (3.38) imply that

$$\|\delta\widetilde{g}\|_{H^{\frac12}_{p,0,\gamma_0}(\dot\Omega\times\mathbb{R})} \lesssim T^{s_{p,q}} L\mathfrak{L}, \quad \text{as long as } \ CT^{\frac{1}{p'}}L<1 \ \text{ for some constant } \ C. \tag{3.72}$$

Finally, the bounds (3.69) and (3.72) above yield

$$\|\delta\widetilde{g}\|_{L_{p,0,\gamma_0}(\mathbb{R};W^1_q(\dot\Omega))\cap H^{\frac12}_{p,0,\gamma_0}(\dot\Omega\times\mathbb{R})} \lesssim T^{\min\{\frac{1}{p'}+\sigma_{p,q},s_{p,q}\}} L\mathfrak{L} \lesssim T^{s_{p,q}} L\mathfrak{L}. \tag{3.73}$$

Bound of $\delta\widetilde{\boldsymbol{R}}$

Going along the proof of (3.43) and recalling $\widetilde{\sigma}_{p,q} = \min\{1/p, 1/2 - N/(2q)\}$, we can bound $\delta\widetilde{\boldsymbol{R}}$ as follows,

$$\|\partial_t\delta\widetilde{\boldsymbol{R}}\|_{L_{p,0,\gamma_0}(\mathbb{R};L_q(\dot\Omega))} \lesssim T^{\widetilde{\sigma}_{p,q}} L\mathfrak{L}, \tag{3.74}$$

thanks to (3.64) and Condition (3.17).

Bound of $\delta\widetilde{\boldsymbol{h}}$

Now we write out $\delta\widetilde{\boldsymbol{h}}$ as below

$$\begin{aligned}\delta\widetilde{\boldsymbol{h}} = {} & \mu(\rho_0)\,\delta\widetilde{\mathbb{H}}\,\boldsymbol{n} + \mu(\rho_0)\,\delta\widetilde{\mathbb{H}}\,E_{(T)}(\overline{\boldsymbol{n}}_{W_2}-\boldsymbol{n}) + \mu(\rho_0)\,\widetilde{\mathbb{H}}_{W_1}\,E_{(T)}\delta\overline{\boldsymbol{n}}\\ & - \mu(\rho_0)\,\mathbb{D}(E_{(T)}\delta\boldsymbol{w})\,E_{(T)}(\overline{\boldsymbol{n}}_{W_2}-\boldsymbol{n}) - \mu(\rho_0)\,\mathbb{D}(\widetilde{\boldsymbol{W}}_1)\,E_{(T)}\delta\overline{\boldsymbol{n}}\\ & - \delta\Pi\,E_{(T)}(\overline{\boldsymbol{n}}_{W_2}-\boldsymbol{n}) + \Pi_1\,E_{(T)}\delta\overline{\boldsymbol{n}}.\end{aligned}$$

and claim that

$$\|\delta\widetilde{\boldsymbol{h}}\|_{H^{1,\frac12}_{q,p,0,\gamma_0}(\dot\Omega\times\mathbb{R})} \lesssim \big(\|\mu\|_{L_\infty(\dot\Omega)}T^{s_{p,q}}L+\|\nabla\mu(\rho_0)\|_{L_q(\dot\Omega)}T^{\frac{1}{p'}+2\sigma_{p,q}}L+T^{\min\{s_{p,q},\frac{1}{p'}\}}L\big)\mathfrak{L}. \tag{3.75}$$

In fact, based on our previous discussions and Lemma 3.9, it is not hard to show (3.75).

For instance, let us consider the bound of $\Pi_1\, E_{(T)}\,\delta\overline{\boldsymbol{n}}$ in $H^{\frac12}_{p,0,\gamma_0}(\dot\Omega\times\mathbb{R})$. Thanks to Lemma 3.9, (3.71), (3.37) and (3.64), we have

$$\|\partial_t\delta\overline{\boldsymbol{n}}\|_{L_{p/\theta}(0,T;L_\beta(\dot\Omega))} \lesssim \mathfrak{L}+T^{\frac{1}{p'}+\sigma_{p,q}}L\mathfrak{L}\lesssim\mathfrak{L}. \tag{3.76}$$

Then combing the bound (3.76) with the arguments for (3.36) and (3.38) gives

$$\|\Pi_1\, E_{(T)}\,\delta\overline{\boldsymbol{n}}\|_{H^{\frac12}_{p,0,\gamma_0}(\mathbb{R};L_q(\dot\Omega))}\lesssim T^{s_{p,q}}L\mathfrak{L}.$$

Bound of $\delta\widetilde{\boldsymbol{k}}$

Similarly, we may study the difference $\delta\widetilde{\boldsymbol{k}}$ and show that

$$\|\delta\widetilde{\boldsymbol{k}}\|_{H^{1,\frac12}_{q,p,0,\gamma_0}(\dot\Omega_+\times\mathbb{R})} \lesssim \big(\|\mu\|_{L_\infty(\dot\Omega)}T^{s_{p,q}}L+\|\nabla\mu(\rho_0)\|_{L_q(\dot\Omega)}T^{\frac{1}{p'}+2\sigma_{p,q}}L+T^{\min\{s_{p,q},\frac{1}{p'}\}}L\big)\mathfrak{L}. \tag{3.77}$$

Finally, combining the estimates (3.68), (3.73), (3.74) (3.75) and (3.77) implies that

$$\begin{aligned}\|(\delta\widetilde{\boldsymbol{f}},\delta\widetilde{g},\delta\widetilde{\boldsymbol{R}},\delta\widetilde{\boldsymbol{h}},\delta\widetilde{\boldsymbol{k}})\|_{\mathcal{Y}_{p,q,\gamma_0}} &\lesssim c\mathfrak{L}+\|\mu\|_{L_\infty(\dot\Omega)}T^{s_{p,q}}L\mathfrak{L}+\|\nabla\mu(\rho_0)\|_{L_q(\dot\Omega)}T^{\frac{1}{p'}+2\sigma_{p,q}}L\mathfrak{L}\\ &\quad+T^{\min\{\widetilde\sigma_{p,q},s_{p,q},\frac{1}{p'}\}}L\mathfrak{L}.\end{aligned}$$

Therefore taking c and T small enough yields (3.62).

3.5 Technical lemma

To display the Lagrangian coordinates approach, let us recall some technical results here. Assume that $\boldsymbol{u}\in W^{2,1}_{q,p}(G\times]0,T[)$ for some open (not necessary bounded) domain $G\subset\mathbb{R}^N$ and $T\in\,]0,\infty]$. Denote that

$$\boldsymbol{X}(\xi,t):=\xi+\int_0^t\boldsymbol{u}(\xi,\tau)\,d\tau\quad\text{for }\ \xi\in G.$$

If $\|\int_0^t\nabla_\xi\boldsymbol{u}(\xi,t')\,dt'\|_{L_\infty(G)}$ is strictly smaller than 1, then the following definition makes sense,

$$\mathscr{A}:=(D_\xi\boldsymbol{X}^{-1})^{tr}=(\nabla_\xi\boldsymbol{X})^{-1}=\sum_{k=0}^\infty\Big(-\int_0^t\nabla_\xi\boldsymbol{u}(\xi,t')\,dt'\Big)^k.$$

In fact, we have the following Lemma concerning the estimates of $\mathscr{A}$ and $\mathscr{A}^{tr}$.

Lemma 3.7. *Assume that $\boldsymbol{u}$ is some smooth enough vector field satisfying*

$$\|\nabla_\xi \boldsymbol{u}\|_{L_1(0,T;L_\infty(G))} \le \kappa < 1,$$

and $\mathfrak{A}$ stands for $\mathscr{A}$ or $\mathscr{A}^{tr}$ as we defined above. Then following assertions hold true.

1. *For some terms of $\mathfrak{A}$, there exists a constant $C_{N,\kappa}$ such that*

$$\|\mathfrak{A}\|_{L_\infty(G\times]0,T[)} \le C_{N,\kappa},$$

$$\|(\mathfrak{A}-\mathbb{I}, \mathfrak{A}\mathfrak{A}-\mathbb{I}, \mathfrak{A}\mathfrak{A}^{tr}-\mathbb{I})\|_{L_\infty(G\times]0,T[)} \le C_{N,\kappa}\|\nabla_\xi \boldsymbol{u}\|_{L_1(0,T;L_\infty(G))},$$

2. *For the first order derivative terms of $\mathfrak{A}$ and $q \in]1,\infty[$, there exists a constant $C_{N,k}$ such that*

$$\|\nabla_\xi(\mathfrak{A}, \mathfrak{A}\mathfrak{A}, \mathfrak{A}\mathfrak{A}^{tr})\|_{L_\infty(0,T;L_q(G))} \le C_{N,\kappa}\|\nabla^2_\xi \boldsymbol{u}\|_{L_1(0,T;L_q(G))}.$$

3. *For the time derivative of $\mathfrak{A}$, we have for any suitable $(\widetilde{p},\widetilde{q}) \in [1,\infty]^2$*

$$\|\partial_t\mathfrak{A}\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))} \le C_{N,\kappa}\|\nabla_\xi \boldsymbol{u}\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))},$$

$$\begin{aligned}\|\partial_t\nabla_\xi\mathfrak{A}\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))} \le C_{N,\kappa}\Big(&\|\nabla^2_\xi \boldsymbol{u}\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))}\\ &+\|\nabla_\xi \boldsymbol{u}\|_{L_{\widetilde{p}}(0,T;L_\infty(G))}\|\nabla^2_\xi \boldsymbol{u}\|_{L_1(0,T;L_{\widetilde{q}}(G))}\Big).\end{aligned}$$

Above all constants $C_{N,\kappa}$ go to ∞ as κ tends to 1.

Proof. For simplicity, we only concentrate to the proof concerning the terms $\mathscr{A}$, $\mathscr{A}-\mathbb{I}$ and $\mathscr{A}^{tr}\mathscr{A}-\mathbb{I}$. Recall the definition of $\mathscr{A}$, we have

$$\|\mathscr{A}\|_{L_\infty(G\times]0,T[)} \le \frac{C_N}{1-\|\int_0^t \nabla_\xi \boldsymbol{u}\,dt'\|_{L_\infty(G\times]0,T[)}} \le \frac{C_N}{1-\kappa}.$$

Note that $\|\int_0^t \nabla_\xi \boldsymbol{u}\,dt'\|_{L_\infty(G\times]0,T[)} \le \|\nabla_\xi \boldsymbol{u}\|_{L_1(0,T;L_\infty(G))} < 1$, then we have

$$\|\mathscr{A}-\mathbb{I}\|_{L_\infty(G\times]0,T[)} \le \frac{\|\int_0^t \nabla_\xi \boldsymbol{u}\,dt'\|_{L_\infty(G\times]0,T[)}}{1-\|\int_0^t \nabla_\xi \boldsymbol{u}\,dt'\|_{L_\infty(G\times]0,T[)}} \le \frac{C_N}{1-\kappa}\|\nabla_\xi \boldsymbol{u}\|_{L_1(0,T;L_\infty(G))}.$$

Rewrite the term $\mathscr{A}^{tr}\mathscr{A}-\mathbb{I}$ by

$$\mathscr{A}^{tr}\mathscr{A}-\mathbb{I} = (\mathscr{A}^{tr}-\mathbb{I})(\mathscr{A}-\mathbb{I}) + (\mathscr{A}^{tr}-\mathbb{I}) + (\mathscr{A}-\mathbb{I}), \tag{3.78}$$

and combine the results of $\mathfrak{A} - \mathbb{I}$ yield the bounds for $\mathscr{A}^{tr}\mathscr{A} - \mathbb{I}$ with taking

$$C_{N,\kappa} = C_N \frac{1+\kappa}{1-\kappa} \max\{\frac{1}{(1-\kappa)}, 1\}.$$

Let us consider the first order derivative term $\nabla_\xi \mathscr{A}$, it is not hard to see from definition of $\mathscr{A}$

$$\|\nabla_\xi \mathscr{A}\|_{L_\infty(0,T;L_q(G))} \leq \frac{C_N}{(1-\kappa)^2} \|\nabla_\xi^2 \boldsymbol{u}\|_{L_1(0,T;L_q(G))}.$$

Combine above estimate and (3.78), we have

$$\|\nabla_\xi (\mathscr{A}^{tr}\mathscr{A} - \mathbb{I})\|_{L_\infty(0,T;L_q(G))} \leq C_N \frac{1+\kappa}{(1-\kappa)^2} \|\nabla_\xi^2 \boldsymbol{u}\|_{L_1(0,T;L_q(G))}.$$

Lastly, for the time derivative, we just take advantage the definition of $\mathscr{A}$,

$$\|\partial_t \mathscr{A}\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))} \leq \frac{C_N}{(1-\kappa)^2} \|\nabla_\xi \boldsymbol{u}\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))}, \quad \text{for} \quad \widetilde{p}, \widetilde{q} \in [1,\infty].$$

$$\begin{aligned} \|\partial_t \nabla_\xi \mathscr{A}\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))} \leq & \frac{C_N}{(1-\kappa)^2} \|\nabla_\xi^2 \boldsymbol{u}\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))} \\ & + \frac{C_N}{(1-\kappa)^3} \|\nabla_\xi \boldsymbol{u}\|_{L_{\widetilde{p}}(0,T;L_\infty(G))} \|\nabla_\xi^2 \boldsymbol{u}\|_{L_1(0,T;L_{\widetilde{q}}(G))}. \end{aligned}$$

□

To study the stability problem, we need the following technical results similar to Lemma 3.7.

