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Analyse de problèmes inverses et directs en théorie du contrôle

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ÉCOLE DOCTORALE MER ET SCIENCES**Laboratoire des Sciences de l'Information et des Systèmes****THÈSE**

présentée par :

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pour obtenir le grade de Docteur en Automatique, signal, productique, robotique

Spécialité : Automatique

**Analyse de problèmes inverses et directs en
théorie du contrôle****THÈSE dirigée par :**

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I have not failed. I've just found 10,000 ways that won't work.

THOMAS A. EDISON

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Table des matières

Introduction générale	1
1. Contexte	1
2. Structure du manuscrit	2
Chapitre 1 : Synthèse temps-minimum pour drone	5
1. Introduction	5
1.1. Description du problème	6
1.2. Réduction du système dynamique	7
2. Préliminaires et outils théoriques	8
2.1. Contrôlabilité et existence de solutions	8
2.2. Principe du maximum de Pontryagin	9
3. Résultats et démonstrations	12
3.1. Construction de la synthèse	12
3.2. Résultats	43
4. Conclusion et discussions	48
Chapitre 2 : Généricité en contrôle optimal inverse	49
1. Introduction	49
1.1. État de l art	50
1.2. Description du problème	51
1.3. Résultat	52
2. Préliminaires et outils théoriques	52
2.1. Produit tensoriel	52
2.2. Différentielle et produit tensoriel	53
2.3. k -jets	53
2.4. Transversalité et théorie de Thom	55

2.5. Méthode des caractéristiques	56
3. Démonstration du résultat	56
4. Discussions	87
4.1. Conclusion	87
4.2. Difficultés par rapport au cas un contrôle	87
4.3. Ouvertures	87
Chapitre 3 : Stabilisation en des points inobservables	89
1. Introduction	89
1.1. Problématique	89
1.2. Exemple considéré	91
2. Préliminaires et outils théoriques	92
2.1. Principe d'invariance de LaSalle	92
2.2. Théorie de la variété centrale	93
3. Contribution scientifique	95
3.1. Résultats	95
3.2. Démonstration	96
4. Conclusion et discussions	108
Conclusion générale	109

Notations

\mathbb{R}	ensemble des nombres réels
\mathbb{R}^n	ensemble des n -uplets ordonnés de nombres réels
$\langle x, y \rangle$	produit scalaire entre deux vecteurs x et y de \mathbb{R}^n ou crochet de dualité
$\ x\ $	norme euclidienne du vecteur x
M'	transposée d une matrice M
$\overset{\circ}{A}$	intérieur d une partie A d un espace topologique
\bar{A}	adhérence d une partie A d un espace topologique
$E \simeq F$	les espaces E et F sont isomorphes
E^*	espace dual de l espace vectoriel E
\dot{x}	dérivée par rapport au temps de la fonction $x(\cdot)$
$Df(x)$	différentielle de l application f au point x

$Df(x)[v]$	différentielle de f au point x appliquée au vecteur v
$D^k f(x)$	différentielle à l'ordre k de l'application f au point x
$D^k f(x)[w]$	différentielle à l'ordre k de f au point x appliquée au vecteur $w = (v_1, \dots, v_k) \in (\mathbb{R}^n)^k$
$\frac{\partial f}{\partial x}(a)$	dérivée partielle de f par rapport à x évaluée en a
$[F, G]$	crochet de Lie de deux champs de vecteurs F et G
C^k	ensemble des fonctions dérivables k fois et dont la $k^{\text{ème}}$ dérivée est continue
$T_x M$	espace tangent en x à la variété différentielle M
TM	fibré tangent à M
\mathbb{L}^∞	ensemble des fonctions mesurables et essentiellement bornées

Introduction générale

1. Contexte

La théorie du contrôle concerne l'étude mathématique des systèmes dynamiques paramétrés. La première étude ayant mené à cette théorie fût celle du célèbre régulateur à boules de J. Watt datant de 1769. Ce n'est cependant qu'en 1868 que la première étude mathématique rigoureuse des régulateurs fût publiée par J. C. Maxwell (voir Maxwell [59]).

Aujourd'hui, la théorie du contrôle est omniprésente dans l'industrie de l'automobile, l'aéronautique ou encore la domotique. Un exemple d'application de la théorie du contrôle est celui du filtre de Kalman qui constitua une grande avancée pour le programme de conquête spatiale Apollo (voir McGee et Schmidt [60]). Pour le grand public, les applications les plus connues sont par exemples les drones de loisir ou encore les voitures autonomes qui connaissent depuis plusieurs années un succès croissant. En marge de cet engouement, de nouvelles applications prometteuses voient le jour dans le domaine des neurosciences ou de la mécanique quantique.

De manière générale, nous avons l'habitude de résoudre des *problèmes directs*, c'est à dire que nous recherchons des effets à partir de causes données. Ainsi, l'étude de l'état futur d'un système physique à partir de son état actuel et éventuellement des actions que nous lui appliquons constitue un problème direct.

Un *problème inverse* consiste à retrouver les causes à partir de l'observation des effets. L'étude de l'état d'un système physique à partir d'une mesure partielle de son état constitue un problème inverse. Cette approche inverse

est très largement utilisée, par exemple en tomographie ou en théorie du contrôle avec l'identification de paramètres. Cependant, on imagine aisément que plusieurs causes peuvent produire le même effet et c'est à dire qu'il n'existe (en général) pas de solution unique à un tel problème. On dira dans ce cas que le problème est *mal posé*. Plus précisément, en se basant sur la définition de Hadamard (Hadamard [34]) on dira qu'un problème est bien posé si :

- la solution existe ;
- elle est unique ;
- elle dépend continûment des données.

La résolution d'un problème inverse nécessite une très bonne connaissance du problème direct, de ses spécificités et des outils nécessaires à sa résolution.

2. Structure du manuscrit

Cette thèse aborde trois problèmes inverses ou directs de la théorie du contrôle. Chaque étude constitue un chapitre de ce manuscrit et est articulée autour d'un article publié ou soumis dans un journal scientifique. Les contextes et motivations de chaque problème sont détaillés dans les chapitres correspondants.

Le premier chapitre concerne l'étude d'un problème de contrôle optimal. Il s'agit d'une approche directe. L'objectif est de calculer un ensemble de trajectoires optimales en temps pour un modèle cinématique de drone. Ce chapitre permettra par ailleurs d'introduire et d'utiliser les outils du contrôle optimal qui seront également utilisés dans le chapitre suivant.

Le deuxième chapitre traite d'une problématique inverse de la précédente puisqu'il apporte des résultats et des réflexions à propos d'un problème de contrôle optimal inverse. Nous supposons dans cette étude que nous avons à notre disposition des contrôles optimaux et leurs trajectoires associées. L'objectif est de retrouver le critère qui est minimisé par ces trajectoires. Nous donnons dans ce chapitre un résultat général en nous basant sur la théorie de la transversalité de Thom, que nous rappelons préalablement.

Enfin, le troisième chapitre aborde la stabilisation par retour de sortie dans le cas où le système dynamique étudié n'est pas uniformément observable. On parle dans ce cas de problème inverse puisqu'il s'agit de retrouver l'état d'un système à partir de mesures partielles. L'étude est réalisée sur un exemple simple provenant de la mécanique quantique. Nous donnons à partir

des résultats obtenus pour cet exemple une méthode constructive générale de retour de sortie stabilisant semi-globalement. Comme dans les chapitres précédents, des outils permettant l'étude de la stabilité des systèmes dynamiques sont rappelés.

Synthèse temps-minimum pour drone

Ce chapitre concerne l'étude d'un problème de contrôle optimal dans lequel le critère à minimiser est le temps. Après une introduction du problème, les résultats seront donnés sous la forme d'un article publié dans un journal scientifique. À la suite de cette publication, il y aura une discussion des résultats et une mise en parallèle avec des résultats similaires existants dans la littérature. Enfin des réflexions sur de possibles suites basées sur ces travaux seront exposées.

1. Introduction

La théorie du contrôle optimal permet de déterminer une loi de commande à appliquer à un système dynamique afin de minimiser un critère donné. Les deux principaux outils sont le principe du maximum de Pontryagin (voir par exemple Agrachev et Sachkov [2]; Trélat [73]; Vinter [75]) et l'équation de Hamilton-Jacobi-Bellman (voir par exemple Trélat [73]; Vinter [75]). Par la suite et après avoir fait un rappel, nous utiliserons le principe du maximum de Pontryagin.

Les travaux réalisés reprennent et approfondissent certains résultats énoncés dans Boscaïn et Piccoli [16]. Cet ouvrage traite de synthèse optimale pour les systèmes à un contrôle en dimension deux. Nous utilisons et étendons certains de ces résultats pour les systèmes à deux contrôles. Nos travaux suivent également une approche similaire à Boscaïn et Piccoli [17] qui traite également d'un problème de contrôle optimal en temps pour un système quantique à deux niveaux.

1.1. Description du problème

L'objectif de ce chapitre est de fournir une synthèse temps minimum (c'est à dire un ensemble de trajectoires temps-optimales) pour un modèle cinématique de drone. Plus précisément, il s'agit de déterminer, pour toute position initiale, comment amener un drone volant à altitude constante sur une trajectoire circulaire de rayon de courbure minimum. Le drone peut-être décrit par le modèle cinématique suivant (voir par exemple Bhatia *et al.* [12]; Ding *et al.* [25]; Walsh *et al.* [76])

$$\begin{cases} \dot{x} = v \cos \theta \\ \dot{y} = v \sin \theta \\ \dot{\theta} = u, \end{cases} \quad (1.1)$$

où $(x, y) \in \mathbb{R}^2$ désigne la position du drone, $\theta \in \mathbb{R}$ son angle de lacet et $(u, v) \in \mathcal{U} = [-u_{\max}, u_{\max}] \times [v_{\min}, v_{\max}]$ avec $u_{\max}, v_{\min}, v_{\max} > 0$ désigne le contrôle.

L'objectif est d'atteindre et de parcourir dans le sens trigonométrique un cercle de rayon de courbure minimum centré en l'origine. Dans les coordonnées (x, y, θ) , cette cible est donnée par

$$\mathcal{C} = \{(x, y, \theta) \mid x = r_{\min} \sin \theta, y = r_{\min} \cos \theta\}.$$

Le problème de contrôle optimal est donc le suivant.

(P₀) Pour toute condition initiale (x_0, y_0, θ_0) , déterminer le contrôle et la trajectoire solution de (1.1) joignant (x_0, y_0, θ_0) et \mathcal{C} en temps minimum.

Notons qu'il existe de nombreuses études sur le modèle de Dubins [27], c'est à dire dans le cas où la vitesse linéaire v est constante. Citons par exemple Souères *et al.* [66]; Souères *et al.* [67] où les auteurs étudient les trajectoires les plus courtes joignant une ligne droite ou encore Balluchi *et al.* [7], qui concerne la stabilité et la robustesse d'une synthèse optimale obtenue pour le suivi de trajectoires. En n les travaux de Chitsaz et LaValle [22] traitent du calcul de trajectoires temps-minimales pour un modèle de Dubins étendu avec l'altitude dans le but d'aller en un point fixé avec une orientation donnée.

Dans le cas où la vitesse linéaire n'est pas constante, citons par exemple Sussmann et Tang [70] où les trajectoires les plus courtes sont étudiées pour $v \in [-1, 1]$. Le fait que la vitesse linéaire puisse être négative ou nul constitue naturellement une difficulté majeure. Notons que cet article démontre également l'intérêt du contrôle géométrique que nous utilisons par la suite.

Concernant la construction de synthèses optimales, nos travaux reprennent les méthodologies de Boscain *et al.* [14] et Bressan et Piccoli [18]. Ces articles concernent en particulier la synthèse temps-minimum de systèmes de dimension deux.

Notre étude s'inscrit dans la continuité de l'article Maillot *et al.* [56], où les auteurs donnent la synthèse temps minimum pour le système (1.1) dans le cas où la vitesse linéaire v est constante. Notre objectif est d'étudier l'influence d'une vitesse pouvant varier sur les solutions de ce problème. Comme nous le verrons, pour atteindre la cible le plus rapidement possible il ne faut pas forcément y aller avec la vitesse maximale.

1.2. Réduction du système dynamique

Comme dans Maillot *et al.* [56], l'invariance par rotation du problème permet de réduire l'étude à un système dynamique en dimension deux. Ensuite, la transformation

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

permet de découpler la variable θ et après une projection parallèlement à θ réduit la cible \mathcal{C} au point $(0, r_{\min})$. De manière équivalente, le problème (P₀) peut être reformulé :

(P₁) Pour toute condition initiale $(\tilde{x}_0, \tilde{y}_0)$, déterminer le contrôle et la trajectoire solution de

$$\begin{cases} \dot{\tilde{x}} = v + u\tilde{y} \\ \dot{\tilde{y}} = -u\tilde{x} \end{cases}, \quad (u, v) \in \mathcal{U}$$

et joignant $(\tilde{x}_0, \tilde{y}_0)$ et $(0, r_{\min})$ en temps minimum.

Puisque la construction de la synthèse temps-minimum consiste à chercher toutes les trajectoires optimales allant vers la cible, il est plus simple d'étudier le problème en temps rétrograde et chercher toutes les trajectoires optimales issues de la cible. Le problème à résoudre devient alors :

(P₂) Pour toute condition finale $(\tilde{x}_f, \tilde{y}_f)$, déterminer le contrôle et la trajectoire solution de

$$\begin{cases} \dot{\tilde{x}} = -v - u\tilde{y} \\ \dot{\tilde{y}} = u\tilde{x} \end{cases}, \quad (u, v) \in \mathcal{U}$$

et joignant $(0, r_{\min})$ et $(\tilde{x}_f, \tilde{y}_f)$ en temps minimum.

2. Préliminaires et outils théoriques

La résolution d'un problème de contrôle optimal se décompose en deux étapes. Il faut dans un premier temps vérifier qu'il existe une trajectoire joignant le point initial et le point final et que le problème admette une solution. Dans un second temps, il faut chercher parmi les trajectoires joignant les deux points, celles qui minimisent le coût.

2.1. Contrôlabilité et existence de solutions

Les systèmes étudiés dans ce chapitre et dans le chapitre 2 étant affines en leurs contrôles, nous bornerons les rappels préliminaires à cette classe de systèmes. Considérons le système dynamique autonome et avec dérive

$$\dot{x} = f_0(x) + u_1 f_1(x) + \cdots + u_m f_m(x), \quad (1.2)$$

où $x \in \mathbb{R}^n$ est l'état, f_0, \dots, f_m sont des champs de vecteurs de classe C^∞ et le contrôle $u = (u_0, \dots, u_m)$ est une fonction \mathbb{L}^∞ de $[0, T]$ dans $\mathcal{U} \subset \mathbb{R}^m$. Soit $C(u)$ le coût associé à la trajectoire $\gamma(\cdot)$ solution de (1.2) défini par

$$C(u) = \int_0^T L(\gamma(\tau), u(\tau)) d\tau,$$

où le critère $L : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ est de classe C^1 .

Le problème de contrôle optimal est donc le suivant.

(P_L) Déterminer les trajectoires $\gamma(\cdot)$ solutions de (1.2) joignant x_0 et x_1 et minimisant le coût $C(u)$.

On dit que x_1 est *accessible* depuis x_0 s'il existe une trajectoire $\gamma : [0, T] \rightarrow \mathbb{R}^n$ telle que $\gamma(0) = x_0$ et $\gamma(T) = x_1$. Le système (1.2) est dit *contrôlable* si pour tout x_0 et tout x_1 , x_1 est accessible depuis x_0 . Le théorème suivant donne un critère classique de contrôlabilité (voir par exemple Agrachev et Sachkov [2]; Hermann et Krener [35]; Jean [44]).

Théorème 1.1 (Jean [44]). *Supposons que l'ensemble de contrôle soit non borné ($\mathcal{U} = \mathbb{R}^m$) et que l'algèbre de Lie engendrée par les champs de vecteurs f_i , $i = 1, \dots, m$ évaluée en tout point soit de dimension n . Alors le système (1.2) est contrôlable.*

Remarquons cependant que le système (1.1) avec dérive ne satisfait pas les hypothèses du théorème 1.1 puisque l'ensemble de contrôle est borné. Dans ce cas, nous avons le théorème suivant.

Théorème 1.2 (Jean [44]). *Supposons que l'enveloppe convexe de \mathcal{U} contient un voisinage de l'origine dans \mathbb{R}^m , que l'algèbre de Lie engendrée par les champs de vecteurs $f_i, i = 0, \dots, m$ évaluée en tout point soit de dimension n et que la famille de champ de vecteurs $\mathcal{F} = \{f_0(x) + u_1 f_1(x) + \dots + u_m f_m(x), u \in U\}$ contient un champ récurrent. Alors le système (1.2) est contrôlable.*

Les deux théorèmes précédents permettent de déterminer s'il existe des trajectoires joignant x_0 et x_1 , pour tout x_0 et x_1 mais ne garantissent pas que (P_L) admet une solution. Pour cela, le théorème de Filippov (voir e.g. Agrachev et Sachkov [2]; Boscaïn et Piccoli [16]) implique le théorème suivant.

Théorème 1.3. *Supposons que les champs de vecteurs f_i sont bornés, \mathcal{U} est compact et l'ensemble $\{(f_0(x) + u_1 f_1(x) + \dots + u_m f_m(x), L(x, u)), u \in U\}$ est convexe. De plus, si $L \geq \alpha$, où $\alpha \in \mathbb{R}_{>0}$ et si x_1 est accessible depuis x_0 alors le problème (P_L) admet une solution.*

Exemple 1.4. Le modèle cinématique du drone (1.1) peut s'écrire sous la forme (1.2), où

$$f_0 = \beta \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}, \quad f_1 = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$\beta = v_{\min} + \frac{v_{\max} - v_{\min}}{2}$ et $\mathcal{U} = [-u_{\max}, u_{\max}] \times [-\frac{v_{\max} - v_{\min}}{2}, \frac{v_{\max} - v_{\min}}{2}]$. L'ensemble de contrôle \mathcal{U} contient un voisinage de l'origine. De plus, le crochet de Lie de f_1 et f_2 est

$$[f_1, f_2] = \begin{pmatrix} \sin \theta \\ -\cos \theta \\ 0 \end{pmatrix}$$

et pour tout $(x, y, \theta) \in \mathbb{R}^2 \times S^1$ l'algèbre de Lie engendrée par f_1 et f_2 est de dimension 3. En n, pour tout contrôle $u \neq 0$, les champs de vecteurs associés sont récurrents puisque les trajectoires sont périodiques. On déduit du théorème 1.2 que le système (1.1) est contrôlable.

Par ailleurs, les hypothèses du théorème (1.3) étant vérifiées pour le critère $L(x, u) = 1$, le problème de contrôle optimal (P_0) admet une solution.

2.2. Principe du maximum de Pontryagin

Une fois l'existence de solution au problème de contrôle optimal déterminée, l'étape suivante est de calculer les trajectoires optimales. Pour cela, le

principe du maximum de Pontryagin donne des conditions nécessaires d'optimalité des trajectoires. Soit H le *hamiltonien* du système (1.2) défini pour tout $(x, p, u, \lambda) \in \mathbb{R}^n \times (\mathbb{R}^n)^* \times U \times \mathbb{R}$ par

$$H(x, p, u, \lambda) = \langle p, f_0(x) + \sum_{i=1}^m u_i f_i(x) \rangle + \lambda L(x, u).$$

Théorème 1.5 (Pontryagin). *Si le contrôle admissible $u(\cdot)$ est optimal sur $[0, T]$ et $x(\cdot)$ est la trajectoire correspondant à u alors il existe un vecteur adjoint absolument continu $p : [0, T] \rightarrow (\mathbb{R}^n)^*$ et un réel $\lambda \leq 0$ tels que $(p(t), \lambda) \neq (0, 0)$ pour tout $t \in [0, T]$ et pour presque tout $t \in [0, T]$*

$$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t), u(t), \lambda) \quad (1.3)$$

$$\dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t), u(t), \lambda) \quad (1.4)$$

$$H(x(t), p(t), u(t), \lambda) = \max_{v \in U} H(x(t), p(t), v, \lambda). \quad (1.5)$$

De plus, si le temps final T est libre,

$$H(x(t), p(t), u(t), \lambda) \equiv 0.$$

On dit qu'un quadruplet $(x(\cdot), p(\cdot), u(\cdot), \lambda)$ est une *extrémale* du problème de contrôle optimal s'il satisfait les équations (1.3), (1.4) et (1.5). L'extrémale est dite *anormale* si $\lambda = 0$ et *normale* sinon.

Exemple 1.6. Supposons que l'on souhaite acheminer un train d'une gare à une autre le plus rapidement possible. L'état du système est sa position sur les rails $x_1 \in \mathbb{R}$ et le contrôle est la force motrice $u \in [-1, 1]$. La position du train satisfait l'équation $\ddot{x}_1 = u$, ou encore sous la forme

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u. \end{cases} \quad (1.6)$$

Le problème de contrôle optimal s'écrit :

(P_{ex}) Déterminer les trajectoires $x(\cdot)$ solutions de (1.6) joignant $x_0 = (0, 0)$ et $x_1 = (x_f, 0)$ le plus rapidement possible.

Les trajectoires les plus rapides sont celles qui minimisent le coût $C(u) = T$, c'est à dire $L(x, u) \equiv 1$. Il est évident que ce système est contrôlable et que le problème de contrôle optimal admet une solution.

Le hamiltonien du système est $H(x, p, u, \lambda) = p_1x_2 + p_2u + \lambda$. Soit $u(\cdot)$ un contrôle optimal sur $[0, T]$ et $x(\cdot)$ sa trajectoire associée. D après le PMP,

$$\begin{aligned} \dot{p}_1(t) &= 0 \\ \dot{p}_2(t) &= -p_1(t) \\ p_1(t)x_2(t) + p_2(t)u(t) + \lambda &= \max_{v \in U} p_1(t)x_2(t) + p_2(t)v. \end{aligned}$$

De plus, le temps final T n étant pas fixé, le hamiltonien est nul le long des trajectoires, c est à dire

$$p_1(t)x_2(t) + p_2(t)u(t) \equiv -\lambda.$$

On en déduit que $p_1(t) = p_{10}$, $p_2(t) = -p_{10}t + p_{20}$ et d après la condition de maximisation $u(t) = \text{sgn}(p_2(t))$. On obtient donc

$$p_{10}x_2(t) + | -p_{10}t + p_{20} | \equiv -\lambda.$$

De plus, puisque $x_2(0) = x_2(T) = 0$,

$$|p_{20}| = -\lambda \quad \text{et} \quad | -p_{10}T + p_{20} | = -\lambda.$$

Remarquons qu il n existe pas d extrémale anormale puisque la condition $\lambda = 0$ implique $p_1(t) = p_2(t) \equiv 0$, ce qui contredit le PMP. On peut donc normaliser le hamiltonien et supposer que $\lambda = -1$ et $p_{20} = \varepsilon = \pm 1$. On en déduit que $p_{10} = 2\varepsilon/T$ et $p_2(t) = (1 - 2t/T)\varepsilon$. Ainsi, le contrôle optimal est

$$u(t) = \begin{cases} \varepsilon & \text{si } t < T/2 \\ -\varepsilon & \text{si } t \geq T/2. \end{cases}$$

En intégrant deux fois le contrôle, nous obtenons

$$\begin{aligned} x_2(t) &= \begin{cases} \varepsilon t & \text{si } t < T/2 \\ \varepsilon T/2 - \varepsilon t & \text{si } t \geq T/2, \end{cases} \\ x_1(t) &= \begin{cases} \varepsilon t^2/2 & \text{si } t < T/2 \\ \varepsilon T^2/8 + \varepsilon T/2t - \varepsilon t^2/2 & \text{si } t \geq T/2. \end{cases} \end{aligned}$$

En particulier, la condition finale $x_1(T) = x_f$ donne $p_{20} = \text{sgn}(x_f)$ et $T = \sqrt{2|x_f|}$. On en déduit que pour aller de l origine au point x_f le plus rapidement possible il faut appliquer le contrôle $u = \text{sgn}(x_f)$ pendant une durée $\sqrt{2|x_f|}/2$ puis $u = -\text{sgn}(x_f)$ pendant une durée $\sqrt{2|x_f|}/2$.

Nous avons donc vu comment traiter un problème de contrôle optimal. Une synthèse consiste à calculer un ensemble de trajectoires optimales, c est à dire à résoudre un ensemble de problèmes de contrôle optimal. L étude sur le cas du drone est présenté dans la section suivante.

3. Résultats et démonstrations

3.1. Construction de la synthèse

L'article présenté dans cette section a été publié dans le journal *Mathematical Control and Related Fields* (Lagache *et al.* [49]) et les résultats ont été exposés à Osaka en 2015 à l'occasion de la *Conference on Decision and Control (CDC)*. La synthèse temps minimum ainsi que l'algorithme pour la construire y sont détaillés.

MINIMAL TIME SYNTHESIS FOR A KINEMATIC DRONE MODEL

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ABSTRACT. In this paper, we consider a (rough) kinematic model for a UAV flying at constant altitude moving forward with positive lower and upper bounded linear velocities and positive minimum turning radius. For this model, we consider the problem of minimizing the time travelled by the UAV starting from a general configuration to connect a specified target being a fixed circle of minimum turning radius. The time-optimal synthesis is presented as a partition of the state space which defines a unique optimal path such that the target can be reached optimally.

1. Introduction. The purpose of this study is to determine the fastest way (in time) to steer a kinematic UAV (or drone) flying at a constant altitude from any starting point to a fixed horizontal circle of minimum turning radius.

The problem is only described from a kinematic point of view. In particular, we do not take into account the inertia of the drone. We consider that the drone velocities are controlled parameters. In consequence, they are allowed to vary arbitrarily fast.

From the kinematic point of view, a rough drone that flies at a constant altitude is governed by the standard Dubins equations (see e.g. [2], [9]):

$$\begin{cases} \dot{x} = v \cos \theta \\ \dot{y} = v \sin \theta \\ \dot{\theta} = u, \end{cases} \quad (1.1)$$

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with $(x, y, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1$ being the state (where $(x, y) \in \mathbb{R}^2$ is the UAV's coordinates in the constant altitude plane, and θ the yaw angle), and $u \in [-u_{\max}, u_{\max}]$, $v \in [v_{\min}, v_{\max}]$ being the control variables. Note that the yaw angle θ is the angle between the aircraft direction and the x -axis.

In this paper, only system (1.1) is studied. Despite the fact that our motivation is UAVs, one could apply our results to other control problems modelled by system (1.1).

These equations express that the drone moves on a perfect plane (perfect constant altitude) in the direction of its velocity vector, and is able to turn right and left.

We assume that the controls on the drone kinematics are its angular velocity u and its linear velocity v .

Moreover, we make the assumptions that the linear velocity v has a positive lower bound v_{\min} and a positive upper bound v_{\max} and that the time derivative u of the drone yaw angle is constrained by an upper positive bound u_{\max} .

The above assumptions imply in particular that no stationary or quasi-stationary flights are allowed and that the drone is kinematically restricted by its minimum turning radius $r_{\min} = v_{\min}/u_{\max} > 0$.

A similar problem with a constant linear velocity has already been addressed in [11]. The purpose of this paper is to study the influence of a non-constant linear velocity. A preliminary version of this work as been published in [10].

2. Minimum time problem under consideration.

2.1. Optimal control problem. We aim to steer a UAV driven by system (1.1) in minimum time from any given initial position point to the target manifold \mathcal{C} which is defined to be the counterclockwise-oriented circular trajectory of minimum turning radius centered at the origin. In the (x, y, θ) -coordinates, \mathcal{C} is given by

$$\mathcal{C} = \{(x, y, \theta) \mid x = r_{\min} \sin \theta, y = -r_{\min} \cos \theta\}.$$

More precisely, we consider the following optimal control problem:

(P₀) For every $(x_0, y_0, \theta_0) \in \mathbb{R}^2 \times \mathbb{S}^1$ find a pair trajectory-control joining (x_0, y_0, θ_0) to \mathcal{C} , which is time-optimal for the control system (1.1).

2.2. Existence of solutions. The following two propositions are well-known and stated without proof (see, e.g. [1]).

Proposition 2.1 (Controllability). *System (1.1) is controllable provided that $u \in [-u_{\max}, u_{\max}]$ and $v \in [v_{\min}, v_{\max}]$ for any choice of $0 < u_{\max} \leq +\infty$ and $0 < v_{\min} \leq v_{\max} \leq +\infty$.*

Also, Filippov's theorem gives.

Proposition 2.2 (Existence of minimizers). *For any point $(x_0, y_0, \theta_0) \in \mathbb{R}^2 \times \mathbb{S}^1$, there exists a time-optimal trajectory joining (x_0, y_0, θ_0) to \mathcal{C} .*

2.3. Dimension reduction of the system. To solve problem **(P₀)** it is convenient to work with a reduced system in dimension two. Indeed, in dimension two, a complete theory to build time-optimal syntheses exists and will be described in Section 3.

Let the control set be defined by $\mathcal{U} = [-u_{\max}, u_{\max}] \times [v_{\min}, v_{\max}] \subset \mathbb{R}^2$.

Also, we introduce the UAV-based coordinates $(\tilde{x}, \tilde{y}, \theta)$ with \tilde{x} and \tilde{y} defined by the transformation (in $SO(2)$):

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The main advantage of this UAV-based coordinate system is that it decouples the variable θ and projects the final manifold \mathcal{C} to the point $\tilde{X}_0 = (0, -r_{\min})$. Therefore, the original time-optimal control problem can be equivalently reformulated in the reduced state space (\tilde{x}, \tilde{y}) as the following minimum-time problem:

(P₁) For every $(\tilde{x}_0, \tilde{y}_0) \in \mathbb{R}^2$ find a pair trajectory-control joining $(\tilde{x}_0, \tilde{y}_0)$ to $\tilde{X}_0 = (0, -r_{\min})$, which is time-optimal for the control system

$$\begin{cases} \dot{\tilde{x}} = v + u\tilde{y} \\ \dot{\tilde{y}} = -u\tilde{x} \end{cases}, \quad (u, v) \in \mathcal{U}. \quad (2.1)$$

Remark 2.3. One can read, in equation (2.1) that $(0, -r_{\min})$ is an equilibrium point corresponding to the control values $u = u_{\max}$ and $v = v_{\min}$.

The family of all solutions to problem **(P₁)** for $(\tilde{x}_0, \tilde{y}_0) \in \mathbb{R}^2$ is called the *time-optimal synthesis*.

Following a standard approach (see [6]) for time-optimal control syntheses, it is convenient to rephrase problem **(P₁)** as an equivalent problem backward in time. Hence, changing the sign of the dynamics, the following equivalent time-optimal problem is considered:

(P₂) For every $(\tilde{x}_f, \tilde{y}_f) \in \mathbb{R}^2$ find a pair trajectory-control joining $\tilde{X}_0 = (0, -r_{\min})$ to $(\tilde{x}_f, \tilde{y}_f)$, which is time-optimal for the control system

$$\begin{cases} \dot{\tilde{x}} = -v - u\tilde{y} \\ \dot{\tilde{y}} = u\tilde{x} \end{cases}, \quad (u, v) \in \mathcal{U}. \quad (2.2)$$

Once problem **(P₂)** is solved, then the time-optimal synthesis (corresponding to problem **(P₁)**) is obtained straightforwardly following the travelled trajectories backward.

Remark 2.4. Note that up to a dilation in the (x, y) -plane and a dilation of time (a time-reparametrization with constant derivative), we may assume that $[-u_{\max}, u_{\max}] \times [v_{\min}, v_{\max}] = [-1, 1] \times [1, \eta]$ ($\eta = v_{\max}/v_{\min}$). This normalization is used to simplify the treatment in Sections 4 and 5.

3. Time-optimal synthesis on \mathbb{R}^2 . In this section, following the same ideas as those developed by Boscaïn, Bressan, Piccoli and Sussmann in [5, 6, 8, 12, 15] for optimal syntheses on two-dimensional manifolds for single input control systems, we introduce important definitions and develop basic facts about optimal syntheses on \mathbb{R}^2 for control-affine systems with two bounded controls (which are different from those studied in [3] and [4]). This part is widely inspired by the book [6] and extends some of its results.

The definitions and results given in Subsections 3.1, 3.2 and 3.3 are valid in \mathbb{R}^n for any n . However, starting from Subsection 3.4 results are valid and make sense only for $n = 2$.

3.1. Pontryagin Maximum Principle. Let F and G be two smooth and complete vector fields on \mathbb{R}^n . Define the control variable $U = (u, v)$ and the control set $\mathcal{U} = [-u_{\max}, u_{\max}] \times [v_{\min}, v_{\max}] \subset \mathbb{R}^2$, where $u_{\max}, v_{\min}, v_{\max}$ are assumed to be positive. Consider the following general control-affine time-optimal problem.

(R) For every X_0 and X_f in \mathbb{R}^n find the pair trajectory-control joining X_0 to X_f , which is time-optimal for the control system

$$\dot{X} = vF(X) + uG(X), \quad X \in \mathbb{R}^n, \quad (u, v) \in \mathcal{U}. \quad (3.1)$$

Definition 3.1 (admissible control/trajectory). An *admissible control* for system (3.1) is an essentially bounded function $U(\cdot) : [t_0, t_1] \rightarrow \mathcal{U}$. An *admissible trajectory* is a solution to (3.1) such that $\dot{X}(t) = v(t)F(X(t)) + u(t)G(X(t))$ a.e. on $[t_0, t_1]$ for some admissible control $U(\cdot)$.

Thanks to the compactness of the set of controls, the convexity of the set of velocities, and the completeness of the vector fields, Filippov's theorem (see, for instance, [1]) yields:

Proposition 3.2. *For any pair of points in \mathbb{R}^n , there exists a time-optimal trajectory joining them.*

The main tool to compute time-optimal trajectories is the Pontryagin Maximum Principle (PMP). A general version of PMP can be found in [1]. The following theorem is a version of PMP for control systems of the form (3.1) that we state in our own context only.

Theorem 3.3 (PMP). *Consider the control system (3.1). For every $(P, X, U) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{U}$, define the Hamiltonian function*

$$H(P, X, U) = v \langle P, F(X) \rangle + u \langle P, G(X) \rangle. \quad (3.2)$$

Let $U(\cdot)$ be an admissible time-optimal control defined on $[t_0, t_1]$ and let $X(\cdot)$ be the corresponding trajectory. Then there exist a never vanishing Lipschitz covector (or adjoint vector) $P(\cdot) : t \in [t_0, t_1] \mapsto P(t) \in \mathbb{R}^n$ and a non negative constant λ such that for almost all $t \in [t_0, t_1]$:

- i. $\dot{X}(t) = \frac{\partial H}{\partial P}(P(t), X(t), U(t))$,
- ii. $\dot{P}(t) = -\frac{\partial H}{\partial X}(P(t), X(t), U(t))$,
- iii. $H(P(t), X(t), U(t)) = \max_{W \in \mathcal{U}} H(P(t), X(t), W)$,
- iv. $H(P(t), X(t), U(t)) = \lambda \geq 0$.

A pair trajectory-control $(X(\cdot), U(\cdot))$ (resp. a triplet $(P(\cdot), X(\cdot), U(\cdot))$) satisfying the conditions given by the PMP is said to be an *extremal trajectory* (resp. an *extremal*). An extremal corresponding to $\lambda = 0$ is said to be *abnormal*, otherwise we call it *normal*.

Remark 3.4. Notice that, up to change $U(\cdot)$ on a set of measure zero, an extremal control can always be chosen so that the function $t \mapsto H(P(t), X(t), U(t))$ is continuous. Consequently, we may always assume (without loss of generality) that condition iv of PMP is valid everywhere.

3.2. Basic definitions.

Definition 3.5 (Switching functions). Let $X(\cdot)$ be an extremal trajectory. The corresponding u - and v -switching functions are (the differentiable functions) defined respectively as

$$\phi_u(t) = \langle P(t), G(X(t)) \rangle \text{ and } \phi_v(t) = \langle P(t), F(X(t)) \rangle.$$

Switching functions are very important since their analysis determine when the corresponding control may change. Unlike in the single input case, the switching functions are differentiable but not necessarily C^1 .

In the following three definitions (Definitions 3.6-3.8), $X(\cdot)$ is an extremal trajectory defined on the time interval $[t_0, t_1]$ and $U(\cdot) : [t_0, t_1] \rightarrow \mathcal{U}$ is the corresponding control.

Definition 3.6 (Bang). $U(\cdot)$ is said to be a u -bang (resp. v -bang) control if, for a.e. $t \in [t_0, t_1]$, $u(t) = -u_{\max}$ (or $u(t) = u_{\max}$) (resp. $v(t) = v_{\min}$ (or $v(t) = v_{\max}$)). $U(\cdot)$ is a bang control if, for a.e. t in $[t_0, t_1]$, it is u -bang and v -bang. A finite concatenation of bang controls is called a bang-bang control.

Definition 3.7 (Singular). We say that $U(\cdot)$ is a u -singular control (resp. v -singular) if the corresponding switching function ϕ_u (resp. ϕ_v) vanishes identically on $[t_0, t_1]$. If ϕ_u and ϕ_v both vanish identically on $[t_0, t_1]$, we say that $U(\cdot)$ is totally singular.

Definition 3.8 (Switching times). A u -switching time of $U(\cdot)$ is a time $\tau \in (t_0, t_1)$ such that, for a sufficiently small $\varepsilon > 0$, $u(t) = u_{\max}$ for a.e. $t \in (\tau - \varepsilon, \tau]$ and $u(t) = -u_{\max}$ for a.e. $t \in (\tau, \tau + \varepsilon]$ or vice-versa. A v -switching time is defined similarly. A (u, v) -switching time is a time that is both a u - and a v -switching time. If τ is a switching time, the corresponding point $X(\tau)$ on the trajectory $X(\cdot)$ is called a switching point.

3.3. Abnormal trajectories. The following lemma gives some information on the nature of abnormal extremals of system (3.1).

Lemma 3.9. Let $\gamma(\cdot) = (P(\cdot), X(\cdot), U(\cdot))$ be an abnormal extremal defined on $[t_0, t_1]$. Assume that $\gamma(\cdot)$ is never totally singular in restriction to any subinterval of $[t_0, t_1]$. Then $\gamma(\cdot)$ is bang with $v(\cdot) = v_{\min}$ on an open dense subset of $[t_0, t_1]$.

Proof. Since the considered extremal is abnormal, we have, for all $t \in [t_0, t_1]$,

$$H(P(t), X(t), U(t)) = v(t)\phi_v(t) + u_{\max}|\phi_u(t)| = 0. \quad (3.3)$$

From (3.3), we conclude that a zero value of $\phi_u(\cdot)$ implies a zero value of $\phi_v(\cdot)$ and conversely. Hence, the extremal being never totally singular in restriction to any subinterval of $[t_0, t_1]$, $\phi_u(\cdot)$ and $\phi_v(\cdot)$ cannot vanish identically on any subinterval of $[t_0, t_1]$. Consequently, the set $E = \{t \in [t_0, t_1] \mid \phi_v(t) \neq 0\}$ is open (since ϕ_v is continuous) and dense in $[t_0, t_1]$. From (3.3) again, we necessarily have $\phi_v(t) < 0$ for all $t \in E$. Consequently, the considered extremal is bang (on E) with $v(\cdot) = v_{\min}$. \square

Remark 3.10. According to the forthcoming Remark 3.14, generically, on \mathbb{R}^2 , an abnormal is not totally singular.

3.4. Singular trajectories. From now and until the end of the paper, we assume $n = 2$, i.e. $X \in \mathbb{R}^2$. Let us introduce the functions¹

$$\begin{aligned}\Delta_A(X) &= \det(F(X), G(X)), \\ \Delta_{Bu}(X) &= \det(G(X), [F, G](X)), \\ \Delta_{Bv}(X) &= \det(F(X), [F, G](X)),\end{aligned}$$

whose zero sets are *fundamental loci* (see [6]) in the construction of the optimal synthesis.

Remark 3.11. Notice that, although the functions Δ_A , Δ_{Bu} and Δ_{Bv} depend on coordinates in \mathbb{R}^2 , the sets $\Delta_A^{-1}(0)$, $\Delta_{Bu}^{-1}(0)$ and $\Delta_{Bv}^{-1}(0)$ do not; indeed, they are intrinsic objects related with the control system (3.1).

The following lemma which is a direct generalization of [6, Theorem 12 page 47] is stated without proof.

Lemma 3.12. *u -singular (resp. v -singular) trajectories are contained in the set $\Delta_{Bu}^{-1}(0)$ (resp. $\Delta_{Bv}^{-1}(0)$).*

Lemma 3.13 (*(u, v) -singular trajectories*). *(u, v) -singular trajectories are contained in the set $\Delta_A^{-1}(0) \cap \Delta_{Bu}^{-1}(0) \cap \Delta_{Bv}^{-1}(0)$.*

Proof. Let $(P(t), X(t), U(t))$ be a (u, v) -singular extremal defined on $[t_0, t_1]$. Then for a.e. $t \in [t_0, t_1]$,

$$\begin{aligned}\phi_v(t) &= \langle P(t), F(X(t)) \rangle = 0, \\ \phi_u(t) &= \langle P(t), G(X(t)) \rangle = 0,\end{aligned}$$

which implies that $P(t)$ is orthogonal to both $F(X(t))$ and $G(X(t))$. But, according to PMP, $P(t)$ cannot vanish, hence, F and G must be parallel along $X(\cdot)$. We thus get $X(\cdot) \subset \Delta_A^{-1}(0)$. Moreover, according to Lemma 3.12, we also have $\Delta_{Bu}^{-1}(0) \cap \Delta_{Bv}^{-1}(0)$. \square

Remark 3.14. Although it is not addressed here, it can be proved that the intersection $\Delta_A^{-1}(0) \cap \Delta_{Bu}^{-1}(0) \cap \Delta_{Bv}^{-1}(0)$ is generically empty. In other words, generically, on \mathbb{R}^2 , there is no totally singular trajectories.

The next lemma describes the kind of switches that may occur along singular arcs.

Lemma 3.15. *Along a u -singular trajectory which is not totally singular, v is a.e. equal to v_{\max} . Along a v -singular trajectory which is not totally singular, a u -switching cannot occur.*

Proof. Let $(X(\cdot), U(\cdot))$ be a u -singular extremal trajectory on $[t_0, t_1]$ which is not totally singular. According to the PMP and since the trajectory is not v -singular, we have

$$H(P(t), X(t), U(t)) = v(t)\phi_v(t) = \lambda > 0$$

for a.e. $t \in [t_0, t_1]$. Hence, $\phi_v(\cdot)$ is positive on $[t_0, t_1]$ and consequently $v(t) = v_{\max}$ for a.e. $t \in [t_0, t_1]$. The proof for a v -singular trajectory is similar but the PMP yields not the value of the control u along it. \square

¹If F_1 and F_2 are two vector fields, $[F_1, F_2]$ denotes their Lie bracket.

3.5. Switchings.

3.5.1. u -switchings.

Lemma 3.16. *Along a normal extremal trajectory, a u -switching can occur if and only if $v = v_{\max}$.*

Proof. Let $X(\cdot)$ be a normal trajectory on $[t_0, t_1]$ and let $U(\cdot)$ be the corresponding control. Let $\tau \in (t_0, t_1)$ be a u -switching time, i.e. $\phi_u(\tau) = 0$. From the PMP, we have

$$H(P(\tau), X(\tau), U(\tau)) = v(\tau)\phi_v(\tau) > 0,$$

with $v(\tau) > 0$. Hence, $\phi_v(\tau) > 0$ and thus for ϵ sufficiently small, $v(t) = v_{\max}$ for a.e. $t \in [\tau - \epsilon, \tau + \epsilon]$. \square

Remark 3.17. The previous lemma implies in particular that, along a normal trajectory, a v -switching from v_{\max} to v_{\min} is necessarily followed by another v -switching from v_{\min} to v_{\max} before a u -switching occurs.

3.5.2. (u, v) -switchings.

Lemma 3.18 ((u, v) -switchings). *A (u, v) -switching cannot occur along an extremal trajectory.*

Proof. Let $(P(\cdot), X(\cdot), U(\cdot))$ be an extremal and let τ be a (u, v) -switching time. Since the Hamiltonian is constant along trajectories with

$$H(P(\tau), X(\tau), U(\tau)) = v(\tau)\phi_v(\tau) + u_{\max}|\phi_u(\tau)| = 0,$$

the extremal is abnormal. But, according to Lemma 3.9, an abnormal extremal admits only $(u(\cdot), v(\cdot)) = (\pm u_{\max}, v_{\min})$ as bang controls and thus cannot have a (u, v) -switching. \square

3.6. Special domains. Using the sets $\Delta_A^{-1}(0)$, $\Delta_{Bu}^{-1}(0)$ and $\Delta_{Bv}^{-1}(0)$ defined in Section 3.4, we can define domains in which a control can switch at most once. Consider a point $X \notin \Delta_A^{-1}(0)$. Then $F(X)$ and $G(X)$ are linearly independent and form a basis. An easy computation shows that

$$[F, G](X) = f(X)F(X) + g(X)G(X), \quad (3.4)$$

with

$$f(X) = -\frac{\Delta_{Bu}(X)}{\Delta_A(X)} \quad \text{and} \quad g(X) = \frac{\Delta_{Bv}(X)}{\Delta_A(X)}.$$

Lemma 3.19. *A normal and non-singular trajectory along which $f > 0$ (resp. $f < 0$) admits at most one u -switching and necessarily from $-u_{\max}$ to u_{\max} (resp. from u_{\max} to $-u_{\max}$). Similarly, a normal and non-singular trajectory along which $g > 0$ (resp. $g < 0$) admits at most one v -switching and necessarily from v_{\max} to v_{\min} (resp. from v_{\min} to v_{\max}).*

Proof. Let $(P(\cdot), X(\cdot), U(\cdot))$ be a normal, non-singular extremal defined on $[t_0, t_1]$ along which $f > 0$. At any $\tau \in [t_0, t_1]$ such that $\phi_u(\tau) = 0$, we have

$$H(P(\tau), X(\tau), U(\tau)) = v(\tau)\phi_v(\tau) = \lambda > 0,$$

which, with equation (3.4), implies

$$\begin{aligned} \dot{\phi}_u(\tau) &= v(\tau) \langle P(\tau), [F, G](X(\tau)) \rangle \\ &= v(\tau)f(X(\tau))\phi_v(\tau) + v(\tau)g(X(\tau))\phi_u(\tau) \\ &= v(\tau)\phi_v(\tau)f(X(\tau)) > 0. \end{aligned} \quad (3.5)$$

Since, ϕ_u is continuous, we conclude that ϕ_u can vanish at most once in $[t_0, t_1]$. Moreover, according to inequality (3.5), $\dot{\phi}_u(\tau)$ and $f(X(\tau))$ have same sign. Consequently, if $\phi_u(\tau) = 0$, then $\dot{\phi}_u(\tau) > 0$ and $u(\cdot)$ switches at τ from $-u_{\max}$ to u_{\max} . Reasoning similarly with ϕ_u replaced by ϕ_v gives the second part of the lemma. \square

4. Construction of time-optimal synthesis for the reduced system. In this section, we apply the results obtained in the previous sections to solve problem (\mathbf{P}_2) . Although the problem is similar to the one studied in [11], the resolution is much more complicated due to the presence of a second control.

