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THÈSE

Présentée en vue
D'obtenir le grade de

DOCTEUR

En

Spécialité: Automatique, Génie Informatique, Traitement du Signal et des Images

Par

Jianxin FANG

DOCTORAT DÉLIVRÉ PAR CENTRALE LILLE

Titre de la thèse:

**Analysis and Computational Algorithms for Assemble-To-Order systems:
Application to W-configuration systems**

Analyse et algorithmes de résolution de systèmes ATO (Assemble-To-Order) :

Applications aux systèmes du type W

Soutenue le 2 octobre devant le jury d'examen :

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To my parents,

To my family,

To my girlfriend,

To my friends,

To my professors.

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Abstract

This thesis analyzes a W -configuration Assemble-To-Order (ATO) system with random lead times, random arrival of demands, and lost sales, in continuous time. Our work contributes to the research stream of ATO systems by characterizing the optimal control policy of a W -configuration ATO system, which is a building block for more general systems. While the vast majority of the literature dealing with integrated production and inventory control tries to characterize the structure of the optimal policy over the entire state space, we used a novel approach that characterizes the structure of the optimal policy within a sub space which we identified as the recurrent region within which the state remains, in the long run. The advantage of such approach is that it allows us to identify a set of structural properties that are satisfied by the cost function (otherwise may not be satisfied outside this region). This approach greatly simplifies the analysis and can be applied to several similar integrated production and inventory control problems. In particular, within this region, the operator associated with the production decision, in the optimality equation, automatically satisfies the set of identified structural properties. This, in addition to greatly simplifying the proof of the structure of the optimal policy, avoids the challenge of having the cost function not satisfying certain properties in the region other than the recurrent. Furthermore, in addition to integrated production and inventory control systems, our approach can be applied to problems where the optimal policy yields both a recurrent and a transient region. Below, we elaborate on the contents of each chapter.

In the first chapter, we give a basic introduction of ATO systems, which includes its configurations and its advantages in various industries applications. We describe the challenges in implementing ATO systems in practice and give the outline of our thesis.

In the second chapter, we review the current literatures dealing with ATO systems. Specifically, we divide the literature based on the inventory review models and number of end products. For the periodic review models, we divide the literature into one-period models and

multiple-period models. For the continuous review models, we divide the literature into exogenous load-independent models and endogenous load-dependent models. We also discuss the optimal policy structures and computational methods in diverse problem settings.

In Chapter 3, we formulate the problem as an infinite-horizon Markov decision process. We deviate from the standard approach by first characterizing a region (which we refer to it as “the recurrent region”) of the state space where all properties of the cost function hold. We apply a linear programming technique to solve the problem, which shows that all states in the recurrent region have a strictly positive limiting probability. Our analysis reveals that the optimal allocation policy is counter-intuitive. For instance, even when one product dominates the other, in terms of lost sale cost and lost sale cost rate (i.e., demand rate times the lost sale cost), its demand may not have absolute priority (in being satisfied) over the other product’s demand. Such a feature has not been observed in many integrated production/inventory settings where inventory allocation follows a fixed priority in satisfying demands.

In Chapter 4, we characterize the optimal policy within the recurrent region. In particular, we show that within the interior of the recurrent region, components are always produced. The optimal component allocation within the recurrent region is characterized by two state-dependent thresholds corresponding to the components used by the product. We also show that the structure of the optimal policy remains for systems with batch production, non-unitary product demand and K-Erlang distributed production times.

In Chapter 5, we propose a heuristic method that is based on system decomposition. We compare this heuristic and two other state-independent heuristic methods to the optimal policy. Extensive numerical experiments show that these three heuristics are efficient with compared to the optimal policy. In particular, the decomposition method works best both with respect to relative percentage error and computational time. Taking advantage of this fact, we use this heuristic policy as a startup policy to the dynamic programming algorithms used to obtain the optimal policy. We show that doing so reduces the CPU time required to obtain the optimal policy, by several orders of magnitude. In addition, in order to verify the effectiveness of the

decomposition method, we apply it to three other larger systems. In each of these three systems, numerical results show it still performs very well.

In last chapter, we conclude our thesis and propose several branches of future research perspectives.

Key words: Assemble-to-order, Markov decision process, Dynamic programming, Linear Programming, Production and inventory control, Inventory rationing, Demand management

Résumé étendue en Français

Les systèmes Assemble-To-Order (ATO) qui comprennent un système d'assemblage et un système de distribution sont difficiles à analyser par nature. Dans cette thèse, nous analysons un type W (3 composants, 2 produits finales) de système ATO avec des délais de livraison aléatoires, l'arrivée aléatoire de la demande et des ventes perdues, en temps continu. Notre travail contribue au flux de recherche des systèmes ATO en caractérisant la politique de contrôle optimale du système ATO avec la configuration W , qui est un bloc de construction pour des systèmes plus généraux. Alors que la grande majorité de la littérature sur la production intégrée et le contrôle des inventaires tente de caractériser la structure de la politique optimale sur l'ensemble de l'espace étatique, nous avons utilisé une approche originale qui caractérise la structure de la politique optimale à la région récurrente dans laquelle l'état reste, à la longue. L'avantage d'une telle approche est qu'elle nous permet d'identifier un ensemble de propriétés structurelles qui sont satisfaites par la fonction de coût (autrement, elles pourraient ne pas être satisfaites en dehors de cette région). Cette approche simplifie grandement l'analyse et peut être appliquée à plusieurs problèmes similaires de production intégrée et de contrôle d'inventaire. En particulier, dans cette région, l'opérateur associé à la décision de production, dans l'équation d'optimalité, satisfait automatiquement l'ensemble des propriétés structurelles identifiées. Ceci, en plus de simplifier grandement la preuve de la structure de la politique optimale, évite le défi d'avoir la fonction de coût ne satisfaisant pas certaines propriétés dans la région autre que le récurrent. En outre, en plus des systèmes intégrés de contrôle de la production et des stocks, notre approche peut être appliquée aux problèmes où la politique optimale produit à la fois une région récurrente et une région transitoire. Ci-dessous, nous développons le contenu de chaque chapitre.

Dans le premier chapitre, nous donnons une introduction basique des systèmes ATO, qui inclut ses configurations, ses avantages et ses applications dans diverses industrielles. Nous décrivons les défis dans la mise en œuvre des systèmes ATO dans la pratique et donnons les grandes lignes de notre thèse.

Dans le deuxième chapitre, nous passons en revue les littératures actuelles traitant des systèmes ATO. Plus précisément, nous divisons la littérature en fonction des modèles de revue d'inventaire et du nombre de produits finaux. Pour les modèles d'examen périodique, nous divisons la littérature en modèles à une période et en modèles à périodes multiples. Pour les modèles de revue continue, nous divisons la littérature en modèles exogènes et indépendants de la charge et en modèles endogènes et dépendants de la charge. Nous discutons également des structures politiques optimales et des méthodes de calcul dans divers contextes de problèmes.

Dans le chapitre 3, nous formulons le problème comme un processus de décision de Markov à l'horizon infini. Nous nous écartons de l'approche standard en caractérisant d'abord une région (que nous appelons «la région récurrente») de l'espace d'état où toutes les propriétés de la fonction de coût sont maintenues. Nous appliquons une technique de programmation linéaire pour résoudre le problème, qui montre que tous les états de la région récurrente ont une probabilité de limitation strictement positive. Notre analyse révèle que la politique d'allocation optimale est contre-intuitive. Par exemple, même lorsqu'un produit domine l'autre, en termes de coût de vente perdu et de perte de prix de vente (c'est-à-dire le taux de demande multiplié par le coût de vente perdu), sa demande peut ne pas être prioritaire. Une telle caractéristique n'a pas été observée dans de nombreux contextes intégrés de production / d'inventaire où la répartition des stocks suit une priorité fixe dans la satisfaction des demandes.