Lemma 3.8. *Assume that $\boldsymbol{u}_k$ with $k = 1, 2$ are some smooth enough vector field satisfying*

$$\|(\nabla_\xi \boldsymbol{u}_1, \nabla_\xi \boldsymbol{u}_2)\|_{L_1(0,T;L_\infty(G))} \leq \kappa < 1.$$

Define the corresponding mapping $\boldsymbol{X}_k(\xi, t) := \xi + \int_0^t \boldsymbol{u}_k(\xi, \tau)\, d\tau$ associated to $\boldsymbol{u}_k$, and $\mathscr{A}_k := (\nabla_\xi \boldsymbol{X}_k)^{-1}$. For simplicity, $\mathfrak{A}_k$ stands for $\mathscr{A}_k$ or $\mathscr{A}_k^{tr}$, and the notations on difference are given by $(\delta \boldsymbol{u}, \delta \mathfrak{A}) := (\boldsymbol{u}_2 - \boldsymbol{u}_1, \mathfrak{A}_2 - \mathfrak{A}_1)$. Then following assertions hold true.

1. *For some terms of $\mathfrak{A}$, there exists a constant $C_{N,\kappa}$ such that*

$$\|(\delta \mathfrak{A}, \mathfrak{A}_2 \mathfrak{A}_2 - \mathfrak{A}_1 \mathfrak{A}_1, \mathfrak{A}_2 \mathfrak{A}_2^{tr} - \mathfrak{A}_1 \mathfrak{A}_1^{tr})\|_{L_\infty(G \times]0,T[)} \leq C_{N,\kappa} \|\nabla_\xi \delta \boldsymbol{u}\|_{L_1(0,T;L_\infty(G))},$$

2. *For the first order derivative terms of $\mathfrak{A}$ and $q \in]1,\infty[$, there exists a constant $C_{N,k}$ such that*

$$\begin{aligned} \|\nabla_\xi (\delta \mathfrak{A}, \mathfrak{A}_2 \mathfrak{A}_2 - \mathfrak{A}_1 \mathfrak{A}_1, \mathfrak{A}_2 \mathfrak{A}_2^{tr} - \mathfrak{A}_1 \mathfrak{A}_1^{tr})\|_{L_\infty(0,T;L_q(G))} \leq C_{N,\kappa} \Big(& \|\nabla_\xi^2 \delta \boldsymbol{u}\|_{L_1(0,T;L_q(G))} \\ & + \|\nabla_\xi^2 (\boldsymbol{u}_1, \boldsymbol{u}_2)\|_{L_1(0,T;L_q(G))} \|\nabla_\xi \delta \boldsymbol{u}\|_{L_1(0,T;L_\infty(G))} \Big), \end{aligned}$$

3. *For the time derivative of* $\mathfrak{A}$, *we have for any suitable* $(\widetilde{p}, \widetilde{q}) \in [1, \infty]^2$

$$\|\partial_t \delta \mathfrak{A}\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))} \leq C_{N,\kappa} \Big(\|\nabla_\xi \delta \boldsymbol{u}\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))} \\ + \|\nabla_\xi (\boldsymbol{u}_1, \boldsymbol{u}_2)\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))} \|\nabla_\xi \delta \boldsymbol{u}\|_{L_1(0,T;L_\infty(G))} \Big),$$

$$\|\partial_t \nabla_\xi \delta \mathscr{A}\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))} \leq C_{N,\kappa} \Big(\|\nabla_\xi^2 \delta \boldsymbol{u}\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))} \\ + \|\nabla_\xi^2 (\boldsymbol{u}_1, \boldsymbol{u}_2)\|_{L_1(0,T;L_{\widetilde{q}}(G))} \|\nabla_\xi \delta \boldsymbol{u}\|_{L_{\widetilde{p}}(0,T;L_\infty(G))} \\ + \|\nabla_\xi (\boldsymbol{u}_1, \boldsymbol{u}_2)\|_{L_{\widetilde{p}}(0,T;L_\infty(G))} \|\nabla_\xi^2 \delta \boldsymbol{u}\|_{L_1(0,T;L_{\widetilde{q}}(G))} \\ + \|\nabla_\xi^2 (\boldsymbol{u}_1, \boldsymbol{u}_2)\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))} \|\nabla_\xi \delta \boldsymbol{u}\|_{L_1(0,T;L_\infty(G))} \\ + \|\nabla_\xi (\boldsymbol{u}_1, \boldsymbol{u}_2)\|_{L_{\widetilde{p}}(0,T;L_\infty(G))} \|\nabla_\xi^2 (\boldsymbol{u}_1, \boldsymbol{u}_2)\|_{L_1(0,T;L_{\widetilde{q}}(G))} \|\nabla_\xi \delta \boldsymbol{u}\|_{L_1(0,T;L_\infty(G))} \Big).$$

Above all constants $C_{N,\kappa}$ *go to* ∞ *as* κ *tends to* 1.

Proof. For simplicity, we only concentrate on the case $\mathfrak{A}_k = \mathscr{A}_k$ for $k = 1, 2$. The proof is based on the following expression of $\delta \mathscr{A}$ (see [36, Appendix] for instance),

$$\delta \mathscr{A}(t) = \Big(\int_0^t \nabla_\xi \delta \boldsymbol{u} \, d\tau \Big) \sum_{\ell \geq 1} (-1)^\ell \sum_{j=0}^{\ell-1} C_1(t)^j C_2^{\ell-1-j}(t), \tag{3.79}$$

where $C_k(t) := \int_0^t \nabla_\xi \boldsymbol{u}_k \, d\tau$ for $k = 1, 2$. Thus our assumption $\kappa < 1$ and (3.79) imply the bound

$$\|\delta \mathscr{A}\|_{L_\infty(G \times]0,T[)} \leq \frac{C_N}{(1-\kappa)^2} \|\nabla_\xi \delta \boldsymbol{u}\|_{L_1(0,T;L_\infty(G))}. \tag{3.80}$$

To complete the proof of the first part, without loss of generality, let us only consider the difference

$$\mathscr{A}_2 \mathscr{A}_2^{tr} - \mathscr{A}_1 \mathscr{A}_1^{tr} = \delta \mathscr{A} \mathscr{A}_2^{tr} + \mathscr{A}_1 \delta \mathscr{A}^{tr}. \tag{3.81}$$

Then Lemma 3.7 (1) and (3.80) yield

$$\|\mathscr{A}_2 \mathscr{A}_2^{tr} - \mathscr{A}_1 \mathscr{A}_1^{tr}\|_{L_\infty(G \times]0,T[)} \leq \frac{C_N}{(1-\kappa)^3} \|\nabla_\xi \delta \boldsymbol{u}\|_{L_1(0,T;L_\infty(G))}.$$

Now, the proof of Part (2) also relies on the formulas (3.79) and (3.81). For instance, we have

$$\|\nabla_\xi \delta \mathscr{A}(t)\|_{L_\infty(0,T;L_q(G))} \leq \frac{C_N}{(1-\kappa)^2} \|\nabla_\xi^2 \delta \boldsymbol{u}\|_{L_1(0,T;L_q(G))} \\ + \frac{C_N}{(1-\kappa)^3} \|\nabla_\xi^2 (\boldsymbol{u}_1, \boldsymbol{u}_2)\|_{L_1(0,T;L_q(G))} \|\nabla_\xi \delta \boldsymbol{u}\|_{L_1(0,T;L_\infty(G))}. \tag{3.82}$$

Then combining above inequalities (3.80), (3.82) and Lemma 3.7 yields

$$\|\nabla_\xi(\mathscr{A}_2\mathscr{A}_2^{tr} - \mathscr{A}_1\mathscr{A}_1^{tr})\|_{L_\infty(0,T;L_q(G))} \le \frac{C_N}{(1-\kappa)^3}\|\nabla_\xi^2\delta\boldsymbol{u}\|_{L_1(0,T;L_q(G))} + \frac{C_N}{(1-\kappa)^4}\|\nabla_\xi^2(\boldsymbol{u}_1,\boldsymbol{u}_2)\|_{L_1(0,T;L_q(G))}\|\nabla_\xi\delta\boldsymbol{u}\|_{L_1(0,T;L_\infty(G))}.$$

The proof for other terms in Part (2) are similar and hence it remains to study $\partial_t\mathscr{A}$ in Part (3). Finally, it is not hard to see that

$$\|\partial_t\delta\mathscr{A}\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))} \le \frac{C_N}{(1-\kappa)^2}\|\nabla_\xi\delta\boldsymbol{u}\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))} + \frac{C_N}{(1-\kappa)^3}\|\nabla_\xi(\boldsymbol{u}_1,\boldsymbol{u}_2)\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))}\|\nabla_\xi\delta\boldsymbol{u}\|_{L_1(0,T;L_\infty(G))}.$$

Apply the operator $\partial_t\nabla_\xi$ to (3.79) and we have

$$\begin{aligned}\|\partial_t\nabla_\xi\delta\mathscr{A}\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))} \le\ & \frac{C_N}{(1-\kappa)^2}\|\nabla_\xi^2\delta\boldsymbol{u}\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))} \\ &+ \frac{C_N}{(1-\kappa)^3}\Big(\|\nabla_\xi^2(\boldsymbol{u}_1,\boldsymbol{u}_2)\|_{L_1(0,T;L_{\widetilde{q}}(G))}\|\nabla_\xi\delta\boldsymbol{u}\|_{L_{\widetilde{p}}(0,T;L_\infty(G))} \\ &\qquad + \|\nabla_\xi(\boldsymbol{u}_1,\boldsymbol{u}_2)\|_{L_{\widetilde{p}}(0,T;L_\infty(G))}\|\nabla_\xi^2\delta\boldsymbol{u}\|_{L_1(0,T;L_{\widetilde{q}}(G))} \\ &\qquad + \|\nabla_\xi^2(\boldsymbol{u}_1,\boldsymbol{u}_2)\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))}\|\nabla_\xi\delta\boldsymbol{u}\|_{L_1(0,T;L_\infty(G))}\Big) \\ &+ \frac{C_N}{(1-\kappa)^4}\|\nabla_\xi(\boldsymbol{u}_1,\boldsymbol{u}_2)\|_{L_{\widetilde{p}}(0,T;L_\infty(G))}\|\nabla_\xi^2(\boldsymbol{u}_1,\boldsymbol{u}_2)\|_{L_1(0,T;L_{\widetilde{q}}(G))} \\ &\times\|\nabla_\xi\delta\boldsymbol{u}\|_{L_1(0,T;L_\infty(G))},\end{aligned}$$

which completes our proof of Lemma 3.8. □

To handle the free boundary condition in Lagrangian coordinate, we need the following estimates.

Lemma 3.9. *Assume that G is a uniform $W_q^{2-1/q}$ connected domain in $\mathbb{R}^N$ and $\boldsymbol{n}$ is the unit normal along some boundary $\Gamma \subset \partial G$. With the same conventions on $(q,\widetilde{p},\widetilde{q},\boldsymbol{u},\boldsymbol{u}_k,\boldsymbol{X},\boldsymbol{X}_k,\mathscr{A},\mathscr{A}_k)$ $(k=1,2)$ as in Lemma 3.7 and Lemma 3.8, we define*

$$(\overline{\boldsymbol{n}},\overline{\boldsymbol{n}}_k)(\xi,t) := \Big(\frac{\mathscr{A}\boldsymbol{n}}{|\mathscr{A}\boldsymbol{n}|},\frac{\mathscr{A}_k\boldsymbol{n}}{|\mathscr{A}_k\boldsymbol{n}|}\Big)(\xi,t) \quad \textit{for any } (\xi,t)\in\Gamma\times]0,T[.$$

For simplicity, we use $\overline{\mathfrak{n}}$ for any element in $\{\overline{\boldsymbol{n}},\overline{\boldsymbol{n}}_1,\overline{\boldsymbol{n}}_2\}$. Then the following assertions hold true as long as $0<\kappa\ll 1$.