In this section, for the sake of clarity and without loss of generality, we assume, according to Remark 2.4, that $\mathcal{U} = [-1, 1] \times [1, \eta]$. Note moreover that, in this case, $r_{\min} = 1$.

4.1. Application of the Pontryagin Maximum Principle. First of all, notice that system (2.2) is of the form (3.1) with

$$\tilde{X} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}, \quad F(\tilde{X}) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \text{and} \quad G(\tilde{X}) = \begin{pmatrix} -\tilde{y} \\ \tilde{x} \end{pmatrix}.$$

We apply PMP to (\mathbf{P}_2) . The control-dependent Hamiltonian function of PMP is

$$H(\tilde{X}, P, U) = -vp + u(q\tilde{x} - p\tilde{y}),$$

with $P = (p, q) \in \mathbb{R}^2$ being the covector. The adjoint system is thus given by

$$\begin{cases} \dot{p}(t) = -u(t)q(t) \\ \dot{q}(t) = u(t)p(t) \end{cases}. \quad (4.1)$$

and the switching functions are

$$\phi_u(t) = q(t)\tilde{x}(t) - p(t)\tilde{y}(t), \quad (4.2)$$

$$\phi_v(t) = -p(t). \quad (4.3)$$

The maximality condition of the PMP reads

$$H(\tilde{X}(t), P(t), U(t)) = \max_{(u,v) \in \mathcal{U}} (u\phi_v(t) + v\phi_u(t)) = \lambda,$$

and yields the controls

$$u(t) = \begin{cases} -1 & \text{if } \phi_u(t) < 0 \\ 1 & \text{if } \phi_u(t) > 0 \end{cases}, \quad v(t) = \begin{cases} 1 & \text{if } \phi_v(t) < 0 \\ \eta & \text{if } \phi_v(t) > 0 \end{cases}. \quad (4.4)$$

Remark 4.1. The cases where the switching functions vanish identically is addressed in the next subsection.

4.2. Singular trajectories. Let us compute the quantities

$$\Delta_A(\tilde{x}, \tilde{y}) = -\tilde{x}, \quad \Delta_{Bu}(\tilde{x}, \tilde{y}) = \tilde{y} \quad \Delta_{Bv}(\tilde{x}, \tilde{y}) = 1,$$

$$f(\tilde{x}, \tilde{y}) = \frac{\tilde{y}}{\tilde{x}}, \quad g(\tilde{x}, \tilde{y}) = -\frac{1}{\tilde{x}}.$$

Lemmas 3.12, 3.13 and 3.15 imply that

- there exists no v -singular trajectory (and consequently no totally singular trajectory) since $\Delta_{Bv}^{-1}(0) = \emptyset$;

- u -singular trajectories are contained in the set²

$$\Delta_{Bu}^{-1}(0) = \{(\tilde{x}, \tilde{y}) \in \mathbb{R}^2 \mid \tilde{y} = 0\}.$$

To compute the corresponding control, we differentiate w.r.t. t the function ϕ_u (which is identically zero). A straightforward calculation yields

$$\dot{\phi}_u(t) = \eta \langle P(t), [F, G](X(t)) \rangle = -\eta q(t) = 0,$$

$$\ddot{\phi}_u(t) = \eta u(t) \langle P(t), [G, [F, G]](X(t)) \rangle = \eta u(t)p(t) = 0,$$

which, taking into account that $(p(t), q(t))$ never vanishes, implies that, along u -singular trajectories, $u(\cdot)$ vanishes identically. Note that this is quite intuitive since $u = 0$ is the only control that allows the trajectory to stay on the x -axis.

4.3. Notation. The following table defines a naming convention for the five possible optimal controls in order to simplify the description of the trajectories. The

Control	Notation
$(-1, 1)$	m
$(1, 1)$	p
$(-1, \eta)$	M
$(1, \eta)$	P
u -singular	s

TABLE 1. Notation of the five possible optimal controls

letter stands for u equals to plus (p, P) or minus (m, M) one or to the singular control. The lower case and upper case correspond to 1 and η respectively. We define γ^i as the trajectory starting from \tilde{X}_0 with the control $i \in \{m, M, p, P, s\}$. Similarly, γ^{ij} denotes the trajectory switching from γ^i at time T^{ij} with the control $j \in \{m, M, p, P, s\} \setminus \{i\}$, and $\zeta_{\phi_u}^{ij}$ (resp. $\zeta_{\phi_v}^{ij}$) denotes a zero of the switching function on u (resp. v) corresponding to the trajectory γ^{ij} .

4.4. Optimal synthesis algorithm. Since \tilde{X}_0 is an equilibrium point of system (1.1) for the control $(1, 1)$ and since $\tilde{X}_0 \notin \Delta_{Bu}^{-1}(0)$, there are, a priori, three possible optimal starting trajectories candidates: γ^M , γ^m and γ^P corresponding to the bang controls $(-1, \eta)$, $(-1, 1)$ and $(1, \eta)$ respectively (see Fig. 1). The time-optimal synthesis is constructed following the three steps described below.

Step 1 For each bang trajectory starting from \tilde{X}_0 , compute the last time at which the trajectory is extremal (or has lost its optimality by intersecting itself) and study which kind of extremal trajectories can bifurcate from it.

Step 2 For each bang or singular trajectory bifurcating from one of the starting trajectories, compute the last time at which it is extremal. If there are intersections among trajectories, we cancel those parts that are not optimal (among trajectories already computed up to this step).

Step 3 For each trajectory computed at the previous step that did not lose its optimality, prolong it with the next bang or singular trajectory up to the last time at which it is extremal. If there are intersections among trajectories, cancel those parts that are not optimal (among trajectories already computed up to this step).

²According to [6], the set $\{(\tilde{x}, \tilde{y}) \in \mathbb{R}^2 \mid \tilde{y} = 0\}$ is a turnpike.

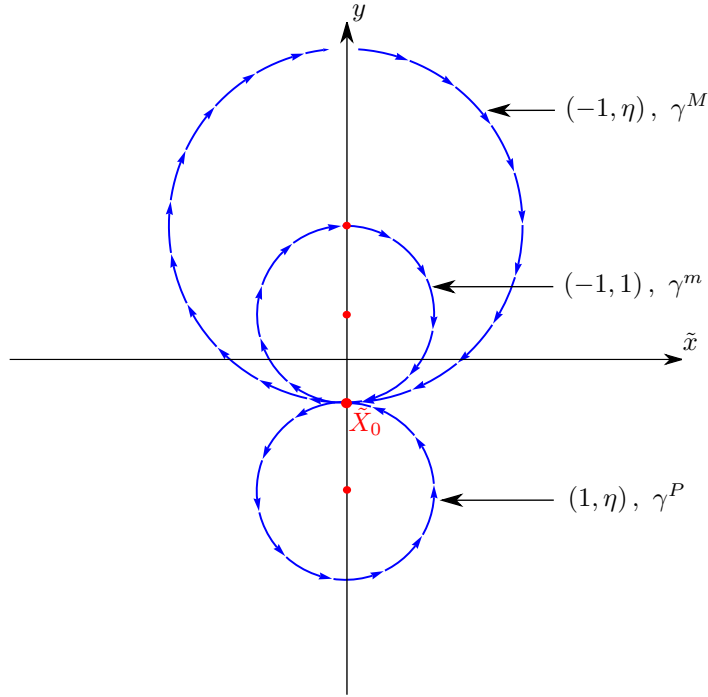


FIGURE 1. Candidate extremal trajectories of problem (\mathbf{P}_2) issued from \tilde{X}_0

Then the synthesis is built recursively, repeating Step 3 until no new trajectories are generated. In our case, four applications of Step 3 are necessary.

Remark 4.2. Notice that, although Step 2 and Step 3 seem similar, Step 2 does not add any trajectory to the synthesis whereas Step 3 does. Step 1 and 2 correspond to an initialisation of the algorithm whereas Step 3 is the step that construct the synthesis recursively.

4.5. Expression of the adjoint vector. Since the starting control $(u_0, v_0) = (u(0), v(0)) \in \{(-1, 1), (-1, \eta), (1, \eta)\}$, the solution to system (4.1) with the normalization $|P(0)| = 1$ is

$$\begin{cases} p(t) = \cos(\alpha + t) \\ q(t) = u_0 \sin(\alpha + t), \quad u_0 \in \{-1, 1\}, \end{cases} \quad (4.5)$$

where α is defined by $P(0) = (\cos \alpha, u_0 \sin \alpha)$. The condition iv of PMP written at the initial point implies that

$$(-v_0 + u_0)p(0) \geq 0.$$

Since $(u_0, v_0) \neq (1, 1)$, $(-v_0 + u_0)$ is non-positive. Hence, $p(0) = \cos \alpha \leq 0$ and $\alpha \in [\frac{\pi}{2}, \frac{3\pi}{2}]$. The following study consists in analyzing the behaviour of the extremal trajectories depending on the value of $(\alpha, u_0, v_0) \in [\frac{\pi}{2}, \frac{3\pi}{2}] \times \{(-1, 1), (-1, \eta), (1, \eta)\}$.

4.6. Step 1: Analysis of the trajectories starting from \tilde{X}_0 . This section details the first step (and only this one) of the algorithm described in Section 4.4.

4.6.1. *Trajectories corresponding to control $(-1, \eta)$.* The trajectory γ^M starting from \tilde{X}_0 with the control $(-1, \eta)$ has coordinates

$$\begin{cases} \tilde{x}(t) = -(\eta + 1) \sin t \\ \tilde{y}(t) = \eta - (\eta + 1) \cos t, \end{cases}$$

and from (4.5), the coordinates of the adjoint vector are

$$\begin{cases} p(t) = \cos(t + \alpha) \\ q(t) = -\sin(t + \alpha), \end{cases}$$

with $\alpha \in [\frac{\pi}{2}, \frac{3\pi}{2}]$. It follows that the switching functions are

$$\begin{aligned} \phi_u(t) &= (\eta + 1) \cos \alpha - \eta \cos(t + \alpha), \\ \phi_v(t) &= -p(t) = -\cos(t + \alpha). \end{aligned}$$

Recall that the sign of each of these functions determines the value of the corresponding control. It is thus fundamental to study when does a switching function change its sign. Depending on the initial value of the covector parameterized by α the following cases have to be distinguished.

- For $\alpha = \pi/2$, ϕ_u starts with value zero and then takes positive values: from (4.4), it cannot correspond to a trajectory starting with control $(-1, \eta)$.
- For $\alpha_{\text{sing}} = \arccos(-\eta/(\eta + 1))$, ϕ_u starts with negative values and has a zero of order two at $t_{\text{sing}} = \pi - \alpha_{\text{sing}}$. ϕ_v starts with positive values and vanishes after ϕ_u does. The trajectory reaches the \tilde{x} -axis at time t_{sing} . In this case, at time $t = t_{\text{sing}}$, either the trajectory becomes u -singular, switches to $(1, \eta)$ or does not switch. In the latter case, there is a v -switching at time $T^{Mm}(\alpha) = -\alpha + 3\pi/2$.
- For every $\alpha \in (\pi/2, \alpha_{\text{sing}})$, ϕ_u starts with negative values and ϕ_v with positive values. The function that vanishes first is ϕ_u . At time $T^{MP}(\alpha) = -\alpha + \arccos(\cos \alpha(\eta + 1)/\eta)$, ϕ_u changes its sign and a u -switching occurs.
- For every $\alpha \in (\alpha_{\text{sing}}, 3\pi/2)$, ϕ_u is always negative. ϕ_v starts with positive values and changes sign at time $T^{Mm}(\alpha) = -\alpha + 3\pi/2$. There is a v -switching at this time.
- For $\alpha = 3\pi/2$, ϕ_v starts with value zero and then takes negative values: from (4.4), it cannot correspond to a trajectory starting with control $(-1, \eta)$.

From this analysis, we define the following families of trajectories issued from γ^M :

Family 1: trajectories γ^{Ms} corresponding to $\alpha = \alpha_{\text{sing}} = \arccos(-\eta/(\eta + 1))$.

Family 2: trajectories γ^{MP} corresponding to $\alpha \in (\pi/2, \alpha_{\text{sing}}]$.

Family 3: trajectories γ^{Mm} corresponding to $\alpha \in [\alpha_{\text{sing}}, 3\pi/2)$.

4.6.2. *Trajectories corresponding to control $(1, \eta)$.* The trajectory γ^P starting from \tilde{X}_0 with the control $(1, \eta)$ has coordinates

$$\begin{cases} \tilde{x}(t) = -(\eta - 1) \sin t \\ \tilde{y}(t) = -\eta + (\eta - 1) \cos t \end{cases}$$

and from (4.5), the coordinates of the corresponding adjoint vector are

$$\begin{cases} p(t) = \cos(t + \alpha) \\ q(t) = \sin(t + \alpha), \end{cases}$$

with $\alpha \in [\frac{\pi}{2}, \frac{3\pi}{2}]$. It follows that the switching functions are

$$\begin{aligned}\phi_u(t) &= -(\eta - 1) \cos \alpha + \eta \cos(t + \alpha), \\ \phi_v(t) &= -\cos(t + \alpha).\end{aligned}$$

Similarly to Section 4.6.1, depending on the value of α the following analysis may be performed.

- For $\alpha \in [\frac{\pi}{2}, \frac{3\pi}{2})$, ϕ_u starts with a nonpositive value and then takes negative values: from (4.4), it cannot correspond to a trajectory starting with control $(1, \eta)$.
- For $\alpha = \frac{3\pi}{2}$, ϕ_v starts with value zero and then takes negative values: from (4.4), it cannot correspond to a trajectory starting with control $(1, \eta)$.

We conclude that there is no optimal trajectory starting from \tilde{X}_0 with the controls $(1, \eta)$.

4.6.3. *Trajectories corresponding to control $(-1, 1)$.* The trajectory γ^m starting from \tilde{X}_0 with the control $(-1, 1)$ has coordinates

$$\begin{cases} \tilde{x}(t) = -2 \sin t \\ \tilde{y}(t) = 1 - 2 \cos t, \end{cases}$$

and from (4.5), the coordinates of the adjoint vector are

$$\begin{cases} p(t) = \cos(t + \alpha) \\ q(t) = -\sin(t + \alpha). \end{cases}$$

It follows that the switching functions are

$$\begin{aligned}\phi_u(t) &= 2 \cos(\alpha) - \cos(t + \alpha), \\ \phi_v(t) &= -\cos(t + \alpha).\end{aligned}$$

The maximization condition iv of PMP implies

$$\begin{aligned}\phi_u(0) &= \cos(\alpha) \leq 0, \\ \phi_v(0) &= -\cos(\alpha) \leq 0,\end{aligned}$$

i.e. $\alpha \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ (in particular, the trajectory is abnormal). Note that ϕ_u and ϕ_v must be negative on a (small) open interval of the form $(0, \epsilon)$ since the starting control is the bang control $(-1, 1)$. Moreover, since $\phi_u(0) = \phi_v(0) = 0$ and

$$\begin{aligned}\dot{\phi}_u(0) &= \sin \alpha, \\ \dot{\phi}_v(0) &= \sin \alpha,\end{aligned}$$

ϕ_u and ϕ_v will be both negative on $(0, \epsilon)$ if and only if $\alpha = 3\pi/2$. Consequently, there is only one extremal (corresponding to $\alpha = 3\pi/2$) starting from \tilde{X}_0 with control $(-1, 1)$. The two switching functions along this extremal ($\phi_u(t) = \phi_v(t) = -\sin t$) change their sign at $t = \pi$ and since a (u, v) -switching cannot occur (see Lemma 3.18) the trajectory γ^m loses its optimality no later than time π .

4.7. Step 2: Analysis of the trajectories bifurcating from γ^M . At this step we study separately all the bifurcating candidate extremals found at the previous step.

4.7.1. *Family 1: Singular trajectory.* This family corresponds to a trajectory entering the turnpike at the time $t_{\text{sing}} = \pi - \alpha_{\text{sing}}$. From Lemma 3.15, the control $(0, \eta)$ remains unchanged and the trajectory exits the turnpike with the control $v = \eta$. The trajectory is obtained from (2.1) and lies on the turnpike, that is

$$\begin{cases} \tilde{x}(t) = -\eta t - \sqrt{(2\eta + 1)} + \eta t_{\text{sing}}, & \forall t > t_{\text{sing}}. \\ \tilde{y}(t) = 0 \end{cases}$$

It follows from (4.1) that the covector is constant along a u -singular trajectory. Evaluating (4.5) at time $t = t_{\text{sing}}$ with $\alpha = \alpha_{\text{sing}}$ gives

$$P(t) = (p(t), q(t)) = (-1, 0), \quad \forall t > t_{\text{sing}}.$$

The u -singular trajectory is then extremal for any time $t > t_{\text{sing}}$.

4.7.2. *Family 2: First switching on u .* This family corresponds to the set of trajectories switching from γ^M with a u -switching. Recall that the trajectories of this family correspond to controls $(1, \eta)$ and the u -switching time is $T^{MP}(\alpha) = -\alpha + \arccos(\cos \alpha(\eta + 1)/\eta)$, with $\alpha \in (\frac{\pi}{2}, \alpha_{\text{sing}}]$. The trajectory, obtained from (2.1), is

$$\begin{cases} \tilde{x}(t) = (\eta + 1) \sin(t - 2T^{MP}(\alpha)) - 2\eta \sin(t - T^{MP}(\alpha)) \\ \tilde{y}(t) = -\eta - (\eta + 1) \cos(t - 2T^{MP}(\alpha)) + 2\eta \cos(t - T^{MP}(\alpha)), \end{cases}$$

for all $t > T^{MP}(\alpha)$. The covector satisfies (4.1), which leads to the coordinates of the adjoint vector:

$$\begin{cases} p(t) = \cos(t - \alpha - 2T^{MP}(\alpha)) \\ q(t) = \sin(t - \alpha - 2T^{MP}(\alpha)), \end{cases}$$

Given the coordinates of the trajectory and the covector, the switching functions are:

$$\begin{aligned} \phi_u(t) &= q(t)\tilde{x}(t) - p(t)\tilde{y}(t) \\ &= \eta \cos(t - \alpha - 2T^{MP}(\alpha)) + (\eta + 1) \cos \alpha - 2\eta \cos(\alpha + T^{MP}(\alpha)) \\ &= -(\eta + 1) \cos \alpha + \eta \cos(t - \alpha - 2T^{MP}(\alpha)) \\ \phi_v(t) &= -p(t) \\ &= -\cos(t - \alpha - 2T^{MP}(\alpha)) \end{aligned}$$

As in the previous section, we study the sign of the switching functions. Let $\zeta_{\phi_u}^{MP}(\alpha)$ and $\zeta_{\phi_v}^{MP}(\alpha)$, both greater than $T^{MP}(\alpha)$, be the first zeros of ϕ_u and ϕ_v respectively. We thus have

$$\phi_u(\zeta_{\phi_u}^{MP}(\alpha)) = -(\eta + 1) \cos \alpha + \eta \cos(\zeta_{\phi_u}^{MP}(\alpha) - \alpha - 2T^{MP}(\alpha)) = 0.$$

Since we seek the first zero of ϕ_u , it follows that

$$\begin{aligned} \zeta_{\phi_u}^{MP}(\alpha) &= \alpha + \arccos\left(\frac{\eta + 1}{\eta} \cos \alpha\right) + 2T^{MP}(\alpha), \\ &= -\alpha + 3 \arccos\left(\frac{\eta + 1}{\eta} \cos \alpha\right). \end{aligned}$$

In the same way, we determine the first zero of ϕ_v , we have

$$\phi_v(\zeta_{\phi_v}^{MP}(\alpha)) = -\cos(\zeta_{\phi_v}^{MP}(\alpha) - \alpha - 2T^{MP}(\alpha)) = 0.$$

which gives the v -switching time

$$\begin{aligned}\zeta_{\phi_v}^{MP}(\alpha) &= -\frac{\pi}{2} + \alpha + 2T^{MP}(\alpha), \\ &= -\frac{\pi}{2} - \alpha + 2 \arccos\left(\frac{\eta+1}{\eta} \cos \alpha\right).\end{aligned}$$

Evaluating the difference between the two switching times, one can determine which switching function changes sign first. Since $\alpha_{\text{sing}} < \pi$, the difference

$$\zeta_{\phi_u}^{MP}(\alpha) - \zeta_{\phi_v}^{MP}(\alpha) = \arccos\left(\frac{\eta+1}{\eta} \cos \alpha\right) + \frac{\pi}{2}$$

is positive for all $\alpha \in (\frac{\pi}{2}, \alpha_{\text{sing}}]$. The next switching is then a v -switching which occurs at a time $T^{MPp}(\alpha) = \zeta_{\phi_v}^{MP}(\alpha)$ for this family of trajectories.

The curve made of v -switching points is called *v-switching curve*. This curve, parametrized by α , is given by

$$\begin{cases} \tilde{x}_{\text{sw}}^{MP}(\alpha) = \tilde{x}(T^{MPp}(\alpha)) = (\eta+1) \cos \alpha \\ \tilde{y}_{\text{sw}}^{MP}(\alpha) = \tilde{y}(T^{MPp}(\alpha)) = -\eta - (\eta+1) \sin \alpha + 2\eta \sqrt{1 - \left(\frac{\eta+1}{\eta} \cos \alpha\right)^2}, \end{cases}$$

with $\alpha \in (\frac{\pi}{2}, \alpha_{\text{sing}}]$. On Fig. 2, this curve is shown in gray.

4.7.3. Family 3: First switching on v . This family corresponds to the set of trajectories switching from γ^M with a v -switching. Recall that the trajectories of this family correspond to a control $(-1, 1)$ and a v -switching time $T^{Mm}(\alpha) = -\alpha + 3\pi/2$, with $\alpha \in [\alpha_{\text{sing}}, \frac{3\pi}{2})$. For all $t > T^{Mm}(\alpha)$, the trajectory is

$$\begin{cases} \tilde{x}(t) = (\eta-1) \cos(\alpha+t) - (\eta+1) \sin t \\ \tilde{y}(t) = 1 - (\eta+1) \cos t - (\eta-1) \sin(\alpha+t) \end{cases}$$

and the covector is given by

$$\begin{cases} p(t) = \sin(t - T^{Mm}(\alpha)) = \cos(t + \alpha) \\ q(t) = \cos(t - T^{Mm}(\alpha)) = -\sin(t + \alpha). \end{cases}$$

Since the Family 3 of extremals results from a switching on v , according to Remark 3.17, the next switching is a v -switching. Moreover, since $\phi_v(t) = -p(t)$ the next switching occurs at time $T^{MmM}(\alpha) = \pi + T^{Mm}(\alpha) = -\alpha + 5\pi/2$. As for the Family 2, we get the corresponding parametric equation of the switching curve:

$$\begin{cases} \tilde{x}_{\text{sw}}^{Mm}(\alpha) = -(\eta+1) \cos \alpha \\ \tilde{y}_{\text{sw}}^{Mm}(\alpha) = (2-\eta) - (\eta+1) \sin \alpha, \quad \alpha \in \left[\alpha_{\text{sing}}, \frac{3\pi}{2}\right). \end{cases}$$

All results of Subsections 4.6 and 4.7 are shown in Fig. 2.

5. Numerical simulations.

5.1. Synthesis in the UAV-based coordinates. Numerical simulations have been made with $\eta = 2$. For the sake of clarity, the detailed construction of the synthesis is postponed to the Appendix.

During the construction of the optimal synthesis some special curves appears, namely

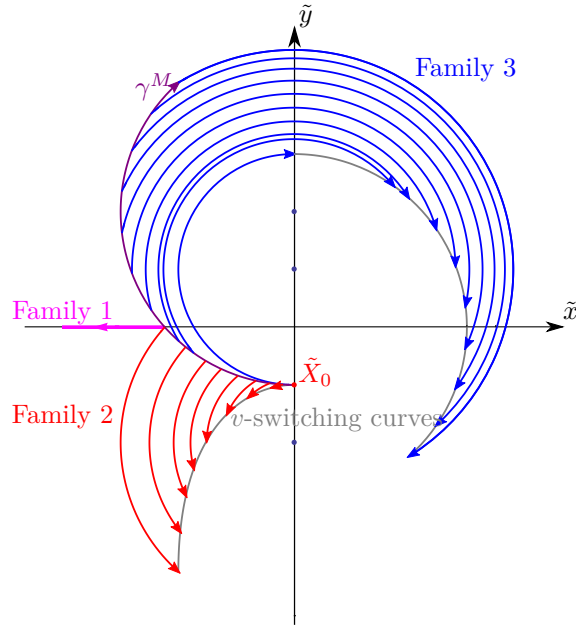


FIGURE 2. Extremal trajectories starting from \tilde{X}_0 and having one switching

- *switching curves*, i.e. curves made of switching points;
- *cut loci*, i.e. sets of points where the extremal curves of problem (\mathbf{P}_2) lose global optimality.

In practice, switching curves and cut loci can be very difficult to compute. In the following, some of them were computed numerically. Following the algorithm described in Section 4.4, the time-optimal synthesis corresponding to the problem (\mathbf{P}_1) has been solved. The corresponding (discontinuous) state-feedback is given in Fig. 3 and Table 2 as a partition of the reduced state space.

Remark 5.1. Note that the first version of the synthesis (Fig. 3) presented without detail in the conference paper [10] is mistaken. Indeed, in [10], the synthesis is incomplete since the last arc γ^{MPpPMm} of Family 2 is missing.

Remark 5.2. Notice that the minimum time function (as a function of \tilde{x} and \tilde{y}) is not continuous along the abnormal trajectory. As a consequence, the optimality of the synthesis cannot be confirmed a posteriori using the verification theorem ([13, Theorem 2.13]) based on the notion of regular synthesis as it was done in [14] for the case of the Dubins' system for tracking a rectilinear route in minimum time. Indeed, it is easy to see that the minimum time function is not weakly upper semicontinuous (w.u.s.c. for short) and thus does not match the hypothesis of [13, Theorem 2.13]. To see this let $\tilde{X}_n \rightarrow \tilde{X}_0$ as $n \rightarrow \infty$ where $\tilde{X}_n = (\tilde{x}_n, \tilde{y}_n)$, with $\tilde{x}_n > 0$, belongs to the v -switching curve passing through \tilde{X}_0 . Denote by $V(X)$ the minimum time to reach \tilde{X}_0 from X . We have $V(\tilde{X}_0) = 0$. The v -switching occurring on the considered switching curve is a switching from η to 1. Then, according to Remark 3.17 and since $\phi_v(t) = -\cos(\alpha + t)$, $V(\tilde{X}_n) > \pi$. Consequently, $\lim_{n \rightarrow \infty} V(\tilde{X}_n) \geq \pi > 0 = V(\tilde{X}_0)$ which shows that V is not w.u.s.c. at \tilde{X}_0 .

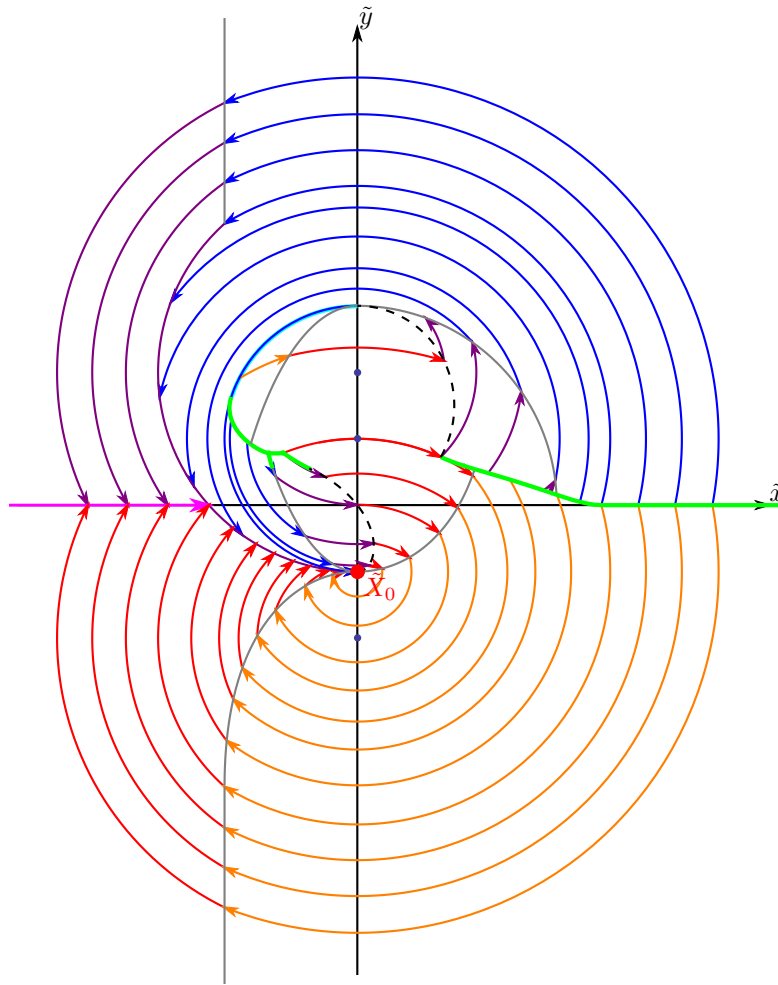


FIGURE 3. Time-optimal synthesis for the problem (\mathbf{P}_1)

$(-1, 1)$ -bang arc	Blue
$(1, 1)$ -bang arc	Orange
$(-1, \eta)$ -bang arc	Purple
$(1, \eta)$ -bang arc	Red
u -singular arc	Magenta
u -switching curves	Dashed black
v -switching curves	Gray
Cut Locus	Green
Abnormal Cut Locus	Cyan

TABLE 2. Color convention of the optimal synthesis

5.2. **Verification of the synthesis.** Using Lemma 3.19 for the problem (\mathbf{P}_1) , one can check the given synthesis. We have the functions:

$$f(\tilde{X}) = -\frac{\Delta_{Bu}(\tilde{X})}{\Delta_A(\tilde{X})} = -\frac{\tilde{y}}{\tilde{x}} \quad \text{and} \quad g(\tilde{X}) = \frac{\Delta_{Bv}(\tilde{X})}{\Delta_A(\tilde{X})} = \frac{1}{\tilde{x}}.$$

We define four domains according to the values of f and g (see Figure 4). We conclude that there is at most one switching on u in each domains and at most one switching on v in the union of domains 1 and 4 and in the union of 2 and 3. Moreover,

- in orthants 1 and 3: only a u -switching from 1 to -1 is allowed,
- in orthants 2 and 4: only a u -switching from -1 to 1 is allowed,
- in the union of orthants 1 and 4: only a v -switching from η to 1 is allowed,
- in the union of orthants 2 and 3: only a v -switching from 1 to η is allowed.

One can easily check that the synthesis given in Figure 3 respects Lemma 3.19.

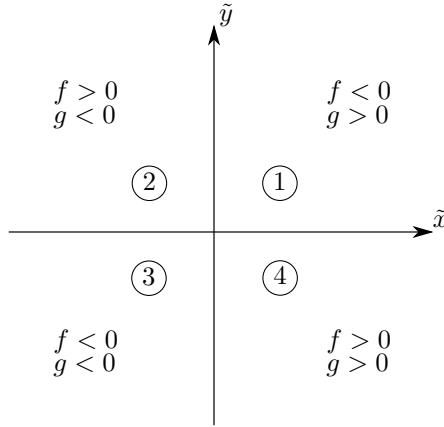


FIGURE 4. Domains on which at most one switching is possible

5.3. **Correspondence with problem (\mathbf{P}_0) .** The solutions of problem (\mathbf{P}_0) can be deduced from the solutions of problem (\mathbf{P}_1) . In this section, we display pairs of figures (Fig. 5 and 6) showing two solutions of problem (\mathbf{P}_1) (that start from the same point in the cut locus) and the corresponding lifted solutions of problem (\mathbf{P}_0) . Notice that the singular trajectory go straight to the center of the target.

5.4. **Stability of the optimal feedback control.** We have shown that the discontinuous feedback control law is a time-optimal feedback for system (2.1). However, the optimality of the feedback does not imply Lyapunov stability (see e.g. the beautiful and simple example in [7]). In our case we have the following proposition.

Proposition 5.3. *The time-optimal feedback law renders system (2.1) globally asymptotically stable.*

Proof. The convergence to the equilibrium \tilde{X}_0 follows from the optimality of the synthesis (actually, the equilibrium is even reached in finite time). We prove the stability by a direct application of Lyapunov's definition (in the reduced space). A time-optimal trajectory starting in the ball $\{Z \mid \|Z - \tilde{X}_0\| \leq \delta \leq 1\}$ may escape this ball if and only if it contains a bang arc corresponding to the control $(u, v) = (1, \eta)$. It

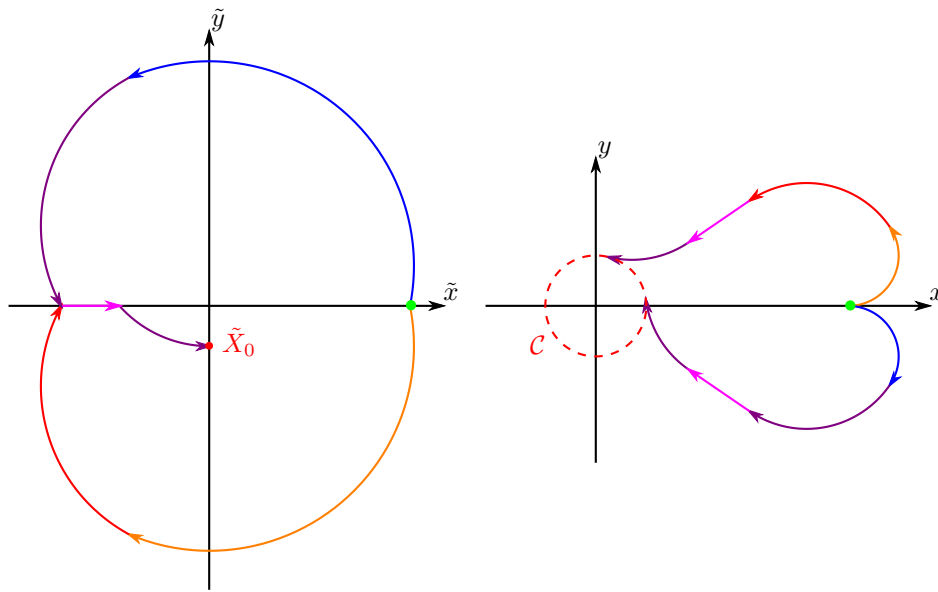


FIGURE 5. Bang-bang-singular-bang optimal trajectories. Two optimal trajectories solutions of problem (\mathbf{P}_1) starting from the same point in the cut locus (left) and the corresponding optimal trajectories solutions of problem (\mathbf{P}_0) (right)

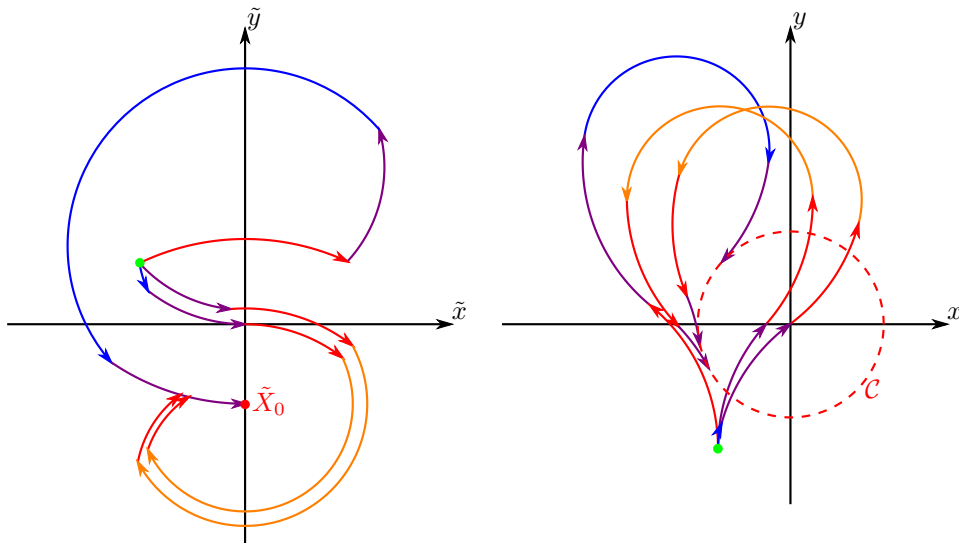


FIGURE 6. Bang-bang-bang-bang-bang optimal trajectories and a bang-bang-bang-bang-bang-bang optimal trajectory. Three optimal trajectories solutions of problem (\mathbf{P}_1) starting from the same point in the cut locus (left) and the corresponding optimal trajectories solutions of problem (\mathbf{P}_0) (right)

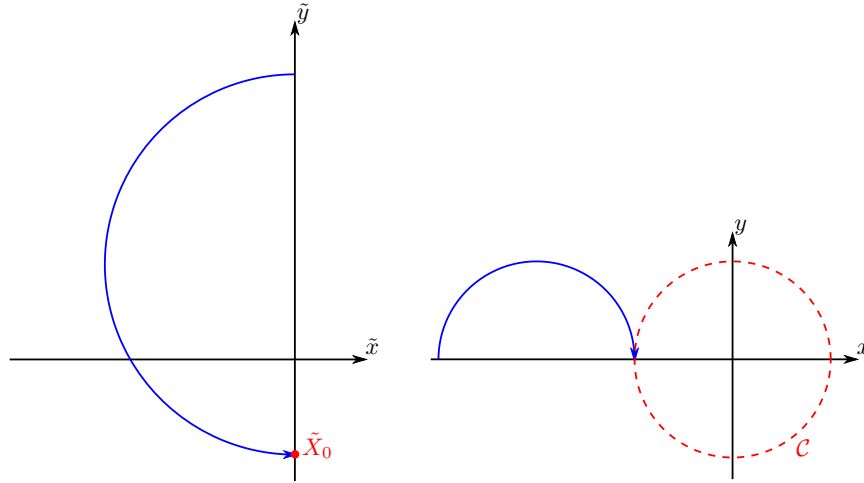


FIGURE 7. The abnormal trajectory of problem (\mathbf{P}_1) and the corresponding trajectory for problem (\mathbf{P}_0) (right)

is an easy matter to see that for such a trajectory the norm $\|X(t; Z) - \tilde{X}_0\|$ increases only along its $(1, \eta)$ -bang arc. Moreover, this bang arc stops at the switching curve defined by (A.3), which is above the horizontal line $\{(\tilde{x}, \tilde{y}) \in \mathbb{R} \mid \tilde{y} = -1\}$. Hence, one easily check that for every $\delta \leq 1$

$$\sup_{\|Z - \tilde{X}_0\| \leq \delta} \sup_{t \in \mathbb{R}_+} \{\|X(t; Z) - \tilde{X}_0\|\} \leq \sqrt{\delta(2(\eta - 1) + \delta)}.$$

Consequently, for every $\epsilon > 0$, choosing either $\delta = 1$ if $\epsilon \geq (2(\eta - 1) + 1)$ or $\delta = \sqrt{((\eta - 1)^2 + \epsilon^2)} - (\eta - 1)$ otherwise we obtain

$$\forall \epsilon > 0 \quad \exists \delta > 0 \mid \|X(t; Z) - \tilde{X}_0\| \leq \epsilon, \quad \forall t \geq 0, \quad \forall Z \in B(\tilde{X}_0, \delta).$$

□

6. Conclusion. In this paper we have solved a time-minimal control problem for a kinematic model describing a UAV flying at constant altitude with controls on the steering angle and on the linear velocity. Thanks to a change of coordinates applied to the three-dimensional Dubins' system, we could simplify the problem and use (and extend) the existing theory of time-optimal syntheses for two-dimensional single input affine control systems to two-dimensional affine control systems with two inputs. We gave the time-optimal synthesis as a state-feedback law such that the target is reached optimally in finite time. Note however that our kinematic model does not actually correspond to a realistic dynamic, as the velocities may not be continuous. A more challenging problem would be to consider a model with controls on accelerations.

Appendix: Construction details of the synthesis.

A.1. 3rd arc.

A.1.1. *Family 1.* In this subsection, we study the trajectories bifurcating from the singular trajectory γ^{Ms} . A trajectory exiting the turnpike has either the control $(-1, \eta)$ or the control $(1, \eta)$.

From each point

$$\begin{cases} \tilde{x}(\tau) = -\eta\tau - \sqrt{(2\eta+1)} + \eta t_{\text{sing}} \\ \tilde{y}(\tau) = 0 \end{cases}, \quad \forall \tau > t_{\text{sing}},$$

two trajectories bifurcate. This family of trajectories is then parametrized by τ instead of α . For $u \in \{-1, 1\}$ and for all $t > \tau$ the corresponding trajectory is

$$\begin{cases} \tilde{x}(t) = \left(-\sqrt{(2\eta+1)} - \eta\tau + \eta t_{\text{sing}}\right) \cos(t - \tau) - \eta \sin(t - \tau) \\ \tilde{y}(t) = u \left(-\eta + \left(-\sqrt{(2\eta+1)} - \eta\tau + \eta t_{\text{sing}}\right) \sin(t - \tau) + \eta \cos(t - \tau)\right), \end{cases}$$

the covector is

$$\begin{cases} p(t) = -u \cos(t - \tau) \\ q(t) = -\sin(t - \tau), \end{cases}$$

the switching functions, defined by (4.2) and (4.3), are

$$\begin{aligned} \phi_u(t) &= \eta - \eta \cos(t - \tau), \\ \phi_v(t) &= u \cos(t - \tau). \end{aligned}$$

Since in each case $\phi_u > 0$, the next switching occurs on the control v at time $T^{MsPp}(\tau) = T^{MsMm}(\tau) = \tau + \pi/2$.

The parametric equations of the corresponding switching curve is

$$\begin{cases} \tilde{x}_{\text{sw}}^{Msi}(\tau) = -\eta \\ \tilde{y}_{\text{sw}}^{Msi}(\tau) = u(-\eta - \sqrt{(2\eta+1)} - \eta\tau + \eta t_{\text{sing}}), \end{cases}$$

with $i = P$ (resp. M) for $u = 1$ (resp. -1).

A.1.2. Family 2. In this subsection, we study the trajectories γ^{MPp} switching from γ^{MP} at a time $T^{MPp}(\alpha)$ with a control $(1, 1)$ and $\alpha \in \left(\frac{\pi}{2}, \alpha_{\text{sing}}\right]$. These trajectories start from the parametrized curve

$$\begin{cases} \tilde{x}_{\text{sw}}^{MP}(\alpha) = (\eta + 1) \cos \alpha \\ \tilde{y}_{\text{sw}}^{MP}(\alpha) = -\eta - (\eta + 1) \sin \alpha + 2\eta \sqrt{1 - \left(\frac{\eta+1}{\eta} \cos \alpha\right)^2}, \end{cases} \quad (\text{A.1})$$

and, for all $t \geq T^{MPp}(\alpha)$, have coordinates

$$\begin{cases} \tilde{x}(t) = \tilde{x}_{\text{sw}}^{MP}(\alpha) \cos(t - T^{MPp}(\alpha)) - (1 + \tilde{y}_{\text{sw}}^{MP}(\alpha)) \sin(t - T^{MPp}(\alpha)) \\ \tilde{y}(t) = (1 + \tilde{y}_{\text{sw}}^{MP}(\alpha)) \cos(t - T^{MPp}(\alpha)) + \tilde{x}_{\text{sw}}^{MP}(\alpha) \sin(t - T^{MPp}(\alpha)) - 1 \end{cases} \quad (\text{A.2})$$

with $T^{MPp}(\alpha) = -\pi/2 - \alpha + 2 \arccos\left(\frac{\eta+1}{\eta} \cos \alpha\right) = -\frac{\pi}{2} + \alpha + 2T^{MP}(\alpha)$, i.e.

$$\left\{ \begin{array}{l} \tilde{x}(t) = (\eta - 1) \cos \left(t + \alpha - 2 \arccos \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \right) \\ \quad - 2\eta \sin \left(t + \alpha - \arccos \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \right) \\ \quad + (\eta + 1) \sin \left(t + 2\alpha - 2 \arccos \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \right) \\ \tilde{y}(t) = -1 - (\eta - 1) \sin \left(t + \alpha - 2 \arccos \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \right) \\ \quad + 2\eta \cos \left(t + \alpha - \arccos \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \right) \\ \quad - (\eta + 1) \cos \left(t + 2\alpha - 2 \arccos \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \right). \end{array} \right.$$

By continuity of the covector, we have $P(T^{MPp}(\alpha)) = (0, 1)$, the covector is then given by

$$\begin{cases} p(t) = -\sin(t - T^{MPp}(\alpha)) \\ q(t) = \cos(t - T^{MPp}(\alpha)) \end{cases}, \quad \forall t \geq T^{MPp}(\alpha).$$

Since the previous switching was a v -switching from η to 1, the next switching will be a v -switching (see Remark 3.17) and will occur after a duration π since $(\phi_u(t) = p(t))$ is a sinusoidal function). We then have $T^{MPpP}(\alpha) = T^{MPp}(\alpha) + \pi = \pi/2 - \alpha + 2 \arccos \left(\frac{\eta+1}{\eta} \cos \alpha \right)$.