Au chapitre 4, nous caractérisons la politique optimale dans la région récurrente. En particulier, nous montrons qu'à l'intérieur de la région récurrente, les composants sont toujours produits. L'allocation optimale des composants dans la région récurrente est caractérisée par deux seuils dépendants de l'état correspondant aux composants utilisés par le produit. Nous montrons également que la structure de la politique optimale reste pour les systèmes avec production par lots, demande de produits non-unitaires et temps de production distribués K-Erlang.

Au chapitre 5, nous proposons une méthode heuristique basée sur la décomposition du système. Nous comparons cette heuristique et deux autres méthodes heuristiques indépendantes de l'état à la politique optimale. Des expériences numériques approfondies montrent que ces trois heuristiques sont efficaces par rapport à la politique optimale. En particulier, la méthode de décomposition fonctionne le mieux en ce qui concerne le pourcentage relatif d'erreur et le temps de calcul. Profitant de ce fait, nous utilisons cette politique heuristique comme une politique de démarrage pour les algorithmes de programmation dynamique utilisés pour obtenir la politique optimale. Nous montrons que cela réduit le temps CPU nécessaire pour obtenir la politique optimale, de plusieurs ordres de grandeur. De plus, afin de vérifier l'efficacité de la méthode de décomposition, nous l'appliquons à trois autres systèmes plus importants. Dans chacun de ces trois systèmes, les résultats numériques montrent qu'il fonctionne encore très bien.

Dans le dernier chapitre, nous terminons notre thèse et proposons plusieurs branches de perspectives de recherche futures.

Mots-clés: Assemble-To-Order, Processus de décision Markov, Programmation dynamique, Contrôle de la production et des stocks, Gestion de la demande

Chapter 1. Introduction

Assemble-To-Order (ATO) systems have the advantages of decreasing operational costs, increasing variety of end products and reducing response time by pooling inventory of components and delaying the assembly stage until demand materialization. It has attracted great deal of attention in industry for a long time. In the first chapter, we will try to understand what an ATO system is, its advantages, disadvantages and its application fields. Then, we will describe the optimal control of ATO systems and lay out the plan of the thesis.

1.1 The ATO system and its applications

1.1.1 What is an ATO system?

An appropriate production strategy is the keystone for a company (especially for a manufacturing company) to integrate effectively its manpower, material resources and capital, so as to provide market competitiveness. Typically, based on the customer order decoupling point (M. Rudberg and J. Wikner, 2004), there are four frequently used production strategies: Engineering-To-Order (ETO), Make-To-Order (MTO), Make-To-Stock (MTS) and Assemble-To-Order (ATO) (cf. Figure 1.1). In General, the earlier the customer order decoupling point, the higher degree of customization. Depending on the category of products, the complexity of production and the specification of demand, different companies may apply different production strategies. For instance, for bulky and complex products (Airplane, High-speed train etc.) or non-standard products (Customized furniture, Tailored dress etc.), customers' opinions are involved at the design phase of products. In this case, an ETO/MTO is a suitable production strategy. However, extra efforts and resources will be needed to meet customers' specifications which increases the cost of the final product. Sometimes, customers could only choose from what is available in the marketplace (such as Agricultural products, Apartments, etc.). In this case, Research and Development (R&D) is done through market investigation and the MTS strategy is chosen for mass production to reduce costs. For example, the Ford model T has always been regarded as the most successful initiative of changing from MTO production to MTS production, which substantially reduced the cost of a car (K.

Williams et al. 1992). The information in the MTS production system transfers to a bottom-up model, which may induce a discrepancy between company's supplies and customers' needs and which may result in a potential loss (H. L. Lee et al. 1997).

Because of customer desire for low cost as well as for customization, companies try to strike a balance between low cost production and product customization. To this end, Mass Customization has been proposed to serve each customer with customized product at a reasonable price (B. J. Pine II, 1993). To obtaining the benefits of Mass Customization, companies need to find an appropriate production stage to take customers' needs into account. Zinn (1990) found that the postponement of the assembly stage is an effective way to reduce operational costs, to avoid unnecessary inventory and to offer a broader assortment of products. In particular, postponement of the assembly stage represents a major strategic alternative to sales forecast-based distribution when errors in demand forecasting are high. As such, the use of the ATO strategy to organize production becomes an attractive alternative, which provides a way to gain competitive advantage.

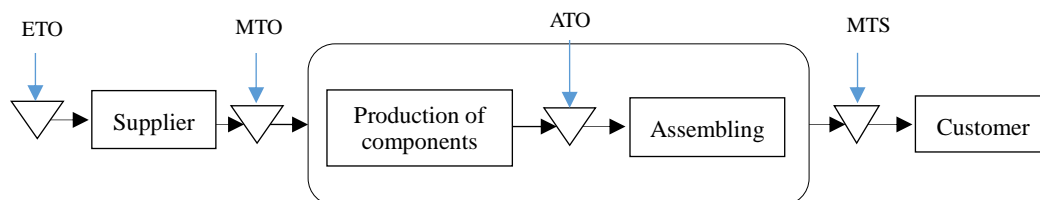


Figure 1.1: Production process of ETO, MTO, ATO and MTS

In general, an ATO system comprises several components or sub-assemblies that are assembled into end products according to a bill of materials (BOM). Components are purchased from suppliers or produced by the company itself. Products are assembled only when customers' demands materialize. The time of assembly can always be neglected, as the time to acquire components is always substantially longer (Song and Zipkin, 2003). This strategy is particularly attractive to firms with long component production/supply lead-times and relatively short assembly time. A basic configuration of an ATO system with n components

and m products is shown in Figure 1.2, where certain components with certain units are assembled to an end product and some of them are shared among different end products. The system manager must decide when to produce/purchase components and when to satisfy demands. If demands cannot be satisfied, they are backlogged or lost, which results in an additional cost.

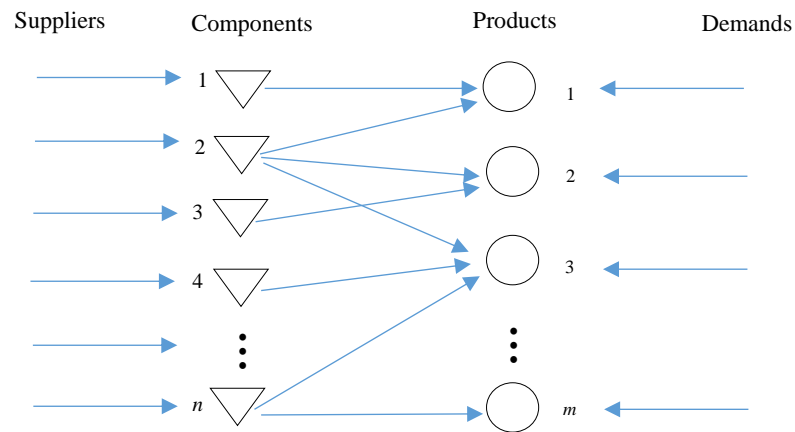


Figure 1.2: Configuration of a basic ATO system

1.1.2 Applications of ATO systems

The ATO strategy has frequently been used in today's marketplace due to its advantages. The applications of ATO production not only allow companies to offer a large number of products with different appearances and performances, but also let them reduce their operational costs. We describe the following three fields where ATO systems are widely applied.

1) Traditional manufacturing

ATO is an appealing strategy for firms in such industries as high-tech, automotive and white goods manufacturing. One well-known ATO system (also refer to as Configure-To-Order (CTO)) is the operation of Dell Computer. Known for its direct sale, Dell lets its customers configure their computers from sets of processors, memories, monitors, hard drives etc., and build a customized personal computer (R. Kapuscinski, 2004). This strategy has become so successful that other personal computer producers are adopting similar strategies, such as

Apple and H&P. In the automotive industry, the ATO system is also referred to as Build-To-Order (BTO) system. For example, BMW allows customers to make changes to their vehicle within 6 days of final assembly. This allows the company to build up to 550,000 permutations of the Z3 vehicle in applying such a BTO system (A. Gunasekaran, 2004).

