Explicit Calculations of Siu's Effective Termination of Kohn's Algorithm and the Hachtroudi-Chern-Moser Tensors in CR Geometry

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Chapter 1

Introduction

Cette thèse consiste en deux parties: la première partie est l'étude d'un article de Siu sur la terminaison effective de l'algorithme de Kohn pour les domaines pseudoconvexes spéciaux dans $\mathbb{C}^3$, tandis que la deuxième partie est l'étude du tenseur de Hachtroudi-Chern-Moser pour les variétés CR.

1.0.1 Première Partie: Méthode de Siu pour l'algorithme de Kohn

En 1979, J.J. Kohn introduit dans son article [Koh79] un algorithme qui établit les relations entre la géométrie du bord d'un domaine, la régularité des solutions au problème de $\bar{\partial}$-Neumann, et les propriétés algébriques des idéaux multiplicateurs.

Soient $(z_1, \ldots, z_n, z_{n+1})$ les coordonnées holomorphes sur $\mathbb{C}^{n+1}$, et soient $F_1(z_1, \ldots, z_n)$, $\ldots$, $F_{\mathrm{N}}(z_1, \ldots, z_n)$ des germes de fonctions holomorphes en les n premières variables qui s'annulent au point origine. Un domaine spécial Ω est décrit par une inéquation définissante donnée par une fonction analytique réelle de la forme

$$r := \mathsf{Re}(z_{n+1}) - \sum_{1 \leqslant k \leqslant \mathrm{N}} |F_k(z_1, \ldots, z_n)|^2 < 0,$$

dont le bord $b\Omega$ de Ω est le lieux d'annulation de r:

$$\mathsf{Re}(z_{n+1}) - \sum_{1 \leqslant k \leqslant \mathrm{N}} |F_k(z_1, \ldots, z_n)|^2 = 0.$$

Une forme différentielle ϕ de type (p, q) peut s'écrire comme

$$\phi = \sum_{|I|=p} \sum_{|J|=q} \phi_{IJ}\, dz^I \wedge d\bar{z}^J,$$

où les ϕ_{IJ} sont des fonctions sur Ω. Si ces ϕ_{IJ} sont $\mathscr{C}^1$, l'opérateur de Dolbeault, noté $\bar{\partial}$, agit sur ϕ par

$$\bar{\partial}\phi = \sum_{1 \leqslant j \leqslant n+1} \sum_{|I|=p} \sum_{|J|=q} \partial_{\bar{z}_j} \phi_{IJ}\, d\bar{z}^j \wedge dz^I \wedge d\bar{z}^J.$$

Cette forme $\bar{\partial}\phi$ est de type $(p, q+1)$. Si f est une $(p, q+1)$-forme lisse, une question importante est la recherche des solutions ϕ qui satisfont l'équation

$$\bar{\partial}\phi = f,$$

assujettie à la condition que $\bar{\partial} f = 0$. Ce problème, qui s'appelle le problème de $\bar{\partial}$-Neumann, est étudié dans un cadre plus général en utilisant la théorie L^2 de Hörmander.

Pour tout $1 \leqslant p, q \leqslant n+1$, l'espace de Hilbert $L^2_{(p,q)}(\Omega)$ consiste en les formes différentielles de type (p, q) dont les coefficients ϕ_{IJ} sont L^2-intégrable sur Ω par rapport à $d\lambda$ la mesure de Lebesgue.

L'opérateur $\bar\partial$ agit sur une telle forme différentielle au sens des distribution, mais $\bar\partial\phi$ n'est pas forcément dans $L^2_{(p,q+1)}(\Omega)$. Donc, l'opérateur $\bar\partial$ est une application linéaire non-bornée de $L^2_{(p,q)}(\Omega)$ vers $L^2_{(p,q+1)}(\Omega)$, et qui est bien définie sur son domaine $\mathsf{Dom}_{p,q}(\bar\partial)$. Ce domaine est dense, parce qu'il contient les (p,q)-formes lisses à support compact.

Cet espace $L^2_{(p,q+1)}(\Omega)$, muni d'un produit scalaire

$$\left(\sum_{|I|=p}\sum_{|J|=q}\phi_{IJ}dz^I\wedge d\bar z^J,\sum_{|I|=p}\sum_{|J|=q}\psi_{IJ}dz^I\wedge d\bar z^J\right)_{L^2}:=\sum_{|I|=p}\sum_{|J|=q}\int_\Omega\phi_{IJ}\overline{\psi_{IJ}}\,d\lambda,$$

fournit un opérateur adjoint $\bar\partial^*$ de $\bar\partial$ au sens de von Neumann. Cet opérateur $\bar\partial^*$ est non-borné de $L^2_{(p,q+1)}(\Omega)$ vers $L^2_{(p,q)}(\Omega)$ défini sur un sous-espace dense $\mathsf{Dom}_{(p,q+1)}(\bar\partial^*)$ de $L^2_{(p,q+1)}(\Omega)$, et qui satisfait la relation de dualité

$$(\bar\partial\phi,\psi)_{L^2}=(\phi,\bar\partial^*\psi)_{L^2}$$

pour tout $\phi\in\mathsf{Dom}_{(p,q)}(\bar\partial)$ et tout $\psi\in\mathsf{Dom}_{(p,q+1)}(\bar\partial^*)$. L'opérateur laplacien

$$\Delta:=\bar\partial\bar\partial^*+\bar\partial^*\bar\partial:\ L^2_{(p,q)}(\Omega)\to L^2_{(p,q)}(\Omega),$$

avec son domaine de définition $\mathsf{Dom}_{(p,q)}(\Delta)$, est fermé au sens du graphe, et auto-adjoint au sens de von Neumann.

La régularité des solutions pour le problème de $\bar\partial$-Neumann se ramène à l'étude de la régularité du laplacien au bord $b\Omega$ de Ω (voir Proposition 1.17). Plus précisément, soit $x\in b\Omega$ un point dans le bord. Existe-t-il un voisinage $U\subset\mathbb{C}^{n+1}$ de x et un nombre strictement positif $\varepsilon>0$ tels que pour tout $\phi\in\mathscr{D}_{p,q}(\Omega)$ (Définition 1.13), l'estimée suivante

$$|||\phi|||^2_\varepsilon\leqslant C(\|\bar\partial\phi\|^2+\|\bar\partial^*\phi\|^2+\|\phi\|^2)\tag{1.0.1}$$

soit satisfaite? Ici, $|||\cdot|||^2$ est la norme de Sobolev tangentielle (Section 1.2.4), et la constante C ne dépend pas de ϕ. C'est à ce moment que Kohn introduit la notion de multiplicateurs sous-elliptiques (Définition 1.19). Ce sont les fonctions-germes lisses $g\in\mathscr{C}^\infty_x$ en x avec des voisinages $U\subset\mathbb{C}^{n+1}$ de x, et des nombres strictement positifs ε, C tels que pour toute forme différentielle $\phi\in\mathscr{D}_{p,q}(\Omega)$, une variante de l'estimée sous-elliptique

$$|||g\phi|||^2_\varepsilon\leqslant C(\|\bar\partial\phi\|^2+\|\bar\partial^*\phi\|^2+\|\phi\|^2)$$

soit satisfaite. Les données U, ε, C dépendent de g. L'ensemble $\mathscr{J}_x$ des multiplicateurs sous-elliptiques est un idéal radical réel de l'anneau $\mathscr{C}^\infty_x$ (Proposition 1.21). Évidemment, l'inégalité (1.0.1) est établie si et seulement si $1\in\mathscr{J}_x$.

Comme la fonction définissante r et le déterminant de la forme de Levi $\mathsf{Lev}(r)$ sont des multiplicateurs avec régularités respectives $\varepsilon=1$ et $\varepsilon=1/2$ (Propositions 1.20, 1.24 et remarque avant le paragraphe 1.2.7), Kohn crée un algorithme qui permet de déduire dans quelles conditions $g=1$ est atteint. L'algorithme pour les domaines spéciaux est le suivant.

Definition 1.0.2. Soit $\langle F_1,\dots,F_{\mathrm{N}}\rangle$ un idéal de l'anneau local $\mathscr{O}_{\mathbb{C}^n,0}$ des fonctions holomorphes. Soient $g_1,\dots,g_n$ des éléments de l'idéal $\langle F_1,\dots,F_{\mathrm{N}}\rangle$, avec le déterminant jacobien

$$\mathsf{det}(g):=\mathsf{Jac}(g_1,\dots,g_n)=\mathsf{det}\begin{pmatrix}\partial_{z_1}g_1&\cdots&\partial_{z_n}g_1\\ \vdots&\ddots&\vdots\\ \partial_{z_1}g_n&\cdots&\partial_{z_n}g_n\end{pmatrix}.$$

L'idéal $I_1^\#$ est engendré par les éléments de la forme $\mathsf{det}(g)$, et I_1 est son radical. Si I_k est déjà construit, l'idéal $I_{k+1}^\#$ est engendré par I_k avec $\mathsf{det}(h_1,\dots,h_n)$, où h_i est une fonction holomorphe qui appartient à I_k ou bien une des fonctions $F_1,\dots,F_{\mathrm{N}}$. Ensuite, soit I_{k+1} le radical de $I_{k+1}^\#$. □

Alors il est évident qu'il y a une suite croissante d'inclusions d'idéaux

$$I_1 \subseteq I_2 \subseteq \cdots,$$

et comme l'anneau $\mathscr{O}_{\mathbb{C}^n,0}$ est noethérien, la suite se stabilise. S'il existe un nombre K tel que $1 \in I_{\mathrm{K}}$, l'algorithme s'arrête, et l'estimée sous-elliptique est obtenue.

Kohn donne aussi une interprétation géométrique des idéaux de multiplicateurs sous-elliptiques qui sont produits par cet algorithme. En utilisant le théorème de Diederich-Fornæss (Théorème 1.35), le fait que l'algorithme se termine avec $1 \in I_{\mathrm{K}}$ pour quelque K équivaut à dire qu'il n'existe pas de germe de variété analytique complexe contenu dans $b\Omega$ et passant par x. Dans un domaine spécial avec $x := 0 \in b\Omega$, cela revient à demander que l'intersection des germes de variétes analytiques complexes

$$\bigcap_{i=1}^{\mathrm{N}} \{F_i = 0\} = \{0\} \tag{1.0.3}$$

consiste uniquement en le point origine. Dans le langage de la géométrie analytique locale, l'intersection totale des variétés définies par les F_i est équivalente à la finitude de la dimension de l'espace vectoriel quotient suivant

$$\mathsf{dim}_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^n,0}/\langle F_1, \ldots, F_{\mathrm{N}} \rangle := s < \infty. \tag{1.0.4}$$

Une question pertinente, c'est l'existence d'un processus effectif qui termine l'algorithme de Kohn si la condition (1.0.4) est satisfaite. Pour un domaine spécial dans $\mathbb{C}^n$, l'énoncé de Siu dans son article [Siu10] est suivant:

Théorème 1.0.5. *Il existe un nombre explicite m qui ne dépend que n et s tel que $I_m = \mathscr{O}_{\mathbb{C}^n,0}$.*

Le but de cette partie de la thèse est la vérification de ce thèorème pour le cas $n+1 = 3$, avec approfondissement de la méthode de Siu. Le théorème suivant exprime la régularité ε en fonction de s:

Théorème 1.0.6. *Soient (z_1, z_2, z_3) les coordonnées holomorphes dans $\mathbb{C}^3$ avec $z_i = x_i + \sqrt{-1}y_i$. Pour $\mathrm{N} \geqslant 2$, soient $F_1, \ldots, F_{\mathrm{N}}$ des germes de fonctions holomorphes en (z_1, z_2) dans $\mathscr{O}_{\mathbb{C}^2,0}$ qui s'annulent à l'origine, tels que*

$$\mathsf{dim}_{\mathbb{C}}\, \mathscr{O}_{\mathbb{C}^2,0}\big/\langle F_1, \ldots, F_{\mathrm{N}} \rangle := s < \infty.$$

Soit $\Omega \subset \mathbb{C}^3$ le domaine spécial défini par

$$\Omega = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 : \ 2\mathsf{Re}(z_3) - \sum_{1 \leqslant i \leqslant \mathrm{N}} |F_i(z_1, z_2)|^2 < 0. \right\}$$

Alors, l'algorithme de Kohn se termine en au plus $4s^2 - 1$ étapes. De plus, pour tout $\phi \in \mathscr{D}_{0,1}(\Omega)$ à support compact,

$$|||\phi|||_{\varepsilon}^2 \lesssim \|\bar{\partial}\phi\|^2 + \|\bar{\partial}^*\phi\|^2 + \|\phi\|^2,$$

où

$$\varepsilon \geqslant \frac{1}{2^{(4s^2-1)s+3} s^2 (4s^2-1)^4 \binom{8s+1}{8s-1}}.$$

En revanche, la même méthode ne peut pas s'appliquer aux dimensions supérieures $n+1 \geqslant 4$. Le problème réside dans l'assertion [Siu10, page 1234]:

Assertion 1.0.7. *Soient $F_1, \ldots, F_{\mathrm{N}}$ des germes de fonctions holomorphes sur $\mathbb{C}^n$ qui s'annulent à l'origine telles que l'idéal engendré par les F_i contient $\mathfrak{m}^E$ pour quelque nombre effectif E (ici, $\mathfrak{m}$ est l'idéal maximal unique de l'anneau local $\mathcal{O}_{\mathbb{C}^n,0}$). Pour tous $1 \leqslant i_1 < \cdots < i_\nu \leqslant \mathrm{N}$ et $1 \leqslant j_1 < \cdots < j_\nu \leqslant n$, soit J_ν l'idéal engendré par*

$$\frac{\partial(F_{i_1}, \ldots, F_{i_\nu})}{\partial(z_{i_i}, \ldots, z_{j_\nu})}.$$

Alors il existe un autre nombre effectif E' tel que cet idéal J_ν contient $\mathfrak{m}^{E'}$.

Cette assertion a un contre-exemple direct: dans $\mathbb{C}^3$ avec ses coordonnées holomorphes (z_1, z_2, z_3), il suffit de considérer l'idéal $\langle z_1, z_2^2, z_3^2 \rangle$.

La dernière sous-section de cette partie donne des exemples de domaines spéciaux dans $\mathbb{C}^3$ avec terminaison effective de l'algorithme en deux étapes. Soient F et G des polynômes homogènes en deux variables (z, w). Sous l'hypothèse que l'intersection des variétés

$$\{F = 0\} \cap \{G = 0\} := \{0\}$$

ne consiste qu'en le point-origine, en utilisant des résultant, avec une hypothèse de généricité, deux étapes suffisent.

Théorème 1.0.8. *Étant donné deux polynômes homogènes F, $G \in \mathbb{C}[z, w]$ génériques tels que*

$$\mathsf{Resultant}(F, G) \neq 0,$$

il existe une transformation linéaire inversible $A : \mathbb{C}^2 \to \mathbb{C}^2$ tel que l'algorithme de Kohn se termine en deux étapes pour $F \circ A$ et $G \circ A$.

1.0.2 Deuxième partie: Tenseur de Hachtroudi-Chern-Moser en géométrie CR

La deuxième partie est consacrée au calcul des invariants des variétés CR dans diverses situations. La première situation est de déterminer l'existence des lieux CR-ombilics pour les ellipsoïdes dans $\mathbb{C}^2$. La deuxième partie est la géométrie des hypersurfaces réelles M dans $\mathbb{C}^{n+1}$ qui sont lorentzienne, d'après le travail de Bryant [Bry82]. Dans cette partie, en cherchant les équations explicites qui permettent de trouver les champs de vecteurs possibles pour les courbes holomorphes plongées dans M, les composantes de l'invariant de Chern-Moser peuvent être calculées. Dans la troisième partie, étant donné les équations aux dérivées partielles:

$$y_{x^\alpha x^\beta} = F^{\alpha\beta}(x^\gamma, y, y_{x^\delta}) \qquad \big(1 \leqslant \alpha, \beta, \gamma, \delta \leqslant n\big),$$

nous reconstruisons l'invariant $S^{\beta\sigma}_{\alpha\rho}$ associé à ces équations, trouvé par Hachtroudi dans sa thèse, soutenue pendant l'entre-deux-guerres sous la direction d'Élie Cartan. Ensuite, le tenseur $S^{\beta\sigma}_{\alpha\rho}$ sera adapté pour le cas où l'hypersurface réelle M est donnée par une équation définissante implicite $r = 0$.

1.0.2.1 Les lieux CR-ombilics des ellipsoïdes dans $\mathbb{C}^2$ Pour $n \geqslant 2$, l'espace complexe $\mathbb{C}^n$ qui s'identifie avec $\mathbb{R}^{2n}$, est équipé des coordonnées holomorphes $(z_1, \ldots, z_n)$ où $z_i = x_i + \sqrt{-1}y_i$. Un ellipsoïde est l'image de la sphère de rayon 1:

$$\mathbb{S}^{2n-1} = \{z \in \mathbb{C}^n : \ |z_1|^2 + \cdots + |z_n|^2 = 1\}$$

sous une transformation affine de $\mathbb{R}^{2n}$. Un tel ellipsoïde est donc défini par une équation de la forme

$$\sum_{1\leqslant i\leqslant n}\big(\alpha_i x_i^2+\beta_i y_i^2\big)=1, \tag{1.0.9}$$

où les $\alpha_i \geqslant \beta_i > 0$ sont des constantes réelles. En changeant les variables

$$z_i \mapsto z_i/\sqrt{\beta_i},$$

puis en posant $a_i := \alpha_i/\beta_i \geqslant 1$, l'equation (1.0.9) se transforme en

$$\sum_{1\leqslant i\leqslant n}\big(a_i x_i^2+y_i^2\big)=1. \tag{1.0.10}$$

Ensuite, avec

$$A_i := \frac{a_i-1}{2a_i+2} \qquad (1\leqslant i\leqslant n).$$

qui satisfont $0 \leqslant A_i \leqslant 1/2$, le deuxième changement de coordonnées

$$z_i \mapsto \sqrt{1-2A_i}z_i, \qquad \bar{z}_i \mapsto \sqrt{1-2A_i}\bar{z}_i,$$

conduit à l'équation finale d'un ellipsoïde considérée par Webster [Web00]

$$\sum_{1\leqslant i\leqslant n}\left(z_i\bar{z}_i+A_i(z_i^2+\bar{z}_i^2)\right)=1.$$

Avec ce formalisme, pour $n \geqslant 3$, et si les A_i sont choisis génériquement avec $0 < A_1 < \cdots < A_n < 1/2$, Webster démontre qu'il n'y a pas de point CR-ombilics.

Dans $\mathbb{C}^2$ avec $z = x+\sqrt{-1}y$ et $w = u+\sqrt{-1}v$, Huang et Ji dans leur article [XS07] ont démontré que les ellipsoïde de $\mathbb{C}^2$ ont toujours au moins 4 points CR-ombilics. La deuxième partie de cette thèse établit le résultat nouveau suivant, montrant que l'ensemble des points CR-ombilics est de cardinal infini. Ce résultat, qui est dans un travail en commun avec Professeur Merker et un doctorant The-Anh Ta, va apparaître dans Comptes Rendus Académie des Sciences:

Théorème 1.0.11 (cf [FMT])**.** *Pour $a \geqslant 1$ et $b \geqslant 1$ avec $(a,a) \neq (1,1)$, soit $\gamma(\theta)$ la courbe parametrée par $\theta \in \mathbb{R}$ à valeur dans $\mathbb{C}^2 \cong \mathbb{R}^4$:*

$$\gamma:\, \theta \longmapsto \big(x(\theta)+\sqrt{-1}y(\theta),\, u(\theta)+\sqrt{-1}v(\theta)\big)$$

où

$$x(\theta)=\sqrt{\frac{a-1}{a(ab-1)}}\mathsf{cos}\theta, \qquad y(\theta)=\sqrt{\frac{b(a-1)}{ab-1}}\mathsf{sin}\theta,$$
$$u(\theta)=\sqrt{\frac{b-1}{b(ab-1)}}\mathsf{sin}\theta, \qquad v(\theta)=-\sqrt{\frac{a(b-1)}{ab-1}}\mathsf{cos}\theta.$$

Alors son image $\gamma(\mathbb{R})$ est contenue dans le lieu CR-ombilic

$$\gamma(\mathbb{R}) \subset \mathsf{Umb}_{\mathsf{CR}}(E_{a,b}) \subset E_{a,b},$$

où $E_{a,b}$ est l'ellipsoïde défini par

$$ax^2+y^2+bu^2+v^2=1.$$

L'idée de la preuve est de considérer les fonctions suivantes,

$$\begin{aligned} H(\rho) &:= \rho_z^2\rho_{ww} - 2\rho_z\rho_w\rho_{zw} + \rho_w^2\rho_{zz}, \\ \mathrm{L}(\rho) &:= \rho_z\rho_{\bar{z}}\rho_{w\bar{w}} - \rho_z\rho_{\bar{w}}\rho_{\bar{z}w} - \rho_{\bar{z}}\rho_w\rho_{z\bar{w}} + \rho_w\rho_{\bar{w}}\rho_{z\bar{z}}. \end{aligned}$$

avec l'invariant

$$\begin{aligned} I_{[w]} = & \left[\frac{\mathrm{L}(\rho)}{\rho_w^2}\right]^3 \bar{L}\left(\frac{H(\rho)}{\rho_w^3}\right) \\ & -6\left[\frac{\mathrm{L}(\rho)}{\rho_w^2}\right]^2 \bar{L}\left(\frac{\mathrm{L}(\rho)}{\rho_w^2}\right)\bar{L}^3\left(\frac{H(\rho)}{\rho_w^3}\right) - 4\left[\frac{\mathrm{L}(\rho)}{\rho_w^2}\right]^2 \bar{L}^2\left(\frac{\mathrm{L}(\rho)}{\rho_w^2}\right)\bar{L}^2\left(\frac{H(\rho)}{\rho_w^3}\right) \\ & -\left[\frac{\mathrm{L}(\rho)}{\rho_w^2}\right]^2 \bar{L}^3\left(\frac{\mathrm{L}(\rho)}{\rho_w^2}\right)\bar{L}\left(\frac{H(\rho)}{\rho_w^3}\right) + 15\frac{\mathrm{L}(\rho)}{\rho_w^2}\left[\mathrm{L}\left(\frac{\mathrm{L}(\rho)}{\rho_w^2}\right)\right]^2 \bar{L}^2\left(\frac{H(\rho)}{\rho_w^3}\right) \\ & +10\frac{\mathrm{L}(\rho)}{\rho_w^2}\bar{L}\left(\frac{\mathrm{L}(\rho)}{\rho_w^2}\right)\bar{L}^2\left(\frac{\mathrm{L}(\rho)}{\rho_w^2}\right)\bar{L}\left(\frac{H(\rho)}{\rho_w^3}\right) - 15\left[\bar{L}\left(\frac{\mathrm{L}(\rho)}{\rho_w^2}\right)\right]^3 \bar{L}\left(\frac{H(\rho)}{\rho_w^3}\right) \end{aligned}$$

qui s'annule sur la courbe $\gamma(\theta)$, et par conséquent, l'ellipsoïde contient des points CR-ombilics.

1.0.2.2 Courbes holomorphes dans les hypersurfaces réelles lorentziennes – d'après Bryant

Soit M une hypersurface réelle dans $\mathbb{C}^{n+1}$ localement définie par une équation définissante $r = 0$, et soient $(z_1, \ldots, z_n, w = u + \sqrt{-1}v)$ les coordonnées holomorphes de $\mathbb{C}^{n+1}$. Dans cette partie, la variété M est supposée rigide, c'est-à-dire qu'elle est de la forme

$$r := u - F(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0.$$

Les champs de vecteurs suivants pour $1 \leqslant k \leqslant n$

$$\begin{aligned} \mathscr{L}_k &:= \frac{\partial}{\partial z_k} + \frac{F_{z_k}}{\sqrt{-1} - F_v}\frac{\partial}{\partial v}, \\ \overline{\mathscr{L}}_k &:= \frac{\partial}{\partial \overline{z}_k} + \frac{F_{\bar{z}_k}}{-\sqrt{-1} - F_v}\frac{\partial}{\partial v} \end{aligned}$$

forment des repères de $T^{1,0}M := \mathbb{C}TM \cap T^{1,0}\mathbb{C}^{n+1}$ et de $T^{0,1}M := \mathbb{C}TM \cap T^{0,1}\mathbb{C}^{n+1}$ respectivement. En revanche, un co-repère de $\mathbb{C}TM$ naturel est défini par

$$\begin{aligned} \theta &= -dv + \sum_{1\leqslant k\leqslant n}\frac{F_{z_k}}{\sqrt{-1} - F_v}dz_k + \sum_{1\leqslant k\leqslant n}\frac{F_{\bar{z}_k}}{-\sqrt{-1} - F_v}d\bar{z}_k, \\ \theta^k &= dz_k, \\ \bar{\theta}^k &= d\bar{z}_k, \end{aligned}$$

où k est compris entre 1 et n. Évidemment, les θ^k engendrent $T^{1,0*}M$ et les $\bar{\theta}^k$ engendrent $T^{0,1*}M$. Par un changement des co-repères, la 2-forme $d\theta$ peut s'écrire

$$d\theta = \sqrt{-1}(\pm\alpha^1 \wedge \bar{\alpha}^1 \pm \cdots \pm \alpha^n \wedge \bar{\alpha}^n) \bmod \theta.$$

La variété M est dite lorentzienne si la signature de $d\theta$ est $(1, n-1)$, ou autrement dit,

$$d\theta = \sqrt{-1}(\alpha^1 \wedge \bar{\alpha}^1 - \cdots - \alpha^n \wedge \bar{\alpha}^n) \bmod \theta.$$

Soient $\mathscr{A}_i$ et $\bar{\mathscr{A}_i}$ les champs de vecteurs qui sont duaux de α^i et $\overline{\alpha}^i$ respectivement. S'il y a une courbe holomorphe $\phi : \mathbb{D} \to M$ qui est contenue dans M, le champ de vecteurs $\mathscr{L}_{\phi(t)}$ qui est tangent à $\phi(\mathbb{D})$ est donné par

$$\mathscr{L}_{\phi(t)} = \sum_{1\leqslant i\leqslant n+1}\phi_i'(t)\frac{\partial}{\partial z_i} = f_1(t,\bar{t})\mathscr{A}_1 + \cdots + f_n(t,\bar{t})\mathscr{A}_n,$$

pour quelques f_i correspondant au changement de repères. Ce champ de vecteurs est dans le noyau de $d\theta$ qui est une forme quadratique, et donc

$$|f_1|^2 - |f_2|^2 - \cdots - |f_n|^2 = 0.$$

Puisque ϕ est une immersion, la fonction f_1 ne s'annule pas à l'origine, et alors en divisant partout par f_1, l'équation d'une sphère est obtenue.

L'idée de Bryant est de traiter les f_i/f_1 comme des inconnues, et d'introduire de nouvelles variables $\lambda_1,\ldots,\lambda_{n-1}$ qui satisfont

$$\sum_{1\leqslant i\leqslant n-1} |\lambda_i|^2 = 1.$$

Donc, ces λ_i constituent des coordornées locales du fibré en sphére $M \times \mathbb{S}^{2n-3}$ au dessus de M. Dans le formalisme des co-repères, le système de Pfaff suivant est établi:

$$\begin{aligned}
\omega^0 &:= \theta,\\
\omega^1 &:= \alpha^1,\\
\omega^k &:= \alpha^k - \lambda_k\alpha^1,\\
\bar\omega^k &:= \bar\alpha^1,\\
\bar\omega^k &:= \bar\alpha^k - \bar\lambda_k\bar\alpha^k.
\end{aligned}$$

Il existe des fonctions L_k telles que si τ est la 1 forme

$$\tau := -\sum_{2\leqslant k\leqslant n} \bar\lambda_k d\lambda_k + \sum_{2\leqslant k\leqslant n} \bar\lambda_k L_k\bar\alpha^1 - \sum_{2\leqslant k\leqslant n} \lambda_k \bar L_k\alpha^1,$$

sa tirée en arrière $\phi^*\tau$ par ϕ au disque $\mathbb{D}$ est zéro. Pour le cas $n = 2$, le fibré en sphères est $M \times \mathbb{S}^1$, avec $|\lambda| = 1$. Si

$$I_+ := \langle\theta, \alpha^2, \ldots, \alpha^n, \overline\alpha^2, \ldots, \overline\alpha^n, \tau\rangle,$$

est l'idéal engendré par ces 1-formes, la 2-forme $d\tau$ modulo I_+

$$d\tau = -[a_2\lambda^2 + 4a_1\lambda + 6a_0 + 4\bar a_1\bar\lambda + \bar a_2\bar\lambda^2]\,\omega^1\wedge\overline\omega^1 \text{ mod } I_+,$$

est un polynôme en $(\lambda, \bar\lambda)$, et par la théorie de Chern-Moser, les a_2, a_1, a_0 font partie du composantes du tenseur de Chern-Moser. La section suivante donne une formule explicite pour ces coefficients.

1.0.3 Tenseur de Hachtroudi-Chern-Moser pour les variétés CR

Dans la dernière partie de cette thèse, soient $(x^1, \ldots, x^n, y)$ les coordonnées de $\mathbb{C}^{n+1}$. Le but est l'étude d'un système d'équations aux derivées partielles

$$y_{x^\alpha x^\beta} = F^{\alpha,\beta}(x^\gamma, y, y_{x^\delta}), \qquad \big(1\leqslant\alpha,\beta,\gamma,\delta\leqslant n\big)$$

avec la condition de compatibilité

$$F^{\beta,\alpha} = F^{\alpha,\beta}, \qquad D_{x^\gamma}(F^{\alpha\beta}) = D_{x^\beta}(F^{\alpha\gamma}),$$

où D_{x^α} est la dérivée totale

$$D_{x^\alpha} = \frac{\partial}{\partial x^\alpha} + y_{x^\alpha}\frac{\partial}{\partial y} + \sum_{1\leqslant\beta\leqslant n} F^{\alpha,\beta}(x^\gamma, y, y_{x^\delta})\frac{\partial}{\partial y_{x^\delta}}.$$

Dans sa thèse, Hachtroudi [Hac37] utilisait la méthode d'Élie Cartan pour obtenir l'invariant suivant

$$\begin{aligned} S^{\beta\sigma}_{\alpha\rho} &= F^{\alpha,\rho}_{y_{x^\beta},y_{x^\sigma}} - \frac{1}{n+2}\left(\delta^\sigma_\rho \sum_\varepsilon F^{\alpha,\varepsilon}_{y_{x^\beta},y_{x^\varepsilon}} + \delta^\sigma_\alpha \sum_\varepsilon F^{\rho,\varepsilon}_{y_{x^\beta},y_{x^\varepsilon}} + \delta^\beta_\rho \sum_\varepsilon F^{\alpha,\varepsilon}_{y_{x^\sigma},y_{x^\varepsilon}} + \delta^\beta_\alpha \sum_\varepsilon F^{\rho,\varepsilon}_{y_{x^\sigma},y_{x^\varepsilon}}\right) \\ &\quad + \frac{1}{(n+1)(n+2)} \sum_\varepsilon \sum_\delta (\delta^\sigma_\rho \delta^\beta_\alpha + \delta^\beta_\rho \delta^\sigma_\alpha) F^{\delta,\varepsilon}_{y_{x^\delta},y_{x^\varepsilon}}. \end{aligned}$$

Pour adapter cet invariant au cas CR, soient $(z_1, \ldots, z_n, z_{n+1} := w)$ les coordonnées holomorphes de $\mathbb{C}^{n+1}$ et M l'hypersurface réelle définie par $r = 0$. Pour $1 \leqslant k, l \leqslant n+1$, définissons

$$\begin{aligned} H_{k,l} &:= -[r_w r_w r_{z_k z_l} - r_{z_l} r_w r_{z_k w} - r_{z_k} r_w r_{z_l w} + r_{z_k} r_{z_l} r_{ww}], \\ H_{k,\bar{l}} &:= -[r_w r_w r_{z_k \bar{z}_l} - r_{\bar{z}_l} r_w r_{z_k w} - r_{z_k} r_w r_{\bar{z}_l w} + r_{z_k} r_{\bar{z}_l} r_{ww}]. \end{aligned}$$

Notons $\Delta(r)$ le determinant de la matrice suivante

$$\Delta(r) := \begin{vmatrix} r_{\bar{z}_1} & \cdots & r_{\bar{z}_n} & r_{\bar{z}_{n+1}} \\ H_{1,\bar{1}} & \cdots & H_{1,\bar{n}} & H_{1,\overline{n+1}} \\ \vdots & \ddots & \vdots & \vdots \\ H_{n,\bar{1}} & \cdots & H_{n,\bar{n}} & H_{n,\overline{n+1}} \end{vmatrix},$$

et soit $\Delta_{i,j+1}(r)$ le déterminant mineur de la matrice $\Delta(r)$ en enlevant la i-ème colonne et la $(j+1)$-ième ligne. En ré-adaptant les raisonnements de la section sur les ellipsoïdes, nous retrouvons l'invariant associé à l'hypersurface réelle M dans $\mathbb{C}^{n+1}$ définie par une équation définissante $r = 0$:

$$\begin{aligned} \tilde{S}^{\beta\sigma}_{\alpha\rho}|_{\text{id}} &= \sum_k \sum_l \frac{r^3_{z_{n+1}} \cdot \Delta_{(k,\sigma+1)}(r)}{\Delta(r)} \partial_{\bar{z}_k}\left[\frac{r^3_{z_{n+1}} \cdot \Delta_{(l,\beta+1)}(r)}{\Delta(r)} \partial_{\bar{z}_l}\left(\frac{H_{\alpha,\sigma}}{r^3_{z_{n+1}}}\right)\right] \\ &\quad - \frac{1}{n+2}\left\{\delta^\sigma_\rho \sum_\varepsilon \left(\sum_k \sum_l \frac{r^3_{z_{n+1}} \cdot \Delta_{(k,\varepsilon+1)}(r)}{\Delta(r)} \partial_{\bar{z}_k}\left[\frac{r^3_{z_{n+1}} \cdot \Delta_{(k,\beta+1)}(r)}{\Delta(r)} \partial_{\bar{z}_l}\left(\frac{H_{\alpha,\varepsilon}}{r^3_{z_{n+1}}}\right)\right]\right)\right. \\ &\quad + \delta^\sigma_\alpha \sum_\varepsilon \left(\sum_k \sum_l \frac{r^3_{z_{n+1}} \cdot \Delta_{(k,\varepsilon+1)}(r)}{\Delta(r)} \partial_{\bar{z}_k}\left[\frac{r^3_{z_{n+1}} \cdot \Delta_{(k,\beta+1)}(r)}{\Delta(r)} \partial_{\bar{z}_l}\left(\frac{H_{\rho,\varepsilon}}{r^3_{z_{n+1}}}\right)\right]\right) \\ &\quad + \delta^\beta_\rho \sum_\varepsilon \left(\sum_k \sum_l \frac{r^3_{z_{n+1}} \cdot \Delta_{(k,\varepsilon+1)}(r)}{\Delta(r)} \partial_{\bar{z}_k}\left[\frac{r^3_{z_{n+1}} \cdot \Delta_{(k,\sigma+1)}(r)}{\Delta(r)} \partial_{\bar{z}_l}\left(\frac{H_{\alpha,\varepsilon}}{r^3_{z_{n+1}}}\right)\right]\right) \\ &\quad \left. + \delta^\beta_\alpha \sum_\varepsilon \left(\sum_k \sum_l \frac{r^3_{z_{n+1}} \cdot \Delta_{(k,\varepsilon+1)}(r)}{\Delta(r)} \partial_{\bar{z}_k}\left[\frac{r^3_{z_{n+1}} \cdot \Delta_{(k,\sigma+1)}(r)}{\Delta(r)} \partial_{\bar{z}_l}\left(\frac{H_{\rho,\varepsilon}}{r^3_{z_{n+1}}}\right)\right]\right)\right\} \\ &\quad + \frac{1}{(n+1)(n+2)} \sum_\varepsilon \sum_\delta \sum_k \sum_l (\delta^\sigma_\rho \delta^\beta_\alpha + \delta^\beta_\rho \delta^\sigma_\alpha) \frac{r^3_{z_{n+1}} \cdot \Delta_{(k,\varepsilon+1)}(r)}{\Delta(r)} \partial_{\bar{z}_k}\left[\frac{r^3_{z_{n+1}} \cdot \Delta_{(k,\delta+1)}(r)}{\Delta(r)} \partial_{\bar{z}_l}\left(\frac{H_{\delta,\varepsilon}}{r^3_{z_{n+1}}}\right)\right]\Bigg) \end{aligned}$$

qui est une formule analogue à celle d'Hachtroudi.

Chapter 2

Kohn's Algorithm and Siu's Effective Methods

1 Kohn's Algorithm – Introduction

1.1 The $\bar{\partial}$-Neumann problem – a survey

1.1.1 The $\bar{\partial}$ equation One of the most important results in analysis in several complex variables is the solution to the Levi problem. In summary, the theorem is as follows:

Theorem 1.1. *Let $\mathscr{O}_{\mathbb{C}^n}$ be the sheaf of holomorphic functions on $\mathbb{C}^n$. The following conditions are equivalent for a domain $\Omega \subset \mathbb{C}^n$:*

1. *Ω is a domain of holomorphy,*
2. *Ω is pseudoconvex,*
3. *for all $q \geqslant 1$, and for all smooth $(0,q)$ forms α such that $\bar{\partial}\alpha = 0$, there exists a smooth $(0,q-1)$ form u such that $\bar{\partial}u = \alpha$. In the language of cohomology, $H^q(\Omega, \mathscr{O}_{\mathbb{C}^n}) = 0$.* □

Such an equation $\bar{\partial}u = \alpha$ is called the Cauchy-Riemann equation, and Kohn's works study the behaviour of the equation near the boundary.

1.1.2 Some settings Let $(z_1, \ldots, z_n)$ be holomorphic coordinates on $\mathbb{C}^n$, and let $\Omega \subset \mathbb{C}^n$ be a domain. Then a (p,q) form α on Ω can be expressed as

$$\alpha = \sum_{1\leqslant i_1<\cdots<i_p\leqslant n} \sum_{1\leqslant j_1<\cdots<j_q\leqslant n} \alpha_{i_1,\ldots,i_p,j_1,\ldots j_q}(z_1,\ldots,z_n)\, dz_{i_i}\wedge\cdots\wedge dz_{i_p}\wedge d\bar{z}_{j_1}\wedge\cdots\wedge d\bar{z}_{j_q},$$

or simply

$$\alpha = {\sum_{|I|=p}}' {\sum_{|J|=q}}' \alpha_{IJ}\, dz_I \wedge d\bar{z}_J,$$

where the α_{IJ}'s are functions on Ω. Then α belongs to $\mathscr{C}^\infty_{p,q}(\Omega)$ the space of smooth (p,q) forms if α_{IJ} 's are smooth. The space $\mathscr{C}^\infty_{p,q}(\bar{\Omega}) \subset \mathscr{C}^\infty_{p,q}(\Omega)$ consists of smooth (p,q) form α that has a smooth extension to a slightly larger open neighbourhood of $\bar{\Omega}$. For f and g that are (p,q) forms which are written as

$$f = {\sum_{|I|=p}}' {\sum_{|J|=q}}' f_{IJ}\, dz_I \wedge d\bar{z}_J \qquad \text{and} \qquad g = {\sum_{|I|=p}}' {\sum_{|J|=q}}' g_{IJ}\, dz_I \wedge d\bar{z}_J,$$

define the inner product

$$(f,g) := {\sum_{|I|=p}}' {\sum_{|J|=q}}' \int_\Omega f_{IJ}\overline{g_{IJ}}\, d\lambda,$$

where $d\lambda$ is the Lebesgue measure on $\mathbb{C}^n$. Then the space $L^2_{(p,q)}(\Omega)$ consists of (p,q) forms α such that

$$\|\alpha\|^2 \;=\; (\alpha,\alpha) \;=\; \sum_{|I|=p}{}' \sum_{|J|=q}{}' \int_\Omega |\alpha_{IJ}|^2 \, d\lambda \;<\; \infty.$$

1.1.3 The $\bar\partial$ Operator Let $\alpha \in \mathscr{C}^\infty_{0,q}(\Omega)$. The $\bar\partial$ operator is defined by

$$\bar\partial\alpha \;:=\; \sum_{|J|=q}{}' \sum_{1\leqslant j\leqslant n} \partial_{\bar z_j}\alpha_J \, d\bar z_j \wedge d\bar z_J.$$

In the L^2 setting, the $\bar\partial$ operator is not a bounded operator as can be seen for $L^2_{0,1}(\Omega)$. For example, if Ω is bounded, let

$$f_n \;= e^{n\bar z_i} \, d\bar z_j, \qquad (i<j).$$

Then

$$\bar\partial f_n \;=\; n e^{n\bar z_i} \, d\bar z_i \wedge d\bar z_j.$$

Therefore,

$$\|\bar\partial f_n\|^2 \;=\; n^2 \|e^{n\bar z_i}\|^2 \;=\; n^2 \|f_n\|^2,$$

and so

$$\frac{\|\bar\partial f_n\|^2}{\|f_n\|^2} = n^2 \longrightarrow +\infty \qquad \text{as} \qquad n \longrightarrow +\infty.$$

On the other hand, it may happen that there are some elements $u \in L^2_{0,1}(\Omega)$ such that $\bar\partial u$ may not be in $L^2_{0,2}(\Omega)$. For example, in $\mathbb{C}^2$ with coordinates (z_1, z_2), write $z_i = x_i + \sqrt{-1}y_i$ for $i = 1,\ 2$. Let Ω be the open set of $\mathbb{C}^2$ given by

$$\Omega \;=\; \{(z_1,z_2) \in \mathbb{C}^2 :\; -1 < x_i < 1,\ -1 < y_i < 1 \text{ for } i = 1,\ 2\}.$$

Let $u \in L^2_{0,1}(\Omega)$ given by

$$u \;=\; \sqrt{1+x_1}\, d\bar z_2.$$

Under the action of the $\bar\partial$-operator,

$$\bar\partial u \;=\; \frac{1}{2\sqrt{1+x_1}} \, d\bar z_1 \wedge d\bar z_2,$$

which is not integrable since

$$\|\bar\partial u\|^2 \;=\; \int_\Omega \left| \frac{1}{4(1+x_1)} \right| \, d\lambda \;=\; +\infty.$$

This requires that $\bar\partial$ be defined on a suitable set $\mathsf{Dom}_{0,q}(\bar\partial) \subseteq L^2_{0,q}(\Omega)$ given by

$$\mathsf{Dom}_{0,q}(\bar\partial) \;:=\; \left\{\, u \in L^2_{0,q}(\Omega) :\; \bar\partial u \in L^2_{0,q+1}(\Omega) \,\right\}.$$

This set is dense in $L^2_{0,q}(\Omega)$ since it contains all smooth $(0,q)$ forms that are compactly supported in Ω. Also the operator $\bar\partial :\; L^2_{0,q}(\Omega) \to L^2_{0,q+1}(\Omega)$ is a closed operator in the sense of graph.

1.1.4 The Hilbert Space Adjoint $\bar\partial^*$ The Hilbert space adjoint $\bar\partial^*$ of $\bar\partial$ on the other hand needs to be defined on a certain set $\mathsf{Dom}_{0,q+1}(\bar\partial^*)$ of $L^2_{0,q+1}(\Omega)$, given by

$$\begin{aligned}\mathsf{Dom}_{0,q+1}(\bar\partial^*) \;=\; \big\{\, v \in L^2_{0,q+1}(\Omega) :\; &\text{the map } T_v : \mathsf{Dom}_{0,q}(\bar\partial) \to \mathbb{C} \\ &\text{given by } u \mapsto (\bar\partial u, v) \text{ is continuous}\big\}.\end{aligned}$$

With this, the action of $\bar\partial^*$ can be found by first applying Hahn-Banach theorem, followed by the Riesz representation theorem. More precisely, Hahn-Banach theorem allows the unique extension of T_v to a continuous operator $\tilde T_v : L^2_{0,q}(\Omega) \to \mathbb{C}$. Then by Riesz representation theorem, there exists the unique element, denoted by $\bar\partial^* v$, such that for all $f \in L^2_{0,q}(\Omega)$, $\tilde T_v(f) = (f, \bar\partial^* v)$. Then for all $u \in \mathsf{Dom}_{0,q}(\bar\partial)$,

$$(\bar\partial u, v) \;=\; T_v(u) \;=\; \tilde T_v(u) \;=\; (u, \bar\partial^* v).$$

1.1.5 Concrete description of $\bar\partial^*$ on $\mathscr{C}^1_{0,1}(\bar\Omega) \cap \mathsf{Dom}_{0,1}(\bar\partial^*)$ on bounded domains Ω Let $\Omega \subset \mathbb{C}^n$ be a bounded domain given by a $\mathscr{C}^\infty$ defining equation $\Omega = \{r < 0\}$, and assume that r is $\mathscr{C}^\infty$. For $\phi \in \mathscr{C}^1_{0,1}(\bar\Omega) \cap \mathsf{Dom}_{0,1}(\bar\partial^*)$ given by $\phi = \sum \phi_i d\bar z_i$, the Hilbert space adjoint $\bar\partial^*$ has a geometric description. In fact,

$$\begin{aligned}(\bar\partial f, \phi) \;&=\; \sum_{1\leqslant j\leqslant n} \int_\Omega \frac{\partial f}{\partial \bar z_j}\, \overline{\phi_j}\, d\lambda \\ &=\; -\sum_{1\leqslant j\leqslant n} \int_\Omega f \overline{\frac{\partial \phi}{\partial z_j}}\, d\lambda \;+\; \sum_{1\leqslant j\leqslant n} \int_{b\Omega} f \overline{\phi_j \frac{\partial r}{\partial z_j}}\, dS,\end{aligned}$$

where dS is the surface measure on $b\Omega$. Take a sequence of smooth functions f_n with compact support so that $f_n \to f$ in $L^2(\Omega)$. By definition that $\phi \in \mathsf{Dom}_{0,1}(\bar\partial^*)$, the map $f \mapsto (\bar\partial f, \phi)$ is continuous on $\mathsf{Dom}_{0,0}(\bar\partial)$, and hence

$$(\bar\partial f_n, \phi) \longrightarrow (\bar\partial f, \phi)$$

as $n \to \infty$. This easily implies that

$$\int_{b\Omega} f \overline{\sum_{1\leqslant j\leqslant n} \phi_j \frac{\partial r}{\partial z_j}}\, dS = 0$$

for all $f \in \mathsf{Dom}_{0,0}(\bar\partial)$. Moreover, since ϕ and $\partial_{z_j} r$ are continuous on $\bar\Omega$, the function

$$\sum_{1\leqslant j\leqslant n} \phi_j \frac{\partial r}{\partial z_j}$$

is therefore in $L^2_{0,0}(\Omega)$ because Ω is bounded. This defines a continuous map

$$\begin{aligned} T : L^2_{0,0}(\bar\Omega) \;&\longrightarrow\; \mathbb{C} \\ g \;&\longmapsto\; \int_{b\Omega} g \overline{\sum_{1\leqslant j\leqslant n} \phi_j \frac{\partial r}{\partial z_j}} dS.\end{aligned}$$

Here $L^2_{0,0}(\bar\Omega) \subset L^2_{0,0}(\Omega)$ is the set of all L^2 integrable functions on Ω such that they can be extended to L^2 integrable functions on a slightly bigger open neighbourhood of $\bar\Omega$. Hence, every element g in $L^2_{0,0}(\bar\Omega)$ may be approximated by elements in $\mathscr{C}^\infty_c(\Omega) \subset L^2_{0,0}(\bar\Omega)$, and in particular, there exists a sequence $g_n \in \mathscr{C}^\infty_c(\Omega) \subset L^2_{0,0}(\bar\Omega)$ such that

$$g_n \longrightarrow \sum_{1\leqslant j\leqslant n} \phi_j \frac{\partial r}{\partial z_j} \qquad \text{in } L^2_{0,0}(\bar\Omega).$$

Consequently, by continuity of T, and the fact that $T(g_n) = 0$ for all n:

$$T\left(\sum_{1\leqslant j\leqslant n}\phi_j\frac{\partial r}{\partial z_j}\right) = \mathsf{lim}_{n\to\infty}\, T(g_n) = 0,$$

or

$$\int_{b\Omega}\left|\sum_{1\leqslant j\leqslant n}\phi_j\frac{\partial r}{\partial z_j}\right|^2\, dS = 0.$$

Hence

$$\sum_{1\leqslant j\leqslant n}\phi_j\frac{\partial r}{\partial z_j} = 0$$

almost everywhere on $b\Omega$. Since $\sum_{1\leqslant j\leqslant n}\phi_j\frac{\partial r}{\partial z_j}$ is continuous on $b\Omega$, it is therefore zero everywhere.

1.1.6 The Laplacian Δ Having introduced $\bar\partial$ and $\bar\partial^*$, the laplacian Δ is given by

$$\Delta \;:=\; \bar\partial\bar\partial^* \;+\; \bar\partial^*\bar\partial :\; L^2_{0,q}(\Omega) \to L^2_{0,q}(\Omega),$$

defined on the domain

$$\begin{aligned}\mathsf{Dom}_{0,q}(\Delta) = \big\{f \in L^2_{0,q}(\Omega) :\quad & f \in \mathsf{Dom}_{0,q}(\bar\partial),\ \bar\partial f \in \mathsf{Dom}_{0,q+1}(\bar\partial^*)\\ & f \in \mathsf{Dom}_{0,q}(\bar\partial^*),\ \bar\partial^* f \in \mathsf{Dom}_{0,q-1}(\bar\partial)\big\},\end{aligned}$$

which is also a dense set. It is to be emphasised that this is not to be seen as a differential operator but rather as an unbounded operator on the Hilbert space $L^2_{0,q}(\Omega)$. For example, it is known that Δ is a closed and self-adjoint operator in the sense of von Neumann. For the self-adjointness, one has to show that not only $\Delta = \Delta^*$ on $\mathsf{Dom}_{0,q}(\Delta) \cap \mathsf{Dom}_{0,q}(\Delta^*)$, one also has to show that $\mathsf{Dom}_{0,q}(\Delta) = \mathsf{Dom}_{0,q}(\Delta^*)$. These difficulties disappear if Ω is a closed, compact, complex manifold as every smooth differential forms on Ω are automatically smooth differential forms with compact support, and hence the Hilbert space adjoint $\bar\partial^*$ is the same as the formal adjoint. Then the laplacian Δ as a Hilbert space operator is in this case the same as the differential operator in the usual sense.

1.1.7 Pseudoconvexity and the closedness of $R(\bar\partial)$ Let $T : X \to Y$ be an unbounded closed operator from a Hilbert space X to a Hilbert space Y defined on $\mathsf{Dom}_X(T) \subseteq X$, which is assumed to be dense in X. Let $R(T)$ denote the range of T. Recall that T has closed range if $\overline{R(T)} = R(T)$. There are several equivalent conditions of closedness of $R(T)$.

Theorem 1.2 (See [CS01], Chapter 4)**.** *Let T be as above. The following statements are equivalent:*

1. *$R(T)$ is closed in X.*
2. *There is a constant C such that $\|f\|_X \leqslant C\|Tf\|_Y$ for all $f \in \mathsf{Dom}_X(T) \cap \overline{R(T^*)}$.*
3. *$R(T^*)$ is closed in Y.*
4. *There exists the same constant C such that $\|g\|_Y \leqslant C\|T^*g\|_X$ for all $g \in \mathsf{Dom}_Y(T^*) \cap \overline{R(T)}$.*

□

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain which this time is assumed to be pseudoconvex. The following result is due to Hörmander:

Theorem 1.3 (Hörmander's Existence Theorem for $\bar\partial$)**.** *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. For every $\alpha \in L^2_{p,q}(\Omega)$, where $1 \leqslant p \leqslant n$ and $1 \leqslant q \leqslant n$, with $\bar\partial\alpha = 0$, there exists $u \in L^2_{p,q-1}(\Omega)$ such that $\bar\partial u = \alpha$ and*

$$q\|u\|^2 \leqslant e\delta\|\alpha\|^2,$$

where e is the Euler constant and δ is the diameter of Ω. □

As a consequence, there is a very important observation:

Corollary 1.4. *Let Ω be a bounded pseudoconvex domain of $\mathbb{C}^n$. The range of $\bar\partial : L^2_{p,q}(\Omega) \to L^2_{p,q+1}(\Omega)$ is closed.*

Proof. Consider the following complex:

$$L^2_{p,q}(\Omega) \xrightarrow{\bar\partial_{p,q}} L^2_{p,q+1}(\Omega) \xrightarrow{\bar\partial_{p,q+1}} L^2_{p,q+2}(\Omega).$$

The theorem of Hörmander above implies that $\ker(\bar\partial_{p,q+1}) = R(\bar\partial_{p,q})$. Since $\bar\partial_{p,q+1}$ is a closed operator in the sense of graph, $\ker(\bar\partial_{p,q+1})$ is a closed subspace of $L^2_{p,q+1}(\Omega)$ and hence so is $R(\bar\partial_{p,q})$. □

1.1.8 Consequence of Hörmander's theorem and closedness of $R(\bar\partial)$ Given that $\Delta : L^2_{0,q}(\Omega) \to L^2_{0,q}(\Omega)$ is a closed operator, its kernel $\ker\Delta$ is a closed subspace of $L^2_{0,q}(\Omega)$. Therefore, basic Hilbert space theory shows that there is a decomposition

$$L^2_{0,q}(\Omega) \;=\; \ker(\Delta) \oplus (\ker\Delta)^\perp.$$

Moreover, $(\ker\Delta)^\perp = \overline{R(\Delta^*)} = \overline{R(\Delta)}$ where the last equality follows from the fact that Δ is self-adjoint in the sense of von Neumann. Therefore,

$$L^2_{0,q}(\Omega) \;=\; \ker(\Delta) \oplus \overline{R(\Delta)}.$$

Hörmander's theorem implies that $\ker(\Delta) = \{0\}$ (i.e. the operator Δ is injective). To see this, observe that $\ker(\Delta) = \ker(\bar\partial) \cap \ker(\bar\partial^*)$ since it follows immediately from

$$(\Delta u, u) \;=\; \|\bar\partial u\|^2 \;+\; \|\bar\partial^* u\|^2.$$

Next, observe that $\ker\bar\partial \cap \ker\bar\partial^* = \{0\}$. This is because if $u \in \ker\bar\partial \cap \ker\bar\partial^*$, then by Hörmander's theorem, there exists $v \in L^2_{0,q-1}(\Omega)$ such that $\bar\partial v = u$. Hence

$$\|u\|^2 \;=\; (u,u) \;=\; (\bar\partial v, u) \;=\; (v, \bar\partial^* u) \;=\; 0,$$

so that $u = 0$. Thus the decomposition may be rewritten as

$$L^2_{0,q}(\Omega) \;=\; \overline{R(\Delta)}.$$

The closedness of $R(\bar\partial)$ implies the closedness of $R(\Delta)$. This is because since $\bar\partial$ is closed, its kernel is a closed subspace of $L^2_{0,q}(\Omega)$ so that

$$L^2_{0,q}(\Omega) \;=\; \ker(\bar\partial) \;\oplus\; (\ker(\bar\partial))^\perp,$$

and $\ker(\bar\partial)^\perp = \overline{R(\bar\partial^*)}$. Since $R(\bar\partial)$ is closed, by Theorem 1.2, $R(\bar\partial^*)$ is also a closed subspace of $L^2_{0,q}(\Omega)$, and hence

$$L^2_{0,q}(\Omega) \;=\; \ker(\bar\partial) \;\oplus R(\bar\partial^*).$$

By Hörmander's theorem, $\ker(\bar\partial) = R(\bar\partial)$ and so

$$L^2_{0,q}(\Omega) \;=\; R(\bar\partial) \;\oplus\; R(\bar\partial^*).$$

Let $f \in \mathsf{Dom}_{0,q}(\bar\partial) \cap \mathsf{Dom}_{0,q}(\bar\partial^*)$ which contains $\mathsf{Dom}_{0,q}(\Delta)$. Hence f may be written as $f = f_1 \oplus f_2$ where $f_1 \in R(\bar\partial)$ and $f_2 \in R(\bar\partial^*)$. Applying $\bar\partial$ and $\bar\partial^*$ to both sides lead to $\bar\partial f = \bar\partial f_2$ and $\bar\partial^* f = \bar\partial^* f_1$.

Hence both f_1 and f_2 belong to $\mathsf{Dom}_{0,q}(\bar\partial) \cap \mathsf{Dom}_{0,q}(\bar\partial^*)$. By Theorem 1.2, there exists a constant C such that

$$\begin{aligned} \|f_1\|^2 &\leqslant C\|\bar\partial^* f_1\|^2 \\ \|f_2\|^2 &\leqslant C\|\bar\partial f_2\|^2, \end{aligned}$$

so that

$$\begin{aligned} \|f\|^2 &= \|f_1\|^2 + \|f_2\|^2 \\ &\leqslant C(\|\bar\partial f_2\|^2 + \|\bar\partial^* f_1\|^2) \\ &= C(\|\bar\partial f\|^2 + \|\bar\partial^* f\|^2) \\ &= C\big((\bar\partial f, \bar\partial f) + (\bar\partial^* f, \bar\partial^* f)\big) \\ &= C\big((\Delta f, f)\big) \leqslant C\|\Delta f\|\|f\|. \end{aligned}$$

Dividing $\|f\|$ from both sides,

$$\|f\| \leqslant C\|\Delta f\|,$$

from which, combining with Theorem 1.2, implies that Δ has closed range. Therefore,

$$L^2_{0,q}(\Omega) = R(\Delta)$$

and $\Delta : \mathsf{Dom}_{0,q}(\Delta) \to L^2_{0,q}(\Omega)$ is a vector space isomorphism.

1.1.9 Canonical solution to the $\bar\partial$-Neumann problem

Given that

$$\Delta : \mathsf{Dom}_{0,q}(\Delta) \longrightarrow L^2_{0,q}(\Omega)$$

is a vector space isomorphism, it has an inverse

$$N : L^2_{0,q}(\Omega) \longrightarrow \mathsf{Dom}_{0,q}(\Delta),$$

so that $\Delta \circ N = \mathrm{id}_{L^2_{0,q}(\Omega)}$ and $N \circ \Delta = \mathrm{id}_{\mathsf{Dom}_{0,q}(\Delta)}$. From,

$$f = \bar\partial\bar\partial^* N f + \bar\partial^*\bar\partial N f.$$

by applying $\bar\partial$ to both sides, and noting that $\bar\partial^2 = 0$,

$$\bar\partial f = \bar\partial\bar\partial^*\bar\partial N f,$$

it follows that

$$N\bar\partial f = N\bar\partial\bar\partial^*\bar\partial N f = N(\bar\partial^*\bar\partial + \bar\partial\bar\partial^*)\bar\partial N f = \bar\partial N f.$$

Therefore, for any α such that $\bar\partial\alpha = 0$, the equation $\bar\partial u = \alpha$ has a solution

$$u = \bar\partial^* N\alpha$$

as can be easily verified from

$$\bar\partial u = \bar\partial\bar\partial^* N\alpha = \bar\partial\bar\partial^* N\alpha + \bar\partial^* N \underbrace{\bar\partial\alpha}_{=0} = \bar\partial\bar\partial^* N\alpha + \bar\partial^*\bar\partial N\alpha = \Delta N\alpha = \alpha.$$

Moreover, the solution u is orthogonal to the kernel of $\bar\partial$ since for every $v \in \ker\bar\partial$,

$$(u, v) = (\bar\partial^* N\alpha, v) = (N\alpha, \bar\partial v) = 0.$$

This solution $\bar\partial^* N\alpha$ is called the Kohn's solution, or the canonical solution to the $\bar\partial$-Neumann problem.

1.2 The Cauchy-Riemann geometry of boundary and subelliptic multipliers

1.2.1 The Cauchy-Riemann Geometry of the boundary $b\Omega$ Let Ω be a domain in $\mathbb{C}^n$, and suppose that the boundary $b\Omega$ is a smooth manifold. Then a smooth real-valued defining function r is a local defining function for Ω if $b\Omega$ is locally given by the zero set of r. More precisely, if $p \in b\Omega$, then r is a local defining function near p if there is a neighbourhood $U_p \subseteq \mathbb{C}^n$ such that

$$r : U_p \longrightarrow \mathbb{R}$$

is $\mathscr{C}^\infty$, $r < 0$ on $\Omega \cap U_p$, $r = 0$ on $b\Omega \cap U_p$, $r > 0$ on $\Omega^c \cap U_p$, and $dr \neq 0$ on U_p.

Assume moreover that r is real-analytic so that r may be expressed in terms of the convergent power series

$$r \;=\; \sum r_{i_1,\ldots,i_n,j_1,\ldots,j_n}\; z_1^{i_1}\cdots z_n^{i_n}\,\bar{z}_1^{j_1}\cdots\bar{z}_n^{j_n},$$

with $r(0) = 0$ (i.e. $0 \in b\Omega$). The fact that r is real implies that

$$\overline{r_{i_1,\ldots,i_n,j_1,\ldots,j_n}} = r_{j_1,\ldots,j_n,i_1,\ldots,i_n}.$$

The first few terms of the expansion of r is given by

$$r \;=\; \sum_{1\leqslant i\leqslant n} r_i z_i \;+\; \sum_{1\leqslant i\leqslant n} \overline{r_i}\,\bar{z}_i \;+\; O(|z|^2).$$

By renumbering if necessary, $r_n \neq 0$ may be assumed. Then by a biholomorphic change of coordinates

$$(z_1,\cdots,z_n) \longmapsto (w_1,\cdots,w_n) := (z_1,\cdots,z_{n-1}, r_1z_1+\cdots+r_nz_n),$$

the function r may be re-expressed as

$$\begin{aligned} r &= w_n \;+\; \bar{w}_n \;+\; h(w,\bar{w}) \\ &= 2\mathsf{Re}\; w_n \;+\; h(w,\bar{w}), \end{aligned}$$

where $h(w,\bar{w}) = O(|w|^2)$. Renaming back to z, the real-valued, real-analytic defining function may be assumed to be of the form

$$r \;=\; 2\mathsf{Re}\; z_n \;+\; h(z,\bar{z}), \tag{1.5}$$

with $h(z,\bar{z}) = O(|z|^2)$. For $1 \leqslant i \leqslant n-1$, both

$$\begin{aligned} r_i &= \partial_{z_i} r = \partial_{z_i} h, \\ r_{\bar{i}} &= \partial_{\bar{z}_i} r = \partial_{\bar{z}_i} h, \end{aligned}$$

vanish at the origin. Based on equation (1.2), for $1 \leqslant i,j \leqslant n-1$, define the following local frames of $\mathbb{C}T\mathbb{C}^n$

$$\begin{aligned} L_i &:= \frac{\partial}{\partial z_i} \;-\; \frac{r_{z_i}}{r_{z_n}}\frac{\partial}{\partial z_n}, \\ \bar{L}_j &:= \overline{L_j}, \\ L_n &:= \frac{1}{r_{z_n}}\frac{\partial}{\partial z_n}, \\ T &:= L_n - \bar{L}_n = \frac{1}{r_{z_n}}\frac{\partial}{\partial z_n} \;-\; \frac{1}{r_{\bar{z}_n}}\frac{\partial}{\partial \bar{z}_n}. \end{aligned}$$

The proposition below describes the commutator properties of the vector fields on $b\Omega$:

Proposition 1.6. *Let M be a real hypersurface of $\mathbb{C}^n$ containing the origin. For $1 \leqslant i \leqslant n-1$, the vector fields L_i, $\overline{L}_i$ and T defined above form a local frame of $\mathbb{C}TM$, and they satisfy the following properties:*

1. *For $1 \leqslant i, j \leqslant n-1$, $[L_i, L_j] = 0$,*
2. *For $1 \leqslant i, j \leqslant n-1$, $[L_i, \overline{L}_j] = \lambda_{ij} T$, where*

$$\lambda_{ij} \ := \ \frac{r_{i\bar{j}} r_n r_{\bar{n}} - r_{i\bar{n}} r_n r_{\bar{j}} - r_{n\bar{j}} r_i r_{\bar{n}} + r_{n\bar{n}} r_i r_{\bar{j}}}{|r_n|^2}.$$

The matrix λ with the coefficients $(\lambda_{ij})_{1 \leqslant i,j \leqslant n}$ is called the **Levi matrix**. For the $n-1$ by $n-1$ matrix minor $\lambda_{n-1} := (\lambda_{ij})_{1 \leqslant i,j \leqslant n-1}$, at $0 \in \mathbb{C}^n$, observe that its coefficients are given by $\lambda_{ij}(0) = r_{i\bar{j}}(0)$ since $r_i(0) = r_{\bar{j}}(0) = 0$ by equation (1.2). At every point $p \in b\Omega$, each $X \in \mathbb{C}T^{1,0}_p \mathbb{C}^n$ may be written in terms of the local frames $L_1, \ldots, L_n$ as

$$X \ = \ \sum_{1 \leqslant i \leqslant n} x_i L_i|_p \qquad (x_i \in \mathbb{C}),$$

or sometimes in vector notation, $X = (x_1, \ldots, x_{n-1}, x_n)$. If X is also tangent to $b\Omega$, then $x_n = 0$. Moreover, the Levi matrix, seen as a bilinear form when restricted to $T^{1,0}_p b\Omega \times T^{0,1}_p b\Omega$, gives for every X and $Y = (y_1, \ldots, y_n)$ in $T^{1,0}_p b\Omega$ that

$$X \lambda Y^* \ = \ (x_1, \ldots, x_{n-1}) \lambda_{n-1} (y_1, \ldots, y_{n-1})^*,$$

which is also called as the Levi form.

The Levi form at a point $p \in b\Omega$ has other descriptions. Let X and Y be $(1,0)$ vector fields on a neighbourhood of p. It is also described as the following map:

$$\begin{aligned} \lambda : T^{1,0}_p b\Omega \times T^{1,0}_p b\Omega \ &\longrightarrow \ \mathbb{C}TM / T^{1,0}_p b\Omega \oplus T^{0,1}_p b\Omega & (1.7) \\ (X_p, Y_p) \ &\longmapsto \ [X, \bar{Y}](p) \ \mathsf{mod} \ T^{1,0}_p b\Omega \oplus T^{0,1}_p b\Omega. & (1.8) \end{aligned}$$

This map is well-defined, in the sense that this is independent of the choice of vector fields X and Y whose evaluation at p are respectively X_p and Y_p. These two definitions of the Levi forms are related by the following 1-form

$$\sqrt{-1}\partial r|_{\mathbb{C}Tb\Omega} = \ \sqrt{-1} \sum_{1 \leqslant k \leqslant n} r_{z_k} \, dz_k \big|_{\mathbb{C}Tb\Omega}.$$

This is a real differential form because on $b\Omega$, the equation is given by $r = 0$, and whose tangent bundle is given by the vanishing of dr. Therefore

$$0 \ = \ dr|_{\mathbb{C}Tb\Omega} \ = \ \partial r|_{\mathbb{C}Tb\Omega} \ + \ \bar{\partial} r|_{\mathbb{C}Tb\Omega}.$$

Hence

$$\overline{\sqrt{-1}\partial r|_{\mathbb{C}Tb\Omega}} \ = \ -\sqrt{-1}\bar{\partial} r|_{\mathbb{C}Tb\Omega} \ = \ \sqrt{-1}\partial r|_{\mathbb{C}Tb\Omega}.$$

The one-form $\sqrt{-1}\partial r$ satisfies the following identities

$$\begin{aligned} \sqrt{-1}\partial r(L_i) \ &= \ \sqrt{-1} \left(\sum_{1 \leqslant k \leqslant n} r_{z_k} \, dz_k \right) \left(\frac{\partial}{\partial z_i} - \frac{r_{z_i}}{r_{z_n}} \frac{\partial}{\partial z_n} \right) \\ &= \ \sqrt{-1} \left(r_{z_i} - \frac{r_{z_i}}{r_{z_n}} r_{z_n} \right) = 0, \\ \sqrt{-1}\partial r(\bar{L}_i) \ &= \ 0, \\ \sqrt{-1}\partial r(T) \ &= \ \left(\sum_{1 \leqslant k \leqslant n} r_{z_k} \, dz_k \right) \left(\frac{1}{r_{z_n}} \frac{\partial}{\partial z_n} \ - \ \frac{1}{r_{\bar{z}_n}} \frac{\partial}{\partial \bar{z}_n} \right) \\ &= \ \sqrt{-1}. \end{aligned}$$

Therefore, for local sections X and Y of the $T^{1,0}b\Omega$ bundle, the Levi form may simply be recovered by

$$\lambda(X_p, Y_p) \;=\; \langle \sqrt{-1}\partial r(p), [X, \bar{Y}](p)\rangle.$$

Note that by the Cartan-Lie formula applied to $\sqrt{-1}\partial r$,

$$\sqrt{-1}d\bar{\partial} r(X \wedge \bar{Y}) \;=\; \sqrt{-1}X\bar{\partial} r(\bar{Y}) - \sqrt{-1}\bar{Y}\bar{\partial}(X) - \sqrt{-1}\bar{\partial} r([X, \bar{Y}]).$$

Recognising that $\bar{\partial} r$ vanishes on $T^{1,0}b\Omega$ and $T^{0,1}b\Omega$ sections, an alternative expression of the Levi form at p can be written as

$$\lambda(X_p, Y_p) \;=\; \langle \sqrt{-1}\partial\bar{\partial} r(p), X_p \wedge \bar{Y}_p\rangle.$$

1.2.2 Kernel of the Levi form for pseudoconvex domains Recall that $b\Omega$ is pseudoconvex at $p \in b\Omega$ if the Levi map in equation (1.7) is non-negative definite at p, and strongly pseudoconvex at p if the matrix is strictly positive definite at p. The following definitions introduce the notions of the kernel of the Levi form, isotropic cone of the Levi form, and the kernel of the Levi matrix.

Definition 1.9 (Kernel of Levi form)**.** Let λ denote the Levi form on the boundary $b\Omega$. At $p \in b\Omega$, the kernel of the Levi form is the subspace of $T^{1,0}_p b\Omega$ given by

$$K^{1,0}_p(b\Omega) \;:=\; \big\{X_p \in T^{1,0}_p b\Omega :\; 0 = \lambda(X_p, Y_p) \text{ for all } Y_p \in T^{1,0}_p b\Omega\big\}.$$

Definition 1.10 (Isotropic cone of the Levi form)**.** At $p \in b\Omega$, the isotropic cone of the Levi form is given by

$$C^{1,0}_p(b\Omega) \;=\; \{X_p \in T^{1,0}_p b\Omega :\; 0 = \lambda(X_p, X_p)\}.$$

Proposition 1.11. *Let $b\Omega$ be the boundary of a domain $\Omega \subset \mathbb{C}^n$. Suppose that it is pseudoconvex at $p \in b\Omega$, then the kernel of the Levi form and the isotropic cone of the Levi form are the same, in other words*

$$K^{1,0}_p(b\Omega) \;=\; C^{1,0}_p(b\Omega).$$

Proof. The containment $K^{1,0}_p(b\Omega) \subset C^{1,0}_p(b\Omega)$ is trivial. For the reverse, since λ is pseudoconvex at p, it follows from the Cauchy-Schwarz inequality

$$|\lambda(X_p, Y_p)|^2 \leqslant |\lambda(X_p, X_p)| \cdot |\lambda(Y_p, Y_p)|$$

that if X_p lies in the isotropic cone of the Levi form at p, then immediately it belongs to the kernel of the Levi form. □

The following proposition is clear and will be stated without proof.

Proposition 1.12. *Let $b\Omega$ be the boundary of a domain $\Omega \subset \mathbb{C}^n$. Suppose that it is pseudoconvex at $p \in b\Omega$, then $X_p \in \mathscr{C}^{1,0}_p(\Omega)$ if and only if the vector $X_p \cdot \lambda$ annihilates the first $n-1$ columns of the Levi matrix.*

Definition 1.13 (Definition of N_x)**.** For the rest of the introduction, to follow the exposition of Kohn's paper [Koh79], N_x will be used to denote the isotropic cone, or the kernel of the Levi matrix, whenever pseudoconvexity is assumed.

1.2.3 Subellipticity of Δ and regularity of the canonical solution As explained earlier, the pseudoconvexity condition of the bounded domain $\Omega \subset \mathbb{C}^n$ implies the existence of Kohn's canonical solution to the $\bar{\partial}$-Neumann problem:

$$\bar{\partial} u \;=\; \alpha, \qquad \big(\bar{\partial}\alpha = 0\big),$$

with u a $(0,1)$-form. Suppose that $x_0 \in \bar{\Omega}$, and $\alpha \in L^2_{0,2}(\Omega)$ (or $L^2_{0,2}(\bar{\Omega})$) so that it is smooth in a neighbourhood of x_0, then the question is whether Kohn's canonical solution to the $\bar{\partial}$ equation is also smooth in some neighbourhood of x_0. A sufficent condition for this to hold is the notion of subelliptic estimates for the Δ operator.

Definition 1.14 (Subelliptic Estimates)**.** If $x_0 \in \bar{\Omega}$, the $\bar{\partial}$-Neumann problem for (p,q) forms satisfies a subelliptic estimate at x_0 if there exist a neighbourhood $U \subseteq \mathbb{C}^n$ of x_0, and constants $\varepsilon > 0$ and $C > 0$, such that for all

$$\phi \in \mathscr{D}_{p,q}(U) \;:=\; \big\{\phi \in \mathsf{Dom}_{p,q}(\bar{\partial}^*) :\; \phi_{IJ} \in \mathscr{C}^\infty_c(U \cap \bar{\Omega})\big\},$$

the following estimate holds:

$$\|\phi\|^2_\varepsilon \;\leqslant\; C\big(\|\bar{\partial}\phi\|^2 \;+\; \|\bar{\partial}^*\phi\|^2 \;+\; \|\phi\|^2\big). \quad \square \tag{1.15}$$

Here $\|\cdot\|^2_\varepsilon$ denotes the Sobolev norm of order ε. To ease some notations, given any two (p,q)-forms ϕ and ψ, let $Q(\phi,\psi)$ denote the bilinear pairing

$$Q(\phi,\psi) \;:=\; (\bar{\partial}\phi, \bar{\partial}\psi) \;+\; (\bar{\partial}^*\phi, \bar{\partial}^*\psi) \;+\; (\phi,\psi).$$

A consequence of the subelliptic estimate in equation (1.15) is the following theorem which answers the question of local regularity of the canonical solution to the $\bar{\partial}$ equation:

Theorem 1.16 (Kohn-Nirenberg, see [Koh79])**.** *Suppose that $\Omega \subset \mathbb{C}^n$ is a bounded pseudoconvex domain with $\mathscr{C}^\infty$ boundary. Assume also that equation* (1.15) *holds at $x_0 \in \bar{\Omega}$. If $\alpha \in L^2_{p,q}(\Omega)$ is smooth in a neighbourhood of x_0, then $N\alpha$ is also smooth in a neighbourhood of x_0. More precisely, if α is in H^s (which is the Sobolov space of order s) in a neighbourhood of x_0, then $N\alpha \in H^{s+2\varepsilon}$ and $\bar{\partial}^* N\alpha \in H^{s+\varepsilon}$.* $\square$

A point to emphasise is that the smoothness of the solution is guaranteed only when such an $\varepsilon > 0$ exists. For $x_0 \in \Omega$, subelliptic estimates always hold with $\varepsilon = 1$, as Δ is elliptic on the interior of Ω. The problem appears when $x_0 \in b\Omega$. The following definition will then be used to explain in the next paragraph the relation between subelliptic estimates and the tangential subelliptic estimates when $x_0 \in b\Omega$:

Definition 1.17. For $\varepsilon > 0$, let $\mathscr{E}^q(\varepsilon)$ denote the subset of $\bar{\Omega}$ consisting of elements x_0 such that there exists a neighbourhood U of x_0 on which equation (1.15) holds.

1.2.4 Tangential Sobolev Norm Kohn, in the paper [Koh79], shows that for $x_0 \in b\Omega$, the subelliptic estimate in equation (1.15) can be reduced to the study of regularity property near the boundary of Ω. For this, the tangential Sobolev norm needs to be introduced. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with smooth boundary, and let $x_0 \in b\Omega$. Assume that in a neighbourhood $U \subseteq \mathbb{C}^n$ of x_0, the boundary $b\Omega \cap U$ may be defined by a defining function $\Omega \cap U = \{r < 0\}$ so that dr does not vanish anywhere on the set $\{r = 0\} = b\Omega \cap U$.

By the Implicit Function Theorem, there exists a change of local coordinates on U such that with the new system $(t_1, \ldots, t_{2n-1}, r) \in \mathbb{R}^{2n-1} \times \mathbb{R} \cong \mathbb{C}^n$, the boundary $b\Omega \cap U$ is given by $r = 0$. The tangential Fourier transform is then given by

$$\hat{f}(\tau, r) \;:=\; \frac{1}{(2\pi)^{\frac{2n-1}{2}}} \int_{\mathbb{R}^{2n-1}} e^{-\sqrt{-1}t\cdot\tau}\, f(t,r)\, dt,$$

where $t := (t_1, \ldots, t_{2n-1})$, $\tau := (\tau_1, \ldots, \tau_{2n-1})$, and $t \cdot \tau := \sum_{1 \leqslant i \leqslant n} t_i \tau_i$.

The tangential pseudodifferential operator of order s is given by

$$\Lambda^s f(t,r) \ := \ \frac{1}{(2\pi)^{\frac{2n-1}{2}}} \int_{\mathbb{R}^{2n-1}} e^{\sqrt{-1}t\cdot\tau}(1+|\tau|^2)^{s/2} \hat{f}(\tau, r)\, d\tau,$$

and the tangential Sobolev norm of order s is defined as

$$|||f|||_s^2 \ := \ \int_{-\infty}^{0} \int_{\mathbb{R}^{2n-1}} |\Lambda^s f(t,r)|^2 \, dt \, dr.$$

For a (p,q) form $\phi = \sum'_{|I|=p} \sum'_{|J|=q} \phi_{IJ} \, dz_I \wedge d\bar{z}_J$, its tangential Sobolev norm of order s is

$$|||\phi|||_s^2 \ = \ \sum_{|I|=p}{}' \ \sum_{|J|=q}{}' \ |||\phi_{IJ}|||_s^2.$$

Near the boundary, the subelliptic estimates as in inequality (1.15) can be expressed entirely in terms of tangential Sobolev norm instead of the Sobolev norm.

Proposition 1.18 (See [Koh79]). *For $\varepsilon > 0$, if $x_0 \in b\Omega$, then $x_0 \in \mathscr{E}^{pq}(\varepsilon)$ if and only if there exists a neighbourhood U of x_0 and a constant $C > 0$ such that for all $\phi \in \mathscr{D}_{p,q}(U)$,*

$$|||\phi|||_\varepsilon^2 \leqslant C \, Q(\phi, \phi). \quad \square \tag{1.19}$$

Since for the rest of the introduction x_0 is always assumed to be in the boundary $b\Omega$, equation (1.19) will also be referred to as the subelliptic estimate of Δ without much ambiguity. This definition appears in several literatures such as in [D'A93].

1.2.5 Subelliptic multipliers

To solve the $\bar{\partial}$-Neumann problem for bounded pseudoconvex domains, Kohn introduced the notion of subelliptic multipliers.

Definition 1.20 (Subelliptic multipliers). Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ and let $x_0 \in b\Omega$ be a point. Let $\mathscr{C}^\infty_{x_0}$ denote the ring of germs of smooth functions at that point. An element $g \in \mathscr{C}^\infty_{x_0}$ is called a subelliptic multiplier for $(0,1)$-forms if there is a neighbourhood $U \subseteq \mathbb{C}^n$ of x_0, and positive constants $C > 0$ and $\varepsilon > 0$, such that

$$|||g\phi|||_\varepsilon^2 \ \leqslant \ CQ(\phi, \phi)$$

for all $\phi \in \mathscr{D}_{0,1}(U)$. The open set U, and the constants C and ε, depend on g. $\square$

An example of a subelliptic multiplier is the following:

Proposition 1.21 (See [D'A93]). *Let x_0 be a point in the smooth boundary $b\Omega$ of the bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$, which has a defining function r defined in a small neighbourhood $U \subseteq \mathbb{C}^n$ of x_0. Then there exists a constant $C > 0$ such that for all $\phi \in \mathscr{D}_{0,1}(U)$,*

$$|||r\phi|||_1^2 \ \leqslant \ CQ(\phi, \phi).$$

Let $J(x_0) \subseteq \mathscr{C}^\infty_{x_0}$ be the collection of all subelliptic multipliers at x_0. Then $J(x_0)$ is an ideal. In fact, it is also a real radical ideal in the following sense: for any ideal $I \subseteq \mathscr{C}^\infty_{x_0}$, the real radical of I, denoted by $\mathrm{rad}_{\mathbb{R}}(I)$, is the set of elements $g \in \mathscr{C}^\infty_{x_0}$ such that there exists a positive integer N, and an element $f \in I$ so that

$$|g|^N \leqslant |f|.$$

More precisely,

Proposition 1.22 (See [D'A93]). *Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain and $x_0 \in b\Omega$. Suppose $f \in J(x_0)$ so that there exist $U \subset \mathbb{C}^n$ a neighbourhood of x_0, and constants $C_f > 0$ and $\varepsilon > 0$, with for all $\phi \in \mathscr{D}_{0,1}(U)$,*

$$|||f\phi|||_\varepsilon^2 \leqslant C_f Q(\phi,\phi).$$

If $g \in \mathscr{C}^\infty_{x_0}$ be such that $|g|^N \leqslant |f|$ for some positive integer N, then there exists a constant $C_g > 0$ such that for all $\phi \in \mathscr{D}_{0,1}(U)$,

$$|||g\phi|||_{\varepsilon/N}^2 \leqslant C_g Q(\phi,\phi). \quad \square$$

1.2.6 Vector and matrix multipliers Similar to subelliptic multipliers, there are also vector and matrix multipliers.

Definition 1.23 (Vector Multipliers). Let $x_0 \in b\Omega$ be a point in the boundary of a bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with smooth boundary. A $(1,0)$-vector field

$$v = \sum_{1\leqslant j\leqslant n} v_j \frac{\partial}{\partial z_j}$$

is a vector multiplier if there is a neighbourhood U of x_0, and positive constants $C > 0$ and $\varepsilon > 0$ such that for all $\phi \in \mathscr{D}_{0,1}(U)$,

$$\left|\left|\left| \sum_{1\leqslant j\leqslant n} v_j\phi_j \right|\right|\right|_\varepsilon^2 \leqslant CQ(\phi,\phi). \quad \square$$

An example of a vector multiplier is the following proposition:

Proposition 1.24 (See [D'A93]). *Let $x_0 \in b\Omega$ be a point in the smooth boundary of a bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$. Suppose that $f \in \mathscr{C}^\infty_{x_0}$ is a subelliptic multiplier, that is there exist $U \subseteq \mathbb{C}^n$ a neighbourhood of x_0, and positive constants $C > 0$ and $\varepsilon > 0$, such that*

$$|||f\phi|||_\varepsilon^2 \leqslant CQ(\phi,\phi)$$

for all $\phi \in \mathscr{D}_{0,1}(U)$, then there exists a constant $C' > 0$ such that for all $\phi \in \mathscr{D}_{0,1}(U)$,

$$\left|\left|\left| \sum_{1\leqslant j\leqslant n} \frac{\partial f}{\partial z_j}\phi_j \right|\right|\right|_{\varepsilon/2}^2 \leqslant C'Q(\phi,\phi). \quad \square$$

In other words, the $(1,0)$-form ∂f is a vector multiplier. For example, since r is a subelliptic multiplier by Proposition 1.21, ∂r is also a multiplier with regularity $\varepsilon \geqslant 1/2$. Another example of a vector multiplier is the first $n-1$ columns of the Levi matrix.

Proposition 1.25 (See [Koh79], page 97). *Assuming the hypothesis as in the definition 1.26. Each of the first $n-1$ columns of the Levi form λ is a vector multiplier. In other words, there exist constant $C > 0$ and an open neighbourhood $U \subseteq \mathbb{C}^n$ of x_0 such that for all $\phi \in \mathscr{D}_{0,1}(U)$,*

$$\left|\left|\left| \sum_{1\leqslant i\leqslant n} \lambda_{ij}\phi_i \right|\right|\right|_{1/2}^2 \leqslant \sum_{1\leqslant j\leqslant n-1} \left|\left|\left| \sum_{1\leqslant i\leqslant n} \lambda_{ij}\phi_i \right|\right|\right|_{1/2}^2 \leqslant CQ(\phi,\phi).$$

Next, let A be an $n \times n$ matrix with entries in $\mathscr{C}^\infty_{x_0}$ given by

$$A = (a_{ij})_{1\leqslant i,j\leqslant n}, \qquad (a_{ij} \in \mathscr{C}^\infty_{x_0}).$$

The action of A on $(0,1)$ forms $\phi = \sum \phi_j dz_j$ can then be defined in the usual way of matrix multiplication

$$A\phi = \sum_{1\leqslant j\leqslant n} \left(\sum_{1\leqslant k\leqslant n} a_{jk}\phi_k \right) d\bar{z}_j. = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix}.$$

Definition 1.26. Let $x_0 \in b\Omega$ be a point in the smooth boundary of a bounded pseudoconvex domain $\Omega \subset \mathbb{C}^\infty$. A matrix $A = (a_{ij})$ is a matrix multiplier if there exist a neighbourhood U of x_0, and positive constants $C > 0$ and $\varepsilon > 0$ such that

$$|||A\phi|||_\varepsilon^2 \leqslant CQ(\phi, \phi)$$

for all $\phi \in \mathscr{D}_{0,1}(U)$. □

Also, given that A is a matrix multiplier in definition 1.26, its determinant $\mathsf{det}A$ is a subelliptic multiplier with

$$|||(\mathsf{det}A)\phi|||_\varepsilon^2 \leqslant C'Q(\phi, \phi).$$

For details, see [D'A93].

1.2.7 Constructing new multipliers from old ones The properties of subelliptic multipliers, vector multipliers and matrix multipliers allow the construction of new multipliers from the old ones. For example, let $f_1, \ldots, f_n$ be subelliptic multipliers. By restricting to a smaller open neighbourhood $U \subseteq \mathbb{C}^n$ of $x_0 \in b\Omega$, and by choosing suitable constants $C > 0$ and $\varepsilon > 0$, for all $\phi \in \mathscr{D}_{0,1}(U)$, and $1 \leqslant i \leqslant n$, it may be assumed that

$$|||f_i\phi|||_\varepsilon^2 \leqslant CQ(\phi, \phi).$$

By Proposition 1.24, each ∂f_i is a vector multiplier with for all $\phi \in \mathscr{D}_{0,1}(U)$,

$$|||\partial f_i \cdot \phi|||_{\varepsilon/2}^2 \leqslant CQ(\phi, \phi) \qquad (1 \leqslant i \leqslant n).$$

Putting each of the vectors ∂f_i as a row of the following matrix

$$A = \begin{pmatrix} \partial_{z_1} f_1 & \cdots & \partial_{z_n} f_1 \\ \vdots & \ddots & \vdots \\ \partial_{z_1} f_n & \cdots & \partial_{z_n} f_n \end{pmatrix},$$

it is evident that for all $\phi \in \mathscr{D}_{0,1}(U)$,

$$|||A\phi|||_{\varepsilon/2}^2 \leqslant CQ(\phi, \phi).$$

Taking the determinant of the matrix,

$$|||(\mathsf{det}\ A)\phi|||_{\varepsilon/2}^2 \leqslant CQ(\phi, \phi).$$

Hence a new subelliptic multiplier $\mathsf{det}\ A$ is constructed from $f_1, \ldots, f_n$. Moreover, some of the rows in A may be replaced by some of the first $n-1$ columns of the Levi matrix, or the vector ∂r. Taking its determinant also constructs a new subelliptic multiplier.

1.2.8 Kohn's algorithm Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain with smooth boundary $b\Omega$, and let $x_0 \in b\Omega$ be a point. Let $U \subseteq \mathbb{C}^n$ be a neighbourhood of x_0 such that $b\Omega \cap U$ has a real-analytic, real-valued defining function $b\Omega \cap U = \{r = 0\}$, and dr does not vanish anywhere on $b\Omega \cap U$. Based on the discussions in paragraph 1.2.7 on constructing new subelliptic multipliers from old ones, Kohn's algorithm provides a systematic approach to construct ideals of multipliers.

Definition 1.27 (Kohn's Algorithm)**.** The Kohn's ideals of subelliptic multipliers are inductively defined by

$$\begin{aligned} J_0(x_0) &= \text{rad}_{\mathbb{R}}(\mathsf{det}(\mu_0)), \\ J_{k+1}(x_0) &= \text{rad}_{\mathbb{R}}(J_k(x_0), \mathsf{det}(\mu_{k+1})), \end{aligned}$$

where μ_0 is a matrix whose rows consist of either one of the first $n-1$ columns of the Levi matrix or the vector ∂r. Let μ_{k+1} denote the set of matrices whose rows are either ∂f for some $f \in J_k(x_0)$, one of the first $n-1$ columns of the the Levi matrix, or the vector ∂r. The notation $\mathsf{det}(\mu_k)$ then represents the ideal generated by all the determinants of this form.

Observe that there is an increasing sequence of ideals in $\mathscr{C}^{\infty}_{x_0}$

$$J_0(x_0) \subseteq J_1(x_0) \subseteq \cdots \subseteq J_k(x_0) \subseteq \cdots \text{ ad infinitum.}$$

Say that Kohn's algorithm terminates if there exists $k \geqslant 0$ such that $1 \in J_k(x_0)$, and therefore clearly by definition, subelliptic estimates hold at the point x_0. The termination of algorithm is related to the presence of a holomorphic variety contained in $b\Omega \cap U$.

1.3 The geometry of Kohn's algorithm – complex-valued, real-analytic case

1.3.1 The complex-valued, real-analytic case In this subsection, Kohn's algorithm is to be seen in the context of germs of real-analytic functions. Let $x_0 \in b\Omega$, and let $\mathscr{A}_{x_0}$ denote the set of germs of (complex-valued) real-analytic functions at x_0. If $S \subset \mathscr{A}_{x_0}$, then let $\langle S \rangle$ denote the ideal of real-analytic functions generated by S, and let $\text{rad}_{\mathbb{R}}(S)$ denote the set of all $g \in \mathscr{A}_{x_0}$ such that there exists a positive integer $m \geqslant 1$, and $f \in S$ such that $|g|^m \leqslant |f|$.

Let $I \subset \mathscr{A}_{x_0}$ be an ideal. Denote $V(I)$ the germ of real-analytic variety defined by I, that is take $f_1,\ldots,f_k$ be generators of I. Let $U \subset \mathbb{C}^n$ be the open neighbourhood of x_0 on which f_i is defined for every i. Then

$$U \cap V(I) = \{x \in U : f_1(x) = \cdots = f_k(x) = 0\}.$$

If $x \in V(I)$, let $\mathscr{I}_x(V(I))$ denote the germs of real-analytic functions at x which vanish on $V(I)$. The following results and definitions, which are analogous to those in complex analytic geometry, will be mentioned without proof, and instead readers are referred to [Koh79]. They will be used to explain the geometry of Kohn's algorithm.

Theorem 1.28 (Lojasiewicz (Nullstellensatz))**.** *If $I \subseteq \mathscr{A}_0$ is an ideal of germs of real-analytic functions at $0 \in \mathbb{R}^n$, then $\mathscr{I}_0(V(I)) = \text{rad}_{\mathbb{R}}(I)$.* □

Proposition 1.29 (Weak form of Coherence)**.** *If $I \subseteq \mathscr{A}_0$ is an ideal of real-analytic functions at $0 \in \mathbb{R}^n$. Assume that $I = \text{rad}_{\mathbb{R}} I$. Then there exists a sequence of points $x_\nu \in V(I)$ such that $x_\nu \to 0$, and such that each x_ν has a neighbourhood U_ν such that if $y \in U_\nu \cap V(I)$, then $\mathscr{I}_y(V(I))$ is generated by finitely many elements of I.* □

Definition 1.30 (Zariski Tangent Space)**.** Let I be an ideal of germs of real-analytic functions at x_0, and let $x \in V(I)$. Define $Z^{1,0}_x(I)$ be the Zariski tangent space of I at x by

$$Z^{1,0}_x(I) = \{L \in T^{1,0}_x \mathbb{C}^n : \langle \partial f(x), L \rangle = 0 \text{ for all } f \in I\}.$$

For a germ of real-analytic variety V at x_0, define

$$Z^{1,0}_x(V) := Z^{1,0}_x(\mathscr{I}_x V). \quad \square$$

An immediate consequence of the definition of Zariski tangent space is the following lemma:

Lemma 1.31. *If I is an ideal of the ring of germs at x_0 and if $x \in V(I)$, then*

$$Z_x^{1,0}(V(I)) \subseteq Z_x^{1,0}(I). \tag{1.32}$$

Moreover, if $\mathscr{I}_x V(I)$ is generated by elements of I, then the inclusion is an equality.

Suppose that I is generated by $f_1,\ldots,f_l$ a finite number of real-analytic functions. Then

$$Z_x^{1,0}(I) \;=\; \{L \in T_x^{1,0}\mathbb{C}^n :\; \langle \partial f_i(x), L\rangle = 0 \text{ for } i = 1,\ldots,l\}.$$

This is because every $f \in I$ may be written as

$$f \;=\; \sum_{1\leqslant i\leqslant l} h_i f_i,$$

and hence

$$\partial f \;=\; \sum_{1\leqslant i\leqslant l} f_i \partial h_i \;+\; h_i \partial f_i.$$

On $x \in V(I)$,

$$\partial f(x) \;=\; \sum_{1\leqslant i\leqslant l} h_i(x) \partial f_i(x),$$

which is a linear combination of $\partial f_i(x)$. Therefore $Z_x^{1,0}(I)$ is determined by only a finite number of linear equations. Without ambiguity, $Z_x^{1,0}(I)$ may sometimes be referred to by $Z_x^{1,0}(f_1,\ldots,f_l)$ if $(f_1,\ldots,f_l)$ generates I.

For the rest of the section, let $J_k(x_0)$ be the ideals generated by Kohn's algorithm for the ring $\mathscr{A}_{x_0}$ instead of $\mathscr{C}^\infty_{x_0}$. The termination of Kohn's algorithm is equivalent to the fact that that there exists k such that $1 \in J_k(x_0)$, which in turn is equivalent to saying that $V(J_k(x_0)) = \emptyset$.

1.3.2 Holomorphic dimension

Observe the following proposition:

Proposition 1.33. *If $x \in V(J_k(x_0))$, then $x \in V(J_{k+1}(x_0))$ if and only if*

$$\mathsf{dim}_{\mathbb{C}}(Z_x^{1,0}(J_k(x_0)) \cap N_x) \;\geqslant\; 1.$$

Proof. Given that $x \in V(J_k(x_0))$, and suppose that $\mathsf{dim}_{\mathbb{C}}(Z_k^{1,0}(J_k(x_0)) \cap N_x) = 0$. Let $L = \sum_{1\leqslant i\leqslant n} \zeta_i L_i|_x \in T_x^{1,0}\mathbb{C}^n$. Then $L \in Z_k^{1,0}(J_k(x_0)) \cap N_x$ if and only if L satisfies the following linear equations

$$\begin{aligned} \langle \partial r(x), L\rangle &= 0, \\ \sum_{1\leqslant i\leqslant n} \lambda_{ij}\zeta_i &= 0, \qquad 1 \leqslant j \leqslant n-1 \\ \sum_{1\leqslant i\leqslant n} L_i(f)\zeta_i &= 0, \qquad f \in J_k(x_0). \end{aligned} \tag{1.34}$$

The first equation says that $L \in T_x^{1,0} b\Omega$, while the first two equation says that $L \in N_x$. The last equation says $L \in Z_x^{1,0}(J_k(x_0))$. The assumption that $Z_k^{1,0}(J_k(x_0)) \cap N_x$ is zero dimensional means that the intersection of the hyperplanes defined by equation in (1.34) consists only of the origin in $T_x^{1,0}\mathbb{C}^n$. Therefore, there exist n hyperplanes from equations in (1.34), say $H_1,\ldots,H_n$, such that their intersection is only the origin. Let $r_1,\ldots,r_n$ be row vectors that are either ∂r, or one of the first $n-1$ columns of the Levi forms, or ∂f for some $f \in J_k(x_0)$ such that $r_i(x)$ defines the hyperplane H_i. Putting the r_i as rows of a matrix A, the discussion above implies that $\mathsf{det}A(x) \neq 0$, hence by the definition of Kohn's algorithm, $\mathsf{det}A \in J_{k+1}(x_0)$. Consequently, there exists an element $\mathsf{det}(A)$ in $J_{k+1}(x_0)$ which does not vanish at x, and so $x \notin J_{k+1}(x_0)$.

Conversely, if $\mathsf{dim}_{\mathbb{C}}(Z_k^{1,0}(J_k(x_0)) \cap N_x) \geqslant 1$, then any of the n hyperplanes given by equation (1.34) cannot be linearly independent at x. Hence for **any** n hyperplanes $H_1,\ldots,H_n$ taken from the same equation, if $r_1,\ldots,r_n$ are row vectors as before, and A be a matrix with row entries r_i, then $\mathsf{det}\ A$ vanishes at x, and therefore $x \in V(J_{k+1}(x_0))$. □

The notion of holomorphic dimension was introduced by Kohn in his paper [Koh79] to further understand the relation between subelliptic estimates and the presence of holomorphic varieties in the boundary. The following is the definition found in the same paper.

Definition 1.35. If V is a real-analytic variety contained in $b\Omega$, the holomorphic dimension of V is defined by

$$\text{hol-dim } V \;=\; \mathsf{min}_{x\in V}\, \mathsf{dim}_{\mathbb{C}}\, Z_x^{1,0}(V)\cap N_x. \quad \square$$

1.3.3 Zero holomorphic dimension implies termination of algorithm

Let Ω be a bounded, pseudoconvex domain with smooth boundary, and let $x_0 \in b\Omega$. Let $U \subset \mathbb{C}^n$ be an open neighbourhood of x_0 so that there is a real-analytic, real-valued function r on U, with $U \cap b\Omega$ being the zero locus of r. Also, by shrinking to a smaller open set U, dr may be assumed to be non-vanishing everywhere.

If for **every** real analytic variety $V \subset U \cap b\Omega$, its holomorphic dimension hol-dim$(V) = 0$, then Kohn's algorithm terminates. Note that there is no assumption that $x_0 \in V$. More precisely, given the hypothesis, suppose that at step k, the variety $V(J_k(x_0))$ defined by the ideal $J_k(x_0) \subset \mathscr{A}_{x_0}$ generated at the k-th step is non-empty, then it is claimed that

$$\mathsf{dim}\; V(J_k(x_0)) \;>\; \mathsf{dim}\; V(J_{k+1}(x_0)).$$

Suppose otherwise, that is $\mathsf{dim}\; V(J_k(x_0)) = \mathsf{dim}\; V(J_{k+1}(x_0))$. By the weak coherence property, there exists a point $x \in V(J_{k+1}(x_0)) \subseteq V(J_k(x_0))$ and a neighbourhood $W_x \subset U \cap b\Omega$ of x in $b\Omega$ such that for every $y \in W_x \cap V(J_{k+1}(x_0))$, the ideal $\mathscr{I}_y V(J_{k+1}(x_0))$ is generated by finitely many elements of the set $J_{k+1}(x_0)$.

Then choose a smaller neighbourhood $W \subset W_x$ which does not necessarily contain x, and such that both $W \cap V(J_k(x_0))$ and $W \cap V(J_{k+1}(x_0))$ are non-empty, and whose dimensions are maximal at all points. This can be done since the set of singular points forms a thin closed set of $W \cap V(J_k(x_0))$ and $W \cap V(J_{k+1}(x_0))$. From the set inclusion $V(J_{k+1}(x_0)) \subseteq V(J_k(x_0))$ on W, and given that the dimension of both spaces are equal, there is therefore a set equality $W \cap V(J_{k+1}(x_0)) = W \cap V(J_k(x_0)) := X$. Immediate from this, by Proposition 1.33, for all $y \in X$,

$$\mathsf{dim}_{\mathbb{C}}\, (Z_y^{1,0}(J_k(x_0)) \cap N_y) \geqslant 1.$$

Now, the proof concludes by showing that this would imply

$$\text{hol-dim } W \cap V(J_k(x_0)) \geqslant 1,$$

and hence X, which does not necessarily contain x_0, is the variety with positive holomorphic dimension that is contained in $U \cap b\Omega$, and this contradicts the hypothesis that there is no variety $V \subset U \cap b\Omega$ with strictly positive holomorphic dimension. Unravelling the definition of hol-dim $W \cap V(J_k(x_0))$, it is equivalent to showing that at all $y \in W \cap V(J_k(x_0))$,

$$\mathsf{dim}_{\mathbb{C}}(Z_y^{1,0}(\mathscr{I}_y(W \cap V(J_k(x_0)))) \cap N_y) \geqslant 1.$$

By the coherence property, the ideal

$$\mathscr{I}_y(V(J_k(x_0)))$$

is generated by $J_k(x_0)$. By Lemma 1.31, there is an equality of vector spaces

$$Z_y^{1,0}(V(J_k(x_0))) = Z_y^{1,0}(J_k(x_0)).$$

By definition 1.30,

$$Z_y^{1,0}(V(J_k(x_0))) := Z_y^{1,0}(\mathscr{I}_y(W \cap V(J_k(x_0)))),$$

and therefore, it follows easily that

$$\mathsf{dim}_{\mathbb{C}}(Z_y^{1,0}(\mathscr{I}_y(W \cap V(J_k(x_0)))) \cap N_y) \;=\; \mathsf{dim}_{\mathbb{C}}(Z_y^{1,0}(J_k(x_0)) \cap N_y) \geqslant 1.$$

After arriving at this inequality, the only possible conclusion is that the dimension of $V(J_k(x_0))$ is strictly decreasing with respect to increasing k, and hence Kohn's algorithm must terminate.

1.3.4 Varieties with positive holomorphic dimension implies non-termination of algorithm

With the same hypothesis as in the previous paragraph, but this time assuming that there exists $V \subset U \cap b\Omega$ of positive holomorphic dimension. The claim is that $V \subset V(J_k(x_0))$ for all $k \geqslant 0$, and the proof proceeds by induction. Since $V \subset U \cap b\Omega$, the variety V lies in the zero set $\{r = 0\}$. On the other hand, since hol-dim $V \geqslant 1$, the dimension of the kernel of the Levi form λ is at least 1 at all $y \in V$. Let A be the matrix whose rows are either one of the first $n-1$ columns of the Levi form, or the vector ∂r. Therefore $\det(A)$ vanishes at y, and $V \subset V(J_0(x_0))$.

It follows that $V \subset V(J_1(x_0))$ and the reason is as follows. Since hol-dim $V \geqslant 1$, at each $y \in V$, there exists a vector $L \in T^{1,0}_y(\mathbb{C}^n)$ which is not zero such that $L \in N_y$ and $\langle \partial g(y), L\rangle = 0$ for all $g \in \mathscr{I}_y(V)$. Since $V \subset V(J_0(x_0))$, every function $f \in J_0(x_0)$ also vanishes on V, and hence $V \subseteq \{f = 0\}$, which implies that L is also tangent to f at y. Thus $\langle \partial f(y), L\rangle = 0$ for any $f \in J_0(x_0)$. For any matrix A whose rows are either ∂r, or any of the first $n-1$ columns of the Levi matrix, or ∂f for some $f \in J_0(x_0)$, by definition of Kohn's algorithm the determinant $\det A \in J_1(x_0)$ lies in the first ideal, and therefore $\det(A)$ vanishes at all $y \in V$. Hence, $V \subset V(J_1(x_0))$ and the initial induction step is done.

Now assume that for a certain k, $V \subseteq V(J_k(x_0))$. If $g \in J_k(x_0)$, then $V \subset \{g = 0\}$ and so for each $y \in V$, $Z^{1,0}_y(V) \subseteq Z^{1,0}_y(V(J_k(x_0)))$. But from equation (1.32), there is also an inclusion $Z^{1,0}_y(V(J_k(x_0))) \subseteq Z^{1,0}_y(J_k(x_0))$. In summary,

$$Z^{1,0}_y(V) \subseteq Z^{1,0}_y(V(J_k(x_0))) \subseteq Z^{1,0}_y(J_k(x_0)).$$

By the assumption that

$$\dim_{\mathbb{C}} \left(Z^{1,0}_y(V) \cap N_y\right) \geqslant \text{hol-dim}(V) \geqslant 1,$$

the inclusion above implies that

$$\dim_{\mathbb{C}} Z^{1,0}_y(J_k(x_0)) \cap N_y \geqslant 1,$$

which by Proposition 1.33, shows that $x \in V(J_{k+1}(x_0))$. This completes the induction process. Hence for all k, the set $V(J_k(x_0))$ can never be empty for all k and therefore the ideal $J_k(x_0)$ can never contain 1. This means that Kohn's algorithm can never terminate. In summary

Theorem 1.36. *Let Ω be a bounded pseudoconvex domain with smooth boundary in $\mathbb{C}^n$, and its boundary is real-analytic near a boundary point p_0. Let $J_k(x_0)$ be the sequence of ideals of subelliptic multipliers at p_0 constructed from Kohn's algorithm. Then there is an integer k such that $1 \in J_k(x_0)$ if and only if there are no varieties of positive holomorphic dimension in any neighbourhood of p_0. Therefore, there is a subelliptic estimate for the $\bar\partial$-Neumann problem at p_0 if there are no complex analytic varieties of positive dimension in any neighbourhood of p_0.* □

1.3.5 The consequences of the Diederich-Fornæss theorem

Another notion of the holomorphic dimension is given in the paper of Diederich-Fornæss [DF78], which is defined as follows. For a variety $V \subset U \cap b\Omega$, and for $p \in V$, let $Z^{1,0}_p(V)$ be the Zariski tangent space as before, and N_p be the kernel of the Levi form at p. The holomorphic dimension in the sense of Diederich-Fornaess at a point p is given by

$$\text{hol-dim}_{\text{DF},p}(V) := \sup_{W=W(p)} \inf_{w \in W \cap V} \dim_{\mathbb{C}}(Z^{1,0}_w(V) \cap N_w),$$

where W is an open neighbourhood of p. The holomorphic dimension of V in the sense of Diederich-Fornaess is then given by

$$\text{hol-dim}_{\text{DF}}(V) := \min_{p \in V} \text{hol-dim}_{\text{DF},p}(V).$$

The Diederich-Fornaess theorem states the following:

Theorem 1.37 (Diederich-Fornaess)**.** *If Ω is pseudoconvex, if r is real-analytic in a neighbourhood U of $x_0 \in b\Omega$, and if there is a real-analytic variety $V \subseteq U \cap b\Omega$ with hol-dim$_{\text{DF}}(V) \geqslant q$, then there exists a complex analytic variety $W \subset U \cap b\Omega$ with* $\mathsf{dim}\, W = q$. □

The condition that the holomorphic dimension in the sense of Kohn is positive, i.e.

$$\text{hol-dim}(V) \;=\; \mathsf{min}_{p\in V}\, \mathsf{dim}_{\mathbb{C}}(Z_p^{1,0}(V) \cap N_p) \geqslant 1,$$

implies anyway for any neighbourhood W of $p \in V$, the following inequality holds:

$$\mathsf{inf}_{w\in W\cap V}\, \mathsf{dim}_{\mathbb{C}}(Z_w^{1,0}(V) \cap N_w) \geqslant 1,$$

and hence it satisfies the Diederich-Fornaess condition that hol-dim$_{\text{DF}}(V) \geqslant 1$.

However, it is important to note that in the theorem of Diederich-Fornaess, it is not assumed that $x_0 \in V$, nor does the conclusion say that $x_0 \in W$. The following theorem of Fornaess gives a sufficient condition for which W contains x_0 in the case where W is a complex analytic variety with strictly positive complex dimension.

Theorem 1.38 (Fornæss)**.** *If W_k is a sequence of complex varieties with* $\mathsf{dim}\, W_k \geqslant q$*, $W_k \subset b\Omega$, and x_0 is a cluster point of this sequence, then there exists a complex variety W such that* $\mathsf{dim}\, W \geqslant q$*, $W \subset b\Omega$, and $x_0 \in W$.* □

Therefore, with these two theorems, the following result of Kohn summarises the discussion on the geometry of subelliptic estimates:

Theorem 1.39 (See [Koh79])**.** *Assume that Ω is bounded, pseudoconvex with smooth boundary, $x_0 \in b\Omega$, and r is real-analytic in a neighbourhood of x_0. Then the following conditions are equivalent (the statement differs from the original result in that it is adapted for $q = 1$):*

1. *$1 \in J_k(x_0))$ for some $k \geqslant 1$,*
2. *There exists a neighbourhood $U \subset \mathbb{C}^n$ of x_0 such that $U \cap b\Omega$ does not contain any complex analytic variety of dimension 1,*
3. *If W is a germ of a complex-analytic variety at x_0 such that $W \subset b\Omega$, then* $\mathsf{dim}\, W = 0$.

Proof. The proof proceeds by $3 \implies 2 \iff 1 \implies 3$. To show 1 implies 2, the termination of Kohn's algorithm is equivalent to the non-existence of real-analytic variety $V \subset U \cap b\Omega$ of positive holomorphic dimension by Theorem 1.36. Since every complex analytic variety $X \subset U \cap b\Omega$ of dimension greater than or equals to 1 is automatically of positive holomorphic dimension, therefore this implies by contraposition the non-existence of complex analytic variety of dimension 1 contained in $U \cap b\Omega$. To prove 2 implies 1, the hypothesis implies that there exists $U \subset \mathbb{C}^n$ of x_0 such that $U \cap b\Omega$ does not contain any variety V of positive holomorphic dimension by the Diederich-Fornaess theorem (or otherwise, if $U \cap b\Omega$ contains a variety with positive holomorphic dimension, then it will contain a complex analytic variety with dimension at least 1). But by the same Theorem 1.36, this implies that Kohn's algorithm terminates, and the first set of equivalent conditions is proved.

Next to prove 3 implies 2, the proof proceeds by contraposition. Assume that every neighbourhood $U \subset \mathbb{C}^n$ of x_0 has $U \cap b\Omega$ containing a complex analytic variety of dimension 1. Choose $U_k \subseteq \mathbb{C}^n$ a sequence of open sets containing x_0 that converges to x_0. Hence $U_k \cap b\Omega$ contains a complex analytic variety W_k with x_0 as the cluster point. By Fornæss' theorem, there exists W a complex analytic variety with $W \subset b\Omega$, $\mathsf{dim}\, W \geqslant 1$ and $x_0 \in W$, and hence the proof is done. Finally to prove 1 implies 3, again proceed by contraposition, and this becomes clear: if there exists $W \subset b\Omega$ a complex analytic variety with $\mathsf{dim}\, W \geqslant 1$ and $x_0 \in W$, then for every neighbourhood $U \subseteq \mathbb{C}^n$ of x_0, $U \cap b\Omega$ will contain a variety with positive holomorphic dimension, and therefore by Theorem 1.36, $1 \notin J_k(x_0)$ for all k. □

1.3.6 Explanation of the Diederich-Fornæss theorem Before the discussion starts, here is an observation:

Proposition 1.40. *On some open subset U of a $\mathscr{C}^3$ hypersurface $b\Omega \subset \mathbb{C}^n$ (pseudoconvexity is not necessary in this case), if the kernel of the Levi forms at various points $p \in U$ gather as a certain complex vector sub-bundles (identifying U with $b\Omega$ upon suitable restriction to a smaller topological subspace)*

$$K^{1,0}(b\Omega) \subset T^{1,0}(b\Omega),$$

then the following three involutiveness conditions are satisfied

$$\begin{aligned}
[K^{1,0}(b\Omega), K^{1,0}(b\Omega)] &\subset K^{1,0}(b\Omega),\\
[K^{0,1}(b\Omega), K^{0,1}(b\Omega)] &\subset K^{0,1}(b\Omega),\\
[K^{1,0}(b\Omega), K^{0,1}(b\Omega)] &\subset K^{1,0}(b\Omega) \oplus K^{0,1}(b\Omega).
\end{aligned}$$

Proof. See [MPS]. □

Introduce the following notation

$$K^c_p(b\Omega) \;:=\; \mathsf{Re}\, K^{1,0}_p(b\Omega).$$

The Diederich-Fornaess theorem may be explained without difficulty $\mathscr{C}^\infty$ submanifold $V \subset b\Omega$ of a real analytic pseudoconvex boundary of $\Omega \subset \mathbb{C}^n$ is assumed to have a positive constant holomorphic dimension (and hence independent of p), that is to say define

$$\text{hol-dim}_1(V) \;:=\; \mathsf{dim}\, T^{1,0}_p V \cap K^{1,0}_p(b\Omega) \geqslant 1.$$

Here $T^{1,0}_p V$ is understood to be the $(1,0)$-tangent space of V at p, and hence the definition of this holomorphic dimension differs slightly from Kohn's definition. On the other hand, if $b\Omega$ is pseudoconvex, by Proposition 1.11, this is equivalent to the null space N_p of the Levi matrix at p, so there is no ambiguity here. The subscript "1" is used to make this distinction, but for convenience, it will be dropped in the rest of this section.

Given this setting, the main point of the Diederich-Fornaess theorem is the involutiveness of the Levi kernel when $p \mapsto \mathsf{dim}_{\mathbb{C}} K^c_p(b\Omega)$ has a constant dimension, that is $K^c_p(b\Omega)$ is a real sub-bundle of $T(b\Omega)$. More precisely,

Proposition 1.41. *In a $\mathscr{C}^\infty$ pseudoconvex boundary $b\Omega \subset \mathbb{C}^n$, assume that a $\mathscr{C}^\infty$ submanifold $V \subset b\Omega$ has the property that the intersection of its tangent bundle with the ambient Levi kernel*

$$T^{\text{DF}}V \;:=\; TV \cap K^c(b\Omega)|_S$$

is a real vector sub-bundle of $T^c(b\Omega)$, then

$$[\Gamma(T^{\text{DF}}V), \Gamma(T^{\text{DF}}V)] \subseteq \Gamma(T^{\text{DF}}V).$$

The proof of this proposition is done in several stages. To begin, the following lemma describes the order of vanishing of the Levi form at a point $p \in b\Omega$.

Lemma 1.42. *Let $b\Omega = \{r = 0\}$ be a $\mathscr{C}^\infty$ boundary, which is pseudoconvex. Let $p \in b\Omega$ and let $\mathscr{X}, \mathscr{Y} \in \Gamma(T^{1,0}(b\Omega))$ be sections of $T^{1,0}b\Omega$ defined in a neighbourhood of p. If*

$$\mathscr{X}_p \in K^{1,0}_p(b\Omega), \qquad \textit{and} \qquad \mathscr{Y}_p \in K^{1,0}_p(b\Omega)$$

then the Levi form

$$b\Omega \ni q \longmapsto \lambda(\mathscr{X}, \mathscr{Y})(q)$$

vanishes at least order 2 at $q = p$.

Proof. The proof reduces to studying the following map:

$$\varphi:\ b\Omega \ni q \longmapsto \lambda(\mathscr{X},\mathscr{X})(q).$$

By the pseudoconvexity condition, $\varphi(q) \geqslant 0$ for all $q \in b\Omega$. Also φ is smooth since r is smooth. As this map reaches the minimum at $q = 0$, φ and its first derivative vanish at p, and this proves the simple case.

For the general case, consider the following formulae

$$\begin{aligned}
\mathsf{Re}\ \lambda(\mathscr{X},\mathscr{Y}) &= \frac{1}{4}\big[\lambda(\mathscr{X}+\mathscr{Y},\ \mathscr{X}+\mathscr{Y}) - \lambda(\mathscr{X}-\mathscr{Y},\ \mathscr{X}-\mathscr{Y})\big],\\
\mathsf{Im}\ \lambda(\mathscr{X},\mathscr{Y}) &= \frac{1}{4}\big[\lambda(\mathscr{X}+\sqrt{-1}\mathscr{Y},\ \mathscr{X}+\sqrt{-1}\mathscr{Y}) - \lambda(\mathscr{X}-\sqrt{-1}\mathscr{Y},\ \mathscr{X}-\sqrt{-1}\mathscr{Y})\big].
\end{aligned}$$

Since $\mathscr{X}+\mathscr{Y}$, $\mathscr{X}-\mathscr{Y}$, $\mathscr{X}+\sqrt{-1}\mathscr{Y}$ and $\mathscr{X}-\sqrt{-1}\mathscr{Y}$ belong to the isotropic cone, by the discussion earlier, the order of vanishing of $\mathsf{Re}\ \lambda(\mathscr{X},\mathscr{Y})$ and $\mathsf{Im}\ \lambda(\mathscr{X},\mathscr{Y})$ at p is at least 2. Hence so is $\lambda(\mathscr{X},\mathscr{Y})$. □

Lemma 1.43. *Let $b\Omega = \{r = 0\}$ be a $\mathscr{C}^\infty$ boundary, which is pseudoconvex. Let $p \in b\Omega$ and let $\mathscr{X},\mathscr{Y} \in \Gamma(T^{1,0}(b\Omega))$ be sections of $T^{1,0}b\Omega$ defined in a neighbourhood of p. If*

$$\mathscr{X}_p \in K_p^{1,0}(b\Omega), \qquad \textit{and} \qquad \mathscr{Y}_p \in K_p^{1,0}(b\Omega),$$

then there exists $\mathscr{Z} \in \Gamma(T^{1,0}(b\Omega))$ defined in a neighbourhood of p so that

$$[\mathscr{X}+\overline{\mathscr{X}},\mathscr{Y}+\overline{\mathscr{Y}}] \ =\ \mathscr{Z}+\overline{\mathscr{Z}}+\mu T,$$

where the order of vanishing of μ at p is at least 2.

Proof. An expansion of the expression on the left side of the equation yields

$$[\mathscr{X}+\overline{\mathscr{X}},\mathscr{Y}+\overline{\mathscr{Y}}] \ =\ [\mathscr{X},\mathscr{Y}]+[\mathscr{X},\overline{\mathscr{Y}}]+[\overline{\mathscr{X}},\mathscr{Y}]+[\overline{\mathscr{X}},\overline{\mathscr{Y}}].$$

Write $\mathscr{Z}_1 := [\mathscr{X},\mathscr{Y}]$ which is a $T^{1,0}(b\Omega)$ section, and hence so is its conjugate $\overline{\mathscr{Z}_1} = [\overline{\mathscr{X}},\overline{\mathscr{Y}}]$. Next, write

$$[\mathscr{X},\overline{\mathscr{Y}}] = \mathscr{A}+\overline{\mathscr{B}}+\mu' T,$$

for some $\mathscr{A}$ and $\mathscr{B}$ sections of the $T^{1,0}(b\Omega)$ bundle. By the previous lemma, μ' vanishes at p up to order 2. Similarly, $[\overline{\mathscr{X}},\mathscr{Y}]$ is just the conjugate of $[\mathscr{X},\overline{\mathscr{Y}}]$. Therefore,

$$\begin{aligned}
[\mathscr{X}+\overline{\mathscr{X}},\mathscr{Y}+\overline{\mathscr{Y}}] &= \mathscr{Z}_1+(\mathscr{A}+\overline{\mathscr{B}}+\mu' T)+(\overline{\mathscr{A}}+\mathscr{B}+\overline{\mu'}T)+\overline{\mathscr{Z}_1}\\
&= (\mathscr{Z}_1+\mathscr{A}+\mathscr{B})+\overline{(\mathscr{Z}_1+\mathscr{A}+\mathscr{B})}+(\mu'+\overline{\mu'})T.
\end{aligned}$$

Setting $\mathscr{Z} := \mathscr{Z}_1+\mathscr{A}+\mathscr{B}$ and $\mu := \mu'+\overline{\mu'}$ finishes the proof. □

Finally with these two lemmas, the proof of Proposition proceeds:

Proof of Proposition 1.41. To end the proof, pick $p \in V$ and let X, Y be tangent vector field near p satisfying

$$X|_q \in T_qV \cap K_q^c(b\Omega) \qquad \text{and} \qquad Y|_q \in T_qV \cap K_q^c(b\Omega)$$

for all q near p. Since V is a manifold, obviously $[X,Y]|_q \in T_qV$. It remains to show that $[X,Y]|_q \in K_q^c(b\Omega)$. To this effect, introduce

$$\mathscr{X} = X - \sqrt{-1}J(X) \qquad \text{and} \qquad \mathscr{Y} = Y - \sqrt{-1}J(Y),$$

where J is the complex structure on $T\mathbb{C}^n$. These are sections of $K^{1,0}(b\Omega)|_V \subset T^{1,0}(b\Omega)|_V$. Let T be the section of the real tangent bundle $T(b\Omega)$ satisfying $1 \equiv \sqrt{-1}\partial r(T)$, so that $T|_q \notin T^c_q(b\Omega)$ for all q near p. Hence

$$\mathbb{C} \otimes_{\mathbb{R}} T_q(b\Omega) = T^{1,0}_q(b\Omega) \oplus T^{1,0}_q(b\Omega) \oplus \mathbb{C}T|_q.$$

Moreover, by Lemma 1.43 that

$$\begin{aligned} [X, Y] &= [\mathscr{X} + \overline{\mathscr{X}}, \mathscr{Y} + \overline{\mathscr{Y}}] \\ &= \mathscr{Z} + \overline{\mathscr{Z}} + \mu T, \end{aligned}$$

where μ vanishes on V with order at least 2. It follows that for $p \in V$,

$$[X, Y]|_p = \mathscr{Z}|_p + \overline{\mathscr{Z}}|_p.$$

It remains to show that $\mathscr{Z}|_V \in \Gamma(K^{1,0}(b\Omega))$ in order to show that $[X, Y] \in \Gamma(K^c(b\Omega))$. But equivalently, this amounts to showing that for any $q \in V$ that are near p,

$$\lambda(\mathscr{Z}, \mathscr{Z})(q) = 0.$$

Writing $\vartheta := \sqrt{-1}\partial r$,

$$\begin{aligned} \lambda(\mathscr{Z}, \mathscr{Z}) &= \vartheta([\mathscr{Z}, \overline{\mathscr{Z}}]) \\ &= \vartheta([\mathscr{Z} + \overline{\mathscr{Z}}], \overline{\mathscr{Z}}]) \\ &= \vartheta\big(\big[[\mathscr{X} + \overline{\mathscr{X}}, \mathscr{Y} + \overline{\mathscr{Y}}], \overline{\mathscr{Z}}\big]\big) - \vartheta([\mu T, \overline{\mathscr{Z}}]). \end{aligned}$$

The second term on the right vanishes on V, as

$$[\mu T, \overline{\mathscr{Z}}] = \mu[T, \overline{\mathscr{Z}}] - \overline{\mathscr{Z}}(\mu)[T, \overline{\mathscr{Z}}],$$

and μ vanishes on V along with its first derivatives. For the remaining term, applying the Jacobi's identity to Lie brackets

$$\begin{aligned} \big[[\mathscr{X} + \overline{\mathscr{X}}, \mathscr{Y} + \overline{\mathscr{Y}}], \overline{\mathscr{Z}}\big] &= -\big[[\mathscr{Y} + \overline{\mathscr{Y}}, \overline{\mathscr{Z}}], \mathscr{X} + \overline{\mathscr{X}}\big] - \big[[\overline{\mathscr{Z}}, \mathscr{X} + \overline{\mathscr{X}}], \mathscr{Y} + \overline{\mathscr{Y}}\big] \\ &= -\big[[\mathscr{Y}, \overline{\mathscr{Z}}], \mathscr{X} + \overline{\mathscr{X}}\big] - \big[[\overline{\mathscr{Y}}, \overline{\mathscr{Z}}], \mathscr{X}\big] - \big[[\overline{\mathscr{Y}}, \overline{\mathscr{Z}}], \overline{\mathscr{X}}\big] \\ &\quad -\big[[\overline{\mathscr{Z}}, \mathscr{X}], \mathscr{Y} + \overline{\mathscr{Y}}\big] - \big[[\overline{\mathscr{Z}}, \overline{\mathscr{X}}], \mathscr{Y}\big] - \big[[\overline{\mathscr{Z}}, \overline{\mathscr{X}}], \overline{\mathscr{Y}}\big]. \end{aligned}$$

Applying the Levi form $\vartheta = \sqrt{-1}\partial r$ to both sides, the second, third, fifth and sixth terms on the right vanishes because they belong to the $T^{1,0}b\Omega \oplus T^{0,1}b\Omega$ bundle. For the first term and the fourth on the right, the argument for both is symmetric, hence it suffices to show the vanishing of the first term. Write

$$[\mathscr{Y}, \overline{\mathscr{Z}}] = \mathscr{A} + \overline{\mathscr{B}} + \nu T.$$

By the assumption that $\mathscr{Y}|_V \in \Gamma(K^{1,0}b\Omega)|_V$, ν vanishes on V so that V lies in the zero set $\{\nu = 0\}$. Also, the assumption $X|_q \in T_qV$ implies that $\mathscr{X}$ is tangent to V at q, and so it is also tangent to the zero set $\{\nu = 0\}$. Consequently, both ν and $\mathscr{X}(\nu)$ vanish on V. Therefore, using $2X = \mathscr{X} + \overline{\mathscr{X}}$,

$$\begin{aligned} \big[\mathscr{X} + \overline{\mathscr{X}}, [\mathscr{Y}, \overline{\mathscr{Z}}]\big] &= [\mathscr{X}, \mathscr{A}] + [\mathscr{X}, \overline{\mathscr{B}}] + [\overline{\mathscr{X}}, \mathscr{A}] + [\overline{\mathscr{X}}, \overline{\mathscr{B}}] \\ &\quad + \nu[2X, T] + 2X(\nu)T. \end{aligned}$$

Applying the Levi form to both sides, the first and the fourth term vanish because they lie in $T^{1,0}b\Omega \oplus T^{0,1}b\Omega$, while the second and the third term vanish because $\mathscr{X}$ is in the Levi kernel. The last two terms vanish because ν and $\mathscr{X}(\nu)$ vanish on V. □

Proposition 1.41 is the main step of the Diederich-Fornaess theorem. Let $S \subset b\Omega$ be a real hypersurface in $\mathbb{C}^n$ which contains V. The rest of the proof involves applying the Frobenius theorem to the bundle $T^{\mathrm{DF}}V$ to obtain a submanifold $V' \subseteq V$. Out of a closed set of V' of lower dimension, there is a variety $V'' \subseteq V'$ that satisfies the property that locally there is a variety of the form $\mathbb{C}^s \times \{0\}$ which is tangent to V'' to infinite order, and hence S contains a complex variety. (c.f. [DF78], proposition 3)

1.4 Examples

1.4.1 Example 1: Strongly pseudoconvex domains Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded pseudoconvex domain with a smooth boundary $b\Omega$, and assume that $0 \in b\Omega$. Let $U \subset \mathbb{C}^n$ be an open neighbourhood of the origin so that $U \cap b\Omega$ has a real-analytic defining function $U \cap \Omega = \{r < 0\}$. By a biholomorphic change of coordinate, r may be assumed to be of the form

$$r \;=\; \mathsf{Re}\; z_n \;+\; h(z, \bar{z}),$$

where $h = O(|z|^2)$ for all $1 \leqslant i \leqslant n-1$, and both $\partial_{z_i} h$ and $\partial_{\bar{z}_i} h$ vanish at the origin. Recall that the coefficients of the Levi form λ are given by

$$\lambda_{ij} \;=\; \frac{r_{i\bar{j}} r_n r_{\bar{n}} \;-\; r_{i\bar{n}} r_n r_{\bar{i}} \;-\; r_{n\bar{j}} r_i r_{\bar{n}} \;+\; r_{n\bar{n}} r_i r_{\bar{j}}}{|r_n|^2}.$$

At the origin, all the terms containing r_i and $r_{\bar{j}}$ vanish and hence

$$\lambda_{ij}(0) \;=\; r_{i\bar{j}}(0).$$

This time, assume that $b\Omega$ is strongly pseudoconvex at the origin 0, that is to say the Levi matrix at 0, given by the coefficients $(\lambda_{ij}(0)) = (r_{i\bar{j}}(0))$, is strictly positive definite. The assertion is that there is no holomorphic curve $\varphi : \mathbb{D} \to b\Omega$ contained in $b\Omega$ and passes through the origin $0 \in b\Omega$. The reason is as follows, write explicitly φ as

$$\varphi :\; t \;\longmapsto\; (\varphi_1(t),\; \ldots,\; \varphi_n(t)),$$

where each $\varphi_i(t)$ is a holomorphic function in t. Then necessarily,

$$r(\varphi_1(t), \ldots, \varphi_n(t), \overline{\varphi_1(t)}, \ldots, \overline{\varphi_n(t)}) \equiv 0.$$

Taking the derivative with respect to t,

$$\begin{aligned} 0 \;&\equiv\; \frac{d}{dt} r(\varphi_1(t), \ldots, \varphi_n(t), \overline{\varphi_1(t)}, \ldots, \overline{\varphi_n(t)}) \\ &=\; \sum_{i=1}^{n} \frac{\partial r}{\partial z_i}(\varphi(t), \overline{\varphi(t)}) \cdot \varphi_i'(t). \end{aligned}$$

Next, taking the derivative of the equation above with respect to $\bar{t}$ to obtain

$$\begin{aligned} 0 \;&\equiv\; \frac{d}{d\bar{t}} \left(\sum_{i=1}^{n} \frac{\partial r}{\partial z_i}(\varphi(t), \overline{\varphi(t)}) \cdot \varphi_i'(t) \right) \\ &=\; \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 r}{\partial z_i \partial \bar{z}_j}(\varphi(t), \overline{\varphi(t)}) \cdot \varphi_i'(t) \overline{\varphi_j'(t)}. \end{aligned}$$

At the origin, the following equations are obtained:

$$\sum_{i=1}^{n} \frac{\partial r}{\partial z_i}(0) \varphi_i'(0) \;=\; 0, \tag{1.44}$$

$$\sum_{i,j=1}^{n} \frac{\partial^2 r}{\partial z_i \partial \bar{z}_j}(0) \cdot \varphi_i'(0) \overline{\varphi_j'(0)} \;=\; 0. \tag{1.45}$$

But equation (1.45) will not be satisfied if $b\Omega$ is strictly pseudoconvex at 0, as it would have been strictly positive instead of being 0. Therefore, if $b\Omega$ is strictly pseudoconvex at the origin 0, then there

is no holomorphic curve passing through the origin. By Theorem 1.39, Kohn's algorithm terminates for strongly pseudoconvex domains and hence there is a subelliptic estimate. In fact, it has been calculated before Kohn created this algorithm that there exists a positive constant C such that for all $\phi \in \mathscr{D}_{0,1}(U)$,

$$\|\phi\|^2_{\frac{1}{2}} \leqslant CQ(\phi, \phi).$$

For details, see [CS01].

1.4.2 Example 2: Special Domains An important class of real hypersurfaces in $\mathbb{C}^n$ for the study of Kohn's algorithm is the class of special domains. Let $F_1(z_1, \ldots, z_n)$,..., $F_N(z_1, \ldots, z_n)$ be holomorphic function germs at $0 \in \mathbb{C}^n$ vanishing at the origin of $\mathbb{C}^n$. Let $\Omega \subset \mathbb{C}^{n+1}$ be an open set defined by

$$\Omega := \left\{(z, z_{n+1}) := (z_1, \ldots, z_n, z_{n+1}) \in \mathbb{C}^{n+1} : \mathsf{2Re}\ z_{n+1} - \sum_{1 \leqslant i \leqslant \mathrm{N}} |F_i|^2 < 0\right\}.$$

Therefore, its boundary

$$b\Omega = \left\{\mathsf{2Re}\ z_{n+1} = \sum_{1 \leqslant i \leqslant \mathrm{N}} |F_i|^2\right\} \tag{1.46}$$

contains the origin $0 \in \mathbb{C}^{n+1}$. Its Levi form is a matrix whose coefficients are given by

$$\lambda_{ij} = \sum_{1 \leqslant m \leqslant \mathrm{N}} \frac{\partial F_m}{\partial z_i} \overline{\frac{\partial F_m}{\partial z_j}}$$

and moreover, $b\Omega$ is pseudoconvex. Its determinant can be expressed as

$$\mathsf{det}(\lambda) = \sum_{1 \leqslant i_1 < \cdots < i_n \leqslant \mathrm{N}} |\mathsf{Jac}(F_{i_1}, \ldots, F_{i_n})|^2.$$

Therefore, $\mathsf{det}(\lambda)(p) = 0$ if and only if for every possible combination $(i_1, \ldots, i_n)$, $\mathsf{Jac}(F_{i_1}, \ldots, F_{i_n})$ vanishes at p. Kohn's algorithm may be modified as follows:

Definition 1.47 (Kohn's algorithm for special domains)**.** Let $\langle F_1, \ldots, F_{\mathrm{N}}\rangle$ be an ideal of $\mathscr{O}_{\mathbb{C}^n,0}$, the ring of holomorphic function germs at the origin. Associate with it a sequence of nested radical ideals

$$I_1 \subseteq I_2 \subseteq \cdots \qquad \text{ad infinitum.}$$

For each choice of $g_1, \ldots, g_n$ in the ideal $\langle F_1, \ldots, F_{\mathrm{N}}\rangle$, consider the Jacobian

$$\mathsf{det}(g) := \mathsf{Jac}(g_1, \ldots, g_n) = \mathsf{det}\begin{pmatrix} \partial_{z_1} g_1 & \cdots & \partial_{z_n} g_n \\ \vdots & \ddots & \vdots \\ \partial_{z_1} g_n & \cdots & \partial_{z_n} g_n \end{pmatrix}.$$

Let

$$I_1^{\#} := \langle\, \mathsf{det}(g) : g_i \in \langle F_1, \ldots, F_{\mathrm{N}}\rangle\rangle$$

and set $I_1 = \sqrt{I_1^{\#}}$ the radical (in the usual algebraic sense) of $I_1^{\#}$. Inductively, once I_k has been constructed, let $I_{k+1}^{\#}$ denote the ideal generated by I_k and $\mathsf{det}\, A$, where A is a matrix whose row is either of the form ∂g for some $g \in I_k$, or ∂F_i for some $1 \leqslant i \leqslant \mathrm{N}$. Then let $I_{k+1} = \sqrt{I_k^{\#}}$ be the radical ideal. Note that the original functions $F_1, \ldots, F_{\mathrm{N}}$ do not belong to the nested ideals. □

The following subelliptic estimates ε are assigned to the following operations: for the vector multipliers ∂F_i, let $\varepsilon = 1/4$ (see [Siu10]), and hence by Proposition 1.25, assign $1/4$ to $\mathsf{Jac}(F_{i_1}, \ldots, F_{i_n})$. If the subelliptic multiplier f is assigned with ε, and if $g^m = f$ for some positive integer m, then assign ε/m to g. Finally, if every $h_1, \ldots, h_n$ is a subelliptic multiplier that is assigned with ε, then assign $\varepsilon/2$ to $\mathsf{Jac}(h_1, \ldots, h_n)$.