The switching curve is then given by

$$\begin{cases} \tilde{x}_{sw}^{MPp}(\alpha) = \tilde{x}(T^{MPpP}(\alpha)) = \tilde{x}(T^{MPp}(\alpha) + \pi) \\ \tilde{y}_{sw}^{MPp}(\alpha) = \tilde{y}(T^{MPpP}(\alpha)) = \tilde{y}(T^{MPp}(\alpha) + \pi). \end{cases}$$

which, using (A.2), becomes

$$\begin{cases} \tilde{x}_{sw}^{MPp}(\alpha) = -\tilde{x}_{sw}^{MP}(\alpha) \\ \tilde{y}_{sw}^{MPp}(\alpha) = -2 - \tilde{y}_{sw}^{MP}(\alpha), \end{cases} \quad (\text{A.3})$$

where \tilde{x}_{sw}^{MP} and \tilde{y}_{sw}^{MP} are given by (A.1).

A.1.3. Family 3. In this subsection, we study the trajectories γ^{MmM} switching from γ^{Mm} at a time $T^{MmM}(\alpha) = \pi + T^{Mm}(\alpha) = -\alpha + 5\pi/2$ with a control $(-1, \eta)$ and $\alpha \in [\alpha_{\text{sing}}, \frac{3\pi}{2})$. These trajectories starts from the parametrized curve

$$\begin{cases} \tilde{x}_{sw}^{Mm}(\alpha) = -(\eta + 1) \cos \alpha \\ \tilde{y}_{sw}^{Mm}(\alpha) = (2 - \eta) - (\eta + 1) \sin \alpha. \end{cases}$$

For all $t \geq T^{MmM}(\alpha)$, the trajectories are

$$\begin{cases} \tilde{x}(t) = \tilde{x}_{sw}^{Mm}(\alpha) \cos(-t + T^{MmM}(\alpha)) + (\eta - \tilde{y}_{sw}^{Mm}(\alpha)) \sin(-t + T^{MmM}(\alpha)) \\ \tilde{y}(t) = \eta - (\eta - \tilde{y}_{sw}^{Mm}(\alpha)) \cos(-t + T^{MmM}(\alpha)) + \tilde{x}_{sw}^{Mm}(\alpha) \sin(-t + T^{MmM}(\alpha)), \end{cases}$$

which, after simplifications, yields

$$\begin{cases} \tilde{x}(t) = -(\eta + 1) \sin t + 2(\eta - 1) \cos(t + \alpha) \\ \tilde{y}(t) = \eta - 2(\eta - 1) \sin(t + \alpha) - (\eta + 1) \cos t. \end{cases}$$

By continuity, $P(T^{MmM}(\alpha)) = (0, 1)$ and for all $t \geq T^{MmM}(\alpha)$ the covector is

$$\begin{cases} p(t) = \sin(t - T^{MmM}(\alpha)) = \sin(t + \alpha - 5\pi/2) = \cos(t + \alpha) \\ q(t) = \cos(t - T^{MmM}(\alpha)) = \cos(t + \alpha - 5\pi/2) = -\sin(t + \alpha). \end{cases}$$

The switching functions, defined by (4.2) and (4.3), are then

$$\begin{aligned} \phi_u(t) &= (\eta + 1) \cos \alpha - \eta \cos(t + \alpha), \\ \phi_v(t) &= -\cos(t + \alpha). \end{aligned}$$

The analysis of these switching functions shows that

- for $\alpha \in [\alpha_{\text{sing}}, 2\pi - \arccos(-\eta/(\eta + 1))]$ ϕ_u is always negative. ϕ_v starts with positive values and changes sign at time $T^{MmMm}(\alpha) = T^{MmM}(\alpha) + \pi$, at which a v -switching occurs.
- for $\alpha \in [2\pi - \arccos(-\eta/(\eta + 1)), 3\pi/2]$ ϕ_u starts with negative values and ϕ_v with positive values. The function that has a zero first is ϕ_u and there is a u -switching at time $T^{MmMP}(\alpha) = T^{MmM}(\alpha) - \arcsin((\eta + 1)/\eta \cos \alpha)$

For $\alpha \in [2\pi - \arccos(-\eta/(\eta + 1)), 3\pi/2]$, the switching curve is given by

$$\begin{cases} \tilde{x}_{\text{sw}}^{MmM}(\alpha) = -(\eta + 1) \cos\left(-\alpha - \arcsin\left(\frac{\eta + 1}{\eta} \cos \alpha\right)\right) + 2\frac{(\eta^2 - 1)}{\eta} \cos \alpha \\ \tilde{y}_{\text{sw}}^{MmM}(\alpha) = \eta - 2(\eta - 1) \cos\left(\arcsin\left(\frac{\eta + 1}{\eta} \cos \alpha\right)\right) \\ \quad + (\eta + 1) \sin\left(-\alpha - \arcsin\left(\frac{\eta + 1}{\eta} \cos \alpha\right)\right). \end{cases}$$

We will see later that the switching curve for the other interval of α is not needed (see Remark A.1, page 283).

A.1.4. *Reduction of γ^{MPp} , γ^{Mm} and γ^{MmM} .* The plot of the trajectories computed previously (see left part of Fig. 8) shows that γ^{MPp} intersects with γ^{Mm} and γ^{MmM} . We then have to compute when the optimality of each trajectory is lost. This computation has been made numerically for $\eta = 2$ and indicates that

- for $\alpha \in [\alpha^{MPp}, \alpha_{\text{sing}}]$, with $\alpha^{MPp} \simeq 2.19947$, the trajectories γ^{MPp} loose their optimality. γ^{MPpP} is then defined for $\alpha \in (\pi/2, \alpha^{MPp})$.
- for $\alpha \in [\alpha_{\text{sing}}, \alpha^{MmM}]$, with $\alpha^{MmM} \simeq 3.84506$, the trajectories γ^{Mm} or γ^{MmM} loose their optimality. γ^{MmMP} is then defined for $\alpha \in (\alpha^{MmM}, 3\pi/2)$.

The times at which trajectories loose optimality, although not detailed here, allow to plot the corresponding cut locus (in green on the right part of Fig. 8). The rest of the synthesis will be restricted to the above-mentioned intervals.

A.2. 4th arc.

A.2.1. *Family 1.* In the following, the computations are made for γ^{MsPp} . Computations for γ^{MsMm} are obtained similarly since these two trajectories are symmetric with respect to the \tilde{x} axis. The trajectories γ^{MsPp} bifurcate at the time $T^{MsPp}(\tau) = \tau + \pi/2$ from

$$\begin{cases} \tilde{x}_{\text{sw}}^{MsP}(\tau) = -\eta \\ \tilde{y}_{\text{sw}}^{MsP}(\tau) = -\eta - \sqrt{(2\eta + 1)} - \eta\tau + \eta t_{\text{sing}}. \end{cases}$$

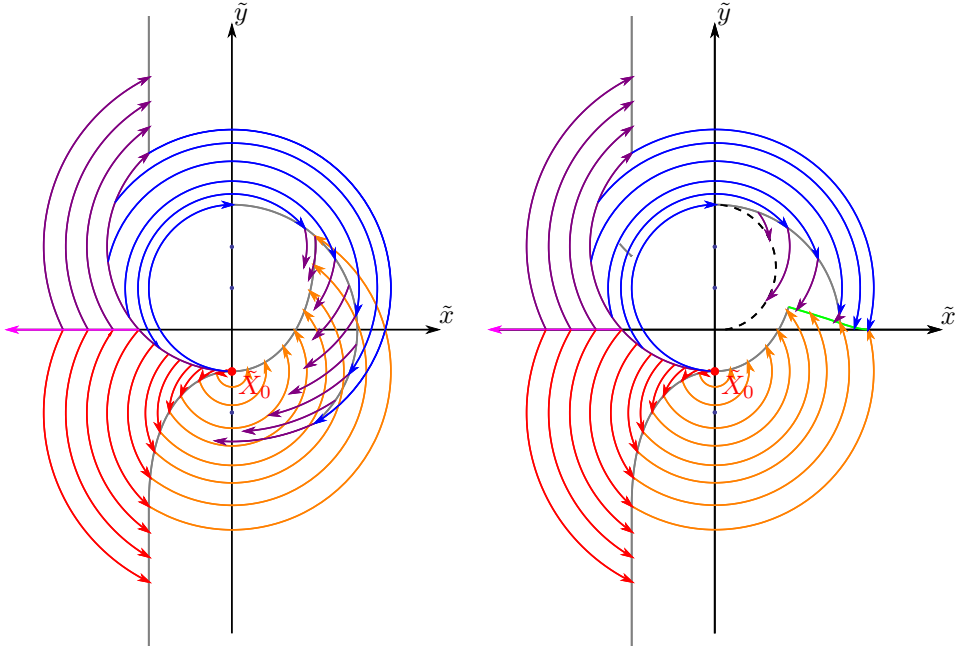


FIGURE 8. Intersection of γ^{MPP} with γ^{Mm} and γ^{MmM} (left) and the corresponding cut locus (right)

Then for all $t \geq T^{MsPp}(\tau)$

$$\begin{cases} \tilde{x}(t) = \tilde{x}_{sw}^{MsP}(\tau) \cos(t - T^{MsPp}(\tau)) - (1 + \tilde{y}_{sw}^{MsP}(\tau)) \sin(t - T^{MsPp}(\tau)) \\ \tilde{y}(t) = -1 + (1 + \tilde{y}_{sw}^{MsP}(\tau)) \cos(t - T^{MsPp}(\tau)) + \tilde{x}_{sw}^{MsP}(\tau) \sin(t - T^{MsPp}(\tau)), \end{cases}$$

which leads to

$$\begin{cases} \tilde{x}(t) = -\eta \sin(t - \tau) + \\ \quad (1 - \eta - \sqrt{(2\eta + 1)} - \eta\tau + \eta t_{\text{sing}}) \cos(t - \tau) \\ \tilde{y}(t) = -1 + \eta \cos(t - \tau) + \\ \quad (1 - \eta - \sqrt{(2\eta + 1)} - \eta\tau + \eta t_{\text{sing}}) \sin(t - \tau). \end{cases}$$

By continuity of the covector $P(T^{MsPp}(\tau)) = (0, -1)$ and the covector is

$$\begin{cases} p(t) = \sin(t - T^{MsPp}(\tau)) = -\cos(t - \tau) \\ q(t) = -\cos(t - T^{MsPp}(\tau)) = -\sin(t - \tau), \quad \forall t \geq T^{MsPp}(\tau). \end{cases}$$

Since the last switching was a v -switching from η to 1, the next switching will be a v -switching. Taking into account the form of ϕ_v , the switching will occur after a duration π . We conclude that $T^{MsPpP}(\tau) = T^{MsPp}(\tau) + \pi = \tau + 3\pi/2$.

The plot of the trajectories shows that trajectories γ^{MsMm} and γ^{MsPp} will necessarily intersect before reaching their next switching curve, which, therefore does not need to be computed.

Moreover, since for a same τ , $\gamma^{MsMm}(t)$ and $\gamma^{MsPp}(t)$ are symmetric with respect to the \tilde{x} axis, they will reach the \tilde{x} axis at the same time. We obtain that the set

$$\left\{ (\tilde{x}, \tilde{y}) \mid \tilde{x} > \eta^2 - 1 + \left(1 - \eta - \sqrt{(2\eta + 1)}\right)^2 \text{ and } \tilde{y} = 0 \right\}$$

forms a cut locus.

A.2.2. *Family 2.* Trajectories of Family 2 bifurcate from γ^{MPp} at time $T^{MPpP}(\alpha) = \pi/2 - \alpha + 2 \arccos\left(\frac{\eta+1}{\eta} \cos \alpha\right)$, with control $(1, \eta)$. Notice that after reduction of this family, we have $\alpha \in (\pi/2, \alpha^{MPp})$. γ^{MPp} starts from the switching curve

$$\begin{cases} \tilde{x}_{sw}^{MPp}(\alpha) = -(\eta + 1) \cos \alpha \\ \tilde{y}_{sw}^{MPp}(\alpha) = (\eta - 2) + (\eta + 1) \sin \alpha - 2\eta \sqrt{1 - \left(\frac{\eta + 1}{\eta} \cos \alpha\right)^2}, \end{cases}$$

and has, for all $t \geq T^{MPpP}(\alpha)$, coordinates

$$\begin{cases} \tilde{x}(t) = \tilde{x}_{sw}^{MPp}(\alpha) \cos(t - T^{MPpP}(\alpha)) - (\eta + \tilde{y}_{sw}^{MPp}(\alpha)) \sin(t - T^{MPpP}(\alpha)) \\ \tilde{y}(t) = -\eta + (\eta + \tilde{y}_{sw}^{MPp}(\alpha)) \cos(t - T^{MPpP}(\alpha)) + \tilde{x}_{sw}^{MPp}(\alpha) \sin(t - T^{MPpP}(\alpha)). \end{cases}$$

which is equivalent to

$$\begin{cases} \tilde{x}(t) = 2(\eta - 1) \cos\left(t + \alpha - 2 \arccos\left(\frac{\eta + 1}{\eta} \cos \alpha\right)\right) \\ \quad - 2\eta \sin\left(t + \alpha - \arccos\left(\frac{\eta + 1}{\eta} \cos \alpha\right)\right) \\ \quad + (\eta + 1) \sin\left(t + 2\alpha - 2 \arccos\left(\frac{\eta + 1}{\eta} \cos \alpha\right)\right) \\ \tilde{y}(t) = -\eta + 2(\eta - 1) \sin\left(t + \alpha - 2 \arccos\left(\frac{\eta + 1}{\eta} \cos \alpha\right)\right) \\ \quad + 2\eta \cos\left(t + \alpha - \arccos\left(\frac{\eta + 1}{\eta} \cos \alpha\right)\right) \\ \quad - (\eta + 1) \cos\left(t + 2\alpha - 2 \arccos\left(\frac{\eta + 1}{\eta} \cos \alpha\right)\right). \end{cases}$$

The covector is then

$$\begin{cases} p(t) = -\sin(t - T^{MPpP}(\alpha)) \\ q(t) = \cos(t - T^{MPpP}(\alpha)), \end{cases}$$

which leads to

$$\begin{cases} p(t) = \cos\left(t + \alpha - 2 \arccos\left(\frac{\eta + 1}{\eta} \cos \alpha\right)\right) \\ q(t) = \sin\left(t + \alpha - 2 \arccos\left(\frac{\eta + 1}{\eta} \cos \alpha\right)\right), \end{cases}$$

The switching functions, defined by (4.2) and (4.3), are then

$$\begin{aligned} \phi_u(t) &= -(\eta + 1) \cos \alpha - \eta \cos\left(t + \alpha - 2 \arccos\left(\frac{\eta + 1}{\eta} \cos \alpha\right)\right), \\ \phi_v(t) &= \cos\left(t + \alpha - 2 \arccos\left(\frac{\eta + 1}{\eta} \cos \alpha\right)\right). \end{aligned}$$

The analysis of these switching functions for $\alpha \in (\pi/2, \alpha^{MPp})$ shows that the next switching will be a u -switching that will occur at time

$$T^{MPpPM}(\alpha) = T^{MPpP}(\alpha) - \arcsin\left(\frac{\eta + 1}{\eta} \cos \alpha\right) = -\alpha + 3 \arccos\left(\frac{\eta + 1}{\eta} \cos \alpha\right).$$

The switching curve is then

$$\left\{ \begin{array}{l} \tilde{x}_{\text{sw}}^{MPpP}(\alpha) = \frac{\eta+1}{\eta} \cos \alpha \\ \quad \left(-3\eta \cos \left(\arcsin \left(\frac{\eta+1}{\eta} \cos \alpha \right) \right) + 2(\eta-1) \sin \alpha + (\eta+1) \sin \alpha \right) \\ \tilde{y}_{\text{sw}}^{MPpP}(\alpha) = -3\eta + (\eta+1) \cos \left(\arcsin \left(\frac{\eta+1}{\eta} \cos \alpha \right) \right) \sin \alpha \\ \quad + 3\eta \left(\frac{\eta+1}{\eta} \cos \alpha \right)^2 + 2(\eta-1) \cos \left(\arcsin \left(\frac{\eta+1}{\eta} \cos \alpha \right) \right). \end{array} \right.$$

A.2.3. *Reduction of γ^{MPpP} and γ^{MmM} .* The simulation shows that γ^{MPpP} and γ^{MmM} intersect themselves. The corresponding cut locus is computed numerically and it yields:

- for $\alpha \in [\alpha^{MPpP}, \alpha_{\text{sing}}]$, with $\alpha^{MPpP} \simeq 2.18628$, the trajectories γ^{MPpP} loose their optimality. γ^{MPpPM} is then defined for $\alpha \in (\pi/2, \alpha^{MPpP})$.
- for $\alpha \in [\alpha_{\text{sing}}, \alpha^{MmM'}]$, with $\alpha^{MmM'} \simeq 4.09691$, the trajectories γ^{MmM} loose their optimality. γ^{MmMP} is then defined for $\alpha \in (\alpha^{MmM'}, 3\pi/2)$.

The remaining part of the synthesis will be restricted to the above-mentioned intervals.

Remark A.1. Recall that the trajectories γ^{MmMm} are defined for $\alpha \in [\alpha_{\text{sing}}, 2\pi - \arccos(-\eta/(\eta+1))]$. Since $\eta = 2$, $\alpha^{MmM'} > 2\pi - \arccos(-\eta/(\eta+1))$, these arcs do not appear because γ^{MmM} loses its optimality before switching to γ^{MmMm} .

A.2.4. *Family 3.* In this subsection, we study the trajectories γ^{MmMP} switching from γ^{MmM} at a time $T^{MmMP}(\alpha) = -\alpha + 5\pi/2 - \arcsin((\eta+1)/\eta \cos \alpha)$ with a control $(1, \eta)$ and $\alpha \in (\alpha^{MmM'}, 3\pi/2)$. The starting points are on the parametrized curve

$$\left\{ \begin{array}{l} \tilde{x}_{\text{sw}}^{MmM}(\alpha) = -(\eta+1) \cos \left(-\alpha - \arcsin \left(\frac{\eta+1}{\eta} \cos \alpha \right) \right) + 2 \frac{(\eta^2-1)}{\eta} \cos \alpha \\ \tilde{y}_{\text{sw}}^{MmM}(\alpha) = \eta - 2(\eta-1) \cos \left(\arcsin \left(\frac{\eta+1}{\eta} \cos \alpha \right) \right) \\ \quad + (\eta+1) \sin \left(-\alpha - \arcsin \left(\frac{\eta+1}{\eta} \cos \alpha \right) \right). \end{array} \right.$$

For all $t \geq T^{MmMP}(\alpha)$, the trajectory is

$$\left\{ \begin{array}{l} \tilde{x}(t) = \tilde{x}_{\text{sw}}^{MmM}(\alpha) \cos(t - T^{MmMP}(\alpha)) - (\eta + \tilde{y}_{\text{sw}}^{MmM}(\alpha)) \sin(t - T^{MmMP}(\alpha)) \\ \tilde{y}(t) = -\eta + (\eta + \tilde{y}_{\text{sw}}^{MmM}(\alpha)) \cos(t - T^{MmMP}(\alpha)) \\ \quad + \tilde{x}_{\text{sw}}^{MmM}(\alpha) \sin(t - T^{MmMP}(\alpha)). \end{array} \right.$$

By continuity of the covector we have

$$\left\{ \begin{array}{l} p(T^{MmMP}(\alpha)) = - \left(\frac{\eta+1}{\eta} \cos \alpha \right) \\ q(T^{MmMP}(\alpha)) = \cos \left(\arcsin \left(\frac{\eta+1}{\eta} \cos \alpha \right) \right). \end{array} \right.$$

The covector is then given by

$$\begin{cases} p(t) = \cos \left(t + \alpha - 2 \arccos \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \right) \\ q(t) = \sin \left(t + \alpha - 2 \arccos \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \right) \end{cases}.$$

The switching functions, defined by (4.2) and (4.3), are then

$$\begin{aligned} \phi_u(t) &= \cos \left(\arcsin \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \right) \tilde{x}_{sw}^{MmM}(\alpha) + \left(\frac{\eta + 1}{\eta} \cos \alpha \right) (\eta + \tilde{y}_{sw}^{MmM}(\alpha)) \\ &\quad - \eta \cos \left(t + \alpha - 2 \arccos \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \right), \\ \phi_v(t) &= -\cos \left(t + \alpha - 2 \arccos \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \right). \end{aligned}$$

The analysis of these switching functions shows that the next switching will be a v -switching that will occur at time

$$\begin{aligned} T^{MmMPp}(\alpha) &= -\alpha + 5\pi/2 - 2 \arcsin \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \\ &= T^{MmMP}(\alpha) - \arcsin \left(\frac{\eta + 1}{\eta} \cos \alpha \right). \end{aligned}$$

This switching time leads to the switching curve:

$$\begin{cases} \tilde{x}_{sw}^{MmMP}(\alpha) = \tilde{x}_{sw}^{MmM}(\alpha) \cos \left(\arcsin \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \right) \\ \quad - (\eta + \tilde{y}_{sw}^{MmM}(\alpha)) \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \\ \tilde{y}_{sw}^{MmMP}(\alpha) = -\eta + (\eta + \tilde{y}_{sw}^{MmM}(\alpha)) \cos \left(\arcsin \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \right) \\ \quad + \tilde{x}_{sw}^{MmM}(\alpha) \left(\frac{\eta + 1}{\eta} \cos \alpha \right). \end{cases}$$

A.2.5. *Family 2.* In this subsection we study the trajectories γ^{MPpPM} switching from γ^{MPpP} at time $T^{MPpPM} = -\alpha + 3 \arccos \left(\frac{\eta + 1}{\eta} \cos \alpha \right)$ with control $(-1, \eta)$ and $\alpha \in (\pi/2, \alpha^{MPpP})$. We recall

$$\begin{cases} \tilde{x}_{sw}^{MPpP}(\alpha) = \frac{\eta + 1}{\eta} \cos \alpha \\ \quad \left(-3\eta \cos \left(\arcsin \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \right) + 2(\eta - 1) \sin \alpha + (\eta + 1) \sin \alpha \right) \\ \tilde{y}_{sw}^{MPpP}(\alpha) = -3\eta + (\eta + 1) \cos \left(\arcsin \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \right) \sin \alpha \\ \quad + 3\eta \left(\frac{\eta + 1}{\eta} \cos \alpha \right)^2 + 2(\eta - 1) \cos \left(\arcsin \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \right). \end{cases}$$

The trajectories are then

$$\begin{cases} \tilde{x}(t) = \tilde{x}_{sw}^{MPpP}(\alpha) \cos(t - T^{MPpPM}(\alpha)) - (\eta - \tilde{y}_{sw}^{MPpP}(\alpha)) \sin(t - T^{MPpPM}(\alpha)) \\ \tilde{y}(t) = \eta - (\eta - \tilde{y}_{sw}^{MPpP}(\alpha)) \cos(t - T^{MPpPM}(\alpha)) \\ \quad - \tilde{x}_{sw}^{MPpP}(\alpha) \sin(t - T^{MPpPM}(\alpha)). \end{cases}$$

The covector satisfies

$$\begin{cases} p(T^{MPpPM}(\alpha)) = \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \\ q(T^{MPpPM}(\alpha)) = \sin \left(\arccos \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \right), \end{cases}$$

which leads to

$$\begin{cases} p(t) = \cos \left(t + \alpha - 4 \arccos \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \right) \\ q(t) = -\sin \left(t + \alpha - 4 \arccos \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \right). \end{cases}$$

We have then the switching functions, defined by (4.2) and (4.3),

$$\begin{aligned} \phi_u(t) &= \tilde{x}_{sw}^{MPpP}(\alpha) \sqrt{1 - \left(\frac{\eta + 1}{\eta} \cos \alpha \right)^2} + (\eta - \tilde{y}_{sw}^{MPpP}(\alpha)) \frac{\eta + 1}{\eta} \cos \alpha \\ &\quad - \eta \cos \left(t + \alpha - 4 \arccos \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \right), \\ \phi_v(t) &= -\cos \left(t + \alpha - 4 \arccos \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \right). \end{aligned}$$

The analysis of these switching functions shows that the next switching will be a v -switching and will occur at time

$$\begin{aligned} T^{MPpPMm}(\alpha) &= T^{MPpPM}(\alpha) - \arcsin \left(\frac{\eta + 1}{\eta} \cos \alpha \right) \\ &= -\alpha - \frac{\pi}{2} + 4 \arccos \left(\frac{\eta + 1}{\eta} \cos \alpha \right). \end{aligned}$$

The switching curve is then

$$\begin{cases} \tilde{x}_{sw}^{MPpPM}(\alpha) = \tilde{x}_{sw}^{MPpP}(\alpha) \sqrt{1 - \left(\frac{\eta + 1}{\eta} \cos \alpha \right)^2} + (\eta - \tilde{y}_{sw}^{MPpP}(\alpha)) \frac{\eta + 1}{\eta} \cos \alpha \\ \tilde{y}_{sw}^{MPpPM}(\alpha) = \eta - (\eta - \tilde{y}_{sw}^{MPpP}(\alpha)) \sqrt{1 - \left(\frac{\eta + 1}{\eta} \cos \alpha \right)^2} \\ \quad + \tilde{x}_{sw}^{MPpP}(\alpha) \frac{\eta + 1}{\eta} \cos \alpha. \end{cases}$$

A.2.6. Reduction of γ^{MPpP} and γ^{MPpPM} . The simulation shows that γ^{MPpP} and γ^{MPpPM} intersect themselves. Notice that this is a particular case because these trajectories belong to the same family (see e.g. Fig. 9). The corresponding cut locus is computed numerically and it yields that for $\alpha \in [\alpha^{MPpPM}, \alpha^{MPpP})$, with $\alpha^{MPpPM} = 2.13033$, the trajectories γ^{MPpP} loose their optimality. γ^{MPpPMm} is then defined for $\alpha \in (\pi/2, \alpha^{MPpPM})$.

The rest of the synthesis will be restricted to the above-mentioned intervals.

A.3. 5th arc:

A.3.1. *Family 3.* In this subsection we study the trajectories γ^{MmMPp} switching from γ^{MmMP} at time $T^{MmMPp} = -\alpha + 5\pi/2 - 2 \arcsin\left(\frac{\eta+1}{\eta} \cos \alpha\right)$ with control $(1, 1)$ and $\alpha \in (\alpha^{MmM'}, 3\pi/2)$. We recall that these trajectories start from

$$\left\{ \begin{array}{l} \tilde{x}_{sw}^{MmMP}(\alpha) = \tilde{x}_{sw}^{MmM}(\alpha) \cos\left(\arcsin\left(\frac{\eta+1}{\eta} \cos \alpha\right)\right) \\ \quad - (\eta + \tilde{y}_{sw}^{MmM}(\alpha)) \left(\frac{\eta+1}{\eta} \cos \alpha\right) \\ \tilde{y}_{sw}^{MmMP}(\alpha) = -\eta + (\eta + \tilde{y}_{sw}^{MmM}(\alpha)) \cos\left(\arcsin\left(\frac{\eta+1}{\eta} \cos \alpha\right)\right) \\ \quad + \tilde{x}_{sw}^{MmM}(\alpha) \left(\frac{\eta+1}{\eta} \cos \alpha\right), \end{array} \right.$$

For all $t \geq T^{MmMPp}(\alpha)$, the trajectories are

$$\left\{ \begin{array}{l} \tilde{x}(t) = \tilde{x}_{sw}^{MmMP}(\alpha) \cos(t - T^{MmMPp}(\alpha)) \\ \quad - (1 + \tilde{y}_{sw}^{MmMP}(\alpha)) \sin(t - T^{MmMPp}(\alpha)) \\ \tilde{y}(t) = -1 + (\eta + \tilde{y}_{sw}^{MmMP}(\alpha)) \cos(t - T^{MmMPp}(\alpha)) \\ \quad + \tilde{x}_{sw}^{MmMP}(\alpha) \sin(t - T^{MmMPp}(\alpha)), \end{array} \right.$$

Since the previous switching occurred on v , the next switching will also be on v and will occur after a time π (following the same reasoning as previously).

At this step of the construction of the synthesis, we immediately see that the next switching will not occur. Indeed, these trajectories will necessarily intersect with trajectories found earlier in the construction of the synthesis before reaching their switching curve.

A.3.2. *Family 2.* In this subsection we study the trajectories γ^{MPpPMm} switching from γ^{MPpPM} at time $T^{MPpPMm} = -\alpha - \frac{\pi}{2} + 4 \arccos\left(\frac{\eta+1}{\eta} \cos \alpha\right)$ with control $(-1, 1)$ and $\alpha \in (\pi/2, \alpha^{MPpPM})$.

The trajectories are

$$\left\{ \begin{array}{l} \tilde{x}(t) = \tilde{x}_{sw}^{MPpPM}(\alpha) \cos(t - T^{MPpPMm}(\alpha)) \\ \quad - (1 - \tilde{y}_{sw}^{MPpPM}(\alpha)) \sin(t - T^{MPpPMm}(\alpha)) \\ \tilde{y}(t) = 1 - (1 - \tilde{y}_{sw}^{MPpPM}(\alpha)) \cos(t - T^{MPpPMm}(\alpha)) \\ \quad - \tilde{x}_{sw}^{MPpPM}(\alpha) \sin(t - T^{MPpPMm}(\alpha)). \end{array} \right.$$

Since the previous switching occurred on v , the next switching will also be on v and will occur after a time π (following the same reasoning as previously).

At this step of the construction of the synthesis, we immediately see that the next switching will not occur. Indeed, these trajectories will necessarily intersect with trajectories found earlier in the construction of the synthesis before reaching their switching curve.

A.3.3. *Reduction of γ^{MmMP} , γ^{MmMPp} , γ^{MPpPM} and γ^{MPpPMm} .* The simulation shows that γ^{MPpPMm} intersects with γ^{MmMP} , γ^{MmMPp} and γ^{MPpPM} (see Fig. 9). The corresponding cut locus is computed numerically.

It also appears that γ^{MmMPp} intersects with the abnormal trajectory γ^m . Clearly, γ^{MmMPp} loses its optimality at this intersection. A part of the abnormal trajectory is then a cut locus (in cyan on Fig. 3).

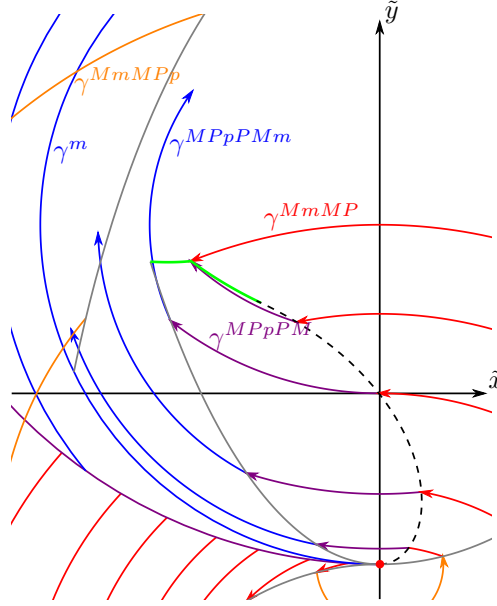


FIGURE 9. Intersection of γ^{MPpPMm} with γ^{MmMP} , γ^{MmMPp} and γ^{MPpPM}

Since no new trajectories can be generated, the construction of the synthesis stops.

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Arc $(-1, 1)$ -bang	Bleu
Arc $(1, 1)$ -bang	Orange
Arc $(-1, \mu)$ -bang	Violet
Arc $(1, \mu)$ -bang	Rouge
Arc u -singulier	Magenta
Courbe de switch sur u	Noir (pointillé)
Courbe de switch sur v	Gris
Cut locus	Vert
Cut locus anormal	Cyan

TABLE 1.1 Code couleur correspondant aux synthèses temps minimum Fig. 1.3 et 1.4.

3.2. Résultats

L'article exposé dans la sous-section 3.1 donne la synthèse temps minimum pour le modèle cinématique de drone dans les coordonnées (\tilde{x}, \tilde{y}) (voir figure 1.3 page 46). De manière équivalente, le résultat est un retour d'état stabilisant en temps minimum ni le drone sur un cercle de rayon de courbure minimum. Les figures 1.1 et 1.2 montrent des exemples de trajectoires optimales partant d'un cut locus dans les coordonnées (\tilde{x}, \tilde{y}) et les trajectoires correspondantes dans les coordonnées (x, y) . Sur chaque figure, les trajectoires mettent le même temps à atteindre la cible. Notons que pour la figure 1.1 la position initiale correspond à un drone démarrant avec la cible "dans le dos".

La figure 1.4 page 47 présente la synthèse temps minimum pour un modèle cinématique de drone à un contrôle ($\mu = v_{\max}/v_{\min} = 1$). Cette figure a été reproduite, avec l'accord des auteurs, en reprenant les résultats exposés dans Maillot *et al.* [56]. Bien que la méthode utilisée soit similaire, la synthèse à deux contrôles est plus complexe puisque certaines trajectoires optimales sont composées de 6 arcs, alors que le maximum était de 3 pour le cas à un contrôle. Par ailleurs, contrairement au cas à un contrôle, il existe une extrémale anormale. Cette extrémale a pour particularité d'être extrémale pour tout critère L car $\lambda = 0$ dans le PMP. Une trajectoire anormale est donc indépendante du critère (mais elle n'est pas forcément optimale). Notons que contrairement au cas sous-riemannien, les trajectoires anormales et singulières ne correspondent pas dans notre cas.

La comparaison des deux synthèses montre des similitudes, en particulier sur la forme générale, la présence d'une trajectoire singulière et l'emplace-

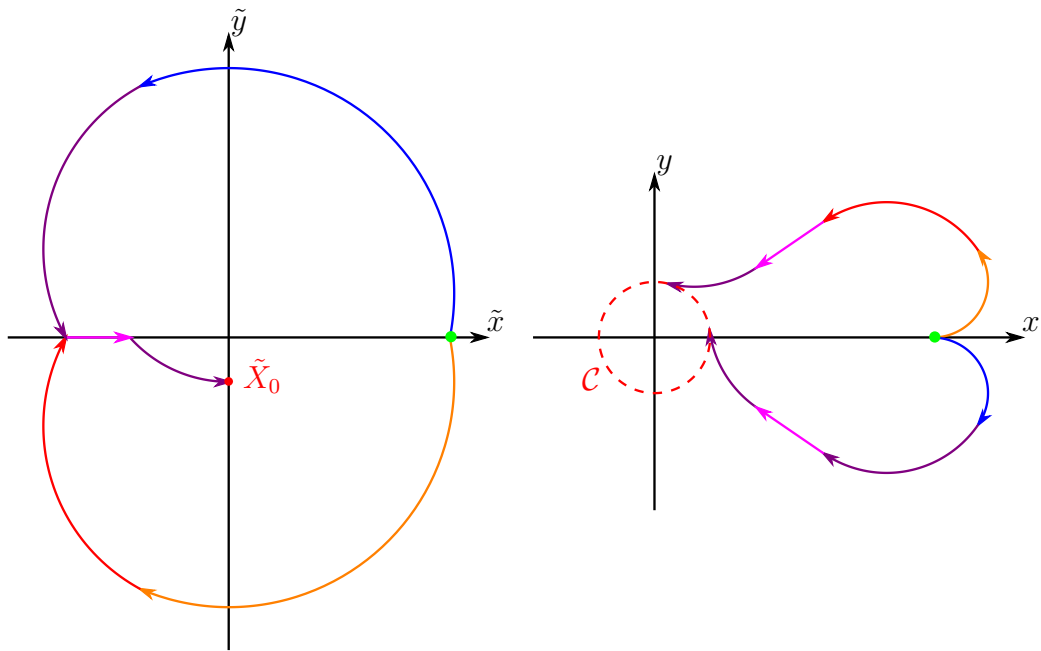


FIGURE 1.1 Exemple de trajectoires optimales partant d'un cut locus dans les coordonnées (\tilde{x}, \tilde{y}) et (x, y)

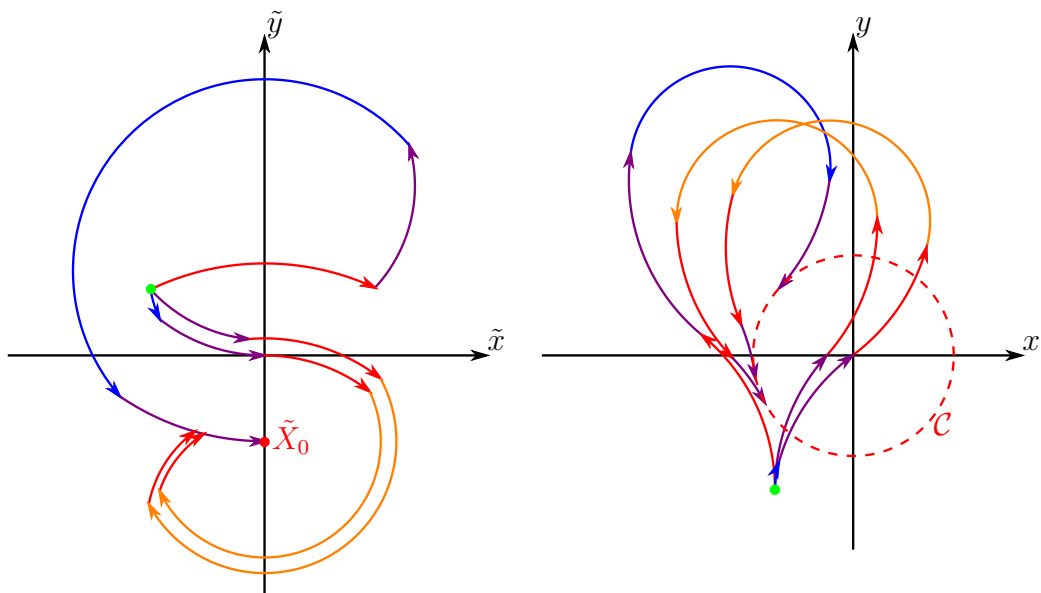


FIGURE 1.2 Exemple de trajectoires optimales partant d'un cut locus dans les coordonnées (\tilde{x}, \tilde{y}) et (x, y)

ment des cut loci. On remarque également la complexité accrue de la synthèse au niveau du cut locus dans le quadrant supérieur gauche. En effet, ce dernier possède une branche supplémentaire, formant le point P_t à partir duquel trois trajectoires optimales démarrent (voir figures 1.2 et 1.3).

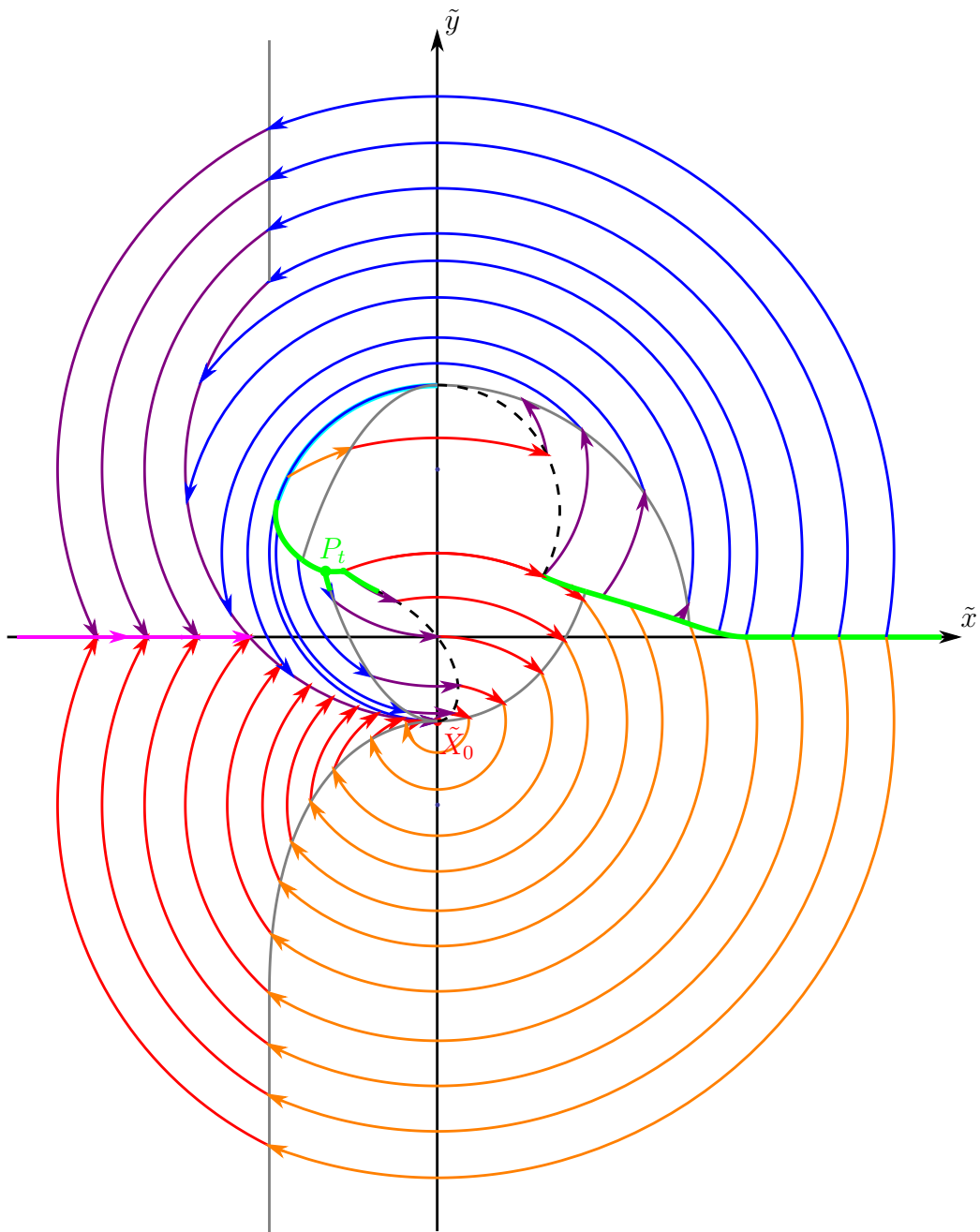


FIGURE 1.3 Synthèse temps minimum pour un modèle cinématique de drone à deux contrôles dans l'espace réduit (\tilde{x}, \tilde{y})

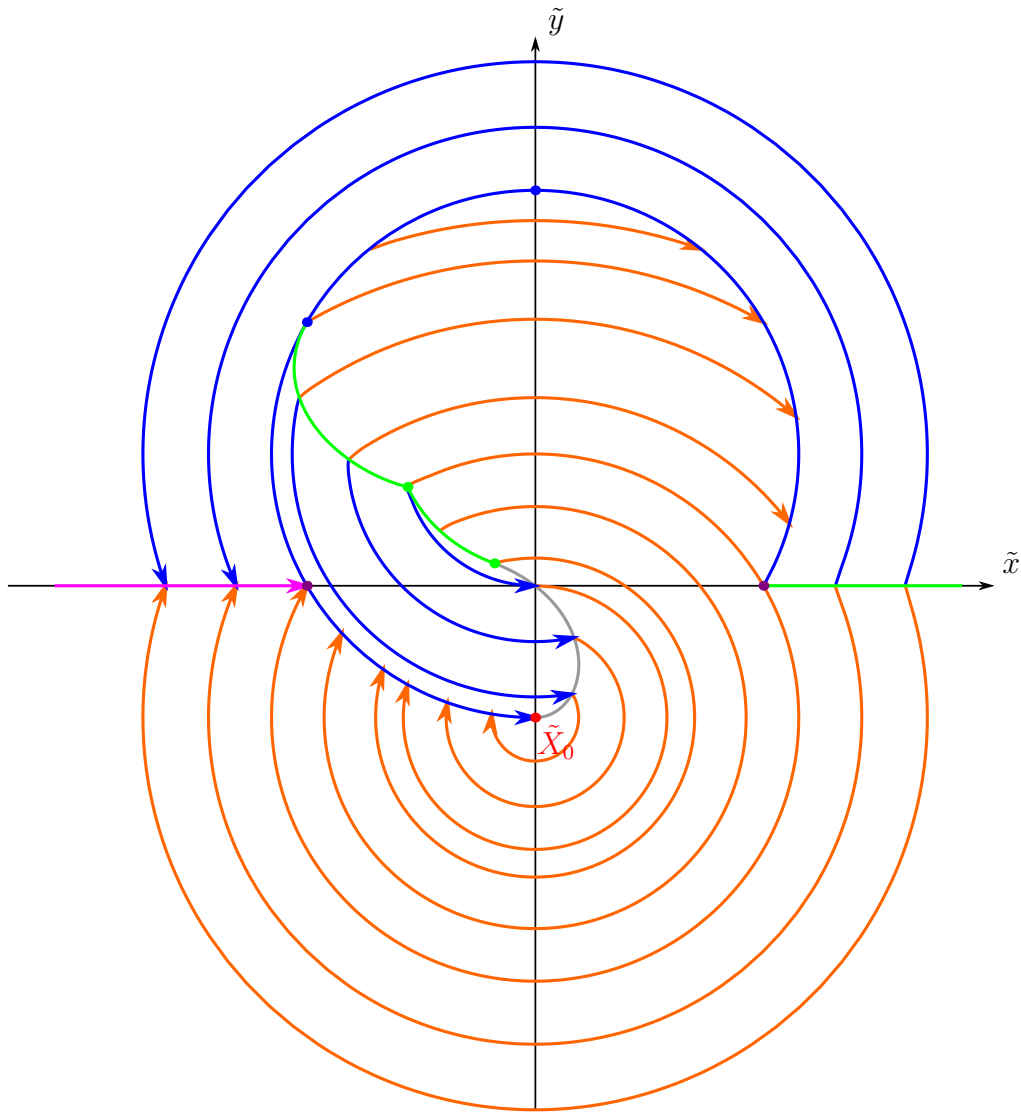


FIGURE 1.4 Synthèse temps minimum pour un modèle cinématique de drone à un contrôle dans l'espace réduit (\tilde{x}, \tilde{y}) (figure reproduite à partir de Maillot *et al.* [56])

4. Conclusion et discussions

Dans ce chapitre, nous avons vu les outils théoriques permettant de résoudre un problème de contrôle optimal et comment les appliquer afin de construire une synthèse temps-minimum pour un modèle cinématique de drone. L'algorithme de construction de la synthèse est également donné.