2) Spare part management

Likewise, the ATO system can be used in the spare part control operation. Due to the uncertain arrival of maintenance jobs that require different spare parts with different units, any shortage of these leads to delayed maintenance jobs. In order to provide short repair turnaround time, the maintenance department needs to keep a high stock level of spare parts. Hence, the manager needs to balance the system cost and service level. In this setting, the spare parts refer to components, while the end product is the maintenance component. There are many practical applications of ATO systems in spare part management. For example, van Jaarsveld (2015) analyzed the maintenance system of Fokker Service, which is a Netherlands-based aircraft repair shop. He provided managerial insights that aim at balancing the operational cost and service quality. Other examples can be found in ASML (Vilegen, I.M.H 2009), Airbus (A. Regattieri et al. 2015) etc.

3) E-commerce

The structure of the ATO strategy also applies to other settings such as mail order catalogs, e-retailers and distributors of multiple finished products. In such settings, customers may order different and possibly overlapping sets of items. In this case, customer service is based on the delivery of the entire order. The assembly of a product refers to pick the items in customers' order and packaging them. For instance, Xu (2006) analyzed the relevant order assignment problem so as to minimize the procurement and transportation costs of Amazon Inc.

1.2 Outline of the thesis

ATO systems however, are difficult to analyze in spite of their popularity due to the inability of identifying the associated optimal operating policies in general. Much of this difficulty can be attributed to several factors: Demand of a component depends on the demand of other components; production/procurement lead times differ from one component to another; and

fulfillment of demand depends on the availability of several components. In multiple-products, multiple-customer classes ATO systems, this difficulty is further compounded since the system manager not only must decide when to produce/procure components but also must decide how to allocate component inventory among competing demands with different priorities (Benjaafar and Elhafsi, 2006). Furthermore, an ATO system can be thought of as a combination of an assembly system and a distribution system (Song and Zipkin, 2003). The main challenge in an assembly system is component coordination and the main challenge in a distribution system is component allocation among multiple products. Thus, the difficulty of concurrently dealing with coordination and allocation issues in an ATO system.

In this thesis, we study a continuous time integrated production and inventory control *W*-configuration system with several features common in the literature. Such a system constitutes a building block for the general multiple-component, multiple-product ATO system. To our knowledge, there is no extant characterization of the optimal component production/procurement and inventory allocation policies for even the basic *W*-configuration system. We believe that studying and characterizing the optimal policy of the *W*-system allows us to gain insight and enhance our understanding of the general multiple-component, multiple-product ATO system. Furthermore, knowing the structure of the optimal policy for the *W*-configuration system allows us to extrapolate and develop efficient algorithms and heuristics to manage the general configuration ATO systems.

The plan of our thesis is organized as follows: In chapter 2, we provide a review of the current literature of ATO systems with different perspectives. In chapter 3, we formulate the *W*-configuration ATO system and use a linear programming model to solve the system. In chapter 4, we characterize the structure of optimal policy of *W*-configuration ATO system with lost sales and extensions. In chapter 5, we propose three heuristic methods that can be used efficiently for larger systems. In chapter 6, we conclude the thesis and provide some suggestions for managing ATO systems.

1.3 Conclusion

In this chapter, we introduced the concept and advantages of an ATO system. We also presented the applications of ATO systems. However, in spite of their popularity, ATO systems are difficult to analyze due to their inherent characteristics. Specifically, the difficulty is due to concurrently dealing with coordination and allocation issues in an ATO system, which leads to the necessity of finding the optimal control policy of such systems.

Chapter 2. Literature Review

In recent years, academicians and practitioners alike have become increasingly interested in analyzing ATO systems, with most of the papers being published in top journals of this field. The first review of literature relating to ATO systems was provided by Song and Zipkin (2003) and the most recent review was provided by Atan et al. (2017). In this chapter, we follow their leads to provide a comprehensive review of the current literature in ATO systems. Depending on the inventory review approach, we divide the literature into periodic review models and continuous review models.

2.1 Inventory periodic review models

In this section, we study ATO systems in the inventory periodic review setting. Based on the number of periods, we divide this stream of literature into one-period models and multi-period models.

2.1.1. One-period models

One-period models are studied because of their simplicity. On one hand, they can serve as a myopic heuristic or an approximation method to more general models. On the other hand, the one-period model can be practical under certain circumstances. For instance, the traditional news-vendor model is a basic one-period model, where certain quantity of newspapers is purchased according to the distribution of sales forecast ahead of the day and the leftover at the end of the day will be discarded. Furthermore, the rapid changes in technology and engineering can force companies to treat each demand separately, which lead to one-period models.

In a one-period setting, the sequence of events within the period is as follows: (1) Components produced or purchased based on anticipated demand for the end products. (2) The realization of customers' demand. (3) Products are assembled from components and delivered to customers. Because lead times vary among components, the main objective in the one-period model is the coordination of the time and quantity of replenishment orders.

Depending on the number of end products, we further divide the literature of one-period models into single end product and multiple end products.

(1) One-period models with single end product

The literature of one-period models with single end product is limited. Most papers attempt to get the optimal order policy under particular assumptions. Fu et al. (2006) consider an inventory and production planning problem in a single product ATO system with uncertain demand quantity. With assumptions of long components procurement lead times and limited assembly capacity, they use a profit-maximization model to find the optimal inventory and production decisions. In addition, the option of outsourcing is considered as an alternative way to satisfy customers' demands. Xiao et al. (2010b) consider a similar system, whereas assuming uncertain assembly capacity and emergency component orders. They establish the structural properties of the optimal solution and find assemble-in-advance strategy should be adopted in certain conditions. Yao et al. (2013) study a single period ATO system for a single product assembled from multiple components and assume the stock-out components can be obtained through additional two sourcing channels that have different prices and lead-times. They use a branch and bound algorithm to solve the system and explore the structure of the optimal solutions.

The other branch of literature in one-period single end product deals with the lead-time dependent pricing problem. Hsu et al. (2006) study the optimal stocking quantities problem under the assumption of uncertain demand, while the price of the final product and the cost of components depend on their delivery lead times. The authors provide an efficient solution procedure that allows multiple shipments of full order quantity and extend it to the option of not delivering the full quantity but instead taking the penalty for a delivery shortage. Hsu et al. (2007) extend this result to consider the situation where the full order quantity must be shipped together. They show that the full-shipment model is more appropriate for industries where the economies of scale in transportation are critical for cost control. Fang et al. (2008) introduce the 'Vendor Managed Consignment Inventory (VMCI)' concept to manage the underlying risk and suppliers' decision on production quantities under demand uncertainty. They formulate the problem using game theory to find suppliers' Pareto-optimal equilibrium and derive the manufacturer's optimal pricing scheme. Fu et al. (2009) assume the price of the end product is

non-increasing in delivery lead time and the supply of components can be expedited at higher prices. They propose a cost-minimization model and develop an efficient polynomial time algorithm to solve the problem.

In conclusion, most researchers studied the optimal ordering problem of one-period single end product model with distinct assumptions on the sourcing mode (multiple sourcing channels, expedited delivery etc.), the delivery mode (full quantity delivery, partial shipment, lost sale etc.) and the assembly capacity (limited, unlimited, outsourcing etc.). In addition, some researchers studied the lead-time dependent pricing problem, where the relation between the price of end product and delivery lead-time need to be fully taken into consideration.

(2) One-period models with multiple end products

Unlike one-period models with single end product, one-period models with multiple end products introduce more complication since components could be demanded by distinct end products. To handle such systems, one needs not only to find the optimal order quantity for each component, but also needs to determine how to allocate the component to products that require it. Component commonality is a special issue in such system.