1. *$\boldsymbol{n}$ and $\overline{\mathfrak{n}}$ can be extended into $W_q^1(G)^N$. Moreover, there is some constant $C_{N,\kappa}$ such that*

$$\|\boldsymbol{n}\|_{L_\infty(G)\cap W_q^1(G)} + \|\overline{\mathfrak{n}}\|_{L_\infty(0,T;L_\infty(G)\cap W_q^1(G))} \le C_{N,\kappa}. \tag{3.83}$$

2. *If we take $\mathfrak{A}$ in $\{\mathscr{A}, \mathscr{A}_1, \mathscr{A}_2\}$ with respect to the corresponding $\overline{\mathfrak{n}}$ as above, then there exists a constant C such that*

$$
\begin{aligned}
\|\overline{\mathfrak{n}} - \boldsymbol{n}\|_{L_\infty(G\times]0,T[)} &\leq C_{N,\kappa}\|\mathfrak{A} - \mathbb{I}\|_{L_\infty(G\times]0,T[)},\\
\|\nabla_\xi(\overline{\mathfrak{n}} - \boldsymbol{n})\|_{L_\infty(0,T;L_q(G))} &\leq C_{N,\kappa}\big(\|\nabla_\xi\mathfrak{A}\|_{L_\infty(0,T;L_q(G))} + \|\mathfrak{A} - \mathbb{I}\|_{L_\infty(G\times]0,T[)}\big),\\
\|\partial_t(\overline{\mathfrak{n}} - \boldsymbol{n})\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))} &\leq C_{N,\kappa}\|\partial_t\mathfrak{A}\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))},\\
\|\partial_t\nabla_\xi(\overline{\mathfrak{n}} - \boldsymbol{n})\|_{L_p(0,T;L_q(G))} &\leq C_{N,\kappa}\big(\|\partial_t\nabla_\xi\mathfrak{A}\|_{L_p(0,T;L_q(G))} + \|\partial_t\mathfrak{A}\|_{L_p(0,T;L_\infty(G))}\\
&\qquad + \|\partial_t\mathfrak{A}\|_{L_p(0,T;L_\infty(G))}\|\nabla_\xi\mathfrak{A}\|_{L_\infty(0,T;L_q(G))}\big).
\end{aligned}
$$

3. *Set $\delta\overline{\boldsymbol{n}} := \overline{\boldsymbol{n}}_2 - \overline{\boldsymbol{n}}_1$ and recall $\delta\mathscr{A} := \mathscr{A}_2 - \mathscr{A}_1$. Then there exists a constant C such that*

$$
\begin{aligned}
\|\delta\overline{\boldsymbol{n}}\|_{L_\infty(G\times]0,T[)} &\leq C_{N,\kappa}\|\delta\mathscr{A}\|_{L_\infty(G\times]0,T[)},\\
\|\nabla_\xi\delta\overline{\boldsymbol{n}}\|_{L_\infty(0,T;L_q(G))} &\leq C_{N,\kappa}\Big(\|\nabla_\xi\delta\mathscr{A}\|_{L_\infty(0,T;L_q(G))} + \|\delta\mathscr{A}\|_{L_\infty(G\times]0,T[)}A_{\infty,q}(T)\Big),\\
\|\partial_t\delta\overline{\boldsymbol{n}}\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))} &\leq C_{N,\kappa}\Big(\|\partial_t\delta\mathscr{A}\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))} + \|\delta\mathscr{A}\|_{L_\infty(G\times]0,T[)}B_{\widetilde{p},\widetilde{q}}(T)\Big),\\
\|\partial_t\nabla_\xi\delta\overline{\boldsymbol{n}}\|_{L_p(0,T;L_q(G))} &\leq C_{N,\kappa}\Big(\|\partial_t\nabla_\xi\delta\mathscr{A}\|_{L_p(0,T;L_q(G))} + \|\partial_t\delta\mathscr{A}\|_{L_p(0,T;L_\infty(G))}A_{\infty,q}(T)\\
&\qquad + \|\nabla_\xi\delta\mathscr{A}\|_{L_\infty(0,T;L_q(G))}B_{p,\infty}(T) + \|\delta\mathscr{A}\|_{L_\infty(G\times]0,T[)}C_{p,q}(T)\Big),
\end{aligned}
$$

where constants $A_{p,q}(T)$, $B_{p,q}(T)$ and $C_{p,q}(T)$ are given by

$$
\begin{aligned}
A_{p,q}(T) &:= 1 + \|\nabla_\xi(\mathscr{A}_1, \mathscr{A}_2)\|_{L_p(0,T;L_q(G))},\\
B_{p,q}(T) &:= \|\partial_t(\mathscr{A}_1, \mathscr{A}_2)\|_{L_p(0,T;L_q(G))},\\
C_{p,q}(T) &:= \|\partial_t\nabla_\xi(\mathscr{A}_1, \mathscr{A}_2)\|_{L_p(0,T;L_q(G))} + A_{\infty,q}(T)\,B_{p,\infty}(T).
\end{aligned}
$$

Proof. The proof of (3.83) is standard and let us verify the second part. Without loss of generality, we just treat the pair $(\overline{\boldsymbol{n}}, \mathscr{A})$ here and set $\mathscr{B} := \mathscr{A} - \mathbb{I}$. Then the difference $\overline{\boldsymbol{n}} - \boldsymbol{n}$ is formulated as follows,

$$
\overline{\boldsymbol{n}} - \boldsymbol{n} = \frac{\mathscr{B}\boldsymbol{n}}{|\mathscr{A}\boldsymbol{n}|} + \Big(\frac{1}{|\mathscr{A}\boldsymbol{n}|} - 1\Big)\boldsymbol{n} = \frac{\mathscr{B}\boldsymbol{n}}{|\mathscr{A}\boldsymbol{n}|} - \frac{2\mathscr{B}\boldsymbol{n}\cdot\boldsymbol{n} + |\mathscr{B}\boldsymbol{n}|^2}{|\mathscr{A}\boldsymbol{n}|(1 + |\mathscr{A}\boldsymbol{n}|)}\boldsymbol{n} \tag{3.84}
$$

Firstly, we know from (3.83) that

$$
\|\mathscr{B}\boldsymbol{n}\|_{L_\infty(G\times]0,T[)} \leq C_{N,\kappa}\|\mathscr{B}\|_{L_\infty(G\times]0,T[)}, \tag{3.85}
$$

which also yields for κ small enough,

$$
0 < c_{N,\kappa} \leq |\boldsymbol{n}| - |\mathscr{B}\boldsymbol{n}| \leq |\mathscr{A}\boldsymbol{n}| \leq \|\mathscr{A}\boldsymbol{n}\|_{L_\infty(G\times]0,T[)} \leq C_{N,\kappa}. \tag{3.86}
$$

Hence, keeping (3.85) and (3.86) in mind, Lemma 3.7 yields our result for $\|\overline{\boldsymbol{n}} - \boldsymbol{n}\|_{L_\infty(G\times]0,T[)}$.

To bound $\nabla_\xi(\overline{\boldsymbol{n}} - \boldsymbol{n})$, we first note by (3.83)

$$\|\nabla_\xi(\mathscr{B}\boldsymbol{n})\|_{L_\infty(0,T;L_q(G))} \leq C_{N,\kappa}\big(\|\nabla_\xi\mathscr{A}\|_{L_\infty(0,T;L_q(G))} + \|\mathscr{B}\|_{L_\infty(G\times]0,T[)}\big). \tag{3.87}$$

Then (3.87) and (3.83), together with Lemma 3.7, imply that

$$\|\nabla_\xi(\mathscr{A}\boldsymbol{n})\|_{L_\infty(0,T;L_q(G))} \leq C_{N,\kappa}\big(1 + \|\nabla_\xi\mathscr{A}\|_{L_\infty(0,T;L_q(G))}\big). \tag{3.88}$$

Thus based on (3.87) and (3.88), direct computations yield the bound of $\nabla_\xi(\overline{\boldsymbol{n}} - \boldsymbol{n})$.

Next, the bound of $\partial_t(\overline{\boldsymbol{n}} - \boldsymbol{n})$ is immediate from the following inequality,

$$\|\partial_t(\mathscr{A}\boldsymbol{n}, \mathscr{B}\boldsymbol{n})\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))} \leq C_{N,\kappa}\|\partial_t\mathscr{A}\|_{L_{\widetilde{p}}(0,T;L_{\widetilde{q}}(G))}. \tag{3.89}$$

Now, combining (3.83) and above inequality (3.89) implies

$$\|\partial_t\nabla_\xi(\mathscr{A}\boldsymbol{n}, \mathscr{B}\boldsymbol{n})\|_{L_p(0,T;L_q(G))} \leq C_{N,\kappa}\big(\|\partial_t\nabla_\xi\mathscr{A}\|_{L_p(0,T;L_q(G))} + \|\partial_t\mathscr{A}\|_{L_p(0,T;L_\infty(G))}\big). \tag{3.90}$$

Therefore we can conclude the bound of $\partial_t\nabla_\xi(\overline{\mathfrak{n}} - \boldsymbol{n})$ by putting (3.85) - (3.90) together.

In fact, the proof of last part is similar to the previous step and is based on the following expression,

$$\begin{aligned}\delta\overline{\boldsymbol{n}} &= \frac{\delta\mathscr{A}\boldsymbol{n}}{|\mathscr{A}_2\boldsymbol{n}|} + \Big(\frac{1}{|\mathscr{A}_2\boldsymbol{n}|} - \frac{1}{|\mathscr{A}_1\boldsymbol{n}|}\Big)\mathscr{A}_1\boldsymbol{n}\\ &= \frac{\delta\mathscr{A}\boldsymbol{n}}{|\mathscr{A}_2\boldsymbol{n}|} - \frac{(\delta\mathscr{A}\boldsymbol{n})\cdot(\mathscr{A}_2\boldsymbol{n}) + (\mathscr{A}_1\boldsymbol{n})\cdot(\delta\mathscr{A}\boldsymbol{n})}{|\mathscr{A}_1\boldsymbol{n}||\mathscr{A}_2\boldsymbol{n}|(|\mathscr{A}_1\boldsymbol{n}| + |\mathscr{A}_2\boldsymbol{n}|)}\mathscr{A}_1\boldsymbol{n}.\end{aligned}$$

The details are left to the readers. □

3.6 Interpolation property

Recall the definition of Stokes operator $\mathcal{A}_q$ with $1 < q < \infty$. We recall that the vector field $\boldsymbol{u}$ satisfies the two phase compatibility condition in $\dot{\Omega}$ if $\boldsymbol{u}$ enjoys

$$[\![\boldsymbol{u}]\!]|_\Gamma = [\![\mathcal{T}_{\boldsymbol{n}}\big(\mu\mathbb{D}(\boldsymbol{u})\boldsymbol{n}\big)]\!]|_\Gamma = \boldsymbol{0}, \;\; \mathcal{T}_{\boldsymbol{n}_+}\big(\mu\mathbb{D}(\boldsymbol{u})\boldsymbol{n}_+\big)|_{\Gamma_+} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{u}|_{\Gamma_-} = 0, \tag{3.91}$$

where the projection operator

$$\mathcal{T}_{\boldsymbol{\nu}}\boldsymbol{h} := \boldsymbol{h} - (\boldsymbol{h}\cdot\boldsymbol{\nu})\boldsymbol{\nu},$$

for any vector $\boldsymbol{\nu}$ and $\boldsymbol{h}$ defined along some surface S. Then the domain of $\mathcal{D}(\mathcal{A}_q)$ can be defined by

$$\mathcal{A}_q\boldsymbol{u} := \eta^{-1}\mathrm{Div}\,\mathbb{T}\big(\boldsymbol{u}, K(\boldsymbol{u})\big), \quad \forall\;\boldsymbol{u}\in\mathcal{D}(\mathcal{A}_q) := \{\boldsymbol{u}\in W^2_q(\dot{\Omega})\cap J_q(\dot{\Omega}) : \boldsymbol{u}\;\text{ satisfies }\;(3.91)\}.$$

If we donote the (real) interpolation space $\mathcal{D}_{q,p}^{2(1-\frac{1}{p})}(\dot{\Omega}) := \big(J_q(\dot{\Omega}), \mathcal{D}(\mathcal{A}_q)\big)_{1-1/p,p}$, then we have the following properties,

$$\mathcal{D}_{q,p}^{2(1-\frac{1}{p})}(\dot{\Omega}) := \begin{cases} \big\{\boldsymbol{u} \in B_{q,p}^{2(1-\frac{1}{p})}(\dot{\Omega}) \cap J_q(\dot{\Omega}) : (3.91) \text{ holds}\big\} & \text{if } \ 2(1-\dfrac{1}{p}) > 1 + \dfrac{1}{q}, \\ \big\{\boldsymbol{u} \in B_{q,p}^{2(1-\frac{1}{p})}(\dot{\Omega}) \cap J_q(\dot{\Omega}) : \boldsymbol{u}|_{\Gamma_-} = 0\big\} & \text{if } \ \dfrac{1}{q} < 2(1-\dfrac{1}{p}) < 1 + \dfrac{1}{q}, \\ B_{q,p}^{2(1-\frac{1}{p})}(\dot{\Omega}) \cap J_q(\dot{\Omega}) & \text{if } \ 0 < 2(1-\dfrac{1}{p}) < \dfrac{1}{q}. \end{cases}$$

Motivated by above discussions, we assume that a general (Stokes) operator A_q is defined in some domain G equipped with general compatibility boundary conditions and $\mathcal{D}_{q,p}^{2(1-\frac{1}{p})}(G) := \big(J_q(G), \mathcal{D}(A_q)\big)_{1-1/p,p}$ for some $1 < p, q < \infty$. Then the following general interpolation properties hold true.