Une comparaison a été faite avec le résultat existant à un contrôle. Compte tenu de la ressemblance entre les deux résultats, il semble naturel de se demander si la synthèse temps-minimal pour le modèle (1.1) est continue par rapport à $\mu = v_{\max}/v_{\min}$ et en particulier en $\mu = 1$. Une étude approfondie est nécessaire pour bien définir et répondre à cette question. En particulier, il semble nécessaire de définir et étudier la continuité d'une synthèse optimale vis à vis de l'intervalle de contrôle du système.

Une autre étude, plus appliquée, serait d'implémenter cette synthèse sur le système embarqué d'un drone. Cependant, deux difficultés apparaissent.

Certains cut loci sont calculés de manière numérique et il est par conséquent difficile de savoir en pratique où se situe le drone par rapport à ces cut loci. De la même manière, les mesures de position ayant une précision limitée, il n'est pas possible de suivre très exactement une trajectoire singulière. Pour palier à ces problèmes il est possible, comme cela est fait dans Prieur et Trélat [65], de définir une loi de commutation et d'utiliser des cycles d'hystérésis.

Les trajectoires optimales données dans cette synthèse correspondent à un modèle cinématique. Le drone ne pouvant pas changer de vitesse instantanément, on ne pourra pas en pratique suivre exactement ces trajectoires. L'étude d'une synthèse temps-minimum pour un modèle dynamique de drone pourrait être intéressante.

Chapitre 2

Généricité en contrôle optimal inverse

Dans ce chapitre, nous étudions un problème inverse du précédent. Contrairement au chapitre 1, où l'objectif était de calculer les trajectoires minimisant un critère donné, l'objectif est maintenant de retrouver le critère minimisé par des trajectoires optimales données. Nous commençons par l'introduction du problème et nous énonçons notre résultat principal, qui est ensuite démontré sous forme d'article scientifique. Nous donnons à la fin du chapitre des remarques sur nos résultats et de possibles relaxations des hypothèses.

1. Introduction

D'après un paradigme largement accepté en neurophysiologie, parmi tous les mouvements possibles ceux effectivement réalisés sont solutions d'un processus d'optimisation (voir par exemple Li *et al.* [54]). On peut alors se demander, à partir d'observations de mouvements humains quel est le critère qu'ils minimisent.

Une application du contrôle optimal inverse peut être de décrire l'Homme ou l'animal comme un système dynamique (mouvements du bras, locomotion...) et de supposer que les mouvements observés sont la solution d'un problème de contrôle optimal. L'objectif est alors de déterminer comment construire des expériences permettant la reconstruction du critère à partir des mouvements observés.

Un des intérêts du contrôle optimal inverse réside dans un premier temps dans une meilleure compréhension des mouvements et des comportements.

Dans un second temps, à partir du critère reconstruit, il est possible d'implémenter sur un système embarqué un algorithme de contrôle optimal afin d'obtenir des systèmes autonomes (robots, drones, véhicules...) dont le comportement est proche de celui de l'Homme.

1.1. État de l'art

Les premières études des mouvements humains ou animaux datent des années 1870 avec Étienne-Jules Marey (voir Marey [58]), qui a en particulier inventé la chronophotographie (précurseur du cinéma) afin d'étudier les mouvements des ailes d'oiseaux et de la marche humaine. Ces études constituent une approche physiologique, qui est aujourd'hui assez bien comprise.

Plus récemment, la locomotion humaine a été étudiée en utilisant la théorie du contrôle. Il a été démontré dans Arechavaleta *et al.* [5]; Bayen *et al.* [8] que la locomotion humaine pouvait être décrite par un système dynamique non holonome. À partir de ce modèle, des trajectoires expérimentales ont été étudiées comme solutions d'un problème de contrôle optimal (voir Arechavaleta *et al.* [6]; Mombaur *et al.* [62]). Ces études font des hypothèses *a priori* sur l'expression du critère minimisé (e.g. polynomial) et se ramène à un problème de dimension finie. Par ailleurs, la résolution du problème de contrôle optimal est faite de manière numérique. Ce type d'approche a permis de s'appliquer expérimentalement une loi de contrôle basée sur le critère reconstruit sur des robots (voir Mombaur *et al.* [61]).

À la suite des études précédentes, la locomotion humaine a également été traitée dans Chitour *et al.* [21]; Chittaro *et al.* [23]; Jean *et al.* [43]. Dans ces études, des hypothèses sont faites sur la forme du critère mais l'objectif reste de reconstruire une fonction (et non un polynôme) et le problème étudié reste donc de dimension infinie.

Dans un cadre plus général, pour les systèmes linéaires à un contrôle, la question de savoir quel critère quadratique était minimisé par une loi de contrôle donnée a été abordée par Kalman [46]. À la suite de cette étude, les résultats de Kalman ont été étendus aux systèmes multivariables (voir par exemple Jameson et Kreindler [42]; Vidyasagar [74]). Des extensions aux systèmes non linéaires ont également été considérées (voir par exemple Thau [72]).

1.2. Description du problème

Les travaux exposés dans ce chapitre reprennent la méthodologie basée sur la transversalité de Thom et décrite dans Ajami *et al.* [3]; Berret *et al.* [9; 10]. En particulier, le but est de généraliser les résultats de Ajami *et al.* [3]. Plus précisément, considérons un système dynamique sous la forme

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = u, \end{cases} \quad (2.1)$$

où $(x, y) \in \mathbb{R}^n \times \mathbb{R}^d$ est l'état et $u \in \mathbb{R}^d$ est le contrôle et un coût

$$C(u(\cdot)) = \int_0^T L(u(t)) dt.$$

Nous supposons que nous avons à notre disposition des trajectoires et des contrôles optimaux de classe C^∞ , c'est à dire des trajectoires solutions de (2.1) dont les contrôles minimisent le coût $C(\cdot)$. L'objectif est d'être capable de reconstruire le critère L à partir des ces données. Nous pouvons d'ores et déjà remarquer que cela constitue un problème mal posé. En effet, il est aisé de voir que si une trajectoire minimise le critère L , elle minimisera également tous les critères de forme aL avec $a > 0$. Il n'y a donc pas unicité de la solution au problème de contrôle optimal inverse. Évidemment, la non unicité de la solution constituera une difficulté à la résolution du problème.

Quant au critère, nous le considérons, comme dans Ajami *et al.* [3], dans la classe des fonctions $L : \mathbb{R}^d \rightarrow \mathbb{R}$ de classe C^∞ , positives, strictement convexes et ayant une croissance super-linéaire. Nous ajoutons également l'hypothèse que nous avons une connaissance *a priori* du critère à reconstruire sur une hypersurface de \mathbb{R}^{d-1} . Cette hypothèse est forte et nous n'avons, pour l'instant, pas réussi à la relaxer. Nous exposons tout de même des idées pouvant mener à la relaxation de cette hypothèse.

Une différence importante entre les cas $d = 1$ et $d > 1$ est l'ensemble des transformations du critère qui laissent invariantes les trajectoires (ou au moins les extrémales). Dans les deux cas les transformations qui s'écrivent $\tilde{L} = aL + \langle b, u \rangle$, avec $a \in \mathbb{R}_{>0}$ et $b \in \mathbb{R}^d$ laissent les trajectoires optimales invariantes. Cependant, alors que pour un contrôle ce sont les seules transformations admissibles, pour plusieurs contrôles il peut en exister d'autres et il est très difficile de les déterminer puisque qu'elles dépendent du critère, qui est une inconnue du problème.

1.3. Résultat

Le résultat principal de Ajami *et al.* [3], qui concerne les systèmes à un contrôle de la forme (2.1) (avec $d = 1$), est (en substance) le suivant.

Théorème 2.1. *Pour reconstruire le critère L , en général trois expériences sont suffisantes pourvu qu'elle visite les mêmes valeurs de contrôle, à certains instants (qui peuvent être différents). Deux expériences ne sont en général pas suffisantes.*

Notre résultat est la généralisation du théorème 2.1 aux systèmes à d contrôles de la forme (2.1) :

Théorème 2.2. *Pour reconstruire le critère L , en général $d + 2$ expériences à $d - 1$ paramètres sont suffisantes pourvu qu'elle visite les mêmes valeurs de contrôle, à certains instants (qui peuvent être différents) et que le critère soit connu a priori sur une hypersurface.*

Ce théorème est détaillé et démontré dans l'article exposé section 3.

2. Préliminaires et outils théoriques

Le contrôle optimal inverse consiste dans un premier temps à étudier le problème de contrôle optimal correspondant. En effet, la résolution du problème direct va permettre de déduire les équations satisfaites par les solutions du problème inverse. Le principe du maximum de Pontryagin est donc un outil primordial dans l'étude qui suit (voir section 2 chapitre 1). Nous exposons dans cette section les autres outils nécessaires à la démonstration de notre résultat, à savoir le produit tensoriel qui permettra dans un second temps de simplifier les calculs différentiels, les k -jets et des éléments de la théorie de la transversalité de Thom.

2.1. Produit tensoriel

Soit E et F deux espaces vectoriels sur un corps commutatif K . Il existe un unique espace vectoriel (à un isomorphisme près), noté $E \otimes F$, et une application bilinéaire

$$\phi : E \times F \rightarrow E \otimes F, \quad (x, y) \mapsto \phi(x, y) = x \otimes y$$

telle que pour tout espace vectoriel G et toute application bilinéaire $g : E \times F \rightarrow G$, il existe une unique application linéaire $\tilde{g} : E \otimes F \rightarrow G$ telle que $g = \tilde{g} \circ \phi$.

2.2. Différentielle et produit tensoriel

Soit E , F et G trois espaces de Banach, $U \subset E$ et $V \subset F$ deux ouverts et $g : U \rightarrow V$ et $f : V \rightarrow G$. D'après la règle de composition énoncée dans Abraham et Robbin [1], si f et g sont de classe C^k alors $f \circ g : U \rightarrow G$ est de classe C^k et pour tout $x \in U$

$$D(f \circ g)(x) = Df \circ g(x)Dg(x)$$

et

$$D^2(f \circ g)(x) = ((D^2f \circ g(x))Dg(x))Dg(x) + (Df \circ g(x))D^2g(x).$$

Comme cela est souligné dans Manton [57], l'application de ces règles n'est pas évidente. En effet, multiplier $D^2f \circ g(x)$ par $Dg(x)$ deux fois "n'a pas de sens". De manière rigoureuse, $D^2(f \circ g)(x) : U \times U \rightarrow G$ est une application bilinéaire et pour tout $(v, w) \in U \times U$

$$D^2(f \circ g)(x)[v, w] = (D^2f \circ g(x)(Dg(x)[v], Dg(x)[w])) + (Df \circ g(x))D^2g(x)(v, w).$$

La différentielle seconde étant une application bilinéaire, le produit tensoriel exposé dans la sous-section 2.1 permet d'alléger les notations (en éliminant les vecteurs v et w) tout en gardant de la rigueur. D'après Manton [57], on obtient alors les règles de dérivation suivantes.

$$\begin{aligned} D(f \circ g)(x) &= (Df(x) \circ g(x))(Dg(x) \otimes I) \\ D(fg)(x) &= (Df(x))(I \otimes g(x)) + f(x)Dg(x) \\ D(f \otimes g)(x) &= (Df(x) \otimes g(x))(f(x) \otimes Dg(x)) \end{aligned}$$

En particulier, d'après ces règles, la différentielle seconde de la composition devient l'application linéaire $D^2(f \circ g)(x) : U \otimes U \rightarrow G$

$$D^2(f \circ g)(x) = (D^2f \circ g(x))(Dg(x) \otimes Dg(x)) + (Df \circ g(x))D^2g(x).$$

On peut de la même manière calculer toutes les différentielles successives.

2.3. k -jets

Dans toute cette sous-section, X et Y désignent des variétés lisses. Les notions de géométrie différentielle ne sont pas rappelées. Pour plus de détails sur les variétés lisses, voir par exemple Lee [53].

Dé nition 2.3 (voir Hirsch [37]). Un k -jet de X dans Y est une classe d'équivalence $[x, f, U]_k$ du triplet (x, f, U) où $U \subset M$ est un ouvert de carte, $x \in U$ et $f : U \rightarrow Y$ est de classe C^∞ . La relation d'équivalence est définie par $[x, f, U]_k = [x', f', U']_k$ si $x = x'$ et f et f' ont les mêmes dérivées au point x jusqu'à l'ordre k .

On notera $[x, f, U]_k = j_x^k f$ le k -jet de f au point x , $J^k(X, Y)$ l'ensemble des k -jets de X dans Y et $J_x^k(X, Y) = \{j_{x'}^k \in J^k(X, Y) \mid x' = x\}$ l'ensemble des k -jets de X dans Y au point x . On définit également le k -jet de f comme étant l'application $j^k f : X \rightarrow J^k(X, Y)$, $x \mapsto j_x^k f$.

Par la suite, nous nous intéresserons au cas particulier où X et Y sont des ouverts connexes de \mathbb{R}^m et \mathbb{R}^n respectivement. On écrira alors $J^k(X, Y) = J^k(m, n)$. Dans ce cas, un élément $j_x^k f \in J^k(m, n)$ peut-être identifié de manière canonique au polynôme de Taylor de f à l'ordre k défini par

$$P_x^k f(z) = f(x) + Df(x)[z] + D^2 f(x)[z^{\otimes 2}] \dots D^k f(x)[z^{\otimes k}].$$

On en déduit que $J_x^k(m, n)$ est un espace vectoriel isomorphe à

$$\prod_{i=0}^k L_{sym}^i(\mathbb{R}^m, \mathbb{R}^n) = \mathbb{R}^n \otimes \mathbb{R}_k[z_1, \dots, z_m],$$

où $L_{sym}^i(\mathbb{R}^m, \mathbb{R}^n)$ désigne l'ensemble des applications i -linéaires symétriques de \mathbb{R}^m dans \mathbb{R}^n et $\mathbb{R}_k[z_1, \dots, z_m]$ désigne l'ensemble des polynômes à m indéterminées et de degré inférieur ou égal à k . Par conséquent, $J^k(m, n)$ est un espace vectoriel isomorphe à $\mathbb{R}^n \otimes \mathbb{R}_k[z_1, \dots, z_m] \times \mathbb{R}^m$.

Notons que la dimension de l'ensemble des polynômes homogènes à m indéterminées et de degré k est égal au nombre de k -combinaisons avec répétition d'un ensemble à n éléments, c'est à dire

$$\binom{m+k-1}{k} = \frac{(m+k-1)!}{k!(m-1)!}.$$

On en déduit que

$$\dim J_x^k(m, n) = n \sum_{i=1}^k \binom{m+i-1}{i} = n \frac{(k+m)!}{m!k!}$$

et

$$\dim J^k(m, n) = m + \dim J_x^k(m, n) = m + n \frac{(k+m)!}{m!k!}.$$

2.4. Transversalité et théorie de Thom

Nous exposons dans cette sous-section quelques éléments concernant la transversalité et la théorie de Thom. Pour plus de détails voir Golubitsky et Guillemin [31]; Goresky et MacPherson [32]; Guillemin et Pollack [33]; Hirsch [37]; Laudénbach [52]. Par la suite nous munissons l'ensemble $C^\infty(X, Y)$ de la topologie C^∞ de Whitney (voir par exemple Golubitsky et Guillemin [31]; Hirsch [37]).

Définition 2.4. Soit X et Y deux variétés lisses, $f : X \rightarrow Y$ une application de classe C^∞ et $W \subset Y$ une sous-variété. On dit que f est *transverse* à W en $x \in X$ si

$$f(x) \notin W \text{ ou } f(x) \in W \text{ et } T_{f(x)}Y = T_{f(x)}W + \text{Im } Df_x,$$

et cela se note $f \pitchfork_x W$. De plus, on dit que f est transverse à W ($f \pitchfork W$) si $f \pitchfork_x W$ pour tout $x \in X$.

Remarque 2.5. Si $f : X \rightarrow Y$ est une submersion, c'est à dire que sa différentielle en tout point est surjective, alors f est transverse à toute sous-variété $W \subset Y$.

Proposition 2.6. Soit X et Y deux variétés lisses et $W \subset Y$ une sous-variété. Supposons que $\dim W + \dim X < \dim Y$ (i.e., $\dim X < \text{codim}_Y W$). Soit $f : X \rightarrow Y$ une application de classe C^∞ telle que $f \pitchfork W$. Alors $f(X) \cap W = \emptyset$.

Théorème 2.7 (Golubitsky et Guillemin [31, lemme 4.6]). Soit A, X et Y des variétés lisses et $W \subset Y$ une sous-variété. Soit $f : A \times X \rightarrow Y$ une application de classe C^∞ . Pour tout $a \in A$, soit $f_a : X \rightarrow Y$ définie par $f_a(x) = f(a, x)$. Si f est transverse à W , alors l'ensemble $\{a \in A : f_a \pitchfork W\}$ est dense dans A .

Théorème 2.8 (Golubitsky et Guillemin [31, théorème 4.9]). Soit W une sous-variété de $J^k(\mathbb{R}^n, \mathbb{R}^m)$. L'ensemble

$$\{f \in C^\infty(\mathbb{R}^n, \mathbb{R}^m) \mid j^k f \pitchfork W\}$$

est résiduel (i.e. contient une intersection dénombrable d'ouverts denses) dans $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$.

Exemple 2.9. Génériquement, deux courbes lisses de \mathbb{R}^3 ne s'intersectent pas.

Exemple 2.10. Génériquement, une courbe lisse de \mathbb{R}^2 n'est tangente à aucune droite passant par l'origine.

2.5. Méthode des caractéristiques

La méthode des caractéristiques est une méthode permettant de résoudre les équations aux dérivées partielles. Le principe général de restreindre l'équation aux dérivées partielles à des courbes dites *caractéristiques* le long desquelles l'équation se réduit à une équation différentielle ordinaire. Pour plus de détails concernant cette méthode, voir par exemple Fritz [29].

3. Démonstration du résultat

L'article présenté dans cette section est en cours de rédaction pour être envoyé pour publication. Cet article précise et démontre notre résultat (théorème 2.2).

Cost reconstruction in inverse optimal control: a generic property

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Abstract

This paper deals with an inverse approach to optimal control. In the last fifteen years, there has been a renewed interest in inverse optimal control, especially in human motor behavior. Indeed, according to a well accepted paradigm in neurophysiology, among all possible movements, those actually accomplished are solutions of an optimization process. We develop theoretical methods in order to reconstruct the minimized criterion in an optimal control problem from a set of solutions to this problem. We prove, using Thom's transversality theory, that for particular control affine systems, cost reconstruction is a generic property. More precisely, we characterize how the experiments need to be and how many of them are enough to reconstruct the cost in generic cases. Finally, we give an example highlighting some difficulties.

Keywords: Inverse optimal control, Pontryagin maximum principle, Thom's transversality theory

AMS classification: 49J15, 34H05, 49K15, 93C15, 93C10

1 Introduction and statement of the problem

1.1 Background

The problem of finding which quadratic cost is minimized by a given control law applied to a single-input linear system was first considered by Kalman in [12]. This study was then extended to multi-input linear systems (see e.g. [11, 18]) and non-linear systems (see e.g. [17]).

Nowadays, inverse optimal control is motivated by the well accepted paradigm that when humans perform a task such as moving, grabbing, driving, etc. they implicitly minimize a cost function (see e.g. [13]). However, this cost function being unknown, the main idea is to reconstruct it by using experimental data, which are assumed to be solution of an optimal control problem. Results about cost reconstruction in human movements can be found in [3, 6, 7]. This inverse approach allows, secondly, to generate human-like behavior on humanoid robots (see e.g. [4, 5]).

While a lot of contributions on inverse optimal control are related to human locomotion or movements, the present work aims to stay as general as possible and does not treat a particular application. The aim of this paper is mainly to extend the results of [3] to multi-input control affine systems.

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1.2 Notations

The following notations will be used throughout this paper. More specific notations will be defined when needed.

- We denote by $\langle \cdot, \cdot \rangle$ the canonical scalar product on \mathbb{R}^n and by $|\cdot|$ the induced Euclidean norm.
- Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$. $Df(q)[v]$ stands for the (total) differential of f at $q \in \mathbb{R}^n$ applied to the vector $v \in \mathbb{R}^n$. Similarly, $D_x f(q)[v]$ stands for the partial differential of f at $q \in \mathbb{R}^n$ with respect the variable x applied to the vector $v \in \mathbb{R}^n$.
- The dual space of a vector space E is denoted by E^* .
- $\mathbb{M}(m \times n, \mathbb{R})$ denotes the set of $m \times n$ matrices with real entries.
- A' denotes the transpose of a matrix A .
- For any multi index $I = (i_1, \dots, i_m) \in \mathbb{N}^m$ (or any subset of \mathbb{N}^m), we set $|I| = \sum_{j=1}^m i_j$.
- For a linear map f , we define $f^{\otimes k} := \underbrace{f \otimes \dots \otimes f}_{k \text{ times}}$ and $f^{\otimes 0} := 1$
- If $Q = X \times Y$ is a product space, we denote by π_x or π_X (resp. π_y or π_Y) the projection onto the first (resp. second) factor of Q .
- Throughout the paper, *smooth* means of class C^∞ .

1.3 Optimal control problem

We consider multi-input control affine systems with $q = (x, y) \in Q = \mathbb{R}^n \times \mathbb{R}^d$ the state variable and $u \in U \subset \mathbb{R}^d$ the control variable. More specifically, we consider the control affine system

$$\dot{q} = F_0(q) + \sum_{i=1}^d u_i F_i(q), \quad (1.1)$$

where $F_0 = (f, 0_d)$ and $F_i = \frac{\partial}{\partial y_i}$ are smooth vector fields on \mathbb{R}^{n+d} .

Let $\mathcal{F} = \mathcal{C}^\infty(Q, TX)$ denote the set of smooth control systems form (1.1), i.e. elements of \mathcal{F} are smooth vector fields F_0 on \mathbb{R}^{n+d} such that $F_0 : \mathbb{R}^{n+d} \rightarrow T\mathbb{R}^n \times \{0_{T\mathbb{R}^d}\}$. We endow \mathcal{F} with the C^∞ Whitney topology. An admissible trajectory is the solution of system (1.1) corresponding to an admissible control $u \in \mathbb{L}_{loc}^\infty([t_1, t_2], U)$. If necessary it is always possible to assume that $t_1 = 0$.

The main assumption is that we suppose that the integral cost

$$C(u(\cdot)) = \int_{t_1}^{t_2} L(u(t)) dt \quad (1.2)$$

is minimized in free time t_2 . More precisely the trajectories observed from system (1.1) are the solution of an optimal control problem:

- (**P_L**) Minimize the integral cost (1.2) among all admissible controls $u(\cdot)$ steering system (1.1) from a source point q_1 to a target point q_2 in free final time t_2 .

We do not consider arbitrary functions L in (1.2). We choose a class \mathcal{L} ensuring that the optimal control problem (\mathbf{P}_L) has a solution for each $L \in \mathcal{L}$. Following [7], \mathcal{L} is defined to be the set of $L : \mathbb{R}^d \rightarrow \mathbb{R}$ meeting the following assumptions:

(A1) L is strictly positive and smooth;

(A2) L is strictly convex;

(A3) $\lim_{\|u\| \rightarrow \infty} \frac{L(u)}{\|u\|} = +\infty$.

1.4 Inverse optimal control problem

Since the purpose of the following study is to reconstruct the cost function $L : \mathbb{R}^d \rightarrow \mathbb{R}$, the controls are assumed to visit an open subset of \mathbb{R}^d , i.e. the controls are parametrized by a parameter s that belongs to an open subset $\mathcal{O} \subset \mathbb{R}^{d-1}$. It follows that t_1 and t_2 also depend on s . We assume that t_1 and t_2 are smooth and we define the domain of u by

$$\mathcal{D} = \{(s, t) \in \mathcal{O} \times \mathbb{R} \mid t \in [t_1(s), t_2(s)]\} \subset \mathbb{R}^d.$$

A *parametrized control* is a smooth function

$$\begin{aligned} u : \mathcal{D} &\rightarrow \mathbb{R}^d \\ (s, t) &\mapsto u(s, t), \end{aligned}$$

such that $t \mapsto u(s, t)$ is an admissible control for every $s \in \pi_s(\mathcal{D})$. Similarly a *parametrized trajectory* (corresponding to a parametrized control u) is a function

$$\begin{aligned} q : \mathcal{D} &\rightarrow \mathbb{R}^{n+d} \\ (s, t) &\mapsto q(s, t), \end{aligned}$$

such that $t \mapsto q(s, t)$ is the admissible trajectory corresponding to the admissible control $t \mapsto u(s, t)$ for every $s \in \pi_s(\mathcal{D})$.

Definition 1.1 (experiment). A pair of admissible parametrized trajectory-control $\gamma = (q, u)$ is called an *experiment* if there exists a cost L such that $(q(s, \cdot), u(s, \cdot))$ solves (\mathbf{P}_L) for all $s \in \pi_s(\mathcal{D})$.

Definition 1.2 (compatible experiments). Two experiments γ_1 and γ_2 are said to be *compatible* if there exists (a common) $L \in \mathcal{L}$ such that each pair trajectory-control of γ_1 and γ_2 solve (\mathbf{P}_L) .

Definition 1.3 (diffeomorphic experiment). An experiment γ is said to be *diffeomorphic* if its parametrized control function u is a smooth diffeomorphism on its domain \mathcal{D} . The inverse of u will be denoted by Φ .

Definition 1.4 (effective experiments). Several experiments (q^α, u^α) , $\alpha = 1, \dots, \ell$ such that $\bigcap_{\alpha=1}^{\ell} \text{Im } u^\alpha$ has non empty interior \tilde{U} , are said to be *effective* if for all $\alpha \neq \beta$ there exist $(s^\alpha, t^\alpha) \in \Phi^\alpha(\tilde{U})$ and $(s^\beta, t^\beta) \in \Phi^\beta(\tilde{U})$ such that the following conditions are satisfied

- (i) $u^\alpha(s^\alpha, t^\alpha) = u^\beta(s^\beta, t^\beta)$,
- (ii) $q^\alpha(s^\alpha, t^\alpha) \neq q^\beta(s^\beta, t^\beta)$.

We define the *effective domain* of effective experiments to be the set of values of u such that for all $\alpha \neq \beta$ there exist $(s^\alpha, t^\alpha) \in \Phi^\alpha(\tilde{U})$ and $(s^\beta, t^\beta) \in \Phi^\beta(\tilde{U})$ such that conditions (i) and (ii) are satisfied.

We then define the following inverse optimal control problem.

($\mathfrak{I}\mathbf{P}$) Given a family Γ_f of experiments, find a cost $L \in \mathcal{L}$ such that every $\gamma \in \Gamma_f$ solves (\mathbf{P}_L) .

2 Study of the optimal control problem

2.1 Existence of solutions

Following the same idea as in [8, Section 3.1] and [3] we state the following.

Theorem 2.1. *For every $q_1, q_2 \in \mathbb{R}^{n+d}$ such that q_2 is reachable from q_1 , there exist a real positive time t_2^* and a control function $u \in \mathbb{L}^1([t_1, t_2^*], U)$ such that the corresponding trajectory q solves problem (\mathbf{P}_L) .*

Notice that Theorem 2.1 does not provide the boundedness of the optimal controls. It will follow from Lemma 2.7.

Remark 2.2. The existence of a bounded positive t_2^* is not trivial and follows from the special form of the cost function L (see [3]).

2.2 Pontryagin's Maximum Principle

Since we do not know yet that optimal controls are bounded in the \mathbb{L}^∞ topology, we cannot apply the Pontryagin's Maximum Principle (or PMP for short) in its classical version. Nonetheless, it is easy to check that problem (\mathbf{P}_L) meets all the hypotheses required in [19, Theorem 8.7.1] (a more refined version of PMP for unbounded controls), and thus we can state the following.

Proposition 2.3. *Every solution of (\mathbf{P}_L) satisfies the PMP.*

Define the Hamiltonian function

$$h(\lambda, p, q, u) = \xi f(q) + \langle \zeta, u \rangle + \lambda L(u),$$

with $p = (\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_d) \in (\mathbb{R}^{n+d})^*$ and $\lambda \leq 0$. The PMP states that if q is an optimal trajectory and u is the corresponding control defined on $[t_1, t_2]$, then there exist an absolutely continuous curve $p : [t_1, t_2] \rightarrow (\mathbb{R}^{n+d})^*$ and $\lambda \leq 0$ such that the pair $(\lambda, p(t))$ never vanishes and for almost all $t \in [t_1, t_2]$, (q, p) satisfies the Hamiltonian system

$$\begin{cases} \dot{q}(t) = \frac{\partial h}{\partial p}(\lambda, p(t), q(t), u(t)) \\ \dot{p}(t) = -\frac{\partial h}{\partial q}(\lambda, p(t), q(t), u(t)), \end{cases} \quad (2.1)$$

and the maximality condition

$$h(\lambda, p(t), q(t), u(t)) = \max_{w \in U} h(\lambda, p(t), q(t), w). \quad (2.2)$$

Moreover, since the final time is free, the Hamiltonian vanishes identically on $[t_1, t_2]$, hence

$$\xi(t)f(q(t)) + \langle \zeta(t), u(t) \rangle + \lambda L(u(t)) = 0. \quad (2.3)$$

A quadruple (λ, p, q, u) satisfying the necessary conditions (2.1), (2.2) and (2.3) is called an *extremal* for problem (\mathbf{P}_L) .

Remark 2.4. An extremal is said to be *abnormal* if $\lambda = 0$ and *normal* otherwise. In the following, we will not deal with abnormal extremals since in this case, the cost vanishes from the Hamiltonian.

Definition 2.5 (parametrized extremal). A *parametrized extremal* (for problem (\mathbf{P}_L)) is a quadruple $(\lambda, p, q, u) : (s, t) \mapsto (\lambda(s), p(s, t), q(s, t), u(s, t))$ satisfying the necessary conditions (2.1), (2.2) and (2.3) for all $s \in \pi_s(\mathcal{D})$.

Remark 2.6. A parametrized extremal is said to be *abnormal* if there exist $s \in \pi_s(\mathcal{D})$ such that $\lambda(s) = 0$ and *normal* otherwise.

2.3 Normal extremals

Let (λ, p, q, u) be a normal parametrized extremal for problem (\mathbf{P}_L) . In particular, we may assume that $\lambda(s) \equiv -1$. Along a normal extremal, the maximality condition (2.2) of the PMP yields

$$\zeta = DL(u) \quad (2.4)$$

which together with (2.3) yields

$$L(u(s, t)) - DL(u(s, t))[u(s, t)] = \xi(s, t)f(q(s, t)). \quad (2.5)$$

Notice that since L is strictly convex, $u \mapsto DL(u)$ is invertible and

$$u = (DL)^{-1}(\zeta).$$

For the control affine system (1.1), the second term of (2.1) gives the adjoint system

$$\begin{cases} \dot{\xi} = -\xi \frac{\partial f}{\partial x}(x, y) \\ \dot{\zeta} = -\xi \frac{\partial f}{\partial y}(x, y). \end{cases} \quad (2.6)$$

We conclude that normal extremals satisfies the following system

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = (DL)^{-1}(\zeta) \\ \dot{\xi} = -\xi \frac{\partial f}{\partial x}(x, y) \\ \dot{\zeta} = -\xi \frac{\partial f}{\partial y}(x, y) \\ L(u) - DL(u)[u] = \xi f(q). \end{cases} \quad (2.7)$$

Since (ξ, ζ) satisfies the smooth differential system (2.6) it is fully determined by its parametrized initial condition $(\xi_0(s), \zeta_0(s))$. In the following we make the hypothesis that (ξ_0, ζ_0) is smooth.

Lemma 2.7. *The parametrized controls u corresponding to normal parametrized extremals of (\mathbf{P}_L) are smooth.*

Proof. By strict convexity, smoothness of L and from (2.4), $DL(\cdot)$ is invertible and u depends smoothly on ζ . Moreover, since ζ is a component of the solution of a smooth differential system with initial condition depending smoothly on s , it is smooth and so is u . ■

Remark 2.8. Lemma 2.7 guarantees that extremal controls are also extremal controls of the standard $(\mathbb{L}_{\text{loc}}^\infty)$ PMP (see [2]).

Since the parametrized control corresponding to a normal parametrized extremal is a smooth diffeomorphism, we define $\mathcal{U} = \text{Diff}(\mathcal{D}, U)$, endowed with the \mathcal{C}^∞ Whitney topology.

2.4 Abnormal extremals

Let $t_0(s) \in [t_1(s), t_2(s)]$ and let $R(s, t, t_0(s))$ be the resolvent of the third equation of the time varying system (2.7), i.e. $R(s, t, t_0(s))$ is the matrix solution to

$$\dot{R} = -R \frac{\partial f}{\partial x}(x, y), \quad R(s, t_0(s), t_0(s)) = \text{Id}$$

for all $s \in \pi_s(\mathcal{D})$. We define $V(s, t) = R(s, t, t_0(s))f(q(s, t))$.

Lemma 2.9. *The trajectory $t \mapsto q(s, t)$ is the projection of an abnormal extremal if and only if $\text{span}\{V(s, t), t \in [t_1(s), t_2(s)]\} \neq \mathbb{R}^n$.*

Proof. Assume that $q(s, \cdot)$ is the projection of an abnormal trajectory. The Hamiltonian is then

$$h(0, p(s, t), q(s, t), u(s, t)) = \max_{w \in U} (\xi(t)f(q(s, t)) + \langle \zeta(s, t), w \rangle) = 0.$$

The maximality condition implies that $\zeta(s, t) = 0$ for all $t \in [t_1(s), t_2(s)]$. It follows that $\xi(s, t) \neq 0$ and

$$\xi(s, t)f(q(s, t)) = \xi_0(s)V(s, t) = 0,$$

which means exactly that $\{V(s, t), t \in [t_1(s), t_2(s)]\}$ does not span \mathbb{R}^n .

Assume now that $\{V(s, t), t \in [t_1(s), t_2(s)]\}$ does not span \mathbb{R}^n . Then for every $s \in \pi_s(\mathcal{D})$ there exists $\xi_0(s) \in (\mathbb{R}^n)^*$ such that

$$\xi_0(s)V(s, t) = \xi(s, t)f(q(s, t)) \equiv 0.$$

The Hamiltonian is then

$$h(\lambda, p(s, t), q(s, t), u(s, t)) = \langle \zeta(s, t), u(s, t) \rangle + \lambda(s)L(u(s, t)) \equiv 0$$

The maximality condition yields

$$\lambda(s)DL(u(s, t)) = -\zeta(s, t)$$

and thus

$$\lambda(s)L(u(s, t)) - \lambda(s)DL(u(s, t))[u(s, t)] = 0$$

for all $t \in [t_1(s), t_2(s)]$. If $\lambda(s) \neq 0$, the last equation contradicts the hypothesis of strict convexity of the cost. We conclude that $\lambda(s) = 0$. \blacksquare

Definition 2.10 (normal-regular experiment). A diffeomorphic experiment (q, u) , defined on \mathcal{D} is *normal-regular* if for all $s \in \pi_s(\mathcal{D})$, $q(s, \cdot)$ is not the projection of an extremal which is abnormal on some (nontrivial) subinterval of $[t_1(s), t_2(s)]$.

In order to characterize normal-regular experiments, we need the following lemma (see [3, Lemma 3.9]).

Lemma 2.11. *Let $\gamma : I \rightarrow \mathbb{R}^n$ be a smooth curve, defined on some nontrivial interval I . Then, the two following assertions are equivalent.*

- For any subinterval $J \subset I$, $\text{span}\{\gamma(t), t \in J\} = \mathbb{R}^n$
- The set of t_0 such that $\text{span}\{\gamma(t_0), \gamma'(t_0), \dots, \gamma^{(n-1)}(t_0)\} = \mathbb{R}^n$ is (open) dense in I .

The following corollary stems directly from Lemma 2.9 and 2.11.

Corollary 2.12. *Let (q, u) be a diffeomorphic experiment. The trajectory $t \mapsto q(s, t)$ is normal-regular if and only if for all $s \in \pi_s(\mathcal{D})$ the set of t such that*

$$\text{span}\{V(s, t), \dot{V}(s, t) \cdots V^{(n-1)}(s, t)\} = \mathbb{R}^n$$

is (open) dense in $[t_1(s), t_2(s)]$.

2.4.1 Test of abnormality

Although the restriction to normal-regular experiments seems strong, there exists a practical test of abnormality. Define the Gram matrix relative to the interval $[t_1(s), t_2(s)]$ as the symmetric semi-definite positive matrix

$$G(s) = \int_{t_1(s)}^{t_2(s)} V(s, \tau) V(s, \tau)' d\tau.$$

One can easily deduce that for all $s \in \pi_s(\mathcal{D})$, $\text{span}\{V(s, t), t \in [t_1(s), t_2(s)]\} = \mathbb{R}^n$ if and only if $G(s)$ is positive definite or equivalently if and only if its smallest eigenvalue is strictly positive. Then, the test resumes to decide whether a matrix is positive definite which is standard in practice (see e.g. [16]).

3 Study of the inverse optimal control problem

3.1 An ill-posed problem

The following lemma shows that, like a lot of inverse problems, the inverse control problem ($\mathfrak{I}\mathfrak{P}$) is an ill-posed problem.

Lemma 3.1. *Let $a > 0$ and $b \in \mathbb{R}^d$. If u is an optimal control for the cost L , then it is an optimal control for the cost $\tilde{L} = aL + \langle b, \cdot \rangle$.*

Proof. Let q_u be the optimal trajectory corresponding to an optimal control u defined on $[t_1, t_2]$. Set $q_1 = q_u(t_1)$ and $q_2 = q_u(t_2)$. We have to prove that

$$\int_{t_1}^{t_2} \tilde{L}(u(t)) dt \leq \int_{t_1}^{t_2} \tilde{L}(v(t)) dt,$$

for any control v whose corresponding trajectory q_v satisfies (1.1) with $q_v(t_1) = q_1$ and $q_v(t_2) = q_2$. We have

$$\begin{aligned} \int_{t_1}^{t_2} \tilde{L}(u(t)) dt &= \int_{t_1}^{t_2} aL(u(t)) dt + \int_{t_1}^{t_2} \langle b, u(t) \rangle dt \\ &= \int_{t_1}^{t_2} aL(u(t)) dt + \int_{t_1}^{t_2} \langle b, \pi_y \dot{q}_u(t) \rangle dt \\ &= \int_{t_1}^{t_2} aL(u(t)) dt + \langle b, \pi_y q_2 - \pi_y q_1 \rangle \\ &\leq \int_{t_1}^{t_2} aL(v(t)) dt + \langle b, \pi_y q_2 - \pi_y q_1 \rangle && \text{(Since } u \text{ is optimal for } L) \\ &= \int_{t_1}^{t_2} \tilde{L}(v(t)) dt, && \text{(Since } q_v(t_1) = q_1 \text{ and } q_v(t_2) = q_2) \end{aligned}$$

which is the desired conclusion. ■

Remark 3.2. Lemma 3.1 shows that a solution of (\mathbf{P}_L) is also a solution of $(\mathbf{P}_{aL + \langle b, \cdot \rangle})$. One can only expect to reconstruct a class representative of the cost function under the equivalence relation \sim , where $\tilde{L} \sim L$ if there exists a real $a > 0$ and a $b \in \mathbb{R}^d$ such that $\tilde{L} = aL + \langle b, \cdot \rangle$.

Remark 3.3. Notice that the invariance of the extremals stated in Lemma 3.1 is closely related to the special form of the vector field that defines system (1.1).

Because, in the proof of Lemma 3.1, the variable of integration is the time, one could think of choosing a function $a : u \mapsto a(u)$ such that $\frac{\partial a(u(s,t))}{\partial t} = 0$ for all $(s, t) \in \mathcal{D}$. Indeed, consider ℓ experiment $\gamma^1, \dots, \gamma^\ell$ and define

$$\tilde{L}(u) = a(u)L(u),$$

where $a(\cdot)$ is a positive function such that $\frac{\partial a(u(s,t))}{\partial t} = 0$ for all $(s, t) \in \mathcal{D}^\alpha$ and each $\alpha \in \{1, \dots, \ell\}$. It follows that

$$Da(u^\alpha(s, t)) [\dot{u}^\alpha(s, t)] = 0. \quad (3.1)$$

We deduce that the existence a function $a(\cdot)$ satisfying the given condition cannot be guaranteed in general and depends on the system (i.e. on f) and on the (unknown) cost. Nevertheless, suppose that there exists a function $a(\cdot)$ satisfying the given condition. The computations made in the proof of Lemma 3.1 are not valid anymore since $a(v(s, t))$ could depend on time and the last inequality could not hold. However, since we use the necessary conditions given by the PMP, it still remains to determine if there exists an extremal $(-1, \tilde{p}, q, u)$ that satisfies the PMP for the cost \tilde{L} . More specifically, since $\tilde{\zeta}(s, t) = D\tilde{L}(u(s, t))$, the equation

$$\frac{\partial}{\partial t} (D\tilde{L}(u(s, t))) = -\tilde{\xi}(s, t) \frac{\partial f}{\partial y}(q(s, t)) \quad (3.2)$$

must be satisfied. It is very difficult to conclude if whether or not there exists a (non constant) function $a(\cdot)$ satisfying equations (3.1) and (3.2). In Section 5 we give a simple linear example for which it is not possible to consider a cost transformation depending on u .

3.2 Existence of the reconstructed cost

In the following, we make the hypothesis that we have an *a priori* knowledge of the cost function on a smooth hypersurface $\Sigma \subset \mathbb{R}^d$. Although we were unable to weaken this condition, some possible improvements are given in Subsection 4.3.3.

Suppose that $\xi_0(s)$ is known for all $s \in \pi_s(\mathcal{D})$. Set $\xi(s, t) = \xi_0(s)R(s, t, t_0(s))$. For the sake of clarity, if $f : (s, t) \mapsto f(s, t)$, we define $\hat{f}(u) = f(\Phi(u))$. We aim to solve using the method of characteristics (see e.g. [9]) the first-order partial differential equation

$$\begin{cases} L(u) - DL(u)[u] = \hat{\xi}_0(u)\hat{V}(u) \\ L|_\Sigma = L_0, \end{cases} \quad (3.3)$$

where Σ is a hypersurface of \mathbb{R}^d defined by $\Sigma = \{\phi(\theta), \theta \in \mathbb{R}^{d-1}\}$ and L_0 is a known boundary condition. Without loss of generality, we assume that Σ is non-characteristic, i.e. Σ is nowhere tangent to any straight line passing through the origin.

In order to solve equation (3.3), define the system of characteristic ODEs,

$$\begin{cases} \frac{dv}{d\rho}(\rho, \theta) = v(\rho, \theta) \\ \frac{dw}{d\rho}(\rho, \theta) = w(\rho, \theta) - \hat{\xi}_0(v(\rho, \theta))\hat{V}(v(\rho, \theta)) \end{cases} \quad (3.4)$$

with initial conditions

$$\begin{cases} v(0, \theta) = \phi(\theta) \\ w(0, \theta) = L_0(\theta). \end{cases} \quad (3.5)$$

This system admits the unique solution

$$\begin{cases} v(\rho, \theta) = \phi(\theta)e^\rho \\ w(\rho, \theta) = L_0(\theta)e^\rho - e^\rho \int_0^\rho \hat{\xi}_0(v(r, \theta))\hat{V}(v(r, \theta))e^{-r} dr. \end{cases} \quad (3.6)$$

Since, by hypothesis, Σ is non-characteristic, the inverse function theorem yields that there exists a neighborhood \mathcal{N} of Σ such that the mapping $v|_{\mathcal{N}} : (\rho, \theta) \mapsto \phi(\theta)e^\rho$ is a diffeomorphism. We denote by ψ the inverse of $v|_{\mathcal{N}}$. It follows that if $\xi_0(s)$ is known for all $s \in \pi_s(\mathcal{D})$, the cost is uniquely determined on \mathcal{N} and is given by

$$\begin{aligned} L(u) &= w \circ \psi(u) \\ &= e^{\psi_1(u)} \left(L_0(\psi_2(u)) - \int_0^{\psi_1(u)} (\hat{\xi}_0 \hat{V}) \circ \psi^{-1}(r, \psi_2(u)) e^{-r} dr \right). \end{aligned} \quad (3.7)$$

4 Main result

The purpose of this section is to prove the following theorem.