To study holomorphic varieties contained in special domains, let $\varphi : \mathbb{D} \to b\Omega$ be a holomorphic curve given by $\varphi(t) = (\varphi_1(t), \ldots, \varphi_{n+1}(t))$. Since $b\Omega$ is a special domain as in equation (1.46), necessarily $\varphi_{n+1}(t) \equiv 0$. The reason is as follows. Since $\varphi(\mathbb{D}) \subset b\Omega$, there is an identity

$$\varphi_{n+1}(t) + \bar{\varphi}_{n+1}(\bar{t}) - \sum_{1 \leqslant i \leqslant \mathrm{N}} F_i(\varphi(t)) \bar{F}_i(\bar{\varphi}(\bar{t})) \equiv 0.$$

If $\bar{\varphi}_j(w)$ is the polarisation of $\overline{\varphi_j(t)}$, then necessarily

$$\varphi_{n+1}(t) + \bar{\varphi}_{n+1}(w) - \sum_{1 \leqslant i \leqslant \mathrm{N}} F_i(\varphi(t)) \bar{F}_i(\bar{\varphi}(w)) \equiv 0.$$

Setting $w = 0$ leads to $\varphi_{n+1}(t) \equiv 0$ and this concludes the claim. Hence any holomorphic curve $\varphi : \mathbb{D} \to b\Omega$ into special domains passing through the origin must necessarily satisfy

$$\sum_{1 \leqslant i \leqslant \mathrm{N}} |F_i(\varphi_1(t), \ldots, \varphi_n(t))|^2 \equiv 0,$$

or equivalently, any holomorphic curve $\varphi(\mathbb{D})$ that contains the origin $0 \in \mathbb{C}^n$ is contained in the common zero set of F_i, i.e.

$$\varphi(\mathbb{D}) \ \subseteq \ \bigcap_{1 \leqslant i \leqslant \mathrm{N}} \{F_i = 0\}.$$

By a result which is to be seen later, this is equivalent to the finiteness of the intersection multiplicity

$$\mathsf{dim}_{\mathbb{C}} \, \mathscr{O}_{\mathbb{C}^n, 0} \big/ \langle F_1, \ldots, F_{\mathrm{N}} \rangle \ := \ s < \infty.$$

Question 1.48. The question about Kohn's algorithm is as follows: are there functions $f(s)$ and $g(s)$ such that Kohn's algorithm terminates in $f(s)$ number of steps, and for some open neighbourhood $U \subset \mathbb{C}^{n+1}$ of the origin

$$|||\phi|||_{\varepsilon}^2 \leqslant CQ(\phi, \phi)$$

for all $\phi \in \mathscr{D}_{0,1}(U)$, with $\varepsilon \geqslant g(s)$. Here the constant C is independent of ϕ. If such f and g can be found, Kohn's algorithm is said to terminate with effectiveness.

1.4.3 Siu's effective method for special domains in $\mathbb{C}^3$, and problems with generalisation

To gain an understanding of Siu's effective method for special domains in $\mathbb{C}^3$, consider the following "model" space

$$b\Omega \ := \ \left\{ \, 2\mathsf{Re} \, z_3 = |z_1|^2 + \left| z_2^{\lambda} + \sum_{0 \leqslant j \leqslant \lambda - 1} a_j(z_1) z_2^j \right|^2 \right\},$$

where $\lambda \geqslant 1$. Here the $a_j(z_1)$'s are holomorphic functions in z_1 that vanish at $z_1 = 0$, in other words, $z_2^{\lambda} + \sum_{0 \leqslant j \leqslant \lambda - 1} a_j(z_1) z_2^j$ is a Weierstrass polynomial. Moreover, the intersection multiplicity can be calculated

$$\mathsf{dim}_{\mathbb{C}} \, \mathscr{O}_{\mathbb{C}^2, 0} \Big/ \left\langle z_1, z_2^{\lambda} + \sum_{0 \leqslant j \leqslant \lambda - 1} a_j(z_1) z_2^j \right\rangle \ = \lambda.$$

The first step in Kohn's algorithm consists of taking the Jacobian

$$\mathsf{Jac} \left(z_2, z_2^{\lambda} + \sum_{0 \leqslant j \leqslant \lambda - 1} a_j(z_1) z_2^j \right) \ = \ \lambda z_2^{\lambda - 1} \ + \ \sum_{1 \leqslant j \leqslant \lambda - 1} j a_j(z_1) z_2^{j-1},$$

whose multiplicity is assigned $\frac{1}{4}$. Then after λ number of steps, the ideal I_λ contains the constant function $\lambda!$ which is invertible, and hence $1 \in I_\lambda$. Moreover, the subelliptic estimate is obtained with $\varepsilon \geqslant \frac{1}{4\cdot 2^{\lambda-1}}$. In this case, Kohn's algorithm terminates with effectiveness.

This idea can be generalised, as shown in the article of Siu [Siu10]. Given $F_1,\ldots,F_{\mathrm{N}}$ holomorphic function germs at $0 \in \mathbb{C}^n$ that vanish at the origin. Assume also that their intersection multiplicity is finite. The first step is to reduce N functions to the case of two functions by taking a suitable generic linear combinations

$$A \;=\; \sum_{1\leqslant i\leqslant \mathrm{N}} \mu_i F_i \qquad \text{and} \qquad B \;=\; \sum_{1\leqslant i\leqslant \mathrm{N}} \lambda_i F_i,$$

where μ_i and λ_i are constant complex numbers. Let $\tilde{h}_2$ be the reduction of $\mathsf{Jac}(A, B)$. Then choose another generic linear combination

$$h_1 \;=\; \sum_{1\leqslant i\leqslant \mathrm{N}} c_i F_i$$

so that

$$\mathsf{dim}_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^2,0} \big/ \langle h_1, \tilde{h}_2 \rangle$$

has effective upper bound. From h_1 and $\tilde{h}_2$, a holomorphic function h_2 may be constructed so that $h_2(0, z_2)$ has an effective vanishing order at $z_2 = 0$. Moreover, $h_2(h_1, z_2)$ has an additional property that it is a subelliptic multiplier, and hence by Weierstrass preparation theorem applied to $h(z_1, z_2)$ (forgetting about the unit),

$$h_2(h_1, z_2) \;=\; z_2^\lambda \;+\; \sum_{1\leqslant j\leqslant \lambda} a_j(h_1) z_2^{\lambda-j}.$$

Here λ is effective. The rest of Siu's argument consists of applying the Kohn's algorithm to the premultiplier h_1, and the subelliptic multiplier $h_2(h_1, z_2)$, and the case is similar to the model case. The following theorem appeared on Arxiv:

Theorem 1.49. *Let (z_1, z_2, z_3) be holomorphic coordinates in $\mathbb{C}^3$ with $z_i = x_i + \sqrt{-1}y_i$. For some $\mathrm{N} \geqslant 2$, let $F_1(z_1, z_2),\ldots,F_{\mathrm{N}}(z_1, z_2)$ be holomorphic function germs in $\mathscr{O}_{\mathbb{C}^2,0}$ vanishing at the origin such that*

$$\mathsf{dim}_{\mathbb{C}}\, \mathscr{O}_{\mathbb{C}^2,0} \big/ \langle F_1, \ldots, F_{\mathrm{N}} \rangle := s < \infty.$$

Let $\Omega \subset \mathbb{C}^3$ be the domain defined by

$$\Omega = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 :\; 2\mathrm{Re}\; z_3 - \sum_{i=1}^{\mathrm{N}} |F_i(z_1, z_2)|^2 < 0 \right\}.$$

Then by Siu's method, Kohn's algorithm terminates in at most $(4s^2 - 1)s$ steps. Moreover, for any $\phi \in \mathscr{D}_{0,1}(\Omega)$ with compact support,

$$\|\!|\phi|\!\|_\varepsilon^2 \lesssim \|\overline{\partial}\phi\|^2 + \|\overline{\partial}^*\phi\|^2 + \|\phi\|^2,$$

where

$$\varepsilon \geqslant \frac{1}{2^{(4s^2-1)s+3} s^2 (4s^2-1)^4 \binom{8s+1}{8s-1}}.$$

Siu's argument in the same paper [Siu10] to generalise this idea to higher dimension has a main difficulty. The problem lies in Proposition A3, page 1234 in Appendix, which he uses in his main argument, and it says the following:

Claim 1.50 (Proposition A3 in Siu)**.** *Let $F_1, \ldots, F_{\mathrm{N}}$ be holomorphic function germs on $\mathbb{C}^n$ at the origin, vanishing at the origin such that the ideal generated by $F_1, \ldots, F_{\mathrm{N}}$ contains an effective power of the maximal ideal of $\mathbb{C}^n$. Let $1 \leqslant \nu \leqslant n$. Let J_ν be the ideal generated by*

$$\frac{\partial(F_{i_1}, \ldots, F_{i_\nu})}{\partial(z_{j_1}, \ldots, z_{j_\nu})}$$

for $1 \leqslant i_1 < \cdots \leqslant i_\nu < \mathrm{N}$ and $1 \leqslant j_i < \cdots < j_\nu \leqslant n$. Then the ideal J_ν contains an effective power of the maximal ideal of $\mathbb{C}^n$ at the origin.

This claim is true only for $\nu = 1$ as a result of the Briançon-Skoda theorem (see later chapter for more information). The problem with $\nu \geqslant 2$ can be illustrated in the following simple counterexample: for $n = 3$, take $F_1 = z_1$, $F_2 = z_2^2$ and $F_3 = z_3^2$. Take $\nu = 2$. Then

$$J_2 \;=\; \langle z_2, z_3, z_2 z_3 \rangle$$

which does not contain any powers of the maximal ideal of $\mathbb{C}^3$ at the origin. Therefore, this thesis only treats the calculations for special domains in $\mathbb{C}^3$.

In the next part of the thesis, the question is whether there is an example where Kohn's algorithm terminates in less than 6 steps for special domains in $\mathbb{C}^3$? This is a natural question since if $J_1 \subseteq J_2 \subseteq \cdots$ are ideals generated from Kohn's algorithm in the real-analytic case, then $\dim V(J_1) > \dim V(J_2) > \cdots$. Hence, Kohn's algorithm should terminate a finite number of steps. For special domains of the form

$$\mathrm{Re}\; z_3 = |F(z_1, z_2)|^2 + |G(z_1, z_2)|^2,$$

where F and G are generic homogenous polynomials so that the intersection $\{F = G = 0\}$ consists only of the origin, Kohn's algorithm terminates in 2 steps. The main idea is find a relationship between the resultants of homogeneous polynomials and the Jacobian of homogeneous polynomials, and keeping track of the resultants at each step of the Kohn's algorithm. It also shows why taking radicals is necessary for Kohn's algorithm to work. Under the assumption that the resultant $\mathsf{Resultant}(F, G)$ is not identically zero (which then leads to $\{F = G = 0\} = \{0\}$), the first step of Kohn's algorithm is to take the Jacobian $\mathsf{Jac}(F, G)$ of these F and G, and let J_A be the reduction of $\mathsf{Jac}(F, G)$ (i.e. the generator of $\mathrm{rad}\langle \mathsf{Jac}(F, G) \rangle$). The next step of Kohn's algorithm is to take the following Jacobians $\mathsf{Jac}(J_A, F)$, and $\mathsf{Jac}(J_A, G)$. Kohn's algorithm terminates if either one of the resultants of any two of the three J_A, $\mathsf{Jac}(J_A, F)$ and $\mathsf{Jac}(J_A, G)$ does not vanish, as this would imply that the solution set of $\{J_A = \mathsf{Jac}(J_A, F) = \mathsf{Jac}(J_A, G) = 0\}$ consists only of the origin, and hence by Nullstellensatz, the radical of the ideal generated by these three elements is the maximal ideal $\langle z, w \rangle$. Then Kohn's algorithm terminates at the next step. Under certain generic conditions, the relationship between the Jacobian and the resultant is the following formula

$$\mathsf{Resultant}(J_A, F) = (-1)^? \frac{\mathsf{Resultant}(F, J_A)\mathsf{Resultant}(\partial_z J_A, J_A)}{\mathsf{Resultant}(w, J_A)},$$

where $?$ depends on the degree of F and G. The definition of the resultant of 2 homogeneous polynomials may be found in chapter 2, section 9, paragraphs 9.1 and 9.2. Here it can be seen that if J_A is not reduced, then $\mathsf{Resultant}(\partial_z J_A, J_A)$, which is the discriminant of J_A, would have been zero, and hence so is $\mathsf{Resultant}(J_A, F)$. Kohn's algorithm might then need to take more than 2 steps.

1.4.4 Other results

For other examples where Kohn's algorithm terminate with effectiveness, D'Angelo and Catlin introduced the triangular system in [CD10], which is defined as follows

Definition 1.51 (Triangular Systems)**.** Let H be a collection of non-zero elements of $\mathfrak{m} \subset \mathscr{O}_{\mathbb{C}^n,0}$. Then H is a triangular system of full rank if possibly after a change of coordinates, there exist $h_1, \ldots, h_n$ in H such that

1. For each $1 \leqslant i \leqslant n$, and whenever $j \geqslant i$, the derivative of h_i with respect to z_j vanishes, that is $\frac{\partial h_i}{\partial z_j} = 0$.

2. For each i with $1 \leqslant i \leqslant n$, $h_i(0, z_i)$ does not completely vanish. □

Given that $\{h_1, \ldots, h_n\} \subset \mathscr{O}_{\mathbb{C}^n,0}$ forms a triangular system, they may be written in the form

$$\begin{aligned} h_1 &= z_1^{m_1}, \\ h_2 &= z_2^{m_2} + z_1 g_{21}(z_1, z_2), \\ h_3 &= z_3^{m_3} + z_1 g_{31}(z_1, z_2, z_3) + z_2 g_{32}(z_1, z_2, z_3), \\ &\vdots \\ h_n &= z_n^{m_n} + \sum_{1 \leqslant j \leqslant n-1} z_j g_{nj}(z_1, \ldots, z_{n-1}). \end{aligned} \tag{1.52}$$

The intersection multiplicity of the h_i's is finite, or more precisely, in [CD10],

$$\mathsf{dim}_{\mathbb{C}}\, \mathscr{O}_{\mathbb{C}^n,0} \big/ \langle h_1, \ldots, h_n \rangle = m_1 \cdots m_n.$$

This is because the basis of the vector space $\mathscr{O}_{\mathbb{C}^n,0} \big/ \langle h_1, \ldots, h_n \rangle$ is spanned by $\{z_1^{\alpha_1}, \ldots, z_n^{\alpha_n} : 0 \leqslant \alpha_i \leqslant m_i\}$. The following theorem describes the effectiveness of the algorithm for the triangular systems:

Theorem 1.53 (See [CD10])**.** *There is an effective algorithm for establishing estimates for triangular systems. Let h_1,…,h_n define a triangular system as in equation* (1.52) *. Then*

1. *There is a finite list of pairs of subelliptic multipliers (B_1, A_1),…,(B_L, A_L) such that $B_1 = A_1 = \mathsf{det}(\partial_{z_j} h_i)$, and $B_L = A_L$ where B_L is invertible in $\mathscr{O}_{\mathbb{C}^n,0}$.*

2. *Each B_j divides a power of A_j, and the power depends on dimension n.*

3. *The length L equals to $m_1 \cdots m_n$.*

Since subelliptic estimates depend on the presence of holomorphic variety contained in the boundary, D'Angelo has defined a way to measure the closedness of the all types of holomorphic varieties that touch the boundary $b\Omega$.

Definition 1.54 ($\Delta^1(b\Omega, p)$, see [D'A82])**.** Let (M, p) be the germ at p of a smooth real hypersurface in $\mathbb{C}^n$. Let r_p denote a generator of the principal ideal in $\mathscr{C}_p^\infty$ of functions that vanish on M. Then the maximal order of contact of complex analytic one-dimensional varieties with M at p is the number

$$\Delta^1(M, p) = \mathsf{sup}_z \frac{\mathrm{ord}_p(r_p \circ z)}{\mathrm{ord}_p z}$$

where the supremum is taken over all possible holomorphic curve $z : \mathbb{D}^1 \to \mathbb{C}^n$ that passes through the p, that is $z(0) = p$.

A point $p \in b\Omega$ is of finite type (see [D'A82]) if there is a bound on the maximum order of contact of one-dimensional varieties with the boundary at p, or in other words, $\Delta^1 < \infty$. With this definition, Catlin has established in [Cat83]; [Cat84]; [Cat87] that the notion of finite type is a necessary and sufficient condition for subelliptic estimates to hold:

Theorem 1.55 (Theorem of Catlin)**.** *Suppose Ω is a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$, and p is a boundary point. Then the $\bar{\partial}$-Neumann problem satisfies a subelliptic estimate on $(0, 1)$ forms if and only if p is a point of finite type.*

Catlin's theorem is a generalisation of Kohn's theorem, as it does not involve anything about the algorithm. The main idea and also the crucial step in proving Theorem is the following

Theorem 1.56 (Theorem of Catlin)**.** *Let Ω be a smoothly bounded domain, defined near a boundary point p by $\{r = 0\}$. Suppose that there is a neighbourhood U of p such that the following holds: for each $\delta > 0$, there is a smooth function Φ_δ satisfying*

1. *$|\Phi_\delta| \leqslant 1$ on U,*
2. *Φ_δ is plurisubharmonic on U, so that its Hessian $H(\Phi_\delta) \geqslant 0$ on U, and*
3. *$H(\Phi_\delta) \geqslant c\delta^{2\varepsilon}$ on $U \cap \{\delta < r \leqslant 0\}$,*

then there is a subelliptic estimate of order ε at p.

While the ε obtained for triangular systems via Kohn's algorithm is probably far from optimal, Catlin and Cho [CC], and independently Khanh and Zampieri [KG14], have established some sharper subelliptic estimates for certain specific triangular systems using the ideas in Theorem 1.55.

The problem with Kohn's algorithm is that it terminates but not always with effectiveness. For example

Proposition 1.57 (See [CD10])**.** *Let (z, w, z_3) be holomorphic coordinates of $\mathbb{C}^3$, and let $\Omega \subset \mathbb{C}^3$ be a pseudoconvex domain whose boundary is defined by the defining function*

$$r(z, \overline{z}) = \mathsf{Re}\ z_3 + |z^M|^2 + |w^N + wz^K|^2$$

where $K \geqslant M \geqslant 2$ and $N \geqslant 2$. Then the root taken in the radical required in the second step of the Kohn's algorithm is at least K, and hence is independent of the intersection multiplicity.

Here, a summary of the reason is given in [CD10] as follows. For simplicity, let $g := w^N + wz^K$. By the assumption that $K \geqslant M$, the intersection multiplicity of z^M and g is given by

$$\mathsf{dim}_{\mathbb{C}}\, \mathscr{O}_{\mathbb{C}^2,0}/\langle z^M, g\rangle = MN.$$

If Kohn's algorithm were to terminate, the expected subelliptic regularity ε would depend only on the intersection multiplicity MN. The first step of the Kohn's algorithm consists in taking the Jacobian of z^M and g, which gives

$$J_1 := \langle z^{M-1} g_w \rangle,$$

and hence its radical I_1 is generated by the element zg_w. Next by choosing the determinant of the 2×2 minor of the matrix

$$\begin{pmatrix} z^{M-1} & 0 \\ g_z & g_w \\ zg_{wz} + g_w & zg_{ww} \end{pmatrix},$$

the ideal generated by zg_w and the two new determinants is given by

$$J_2 = \langle zg_w,\ z^M g_{ww},\ zg_z g_{ww} - zg_w g_{zw} \rangle.$$

Let I_2 be the radical of J_2, and it turns out that

$$I_2 \ = \ \langle z, w \rangle$$

which is the maximal ideal in $\mathscr{O}_{\mathbb{C}^2,0}$, but z^{K-1} does not lie in J_2 (see [CD10] for more details). To calculate the ε, note that for all $(0,1)$-forms in $\mathscr{D}_{0,1}(U)$ for some suitable set U (for the rest of the calculation, U will be ignored),

$$\| |z^M g_w \phi| \|^2_{\frac{1}{4}} \lesssim Q(\phi, \phi).$$

Taking the radical means

$$\|zg_w\phi\|^2_{\frac{1}{4M}} \lesssim Q(\phi,\phi).$$

The next two determinants give also

$$\|z^M g_{ww}\phi|\|^2_{\frac{1}{4M}} \lesssim Q(\phi,\phi)$$

and

$$\|(zg_zg_{ww} - zg_wg_{zw})\phi|\|^2_{\frac{1}{4M}} \lesssim Q(\phi,\phi).$$

Since the radical of J_2 is the maximal ideal, there exists a large R such that $z^R \in J_2$. Therefore,

$$\|\,|z^R\phi|\,\|^2_{\frac{1}{4M}} \lesssim Q(\phi,\phi),$$

with $R \geqslant K-1$. Taking the radical,

$$\|\,|z\phi|\,\|^2_{\frac{1}{4MR}} \lesssim Q(\phi,\phi).$$

The rest of the calculation is not necessary, as the ε obtained from the algorithm would satisfy

$$\varepsilon \geqslant \frac{1}{4MR} \qquad \big(R\geqslant K-1\big).$$

Since K is arbitrary, the ε obtained here is not effective.

2 Local Geometry of Complex Spaces and Local Intersection Theory

2.0.5 Throughout this section, we will study study the geometry of analytic varieties near the origin.

2.0.6 Let $\mathscr{O}_{\mathbb{C}^n,0}$ denote the ring of holomorphic function germs at the origin. It can be canonically identified with $\mathbb{C}\{z_1,\ldots,z_n\}$ the ring of convergent power series.

2.0.7 The ring $\mathscr{O}_{\mathbb{C}^n,0}$ is local and let $\mathfrak{m}$ denote its unique maximal ideal, which can be characterised by one of the following equivalent properties:

(i)

$$\mathfrak{m} := \{f \in \mathscr{O}_{\mathbb{C}^n,0} :\ f(0)=0\};$$

(ii)

$$\mathfrak{m} = \langle z_1,\ldots,z_n\rangle;$$

(iii) every holomorphic function germ f may be written as $f = \sum_{k\geq 1} f_k$ a sum of homogeneous polynomials f_k of degree k.

2.0.8 For each $l \in \mathbb{N}$, we define $\mathfrak{m}^{l+1}$ recursively by

$$\begin{aligned}\mathfrak{m}^{l+1} &= \mathfrak{m} \cdot \mathfrak{m}^l \\ &= \left\{ \sum_{k=1}^{N} f_k g_k : \ N \in \mathbb{N}, \ f_k \in \mathfrak{m}, \ g_k \in \mathfrak{m}^l \right\}.\end{aligned}$$

For any fixed $l \geqslant 1$, the following conditions are equivalent:
(i) $h \in \mathfrak{m}^l$;
(ii) for each $(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ such that $\alpha_1 + \cdots + \alpha_n \leqslant l-1$,

$$(\partial_{z_1}^{\alpha_1} \cdots \partial_{z_n}^{\alpha_n} h)(0) = 0;$$

(iii)

$$\mathfrak{m}^l = \langle z_1^{k_1} \cdots z_n^{k_n} : \ k_1 + \cdots k_n = l \rangle;$$

(iv) $h = \sum_{k \geqslant l} h_k$ where either h_k vanishes or is a homogeneous polynomial of degree k.

2.0.9 Multiplicity

Definition 2.1. Let $h \in \mathscr{O}_{\mathbb{C}^n,0}$ be a holomorphic function germ, which can be written as

$$h = \sum_{k=0}^{\infty} h_k$$

a sum of homogeneous polynomials h_k of degree k. The multiplicity of h, which will be denoted by $\mathsf{mult}_0\ h$, is the smallest positive integer k for which $h_k \not\equiv 0$.

2.0.10 By Paragraph 2.0.8((i) $\iff$ (iv)), the holomorphic function h lies in $\mathfrak{m}^l$ if and only if $\mathsf{mult}_0\ h \geqslant l$. In section 5, we will study the geometric characterisation of multiplicity of a holomorphic function, and the extension of this notion to certain ideals.

2.1 Local Analytic Geometry

2.1.1 In this subsection, we let $F_1, \ldots, F_{\mathrm{N}}$ be holomorphic function germs in $\mathscr{O}_{\mathbb{C}^n,0}$ vanishing at the origin. For easier exposition, we will not specify the domain of definition of the holomorphic function germs.

2.1.2 Local Analytic Set, Germs of Analytic Space

Definition 2.2. A set $X \subseteq \mathbb{C}^n$ is *locally analytic* if for any point $p \in X$, there exists an open subset V of p in $\mathbb{C}$, and finitely many holomorphic functions $f_1, \ldots, f_s$ defined on V, such that

$$X \cap V = \{x \in V : \ f_1(x) = \cdots = f_s(x) = 0\}.$$

Definition 2.3. A *germ of analytic space* $(X, 0)$ is a germ at 0 of a locally analytic subset of $\mathbb{C}^n$.

2.1.3 Any germ of an analytic space $(X, 0)$ may be uniquely written as

$$(X, 0) = (X_1, 0) \cup \cdots \cup (X_r, 0)$$

a union of irreducible components[1], each of which is also a germ of an analytic space ([JP00, Corollary 3.4.18, p 118]).

[1] A germ of an analytic space (X, x) is irreducible if whenever $(X, x) = (X_1, x) \cup (X_2, x)$ with (X_1, x) and (X_2, x) germs of analytic spaces, either $(X, x) = (X_1, x)$ or $(X, x) = (X_2, x)$.

2.1.4 $(V(F),0)$, $(V(\mathscr{I}_F),0)$ and $\mathscr{I}(X,0)$.

Definition 2.4. Let $F \in \mathscr{O}_{\mathbb{C}^n,0}$. The germ of an analytic hypersurface $(V(F),0)$ is defined as follows. Let U be an open neighbourhood of the origin on which F seen as a power series converges. Consider $V(F) = \{p \in U :\ F(p) = 0\}$. Then $(V(F),0)$ is the germ of $V(F)$ at zero, and is called the *zero set* of F.

Definition 2.5. Let $\mathscr{I}_F = \langle F_1, \ldots, F_{\mathrm{N}}\rangle$ be an ideal of $\mathscr{O}_{\mathbb{C}^n,0}$. The germ of analytic space $(V(\mathscr{I}_F),0)$ is defined by

$$(V(\mathscr{I}_F),0) = \bigcap_{i=1}^{\mathrm{N}} (V(F_i),0).$$

Definition 2.6. Let $(X,0)$ be a germ of an analytic space. Then define

$$\mathscr{I}(X,0) = \{f \in \mathscr{O}_{\mathbb{C}^n,0} :\ (X,0) \subseteq (V(f),0)\}.$$

2.1.5 Properties of $(V(\mathscr{I}_F),0)$ and $\mathscr{I}(X,0)$ Let $F_1,\ldots,F_{\mathrm{N}}$ and $G_1,\ldots,G_{\mathrm{M}}$ be holomorphic function germs in $\mathscr{O}_{\mathbb{C}^n,0}$. Let $\mathscr{I}_F = \langle F_1,\ldots,F_{\mathrm{N}}\rangle$ and $\mathscr{I}_G = \langle G_1,\ldots,G_{\mathrm{M}}\rangle$ be the corresponding ideals they generate. Let $(X,0)$ and $(Y,0)$ be germs of analytic spaces.

(i) $\mathscr{I}_F \subseteq \mathscr{I}_G$ implies that $(V(\mathscr{I}_G),0) \subseteq (V(\mathscr{I}_F),0)$;

(ii) $(X,0) \subseteq (Y,0)$ implies that $\mathscr{I}(Y,0) \subseteq \mathscr{I}(X,0)$;

(iii) for any $k \in \mathbb{N}_{\geqslant 1}$, and for any ideal $\mathscr{I}_F$, $(V(\mathscr{I}_F^k),0) = (V(\mathscr{I}_F),0)$;

(iv) for any germ of analytic space $(X,0)$, $(V(\mathscr{I}(X,0)),0) = (X,0)$

(v) (Nullstellensatz) $\mathscr{I}(V(\mathscr{I}_F),0)) = \sqrt{\mathscr{I}_F}$.

For ease of notation, let $V(F_1,\ldots,F_{\mathrm{N}})$ or $V(\mathscr{I}_F)$ denote $(V(\mathscr{I}_F),0)$.

2.2 Local Intersection Theory I

2.2.1 We begin with the characterisation of complete intersections of germs of analytic varieties at the origin.

Theorem 2.7. *Let $F_1,\ldots,F_{\mathrm{N}}$ be holomorphic function germs in $\mathscr{O}_{\mathbb{C}^n,0}$ at the origin. The following statements are equivalent.*

(i) $V(F_1,\ldots,F_{\mathrm{N}}) = \{0\}$*;*

(ii) *there exists a positive integer $q \geqslant 1$ such that $\mathfrak{m}^q \subseteq \mathscr{I}_F$;*

(iii) *the number*

$$\dim_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^n,0}\big/\mathscr{I}_F =: s$$

is finite;

(iv) *there exists a positive integer p such that locally*

$$|z|^p \lesssim \sum_{i=1}^{\mathrm{N}} |F_i|.$$

Proof. The proof proceeds in the following manner: (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i), and (ii) $\Longleftrightarrow$ (iv).

For (i) $\Longrightarrow$ (ii), since $V(F_1,\ldots,F_{\mathrm{N}}) = \{0\} = V(\mathfrak{m})$, there is an equality of ideals $\mathscr{I}(V(F_1,\ldots,F_{\mathrm{N}})) = \mathscr{I}(V(\mathfrak{m}))$. By Nullstellensatz, therefore $\mathfrak{m} = \sqrt{\mathfrak{m}} = \sqrt{\mathscr{I}_F}$. Hence there exists $q \in \mathbb{N}_{\geqslant 1}$ such that $\mathfrak{m}^q \subseteq \mathscr{I}_F$.

For (ii) $\Longrightarrow$ (iii), the condition that $\mathfrak{m}^q \subseteq \mathscr{I}_F$ implies that there is a surjective map of $\mathbb{C}$-vector space

$$\begin{array}{ccc} \mathscr{O}_{\mathbb{C}^n,0}\big/\mathfrak{m}^q & \longrightarrow & \mathscr{O}_{\mathbb{C}^n,0}\big/\mathscr{I}_F \\ f \bmod \mathfrak{m}^q & \longmapsto & f \bmod \mathscr{I}_F. \end{array}$$

Hence,

$$\dim_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^n,0}\big/\mathscr{I}_F \leqslant \dim_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^n,0}\big/\mathfrak{m}^q,$$

and the proof is complete since $\dim_{\mathbb{C}}\mathscr{O}_{\mathbb{C}^n,0}\big/\mathfrak{m}^q$ is always finite for $q \in \mathbb{N}_{\geqslant 1}$.

For (iii) $\Longrightarrow$ (i), it is needed to show that the set

$$\{(\alpha_1,\ldots,\alpha_n) \in \mathbb{C}^n :\ F_k(\alpha_1,\ldots,\alpha_n) = 0 \text{ for all } 1 \leqslant k \leqslant \mathrm{N}\}$$

is finite. To this effect, it suffices to show that there can only be finitely many choices for each α_i. Since $\mathscr{O}_{\mathbb{C}^n,0}\big/\mathscr{I}_F$ is finite dimensional, for each $1 \leqslant i \leqslant n$, there exists $k_i \in \mathbb{N}_{\geqslant 1}$ such that the classes

$$\{1,\ z_i,\ \ldots,\ z_i^{k_i}\}$$

form a linearly dependent set in $\mathscr{O}_{\mathbb{C}^n,0}\big/\mathscr{I}_F$. Hence there exist constants $\{c_{i,0},\ldots,c_{i,k_i}\}$ such that

$$\sum_{j=0}^{k_i} c_{ij} z_i^j \equiv 0 \bmod \mathscr{I}_F.$$

Thus there exists a holomorphic function $h_i(z_1,\ldots,z_n) \in \mathscr{I}_F$ such that

$$\sum_{j=0}^{k_i} c_{ij} z_i^j - h_i(z_1,\ldots,z_n) \equiv 0 \qquad (1 \leqslant i \leqslant n).$$

If $(\alpha_1,\ldots,\alpha_n) \in \mathscr{I}_F$, then for all $1 \leqslant i \leqslant n$, one has $h_i(\alpha_1,\ldots,\alpha_n) = 0$. Hence

$$\sum_{j=0}^{k_i} c_{ij} \alpha_i^j = 0.$$

The equation above is a polynomial equation in degree k_i, and so there are at most k_i distinct solutions for α_i. This holds for all i, and therefore $V(\mathscr{I}_F)$ is a finite set. The proof is complete.

The implication (ii) $\Longrightarrow$ (iv) is immediate. The converse will be proved after Skoda's theorem is introduced. The proof is reproduced from [Siu10, p 1179] □

Theorem 2.8 (Theorem of Henri Skoda)**.** *Let D be a pseudoconvex domain in $\mathbb{C}^n$ and let χ be a plurisubharmonic function on D. Let $g_1,\ldots,g_m$ be holomorphic functions on D. Let $\alpha > 1$ and $l = \min\{n, m-1\}$. Then for every holomorphic function F on D such that*

$$\int_D |F|^2 |g|^{-2\alpha l - 2} e^{-\chi} < \infty,$$

there exist holomorphic functions $h_1,\ldots,h_m$ on D such that

$$F = \sum_{i=1}^m h_i g_i,$$

and

$$\int_D |h|^2 |g|^{-2\alpha l - 2} e^{-\chi} \leqslant \frac{\alpha}{\alpha - 1} \int_D |F|^2 |g|^{-2\alpha l - 2} e^{-\chi},$$

where $|g| = \left(\sum_{i=1}^m |g_i|^2\right)^{1/2}$ and $|h| = \left(\sum_{i=1}^m |h_i|^2\right)^{1/2}$.

Finishing the proof of Theorem. For any non-negative numbers $\gamma_1,\ldots,\gamma_n$ with $\gamma_1+\cdots+\gamma_n=(n+2)p$, Skoda's theorem is applied with the following variables: $F=z_1^{\gamma_1}\cdots z_n^{\gamma_n}$, $m=\mathrm{N}+n$, $\chi\equiv 0$, $(F_1,\ldots,F_{\mathrm{N}},0,\ldots,0)=(g_1,\ldots,g_m)$, $l=n$ and $\alpha=\frac{n+1}{n}$. By the hypothesis in (iv),

$$|z_1^{\gamma_1}\cdots z_n^{\gamma_n}|^2\lesssim|z|^{2(n+2)p}\lesssim\left(\sum_{i=1}^{\mathrm{N}}|F_i|\right)^{2(n+2)}\lesssim\left(\sum_{i=1}^{\mathrm{N}}|F_i|^2\right)^{(n+2)}$$

where the last inequality follows from Jensen's inequality. Hence over a small pseudoconvex domain D,

$$\int_D\frac{|z_1^{\gamma_1}\cdots z_n^{\gamma_n}|^2}{\left(\sum_{j=1}^{2}|F_j|^2\right)^{n+2}}\lesssim\int_D\frac{\left(\sum_{i=1}^{\mathrm{N}}|F_i|^2\right)^{(n+2)}}{\left(\sum_{i=1}^{\mathrm{N}}|F_i|^2\right)^{(n+2)}}=\int_D 1<\infty.$$

Skoda's theorem applies and therefore $z_1^{\gamma_1}\cdots z_n^{\gamma_n}\in\mathscr{I}_F$. Consequently, $\mathfrak{m}^{(n+2)p}\subseteq\mathscr{I}_F$. □

From the proof above, we obtain the following corollary.

Corollary 2.9. *Let $F_1,\ldots,F_{\mathrm{N}}$ be holomorphic function germs in $\mathscr{O}_{\mathbb{C}^n,0}$ at the origin, and suppose there exists $p\in\mathbb{N}_{\geqslant 1}$ such that*

$$|z|^p\lesssim\sum_{i=1}^{\mathrm{N}}|F_i|$$

in a small neighbourhood 0, then $\mathfrak{m}^{(n+2)p}\subseteq\mathscr{I}_F$.

2.2.2 The intersection invariants (p,q,s).

Definition 2.10. Let $F_1,\ldots,F_{\mathrm{N}}$ be holomorphic function germs in $\mathscr{O}_{\mathbb{C}^n,0}$ at the origin. The ideal $\mathscr{I}_F=\langle F_1,\ldots,F_{\mathrm{N}}\rangle$ is said to have finite intersection multiplicity with data (p,q,s) if

(i) p is the smallest strictly postive integer satsifying

$$|z|^p\lesssim\sum_{i=1}^{\mathrm{N}}|F_i|;$$

(ii) q is the smallest strictly positive integer satisfying

$$\mathfrak{m}^q\subseteq\mathscr{I}_F;$$

(iii) s is following number below

$$\mathsf{dim}_{\mathbb{C}}\,\mathscr{O}_{\mathbb{C}^n,0}\big/\mathscr{I}_F=:s.$$

2.2.3 The relations between the intersection invariants.

Proposition 2.11. *Let $F_1,\ldots,F_{\mathrm{N}}$ be holomorphic function germs in $\mathscr{O}_{\mathbb{C}^n,0}$ at the origin so that the ideal they generate $\mathscr{I}_F$ has finite intersection multiplicity with data (p,q,s). Then we have the following inequalities:*

(i) $q\leqslant s\leqslant\binom{n+q-1}{q-1}$,

(ii) $p\leqslant q\leqslant(n+2)p$.

Proof. To prove $q \leqslant s$, it is first observed that $\mathscr{O}_{\mathbb{C}^n,0}/\mathscr{I}_F$ is also a local ring with the maximal ideal $\mathfrak{m}/\mathscr{I}_F$. In the chain of inclusion of vector spaces with

$$\frac{\mathscr{O}_{\mathbb{C}^n,0}}{\mathscr{I}_F} \supseteq \frac{\mathfrak{m}}{\mathscr{I}_F} \supseteq \left(\frac{\mathfrak{m}}{\mathscr{I}_F}\right)^2 \supseteq \cdots \supseteq \left(\frac{\mathfrak{m}}{\mathscr{I}_F}\right)^{s+1},$$

since $\mathscr{O}_{\mathbb{C}^n,0}/\mathscr{I}_F$ is an s-dimensional complex vector space, there exists a positive integer $1 \leqslant k \leqslant s$ such that

$$\left(\frac{\mathfrak{m}}{\mathscr{I}_F}\right)^k = \left(\frac{\mathfrak{m}}{\mathscr{I}_F}\right)^{k+1} = \left(\frac{\mathfrak{m}}{\mathscr{I}_F}\right)\left(\frac{\mathfrak{m}}{\mathscr{I}_F}\right)^k.$$

By Nakayama's lemma[2],

$$\left(\frac{\mathfrak{m}}{\mathscr{I}_F}\right)^k \equiv 0 \qquad \text{in } \frac{\mathscr{O}_{\mathbb{C}^n,0}}{\mathscr{I}_F}.$$

Therefore, if $g_1,\ldots,g_k$ are elements of $\mathfrak{m}$ in $\mathscr{O}_{\mathbb{C}^n,0}$, then the class $g_1 \cdots g_k$ belongs to $(\mathfrak{m}/\mathscr{I}_F)^k$ which is the zero vector space. Hence the holomorphic function $g_1 \cdots g_k$ lies in $\mathscr{I}_F$. Since the $g_1 \cdots g_k$ generate $\mathfrak{m}^k$, the ideal $\mathfrak{m}^k$ is contained in $\mathscr{I}_F$. By the definition of q, the inequality $q \leqslant k \leqslant s$ holds.

Next, for $s \leqslant \binom{n+q-1}{q-1}$, this follows directly from $\mathfrak{m}^q \subseteq \mathscr{I}_F$.

In the second set of inequalities, to prove $p \leqslant q$, observe that since $\mathfrak{m}^q \subseteq \mathscr{I}_F$,

$$|z|^q \lesssim \sum_{i=1}^{\mathrm{N}} |F_i|.$$

Hence, by the definition of p, $p \leqslant q$.

To prove $q \leqslant (n+2)p$, it follows from Corollary 2.9. □

2.2.4 Application of the relations of the invariants.

Lemma 2.12. *Let $F_1, \ldots, F_{\mathrm{N}}$ be holomorphic function germs such that the intersection multiplicity of $\mathscr{I}_F = \langle F_1, \ldots, F_{\mathrm{N}}\rangle$ is finite with data (p,q,s). If $h \in \mathscr{O}_{\mathbb{C}^n,0}$ is a holomorphic function germ with $h(0) = 0$, then $h^s \in \mathscr{I}_F$.*

Proof. Since $h(0) = 0$, the function h lies in $\mathfrak{m}$. Consequently, $h^s \in \mathfrak{m}^s$. By $q \leqslant s$, there is an inclusion of ideals $\mathfrak{m}^s \subseteq \mathfrak{m}^q$. Therefore, $h^s \in \mathfrak{m}^s \subseteq \mathfrak{m}^q \subseteq \mathscr{I}_F$. □

2.3 Local Intersection Theory II

2.3.1 The case where $\mathrm{N} = n = \dim \mathbb{C}^n$ brings another set of equivalent conditions for complete local intersection of n holomorphic function germs $F_1, \ldots, F_{\mathrm{N}=n}$.

2.3.2

Theorem 2.13. *Let $F_1, \ldots, F_n$ be holomorphic function germs in $\mathscr{O}_{\mathbb{C}^n,0}$ such that $F_i(0) = 0$ for all $1 \leqslant i \leqslant n$. The following are equivalent:*

(i) $\dim_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^n,0}/\langle F_1, \ldots, F_n\rangle =: s < \infty$;

(ii) *the holomorphic map of germs of analytic spaces*

$$\begin{aligned} F : (\mathbb{C}^n, 0) &\longrightarrow (\mathbb{C}^n, 0) \\ (z_1, \ldots, z_n) &\longmapsto (F_1, \ldots, F_n) \end{aligned}$$

[2]The following version of Nakayama's lemma is used: let A be a commutative local ring with 1, and $\mathfrak{m}$ its maximal ideal. For any finitely generated A-module M, if $\mathfrak{m}M = M$, then $M = 0$

defines a ramified s-sheeted analytic covering;
(iii) *for each $1 \leqslant i \leqslant n$, let ε_i be a small strictly positive number, and Γ be given by*

$$\Gamma = \{z : \ |F_i| = \varepsilon_i\}.$$

Then the residue map of F at the origin equals to s:

$$\mathsf{Res}_0\ F = \int_\Gamma \frac{dF_1 \wedge \cdots \wedge dF_n}{F_1 \cdots F_n} = s.$$

Proof. See [D'A93, p 60], [GH94, p 666–667], and [Chi89, p 140, Proposition 1] for discussion. □

2.3.3 We will show that given Theorem 2.13, one has $\mathsf{mult}_0\ \mathsf{Jac}(F) \leqslant s - 1$.

Theorem 2.14. *Let h be a holomorphic function germ. If $h \in \mathscr{I}_F$, then*

$$\int_\Gamma \frac{h\, dz_1 \wedge \cdots \wedge dz_n}{F_1 \cdots F_n} = 0.$$

Proof. See [D'A93, p 64]. □

Corollary 2.15. *Let $F_1, \ldots, F_n$ be holomorphic function germs in $\mathscr{O}_{\mathbb{C}^n,0}$ vanishing at the origin, whose varieties they define have complete intersection at the origin. Let F be the map in Theorem 2.13(ii). Then* $\mathsf{Jac}(F) \notin \langle F_1, \ldots, F_{\mathrm{N}}\rangle$.

Proof. By Theorem 2.13(iii),

$$0 \neq s = \int_\Gamma \frac{dF_1 \wedge \cdots \wedge dF_n}{F_1 \cdots F_n} = \int_\Gamma \frac{\mathsf{Jac}(F)\, dz_1 \wedge \cdots \wedge dz_n}{F_1 \cdots F_n}.$$

Hence $\mathsf{Jac}(F) \notin \langle F_1, \ldots, F_n\rangle$ by Theorem 2.14. □

Corollary 2.16. *Let $F_1, \ldots, F_n$ be holomorphic function germs in $\mathscr{O}_{\mathbb{C}^n,0}$ vanishing at the origin so that the ideal $\mathscr{I}_F$ has finite intersection multiplicity with data (p, q, s). Then the multiplicity of* $\mathsf{Jac}(F)$ *at the origin cannot be greater than or equal to s.*

Proof. Suppose otherwise that $\mathsf{mult}_0\ \mathsf{Jac}(F) \geqslant s$, then $\mathsf{Jac}(F) \in \mathfrak{m}^s$. From the inequality $q \leqslant s$, there is an inclusion of ideals $\mathfrak{m}^s \subseteq \mathfrak{m}^q \subseteq \langle F_1, \ldots, F_n\rangle$. Hence $\mathsf{Jac}(F) \in \langle F_1, \ldots, F_n\rangle$, which contradicts Corollary 2.15. □

2.3.4 Miscellaneous Result We will state the following result which will be used later.

Proposition 2.17. *Let $f_1, \ldots, f_{n-1}$, f, and g be holomorphic function germs in $\mathscr{O}_{\mathbb{C}^n,0}$ such that*

$$\mathsf{dim}_{\mathbb{C}}\ \mathscr{O}_{\mathbb{C}^n,0}/\langle f_1, \ldots, f_{n-1}, fg\rangle < \infty.$$

Then

$$\mathsf{dim}_{\mathbb{C}}\ \mathscr{O}_{\mathbb{C}^n,0}/\langle f_1, \ldots, f_{n-1}, fg\rangle = \mathsf{dim}_{\mathbb{C}}\ \mathscr{O}_{\mathbb{C}^n,0}/\langle f_1, \ldots, f_{n-1}, f\rangle + \mathsf{dim}_{\mathbb{C}}\ \mathscr{O}_{\mathbb{C}^n,0}/\langle f_1, \ldots, f_{n-1}, g\rangle.$$

Proof. See [D'A93, p 60, Theorem 1] □

3 Ideals Generated by the Components of Gradient

3.0.5 In this section we shall study the ideals generated by the components of the gradient of a holomorphic function. Let $f \in \mathscr{O}_{\mathbb{C}^n,0}$ be a holomorphic function germ such that $f(0) = 0$. In a first moment, it will be shown that there exists a positive integer k with

$$f^k \in \left\langle \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right\rangle.$$

In a second moment, more accurately, it will be shown that $k = \mathsf{dim}\ \mathbb{C}^n = n$ works (optimally) rendering k effective.

3.0.6 Example In 1-dimensional complex analysis, every holomorphic function $f(\zeta)$ with $f(0) = 0$ may be factorised as

$$f(\zeta) = \zeta^k g(\zeta),$$

where $g(0) \neq 0$. A differentiation yields

$$f'(\zeta) = \zeta^{k-1}(kg(\zeta) + \zeta g'(\zeta)),$$

and hence $f \in \langle f' \rangle$. Therefore, $k = 1$ works in this case. In the next few paragraphs we will recall some notions in algebraic geometry.

3.0.7 Spec, Zariski Topology Let A be a commutative ring with 1. We let

$$\mathsf{Spec}\ A := \{\mathfrak{p} \subset A :\ \mathfrak{p} \text{ is a prime ideal in } A\}.$$

For every ideal $I \subset A$, set

$$V_A(I) = \{\mathfrak{p} \in \mathsf{Spec} A :\ I \subseteq \mathfrak{p}\}.$$

The sets $V_A(I)$ are defined as closed sets in $\mathsf{Spec}\ A$, and the collection

$$\{V_A(I) :\ I \text{ is an ideal of } A\}$$

defines the *Zariski topology* of $\mathsf{Spec}\ A$. For principal ideals $\langle f \rangle$, $V_A(\langle f \rangle)$ may be written as $V_A(f)$. Therefore,

$$D_A(f) := \mathsf{Spec}\ A \backslash V_A(f) = \{\mathfrak{p} \in \mathsf{Spec}\ A :\ f \notin \mathfrak{p}\}$$

is open in $\mathsf{Spec}\ A$. The collection

$$\{D_A(f) :\ f \in A\}$$

forms a basis for the open set in the Zariski topology. To see this, for any ideal I, one has

$$\mathsf{Spec}\ A \backslash V_A(I) = \bigcup_{f \in I} D_A(f).$$

3.0.8 For any ideal $I \subset A$, there is a one-to-one correspondance between $\mathsf{Spec}\ A/I$ and $V_A(I)$. On the other hand, let $f \in A$ and A_f be its localisation. Every element in A_f is a class with representative a/f^k for some $a \in A$ and $k \in \mathbb{N}$. The two representatives a/f^k and b/f^l are equal if there exists $j \geqslant 0$ such that

$$f^j(af^l - bf^k) = 0 \qquad \text{(in } A\text{)}.$$

It is easily seen that like $\mathbb{Q}$ the quotient numbers of $\mathbb{Z}$, A_f has a ring structure, and there is a one-to-one correspondance (as sets) between $\mathsf{Spec}\ A_f$ and $D_A(f)$.

3.0.9 Recall that a commutative ring with 1 is *semi-local* if it has only finitely many maximal ideals. We state the following Artin-Tate theorem.

Theorem 3.1. *Let A be a Noetherian integral domain. Then A is semi-local with* dim $A \leqslant 1$ *if and only if there exists $f \in A$ such that A_f is a field.*

Proof. See [GW10], page 562, Corollary B62. □

3.0.10 Recall that the *Krull dimension* of A is given by

$$\text{dim } A := \text{sup}\{k : \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_k\}.$$

Moreover, if A is local, Artin-Tate's theorem may be restated as follows: there exists $f \in A$ such that A_f is a field if and only if dim $A \leqslant 1$.

3.0.11 For any germ variety $V(\mathscr{I})$ defined by an ideal $\mathscr{I} \subset \mathscr{O}_{\mathbb{C}^n,0}$, the Krull dimension of $\mathscr{O}_{\mathbb{C}^n,0}/\mathscr{I}$ coincides with the usual intuition of dimension.

3.0.12 To see this, recall that the *Weierstrass dimension* of a germ of complex space (X, x) is the least number k such that there exists a Noether normalisation $\pi^* : \mathscr{O}_{\mathbb{C}^k,x} \hookrightarrow \mathscr{O}_{X,x}$. Both the Weierstrass dimension of (X, x) and the Krull dimension of $\mathscr{O}_{X,x} := \mathscr{O}_{\mathbb{C}^n,x}/\mathscr{I}(X, x)$ coincide ([JP00, Theorem 4.1.9, pp 131]). The Noether normalisation π^* is uniquely induced by the projection

$$\pi : (X, x) \to (\mathbb{C}^k, x)$$

of the germ variety (X, x) onto $(\mathbb{C}^n, x)$ with finite fibres. By [JP00, p 129, Lemma 4.14], dim $(X, x) =$ dim $(\mathbb{C}^k, x) = k$. Hence to say that dim $\mathscr{O}_{X,x} \leqslant 1$ is to say that either X is of dimension 1 or 0 (but the dimension need not be pure).

3.0.13 The following lemma is a restatement of the Artin-Tate's theorem in more geometric terms.

Lemma 3.2. *Let A be an integral Noetherian ring, and let (0) be a point in* Spec A. *Then the set $\{(0)\}$ is open in the Zariski topology of* Spec A *if and only if* Spec A *is a finite set and* dim $A \leqslant 1$.

Proof. First, it will be shown that the singleton $\{(0)\}$ is open in Spec A if and only if there exists $f \in A$ such that A_f is a field. Then secondly, it will be shown that Spec A is a finite set and dim $A \leqslant 1$ if and only if A is semi-local (finitely many maximal ideals) and dim $A \leqslant 1$.

For the first assertion, suppose that $\{(0)\}$ is open, then for some index Λ,

$$\{(0)\} = \bigcup_{i \in \Lambda} D_A(f_i).$$

Therefore, $(0) \in D_A(f_i)$ for some $i \in \Lambda$ and hence

$$\{(0)\} \subseteq D_A(f_i) \subseteq \bigcup_{i \in \Lambda} D_A(f_i) = \{(0)\},$$

so that $\{(0)\} = D_A(f_i)$. By paragraph 3.0.8, this means that the ring A_f has only (0) as its prime ideal, and so A_f is a field. Conversely, if there exists $f \in A$ such that A_f is a field, then Spec $A_f = D_A(f) = \{(0)\}$ as sets. Consequently, the singleton $\{(0)\}$ is open in Spec A.

For the second assertion, suppose that Spec A is a finite set and dim $A \leqslant 1$. The first condition implies that A only has finitely many prime ideals. On the other hand, for any maximal chain of prime ideals

$$(0) := \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_k$$

one has $k \leqslant 1$ by the second condition. If $k = 1$, then $\mathfrak{p}_k$ is a maximal ideal. If $k = 0$, then $\mathfrak{p}_0$ is just the zero ideal. Hence any non-zero prime ideal $\mathfrak{p}_1$ is maximal, and since A has finitely many prime ideals, A is semi-local. Conversely, suppose that A is semi-local and $\mathsf{dim}\ A \leqslant 1$. By the chain of inclusion of prime ideals above, any non-zero prime ideal is maximal by second condition. Moreover, A is semi-local means that there are only finitely many maximal ideals. Hence $\mathsf{Spec}\ A$ is a finite set and $\mathsf{dim}\ A \leqslant 1$. □

3.0.14 The lemma above is used to prove the following lemma.

Lemma 3.3. *Let A be a local integral domain, $(0) \neq \mathfrak{m}$, and $\mathsf{dim}\ A = n \geqslant 1$. Let $f \in A\backslash\{0\}$ be a non-zero element with $f \in \mathfrak{m}$. Then there exists a prime ideal $\mathfrak{p}$ such that $\mathsf{dim}\ A/\mathfrak{p} = 1$ and $f \notin \mathfrak{p}$.*

Proof. We will construct by induction on k a sequence of inclusion of prime ideals

$$(0) := \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_k \subsetneq \mathfrak{m}$$

with the following conditions:

(i) $f \notin \mathfrak{p}_k$;

(ii) for every $1 \leqslant i \leqslant k$, the prime ideal $\mathfrak{p}_i$ is of height one over $\mathfrak{p}_{i-1}$. In other words, there is no prime ideal $\mathfrak{q}$ with $\mathfrak{p}_{i-1} \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}_i$;

(iii) either $\mathsf{dim}\ A/\mathfrak{p}_k = 1$ or $\mathsf{dim}\ A/\mathfrak{p}_k \leqslant (\mathsf{dim}\ A) - k$.

Suppose such a $\mathfrak{p}_k$ is constructed, we see that the second condition in (iii) is always a consequence of (ii). This is because from the definition of height of a prime, one always has $k \leqslant \mathsf{Ht}(\mathfrak{p}_k)$. By [JP00, p 133, Remark 4.1.15], one always has

$$\mathsf{Ht}(\mathfrak{p}) + \mathsf{dim}(A/\mathfrak{p}_k) \leqslant \mathsf{dim}\ A.$$

Therefore,

$$k + \mathsf{dim}(A/\mathfrak{p}_k) \leqslant \mathsf{Ht}(\mathfrak{p}) + \mathsf{dim}(A/\mathfrak{p}_k) \leqslant \mathsf{dim}\ A,$$

and consequently $\mathsf{dim}(A/\mathfrak{p}_k) \leqslant (\mathsf{dim}\ A) - k$.

STEP 1: We let $A_0 := A/\mathfrak{p}_0 = A$. By hypothesis that $f \neq 0$ in A, therefore $(0) \in D_A(f)$, and so the open set $D_A(f)$ is non-empty. There are two cases according to whether $D_A(f)$ has exactly one or more than one elements:

(1) Suppose that $D_A(f) = \{(0)\}$, then set $\{(0)\}$ is Zariski open in $\mathsf{Spec}\ A$. By Lemma 3.2 and remark in paragraph 3.0.10, one has $\mathsf{dim}\ A \leqslant 1$. By the hypothesis that $(0) \neq \mathfrak{m}$, hence $\mathsf{dim}\ A = 1$ and the proof is finished.

(2) Otherwise, the Zariski open set $D_A(f)$ contains another prime ideal $\mathfrak{p}_1'$ in A such that $\mathfrak{p}_1' \neq (0)$. Moreover, since $\mathsf{dim}\ A$ is finite, $\mathsf{Ht}(\mathfrak{p}')$ is finite, say h_1. Consider a maximal chain of prime ideals

$$(0) = \mathfrak{p}_0 \subsetneq \mathfrak{q}_{1,1} \subsetneq \cdots \subsetneq \mathfrak{q}_{1,h_1} = \mathfrak{p}_1'$$

whose length is $\mathsf{Ht}(\mathfrak{p}_1')$. By maximality of the chain, $\mathsf{Ht}(\mathfrak{q}_{1,1}) = 1$. Moreover, $f \notin \mathfrak{p}_1'$ implies that $f \notin \mathfrak{q}_{1,1}$. Therefore, we may let $\mathfrak{p}_1 := \mathfrak{q}_{1,1}$.

INDUCTIVE STEP: Once the prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ have been constructed, the existence of $\mathfrak{p}_{k+1}$ that satisfies the first two conditions (i) and (ii) will be constructed. The idea is to pass through the quotient $\pi_k : A \to A/\mathfrak{p}_k := A_k$ and repeat the steps as in **STEP 1** . This time, since $f \in \mathfrak{m}$ but $f \notin \mathfrak{p}_k$, hence $\mathfrak{p}_k \neq \mathfrak{m}$. So in A_k, the zero ideal (0) is not maximal. Also, the hypothesis that $f \notin \mathfrak{p}_k$ implies that the class f is not zero in A_k. Therefore, the open set $D_{A_k}(f)$ is not empty since it contains the zero ideal (0). We study the open set $D_A(f)$ with the two following cases just as before:

(1) Either $D_{A_k}(f)$ is a singleton, meaning $D_{A_k}(f) = \{(0)\}$. Then $\{(0)\}$ is open in $\mathsf{Spec}\ A_k$. By Lemma 3.2, one has $\mathsf{dim}\ A_k \leqslant 1$. But since $(0) \neq \mathfrak{m}$, $\mathsf{dim}\ A_k = 1$.

(2) Otherwise, in A_k we may find a prime ideal $\mathfrak{P}_{k+1} \subseteq A_k$ such that $\mathfrak{P}_k \in D_{A_k}(f)$ and $\mathfrak{P}_{k+1} \neq (0)$. Since $\mathsf{dim}\ A_k$ is finite, so is $\mathsf{Ht}(\mathfrak{P}_{k+1})$, which is for example h_k. We let

$$(0) := \mathfrak{P}_0 \subsetneq \mathfrak{Q}_{k+1,1} \subsetneq \cdots \subsetneq \mathfrak{Q}_{h_{k+1},1} := \mathfrak{P}_{k+1}$$

be a maximal chain of prime ideals in A_k corresponding whose length corresponds to the height of $\mathfrak{P}_{k+1}$. Therefore $\mathfrak{Q}_{k+1,1}$ is of height one by the maximality of the chain, and therefore we may let $\mathfrak{p}_{k+1} = \pi^{-1}(\mathfrak{Q}_{k+1,1})$. Moreover, since the class f does not belong to $\mathfrak{P}_{k+1}$, the element f does not belong to $\mathfrak{p}_{k+1}$. Hence $\mathfrak{p}_{k+1}$ satisfies the two conditions and the inductive step is complete.

To conclude, since $\mathsf{dim}\ A = n$, the length of the chain k is at most $n-1$. In this case one has

$$\mathsf{dim}\ A/\mathfrak{p}_{n-1} \leqslant 1.$$

But since $f \in \mathfrak{m}$ and $f \notin \mathfrak{p}_{n-1}$, so $\mathsf{dim}\ A/\mathfrak{p}_{n-1} = 1$, and the proof is complete. □

3.0.15

Lemma 3.4. *Let $\mathscr{I} \subseteq \mathscr{O}_{\mathbb{C}^n,0}$ be an ideal and $Y := V(\mathscr{I})$ be the variety defined by $\mathscr{I}$. Let f be a holomorphic function germ vanishing at the origin and $X = V(\mathscr{I} + (f))$ be the variety defined by $\mathscr{I} + (f)$. If the inclusion $X \subsetneq Y$ is strict, then there exists an irreducible curve $C \subseteq Y$ passing through the origin such that $f|_C \not\equiv 0$.*

Proof. Almost half of the proof is done by the previous lemma. It suffices to observe that if the inclusion $X \subsetneq Y$ is strict, then there exists an irreducible component $W \subseteq Y$ passing through the origin on which $f|_W \not\equiv 0$. Once this is proved, then the condition that $f|_W \not\equiv 0$ implies that $f \notin \mathscr{I}(W,0)$. Since W is irreducible, the ideal $\mathscr{I}(W,0)$ is prime. Moreover, the dimension of W is greater than 1, otherwise W will just be the origin but $f(0) = 0$, which contradicts that $f|_W \not\equiv 0$. Put differently, one has f a non-zero element in $\mathscr{O}_{W,0} = \mathscr{O}_{\mathbb{C}^n,0}/\mathscr{I}(W,0)$, with $f \in \mathfrak{m}$. Moreover, $\mathscr{O}_{W,0}$ is a local integral domain with $(0) \neq \mathfrak{m}$ since $\mathsf{dim}\ W \geqslant 1$. By previous lemma, there exists a prime ideal $\mathfrak{P}$ in $\mathscr{O}_{W,0}$ such that the class f does not lie in $\mathfrak{P}$ and $\mathsf{dim}\ \mathscr{O}_{W,0}/\mathfrak{P} = 1$. If we let $\pi : \mathscr{O}_{\mathbb{C}^n,0} \to \mathscr{O}_{W,0}$ be the usual ring homomorphism by quotient, one has the prime ideal $\mathfrak{p} := \pi^{-1}\mathfrak{P}$ such that $\mathsf{dim}\mathscr{O}_{\mathbb{C}^n,0}/\mathfrak{p} = \mathsf{dim}\ \mathscr{O}_{W,0}/\mathfrak{P} = 1$, with $f \notin \mathfrak{p}$. Hence $C = V(\mathfrak{p})$ is an irreducible one-dimensional variety on which f does not totally vanish.

It remains to prove the observation. By Primary decomposition theorem, the ideal $\mathscr{I}$ may be decomposed as an intersection

$$\mathscr{I} = \bigcap_{i=1}^{M} P_i$$

of primary ideals P_i, whose radical $\sqrt{P_i} := \mathfrak{p}_i$ is prime. Hence

$$Y = V(\mathscr{I}) = \bigcup_{i=1}^{M} V(P_i) = \bigcup_{i=1}^{M} V(\mathfrak{p}_i)$$

where the last equality follows from Nullstellensatz.

We claim that there exists i such that $f|_{V(\mathfrak{p}_i)} \not\equiv 0$. Otherwise, for every $1 \leqslant i \leqslant M$, one has $f \in \mathscr{I}(V(\mathfrak{p}_i),0) = \mathfrak{p}_i$ by Nullstellensatz. Therefore, there exists a positive integer k such that for all $1 \leqslant i \leqslant M$, one has $f^k \in P_i$, and hence $f^k \in \mathscr{I}$. Therefore,

$$X = V(\mathscr{I} + (f)) = V(\mathscr{I}) \cap V(f) = V(\mathscr{I}) \cap V(f^k) = V(\mathscr{I} + (f^k)) = V(\mathscr{I}) = Y,$$

which contradicts our assumption that $X \subsetneq Y$. The proof is complete. □

3.0.16 We have therefore arrived at one of the main results of this section.

Proposition 3.5. *Let $f \in \mathscr{O}_{\mathbb{C}^n,0}$ with $f(0) = 0$. Then there exists $k \in \mathbb{N}$ such that*

$$f^k \in \left\langle \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right\rangle.$$

Proof. Let $X = V(\partial_{z_1} f, \ldots, \partial_{z_n} f, f)$ and $Y = V(\partial_{z_1} f, \ldots, \partial_{z_n} f)$. If $X \subsetneq Y$, by previous proposition, there exists an irreducible curve $C \subseteq Y$ such that $f|_C \not\equiv 0$. Let

$$n : (\mathbb{C}, 0) \longrightarrow (C, 0)$$

be a local normalisation of C. Hence

$$\frac{df \circ n(\zeta)}{d\zeta} = \sum_{k=1}^{n} \frac{\partial f}{\partial z_k} \circ n(\zeta) n'(\zeta).$$

Since $n(\mathbb{C}, 0) \subseteq C$, so $(\partial_{z_k} f) \circ n(\zeta) \equiv 0$ for all $1 \leqslant k \leqslant n$. Therefore, $\frac{d}{d\zeta}(f \circ n) \equiv 0$, and hence $f \circ n$ is a constant on C. Since $f \circ n(0) = 0$, so $f \circ n$ vanishes identically on C, which contradicts our hypothesis that $f|_C \not\equiv 0$. Therefore, $X = Y$ and by Nullstellensatz, there exists an integer k such that $f^k \in \mathscr{I}$ and the proof is complete. □

3.1 Ideals Generated by Components of Gradients: Effective Aspects

3.1.1 In fact, the exponent k in the previous proposition may be taken to be $k = n = \dim \mathbb{C}^n$. We summarise some of the details in [JP96, p 59].

3.1.2

Definition 3.6. Let $\mathscr{I}$ be an ideal in $\mathscr{O}_{\mathbb{C}^n,0}$. The *integral closure* of $\mathscr{I}$, denoted by $\bar{\mathscr{I}}$, is the set of germs $u \in \mathscr{O}_{\mathbb{C}^n,0}$ such that there exist $d \in \mathbb{N}_{\geqslant 1}$ and $\alpha_s \in \mathscr{I}^s$ for $1 \leqslant s \leqslant d$ with:

$$u^d + a_1 u^{d-1} + \cdots + a_d = 0.$$

Definition 3.7. Let $\mathscr{I} = \langle F_1, \ldots, F_{\mathrm{N}} \rangle$ be an ideal of $\mathscr{O}_{\mathbb{C}^n,0}$ generated by N elements. Let $u \in \mathbb{R}_+$. The ideal $\overline{\mathscr{I}}^{(k)}$ is defined by

$$\overline{\mathscr{I}}^{(k)} = \{u \in \mathscr{O}_{\mathbb{C}^n,0} : \ |u| \leqslant C|F|^k\}$$

for some constants $C \geqslant 0$, and where $|F|^2 = |F_1|^2 + \cdots + |F_{\mathrm{N}}|^2$.

3.1.3 By [JP96, p 60, Proposition 12.2], for every k, l positive real numbers, $\overline{\mathscr{I}}^{(k)} \cdot \overline{\mathscr{I}}^{(l)} \subseteq \overline{\mathscr{I}}^{(k+l)}$. Moreover, $\overline{\mathscr{I}}^{(1)} = \overline{\mathscr{I}}$ which is the integral closure of the ideal $\mathscr{I}$ ([JP96, p 61, Corollary 12.5]).

3.1.4 By the Briançon-Skoda theorem (1974), if $p = \min\{n - 1, \mathrm{N} - 1\}$, then $\overline{\mathscr{I}}^{(k+p)} \subseteq \mathscr{I}^k$ for all $k \in \mathbb{N}$.

3.1.5 Let $f \in \mathscr{O}_{\mathbb{C}^n,0}$ be a holomorphic function germ at the origin, and let $\mathscr{I}_f$ and $J(f)$ denote the following ideals:

$$\begin{aligned} \mathscr{I}_f &:= \left\langle z_1\frac{\partial f}{\partial z_1}, \ldots, z_n\frac{\partial f}{\partial z_n} \right\rangle, \\ J(f) &:= \left\langle \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right\rangle. \end{aligned}$$

It is evident that $\mathscr{I}_f \subset J(f)$. Moreover, one has $f \in \overline{\mathscr{I}_f} = \mathscr{I}^{(1)}$ ([JP96, p 62, Corollary 12.6]). Therefore, by Paragraph 3.1.3, $f^{k+n-1} \in \overline{\mathscr{I}}^{(k+n-1)}$. By Briançon-Skoda theorem, for all $k \in \mathbb{N}_{\geqslant 1}$

$$f^{k+n-1} \in \mathscr{I}_f^k.$$

By setting $k = 1$, $f^n \in J(f)$. This completes the proof of the following proposition which is well-known in other literatures:

Proposition 3.8. *Let $f \in \mathscr{O}_{\mathbb{C}^n,0}$ with $f(0) = 0$. Then*

$$f^n \in \left\langle \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right\rangle.$$

3.2 Application

3.2.1 Let $F_1,\ldots,F_{\mathrm{N}}$ be holomorphic function germs in $\mathscr{O}_{\mathbb{C}^n,0}$ such that intersection multiplicity of the ideal $\langle F_1, \ldots, F_{\mathrm{N}}\rangle$ is finite with data (p, q, s). We will show that the ideal

$$\left\langle \frac{\partial F_i}{\partial z_j} : \ 1 \leqslant i \leqslant \mathrm{N},\ 1 \leqslant j \leqslant n \right\rangle$$

has an effectively bounded intersection multiplicity. More precisely,

Proposition 3.9. *For any $n \in \mathbb{N}_{\geqslant 1}$,let $F_1,\ldots,F_{\mathrm{N}}$ be holomorphic function germs in $\mathscr{O}_{\mathbb{C}^n,0}$ vanishing at the origin such that the ideal $\langle F_1, \ldots, F_{\mathrm{N}}\rangle$ has finite intersection multiplicity with data (p, q, s). Then*

$$\mathsf{dim}_{\mathbb{C}}\, \mathscr{O}_{\mathbb{C}^n,0}\Big/ \left\langle \frac{\partial F_i}{\partial z_j} : \ 1 \leqslant i \leqslant \mathrm{N},\ 1 \leqslant j \leqslant n \right\rangle \leqslant \binom{(n^2+2n)s+n-1}{(n^2+2n)s-1}.$$

Proof. By Proposition 3.8, for each $1 \leqslant i \leqslant \mathrm{N}$,

$$F_i^n \in \left\langle \frac{\partial F_i}{\partial z_1}, \ldots, \frac{\partial F_i}{\partial z_n} \right\rangle.$$

Evidently,

$$\langle F_1^n, \ldots, F_{\mathrm{N}}^n\rangle \subseteq \left\langle \frac{\partial F_i}{\partial z_j} : \ 1 \leqslant i \leqslant \mathrm{N},\ 1 \leqslant j \leqslant n \right\rangle.$$

As a result,

$$\mathsf{dim}_{\mathbb{C}}\, \mathscr{O}_{\mathbb{C}^n,0}\Big/ \left\langle \frac{\partial F_i}{\partial z_j} : \ 1 \leqslant i \leqslant \mathrm{N},\ 1 \leqslant j \leqslant n \right\rangle \leqslant \mathsf{dim}_{\mathbb{C}}\, \mathscr{O}_{\mathbb{C}^n,0}\big/\langle F_1^n, \ldots, F_{\mathrm{N}}^n\rangle.$$

It suffices to estimate the term on the right. By the hypothesis that

$$|z|^p \lesssim \sum_{i=1}^{\mathrm{N}} |F_i|,$$

Jensen's inequality yields

$$\begin{aligned} |z^{np}| &\lesssim \left(\sum_{i=1}^{\mathrm{N}} |F_i|\right)^n \\ &= \mathrm{N}^n \left(\sum_{i=1}^{\mathrm{N}} \frac{1}{\mathrm{N}} |F_i|\right)^n \\ &\leqslant \mathrm{N}^n \sum_{i=1}^{\mathrm{N}} \frac{1}{\mathrm{N}} |F_i|^n \\ &= \mathrm{N}^{n-1} \sum_{i=1}^{\mathrm{N}} |F_i|^n \lesssim \sum_{i=1}^{\mathrm{N}} |F_i^n|. \end{aligned}$$

By Theorem 2.7, the ideal $\langle F_1^n, \ldots, F_{\mathrm{N}}^n \rangle$ has finite intersection multiplicity with data (p', q', s'). By the definition of p', one has $p' \leqslant np$. At the same time, by Proposition 2.11,

$$s' \leqslant \binom{n+q'+1}{q'-1}.$$

Also by Proposition 2.11, using $p' \leqslant np$, the q' in the inequality above has a bound

$$q' \leqslant (n+2)p' \leqslant (n+2)np \leqslant (n+2)nq \leqslant (n+2)ns.$$

Hence

$$s' \leqslant \binom{n+(n+2)ns-1}{(n+2)ns-1}$$

and the proof is complete. □

3.2.2 Remark The estimate can be made more precise in $\mathrm{N} = n$. In this case by repeated application of Proposition 2.17 ,

$$\mathsf{dim}_{\mathbb{C}}\, \mathscr{O}_{\mathbb{C}^n,0} \big/ \langle F_1^n, \ldots, F_n^n \rangle = n^n \mathsf{dim}_{\mathbb{C}}\, \mathscr{O}_{\mathbb{C}^n,0} \big/ \langle F_1, \ldots, F_n \rangle = n^n s.$$

Therefore,

$$\mathsf{dim}_{\mathbb{C}}\, \mathscr{O}_{\mathbb{C}^n,0} \Big/ \left\langle \frac{\partial F_i}{\partial z_j} : \ 1 \leqslant i,j \leqslant n \right\rangle \leqslant n^n s.$$

4 Multiplicity of an ideal

4.0.3 Following [Chi89], we will present the notion of multiplicity of an ideal of holomorphic functions defining a pure dimensional variety. Let $F \in \mathscr{O}_{\mathbb{C}^n,0}$ be a holomorphic function germ. We may write F as an infinite sum of homogeneous polynomials

$$F = \sum_{k=m}^{\infty} F_k,$$

where each F_k is of degree k, with $F_m \not\equiv 0$.The multiplicity of F at 0 is then equal to m.

4.0.4 Another way to characterise multiplicity is to look at the order of vanishing of F along generic lines. Indeed, let

$$\begin{aligned} \varphi : \mathbb{C} &\longrightarrow \mathbb{C}^n \\ \zeta &\longmapsto (c_1\zeta, \ldots, c_n\zeta) \end{aligned}$$

be a parametrisation of a line. Composing φ with F gives

$$F \circ \varphi = \sum_{k=m}^{\infty} F_k(c_1\zeta, \ldots, c_n\zeta) = \sum_{k=m}^{\infty} \zeta^k F_k(c_1, \ldots, c_n).$$

Therefore, any vector $(c_1, \ldots, c_n)$ satisfying

$$F_m(c_1, \ldots, c_n) \neq 0$$

will imply that $\mathsf{ord}_0\ F \circ \varphi = m = \mathsf{mult}_0\ F$. Since every line is a complete intersection of $n-1$ hyperplanes H_1,...,H_{n-1}, the multiplicity of F is the intersection multiplicity of $V(F) = \{z \in \mathbb{C}^n : F(z) = 0\}$ with $n-1$ generic hyperplanes. In other words, if $H_i = \{z \in \mathbb{C}^n : L_i = 0\}$ for some linear function L_i, then

$$\mathsf{mult}_0 F = \mathsf{dim}_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^n,0} \big/ \langle F, L_1, \ldots, L_{n-1} \rangle.$$

4.0.5 More generally at 0, for $1 \leqslant q \leqslant n-1$, let $\mathscr{I}_F = \langle F_1, \ldots, F_q \rangle$ be the ideal generated by q holomorphic function germs and assume that it forms a *regular sequence*[3]. We would like to find a positive integer m analogous to the multiplicity of a function such that for $(n-q)$ generic hyperplanes,

$$m = \mathsf{dim}_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^n,0} \big/ \langle F_1, \ldots, F_q, L_1, \ldots, L_{n-q} \rangle.$$

The important point about F_1,...,F_q being a regular sequence is that the variety $V(\mathscr{I}_F)$ defined by the ideal $\mathscr{I}_F$ is of *pure* dimension $n-q$, due to the property of *Cohen-Macaulayness*. This allows us to apply the results in [Chi89, Chapter 2].

Definition 4.1 (Tangent Cones, [Chi89, p. 79])**.** Let E be an arbitrary set in $\mathbb{R}^n$. A vector $v \in \mathbb{R}^n$ is called *tangent* to E at a point of the closure $a \in \bar{E}$ if there exist a sequence of points $a_j \in E$ and positive numbers $t_j > 0$ such that $a_j \to a$ and

$$t_j(a_j - a) \to v \qquad (j \to \infty).$$

The set of all such tangent vectors at a is denoted by $C(E, a)$, and is called the *tangent cone to E at the point a*.

4.0.6 The set $C(E, a)$ is a cone with vertex 0: if $v \in C(E, a)$, then the vectors tv lie in $C(E, a)$ for all $t > 0$. Geometrically, the cone is a set of limit positions of secants of E passing through a.

4.0.7 If V is a pure one-dimensional analytic set in $\mathbb{C}^n$, the tangent cone at any $a \in V$ is a finite union of complex lines ([Chi89, p. 80, Corollary]).

4.0.8 In general, if $0 \in V$ is a pure analytic subset of a domain D in $\mathbb{C}^n$, then $C(V, 0)$ is a pure p-dimensional algebraic set in $\mathbb{C}^n$ (c.f. [Chi89, p. 83, Corollary]).

[3]Let R be a local ring. A sequence of non-units f_1,...,f_k is called a regular sequence if for all $1 \leqslant i \leqslant k$, the class f_i is not a zero divisor of $R/\langle f_1, \ldots, f_{i-1} \rangle$

4.0.9 We recall that if a $n-p$-dimensional variety V is defined by p holomorphic functions $F_1,\ldots,$ F_p which form a regular sequence, then V is a pure $(n-p)$-dimensional analytic variety.

4.1 Multiplicities of Analytic Sets

We refer the readers to [Chi89, p. 120] for more details.

4.1.1 Let V be a pure p dimensional analytic set in $\mathbb{C}^n$, and let $a \in V$. Let L be an $(n-p)$-dimensional complex subspace in $\mathbb{C}^n$, such that a is an isolated point of the set $V \cap (a+L)$. Then there is a domain $U \ni a$ in $\mathbb{C}^n$ of the form $U = U' \times U'' \subseteq \mathbb{C}^{n-p} \times \mathbb{C}^p$ such that $V \cap U \cap (a+L) = \{a\}$, and the projection

$$\pi_L : V \cap U \to U' \subseteq L^\perp$$

along L is a ramified k-sheeted analytic cover. This number k is the multiplicity of the projection $\pi_L|_V$ at a, denoted by $\mu_a(\pi_L|_V)$.

4.1.2 For simplicity, suppose that $0 \in V$ and $a = 0$ in the previous paragraph, the multiplicity of intersection of V with L is $\mu_0(\pi_L|_V)$. See [Chi89, p 139, Corollary] and [Chi89, p 140, Proposition 1].

Definition 4.2 (Multiplicity of an Analytic Set at a point)**.** Let V be a pure p-dimensional analytic set in $\mathbb{C}^n$ and let $a \in V$. For every $(n-p)$-dimensional plane L which contains the origin such that

$$V \cap (a+L) = \{a\},$$

the multiplicity of the projection $\mu_a(\pi_L|_V)$ is finite. The multiplicity of V at a is given by

$$\mu_a(V) := \mathsf{min}\{\mu_a(\pi_L|_V) : L \in G(n-p,n)\}.$$

Example 4.3. Suppose $V = \{F = 0\}$ is a principal analytic set in a neighbourhood of $0 \in \mathbb{C}^n$, and F is the minimum defining function for V. Write F

$$F = \sum_{k=\mathsf{ord}_0 F}^{\infty} F_k$$

as a sum of homogeneous polynomials F_k of degree k. Then by [Chi89, p. 83, Proposition 1],

$$C(V,0) = \{F_{\mathsf{ord}_0 F} = 0\}.$$

For any complex *line* L containing 0, by [Chi89, p. 121, Proposition 1],

$$\mu_0(\pi_L|_V) = \mathsf{ord}_0 F|_L \geqslant \mathsf{ord}_0 F,$$

with equality if and only if $L \cap C(V,0) = \{0\}$. In other words, if

$$\zeta \mapsto (c_1\zeta, \ldots, c_n\zeta)$$

is a parametrisation of the line L, then the line has trivial intersection with $C(V,0)$ if and only if

$$F_{\mathsf{ord}_0 F}(c_1,\ldots,c_n) \neq 0.$$

This agrees with our intuition in paragraph 4.0.4.

4.1.3 More generally,

Proposition 4.4. *Let V be a pure p-dimensional analytic set in a neighbourhood of $0 \in \mathbb{C}^n$, and let $L \in G(n-p, n)$. The equality $\mu_0(\pi_L|_V) = \mu_0(V)$ holds if and only if the plane L is transversal to V at 0. In other words,*

$$L \cap C(V, 0) = \{0\}.$$

Proof. See [Chi89, p. 122, Proposition 2]. □

4.1.4 Combining paragraphs 4.0.8, 4.0.9, 4.1.2, and Proposition 4.4, we obtain

Proposition 4.5. *Let $F_1, \ldots, F_p$ be holomorphic function germs at the origin so that $F_1(0) = \cdots = F_p(0) = 0$. Suppose that the sequence $F_1, \ldots, F_p$ is regular so that the variety defined by the intersection $V := V(F_1, \ldots, F_p)$ is a pure $(n-p)$-dimensional analytic variety. Then there exists an integer $\mu_0(V)$ such that for a generic choice of p hyperplanes given by the zeros of $n-p$ linear functions $L_1, \ldots, L_{n-p}$, one has*

$$\mathsf{dim}_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^n,0} \big/ \langle F_1, \ldots, F_p, L_1, \ldots, L_{n-p} \rangle = \mu_0(V).$$

4.2 Multiplicity of an Ideal – Case of a Curve

4.2.1 In this section, we will discuss more in depth of Proposition 4.5 in the case where $p = n-1$. In other words, the ideal $\mathscr{I}_F = \langle F_1, \ldots, F_{n-1} \rangle$ forms a regular sequence in $\mathscr{O}_{\mathbb{C}^n,0}$ so that the variety $V(F_1, \ldots, F_{n-1})$ is a pure 1-dimensional analytic variety, which is a union

$$V(F_1, \ldots, F_{n-1}) = \bigcup_{k=1}^{\mathrm{M}} Z_k$$

of its irreducible components Z_k.

4.2.2 For $1 \leqslant k \leqslant \mathrm{M}$, since each Z_k is an irreducible curve, there exists a parametrisation

$$\begin{array}{rcl} n_k : (\mathbb{C}, 0) & \to & (Z_k, 0) \\ \zeta & \mapsto & (\zeta^{\mu_{k1}} a_{k1}(\zeta), \ldots, \zeta^{\mu_{kn}} a_{kn}(\zeta)), \end{array}$$

where for all $1 \leqslant j \leqslant n$, $a_{kj}(0) \neq 0$ (c.f. [JP00, p 164, Theorem 4.4.8]) and [JP00, p 165, Theorem 4.4.10]).

Theorem 4.6. *There exist positive integers $m_1, \ldots, m_{\mathrm{M}}$ such that for any holomorphic function f with*

$$\mathsf{dim}_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^n,0} / \langle F_1, \ldots, F_{n-1}, f \rangle < \infty,$$

the equality holds

$$\mathsf{dim}_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^n,0} / \langle F_1, \ldots, F_{n-1}, f \rangle = \sum_{k=1}^{\mathrm{M}} m_k \, \mathsf{ord}_0 f \circ n_k.$$

Proof. See [D'A93, p 78, Theorem 3] for further discussion. □

4.2.3 We begin discussion with a small lemma.

Lemma 4.7. *Let Z_k be an irreducible 1-dimensional analytic variety and $n_k : (\mathbb{C}, 0) \to (Z_k, 0)$ be its normalisation. Let f be a holomorphic function germ vanishing at the origin such that $\mathsf{ord}_0 f \circ n_k$ is finite. Then the intersection $Z_k \cap \{f = 0\}$ is discrete, and hence the origin is an isolated point in the intersection.*

Proof. There is an equality of sets:

$$\{f = 0\} \cap Z_k = \{n_k(\zeta) : \ f \circ n_k(\zeta) = 0\}.$$

Now the set on the right is just simply $\{0\}$. This is because by hypothesis on the vanishing order of $f \circ n_k$,

$$f \circ n_k(\zeta) = \zeta^m g(\zeta),$$

where $g(0) \neq 0$ and $m = \mathsf{ord}_0 \ f \circ n_k < \infty$. Hence

$$f \circ n_k(\zeta) = 0 \implies \zeta = 0 \implies n_k(\zeta) = 0. \qquad \square$$

Proposition 4.8. *Let f be the holomorphic function such that for each $1 \leqslant k \leqslant \mathrm{M}$, the vanishing order $\mathsf{ord}_0 f \circ n_k$ is finite. Then the intersection multiplicity of the ideal $\langle F_1, \ldots, F_{n-1}, f\rangle$ is finite.*

Proof. The previous lemma implies that

$$Z_k \cap \{f = 0\} = \{0\}.$$

Hence

$$V(F_1, \ldots, F_{n-1}, f) = C \cap \{f = 0\} = \left(\bigcup_{k=1}^{\mathrm{M}} Z_k\right) \cap \{f = 0\} = \bigcup_{k=1}^{\mathrm{M}} (Z_k \cap \{f = 0\}) = \{0\}. \qquad \square$$

4.2.4 We are now in a position to prove the following lemma.

Proposition 4.9. *Let F_1,…,F_{n-1} be a regular sequence such that $V(F_1, \ldots, F_{n-1})$ is a pure 1-dimensional variety. For a generic choice of hyperplane defined by a linear function L,*

$$\mathsf{dim}_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^n,0}/\langle F_1, \ldots, F_{n-1}, L\rangle = \sum_{k=1}^{\mathrm{M}} m_k \ \mathsf{min}\{\mu_{k1}, \ldots, \mu_{kn}\}.$$

Proof. First of all, L may be written as

$$L = \sum_{j=1}^{n} c_j z_j \qquad (c_k \in \mathbb{C}).$$

Suppose that the intersection multiplicity of the ideal $\langle F_1, \ldots, F_{n-1}, L\rangle$ is finite, by Theorem 4.6,

$$\mathsf{dim}_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^n,0}/\langle F_1, \ldots, F_{n-1}, L\rangle = \sum_{k=1}^{\mathrm{M}} m_k \ \mathsf{ord}_0 L \circ n_k.$$

By Proposition 4.8, it suffices to choose an appropriate L such that $\mathsf{ord}\ L \circ n_k < \infty$. First, observe that

$$\begin{aligned} L \circ n_k &= \sum_{j=1}^{n} c_j \zeta^{\mu_{kj}} a_{kj}(\zeta) \\ &= \zeta^{\mathsf{min}\{\mu_{k1},\ldots,\mu_{kn}\}} \sum_{\{j:\mu_{kj}=\mathsf{min}\{\mu_{k1},\ldots,\mu_{kn}\}\}} c_j a_{kj}(\zeta) + O(\zeta^{\mathsf{min}\{\mu_{k1},\ldots,\mu_{kn}\}+1}). \end{aligned} \tag{4.10}$$

If

$$(c_1,\dots,c_n)\in\mathbb{C}^n-\bigcup_{k=1}^{\mathrm{M}}\left\{(d_1,\dots,d_n)\in\mathbb{C}^n:\ \sum_{j=1}^{n}d_ja_{kj}(0)=0\right\},$$

then by equation (4.10),

$$\mathsf{ord}_0L\circ n_k=\mathsf{min}\{\mu_{k1},\dots,\mu_{kn}\}<\infty.$$

This completes the proof. □

5 Generic Selection of Linear Combinations for Effective Termination

The following proposition appears in [Siu10, p 1190].

Proposition 5.1. *Let $0\leqslant q\leqslant n$, and $f_1,\dots,f_q$ be holomorphic function germs on $\mathbb{C}^n$ at the origin such that the common zero set $\{f_1=\cdots=f_q=0\}$ is a pure $(n-q)$-dimensional variety germ in $\mathbb{C}^n$ at the origin. Let m be the multiplicity of the ideal $\langle f_1,\dots,f_q\rangle$ in the sense that for any $(n-q)$ generic homogeneous linear functions $L_1,\dots,L_{n-q}$,*

$$\mathsf{dim}_{\mathbb{C}}\ \mathscr{O}_{\mathbb{C}^n,0}/\langle f_1,\dots,f_q,L_1,\dots,L_{n-q}\rangle=m.$$

Let $V(f_1,\dots,f_q,L_1,\dots,L_{n-q})$ be a pure 1 dimensional analytic variety and let

$$V(f_1,\dots,f_q,L_1,\dots,L_{n-q})=\bigcup_{k=1}^{\mathrm{M}}Z_k$$

be the irreducible decomposition of $V(f_1,\dots,f_q,L_1,\dots,L_{n-q})$. Let $F_1,\dots,F_{\mathrm{N}}$ be holomorphic function germs in $\mathscr{O}_{\mathbb{C}^n,0}$ vanishing at the origin and $p\geqslant 1$ be an integer such that

$$|z|^p\lesssim\sum_{i=1}^{\mathrm{N}}|F_i|.$$

Then there exist M hyperplanes, $H_1,\dots,H_{\mathrm{M}}$, in $\mathbb{C}^{\mathrm{N}}$ such that for any

$$(c_1,\dots,c_{\mathrm{N}})\in\mathbb{C}^{\mathrm{N}}-\bigcup_{i=1}^{\mathrm{M}}H_i,$$

and for any generic $(n-q-1)$ homogeneous linear functions $L_1,\dots,L_{n-q-1}$ the following inequality holds

$$\mathsf{dim}_{\mathbb{C}}\mathscr{O}_{\mathbb{C}^n,0}\Big/\left\langle f_1,\dots,f_q,\sum_{j=1}^{\mathrm{N}}c_jF_j,L_1,\dots,L_{n-q-1}\right\rangle\leqslant mp.$$

Proof. As in the statement of the proof, let

$$V(f_1,\dots,f_q,L_1,\dots,L_{n-q})=\bigcup_{k=1}^{\mathrm{M}}Z_k$$

be the irreducible decomposition of the pure 1-dimensional analytic variety, and let

$$n_k:(\mathbb{C},0)\longrightarrow(Z_k,0)$$

be normalisations of Z_k. By Theorem 4.6 and Proposition 4.8, there exist strictly positive integers $m_1,\ldots,m_{\mathrm{M}}$ such that for any holomorphic function germs $g \in \mathscr{O}_{\mathbb{C}^n,0}$ with $\mathsf{ord}_0\, g \circ n_k < \infty$ for all k,

$$\mathsf{dim}_{\mathbb{C}}\, \mathscr{O}_{\mathbb{C}^n,0}\big/\langle f_1,\ldots,f_q,g,L_1,\ldots,L_{n-q-1}\rangle = \sum_{k=1}^{\mathrm{M}} m_k\, \mathsf{ord}_0\, g\circ n_k.$$

It suffices to find suitable constants $(c_1,\ldots,c_{\mathrm{N}})$ such that the order of vanishing of the following function

$$g\circ n_k := \sum_{j=1}^{\mathrm{N}} c_j F_j \circ n_k$$

is finite for all k. For each fixed $1 \leqslant k \leqslant \mathrm{M}$, the map n_k may be explicitly written as

$$n_k : \zeta \longmapsto (\zeta^{\mu_{k,1}} a_{k,1}(\zeta),\ldots,\zeta^{\mu_{k,n}} a_{k,n}(\zeta)),$$

where for each $1 \leqslant l \leqslant n$, $a_{k,l}(0) \neq 0$. Let

$$s_k := \mathsf{min}\,\{\mu_{k,1},\ldots,\mu_{k,n}\} \qquad (1\leqslant k\leqslant \mathrm{M}).$$

Pulling back the inequality

$$|z|^p \lesssim \sum_{j=1}^{\mathrm{N}} |F_j|$$

by the normalisations give

$$\begin{aligned} |\zeta|^{s_k p} &\lesssim |(\zeta^{\mu_{k,1}} a_{k,1}(\zeta),\ldots,\zeta^{\mu_{k,n}} a_{k,n}(\zeta))|^p \\ &\leqslant A\sum_{j=1}^{\mathrm{N}} |F_j\circ n_k(\zeta)| \qquad (1\leqslant k\leqslant \mathrm{M}). \end{aligned}$$

Consequently, not all $F_j\circ n_k$ vanish at the same time. For any $F_j\circ n_k \not\equiv 0$, the one-variable holomorphic function may be expanded into power series

$$F_j\circ n_k = \sum_{l=t_{j,k}}^{\infty} F_{j,k,l}\zeta^l \qquad (1\leqslant j\leqslant \mathrm{N},\ 1\leqslant k\leqslant \mathrm{M}),$$

where $t_{j,k} = \mathsf{ord}_0\, F_j\circ n_k$ and $F_{j,k,l} \in \mathbb{C}$. By convention, $t_{j,k} = \infty$ if $F_j\circ n_k \equiv 0$. For a fixed $1 \leqslant k \leqslant \mathrm{M}$, let

$$t_k := \mathsf{min}\{t_{j,k} :\ 1\leqslant j\leqslant \mathrm{N},\ F_j\circ n_k \not\equiv 0\} < \infty.$$

Hence

$$|\zeta|^{s_k p} \lesssim |\zeta|^{t_k} \qquad (1\leqslant k\leqslant \mathrm{M}),$$

which implies that $t_k \leqslant s_k p$, since $|\zeta| \ll 1$. For any $(c_1,\ldots,c_{\mathrm{N}}) \in \mathbb{C}^{\mathrm{N}}$,

$$\begin{aligned} \sum_{j=1}^{\mathrm{N}} c_j F_j\circ n_k &= \sum_{j=1}^{\mathrm{N}}\sum_{l\geqslant t_{j,k}}^{\infty} c_j F_{j,k,l}\zeta^l \\ &= \left(\sum_{\{j:\ t_{j,k}=t_k\}} c_j F_{j,k,t_k}\right)\zeta^{t_k} + O(\zeta^{t_k+1}) \qquad (1\leqslant k\leqslant \mathrm{M}). \end{aligned}$$

Therefore, the order of vanishing of $\sum_{j=1}^{\mathrm{N}} c_j F_j\circ n_k$ is exactly t_k if $\sum_{\{j:t_{j,k}=t_k\}} c_j F_{j,k,t_k} \neq 0$. If

$$(c_1,\ldots,c_{\mathrm{N}}) \in \mathbb{C}^{\mathrm{N}} - \bigcup_{k=1}^{\mathrm{M}} \left\{(d_1,\ldots,d_{\mathrm{N}}) \in \mathbb{C}^{\mathrm{N}} :\ \sum_{\{j:\ t_{j,k}=t_k\}} d_j F_{k,j,t_k} = 0\right\}$$

which in the complement of the union of M hyperplanes, then for each $1 \leqslant k \leqslant \mathrm{M}$,

$$\mathsf{ord}_0 F \circ n_k = t_k \leqslant s_k p.$$

Consequently,

$$\begin{aligned}
&\mathsf{dim}_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^n,0} \Big/ \left\langle f_1, \ldots, f_q, \tilde{F}_1, \ldots, \tilde{F}_\nu, \sum_{j=1}^{\mathrm{N}} c_j F_j, L_1, \ldots, L_{n-q-\nu-1} \right\rangle \\
= &\sum_{k=1}^{\mathrm{M}} m_k \, \mathsf{ord}_0 \sum_{j=1}^{\mathrm{N}} c_j F_j \circ n_k \\
= &\sum_{k=1}^{\mathrm{M}} m_k t_k \leqslant \sum_{k=1}^{\mathrm{M}} m_k s_k p \\
&= p \sum_{k=1}^{\mathrm{M}} m_k s_k \\
&= p \sum_{k=1}^{\mathrm{M}} m_k \mathsf{min}\{\mu_{k,1}, \ldots, \mu_{k,n}\}.
\end{aligned}$$

By Proposition 4.9, the number $\sum_{k=1}^{\mathrm{M}} m_k \mathsf{min}\{\mu_{k,1}, \ldots, \mu_{k,n}\}$ is the intersection multiplicity of the curve $V(f_1, \ldots, f_q, L_1, \ldots, L_{n-q-1})$ with a generic hyperplane defined by $\{L_{n-q} = 0\}$. By hypothesis,

$$\sum_{k=1}^{\mathrm{M}} m_k \mathsf{min}\{\mu_{k,1}, \ldots, \mu_{k,n}\} = m$$

and this completes the proof. □

5.0.5 In dimension 2

We will state the corollary of Proposition 5.1 in the case of dimension 2.

Corollary 5.2. *Let $F_1, \ldots, F_{\mathrm{N}}$ be holomorphic functions in $\mathscr{O}_{\mathbb{C}^2,0}$ such that the ideal $\mathscr{I}_F = \langle F_1, \ldots, F_{\mathrm{N}} \rangle$ has finite intersection multiplicity with data (p, q, s). Then there exist generic constants $(c_1, \ldots, c_{\mathrm{N}}) \in \mathbb{C}^{\mathrm{N}}$ such that*

$$\mathsf{mult}_0 \left(\sum_{j=1}^{\mathrm{N}} c_j F_j \right) \leqslant q \leqslant 4p.$$

Moreover, let $V(\tilde{F}_1) = \cup_{k=1}^{\mathrm{M}} Z_k$ be the irreducible decomposition of the variety. Then there exist M *hyperplanes $H_1, \ldots, H_{\mathrm{M}}$ in $\mathbb{C}^{\mathrm{N}}$ such that for all $(d_1, \ldots, d_{\mathrm{N}}) \in \mathbb{C}^{\mathrm{N}} - \cup_{i=1}^{\mathrm{M}} H_i$,*

$$\mathsf{dim}_{\mathbb{C}} \, \mathscr{O}_{\mathbb{C}^n,0} \Big/ \left\langle \sum_{j=1}^{\mathrm{N}} c_j F_j, \, \sum_{j=1}^{\mathrm{N}} d_j F_j \right\rangle \leqslant 4p^2 \leqslant 4s^2.$$

Proof. First, there exists $1 \leqslant i \leqslant \mathrm{N}$ such that $\mathsf{mult}_0 \, F_i \leqslant q$. Otherwise, if $\mathsf{mult}_0 \, F_i \geqslant q+1$ for every $1 \leqslant i \leqslant \mathrm{N}$, then

$$\mathfrak{m}^q \subseteq \langle F_1, \ldots, F_{\mathrm{N}} \rangle \subseteq \mathfrak{m}^{q+1},$$

which is a contradiction. So let $(c_1, \ldots, c_{\mathrm{N}})$ be constants so that

$$\mathsf{mult}_0 \left(\sum_{j=1}^{\mathrm{N}} c_j F_j \right) \leqslant q \leqslant 4p,$$

where the last inequality follows from Proposition 2.11. Then the existence of M hyperplanes in $\mathbb{C}^{\mathrm{N}}$ and constants $(d_1, \ldots, d_{\mathrm{N}})$ so that the conclusion holds follow directly from the previous propostion, and the proof is complete. □

6 Proper maps and projections

6.0.6 In this section, let $h_1,\ldots,h_n$ be holomorphic function germs in $\mathscr{O}_{\mathbb{C}^n,0}$ vanishing at the origin with

$$\mathsf{dim}_{\mathbb{C}}\mathscr{O}_{\mathbb{C}^n,0}\big/\langle h_1,\ldots,h_{n-1}\rangle =: s < \infty.$$

Hence the $(n-1)$-tuple $(h_1,\ldots,h_{n-1})$ forms a regular sequence. By Proposition 4.9, and by a suitable linear change of coordinates, there exists a positive integer m such that

$$\mathsf{dim}_{\mathbb{C}}\mathscr{O}_{\mathbb{C}^n,0}\big/\langle h_1,\ldots,h_{n-1},z_n\rangle =: m$$

which is the multiplicity of the ideal $\langle h_1,\ldots,h_{n-1}\rangle$.

6.0.7 The map

$$\begin{array}{rcl}\varphi : (\mathbb{C}^n,0) & \longrightarrow & (\mathbb{C}^n,0)\\ (z_1,\ldots,z_n) & \longmapsto & (h_1(z),\ldots,h_{n-1}(z),z_n) =: (w_1,\ldots,w_{n-1},w_n)\end{array}$$

is proper and open with finite fibres. Let $H=\{h_n=0\}$ be the hypersurface defined by the zeros of h_n. By Remmert's proper mapping theorem[4], the image $\varphi(H)$ is also an analytic set. Since the map restricted to the hypersurface H:

$$\varphi|_H : H \longrightarrow \varphi(H)$$

is surjective with finite fibres, by section 4, paragraph 3.0.12 (or [JP00, p 129, Lemma 4.1.4]), one has $\mathsf{dim}\ H = \mathsf{dim}\ \varphi(H) = n-1$.

6.0.8 Since $\varphi(H)$ is of dimension $n-1$, it is a hypersurface locally defined at the origin by a certain holomorphic function $\tilde{h}_n(w_1,\ldots,w_n)$, which will be shown to have the following properties:
(i) $\tilde{h}_n(0,\ldots,0,w_n)\not\equiv 0$ with certain order of vanishing $\lambda := \mathsf{ord}_0(\tilde{h}_n(0,z_n))$. By the Weierstrass Preparation Theorem, $\tilde{h}_n$ may be expressed as

$$\tilde{h}_n(w) = u(w)\left(w_n^{\lambda}+\sum_{j=0}^{\lambda-1}a_j(w_1,\ldots,w_{n-1})w_n^j\right),$$

for some unit $u(w)$, and $a_j(0)=0$ for all $0\leqslant j\leqslant \lambda-1$.
(ii) $\lambda\leqslant s$.

6.0.9

Lemma 6.1. *Let $h_1,\ldots,h_n$ be holomorphic function germs in $\mathscr{O}_{\mathbb{C}^n,0}$ vanishing at the origin such that the intersection multiplicity of the ideal $\langle h_1,\ldots,h_n\rangle$ is finite with data (p,q,s). Let $H:=\{h_n=0\}$ be the hypersurface defined as the vanishing locus of h_n. Consider the map:*

$$\begin{array}{rcl}\psi : H & \longrightarrow & \mathbb{C}^{n-1}\\ z:=(z_1,\ldots,z_n) & \longmapsto & (h_1(z),\ldots,h_{n-1}(z)).\end{array}$$

Then there exists a open neighbourhood $U\subseteq\mathbb{C}^{n-1}$ of the origin $0\in\mathbb{C}^{n-1}$ such that for every $\alpha:=(\alpha_1,\ldots,\alpha_{n-1})\in U$, there are at most s distinct elements in $\psi^{-1}(\alpha)$.

[4]Remmert's proper mapping theorem may be stated as follows: if M and N are complex manifolds, $f: M\to N$ a holomorphic map and $V\subset M$ an analytic variety such that $f|_V$ is proper, then $f(V)$ is an analytic subvariety of N.

Proof. We prove by contradiction. Suppose for every open neighbourhood $U \subseteq \mathbb{C}^{n-1}$ of the origin $0 \in \mathbb{C}^{n-1}$, there exists a point $\alpha \in U$ such that the number of distinct elements in $\psi^{-1}(\alpha)$ is at least $s+1$.

By hypothesis, the map

$$\begin{array}{rcl} \Psi : \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ z & \longmapsto & (h_1(z), \ldots, h_n(z)) \end{array}$$

is a ramified s-sheeted analytic covering map. Hence, there exists a neighbourhood $V = V' \times V'' \subseteq \mathbb{C}^{n-1} \times \mathbb{C}$ of the origin $0 \in \mathbb{C}^n$ such that for every $\beta := (\beta_1, \ldots, \beta_n) \in V$, the number of distinct points in $\Psi^{-1}(\beta)$ is at most s.

But by our assumption, given V' a neighbourhood of the origin $0 \in \mathbb{C}^{n-1}$, there exists a point $\alpha := (\alpha_1, \ldots, \alpha_{n-1}) \in V'$ such that there are at least $s+1$ distinct points in $\psi^{-1}(\alpha)$. Since $(\alpha, 0) \in V$ and

$$\Psi^{-1}(\alpha, 0) = \psi^{-1}(\alpha),$$

there are at least $s+1$ distinct points in $\Psi^{-1}(\alpha, 0)$, which is a contradiction. □

6.0.10 We will therefore answer the first claim in paragraph 3.

Proposition 6.2. *Let $h_1, \ldots, h_n$ be holomorphic function germs in $\mathscr{O}_{\mathbb{C}^n,0}$ vanishing at the origin such that the multiplicity of the ideal $\langle h_1, \ldots, h_n \rangle$ is $s \in \mathbb{N}_{\geqslant 1}$. Suppose that the holomorphic map*

$$\begin{array}{rcl} \varphi : \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ (z_1, \ldots, z_n) & \longmapsto & (h_1(z), \ldots, h_{n-1}(z), z_n) \end{array}$$

defines a ramified k-sheeted covering for some positive integer k. Let $\tilde{h}_n$ be a holomorphic function such that $\varphi(\{h_n = 0\}) = \{\tilde{h}_n = 0\}$. Then $\tilde{h}_n(0, z_n) \not\equiv 0$.

Proof. Suppose on the contrary that $\tilde{h}_n(0, z_n) \equiv 0$. Consider the composition of maps

$$\begin{array}{ccccc} H & \xrightarrow{\varphi} & \varphi(H) & \xrightarrow{\mathsf{proj}} & \mathbb{C}^{n-1} \\ (z_1, \ldots, z_n) & \longmapsto & (h_1(z), \ldots, h_{n-1}(z), z_n) & & \\ & & (w_1, \ldots, w_n) & \longmapsto & (w_1, \ldots, w_{n-1}). \end{array}$$

Here φ is the map in the statement of the proposition and proj is the projection onto the first $n-1$ coordinates. Above $0 \in \mathbb{C}^{n-1}$, since $\tilde{h}_n(0, z_n) \equiv 0$,

$$\{(0, z_n) \in \mathbb{C}^n : \ z_n \in \mathbb{C}\} \subseteq \{\tilde{h}_n = 0\} = \varphi(H).$$

Moreover, since $\mathsf{proj}(0, z_n) = 0 \in \mathbb{C}^{n-1}$,

$$\{(0, z_n) \in \mathbb{C}^n : \ z_n \in \mathbb{C}\} \subseteq \mathsf{proj}^{-1}(0).$$

Therefore, $\mathsf{proj}^{-1}(0)$ has infinitely many distinct fibre points. Consequently, $(\mathsf{proj} \circ \varphi)^{-1}(0)$ has infinitely many distinct fibre points. But $\mathsf{proj} \circ \varphi = \psi$ in the previous lemma, has finite distinct fibres, contradiction. □

6.0.11 Next, we will show that $\mathsf{ord}_0\ \tilde{h}_n(0, z_n) \leqslant s$.

Lemma 6.3. *Let $\tilde{h}_n$ be a holomorphic function germ in $\mathscr{O}_{\mathbb{C}^n,0}$ with $\tilde{h}_n(0) = 0$, and $\tilde{h}_n(0, z_n) \not\equiv 0$ so that $\mathsf{ord}_0\ \tilde{h}_n(0, z_n) < \infty$. If the projection*

$$\begin{array}{rcl} \pi : \{\tilde{h}_n = 0\} & \longrightarrow & (\mathbb{C}^{n-1}, 0) \\ (\alpha_1, \ldots, \alpha_n) & \longmapsto & (\alpha_1, \ldots, \alpha_{n-1}) \end{array}$$

is a finite surjective map with at most s distinct fibre points above each point in $(\mathbb{C}^{n-1}, 0)$, then $\mathsf{ord}_0\ \tilde{h}_n(0, z_n) \leqslant s$.

Proof. Suppose on the contrary that $\lambda := \mathsf{ord}_0\ \tilde{h}_n(0, z_n) \geqslant s + 1$. By the hypothesis that $\lambda < \infty$, Weierstrass Preparation Theorem implies the existence of a unit $u(z_1, \ldots, z_n)$ and $\mathsf{ord}_0\ \tilde{h}_n(0, z_n)$ holomorphic functions $a_j(z_1, \ldots, z_{n-1})$ vanishing at $(z_1, \ldots, z_{n-1}) = (0, \ldots, 0)$ such that

$$h(z_1, \ldots, z_n) = u(z_1, \ldots, z_n) \left(z_n^\lambda + \sum_{j=0}^{\lambda-1} a_j(z_1, \ldots, z_{n-1})\ z_n^j \right).$$

Therefore, above a generic point $(\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{C}^{n-1}$, the preimages $(\alpha_1, \ldots, \alpha_{n-1}, z_n)$ of π which must satisfy the following polynomial equation

$$z_n^\lambda + \sum_{j=0}^{\lambda-1} a_j(\alpha_1, \ldots, \alpha_{n-1})\ z_n^j = 0$$

has $\lambda \geqslant s + 1$ distinct solutions in z_n. This contradicts the hypothesis in the statement of the lemma. □

Proposition 6.4. *Let h_1,…,h_n be holomorphic function germs in $\mathscr{O}_{\mathbb{C}^n,0}$ vanishing at the origin such that*

$$\mathsf{dim}_{\mathbb{C}}\ \mathscr{O}_{\mathbb{C}^n,0} \big/ \langle h_1, \ldots, h_n \rangle = s < \infty.$$

Let $H = \{h_n = 0\}$. Suppose that the holomorphic map

$$\begin{array}{rcl} \varphi : \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ (z_1, \ldots, z_n) & \longmapsto & (h_1(z), \ldots, h_{n-1}(z), z_n) \end{array}$$

is proper, open so that there exists a holomorphic function $\tilde{h}_n(w_1, \ldots, w_n)$ with $\varphi(H) = \{\tilde{h}_n = 0\}$. Then $\mathsf{ord}_0\ \tilde{h}_n(0, \ldots, 0, w_n) \leqslant s$.

Proof. Consider the map

$$\begin{array}{ccccc} H & \xrightarrow{\varphi|_H} & \varphi(H) & \xrightarrow{\mathsf{proj}} & \mathbb{C}^{n-1} \\ (z_1, \ldots, z_n) & \longmapsto & (h_1(z), \ldots, h_{n-1}(z), z_n) & & \\ & & (w_1, \ldots, w_n) & \longmapsto & (w_1, \ldots, w_{n-1}). \end{array}$$

By lemma 6.1, there exists a neighbourhood U of the origin $0 \in \mathbb{C}^{n-1}$ such that for all $\alpha \in U$, there are at most s distinct points in $\psi^{-1}(\alpha) = (\mathsf{proj} \circ \varphi|_H)^{-1}(\alpha)$. Choose a generic point $\alpha \in \mathbb{C}^{n-1}$ as in the lemma 6.3. Therefore, above α, there are $\mathsf{ord}_0\ \tilde{h}_n(0, w_n)$ distinct fibre points in $\mathsf{proj}^{-1}(\alpha)$, and hence

$$\begin{array}{rl} & \mathsf{ord}_0\ \tilde{h}_n(0, w_n) \\ = & \text{number of distinct points in } \mathsf{proj}^{-1}(\alpha) \\ \leqslant & \text{number of distinct points in } (\mathsf{proj} \circ \varphi|_H)^{-1}(\alpha) \leqslant s. \quad \square \end{array}$$

7 Calculation of Explicit ε in Dimension 2 (Preliminaries)

7.0.12 In this section we will use some of the results in the earlier sections to establish some preliminary results for the calculation of explicit ε in the case of dimension 2.

7.0.13 Let $F_1,\ldots,F_{\mathrm{N}}$ be holomorphic function germs in $\mathscr{O}_{\mathbb{C}^2,0}$ vanishing at the origin such that the ideal they generate $\langle F_1,\ldots,F_{\mathrm{N}}\rangle$ has finite intersection multiplicity with data (p,q,s).

7.1 Ideal Generated by Gradient and Generic Selection in Dimension 2

7.1.1 In $\mathbb{C}^2$, Proposition 3.9 implies that

$$\dim_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^2,0}\Big/\left\langle \frac{\partial F_i}{\partial z_j} :\ 1\leqslant i\leqslant \mathrm{N},\ 1\leqslant j\leqslant 2\right\rangle \leqslant \binom{8s+1}{8s-1}.$$

Moreover, if $\mathrm{N}=2$, there is a better upper bound

$$\dim_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^2,0}\Big/\left\langle \frac{\partial F_i}{\partial z_j} :\ 1\leqslant i,j\leqslant 2\right\rangle \leqslant 4s.$$

7.1.2 Let $\tilde{h}_2$ be any holomorphic function germ in $\mathscr{O}_{\mathbb{C}^2,0}$ with multiplicity $\tilde{m}_2$. Let

$$V(\tilde{h}_2)=\bigcup_{k=1}^{\tilde{r}} Z_k$$

be the irreducible decomposition of the pure 1-dimensional analytic variety. By Proposition 5.1, there exist $\tilde{r}$ hyperplanes $H_1,\ldots,H_{\tilde{r}}$ in $\mathbb{C}^{2\mathrm{N}}$ such that for all

$$(\lambda_1,\ldots,\lambda_{\mathrm{N}},\theta_1,\ldots,\theta_{\mathrm{N}})\in\mathbb{C}^{2\mathrm{N}}-\bigcup_{i=1}^{\tilde{r}} H_i,$$

there is an effective upper bound on the intersection multiplicity

$$\dim_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^2,0}\Big/\left\langle \tilde{h}_2,\ \sum_{j=1}^{\mathrm{N}}\lambda_j\frac{\partial F_j}{\partial z_1}+\theta_j\frac{\partial F_j}{\partial z_2}\right\rangle \leqslant \tilde{m}_2\binom{8s+1}{8s-1}.$$

7.1.3

Lemma 7.1. *Let $\tilde{h}_2$ be any holomorphic function germ in $\mathscr{O}_{\mathbb{C}^2,0}$ that vanishes at the origin, whose multiplicity is $\tilde{m}_2$. Suppose that the vanishing locus $\{\tilde{h}_2=0\}$ is a union of $\tilde{r}$ irreducible components (not counting multiplicity). Then there exist $2\tilde{r}$ hyperplanes $H_1,\ldots,H_{2\tilde{r}}$ in $\mathbb{C}^{\mathrm{N}}$ so that whenever*

$$(c_1,\ldots,c_{\mathrm{N}})\in\mathbb{C}^{\mathrm{N}}-\bigcup_{k=1}^{2\tilde{r}} H_k$$

there are $\tilde{r}$ hyperplanes $\tilde{H}_1,\ldots,\tilde{H}_{\tilde{r}}$ in $\mathbb{C}^2$ such that if

$$(\alpha,\gamma)\in\mathbb{C}^2-\bigcup_{k=1}^{\tilde{r}}\tilde{H}_k,$$

then it holds that

$$\dim_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^2,0}\Big/\left\langle \tilde{h}_2,\ \sum_{j=1}^{\mathrm{N}}c_j\alpha\frac{\partial F_j}{\partial z_1}+c_j\gamma\frac{\partial F_j}{\partial z_2}\right\rangle \leqslant \tilde{m}_2\binom{8s+1}{8s-1}.$$

Proof. By paragraph 7.1.1, the ideal

$$\left\langle \frac{\partial F_i}{\partial z_j} : \ 1 \leqslant i \leqslant \mathrm{N}, \ 1 \leqslant j \leqslant 2 \right\rangle$$

has finite intersection multiplicity with data (p', q', s').

By Proposition 5.1, there exist $\tilde{r}$ hyperplanes in $\mathbb{C}^{2\mathrm{N}}$ of the form

$$H'_l = \left\{ (v_1, \ldots, v_{\mathrm{N}}, w_1, \ldots, w_{\mathrm{N}}) \in \mathbb{C}^{2\mathrm{N}} : \ \sum_{k=1}^{\mathrm{N}} \sigma_{lk} v_k + \mu_{lk} w_k = 0 \right\} \qquad (1 \leqslant l \leqslant \tilde{r}),$$

such that if $(v_1, \ldots, v_{\mathrm{N}}, w_1, \ldots, w_{\mathrm{N}}) \in \mathbb{C}^{2\mathrm{N}} - \cup_{l=1}^{\tilde{r}} H'_l$, then

$$\mathsf{dim}_{\mathbb{C}} \, \mathscr{O}_{\mathbb{C}^2,0} \Big/ \left\langle \tilde{h}_2, \ \sum_{k=1}^{2\mathrm{N}} v_k \frac{\partial F_k}{\partial z_1} + w_k \frac{\partial F_k}{\partial z_2} \right\rangle \leqslant \tilde{m}_2 \binom{8s+1}{8s-1}.$$

To conclude the proof, it suffices to choose $(c_1\alpha, \ldots, c_{\mathrm{N}}\alpha, c_1\gamma, \ldots, c_{\mathrm{N}}\gamma) \in \mathbb{C}^{2\mathrm{N}} - \cup_{l=1}^{\tilde{r}} H'_l$, or equivalently for every $1 \leqslant l \leqslant \tilde{r}$,

$$\sum_{k=1}^{\mathrm{N}} \sigma_{lk} c_k \alpha + \mu_{lk} c_k \gamma \neq 0. \tag{7.2}$$

To this aim, write

$$\sum_{k=1}^{\mathrm{N}} \sigma_{lk} c_k \alpha + \mu_{lk} c_k \gamma = \left(\sum_{k=1}^{\mathrm{N}} \sigma_{lk} c_k \right) \alpha + \left(\sum_{k=1}^{\mathrm{N}} \mu_{lk} c_k \right) \gamma. \tag{7.3}$$

If

$$\begin{aligned} (c_1, \ldots, c_{\mathrm{N}}) \in \mathbb{C}^{\mathrm{N}} \ &- \ \bigcup_{l=1}^{\tilde{r}} \left\{ (d_1, \ldots, d_{\mathrm{N}}) \in \mathbb{C}^{\mathrm{N}} : \ \sum_{k=1}^{\mathrm{N}} \sigma_{lk} d_k = 0 \right\} \\ &- \ \bigcup_{l=1}^{\tilde{r}} \left\{ (d_1, \ldots, d_{\mathrm{N}}) \in \mathbb{C}^{\mathrm{N}} : \ \sum_{k=1}^{\mathrm{N}} \mu_{lk} d_k = 0 \right\}, \end{aligned}$$

which is in a complement of $2\tilde{r}$ hyperplanes, the coefficients of α and γ in the equation (7.3) do not vanish. Once $(c_1, \ldots, c_{\mathrm{N}})$ is chosen, if

$$(\alpha, \gamma) \in \mathbb{C}^2 - \bigcup_{l=1}^{\tilde{r}} \left\{ \left(\sum_{k=1}^{\mathrm{N}} \sigma_{lk} c_k \right) \alpha + \left(\sum_{k=1}^{\mathrm{N}} \mu_{lk} c_k \right) \gamma = 0 \right\},$$

which lies in the complement of $\tilde{r}$ hyperplanes in $\mathbb{C}^2$, then equation (7.2) holds. Hence the proof is complete. □

7.1.4

Proposition 7.4. *Let $(z_1, z_2) \in \mathbb{C}^2$ be holomorphic coordinates in $\mathbb{C}^2$. Let $\tilde{h}_2$ be a holomorphic function germ in $\mathscr{O}_{\mathbb{C}^2,0}$ vanishing at the origin with multiplicity $\tilde{m}_2$, and suppose that its vanishing locus $\{\tilde{h}_2 = 0\}$ has $\tilde{r}$ irreducible components (not counting multiplicity).*

Let $F_1, \ldots, F_{\mathrm{N}}$ be holomorphic function germs which generate an ideal $\langle F_1, \ldots, F_{\mathrm{N}} \rangle$ having finite intersection multiplicity with data (p, q, s).

Let

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

be an invertible linear change of coordinates. Then there are $3\tilde{r}$ hyperplanes $H_1,\ldots,H_{3\tilde{r}}$ in $\mathbb{C}^{\mathrm{N}}$ such that for each $(c_1,\ldots,c_{\mathrm{N}}) \in \mathbb{C}^{\mathrm{N}} - \cup_{k=1}^{3\tilde{r}} H_k$, there exist $\tilde{r}$ hyperplanes $\tilde{H}_1,\ldots,\tilde{H}_{\tilde{r}}$ and a hypersurface defined by a homogeneous polynomial $\{P=0\}$ such that whenever

$$(\alpha,\gamma) \in \mathbb{C}^2 - \bigcup_{k=1}^{\tilde{r}} \tilde{H}_k - \{P=0\},$$

the linear combination

$$h_1(z_1,z_2) = \sum_{j=1}^{\mathrm{N}} c_j F_j(z_1,z_2)$$

will satisfies the following conditions:

(i) *the intersection multiplicity of the ideal $\langle h_1, \tilde{h}_2\rangle$ has an effective bound:*

$$\mathsf{dim}_{\mathbb{C}}\, \mathscr{O}_{\mathbb{C}^2,0}\Big/\langle h_1,\tilde{h}_2\rangle \leqslant \tilde{m}_2 p \leqslant \tilde{m}_2 s;$$

(ii) *in the new coordinates (w_1,w_2),*

$$\begin{aligned} & \mathsf{dim}_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^2,0}\Big/\left\langle \tilde{h}_2(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2),\ \frac{\partial h_1(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2)}{\partial w_1}\right\rangle \\ \leqslant\ & \tilde{m}_2 \binom{8s+1}{8s-1}; \end{aligned}$$

(iii) *the holomorphic map induced from the change of coordinates*

$$\begin{aligned} \varphi : \mathbb{C}^2 &\longrightarrow \mathbb{C}^2 \\ (w_1,w_2) &\longmapsto \big(h_1(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2), w_2\big) \end{aligned}$$

is a covering map with finite fibres.

Proof. **(i)** By Proposition 5.1, there exist $\tilde{r}$ hyperplanes $H_1,\ldots,H_{\tilde{r}}$ in $\mathbb{C}^{\mathrm{N}}$ so that for all

$$(c_1,\ldots,c_{\mathrm{N}}) \in \mathbb{C}^{\mathrm{N}} - \bigcup_{k=1}^{\tilde{r}} H_k,$$

one has (in variables (z_1,z_2))

$$\mathsf{dim}_{\mathbb{C}}\, \mathscr{O}_{\mathbb{C}^2,0}\Big/\left\langle \tilde{h}_2,\ \sum_{j=1}^{\mathrm{N}} c_j F_j\right\rangle \leqslant \tilde{m}_2 p \leqslant \tilde{m}_2 s.$$

This satisfies the first condition, which remains unchanged even after a linear change of coordinates $(z_1,z_2) \leftrightarrow (w_1,w_2)$.

(ii) After a change of variables,

$$\begin{aligned} \frac{\partial h_1(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2)}{\partial w_1} &= \frac{\partial h_1(z_1,z_2)}{\partial z_1}\frac{\partial z_1}{\partial w_1} + \frac{\partial h_1(z_1,z_2)}{\partial z_2}\frac{\partial z_2}{\partial w_1} \\ &= \alpha\frac{\partial h_1(z_1,z_2)}{\partial z_1} + \gamma\frac{\partial h_1(z_1,z_2)}{\partial z_2} \\ &= \sum_{j=1}^{\mathrm{N}} c_j\alpha\frac{\partial F_j(z_1,z_2)}{\partial z_1} + c_j\gamma\frac{\partial F_j(z_1,z_2)}{\partial z_2}. \end{aligned}$$

By Lemma 7.1, there exist $2\tilde{r}$ hyperplanes $H_{\tilde{r}+1},\ldots,H_{3\tilde{r}}$ in $\mathbb{C}^{\mathrm{N}}$ such that whenever

$$(c_1,\ldots,c_{\mathrm{N}}) \in \mathbb{C}^{\mathrm{N}} - \bigcup_{k=\tilde{r}+1}^{3\tilde{r}} H_k,$$

there are $\tilde{r}$ hyperplanes $\tilde{H}_1,\ldots,\tilde{H}_{\tilde{r}}$ in $\mathbb{C}^2$ so that if

$$(\alpha,\gamma) \in \mathbb{C}^2 - \bigcup_{k=1}^{\tilde{r}} \tilde{H}_k,$$

then

$$\mathsf{dim}_{\mathbb{C}}\ \mathscr{O}_{\mathbb{C}^2,0}\Big/\left\langle \tilde{h}_2(z_1,z_2),\ \sum_{j=1}^{\mathrm{N}} c_j\alpha\frac{\partial F_j(z_1,z_2)}{\partial z_1} + c_j\gamma\frac{\partial F_j(z_1,z_2)}{\partial z_2}\right\rangle \leqslant \tilde{m}_2\binom{8s+1}{8s-1},$$

or in other words,

$$\mathsf{dim}_{\mathbb{C}}\ \mathscr{O}_{\mathbb{C}^2,0}\Big/\left\langle \tilde{h}_2(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2),\ \frac{\partial h_1(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2)}{\partial w_1}\right\rangle \leqslant \tilde{m}_2\binom{8s+1}{8s-1},$$

and hence the second condition is attained.

(iii) For the last condition, in order for φ to be a covering map with finite fibres, it suffices to find $(\alpha,\gamma)\in\mathbb{C}^2$ so that the holomorphic function of one variable

$$h_1(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2)|_{\{w_2=0\}} = h_1(\alpha w_1,\gamma w_1)$$

has a finite order of vanishing at $w_1=0$. To this effect, $h_1(z_1,z_2)$ may be written as an infinite sum

$$h_1(z_1,z_2) = P_{m_1} + \sum_{k\geqslant m_1+1} P_k$$

of homogeneous polynomials P_k of degree k, with $m_1 = \mathsf{mult}_0\, h_1$. Hence,

$$\begin{aligned} h_1(\alpha w_1,\gamma w_1) &= P_{m_1}(\alpha w_1,\gamma w_1) + O(w_1^{m_1+1}) \\ &= w_1^{m_1}P_m(\alpha,\gamma) + O(w_1^{m_1+1}). \end{aligned}$$

If $(\alpha,\gamma)\in\mathbb{C}^2-\{P_{m_1}=0\}$, then $h_1(\alpha w_1,\gamma w_1)\not\equiv 0$ and hence φ defines a ramified m_1-sheeted analytic covering.

In summary, there exist $3\tilde{r}$ hyperplanes $H_1,\ldots,H_{3\tilde{r}}$ in $\mathbb{C}^{\mathrm{N}}$ so that for every

$$(c_1,\ldots,c_{\mathrm{N}}) \in \mathbb{C}^{\mathrm{N}} - \bigcup_{k=1}^{3\tilde{r}} H_k,$$

there are $\tilde{r}$ hyperplanes $\tilde{H}_1,\ldots,\tilde{H}_{\tilde{r}}$ and a hypersurface $\{P_{m_1}=0\}$ in $\mathbb{C}^2$ such that whenever

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

is an invertible linear change of coordinate satisfying

$$(\alpha,\gamma) \in \mathbb{C}^2 - \bigcup_{k=1}^{\tilde{r}} \tilde{H}_k - \{P_{m_1}=0\},$$

the three conditions **(i)**, **(ii)**, **(iii)** are satisfied. □

8 Calculation of Explicit ε in Dimension 2

8.0.5 As before, we work in $\mathbb{C}^2$. Let $F_1,\ldots,F_N$ be holomorphic function germs in $\mathscr{O}_{\mathbb{C}^2,0}$ vanishing at the origin whose ideal $\mathscr{I}_F = \langle F_1,\ldots,F_N\rangle$ has finite intersection multiplicity with data (p,q,s).

8.0.6 By [Siu10, p 1182], one has for all $\phi \in \mathscr{D}_{0,1}(\Omega)$ with compact support that

$$\||dF_j\cdot\phi|\|^2_{\frac{1}{4}} \lesssim Q(\phi,\phi) \qquad (1\leqslant j\leqslant N).$$

8.0.7 For any two vectors $(\lambda_1,\ldots,\lambda_N)$, $(\mu_1,\ldots,\mu_N)$ in $\mathbb{C}^N$, if

$$A=\sum_{i=1}^{N}\lambda_i F_i \qquad \text{and} \qquad B=\sum_{i=1}^{N}\mu_i F_i,$$

then by Proposition 1.24

$$\||\mathsf{Jac}(A,B)\phi|\|^2_{\frac{1}{4}} \lesssim Q(\phi,\phi).$$

8.0.8 By Corollary 5.2, there exist vectors $(\lambda_1,\ldots,\lambda_N)$ and $(\mu_1,\ldots,\mu_N)$ such that

$$\mathsf{dim}_{\mathbb{C}}\ \mathscr{O}_{\mathbb{C}^2,0}\big/\langle A,B\rangle \leqslant 4s^2.$$

By Corollary 2.16,

$$\mathsf{mult}_0\ \mathsf{Jac}(A,B)\leqslant 4s^2-1.$$

8.0.9 Write

$$\mathsf{Jac}(A,B)=f_1^{\alpha_1}\cdots f_{\tilde{r}}^{\alpha_{\tilde{r}}} \tag{8.1}$$

as a product of prime elements, and let $\alpha := \mathsf{max}\{\alpha_1,\ldots,\alpha_{\tilde{r}}\}$. The holomorphic function

$$\tilde{h}_2 := f_1\cdots f_{\tilde{r}}$$

is also a subelliptic multiplier since

$$\tilde{h}_2^{\alpha}=f_1^{\alpha}\cdots f_{\tilde{r}}^{\alpha}=f_1^{\alpha-\alpha_1}\cdots f_{\tilde{r}}^{\alpha-\alpha_{\tilde{r}}}\mathsf{Jac}(A,B)$$

is a multiple of a subelliptic multiplier. Consequently, by radical property of subelliptic multipliers Proposition 1.22,

$$\||\tilde{h}_2\phi|\|^2_{\frac{1}{4\alpha}} \lesssim Q(\phi,\phi).$$

Moreover, by equation (8.1),

$$\mathsf{mult}_0\ \mathsf{Jac}(A,B)=\sum_{i=1}^{\tilde{r}}\alpha_i\ \mathsf{mult}_0\ (f_i)\geqslant\alpha.$$

Hence

$$\frac{1}{4\alpha}\geqslant\frac{1}{4\ \mathsf{mult}_0\mathsf{Jac}(A,B)}\geqslant\frac{1}{4(4s^2-1)},$$

and

$$\||\tilde{h}_2\phi|\|^2_{\frac{1}{4(4s^2-1)}} \lesssim \||\tilde{h}_2\phi|\|^2_{\frac{1}{4\alpha}} \lesssim Q(\phi,\phi).$$

8.0.10 As a remark,

$$\mathsf{mult}_0\ \tilde{h}_2 = \sum_{i=1}^{\tilde{r}} \mathsf{mult}_0\ f_i \leqslant \sum_{i=1}^{\tilde{r}} \alpha_i \mathsf{mult}_0\ f_i = \mathsf{mult}_0\ \mathsf{Jac}(A,B) \leqslant 4s^2-1.$$

8.0.11 By Proposition 7.4, there exists $(c_1,\ldots,c_{\mathrm{N}}) \in \mathbb{C}^{\mathrm{N}}$ and a linear change of coordinate $(z_1,z_2)\mapsto(w_1,w_2)$ via

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

such that if $h_1 = \sum_{k=1}^{\mathrm{N}} c_k F_k$, one has

(i)

$$\begin{aligned} & \mathsf{dim}_{\mathbb{C}}\ \mathscr{O}_{\mathbb{C}^2,0}\big/\langle h_1(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2), \tilde{h}_2(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2)\rangle \\ \leqslant\ & (\mathsf{mult}_0\ \tilde{h}_2)s \leqslant (4s^2-1)s; \end{aligned}$$

(ii)

$$\begin{aligned} & \mathsf{dim}_{\mathbb{C}}\ \mathscr{O}_{\mathbb{C}^2,0}\Big/ \left\langle \tilde{h}_2(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2), \frac{\partial h_1(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2)}{\partial w_1} \right\rangle \\ \leqslant\ & (\mathsf{mult}_0\ \tilde{h}_2)\begin{pmatrix} 8s+1 \\ 8s-1 \end{pmatrix}; \end{aligned}$$

(iii) if we let $(\mathbb{C}^2,(w_1,w_1))$ [resp. $(\mathbb{C}^2,(x,y))$] denote $\mathbb{C}^2$ with coordinate system (w_1,w_2) [resp. (x,y)], the holomorphic map

$$\begin{aligned} \varphi : (\mathbb{C}^2,(w_1,w_2)) &\longrightarrow (\mathbb{C}^2,(x,y)) \\ (w_1,w_2) &\longmapsto (h_1(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2), w_2) \end{aligned}$$

defines a ramified $\mathsf{ord}_{w_1=0}\ \tilde{h}_1(\alpha w_1,\beta w_1)$-cover over $\mathbb{C}^2$, which is therefore open and proper with finite fibres.

8.0.12 Since h_1 vanishes at the origin, by Lemma 2.12,

$$\begin{aligned} & h_1(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2)^{(4s^2-1)\binom{8s+1}{8s-1}} \\ \in\ & \left\langle \frac{\partial h_1(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2)}{\partial z_1}, \tilde{h}_2(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2) \right\rangle. \end{aligned}$$

8.0.13 Let $C_2 := \{\tilde{h}_2(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2)=0\}$ be the reduced curve. Since φ is a proper map, by Remmert's proper mapping theorem, the image $\tilde{C}_2 := \varphi(C_2)$ is an analytic set of dimension 1. There exists an analytic function h_2 on $\mathbb{C}^2$ such that $\varphi(C_2)=\{h_2(x,y)=0\}$. By Proposition 6.4, $\lambda := \mathsf{ord}_0\ h_2(0,y) \leqslant (4s^2-1)s$. Hence, by Weierstrass' preparation theorem, there exist a unit $u(x,y)$, and holomorphic functions $a_1(x),\ldots,a_{(4s^2-1)s-1}(x)$ that vanish at $x=0$ such that $h_2(x,y)$ may be expressed as a Weierstrass polynomial

$$h_2(x,y) = u(x,y)\left(y^\lambda + \sum_{k=0}^{\lambda-1} a_j(x)y^j\right) \qquad \lambda\leqslant(4s^2-1)s.$$

8.0.14 The holomorphic function $h_2(h_1(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2),w_2)$ is also a subelliptic multiplier. More precisely, $h_2(h_1(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2),w_2)$ is a multiple of $\tilde{h}_2(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2)$ which is a subelliptic multiplier by paragraph 8.0.9. This follows from the fact (which will be explained below) that $V(\tilde{h}_2)\subseteq V(h_2(h_1,w_2))$ and hence by the Nullstellensatz,

$$\langle h_2(h_1,w_2)\rangle \subseteq \sqrt{\langle h_2(h_1,w_2)\rangle}\subseteq\sqrt{\langle\tilde{h}_2\rangle}=\langle\tilde{h}_2\rangle,$$

where the equality follows from the fact that $\tilde{h}_2$ is reduced.