One branch of the literature in such systems is to determine the value of component commonality. Inman and Schmeling (2003) investigate the value of decoupled assembly that allows the automotive plant to change customer orders associated with a particular physical vehicle in traditional assembly initiation. Using a simulation method, they show that the decoupled assembly can simultaneously reduce material usage unevenness and worst case lead-times. In particular, they claim that the benefit of decoupled orders increases with product variety. Fong et al. (2004) provide analytical results for a commonality problem under a general cost structure when demands follow an Erlang distribution. They show that the effect of component commonality is significant when the budget level is high even if the common component is much more expensive, whereas the effect is small or negative when the budget level is low relative to the demand requirements for the end products. Chod et al. (2010) study component procurement, assembly and product pricing under an MTO system. Specifically, they show that the benefits of flexibility increase in demand correlation if the corresponding products have a relatively low degree of commonality. Other results can be found in Van Mieghem (2004), Zhang et al. (2008) etc.

The other stream of the literature deals with comparing different allocation rules and finding their effect on system performance. Xiao et al. (2010a) analyze an ATO system that produces two end products with a common component. They apply a profit-maximization model to establish the structural properties for the optimal stocking solutions and investigate with scenarios of how to fulfill low priority customer's order. Bernstein et al. (2011) consider an ATO system in which multiple products are assembled from a common component and a set of product-specific components. In comparing a collection of allocation mechanisms involving varying degrees of demand aggregation, they show that the benefit from increased demand aggregation is closely linked to the degree of capacity imbalance. Tsai et al. (2013) propose a two stages cost-based module mining method for an ATO system, where the first stage determines the set of components to be formed as modules and the second stage determines how to allocate modules among end products.

Bernstein et al. (2007) study the pricing problem in a decentralized multiproduct assembly system and show that there exists a unique Pareto-optimal equilibrium in the supplier's capacity game for any set of wholesale prices.

Correspondingly, in one-period models with multiple end products, most of the research focused on determining the value of component commonality and comparing the effect of different allocation rules on system performance. Zipkin (2016) studied the relative integer linear programming formulation of one-period models with multiple end products. Assuming increasing acquisition cost, he proved that the problem holds a cover- $L^\#$ -convexity property and the feasible set has a polymatroid form.

2.1.2. Multi-period models

In multi-period models, the decisions and sequence of events within a single period are the same as in one-period models. However, complications arise by linking subsequent periods together. In general, there are two sources of complications. One is that the state at the end of the current period will be the beginning state at next period. Besides, if the unsatisfied product is backlogged, the backlogged demand need to be satisfied in subsequent periods and the rule to clear the backlog is crucial. The other complication is due to the lead-times for components replenishment. Song and Zipkin (2003) point out that if the lead times for different components

are different, the replenishment decision in one period will affect inventory levels in other periods in the future. Depending on the number of end products, we divide the literature on multi-period models into single end product and multiple end products.

(1) Multi-period models with single end product

In this setting, the major problem is to find the optimal or near optimal component replenishment policy. Rosling (1989) shows that simple reorder policies are optimal under a mild condition of components' initial stock levels. Karaaslan et al. (2013) study an ATO system with one final product that is composed of one long lead-time component with short review period and one short lead-time component with long review period. They compare the balanced base-stock policy with pure base-stock policy and show that the balanced base stock policy performs better under low service levels, low holding cost ratios and high demand uncertainty.

Some researchers study the effect of different base-stock policies on system performance. Glasserman and Wang (1998) find there is a linear trade-off between inventory levels and the delivery lead time that is subject to a service level target. Song and Yao (2002) show that the performance analysis for any base-stock policy reduces to the evaluation of a set of $M/G/\infty$ queues with a common arrival stream in a single-product assembly system. In addition, they show that it is more desirable to keep higher base-stock levels for components with longer mean lead times and lower unit costs. Xu and Li (2007) investigate the strategic level and operational level technology-inventory coordination for short life cycle products, in which they show the strategic-level technology-inventory coordination is generally sufficient, but the operational level coordination becomes necessary when demand variability is high and salvage loss is heavy. To extend the corresponding joint price-inventory control problem to multi-period models with single end product, Pang et al. (2012) partially characterize the structure of optimal joint ordering and pricing policies.

As a result, in multi-period models with single end product, researchers mainly focus on optimal replenishment policies and the effect of distinct base-stock policies on system performance. However, the optimal replenishment policy is still an open problem for stochastic demand process.

(2) Multi-period models with multiple end products

In this setting, the complications are further exacerbated by taking the allocation policy into account as in one-period models with multiple end products. Since components may be shared among different products, the value of component commonality is one of the research topics. Mohebbi and Choobineh (2005) study the impact of component commonality in an ATO system under supply and demand uncertainty. They apply a simulation method and show that the introduction of common components significantly increases the average percentage of products' on-time delivery, while effects on the reduction of average total inventory of components and average total backorder of products are not significant.

To fully take the risk pooling advantages of component commonality, one must consider the replenishment and allocation decisions jointly. However, Akcay and Xu (2004) show that the component allocation problem is a general multidimensional knapsack problem and is NP-hard, which means that the effort to find the optimal allocation policy in general ATO systems is enormous. Some researchers tried to bypass this difficulty by finding a near optimal or an asymptotically optimal allocation policy under certain assumptions. Plambeck and Ward (2006) study an ATO system with a high volume of demand arriving. Taking product prices, component production capacities and assembly sequences into consideration, they prove that a myopic discrete-review sequencing policy is asymptotically optimal. Plambeck and Ward (2008) extend their previous model by accepting expedite and salvage and derive the relative asymptotical policy. Huang and de Kok (2015) consider a periodic review ATO system under linear holding and backloging costs and show that the First-Come-First-Service (FCFS) allocation policy decouples the optimal component allocation problem over time into deterministic period-by-period optimal component allocation problems, which results in a multi-matching problem.

The benefits of ATO systems are undermined by the difficulty of finding the optimal control policy. Some researchers examine the efficiency of combining an ATO system with other strategies. Eynan and Rosenblatt (2007) show it is beneficial to employ the Assemble-In-Advance (AIA) and ATO simultaneously. Furthermore, they state that the use of component commonality reduces the number of units to be AIA and increase the number of units to be ATO. Tsai and Wang (2009) develop a three-stage multi-site Available-To-Promise (ATP) mechanism for ATO manufacturing and experiment on a TFT-LCD manufacturer. Their

results show this system could make good use of scarce capacity and provide customers with high service level. Chen and Dong (2014) consider an ATP model with the consideration of customers' priority and show it is better than the ATP model that does not consider customers' priority. For components' ordering policy, Huang et al. (2014) consider a mixed replenishment policy that adopts JIT and (Q, r) replenishment mode simultaneously. They use a hybrid genetic simulated annealing algorithm to solve the system and show the good quality of optimal mixed replenishment policy.

Most papers assume the instantaneous assembly in ATO systems. However, Guhlich and Fleischmann (2015) explicitly model the intermediate materials and assembly capacity in an ATO system. By taking scheduling and due date quoting decisions into account, they show that the result is close to that of the ex-post solution, which contains full knowledge of all incoming demand.

For the joint control of pricing and production decisions, Oh et al. (2014) show that a state-dependent base-stock list-price is optimal, in which the optimal state-dependent base-stock levels and list prices may increase or decrease as demand backlogs increase, whereas demand backlogs always improve the optimal expected profit by providing additional flexibilities in component allocation.

Since it's difficult to find the optimal base-stock and allocation policies in the multi-period model with multiple end products, most papers in this section focus on testing and improving the performance of heuristic policies. One way to conquer this problem is to find effective heuristic production and allocation policies. The other way is to combine ATO with other strategies, such as AIA, ATP and JIT etc.