Theorem 4.1. *There exists a residual (for the Whitney topology) subset $\mathcal{G} \subset \mathcal{F}$ of systems such that $d+2$ diffeomorphic, compatible, effective and normal-regular experiments are enough to reconstruct the cost function L , provided that the latter is known on a hypersurface nowhere tangent to any straight line passing through the origin. The cost function is reconstructed over (the closure of) the optimal control ranges intersecting the effective domain of the $d+2$ experiments.*

4.1 Jets spaces

The set of all k -jets from \mathbb{R}^m to \mathbb{R}^n is denoted by $J^k(m, n)$. The subsets $J_x^k(m, n)$, $J^k(m, n)_y$ and $J_x^k(m, n)_y$ will denote the subset of k -jets with source x , with target y , and with source x and target y respectively. The mapping $\sigma : J_x^k(m, n) \rightarrow \mathbb{R}^m$ given by $\sigma : j^k f \mapsto \sigma(j^k f) = x$ is called the source map and the mapping $\tau : J_x^k(m, n) \rightarrow \mathbb{R}^n$ given by $\tau : j^k f \mapsto \tau(j^k f) = f(x)$ is called the target map. We have

$$\begin{aligned} J^k(m, n) &= \bigsqcup_{x \in \mathbb{R}^m} J_x^k(m, n) = \mathbb{R}^m \times J_x^k(m, n) \\ &= \bigsqcup_{y \in \mathbb{R}^n} J^k(m, n)_y = J^k(m, n)_y \times \mathbb{R}^m \\ &= \bigsqcup_{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n} J_x^k(m, n)_y = \mathbb{R}^m \times J_x^k(m, n)_y \times \mathbb{R}^n \end{aligned}$$

Because the k -jet of a map f at $x \in \mathbb{R}^m$ has, as canonical representative, the Taylor polynomial of f of order k , we have the natural identification

$$J_x^k(m, n) = \bigoplus_{i=0}^k L_{\text{sym}}^i(\mathbb{R}^m, \mathbb{R}^n) \quad (4.1)$$

$$= \mathbb{R}_k[X_1, X_2, \dots, X_m] \otimes \mathbb{R}^n, \quad (4.2)$$

where $\mathbb{R}_k[X_1, X_2, \dots, X_m]$ denotes the formal polynomials of degree no greater than k in the indeterminates X_1, X_2, \dots, X_m . We shall use freely those identifications in the sequel.

With a little abuse of notation and because our purpose is essentially local, we still denote by $J^k(m, n)$ the bundle of k -jets of applications in $C^k(U, \mathbb{R}^n)$ where U is a connected and simply connected open subset of \mathbb{R}^m .

4.2 Preliminary lemmas

Lemma 4.2. *Let (q, u) be a diffeomorphic experiment defined on \mathcal{D} . Then the subset of \mathcal{D} at which y is a diffeomorphism is (open) dense.*

Proof. Suppose, contrary to our claim, that there exists an open subset $\mathcal{O} \subset \mathcal{D}$ such that

$$\text{rank}(Dy(s, t)) < d,$$

for all $(s, t) \in \mathcal{O}$. Let $r < d$ be defined by $r = \max_{(s, t) \in \mathcal{O}} \{\text{rank}(Dy(s, t))\}$. Then the lower semicontinuity of the rank function implies that $\mathcal{O}' = \{(s, t) \in \mathcal{O} \mid \text{rank}(Dy(s, t)) = r\}$ is an open subset of \mathcal{O} . Because $u(s, t) = \frac{\partial y}{\partial t}(s, t)$, we have $Du(s, t) = \frac{\partial}{\partial t} Dy(s, t)$. It follows that

$$\text{rank } Du(s, t) \leq \text{rank } Dy(s, t) < d,$$

which contradicts the hypothesis that the experiment is diffeomorphic. ■

Lemma 4.3. *Let (λ, q, p, u) be a parametrized extremal for problem (\mathbf{P}_L) defined on \mathcal{D} and let $\varphi : \mathbb{R}^d \rightarrow \mathcal{D}$ be a smooth diffeomorphism. The quadruple $(\lambda, q \circ \varphi, p \circ \varphi, u \circ \varphi)$ is a parametrized extremal for problem (\mathbf{P}_L) if and only if $\varphi(s, t) = (\varphi_1(s), t + \varphi_2(s))$, where $\varphi_1 : \mathbb{R}^{d-1} \rightarrow \pi_s(\mathcal{D})$ and $\varphi_2 : \mathbb{R}^{d-1} \rightarrow \pi_t(\mathcal{D})$.*

Proof. Let (λ, q, p, u) be a parametrized extremal for problem (\mathbf{P}_L) . One easily checks that if $\varphi(s, t) = (\varphi_1(s), t + \varphi_2(s))$, then the quadruple $(\lambda, q \circ \varphi, p \circ \varphi, u \circ \varphi)$ satisfies all the conditions given by the PMP and is consequently a parametrized extremal for problem (\mathbf{P}_L) . Conversely, suppose that the quadruple $(\lambda, q \circ \varphi, p \circ \varphi, u \circ \varphi)$ is a parametrized extremal for problem (\mathbf{P}_L) . Set $\varphi(s, t) = (\varphi_1(s, t), \varphi_2(s, t))$, where $\varphi_1 : \mathbb{R}^d \rightarrow \pi_s(\mathcal{D})$ and $\varphi_2 : \mathbb{R}^d \rightarrow \pi_t(\mathcal{D})$. The dynamic of the system, which is the same for q and $q \circ \varphi$, yields

$$\frac{\partial(y \circ \varphi)}{\partial t}(s, t) = u(\varphi(s, t)) = \frac{\partial y}{\partial t}(\varphi(s, t)).$$

On the other hand, the chain rule yields

$$\frac{\partial(y \circ \varphi)}{\partial t}(s, t) = D_s y(\varphi(s, t)) \left[\frac{\partial \varphi_1}{\partial t}(s, t) \right] + D_t y(\varphi(s, t)) \left[\frac{\partial \varphi_2}{\partial t}(s, t) \right]$$

and we obtain

$$Dy(\varphi(s, t)) \left[\frac{\partial \varphi_1}{\partial t}(s, t) \quad \frac{\partial \varphi_2}{\partial t}(s, t) - 1 \right] = 0. \quad (4.3)$$

Moreover, from Lemma 4.2, the subset \mathcal{O} of \mathcal{D} at which y is a diffeomorphism is (open) dense. We conclude that for all $(s, t) \in \mathcal{O}$, $\frac{\partial \varphi_1}{\partial t}(s, t) = 0$ and $\frac{\partial \varphi_2}{\partial t}(s, t) = 1$ and by continuity of φ that $\frac{\partial}{\partial t} \varphi_1$ and $\frac{\partial}{\partial t} \varphi_2$ have a unique prolongation to \mathcal{D} , i.e. $\frac{\partial \varphi_1}{\partial t}(s, t) = 0$ and $\frac{\partial \varphi_2}{\partial t}(s, t) = 1$ for all $(s, t) \in \mathcal{D}$. Integrating with respect to t gives the desired result. ■

Remark 4.4. An admissible transformation φ is given by d smooth functions of $d - 1$ parameters.

The next lemma, which is used in the proofs of Lemma 4.14 and Corollary 4.15, is given without proof since it easily follows from a direct application of the implicit function theorem.

Lemma 4.5. *Let $\ell \in \mathbb{N} \setminus \{0\}$ and $t_0 \in \mathbb{R}$ be fixed. Assume that u^1, \dots, u^ℓ are experiments with respective domains $\mathcal{D}^1, \dots, \mathcal{D}^\ell$ such that the interior \tilde{U} of $\bigcap_{\alpha=1}^\ell u^\alpha(\mathcal{D}^\alpha)$ is non empty. Fix a smooth hypersurface $\Sigma \subset \tilde{U}$. Then there exist a positive ε , an open domain $\bar{\mathcal{D}} \in \mathbb{R}^d$ and ℓ local diffeomorphisms $\varphi^\alpha : \bar{\mathcal{D}} \rightarrow \varphi^\alpha(\bar{\mathcal{D}}) \subset \mathcal{D}^\alpha$ having the form of the diffeomorphism described in Lemma 4.3 such that*

- $[t_0 - \varepsilon, t_0 + \varepsilon] \subset \pi_t \bar{\mathcal{D}}$,
- $\Sigma = \text{Im} \left(u^\alpha \circ \varphi^\alpha \left(\pi_s(\bar{\mathcal{D}}), t_0 \right) \right), \quad \forall \alpha \in \{1, \dots, \ell\}.$

4.3 The equation to be solved

4.3.1 How to proceed: computation in (s, t) variables

Consider ℓ effective experiments $\gamma^1, \dots, \gamma^\ell$. On \tilde{U} (see Definition 1.4), the following holds

$$\hat{\xi}_0^1(u) \hat{V}^1(u) = \hat{\xi}_0^2(u) \hat{V}^2(u) = \dots = \hat{\xi}_0^\ell(u) \hat{V}^\ell(u) = L(u) - DL(u)[u]. \quad (4.4)$$

Hence,

$$\hat{\xi}_0^\alpha(u) \hat{V}^\alpha(u) - \hat{\xi}_0^\beta(u) \hat{V}^\beta(u) = 0, \quad \forall \alpha, \beta \in \{1, \dots, \ell\},$$

or, equivalently

$$(\xi_0^\alpha V^\alpha) \circ \Phi^\alpha(u) - (\xi_0^\beta V^\beta) \circ \Phi^\beta(u) = 0, \quad \forall \alpha, \beta \in \{1, \dots, \ell\}, \quad (4.5)$$

where the maps Φ^α and Φ^β denote the inverse of u^α and u^β respectively. According to the rule of multiplication of jets, equation (4.5) implies that

$$j_{(s^\alpha, t^\alpha)}^k(\xi_0^\alpha V^\alpha) \cdot j_u^k \Phi^\alpha - j_{(s^\beta, t^\beta)}^k(\xi_0^\beta V^\beta) \cdot j_u^k \Phi^\beta = 0, \quad \forall \alpha, \beta \in \{1, \dots, \ell\}, \quad \forall k \in \mathbb{N}. \quad (4.6)$$

More precisely, following the same idea as in [14], the k -th differential of the composition $(\xi_0 V) \circ \Phi$ is

$$D^k ((\xi_0 V) \circ \Phi)(u) = \sum_{1 \leq j \leq k} \sum \sigma_k D^j(\xi_0 V)(\Phi(u)) \left(D^{i_1} \Phi(u) \otimes \dots \otimes D^{i_j} \Phi(u) \right), \quad (4.7)$$

where the second sum is over all j -tuples of integers $I = (i_1, \dots, i_j)$ such that $|I| = k$ and σ_k is defined recursively by $\sigma_k(k) = 1$ and (see [1])

$$\sigma_{k+1}(i_1, \dots, i_j, k - \iota + 1) = \binom{k}{\iota} \sigma_k(i_1, \dots, i_j).$$

Noticing that for all k , $\sigma_k(1, \dots, 1) = 1$, we obtain

$$\underbrace{\left((\xi_0 V) \circ \Phi(u) \quad D((\xi_0 V) \circ \Phi)(u) \cdots D^k((\xi_0 V) \circ \Phi)(u) \right)}_{J_u^k((\xi_0 V) \circ \Phi(u))} = \underbrace{\left((\xi_0 V) \circ \Phi(u) \quad D(\xi_0 V)(\Phi(u)) \quad \cdots \quad D^k(\xi_0 V)(\Phi(u)) \right)}_{J_{\Phi(u)}^k(\xi_0 V)} \underbrace{\begin{pmatrix} 1 & * & \cdots & * \\ 0 & D\Phi(u) & & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & 0 & D\Phi(u)^{\otimes k} \end{pmatrix}}_B$$

Since Φ is a diffeomorphism, $D\Phi(u)^{\otimes k}$ has full rank for all $k \in \mathbb{N}$. The matrix representation of B is then invertible. These computations justify that all the following computations are performed in the variables (s, t) .

4.3.2 The homogeneous equation

Set

- $\mu = \dim \left(J_{s_0}^k(d-1, n) \right) = n \frac{(k+d-1)!}{(d-1)!k!} = n \frac{k^{d-1}}{(d-1)!} + o(k^{d-1})$ when $k \rightarrow \infty$,
- $\nu = \dim \left(J_{(s_0, t_0)}^k(d, 1) \right) = \frac{(k+d)!}{d!k!} = \frac{k^d}{d!} + o(k^d)$ when $k \rightarrow \infty$.

In this section, our aim is to write a formula for the k -jet of the map

$$(s, t) \mapsto (\xi_0 V)(s, t)$$

in terms of the k -jets of ξ_0 and V . More precisely, we aim to give an expression of the form

$$j_{(s,t)}^k(\xi_0 V) = PA, \quad (4.8)$$

where $P \in (\mathbb{R}^\mu)^*$ is a vector representation of the $j_s^k \xi_0$, and $A \in \mathbb{M}(\nu \times \mu, \mathbb{R})$ is a matrix that depends on $j_{(s,t)}^k V$.

To this aim, we first need to find a good manner to present the computational result of the successive derivative of the product map $(s, t) \mapsto (\xi_0 V)(s, t)$. Let us begin with the following lemma whose proof is omitted.

Lemma 4.6. *Let E and F be finite-dimensional real vector spaces. Let $\varphi : E \rightarrow F$ be a linear map. Assume that E and F decompose as $E = \bigoplus_{i=0}^k E_i$ and $F = \bigoplus_{i=0}^k F_i$. Assume moreover, that $\varphi(E_i) \subset \bigoplus_{j=i}^k F_j$. Then,*

$$\text{rank } \varphi \geq \sum_{i=0}^k \dim \left(\varphi(E_i) \cap F_i \right).$$

According to Lemma 2.12, we may assume that

$$\text{span} \left\{ V(s_0, t_0), \frac{\partial V}{\partial t}(s_0, t_0), \dots, \frac{\partial^{n-1} V}{\partial t^{n-1}}(s_0, t_0) \right\} = \mathbb{R}^n. \quad (4.9)$$

Let $k \in \mathbb{N}$ and define the mapping

$$\begin{aligned} \varphi : J_{s_0}^k(d-1, n) &\rightarrow J_{(s_0, t_0)}^k(d, 1) \\ j_{s_0}^k \xi_0 &\mapsto j_{(s_0, t_0)}^k(\xi_0 V). \end{aligned}$$

Recall that, we have the natural identifications

$$J_{s_0}^k(d-1, n) = \bigoplus_{i=0}^k L_{\text{sym}}^i(\mathbb{R}^{d-1}, (\mathbb{R}^n)^*) = \mathbb{R}_k[S_1, S_2, \dots, S_{d-1}] \otimes (\mathbb{R}^n)^*$$

and

$$J_{(s_0, t_0)}^k(d, 1) = \bigoplus_{i=0}^k L_{\text{sym}}^i(\mathbb{R}^d, \mathbb{R}) = \mathbb{R}_k[S_1, S_2, \dots, S_{d-1}, T]$$

where $\mathbb{R}_k[S_1, S_2, \dots, S_{d-1}]$ (resp. $\mathbb{R}_k[S_1, S_2, \dots, S_{d-1}, T]$) denotes the formal polynomials of degree no greater than k in the indeterminates S_1, S_2, \dots, S_{d-1} (resp. $S_1, S_2, \dots, S_{d-1}, T$). Fix $\{e_1, \dots, e_n\}$ a basis of \mathbb{R}^n and let $\{e_1^*, \dots, e_n^*\}$ be its dual basis, so that $V = \sum_{a=1}^n V_a e_a$ and $\xi_0 = \sum_{a=1}^n \xi_{0,a} e_a^*$. According to these identifications,

$$J_{s_0}^k(d-1, n) = \text{span} \left\{ S^I \otimes e_a^* \mid 0 \leq |I| \leq k, \quad 1 \leq a \leq n \right\}$$

and

$$J_{s_0}^k(d, 1) = \text{span} \left\{ S^I T^j \mid 0 \leq |I| + j \leq k \right\},$$

where $I = (i_1, \dots, i_{d-1})$ is a multi-index and $S^I = S_1^{i_1} \cdots S_{d-1}^{i_{d-1}}$. Moreover, in term of this basis, an element $j_{s_0}^k f \in J_{s_0}^k(d-1, n)$ reads

$$j_{s_0}^k f = \sum_{\substack{I \in \mathbb{N}^{d-1} \\ |I| \leq k}} \partial^I f(s_0) S^I = \sum_{a=1}^n \sum_{\substack{I \in \mathbb{N}^{d-1} \\ |I| \leq k}} \partial^I f_a(s_0) S^I \otimes e_a^*$$

and

$$\varphi(j_{s_0}^k f) = \sum_{\substack{(I,j) \in \mathbb{N}^d \\ |I|+j \leq k}} \partial^{I+j}(fV)(s_0, t_0) S^I T^j,$$

with

$$\partial^I f = \frac{\partial^{|I|} f}{\partial s_1^{i_1} \cdots \partial s_{d-1}^{i_{d-1}}}, \quad I = (i_1, \dots, i_{d-1}), \quad |I| = i_1 + \cdots + i_{d-1}.$$

Note that, $J_{s_0}^k(d-1, n)$ and $J_{s_0}^k(d, 1)$ decompose as the direct sums

$$J_{s_0}^k(d-1, n) = \bigoplus_{\substack{I \in \mathbb{N}^{d-1} \\ |I| \leq k}} E_I, \quad J_{s_0}^k(d, 1) = \bigoplus_{\substack{I \in \mathbb{N}^{d-1} \\ |I| \leq k}} F_I,$$

where

$$\begin{aligned} E_I &= \text{span} \left\{ S^I \otimes e_a^* \mid I = (i_1, \dots, i_{d-1}) \text{ is fixed, } |I| = i, \quad 1 \leq a \leq n \right\} \\ &= \mathbb{R} S^I \otimes (\mathbb{R}^n)^* \simeq (\mathbb{R}^n)^* \\ F_I &= \text{span} \left\{ S^I T^j \mid I = (i_1, \dots, i_{d-1}) \text{ is fixed, } |I| = i, \quad 0 \leq j \leq k - |I| \right\} \\ &= \mathbb{R} S^I \otimes \mathbb{R}_{k-|I|}[T] \simeq \mathbb{R}_{k-|I|}[T] \end{aligned}$$

We denote

$$E_i = \bigoplus_{\substack{I \in \mathbb{N}^{d-1} \\ |I|=i, \quad i \leq k}} E_I = \mathbb{R}^i[S_1, S_2, \dots, S_{d-1}] \otimes (\mathbb{R}^n)^*, \quad F_i = \bigoplus_{\substack{I \in \mathbb{N}^{d-1} \\ |I|=i, \quad i \leq k}} F_I = \mathbb{R}^i[S_1, S_2, \dots, S_{d-1}] \otimes \mathbb{R}_{k-i}[T], \quad (4.10)$$

where $\mathbb{R}^k[S_1, S_2, \dots, S_{d-1}] \subset \mathbb{R}_k[S_1, S_2, \dots, S_{d-1}]$ denotes the subset of homogeneous polynomials of degree k .

We will now prove that the map φ satisfies the hypothesis of Lemma 4.6 and we will estimate its rank. Let $I = (i_1, \dots, i_{d-1}) \in \mathbb{N}^d$ with $|I| \leq k$ be fixed. Set $i = |I|$. Define the function $\mathcal{S}^I : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ by

$$\mathcal{S}^I(s) = \frac{(s - s_0)^I}{I!} = \frac{(s_1 - s_{01})^{i_1}}{i_1!} \cdots \frac{(s_{d-1} - s_{0,d-1})^{i_{d-1}}}{i_{d-1}!}, \quad s_0 = (s_{01}, \dots, s_{0,d-1}).$$

Notice that,

$$\partial^K \mathcal{S}^I|_{s=s_0} = \begin{cases} 1 & \text{if } K = I, \\ 0 & \text{otherwise.} \end{cases} \quad (4.11)$$

We claim that

$$\begin{cases} \dim(\varphi(\mathbb{R}\mathcal{S}^I \otimes (\mathbb{R}^n)^*)) = n & \text{if } k - |I| \geq n - 1, \\ \dim(\varphi(\mathbb{R}\mathcal{S}^I \otimes (\mathbb{R}^n)^*)) \geq k - |I| & \text{otherwise.} \end{cases} \quad (4.12)$$

Multi-indices in \mathbb{N}^d are denoted with a hat and we decompose any such multi-index $\hat{J} \in \mathbb{N}^d$ as $\hat{J} = (J, j_d)$ where $J \in \mathbb{N}^{d-1}$ and $j_d \in \mathbb{N}$. We have

$$\begin{aligned} \varphi(\mathcal{S}^I \otimes e_a^*) &= j_{(s_0, t_0)}^k (\mathcal{S}^I V_a) \\ &= \sum_{\substack{\hat{J} \in \mathbb{N}^d \\ 0 \leq |\hat{J}| \leq k}} \partial^{\hat{J}} (\mathcal{S}^I V_a)|_{(s_0, t_0)} (S, T)^{\hat{J}} \\ &= \sum_{|J|=0}^k \sum_{j_d=0}^{k-|J|} \partial^{(J, j_d)} (\mathcal{S}^I V_a)|_{(s_0, t_0)} S^J T^{j_d}. \end{aligned} \quad (\text{Since } \hat{J} = (J, j_d))$$

For any $\hat{J} = (J, j_d) \in \mathbb{N}^d$ such that $|J| + j_d \leq k$, we have

$$\begin{aligned} \partial^{\hat{J}} (\mathcal{S}^I V_a)|_{(s_0, t_0)} &= \sum_{\substack{\hat{K} \in \mathbb{N}^d \\ \hat{K} \leq \hat{J}}} \binom{\hat{J}}{\hat{K}} \partial^{\hat{K}} \mathcal{S}^I \partial^{\hat{J}-\hat{K}} V_a|_{(s_0, t_0)} \\ &= \sum_{\substack{K \in \mathbb{N}^{d-1} \\ K \leq J}} \binom{\hat{J}}{(K, 0)} \partial^{(K, 0)} \mathcal{S}^I \partial^{\hat{J}-(K, 0)} V_a|_{(s_0, t_0)} \\ &\quad (\text{Since } \mathcal{S}^I \text{ does not depend on } t) \\ &= \sum_{\substack{K \in \mathbb{N}^{d-1} \\ K \leq J}} \binom{(J, j_d)}{(K, 0)} \partial^{(K, 0)} \mathcal{S}^I \partial^{(J-K, j_d)} V_a|_{(s_0, t_0)} \\ &= \sum_{\substack{K \in \mathbb{N}^{d-1} \\ K \leq J}} \binom{J}{K} (\partial^K \mathcal{S}^I)|_{s_0} (\partial^{(J-K, j_d)} V_a)|_{(s_0, t_0)} \\ &= \sum_{\substack{K \in \mathbb{N}^{d-1} \\ K \leq J}} \binom{J}{K} (\partial^K \mathcal{S}^I)|_{s_0} \left(\frac{\partial^{j_d}}{\partial t^{j_d}} \partial^{J-K} V_a \right)|_{(s_0, t_0)} \\ &= \begin{cases} \binom{J}{I} (\partial^{(J-I, j_d)} V_a)|_{(s_0, t_0)} & J \geq I, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (\text{According to (4.11)})$$

It follows that for any $\mathcal{S}^I \otimes e_a^* \in E_i$ (in particular $|I| = i$)

$$\varphi(\mathcal{S}^I \otimes e_a^*) = \sum_{\substack{(J, j_d) \in \mathbb{N}^d \\ 0 \leq |J| + j_d \leq k \\ I \leq J}} \binom{J}{I} (\partial^{(J-I, j_d)} V_a)|_{(s_0, t_0)} S^J T^{j_d}$$

$$= \sum_{j_d=0}^{k-|I|} \sum_{\substack{J \in \mathbb{N}^{d-1} \\ 0 \leq |J| + j_d \leq k \\ I \leq J}} \binom{J}{I} \left(\partial^{(J-I, j_d)} V_a \right) \Big|_{(s_0, t_0)} S^J T^{j_d}. \quad (4.13)$$

which shows that $\varphi(S^I \otimes e_a) \in \bigoplus_{J \geq I} F_J$ for every $a \in \{1, \dots, n\}$. Hence,

$$\varphi(E_I) \subset \bigoplus_{J \geq I} F_J, \quad \forall I, I \in \mathbb{N}^{d-1}, |I| \leq k,$$

and by consequence

$$\varphi(E_i) \subset F_i, \quad \forall i \in \mathbb{N}, 0 \leq i \leq k.$$

This shows that φ satisfies the hypothesis of Lemma 4.6. It follows that the matrix of the decomposition of the vectors $\pi_{F_I}(\varphi(S^I \otimes e_1^*)), \dots, \pi_{F_I}(\varphi(S^I \otimes e_n^*))$ over the basis $S^I, S^I T, \dots, S^I T^{k-|I|}$ of F_I is

$$M = \left(\pi_{F_I}(\varphi(S^I \otimes e_1^*)) \quad \dots \quad \pi_{F_I}(\varphi(S^I \otimes e_n^*)) \right) = \begin{pmatrix} V_1 & \dots & V_n \\ \frac{\partial}{\partial t} V_1 & \dots & \frac{\partial}{\partial t} V_n \\ \vdots & & \vdots \\ \frac{\partial^{k-|I|}}{\partial t^{k-|I|}} V_1 & \dots & \frac{\partial^{k-|I|}}{\partial t^{k-|I|}} V_n \end{pmatrix}.$$

Hence,

$$\dim \left(\varphi(\mathbb{R}S^I \otimes (\mathbb{R}^n)^*) \right) \geq \text{rank } M = \text{rank} \left\{ V, \frac{\partial}{\partial t} V, \dots, \frac{\partial^{k-|I|}}{\partial t^{k-|I|}} V \right\}.$$

Taking into account equality (4.9), our claim follows.

We now aim to compute an estimation of the rank of φ . Notice that for the case $d = 1$, our claim implies that $\text{rank } \varphi = \dim(\varphi((\mathbb{R}^n)^*)) = n$ as soon as $k \geq n - 1$. If $d \geq 2$, we have

$$\begin{aligned} \text{rank } \varphi &\geq \sum_{i=0}^k \dim \varphi(E_i) && \text{(According to Lemma 4.6)} \\ &= \sum_{|I|=0}^k \dim \varphi(E_I) \\ &= \sum_{|I|=0}^{k-n+1} \dim \varphi(E_I) + \sum_{|I|=k-n+2}^k \dim \varphi(E_I) \\ &\geq \sum_{|I|=0}^{k-n+1} n + \sum_{|I|=k-n+2}^k k - |I| + \sum_{|I|=k-n+2}^k n - \sum_{|I|=k-n+2}^k n && \text{(According to (4.12))} \\ &= \mu - \sum_{|I|=k-n+2}^k n - k + |I| \\ &= \mu - \sum_{i=k-n+2}^k \binom{d+i-2}{i} (n - k + i). \end{aligned} \quad (4.14)$$

Let r_{\min} be defined by the right hand side of (4.14). It is the minimum possible value of $\text{rank } \varphi$. We now identify $J_{s_0}^k(d-1, n)$ with

$$(\mathbb{R}^\mu)^* = \bigoplus_{|I| \leq k} \mathbb{R}S^I \otimes e_1^* \oplus \dots \oplus S^I \otimes e_n^*,$$

and $J_{(s_0, t_0)}^k(d, 1)$ with

$$\mathbb{R}^\nu = \bigoplus_{|I| \leq k} \mathbb{R} S^I T^0 \oplus \dots \oplus S^I T^{k-|I|},$$

where the multi-indices I are order by the lexicographical order. Viewing an element of $J_{(s_0, t_0)}^k f \in J_{(s_0, t_0)}^k(d-1, n)$ as a vector $P \in (\mathbb{R}^\mu)^*$, we may write $\varphi(j_{s_0}^k f)$ in matrix form as

$$\varphi(j_{s_0}^k f) = PA, \quad A \in \mathbb{M}(\mu \times \nu, \mathbb{R}).$$

Notice that

- every element $P \in (\mathbb{R}^\mu)^*$ decomposes as $P = (P_0, \eta)$, with $P_0 = \xi_0(s) = \tau(P) \in (\mathbb{R}^n)^*$, $\eta \in (\mathbb{R}^{\mu-n})^*$, and $s \mapsto \xi_0(s)$ is such that $j_s^k \xi_0 = P$;
- the matrix $A \in \mathbb{M}(\mu \times \nu, \mathbb{R})$ has the form

$$A = \begin{pmatrix} A_0 \\ \psi(A_0) \end{pmatrix}, \quad (4.15)$$

where $A_0 \in \mathbb{M}(n \times \nu, \mathbb{R})$ is the matrix representation of $j_{(s,t)}^k V$ and $\psi(A_0) \in \mathbb{M}((\mu-n) \times \nu, \mathbb{R})$.

These computations being done, we may equivalently rewrite the system of linear equations (4.6) in matrix form as

$$\mathcal{X}\mathcal{A} = 0, \quad (4.16)$$

with $\mathcal{X} = (P^1, \dots, P^\ell) \in ((\mathbb{R}^\mu)^*)^\ell$ and $\mathcal{A} \in \mathcal{A}_{k,\ell}$, where $\mathcal{A}_{k,\ell} \subset \mathbb{M}(\ell\mu \times (\ell-1)\nu, \mathbb{R})$ is defined to be the subset of structured matrices of the form

$$\mathcal{A} = \begin{pmatrix} A^1 B^1 & & & 0 \\ -A^2 B^2 & A^2 B^2 & & \\ & -A^3 B^3 & \ddots & \\ & & \ddots & A^{\ell-1} B^{\ell-1} \\ 0 & & & -A^\ell B^\ell \end{pmatrix}$$

where every submatrix A^α belongs to $\mathbb{M}(\mu \times \nu, \mathbb{R})$ and is of the form (4.15) and every submatrix $B^\alpha \in \text{GL}(\mathbb{R}^\nu)$.

Remark 4.7. Notice that $\mathcal{A}_{k,\ell}$ is a subset of $\mathbb{M}(\ell\mu \times (\ell-1)\nu, \mathbb{R})$ which, according to Remark 4.4, is diffeomorphic to a semi-algebraic subset of $(J_{(s,t)}^k(\mathbb{R}^d, \mathbb{R}^n))^\ell \times (J_s^k(\mathbb{R}^{d-1}, \mathbb{R}^d))^\ell$.

Lemma 4.8. $\mathcal{A}_{k,\ell}$ is a properly embedded submanifold of dimension $\ell n \nu$ of $\mathbb{M}(\ell\mu \times (\ell-1)\nu, \mathbb{R})$.

Proof. The statement follows directly from the fact that, for every $\alpha \in \{1, \dots, \ell\}$, $A^\alpha = \begin{pmatrix} A_0^\alpha \\ \psi(A_0^\alpha) \end{pmatrix}$ is a graph over A_0^α and from the fact that the matrices B^α do not play any role in the computation of the dimension of $\mathcal{A}_{k,\ell}$. ■

4.3.3 Inhomogeneous equations

Let us show that if $(-1, q, p, u)$ is a parametrized extremal for the problem (\mathbf{P}_L) , where L is the cost defined by equation (3.7), then additional conditions on ξ_0 must be satisfied. Since

$$\zeta(s, t) = DL(u(s, t)),$$

one gets

$$\frac{\partial \zeta}{\partial t}(s, t) = D^2L(u(s, t)) [\dot{u}(s, t) \otimes \text{Id}].$$

Moreover, from the PMP

$$\frac{\partial \zeta}{\partial t}(s, t) = -\xi(s, t) \frac{\partial f}{\partial y}(q(s, t)).$$

This yields

$$-\xi_0(s)R(s, t, t_0) \frac{\partial f}{\partial y}(q(s, t)) = D^2L(u(s, t)) [\dot{u}(s, t) \otimes \text{Id}], \quad (4.17)$$

where $D^2L(u(s, t))$ can be computed from (3.7) and depends on ξ_0 , $D\xi_0$ and L_0 . The resolution of equation (4.17) in the variable ξ_0 for all $(s, t) \in \mathcal{D}$ is however very difficult, even numerically for simple examples. The difficulty arises from the fact that when differentiating twice equation (3.7), the integral terms remain and equation (4.17) is then a partial integro-differential equation. However, suppose that there exists $s_0 \in \pi_s(\mathcal{D})$ such that Σ is not orthogonal to the optimal control curve $\{u(s_0, t), t \in [t_1(s_0), t_2(s_0)]\}$ at some $u_0 \neq 0$ (this is a generic condition). Under this hypothesis, the implicit function theorem yields that there exists $\alpha : \pi_s(\mathcal{D}) \rightarrow \mathbb{R}$ such that $\psi_1(u(s, \alpha(s))) = 0$ for all s in a neighborhood of s_0 . The integral terms in equation (4.17) then vanish and we obtain d linear inhomogeneous equations in the variables $\xi_0(s)$ and $D\xi_0(s)$. The resolution of these inhomogeneous equations remains very difficult and is not treated here. However, this is done for an example in Section 5.

Remark 4.9. If the boundary condition L_0 is unknown, we believe that the direct resolution of the partial integro-differential equation (4.17) in the variables ξ_0 and L_0 for all $(s, t) \in \mathcal{D}$ gives informations about this boundary condition.

Remark 4.10. If $n \leq d$, it seems that the inhomogeneous equation (4.17) allows, generically, to find $\xi_0(s)$ directly. The idea of the proof is the same as the one concerning the homogeneous part. This means that only one experiment could be needed in this case. However, this has not been studied and will be done in a future work.

4.4 A corank-type bound

We set $c_{\max} = \mu - r_{\min}$ and $\kappa_{\max} = (\ell - 1)c_{\max}$. In this section, we assume that k is taken such that $\ell\mu < (\ell - 1)\nu$. Hence,

$$\text{corank } A = \mu - \text{rank } A \quad \text{and} \quad \text{corank } \mathcal{A} = \ell\mu - \text{rank } \mathcal{A}.$$

For $\mathcal{A} \in \mathcal{A}_{k, \ell}$, let κ denote its corank, and let r^α (resp. c^α) denote the rank (resp. the corank) of every $(\mu \times \nu)$ -submatrix A^α of \mathcal{A} , $\alpha = 1, \dots, \ell$. For simplicity, the elements of $((\mathbb{R}^\mu)^*)^\ell$ are denoted by $\mathcal{X} = (P^1, \dots, P^\ell)$. Let $\tau^\ell : ((\mathbb{R}^\mu)^*)^\ell \rightarrow ((\mathbb{R}^n)^*)^\ell$ denote the (multi-)target mapping given by $\mathcal{X} \mapsto (P_0^1, \dots, P_0^\ell)$.

For $\mathcal{A} \in \mathcal{A}_{k, \ell}$, let $\mathcal{P}(\mathcal{A}) \subset ((\mathbb{R}^\mu)^*)^\ell$ be the subset defined by

$$\mathcal{P}(\mathcal{A}) = \left\{ \mathcal{X} \in ((\mathbb{R}^\mu)^*)^\ell \mid \mathcal{X}\mathcal{A} = 0 \right\}.$$

Define also

$$\mathcal{W}_{k,\ell}^\kappa = \left\{ \mathcal{A} \in \mathcal{A}_{k,\ell} \mid \dim \left(\tau^\ell(\mathcal{P}(\mathcal{A})) \right) = \kappa \right\}, \quad \forall \kappa \in \{0, \dots, \ell n\},$$

and

$$\mathcal{W}_{k,\ell}^{\geq \kappa} = \bigcup_{i=\kappa}^{\ell n} \mathcal{W}_{k,\ell}^i, \quad \kappa \in \{0, \dots, \ell n\}. \quad (4.18)$$

Notice that $\mathcal{W}_{k,\ell}^\kappa$ could be defined for all $\kappa \in \{0, \dots, \kappa_{\max}\}$. In that case, we would have $\mathcal{W}_{k,\ell}^\kappa = \emptyset$ for all $\kappa \in \{\ell n + 1, \dots, \kappa_{\max}\}$. Our main result (Theorem 4.1) will follow from the fact that, generically, the projection onto $((\mathbb{R}^n)^*)^\ell$ of space of solutions of system (4.16) is of dimension $d + 1$ if $\ell \geq d + 2$, or equivalently, that, generically, $\mathcal{W}_{k,\ell}^{\geq d+1} = \emptyset$ if $\ell \geq d + 2$.

Our goal in this section is to provide a corank-type lower bound of the codimension (in $\mathcal{A}_{k,\ell}$) of $\mathcal{W}_{k,\ell}^{\geq \kappa}$. First of all, notice that, because the set $\mathcal{W}_{k,\ell}^{\geq \kappa}$ is defined after projecting the solutions of equation (4.16) onto $((\mathbb{R}^n)^*)^\ell$, and because the A^α 's may not have full rank, the corank bound given in [3, Lemma 4.9]¹ cannot be applied directly. Indeed, we will prove a generalization of [3, Lemma 4.9].

Set $\mathbb{N}_{c_{\max}} = \{0, \dots, c_{\max}\}$. For all $C = (c^1, \dots, c^\ell) \in \mathbb{N}_{c_{\max}}^\ell$ and all $\kappa \in \{0, \dots, \kappa_{\max}\}$, define

- $\mathcal{A}_{k,\ell,C} = \{ \mathcal{A} \in \mathcal{A}_{k,\ell} \mid \text{corank } A^\alpha = c^\alpha, \forall \alpha \in \{1, \dots, \ell\} \},$
- $\mathcal{A}_{k,\ell}^\kappa = \{ \mathcal{A} \in \mathcal{A}_{k,\ell} \mid \text{corank } \mathcal{A} = \kappa \} (= \{ \mathcal{A} \in \mathcal{A}_{k,\ell} \mid \dim(\mathcal{P}(\mathcal{A})) = \kappa \}),$
- $\mathcal{A}_{k,\ell}^{\geq \kappa} = \bigcup_{i=\kappa}^{\kappa_{\max}} \mathcal{A}_{k,\ell}^i,$
- $\mathcal{A}_{k,\ell,C}^\kappa = \mathcal{A}_{k,\ell}^\kappa \cap \mathcal{A}_{k,\ell,C}.$

Obviously,

$$\mathcal{W}_{k,\ell}^{\geq \kappa} \subset \mathcal{A}_{k,\ell}^{\geq \kappa}, \quad \forall \kappa \in \{1, \dots, \ell n\},$$

which yields

$$\text{codim}_{\mathcal{A}_{k,\ell}} \mathcal{W}_{k,\ell}^{\geq \kappa} \geq \text{codim}_{\mathcal{A}_{k,\ell}} \mathcal{A}_{k,\ell}^{\geq \kappa}. \quad \forall \kappa \in \{1, \dots, \ell n\}. \quad (4.19)$$

In particular, a lower bound of $\text{codim}_{\mathcal{A}_{k,\ell}}(\mathcal{A}_{k,\ell}^{\geq \kappa})$ will yield a lower bound of $\text{codim}_{\mathcal{A}_{k,\ell}}(\mathcal{W}_{k,\ell}^{\geq \kappa})$. Since

$$\mathcal{A}_{k,\ell}^{\geq \kappa} = \bigcup_{\substack{C \in \mathbb{N}_{c_{\max}}^\ell \\ \kappa \leq i \leq \kappa_{\max}}} \mathcal{A}_{k,\ell,C}^i,$$

we get

$$\begin{aligned} \text{codim}_{\mathcal{A}_{k,\ell}}(\mathcal{A}_{k,\ell}^{\geq \kappa}) &= \min_{\substack{C \in \mathbb{N}_{c_{\max}}^\ell \\ \kappa \leq i \leq \kappa_{\max}}} \text{codim}_{\mathcal{A}_{k,\ell}}(\mathcal{A}_{k,\ell,C}^i) \\ &\geq \min_{\substack{C \in \mathbb{N}_{c_{\max}}^\ell \\ \kappa \leq i \leq \kappa_{\max}}} \text{codim}_{\mathcal{A}_{k,\ell,C}}(\mathcal{A}_{k,\ell,C}^i), \quad (\text{Since } \mathcal{A}_{k,\ell,C}^i \subset \mathcal{A}_{k,\ell,C} \subset \mathcal{A}_{k,\ell}) \end{aligned}$$

which, employing (4.19) yields

$$\text{codim}_{\mathcal{A}_{k,\ell}}(\mathcal{W}_{k,\ell}^{\geq \kappa}) \geq \min_{\substack{C \in \mathbb{N}_{c_{\max}}^\ell \\ \kappa \leq i \leq \kappa_{\max}}} \text{codim}_{\mathcal{A}_{k,\ell,C}}(\mathcal{A}_{k,\ell,C}^i) \quad \forall \kappa \in \{0, \dots, \ell n\}. \quad (4.20)$$

¹There is a misprint in the corank inequality given in the statement of [3, Lemma 4.9]. The right-handside of [3, Lemma 4.9, inequality (4.5)] should be changed to $r((\ell - 1)(k + 1) - (\ell n - r))$.

We now aim to find an estimate of the lower bound of $\text{codim}_{\mathcal{A}_{k,\ell,C}}(\mathcal{A}_{k,\ell,C}^i)$. To this aim we recall that the mapping φ^α is defined by

$$\varphi^\alpha : j_{s_0}^k \xi_0^\alpha \mapsto j_{(s_0,t_0)}^k(\xi_0^\alpha V^\alpha).$$

For every $\alpha \in \{1, \dots, \ell\}$, φ^α induces a one-to-one linear map according to the following commutative diagram.

$$\begin{array}{ccc} (\mathbb{R}^\mu)^* & \xrightarrow{\varphi^\alpha} & \mathbb{R}^\nu \\ \downarrow \pi_{c^\alpha} & \nearrow \tilde{\varphi}^\alpha & \\ (\mathbb{R}^\mu)^* / \ker \varphi^\alpha & & \end{array} \quad (4.21)$$

Since $A^\alpha = \begin{pmatrix} A_0^\alpha \\ \psi(A_0^\alpha) \end{pmatrix} = \begin{pmatrix} A_{0,0}^\alpha & A_{0,1}^\alpha \\ 0 & \psi(A_0^\alpha) \end{pmatrix}$ and if we write $(\mathbb{R}^\mu)^* = (\mathbb{R}^n)^* \times (\mathbb{R}^{\mu-n})^*$ and $P^\alpha = (P_0^\alpha, \eta^\alpha) \in (\mathbb{R}^n)^* \times (\mathbb{R}^{\mu-n})^*$, it is clear that, due to the form of A^α and since $A_{0,0}^\alpha$ has full rank, $\text{coker } A^\alpha \subset \{P^\alpha \in (\mathbb{R}^\mu)^* \mid P_0^\alpha = 0\}$. Consequently, $(\mathbb{R}^\mu)^* / \ker \varphi^\alpha = (\mathbb{R}^n)^* \times (\mathbb{R}^{\mu-n})^* / \ker \varphi^\alpha = (\mathbb{R}^n)^* \times (\mathbb{R}^{\mu-n-c^\alpha})^*$, and

$$\tilde{\varphi}^\alpha : (\mathbb{R}^n)^* \times (\mathbb{R}^{\mu-n})^* / \ker \varphi^\alpha \rightarrow \mathbb{R}^\nu.$$

If \tilde{P}^α denotes an element of $(\mathbb{R}^\mu)^* / \ker \varphi^\alpha$, then $\tilde{P}^\alpha = (P_0^\alpha, \tilde{\eta}^\alpha)$, where the $\tilde{}$ means the equivalent class of a vector in $(\mathbb{R}^{\mu-n})^*$.

Obviously, the ℓ factorizations (4.21) yields a factorization $\tilde{\phi}$ of the linear map $\phi : ((\mathbb{R}^\mu)^*)^\ell \rightarrow (\mathbb{R}^\nu)^\ell$ given by $\phi(\mathcal{X}) = \mathcal{X}\mathcal{A}$ such that

$$\tilde{\phi}(\tilde{\mathcal{X}}) = \phi(\mathcal{X}).$$

And consequently, induce a well-defined map

$$\Psi : \mathcal{A}_{k,\ell,C} \rightarrow \tilde{\mathcal{A}}_{k,\ell,C},$$

where $\tilde{\mathcal{A}}_{k,\ell,C} = \Psi(\mathcal{A}_{k,\ell,C})$ denotes the subset of $\mathbb{M}((\ell\mu - |C|) \times \nu, \mathbb{R})$ of structured matrices of the form

$$\begin{pmatrix} \tilde{A}^1 B^1 & 0 & \dots & 0 \\ -\tilde{A}^2 B^2 & \tilde{A}^2 B^2 & \ddots & \vdots \\ 0 & -\tilde{A}^3 B^3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \tilde{A}^{\ell-1} B^{\ell-1} \\ 0 & \dots & 0 & -\tilde{A}^\ell B^\ell \end{pmatrix}$$

where every submatrix $\tilde{A}^\alpha \in \mathbb{M}((\mu - c^\alpha) \times \nu, \mathbb{R})$, $\alpha = 1, \dots, \ell$, is a matrix representation of the mapping $\tilde{\varphi}^\alpha$. In particular, \tilde{A}^α has full rank (equal to $\mu - c^\alpha$). Moreover, notice that because $\tilde{P}^\alpha = (P_0^\alpha, \tilde{\eta}^\alpha)$, we have $\tilde{A}^\alpha = \begin{pmatrix} A_0^\alpha \\ \tilde{\psi}(A_0^\alpha) \end{pmatrix}$. It is easy to see that the map ϕ is indeed a surjective submersion. Hence, Ψ respects the stratification of $\mathcal{A}_{k,\ell,C}$ by the corank, i.e. $\Psi(\mathcal{A}_{k,\ell,C}^\kappa) = \tilde{\mathcal{A}}_{k,\ell,C}^\kappa$. Since moreover, Ψ is a submersion,

$$\text{codim}_{\mathcal{A}_{k,\ell,C}} \mathcal{A}_{k,\ell,C}^\kappa = \text{codim}_{\tilde{\mathcal{A}}_{k,\ell,C}} \tilde{\mathcal{A}}_{k,\ell,C}^\kappa. \quad (4.22)$$

One of the key points in the proof of our main result (Theorem 4.1) is Lemma 4.12. whose proofs needs some preliminary results which are stated and proved below. Following the same idea as in [3], we have the following lemma.