Now to show that $V(\tilde{h}_2(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2))\subseteq V(h_2(h_1(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2),w_2))$, if $(\sigma,\mu)\in\mathbb{C}^2$ satisfies $\tilde{h}_2(\alpha\sigma+\beta\mu,\gamma\sigma+\delta\mu)=0$, then

$$\varphi(\sigma,\mu)=(h_1(\alpha\sigma+\beta\mu,\gamma\sigma+\delta\mu),\mu)\in\{h_2(x,y)=0\}.$$

Hence

$$0=h_2(\varphi(\sigma,\mu))=h_2(h_1(\alpha\sigma+\beta\mu,\gamma\sigma+\delta\mu),\mu),$$

from which we have proved the set inclusion. Consequently,

$$\|\|h_2(h_1(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2),w_2)\phi\|\|^2_{\frac{1}{4(4s^2-1)}}\lesssim Q(\phi,\phi).$$

Moreover, since $u(h_1(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2),w_2)$ is a unit,

$$\left\|\left\|\left(w_2^\lambda+\sum_{j=1}^{\lambda-1}a_j(h_1(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2))w_2^j\right)\phi\right\|\right\|^2_{\frac{1}{4(4s^2-1)}}\lesssim Q(\phi,\phi),$$

where $\lambda\leqslant(4s^2-1)s$.

8.0.15 To declutter notations, we will set

$$\begin{aligned}
h_1(w_1,w_2) &:= h_1(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2)\\
\frac{\partial h_1(w_1,w_2)}{\partial w_1} &:= \frac{\partial h_1(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2)}{\partial w_1}\\
\tilde{h}_2(w_1,w_2) &:= \tilde{h}_2(\alpha w_1+\beta w_2,\gamma w_1+\delta w_2)\\
\eta &:= (4s^2-1)\binom{8s+1}{8s-1}\\
\lambda &:= \mathsf{ord}_0\, h_2(0,y)\leqslant(4s^2-1)s.
\end{aligned}$$

By Paragraph 8.0.12, since

$$h_1^\eta\in\left\langle\frac{\partial h_1}{\partial w_1},\,\tilde{h}_2\right\rangle,$$

there is an estimate

$$|h_1^\eta|\lesssim\left|\frac{\partial h_1}{\partial w_1}\right|+|\tilde{h}_2|.$$

8.1 Siu's method: Starting Point

8.1.1 Since h_1 is a pre-multiplier and

$$dh_1\wedge dh_2=\frac{\partial h_1}{\partial w_1}\left(\lambda w_2^{\lambda-1}+\sum_{j=1}^{\lambda-1}ja_j(h_1)w_2^{j-1}\right)dw_1\wedge dw_2,$$

the holomorphic function

$$\frac{\partial h_1}{\partial w_1}\left(\lambda w_2^{\lambda-1}+\sum_{j=1}^{\lambda-1} j a_j(h_1) w_2^{j-1}\right)$$

is also a subelliptic multiplier and we will estimate its regularity property. Since

$$\left|\left|\left|\left(w_2^{\lambda}+\sum_{j=0}^{\lambda-1} a_j(h_1) w_2^{j}\right)\phi\right|\right|\right|^2_{\frac{1}{4(4s^2-1)}} \lesssim Q(\phi,\phi),$$

using Proposition 1.24,

$$\left|\left|\left|d\left(w_2^{\lambda}+\sum_{j=0}^{\lambda-1} a_j(h_1) w_2^{j}\right)\cdot\phi\right|\right|\right|^2_{\frac{1}{8(4s^2-1)}} \lesssim Q(\phi,\phi).$$

Also,

$$|||dh_1\cdot\phi|||^2_{\frac{1}{8(4s^2-1)}} \lesssim |||dh_1\cdot\phi|||^4_{\frac{1}{4}} \lesssim Q(\phi,\phi),$$

where the last inequality comes from Paragraph 8.0.6 and the fact that h_1 is a linear combination of the F_i. By the remark after Definition 1.26, the regularity of the subelliptic multiplier is obtained below

$$\left|\left|\left|\frac{\partial h_1}{\partial w_1}\left(\lambda w_2^{\lambda-1}+\sum_{j=1}^{\lambda-1} j a_j(h_1) w_2^{j-1}\right)\phi\right|\right|\right|^2_{\frac{1}{8(4s^2-1)}} \lesssim Q(\phi,\phi).$$

8.1.2 Since $\tilde{h}_2$ is also a subelliptic multiplier, so is

$$\tilde{h}_2\left(\lambda w_2^{\lambda-1}+\sum_{j=1}^{\lambda-1} j a_j(h_1) w_2^{j-1}\right).$$

Hence by paragraph 8.0.9,

$$\left|\left|\left|\tilde{h}_2\left(\lambda w_2^{\lambda-1}+\sum_{j=1}^{\lambda-1} j a_j(h_1) w_2^{j-1}\right)\phi\right|\right|\right|^2_{\frac{1}{4(4s^2-1)}} \lesssim Q(\phi,\phi).$$

8.1.3 By the previous two paragraphs and the inequality in 8.0.15,

$$|h_1^{\eta}| \lesssim \left|\frac{\partial h_1}{\partial z_1}\right| + |\tilde{h}_2|,$$

there is an estimate

$$\begin{aligned}
&\left|\left|\left|h_1^{\eta}\left(\lambda w_2^{\lambda-1}+\sum_{j=1}^{\lambda-1} j a_j(h_1) w_2^{j-1}\right)\phi\right|\right|\right|^2_{\frac{1}{8(4s^2-1)}}\\
\lesssim\ &\left|\left|\left|\frac{\partial h_1}{\partial w_1}\left(\lambda w_2^{\lambda-1}+\sum_{j=1}^{\lambda-1} j a_j(h_1) w_2^{j-1}\right)\phi\right|\right|\right|^2_{\frac{1}{8(4s^2-1)}} + \left|\left|\left|\tilde{h}_2\left(\lambda w_2^{\lambda-1}+\sum_{j=1}^{\lambda-1} j a_j(h_1) w_2^{j-1}\right)\phi\right|\right|\right|^2_{\frac{1}{8(4s^2-1)}}\\
\lesssim\ & Q(\phi,\phi).
\end{aligned}$$

8.2 Siu's method: Inductive Step

8.2.1 Let $h_2^{(0)} := h_2$ and

$$h_2^{(1)} := h_1^{\eta}\left(\lambda w_2^{\lambda-1} + \sum_{j=1}^{\lambda-1} j a_j(h_1) w_2^{j-1}\right).$$

For $1 \leqslant \nu \leqslant \lambda$, define

$$h_2^{(\nu)} := h_1^{\nu\eta}\left(\frac{\lambda!}{(\lambda-\nu)!} w_2^{\lambda-\nu} + \sum_{j=\nu}^{\lambda-1} \frac{j!}{(j-\nu)!} a_j(h_1) w_2^{j-\nu}\right),$$

which will be shown that it is also a subelliptic multiplier and

$$\||| h_2^{(\nu)}\phi |||^2_{\frac{1}{2^\nu \cdot 4(4s^2-1)}} \lesssim Q(\phi,\phi).$$

8.2.2 We will first calculate something analogous to the first paragraph of the previous subsection. Suppose that the induction statement is true for $\nu - 1$, meaning that

$$\||| h_2^{(\nu-1)}\phi |||^2_{\frac{1}{2^{\nu-1} \cdot 4(4s^2-1)}} \lesssim Q(\phi,\phi).$$

Then

$$\||| dh_2^{(\nu-1)}\phi |||^2_{\frac{1}{2^\nu \cdot 4(4s^2-1)}} \lesssim Q(\phi,\phi).$$

Moreover,

$$\||| dh_1 \cdot \phi |||^4_{\frac{1}{2}} \lesssim Q(\phi,\phi).$$

Therefore,

$$dh_1 \wedge dh_2^{(\nu-1)} = \frac{\partial h_1}{\partial w_1}\left(h_1^{\eta(\nu-1)}\left(\frac{\lambda!}{(\lambda-\nu)!} w_2^{\lambda-\nu} + \sum_{j=\nu}^{\lambda-1} \frac{j!}{(j-\nu)!} a_j(h_1) w_2^{j-\nu}\right)\right) dw_1 \wedge dw_2,$$

whose coefficient is also a subelliptic multiplier with

$$\left\| \frac{\partial h_1}{\partial w_1}\left(h_1^{\eta(\nu-1)}\left(\frac{\lambda!}{(\lambda-\nu)!} w_2^{\lambda-\nu} + \sum_{j=\nu}^{\lambda-1} \frac{j!}{(j-\nu)!} a_j(h_1) w_2^{j-\nu}\right)\right)\phi \right\|^2_{\frac{1}{2^\nu \cdot 4(4s^2-1)}} \lesssim Q(\phi,\phi).$$

8.2.3 Since $\tilde{h}_2$ is also a subelliptic multiplier,

$$\left\| \tilde{h}_2\left(h_1^{\eta(\nu-1)}\left(\frac{\lambda!}{(\lambda-\nu)!} w_2^{\lambda-\nu} + \sum_{j=\nu}^{\lambda-1} \frac{j!}{(j-\nu)!} a_j(h_1) w_2^{j-\nu}\right)\right)\phi \right\|^2_{\frac{1}{2^\nu \cdot 4(4s^2-1)}} \lesssim Q(\phi,\phi).$$

8.2.4 Combining the inequalities in the last two paragraphs, and using the fact that $|h_1^\eta| \lesssim |\partial_{z_1} h_1| + |\tilde{h}_2|$,

$$\begin{aligned}
& |||h_2^{(\nu)}\phi|||^2_{\frac{1}{2^\nu\cdot 4(4s^2-1)}} \\
= & \left|\left|\left| h_1^{\nu\eta}\left(\frac{\lambda!}{(\lambda-\nu)!}w_2^{\lambda-\nu} + \sum_{j=\nu}^{\lambda-1}\frac{j!}{(j-\nu)!}a_j(h_1)w_2^{j-\nu}\right)\phi\right|\right|\right|^2_{\frac{1}{2^\nu\cdot 4(4s^2-1)}} \\
= & \left|\left|\left| h_1^{\eta}\left(h_1^{\eta(\nu-1)}\left(\frac{\lambda!}{(\lambda-\nu)!}w_2^{\lambda-\nu} + \sum_{j=\nu}^{\lambda-1}\frac{j!}{(j-\nu)!}a_j(h_1)w_2^{j-\nu}\right)\right)\phi\right|\right|\right|^2_{\frac{1}{2^\nu\cdot 4(4s^2-1)}} \\
\lesssim & \left|\left|\left| \frac{\partial h_1}{\partial w_1}\left(h_1^{\eta(\nu-1)}\left(\frac{\lambda!}{(\lambda-\nu)!}w_2^{\lambda-\nu} + \sum_{j=\nu}^{\lambda-1}\frac{j!}{(j-\nu)!}a_j(h_1)w_2^{j-\nu}\right)\right)\phi\right|\right|\right|^2_{\frac{1}{2^\nu\cdot 4(4s^2-1)}} \\
& + \left|\left|\left| \tilde{h}_2\left(h_1^{\eta(\nu-1)}\left(\frac{\lambda!}{(\lambda-\nu)!}w_2^{\lambda-\nu} + \sum_{j=\nu}^{\lambda-1}\frac{j!}{(j-\nu)!}a_j(h_1)w_2^{j-\nu}\right)\right)\phi\right|\right|\right|^2_{\frac{1}{2^\nu\cdot 4(4s^2-1)}} \\
\lesssim & \ Q(\phi,\phi).
\end{aligned}$$

This finishes the induction process.

8.2.5 Setting $\nu = \lambda$, we get

$$|||h_2^{(\lambda)}\phi|||^2_{\frac{1}{2^\lambda\cdot 4(4s^2-1)}} \lesssim Q(\phi,\phi).$$

But $h_2^{(\lambda)} = h_1^{\eta\lambda}\lambda!$, therefore

$$|||h_1^{\lambda\eta}\phi|||^2_{\frac{1}{2^\lambda\cdot 4(4s^2-1)}} \lesssim Q(\phi,\phi).$$

8.3 Siu's method: Conclusion and End of Calculation

8.3.1 Since

$$\dim_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^2,0}/\langle h_1, \tilde{h}_2\rangle \leqslant (4s^2-1)s,$$

by Proposition 2.17,

$$\dim_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^2,0}/\langle h_1^{\lambda\eta}, \tilde{h}_2\rangle = \eta\lambda \dim_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^2,0}/\langle h_1, \tilde{h}_2\rangle \leqslant (4s^2-1)s\eta\lambda.$$

For $i = 1,\ 2$, by the Lemma 2.12,

$$w_i^{(4s^2-1)s\lambda\eta} \in \left\langle h_1^{\lambda\eta}, \tilde{h}_2\right\rangle.$$

Thus $w_i^{(4s^2-1)s\lambda\eta}$ is also a multiplier with

$$|w_i^{(4s^2-1)s\lambda\eta}| \lesssim |h_1^{\eta\lambda}| + |\tilde{h}_2|.$$

Hence

$$\begin{aligned}
& |||w_i^{(4s^2-1)s\lambda\eta}\phi|||^2_{\frac{1}{2^\lambda\cdot 4(4s^2-1)}} \\
\lesssim & \ |||h_1^{\eta\lambda}\phi|||^2_{\frac{1}{2^\lambda\cdot 4(4s^2-1)}} + |||\tilde{h}_2\phi|||^2_{\frac{1}{2^\lambda\cdot 4(4s^2-1)}} \\
\lesssim & \ |||h_1^{\eta\lambda}\phi|||^2_{\frac{1}{2^\lambda\cdot 4(4s^2-1)}} + |||\tilde{h}_2\phi|||^2_{\frac{1}{4(4s^2-1)}} \lesssim Q(\phi,\phi).
\end{aligned}$$

By radical property of subelliptic multipliers Proposition 1.22, one has for each $i = 1,\ 2$ that

$$\||w_i\phi|\|^2_{\frac{1}{2^\lambda\cdot 4s\eta\lambda(4s^2-1)^2}} \lesssim Q(\phi,\phi).$$

Taking the Jacobian, one obtains by Propositions 1.22 and 1.24 that

$$\||\phi|\|^2_{\frac{1}{2^{\lambda+1}\cdot 4s\eta\lambda(4s^2-1)^2}} \lesssim Q(\phi,\phi),$$

and this terminates the calculation, and therefore the following theorem is obtained:

Theorem 8.2. *Let (z_1, z_2, z_3) be holomorphic coordinates in $\mathbb{C}^3$ with $z_i = x_i + \sqrt{-1}y_i$. For some $\mathrm{N} \geqslant 2$, let $F_1(z_1, z_2),\ldots,F_\mathrm{N}(z_1, z_2)$ be holomorphic function germs in $\mathscr{O}_{\mathbb{C}^2,0}$ vanishing at the origin such that*

$$\mathsf{dim}_{\mathbb{C}}\ \mathscr{O}_{\mathbb{C}^2,0}\big/\langle F_1,\ldots,F_\mathrm{N}\rangle := s < \infty.$$

Let $\Omega \subset \mathbb{C}^3$ be the domain defined by

$$\Omega = \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 :\ 2\mathrm{Re}\ z_3 - \sum_{i=1}^{\mathrm{N}} |F_i(z_1, z_2)|^2 < 0\right\}.$$

Then by Siu's method, Kohn's algorithm terminates in at most $(4s^2 - 1)s$ steps. Moreover, for any $\phi \in \mathscr{D}_{0,1}(\Omega)$ with compact support,

$$\||\phi|\|^2_\varepsilon \lesssim \|\overline{\partial}\phi\|^2 + \|\overline{\partial}^*\phi\|^2 + \|\phi\|^2,$$

where

$$\varepsilon \geqslant \frac{1}{2^{(4s^2-1)s+3}s^2(4s^2-1)^4\binom{8s+1}{8s-1}}.$$

9 Homogeneous Polynomials in Two Variables

9.1 Some Properties of Homogeneous Polynomials

9.1.1 In this subsection, we will let F and G denote any homogeneous polynomials in variables (z, w) of respective degrees $\mathsf{deg}\ F = m$ and $\mathsf{deg}\ G = n$. They may be expressed as

$$\begin{aligned} F(z,w) &= a_0z^mw^0 + \cdots + a_mz^0w^m \\ G(z,w) &= b_0z^nw^n + \cdots + b_nz^0w^n. \end{aligned} \tag{9.1}$$

There are two ways to dehomogenise F and G:

$$\begin{aligned} F(z,1) := f &= a_0z^m + a_1z^{m-1} + \cdots + a_m, \\ G(z,1) := g &= b_0z^n + b_1z^{n-1} + \cdots + b_n, \end{aligned} \tag{9.2}$$

and

$$\begin{aligned} F(1,w) := \tilde{f} &= a_0 + a_1w + \cdots + a_mw^m, \\ G(1,w) := \tilde{g} &= b_0 + b_1w + \cdots + b_nw^n. \end{aligned} \tag{9.3}$$

9.1.2 Factorisation of Homogeneous Polynomials

Lemma 9.4. *Let $F \in \mathbb{C}[z,w]$ be a homogeneous polynomial of degree m. Then F can be written as a product of linear factors:*

$$F(z,w) = w^{m-r}\prod_{i=1}^{k_F}(a_i z - b_i w)^{r_i}.$$

where $\sum_{i=1}^{k_F} r_i = r$.

Proof. By dehomogenising F,

$$F(X,Y) = Y^m F(X/Y,1) := Y^m Q(X/Y),$$

where Q is of degree less than or equals to m. The proof follows by the Fundamental Theorem of Algebra. □

9.1.3 The following lemma gives the expressions of $\mathsf{Jac}(F,G)$ in a form which simplifies calculations in the later sections.

Lemma 9.5. *Let F and G be homogeneous polynomials of degrees m and n in equation* (9.1)*. Then*

$$\begin{aligned} zw\mathsf{Jac}(F,G) &= w(mFG_w - nGF_w) \\ &= z(nGF_z - mFG_z). \end{aligned} \tag{9.6}$$

Proof. From the Euler's identity applied to F and G,

$$\begin{aligned} zF_z + wF_w &= mF, \\ zG_z + wG_w &= nG. \end{aligned}$$

Multiplying the two expressions together,

$$\begin{aligned} (mF)(nG) &= z^2F_zG_z + zwF_zG_w + wzF_wG_z + w^2F_wG_w \\ &= zw(F_zG_w - F_wG_z) + (z^2F_zG_z + zwF_wG_z) + (w^2F_wG_w + zwF_wG_z) \\ &= zw\mathsf{Jac}(F,G) + zG_z(zF_z + wF_w) + wF_w(zG_z + wG_w) \\ &= zw\mathsf{Jac}(F,G) + zG_zmF + wF_wnG. \end{aligned}$$

Bringing $zG_zmF + wF_wnG$ to the left side of the equation,

$$zw\mathsf{Jac}(F,G) = (mF)(nG) - zG_zmF - wF_znG. \tag{9.7}$$

Using the Euler's identity for G, in equation (9.7),

$$\begin{aligned} zw\mathsf{Jac}(F,G) &= (mF)(nG) - zG_zmF - wF_wnG \\ &= mF(nG - zG_z) - wF_wnG \\ &= mFwG_w - wF_wnG \\ &= w(mFG_w - nGF_w), \end{aligned}$$

which gives the first formula in equation (9.6). For the second formula in equation (9.6), applying Euler's identity for F in (9.7) gives

$$\begin{aligned} zw\mathsf{Jac}(F,G) &= (mF)(nG) - zG_zmF - wF_wnG \\ &= nG(mF - wF_w) - zG_zmF \\ &= nGzF_z - zG_zmF \\ &= z(nGF_z - mFG_z). \end{aligned}$$

The proof is then complete. □

9.2 Resultants

9.2.1 Let $F,\ G \in \mathbb{C}[z,w]$ be homogeneous polynomials given as before by

$$\begin{aligned} F(z,w) &= a_0 z^m w^0 + \cdots + a_{m-1} z w^{m-1} + a_m w^m, \\ G(z,w) &= b_0 z^n w^0 + \cdots + b_{n-1} z w^{n-1} + b_n w^n, \end{aligned}$$

The resultant of these two polynomials is given by the following $m+n$ by $m+n$ Sylvester matrix

$$\mathsf{Resultant}_{m,n}(F,G) := \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & a_m & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & \cdots & a_{m-1} & a_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & a_0 & a_1 & \cdots & \cdots & a_{m-1} & a_m \\ b_0 & b_1 & b_2 & \cdots & \cdots & b_n & 0 & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & \cdots & b_{n-1} & b_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & b_0 & b_1 & \cdots & \cdots & b_{n-1} & b_n \end{vmatrix}.$$

Over the field of complex numbers, $\mathsf{Resultant}_{m,n}(F,G) = 0$ if and only if the equations $F = G = 0$ have a solution $(x,y) \neq (0,0)$ in $\mathbb{C}^2$ (See [CLO05, page 81]). Direct from the definition, therefore

$$\begin{aligned} \mathsf{Resultant}_{1,m}(z,F) &= a_m \\ \mathsf{Resultant}_{1,m}(w,F) &= a_0. \end{aligned}$$

Moreover, by a result of Jouanolou [Jou91], the resultant of homogeneous polynomials satisfies multiplicative property:

Proposition 9.8 (Section 5.7, page 154 in [Jou91])**.** *Let F, G and H be homogenenous polynomials in $\mathbb{C}[z,w]$ of respective degrees d_1, d_2 and d_3. Then*

$$\mathsf{Resultant}_{d_1+d_2,d_3}(FG,H) = \mathsf{Resultant}_{d_1,d_3}(F,H)\mathsf{Resultant}_{d_2,d_3}(G,H).$$

9.2.2 Resultant by dehomogenised polynomials Let $f(z) = F(z,1)$ and $g(z) = G(z,1)$ be the dehomogenisation of the homogeneous polynomials F and G respectively. Provided that $a_0 \neq 0$ and $b_0 \neq 0$, the resultant of F and G is then given by

$$\mathsf{Resultant}_{m,n}(F,G) =: \mathsf{resultant}_{m,n}(f,g,z).$$

Under this assumption, by [CLO05, Chapter 3, page 78], $\mathsf{resultant}_{m,n}(f,g,z) = 0$ if and only if f and g have common factors. Moreover, by [CLO05, Chapter 3, page 81, Proposition 1.7], $\mathsf{Resultant}_{m,n}(F,G) \not\equiv 0$ if and only if $\{F = G = 0\} = \{(0,0)\}$.

9.2.3 The following lemma is crucial for our calculations.

Lemma 9.9. *Let $a(z)$, $b(z)$ and $c(z)$ be polynomials in $\mathbb{C}[z]$ given by*

$$\begin{aligned} a(z) &:= a_0 z^m + \cdots + a_{m-1} z + a_m, \\ b(z) &= b_0 z^n + \cdots + b_{n-1} z + b_n, \\ c(z) &= c_0 z^d + \cdots + c_{d-1} z + c_d, \end{aligned}$$

with $m \geqslant n$ and $d = m - n$. Suppose $a_0 \neq 0$, $b_0 \neq 0$ and $a_0 - b_0 c_0 \neq 0$, then

$$\mathsf{resultant}_{m,n}(a - bc, b, z) = \mathsf{resultant}_{m,n}(a,b,z).$$

The proof of this lemma is long and complicated, but an example will explain how the proof should proceed. For the purpose, let A, B and C be homogeneous polynomials of respective degrees 3, 2 and 1, so that

$$\begin{aligned} a &= a_0z^3 + a_1z^2 + a_2z + a_3, \\ b &= b_0z^2 + b_1z + b_2, \\ c &= c_0z + c_1, \end{aligned}$$

and hence

$$a - bc = (a_0 - b_0c_0)z^3 + (a_1 - b_0c_1 - b_1c_0)z^2 + (a_2 - b_1c_1 - b_2c_0)z + (a_3 - b_2c_1).$$

The resultant of $a - bc$ and b is the determinant of the following 5 by 5 matrix:

$$\begin{vmatrix} b_0 & b_1 & b_2 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 \\ a_0 - b_0c_0 & a_1 - b_0c_1 - b_1c_0 & a_2 - b_1c_1 - b_2c_0 & a_3 - b_2c_1 & 0 \\ 0 & a_0 - b_0c_0 & a_1 - b_0c_1 - b_1c_0 & a_2 - b_1c_1 - b_2c_0 & a_3 - b_2c_1 \end{vmatrix}$$

$$\overset{\substack{R4+c_0R1 \\ R5+c_0R2}}{=} \begin{vmatrix} b_0 & b_1 & b_2 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 \\ a_0 & a_1 - b_0c_1 & a_2 - b_1c_1 & a_3 - b_2c_1 & 0 \\ 0 & a_0 & a_1 - b_0c_1 & a_2 - b_1c_1 & a_3 - b_2c_1 \end{vmatrix}$$

$$\overset{\substack{R4+c_1R2 \\ R5+c_1R3}}{=} \begin{vmatrix} b_0 & b_1 & b_2 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 \\ a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 \end{vmatrix} = \mathsf{resultant}(a, b, z).$$

This lemma has an easy generalisation to homogeneous polynomials.

Lemma 9.10. *Let $a(z)$, $b(z)$ and $c(z)$ be polynomials above with $m \geqslant n$, $d = m - n$, $a_0 \neq 0$, $b_0 \neq 0$ and $a_0 - b_0c_0 \neq 0$. Let $A(z, w)$, $B(z, w)$ and $C(z, w)$ be homogenisation of $a(z)$, $b(z)$ and $c(z)$ respectively so that*

$$\begin{aligned} A(z) &:= a_0z^mw^0 + \cdots + a_{m-1}zw^{m-1} + a_mz^0w^m, \\ B(z) &= b_0z^nw^0 + \cdots + b_{n-1}zw^{n-1} + b_nz^0w^n, \\ C(z) &= c_0z^dw^0 + \cdots + c_{d-1}zw^{d-1} + c_dz^0w^d, \end{aligned}$$

then

$$\mathsf{Resultant}_{m,n}(A - BC, B) = \mathsf{Resultant}_{m,n}(A, B).$$

Proof. By the definition of resultant of homogeneous polynomials and applying the previous lemma

$$\begin{aligned} \mathsf{Resultant}_{m,n}(A - BC, B) &= \mathsf{resultant}_{m,n}(A(z, 1) - B(z, 1)C(z, 1), B(z, 1), z) \\ &= \mathsf{resultant}_{m,n}(a - bc, b, z) \\ &= \mathsf{resultant}_{m,n}(a, b, z) = \mathsf{Resultant}_{m,n}(A, B). \end{aligned}$$

Hence the lemma is proved. □

9.3 Resultants and Jacobians

9.3.1 The relationship between the resultant and the Jacobian The following proposition will be used for Kohn's algorithm on homogeneous polynomials.

Proposition 9.11. *Let F and G be homogeneous polynomials with respective degrees m and n. If $a_0 \neq 0$, $a_1 \neq 0$, $b_0 \neq 0$, and*

$$\mathsf{Jac}(F,G)(1,0) = nb_0a_1 - ma_0b_1 \neq 0,$$

then

$$a_m \mathsf{Resultant}_{m+n-2,m}(\mathsf{Jac}(F,G),F) = (-n)^m \mathsf{Resultant}_{m,n}(F,G)\ \mathsf{Resultant}_{m-1,m}(F_w,F).$$

Proof. Using the formula of the Jacobian for homogeneous polynomials,

$$\begin{aligned}
\mathsf{Resultant}_{n+m,m}(zw\mathsf{Jac}(F,G),F) &= \mathsf{Resultant}_{n+m,m}(mFwG_w - nGwF_w, F) \\
&= \mathsf{Resultant}_{1,m}(w,F)\mathsf{Resultant}_{n+m-1,m}(mFG_w - nGF_w, F) \\
&= a_0 \mathsf{Resultant}_{n+m-1,m}(mFG_w - nGF_w, F)
\end{aligned}$$

The coefficient of z^{n+m-2} in $mFG_w - nGF_w$ is $ma_0b_1 - nb_0a_1$, which is non-zero by hypothesis. Set as in lemma 9.10 the following polynomials

$$\begin{aligned}
A &= nGF_w, \\
B &= F, \\
C &= G_w,
\end{aligned}$$

and observe that the coefficient of z^{n+m-1} in A is nb_0a_1 which is non-zero by hypothesis. Moreover, the coefficient of z^m in F is a_0 which is also non-zero by hypothesis. Therefore,

$$\begin{aligned}
\mathsf{deg}\ A &= \mathsf{deg}\ G + \mathsf{deg}\ F - 1, \\
\mathsf{deg}\ B &= \mathsf{deg}\ F, \\
\mathsf{deg}\ C &= \mathsf{deg}\ G - 1,
\end{aligned}$$

from which it is clear that $\mathsf{deg}C = \mathsf{deg}A - \mathsf{deg}B$. Furthermore, the hypothesis $nb_0a_1 - ma_0b_1 \neq 0$ implies that $\mathsf{deg}(A - BC) = \mathsf{deg}A$. By lemma 9.10,

$$\begin{aligned}
\mathsf{Resultant}_{n+m-1,m}(mFG_w - nGF_w, F) &= \mathsf{Resultant}_{n+m-1,m}(-nGF_w, F) \\
&= (-n)^m \mathsf{Resultant}_{n+m-1,m}(GF_w, F),
\end{aligned}$$

from which one obtains

$$\mathsf{Resultant}_{n+m,m}(zw\mathsf{Jac}(F,G),F) = a_0(-n)^m \mathsf{Resultant}_{n+m-1,m}(GF_w,F). \tag{9.12}$$

From the multiplicative property of the resultant, therefore

$$\mathsf{Resultant}_{n+m,m}(zw\mathsf{Jac}(F,G),F) = a_0a_m \mathsf{Resultant}_{n+m-2,m}(\mathsf{Jac}(F,G),F) \tag{9.13}$$

and

$$\mathsf{Resultant}_{n+m-1,m}(GF_w,F) = \mathsf{Resultant}_{n,m}(G,F)\mathsf{Resultant}_{m-1,m}(F_w,F). \tag{9.14}$$

Hence substituting equations (9.13) and (9.14) into (9.12) proves the identity.

Remark: The condition that both $a_0 \neq 0$ and $a_1 \neq 0$ is equivalent to the fact that both

$$F(z,0) \qquad \text{and} \qquad (\partial_w F)(z,0)$$

do not vanish.

□

9.3.2 Some generic conditions In view of the previous result, the following lemma ensures generic conditions:

Lemma 9.15. *Let $F_1, \ldots, F_N$ be homogeneous polynomials in two variables z and w. Let $(z, w) \mapsto (\lambda z + \theta w, \mu z + \sigma w)$ be an invertible linear change of coordinates. Then for a general matrix*

$$A = \begin{pmatrix} \lambda & \theta \\ \mu & \sigma \end{pmatrix},$$

the new homogeneous polynomials $G_i(z, w) := F_i(\lambda z + \theta w, \mu z + \sigma w)$ satisfy $G_i(z, 0) \neq 0$ for all i.

Proof. Simply choose $(\lambda, \theta, \mu, \sigma)$ such that for each i,

$$\begin{aligned} F_i(\lambda, \mu) &\neq 0, \\ \mathsf{det}(A) = \lambda\sigma - \theta\mu &\neq 0, \end{aligned}$$

which completes the proof. □

Proposition 9.16. *Let F, G and H be homogeneous polynomials in z and w each of degree greater than or equals to 1. Then there exists a general change of coordinates $A : (z, w) \mapsto (\lambda z + \theta w, \mu z + \sigma w)$ such that if*

$$\begin{aligned} P &= F(\lambda z + \theta w, \mu z + \sigma w), \\ Q &= G(\lambda z + \theta w, \mu z + \sigma w), \\ R &= H(\lambda z + \theta w, \mu z + \sigma w), \end{aligned}$$

the following properties are satisfied:

$$P(z, 0) \neq 0, \qquad \partial_w P(z, 0) \neq 0, \qquad Q(z, 0) \neq 0, \qquad R(z, 0) \neq 0.$$

Proof. Let θ and σ be chosen so that

$$\theta \partial_z F(z, w) + \sigma \partial_w F(z, w)$$

doesn't vanish identically. This is a generic condition on $(\theta, \sigma) \in \mathbb{C}^2$. Then apply the previous proposition to F, G, H and $\theta \partial_z F(z, w) + \sigma \partial_w F(z, w)$ to obtain the conclusion, while observing that

$$\partial_w \big(F(\lambda z + \theta w, \mu z + \sigma w) \big) = \theta \partial_z F(\lambda z + \theta w, \mu z + \sigma w) + \sigma \partial_w F(\lambda z + \theta w, \mu z + \sigma w).$$

□

9.4 Kohn's Algorithm Applied to Homogeneous Polynomials in 2 Variables

9.4.1 A recall of Kohn's Algorithm Let F and G be homogeneous polynomials in (z, w), and suppose that the common intersection at the origin is 0 dimensional, that is

$$\{F = G = 0\} = \{0\}.$$

The degree of each of F and G may assumed to be at least 2, otherwise if F is linear, a suitable change of holomorphic coordinates gives a triangular system in the sense of Catlin-D'Angelo, and Kohn's algorithm terminates in 2 steps. Recall the Kohn's ideal of sub-elliptic multipliers generated at first step is given by the radical of the Jacobian of the two polynomials

$$\langle \mathsf{radical}(\mathsf{Jac}(F, G)) \rangle$$

Then the next step involves the following Jacobians

$$\mathsf{Jac}\big(F, \mathrm{radical}(\mathsf{Jac}(F,G))\big), \qquad \mathsf{Jac}\big(G, \mathrm{radical}(\mathsf{Jac}(F,G))\big).$$

For Kohn's algorithm to terminate at this step, it suffices to show that any of the common intersections

$$\begin{aligned}\{\mathrm{radical}(\mathsf{Jac}(F,G)) = \mathsf{Jac}\big(F, \mathrm{radical}(\mathsf{Jac}(F,G))\big) = 0\} &= \{0\},\\ \{\mathrm{radical}(\mathsf{Jac}(F,G)) = \mathsf{Jac}\big(G, \mathrm{radical}(\mathsf{Jac}(F,G))\big) = 0\} &= \{0\},\end{aligned}$$

consists only of the origin. To this end, by the property of resultant, it therefore suffices to show that either

$$\mathsf{Resultant}\big(\mathrm{radical}(\mathsf{Jac}(F,G)), \mathsf{Jac}\big(F, \mathrm{radical}(\mathsf{Jac}(F,G))\big)\big) \neq 0$$

or

$$\mathsf{Resultant}\big(\mathrm{radical}(\mathsf{Jac}(F,G)), \mathsf{Jac}\big(G, \mathrm{radical}(\mathsf{Jac}(F,G))\big)\big) \neq 0$$

holds. This is exactly the case in Proposition 9.11, except that the generic conditions need to be fulfilled. This can be overcome by applying a suitable linear change of coordinates $A : \mathbb{C}^2 \to \mathbb{C}^2$ by a general matrix. More details will be explained below.

9.4.2 Intersection multiplicities of homogeneous polynomials at the origin Let $F(z,w)$ and $G(z,w)$ be homogeneous polynomials with degrees $\mathsf{deg}\ F = m$ and $\mathsf{deg}\ G = n$. Suppose that the coefficients of F and G are chosen so that a sufficient condition

$$\mathsf{Resultant}_{m,n}(F,G) \neq 0$$

is satisfied. By Lemma 9.4, the polynomials F and G may be written as a product of linear factors:

$$\begin{aligned}F(z,w) &= \prod_{i=1}^{k_1}(a_i z - b_i w)^{\lambda_i},\\ G(z,w) &= \prod_{i=1}^{k_2}(c_i z - d_i w)^{\mu_i},\end{aligned}$$

with $\sum_{i=1}^{k_1}\lambda_i = m$ and $\sum_{i=1}^{k_2}\mu_i = n$. The condition that $\mathsf{Resultant}_{m,n}(F,G) \neq 0$ implies that

$$\{(z,w) \in \mathbb{C}^2 :\ F(z,w) = G(z,w) = 0\} = \{(0,0)\}.$$

Therefore, for any $1 \leqslant i \leqslant k_1$ and $1 \leqslant j \leqslant k_2$, the linear factors

$$(a_i z - b_i w), \qquad \text{and} \qquad (c_j z - d_j w)$$

are not constant multiples of one another. Otherwise, this means that there exists $\lambda > 0$ such that $a_i = \lambda c_j$ and $b_i = \lambda d_j$, and hence $F(b_i, a_i) = 0 = G(d_i, c_i) = \frac{1}{\lambda^n}G(b_i, a_i)$ is a non-trivial common solution. Therefore

$$\begin{aligned}\mathsf{dim}_{\mathbb{C}}\mathscr{O}_{\mathbb{C}^2,0}\big/\langle F(z,w), G(z,w)\rangle &= \sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\mathsf{dim}_{\mathbb{C}}\mathscr{O}_{\mathbb{C}^2,0}\big/\langle (a_i z - b_i w)^{\lambda_i},\ (c_j z - d_j w)^{\mu_j}\rangle\\ &= \sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\lambda_i\mu_j\mathsf{dim}_{\mathbb{C}}\mathscr{O}_{\mathbb{C}^2,0}\big/\langle (a_i z - b_i w),\ (c_j z - d_j w)\rangle\\ &= \sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\lambda_i\mu_j = \left(\sum_{i=1}^{k_1}\lambda_i\right)\left(\sum_{j=1}^{k_2}\mu_j\right) = mn.\end{aligned}$$

9.4.3 Let $A : \mathbb{C}^2 \to \mathbb{C}^2$ be an invertible change of coordinates given by

$$A := \begin{pmatrix} \lambda & \theta \\ \mu & \sigma \end{pmatrix} \in \mathsf{GL}(2, \mathbb{C}).$$

Firstly, consider $F \circ A$ and $G \circ A$. The first step in Kohn's algorithm consists in taking the Jacobian of $F \circ A$ and $G \circ A$, followed by taking its radical. Observe that

$$\mathsf{Jac}(F \circ A, G \circ A) = \mathsf{det}(A)\mathsf{Jac}(F, G) \circ A.$$

Set $\tilde{J}_A := \mathsf{Jac}(F \circ A, G \circ A)$ and let $J_A := \text{radical}(\mathsf{Jac}(F \circ A, G \circ A))$. Let I be the identity matrix.

Lemma 9.17. *One has*

$$J_A = \mathsf{det}(A) J_I \circ A.$$

Proof. The first step involves computing J_A:

$$\begin{aligned} J_A &= \text{radical}(\mathsf{Jac}(F \circ A, G \circ A)) \\ &= \text{radical}(\mathsf{det}(A)\mathsf{Jac}(F, G) \circ A) \\ &= \mathsf{det}(A)\text{radical}(\mathsf{Jac}(F, G) \circ A). \end{aligned}$$

Let $\mathsf{Jac}(F, G) = h_1^{\alpha_1} \cdots h_k^{\alpha_k}$ be a decomposition of the Jacobian into product of primes (in fact they are linear factors). Hence

$$\begin{aligned} \text{radical}(\mathsf{Jac}(F, G) \circ A) &= \text{radical}((h_1 \circ A)^{\alpha_1} \cdots (h_k \circ A)^{\alpha_k}) \\ &= (h_1 \circ A) \cdots (h_k \circ A) \\ &= \text{radical}(\mathsf{Jac}(F, G)) \circ A = J_I \circ A. \end{aligned}$$

The proof is then complete. □

9.4.4 The next step in Kohn's algorithm consists of taking the Jacobians

$$\begin{aligned} H_1 &:= \mathsf{Jac}(J_A, F \circ A), \\ H_2 &:= \mathsf{Jac}(J_A, G \circ A). \end{aligned}$$

Observe that H_1 and H_2 cannot vanish simultaneously, or otherwise Kohn's algorithm will not terminate despite $\{F = 0\}$ and $\{G = 0\}$ having complete intersection, which is a contradiction. Assume without loss of generality that H_1 does not vanish. Next, by the lemma in the previous paragraph,

$$\begin{aligned} H_1 &= \mathsf{Jac}(J_A, F \circ A) \\ &= \mathsf{Jac}(\mathsf{det}(A) J_I \circ A, F \circ A) \\ &= \mathsf{det}(A)\mathsf{Jac}(J_I \circ A, F \circ A) = (\mathsf{det}(A))^2 \mathsf{Jac}(J_I, F) \circ A. \end{aligned}$$

9.4.5 Kohn's algorithm terminates at this point if it can be shown that

$$\mathsf{Resultant}(\mathsf{Jac}(J_A, F \circ A), J_A)$$

does not vanish. In order to apply the results in the previous section, the following conditions need to be satisfied:

$$\begin{aligned} \mathsf{Jac}(J_A, F \circ A)(z, 0) &\neq 0, \\ J_A(z, 0) &\neq 0, \\ \partial_w J_A(z, 0) &\neq 0, \\ F \circ A(z, 0) &\neq 0. \end{aligned}$$

In order to achieve this, observe that conditions above are equivalent to the following conditions using the formula in the previous paragraph and Lemma 9.17

$$\begin{aligned}(\mathsf{det}(A))^2\mathsf{Jac}(J_I, F)\circ A(z,0) &\neq 0,\\ \mathsf{det}(A)J_I\circ A(z,0) &\neq 0,\\ \mathsf{det}(A)\partial_w(J_I\circ A)(z,0) &\neq 0,\\ F\circ A(z,0) &\neq 0.\end{aligned}$$

This can be done by a suitable change of coordinates A by Proposition 9.16. Therefore,

$$\mathsf{Resultant}(\mathsf{Jac}(J_A, F\circ A), J_A) = (-n)^? \frac{\mathsf{Resultant}(F\circ A, J_A)\mathsf{Resultant}(\partial_z J_A, J_A)}{\mathsf{Resultant}(w, J_A)}$$

9.4.6 Next, it is claimed that $\mathsf{Resultant}(\partial_z J_A, J_A)$ is not zero. It suffices to prove that J_A is reduced. But from Lemma 9.17, $J_A = \mathsf{det}(A)J_I\circ A$, and since J_I is reduced, so is J_A.

9.4.7 On the other hand, to show that

$$\mathsf{Resultant}(F\circ A, J_A) = \mathsf{Resultant}(F\circ A, \mathrm{radical}(\mathsf{Jac}(F\circ A, G\circ A)))$$

does not vanish, this is equivalent to proving that

$$\mathsf{Resultant}(F\circ A, \mathsf{Jac}(F\circ A, G\circ A))$$

does not vanish. This is because by the property of resultant and using the decomposition of $\mathsf{Jac}(F\circ A, G\circ A)$ in the proof of lemma 1.14,

$$\begin{aligned}\mathsf{det}(A)^{-m}\mathsf{Resultant}(F\circ A, \mathsf{Jac}(F\circ A, G\circ A)) &= \mathsf{Resultant}(F\circ A, \mathsf{Jac}(F, G)\circ A)\\ &= \mathsf{Resultant}(F\circ A, (h_1\circ A)^{\alpha_1}\cdots(h_k\circ A)^{\alpha_k})\\ &= \mathsf{Resultant}(F\circ A, (h_1\circ A)^{\alpha_1})\cdots\mathsf{Resultant}(F\circ A, (h_k\circ A)^{\alpha_k})\\ &= \mathsf{Resultant}(F\circ A, (h_1\circ A))^{\alpha_1}\cdots\mathsf{Resultant}(F\circ A, (h_k\circ A))^{\alpha_k}.\end{aligned}$$

Hence, $\mathsf{Resultant}(F\circ A, \mathsf{Jac}(F\circ A, G\circ A))$ does not vanish if and only if each of the terms $\mathsf{Resultant}(F\circ A, h_i\circ A)$ does not vanish. Consequently,

$$\begin{aligned}\mathsf{det}(A)^m\prod_{i=1}^{k}\mathsf{Resultant}(F\circ A, h_i\circ A) &= \mathsf{Resultant}(F\circ A, \mathsf{det}(A)(h_1\circ A)\cdots(h_k\circ A))\\ &= \mathsf{Resultant}(F\circ A, \mathsf{det}(A)J_I\circ A)\\ &= \mathsf{Resultant}(F\circ A, J_A)\end{aligned}$$

does not vanish.

9.4.8 For the non-vanishing of $\mathsf{Resultant}(F\circ A, \mathsf{Jac}(F\circ A, G\circ A))$, Proposition 9.11 will be used again, and hence the following conditions are needed:

$$\begin{aligned}\mathsf{det}(A)\mathsf{Jac}(F, G)\circ A(z,0) = \mathsf{Jac}(F\circ A, G\circ A)(z,0) &\not\equiv 0,\\ F\circ A(z,0) &\neq 0,\\ \partial_w(F\circ A)(z,0) &\neq 0,\\ G\circ A(z,0) &\neq 0.\end{aligned}$$

Again, this can be achieved by a suitable change of coordinates A. Therefore,

$$\mathsf{Resultant}(\mathsf{Jac}(F\circ A, G\circ A), F\circ A) \;=\; n^m \frac{\mathsf{Resultant}(G\circ A, F\circ A)\mathsf{Resultant}(\partial_z(F\circ A), F\circ A)}{\mathsf{Resultant}(w, F\circ A)}.$$

By hypothesis, the fact that $\{G=0\}$ and $\{F=0\}$ have complete intersection at the origin means that $\mathsf{Resultant}(G\circ A, F\circ A)$ does not vanish. To show that $\mathsf{Resultant}(\partial_z(F\circ A), F\circ A)$ does not vanish, this is equivalent to the fact that the discriminant of the polynomial in one-variable $F(z,1)$ does not vanish. This is because

$$\partial_z(F\circ A)(z,1) = \frac{dF\circ A(z,1)}{dz},$$

and therefore

$$\begin{aligned}\mathsf{Resultant}(\partial_z(F\circ A), F\circ A) &= \mathsf{resultant}(\partial_z(F\circ A)(z,1), F\circ A(z,1)\\ &= \mathsf{resultant}\left(\frac{dF\circ A(z,1)}{dz}, F\circ A(z,1)\right).\end{aligned}$$

Hence the discriminant of $F\circ A(z,1)$, which is

$$\mathsf{resultant}\left(\frac{dF\circ A(z,1)}{dz}, F\circ A(z,1)\right),$$

does not vanish if and only if $F\circ A(z,1)$ does not have repeated factors, and thus it is a generic condition on the coefficients of F.

9.4.9 The discussions in paragraph 9.4.8 leads to the following theorem:

Theorem 9.18. *Let F and G be homogeneous polynomials in $\mathbb{C}[z,w]$ with respective degrees $\mathsf{deg}F = m$ and $\mathsf{deg}G = n$ such that*

$$\mathsf{Resultant}(F,G) \neq 0.$$

Then for generic choices of F and G, there exists a generic choice of invertible linear transformation $A:\mathbb{C}^2\to\mathbb{C}^2$, such that Kohn's algorithm terminates in 2 steps for $F\circ A$ and $G\circ A$.

Corollary 9.19. *Let (z,w,v) be holomorphic coordinates on $\mathbb{C}^3$. For generic choices of homogeneous polynomials F and G in $\mathbb{C}[z,w]$ with $m = \mathsf{deg}F$, and $n = \mathsf{deg}G$, let Ω be the special domain given by*

$$\mathsf{Re}(v) - |F(z,w)|^2 + |G(z,w)|^2 < 0.$$

Then at the origin $0\in\mathbb{C}^3$, there exists a neighbourhood $V\subseteq\mathbb{C}^3$ of 0 such that for all $\phi\in\mathscr{D}^{0,1}(V)$,

$$|||\phi|||_\varepsilon^2 \lesssim \|\overline{\partial}\phi\|^2 + \|\overline{\partial}^*\phi\|^2 + \|\phi\|^2,$$

where

$$\varepsilon \geqslant \frac{1}{16(m+n-2)^2(2m+n-2)}.$$

Proof. The first step involves taking the Jacobian $\mathsf{Jac}(F,G)$, which gives

$$|||\mathsf{Jac}(F,G)\varphi|||_{1/4}^2 \lesssim Q(\varphi,\varphi).$$

Observe that

$$\mathrm{radical}(\mathsf{Jac}(F,G))^{m+n-2} \in \langle \mathsf{Jac}(F,G)\rangle,$$

therefore

$$|||\mathrm{radical}(\mathsf{Jac}(F,G))|||^2_{\frac{1}{4(m+n-2)}} \lesssim Q(\varphi,\varphi).$$

Next, taking the Jacobian of radical($\mathsf{Jac}(F,G)$) and F gives the following estimates

$$\begin{aligned} \||\mathsf{Jac}(\text{radical}(\mathsf{Jac}(F,G)),F)\varphi|\|^2_{\frac{1}{8(m+n-2)}} &\lesssim Q(\varphi,\varphi), \\ \||\text{radical}(\mathsf{Jac}(F,G))\varphi|\|^2_{\frac{1}{8(m+n-2)}} &\lesssim Q(\varphi,\varphi). \end{aligned}$$

By considering the degrees, the intersection multiplicities of $\mathsf{Jac}(\text{radical}(\mathsf{Jac}(F,G)),F)$ and radical($\mathsf{Jac}(F,G)$) is at most $(2m+n-2)(m+n-2)$. Hence

$$\begin{aligned} \||z\varphi|\|^2_{\frac{1}{8(m+n-2)(2m+n-2)(m+n-2)}} &\lesssim Q(\varphi,\varphi) \\ \||w\varphi|\|^2_{\frac{1}{8(m+n-2)(2m+n-2)(m+n-2)}} &\lesssim Q(\varphi,\varphi). \end{aligned}$$

Taking the Jacobian of z and w in the final step yields the subelliptic estimate with

$$\varepsilon \geqslant \frac{1}{16(m+n-2)^2(2m+n-2)}.$$

□

This therefore provides a special domain in $\mathbb{C}^4$ which is a triangular system in the sense of D'Angelo-Catlin, and yet there is an effective termination of Kohn's algorithm:

Corollary 9.20. *Let (z_1,z_2,z_3,z_4) be holomorphic coordinates of $\mathbb{C}^4$, and let $P(z_1,z_2)$, $Q(z_1,z_2)$ be homogeneous polynomials of respective degrees m and n so that*

$$\mathsf{dim}_{\mathbb{C}}\mathbb{C}\{z_1,z_2\}\big/\langle P,Q\rangle = mn < \infty.$$

Then for each $k,\ l,\ m \in \mathbb{N}_{\geqslant 1}$ and for generic choices of P and Q, Kohn's algorithm terminates with effectiveness for the following special domain defined by

$$\mathsf{Re}(z_4) - |z_3^k|^2 - \left|z_3^l - P(z_1,z_2)\right|^2 - \left|z_3^m - Q(z_1,z_2)\right|^2.$$

Proof. The first step in Kohn's algorithm consists in taking the Jacobian of the functions $F := z_3^k$, $G := z_3^l - P$ and $H := z_3^m - Q$, which gives

$$\mathsf{Jac}(F,G,H) = kz_3^{k-1}\frac{\partial(P,Q)}{\partial(z_1,z_2)}.$$

Let A_1 be the radical of $\frac{\partial(P,Q)}{\partial(z_1,z_2)}$. Therefore, the first Kohn's ideal of sub-elliptic multipliers is generated by $z_3 \cdot A_1$. Now to generate the next ideal, consider

$$\mathsf{Jac}(z_3^k, z_3^l - P, z_3\cdot A_1) = kz_3^k\frac{\partial(P,A_1)}{\partial(z_1,z_2)}$$

which is also a sub-elliptic multiplier. By the previous result, the ideal in $\mathbb{C}\{z_1,z_2\}$

$$\left\langle A_1, \frac{\partial(P,A_1)}{\partial(z_1,z_2)}\right\rangle$$

contains $\langle z_1,z_2\rangle^{\mathrm{N}}$ for some effective number N. Therefore, both z_1z_3 and z_1z_2 are subelliptic multipliers. Taking the following Jacobian implies that

$$\mathsf{Jac}(z_3^k, z_1z_3, z_2z_3) = z_3^{k+1}$$

is a subelliptic multiplier, and hence by the radical property, so is z_3. It follows that

$$\mathsf{Jac}(z_3, z_3^l - P, z_3^m - Q) = \frac{\partial(P,Q)}{\partial(z_1,z_2)}$$

is a sub-elliptic multiplier, and so is A_1. Moreover,

$$\mathsf{Jac}(z_3, z_3^l - P, A_1) = \frac{\partial(P,A_1)}{\partial(z_1,z_2)}$$

is also a sub-elliptic multiplier. Again by the previous result, this implies that z_1^{N} and z_2^{N} are sub-elliptic multipliers. Since z_3 has been shown to be a sub-elliptic multiplier, the proof concludes. □

Another effective example which can be mentioned is the following. Let (x,y,z) be holomorphic coordinates in $\mathbb{C}^3$ and let F, G and H be holomorphic function germs at the origin so that there exists an effective number k with

$$\langle x,y,z\rangle^k \subset \langle F_1, F_2, F_3\rangle.$$

Hence there exist holomorphic function germs A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3 such that

$$\begin{aligned} x^k &= A_1F_1 + A_2F_2 + A_3F_3, && (9.21)\\ y^k &= B_1F_1 + B_2F_2 + B_3F_3, && (9.22)\\ z^k &= C_1F_1 + C_2F_2 + C_3F_3. && (9.23) \end{aligned}$$

Proposition 9.24. *Let (x,y,z,w) be holomorphic coordinates in $\mathbb{C}^4$ and let Ω be the special domain in $\mathbb{C}^4$ given by the defining function*

$$r := \mathsf{Re}(w) - |F_1|^2 - |F_2|^2 - |F_3|^2 - \left(\sum_{1\leqslant i\leqslant 3} |A_i|^2 + |B_i|^2 + |C_i|^2\right),$$

then Kohn's algorithm terminates with effectiveness.

Proof. The element $\mathsf{Jac}(x^k, y^k, z^k)$ lies in the first subelliptic ideal J_1 because

$$\begin{aligned} \mathsf{Jac}(x^k,y^k,z^k) &= \mathsf{Jac}\left(\sum_{1\leqslant i\leqslant 3} A_iF_i, \sum_{1\leqslant j\leqslant 3} B_jF_j, \sum_{1\leqslant k\leqslant 3} C_kF_k\right)\\ &= \sum_{1\leqslant i,j,k\leqslant 3} \mathsf{Jac}(A_iF_i, B_jF_j, C_kF_k)\\ &= \sum_{1\leqslant i,j,k\leqslant 3}\Big(F_iB_jC_k\mathsf{Jac}(A_i,F_j,F_k) + F_iB_jF_k\mathsf{Jac}(A_i,F_j,C_k)\\ &\quad + F_iF_jF_k\mathsf{Jac}(A_i,B_j,C_k) + F_iF_jC_k\mathsf{Jac}(A_i,B_j,F_k)\\ &\quad + A_iB_jC_k\mathsf{Jac}(F_i,F_j,F_k) + A_iB_jF_k\mathsf{Jac}(F_i,F_j,C_k)\\ &\quad + A_iF_jC_k\mathsf{Jac}(F_i,B_j,F_k) + A_iF_jF_k\mathsf{Jac}(F_i,B_j,C_k)\Big) \end{aligned}$$

which lies in the first ideal of subelliptic multipliers J_1, and hence so is its radical xyz. The next step involves

$$\begin{aligned} \mathsf{Jac}(x^k,y^k,xyz) &= \mathsf{Jac}\left(\sum_{1\leqslant i\leqslant 3} A_iF_i, \sum_{1\leqslant j\leqslant 3} B_iF_i, xyz\right)\\ &= \sum_{1\leqslant i,j\leqslant 3}\big(A_iB_j\mathsf{Jac}(F_i,F_j,xyz) + A_iF_j\mathsf{Jac}(F_i,B_j,xyz)\\ &\quad + F_iB_j\mathsf{Jac}(A_i,F_j,xyz) + F_iF_j\mathsf{Jac}(A_i,B_j,xyz)\big) \end{aligned}$$

which lies in the second subelliptic multiplier ideal J_2, and hence so is its radical xy. In a similar manner, so are xz and yz. Finally

$$x^k = \mathsf{Jac}(x^k, xy, xz) = \sum_{1 \leqslant i \leqslant 3} A_i \mathsf{Jac}(F_i, xy, yz) + F_i \mathsf{Jac}(A_i, xy, yz)$$

which lies in the third subelliptic multiplier ideal J_3, and hence so it x. Similarly, so are y and z, and Kohn's algorithm terminates at the next step. □

Chapter 3

The Hachtroudi-Chern-Moser invariants in CR geometry (Introduction)

This chapter serves as an introduction to the rest of the thesis, and it consists mainly of the differential geometry aspects of CR manifolds.

1 Umbilical points in CR ellipsoids in $\mathbb{C}^2$

The second part of the thesis consists of the calculation of local invariants of CR manifolds. The first part being the calculation of the umbilical locus for certain real hypersurfaces in $\mathbb{C}^2$. In 1932, Elie Cartan [Car]; [Car]; [Car32b] has shown that any local real-analytic hypersurface $M \subset \mathbb{C}^2$ is determined up to local biholomorphic equivalence, by a single local invariant

$$\mathscr{I}^{M}_{\mathsf{Cartan}} : M \to \mathbb{C}.$$

By invariance, it means that any local biholomorphism $h : \mathbb{C}^2 \to \mathbb{C}^2$ transforms $\mathscr{I}^{M}_{\mathsf{Cartan}}$ by a factor of a non-vanishing function $v : M \to \mathbb{C}$ so that

$$\mathscr{I}^{h(M)}_{\mathsf{Cartan}}(h(p)) = v(p)\mathscr{I}^{M}_{\mathsf{Cartan}}(p).$$

This guarantees that the vanishing set

$$\mathsf{Umb}_{\mathsf{CR}}(\mathsf{M}) = \{\mathsf{p} \in \mathsf{M} : \ \mathscr{I}^{\mathsf{M}}_{\mathsf{Cartan}}(\mathsf{p}) = 0\}$$

called the *umbilical locus*, is intrinsic. It has been known for a while that the explicit calculation of such a locus can be notoriously difficult for many cases.

Let $(z, w) = (x + \sqrt{-1}y, u + \sqrt{-1}v)$ be holomorphic coordinates for $\mathbb{C}^2$, and assume that locally M is given by a real-analytic, real-defining function of the form $\{r(z, w, \bar{z}, \bar{w}) = 0\}$. Assume also $0 \in M$ and $r_w(0) \neq 0$, so that M is smooth at 0.

Let $r(z, w, s, t)$ be the polarisation of $r(z, w, \bar{z}, \bar{w})$, which replaces $\bar{z}$ and $\bar{w}$ by holomorphic coordinates s and t. Therefore the vanishing locus of $r(z, w, s, t)$ in $\mathbb{C}^4$ defines a holomorphic hypersurface, which is smooth at 0 by $r_w(0) \neq 0$. Holomorphic implicit funtion theorem then implies the existence of a holomorphic function $\Theta(z, s, t)$ such that $r(z, w, s, t) = 0$ if and only if $w = \Theta(z, s, t)$. By restriction to $\bar{z} = s$ and $\bar{w} = t$, the real hypersurface M can be alternatively described as

$$M = \{w = \Theta(z, \bar{z}, \bar{w})\}.$$

Let $\mathrm{L}(r)$ be the Levi determinant

$$\mathrm{L}(r) = \det \begin{pmatrix} 0 & r_z & r_w \\ r_{\bar{z}} & r_{z\bar{z}} & r_{w\bar{z}} \\ r_{\bar{w}} & r_{z\bar{w}} & r_{w\bar{w}} \end{pmatrix}$$

which this time is assumed to be Levi non-zero. Let Δ be the following expression

$$\Delta = -\Theta_{\bar{w}}\Theta_{z\bar{z}} + \Theta_{\bar{z}}\Theta_{z\bar{w}}.$$

A direct calculation can be done to show that the Levi non-degeneracy condition translates to the non-vanishing of Δ. If $\bar{\mathscr{L}}$ is the vector field given by

$$\bar{\mathscr{L}} := -\frac{1}{r_w}\underbrace{\left(-r_{\bar{w}}\frac{\partial}{\partial\bar{z}} + r_{\bar{z}}\frac{\partial}{\partial\bar{w}}\right)}_{:=\bar{L}} = \frac{\partial}{\partial\bar{z}} - \frac{\Theta_{\bar{z}}}{\Theta_{\bar{w}}}\frac{\partial}{\partial\bar{w}},$$

a theorem in [Mer10] shows that

Theorem 1.1. *At a point $p \in \{\Theta_{\bar{w}} \neq 0\}$, the hypersurface M is spherical if and only if*

$$0 = \left(\frac{1}{\Box}\bar{\mathscr{L}}\right)^4(\Theta_{zz}), \tag{1.2}$$

where $\Box = -\frac{\Delta}{\Theta_w}$.

To recall, a real hypersurface M is spherical if locally it is biholomorphic to $|z|^2 + |w|^2 = 1$. It turns out that the Cartan invariant $\mathscr{I}^M_{\mathsf{Cartan}}$ is a multiple of $\left(\frac{1}{\Box}\bar{\mathscr{L}}\right)^4(\Theta_{zz})$ by a non-vanishing factor. An expansion of (1.2), followed by the substitution of the following identities

$$\Theta_{zz} = \frac{-H(r)}{r_w^3}, \qquad H(r) = r_z r_z r_{ww} - 2r_z r_w r_{zw} + r_w r_w r_{zz},$$

provides the following formula for the Cartan invariant

$$\begin{aligned}
I_{[w]} \;=\; & \left[\frac{\mathrm{L}(r)}{r_w^2}\right]^3 \bar{L}\left(\frac{H(r)}{r_w^3}\right) \\
& -6\left[\frac{\mathrm{L}(r)}{r_w^2}\right]^2 \bar{L}\left(\frac{\mathrm{L}(r)}{r_w^2}\right)\bar{L}^3\left(\frac{H(r)}{r_w^3}\right) - 4\left[\frac{\mathrm{L}(r)}{r_w^2}\right]^2 \bar{L}^2\left(\frac{\mathrm{L}(r)}{r_w^2}\right)\bar{L}^2\left(\frac{H(r)}{r_w^3}\right) \\
& -\left[\frac{\mathrm{L}(r)}{r_w^2}\right]^2 \bar{L}^3\left(\frac{\mathrm{L}(r)}{r_w^2}\right)\bar{L}\left(\frac{H(r)}{r_w^3}\right) + 15\frac{\mathrm{L}(r)}{r_w^2}\left[\mathrm{L}\left(\frac{\mathrm{L}(r)}{r_w^2}\right)\right]^2 \bar{L}^2\left(\frac{H(r)}{r_w^3}\right) \\
& +10\frac{\mathrm{L}(r)}{r_w^2}\bar{L}\left(\frac{\mathrm{L}(r)}{r_w^2}\right)\bar{L}^2\left(\frac{\mathrm{L}(r)}{r_w^2}\right)\bar{L}\left(\frac{H(r)}{r_w^3}\right) - 15\left[\bar{L}\left(\frac{\mathrm{L}(r)}{r_w^2}\right)\right]^3 \bar{L}\left(\frac{H(r)}{r_w^3}\right).
\end{aligned}$$

In the special case where M is an ellipsoid given by

$$ax^2 + y^2 + bu^2 + v^2 = 1 \qquad \big(a\geqslant 1,\ b\geqslant 1,\ (a,b)\neq(1,1)\big),$$

a joint work with Professor Joël Merker and a Ph.D student The-Anh Ta, which is accepted for publication in Comptes Rendus Académie des Sciences, shows that $I_{[w]}$ vanishes on a certain curve contained in M, and thus verifying the existence of umbilical locus:

Theorem 1.3 ([FMT])**.** *For every real numbers $a \geqslant 1$, $b \geqslant 1$ with $(a,b) \neq (1,1)$, the curve parametrised by $\theta \in \mathbb{R}$ valued in $\mathbb{C}^2$:*

$$\gamma:\ \theta :\longmapsto (x(\theta) + \sqrt{-1}y(\theta),\ u(\theta) + \sqrt{-1}v(\theta)),$$

with

$$x(\theta) := \sqrt{\frac{a-1}{a(ab-1)}}\cos\theta, \qquad y(\theta) := \sqrt{\frac{b(a-1)}{ab-1}}\sin\theta$$

and

$$u(\theta) := \sqrt{\frac{b-1}{(ab-1)}}\sin\theta, \qquad v(\theta) := -\sqrt{\frac{a(b-1)}{ab-1}}\cos\theta.$$

has image contained in the CR-umbilical locus $\{I_{[w]} = 0\}$ *of the ellipsoid given by* $ax^2 + y^2 + bu^2 + v^2 = 1$.

2 Holomorphic curves in Lorentzian CR manifolds

Let $(z_1, z_2, w = u + \sqrt{-1}v)$ be holomorphic coordinates of $\mathbb{C}^3$, and M be a real analytic real hypersurface passing through the origin of the form

$$u = F(z_1, z_2, \bar{z}_1, \bar{z}_2).$$

Recall that its $T^{1,0}M$ and $T^{0,1}M$ sections are respectively given by (for $i = 1, 2$),

$$\begin{aligned} \mathscr{L}_i &= \frac{\partial}{\partial z_i} - \sqrt{-1}F_{z_i}\frac{\partial}{\partial v}, \\ \bar{\mathscr{L}_i} &= \frac{\partial}{\partial \bar{z}_i} + \sqrt{-1}F_{\bar{z}_i}\frac{\partial}{\partial v}. \end{aligned}$$

By means of taking the Lie brackets between the $T^{1,0}M$ and the $T^{0,1}M$ sections, the Levi matrix of M^5 is therefore given by

$$\Lambda = 2\begin{pmatrix} F_{z_1\bar{z}_1} & F_{z_1\bar{z}_2} \\ F_{z_2\bar{z}_1} & F_{z_2\bar{z}_2} \end{pmatrix}.$$

Definition 2.1. The real hypersurface $M^5 \subseteq \mathbb{C}^3$ is *Lorentzian* at $0 \in M^5$ if the Levi form at 0 has 1 strictly positive eigenvalue and 1 strictly negative eigenvalue. Moreover the manifold is said to be Lorentzian if this happens at every point.

The main objective of this section is to study the holomorphic immersions of curves

$$\phi : \mathbb{D} \to M$$

passing through the origin. To this end, it is necessary to look at the contact form of M^5, which is a one form given by

$$\theta = -dv + \sum_{k=1,2}(-\sqrt{-1}F_{z_k})dz_k + \sum_{k=1,2}(\sqrt{-1}F_{\bar{z}_k})d\bar{z}_k.$$

Therefore,

$$d\theta = 2\sqrt{-1}\sum_{i,j=1,2} F_{z_i\bar{z}_j}\, dz_i \wedge d\bar{z}_j,$$

which recovers the Levi matrix. By the Gram-Schmidt process, there is a change of local frames of $T^{1,0}M$ and $T^{0,1}M$ that diagonalises $d\theta$ so that

$$d\theta = \sqrt{-1}(\alpha^1 \wedge \bar{\alpha}^1 - \alpha^2 \wedge \bar{\alpha}^2) \bmod \theta.$$

Let $\mathscr{A}_1$, $\mathscr{A}_2$ (resp. $\bar{\mathscr{A}_1}$, $\bar{\mathscr{A}_2}$) be the respective duals of α^1 and α^2 (resp. $\bar{\alpha}^1$, $\bar{\alpha}^2$). Since a holomorphic curve $\phi(\mathbb{D})$ has tangent space lying in $\ker\theta$, its tangent vector may in general be written as

$$\mathscr{L}_{\phi(t)} = f_1(t, \bar{t})\mathscr{A}_1 + f_2(t, \bar{t})\mathscr{A}_2.$$

This tangent vector field must also satisfy the compatibility condition

$$d\theta(\mathscr{L}_{\phi(t)} \wedge \overline{\mathscr{L}_{\phi(t)}}) = 0,$$

and therefore

$$|f_1(t,\bar{t})|^2 - |f_2(t,\bar{t})|^2 = 0.$$

By the assumption that ϕ is an immersion, f_1 cannot vanish anywhere at all $t \in \mathbb{D}$, and hence

$$\left|\frac{f_2(t,\bar{t})}{f_1(t,\bar{t})}\right|^2 = 1,$$

which implies that $f_2 = \lambda f_1$ for some $\lambda \in \mathbb{S}^1$. Since the direction λ is not known, therefore it is natural to set up the following system of differential forms:

$$\begin{aligned}
\omega^0 &= \theta \\
\omega^1 &= \alpha^1 \qquad \bar{\omega}^1 = \bar{\alpha}^1 \\
\omega^2 &= \alpha^2 - \lambda\alpha^1 \qquad \bar{\omega}^2 = \bar{\alpha}^2 - \bar{\lambda}\bar{\alpha}^1,
\end{aligned}$$

and let $I = \langle \omega^0, \omega^2, \bar{\omega}^2 \rangle$.

By a reason which will be explained later, Byrant introduces a second prolongation by adding additional variables μ and thus obtaining the following system of differential forms

$$\begin{aligned}
\omega^0 &= \theta \\
\omega^1 &= \alpha^1 \qquad \omega^2 = \alpha^2 - \lambda\alpha^1 \qquad \pi^2 = d\lambda - L\bar{\alpha}^1 - \mu\alpha^1 \\
\bar{\omega}^1 &= \bar{\alpha}^1 \qquad \bar{\omega}^2 = \bar{\alpha}^2 - \bar{\lambda}\bar{\alpha}^1 \qquad \bar{\pi}^2 = d\bar{\lambda} - \bar{L}\alpha^1 - \bar{\mu}\bar{\alpha}^1,
\end{aligned}$$

with $I_2 = \langle \omega^0, \omega^2, \bar{\omega}^2, \pi^2, \bar{\pi}^2 \rangle$. The new variable μ will have to satisfy the equation of sphere and a hyperplane

$$\begin{aligned}
\bar{\lambda}\mu &= \lambda\bar{L}, &(2.2)\\
|\mu - B|^2 &= |B|^2 + AL + \bar{A}\bar{L} + |L|^2 - C. &(2.3)
\end{aligned}$$

By substituting equation (2.2) into (2.3), this implies that λ will have to satisfy the following equation

$$|\lambda^2\bar{L} - B|^2 = |B|^2 + AL + \bar{A}\bar{L} + |L|^2 - C \tag{2.4}$$

subjected to $|\lambda|^2 = 1$. Equation (2.4) describes a hyperbola in $\mathbb{R}^2$ and hence its intersection with the circle $|\lambda|^2 = 1$ has at most 4 points of intersection. Finally define the following 1-form $\tau = \bar{\lambda}d\lambda + \bar{\lambda}L\bar{\alpha}^1 - \lambda\bar{L}\alpha^1$, and let $I_+ := \langle I, \tau \rangle$ be the ideal generated by I and τ, the 1-from $d\tau$ can be written as

$$d\tau = -[S_{22\bar{1}\bar{1}}\lambda^2 + 4S_{21\bar{1}\bar{1}}\lambda + 6S_{11\bar{1}\bar{1}} + 4\overline{S_{21\bar{1}\bar{1}}}\bar{\lambda} + \overline{S_{22\bar{1}\bar{1}}}\bar{\lambda}^2] \text{ mod } I_+.$$

where the coefficients are components of Hachtroudi-Chern-Moser's S tensors.

3 The Hachtroudi-Chern-Moser Tensor

The computation of CR invariants in higher dimensions involves the study of the S-tensor, $S^{\beta\sigma}_{\alpha\rho}$, which has its origin traced back to the 1939 Ph.D thesis (thèse de l'entre-deux-guerres) of Mosen Hachtroudi [Hac37], who was a student of Élie Cartan.

Let $\mathbb{K}$ be either $\mathbb{C}$ or $\mathbb{R}$, and let $(x^1, \ldots, x^n, y) \in \mathbb{K}^{n+1}$ be its coordinates. The first and second jet space, denoted respectively by $J^1(\mathbb{K}^n, \mathbb{K})$ and $J^2(\mathbb{K}^n, \mathbb{K})$, of the graphing function $y(x^1, \ldots, x^n)$, are equipped with the following coordinates

$$\begin{aligned}
(x^\alpha, y, y_{x^\beta}) &\in \mathbb{K}^{n+1+n}, \\
(x^\alpha, y, y_{x^\beta}, y_{x^\gamma x^\delta}) &\in \mathbb{K}^{n+1+n+\frac{n(n+1)}{2}}.
\end{aligned}$$

The starting point is to consider the following system of partial differential equations

$$y_{x^\alpha x^\beta} = F^{\alpha,\beta}(x^\gamma, y, y_{x^\delta}) \qquad (1\leqslant\alpha,\beta\leqslant n),$$

with the compatibility condition that

$$F^{\alpha,\beta} = F^{\beta,\alpha}.$$

Moreover, assume that this system is completely integrable. This means that for any choice of constants

$$(a^1, \ldots, a^n, b) \in \mathbb{K}^{n+1},$$

there exists a unique $\mathbb{K}$-analytic solution

$$y = Q(x, a^1, \ldots, a^n, b)$$

satisfying

$$Q_{x^\alpha x^\beta} = F^{\alpha,\beta}(x^\gamma, Q(x,a,b), Q_{x^\delta}(x,a,b))$$

with the initial conditions

$$Q_{x^\alpha x^\beta}(0,a,b) = b, \qquad Q_{x^\gamma}(0,a,b) = a^\gamma, \qquad (1\leqslant\gamma\leqslant n).$$

If

$$\mathsf{D}_{x^\alpha} := \frac{\partial}{\partial x^\alpha} + y_{x^\alpha}\frac{\partial}{\partial y} + \sum_{\beta=1}^{n} y_{x^\alpha x^\beta}\frac{\partial}{\partial y_{x^\beta}}.$$

is the vector field on $J^1(\mathbb{K}^n, \mathbb{K})$, the complete integrability condition is equivalent by Frobenius' theorem to the following compatibility equation:

$$\mathsf{D}_{x^\gamma} F^{\alpha\beta} = \mathsf{D}_{x^\beta} F^{\alpha\gamma}.$$

Using Cartan's method, the following formula of the Hachtroudi-Chern-Moser S-tensor (See [Bie07]) is therefore reconstructed:

$$\begin{aligned} S^{\beta\sigma}_{\alpha\rho} &= F^{\alpha,\rho}_{y_{x^\beta}, y_{x^\sigma}} - \frac{1}{n+2}\left(\delta^\sigma_\rho \sum_\varepsilon F^{\alpha,\varepsilon}_{y_{x^\beta}, y_{x^\varepsilon}} + \delta^\sigma_\alpha \sum_\varepsilon F^{\rho,\varepsilon}_{y_{x^\beta}, y_{x^\varepsilon}} + \delta^\beta_\rho \sum_\varepsilon F^{\alpha,\varepsilon}_{y_{x^\sigma}, y_{x^\varepsilon}} + \delta^\beta_\alpha \sum_\varepsilon F^{\rho,\varepsilon}_{y_{x^\sigma}, y_{x^\varepsilon}}\right) \\ &\quad + \frac{1}{(n+1)(n+2)} \sum_\varepsilon \sum_\delta (\delta^\sigma_\rho \delta^\beta_\alpha + \delta^\beta_\rho \delta^\sigma_\alpha) F^{\delta,\varepsilon}_{y_{x^\delta}, y_{x^\varepsilon}}. \end{aligned}$$

To adapt the tensor above to CR hypersurface case, similar argument in the section on ellipsoid in $\mathbb{C}^2$ may be applied to real hypersurfaces in higher dimensions.

To this effect, let $(z_1, \ldots, z_n, w)$ be holomorphic coordinates in $\mathbb{C}^{n+1}$ and let $z := (z_1, \ldots, z_n)$. Suppose that the real hypersurface $M \subset \mathbb{C}^{n+1}$ is given by the local real-valued, real-analytic defining function $\{r(z, w, \bar{z}, \bar{w}) = 0\}$, along with the condition that $r = \bar{r}$ and $r_w(0) \neq 0$. Let $(s,t) := (s_1, \ldots, s_n, w)$ be the holomorphic coordinates, and $r(z,w,s,t)$ be the polarisation of r. Then $r(z,w,s,t)$ defines a holomorphic hypersurface in $\mathbb{C}^{2n+2}$. By the holomorphic implicit function theoreom, there exists a holomorphic function $\Theta(z,s,t)$ such that $r(z,w,s,t) = 0$ if and only if $w = \Theta(z,s,t)$. Hence by restricting to the subspaces $s = \bar{z}$ and $t = \bar{w}$, the manifold has an alternative description $M = \{w = \Theta(z, \bar{z}, \bar{w})\}$.

Next, fix the following notation (z, w, w_z) which denotes the holomorphic coordinates in $\mathbb{C}^{2n+1}$. Let $\varphi : (\mathbb{C}^{2n+1}, (z,s,t)) \to (\mathbb{C}^{2n+1}, (z,w,w_z))$ be the holomorphic map given by

$$\begin{aligned} \varphi : \mathbb{C}^{2n+1} &\longrightarrow \mathbb{C}^{2n+1} \\ (z,s,t) &\longmapsto (z, \Theta, \Theta_z). \end{aligned}$$

An additional assumption that M is Levi non-degenerate means that the determinant of the following matrix

$$\mathrm{L}(r) := \det \underbrace{\begin{pmatrix} 0 & r_{z_1} & \cdots & r_{z_n} & r_w \\ r_{\bar{z}_1} & r_{z_1\bar{z}_1} & \cdots & r_{z_n\bar{z}_1} & r_{w\bar{z}_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{\bar{z}_n} & r_{z_1\bar{z}_n} & \cdots & r_{z_n\bar{z}_n} & r_{w\bar{z}_n} \\ r_{\bar{w}} & r_{z_1\bar{w}} & \cdots & r_{z_n\bar{w}} & r_{w\bar{w}} \end{pmatrix}}_{:=\hat{\mathrm{L}}(r)}$$

vanishes nowhere. This condition is equivalent to the non-vanishing of the Jacobian of the map φ, and thus the non-vanishing of the determinant

$$\Delta(\Theta) = \det \underbrace{\begin{pmatrix} \Theta_{s_1} & \cdots & \Theta_{s_n} & \Theta_t \\ \Theta_{z_1 s_1} & \cdots & \Theta_{z_1 s_n} & \Theta_{z_1 t} \\ \vdots & \ddots & \vdots & \vdots \\ \Theta_{z_n s_1} & \cdots & \Theta_{z_n s_n} & \Theta_{z_n t} \end{pmatrix}}_{:=\mathrm{j}(\Theta)}.$$

Therefore, φ is a local biholomorphism with the inverse

$$\varphi^{-1}(z, w, w_z) = (z, \Lambda_1(z, w, w_z), \ldots, \Lambda_n(z, w, w_z), \Pi(z, w, w_z)).$$

If $\Delta_{(i,j)}(\Theta)$ is the determinant of the matrix $\Delta_{(i,j)}(\Theta) = (-1)^{i+j}\det(\mathrm{j}(\Theta_{(j,i)}))$, where $\mathrm{j}(\Theta_{(i,j)})$ is the matrix with i-th row and j-th column removed, then for $1 \leqslant i, j \leqslant n$, an application of Cramer's rule shows that the vector field $\partial_{w_{z_j}}$ may be given by

$$\partial_{w_{z_j}} = \sum_{1 \leqslant l \leqslant n} \frac{\Delta_{(l,j+1)}(\Theta)}{\Delta(\Theta)} \partial_{s_l} + \frac{\Delta_{(n+1,j+1)}(\Theta)}{\Delta(\Theta)} \partial_t.$$

At this point, Cartan's method applied to

$$\begin{aligned} w_{z_i z_j} &= \Theta_{z_i z_j}(z, s, t) \\ &= \Theta_{z_i, z_j}(z, \Lambda_1(z, w, w_z), \ldots, \Lambda_n(z, w, w_z), \Pi(z, w, w_z)) \\ &:= G^{i,j}(z, w, w_z) \end{aligned}$$

provides the following Hachtroudi-Chern-Moser tensor

$$\begin{aligned} S^{\beta\sigma}_{\alpha\rho} &= G^{\alpha,\rho}_{w_{z_\beta}, w_{z_\sigma}} - \frac{1}{n+2}\left(\delta^\sigma_\rho \sum_\varepsilon G^{\alpha,\varepsilon}_{w_{z_\beta}, w_{z_\varepsilon}} + \delta^\sigma_\alpha \sum_\varepsilon G^{\rho,\varepsilon}_{w_{z_\beta}, w_{z_\varepsilon}} + \delta^\beta_\rho \sum_\varepsilon G^{\alpha,\varepsilon}_{w_{z_\sigma}, w_{z_\varepsilon}} + \delta^\beta_\alpha \sum_\varepsilon G^{\rho,\varepsilon}_{w_{z_\sigma}, w_{z_\varepsilon}} \right) \\ &\quad + \frac{1}{(n+1)(n+2)} \sum_\varepsilon \sum_\delta (\delta^\sigma_\rho \delta^\beta_\alpha + \delta^\beta_\rho \delta^\sigma_\alpha) G^{\delta,\varepsilon}_{w_{z_\delta}, w_{z_\varepsilon}}. \end{aligned}$$

One of the results in my thesis is to translate the formula above in terms of the implicit function $r = 0$ instead of the graphing function $w = \Theta(z, \bar{z}, \bar{w})$. For $1 \leqslant i \leqslant n+1$ and $1 \leqslant l \leqslant n$, let $\hat{L}_{i,l}(r)$ be the matrix with $i+1$-th row and $l+1$-th column of $\hat{L}(r)$ deleted, and let $L_{i,l}(r)$ be its determinant. By exploiting the relations $r(z, \Theta(z, \bar{z}, \bar{w}), \bar{z}, \bar{w}) \equiv 0$, the final form of the vector field $\partial_{w_{z_k}}$ is given by

$$\partial_{w_{z_k}} = \sum_{1 \leqslant l \leqslant n} r_w \frac{(-1)^{l+k+1} L_{l,k+1}(r)}{L(r)} \partial_{\bar{z}_l} + r_w \frac{(-1)^{n+1+k+1} L_{n+1,k+1}(r)}{L(r)} \partial_{\bar{w}},$$

and thus the following theorem:

Theorem 3.1 (Theorem 13.20 in page 234)**.** *For $n \geqslant 2$, $1 \leqslant k_1, k_2 \leqslant n$ and $1 \leqslant i_1, i_2 \leqslant n$, the CR-umbilical locus is the zero set*

$$\begin{aligned} 0 \;=\; & \partial_{w_{z_{i_1}}}\left(\partial_{w_{z_{i_2}}}\frac{H_{k_1,k_2}}{r_w^3}\right) - \frac{1}{n+2}\sum_{l=1}^{n}\left(\delta_{k_1,i_1}\partial_{w_{z_l}}\left(\partial_{w_{z_{i_2}}}\frac{H_{l,k_2}}{r_w^3}\right) + \delta_{k_1,i_2}\partial_{w_{z_{i_1}}}\left(\partial_{w_{z_l}}\frac{H_{l,k_2}}{r_w^3}\right)\right. \\ & \left. +\delta_{k_2,i_1}\partial_{w_{z_l}}\left(\partial_{w_{z_{i_2}}}\frac{H_{k_1,l}}{r_w^3}\right) + \delta_{k_2,i_2}\partial_{w_{z_{i_1}}}\left(\partial_{w_{z_l}}\frac{H_{k_1,l}}{r_w^3}\right)\right) \\ & + \frac{1}{(n+1)(n+2)}\left(\delta_{k_1,i_1}\delta_{k_2,i_2} + \delta_{k_2,i_1}\delta_{k_1 i_2}\right)\sum_{l_1=1}^{n}\sum_{l_2=1}^{n}\partial_{w_{l_1}}\left(\partial_{w_{l_2}}\frac{H_{l_1,l_2}}{r_w^3}\right), \end{aligned}$$

where

$$\Theta_{z_i z_j} = \frac{H_{ij}}{r_w^3}, \qquad H_{ij} = -\left[r_w r_w r_{z_i z_j} - r_{z_j} r_w r_{z_i w} - r_{z_i} r_w r_{z_j w} + r_{z_i} r_{z_j} r_{ww}\right].$$

Chapter 4

Umbilic Points in Ellipsoid in $\mathbb{C}^2$

0.0.10 The real-analytic function ρ. Let $(z,w) = (x+\sqrt{-1}y, u+i\sqrt{-1}v)$ be holomorphic coordinates of $\mathbb{C}^2$, and let

$$\begin{array}{rcl} \rho : \mathbb{C}^2 & \longrightarrow & \mathbb{R} \\ (z,w) & \longmapsto & \rho(z,w,\bar{z},\bar{w}) \end{array}$$

be a real analytic, real-valued function. The real analyticity of ρ means that in a neighbourhood V of the origin such that $\overline{V}$ is compact, ρ may be expressed as a power series that converges normally on V:

$$\rho(z,w,\bar{z},\bar{w}) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_{ijkl}\, z^i w^j \bar{z}^k \bar{w}^l. \qquad (a_{ijkl}\in\mathbb{C}).$$

The condition that ρ is real implies that the coefficients a_{ijkl} satisfy the following property

$$\overline{a_{ijkl}} = a_{klij}. \qquad (0.1)$$

Taking derivatives of ρ with respect to z and w:

$$\begin{array}{rcl} \rho_z(z,w,\bar{z},\bar{w}) & = & \displaystyle\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} i\cdot a_{ijkl}\, z^{i-1} w^j \bar{z}^k \bar{w}^l, \\ \rho_w(z,w,\bar{z},\bar{w}) & = & \displaystyle\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} j\cdot a_{ijkl}\, z^{i} w^{j-1} \bar{z}^k \bar{w}^l. \end{array}$$

Taking conjugates on both sides,

$$\begin{array}{rcl} \overline{\rho_z(z,w,\bar{z},\bar{w})} & = & \displaystyle\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} i\cdot \overline{a_{ijkl}}\, \bar{z}^{i-1} \bar{w}^j z^k w^l, \\ \overline{\rho_w(z,w,\bar{z},\bar{w})} & = & \displaystyle\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} j\cdot \overline{a_{ijkl}}\, \bar{z}^{i} \bar{w}^{j-1} z^k w^l. \end{array}$$

By using (0.1), the identities above become

$$\begin{array}{rcl} \overline{\rho_z(z,w,\bar{z},\bar{w})} & = & \displaystyle\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} i\cdot a_{klij}\, \bar{z}^{i-1} \bar{w}^j z^k w^l, \\ \overline{\rho_w(z,w,\bar{z},\bar{w})} & = & \displaystyle\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} j\cdot a_{klij}\, \bar{z}^{i} \bar{w}^{j-1} z^k w^l. \end{array}$$

Switching the indicies i with k, and j with l:

$$\begin{aligned}\overline{\rho_z(z,w,\bar{z},\bar{w})} &= \sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} k\cdot a_{ijkl}\, z^i w^j \bar{z}^{k-1}\bar{w}^l,\\ \overline{\rho_w(z,w,\bar{z},\bar{w})} &= \sum_{k=0}^{\infty}\sum_{l=1}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} l\cdot a_{ijkl}\, z^i w^j \bar{z}^{k}\bar{w}^{l-1}.\end{aligned}$$

The equations above imply the following identities

$$\begin{aligned}\overline{\rho_z(z,w,\bar{z},\bar{w})} &= \rho_{\bar{z}}(z,w,\bar{z},\bar{w}),\\ \overline{\rho_w(z,w,\bar{z},\bar{w})} &= \rho_{\bar{w}}(z,w,\bar{z},\bar{w}).\end{aligned} \tag{0.2}$$

0.0.11 Real manifold M in $\mathbb{C}^2$. Let M be the real hypersurface in $\mathbb{C}^2$ given by

$$M = \{(z,w)\in\mathbb{C}^2 : \rho(z,w,\bar{z},\bar{w}) = 0\},$$

with the assumption that

$$d\rho|_M = \rho_z|_M\, dz + \rho_w|_M\, dw + \rho_{\bar{z}}|_M d\bar{z} + \rho_{\bar{w}}|_M d\bar{w}$$

does not vanish anywhere on M, and therefore is a manifold of real dimension 3 in $\mathbb{C}^2\cong\mathbb{R}^4$. Moreover, ρ_z, ρ_w, $\rho_{\bar{z}}$ and $\rho_{\bar{w}}$ do not simultaneously vanish on M. By equation (0.2), this means that ρ_z and ρ_w do not vanish simultaneously on M, and the same conclusion applies to the pair $\rho_{\bar{z}}$ and $\rho_{\bar{w}}$.

Such a defining function of M is not unique. In fact, if $\mu(z,w,\bar{z},\bar{w})$ is another real-valued defining function that does not vanish anywhere on M, then $M=\{\mu\rho=0\}$. Conversely, if $\tilde{\rho}$ is another defining function of M, then there exists a real-valued function μ that vanishes nowhere on $\mathbb{C}^2$ such that $\tilde{\rho}=\mu\rho$, as shown in the following lemma below:

Lemma 0.3. *Let $n\geqslant 2$ and let r_1 and r_2 be two local real analytic, real valued defining functions for M such that both of them vanish at the origin. Assume also that $dr_1(0)$ and $dr_2(0)$ do not vanish. Then there exists an open neighbourhood $U\subseteq\mathbb{C}^n$ of the origin, and a non-vanishing function h on U such that $r_1 = hr_2$. Moreover, for $x\in U\cap M$, $dr_1 = h\cdot dr_2$ for $x\in U\cap M$.*

Proof. Since $dr_2(0)\neq 0$, there exists a neighbourhood of the origin U such that after a local change of coordinates, $M=\{r_2=0\}=\{x_n=0\}$. Let $x'=(x_1,\ldots,x_{n-1})$. Then $r_1(x',0)=0$ and by the Fundamental Theorem of calculus,

$$r_1(x',x) \;=\; r_1(x',x_n) - r_1(x',0) \;=\; x_n\int_0^1 r_{1,x_n}(x',tx_n)\,dt. \qquad \square$$

Therefore, h is found, which is real-analytic, and

$$dr_1 \;=\; d(hr_2) \;=\; h\cdot dr_2 \;+\; r_2\cdot dh,$$

so that if $(x',x)\in M$, $dr_1 = h\cdot dr_2$. At the origin, $dr_1(0)=h(0)\cdot dr_2(0)$, and by hypothesis, h does not vanish at the origin. By further restriction to a smaller open set of the origin, h may be assumed to be non-vanishing everywhere.

0.0.12 The tangent bundle of M. Let TM denote the real tangent bundle on M, and let $\mathbb{C}TM = TM \otimes_{\mathbb{R}} \mathbb{C}$ be its complexified tangent bundle. Designate at each $p \in M$ the following vector spaces

$$\begin{aligned} T^{1,0}_p M &= \mathbb{C}T_pM \cap T^{1,0}_p\mathbb{C}, \\ T^{0,1}_p M &= \mathbb{C}T_pM \cap T^{0,1}_p\mathbb{C}. \end{aligned}$$

In terms of sets,

$$\begin{aligned} T^{1,0}_p M &= \{a\partial_z + b\partial_w : \ (a,b) \in \mathbb{C}^2, \ a\rho_z(p) + b\rho_w(p) = 0\}, \\ T^{0,1}_p M &= \{a\partial_{\bar z} + b\partial_{\bar w} : \ (a,b) \in \mathbb{C}^2, \ a\rho_{\bar z}(p) + b\rho_{\bar w}(p) = 0\}. \end{aligned} \tag{0.4}$$

Let $T^{1,0}M$ and $T^{0,1}M$ be the following complex vector bundles

$$\begin{aligned} T^{1,0}M &= \sqcup_{p\in M} T^{1,0}_p M, \\ T^{0,1}M &= \sqcup_{p\in M} T^{0,1}_p M. \end{aligned}$$

A $T^{1,0}M$ vector field (or section) is a vector field X of the form

$$X = a(z,w,\bar z,\bar w)\partial_z + b(z,w,\bar z,\bar w)\partial_w$$

where a and b are functions on M such that for all $p \in M$, $X(p) = a(p)\partial_z + b(p)\partial_w \in T^{1,0}_p M$. The $T^{0,1}M$ vector field is defined similarly by replacing ∂_z (respectively ∂_w) by $\partial_{\bar z}$ (respectively $\partial_{\bar w}$). Set

$$\begin{aligned} L &:= -\rho_w \frac{\partial}{\partial z} + \rho_z \frac{\partial}{\partial w}, \\ \bar L &:= -\rho_{\bar w} \frac{\partial}{\partial \bar z} + \rho_{\bar z} \frac{\partial}{\partial \bar w}. \end{aligned}$$

The assumption that $d\rho|_M$ does not vanish anywhere on M implies that both L and $\bar L$ do not vanish at any point on M. Moreover, they are complex conjugates of each other by (0.2). They also easily verify

$$\begin{aligned} L\rho &= -\rho_w\rho_z + \rho_z\rho_w = 0, \\ \bar L\rho &= -\rho_{\bar w}\rho_{\bar z} + \rho_{\bar z}\rho_{\bar w} = 0. \end{aligned}$$

By equation (0.4), L (respectively $\bar L$) is a section of the $T^{1,0}M$ (respectively $T^{0,1}M$) bundle. Since $T^{1,0}M$ is a rank 1 complex line bundle, and L does not vanish anywhere on M, L generates $T^{1,0}M$ at each point of M. Similarly, $\bar L$ generates $T^{0,1}M$ at each point of M.

0.0.13 The Levi form $\mathrm{L}(\rho)$ By the previous paragraph, the $T^{1,0}M$ sections X and Y can be expressed as

$$X = a(z,w,\bar z,\bar w)L \qquad \text{and} \qquad Y = b(z,w,\bar z,\bar w)L$$

for some complex-valued functions a and b. Assume that X and Y are $\mathscr{C}^1$. At each point $p \in M$, there is a bilinear form

$$\begin{aligned} \mathrm{L}_p(\rho) : T^{1,0}_p M \times T^{0,1}_p M &\longrightarrow \mathbb{C}T_pM \mod T^{1,0}_p M \oplus T^{0,1}_p M \\ (X(p), Y(p)) &\longmapsto [X, \bar Y](p) \mod T^{1,0}_p M \oplus T^{0,1}_p M. \end{aligned}$$

This map is well-defined for two reasons. First, if at $p \in M$, two vectors are given

$$A_p := \sigma L(p) \qquad \text{and} \qquad B_p := \theta L(p) \qquad (\sigma,\theta\in\mathbb{C}).$$

Therefore, it is possible to choose two $\mathscr{C}^1$ vector fields

$$X := \sigma L \qquad \text{and} \qquad Y := \theta L \qquad (\sigma, \theta \in \mathbb{C})$$

so that $\mathrm{L}_p(\rho)(A_p, B_p)$ can be defined for every θ and σ. Secondly, the map $\mathrm{L}_p(\rho)$ is independent of the choice of vector fields X and Y such that $X(p) = \sigma L(p)$ and $Y(p) = \theta L(p)$. Suppose another set of $\mathscr{C}^1$ vector fields $\tilde{X}$ and $\tilde{Y}$ are given so that

$$\tilde{X}(p) = \sigma L(p) \qquad \text{and} \qquad \tilde{Y}(p) = \theta L(p).$$

There exist $\mathscr{C}^1$ functions $\tilde{a}$ and $\tilde{b}$ such that

$$\tilde{X} = \tilde{a}(z, w, \bar{z}, \bar{w})L \qquad \text{and} \qquad \tilde{Y} = \tilde{b}(z, w, \bar{z}, \bar{w})L$$

with

$$\tilde{a}(p) = \sigma = a(p) \qquad \text{and} \qquad \tilde{b}(p) = \theta = b(p).$$

Taking the Lie bracket $[X, \bar{Y}]$,

$$\begin{aligned} [X, \bar{Y}] &= [aL, \bar{b}\bar{L}] \\ &= aL(\bar{b})\bar{L} + a\bar{b}L\bar{L} - \bar{b}\bar{L}(a)L - \bar{b}a\bar{L}L. \end{aligned}$$

At the point $p \in M$, taking modulo $T^{1,0}_p M \oplus T^{0,1}_p M$,

$$[X, \bar{Y}] \equiv a(p)\bar{b}(p)[L, \bar{L}] \qquad \mathsf{mod} \quad T^{1,0}_p M \oplus T^{0,1}_p M.$$

By formally replacing a by $\tilde{a}$ and b by $\tilde{b}$ in the calculations above,

$$\begin{aligned} [\tilde{X}, \overline{\tilde{Y}}](p) &\equiv \tilde{a}(p)\overline{\tilde{b}(p)}[L, \bar{L}] \qquad \mathsf{mod} \qquad T^{1,0}_p M \oplus T^{0,1}_p M \\ &\equiv [X, \bar{Y}](p) \qquad \mathsf{mod} \qquad T^{1,0}_p M \oplus T^{0,1}_p M. \end{aligned}$$

Therefore, the Levi map $\mathrm{L}_p(\rho)$ is well-defined at every point $p \in M$.

0.0.14 Explicit Calculations of $[L, \bar{L}]$ To obtain more information about the bilinear map $\mathrm{L}(\rho)$, $[L, \bar{L}]$ needs to be calculated.

$$\begin{aligned} [L, \bar{L}] &= [-\rho_w\partial_z + \rho_z\partial_w,\ -\rho_{\bar{w}}\partial_{\bar{z}} + \rho_{\bar{z}}\partial_{\bar{w}}] \\ &= [-\rho_w\partial_z,\ -\rho_{\bar{w}}\partial_{\bar{z}}] - [\rho_w\partial_z,\ \rho_{\bar{z}}\partial_{\bar{w}}] - [\rho_z\partial_w,\ \rho_{\bar{w}}\partial_{\bar{z}}] + [\rho_z\partial_w,\ \rho_{\bar{z}}\partial_{\bar{w}}] \\ &= \rho_w\rho_{\bar{w}z}\partial_{\bar{z}} + \rho_w\rho_{\bar{w}}\partial^2_{z\bar{z}} - \rho_{\bar{w}}\rho_{w\bar{z}}\partial_z - \rho_w\rho_{\bar{w}}\partial^2_{z\bar{z}} \\ &\quad -\rho_w\rho_{z\bar{z}}\partial_{\bar{w}} - \rho_w\rho_{\bar{z}}\partial^2_{z\bar{w}} + \rho_{\bar{z}}\rho_{w\bar{w}}\partial_z + \rho_{\bar{z}}\rho_w\partial^2_{z\bar{w}} \\ &\quad -\rho_z\rho_{\bar{w}w}\partial_{\bar{z}} - \rho_z\rho_{\bar{w}}\partial^2_{w\bar{z}} + \rho_{\bar{w}}\rho_{z\bar{z}}\partial_w + \rho_{\bar{w}}\rho_z\partial^2_{w\bar{z}} \\ &\quad +\rho_z\rho_{\bar{z}w}\partial_{\bar{w}} + \rho_z\rho_{\bar{z}}\partial^2_{w\bar{w}} - \rho_{\bar{z}}\rho_{z\bar{w}}\partial_w - \rho_{\bar{z}}\rho_z\partial_{w\bar{w}} \\ &= \rho_w\rho_{\bar{w}z}\partial_{\bar{z}} - \rho_{\bar{w}}\rho_{w\bar{z}}\partial_z - \rho_w\rho_{z\bar{z}}\partial_{\bar{w}} + \rho_{\bar{z}}\rho_{w\bar{w}}\partial_z \\ &\quad -\rho_z\rho_{\bar{w}w}\partial_{\bar{z}} + \rho_{\bar{w}}\rho_{z\bar{z}}\partial_w + \rho_z\rho_{\bar{z}w}\partial_{\bar{w}} - \rho_{\bar{z}}\rho_{z\bar{w}}\partial_w \\ &= \frac{\rho_w\rho_{\bar{w}z}}{\rho_{\bar{w}}}\rho_{\bar{w}}\partial_{\bar{z}} - \frac{\rho_{\bar{w}}\rho_{w\bar{z}}}{\rho_w}\rho_w\partial_z + \frac{\rho_{\bar{z}}\rho_{w\bar{w}}}{\rho_w}\rho_w\partial_z - \frac{\rho_z\rho_{w\bar{w}}}{\rho_{\bar{w}}}\rho_{\bar{w}}\partial_{\bar{z}} \\ &\quad -\rho_w\rho_{z\bar{z}}\partial_{\bar{w}} + \rho_{\bar{w}}\rho_{z\bar{z}}\partial_w + \rho_z\rho_{\bar{z}w}\partial_{\bar{w}} - \rho_{\bar{z}}\rho_{z\bar{w}}\partial_w. \end{aligned}$$

Since L and $\bar{L}$ are sections of the $T^{1,0}M \oplus T^{0,1}M$ bundle, using

$$\begin{aligned} \rho_w\partial_z &\equiv \rho_z\partial_w \qquad \mathsf{mod}\ T^{1,0}M \oplus T^{0,1}M \\ \rho_{\bar{w}}\partial_{\bar{z}} &\equiv \rho_{\bar{z}}\partial_{\bar{w}} \qquad \mathsf{mod}\ T^{1,0}M \oplus T^{0,1}M, \end{aligned}$$

to replace ∂_z (resp $\partial_{\bar z}$) by ∂_w (resp $\partial_{\bar w}$), the calculation continues as follows

$$\begin{aligned}
[L,\bar L] &\equiv \frac{\rho_w\rho_{\bar w z}}{\rho_{\bar w}}\rho_{\bar z}\partial_{\bar w} - \frac{\rho_{\bar w}\rho_{w\bar z}}{\rho_w}\rho_z\partial_w + \frac{\rho_{\bar z}\rho_{w\bar w}}{\rho_w}\rho_z\partial_w - \frac{\rho_z\rho_{\bar w w}}{\rho_{\bar w}}\rho_{\bar z}\partial_{\bar w}\\
&\quad -\rho_w\rho_{z\bar z}\partial_{\bar w} + \rho_{\bar w}\rho_{z\bar z}\partial_w + \rho_z\rho_{\bar z w}\partial_{\bar w} - \rho_{\bar z}\rho_{z\bar w}\partial_w\\
&\equiv \frac{1}{\rho_{\bar w}}[\rho_w\rho_{\bar w z} - \rho_z\rho_{\bar z}\rho_{\bar w w} - \rho_{\bar w}\rho_w\rho_{z\bar z} + \rho_{\bar w}\rho_z\rho_{\bar z w}]\partial_{\bar w}\\
&\quad -\frac{1}{\rho_w}[\rho_w\rho_{\bar z}\rho_{z\bar w} - \rho_z\rho_{\bar z}\rho_{\bar w w} - \rho_w\rho_{\bar w}\rho_{z\bar z} + \rho_{\bar w}\rho_z\rho_{w\bar z}]\partial_w\\
&\equiv [\rho_w\rho_{\bar w z}\rho_{\bar z} - \rho_z\rho_{\bar z}\rho_{\bar w w} - \rho_{\bar w}\rho_w\rho_{z\bar z} + \rho_{\bar w}\rho_z\rho_{\bar z w}]\cdot\\
&\quad \left(\frac{1}{\rho_{\bar w}}\partial_{\bar w} - \frac{1}{\rho_w}\partial_w\right) \mod T^{1,0}M\oplus T^{0,1}M.
\end{aligned}$$

Definition 0.5. A smooth hypersurface $M^3\subset\mathbb{C}^2$ is called Levi non-degenerate at $p\in M$ if

$$\mathrm{L}(\rho) := -(\rho_w\rho_{\bar w z}\rho_{\bar z} - \rho_z\rho_{\bar z}\rho_{\bar w w} - \rho_{\bar w}\rho_w\rho_{z\bar z} + \rho_{\bar w}\rho_z\rho_{\bar z w})$$

does not vanish at p.

The expression $\mathrm{L}(\rho)$ coincides with the determinant of a certain matrix

$$\mathrm{L}(\rho) = -\begin{vmatrix} 0 & \rho_z & \rho_w\\ \rho_{\bar z} & \rho_{z\bar z} & \rho_{w\bar z}\\ \rho_{\bar w} & \rho_{z\bar w} & \rho_{w\bar w}\end{vmatrix}.$$

and its general expression in higher dimensions will be presented in the last chapter.

0.0.15 Action on M by biholomorphism of $\mathbb{C}^2$.

Let

$$\begin{aligned} h:\mathbb{C}^2 &\longrightarrow \mathbb{C}^2\\ (z,w) &\longmapsto (f(z,w),g(z,w)) := (z',w')\end{aligned}$$

be a local biholomorphism. The vector fields ∂_z, ∂_w, $\partial_{z'}$ and $\partial_{w'}$ are related by

$$\begin{aligned}\partial_z &= f_z\partial_{z'} + g_z\partial_{w'},\\ \partial_w &= f_w\partial_{z'} + g_w\partial_{w'}.\end{aligned} \tag{0.6}$$

Let $M=\{\rho=0\}$ and $M'=\{\rho'=0\}$ be real analytic hypersurfaces such that $h(M)\subseteq M'$. The $T^{1,0}M'$ vector field on M' is given by

$$L' := -\rho'_{w'}\partial_{z'} + \rho'_{z'}\partial_{w'}.$$

Upon restriction to a smaller domain of h, $h(M)=M'$. Thus,

$$M=\{(z,w)\in\mathbb{C}^2 : \rho'(f(z,w),g(z,w),\bar f(\bar z,\bar w),\bar g(\bar z,\bar w))=0\}.$$

By Lemma 0.3, there exists a nowhere vanishing function $\mu: M\to\mathbb{C}\backslash\{0\}$ such that

$$\mu(z,w,\bar z,\bar w)\rho(z,w,\bar z,\bar w) = \rho'(f(z,w),g(z,w),\bar f(\bar z,\bar w),\bar g(\bar z,\bar w)).$$

The $T^{1,0}M$ vector field tangent to M defined by the function $\rho'(f,g,\bar f,\bar g)$ is given by

$$L_1 := -(\rho'(f,g,\bar f,\bar g))_w\partial_z + (\rho'(f,g,\bar f,\bar g))_z\partial_w.$$

Using the chain rule

$$\begin{aligned}\partial_z\rho'(f,g,\bar f,\bar g) &= \rho'_{z'}(f,g,\bar f,\bar g)f_z+\rho'_{w'}(f,g,\bar f,\bar g)g_z,\\ \partial_w\rho'(f,g,\bar f,\bar g) &= \rho'_{z'}(f,g,\bar f,\bar g)f_w+\rho'_{w'}(f,g,\bar f,\bar g)g_w,\end{aligned}$$

and the relations between ∂_z, ∂_w, $\partial_{z'}$ and $\partial_{w'}$ in equation (0.6) to obtain

$$\begin{aligned}L_1 &:= -(\rho'_{z'}(f,g,\bar f,\bar g)f_w+\rho'_{w'}(f,g,\bar f,\bar g)g_w)(f_z\partial_{z'}+g_z\partial_{w'})\\ &\quad +(\rho'_{z'}(f,g,\bar f,\bar g)f_z+\rho'_{w'}(f,g,\bar f,\bar g)g_z)(f_w\partial_{z'}+g_w\partial_{w'})\\ &= -\rho'_{z'}(f,g,\bar f,\bar g)f_wf_z\partial_{z'}-\rho'_{w'}(f,g,\bar f,\bar g)g_wf_z\partial_{z'}\\ &\quad -\rho'_{z'}(f,g,\bar f,\bar g)f_wg_z\partial_{w'}-\rho'_{w'}(f,g,\bar f,\bar g)g_wg_z\partial_{w'}\\ &\quad +\rho'_{z'}(f,g,\bar f,\bar g)f_zf_w\partial_{z'}+\rho'_{w'}(f,g,\bar f,\bar g)g_zf_w\partial_{z'}\\ &\quad +\rho'_{z'}(f,g,\bar f,\bar g)f_zg_w\partial_{w'}+\rho'_{w'}(f,g,\bar f,\bar g)g_zg_w\partial_{w'}\\ &= (f_zg_w-f_wg_z)\big(-\rho'_{w'}(f,g,\bar f,\bar g)\partial_{z'}+\rho'_{z'}(f,g,\bar f,\bar g)\partial_{w'}\big).\end{aligned}$$

On the other hand, for the $T^{1,0}M$ vector field on M defined by the function $\mu\rho$,

$$\begin{aligned}L_2 &:= -(\mu\rho)_w\partial_z+(\mu\rho)_z\partial_w\\ &= -\mu_w\rho\partial_z-\mu\rho_w\partial_z+\mu_z\rho\partial_w+\mu\rho_z\partial_w\\ &= \mu(-\rho_w\partial_z+\rho_z\partial_w)+\rho(-\mu_w\partial_z+\mu_z\partial_w)\\ &= \mu L+\rho(-\mu_w\partial_z+\mu_z\partial_w).\end{aligned}$$

Since $L_1=L_2$, upon restriction to $M=\{\rho=0\}$,

$$\mu(-\rho_w\partial_z+\rho_z\partial_w)=(f_zg_w-f_wg_z)(-\rho'_{w'}\partial_{z'}+\rho'_{z'}\partial_{w'}) \qquad (\text{on } M).$$

Therefore

$$\mu\cdot L=\mathsf{Jac}(h)\cdot L' \qquad (\text{on } M).$$

1 M as a graph of a function

1.0.16 Implicit function theorem and the complex graph of M Let $M\subseteq\mathbb{C}^2$ be a real hypersurface defined by a real-analytic, real valued defining function ρ such that $d\rho$ does not vanish anywhere on M, or equivalently, M is a smooth real submanifold of $\mathbb{C}^2$. Without loss of generality, ρ_w is assumed to be non-vanishing everywhere on M. Throughout this section, M is also assumed to be Levi non-degenerate (see Definition 0.5).

Upon polarisation, that is replacing $\bar z$ and $\bar t$ by holomorphic function coordinates s and t respectively, $\rho(z,w,s,t)$ is a holomorphic function on $\mathbb{C}^4$ since ρ is real-analytic. By assumption, and by restricting to a smaller open set if necessary, $\rho_w(z,w,s,t)$ does not vanish anywhere. Hence by the Holomorphic Implicit Function Theorem, there exists a holomorphic function $\Theta(z,s,t)$ on $\mathbb{C}^3$ such that

$$\rho(a,b,c,d)=0 \qquad \text{if and only if} \qquad b=\Theta(a,c,d). \qquad (a,b\,c,d\in\mathbb{C}).$$

Therefore, by restricting to the subspace $s=\bar z$ and $t=\bar w$,

$$\{(z,w)\in\mathbb{C}^2:\ \rho(z,w,\bar z,\bar w)=0\}=\{(z,w)\in\mathbb{C}^2:\ w=\Theta(z,\bar z,\bar w)\},$$

and hence the following identity holds clearly by definition:

$$\rho(z,\Theta(z,\bar z,\bar w),\bar z,\bar w)\equiv 0. \qquad (1.1)$$

1.0.17 Some Identities Denote $F(z,w,\bar z,\bar w) = (z,\Theta(z,\bar z,\bar w),\bar z,\bar w)$. Differentiating equation (1.1) with respect to z, $\bar z$ and $\bar w$ yields the following set of identities:

$$\begin{aligned} 0 &\equiv \rho_z\circ F + \Theta_z\rho_w\circ F,\\ 0 &\equiv \rho_{\bar z}\circ F + \Theta_{\bar z}\rho_w\circ F,\\ 0 &\equiv \rho_{\bar w}\circ F + \Theta_{\bar w}\rho_w\circ F, \end{aligned} \tag{1.2}$$

while a further differentiation of equation (1.2) gives another set of identities:

$$\begin{aligned} 0 &\equiv \rho_{zz}\circ F + 2\Theta_z\rho_{zw}\circ F + \Theta_z\Theta_z\rho_{ww}\circ F + \Theta_{zz}\rho_w\circ F,\\ 0 &\equiv \rho_{z\bar z}\circ F + \Theta_z\rho_{\bar z w}\circ F + \Theta_{\bar z}\rho_{zw}\circ F + \Theta_z\Theta_{\bar z}\rho_{ww}\circ F + \Theta_{z\bar z}\rho_w\circ F,\\ 0 &\equiv \rho_{z\bar w}\circ F + \Theta_z\rho_{w\bar w}\circ F + \Theta_{\bar w}\rho_{zw}\circ F + \Theta_z\Theta_{\bar w}\rho_{ww}\circ F + \Theta_{z\bar w}\rho_w\circ F,\\ 0 &\equiv \rho_{\bar z\bar z}\circ F + 2\Theta_{\bar z}\rho_{\bar z w}\circ F + \Theta_{\bar z}\Theta_{\bar z}\rho_{ww}\circ F + \Theta_{\bar z\bar z}\rho_w\circ F,\\ 0 &\equiv \rho_{\bar z\bar w}\circ F + \Theta_{\bar z}\rho_{w\bar w}\circ F + \Theta_{\bar w}\rho_{\bar z w}\circ F + \Theta_{\bar z}\Theta_{\bar w}\rho_{ww}\circ F + \Theta_{\bar z\bar w}\rho_w\circ F,\\ 0 &\equiv \rho_{\bar w\bar w}\circ F + 2\Theta_{\bar w}\rho_{w\bar w}\circ F + \Theta_{\bar w}\Theta_{\bar w}\rho_{ww}\circ F + \Theta_{\bar w\bar w}\rho_w\circ F. \end{aligned} \tag{1.3}$$

These identities will be useful later in translating invariants from the representation of M by graph to the representation of M by ρ.

1.0.18 The expression Δ Let Δ denote

$$\Delta := \Theta_{\bar z}\Theta_{z\bar w} - \Theta_{\bar w}\Theta_{z\bar z}.$$

Lemma 1.4. *The following identity holds*

$$\mathrm{L}(\rho)\circ F = -(\rho_w\circ F)^3\Delta.$$

Proof. For the rest of the calculation, the notation $\circ F$ will be suppressed. Starting from $\mathrm{L}(\rho)\circ F$,

$$\mathrm{L}(\rho)\circ F = \rho_z\rho_{\bar z}\rho_{w\bar w} - \rho_z\rho_{\bar w}\rho_{\bar z w} - \rho_{\bar z}\rho_w\rho_{z\bar w} + \rho_w\rho_{\bar w}\rho_{z\bar z},$$

each of the terms on the right hand side will be replaced by equations (1.2) and (1.3). For example, for the first term:

$$\rho_z\rho_{\bar z}\rho_{w\bar w} = (-\Theta_z\rho_w)(-\Theta_{\bar z}\rho_w)\rho_{w\bar w} = \Theta_z\Theta_{\bar z}\,\rho_w\rho_w\rho_{w\bar w}.$$

For the second term:

$$\rho_z\rho_{\bar w}\rho_{\bar z w} = (-\Theta_z\rho_w)(-\Theta_{\bar w}\rho_w)\rho_{\bar z w} = \Theta_z\Theta_{\bar w}\,\rho_w\rho_w\rho_{\bar z w}.$$

For the third term:

$$\begin{aligned} \rho_{\bar z}\rho_w\rho_{z\bar w} &= (-\Theta_{\bar z}\rho_w)\rho_w(-\Theta_z\rho_{w\bar w} - \Theta_{\bar w}\rho_{zw} - \Theta_z\Theta_{\bar w}\rho_{ww} - \Theta_{z\bar w}\rho_w)\\ &= \Theta_{\bar z}\Theta_z\rho_w\rho_w\rho_{w\bar w} + \Theta_{\bar z}\Theta_{\bar w}\rho_w\rho_w\rho_{zw} + \Theta_z\Theta_{\bar z}\Theta_{\bar w}\rho_w\rho_w\rho_{ww} + \Theta_{\bar z}\Theta_{z\bar w}\rho_w\rho_w\rho_w. \end{aligned}$$

Finally for the last term,

$$\begin{aligned} \rho_w\rho_{\bar w}\rho_{z\bar z} &= \rho_w(-\Theta_{\bar w}\rho_w)(-\Theta_z\rho_{\bar z w} - \Theta_{\bar z}\rho_{zw} - \Theta_z\Theta_{\bar z}\rho_{ww} - \Theta_{z\bar z}\rho_w)\\ &= \Theta_{\bar w}\Theta_z\rho_w\rho_w\rho_{\bar z w} + \Theta_{\bar w}\Theta_{\bar z}\rho_w\rho_w\rho_{zw} + \Theta_{\bar w}\Theta_z\Theta_{\bar z}\rho_w\rho_w\rho_{ww} + \Theta_{\bar w}\Theta_{z\bar z}\rho_w\rho_w\rho_w. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{L}(\rho)\circ F &= \Theta_z\Theta_{\bar z}\rho_w\rho_w\rho_{w\bar w} - \Theta_z\Theta_{\bar w}\rho_w\rho_w\rho_{\bar z w} - \Theta_{\bar z}\Theta_z\rho_w\rho_w\rho_{w\bar w} - \Theta_{\bar z}\Theta_{\bar w}\rho_w\rho_w\rho_{zw}\\ &\quad -\Theta_z\Theta_{\bar z}\Theta_{\bar w}\rho_w\rho_w\rho_{ww} - \Theta_{\bar z}\Theta_{z\bar w}\rho_w\rho_w\rho_w + \Theta_{\bar w}\Theta_z\rho_w\rho_w\rho_{\bar z w} + \Theta_{\bar w}\Theta_{\bar z}\rho_w\rho_w\rho_{zw}\\ &\quad +\Theta_{\bar w}\Theta_z\Theta_{\bar z}\rho_w\rho_w\rho_{ww} + \Theta_{\bar w}\Theta_{z\bar z}\rho_w\rho_w\rho_w\\ &= \rho_w^3(\Theta_{\bar w}\Theta_{z\bar z} - \Theta_{\bar z}\Theta_{z\bar w})\\ &= -(\rho_w\circ F)^3\Delta. \end{aligned}$$

□

It is immediate from the calculation above, and from the definition of a Levi non-degenerate real hypersurface is the following:

Corollary 1.5. *M is non-degenerate at p if and only if Δ does not vanish at p.*

1.0.19 The vector field $\frac{1}{\Box}\mathscr{L}$

Set

$$\Box := -\frac{\Delta}{\Theta_{\bar{w}}},$$

and let

$$\overline{\mathscr{L}} := \frac{\partial}{\partial \bar{z}} - \frac{\Theta_{\bar{z}}}{\Theta_{\bar{w}}}\frac{\partial}{\partial \bar{w}}.$$

The vector field $\frac{1}{\Box}\overline{\mathscr{L}}$ comes from the transfer of a certain map between the jet spaces. Let (z,s,t) and (z,w,w_z) be holomorphic coordinates of $\mathbb{C}^3$. Here, s and t will represent $\bar{z}$ and $\bar{w}$ respectively. Let $\varphi : \mathbb{C}^3 \to \mathbb{C}^3$ be the holomorphic map given by

$$\begin{array}{rcl} \varphi : \mathbb{C}^3 & \longrightarrow & \mathbb{C}^3 \\ (z,s,t) & \longmapsto & (z, \Theta(z,s,t), \Theta_z(z,s,t)). \end{array}$$

The Jacobian of this map is

$$\mathsf{Jac}(\varphi) = \begin{vmatrix} 1 & 0 & 0 \\ \Theta_z & \Theta_s & \Theta_t \\ \Theta_{zz} & \Theta_{zs} & \Theta_{zt} \end{vmatrix} = \Theta_s\Theta_{zt} - \Theta_t\Theta_{zs}$$

which is recognised as the polarisation of Δ. By assumption that M is Levi non-degenerate at the origin, upon restriction to a suitable smaller open neighbourhood of the origin in $\mathbb{C}^3$, and by Corollary 1.5, $\mathsf{Jac}(\varphi)$ can be assumed to be non-vanishing everywhere. By the inverse function theorem, there exist functions $\Lambda(z,w,w_z)$ and $\Pi(z,w,w_z)$ on $\mathbb{C}^3$ such that

$$\begin{array}{rcl} s = & \Lambda(z, \Theta(z,s,t), \Theta_z(z,s,t)), \\ t = & \Pi(z, \Theta(z,s,t), \Theta_z(z,s,t)). \end{array} \tag{1.6}$$

Substituting the equations above into the expressions of w and w_z to get

$$w = \Theta(z, \Lambda(z,w,w_z), \Pi(z,w,w_z)), \tag{1.7}$$

$$w_z = \Theta_z(z, \Lambda(z,w,w_z), \Pi(z,w,w_z)). \tag{1.8}$$

Differentiating equations (1.7) and (1.8) with respect to w and w_z and give respectively

$$\begin{array}{l} 0 = \Theta_z + \Theta_s\Lambda_z + \Theta_t\Pi_z, \\ 1 = \Theta_s\Lambda_w + \Theta_t\Pi_w, \\ 0 = \Theta_s\Lambda_{w_z} + \Theta_t\Pi_{w_z}, \end{array} \tag{1.9}$$

and

$$\begin{array}{l} 0 = \Theta_{zz} + \Theta_{zs}\Lambda_z + \Theta_{zt}\Pi_z, \\ 0 = \Theta_{zs}\Lambda_w + \Theta_{zt}\Pi_w, \\ 1 = \Theta_{zs}\Lambda_{w_z} + \Theta_{zt}\Pi_{w_z}. \end{array} \tag{1.10}$$

Comparing the first line of (1.9) with the first line of (1.10) and etc, the following system of linear equations are obtained:

$$\begin{array}{l} \begin{pmatrix} \Theta_s & \Theta_t \\ \Theta_{zs} & \Theta_{zt} \end{pmatrix} \begin{pmatrix} \Lambda_z \\ \Pi_z \end{pmatrix} = \begin{pmatrix} -\Theta_z \\ -\Theta_{zz} \end{pmatrix}, \\ \begin{pmatrix} \Theta_s & \Theta_t \\ \Theta_{zs} & \Theta_{zt} \end{pmatrix} \begin{pmatrix} \Lambda_w \\ \Pi_w \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ \begin{pmatrix} \Theta_s & \Theta_t \\ \Theta_{zs} & \Theta_{zt} \end{pmatrix} \begin{pmatrix} \Lambda_{w_z} \\ \Pi_{w_z} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{array} \tag{1.11}$$

Applying the inverse matrix

$$\begin{pmatrix} \Theta_s & \Theta_t \\ \Theta_{zs} & \Theta_{zt} \end{pmatrix}^{-1} = \frac{1}{\Theta_s\Theta_{zt} - \Theta_t\Theta_{zs}} \begin{pmatrix} \Theta_{zt} & -\Theta_t \\ -\Theta_{zs} & \Theta_s \end{pmatrix}$$

to both sides of equations (1.11) gives the following set of formulae concerning the differentiation of Λ and Π by z, w and w_z:

$$\begin{aligned} \Lambda_z &= \frac{\Theta_t\Theta_{zz} - \Theta_z\Theta_{zt}}{\Theta_s\Theta_{zt} - \Theta_t\Theta_{zs}}, & \Pi_z &= \frac{\Theta_z\Theta_{zs} - \Theta_s\Theta_{zz}}{\Theta_s\Theta_{zt} - \Theta_t\Theta_{zs}}, \\ \Lambda_w &= \frac{\Theta_{zt}}{\Theta_s\Theta_{zt} - \Theta_t\Theta_{zs}}, & \Pi_w &= \frac{-\Theta_{zs}}{\Theta_s\Theta_{zt} - \Theta_t\Theta_{zs}}, \\ \Lambda_{w_z} &= \frac{-\Theta_t}{\Theta_s\Theta_{zt} - \Theta_t\Theta_{zs}}, & \Pi_{w_z} &= \frac{\Theta_s}{\Theta_s\Theta_{zt} - \Theta_t\Theta_{zs}}. \end{aligned}$$

Let $G(z, s, t)$ be a function on $\mathbb{C}^3$. As above, from equation (1.6), $G(z, s, t)$ may be written as

$$G(z, s, t) = G\left(z, \Lambda(z, w, w_z), \Pi(z, w, w_z)\right).$$

Therefore,

$$\begin{aligned} \partial_z G(z, s, t) &= \partial_z G + \Lambda_z \partial_s G + \Pi_z \partial_t G \\ \partial_w G(z, s, t) &= \Lambda_w \partial_s G + \Pi_w \partial_t G \\ \partial_{w_z} G(z, s, t) &= \Lambda_{w_z} \partial_s G + \Pi_{w_z} \partial_t G. \end{aligned}$$

As a result, the vector field ∂_{w_z} may be expressed as

$$\begin{aligned} \partial_{w_z} &= \frac{1}{\Theta_s\Theta_{zt} - \Theta_t\Theta_{zs}} \left(-\Theta_t\partial_s + \Theta_s\partial_t\right) \\ &= \frac{-\Theta_t}{\Theta_s\Theta_{zt} - \Theta_t\Theta_{zs}} \left(\partial_s - \frac{\Theta_s}{\Theta_t}\partial_t\right). \end{aligned}$$

For any $G(z, \bar{z}, \bar{w})$ a real analytic function on $\mathbb{C}^2$, let $G(z, s, t)$ be the polarisation of $G(z, \bar{z}, \bar{w})$. Therefore

$$\partial_{w_z} G(z, s, t)$$

is the polarisation of

$$\frac{1}{\Box} \mathscr{L}\left(G(z, \bar{z}, \bar{w})\right)$$

simply by looking at the power series expansion of G (in both variables (z, s, t) and $(z, \bar{z}, \bar{w})$).