2.2 Inventory continuous review models

In this section, we study ATO systems in continuous review setting. Unlike periodic-review, continuous review lets manufacturers continuously monitor their system and make decisions immediately. The development of Enterprise Resource Planning (ERP) software allows this to be accomplished easily. We divide this section into the following three parts depending on the

characterisation of optimal policies, the use of heuristics policies, performance metrics, and approximation methods.

2.2.1 Optimal policies under several ATO configurations

In general, it may be hopeless to find the optimal policy for multiple-product multiple-component ATO systems due to the high dimensionality of the system in one hand and the lack of special properties of the cost function on the other hand. However, under some assumptions on the demand and the supply processes and for special configurations of ATO systems, one can characterize, fully or partially, the structure of an optimal policy. In the ATO literature, such special configurations have been classified based on the number of components, number of products, and the degree of commonality of the components. Lu et al. (2010) identify four special configurations: (1) The *N*-configuration is a 2-component, 2-product system where one product is made only from one of the components and the other product is assembled using both components. (2) The *M*-configuration is a 2-component, 3-product system where one product is assembled using both components while the other two products use different components. (3) The *W*-configuration is a 3-component, 2-product system where each product is assembled using a common component and a product-specific component. (4) The Nested-configuration is a multiple-component, multiple product system where the “smallest” product uses one component only, the “largest” product uses all components, and the other products use one component less than the next “larger” product. These systems are named after the shape of each configuration, which is shown in Figure 2.1. Depending on components’ supply mode, we divide the literature on continuous review setting into endogenous and load-dependent systems and exogenous and load-independent systems.

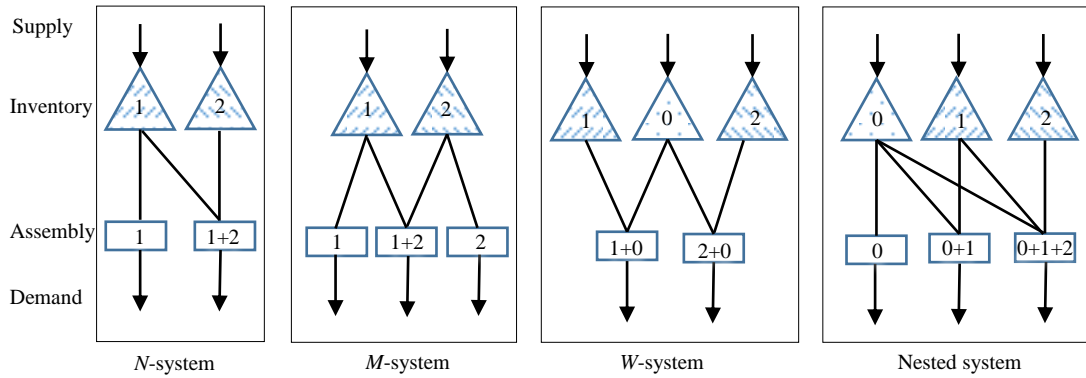


Figure 2.1: Configurations of simple ATO systems

(1) Endogenous and load-dependent continuous review systems

In such systems, replenishment lead-times are endogenous and depend on the load of the system. This includes the work of Ha (1997a, 1997b, 2000), Carr and Duenyas (2000), de Véricourt et al. (2002), Teunter and Klein Haneveld (2008), Zhao et al. (2009), Benjaafar et al. (2010), Benjaafar and Elhafsi (2012) and Pang (2015). In such literature, typically, the system produces a single product and faces Poisson demands which can be for a single unit or multiple units and consist of several classes. Demand is either lost or backordered if not fulfilled immediately. Items are produced either one at a time or in batches with exponentially distributed production times. In this case, the problem is formulated as a continuous time Markov Decision Process (MDP). The aim is to determine the optimal production and inventory rationing policy.

As far as ATO systems are concerned, Benjaafar and Elhafsi (2006) were the first to consider an MDP formulation of an ATO system. They studied a single-product multiple-demand class ATO system under the assumption of exponential component production times and Poisson demand arrivals. They showed that the optimal component production and the optimal inventory rationing policies are both threshold type policies where the thresholds depend on the inventory level of all components. Feng et al. (2008) study the optimal pricing of a 2-component, single-product ATO system and showed that the optimal component production and product pricing policies are both threshold type as in Benjaafar and Elhafsi (2006). Elhafsi (2009) extended the results of Benjaafar and Elhafsi (2006) to the case of compound Poisson demands and showed that using the optimal rationing policy results in substantial savings compared to the First Come First Serve (FCFS) policy. Cheng et al. (2011)

study the same system as Benjaafar and Elhafsi (2006), however they assume that the production facilities are failure prone. They show that the production and inventory allocation policy with respect to a component is not only influenced by the state of the other components but also by the state of the production facilities as well. Keblis and Feng (2012) studied the same system as Feng et al. (2008) with a general stock-out cost function and proved that the pricing policy in Feng et al. (2008) continues to be optimal.

Elhafsi et al. (2009) study the optimal control of a nested multiple products ATO system, which shows that components' optimal production policy is of the base-stock type with the base-stock level for each component non-decreasing in the inventory level of other components and components' optimal inventory allocation is a multi-level rationing policy with the rationing level for each component depending on the inventory level of all other components.

Elhafsi et al. (2015) characterize the optimal policy structure for an M -configuration ATO system with lost sales, which exhibits the state-dependent base-stock policy and state-dependent rationing policy. Nadar et al. (2014) study a lost sales general M -configuration ATO system. Assuming a certain production batch size, the authors show that the optimal component production and inventory allocation policies are both lattice-dependent. ElHafsi and Hamouda (2015) study a general M -configuration ATO system where in addition to the final product, components are also available for sale. Assuming that demand is backordered if it cannot be fulfilled immediately, they showed that the optimal rationing policy is not a strict priority rule, as is usually the case for integrated production and inventory control problems, but rather depends on the inventory levels of components.

(2) Exogenous and load-independent continuous review systems

In such systems, components lead-times are assumed to be exogenous and the system is modeled as a set of queues with infinite servers and correlated arrivals. This includes the work of Song et al. (1999, 2000, 2002 etc.), Song and Yao (2002), Gallien and Wein (2001), Dayanik et al. (2003), Zhao (2009), Lu et al. (2010) and the references therein.

Lu et al. (2010) studied an N -configuration ATO system with general component lead times and fixed base-stock replenishment policy. They show that a no-holdback allocation policy (a no-holdback allocation policy does not reserve components for future product demands) is optimal among all allocation policies.

Dogru et al. (2010) study a W -configuration ATO system where demand is backlogged if not fulfilled immediately and component replenishment times are identical and deterministic. Using a stochastic programming framework, they prove that a base stock component replenishment policy combined with a priority based allocation rule is optimal under some symmetry assumptions. Namely, balanced capacities and identical holding costs allow the optimality of this intuitive policy. Lu et al. (2015) considered both an N -configuration and a W -configuration system assuming a symmetric cost structure and general component procurement lead times. Using a fixed base stock component replenishment policy, they obtained similar results to Dogru et al. (2010).

In summary, the optimal control policy has been identified for N -, M - and nested configurations of ATO systems. In such cases, the optimal policy is characterized by state-dependent base-stock and state-dependent rationing thresholds. If components are produced in batches and the end products require non-unit size of components, the optimal policy is characterized by a lattice-dependent type for M -configuration ATO systems. If the replenishment lead-times are identical and costs satisfy the symmetry condition in W -configuration, the optimal replenishment policy is characterized by the coordinated base-stock rule and the optimal allocation policy is characterized by the no-hold back allocation rule. However, because of the curse of dimensionality in the MDP formulation, the optimal control policy has not been determined for the general W -configuration ATO system.