Lemma 4.11. *If $\tilde{A} \in \tilde{\mathcal{A}}_{k,\ell,C}^i$ ($0 \leq i \leq \kappa_{\max}$), then its cokernel has no non zero element of the form $(0, \tilde{P}^2, \dots, \tilde{P}^\ell)$, $(\tilde{P}^1, 0, \tilde{P}^3, \dots, \tilde{P}^\ell)$, \dots , or $(\tilde{P}^1, \dots, \tilde{P}^{\ell-1}, 0)$.*

Proof. Assume without loss of generality that $(0, \tilde{P}^2, \dots, \tilde{P}^\ell)$ belongs to $\text{coker } \tilde{A}$. It means that $(0, \tilde{P}^2, \dots, \tilde{P}^\ell)\tilde{A} = 0$. Consequently, $\tilde{P}^2\tilde{A}^2 = 0$, which implies that $\tilde{P}^2 = 0$ since \tilde{A}^2 has full rank. By induction, we get $\tilde{P}^\alpha = 0$ for all $\alpha \in \{1, \dots, \ell\}$. \blacksquare

In the following elements of $(\mathbb{R}^{\mu-c^\alpha})^*$ are denoted by \tilde{P}^α , and elements of $(\mathbb{R}^{\ell\mu-|C|})^*$ are denoted by $\tilde{\mathcal{X}} = (\tilde{P}^1, \dots, \tilde{P}^\ell)$. Let $\text{Gr}(i, \ell\mu - |C|)$ denotes the Grassmannian of i -dimensional subspaces of $(\mathbb{R}^{\ell\mu-|C|})^*$. With a little abuse of notation, we identify a i -dimensional linear space $\Pi \in \text{Gr}(i, \ell\mu - |C|)$ with any set of vectors $\{\tilde{\mathcal{X}}_1, \dots, \tilde{\mathcal{X}}_i\} \subset (\mathbb{R}^{\ell\mu-|C|})^*$ such that $\Pi = \text{span}\{\tilde{\mathcal{X}}_1, \dots, \tilde{\mathcal{X}}_i\}$. Finally, let $\mathcal{B}_{k,\ell,C}^i$ be the subset of $\text{Gr}(i, \ell\mu - |C|) \times \tilde{\mathcal{A}}_{k,\ell,C}$ defined by

$$\mathcal{B}_{k,\ell,C}^i = \left\{ (\tilde{\mathcal{X}}_1, \dots, \tilde{\mathcal{X}}_i, \tilde{A}) \in \text{Gr}(i, \ell\mu - |C|) \times \tilde{\mathcal{A}}_{k,\ell,C} \mid \tilde{\mathcal{X}}_j \tilde{A} = 0, \quad \forall j \in \{1, \dots, i\} \right\}.$$

Lemma 4.12 (Generalization of [3, Lemma 4.9]). *For every $C \in \mathbb{N}_{c_{\max}}^\ell$ and every $i \in \{0, \dots, \kappa_C\}$, the semi-algebraic set $\tilde{\mathcal{A}}_{k,\ell,C}$ satisfies*

$$\text{codim}_{\tilde{\mathcal{A}}_{k,\ell,C}} \mathcal{B}_{k,\ell,C}^i \geq \kappa^* ((\ell - 1)\nu - (\ell\mu - \kappa^*)), \quad \kappa^* = \min(i, n).$$

Proof. According to equation (4.16), if $\tilde{A} \in \tilde{\mathcal{A}}_{k,\ell,C}^i$, then there exist i independent vectors $\tilde{\mathcal{X}}_1 = (P_1^1, \dots, P_1^\ell)$, \dots , $\tilde{\mathcal{X}}_i = (P_i^1, \dots, P_i^\ell)$ in the cokernel of \tilde{A} such that

$$P_j^\alpha \tilde{A}^\alpha B^\alpha - P_j^{\alpha+1} \tilde{A}^{\alpha+1} B^{\alpha+1} = 0, \quad \forall j \in \{1, \dots, i\}, \quad \forall \alpha \in \{1, \dots, \ell - 1\},$$

or equivalently, such that

$$F_i^\alpha := M_i^\alpha \tilde{A}^\alpha - M_i^{\alpha+1} \tilde{A}^{\alpha+1} B^{\alpha+1} (B^\alpha)^{-1} = 0, \quad M_i^\alpha = \begin{pmatrix} P_1^\alpha \\ \vdots \\ P_i^\alpha \end{pmatrix}, \quad \forall \alpha \in \{1, \dots, \ell - 1\}. \quad (4.23)$$

We may assume without loss of generality that, for any fixed $\alpha \in \{1, \dots, \ell - 1\}$, the P_j^α 's ($j \in \{1, \dots, i\}$) are the i -th first vectors of the canonical basis of $(\mathbb{R}^{\mu-c^\alpha})^*$, namely that, $P_j^\alpha = (0, \dots, 0, 1, 0, \dots, 0)$. Indeed, if for $\alpha \in \{1, \dots, \ell\}$, P_1^α, \dots , and P_i^α were dependent, there would exist $(\lambda_1, \dots, \lambda_i) \neq (0, \dots, 0)$ so that $\lambda_1 P_1^\alpha + \dots + \lambda_i P_i^\alpha = 0$. Consequently, the vector $\lambda_1 \tilde{\mathcal{X}}_1 + \dots + \lambda_i \tilde{\mathcal{X}}_i$ is of the form $(P^1, \dots, P^{\alpha-1}, 0, P^{\alpha+1}, \dots, P^\ell)$ and belongs to the cokernel of \tilde{A} which contradicts Lemma 4.11. Set $\mathcal{F}_i = F_i^1 \times \dots \times F_i^{\ell-1}$. It is easy to see that the differential of \mathcal{F}_i with respect to $a = (A_0^1, \dots, A_0^{\ell-1})$ is

$$\frac{\partial \mathcal{F}_i}{\partial a} = \begin{pmatrix} \text{Id}_{\mathbb{R}^{i \times \nu}} & * & \dots & * \\ * & & & \vdots \\ 0 & & \ddots & * \\ \vdots & & & \text{Id}_{\mathbb{R}^{i \times \nu}} \\ 0 & \dots & 0 & * \end{pmatrix}.$$

Hence, by the implicit function theorem, we can smoothly solve system (4.23) in a neighborhood of \tilde{A} . Therefore the set of solutions to system (4.23) is contained in a manifold of codimension $\min\{i(\ell - 1)\nu, \dim \text{Gr}(i, \ell\mu - |C|) + \dim \tilde{\mathcal{A}}_{k,\ell,C}\}$, namely,

$$\text{codim}_{\text{Gr}(i, \ell\mu - |C|) \times \tilde{\mathcal{A}}_{k,\ell,C}} \mathcal{B}_{k,\ell,C}^i \geq \min\{i(\ell - 1)\nu, \dim \text{Gr}(i, \ell\mu - |C|) + \dim \tilde{\mathcal{A}}_{k,\ell,C}\}.$$

Since $\tilde{\mathcal{A}}_{k,\ell,C}^i$ is the projection of $\mathcal{B}_{k,\ell,C}^i$ on $\tilde{\mathcal{A}}_{k,\ell,C}$ parallel to the Grassmannian $\text{Gr}(i, \ell\mu - |C|)$, it yields

$$\begin{aligned} \text{codim}_{\tilde{\mathcal{A}}_{k,\ell,C}} \tilde{\mathcal{A}}_{k,\ell,C}^i &\geq \text{codim}_{\text{Gr}(i, \ell\mu - |C|) \times \tilde{\mathcal{A}}_{k,\ell,C}} \mathcal{B}_{k,\ell,C}^i - \dim \text{Gr}(i, \ell\mu - |C|) \\ &= \min \left\{ i(\ell - 1)\nu - \dim \text{Gr}(i, \ell\mu - |C|), \dim \tilde{\mathcal{A}}_{k,\ell,C} \right\} \\ &= \min \{ i(\ell - 1)\nu - i(\ell\mu - i), \ell n \nu \}. \end{aligned}$$

The required inequality follows since $\ell n \nu \geq n(\ell - 1)\nu - n(\ell\mu - n)$ if k is taken large enough. \blacksquare

Corollary 4.13. *For all $\kappa \in \{0, \dots, \ell n\}$, the semi-algebraic set $\mathcal{W}_{k,\ell}^{\geq \kappa}$ satisfies*

$$\text{codim}_{\mathcal{A}_{k,\ell}} \mathcal{W}_{k,\ell}^{\geq \kappa} \geq \kappa^* ((\ell - 1)\nu - (\ell\mu - \kappa^*)), \quad \kappa^* = \min(\kappa, n).$$

Proof. This is a direct consequence of equations (4.20), (4.22) and Lemma 4.12. \blacksquare

4.5 Construction of a surjective submersion

Let $\mathcal{Q} = \mathcal{C}^\infty(\mathbb{R}^{d-1}, \mathbb{R}^{n+d}) = \{q_0 : \mathbb{R}^{d-1} \mapsto \mathbb{R}^{n+d} \text{ of class } \mathcal{C}^\infty\}$. Let $J^k \mathcal{U}$ denotes the space of k -jets of elements of $\mathcal{U} = \text{Diff}(\mathcal{D}, U)$. Note that $J^k \mathcal{U}$ is an open subset of $J^k(\mathcal{D}, U)$ since

$$J^k \mathcal{U} = \{j^k u \in J^k(\mathcal{D}, U) \mid Du(s, t) \in \text{GL}(\mathbb{R}^d), \forall (s, t) \in \mathcal{D}\}.$$

Hence $\dim J^k \mathcal{U} = \dim J^k(d, d)$. For simplicity in the notations, we are treating the k -jets of a map as the map itself. This is justified by the fact that all the manipulations we do depend only on the k -jets of the objects. Note that we have

$$\begin{aligned} &\left\{ (f, q_0, u) \in J^k \mathcal{F} \times J_s^k \mathcal{Q} \times J_s^k \mathcal{U} \mid \right. \\ &\quad \left. \sigma(f) = \tau(q_0), \det(u(s, t_0), D(\pi_y q_0)(s)) \neq 0 \right\} \simeq J^k \mathcal{F} \times \mathcal{E}_k^s, \end{aligned} \quad (4.24)$$

where

$$\mathcal{E}_k^s = \left\{ (q_0, u) \in J_s^k \mathcal{Q} \setminus J_s^0 \mathcal{Q} \times J_s^k \mathcal{U} \mid \det(u(s, t_0), D(\pi_y q_0)(s)) \neq 0 \right\}.$$

The reason why we consider $\det(u(s, t_0), D(\pi_y q_0)(s)) \neq 0$ is that we will need $Dy(s, t_0)$ to be invertible in the proof of the next lemma. This hypothesis can be made without loss of generality according to Lemma 4.2.

Lemma 4.14. *The mapping*

$$\begin{aligned} \Xi_k^s &: J^k \mathcal{F} \times \mathcal{E}_k^s \rightarrow J^k(d, n) \\ (j_{\gamma_0(s)}^k f, j_s^k \gamma_0, j_{(s,t_0)}^k u) &\mapsto j_{(s,t_0)}^k V \end{aligned}$$

is a surjective submersion.

Proof. For the sake of clarity, let us recall that

$$\dot{q} = F_{u(s,t)}(q) = F_0(q) + \sum_{i=1}^d u_i(s, t) F_i(q), \quad (4.25)$$

$$\dot{p} = -p \frac{\partial F_{u(s,t)}}{\partial q}(q). \quad (4.26)$$

Let $Q_{t_0,t}^s$ and $P_{t_0,t}^s$ denote the flows of (4.25) and (4.26) respectively. They satisfy

$$\frac{d}{dt}Q_{t_0,t}^s = F_{u(s,t)} \circ Q_{t_0,t}^s, \quad Q_{t_0,t_0}^s = \text{Id}, \quad \forall s \in \pi_s(\mathcal{D}) \quad (4.27)$$

$$\frac{d}{dt}P_{t_0,t}^s = -P_{t_0,t}^s \frac{\partial F_{u(s,t)}}{\partial q} \circ Q_{t_0,t}^s, \quad P_{t_0,t_0}^s = \text{Id}, \quad \forall s \in \pi_s(\mathcal{D}), \quad (4.28)$$

where the second equalities in (4.27) and (4.28) can be assumed after a direct application of Lemma 4.5. We recall that

$$V(s,t) = P_{t_0,t}^s F_0(q(s,t)), \quad q(s,t) = Q_{t_0,t}^s(q_0(s)), \quad q_0(s) = q(s,t_0). \quad (4.29)$$

It follows from equations (4.25), (4.26), (4.27), (4.28) and (4.29),

$$\begin{aligned} \frac{\partial V}{\partial t} \Big|_{(s,t_0)} &= \sum_{i=1}^d u_i(s,t_0) [F_i, F_0] \\ &= D_y F_0(q_0(s)) [u(s,t_0)] \\ &= D_y F_0(q_0(s)) \left[\frac{\partial y}{\partial t}(s,t_0) \right] \\ &= D_y F_0(q_0(s)) \circ D_y(s,t_0) \left[\frac{\partial}{\partial t} \right]. \end{aligned}$$

Also we have

$$\begin{aligned} D_s V(s,t_0) &= \frac{\partial}{\partial s} F_0(q(s,t_0)) \\ &= D F_0(q_0(s)) \circ D q(s,t_0) \left[\frac{\partial}{\partial s} \right] \\ &= D_y F_0(q_0(s)) \circ D y(s,t_0) \left[\frac{\partial}{\partial s} \right] + D_x F_0(q_0(s)) \circ D x(s,t_0) \left[\frac{\partial}{\partial s} \right]. \end{aligned}$$

Summing up,

$$D V(s,t_0) = D_y F_0(q_0(s)) \circ D y(s,t_0) + R_1,$$

and by an easy induction one gets that for all $k \in \mathbb{N}$,

$$D^k V(s,t_0) = D_y^k F_0(q_0(s)) \circ (D y(s,t_0))^{\otimes k} + R_k,$$

where the remainder term R_k depends on the k -jet $j_s^k q_0$ and on successive derivatives of F_0 (or equivalently f) with respect to the variable y up to order less than k . Since the maps $(D y(s,t_0))^{\otimes i}$, $i \in \{1, \dots, k\}$ are all invertible, the surjectivity of Ξ_k^s follows by induction.

The fact that Ξ_k^s is a submersion follows easily since the partial differential of Ξ_k^s with respect to the partial derivatives of F_0 with respect to y is

$$\begin{pmatrix} \text{Id} & * & \cdots & * \\ 0 & D y(s,t_0) & & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & 0 & (D y(s,t_0))^{\otimes k} \end{pmatrix},$$

which has full rank. ■

Let $\mathcal{Q}_{k,\ell} = \{\mathcal{Q} = (q^1, \dots, q^\ell) \in (J^k \mathcal{Q})^\ell \mid \tau(q^\alpha) \neq \tau(q^\beta) \text{ for } 1 \leq \alpha < \beta \leq \ell\}$ and $\mathcal{F}_{k,\ell} = (J^k \mathcal{F})^\ell \big|_{\mathcal{Q}_{k,\ell}}$ denote the restriction of the product bundle $(J^k \mathcal{F})^\ell$ to $\mathcal{Q}_{k,\ell}$, where $(J^k \mathcal{F})^\ell$ denotes the bundle over \mathbb{R}^{n+d} of k -jets of elements of \mathcal{F} . Typical elements of $\mathcal{F}_{k,\ell}$ are tuples

$$\mathcal{F} = (j_{q^1}^k f, \dots, j_{q^\ell}^k f), (q^1, \dots, q^\ell) \in \mathcal{Q}_{k,\ell}.$$

Moreover, we define

$$\mathcal{U}_{k,\ell} = \{(\mathcal{U}^1, \dots, \mathcal{U}^\ell) \in (J^k \mathcal{U})^\ell \mid \tau(\mathcal{U}^\alpha) = \tau(\mathcal{U}^\beta), 1 \leq \alpha < \beta \leq \ell\}$$

and

$$\mathcal{E}_{k,\ell}^s = \{(\mathcal{Q}, \mathcal{U}) \in \mathcal{Q}_{k,\ell} \setminus \mathcal{Q}_{0,\ell} \times \mathcal{U}_{k,\ell} \mid (q^i, u^i) \in \mathcal{E}_k^s, i \in \{1, \dots, \ell\}\}.$$

From the definition of $\mathcal{U}_{k,\ell}$, one easily gets that it is a properly embedded submanifold of dimension $\ell d\nu - (\ell - 1)d$ of $(J^k(d, d))^\ell$.

Corollary 4.15. *The mapping*

$$\begin{aligned} \Xi_{k,\ell}^s : \mathcal{F}_{k,\ell} \times \mathcal{E}_{k,\ell}^s &\rightarrow \mathcal{A}_{k,\ell} \\ (\mathcal{F}, \mathcal{Q}, \mathcal{U}) &\mapsto \mathcal{A} \end{aligned}$$

is a surjective submersion.

Proof. Since for every $\mathcal{Q} \in \mathcal{Q}_{k,\ell}$, the target points $\tau(q^1) \dots \tau(q^\ell)$ are distinct, the proof is an immediate consequence of Lemma 4.14 thanks to the structure of the matrices $\mathcal{A} \in \mathcal{A}_{k,\ell}$. \blacksquare

4.5.1 Application of Thom's transversality theorem

The set $\mathcal{N}_{k,\ell,\geq\kappa}^s \subset \mathcal{F}_{k,\ell}$ that we will construct in the following, will, in some sense, represents $\mathcal{W}_{k,\ell}^{\geq\kappa}$ in $\mathcal{F}_{k,\ell}$. Since, according to Corollary 4.15, the map $\Xi_{k,\ell}^s$ is a surjective submersion, it is transversal to everything. It follows that $(\Xi_{k,\ell}^s)^{-1}(\mathcal{W}_{k,\ell}^{\geq\kappa})$ is a semi-algebraic Whitney-b stratified set of codimension $c_{k,\ell}$, with $c_{k,\ell} \geq \kappa^* ((\ell - 1)\nu - (\ell\mu - \kappa^*))$, $\kappa^* = \min(\kappa, n)$ by Lemma 4.13. Define the set $\mathcal{N}_{k,\ell,\geq\kappa}^s$ by

$$\mathcal{N}_{k,\ell,\geq\kappa}^s = \pi_{\mathcal{F}_{k,\ell}} \left((\Xi_{k,\ell}^s)^{-1}(\mathcal{W}_{k,\ell}^{\geq\kappa}) \right).$$

Consequently, $\mathcal{N}_{k,\ell,\geq\kappa}^s$ is a stratified set of codimension

$$\text{codim } \mathcal{N}_{k,\ell,\geq\kappa}^s \geq \text{codim } \mathcal{W}_{k,\ell}^{\geq\kappa} - \dim \mathcal{E}_{k,\ell}^s,$$

which leads to

$$\text{codim } \mathcal{N}_{k,\ell,\geq\kappa}^s \geq \kappa^* ((\ell - 1)\nu - (\ell\mu - \kappa^*)) - n\ell\mu - \ell n - d(\ell\nu - (\ell - 1)),$$

where $\kappa^* = \min(\kappa, n)$. If k is large enough, $\nu \gg \mu$ and we get the estimate

$$\text{codim } \mathcal{N}_{k,\ell,\geq\kappa}^s \geq (\kappa^*(\ell - 1) - d\ell)\nu.$$

According to [3, Theorem 4.14], the set \mathcal{G}^s of $f \in \mathcal{F}$ such that \mathcal{F} is transversal to $\mathcal{N}_{k,\ell,\geq\kappa}^s$ is residual for the Whitney topology. Notice that $\text{codim } \mathcal{N}_{k,\ell,\geq\kappa}^s > \ell \dim \mathcal{Q} = \ell(n + d)$ means that \mathcal{F} avoids $\mathcal{N}_{k,\ell,\geq\kappa}^s$. Moreover $\text{codim } \mathcal{N}_{k,\ell,\geq\kappa}^s > \ell n$ if $(\kappa^*(\ell - 1) - d\ell) > 0$ (since $\nu \gg \ell(n + d)$), i.e. if

$$\kappa^*(\ell - 1) > d\ell \Leftrightarrow \frac{\ell - 1}{\ell} > \frac{d}{\kappa^*}.$$

Since $(\ell - 1)/\ell < 1$, we get that $\kappa^* > d$ and $\ell > \kappa^*/(\kappa^* - d)$. The condition $\kappa^* > d$ yields that both κ and n must be greater than d . Suppose $n > d$ and let us choose $\kappa = d + 1$ and $\ell = d + 2$. It follows that for each $s \in \pi_s(\mathcal{D})$, the set \mathcal{G}^s of $f \in \mathcal{F}$ such that \mathcal{F} avoids $\mathcal{N}_{k,d+2,\geq d+1}^s$ is residual for the Whitney topology, which means that generically, $d + 2$ experiments of $d - 1$ parameters are enough to reconstruct $\xi_0(s)$ (since we have d inhomogeneous conditions). If $n \leq d$, one experiment is sufficient to reconstruct the cost thanks to the d inhomogeneous conditions (see Remark 4.10).

Now, let \mathcal{S} be a countable and dense subset of $\pi_s(\mathcal{D})$. Since the countable intersection of residual sets is a residual set, we conclude that the set \mathcal{G} of $f \in \mathcal{F}$ such that we can reconstruct ξ_0 on \mathcal{S} is residual.

Lastly, since ξ_0 is smooth and \mathcal{S} is a dense subset of $\pi_s(\mathcal{D})$ there exists a unique prolongation by continuity of ξ_0 on $\pi_s(\mathcal{D})$. This ends the proof of our main result.

4.6 Openness properties of the sets \mathcal{G}

In this section we aim to understand whether the set \mathcal{G} given by Theorem 4.1 is open or not or, if it contains an open and dense subset of \mathcal{F} .

For simplicity, we also assume that $d = 1$. We assume that k and ℓ are fixed and such that $d > n$, $\ell \geq d + 2$ and k is large enough. In this case, the parameter s is zero-dimensional (we thus omit it in all notations), $\mathcal{Q}_{k,\ell} = \{\mathcal{Q} = (q^1, \dots, q^\ell) \in Q^\ell \mid q^\alpha \neq q^\beta \text{ for } 1 \leq \alpha < \beta \leq \ell\}$ and the set $\mathcal{W}_{k,\ell}^{\geq 2}$ defined by (4.18) may equivalently and simply be described as the elements $\mathcal{A} \in \mathcal{A}_{k,\ell}$ that satisfy:

- $\text{corank } \mathcal{A} \geq 2$;
- every submatrix A^α satisfies $\text{rank } A^\alpha = n$ for every $\alpha \in \{1, \dots, \ell\}$.

Define the following mapping

$$\begin{aligned} r^\ell : \mathcal{F} \times \mathcal{E}_{k,\ell} &\rightarrow \mathcal{A}_{k,\ell} \\ (\mathcal{F}, \mathcal{Q}, \mathcal{U}) &\mapsto r^\ell(\mathcal{F}, \mathcal{Q}, \mathcal{U}) = \Xi_{k,\ell}(\mathcal{F}, \mathcal{Q}, \mathcal{U}) = \mathcal{A}. \end{aligned}$$

Theorem 4.1 means that the set

$$\begin{aligned} \mathcal{G} &= \{f \in \mathcal{F} \mid r^\ell(\mathcal{F}, \mathcal{Q}, \mathcal{U}) \text{ avoids } \mathcal{W}_{k,\ell}^{\geq 2}\} \\ &= \underbrace{\{f \in \mathcal{F} \mid \text{corank } r^\ell(\mathcal{F}, \mathcal{Q}, \mathcal{U}) \leq 1\}}_{=: \mathcal{G}_a} \cup \underbrace{\{f \in \mathcal{F} \mid \exists \alpha, \text{rank } A_k^\alpha < n\}}_{=: \mathcal{G}_b} \end{aligned}$$

is a residual subset in \mathcal{F} .

Lemma 4.16. *The mapping r^ℓ is continuous.*

Proof. According to [10, Propositions 3.4 and 3.10], the maps

$$\begin{aligned} j^k : \mathcal{C}^\infty(X, Y) &\rightarrow \mathcal{C}^\infty(X, J^k(X, Y)) \\ f &\mapsto j^k f \end{aligned}$$

and

$$\begin{aligned} \delta_\ell : \mathcal{C}^\infty(X, Y)^\ell &\rightarrow \mathcal{C}^\infty(X^\ell, Y^\ell) \\ (f_1, \dots, f_\ell) &\mapsto f_1 \times \dots \times f_\ell : (x_1, \dots, x_\ell) \mapsto (f_1(x_1), \dots, f_\ell(x_\ell)) \end{aligned}$$

are continuous. Also, according to [15, Theorem 46.10, page 286] the evaluation map

$$\begin{aligned} \mathcal{C}^\infty \left(Q^\ell, (J^k \mathcal{F})^\ell \right) \times Q^\ell &\rightarrow (J^k \mathcal{F})^\ell \\ \left((j^k f_1, \dots, j^k f_\ell), (q_1, \dots, q_\ell) \right) &\mapsto (j_{q_1}^k f_1, \dots, j_{q_\ell}^k f_\ell) \end{aligned}$$

is continuous for the compact-open topology and consequently for the Whitney \mathcal{C}^∞ topology (which is finer). By restriction to $\mathcal{C}^\infty \left(Q^\ell, (J^k \mathcal{F})^\ell \right) \times \mathcal{Q}_{k,\ell}$ the evaluation mapping

$$\text{ev} : \mathcal{C}^\infty \left(Q^\ell, (J^k \mathcal{F})^\ell \right) \times \mathcal{Q}_{k,\ell} \rightarrow \mathcal{F}_{k,\ell}$$

is continuous. Also the following mapping is continuous:

$$\begin{aligned} \text{Id}^\ell : \mathcal{C}^\infty(X, Y) &\rightarrow \mathcal{C}^\infty(X, Y)^\ell \\ f &\mapsto \underbrace{(f, \dots, f)}_{\ell \text{ times}}. \end{aligned}$$

We deduce from above that the following mapping is continuous

$$\delta_\ell \circ \text{Id}^\ell \circ j^k : \mathcal{F} \xrightarrow{j^k} \mathcal{C}^\infty(Q, J^k \mathcal{F}) \xrightarrow{\text{Id}^\ell} \mathcal{C}^\infty(Q, (J^k \mathcal{F})^\ell) \xrightarrow{\delta_\ell} \mathcal{C}^\infty(Q^\ell, (J^k \mathcal{F})^\ell).$$

Consequently, the mapping

$$\text{EV} = \text{ev} \circ \left((\delta_\ell \circ \text{Id}^\ell \circ j^k) \times \text{Id}_{\mathcal{Q}_{k,\ell}} \right) : \mathcal{F} \times \mathcal{Q}_{k,\ell} \rightarrow (J^k \mathcal{F})^\ell$$

given by $(f, \mathcal{Q}) \mapsto (j_{q_1}^k f^\ell, \dots, j_{q_\ell}^k f^\ell)$ is continuous. Notice that $\text{Im}(\text{EV}) \subset \mathcal{F}_{k,\ell}$. Consequently, the mapping $r^\ell = \Xi^\ell \circ (\text{EV} \times \text{Id}_{\mathcal{Q}_{k,\ell}})$ is continuous. ■

Lemma 4.17. *The set \mathcal{G}_a is an open subset of \mathcal{F} .*

Proof. From Lemma 4.16 and from the openness of the set $\mathcal{D} = \{A \in \mathcal{A}_{k,\ell} \mid \text{corank } A \leq 1\}$ that the set $\mathcal{S} = (r^\ell)^{-1}(\mathcal{D})$ is an open subset of $\mathcal{F} \times \mathcal{E}_{k,\ell}$. Hence, $\mathcal{G}_a = \pi_{\mathcal{F}}(\mathcal{S})$ is an open subset of \mathcal{F} . ■

Let the subset $\mathcal{T} \subset \mathcal{F} \times \mathcal{E}_{k,\ell}$ be defined by

$$\mathcal{T} = \{(\mathcal{F}, \mathcal{Q}, \mathcal{U}) \in \mathcal{F} \times \mathcal{E}_{k,\ell} \mid A_k^\alpha \text{ has full rank, } \alpha = 1, \dots, \ell\}.$$

Lemma 4.18. *\mathcal{T} is an open and dense subset of $\{\mathcal{F} \times \mathcal{E}_{k,\ell}\}$.*

Proof. The set $\mathcal{Z} = \{A \in \mathcal{A}_{k,\ell} \mid A_k^\alpha \text{ has full rank, } \alpha = 1, \dots, \ell\}$ is an open and dense subset of $\mathcal{A}_{k,\ell}$. Because r^ℓ is continuous, it follows that $\mathcal{T} = (r^\ell)^{-1}(\mathcal{Z})$ is an open subset of $\mathcal{F} \times \mathcal{E}_{k,\ell}$.

Let us now show the density of \mathcal{T} . Let $U \subset \mathcal{F} \times \mathcal{E}_{k,\ell}$ be an open set. We have to show that $U \cap \mathcal{T} \neq \emptyset$. Because r^ℓ is a (surjective) submersion, r^ℓ is an open mapping. Consequently, $O = r^\ell(U)$ is an open subset of $\mathcal{A}_{k,\ell}$. Hence, it follows from the density of \mathcal{Z} in $\mathcal{A}_{k,\ell}$ that $O \cap \mathcal{Z} \neq \emptyset$. Thus, there exists $y \in r^\ell(U) \cap \mathcal{Z}$ with $y = r^\ell(x)$, $x \in U$. It follows that $x \in U \cap (r^\ell)^{-1}(\mathcal{Z}) = U \cap \mathcal{T}$. ■

Remark 4.19. Lemma 4.18 implies that the set

$$\pi_{\mathcal{F}}(\mathcal{T}) = \left\{ f \in \mathcal{F} \mid \exists (\mathcal{Q}, \mathcal{U}) \in \mathcal{E}_{k,\ell}, \text{ every submatrix } A_k^\alpha \text{ of } r^\ell(\mathcal{F}, \mathcal{Q}, \mathcal{U}) \text{ has full rank} \right\}$$

is an open and dense subset in \mathcal{F} .

Because the set \mathcal{G}_b is not open, the residual set \mathcal{G} in Theorem 4.1 may not be open. However, we shall prove below that \mathcal{G} contains an open and dense set. Indeed the mapping $r^\ell|_{\mathcal{F}}$ is a surjective submersion on its image \mathcal{S} . Set

$$\begin{aligned}\widehat{\mathcal{G}} &= \{f \in \pi_{\mathcal{F}}(\mathcal{T}) \mid r^\ell|_{\mathcal{F}}(\mathcal{F}, \mathcal{Q}, \mathcal{U}) \text{ avoids } \mathcal{W}_{k,\ell}^{\geq 2} \cap \mathcal{Z}\} \\ &= \{f \in \pi_{\mathcal{F}}(\mathcal{T}) \mid \text{corank } r^\ell|_{\mathcal{F}}(\mathcal{F}, \mathcal{Q}, \mathcal{U}) \leq 1\}.\end{aligned}$$

If the set $\widehat{\mathcal{G}}$ is residual, we are done since $\mathcal{G} \supset \widehat{\mathcal{G}}$. To prove that the set $\widehat{\mathcal{G}}$ is residual, we follow the same steps as in the proof of Corollary 4.15 which is possible since the mapping Ξ^ℓ is still a surjective submersion in restriction to the open set $(\Xi^\ell)^{-1}(\mathcal{Z})$.

We have thus proved the following lemma.

Lemma 4.20. *If $d = 1$ the residual set \mathcal{G} given by Theorem 4.1 contains an open and dense set.*

Remark 4.21. A similar reasoning could be done for $d > 1$ in order to prove that for s fixed the set \mathcal{G}^s contains an open and dense set. However this would not yield the same conclusion as the preceding lemma since \mathcal{G} is a countable but not finite intersection of sets \mathcal{G}^s .

4.7 Some remarks

A sufficient number of experiments

Notice that, contrary to the one control case, less experiments could be enough in order to reconstruct the cost. This comes from the fact that we compute not only $\xi_0(s)$ but its full k -jet. We could imagine generic cases for which less experiments could be sufficient in order to reconstruct only $\xi_0(s)$ and not its full k -jet.

Fixed final time case

We supposed that the experiments were with a free final time. Our result also holds if the final time is fixed but the proof needs to be slightly modified. Indeed, if the final time is fixed, we cannot state that the Hamiltonian vanishes along trajectories but only that it is constant (with respect to time). It follows that the Hamiltonians along experiments need not be the same and the computation (4.4) does not hold anymore. The idea is to consider, for each extremal, the constant value of the Hamiltonian as variable in order to reconstruct it along with the covector using the same proof as before.

Non special form

In this paper, although we considered only systems of the form (1.1), we believe that our result applies for general d -control affine systems. Indeed, the considered special form is used to simplify the computations in the proof of Lemma 4.14, which we believe would stay valid in general.

5 Linear example

In this section, we treat a simple example in order to show the computations of the inhomogeneous equations. This is a particular case in which $n \leq d$ and one experiment is sufficient to reconstruct the cost.

5.1 Optimal control problem

By way of illustration, consider the dynamical system

$$\begin{cases} \dot{x} = y_1 \\ \dot{y}_1 = u_1 \\ \dot{y}_2 = u_2, \end{cases} \quad (5.1)$$

where $x \in \mathbb{R}$ denotes the state and $u = (u_1, u_2) \in \mathbb{R}^2$ the control variable. Define the cost function $L(u) = \frac{1}{2}(u_1^2 + u_2^2 + 1)$ and the parametrized target points $q_f(s) = (\frac{\cos(s)}{3} \quad 0 \quad \sin(s))'$, $s \in (-1, 1)$. We aim to solve the optimal control problem:

(\mathbf{P}_L) For all $s \in (-1, 1)$, minimize the integral cost (1.2) among all admissible controls u steering system (5.1) from a source point q_0 to the target points $q_f(s)$ in free final time $T(s)$.

In order to solve problem (\mathbf{P}_L) using the PMP, define the Hamiltonian function

$$h(\lambda, p, q, u) = \xi y_1 + \zeta_1 u_1 + \zeta_2 u_2 - \frac{\lambda}{2}(u_1^2 + u_2^2 + 1).$$

Let $\gamma = (q, u)$ be an experiment relative to the cost L . It follows from Proposition 2.3 that there exists a parametrized extremal (λ, p, q, u) . Without loss of generality, we suppose that for each $s \in (-1, 1)$ the initial time is zero and $\lambda = 1$ (since this system does not admit any abnormal extremal). The adjoint system is then

$$\begin{cases} \dot{\xi} = 0 \\ \dot{\zeta}_1 = -\xi \\ \dot{\zeta}_2 = 0 \end{cases} \quad (5.2)$$

and it follows from the maximization condition that

$$\zeta_i = D_{u_i} L(u) = u_i. \quad (5.3)$$

Moreover since the final time is free, the Hamiltonian function is zero along trajectories and since $q(s, 0) = q_0(s) \equiv 0$, the initial condition of the covector must satisfy $\zeta_{10}^2(s) + \zeta_{20}^2(s) = 1$. In particular, there exists a function α_0 such that $\zeta_{10}(s) = \cos(\alpha_0(s))$ and $\zeta_{20}(s) = \sin(\alpha_0(s))$. Equation (5.1) together with equations (5.2) and (5.3) yield

$$\begin{cases} \dot{x} = y_1 \\ \dot{y}_1 = \zeta_1 \\ \dot{y}_2 = \zeta_2 \\ \dot{\xi} = 0 \\ \dot{\zeta}_1 = -\xi \\ \dot{\zeta}_2 = 0, \end{cases}$$

whose solution with initial condition $q_0 = 0$ is

$$\begin{cases} x(s, t) = -\xi_0(s) \frac{t^3}{6} + \zeta_{10}(s) \frac{t^2}{2} \\ y_1(s, t) = -\xi_0(s) \frac{t^2}{2} + \zeta_{10}(s)t \\ y_2(s, t) = \zeta_{20}(s)t \\ \xi(s, t) = \xi_0(s) \\ \zeta_1(s, t) = -\xi_0(s)t + \zeta_{10}(s) \\ \zeta_2(s, t) = \zeta_{20}(s). \end{cases} \quad (5.4)$$

Evaluating the first three equations of (5.4) at $t = T(s)$ yields

$$\begin{cases} -\xi_0(s) \frac{T(s)^3}{6} + \cos(\alpha_0(s)) \frac{T(s)^2}{2} = \frac{\cos(s)}{3} \\ -\xi_0(s) \frac{T(s)^2}{2} + \cos(\alpha_0(s))T(s) = 0 \\ \sin(\alpha_0(s))T(s) = \sin(s), \end{cases}$$

which gives $\alpha_0(s) = s$, $T(s) = 1$ and $\xi_0(s) = 2 \cos(s)$. In conclusion, the solution of problem (\mathbf{P}_L) is the parametrized optimal control

$$\begin{cases} u_1(s, t) = -2 \cos(s)t + \cos(s) \\ u_2(s, t) = \sin(s). \end{cases} \quad (5.5)$$

One easily check that u is a diffeomorphism and therefore the experiment γ is diffeomorphic.

Remark 5.1. For this example, $\mathcal{D} = (-1, 1) \times [0, 1]$.

5.2 Inverse optimal control problem

Using the experiment generated in the last subsection, we aim to find a cost L such that (q, u) is a solution of the optimal control problem (\mathbf{P}_L) . The test of abnormality yields, $G(s) = \int_0^1 y_1(s, \tau)^2 d\tau \neq 0$ and the constructed experiment is normal. As explained in Subsection 3.2, we must give ourselves a knowledge of the cost on a one-dimensional manifold Σ . Suppose that $\Sigma = \{(\phi_1(\theta), \phi_2(\theta)), \theta \in [0, 2\pi]\}$ is the unit circle (which is non-characteristic for equation (3.3)), i.e.

$$\begin{cases} \phi_1(\theta) = \cos(\theta) \\ \phi_2(\theta) = \sin(\theta) \end{cases} \quad (5.6)$$

and let $L_0(\theta) = 1$ be the boundary condition. One get from (3.6) and (5.6) that

$$\begin{cases} \rho = \ln(\sqrt{v_1^2 + v_2^2}) \\ \theta = \arctan(v_2/v_1) \end{cases} \quad (5.7)$$

and the solution of (3.3) is

$$L(u_1, u_2) = \|u\| \left(1 - \int_0^{\ln(\|u\|)} (\hat{\xi}_0 \hat{V})(\cos(\arctan(u_2/u_1))e^r, \sin(\arctan(u_2/u_1))e^r) e^{-r} dr \right). \quad (5.8)$$

Equivalently, using the change of variable $r \mapsto \ln(r)$ in the integral, one gets

$$L(u_1, u_2) = \|u\| \left(1 - \int_1^{\|u\|} \frac{(\hat{\xi}_0 \hat{V})(\cos(\arctan(u_2/u_1))r, \sin(\arctan(u_2/u_1))r)}{r^2} dr \right). \quad (5.9)$$

We conclude that if ξ_0 is known the cost L will be fully determined. Moreover, as explained in Subsection 4.3.3, the inhomogeneous equation (4.17) must be satisfied. Notice that from equation (5.5), one deduce that $\|u(s, 1)\| = 1$, i.e. the integral terms of equation (4.17) vanish along the curve $(s, \tau(s)) = (s, 1)$. Although it is not detailed here for the sake of clarity, the resolution of equation (4.17) in the variable $\xi_0(s)$ along the curve $(s, \tau(s)) = (s, 1)$ using Wolfram Mathematica yields $\xi_0(s) = 2 \cos(s)$ and from equation (5.9)

$$L(u) = \frac{1}{2}(u_1^2 + u_2^2 + 1).$$

Remark 5.2. The present example strengthen our belief described in Remark 4.9 since if we choose another (wrong) boundary condition L_0 and solve (4.17) along a curve in the variables ξ_0 (as it is done for the example), we obtain another cost. However, it turns out that equation (4.17) is not satisfied for all $(s, t) \in \mathcal{D}$ for this newly reconstructed cost.

5.3 Some remarks about the example

First, in this particular simple example, only one experiment is needed in order to reconstruct the cost. According to Remark 4.10, this comes from the fact that the dimension of the state x is lower than the number of controls (i.e. $n < d$).

Secondly, it follows straightforwardly from equation (5.2) and (5.3) of Subsection 5.1 that for this example the control u_2 is time invariant no matter how the experiment is constructed. It is then possible to find a positive function $a(\cdot)$ such that $\frac{\partial a(u(s, t))}{\partial t} = 0$ for any optimal control $u(s, \cdot)$. Considering the cost $\tilde{L}(u) = a(u_2)L(u)$, one can prove that equation (3.2) is not satisfied, i.e. a trajectory which is a projection of an extremal for the cost L is not a projection of an extremal for the cost \tilde{L} .

6 Conclusion

In this paper, we defined and analyzed an inverse optimal control problem for a class of control affine systems with d inputs. Our result states that, generically, $d + 2$ families of optimal trajectories parametrized by a $(d - 1)$ -dimensional parameter are enough to reconstruct the cost function minimized by these trajectories. The main limitation of this study is that the cost function needs to be known on a $(d - 1)$ -dimensional surface. Although it has not been studied properly yet, we give a partial integro-differential equation that should allow to weaken this hypothesis. In particular, an example highlighting the importance of this equation is studied.

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4. Discussions

4.1. Conclusion

Le résultat principal de ce chapitre (théorème 2.2) indique que, en général, $d + 2$ expériences sont suffisantes pour reconstruire le critère et ainsi résoudre le problème de contrôle optimal inverse. Contrairement au théorème 2.1, nous ne pouvons pas affirmer que $d + 2$ expériences sont nécessaires (en général). En effet, la reconstruction du critère se fait à partir de la connaissance du covecteur du PMP. Or, avec la méthode que nous proposons, nous ne pouvons pas calculer seulement le covecteur mais tout son k -jet. On pourrait alors imaginer des cas (génériques) pour lesquelles moins d'expériences permettraient de reconstruire le covecteur de manière unique mais pas son k -jet. Nous n'avons cependant pas approfondi cette question.

4.2. Difficultés par rapport au cas un contrôle

Notons cependant une différence importante entre le cas à un contrôle et le cas à d contrôles. Pour $d = 1$, l'invariance des trajectoires optimales vis à vis des transformations $L \mapsto aL + bu$ permet de fixer la valeur du critère en un point et ainsi de reconstruire le critère de manière unique. À d contrôles, l'unicité de la solution nécessite la connaissance du critère sur une sous-variété de dimension $d - 1$. Malheureusement, l'invariance des trajectoires optimales vis à vis des transformations admissibles sur le critère ne permet plus de se donner cette connaissance. C'est pourquoi une connaissance *a priori* sur le critère est nécessaire. Nous donnons cependant certaines pistes afin de relaxer cette hypothèse.

4.3. Ouvertures

Deux autres améliorations pourraient également être apportées à ce résultat. La première concerne la forme spéciale du système. Il serait intéressant de réaliser la même étude pour des systèmes affines. La seconde amélioration concerne la forme du critère, qui ne dépend que du contrôle dans cette étude. Notons cependant que l'introduction de l'état dans le critère augmente de manière significative la difficulté du problème. En effet, dans ce cas, l'équation différentielle satisfaite par le covecteur dépend du critère et est par conséquent inconnue.

En n, nous n avons pas encore traité d application de nos résultats. Il serait possible d étudier le pilotage d un avion comme cela a été fait dans Ajami *et al.* [3]. A deux contrôles, le modèle considéré pourrait être une extension du modèle de Markov-Dubins (voir par exemple Sussmann [68]) où la vitesse linéaire ne serait pas supposée constante, c est à dire

$$\begin{cases} \dot{x} = v \cos \theta \\ \dot{y} = v \sin \theta \\ \dot{v} = a \\ \dot{\theta} = u, \end{cases}$$

où a et u désignent les deux contrôles à disposition du pilote.

Une autre approche pourrait consister à ne reconstruire qu une approximation du critère, c est à dire sa série de Taylor. On peut avec cette approche espérer avoir besoin de moins d expériences. Cependant, il n est pas garanti que les extrémals qui correspondent au critère soit les mêmes que celles de son approximation. Il faut alors garantir que ces extrémals sont dans un certain sens assez proches. Par ailleurs, il semble naturel de penser que le nombre d expériences pour reconstruire l approximation doit tendre vers $d+2$ lorsque l ordre de l approximation tend vers l in ni. Ceci pourrait constituer une future étude intéressante.

Chapitre 3

Stabilisation en des points inobservables

Dans ce chapitre, nous traitons à travers un exemple académique un problème de stabilisation par retour de sortie. Comme dans les chapitres précédents, après une introduction et la description du problème, certains outils nécessaires à la résolution du problème sont rappelés. Nos résultats sont donnés et démontrés sous forme d'article scientifique. Nous concluons en donnant de possibles suites à nos travaux.

1. Introduction

Certaines notions classiques d'observabilité ou de stabilité ne sont pas rappelés ou approfondies. Pour plus de détails, voir par exemple Gauthier et Kupka [30]; Isidori [41]; Khalil [48].

1.1. Problématique

Il arrive souvent en théorie du contrôle de vouloir stabiliser un système dynamique dans un état donné. Cependant, la stabilisation nécessite souvent la connaissance d'états non mesurés. Il est alors nécessaire d'introduire un observateur afin d'estimer les états manquants.