1.0.20 Sphericity of M

Definition 1.12. Call M spherical if M is locally biholomorphic to

$$\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 : \ |z|^2 + |w|^2 = 1\}.$$

The following (due to Lie) attributes an equivalent condition to sphericity of M (see [Mer10, page 20]):

Proposition 1.13. *Let $M \subset \mathbb{C}^2$ be a smooth, real-analytic, real hypersurface which is Levi non-degenerate at one point $p \in M$, and has complex defining equation of the form*

$$w = \Theta(z, \bar{z}, \bar{w})$$

for some system of local holomorphic coordinates $(z, w) \in \mathbb{C}^2$ centred at p. Then M is spherical at p if and only if the right side of the associated second-order complex differential equation:

$$w_{zz} = \Theta_{zz}(z, s, t) = \Theta_{zz}(z, \Lambda(z, w, w_z), \Pi(z, w, w_z))$$

satisfies the single fourth order partial differential equation

$$\partial_{w_z}(\partial_{w_z}(\partial_{w_z}(\partial_{w_z}(\Theta_{zz}(z, s, t))))) \equiv 0. \tag{1.14}$$

Obviously, equation (1.14) implies the vanishing of

$$0 \equiv \frac{1}{\Box}\overline{\mathscr{L}}\left(\frac{1}{\Box}\overline{\mathscr{L}}\left(\frac{1}{\Box}\overline{\mathscr{L}}\left(\frac{1}{\Box}\overline{\mathscr{L}}\Big(\Theta_{zz}\Big)\right)\right)\right). \tag{1.15}$$

On the other hand, as remarked earlier,

$$\partial_{w_z}(\partial_{w_z}(\partial_{w_z}(\partial_{w_z}(\Theta_{zz}(z, s, t))))))$$

is the polarisation of

$$\frac{1}{\Box}\overline{\mathscr{L}}\left(\frac{1}{\Box}\overline{\mathscr{L}}\left(\frac{1}{\Box}\overline{\mathscr{L}}\left(\frac{1}{\Box}\overline{\mathscr{L}}\Big(\Theta_{zz}(z, \bar{z}, \bar{w})\Big)\right)\right)\right).$$

It is known that if G is real-analytic and if $G(z, s, t)$ is the polarisation of $G(z, \bar{z}, \bar{w})$, then $G(z, s, t) \equiv 0$ if and only if $G(z, \bar{z}, \bar{w}) \equiv 0$. Based on this fact,

Theorem 1.16 (See [Mer10, page 18])**.** *At a point $p \in \{\Theta_{\bar{w}} \neq 0\}$, the real hypersurface $M \subset \mathbb{C}^2$ is spherical if and only if near p,*

$$0 \equiv \frac{1}{\Box}\overline{\mathscr{L}}\left(\frac{1}{\Box}\overline{\mathscr{L}}\left(\frac{1}{\Box}\overline{\mathscr{L}}\left(\frac{1}{\Box}\overline{\mathscr{L}}\Big(\Theta_{zz}(z, \bar{z}, \bar{w})\Big)\right)\right)\right).$$

A direct expansion of the expression above gives

Corollary 1.17. *On the open set $\{\Theta_{\bar{w}} \neq 0\}$, the real hypersurface $M \subset \mathbb{C}^2$ is spherical if and only if*

$$\begin{aligned} 0 \;=\; & \frac{\overline{\mathscr{L}}^4(\Theta_{zz})}{\Box^4} - \\ & 6\frac{\overline{\mathscr{L}}(\Box)\overline{\mathscr{L}}^3(\Theta_{zz})}{\Box^5} - 4\frac{\overline{\mathscr{L}}^2(\Box)\overline{\mathscr{L}}^2(\Theta_{zz})}{\Box^5} - \frac{\overline{\mathscr{L}}^3(\Box)\overline{\mathscr{L}}(\Theta_{zz})}{\Box^5} + \\ & \frac{[\overline{\mathscr{L}}(\Box)]^2\overline{\mathscr{L}}^2(\Theta_{zz})}{\Box^6} + 10\frac{\overline{\mathscr{L}}(\Box)\overline{\mathscr{L}}(\Box)^2\overline{\mathscr{L}}(\Theta_{zz})}{\Box^6} \\ & -15\frac{[\overline{\mathscr{L}}(\Box)]^3\overline{\mathscr{L}}(\Theta_{zz})}{\Box^7}. \end{aligned}$$

Proof. The calculations presented below will be done in steps. First,

$$\frac{1}{\Box}\overline{\mathscr{L}}\left(\frac{1}{\Box}\overline{\mathscr{L}}(\Theta_{zz})\right) = -\frac{\overline{\mathscr{L}}(\Box)\overline{\mathscr{L}}(\Theta_{zz})}{\Box^3} + \frac{\overline{\mathscr{L}}^2(\Theta_{zz})}{\Box^2}.$$

Next for the second application of $\frac{1}{\Box}\overline{\mathscr{L}}$,

$$\begin{aligned}\overline{\mathscr{L}}\left(\frac{1}{\Box}\overline{\mathscr{L}}\left(\frac{1}{\Box}\overline{\mathscr{L}}(\Theta_{zz})\right)\right) &= -\left[\frac{\Box^3\overline{\mathscr{L}}(\overline{\mathscr{L}}(\Box)\overline{\mathscr{L}}(\Theta_{zz}))-3\overline{\mathscr{L}}(\Box)\overline{\mathscr{L}}(\Theta_{zz})\overline{\mathscr{L}}(\Box)\Box^2)}{\Box^6}\right]\\ &\quad+\frac{\Box^2\overline{\mathscr{L}}^3(\Theta_{zz})-2\overline{\mathscr{L}}^2(\Theta_{zz})\overline{\mathscr{L}}(\Box)\Box}{\Box^4}\\ &= \frac{-\Box^3\overline{\mathscr{L}}^2(\Box)\overline{\mathscr{L}}(\Theta_{zz})-\Box^3\overline{\mathscr{L}}(\Box)\overline{\mathscr{L}}^2(\Theta_{zz})}{\Box^6}\\ &\quad+\frac{3\overline{\mathscr{L}}(\Box)^2\overline{\mathscr{L}}(\Theta_{zz})\Box^2}{\Box^6}+\frac{\Box^2\overline{\mathscr{L}}^3(\Theta_{zz})-2\overline{\mathscr{L}}^2(\Theta_{zz})\overline{\mathscr{L}}(\Box)\Box}{\Box^4}\\ &= \frac{-\Box\overline{\mathscr{L}}^2(\Box)\overline{\mathscr{L}}(\Theta_{zz})-\Box\overline{\mathscr{L}}(\Box)\overline{\mathscr{L}}^2(\Theta_{zz})}{\Box^4}\\ &\quad+\frac{3\overline{\mathscr{L}}(\Box)^2\overline{\mathscr{L}}(\Theta_{zz})}{\Box^4}+\frac{\Box\overline{\mathscr{L}}^3(\Theta_{zz})-2\overline{\mathscr{L}}^2(\Theta_{zz})\overline{\mathscr{L}}(\Box)}{\Box^3}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{\Box}\overline{\mathscr{L}}\left(\frac{1}{\Box}\overline{\mathscr{L}}\left(\frac{1}{\Box}\overline{\mathscr{L}}(\Theta_{zz})\right)\right) &= -\frac{\overline{\mathscr{L}}^2(\Box)\overline{\mathscr{L}}(\Theta_{zz})}{\Box^4}-3\frac{\overline{\mathscr{L}}(\Box)\overline{\mathscr{L}}^2(\Theta_{zz})}{\Box^4}\\ &\quad+3\frac{\overline{\mathscr{L}}(\Box)^2\overline{\mathscr{L}}(\Theta_{zz})}{\Box^5}+\frac{\overline{\mathscr{L}}^3(\Theta_{zz})}{\Box^3}.\end{aligned}$$

Finally for the last application of the vector field $\frac{1}{\Box}\overline{\mathscr{L}}$, the application of $\overline{\mathscr{L}}$ to each of the terms above are calculated as below. For the first term:

$$\begin{aligned}\frac{1}{\Box}\overline{\mathscr{L}}\left(\frac{\overline{\mathscr{L}}(\Theta_{zz})}{\Box^3}\right) &= \frac{\Box^3\overline{\mathscr{L}}^4(\Theta_{zz})-3\Box^2\overline{\mathscr{L}}^3(\Theta_{zz})\overline{\mathscr{L}}(\Box)}{\Box^7}\\ &= \frac{\overline{\mathscr{L}}^4(\Theta_{zz})}{\Box^4}-\frac{3\overline{\mathscr{L}}^2(\Theta_{zz})\overline{\mathscr{L}}(\Box)}{\Box^5}.\end{aligned}$$

For the second term:

$$\begin{aligned}\frac{1}{\Box}\overline{\mathscr{L}}\left(\frac{\overline{\mathscr{L}}(\Box)^2\overline{\mathscr{L}}(\Theta_{zz})}{\Box^5}\right) &= \frac{2\Box^5\overline{\mathscr{L}}(\Box)\overline{\mathscr{L}}^2(\Box)\overline{\mathscr{L}}(\Theta_{zz})}{\Box^{11}}+\frac{\Box^5\overline{\mathscr{L}}(\Box)^2\overline{\mathscr{L}}^2(\Theta_{zz})}{\Box^{11}}\\ &\quad-\frac{5\overline{\mathscr{L}}(\Box)^2\overline{\mathscr{L}}(\Theta_{zz})\overline{\mathscr{L}}(\Box)\Box^4}{\Box^{11}}\\ &= \frac{2\overline{\mathscr{L}}(\Box)\overline{\mathscr{L}}^2(\Box)\overline{\mathscr{L}}(\Theta_{zz})}{\Box^5}+\frac{\overline{\mathscr{L}}(\Box)^2\overline{\mathscr{L}}^2(\Theta_{zz})}{\Box^5}\\ &\quad-\frac{5\overline{\mathscr{L}}(\Box)^3\overline{\mathscr{L}}(\Theta_{zz})}{\Box^6}.\end{aligned}$$

For the third term:

$$\begin{aligned}\frac{1}{\Box}\overline{\mathscr{L}}\left(\frac{\overline{\mathscr{L}}(\Box)\overline{\mathscr{L}}^2(\Theta_{zz})}{\Box^4}\right) &= \frac{\Box^4\overline{\mathscr{L}}^2(\Box)\overline{\mathscr{L}}^2(\Theta_{zz})}{\Box^9}+\frac{\Box^4\overline{\mathscr{L}}(\Box)\overline{\mathscr{L}}^3(\Theta_{zz})}{\Box^9}\\ &\quad-\frac{4\Box^3\overline{\mathscr{L}}(\Box)\overline{\mathscr{L}}(\Theta_{zz})}{\Box^9}\\ &= \frac{\overline{\mathscr{L}}^2(\Box)\overline{\mathscr{L}}^2(\Theta_{zz})}{\Box^5}+\frac{\overline{\mathscr{L}}(\Box)\overline{\mathscr{L}}^3(\Theta_{zz})}{\Box^5}\\ &\quad-\frac{4\overline{\mathscr{L}}(\Box)^2\overline{\mathscr{L}}^2(\Theta_{zz})}{\Box^6}.\end{aligned}$$

For the final term

$$\begin{aligned}\frac{1}{\Box}\overline{\mathscr{L}}\left(\frac{\overline{\mathscr{L}}^2(\Box)\overline{\mathscr{L}}(\Theta_{zz})}{\Box^4}\right) &= \frac{\Box^4\overline{\mathscr{L}}^3(\Box)\overline{\mathscr{L}}(\Theta_{zz})}{\Box^9}+\frac{\Box^4\overline{\mathscr{L}}^2(\Box)\overline{\mathscr{L}}^2(\Theta_{zz})}{\Box^9}\\ &\quad -\frac{4\Box^3\overline{\mathscr{L}}(\Box)\overline{\mathscr{L}}^2(\Box)\overline{\mathscr{L}}(\Theta_{zz})}{\Box^9}\\ &= \frac{\overline{\mathscr{L}}^3(\Box)\overline{\mathscr{L}}(\Theta_{zz})}{\Box^5}+\frac{\overline{\mathscr{L}}^2(\Box)\overline{\mathscr{L}}^2(\Theta_{zz})}{\Box^5}\\ &\quad -\frac{4\overline{\mathscr{L}}(\Box)\overline{\mathscr{L}}^2(\Box)\overline{\mathscr{L}}(\Theta_{zz})}{\Box^6}.\end{aligned}$$

Therefore,

$$\begin{aligned}\left(\frac{1}{\Box}\overline{\mathscr{L}}\right)^4(\Theta_{zz}) &= \frac{\overline{\mathscr{L}}^4(\Theta_{zz})}{\Box^4}-3\frac{\overline{\mathscr{L}}^3(\Theta_{zz})\overline{\mathscr{L}}(\Box)}{\Box^5}+6\frac{\overline{\mathscr{L}}(\Box)\overline{\mathscr{L}}^2(\Box)\overline{\mathscr{L}}(\Theta_{zz})}{\Box^6}\\ &\quad +3\frac{\overline{\mathscr{L}}(\Box)^2\overline{\mathscr{L}}^2(\Theta_{zz})}{\Box^6}-15\frac{\overline{\mathscr{L}}(\Box)^3\overline{\mathscr{L}}(\Theta_{zz})}{\Box^7}-3\frac{\overline{\mathscr{L}}^2(\Box)\overline{\mathscr{L}}^2(\Theta_{zz})}{\Box^5}\\ &\quad -3\frac{\overline{\mathscr{L}}(\Box)\overline{\mathscr{L}}^3(\Theta_{zz})}{\Box^5}+12\frac{\overline{\mathscr{L}}(\Box)^2\overline{\mathscr{L}}^2(\Theta_{zz})}{\Box^6}-\frac{\overline{\mathscr{L}}^3(\Box)\overline{\mathscr{L}}(\Theta_{zz})}{\Box^5}\\ &\quad -\frac{\overline{\mathscr{L}}^2(\Box)\overline{\mathscr{L}}^2(\Theta_{zz})}{\Box^5}+4\frac{\overline{\mathscr{L}}(\Box)\overline{\mathscr{L}}^2(\Box)\overline{\mathscr{L}}(\Theta_{zz})}{\Box^6}\\ &= \frac{\overline{\mathscr{L}}^4(\Theta_{zz})}{\Box^4}-6\frac{\overline{\mathscr{L}}(\Box)\overline{\mathscr{L}}^3(\Theta_{zz})}{\Box^5}-4\frac{\overline{\mathscr{L}}^2(\Box)\overline{\mathscr{L}}^2(\Theta_{zz})}{\Box^5}-\frac{\overline{\mathscr{L}}^3(\Box)\overline{\mathscr{L}}(\Theta_{zz})}{\Box^5}\\ &\quad +10\frac{\overline{\mathscr{L}}(\Box)\overline{\mathscr{L}}^2(\Box)\overline{\mathscr{L}}(\Theta_{zz})}{\Box^6}+15\frac{\overline{\mathscr{L}}(\Box)^2\overline{\mathscr{L}}^2(\Theta_{zz})}{\Box^6}-15\frac{\overline{\mathscr{L}}(\Box)^3\overline{\mathscr{L}}(\Theta_{zz})}{\Box^7}.\end{aligned}$$

□

1.0.21 Explanation by Cartan's Equivalence Method This subsection gives a very brief summary of the following differential equation

$$\left(\frac{1}{\Box}\mathscr{L}\right)^4(\Theta_{zz})\equiv 0$$

in relation to Cartan's equivalence method. Interested readers may consult [Olv95]. Set up the following coframes from the defining function in equation

$$\begin{aligned}\omega^1 &= dw-w_z dz,\\ \omega^2 &= dw_z-\Phi(z,w,w_z)dz,\\ \omega^3 &= dz.\end{aligned}$$

Here, Φ is the function $\Theta_{zz}(z,\Lambda(z,w,w_z),\Pi(z,w,w_z))$ in the previous paragraphs. The reduced G structure (after normalisation) is given by

$$g:=\hat{G}_p=\left\{\begin{pmatrix}a_1 & 0 & 0\\ a_2 & a_1a_6^{-1} & 0\\ a_4 & 0 & a_6\end{pmatrix}:\ a_1a_6\neq 0\right\}.$$

The lifted coframes are therefore

$$\begin{aligned}\theta^1 &= a_1\omega^1,\\ \theta^2 &= a_2\omega^1+a_1a_6^{-1}\omega^2,\\ \theta^3 &= a_4\omega^1+a_6\omega^3.\end{aligned}$$

Upon computing $d\theta^1$, $d\theta^2$ and $d\theta^3$,

$$\begin{aligned} d\theta^1 &= \omega^1 \wedge \theta^1 - \theta^2 \wedge \theta^3, \\ d\theta^2 &= \omega^2 \wedge \theta^2 - (\omega^1 - \omega^6) \wedge \theta^2, \\ d\theta^3 &= \omega^4 \wedge \theta^1 + \omega^6 \wedge \theta^3, \end{aligned} \tag{1.18}$$

where

$$\begin{aligned} \omega^1 &= \alpha^1 + \frac{a_4}{a_1}\theta^2 - \frac{a_2}{a_1}\theta^3, \\ \omega^2 &= \alpha^2 - \frac{a_1 a_4 \Phi_{w_z} + a_2 a_4 a_6}{a_1^2 a_6}\theta^2 + \frac{a_1^2 \Phi_w - a_1 a_2 a_6 \Phi_{w_z} - a_2^2 a_6^2}{a_1^2 a_6^2}\theta^3, \\ \omega^4 &= \alpha^4 + \frac{a_4^2}{a_1^2}\theta^2 - \frac{a_2 a_4}{a_1^2}\theta^3, \\ \omega^6 &= \alpha^6 - \frac{a_4}{a_1}\theta^2 - \frac{2a_2 a_6 + a_1 \Phi_{w_z}}{a_1 a_6}\theta^3. \end{aligned}$$

Here, α^1, α^2, α^3 and α^4 are coefficients that appear in the Maurer-Cartan form $dg \cdot g^{-1}$. At this stage the prologation process proceeds. For $i = 1, 2, 4, 6$, and $1 \leqslant j \leqslant 4$, introduce the modified Maurer-Cartan forms π^i that satisfy

$$\omega^i = \pi^i - \sum_{1 \leqslant j \leqslant 4} z_j^i \theta^j. \tag{1.19}$$

Substituting equation into equation, and setting the torsions in $d\theta^1$, $d\theta^2$ and $d\theta^3$ to zero, the new Maurer-Cartan forms π^i are therefore

$$\begin{aligned} \pi^1 &= \omega^1 + z_1^1 \theta^1, \\ \pi^2 &= \omega^2 + z_1^2 \theta^1 + (z_1^1 - z_1^6)\theta^2, \\ \pi^4 &= \omega^4 + z_1^4 \theta^1 + z_1^6 \theta^3, \\ \pi^6 &= \omega^6 + z_1^6 \theta^1 \end{aligned} \tag{1.20}$$

The prolonged structure group is therefore a 4-dimensional abelian group with 7×7 matrix representation

$$\hat{G}^1 = \left\{ \begin{pmatrix} I & 0 \\ R & I \end{pmatrix} : \; R = \begin{pmatrix} z_1^1 & 0 & 0 \\ z_1^2 & z_1^1 - z_1^6 & 0 \\ z_1^4 & 0 & z_1^6 \\ z_1^6 & 0 & 0 \end{pmatrix} \right\}.$$

From equation, the $d\pi^i$ for $i = 1, 2, 4, 6$ take the following expression after absorption:

$$\begin{aligned} d\pi^1 &= \rho^1 \wedge \theta^1 + \pi^4 \wedge \theta^2 - \pi^2 \wedge \theta^3, \\ d\pi^2 &= \rho^2 \wedge \theta^1 + (\rho^1 - \rho^3) \wedge \theta^2 + \pi^2 \wedge \pi^6, \\ d\pi^4 &= \rho^4 \wedge \theta^1 + \rho^3 \wedge \theta^3 + \pi^4 \wedge (\pi^1 - \pi^6), \\ d\pi^6 &= \rho^3 \wedge \theta^1 - \pi^4 \wedge \theta^2 - 2\pi^2 \wedge \theta^3 + T\theta^1 \wedge \theta^2, \end{aligned}$$

where T is the torsion that cannot be absorbed, and has the formula

$$T = \frac{2z_1^1 - 4z_1^6 + (2a_4 \Phi_{w_z} - a_6 \Phi_{w_z w_z})}{a_1 a_6}.$$

By setting

$$z_1^6 = \frac{1}{2} z_1^1 + \frac{2a_4 \Phi_{w_z} - a_6 \Phi_{w_z w_z}}{4a_1 a_6}, \tag{1.21}$$

T is normalised to 0. Substituting equation into the expression for π^i in equation, and recalculating their differentials $d\pi^i$, the following system of one forms is obtained:

$$\begin{aligned} d\pi^1 &= \rho^1 \wedge \theta^1 + \pi^4 \wedge \theta^2 - \pi^2 \wedge \theta^2, \\ d\pi^2 &= \rho^2 \wedge \theta^1 + \frac{1}{2}\rho^1 \wedge \theta^2 + \pi^2 \wedge \pi^6 - U\theta^2 \wedge \theta^3, \\ d\pi^4 &= \rho^4 \wedge \theta^1 + \frac{1}{2}\rho^1 \wedge \theta^3 + \pi^4 \wedge (\pi^1 - \pi^6) - V\theta^2 \wedge \theta^3, \\ d\pi^6 &= \frac{1}{2}\rho^1 \wedge \theta^1 - \pi^4 \wedge \theta^2 - 2\pi^2 \wedge \theta^3 + V\theta^1 \wedge \theta^2 + U\theta^1 \wedge \theta^3. \end{aligned} \tag{1.22}$$

The essential torsion coefficients U and V are respectively

$$\begin{aligned} U &= \frac{3}{2}z_1^2 + \frac{12a_2^2a_4a_6^2 + 6a_1^2a_4\Phi_w + 3a_1a_2a_6^2\Phi_{w_zw_z} + a_1^2a_6D_x\Phi_{w_zw_z} - 4a_1^2a_6\Phi_{w_zw}}{4a_1^3a_6^2} \\ V &= \frac{3}{2}z_1^4 + \frac{6a_2a_4^2a_6 + 6a_1a_4^2\Phi_{w_z} - 3a_1a_4a_6\Phi_{w_zw_z} + a_1a_6^2\Phi_{w_zw_zw_z}}{4a_1^3a_6^2}, \end{aligned} \tag{1.23}$$

where D_x is the total differential

$$D_x = \partial_z + w_z\partial_w + \Phi(z, w, w_z)\partial_{w_z}.$$

These torsions U and V can be normalised by choosing the appropriate values for z_1^2 and z_1^4. With the values z_1^6, z_1^2 and z_1^4 chosen, these are substituted back to the expression of π^i in equation, and hence there exist a modified Maurer-Cartan form ρ such that the following structure equations are obtained:

$$\begin{aligned} d\theta^1 &= \pi^1 \wedge \theta^1 - \theta^2 \wedge \theta^3, \\ d\theta^2 &= \pi^2 \wedge \theta^1 + (\pi^1 - \pi^6) \wedge \theta^2, \\ d\theta^3 &= \pi^4 \wedge \theta^1 + \pi^6 \wedge \theta^3, \\ d\pi^1 &= \rho \wedge \theta^1 + \pi^4 \wedge \theta^2 - \pi^2 \wedge \theta^3, \\ d\pi^2 &= \frac{1}{2}\rho \wedge \theta^2 + \pi^2 \wedge \pi^6 + K_2\theta^1 \wedge \theta^2, \\ d\pi^4 &= \frac{1}{2}\rho \wedge \theta^3 + \pi^4 \wedge (\pi^1 - \pi^6) + K_1\theta^1 \wedge \theta^3, \\ d\pi^6 &= \frac{1}{2}\rho \wedge \theta^1 - \pi^4 \wedge \theta^2 - 2\pi^2 \wedge \theta^3. \end{aligned}$$

The only group variable, r_1, cannot be absorbed, and hence by adding the final equation

$$d\rho = \rho \wedge \pi^1 - 2\pi^2 \wedge \pi^4 - K_3\theta^1 \wedge \theta^2 - K_4\theta^1 \wedge \theta^3$$

to the structure equation above completes Cartan's method. The torsion K_1 is given by

$$\begin{aligned} K_1 &= \frac{a_6^2}{6a_1^3}\Phi_{w_zw_zw_zw_z} \\ &= \frac{a_6^2}{6a_1^3}\partial_{w_z}(\partial_{w_z}(\partial_{w_z}(\partial_{w_z}(\Theta_{zz}(z, s, t))))) \end{aligned}$$

with $s = \Lambda(z, w, w_z)$ and $t = \Pi(z, w, w_z)$, is exactly (up to some multiplicative constant) the polarisation of $(\frac{1}{\Box}\mathscr{L})^4(\Theta_{zz})$.

1.0.22 Translation between $\{\rho = 0\}$ and $\{w = \Theta\}$ Starting from a defining function ρ of M, it is possible to derive Θ_{zz}, which is useful for determining if M is spherical. Recall that $F(z, w, \bar{z}, \bar{w}) = (z, \Theta(z, \bar{z}, \bar{w}), \bar{z}, \bar{w})$. Set

$$H(\rho) = (\rho_z \circ F)^2(\rho_{ww} \circ F) - 2(\rho_z \circ F)(\rho_w \circ F)(\rho_{zw} \circ F) + (\rho_w \circ F)^2(\rho_{zz} \circ F).$$

Lemma 1.24. *The following identity holds:*

$$\Theta_{zz}(z, \bar{z}, \bar{w}) \equiv \frac{H(\rho)}{\rho_w \circ F}.$$

Proof. Starting from the first identity in equation (1.3),

$$0 \equiv \rho_{zz} \circ F + 2\Theta_z \rho_{zw} \circ F + \Theta_z \Theta_z \rho_{ww} \circ F + \Theta_{zz} \rho_w \circ F,$$

bringing the last term over to the left side of the identity,

$$-\Theta_{zz} \rho_w \circ F \equiv \rho_{zz} \circ F + 2\Theta_z \rho_{zw} \circ F + \Theta_z \Theta_z \rho_{ww} \circ F.$$

Multiplying on both sides of the equation by $(\rho_w \circ F)^2$ to obtain

$$-(\rho_w \circ F)^3 \Theta_{zz} \equiv (\rho_w \circ F)^2(\rho_{zz} \circ F) + 2(\Theta_z \rho_w \circ F)(\rho_w \circ F)(\rho_{wz} \circ F) + (\Theta_z \rho_w \circ F)^2(\rho_{ww} \circ F).$$

Using the identities in (1.2), the conclusion follows. □

The next proposition is the main ingredient for the study of the umbilics for ellipsoids.

Proposition 1.25. *Set*

$$\begin{aligned} I_{[w]} \;=\; & \left[\frac{\mathrm{L}(\rho)}{\rho_w^2}\right]^3 \bar{L}\left(\frac{H(\rho)}{\rho_w^3}\right) \\ & -6\left[\frac{\mathrm{L}(\rho)}{\rho_w^2}\right]^2 \bar{L}\left(\frac{\mathrm{L}(\rho)}{\rho_w^2}\right) \bar{L}^3\left(\frac{H(\rho)}{\rho_w^3}\right) - 4\left[\frac{\mathrm{L}(\rho)}{\rho_w^2}\right]^2 \bar{L}^2\left(\frac{\mathrm{L}(\rho)}{\rho_w^2}\right) \bar{L}^2\left(\frac{H(\rho)}{\rho_w^3}\right) \\ & -\left[\frac{\mathrm{L}(\rho)}{\rho_w^2}\right]^2 \bar{L}^3\left(\frac{\mathrm{L}(\rho)}{\rho_w^2}\right) \bar{L}\left(\frac{H(\rho)}{\rho_w^3}\right) + 15\frac{\mathrm{L}(\rho)}{\rho_w^2}\left[\mathrm{L}\left(\frac{\mathrm{L}(\rho)}{\rho_w^2}\right)\right]^2 \bar{L}^2\left(\frac{H(\rho)}{\rho_w^3}\right) \\ & +10\frac{\mathrm{L}(\rho)}{\rho_w^2}\bar{L}\left(\frac{\mathrm{L}(\rho)}{\rho_w^2}\right) \bar{L}^2\left(\frac{\mathrm{L}(\rho)}{\rho_w^2}\right) \bar{L}\left(\frac{H(\rho)}{\rho_w^3}\right) - 15\left[\bar{L}\left(\frac{\mathrm{L}(\rho)}{\rho_w^2}\right)\right]^3 \bar{L}\left(\frac{H(\rho)}{\rho_w^3}\right). \end{aligned}$$

Then

$$(\Delta(\rho_w \circ F))^7 \left(\frac{1}{\Box}\overline{\mathscr{L}}\right)^4 (\Theta_{zz}) = I_{[w]}.$$

Proof. Observe that

$$\begin{aligned} \frac{1}{\Box}\overline{\mathscr{L}} \;&=\; -\frac{\Theta_{\bar{w}}}{\Delta}\left(\partial_{\bar{z}} - \frac{\Theta_{\bar{z}}}{\Theta_{\bar{w}}}\partial_{\bar{w}}\right) \\ &=\; -\frac{\Theta_{\bar{w}}}{\Delta \cdot (\rho_{\bar{w}} \circ F)}\left((\rho_{\bar{w}} \circ F)\partial_{\bar{z}} - \frac{\Theta_{\bar{z}} \cdot (\rho_{\bar{w}} \circ F)}{\Theta_{\bar{w}}}\partial_{\bar{w}}\right) \end{aligned}$$

Substituting

$$\begin{aligned} \Theta_{\bar{z}} \;&=\; \frac{\rho_{\bar{z}} \circ F}{\rho_w \circ F}, \\ \Theta_{\bar{w}} \;&=\; \frac{\rho_{\bar{w}} \circ F}{\rho_w \circ F}, \end{aligned}$$

which can be found in equations, into the expression above,

$$\begin{aligned}\frac{1}{\Box}\overline{\mathscr{L}} &= \frac{\Theta_{\bar{w}}}{\Delta\cdot(\rho_{\bar{w}}\circ F)}\left((\rho_{\bar{w}}\circ F)\partial_{\bar{z}}-\frac{\Theta_{\bar{z}}\cdot(\rho_{\bar{w}}\circ F)}{\Theta_{\bar{w}}}\partial_{\bar{w}}\right)\\ &= \frac{1}{\Delta(\rho\circ F)}\bar{L}.\end{aligned}$$

In the calculation of $(\frac{1}{\Box}\bar{L})^4(\Theta_{zz})$ as in paragraph, the expression $(\frac{1}{\Delta(\rho\circ F)}\bar{L}^4\Theta_{zz}$ can calculated by formally replacing $\Box$ by $\Delta(\rho_w\circ F)$, and $\overline{\mathscr{L}}$ by $\bar{L}$. As a consequence,

$$\begin{aligned}\left(\frac{1}{\Box}\overline{\mathscr{L}}\right)^4(\Theta_{zz}) &= \left(\frac{1}{\Delta(\rho_w\circ F)}\right)^4(\Theta_{zz})\\ &= \frac{\bar{L}^4\Theta_{zz}}{(\Delta(\rho_w\circ F))^4}-6\frac{\bar{L}(\Delta(\rho_w\circ F))\bar{L}^3(\Theta_{zz})}{(\Delta(\rho_w\circ F))^5}-4\frac{\bar{L}^2(\Delta(\rho_w\circ F))\bar{L}^2(\Theta_{zz})}{(\Delta(\rho_w\circ F))^5}\\ &\quad-\frac{\bar{L}^3(\Delta(\rho_w\circ F))\bar{L}(\Theta_{zz})}{(\Delta(\rho_w\circ F))^5}+15\frac{[\bar{L}(\Delta(\rho_w\circ F))]^2\bar{L}^2(\Theta_{zz})}{(\Delta(\rho_w\circ F))^6}\\ &\quad 10\frac{\bar{L}(\Delta(\rho_w\circ F))\bar{L}^2(\Delta(\rho_w\circ F))\bar{L}(\Theta_{zz})}{(\Delta(\rho_w\circ F))^6}-15\frac{[\bar{L}(\Delta(\rho_w\circ F))]^3\bar{L}(\Theta_{zz})}{(\Delta(\rho_w\circ F))^7}.\end{aligned}$$

Multiplying on both sides by $(\Delta(\rho_w\circ F))^7$ (suppressing the notation $\circ F$ as usual),

$$\begin{aligned}(\Delta(\rho_w))^7\left(\frac{1}{\Box}\overline{\mathscr{L}}\right)^4(\Theta_{zz}) &= (\Delta\rho_w)^3\bar{L}^4(\Theta_{zz})-6(\Delta\rho_w)^2\bar{L}(\Delta\rho_w)^2\bar{L}^3(\Theta_{zz})\\ &\quad-4(\Delta\rho_w)^2\bar{L}^2(\Delta\rho_w)\bar{L}^2(\Theta)_{zz}-(\Delta\rho_w)^2\bar{L}^3(\Delta\rho_w)\bar{L}(\Theta_{zz})\\ &\quad+15(\Delta\rho_w)[L(\Delta\rho_w)]^2\bar{L}^2(\Theta_{zz})\\ &\quad+10(\Delta\rho_w)\bar{L}(\Delta\rho_w)\bar{L}^2(\Delta\rho_w)\bar{L}(\Theta_{zz})-15[\bar{L}(\Delta\rho_w)]^3\bar{L}(\Theta_{zz}).\end{aligned}$$

Finally, using $\rho_w\Delta=\frac{\mathrm{L}(\rho)}{\rho_w^2}$ and $\Theta_{zz}=\frac{H(\rho)}{\rho_w^3}$, the right side of the equation above is exactly $I_{[w]}$, and the proof is complete. □

Corollary 1.26. *Let M be a smooth real hypersurface in $\mathbb{C}^2$ with the defining functions $\{\rho=0\}=\{w-\Theta=0\}$. Then $(\frac{1}{\Box}\overline{\mathscr{L}})^4(\Theta_{zz})\equiv 0$ if and only if $I_{[w]}\equiv 0$.*

2 Ellipsoids in $\mathbb{C}^n$

2.0.23 Some background For $n\geqslant 2$, in $\mathbb{C}^n\cong\mathbb{R}^{2n}$ equipped with the coordinates $z_i=x_i+\sqrt{-1}y_i$, an ellipsoid is the image of the unit sphere

$$\mathbb{S}^{2n-1} \;=\; \{z\in\mathbb{C}^n:\;|z_1|^2\;+\;\cdots\;+\;|z_n|^2\;=\;1\}.$$

under the affine transformation of $\mathbb{R}^{2n}$ and hence has the equation of the form

$$\sum_{1\leqslant i\leqslant n}(\alpha_i x_i^2\;+\;\beta_i y_i^2)\;=\;1 \qquad (E_{\alpha,\beta}),$$

with real constants $\alpha_i\geqslant\beta_i>0$. By Webster, two ellipsoids $E_{\alpha,\beta}\cong E_{\alpha',\beta'}$ are biholomorphically equivalent if and only if the following relation is satisfied:

$$\frac{\alpha_i-\beta_i}{\alpha_i+\beta_i}\;=\;\frac{\alpha_i'-\beta_i'}{\alpha_i'+\beta_i'}.$$

By replacing z_i by $z_i/\sqrt{\beta_i}$, and setting $a_i := \alpha_i/\beta_i$, the equation transforms into

$$\sum_{1\leqslant i\leqslant n} (a_i x_i^2 \,+\, y_i^2) \,=\, 1.$$

For $0 \leqslant A_i \leqslant 1/2$, set $A_i := \frac{a_i-1}{2a_i+2}$ so that $a_i = \frac{1+2A_i}{1-2A_i}$. Subsituting the expression for a_i into the equation above,

$$\begin{aligned}
1 \;&=\; \sum_{1\leqslant i\leqslant n} (a_i x_i^2 \,+\, y_i^2)\\
&=\; \sum_{1\leqslant i\leqslant n} \left(\frac{1+2A_i}{1-2A_i}\left(\frac{z_i+\bar z_i}{2}\right)^2 \,+\, \left(\frac{z_i-\bar z_i}{2\sqrt{-1}}\right)^2\right)\\
&=\; \sum_{1\leqslant i\leqslant n} \left(\frac{1+2A_i}{1-2A_i}\left(\frac{z_i^2 \,+\, 2z_i\bar z_i \,+\, \bar z_i^2}{4}\right) - \frac{z_i^2 \,-\, 2z_i\bar z_i \,+\, \bar z_i^2}{4}\right)\\
&=\; \sum_{1\leqslant i\leqslant n} \frac{1}{2}\left(\frac{1+2A_i}{1-2A_i}+1\right) z_i\bar z_i \,+\, \frac{1}{4}\left(\frac{1+2A_i}{1-2A_i}-1\right) z_i^2 \,+\, \frac{1}{4}\left(\frac{1+2A_i}{1-2A_i}-1\right)\bar z_i^2\\
&=\; \sum_{1\leqslant i\leqslant n} \frac{1}{2}\left(\frac{2}{1-2A_i}\right) z_i\bar z_i \,+\, \frac{1}{4}\left(\frac{4A_i}{1-2A_i}\right)(z_i^2+\bar z_i^2)\\
&=\; \sum_{1\leqslant i\leqslant n} \frac{z_i\bar z_i}{1-2A_i} \,+\, \left(\frac{A_i}{1-2A_i}\right)(z_i^2+\bar z_i^2).
\end{aligned}$$

By a change of variables $z_i \mapsto \sqrt{1-2A_i}z_i$ and $\bar z_i \mapsto \sqrt{1-2A_i}\bar z_i$,

$$\sum_{1\leqslant i\leqslant n} z_i\bar z_i \,+\, A_i(z_i^2 \,+\, \bar z_i^2) \,=\, 1. \tag{2.1}$$

When $n \geqslant 3$ the Hachtroudi-Chern-Moser invariant $\mathscr{I}^M_{\text{Cartan}}$ is the tensor $S^{\alpha\beta}_{\rho\sigma}$ with $1 \leqslant \alpha, \beta, \gamma, \delta \leqslant n$. The CR umbilical locus is defined by

$$\mathsf{Umb}_{\mathsf{CR}}(M) \,:=\, \{p \in M :\; S^{\alpha\beta}_{\rho\sigma} = 0,\; 1 \leqslant \alpha, \beta, \gamma, \delta \leqslant n\}.$$

A theorem of Webster [Web00] shows that for $n \geqslant 3$, and if $0 < A_1 < \cdots < A_n < \frac{1}{2}$,

$$\mathsf{Umb}_{\mathsf{CR}}(M)(E_{A_1,\ldots,A_n}) \,=\, \emptyset.$$

This motivated Huang-Ji to study real-analytic hypersurface $M \subset \mathbb{C}^2$, and a result in [XS07] shows that the umbilical locus of every real ellipsoid $E_{a,b} \subset \mathbb{C}^2$ whose equation is

$$ax^2 \,+\, y^2 \,+\, bu^2 \,+\, v^2 \,=\, 1,$$

where $z = x + \sqrt{-1}y$ and $w = u + \sqrt{-1}v$, contains a real algbraic curve parametrised by

$$\gamma :\; \theta :\longmapsto (x(\theta) + \sqrt{-1}y(\theta),\; u(\theta) + \sqrt{-1}v(\theta)),$$

with

$$x(\theta) := \sqrt{\frac{a-1}{a(ab-1)}}\cos\theta, \qquad y(\theta) := \sqrt{\frac{b(a-1)}{ab-1}}\sin\theta$$

and

$$u(\theta) := \sqrt{\frac{b-1}{(ab-1)}}\sin\theta, \qquad v(\theta) := -\sqrt{\frac{a(b-1)}{ab-1}}\cos\theta.$$

2.0.24 The case of the invariant $(\frac{1}{\Box}\overline{\mathscr{L}})^4(\Theta_{zz})$ Analogue to the situation above, it can be shown that the locus of $(\frac{1}{\Box}\overline{\mathscr{L}})^4(\Theta_{zz})$ also vanishes on the same curve γ. This time set

$$\rho \;=\; ax^2 \;+\; y^2 \;+\; bu^2 \;+\; v^2 \;-\; 1.$$

By corollary, it suffices to prove the vanishing for $I_{[w]}$. The Levi form of the ellipsoid is given by

$$\mathrm{L}(\rho) \;=\; \frac{1}{2}\big(a^2bx^2 \;+\; ab^2u^2 \;+\; a^2x^2 \;+\; b^2u^2 \;+\; av^2 \;+\; by^2 \;+\; v^2 \;+\; y^2\big).$$

Also the Hessian $H(\rho)$ has the following expression

$$\begin{aligned} H(\rho) \;=\; & \frac{1}{2}ab^2u^2 \;-\; \frac{1}{2}b^2u^2 \;-\; \sqrt{-1}abuv \;+\; \sqrt{-1}buv \;-\; \frac{1}{2}av^2 \;+\; \frac{1}{2}v^2 \\ & -\frac{1}{2}a^2bx^2 \;-\; \frac{1}{2}a^2x^2 \;-\; \sqrt{-1}abxy \;+\; \sqrt{-1}axy \;-\; \frac{1}{2}by^2 \;+\; \frac{1}{2}y^2. \end{aligned}$$

Finally,

$$\rho_w \;=\; bu - \sqrt{-1}v.$$

Substituting these equations into the expression for $I_{[w]}$, a quick calculation by Maple gives the following results of each term that appears in $I_{[w]}$. By substituting the expressions of the curve for x, y, u and v, the following expressions are obtained

$$\begin{aligned} \gamma^*T_1 &= \frac{1}{8}\sqrt{-1}(a-1)\frac{N_1}{D} \\ \gamma^*T_2 &= \frac{3}{4}\sqrt{-1}(a-1)\frac{N_2}{D} \\ \gamma^*T_3 &= \frac{1}{2}\sqrt{-1}(a-1)\frac{N_3}{D} \\ \gamma^*T_4 &= \frac{1}{8}\sqrt{-1}(a-1)\frac{N_4}{D} \\ \gamma^*T_5 &= \frac{15}{8}\sqrt{-1}(a-1)\frac{N_5}{D} \\ \gamma^*T_6 &= \frac{5}{4}\sqrt{-1}(a-1)\frac{N_6}{D} \\ \gamma^*T_7 &= \frac{15}{8}\sqrt{-1}(a-1)\frac{N_7}{D} \end{aligned}$$

with the denominator

$$D \;:=\; \big(\sqrt{a}\theta - \sqrt{-1}\sqrt{b}\mathsf{sin}\theta\big)^8(ab-1)\left(\frac{b-1}{ab-1}\right)^{\frac{11}{2}},$$

and the following numerators N_i:

$$\begin{aligned} N_1 :=\; & \mathsf{cos}^7\theta[499a^{3/2}b^3 + 625a^{9/2}b^2 - 233a^{7/2}b^3 + 205a^{9/2}b - 631a^{7/2}b^2 + 15a^{9/2} - 415a^{7/2}b - 65a^{7/2}] \\ & +\sqrt{-1}\mathsf{cos}^6\theta\mathsf{sin}\theta[2887a^4b^{7/2} + 4401a^4b^{5/2} - 1297a^3b^{7/2} + 1905a^4b^{3/2} - 4059a^3b^{5/2} + 215a^4b^{1/2} - 3327a^3b^{3/2} - 725a^3b^{1/2}] \\ & +\mathsf{cos}^5\theta\mathsf{sin}^2\theta[-7023a^{7/2}b^4 - 13021a^{7/2}b^3 + 3013a^{5/2}b^4 - 7105a^{7/2}b^2 + 11011a^{5/2}b^3 - 1075a^{7/2}b + 11059a^{5/2}b^2 + 3141a^{5/2}b] \\ & \sqrt{-1}\mathsf{cos}^4\theta\mathsf{sin}^3\theta[-9267a^3b^{9/2} - 20989a^3b^{7/2} + 3757a^2b^{9/2} - 14101a^3b^{5/2} + 16279a^2b^{7/2} - 2683a^3b^{3/2} + 19891a^2b^{5/2} + 7113a^2b^{3/2}] \\ & +\mathsf{cos}^3\theta\mathsf{sin}^4\theta[7113a^{5/2}b^5 + 19891a^{5/2}b^4 - 2683a^{3/2}b^5 + 16279a^{5/2}b^3 - 14101a^{3/2}b^4 + 3757a^{5/2}b^2 - 20989a^{3/2}b^3 - 9267a^{3/2}b^2] \\ & +\sqrt{-1}\mathsf{cos}^2\theta\mathsf{sin}^5\theta[3141a^2b^{11/2} + 11059a^2b^{9/2} - 1075ab^{11/2} + 11011a^2b^{7/2} - 7105ab^{9/2} + 3013a^2b^{5/2} - 13021ab^{7/2} - 7023ab^{5/2}] \\ & +\mathsf{cos}\theta\mathsf{sin}^6\theta[-725a^{3/2}b^6 - 3327a^{3/2}b^5 + 215a^{1/2}b^6 - 4059a^{3/2}b^4 + 1905a^{1/2}b^5 - 1297a^{3/2}b^3 + 4401a^{1/2}b^4 + 2287a^{1/2}b^3] \\ & +\sqrt{-1}\mathsf{sin}^7\theta[-65ab^{13/2} - 415ab^{11/2} + 15b^{13/2} - 631ab^{9/2} + 205b^{11/2} - 233ab^{7/2} + 625b^{9/2} + 499b^{7/2}]. \end{aligned}$$

$$
\begin{aligned}
N_2 :=\ & \cos^7[-165a^{9/2}b^3-193a^{9/2}b^2+93a^{7/2}b^3-67a^{9/2}b+205a^{7/2}b^2-7a^{9/2}+115a^{7/2}b+19a^{7/2}]\\
&+\sqrt{-1}\cos^6\theta\sin\theta[-925a^4b^{7/2}-1389a^4b^{5/2}+505a^3b^{7/2}-627a^4b^{3/2}+1341a^3b^{5/2}-83a^4b^{1/2}+975a^3b^{3/2}+203a^3b^{1/2}]\\
&+\cos^5\theta\sin^2\theta[2177a^{7/2}b^4+4141a^{7/2}b^3-1145a^{5/2}b^4+2359a^{7/2}b^2-3673a^{5/2}b^3+395a^{7/2}b-3367a^{5/2}b^2-887a^{5/2}b]\\
&+\sqrt{-1}\cos^4\theta\sin^3\theta[2777a^3b^{9/2}+6649a^3b^{7/2}-1397a^2b^{9/2}+4711a^3b^{5/2}-5499a^2b^{7/2}+983a^3b^{3/2}-6211a^2b^{5/2}-2063a^2b^{3/2}]\\
&+\cos^3\theta\sin^4\theta[-2063a^{5/2}b^5-6211a^{5/2}b^4+983a^{3/2}b^5-5449a^{5/2}b^3+4711a^{3/2}b^4-1397a^{5/2}b^2+6649a^{3/2}b^3+2777a^{3/2}b^2]\\
&+\sqrt{-1}\cos^2\theta\sin^5\theta[-887a^2b^{11/2}-3367a^2b^{9/2}+395ab^{11/2}-3673a^2b^{7/2}+2359ab^{9/2}-1145a^2b^{5/2}+4141ab^{7/2}+2177ab^{5/2}]\\
&+\cos\theta\sin^6\theta[203a^{3/2}b^6+975a^{3/2}b^5-83a^{1/2}b^6+1341a^{3/2}b^4-627a^{1/2}b^5+505a^{3/2}b^3-1389a^{1/2}b^4-925a^{1/2}b^3]\\
&+\sqrt{-1}\sin^7\theta[19ab^{13/2}+115ab^{11/2}-7b^{13/2}+205ab^{9/2}-67b^{11/2}+93ab^{7/2}-193b^{9/2}-165b^{7/2}].
\end{aligned}
$$

$$
\begin{aligned}
N_3 :=\ & \cos^7\theta[-91a^{9/2}b^3-109a^{9/2}b^2+65a^{7/2}b^3-37a^{9/2}b+115a^{7/2}b^2-3a^{9/2}+55a^{7/2}b+5a^{7/2}]\\
&+\sqrt{-1}\cos^6\theta\sin\theta[-499a^4b^{7/2}-777a^4b^{5/2}+349a^3b^{7/2}-357a^4b^{3/2}+771a^3b^{5/2}-47a^4b^{1/2}+483a^3b^{3/2}+77a^3b^{1/2}]\\
&+\cos^5\theta\sin^2\theta[1143a^{7/2}b^4+2281a^{7/2}b^3-781a^{5/2}b^4+1369a^{7/2}b^2-2143a^{5/2}b^3+247a^{7/2}b-1723a^{5/2}b^2-393a^{5/2}b]\\
&+\sqrt{-1}\cos^4\theta\sin^3\theta[1407a^3b^{9/2}+3589a^3b^{7/2}-937a^2b^{9/2}+2761a^3b^{5/2}-3199a^2b^{7/2}+643a^3b^{3/2}-3271a^2b^{5/2}-993a^2b^{3/2}]\\
&+\cos^3\theta\sin^4\theta[-993a^{5/2}b^5-3271a^{5/2}b^4+643a^{3/2}b^5-3199a^{5/2}b^3+2761a^{3/2}b^4-937a^{5/2}b^2+3589a^{3/2}b^3+1407a^{3/2}b^2]\\
&+\sqrt{-1}\cos^2\theta\sin^5\theta[-393a^2b^{11/2}-1723a^2b^{9/2}+247ab^{11/2}-2143a^2b^{7/2}+1369ab^{9/2}-781a^2b^{5/2}+2281ab^{7/2}+1143ab^{5/2}]\\
&+\cos\theta\sin^6\theta[77a^{3/2}b^6+483a^{3/2}b^5-47a^{1/2}b^6+771a^{3/2}b^4-357a^{1/2}b^5+349a^{3/2}b^3-777a^{1/2}b^4-499a^{1/2}b^3]\\
&+\sqrt{-1}\sin^7\theta[5ab^{13/2}+55ab^{11/2}-3b^{13/2}+115ab^{9/2}-37b^{11/2}+65ab^{7/2}-109b^{9/2}-91b^{7/2}].
\end{aligned}
$$

$$
\begin{aligned}
N_4 :=\ & \cos^7\theta[-75a^{9/2}b^3-91a^{9/2}b^2+75a^{7/2}b^3-25a^{9/2}b+91a^{7/2}b^2-a^{9/2}+25a^{7/2}b+a^{7/2}]\\
&+\sqrt{-1}\cos^6\theta\sin\theta[-391a^4b^{7/2}-639a^4b^{5/2}+391a^3b^{7/2}-285a^4b^{3/2}+639a^3b^{5/2}-29a^4b^{1/2}+285a^3b^{3/2}+29a^3b^{1/2}]\\
&+\cos^5\sin^2\theta[839a^{7/2}b^4+1831a^{7/2}b^3-839a^{5/2}b^4+1165a^{7/2}b^2-1831a^{5/2}b^3+197a^{7/2}b-1165a^{5/2}b^2-197a^{5/2}b]\\
&+\sqrt{-1}\cos^4\theta\sin^3\theta[947a^3b^{9/2}+2779a^3b^{7/2}-947a^2b^{9/2}+2401a^3b^{5/2}-2779a^2b^{7/2}+593a^3b^{3/2}-2401a^2b^{5/2}-593a^2b^{3/2}]\\
&+\cos^3\theta\sin^4\theta[-593a^{5/2}b^5-2401a^{5/2}b^4+593a^{3/2}b^5-2779a^{5/2}b^3+2401a^{3/2}b^4-947a^{5/2}b^2+2779a^{3/2}b^3+947a^{3/2}b^2]\\
&+\sqrt{-1}\cos^2\theta\sin^5\theta[-197a^2b^{11/2}-1165a^2b^{9/2}+197ab^{11/2}-1831a^2b^{7/2}+1165ab^{9/2}-839a^2b^{5/2}+1831ab^{7/2}+839ab^{5/2}]\\
&+\cos\theta\sin^6\theta[29a^{3/2}b^6+285a^{3/2}b^5-29a^{1/2}b^6+639a^{3/2}b^4-285a^{1/2}b^5+391a^{3/2}b^3-639a^{1/2}b^4-391a^{1/2}b^3]\\
&+\sqrt{-1}\sin^7\theta[ab^{13/2}+25ab^{11/2}-b^{13/2}+91ab^{9/2}-25b^{11/2}+75ab^{7/2}-91b^{9/2}-75b^{7/2}].
\end{aligned}
$$

$$
\begin{aligned}
N_5 :=\ & \cos^7\theta[63a^{9/2}b^3+69a^{9/2}b^2-45a^{7/2}b^3+25a^{9/2}b-75a^{7/2}b^2+3a^{9/2}-35a^{7/2}b-5a^{7/2}]\\
&+\sqrt{-1}\cos^6\theta\sin\theta[339a^4b^{7/2}+509a^4b^{5/2}-237a^3b^{7/2}+237a^4b^{3/2}-511a^3b^{5/2}+35a^4b^{1/2}-315a^3b^{3/2}-57a^3b^{1/2}]\\
&+\cos^5\theta\sin^2\theta[-763a^{7/2}b^4-1521a^{7/2}b^3+521a^{5/2}b^4-909a^{7/2}b^2+1431a^{5/2}b^3-167a^{7/2}b+1143a^{5/2}b^2+265a^{5/2}b]\\
&+\sqrt{-1}\cos^4\theta\sin^3\theta[-927a^3b^{9/2}-2409a^3b^{7/2}+617a^2b^{9/2}-1841a^3b^{5/2}+2139a^2b^{7/2}-423a^2b^{3/2}+2191a^2b^{5/2}+653a^2b^{3/2}]\\
&+\cos^3\theta\sin^4\theta[653a^{5/2}b^5+2191a^{5/2}b^4-423a^{3/2}b^5+2139a^{5/2}b^3-1841a^{3/2}b^4+617a^{5/2}b^2-2409a^{3/2}b^3-927a^{3/2}b^2]\\
&+\sqrt{-1}\cos^2\theta\sin^5\theta[265a^2b^{11/2}+1143a^2b^{9/2}-167ab^{11/2}+1431a^2b^{7/2}-909ab^{9/2}+521a^2b^{5/2}-1521ab^{7/2}-763ab^{5/2}]\\
&+\cos\theta\sin^6\theta[-57a^{3/2}b^6-315a^{3/2}b^5+35a^{1/2}b^6-511a^{3/2}b^4+237a^{1/2}b^5-237a^{3/2}b^3+509a^{1/2}b^4+339a^{1/2}b^3]\\
&+\sqrt{-1}\sin^7\theta[-5ab^{13/2}-35ab^{11/2}+3b^{13/2}-75ab^{9/2}+25b^{11/2}-45ab^{7/2}+69b^{9/2}+63b^{7/2}].
\end{aligned}
$$

$$
\begin{aligned}
N_6 :=\ & \cos^7\theta[39a^{9/2}b^3+43a^{9/2}b^2-39a^{7/2}b^3+13a^{9/2}b-43a^{7/2}b^2+a^{9/2}-13a^{7/2}b-a^{7/2}]\\
&+\sqrt{-1}\cos^6\theta\sin\theta[199a^4b^{7/2}+315a^4b^{5/2}-199a^3b^{7/2}+141a^4b^{3/2}-315a^3b^{5/2}+17a^4b^{1/2}-141a^3b^{3/2}-17a^3b^{1/2}]\\
&+\cos^5\theta\sin^2\theta[-419a^{7/2}b^4-919a^{7/2}b^3+419a^{5/2}b^4-577a^{7/2}b^2+919a^{5/2}b^3-101a^{7/2}b+577a^{5/2}b^2+101a^{5/2}b]\\
&+\sqrt{-1}\cos^4\theta\sin^3\theta[-467a^3b^{9/2}-1399a^3b^{7/2}+467a^2b^{9/2}-1201a^3b^{5/2}+1399a^2b^{7/2}-293a^3b^{3/2}+1201a^2b^{5/2}+293a^2b^{3/2}]\\
&+\cos^3\theta\sin^4\theta[293a^{5/2}b^5+1201a^{5/2}b^4-293a^{3/2}b^5+1399a^{5/2}b^3-1201a^{3/2}b^4+467a^{5/2}b^2-1399a^{3/2}b^3-467a^{3/2}b^2]\\
&+\sqrt{-1}\cos^2\theta\sin^5\theta[101a^2b^{11/2}+577a^2b^{9/2}-101ab^{11/2}+919a^2b^{7/2}-577ab^{9/2}+419a^2b^{5/2}-919ab^{7/2}-419ab^{5/2}]\\
&\cos\theta\sin^6\theta[-17a^{3/2}b^6-141a^{3/2}b^5+17a^{1/2}b^6-315a^{3/2}b^4+141a^{1/2}b^5-199a^{3/2}b^3+315a^{1/2}b^4+199a^{1/2}b^3]\\
&+\sqrt{-1}\sin^7\theta[-ab^{13/2}-13ab^{11/2}+b^{13/2}-43ab^{9/2}+13b^{11/2}-39ab^{7/2}+43b^{9/2}+39b^{7/2}].
\end{aligned}
$$

$$
\begin{aligned}
N_7 :=\ & \cos^7\theta[-27a^{9/2}b^3-27a^{9/2}b^2+27a^{7/2}b^3-9a^{9/2}b+27a^{7/2}b^2-a^{9/2}+9a^{7/2}b+a^{7/2}]\\
&+\sqrt{-1}\cos^6\theta\sin\theta[-135a^4b^{7/2}-207a^4b^{5/2}+135a^3b^{7/2}-93a^4b^{3/2}+207a^3b^{5/2}-13a^4b^{1/2}+93a^3b^{3/2}+13a^3b^{1/2}]\\
&+\cos^5\theta\sin^2\theta[279a^{7/2}b^4+615a^{7/2}b^3-279a^{5/2}b^4+381a^{7/2}b^2-615a^{5/2}b^3+69a^{7/2}b-381a^{5/2}b^2-69a^{5/2}b]\\
&+\sqrt{-1}\cos^4\theta\sin^2\theta[307a^3b^{9/2}+939a^3b^{7/2}-307a^2b^{9/2}+801a^3b^{5/2}-939a^2b^{7/2}+193a^3b^{3/2}-801a^2b^{5/2}-193a^2b^{3/2}]\\
&+\cos^3\theta\sin^4\theta[-193a^{5/2}b^5-801a^{5/2}b^4+193a^{3/2}b^5-939a^{5/2}b^3+801a^{3/2}b^4-307a^{5/2}b^2+939a^{3/2}b^3+307a^{3/2}b^2]\\
&+\sqrt{-1}\cos^2\theta\sin^5\theta[-69a^2b^{11/2}-381a^2b^{9/2}+69ab^{11/2}-615a^2b^{7/2}+381ab^{9/2}-279a^2b^{5/2}+615ab^{7/2}+279ab^{5/2}]\\
&+\cos\theta\sin^6\theta[13a^{3/2}b^6+93a^{3/2}b^5-13a^{1/2}b^6+207a^{3/2}b^4-93a^{1/2}b^5+135a^{3/2}b^3-207a^{1/2}b^4-135a^{1/2}b^3]\\
&+\sqrt{-1}\sin^7\theta[ab^{13/2}+9ab^{11/2}-b^{13/2}+27ab^{9/2}-9b^{11/2}+27ab^{7/2}-27b^{9/2}-27b^{7/2}]
\end{aligned}
$$

Finally, with these expression, the following is obtained

$$\frac{1}{8}N_1(\theta)+\frac{3}{4}N_2(\theta)+\frac{1}{2}N_3(\theta)+\frac{1}{8}N_4(\theta)+\frac{15}{8}N_5(\theta)+\frac{5}{4}N_6(\theta)+\frac{15}{8}N_7(\theta)\equiv 0,$$

and hence the curve is contained in the invariant $(\frac{1}{\square}\bar{\mathscr{L}})^4\Theta_{zz}$

Chapter 5

Holomorphic curves in Lorentzian rigid hypersurfaces in $\mathbb{C}^3$

1 CR Geometry of Real-Analytic Hypersurfaces $M^{2n+1} \subset \mathbb{C}^{n+1}$

1.1 Some Remarks

The purpose of this paper is to study the presence of holomorphic curve in Lorentzian rigid real-analytic hypersurfaces $M^5 \subseteq \mathbb{C}^3$. It has been shown in [Bry82] that out of the umbilical locus any Lorentzian real-analytic real hypersurface M^5, there are at most 4 holomorphic curves contained in M^5. This paper follows through the argument of the paper and gives an explicit expression of the equation that defines the possible tangent vectors of the imbedded holomorphic curve, explaining why there are at most 4 imbedded holomorphic curves for general cases. Through these calculations, the explicit expressions of some components of the Hachtroudi-Chern-Moser tensor can be obtained.

Since this paper concerns only rigid real hypersurfaces in $\mathbb{C}^3$, only two prolongations are needed, which simplifies many of the calculations. Therefore, this paper only presents a part of what is presented in [Bry82], which is sufficient for the purpose.

1.2 Recall on Real-Analytic Functions

Let $\mathrm{N} \in \mathbb{N}_{\geqslant 1}$. On an open subset $\Omega \subseteq \mathbb{C}^{\mathrm{N}}$, a function $F : \Omega \to \mathbb{C}$ is real-analytic if locally it can be expanded as Taylor series. If $z := (z_1, \ldots, z_{\mathrm{N}})$ are holomorphic coordinates of $\mathbb{C}^{\mathrm{N}}$, then F may be written as

$$F(z, \bar{z}) = \sum_{a \in \mathbb{N}^{\mathrm{N}},\ b \in \mathbb{N}^{\mathrm{N}}} F_{a,b} z^a \bar{z}^b,$$

where

$$F_{a,b} \in \mathbb{C}, \qquad z^a := z_1^{a_1} \cdots z_{\mathrm{N}}^{a_{\mathrm{N}}}, \qquad \bar{z}^b := \bar{z}_1^{b_1} \cdots \bar{z}_{\mathrm{N}}^{b_{\mathrm{N}}}.$$

If F is real-valued, then the coefficients satisfy the relation

$$\overline{F_{a,b}} = F_{b,a}.$$

1.3 The Geometry of $\mathbb{C}^{\mathrm{N}}$

For $\mathrm{N} \in \mathbb{N}_{\geqslant 2}$, let z be the holomorphic coordinates as before, and for $1 \leqslant i \leqslant \mathrm{N}$, $z_i = x_i + \sqrt{-1} y_i$. Therefore $\mathbb{C}^{\mathrm{N}}$ is to be seen as $\mathbb{R}^{2\mathrm{N}}$ equipped with coordinates $(x_1, y_1, \ldots, x_{\mathrm{N}}, y_{\mathrm{N}})$. The tangent bundle $T\mathbb{C}^{\mathrm{N}}$ is a real vector bundle generated by the sections

$$\frac{\partial}{\partial x_i}, \qquad \frac{\partial}{\partial y_i}, \qquad \big(1 \leqslant i \leqslant \mathrm{N}\big).$$

This tangent bundle is equipped with a complex structure $J : T\mathbb{C}^{\mathrm{N}} \to T\mathbb{C}^{\mathrm{N}}$, given at each point in $\mathbb{C}^{\mathrm{N}}$ by

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i} \qquad \text{and} \qquad J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}.$$

The complexified tangent bundle $\mathbb{C}T\mathbb{C}^{\mathrm{N}} := T\mathbb{C}^{\mathrm{N}} \otimes_{\mathbb{R}} \mathbb{C}$ decomposes into a direct sum of complex sub-bundles

$$\mathbb{C}T\mathbb{C}^{\mathrm{N}} = T^{1,0}\mathbb{C}^{\mathrm{N}} \oplus T^{0,1}\mathbb{C}^{\mathrm{N}},$$

where at each point $p \in \mathbb{C}^{\mathrm{N}}$,

$$\begin{aligned} T^{1,0}_p\mathbb{C}^{\mathrm{N}} &:= \{X_p - \sqrt{-1}J(X_p) : \ X_p \in T_p\mathbb{C}^{\mathrm{N}}\}, \\ T^{0,1}_p\mathbb{C}^{\mathrm{N}} &:= \{X_p + \sqrt{-1}J(X_p) : \ X_p \in T_p\mathbb{C}^{\mathrm{N}}\}. \end{aligned}$$

For $1 \leqslant i \leqslant \mathrm{N}$, the vector fields

$$\frac{\partial}{\partial z_i} := \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right), \qquad \frac{\partial}{\partial \bar{z}_i} := \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right).$$

generate respectively the bundles $T^{1,0}\mathbb{C}^{\mathrm{N}}$ and $T^{0,1}\mathbb{C}^{\mathrm{N}}$. Dually, the differential forms

$$dz_i := dx_i + \sqrt{-1}dy_i, \qquad d\bar{z}_i := dx_i - \sqrt{-1}dy_i$$

generate the respective cotangent bundles $T^{*1,0}\mathbb{C}^{\mathrm{N}}$ and $T^{*0,1}\mathbb{C}^{\mathrm{N}}$.

For $1 \leqslant p, q \leqslant \mathrm{N}$, a (p, q) form is given by

$$\eta := \sum_{1\leqslant i_1<\cdots<i_p\leqslant \mathrm{N}} \sum_{1\leqslant j_1<\cdots<j_q\leqslant \mathrm{N}} \omega_{i_1,\ldots,i_p,j_1,\ldots,j_q}\, dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q},$$

where $\omega_{i_1,\ldots,j_q}$ are functions on $\mathbb{C}^{\mathrm{N}}$. The de-Rham differential operator d is defined on η by

$$\begin{aligned} d\eta \ = \ & \sum_{\substack{1\leqslant i_1<\cdots<i_p\leqslant \mathrm{N} \\ 1\leqslant j_1<\cdots<j_q\leqslant \mathrm{N}}} \sum_{1\leqslant i\leqslant \mathrm{N}} \partial_{z_i}\omega_{i_1,\ldots,i_p,j_1,\ldots,j_q}\, dz_i \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q} \\ & + \sum_{\substack{1\leqslant i_1<\cdots<i_p\leqslant \mathrm{N} \\ 1\leqslant j_1<\cdots<j_q\leqslant \mathrm{N}}} \sum_{1\leqslant j\leqslant \mathrm{N}} \partial_{\bar{z}_j}\omega_{i_1,\ldots,i_p,j_1,\ldots,j_q}\, d\bar{z}_j \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}. \end{aligned}$$

By the anti-symmetry of the wedge product and commutativity of the vector fields ∂_{z_i}, $\partial_{\bar{z}_j}$, the differential operator is nilpotent, i.e. $d^2 = 0$.

1.4 Defining function of M

Let $M \subseteq \mathbb{C}^{n+1}$ be a real-analytic real hypersurface passing through the origin. Fix a reference point $0 \in M$. There exists affine holomorphic coordinates in $\mathbb{C}^{n+1}$

$$(z_1, \ldots, z_n, w) \ = \ (x_1 + \sqrt{-1}y_1, \ldots, x_n + \sqrt{-1}y_n, u + \sqrt{-1}v)$$

and a real analytic, real-valued function $r(x, u, y, v)$ such that M may be described by the zero locus of r:

$$M \ = \ \{r = 0\}.$$

Therefore $r(0) = 0$. If moreover that M is smooth at 0, that is $dr(0) \neq 0$, by a change of coordinates, without loss of generality $r_u(0) \neq 0$. By the Implicit Function Theorem, there exists a real-analytic, real-valued function $F(x, y, v)$ such that M is described by the following equation

$$u = F(x_1, \ldots, x_n, y_1, \ldots, y_n, v).$$

Therefore, the defining function r may be re-expressed as

$$r = -u + F.$$

Such a defining function is not unique, as it can be multiplied by a function on $\mathbb{C}^{n+1}$ which does not vanish anywhere. By a bihilomorphic change of coordinates in $\mathbb{C}^{n+1}$, the defining function $-u + F$ may undergo further simiplification so that M may be defined by

$$u = \sum_{1 \leqslant i \leqslant n} \sum_{1 \leqslant j \leqslant n} a_{ij} z_i \bar{z}_j + O_{z,\bar{z}}(3) + v O_{z,\bar{z},v}(1),$$

where $a_{ij} \in \mathbb{C}$.

Definition 1.1. Let M be a real-analytic, real hypersurface in $\mathbb{C}^3$ given by

$$u = F(x, y, v).$$

Then M is called rigid if F is independent of v.

For the rest of the section, M is not supposed to be rigid unless otherwise stated.

1.5 CR bundles induced on M

Over $M^{2n+1} \subset \mathbb{C}^{n+1}$, let ,

$$T^{1,0}M := T^{1,0}\mathbb{C}^{n+1}|_M \cap (\mathbb{C} \otimes_{\mathbb{R}} TM) \qquad \text{and} \qquad T^{0,1}M := (T^{0,1}\mathbb{C}^{n+1}|_M) \cap (\mathbb{C} \otimes_{\mathbb{R}} TM).$$

They form complex vector bundles over M, with $\overline{T^{1,0}M} = T^{0,1}M$. The common real part of these two bundles

$$T^cM \ := \ \mathsf{Re}(T^{1,0}M) \ = \ \mathsf{Re}(T^{0,1}M)$$

is a real sub-bundle of TM of real rank $2n$. It may alternatively be defined as

$$T^cM \ = \ TM \cap J(TM).$$

At each $p \in M$,

$$\begin{aligned} T^{1,0}_p M \ &= \ \{X_p - \sqrt{-1}J(X_p) : \ X_p \in T^c_p M\}, \\ T^{0,1}_p M \ &= \ \{X_p + \sqrt{-1}J(X_p) : \ X_p \in T^c_p M\}. \end{aligned}$$

1.6 Frames of $T^{1,0}M$ and $T^{0,1}M$

For a real manifold given by

$$u = F(z, \bar{z}, v),$$

its extrinsic $T^{1,0}M$ sections are given by

$$L_k := \frac{\partial}{\partial z_k} + A_k \frac{\partial}{\partial w},$$

with

$$\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - \sqrt{-1}\frac{\partial}{\partial y_k}\right), \qquad \frac{\partial}{\partial w} = \frac{1}{2}\left(\frac{\partial}{\partial u} - \sqrt{-1}\frac{\partial}{\partial v}\right).$$

By requiring that $L_k(u - F) \equiv 0$, it follows that

$$A_k = \frac{2F_{z_k}}{1 + \sqrt{-1}F_v}.$$

The projection of $\mathbb{C}^{n+1}$ onto $\mathbb{R}^{2n+1}$ by

$$\pi : (x, u, y, v) \mapsto (x, y, v)$$

gives a local chart on M^{2n+1} and hence the intrinsic $(1,0)$ vector fields on M^{2n+1} become

$$\mathscr{L}_k := \pi_*(L_k) = \frac{\partial}{\partial z_k} + \frac{F_{z_k}}{\sqrt{-1} - F_v}\frac{\partial}{\partial v},$$

and along with its conjugates

$$\bar{\mathscr{L}}_k := \frac{\partial}{\partial \bar{z}_k} + \frac{F_{\bar{z}_k}}{-\sqrt{-1} - F_v}\frac{\partial}{\partial v} \qquad (1 \leqslant k \leqslant n)$$

which form a frame of the $T^{0,1}M$ bundle. These vector fields satisfy the involutive properties

$$[\mathscr{L}_i, \mathscr{L}_j] = [\bar{\mathscr{L}}_i, \bar{\mathscr{L}}_j] = 0,$$

while a lengthy calculation (see **MPS**) gives

$$\begin{aligned}[\mathscr{L}_i, \bar{\mathscr{L}}_j] \;=\; & \frac{2\sqrt{-1}}{|\sqrt{-1} + F_v|^2}(F_{z_i\bar{z}_j} + F_v^2 F_{z_i\bar{z}_j} - F_v F_{\bar{z}_j} F_{z_i v} - F_v F_{z_i} F_{v\bar{z}_j} + F_{z_i} F_{\bar{z}_j} F_{vv} \\ & + \sqrt{-1} F_{\bar{z}_j} F_{z_i v} - \sqrt{-1} F_{z_i} F_{v\bar{z}_j}).\end{aligned}$$

If M is rigid, then

$$[\mathscr{L}_i, \bar{\mathscr{L}}_j] = 2\sqrt{-1} F_{z_i\bar{z}_j}.$$

1.7 Contact Form

A differentiation of the defining function

$$r = -u + F(x, y, v)$$

yields

$$0 \;\equiv dr|_{TM} \;=\; \partial r|_{TM} + \bar{\partial} r|_{TM}.$$

Therefore $-\sqrt{-1}\partial r|_{TM} = \sqrt{-1}\bar{\partial} r|_{TM}$ is a real-valued differential form on M. For any nowhere vanishing function $c(z, \bar{z}, v)$ on M,

$$\partial(cr)|_{TM} = r|_M \partial c|_{TM} + c|_M \partial r|_{TM} = c|_M \partial r|_{TM}.$$

Therefore, this allows a modification of the defining function r to facilitate calculations later. In fact, r may be redefined as

$$r \;:=\; \frac{2}{1 + F_v^2}(-u + F).$$

With this new defining function, set

$$\theta \;:=\; -\sqrt{-1}\partial r|_{TM}.$$

Then

$$\theta \;=\; -dv + \sum_{1 \leqslant k \leqslant n} \frac{F_{z_k}}{\sqrt{-1} - F_v}\, dz_k + \sum_{1 \leqslant k \leqslant n} \frac{F_{\bar{z}_k}}{-\sqrt{-1} - F_v}\, d\bar{z}_k.$$

This permits the definition of the CR coframe of M.

1.8 CR frame and CR coframe

In local intrinsic coordinates (x, y, v) on M, the following differential 1 forms form a coframe for $\mathbb{C}TM$:

$$\begin{aligned}\theta &= -dv + \sum_{1\leqslant k\leqslant n} \frac{F_{z_k}}{\sqrt{-1} - F_v}\, dz_k + \sum_{1\leqslant k\leqslant n} \frac{F_{\bar{z}_k}}{-\sqrt{-1} - F_v}\, d\bar{z}_k,\\ \theta^k &= dz_k,\\ \bar{\theta}^k &= d\bar{z}_k,\end{aligned}$$

where $1 \leqslant k \leqslant n$. If $T := -\frac{\partial}{\partial v}$, then the following duality relations are obtained:

$$\begin{array}{lll}\theta(T) = 1, & \theta(\mathscr{L}_k) = 0, & \theta(\bar{\mathscr{L}}_k) = 0,\\ \theta^k(T) = 0, & \theta^i(\mathscr{L}_k) = \delta_{ik}, & \theta^i(\bar{\mathscr{L}}_k) = 0,\\ \bar{\theta}^k(T) = 0, & \bar{\theta}^i(\mathscr{L}_k) = 0, & \bar{\theta}^i(\bar{\mathscr{L}}_k) = \delta_{ik}.\end{array}$$

Given these relations, therefore the bundles $T^{1,0}M$ and $T^{0,1}M$ may be re-expressed in terms of differential forms: at each point $p \in M$,

$$\begin{aligned}T^{1,0}_p M &= \{X_p \in T_pM \otimes_{\mathbb{R}} \mathbb{C} :\ 0 = \theta(X_p) = \bar{\theta}^1(X_p) = \cdots = \bar{\theta}^n(X_p)\},\\ T^{0,1}_p M &= \{X_p \in T_pM \otimes_{\mathbb{R}} \mathbb{C} :\ 0 = \theta(X_p) = \theta^1(X_p) = \cdots = \theta^n(X_p)\}.\end{aligned}$$

Moreover,

$$[\mathscr{L}_i, \bar{\mathscr{L}}_j] = -\left(\mathscr{L}_i\left(\frac{F_{\bar{z}_j}}{-\sqrt{-1} - F_v}\right) - \bar{\mathscr{L}}_j\left(\frac{F_{z_i}}{\sqrt{-1} - F_v}\right)\right)T,$$

Applying the de Rham operator d to θ, the following formula comes directly from the Cartan-Lie formula:

$$d\theta = \sum_{1\leqslant i,j\leqslant n} \left(\mathscr{L}_i\left(\frac{F_{\bar{z}_j}}{-\sqrt{-1} - F_v}\right) - \bar{\mathscr{L}}_j\left(\frac{F_{z_i}}{\sqrt{-1} - F_v}\right)\right) \theta^i \wedge \bar{\theta}^j \ \mathsf{mod}\ \theta.$$

Setting

$$h_{ij} := -\sqrt{-1}\left(\mathscr{L}_i\left(\frac{F_{\bar{z}_j}}{-\sqrt{-1} - F_v}\right) - \bar{\mathscr{L}}_j\left(\frac{F_{z_i}}{\sqrt{-1} - F_v}\right)\right) = \theta(\sqrt{-1}[\mathscr{L}_i, \overline{\mathscr{L}}_j]),$$

therefore,

$$d\theta = \sqrt{-1} \sum_{1\leqslant i,j\leqslant n} h_{ij}\, \theta^i \wedge \bar{\theta}^j \ \mathsf{mod}\ \theta.$$

The matrix (h_{ij}) is then Hermitian, with the diagonal entries being real-valued functions on M.

In the case of rigid real hypersurfaces,

$$\mathscr{L}_i(\sqrt{-1}F_{\bar{z}_j}) - \bar{\mathscr{L}}_j(-\sqrt{-1}F_{z_i}) = 2\sqrt{-1}F_{z_i\bar{z}_j},$$

so that

$$d\theta = \sum_{1\leqslant i,j\leqslant n} 2F_{z_i\bar{z}_j}\, \theta^i \wedge \bar{\theta}^j \ \mathsf{mod}\ \theta.$$

1.9 The Levi form

There exists a bilinear form on $T^{1,0}M$ at each point $p \in M$ given as follows: for any two vector fields $\mathscr{X}, \mathscr{Y}$

$$\begin{aligned}\mathsf{Lev} : T^{1,0}_p M \times T^{1,0}_p M &\longrightarrow \mathbb{C}T_pM/T^{1,0}_p M \oplus T^{0,1}_p M\\ (\mathscr{X}(p), \mathscr{Y}(p)) &\longmapsto [\mathscr{X}, \bar{\mathscr{Y}}](p) \qquad \mathsf{mod}\ T^{1,0}_p M \oplus T^{0,1}_p M.\end{aligned}$$

This map is independent of the choice of sections, that is if $\mathscr{X}'$ and $\mathscr{Y}'$ are another $T^{1,0}M$ sections such that $\mathscr{X}'(p) = \mathscr{X}(p)$ and $\mathscr{Y}'(p) = \mathscr{Y}'(p)$, then

$$\mathsf{Lev}(\mathscr{X}(p), \mathscr{Y}(p)) = \mathsf{Lev}(\mathscr{X}'(p), \mathscr{Y}'(p)).$$

Any two sections $\mathscr{X}$, $\mathscr{Y}$ can be written as

$$\begin{aligned} \mathscr{X} &= \mu_1 \mathscr{L}_1 + \cdots + \mu_n \mathscr{L}_n, \\ \mathscr{Y} &= \nu_1 \mathscr{L}_1 + \cdots + \nu_n \mathscr{L}_n. \end{aligned}$$

Then the Levi form may be expressed as a Levi matrix which is denoted by $\Lambda := (l_{ij})$:

$$\mathsf{Lev}(\mathscr{X}, \mathscr{Y}) = (\bar{\nu}_1, \ldots, \bar{\nu}_n) \begin{pmatrix} l_{11} & \cdots & l_{1n} \\ \vdots & \ddots & \vdots \\ l_{n1} & \cdots & l_{nn} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix},$$

where the (i, j)-th entry l_{ij} is given by

$$l_{ij} = \theta(\sqrt{-1}[\mathscr{L}_j, \bar{\mathscr{L}_i}]) = h_{ji}.$$

The Levi form also has another representation by the contact form. Let

$$r = \frac{2}{1 + F_v^2}(-u + F)$$

be the defining function for M as before, and let

$$\theta = -\sqrt{-1}\partial r.$$

By the Cartan-Lie formula,

$$d\theta(\mathscr{X} \wedge \bar{\mathscr{Y}}) = \mathscr{X}\theta(\bar{\mathscr{Y}}) - \bar{\mathscr{Y}}\theta(\mathscr{X}) - \theta([\mathscr{X}, \bar{\mathscr{Y}}]).$$

Since θ vanishes on $T^{1,0}M \oplus T^{0,1}M$ sections,

$$-\theta([\mathscr{X}, \bar{\mathscr{Y}}]) = d\theta(\mathscr{X} \wedge \bar{\mathscr{Y}}),$$

and multiplying on both sides by $-\sqrt{-1}$, the Levi form may therefore be represented by $-\sqrt{-1}\, d\theta$.

1.10 Diagonalisation of the Levi form

Let

$$\mathscr{A}_j = \sum_{m=1}^{n} a_{mj} \mathscr{L}_m$$

be a change of frame so that

$$\varepsilon_i \delta_{ij} = \theta(\sqrt{-1}[\mathscr{A}_j, \bar{\mathscr{A}_i}]),$$

where δ_{ij} is the usual Kronecker delta, and $\varepsilon_i \in \{1, 0, -1\}$. Expanding the expression on the right,

$$\begin{aligned} \theta(\sqrt{-1}[\mathscr{A}_j, \bar{\mathscr{A}_i}]) &= \theta\left(\sqrt{-1}\left[\sum_{m_1} a_{m_1,j}\mathscr{L}_{m_1}, \sum_{m_2} \overline{a_{m_2,i}}\bar{\mathscr{L}}_{m_2}\right]\right) \\ &= \sum_{m_1}\sum_{m_2} \overline{a_{m_2,i}}\, \theta(\sqrt{-1}[\mathscr{L}_{m_1}, \bar{\mathscr{L}}_{m_2}])\, a_{m_1,j} \\ &= \sum_{m_1}\sum_{m_2} \overline{a_{m_2,i}}\, l_{m_2,m_1}\, a_{m_1,j}. \end{aligned}$$

Let $a := (a_{ij})$ denote the matrix, and E be the diagonal matrix whose entries are either 1, 0 or -1. Then the purpose is to seek the matrix a such that

$$E = {}^T\bar{a}\Lambda a.$$

Let $\alpha^i = \sum_m b_{j,m}\theta^m$ be a change of coframe. Then

$$\theta^j = \sum_{1\leqslant m\leqslant n} b^{-1}_{j,m}\alpha^m,$$

where $b^{-1}_{i,j}$ are the coefficients of the inverse matrix b^{-1}. Observe that

$$\begin{aligned} d\theta &\equiv \sqrt{-1}(\theta^1,\cdots,\theta^n)\begin{pmatrix} h_{1,1} & \cdots & h_{1,n}\\ \vdots & \ddots & \vdots\\ h_{n,1} & \cdots & h_{n,n}\end{pmatrix}\begin{pmatrix}\bar\theta^1\\ \vdots\\ \bar\theta^n\end{pmatrix} \qquad \mathsf{mod}\ \theta\\ &\equiv \sqrt{-1}(\bar\theta^1,\cdots,\bar\theta^n)\begin{pmatrix} l_{11} & \cdots & l_{1,n}\\ \vdots & \ddots & \vdots\\ l_{n,1} & \cdots & l_{n,n}\end{pmatrix}\begin{pmatrix}\theta^1\\ \vdots\\ \theta^n\end{pmatrix} \qquad \mathsf{mod}\ \theta\\ &\equiv \sqrt{-1}(\bar\alpha_1,\ldots,\bar\alpha_n)\overline{{}^Tb^{-1}}\cdot(l_{ij})\cdot b^{-1}(\alpha_1,\ldots,\alpha_n)^T \qquad \mathsf{mod}\ \theta\end{aligned}$$

If $b = (b_{ij})$ is chosen so that for the same matrix E,

$$E = {}^T\overline{b^{-1}}\ \Lambda\ b^{-1},$$

then the choice $a = b^{-1}$ works.

1.11 Explicit Diagonalisation of the Levi form

The process of diagonalising the Levi form involves seeking the new frame $\mathscr{A}_i$ $(1 \leqslant i \leqslant n)$ and the following triangular linear system of equations

$$\begin{aligned}\mathscr{A}_1 &= a_{1,1}\mathscr{L}_1,\\ \mathscr{A}_2 &= a_{1,2}\mathscr{L}_1 + a_{2,2}\mathscr{L}_2,\\ &\vdots\\ \mathscr{A}_n &= a_{1,n}\mathscr{L}_1 + \cdots + a_{n,n}\mathscr{L}_n,\end{aligned}$$

so that with respect to the Levi form Lev,

$$\mathsf{Lev}(\mathscr{A}_i,\mathscr{A}_j) = \varepsilon_i\delta_{ij},$$

where $\varepsilon \in \{1,0,-1\}$ are the signatures of the eigenvalues of the Levi form. At each point $p \in M$, define the minor determinant $\Delta_j(p)$ by

$$\Delta_j(p) = \mathsf{det}\begin{pmatrix} l_{1,1}(p) & \cdots & l_{1,j}(p)\\ \vdots & \ddots & \vdots\\ l_{j,1}(p) & \cdots & l_{j,j}(p)\end{pmatrix}.$$

By Gram-Schmidt process, the following proposition gives the explicit diagonalisation of the Levi form:

Proposition 1.2. *The closed formulae for a $T^{1,0}M$ frame which is orthonormal for the Levi form, i.e.*

$$\mathsf{Lev}(\mathscr{A}_i, \mathscr{A}_j) = \varepsilon_i \delta_{ij},$$

are given by

$$\mathscr{A}_j = \frac{1}{\sqrt{\varepsilon_j \Delta_{j-1} \Delta_j}} \begin{vmatrix} l_{1,1} & l_{1,2} & \cdots & l_{1,j} \\ \vdots & \vdots & \ddots & \vdots \\ l_{j-1,1} & l_{j-1,2} & \cdots & l_{j-1,j} \\ \mathscr{L}_1 & \mathscr{L}_2 & \cdots & \mathscr{L}_j \end{vmatrix}.$$

2 The Geometry of Lorentzian Real Hypersurfaces in $\mathbb{C}^{n+1}$

2.1 Holomorphic Curves in Hypersurfaces

Let $M^{2n+1} \subset \mathbb{C}^{n+1}$ be a real hypersurfaces and let $\mathbb{D} := \{t \in \mathbb{C} : \ |t| < 1\}$ be the unit disk in $\mathbb{C}$. Let ϕ be the map

$$\begin{array}{rcl} \phi : \mathbb{D} & \longrightarrow & \mathbb{C}^{n+1} \\ t & \longmapsto & (\phi_1(t), \ldots, \phi_n(t), \phi_{n+1}(t)), \end{array}$$

and suppose that $\phi(\mathbb{D}) \subset M$ so that for all $t \in \mathbb{D}$,

$$r(\phi_1(t), \ldots, \phi_n(t), \phi_{n+1}(t), \overline{\phi_1(t)}, \ldots, \overline{\phi_n(t)}, \overline{\phi_{n+1}(t)}) \equiv 0.$$

Differentiating the equation above with respect to t,

$$\begin{array}{rcl} 0 \equiv \dfrac{\partial}{\partial t} r(\phi(t), \overline{\phi(t)}) = & \displaystyle\sum_{i=1}^{n+1} r_{z_i}(\phi(t), \overline{\phi(t)}) \phi_i'(t) \\ = & \partial r \Big(\displaystyle\sum_{i=1}^{n+1} \phi_i'(t) \frac{\partial}{\partial z_i} \Big), \end{array} \tag{2.1}$$

from which it is deduced that at each $t \in \mathbb{D}$, the vector field

$$\mathscr{L}_{\phi(t)} = \sum_{i=1}^{n+1} \phi_i'(t) \frac{\partial}{\partial z_i}$$

lies in $T^{1,0}_{\phi(t)} M \oplus T^{0,1}_{\phi(t)} M$. But since the vector field involves ∂_{z_i} and not its conjugates, therefore it lies in $T^{1,0}_{\phi(t)} M$. Moreover,

$$0 \equiv \sqrt{-1} d\theta(\mathscr{L}_{\phi(t)} \wedge \overline{\mathscr{L}}_{\phi(t)}) = \sqrt{-1} \partial\bar{\partial} r(\mathscr{L}_{\phi(t)} \wedge \overline{\mathscr{L}_{\phi(t)}}) = \lambda(\mathscr{L}_{\phi(t)}, \mathscr{L}_{\phi(t)}). \tag{2.2}$$

Definition 2.3. At each $p \in M$, the isotropic cone of the Levi form is given by

$$C^{1,0}_p M := \{\mathscr{C}_p \in T^{1,0}_p M : \ \mathsf{Lev}(\mathscr{C}_p, \mathscr{C}_p) = 0\}.$$

The common part of the isotropic cone is then given by

$$C^c_p M \ := \ \mathsf{Re}\, C^{1,0}_p M \ = \ \mathsf{Re}\, C^{0,1}_p M \subset T^c_p M.$$

2.2 Holomorphic curves in Lorentzian real hypersurfaces

Definition 2.4. The real hypersurface $M^{2n+1} \subset \mathbb{C}^{n+1}$ is Lorentzian at $0 \in M^{2n+1}$ if the Levi form at 0 has 1 strictly positive eigenvalue and $n-1$ strictly negative eigenvalues. Moreover, M^{2n+1} is Lorentzian if the Levi form has 1 strictly positive eigenvalue and $n-1$ strictly negative eigenvalues at every point $p \in M$.

When M is Lorentzian, by a change of frames and coframes in Proposition 1.2, there exist vector fields $\mathscr{A}_i$ and differential $(T^{1,0}M)^*$-forms α^i such that

$$d\theta = \sqrt{-1}(\alpha^1 \wedge \bar{\alpha}^1 - \alpha^2 \wedge \bar{\alpha}^2 - \cdots - \alpha^n \wedge \bar{\alpha}^n) \qquad \mathsf{mod}\ \theta, \tag{2.5}$$

along with the following duality relations

$$\begin{aligned} \theta(T) &= 1, & \theta(\mathscr{A}_j) &= 0, & \theta(\bar{\mathscr{A}_j}) &= 0, \\ \alpha^i(T) &= 0, & \alpha^i(\mathscr{A}_j) &= \delta_{ij}, & \alpha^i(\bar{\mathscr{A}_j}) &= 0, \\ \bar{\alpha}^i(T) &= 0, & \bar{\alpha}^i(\mathscr{A}_j) &= 0, & \bar{\alpha}^i(\bar{\mathscr{A}_j}) &= \delta_{ij}. \end{aligned}$$

If $\phi : \mathbb{D} \to \mathbb{C}^{n+1}$ is a holomorphic curve as before, then the vector field $\mathscr{L}_{\phi(t)}$ is a $T^{1,0}M$ vector at each point $\phi(t)$, and hence may be written as

$$\mathscr{L}_{\phi(t)} = f_1(t,\bar{t})\mathscr{A}_1 + \cdots + f_n(t,\bar{t})\mathscr{A}_n$$

for some real-analytic functions $f_i(t,\bar{t})$ for $1 \leqslant i \leqslant n$. By equations (2.1) and (2.2), $d\theta(\mathscr{L}_{\phi(t)} \wedge \overline{\mathscr{L}_{\phi(t)}}) \equiv 0$ and $\theta(\mathscr{L}_{\phi(t)}) \equiv 0$ for all t. Therefore by equation (2.5),

$$|f_1|^2 - |f_2|^2 - \cdots - |f_n|^2 \equiv 0 \qquad (t \in \mathbb{D}).$$

If ϕ is an immersion, then for all $t \in \mathbb{D}$, $f_1(t,\bar{t}) \neq 0$, or otherwise then

$$|f_2(t,\bar{t})|^2 + \cdots + |f_n(t,\bar{t})|^2 = 0,$$

from which one concludes that $f_2(t,\bar{t}) = \cdots = f_n(t,\bar{t}) = 0$, which is a contradiction. Consequently, f_1 does not vanish anywhere. Dividing throughout by f_1,

$$\left|\frac{f_2}{f_1}\right|^2 + \cdots + \left|\frac{f_n}{f_1}\right|^2 = 1,$$

which is an equation of the sphere. More generally, if $(\mu_2, \ldots, \mu_n) \in \mathbb{C}^{n-1}$ be such that $|\mu_2|^2 + \cdots + |\mu_n|^2 = 1$, then the vector fields

$$\mathscr{A}_1 - \mu_2\mathscr{A}_2 - \cdots - \mu_n\mathscr{A}_n$$

lie in the isotropic cone of the Levi form. Conversely, the isotropic cones are generated by vector fields of the form above, i.e.

$$C^{1,0}_p M = \bigcup_{\lambda \in \mathbb{S}^{2n-3}} \{\mathscr{X}_p \in T^{1,0}_p M :\ 0 = (\alpha^k - \lambda_k\alpha_1)(\mathscr{X}_p),\ 1 \leqslant k \leqslant n-1\}.$$

For the rest, let $L^{1,0}_x M$ and $L^{0,1}_x M$ be the folowing sets at each $x \in M$ and for each fixed $(\lambda_2, \ldots, \lambda_n) \in \mathbb{S}^{2n-3}$

$$\begin{aligned} L^{1,0}_{p,(\lambda_2,\ldots,\lambda_n)} M &:= \{\mathscr{X}_p \in T^{1,0}_p M :\ 0 = (\alpha^k - \lambda_k\alpha_1)(\mathscr{X}_p),\ 1 \leqslant k \leqslant n-1\}, \\ L^{0,1}_{p,(\lambda_2,\ldots,\lambda_n)} M &:= \{\mathscr{X}_p \in T^{0,1}_p M :\ 0 = (\bar{\alpha}^k - \bar{\lambda}_k\bar{\alpha}_1)(\mathscr{X}_p),\ 1 \leqslant k \leqslant n-1\}. \end{aligned}$$

2.3 The Sphere Bundle and the First Prolongation

The main idea of [Bry82] is to introduce new unknown variables $(\lambda_2, \ldots, \lambda_n) \in \mathbb{S}^{2n-3}$ to a system of differential forms, which is known as prolongation in Cartan's method. Later it will be found that the variables satisfy certain equations in $\mathbb{R}^n$. Let α^1,..., α^n be sections of $T^{*1,0}M$ that diagonalise $d\theta$ as in equation (2.5). Let $\phi : \mathbb{D} \to \mathbb{C}^{n+1}$ be a non-degenerate holomorphic curve such that $\phi(\mathbb{D}) \subset M^{2n+1}$. Then for each i, there exist real-analytic complex-valued functions $a_j(t, \bar{t})$ such that

$$\phi^*(\alpha^j) \;=\; a_j(t, \bar{t})\, dt$$

and

$$\phi^*(\bar{\alpha}^j) \;=\; \overline{a_j(t, \bar{t})}\, d\bar{t}.$$

Moreover, in paragraph 2.1, it has been established that $\phi^*\theta \equiv 0$. Therefore

$$\sqrt{-1}(|a_1(t, \bar{t})|^2 - |a_2(t, \bar{t})|^2 - \cdots - |a_n(t, \bar{t})|^2) dt \wedge d\bar{t} = \phi^*(d\theta) = d(\phi^*\theta) = 0.$$

Since $dt \wedge d\bar{t}$ is the volume form on $\mathbb{D}$, therefore

$$|a_1(t, \bar{t})|^2 - |a_2(t, \bar{t})|^2 - \cdots - |a_n(t, \bar{t})|^2 \equiv 0$$

for all $t \in \mathbb{D}$.

Lemma 2.6. *Let $\phi : \mathbb{D} \to M^{2n+1}$ be an immersive holomorphic curve into a Lorentzian real hypersurface in $\mathbb{C}^{n+1}$. If $a_1(t, \bar{t}) = 0$ at some point $t \in \mathbb{D}$, then $\phi'(t) = 0$.*

Proof. By the same reasoning as before, if $a_1(t, \bar{t}) = 0$, then so are the rest. Since θ, α^1,..., α^n, $\bar{\alpha}^1$,...,$\bar{\alpha}^n$ form a frame for $\mathbb{C}TM$ at every point in M, and the pullback of each of these differential 1-forms is zero, therefore $\phi'(t) = 0$. □

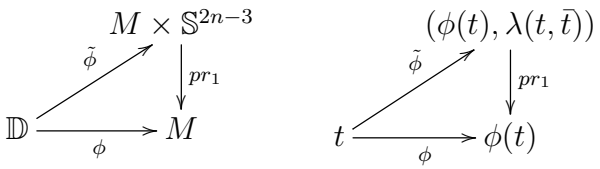
Therefore, by setting

$$\lambda_k := \frac{a_k(t, \bar{t})}{a_1(t, \bar{t})},$$

the equation of sphere is obtained

$$|\lambda_2|^2 + \cdots + |\lambda_n|^2 = 1.$$

The functions $\lambda_k = \alpha_k/\alpha_1$ also allow the lifting of the holomorphic curves to a bigger space:



Then for $2 \leqslant k \leqslant n$, introduce the 1 forms

$$\begin{aligned}
\omega^0 &:= \theta, \\
\omega^1 &:= \alpha^1, \\
\omega^k &:= \alpha^k - \lambda_k \alpha^1, \\
\bar{\omega}^1 &:= \bar{\alpha}^1, \\
\bar{\omega}^k &:= \bar{\alpha}^k - \bar{\lambda}_k \bar{\alpha}^1,
\end{aligned}$$

which is the first prolongation by the new variables $(\lambda_2, \ldots, \lambda_n) \in \mathbb{S}^{2n-3}$.

Lemma 2.7. *The lift $\tilde{\phi} : \mathbb{D} \to M \times \mathbb{S}^{2n-3}$ of the holomorphic curve $\phi : \mathbb{D} \to M$ satisfies*

$$0 = \tilde{\phi}^*(\omega^0) = \tilde{\phi}^*(\omega^k) = \tilde{\phi}^*(\bar{\omega}^k).$$

Conversely, if a real-analytic map

$$\tilde{\phi}(t, \bar{t}) = (\phi_1, \ldots, \phi_n, \mathsf{Im}\phi_{n+1}, \lambda_2, \ldots, \lambda_n)$$

with $\tilde{\phi}$ immersive from $\mathbb{D}$ to $\mathbb{C}^n \times \mathbb{R} \times \mathbb{S}^{2n-3}$ annihilates ω^0, ω^k and $\bar{\omega}^k$ after pullback, then its projected image

$$\phi(\mathbb{D}) = pr_1(\tilde{\phi}(\mathbb{D})) \subset M$$

is a holomorphic curve in M.

Proof. The direct sense holds by definition of $(\lambda_2, \ldots, \lambda_n) \in \mathbb{S}^{2n-3}$. Conversely, the condition that $0 = \tilde{\phi}^*(\omega^0) = \phi^*(\theta)$ implies that

$$\phi_*(T\mathbb{D}) \subseteq \mathsf{Re}\, T^{1,0}M|_{\phi(\mathbb{D})}.$$

For any fixed $(\lambda_2, \ldots, \lambda_n) \in \mathbb{S}^{2n-3}$, and for all $(2 \leqslant k \leqslant n)$, the conditions

$$\begin{aligned} 0 &= \tilde{\phi}^*(\omega^k) = \phi^*(\alpha^k) - \lambda_k \phi^*(\alpha^1), \\ 0 &= \tilde{\phi}^*(\bar{\omega}^k) = \phi^*(\bar{\alpha}^k) - \bar{\lambda}_k \phi^*(\bar{\alpha}^1), \end{aligned}$$

imply that

$$\phi_*(T_t\mathbb{D} \otimes_{\mathbb{R}} \mathbb{C}) \subset L^{1,0}_{\phi(t,\bar{t}),(\lambda_2,\ldots,\lambda_n)}M \oplus L^{0,1}_{\phi(t,\bar{t}),(\lambda_2,\ldots,\lambda_n)}M.$$

Taking the real part, implies that

$$\phi_*(T_t(\mathbb{D})) \subset L^{c}_{\phi(t,\bar{t}),(\lambda_2,\ldots,\lambda_n)}M = \mathsf{Re}(L^{1,0}_{\phi(t,\bar{t}),(\lambda_2,\ldots,\lambda_n)}M).$$

which guarantees that the real 2-dimensional locally embedded submanifold $\phi(\mathbb{D}) \subset M$ has a tangent bundle that is invariant under multiplication by $\sqrt{-1}$, and hence by the theorem of Levi-Civita [Sha92, page 99], $\phi(\mathbb{D})$ is a holomorphic curve. □

2.4 Partial Pullback

At this stage it will be convenient to introduce the notion of partial pullbacks. If $\tilde{\phi} : \mathbb{D} \to M \times \mathbb{S}^{2n-3}$ is the lifting as in the previous paragraph

$$t \longmapsto (\phi_1, \ldots, \phi_n, \nu, \lambda_2, \ldots, \lambda_n),$$

where $\nu(t, \bar{t}) = \mathsf{Im}\phi_{n+1}(t)$.

Definition 2.8. The partial pullback of differential forms on $M \times \mathbb{S}^{2n-3}$ replaces $x := (\phi_1(t), \ldots, \phi_n(t), \nu(t, \bar{t}))$ but keeps $(\lambda_2, \ldots, \lambda_n) \in \mathbb{S}^{2n-3}$ as untouched variables.

The same notation ϕ^* will be reserved for partial pullbacks. The partial pullbacks of ω^k are

$$\begin{aligned} \phi^*(\omega^k) &= \phi^*(\alpha^k) - \lambda_k \phi^*(\alpha^1) \\ &= (a_k(t, \bar{t}) - \lambda_k a_1(t, \bar{t}))\, dt, \end{aligned} \tag{2.9}$$

while the full-pullback requires λ_k be replaced by a_k/a_1, from which it is deduced that the full pullback of ω^k is identically 0.

2.5 The Second Prolongation and the Second Lifted Space

Let I denote the Pfaffian system

$$I := \langle \omega^0, \omega^k, \bar{\omega}^k \rangle \qquad (2 \leqslant k \leqslant n)$$

on $M \times \mathbb{S}^{2n-3}$, which consists of linear combinations of the 1-forms with real-analytic coefficients. When doing computations, taking modulo the system I is synonymous to pulling back the differential forms by ϕ to the disk $\mathbb{D}$. By construction,

$$\begin{aligned} d\omega^0 &= d\theta \\ &\equiv \sqrt{-1}(\alpha^1 \wedge \bar{\alpha}^1 - \alpha^2 \wedge \bar{\alpha}^2 - \cdots - \alpha^n \wedge \bar{\alpha}^n) \qquad \text{mod } I \\ &\equiv \sqrt{-1}(1 - |\lambda_2|^2 - \cdots - |\lambda_n|^2)\alpha^1 \wedge \bar{\alpha}^1 \qquad \text{mod } I \\ &= 0 \qquad \text{mod } I, \end{aligned}$$

which is analogous to the fact that $\phi^*(d\theta) \equiv 0$. In the later calculations, observe that for every 1-form β on M, its exterior derivative $d\beta$ may be expressed as

$$d\beta \equiv B(x, \lambda) \cdot \alpha^1 \wedge \bar{\alpha}^1 \qquad \text{mod } I,$$

where here x denotes the coordinate system on M, and B is a function. Therefore, for each $2 \leqslant k \leqslant n$,

$$\begin{aligned} d\omega^k &= -d\lambda_k \wedge \alpha^1 + d\alpha^k - \lambda_k d\alpha^1 \\ &\equiv -(d\lambda_k - L_k \bar{\alpha}^1) \wedge \alpha^1 \qquad \text{mod } I, \end{aligned}$$

for some function L_k. From equation (2.9), the partial pullback $\phi^* d\omega^k = d(\phi^* \omega^k)$ gives

$$\begin{aligned} d(\phi^*\omega^k) &= a_{k,\bar{t}}\, d\bar{t} \wedge dt - d\lambda_k \wedge a_1 dt - \lambda_k a_{1,\bar{t}}\, d\bar{t} \wedge dt \\ &= -d\lambda_k \wedge (a_1 dt) + \frac{a_{k,\bar{t}}}{a_1 \bar{a}_1}(\bar{a}_1 d\bar{t}) \wedge (a_1 dt) - \lambda_k \frac{a_{1,\bar{t}}}{a_1 \bar{a}_1}(\bar{a}_1 d\bar{t}) \wedge (a_1 dt) \\ &= -\left(d\lambda_k - \left[\frac{a_{k,\bar{t}}}{a_1 \bar{a}_1} - \lambda_k \frac{a_{1,\bar{t}}}{a_1 \bar{a}_1} \right] \bar{a}_1 d\bar{t} \right) \wedge (a_1 dt). \end{aligned} \qquad (2.10)$$

When λ_k is replaced by a_k/a_1, $d(\phi^*\omega^k) \equiv 0$. But

$$d\lambda_k = \lambda_{k,t} dt + \lambda_{k,\bar{t}}\, d\bar{t}$$

involves the term $\lambda_{k,t}$, which is missing from equation (2.10) since $dt \wedge dt = 0$. Therefore it necessitates a second prolongation to compensate the loss. For example,

$$d(\tilde{\phi}^* \lambda_k) = l_k(t, \bar{t})\, \phi^* \bar{\alpha}^1 + m_k\, \phi^* \alpha^1,$$

where

$$l_k(t, \bar{t}) = \frac{\lambda_{k,\bar{t}}}{\bar{a}_1(\bar{t}, t)}, \qquad \text{and} \qquad m_k(t, \bar{t}) = \frac{\lambda_{k,t}(t, \bar{t})}{a_1(t, \bar{t})}, \qquad (2.11)$$

has to be compared with the corresponding two form

$$\begin{aligned} d\omega^k &\equiv -(d\lambda_k - L_k \bar{\alpha}^1) \wedge \alpha^1 \qquad \text{mod } I \\ &\equiv -(d\lambda_k - L_k \bar{\alpha}^1 - M_k \alpha^1) \wedge \alpha^1 \qquad \text{mod } I, \end{aligned}$$

for some function M_k. The second prolongation then introduces new variables

$$\mu = (\mu_2, \ldots, \mu_n) \in \mathbb{C}^n$$

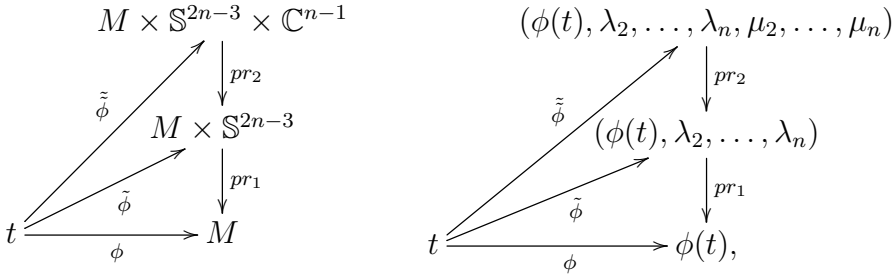
together with the new one forms

$$\pi^k := d\lambda_k - L_k\bar{\alpha}^1 - \mu_k\alpha^1$$

on the second lifted space

$$(x, \lambda_2, \dots \lambda_n, \mu_2, \dots, \mu_n) \in M \times \mathbb{S}^{2n-3} \times \mathbb{C}^{n-1}.$$

The second lift of holomorphic curve also exists



with

$$\big(\tilde{\tilde{\phi}}^*\mu_k\big)(t,\bar{t}) := \frac{\lambda_{k,t}(t,\bar{t})}{a_1(t,\bar{t})} = m_k(t,\bar{t}). \tag{2.12}$$

Lemma 2.13. *The pullback of L_k by $\tilde{\phi}$ gives*

$$\tilde{\phi}^*(L_k) = l_k(t,\bar{t}).$$

Proof. Starting from

$$d\omega^k = -(d\lambda_k - L_k\bar{\alpha}^1) \wedge \alpha^1,$$

its full pullback is given by

$$\begin{aligned} 0 &= \tilde{\phi}^*(d\omega^k) \\ &= -(d\tilde{\phi}^*(\lambda_k) - \tilde{\phi}^*(L_k)\bar{a}_1 d\bar{t}) \wedge a_1 dt \\ &= -(l_k\bar{a}_1 d\bar{t} + m_k a_1 dt - \tilde{\phi}^*(L_k)\bar{a}_1 d\bar{t}) \wedge a_1 dt. \end{aligned}$$

From the fact that $dt \wedge d\bar{t}$ vanishes nowhere on $\mathbb{D}$, therefore

$$l_k(t,\bar{t}) = \tilde{\phi}^*(L_k). \qquad \square$$

2.6 The hyperplane and sphere equation on the fibres $(\mu_2, \dots, \mu_n) \in \mathbb{C}^{n-1}$

The new variables μ_k must satisfy a hyperplane equation:

Proposition 2.14. *After pullback to $\mathbb{D}$, a non-trivial affine hyperplane relation is satisfied by the fibres $(\mu_2, \dots, \mu_n)$ of $pr_2 : M \times \mathbb{S}^{2n-3} \times \mathbb{C}^{n-1} \to M \times \mathbb{S}^{2n-3}$:*

$$\sum_{2\leqslant k\leqslant n} \bar{\lambda}_k\mu_k = -\sum_{2\leqslant k\leqslant n} \lambda_k\bar{L}_k.$$

Proof. From equation (2.12), and the fact that $\tilde{\phi}(\bar{L}_k) = \bar{l}_k(\bar{t}, t)$,

$$\begin{aligned} \sum_k \tilde{\phi}^*(\bar{\lambda}_k)\tilde{\tilde{\phi}}^*(\mu_k) + \sum_k \tilde{\phi}^*(\lambda_k)\tilde{\phi}^*(\bar{L}_k) &= \sum_k \bar{\lambda}_k(t,\bar{t})m_k(t,\bar{t}) + \sum_k \lambda_k(t,\bar{t})\bar{l}_k(\bar{t},t) \\ &= \sum_k \bar{\lambda}_k(\bar{t},t)\frac{\lambda_{k,t}(t,\bar{t})}{a_1(t,\bar{t})} + \sum_k \lambda_k(t,\bar{t})\frac{\overline{\lambda_{k,\bar{t}}(t,\bar{t})}}{a_1(t,\bar{t})} \\ &= \frac{1}{a_1(t,\bar{t})}\frac{\partial}{\partial t}\sum_k |\lambda_k(t,\bar{t})|^2, \end{aligned}$$

and since $\sum_k |\lambda_k|^2 = 1$, its derivative $\partial_t\sum_k |\lambda_k|^2 = 0$ vanishes. $\square$

Next let τ be the one-form on $M \times \mathbb{S}^{2n-3}$ given by

$$\tau := -\sum_{2\leqslant k\leqslant n} \bar{\lambda}_k d\lambda_k + \sum_{2\leqslant k\leqslant n} \bar{\lambda}_k L_k \bar{\alpha}^1 - \sum_{2\leqslant k\leqslant n} \lambda_k \bar{L}_k \alpha^1.$$

From the relation

$$\sum_{2\leqslant k\leqslant n} \bar{\lambda}_k d\lambda_k = -\sum_{2\leqslant k\leqslant n} \lambda_k d\bar{\lambda}_k,$$

one has therefore

$$\tau + \bar{\tau} = 0.$$

Lemma 2.15. *The full pullback of τ is zero.*

Proof. Directly from the expression of τ, using Lemma 2.13 and l_k in (2.11), is pullback turns out to be zero:

$$\begin{aligned}
& -\sum_k \tilde{\phi}^*(\bar{\lambda}_k)\tilde{\phi}^*(d\lambda_k) + \sum_k \tilde{\phi}^*(\bar{\lambda}_k)\tilde{\phi}^*(L_k)\phi^*(\bar{\alpha}^1) - \sum_k \tilde{\phi}^*(\lambda_k)\tilde{\phi}^*(\bar{L}_k)\phi^*(\alpha^1) \\
= & -\sum_k \left(\bar{\lambda}_k [\lambda_{k,\bar{t}} d\bar{t} + \lambda_{k,t} dt] \right) + \sum_k \bar{\lambda}_k \frac{\lambda_{k,\bar{t}}}{\bar{a}_1} \bar{a}_1 d\bar{t} - \sum_k \lambda_k \frac{\bar{\lambda}_{k,t}}{a_1} a_1 dt \\
= & -dt \left(\sum_k (\lambda_{k,t}\bar{\lambda}_k + \lambda_k \bar{\lambda}_{k,t}) \right) + 0 \\
= & \ 0. \quad \square
\end{aligned}$$

Since the pullback of π^k and $\bar{\pi}^k$ by ϕ to the disk $\mathbb{D}$ is zero, it is natural to introduce an enlarged Pfaffian system on $M \times \mathbb{S}^{2n-3} \times \mathbb{C}^{n-1}$:

$$I_2 = \langle \omega^0, \omega^k, \bar{\omega}^k, \pi^k, \bar{\pi}^k \rangle.$$

The following proposition describes the equation of the sphere that is satisfied by $(\mu_2, \ldots, \mu_n) \in \mathbb{C}^{n-1}$:

Proposition 2.16. *After pulling back to $\mathbb{D}$, a non-trivial affine sphere relation is satisfied by the fibres $(\mu_2, \ldots, \mu_n)$ of $pr_2 : M \times \mathbb{S}^{2n-3} \times \mathbb{C} \to M \times \mathbb{S}^{2n-3}$ given by*

$$\sum_{2\leqslant k\leqslant n} |\mu_k - B_k|^2 - E = 0$$

for functions B and E on $M \times \mathbb{S}^{2n-3}$.

Proof. By applying the exterior differential operator d to τ,

$$\begin{aligned}
d\tau = & \sum_k d\lambda_k \wedge d\bar{\lambda}_k \\
& + \sum_k L_k d\bar{\lambda}_k \wedge \bar{\alpha}^1 + \sum_k \bar{\lambda}_k dL_k \wedge \bar{\alpha}^1 + \sum_k \bar{\lambda}_k L_k d\bar{\alpha}^1 \\
& - \sum_k \bar{L}_k d\lambda_k \wedge \alpha^1 - \sum_k \lambda_k d\bar{L}_k \wedge \alpha^1 - \sum_k \lambda_k \bar{L}_k d\alpha^1.
\end{aligned}$$

Modulo I the Pfaffian system, there exist complex-valued functions A, B and C such that

$$\begin{aligned}
d\tau \equiv & \sum_k d\lambda_k \wedge d\bar{\lambda}_k \\
& + \sum_k A_k d\lambda_k \wedge \alpha^1 - \sum_k \bar{A}_k d\bar{\lambda}_k \wedge \bar{\alpha}^1 \\
& + \sum_k B_k d\bar{\lambda}_k \wedge \alpha^1 - \sum_k \bar{B}_k d\lambda_k \wedge \bar{\alpha}^1 \\
& + C\alpha^1 \wedge \bar{\alpha}^1 \qquad \text{mod } I.
\end{aligned}$$

Modulo the enlarged Pfaffian system I_2, replacing in effect $d\lambda_k$ by $L_k\bar{\alpha}^1 + \mu_k\alpha^1$ and similarly for $d\bar{\lambda}_k$:

$$\begin{aligned} d\tau \;\equiv\; & \sum_k \Big(L_k\bar{\alpha}^1 + \mu_k\alpha^1\Big) \wedge \Big(\bar{L}_k\alpha^1 + \bar{\mu}_k\bar{\alpha}^1\Big) \\ & + \Big(\sum_k A_kL_k\bar{\alpha}^1 + \sum_k A_k\mu_k\alpha^1\Big) \wedge \alpha^1 - \Big(\sum_k \bar{A}_k\bar{L}_k\alpha^1 + \sum_k \bar{A}_k\bar{\mu}_k\bar{\alpha}^1\Big) \wedge \bar{\alpha}^1 \\ & + \Big(\sum_k B_k\bar{L}_k\alpha^1 + \sum_k B_k\bar{\mu}_k\bar{\alpha}^1\Big) \wedge \alpha^1 - \Big(\sum_k \bar{B}_kL_k\bar{\alpha}^1 + \sum_k \bar{B}_k\mu_k\alpha^1\Big) \wedge \bar{\alpha}^1 \\ & + C\alpha^1 \wedge \bar{\alpha}^1 \qquad \mathsf{mod}\ I_2, \end{aligned}$$

and after some reorganisation,

$$d\tau \equiv \Big\{\sum_k |\mu_k - B_k|^2 - \underbrace{\sum_k \Big(|B_k|^2 + A_kL_k + \bar{A}_k\bar{L}_k + |L_k|^2\Big) + C}_{:=E}\Big\}\alpha^1 \wedge \bar{\alpha}^1 \qquad \mathsf{mod}\ I_2.$$

In order that $d\tau$ to vanish while requiring that $\alpha^1 \wedge \bar{\alpha}^1$ to remain non-zero, the equation of the sphere is obtained. □

2.7 Rigid Real hypersurface M^5 and the Hachtroudi-Chern-Moser Tensor in $\mathbb{C}^3$

Let $M^5 \subset \mathbb{C}^3$ be a real-analytic, rigid real hypersurface that contains the origin, and suppose that it is smooth. Let (z_1, z_2, w) be holomorphic coordinates with $z_i = x_i + \sqrt{-1}y_i$ and $w = u + \sqrt{-1}v$. Then by a biholomorphic change of coordinates, there exists a real-analytic function $F(z, \bar{z})$ (here $z := (z_1, z_2)$) such that

$$u = F(z, \bar{z}).$$

The fact that M is rigid means that F is independent of v. The intrinsic $T^{1,0}M$ sections (see section 1.6) are generated in local coordinates $(z, \bar{z}, v)$ by

$$\mathscr{L}_i = \frac{\partial}{\partial z_i} - \sqrt{-1}F_{z_i}\frac{\partial}{\partial v}, \qquad (i=1,2)$$

while their conjugates generate the corresponding $T^{0,1}M$.

The Levi form of M^5 is given by the matrix

$$\Lambda := \begin{pmatrix} l_{1,1} & l_{1,2} \\ l_{2,1} & l_{2,2} \end{pmatrix} = 2\begin{pmatrix} F_{z_1\bar{z}_1} & F_{z_2\bar{z}_1} \\ F_{z_1\bar{z}_2} & F_{z_2\bar{z}_2} \end{pmatrix} = \begin{pmatrix} h_{1,1} & h_{2,1} \\ h_{1,2} & h_{2,2} \end{pmatrix}.$$

The coefficients of the Levi matrix, and hence the Levi determinant Δ, are independent of the variable v. Since M is Lorentzian, $-\Delta$ is also strictly positive and hence there is no ambiguity when taking the square roots.

The contact form of M^5 is given by

$$\theta = -dv + \sum_{k=1,2}(-\sqrt{-1}F_{z_i})dz_i + \sum_{k=1,2}(\sqrt{-1}F_{\bar{z}_i})d\bar{z}_i,$$

with $\{\theta, dz_1, dz_2, d\bar{z}_1, d\bar{z}_2\}$ forming a coframe of $\mathbb{C}T^*M$. By Proposition 1.2 on explicit diagonalisation of the Levi form, there exist $T^{1,0}M$ sections $\mathscr{A}_1$ and $\mathscr{A}_2$ that diagonalise the Levi form. Dually, there exist $T^{*1,0}M$ sections α^1 and α^2 such that

$$d\theta = \sqrt{-1}(\alpha^1 \wedge \bar{\alpha}^1 - \alpha^2 \wedge \bar{\alpha}^2) \qquad \mathsf{mod}\ \theta.$$

Let $\lambda \in \mathbb{S}^1$ be an element in the circle in $\mathbb{R}^2$, and set up the following system

$$\begin{array}{ll} \omega^0 := \theta, & \\ \omega^1 := \alpha^1, & \bar{\omega}^1 := \bar{\alpha}^1, \\ \omega^2 := \alpha^2 - \lambda\alpha^1, & \bar{\omega}^2 := \bar{\alpha}^2 - \bar{\lambda}\bar{\alpha}^1. \end{array}$$

and let $I = \langle \omega^2, \bar{\omega}^2 \rangle$ be the Pfaffian system. Then by earlier discussions, there exists a complex valued function L such that

$$d\omega^2 \equiv -(d\lambda - L\bar{\alpha}^1) \wedge \alpha^1 \qquad \mathsf{mod}\ I.$$

Set the following 1 form

$$\tau = -\bar{\lambda}d\lambda + \bar{\lambda}L\bar{\alpha}^1 - \lambda\bar{L}\alpha^1.$$

Then it is clear that $\tau + \bar{\tau} = 0$. Taking the differential $d\tau$, one obtains functions A, B, C such that

$$\begin{array}{rcl} d\tau & \equiv & d\lambda \wedge d\bar{\lambda} + A\, d\lambda \wedge \alpha^1 - \bar{A}\, d\bar{\lambda} \wedge \bar{\alpha}^1 \\ & & +B\, d\bar{\lambda} \wedge \alpha^1 - \bar{B}\, d\lambda \wedge \bar{\alpha}^1 \\ & & +C\, \alpha^1 \wedge \bar{\alpha}^1 \qquad \mathsf{mod}\ I. \end{array}$$

The second prolongation gives the following system of differential one-forms:

$$\begin{array}{lll} \omega^0 := \theta, & & \\ \omega^1 := \alpha^1, & \bar{\omega}^1 := \bar{\alpha}^1, & \pi^2 := d\lambda - L\bar{\alpha}^1 - \mu\alpha^1, \\ \omega^2 := \alpha^2 - \lambda\alpha^1, & \bar{\omega}^2 := \bar{\alpha}^2 - \bar{\lambda}\bar{\alpha}^1, & \bar{\pi}^2 = d\bar{\lambda} - \bar{L}\alpha^1 - \bar{\mu}\bar{\alpha}^1. \end{array}$$

The equations of the hyperplane and the sphere are therefore given by

$$\begin{array}{rcl} \bar{\lambda}\mu & = & \lambda\bar{L}, \\ |\mu - B|^2 & = & |B|^2 + AL + \bar{A}\bar{L} + |L|^2 - C. \end{array}$$

This time, let I_+ be the ideal generated by I and τ. Then it will be seen later that $d\tau$ may be written as

$$d\tau \equiv -[a_2\lambda^2 + 4a_1\lambda + 6a_0 + 4\bar{a}_1\bar{\lambda} + \bar{a}_2\bar{\lambda}^2]\, \omega^1 \wedge \bar{\omega}^1 \qquad \mathsf{mod}\ I_+,$$

where a_2, a_1, a_0 and their conjugates are independent of λ. By [Bry82, page 217],

$$a_0 = S_{11\bar{1}\bar{1}} = \bar{a}_0, \qquad a_1 = S_{21\bar{1}\bar{1}}, \qquad a_2 = S_{22\bar{1}\bar{1}}$$

are the components of the Hachtroudi-Chern-Moser tensor S. Let

$$K(\lambda) := a_2\lambda^2 + 4a_1\lambda + 6a_0 + 4\bar{a}_1\bar{\lambda} + \bar{a}_2\bar{\lambda}^2.$$

In classical literature, the points where $K(\lambda) \equiv 0$ for all $\lambda \in \mathbb{S}^1$ are called the umbilical points of M. In the later sections that deal with explicit calculations dealing with intersection of lines with circles for the case of 5-dimensional CR real-hypersurfaces in $\mathbb{C}^3$, we will explain the following:

Proposition 2.17 (See Proposition 3.8)**.** *Let M be a Lorentzian CR manifold. At each point of M, the number of possible holomorphic curves that passes through it and contained in M is at most 4.*

3 Lorentzian rigid real hypersurfaces M^5 in $\mathbb{C}^3$-Calculations

3.1 Setting

Let M be a Lorentzian rigid hypersurface in $\mathbb{C}^3$ containing the origin. In coordinates $(z_1, z_2, z_3 := w)$ where $w = u + \sqrt{-1}v$, this means that M may be defined by

$$v = F(z_1, z_2, \bar{z}_1, \bar{z}_2),$$

where F is independent of u. By Proposition 1.2, if $\mathscr{L}_1$ and $\mathscr{L}_2$ are $T^{1,0}(M)$ sections, the following vector fields

$$\begin{aligned} \mathscr{A}_1 &= \frac{1}{\sqrt{l_{11}}}\mathscr{L}_1, \\ \mathscr{A}_2 &= \frac{-1}{\sqrt{-\Delta l_{11}}}\langle \mathscr{L}_2, \mathscr{L}_1\rangle \mathscr{L}_1 + \frac{1}{\sqrt{-\Delta l_{11}}}\langle \mathscr{L}_1, \mathscr{L}_1\rangle \mathscr{L}_2 \end{aligned}$$

diagonalise the Levi matrix. The following 2-forms

$$\alpha^1 = \frac{\langle \mathscr{L}_1, \mathscr{L}_1\rangle}{\sqrt{l_{11}}}\mathscr{L}_1\, dz_1 + \frac{\langle \mathscr{L}_2, \mathscr{L}_1\rangle}{\sqrt{l_{11}}}\, dz_2, \tag{3.1}$$

$$\alpha^2 = \frac{\sqrt{-\Delta}}{\sqrt{l_{11}}}\, dz_2 \tag{3.2}$$

diagonalise the 1-form $d\theta$ so that

$$d\theta \equiv \sqrt{-1}(\alpha^1 \wedge \overline{\alpha}^1 - \alpha^2 \wedge \overline{\alpha}^2) \qquad \text{mod } \theta.$$

For simplicity, let P, Q, R denote the following coefficients of α^1 and α^2:

$$P := \frac{\langle \mathscr{L}_1, \mathscr{L}_1\rangle}{\sqrt{l_{11}}}, \qquad Q := \frac{\langle \mathscr{L}_2, \mathscr{L}_1\rangle}{\sqrt{l_{11}}}, \qquad R := \frac{\sqrt{-\Delta}}{\sqrt{l_{11}}},$$

and observe that each of them is independent of u. Moreover, P and R are real-valued functions. Equations (3.1) and (3.2) allow the differential 1-forms be written as

$$\begin{aligned} \alpha^1 &= P\, dz_1 + Q\, dz_2, \\ \alpha^2 &= R\, dz_2, \end{aligned}$$

with its inversion

$$\begin{aligned} dz_1 &= \frac{1}{P}\alpha^1 - \frac{Q}{PR}\alpha^2, \\ dz_2 &= \frac{1}{R}\alpha^2, \end{aligned} \tag{3.3}$$

3.2 The Exterior Derivative

Let $G(z_1, z_2, \bar{z}_1, \bar{z}_2)$ be any function that is independent of u. An application of the exterior derivative to G gives

$$dG = \sum_{k=1,\,2} \frac{\partial G}{\partial z_k}\, dz_k + \sum_{k=1,\,2} \frac{\partial G}{\partial \bar{z}_k}\, d\bar{z}_k = \sum_{k=1,\,2} \mathscr{L}_k(G)\, dz_k + \sum_{k=1,\,2} \overline{\mathscr{L}}_k(G)\, d\bar{z}_k.$$

A direct calculation shows that dG may be expressed in terms of $\mathscr{A}_k$ and α^k:

$$dG = \mathscr{A}_1(G)\alpha^1 + \mathscr{A}_2(G)\alpha^2 + \overline{\mathscr{A}}_1(G)\bar{\alpha}^1 + \overline{\mathscr{A}_2}(G)\bar{\alpha}^2. \tag{3.4}$$

3.3 The Pfaffian system

Let $\lambda \in \mathbb{S}^1$ be an element in the unit circle. The following system of differential forms

$$\begin{aligned}
\omega^0 &:= \theta, \\
\omega^1 &:= \alpha^1, \qquad\qquad \bar{\omega}^1 := \bar{\alpha}^1, \\
\omega^2 &:= \alpha^2 - \lambda\alpha^1, \qquad \bar{\omega}^2 := \bar{\alpha}^2 - \bar{\lambda}\bar{\alpha}^1,
\end{aligned}$$

along with the ideal

$$I := \langle \omega^0, \omega^2, \bar{\omega}^2 \rangle$$

sets up the Pfaffian system for the existence of holomorphic curves in M.

3.4 Calculations of $d\alpha^1$ and $d\alpha^2$

Using equation (3.3), an application of the exterior differentiation to α^1 and α^2 gives

$$\begin{aligned}
d\alpha^1 &= \sum_{k=1,2} \mathscr{A}_k(P)\, \alpha^k \wedge dz_1 + \sum_{k=1,2} \mathscr{A}_k(Q)\, \alpha^k \wedge dz_2 \\
&\quad + \sum_{k=1,2} \overline{\mathscr{A}}_k(P)\, \bar{\alpha}^k \wedge dz_1 + \sum_{k=1,2} \overline{\mathscr{A}}_k(Q)\, \bar{\alpha}^k \wedge dz_2, \\
d\alpha^2 &= \sum_{k=1,2} \mathscr{A}_k(R)\, \alpha^k \wedge dz_2 + \sum_{k=1,2} \overline{\mathscr{A}}_k(R)\, \bar{\alpha}^k \wedge dz_2.
\end{aligned}$$

Substituting the dz_1 and dz_2 by the inversion formula in equation (3.3) and reorganise:

$$\begin{aligned}
d\alpha^1 &= \left(\frac{1}{R}\mathscr{A}_1(Q) - \frac{Q}{PR}\mathscr{A}_1(R) - \frac{1}{P}\mathscr{A}_2(P)\right) \alpha^1 \wedge \alpha^2 + \left(\frac{-1}{P}\overline{\mathscr{A}}_1(P)\right)\alpha^1 \wedge \bar{\alpha}^1 \\
&\quad + \left(\frac{Q}{PR}\overline{\mathscr{A}}_1(P) - \frac{1}{R}\overline{\mathscr{A}}_1(Q)\right)\alpha^2 \wedge \bar{\alpha}^1 + \left(\frac{-1}{P}\overline{\mathscr{A}}_2(P)\right)\alpha^1 \wedge \bar{\alpha}^2 \\
&\quad + \left(\frac{Q}{PR}\overline{\mathscr{A}}_2(P) - \frac{1}{R}\overline{\mathscr{A}}_2(Q)\right)\alpha^2 \wedge \bar{\alpha}^2, \\
d\alpha^2 &= \frac{1}{R}\mathscr{A}_1(R)\alpha^1 \wedge \alpha^2 - \frac{1}{R}\overline{\mathscr{A}}_1(R)\alpha^2 \wedge \bar{\alpha}^1 - \frac{1}{R}\overline{\mathscr{A}}_2(R)\alpha^2 \wedge \bar{\alpha}^2.
\end{aligned}$$

Modulo the ideal I, the form α^2 is replaced by $\lambda\alpha^1$ and $\bar{\alpha}^2$ by $\bar{\lambda}\bar{\alpha}^1$. These replacements transform $d\alpha^1$ and $d\alpha^2$ into

$$\begin{aligned}
d\alpha^1 &\equiv \alpha^1 \wedge \bar{\alpha}^1 \left[\left(\frac{-1}{P}\overline{\mathscr{A}}(P)\right)\bar{\lambda} + \left(\frac{Q}{PR}\overline{\mathscr{A}}_2(P) - \frac{1}{R}\overline{\mathscr{A}}_2(Q) - \frac{1}{P}\overline{\mathscr{A}}_1(P)\right)\right. \\
&\qquad \left. + \left(\frac{Q}{PR}\overline{\mathscr{A}}_1(P) - \frac{1}{R}\overline{\mathscr{A}}_1(Q)\right)\lambda\right] \qquad \text{mod } I, \qquad (3.5) \\
d\alpha^2 &\equiv \alpha^1 \wedge \bar{\alpha}^1 \left[\frac{-1}{R}\overline{\mathscr{A}}_2(R) - \frac{1}{R}\overline{\mathscr{A}}_1(R)\lambda\right] \qquad \text{mod } I.
\end{aligned}$$

Recalling that $\bar{P} = P$ and $\bar{R} = R$, their respective conjugates are therefore

$$\begin{aligned}
d\bar{\alpha}^1 &\equiv \alpha^1 \wedge \bar{\alpha}^1 \left[\left(-\frac{\bar{Q}}{PR}\mathscr{A}_1(P) + \frac{1}{R}\mathscr{A}_1(\bar{Q})\right)\bar{\lambda} + \left(-\frac{\bar{Q}}{PR}\mathscr{A}_2(P) + \frac{1}{R}\mathscr{A}_2(\bar{Q}) + \frac{1}{P}\mathscr{A}_1(P)\right)\right. \\
&\qquad \left. + \left(\frac{1}{P}\mathscr{A}_2(P)\right)\lambda\right] \qquad \text{mod } I, \\
d\bar{\alpha}^2 &\equiv \alpha^1 \wedge \bar{\alpha}^1 \left[\frac{1}{R}\mathscr{A}_1(R)\bar{\lambda} + \frac{1}{R}\mathscr{A}_2(R)\right] \qquad \text{mod } I.
\end{aligned}$$

3.5 Calculation of $d\omega^2$ and the expression of L

The exterior differentiation applied to

$$\omega^2 = \alpha^2 - \lambda\alpha^1$$

yields

$$\begin{aligned} d\omega^2 &= d\alpha^2 - d\lambda \wedge \alpha^1 - \lambda d\alpha^1 \\ &= -d\lambda \wedge \alpha^1 + (d\alpha^2 - \lambda d\alpha^1). \end{aligned}$$

Using the expressions of $d\alpha^1$ and $d\alpha^2$ modulo I in equation (3.5), it follows that

$$\begin{aligned} d\omega^2 \equiv\ & -d\lambda \wedge \alpha^1 \\ & +\alpha^1 \wedge \bar{\alpha}^1 \Bigg[\left(\frac{1}{R}\overline{\mathscr{A}}_2(R) - \frac{1}{P}\overline{\mathscr{A}}_2(P) \right) + \left(\frac{Q}{PR}\overline{\mathscr{A}}_2(P) - \frac{1}{R}\overline{\mathscr{A}}_2(Q) - \frac{1}{P}\overline{\mathscr{A}}_1(P) + \frac{1}{R}\overline{\mathscr{A}}_1(R) \right)\lambda \\ & + \left(\frac{Q}{PR}\overline{\mathscr{A}}_1(P) - \frac{1}{R}\overline{\mathscr{A}}_1(Q) \right)\lambda^2 \Bigg] \qquad \text{mod } I, \end{aligned}$$

and hence the expression of L may be obtained:

$$\begin{aligned} L :=\ & \left(\frac{1}{R}\overline{\mathscr{A}}_2(R) - \frac{1}{P}\overline{\mathscr{A}}_2(P) \right) + \left(\frac{Q}{PR}\overline{\mathscr{A}}_2(P) - \frac{1}{R}\overline{\mathscr{A}}_2(Q) - \frac{1}{P}\overline{\mathscr{A}}_1(P) + \frac{1}{R}\overline{\mathscr{A}}_1(R) \right)\lambda \\ & + \left(\frac{Q}{PR}\overline{\mathscr{A}}_1(P) - \frac{1}{R}\overline{\mathscr{A}}_1(Q) \right)\lambda^2, \end{aligned}$$

along with its conjugate:

$$\begin{aligned} \overline{L} =\ & \left(\frac{\bar{Q}}{PR}\mathscr{A}_1(P) - \frac{1}{R}\mathscr{A}_1(\bar{Q}) \right)\bar{\lambda}^2 + \left(\frac{\bar{Q}}{PR}\mathscr{A}_2(P) - \frac{1}{R}\mathscr{A}_2(\bar{Q}) - \frac{1}{P}\mathscr{A}_1(P) + \frac{1}{R}\mathscr{A}_1(R) \right)\bar{\lambda} \\ & + \left(\frac{1}{R}\mathscr{A}_2(R) - \frac{1}{P}\mathscr{A}_2(P) \right). \end{aligned}$$

3.6 Equation of the hyperplane

From section 2.7, the equation of the hyperplane is therefore given by

$$\bar{\lambda}\mu = \lambda\overline{L},$$

whose solution in u is of the form

$$\begin{aligned} \mu &= \lambda^2\overline{L} \\ &= \left(\frac{\bar{Q}}{PR}\mathscr{A}_1(P) - \frac{1}{R}\mathscr{A}_1(\bar{Q}) \right) + \left(\frac{\bar{Q}}{PR}\mathscr{A}_2(P) - \frac{1}{R}\mathscr{A}_2(\bar{Q}) - \frac{1}{P}\mathscr{A}_1(P) + \frac{1}{R}\mathscr{A}_1(R) \right)\lambda \\ &\quad + \left(\frac{1}{R}\mathscr{A}_2(R) - \frac{1}{P}\mathscr{A}_2(P) \right)\lambda^2. \end{aligned}$$

3.7 The differential forms τ and $d\tau$

Recall that previously, the differential forms τ and $d\tau$ are given by

$$\begin{aligned} \tau &= -\bar{\lambda}d\lambda + \bar{\lambda}L\bar{\alpha}^1 - \lambda\overline{L}\alpha^1, \\ d\tau &= d\lambda \wedge d\bar{\lambda} + Ld\bar{\lambda} \wedge \bar{\alpha}^1 + \bar{\lambda}dL \wedge \bar{\alpha}^1 + \bar{\lambda}Ld\bar{\alpha}^1 \\ &\qquad -\overline{L}d\lambda \wedge \alpha^1 - \lambda d\overline{L} \wedge \alpha^1 - \lambda\overline{L}d\alpha^1. \end{aligned}$$

To obtain the explicit expressions of $d\tau$ modulo the Pfaffian system I, the following terms will be calculated in the following order:

1. $Ld\bar{\lambda}\wedge\bar{\alpha}^1$ and $\overline{L}d\lambda\wedge\alpha^1$,

2. $\bar{\lambda}dL\wedge\bar{\alpha}^1$ and $\lambda d\overline{L}\wedge\alpha^1$,

3. $\bar{\lambda}Ld\bar{\alpha}^1$ and $\lambda\overline{L}d\alpha^1$.