2.2.2 Heuristic policies and performance metrics

Though the optimal control policies have been identified for simple ATO systems, it is difficult to characterize it in general cases. In addition, the optimal policy is strenuous to be applied in practice. Therefore, finding effective and easy to implement heuristic policies is an important alternative. Another stream of the literature that is also related to our problem studies ATO systems assuming component production/procurement is managed through base-stock levels that are independent of the inventory level of other components and inventory allocation is managed through specific rules. The aim is to study the performance of the system under such rules. In this vein, Gao et al (2010) study a multi-product ATO system where product demand occur according to independent Poisson processes and inventory of components is managed by

fixed base stock levels. They derive the joint distribution of component inventories which allows obtaining performance metrics such as order and item based fill rates. Fu et al. (2011) propose approximate methods for obtaining bounds on similar performance metrics. Feng et al. (2012) identify a necessary and sufficient condition to evaluate the order-based backorder level. Zhou and Chao (2012) study the same system as Gao et al. (2010) and Fu et al. (2011) and develop approximation methods to calculate the order-based fill rate of the system.

In general, most companies use independent base-stock level as components replenishment rule. However, Benjaafar and Elhafsi (2006) show that the fixed base-stock policy may not be able to work well in certain cases and that the coordinated base-stock rule is more suitable. Coincidentally, Lu and Song (2015) show the coordinated base-stock policy is optimal for W -configuration under cost symmetry condition. Van Jaarsveld et al. (2015) utilize a (s, S) replenishment rule and show the policy works well in a practical example. For components allocation rule, Huang and de Kok (2015) show it is of general importance. Though FCFS rule is frequently applied, it is not optimal and sometimes it may incur great loss potential. Elhafsi and Hamouda (2015) use a priority based allocation rule, which works well when the demands have large differences. When the unsatisfied demands are backlogged, Lu et al. (2010) show that the non-holdback rule is proved to be optimal in some circumstances and outperforms all other component allocation rules.

The other branch of literature is to analyze the heuristic policies under certain performance metrics. Lu et al. (2005) use the weighted average of backorders over product types as the performance metric and derive easy-to-compute bounds for it. Similar to Lu et al. (2005), Zhao (2009) study a multi-product multi-component ATO system where the external demand follows compound Poisson processes and component inventories are controlled by continuous time batch ordering policies. The author derives the exact expression for the expected delivery lead-times and the order-based fill-rates metrics and estimates it under an efficient sampling method. Iravani et al. (2004) consider an ATO system with consideration of key and non-key items, in which they evaluate different levels of customer satisfactions using a quasi-birth-and-death process. Lu (2008) evaluate the average inventory and immediate order fill rate metrics and obtain qualitative results that reveal the impacts of changes in demand patterns and lead-time variability upon the performance of the systems.

The exact evaluation of any performance metric for complex ATO system faces a huge numerical difficulty. To overcome this problem, many researchers tried to obtain performance bounds. Dayanik et al. (2003) present the order fill rate metric involving evaluation of multivariate probability distributions, which is computationally demanding. In alternative to the exact evaluation of the performance metric, the authors use tailored approximation ideas to derive performance bounds. Basic et al. (2013) present a new bounding method for Markov chains and apply it to analyze the out-of-stock situation in an ATO system.

In conclusion, many easy-to-implement heuristic policies have been developed for ATO systems. One way to determine the efficiency of heuristic policies is to compare it with the optimal policy. However, it is difficult to obtain optimal policies for complex ATO systems. In order to bypass this difficulty, performance metrics have been developed to ease the evaluation of heuristic policies, such as delivery lead time, expected backorders, order-based immediate fill rate, order-based fill rate within a time window and a response-time based order fill rate. In addition to exact evaluation, approximation methods have also been proposed for determining performance bounds.

2.2.3 Approximation methods

The exact solution algorithms always face the curse of dimensionality problem, which means that these methods cannot be used for large-scale problems. More recent literature focuses on the development of efficient computational and solution methodologies for general ATO systems. In this section, we review the studies on approximation methods to analyze large-scale ATO systems.

Ceryan et al. (2012) propose to decompose an n -component, $n+1$ -product ATO system into $N+1$ independent problems. By solving each sub-problem, they can determine the base-stock level of each component. Horng and Yang (2012) study a lost sale ATO system with Normally-distributed component lead times and fixed base stock levels. They use combinatorial optimization to formulate the problem. Using a genetic search algorithm, they analyze the computational efficiency and solution quality of the algorithm for a 10-component, 6-product ATO system. Huang et al (2014) study an ATO system with setup costs and simultaneous Kanban and (r, Q) component replenishment policies. Using a combined genetic

search and simulated annealing algorithm, for a 20-component, 2-product system, they show that such algorithm is very efficient in finding the global optimal solution. van Jaarsveld and Scheller-Wolf (2015) use a stochastic programming formulation to determine the state-independent base-stock levels of a large-scale ATO system. They assume product demands are fulfilled in FCFS fashion and when demand cannot be fulfilled immediately it is backordered. By using the column generation method in the integer programming formulation, they show their method can be applied in an industrial scale problem. Nadar et al. (2016) propose a mixed-integer linear programming method to evaluate the performance of lattice-dependent policy, state-dependent policy and fixed policy, in which they show that lattice-dependent policies outperform state-dependent and fixed ones with respect to both objective value and computation time. Reiman et al. (2015) applied a two-stage stochastic programming to set a lower bound on the average inventory cost for general ATO systems with identical component lead times and develop the relevant inventory control policies, which proved to be asymptotically optimal on the diffusion scale.

In conclusion, because of the curse of dimensionality afflicting exact algorithms, researchers propose approximation methods for large-scale ATO problems, which can be solved in polynomial time. These approximation methods depend greatly on the researcher's ingenuity and do not have the same format.

2.3 Conclusion

In this chapter, we compared and analyzed the ATO literature in a systematic way. This helped us to gain insight into the current research progress and challenges facing ATO systems. Specifically, we divided the literature into inventory periodic review ATO models and inventory continuous review ATO models. In periodic review ATO models, we split this stream into one-period ATO models and multi-period ATO models, with single end product or multiple end products, respectively. For continuous review ATO models, we analyzed the literature from optimal policy, heuristic policy and performance metrics, and approximation methods standpoints.

Chapter 3. Problem formulation and numerical example of continuous-time W -configuration ATO systems

The W -configuration ATO system combines both an assembly system and a distribution system, which constitutes a building block for the general multiple-component, multiple-product ATO system. In this chapter, we will analyze such W -configuration ATO systems under the condition that unsatisfied demand is lost. Assuming the production processes follow exponential distributions and demand processes follow Poisson distributions, we will formulate the problem as a MDP and will adopt the Linear Programming (LP) method to solve it. In addition, we will provide a numerical example and will show the form of the optimal policy.

3.1 Introduction

We consider a continuous-time W -configuration ATO system (cf. Figure 3.1). The system consists of three production facilities, each producing a different component. Component 0 is common to both products. Thus, Product j , $j \in \{1, 2\}$, is assembled from Components j and 0. Component i , $i \in \{0, 1, 2\}$, is produced ahead of demand in a make-to-stock fashion. Component i 's production times are independent of the state of the system, are independent of the production status of other components, and are exponentially distributed with mean μ_i^{-1} . Therefore, each production facility can be viewed as a single server queue with finite service rate μ_i . Demand for Product j takes place continuously over time according to an independent Poisson process with rate λ_j . Thus, the inter-arrival times between consecutive demands for each product are exponentially distributed with mean λ_j^{-1} . The assumption that demand inter-arrival times and production times are exponentially distributed, is common in the integrated production/inventory systems literature (for example, Buzacott and Shanthikumar (1993), Ha (1997a, 1997b), Zipkin (2000), and de Véricourt et al. (2002)); and is partly made for mathematical tractability since it allows us to formulate the problem as a Markov Decision Process (MDP) which enables us to characterize the structure of the optimal operating policy.