Proposition 3.10. *Let G be a uniform $W_r^{2-\frac{1}{r}}$ domain with $r \in]N, \infty[$ and assume (θ, β, q, p) satisfying the following conditions*

$$(\theta, \beta, q, p) \in]0,1[\times]1,\infty]\times]1,\infty[\times[1,\infty[, \quad \beta \geq q \quad \textit{and} \quad 1 - \frac{2(1-\theta)}{p} = \frac{N}{q} - \frac{N}{\beta}.$$

Then the following inequality holds,

$$\|\nabla \boldsymbol{u}\|_{L_\beta(G)} \lesssim \|\boldsymbol{u}\|^{1-\theta}_{\mathcal{D}_{q,p}^{2-\frac{2}{p}}(G)} \|\boldsymbol{u}\|^{\theta}_{\mathcal{D}(A_q)}. \tag{3.92}$$

When we consider smooth enough time-depend function $\boldsymbol{u}$, take $L^{\frac{p}{\theta}}(0,t)$ on both side of (3.92), we know that

$$\|\nabla \boldsymbol{u}\|_{L_{\frac{p}{\theta}}(0,T;L_\beta(G))} \lesssim \|\boldsymbol{u}\|^{1-\theta}_{L_\infty\big(0,T;\mathcal{D}_{q,p}^{2-\frac{2}{p}}(G)\big)} \|\nabla^2 \boldsymbol{u}\|^{\theta}_{L_p(0,T;L_q(G))},$$

which, for example, gives the decay of $\|\nabla \boldsymbol{u}\|_{L_p(0,T;L_\beta(\Omega))}$ as in [31, Lemma 4.1]:

$$\|\nabla \boldsymbol{u}\|_{L_p(0,T;L_\beta(G))} \lesssim T^{\frac{1}{2}(1-\frac{N}{q}+\frac{N}{\beta})} \|\boldsymbol{u}\|^{1-\theta}_{L_\infty\big(0,T;\mathcal{D}_{q,p}^{2-\frac{2}{p}}(G)\big)} \|\nabla^2 \boldsymbol{u}\|^{\theta}_{L_p(0,T;L_q(G))}. \tag{3.93}$$

Proof of Proposition 3.10. Recall the definition of $\mathcal{D}_{q,p}^{2-\frac{2}{p}}(G)$, we have

$$\mathcal{D}_{q,p}^{2-\frac{2}{p}}(G) = \big(J_q(G), \mathcal{D}(A_q)\big)_{1-\frac{1}{p},p}, \quad \forall p \in]1,\infty[.$$

For any $(\theta, r) \in]0,1[\times[1,\infty[$, we obtain from Reiteration Theorem (see [90]),

$$\big(\mathcal{D}_{q,p}^{2-\frac{2}{p}}(G), \mathcal{D}(A_q)\big)_{\theta,r} = \big(J_q(G), \mathcal{D}(A_q)\big)_{(1-\theta)(1-\frac{1}{p})+\theta,r} = \mathcal{D}_{q,r}^{2(1-\theta)(1-\frac{1}{p})+2\theta}(G),$$

where we note that $(1-\theta)(1-\frac{1}{p})+\theta = 1-\frac{1-\theta}{p} \in]0,1[$. Moreover, we have

$$\|\boldsymbol{u}\|_{\mathcal{D}_{q,r}^{2(1-\theta)(1-\frac{1}{p})+2\theta}(G)} \lesssim \|\boldsymbol{u}\|^{1-\theta}_{\mathcal{D}_{q,p}^{2-\frac{2}{p}}(G)} \|\boldsymbol{u}\|^{\theta}_{\mathcal{D}(A_q)}.$$

By definition of β, above inequality, we have

$$\|\nabla \boldsymbol{u}\|_{B^0_{\beta,r}(G)} \lesssim \|\nabla \boldsymbol{u}\|_{B_{q,r}^{1-\frac{2(1-\theta)}{p}}(G)} \lesssim \|\boldsymbol{u}\|^{1-\theta}_{\mathcal{D}_{q,p}^{2-\frac{2}{p}}(G)} \|\boldsymbol{u}\|^{\theta}_{\mathcal{D}(A_q)}.$$

Thus that taking $r=1$ and the embedding $B^0_{\beta,1}(G) \hookrightarrow L^\beta(G)$ (e.g. see [124]) yield the desired result. $\square$

To study the fractional derivative with respect to time variable, we need following Lemma by Shibata and Shimizu in [108, Lemma 2.7]. Let us define for $(s,\sigma,p,q) \in \mathbb{R}_+^2 \times [1,\infty]^2$

$$H^{s;\sigma}_{q,p}(G\times\mathbb{R}) := H^\sigma_p\big(\mathbb{R}; L_q(G)\big) \cap L_p\big(\mathbb{R}; H^s_q(G)\big)$$

Lemma 3.11. *Let $(p,q,T) \in]1,\infty[\times]N,\infty[\times]0,1]$. Assume g and f satisfy the following conditions,*

$$(g,f) \in H^{\frac{1}{2},\frac{1}{2}}_{q,p}(G\times\mathbb{R}) \times H^{1,1}_{q,\infty}(G\times\mathbb{R}),$$
$$f \equiv 0 \textit{ for } t \notin]0,2T[\;\textit{ and }\; \partial_t f \in L_p\big(0,2T; H^1_q(G)\big).$$

Then there exists a constant $C_{p,q}$ such that

$$\begin{aligned}\|fg\|_{H^{\frac{1}{2},\frac{1}{2}}_{q,p}(G\times\mathbb{R})} \le C_{p,q}\|f\|^{\frac{1}{2}}_{L_\infty(G\times]0,2T[)}\Big(&\|\nabla f\|_{L_\infty(0,2T;L_q(G))} + \|f\|_{L_\infty(G\times]0,2T[)}\\ &+ T^{\frac{q-N}{pq}}\|\partial_t f\|^{1-\frac{N}{2q}}_{L_\infty(0,2T;L_q(G))}\|\partial_t f\|^{\frac{N}{2q}}_{L_p(0,2T;H^1_q(G))}\Big)^{\frac{1}{2}}\|g\|_{H^{\frac{1}{2},\frac{1}{2}}_{q,p}(G\times\mathbb{R})}.\end{aligned}$$

Moreover, for the case $\frac{N}{q}+\frac{2}{p} > 1$, we have another result like Lemma 3.11 based on the interpolation property proved above.

Lemma 3.12. *Let $(\theta_1,\theta_2,\alpha,\beta,q,p,T) \in]0,1[^2\times[q,\infty]^2\times]N,\infty[\times[1,2]\times]0,1]$ satisfy the following conditions*

$$\theta_1+\theta_2 \in]0,1], \quad \frac{1}{q} = \frac{1}{\alpha}+\frac{1}{\beta}, \quad 1-\frac{1-\theta_1}{p} = \frac{N}{q}-\frac{N}{\alpha} \;\textit{ and }\; 1-\frac{2(1-\theta_2)}{p} = \frac{N}{q}-\frac{N}{\beta}. \tag{3.94}$$

Assume $g \in H^{\frac{1}{2},\frac{1}{2}}_{q,p}(G\times\mathbb{R})$ and $f \in L_\infty(\mathbb{R}; W^1_q(G))$ satisfies

$$f \equiv 0 \textit{ for } t \notin]0,2T[\;\textit{ and }\; \partial_t f \in L_{p/\theta_2}\big(0,2T; L_\beta(G)\big).$$

Then there exists a constant $C_{p,q}$ such that

$$\|fg\|_{H^{\frac{1}{2},\frac{1}{2}}_{q,p}(G\times\mathbb{R})} \leq C_{p,q}\|f\|^{\frac{1}{2}}_{L_\infty(G\times]0,2T[)}\Big(\|\nabla f\|_{L_\infty(0,2T;L_q(G))} + \|f\|_{L_\infty(G\times]0,2T[)} + T^{\frac{3}{2}-\frac{1}{p}-\frac{N}{2q}-\frac{N}{2\beta}}\|\partial_t f\|_{L_{p/\theta_2}(0,2T;L_\beta(G))}\Big)^{\frac{1}{2}}\|g\|_{H^{\frac{1}{2},\frac{1}{2}}_{q,p}(G\times\mathbb{R})}.$$

Proof. Let g belong to $H^{1,1}_{q,p}(G\times\mathbb{R})$ and we first obtain

$$\|fg\|_{L_p(\mathbb{R};L_q(G))} \leq C_N\|f\|_{L_\infty(G\times]0,2T[)}\|g\|_{L_p(\mathbb{R};L_q(G))}, \tag{3.95}$$

Now, the definition of $H^{1,1}_{q,p}(G\times\mathbb{R})$ yields

$$\|fg\|_{H^{1,1}_{q,p}(G\times\mathbb{R})} \leq C_N\Big(\|\nabla f\|_{L_\infty(0,2T;L_q(G))} + \|f\|_{L_\infty(G\times]0,2T[)}\Big)\|g\|_{H^{1,1}_{q,p}(G\times\mathbb{R})} + \|(\partial_t f)\,g\|_{L_p(0,2T;L_q)}. \tag{3.96}$$

In the following, we will mainly study the last term on the right hand side of (3.96). To this end, recall the standard embedding $H^{1,1}_{q,p}(G\times\mathbb{R}) \hookrightarrow \mathcal{BUC}\big(\mathbb{R}_+;B^{1-1/p}_{q,p}(G)\big)$. Then, thanks to Reiteration Theorem (see [90]), we obtain for any $(\theta,r)\in]0,1[\times[1,\infty[$,

$$\big(B^{1-\frac{1}{p}}_{q,p}(G),W^1_q(G)\big)_{\theta,r} = \big(L_q(G),W^1_q\big)_{(1-\theta)(1-\frac{1}{p})+\theta,r} = B^{1-\frac{1-\theta}{p}}_{q,r}(G).$$

Assuming $\alpha\geq q$ and θ_1 satisfy $1-(1-\theta_1)/p = N/q - N/\alpha$, we have the following estimate

$$\|g\|_{L_\alpha(G)} \lesssim \|g\|_{B^{1-\frac{1-\theta_1}{p}}_{q,1}(G)} \lesssim \|g\|^{1-\theta_1}_{B^{1-\frac{1}{p}}_{q,p}(G)}\|g\|^{\theta_1}_{W^1_q(G)},$$

which yields

$$\|g\|_{L_{p/\theta_1}(\mathbb{R}_+;L_\alpha(G))} \lesssim \|g\|_{H^{1,1}_{q,p}(G\times\mathbb{R})}.$$

Therefore, we infer from above bound and Conditions (3.94)

$$\begin{aligned}\|(\partial_t f)\,g\|_{L_p(0,2T;L_q)} &\lesssim T^{\frac{1-\theta_1-\theta_2}{p}}\|\partial_t f\|_{L_{p/\theta_2}(0,2T;L_\beta)}\|g\|_{L_{p/\theta_1}(0,2T;L_\alpha)} \\ &\lesssim T^{\frac{3}{2}-\frac{1}{p}-\frac{N}{2q}-\frac{N}{2\beta}}\|\partial_t f\|_{L_{p/\theta_2}(0,2T;L_\beta(G))}\|g\|_{H^{1,1}_{q,p}(G\times\mathbb{R})}.\end{aligned} \tag{3.97}$$