Almost nothing can be for the first pair at this moment. For the second pair,

$$\begin{aligned}
\bar{\lambda}dL\wedge\bar{\alpha}^1 \;\equiv\; & \left[\left(\frac{Q}{PR}\bar{\mathscr{A}_2}(P)-\frac{1}{R}\bar{\mathscr{A}_2}(Q)-\frac{1}{P}\bar{\mathscr{A}_1}(P)+\frac{1}{R}\bar{\mathscr{A}_1}(R)\right)\bar{\lambda}+\left(\frac{Q}{PR}\bar{\mathscr{A}_1}(P)-\frac{1}{R}\bar{\mathscr{A}_1}(Q)\right)\right]d\lambda\wedge\bar{\alpha}^1\\
&+\alpha^1\wedge\bar{\alpha}^1\\
&\left\{\left(\frac{\mathscr{A}_1(\overline{\mathscr{A}}_2(R))}{R}-\frac{\mathscr{A}_1(\overline{\mathscr{A}}_2(P))}{P}+\frac{\mathscr{A}_1(P)\overline{\mathscr{A}}_2(P)}{P^2}-\frac{\mathscr{A}_1(R)\overline{\mathscr{A}}_2(R)}{R^2}\right)\bar{\lambda}\right.\\
&+\left(\frac{\mathscr{A}_1(\overline{\mathscr{A}}_2(R))}{R}-\frac{\mathscr{A}_1(\overline{\mathscr{A}}_1(P))}{P}+\frac{Q\mathscr{A}_1(\overline{\mathscr{A}}_2(P))}{PR}-\frac{\mathscr{A}_1(\overline{\mathscr{A}}_2(Q))}{R}\right.\\
&+\frac{\mathscr{A}_2(\overline{\mathscr{A}}_2(R))}{R}-\frac{\mathscr{A}_2(\overline{\mathscr{A}}_2(P))}{P}+\frac{\mathscr{A}_1(P)\overline{\mathscr{A}}_1(P)}{P^2}-\frac{\mathscr{A}_1(R)\overline{\mathscr{A}}_1(R)}{R^2}\\
&+\frac{\mathscr{A}_1(R)\overline{\mathscr{A}}_2(Q)}{R^2}-\frac{Q\mathscr{A}_1(R)\overline{\mathscr{A}}_2(P)}{PR^2}+\frac{\mathscr{A}_1(Q)\overline{\mathscr{A}}_2(P)}{PR}-\frac{Q\mathscr{A}_1(P)\overline{\mathscr{A}}_2(P)}{P^2R}\\
&\left.+\frac{\mathscr{A}_2(P)\overline{\mathscr{A}}_2(P)}{P^2}+\frac{\mathscr{A}_2(R)\overline{\mathscr{A}}_2(R)}{R^2}\right)\\
&+\left(-\frac{\mathscr{A}(\overline{\mathscr{A}}_1(Q))}{R}+\frac{Q\mathscr{A}_1(\overline{\mathscr{A}}_1(P))}{PR}+\frac{\mathscr{A}_2(\overline{\mathscr{A}}_1(R))}{R}-\frac{\mathscr{A}_2(\overline{\mathscr{A}}_1(P))}{P}\right.\\
&-\frac{\mathscr{A}_2(\overline{\mathscr{A}}_2(Q))}{R}+\frac{Q\mathscr{A}_2(\overline{\mathscr{A}}_2(P))}{PR}+\frac{\overline{\mathscr{A}}_1(Q)\mathscr{A}_1(R)}{R^2}-\frac{Q\mathscr{A}_1(P)\overline{\mathscr{A}}_1(P)}{P^2R}\\
&-\frac{Q\mathscr{A}_1(R)\overline{\mathscr{A}}_1(P)}{PR^2}+\frac{\mathscr{A}_1(Q)\overline{\mathscr{A}}_1(P)}{PR}-\frac{\mathscr{A}_2(R)\overline{\mathscr{A}}_1(R)}{R^2}+\frac{\mathscr{A}_2(P)\overline{\mathscr{A}}_1(P)}{P^2}\\
&+\frac{\mathscr{A}_2(R)\overline{\mathscr{A}}_2(Q)}{R^2}+\frac{\mathscr{A}_2(Q)\overline{\mathscr{A}}_2(P)}{PR}-\frac{Q\mathscr{A}_2(R)\overline{\mathscr{A}}_2(P)}{PR^2}-\frac{Q\mathscr{A}_2(P)\overline{\mathscr{A}}_2(P)}{P^2R}\left.\right)\lambda\\
&+\left(\frac{Q\mathscr{A}_2(\overline{\mathscr{A}}_1(P))}{PR}-\frac{\mathscr{A}_2(\overline{\mathscr{A}}_1(Q))}{R}+\frac{\mathscr{A}_2(Q)\overline{\mathscr{A}}_1(P)}{PR}-\frac{Q\mathscr{A}_2(P)\overline{\mathscr{A}}_1(P)}{P^2R}\right.\\
&\left.\left.-\frac{Q\mathscr{A}_2(R)\overline{\mathscr{A}}_1(P)}{PR^2}+\frac{\mathscr{A}_2(R)\overline{\mathscr{A}}_1(Q)}{R^2}\right)\lambda^2\right\}\qquad \mathsf{mod}\ I,
\end{aligned}$$

and

$$\begin{aligned}
\lambda d\overline{L}\wedge\alpha^1 \;\equiv\; & \left[\left(\frac{\bar{Q}}{PR}\mathscr{A}_1(P)-\frac{1}{R}\mathscr{A}_1(\bar{Q})\right)+\left(\frac{\bar{Q}}{PR}\mathscr{A}_2(P)-\frac{1}{R}\mathscr{A}_2(\bar{Q})-\frac{1}{P}\mathscr{A}_1(P)+\frac{1}{R}\mathscr{A}_1(R)\right)\lambda\right]d\bar{\lambda}\wedge\alpha^1\\
&+\alpha^1\wedge\bar{\alpha}^1\\
&\left\{\left(\frac{\overline{\mathscr{A}}_2(\mathscr{A}_1(\bar{Q}))}{R}-\frac{\bar{Q}\overline{\mathscr{A}}_2(\mathscr{A}_1(P))}{PR}-\frac{\overline{\mathscr{A}}_2(\bar{Q})\mathscr{A}_1(P)}{PR}+\frac{\bar{Q}\overline{\mathscr{A}}_2(P)\mathscr{A}_1(P)}{P^2R}\right.\right.\\
&\left.\left.+\frac{\bar{Q}\mathscr{A}_2(R)\overline{\mathscr{A}}_1(P)}{PR^2}-\frac{\overline{\mathscr{A}}_2(R)\mathscr{A}_1(\bar{Q})}{R^2}\right)\bar{\lambda}^2+\ldots
\end{aligned}$$

$$
\begin{aligned}
\dots \quad & \Bigg(\frac{\overline{\mathscr{A}}_1(\overline{\mathscr{A}}_1(\bar{Q}))}{R} - \frac{\bar{Q}\overline{\mathscr{A}}_1(\mathscr{A}_1(P))}{PR} - \frac{\overline{\mathscr{A}}_2(\mathscr{A}_1(R))}{R} + \frac{\overline{\mathscr{A}}_2(\mathscr{A}_1(P))}{P} \\
& + \frac{\overline{\mathscr{A}}_2(\mathscr{A}_2(\bar{Q}))}{R} - \frac{\bar{Q}\overline{\mathscr{A}}_2(\mathscr{A}_2(P))}{PR} - \frac{\mathscr{A}_1(\bar{Q})\overline{\mathscr{A}}_1(R)}{R^2} + \frac{\bar{Q}\mathscr{A}_1(P)\overline{\mathscr{A}}_1(P)}{P^2R} \\
& + \frac{\bar{Q}\mathscr{A}_1(P)\overline{\mathscr{A}}_1(R)}{PR^2} - \frac{\overline{\mathscr{A}}_1(\bar{Q})\mathscr{A}_1(P)}{PR} + \frac{\mathscr{A}_1(R)\overline{\mathscr{A}}_2(R)}{R^2} - \frac{\mathscr{A}_1(P)\overline{\mathscr{A}}_2(P)}{P^2} \\
& - \frac{\mathscr{A}_2(\bar{Q})\overline{\mathscr{A}}_2(R)}{R^2} - \frac{\overline{\mathscr{A}}_2(\bar{Q})\mathscr{A}_2(P)}{PR} + \frac{\bar{Q}\overline{\mathscr{A}}_2(R)\mathscr{A}_2(P)}{PR^2} + \frac{\bar{Q}\mathscr{A}_2(P)\overline{\mathscr{A}}_2(P)}{P^2R} \Bigg)\bar{\lambda} \\
& + \Bigg(- \frac{\overline{\mathscr{A}}_1(\mathscr{A}_1(R))}{R} + \frac{\overline{\mathscr{A}}_1(\mathscr{A}_1(P))}{P} - \frac{\bar{Q}\overline{\mathscr{A}}_1(\mathscr{A}_2(P))}{PR} + \frac{\overline{\mathscr{A}}_1(\mathscr{A}_2(Q))}{R} \\
& - \frac{\mathscr{A}_2(\overline{\mathscr{A}}_2(R))}{R} + \frac{\overline{\mathscr{A}}_2(\mathscr{A}_2(P))}{P} - \frac{\mathscr{A}_1(P)\overline{\mathscr{A}}_1(P)}{P^2} + \frac{\mathscr{A}_1(R)\overline{\mathscr{A}}_1(R)}{R^2} \\
& - \frac{\overline{\mathscr{A}}_1(R)\mathscr{A}_2(\bar{Q})}{R^2} + \frac{\bar{Q}\overline{\mathscr{A}}_1(R)\mathscr{A}_2(P)}{PR^2} - \frac{\overline{\mathscr{A}}_1(\bar{Q})\mathscr{A}_2(P)}{PR} + \frac{\bar{Q}\mathscr{A}_2(P)\overline{\mathscr{A}}_1(P)}{P^2R} \\
& - \frac{\mathscr{A}_2(P)\overline{\mathscr{A}}_2(P)}{P^2} + \frac{\mathscr{A}_2(R)\overline{\mathscr{A}}_2(R)}{R^2} \Bigg) \\
& + \Bigg(- \frac{\overline{\mathscr{A}}_1(\mathscr{A}_2(R))}{R} + \frac{\overline{\mathscr{A}}_1(\mathscr{A}_2(P))}{P} - \frac{\overline{\mathscr{A}}_1(P)\mathscr{A}_2(P)}{P^2} + \frac{\overline{\mathscr{A}}_1(R)\mathscr{A}_2(R)}{R^2} \Bigg)\lambda \Bigg\} \\
& \qquad \mod I.
\end{aligned}
$$

For the final pair,

$$
\begin{aligned}
\bar{\lambda}Ld\bar{\alpha}^1 \;=\; & \alpha^1 \wedge \bar{\alpha}^1 \\
& \Bigg\{ \Bigg(\frac{\overline{\mathscr{A}}_2(R)\mathscr{A}_1(\bar{Q})}{R^2} + \frac{\bar{Q}\overline{\mathscr{A}}_2(P)\mathscr{A}_1(P)}{P^2R} - \frac{\overline{\mathscr{A}}_2(P)\mathscr{A}_1(\bar{Q})}{PR} - \frac{\bar{Q}\overline{\mathscr{A}}_2(R)\mathscr{A}_1(P)}{PR^2} \Bigg)\bar{\lambda}^2 \\
& + \Bigg(\frac{\overline{\mathscr{A}}_2(R)\mathscr{A}_1(P)}{PR} + \frac{\bar{Q}\overline{\mathscr{A}}_1(R)\mathscr{A}_1(P)}{P^2R} + \frac{\bar{Q}\overline{\mathscr{A}}_2(P)\mathscr{A}_2(P)}{P^2R} - \frac{\bar{Q}\overline{\mathscr{A}}_1(R)\mathscr{A}_1(P)}{PR^2} \\
& - \frac{\overline{\mathscr{A}}_2(P)\mathscr{A}_2(\bar{Q})}{PR} - \frac{\overline{\mathscr{A}}_1(P)\mathscr{A}_1(\bar{Q})}{PR} - \frac{Q\bar{Q}\overline{\mathscr{A}}_2(P)\mathscr{A}_1(P)}{P^2R^2} + \frac{\overline{\mathscr{A}}_1(R)\mathscr{A}_1(\bar{Q})}{R^2} \\
& + \frac{\bar{Q}\overline{\mathscr{A}}_2(Q)\mathscr{A}_1(P)}{PR^2} + \frac{\overline{\mathscr{A}}_2(R)\mathscr{A}_2(\bar{Q})}{R^2} - \frac{\bar{Q}\overline{\mathscr{A}}_2(R)\mathscr{A}_2(P)}{PR^2} - \frac{\overline{\mathscr{A}}_2(P)\mathscr{A}_1(P)}{P^2} \\
& + \frac{\overline{\mathscr{A}}_2(P)\mathscr{A}_1(\bar{Q})}{PR^2} - \frac{\overline{\mathscr{A}}_2(Q)\mathscr{A}_1(\bar{Q})}{R^2} \Bigg)\bar{\lambda} \\
& + \Bigg(- \frac{\overline{\mathscr{A}}_1(P)\mathscr{A}_1(P)}{P^2} + \frac{\bar{Q}\overline{\mathscr{A}}_1(P)\mathscr{A}_2(P)}{P^2R} - \frac{\overline{\mathscr{A}}_2(Q)\mathscr{A}_2(\bar{Q})}{R^2} - \frac{Q\bar{Q}\overline{\mathscr{A}}_2(P)\mathscr{A}_2(P)}{P^2R^2} \\
& + \frac{\overline{\mathscr{A}}_2(P)\mathscr{A}_1(P)}{P^2R} + \frac{\overline{\mathscr{A}}_1(R)\mathscr{A}_1(P)}{PR} - \frac{\overline{\mathscr{A}}_1(P)\mathscr{A}_2(\bar{Q})}{PR} - \frac{\bar{Q}\overline{\mathscr{A}}_1(R)\mathscr{A}_2(P)}{PR^2} \\
& - \frac{\overline{\mathscr{A}}_2(P)\mathscr{A}_1(P)}{P^2} + \frac{\overline{\mathscr{A}}_1(R)\mathscr{A}_2(\bar{Q})}{R^2} - \frac{Q\bar{Q}\overline{\mathscr{A}}_1(P)\mathscr{A}_1(P)}{P^2R^2} - \frac{\overline{\mathscr{A}}_1(Q)\mathscr{A}_1(\bar{Q})}{R^2} \\
& + \frac{Q\overline{\mathscr{A}}_1(P)\mathscr{A}_1(\bar{Q})}{PR^2} + \frac{\bar{Q}\overline{\mathscr{A}}_1(Q)\mathscr{A}_1(P)}{PR^2} + \frac{\bar{Q}\overline{\mathscr{A}}_2(Q)\mathscr{A}_2(P)}{PR^2} + \frac{\overline{\mathscr{A}}_2(R)\mathscr{A}_2(P)}{PR} \\
& - \frac{\overline{\mathscr{A}}_2(Q)\mathscr{A}_1(P)}{PR} + \frac{Q\overline{\mathscr{A}}_2(P)\mathscr{A}_2(\bar{Q})}{PR^2} \Bigg) + \dots
\end{aligned}
$$

$$
\begin{aligned}
\cdots \quad & +\Big(\frac{Q\overline{\mathscr{A}}_2(P)\mathscr{A}_2(Q)}{P^2R}+\frac{\overline{\mathscr{A}}_1(R)\mathscr{A}_2(P)}{PR}+\frac{Q\overline{\mathscr{A}}_1(P)\mathscr{A}_2(\bar{Q})}{PR^2}+\frac{\bar{Q}\overline{\mathscr{A}}_1(Q)\mathscr{A}_2(P)}{PR^2}\\
& -\frac{\overline{\mathscr{A}}_1(Q)\mathscr{A}_1(P)}{PR}-\frac{\overline{\mathscr{A}}_1(Q)\mathscr{A}_2(\bar{Q})}{R^2}+\frac{Q\overline{\mathscr{A}}_1(P)\mathscr{A}_1(P)}{P^2R)}-\frac{Q\bar{Q}\overline{\mathscr{A}}_1(P)\mathscr{A}_2(P)}{P^2R^2}\\
& -\frac{\overline{\mathscr{A}}_2(Q)\mathscr{A}_2(P)}{PR}-\frac{\overline{\mathscr{A}}_1(P)\mathscr{A}_2(P)}{P^2}\Big)\lambda\\
& +\Big(-\frac{\overline{\mathscr{A}}_1(Q)\mathscr{A}_2(P)}{PR}+\frac{Q\overline{\mathscr{A}}_1(P)\mathscr{A}_2(P)}{P^2R}\Big)\lambda^2\Big\} \qquad \mathrm{mod}\ I,
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda\overline{L}d\alpha^1 \;=\; & \alpha^1\wedge\bar{\alpha}^1\\
& \Big\{\Big(\frac{\overline{\mathscr{A}}_2(P)\mathscr{A}_1(\bar{Q})}{PR}-\frac{\bar{Q}\overline{\mathscr{A}}_2(P)\mathscr{A}_1(P)}{P^2R}\Big)\bar{\lambda}^2\\
& +\Big(\frac{\overline{\mathscr{A}}_2(P)\mathscr{A}_1(P)}{P^2}-\frac{\bar{Q}\overline{\mathscr{A}}_2(Q)\mathscr{A}_1(P)}{PR^2}+\frac{\overline{\mathscr{A}}_2(Q)\mathscr{A}_1(\bar{Q})}{R^2}-\frac{\bar{Q}\overline{\mathscr{A}}_2(P)\mathscr{A}_2(P)}{P^2R}\\
& -\frac{\bar{Q}\overline{\mathscr{A}}_1(P)\mathscr{A}_1(P)}{P^2R}+\frac{\overline{\mathscr{A}}_1(P)\mathscr{A}_1(\bar{Q})}{PR}-\frac{\overline{\mathscr{A}}_2(P)\mathscr{A}_1(R)}{PR}+\frac{Q\bar{Q}\overline{\mathscr{A}}_2(P)\mathscr{A}_1(P)}{P^2R^2}\\
& +\frac{\overline{\mathscr{A}}_2(P)\mathscr{A}_2(\bar{Q})}{PR}-\frac{Q\overline{\mathscr{A}}_2(P)\mathscr{A}_1(\bar{Q})}{PR^2}\Big)\bar{\lambda}\\
& +\Big(\frac{Q\bar{Q}\overline{\mathscr{A}}_2(P)\mathscr{A}_2(P)}{P^2R^2}-\frac{\overline{\mathscr{A}}_2(P)\mathscr{A}_2(R)}{PR}-\frac{Q\overline{\mathscr{A}}_2(P)\mathscr{A}_1(P)}{P^2R}+\frac{\overline{\mathscr{A}}_2(Q)\mathscr{A}_1(P)}{PR}\\
& -\frac{\overline{\mathscr{A}}_2(Q)\mathscr{A}_1(R)}{R^2}+\frac{\overline{\mathscr{A}}_1(P)\mathscr{A}_1(P)}{P^2}+\frac{\overline{\mathscr{A}}_1(Q)\mathscr{A}_1(\bar{Q})}{R^2}+\frac{\overline{\mathscr{A}}_2(P)\mathscr{A}_2(P)}{P^2}\\
& +\frac{Q\bar{Q}\overline{\mathscr{A}}_1(P)\mathscr{A}_1(P)}{P^2R^2}-\frac{\bar{Q}\overline{\mathscr{A}}_2(Q)\mathscr{A}_2(P)}{PR^2}+\frac{\overline{\mathscr{A}}_1(P)\mathscr{A}_2(\bar{Q})}{PR}-\frac{\bar{Q}\overline{\mathscr{A}}_1(P)\mathscr{A}_2(P)}{P^2R}\\
& -\frac{Q\overline{\mathscr{A}}_1(P)\mathscr{A}_1(\bar{Q})}{PR^2}+\frac{\overline{\mathscr{A}}_2(Q)\mathscr{A}_2(\bar{Q})}{R^2}-\frac{\overline{\mathscr{A}}_1(P)\mathscr{A}_1(R)}{PR}-\frac{\bar{Q}\overline{\mathscr{A}}_1(Q)\mathscr{A}_1(P)}{PR^2}\\
& -\frac{Q\overline{\mathscr{A}}_2(P)\mathscr{A}_2(\bar{Q})}{PR^2}+\frac{Q\overline{\mathscr{A}}_2(P)\mathscr{A}_1(R)}{PR^2}\Big)\\
& +\Big(\frac{Q\bar{Q}\overline{\mathscr{A}}_1(P)\mathscr{A}_2(P)}{P^2R^2}-\frac{Q\overline{\mathscr{A}}_1(P)\mathscr{A}_2(\bar{Q})}{PR^2}+\frac{\overline{\mathscr{A}}_2(Q)\mathscr{A}_2(P)}{PR}+\frac{\overline{\mathscr{A}}_1(Q)\mathscr{A}_2(\bar{Q})}{R^2}\\
& -\frac{\bar{Q}\overline{\mathscr{A}}_1(Q)\mathscr{A}_2(P)}{PR^2}-\frac{\overline{\mathscr{A}}_2(Q)\mathscr{A}_2(R)}{R^2}+\frac{\overline{\mathscr{A}}_1(P)\mathscr{A}_2(P)}{P^2}-\frac{\overline{\mathscr{A}}_1(Q)\mathscr{A}_1(R)}{R^2}\\
& -\frac{Q\overline{\mathscr{A}}_1(P)\mathscr{A}_1(P)}{P^2R}+\frac{\overline{\mathscr{A}}_1(Q)\mathscr{A}_1(P)}{PR}+\frac{Q\overline{\mathscr{A}}_1(P)\mathscr{A}_1(R)}{PR^2}-\frac{\overline{\mathscr{A}}_1(P)\mathscr{A}_2(R)}{PR}\\
& -\frac{Q\overline{\mathscr{A}}_2(P)\mathscr{A}_2(P)}{P^2R}+\frac{Q\overline{\mathscr{A}}_2(P)\mathscr{A}_2(R)}{PR^2}\Big)\lambda\\
& +\Big(\frac{\overline{\mathscr{A}}_1(Q)\mathscr{A}_2(R)}{R^2}+\frac{Q\overline{\mathscr{A}}_1(P)\mathscr{A}_2(R)}{PR^2}-\frac{Q\overline{\mathscr{A}}_1(P)\mathscr{A}_2(P)}{P^2R}+\frac{\overline{\mathscr{A}}_1(Q)\mathscr{A}_2(P)}{PR}\Big)\lambda^2\Big\}
\end{aligned}
$$

3.8 The expressions A, B, C and the equation of the Sphere

The expressions of A, B and C in section 2.7 are therefore explicitly given by

$$\begin{aligned} A &= -\bar{L}, \\ B &= -2\left(\frac{\bar{Q}}{PR}\mathscr{A}_1(P) - \frac{1}{R}\mathscr{A}_1(\bar{Q})\right) + \lambda\left(\frac{\bar{Q}}{PR}\mathscr{A}_2(P) - \frac{1}{R}\mathscr{A}_2(\bar{Q}) - \frac{1}{P}\mathscr{A}_1(P) + \frac{1}{R}\mathscr{A}_1(R)\right) \\ C &= c_{-2}\bar{\lambda}^2 + c_{-1}\bar{\lambda} + c_0 + c_1\lambda + c_2\lambda^2, \end{aligned}$$

where

$$\begin{aligned} c_{-2} &= -\frac{\overline{\mathscr{A}}_2(\mathscr{A}_1(\bar{Q}))}{R} + \frac{\bar{Q}\overline{\mathscr{A}}_2(\mathscr{A}_1(P))}{PR} + \frac{2\overline{\mathscr{A}}_2(R)\mathscr{A}_1(\bar{Q})}{R^2} - \frac{2\bar{Q}\overline{\mathscr{A}}_2(R)\mathscr{A}_1(P)}{PR^2} \\ &\quad + \frac{\bar{Q}\overline{\mathscr{A}}_2(P)\mathscr{A}_1(P)}{P^2R} + \frac{\mathscr{A}_1(P)\overline{\mathscr{A}}_2(\bar{Q})}{PR} - \frac{2\overline{\mathscr{A}}_2(P)\mathscr{A}_1(\bar{Q})}{PR}, \end{aligned}$$

$$\begin{aligned} c_{-1} &= -\frac{\mathscr{A}_1(\overline{\mathscr{A}}_2(P))}{P} + \frac{\mathscr{A}_1(\overline{\mathscr{A}}_2(R))}{R} + \frac{\bar{Q}\overline{\mathscr{A}}_1(\mathscr{A}_1(P))}{PR} + \frac{\bar{Q}\overline{\mathscr{A}}_2(\mathscr{A}_2(P))}{PR} \\ &\quad - \frac{\overline{\mathscr{A}}_1(\mathscr{A}_1(\bar{Q}))}{R} - \frac{\overline{\mathscr{A}}_2(\mathscr{A}_2(\bar{Q}))}{R} + \frac{\overline{\mathscr{A}}_2(\mathscr{A}_1(R))}{R} - \frac{\overline{\mathscr{A}}_2(\mathscr{A}_1(P))}{P} + \frac{\mathscr{A}_2(P)\overline{\mathscr{A}}_2(\bar{Q})}{PR} \\ &\quad - \frac{2\overline{\mathscr{A}}_2(Q)\mathscr{A}_1(\bar{Q})}{R^2} + \frac{2\overline{\mathscr{A}}_1(R)\mathscr{A}_1(\bar{Q})}{R^2} + \frac{2\overline{\mathscr{A}}_2(R)\mathscr{A}_2(\bar{Q})}{R^2} - \frac{2\overline{\mathscr{A}}_2(R)\mathscr{A}_1(R)}{R^2} \\ &\quad - \frac{2\overline{\mathscr{A}}_1(P)\mathscr{A}_1(\bar{Q})}{PR} - \frac{2\overline{\mathscr{A}}_2(P)\mathscr{A}_2(\bar{Q})}{PR} + \frac{\overline{\mathscr{A}}_2(P)\mathscr{A}_1(R)}{PR} + \frac{\overline{\mathscr{A}}_2(R)\mathscr{A}_1(P)}{PR} \\ &\quad + \frac{\bar{Q}\overline{\mathscr{A}}_1(P)\mathscr{A}_1(P)}{P^2R} + \frac{2\bar{Q}\overline{\mathscr{A}}_2(Q)\mathscr{A}_1(P)}{R^2P} - \frac{2\bar{Q}\overline{\mathscr{A}}_1(R)\mathscr{A}_1(P)}{R^2P} - \frac{2Q\bar{Q}\overline{\mathscr{A}}_2(P)\mathscr{A}_1(P)}{P^2R^2} \\ &\quad + \frac{2Q\overline{\mathscr{A}}_2(P)\mathscr{A}_1(\bar{Q})}{PR^2} + \frac{\bar{Q}\overline{\mathscr{A}}_2(P)\mathscr{A}_2(P)}{P^2R} - \frac{2\bar{Q}\overline{\mathscr{A}}_2(R)\mathscr{A}_2(P)}{R^2P} + \frac{\mathscr{A}_1(P)\overline{\mathscr{A}}_1(\bar{Q})}{PR}, \end{aligned}$$

$$\begin{aligned} c_0 &= -\frac{\mathscr{A}_1(\overline{\mathscr{A}}_1(P))}{P} - \frac{\mathscr{A}(\overline{\mathscr{A}}_2(Q))}{R} + \frac{\mathscr{A}_1(\overline{\mathscr{A}}_1(R))}{R} + \frac{Q\mathscr{A}_1(\overline{\mathscr{A}}_2(P))}{PR} \\ &\quad - \frac{\mathscr{A}_2(\overline{\mathscr{A}}_2(P))}{P} + \frac{\mathscr{A}_2(\overline{\mathscr{A}}_2(R))}{R} - \frac{\overline{\mathscr{A}}_2(\mathscr{A}_2(R))}{R} - \frac{\overline{\mathscr{A}}_1(\mathscr{A}_1(P))}{P} \\ &\quad - \frac{\overline{\mathscr{A}}_2(\mathscr{A}_2(P))}{P} - \frac{\overline{\mathscr{A}}_1(\mathscr{A}_2(\bar{Q}))}{R} + \frac{\overline{\mathscr{A}}_1(\mathscr{A}_1(R))}{R} + \frac{\bar{Q}\overline{\mathscr{A}}_1(\mathscr{A}_2(P))}{PR} \\ &\quad - \frac{2\overline{\mathscr{A}}_2(R)\mathscr{A}_2(R)}{R^2} - \frac{2\overline{\mathscr{A}}_1(Q)\mathscr{A}_1(\bar{Q})}{R^2} - \frac{2Q\bar{Q}\overline{\mathscr{A}}_1(P)\mathscr{A}_1(P)}{P^2R^2} + \frac{2Q\overline{\mathscr{A}}_1(P)\mathscr{A}_1(\bar{Q})}{PR^2} \\ &\quad + \frac{2\bar{Q}\overline{\mathscr{A}}_1(Q)\mathscr{A}_1(P)}{PR^2} - \frac{2\overline{\mathscr{A}}_2(Q)\mathscr{A}_2(\bar{Q})}{R^2} + \frac{2\overline{\mathscr{A}}_2(Q)\mathscr{A}_1(R)}{R^2} + \frac{2\overline{\mathscr{A}}_1(R)\mathscr{A}_2(\bar{Q})}{R^2} \\ &\quad - \frac{2\overline{\mathscr{A}}_1(R)\mathscr{A}_1(R)}{R^2} + \frac{\overline{\mathscr{A}}_2(P)\mathscr{A}_2(R)}{PR} + \frac{\overline{\mathscr{A}}_2(R)\mathscr{A}_2(P)}{PR} + \frac{\bar{Q}\overline{\mathscr{A}}_1(P)\mathscr{A}_2(P)}{P^2R} \\ &\quad - \frac{2\overline{\mathscr{A}}_1(P)\mathscr{A}_2(\bar{Q})}{PR} + \frac{\overline{\mathscr{A}}_1(P)\mathscr{A}_1(R)}{PR} - \frac{2\overline{\mathscr{A}}_2(Q)\mathscr{A}_1(P)}{PR} + \frac{2\bar{Q}\overline{\mathscr{A}}_2(Q)\mathscr{A}_2(P)}{PR^2} \\ &\quad + \frac{\overline{\mathscr{A}}_1(R)\mathscr{A}_1(P)}{PR} - \frac{2\bar{Q}\overline{\mathscr{A}}_1(R)\mathscr{A}_2(P)}{PR^2} + \frac{Q\overline{\mathscr{A}}_2(P)\mathscr{A}_1(P)}{P^2R} - \frac{2Q\bar{Q}\overline{\mathscr{A}}_2(P)\mathscr{A}_2(P)}{P^2R^2} \\ &\quad + \frac{2Q\overline{\mathscr{A}}_2(P)\mathscr{A}_2(\bar{Q})}{PR^2} - \frac{2Q\overline{\mathscr{A}}_2(P)\mathscr{A}_1(R)}{PR^2} + \frac{\overline{\mathscr{A}}_2(P)\mathscr{A}_1(Q)}{PR} + \frac{\mathscr{A}_2(P)\overline{\mathscr{A}}_1(\bar{Q})}{PR}, \end{aligned}$$

$$
\begin{aligned}
c_1 \;=\; & -\frac{\mathscr{A}_2(\overline{\mathscr{A}}_2(Q))}{R} - \frac{\mathscr{A}_2(\overline{\mathscr{A}}_1(P))}{P} + \frac{\mathscr{A}_2(\overline{\mathscr{A}}_1(R))}{R} + \frac{Q\mathscr{A}_2(\overline{\mathscr{A}}_2(P))}{PR} \\
& + \frac{Q\mathscr{A}_1(\overline{\mathscr{A}}_1(P))}{PR} - \frac{\mathscr{A}_1(\overline{\mathscr{A}}_1(Q))}{R} + \frac{\overline{\mathscr{A}}_1(\mathscr{A}_2(R))}{R} - \frac{\overline{\mathscr{A}}_1(\mathscr{A}_2(P))}{P} \\
& + \frac{\overline{\mathscr{A}}_2(P)\mathscr{A}_2(Q)}{PR} - \frac{2\overline{\mathscr{A}}_1(Q)\mathscr{A}_2(\bar{Q})}{R^2} + \frac{2\overline{\mathscr{A}}_1(Q)\mathscr{A}_1(R)}{R^2} + \frac{2\overline{\mathscr{A}}_2(Q)\mathscr{A}_2(R)}{R^2} \\
& - \frac{2\overline{\mathscr{A}}_1(R)\mathscr{A}_2(R)}{R^2} - \frac{2\overline{\mathscr{A}}_1(Q)\mathscr{A}_1(P)}{PR} + \frac{\overline{\mathscr{A}}_1(P)\mathscr{A}_2(R)}{PR} - \frac{2\overline{\mathscr{A}}_2(Q)\mathscr{A}_2(P)}{PR} \\
& + \frac{\overline{\mathscr{A}}_1(R)\mathscr{A}_2(P)}{PR} + \frac{Q\overline{\mathscr{A}}_1(P)\mathscr{A}_1(P)}{P^2R} - \frac{-2Q\bar{Q}\overline{\mathscr{A}}_1(P)\mathscr{A}_2(P)}{P^2R^2} + \frac{2Q\overline{\mathscr{A}}_1(P)\mathscr{A}_2(\bar{Q})}{PR^2} \\
& - \frac{2Q\overline{\mathscr{A}}_1(P)\mathscr{A}_1(R)}{PR^2} + \frac{2\bar{Q}\overline{\mathscr{A}}_1(Q)\mathscr{A}_2(P)}{PR^2} + \frac{Q\overline{\mathscr{A}}_2(P)\mathscr{A}_2(P)}{P^2R} - \frac{2Q\overline{\mathscr{A}}_2(P)\mathscr{A}_2(R)}{PR^2} \\
& + \frac{\overline{\mathscr{A}}_1(P)\mathscr{A}_1(Q)}{PR},
\end{aligned}
$$

$$
\begin{aligned}
c_2 \;=\; & \frac{Q\mathscr{A}_2(\overline{\mathscr{A}}_1(P))}{PR} - \frac{\mathscr{A}_2(\overline{\mathscr{A}}_1(Q))}{R} - \frac{2\overline{\mathscr{A}}_1(Q)\mathscr{A}_2(P)}{PR} + \frac{Q\overline{\mathscr{A}}_1(P)\mathscr{A}_2(P)}{P^2R} \\
& + \frac{2\overline{\mathscr{A}}_1(Q)\mathscr{A}_2(R)}{R^2} - \frac{2Q\overline{\mathscr{A}}_1(P)\mathscr{A}_2(R)}{PR^2} + \frac{\overline{\mathscr{A}}_1(P)\mathscr{A}_2(Q)}{PR}.
\end{aligned}
$$

Observe that $c_2 = \overline{c_{-2}}$, $c_1 = \overline{c_{-1}}$ and c_0 is real-valued function on M. Therefore, C may be re-expressed as

$$C = 2\mathsf{Re}(c_2\lambda^2 + c_1\lambda + c_0).$$

On the other hand, μ B and L may be expressed as

$$
\begin{aligned}
\mu &= \mu_2\lambda^2 + \mu_1\lambda + \mu_0, \\
B &= B_1\lambda + B_0, \\
L &= L_2\lambda^2 + L_1\lambda + L_0.
\end{aligned}
$$

The following expressions are therefore obtained:

$$
\begin{aligned}
|B|^2 &= 2\mathsf{Re}(B_1\overline{B_0}\lambda + 0.5(|B_1|^2 + |B_0|^2)), \\
|L|^2 &= 2\mathsf{Re}(L_2\overline{L_0}\bar{\lambda} + (L_2\overline{L_1} + L_1\overline{L_0})\lambda + 0.5(|L_2|^2 + |L_1|^2 + |L_0|^2)), \\
|\mu - B|^2 &= \mathsf{Re}(k_2\lambda^2 + k_1\lambda + k_0),
\end{aligned}
$$

where here k_0, k_1 and k_2 are complex-valued functions whose calculations will not be presented here. Therefore, the equation of the sphere in the previous section

$$|\mu - B|^2 = |B|^2 + |L|^2 - C, \tag{3.6}$$

can be in general written as

$$\mathsf{Re}(P_2\lambda^2 + P_1\lambda + P_0) = 0, \tag{3.7}$$

where again P_2, P_1 and P_0 are complex-valued functions which not be made explicit here.

Suppose that in some neighbourhood $U \subset M$ of a fixed point $x \in M$, equation (3.7) is not identically zero, then it is of interest to find values of λ satisfying the equation of the sphere subjected to the condition $|\lambda|^2 = 1$.

Writing λ as $\lambda = s + \sqrt{-1}t$; and at $x \in M$, write also

$$\begin{aligned} P_2(x) &= X_1 + \sqrt{-1}X_2, \\ P_1(x) &= Y_1 + \sqrt{-1}Y_2, \\ P_0(x) &= Z_1 + \sqrt{-1}Z_2. \end{aligned}$$

for some real numbers X_1, X_2, Y_1, Y_2, Z_1, Z_2. Hence

$$\begin{aligned} 0 &= \mathsf{Re}(P_2(x)\lambda^2 + P_1(x)\lambda + P_0(x)) \\ &= X_1 s^2 - 2X_2 st - X_1 t^2 + Y_1 s - Y_2 t + Z_1, \end{aligned}$$

which can be recognised as the equation of a hyperbola in $\mathbb{R}^2$, whose intersection with the unit sphere $|\lambda|^2 = s^2 + t^2 = 1$ provides at most 4 intersection points. This leads to the following proposition:

Proposition 3.8. *At each point $x \in M$ where*

$$|\mu - B|^2 - (|B|^2 + |L|^2 - C)$$

does not identically vanish, there are at most 4 values of λ satisfying

$$\begin{aligned} |\mu - B|^2 - (|B|^2 + |L|^2 - C) &= 0, \\ |\lambda|^2 &= 1. \quad \square \end{aligned}$$

4 The Hachtroudi-Chern-Moser tensor components

4.1 The components of the invariant tensor

Let I_+ be the ideal of differential forms given by

$$I_+ = \langle \omega^0, \omega^2, \overline{\omega}^2, \tau \rangle.$$

The reason why τ is taken into consideration is that its pullback to the unit disk is zero. Hence two equations are obtained:

$$d\lambda \equiv L\bar{\alpha}^1 - \lambda^2 \bar{L}\alpha^1 \qquad \mathsf{mod}\ I_+, \qquad d\bar{\lambda} \equiv \bar{L}\alpha^1 - \bar{\lambda}^2 L\bar{\alpha}^1 \qquad \mathsf{mod}\ I_+.$$

Thus

$$\begin{aligned} d\tau &\equiv \alpha^1 \wedge \bar{\alpha}^1 \Big[-AL - \bar{A}L + B\bar{\lambda}^2 L + \bar{B}\lambda^2 \bar{L} + C \Big] \qquad \mathsf{mod}\ I_+ \\ &\equiv \alpha^1 \wedge \overline{\alpha}^1 \big(\tau_{-2}\bar{\lambda}^2 + \tau_{-1}\bar{\lambda} + \tau_0 + \tau_1 \lambda + \tau_2 \lambda^2 \big) \qquad \mathsf{mod}\ I_+. \end{aligned}$$

4.2 Relation to the defining function $u = F(z, \bar{z})$

The final expressions of the coefficients τ_{-2}, τ_{-1}, τ_0, τ_1 and τ_2 may be expressed in terms of F as follows:

$$
\begin{aligned}
&\tau_{-2} \\
=\ & \frac{1}{F_{z_1\bar{z}_1}^2(F_{z_1\bar{z}_1}F_{z_2\bar{z}_2} - F_{z_2\bar{z}_1}F_{z_1\bar{z}_2})} \\
& \left(F_{z_1z_1\bar{z}_1\bar{z}_1}F_{z_1\bar{z}_2}^2 - 2F_{z_1z_1\bar{z}_1\bar{z}_2}F_{z_1\bar{z}_1}F_{z_1\bar{z}_2} + F_{z_1z_1\bar{z}_2\bar{z}_2}F_{z_1\bar{z}_1}^2 \right) \\
& + \frac{1}{F_{z_1\bar{z}_1}^2(F_{z_1\bar{z}_1}F_{z_2\bar{z}_2} - F_{z_2\bar{z}_1}F_{z_1\bar{z}_2})^2} \\
& \Big(- F_{z_1z_1\bar{z}_1}F_{z_1\bar{z}_1\bar{z}_1}F_{z_1\bar{z}_2}^2F_{z_2\bar{z}_2} + 2F_{z_1z_1\bar{z}_1}F_{z_1\bar{z}_1\bar{z}_2}F_{z_1\bar{z}_1}F_{z_1\bar{z}_2}F_{z_2\bar{z}_2} - F_{z_1z_1\bar{z}_2}F_{z_2\bar{z}_2\bar{z}_2}F_{z_1\bar{z}_1}^3 \\
& -2F_{z_1z_1\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_2}F_{z_1\bar{z}_1}F_{z_1\bar{z}_2}F_{z_2\bar{z}_1} + F_{z_1\bar{z}_1\bar{z}_1}F_{z_1z_1\bar{z}_2}F_{z_1\bar{z}_2}^2F_{z_2\bar{z}_1} - F_{z_1z_1\bar{z}_1}F_{z_1\bar{z}_2\bar{z}_2}F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_2} \\
& +2F_{z_1z_1\bar{z}_2}F_{z_2\bar{z}_1\bar{z}_2}F_{z_1\bar{z}_1}^2F_{z_1\bar{z}_2} - F_{z_1z_1\bar{z}_2}F_{z_2\bar{z}_1\bar{z}_1}F_{z_1\bar{z}_1}F_{z_1\bar{z}_2}^2 + F_{z_1z_1\bar{z}_2}F_{z_1\bar{z}_2\bar{z}_2}F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1} \\
& -F_{z_1z_1\bar{z}_1}F_{z_2\bar{z}_1\bar{z}_2}F_{z_1\bar{z}_1}F_{z_1\bar{z}_2}^2 + F_{z_1z_1\bar{z}_1}F_{z_2\bar{z}_1\bar{z}_1}F_{z_1\bar{z}_2}^3 + F_{z_1z_1\bar{z}_1}F_{z_2\bar{z}_2\bar{z}_2}F_{z_1\bar{z}_1}^2F_{z_1\bar{z}_2} \Big),
\end{aligned}
$$

$$
\begin{aligned}
&\tau_{-1} \\
=\ & \frac{2}{(-F_{z_1\bar{z}_1}F_{z_2\bar{z}_2} + F_{z_2\bar{z}_1}F_{z_1\bar{z}_2})^{5/2}F_{z_1\bar{z}_1}^2)} \\
& \Big(4F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}F_{z_2\bar{z}_2}F_{z_1z_1\bar{z}_1\bar{z}_2} - 3F_{z_1\bar{z}_1}F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}^2F_{z_2\bar{z}_2}F_{z_1z_1\bar{z}_1\bar{z}_1} - F_{z_1\bar{z}_1}^3F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}F_{z_1z_2\bar{z}_2\bar{z}_2} \\
& F_{z_1\bar{z}_1}^3F_{z_2\bar{z}_1}F_{z_2\bar{z}_2}F_{z_1z_1\bar{z}_2\bar{z}_2} - 2F_{z_1\bar{z}_1}^3F_{z_1\bar{z}_2}F_{z_2\bar{z}_2}F_{z_1z_2\bar{z}_1\bar{z}_2} + F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}^2F_{z_1\bar{z}_2}F_{z_1z_1\bar{z}_2\bar{z}_2} \\
& +2F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}^2F_{z_1z_2\bar{z}_1\bar{z}_2} + F_{z_1\bar{z}_1}^2F_{z_1\bar{z}_2}^2F_{z_2\bar{z}_2}F_{z_1z_2\bar{z}_1\bar{z}_1} + F_{z_1\bar{z}_1}^2F_{z_1\bar{z}_2}F_{z_2\bar{z}_2}^2F_{z_1z_1\bar{z}_1\bar{z}_1} \\
& -3F_{z_1\bar{z}_1}F_{z_2\bar{z}_1}^2F_{z_1\bar{z}_2}^2F_{z_1z_1\bar{z}_1\bar{z}_2} - F_{z_1\bar{z}_1}F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}^3F_{z_1z_2\bar{z}_1\bar{z}_1} + F_{z_1\bar{z}_1}^4F_{z_2\bar{z}_2}F_{z_1z_2\bar{z}_2\bar{z}_2} \\
& -F_{z_1\bar{z}_1}^3F_{z_2\bar{z}_2}^2F_{z_1z_1\bar{z}_1\bar{z}_2} + 2F_{z_2\bar{z}_1}^2F_{z_1\bar{z}_2}^3F_{z_1z_1\bar{z}_1\bar{z}_1} + F_{z_1\bar{z}_1}F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}F_{z_2\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_1}F_{z_1z_1\bar{z}_2} \\
& -3F_{z_1\bar{z}_1}F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}F_{z_2\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_2}F_{z_1z_1\bar{z}_1} + 4F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}F_{z_2\bar{z}_2}F_{z_1z_1\bar{z}_1\bar{z}_2} - 2F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_2}F_{z_1z_2\bar{z}_2} \\
& -3F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}F_{z_2\bar{z}_1\bar{z}_2}F_{z_1z_1\bar{z}_2} - F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}F_{z_2\bar{z}_2\bar{z}_2}F_{z_1z_1\bar{z}_1} - F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}F_{z_2\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_2}F_{z_1z_1\bar{z}_2} + \\
& F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}F_{z_2\bar{z}_2}F_{z_1\bar{z}_2\bar{z}_2}F_{z_1z_1\bar{z}_1} - F_{z_1\bar{z}_1}^2F_{z_1\bar{z}_2}F_{z_2\bar{z}_2}F_{z_2\bar{z}_1\bar{z}_1}F_{z_1z_1\bar{z}_2} + 2F_{z_1\bar{z}_1}^2F_{z_1\bar{z}_2}F_{z_2\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_2}F_{z_1z_2\bar{z}_1} \\
& -F_{z_1\bar{z}_1}^2F_{z_1\bar{z}_2}F_{z_2\bar{z}_2}F_{z_2\bar{z}_1\bar{z}_1}F_{z_1z_1\bar{z}_2} + 3F_{z_1\bar{z}_1}F_{z_2\bar{z}_1}^2F_{z_1\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_2}F_{z_1z_1\bar{z}_2} + F_{z_1\bar{z}_1}F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}^2F_{z_1\bar{z}_1\bar{z}_1}F_{z_1z_2\bar{z}_2} \\
& +2F_{z_1\bar{z}_1}F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}^2F_{z_2\bar{z}_1\bar{z}_1}F_{z_1z_1\bar{z}_2} + 3F_{z_1\bar{z}_1}F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}^2F_{z_2\bar{z}_1\bar{z}_2}F_{z_1z_1\bar{z}_1} - F_{z_1\bar{z}_1}F_{z_1\bar{z}_2}^2F_{z_2\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_1}F_{z_1z_2\bar{z}_1} \\
& +F_{z_1\bar{z}_1}F_{z_1\bar{z}_2}^2F_{z_2\bar{z}_2}F_{z_2\bar{z}_1\bar{z}_1}F_{z_1z_1\bar{z}_1} - F_{z_1\bar{z}_1}F_{z_1\bar{z}_2}F_{z_2\bar{z}_2}^2F_{z_1\bar{z}_1\bar{z}_1}F_{z_1z_1\bar{z}_2} + 2F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}^2F_{z_2\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_1}F_{z_1z_1\bar{z}_1} \\
& +F_{z_1\bar{z}_1}^3F_{z_2\bar{z}_1}F_{z_1\bar{z}_2\bar{z}_2}F_{z_1z_2\bar{z}_2} + F_{z_1\bar{z}_1}^3F_{z_2\bar{z}_1}F_{z_2\bar{z}_2\bar{z}_2}F_{z_1z_1\bar{z}_2} + 2F_{z_1\bar{z}_1}^3F_{z_1\bar{z}_2}F_{z_2\bar{z}_1\bar{z}_2}F_{z_1z_2\bar{z}_2} \\
& +F_{z_1\bar{z}_1}^3F_{z_1\bar{z}_2}F_{z_1z_2\bar{z}_1}F_{z_2\bar{z}_2\bar{z}_2} + F_{z_1\bar{z}_1}^3F_{z_2\bar{z}_2}F_{z_2\bar{z}_1\bar{z}_2}F_{z_1z_1\bar{z}_2} - F_{z_1\bar{z}_1}^3F_{z_2\bar{z}_2}F_{z_1z_2\bar{z}_1}F_{z_1\bar{z}_2\bar{z}_2} \\
& -F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}^2F_{z_1\bar{z}_2\bar{z}_2}F_{z_1z_1\bar{z}_2} - F_{z_1\bar{z}_1}^2F_{z_1\bar{z}_2}^2F_{z_2\bar{z}_1\bar{z}_1}F_{z_1z_2\bar{z}_2} - 2F_{z_1\bar{z}_1}^2F_{z_1\bar{z}_2}^2F_{z_2\bar{z}_1\bar{z}_2}F_{z_1z_2\bar{z}_1} \\
& +F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_2}^2F_{z_1\bar{z}_2\bar{z}_2}F_{z_1z_1\bar{z}_1} + F_{z_1\bar{z}_1}F_{z_1\bar{z}_2}^3F_{z_2\bar{z}_1\bar{z}_1}F_{z_1z_2\bar{z}_1} - 2F_{z_2\bar{z}_1}^2F_{z_1\bar{z}_2}^2F_{z_1\bar{z}_1\bar{z}_1}F_{z_1z_1\bar{z}_2} \\
& -2F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}^3F_{z_2\bar{z}_1\bar{z}_1}F_{z_1z_1\bar{z}_1} - F_{z_1\bar{z}_1}^4F_{z_2\bar{z}_2\bar{z}_2}F_{z_1z_2\bar{z}_2} \Big),
\end{aligned}
$$

$$
\begin{aligned}
&\tau_0\\
=\ &\frac{1}{F_{z_1\bar{z}_1}^2(F_{z_1\bar{z}_1}F_{z_2\bar{z}_2}-F_{z_1\bar{z}_2}F_{z_2\bar{z}_1})^3}\\
&\Big(-6F_{z_1\bar{z}_1}^3F_{z_1\bar{z}_2}^3F_{z_1z_1\bar{z}_1\bar{z}_1}+12F_{z_1\bar{z}_1}^3F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}F_{z_2\bar{z}_2}F_{z_1z_2\bar{z}_1\bar{z}_2}-10F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}^2F_{z_1\bar{z}_2}F_{z_2\bar{z}_2}F_{z_1z_1\bar{z}_1\bar{z}_2}\\
&-10F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}^2F_{z_2\bar{z}_2}F_{z_1z_2\bar{z}_1\bar{z}_1}-7F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}F_{z_2\bar{z}_2}^2F_{z_1z_1\bar{z}_1\bar{z}_1}+12F_{z_1\bar{z}_1}F_{z_2\bar{z}_1}^2F_{z_1\bar{z}_2}^2F_{z_2\bar{z}_2}F_{z_1z_1\bar{z}_1\bar{z}_1}\\
&-F_{z_1\bar{z}_1}^4F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}F_{z_2z_2\bar{z}_2\bar{z}_2}-2F_{z_1\bar{z}_1}^4F_{z_2\bar{z}_1}F_{z_2\bar{z}_2}F_{z_1z_2\bar{z}_2\bar{z}_2}-2F_{z_1\bar{z}_1}^4F_{z_1\bar{z}_2}F_{z_2\bar{z}_2}F_{z_2z_2\bar{z}_1\bar{z}_2}\\
&+2F_{z_1\bar{z}_1}^3F_{z_2\bar{z}_1}^2F_{z_1\bar{z}_2}F_{z_1z_2\bar{z}_2\bar{z}_2}+F_{z_1\bar{z}_1}^3F_{z_2\bar{z}_1}^2F_{z_2\bar{z}_2}F_{z_1z_1\bar{z}_2\bar{z}_2}+2F_{z_1\bar{z}_1}^3F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}^2F_{z_2z_2\bar{z}_1\bar{z}_2}\\
&+4F_{z_1\bar{z}_1}^3F_{z_2\bar{z}_1}F_{z_2\bar{z}_2}^2F_{z_1z_1\bar{z}_1\bar{z}_2}+F_{z_1\bar{z}_1}^3F_{z_1\bar{z}_2}^2F_{z_2\bar{z}_2}F_{z_2z_2\bar{z}_1\bar{z}_1}+4F_{z_1\bar{z}_1}^3F_{z_1\bar{z}_2}F_{z_2\bar{z}_2}^2F_{z_1z_2\bar{z}_1\bar{z}_1}\\
&-F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}^3F_{z_1\bar{z}_2}F_{z_1z_1\bar{z}_2\bar{z}_2}-8F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}^2F_{z_1\bar{z}_2}^2F_{z_1z_2\bar{z}_1\bar{z}_2}-F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}^3F_{z_2z_2\bar{z}_1\bar{z}_1}\\
&+6F_{z_1\bar{z}_1}F_{z_2\bar{z}_1}^3F_{z_1\bar{z}_2}^2F_{z_1z_1\bar{z}_1\bar{z}_2}+6F_{z_1\bar{z}_1}F_{z_2\bar{z}_1}^2F_{z_1\bar{z}_2}^3F_{z_1z_2\bar{z}_1\bar{z}_1}+F_{z_1\bar{z}_1}^5F_{z_1\bar{z}_2}F_{z_2z_2\bar{z}_2\bar{z}_2}\\
&-4F_{z_1\bar{z}_1}^4F_{z_2\bar{z}_2}^2F_{z_1z_2\bar{z}_1\bar{z}_2}+F_{z_1\bar{z}_1}^3F_{z_1\bar{z}_2}^3F_{z_1z_1\bar{z}_1\bar{z}_1}\\
&4F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}F_{z_2\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_1}F_{z_1z_2\bar{z}_2}+6F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}F_{z_2\bar{z}_2}F_{z_2\bar{z}_1\bar{z}_1}F_{z_1z_1\bar{z}_2}\\
&-8F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}F_{z_2\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_2}F_{z_1z_2\bar{z}_1}+4F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}F_{z_2\bar{z}_2}F_{z_2\bar{z}_1\bar{z}_2}F_{z_1z_1\bar{z}_1}\\
&-6F_{z_2\bar{z}_1}^2F_{z_1\bar{z}_1}F_{z_1\bar{z}_2}F_{z_2\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_1}F_{z_1z_1\bar{z}_2}+6F_{z_2\bar{z}_1}^2F_{z_1\bar{z}_2}F_{z_2\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_1}F_{z_1z_2\bar{z}_1}\\
&+6F_{z_1\bar{z}_2}^2F_{z_1\bar{z}_1}F_{z_2\bar{z}_1}F_{z_2\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_1}F_{z_1z_1\bar{z}_1}-6F_{z_1\bar{z}_2}^2F_{z_1\bar{z}_1}F_{z_2\bar{z}_1}F_{z_2\bar{z}_2}F_{z_2\bar{z}_1\bar{z}_1}F_{z_1z_1\bar{z}_1}\\
&+6F_{z_2\bar{z}_2}^2F_{z_1\bar{z}_1}F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_1}F_{z_1z_1\bar{z}_1}-2F_{z_1\bar{z}_1}^3F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_2}F_{z_2z_2\bar{z}_2}\\
&-8F_{z_1\bar{z}_1}^3F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}F_{z_2\bar{z}_1\bar{z}_2}F_{z_1z_2\bar{z}_2}-2F_{z_1\bar{z}_1}^3F_{z_2\bar{z}_1}F_{z_1\bar{z}_2}F_{z_1z_2\bar{z}_1}F_{z_2\bar{z}_2\bar{z}_2}\\
&-4F_{z_1\bar{z}_1}^3F_{z_2\bar{z}_1}F_{z_2\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_2}F_{z_1z_2\bar{z}_2}-4F_{z_1\bar{z}_1}^3F_{z_2\bar{z}_1}F_{z_2\bar{z}_2}F_{z_2\bar{z}_1\bar{z}_2}F_{z_1z_1\bar{z}_2}\\
&+2F_{z_1\bar{z}_1}^3F_{z_2\bar{z}_1}F_{z_2\bar{z}_2}F_{z_1z_2\bar{z}_1}F_{z_1\bar{z}_2\bar{z}_2}-4F_{z_1\bar{z}_1}^3F_{z_1\bar{z}_2}F_{z_2\bar{z}_2}F_{z_2\bar{z}_1\bar{z}_1}F_{z_1z_2\bar{z}_2}\\
&+2F_{z_1\bar{z}_1}^3F_{z_1\bar{z}_2}F_{z_2\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_2}F_{z_2z_2\bar{z}_1}-4F_{z_1\bar{z}_1}^3F_{z_1\bar{z}_2}F_{z_2\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_2}F_{z_1z_2\bar{z}_1}\\
&+8F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}^2F_{z_1\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_2}F_{z_1z_2\bar{z}_2}+6F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}^2F_{z_1\bar{z}_2}F_{z_2\bar{z}_1\bar{z}_2}F_{z_1z_1\bar{z}_2}\\
&+F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}^2F_{z_1\bar{z}_2}F_{z_1z_2\bar{z}_1}F_{z_2\bar{z}_2\bar{z}_2}+4F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}^2F_{z_2\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_2}F_{z_1z_2\bar{z}_2}\\
&-F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1}^2F_{z_2\bar{z}_2}F_{z_1z_1\bar{z}_1}F_{z_1\bar{z}_2\bar{z}_2}+F_{z_1\bar{z}_1}^2F_{z_1\bar{z}_2}^2F_{z_2\bar{z}_1}F_{z_1\bar{z}_1\bar{z}_1}F_{z_2z_2\bar{z}_2}+6F_{z_1\bar{z}_1}^2F_{z_1\bar{z}_2}^2F_{z_2\bar{z}_1}F_{z_2\bar{z}_1\bar{z}_1}F_{z_1z_2\bar{z}_2}\\
&+8F_{z_1\bar{z}_1}^2F_{z_1\bar{z}_2}^2F_{z_2\bar{z}_1}F_{z_2\bar{z}_1\bar{z}_2}F_{z_1z_2\bar{z}_1}+F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_2}^2F_{z_2\bar{z}_1}F_{z_1\bar{z}_1\bar{z}_1}F_{z_1z_2\bar{z}_2}-4F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_2}^2F_{z_2\bar{z}_1}F_{z_1\bar{z}_1\bar{z}_2}F_{z_1z_2\bar{z}_1}\\
&-F_{z_1\bar{z}_1}^2F_{z_1\bar{z}_2}^2F_{z_2\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_1}F_{z_2z_2\bar{z}_1}+4F_{z_1\bar{z}_1}^2F_{z_1\bar{z}_2}^2F_{z_2\bar{z}_2}F_{z_2\bar{z}_1\bar{z}_1}F_{z_1z_2\bar{z}_1}-4F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_2}^2F_{z_1\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_1}F_{z_1z_2\bar{z}_1}\\
&+F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_2}^2F_{z_2\bar{z}_1}F_{z_2\bar{z}_1\bar{z}_1}F_{z_1z_1\bar{z}_1}-6F_{z_1\bar{z}_1}F_{z_2\bar{z}_1}^3F_{z_1\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_2}F_{z_1z_1\bar{z}_2}-6F_{z_2\bar{z}_1}^2F_{z_1\bar{z}_2}^2F_{z_1\bar{z}_1}F_{z_1\bar{z}_1\bar{z}_1}F_{z_1z_2\bar{z}_2}\\
&-6F_{z_2\bar{z}_1}^2F_{z_1\bar{z}_2}^2F_{z_1\bar{z}_1}F_{z_2\bar{z}_1\bar{z}_1}F_{z_1z_1\bar{z}_2}-6F_{z_2\bar{z}_1}^2F_{z_1\bar{z}_2}^2F_{z_1\bar{z}_1}F_{z_2\bar{z}_1\bar{z}_2}F_{z_1z_1\bar{z}_1}-6F_{z_1\bar{z}_2}^3F_{z_1\bar{z}_1}F_{z_2\bar{z}_1}F_{z_2\bar{z}_1\bar{z}_1}F_{z_1z_2\bar{z}_1}\\
&-6F_{z_2\bar{z}_1}^2F_{z_1\bar{z}_2}^2F_{z_2\bar{z}_2}F_{z_1\bar{z}_1\bar{z}_1}F_{z_1z_1\bar{z}_1}+F_{z_1\bar{z}_1}^4F_{z_2\bar{z}_1}F_{z_1\bar{z}_2\bar{z}_2}F_{z_2z_2\bar{z}_2}+2F_{z_1\bar{z}_1}^4F_{z_2\bar{z}_1}F_{z_2\bar{z}_2\bar{z}_2}F_{z_1z_2\bar{z}_2}\\
&+2F_{z_1\bar{z}_1}^4F_{z_1\bar{z}_2}F_{Z_2\bar{z}_1\bar{z}_2}F_{z_2z_2\bar{z}_2}+F_{z_1\bar{z}_1}^4F_{z_1\bar{z}_2}F_{z_2z_2\bar{z}_1}F_{z_2\bar{z}_2\bar{z}_2}+4F_{z_1\bar{z}_1}^4F_{z_2\bar{z}_2}F_{z_2\bar{z}_1\bar{z}_2}F_{z_1z_2\bar{z}_2}\\
&-F_{z_1\bar{z}_1}^4F_{z_2\bar{z}_2}F_{z_2z_2\bar{z}_1}F_{z_1\bar{z}_2\bar{z}_2}-2F_{z_1\bar{z}_1}^3F_{z_2\bar{z}_1}^2F_{z_1\bar{z}_2\bar{z}_2}F_{z_1z_2\bar{z}_2}-F_{z_1\bar{z}_1}^3F_{z_2\bar{z}_1}^2F_{z_2\bar{z}_2\bar{z}_2}F_{z_1z_1\bar{z}_2}\\
&-F_{z_1\bar{z}_1}^3F_{z_2\bar{z}_1}^2F_{z_2\bar{z}_1\bar{z}_1}F_{z_1z_2\bar{z}_2}-2F_{z_1\bar{z}_1}^3F_{z_2\bar{z}_2}^2F_{z_1\bar{z}_2}F_{z_2\bar{z}_1\bar{z}_1}F_{z_1z_2\bar{z}_1}-F_{z_1\bar{z}_1}^3F_{z_2\bar{z}_2}^2F_{z_2\bar{z}_1\bar{z}_1}F_{z_1z_1\bar{z}_2}\\
&+4F_{z_1\bar{z}_1}^3F_{z_2\bar{z}_2}^2F_{z_1\bar{z}_1\bar{z}_2}F_{z_1z_2\bar{z}_1}+F_{z_1\bar{z}_1}^3F_{z_1\bar{z}_2}^2F_{z_1\bar{z}_1}F_{z_2\bar{z}_2\bar{z}_2}F_{z_1z_1\bar{z}_2}+F_{z_1\bar{z}_2}^3F_{z_1\bar{z}_1}^2F_{z_2\bar{z}_1\bar{z}_1}F_{z_2z_2\bar{z}_1}\\
&-F_{z_2\bar{z}_2}^3F_{z_1\bar{z}_1}^2F_{z_1\bar{z}_1\bar{z}_1}F_{z_1z_1\bar{z}_1}+6F_{z_2\bar{z}_1}^3F_{z_1\bar{z}_2}^2F_{z_1\bar{z}_1\bar{z}_1}F_{z_1z_1\bar{z}_2}+6F_{z_1\bar{z}_2}^3F_{z_2\bar{z}_1}^2F_{z_2\bar{z}_1\bar{z}_1}F_{z_1z_1\bar{z}_1}\\
&-F_{z_1\bar{z}_1}^5F_{z_2\bar{z}_2\bar{z}_2}F_{z_2z_2\bar{z}_2}\Big),
\end{aligned}
$$

while τ_2 and τ_1 are conjugates of τ_{-2} and τ_{-1} respectively.

The following next few pages are the Maple codes for the calculations. The first Maple file calculates the expression of C in $d\tau$ modulo I, whereas the second maple file calculates the S tensor components in terms of F.

```
> restart:

  ###########################################################
  ######### 1) Load differential geometry package ###########
  ######### 2) Define the coframes and their conjugates #####
  ###########################################################

  with(difforms):

  alias(conjugate(alpha[1])=alpha1bar);
  alias(conjugate(alpha[2])=alpha2bar);
  alias(conjugate(lambda)=lambdabar);
  alias(alpha[1]=alpha1);
  alias(alpha[2]=alpha2);
```

$$\overline{\alpha_1}$$
$$\overline{\alpha_1}, \overline{\alpha_2}$$
$$\overline{\alpha_1}, \overline{\alpha_2}, \overline{\lambda}$$
$$\overline{\alpha_1}, \overline{\alpha_2}, \overline{\lambda}, \alpha_1$$
$$\overline{\alpha_1}, \overline{\alpha_2}, \overline{\lambda}, \alpha_1, \alpha_2 \qquad (1)$$

```
>
> ######################################################
  # 1) defform sets the degree of differential forms ##
  #2) test that wedge of two same 1 forms is 0 #######
  ######################################################


  defform(alpha1=1, alpha1bar=1, alpha2=1, alpha2bar=1, P=scalar, Q=
  scalar, R=scalar, lambda=scalar, lambdabar=scalar, dlambda=1,
  dlambda1bar=1);

  simpform(alpha1bar &^ alpha2bar);
  simpform(alpha1bar &^ alpha1bar);
```

$$\overline{\alpha_1} \,\&\wedge\, \overline{\alpha_2}$$
$$0 \qquad (2)$$

```
> ######################################################
  # Formally assign algebraic properties of 1st order ####
  # differential operator to a symbol A1 etc #############
  ######################################################

  A1  := proc (x)
         local y; y := op(1, x);
                     if type(x, `+`) = true then add(A1(op(i,x)), i
```

```
= 1 .. nops(x))
                  elif type(x, `*`) = true then expand(x*A1(y)/y+y*
A1(x/y))
                  elif type(x, `^`) = true then op(2,x)*y^(op(2,x)
-1)*A1(y)
                  elif type(x, function) = true or type(x, symbol)
= true
                  then A[1](x) else 0
                  end if
       end proc:

A2  := proc (x)
       local y; y := op(1, x);
                    if type(x, `+`) = true then add(A2(op(i,x)), i
= 1 .. nops(x))
                  elif type(x, `*`) = true then expand(x*A2(y)/y+y*
A2(x/y))
                  elif type(x, `^`) = true then op(2,x)*y^(op(2,x)
-1)*A2(y)
                  elif type(x, function) = true or type(x, symbol)
= true
                  then A[2](x) else 0
                  end if
       end proc:

A1bar  := proc (x)
          local y; y := op(1, x);
                       if type(x, `+`) = true then add(A1bar(op(i,
x)), i = 1 .. nops(x))
                     elif type(x, `*`) = true then expand(x*A1bar
(y)/y+y*A1bar(x/y))
                     elif type(x, `^`) = true then op(2,x)*y^(op(2,
x)-1)*A1bar(y)
                     elif type(x, function) = true or type(x,
symbol) = true
                     then conjugate(A[1])(x) else 0
                     end if
       end proc:

A2bar  := proc (x)
          local y; y := op(1, x);
                       if type(x, `+`) = true then add(A2bar(op(i,
x)), i = 1 .. nops(x))
                     elif type(x, `*`) = true then expand(x*A2bar
(y)/y+y*A2bar(x/y))
                     elif type(x, `^`) = true then op(2,x)*y^(op(2,
x)-1)*A2bar(y)
                     elif type(x, function) = true or type(x,
```

```
symbol) = true
                    then conjugate(A[2])(x) else 0
                    end if
       end proc:

Z1  := proc (x)
          local y; y := op(1, x);
                     if type(x, `+`) = true then add(Z1(op(i,x)),
i = 1 .. nops(x))
                    elif type(x, `*`) = true then expand(x*Z1(y)
/y+y*Z1(x/y))
                    elif type(x, `^`) = true then op(2,x)*y^(op(2,
x)-1)*Z1(y)
                    elif type(x, function) = true or type(x,
symbol) = true
                    then Z[1](x) else 0
                    end if
       end proc:

Z2  := proc (x)
          local y; y := op(1, x);
                     if type(x, `+`) = true then add(Z2(op(i,x)),
i = 1 .. nops(x))
                    elif type(x, `*`) = true then expand(x*Z2(y)
/y+y*Z2(x/y))
                    elif type(x, `^`) = true then op(2,x)*y^(op(2,
x)-1)*Z2(y)
                    elif type(x, function) = true or type(x,
symbol) = true
                    then Z[2](x) else 0
                    end if
       end proc:
>
> ##########################################################
  # Set up the Pfaffian system (see page [Bryant, page 208) ##
  #Set the condition that |lambda|^2=1 ######################
  ##########################################################

  assign(alpha2=lambda*alpha1);
  assign(alpha2bar=lambdabar*alpha1bar);
  assign('lambdabar'=1/lambda);
>
> ################################################
  # Calculation of exterior differential of alpha1 #
  # Simplification of the expression d(alpha1) #####
  ################################################

  dalpha1A:= simpform(
```

```
              expand(
   ((1/R)*A1(Q)-(Q/(P*R))*A1(P)-(1/P)*A2(P))*alpha1 &^ alpha2
   +
   ((-1/P)*A1bar(P))*alpha1 &^ alpha1bar
   +
   ((Q/(P*R))*A1bar(P)-(1/R)*A1bar(Q))*alpha2 &^ alpha1bar
   +
   ((-1/P)*A2bar(P))*alpha1 &^ alpha2bar
   +
   ((Q/(P*R))*A2bar(P)-(1/R)*A2bar(Q))*alpha2 &^ alpha2bar
   )
   );
   dalpha1:=collect(dalpha1A, alpha[1] &^ conjugate(alpha[1]));
```

$$dalpha1A := \frac{(-R+\lambda\,Q)\,\overline{A_1}(P)\,\alpha_1\,\&\hat{}\,\overline{\alpha_1}}{PR} - \frac{\lambda\,\alpha_1\,\&\hat{}\,\overline{\alpha_1}\,\overline{A_1}(Q)}{R} + \frac{(-R+\lambda\,Q)\,\overline{A_2}(P)\,\alpha_1\,\&\hat{}\,\overline{\alpha_1}}{P\lambda\,R} - \frac{\alpha_1\,\&\hat{}\,\overline{\alpha_1}\,\overline{A_2}(Q)}{R}$$

$$dalpha1 := \left(\frac{(-R+\lambda\,Q)\,\overline{A_1}(P)}{PR} - \frac{\lambda\,\overline{A_1}(Q)}{R} + \frac{(-R+\lambda\,Q)\,\overline{A_2}(P)}{P\lambda\,R} - \frac{\overline{A_2}(Q)}{R}\right)\alpha_1\,\&\hat{}\,\mathrm{conjugate}(\alpha_1) \tag{3}$$

```
>
> ################################################
   # Calculation of exterior differential of alpha2 ##
   #Simplification of the expression d(alpha2) #######
   ################################################

   dalpha2A:=simpform(expand(
   (1/R)*A1(R)*alpha1 &^ alpha2
   -
   (1/R)*A1bar(R)* alpha2 &^ alpha1bar
   -
   (1/R)*A2bar(R)* alpha2 &^ alpha2bar
   ));
   dalpha2:=collect(dalpha2A, alpha[1] &^ conjugate(alpha[1]));
```

$$dalpha2A := -\frac{\overline{A_1}(R)\,\lambda\,\alpha_1\,\&\hat{}\,\overline{\alpha_1}}{R} - \frac{\overline{A_2}(R)\,\alpha_1\,\&\hat{}\,\overline{\alpha_1}}{R}$$

$$dalpha2 := \left(-\frac{\overline{A_1}(R)\,\lambda}{R} - \frac{\overline{A_2}(R)}{R}\right)\alpha_1\,\&\hat{}\,\overline{\alpha_1} \tag{4}$$

```
>
> ########################################
   # 1) calculation of d(alpha1bar) #######
   # 2) simplify to factor out the 2-form #
   #    (whose order is not right!) #######
```

```
# 3) copy out the coeffcient ##########
# 4) rewrite d(alpha1bar) so that the ##
#    2-form is in right order ##########
########################################

dalpha1barA:=simpform(expand(
((1/R)*A1bar(conjugate(Q))-((conjugate(Q))/(P*R))*A1bar(P)-(1/P)*
A2bar(P))*alpha1bar &^ alpha2bar
+((-1/P)*A1(P))*alpha1bar &^ alpha1
+(((conjugate(Q))/(P*R))*A1(P)-(1/R)*A1(conjugate(Q)))*alpha2bar
&^ alpha1
+((-1/P)*A2(P))*alpha1bar &^ alpha2
+((conjugate(Q)/(P*R))*A2(P)-(1/R)*A2(conjugate(Q)))*alpha2bar &^
alpha2
));
```

$$dalpha1barA := -\frac{A_1(P)\,\overline{\alpha_1}\ \&\wedge\ \alpha_1}{P} + \frac{\overline{\alpha_1}\ \&\wedge\ \alpha_1\,\overline{Q}\,A_1(P)}{\lambda\,P\,R} - \frac{\overline{\alpha_1}\ \&\wedge\ \alpha_1\,A_1(\overline{Q})}{\lambda\,R} - \frac{A_2(P)\,\lambda\,\overline{\alpha_1}\ \&\wedge\ \alpha_1}{P} + \frac{\overline{\alpha_1}\ \&\wedge\ \alpha_1\,\overline{Q}\,A_2(P)}{P\,R} - \frac{\overline{\alpha_1}\ \&\wedge\ \alpha_1\,A_2(\overline{Q})}{R} \tag{5}$$

```
> dalpha1barB := collect(dalpha1barA, conjugate(alpha[1]) &^ alpha
  [1]);
  coeffdalpha1B :=dalpha1barB/(alpha1bar &^ alpha1);
  dalpha1barBB := simpform(-coeffdalpha1B*(alpha1 &^ alpha1bar));
  dalpha1bar :=collect(dalpha1barBB, alpha1 &^ alpha1bar);
```

$$dalpha1barB := \left(-\frac{A_1(P)}{P} + \frac{\overline{Q}\,A_1(P)}{\lambda\,P\,R} - \frac{A_1(\overline{Q})}{\lambda\,R} - \frac{A_2(P)\,\lambda}{P} + \frac{\overline{Q}\,A_2(P)}{P\,R} - \frac{A_2(\overline{Q})}{R}\right) \mathrm{conjugate}\left(\alpha_1\right)\ \&\wedge\ \alpha_1$$

$$coeffdalpha1B := -\frac{A_1(P)}{P} + \frac{\overline{Q}\,A_1(P)}{\lambda\,P\,R} - \frac{A_1(\overline{Q})}{\lambda\,R} - \frac{A_2(P)\,\lambda}{P} + \frac{\overline{Q}\,A_2(P)}{P\,R} - \frac{A_2(\overline{Q})}{R}$$

$$dalpha1barBB := \frac{A_1(P)\,\alpha_1\ \&\wedge\ \overline{\alpha_1}}{P} - \frac{\overline{Q}\,A_1(P)\,\alpha_1\ \&\wedge\ \overline{\alpha_1}}{\lambda\,P\,R} + \frac{A_1(\overline{Q})\,\alpha_1\ \&\wedge\ \overline{\alpha_1}}{\lambda\,R} + \frac{A_2(P)\,\lambda\,\alpha_1\ \&\wedge\ \overline{\alpha_1}}{P} - \frac{\overline{Q}\,A_2(P)\,\alpha_1\ \&\wedge\ \overline{\alpha_1}}{P\,R} + \frac{A_2(\overline{Q})\,\alpha_1\ \&\wedge\ \overline{\alpha_1}}{R}$$

$$dalpha1bar := \left(\frac{A_1(P)}{P} - \frac{\overline{Q}\,A_1(P)}{\lambda\,P\,R} + \frac{A_1(\overline{Q})}{\lambda\,R} + \frac{A_2(P)\,\lambda}{P} - \frac{\overline{Q}\,A_2(P)}{P\,R} + \frac{A_2(\overline{Q})}{R}\right) \alpha_1\ \&\wedge\backslash\ \overline{\alpha_1} \tag{6}$$

```
>
> ##############################
  #Calculation of d(alpha2bar) #
  ##############################

  dalpha2bar:=((conjugate(lambda)/R)*A1(R)+(1/R)*A2(R))*alpha1 &^
  alpha1bar;
```

$$dalpha2bar := \left(\frac{A_1(R)}{\lambda R} + \frac{A_2(R)}{R} \right) \alpha_1 \,\&\wedge\, \overline{\alpha_1} \tag{7}$$

```
>
> ##################################################
  # 1) Calculation of d(omega2) ####################
  # 2) Simplify by grouping according to diff forms ##
  # 3) Copy out the coefficient L ##################
  ##################################################


  domega2:= simpform(expand(
  -dlambda &^ alpha1-lambda*dalpha1+dalpha2
  ));
```

$$\begin{aligned} domega2 := & -dlambda \,\&\wedge\, \alpha_1 - \frac{\lambda\,(-R+\lambda\,Q)\,\overline{A_1}(P)\,\alpha_1 \,\&\wedge\, \overline{\alpha_1}}{PR} \\ & + \frac{\lambda^2\,\alpha_1 \,\&\wedge\, \overline{\alpha_1}\,\overline{A_1}(Q)}{R} - \frac{(-R+\lambda\,Q)\,\overline{A_2}(P)\,\alpha_1 \,\&\wedge\, \overline{\alpha_1}}{PR} \\ & + \frac{\lambda\,\alpha_1 \,\&\wedge\, \overline{\alpha_1}\,\overline{A_2}(Q)}{R} - \frac{\overline{A_1}(R)\,\lambda\,\alpha_1 \,\&\wedge\, \overline{\alpha_1}}{R} - \frac{\overline{A_2}(R)\,\alpha_1 \,\&\wedge\, \overline{\alpha_1}}{R} \end{aligned} \tag{8}$$

```
> domega2A := collect(domega2, alpha[1] &^ conjugate(alpha[1]));
```

$$\begin{aligned} domega2A := & \left(-\frac{\lambda\,(-R+\lambda\,Q)\,\overline{A_1}(P)}{PR} + \frac{\lambda^2\,\overline{A_1}(Q)}{R} - \frac{(-R+\lambda\,Q)\,\overline{A_2}(P)}{PR} \right. \\ & \left. + \frac{\lambda\,\overline{A_2}(Q)}{R} - \frac{\overline{A_1}(R)\,\lambda}{R} - \frac{\overline{A_2}(R)}{R} \right) \alpha_1 \,\&\wedge\, \overline{\alpha_1} - dlambda \,\&\wedge\, \alpha_1 \end{aligned} \tag{9}$$

```
> L:= collect(-(domega2A+(dlambda &^ alpha1))/(alpha1 &^ alpha1bar),
  lambda);
```

$$\begin{aligned} L := & \left(\frac{Q\,\overline{A_1}(P)}{PR} - \frac{\overline{A_1}(Q)}{R} \right) \lambda^2 \\ & + \left(-\frac{\overline{A_1}(P)}{P} - \frac{\overline{A_2}(Q)}{R} + \frac{\overline{A_1}(R)}{R} + \frac{Q\,\overline{A_2}(P)}{PR} \right) \lambda - \frac{\overline{A_2}(P)}{P} + \frac{\overline{A_2}(R)}{R} \end{aligned} \tag{10}$$

```
>
> ##################################################
  #Conjugate of d(omega2bar) to check conjugate of L #
  ##################################################
```

```
domega2bar:=simpform(-dlambdabar &^ alpha1bar - \lambdabar*
dalpha1bar + dalpha2bar);
```

$$domega2bar := -dlambdabar \,\&\wedge\, \overline{\alpha_1} - \frac{A_1(P)\,\alpha_1 \,\&\wedge\, \overline{\alpha_1}}{\lambda P} \tag{11}$$
$$+ \frac{\overline{Q} A_1(P)\,\alpha_1 \,\&\wedge\, \overline{\alpha_1}}{\lambda^2 PR} - \frac{A_1(\overline{Q})\,\alpha_1 \,\&\wedge\, \overline{\alpha_1}}{\lambda^2 R} - \frac{A_2(P)\,\alpha_1 \,\&\wedge\, \overline{\alpha_1}}{P}$$
$$+ \frac{\overline{Q} A_2(P)\,\alpha_1 \,\&\wedge\, \overline{\alpha_1}}{\lambda PR} - \frac{A_2(\overline{Q})\,\alpha_1 \,\&\wedge\, \overline{\alpha_1}}{\lambda R} + \frac{A_1(R)\,\alpha_1 \,\&\wedge\, \overline{\alpha_1}}{\lambda R} + \frac{A_2(R)\,\alpha_1 \,\&\wedge\, \overline{\alpha_1}}{R}$$

```
> domega2barA := collect(domega2bar, alpha1 &^ alpha1bar);
```

$$domega2barA := \left(-\frac{A_1(P)}{\lambda P} + \frac{\overline{Q} A_1(P)}{\lambda^2 PR} - \frac{A_1(\overline{Q})}{\lambda^2 R} - \frac{A_2(P)}{P} + \frac{\overline{Q} A_2(P)}{\lambda PR} - \frac{A_2(\overline{Q})}{\lambda R} \right. \tag{12}$$
$$\left. + \frac{A_1(R)}{\lambda R} + \frac{A_2(R)}{R} \right) \alpha_1 \,\&\wedge\, \overline{\alpha_1} - dlambdabar \,\&\wedge\, \overline{\alpha_1}$$

```
> Lbar := collect((domega2barA+(dlambdabar &^ alpha1bar))/(alpha1 &^
alpha1bar),lambda);
```

$$Lbar := -\frac{A_2(P)}{P} + \frac{A_2(R)}{R} + \frac{-\frac{A_1(P)}{P} + \frac{\overline{Q} A_2(P)}{PR} - \frac{A_2(\overline{Q})}{R} + \frac{A_1(R)}{R}}{\lambda} \tag{13}$$
$$+ \frac{\frac{\overline{Q} A_1(P)}{PR} - \frac{A_1(\overline{Q})}{R}}{\lambda^2}$$

```
>
> ##############################################
  # Find the norm of L ##########################
  # Organise them in terms of degree of lambda #
  ##############################################

  series(expand(L*Lbar),lambda);
```

$$\frac{-\frac{\overline{A_2}(P)\,\overline{Q} A_1(P)}{P^2 R} + \frac{\overline{A_2}(P)\,A_1(\overline{Q})}{PR} - \frac{\overline{A_2}(R)\,A_1(\overline{Q})}{R^2} + \frac{\overline{A_2}(R)\,\overline{Q} A_1(P)}{R^2 P}}{\lambda^2} + \frac{1}{\lambda}\left(- \right. \tag{14}$$
$$\frac{\overline{A_2}(Q)\,\overline{Q} A_1(P)}{R^2 P} + \frac{\overline{A_2}(Q)\,A_1(\overline{Q})}{R^2}$$
$$+ \frac{\overline{A_1}(R)\,\overline{Q} A_1(P)}{R^2 P} - \frac{\overline{A_1}(R)\,A_1(\overline{Q})}{R^2} - \frac{\overline{A_2}(R)\,A_1(P)}{RP} - \frac{\overline{A_2}(P)\,A_1(R)}{PR}$$

$$+\frac{Q\overline{A_2}(P)\,\overline{Q}A_1(P)}{P^2R^2}+\frac{\overline{A_2}(P)\,A_1(P)}{P^2}-\frac{Q\overline{A_2}(P)\,A_1(\overline{Q})}{PR^2}$$
$$+\frac{\overline{A_2}(P)\,A_2(\overline{Q})}{PR}-\frac{\overline{A_2}(P)\,\overline{Q}A_2(P)}{P^2R}-\frac{\overline{A_2}(R)\,A_2(\overline{Q})}{R^2}$$
$$+\frac{\overline{A_2}(R)\,A_1(R)}{R^2}-\frac{\overline{A_1}(P)\,\overline{Q}A_1(P)}{P^2R}+\frac{\overline{A_2}(R)\,\overline{Q}A_2(P)}{R^2P}+\frac{\overline{A_1}(P)\,A_1(\overline{Q})}{PR}\Bigg)+$$
$$\frac{\overline{A_1}(Q)\,A_1(\overline{Q})}{R^2}+\frac{\overline{A_1}(R)\,A_1(R)}{R^2}-\frac{Q\overline{A_2}(P)\,A_2(\overline{Q})}{PR^2}-\frac{\overline{A_2}(P)\,A_2(R)}{PR}$$
$$+\frac{Q\overline{A_2}(P)\,A_1(R)}{PR^2}-\frac{\overline{A_2}(Q)\,\overline{Q}A_2(P)}{R^2P}-\frac{\overline{A_1}(P)\,A_1(R)}{PR}+\frac{Q\overline{A_2}(P)\,\overline{Q}A_2(P)}{P^2R^2}$$
$$+\frac{Q\overline{A_1}(P)\,\overline{Q}A_1(P)}{P^2R^2}-\frac{\overline{A_1}(R)\,A_2(\overline{Q})}{R^2}+\frac{\overline{A_2}(Q)\,A_2(\overline{Q})}{R^2}+\frac{\overline{A_1}(P)\,A_1(P)}{P^2}$$
$$+\frac{\overline{A_2}(Q)\,A_1(P)}{RP}-\frac{\overline{A_1}(P)\,\overline{Q}A_2(P)}{P^2R}-\frac{Q\overline{A_1}(P)\,A_1(\overline{Q})}{PR^2}-\frac{\overline{A_2}(Q)\,A_1(R)}{R^2}-$$
$$\frac{\overline{A_1}(Q)\,\overline{Q}A_1(P)}{R^2P}-\frac{\overline{A_1}(R)\,A_1(P)}{RP}-\frac{\overline{A_2}(R)\,A_2(P)}{RP}+\frac{\overline{A_1}(P)\,A_2(\overline{Q})}{PR}$$
$$+\frac{\overline{A_1}(R)\,\overline{Q}A_2(P)}{R^2P}-\frac{Q\overline{A_2}(P)\,A_1(P)}{P^2R}+\frac{\overline{A_2}(R)\,A_2(R)}{R^2}+\frac{\overline{A_2}(P)\,A_2(P)}{P^2}+\Bigg(-$$
$$\frac{Q\overline{A_2}(P)\,A_2(P)}{P^2R}+\frac{Q\overline{A_1}(P)\,\overline{Q}A_2(P)}{P^2R^2}+\frac{\overline{A_2}(Q)\,A_2(P)}{RP}+\frac{Q\overline{A_2}(P)\,A_2(R)}{PR^2}$$
$$+\frac{\overline{A_1}(Q)\,A_1(P)}{RP}-\frac{\overline{A_1}(Q)\,A_1(R)}{R^2}$$
$$+\frac{Q\overline{A_1}(P)\,A_1(R)}{PR^2}-\frac{\overline{A_1}(Q)\,\overline{Q}A_2(P)}{R^2P}-\frac{\overline{A_1}(R)\,A_2(P)}{RP}-\frac{Q\overline{A_1}(P)\,A_2(\overline{Q})}{PR^2}$$
$$+\frac{\overline{A_1}(P)\,A_2(P)}{P^2}+\frac{\overline{A_1}(Q)\,A_2(\overline{Q})}{R^2}-\frac{\overline{A_1}(P)\,A_2(R)}{PR}-\frac{\overline{A_2}(Q)\,A_2(R)}{R^2}-$$
$$\frac{Q\overline{A_1}(P)\,A_1(P)}{P^2R}+\frac{\overline{A_1}(R)\,A_2(R)}{R^2}\Bigg)\lambda+\Bigg(-\frac{Q\overline{A_1}(P)\,A_2(P)}{P^2R}$$
$$+\frac{Q\overline{A_1}(P)\,A_2(R)}{PR^2}-\frac{\overline{A_1}(Q)\,A_2(R)}{R^2}+\frac{\overline{A_1}(Q)\,A_2(P)}{RP}\Bigg)\lambda^2$$

```
>
> ##############################################
  # Calculation of dL and dLbar for finding A,B,C #
  #in the expression of dtau mod I ###############
  ##############################################
```

```
##############################################
#Tell the software that lambda is a constant #
##############################################

assign(A[1](lambda)=0, A[2](lambda)=0);
A1(lambda);
A2(lambda);
```

$$0$$
$$0 \tag{15}$$

```
>
> ############################################################
  # Calculate the exterior derivative of L, step 1 ###########
  # Only holomorphic part is needed because it will ##########
  # wedge with alpha1bar then taking mod I (Pfaffian system) #
  # The result here will be multiples of alpha1 because ######
  # alpha2 = lambda alpha1 mod I ##############################
  ############################################################

  dLxx:= simpform(A1(L)*alpha1+A2(L)*alpha2);
```

$$dLxx := \frac{A_1(\overline{A_2}(R))\,\alpha_1}{R} + \frac{A_2(\overline{A_2}(R))\,\lambda\,\alpha_1}{R} \tag{16}$$
$$+ \frac{\lambda^2\,\overline{A_2}(P)\,A_2(Q)\,\alpha_1}{PR} - \frac{\overline{A_2}(R)\,A_1(R)\,\alpha_1}{R^2} - \frac{(-R+\lambda\,Q)\,\overline{A_2}(P)\,A_1(P)\,\alpha_1}{P^2\,R}$$
$$+ \frac{\lambda^2\,\overline{A_2}(Q)\,A_2(R)\,\alpha_1}{R^2} - \frac{\lambda^2\,Q\,\overline{A_2}(P)\,A_2(R)\,\alpha_1}{PR^2}$$
$$+ \frac{\lambda^3\,\overline{A_1}(P)\,A_2(Q)\,\alpha_1}{PR} - \frac{\lambda^3\,A_2(\overline{A_1}(Q))\,\alpha_1}{R} + \frac{(-R+\lambda\,Q)\,A_1(\overline{A_2}(P))\,\alpha_1}{PR}$$
$$+ \frac{\lambda\,(-R+\lambda\,Q)\,A_2(\overline{A_2}(P))\,\alpha_1}{PR} + \frac{\lambda^3\,\overline{A_1}(Q)\,A_2(R)\,\alpha_1}{R^2}$$
$$+ \frac{\lambda^2\,(-R+\lambda\,Q)\,A_2(\overline{A_1}(P))\,\alpha_1}{PR} - \frac{\lambda^3\,Q\,\overline{A_1}(P)\,A_2(R)\,\alpha_1}{PR^2} - \frac{\overline{A_2}(R)\,A_2(R)\,\lambda\,\alpha_1}{R^2} -$$
$$\frac{\lambda^2\,A_2(\overline{A_2}(Q))\,\alpha_1}{R} + \frac{\lambda^2\,A_2(\overline{A_1}(R))\,\alpha_1}{R} - \frac{\lambda^2\,(-R+\lambda\,Q)\,\overline{A_1}(P)\,A_2(P)\,\alpha_1}{P^2\,R} -$$
$$\frac{\lambda\,(-R+\lambda\,Q)\,\overline{A_2}(P)\,A_2(P)\,\alpha_1}{P^2\,R}$$
$$+ \frac{\lambda\,(-R+\lambda\,Q)\,A_1(\overline{A_1}(P))\,\alpha_1}{PR} - \frac{\lambda^2\,A_1(\overline{A_1}(Q))\,\alpha_1}{R} - \frac{\lambda\,A_1(\overline{A_2}(Q))\,\alpha_1}{R}$$

$$+\frac{\lambda A_1(\overline{A_1}(R))\,\alpha_1}{R}-\frac{\overline{A_1}(R)\,\lambda^2 A_2(R)\,\alpha_1}{R^2}-\frac{\lambda(-R+\lambda Q)\,\overline{A_1}(P)\,A_1(P)\,\alpha_1}{P^2 R}$$
$$+\frac{\lambda^2\overline{A_1}(P)\,A_1(Q)\,\alpha_1}{PR}-\frac{\lambda^2 Q\overline{A_1}(P)\,A_1(R)\,\alpha_1}{PR^2}+\frac{\lambda^2\overline{A_1}(Q)\,A_1(R)\,\alpha_1}{R^2}$$
$$+\frac{\lambda\overline{A_2}(Q)\,A_1(R)\,\alpha_1}{R^2}-\frac{\lambda\overline{A_1}(R)\,A_1(R)\,\alpha_1}{R^2}$$
$$+\frac{\lambda\overline{A_2}(P)\,A_1(Q)\,\alpha_1}{PR}-\frac{\lambda Q\overline{A_2}(P)\,A_1(R)\,\alpha_1}{PR^2}$$

```
>
> ######################################
  # Calculate one of the terms of dtau #
  # (1/lambda)*dL * alpha1bar ##########
  # And gather only the coefficients ###
  ######################################

  dL := collect(expand((1/lambda)*dLxx), alpha1);
```

$$dL := \left(-\frac{A_1(\overline{A_2}(P))}{\lambda P}-\frac{\lambda^2 A_2(\overline{A_1}(Q))}{R}-\frac{\lambda A_2(\overline{A_1}(P))}{P}+\frac{\lambda\overline{A_1}(P)\,A_1(Q)}{PR}\right. \tag{17}$$
$$+\frac{\lambda Q A_1(\overline{A_1}(P))}{PR}-\frac{\lambda Q\overline{A_1}(P)\,A_1(P)}{P^2 R}-\frac{\lambda Q\overline{A_1}(P)\,A_1(R)}{PR^2}$$
$$+\frac{A_1(\overline{A_2}(R))}{\lambda R}-\frac{\lambda A_1(\overline{A_1}(Q))}{R}+\frac{\overline{A_2}(P)\,A_1(P)}{\lambda P^2}$$
$$+\frac{\lambda\overline{A_1}(Q)\,A_1(R)}{R^2}-\frac{\overline{A_2}(R)\,A_1(R)}{\lambda R^2}+\frac{\lambda^2\overline{A_1}(Q)\,A_2(R)}{R^2}+\frac{\overline{A_1}(P)\,A_1(P)}{P^2}$$
$$+\frac{\lambda\overline{A_1}(P)\,A_2(P)}{P^2}+\frac{\overline{A_2}(Q)\,A_1(R)}{R^2}$$
$$+\frac{\lambda\overline{A_2}(Q)\,A_2(R)}{R^2}-\frac{\overline{A_1}(R)\,A_1(R)}{R^2}-\frac{\overline{A_1}(R)\,\lambda A_2(R)}{R^2}+\frac{Q A_1(\overline{A_2}(P))}{PR}$$
$$+\frac{\overline{A_2}(P)\,A_2(P)}{P^2}-\frac{\overline{A_2}(R)\,A_2(R)}{R^2}-\frac{\lambda^2 Q\overline{A_1}(P)\,A_2(P)}{P^2 R}-\frac{\lambda^2 Q\overline{A_1}(P)\,A_2(R)}{PR^2}-$$
$$\frac{Q\overline{A_2}(P)\,A_1(P)}{P^2 R}-\frac{\lambda Q\overline{A_2}(P)\,A_2(P)}{P^2 R}-\frac{Q\overline{A_2}(P)\,A_1(R)}{PR^2}-\frac{\lambda Q\overline{A_2}(P)\,A_2(R)}{PR^2}-$$
$$\left.\frac{A_1(\overline{A_1}(P))}{P}-\frac{A_1(\overline{A_2}(Q))}{R}+\frac{A_1(\overline{A_1}(R))}{R}+\frac{\overline{A_2}(P)\,A_1(Q)}{PR}+\frac{\lambda^2\overline{A_1}(P)\,A_2(Q)}{PR}\right.$$

$$+\frac{\lambda^2 Q A_2(\overline{A_1}(P))}{PR}-\frac{A_2(\overline{A_2}(P))}{P}+\frac{A_2(\overline{A_2}(R))}{R}-\frac{\lambda A_2(\overline{A_2}(Q))}{R}$$
$$+\frac{\lambda A_2(\overline{A_1}(R))}{R}+\frac{\lambda \overline{A_2}(P) A_2(Q)}{PR}+\frac{\lambda Q A_2(\overline{A_2}(P))}{PR}\Bigg)\alpha_1$$

```
> Y1 := collect(dL/alpha1,lambda);
```

$$Y1 := \left(\frac{\overline{A_1}(P) A_2(Q)}{PR}-\frac{Q\overline{A_1}(P) A_2(P)}{P^2 R}-\frac{Q\overline{A_1}(P) A_2(R)}{PR^2}+\frac{Q A_2(\overline{A_1}(P))}{PR}\right.$$
$$\left.+\frac{\overline{A_1}(Q) A_2(R)}{R^2}-\frac{A_2(\overline{A_1}(Q))}{R}\right)\lambda^2+\left(-\frac{A_2(\overline{A_2}(Q))}{R}+\frac{\overline{A_1}(P) A_1(Q)}{PR}\right.$$
$$+\frac{A_2(\overline{A_1}(R))}{R}-\frac{\overline{A_1}(R) A_2(R)}{R^2}-\frac{Q\overline{A_2}(P) A_2(R)}{PR^2}$$
$$+\frac{\overline{A_1}(P) A_2(P)}{P^2}-\frac{A_1(\overline{A_1}(Q))}{R}-\frac{Q\overline{A_1}(P) A_1(P)}{P^2 R}-\frac{Q\overline{A_1}(P) A_1(R)}{PR^2}$$
$$+\frac{\overline{A_1}(Q) A_1(R)}{R^2}+\frac{Q A_1(\overline{A_1}(P))}{PR}-\frac{A_2(\overline{A_1}(P))}{P}+\frac{Q A_2(\overline{A_2}(P))}{PR}$$
$$\left.+\frac{\overline{A_2}(Q) A_2(R)}{R^2}-\frac{Q\overline{A_2}(P) A_2(P)}{P^2 R}+\frac{\overline{A_2}(P) A_2(Q)}{PR}\right)\lambda-\frac{\overline{A_2}(R) A_2(R)}{R^2}$$
$$+\frac{\overline{A_2}(Q) A_1(R)}{R^2}+\frac{A_1(\overline{A_1}(R))}{R}+\frac{Q A_1(\overline{A_2}(P))}{PR}$$
$$+\frac{A_2(\overline{A_2}(R))}{R}-\frac{A_2(\overline{A_2}(P))}{P}-\frac{\overline{A_1}(R) A_1(R)}{R^2}-\frac{Q\overline{A_2}(P) A_1(P)}{P^2 R}-\frac{A_1(\overline{A_2}(Q))}{R}$$
$$+\frac{\overline{A_2}(P) A_2(P)}{P^2}-\frac{Q\overline{A_2}(P) A_1(R)}{PR^2}+\frac{\overline{A_1}(P) A_1(P)}{P^2}-\frac{A_1(\overline{A_1}(P))}{P}$$
$$+\frac{\overline{A_2}(P) A_1(Q)}{PR}+$$
$$\frac{-\frac{A_1(\overline{A_2}(P))}{P}+\frac{A_1(\overline{A_2}(R))}{R}-\frac{\overline{A_2}(R) A_1(R)}{R^2}+\frac{\overline{A_2}(P) A_1(P)}{P^2}}{\lambda} \tag{18}$$

```
>
> ######################################
  # Calculate one of the terms of dtau #
  # lambda*dLbar * alpha1 ##############
  # And gather only the coefficients ###
  ######################################
>
> ##########################################
  # Assign doesnt work well with conjugates###
```

```
# Introduce an auxiliary vector field Z1 Z2#
# to replace A1bar and A2bar for the moment#
##########################################
assign(Z1(lambda)=0);
assign(Z2(lambda)=0);
Z1(lambda);
Z2(lambda);
```

$$0$$
$$0 \tag{19}$$

```
>
> ########################
  # Calculation of dLbar #
  ########################

  dLbarxxx:=simpform(Z1(Lbar)*alpha1bar+Z2(Lbar)*alpha2bar);
```

$$dLbarxxx := -\frac{Z_1(A_2(P))\,\overline{\alpha_1}}{P} - \frac{\overline{Q}A_1(P)\,Z_2(P)\,\overline{\alpha_1}}{\lambda^3 P^2 R} + \frac{\overline{Q}Z_2(A_1(P))\,\overline{\alpha_1}}{\lambda^3 PR} \tag{20}$$
$$+ \frac{A_2(P)\,Z_1(P)\,\overline{\alpha_1}}{P^2} - \frac{A_2(R)\,Z_1(R)\,\overline{\alpha_1}}{R^2} + \frac{\overline{Q}Z_1(A_1(P))\,\overline{\alpha_1}}{\lambda^2 PR} - \frac{\overline{Q}A_2(P)\,Z_2(P)\,\overline{\alpha_1}}{\lambda^2 P^2 R}$$
$$+ \frac{A_1(P)\,Z_1(\overline{Q})\,\overline{\alpha_1}}{\lambda^2 PR} + \frac{A_1(P)\,Z_2(\overline{Q})\,\overline{\alpha_1}}{\lambda^3 PR} + \frac{A_1(P)\,Z_2(P)\,\overline{\alpha_1}}{\lambda^2 P^2} + \frac{A_2(\overline{Q})\,Z_2(R)\,\overline{\alpha_1}}{\lambda^2 R^2}$$
$$+ \frac{Z_2(A_1(R))\,\overline{\alpha_1}}{\lambda^2 R} - \frac{\overline{Q}A_2(P)\,Z_1(P)\,\overline{\alpha_1}}{\lambda P^2 R} + \frac{A_2(P)\,Z_2(\overline{Q})\,\overline{\alpha_1}}{\lambda^2 PR} - \frac{\overline{Q}A_2(P)\,Z_2(R)\,\overline{\alpha_1}}{\lambda^2 PR^2}$$
$$+ \frac{\overline{Q}Z_2(A_2(P))\,\overline{\alpha_1}}{\lambda^2 PR} - \frac{A_2(R)\,Z_2(R)\,\overline{\alpha_1}}{R^2\lambda} - \frac{\overline{Q}A_2(P)\,Z_1(R)\,\overline{\alpha_1}}{\lambda PR^2}$$
$$+ \frac{\overline{Q}Z_1(A_2(P))\,\overline{\alpha_1}}{\lambda PR} - \frac{Z_2(A_1(\overline{Q}))\,\overline{\alpha_1}}{\lambda^3 R} - \frac{\overline{Q}A_1(P)\,Z_2(R)\,\overline{\alpha_1}}{\lambda^3 PR^2}$$
$$+ \frac{A_2(\overline{Q})\,Z_1(R)\,\overline{\alpha_1}}{\lambda R^2} - \frac{A_1(R)\,Z_2(R)\,\overline{\alpha_1}}{\lambda^2 R^2} - \frac{Z_1(A_1(P))\,\overline{\alpha_1}}{\lambda P} - \frac{Z_1(A_2(\overline{Q}))\,\overline{\alpha_1}}{\lambda R}$$
$$+ \frac{Z_1(A_1(R))\,\overline{\alpha_1}}{\lambda R} - \frac{Z_1(A_1(\overline{Q}))\,\overline{\alpha_1}}{\lambda^2 R} + \frac{A_2(P)\,Z_2(P)\,\overline{\alpha_1}}{P^2\lambda}$$
$$+ \frac{A_1(P)\,Z_1(P)\,\overline{\alpha_1}}{\lambda P^2} - \frac{A_1(R)\,Z_1(R)\,\overline{\alpha_1}}{\lambda R^2} - \frac{\overline{Q}A_1(P)\,Z_1(P)\,\overline{\alpha_1}}{\lambda^2 P^2 R} - \frac{Z_2(A_2(P))\,\overline{\alpha_1}}{P\lambda}$$
$$+ \frac{Z_2(A_2(R))\,\overline{\alpha_1}}{R\lambda} - \frac{Z_2(A_1(P))\,\overline{\alpha_1}}{\lambda^2 P} - \frac{Z_2(A_2(\overline{Q}))\,\overline{\alpha_1}}{\lambda^2 R} + \frac{A_1(\overline{Q})\,Z_1(R)\,\overline{\alpha_1}}{\lambda^2 R^2}$$
$$+ \frac{A_2(P)\,Z_1(\overline{Q})\,\overline{\alpha_1}}{\lambda PR} + \frac{A_1(\overline{Q})\,Z_2(R)\,\overline{\alpha_1}}{\lambda^3 R^2} - \frac{\overline{Q}A_1(P)\,Z_1(R)\,\overline{\alpha_1}}{\lambda^2 PR^2} + \frac{Z_1(A_2(R))\,\overline{\alpha_1}}{R}$$

```
> ##########################################
  # Simplify to collect the ################
  # coefficients of lambda*dLbar*alpha1bar ##
  ##########################################

  dLbar:=-collect(expand(lambda*dLbarxxx),alpha1bar);
```

$$dLbar := -\left(\frac{A_1(P)\,Z_1(P)}{P^2} + \frac{A_2(\overline{Q})\,Z_1(R)}{R^2} + \frac{A_1(P)\,Z_2(P)}{\lambda P^2} \right. \tag{21}$$
$$+ \frac{A_2(\overline{Q})\,Z_2(R)}{\lambda R^2} - \frac{A_1(R)\,Z_2(R)}{\lambda R^2}$$
$$+ \frac{A_1(\overline{Q})\,Z_2(R)}{\lambda^2 R^2} - \frac{\overline{Q}\,A_1(P)\,Z_1(P)}{\lambda P^2 R} - \frac{\overline{Q}\,A_1(P)\,Z_1(R)}{\lambda P R^2} + \frac{\overline{Q}\,Z_1(A_1(P))}{\lambda P R}$$
$$+ \frac{A_1(P)\,Z_1(\overline{Q})}{\lambda P R} + \frac{A_2(P)\,Z_2(P)}{P^2}$$
$$+ \frac{\lambda\,Z_1(A_2(R))}{R} - \frac{\overline{Q}\,A_1(P)\,Z_2(P)}{\lambda^2 P^2 R} - \frac{\overline{Q}\,A_1(P)\,Z_2(R)}{\lambda^2 P R^2}$$
$$+ \frac{\overline{Q}\,Z_2(A_1(P))}{\lambda^2 P R} - \frac{A_1(R)\,Z_1(R)}{R^2} - \frac{A_2(R)\,Z_2(R)}{R^2}$$
$$+ \frac{A_2(P)\,Z_1(\overline{Q})}{P R} - \frac{\overline{Q}\,A_2(P)\,Z_1(P)}{P^2 R} - \frac{\overline{Q}\,A_2(P)\,Z_1(R)}{P R^2} + \frac{\overline{Q}\,Z_1(A_2(P))}{P R}$$
$$+ \frac{Z_2(A_1(R))}{\lambda R} - \frac{Z_2(A_2(\overline{Q}))}{\lambda R} - \frac{Z_2(A_1(P))}{\lambda P} - \frac{\lambda\,Z_1(A_2(P))}{P} - \frac{Z_1(A_1(\overline{Q}))}{\lambda R}$$
$$+ \frac{A_1(\overline{Q})\,Z_1(R)}{\lambda R^2} + \frac{\lambda\,A_2(P)\,Z_1(P)}{P^2} - \frac{\lambda\,A_2(R)\,Z_1(R)}{R^2} - \frac{Z_2(A_1(\overline{Q}))}{\lambda^2 R} -$$
$$\frac{Z_1(A_1(P))}{P} - \frac{Z_1(A_2(\overline{Q}))}{R} + \frac{Z_1(A_1(R))}{R}$$
$$+ \frac{A_2(P)\,Z_2(\overline{Q})}{\lambda P R} - \frac{\overline{Q}\,A_2(P)\,Z_2(P)}{\lambda P^2 R} - \frac{\overline{Q}\,A_2(P)\,Z_2(R)}{\lambda P R^2}$$
$$\left. + \frac{\overline{Q}\,Z_2(A_2(P))}{\lambda P R} - \frac{Z_2(A_2(P))}{P} + \frac{Z_2(A_2(R))}{R} + \frac{A_1(P)\,Z_2(\overline{Q})}{\lambda^2 P R} \right) \overline{\alpha_1}$$

```
> assign(Z[1]=conjugate(A[1]));
  assign(Z[2]=conjugate(A[2]));
  Y2:=collect(dLbar/alpha1bar,lambda);
```

$$Y2 := \left(\frac{\overline{A_1}(A_2(P))}{P} - \frac{\overline{A_1}(A_2(R))}{R} + \frac{\overline{A_1}(R)\,A_2(R)}{R^2} - \frac{\overline{A_1}(P)\,A_2(P)}{P^2} \right) \lambda \tag{22}$$

$$+\frac{\overline{A_2}(R)\,A_2(R)}{R^2}+\frac{\overline{A_2}(A_2(P))}{P}-\frac{\overline{A_2}(A_2(R))}{R}-\frac{\overline{A_2}(P)\,A_2(P)}{P^2}-\frac{\overline{A_1}(P)\,A_1(P)}{P^2}$$
$$+\frac{\overline{A_1}(R)\,A_1(R)}{R^2}-\frac{\overline{A_1}(A_1(R))}{R}-\frac{\overline{A_1}(R)\,A_2(\overline{Q})}{R^2}-\frac{\overline{Q}\,\overline{A_1}(A_2(P))}{PR}+\frac{\overline{A_1}(A_1(P))}{P}$$
$$+\frac{\overline{A_1}(A_2(\overline{Q}))}{R}-\frac{A_2(P)\,\overline{A_1}(\overline{Q})}{PR}+\frac{\overline{A_1}(P)\,\overline{Q}A_2(P)}{P^2R}+\frac{\overline{A_1}(R)\,\overline{Q}A_2(P)}{R^2P}$$
$$+\frac{1}{\lambda}\left(-\frac{\overline{Q}\,\overline{A_2}(A_2(P))}{PR}+\frac{\overline{A_2}(P)\,\overline{Q}A_2(P)}{P^2R}-\frac{\overline{A_2}(A_1(R))}{R}+\frac{\overline{A_1}(A_1(\overline{Q}))}{R}\right.$$
$$+\frac{\overline{A_2}(A_1(P))}{P}+\frac{\overline{A_1}(P)\,\overline{Q}A_1(P)}{P^2R}+\frac{\overline{A_1}(R)\,\overline{Q}A_1(P)}{R^2P}$$
$$+\frac{\overline{A_2}(A_2(\overline{Q}))}{R}-\frac{A_1(P)\,\overline{A_1}(\overline{Q})}{PR}-\frac{\overline{Q}\,\overline{A_1}(A_1(P))}{PR}-\frac{\overline{A_2}(R)\,A_2(\overline{Q})}{R^2}$$
$$+\frac{\overline{A_2}(R)\,\overline{Q}A_2(P)}{R^2P}+\frac{\overline{A_2}(R)\,A_1(R)}{R^2}-\frac{\overline{A_1}(R)\,A_1(\overline{Q})}{R^2}-\frac{\overline{A_2}(P)\,A_1(P)}{P^2}-$$
$$\left.\frac{A_2(P)\,\overline{A_2}(\overline{Q})}{PR}\right)+\frac{1}{\lambda^2}\left(-\frac{\overline{Q}\,\overline{A_2}(A_1(P))}{PR}-\frac{A_1(P)\,\overline{A_2}(\overline{Q})}{PR}+\frac{\overline{A_2}(P)\,\overline{Q}A_1(P)}{P^2R}\right.$$
$$\left.+\frac{\overline{A_2}(R)\,\overline{Q}A_1(P)}{R^2P}-\frac{\overline{A_2}(R)\,A_1(\overline{Q})}{R^2}+\frac{\overline{A_2}(A_1(\overline{Q}))}{R}\right)$$

```
>
> ##########################################
  ## Computation of lambdabar*L*dalpha1bar  #
  ## in dtau mod , step 3 ###################
  ##########################################
```

```
  dalpha1bar;
```

$$\left(\frac{A_1(P)}{P}-\frac{\overline{Q}A_1(P)}{\lambda PR}+\frac{A_1(\overline{Q})}{\lambda R}+\frac{A_2(P)\,\lambda}{P}-\frac{\overline{Q}A_2(P)}{PR}+\frac{A_2(\overline{Q})}{R}\right)\alpha_1\,\&\wedge\,\overline{\alpha_1} \tag{23}$$

```
> Y3A:=collect(expand(lambdabar*L*dalpha1bar), alpha1 &^ alpha1bar);
```

$$Y3A:=\left(\frac{\overline{Q}A_1(P)\,\overline{A_2}(P)}{\lambda^2P^2R}-\frac{\overline{Q}A_1(P)\,\overline{A_2}(R)}{\lambda^2PR^2}\right. \tag{24}$$
$$+\frac{\lambda\,Q\,\overline{A_1}(P)\,A_1(P)}{P^2R}-\frac{\lambda\,\overline{A_1}(Q)\,A_2(\overline{Q})}{R^2}-\frac{\overline{A_2}(Q)\,A_1(\overline{Q})}{\lambda R^2}$$
$$\left.+\frac{\overline{Q}A_2(P)\,\overline{A_2}(P)}{\lambda P^2R}-\frac{\overline{Q}A_2(P)\,\overline{A_2}(R)}{\lambda PR^2}+\frac{\overline{Q}A_1(P)\,\overline{A_1}(P)}{\lambda P^2R}-\frac{\overline{Q}A_1(P)\,\overline{A_1}(R)}{\lambda PR^2}\right.$$

$$+\frac{A_2(\overline{Q})\,\overline{A_2}(R)}{\lambda R^2}+\frac{A_1(\overline{Q})\,\overline{A_2}(R)}{\lambda^2 R^2}$$
$$+\frac{A_1(\overline{Q})\,\overline{A_1}(R)}{\lambda R^2}-\frac{\overline{A_2}(P)\,A_1(P)}{\lambda P^2}-\frac{\overline{A_1}(Q)\,A_1(\overline{Q})}{R^2}-\frac{\overline{A_1}(P)\,A_1(P)}{P^2}-$$
$$\frac{\lambda\,\overline{A_1}(P)\,A_2(P)}{P^2}-\frac{\overline{A_2}(Q)\,A_2(\overline{Q})}{R^2}+\frac{\overline{A_1}(R)\,A_2(\overline{Q})}{R^2}$$
$$+\frac{\overline{A_2}(R)\,A_2(P)}{R\,P}-\frac{\overline{A_2}(P)\,A_2(P)}{P^2}-\frac{Q\overline{A_1}(P)\,\overline{Q}A_1(P)}{P^2 R^2}+\frac{Q\overline{A_1}(P)\,A_1(\overline{Q})}{PR^2}$$
$$+\frac{\lambda^2\,Q\overline{A_1}(P)\,A_2(P)}{P^2 R}+\frac{\overline{A_1}(Q)\,\overline{Q}A_1(P)}{R^2 P}$$
$$+\frac{\overline{A_1}(P)\,\overline{Q}A_2(P)}{P^2 R}-\frac{\overline{A_1}(P)\,A_2(\overline{Q})}{PR}-\frac{\overline{A_2}(Q)\,A_1(P)}{R\,P}+\frac{\overline{A_2}(Q)\,\overline{Q}A_2(P)}{R^2 P}$$
$$+\frac{\overline{A_1}(R)\,A_1(P)}{R\,P}-\frac{\overline{A_1}(R)\,\overline{Q}A_2(P)}{R^2 P}+\frac{Q\overline{A_2}(P)\,A_1(P)}{P^2 R}$$
$$+\frac{\lambda\,Q\overline{A_2}(P)\,A_2(P)}{P^2 R}-\frac{Q\overline{A_2}(P)\,\overline{Q}A_2(P)}{P^2 R^2}+\frac{Q\overline{A_2}(P)\,A_2(\overline{Q})}{PR^2}+\frac{\overline{A_2}(Q)\,\overline{Q}A_1(P)}{\lambda R^2 P}$$
$$+\frac{\overline{A_2}(R)\,A_1(P)}{\lambda R\,P}+\frac{\lambda\,Q\overline{A_1}(P)\,A_2(\overline{Q})}{PR^2}+\frac{Q\overline{A_2}(P)\,A_1(\overline{Q})}{\lambda PR^2}+\frac{\lambda\,\overline{A_1}(R)\,A_2(P)}{R\,P}$$
$$+\frac{\lambda\,\overline{A_1}(Q)\,\overline{Q}A_2(P)}{R^2 P}-\frac{\lambda\,Q\overline{A_1}(P)\,\overline{Q}A_2(P)}{P^2 R^2}-\frac{Q\overline{A_2}(P)\,\overline{Q}A_1(P)}{\lambda P^2 R^2}-$$
$$\frac{\lambda\,\overline{A_2}(Q)\,A_2(P)}{R\,P}-\frac{\overline{A_1}(P)\,A_1(\overline{Q})}{\lambda PR}-\frac{\overline{A_2}(P)\,A_2(\overline{Q})}{\lambda PR}-\frac{\lambda\,\overline{A_1}(Q)\,A_1(P)}{R\,P}-$$
$$\left.\frac{\lambda^2\,\overline{A_1}(Q)\,A_2(P)}{R\,P}-\frac{\overline{A_2}(P)\,A_1(\overline{Q})}{\lambda^2 PR}\right)\alpha_1\,\&\wedge\,\overline{\alpha_1}$$

```
> Y3:=collect(Y3A/(alpha1 &^ alpha1bar),lambda);
```

$$Y3:=\left(-\frac{\overline{A_1}(Q)\,A_2(P)}{R\,P}+\frac{Q\overline{A_1}(P)\,A_2(P)}{P^2 R}\right)\lambda^2 \tag{25}$$
$$+\left(\frac{Q\overline{A_2}(P)\,A_2(P)}{P^2 R}+\frac{\overline{A_1}(R)\,A_2(P)}{R\,P}+\frac{\overline{A_1}(Q)\,\overline{Q}A_2(P)}{R^2 P}-\frac{\overline{A_1}(Q)\,A_2(\overline{Q})}{R^2}\right.$$
$$+\frac{Q\overline{A_1}(P)\,A_2(\overline{Q})}{PR^2}-\frac{\overline{A_1}(P)\,A_2(P)}{P^2}-\frac{\overline{A_2}(Q)\,A_2(P)}{R\,P}$$
$$\left.+\frac{Q\overline{A_1}(P)\,A_1(P)}{P^2 R}-\frac{\overline{A_1}(Q)\,A_1(P)}{R\,P}-\frac{Q\overline{A_1}(P)\,\overline{Q}A_2(P)}{P^2 R^2}\right)\lambda-\frac{\overline{A_2}(Q)\,A_1(P)}{R\,P}-$$

$$\frac{\overline{A_1}(R)\,\overline{Q}A_2(P)}{R^2\,P}-\frac{Q\overline{A_2}(P)\,\overline{Q}A_2(P)}{P^2\,R^2}-\frac{\overline{A_2}(Q)\,A_2(\overline{Q})}{R^2}+\frac{\overline{A_1}(Q)\,\overline{Q}A_1(P)}{R^2\,P}$$
$$+\frac{\overline{A_2}(R)\,A_2(P)}{R\,P}+\frac{\overline{A_2}(Q)\,\overline{Q}A_2(P)}{R^2\,P}$$
$$+\frac{\overline{A_1}(R)\,A_1(P)}{R\,P}-\frac{\overline{A_1}(P)\,A_2(\overline{Q})}{PR}-\frac{\overline{A_1}(Q)\,A_1(\overline{Q})}{R^2}$$
$$+\frac{\overline{A_1}(R)\,A_2(\overline{Q})}{R^2}-\frac{\overline{A_2}(P)\,A_2(P)}{P^2}+\frac{\overline{A_1}(P)\,\overline{Q}A_2(P)}{P^2\,R}+\frac{Q\overline{A_1}(P)\,A_1(\overline{Q})}{PR^2}$$
$$+\frac{Q\overline{A_2}(P)\,A_1(P)}{P^2\,R}+\frac{Q\overline{A_2}(P)\,A_2(\overline{Q})}{PR^2}-\frac{Q\overline{A_1}(P)\,\overline{Q}A_1(P)}{P^2\,R^2}-\frac{\overline{A_1}(P)\,A_1(P)}{P^2}$$
$$+\frac{1}{\lambda}\left(-\frac{Q\overline{A_2}(P)\,\overline{Q}A_1(P)}{P^2\,R^2}-\frac{\overline{A_2}(R)\,\overline{Q}A_2(P)}{R^2\,P}+\frac{\overline{A_2}(P)\,\overline{Q}A_2(P)}{P^2\,R}-\frac{\overline{A_2}(P)\,A_1(P)}{P^2}\right.$$
$$+\frac{\overline{A_1}(P)\,\overline{Q}A_1(P)}{P^2\,R}-\frac{\overline{A_1}(R)\,\overline{Q}A_1(P)}{R^2\,P}-\frac{\overline{A_2}(P)\,A_2(\overline{Q})}{PR}$$
$$+\frac{\overline{A_2}(R)\,A_2(\overline{Q})}{R^2}-\frac{\overline{A_2}(Q)\,A_1(\overline{Q})}{R^2}+\frac{\overline{A_1}(R)\,A_1(\overline{Q})}{R^2}+\frac{\overline{A_2}(Q)\,\overline{Q}A_1(P)}{R^2\,P}$$
$$\left.+\frac{\overline{A_2}(R)\,A_1(P)}{R\,P}-\frac{\overline{A_1}(P)\,A_1(\overline{Q})}{PR}+\frac{Q\overline{A_2}(P)\,A_1(\overline{Q})}{PR^2}\right)+$$
$$\frac{\frac{\overline{A_2}(P)\,\overline{Q}A_1(P)}{P^2\,R}-\frac{\overline{A_2}(R)\,\overline{Q}A_1(P)}{R^2\,P}-\frac{\overline{A_2}(P)\,A_1(\overline{Q})}{PR}+\frac{\overline{A_2}(R)\,A_1(\overline{Q})}{R^2}}{\lambda^2}$$

>

```
> ########################################
  ## Computation of lambda*Lbar*dalpha1  #
  ## in dtau mod , step 4 ################
  ########################################

  Y4A:=collect(expand(lambda*Lbar*dalpha1),alpha1 &^ alpha1bar);
```

$$Y4A:=\left(-\frac{\overline{Q}A_1(P)\,\overline{A_2}(P)}{\lambda^2\,P^2\,R}-\frac{\lambda\,Q\overline{A_1}(P)\,A_1(P)}{P^2\,R}+\frac{\lambda\,Q\overline{A_1}(P)\,A_1(R)}{PR^2}\right. \tag{26}$$
$$+\frac{\lambda\,\overline{A_1}(Q)\,A_2(\overline{Q})}{R^2}+\frac{\overline{A_2}(Q)\,A_1(\overline{Q})}{\lambda\,R^2}-\frac{\overline{Q}A_2(P)\,\overline{A_2}(P)}{\lambda\,P^2\,R}-\frac{\overline{Q}A_1(P)\,\overline{A_1}(P)}{\lambda\,P^2\,R}$$
$$+\frac{\overline{A_2}(P)\,A_1(P)}{\lambda\,P^2}-\frac{\lambda\,\overline{A_1}(Q)\,A_1(R)}{R^2}+\frac{\overline{A_1}(Q)\,A_1(\overline{Q})}{R^2}-\frac{\lambda^2\,\overline{A_1}(Q)\,A_2(R)}{R^2}$$
$$\left.+\frac{\overline{A_1}(P)\,A_1(P)}{P^2}+\frac{\lambda\,\overline{A_1}(P)\,A_2(P)}{P^2}\right.$$

$$+\frac{\overline{A_2}(Q)\,A_2(\overline{Q})}{R^2}-\frac{\overline{A_2}(Q)\,A_1(R)}{R^2}-\frac{\lambda\,\overline{A_2}(Q)\,A_2(R)}{R^2}-\frac{\overline{A_2}(P)\,A_2(R)}{PR}$$
$$+\frac{\overline{A_2}(P)\,A_2(P)}{P^2}+\frac{Q\overline{A_1}(P)\,\overline{Q}A_1(P)}{P^2R^2}-\frac{Q\overline{A_1}(P)\,A_1(\overline{Q})}{PR^2}-\frac{\lambda^2\,Q\overline{A_1}(P)\,A_2(P)}{P^2R}$$
$$+\frac{\lambda^2\,Q\overline{A_1}(P)\,A_2(R)}{PR^2}-\frac{\overline{A_1}(Q)\,\overline{Q}A_1(P)}{R^2P}-\frac{\overline{A_1}(P)\,\overline{Q}A_2(P)}{P^2R}$$
$$+\frac{\overline{A_1}(P)\,A_2(\overline{Q})}{PR}-\frac{\overline{A_1}(P)\,A_1(R)}{PR}$$
$$+\frac{\overline{A_2}(Q)\,A_1(P)}{RP}-\frac{\overline{A_2}(Q)\,\overline{Q}A_2(P)}{R^2P}-\frac{Q\overline{A_2}(P)\,A_1(P)}{P^2R}-\frac{\lambda\,Q\overline{A_2}(P)\,A_2(P)}{P^2R}$$
$$+\frac{Q\overline{A_2}(P)\,\overline{Q}A_2(P)}{P^2R^2}-\frac{Q\overline{A_2}(P)\,A_2(\overline{Q})}{PR^2}+\frac{Q\overline{A_2}(P)\,A_1(R)}{PR^2}$$
$$+\frac{\lambda\,Q\overline{A_2}(P)\,A_2(R)}{PR^2}-\frac{\overline{A_2}(Q)\,\overline{Q}A_1(P)}{\lambda R^2P}-\frac{\lambda\,Q\overline{A_1}(P)\,A_2(\overline{Q})}{PR^2}-\frac{Q\overline{A_2}(P)\,A_1(\overline{Q})}{\lambda PR^2}-$$
$$\frac{\lambda\,\overline{A_1}(Q)\,\overline{Q}A_2(P)}{R^2P}+\frac{\lambda\,Q\overline{A_1}(P)\,\overline{Q}A_2(P)}{P^2R^2}+\frac{Q\overline{A_2}(P)\,\overline{Q}A_1(P)}{\lambda P^2R^2}$$
$$+\frac{\lambda\,\overline{A_2}(Q)\,A_2(P)}{RP}+\frac{\overline{A_1}(P)\,A_1(\overline{Q})}{\lambda PR}+\frac{\overline{A_2}(P)\,A_2(\overline{Q})}{\lambda PR}+\frac{\lambda\,\overline{A_1}(Q)\,A_1(P)}{RP}$$
$$+\frac{\lambda^2\,\overline{A_1}(Q)\,A_2(P)}{RP}+\frac{\overline{A_2}(P)\,A_1(\overline{Q})}{\lambda^2PR}-\frac{\lambda\,\overline{A_1}(P)\,A_2(R)}{PR}-\frac{A_1(R)\,\overline{A_2}(P)}{\lambda RP}\Bigg)$$
$$\alpha_1\,\&\wedge\,\overline{\alpha_1}$$

```
> Y4:=collect(Y4A/(alpha1 &^ alpha1bar),lambda);
```

$$Y4:=\left(-\frac{Q\overline{A_1}(P)\,A_2(P)}{P^2R}+\frac{Q\overline{A_1}(P)\,A_2(R)}{PR^2}-\frac{\overline{A_1}(Q)\,A_2(R)}{R^2}+\frac{\overline{A_1}(Q)\,A_2(P)}{RP}\right)\lambda^2 \tag{27}$$
$$+\left(\frac{\overline{A_1}(P)\,A_2(P)}{P^2}-\frac{Q\overline{A_1}(P)\,A_2(\overline{Q})}{PR^2}+\frac{Q\overline{A_2}(P)\,A_2(R)}{PR^2}\right.$$
$$+\frac{Q\overline{A_1}(P)\,A_1(R)}{PR^2}-\frac{\overline{A_1}(Q)\,A_1(R)}{R^2}+\frac{\overline{A_1}(Q)\,A_1(P)}{RP}-\frac{\overline{A_2}(Q)\,A_2(R)}{R^2}$$
$$+\frac{\overline{A_2}(Q)\,A_2(P)}{RP}-\frac{\overline{A_1}(P)\,A_2(R)}{PR}$$
$$+\frac{Q\overline{A_1}(P)\,\overline{Q}A_2(P)}{P^2R^2}-\frac{Q\overline{A_1}(P)\,A_1(P)}{P^2R}-\frac{\overline{A_1}(Q)\,\overline{Q}A_2(P)}{R^2P}-\frac{Q\overline{A_2}(P)\,A_2(P)}{P^2R}$$
$$\left.+\frac{\overline{A_1}(Q)\,A_2(\overline{Q})}{R^2}\right)\lambda+\frac{\overline{A_2}(Q)\,A_2(\overline{Q})}{R^2}+\frac{\overline{A_2}(P)\,A_2(P)}{P^2}$$

$$+\frac{\overline{A_1}(P)\,A_1(P)}{P^2}-\frac{Q\overline{A_1}(P)\,A_1(\overline{Q})}{PR^2}-\frac{\overline{A_2}(P)\,A_2(R)}{PR}-\frac{\overline{A_2}(Q)\,A_1(R)}{R^2}-$$
$$\frac{\overline{A_1}(Q)\,\overline{Q}A_1(P)}{R^2\,P}+\frac{\overline{A_2}(Q)\,A_1(P)}{R\,P}+\frac{Q\overline{A_2}(P)\,A_1(R)}{PR^2}+\frac{\overline{A_1}(P)\,A_2(\overline{Q})}{PR}$$
$$+\frac{\overline{A_1}(Q)\,A_1(\overline{Q})}{R^2}-\frac{\overline{A_1}(P)\,A_1(R)}{PR}-\frac{\overline{A_1}(P)\,\overline{Q}A_2(P)}{P^2\,R}-\frac{Q\overline{A_2}(P)\,A_1(P)}{P^2\,R}-$$
$$\frac{Q\overline{A_2}(P)\,A_2(\overline{Q})}{PR^2}+\frac{Q\overline{A_2}(P)\,\overline{Q}A_2(P)}{P^2\,R^2}+\frac{Q\overline{A_1}(P)\,\overline{Q}A_1(P)}{P^2\,R^2}-\frac{\overline{A_2}(Q)\,\overline{Q}A_2(P)}{R^2\,P}$$
$$+\frac{1}{\lambda}\left(-\frac{\overline{A_1}(P)\,\overline{Q}A_1(P)}{P^2\,R}+\frac{Q\overline{A_2}(P)\,\overline{Q}A_1(P)}{P^2\,R^2}-\frac{\overline{A_2}(Q)\,\overline{Q}A_1(P)}{R^2\,P}\right.$$
$$+\frac{\overline{A_2}(P)\,A_2(\overline{Q})}{PR}-\frac{\overline{A_2}(P)\,A_1(R)}{PR}+\frac{\overline{A_2}(Q)\,A_1(\overline{Q})}{R^2}-\frac{\overline{A_2}(P)\,\overline{Q}A_2(P)}{P^2\,R}$$
$$\left.+\frac{\overline{A_1}(P)\,A_1(\overline{Q})}{PR}+\frac{\overline{A_2}(P)\,A_1(P)}{P^2}-\frac{Q\overline{A_2}(P)\,A_1(\overline{Q})}{PR^2}\right)+$$
$$\frac{\frac{\overline{A_2}(P)\,A_1(\overline{Q})}{PR}-\frac{\overline{A_2}(P)\,\overline{Q}A_1(P)}{P^2\,R}}{\lambda^2}$$

```
> ####################################
  # Finally explicit expression of C #
  # in dtau mod I ####################
  ####################################

  C:=series(Y1-Y2+Y3-Y4,lambda,5):
  C[-2] := coeff(C,lambda,-2);
```

$$C_{-2} := \frac{\overline{A_2}(P)\,\overline{Q}A_1(P)}{P^2\,R}-\frac{2\,\overline{A_2}(R)\,\overline{Q}A_1(P)}{R^2\,P}-\frac{2\,\overline{A_2}(P)\,A_1(\overline{Q})}{PR}+\frac{2\,\overline{A_2}(R)\,A_1(\overline{Q})}{R^2}$$
$$+\frac{\overline{Q}\,\overline{A_2}(A_1(P))}{PR}+\frac{A_1(P)\,\overline{A_2}(\overline{Q})}{PR}-\frac{\overline{A_2}(A_1(\overline{Q}))}{R} \tag{28}$$

```
> C:=series(Y1-Y2+Y3-Y4,lambda,5):
  C[-1] := coeff(C,lambda,-1);
```

$$C_{-1} := \frac{\overline{A_2}(A_1(R))}{R}-\frac{\overline{A_1}(A_1(\overline{Q}))}{R}-\frac{\overline{A_2}(A_1(P))}{P}-\frac{\overline{A_2}(A_2(\overline{Q}))}{R}+\frac{\overline{Q}\,\overline{A_2}(A_2(P))}{PR}$$
$$+\frac{A_1(P)\,\overline{A_1}(\overline{Q})}{PR}+\frac{\overline{Q}\,\overline{A_1}(A_1(P))}{PR}$$
$$+\frac{A_2(P)\,\overline{A_2}(\overline{Q})}{PR}-\frac{2\,\overline{A_1}(P)\,A_1(\overline{Q})}{PR}-\frac{2\,\overline{A_2}(P)\,A_2(\overline{Q})}{PR}+\frac{\overline{A_2}(P)\,A_1(R)}{PR} \tag{29}$$

$$+\frac{\overline{A_2}(R)\,A_1(P)}{R\,P}-\frac{2\,\overline{A_2}(Q)\,A_1(\overline{Q})}{R^2}+\frac{2\,\overline{A_1}(R)\,A_1(\overline{Q})}{R^2}$$
$$+\frac{2\,\overline{A_2}(R)\,A_2(\overline{Q})}{R^2}-\frac{2\,\overline{A_2}(R)\,A_1(R)}{R^2}+\frac{\overline{A_1}(P)\,\overline{Q}A_1(P)}{P^2\,R}$$
$$+\frac{2\,\overline{A_2}(Q)\,\overline{Q}A_1(P)}{R^2\,P}-\frac{2\,\overline{A_1}(R)\,\overline{Q}A_1(P)}{R^2\,P}-\frac{2\,Q\overline{A_2}(P)\,\overline{Q}A_1(P)}{P^2\,R^2}$$
$$+\frac{2\,Q\overline{A_2}(P)\,A_1(\overline{Q})}{P\,R^2}+\frac{\overline{A_2}(P)\,\overline{Q}A_2(P)}{P^2\,R}-\frac{2\,\overline{A_2}(R)\,\overline{Q}A_2(P)}{R^2\,P}-\frac{A_1(\overline{A_2}(P))}{P}$$
$$+\frac{A_1(\overline{A_2}(R))}{R}$$

```
> C:=series(Y1-Y2+Y3-Y4,lambda,5):
  M0 := coeff(C,lambda,0);
```

$$M0:=-\frac{\overline{A_1}(A_1(P))}{P}-\frac{\overline{A_1}(A_2(Q))}{R}+\frac{\overline{A_1}(A_1(R))}{R}-\frac{\overline{A_2}(A_2(P))}{P}+\frac{\overline{A_2}(A_2(R))}{R} \tag{30}$$
$$+\frac{A_2(P)\,\overline{A_1}(\overline{Q})}{P\,R}+\frac{\overline{Q}\,\overline{A_1}(A_2(P))}{P\,R}-\frac{2\,\overline{A_1}(Q)\,A_1(\overline{Q})}{R^2}-\frac{2\,\overline{A_2}(Q)\,A_2(\overline{Q})}{R^2}$$
$$+\frac{2\,\overline{A_2}(Q)\,A_1(R)}{R^2}+\frac{2\,\overline{A_1}(R)\,A_2(\overline{Q})}{R^2}-\frac{2\,\overline{A_1}(R)\,A_1(R)}{R^2}+\frac{\overline{A_2}(P)\,A_2(R)}{P\,R}$$
$$+\frac{\overline{A_2}(R)\,A_2(P)}{R\,P}+\frac{Q\,A_1(\overline{A_2}(P))}{P\,R}-\frac{2\,\overline{A_2}(R)\,A_2(R)}{R^2}-\frac{2\,Q\overline{A_1}(P)\,\overline{Q}A_1(P)}{P^2\,R^2}$$
$$+\frac{2\,Q\overline{A_1}(P)\,A_1(\overline{Q})}{P\,R^2}+\frac{2\,\overline{A_1}(Q)\,\overline{Q}A_1(P)}{R^2\,P}+\frac{\overline{A_1}(P)\,\overline{Q}A_2(P)}{P^2\,R}-\frac{2\,\overline{A_1}(P)\,A_2(\overline{Q})}{P\,R}$$
$$+\frac{\overline{A_1}(P)\,A_1(R)}{P\,R}-\frac{2\,\overline{A_2}(Q)\,A_1(P)}{R\,P}+\frac{2\,\overline{A_2}(Q)\,\overline{Q}A_2(P)}{R^2\,P}$$
$$+\frac{\overline{A_1}(R)\,A_1(P)}{R\,P}-\frac{2\,\overline{A_1}(R)\,\overline{Q}A_2(P)}{R^2\,P}+\frac{Q\overline{A_2}(P)\,A_1(P)}{P^2\,R}-\frac{2\,Q\overline{A_2}(P)\,\overline{Q}A_2(P)}{P^2\,R^2}$$
$$+\frac{2\,Q\overline{A_2}(P)\,A_2(\overline{Q})}{P\,R^2}-\frac{2\,Q\overline{A_2}(P)\,A_1(R)}{P\,R^2}-\frac{A_1(\overline{A_1}(P))}{P}-\frac{A_1(\overline{A_2}(Q))}{R}$$
$$+\frac{A_1(\overline{A_1}(R))}{R}+\frac{\overline{A_2}(P)\,A_1(Q)}{P\,R}-\frac{A_2(\overline{A_2}(P))}{P}+\frac{A_2(\overline{A_2}(R))}{R}$$

```
> C:=series(Y1-Y2+Y3-Y4,lambda,5):
  M1 := coeff(C,lambda,1);
```

$$M1:=-\frac{\overline{A_1}(A_2(P))}{P}-\frac{A_1(\overline{A_1}(Q))}{R}-\frac{2\,\overline{A_1}(Q)\,A_1(P)}{R\,P} \tag{31}$$
$$+\frac{\overline{A_1}(P)\,A_2(R)}{P\,R}-\frac{2\,\overline{A_2}(Q)\,A_2(P)}{R\,P}+\frac{\overline{A_1}(R)\,A_2(P)}{R\,P}-\frac{2\,\overline{A_1}(Q)\,A_2(\overline{Q})}{R^2}$$

$$+\frac{2\,\overline{A_1}(Q)\,A_1(R)}{R^2}+\frac{2\,\overline{A_2}(Q)\,A_2(R)}{R^2}-\frac{2\,\overline{A_1}(R)\,A_2(R)}{R^2}$$
$$+\frac{Q\,\overline{A_1}(P)\,A_1(P)}{P^2\,R}-\frac{2\,Q\,\overline{A_1}(P)\,\overline{Q}\,A_2(P)}{P^2\,R^2}$$
$$+\frac{2\,Q\,\overline{A_1}(P)\,A_2(\overline{Q})}{PR^2}-\frac{2\,Q\,\overline{A_1}(P)\,A_1(R)}{PR^2}+\frac{2\,\overline{A_1}(Q)\,\overline{Q}\,A_2(P)}{R^2\,P}$$
$$+\frac{Q\,\overline{A_2}(P)\,A_2(P)}{P^2\,R}-\frac{2\,Q\,\overline{A_2}(P)\,A_2(R)}{PR^2}$$
$$+\frac{\overline{A_1}(A_2(R))}{R}-\frac{A_2(\overline{A_1}(P))}{P}-\frac{A_2(\overline{A_2}(Q))}{R}+\frac{A_2(\overline{A_1}(R))}{R}+\frac{Q\,A_2(\overline{A_2}(P))}{PR}$$
$$+\frac{\overline{A_2}(P)\,A_2(Q)}{PR}+\frac{Q\,A_1(\overline{A_1}(P))}{PR}+\frac{\overline{A_1}(P)\,A_1(Q)}{PR}$$

```
> C:=series(Y1-Y2+Y3-Y4,lambda,5):
  M2 := coeff(C,lambda,2);
```

$$M2:=-\frac{2\,\overline{A_1}(Q)\,A_2(P)}{R\,P}+\frac{Q\,\overline{A_1}(P)\,A_2(P)}{P^2\,R}-\frac{2\,Q\,\overline{A_1}(P)\,A_2(R)}{PR^2}+\frac{2\,\overline{A_1}(Q)\,A_2(R)}{R^2}$$
$$+\frac{\overline{A_1}(P)\,A_2(Q)}{PR}+\frac{Q\,A_2(\overline{A_1}(P))}{PR}-\frac{A_2(\overline{A_1}(Q))}{R} \qquad (32)$$

```
>
> ################################
  # Check against old manuscript #
  ################################

  C2:= expand(A2((Q/(P*R))*A1bar(P)-(1/R)*A1bar(Q))-((Q/(P*R))*A1bar
  (P)-(1/R)*A1bar(Q))*((-1/P)*A2(P))-((1/R)*A2(R)-(1/P)*A2(P))*((Q/
  (P*R))*A1bar(P)-(1/R)*A1bar(Q)));
```

$$C2:=-\frac{2\,\overline{A_1}(Q)\,A_2(P)}{R\,P}+\frac{Q\,\overline{A_1}(P)\,A_2(P)}{P^2\,R}-\frac{2\,Q\,\overline{A_1}(P)\,A_2(R)}{PR^2}+\frac{2\,\overline{A_1}(Q)\,A_2(R)}{R^2}$$
$$+\frac{\overline{A_1}(P)\,A_2(Q)}{PR}+\frac{Q\,A_2(\overline{A_1}(P))}{PR}-\frac{A_2(\overline{A_1}(Q))}{R} \qquad (33)$$

```
> C1:= expand(A1((Q/(P*R))*A1bar(P)-(1/R)*A1bar(Q))+A2((Q/(P*R))*
  A2bar(P)-(1/R)*A2bar(Q)-(1/P)*A1bar(P)+(1/R)*A1bar(R))+A1bar((1/R)
  *A2(R)-(1/P)*A2(P))-((Q/(P*R))*A1bar(P)-(1/R)*A1bar(Q))*(
  (conjugate(Q)/(P*R))*A2(P)-(1/R)*A2(conjugate(Q))-(1/P)*A1(P))-(
  (Q/(P*R))*A2bar(P)-(1/R)*A2bar(Q)-(1/P)*A1bar(P)+(1/R)*A1bar(R))*(
  (-1/P)*A2(P))-((conjugate(Q)/(P*R))*A2(P)-(1/R)*A2(conjugate(Q))-
  (1/P)*A1(P)+(1/R)*A1(R))*((Q/(P*R))*A1bar(P)-(1/R)*A1bar(Q))-(
  (1/R)*A2(R)-(1/P)*A2(P))*((Q/(P*R))*A2bar(P)-(1/R)*A2bar(Q)-(1/P)*
  A1bar(P)));
```