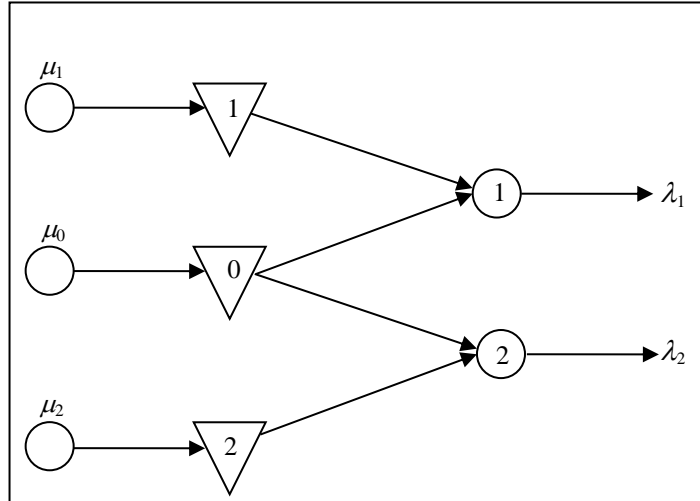


Figure 3.1: A W -configuration ATO system

3.2 Lost sale case

3.2.1 Model formulation of expected total discounted cost criterion

In this section, we assume any unfulfilled demand is considered lost (or must be expedited through other means such as overtime or outsourcing to a third party) incurring a lost sale cost L_j per unit for product j . Demand for, any product, can be satisfied only if there is positive inventory available for its components. Without loss of generality, we assume $L_1 > L_2$.

Since the two products require a common component (Component 0) and their lost sale penalties are different, it might be desirable to reject the demand of one product in order to preserve the common component's available inventory for future demand of the product with the greater lost sale penalty. Thus, when demand for a product materializes, the system manager must decide whether to satisfy it, if the needed components are available, or reject it. Furthermore, at any point in time, if a component is not currently in production, the system manager must decide whether to start its production. If a component is currently in production, the system manager can either continue or interrupt its production. Since component production times are exponentially distributed, and because of the memoryless property of the exponential distribution, continuing production from where it was interrupted is equivalent to starting it from the beginning. Hence, if the production of a component is interrupted, it can be resumed next time the production of that component is initiated. Also, since the decision is

whether to produce Component i , there is at most one outstanding production order, for each component, at any point in time. These assumptions are common in the integrated production/inventory systems literature (for example, Veatch and Wein (1996) and Ha (1997a, 1997b)).

Let $\mathbf{X}(t) = (X_0(t), X_1(t), X_2(t))$ denote the state of the system at time t , where $X_i(t)$ denotes Component i 's inventory at time t . Let $h(\mathbf{X}(t)) = \sum_{i=0}^2 h_i(X_i(t))$, where h_i is an increasing convex function, denote the total inventory holding cost rate when the system is in state $\mathbf{X}(t)$. Given our assumption $L_1 > L_2$, we naturally assume that $h_1(\cdot) > h_2(\cdot)$. Since both demand inter-arrival times and production times are exponentially distributed, the system has no memory. As such, we can restrict decision epochs to only the times when the state of the system changes. Such state changes occur at production completion of a component or the fulfillment of a demand for a product. Using the memoryless property, we model the problem as an MDP and confine our treatment to Markovian policies for which actions taken at each decision epoch depend only on the current state of the system. For each state $\mathbf{x} = (x_0, x_1, x_2)$, a control policy π specifies an action $\mathbf{a} = (\mathbf{u}, \mathbf{w})$, where $\mathbf{u} = (u_0, u_1, u_2)$ and $\mathbf{w} = (w_1, w_2)$. Here, $u_i = 1$ denotes the action ‘‘produce Component i ’’, $u_i = 0$ denotes the action ‘‘do not produce Component i ’’, $w_j = 1$ denotes the action ‘‘satisfy an incoming Product j demand’’, and $w_j = 0$ denotes the action ‘‘reject an incoming Product j demand’’. Let $\mathbf{A}(\mathbf{x})$ denote the set of admissible actions in state \mathbf{x} . Also, for any action $\mathbf{a} \in \mathbf{A}(\mathbf{x})$, we must have (i) $u_i \in \{0, 1\}$, for $i \in \{0, 1, 2\}$, and (ii) $w_j = 0$, if $x_0 x_j = 0$, otherwise $w_j \in \{0, 1\}$, for $j \in \{1, 2\}$.

For a given policy π , we use the expected total discounted cost, over an infinite horizon, as our evaluation criterion. Our treatment extends to the average cost criterion, over an infinite horizon, as will be shown in section 3.2.2.

Let $n_j(t)$, $j \in \{1, 2\}$, denote the cumulative number of Product j demands that have not been satisfied up to time t . The expected total discounted cost $v^\pi(\mathbf{x})$, for a given policy π

and initial state $\mathbf{x} = (x_0, x_1, x_2)$, can be written as

$$v^\pi(\mathbf{x}) = E \left[\int_0^\infty e^{-\alpha t} h(\mathbf{X}(t)) dt + \sum_{j=1}^2 \int_0^\infty e^{-\alpha t} L_j dn_j(t) \mid \mathbf{X}(0) = \mathbf{x} \right] \quad (3.1)$$

where $0 < \alpha < 1$ is the discount rate. Here, $v^\pi(\cdot)$ denotes a real-valued function defined on \mathbb{Z}_+^3 where \mathbb{Z}_+ is the set of nonnegative integers and \mathbb{Z}_+^3 is its three-dimensional cross product.

Since production and demand inter-arrival times are both exponentially distributed, the transition time from a state to another, which represents either a component production completion or a demand arrival, is the minimum of two exponentially distributed random variables, and thus it is exponentially distributed. Also, because a transition to the next state depends only on the current state, we denote by $\beta(\mathbf{a}(\mathbf{x}))$, for $\mathbf{a} \in \mathbf{A}(\mathbf{x})$, the transition rate from state \mathbf{x} to state \mathbf{x}' . Let t_k denote the time of the k^{th} , $k = 0, 1, \dots$, transition occurs with $t_0 = 0$. Then, the state of the system, $\mathbf{X}(t)$, remains the same in the time interval $[t_k, t_{k+1})$. Following Lippman (1975), we use a uniformized transition rate, $\beta = \sum_{i=0}^2 \mu_i + \sum_{j=1}^2 \lambda_j$, such that the times between transitions, $\{t_{k+1} - t_k : k \geq 0\}$, form a sequence of *i.i.d* exponentially distributed random variables, each with mean $1/\beta$. As such, the process $\{\mathbf{Y}_k = \mathbf{X}(t_k) : k \geq 0\}$ is a Markov chain with transition probabilities given by:

$$p_{\mathbf{x}, \mathbf{x}'}(\mathbf{a}(\mathbf{x})) = \begin{cases} \mu_i / \beta & \text{if } \mathbf{x}' = \mathbf{x} + u_i \mathbf{e}_i, \quad i \in \{0, 1, 2\} \\ \lambda_j / \beta & \text{if } \mathbf{x}' = \mathbf{x} - w_j (\mathbf{e}_0 + \mathbf{e}_j), \quad j \in \{1, 2\} \\ 1 - (\mu_i + \lambda_j) / \beta & \text{otherwise} \end{cases} \quad (3.2)$$

Here, \mathbf{e}_i (\mathbf{e}_j) is a three-dimensional vector with 1 in the i^{th} (j^{th}) element and 0 elsewhere.