Finally, combining the bounds (3.95), (3.96), (3.97) and interpolation property

$$H^{\frac{1}{2},\frac{1}{2}}_{q,p}(G\times\mathbb{R}) = \Big(L_p\big(\mathbb{R};L_q(G)\big),H^{1,1}_{q,p}(G\times\mathbb{R})\Big)_{[\frac{1}{2}]},$$

yield the desired result. □

Appendix A

Vortex patch problem for ideal fluid

To address the *vortex patch problem*, let us recall the following Cauchy problem for Euler system over $\mathbb{R}^2$,

$$\begin{cases} \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla P = \boldsymbol{0} & \text{in} \quad \mathbb{R}^2 \times]0, T[, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \mathbb{R}^2 \times]0, T[, \\ \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0 & \text{in} \quad \mathbb{R}^2. \end{cases} \tag{E}$$

Above the unknowns $\boldsymbol{u}$ and P stand for the velocity field and the pressure respectively. Given initial divergence-free field $\boldsymbol{u}_0$, the key to solving this hyperbolic type system (E) in classical (or smooth) sense is to retain the Lipschtiz regularity of $\boldsymbol{u}$ [1]. Thus in this smooth situation, the vorticity $\omega \equiv \operatorname{rot} \boldsymbol{u} := \partial_1 u^2 - \partial_2 u^1$ necessarily lies in $L_\infty(\mathbb{R}^2)$ for $t \geq 0$. In fact, for two-dimensional case, ω is advected by $\boldsymbol{u}$ by the so-called *vorticity-stream formulation*,

$$\begin{cases} \partial_t \omega + \boldsymbol{u} \cdot \nabla \omega = 0 & \text{in} \quad \mathbb{R}^2 \times]0, T[, \\ \boldsymbol{u} = -\nabla^\perp (-\Delta)^{-1} \omega & \text{in} \quad \mathbb{R}^2 \times]0, T[, \\ \omega|_{t=0} = \omega_0 & \text{in} \quad \mathbb{R}^2. \end{cases} \tag{V_E}$$

Here the equations $(V_E)_2$ is called the *Biot-Savart Law* given by

$$\boldsymbol{u}(x,t) = -\nabla^\perp (-\Delta)^{-1} \omega(x,t) := \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(y,t)\, dy.$$

Based on this new formulation (V_E), we can construct a class of weak solution of (V_E) with bounded vorticity. Here and subsequently, $D_t := \partial_t + \boldsymbol{u} \cdot \nabla$ stands for the material derivative if there is no confusion on $\boldsymbol{u}$.

Definition A.1. *Given $\omega_0 \in L_1(\mathbb{R}^2) \cap L_\infty(\mathbb{R}^2)$, $(\boldsymbol{u}, \omega)$ is a weak solution to (V_E) with $\omega_0 = \operatorname{rot} \boldsymbol{u}_0$, provided that*

[1] See the standard theory for linear transport equation in [6, Chapter 3].

1. *$\omega = \operatorname{rot} \boldsymbol{u} \in L_\infty\big(0,T; L_1(\mathbb{R}^2)\cap L_\infty(\mathbb{R}^2)\big)$ and $\boldsymbol{u} = -\nabla^\perp(-\Delta)^{-1}\omega$;*

2. *For some $T>0$ and all test function φ in $\mathcal{C}^1\big([0,T];\mathcal{C}_c^1(\mathbb{R}^2)\big)$, we have*

$$\int_{\mathbb{R}^2} \big((\varphi\omega)(x,T) - \varphi(0,x)\omega_0(x)\big)\,dx = \int_0^T\int_{\mathbb{R}^2} \big((D_t\varphi)\omega\big)(x,t)\,dxdt.$$

With assuming ω_0 bounded, V.I.Yudovich in [128] first investigated the motion of ideal fluid in some smooth bounded vessel and constructed unique global weak solution. Then A.J.Majda in [91] pointed out that the proof in [128] could be extended to the whole plane with ω_0 bounded and compactly supported. More generally, J.-Y. Chemin established the following result in [18]:

Theorem A.2 (Yudovich). *Assume that the divergence-free vector $\boldsymbol{u}_0$ satisfies* [2]

$$\boldsymbol{u}_0 \in E_m \quad \text{and} \;\; \omega_0 = \operatorname{rot}\boldsymbol{u}_0 \in L_p(\mathbb{R}^2)\cap L_\infty(\mathbb{R}^2) \quad \text{for some} \;\; (m,p)\in\mathbb{R}\times]1,\infty[. \tag{A.1}$$

Then there exists a unique global solution $(\boldsymbol{u},P)$ of (E) such that

$$\boldsymbol{u}\in\mathcal{C}(\mathbb{R};E_m)\cap L_{\infty,\mathrm{loc}}\big(\mathbb{R};L_2(\mathbb{R}^2)^2\big) \;\; \text{and} \;\; \omega := \operatorname{rot}\boldsymbol{u}\in L_\infty\big(\mathbb{R};L_p(\mathbb{R}^2)\cap L_\infty(\mathbb{R}^2)\big).$$

Moreover, there is a unique (generalized) flow $\boldsymbol{\psi}_u$ associated to the solution $\boldsymbol{u}$ of (E) satisfying,

$$\boldsymbol{\psi}_u(x,t) := x + \int_0^t \boldsymbol{u}\big(\boldsymbol{\psi}_u(x,t'),t'\big)\,dt'. \tag{A.2}$$

In addition, the flow $\boldsymbol{\psi}_u(\cdot,t)$ is only $\mathcal{C}^{0,\varepsilon(t)}$ regular with $\varepsilon(t) := \exp(-Ct\|\omega_0\|_{L_p(\mathbb{R}^2)\cap L_\infty(\mathbb{R}^2)})$ for some constant C.

Let us make some comments on Theorem A.2. Firstly, from the standard singular integral theory, the operator $\nabla\otimes\nabla^\perp(-\Delta)^{-1}$ is bounded from $L_\infty(\mathbb{R}^2)$ into BMO $(\mathbb{R}^2)$, which yields $\nabla\boldsymbol{u}$ is generally unbounded by the Biot-Savart Law. Hence the flow $\boldsymbol{\psi}_u$ (or $\boldsymbol{\psi}$ if no confusion) in Theorem A.2 is no longer $\mathcal{C}^1$ regular. Consequently, $\boldsymbol{u}$ is only Log-Lipschitz (LL for brevity). More precisely, we can prove for some constant C,

$$\|\boldsymbol{u}\|_{LL} := \sup_{0<|x-x'|\le 1} \frac{|\boldsymbol{u}(x,t)-\boldsymbol{u}(x',t)|}{|x-x'|(1-\log|x-x'|)} \le C\|\omega\|_{L_p(\mathbb{R}^2)\cap L_\infty(\mathbb{R}^2)} < \infty. \tag{A.3}$$

Now, one may ask whether this LL-type bound is optimal for (E). To get rid of this doubt, we can consider some special initial state. Suppose $\omega_0 := \operatorname{rot}\boldsymbol{u}_0 = \mathbb{1}_{\mathcal{D}_0}$ for some bounded connected domain $\mathcal{D}_0$ in $\mathbb{R}^2$. Thanks to Theorem A.2, we know that (E) admits a unique global weak solution $(\boldsymbol{u},P)$ and

[2]Here E_m is the space of *radial-energy decomposition*, i.e. some perturbation space with respect to $L_2(\mathbb{R}^2)^2$. For more details, see [20, Chapter 1] or [92, Chapter 3].

$$\omega(\cdot,t) = \operatorname{rot} \boldsymbol{u}(\cdot,t) = \mathbb{1}_{\mathcal{D}_t} \text{ with } \mathcal{D}_t := \boldsymbol{\psi}_u(\mathcal{D}_0,t). \tag{A.4}$$

In fact, H.Bahouri and J.-Y. Chemin in [5] showed that if $\omega_0 := \mathbb{1}_{\mathcal{D}_0}$ with $\mathcal{D}_0 :=]-1,1[^2$, then the growth scale of $|\nabla \boldsymbol{u}_0(x)|$ is at least $1 - log\,|x|$ in some neighborhood of the origin. For more general (Banach valued) theory of non-Lipschitz vector field, the reader can find details in [23].

Furthermore, by this natural example (A.4), more intriguing *vortex patch* problem was raised in [91]: if given initially $\omega_0 := \mathbb{1}_{\mathcal{D}_0}$ for some smooth bounded connected domain $\mathcal{D}_0 \subset \mathbb{R}^2$, one asks whether the smoothness $\partial\mathcal{D}_t$ will be conserved as the initial pattern? If so, this property holds locally or globally with respect to time?

In early numerical experiments, N.J.Zabusky et al. in [129] addressed the contour dynamics method for vortex patch problem. As ω in (A.4) is determined by the curve $\partial\mathcal{D}_t = \boldsymbol{\psi}(\partial\mathcal{D}_0,t)$, we may parametrize the curve $\partial\mathcal{D}_t$ by $\boldsymbol{\gamma}(t,\sigma)$ with $0 \le \sigma < 2\pi$ and $t \ge 0$. Then (A.4) is reduced to

$$\partial_t \boldsymbol{\gamma}(t,s) = \frac{1}{2\pi}\int_0^{2\pi} \log|\boldsymbol{\gamma}(t,s) - \boldsymbol{\gamma}(t,\sigma)| \partial_\sigma \boldsymbol{\gamma}(t,\sigma)\, d\sigma. \tag{A.5}$$

By this great simplification (A.5), the authors in [129] found the amplitude of deformation of $\partial\mathcal{D}_t$ was large and the self intersection of $\boldsymbol{\gamma}(t,\sigma)$ was possible to appear by numerical simulations. Thus combining the fact (A.3), A.J. Majda conjectured in [91] the smoothness of $\mathcal{D}_t$ would break down in finite time and form cusps even if $\mathcal{D}_0$ was smooth. However, J.-Y.Chemin made a breakthrough in [19] and proved that:

Theorem A.3. *Suppose ε belongs to the interval $]0,1[$ and k is some positive integer. Assume that γ_0 in $\mathcal{C}^{k,\varepsilon}(\mathbb{S}^1;\mathbb{R}^2)$ is some one-to-one nondegenerate parametrization of Jordan curve $\partial\mathcal{D}_0$. Then there is a unique solution $\boldsymbol{\gamma}(t,s)$ of (A.5) such that*

$$\boldsymbol{\gamma}(t,s) \in L_{\infty,loc}\big(\mathbb{R};\mathcal{C}^{k,\varepsilon}(\mathbb{S}^1;\mathbb{R}^2)\big) \cap \mathcal{C}^\infty\big(\mathbb{R};\mathcal{C}^{k,\varepsilon'}(\mathbb{S}^1;\mathbb{R}^2)\big) \quad \textit{for any } \varepsilon' < \varepsilon.$$

Shortly after [19], A.L.Bertozzi and P.Constantin [10] gave a shorter proof of Theorem A.3 by directly treating (A.5). Obviously, this contour dynamics formula (A.5) works well only for the 2-D problem. Although J.-Y. Chemin's approach is more abstract, it is more flexible for higher dimensional case (see [55]) and allows to propagate much more general geometric structures. Here we will only explain the strategy in [19] as it inspired our results in general dimensional setting.

For simplicity, assuming $k = 1$ in Theorem A.3 hereafter, there are two important points in J.-Y.Chemin's proof: some log-type interpolation and persistence of tangential regularity of the vorticity. To be more precise, let us introduce some notations.

Definition A.4. *We say $\mathcal{X} := \{\boldsymbol{X}_\lambda\}_{\lambda\in\Lambda}$ is an admissible family of vector fields over $\mathbb{R}^2$ if*

- *($\mathcal{C}^{0,\varepsilon}$ regular) For some ε in $]0,1[$ and any $\lambda \in \Lambda$, $\boldsymbol{X}_\lambda$ belongs to $\mathcal{C}^{0,\varepsilon}(\mathbb{R}^2;\mathbb{R}^2)$ with* $\operatorname{div} \boldsymbol{X}_\lambda$

bounded.