Considérons le système dynamique

$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x), \end{cases} \quad (3.1)$$

où $x \in \mathbb{R}^n$ est l'état, $u \in U \subset \mathbb{R}^m$ est l'entrée, $y \in \mathbb{R}^p$ est la sortie. A une entrée $u : [0, T] \rightarrow U$ continue par morceaux et une condition initiale x_0 corres-

pond une trajectoire $t \mapsto \gamma(u, x_0; t)$ solution du système $\dot{x} = f(x, u)$, $x(0) = x_0$. D'après Sussmann [69], on dit que le système (3.1) est *observable* si pour toutes conditions initiales x_1 et x_2 tels que $x_1 \neq x_2$, il existe une entrée continue par morceaux $u : [0, T] \rightarrow U$ telle que $h(\gamma(u, x_1; t)) \neq h(\gamma(u, x_2; t))$ pour au moins un $t \in [0, T]$. Pour les systèmes linéaires

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx, \end{cases}$$

l'observabilité ne dépend ni de l'entrée ni des conditions initiales et le système est observable si la matrice

$$\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

est de rang plein n (voir Kalman [45]). Pour les systèmes non linéaires cependant, il existe également un critère (voir Hermann et Krener [35]) mais, en général, il existe des entrées qui rendent le système inobservable (voir Celle *et al.* [20]).

Par ailleurs, on dit qu'un point d'équilibre x^* du système (3.1) est *stabilisable* s'il existe une fonction $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^d$ telle que x^* soit un point d'équilibre globalement asymptotiquement stable du système

$$\dot{x} = f(x, \lambda(x)).$$

Pour les systèmes linéaires, si x^* est stabilisable et si le système est observable, alors x^* est stabilisable par retour de sortie dynamique (voir Luenberger [55]). Dans le cas des systèmes non linéaires, il n'y a malheureusement pas d'équivalent à ce résultat. Cependant, d'après Teel et Praly [71], pour les systèmes à un contrôle et une sortie, si le x^* est stabilisable et le système est complètement uniformément observable alors x^* est semi-globalement stabilisable par retour dynamique de sortie. Néanmoins, un problème survient lorsque l'on souhaite stabiliser le système dans une configuration où il devient inobservable, c'est à dire qu'il n'est pas possible d'estimer les états du système à partir de la sortie.

Le but de ce chapitre est d'étudier la stabilisation par retour dynamique de sortie dans le cas où l'on cherche à stabiliser en des points qui rendent le système inobservable. Bien que la littérature sur les observateurs soit assez complète (voir par exemple Gauthier et Kupka [30]; Hermann et Krener

[35]; Luenberger [55]; Sussmann [69]), celle sur la stabilisation par retour de sortie des systèmes non linéaires est plus mince et concerne souvent les systèmes uniformément observables (voir par exemple Andrieu et Praly [4]; Praly et Astol [64]; Teel et Praly [71]). Parmi les études sur les systèmes non uniformément observables, citons en particulier Besançon et Hammouri [11], où les auteurs étudient la stabilisation par retour de sortie sur pour la classe des systèmes en cascades. Citons également Coron [24], où une condition suffisante de stabilisation locale en temps petit par retour de sortie dynamique, périodique et variant dans le temps est donnée. Le système étudié dans ce chapitre satisfait les hypothèses données dans Coron [24], cependant notre objectif est de fournir un résultat de stabilité asymptotique en utilisant un retour dynamique de sortie lisse et invariant dans le temps.

1.2. Exemple considéré

L'étude réalisée dans ce chapitre est articulée autour d'un exemple académique provenant de la mécanique quantique (voir par exemple Boscain et Long [13]; Boscain *et al.* [15]) afin de cerner les difficultés qui apparaissent. Plus précisément, nous considérons le système dynamique

$$\begin{cases} \dot{x} = A(u)x \\ y = Cx = x_3, \end{cases} \quad (3.2)$$

où $x = (x_1, x_2, x_3) \in S^2$ est l'état, $y \in \mathbb{R}$ est la sortie mesurée, $u = (u_1, u_2) \in \mathbb{R}^2$ est le contrôle, $C = (0 \ 0 \ 1)$ et

$$A(u) = \begin{pmatrix} 0 & 1 & u_1 \\ -1 & 0 & u_2 \\ -u_1 & -u_2 & 0 \end{pmatrix}.$$

Le problème est de stabiliser le système (3.2) au point d'équilibre $x_t = (0 \ 0 \ -1)$ en utilisant uniquement la sortie mesurée. Remarquons que le contrôle correspondant au point d'équilibre x_t est $u \equiv 0$, contrôle qui rend le système inobservable. En effet, si $x_3 \neq \pm 1$ et $u \equiv 0$, la sortie mesurée est constante et ne permet pas de distinguer les états variants x_1 et x_2 .

Remarquons également que le retour d'état

$$\lambda^s(x) = r_1(x_1, x_2), \quad r_1 \in \mathbb{R}_+$$

stabilise le système au point x_t . Cependant, $\lambda^s(x_t) = (0, 0)$ et le système devient de moins en moins observable au fur et à mesure que l'état se rapproche

de la cible. Cela va naturellement poser un problème lorsque l'on va vouloir boucler le retour d'état avec un observateur.

2. Préliminaires et outils théoriques

Les travaux exposés par la suite font appel au principe d'invariance de LaSalle et à la théorie de la variété centrale. Ces deux outils sont rappelés dans cette section.

2.1. Principe d'invariance de LaSalle

Le principe d'invariance de LaSalle (voir par exemple LaSalle [50]) est un outil très utile en théorie du contrôle pour démontrer la stabilité d'un système dynamique. Soit le système dynamique autonome

$$\dot{x} = f(x), \tag{3.3}$$

où $x \in \mathbb{R}^n$ et f est une fonction C^1 sur un domaine $D \subset \mathbb{R}^n$ telle que $f(0) = 0$.

Soit $V : \mathbb{R}^n \rightarrow \mathbb{R}$ une fonction de classe C^1 . Suivant la définition de LaSalle [51], on dit que V est une fonction de Lyapunov sur $G \subset \mathbb{R}^n$ pour le système (3.3) si $\dot{V}(x) = DV(x)[f(x)]$ ne change pas de signe sur G . De plus soit $E = \{x \in \Omega \mid \dot{V}(x) = 0\}$ et M le plus grand invariant contenu dans E .

Théorème 3.1 (Principe d'invariance de LaSalle). *Soit Ω un compact positivement invariant pour le système (3.3). Supposons qu'il existe une fonction de Lyapunov V telle que $\dot{V}(x) \leq 0$ sur Ω . Alors toute trajectoire du système (3.3) initialisée dans Ω tend vers M lorsque $t \rightarrow \infty$.*

Ce théorème donne un résultat quant à l'attractivité d'un ensemble invariant. Le corollaire suivant donne également des informations concernant la stabilité et le bassin d'attraction de l'attracteur.

Corollaire 3.2. *Soit $G \subset \{x \mid V(x) < \ell\}$. Supposons que G est borné et que $\dot{V}(x) \leq 0$ sur G . Alors $A = \overline{M \cap G}$ est un attracteur et G est son bassin d'attraction. De plus, si V est constante sur A alors A est un attracteur stable.*

Remarque 3.3. Dans le corollaire précédent, si A est réduit à un point alors A est asymptotiquement stable et G est une estimation de son bassin d'attraction.

2.2. Théorie de la variété centrale

Le principe de LaSalle est un outil très puissant pour étudier la stabilité d'un système dynamique. Cependant, il nécessite de trouver une fonction de Lyapunov, ce qui est parfois complexe. Une autre méthode consiste à étudier la stabilité locale d'un point d'équilibre à partir de son linéarisé, en utilisant le théorème d'Hartman-Grobman (voir par exemple Perko [63]). Cependant, ce théorème ne s'applique que lorsque les valeurs propres du linéarisé sont à partie réelle non nulle.

La théorie de la variété centrale permet d'étudier la stabilité lorsque des valeurs propres du linéarisé sont à partie réelle nulle. Considérons le système dynamique

$$\begin{cases} \dot{x} = Ax + F(x, y) \\ \dot{y} = By + G(x, y) \end{cases} \quad (3.4)$$

où $x \in \mathbb{R}^c$, $y \in \mathbb{R}^s$, A et B sont des matrices réelles à valeurs propres à partie réelle nulle et strictement négatives respectivement et F et G sont des fonctions C^k dans un voisinage N_δ de l'origine telles que $F(0) = 0$, $G(0) = 0$, $DF(0) = 0$ et $DG(0) = 0$.

Pour les démonstrations des théorèmes suivants, voir Carr [19].

Théorème 3.4. *Pour $2 \leq k < \infty$, le système (3.4) admet une variété invariante, appelée variété centrale,*

$$W^c = \{(x, y) \mid |x| < \varepsilon, y = h(x)\},$$

où $h : \mathbb{R}^c \mapsto \mathbb{R}^s$ est une fonction C^k dans un voisinage N_ε de l'origine, $h(0) = 0$, $Dh(0) = 0$ et pour tout $x \in N_\varepsilon$,

$$N(h(x)) := Dh(x)[Ax + F(x, h(x))] - Bh(x) - G(x, h(x)) = 0. \quad (3.5)$$

De plus, la dynamique sur W^c est donnée par l'équation

$$\dot{x} = Ax + F(x, h(x)). \quad (3.6)$$

Remarque 3.5. En général, la variété centrale n'est pas unique.

Théorème 3.6. *Si le point $x = 0$ est stable (asymptotiquement stable) (instable) pour le système (3.6), alors le point $(x, y) = (0, 0)$ est stable (asymptotiquement stable) (instable) pour le système (3.4).*

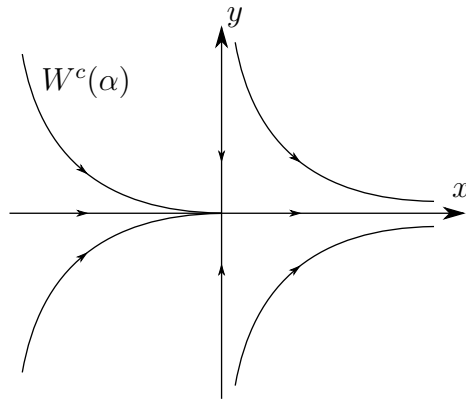


FIGURE 3.1 Portrait de phase du système (3.7).

Il est en général très difficile de résoudre l'équation (3.5) afin de calculer h . Cependant, le théorème suivant stipule que qu'il est possible d'approximer h aussi précisément qu'on le désire en utilisant son développement en série de Taylor.

Théorème 3.7. *S'il existe une fonction ϕ telle que $\phi(0) = 0$, $D\phi(0) = 0$ et $N(\phi(x)) = O(|x|^m)$ quand $|x| \rightarrow 0$ pour $m > 1$ alors $|h(x) - \phi(x)| = O(|x|^m)$ quand $|x| \rightarrow 0$.*

Exemple 3.8 (Section 4, Kelley [47]). Afin de montrer que la variété centrale peut ne pas être unique, considérons le système suivant

$$\begin{cases} \dot{x} = x^2 \\ \dot{y} = -y, \end{cases} \quad (3.7)$$

dont le portrait de phase est donné sur la figure 3.1. D'après le théorème 3.4, il existe une fonction $h : \mathbb{R} \rightarrow \mathbb{R}$ telle que

$$h'(x)x^2 - h(x) = 0, \quad h(0) = 0 \quad \text{et} \quad h'(0) = 0.$$

La résolution de cette équation donne comme solution

$$h(\alpha, x) = \begin{cases} \alpha e^{1/x} & \text{si } x < 0 \\ 0 & \text{si } x \geq 0, \end{cases}$$

où $\alpha \in \mathbb{R}$. Cela montre que la variété centrale n'est pas unique et on obtient ainsi pour chaque α une variété centrale

$$W^c(\alpha) = \{(x, y) \mid y = h(\alpha, x)\}.$$

Exemple 3.9. Considérons le système

$$\begin{cases} \dot{x}_1 = x_1 y - x_1 x_2^2 \\ \dot{x}_2 = x_2 y - x_2 x_1^2 \\ \dot{y} = -y + x_1^2 + x_2^2 \end{cases} \quad (3.8)$$

et étudions la stabilité du point d'équilibre $x_1 = x_2 = y = 0$. Ce système est sous la forme (3.4). D'après le théorème 3.4, il existe une fonction $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ telle que

$$Dh(x)[Ax + F(x, h(x))] - Bh(x) - G(x, h(x)) = 0, \quad h(0) = 0 \quad \text{et} \quad Dh(0) = 0.$$

Cette équation aux dérivées partielles étant très difficile à résoudre, cherchons une approximation de h , à commencer par le second ordre. Posons

$$h(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2 + O(|x|^3),$$

de sorte que $h(0) = 0$ et $Dh(0) = 0$. La résolution de l'équation d'invariance donne alors $a = c = 1$ et $b = 0$. On en déduit que la dynamique sur la variété centrale est donc donnée par

$$\begin{cases} \dot{x}_1 = x_1^3 + O(|x|^3) \\ \dot{x}_2 = x_2^3 + O(|x|^3), \end{cases}$$

dont l'origine est un point d'équilibre instable. D'après les théorèmes 3.6 et 3.7, l'origine $x_1 = x_2 = y = 0$ est instable pour le système (3.8).

3. Contribution scientifique

3.1. Résultats

L'idée générale de l'étude qui suit est que si une entrée rend le système (3.2) inobservable alors il suffira de la perturber légèrement pour qu'elle conserve l'observabilité du système. En effet, d'après Celle *et al.* [20], pour les systèmes bilinéaires, l'ensemble des entrées persistantes (i.e. les entrées qui sont "assez loin des mauvaises entrées") forme un sous-ensemble ouvert dense dans l'ensemble des entrées admissibles. Notre méthode va donc consister à perturber le retour d'état stabilisant bouclé avec un observateur afin qu'il conserve un certain niveau d'observabilité du système.

Afin de reconstruire l'état système (3.2), nous considérons l'observateur

$$\dot{\hat{x}} = A(u)\hat{x} - r_2 C'(C\hat{x} - y), \quad (3.9)$$

où $r_2 \in \mathbb{R}_+$. Nos deux résultats concernant le système (3.2) sont les suivants.

Théorème 3.10. *Le retour de sortie dynamique $u = \lambda_\delta^s(\hat{x}) = r_1(\hat{x}_1, \hat{x}_2) + (\delta, \delta)$, $\delta \in \mathbb{R}$ stabilise semi-globalement et asymptotiquement le système (3.2) sur un point d'équilibre arbitrairement proche de la cible. Ce point d'équilibre se rapproche de la cible lorsque δ tend vers zéro.*

Théorème 3.11. *Soit $\lambda_{\delta,\alpha}^s(\hat{x}) = r_1(\hat{x}_1, \hat{x}_2) + (\delta, \delta)((\hat{x}_3)^\alpha - 1)$ où α est un nombre pair strictement positif et $\delta \in (-\sqrt{2}/\alpha, \sqrt{2}/\alpha)$. Le retour de sortie dynamique $u = \lambda_{\delta,\alpha}^s(\hat{x})$ stabilise semi-globalement et asymptotiquement le système (3.2) sur la cible.*

Nous donnons également un résultat général utile pour démontrer les deux résultats précédents :

Théorème 3.12. *Si le linéarisé d'un système dynamique en un point d'équilibre possède au moins une valeur propre à partie réelle positive, alors l'ensemble des trajectoires qui convergent vers ce point d'équilibre instable est de mesure nulle.*

Remarque 3.13. Ce théorème peut paraître évident, surtout dans le cas où le linéarisé ne possède pas de valeur propre à partie réelle nulle. Cependant, dans le cas contraire le système admet une variété centrale. C est en particulier la non unicité de la variété centrale qui pose problème.

3.2. Démonstration

Nos résultats sont détaillés et démontrés dans l'article suivant, qui a été envoyé pour publication dans le journal IEEE Transactions on Automatic Control. Les résultats concernant la stabilisation pratique ont été exposés à Buenos Aires en 2016 à l'occasion de la Multi-Conference on Systems and Control (MSC) et les résultats concernant la stabilisation exacts seront exposés à Melbourne en décembre 2017 à l'occasion de la Conference on Decision and Control (CDC).

Practical and exact output feedback stabilizations at unobservable points

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Abstract—In this paper, we analyze, via a simple academic example from quantum control, the problem of dynamic output feedback stabilization, when the point where we want to stabilize corresponds to a control value that makes the system unobservable.

The stability analysis is performed in three steps. We have validated the method for a specific case, but the idea can be extended to more general problems. In particular, the method can be applied to the case of Killing systems that are injectable into bilinear skew-symmetric (or skew-adjoint) systems.

I. INTRODUCTION

It is usually not possible to measure the whole state of a system in practice, but only a part of it. This part is called the observation, and only this can be measured. Problems arise when one wants to stabilize a system to some target point since, in most cases, the knowledge of the whole state might be needed. In [1, Chapters 6 and 7] (see also [2]), the authors give different methods in order to reconstruct the whole state space and stabilize the system provided it is observable. However, for nonlinear systems, there exist in general inputs that make the system unobservable (see [3]). Here, we are interested in the case where the point we want to stabilize the system to corresponds to a control value that renders it unobservable. Although similar problems were already addressed in [4], [5], [6], [7], the literature on output feedback stabilization for non-uniformly observable systems is scarce. In [5] a sufficient condition for local stabilization in small time by means of a dynamic periodic time-varying output feedback law is given. Whereas the system treated in the present paper meets the conditions given in [5], our work aims to give (semi-global) asymptotic stability of the target by means of a smooth time-invariant dynamic output feedback.

The present paper deals with the same problem as the one exposed in [7]. However, [7, Theorem 2.10] includes a mistake since the result should not be stated as semi-global. Theorem 3.10 corrects this mistake. Moreover, while in [7] the authors gave a result on practical asymptotic stabilization of an arbitrarily small neighborhood of the target point, the present work adds a result about the

asymptotic stability of the target point (exactly) and not only of a neighborhood of it.

The method we present here is illustrated on a specific example borrowed from quantum control that we treat in all details. Consider the following (three dimensional) typically observed closed quantum system (see e.g. [8], [9], [10]).

$$\begin{cases} \dot{x} = A(u)x \\ y = Cx, \end{cases} \quad (1)$$

where $(x_1, x_2, x_3) = x \in S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$ is the state, $y \in \mathbb{R}$ is the measured output (or, the observation), $u = (u_1, u_2) \in \mathbb{R}^2$ is the control variable, $C = (0 \ 0 \ 1)$ is in $\mathbb{R}^{1 \times 3}$, and $A(u)$ is a control-dependent matrix in $\mathbb{R}^{3 \times 3}$ given by

$$A(u) = \begin{pmatrix} 0 & e & u_1 \\ -e & 0 & u_2 \\ -u_1 & -u_2 & 0 \end{pmatrix}.$$

In quantum mechanics, x corresponds to the state of a qubit and e is a constant energy gap which will be set to one. Even though the validity of the continuous measurement $y(t)$ is unclear in the quantum context, system (1) is a typical example of a more general situation that will be exposed in Section IV.

The problem is to stabilize system (1) at the target point $x_t = (0, 0, -1)$, which is an equilibrium point corresponding to the control value $u = (u_1, u_2) = (0, 0)$. Notice that system (1) is unobservable for this control since indistinguishable points are those that are on the same parallel $\{x_3 = \text{cst}\} \cap S^2$.

The paper is organized as follows. In Section II we characterize the set of trajectories that converge to an unstable equilibrium point. These preliminary results will be needed in Section III since the instability of an equilibrium point does not guarantee its repulsiveness.

Section III is devoted to the complete analysis of the problem. The main idea is to perturb a stabilizing control in order to preserve a small level of observability. Two approaches are studied. The first one does not stabilize system (1) exactly at the target point x_t , but at an equilibrium point arbitrarily close to it. In the second approach, the perturbation is chosen not only to preserve a small level of observability but also to stabilize (1) at the target point.

In Section IV we describe how the analysis done in Section III can be extended to more general situations. Subsection IV-A details how the method built through

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the example of system (1) can be extended to general bilinear skew-symmetric systems. Following [3], in Subsection IV-B, we give conditions under which more general systems can be immersed into skew-adjoint systems. The study of skew-adjoint systems is then only the first step towards the study of output feedback stabilization for a wider class of systems.

Due to space limitation, the detailed analysis of the two general situations presented in Section IV will be reported elsewhere.

Notation

We denote by $\|\cdot\|$ the Euclidean norm on \mathbb{R}^n and by $B(x, r)$ the open ball centered at x of radius r for this norm. If f is a function from \mathbb{R}^n to \mathbb{R}^m , the notation $Df(x)[v]$ stands for the differential at $x \in \mathbb{R}^n$ applied to the vector $v \in \mathbb{R}^n$ of the function f . Also, we use the symbol $'$ to denote the transposition operation.

II. PRELIMINARY RESULT

Consider the dynamical system

$$\dot{x} = X(x), \quad x \in M, \quad (2)$$

where M is a m -dimensional smooth Riemannian manifold and X is a smooth vector field on M . Let x^* be an equilibrium point for system (2) and let (O, φ) be a local coordinate chart on M centered at x^* . Using the coordinate map φ , system (2) can be locally rewritten as

$$\begin{cases} \dot{\xi} = F\xi + f(\xi, \eta) \\ \dot{\eta} = G\eta + g(\xi, \eta), \end{cases} \quad (3)$$

where $F \in \mathbb{R}^{m_1 \times m_1}$ has m_1 eigenvalues with non-positive real parts, $G \in \mathbb{R}^{m_2 \times m_2}$ has m_2 eigenvalues with positive real parts and, f and g are smooth maps in a neighborhood N of $0 \in \mathbb{R}^m = \mathbb{R}^{m_1+m_2}$ such that $f(0, 0) = 0$, $g(0, 0) = 0$, $Df(0, 0) = 0$ and $Dg(0, 0) = 0$. Since the matrix F has eigenvalues with non-positive real parts, there may exist trajectories converging to the origin.

Kelley proved in [11] that for system (3) there exists an invariant manifold

$$W^{cs} = \{(\xi, \eta) \mid \|\xi\| < \delta, \eta = h(\xi)\},$$

where, for $\delta > 0$ small enough, h can be chosen to be C^k for any arbitrarily but fixed $k \in \mathbb{N}$. Moreover, h satisfies $h(0) = 0$. W^{cs} is called a (local) center-stable manifold and is a priori not unique.

Although the idea of the proof of the following two lemmas was found in [11], we did not find these general results in the literature.

Lemma 2.1: Any trajectory of system (3) converging towards the origin belongs to a center-stable manifold.

Proof: Since W^{cs} is an invariant manifold for system (3), h satisfies

$$Dh(\xi)[F\xi + f(\xi, h(\xi))] = Gh(\xi) + g(\xi, h(\xi)). \quad (4)$$

$$\begin{aligned} \text{Set } \tilde{\eta} &= \eta - h(\xi). \text{ With equations (3) and (4) it yields} \\ \dot{\tilde{\eta}} &= \dot{\eta} - Dh(\xi)[\dot{\xi}] = G\eta + g(\xi, \eta) - Dh(\xi)[F\xi + f(\xi, \eta)] \\ &= G\eta + g(\xi, \tilde{\eta} + h(\xi)) - Dh(\xi)[F\xi + f(\xi, \tilde{\eta} + h(\xi))] \\ &\quad + Dh(\xi)[f(\xi, h(\xi))] - Dh(\xi)[f(\xi, h(\xi))] \\ &= G\eta - Gh(\xi) - g(\xi, h(\xi)) + g(\xi, \tilde{\eta} + h(\xi)) \\ &\quad + Dh(\xi)[f(\xi, h(\xi)) - f(\xi, \tilde{\eta} + h(\xi))] \\ &= G\tilde{\eta} + \tilde{g}(\xi, \tilde{\eta}), \end{aligned}$$

where

$$\begin{aligned} \tilde{g}(\xi, \tilde{\eta}) &= -g(\xi, h(\xi)) + g(\xi, \tilde{\eta} + h(\xi)) \\ &\quad + Dh(\xi)[f(\xi, h(\xi)) - f(\xi, \tilde{\eta} + h(\xi))]. \end{aligned}$$

Recall that, up to a linear change of coordinates, there exists $\mu > 0$ such that $\langle G\tilde{\eta}, \tilde{\eta} \rangle \geq 2\mu\|\tilde{\eta}\|^2$ (see e.g. [11]). By construction, \tilde{g} is a C^r map for any $r < \infty$ and

$$\tilde{g}(\xi, \tilde{\eta}) - \tilde{g}(\xi, 0) = \int_0^1 D\tilde{g}(\xi, s\tilde{\eta})ds[(0, \tilde{\eta})].$$

Because $\tilde{g}(\xi, 0) = 0$ for all $\xi \in N_\delta$, $D\tilde{g}(\cdot, \cdot)$ is continuous and $D\tilde{g}(0, 0) = 0$, there exists a sufficiently small neighborhood of the origin such that $\|\tilde{g}(\xi, \tilde{\eta})\| \leq \mu\|\tilde{\eta}\|$. It follows that

$$\begin{aligned} \frac{d}{dt}\|\tilde{\eta}\|^2 &= 2\langle G\tilde{\eta}, \tilde{\eta} \rangle + 2\langle \tilde{g}(\xi, \tilde{\eta}), \tilde{\eta} \rangle \\ &\geq 4\mu\|\tilde{\eta}\|^2 - 2\mu\|\tilde{\eta}\|^2 \geq 2\mu\|\tilde{\eta}\|^2. \end{aligned}$$

We conclude that any trajectory starting with an initial condition $\tilde{\eta}(0) \neq 0$ must leave any sufficiently small neighborhood of the origin. Equivalently, a trajectory converging to the origin must belong to a center-stable manifold. ■

Lemma 2.2: Any trajectory of system (3) converging towards the origin belongs to the intersection of all center-stable manifolds.

Proof: Consider two center-stable manifolds W_α^{cs} and W_β^{cs} and a trajectory $(\xi(t), \eta(t)) \in W_\alpha^{cs}$ converging to the origin. Set $\tilde{\eta} = \eta - h_\beta(\xi)$. The same computations as in the proof of Lemma 2.1 yield

$$\frac{d}{dt}\|\tilde{\eta}\|^2 \geq 2\mu\|\tilde{\eta}\|^2. \quad (5)$$

Since $\xi(t)$ goes to zero as t goes to infinity, $\tilde{\eta}(t)$ must converge to zero. However, from (5), the norm of $\tilde{\eta}(t)$ cannot decrease. We conclude that $\tilde{\eta}(t)$ must be identically zero. In particular $\eta(t) = h_\alpha(\xi(t)) = h_\beta(\xi(t))$ for any $t \geq 0$, from which the lemma follows. ■

Corollary 2.3: Any center-stable manifold contains all the trajectories converging to the origin.

Let ϕ_t denote the flow of system (2) and let E be the set defined by

$$E = \{x \in M \mid \phi_t(x) \rightarrow x^* \text{ as } t \rightarrow +\infty\}.$$

Lemma 2.4: If $m_2 > 0$, the set of all trajectories of system (2) converging to the unstable equilibrium point x^* have zero measure.

Proof: Let W^{cs} be a local center-stable manifold. Denote by d the distance on M . If $\delta > 0$ is chosen

small enough, then $N = \varphi^{-1}(W^{cs}) \cap \varphi^{-1}(\partial B(0, \delta))$ is a codimension m_2+1 smooth submanifold of $\varphi^{-1}(\partial B(0, \delta))$ diffeomorphic to the sphere S^{m_1-1} . Set

$$N_q^R = \{x \in N \mid \forall t \in [q, 0] \quad d(\phi_t(x), x^*) < R\}.$$

Notice that, because $N_q^R \subset N$, N_q^R has zero measure. We claim that

$$E \subset (\varphi^{-1}(W^{cs}) \cap \varphi^{-1}(B(0, \delta))) \cup \bigcup_Q \psi_q^R([q, 0] \times N_q^R),$$

where $Q = \{(q, R) \in \mathbb{Q} \times \mathbb{N} : q < 0, N_q^R \neq \emptyset\}$ and, for a nonempty N_q^R , $\psi_q^R : [q, 0] \times N_q^R \rightarrow M$ is the mapping given by $\psi_q^R(t, x) = \phi_t(x)$. We now prove the claim. Let $a \in E$. If $a \in \varphi^{-1}(B(0, \delta))$, it follows from Lemmas 2.1 and 2.2 that $a \in \varphi^{-1}(W^{cs})$. On the other hand, if $a \notin \varphi^{-1}(B(0, \delta))$, then there exists $t > 0$ such that $x_0 = \phi_t(a) \in \varphi^{-1}(\partial B(0, \delta))$. It follows from the existence theorem for flows that there exist $\eta, \varepsilon > 0$ and $q \in \mathbb{Q}$ such that ϕ is well defined on $[-\varepsilon - t, 0] \times \varphi^{-1}(B(x_0, \eta))$. Choose $q \in \mathbb{Q} \cap [-\varepsilon - t, -t]$ and let $R \in \mathbb{N}$ be such that $\|\phi_t(x)\| < R$ for all $t \in [q, 0]$ and all $x \in \varphi^{-1}(B(x_0, \eta))$. By construction of q and R , $a \in \psi_q^R([q, 0] \times N_q^R)$ and the claim follows.

Since $[q, 0] \times N_q^R$ has zero measure and since ψ_q^R is Lipschitz, it follows that $\text{meas}(\psi_q^R([q, 0] \times N_q^R)) = 0$. The lemma follows since E is included in the union of countably many such sets and $\varphi^{-1}(W^{cs}) \cap \varphi^{-1}(B(0, \delta)) \in \mathbb{R}^m$ whose measure is also zero as being a submanifold of positive codimension. ■

Remark 2.5: Lemma 2.4 is a general result. As soon as an equilibrium point has an unstable manifold, the set of all trajectories converging to the equilibrium point has zero measure.

III. ANALYSIS OF THE EXAMPLE

A. Stabilizing state feedback and observer design

1) *Feedback stabilization of system (1):* Consider the feedback control function $\lambda^s : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ given by

$$\lambda^s(x) = r_1(x_1, x_2), \quad (6)$$

where r_1 is an arbitrary positive constant.

Proposition 3.1: The target point x_t is an asymptotically stable equilibrium for the closed-loop system resulting from applying the feedback control $u = \lambda^s(x)$ (given by equation (6)) to system (1). Moreover, its basin of attraction is $S^2 \setminus \{-x_t\}$.

Proof: Let the mapping $V : S^2 \rightarrow \mathbb{R}$ given by

$$V(x) = x_3,$$

be the candidate Lyapunov function and

$$E = \{x \in S^2 \mid V(x) < 1\} = S^2 \setminus \{-x_t\}.$$

One has

$$\dot{V}(x) = \dot{x}_3 = -r_1(x_1^2 + x_2^2),$$

which is ≤ 0 on E . Let I be the largest invariant set contained in

$$\{x \in S^2 \mid \dot{V}(x) = 0\}.$$

We have $I = \{-x_t, x_t\}$, where x_t is the target point. Since $I \cap E = \{x_t\}$ and since $E \subset S^2$ is bounded, according to LaSalle's principle (see e.g. [12, Corollary 2]), we get the desired result. ■

2) *The observer:* The chosen Luenberger-like observer for system (1) is

$$\dot{\hat{x}} = A(u)\hat{x} - r_2 C'(C\hat{x} - y), \quad \hat{x} \in \mathbb{R}^3, \quad (7)$$

where r_2 is a given positive constant. Hence, the observation error defined by $\varepsilon = \hat{x} - x$ satisfies

$$\dot{\varepsilon} = (A(u) - r_2 C' C) \varepsilon, \quad \varepsilon \in \mathbb{R}^3. \quad (8)$$

Remark 3.2: The constants r_1 and r_2 could be used as tuning parameters. However, in order to simplify the computations, from now these constants are set to one.

The Gram observability matrix relative to an input $u : [0, T] \rightarrow \mathbb{R}^2$ is defined by

$$G(u, T) = \int_0^T \Phi_u(t)' C' C \Phi_u(t) dt,$$

where $\Phi_u(t)$ is the resolvent matrix solution of equation (8), i.e.

$$\begin{aligned} \dot{\Phi}_u(t) &= (A(u(t)) - C C') \Phi_u(t) \\ \Phi_u(0) &= \text{Id}, \end{aligned}$$

and we denote by $\text{ind}(u, T)$ the observability index of u , that is, the smallest eigenvalue of $G(u, T)$. Clearly, $\text{ind}(u, T) \geq 0$. For $u \in L^\infty(\mathbb{R}_+, \mathbb{R}^2)$, we denote by $u_{[\theta]}$ the shifted input defined by $u_{[\theta]}(t) = u(\theta + t)$. As in [3], we call an input $u \in L^\infty(\mathbb{R}_+, \mathbb{R}^2)$ *persistent*, if there exists a time interval $[0, T]$, $T > 0$ such that

$$\limsup_{\theta \rightarrow +\infty} \text{ind}(u_{[\theta]}, T) > 0.$$

According to [3, Theorem 3 page 300], we have the following theorem.

Theorem 3.3: If $u \in L^\infty(\mathbb{R}_+, \mathbb{R}^2)$ is a persistent input, then the observation error tends to zero, i.e.

$$\lim_{t \rightarrow +\infty} \varepsilon(t) = 0.$$

3) *The coupled system controller-observer:* The equations of the controller-observer system are, as usual

$$\begin{cases} \dot{\hat{x}} = A(\lambda^s(\hat{x})) \hat{x} - C' C \varepsilon \\ \dot{\varepsilon} = (A(\lambda^s(\hat{x})) - C' C) \varepsilon, \quad (\hat{x}, \varepsilon) \in M, \end{cases} \quad (9)$$

where λ^s is the stabilizing feedback (6) of Section III-A.1 and, since $x \in S^2$, $M = \{(\hat{x}, \varepsilon) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \|\hat{x} - \varepsilon\| = 1\}$. It is clear that system (9) is not a stabilizing device since all the domains

$$\mathcal{D}_{a,b} = \{(\hat{x}, \varepsilon) \in M \mid \hat{x} = (0, 0, a), \varepsilon \in C_b\}, \quad (10)$$

where C_b is any circle in \mathbb{R}^3 of the form $\{\varepsilon \in \mathbb{R}^3 \mid \varepsilon_1^2 + \varepsilon_2^2 = b^2, \varepsilon_3 = 0\}$ are invariant sets for the coupled system (9). Indeed, all the inputs that make system (1) unobservable are those for which $y = Cx$ is constant, i.e. such that

$$-u_1(t)x_1(t) - u_2(t)x_2(t) = 0, \quad \forall t \in \mathbb{R}_+.$$

In particular, the null input $u(\cdot) = 0$ makes system (1) unobservable. A natural and simple idea is then to choose a small perturbation of the feedback control that makes it observable.

B. Practical asymptotic stabilization

In order to make the input $\lambda^s(\hat{x}(\cdot))$ observable, a first simple choice is to add a (small) nonzero constant δ to the feedback, i.e. we consider

$$\lambda_\delta^s(\hat{x}) = \lambda^s(\hat{x}) + (\delta, \delta). \quad (11)$$

We shall prove the following lemma which is crucial for our purpose.

Lemma 3.4: All the inputs of the form (11) applied to the full coupled system (9) make system (1) observable on any time interval $[0, T]$, $T > 0$.

Proof: Suppose, contrary to our claim, that there exist a positive T and an input $t \mapsto u(t) = \lambda_\delta^s(\hat{x}(t))$ (with λ_δ^s given by equation (11)) that renders system (1) unobservable on $[0, T]$. Such an input is solution of system (9). This means that there exists a non-identically vanishing function $t \mapsto \omega(t) \in \mathbb{R}^3$ such that, for all $t \in [0, T]$, we have

$$\begin{aligned} \dot{\omega}(t) &= A(u(t))\omega(t) = A(\lambda_\delta^s(\hat{x}(t)))\omega(t), \\ y(t) &= C\omega(t) = \omega_3(t) = 0. \end{aligned} \quad (12)$$

Then, differentiating with respect to time equation (12), we get that

$$\begin{aligned} 0 &= \dot{\omega}_3 \\ &= -u_1\omega_1 - u_2\omega_2 \\ &= -(\hat{x}_1 + \delta)\omega_1 - (\hat{x}_2 + \delta)\omega_2. \end{aligned} \quad (13)$$

Differentiating once again with respect to time yields

$$\begin{aligned} 0 &= \dot{\hat{x}}_1\omega_1 + \hat{x}_1\dot{\omega}_1 + \dot{\hat{x}}_2\omega_2 + \hat{x}_2\dot{\omega}_2 + \delta(\dot{\omega}_1 + \dot{\omega}_2) \\ &= (e\hat{x}_2 + u_1\hat{x}_3)\omega_1 + \hat{x}_1e\omega_2 \\ &\quad + (-e\hat{x}_1 + u_2\hat{x}_3)\omega_2 - \hat{x}_2e\omega_1 + \delta(e\omega_2 - e\omega_1) \\ &= \delta e(\omega_2 - \omega_1). \end{aligned} \quad (\text{According to (13)})$$

Hence, we have

$$\omega_2 - \omega_1 = 0, \quad (14)$$

which implies that $0 = \dot{\omega}_2 - \dot{\omega}_1 = -e(\omega_1 + \omega_2)$, i.e.

$$\omega_1 + \omega_2 = 0. \quad (15)$$

Therefore, according to equations (12), (14) and (15), $t \mapsto \omega(t)$ vanishes identically on $[0, T]$ which is a contradiction. Consequently, system (1) is observable for the chosen input $t \mapsto \lambda_\delta^s(\hat{x}(t))$ on any time interval. ■

With this lemma in hand, the proof of our stabilization result will follow after three steps.

Remark 3.5: The proof of Lemma 3.4 depends on the chosen perturbation of the stabilizing control, and in a general context, the chosen kind of perturbation is specific to each system. For example, if $A(u)$ is replaced

by

$$\begin{pmatrix} 0 & e\varphi(u) & u_1 \\ -e\varphi(u) & 0 & u_2 \\ -u_1 & -u_2 & 0 \end{pmatrix}, \quad \varphi: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad \varphi(0) = 0, \quad (16)$$

in system (1), a constant perturbation does not work while a perturbation $t \mapsto \delta(t)$ that satisfies $\dot{\delta} = -\delta$ does.

1) *Step 1: proof that the estimation error goes to zero:* Using the perturbed feedback as an input, the fully coupled system (9) is rewritten as

$$\begin{cases} \dot{\hat{x}} = A(\lambda_\delta^s(\hat{x}))\hat{x} - r_2C'C\varepsilon \\ \dot{\varepsilon} = (A(\lambda_\delta^s(\hat{x})) - r_2C'C)\varepsilon \end{cases} \quad (\hat{x}, \varepsilon) \in M. \quad (17)$$

Let $T > 0$ be fixed. It follows straightforwardly from the continuous dependence on initial data of the solutions of an ordinary differential equation, that the map $(\hat{x}_0, \varepsilon_0) \mapsto \lambda_\delta^s(\hat{x}(\cdot))$ from M to $C^0([0, T], \mathbb{R}^2)$ is continuous. Then, so is the map $\lambda_\delta^s(\hat{x}(\cdot)) \mapsto \text{ind}(\lambda_\delta^s(\hat{x}(\cdot)), T)$. Consequently, the mapping $F: M \rightarrow \mathbb{R}_+$ given by $F(\hat{x}_0, \varepsilon_0) = \text{ind}(\lambda_\delta^s(\hat{x}(\cdot)), T)$ is continuous.

Lemma 3.6: All the inputs $\lambda_\delta^s(\hat{x}(\cdot))$ are persistent.

Proof: According to (8), we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\varepsilon\|^2 &= \varepsilon' \dot{\varepsilon} \\ &= \varepsilon' (A(\lambda_\delta^s(\hat{x})) - r_2C'C)\varepsilon \\ &= -r_2(C\varepsilon)^2, \end{aligned} \quad (\text{since } A(\cdot)' = -A(\cdot))$$

which shows that $t \mapsto \varepsilon(t)$ is bounded if $t \geq 0$. Consequently,

$$\begin{aligned} \|\hat{x}(t)\| &\leq \|\hat{x}(t) - x(t)\| + \|x(t)\| \\ &= \|\varepsilon(t)\| + 1. \end{aligned} \quad (\text{since } x(t) \in S^2)$$

Therefore, $t \mapsto \hat{x}(t)$ is also bounded if $t \geq 0$. It follows that the solution of system (17) are defined for all $t \geq 0$ and that $\{(\hat{x}(t), \varepsilon(t)) \mid t \geq 0\} \subset K$, where $K = \bar{B}(0, \|\varepsilon_0\| + 1) \times \bar{B}(0, \|\varepsilon_0\|)$. In particular,

$$\{(\hat{x}(t), \varepsilon(t)) \mid t \geq \theta\} \subset K, \quad \forall \theta \geq 0.$$

The function F defined above being nonnegative, we infer that $\inf_K F \leq \inf\{F(\hat{x}(t), \varepsilon(t)) \mid t \geq \theta\}$, for all $\theta \geq 0$. It follows that for every fixed positive T ,

$$\begin{aligned} \inf_K F &\leq \limsup_{\theta \rightarrow +\infty} F(\hat{x}(\theta), \varepsilon(\theta)) \\ &= \limsup_{\theta \rightarrow +\infty} \text{ind}(\lambda_\delta^s(\hat{x}(\cdot + \theta)), T). \end{aligned}$$

By continuity, the infimum of F over K is reached, and is positive by the crucial Lemma 3.4. ■

We have the following corollary whose proof is an immediate consequence of Theorem 3.3

Corollary 3.7: For any trajectory of the coupled system (17), we have $\lim_{t \rightarrow +\infty} \varepsilon(t) = 0$.

2) *Step 2: Asymptotic stability of an equilibrium point close to x_t :* In the present subsection, we prove that system (17) admits an asymptotically stable equilibrium point and an unstable equilibrium point that are respectively close to $(x_t, 0)$ and $(-x_t, 0)$.

Lemma 3.8: For δ small enough, system (17) admits a locally asymptotically stable equilibrium point $(x_\delta^s, 0)$ arbitrarily close to $(x_t, 0)$.

Proof: Let (x^*, ε^*) be an equilibrium point of system (17). Then (x^*, ε^*) satisfies

$$\begin{cases} A(\lambda_\delta^s(x^*))x^* - r_2 C' C \varepsilon^* = 0 \\ (A(\lambda_\delta^s(x^*)) - r_2 C' C)\varepsilon^* = 0. \end{cases}$$

A straightforward resolution of the previous system leads to three solutions : the zero solution (which is not admissible since $\|x\| = \|\hat{x} - \varepsilon\| = 1$) and two solutions arbitrarily close to $(x_t, 0)$ and $(-x_t, 0)$ respectively (as soon as δ is chosen small enough). In particular, the solution close to $(x_t, 0)$ is $(x_\delta^s, 0) = (x_1^s, x_2^s, x_3^s, 0, 0, 0)$ where

$$\begin{aligned} x_1^s &= -\frac{\delta((x_3^s)^2 - x_3^s)}{(x_3^s)^2 + 1} \\ x_2^s &= -\frac{\delta((x_3^s)^2 + x_3^s)}{(x_3^s)^2 + 1} \\ x_3^s &= -\sqrt{-\delta^2 + \sqrt{\delta^4 + 1}}, \end{aligned}$$

Let us write ε_3 as a function of $\varepsilon_1, \varepsilon_2$ and \hat{x} . Since $x \in S^2$, we have

$$\|x\| = \|\hat{x} - \varepsilon\| = 1 \quad (18)$$

which is a polynomial equation of degree two in ε_3 . The resolution of equation (18) yields

$$\varepsilon_3 = \hat{x}_3 \pm \sqrt{1 - (\hat{x}_1 - \varepsilon_1)^2 - (\hat{x}_2 - \varepsilon_2)^2}. \quad (19)$$

In the following, we are going to study the stability of $(x_\delta^s, 0)$ by linearization. Hence, we assume that (\hat{x}, ε) is close to $(x_\delta^s, 0)$ which is close to $(x_t, 0)$. Consequently, in equation (19) we are only interested by the solution having the plus sign in its expression.

Now, substituting the expression given by (19) in system (17) yields a new system of the form

$$(\dot{\hat{x}}, \dot{\bar{\varepsilon}})' = F_\delta(\hat{x}, \bar{\varepsilon}), \quad F_\delta : \mathbb{R}^5 \rightarrow \mathbb{R}^5, \quad (20)$$

where $\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2)$. The linearization of system (20) around $(x_\delta^s, 0)$ is

$$(\dot{\bar{x}}, \dot{\bar{\varepsilon}})' = DF_\delta(x_\delta^s, 0)[(\bar{x}, \bar{\varepsilon})], \quad (21)$$

where $\bar{x} = \hat{x} - x_\delta^s$. A standard analysis shows that, for δ small enough, the eigenvalues of $DF_\delta(x_\delta^s, 0)$ have negative real parts. Consequently, $(x_\delta^s, 0)$ is a locally asymptotically stable equilibrium point of (17). ■

Lemma 3.9: For δ small enough, system (17) admits an unstable equilibrium point $(x_\delta^u, 0)$ arbitrarily close to $(-x_t, 0)$ and the set of all trajectories converging to $(x_\delta^u, 0)$ has zero measure.