(34)

$$C1 := -\frac{\overline{A_1}(A_2(P))}{P} - \frac{A_1(\overline{A_1}(Q))}{R} - \frac{2\,\overline{A_1}(Q)\,A_1(P)}{R\,P} \tag{34}$$
$$+ \frac{\overline{A_1}(P)\,A_2(R)}{PR} - \frac{2\,\overline{A_2}(Q)\,A_2(P)}{R\,P} + \frac{\overline{A_1}(R)\,A_2(P)}{R\,P} - \frac{2\,\overline{A_1}(Q)\,A_2(\overline{Q})}{R^2}$$
$$+ \frac{2\,\overline{A_1}(Q)\,A_1(R)}{R^2} + \frac{2\,\overline{A_2}(Q)\,A_2(R)}{R^2} - \frac{2\,\overline{A_1}(R)\,A_2(R)}{R^2}$$
$$+ \frac{Q\,\overline{A_1}(P)\,A_1(P)}{P^2\,R} - \frac{2\,Q\,\overline{A_1}(P)\,\overline{Q}\,A_2(P)}{P^2\,R^2}$$
$$+ \frac{2\,Q\,\overline{A_1}(P)\,A_2(\overline{Q})}{P\,R^2} - \frac{2\,Q\,\overline{A_1}(P)\,A_1(R)}{P\,R^2} + \frac{2\,\overline{A_1}(Q)\,\overline{Q}\,A_2(P)}{R^2\,P}$$
$$+ \frac{Q\,\overline{A_2}(P)\,A_2(P)}{P^2\,R} - \frac{2\,Q\,\overline{A_2}(P)\,A_2(R)}{P\,R^2}$$
$$+ \frac{\overline{A_1}(A_2(R))}{R} - \frac{A_2(\overline{A_1}(P))}{P} - \frac{A_2(\overline{A_2}(Q))}{R} + \frac{A_2(\overline{A_1}(R))}{R} + \frac{Q\,A_2(\overline{A_2}(P))}{P\,R}$$
$$+ \frac{\overline{A_2}(P)\,A_2(Q)}{PR} + \frac{Q\,A_1(\overline{A_1}(P))}{PR} + \frac{\overline{A_1}(P)\,A_1(Q)}{PR}$$

```
> C0:=expand(A1((Q/(P*R))*A2bar(P)-(1/R)*A2bar(Q)-(1/P)*A1bar(P)+
  (1/R)*A1bar(R))+A1bar((conjugate(Q)/(P*R))*A2(P)-(1/R)*A2
  (conjugate(Q))-(1/P)*A1(P)+(1/R)*A1(R))+A2((1/R)*A2bar(R)-(1/P)*
  A2bar(P))+A2bar((1/R)*A2(R)-(1/P)*A2(P))-2*((Q/(P*R))*A1bar(P)-
  (1/R)*A1bar(Q))*((conjugate(Q)/(P*R))*A1(P)-(1/R)*A1(conjugate(Q))
  )-((1/R)*A2bar(R)-(1/P)*A2bar(P))*((-1/P)*A2(P))-((1/R)*A2(R)-
  (1/P)*A2(P))*((-1/P)*A2bar(P))-((Q/(P*R))*A2bar(P)-(1/R)*A2bar(Q)-
  (1/P)*A1bar(P)+(1/R)*A1bar(R))*((conjugate(Q)/(P*R))*A2(P)-(1/R)*
  A2(conjugate(Q))-(1/P)*A1(P))-((conjugate(Q)/(P*R))*A2(P)-(1/R)*A2
  (conjugate(Q))-(1/P)*A1(P)+(1/R)*A1(R))*((Q/(P*R))*A2bar(P)-(1/R)*
  A2bar(Q)-(1/P)*A1bar(P)));
```

$$C0 := -\frac{\overline{A_1}(A_1(P))}{P} - \frac{\overline{A_1}(A_2(Q))}{R} + \frac{\overline{A_1}(A_1(R))}{R} - \frac{\overline{A_2}(A_2(P))}{P} + \frac{\overline{A_2}(A_2(R))}{R} \tag{35}$$
$$+ \frac{A_2(P)\,\overline{A_1}(\overline{Q})}{PR} + \frac{\overline{Q}\,\overline{A_1}(A_2(P))}{PR} - \frac{2\,\overline{A_1}(Q)\,A_1(\overline{Q})}{R^2} - \frac{2\,\overline{A_2}(Q)\,A_2(\overline{Q})}{R^2}$$
$$+ \frac{2\,\overline{A_2}(Q)\,A_1(R)}{R^2} + \frac{2\,\overline{A_1}(R)\,A_2(\overline{Q})}{R^2} - \frac{2\,\overline{A_1}(R)\,A_1(R)}{R^2} + \frac{\overline{A_2}(P)\,A_2(R)}{PR}$$
$$+ \frac{\overline{A_2}(R)\,A_2(P)}{R\,P} + \frac{Q\,A_1(\overline{A_2}(P))}{PR} - \frac{2\,\overline{A_2}(R)\,A_2(R)}{R^2} - \frac{2\,Q\,\overline{A_1}(P)\,\overline{Q}\,A_1(P)}{P^2\,R^2}$$
$$+ \frac{2\,Q\,\overline{A_1}(P)\,A_1(\overline{Q})}{P\,R^2} + \frac{2\,\overline{A_1}(Q)\,\overline{Q}\,A_1(P)}{R^2\,P} + \frac{\overline{A_1}(P)\,\overline{Q}\,A_2(P)}{P^2\,R} - \frac{2\,\overline{A_1}(P)\,A_2(\overline{Q})}{PR}$$

$$+\frac{\overline{A_1}(P)\,A_1(R)}{PR}-\frac{2\,\overline{A_2}(Q)\,A_1(P)}{R\,P}+\frac{2\,\overline{A_2}(Q)\,\overline{Q}A_2(P)}{R^2\,P}$$
$$+\frac{\overline{A_1}(R)\,A_1(P)}{R\,P}-\frac{2\,\overline{A_1}(R)\,\overline{Q}A_2(P)}{R^2\,P}+\frac{Q\overline{A_2}(P)\,A_1(P)}{P^2\,R}-\frac{2\,Q\overline{A_2}(P)\,\overline{Q}A_2(P)}{P^2\,R^2}$$
$$+\frac{2\,Q\overline{A_2}(P)\,A_2(\overline{Q})}{PR^2}-\frac{2\,Q\overline{A_2}(P)\,A_1(R)}{PR^2}-\frac{A_1(\overline{A_1}(P))}{P}-\frac{A_1(\overline{A_2}(Q))}{R}$$
$$+\frac{A_1(\overline{A_1}(R))}{R}+\frac{\overline{A_2}(P)\,A_1(Q)}{PR}-\frac{A_2(\overline{A_2}(P))}{P}+\frac{A_2(\overline{A_2}(R))}{R}$$

```
>
> ##############################################
  #### Final verification of C ################
  ##############################################

  M2-C2;
  M1-C1;
  M0-C0;
```

$$\begin{matrix}0\\0\\0\end{matrix} \tag{36}$$

```
>
```

```
> restart:

  with(linalg):
  with(PDEtools, casesplit, declare):
  with(DEtools, gensys):
```

```
> declare(F(z[1],z[2],a[1],a[2]));
```

$$F(z_1, z_2, a_1, a_2) \text{ will now be displayed as } F \tag{1}$$

```
> L1 := proc(fonction) diff(fonction,z[1]) end proc;
  L2 := proc(fonction) diff(fonction,z[2]) end proc;

  L1bar := proc(fonction) diff(fonction,(a[1])) end proc;
  L2bar := proc(fonction) diff(fonction,(a[2])) end proc;
```

$$L1 := \mathbf{proc}(fonction)\ diff(fonction, z[1])\ \mathbf{end\ proc}$$
$$L2 := \mathbf{proc}(fonction)\ diff(fonction, z[2])\ \mathbf{end\ proc}$$
$$L1bar := \mathbf{proc}(fonction)\ diff(fonction, a[1])\ \mathbf{end\ proc}$$
$$L2bar := \mathbf{proc}(fonction)\ diff(fonction, a[2])\ \mathbf{end\ proc} \tag{2}$$

```
> Fz1a1 := L1(L1bar(F(z[1],z[2],a[1],a[2])));
  Fz2a1 := L2(L1bar(F(z[1],z[2],a[1],a[2])));
  Fz1a2 := L1(L2bar(F(z[1],z[2],a[1],a[2])));
  Fz2a2 := L2(L2bar(F(z[1],z[2],a[1],a[2])));
```

$$Fz1a1 := F_{a_1, z_1}$$
$$Fz2a1 := F_{a_1, z_2}$$
$$Fz1a2 := F_{a_2, z_1}$$
$$Fz2a2 := F_{a_2, z_2} \tag{3}$$

```
> ## DECLARER P, Q, R
  ## DECLARER J, K, L LEURS CONJUGUES COMPLEXIFIES
```

```
> Delta := Fz1a1*Fz2a2 - Fz2a1*Fz1a2;
```

$$\Delta := F_{a_1, z_1} F_{a_2, z_2} - F_{a_1, z_2} F_{a_2, z_1} \tag{4}$$

```
> P := sqrt(Fz1a1);
  Q := Fz2a1/sqrt(Fz1a1);
  R := sqrt(-Delta)/sqrt(Fz1a1);
```

$$P := \sqrt{F_{a_1, z_1}}$$

$$Q := \frac{F_{a_1, z_2}}{\sqrt{F_{a_1, z_1}}}$$

$$R := \frac{\sqrt{-F_{a_1, z_1} F_{a_2, z_2} + F_{a_1, z_2} F_{a_2, z_1}}}{\sqrt{F_{a_1, z_1}}} \tag{5}$$

```
> J := P;
  K := Fz1a2/sqrt(Fz1a1);
  L := R;
```

$$J := \sqrt{F_{a_1, z_1}}$$

$$K := \frac{F_{a_2, z_1}}{\sqrt{F_{a_1, z_1}}}$$

$$L := \frac{\sqrt{-F_{a_1, z_1} F_{a_2, z_2} + F_{a_1, z_2} F_{a_2, z_1}}}{\sqrt{F_{a_1, z_1}}} \tag{6}$$

```
> ## CHAMPS DE VECTEURS ORTHOGONORMES
```

```
> U := 1/P;
  V := - Q/(P*R);
  W := 1/R;
```

$$U := \frac{1}{\sqrt{F_{a_1, z_1}}}$$

$$V := -\frac{F_{a_1, z_2}}{\sqrt{F_{a_1, z_1}}\sqrt{-F_{a_1, z_1} F_{a_2, z_2} + F_{a_1, z_2} F_{a_2, z_1}}}$$

$$W := \frac{\sqrt{F_{a_1, z_1}}}{\sqrt{-F_{a_1, z_1} F_{a_2, z_2} + F_{a_1, z_2} F_{a_2, z_1}}} \quad (7)$$

```
> X := 1/J;
  Y := - K/(J*L);
  Z := 1/L;
```

$$X := \frac{1}{\sqrt{F_{a_1, z_1}}}$$

$$Y := -\frac{F_{a_2, z_1}}{\sqrt{F_{a_1, z_1}}\sqrt{-F_{a_1, z_1} F_{a_2, z_2} + F_{a_1, z_2} F_{a_2, z_1}}}$$

$$Z := \frac{\sqrt{F_{a_1, z_1}}}{\sqrt{-F_{a_1, z_1} F_{a_2, z_2} + F_{a_1, z_2} F_{a_2, z_1}}} \quad (8)$$

```
> Z1 := proc(fonction) U*diff(fonction,z[1])
    end proc;
  Z2 := proc(fonction) V*diff(fonction,z[1]) + W*diff(fonction,z
  [2]) end proc;

  A1 := proc(fonction) X*diff(fonction,a[1])
    end proc;
  A2 := proc(fonction) Y*diff(fonction,a[1]) + Z*diff(fonction,a
  [2]) end proc;
```

$$Z1 := \mathbf{proc}(fonction)\ U*(diff(fonction, z[1]))\ \mathbf{end\ proc}$$

$$Z2 := \mathbf{proc}(fonction)\ V*(diff(fonction, z[1])) + W*(diff(fonction, z[2]))\ \mathbf{end\ proc}$$

$$A1 := \mathbf{proc}(fonction)\ X*(diff(fonction, a[1]))\ \mathbf{end\ proc}$$

$$A2 := \mathbf{proc}(fonction)\ Y*(diff(fonction, a[1])) + Z*(diff(fonction, a[2]))\ \mathbf{end\ proc} \quad (9)$$

```
> ## TESTS DIVERS

  simplify(  A1(A2(Z1(F(z[1],z[2],a[1],a[2]))))  ):

  simplify( Q*Z1(Z2(P))/(P*R) ):

  simplify( A1(Z1(R))/R ):
```

```
> ## CALCULS EN FONCTION DE F
```

```
> newCT[-2] := sort(factor(simplify(K*A2(Z1(P))/(P*R)+Z1(P)*A2(K)/
  (P*R)-2*Z1(K)*A2(P)/(P*R)-2*K*Z1(P)*A2(R)/(P*R^2)+K*Z1(P)*A2(P)/
  (P^2*R)-A2(Z1(K))/R+2*Z1(K)*A2(R)/R^2)));
  nops(op(1,newCT[-2]));
```

$$newCT_{-2} := \frac{1}{F_{a_1,z_1}^2\left(F_{a_1,z_1}F_{a_2,z_2}-F_{a_1,z_2}F_{a_2,z_1}\right)^2}\left(-F_{a_2,z_1}^2F_{a_1,z_1,z_1}F_{a_1,a_1,z_1}F_{a_2,z_2}\right.$$
$$+2F_{a_1,z_1,z_1}F_{a_2,z_1}F_{a_1,a_2,z_1}F_{a_1,z_1}F_{a_2,z_2}+F_{a_1,z_1}^3F_{a_2,a_2,z_1,z_1}F_{a_2,z_2}-$$
$$F_{a_1,z_1}^3F_{a_2,z_1,z_1}F_{a_2,a_2,z_2}-2F_{a_1,z_1}F_{a_2,z_1,z_1}F_{a_1,a_2,z_1}F_{a_1,z_2}F_{a_2,z_1}+F_{a_2,z_1,z_1}$$
$$F_{a_2,z_1}^2F_{a_1,a_1,z_1}F_{a_1,z_2}+F_{a_2,z_1}^2F_{a_1,z_1}F_{a_1,a_1,z_1,z_1}F_{a_2,z_2}-F_{a_2,z_1}^3F_{a_1,a_1,z_1,z_1}F_{a_1,z_2}-2F_{a_2,z_1}$$
$$F_{a_1,z_1}^2F_{a_1,a_2,z_1,z_1}F_{a_2,z_2}+2F_{a_2,z_1}^2F_{a_1,z_1}F_{a_1,a_2,z_1,z_1}F_{a_1,z_2}-F_{a_1,z_1,z_1}F_{a_1,z_1}^2F_{a_2,a_2,z_1}F_{a_2,z_2}$$
$$+2F_{a_1,z_1}^2F_{a_2,z_1,z_1}F_{a_2,z_1}F_{a_1,a_2,z_2}-F_{a_1,z_1}F_{a_2,z_1,z_1}F_{a_2,z_1}^2F_{a_1,a_1,z_2}+$$
$$F_{a_1,z_1}^2F_{a_2,z_1,z_1}F_{a_1,z_2}F_{a_2,a_2,z_1}-2F_{a_1,z_1}F_{a_1,z_1,z_1}F_{a_2,z_1}^2F_{a_1,a_2,z_2}+F_{a_1,z_1,z_1}F_{a_2,z_1}^3F_{a_1,a_1,z_2}$$
$$\left.+F_{a_1,z_1}^2F_{a_1,z_1,z_1}F_{a_2,z_1}F_{a_2,a_2,z_2}-F_{a_1,z_1}^2F_{a_2,a_2,z_1,z_1}F_{a_1,z_2}F_{a_2,z_1}\right)$$

(10)

```
> newCT[-1] := sort(factor(simplify(3*Q*Z1(K)*A2(P)/(P*R^2)+K*A1
  (Z1(P))/(P*R)+K*A2(Z2(P))/(P*R)+Z1(P)*A1(K)/(P*R)-3*Z1(K)*A1(P)/
  (P*R)+Z2(P)*A2(K)/(P*R)-Z2(K)*A2(P)/(P*R)-3*K*Z1(P)*A1(R)/(P*
  R^2)+3*K*Z1(P)*A2(Q)/(P*R^2)-K*Z2(P)*A2(R)/(P*R^2)-Z1(A2(P))/P-
  A2(Z1(P))/P-3*Q*K*Z1(P)*A2(P)/(P^2*R^2)+2*K*Z1(P)*A1(P)/(P^2*R)+
  Z1(P)*A2(P)/P^2+Z1(A2(R))/R-A1(Z1(K))/R+A2(Z1(R))/R-A2(Z2(K))/R-
  Z1(R)*A2(R)/R^2+3*Z1(K)*A1(R)/R^2-3*Z1(K)*A2(Q)/R^2+Z2(K)*A2(R)
  /R^2)));
  nops(op(2,newCT[-1]));
```

$$
\begin{aligned}
&newCT_{-1} := \\
&\frac{1}{\left(-F_{a_1,z_1}F_{a_2,z_2}+F_{a_1,z_2}F_{a_2,z_1}\right)^{5/2}F_{a_1,z_1}^2}\Big(2\Big(F_{a_1,z_1}F_{a_1,z_2}F_{a_2,z_1}F_{a_2,z_2}F_{a_1,a_1,z_1}F_{a_2,z_1,z_1} \\
&-3F_{a_1,z_1}F_{a_1,z_2}F_{a_2,z_1}F_{a_2,z_2}F_{a_1,a_2,z_1}F_{a_1,z_1,z_1}+4F_{a_1,z_1}^2F_{a_1,z_2}F_{a_2,z_1}F_{a_2,z_2}F_{a_1,a_2,z_1,z_1}-2 \\
&F_{a_1,z_1}^2F_{a_1,z_2}F_{a_2,z_1}F_{a_1,a_2,z_1}F_{a_2,z_1,z_2}-3F_{a_1,z_1}^2F_{a_1,z_2}F_{a_2,z_1}F_{a_1,a_2,z_2}F_{a_2,z_1,z_1}- \\
&F_{a_1,z_1}^2F_{a_1,z_2}F_{a_2,z_1}F_{a_1,z_1,z_1}F_{a_2,a_2,z_2}-F_{a_1,z_1}^2F_{a_1,z_2}F_{a_2,z_2}F_{a_1,a_2,z_1}F_{a_2,z_1,z_1}+ \\
&F_{a_1,z_1}^2F_{a_1,z_2}F_{a_2,z_2}F_{a_1,z_1,z_1}F_{a_2,a_2,z_1}-F_{a_1,z_1}^2F_{a_2,z_1}F_{a_2,z_2}F_{a_1,a_1,z_2}F_{a_2,z_1,z_1}+2 \\
&F_{a_1,z_1}^2F_{a_2,z_1}F_{a_2,z_2}F_{a_1,a_2,z_1}F_{a_1,z_1,z_2}-F_{a_1,z_1}^2F_{a_2,z_1}F_{a_2,z_2}F_{a_1,a_2,z_2}F_{a_1,z_1,z_1}+3F_{a_1,z_1} \\
&F_{a_1,z_2}^2F_{a_2,z_1}F_{a_1,a_2,z_1}F_{a_2,z_1,z_1}-3F_{a_1,z_1}F_{a_1,z_2}F_{a_2,z_1}^2F_{a_2,z_2}F_{a_1,a_1,z_1,z_1}+F_{a_1,z_1}F_{a_1,z_2} \\
&F_{a_2,z_1}^2F_{a_1,a_1,z_1}F_{a_2,z_1,z_2}+2F_{a_1,z_1}F_{a_1,z_2}F_{a_2,z_1}^2F_{a_1,a_1,z_2}F_{a_2,z_1,z_1}+3F_{a_1,z_1}F_{a_1,z_2} \\
&F_{a_2,z_1}^2F_{a_1,a_2,z_2}F_{a_1,z_1,z_1}-F_{a_1,z_1}F_{a_2,z_1}^2F_{a_2,z_2}F_{a_1,a_1,z_1}F_{a_1,z_1,z_2}+F_{a_1,z_1} \\
&F_{a_2,z_1}^2F_{a_2,z_2}F_{a_1,a_1,z_2}F_{a_1,z_1,z_1}-F_{a_1,z_1}F_{a_2,z_1}F_{a_2,z_1}^2F_{a_2,z_2}F_{a_1,a_1,z_1,z_1}+2F_{a_1,z_2} \\
&F_{a_2,z_1}^2F_{a_2,z_2}F_{a_1,a_1,z_1}F_{a_1,z_1,z_1}-F_{a_1,z_1}^3F_{a_1,z_2}F_{a_2,z_1}F_{a_2,a_2,z_1,z_2}- \\
&F_{a_1,z_1}^3F_{a_1,z_2}F_{a_2,z_2}F_{a_2,a_2,z_1,z_1}+F_{a_1,z_1}^3F_{a_1,z_2}F_{a_2,a_2,z_1}F_{a_2,z_1,z_2}+ \\
&F_{a_1,z_1}^3F_{a_1,z_2}F_{a_2,a_2,z_2}F_{a_2,z_1,z_1}-2F_{a_1,z_1}^3F_{a_2,z_1}F_{a_2,z_2}F_{a_1,a_2,z_1,z_2}+2 \\
&F_{a_1,z_1}^3F_{a_2,z_1}F_{a_1,a_2,z_2}F_{a_2,z_1,z_2}+F_{a_1,z_1}^3F_{a_2,z_1}F_{a_1,z_1,z_2}F_{a_2,a_2,z_2}+ \\
&F_{a_1,z_1}^3F_{a_2,z_2}F_{a_1,z_1,z_2}F_{a_2,a_2,z_1}-F_{a_1,z_1}^3F_{a_2,z_2}F_{a_1,a_2,z_1}F_{a_2,z_1,z_1}+F_{a_1,z_1}^2 \\
&F_{a_1,z_2}^2F_{a_2,z_1}F_{a_2,a_2,z_1}F_{a_2,z_1,z_1}-F_{a_1,z_1}^2F_{a_1,z_2}^2F_{a_2,a_2,z_1}F_{a_2,z_1,z_1}+2F_{a_1,z_1}^2F_{a_1,z_2}F_{a_2,z_1}^2F_{a_1,a_2,z_1,z_2} \\
&+F_{a_1,z_1}^2F_{a_2,z_1}^2F_{a_2,z_2}F_{a_1,a_1,z_1,z_2}-F_{a_1,z_1}^2F_{a_2,z_1}^2F_{a_1,a_1,z_2}F_{a_2,z_1,z_2}-2F_{a_1,z_1}^2 \\
&F_{a_2,z_1}^2F_{a_1,a_2,z_1}F_{a_2,z_1,z_2}+F_{a_1,z_1}^2F_{a_2,z_1}F_{a_2,z_2}^2F_{a_1,a_1,z_1,z_1}+F_{a_1,z_1}^2F_{a_2,z_2}^2F_{a_1,a_2,z_1}F_{a_1,z_1,z_1} \\
&-3F_{a_1,z_1}F_{a_1,z_2}^2F_{a_2,z_1}^2F_{a_1,a_2,z_1}F_{a_2,z_1,z_1}-F_{a_1,z_1}F_{a_1,z_2}F_{a_2,z_1}^3F_{a_2,z_2}F_{a_1,a_1,z_1,z_2}+F_{a_1,z_1} \\
&F_{a_2,z_1}^3F_{a_1,a_1,z_2}F_{a_1,z_1,z_2}-2F_{a_1,z_2}^2F_{a_2,z_1}^2F_{a_1,a_1,z_1}F_{a_2,z_1,z_1}-2F_{a_1,z_2}F_{a_2,z_1}^3F_{a_1,a_1,z_2}F_{a_1,z_1,z_1} \\
&+F_{a_1,z_1}^4F_{a_2,z_2}F_{a_2,a_2,z_1,z_2}-F_{a_1,z_1}^4F_{a_2,a_2,z_2}F_{a_2,z_1,z_2}-F_{a_1,z_1}^3F_{a_2,z_2}^2F_{a_1,a_2,z_1,z_1}+2F_{a_1,z_2}^2
\end{aligned}
$$

$$F^3_{a_2,z_1} F_{a_1,a_1,z_1,z_1}\Big)\Big)$$

(11)

```
> newCT[0] := sort(factor(simplify(2*Z1(R)*A2(Q)/R^2-4*Z1(K)*A1(Q)
  /R^2-2*Z1(R)*A1(R)/R^2-Z1(A2(Q))/R-A1(Z1(P))/P+A1(Z1(R))/R-Z1(A1
  (P))/P+Z1(A1(R))/R+Q*Z1(A2(P))/(P*R)-2*Z1(P)*A2(Q)/(P*R)+Z1(Q)*
  A2(P)/(P*R)+Z1(P)*A1(R)/(P*R)+Z1(R)*A1(P)/(P*R)+Q*Z1(P)*A2(P)/
  (P^2*R)-2*Q*Z1(R)*A2(P)/(P*R^2)+4*K*Z1(P)*A1(Q)/(P*R^2)+4*Q*Z1
  (K)*A1(P)/(P*R^2)-4*Q*K*Z1(P)*A1(P)/(P^2*R^2)+Z2(A2(R))/R-Z2(A2
  (P))/P+A2(Z2(R))/R-A2(Z2(P))/P-A1(Z2(K))/R-2*Q*K*Z2(P)*A2(P)/
  (P^2*R^2)-Z2(R)*A2(P)/(P*R)-Z2(P)*A2(R)/(P*R)+K*A1(Z2(P))/(P*R)
  -2*Z2(K)*A1(P)/(P*R)+Z2(P)*A1(K)/(P*R)+2*Z2(P)*A2(P)/P^2+2*Z2(K)
  *A1(R)/R^2-2*Z2(K)*A2(Q)/R^2+2*K*Z2(P)*A2(Q)/(P*R^2)+2*Q*Z2(K)*
  A2(P)/(P*R^2)+K*Z2(P)*A1(P)/(P^2*R)-2*K*Z2(P)*A1(R)/(P*R^2))));
  nops(op(2,newCT[0]));
```

$$newCT_0 := -\frac{1}{\left(F_{a_1,z_1}F_{a_2,z_2}-F_{a_1,z_2}F_{a_2,z_1}\right)^3 F^2_{a_1,z_1}}\Big(-6F^3_{a_1,z_2}F^3_{a_2,z_1}F_{a_1,a_1,z_1,z_1}+4$$
$$F^2_{a_1,z_1}F_{a_1,z_2}F_{a_2,z_1}F_{a_2,z_2}F_{a_1,a_1,z_1}F_{a_2,z_1,z_2}+6F^2_{a_1,z_1}F_{a_1,z_2}F_{a_2,z_1}F_{a_2,z_2}F_{a_1,a_1,z_2}F_{a_2,z_1,z_1}$$
$$-8F^2_{a_1,z_1}F_{a_1,z_2}F_{a_2,z_1}F_{a_2,z_2}F_{a_1,a_2,z_1}F_{a_1,z_1,z_2}+4$$
$$F^2_{a_1,z_1}F_{a_1,z_2}F_{a_2,z_1}F_{a_2,z_2}F_{a_1,a_2,z_2}F_{a_1,z_1,z_1}-6F_{a_1,z_1}F^2_{a_1,z_2}F_{a_2,z_1}F_{a_2,z_2}F_{a_1,a_1,z_1}F_{a_2,z_1,z_1}$$
$$+6F_{a_1,z_1}F^2_{a_1,z_2}F_{a_2,z_1}F_{a_2,z_2}F_{a_1,a_2,z_1}F_{a_1,z_1,z_1}+6F_{a_1,z_1}F_{a_1,z_2}$$
$$F^2_{a_2,z_1}F_{a_2,z_2}F_{a_1,a_1,z_1}F_{a_1,z_1,z_2}-6F_{a_1,z_1}F_{a_1,z_2}F^2_{a_2,z_1}F_{a_2,z_2}F_{a_1,a_1,z_2}F_{a_1,z_1,z_1}$$
$$+6F_{a_1,z_1}F_{a_1,z_2}F_{a_2,z_1}F^2_{a_2,z_2}F_{a_1,a_1,z_1}F_{a_1,z_1,z_1}+12F^3_{a_1,z_1}F_{a_1,z_2}F_{a_2,z_1}F_{a_2,z_2}F_{a_1,a_2,z_1,z_2}$$
$$-2F^3_{a_1,z_1}F_{a_1,z_2}F_{a_2,z_1}F_{a_1,a_2,z_1}F_{a_2,z_2,z_2}-8F^3_{a_1,z_1}F_{a_1,z_2}F_{a_2,z_1}F_{a_1,a_2,z_2}F_{a_2,z_1,z_2}-2$$
$$F^3_{a_1,z_1}F_{a_1,z_2}F_{a_2,z_1}F_{a_1,z_1,z_2}F_{a_2,a_2,z_2}-4F^3_{a_1,z_1}F_{a_1,z_2}F_{a_2,z_2}F_{a_1,a_2,z_1}F_{a_2,z_1,z_2}-4$$
$$F^3_{a_1,z_1}F_{a_1,z_2}F_{a_2,z_2}F_{a_1,a_2,z_2}F_{a_2,z_1,z_1}+2F^3_{a_1,z_1}F_{a_1,z_2}F_{a_2,z_2}F_{a_1,z_1,z_2}F_{a_2,a_2,z_1}-4$$
$$F^3_{a_1,z_1}F_{a_2,z_1}F_{a_2,z_2}F_{a_1,a_1,z_2}F_{a_2,z_1,z_2}+2F^3_{a_1,z_1}F_{a_2,z_1}F_{a_2,z_2}F_{a_1,a_2,z_1}F_{a_1,z_2,z_2}-4$$
$$F^3_{a_1,z_1}F_{a_2,z_1}F_{a_2,z_2}F_{a_1,a_2,z_2}F_{a_1,z_1,z_2}-10F^2_{a_1,z_1}F^2_{a_1,z_2}F_{a_2,z_1}F_{a_2,z_2}F_{a_1,a_2,z_1,z_1}+8F^2_{a_1,z_1}$$
$$F^2_{a_1,z_2}F_{a_2,z_1}F_{a_1,a_2,z_1}F_{a_2,z_1,z_2}+6F^2_{a_1,z_1}F^2_{a_1,z_2}F_{a_2,z_1}F_{a_1,a_2,z_2}F_{a_2,z_1,z_1}+F^2_{a_1,z_1}$$
$$F^2_{a_1,z_2}F_{a_2,z_1}F_{a_1,z_1,z_1}F_{a_2,a_2,z_2}+4F^2_{a_1,z_1}F^2_{a_1,z_2}F_{a_2,z_2}F_{a_1,a_2,z_1}F_{a_2,z_1,z_1}-F^2_{a_1,z_1}$$

$$F_{a_1,z_2}^2 F_{a_2,z_2} F_{a_1,z_1,z_1} F_{a_2,a_2,z_1} - 10 F_{a_1,z_1}^2 F_{a_1,z_2} F_{a_2,z_1}^2 F_{a_2,z_2} F_{a_1,a_1,z_1,z_2} + F_{a_1,z_1}^2 F_{a_1,z_2}$$

$$F_{a_2,z_1}^2 F_{a_1,a_1,z_1} F_{a_2,z_2,z_2} + 6 F_{a_1,z_1}^2 F_{a_1,z_2} F_{a_2,z_1}^2 F_{a_1,a_1,z_2} F_{a_2,z_1,z_2} + 8 F_{a_1,z_1}^2 F_{a_1,z_2}$$

$$F_{a_2,z_1}^2 F_{a_1,a_2,z_2} F_{a_1,z_1,z_2} - 7 F_{a_1,z_1}^2 F_{a_1,z_2} F_{a_2,z_1} F_{a_2,z_2}^2 F_{a_1,a_1,z_1,z_1} + F_{a_1,z_1}^2 F_{a_1,z_2}$$

$$F_{a_2,z_2}^2 F_{a_1,a_1,z_1} F_{a_2,z_1,z_1} - 4 F_{a_1,z_1}^2 F_{a_1,z_2} F_{a_2,z_2}^2 F_{a_1,a_2,z_1} F_{a_1,z_1,z_1} - F_{a_1,z_1}^2$$

$$F_{a_2,z_1}^2 F_{a_2,z_2} F_{a_1,a_1,z_1} F_{a_1,z_2,z_2} + 4 F_{a_1,z_1}^2 F_{a_2,z_1}^2 F_{a_2,z_2} F_{a_1,a_1,z_2} F_{a_1,z_1,z_2} - 4 F_{a_1,z_1}^2 F_{a_2,z_1}$$

$$F_{a_2,z_2}^2 F_{a_1,a_1,z_1} F_{a_1,z_1,z_2} + F_{a_1,z_1}^2 F_{a_2,z_1} F_{a_2,z_2}^2 F_{a_1,a_1,z_2} F_{a_1,z_1,z_1} - 6 F_{a_1,z_1}$$

$$F_{a_1,z_2}^3 F_{a_2,z_1} F_{a_1,a_2,z_1} F_{a_2,z_1,z_1} + 12 F_{a_1,z_1} F_{a_1,z_2}^2 F_{a_2,z_1}^2 F_{a_2,z_2} F_{a_1,a_1,z_1,z_1} - 6 F_{a_1,z_1} F_{a_1,z_2}^2$$

$$F_{a_2,z_1}^2 F_{a_1,a_1,z_1} F_{a_2,z_1,z_2} - 6 F_{a_1,z_1} F_{a_1,z_2}^2 F_{a_2,z_1}^2 F_{a_1,a_1,z_2} F_{a_2,z_1,z_1} - 6 F_{a_1,z_1} F_{a_1,z_2}^2$$

$$F_{a_2,z_1}^2 F_{a_1,a_2,z_2} F_{a_1,z_1,z_1} - 6 F_{a_1,z_1} F_{a_1,z_2} F_{a_2,z_1}^3 F_{a_1,a_1,z_2} F_{a_1,z_1,z_2} - 6 F_{a_1,z_2}^2$$

$$F_{a_2,z_1}^2 F_{a_2,z_2} F_{a_1,a_1,z_1} F_{a_1,z_1,z_1} - F_{a_1,z_1}^4 F_{a_1,z_2} F_{a_2,z_1} F_{a_2,a_2,z_2,z_2} - 2$$

$$F_{a_1,z_1}^4 F_{a_1,z_2} F_{a_2,z_2} F_{a_2,a_2,z_1,z_2} + F_{a_1,z_1}^4 F_{a_1,z_2} F_{a_2,a_2,z_1} F_{a_2,z_2,z_2} + 2$$

$$F_{a_1,z_1}^4 F_{a_1,z_2} F_{a_2,a_2,z_2} F_{a_2,z_1,z_2} - 2 F_{a_1,z_1}^4 F_{a_2,z_1} F_{a_2,z_2} F_{a_1,a_2,z_2,z_2} + 2$$

$$F_{a_1,z_1}^4 F_{a_2,z_1} F_{a_1,a_2,z_2} F_{a_2,z_2,z_2} + F_{a_1,z_1}^4 F_{a_2,z_1} F_{a_1,z_2,z_2} F_{a_2,a_2,z_2} + 4$$

$$F_{a_1,z_1}^4 F_{a_2,z_2} F_{a_1,a_2,z_2} F_{a_2,z_1,z_2} - F_{a_1,z_1}^4 F_{a_2,z_2} F_{a_1,z_2,z_2} F_{a_2,a_2,z_1} + 2 F_{a_1,z_1}^3$$

$$F_{a_1,z_2}^2 F_{a_2,z_1} F_{a_2,a_2,z_1,z_2} + F_{a_1,z_1}^3 F_{a_1,z_2}^2 F_{a_2,z_2} F_{a_2,a_2,z_1,z_1} - 2 F_{a_1,z_1}^3 F_{a_1,z_2}^2 F_{a_2,a_2,z_1} F_{a_2,z_1,z_2}$$

$$- F_{a_1,z_1}^3 F_{a_1,z_2}^2 F_{a_2,a_2,z_2} F_{a_2,z_1,z_1} + 2 F_{a_1,z_1}^3 F_{a_1,z_2} F_{a_2,z_1}^2 F_{a_1,a_2,z_2,z_2} + 4 F_{a_1,z_1}^3 F_{a_1,z_2}$$

$$F_{a_2,z_2}^2 F_{a_1,a_2,z_1,z_1} + F_{a_1,z_1}^3 F_{a_2,z_1}^2 F_{a_2,z_2} F_{a_1,a_1,z_2,z_2} - F_{a_1,z_1}^3 F_{a_2,z_1}^2 F_{a_1,a_1,z_2} F_{a_2,z_2,z_2} - 2$$

$$F_{a_1,z_1}^3 F_{a_2,z_1}^2 F_{a_1,a_2,z_2} F_{a_1,z_2,z_2} + 4 F_{a_1,z_1}^3 F_{a_2,z_1} F_{a_2,z_2}^2 F_{a_1,a_1,z_1,z_2} - F_{a_1,z_1}^3$$

$$F_{a_2,z_2}^2 F_{a_1,a_1,z_2} F_{a_2,z_1,z_1} + 4 F_{a_1,z_1}^3 F_{a_2,z_2}^2 F_{a_1,a_2,z_1} F_{a_1,z_1,z_2} - F_{a_1,z_1}^2 F_{a_1,z_1}^3 F_{a_2,z_1} F_{a_2,a_2,z_1,z_1}$$

$$+ F_{a_1,z_1}^2 F_{a_1,z_2}^3 F_{a_2,a_2,z_1} F_{a_2,z_1,z_1} - 8 F_{a_1,z_1}^2 F_{a_1,z_2}^2 F_{a_2,z_1}^2 F_{a_1,a_2,z_1,z_2} - F_{a_1,z_1}^2 F_{a_1,z_2}$$

$$F_{a_2,z_1}^3 F_{a_1,a_1,z_2,z_2} + F_{a_1,z_1}^2 F_{a_2,z_1}^3 F_{a_1,a_1,z_2} F_{a_1,z_2,z_2} - F_{a_1,z_1}^2 F_{a_2,z_2}^3 F_{a_1,a_1,z_1} F_{a_1,z_1,z_1}$$

$$+ 6 F_{a_1,z_1} F_{a_1,z_2}^3 F_{a_2,z_1}^2 F_{a_1,a_2,z_1,z_1} + 6 F_{a_1,z_1} F_{a_1,z_2}^2 F_{a_2,z_1}^3 F_{a_1,a_1,z_1,z_2} + 6 F_{a_1,z_2}^3$$

$$F_{a_2,z_1}^2 F_{a_1,a_1,z_1} F_{a_2,z_1,z_1} + 6 F_{a_1,z_2}^2 F_{a_2,z_1}^3 F_{a_1,a_1,z_2} F_{a_1,z_1,z_1} + F_{a_1,z_1}^5 F_{a_2,z_2} F_{a_2,a_2,z_2,z_2} -$$
$$F_{a_1,z_1}^5 F_{a_2,a_2,z_2} F_{a_2,z_2,z_2} - 4 F_{a_1,z_1}^4 F_{a_2,z_2}^2 F_{a_1,a_2,z_1,z_2} + F_{a_1,z_1}^3 F_{a_2,z_2}^3 F_{a_1,a_1,z_1,z_1}\Big)$$

(12)

```
> newCT[1] := sort(factor(simplify(Q*Z1(A1(P))/(P*R)+Q*Z2(A2(P))/
(P*R)-3*Q*Z1(R)*A1(P)/(P*R^2)-Q*Z2(R)*A2(P)/(P*R^2)+3*Q*Z2(K)*A1
(P)/(P*R^2)-3*Z1(P)*A1(Q)/(P*R)+Z1(Q)*A1(P)/(P*R)-Z2(P)*A2(Q)/
(P*R)+Z2(Q)*A2(P)/(P*R)+3*K*Z2(P)*A1(Q)/(P*R^2)-Z2(A1(P))/P-A1
(Z2(P))/P+2*Q*Z1(P)*A1(P)/(P^2*R)-3*Q*K*Z2(P)*A1(P)/(P^2*R^2)+Z2
(P)*A1(P)/P^2-Z1(A1(Q))/R+Z2(A1(R))/R-Z2(A2(Q))/R+A1(Z2(R))/R+3*
Z1(R)*A1(Q)/R^2-Z2(R)*A1(R)/R^2+Z2(R)*A2(Q)/R^2-3*Z2(K)*A1(Q)
/R^2)));
nops(op(2,newCT[1]));
```

$$newCT_1 := \frac{1}{\left(-F_{a_1,z_1} F_{a_2,z_2} + F_{a_1,z_2} F_{a_2,z_1}\right)^{5/2} F_{a_1,z_1}^2} \Big(2 \Big($$

$$-3 F_{a_1,z_1} F_{a_1,z_2} F_{a_2,z_1} F_{a_2,z_2} F_{a_1,a_1,z_1} F_{a_1,z_1,z_2} + F_{a_1,z_1} F_{a_1,z_2} F_{a_2,z_1} F_{a_2,z_2} F_{a_1,a_1,z_2} F_{a_1,z_1,z_1}$$
$$- F_{a_1,z_1}^3 F_{a_1,z_2} F_{a_2,z_1} F_{a_1,a_2,z_2,z_2} - 2 F_{a_1,z_1}^3 F_{a_1,z_2} F_{a_2,z_2} F_{a_1,a_2,z_1,z_2} +$$
$$F_{a_1,z_1}^3 F_{a_1,z_2} F_{a_1,a_2,z_1} F_{a_2,z_2,z_2} + 2 F_{a_1,z_1}^3 F_{a_1,z_2} F_{a_1,a_2,z_2} F_{a_2,z_1,z_2} -$$
$$F_{a_1,z_1}^3 F_{a_2,z_1} F_{a_2,z_2} F_{a_1,a_1,z_2,z_2} + F_{a_1,z_1}^3 F_{a_2,z_1} F_{a_1,a_1,z_2} F_{a_2,z_2,z_2} + F_{a_1,z_1}^4 F_{a_2,z_2} F_{a_1,a_2,z_2,z_2} -$$
$$F_{a_1,z_1}^4 F_{a_1,a_2,z_2} F_{a_2,z_2,z_2} - F_{a_1,z_1}^3 F_{a_2,z_2}^2 F_{a_1,a_1,z_1,z_2} + 2 F_{a_1,z_2}^3 F_{a_2,z_1}^2 F_{a_1,a_1,z_1,z_1} + 4$$
$$F_{a_1,z_1}^2 F_{a_1,z_2} F_{a_2,z_1} F_{a_2,z_2} F_{a_1,a_1,z_1,z_2} - F_{a_1,z_1}^2 F_{a_1,z_2} F_{a_2,z_1} F_{a_1,a_1,z_1} F_{a_2,z_2,z_2} - 3$$
$$F_{a_1,z_1}^2 F_{a_1,z_2} F_{a_2,z_1} F_{a_1,a_1,z_2} F_{a_2,z_1,z_2} - 2 F_{a_1,z_1}^2 F_{a_1,z_2} F_{a_2,z_1} F_{a_1,a_2,z_2} F_{a_1,z_1,z_2} -$$
$$F_{a_1,z_1}^2 F_{a_1,z_2} F_{a_2,z_2} F_{a_1,a_1,z_1} F_{a_2,z_1,z_2} - F_{a_1,z_1}^2 F_{a_1,z_2} F_{a_2,z_2} F_{a_1,a_1,z_2} F_{a_2,z_1,z_1} + 2$$
$$F_{a_1,z_1}^2 F_{a_1,z_2} F_{a_2,z_2} F_{a_1,a_2,z_1} F_{a_1,z_1,z_2} + F_{a_1,z_1}^2 F_{a_2,z_1} F_{a_2,z_2} F_{a_1,a_1,z_1} F_{a_1,z_2,z_2} -$$
$$F_{a_1,z_1}^2 F_{a_1,z_2} F_{a_2,z_1} F_{a_1,a_1,z_2} F_{a_1,z_1,z_2} - 3 F_{a_1,z_1} F_{a_1,z_2}^2 F_{a_2,z_1} F_{a_2,z_2} F_{a_1,a_1,z_1,z_1} + 3 F_{a_1,z_1}$$
$$F_{a_1,z_2}^2 F_{a_2,z_1} F_{a_1,a_1,z_1} F_{a_2,z_1,z_2} + 2 F_{a_1,z_1} F_{a_1,z_2}^2 F_{a_2,z_1} F_{a_1,a_1,z_2} F_{a_2,z_1,z_1} + F_{a_1,z_1}$$
$$F_{a_1,z_2}^2 F_{a_2,z_1} F_{a_1,a_2,z_2} F_{a_1,z_1,z_1} + F_{a_1,z_1} F_{a_1,z_2}^2 F_{a_2,z_2} F_{a_1,a_1,z_1} F_{a_2,z_1,z_1} - F_{a_1,z_1}$$
$$F_{a_1,z_2}^2 F_{a_2,z_2} F_{a_1,a_2,z_1} F_{a_1,z_1,z_1} + 3 F_{a_1,z_1} F_{a_1,z_2} F_{a_2,z_1}^2 F_{a_1,a_1,z_2} F_{a_1,z_1,z_2} - F_{a_1,z_1} F_{a_1,z_2}$$

$$F_{a_2,z_2}^2 F_{a_1,a_1,z_1} F_{a_1,z_1,z_1} + 2 F_{a_1,z_2}^2 F_{a_2,z_1} F_{a_2,z_2} F_{a_1,a_1,z_1} F_{a_1,z_1,z_1} +$$
$$F_{a_1,z_1}^3 F_{a_2,z_1} F_{a_1,a_2,z_2} F_{a_1,z_2,z_2} + F_{a_1,z_1}^3 F_{a_2,z_2} F_{a_1,a_1,z_2} F_{a_2,z_1,z_2} -$$
$$F_{a_1,z_1}^3 F_{a_2,z_2} F_{a_1,a_2,z_1} F_{a_1,z_2,z_2} + 2 F_{a_1,z_1}^2 F_{a_1,z_2}^2 F_{a_2,z_1} F_{a_1,a_2,z_1,z_2} + F_{a_1,z_1}^2$$
$$F_{a_1,z_2}^2 F_{a_2,z_2} F_{a_1,a_2,z_1,z_1} - 2 F_{a_1,z_1}^2 F_{a_1,z_2}^2 F_{a_1,a_2,z_1} F_{a_2,z_1,z_2} - F_{a_1,z_1}^2 F_{a_1,z_2}^2 F_{a_1,a_2,z_2} F_{a_2,z_1,z_1}$$
$$+ F_{a_1,z_1}^2 F_{a_1,z_2} F_{a_2,z_1}^2 F_{a_1,a_1,z_2,z_2} + F_{a_1,z_1}^2 F_{a_1,z_2} F_{a_2,z_2}^2 F_{a_1,a_1,z_1,z_1} - F_{a_1,z_1}^2$$
$$F_{a_2,z_1}^2 F_{a_1,a_1,z_2} F_{a_1,z_2,z_2} + F_{a_1,z_1}^2 F_{a_2,z_2}^2 F_{a_1,a_1,z_1} F_{a_1,z_1,z_2} - F_{a_1,z_1} F_{a_1,z_2}^3 F_{a_2,z_1} F_{a_1,a_2,z_1,z_1}$$
$$+ F_{a_1,z_1} F_{a_1,z_2}^3 F_{a_1,a_2,z_1} F_{a_2,z_1,z_1} - 3 F_{a_1,z_1} F_{a_1,z_2}^2 F_{a_2,z_1}^2 F_{a_1,a_1,z_1,z_2} - 2$$
$$F_{a_1,z_2}^3 F_{a_2,z_1} F_{a_1,a_1,z_1} F_{a_2,z_1,z_1} - 2 F_{a_1,z_2}^2 F_{a_2,z_1}^2 F_{a_1,a_1,z_2} F_{a_1,z_1,z_1} \Big) \Big)$$

(13)

```
> newCT[2] := sort(factor(simplify(Q*Z2(A1(P))/(P*R)-2*Q*Z2(R)*A1
  (P)/(P*R^2)-2*Z2(P)*A1(Q)/(P*R)+Z2(Q)*A1(P)/(P*R)+Q*Z2(P)*A1(P)/
  (P^2*R)-Z2(A1(Q))/R+2*Z2(R)*A1(Q)/R^2)));
  nops(op(1,newCT[2]));
```

$$newCT_2 := \frac{1}{F_{a_1,z_1}^2 \left(F_{a_1,z_1} F_{a_2,z_2} - F_{a_1,z_2} F_{a_2,z_1}\right)^2} \Big(F_{a_1,z_1}^3 F_{a_1,a_1,z_2,z_2} F_{a_2,z_2} -$$
$$F_{a_1,z_1}^3 F_{a_2,z_2,z_2} F_{a_1,a_1,z_2} - F_{a_1,z_2}^2 F_{a_1,z_1,z_1} F_{a_1,a_1,z_1} F_{a_2,z_2}$$
$$+ 2 F_{a_1,z_1,z_2} F_{a_1,z_1} F_{a_1,z_2} F_{a_1,a_1,z_1} F_{a_2,z_2} - 2 F_{a_1,z_1} F_{a_1,z_1,z_2} F_{a_1,a_1,z_2} F_{a_1,z_2} F_{a_2,z_1} +$$
$$F_{a_1,z_2}^2 F_{a_1,z_1,z_1} F_{a_1,a_1,z_2} F_{a_2,z_1} + F_{a_1,z_2}^2 F_{a_1,z_1} F_{a_1,a_1,z_1,z_1} F_{a_2,z_2} - F_{a_1,z_2}^3 F_{a_1,a_1,z_1,z_1} F_{a_2,z_1}$$
$$- 2 F_{a_1,z_2} F_{a_1,z_1}^2 F_{a_1,a_1,z_1,z_2} F_{a_2,z_2} + 2 F_{a_1,z_2}^2 F_{a_1,z_1} F_{a_1,a_1,z_1,z_2} F_{a_2,z_1} - 2$$
$$F_{a_1,z_2}^2 F_{a_1,a_1,z_1} F_{a_1,z_1} F_{a_2,z_1,z_2} + F_{a_1,z_2}^3 F_{a_1,a_1,z_1} F_{a_2,z_1,z_1} + F_{a_1,z_2} F_{a_1,a_1,z_1} F_{a_1,z_1}^2 F_{a_2,z_2,z_2}$$
$$- F_{a_1,a_1,z_1} F_{a_1,z_1}^2 F_{a_1,z_2,z_2} F_{a_2,z_2} - F_{a_1,z_1}^2 F_{a_1,a_1,z_2,z_2} F_{a_1,z_2} F_{a_2,z_1} + 2$$
$$F_{a_1,z_1}^2 F_{a_1,z_2} F_{a_2,z_1,z_2} F_{a_1,a_1,z_2} - F_{a_1,z_1} F_{a_1,z_2}^2 F_{a_2,z_1,z_1} F_{a_1,a_1,z_2} +$$
$$F_{a_1,z_1}^2 F_{a_1,z_2,z_2} F_{a_2,z_1} F_{a_1,a_1,z_2} \Big)$$

(14)

Chapter 6

Hachtroudi-Chern-Moser tensor in CR geometry

1 First and Second Jet Lifts of Equivalences

Let $\mathbb{K} = \mathbb{R}$ be the field of real numbers, or the field $\mathbb{C}$ of complex numbers. To fix ideas, only $\mathbb{K}$-analytic objects will be dealt with, although everything will appear *a posteriori* to be true in the $\mathscr{C}^4$-smoothness category.

Let $n \geqslant 1$ be an integer, and on $\mathbb{K}^{n+1}$, consider coordinates:

$$\big(x^1, \ldots, x^n, y\big).$$

Similarly, on another space $\mathbb{K}^{n+1}$, consider coordinates:

$$\big(X^1, \ldots, X^n, Y\big),$$

and assume that a local $\mathbb{K}$-analytic diffeomorphism is given:

$$\begin{aligned} E\colon \quad \big(x^1, \ldots, x^n, y\big) &\longmapsto \big(X^1, \ldots, X^n, Y\big) \\ &\qquad \Big(X^1\big(x^1, \ldots, x^n, y\big), \ldots, X^n\big(x^1, \ldots, x^n, y\big), Y\big(x^1, \ldots, x^n, y\big)\Big). \end{aligned}$$

By *local* is meant that E is defined in some small open neighborhood of the origin in $\mathbb{K}^{n+1}$. To lighten the formalism, open sets and subsets will never be mentioned.

Provided E is close enough to the identity, it transfers graphed $\mathbb{K}$-analytic hypersurfaces $\{y = y(x^1, \ldots, x^n)\}$ to similar graphed $\mathbb{K}$-analytic hypersurfaces $\{Y = Y(X^1, \ldots, X^n)\}$.

Notational Convention 1.1. *Integer indices running in $\{1, 2, \ldots, n\}$ will be denoted using the (first letters of the) Greek alphabet:*

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \eta,\ \theta,\ \iota,\ \kappa,\ \lambda,\ \mu,\ \nu,\ \xi,\ o,\ \pi,\ \rho,\ \sigma,\ \tau,\ \upsilon,\ \varphi,\ \chi,\ \psi,\ \omega.$$

The first and second jet spaces of such graphing functions $y(x^1, \ldots, x^n)$:

$$\begin{aligned} &J^1(\mathbb{K}^n, \mathbb{K}), \\ &J^2(\mathbb{K}^n, \mathbb{K}), \end{aligned}$$

come equipped with *independent* coordinates:

$$\begin{aligned} \big(x^\alpha, y, y_{x^\beta}\big) &\in \mathbb{K}^{n+1+n}, \\ \big(x^\alpha, y, y_{x^\beta}, y_{x^\gamma x^\delta}\big) &\in \mathbb{K}^{n+1+n+\frac{n(n+1)}{2}}, \end{aligned}$$

which are independent, and correspond of course to partial derivatives of graphing functions. Similarly, the target jet spaces come equipped with coordinates:

$$\begin{aligned}\big(X^\alpha, Y, Y_{X^\beta}\big) &\in \mathbb{K}^{n+1+n},\\ \big(X^\alpha, Y, Y_{X^\beta}, Y_{X^\gamma X^\delta}\big) &\in \mathbb{K}^{n+1+n+\frac{n(n+1)}{2}}.\end{aligned}$$

The way partial derivatives of such graphing functions are transferred through E:

$$\begin{array}{ccc} J^2(\mathbb{K}^n,\mathbb{K}) & \xrightarrow{J^2E} & J^2(\mathbb{K}^n,\mathbb{K}) \\ \downarrow & & \downarrow \\ J^1(\mathbb{K}^n,\mathbb{K}) & \xrightarrow{J^1E} & J^1(\mathbb{K}^n,\mathbb{K}) \\ \downarrow & & \downarrow \\ \mathbb{K}^{n+1} & \xrightarrow{\quad E \quad} & \mathbb{K}^{n+1}, \end{array} \qquad \begin{array}{ccc} \big(x^\alpha, y, y_{x^\beta}, y_{x^\gamma x^\delta}\big) & \longrightarrow & \big(X^\alpha, Y, Y_{X^\beta}, Y_{X^\gamma X^\delta}\big) \\ \downarrow & & \downarrow \\ \big(x^\alpha, y, y_{x^\beta}\big) & \longrightarrow & \big(X^\alpha, Y, Y_{X^\beta}\big) \\ \downarrow & & \downarrow \\ (x^\alpha, y) & \longrightarrow & (X^\alpha, Y), \end{array}$$

is provided by classical formulas [ML15]; [BK89]; [Mer08], the proof of which will not be redone here. To present them, introduce the

Definition 1.2. On $J^1(\mathbb{K}^n,\mathbb{K})$, for each index $1 \leqslant \alpha \leqslant n$, the *total differentiation* along the x^α axis is the operator:

$$\mathsf{D}_{x^\alpha} \;:=\; \frac{\partial}{\partial x^\alpha} + y_{x^\alpha}\,\frac{\partial}{\partial y} + \sum_{\beta=1}^{n} y_{x^\alpha x^\beta}\,\frac{\partial}{\partial y_{x^\beta}}.$$

In the concerned formulas below, a matrix must be inverted.

Proposition 1.3. *The first prolongation J^1E of the diffeomorphism E to the first jet space expresses as:*

$$\begin{pmatrix} Y_{X^1} \\ \vdots \\ Y_{X^n} \end{pmatrix} = \begin{pmatrix} \mathsf{D}_{x^1}(X^1) & \cdots & \mathsf{D}_{x^1}(X^n) \\ \vdots & \ddots & \vdots \\ \mathsf{D}_{x^n}(X^1) & \cdots & \mathsf{D}_{x^n}(X^n) \end{pmatrix}^{-1} \begin{pmatrix} \mathsf{D}_{x^1}(Y) \\ \vdots \\ \mathsf{D}_{x^n}(Y) \end{pmatrix},$$

and the second prolongation J^2E, for $1 \leqslant \alpha \leqslant n$, as:

$$\begin{pmatrix} Y_{X^\alpha X^1} \\ \vdots \\ Y_{X^\alpha X^n} \end{pmatrix} = \begin{pmatrix} \mathsf{D}_{x^1}(X^1) & \cdots & \mathsf{D}_{x^1}(X^n) \\ \vdots & \ddots & \vdots \\ \mathsf{D}_{x^n}(X^1) & \cdots & \mathsf{D}_{x^n}(X^n) \end{pmatrix}^{-1} \begin{pmatrix} \mathsf{D}_{x^1}(Y_{X^\alpha}) \\ \vdots \\ \mathsf{D}_{x^n}(Y_{X^\alpha}) \end{pmatrix}. \qquad \square$$

In fact, it can be realized that the diffeomorphism $(x,y) \overset{E}{\longmapsto} (X,Y)$ transfers horizontal graphs $\{y = y(x)\}$ to horizontal graphs $\{Y = Y(X)\}$ if and only if the above $n \times n$ matrix is invertible.

Because of the presence of such an inverse matrix, these formulas are not handy as soon as $n \geqslant 2$. In the case $n = 1$, they become after reorganization:

$$Y_X \;=\; \frac{Y_x + y_x\,Y_y}{X_x + y_x\,X_y},$$

and:

$$\begin{aligned} Y_{XX} \;=\; \frac{1}{[X_x + y_x\,X_y]^3}\,\Bigg(& y_{xx}\begin{vmatrix} X_x & X_y \\ Y_x & Y_y \end{vmatrix} + \begin{vmatrix} X_x & X_{xx} \\ Y_x & Y_{xx} \end{vmatrix} + y_x\left\{2\begin{vmatrix} X_x & X_{xy} \\ Y_x & Y_{xy} \end{vmatrix} - \begin{vmatrix} X_{xx} & X_y \\ Y_{xx} & Y_y \end{vmatrix}\right\} + \\ & + y_x y_x\left\{\begin{vmatrix} X_x & X_{yy} \\ Y_x & Y_{yy} \end{vmatrix} - 2\begin{vmatrix} X_{xy} & X_y \\ Y_{xy} & Y_y \end{vmatrix}\right\} - y_x y_x y_x \begin{vmatrix} X_{yy} & X_y \\ Y_{yy} & Y_y \end{vmatrix}\Bigg). \end{aligned}$$

Then a direct computation shows after pulling back through E:

$$\begin{aligned} J^1E^*\big(dY - Y_X\,dX\big) \;&=\; Y_x\,dx + Y_y\,dy - \frac{Y_x + y_x\,Y_y}{X_x + y_x\,X_y}.\big[X_x\,dx + X_y\,dy\big] \\ &=\; u\,\big(dy - y_x\,dx\big), \end{aligned}$$

with the nowhere vanishing function:

$$u \;:=\; \frac{X_xY_y - X_yY_x}{X_x + y_x\,X_y}.$$

Furthermore, using:

$$J^2E^*\big(dY_X\big) \;=\; \frac{\partial Y_X}{\partial x}\,dx + \frac{\partial Y_X}{\partial y}\,dy + \frac{\partial Y_X}{\partial y_x}\,dy_x,$$

a direct, although more delicate, computation gives:

$$J^2E^*\big(dY_X - Y_{XX}\,dX\big) \;=\; v\cdot\big(dy - y_x\,dx\big) + w\cdot\big(dy_x - y_{xx}\,dx\big),$$

with the function:

$$\begin{aligned} v \;:=\; \frac{1}{[X_x + y_x\,X_y]^3}\,\Big\{ & X_xX_xY_{xy} - X_xX_yY_{xx} - X_xX_{xy}Y_x + X_yX_{xx}Y_x + \\ & + y_x\,\Big(X_xX_xY_{yy} - X_xX_{xy}Y_y - X_xX_{yy}Y_x - X_yX_yY_{xx} + X_yX_{xx}Y_y + X_yX_{xy}Y_x\Big) + \\ & + y_xy_x\,\Big(X_xX_yY_{yy} - X_xX_{yy}Y_y - X_yX_yY_{xy} + X_yX_{xy}Y_y\Big) + \\ & + y_xy_xy_x\,\Big(-X_xX_yY_y + X_yX_yY_x\Big)\Big\}, \end{aligned}$$

and with the nowhere vanishing function:

$$w \;:=\; \frac{X_xY_y - X_yX_x}{[X_x + y_x\,X_y]^3}.$$

In the case of $n \geqslant 2$ independent variables $(x^1, \ldots, x^n)$, similar computations can be achieved, with some mastered efforts in the algebra of formal matrices.

But it is known [ML15]; [BK89], [Olv95, p. 125] (without much computations) that the corresponding ideals of *contact* differential forms:

$$\Big\langle dY - \sum_\alpha\, Y_{X^\alpha}\,dX^\alpha\Big\rangle,$$

and:

$$\Big\langle dY - \sum_\alpha\, Y_{X^\alpha}\,dX,\;\; dY_{X^\alpha} - \sum_\beta\, Y_{X^\alpha X^\beta}\,dX^\beta\Big\rangle,$$

have kernels invariant under pullbacks, as stated in the following (admitted)

Lemma 1.4. *On the source jet spaces $J^1(\mathbb{K}^n, \mathbb{K})$ and $J^2(\mathbb{K}^n, \mathbb{K})$, there exist $1 + n + n^2$ functions:*

$$u \;=\; u\big(x^\gamma, y, y_{x^\delta}\big) \qquad \text{and} \qquad \begin{aligned} u_\alpha \;&=\; u_\alpha\big(x^\gamma, y, y_{x^\delta}, y_{x^\varepsilon x^\zeta}\big), \\ v_\alpha^\beta \;&=\; v_\alpha^\beta\big(x^\gamma, y, y_{x^\delta}, y_{x^\varepsilon x^\zeta}\big), \end{aligned}$$

with $u \neq 0$ and $0 \neq \det v_\alpha^\beta$ nowhere vanishing, such that:

$$\begin{aligned} J^1E^*\Big(dY - \sum_\alpha\, Y_{X^\alpha}\,dX^\alpha\Big) \;&=\; u\,\Big(dy - \sum_\beta\, y_{x^\beta}\,dx^\beta\Big), \\ J^2E^*\Big(dY_{X^\alpha} - \sum_\beta\, Y_{X^\alpha X^\beta}\,dX^\beta\Big) \;&=\; u_\alpha\,\Big(dy - \sum_\beta\, y_{x^\beta}\,dx^\beta\Big) + \sum_\beta\, v_\alpha^\beta\,\Big(dy_{x^\beta} - \sum_\gamma\, y_{x^\beta x^\gamma}\,dx^\gamma\Big). \quad \square \end{aligned}$$

To complete the picture, it is plain that there are functions u^α — not to be confused with the u_α — and u^α_β such that:

$$\begin{aligned} E^*\big(dX^\alpha\big) \ &= \ X^\alpha_y\,dy+\sum_\beta\,X^\alpha_{x^\beta}\,dx^\beta \\ &= \ X^\alpha_y\,\Big(dy-\sum_\beta\,y_{x^\beta}\,dx^\beta\Big)+\sum_\beta\,\mathsf{D}_{x^\beta}\big(X^\alpha\big)\,dx^\beta \\ &=: \ u^\alpha\,\Big(dy-\sum_\beta\,y_{x^\beta}\,dx^\beta\Big)+\sum_\beta\,u^\alpha_\beta\,dx^\beta, \end{aligned}$$

with a nowhere vanishing determinant, as follows from an already mentioned hypothesis:

$$0\ \neq\ \det u^\alpha_\beta\ =\ \det\Big(D_{x^\beta}\big(X^\alpha\big)\Big).$$

These three pullback formulas will justify the shape of the initial G-structure associated to equivalences of certain systems of partial differential equations.

2 Completely Integrable Second Order Systems

Still in $\mathbb{K}^{n+1}$, consider a system of partial differential equations:

$$y_{x^\alpha x^\beta}\ =\ F^{\alpha,\beta}\big(x^\gamma,y,y_{x^\delta}\big) \qquad (1\leqslant\alpha,\beta\leqslant n),$$

with $\mathbb{K}$-analytic right-hand sides:

$$F^{\beta,\alpha}\ =\ F^{\alpha,\beta}.$$

Assume that this system is *completely integrable*, in the sense that for every choice of constants:

$$b\in\mathbb{K},\ \ a^1\in\mathbb{K},\ \ldots\ldots,\ a^n\in\mathbb{K},$$

there exists a unique $\mathbb{K}$-analytic solution:

$$y\ =\ Q(x,a^1,\ldots,a^n,b),$$

which satisfies identically:

$$Q_{x^\alpha x^\beta}(x,a,b)\ \equiv\ F^{\alpha,\beta}\Big(x^\gamma,Q(x,a,b),Q_{x^\delta}(x,a,b)\Big),$$

together with the initial conditions:

$$Q(0,a,b)\ =\ b, \qquad\qquad Q_{x^\gamma}(0,a,b)\ =\ a^\gamma \qquad (1\leqslant\gamma\leqslant n).$$

A simple differential characterization of complete integrability exists by means of total differentiations.

Lemma 2.1. *The system $y_{x^\alpha x^\beta}=F^{\alpha,\beta}$ is completely integrable if and only if:*

$$\mathsf{D}_{x^\gamma}\big(F^{\alpha,\beta}\big)\ =\ \mathsf{D}_{x^\beta}\big(F^{\alpha,\gamma}\big) \qquad (1\leqslant\alpha,\beta,\gamma\leqslant n).$$

Proof. The known argument — details are skipped — consists in applying the Frobenius theorem within the second-order jet space $J^2(\mathbb{K}^n,\mathbb{K})$. □

Next, with the coordinates:

$$\big(x^1,\dots,x^n,y,y_{x^1},\dots,y_{x^n}\big)\ \in\ \mathbb{K}^{2n+1}$$

on the $(2n+1)$-dimensional first jet space $J^1(\mathbb{K}^n,\mathbb{K})$, introduce the $1+n+n$ differential 1-forms (mind index positions):

$$\begin{aligned}
\varpi\ &:=\ dy-\sum_\beta\, y_{x^\beta}\,dx^\beta,\\
\varpi^\alpha\ &:=\ dx^\alpha && (1\leqslant\alpha\leqslant n),\\
\varpi_\alpha\ &:=\ dy_{x^\alpha}-\sum_\beta\, F^{\alpha,\beta}\big(x^\gamma,y,y_{x^\delta}\big)\,dx^\beta && (1\leqslant\alpha\leqslant n),
\end{aligned}$$

which are visibly independent at every point, hence constitute a coframe. By assumption of complete integrability, the Pfaffian system (zero-set):

$$\big\{\varpi=\varpi_1=\cdots=\varpi_n=0\big\}$$

has the family of integral manifolds $\big\{y=Q(x,a,b)\big\}_{a,b}$ parametrized by (a,b), while the 1-forms ϖ^α are added to have a coframe.

In what follows, $y_{x^\alpha x^\beta}$ will *always* be replaced by $F^{\alpha,\beta}\big(x^\gamma,y,y_{x^\delta}\big)$, which means pullbacking to the graphed submanifold of $J^2(\mathbb{K}^n,\mathbb{K})$ defined by:

$$\Big\{\big(x^\gamma,y,y_{x^\delta},y_{x^\alpha x^\beta}\big)\in J^2(\mathbb{K}^n,\mathbb{K})\colon\ y_{x^\alpha x^\beta}=F^{\alpha,\beta}\big(x^\gamma,y,y_{x^\delta}\big)\Big\},$$

that corresponds naturally to the considered system of partial differential equations. In particular:

$$\mathsf{D}_{x^\alpha}\ =\ \frac{\partial}{\partial x^\alpha}+y_{x^\alpha}\,\frac{\partial}{\partial y}+\sum_{\beta=1}^n\,F^{\alpha,\beta}\big(x^\gamma,y,y_{x^\delta}\big)\,\frac{\partial}{\partial y_{x^\beta}}\qquad(1\leqslant\alpha\leqslant n).$$

Lemma 2.2. *The exterior differential dG of any function $G=G\big(x^\alpha,y,y_{x^\beta}\big)$:*

$$dG\ =\ \frac{\partial G}{\partial y}\cdot dy+\sum_{\alpha=1}^n\,\frac{\partial G}{\partial x^\alpha}\cdot dx^\alpha+\sum_{\alpha=1}^n\,\frac{\partial G}{\partial y_{x^\alpha}}\cdot dy_{x^\alpha}$$

re-expresses in terms of the coframe $\big\{\varpi,\varpi^\alpha,\varpi_\alpha\big\}$ as:

$$dG\ =\ G_y\,\varpi+\sum_\alpha\,\mathsf{D}_{x^\alpha}(G)\cdot\varpi^\alpha+\sum_\alpha\,G_{y_{x^\alpha}}\cdot\varpi_\alpha.$$

Proof. Indeed, it suffices to plug the inverse formulas:

$$\begin{aligned}
dy\ &=\ \varpi+\sum_\beta\,y_{x^\beta}\cdot\varpi^\beta,\\
dx^\alpha\ &=\ \varpi^\alpha,\\
dy_{x^\alpha}\ &=\ \sum_\beta\,F^{\alpha,\beta}\,\varpi^\beta+\varpi_\alpha,
\end{aligned}$$

in:

$$\begin{aligned}
dG\ &=\ G_y\,\Big[\varpi+\sum_\alpha\,y_{x^\alpha}\,\varpi^\alpha\Big]+\sum_\alpha\,G_{x^\alpha}\,\big[\varpi^\alpha\big]+\sum_\alpha\,G_{y_{x^\alpha}}\,\Big[\sum_\beta\,F^{\alpha,\beta}\,\varpi^\beta+\varpi_\alpha\Big]\\
&=\ G_y\cdot\varpi+\sum_\alpha\,\Big[G_{x^\alpha}+y_{x^\alpha}\,G_y+\sum_\beta\,F^{\beta,\alpha}\,G_{y_{x^\beta}}\Big]\cdot\varpi^\alpha+\sum_\alpha\,G_{y_{x^\alpha}}\cdot\varpi_\alpha,
\end{aligned}$$

and to recognize the presence of the total differentiation operator. □

Lemma 2.3. *The initial Darboux-Cartan structure of the coframe* $\{\varpi, \varpi^\alpha, \varpi_\alpha\}$ *is:*

$$\begin{aligned} d\varpi &= \sum_\beta \varpi^\beta \wedge \varpi_\beta, \\ d\varpi^\alpha &= 0, \\ d\varpi_\alpha &= -\sum_\beta F_y^{\alpha,\beta} \cdot \varpi \wedge \varpi^\beta + \sum_\beta \sum_\gamma F_{y_{x^\gamma}}^{\alpha,\beta} \cdot \varpi^\beta \wedge \varpi_\gamma. \end{aligned}$$

Proof. Starting with the last formula, compute:

$$\begin{aligned} d\varpi_\alpha &= \sum_\beta dx^\beta \wedge dF^{\alpha,\beta} \\ &= \sum_\beta \varpi^\beta \wedge \Big[F_y^{\alpha,\beta} \cdot \varpi + \sum_\gamma \mathsf{D}_{x^\gamma}\big(F^{\alpha,\beta}\big) \cdot \varpi^\gamma + \sum_\gamma F_{y_{x^\gamma}}^{\alpha,\beta} \cdot \varpi_\gamma \Big] \\ &= -\sum_\beta \varpi \wedge \varpi^\beta \left[F_y^{\alpha,\beta}\right] + \underline{\sum_\beta \sum_\gamma \varpi^\beta \wedge \varpi^\gamma \left[D_{x^\gamma}\big(F^{\alpha,\beta}\big)\right]}_{\circ} + \sum_\beta \sum_\gamma \varpi^\beta \wedge \varpi_\gamma \left[F_{y_{x^\gamma}}^{\alpha,\beta}\right], \end{aligned}$$

and observe that the middle term:

$$\sum_{1 \leqslant \beta < \gamma \leqslant n} \varpi^\beta \wedge \varpi^\gamma \Big[\underline{D_{x^\gamma}\big(F^{\alpha,\beta}\big) - D_{x^\beta}\big(F^{\alpha,\gamma}\big)}_{\circ} \Big],$$

vanishes due to complete integrability.

Then the first formula:

$$\begin{aligned} d\varpi &= \sum_\beta dx^\beta \wedge dy_{x^\beta} \\ &= \sum_\beta \varpi^\beta \wedge \Big[\varpi_\beta + \sum_\gamma F^{\beta,\gamma} \cdot \varpi^\gamma \Big] \end{aligned}$$

follows from the index symmetry $F^{\gamma,\beta} = F^{\beta,\gamma}$. □

Corollary 2.4. *The complete integrability hypothesis* $\mathsf{D}_{x^\gamma}\big(F^{\alpha,\beta}\big) = \mathsf{D}_{x^\beta}\big(F^{\alpha,\gamma}\big)$ *reads as:*

$$\begin{aligned} &d\varpi \equiv 0 \mod \langle \varpi, \varpi^\beta \rangle \qquad \textit{and} \qquad d\varpi^\alpha \equiv 0 \mod \langle \varpi, \varpi^\beta \rangle, \\ &d\varpi \equiv 0 \mod \langle \varpi, \varpi_\beta \rangle \qquad \textit{and} \qquad d\varpi_\alpha \equiv 0 \mod \langle \varpi, \varpi_\beta \rangle. \end{aligned}$$ □

3 Initial G_1-structure and Its First Reduction

Lemma 1.4 provides the general form of how such a coframe $\{\varpi, \varpi^\alpha, \varpi_\alpha\}$ transfers through any local $\mathbb{K}$-analytic equivalence. According to the general procedure of Cartan's method of equivalence, it is therefore advisable to replace such functions by independent (fiber) coordinates:

$$\begin{aligned} &u \in \mathbb{K}, \\ &u^\alpha \in \mathbb{K}, \qquad u_\beta^\alpha \in \mathbb{K}, \\ &u_\alpha \in \mathbb{K}, \qquad\qquad v_\alpha^\beta \in \mathbb{K}, \end{aligned}$$

and to introduce the following $1 + n + n$ *lifted* 1*-forms:*

$$\begin{aligned} \omega &:= u \cdot \varpi, \\ \omega^\alpha &:= u^\alpha \cdot \varpi + \sum_\beta u_\beta^\alpha \cdot \varpi^\beta, \\ \omega_\alpha &:= u_\alpha \cdot \varpi + \sum_\beta v_\alpha^\beta \cdot \varpi_\beta, \end{aligned} \tag{3.1}$$

which live on the product space $\mathbb{K}^{n+1+n} \times \mathbb{K}^{1+n^2+n^2+n+n}$ equipped with coordinates:

$$\big(x^\alpha, y, y_{x^\beta};\ u, u^\alpha_\beta, v^\beta_\alpha, u^\alpha, u_\alpha\big).$$

As is standard, the fiber space with its coordinates $(u, u^\alpha_\beta, v^\beta_\alpha, u^\alpha, u_\alpha)$ has the structure of a matrix subgroup of $\mathsf{GL}_{2n+1}(\mathbb{K})$:

$$G_1 := \left\{ \begin{pmatrix} u & 0 & 0 \\ u^\alpha & u^\alpha_\beta & 0 \\ u_\alpha & 0 & v^\beta_\alpha \end{pmatrix} \in \mathsf{GL}_{2n+1}(\mathbb{K}) \colon\ u, u^\alpha_\beta, v^\beta_\alpha, u^\alpha, u_\alpha \in \mathbb{K},\ 0 \neq u,\ 0 \neq \mathsf{det}\,\big(u^\alpha_\beta\big),\ 0 \neq \mathsf{det}\,\big(v^\beta_\alpha\big) \right\},$$

for which the stability under matrix multiplication can easily be checked, minding how indices refer to either rows R or columns C:

$$\begin{pmatrix} u & 0 & 0 \\ u^{\alpha\mathrm{R}} & u^{\alpha\mathrm{R}}_{\beta\mathrm{C}} & 0 \\ u_{\alpha\mathrm{R}} & 0 & v^{\beta\mathrm{C}}_{\alpha\mathrm{R}} \end{pmatrix},$$

so that the placement, bottom and up, of row and column indices, is *not* the same in u^α_β and in v^β_α!

Later computations will require to express inversely the $\varpi, \varpi^\alpha, \varpi_\alpha$ in terms of the $\omega, \omega^\alpha, \omega_\alpha$, and for this, it is necessary to introduce the two inverse $n \times n$ matrices:

$$\big(\bar{u}^{1\alpha}_{\ \beta}\big) \ =\ \big(\bar{u}^{1\alpha\mathrm{R}}_{\ \ \beta\mathrm{C}}\big) \qquad \text{and} \qquad \big(\bar{v}^{1\beta}_{\ \alpha}\big) \ =\ \big(\bar{v}^{1\beta\mathrm{C}}_{\ \ \alpha\mathrm{R}}\big),$$

also specifying their row R and column C indices. They satisfy by definition:

$$\begin{aligned} \boldsymbol{\delta}^\alpha_\beta \ &=\ \sum_\gamma\ \bar{u}^{1\alpha}_{\ \gamma}\, u^\gamma_\beta \ =\ \sum_\gamma\ u^\alpha_\gamma\, \bar{u}^{1\gamma}_{\ \beta}, \\ \boldsymbol{\delta}^\beta_\alpha \ &=\ \sum_\gamma\ \bar{v}^{1\gamma}_{\ \alpha}\, v^\beta_\gamma \ =\ \sum_\gamma\ v^\gamma_\alpha\, \bar{v}^{1\beta}_{\ \gamma}, \end{aligned}$$

employing the Kronecker symbol $\boldsymbol{\delta}^{\bullet}_{\bullet}$ in a special font to avoid status confusion with the already used running Greek index $1 \leqslant \delta \leqslant n$, which will also several times be used later on.

Back to (3.1), the first line inverts as:

$$\varpi \ =\ u^{-1} \cdot \omega.$$

For what concerns then the middle n lines:

$$\sum_\beta\ u^\alpha_\beta\, \varpi^\beta \ =\ -\, u^{-1}\, u^\alpha \cdot \omega + \omega^\alpha,$$

a multiplication-summation by $\sum_\alpha\ \bar{u}^{1\gamma}_{\ \alpha}\,(\cdot)$ gives:

$$\begin{aligned} \varpi^\gamma \ &=\ \sum_\alpha \sum_\beta\ \bar{u}^{1\gamma}_{\ \alpha}\, u^\alpha_\beta\, \varpi^\beta \\ &=\ -\Big(\sum_\alpha\ \bar{u}^{1\gamma}_{\ \alpha}\, u^{-1} u^\alpha\Big) \cdot \omega + \sum_\alpha\ \bar{u}^{1\gamma}_{\ \alpha} \cdot \omega^\alpha, \end{aligned}$$

hence changing indices:

$$\varpi^\alpha \ =\ -\Big(\sum_\beta\ u^{-1} \bar{u}^{1\alpha}_{\ \beta}\, u^\beta\Big) \cdot \omega + \sum_\beta\ \bar{u}^{1\alpha}_{\ \beta} \cdot \omega^\beta.$$

Lastly, for what concerns the last n lines:

$$\sum_\beta\ v^\beta_\alpha \cdot \varpi_\beta \ =\ -\, u^{-1} u_\alpha \cdot \omega + \omega_\alpha,$$

a multiplication-summation by $\sum_\alpha \bar{v}^{1\alpha}_{\gamma}(\cdot)$ gives:

$$\begin{aligned}\varpi_\gamma &= \sum_\alpha \sum_\beta \bar{v}^{1\alpha}_{\gamma}\, v^\beta_\alpha \cdot \varpi_\beta\\ &= -\Big(\sum_\alpha \bar{v}^{1\alpha}_{\gamma}\, u^{-1} u_\alpha\Big)\cdot\omega + \sum_\alpha \bar{v}^{1\alpha}_{\gamma}\,\omega_\alpha,\end{aligned}$$

hence changing indices:

$$\varpi_\alpha = -\Big(\sum_\beta u^{-1}\bar{v}^{1\beta}_{\alpha}\, u_\beta\Big)\cdot\omega + \sum_\beta \bar{v}^{1\beta}_{\alpha}\,\omega_\beta.$$

In summary, the inversion of the lifting is:

$$\begin{aligned}\varpi &= u^{-1}\cdot\omega,\\ \varpi^\alpha &= -\Big(\sum_\beta u^{-1}\bar{u}^{1\alpha}_{\beta}\, u^\beta\Big)\cdot\omega + \sum_\beta \bar{u}^{1\alpha}_{\beta}\cdot\omega^\beta,\\ \varpi_\alpha &= -\Big(\sum_\beta u^{-1}\bar{v}^{1\beta}_{\alpha}\, u_\beta\Big)\cdot\omega + \sum_\beta \bar{v}^{1\beta}_{\alpha}\,\omega_\beta.\end{aligned}\tag{3.2}$$

Now, following the general Cartan procedure, the 2-forms $d\omega$, $d\omega^\alpha$, $d\omega_\alpha$ should be expressed in terms of the lifted coframe $\{\omega, \omega^\alpha, \omega_\alpha\}$ in order to chase some *essential torsion coefficients*.

However here, it is known that such essential (non-absorbable) torsion coefficients already appear when computing $d\omega$, hence it is not necessary for the moment to touch $d\omega^\alpha$ and $d\omega_\alpha$.

Indeed, applying the exterior differentiation operator to the first line of (3.1):

$$\begin{aligned}d\omega &= du\wedge\varpi + u\cdot d\varpi\\ \text{[Lemma 2.3]}\qquad &= du\wedge\varpi + u\cdot\sum_\alpha \varpi^\alpha\wedge\varpi_\alpha\\ \text{[Equations (3.2)]}\qquad &= u^{-1}du\wedge\omega +\\ &\quad + u\sum_\alpha\Big(-\sum_\beta u^{-1}\bar{u}^{1\alpha}_{\beta}\, u^\beta\cdot\omega + \sum_\beta \bar{u}^{1\alpha}_{\beta}\cdot\omega^\beta\Big)\wedge\Big(-\sum_\gamma u^{-1}\bar{v}^{1\gamma}_{\alpha}\, u_\gamma\cdot\omega + \sum_\gamma \bar{v}^{1\gamma}_{\alpha}\cdot\omega_\gamma\Big),\end{aligned}$$

hence the result can be abbreviated as:

$$d\omega = \Big(u^{-1}du + \mathsf{something}\Big)\wedge\omega + \sum_\beta\sum_\gamma\Big(\sum_\alpha u\,\bar{u}^{1\alpha}_{\beta}\,\bar{v}^{1\gamma}_{\alpha}\Big)\cdot\omega^\beta\wedge\omega_\gamma.$$

Here, none of the coefficients of the 2-forms $\omega^\beta\wedge\omega_\gamma$ can be absorbed in $\big(\mathsf{something}\big)\wedge\omega$, and this proves that these coefficients are *essential torsion coefficients*.

According to the general Cartan procedure, they can be *normalized* by setting:

$$\boxed{\bar{v}^{1\gamma}_{\alpha} := u^{-1}u^\gamma_\alpha \quad\Longleftrightarrow\quad v^\beta_\alpha = u\,\bar{u}^{1\beta}_{\alpha},}$$

so that the $v^\bullet_\bullet$ fiber variables disappear, and the above result simplifies as:

$$\begin{aligned}d\omega &= \Big(u^{-1}du + \mathsf{something}\Big)\wedge\omega + \sum_\beta\sum_\gamma\Big(\sum_\alpha \underline{u}_\circ\,\bar{u}^{1\alpha}_{\beta}\,\underline{u^{-1}}_\circ u^\gamma_\alpha\Big)\cdot\omega^\beta\wedge\omega_\gamma\\ &= \Big(u^{-1}du + \mathsf{something}\Big)\wedge\omega + \sum_\beta\omega^\beta\wedge\omega_\beta.\end{aligned}$$

The computation of the unshown 1-form in parenthesis will be done later.

Thus, the initial G_1-structure reduces to a G_2-structure with new group:

$$G_2 := \left\{\begin{pmatrix} u & 0 & 0\\ u^\alpha & u^\alpha_\beta & 0\\ u_\alpha & 0 & u\bar{u}^{1\beta}_{\alpha}\end{pmatrix}\in \mathsf{GL}_{2n+1}(\mathbb{K})\colon\ u, u^\alpha_\beta, u^\alpha, u_\alpha\in\mathbb{K},\ 0\neq u,\ 0\neq\mathsf{det}\left(u^\alpha_\beta\right)\right\},$$

of smaller dimension:

$$\mathsf{dim}_{\mathbb{K}}\, G_2 \;=\; 1+n^2+n+n \;\leqslant\; 1+n^2+n^2+n+n \;=\; \mathsf{dim}_{\mathbb{K}}\, G_1.$$

After elimination of v_α^β, the formulas for the lifted coframe become:

$$\begin{aligned} \omega \;&:=\; u\cdot\varpi,\\ \omega^\alpha \;&:=\; u^\alpha\cdot\varpi+\sum_\beta\, u_\beta^\alpha\cdot\varpi^\beta,\\ \omega_\alpha \;&:=\; u_\alpha\cdot\varpi+\sum_\beta\, u\,\bar{u}^{-1\beta}_{\ \ \alpha}\cdot\varpi_\beta, \end{aligned} \tag{3.3}$$

and the useful inverse formulas are:

$$\begin{aligned} \varpi \;&=\; u^{-1}\cdot\omega,\\ \varpi^\alpha \;&=\; -\Big(\sum_\beta\, u^{-1}\bar{u}^{-1\alpha}_{\ \ \beta}\,u^\beta\Big)\cdot\omega+\sum_\beta\,\bar{u}^{-1\alpha}_{\ \ \beta}\cdot\omega^\beta,\\ \varpi_\alpha \;&=\; -\Big(\sum_\beta\, u^{-2}u_\alpha^\beta\,u_\beta\Big)\cdot\omega+\sum_\beta\, u^{-1}u_\alpha^\beta\,\omega_\beta. \end{aligned} \tag{3.4}$$

In matrix form:

$$\begin{pmatrix}\omega\\ \omega^\alpha\\ \omega_\alpha\end{pmatrix} \;=\; \begin{pmatrix} u & 0 & 0\\ u^\alpha & u_\beta^\alpha & 0\\ u_\alpha & 0 & u\bar{u}^{-1\beta}_{\ \ \alpha}\end{pmatrix}\begin{pmatrix}\varpi\\ \varpi^\beta\\ \varpi_\beta\end{pmatrix},$$

with inverse:

$$\begin{pmatrix}\varpi\\ \varpi^\alpha\\ \varpi_\alpha\end{pmatrix} \;=\; \begin{pmatrix} u^{-1} & 0 & 0\\ -\sum_\beta u^{-1}\bar{u}^{-1\alpha}_{\ \ \beta}u^\beta & \bar{u}^{-1\alpha}_{\ \ \beta} & 0\\ -\sum_\beta u^{-2}u_\alpha^\beta u_\beta & 0 & u^{-1}u_\alpha^\beta\end{pmatrix}\begin{pmatrix}\omega\\ \omega^\beta\\ \omega_\beta\end{pmatrix},$$

where:

$$\boldsymbol{\delta}_\beta^\alpha \;=\; \sum_\gamma\, \bar{u}^{-1\alpha}_{\ \ \gamma}\,u_\beta^\gamma \;=\; \sum_\gamma\, u_\gamma^\alpha\,\bar{u}^{-1\gamma}_{\ \ \beta}, \tag{3.5}$$

4 Reduced G_2-structure and Stabilization

After this first group-parameter normalization $v_\alpha^\beta = u\,\bar{u}^{-1\beta}_{\ \ \alpha}$, the Cartan procedure requires to restart the computations, to calculate the exterior differentials:

$$\begin{aligned} d\omega \;&=\; du\wedge\varpi+u\cdot d\varpi,\\ d\omega^\alpha \;&=\; du^\alpha\wedge\varpi+u^\alpha\cdot d\varpi+\sum_\beta\, du_\beta^\alpha\wedge\varpi^\beta+\sum_\beta\, u_\beta^\alpha\cdot\underline{d\varpi^\beta}_{\circ},\\ d\omega_\alpha \;&=\; du_\alpha\wedge\varpi+u_\alpha\cdot d\varpi+\sum_\beta\, du\wedge\bar{u}^{-1\beta}_{\ \ \alpha}\varpi_\beta+\\ &\qquad\qquad +\sum_\beta\, u\,d\bar{u}^{-1\beta}_{\ \ \alpha}\wedge\varpi_\beta+\sum_\beta\, u\,\bar{u}^{-1\beta}_{\ \ \alpha}\cdot d\varpi_\beta, \end{aligned} \tag{4.1}$$

and to re-express the right-hand sides in terms of the lifted coframe $\big\{\omega,\omega^\alpha,\omega_\alpha\big\}$.

Consequently, beyond Lemma 2.3, it is necessary to re-express at first $d\varpi$, $d\varpi^\alpha$, $d\varpi_\alpha$ in the lifted coframe. To this aim, in continuation with Lemma 2.2, a preliminary is needed.

Lemma 4.2. *The exterior differential dG of any $\mathbb{K}$-analytic function $G = G\big(x^\alpha, y, y_{x^\beta}\big)$ expresses as:*

$$\begin{aligned} dG \,&=\, \omega \cdot \Big[u^{-1} G_y - \sum_\alpha \sum_\beta u^{-1} \bar{u}^{1\beta}_\alpha \, u^\alpha \, \mathsf{D}_{x^\beta}(G) - \sum_\alpha \sum_\beta u^{-2} u^\alpha_\beta \, u_\alpha \, G_{y_{x^\beta}} \Big] + \\ &+ \sum_\alpha \omega^\alpha \cdot \Big[\sum_\beta \bar{u}^{1\beta}_\alpha \, \mathsf{D}_{x^\beta}(G) \Big] + \sum_\alpha \omega_\alpha \cdot \Big[\sum_\beta u^{-1} u^\alpha_\beta \, G_{y_{x^\beta}} \Big]. \end{aligned}$$

Proof. Back to the expression, given by Lemma 2.2, of dG in the coframe $\big\{\varpi, \varpi^\alpha, \varpi_\alpha\big\}$, it suffices to replace, using (3.4):

$$\begin{aligned} dG \,&=\, G_y \cdot \varpi + \sum_\alpha \mathsf{D}_{x^\alpha}(G) \cdot \varpi^\alpha + \sum_\alpha G_{y_{x^\alpha}} \cdot \varpi_\alpha \\ &=\, u^{-1} \, G_y \cdot \omega - \sum_\alpha \sum_\beta u^{-1} \bar{u}^{1\alpha}_\beta \, u^\beta \, \mathsf{D}_{x^\alpha}(G) \cdot \omega + \sum_\alpha \sum_\beta \bar{u}^{1\alpha}_\beta \, \mathsf{D}_{x^\alpha}(G) \cdot \omega^\beta - \\ &- \sum_\alpha \sum_\beta u^{-2} u^\beta_\alpha \, u_\beta \, G_{y_{x^\alpha}} \cdot \omega + \sum_\alpha \sum_\beta u^{-1} u^\beta_\alpha \, G_{y_{x^\alpha}} \cdot \omega_\beta, \end{aligned}$$

to collect terms:

$$\begin{aligned} dG \,&=\, \omega \cdot \Big[u^{-1} G_y - \sum_\alpha \sum_\beta u^{-1} \bar{u}^{1\alpha}_\beta \, u^\beta \, \mathsf{D}_{x^\alpha}(G) - \sum_\alpha \sum_\beta u^{-2} u^\beta_\alpha \, u_\beta \, G_{y_{x^\alpha}} \Big] + \\ &+ \sum_\beta \omega^\beta \cdot \Big[\sum_\alpha \bar{u}^{1\alpha}_\beta \, \mathsf{D}_{x^\alpha}(G) \Big] + \sum_\beta \omega_\beta \cdot \Big[\sum_\alpha u^{-1} u^\beta_\alpha \, G_{y_{x^\alpha}} \Big], \end{aligned}$$

and to change indices $\alpha \longleftrightarrow \beta$. □

Proposition 4.3. *In the coframe $\big\{\omega, \omega^\alpha, \omega_\alpha\big\}$:*

$$\begin{aligned} d\varpi \,&=\, \sum_\beta u^{-2} u_\beta \cdot \omega \wedge \omega^\beta - \sum_\beta u^{-2} u^\beta \cdot \omega \wedge \omega_\beta + \sum_\beta u^{-1} \cdot \omega^\beta \wedge \omega_\beta, \\ d\varpi^\alpha \,&=\, 0, \\ d\varpi_\alpha \,&=\, \sum_\beta \omega \wedge \omega^\beta \cdot \Big[- \sum_\gamma u^{-1} \bar{u}^{1\gamma}_\beta \, F^{\alpha,\gamma}_y + \sum_\gamma \sum_\delta \sum_\varepsilon u^{-2} u^\varepsilon_\gamma \, \bar{u}^{1\delta}_\beta \, u_\varepsilon \, F^{\alpha,\delta}_{y_{x^\gamma}} \Big] + \\ &+ \sum_\beta \omega \wedge \omega_\beta \cdot \Big[- \sum_\gamma \sum_\delta \sum_\varepsilon u^{-2} u^\beta_\gamma \, \bar{u}^{1\varepsilon}_\delta \, u^\delta \, F^{\alpha,\varepsilon}_{y_{x^\gamma}} \Big] + \\ &+ \sum_\beta \sum_\gamma \omega^\beta \wedge \omega_\gamma \cdot \Big[\sum_\delta \sum_\varepsilon u^{-1} u^\gamma_\varepsilon \, \bar{u}^{1\delta}_\beta \, F^{\alpha,\delta}_{y_{x^\varepsilon}} \Big]. \end{aligned} \tag{4.4}$$

Proof. Back to Lemma 2.3, for the first work, it is necessary to replace and to expand:

$$\begin{aligned} d\varpi \,&=\, \sum_\beta \varpi^\beta \wedge \varpi_\beta \\ &=\, \sum_\beta \Big(- \sum_\gamma u^{-1} \bar{u}^{1\beta}_\gamma \, u^\gamma \cdot \omega + \sum_\gamma \bar{u}^{1\beta}_\gamma \cdot \omega^\gamma \Big) \wedge \Big(- \sum_\delta u^{-2} u^\delta_\beta \, u_\delta \cdot \omega + \sum_\delta u^{-1} u^\delta_\beta \cdot \omega_\delta \Big) \\ &=\, - \sum_\beta \sum_\gamma \sum_\delta u^{-1} \bar{u}^{1\beta}_\gamma \, u^\gamma \, u^{-1} u^\delta_\beta \cdot \omega \wedge \omega_\delta - \\ &- \sum_\beta \sum_\gamma \sum_\delta \bar{u}^{1\beta}_\gamma \, u^{-2} u^\delta_\beta \, u_\delta \cdot \omega^\gamma \wedge \omega + \\ &+ \sum_\beta \sum_\gamma \sum_\delta \bar{u}^{1\beta}_\gamma \, u^{-1} u^\delta_\beta \cdot \omega^\gamma \wedge \omega_\delta, \end{aligned}$$

to reorganize:

[Equations (3.5)]

$$\begin{aligned} d\varpi \,&=\, - \sum_\gamma \sum_\delta u^{-2} \Big(\sum_\beta u^\delta_\beta \, \bar{u}^{1\beta}_\gamma \Big) \, u^\gamma \cdot \omega \wedge \omega_\delta - \\ &- \sum_\gamma \sum_\delta u^{-2} \underbrace{\Big(\sum_\beta u^\delta_\beta \, \bar{u}^{1\beta}_\gamma \Big)}_{= \boldsymbol{\delta}^\delta_\gamma} u_\delta \cdot \omega^\gamma \wedge \omega + \\ &+ \sum_\gamma \sum_\delta u^{-1} \Big(\sum_\beta u^\delta_\beta \, \bar{u}^{1\beta}_\gamma \Big) \cdot \omega^\gamma \wedge \omega_\delta, \end{aligned}$$

and to realize three sum cancellations:

$$d\varpi \;=\; -\sum_{\gamma}\, u^{-2}u^{\gamma}\cdot\omega\wedge\omega_{\gamma} - \sum_{\gamma}\, u^{-2}u_{\gamma}\cdot\omega^{\gamma}\wedge\omega + \sum_{\gamma}\, u^{-1}\cdot\omega^{\gamma}\wedge\omega_{\gamma},$$

to reach the first line of (4.4).

Since $d\varpi^{\alpha}=0$ trivially, the third and last work starts from Lemma 2.3 as:

$$\begin{aligned}
d\varpi_{\alpha} \;&=\; -\sum_{\beta}\, F_{y}^{\alpha,\beta}\cdot\varpi\wedge\varpi^{\beta} + \sum_{\beta}\sum_{\gamma}\, F_{y_{x}\gamma}^{\alpha,\beta}\cdot\varpi^{\beta}\wedge\varpi_{\gamma}\\
&=\; -\sum_{\beta}\, F_{y}^{\alpha,\beta}\cdot\big(u^{-1}\cdot\omega\big)\wedge\Big(-\sum_{\gamma}\, u^{-1}\bar{u}^{1\beta}_{\gamma}\,u^{\gamma}\cdot\omega + \sum_{\gamma}\,\bar{u}^{1\beta}_{\gamma}\cdot\omega^{\gamma}\Big) +\\
&\quad\; +\sum_{\beta}\sum_{\gamma}\, F_{y_{x}\gamma}^{\alpha,\beta}\cdot\Big(-\sum_{\delta}\, u^{-1}\bar{u}^{1\beta}_{\delta}\,u^{\delta}\cdot\omega + \sum_{\delta}\,\bar{u}^{1\beta}_{\delta}\cdot\omega^{\delta}\Big)\wedge\Big(-\sum_{\varepsilon}\, u^{-2}u^{\varepsilon}_{\gamma}\,u_{\varepsilon}\cdot\omega + \sum_{\varepsilon}\, u^{-1}u^{\varepsilon}_{\gamma}\cdot\omega_{\varepsilon}\Big),
\end{aligned}$$

and this becomes:

$$\begin{aligned}
[\gamma\longleftrightarrow\beta]\qquad\qquad d\varpi_{\alpha} \;&=\; -\sum_{\beta}\sum_{\gamma}\, u^{-1}\bar{u}^{1\beta}_{\gamma}\,F_{y}^{\alpha,\beta}\cdot\omega\wedge\omega^{\gamma} -\\
[\varepsilon\longleftrightarrow\beta]\qquad\qquad &\quad\; -\sum_{\beta}\sum_{\gamma}\sum_{\delta}\sum_{\varepsilon}\, u^{-1}\bar{u}^{1\beta}_{\delta}\,u^{\delta}\,u^{-1}u^{\varepsilon}_{\gamma}\,F_{y_{x}\gamma}^{\alpha,\beta}\cdot\omega\wedge\omega_{\varepsilon} -\\
[\delta\longleftrightarrow\beta]\qquad\qquad &\quad\; -\sum_{\beta}\sum_{\gamma}\sum_{\delta}\sum_{\varepsilon}\,\bar{u}^{1\beta}_{\delta}\,u^{-2}u^{\varepsilon}_{\gamma}\,u_{\varepsilon}\,F_{y_{x}\gamma}^{\alpha,\beta}\cdot\omega^{\delta}\wedge\omega +\\
[\delta\longleftrightarrow\beta\text{ and }\varepsilon\longleftrightarrow\gamma]\qquad &\quad\; +\sum_{\beta}\sum_{\gamma}\sum_{\delta}\sum_{\varepsilon}\,\bar{u}^{1\beta}_{\delta}\,u^{-1}u^{\varepsilon}_{\gamma}\,F_{y_{x}\gamma}^{\alpha,\beta}\cdot\omega^{\delta}\wedge\omega_{\varepsilon},
\end{aligned}$$

while the left-mentioned changes of indices followed by final reordering concludes. □

Corollary 4.5. *The complete integrability of Corollary 2.4 reads as:*

$$\begin{aligned}
d\omega \;&\equiv\; 0 \mod\big\langle\omega,\omega^{\beta}\big\rangle \qquad \textit{and} \qquad d\omega^{\alpha} \;\equiv\; 0 \mod\big\langle\omega,\omega^{\beta}\big\rangle,\\
d\omega \;&\equiv\; 0 \mod\big\langle\omega,\omega_{\beta}\big\rangle \qquad \textit{and} \qquad d\omega_{\alpha} \;\equiv\; 0 \mod\big\langle\omega,\omega_{\beta}\big\rangle.
\end{aligned}$$

Proof. A look at formulas (4.1), (3.4), (4.4):

$$\begin{array}{ccccc}
d\omega & \in & \big\langle\varpi,d\varpi\big\rangle & \subset & \big\langle\omega,\omega^{\beta}\big\rangle\cap\big\langle\omega,\omega_{\beta}\big\rangle,\\
d\omega^{\alpha} & \in & \big\langle\varpi,d\varpi,\varpi^{\beta}\big\rangle & \subset & \big\langle\omega,\omega^{\beta}\big\rangle,\\
d\omega_{\alpha} & \in & \big\langle\varpi,d\varpi,\varpi_{\beta},d\varpi_{\beta}\big\rangle & \subset & \big\langle\omega,\omega_{\beta}\big\rangle,
\end{array}$$

makes the claims transparent. □

With these preliminaries at hand, the computation of $d\omega$, unfinished in Section **3**, can now be finalized:

$$\begin{aligned}
d\omega \;&=\; du\wedge\varpi + u\cdot d\varpi\\
&=\; u^{-1}\,du\wedge\omega + u\cdot\Big(\sum_{\beta}\, u^{-2}u_{\beta}\cdot\omega\wedge\omega^{\beta} - \sum_{\beta}\, u^{-2}u^{\beta}\cdot\omega\wedge\omega_{\beta} + \sum_{\beta}\, u^{-1}\cdot\omega^{\beta}\wedge\omega_{\beta}\Big)\\
&=:\; \omega\wedge\varphi + \sum_{\beta}\,\omega^{\beta}\wedge\omega_{\beta},
\end{aligned}$$

with:

$$\varphi \;:=\; -\,u^{-1}\,du + \sum_{\beta}\, u^{-1}u_{\beta}\cdot\omega^{\beta} - \sum_{\beta}\, u^{-1}u^{\beta}\cdot\omega_{\beta}.$$

5 Unparametric Cartan Lemma Reasonings

Since subsequent computations start to become harder, it is appropriate to employ unparametric reasonings to anticipate and to economize completely explicit computations. All differential forms now live in the space of variables:

$$\big(x^\alpha, y, y_{x^\beta};\ u, u^\alpha_\beta, u^\alpha, u_\alpha\big).$$

Proposition 5.1. *After absorption of torsion, there exist modified Maurer-Cartan* 1*-forms*

$$\varphi, \quad \varphi^\alpha_\beta, \quad \varphi^\alpha, \quad \varphi_\alpha,$$

such that the structure equations for the G_2*-lifted coframe read as:*

$$\begin{aligned} d\omega &= \sum_\alpha \omega^\alpha \wedge \omega_\alpha + \omega \wedge \varphi, \\ d\omega^\alpha &= \sum_\beta \omega^\beta \wedge \varphi^\alpha_\beta + \omega \wedge \varphi^\alpha, \\ d\omega_\alpha &= \sum_\beta \varphi^\beta_\alpha \wedge \omega_\beta + \omega_\alpha \wedge \varphi + \omega \wedge \varphi_\alpha. \end{aligned}$$

The thing is that no (essential) torsion coefficients remain.

Proof. The first structure equation was already obtained above:

$$d\omega = \omega \wedge \varphi + \sum_\alpha \omega^\alpha \wedge \omega_\alpha. \tag{5.2}$$

Next, back to (4.1), replace ϖ, $d\varpi$, ϖ^β:

$$\begin{aligned} d\omega^\alpha &= du^\alpha \wedge \varpi + u^\alpha\, d\varpi + \sum_\beta du^\alpha_\beta \wedge \varpi^\beta + 0 \\ &= du^\alpha \wedge \big(u^{-1}\omega\big) + u^\alpha \Big(\sum_\beta u^{-2}u_\beta \cdot \omega \wedge \omega^\beta - \sum_\beta u^{-2}u^\beta \cdot \omega \wedge \omega_\beta + \sum_\beta u^{-1} \cdot \omega^\beta \wedge \omega_\beta\Big) + \\ &\qquad + \sum_\beta du^\alpha_\beta \wedge \Big(- \sum_\gamma u^{-1}\bar{u}^{1\beta}_{\ \gamma}\, u^\gamma \cdot \omega + \sum_\gamma \bar{u}^{1\beta}_{\ \gamma} \cdot \omega^\gamma\Big). \end{aligned}$$

All terms here are either multiple of ω or of some ω^β, hence there are 1-forms φ^α_β, φ^α such that:

$$d\omega^\alpha = \sum_\beta \omega^\beta \wedge \varphi^\alpha_\beta + \omega \wedge \varphi^\alpha. \tag{5.3}$$

Without finishing the computation (a task that will be done later), it is visible here that:

$$\begin{aligned} \varphi^\alpha_\beta &= -\sum_\gamma \bar{u}^{1\gamma}_{\ \beta} \cdot du^\alpha_\gamma + \text{something}, \\ \varphi^\alpha &= -u^{-1} \cdot du^\alpha + \sum_\beta \sum_\gamma u^{-1}\bar{u}^{1\beta}_{\ \gamma}\, u^\gamma \cdot du^\alpha_\beta + \text{something}, \end{aligned}$$

but the choice of φ^α_β, φ^α is not unique, as will be discussed later.

The last work concerns $d\omega_\alpha$, and is a bit delicate. Complete computations will be done in the next Section **6**. However, it is interesting to see how expanded computations can be somehow avoided, getting some information about the shape of $d\omega_\alpha$ *without replacing* ϖ, $d\varpi$, ϖ_β, $d\varpi_\beta$ in the third group of equations (4.1).

The (indirect) trick is to differentiate (5.2) and to replace $d\omega$, $d\omega^\alpha$, but *not* $d\omega_\alpha$, about which an information is wanted:

$$\begin{aligned} 0 &= dd\omega \\ &= \underset{\text{replace}}{d\omega} \wedge \varphi - \omega \wedge d\varphi + \sum_\alpha \underset{\text{replace}}{d\omega^\alpha} \wedge \omega_\alpha - \sum_\alpha \omega^\alpha \wedge d\omega_\alpha \\ &= \Big(\underline{\omega \wedge \varphi_\circ} + \sum_\alpha \omega^\alpha \wedge \omega_\alpha\Big) \wedge \varphi - \omega \wedge d\varphi + \\ &\qquad + \sum_\alpha \Big(\sum_\beta \omega^\beta \wedge \varphi^\alpha_\beta + \omega \wedge \varphi^\alpha\Big) \wedge \omega_\alpha - \sum_\alpha \omega^\alpha \wedge d\omega_\alpha \end{aligned}$$

which is:

$$0 \;=\; \sum_\alpha \omega^\alpha \wedge \Big(\underbrace{-\,d\omega_\alpha + \sum_\beta \varphi^\beta_\alpha \wedge \omega_\beta + \omega_\alpha \wedge \varphi}_{=:\,\Omega_\alpha} \Big) + \omega \wedge \Big(-\,d\varphi + \sum_\alpha \varphi^\alpha \wedge \omega_\alpha \Big). \tag{5.4}$$

To this identity, with the ideal of 1-forms $\langle \omega^\alpha, \omega \rangle$ on the manifold $\mathbb{K}^{2n+1} \times \mathbb{K}^{n^2+2n+1}$ equipped with the coframe $\big\{\omega, \omega^\alpha, \omega_\alpha;\; du, du^\alpha_\beta, du^\alpha, du_\alpha\big\}$, the Cartan Lemma 12.1 applies and shows that:

$$\begin{aligned} \Omega_\alpha \;&:=\; -\,d\omega_\alpha + \sum_\beta \varphi^\beta_\alpha \wedge \omega_\beta + \omega_\alpha \wedge \varphi \\ &\equiv\; 0 \mod \big\langle \omega, \omega^\gamma \big\rangle, \end{aligned}$$

the same being true for:

$$0 \;\equiv\; -\,d\varphi + \sum_\alpha \varphi^\alpha \wedge \omega_\alpha \mod \big\langle \omega, \omega^\gamma \big\rangle,$$

although this will not be useful for the moment.

On the other hand, since Corollary 4.5 showed that $d\omega_\alpha \equiv 0$ modulo $\langle \omega, \omega_\gamma \rangle$, it comes:

$$-\,d\omega_\alpha + \sum_\beta \varphi^\beta_\alpha \wedge \omega_\beta + \omega_\alpha \wedge \varphi \;\equiv\; 0 \mod \big\langle \omega, \omega_\gamma \big\rangle,$$

hence from these two conditions:

$$\Omega_\alpha \mod \langle \omega \rangle \;\equiv\; \begin{cases} 0 \mod \langle \omega^\gamma \rangle, \\ 0 \mod \langle \omega_\gamma \rangle, \end{cases}$$

it comes that there are functions $A^\beta_{\alpha,\gamma}$ on $\mathbb{K}^{2n+1} \times \mathbb{K}^{n^2+2n+1}$ such that:

$$\Omega_\alpha \;\equiv\; \sum_\beta \sum_\gamma A^\beta_{\alpha,\gamma} \cdot \omega^\gamma \wedge \omega_\beta \mod \langle \omega \rangle.$$

Assertion 5.5. *Index symmetries hold:*

$$A^\beta_{\gamma,\alpha} \;=\; A^\beta_{\alpha,\gamma} \qquad (1 \leqslant \alpha, \beta, \gamma \leqslant n).$$

Proof. Indeed, back to (5.4) which becomes:

$$0 \;\equiv\; \sum_\alpha \omega^\alpha \wedge \Omega_\alpha \mod \langle \omega \rangle,$$

insert:

$$0 \;\equiv\; \sum_\alpha \sum_\beta \sum_\gamma A^\beta_{\alpha,\gamma} \cdot \omega^\alpha \wedge \omega^\gamma \wedge \omega_\beta \mod \langle \omega \rangle,$$

collect $\sum_{\alpha<\gamma}$, and identify to 0 the coefficients of $\omega^\alpha \wedge \omega^\gamma$. □

Next, replace:

$$\begin{aligned} \varphi^\beta_\alpha \;&\longmapsto\; \varphi^\beta_\alpha - \sum_\gamma A^\beta_{\alpha,\gamma} \cdot \omega^\gamma \\ &=:\; \widetilde{\varphi}^\beta_\alpha, \end{aligned}$$

which, thanks to the above symmetries:

$$\begin{aligned} d\omega^\alpha \;&=\; \sum_\beta \omega^\beta \wedge \varphi^\alpha_\beta + \omega \wedge \varphi^\alpha \\ &=\; \sum_\beta \omega^\beta \wedge \Big(\widetilde{\varphi}^\alpha_\beta + \sum_\gamma A^\alpha_{\beta,\gamma} \cdot \omega^\gamma \Big) + \omega \wedge \varphi^\alpha \\ &=\; \sum_\beta \omega^\beta \wedge \widetilde{\varphi}^\alpha_\beta + \omega \wedge \varphi^\alpha, \end{aligned}$$

leaves unchanged the previously obtained structure equations (5.3), in order that the new 2-form:

$$\begin{aligned}\widetilde{\Omega}_\alpha \,&:=\, \Omega_\alpha - \sum_\beta \sum_\gamma A^\beta_{\alpha,\gamma}\cdot\omega^\gamma\wedge\omega_\beta\\ &=\, -d\omega_\alpha + \sum_\beta \varphi^\beta_\alpha\wedge\omega_\beta + \omega_\alpha\wedge\varphi - \sum_\beta\sum_\gamma A^\beta_{\alpha,\gamma}\cdot\omega^\gamma\wedge\omega_\beta\\ &=\, -d\omega_\alpha + \sum_\beta \widetilde{\varphi}^\beta_\alpha\wedge\omega_\beta + \omega_\alpha\wedge\varphi,\end{aligned}$$

satisfy:

$$\widetilde{\Omega}_\alpha \,\equiv\, 0 \mod \langle\omega\rangle,$$

whence there exist differential 1-forms φ_α such that:

$$-d\omega_\alpha + \sum_\beta \widetilde{\varphi}^\beta_\alpha\wedge\omega_\beta + \omega_\alpha\wedge\varphi \,=\, -\omega\wedge\varphi_\alpha.$$

Renaming $\widetilde{\varphi}^\beta_\alpha$ simply as φ^β_α concludes the proof of the third group of n equations in Proposition 5.1. □

In fact, this work — and the previous as well! — is not achieved properly, because Maurer-Cartan forms should be pointed out precisely, and because starting from:

$$\begin{aligned}d\omega_\alpha \,&=\, du_\alpha\wedge\varpi + u_\alpha\, d\varpi + \sum_\beta du\wedge \bar{u}^{1\beta}_\alpha\varpi_\beta + \sum_\beta u\, d\bar{u}^{1\beta}_\alpha\wedge\varpi_\beta + \sum_\beta u\,\bar{u}^{1\beta}_\alpha\cdot d\varpi_\beta\\ &=\, du_\alpha\wedge\big(u^{-1}\omega\big) + \sum_\beta \bar{u}^{1\beta}_\alpha\, du\wedge\Big(-\sum_\gamma u^{-2}u^\gamma_\beta\, u_\gamma\cdot\omega + \sum_\gamma u^{-1}u^\gamma_\beta\cdot\omega_\gamma\Big) +\\ &\qquad + \sum_\beta u\, d\bar{u}^{1\beta}_\alpha\wedge\Big(-\sum_\gamma u^{-2}u^\gamma_\beta\, u_\gamma\cdot\omega + \sum_\gamma u^{-1}u^\gamma_\beta\cdot\omega_\gamma\Big),\end{aligned}$$

torsion (remainder) terms should be computed before absorption in order to know what φ, φ^α_β, φ^α, φ_α really are.

In particular, it will be necessary to compute:

$$d\bar{u}^{1\beta}_\alpha.$$

6 Parametric Determination of φ, φ^α_β, φ^α, φ_α

Abbreviating (3.3) as:

$$\omega \,=\, g\cdot\varpi,$$

where $g\in G_2$ is a general element of the group G_2 parametrized by u, u^α_β, u^α, u_α, equations (4.1) contract as:

$$\begin{aligned}d\omega \,&=\, dg\wedge\varpi + g\cdot d\varpi\\ &=\, dg\cdot g^{-1}\wedge\omega + g\cdot d\varpi,\end{aligned}$$

hence the very first natural task is to compute the *Maurer-Cartan matrix:*

$$\begin{aligned}dg\cdot g^{-1} \,&=\, \begin{pmatrix} du & 0 & 0\\ du^\alpha & du^\alpha_{\beta\mathrm{C}} & 0\\ du_\alpha & 0 & d\big(u\,\bar{u}^{1\beta\mathrm{C}}_\alpha\big)\end{pmatrix}\cdot\begin{pmatrix} u^{-1} & 0 & 0\\ -\sum_\gamma u^{-1}\bar{u}^{1\beta\mathrm{R}}_\gamma u^\gamma & \bar{u}^{1\beta\mathrm{R}}_\delta & 0\\ -\sum_\gamma u^{-2}u^\gamma_{\beta\mathrm{R}}u_\gamma & 0 & u^{-1}u^\delta_{\beta\mathrm{R}}\end{pmatrix}\\ &=\, \begin{pmatrix} u^{-1}du & 0 & 0\\ u^{-1}du^\alpha - \sum_\beta\sum_\gamma u^{-1}\bar{u}^{1\beta}_\gamma u^\gamma\cdot du^\alpha_\beta & \sum_\beta \bar{u}^{1\beta}_\delta\cdot du^\alpha_\beta & 0\\ u^{-1}du_\alpha - \sum_\beta\sum_\gamma u^{-2}u^\gamma_\beta u_\gamma\cdot d\big(u\bar{u}^{1\beta}_\alpha\big) & 0 & \sum_\beta u^{-1}u^\delta_\beta\cdot d\big(u\bar{u}^{1\beta}_\alpha\big)\end{pmatrix},\end{aligned}$$

specifying rows R of the right factor that couple up with columns C of left factor, simply here are as a $\sum_\beta$. Two entries of the last row incorporate differentials:

$$d\big(u\bar{u}^{1\beta}_{\ \alpha}\big),$$

hence the computation is unfinished.

Lemma 6.1. *For all* $1 \leqslant \alpha, \beta \leqslant n$*:*

$$d\bar{u}^{1\alpha}_{\ \beta} \ = \ -\sum_\gamma \sum_\delta \ \bar{u}^{1\alpha}_{\ \delta} \cdot du^\delta_\gamma \cdot \bar{u}^{1\gamma}_{\ \beta}.$$

Proof. Start from (3.5), rewritten as:

$$\boldsymbol{\delta}^\alpha_\beta \ = \ \sum_\gamma \ u^\alpha_\gamma \, \bar{u}^{1\gamma}_{\ \beta},$$

differentiate it:

$$0 \ = \ \sum_\gamma \ du^\alpha_\gamma \cdot \bar{u}^{1\gamma}_{\ \beta} + \sum_\gamma \ u^\alpha_\gamma \cdot d\bar{u}^{1\gamma}_{\ \beta},$$

apply summation $\sum_\alpha \ \bar{u}^{1\delta}_{\ \alpha}(\cdot)$:

$$\begin{aligned} 0 \ &= \ \sum_\alpha \sum_\gamma \ \bar{u}^{1\delta}_{\ \alpha} \cdot du^\alpha_\gamma \cdot \bar{u}^{1\gamma}_{\ \beta} + \sum_\gamma \ \underbrace{\sum_\alpha \ \bar{u}^{1\delta}_{\ \alpha} \, u^\alpha_\gamma}_{=\boldsymbol{\delta}^\delta_\gamma} \cdot d\bar{u}^{1\gamma}_{\ \beta} \\ [\alpha \longleftrightarrow \delta] \qquad &= \ -\sum_\alpha \sum_\gamma \ \bar{u}^{1\delta}_{\ \alpha} \cdot du^\alpha_\gamma \cdot \bar{u}^{1\gamma}_{\ \beta} + d\bar{u}^{1\delta}_{\ \beta}, \end{aligned}$$

and change indices as indicated on the left to conclude. □

Consequently:

$$dg \cdot g^{-1} \cdot \omega \ = \ \begin{pmatrix} u^{-1}du & 0 & 0 \\ u^{-1}du^\alpha - \sum\limits_\beta \sum\limits_\gamma u^{-1}\bar{u}^{1\beta}_{\ \gamma}u^\gamma \cdot du^\alpha_\beta & \sum\limits_\beta \bar{u}^{1\beta}_{\ \delta} \cdot du^\alpha_\beta & 0 \\ u^{-1}du_\alpha - \sum\limits_\beta \sum\limits_\gamma u^{-2}u^\gamma_\beta u_\gamma \cdot d\big(u\bar{u}^{1\beta}_{\ \alpha}\big) & 0 & \sum\limits_\beta u^{-1}u^\delta_\beta \cdot d\big(u\bar{u}^{1\beta}_{\ \alpha}\big) \end{pmatrix} \wedge \begin{pmatrix} \omega \\ \omega^\delta \\ \omega_\delta \end{pmatrix},$$

which shows:

$$\begin{aligned} d\omega \ &= \ u^{-1}du \wedge \omega + \mathsf{torsion}, \\ d\omega^\alpha \ &= \ \Big(u^{-1}\, du^\alpha - \sum_\beta \sum_\gamma \ u^{-1}\bar{u}^{1\beta}_{\ \gamma}\, u^\gamma \cdot du^\alpha_\beta\Big) \wedge \omega + \sum_\beta \sum_\delta \ \bar{u}^{1\beta}_{\ \delta} \cdot du^\alpha_\beta \wedge \omega^\delta + \mathsf{torsion}^\alpha, \\ d\omega_\alpha \ &= \ \Big(u^{-1}\, du_\alpha - \sum_\beta \sum_\gamma \ u^{-2}\, u^\gamma_\beta\, u_\gamma \cdot d\big(u\, \bar{u}^{1\beta}_{\ \alpha}\big)\Big) \wedge \omega + \sum_\beta \sum_\delta \ u^{-1}\, u^\delta_\beta \cdot d\big(u\bar{u}^{1\beta}_{\ \alpha}\big) \wedge \omega_\delta + \mathsf{torsion}_\alpha, \end{aligned}$$

so that Lemma 6.1 helps to finish the computation of the third line:

$$\begin{aligned} d\omega_\alpha \ &= \ \Big(u^{-1} \cdot du_\alpha - \sum_\beta \sum_\gamma \ u^{-2}u^\gamma_\beta\, u_\gamma \, \big(\bar{u}^{1\beta}_{\ \alpha} \cdot du + u \cdot d\bar{u}^{1\beta}_{\ \alpha}\big)\Big) \wedge \omega + \\ &\quad + \sum_\beta \sum_\delta \ u^{-1}u^\delta_\beta \, \big(\bar{u}^{1\beta}_{\ \alpha} \cdot du + u \cdot d\bar{u}^{1\beta}_{\ \alpha}\big) \wedge \omega_\delta + \mathsf{torsion}_\alpha \\ &= \ \Big(u^{-1}du_\alpha - \sum_\gamma \ u^{-2}u_\gamma \, \big(\sum_\beta \ u^\gamma_\beta \, \bar{u}^{1\beta}_{\ \alpha}\big) \cdot du + \sum_\beta \sum_\gamma \sum_\delta \sum_\varepsilon \ u^{-1}u^\gamma_\beta \, u_\gamma \, \bar{u}^{1\beta}_{\ \delta} \cdot du^\delta_\varepsilon \cdot \bar{u}^{1\varepsilon}_{\ \alpha}\Big) \wedge \omega + \\ &\quad + \sum_\delta \ \Big(u^{-1} \sum_\beta \ u^\delta_\beta \, \bar{u}^{1\beta}_{\ \alpha} \cdot du - \sum_\beta \sum_\varepsilon \sum_\zeta \ u^\delta_\beta \, \bar{u}^{1\beta}_{\ \varepsilon} \cdot du^\varepsilon_\zeta \cdot \bar{u}^{1\zeta}_{\ \alpha}\Big) \wedge \omega_\delta + \mathsf{torsion}_\alpha \\ &= \ \Big(u^{-1} \cdot du_\alpha - u^{-2}u_\alpha \cdot du + \sum_\beta \sum_\gamma \ u^{-1}\bar{u}^{1\gamma}_{\ \alpha}\, u_\beta \cdot du^\beta_\gamma\Big) \wedge \omega + \\ &\quad + \sum_\delta \ \Big(\boldsymbol{\delta}^\delta_\alpha \, u^{-1} \cdot du - \sum_\zeta \ \bar{u}^{1\zeta}_{\ \alpha} \cdot du^\delta_\zeta\Big) \wedge \omega_\delta + \mathsf{torsion}_\alpha, \end{aligned}$$

hence changing indices conducts to a finalization of the Maurer-Cartan part:

$$
\begin{aligned}
d\omega &= \omega \wedge \Big(- u^{-1} \cdot du \Big) + \mathsf{torsion},\\
d\omega^\alpha &= \sum_\beta \omega^\beta \wedge \Big(- \sum_\gamma \bar{u}^{1\gamma}_\beta \cdot du^\alpha_\gamma \Big) +\\
&\qquad + \omega \wedge \Big(- u^{-1} \cdot du^\alpha + \sum_\beta \sum_\gamma u^{-1} \bar{u}^{1\beta}_\gamma \, u^\gamma \cdot du^\alpha_\beta \Big) + \mathsf{torsion}^\alpha,\\
d\omega_\alpha &= \sum_\beta \omega_\beta \wedge \Big(- \boldsymbol{\delta}^\beta_\alpha \, u^{-1} \cdot du + \sum_\gamma \bar{u}^{1\gamma}_\alpha \cdot du^\beta_\gamma \Big) +\\
&\qquad + \omega \wedge \Big(- u^{-1} \cdot du_\alpha + u^{-2} u_\alpha \cdot du - \sum_\beta \sum_\gamma u^{-1} \bar{u}^{1\gamma}_\alpha \, u_\beta \cdot du^\beta_\gamma \Big) + \mathsf{torsion}_\alpha.
\end{aligned}
\tag{6.2}
$$

The next task is to determine the torsion remainders. For the first (easy) line, this has already been done at the end of Section **4**.

For the second n lines, thanks to (4.4), it is similarly easy:

$$
\begin{aligned}
\mathsf{torsion}^\alpha &= u^\alpha \cdot d\varpi\\
&= \sum_\beta u^{-2} u^\alpha \, u_\beta \cdot \omega \wedge \omega^\beta - \sum_\beta u^{-2} u^\alpha \, u^\beta \cdot \omega \wedge \omega_\beta + \sum_\beta u^{-1} u^\alpha \cdot \omega^\beta \wedge \omega_\beta .
\end{aligned}
$$

For the last n lines, the use of (4.4) requires more efforts:

$$
\begin{aligned}
\mathsf{torsion}_\alpha &= u_\alpha \cdot d\varpi + \sum_\beta u \, \bar{u}^{1\beta}_\alpha \cdot d\varpi_\beta\\
&= \sum_\beta u^{-2} u_\alpha \, u_\beta \cdot \omega \wedge \omega^\beta - \sum_\beta u^{-2} u_\alpha \, u^\beta \cdot \omega \wedge \omega_\beta + \sum_\beta u^{-1} u_\alpha \cdot \omega^\beta \wedge \omega_\beta +\\
&\qquad [\beta \longleftrightarrow \gamma] \qquad + \sum_\beta \sum_\gamma \omega \wedge \omega^\gamma \cdot \Big(- \sum_\delta \bar{u}^{1\beta}_\alpha \, \bar{u}^{1\delta}_\gamma \, F^{\beta,\delta}_y + \sum_\delta \sum_\varepsilon \sum_\zeta u^{-1} \bar{u}^{1\beta}_\alpha \, u^\zeta_\delta \, \bar{u}^{1\varepsilon}_\gamma \, u_\zeta \, F^{\beta,\varepsilon}_{y_{x^\delta}} \Big) +\\
&\qquad [\beta \longleftrightarrow \gamma] \qquad + \sum_\beta \sum_\gamma \omega \wedge \omega_\gamma \cdot \Big(- \sum_\delta \sum_\varepsilon \sum_\zeta u^{-1} \bar{u}^{1\beta}_\alpha \, u^\gamma_\delta \, \bar{u}^{1\zeta}_\varepsilon \, u^\varepsilon \, F^{\beta,\zeta}_{y_{x^\delta}} \Big) +\\
&\qquad (\beta,\gamma,\delta) \longmapsto (\delta,\beta,\gamma)] \qquad + \sum_\beta \sum_\gamma \sum_\delta \omega^\gamma \wedge \omega_\delta \cdot \Big(\sum_\varepsilon \sum_\zeta \bar{u}^{1\beta}_\alpha \, u^\delta_\zeta \, \bar{u}^{1\varepsilon}_\gamma \, F^{\beta,\varepsilon}_{y_{x^\zeta}} \Big)\\
&= \sum_\beta \omega \wedge \omega^\beta \cdot \Big(u^{-2} u_\alpha \, u_\beta - \sum_\gamma \sum_\delta \bar{u}^{1\gamma}_\alpha \, \bar{u}^{1\delta}_\beta \, F^{\gamma,\delta}_y + \sum_\gamma \sum_\delta \sum_\varepsilon \sum_\zeta u^{-1} \bar{u}^{1\gamma}_\alpha \, u^\zeta_\delta \, \bar{u}^{1\varepsilon}_\beta \, u_\zeta \, F^{\gamma,\varepsilon}_{y_{x^\delta}} \Big) +\\
&\qquad + \sum_\beta \omega \wedge \omega_\beta \cdot \Big(- u^{-2} u_\alpha \, u^\beta - \sum_\gamma \sum_\delta \sum_\varepsilon \sum_\zeta u^{-1} \bar{u}^{1\gamma}_\alpha \, u^\beta_\delta \, \bar{u}^{1\zeta}_\varepsilon \, u^\varepsilon \, F^{\gamma,\zeta}_{y_{x^\delta}} \Big) +\\
&\qquad + \sum_\beta \sum_\gamma \omega^\beta \wedge \omega_\gamma \cdot \Big(\boldsymbol{\delta}^\gamma_\beta \, u^{-1} u_\alpha + \sum_\delta \sum_\varepsilon \sum_\zeta \bar{u}^{1\delta}_\alpha \, u^\gamma_\zeta \, \bar{u}^{1\varepsilon}_\beta \, F^{\delta,\varepsilon}_{y_{x^\zeta}} \Big).
\end{aligned}
$$

When summing up what has been done, put in advance two "$+0$" terms which are for the moment inexistent in the computations above, but will soon be replaced with terms that *are zero*, and *are useful* for subsequent computations.