- *(Nondegenerate) The quantity $I(\mathcal{X})$ is always strictly positive with $I(\mathcal{X})$ given by*

$$I(\mathcal{X}) := \inf_{x\in\mathbb{R}^2} \sup_{\lambda\in\Lambda} |\boldsymbol{X}_\lambda(x)|.$$

For any admissible family $\mathcal{X}$, we can define a Banach space $\mathcal{C}^\varepsilon_{\mathcal{X}}(\mathbb{R}^2)$ as follows [3]

$$\mathcal{C}^\varepsilon_{\mathcal{X}}(\mathbb{R}^2) := \Big\{\omega \in L_\infty : \|\omega\|_{\mathcal{C}^\varepsilon_{\mathcal{X}}} := \sup_{\lambda\in\Lambda} \frac{\|\omega\|_{L_\infty}\|\boldsymbol{X}_\lambda\|_{\mathcal{C}^{0,\varepsilon}} + \|\mathrm{div}\,(\boldsymbol{X}_\lambda\omega)\|_{B^{\varepsilon-1}_{\infty,\infty}}}{I(\mathcal{X})} < \infty\Big\}.$$

More importantly, from $\omega := \mathrm{rot}\,\boldsymbol{u}$, we can gain Lipschitz regularity of $\boldsymbol{u}$ by the following interpolation property

$$\|\nabla\boldsymbol{u}\|_{L_\infty} \lesssim \|\omega\|_{L_p} + \|\omega\|_{L_\infty} \log\Big(e + \frac{\|\omega\|_{\mathcal{C}^\varepsilon_{\mathcal{X}}}}{\|\omega\|_{L_\infty}}\Big) \quad \text{for some finite } p. \tag{A.6}$$

Now, if we start with some admissible family $\mathcal{X}_0 := \{\boldsymbol{X}_{\lambda,0}\}_{\lambda\in\Lambda}$ such that $\omega_0 := \mathbb{1}_{\mathcal{D}_0} \in \mathcal{C}^\varepsilon_{\mathcal{X}_0}(\mathbb{R}^2)$ for some $\mathcal{C}^{1,\varepsilon}$ bounded connected domain $\mathcal{D}_0$ in $\mathbb{R}^2$, then one can prove that the family of vector fields

$$\mathcal{X}_t := \{\boldsymbol{X}_{t,\lambda}(x,t) := \big(\partial_{X_{0,\lambda}}\boldsymbol{\psi}_u(\cdot,t)\big)\circ\boldsymbol{\psi}_u^{-1}(x,t)\}_{\lambda\in\Lambda}$$

is still admissible and that $\omega(t,\cdot)$ belongs to $\mathcal{C}^\varepsilon_{\mathcal{X}_t}(\mathbb{R}^2)$. Here for any $\mathcal{C}^1$ function f and any vector $\boldsymbol{Y}$ over $\mathbb{R}^N$, we adopt the standard summation convention and the notation for directional derivative

$$\partial_Y f := \boldsymbol{Y}\cdot\nabla f = Y^k\partial_k f.$$

Apart from the log-type interpolation (A.6), the fact that

$$D_t\boldsymbol{X} = \partial_X\boldsymbol{u} \quad \text{for any } \boldsymbol{X}\in\mathcal{X}_t, \tag{A.7}$$

plays a vital role, which implies

$$D_t\mathrm{div}\,(\boldsymbol{X}\omega) = 0 \quad \text{for any } \boldsymbol{X}\in\mathcal{X}_t, \tag{A.8}$$

as ω satisfies the free transport equation $(V_E)_1$. Combining (A.6),(A.7) and (A.8), the standard estimates of transport equations yield the desired results on the persistence of directional regularity of $\omega(\cdot,t)$ along $\mathcal{X}_t$. In fact, these more general results imply Theorem A.3 by choosing suitable initial family $\mathcal{X}_0$. For more details, see [20, Chapter 5] and [6, Chapter 7].

Now, let us end up this subsection with a brief review of further development of striated problem

[3] In the whole text, we will follow the notations of (homogeneous or non-homogeneous) Besov and Sobolev spaces over $\mathbb{R}^N$ in [6, Chapter 2].

after [19].

- (3-D vortex patch) As we already mentioned, P.Gamblin and X.Saint-Raymond first extended this framework to three dimensional vortex patch problem in [55] but only for short time essentially (also see [118, 130, 131] for related discussions);
- (Singular vortex patch) For the nonsmooth vortex pattern, R.Danchin investigated in [26, 29] where the author studied the stability of smooth region and cusp type singularity respectively;
- (Viscous vortex patch) For the viscous incompressible Navier-Stokes equation, we can still consider striated regularity as in [27, 68] but the vortex patch structure is not preserved any longer;
- (Inhomogeneous inviscid flow) F.Fanelli in [50] proved the persistence of conormal regularity for inhomogeneous incompressible Euler system.

Appendix B

Navier-Stokes equations in critical spaces

As this thesis is devoted to the mathematical study of viscous flow, it is suitable to recall the state-of-the-art for the following *homogeneous (or classic) incompressible Navier-Stokes* equations over some open subset $\Omega \subset \mathbb{R}^N$,

$$\begin{cases} \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla P = \boldsymbol{f} & \text{in} \quad \Omega \times]0, T[, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega \times]0, T[, \\ \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0 & \text{in} \quad \Omega. \end{cases} \tag{NS_ν}$$

Compared with (E) for inviscid liquid in Appendix A, here we can take advantage of the smoothing effect of $-\nu\Delta$ for some viscosity constant $\nu > 0$. In this viscous situation, the roles of initial state $\boldsymbol{u}_0$ and the external force $\boldsymbol{f}$ are, as a rule, different (for example see Chapter 3). Besides, different mathematical tools have to be applied according to different type of domain Ω. Roughly speaking, let us consider [1]

(Ω_1) Ω is the whole space $\mathbb{R}^N$ or the half space $\mathbb{R}^N_+ := \{x = (x_1, \ldots, x_{N-1}, x_N) \in \mathbb{R}^N : x_N > 0\}$;

(Ω_2) Ω is some connected $\mathcal{C}^2$ bounded, exterior domain in $\mathbb{R}^N$ or some uniform $\mathcal{C}^2$ general unbounded domain.

For simplicity, we only focus on the Dirichlet (or no-slip) boundary condition so long as $\partial\Omega \neq \emptyset$. In other words, we supplement (NS_ν) with the boundary condition

$$\boldsymbol{u}|_{\partial\Omega} = 0, \quad \text{on} \quad \partial\Omega \times]0, T[. \tag{B.1}$$

[1] see definition in Chapter 3 for more general regular domain

Whenever Ω is unbounded, we need to impose some decay property of the solution at infinity to avoid troubles on uniqueness issue, i.e.

$$\lim_{|x|\to\infty} \boldsymbol{u}(x,t) = 0. \tag{B.2}$$

The reason for clarifying the domain as above is due to whether one can employ *Fourier Analysis* directly. Furthermore, (Ω_1) are fundamental model problems for (Ω_2). However, Fourier Analysis tells more precise information, which brings more freedom for the estimates over (Ω_1) than the abstract approach to the case (Ω_2). In the following, we will recall some results when $\Omega = \mathbb{R}^N$ and go to (Ω_2) with brief review. For simplicity, we introduce the notation for some (hydrodynamic) Banach space. For any Banach space E defined in $\Omega \subset \mathbb{R}^N$, we set

$$E_\sigma(\Omega) := \{\boldsymbol{u} \in E(\Omega)^N : \operatorname{div} \boldsymbol{u} = 0\} \quad \text{and} \quad \|\boldsymbol{u}\|_{E_\sigma(\Omega)} := \|\boldsymbol{u}\|_{E(\Omega)^N}.$$

For instance, $L_{p,\sigma}(\Omega)$ stands for solenoidal vector fields in $L_p(\Omega)^N$ over Ω for some $1 \leq p \leq \infty$.

(NS_ν) in the whole space

Before the milestone [85] by J. Leray, we were not sure how to construct the global solution of (NS_ν), and the works before [85] were mostly dedicated to seeking explicit solution formula for linear Stokes system. By the modern setting on distribution theory, let us define the weak solution of (NS_ν) originating from [85] as below [2].

Definition B.1. *(Weak solution) A time-dependent vector field $\boldsymbol{u}$ in $L_{2,loc}(\mathbb{R}^N \times]0,T[)^N$ is called a weak solution of (NS_ν), if for any $\boldsymbol{\phi} \in \mathcal{C}^1\big([0,T[; \mathcal{C}^\infty_{0,\sigma}(\mathbb{R}^N)\big)$, we have*

$$\int_{\mathbb{R}^N} \boldsymbol{u}(x,t)\boldsymbol{\phi}(x,t)\,dx = \int_0^t \int_{\mathbb{R}^N} \big(\boldsymbol{u}(\partial_{t'} + \nu\Delta)\boldsymbol{\phi} + (\boldsymbol{u}\otimes\boldsymbol{u})\cdot D\boldsymbol{\phi} + \boldsymbol{f}\boldsymbol{\phi}\big)(x,t')\,dxdt'.$$

The work by J. Leray essentially implies that if taking $\boldsymbol{u}_0 \in L_{2,\sigma}(\mathbb{R}^N)$, we can construct a global weak solution $\boldsymbol{u}$ such that

$$\boldsymbol{u} \in L_\infty\big(\mathbb{R}_+; L_{2,\sigma}(\mathbb{R}^N)\big) \cap L_2\big(\mathbb{R}_+; \dot{H}^1(\mathbb{R}^N)^N\big),$$

and the natural energy inequality holds true

$$\frac{1}{2}\|\boldsymbol{u}(t)\|^2_{L_2} + \nu \int_0^t \|\nabla \boldsymbol{u}(t')\|^2_{L_2}\,dt' \leq \frac{1}{2}\|\boldsymbol{u}_0\|_{L_2} + \int_0^t \langle \boldsymbol{f}(t'), \boldsymbol{u}(t')\rangle_{L_2\times L_2}\,dt'.$$

The reader may also see the proof of the above result via a functional analysis approach in [22, Chapter 2,3] for instance.

[2] Almost same time, the talented Soviet mathematician S.L. Sobolev introduced generalized solution in some hyperbolic problem (see [110]).

As the regularity in Definition B.1 is too weak to guarantee the uniqueness ($N \geq 3$), let us consider some more regular (mild) solution [3] as in [53, 76] by T.Kato and H.Fujita. By this new scheme beyond the weak solution framework in [85], we can fix the uniqueness issue for (NS_ν) in certain situations.

To save the cost in producing T.Kato's scheme, we formulate the mild solution in the whole space case without any external force $\boldsymbol{f}$ involved. For convenience, introduce the Leray projector [4]

$$\mathcal{F}_{x\to\xi}(\mathbb{P}\boldsymbol{v})^j(\xi) := \sum_{k=1}^{N}(\delta_k^j - 1)\frac{\xi_j\xi_k}{|\xi|^2}(\mathcal{F}_{x\to\xi}v^k)(\xi), \quad \forall\ \boldsymbol{v} \in \mathcal{S}(\mathbb{R}^N)^N.$$

Then (NS_ν) is at least formally reduced to [5]

$$\begin{cases} \partial_t\boldsymbol{u} - \nu\Delta\boldsymbol{u} = -\mathbb{P}\,\mathrm{div}\,(\boldsymbol{u}\otimes\boldsymbol{u}) & \text{in} \quad \mathbb{R}^N\times]0,T[, \\ P = \mathcal{R}_j\mathcal{R}_k(u^ju^k) & \text{in} \quad \mathbb{R}^N\times]0,T[, \\ \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0 & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{NS_ν^0}$$

where $\mathcal{R}_j$ is standard $j\,th$ Riesz transform. Assume that $\nu\Delta$ is the infinitesimal generator of some C_0 semigroup of bounded operators ($e^{\nu t\Delta}$) over some Banach space. For smooth enough $\boldsymbol{u}_0$, the following solution formula makes sense

$$\begin{aligned} \boldsymbol{u} &= e^{\nu t\Delta}u_0 + \mathcal{B}(\boldsymbol{u},\boldsymbol{u}), \\ \mathcal{B}(\boldsymbol{u},\boldsymbol{v}) &:= -\int_0^t e^{\nu(t-t')\Delta}\mathbb{P}\,\mathrm{div}\,(\boldsymbol{u}\otimes\boldsymbol{v})\,dt'. \end{aligned}$$

Thus the existence of solution (NS_ν^0) is reduced to seeking some fixed point for the above mapping. This is based on the following lemma.

Lemma B.2. *Let X be a Banach space, $\mathcal{B}(\cdot,\cdot)$ a continuous bilinear map from $X\times X$ to X, and α a positive real number such that*

$$\alpha < \frac{1}{4\|\mathcal{B}\|} \quad \textit{with } \|\mathcal{B}\| := \sup_{\|u\|_X,\|v\|_X\leq 1}\|\mathcal{B}(u,v)\|_X.$$

For any a in the ball $B(0,\alpha)\subset X$, there exists a unique x in $B(0,2\alpha)$ such that

$$x = a + \mathcal{B}(x,x).$$

Now one may be wondering what type of functional space is suitable for this T.Kato's algorithm.

[3] T.Kato and H.Fujita called strong solution in [76] compared with the weak solution in [85], and they studied (NS_ν) over some bounded domain by adopting fractional power theory in Hilbert space.

[4] For case of domain with non-trivial boundary, Leray projector corresponds to the so-called *Helmholtz* decomposition in certain Banach space.

[5] This idea goes back to [79, 111].