Proof: Similar computations as in the proof of Lemma 3.8 yields that system (17) can be rewritten under the form

$$(\dot{\hat{x}}, \dot{\bar{\varepsilon}})' = F_\delta(\hat{x}, \bar{\varepsilon}), \quad F_\delta : \mathbb{R}^5 \rightarrow \mathbb{R}^5, \quad (22)$$

where $\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2)$. The linearization of (22) around the

equilibrium point $(x_\delta^u, 0)$ arbitrarily close to $(-x_t, 0)$ is

$$(\dot{\bar{x}}, \dot{\bar{\varepsilon}})' = DF_\delta(x_\delta^u, 0)[(\bar{x}, \bar{\varepsilon})], \quad (23)$$

where $\bar{x} = \hat{x} - x_\delta^u$. A standard analysis shows that, for δ small enough, $DF_\delta(x_\delta^u, 0)$ has two eigenvalues with positive real parts and three with negative real parts. The result follows from Lemma 2.4. ■

3) Step 3 : The main result: We are now ready to state and prove our main result.

Theorem 3.10: The point $(x_\delta^s, 0) \in M$ is asymptotically stable for system (17), and its region of attraction is an open, dense and of full measure subset of M .

Proof: From Corollary 3.7 we know that the error converges to zero. The ω -limit points of system (17) are then of the form $(\hat{x}, 0)$, with $\hat{x} \in S^2$. In this case, a trajectory in the ω -limit set satisfies the differential equation

$$\begin{cases} \dot{\hat{x}} = A(\lambda_\delta^s(\hat{x}))\hat{x} \\ \dot{\varepsilon} = 0 \\ \dot{\delta} = 0. \end{cases} \quad (24)$$

Set $C^- = \{x \in S^2 \mid x_3 \leq 0\}$ and consider the function $L : C^- \rightarrow \mathbb{R}_+$ defined by

$$L(\hat{x}) = \frac{1}{2} \|\hat{x} - x_\delta^s\|^2.$$

Along the trajectories of system (24), we have (since $A(\cdot)$ is skew-symmetric)

$$\dot{L}(\hat{x}) = -(x_\delta^s)' A(\lambda_\delta^s(\hat{x})) \hat{x}.$$

The analysis of the maximum value of \dot{L} on C^- indicates that for all $\hat{x} \in C^-$, $\dot{L}(\hat{x}) \leq 0$ and $\dot{L}(\hat{x}) = 0$ if and only if $\hat{x} = x_\delta^s$. From LaSalle's invariant principle, we get that x_δ^s is asymptotically stable for system (24) and its region of attraction contains C^- .

A similar reasoning with C^- replaced by $C^+ = \{x \in S^2 \mid x_3 \geq 0\}$, $\dot{L} \leq 0$ replaced by $\dot{L} \geq 0$, and x_δ^s replaced by x_δ^u proves that any trajectory of (24) starting in $C^+ \setminus \{x_\delta^u\}$ will enter C^- , i.e. the basin of attraction of x_δ^s .

It follows that the ω -limit set of any trajectory of system (17) is contained in the set $\{(x_\delta^s, 0), (x_\delta^u, 0)\}$. However, from Lemma 3.8, the point $(x_\delta^s, 0)$ is locally asymptotically stable. It follows that any trajectory goes either to $(x_\delta^s, 0)$ or to $(x_\delta^u, 0)$. Since, from Lemma 3.9, the set of trajectories converging to $(x_\delta^u, 0)$ has zero measure, the Lemma follows. ■

Theorem 3.10 immediately yields the following Corollary.

Corollary 3.11: For every neighborhood \mathcal{N} of $(x_t, 0)$, there exists $\delta_{\max} > 0$ such that for all $\delta \in (0, \delta_{\max})$, \mathcal{N} contains an asymptotically stable attractor \mathcal{A} (a point in our case) for system (17). Moreover, the region of attraction of \mathcal{A} is an open, dense and of full measure subset of M .

Remark 3.12: In more general situations (for example, system (16) in Remark 3.5), Theorem 3.10 might not be true, but Corollary 3.11 remains valid. Note that in such a situation, the perturbation may depend on time but

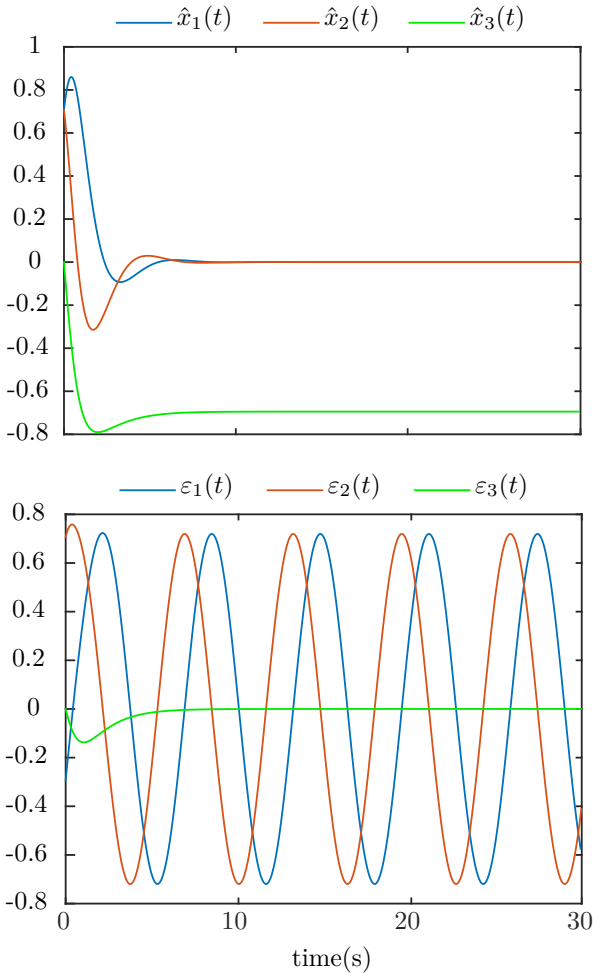


Fig. 1. State estimations and observation errors of system (9).

system (24) has to remain autonomous in order to apply LaSalle's Principle¹.

4) *Numerical simulations:* The theoretical results presented in the previous section have been tested by means of a numerical simulation. The case of the quantum system (1) has been treated and the observer was constructed using equation (7). The system has been simulated with e , r_1 and r_2 set to 1. The initial conditions were arbitrarily set to $x_0 = (1, 0, 0)$ and $\hat{x}_0 = (1/\sqrt{2}, 1/\sqrt{2}, 0)^2$.

Fig. 1 shows the state variables of system (9) which is clearly not a stabilizing device because, we see that the error does not converge to zero but rather to a circle in \mathbb{R}^3 , and the estimation converges to a point $(0, 0, a)$, i.e. (\hat{x}, ε) lies indeed in a domain $\mathcal{D}_{a,b}$ (defined by (10)).

We now take a feedback control in order to make the system observable, that is, we choose a perturbed feedback control of the form (11). The simulations have been made for two different values of the perturbation

¹This difficulty can be overcome by considering a perturbation solution of a finite dimensional ordinary differential equation.

²The choice of $\hat{x}_0 \in S^2$ is legitimated by the fact that the state of system (1) lies in S^2 .

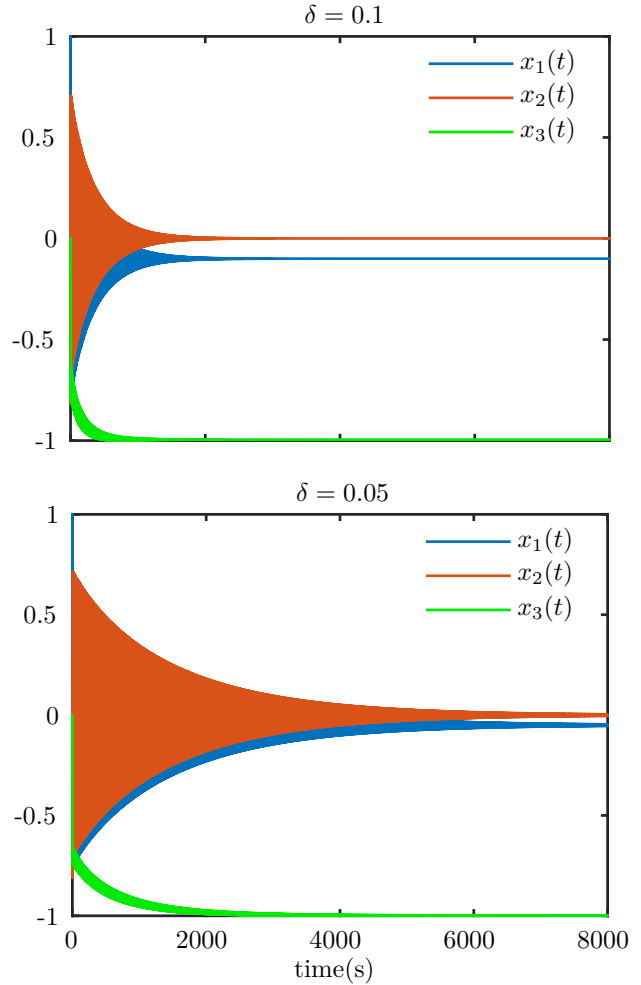


Fig. 2. State variables of system (1) with input $u = s(\hat{x})$ and for $\delta = 0.1$ and $\delta = 0.05$.

of the stabilizing feedback control: $\delta = 0.1$ and $\delta = 0.05$. Fig. 2 shows, for both cases, the evolution of the state variables, and Fig. 3 shows the evolution of the observation errors. Notice that the thick aspect of the drawn curves is due to the observation of an oscillating system over a very long time.

In both cases, the simulations show that the observation errors converge to zero, which corroborates Corollary 3.7 as well as the idea that perturbing the stabilizing feedback makes the system observable. It is also clear that the state converges to a point close to x_t and that this point is closer to x_t for a smaller perturbation δ . This is exactly the statement of Theorem 3.10 which asserts that it is possible to asymptotically stabilize system (1) to a point arbitrarily close to the target point x_t .

However, notice that choosing a smaller perturbation leads the system to converge closer but slower to the target. The convergence can therefore be very slow.

C. Exact asymptotic stabilization

In order to stabilize (9) to the target point $(x_t, 0)$, we add to the feedback a (small) perturbation that decreases

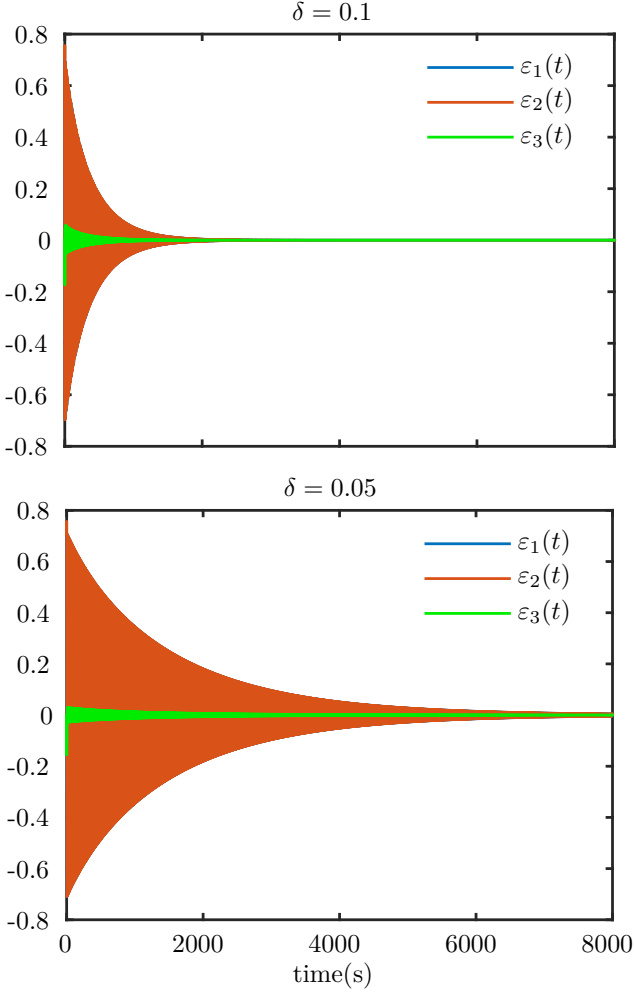


Fig. 3. Observation errors of system (17) for $\delta = 0.1$ and $\delta = 0.05$.

as \hat{x} goes to x_t , i.e. we consider

$$\lambda_{\delta,\alpha}^s(\hat{x}) = \lambda^s(\hat{x}) + (\delta, \delta)((\hat{x}_3)^\alpha - 1), \quad (25)$$

where δ is a positive constant and α is a positive even number. We shall prove the following lemma which is crucial for our purpose.

Lemma 3.13: All the inputs of the form (25) applied to the full coupled system (9) make system (1) observable on any time interval $[0, T]$, $T > 0$.

Proof: Suppose, contrarily to our claim, that there exist a positive T and an input $t \mapsto u(t) = \lambda_{\delta,\alpha}^s(\hat{x}(t))$ (with $\lambda_{\delta,\alpha}^s$ given by equation (25)) that renders system (1) unobservable on $[0, T]$. Such an input is solution of system (9). This means that there exists a non-identically vanishing function $t \mapsto \omega(t) \in \mathbb{R}^3$ such that, for all $t \in [0, T]$, we have

$$\begin{aligned} \dot{\omega}(t) &= A(u(t))\omega(t) = A(\lambda_{\delta,\alpha}^s(\hat{x}(t)))\omega(t), \\ y(t) &= C\omega(t) = \omega_3(t) = 0. \end{aligned} \quad (26)$$

Differentiate equation (26) with respect to time to obtain

$$\begin{aligned} 0 &= \dot{\omega}_3 \\ &= -u_1\omega_1 - u_2\omega_2 \end{aligned} \quad (27)$$

$$\begin{aligned} &= -(\hat{x}_1 + \delta((\hat{x}_3)^\alpha - 1))\omega_1 - (\hat{x}_2 + \delta((\hat{x}_3)^\alpha - 1))\omega_2 \\ &= -\hat{x}_1\omega_1 - \hat{x}_2\omega_2 - \delta((\hat{x}_3)^\alpha - 1)(\omega_1 + \omega_2). \end{aligned} \quad (28)$$

Differentiating once again yields

$$\begin{aligned} 0 &= \dot{\hat{x}}_1\omega_1 + \hat{x}_1\dot{\omega}_1 + \dot{\hat{x}}_2\omega_2 + \hat{x}_2\dot{\omega}_2 \\ &\quad + \delta\alpha(\hat{x}_3)^{\alpha-1}\dot{\hat{x}}_3(\omega_1 + \omega_2) \\ &\quad + \delta((\hat{x}_3)^\alpha - 1)(\dot{\omega}_1 + \dot{\omega}_2) \\ &= (\hat{x}_2 + u_1\hat{x}_3)\omega_1 + \hat{x}_1\omega_2 + (-\hat{x}_1 + u_2\hat{x}_3)\omega_2 - \hat{x}_3\omega_1 \\ &\quad + \delta\alpha(\hat{x}_3)^{\alpha-1}(-u_1\hat{x}_3 - u_2\hat{x}_2 - \varepsilon_3)(\omega_1 + \omega_2) \\ &\quad + \delta((\hat{x}_3)^\alpha - 1)(\omega_2 - \omega_1) \\ 3 &= \delta\alpha(\hat{x}_3)^{\alpha-1}(-\hat{x}_1^2 - \hat{x}_2^2 - \varepsilon_3 - (\hat{x}_1 + \hat{x}_2)\delta((\hat{x}_3)^\alpha - 1)) \\ &\quad \times (\omega_1 + \omega_2) + \delta((\hat{x}_3)^\alpha - 1)(\omega_2 - \omega_1) \end{aligned} \quad (29)$$

Equations (28) and (29) are rewritten as $B\bar{\omega} = 0$, where $\bar{\omega} = (\omega_1, \omega_2)$ and $\mathbb{R}^{2 \times 2} \ni B$ depends on ε and \hat{x} . Solving $\det(B) = 0$ being cumbersome, it is not detailed here. However, one gets that $\det(B) = 0$ if and only if $(\varepsilon, \hat{x}) = (0, x_t)$ or $(\varepsilon, \hat{x}) = (0, -x_t)$. This means that provided that $(\varepsilon, \hat{x}) \neq (0, \pm x_t)$, $\omega(\cdot)$ vanishes identically on $[0, T]$ which is a contradiction.

We conclude that all the inputs of the form $\lambda_{\delta,\alpha}^s(\hat{x}(\cdot))$ make system (1) observable provided that $(\varepsilon, \hat{x}) \neq (0, \pm x_t)$. If $(\varepsilon, \hat{x}) = (0, \pm x_t)$, $x_3 = \pm 1$ and since $\|x\| = 1$ we get $x = \pm x_t$. ■

With this lemma in hand, the proof of our stabilization result will follow after three steps.

Remark 3.14: Although the computations are more complicated, one can choose any $\alpha > 1$ and consider $|\hat{x}_3|$ instead of \hat{x}_3 in (25).

1) *Step 1: proof that the estimation error goes to zero:* Using the perturbed feedback as an input, the fully coupled system (9) is rewritten as

$$\begin{cases} \dot{\hat{x}} = A(\lambda_{\delta,\alpha}^s(\hat{x}))\hat{x} - C'C\varepsilon \\ \dot{\varepsilon} = (A(\lambda_{\delta,\alpha}^s(\hat{x})) - C'C)\varepsilon, \quad (\hat{x}, \varepsilon) \in M. \end{cases} \quad (30)$$

Fix $T > 0$. The same reasoning as in Subsection III-B.1 yields that the mapping $F : M \rightarrow \mathbb{R}_+$ given by $F(\hat{x}_0, \varepsilon_0) = \text{ind}(\lambda_{\delta,\alpha}^s(\hat{x}(\cdot)), T)$ is continuous.

Lemma 3.15: For any trajectory of the coupled system (30), $\lim_{t \rightarrow +\infty} \varepsilon(t) = 0$.

Proof: In much the same way as in the proof of Lemma 3.6, it follows that

$$\{(\hat{x}(t), \varepsilon(t)) \mid t \geq 0\} \subset \bar{B}(0, \|\varepsilon_0\| + 1) \times \bar{B}(0, \|\varepsilon_0\|).$$

Define the compact set

$$K_n = \bar{B}(0, \|\varepsilon_0\| + 1) \times (\bar{B}(0, \|\varepsilon_0\|) \setminus B(0, \|\varepsilon_0\|/2^n))$$

and suppose that there exists $n \in \mathbb{N}$ such that

$$\{(\hat{x}(t), \varepsilon(t)) \mid t \geq 0\} \subset K_n.$$

³According to (27)

In particular,

$$\{(\hat{x}(t), \varepsilon(t)) \mid t \geq \theta\} \subset K_n, \quad \forall \theta \geq 0.$$

We infer that $\inf_{K_n} F \leq \inf\{F(\hat{x}(t), \varepsilon(t)) \mid t \geq \theta\}$, for all $\theta \geq 0$. It follows that for every fixed positive T ,

$$\begin{aligned} \inf_{K_n} F &\leq \limsup_{\theta \rightarrow +\infty} F(\hat{x}(\theta), \varepsilon(\theta)) \\ &= \limsup_{\theta \rightarrow +\infty} \text{ind}(\lambda_{\delta, \alpha}^s(\hat{x}(\cdot + \theta)), T). \end{aligned}$$

By continuity, the minimum of F over K_n is reached, and is positive by the crucial Lemma 3.13. This means that $\lambda_{\delta, \alpha}^s(\hat{x}(\cdot))$ is a persistent input and by Theorem 3.3 that the error goes to zero, which contradicts the hypothesis that the trajectory stays in K_n . Consequently, there exists a strictly increasing sequence $(t_n)_{n \in \mathbb{N}}$ such that for all $t \geq t_n$, $\varepsilon(t) \in B(\|\varepsilon_0\|/2^n)$. Since $\bigcap_{n \in \mathbb{N}} B(\|\varepsilon_0\|/2^n) = \{0\}$, we conclude that $\lim_{t \rightarrow +\infty} \varepsilon(t) = 0$ ■

Remark 3.16: In Subsection III-B, the input $\lambda_{\delta}^s(\hat{x}(\cdot))$ is persistent and the convergence of the error towards zero follows immediately from Theorem 3.3. In the present subsection, the input $\lambda_{\delta, \alpha}^s(\hat{x}(\cdot))$ is not persistent and yet the error still goes to zero.

2) *Step 2 : Local stability of $(x_t, 0)$ and instability of $(-x_t, 0)$:* In the present subsection, we prove that $(x_t, 0)$ is a locally asymptotically stable equilibrium point for system (30). We also prove that $(-x_t, 0)$ is an unstable equilibrium point for system (30) and that the set of all trajectories converging to $(-x_t, 0)$ has zero measure.

Lemma 3.17: The point $(x_t, 0)$ is locally asymptotically stable for system (30).

Proof: Since $x \in M$, we have $\|x\| = \|\hat{x} - \varepsilon\| = 1$, which leads for \hat{x} close (enough) to $(x_t, 0)$,

$$\hat{x}_3 = \varepsilon_3 - \sqrt{1 - (\hat{x}_1 - \varepsilon_1)^2 - (\hat{x}_2 - \varepsilon_2)^2}.$$

Substituting \hat{x}_3 in (30) we obtain the system in \mathbb{R}^5 :

$$\begin{cases} \dot{Z}_1 = FZ_1 + f(Z_1, Z_2) \\ \dot{Z}_2 = GZ_2 + g(Z_1, Z_2) \end{cases}, \quad (31)$$

where $Z_1 = (\hat{x}_1, \hat{x}_2, \varepsilon_3)$, $Z_2 = (\varepsilon_1, \varepsilon_2)$,

$$F = \begin{pmatrix} -1 & 1 & \alpha\delta \\ -1 & -1 & \alpha\delta \\ 0 & 0 & -1 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and $f : \mathbb{R}^5 \rightarrow \mathbb{R}^3$ and $g : \mathbb{R}^5 \rightarrow \mathbb{R}^2$ are C^∞ functions such that $f(0) = 0$, $Df(0) = 0$, $g(0) = 0$ and $Dg(0) = 0$. Since the eigenvalues of G have zero real parts we cannot conclude directly about the local stability of the origin. However, the center manifold theory (see [13]) yields our result. After simplifications, one gets that the dynamic on the center manifold is given by

$$\begin{cases} \dot{\varepsilon}_1 = \varepsilon_2 + o(\|(\varepsilon_1, \varepsilon_2)\|^5) \\ \dot{\varepsilon}_2 = -\varepsilon_1 - 4\beta(\varepsilon_1 + \varepsilon_2)(\varepsilon_1^2 + \varepsilon_2^2) + o(\|(\varepsilon_1, \varepsilon_2)\|^5) \end{cases}, \quad (32)$$

with $\beta = \alpha^2\delta^2/32$. Applying the change of variables

$$\begin{cases} z_1 = \varepsilon_1 - \varepsilon_2 \\ z_2 = \varepsilon_1 + \varepsilon_2 \end{cases},$$

one gets

$$\begin{cases} \dot{z}_1 = z_2 + \beta z_2 \|z\|^4 + o(\|z\|^5) \\ \dot{z}_2 = -z_1 - \beta z_2 \|z\|^4 + o(\|z\|^5) \end{cases}, \quad (33)$$

with $z = (z_1, z_2)$. Let the mapping $W : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$W(z) = z_1^2 + \frac{\beta}{2} z_1 z_2 \|z\|^4 + (1 + \beta \|z\|^4) z_2^2$$

be the candidate Lyapunov function. One has

$$\dot{W}(z) = -\frac{\beta}{2} \|z\|^4 (z_1^2 + 3z_2^2) + o(\|z\|^6),$$

which is negative definite on a neighborhood of the origin. From LaSalle's principle, we conclude that the origin is a locally asymptotically stable equilibrium point for system (32). The result then follows using the center manifold theorem [13, Theorem 2]. ■

Lemma 3.18: The set of all trajectories of system (30) converging to $(x_t, 0)$ has zero measure.

Proof: A similar computation as in the proof of Lemma 3.17 yields that in a neighborhood of the point $(x_t, 0)$ system (30) can be written

$$\begin{cases} \dot{Z}_1 = FZ_1 + f(Z_1, Z_2) \\ \dot{Z}_2 = GZ_2 + g(Z_1, Z_2) \end{cases}, \quad (34)$$

where F has an eigenvalue with negative real part and two eigenvalues with zero real parts, G has eigenvalues with positive real parts, Z_1 and Z_2 are vectors in \mathbb{R}^3 and \mathbb{R}^2 respectively and f and g are smooth maps in a neighborhood \mathcal{N} of $0 \in \mathbb{R}^5$ such that $f(0, 0) = 0$, $g(0, 0) = 0$, $Df(0, 0) = 0$ and $Dg(0, 0) = 0$. Since the matrix G has eigenvalues with positive real parts, according to Lemma 2.4, the result follows. ■

3) *Step 3 : The main result:* We are now ready to state and prove our main result.

Theorem 3.19: Provided that $\delta \in (-\sqrt{2}/\alpha, \sqrt{2}/\alpha)$, the point $(x_t, 0) \in M$ is asymptotically stable for system (30) and its region of attraction is an open, dense and of full measure subset of M .

Proof: According to Lemma 3.15 the error converges to zero. Hence, the ω -limit points of any trajectory of system (30) are of the form $(\hat{x}, 0)$, with $\hat{x} \in S^2$. In this case, a trajectory in the ω -limit set satisfies the differential system

$$\begin{cases} \dot{\hat{x}} = A(\lambda_{\delta, \alpha}^s(\hat{x})) \hat{x} \\ \dot{\varepsilon} = 0 \\ \dot{\delta} = 0. \end{cases} \quad (35)$$

Let the mapping $V : S^2 \rightarrow \mathbb{R}$ defined in Section III-A.1 be a candidate Lyapunov function for system (35). One has

$$\dot{V}(\hat{x}) = \dot{\hat{x}}_3 = -(\hat{x}_1^2 + \hat{x}_2^2) - \delta((\hat{x}_3)^\alpha - 1)(\hat{x}_1 + \hat{x}_2).$$

Using $\|\hat{x}\| = 1$ and Bernoulli's inequality one gets

$$(\hat{x}_3)^\alpha = (1 - \hat{x}_1^2 - \hat{x}_2^2)^{\alpha/2} \geq 1 - \frac{\alpha}{2}(\hat{x}_1^2 + \hat{x}_2^2),$$

which yields

$$\dot{V}(\hat{x}) \leq -(\hat{x}_1^2 + \hat{x}_2^2) \left(1 - \frac{\alpha\delta}{2}(\hat{x}_1 + \hat{x}_2)\right).$$

Since $-\sqrt{2} \leq \hat{x}_1 + \hat{x}_2 \leq \sqrt{2}$, there exists $\delta_{\max} \geq \sqrt{2}/\alpha$ such that for all $\delta \in (-\delta_{\max}, \delta_{\max})$, $\dot{V}(\hat{x}) \leq 0$ on S^2 . The largest invariant set contained in $\{\hat{x} \in S^2, \dot{V}(\hat{x}) = 0\}$ is then $\{-x_t, x_t\}$. It follows from LaSalle's principle (see e.g. [12, Theorem 2]) that the ω -limit set of any trajectory of system (30) is contained in the set $\{(-x_t, 0), (x_t, 0)\}$. However, from Lemma 3.17, the point $(x_t, 0)$ is locally asymptotically stable. It follows that any trajectory goes either to $(-x_t, 0)$ or to $(x_t, 0)$. Since, from Lemma 3.18, the set of trajectories converging to $(-x_t, 0)$ has zero measure, the Lemma follows. ■

4) *Numerical simulations:* The theoretical results presented in the previous section have been tested by means of a numerical simulation. The case of system (1) has been treated and the observer was constructed using equation (7). The initial conditions were arbitrarily set to $x_0 = (1, 0, 0)$ and $\hat{x}_0 = (1/\sqrt{2}, 1/\sqrt{2}, 0)^4$. We took a feedback control in order to make the system observable, that is, we chose a perturbed feedback control of the form (25). The simulations have been made for two different parameters of the perturbation of the stabilizing feedback control: $(\delta, \alpha) = (0.1, 2)$ and $(\delta, \alpha) = (0.1, 6)$. Fig. 4 shows, for both cases, the evolution of the state variables, and Fig. 5 shows the evolution of the observation errors. Notice that the thick aspect of the drawn curves is due to the observation of an oscillating system over a very long time.

In both cases, the simulations show that the observation errors converge to zero, which corroborates Corollary 3.15 as well as the idea that perturbing the stabilizing feedback enables the observer to fulfill his task. It is also clear that the state converges to x_t , which is exactly the statement of Theorem 3.19. Notice moreover that the convergence is faster as α increases.

A comparison between the present simulations and those given in Subsection III-B enlightens that the convergence is much slower in the present case. However, the influence of the tuning parameters r_1, r_2, α and δ on the convergence rate has not been explored and a thorough study could be interesting.

IV. GENERALIZATIONS

In Section III we tested, through a particular example, our idea to perturb a feedback control in order to stabilize system (1) to the target point x_t by means of an output feedback. We believe that this idea is valid for the more general situations exposed below.

⁴The choice of $\hat{x}_0 \in S^2$ is legitimated by the fact that the state of system (1) lies in S^2 .

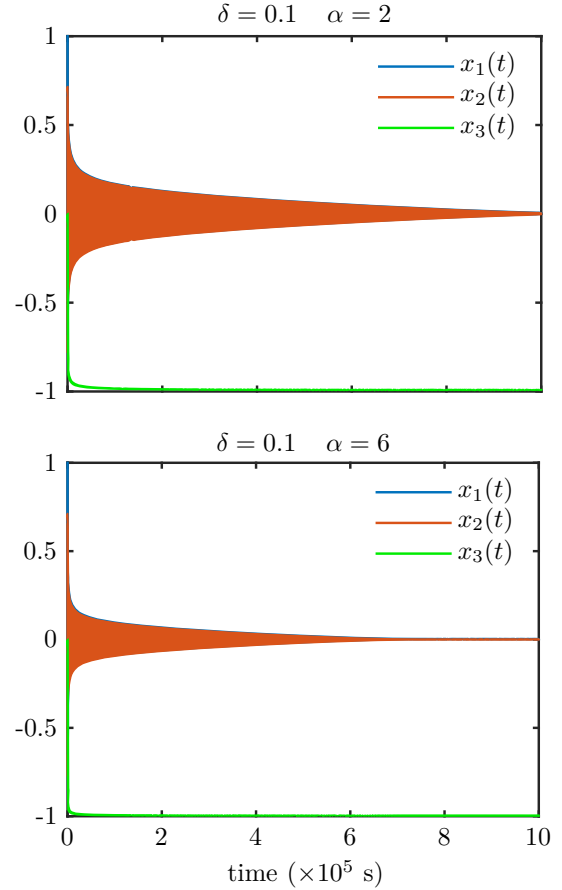


Fig. 4. State variables of system (1) with input $u = \frac{s}{\delta, \alpha}(\hat{x})$ and for $(\delta, \alpha) = (0.1, 2)$ and $(\delta, \alpha) = (0.1, 6)$.

A. Bilinear skew-symmetric systems

The method exposed in Section III can be extended to all bilinear skew-symmetric systems of the form

$$\begin{cases} \dot{\chi} = A(u)\chi \\ \eta = C\chi, \end{cases} \quad (36)$$

where $\chi \in S^{n-1}$ is the state, $\eta \in \mathbb{R}^p$ is the measured output, $u \in \mathbb{R}^d$ is the control variable, $C \in \mathbb{R}^{1 \times n}$ and $A(u)$ is a skew-symmetric matrix affine in u . The goal is to stabilize system (36) at an equilibrium point χ_{eq} whose corresponding control renders the system unobservable. Of course, system (36) may admit other invariant sets.

Using the same Luenberger-like observer as in Section III-A.2, the controller-observer system corresponding to system (36) is then

$$\begin{cases} \dot{\hat{\chi}} = A(\lambda_\delta^s(\hat{\chi}))\hat{\chi} - C'C\xi \\ \dot{\xi} = (A(\lambda_\delta^s(\hat{\chi})) - C'C)\xi, \quad (\hat{\chi}, \xi) \in M, \end{cases} \quad (37)$$

where $\hat{\chi}$ is the state estimate, ξ is the observation error and $M = \{(\hat{\chi}, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid \|\hat{\chi} - \xi\| = 1\}$.

Suppose that there exists a state feedback $\lambda^s(\cdot)$ stabilizing system (36) at χ_{eq} . Denote by \mathcal{A} the region of attraction of χ_{eq} . The general steps are the following.

- 1) Find a (small) perturbation $\tilde{\lambda}(\cdot)$ of $\lambda^s(\cdot)$ such that

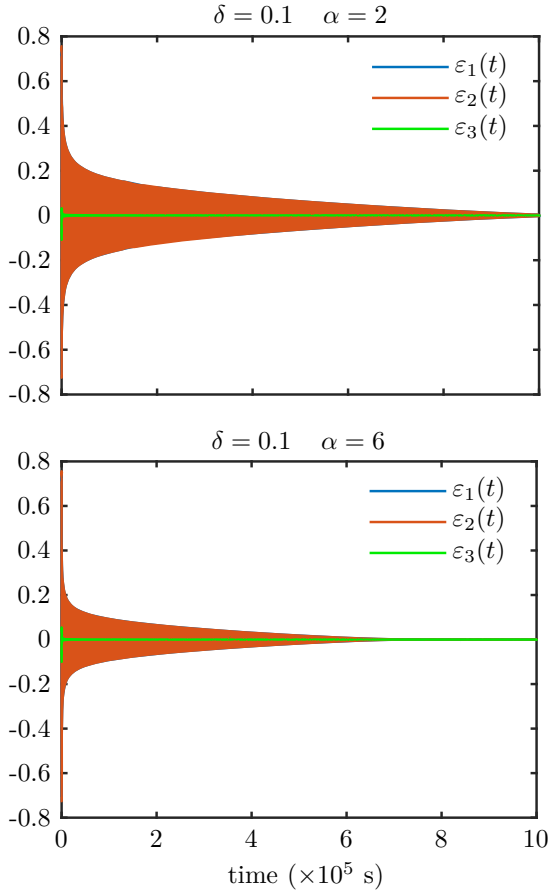


Fig. 5. Observation errors of system (30) for $(\cdot) = (0 \ 1 \ 2)$ and $(\cdot) = (0 \ 1 \ 6)$.

$\tilde{\lambda}(\cdot)$ used as a state feedback stabilizes system (36) at χ_{eq} and such that the input $\tilde{\lambda}(\tilde{\chi}(\cdot))$ makes the observation error go to zero.

- 2) Prove that the equilibrium point $(\chi_{\text{eq}}, 0)$ is (Lyapunov) stable for system (37) and prove that the invariant subsets of $\mathcal{B} = \{(\hat{\chi}, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid \hat{\chi} - \xi \in \mathcal{A}\}$ have zero measure.
- 3) Using ω -limit arguments, prove that $(-\chi_{\text{eq}}, 0)$ is asymptotically stable for system (37) and that its region of attraction is an open and dense subset of \mathcal{B} that has full measure.

This generalization to bilinear skew-symmetric systems is only the very first step toward a far more general situation described below.

B. Immersion into bilinear skew-adjoint systems

Consider an analytic affine control system on a connected smooth manifold X

$$\begin{cases} \dot{x} = F_0(x) + \sum_{i=1}^d u_i F_i(x) \\ y = h(x), \end{cases} \quad (\Sigma)$$

where $x \in X$ is the state, $y \in \mathbb{R}^p$ is the observation, $u = (u_1, \dots, u_d) \in \mathbb{R}^d$ is the control variable and the

vector fields F_i , $i = 1, \dots, d$ are complete. The problem is to stabilize system (Σ) at some equilibrium point x_{eq} (supposed to exist).

Set $\mathcal{F} = \{F_0 + \sum_{i=1}^d u_i F_i, u \in \mathbb{R}^d\}$. Denote by $\text{Lie}(\mathcal{F})$ the Lie algebra generated by the family \mathcal{F} and denote $\text{Gr}(\mathcal{F}) = \{e^{t_k f_k} \circ \dots \circ e^{t_1 f_1} \mid t_i \in \mathbb{R}, f_i \in \mathcal{F}, k \in \mathbb{N}\}$ to be the *group of the system*. Also, following [3], let $\text{Obs}(\Sigma)$ denote the observation space of (Σ) , i.e. the smallest vector subspace of $C^\infty(X)$ containing the observation function h and closed under the Lie derivation along elements of \mathcal{F} .

Suppose that $\text{Lie}(\mathcal{F})$ is a finite dimensional Lie algebra and that system (Σ) is observable and orbit minimal (i.e. $\text{Gr}(\mathcal{F})$ acts transitively on X). Then, according to [3, Theorem 6], system (Σ) can be immersed into a finite dimensional bilinear skew-adjoint system if and only if $\text{Obs}(\Sigma)$ is finite dimensional and a lift of h on $\text{Gr}(\mathcal{F})$ is almost periodic (for precise definitions, see [14]).

There is another possible generalization when $\text{Lie}(\mathcal{F})$ is finite dimensional but $\text{Obs}(\Sigma)$ is not. In this case, if a lift of h on $\text{Gr}(\mathcal{F})$ is a coefficient of a unitary representation of $\text{Gr}(\mathcal{F})$ then system (Σ) can be immersed into an infinite dimensional skew-adjoint bilinear system. Alternatively, if a lift of h on $\text{Gr}(\mathcal{F})$ is not a coefficient of a unitary representation of $\text{Gr}(\mathcal{F})$, it is always possible to approximate the lift of h by a linear combination of pure positive-definite functions (see [14, Corollary 13.6.5]) and immerse the resulting system into an infinite dimensional skew-adjoint bilinear system. Notice that this approximation necessarily yields to practical stabilization and thus to a theorem having the taste of Corollary 3.11.

The generalization exposed in Subsection IV-A to infinite dimensional systems is more involved and its extension will not be exposed here (for an infinite dimensional version of the center manifold theorem, see e.g. [15]).

C. Immersion into bilinear non skew-adjoint systems

If the observation space is finite dimensional, then (Σ) can be immersed into a finite dimensional bilinear (not necessarily skew-adjoint) system (see [16]). For general bilinear control systems, there is another natural generalization of the above situation. The observer is (the deterministic version of) the (time dependent) Kalman filter, i.e.

$$\begin{cases} \dot{S} = -A'(u(\hat{x}))S - SA(u(\hat{x})) + C'R^{-1}C - SQS \\ \dot{\hat{x}} = A(u(\hat{x}))\hat{x} - S^{-1}C'C\varepsilon, \end{cases}$$

for some positive definite matrices R and Q .

Again, since S is bounded from below by the observability index, the same scheme of reasoning could be applied. Notice that the dimension of the observer, although finite, can be very big. A natural question then arises: if the observation space is finite dimensional and a lift of h on $\text{Gr}(\mathcal{F})$ is a coefficient of a unitary representation of $\text{Gr}(\mathcal{F})$, is it better to immerse (Σ) in an infinite dimensional bilinear skew-adjoint system or in a finite dimensional bilinear non skew-adjoint one?

V. CONCLUSION AND REMARKS

In this paper, we have analysed via an example the problem of output feedback stabilization when the point where we want to stabilize the system corresponds to a control value that makes it unobservable.

Our main result (Theorem 3.10) show that adding a small well-chosen perturbation to the stabilizing feedback control allows to asymptotically stabilize the system to the target point.

We do believe that although the example treated in Section III satisfies the hypothesis of null-observability introduced by Coron (see [5, Definition 2.3] for the precise definition) which is necessary (for analytic systems) and sufficient for local stabilizability in small time by means of a continuous dynamic periodic time-varying output feedback law, our result should apply to a wider class of systems.

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4. Conclusion et discussions

Les résultats présentés dans ce chapitre, bien que traitant un exemple simple, montrent que la stabilisation par retour de sortie de systèmes non linéaires peut très vite s'avérer complexe. Nous ne donnons pas de théorème général mais nous exposons une méthode constructive qui peut se généraliser à une grande classe de systèmes à l'aide de la théorie des fonctions représentatives (voir par exemple Hochschild et Mostow [38; 39; 40]) et de l'analyse harmonique (voir par exemple Dixmier [26]; Folland [28]; Hewitt et Ross [36]). En effet, d'après Celle *et al.* [20, théorème 6], un système de contrôle à une Killing qui est observable et d'orbite minimale peut-être immergé dans un système bilinéaire anti-symétrique observable si et seulement si l'espace d'observation du système est de dimension finie et le relevé de la sortie sur le groupe du système est presque périodique. Nous pouvons alors appliquer notre méthode sur ce nouveau système bilinéaire anti-symétrique.

Bien que cela n'ait pas encore été approfondi, il semble possible de démontrer, en utilisant la théorie de la transversalité de Thom exposée au chapitre 2, qu'il existe une perturbation du retour d'état stabilisant qui rend le système observable (démonstration des lemmes 2.4 et 3.13 de l'article).

D'autres approches intéressantes pourraient être d'appliquer notre méthode avec une perturbation au niveau de la sortie ou encore de considérer une perturbation stochastique. Dans ce dernier cas cependant, des outils d'analyse stochastique seraient nécessaires.

Conclusion générale

Dans cette thèse, trois problèmes de la théorie du contrôle ont été abordés. Leurs natures différentes, a ainsi permis l'utilisation d'outils variés des mathématiques et de la théorie du contrôle. Chaque chapitre comportait donc une conclusion et détaillait les ouvertures possibles. Nous exposons dans ce chapitre une conclusion générale pour cette thèse.

Le premier problème, concerne la synthèse temps minimal pour un modèle cinématique de drone. Notre résultat principal est la synthèse temps minimum donné Figure 1.3 p. 46. Cette synthèse peut également être vue comme un retour d'état stabilisant le drone en temps minimum sur une trajectoire circulaire de rayon minimum. La comparaison de notre résultat avec celui donné dans Maillot *et al.* [56] est très intéressante et soulève naturellement la question de la continuité de la synthèse vis-à-vis de la vitesse linéaire du drone.

Le second aspect de cette thèse concerne un problème de contrôle optimal inverse. Notre résultat principal, généralisant celui de Ajami *et al.* [3], nécessite cependant des hypothèses assez fortes. En particulier, il nécessite d'avoir une connaissance *a priori* du critère sur une sous-variété de dimension $d - 1$. Nous donnons cependant des réflexions pouvant amener à la relaxation de cette hypothèse.

Le dernier problème, concernant la stabilisation par retour de sortie en des points où le système est inobservable, est traité à travers un exemple. Bien que l'exemple traité soit assez simple, nous nous apercevons que la résolution d'un tel problème devient vite complexe. Cette étude permet en particulier de montrer que l'introduction d'une perturbation peut à la fois stabiliser le

ystème et conserver un certain niveau d'observabilité. Nous donnons également une méthode générale ainsi qu'une généralisation à une classe plus large de systèmes grâce aux outils de l'analyse harmonique.

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Analyse de problèmes inverses et directs en théorie du contrôle

Le contexte général de cette thèse est l'étude de problèmes inverses et directs en théorie du contrôle. Plus précisément, les trois problèmes étudiés sont les suivants.

Le premier est un problème de contrôle optimal (approche directe). Il s'agit de fournir la synthèse temps minimum du modèle cinématique d'un drone volant à altitude constante, de vitesse linéaire non nécessairement constante voulant rejoindre une trajectoire circulaire de rayon de courbure minimum.

Le deuxième problème concerne une approche inverse du contrôle optimal. Il s'agit d'élaborer des méthodes théoriques de reconstruction du critère optimisé dans un problème de contrôle optimal à partir d'un ensemble de solutions à ce problème, ainsi que caractériser les "bons" ensembles de trajectoires permettant la reconstruction du critère. Le contrôle optimal inverse connaît un regain d'intérêt depuis une quinzaine d'années, en particulier dans l'étude des comportements moteurs humains. En effet, selon un paradigme largement accepté en neurophysiologie, parmi tous les mouvements possibles ceux effectivement réalisés sont solutions d'un processus d'optimisation.

Le troisième problème traite de stabilisation par retour de sortie. Nous analysons, à travers un exemple académique tiré du contrôle quantique, le problème de stabilisation par retour de sortie (à l'aide d'un observateur) lorsque le point où l'on souhaite stabiliser le système correspond à un contrôle qui rend le système inobservable. L'idée générale est de perturber le retour d'état stabilisant afin de garantir l'observabilité du système tout en stabilisant le système sur la cible. L'analyse de cet exemple académique nous permet dans un second temps de dégager une méthode générale pouvant s'appliquer à une classe de système beaucoup plus large.

Mot clés : Problème inverse, contrôle optimal, principe du maximum de Pontryagin, transversalité de Thom, observateur, stabilisation, retour de sortie.

Analysis of inverse and direct problems in control theory

The overall context of this thesis is the study of inverse and direct problems in control theory. More specifically, the following three problems are studied.

The first one is an optimal control problem (direct approach). The aim is to give a time minimum synthesis for a kinematic model of a UAV flying at constant altitude with positive (non-necessarily constant) linear velocity in order to steer it to a fixed circle of minimum turning radius.

The second problem deals with an inverse approach of optimal control. The aim is to develop theoretical methods in order to reconstruct the minimized criterion in an optimal control problem from a set of solution to this problem. The aim is also to characterize the « good » sets of trajectories leading to the reconstruction of the criterion. In the last fifteen years, there has been a renewed interest in inverse optimal control, especially in human motor behavior. Indeed, according to a well accepted paradigm in neurophysiology, among all possible movements, those actually accomplished are solutions of an optimization process.

The third problem tackles output feedback stabilization. We analyze, via a simple academic example from quantum control, the problem of dynamic output feedback stabilization, when the point where we want to stabilize corresponds to a control value that makes the system unobservable. The general idea is to perturb the stabilizing state feedback in order to ensure the observability of the system while stabilizing it to the target. The analysis of this example allows, secondly, to identify a general procedure that can be applied to a wider class of systems.

Keywords: Inverse problems, optimal control, Pontryagin maximum principle, Thom's transversality, observers, stabilization, output feedback.