Lemma 6.3. *After absorption of torsion, the structure equations are:*

$$
\begin{aligned}
d\omega &= \sum_\beta \omega^\beta \wedge \omega_\beta + \omega \wedge \Big(- u^{-1} \cdot du + \sum_\beta u^{-1} u_\beta \cdot \omega^\beta - \sum_\beta u^{-1} u^\beta \cdot \omega_\beta \Big), \\
d\omega^\alpha &= \sum_\beta \omega^\beta \wedge \Big(- \sum_\gamma \bar{u}^{1\gamma}_{\ \beta} \cdot du^\alpha_\gamma + u^{-1} u^\alpha \cdot \omega_\beta + \mathbf{0} \Big) + \\
& \quad + \omega \wedge \Big(- u^{-1} \cdot du^\alpha + \sum_\beta \sum_\gamma u^{-1} \bar{u}^{1\beta}_{\ \gamma} u^\gamma \cdot du^\alpha_\beta + \sum_\beta u^{-2} u^\alpha u_\beta \cdot \omega^\beta - \sum_\beta u^{-2} u^\alpha u^\beta \cdot \omega_\beta \Big), \\
d\omega_\alpha &= \sum_\beta \omega_\beta \wedge \Big(- \boldsymbol{\delta}^\beta_\alpha u^{-1} \cdot du + \sum_\gamma \bar{u}^{1\gamma}_{\ \alpha} \cdot du^\beta_\gamma + \sum_\gamma \Big[- \boldsymbol{\delta}^\beta_\gamma u^{-1} u_\alpha - \sum_\delta \sum_\varepsilon \sum_\zeta \bar{u}^{1\delta}_{\ \alpha} u^\beta_\zeta \bar{u}^{1\varepsilon}_{\ \gamma} F^{\delta,\varepsilon}_{y_x \zeta} \Big] \cdot \omega^\gamma + \mathbf{0} \Big) + \\
& \quad + \omega \wedge \Big(- u^{-1} \cdot du_\alpha + u^{-2} u_\alpha \cdot du - \sum_\beta \sum_\gamma u^{-1} \bar{u}^{1\gamma}_{\ \alpha} u_\beta \cdot du^\beta_\gamma + \\
& \qquad + \sum_\beta \Big[u^{-2} u_\beta u_\alpha - \sum_\gamma \sum_\delta \bar{u}^{1\gamma}_{\ \alpha} \bar{u}^{1\delta}_{\ \beta} F^{\gamma,\delta}_{y} + \sum_\gamma \sum_\delta \sum_\varepsilon \sum_\zeta u^{-1} \bar{u}^{1\gamma}_{\ \alpha} u^\zeta_\delta \bar{u}^{1\varepsilon}_{\ \beta} u_\zeta F^{\gamma,\varepsilon}_{y_x \delta} \Big] \cdot \omega^\beta + \\
& \qquad + \sum_\beta \Big[- u^{-2} u^\beta u_\alpha - \sum_\gamma \sum_\delta \sum_\varepsilon \sum_\zeta u^{-1} \bar{u}^{1\gamma}_{\ \alpha} \bar{u}^{1\zeta}_{\ \varepsilon} u^\beta_\delta u^\varepsilon F^{\gamma,\zeta}_{y_x \delta} \Big] \cdot \omega_\beta \Big). \qquad \square
\end{aligned}
$$

Indeed, to get an appropriate collection of modified Maurer-Cartan 1-forms φ, φ^α_β, φ^α, φ_α as in Proposition 5.1, a final adaptation must be done. The choice is not unique, but a natural choice is as follows. The first "$+\mathbf{0}$" term will be:

$$
\begin{aligned}
\sum_\beta \omega^\beta \wedge \big(+ \mathbf{0} \big) &= \sum_\beta \omega^\beta \wedge \Big(\boldsymbol{\delta}^\alpha_\beta \sum_\gamma u^{-1} u_\gamma \cdot \omega^\gamma + u^{-1} u_\beta \cdot \omega^\alpha + \sum_\gamma \sum_\delta \sum_\varepsilon \sum_\zeta \bar{u}^{1\varepsilon}_{\ \gamma} u^\alpha_\zeta \bar{u}^{1\delta}_{\ \beta} F^{\delta,\varepsilon}_{y_x \zeta} \, \omega^\gamma, \Big) \\
&= \omega^\alpha \wedge \big(\sum_\gamma u^{-1} u_\gamma \cdot \omega^\gamma \big) + \sum_\beta u^{-1} u_\beta \cdot \omega^\beta \wedge \omega^\alpha + \\
{\scriptstyle [F^{\varepsilon,\delta}_{y_x \zeta} = F^{\delta,\varepsilon}_{y_x \zeta}]} \qquad & \quad + \sum_{1 \leqslant \beta < \gamma \leqslant n} \Big(\underline{\sum_\delta \sum_\varepsilon \sum_\zeta \bar{u}^{1\varepsilon}_{\ \gamma} u^\alpha_\zeta \bar{u}^{1\delta}_{\ \beta} F^{\delta,\varepsilon}_{y_x \zeta} - \sum_\delta \sum_\varepsilon \sum_\zeta \bar{u}^{1\varepsilon}_{\ \beta} u^\alpha_\zeta \bar{u}^{1\delta}_{\ \gamma} F^{\delta,\varepsilon}_{y_x \zeta}}_{\circ} \Big) \cdot \omega^\beta \wedge \omega^\gamma \\
&= \mathbf{0} + \mathbf{0}
\end{aligned}
$$

which is indeed zero, and the second term will also be null:

$$
\begin{aligned}
\sum_\beta \omega_\beta \wedge \big(+ \mathbf{0} \big) &= \sum_\beta \omega_\beta \wedge \Big(- \boldsymbol{\delta}^\beta_\alpha \sum_\gamma u^{-1} u^\gamma \cdot \omega_\gamma - u^{-1} u^\beta \cdot \omega_\alpha \Big) \\
&= \mathbf{0}.
\end{aligned}
$$

Remind that the 1-form φ has already been introduced:

$$
d\omega = \sum_\beta \omega^\beta \wedge \omega_\beta + \omega \wedge \Big(\underbrace{-u^{-1} \cdot du + \sum_\beta u^{-1} u_\beta \cdot \omega^\beta - \sum_\beta u^{-1} u^\beta \cdot \omega_\beta}_{=:\varphi} \Big)
$$

has already been defined. Next, adding the mentioned first term:

$$
\begin{aligned}
d\omega^\alpha &= \sum_\beta \omega^\beta \wedge \Big(\underbrace{-\sum_\gamma \bar{u}^{1\gamma}_{\ \beta} \cdot du^\alpha_\gamma + u^{-1} u^\alpha \cdot \omega_\beta + \boldsymbol{\delta}^\alpha_\beta \sum_\gamma u^{-1} u_\gamma \cdot \omega^\gamma + u^{-1} u_\beta \cdot \omega^\alpha + \sum_\gamma \sum_\delta \sum_\varepsilon \sum_\zeta \bar{u}^{1\varepsilon}_{\ \gamma} u^\alpha_\zeta \bar{u}^{1\delta}_{\ \beta} F^{\delta,\varepsilon}_{y_x \zeta} \cdot \omega^\gamma}_{=:\, \varphi^\alpha_\beta} \Big) + \\
& \quad + \omega \wedge \Big(\underbrace{-u^{-1} \cdot du^\alpha + \sum_\beta \sum_\gamma u^{-1} \bar{u}^{1\beta}_{\ \gamma} u^\gamma \cdot du^\alpha_\beta + \sum_\beta u^{-2} u^\alpha u_\beta \cdot \omega^\beta - \sum_\beta u^{-2} u^\alpha u^\beta \cdot \omega_\beta}_{=:\, \varphi^\alpha} \Big),
\end{aligned}
$$

provides appropriate 1-forms φ^α_β, φ^α.

Lastly, adding the mentioned second term in a supplementary line:

$$\begin{aligned}
d\omega_\alpha \;=\; & \sum_\beta \omega_\beta \wedge \Big(-\boldsymbol{\delta}^\beta_\alpha\, u^{-1}\cdot du + \sum_\gamma \bar{u}^{1\gamma}_\alpha \cdot du^\beta_\gamma + \sum_\gamma \Big[-\boldsymbol{\delta}^\beta_\gamma u^{-1}u_\alpha - \sum_\delta \sum_\varepsilon \sum_\zeta \bar{u}^{1\delta}_\alpha\, u^\beta_\zeta\, \bar{u}^{1\varepsilon}_\gamma\, F^{\delta,\varepsilon}_{y_{x^\zeta}} \Big] \cdot \omega^\gamma - \\
& - \boldsymbol{\delta}^\beta_\alpha \sum_\gamma u^{-1}u^\gamma \cdot \omega_\gamma - u^{-1}u^\beta \cdot \omega_\alpha \Big) + \\
& + \omega \wedge \Big(-u^{-1}\cdot du_\alpha + u^{-2}u_\alpha \cdot du - \sum_\beta \sum_\gamma u^{-1}\bar{u}^{1\gamma}_\alpha\, u_\beta \cdot du^\beta_\gamma + \\
& + \sum_\beta \Big[u^{-2}u_\beta\, u_\alpha - \sum_\gamma \sum_\delta \bar{u}^{1\gamma}_\alpha\, \bar{u}^{1\delta}_\beta\, F^{\gamma,\delta}_y + \sum_\gamma \sum_\delta \sum_\varepsilon \sum_\zeta u^{-1}\bar{u}^{1\gamma}_\alpha\, u^\zeta_\delta\, \bar{u}^{1\varepsilon}_\beta\, u_\zeta\, F^{\gamma,\varepsilon}_{y_{x^\delta}} \Big] \cdot \omega^\beta + \\
& + \sum_\beta \Big[-u^{-2}u^\beta\, u_\alpha - \sum_\gamma \sum_\delta \sum_\varepsilon \sum_\zeta u^{-1}\bar{u}^{1\gamma}_\alpha\, \bar{u}^{1\zeta}_\varepsilon\, u^\beta_\delta\, u^\varepsilon\, F^{\gamma,\zeta}_{y_{x^\delta}} \Big] \cdot \omega_\beta \Big) \\
= \; & \sum_\beta \omega_\beta \wedge \big(\underbrace{\boldsymbol{\delta}^\beta_\alpha\, \varphi - \varphi^\beta_\alpha}_{?} \big) + \omega \wedge \varphi_\alpha,
\end{aligned}$$

provides without ambiguity appropriate 1-forms φ^α, but it must be checked that, with the definitions of φ, φ^α_β, φ^α just given, what is underbraced matches (visually) correctly with:

$$\begin{aligned}
\boldsymbol{\delta}^\beta_\alpha\, \varphi - \varphi^\beta_\alpha \;=\; & -\boldsymbol{\delta}^\beta_\alpha\, u^{-1}\cdot du + \underline{\boldsymbol{\delta}^\beta_\alpha \sum_\gamma u^{-1}u_\gamma \cdot \omega^\gamma}\circ \; - \boldsymbol{\delta}^\beta_\alpha \sum_\gamma u^{-1}u^\gamma \cdot \omega_\gamma + \\
& + \sum_\gamma \bar{u}^{1\gamma}_\alpha \cdot du^\beta_\gamma - u^{-1}u^\beta \cdot \omega_\alpha - \underline{\boldsymbol{\delta}^\beta_\alpha \sum_\gamma u^{-1}u_\gamma \cdot \omega^\gamma}\circ \; - \\
& - u^{-1}u_\alpha \cdot \omega^\beta - \sum_\gamma \sum_\delta \sum_\varepsilon \sum_\zeta \bar{u}^{1\varepsilon}_\gamma\, u^\beta_\zeta\, \bar{u}^{1\delta}_\alpha\, F^{\delta,\varepsilon}_{y_{x^\zeta}} \cdot \omega^\gamma,
\end{aligned}$$

thanks to a cancellation of two terms.

The gain of this pain is a parametric determination improving much Proposition 5.1. The 1-forms constructed above will now be denoted:

$$\mathring{\varphi}, \qquad \mathring{\varphi}^\alpha_\beta, \qquad \mathring{\varphi}^\alpha, \qquad \mathring{\varphi}_\alpha.$$

Proposition 6.4. *After absorption of torsion:*

$$\begin{aligned}
d\omega \;&=\; \sum_\alpha \omega^\alpha \wedge \omega_\alpha + \omega \wedge \mathring{\varphi}, \\
d\omega^\alpha \;&=\; \sum_\beta \omega^\beta \wedge \mathring{\varphi}^\alpha_\beta + \omega \wedge \mathring{\varphi}^\alpha, \\
d\omega_\alpha \;&=\; \sum_\beta \mathring{\varphi}^\beta_\alpha \wedge \omega_\beta + \omega_\alpha \wedge \mathring{\varphi} + \omega \wedge \mathring{\varphi}_\alpha,
\end{aligned}$$

with the explicit modified Maurer-Cartan 1*-forms:*

$$\begin{aligned}
\mathring{\varphi} \;:=\; & -u^{-1}\cdot du + \\
& + \sum_\beta u^{-1}u_\beta \cdot \omega^\beta - \sum_\beta u^{-1}u^\beta \cdot \omega_\beta, \\
\mathring{\varphi}^\alpha_\beta \;=\; & -\sum_\gamma \bar{u}^{1\gamma}_\beta \cdot du^\alpha_\gamma + \\
& + u^{-1}u^\alpha \cdot \omega_\beta + \boldsymbol{\delta}^\alpha_\beta \sum_\gamma u^{-1}u_\gamma \cdot \omega^\gamma + u^{-1}u_\beta \cdot \omega^\alpha + \sum_\gamma \sum_\delta \sum_\varepsilon \sum_\zeta \bar{u}^{1\varepsilon}_\gamma\, u^\alpha_\zeta\, \bar{u}^{1\delta}_\beta\, F^{\delta,\varepsilon}_{y_{x^\zeta}} \cdot \omega^\gamma, \\
\mathring{\varphi}^\alpha \;=\; & -u^{-1}\cdot du^\alpha + \sum_\beta \sum_\gamma u^{-1}\bar{u}^{1\beta}_\gamma\, u^\gamma \cdot du^\alpha_\beta + \\
& + \sum_\beta u^{-2}u^\alpha\, u_\beta \cdot \omega^\beta - \sum_\beta u^{-2}u^\alpha\, u^\beta \cdot \omega_\beta, \\
\mathring{\varphi}_\alpha \;=\; & -u^{-1}\cdot du_\alpha + u^{-2}u_\alpha \cdot du - \sum_\beta \sum_\gamma u^{-1}\bar{u}^{1\gamma}_\alpha\, u_\beta \cdot du^\beta_\gamma + \\
& + \sum_\beta \Big[u^{-2}u_\beta\, u_\alpha - \sum_\gamma \sum_\delta \bar{u}^{1\gamma}_\alpha\, \bar{u}^{1\delta}_\beta\, F^{\gamma,\delta}_y + \sum_\gamma \sum_\delta \sum_\varepsilon \sum_\zeta u^{-1}\bar{u}^{1\gamma}_\alpha\, u^\zeta_\delta\, \bar{u}^{1\varepsilon}_\beta\, u_\zeta\, F^{\gamma,\varepsilon}_{y_{x^\delta}} \Big] \cdot \omega^\beta + \\
& + \sum_\beta \Big[-u^{-2}u^\beta\, u_\alpha - \sum_\gamma \sum_\delta \sum_\varepsilon \sum_\zeta u^{-1}\bar{u}^{1\gamma}_\alpha\, \bar{u}^{1\zeta}_\varepsilon\, u^\beta_\delta\, u^\varepsilon\, F^{\gamma,\zeta}_{y_{x^\delta}} \Big] \cdot \omega_\beta. \qquad \square
\end{aligned}$$

7 The 1-form ψ

Proposition 7.1. *The most general* 1*-forms* φ, φ^α_β, φ^α *and* φ_α *that satisfy*

$$\begin{aligned} d\omega &= \sum_\alpha \omega^\alpha \wedge \omega_\alpha + \omega \wedge \varphi, \\ d\omega^\alpha &= \sum_\beta \omega^\beta \wedge \varphi^\alpha_\beta + \omega \wedge \varphi^\alpha, \\ d\omega_\alpha &= \sum_\beta \varphi^\beta_\alpha \wedge \omega_\beta + \omega_\alpha \wedge \varphi + \omega \wedge \varphi_\alpha, \end{aligned}$$

express in terms of $\mathring{\varphi}$, $\mathring{\varphi}^\alpha_\beta$, $\mathring{\varphi}^\alpha$ *and* $\mathring{\varphi}_\alpha$ *as*

$$\begin{aligned} \varphi &= t \cdot \omega + \mathring{\varphi}, \\ \varphi^\alpha_\beta &= b^\alpha_\beta \omega + \mathring{\varphi}^\alpha_\beta, \\ \varphi^\alpha &= c^\alpha \cdot \omega + \sum_\beta b^\alpha_\beta \cdot \omega^\beta + \mathring{\varphi}^\alpha, \\ \varphi_\alpha &= d_\alpha \cdot \omega + \sum_\beta (\delta^\beta_\alpha t - b^\beta_\alpha) \cdot \omega_\beta + \mathring{\varphi}_\alpha. \end{aligned} \tag{7.2}$$

Proof. The proposition is proved by repeated use of Cartan's lemma using the two sets of equations:

$$d\omega = \sum_\alpha \omega^\alpha \wedge \omega_\alpha + \omega \wedge \mathring{\varphi}, \tag{7.3}$$

$$d\omega^\alpha = \sum_\beta \omega^\beta \wedge \mathring{\varphi}^\alpha_\beta + \omega \wedge \mathring{\varphi}^\alpha, \tag{7.4}$$

$$d\omega_\alpha = \sum_\beta \mathring{\varphi}^\beta_\alpha \wedge \omega_\beta + \omega_\alpha \wedge \mathring{\varphi} + \omega \wedge \mathring{\varphi}_\alpha, \tag{7.5}$$

and

$$d\omega = \sum_\alpha \omega^\alpha \wedge \omega_\alpha + \omega \wedge \varphi, \tag{7.6}$$

$$d\omega^\alpha = \sum_\beta \omega^\beta \wedge \varphi^\alpha_\beta + \omega \wedge \varphi^\alpha, \tag{7.7}$$

$$d\omega_\alpha = \sum_\beta \varphi^\beta_\alpha \wedge \omega_\beta + \omega_\alpha \wedge \varphi + \omega \wedge \varphi_\alpha. \tag{7.8}$$

First, subtracting (7.6) from (7.3) yields,

$$0 = \omega \wedge (\mathring{\varphi} - \varphi).$$

By Cartan's lemma, there exists some function t such that

$$\varphi = t \cdot \omega + \mathring{\varphi}.$$

Next, subtracting equation (7.7) from (7.4),

$$0 = \sum_\beta \omega^\beta \wedge (\mathring{\varphi}^\alpha_\beta - \varphi^\alpha_\beta) + \omega \wedge (\mathring{\varphi}^\alpha - \varphi^\alpha).$$

Therefore, by Cartan's lemma again, there exist functions b^α_β, r^α_β, c^α, d^α_β, such that

$$\varphi^\alpha_\beta = b^\alpha_\beta \cdot \omega + \sum_\beta r^\alpha_\beta \cdot \omega^\beta + \mathring{\varphi}^\alpha_\beta, \tag{7.9}$$

$$\varphi^\alpha = c^\alpha \cdot \omega + \sum_\beta d^\alpha_\beta \cdot \omega^\beta + \mathring{\varphi}^\alpha. \tag{7.10}$$

Similarly, substracting equation (7.8) from (7.5) gives

$$0 = \sum_\beta (\mathring{\varphi}^\beta_\alpha - \varphi^\beta_\alpha) \wedge \omega_\beta \ + \ \omega_\alpha \wedge (\mathring{\varphi} - \varphi) \ + \ \omega \wedge (\mathring{\varphi}_\alpha - \varphi_\alpha).$$

By Cartan's lemma, there exist functions g^β_α, h^β_α, d_α, e^β_α such that

$$\begin{aligned} \varphi^\beta_\alpha &= \sum_\beta g^\beta_\alpha \cdot \omega_\beta \ + \ h^\beta_\alpha \omega \ + \ \mathring{\varphi}^\beta_\alpha, \\ \varphi_\alpha &= d_\alpha \cdot \omega \ + \ \sum_\beta e^\beta_\alpha \cdot \omega_\beta \ + \ \mathring{\varphi}_\alpha, \end{aligned}$$

or, by switching the indices $\alpha \leftrightarrow \beta$ only in the first equation above,

$$\varphi^\alpha_\beta = \sum_\alpha g^\alpha_\beta \cdot \omega_\alpha \ + \ h^\alpha_\beta \cdot \omega \ + \ \mathring{\varphi}^\alpha_\beta, \tag{7.11}$$

$$\varphi_\alpha = d_\alpha \cdot \omega \ + \ \sum_\beta e^\beta_\alpha \cdot \omega_\beta \ + \ \mathring{\varphi}_\alpha. \tag{7.12}$$

Next, subtracting equation (7.11) from (7.9) yields

$$0 = -\sum_\alpha g^\alpha_\beta \cdot \omega_\alpha \ + \ \sum_\beta r^\alpha_\beta \cdot \omega^\beta \ + (b^\alpha_\beta - h^\alpha_\beta) \cdot \omega.$$

From the linear independence of coframes, it follows that

$$g^\alpha_\beta = r^\alpha_\beta = 0 \qquad \text{and} \qquad b^\alpha_\beta = h^\alpha_\beta.$$

Therefore, the expressions of φ^α_β, φ_α, φ^α are obtained below

$$\varphi = t \cdot \omega \ + \ \mathring{\varphi}, \tag{7.13}$$

$$\varphi^\alpha_\beta = b^\alpha_\beta \cdot \omega \ + \ \mathring{\varphi}^\alpha_\beta, \tag{7.14}$$

$$\varphi^\alpha = c^\alpha \cdot \omega \ + \ \sum_\beta d^\alpha_\beta \cdot \omega^\beta \ + \ \mathring{\varphi}^\alpha, \tag{7.15}$$

$$\varphi_\alpha = d_\alpha \cdot \omega \ + \ \sum_\beta e^\beta_\alpha \cdot \omega_\beta \ + \ \mathring{\varphi}_\alpha. \tag{7.16}$$

It remains to find e^β_α and d^α_β. To this end, substitute equations (7.14) and (7.15) into (7.7) to get

$$\begin{aligned} d\omega^\alpha &= \sum_\beta \omega^\beta \wedge (b^\alpha_\beta \cdot \omega \ + \ \mathring{\varphi}^\alpha_\beta) + \omega \wedge \left(c^\alpha \cdot \omega \ + \ \sum_\beta d^\alpha_\beta \cdot \omega^\beta + \mathring{\varphi}^\alpha \right) \\ &= \sum_\beta (b^\alpha_\beta - d^\alpha_\beta)\, \omega^\beta \wedge \omega + \sum_\beta \omega^\beta \wedge \mathring{\varphi}^\alpha_\beta + \omega \wedge \mathring{\varphi}^\alpha \\ &= \sum_\beta (b^\alpha_\beta - d^\alpha_\beta)\, \omega^\beta \wedge \omega + d\omega^\alpha, \end{aligned}$$

and notice that $d\omega^\alpha$ cancels on both sides of the equation. This results in

$$0 = \sum_\beta (b^\alpha_\beta - d^\alpha_\beta) \omega^\beta \wedge \omega,$$

and hence by linear independence of coframes, it comes instantly that $d^\alpha_\beta = b^\alpha_\beta$.

Finally, replacing the φ^β_α, φ and φ_α in equation (7.8) by their corresponding expressions in equations (7.13) to (7.16), the term $d\omega_\alpha$ becomes

$$\begin{aligned} d\omega_\alpha &= \sum_\beta \varphi^\beta_\alpha \wedge \omega_\beta \ + \ \omega_\alpha \wedge \varphi + \ \omega \wedge \varphi_\alpha \\ &= \sum_\beta (b^\beta_\alpha \cdot \omega \ + \ \mathring{\varphi}^\beta_\alpha) \wedge \omega_\beta \ + \ \omega_\alpha \wedge (t \cdot \omega + \mathring{\varphi}) \ + \omega \wedge \left(d_\alpha \cdot \omega \ + \ \sum_\beta e^\beta_\alpha \cdot \omega_\beta \ + \ \mathring{\varphi}_\alpha \right) \\ &= \sum_\beta b^\alpha_\beta \cdot \omega \wedge \omega_\beta \ + \ t \cdot \omega_\alpha \wedge \omega \ + \sum_\beta e^\beta_\alpha \cdot \omega \wedge \omega_\beta + \sum_\beta \mathring{\varphi}^\beta_\alpha \wedge \omega_\beta \ + \ \omega_\alpha \wedge \mathring{\varphi} \ + \ \omega \wedge \mathring{\varphi}_\alpha \\ &= \sum_\beta b^\beta_\alpha \cdot \omega \wedge \omega_\beta \ + \ t \cdot \omega_\alpha \wedge \omega \ + \ \sum_\beta e^\beta_\alpha \cdot \omega \wedge \omega_\beta + d\omega_\alpha. \end{aligned}$$

By a similar observation as before, the term $d\omega_\alpha$ cancels on both sides of the equation to obtain the following linear relation

$$\begin{aligned} 0 &= \sum_\beta b_\alpha^\beta \cdot \omega \wedge \omega_\beta \ + \ t\omega_\alpha \wedge \omega \ + \ \sum_\beta e_\alpha^\beta \cdot \omega \wedge \omega_\beta \\ &= \sum_\beta b_\alpha^\beta \cdot \omega \wedge \omega_\beta \ - \ \sum_\beta t\delta_\alpha^\beta \cdot \omega \wedge \omega_\beta \ + \sum_\beta e_\alpha^\beta \cdot \omega \wedge \omega_\beta \\ &= \sum_\beta \left(e_\alpha^\beta - (t\delta_\alpha^\beta - b_\alpha^\beta)\right) \cdot \omega \wedge \omega_\beta. \end{aligned}$$

By linear independence of $\omega \wedge \omega_\beta$, it follows that

$$e_\alpha^\beta \ = \ t\delta_\alpha^\beta - b_\alpha^\beta.$$

Then replacing $b_\beta^\alpha = d_\beta^\alpha$, and $e_\alpha^\beta \ = \ t\delta_\alpha^\beta - b_\alpha^\beta$, the proposition is proved. □

Proposition 7.17. *There is a* 1*-form* ψ *such that*

$$d\varphi \ = \ \sum_\alpha \omega^\alpha \wedge \varphi_\alpha \ - \ \sum_\alpha \omega_\alpha \wedge \varphi^\alpha \ + \ \omega \wedge \psi.$$

Proof. The argument again relies on Cartan's lemma. From

$$d\omega = \sum_\alpha \omega^\alpha \wedge \omega_\alpha \ + \ \omega \wedge \varphi,$$

it is inferred that

$$0 \ = \ d^2\omega \ = \ \sum_\alpha d\omega^\alpha \wedge \omega_\alpha \ - \ \sum_\alpha \omega^\alpha \wedge d\omega_\alpha + \ d\omega \wedge \varphi \ - \ \omega \wedge d\varphi.$$

Then put the last term over to the left,

$$\omega \wedge d\varphi \ = \ \sum_\alpha d\omega^\alpha \wedge \omega_\alpha \ - \ \sum_\alpha \omega^\alpha \wedge d\omega_\alpha + d\omega \wedge \varphi.$$

By replacing $d\omega^\alpha$, $d\omega_\alpha$ and $d\omega$ above by their corresponding expressions in equations (7.6) to (7.8), the right side of the equation may also be expressed as a multiple of ω in a different manner:

$$\begin{aligned} \omega \wedge d\varphi &= \sum_\alpha \left(\sum_\beta \omega^\beta \wedge \varphi_\beta^\alpha\right) \wedge \omega_\alpha + \sum_\alpha \omega \wedge \varphi^\alpha \wedge \omega_\alpha \\ &\quad -\sum_\alpha \omega^\alpha \wedge \left(\sum_\beta \varphi_\alpha^\beta \wedge \omega_\beta + \omega_\alpha \wedge \varphi + \omega \wedge \varphi_\alpha\right) + \left(\sum_\alpha \omega^\alpha \wedge \omega_\alpha + \omega \wedge \varphi\right) \wedge \varphi \\ &= \sum_\alpha \sum_\beta \omega^\beta \wedge \varphi_\beta^\alpha \wedge \omega_\alpha \ - \ \sum_\alpha \sum_\beta \omega^\alpha \wedge \varphi_\alpha^\beta \wedge \omega_\beta \ - \ \sum_\alpha \omega^\alpha \wedge \omega_\alpha \wedge \varphi \\ &\quad + \sum_\alpha \omega^\alpha \wedge \omega_\alpha \wedge \varphi \ + \ \omega \wedge \left(\sum_\alpha \varphi^\alpha \wedge \omega_\alpha \ + \ \sum_\alpha \omega^\alpha \wedge \varphi_\alpha\right) \\ &= \omega \wedge \left(\sum_\alpha \varphi^\alpha \wedge \omega_\alpha \ + \ \sum_\alpha \omega^\alpha \wedge \varphi_\alpha\right). \end{aligned}$$

Taking the difference between both sides of the equation above, it follows that

$$\omega \wedge \left(d\varphi - \left[\sum_\alpha \omega^\alpha \wedge \varphi_\alpha - \sum_\alpha \omega_\alpha \wedge \varphi^\alpha\right]\right) = 0,$$

allowing the application of Cartan's lemma to deduce the existence of a 1-form, ψ, such that

$$d\varphi \ = \ \sum_\alpha \omega^\alpha \wedge \varphi_\alpha \ - \ \sum_\alpha \omega_\alpha \wedge \varphi^\alpha \ + \ \omega \wedge \psi,$$

and hence concluding the proof. □

Lemma 7.18. *The 1-form ψ can be chosen to take on the following explicit expression:*

$$\psi \;=\; -dt \;+\; t\overset{\circ}{\varphi} \;+\; \sum_\alpha d_\alpha \cdot \omega^\alpha \;-\; \sum_\alpha c^\alpha \cdot \omega_\alpha \;+\; \sum_\beta u^{-1}u_\beta \cdot \overset{\circ}{\varphi}^\beta - \sum_\beta u^{-1}u^\beta \cdot \overset{\circ}{\varphi}_\beta.$$

Before starting the proof of the lemma, there is another observation to be made. It is known that from the previous proposition, there exist 1-forms, ψ and $\overset{\circ}{\psi}$, such that

$$\begin{aligned} d\varphi &= \sum_\alpha \omega^\alpha \wedge \varphi_\alpha \;-\; \sum_\alpha \omega_\alpha \wedge \varphi^\alpha \;+\; \omega \wedge \psi, \\ d\overset{\circ}{\varphi} &= \sum_\alpha \omega^\alpha \wedge \overset{\circ}{\varphi}_\alpha \;-\; \sum_\alpha \omega_\alpha \wedge \overset{\circ}{\varphi}_\alpha \;+\; \omega \wedge \overset{\circ}{\psi}. \end{aligned}$$

It follows that their difference is a multiple of ω:

$$\begin{aligned} & \left(d\varphi \;-\; \sum_\alpha \omega^\alpha \wedge \varphi_\alpha \;+\; \sum_\alpha \omega_\alpha \wedge \varphi^\alpha\right) - \left(d\overset{\circ}{\varphi} \;-\; \sum_\alpha \omega^\alpha \wedge \overset{\circ}{\varphi}_\alpha \;+\; \sum_\alpha \omega_\alpha \wedge \overset{\circ}{\varphi}^\alpha\right) \\ = \;& \omega \wedge (\psi - \overset{\circ}{\psi}) \\ \equiv \;& 0 \qquad \mathsf{mod}\ \langle\omega\rangle. \end{aligned}$$

The following lemma shows that it is possible to compute $\omega \wedge (\psi - \overset{\circ}{\psi})$ without calculating explicitly either ψ or $\overset{\circ}{\psi}$.

Lemma 7.19. *The following expression holds:*

$$\omega \wedge (\psi - \overset{\circ}{\psi}) \;=\; \omega \wedge \left(-dt \;+\; t\overset{\circ}{\varphi} \;+\; \sum_\alpha d_\alpha \cdot \omega^\alpha \;-\; \sum_\alpha c^\alpha \cdot \omega_\alpha\right).$$

in which the first four terms of ψ appear.

Proof. From $\varphi = t \cdot \omega + \overset{\circ}{\varphi}$ in equation (7.13), taking its differential $d\varphi$ and then replacing $d\omega$ by the expression in (7.3):

$$\begin{aligned} d\varphi &= dt \wedge \omega \;+\; t \cdot d\omega \;+\; d\overset{\circ}{\varphi} \\ &= dt \wedge \omega \;+\; t\left(\sum_\alpha \omega^\alpha \wedge \omega_\alpha \;+\; \omega \wedge \overset{\circ}{\varphi}\right) \;+\; d\overset{\circ}{\varphi} \\ &= \sum_\alpha \omega^\alpha \wedge (t \cdot \omega_\alpha) \;+\; \omega \wedge (-dt \;+\; t\overset{\circ}{\varphi}) \;+\; d\overset{\circ}{\varphi}. \end{aligned}$$

Using φ_α in Proposition 7.1, the term $t \cdot \omega_\alpha$ may be replaced by

$$t \cdot \omega_\alpha \;=\; \varphi_\alpha \;-\; d_\alpha \cdot \omega \;+\; \sum_\beta b_\alpha^\beta \cdot \omega_\beta \;-\; \overset{\circ}{\varphi}_\alpha$$

to obtain

$$\begin{aligned} d\varphi &= \sum_\alpha \omega^\alpha \wedge \left(\varphi_\alpha \,-\, d_\alpha \cdot \omega \,+\, \sum_\beta b_\alpha^\beta \cdot \omega_\beta - \mathring{\varphi}_\alpha\right) \\ &\quad +\omega \wedge (-dt + t \cdot \mathring{\varphi}) \,+\, d\mathring{\varphi} \\ &= \sum_\alpha \omega^\alpha \wedge \varphi_\alpha \,+\, \sum_\alpha \sum_\beta \omega^\alpha \wedge \left(b_\alpha^\beta \cdot \omega_\beta\right) \,+\, \omega \wedge \left(\sum_\alpha d_\alpha \cdot \omega^\alpha\right) \\ &\quad -\sum_\alpha \omega^\alpha \wedge \mathring{\varphi}_\alpha \,+\, \omega \wedge (-dt \,+\, t \cdot \mathring{\varphi}) \,+\, d\mathring{\varphi} \\ &= \sum_\alpha \omega^\alpha \wedge \varphi_\alpha \,+\, \sum_\alpha \sum_\beta \omega^\beta \wedge \left(b_\beta^\alpha \cdot \omega_\alpha\right) \\ &\quad + \omega \wedge \left(-dt \,+\, t \cdot \mathring{\varphi} \,+\, \sum_\alpha d_\alpha \cdot \omega^\alpha\right) \,-\, \sum_\alpha \omega^\alpha \wedge \mathring{\varphi}_\alpha \,+\, d\mathring{\varphi} \\ &= \sum_\alpha \omega^\alpha \wedge \varphi_\alpha \,-\, \sum_\alpha \omega_\alpha \wedge \left(\sum_\beta b_\beta^\alpha \cdot \omega^\beta\right) \\ &\quad + \omega \wedge \left(-dt \,+\, t \cdot \mathring{\varphi} \,+\, \sum_\alpha d_\alpha \cdot \omega^\alpha\right) \,-\, \sum_\alpha \omega^\alpha \wedge \mathring{\varphi}_\alpha \,+\, d\mathring{\varphi}. \end{aligned}$$

The next step is to use φ^α in Proposition 7.1 to replace $\sum_\beta b_\beta^\alpha \omega^\beta$ by

$$\sum_\beta b_\beta^\alpha \cdot \omega^\beta \,=\, \varphi^\alpha \,-\, c^\alpha \cdot \omega \,-\, \mathring{\varphi}^\alpha$$

to get the conclusion

$$\begin{aligned} d\varphi =& \sum_\alpha \omega^\alpha \wedge \varphi_\alpha \,-\, \sum_\alpha \omega_\alpha \wedge \varphi^\alpha \,+\, \sum_\alpha \omega_\alpha \wedge c^\alpha \cdot \omega \,+\, \sum_\alpha \omega_\alpha \wedge \mathring{\varphi}^\alpha \\ &+ \omega \wedge \left(-dt \,+\, t \cdot \mathring{\varphi} \,+\, \sum_\alpha d_\alpha \cdot \omega^\alpha\right) \,-\, \sum_\alpha \omega^\alpha \wedge \mathring{\varphi}_\alpha \,+\, d\mathring{\varphi} \\ =& \sum_\alpha \omega^\alpha \wedge \varphi_\alpha \,-\, \sum_\alpha \omega_\alpha \wedge \varphi^\alpha \,+\, \omega \wedge \left(-dt \,+\, t \cdot \mathring{\varphi} \,+\, \sum_\alpha d_\alpha \cdot \omega^\alpha \,-\, \sum_\alpha c^\alpha \cdot \omega_\alpha\right) \\ &+ \left[d\mathring{\varphi} \,-\, \left(\sum_\alpha \omega^\alpha \wedge \mathring{\varphi}_\alpha \,-\sum_\alpha \omega_\alpha \wedge \mathring{\varphi}^\alpha\right)\right], \end{aligned} \tag{7.20}$$

and this terminates the proof. □

Lemma 7.19 completes part of the proof of lemma 7.18. The rest of the proof lies in computing

$$d\mathring{\varphi} \,-\, \left(\sum_\alpha \omega^\alpha \wedge \mathring{\varphi}_\alpha \,-\, \sum_\alpha \omega_\alpha \wedge \mathring{\varphi}^\alpha\right)$$

to obtain further terms that are multiples of ω which do not show up in $\omega \wedge (\psi - \mathring{\psi})$. To this end, the terms in the parenthesis will be calculated first, then $d\mathring{\varphi}$.

7.1 The term $\sum_\alpha \omega^\alpha \wedge {}^\circ\varphi_\alpha - \sum_\alpha \omega_\alpha \wedge {}^\circ\varphi^\alpha$

Lemma 7.21. *The following identities hold:*

$$\begin{aligned} 0 &= \sum_\alpha \sum_\beta u^{-2} u_\beta u_\alpha \, \omega^\alpha \wedge \omega^\beta, \\ 0 &= \sum_\alpha \sum_\beta \sum_\gamma \sum_\delta \bar{u}^{1\gamma}_{\ \alpha} \bar{u}^{1\delta}_{\ \beta} \, F_y^{\gamma,\delta} \, \omega^\alpha \wedge \omega^\beta \\ 0 &= \sum_\alpha \sum_\beta \sum_\gamma \sum_\delta \sum_\varepsilon \sum_\zeta u^{-1} \bar{u}^{1\gamma}_{\ \alpha} u^\zeta_\delta \bar{u}^{1\varepsilon}_{\ \beta} u_\zeta \, F_{y^\delta_x}^{\gamma,\varepsilon} \, \omega^\alpha \wedge \omega^\beta. \end{aligned}$$

Proof. The proof of the first identity relies on the anti-commutativity $\omega^\alpha \wedge \omega^\beta = -\omega^\beta \wedge \omega^\alpha$:

$$\sum_\alpha \sum_\beta \; u^{-2} u_\beta u_\alpha \, \omega^\alpha \wedge \omega^\beta \;\; = \;\; \sum_{\alpha<\beta} u^{-2}(u_\alpha u_\beta - u_\beta u_\alpha)\, \omega^\alpha \wedge \omega^\beta = 0.$$

For the second identity, the calculation proceeds in a similar manner, using the symmetry property $F^{\gamma,\delta} = F^{\delta,\gamma}$, to obtain

$$\begin{aligned}
\sum_\alpha \sum_\beta \sum_\gamma \sum_\delta \; \bar{u}^{1\gamma}_{\;\alpha} \bar{u}^{1\delta}_{\;\beta} F_y^{\gamma,\delta} \, \omega^\alpha \wedge \omega^\beta \;\; &= \;\; \sum_{\alpha<\beta} \sum_\gamma \sum_\delta \; (\bar{u}^{1\gamma}_{\;\alpha} \bar{u}^{1\delta}_{\;\beta} - \bar{u}^{1\gamma}_{\;\beta} \bar{u}^{1\delta}_{\;\alpha}) \; F_y^{\gamma,\delta} \, \omega^\alpha \wedge \omega^\beta \\
&= \sum_{\alpha<\beta} \sum_{\gamma=\delta} (\bar{u}^{1\gamma}_{\;\alpha} \bar{u}^{1\delta}_{\;\beta} - \bar{u}^{1\gamma}_{\;\beta} \bar{u}^{1\delta}_{\;\alpha}) \; F_y^{\gamma,\delta} \, \omega^\alpha \wedge \omega^\beta \\
&\quad + \sum_{\alpha<\beta} \sum_{\gamma<\delta} \Big[(\bar{u}^{1\gamma}_{\;\alpha} \bar{u}^{1\delta}_{\;\beta} - \bar{u}^{1\gamma}_{\;\beta} \bar{u}^{1\delta}_{\;\alpha}) \; F_y^{\gamma,\delta} \\
&\qquad\qquad + (\bar{u}^{1\delta}_{\;\alpha} \bar{u}^{1\gamma}_{\;\beta} - \bar{u}^{1\delta}_{\;\beta} \bar{u}^{1\gamma}_{\;\alpha}) \; F_y^{\delta,\gamma} \Big] \, \omega^\alpha \wedge \omega^\beta \\
&= \;\; 0,
\end{aligned}$$

Finally for the last identity, the vanishing follows from

$$\begin{aligned}
&\sum_\alpha \sum_\beta \sum_\gamma \sum_\delta \sum_\varepsilon \sum_\zeta \; u^{-1} \bar{u}^{1\gamma}_{\;\alpha} u^\zeta_\delta \bar{u}^1 \beta^\varepsilon u_\zeta \; F^{\gamma,\varepsilon}_{y^\delta_x} \, \omega^\alpha \wedge \omega^\beta \\
&\qquad = \sum_\delta \sum_\zeta \; u^{-1} u^\zeta_\delta u_\zeta \underbrace{\Big(\sum_\alpha \sum_\beta \sum_\gamma \sum_\varepsilon \; \bar{u}^{1\gamma}_{\;\alpha} \bar{u}^{1\varepsilon}_{\;\beta} \; F^{\gamma,\varepsilon}_{y_{x^\delta}} \, \omega^\alpha \wedge \omega^\beta \Big)}_{=0},
\end{aligned} \tag{7.22}$$

where the expression above the underbrace is zero because it is similar to the second identity. The lemma is proved. □

7.1.1 Calculation of $\sum_\alpha \omega^\alpha \wedge \mathring{\varphi}_\alpha$ Using $\mathring{\varphi}_\alpha$ in Proposition 6.4, and by the use of Lemma 7.21 that the terms in $\sum_\alpha \omega^\alpha \wedge \mathring{\varphi}_\alpha$ containing $\omega^\alpha \wedge \omega^\beta$ vanish, the following expression is obtained:

$$\begin{aligned}
\sum_\alpha \omega^\alpha \wedge \mathring{\varphi}_\alpha \;\; &= \;\; \sum_\alpha \; u^{-1} \, du_\alpha \wedge \omega^\alpha - \sum_\alpha \; u^{-2} u_\alpha \, du \wedge \omega^\alpha + \sum_\alpha \sum_\beta \sum_\gamma \; u^{-1} \bar{u}^{1\gamma}_{\;\alpha} u_\beta \, du^\beta_\gamma \wedge \omega^\alpha \\
&\quad + \sum_\alpha \sum_\beta \Big[u^{-2} u_\beta u_\alpha \; - \; \sum_\gamma \sum_\delta \bar{u}^{1\gamma}_{\;\alpha} \bar{u}^{1\delta}_{\;\beta} \; F_y^{\gamma,\delta} \\
&\qquad\qquad + \sum_\gamma \sum_\delta \sum_\varepsilon \sum_\zeta u^{-1} \bar{u}^{1\gamma}_{\;\alpha} u^\zeta_\delta \bar{u}^{1\varepsilon}_{\;\beta} u_\zeta \; F^{\gamma,\delta}_{y_{x^\delta}} \Big] \, \omega^\alpha \wedge \omega^\beta \\
&\quad + \sum_\alpha \sum_\beta \Big[\; - u^{-2} u^\beta u_\alpha \; - \; \sum_\gamma \sum_\delta \sum_\varepsilon \sum_\zeta \; u^{-1} \bar{u}^{1\gamma}_{\;\alpha} \bar{u}^{1\zeta}_{\;\varepsilon} u^\beta_\delta u^\varepsilon \; F^{\gamma,\zeta}_{y_{x^\delta}} \Big] \, \omega^\alpha \wedge \omega_\beta \\
&= \;\; \sum_\alpha \; u^{-1} \, du_\alpha \wedge \omega^\alpha \; - \; \sum_\alpha \; u^{-2} u_\alpha \, du \wedge \omega^\alpha \; + \; \sum_\alpha \sum_\beta \sum_\gamma u^{-1} \bar{u}^{1\gamma}_{\;\alpha} u_\beta \, du^\beta_\gamma \wedge \omega^\alpha \\
&\quad - \sum_\alpha \sum_\beta \; u^{-2} u^\beta u_\alpha \, \omega^\alpha \wedge \omega_\beta \; - \; \sum_\alpha \sum_\beta \sum_\gamma \sum_\delta \sum_\varepsilon \sum_\zeta \; u^{-1} \bar{u}^{1\gamma}_{\;\alpha} \bar{u}^{1\zeta}_{\;\varepsilon} u^\beta_\delta u^\varepsilon F^{\gamma,\zeta}_{y_{x^\delta}} \, \omega^\alpha \wedge \omega_\beta.
\end{aligned}$$

7.1.2 Calculation of $\sum_\alpha \omega_\alpha \wedge \overset{\circ}{\varphi}{}^\alpha$ On the other hand, using $\overset{\circ}{\varphi}{}^\alpha$ in Proposition 6.4 gives

$$\begin{aligned}\sum_\alpha \omega_\alpha \wedge \overset{\circ}{\varphi}{}^\alpha &= \sum_\alpha \omega_\alpha \wedge \Bigg(- u^{-1}\, du^\alpha + \sum_\beta \sum_\gamma u^{-1} \bar{u}^{1\beta}_{\ \gamma} u^\gamma\, du^\alpha_\beta \\ &\qquad + \sum_\beta u^{-2} u^\alpha u_\beta\, \omega^\beta - \sum_\beta u^{-2} u^\alpha u^\beta\, \omega_\beta \Bigg) \\ &= \sum_\alpha u^{-1}\, du^\alpha \wedge \omega_\alpha - \sum_\alpha \sum_\beta \sum_\gamma u^{-1} \bar{u}^{1\beta}_{\ \gamma} u^\gamma\, du^\alpha_\beta \wedge \omega_\alpha \\ &\quad + \sum_\alpha \sum_\beta u^{-2} u^\alpha u_\beta\, \omega_\alpha \wedge \omega^\beta - \underbrace{\sum_\alpha \sum_\beta u^{-2} u^\alpha u^\beta\, \omega_\alpha \wedge \omega_\beta}_{=0},\end{aligned}$$

where the last term in the equation above is zero by anti-symmetry of the differential form $\omega_\alpha \wedge \omega_\beta = -\omega_\beta \wedge \omega_\alpha$.

7.1.3 Calculation of $\sum_\alpha \omega^\alpha \wedge \overset{\circ}{\varphi}_\alpha - \sum_\alpha \omega_\alpha \wedge \overset{\circ}{\varphi}{}^\alpha$ Taking the difference of the two terms, it follows that

$$\begin{aligned}\sum_\alpha \omega^\alpha \wedge \overset{\circ}{\varphi}_\alpha - \sum_\alpha \omega_\alpha \wedge \overset{\circ}{\varphi}{}^\alpha = &\sum_\alpha u^{-1}\, du_\alpha \wedge \omega^\alpha - \sum_\alpha u^{-2} u_\alpha\, du \wedge \omega^\alpha \\ &+ \sum_\alpha \sum_\beta \sum_\gamma u^{-1} \bar{u}^{1\gamma}_{\ \alpha} u_\beta\, du^\beta_\gamma \wedge \omega^\alpha - \sum_\alpha u^{-1}\, du^\alpha \wedge \omega_\alpha \\ &+ \sum_\alpha \sum_\beta \sum_\gamma u^{-1} \bar{u}^{1\beta}_{\ \gamma} u^\gamma\, du^\alpha_\beta \wedge \omega_\alpha \\ &- \sum_\alpha \sum_\beta \sum_\gamma \sum_\delta \sum_\varepsilon \sum_\zeta u^{-1} \bar{u}^{1\gamma}_{\ \alpha} \bar{u}^{1\zeta}_{\ \varepsilon} u^\beta_\delta u^\varepsilon F^{\gamma,\zeta}_{y_{x^\delta}}\, \omega^\alpha \wedge \omega_\beta,\end{aligned} \tag{7.23}$$

which terminates a part of the computation, and will be reserved for the computation of $d\overset{\circ}{\varphi}$.

7.2 The term $d^\circ\varphi$

The next step, which is also more difficult, is the computation of

$$d\overset{\circ}{\varphi} = d\left(-u^{-1}\, du + \sum_\beta u^{-1} u_\beta \omega^\beta - \sum_\beta u^{-1} u^\beta\, \omega_\beta\right).$$

The computation of each of the terms that appears on the right side of the equation above will be presented here. Notice that

$$du^{-1} = -\frac{du}{u^2},$$

which gives rise to the following identities concerning the three terms of $d\overset{\circ}{\varphi}$:

$$d(u^{-1} du) = -\frac{du}{u^2} \wedge du = 0, \tag{7.24}$$

$$\begin{aligned} d(u^{-1} u_\beta) &= u_\beta\, du^{-1} + u^{-1}\, du_\beta \\ &= -u^{-2} u_\beta\, du + u^{-1}\, du_\beta, \end{aligned} \tag{7.25}$$

$$d(u^{-1} u^\beta) = -u^{-2} u^\beta\, du + u^{-1}\, du^\beta. \tag{7.26}$$

Therefore, the term $d\overset{\circ}{\varphi}$ may be re-written as

$$\begin{aligned} d\overset{\circ}{\varphi} = &\sum_\beta - u^{-2} u_\beta\, du \wedge \omega^\beta + \sum_\beta u^{-1}\, du_\beta \wedge \omega^\beta + \sum_\beta u^{-1} u_\beta\, d\omega^\beta \\ &+ \sum_\beta u^{-2} u^\beta\, du \wedge \omega_\beta - \sum_\beta u^{-1}\, du^\beta \wedge \omega_\beta + \sum_\beta u^{-1} u^\beta\, d\omega_\beta. \end{aligned}$$

Replacing $d\omega^\beta$ and $d\omega_\beta$ by the expressions in equations (7.5) and (7.6), the term which contains ω may be partially obtained as below:

$$\begin{aligned}
d\mathring{\varphi} &= \sum_\beta -u^{-2}u_\beta\, du\wedge\omega^\beta + \sum_\beta u^{-1}\, du_\beta\wedge\omega^\beta + \sum_\beta u^{-1}u_\beta\left(\sum_\gamma \omega^\gamma\wedge\mathring{\varphi}^\beta_\gamma + \omega\wedge\mathring{\varphi}^\beta\right)\\
&\quad + \sum_\beta u^{-2}u^\beta\, du\wedge\omega_\beta - \sum_\beta u^{-1}\, du^\beta\wedge\omega_\beta\\
&\quad - \sum_\beta u^{-1}u^\beta\left(\sum_\gamma \mathring{\varphi}^\gamma_\beta\wedge\omega_\gamma + \omega_\beta\wedge\mathring{\varphi} + \omega\wedge\mathring{\varphi}_\beta\right)\\
&= \sum_\beta -u^{-2}u_\beta\, du\wedge\omega^\beta + \sum_\beta u^{-1}\, du_\beta\wedge\omega^\beta + \sum_\beta\sum_\gamma u^{-1}u_\beta\,\omega^\gamma\wedge\mathring{\varphi}^\beta_\gamma\\
&\quad + \sum_\beta u^{-2}u^\beta\, du\wedge\omega_\beta - \sum_\beta u^{-1}\, du^\beta\wedge\omega_\beta - \sum_\beta\sum_\gamma u^{-1}u^\beta\,\mathring{\varphi}^\gamma_\beta\wedge\omega_\gamma\\
&\quad -\sum_\beta u^{-1}u^\beta\omega_\beta\wedge\mathring{\varphi} + \omega\wedge A,
\end{aligned}$$

where

$$A = \sum_\beta u^{-1}u_\beta\,\mathring{\varphi}^\beta - \sum_\beta u^{-1}u^\beta\,\mathring{\varphi}_\beta,$$

which gives the last two terms of ψ. For the rest of the terms of ψ, it is necessary to use the full expression of $\mathring{\varphi}^\beta_\gamma$, $\mathring{\varphi}^\gamma_\beta$, and $\mathring{\varphi}$. Using equations in Proposition 6.4, therefore

$$\begin{aligned}
d\mathring{\varphi} &= -\sum_\beta u^{-2}u_\beta\, du\wedge\omega^\beta + \sum_\beta u^{-1}\, du_\beta\wedge\omega^\beta\\
&\quad + \sum_\beta\sum_\gamma u^{-1}u_\beta\,\omega^\gamma\wedge\Bigg(-\sum_\delta \bar{u^1}^\delta_\gamma\, du^\beta_\delta + u^{-1}u^\beta\omega_\gamma + \sum_\delta \delta^\beta_\gamma u^{-1}u_\delta\,\omega^\delta + u^{-1}u_\gamma\,\omega^\beta\\
&\qquad\qquad + \sum_\delta\sum_\varepsilon\sum_\zeta\sum_\eta \bar{u^1}^\zeta_\delta u^\beta_\eta \bar{u^1}^\varepsilon_\gamma F^{\varepsilon,\zeta}_{y_x\eta}\,\omega^\delta\Bigg)\\
&\quad + \sum_\beta u^{-2}u^\beta\, du\wedge\omega_\beta - \sum_\beta u^{-1}\, du^\beta\wedge\omega_\beta\\
&\quad - \sum_\beta\sum_\gamma u^{-1}u^\beta\Bigg(-\sum_\delta \bar{u^1}^\delta_\beta\, du^\gamma_\delta + u^{-1}u^\gamma\,\omega_\beta + \delta^\gamma_\beta\sum_\delta u^{-1}u_\delta\,\omega^\delta + u^{-1}u_\beta\,\omega^\gamma\\
&\qquad\qquad + \sum_\delta\sum_\varepsilon\sum_\zeta\sum_\eta \bar{u^1}^\zeta_\delta u^\gamma_\eta \bar{u^1}^\varepsilon_\beta F^{\varepsilon,\zeta}_{y_x\delta}\,\omega^\delta\Bigg)\wedge\omega_\gamma\\
&\quad -\sum_\beta u^{-1}u^\beta\omega_\beta\wedge(-u^{-1}\, du + \sum_\gamma u^{-1}u_\gamma\,\omega^\gamma - \sum_\gamma u^{-1}u^\gamma\,\omega_\gamma) + \omega\wedge A.
\end{aligned}$$

Expanding the expressions above, therefore

$$\begin{aligned}
d\mathring{\varphi} \;=\;& -\sum_{\beta} u^{-2}u_\beta\, du\wedge\omega^\beta \;+\; \sum_{\beta} u^{-1}\, du_\beta\wedge\omega^\beta \\
&- \sum_{\beta}\sum_{\gamma}\sum_{\delta} u^{-1}u_\beta \bar{u}^{1\delta}_{\ \gamma}\,\omega^\gamma\wedge du^\beta_\delta \;+\; \sum_{\beta}\sum_{\gamma} u^{-2}u_\beta u^\beta\,\omega^\gamma\wedge\omega_\gamma \\
&+ \sum_{\beta}\sum_{\gamma}\sum_{\delta} \delta^\beta_\gamma u^{-2}u_\beta u_\delta\,\omega^\gamma\wedge\omega^\delta \;+\; \sum_{\beta}\sum_{\gamma} u^{-2}u_\beta u_\gamma\,\omega^\gamma\wedge\omega^\beta \\
&+ \sum_{\beta}\sum_{\gamma}\sum_{\delta}\sum_{\varepsilon}\sum_{\zeta}\sum_{\eta} u^{-1}u_\beta \bar{u}^{1\zeta}_{\ \delta}u^\beta_\eta \bar{u}^{1\varepsilon}_{\ \gamma}F^{\varepsilon,\zeta}_{y_x\eta}\,\omega^\gamma\wedge\omega^\delta \\
&+ \sum_{\beta} u^{-2}u^\beta\, du\wedge\omega_\beta \;-\; \sum_{\beta} u^{-1}\, du^\beta\wedge\omega_\beta \\
&+ \sum_{\beta}\sum_{\gamma}\sum_{\delta} u^{-1}u^\beta\delta^\gamma_\beta u^{-1}u_\delta\,\omega^\delta\wedge\omega_\gamma \;-\; \sum_{\beta}\sum_{\gamma} u^{-2}u^\beta u^\gamma\,\omega_\beta\wedge\omega_\gamma \\
&- \sum_{\beta}\sum_{\gamma}\sum_{\delta} u^{-1}u^\beta\delta^\gamma_\beta u^{-1}u_\delta\,\omega^\delta\wedge\omega_\gamma \;-\; \sum_{\beta}\sum_{\gamma} u^{-2}u_\beta u^\beta\,\omega^\gamma\wedge\omega_\gamma \\
&- \sum_{\beta}\sum_{\gamma}\sum_{\delta}\sum_{\varepsilon}\sum_{\zeta}\sum_{\eta} u^{-1}u^\beta \bar{u}^{1\zeta}_{\ \delta}u^\gamma_\eta \bar{u}^{1\varepsilon}_{\ \beta}F^{\varepsilon,\zeta}_{y_x\eta}\,\omega^\delta\wedge\omega_\gamma \\
&+ \sum_{\beta} u^{-2}u^\beta\,\omega_\beta\wedge du \;-\; \sum_{\beta}\sum_{\gamma} u^{-2}u^\beta u_\gamma\,\omega_\beta\wedge\omega^\gamma \;+\; \sum_{\beta}\sum_{\gamma} u^{-2}u^\beta u^\gamma\,\omega_\beta\wedge\omega_\gamma \;+\; \omega\wedge A.
\end{aligned}$$

Recognising that some terms cancel each other out, the term simplifies further to give

$$\begin{aligned}
d\mathring{\varphi} \;=\;& -\sum_{\beta} u^{-2}u_\beta\, du\wedge\omega^\beta \;+\; \sum_{\beta} u^{-1}\, du_\beta\wedge\omega^\beta - \sum_{\beta}\sum_{\gamma}\sum_{\delta} u^{-1}u_\beta \bar{u}^{1\delta}_{\ \gamma}\,\omega^\gamma\wedge du^\beta_\delta \\
&- \sum_{\beta} u^{-1}\, du^\beta\wedge\omega_\beta \;+\; \sum_{\beta}\sum_{\gamma}\sum_{\delta} u^{-1}u^\beta \bar{u}^{1\delta}_{\ \beta}\, du^\gamma_\delta\wedge\omega_\gamma \\
&- \sum_{\beta}\sum_{\gamma}\sum_{\delta}\sum_{\varepsilon}\sum_{\zeta}\sum_{\eta} u^{-1}u^\beta \bar{u}^{1\zeta}_{\ \delta}u^\gamma_\eta \bar{u}^{1\varepsilon}_{\ \beta}F^{\varepsilon,\zeta}_{y_x\eta}\,\omega^\delta\wedge\omega_\gamma \;+\; \omega\wedge A \\
=\;& \sum_{\alpha} u^{-1}u_\alpha\, du_\alpha\wedge\omega^\alpha \;-\; \sum_{\alpha} u^{-2}u_\alpha\, du\wedge\omega^\alpha \;+\; \sum_{\alpha}\sum_{\beta}\sum_{\gamma} u^{-1}u_\beta \bar{u}^{1\gamma}_{\ \alpha}\, du^\beta_\gamma\wedge\omega^\alpha \\
&- \sum_{\alpha} u^{-1}\, du^\alpha\wedge\omega_\alpha \;+\; \sum_{\alpha}\sum_{\beta}\sum_{\gamma} u^{-1}u^\gamma \bar{u}^{1\beta}_{\ \gamma}\, du^\alpha_\beta\wedge\omega_\alpha \\
&- \sum_{\alpha}\sum_{\beta}\sum_{\gamma}\sum_{\delta}\sum_{\varepsilon}\sum_{\zeta} u^{-1}\bar{u}^{1\gamma}_{\ \alpha}\bar{u}^{1\zeta}_{\ \varepsilon}u^\beta_\delta u^\varepsilon F^{\gamma,\zeta}_{y_x\delta}\,\omega^\alpha\wedge\omega_\beta \;+\; \omega\wedge A.
\end{aligned}$$

By equation (7.23) in section 7.1, the first 6 terms on the right may be replaced by

$$\begin{aligned}
&\sum_{\alpha}\omega^\alpha\wedge\mathring{\varphi}_\alpha - \sum_{\alpha}\omega_\alpha\wedge\mathring{\varphi}^\alpha \\
=\;& \sum_{\alpha} u^{-1}u_\alpha\, du_\alpha\wedge\omega^\alpha \;-\; \sum_{\alpha} u^{-2}u_\alpha\, du\wedge\omega^\alpha \;+\; \sum_{\alpha}\sum_{\beta}\sum_{\gamma} u^{-1}u_\beta \bar{u}^{1\gamma}_{\ \alpha}\, du^\beta_\gamma\wedge\omega^\alpha \\
&- \sum_{\alpha} u^{-1}\, du^\alpha\wedge\omega_\alpha \;+\; \sum_{\alpha}\sum_{\beta}\sum_{\gamma} u^{-1}u^\gamma \bar{u}^{1\beta}_{\ \gamma}\, du^\alpha_\beta\wedge\omega_\alpha \\
&- \sum_{\alpha}\sum_{\beta}\sum_{\gamma}\sum_{\delta}\sum_{\varepsilon}\sum_{\zeta} u^{-1}\bar{u}^{1\gamma}_{\ \alpha}\bar{u}^{1\zeta}_{\ \varepsilon}u^\beta_\delta u^\varepsilon F^{\gamma,\zeta}_{y_x\delta}\,\omega^\alpha\wedge\omega_\beta \;+\; \omega\wedge A,
\end{aligned}$$

to obtain the final form of $d\mathring{\varphi}$:

$$d\mathring{\varphi} = \sum_{\alpha}\omega^\alpha\wedge\mathring{\varphi}_\alpha - \sum_{\alpha}\omega_\alpha\wedge\mathring{\varphi}^\alpha + \omega\wedge A. \tag{7.27}$$

Replacing $d\mathring{\varphi}$ by equation (7.27) in the expansion of $d\varphi$ that appears in equation (7.20),

$$\begin{aligned}
d\varphi &= \sum_\alpha \omega^\alpha \wedge \varphi_\alpha - \sum_\alpha \omega_\alpha \wedge \varphi^\alpha \\
&\quad + \omega \wedge \left(-dt + t\cdot\mathring{\varphi} + \sum_\alpha d_\alpha \cdot \omega^\alpha - \sum_\alpha c^\alpha \cdot \omega_\alpha \right) \\
&\quad + \left[d\mathring{\varphi} - \left(\sum_\alpha \omega^\alpha \wedge \mathring{\varphi}_\alpha - \sum_\alpha \omega_\alpha \wedge \mathring{\varphi}^\alpha \right) \right] \\
&= \sum_\alpha \omega^\alpha \wedge \varphi_\alpha - \sum_\alpha \omega_\alpha \wedge \varphi^\alpha \\
&\quad + \omega \wedge \left(-dt + t\cdot\mathring{\varphi} + \sum_\alpha d_\alpha \cdot \omega^\alpha - \sum_\alpha c^\alpha \cdot \omega_\alpha \right) \\
&\quad + \omega \wedge \left(\sum_\beta u^{-1}u_\beta \cdot \mathring{\varphi}^\beta - \sum_\beta u^{-1}u^\beta \cdot \mathring{\varphi}_\beta \right) \\
&= \sum_\alpha \omega^\alpha \wedge \varphi_\alpha - \sum_\alpha \omega_\alpha \wedge \varphi^\alpha \\
&\quad + \omega \wedge \underbrace{\left(-dt + t\cdot\mathring{\varphi} + \sum_\alpha d_\alpha \cdot \omega^\alpha - \sum_\alpha c^\alpha \cdot \omega_\alpha + \sum_\beta u^{-1}u_\beta \cdot \mathring{\varphi}^\beta - \sum_\beta u^{-1}u^\beta \cdot \mathring{\varphi}_\beta \right)}_{:=\psi},
\end{aligned}$$

and hence Lemma 7.18 is proved.

8 The S tensor – a review

The following equations are given before:

$$\begin{aligned}
d\omega^\alpha &= \sum_\beta \omega^\beta \wedge \varphi^\alpha_\beta + \omega \wedge \varphi^\alpha, \\
d\omega_\alpha &= \sum_\beta \varphi^\beta_\alpha \wedge \omega_\beta + \omega_\alpha \wedge \varphi + \omega \wedge \varphi_\alpha.
\end{aligned}$$

Proposition 8.1. *There exist* 1*-forms* Φ^β_α, Φ^α *and* Φ_α *of the following form*

$$\begin{aligned}
\Phi^\beta_\alpha &= d\varphi^\beta_\alpha - \sum_\gamma \varphi^\gamma_\alpha \wedge \varphi^\beta_\gamma - \omega_\alpha \wedge \varphi^\beta + \varphi_\alpha \wedge \omega^\beta + \delta^\beta_\alpha \sum_\sigma \varphi_\sigma \wedge \omega^\sigma + \frac{1}{2}\delta^\beta_\alpha \psi \wedge \omega, \\
\Phi^\alpha &= d\varphi^\alpha - \varphi \wedge \varphi^\alpha - \sum_\beta \varphi^\beta \wedge \varphi^\alpha_\beta + \frac{1}{2}\psi \wedge \omega^\alpha, \\
\Phi_\alpha &= d\varphi_\alpha - \sum_\beta \varphi^\beta_\alpha \wedge \varphi_\beta + \frac{1}{2}\psi \wedge \omega_\alpha,
\end{aligned}$$

that satisfy the following relations

$$\begin{aligned}
\sum_\beta \omega^\beta \wedge \Phi^\alpha_\beta + \omega \wedge \Phi^\alpha &= 0 \\
\sum_\beta \Phi^\beta_\alpha \wedge \omega_\beta - \omega \wedge \Phi_\alpha &= 0.
\end{aligned}$$

Proof. Taking the exterior derivative of $d\omega^\alpha$, then replacing $d\omega^\alpha$ and $d\omega$ by equations (7.6) and (7.7),

recovers some of the terms of Φ^α_β and Φ^α:

$$\begin{aligned}
0 &= d^2\omega^\alpha \\
&= \sum_\beta d\omega^\beta \wedge \varphi^\alpha_\beta \;-\; \sum_\beta \omega^\beta \wedge d\varphi^\alpha_\beta \;+\; d\omega \wedge \varphi^\alpha \;-\; \omega \wedge d\varphi^\alpha \\
&= \sum_\beta \left(\sum_\gamma \omega^\gamma \wedge \varphi^\beta_\gamma + \omega \wedge \varphi^\beta \right) \wedge \varphi^\alpha_\beta \;-\; \sum_\beta \omega^\beta \wedge d\varphi^\alpha_\beta \\
&\quad + \left(\sum_\gamma \omega^\gamma \wedge \omega_\gamma + \omega \wedge \varphi \right) \wedge \varphi^\alpha \;-\; \omega \wedge d\varphi^\alpha \\
&= \sum_\beta \sum_\gamma \omega^\gamma \wedge \varphi^\beta_\gamma \wedge \varphi^\alpha_\beta \;+\; \sum_\beta \omega \wedge \varphi^\beta \wedge \varphi^\alpha_\beta \;-\; \sum_\beta \omega^\beta \wedge d\varphi^\alpha_\beta \\
&\quad + \sum_\gamma \omega^\gamma \wedge \omega_\gamma \wedge \varphi^\alpha \;+\; \omega \wedge \varphi \wedge \varphi^\alpha \;-\; \omega \wedge d\varphi^\alpha \\
&= \sum_\beta \sum_\gamma \omega^\beta \wedge \varphi^\gamma_\beta \wedge \varphi^\alpha_\gamma \;+\; \omega \wedge \left(\sum_\beta \varphi^\beta \wedge \varphi^\alpha_\beta \right) \;-\; \sum_\beta \omega^\beta \wedge d\varphi^\alpha_\beta \\
&\quad + \sum_\beta \omega^\beta \wedge \omega_\beta \wedge \varphi^\alpha \;+\; \omega \wedge (\varphi \wedge \varphi^\alpha - d\varphi^\alpha) \\
&= \sum_\beta \omega^\beta \wedge \left(-d\varphi^\alpha_\beta + \sum_\gamma \varphi^\gamma_\beta \wedge \varphi^\alpha_\gamma + \omega_\beta \wedge \varphi^\alpha \right) + \omega \wedge \left(-d\varphi^\alpha + \varphi \wedge \varphi^\alpha + \sum_\beta \varphi^\beta \wedge \varphi^\alpha_\beta \right).
\end{aligned}$$

On the other hand, to get the rest of the terms, it is necessary to take the exterior derivative of $d\omega_\alpha$:

$$\begin{aligned}
0 &= d^2\omega_\alpha \\
&= \sum_\beta d\varphi^\beta_\alpha \wedge \omega_\beta \;-\; \sum_\beta \varphi^\beta_\alpha \wedge d\omega_\beta \;+\; d\omega_\alpha \wedge \varphi \;-\; \omega_\alpha \wedge d\varphi \;+\; d\omega \wedge \varphi_\alpha \;-\; \omega \wedge d\varphi_\alpha.
\end{aligned}$$

Replacing $d\omega_\alpha$ and $d\omega$ by the expressions in equations (7.6) to (7.8), and $d\varphi$ by the expression in Proposition 7.17, gives

$$\begin{aligned}
0 &= \sum_\beta d\varphi^\beta_\alpha \wedge \omega_\beta \;-\; \sum_\beta \varphi^\beta_\alpha \wedge \left(\sum_\gamma \varphi^\gamma_\beta \wedge \omega_\gamma + \omega_\beta \wedge \varphi + \omega \wedge \varphi_\beta \right) \\
&\quad + \left(\sum_\beta \varphi^\beta_\alpha \wedge \omega_\beta + \omega_\alpha \wedge \varphi + \omega \wedge \varphi_\alpha \right) \wedge \varphi - \omega_\alpha \wedge \left(\sum_\beta \omega^\beta \wedge \varphi_\beta - \sum_\beta \omega_\beta \wedge \varphi^\beta + \omega \wedge \psi \right) \\
&\quad + \left(\sum_\beta \omega^\beta \wedge \omega_\beta + \omega \wedge \varphi \right) \wedge \varphi_\alpha - \omega \wedge d\varphi_\alpha \\
&= \sum_\beta d\varphi^\beta_\alpha \wedge \omega_\beta \;-\; \sum_\beta \sum_\gamma \varphi^\beta_\alpha \wedge \varphi^\gamma_\beta \wedge \omega_\gamma \;-\; \sum_\beta \varphi^\beta_\alpha \wedge \omega_\beta \wedge \varphi \;-\; \sum_\beta \varphi^\beta_\alpha \wedge \omega \wedge \varphi_\beta \\
&\quad + \sum_\beta \varphi^\beta_\alpha \wedge \omega_\beta \wedge \varphi \;+\; \omega \wedge \varphi_\alpha \wedge \varphi \;-\; \sum_\gamma \omega_\alpha \wedge \omega^\gamma \wedge \varphi_\gamma \;+\; \sum_\beta \omega_\alpha \wedge \omega_\beta \wedge \varphi^\beta \\
&\quad - \omega_\alpha \wedge \omega \wedge \psi \;+\; \sum_\beta \omega^\beta \wedge \omega_\beta \wedge \varphi_\alpha \;+\; \omega \wedge \varphi \wedge \varphi_\alpha \;-\; \omega \wedge d\varphi_\alpha \\
&= \sum_\beta d\varphi^\beta_\alpha \wedge \omega_\beta \;-\; \sum_\beta \sum_\gamma \varphi^\gamma_\alpha \wedge \varphi^\beta_\gamma \wedge \omega_\beta \;+\; \sum_\beta \varphi^\beta_\alpha \wedge \varphi \wedge \omega_\beta \;+\; \omega \wedge \left(\sum_\beta \varphi^\beta_\alpha \wedge \varphi_\beta \right) \\
&\quad - \sum_\beta \varphi^\beta_\alpha \wedge \varphi \wedge \omega_\beta \;+\; \omega \wedge (\varphi_\alpha \wedge \varphi) \;-\; \sum_\beta \left(\delta^\alpha_\beta \sum_\gamma \omega^\gamma \wedge \varphi_\gamma \right) \wedge \omega_\beta \;-\; \sum_\beta (\omega_\alpha \wedge \varphi^\beta) \wedge \omega_\beta \\
&\quad + \omega \wedge (\omega_\alpha \wedge \psi) \;-\; \sum_\beta (\omega^\beta \wedge \varphi_\alpha) \wedge \omega_\beta \;+\; \omega \wedge (\varphi \wedge \varphi_\alpha - \omega \wedge d\varphi_\alpha) \\
&= \sum_\beta \left(d\varphi^\beta_\alpha - \sum_\gamma \varphi^\gamma_\alpha \wedge \varphi^\beta_\gamma - \delta^\alpha_\beta \sum_\gamma \omega^\gamma \wedge \varphi_\gamma - \omega_\alpha \wedge \varphi^\beta - \omega^\beta \wedge \varphi_\alpha \right) \wedge \omega_\beta \\
&\quad + \omega \wedge \left(\sum_\beta \varphi^\beta_\alpha \wedge \varphi_\beta + \varphi_\alpha \wedge \varphi + \varphi \wedge \varphi_\alpha - d\varphi_\alpha \right) \;+\; \omega \wedge \omega_\alpha \wedge \psi.
\end{aligned}$$

By splitting

$$\omega \wedge \omega_\alpha \wedge \psi = \frac{1}{2}\omega \wedge \omega_\alpha \wedge \psi - \frac{1}{2}\sum_\beta (\delta^\alpha_\beta \omega \wedge \psi) \wedge \omega_\beta,$$

into sum of two halves, therefore,

$$\sum_\beta \underbrace{\left(d\varphi^\beta_\alpha - \sum_\beta \varphi^\gamma_\alpha \wedge \varphi^\beta_\gamma - \omega_\alpha \wedge \varphi^\beta + \varphi_\alpha \wedge \omega^\beta + \delta^\beta_\alpha \sum_\gamma \varphi_\gamma \wedge \omega^\gamma + \frac{1}{2}\delta^\alpha_\beta \psi \wedge \omega \right)}_{:=\Phi^\beta_\alpha} \wedge \omega_\beta$$
$$-\omega \wedge \underbrace{\left(d\varphi_\alpha - \sum_\beta \varphi^\beta_\alpha \wedge \varphi_\beta + \frac{1}{2}\psi \wedge \omega_\alpha \right)}_{:=\Phi_\alpha} = 0.$$

Hence the expression for Φ^β_α and Φ_α are recovered. To obtain Φ^α, the term Φ^β_α found has to be compatible with the following relation

$$\sum_\beta \omega^\beta \wedge \Phi^\alpha_\beta + \omega \wedge \Phi^\alpha = 0,$$

which can be obtained from

$$\begin{aligned}
0 &= d^2\omega^\alpha \\
&= \sum_\beta \omega^\beta \wedge \left(-d\varphi^\alpha_\beta + \sum_\gamma \varphi^\gamma_\beta \wedge \varphi^\alpha_\gamma + \omega_\beta \wedge \varphi^\alpha \right) + \omega \wedge \left(-d\varphi^\alpha + \varphi \wedge \varphi^\alpha + \sum_\beta \varphi^\beta \wedge \varphi^\alpha_\beta \right) \\
&= \sum_\beta \omega^\beta \wedge \left(-d\varphi^\alpha_\beta + \sum_\gamma \varphi^\gamma_\beta \wedge \varphi^\alpha_\gamma + \omega_\beta \wedge \varphi^\alpha \right) + \omega \wedge \left(\frac{1}{2}\psi \wedge \omega^\alpha \right) \\
&\quad + \omega \wedge \underbrace{\left(-d\varphi^\alpha + \varphi \wedge \varphi^\alpha + \sum_\beta \varphi^\beta \wedge \varphi^\alpha_\beta - \frac{1}{2}\psi \wedge \omega^\alpha \right)}_{:=-\Phi^\alpha},
\end{aligned}$$

while recognising that in the first two terms of the right side of the equation above can be further reorganised:

$$\begin{aligned}
&\sum_\beta \omega^\beta \wedge \left(-d\varphi^\alpha_\beta + \sum_\gamma \varphi^\gamma_\beta \wedge \varphi^\alpha_\gamma + \omega_\beta \wedge \varphi^\alpha \right) + \omega \wedge \left(\frac{1}{2}\psi \wedge \omega^\alpha \right) \\
= &\sum_\beta \omega^\beta \wedge \left(-d\varphi^\alpha_\beta + \sum_\gamma \varphi^\gamma_\beta \wedge \varphi^\alpha_\gamma + \omega_\beta \wedge \varphi^\alpha \right) - \sum_\beta \omega^\beta \wedge \left(\frac{1}{2}\delta^\alpha_\beta \psi \wedge \omega \right) \\
= &\sum_\beta \omega^\beta \wedge \left(-d\varphi^\alpha_\beta + \sum_\gamma \varphi^\gamma_\beta \wedge \varphi^\alpha_\gamma + \omega_\beta \wedge \varphi^\alpha - \frac{1}{2}\delta^\alpha_\beta \psi \wedge \omega \right) + 0.
\end{aligned}$$

Replacing the 0 above by the following identity

$$\sum_\beta \omega^\beta \wedge \varphi_\beta \wedge \omega^\alpha + \sum_\beta \omega^\beta \wedge \left(\sum_\gamma \delta^\alpha_\beta \varphi_\gamma \wedge \omega^\gamma \right) = 0,$$

recovers the 1-form Φ^α_β:

$$\begin{aligned}
&\sum_\beta \omega^\beta \wedge \left(-d\varphi^\alpha_\beta + \sum_\gamma \varphi^\gamma_\beta \wedge \varphi^\alpha_\gamma + \omega_\beta \wedge \varphi^\alpha - \frac{1}{2}\delta^\alpha_\beta \psi \wedge \omega \right) \\
= &\sum_\beta \omega^\beta \wedge \left(-d\varphi^\alpha_\beta + \sum_\gamma \varphi^\gamma_\beta \wedge \varphi^\alpha_\gamma + \omega_\beta \wedge \varphi^\alpha - \frac{1}{2}\delta^\alpha_\beta \psi \wedge \omega \right) \\
&+ \sum_\beta \omega^\beta \wedge \varphi_\beta \wedge \omega^\alpha + \sum_\beta \omega^\beta \wedge \sum_\gamma \delta^\alpha_\beta \varphi_\gamma \wedge \omega^\gamma \\
= &\sum_\beta \omega^\beta \wedge \underbrace{\left(-d\varphi^\alpha_\beta + \sum_\gamma \varphi^\gamma_\beta \wedge \varphi^\alpha_\gamma + \omega_\beta \wedge \varphi^\alpha - \frac{1}{2}\delta^\alpha_\beta \psi \wedge \omega + \varphi_\beta \wedge \omega^\alpha + \delta^\alpha_\beta \sum_\gamma \varphi_\gamma \wedge \omega^\gamma \right)}_{=-\Phi^\alpha_\beta}.
\end{aligned}$$

Therefore,

$$\omega^\beta \wedge \Phi^\alpha_\beta + \omega \wedge \Phi^\alpha = 0,$$

and this concludes the proof of the Proposition. □

Cartan's lemma applied to

$$\omega^\beta \wedge \Phi^\alpha_\beta + \omega \wedge \Phi^\alpha = 0,$$

implies the existence of the S-tensor $S^{\beta\sigma}_{\alpha\rho}$ (which is also known as the Hachtroudi-Chern-Moser tensor), and a one form ψ^β_α such that

$$\Phi^\beta_\alpha = S^{\beta\sigma}_{\alpha\rho}\omega^\rho \wedge \omega_\sigma \ + \ \omega \wedge \psi^\beta_\alpha.$$

The next section will be devoted to the explicit calculation of this tensor.

9 The S tensor – an explicit calculation

The S-tensor involves extracting the coefficients of $\omega^\rho \wedge \omega_\sigma$ in

$$\Phi^\beta_\alpha = \underline{d\varphi^\beta_\alpha}_{\mathsf{E}} - \underbrace{\sum_\gamma \varphi^\gamma_\alpha \wedge \varphi^\beta_\gamma}_{\mathsf{D}} - \underline{\omega_\alpha \wedge \varphi^\beta}_{\mathsf{A}} + \underline{\varphi_\alpha \wedge \omega^\beta}_{\mathsf{B}} + \underbrace{\delta^\beta_\alpha \sum_\sigma \varphi_\sigma \wedge \omega^\sigma}_{\mathsf{C}} + \frac{1}{2}\delta^\beta_\alpha \psi \wedge \omega,$$

where the letters that underlined each of the first 5 terms are used for labelling purposes. The calculations will start with the following easier terms:

$$\omega_\alpha \wedge \varphi^\beta, \qquad \varphi_\alpha \wedge \omega^\beta, \qquad \delta^\beta_\alpha\left(\sum_\sigma \varphi_\sigma \wedge \omega^\sigma\right), \qquad \text{and} \qquad \sum_\gamma \varphi^\gamma_\alpha \wedge \varphi^\beta_\gamma.$$

The term that is the most difficult to calculate here, $d\varphi^\beta_\alpha$, will be reserved for the last. The underlined terms are the ones that will be needed for the S-tensor.

9.1 The term $\omega_\alpha \wedge \varphi^\beta$

Replacing $\mathring{\varphi}^\beta$ in

$$\varphi^\beta \ = \ c^\beta \cdot \omega + \sum_\eta b^\beta_\eta + \mathring{\varphi}^\beta$$

by the expansion of $\mathring{\varphi}^\beta$ in Proposition 6.4:

$$\mathring{\varphi}^\beta \ = \ -u^{-1}du^\beta \ + \ \sum_\eta \sum_\theta u^{-1}\bar{u}^{-1\eta}_\theta u^\theta du^\beta_\eta \ + \ \sum_\eta u^{-2}u^\beta u_\eta \omega^\eta \ - \ \sum_\eta u^{-2}u^\beta u^\eta \omega_\eta,$$

gives

$$\begin{aligned}\varphi^\beta \ = \ & c^\beta\omega \ + \ \sum_\eta b^\beta_\eta \omega^\eta \ - \ u^{-1}du^\beta \ + \ \sum_\eta \sum_\theta u^{-1}\bar{u}^{-1\eta}_\theta u^\theta du^\beta_\eta \\ & + \sum_\eta u^{-2}u^\beta u_\eta \omega^\eta \ - \ \sum_\eta u^{-2}u^\beta u^\eta \omega_\eta.\end{aligned}$$

Hence, wedging on both sides by ω_α, taking note of the terms of the form $\omega^* \wedge \omega_*$ (or $\omega_* \wedge \omega^*$), yields

$$\begin{aligned}\omega_\alpha \wedge \varphi^\beta \ = \ & c^\beta \omega_\alpha \wedge \omega \ + \ \underbrace{\sum_\eta b^\beta_\eta \ \omega_\alpha \wedge \omega^\eta}_{\text{(A.ii)}} \ - \ u^{-1} \ \omega_\alpha \wedge du^\beta \\ & + \sum_\eta \sum_\theta u^{-1}\bar{u}^{-1\eta}_\theta u^\theta \ \omega_\alpha \wedge du^\beta_\eta \ + \ \underbrace{\sum_\eta u^{-2}u^\beta u_\eta \ \omega_\alpha \wedge \omega^\eta}_{\text{(A.i)}} \ - \ \sum_\eta u^{-2}u^\beta u^\eta \ \omega_\alpha \wedge \omega_\eta.\end{aligned}$$

9.2 The term $\varphi_\alpha \wedge \omega^\beta$

Next, consider the equation of φ_α that appears in Proposition 7.1, and the equation of $\mathring{\varphi}_\alpha$ that appears in Proposition 6.4:

$$\begin{aligned}
\varphi_\alpha &= d_\alpha\omega + \sum_\eta (\delta^\eta_\alpha t - b^\eta_\alpha)\omega_\eta + \mathring{\varphi}_\alpha,\\
\mathring{\varphi}_\alpha &= -u^{-1}\, du_\alpha + u^{-2}u_\alpha\, du - \sum_\eta\sum_\theta u^{-1}\bar{u}^{1\theta}_{\ \alpha}u_\eta\, du^\eta_\theta\\
&\quad + \sum_\eta u^{-2}u_\eta u_\alpha\,\omega^\eta - \sum_\eta\sum_\theta\sum_\iota \bar{u}^{1\theta}_{\ \alpha}\bar{u}^{1\iota}_{\ \eta}F^{\theta,\iota}_{y}\,\omega^\eta + \sum_\eta\sum_\theta\sum_\iota\sum_\kappa\sum_\lambda u^{-1}\bar{u}^{1\theta}_{\ \alpha}u^\lambda_\iota\bar{u}^{1\kappa}_{\ \eta}u_\lambda F^{\theta,\kappa}_{y_{x^\iota}}\,\omega^\eta\\
&\quad + \sum_\eta -u^{-2}u^\eta u_\alpha\,\omega_\eta - \sum_\eta\sum_\theta\sum_\iota\sum_\kappa\sum_\lambda u^{-1}\bar{u}^{1\theta}_{\ \alpha}\bar{u}^{1\lambda}_{\ \kappa}u^\eta_\iota u^k F^{\theta,\lambda}_{y_{x^\iota}}\,\omega_\eta.
\end{aligned}$$

Wedging $\mathring{\varphi}_\alpha$ with ω^β, and φ_α with ω^β, give

$$\begin{aligned}
\varphi_\alpha\wedge\omega^\beta &= d_\alpha\omega\wedge\omega^\beta + \sum_\eta(\delta^\eta_\alpha t - b^\eta_\alpha)\omega_\eta\wedge\omega^\beta + \mathring{\varphi}_\alpha\wedge\omega^\beta,\\
\mathring{\varphi}_\alpha\wedge\omega^\beta &= -u^{-1}\, du_\alpha\wedge\omega^\beta + u^{-2}u_\alpha\, du\wedge\omega^\beta - \sum_\eta\sum_\theta u^{-1}\bar{u}^{1\theta}_{\ \alpha}u_\eta\, du^\eta_\theta\wedge\omega^\beta\\
&\quad + \sum_\eta u^{-2}u_\eta u_\alpha\omega^\eta\wedge\omega^\beta - \sum_\eta\sum_\theta\sum_\iota \bar{u}^{1\theta}_{\ \alpha}\bar{u}^{1\iota}_{\ \eta}F^{\theta,\iota}_{y}\,\omega^\eta\wedge\omega^\beta\\
&\quad + \sum_\eta\sum_\theta\sum_\iota\sum_\kappa\sum_\lambda u^{-1}\bar{u}^{1\theta}_{\ \alpha}u^\lambda_\iota\bar{u}^{1\kappa}_{\ \eta}u_\lambda F^{\theta,\kappa}_{y_{x^\iota}}\,\omega^\eta\wedge\omega^\beta\\
&\quad + \sum_\eta -u^{-2}u^\eta u_\alpha\,\omega_\eta\wedge\omega^\beta - \sum_\eta\sum_\theta\sum_\iota\sum_\kappa\sum_\lambda\, u^{-1}\bar{u}^{1\theta}_{\ \alpha}\bar{u}^{1\lambda}_{\ \kappa}u^\eta_\iota u^\kappa F^{\theta,\lambda}_{y_{x^\iota}}\,\omega_\eta\wedge\omega^\beta.
\end{aligned}$$

Hence, by substituting the full expression of $\mathring{\varphi}\wedge\omega^\beta$ into $\varphi_\alpha\wedge\omega^\beta$, the following equation is obtained

$$\begin{aligned}
\varphi_\alpha\wedge\omega^\beta &= d_\alpha\,\omega\wedge\omega^\beta + \underline{\sum_\eta(\delta^\eta_\alpha t - b^\eta_\alpha)\,\omega_\eta\wedge\omega^\beta}_{\text{(B.i)}} - u^{-1}\, du_\alpha\wedge\omega^\beta + u^{-2}u_\alpha\, du\wedge\omega^\beta\\
&\quad - \sum_\eta\sum_\theta u^{-1}\bar{u}^{1\theta}_{\ \alpha}u_\eta\, du^\eta_\theta\wedge\omega^\beta + \sum_\eta u^{-2}u_\eta u_\alpha\,\omega^\eta\wedge\omega^\beta - \sum_\eta\sum_\theta\sum_\iota \bar{u}^{1\theta}_{\ \alpha}\bar{u}^{1\iota}_{\ \eta}F^{\theta,\iota}_{y}\,\omega^\eta\wedge\omega^\beta\\
&\quad + \sum_\eta\sum_\theta\sum_\iota\sum_\kappa\sum_\lambda u^{-1}\bar{u}^{1\theta}_{\ \alpha}u^\lambda_\iota\bar{u}^{1\kappa}_{\ \eta}u_\lambda F^{\theta,\kappa}_{y_{x^\iota}}\omega^\eta\,\omega\wedge\omega^\beta + \underline{\sum_\eta -u^{-2}u^\eta u_\alpha\,\omega_\eta\wedge\omega^\beta}_{\text{(B.ii)}}\\
&\quad \underbrace{- \sum_\eta\sum_\theta\sum_\iota\sum_\kappa\sum_\lambda u^{-1}\bar{u}^{1\theta}_{\ \alpha}\bar{u}^{1\lambda}_{\ \kappa}u^\eta_\iota u^\kappa F^{\theta,\lambda}_{y_{x^\iota}}\,\omega_\eta\wedge\omega^\beta}_{\text{(B.iii)}},
\end{aligned}$$

with the necessary terms underlined.

9.3 The term $\delta^\beta_\alpha\sum_\sigma\varphi_\sigma\wedge\omega^\sigma$

The term $\delta^\beta_\alpha\sum_\sigma\varphi_\sigma\wedge\omega^\sigma$ is the most straightforward, simply by using $\mathring{\varphi}_\alpha$ in Proposition 6.4 and φ_α in Proposition 7.1 to obtain the expressions of $\varphi_\alpha\wedge\omega^\beta$, then changing α and β to σ, while taking note of the sum $\delta^\beta_\alpha\sum_\sigma$:

$$
\begin{aligned}
&\delta^\beta_\alpha \sum_\sigma \varphi_\sigma \wedge \omega^\sigma \\
= \;& \delta^\beta_\alpha \Bigg(\sum_\sigma d_\sigma\, \omega \wedge \omega^\sigma \;+\; \underline{\sum_\sigma \sum_\eta (\delta^\eta_\sigma t - b^\eta_\sigma)\, \omega_\eta \wedge \omega^\sigma}_{\text{(C.i)}} \;-\; \sum_\sigma u^{-1}\, du_\sigma \wedge \omega^\sigma \\
&+ \sum_\sigma u^{-2} u_\sigma\, du \wedge \omega^\sigma \;-\; \sum_\sigma \sum_\eta \sum_\theta u^{-1} \bar{u}^{-1\theta}_{\ \sigma} u_\eta\, du^\eta_\sigma \wedge \omega^\sigma \\
&+ \sum_\sigma \sum_\eta u^{-1} u_\eta u_\sigma\, \omega^\eta \wedge \omega^\sigma \;-\; \sum_\sigma \sum_\eta \sum_\theta \sum_\iota \bar{u}^{-1\theta}_{\ \sigma} \bar{u}^{-1\iota}_{\ \eta} F^{\theta,\iota}_y\, \omega^\eta \wedge \omega^\sigma \\
&+ \sum_\sigma \sum_\eta \sum_\theta \sum_\iota \sum_\kappa \sum_\lambda u^{-1} \bar{u}^{-1\theta}_{\ \sigma} u^\lambda_\iota \bar{u}^{-1\kappa}_{\ \eta} u_\lambda F^{\theta,\kappa}_{y_{x^i}}\, \omega^\eta \wedge \omega^\sigma \\
&+ \underline{\sum_\sigma \sum_\eta - u^{-2} u^\eta u_\sigma\, \omega_\eta \wedge \omega^\sigma}_{\text{(C.ii)}} \;-\; \underline{\sum_\sigma \sum_\eta \sum_\theta \sum_\iota \sum_\kappa \sum_\lambda u^{-1} \bar{u}^{-1\theta}_{\ \sigma} \bar{u}^{-1\lambda}_{\ \kappa} u^\eta_\iota u^\kappa F^{\theta,\lambda}_{y_{x^\iota}}\, \omega_\eta \wedge \omega^\sigma}_{\text{(C.iii)}} \Bigg).
\end{aligned}
$$

9.4 Collecting terms with $\omega^* \wedge \omega_*$ from $\omega_\alpha \wedge \varphi^\beta$, $\varphi_\alpha \wedge \omega^\beta$ and $\delta^\beta_\alpha \sum_\sigma \varphi_\sigma \wedge \omega^\sigma$

Collecting the underlined terms, one has:

In $\omega_\alpha \wedge \varphi^\beta$:

$$\sum_\eta u^{-2} u^\beta u_\eta\, \omega_\alpha \wedge \omega^\eta = -\sum_\eta u^{-2} u^\beta u_\eta\, \omega^\eta \wedge \omega_\alpha, \tag{A.i}$$

$$\sum_\eta b^\beta_\eta\, \omega_\alpha \wedge \omega^\eta = -\sum_\eta b^\beta_\eta \omega^\eta \wedge \omega_\alpha. \tag{A.ii}$$

In $\varphi_\alpha \wedge \omega^\beta$:

$$\sum_\eta (\delta^\eta_\alpha t - b^\eta_\alpha)\, \omega_\eta \wedge \omega^\beta = -\sum_\eta (\delta^\eta_\alpha t - b^\eta_\alpha)\, \omega^\beta \wedge \omega_\eta, \tag{B.i}$$

$$-\sum_\eta u^{-2} u^\eta u_\alpha\, \omega_\eta \wedge \omega^\beta = \sum_\eta u^{-2} u^\eta u_\alpha\, \omega^\beta \wedge \omega_\eta, \tag{B.ii}$$

$$-\sum_\eta \sum_\theta \sum_\iota \sum_\kappa \sum_\lambda u^{-1} \bar{u}^{-1\theta}_{\ \alpha} \bar{u}^{-1\lambda}_{\ \kappa} u^\eta_\iota u^\kappa F^{\theta,\lambda}_{y_{x^\iota}}\, \omega_\eta \wedge \omega^\beta = \sum_\eta \sum_\theta \sum_\iota \sum_\kappa \sum_\lambda u^{-1} \bar{u}^{-1\theta}_{\ \alpha} \bar{u}^{-1\lambda}_{\ \kappa} u^\eta_\iota u^\kappa F^{\theta,\lambda}_{y_{x^\iota}}\, \omega^\beta \wedge \omega_\eta. \tag{B.iii}$$

Finally in $\delta^\beta_\alpha \sum_\sigma \varphi_\sigma \wedge \omega^\sigma$:

$$\delta^\beta_\alpha \sum_\sigma \sum_\eta (\delta^\eta_\sigma t - b^\eta_\sigma)\, \omega_\eta \wedge \omega^\sigma = -\delta^\beta_\alpha \sum_\sigma \sum_\eta (\delta^\eta_\sigma t - b^\eta_\sigma)\, \omega^\sigma \wedge \omega_\eta, \tag{C.i}$$

$$-\delta^\beta_\alpha \sum_\sigma \sum_\eta u^{-2} u^\eta u_\sigma\, \omega_\eta \wedge \omega^\sigma = \delta^\beta_\alpha \sum_\sigma \sum_\eta u^{-2} u^\eta u_\sigma\, \omega^\sigma \wedge \omega_\eta, \tag{C.ii}$$

$$-\delta^\beta_\alpha \sum_\sigma \sum_\eta \sum_\theta \sum_\iota \sum_\kappa \sum_\lambda u^{-1} \bar{u}^{-1\theta}_{\ \sigma} \bar{u}^{-1\lambda}_{\ \kappa} u^\eta_\iota u^\kappa F^{\theta,\lambda}_{y_{x^\iota}}\, \omega_\eta \wedge \omega^\sigma = \delta^\beta_\alpha \sum_\sigma \sum_\eta \sum_\theta \sum_\iota \sum_\kappa \sum_\lambda u^{-1} \bar{u}^{-1\theta}_{\ \sigma} \bar{u}^{-1\lambda}_{\ \kappa} u^\eta_\iota u^\kappa F^{\theta,\lambda}_{y_{x^\iota}}\, \omega^\sigma \wedge \omega_\eta. \tag{C.iii}$$

By renaming the indices so that each of the terms above are expressed in the following form $\omega^\rho \wedge \omega_\sigma$ (which means that the use of Kronecker's delta is necessary here), the corresponding terms become

$$-\sum_\rho \sum_\sigma \delta^\sigma_\alpha u^{-2} u^\beta u_\rho\, \omega^\rho \wedge \omega_\sigma, \tag{A.i}$$

$$-\sum_{\rho}\sum_{\sigma}\delta^{\sigma}_{\alpha}b^{\beta}_{\rho}\,\omega^{\rho}\wedge\omega_{\sigma}, \tag{A.ii}$$

$$-\sum_{\rho}\sum_{\sigma}\delta^{\beta}_{\rho}(\delta^{\sigma}_{\alpha}t-b^{\sigma}_{\alpha})\,\omega^{\rho}\wedge\omega_{\sigma}, \tag{B.i}$$

$$\sum_{\rho}\sum_{\sigma}\delta^{\beta}_{\rho}u^{-2}u^{\sigma}u_{\alpha}\omega^{\rho}\wedge\omega_{\sigma}, \tag{B.ii}$$

$$\sum_{\rho}\sum_{\sigma}\left(\sum_{\theta}\sum_{\iota}\sum_{\kappa}\sum_{\lambda}\delta^{\beta}_{\rho}u^{-1}\bar{u}^{1\theta}_{\alpha}\bar{u}^{1\lambda}_{\kappa}u^{\sigma}_{\iota}u^{\kappa}F^{\theta,\lambda}_{y_{x^{\iota}}}\right)\omega^{\rho}\wedge\omega_{\sigma}, \tag{B.iii}$$

$$-\delta^{\beta}_{\alpha}\sum_{\rho}\sum_{\sigma}(\delta^{\sigma}_{\rho}t-b^{\sigma}_{\rho})\,\omega^{\rho}\wedge\omega_{\sigma}, \tag{C.i}$$

$$\delta^{\beta}_{\alpha}\sum_{\rho}\sum_{\sigma}u^{-2}u^{\sigma}u_{\rho}\,\omega^{\rho}\wedge\omega_{\sigma}, \tag{C.ii}$$

$$\delta^{\beta}_{\alpha}\sum_{\rho}\sum_{\sigma}\left(\sum_{\theta}\sum_{\iota}\sum_{\kappa}\sum_{\lambda}u^{-1}\bar{u}^{1\theta}_{\rho}\bar{u}^{1\lambda}_{\kappa}u^{\sigma}_{\iota}u^{\kappa}F^{\theta,\lambda}_{y_{x^{\iota}}}\right)\,\omega^{\rho}\wedge\omega_{\sigma}. \tag{C.iii}$$

9.5 The term $\sum_{\gamma}\varphi^{\gamma}_{\alpha}\wedge\varphi^{\beta}_{\gamma}$

As the expression for φ^{*}_{*} is complicated, the calculations can be very involved, starting with

$$\begin{aligned}\varphi^{\gamma}_{\alpha} &= b^{\gamma}_{\alpha}\omega+\mathring{\varphi}^{\gamma}_{\alpha}\\ &= b^{\gamma}_{\alpha}\omega+\Bigg(-\sum_{\eta}\bar{u}^{1\eta}_{\alpha}du^{\gamma}_{\eta}+u^{-1}u^{\gamma}\omega_{\alpha}+\delta^{\gamma}_{\alpha}\sum_{\eta}u^{-1}u_{\eta}\omega^{\eta}+u^{-1}u_{\alpha}\omega^{\gamma}\\ &\qquad+\sum_{\eta}\sum_{\theta}\sum_{\iota}\sum_{\kappa}\bar{u}^{1\iota}_{\eta}u^{\gamma}_{\kappa}\bar{u}^{1\theta}_{\alpha}F^{\theta,\iota}_{y_{x^{\kappa}}}\,\omega^{\eta}\Bigg),\end{aligned}$$

and

$$\begin{aligned}\varphi^{\beta}_{\gamma} &= b^{\beta}_{\gamma}\omega+\mathring{\varphi}^{\beta}_{\gamma}\\ &= b^{\beta}_{\gamma}\omega+\Bigg(-\sum_{\lambda}\bar{u}^{1\lambda}_{\gamma}du^{\beta}_{\lambda}+u^{-1}u^{\beta}\omega_{\gamma}+\delta^{\beta}_{\gamma}\sum_{\lambda}u^{-1}u_{\lambda}\omega^{\lambda}+u^{-1}u_{\gamma}\omega^{\beta}\\ &\qquad+\sum_{\lambda}\sum_{\mu}\sum_{\nu}\sum_{\xi}\bar{u}^{1\nu}_{\lambda}u^{\beta}_{\xi}\bar{u}^{1\mu}_{\gamma}F^{\mu,\nu}_{y_{x^{\xi}}}\omega^{\lambda}\Bigg).\end{aligned}$$

Hence a direct expansion yields the wedge product of the two expressions

$$
\begin{aligned}
&\varphi^\gamma_\alpha \wedge \varphi^\beta_\gamma \\
=\;& b^\gamma_\alpha b^\beta_\gamma\, \omega\wedge\omega \;-\; \sum_\lambda b^\gamma_\alpha \bar{u}^{1\lambda}_\gamma\, \omega\wedge du^\beta_\lambda \;+\; u^{-1} b^\gamma_\alpha u^\beta\, \omega\wedge\omega_\gamma \;+\; \delta^\beta_\gamma \sum_\lambda u^{-1}u_\lambda b^\gamma_\alpha\, \omega\wedge\omega^\lambda \\
&+ u^{-1} b^\gamma_\alpha u_\gamma\, \omega\wedge\omega^\beta \;+\; \sum_\lambda\sum_\mu\sum_\nu\sum_\xi b^\gamma_\alpha \bar{u}^{1\nu}_\lambda u^\beta_\xi \bar{u}^{1\mu}_\gamma F^{\mu,\nu}_{y_{x^\xi}}\, \omega\wedge\omega^\lambda \\
&- \sum_\eta \bar{u}^{1\eta}_\alpha b^\beta_\gamma\, du^\gamma_\eta\wedge\omega \;+\; \sum_\eta\sum_\lambda \bar{u}^{1\eta}_\alpha \bar{u}^{1\lambda}_\gamma\, du^\gamma_\eta\wedge du^\beta_\lambda \;-\; \sum_\eta \bar{u}^{1\eta}_\alpha u^{-1}u^\beta\, du^\gamma_\eta\wedge\omega_\gamma \\
&- \delta^\beta_\gamma \sum_\eta\sum_\lambda \bar{u}^{1\eta}_\alpha u^{-1}u_\lambda\, du^\gamma_\eta\wedge\omega^\lambda \;-\; \sum_\eta \bar{u}^{1\eta}_\alpha u^{-1}u_\gamma\, du^\gamma_\eta\wedge\omega^\beta \;-\; \sum_\eta\sum_\lambda\sum_\mu\sum_\nu\sum_\xi \bar{u}^{1\eta}_\alpha \bar{u}^{1\nu}_\lambda u^\beta_\xi \bar{u}^{1\mu}_\gamma F^{\mu,\nu}_{y_{x^\xi}}\, du^\gamma_\eta\wedge\omega^\lambda \\
&+ u^{-1}u^\gamma b^\beta_\gamma\, \omega_\alpha\wedge\omega \;-\; \sum_\lambda \bar{u}^{1\lambda}_\gamma u^{-1}u^\gamma\, \omega_\alpha\wedge du^\beta_\lambda \;+\; u^{-2}u^\gamma u^\beta\, \omega_\alpha\wedge\omega_\gamma \\
&+ \underbrace{\delta^\beta_\gamma \sum_\lambda u^{-2}u^\gamma u_\lambda\, \omega_\alpha\wedge\omega^\lambda}_{\text{(D.i)}} \;+\; \underbrace{u^{-2}u^\gamma u_\gamma\, \omega_\alpha\wedge\omega^\beta}_{\text{(D.ii)}} \;+\; \underbrace{\sum_\lambda\sum_\mu\sum_\nu\sum_\xi u^{-1}u^\gamma \bar{u}^{1\nu}_\lambda u^\beta_\xi \bar{u}^{1\mu}_\gamma F^{\mu,\nu}_{y_{x^\xi}}\, \omega_\alpha\wedge\omega^\lambda}_{\text{(D.iii)}} \\
&+ \delta^\gamma_\alpha \sum_\eta u^{-1}u_\eta b^\beta_\gamma\, \omega^\eta\wedge\omega \;-\; \delta^\gamma_\alpha \sum_\eta\sum_\lambda u^{-1}u_\eta \bar{u}^{1\lambda}_\gamma\, \omega^\eta\wedge du^\beta_\lambda \;+\; \underbrace{\delta^\gamma_\alpha \sum_\eta u^{-2}u_\eta u^\beta\, \omega^\eta\wedge\omega_\gamma}_{\text{(D.iv)}} \\
&+ \delta^\beta_\gamma\delta^\gamma_\alpha \sum_\eta\sum_\lambda u^{-2}u_\eta u_\lambda\, \omega^\eta\wedge\omega^\lambda \;+\; \delta^\gamma_\alpha \sum_\eta u^{-2}u_\eta u_\gamma\, \omega^\eta\wedge\omega^\beta \\
&+ \delta^\gamma_\alpha \sum_\eta\sum_\lambda\sum_\mu\sum_\xi\sum_\nu u^{-1}u_\eta \bar{u}^{1\nu}_\lambda u^\beta_\xi \bar{u}^{1\mu}_\gamma F^{\mu,\nu}_{y_{x^\xi}}\, \omega^\eta\wedge\omega^\lambda \;+\; u^{-1}u_\alpha b^\beta_\gamma\, \omega^\gamma\wedge\omega \\
&- \sum_\lambda u^{-1}u_\alpha \bar{u}^{1\lambda}_\gamma\, \omega^\gamma\wedge du^\beta_\lambda \;+\; \underbrace{u^{-2}u_\alpha u^\beta\, \omega^\gamma\wedge\omega_\gamma}_{\text{(D.v)}} \;+\; \delta^\beta_\gamma \sum_\lambda u^{-2}u_\alpha u_\lambda\, \omega^\gamma\wedge\omega^\lambda \;+\; u^{-2}u_\alpha u_\gamma\, \omega^\gamma\wedge\omega^\beta \\
&+ \sum_\lambda\sum_\mu\sum_\nu\sum_\xi u^{-1}u_\alpha \bar{u}^{1\nu}_\lambda u^\beta_\xi \bar{u}^{1\mu}_\gamma F^{\mu,\nu}_{y_{x^\xi}}\, \omega^\gamma\wedge\omega^\lambda \;+\; \sum_\eta\sum_\theta\sum_\iota\sum_\kappa \bar{u}^{1\iota}_\eta u^\gamma_\kappa \bar{u}^{1\theta}_\alpha F^{\theta,\iota}_{y_{x^\kappa}} b^\beta_\gamma\, \omega^\eta\wedge\omega \\
&- \sum_\eta\sum_\theta\sum_\iota\sum_\kappa\sum_\lambda \bar{u}^{1\iota}_\eta u^\gamma_\kappa \bar{u}^{1\theta}_\alpha F^{\theta,\iota}_{y_{x^\kappa}} \bar{u}^{1\lambda}_\gamma\, \omega^\eta\wedge du^\beta_\lambda \;+\; \underbrace{\sum_\eta\sum_\theta\sum_\iota\sum_\kappa \bar{u}^{1\iota}_\eta u^\gamma_\kappa \bar{u}^{1\theta}_\alpha F^{\theta,\iota}_{y_{x^\kappa}} u^{-1}u^\beta\, \omega^\eta\wedge\omega_\gamma}_{\text{(D.vi)}} \\
&+ \delta^\beta_\gamma \sum_\eta\sum_\theta\sum_\iota\sum_\kappa\sum_\lambda \bar{u}^{1\iota}_\eta u^\gamma_\kappa \bar{u}^{1\theta}_\alpha F^{\theta,\iota}_{y_{x^\kappa}} u^{-1}u_\lambda\, \omega^\eta\wedge\omega^\lambda \;+\; \sum_\eta\sum_\theta\sum_\iota\sum_\kappa \bar{u}^{1\iota}_\eta u^\gamma_\kappa \bar{u}^{1\theta}_\alpha F^{\theta,\iota}_{y_{x^\kappa}} u^{-1}u_\gamma\, \omega^\eta\wedge\omega^\beta \\
&+ \sum_\eta\sum_\theta\sum_\iota\sum_\kappa\sum_\lambda\sum_\mu\sum_\nu\sum_\xi \bar{u}^{1\iota}_\eta u^\gamma_\kappa \bar{u}^{1\theta}_\alpha F^{\theta,\iota}_{y_{x^\kappa}} \bar{u}^{1\nu}_\lambda u^\beta_\xi \bar{u}^{1\mu}_\gamma F^{\mu,\nu}_{y_{x^\xi}}\, \omega^\eta\wedge\omega^\lambda.
\end{aligned}
$$

9.6 Collecting terms with $\omega^*\wedge\omega_*$ in $\sum_\gamma \varphi^\gamma_\alpha\wedge\varphi^\beta_\gamma$

Collecting the terms of the form $\omega^*\wedge\omega_*$ (or $\omega_*\wedge\omega^*$) in $\varphi^\gamma_\alpha\wedge\varphi^\beta_\gamma$, the following terms are obtained

$$\delta^\beta_\gamma \sum_\lambda u^{-2}u^\gamma u_\lambda\, \omega_\alpha\wedge\omega^\lambda = -\delta^\beta_\gamma \sum_\lambda u^{-2}u^\gamma u_\lambda\, \omega^\lambda\wedge\omega_\alpha, \tag{D.i}$$

$$u^{-2}u^\gamma u_\gamma\, \omega_\alpha\wedge\omega^\beta = -u^{-2}u^\gamma u_\gamma\, \omega^\beta\wedge\omega_\alpha, \tag{D.ii}$$

$$\sum_\lambda\sum_\mu\sum_\nu\sum_\xi u^{-1}u^\gamma \bar{u}^{1\nu}_\lambda u^\beta_\xi \bar{u}^{1\mu}_\gamma F^{\mu,\nu}_{y_{x^\xi}}\, \omega_\alpha\wedge\omega^\lambda = -\sum_\lambda\sum_\mu\sum_\nu\sum_\xi u^{-1}u^\gamma \bar{u}^{1\nu}_\lambda u^\beta_\xi \bar{u}^{1\mu}_\gamma F^{\mu,\nu}_{y_{x^\xi}}\, \omega^\lambda\wedge\omega_\alpha, \tag{D.iii}$$

$$\delta^\gamma_\alpha \sum_\eta u^{-2}u_\eta u^\beta\, \omega^\eta\wedge\omega_\gamma, \tag{D.iv}$$

$$u^{-2}u_\alpha u^\beta\, \omega^\gamma\wedge\omega_\gamma \tag{D.v}$$

$$\sum_\eta\sum_\theta\sum_\iota\sum_\kappa \bar{u}^{1\iota}_\eta u^\gamma_\kappa \bar{u}^{1\theta}_\alpha F^{\theta,\iota}_{y_{x^\kappa}} u^{-1}u^\beta\, \omega^\eta\wedge\omega_\gamma. \tag{D.vi}$$

Summing over γ, and re-expressing the terms in the form $\omega^\rho \wedge \omega_\sigma$,

$$-\sum_\lambda u_\lambda \left(\sum_\gamma u^{-2} u^\gamma \delta^\beta_\gamma \right) \omega^\lambda \wedge \omega_\alpha = -\sum_\rho \sum_\sigma u_\rho \delta^\sigma_\alpha u^{-2} u^\beta \, \omega^\rho \wedge \omega_\sigma, \tag{D.i}$$

$$\left(\sum_\gamma - u^{-2} u^\gamma u_\gamma \right) \omega^\beta \wedge \omega_\alpha = \sum_\rho \sum_\sigma \left[\delta^\beta_\rho \delta^\sigma_\alpha \left(\sum_\gamma - u^{-2} u^\gamma u_\gamma \right) \right] \omega^\rho \wedge \omega_\sigma, \tag{D.ii}$$

$$-\sum_\lambda \left(\sum_\gamma \sum_\mu \sum_\nu \sum_\xi u^{-1} u^\gamma \bar{u}^{1\nu}_\lambda u^\beta_\xi \bar{u}^{1\mu}_\gamma F^{\mu,\nu}_{y_{x\xi}} \right) \omega^\lambda \wedge \omega_\alpha = -\sum_\rho \sum_\sigma \delta^\sigma_\alpha \left(\sum_\gamma \sum_\mu \sum_\nu \sum_\xi u^{-1} u^\gamma \bar{u}^{1\nu}_\rho u^\beta_\xi \bar{u}^{1\mu}_\gamma F^{\mu,\nu}_{y_{x\xi}} \right) \omega^\rho \wedge \omega_\sigma, \tag{D.iii}$$

$$\sum_\eta \left(\sum_\gamma \delta^\gamma_\alpha u^{-2} u_\eta u^\beta \right) \omega^\eta \wedge \omega_\gamma = \sum_\rho \sum_\sigma \delta^\sigma_\alpha u^{-2} u_\rho u^\beta \, \omega^\rho \wedge \omega_\sigma, \tag{D.iv}$$

$$\sum_\gamma u^{-2} u_\alpha u^\beta \, \omega^\gamma \wedge \omega_\gamma = \sum_\rho \sum_\sigma \delta^\sigma_\rho u^{-2} u_\alpha u^\beta \, \omega^\rho \wedge \omega_\sigma, \tag{D.v}$$

$$\sum_\eta \sum_\gamma \left(\sum_\theta \sum_\iota \sum_\kappa \bar{u}^{1\iota}_\eta u^\gamma_\kappa \bar{u}^{1\theta}_\alpha F^{\theta,\iota}_{y_{x\kappa}} u^{-1} u^\beta \right) \omega^\eta \wedge \omega_\gamma = \sum_\rho \sum_\sigma \left(\sum_\theta \sum_\iota \sum_\kappa \bar{u}^{1\iota}_\rho u^\sigma_\kappa \bar{u}^{1\theta}_\alpha F^{\theta,\iota}_{y_{x\kappa}} u^{-1} u^\beta \right) \omega^\rho \wedge \omega_\sigma. \tag{D.vi}$$

9.7 The term $d\varphi^\beta_\alpha$

The calculation of this term is very complicated, and will be broken down into steps. From

$$\varphi^\beta_\alpha = \mathring{\varphi}^\beta_\alpha + b^\beta_\alpha \omega,$$

and using the expression for $\mathring{\varphi}^\beta_\alpha$,

$$\begin{aligned} d(\varphi^\beta_\alpha) \;=\; & \underbrace{d\left(-\sum_\gamma \bar{u}^{1\gamma}_\alpha \, du^\beta_\gamma \right)}_{\text{Step 1}} + \underbrace{d\left(u^{-1} u^\beta \omega_\alpha \right)}_{\text{Step 2}} + \underbrace{d\left(\delta^\alpha_\beta \sum_\gamma u^{-1} u_\gamma \, \omega^\gamma \right)}_{\text{Step 3}} + \underbrace{d\left(u^{-1} u_\alpha \omega^\beta \right)}_{\text{Step 4}} \\ & + \underbrace{d\left(\sum_\gamma \sum_\delta \sum_\varepsilon \sum_\zeta \bar{u}^{1\varepsilon}_\gamma u^\beta_\zeta \bar{u}^{1\delta}_\alpha F^{\delta,\varepsilon}_{y_{x\zeta}} \, \omega^\gamma \right)}_{\text{Step 5}} + \underbrace{d\big(b^\beta_\alpha \omega \big)}_{\text{Step 6}}, \end{aligned}$$

the calculation will be broken down into 6 steps.

9.7.1 Step 1 From

$$d\left(-\sum_\gamma \bar{u}^{1\gamma}_\alpha \, du^\beta_\gamma \right) = -\sum_\gamma d\bar{u}^{1\gamma}_\alpha \wedge du^\beta_\gamma,$$

and using Lemma,

$$d\bar{u}^{1\gamma}_\alpha = -\sum_\eta \sum_\theta \bar{u}^{1\gamma}_\theta \bar{u}^{1\eta}_\alpha \, du^\theta_\eta,$$

therefore

$$\begin{aligned} d\left(-\sum_\gamma \bar{u}^{1\gamma}_\alpha \, du^\beta_\gamma \right) \;&=\; -\sum_\gamma \left(-\sum_\eta \sum_\theta \bar{u}^{1\gamma}_\theta \bar{u}^{1\eta}_\alpha \, du^\theta_\eta \right) \wedge du^\beta_\gamma \\ &=\; \sum_\gamma \sum_\eta \sum_\theta \bar{u}^{1\gamma}_\theta \bar{u}^{1\eta}_\alpha \, du^\theta_\eta \wedge du^\beta_\gamma. \end{aligned}$$

Hence there is no term of the form $\omega^* \wedge \omega_*$.

9.7.2 Step 2 For step 2, replacing $d\omega_\alpha$ by equation (7.8) in Proposition 7.1 to obtain:

$$\begin{aligned} d(u^{-1}u^\beta\omega_\alpha) &= (du^{-1})u^\beta\wedge\omega_\alpha + u^{-1}du^\beta\wedge\omega_\alpha + u^{-1}u^\beta d\omega_\alpha \\ &= -u^{-2}u^\beta\, du\wedge\omega_\alpha + u^{-1}\, du^\beta\wedge\omega_\alpha \\ &\quad +u^{-1}u^\beta\left(\sum_\eta \varphi^\eta_\alpha\wedge\omega_\eta + \omega_\alpha\wedge\varphi + \omega\wedge\varphi_\alpha\right) \\ &= -u^{-2}u^\beta\, du\wedge\omega_\alpha + u^{-1}\, du^\beta\wedge\omega_\alpha + u^{-1}u^\beta\sum_\eta \varphi^\eta_\alpha\wedge\omega_\eta \\ &\quad +u^{-1}u^\beta\,\omega_\alpha\wedge\varphi + u^{-1}u^\beta\,\omega\wedge\varphi_\alpha. \end{aligned}$$

Any terms that contain ω may be safely ignored, and hence the expression of $\omega\wedge\varphi_\alpha$ may be left unexpanded. Using equation (7.2) to replace φ^η_α and φ by

$$\begin{aligned} \varphi^\eta_\alpha &= b^\eta_\alpha\omega + \mathring{\varphi}^\eta_\alpha, \\ \varphi &= t\omega + \mathring{\varphi}, \end{aligned}$$

in the expression of $d(u^{-1}u^\beta\omega_\alpha)$, it follows that

$$\begin{aligned} d(u^{-1}u^\beta\omega_\alpha) &= -u^{-2}u^\beta\, du\wedge\omega_\alpha \;+\; u^{-1}\, du^\beta\wedge\omega_\alpha \;+\; u^{-1}u^\beta\sum_\eta\left(b^\eta_\alpha\omega + \mathring{\varphi}^\eta_\alpha\right)\wedge\omega_\eta \\ &\quad + u^{-1}u^\beta\omega_\alpha\wedge(t\omega+\mathring{\varphi}) \;+\; u^{-1}u^\beta\,\omega\wedge\varphi_\alpha \\ &= -u^{-2}u^\beta\, du\wedge\omega_\alpha \;+\; u^{-1}\, du^\beta\wedge\omega_\alpha \;+\; u^{-1}u^\beta\sum_\eta\;\mathring{\varphi}^\eta_\alpha\wedge\omega_\eta \;+\; u^{-1}u^\beta\,\omega_\alpha\wedge\mathring{\varphi} \\ &\quad +\omega\wedge\left(u^{-1}u^\beta\varphi_\alpha + u^{-1}u^\beta\sum_\eta b^\eta_\alpha\omega_\eta - u^{-1}u^\beta t\omega_\alpha\right). \end{aligned}$$

To get all the terms of the form $\omega^*\wedge\omega_*$, the terms $\mathring{\varphi}^\eta_\alpha$ and $\mathring{\varphi}$ will be replaced by

$$\begin{aligned} \mathring{\varphi}^\eta_\alpha &= -\sum_\lambda \bar{u}^{1\lambda}_\alpha\, du^\eta_\lambda \;+\; u^{-1}u^\eta\omega_\alpha \;+\; \delta^\eta_\alpha\sum_\lambda u^{-1}u_\lambda\omega^\lambda + u^{-1}u_\alpha\omega^\eta \;+\; \sum_\lambda\sum_\theta\sum_\iota\sum_\kappa \bar{u}^{1\iota}_\lambda u^\eta_\kappa \bar{u}^{1\theta}_\alpha F^{\theta,\iota}_{y_x\kappa}\omega^\lambda, \\ \mathring{\varphi} &= -u^{-1}\, du \;+\; \sum_\eta u^{-1}u_\eta\omega^\eta \;-\; \sum_\eta u^{-1}u^\eta\omega_\eta, \end{aligned}$$

to get

$$\begin{aligned}
d(u^{-1}u^\beta\omega_\alpha) &= -u^{-2}u^\beta\, du\wedge\omega_\alpha \;+\; u^{-1}du^\beta\wedge\omega_\alpha \\
&\quad + u^{-1}u^\beta\sum_\eta\Big(-\sum_\lambda \bar{u}^{1\lambda}_{\alpha}\, du^\eta_\lambda \;+\; u^{-1}u^\eta\omega_\alpha \;+\; \delta^\eta_\alpha\sum_\lambda u^{-1}u_\lambda\omega^\lambda \\
&\qquad\qquad + u^{-1}u_\alpha\omega^\eta \;+\; \sum_\lambda\sum_\theta\sum_\iota\sum_\kappa \bar{u}^{1\iota}_{\lambda}u^\eta_\kappa\bar{u}^{1\theta}_{\alpha}F^{\theta,\iota}_{y_{x^\kappa}}\omega^\lambda\Big)\wedge\omega_\eta \\
&\quad + u^{-1}u^\beta\,\omega_\alpha\wedge\Big(-u^{-1}du+\sum_\eta u^{-1}u_\eta\omega^\eta-\sum_\eta u^{-1}u^\eta\omega_\eta\Big) \\
&\quad + \omega\wedge\Big(u^{-1}u^\beta\varphi_\alpha+u^{-1}u^\beta\sum_\eta b^\eta_\alpha\omega_\eta-u^{-1}u^\beta t\omega_\alpha\Big) \\
&= -u^{-2}u^\beta\, du\wedge\omega_\alpha \;+\; u^{-1}\,du^\beta\wedge\omega_\alpha \;+\; u^{-1}u^\beta\sum_\eta\sum_\lambda -\bar{u}^{1\lambda}_{\alpha}\,du^\eta_\lambda\wedge\omega_\eta \\
&\quad + u^{-2}u^\beta\sum_\eta u^\eta\omega_\alpha\wedge\omega_\eta \;+\; \underbrace{\sum_\eta\sum_\lambda \delta^\eta_\alpha u^{-2}u^\beta u_\lambda\,\omega^\lambda\wedge\omega_\eta}_{\text{(E.i)}} \quad + \underbrace{\sum_\eta u^{-2}u^\beta u_\alpha\,\omega^\eta\wedge\omega_\eta}_{\text{(E.ii)}} \\
&\quad + \underbrace{\sum_\eta\sum_\lambda\sum_\theta\sum_\iota\sum_\kappa u^{-1}u^\beta\bar{u}^{1\iota}_{\lambda}u^\eta_\kappa\bar{u}^{1\theta}_{\alpha}F^{\theta,\iota}_{y_{x^\kappa}}\,\omega^\lambda\wedge\omega_\eta}_{\text{(E.iii)}} \;-\; u^{-2}u^\beta\,\omega_\alpha\wedge du \\
&\quad + \underbrace{\sum_\eta u^{-2}u^\beta u_\eta\,\omega_\alpha\wedge\omega^\eta}_{\text{(E.iv)}} \;-\; \sum_\eta u^{-2}u^\beta u^\eta\,\omega_\alpha\wedge\omega_\eta \\
&\quad + \omega\wedge\Big(u^{-1}u^\beta\varphi_\alpha+u^{-1}u^\beta\sum_\eta b^\eta_\alpha\omega_\eta-u^{-1}u^\beta t\omega_\alpha\Big).
\end{aligned}$$

9.7.3 Step 3

For Step 3,

$$\begin{aligned}
d\Big(\delta^\alpha_\beta\sum_\gamma u^{-1}u_\gamma\omega^\gamma\Big) &= \delta^\alpha_\beta\sum_\gamma d(u^{-1}u_\gamma\omega^\gamma) \\
&= \delta^\alpha_\beta\sum_\gamma\Big(-u^{-2}u_\gamma\,du\wedge\omega^\gamma+u^{-1}du_\gamma\wedge\omega^\gamma+u^{-1}u_\gamma d\omega^\gamma\Big).
\end{aligned}$$

Replacing $d\omega^\gamma$ above by

$$d\omega^\gamma=\sum_\eta\omega^\eta\wedge\varphi^\gamma_\eta+\omega\wedge\varphi^\gamma$$

in Proposition yields 7.1

$$\begin{aligned}
d\Big(\delta^\alpha_\beta\sum_\gamma u^{-1}u_\gamma\omega^\gamma\Big) &= -\delta^\alpha_\beta\sum_\gamma u^{-2}u_\gamma\,du\wedge\omega^\gamma \;+\; \delta^\alpha_\beta\sum_\gamma u^{-1}\,du_\gamma\wedge\omega^\gamma \\
&\quad + \delta^\alpha_\beta\sum_\gamma u^{-1}u_\gamma\Big(\sum_\eta\omega^\eta\wedge\varphi^\gamma_\eta+\omega\wedge\varphi^\gamma\Big) \\
&= -\delta^\alpha_\beta\sum_\gamma u^{-2}u_\gamma\,du\wedge\omega^\gamma \;+\; \delta^\alpha_\beta\sum_\gamma u^{-1}\,du_\gamma\wedge\omega^\gamma \\
&\quad + \delta^\alpha_\beta\sum_\gamma\sum_\eta u^{-1}u_\gamma\omega^\eta\wedge\varphi^\gamma_\eta \;+\; \omega\wedge\Big(\delta^\alpha_\beta\sum_\gamma u^{-1}u_\gamma\varphi^\gamma\Big).
\end{aligned}$$

Then replacing φ^γ_η above by

$$\varphi^\gamma_\eta=b^\gamma_\eta\omega+\mathring{\varphi}^\gamma_\eta,$$

the term takes on the following expression.

$$\begin{aligned}
d\Big(\delta^\alpha_\beta\sum_\gamma u^{-1}u_\gamma\omega^\gamma\Big) &= -\delta^\alpha_\beta\sum_\gamma u^{-2}u_\gamma\,du\wedge\omega^\gamma \;+\; \delta^\alpha_\beta\sum_\gamma u^{-1}du_\gamma\wedge\omega^\gamma \\
&\quad + \delta^\alpha_\beta\sum_\gamma\sum_\eta u^{-1}u_\gamma\omega^\eta\wedge\mathring{\varphi}^\gamma_\eta \;+\; \omega\wedge\Big(\delta^\alpha_\beta\sum_\gamma u^{-1}u_\gamma\varphi^\gamma-\delta^\alpha_\beta\sum_\gamma\sum_\eta u^{-1}u_\gamma b^\gamma_\eta\omega^\eta\Big).
\end{aligned}$$

Making the final substitution using

$$\mathring{\varphi}^\gamma_\eta \;=\; -\sum_\lambda \bar{u}^{1\lambda}_\eta\, du^\gamma_\lambda \;+\; u^{-1}u^\gamma\omega_\eta \;+\; \delta^\gamma_\eta\sum_\lambda u^{-1}u_\lambda\,\omega^\lambda \;+\; u^{-1}u_\eta\omega^\gamma \;+\; \sum_\lambda\sum_\theta\sum_\iota\sum_\kappa \bar{u}^{1\iota}_\lambda u^\gamma_\kappa \bar{u}^{1\theta}_\eta F^{\theta,\iota}_{y_{x^\kappa}}\,\omega^\lambda,$$

the following full expression

$$\begin{aligned}
d\Big(\delta^\alpha_\beta\sum_\gamma u^{-1}u_\gamma\omega^\gamma\Big) \;&=\; -\delta^\alpha_\beta\sum_\gamma u^{-2}u_\gamma\, du\wedge\omega^\gamma \;+\; \delta^\alpha_\beta\sum_\gamma u^{-1}\, du_\gamma\wedge\omega^\gamma\\
&\quad +\delta^\alpha_\beta\sum_\gamma\sum_\eta u^{-1}u_\gamma\omega^\eta\wedge\Big(-\sum_\lambda \bar{u}^{1\lambda}_\eta\, du^\gamma_\lambda + u^{-1}u^\gamma\,\omega_\eta + \delta^\gamma_\eta\sum_\lambda u^{-1}u_\lambda\,\omega^\lambda\\
&\qquad\qquad +u^{-1}u_\eta\,\omega^\gamma + \sum_\lambda\sum_\theta\sum_\iota\sum_\kappa \bar{u}^{1\iota}_\lambda u^\gamma_\kappa\bar{u}^{1\theta}_\eta F^{\theta,\iota}_{y_{x^\kappa}}\,\omega^\lambda\Big)\\
&\quad +\omega\wedge\Big(\delta^\alpha_\beta\sum_\gamma u^{-1}u_\gamma\varphi^\gamma - \delta^\alpha_\beta\sum_\gamma\sum_\eta u^{-1}u_\gamma b^\gamma_\eta\omega^\eta\Big)\\
&=\; -\delta^\alpha_\beta\sum_\gamma u^{-2}u_\gamma\, du\wedge\omega^\gamma \;+\; \delta^\alpha_\beta\sum_\gamma u^{-1}\, du_\gamma\wedge\omega^\gamma\\
&\quad -\delta^\alpha_\beta\sum_\gamma\sum_\eta\sum_\lambda u^{-1}u_\gamma\bar{u}^{1\lambda}_\eta\,\omega^\eta\wedge\, du^\gamma_\lambda \;+\; \underbrace{\delta^\alpha_\beta\sum_\gamma\sum_\eta u^{-2}u_\gamma u^\gamma\,\omega^\eta\wedge\omega_\eta}_{\text{(E.v)}}\\
&\quad +\delta^\alpha_\beta\sum_\gamma\sum_\eta\sum_\lambda\delta^\gamma_\eta u^{-2}u_\gamma u_\lambda\,\omega^\eta\wedge\omega^\lambda \;+\; \delta^\alpha_\beta\sum_\gamma\sum_\eta u^{-2}u_\gamma u_\eta\,\omega^\eta\wedge\omega^\gamma\\
&\quad +\delta^\alpha_\beta\sum_\gamma\sum_\eta\sum_\lambda\sum_\theta\sum_\iota\sum_\kappa u^{-1}u_\gamma\bar{u}^{1\iota}_\lambda u^\gamma_\kappa\bar{u}^{1\theta}_\eta F^{\theta,\iota}_{y_{x^\kappa}}\,\omega^\eta\wedge\omega^\lambda\\
&\quad +\omega\wedge\Big(\delta^\alpha_\beta\sum_\gamma u^{-1}u_\gamma\varphi^\gamma - \delta^\alpha_\beta\sum_\gamma\sum_\eta u^{-1}u_\gamma b^\gamma_\eta\omega^\eta\Big).
\end{aligned}$$

shows that there is only one term that contains $\omega^*\wedge\omega_*$.

9.7.4 Step 4: The term $d(u^{-1}u_\alpha\omega^\beta)$

To calculate $d(u^{-1}u_\alpha\omega^\beta)$, using Proposition 7.1

$$d\omega^\beta \;=\; \sum_\eta \omega^\eta\wedge\varphi^\beta_\eta + \omega\wedge\varphi^\beta,$$

yields the expansion of $d(u^{-1}u_\alpha\omega^\beta)$:

$$\begin{aligned}
d(u^{-1}u_\alpha\omega^\beta) \;&=\; (du^{-1})u_\alpha\wedge\omega^\beta \;+\; u^{-1}du_\alpha\wedge\omega^\beta \;+\; u^{-1}u_\alpha d\omega^\beta\\
&=\; -u^{-2}u_\alpha\, du\wedge\omega^\beta \;+\; u^{-1}\, du_\alpha\wedge\omega^\beta \;+\; u^{-1}u_\alpha\, d\omega^\beta\\
&=\; -u^{-2}u_\alpha\, du\wedge\omega^\beta \;+\; u^{-1}du_\alpha\wedge\omega^\beta + u^{-1}u_\alpha\Big(\sum_\eta\omega^\eta\wedge\varphi^\beta_\eta + \omega\wedge\varphi^\beta\Big)\\
&=\; -u^{-2}u_\alpha\, du\wedge\omega^\beta \;+\; u^{-1}\, du_\alpha\wedge\omega^\beta + u^{-1}u_\alpha\sum_\eta\omega^\eta\wedge\varphi^\beta_\eta \;+\; \omega\wedge\big(u^{-1}u_\alpha\varphi^\beta\big).
\end{aligned}$$

Then by using $\varphi^\beta_\eta = b^\beta_\eta\omega + \mathring{\varphi}^\beta_\eta$, the calculation continues:

$$\begin{aligned}
d(u^{-1}u_\alpha\omega^\beta) \;&=\; -u^{-2}u_\alpha\, du\wedge\omega^\beta \;+\; u^{-1}\, du_\alpha\wedge\omega^\beta + u^{-1}u_\alpha\sum_\eta\omega^\eta\wedge\Big(b^\beta_\eta\omega + \mathring{\varphi}^\beta_\eta\Big)\\
&\quad +\omega\wedge\big(u^{-1}u_\alpha\varphi^\beta\big)\\
&=\; -u^{-2}u_\alpha\, du\wedge\omega^\beta \;+\; u^{-1}\, du_\alpha\wedge\omega^\beta \;+\; u^{-1}u_\alpha\sum_\eta\omega^\eta\wedge\mathring{\varphi}^\beta_\eta\\
&\quad +\omega\wedge\Big(u^{-1}u_\alpha\varphi^\beta - u^{-1}u_\alpha\sum_\eta b^\beta_\eta\omega^\eta\Big).
\end{aligned}$$

Making a final substitution of $\mathring{\varphi}$ by

$$\mathring{\varphi}^\beta_\eta \;=\; -\sum_\lambda \bar{u}^{1\lambda}_{\ \eta}\, du^\beta_\lambda \;+\; u^{-1}u^\beta\,\omega_\eta \;+\; \delta^\beta_\eta \sum_\lambda u^{-1}u_\lambda\,\omega^\lambda + u^{-1}u_\eta\,\omega^\beta \;+\; \sum_\lambda\sum_\theta\sum_\iota\sum_\kappa \bar{u}^{1\iota}_{\ \lambda}u^\beta_\kappa \bar{u}^{1\theta}_{\ \eta}F^{\theta,\iota}_{y_{x^\kappa}}\,\omega^\lambda,$$

it follows that the following term

$$\begin{aligned}
d(u^{-1}u_\alpha\omega^\beta) \;=\;& -u^{-2}u_\alpha\, du\wedge\omega^\beta \;+\; u^{-1}\,du_\alpha\wedge\omega^\beta \\
& +u^{-1}u_\alpha\sum_\eta\omega^\eta\wedge\Bigg(-\sum_\lambda \bar{u}^{1\lambda}_{\ \eta}du^\beta_\lambda + u^{-1}u^\beta\omega_\eta + \delta^\beta_\eta\sum_\lambda u^{-1}u_\lambda\omega^\lambda \\
& \qquad +u^{-1}u_\eta\omega^\beta + \sum_\lambda\sum_\theta\sum_\iota\sum_\kappa \bar{u}^{1\iota}_{\ \lambda}u^\beta_\kappa\bar{u}^{1\theta}_{\ \eta}F^{\theta,\iota}_{y_{x^\kappa}}\omega^\lambda\Bigg) \\
& +\omega\wedge(u^{-1}u_\alpha\varphi^\beta - u^{-1}u_\alpha\sum_\eta b^\beta_\eta\omega^\eta) \\
=\;& -u^{-2}u_\alpha\, du\wedge\omega^\beta \;+\; u^{-1}\,du_\alpha\wedge\omega^\beta \;-\; u^{-1}u_\alpha\sum_\eta\sum_\lambda\ \bar{u}^{1\lambda}_{\ \eta}\omega^\eta\wedge du^\beta_\lambda \\
& \underbrace{+u^{-2}u_\alpha\sum_\eta u^\beta\,\omega^\eta\wedge\omega_\eta}_{\text{(E.vi)}} \;+\; \sum_\eta\sum_\lambda\delta^\beta_\eta u^{-2}u_\alpha u_\lambda\,\omega^\eta\wedge\omega^\lambda \;+\; \sum_\eta u^{-2}u_\alpha u_\eta\,\omega^\eta\wedge\omega^\beta \\
& +\sum_\eta\sum_\lambda\sum_\theta\sum_\iota\sum_\kappa u^{-1}u_\alpha\bar{u}^{1\iota}_{\ \lambda}u^\beta_\kappa\bar{u}^{1\theta}_{\ \eta}F^{\theta,\iota}_{y_{x^\kappa}}\,\omega^\eta\wedge\omega^\lambda + \omega\wedge\left(u^{-1}u_\alpha\varphi^\beta - u^{-1}u_\alpha\sum_\eta b^\beta_\eta\omega^\eta\right).
\end{aligned}$$

also has only one needed term.