Here we respect some *scaling invariance* structure of (NS_ν^0). Note that if $(\boldsymbol{u}, P)$ is some solution of (NS_ν^0), then

$$(\boldsymbol{u}_\lambda, P_\lambda)(x,t) := \big(\lambda \boldsymbol{u}, \lambda^2 P\big)(\lambda x, \lambda^2 t) \quad \forall\, \lambda > 0, \tag{B.3}$$

is still a solution of (NS_ν^0). We aim at finding some solution $\boldsymbol{u}$ of (NS_ν^0) lying in

$$X \subset \mathcal{C}(\mathbb{R}_+; E) \quad with \ \|\boldsymbol{u}_\lambda\|_X \approx \|\boldsymbol{u}\|_X,$$

for some *adaptable* Banach space E (in sense of [95, Section 8]). By taking $t = 0$ in (B.3), it is not hard to find that E can be taken as $L_N(\mathbb{R}^N)$, $\dot{H}^{\frac{N}{2}-1}(\mathbb{R}^N)$ or $\dot{B}^{\frac{N}{p}-1}_{p,r}(\mathbb{R}^N)$ for some (p,r) in $[1,\infty]^2$. Thus we call these functional spaces (or solutions) obeying this scaling property *critical* for (NS_ν).

With some suitable functional space X in hand, the main steps in applying Lemma B.2 to (NS_ν^0) are the following two assertions:

- $e^{\nu t\Delta}$ is bounded from E into X;
- $\mathcal{B}$ maps $X \times X$ continuously into X.

For example in dimension $N = 3$, Y. Meyer found the Lorentz space $X = \mathcal{C}\big(\mathbb{R}_+; L_{3,\infty}(\mathbb{R}^3)^3\big)$ works well within T.Kato's strategy, which is also one cornerstone in our results for Boussinesq system in Chapter 1 of this thesis. However, in the case of Lebesgue space $E := L_p(\mathbb{R}^3)$ with $p \geq 3$ (inspired by [75]), it is not suitable to just take $X = \mathcal{C}\big(\mathbb{R}_+; L_p(\mathbb{R}^3)^3\big)$. Instead, we have to restrict to the following subspace in $\mathcal{C}\big(\mathbb{R}_+; L_p(\mathbb{R}^3)^3\big)$,

$$X := \Big\{\boldsymbol{u} \in \mathcal{C}\big(]0,T]; L_p(\mathbb{R}^3)^3\big) : \sup_{t\in]0,T]} (\nu t)^{\frac{1}{2}(1-\frac{3}{p})} \|\boldsymbol{u}\|_{L_p(\mathbb{R}^3)} < \infty\Big\}.$$

Additionally, with smallness on the norm $\|\boldsymbol{u}_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,\infty}(\mathbb{R}^3)}$ for $p \in [1,\infty[$, one can construct a unique global solution of (NS_ν^0) by [15]. Moreover, H.Koch and D.Tataru in [77] found the most general space BMO $^{-1}$ to search the global solution by T. Kato's scheme. For more detailed presentations of similar results, one may refer to [6, 54, 84, 95].

(NS_ν) in smooth domains

In the last part of this thesis, we study the free boundary value problem of density-dependent Navier-Stokes equation, where the boundary condition is more complicated than (B.1). On one hand, as our result is based on some abstract approach (the $\mathcal{R}$-boundedness theory introduced in Appendix C), we in some sense benefit from previous works in the case of this no-slip condition (B.1). On the other hand, one may utilize the $\mathcal{R}$-boundedness theory to study Dirichlet problem, but the corresponding result seems not be better than the works before XXI century. Thus we recall some earlier results here concerning the cases (Ω_2), where other nice analysis tools are involved.

Motivated by Hilbert space settings in [85], the further extension to Sobolev (or Hölder) spaces was made by Russian school leaded by O.A.Ladyzhenskaya and V.A.Solonnikov (see [6] [82]). Their observations in early years were based on the direct but powerful potential theory and elliptic (or parabolic) estimates. For instance, V.A. Solonnikov in [112] studied the solvability of (NS_ν) over some smooth exterior domain in certain Sobolev spaces and Hölder spaces.

As we already mentioned, the works [53, 76] were based on the operator theory in Hilbert spaces. Later, Y. Giga et al. in [58–60] extended this Dirichlet problem over smooth ($\mathcal{C}^{2,\varepsilon}$ regular) bounded domain into L_p-framework by combining the pseudo-differential operator theory and fractional power of operators theory in Banach spaces. Besides, in the famous survey [61], Y.Giga and H.Sohr successfully extended the technique in [48] and established the $L_p - L_q$ type maximal regularity for the Stokes equations over smooth exterior domain. For simplicity, let us consider the Stokes equations with null initial data and $\nu \equiv 1$,

$$\begin{cases} \partial_t \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla P = \boldsymbol{f} & \text{in} \quad \Omega \times]0,T[, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega \times]0,T[, \\ \boldsymbol{u}|_{\partial\Omega} = 0, & \text{on} \quad \partial\Omega \times]0,T[, \\ \boldsymbol{u}|_{t=0} = \boldsymbol{0} & \text{in} \quad \Omega. \end{cases} \tag{S_{ext}}$$

By means of decay estimates in [74], the authors in [61] proved that for some constant C (independent on time) and $p, q \in]1,\infty[\times]1, \frac{N}{2}[$, the following estimates hold true

$$\|(\partial_t \boldsymbol{u}, \Delta \boldsymbol{u}, \nabla P)\|_{L_p(0,T;L_q(\Omega))} \leq C \|\boldsymbol{f}\|_{L_p(0,T;L_q(\Omega))}. \tag{B.4}$$

One key point to get (B.4) in [61] is to verify the so-called *Bounded Imaginary Power* (BIP) property of the Stokes operator in some Banach space (see [104]). However, one can avoid this abstract BIP approach as in [93] by more elementary hydrodynamic potential theory. Moreover, the authors pointed out in [93] that the condition $q < \frac{N}{2}$ in (B.4) cannot be relaxed.

Lastly, whenever Ω is some general unbounded domain in $\mathbb{R}^3$, i.e. the boundary of Ω not necessarily compact, the authors in [51] found that the corresponding Stokes operator has maximal regularity in $L_q(0,T;\widetilde{L}_q(\Omega)^3)$. Here $\widetilde{L}_q(\Omega)$ is given by [7]

$$\widetilde{L}_q(\Omega) := \begin{cases} (L_2 \cap L_q)(\Omega) & \text{for } \ 2 \leq q < \infty; \\ (L_2 + L_q)(\Omega) & \text{for } \ 1 < q < 2. \end{cases}$$

[6] [82] might be the fist monograph on the boundary value problem of Navier-Stokes equations.

[7] For any suitable pair of Banach spaces E and F, we can define $E \cap F$ and $E + F$, which are standard in the interpolation theory (see [9]).

Appendix C

L_p Maximal regularity and $\mathcal{R}$-boundedness

The terminology L_p-maximal regularity comes from considering the following Cauchy problem in some Banach space X,

$$\partial_t \boldsymbol{u} + A\boldsymbol{u} = \boldsymbol{f}, \quad \text{and} \quad \boldsymbol{u}|_{t=0} = \boldsymbol{0}, \tag{C.1}$$

where $-A$ is some closed densely defined (i.e. the domain $D(A)$ dense in X) linear operator on X. We call the problem (C.1) has L_p-maximal regularity for some $p \in]1,\infty[$ if the following a priori estimate of the solution $\boldsymbol{u}$ is valid for any given $\boldsymbol{f}$ in $L_p(0,T;X)$,

$$\|\partial_t \boldsymbol{u}\|_{L_p(0,T;X)} + \|A\boldsymbol{u}\|_{L_p(0,T;X)} \leq C\|\boldsymbol{f}\|_{L_p(0,T;X)}.$$

For our purpose of use, we shall mainly focus on very concrete case where $X = \big(L_q(\Omega)\big)^N$ for some domain $\Omega \subset \mathbb{R}^N$ and $1 < q < \infty$. Hence we call such property the $L_p - L_q$ *maximal regularity* for the problem (C.1).

Next, denote $T(z) \equiv e^{-Az}$ the bounded analytic semigroup of continuous operators on X over some sector Σ_ϑ ($\vartheta > 0$), where we adopt the notation

$$\Sigma_\varphi := \big\{\lambda \in \mathbb{C}\backslash\{\boldsymbol{0}\} : |\arg\lambda| < \varphi\big\} \quad \text{for any} \ \varphi \in]0,\pi].$$

As in the classical monographs on the theory of the semigroup of bounded linear operators (see [49, 67, 97] for instance), we reserve the symbol $R(\lambda, A)$ for the resolvent $(\lambda - A)^{-1}$ of operator A if λ is located in the resolvent set $\rho(A)$ of A. In addition, $\mathcal{L}(X;Y)$ stands for all the continuous linear operators from some Banach space X into another Banach space Y. Sometimes we write $\mathcal{L}(X)$ if $X = Y$. To pave a way for Weis Theorem later, let us recall here the classical results on the characterization of $T(z) = e^{-Az}$.

Proposition C.1. *For a densely defined closed operator A on a Banach space X, the following assertions are equivalent:*

1. *(Sectorial) the set $\{\lambda(\lambda+A)^{-1} : \lambda \in \Sigma_\theta\}$ is uniformly bounded for some $\theta > \frac{\pi}{2}$;*
2. *(Analyticity) the set $\{e^{-Az} \text{analytic} : z \in \Sigma_\vartheta\}$ is uniformly bounded for some $\vartheta > 0$;*
3. *e^{-At} is differentiable for $t \in \mathbb{R}_+$, and the set $\{tAe^{-At} : t \in \mathbb{R}_+\}$ is uniformly bounded.*

In fact, the sectorial property in Proposition C.1 implies the following natural extension of the operator class.

Definition C.2. *We call a linear operator A on a Banach space X is* sectorial (or nonegative), *written by $A \in Sect(X)$, if the following two conditions are satisfied:*

1. *the domain $D(A)$ and the range $R(A)$ are both dense in X, i.e. $\overline{D(A)} = \overline{R(A)} = X$;*
2. *the resolvent set $\rho(A)$ contains $]-\infty, 0[$ and $\sup_{t>0} \|t(t+A)^{-1}\|_{\mathcal{L}(X)} < \infty$.*

Remark C.3. *Here we focus on the case of injective operator (since $\overline{R(A)} = X$) for the later application to Stokes operator, although this definition can be extended to multivalued mappings. Of course, the Condition $\overline{D(A)} = X$ is also necessary here if A is the infinitesimal generator of some $\mathcal{C}_0$ semigroup.*

From the definition of sectorial operator, we know there exists $\phi \in [0, \pi[$ such that $\Sigma_{\pi-\phi}$ is contained in the resolvent set $\rho(-A)$ of $-A$. Thus it is reasonable to define the *spectral angle* of $A \in Sect(X)$ by

$$\phi_A := \inf\big\{\phi : \Sigma_{\pi-\phi} \subset \rho(-A), \sup_{\lambda \in \Sigma_{\pi-\phi}} \|\lambda(\lambda+A)^{-1}\|_{\mathcal{L}(X)} < \infty\big\}.$$

Whenever X is a Hilbert space, (C.1) has the L_p-maximal property ($1 < p < \infty$) in X as long as $e^{-Az} : \Sigma_\delta \to \mathcal{L}(X)$ is analytic for some $\delta > 0$. However, to gain the L_p-maximal regularity in general Banach space X, the assumption of analyticity of e^{-Az} is not sufficient.

In fact, in order to search more general L_p-maximal regularity, we have to restrict to some Banach space X admitting some additional structure, namely, the *unconditionality property for martingale differences* (UMD). Suppose $(\Omega, \mathscr{F}, \mathscr{P})$ is some probability space, $d := \{d_j\}_{j\geq 1}$ is a $X-$valued martingale difference sequence [1] and $\varepsilon := \{\varepsilon_j\}_{k\geq 1} \in \{-1, 1\}^{\mathbb{N}}$. X is called UMD if for any fixed $N \in \mathbb{N}$, the following estimate is valid:

$$\|\sum_{j=1}^N \varepsilon_j d_j\|_{L_p(\Omega;X)} \leq C_p \|\sum_{j=1}^N d_j\|_{L_p(\Omega;X)}, \quad \text{for some } p \in]1, \infty[.$$

[1] See more detailed probability theory in [13, 99].

Above C_p only depends on p but not on the choice of N, d or ε. Except the characterization by means of martingale difference, there are other two approaches: the geometric aspect (ζ-convex) and the analytic property (L_p-boundedness of vector-valued Hilbert transform (HT)). Sometimes these new points of view are more convenient to study some Banach spaces. Thus let us recall the definitions of ζ-convex and HT.

A Banach space X is ζ-*convex* if there is a biconvex function $\zeta : X \times X \to \mathbb{R}$ such that $\zeta(0,0) > 0$ and

$$\zeta(x,y) \leq \|x+y\|_X, \quad \forall\ \|x\|_X = \|y\|_X = 1.$$

The biconvexity means $\zeta(x,\cdot)$ and $\zeta(\cdot,y)$ are convex on X for any fixed x and y respectively. Besides, we call a Banach space X is *HT* if the following (Banach or vector valued) Hilbert transform H

$$Hf(t) := \text{p.v.} \int_{\mathbb{R}} \frac{f(s)}{t-s}\, ds, \quad \forall f \in \mathcal{S}(\mathbb{R};X),$$

can be extended to a bounded operator on $L_p(\mathbb{R};X)$ for some (or equivalently for any) $p \in]1,\infty[$. Above $\mathcal{S}(\mathbb{R};X)$ denotes the $X-$valued Schwarz class. Hilbert transform is the most fundamental model for singular integral theory which originates from complex analysis, and Hf can be regarded as the *conjugate* part of f (for more details see [62, 105, 117]).

More importantly, all three additional structures above on Banach space are the same, i.e.

$$X \text{ is UMD} \Longleftrightarrow X \text{ is } \zeta\text{-convex} \Longleftrightarrow X \text{ is HT}. \tag{C.2}$$

The first equivalence in (C.2) was found by D.L.Burkholder in [13] and "UMD $\Longrightarrow$ HT" is due to [14] by the same author. The remaining relation "HT $\Longrightarrow$ UMD" goes back to the work [12] by J.Bourgain. With (C.2), one can easily verify that Hilbert space is $\zeta-$convex, and thus UMD.

Apart from the UMD structure on the Banach spaces, we also need a stronger sectorial assumption on the operator A in order to gain the L_p-maximal regularity. To this end, let us first recall the definition of $\mathcal{R}$-boundedness.

Definition C.4. *Let X, Y be two Banach spaces and $\mathcal{L}(X,Y)$ be the collection of all bounded linear operators from X to Y. We say that some family of bounded operators $\tau \subset \mathcal{L}(X,Y)$ is $\mathcal{R}$-bounded if for any $N \in \mathbb{N}$, $T_j \in \tau$, $x_j \in X$ and any independent symmetric (or mean-free) $\{-1,1\}$-valued random variables ε_j on some probability space $(\Omega, \mathscr{F}, \mathscr{P})$ with $1 \leq j \leq N$, then the following inequality holds,*

$$\Big\| \sum_{j=1}^{N} \varepsilon_j T_j x_j \Big\|_{L_p(\Omega;Y)} \leq C_p \Big\| \sum_{j=1}^{N} \varepsilon_j x_j \Big\|_{L_p(\Omega;X)}, \quad \textit{for some } p \in [1.\infty[. \tag{C.3}$$

C_p only depend on the choice of p but not N, T_j, x_j and ε_j. The optimal (or smallest) C_p in (C.3) is called $\mathcal{R}$-bound of τ, denoted by $R_p(\tau)$.

Remark C.5. *Let us state some basic consequences of the above definition.*

1. *The first comment is that the definition of $\mathcal{R}$-boundedness is independent of p, i.e. one can replace "for some $p \in [1,\infty[$" in (C.3) by "for any $q \in [1,\infty[$". Indeed, there exists a constant $C_{p,q,X,Y}$ such that*
$$C^{-1}_{p,q,X,Y}\, R_q(\tau) \le R_p(\tau) \le C_{p,q,X,Y}\, R_q(\tau),$$
which is a direct consequence of Kahane's inequality*(see [81, Theorem 2.4]). Thus we use $R(\tau)$ instead of $R_p(\tau)$ hereafter.*

2. *If $\tau \subset \mathcal{L}(X,Y)$ is $\mathcal{R}$-bounded then τ is uniformly bounded with respect to operator norm, which is an immediate result from (C.3). In other words, $\mathcal{R}$-boundedness is stronger than the uniform boundedness. However, for Hilbert space, $\mathcal{R}$-boundedness is nothing but uniform boundedness if one takes $p = 2$ in above definition.*

3. *Consider $X = Y = L_q(\Omega)$ with the open set $\Omega \subset \mathbb{R}^N$ and $q \in [1,\infty[$. In this case $\tau \subset \mathcal{L}(X)$ is $\mathcal{R}$-bounded iff. there exists a constant C such that*
$$\Big\|\Big(\sum_{j=1}^{N} |T_j f_j|^2\Big)^{\frac{1}{2}}\Big\|_{L_q(\Omega)} \le C \Big\|\Big(\sum_{j=1}^{N} |f_j|^2\Big)^{\frac{1}{2}}\Big\|_{L_q(\Omega)},$$
For any $N \in \mathbb{N}$, $f_j \in L_q(\Omega)$ and $T_j \in \tau$ (for instance see [45, Sect.3]).

Now let us make some other useful conventions. Motivated by Definition C.2 and C.4, we call the sectorial operator A (defined in X) $\mathcal{R}$-*sectorial* if

$$\mathcal{R}_A(0) := \mathcal{R}\big(\{t(t+A)^{-1} : t > 0\}\big) < \infty.$$

Setting $\mathcal{R}_A(\varphi) := \mathcal{R}\big(\{\lambda(\lambda+A)^{-1} : \lambda \in \Sigma_\varphi\}\big)$, we denote the $\mathcal{R}$-*angle* for some $\mathcal{R}$-sectorial operator A by
$$\phi^R_A := \inf\big\{\phi \in]0,\pi[: \mathcal{R}_A(\pi-\phi) < \infty\big\}.$$

As the assumption of $\mathcal{R}$-boundedness is stronger than the uniform boundedness (in norm topology), one easily see that $\phi_A \le \phi^R_A$. With above preparations, the characterization of L_p-maximal regularity in UMD spaces by L.Weis in [126] reads

Theorem C.6 (Weis)**.** *Suppose the Banach space X is UMD, $p \in]1,\infty[$ and A is some sectorial operator in X with spectral angle $\varphi_A < \frac{\pi}{2}$. The Cauchy problem (C.1) has L_p-maximal regularity if and only if A is $\mathcal{R}$-sectorial with $\varphi^R_A < \frac{\pi}{2}$. More precisely, the Cauchy problem (C.1) has L_p-maximal regularity is equivalent to any of the following assertions:*

1. *($\mathcal{R}$-sectorial) the set $\{\lambda(\lambda+A)^{-1} : \lambda \in \Sigma_\theta\}$ is $\mathcal{R}$-bounded for some $\theta > \frac{\pi}{2}$;*

2. *($\mathcal{R}$-analyticity) the set $\{e^{-Az} : z \in \Sigma_\vartheta\}$ is $\mathcal{R}$-bounded for some $\vartheta > 0$;*

3. *the set* $\big\{t^n(it+A)^{-n} : t \in \mathbb{R}\backslash\{0\}\big\}$ *is* $\mathcal{R}$*-bounded for some* $n \in \mathbb{N}$;

4. *the set* $\{e^{-At}, tAe^{-At} : t > 0\}$ *is* $\mathcal{R}$*-bounded;*

5. *the set* $\{\frac{1}{z}\int_0^z e^{-A\lambda} d\lambda : z \in \Sigma_\vartheta\}$ *is* $\mathcal{R}$*-bounded.*

The proof of Theorem C.6 can be regarded as a counterpart of characterization of the bounded analytic semigroup on the general Banach spaces. But one should be careful with the verification of $\mathcal{R}$-boundedness for vector valued integral. The essential trick is the application of Convexity Theorem (for instance, see [25, Lemma 3.2]). Of course, it is not suitable to give the lengthy demonstration of Theorem C.6, but we can see from some simple computations that these $\mathcal{R}$-bounded properties are related to maximal regularity.

It is not hard to find that the most profound assertion of Theorem C.6 is the equivalence between L_p-maximal regularity property and $\mathcal{R}$-sectorial infinitesimal generator. Let us explain formally the key observation by L.Weis in [126]. Assume $\boldsymbol{f}$ in (C.1) is smooth enough, e.g. $C_c^\infty\big(\mathbb{R}_+; D(A)\big)$, then the mild solution $\boldsymbol{u}(t)$, if it exists, can be represented via

$$\boldsymbol{u}(t) = \int_0^t e^{-(t-s)A}\boldsymbol{f}(s)\,ds.$$

Thanks to $\partial_t e^{-At} = -Ae^{-At}$, we have

$$\partial_t\boldsymbol{u}(t) = \boldsymbol{f}(t) - \int_0^t Ae^{-(t-s)A}\boldsymbol{f}(s)\,ds.$$

Thus for some $p \in]1,\infty[$, the L_p-maximal regularity is equivalent to the fact that the following convolution type operator

$$K\boldsymbol{f}(t) := \int_0^t Ae^{-(t-s)A}\boldsymbol{f}(s)\,ds = \int_{-\infty}^{\infty} Ae^{-A(t-s)}\mathbb{1}_{]0,\infty[}(t-s)\boldsymbol{f}^{(0)}(s)\,ds,$$

extends to a bounded operator from $L_p(\mathbb{R};X)$ to itself. Above $\boldsymbol{f}^{(0)}(t)$ is the zero extension of $\boldsymbol{f}$, namely

$$\boldsymbol{f}^{(0)}(t) := \begin{cases} \boldsymbol{f}(t), & \text{if} \quad t > 0; \\ \boldsymbol{0}, & \text{if} \quad t < 0. \end{cases}$$

This motivates the following *Mihlin-Weis Multiplier Theorem* in UMD spaces.

Theorem C.7 (Mihlin-Weis). *Suppose* X *and* Y *are two UMD spaces and* $p \in]1,\infty[$. *Let* $m \in \mathcal{C}^1\big(\mathbb{R}\backslash\{0\}; \mathcal{L}(X,Y)\big)$ *satisfying the following conditions:*

1. $\tau_1 := \big\{m(\xi) : \xi \in \mathbb{R}\backslash\{0\}\big\}$ *is* $\mathcal{R}$*-bounded with* $\mathcal{R}$*-bound* $R_p(\tau_1)$;

2. $\tau_2 := \big\{\xi m'(\xi) : \xi \in \mathbb{R}\backslash\{0\}\big\}$ *is* $\mathcal{R}$*-bounded with* $\mathcal{R}$*-bound* $R_p(\tau_2)$.

Then the operator

$$m(D)\boldsymbol{f}(t) := \mathcal{F}^{-1}_{\xi\to t}\big(m(\xi)\hat{\boldsymbol{f}}(\xi)\big)(t), \quad \forall \boldsymbol{f} \in \mathcal{S}(\mathbb{R}; X),$$

extends to a bounded operator from $L_p(\mathbb{R}; X)$ *to* $L_p(\mathbb{R}; Y)$*. Moreover, we have*

$$\|m(D)\|_{\mathcal{L}\left(L_p(\mathbb{R};X), L_p(\mathbb{R};Y)\right)} \le C_{p,X,Y} \max\left\{R_p(\tau_1), R_p(\tau_2)\right\}.$$

Thus to apply Theorem C.7, let us return to $Kf(t)$ and take Fourier transform $\mathcal{F}_{t\to\xi}$,

$$\mathcal{F}_{t\to\xi}\big(K\boldsymbol{f}(\cdot)\big)(\xi) = \mathcal{F}_{t\to\xi}\big(AT(\cdot)\mathbb{1}_{]0,\infty[}(\cdot)\big)(\xi)\ \mathcal{F}_{t\to\xi}\big(\boldsymbol{f}^{(0)}(\cdot)\big)(\xi).$$

Recall the Laplace transform

$$(\lambda + A)^{-1} = \int_0^\infty e^{-\lambda t} e^{-tA}\, dt, \quad \Re\lambda > 0.$$

Thanks to the analyticity of $\exp(-Az)$ in the sector Σ_ϑ for some $\vartheta > 0$,

$$\mathcal{F}_{t\to\xi}\big(AT(\cdot)\mathbb{1}_{]0,\infty[}(\cdot)\big)(\xi) = \int_0^\infty e^{-it\xi}\, A\, e^{-tA}\, dt = A(i\xi + A)^{-1}.$$

Obviously $A(i\xi + A)^{-1} = I - i\xi(i\xi + A)^{-1}$ and the third assertion (with taking $n = 1$) in Theorem C.6 yields the validity of the first assumption in Theorem C.7. On the other hand,

$$\xi\frac{dA(i\xi + A)^{-1}}{d\,\xi} = -i\xi(i\xi + A)^{-1} - \big(\xi(i\xi + A)^{-1}\big)^2.$$

Thus applying Theorem C.6 again, we conclude that $K \in \mathcal{L}\big(L_p(\mathbb{R}; X)\big)$.

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