9.7.5 Step 5

Finally, the following term will be calculated

$$d\left(\sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta \bar{u}^{1\varepsilon}_{\ \gamma}u^\beta_\zeta\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x^\zeta}}\,\omega^\gamma\right),$$

with

$$\begin{aligned}
& d\left(\sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta \bar{u}^{1\varepsilon}_{\ \gamma}u^\beta_\zeta\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x^\zeta}}\,\omega^\gamma\right) \\
=\;& \sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta\Bigg(\underbrace{u^\beta_\zeta\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x^\zeta}}\,d\bar{u}^{1\varepsilon}_{\ \gamma}\wedge\omega^\gamma}_{\mathsf{A}} + \underbrace{\bar{u}^{1\varepsilon}_{\ \gamma}\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x^\zeta}}\,du^\beta_\zeta\wedge\omega^\gamma}_{\mathsf{B}} \\
& \qquad +\underbrace{\bar{u}^{1\varepsilon}_{\ \gamma}u^\beta_\zeta F^{\delta,\varepsilon}_{y_{x^\zeta}}\,d\bar{u}^{1\delta}_{\ \alpha}\wedge\omega^\gamma}_{\mathsf{C}} + \underbrace{\bar{u}^{1\varepsilon}_{\ \gamma}u^\beta_\zeta\bar{u}^{1\delta}_{\ \alpha}\,d(F^{\delta,\varepsilon}_{y_{x^\zeta}})\wedge\omega^\gamma}_{\mathsf{D}} + \underbrace{\bar{u}^{1\varepsilon}_{\ \gamma}u^\beta_\zeta\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x^\zeta}}\,d\omega^\gamma}_{\mathsf{E}}\Bigg).
\end{aligned}$$

Before computing each of the terms A to E, the following formulae will be used

$$\begin{aligned}
d\bar{u}^{1\varepsilon}_{\ \gamma} \;&=\; -\sum_\eta\sum_\theta \bar{u}^{1\varepsilon}_{\ \theta}\bar{u}^{1\eta}_{\ \gamma}\,du^\theta_\eta, \\
d\bar{u}^{1\delta}_{\ \alpha} \;&=\; -\sum_\eta\sum_\theta \bar{u}^{1\delta}_{\ \theta}\bar{u}^{1\eta}_{\ \alpha}\,du^\theta_\eta, \\
d(F^{\delta,\varepsilon}_{y_{x^\zeta}}) \;&=\; \sum_\eta\omega^\eta\wedge\left[\sum_\theta \bar{u}^{1\theta}_{\ \eta}\mathsf{D}_{x^\theta}(F^{\delta,\varepsilon}_{y_{x^\zeta}})\right] + \sum_\eta\omega_\eta\wedge\left[\sum_\theta u^{-1}u^\eta_\theta F^{\delta,\varepsilon}_{y_{x^\zeta},y_{x^\theta}}\right] \\
& \qquad +\omega\wedge\left[u^{-1}F^{\delta,\varepsilon}_{y_{x^\zeta},y} - \sum_\eta\sum_\theta u^{-1}\bar{u}^{1\theta}_{\ \eta}u^\eta\mathsf{D}_{x^\theta}(F^{\delta,\varepsilon}_{y^\zeta}) - \sum_\eta\sum_\theta u^{-2}u^\eta_\theta u_\eta F^{\delta,\varepsilon}_{y_{x^\zeta},y_{x^\theta}}\right], \\
d\omega^\gamma \;&=\; \sum_\eta\omega^\eta\wedge\omega^\gamma_\eta + \omega\wedge\varphi^\gamma \\
&=\; \sum_\eta\omega^\eta\wedge\omega^\gamma_\eta + \omega\wedge\left(c^\gamma\omega + \sum_\theta b^\gamma_\theta\omega^\theta + \mathring{\varphi}^\gamma\right), \\
\varphi^\gamma_\eta \;&=\; b^\gamma_\eta\omega + \mathring{\varphi}^\eta_\eta.
\end{aligned}$$

Hence the terms corresponding to (A) to (E) yields by substitution of the terms above the following expressions:

$$\begin{aligned}
(\mathsf{A}) \qquad u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x\zeta}}\, d\bar{u}^{1\varepsilon}_{\ \gamma}\wedge\omega^{\gamma} &= u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x\zeta}}\Big(-\sum_{\eta}\sum_{\theta}\bar{u}^{1\varepsilon}_{\ \theta}\bar{u}^{1\eta}_{\ \gamma}\, du^{\theta}_{\eta}\Big)\wedge\omega^{\gamma}\\
&= -\sum_{\eta}\sum_{\theta}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x\zeta}}\bar{u}^{1\varepsilon}_{\ \theta}\bar{u}^{1\eta}_{\ \gamma}\, du^{\theta}_{\eta}\wedge\omega^{\gamma},
\end{aligned}$$

$$(\mathsf{B}) \qquad \bar{u}^{1\varepsilon}_{\ \gamma}\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x\zeta}}\, du^{\beta}_{\zeta}\wedge\omega^{\gamma},$$

$$\begin{aligned}
(\mathsf{C}) \qquad \bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}F^{\delta,\varepsilon}_{y_{x\zeta}}\, d\bar{u}^{1\delta}_{\ \alpha}\wedge\omega^{\gamma} &= \bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}F^{\delta,\varepsilon}_{y_{x\zeta}}\Big(-\sum_{\eta}\sum_{\theta}\bar{u}^{1\delta}_{\ \theta}\bar{u}^{1\eta}_{\ \alpha}\, du^{\theta}_{\eta}\Big)\wedge\omega^{\gamma}\\
&= -\sum_{\eta}\sum_{\theta}\bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \theta}\bar{u}^{1\eta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x\zeta}}\, du^{\theta}_{\eta}\wedge\omega^{\gamma},
\end{aligned}$$

$$\begin{aligned}
(\mathsf{D}) \qquad \bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}d\big(F^{\delta,\varepsilon}_{y_{x\zeta}}\big)\wedge\omega^{\gamma} &= \sum_{\eta}\sum_{\theta}\bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}\bar{u}^{1\theta}_{\ \eta}\mathsf{D}_{x^{\theta}}\big(F^{\delta,\varepsilon}_{y_{x\zeta}}\big)\,\omega^{\eta}\wedge\omega^{\gamma}\\
&\quad+\sum_{\eta}\sum_{\theta}\bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}u^{-1}u^{\eta}_{\theta}F^{\delta,\varepsilon}_{y_{x\zeta},y_{x^{\theta}}}\,\omega_{\eta}\wedge\omega^{\gamma}\\
&\quad+\omega\wedge\omega^{\gamma}\Bigg[\bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}\Bigg(u^{-1}F^{\delta,\varepsilon}_{y_{x\zeta},y}-\sum_{\eta}\sum_{\theta}u^{-1}\bar{u}^{1\theta}_{\ \eta}u^{\eta}\mathsf{D}_{x^{\theta}}\big(F^{\delta,\varepsilon}_{y_{x\zeta}}\big)\\
&\qquad\qquad-\sum_{\eta}\sum_{\theta}u^{-2}u^{\eta}_{\theta}u_{\eta}F^{\delta,\varepsilon}_{y_{x\zeta},y_{x^{\theta}}}\Bigg)\Bigg],
\end{aligned}$$

$$\begin{aligned}
(\mathsf{E}) \qquad \bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x\zeta}}\, d\omega^{\gamma} &= \bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x\zeta}}\Big(\sum_{\eta}\omega^{\eta}\wedge\varphi^{\gamma}_{\eta}+\omega\wedge\varphi^{\gamma}\Big)\\
&= \bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x\zeta}}\Big(\sum_{\eta}\omega^{\eta}\wedge(b^{\gamma}_{\eta}\omega+\mathring{\varphi}^{\gamma}_{\eta})\Big)+\omega\wedge\Big(\bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x\zeta}}\varphi^{\gamma}\Big)\\
&= \bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x\zeta}}\sum_{\eta}\omega^{\eta}\wedge\mathring{\varphi}^{\gamma}_{\eta}+\omega\wedge\Big(\bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x\zeta}}\varphi^{\gamma}-\bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x\zeta}}\sum_{\eta}b^{\gamma}_{\eta}\omega^{\eta}\Big).
\end{aligned}$$

In the expression (E), using $\mathring{\varphi}^{\gamma}_{\eta}$ in Proposition 6.4:

$$\mathring{\varphi}^{\gamma}_{\eta} = -\sum_{\lambda}\bar{u}^{1\lambda}_{\ \eta}\, du^{\gamma}_{\lambda}+u^{-1}u^{\gamma}\omega_{\eta}+\delta^{\gamma}_{\eta}\sum_{\lambda}u^{-1}u_{\lambda}\,\omega^{\lambda}+u^{-1}u_{\eta}\omega^{\gamma}+\sum_{\lambda}\sum_{\theta}\sum_{\iota}\sum_{\kappa}\bar{u}^{1\iota}_{\ \lambda}u^{\gamma}_{\kappa}\bar{u}^{1\theta}_{\ \eta}F^{\theta,\iota}_{y_{x^{\kappa}}}\,\omega^{\lambda}$$

yields

$$\begin{aligned}
(\mathsf{E}) \qquad \bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x\zeta}}\, d\omega^{\gamma} &= \bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x\zeta}}\sum_{\eta}\omega^{\eta}\wedge\Big(-\sum_{\lambda}\bar{u}^{1\lambda}_{\ \eta}\, du^{\gamma}_{\lambda}+u^{-1}u^{\gamma}\omega_{\eta}+\delta^{\gamma}_{\eta}\sum_{\lambda}u^{-1}u_{\lambda}\omega^{\lambda}\\
&\qquad\qquad+u^{-1}u_{\eta}\omega^{\gamma}+\sum_{\lambda}\sum_{\theta}\sum_{\iota}\sum_{\kappa}\bar{u}^{1\iota}_{\ \lambda}u^{\gamma}_{\kappa}\bar{u}^{1\theta}_{\ \eta}F^{\theta,\iota}_{y_{x^{\kappa}}}\,\omega^{\lambda}\Big)\\
&\quad+\omega\wedge\Big(\bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x\zeta}}\varphi^{\gamma}-\bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x\zeta}}\sum_{\eta}b^{\gamma}_{\eta}\omega^{\eta}\Big)\\
&= -\sum_{\eta}\sum_{\lambda}\bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x\eta}}\bar{u}^{1\lambda}_{\ \eta}\,\omega^{\eta}\wedge du^{\gamma}_{\lambda}+\bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x\zeta}}u^{-1}u^{\gamma}\sum_{\eta}\omega^{\eta}\wedge\omega_{\eta}\\
&\quad+\sum_{\eta}\sum_{\lambda}\delta^{\gamma}_{\eta}u^{-1}u_{\lambda}\bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x\zeta}}\,\omega^{\eta}\wedge\omega^{\lambda}+\sum_{\eta}\bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x\zeta}}u^{-1}u_{\eta}\omega^{\eta}\wedge\omega^{\gamma}\\
&\quad+\sum_{\eta}\sum_{\lambda}\sum_{\theta}\sum_{\iota}\sum_{\kappa}\bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x\zeta}}\bar{u}^{1\iota}_{\ \lambda}u^{\gamma}_{\kappa}\bar{u}^{1\theta}_{\ \eta}F^{\theta,\iota}_{y_{x^{\kappa}}}\,\omega^{\eta}\wedge\omega^{\lambda}\\
&\quad+\omega\wedge\Big(\bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x\zeta}}\varphi^{\gamma}-\bar{u}^{1\varepsilon}_{\ \gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x\zeta}}\sum_{\eta}b^{\gamma}_{\eta}\omega^{\eta}\Big).
\end{aligned}$$

Combining the terms A to E, while not forgetting the sums $\sum_\gamma \sum_\delta \sum_\varepsilon \sum_\zeta$, therefore,

$$
\begin{aligned}
& d\Big(\sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta \bar{u}^{1\varepsilon}_{\ \gamma} u^\beta_\zeta \bar{u}^{1\delta}_{\ \alpha} F^{\delta,\varepsilon}_{y_{x^\zeta}} \omega^\gamma\Big) \\
= \ & -\sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta\sum_\eta\sum_\theta u^\beta_\zeta \bar{u}^{1\delta}_{\ \alpha} F^{\delta,\varepsilon}_{y_{x^\zeta}} \bar{u}^{1\varepsilon}_{\ \theta}\bar{u}^{1\eta}_{\ \gamma}\, du^\theta_\eta \wedge \omega^\gamma + \sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta \bar{u}^{1\varepsilon}_{\ \gamma}\bar{u}^{1\delta}_{\ \alpha} F^{\delta,\varepsilon}_{y_{x^\zeta}}\, du^\beta_\zeta \wedge \omega^\gamma \\
& -\sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta\sum_\eta\sum_\theta \bar{u}^{1\varepsilon}_{\ \gamma} u^\beta_\zeta \bar{u}^{1\delta}_{\ \theta}\bar{u}^{1\eta}_{\ \alpha} F^{\delta,\varepsilon}_{y_{x^\zeta}}\, du^\theta_\eta \wedge \omega^\gamma + \sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta\sum_\eta\sum_\theta \bar{u}^{1\varepsilon}_{\ \gamma} u^\beta_\zeta \bar{u}^{1\delta}_{\ \alpha}\bar{u}^{1\theta}_{\ \eta} \mathsf{D}_{x^\theta}\left(F^{\delta,\varepsilon}_{y_{x^\zeta}}\right)\omega^\eta \wedge \omega^\gamma \\
& \underbrace{+\sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta\sum_\eta\sum_\theta \bar{u}^{1\varepsilon}_{\ \gamma} u^\beta_\zeta \bar{u}^{1\delta}_{\ \alpha} u^{-1} u^\eta_\theta F^{\delta,\varepsilon}_{y_{x^\zeta}, y_{x^\theta}}\, \omega_\eta \wedge \omega^\gamma}_{\text{(E.vii)}} - \sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta\sum_\eta\sum_\lambda \bar{u}^{1\varepsilon}_{\ \gamma} u^\beta_\zeta \bar{u}^{1\delta}_{\ \alpha} F^{\delta,\varepsilon}_{y_{x^\zeta}} \bar{u}^{1\lambda}_{\ \eta}\, \omega^\lambda \wedge du^\gamma_\lambda \\
& \underbrace{+\sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta\sum_\eta \bar{u}^{1\varepsilon}_{\ \gamma} u^\beta_\zeta \bar{u}^{1\delta}_{\ \alpha} F^{\delta,\varepsilon}_{y_{x^\zeta}} u^{-1} u^\gamma\, \omega^\eta \wedge \omega_\eta}_{\text{(E.viii)}} + \sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta\sum_\eta\sum_\lambda \delta^\gamma_\eta u^{-1} u_\lambda \bar{u}^{1\varepsilon}_{\ \gamma} u^\beta_\zeta \bar{u}^{1\delta}_{\ \alpha} F^{\delta,\varepsilon}_{y_{x^\zeta}}\, \omega^\eta \wedge \omega^\lambda \\
& +\sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta\sum_\eta \bar{u}^{1\varepsilon}_{\ \gamma} u^\beta_\zeta \bar{u}^{1\delta}_{\ \alpha} F^{\delta,\varepsilon}_{y_{x^\zeta}} u^{-1} u_\eta\, \omega^\eta \wedge \omega^\gamma \\
& +\sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta\sum_\eta\sum_\lambda\sum_\theta\sum_\iota\sum_\kappa \bar{u}^{1\varepsilon}_{\ \gamma} u^\beta_\zeta \bar{u}^{1\delta}_{\ \alpha} F^{\delta,\varepsilon}_{y_{x^\zeta}} \bar{u}^{1\iota}_{\ \lambda} u^\gamma_\kappa \bar{u}^{1\theta}_{\ \eta} F^{\theta,\iota}_{y_{x^\kappa}}\, \omega^\eta \wedge \omega^\lambda + \omega \wedge \mathsf{X},
\end{aligned}
$$

where X is a one-form which will not concern the calculation of S.

9.7.6 Step 6 Finally, to calculate $d(b^\beta_\alpha\omega)$, using the formula for $d\omega$,

$$
\begin{aligned}
d(b^\beta_\alpha\omega) &= db^\beta_\alpha \wedge \omega + b^\beta_\alpha d\omega \\
&= db^\beta_\alpha \wedge \omega + b^\beta_\alpha \sum_\eta \omega^\eta \wedge \omega_\eta + \omega \wedge \varphi b^\beta_\alpha \\
&= \underbrace{b^\beta_\alpha \sum_\eta \omega^\eta \wedge \omega_\eta}_{\text{(E.ix)}} + \omega \wedge (-db^\beta_\alpha + \varphi b^\beta_\alpha).
\end{aligned}
$$

9.7.7 The term $d\varphi^\beta_\alpha$ Combining Steps 1 to 6,

$$
\begin{aligned}
& d\varphi^\beta_\alpha \\
= \ & \sum_\gamma\sum_\eta\sum_\theta \bar{u}^{1\gamma}_{\ \theta}\bar{u}^{1\eta}_{\ \alpha}\, du^\theta_\eta \wedge du^\beta_\gamma \\
& -u^{-2}u^\beta\, du \wedge \omega_\alpha \ + \ u^{-1}\, du^\beta \wedge \omega_\alpha \ + \ u^{-1}u^\beta \sum_\eta\sum_\lambda \ - \bar{u}^{1\lambda}_{\ \alpha}\, du^\eta_\lambda \wedge \omega_\eta \\
& + u^{-2}u^\beta \sum_\eta u^\eta \omega_\alpha \wedge \omega_\eta \ + \underbrace{\sum_\eta\sum_\lambda \delta^\eta_\alpha u^{-2}u^\beta u_\lambda\, \omega^\lambda \wedge \omega_\eta}_{\text{(E.i)}} \ + \underbrace{\sum_\eta u^{-2}u^\beta u_\alpha\, \omega^\eta \wedge \omega_\eta}_{\text{(E.ii)}} \\
& \underbrace{+\sum_\eta\sum_\lambda\sum_\theta\sum_\iota\sum_\kappa u^{-1}u^\beta \bar{u}^{1\iota}_{\ \lambda} u^\eta_\kappa \bar{u}^{1\theta}_{\ \alpha} F^{\theta,\iota}_{y_{x^\kappa}}\, \omega^\lambda \wedge \omega_\eta}_{\text{(E.iii)}} \ - \ u^{-2}u^\beta\, \omega_\alpha \wedge du \\
& \underbrace{+\sum_\eta u^{-2}u^\beta u_\eta\, \omega_\alpha \wedge \omega^\eta}_{\text{(E.iv)}} \ - \ \sum_\eta u^{-2}u^\beta u^\eta\, \omega_\alpha \wedge \omega_\eta \\
& -\delta^\alpha_\beta \sum_\gamma u^{-2}u_\gamma\, du \wedge \omega^\gamma \ + \ \delta^\alpha_\beta \sum_\gamma u^{-1}\, du_\gamma \wedge \omega^\gamma \\
& - \delta^\alpha_\beta \sum_\gamma\sum_\eta\sum_\lambda u^{-1}u_\gamma \bar{u}^{1\lambda}_{\ \eta}\, \omega^\eta \wedge \ du^\gamma_\lambda \ + \ \underbrace{\delta^\alpha_\beta \sum_\gamma\sum_\eta u^{-2}u_\gamma u^\gamma\, \omega^\eta \wedge \omega_\eta}_{\text{(E.v)}} \\
& + \delta^\alpha_\beta \sum_\gamma\sum_\eta\sum_\lambda \delta^\gamma_\eta u^{-2}u_\gamma u_\lambda\, \omega^\eta \wedge \omega^\lambda \ + \ \delta^\alpha_\beta \sum_\gamma\sum_\eta u^{-2}u_\gamma u_\eta\, \omega^\eta \wedge \omega^\gamma \\
& + \delta^\alpha_\beta \sum_\gamma\sum_\eta\sum_\lambda\sum_\theta\sum_\iota\sum_\kappa u^{-1}u_\gamma \bar{u}^{1\iota}_{\ \lambda} u^\gamma_\kappa \bar{u}^{1\theta}_{\ \eta} F^{\theta,\iota}_{y_{x^\kappa}}\, \omega^\eta \wedge \omega^\lambda \ldots
\end{aligned}
$$

$$
\begin{aligned}
&\ldots - u^{-2}u_\alpha\, du\wedge\omega^\beta \;+\; u^{-1}\, du_\alpha\wedge\omega^\beta \;-\; u^{-1}u_\alpha\sum_\eta\sum_\lambda\; \bar u^{1\lambda}_{\ \eta}\omega^\eta\wedge du^\beta_\lambda \;+\; \underbrace{u^{-2}u_\alpha\sum_\eta u^\beta\,\omega^\eta\wedge\omega_\eta}_{\text{(E.vi)}}\\
&+\sum_\eta\sum_\lambda\delta^\beta_\eta u^{-2}u_\alpha u_\lambda\,\omega^\eta\wedge\omega^\lambda \;+\; \sum_\eta u^{-2}u_\alpha u_\eta\,\omega^\eta\wedge\omega^\beta+\sum_\eta\sum_\lambda\sum_\theta\sum_\iota\sum_\kappa u^{-1}u_\alpha\bar u^{1\iota}_{\ \lambda}u^\beta_\kappa\bar u^{1\theta}_{\ \eta}F^{\theta,\iota}_{y_{x^\kappa}}\,\omega^\eta\wedge\omega^\lambda\\
&-\sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta\sum_\eta\sum_\theta u^\beta_\zeta\bar u^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x^\zeta}}\bar u^{1\varepsilon}_{\ \theta}\bar u^{1\eta}_{\ \gamma}\,du^\theta_\eta\wedge\omega^\gamma+\sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta\bar u^{1\varepsilon}_{\ \gamma}\bar u^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x^\zeta}}\,du^\beta_\zeta\wedge\omega^\gamma\\
&-\sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta\sum_\eta\sum_\theta\bar u^{1\varepsilon}_{\ \gamma}u^\beta_\zeta\bar u^{1\delta}_{\ \theta}\bar u^{1\eta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x^\zeta}}\,du^\theta_\eta\wedge\omega^\gamma+\sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta\sum_\eta\sum_\theta\bar u^{1\varepsilon}_{\ \gamma}u^\beta_\zeta\bar u^{1\delta}_{\ \alpha}\bar u^{1\theta}_{\ \eta}\mathsf{D}_{x^\theta}\left(F^{\delta,\varepsilon}_{y_{x^\zeta}}\right)\omega^\eta\wedge\omega^\gamma\\
&+\underbrace{\sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta\sum_\eta\sum_\theta\bar u^{1\varepsilon}_{\ \gamma}u^\beta_\zeta\bar u^{1\delta}_{\ \alpha}u^{-1}u^\eta_\theta F^{\delta,\varepsilon}_{y_{x^\zeta},y_{x^\theta}}\,\omega_\eta\wedge\omega^\gamma}_{\text{(E.vii)}}\qquad-\sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta\sum_\eta\sum_\lambda\bar u^{1\varepsilon}_{\ \gamma}u^\beta_\zeta\bar u^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x^\zeta}}\bar u^{1\lambda}_{\ \eta}\,\omega^\lambda\wedge du^\gamma_\lambda\\
&+\underbrace{\sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta\sum_\eta\bar u^{1\varepsilon}_{\ \gamma}u^\beta_\zeta\bar u^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x^\zeta}}u^{-1}u^\gamma\,\omega^\eta\wedge\omega_\eta}_{\text{(E.viii)}}\qquad+\sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta\sum_\eta\sum_\lambda\delta^\gamma_\eta u^{-1}u_\lambda\bar u^{1\varepsilon}_{\ \gamma}u^\beta_\zeta\bar u^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x^\zeta}}\,\omega^\eta\wedge\omega^\lambda\\
&+\sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta\sum_\eta\bar u^{1\varepsilon}_{\ \gamma}u^\beta_\zeta\bar u^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x^\zeta}}u^{-1}u_\eta\,\omega^\eta\wedge\omega^\gamma\\
&+\sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta\sum_\eta\sum_\lambda\sum_\theta\sum_\iota\sum_\kappa\bar u^{1\varepsilon}_{\ \gamma}u^\beta_\zeta\bar u^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x^\zeta}}\bar u^{1\iota}_{\ \lambda}u^\gamma_\kappa\bar u^{1\theta}_{\ \eta}F^{\theta,\iota}_{y_{x^\kappa}}\,\omega^\eta\wedge\omega^\lambda\\
&\underbrace{b^\beta_\alpha\sum_\eta\omega^\eta\wedge\omega_\eta}_{\text{(E.ix)}}\qquad+\omega\wedge\mathsf{Y},
\end{aligned}
$$

for some one-form Y which will not be concerned in the calculation of the S-tensor.

9.7.8 Terms of the form $\omega^\rho\wedge\omega_\sigma$ in $d\varphi^\beta_\alpha$ First, the terms of the form $\omega^*\wedge\omega_*$ are as follows

$$\sum_\eta\sum_\lambda\delta^\eta_\alpha u^{-2}u^\beta u_\lambda\,\omega^\lambda\wedge\omega_\eta, \tag{E.i}$$

$$\sum_\eta u^{-2}u^\beta u_\alpha\,\omega^\eta\wedge\omega_\eta, \tag{E.ii}$$

$$\sum_\eta\sum_\lambda\sum_\theta\sum_\iota\sum_\kappa u^{-1}u^\beta\bar u^{1\iota}_{\ \lambda}u^\eta_\kappa\bar u^{1\theta}_{\ \alpha}F^{\theta,\iota}_{y_{x^\kappa}}\,\omega^\lambda\wedge\omega_\eta, \tag{E.iii}$$

$$\sum_\eta u^{-2}u^\beta u_\eta\,\omega_\alpha\wedge\omega^\eta=-\sum_\eta u^{-2}u^\beta u_\eta\,\omega^\eta\wedge\omega_\alpha, \tag{E.iv}$$

$$\delta^\alpha_\beta\sum_\gamma\sum_\eta u^{-2}u_\gamma u^\gamma\,\omega^\eta\wedge\omega_\eta, \tag{E.v}$$

$$u^{-2}u_\alpha\sum_\eta u^\beta\,\omega^\eta\wedge\omega_\eta, \tag{E.vi}$$

$$-\sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta\sum_\eta\sum_\theta\bar u^{1\varepsilon}_{\ \gamma}u^\beta_\zeta\bar u^{1\delta}_{\ \alpha}u^{-1}u^\eta_\theta F^{\delta,\varepsilon}_{y_{x^\zeta},y_{x^\theta}}\,\omega^\gamma\wedge\omega_\eta, \tag{E.vii}$$

$$\sum_\gamma\sum_\delta\sum_\varepsilon\sum_\zeta\sum_\eta\bar u^{1\varepsilon}_{\ \gamma}u^\beta_\zeta\bar u^{1\delta}_{\ \alpha}F^{\delta,\varepsilon}_{y_{x^\zeta}}u^{-1}u^\gamma\,\omega^\eta\wedge\omega_\eta, \tag{E.viii}$$

$$b^\beta_\alpha\sum_\eta\omega^\eta\wedge\omega_\eta. \tag{E.ix}$$

Rewriting them in the form $\omega^\rho\wedge\omega_\sigma$, therefore the 9 terms are

$$\sum_\rho\sum_\sigma\delta^\sigma_\alpha u^{-2}u^\beta u_\rho\,\omega^\rho\wedge\omega_\sigma, \tag{E.i}$$

$$\sum_\rho\sum_\sigma\delta^\sigma_\rho u^{-2}u^\beta u_\alpha\,\omega^\rho\wedge\omega_\sigma, \tag{E.ii}$$

$$\sum_\rho\sum_\sigma\left(\sum_\theta\sum_\iota\sum_\kappa u^{-1}u^\beta\bar u^{1\iota}_{\ \rho}u^\sigma_\kappa\bar u^{1\theta}_{\ \alpha}F^{\theta,\iota}_{y_{x^\kappa}}\right)\omega^\rho\wedge\omega_\sigma, \tag{E.iii}$$

$$-\sum_{\rho}\sum_{\sigma}\delta^{\sigma}_{\alpha}u^{-2}u^{\beta}u_{\rho}\,\omega^{\rho}\wedge\omega_{\sigma}, \tag{E.iv}$$

$$\delta^{\alpha}_{\beta}\sum_{\rho}\sum_{\sigma}\sum_{\gamma}\delta^{\sigma}_{\rho}u^{-2}u_{\gamma}u^{\gamma}\,\omega^{\rho}\wedge\omega_{\sigma}, \tag{E.v}$$

$$u^{-2}u_{\alpha}u^{\beta}\sum_{\rho}\sum_{\sigma}\delta^{\sigma}_{\rho}\omega^{\rho}\wedge\omega_{\sigma}, \tag{E.vi}$$

$$-\sum_{\rho}\sum_{\sigma}\left(\sum_{\delta}\sum_{\varepsilon}\sum_{\zeta}\sum_{\theta}\bar{u}^{1\varepsilon}_{\rho}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\alpha}u^{\sigma}_{\theta}u^{-1}F^{\delta,\varepsilon}_{y_{x^{\zeta}},y_{x^{\theta}}}\right)\omega^{\rho}\wedge\omega_{\sigma}, \tag{E.vii}$$

$$\sum_{\rho}\sum_{\sigma}\left(\delta^{\sigma}_{\rho}\sum_{\gamma}\sum_{\delta}\sum_{\varepsilon}\sum_{\zeta}\bar{u}^{1\varepsilon}_{\gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\alpha}u^{-1}u^{\gamma}F^{\delta,\varepsilon}_{y_{x^{\zeta}}}\right)\omega^{\rho}\wedge\omega_{\sigma}, \tag{E.viii}$$

$$b^{\beta}_{\alpha}\sum_{\rho}\sum_{\sigma}\delta^{\sigma}_{\rho}\,\omega^{\rho}\wedge\omega_{\sigma}. \tag{E.ix}$$

9.8 The S tensor

From the calculations above, $S^{\beta\sigma}_{\alpha\rho}$ comes from the following

$$\begin{aligned}\Phi^{\beta}_{\alpha} \;=\; & \sum_{\sigma}\sum_{\rho}\Big((\text{E.i}+\text{E.ii}+\text{E.iii}+\text{E.iv}+\text{E.v}+\text{E.iv}+\text{E.vii}+\text{E.viii}+\text{E.ix})\\ & -(\text{D.i}+\text{D.ii}+\text{D.iii}+\text{D.iv}+\text{D.v}+\text{D.vi})\\ & -(\text{A.i}+\text{A.ii})+(\text{B.i}+\text{B.ii}+\text{B.iii})+(\text{C.i}+\text{C.ii}+\text{C.iii})\Big)\omega^{\rho}\wedge\omega_{\sigma}\\ & +\omega\wedge\psi^{\beta}_{\alpha}\end{aligned}$$

Therefore,

$$\begin{aligned}& S^{\beta\sigma}_{\alpha\rho}\\ =\;& \delta^{\sigma}_{\alpha}u^{-2}u^{\beta}u_{\rho}+\delta^{\sigma}_{\rho}u^{-2}u^{\beta}u_{\alpha}+\sum_{\theta}\sum_{\iota}\sum_{\kappa}u^{-1}u^{\beta}\bar{u}^{1\iota}_{\rho}u^{\sigma}_{\kappa}\bar{u}^{1\theta}_{\alpha}F^{\theta,\iota}_{y_{x^{\kappa}}}-\delta^{\sigma}_{\alpha}u^{-2}u^{\beta}u_{\rho}\\ &+\delta^{\alpha}_{\beta}\delta^{\sigma}_{\rho}\sum_{\gamma}u^{-2}u_{\gamma}u^{\gamma}+\delta^{\sigma}_{\rho}u^{-2}u_{\alpha}u^{\beta}-\sum_{\varepsilon}\sum_{\zeta}\sum_{\delta}\sum_{\theta}\bar{u}^{1\varepsilon}_{\rho}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\alpha}u^{\sigma}_{\theta}u^{-1}F^{\delta,\varepsilon}_{y_{x^{\zeta}},y_{x^{\theta}}}\\ &+\delta^{\sigma}_{\rho}\sum_{\gamma}\sum_{\delta}\sum_{\varepsilon}\sum_{\zeta}u^{-1}u^{\gamma}\bar{u}^{1\varepsilon}_{\gamma}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\alpha}F^{\delta,\varepsilon}_{y_{x^{\zeta}}}+b^{\beta}_{\alpha}\delta^{\sigma}_{\rho}\\ &+u_{\rho}\delta^{\sigma}_{\alpha}u^{-2}u^{\beta}+\delta^{\beta}_{\rho}\delta^{\sigma}_{\alpha}\sum_{\gamma}u^{-2}u^{\gamma}u_{\gamma}+\delta^{\sigma}_{\alpha}\sum_{\gamma}\sum_{\mu}\sum_{\nu}\sum_{\xi}u^{-1}u^{\gamma}\bar{u}^{1\nu}_{\rho}u^{\beta}_{\xi}\bar{u}^{1\mu}_{\gamma}F^{\mu,\nu}_{y_{x^{\xi}}}\\ &-\delta^{\sigma}_{\alpha}u^{-2}u_{\rho}u^{\beta}-\sum_{\theta}\sum_{\iota}\sum_{\kappa}u^{-1}u^{\beta}\bar{u}^{1\iota}_{\rho}u^{\sigma}_{\kappa}\bar{u}^{1\theta}_{\alpha}F^{\theta,\iota}_{y_{x^{\kappa}}}-\delta^{\sigma}_{\rho}u^{-2}u_{\alpha}u^{\beta}\\ &+\delta^{\sigma}_{\alpha}u^{-2}u^{\beta}u_{\rho}+\delta^{\sigma}_{\alpha}b^{\beta}_{\rho}-\delta^{\beta}_{\rho}(\delta^{\sigma}_{\alpha}t-b^{\sigma}_{\alpha})+\delta^{\beta}_{\rho}u^{-2}u^{\sigma}u_{\alpha}\\ &+\sum_{\theta}\sum_{\iota}\sum_{\kappa}\sum_{\lambda}\delta^{\beta}_{\rho}u^{-1}\bar{u}^{1\theta}_{\alpha}\bar{u}^{1\lambda}_{\kappa}u^{\sigma}_{\iota}u^{\kappa}F^{\theta,\lambda}_{y_{x^{\iota}}}-\delta^{\beta}_{\alpha}(\delta^{\sigma}_{\rho}t-b^{\sigma}_{\rho})+\delta^{\beta}_{\alpha}u^{-2}u^{\sigma}u_{\rho}\\ &+\sum_{\theta}\sum_{\iota}\sum_{\kappa}\sum_{\lambda}\delta^{\beta}_{\alpha}u^{-1}\bar{u}^{1\theta}_{\rho}\bar{u}^{1\lambda}_{\kappa}u^{\sigma}_{\iota}u^{\kappa}F^{\theta,\lambda}_{y_{x^{\iota}}}.\end{aligned}$$

After cancellations, rearranging of terms, and relabelling the indices,

$$\begin{aligned}
&S^{\beta\sigma}_{\alpha\rho}\\
&= u^{-2}(\delta^\sigma_\rho u^\beta u_\alpha + \delta^\sigma_\alpha u^\beta u_\rho + \delta^\beta_\rho u^\sigma u_\alpha + \delta^\beta_\alpha u^\sigma u_\rho) + (\delta^\alpha_\beta\delta^\rho_\sigma + \delta^\beta_\rho\delta^\sigma_\alpha)\sum_\gamma u^{-2}u^\gamma u_\gamma\\
&- \sum_\varepsilon\sum_\zeta\sum_\delta\sum_\theta \bar{u^{-1}}^{\varepsilon}_{\rho} u^\beta_\zeta \bar{u^{-1}}^{\delta}_{\alpha} u^\sigma_\theta u^{-1} F^{\delta,\varepsilon}_{y_{x\zeta},y_{x\theta}}\\
&+ \Bigg[\delta^\sigma_\rho\sum_\theta\sum_\iota\sum_\kappa\sum_\lambda u^{-1}u^\kappa \bar{u^{-1}}^{\theta}_{\alpha} u^\beta_\iota \bar{u^{-1}}^{\lambda}_{\kappa} F^{\theta,\lambda}_{y_{x\iota}} + \delta^\sigma_\alpha\sum_\theta\sum_\iota\sum_\kappa\sum_\lambda u^{-1}u^\kappa \bar{u^{-1}}^{\theta}_{\rho} u^\beta_\iota \bar{u^{-1}}^{\lambda}_{\kappa} F^{\theta,\lambda}_{y_{x\iota}}\\
&+ \delta^\beta_\rho\sum_\theta\sum_\iota\sum_\kappa\sum_\lambda u^{-1}u^\kappa \bar{u^{-1}}^{\theta}_{\alpha} u^\sigma_\iota \bar{u^{-1}}^{\lambda}_{\kappa} F^{\theta,\lambda}_{y_{x\iota}} + \delta^\beta_\alpha\sum_\theta\sum_\iota\sum_\kappa\sum_\lambda u^{-1}u^\kappa \bar{u^{-1}}^{\theta}_{\rho} u^\sigma_\iota \bar{u^{-1}}^{\lambda}_{\kappa} F^{\theta,\lambda}_{y_{x\iota}}\Bigg]\\
&- (\delta^\beta_\rho\delta^\sigma_\alpha + \delta^\beta_\alpha\delta^\sigma_\rho)t + (\delta^\sigma_\rho b^\beta_\alpha + \delta^\sigma_\alpha b^\beta_\rho + \delta^\beta_\rho b^\sigma_\alpha + \delta^\beta_\alpha b^\sigma_\rho).
\end{aligned} \tag{9.1}$$

In Proposition 7.1, when $t = b^\alpha_\beta = d_\alpha = c^\alpha = 0$ so that

$$\varphi = \mathring{\varphi}, \qquad \varphi^\alpha_\beta = \mathring{\varphi}^\alpha_\beta, \qquad \varphi^\alpha = \mathring{\varphi}^\alpha, \qquad \varphi_\alpha = \mathring{\varphi}_\alpha,$$

then immediate from $S^{\beta\sigma}_{\alpha\rho}$, the following tensor is obtained

$$\begin{aligned}
\mathring{S}^{\beta\sigma}_{\alpha\rho} &:= S^{\beta\sigma}_{\alpha\rho}|_{t=b^\alpha_\beta=d_\alpha=c^\alpha=0}\\
&= u^{-2}(\delta^\sigma_\rho u^\beta u_\alpha + \delta^\sigma_\alpha u^\beta u_\rho + \delta^\beta_\rho u^\sigma u_\alpha + \delta^\beta_\alpha u^\sigma u_\rho) + (\delta^\alpha_\beta\delta^\rho_\sigma + \delta^\beta_\rho\delta^\sigma_\alpha)\sum_\gamma u^{-2}u^\gamma u_\gamma\\
&\quad -\sum_\varepsilon\sum_\zeta\sum_\delta\sum_\theta \bar{u^{-1}}^{\varepsilon}_{\rho} u^\beta_\zeta \bar{u^{-1}}^{\delta}_{\alpha} u^\sigma_\theta u^{-1} F^{\delta,\varepsilon}_{y_{x\zeta},y_{x\theta}}\\
&\quad +\Bigg[\delta^\sigma_\rho\sum_\theta\sum_\iota\sum_\kappa\sum_\lambda u^{-1}u^\kappa \bar{u^{-1}}^{\theta}_{\alpha} u^\beta_\iota \bar{u^{-1}}^{\lambda}_{\kappa} F^{\theta,\lambda}_{y_{x\iota}} + \delta^\sigma_\alpha\sum_\theta\sum_\iota\sum_\kappa\sum_\lambda u^{-1}u^\kappa \bar{u^{-1}}^{\theta}_{\rho} u^\beta_\iota \bar{u^{-1}}^{\lambda}_{\kappa} F^{\theta,\lambda}_{y_{x\iota}}\\
&\quad +\delta^\beta_\rho\sum_\theta\sum_\iota\sum_\kappa\sum_\lambda u^{-1}u^\kappa \bar{u^{-1}}^{\theta}_{\alpha} u^\sigma_\iota \bar{u^{-1}}^{\lambda}_{\kappa} F^{\theta,\lambda}_{y_{x\iota}} + \delta^\beta_\alpha\sum_\theta\sum_\iota\sum_\kappa\sum_\lambda u^{-1}u^\kappa \bar{u^{-1}}^{\theta}_{\rho} u^\sigma_\iota \bar{u^{-1}}^{\lambda}_{\kappa} F^{\theta,\lambda}_{y_{x\iota}}\Bigg].
\end{aligned}$$

10 Normalisation of the S tensor

Therefore, the transformation in Proposition 7.1 changes $\mathring{S}^{\beta\sigma}_{\alpha\rho}$ into $S^{\beta\sigma}_{\alpha\rho}$ in the following way:

$$S^{\beta\sigma}_{\alpha\rho} = \mathring{S}^{\beta\sigma}_{\alpha\rho} + \delta^\sigma_\rho b^\beta_\alpha + \delta^\sigma_\alpha b^\beta_\rho + \delta^\beta_\rho b^\sigma_\alpha + \delta^\beta_\alpha b^\sigma_\rho - \delta^\beta_\alpha\delta^\sigma_\rho t - \delta^\beta_\rho\delta^\sigma_\alpha t, \tag{10.1}$$

The normalisation of $S^{\beta\sigma}_{\alpha\rho}$ involves finding an appropriate b^β_α so that the trace is zero

$$\sum_\sigma S^{\beta\sigma}_{\alpha\sigma} = 0.$$

The term $\sum_\sigma S^{\beta\sigma}_{\alpha\sigma}$ can be expressed as

$$\begin{aligned}
\sum_\sigma S^{\beta\sigma}_{\alpha\sigma} &= \sum_\sigma (\mathring{S}^{\beta\sigma}_{\alpha\sigma} + \delta^\sigma_\sigma b^\beta_\alpha + \delta^\sigma_\alpha b^\beta_\sigma + \delta^\beta_\sigma b^\sigma_\alpha + \delta^\beta_\alpha b^\sigma_\sigma - \delta^\beta_\alpha\delta^\sigma_\sigma t - \delta^\beta_\sigma\delta^\sigma_\alpha t)\\
&= \left(\sum_\sigma \mathring{S}^{\beta\sigma}_{\alpha\sigma}\right) + n b^\beta_\alpha + b^\beta_\alpha + b^\beta_\alpha + \delta^\beta_\alpha\sum_\tau b^\tau_\tau - n\delta^\beta_\alpha t - \delta^\beta_\alpha t\\
&= \left(\sum_\sigma \mathring{S}^{\beta\sigma}_{\alpha\sigma}\right) + (n+2) b^\beta_\alpha + \delta^\beta_\alpha\sum_\tau b^\tau_\tau - (n+1)\delta^\beta_\alpha t,
\end{aligned} \tag{10.2}$$

where τ is used to avoid confusion of notations. To choose an appropriate b_α^β so that $\sum_\sigma S_{\alpha\sigma}^{\beta\sigma} = 0$, first $\sum_\tau b_\tau^\tau$ has to be calculated, and this can be done by taking trace with respect to α and β on $\sum_\sigma S_{\alpha\sigma}^{\beta\sigma}$:

$$\begin{aligned}\sum_\sigma\sum_\rho S_{\rho\sigma}^{\rho\sigma} &= \left(\sum_\sigma\sum_\rho \overset{\circ}{S}{}_{\rho\sigma}^{\rho\sigma}\right) + (n+2)\sum_\rho b_\rho^\rho + \sum_\rho \delta_\rho^\rho \sum_\tau b_\tau^\tau - (n+1)\sum_\rho \delta_\rho^\rho t \\ &= \sum_\pi\sum_\tau \overset{\circ}{S}{}_{\pi\tau}^{\pi\tau} + 2(n+1)\sum_\tau b_\tau^\tau - (n+1)nt,\end{aligned}$$

where π is used so as to avoid confusion of notations. Since b_α^β is assumed to satisfy $\sum_\sigma S_{\alpha\sigma}^{\beta\sigma} = 0$, therefore $\sum_\rho\sum_\sigma S_{\rho\sigma}^{\rho\sigma} = 0$ and hence

$$\sum_\tau b_\tau^\tau = \frac{1}{2(n+1)}\left(-\sum_\pi\sum_\tau \overset{\circ}{S}{}_{\pi\tau}^{\pi\tau} + (n+1)nt\right) = \frac{-1}{2(n+1)}\left(\sum_\pi\sum_\tau \overset{\circ}{S}{}_{\pi\tau}^{\pi\tau}\right) + \frac{n}{2}t. \tag{10.3}$$

Substitute equation (10.3) into (10.2),

$$\begin{aligned}0 = \sum_\sigma S_{\alpha\sigma}^{\beta\sigma} &= \left(\sum_\sigma \overset{\circ}{S}{}_{\alpha\sigma}^{\beta\sigma}\right) + (n+2)b_\alpha^\beta + \delta_\alpha^\beta\left(\frac{-1}{2(n+1)}\left[\sum_\pi\sum_\tau \overset{\circ}{S}{}_{\pi\tau}^{\pi\tau}\right] + \frac{n}{2}t\right) - (n+1)\delta_\alpha^\beta t \\ &= \left(\sum_\tau \overset{\circ}{S}{}_{\alpha\tau}^{\beta\tau}\right) + (n+2)b_\alpha^\beta - \delta_\alpha^\beta\frac{1}{2(n+1)}\left(\sum_\pi\sum_\tau \overset{\circ}{S}{}_{\pi\tau}^{\pi\tau}\right) - \frac{n+2}{2}\delta_\alpha^\beta t,\end{aligned}$$

and therefore the expression of b_α^β is obtained:

$$b_\alpha^\beta = \frac{-1}{n+2}\left(\sum_\tau \overset{\circ}{S}{}_{\alpha\tau}^{\beta\tau}\right) + \delta_\alpha^\beta\frac{1}{2(n+1)(n+2)}\left(\sum_\pi\sum_\tau \overset{\circ}{S}{}_{\pi\tau}^{\pi\tau}\right) + \delta_\alpha^\beta\frac{t}{2}, \tag{10.4}$$

along with the other set of coefficients

$$\begin{aligned}b_\rho^\beta &= \frac{-1}{n+2}\left(\sum_\tau \overset{\circ}{S}{}_{\rho\tau}^{\beta\tau}\right) + \delta_\rho^\beta\frac{1}{2(n+1)(n+2)}\left(\sum_\pi\sum_\tau \overset{\circ}{S}{}_{\pi\tau}^{\pi\tau}\right) + \delta_\rho^\beta\frac{t}{2}, \\ b_\alpha^\sigma &= \frac{-1}{n+2}\left(\sum_\tau \overset{\circ}{S}{}_{\alpha\tau}^{\sigma\tau}\right) + \delta_\alpha^\sigma\frac{1}{2(n+1)(n+2)}\left(\sum_\pi\sum_\tau \overset{\circ}{S}{}_{\pi\tau}^{\pi\tau}\right) + \delta_\alpha^\sigma\frac{t}{2}, \\ b_\rho^\sigma &= \frac{-1}{n+2}\left(\sum_\tau \overset{\circ}{S}{}_{\rho\tau}^{\sigma\tau}\right) + \delta_\rho^\sigma\frac{1}{2(n+1)(n+2)}\left(\sum_\pi\sum_\tau \overset{\circ}{S}{}_{\pi\tau}^{\pi\tau}\right) + \delta_\rho^\sigma\frac{t}{2}.\end{aligned} \tag{10.5}$$

Substitute equations (10.5) and (10.4) into (10.1),

$$\begin{aligned}
S^{\beta\sigma}_{\alpha\rho} = \overset{\circ}{S}{}^{\beta\sigma}_{\alpha\rho} &+ \delta^{\sigma}_{\rho}\left(\frac{-1}{n+2}\left(\sum_{\tau}\overset{\circ}{S}{}^{\beta\tau}_{\alpha\tau}\right)+\delta^{\beta}_{\alpha}\frac{1}{2(n+1)(n+2)}\left(\sum_{\pi}\sum_{\tau}\overset{\circ}{S}{}^{\pi\tau}_{\pi\tau}\right)+\delta^{\beta}_{\alpha}\frac{t}{2}\right)\\
&+ \delta^{\sigma}_{\alpha}\left(\frac{-1}{n+2}\left(\sum_{\tau}\overset{\circ}{S}{}^{\beta\tau}_{\rho\tau}\right)+\delta^{\beta}_{\rho}\frac{1}{2(n+1)(n+2)}\left(\sum_{\pi}\sum_{\tau}\overset{\circ}{S}{}^{\pi\tau}_{\pi\tau}\right)+\delta^{\beta}_{\rho}\frac{t}{2}\right)\\
&+ \delta^{\beta}_{\rho}\left(\frac{-1}{n+2}\left(\sum_{\tau}\overset{\circ}{S}{}^{\sigma\tau}_{\alpha\tau}\right)+\delta^{\sigma}_{\alpha}\frac{1}{2(n+1)(n+2)}\left(\sum_{\pi}\sum_{\tau}\overset{\circ}{S}{}^{\pi\tau}_{\pi\tau}\right)+\delta^{\sigma}_{\alpha}\frac{t}{2}\right)\\
&+ \delta^{\beta}_{\alpha}\left(\frac{-1}{n+2}\left(\sum_{\tau}\overset{\circ}{S}{}^{\sigma\tau}_{\rho\tau}\right)+\delta^{\sigma}_{\rho}\frac{1}{2(n+1)(n+2)}\left(\sum_{\pi}\sum_{\tau}\overset{\circ}{S}{}^{\pi\tau}_{\pi\tau}\right)+\delta^{\sigma}_{\rho}\frac{t}{2}\right)\\
&- \delta^{\beta}_{\alpha}\delta^{\sigma}_{\rho}t-\delta^{\beta}_{\rho}\delta^{\sigma}_{\alpha}t\\
= \overset{\circ}{S}{}^{\beta\sigma}_{\alpha\rho} &- \frac{1}{n+2}\delta^{\sigma}_{\rho}\left(\sum_{\tau}\overset{\circ}{S}{}^{\beta\tau}_{\alpha\tau}\right)+\delta^{\sigma}_{\rho}\delta^{\beta}_{\alpha}\frac{1}{2(n+1)(n+2)}\left(\sum_{\pi}\sum_{\tau}\overset{\circ}{S}{}^{\pi\tau}_{\pi\tau}\right)\\
&- \frac{1}{n+2}\delta^{\sigma}_{\alpha}\left(\sum_{\tau}\overset{\circ}{S}{}^{\beta\tau}_{\rho\tau}\right)+\delta^{\sigma}_{\alpha}\delta^{\beta}_{\rho}\frac{1}{2(n+1)(n+2)}\left(\sum_{\pi}\sum_{\tau}\overset{\circ}{S}{}^{\pi\tau}_{\pi\tau}\right)\\
&- \frac{1}{n+2}\delta^{\beta}_{\rho}\left(\sum_{\tau}\overset{\circ}{S}{}^{\sigma\tau}_{\alpha\tau}\right)+\delta^{\beta}_{\rho}\delta^{\sigma}_{\alpha}\frac{1}{2(n+1)(n+2)}\left(\sum_{\pi}\sum_{\tau}\overset{\circ}{S}{}^{\pi\tau}_{\pi\tau}\right)\\
&- \frac{1}{n+2}\delta^{\beta}_{\alpha}\left(\sum_{\tau}\overset{\circ}{S}{}^{\sigma\tau}_{\rho\tau}\right)+\delta^{\beta}_{\alpha}\delta^{\sigma}_{\rho}\frac{1}{2(n+1)(n+2)}\left(\sum_{\pi}\sum_{\tau}\overset{\circ}{S}{}^{\pi\tau}_{\pi\tau}\right).
\end{aligned}$$

11 Explicit calculation of $S^{\beta\rho}_{\alpha\sigma}|_{\mathsf{id}}$

The calculation starts with the explicit expression of $\sum_{\tau}\overset{\circ}{S}{}^{\beta\tau}_{\alpha\tau}$ and $\sum_{\pi,\tau}\overset{\circ}{S}{}^{\pi\tau}_{\pi\tau}$. For the term $\sum_{\tau}\overset{\circ}{S}{}^{\beta\tau}_{\alpha\tau}$ which is the first trace of $\overset{\circ}{S}{}^{\beta\rho}_{\alpha\sigma}$, using equation (9.1),

$$\begin{aligned}
\sum_{\tau}\overset{\circ}{S}{}^{\beta\tau}_{\alpha\tau} =& u^{-2}\sum_{\tau}(\delta^{\tau}_{\tau}u^{\beta}u_{\alpha}+\delta^{\tau}_{\alpha}u^{\beta}u_{\tau}+\delta^{\beta}_{\tau}u^{\tau}u_{\alpha}+\delta^{\beta}_{\alpha}u^{\tau}u_{\tau})+\sum_{\tau}(\delta^{\alpha}_{\beta}\delta^{\tau}_{\tau}+\delta^{\beta}_{\tau}\delta^{\tau}_{\alpha})\sum_{\gamma}u^{-2}u^{\gamma}u_{\gamma}\\
&-\sum_{\varepsilon}\sum_{\zeta}\sum_{\delta}\sum_{\theta}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\alpha}\left(\sum_{\tau}\bar{u}^{1\varepsilon}_{\tau}u^{\tau}_{\theta}\right)u^{-1}F^{\delta,\varepsilon}_{y_{x\zeta},y_{x\theta}}\\
&+\Bigg[\sum_{\tau}\sum_{\theta}\sum_{\iota}\sum_{\kappa}\sum_{\lambda}\delta^{\tau}_{\tau}u^{-1}u^{\kappa}\bar{u}^{1\theta}_{\alpha}u^{\beta}_{\iota}\bar{u}^{1\lambda}_{\kappa}F^{\theta,\lambda}_{y_{x\iota}}+\sum_{\tau}\sum_{\theta}\sum_{\iota}\sum_{\kappa}\sum_{\lambda}\delta^{\tau}_{\alpha}u^{-1}u^{\kappa}\bar{u}^{1\theta}_{\tau}u^{\beta}_{\iota}\bar{u}^{1\lambda}_{\kappa}F^{\theta,\lambda}_{y_{x\iota}}\\
&\quad+\sum_{\tau}\sum_{\theta}\sum_{\iota}\sum_{\kappa}\sum_{\lambda}\delta^{\beta}_{\tau}u^{-1}u^{\kappa}\bar{u}^{1\theta}_{\alpha}u^{\tau}_{\iota}\bar{u}^{1\lambda}_{\kappa}F^{\theta,\lambda}_{y_{x\iota}}+\delta^{\beta}_{\alpha}\sum_{\theta}\sum_{\iota}\sum_{\kappa}\sum_{\lambda}u^{-1}u^{\kappa}\left(\sum_{\tau}\bar{u}^{1\theta}_{\tau}u^{\tau}_{\iota}\right)\bar{u}^{1\lambda}_{\kappa}F^{\theta,\lambda}_{y_{x\iota}}\Bigg]\\
=&(n+2)u^{-2}u^{\beta}u_{\alpha}+(n+2)u^{-2}\delta^{\alpha}_{\beta}\sum_{\gamma}u^{\gamma}u_{\gamma}-\sum_{\varepsilon}\sum_{\zeta}\sum_{\delta}u^{\beta}_{\zeta}\bar{u}^{1\delta}_{\alpha}u^{-1}F^{\delta,\varepsilon}_{y_{x\zeta},y_{x\varepsilon}}\\
&+(n+2)\sum_{\theta}\sum_{\iota}\sum_{\kappa}\sum_{\lambda}u^{-1}u^{\kappa}\bar{u}^{1\theta}_{\alpha}u^{\beta}_{\iota}\bar{u}^{1\lambda}_{\kappa}F^{\theta,\lambda}_{y_{x\iota}}+\delta^{\beta}_{\alpha}\sum_{\theta}\sum_{\kappa}\sum_{\lambda}u^{-1}u^{\kappa}\bar{u}^{1\lambda}_{\kappa}F^{\theta,\lambda}_{y_{x\theta}}.
\end{aligned}$$

The expressions $\sum_{\tau}\overset{\circ}{S}{}^{\beta\tau}_{\rho\tau}$, $\sum_{\tau}\overset{\circ}{S}{}^{\sigma\tau}_{\alpha\tau}$ and $\sum_{\tau}\overset{\circ}{S}{}^{\sigma\tau}_{\rho\tau}$ are obtained by changing appropriate letters. Next, taking the second trace,

$$\begin{aligned}
\sum_{\pi}\sum_{\tau}\overset{\circ}{S}{}^{\pi\tau}_{\pi\tau} &= u^{-2}(n+2)\sum_{\pi}u^{\pi}u_{\pi}+(n+2)u^{-2}\sum_{\gamma}u^{\gamma}u_{\gamma}\left(\sum_{\pi}\delta^{\pi}_{\pi}\right)-\sum_{\varepsilon}\sum_{\zeta}\sum_{\delta}\left(\sum_{\pi}u^{\pi}_{\zeta}\bar{u}^{1\delta}_{\pi}\right)u^{-1}F^{\delta,\varepsilon}_{y_{x\zeta},y_{x\varepsilon}}\\
&\quad+(n+2)\sum_{\theta}\sum_{\iota}\sum_{\kappa}\sum_{\lambda}u^{-1}u^{\kappa}\left(\sum_{\pi}\bar{u}^{1\theta}_{\pi}u^{\pi}_{\iota}\right)\bar{u}^{1\lambda}_{\kappa}F^{\theta,\lambda}_{y_{x\iota}}+\left(\sum_{\pi}\delta^{\pi}_{\pi}\right)\left(\sum_{\theta}\sum_{\kappa}\sum_{\lambda}u^{-1}u^{\kappa}\bar{u}^{1\lambda}_{\kappa}F^{\theta,\lambda}_{y_{x\theta}}\right)\\
&= u^{-2}(n+2)(n+1)\sum_{\gamma}u^{\gamma}u_{\gamma}-\sum_{\varepsilon}\sum_{\delta}u^{-1}F^{\delta,\varepsilon}_{y_{x\delta},y_{x\varepsilon}}+2(n+1)\sum_{\theta}\sum_{\kappa}\sum_{\lambda}u^{-1}u^{\kappa}\bar{u}^{1\lambda}_{\kappa}F^{\theta,\lambda}_{y_{x\theta}}.
\end{aligned}$$

By using the explicit expressions of $\sum\limits_{\tau} \overset{\circ}{S}{}^{\beta\tau}_{\rho\tau}$, $\sum\limits_{\tau} \overset{\circ}{S}{}^{\sigma\tau}_{\alpha\tau}$, $\sum\limits_{\tau} \overset{\circ}{S}{}^{\sigma\tau}_{\rho\tau}$, and $\sum\limits_{\pi}\sum\limits_{\tau} \overset{\circ}{S}{}^{\pi\tau}_{\pi\tau}$, after much cancellations and reorganisation of the terms,

$$
\begin{aligned}
S^{\beta\sigma}_{\alpha\rho} &= -\sum_{\varepsilon}\sum_{\zeta}\sum_{\delta}\sum_{\theta} \bar{u}^{-1\varepsilon}_{\ \ \rho} u^{\beta}_{\zeta} \bar{u}^{-1\delta}_{\ \ \alpha} u^{\sigma}_{\theta} u^{-1} F^{\delta,\varepsilon}_{y_{x^\zeta},y_{x^\theta}} \\
&\quad + \frac{u^{-1}}{n+2}\sum_{\varepsilon}\sum_{\zeta}\sum_{\delta} \big(\delta^{\sigma}_{\rho} u^{\beta}_{\zeta} \bar{u}^{-1\delta}_{\ \ \alpha} + \delta^{\sigma}_{\alpha} u^{\beta}_{\zeta} \bar{u}^{-1\delta}_{\ \ \rho} + \delta^{\beta}_{\rho} u^{\sigma}_{\zeta} \bar{u}^{-1\delta}_{\ \ \alpha} + \delta^{\beta}_{\alpha} u^{\sigma}_{\zeta} \bar{u}^{-1\delta}_{\ \ \rho}\big) F^{\delta,\varepsilon}_{y_{x^\zeta},y_{x^\varepsilon}} \\
&\quad - \left[\frac{u^{-1}}{(n+1)(n+2)}(\delta^{\sigma}_{\rho}\delta^{\beta}_{\alpha} + \delta^{\sigma}_{\alpha}\delta^{\beta}_{\rho})\right] \sum_{\varepsilon}\sum_{\delta} F^{\delta,\varepsilon}_{y_{x^\delta},y_{x^\varepsilon}}.
\end{aligned}
$$

Finally, let μ, κ, λ and ν be new indices. Then

$$
\begin{aligned}
S^{\lambda\mu}_{\kappa\nu} &:= -u\sum_{\alpha}\sum_{\beta}\sum_{\sigma}\sum_{\rho} u^{\rho}_{\nu} \bar{u}^{-1\lambda}_{\ \ \beta} u^{\alpha}_{\kappa} \bar{u}^{-1\mu}_{\ \ \sigma} S^{\beta\sigma}_{\alpha\rho} \\
&= \sum_{\varepsilon}\sum_{\zeta}\sum_{\delta}\sum_{\theta} \left(\sum_{\rho} \bar{u}^{-1\varepsilon}_{\ \ \rho} u^{\rho}_{\nu}\right)\left(\sum_{\beta} \bar{u}^{-1\lambda}_{\ \ \beta} u^{\beta}_{\zeta}\right)\left(\sum_{\alpha} \bar{u}^{-1\delta}_{\ \ \alpha} u^{\alpha}_{\kappa}\right)\left(\sum_{\sigma} \bar{u}^{-1\mu}_{\ \ \sigma} u^{\sigma}_{\theta}\right) F^{\delta,\varepsilon}_{y_{x^\zeta},y_{x^\theta}} \\
&\quad -\frac{1}{n+2}\sum_{\varepsilon}\sum_{\zeta}\sum_{\delta}\left(\sum_{\beta} \bar{u}^{-1\lambda}_{\ \ \beta} u^{\beta}_{\zeta}\right)\left(\sum_{\alpha} \bar{u}^{-1\delta}_{\ \ \alpha} u^{\alpha}_{\kappa}\right)\left(\sum_{\sigma}\sum_{\rho} \delta^{\sigma}_{\rho} u^{\rho}_{\nu} \bar{u}^{-1\mu}_{\ \ \sigma}\right) F^{\delta,\varepsilon}_{y_{x^\zeta},y_{x^\varepsilon}} \\
&\quad -\frac{1}{n+2}\sum_{\varepsilon}\sum_{\zeta}\sum_{\delta}\left(\sum_{\beta} \bar{u}^{-1\lambda}_{\ \ \beta} u^{\beta}_{\zeta}\right)\left(\sum_{\rho} \bar{u}^{-1\delta}_{\ \ \rho} u^{\rho}_{\nu}\right)\left(\sum_{\sigma}\sum_{\alpha} \delta^{\sigma}_{\alpha} u^{\alpha}_{\kappa} \bar{u}^{-1\mu}_{\ \ \sigma}\right) F^{\delta,\varepsilon}_{y_{x^\zeta},y_{x^\varepsilon}} \\
&\quad -\frac{1}{n+2}\sum_{\varepsilon}\sum_{\zeta}\sum_{\delta}\left(\sum_{\sigma} \bar{u}^{-1\mu}_{\ \ \sigma} u^{\sigma}_{\zeta}\right)\left(\sum_{\alpha} \bar{u}^{-1\delta}_{\ \ \alpha} u^{\alpha}_{\kappa}\right)\left(\sum_{\rho}\sum_{\beta} \delta^{\beta}_{\rho} u^{\rho}_{\nu} \bar{u}^{-1\lambda}_{\ \ \beta}\right) F^{\delta,\varepsilon}_{y_{x^\zeta},y_{x^\varepsilon}} \\
&\quad -\frac{1}{n+2}\sum_{\varepsilon}\sum_{\zeta}\sum_{\delta}\left(\sum_{\sigma} \bar{u}^{-1\mu}_{\ \ \sigma} u^{\sigma}_{\zeta}\right)\left(\sum_{\rho} \bar{u}^{-1\delta}_{\ \ \rho} u^{\rho}_{\nu}\right)\left(\sum_{\alpha}\sum_{\beta} \delta^{\beta}_{\alpha} u^{\alpha}_{\kappa} \bar{u}^{-1\lambda}_{\ \ \beta}\right) F^{\delta,\varepsilon}_{y_{x^\zeta},y_{x^\varepsilon}} \\
&\quad +\frac{1}{(n+1)(n+2)}\sum_{\varepsilon}\sum_{\delta}\left(\sum_{\rho}\sum_{\sigma} \delta^{\sigma}_{\rho} \bar{u}^{-1\mu}_{\ \ \sigma} u^{\rho}_{\nu}\right)\left(\sum_{\alpha}\sum_{\beta} \delta^{\beta}_{\alpha} \bar{u}^{-1\lambda}_{\ \ \beta} u^{\alpha}_{\kappa}\right) F^{\delta,\varepsilon}_{y_{x^\delta},y_{x^\varepsilon}} \\
&\quad +\frac{1}{(n+1)(n+2)}\sum_{\varepsilon}\sum_{\delta}\left(\sum_{\sigma}\sum_{\alpha} \delta^{\sigma}_{\alpha} \bar{u}^{-1\mu}_{\ \ \sigma} u^{\alpha}_{\kappa}\right)\left(\sum_{\rho}\sum_{\beta} \delta^{\beta}_{\rho} \bar{u}^{-1\lambda}_{\ \ \beta} u^{\rho}_{\nu}\right) F^{\delta,\varepsilon}_{y_{x^\delta},y_{x^\varepsilon}},
\end{aligned}
$$

from which one obtains

$$
\begin{aligned}
S^{\lambda\mu}_{\kappa\nu} &:= -u\sum_{\alpha}\sum_{\beta}\sum_{\sigma}\sum_{\rho} u^{\rho}_{\nu} \bar{u}^{-1\lambda}_{\ \ \beta} u^{\alpha}_{\kappa} \bar{u}^{-1\mu}_{\ \ \sigma} S^{\beta\sigma}_{\alpha\rho} \\
&= F^{\kappa,\nu}_{y_{x^\lambda},y_{x^\mu}} - \frac{1}{n+2}\left(\delta^{\mu}_{\nu}\sum_{\varepsilon} F^{\kappa,\varepsilon}_{y_{x^\lambda},y_{x^\varepsilon}} + \delta^{\mu}_{\kappa}\sum_{\varepsilon} F^{\nu,\varepsilon}_{y_{x^\lambda},y_{x^\varepsilon}} + \delta^{\lambda}_{\nu}\sum_{\varepsilon} F^{\kappa,\varepsilon}_{y_{x^\mu},y_{x^\varepsilon}} + \delta^{\lambda}_{\kappa}\sum_{\varepsilon} F^{\nu,\varepsilon}_{y_{x^\mu},y_{x^\varepsilon}}\right) \\
&\quad +\frac{1}{(n+1)(n+2)}\sum_{\varepsilon}\sum_{\delta}(\delta^{\mu}_{\nu}\delta^{\lambda}_{\kappa} + \delta^{\lambda}_{\nu}\delta^{\mu}_{\kappa}) F^{\delta,\varepsilon}_{y_{x^\delta},y_{x^\varepsilon}},
\end{aligned}
$$

or in the standard notation in [CM75],

$$
\begin{aligned}
S^{\beta\sigma}_{\alpha\rho} &= F^{\alpha,\rho}_{y_{x^\beta},y_{x^\sigma}} - \frac{1}{n+2}\left(\delta^{\sigma}_{\rho}\sum_{\varepsilon} F^{\alpha,\varepsilon}_{y_{x^\beta},y_{x^\varepsilon}} + \delta^{\sigma}_{\alpha}\sum_{\varepsilon} F^{\rho,\varepsilon}_{y_{x^\beta},y_{x^\varepsilon}} + \delta^{\beta}_{\rho}\sum_{\varepsilon} F^{\alpha,\varepsilon}_{y_{x^\sigma},y_{x^\varepsilon}} + \delta^{\beta}_{\alpha}\sum_{\varepsilon} F^{\rho,\varepsilon}_{y_{x^\sigma},y_{x^\varepsilon}}\right) \\
&\quad +\frac{1}{(n+1)(n+2)}\sum_{\varepsilon}\sum_{\delta}(\delta^{\sigma}_{\rho}\delta^{\beta}_{\alpha} + \delta^{\beta}_{\rho}\delta^{\sigma}_{\alpha}) F^{\delta,\varepsilon}_{y_{x^\delta},y_{x^\varepsilon}},
\end{aligned}
$$

which is the expression of the invariant.

12 Appendix: Cartan Lemma for 1-Forms and for 2-Forms

Lemma 12.1. *On $\mathbb{K}^{\mathrm{N}\geqslant 1}$ equipped with a number $1 \leqslant \mathrm{M} \leqslant \mathrm{N}$ of* independent *(local) differential 1-forms $\{\tau_1, \dots, \tau_{\mathrm{M}}\}$, if given differential 1-forms $\{\lambda_1, \dots, \lambda_{\mathrm{M}}\}$ satisfy:*

$$0 \,=\, \tau_1 \wedge \lambda_1 + \cdots + \tau_{\mathrm{M}} \wedge \lambda_{\mathrm{M}} \,=\, \sum_{1 \leqslant i \leqslant \mathrm{M}} \tau_i \wedge \lambda_i,$$

then:

$$\lambda_i \,\equiv\, 0 \mod \langle \tau_1, \dots, \tau_{\mathrm{M}} \rangle \qquad (1 \leqslant i \leqslant \mathrm{M}),$$

and more precisely, there exist functions $A_{i,j}$ such that:

$$\lambda_i \,=\, \sum_{1 \leqslant j \leqslant \mathrm{M}} A_{i,j} \cdot \tau_j.$$

Observe that when $\mathrm{M} = \mathrm{N}$, the statement is void, since $\{\tau_1, \dots, \tau_{\mathrm{N}}\}$ is then a coframe along which any 1-form decomposes.

Proof. Complete $\{\tau_1, \dots, \tau_{\mathrm{M}}\}$ as a coframe $\{\tau_1, \dots, \tau_{\mathrm{M}}, \chi_1, \dots, \chi_{\mathrm{N}-\mathrm{M}}\}$, decompose:

$$\lambda_i \,=\, \sum_{1 \leqslant j \leqslant \mathrm{M}} A_{i,j} \cdot \tau_j + \sum_{1 \leqslant k \leqslant \mathrm{N}-\mathrm{M}} B_{i,k} \cdot \chi_k,$$

insert in the hypothesis:

$$\begin{aligned} 0 \,&=\, \sum_{1 \leqslant i \leqslant \mathrm{M}} \tau_i \wedge \lambda_i \\ &=\, \sum_{1 \leqslant i \leqslant \mathrm{M}} \sum_{1 \leqslant j \leqslant \mathrm{M}} A_{i,j} \cdot \tau_i \wedge \tau_j + \sum_{1 \leqslant i \leqslant \mathrm{M}} \sum_{1 \leqslant k \leqslant \mathrm{N}-\mathrm{M}} \underline{B_{i,k} \cdot \tau_i \wedge \chi_k}_{\circ}, \end{aligned}$$

and conclude that all $B_{i,k} = 0$, since all 2-forms $\tau_i \wedge \chi_k$ are independent, and do not mix linearly with the others $\tau_i \wedge \tau_j$. □

Lemma 12.2. *On $\mathbb{K}^{\mathrm{N}\geqslant 1}$ equipped with a number $1 \leqslant \mathrm{M} \leqslant \mathrm{N}$ of* independent *(local) differential 1-forms $\{\tau_1, \dots, \tau_{\mathrm{M}}\}$, if given differential 2-forms $\{\Omega_1, \dots, \Omega_{\mathrm{M}}\}$ satisfy:*

$$0 \,=\, \tau_1 \wedge \Omega_1 + \cdots + \tau_{\mathrm{M}} \wedge \Omega_{\mathrm{M}} \,=\, \sum_{1 \leqslant i \leqslant \mathrm{M}} \tau_i \wedge \Omega_i,$$

then:

$$\Omega_i \,\equiv\, 0 \mod \langle \tau_1, \dots, \tau_{\mathrm{M}} \rangle \qquad (1 \leqslant i \leqslant \mathrm{M}),$$

and more precisely, completing $\{\tau_1, \dots, \tau_{\mathrm{M}}\}$ as a coframe $\{\tau_1, \dots, \tau_{\mathrm{M}}, \chi_1, \dots, \chi_{\mathrm{N}-\mathrm{M}}\}$, there exist functions A_{i,j_1,j_2} and $B_{i,j,k}$ such that:

$$\Omega_i \,=\, \sum_{1 \leqslant j_1 < j_2 \leqslant \mathrm{M}} A_{i,j_1,j_2} \cdot \tau_{j_1} \wedge \tau_{j_2} + \sum_{1 \leqslant j \leqslant \mathrm{M}} \sum_{1 \leqslant k \leqslant \mathrm{N}-\mathrm{M}} B_{i,j,k} \cdot \tau_j \wedge \chi_k.$$

Proof. Observe that wedge products between its elements are:

$$\begin{aligned} \langle \tau_{j_1}, \chi_{k_1} \rangle \wedge \langle \tau_{j_2}, \chi_{k_2} \rangle \,&=\, \langle \tau_{j_1} \wedge \tau_{j_2} \rangle + \langle \tau_{j_1} \wedge \chi_{k_2} \rangle + \underline{\langle \chi_{k_1} \wedge \tau_{j_2} \rangle}_{\circ} + \langle \chi_{k_1} \wedge \chi_{k_2} \rangle \\ &=\, \langle \tau_{j_1} \wedge \tau_{j_2} \rangle + \langle \tau_j \wedge \chi_k \rangle + \langle \chi_{k_1} \wedge \chi_{k_2} \rangle, \end{aligned}$$

whence there are functions A_{i,j_1,j_2}, $B_{i,j,k}$, C_{i,k_1,k_2} such that:

$$\Omega_i = \sum_{1\leqslant j_1<j_2\leqslant \mathrm{M}} A_{i,j_1,j_2}\cdot\tau_{j_1}\wedge\tau_{j_2} + \sum_{1\leqslant j\leqslant \mathrm{M}}\sum_{1\leqslant k\leqslant \mathrm{N}-\mathrm{M}} B_{i,j,k}\cdot\tau_j\wedge\chi_k + \sum_{1\leqslant k_1<k_2\leqslant \mathrm{N}-\mathrm{M}} C_{i,k_1,k_2}\cdot\chi_{k_1}\wedge\chi_{k_2},$$

insert this in the hypothesis:

$$\begin{aligned} 0 &= \sum_{1\leqslant i\leqslant \mathrm{M}} \tau_i\wedge\Omega_i \\ &= \sum_{1\leqslant i\leqslant \mathrm{M}}\sum_{1\leqslant j_1<j_2\leqslant \mathrm{M}} A_{i,j_1,j_2}\cdot\tau_i\wedge\tau_{j_1}\wedge\tau_{j_2} + \sum_{1\leqslant i\leqslant \mathrm{M}}\sum_{1\leqslant j\leqslant \mathrm{M}}\sum_{1\leqslant k\leqslant \mathrm{N}-\mathrm{M}} B_{i,j,k}\cdot\tau_i\wedge\tau_j\wedge\chi_k + \\ &\qquad + \sum_{1\leqslant i\leqslant \mathrm{M}}\sum_{1\leqslant k_1<k_2\leqslant \mathrm{N}-\mathrm{M}} \underline{C_{i,k_1,k_2}\cdot\tau_i\wedge\chi_{k_1}\wedge\chi_{k_2}}, \end{aligned}$$

and conclude that all $C_{i,k_1,k_2}=0$, since all 3-forms $\tau_i\wedge\chi_{k_1}\wedge\chi_{k_2}$ with $k_1<k_2$ are independent, and do not mix linearly with the others $\tau_i\wedge\tau_{j_1}\wedge\tau_{j_2}$, $\tau_i\wedge\tau_j\wedge\chi_k$. □

13 Hachtroudi-Chern-Moser tensor in CR geometry

In the previous section, the S-tensor is an invariant associated to the second order PDE systems

$$y_{x^\alpha x^\beta} = F^{\alpha,\beta}(x^\gamma, y, y_{x^\delta}).$$

This chapter will discuss the application of the S tensor to CR geometry, notably the case of real hypersurfaces in $\mathbb{C}^{n+1}$. The exposition closely follows the previous chapter on the umbilic locus of the CR ellipsoids in $\mathbb{C}^2$, except with a few differences.

13.1 Some preliminaries

Let $(z_1,\ldots,z_n,w)$ be holomorphic coordinates on $\mathbb{C}^{n+1}$, with

$$\begin{aligned} z_i &= x_i+\sqrt{-1}y_i \\ w &= u+\sqrt{-1}v \end{aligned} \qquad \big(1\leqslant i\leqslant n\big),$$

which at the end, w will be denoted by z_{n+1}. Also, set $z:=(z_1,\ldots,z_n)$, and let

$$\begin{aligned} r:\mathbb{C}^{n+1} &\longrightarrow \mathbb{R} \\ (z,w) &\longmapsto r(z,w,\bar z,\bar w), \end{aligned}$$

be a real-analytic, real-valued function with $r(0)=0$. By analyticity condition, r may be expressed as a converging power series in a neighbourhood of the origin

$$r = \sum_{(i_1,\ldots,i_{n+1})\in\mathbb{N}^{n+1}}\ \sum_{(j_1,\ldots,j_{n+1})\in\mathbb{N}^{n+1}} a_{i_1,\ldots,i_{n+1},j_1,\ldots,j_{n+1}}\, z_1^{i_1}\cdots z_n^{i_n} w^{i_{n+1}}\bar z_1^{j_1}\cdots\bar z_n^{j_n}\bar w^{j_{n+1}}.$$

Since r is real, the conjugation of the coefficients permutes the indices:

$$\overline{a_{i_1,\ldots,i_{n+1},j_1,\ldots,j_{n+1}}} = a_{j_1,\ldots,j_{n+1},i_1,\ldots,i_{n+1}}, \tag{13.1}$$

and hence just as before, for every $1\leqslant k\leqslant n$, the following relations are obtained:

$$\begin{aligned} \overline{r_{z_k}(z,w,\bar z,\bar w)} &= r_{\bar z_k}(z,w,\bar z,\bar w), \\ \overline{r_w(z,w,\bar z,\bar w)} &= r_{\bar w}(z,w,\bar z,\bar w). \end{aligned} \tag{13.2}$$

13.2 Real Manifold $M^{2n+1} \subset \mathbb{C}^{n+1}$

Let r be a real-analytic, real-valued function as before, and let $M^{2n+1} \subset \mathbb{C}^{n+1}$ be the real hypersurface passing through the origin defined by

$$M^{2n+1} = \{r = 0\}.$$

Assume that the following differential form restricted to M

$$dr|_M = \sum_{1 \leqslant k \leqslant n} r_{z_k}|_M \, dz_k + r_w|_M \, dw + \sum_{1 \leqslant k \leqslant n} r_{\bar{z}_k}|_M \, d\bar{z}_k + r_{\bar{w}}|_M \, d\bar{w}$$

does not vanish anywhere, therefore M is a real manifold of real dimension $2n+1$. By renumbering the coordinates and restricting $dr|_M$ to a smaller subset if necessary, without loss of generality, r_w may be assumed to be non-vanishing anywhere on M.

13.3 Tangent bundle on M, extrinsic version

Let TM be the real tangent bundle on M, and let $\mathbb{C}TM = TM \otimes_{\mathbb{R}} \mathbb{C}$ be its complexified tangent bundle. At each $p \in M$, denote the following vector spaces

$$\begin{aligned} T^{1,0}_p M &:= \mathbb{C}T_pM \cap T^{1,0}_p \mathbb{C}^{n+1}, \\ T^{0,1}_p M &:= \mathbb{C}T_pM \cap T^{0,1}_p \mathbb{C}^{n+1}, \end{aligned}$$

or more explicitly, they can be written as

$$\begin{aligned} T^{1,0}_p M &= \left\{ \sum_{1 \leqslant k \leqslant n} a_k \partial_{z_k} + a\partial_w : \, (a_1, \ldots, a_n, a) \in \mathbb{C}^{n+1}, \, \sum_{1 \leqslant k \leqslant n} a_k r_{z_k}(p) + a r_w(p) = 0 \right\}, \\ T^{0,1}_p M &= \left\{ \sum_{1 \leqslant k \leqslant n} a_k \partial_{\bar{z}_k} + a\partial_{\bar{w}} : \, (a_1, \ldots, a_n, a) \in \mathbb{C}^{n+1}, \, \sum_{1 \leqslant k \leqslant n} a_k r_{z_k}(p) + a r_w(p) = 0 \right\}. \end{aligned}$$

Since each of the vector spaces above is defined by a non-degenerate (because r_w does not vanish everywhere) hyperplane equation in $\mathbb{C}^{n+1}$, the complex dimension of each of them is n, and hence the rank of each of the following vector bundles

$$\begin{aligned} T^{1,0}M &= \bigsqcup_{p \in M} T^{1,0}_p M, \\ T^{0,1}M &= \bigsqcup_{p \in M} T^{0,1}_p M, \end{aligned}$$

is n. The vector fields

$$L_k := -r_w \partial_{z_k} + r_{z_k} \partial_w \qquad (1 \leqslant k \leqslant n)$$

form a frame of $T^{1,0}M$ bundle because r_w is assumed to be non-vanishing everywhere, and they annihilate r at every point in M. Their respective conjugates $\bar{L}_k$ form a frame for the $T^{0,1}M$ bundle.

13.4 Commutator properties and the Levi matrix

A quick calculation shows that for $1 \leqslant i, j \leqslant n$,

$$[L_i, L_j] = [\bar{L}_i, \bar{L}_j] = 0.$$

In contrast, the commutators of $T^{1,0}M$ and $T^{0,1}M$ vector fields are not zero in general:

$$\begin{aligned}
[L_i, \bar{L}_j] &= [-r_w\partial_{z_i} + r_{z_i}\partial_{z_i}, -r_{\bar{w}}\partial_{\bar{z}_j} + r_{\bar{z}_j}\partial_{\bar{w}}] \\
&= [-r_w\partial_{z_i}, -r_{\bar{w}}\partial_{\bar{z}_j}] - [r_w\partial_{z_i}, r_{\bar{z}_j}\partial_{\bar{w}}] - [r_{z_i}\partial_w, r_{\bar{w}}\partial_{\bar{z}_j}] + [r_{z_i}\partial_w, r_{\bar{z}_j}\partial_{\bar{w}}] \\
&= r_w r_{z_i\bar{w}}\partial_{\bar{z}_j} - r_{\bar{w}} r_{\bar{z}_j w}\partial_{z_i} - r_w r_{z_i\bar{z}_j}\partial_{\bar{w}} + r_{\bar{z}_j} r_{w\bar{w}}\partial_{z_i} \\
&\quad -r_{z_i} r_{w\bar{w}}\partial_{\bar{z}_j} + r_{\bar{w}} r_{\bar{z}_j z_i}\partial_w + r_{z_i} r_{w\bar{z}_j}\partial_{\bar{w}} - r_{\bar{z}_j} r_{z_i\bar{w}}\partial_w \\
&= \frac{r_w r_{z_i\bar{w}}}{r_{\bar{w}}} r_{\bar{w}}\partial_{\bar{z}_j} - \frac{r_{\bar{w}} r_{\bar{z}_j w}}{r_w} r_w\partial_{z_i} - r_w r_{z_i\bar{z}_j}\partial_{\bar{w}} \\
&\quad + \frac{r_{\bar{z}_j} r_{w\bar{w}}}{r_w} r_w\partial_{z_i} - \frac{r_{z_i} r_{w\bar{w}}}{r_{\bar{w}}} r_{\bar{w}}\partial_{\bar{z}_j} + r_{\bar{w}} r_{\bar{z}_j z_i}\partial_w + r_{z_i} r_{w\bar{z}_j}\partial_{\bar{w}} - r_{\bar{z}_j} r_{z_i\bar{w}}\partial_w.
\end{aligned}$$

By using the following relations

$$\begin{aligned}
r_w\partial_{z_i} &\equiv r_{z_i}\partial_w \qquad \mathsf{mod}\ T^{1,0}M \oplus T^{0,1}M, \\
r_{\bar{w}}\partial_{\bar{z}_i} &\equiv r_{\bar{z}_i}\partial_{\bar{w}} \qquad \mathsf{mod}\ T^{1,0}M \oplus T^{0,1}M,
\end{aligned}$$

the Lie bracket of the two vector fields takes on the final expression:

$$[L_i, \bar{L}_j] \equiv [r_w r_{\bar{w}} r_{z_i\bar{z}_j} - r_w r_{\bar{z}_j} r_{z_i\bar{w}} - r_{z_i} r_{\bar{w}} r_{w\bar{z}_j} + r_{z_i} r_{\bar{z}_j} r_{w\bar{w}}]\left(\frac{1}{r_w}\partial_w - \frac{1}{r_{\bar{w}}}\partial_{\bar{w}}\right) \qquad \mathsf{mod}\ T^{1,0}M \oplus T^{0,1}M.$$

Definition 13.3. The Levi matrix $\mathrm{L}(r) = (\lambda_{ij})$ is a matrix whose entries are given by

$$\lambda_{ij} = r_w r_{\bar{w}} r_{z_i\bar{z}_j} - r_w r_{\bar{z}_j} r_{z_i\bar{w}} - r_{z_i} r_{\bar{w}} r_{w\bar{z}_j} + r_{z_i} r_{\bar{z}_j} r_{w\bar{w}}.$$

Definition 13.4. A smooth real manifold $M^{2n+1} \subset \mathbb{C}^{n+1}$ defined by a real-analytic, real-valued defining function r is Levi non-degenerate at $p \in M$ if $\det(\mathrm{L}(r))$ does not vanish at p. The manifold is Levi non-degenerate if $\det(\mathrm{L}(r))$ does not vanish anywhere.

13.5 M as a graph of complex-valued function

The assumption that r_w does not vanish anywhere on M means that M may be expressed as a graph of a complex-valued function, like in the case of $\mathbb{C}^2$. More precisely, let $(s,t) := (s_1, \ldots, s_n, t)$ be holomorphic coordinates that complexifies $(\bar{z}_1, \ldots, \bar{z}_n, \bar{w})$. Then $r(z,w,s,t)$ is the polarisation of $r(z,w,\bar{z},\bar{w})$. Since r_w does not vanish anywhere on M, its polarisation $r_w(z,w,s,t)$ does not vanish anywhere in a small neighbourhood of the origin in $\mathbb{C}^{2n+2}$. By the Holomorphic Implicit Function theorem, there exists a holomorphic function $\Theta(z,s,t)$ such that

$$\{r(z,w,s,t) = 0\} = \{w = \Theta(z,s,t)\}.$$

Upon restriction to $\{s = \bar{z}, t = \bar{w}\}$, the manifold M may be represented in two ways:

$$M = \{r(z,w,\bar{z},\bar{w}) = 0\} = \{w = \Theta(z,\bar{z},\bar{w})\}.$$

Based on the definition, the following trivial identity holds:

$$0 \equiv r(z, \Theta(z,\bar{z},\bar{w}), \bar{z}, \bar{w}). \tag{13.5}$$

13.6 Some identities between r and Θ.

Let $F(z,w,\bar{z},\bar{w}) := (z, \Theta(z,\bar{z},\bar{w}), \bar{z}, \bar{w})$. By differentiating equation (13.5) with respect to z_i, $\bar{z}_i$ and $\bar{w}$, the following identities are obtained:

$$0 \equiv r_{z_i} \circ F + \Theta_{z_i} r_w \circ F, \tag{13.6}$$

$$0 \equiv r_{\bar{z}_i} \circ F + \Theta_{\bar{z}_i} r_w \circ F, \tag{13.7}$$

$$0 \equiv r_{\bar{w}} \circ F + \Theta_{\bar{w}} r_w \circ F. \tag{13.8}$$

Since the assumption that r_w vanishes nowhere, by equation (13.8), the function $\Theta_{\bar{w}}$ vanishes nowhere. Differentiating equation (13.6) with respect to $\bar{z}_j$ and $\bar{w}$ yields

$$\begin{aligned} 0 &\equiv r_{z_i\bar{z}_j}\circ F + \Theta_{\bar{z}_j} r_{z_i w}\circ F + \Theta_{z_i} r_{w\bar{z}_j}\circ F + \Theta_{z_i}\Theta_{\bar{z}_j} r_{ww}\circ F + \Theta_{z_i\bar{z}_j} r_w \circ F, \qquad (13.9)\\ 0 &\equiv r_{z_i\bar{w}}\circ F + \Theta_{\bar{w}} r_{z_i w}\circ F + \Theta_{z_i} r_{w\bar{w}}\circ F + \Theta_{z_i}\Theta_{\bar{w}} r_{ww}\circ F + \Theta_{z_i\bar{w}} r_w \circ F. \qquad (13.10)\end{aligned}$$

There is also another identity coming from differentiating equation (13.6) with respect to z_j, which will be used later:

$$0 \equiv r_{z_i z_j}\circ F + \Theta_{z_j} r_{z_i w}\circ F + \Theta_{z_i z_j} r_w \circ F + \Theta_{z_i} r_{z_j w}\circ F + \Theta_{z_i}\Theta_{z_j} r_{ww}\circ F. \qquad (13.11)$$

In what follows, the notation $\circ F$ may sometimes be suppressed, and at the end of this chapter the variable w may be replaced by a new notation z_{n+1}.

13.7 The expression δ_{ij}.

Just as in the previous chapter, let δ_{ij} denote the following expression

$$\delta_{ij} := \Theta_{\bar{z}_j}\Theta_{z_i,\bar{w}} - \Theta_{\bar{w}}\Theta_{z_i\bar{z}_j}.$$

A similar proof shows that

Lemma 13.12. *The following identity holds*

$$\lambda_{ij}\circ F = -(r_w\circ F)^3\delta_{ij}.$$

Proof. The proof is carried out verbatim as in the $\mathbb{C}^2$ case, replacing z by z_i and $\bar{z}$ by $\bar{z}_j$. Starting from

$$\lambda_{ij}\circ F = r_{z_i} r_{\bar{z}_j} r_{w\bar{w}} - r_{z_i} r_{\bar{w}} r_{w\bar{z}_j} - r_{\bar{z}_j} r_w r_{z_i\bar{w}} + r_w r_{\bar{w}} r_{z_i\bar{z}_j},$$

using equations (13.6) to (13.9) concludes the proof. □

13.8 The Levi non-degenerate condition

Fix the following notation $(z, w, w_z) := (z_1, \ldots, z_n, w, w_{z_1}, \ldots, w_{z_n})$ which will be one of the holomorphic coordinates on $\mathbb{C}^{2n+1}$. Let $\varphi : (\mathbb{C}^{2n+1}, (z, s, t)) \to (\mathbb{C}^{2n+1}, (z, w, w_z))$ be the holomorphic map given by

$$\begin{aligned} \varphi : \mathbb{C}^{2n+1} &\longrightarrow \mathbb{C}^{2n+1}\\ (z, s, t) &\longmapsto (z_1, \ldots, z_n, \Theta(z, s, t), \Theta_{z_1}(z, s, t), \ldots, \Theta_{z_n}(z, s, t)),\end{aligned}$$

whose Jacobian matrix is

$$\mathsf{J}(\Theta) = \begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & 1 & 0 & \cdots & 0 & 0\\ \Theta_{z_1} & \cdots & \Theta_{z_n} & \Theta_{s_1} & \cdots & \Theta_{s_n} & \Theta_t\\ \Theta_{z_1 z_1} & \cdots & \Theta_{z_1 z_n} & \Theta_{z_1 s_1} & \cdots & \Theta_{z_1 s_n} & \Theta_{z_1 t}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \Theta_{z_n z_1} & \cdots & \Theta_{z_n z_n} & \Theta_{z_n s_1} & \cdots & \Theta_{z_n s_n} & \Theta_{z_n t}\end{pmatrix}.$$

Let $\mathsf{j}(\Theta)$ be the lower right block of the matrix

$$\mathsf{j}(\Theta) := \begin{pmatrix} \Theta_{s_1} & \cdots & \Theta_{s_n} & \Theta_t\\ \Theta_{z_1 s_1} & \cdots & \Theta_{z_1 s_n} & \Theta_{z_1 t}\\ \vdots & \ddots & \vdots & \vdots\\ \Theta_{z_n s_1} & \cdots & \Theta_{z_n s_n} & \Theta_{z_n t}\end{pmatrix}. \qquad (13.13)$$

Therefore, the Jacobian of the map φ is the determinant of the lower right block matrix

$$\Delta(\Theta) := \mathsf{Jac}(\varphi) = \det \mathsf{j}(\Theta).$$

Proposition 13.14. *Let $M^{2n+1} \subset \mathbb{C}^{n+1}$ be a smooth, real hypersurface given by the vanishing locus of a real analytic, real-valued function r. Then the following identity holds on M*

$$\det\big((\lambda_{ij})_{1\leqslant i,j\leqslant n}\big) = (-1)^n (r_w)^{3n} (\Theta_{\bar w})^{n-1} \Delta(\Theta|_M).$$

Therefore, M is Levi non-degenerate if and only if $\Delta(\Theta|_M)$ vanishes nowhere on M.

Proof. The proof involves formal manipulation of matrices. By lemma 13.12,

$$\det\big((\lambda_{ij})_{1\leqslant i,j\leqslant n}\big) = \det\big((-r_w^3 \delta_{ij})_{1\leqslant i,j\leqslant n}\big) = (-1)^n (r_w)^{3n} \det\big((\delta_{ij})_{1\leqslant i,j\leqslant n}\big).$$

It remains to calculate $\det\big((\delta_{ij})_{1\leqslant i,j\leqslant n}\big)$. Observe that

$$\begin{aligned}
\det\big((\delta_{ij})_{1\leqslant i,j\leqslant n}\big) &= \begin{vmatrix} \Theta_{\bar z_1}\Theta_{z_1\bar w} - \Theta_{\bar w}\Theta_{z_1\bar z_1} & \cdots & \Theta_{\bar z_n}\Theta_{z_1\bar w} - \Theta_{\bar w}\Theta_{z_1\bar z_n} \\ \vdots & \ddots & \vdots \\ \Theta_{\bar z_1}\Theta_{z_n\bar w} - \Theta_{\bar w}\Theta_{z_n\bar z_1} & \cdots & \Theta_{\bar z_n}\Theta_{z_n\bar w} - \Theta_{\bar w}\Theta_{z_n\bar z_n} \end{vmatrix} \\
&= (-1)^n \Theta_{\bar w}^n \begin{vmatrix} \Theta_{z_1\bar z_1} - \frac{\Theta_{\bar z_1}}{\Theta_{\bar w}}\Theta_{z_1\bar w} & \cdots & \Theta_{z_1\bar z_n} - \frac{\Theta_{\bar z_n}}{\Theta_{\bar w}}\Theta_{z_1\bar w} \\ \vdots & \ddots & \vdots \\ \Theta_{z_n\bar z_1} - \frac{\Theta_{\bar z_1}}{\Theta_{\bar w}}\Theta_{z_n\bar w} & \cdots & \Theta_{z_n\bar z_n} - \frac{\Theta_{\bar z_n}}{\Theta_{\bar w}}\Theta_{z_n\bar w} \end{vmatrix} \\
&= \Theta_{\bar w}^n \begin{vmatrix} \frac{\Theta_{\bar z_1}}{\Theta_{\bar w}} & \cdots & \frac{\Theta_{\bar z_n}}{\Theta_{\bar w}} & 1 \\ \Theta_{z_1\bar z_1} - \frac{\Theta_{\bar z_1}}{\Theta_{\bar w}}\Theta_{z_1\bar w} & \cdots & \Theta_{z_1\bar z_n} - \frac{\Theta_{\bar z_n}}{\Theta_{\bar w}}\Theta_{z_1\bar w} & 0 \\ \vdots & \ddots & \vdots & \\ \Theta_{z_n\bar z_1} - \frac{\Theta_{\bar z_1}}{\Theta_{\bar w}}\Theta_{z_n\bar w} & \cdots & \Theta_{z_n\bar z_n} - \frac{\Theta_{\bar z_n}}{\Theta_{\bar w}}\Theta_{z_n\bar w} & 0 \end{vmatrix} \\
&= \Theta_{\bar w}^n \begin{vmatrix} \frac{\Theta_{\bar z_1}}{\Theta_{\bar w}} & \cdots & \frac{\Theta_{\bar z_n}}{\Theta_{\bar w}} & 1 \\ \Theta_{z_1\bar z_1} & \cdots & \Theta_{z_1\bar z_n} & \Theta_{z_1\bar w} \\ \vdots & \ddots & \vdots & \\ \Theta_{z_n\bar z_1} & \cdots & \Theta_{z_n\bar z_n} & \Theta_{z_1\bar w} \end{vmatrix} \\
&= \Theta_{\bar w}^{n-1} \begin{vmatrix} \Theta_{\bar z_1} & \cdots & \Theta_{\bar z_n} & \Theta_{\bar w} \\ \Theta_{z_1\bar z_1} & \cdots & \Theta_{z_1\bar z_n} & \Theta_{z_1\bar w} \\ \vdots & \ddots & \vdots & \\ \Theta_{z_n\bar z_1} & \cdots & \Theta_{z_n\bar z_n} & \Theta_{z_1\bar w} \end{vmatrix} = \Theta_{\bar w}^{n-1} \Delta(\Theta|_M),
\end{aligned}$$

which proves the proposition. □

Under the assumption that M is Levi non-degenerate, by restricting φ to a smaller subset of $\mathbb{C}^{2n+1}$, the map φ is a local biholomorphism. Hence there exist $n+1$ functions $\Lambda_1, \ldots, \Lambda_n, \Pi$ such that for $1 \leqslant i \leqslant n$,

$$\begin{aligned}
s_i &= \Lambda_i\big(z, \Theta(z,s,t), \Theta_{z_1}(z,s,t), \ldots, \Theta_{z_n}(z,s,t)\big), \\
t &= \Pi\big(z, \Theta(z,s,t), \Theta_{z_1}(z,s,t), \ldots, \Theta_{z_n}(z,s,t)\big),
\end{aligned}$$

as well as

$$w = \Theta\big(z, \Lambda_1(z,w,w_z), \ldots, \Lambda_n(z,w,w_z), \Pi(z,w,w_z)\big), \tag{13.15}$$

$$w_{z_i} = \Theta_{z_i}\big(z, \Lambda_1(z,w,w_z), \ldots, \Lambda_n(z,w,w_z), \Pi(z,w,w_z)\big). \tag{13.16}$$

By differentiating equations (13.15) and (13.16) with respect to w and w_{z_i}, the following system of equations is obtained for $1 \leqslant i, j \leqslant n$:

$$\begin{aligned}
1 &= \Theta_{s_1}\Lambda_{1,w} + \cdots + \Theta_{s_n}\Lambda_{n,w} + \Theta_t\Pi_w,\\
0 &= \Theta_{s_1}\Lambda_{1,w_{z_j}} + \cdots + \Theta_{s_n}\Lambda_{n,w_{z_j}} + \Theta_t\Pi_{w_{z_j}},\\
0 &= \Theta_{z_i s_1}\Lambda_{1,w} + \cdots + \Theta_{z_i s_n}\Lambda_{n,w} + \Theta_{z_i t}\Pi_w,\\
\delta_{ij} &= \Theta_{z_i s_1}\Lambda_{1,w_{z_j}} + \cdots + \Theta_{z_i s_n}\Lambda_{n,w_{z_j}} + \Theta_{z_i t}\Pi_{w_{z_j}}.
\end{aligned}$$

Consequently, the equations above may be expressed in terms of matrix multiplication:

$$\begin{pmatrix}
\Theta_{s_1} & \cdots & \Theta_{s_n} & \Theta_t\\
\Theta_{z_1 s_1} & \cdots & \Theta_{z_1 s_n} & \Theta_{z_1 t}\\
\vdots & \ddots & \vdots & \vdots\\
\Theta_{z_{n-1} s_1} & \cdots & \Theta_{z_{n-1} s_n} & \Theta_{z_{n-1} t}\\
\Theta_{z_n s_1} & \cdots & \Theta_{z_n s_n} & \Theta_{z_n t}
\end{pmatrix}
\begin{pmatrix}
\Lambda_{1,w} & \Lambda_{1,w_{z_1}} & \cdots & \Lambda_{1,w_{z_n}}\\
\Lambda_{2,w} & \Lambda_{2,w_{z_1}} & \cdots & \Lambda_{2,w_{z_n}}\\
\vdots & \vdots & \ddots & \vdots\\
\Lambda_{n,w} & \Lambda_{n,w_{z_1}} & \cdots & \Lambda_{n,w_{z_n}}\\
\Pi_w & \Pi_{w_{z_1}} & \cdots & \Pi_{w_{z_n}}
\end{pmatrix} = \mathsf{Id}_{n\times n}.$$

Let $\Delta_{(i,j)}(\Theta)$ be the determinant of the matrix

$$\Delta_{(i,j)}(\Theta) := (-1)^{i+j}\mathsf{det}\left(\mathsf{j}(\Theta)_{(j,i)}\right),$$

where $(\mathsf{j}(\Theta))_{(i,j)}$ is the matrix with i-th row and j-th column of $\mathsf{j}(\Theta)$ removed. Therefore by Cramer's rule, for $1 \leqslant i, j \leqslant n$,

$$\begin{aligned}
\Lambda_{i,w} &= \frac{\Delta_{(i,1)}(\Theta)}{\Delta(\Theta)},\\
\Pi_w &= \frac{\Delta_{(n+1,1)}(\Theta)}{\Delta(\Theta)},\\
\Lambda_{i,w_{z_j}} &= \frac{\Delta_{(i,j+1)}(\Theta)}{\Delta(\Theta)},\\
\Pi_{w_{z_j}} &= \frac{\Delta_{(n+1,j+1)}(\Theta)}{\Delta(\Theta)}.
\end{aligned}$$

13.9 Translation between Θ and r

The functions $\Lambda_{i,w}$, Π_w, $\Lambda_{i,w_{z_j}}$, $\Pi_{w_{z_j}}$ and $\Theta_{z_i z_j}$ may be expressed in terms of the defining function r. To this end, let $z_{n+1} := w$ (in some places w may also appear), and for $1 \leqslant k, l \leqslant n+1$, define on M the following functions

$$\begin{aligned}
H_{k,l} &:= -[r_w r_w r_{z_k z_l} - r_{z_l} r_w r_{z_k w} - r_{z_k} r_w r_{z_l w} + r_{z_k} r_{z_l} r_{ww}],\\
H_{k,\bar{l}} &= -[r_w r_w r_{z_k \bar{z}_l} - r_{\bar{z}_l} r_w r_{z_k w} - r_{z_k} r_w r_{\bar{z}_l w} + r_{z_k} r_{\bar{z}_l} r_{ww}],
\end{aligned}$$

Proposition 13.17. *For $1 \leqslant i, j \leqslant n$ and $1 \leqslant k, l \leqslant n+1$, the following identities hold on M:*

$$\begin{aligned}
\Theta_{z_i z_j} &= \frac{H_{i,j}}{r_w^3},\\
\Theta_{z_k \bar{z}_l} &= \frac{H_{k,\bar{l}}}{r_w^3},
\end{aligned}$$

Proof. Moving $\Theta_{z_i z_j} r_w$ in equation (13.11)

$$0 \equiv r_{z_i z_j} + \Theta_{z_j} r_{z_i w} + \Theta_{z_i z_j} r_w + \Theta_{z_i} r_{z_j w} + \Theta_{z_i} \Theta_{z_j} r_{ww}$$

over to the left gives

$$-\Theta_{z_i z_j} r_w \equiv r_{z_i z_j} + \Theta_{z_j} r_{z_i w} + \Theta_{z_i} r_{z_j w} + \Theta_{z_i} \Theta_{z_j} r_{ww}.$$

Multiplying on both sides by r_w^2, and then using equations (13.6) to (13.8), yields

$$-\Theta_{z_i z_j} r_w^3 \equiv r_{z_i z_j} r_w^2 + -r_{z_j} r_{z_i w} r_w - r_{z_i} r_{z_j w} r_w + r_{z_i} r_{z_j} r_{ww}.$$

Therefore, the first identity is proved. The rest of the proof is done in the same way. □

Let $\Delta(r)$ denote the determinant of the following matrix:

$$\Delta(r) := \begin{vmatrix} r_{\bar{z}_1} & \cdots & r_{\bar{z}_n} & r_{\bar{z}_{n+1}} \\ H_{1,\bar{1}} & \cdots & H_{1,\bar{n}} & H_{1,\overline{n+1}} \\ \vdots & \ddots & \vdots & \vdots \\ H_{n,\bar{1}} & \cdots & H_{n,\bar{n}} & H_{n,\overline{n+1}} \end{vmatrix}.$$

Based on the identities above, the determinants $\Delta_{(i,j+1)}(\Theta)$ and $\Delta(\Theta)$ can be expressed in terms of r when restricted to M. More explicitly, applying equations (13.6) to (13.8) and proposition 13.17 to each of the coefficients in $\Delta_{(i,j+1)}(\Theta|_M)$:

$$\begin{aligned} \Delta_{(i,j+1)}(\Theta|_M) &= (-1)^{i+j} \begin{vmatrix} \Theta_{\bar{z}_1} & \cdots & \Theta_{\bar{z}_{i-1}} & \Theta_{\bar{z}_{i+1}} & \cdots & \Theta_{\bar{z}_n} & \Theta_{\bar{w}} \\ \Theta_{z_1\bar{z}_1} & \cdots & \Theta_{z_1\bar{z}_{i-1}} & \Theta_{z_1\bar{z}_{i+1}} & \cdots & \Theta_{z_n\bar{z}_n} & \Theta_{z_1\bar{w}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta_{z_{j-1}\bar{z}_1} & \cdots & \Theta_{z_{j-1}\bar{z}_{i-1}} & \Theta_{z_{j-1}\bar{z}_{i+1}} & \cdots & \Theta_{z_{j-1}\bar{z}_n} & \Theta_{z_{j-1}\bar{w}} \\ \Theta_{z_{j+1}\bar{z}_1} & \cdots & \Theta_{z_{j+1}\bar{z}_{i-1}} & \Theta_{z_{j+1}\bar{z}_{i+1}} & \cdots & \Theta_{z_{j+1}\bar{z}_n} & \Theta_{z_{j+1}\bar{w}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta_{z_n\bar{z}_1} & \cdots & \Theta_{z_n\bar{z}_{i-1}} & \Theta_{z_n\bar{z}_{i+1}} & \cdots & \Theta_{z_n\bar{z}_n} & \Theta_{z_n\bar{w}} \end{vmatrix} \\ &= \frac{-(-1)^{i+j}}{(r_w)^{3n-2}} \begin{vmatrix} r_{\bar{z}_1} & \cdots & r_{\bar{z}_{i-1}} & r_{\bar{z}_{i+1}} & \cdots & r_{\bar{z}_n} & r_{\bar{z}_{n+1}} \\ H_{1,\bar{1}} & \cdots & H_{1,\overline{i-1}} & H_{1,\overline{i+1}} & \cdots & H_{1,\bar{n}} & H_{1,\overline{n+1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ H_{j-1,\bar{1}} & \cdots & H_{j-1\overline{i-1}} & H_{j-1,\overline{i+1}} & \cdots & H_{j-1,\bar{n}} & H_{j-1,\overline{n+1}} \\ H_{j+1,\bar{1}} & \cdots & H_{j+1,\overline{i-1}} & H_{j+1,\overline{i+1}} & \cdots & H_{j+1,\bar{n}} & H_{j+1,\overline{n+1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ H_{n,\bar{1}} & \cdots & H_{n,\overline{i-1}} & H_{n,\overline{i+1}} & \cdots & H_{n,\bar{n}} & H_{n,\overline{n+1}} \end{vmatrix} \\ &=: \frac{-1}{(r_{z_{n+1}})^{3n-2}} \Delta_{(i,j+1)}(r). \end{aligned}$$

The restriction of $\Delta(\Theta)$ to M is also written in a similar way:

$$\Delta(\Theta|_M) = \frac{-1}{r_{z_{n+1}}^{3n+1}} \Delta(r).$$

Consequently, their quotients are related by a factor of a non-vanishing function on M:

$$\frac{\Delta_{(i,j+1)}(\Theta|_M)}{\Delta(\Theta|_M)} = \frac{\Delta_{(i,j+1)}(r)}{\Delta(r)} r_{z_{n+1}}^3.$$

13.10 The vector field $\partial_{w_{z_k}}$ and the Hachtroudi-Chern-Moser tensor for CR geometry

For any differentiable function

$$G(z,s,t)=G\big(z,\Lambda_1(z,w,w_z),\dots,\Lambda_n(z,w,w_z),\Pi(z,w,w_z)\big),$$

its derivative with respect to w_{z_j} for each $1\leqslant j\leqslant n$ gives

$$\begin{aligned}\partial_{w_{z_j}}G(z,s,t) &= \Lambda_{1,w_{z_j}}\partial_{s_1}G(z,s,t)+\cdots+\Lambda_{n,w_{z_j}}\partial_{s_n}G(z,s,t)+\Pi_{w_{z_j}}\partial_tG(z,s,t)\\ &= \sum_{1\leqslant l\leqslant n}\frac{\Delta_{(l,j+1)}(\Theta)}{\Delta(\Theta)}\partial_{s_l}G(z,s,t)+\frac{\Delta_{(n+1,j+1)}(\Theta)}{\Delta(\Theta)}\partial_tG(z,s,t).\end{aligned}$$

By restriction to M and again adopting the notation $z_{n+1}:=w$ yields

$$\begin{aligned}\partial_{w_{z_j}}G(z,\bar z,\bar w) &= (r_w)^3\bigg(\sum_{1\leqslant l\leqslant n}\frac{\Delta_{(l,j+1)}(r)}{\Delta(r)}\partial_{\bar z_l}G(z,\bar z,\bar w)+\frac{\Delta_{(n+1,j+1)}(r)}{\Delta(r)}\partial_{\bar w}G(z,\bar z,\bar w)\bigg)\\ &= (r_{z_{n+1}})^3\sum_{1\leqslant l\leqslant n+1}\frac{\Delta_{(l,j+1)}(r)}{\Delta(r)}\partial_{\bar z_l}G(z,\bar z,\bar z_{n+1}).\end{aligned}$$

At this point, Cartan's method applied to

$$w=\Theta(z_1,\dots,z_n,s_1,\dots,s_n,t)$$

along with the associated second order PDE

$$\begin{aligned}w_{z_iz_j} &= \Theta_{z_iz_j}(z_1,\dots,z_n,s_1,\dots,s_n,t)\\ &= \Theta_{z_iz_j}(z,\Lambda_1(z,w,w_z),\dots,\Lambda_n(z,w,w_z),\Pi(z,w,w_z))\\ &:= G^{ij}(z,w,w_z)\end{aligned}$$

provides the following Hachtroudi-Chern-Moser tensor

$$\begin{aligned}S^{\beta\sigma}_{\alpha\rho}|_{\mathrm{id}} &= G^{\alpha,\rho}_{w_{z_\beta},w_{z_\sigma}}-\frac{1}{n+2}\bigg(\delta^\sigma_\rho\sum_\varepsilon G^{\alpha,\varepsilon}_{w_{z_\beta},w_{z_\varepsilon}}+\delta^\sigma_\alpha\sum_\varepsilon G^{\rho,\varepsilon}_{w_{z_\beta},w_{z_\varepsilon}}+\delta^\beta_\rho\sum_\varepsilon G^{\alpha,\varepsilon}_{w_{z_\sigma},w_{z_\varepsilon}}+\delta^\beta_\alpha\sum_\varepsilon G^{\rho,\varepsilon}_{w_{z_\sigma},w_{z_\varepsilon}}\bigg)\\ &\quad+\frac{1}{(n+1)(n+2)}\sum_\varepsilon\sum_\delta(\delta^\sigma_\rho\delta^\beta_\alpha+\delta^\beta_\rho\delta^\sigma_\alpha)G^{\delta,\varepsilon}_{w_{z_\delta},w_{z_\varepsilon}}.\end{aligned}$$

whose restriction to M gives

$$\begin{aligned}\tilde S^{\beta\sigma}_{\alpha\rho}|_{\mathrm{id}} &= \sum_k\sum_l\frac{r^3_{z_{n+1}}\cdot\Delta_{(k,\sigma+1)}(r)}{\Delta(r)}\partial_{\bar z_k}\bigg[\frac{r^3_{z_{n+1}}\cdot\Delta_{(l,\beta+1)}(r)}{\Delta(r)}\partial_{\bar z_l}\bigg(\frac{H_{\alpha,\sigma}}{r^3_{z_{n+1}}}\bigg)\bigg]\\ &\quad-\frac{1}{n+2}\bigg\{\delta^\sigma_\rho\sum_\varepsilon\bigg(\sum_k\sum_l\frac{r^3_{z_{n+1}}\cdot\Delta_{(k,\varepsilon+1)}(r)}{\Delta(r)}\partial_{\bar z_k}\bigg[\frac{r^3_{z_{n+1}}\cdot\Delta_{(k,\beta+1)}(r)}{\Delta(r)}\partial_{\bar z_l}\bigg(\frac{H_{\alpha,\varepsilon}}{r^3_{z_{n+1}}}\bigg)\bigg]\bigg)\\ &\quad+\delta^\sigma_\alpha\sum_\varepsilon\bigg(\sum_k\sum_l\frac{r^3_{z_{n+1}}\cdot\Delta_{(k,\varepsilon+1)}(r)}{\Delta(r)}\partial_{\bar z_k}\bigg[\frac{r^3_{z_{n+1}}\cdot\Delta_{(k,\beta+1)}(r)}{\Delta(r)}\partial_{\bar z_l}\bigg(\frac{H_{\rho,\varepsilon}}{r^3_{z_{n+1}}}\bigg)\bigg]\bigg)\\ &\quad+\delta^\beta_\rho\sum_\varepsilon\bigg(\sum_k\sum_l\frac{r^3_{z_{n+1}}\cdot\Delta_{(k,\varepsilon+1)}(r)}{\Delta(r)}\partial_{\bar z_k}\bigg[\frac{r^3_{z_{n+1}}\cdot\Delta_{(k,\sigma+1)}(r)}{\Delta(r)}\partial_{\bar z_l}\bigg(\frac{H_{\alpha,\varepsilon}}{r^3_{z_{n+1}}}\bigg)\bigg]\bigg)\\ &\quad+\delta^\beta_\alpha\sum_\varepsilon\bigg(\sum_k\sum_l\frac{r^3_{z_{n+1}}\cdot\Delta_{(k,\varepsilon+1)}(r)}{\Delta(r)}\partial_{\bar z_k}\bigg[\frac{r^3_{z_{n+1}}\cdot\Delta_{(k,\sigma+1)}(r)}{\Delta(r)}\partial_{\bar z_l}\bigg(\frac{H_{\rho,\varepsilon}}{r^3_{z_{n+1}}}\bigg)\bigg]\bigg)\bigg\}\\ &\quad+\frac{1}{(n+1)(n+2)}\sum_\varepsilon\sum_\delta\sum_k\sum_l(\delta^\sigma_\rho\delta^\beta_\alpha+\delta^\beta_\rho\delta^\sigma_\alpha)\frac{r^3_{z_{n+1}}\cdot\Delta_{(k,\varepsilon+1)}(r)}{\Delta(r)}\partial_{\bar z_k}\bigg[\frac{r^3_{z_{n+1}}\cdot\Delta_{(k,\delta+1)}(r)}{\Delta(r)}\partial_{\bar z_l}\bigg(\frac{H_{\delta,\varepsilon}}{r^3_{z_{n+1}}}\bigg)\bigg]\bigg).\end{aligned}$$

13.11 An alternative formulation

There is in fact a more natural expression of the vector field $\partial_{w_{z_i}}$. For $1 \leqslant i, j \leqslant n$, let $\lambda_{ij} = r_{z_i\bar{z}_j}r_{w\bar{w}} - r_{z_i}r_{\bar{w}}r_{w\bar{z}_j} - r_{\bar{z}_j}r_w r_{z_i w} + r_w r_{\bar{w}} r_{z_i\bar{z}_j}$ be the components of the Levi form as previously mentioned, and let (λ_{ij}) denote the matrix with coefficients λ_{ij}. (Here the convention $w = z_{n+1}$ is adopted).

Proposition 13.18. *Let $\hat{L}(r)$ be the following matrix*

$$\hat{L}(r) := \begin{pmatrix} 0 & r_{z_1} & \cdots & r_{z_n} & r_w \\ r_{\bar{z}_1} & r_{z_1\bar{z}_1} & \cdots & r_{z_n\bar{z}_1} & r_{w\bar{z}_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{\bar{z}_n} & r_{z_1\bar{z}_n} & \cdots & r_{z_n\bar{z}_n} & r_{w\bar{z}_n} \\ r_{\bar{w}} & r_{z_1\bar{w}} & \cdots & r_{z_n\bar{w}} & r_{w\bar{w}} \end{pmatrix},$$

and let $L(r) := \det(\hat{L}(r))$. Then

$$\det(\lambda_{ij}) = -(r_w r_{\bar{w}})^{n-1} L(r).$$

Proof. The first and the important step is to observe that

$$\det(\lambda_{ij}) = \frac{-1}{r_w r_{\bar{w}}} \det \begin{pmatrix} 0 & r_{z_1} & \cdots & r_{z_n} & r_w \\ r_{\bar{z}_1} & \lambda_{11} & \cdots & \lambda_{n1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{\bar{z}_n} & \lambda_{1n} & \cdots & \lambda_{nn} & 0 \\ r_{\bar{w}} & 0 & \cdots & 0 & 0 \end{pmatrix},$$

then the rest follows from repeated use of column and row operations of the matrix to remove multiplies of r_{z_i} and $r_{\bar{z}_j}$ in λ_{ij} to obtain $r_{z_i\bar{z}_j}$. For example,

$$\begin{aligned} & \det \begin{pmatrix} 0 & r_{z_1} & \cdots & r_{z_n} & r_w \\ r_{\bar{z}_1} & \begin{smallmatrix} r_{z_1}r_{\bar{z}_1}r_{w\bar{w}} - r_{z_1}r_{\bar{w}}r_{\bar{z}_1 w} \\ -r_{\bar{z}_1}r_w r_{z_1\bar{w}} + r_w r_{\bar{w}} r_{z_1\bar{z}_1} \end{smallmatrix} & \cdots & \begin{smallmatrix} r_{z_n}r_{\bar{z}_1}r_{w\bar{w}} - r_{z_n}r_{\bar{w}}r_{\bar{z}_1 w} \\ -r_{\bar{z}_1}r_w r_{z_n\bar{w}} + r_w r_{\bar{w}} r_{z_n\bar{z}_1} \end{smallmatrix} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{\bar{z}_n} & \begin{smallmatrix} r_{z_1}r_{\bar{z}_n}r_{w\bar{w}} - r_{z_1}r_{\bar{w}}r_{\bar{z}_n w} \\ -r_{\bar{z}_n}r_w r_{z_1\bar{w}} + r_w r_{\bar{w}} r_{z_1\bar{z}_n} \end{smallmatrix} & \cdots & \begin{smallmatrix} r_{z_n}r_{\bar{z}_n}r_{w\bar{w}} - r_{z_n}r_{\bar{w}}r_{\bar{z}_n w} \\ -r_{\bar{z}_n}r_w r_{z_n\bar{w}} + r_w r_{\bar{w}} r_{z_n\bar{z}_n} \end{smallmatrix} & 0 \\ r_{\bar{w}} & 0 & \cdots & 0 & 0 \end{pmatrix} \\ = & \det \begin{pmatrix} 0 & r_{z_1} & \cdots & r_{z_n} & r_w \\ r_{\bar{z}_1} & -r_{\bar{z}_1}r_w r_{z_1\bar{w}} + r_w r_{\bar{w}} r_{z_1\bar{z}_1} & \cdots & -r_{\bar{z}_1}r_w r_{z_n\bar{w}} + r_w r_{\bar{w}} r_{z_n\bar{z}_1} & r_w r_{\bar{w}} r_{\bar{z}_1 w} - r_w r_{\bar{z}_1} r_{w\bar{w}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{\bar{z}_n} & -r_{\bar{z}_n}r_w r_{z_1\bar{w}} + r_w r_{\bar{w}} r_{z_1\bar{z}_n} & \cdots & -r_{\bar{z}_n}r_w r_{z_n\bar{w}} + r_w r_{\bar{w}} r_{z_n\bar{z}_n} & r_w r_{\bar{w}} r_{\bar{z}_n w} - r_w r_{\bar{z}_n} r_{w\bar{w}} \\ r_{\bar{w}} & 0 & \cdots & 0 & 0 \end{pmatrix} \\ = & \det \begin{pmatrix} 0 & r_{z_1} & \cdots & r_{z_n} & r_w \\ r_{\bar{z}_1} & r_w r_{\bar{w}} r_{z_1\bar{z}_1} & \cdots & r_w r_{\bar{w}} r_{z_n\bar{z}_1} & r_w r_{\bar{w}} r_{\bar{z}_1 w} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{\bar{z}_n} & r_w r_{\bar{w}} r_{z_1\bar{z}_n} & \cdots & r_w r_{\bar{w}} r_{z_n\bar{z}_n} & r_w r_{\bar{w}} r_{\bar{z}_n w} \\ r_{\bar{w}} & r_w r_{\bar{w}} r_{z_1\bar{w}} & \cdots & r_w r_{\bar{w}} r_{z_n\bar{w}} & r_w r_{\bar{w}} r_{w\bar{w}} \end{pmatrix} \\ = & r_w^n r_{\bar{w}}^n \det \begin{pmatrix} 0 & r_{z_1} & \cdots & r_{z_n} & r_w \\ r_{\bar{z}_1} & r_{z_1\bar{z}_1} & \cdots & r_{z_n\bar{z}_1} & r_{\bar{z}_1 w} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{\bar{z}_n} & r_{z_1\bar{z}_n} & \cdots & r_{z_n\bar{z}_n} & r_{\bar{z}_n w} \\ r_{\bar{w}} & r_{z_1\bar{w}} & \cdots & r_{z_n\bar{w}} & r_{w\bar{w}} \end{pmatrix}. \quad \square \end{aligned}$$

By Proposition 13.14,

$$\begin{aligned}
-(r_w r_{\bar w})^{n-1} L(r) &= \det(\lambda_{ij}) \\
&= (-1)^n (r_w)^{3n} \Theta_{\bar w}^{n-1} \Delta(\Theta|_M) \\
&= (-1)(r_w)^{3n} \frac{r_{\bar w}^{n-1}}{r_w^{n-1}} \Delta(\Theta|_M) \\
&= -(r_w)^{2n+1} r_{\bar w}^{n-1} \Delta(\Theta|_M).
\end{aligned}$$

Consequently,

$$L(r) = r_w^{n+2} \Delta(\Theta|_M).$$

13.12 A direct approach to the alternative formulation

A direct approach may be done using the following formulae

$$\begin{aligned}
0 &\equiv r_{z_i \bar z_j} + r_{z_i w} \Theta_{\bar z_j} + r_{w \bar z_j} \Theta_{z_i} + r_{ww} \Theta_{z_i} \Theta_{\bar z_j} + r_w \Theta_{z_i \bar z_j}, \\
0 &\equiv r_{z_i \bar w} + r_{z_i w} \Theta_{\bar w} + r_{w \bar w} \Theta_{z_i} + r_{ww} \Theta_{z_i} \Theta_{\bar w} + r_w \Theta_{z_i \bar w}.
\end{aligned}$$

Therefore

$$\begin{aligned}
L(r) &= \det \begin{pmatrix} 0 & r_{z_1} & \cdots & r_{z_n} & r_w \\ r_{\bar z_1} & r_{z_1 \bar z_1} & \cdots & r_{z_n \bar z_1} & r_{w \bar z_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{\bar z_n} & r_{z_1 \bar z_n} & \cdots & r_{z_n \bar z_n} & r_{w \bar z_n} \\ r_{\bar w} & r_{z_1 \bar w} & \cdots & r_{z_n \bar w} & r_{w \bar w} \end{pmatrix} \\
&= r_w^{n+2} \det \begin{pmatrix} 0 & -\Theta_{z_1} & \cdots & -\Theta_{z_n} & 1 \\ -\Theta_{\bar z_1} & \begin{matrix} -\frac{r_{z_1 w}}{r_w}\Theta_{\bar z_1} - \frac{r_{w \bar z_1}}{r_w}\Theta_{z_1} \\ -\frac{r_{ww}}{r_w}\Theta_{z_1}\Theta_{\bar z_1} - \Theta_{z_1 \bar z_1} \end{matrix} & \cdots & \begin{matrix} -\frac{r_{z_n w}}{r_w}\Theta_{\bar z_1} - \frac{r_{w \bar z_1}}{r_w}\Theta_{z_n} \\ -\frac{r_{ww}}{r_w}\Theta_{\bar z_1}\Theta_{z_n} - \Theta_{z_n \bar z_1} \end{matrix} & \frac{r_{w \bar z_1}}{r_w} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\Theta_{\bar z_n} & \begin{matrix} -\frac{r_{z_1 w}}{r_w}\Theta_{\bar z_n} - \frac{r_{w \bar z_n}}{r_w}\Theta_{z_1} \\ -\frac{r_{ww}}{r_w}\Theta_{z_1}\Theta_{\bar z_n} - \Theta_{z_1 \bar z_n} \end{matrix} & \cdots & \begin{matrix} -\frac{r_{z_n w}}{r_w}\Theta_{\bar z_n} - \frac{r_{w \bar z_n}}{r_w}\Theta_{z_n} \\ -\frac{r_{ww}}{r_w}\Theta_{z_n}\Theta_{\bar z_n} - \Theta_{z_n \bar z_n} \end{matrix} & \frac{r_{w \bar z_n}}{r_w} \\ -\Theta_{\bar w} & \begin{matrix} -\frac{r_{z_1 w}}{r_w}\Theta_{\bar w} - \frac{r_{w \bar w}}{r_w}\Theta_{z_1} \\ -\frac{r_{ww}}{r_w}\Theta_{z_1}\Theta_{\bar w} - \Theta_{z_1 \bar w} \end{matrix} & \cdots & \begin{matrix} -\frac{r_{z_n w}}{r_w}\Theta_{\bar w} - \frac{r_{w \bar w}}{r_w}\Theta_{z_n} \\ -\frac{r_{ww}}{r_w}\Theta_{z_n}\Theta_{\bar w} - \Theta_{z_n \bar w} \end{matrix} & \frac{r_{w \bar w}}{r_w} \end{pmatrix} \\
&= r_w^{n+2} \det \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ -\Theta_{\bar z_1} & \begin{matrix} -\frac{r_{z_1 w}}{r_w}\Theta_{\bar z_1} - \frac{r_{ww}}{r_w}\Theta_{z_1}\Theta_{\bar z_1} \\ -\Theta_{z_1 \bar z_1} \end{matrix} & \cdots & \begin{matrix} -\frac{r_{z_n w}}{r_w}\Theta_{\bar z_1} - \frac{r_{ww}}{r_w}\Theta_{\bar z_1}\Theta_{z_n} \\ -\Theta_{z_n}\Theta_{\bar z_1} - \Theta_{z_n \bar z_1} \end{matrix} & \frac{r_{w \bar z_1}}{r_w} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\Theta_{\bar z_n} & \begin{matrix} -\frac{r_{z_1 w}}{r_w}\Theta_{\bar z_n} - \frac{r_{ww}}{r_w}\Theta_{z_1}\Theta_{\bar z_n} \\ -\Theta_{z_1 \bar z_n} \end{matrix} & \cdots & \begin{matrix} -\frac{r_{z_n w}}{r_w}\Theta_{\bar z_n} - \frac{r_{ww}}{r_w}\Theta_{z_n}\Theta_{\bar z_n} \\ -\Theta_{z_n \bar z_n} \end{matrix} & \frac{r_{w \bar z_n}}{r_w} \\ -\Theta_{\bar w} & \begin{matrix} -\frac{r_{z_1 w}}{r_w}\Theta_{\bar w} - \frac{r_{ww}}{r_w}\Theta_{z_1}\Theta_{\bar w} \\ -\Theta_{z_1 \bar w} \end{matrix} & \cdots & \begin{matrix} -\frac{r_{z_n w}}{r_w}\Theta_{\bar w} - \frac{r_{ww}}{r_w}\Theta_{z_n}\Theta_{\bar w} \\ -\Theta_{z_n \bar w} \end{matrix} & \frac{r_{w \bar w}}{r_w} \end{pmatrix} \\
&= r_w^{n+2} \det \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ -\Theta_{\bar z_1} & -\Theta_{z_1 \bar z_1} & \cdots & -\Theta_{z_n \bar z_1} & \frac{r_{w \bar z_1}}{r_w} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\Theta_{\bar z_n} & \Theta_{z_1 \bar z_n} & \cdots & -\Theta_{z_n \bar z_n} & \frac{r_{w \bar z_n}}{r_w} \\ -\Theta_{\bar w} & -\Theta_{z_1 \bar w} & \cdots & -\Theta_{z_n \bar w} & \frac{r_{w \bar w}}{r_w} \end{pmatrix} \\
&= r_w^{n+2} (-1)^{n+3} (-1)^{n+1} \Delta(\Theta|_M) \\
&= r_w^{n+2} \Delta(\Theta|_M).
\end{aligned}$$

On the other hand, for $1 \leqslant i \leqslant n+1$ and $1 \leqslant l \leqslant n+1$, let $\hat L_{il}(r)$ be the matrix with $i+1$-th row and the l-th column deleted, and let $L_{i,l}(r)$ be its determinant. By a slight modification of the calculations above, directly the following proposition holds:

Proposition 13.19. *The following identity holds for* $1 \leqslant i \leqslant n+1$ *and* $2 \leqslant l \leqslant n+1$*:*

$$L_{i,l}(r) = r_w^{n+1}\mathsf{det}\big(\mathsf{j}(\Theta|_M)_{(l,i)}\big),$$

This follows immediately from inspecting the matrix in the calculations of the previous page. Therefore,

$$(-1)^{i+l}L_{i,l}(r) = r_w^{n+1}(-1)^{l+i}\mathsf{det}\big(\mathsf{j}(\Theta|_M)_{(l,i)}\big) = r_w^{n+1}\Delta_{(i,l)}(\Theta|_M).$$

Hence,

$$\frac{r_w^{n+1}\Delta_{(i,l)}(\Theta|_M)}{r_w^{n+2}\Delta(\Theta|_M)} = \frac{(-1)^{i+l}L_{i,l}(r)}{L(r)},$$

or

$$\frac{\Delta_{(i,l)}(\Theta|_M)}{\Delta(\Theta|_M)} = r_w\frac{(-1)^{i+l}L_{i,l}(r)}{L(r)}.$$

Therefore, the vector fields $\partial_{w_{z_k}}$ are given by

$$\begin{aligned}\partial_{w_{z_k}} &= \sum_{1\leqslant l\leqslant n}\frac{\Delta_{(l,k+1)}(\Theta|_M)}{\Delta(\Theta|_M)}\partial_{\bar{z}_l} + \frac{\Delta_{(n+1,k+1)}(\Theta|_M)}{\Delta(\Theta|_M)}\partial_{\bar{w}}\\ &= \sum_{1\leqslant l\leqslant n} r_w\frac{(-1)^{l+k+1}L_{l,k+1}(r)}{L(r)}\partial_{\bar{z}_l} + r_w\frac{(-1)^{n+1+k+1}L_{n+1,k+1}(r)}{L(r)}\partial_{\bar{w}}.\end{aligned}$$

Theorem 13.20. *For* $n \geqslant 2$*,* $1 \leqslant k_1, k_2 \leqslant n$ *and* $1 \leqslant i_1, i_2 \leqslant n$*, the CR-umbilical locus is the zero set*

$$\begin{aligned}0 &= \partial_{w_{z_{i_1}}}\left(\partial_{w_{z_{i_2}}}\frac{H_{k_1,k_2}}{r_w^3}\right)\\ &\quad -\frac{1}{n+2}\sum_{l=1}^{n}\left(\delta_{k_1,i_1}\partial_{w_{z_l}}\left(\partial_{w_{z_{i_2}}}\frac{H_{l,k_2}}{r_w^3}\right) + \delta_{k_1,i_2}\partial_{w_{z_{i_1}}}\left(\partial_{w_{z_l}}\frac{H_{l,k_2}}{r_w^3}\right)\right.\\ &\qquad\qquad \left. +\delta_{k_2,i_1}\partial_{w_{z_l}}\left(\partial_{w_{z_{i_2}}}\frac{H_{k_1,l}}{r_w^3}\right) + \delta_{k_2,i_2}\partial_{w_{z_{i_1}}}\left(\partial_{w_{z_l}}\frac{H_{k_1,l}}{r_w^3}\right)\right)\\ &\quad +\frac{1}{(n+1)(n+2)}\big(\delta_{k_1,i_1}\delta_{k_2,i_2}+\delta_{k_2,i_1}\delta_{k_1 i_2}\big)\sum_{l_1=1}^{n}\sum_{l_2=1}^{n}\partial_{w_{l_1}}\left(\partial_{w_{l_2}}\frac{H_{l_1,l_2}}{r_w^3}\right).\end{aligned}$$

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université PARIS-SACLAY | École doctorale de mathématiques Hadamard (EDMH)

Titre: Calculs explicites pour la terminaison effective de l'algorithme de Kohn d'après Siu, et tenseurs de Hachtroudi-Chern-Moser en géométrie CR.

Résumé: La première partie présente des calculs explicites de terminaison effective de l'algorithme de Kohn proposée par Siu. Dans la deuxième partie, nous étudions la géométrie des hypersurfaces réelles dans $\mathbb{C}^n$, et nous calculons des invariants explicites avec la méthode d'équivalences de Cartan pour déterminer les lieux CR-ombilics.

Mots-clefs: L'algorithme de Kohn, analyse complexe à plusieurs variables, géométrie CR, la méthode d'équivalences de Cartan, CR-ombilics, calcul tensoriel

Title: Explicit Calculations of Siu's Effective Termination of Kohn's Algorithm, and the Hachtroudi-Chern-Moser Tensors in CR Geometry.

Abstract: The first part of the thesis consists of calculations around Siu's effective termination of Kohn's algorithm. The second part of the thesis studies the CR real hypersurfaces in complex spaces and calculates various explicit invariants using Cartan's equivalence method to study CR-umbilical points.

Key words: Kohn's algorithm, several complex variables, CR geometry, Cartan's equivalence method, CR umbilics, tensor calculus.