Using a uniform transition rate allows us to convert the continuous-time problem into an equivalent discrete-time one. In this case, $n_j(t_k)$ represents the cumulative number of product j 's unfulfilled demands at the k^{th} transition, and $h(\mathbf{X}(t_k))$ is the total inventory holding cost incurred over the time interval $[t_k, t_{k+1})$. Because states remain constant between transitions, the value function $v^\pi(\mathbf{x})$ can be written as follows:

$$\begin{aligned}
v^\pi(\mathbf{x}) &= \sum_{k=0}^{\infty} E \left[\int_{t_k}^{t_{k+1}} e^{-\alpha t} h(\mathbf{X}(t)) dt + \sum_{j=1}^2 \int_{t_k}^{t_{k+1}} e^{-\alpha t} L_j dn_j(t) \mid \mathbf{X}(0) = \mathbf{x} \right], \\
&= \sum_{k=0}^{\infty} E \left\{ \int_{t_k}^{t_{k+1}} e^{-\alpha t} dt \right\} E \left\{ h(\mathbf{X}(t_k)) \mid \mathbf{X}(0) = \mathbf{x} \right\} + \sum_{k=0}^{\infty} E \left\{ \int_{t_k}^{t_{k+1}} e^{-\alpha t} dt \right\} E \left\{ \sum_{j=1}^2 L_j d(n_j(t_k) - n_j(t_{k-1})) \right\}.
\end{aligned}$$

Using the fact that transition time intervals, $[t_k, t_{k+1})$, are independent and exponentially distributed, we can write the value function $v^\pi(\mathbf{x})$ as follows:

$$v^\pi(\mathbf{x}) = E \left[\sum_{k=0}^{\infty} \left(\frac{\beta}{\alpha + \beta} \right)^k \frac{h(\mathbf{X}(t_k))}{\alpha + \beta} + \sum_{k=1}^{\infty} \left(\frac{\beta}{\alpha + \beta} \right)^k \sum_{j=1}^2 L_j (n_j(t_k) - n_j(t_{k-1})) \mid \mathbf{X}(0) = \mathbf{x} \right] \quad (3.3)$$

Let π^* be the optimal policy that minimizes the expected total discounted cost over an infinite horizon and $v^* = v^{\pi^*}$. Thus, the optimal cost function, v^* , can be shown to satisfy the following Bellman's optimality equation:

$$v^*(\mathbf{x}) = \min_{\mathbf{a} \in \mathbf{A}(\mathbf{x})} \left[\frac{h(\mathbf{x})}{\alpha + \beta} + \sum_{j=1}^2 \frac{\lambda_j L_j (1 - w_j)}{\alpha + \beta} + \left(\frac{\beta}{\alpha + \beta} \right) \sum_{\mathbf{x}'} P_{\mathbf{x}, \mathbf{x}'}(\mathbf{a}(\mathbf{x})) v^*(\mathbf{x}') \right] \quad (3.4)$$

Without loss of generality, we rescale time by letting $\alpha + \beta = 1$. Using the corresponding transition probabilities and time scaling, we can rewrite (3.4) as follows:

$$v^*(\mathbf{x}) = \min_{\mathbf{a} \in \mathbf{A}(\mathbf{x})} \left[h(\mathbf{x}) + \sum_{j=1}^2 \lambda_j (v^*(\mathbf{x} - w_j(\mathbf{e}_0 + \mathbf{e}_j)) + L_j(1 - w_j)) + \sum_{i=0}^2 \mu_i v^*(\mathbf{x} + u_i \mathbf{e}_i) \right] \quad (3.5)$$

Or equivalently,

$$v^*(\mathbf{x}) = h(\mathbf{x}) + \sum_{j=1}^2 \lambda_j A_j v^*(\mathbf{x}) + \sum_{i=0}^2 \mu_i P_i v^*(\mathbf{x}) \quad (3.6)$$

where, for any real-valued function, $v(\mathbf{x})$, defined on \mathbb{Z}_+^3 , Operator A_j is defined as

$$A_j v(\mathbf{x}) = \begin{cases} v(\mathbf{x}) + L_j & \text{if } x_0 x_j = 0 \\ \min\{v(\mathbf{x} - \mathbf{e}_0 - \mathbf{e}_j), v(\mathbf{x}) + L_j\} & \text{otherwise} \end{cases} \quad (3.7)$$

and Operator P_i is defined as

$$P_i v(\mathbf{x}) = \min\{v(\mathbf{x}), v(\mathbf{x} + \mathbf{e}_i)\} \quad (3.8)$$

For a given state \mathbf{x} , Operator P_i is associated with the decision of whether or not to produce Component i . Operator A_j , upon arrival of Product j 's demand, is associated with the decision of whether or not to fulfill it, if sufficient inventory of Components 0 and j exists. Note that it

is optimal to produce Component i if $v^*(\mathbf{x} + \mathbf{e}_i) \leq v^*(\mathbf{x})$. Similarly, it is optimal to fulfill Product j 's demand if $v^*(\mathbf{x} - \mathbf{e}_0 - \mathbf{e}_j) \leq v^*(\mathbf{x}) + L_j$, provided stock of Components 0 and j is available.

In order to simplify the notation, we define Operator T as follows:

$$\begin{aligned} Tv(\mathbf{x}) &= \min_{\mathbf{a} \in \mathbf{A}(\mathbf{x})} \left[h(\mathbf{x}) + \sum_{j=1}^2 \lambda_j \left(v(\mathbf{x} - w_j \mathbf{E}_j) + L_j(1 - w_j) \right) + \sum_{i=0}^2 \mu_i v(\mathbf{x} + u_i \mathbf{e}_i) \right] \\ &= h(\mathbf{x}) + \sum_{j=1}^2 \lambda_j A_j v(\mathbf{x}) + \sum_{i=0}^2 \mu_i P_i v(\mathbf{x}) \end{aligned} \quad (3.9)$$

where $\mathbf{E}_j = \mathbf{e}_0 + \mathbf{e}_j$, for $j \in \{1, 2\}$. We also let $\mathbf{E} = \mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2$.

For any real valued functions $v(\mathbf{x})$ on \mathbb{Z}_+^3 , we introduce the following difference operators:

$$\Delta_j v(\mathbf{x}) = v(\mathbf{x} + \mathbf{e}_j) - v(\mathbf{x}),$$

$$\Delta_{\mathbf{E}_j} v(\mathbf{x}) = v(\mathbf{x} + \mathbf{E}_j) - v(\mathbf{x}).$$

Here, $\Delta_j v(\mathbf{x})$ represents the cost changes resulting from producing a unit of Component j . It also can be interpreted as the marginal cost of producing a unit of Component j . Thus, $\Delta_{\mathbf{E}_j} v(\mathbf{x})$ represents the marginal cost of simultaneously producing Components 0 and j (or equivalently, ‘‘producing’’ a unit of Product j).

3.2.2 Model formulation of average cost criterion

In this section, we briefly discuss the case where the optimization criterion is the average cost rate over an infinite planning horizon. Under a control policy π and an initial state \mathbf{x} , the average cost rate is given by:

$$v^\pi(\mathbf{x}) = \limsup_{\tau \rightarrow \infty} \frac{1}{\tau} E \left[\int_0^\tau h(\mathbf{X}(t)) dt + \sum_{j=1}^2 \int_0^\tau L_j dn_j(t) \right] \quad (3.10)$$

The objective is to identify a policy π^* that yields $v^*(\mathbf{x}) = \inf_\pi v^\pi(\mathbf{x})$ for all states \mathbf{x} .

The optimal average cost rate, γ^* , is the unique solution together with an associated differential cost function f^* that satisfies the following optimality equations:

$$\gamma^* + f^*(\mathbf{x}) = \min_{\mathbf{a}(\mathbf{x}) \in \mathbf{A}(\mathbf{x})} \left[g(\mathbf{x}, \mathbf{a}(\mathbf{x})) + \sum_{\mathbf{x}'} P_{\mathbf{x}, \mathbf{x}'}(\mathbf{a}(\mathbf{x})) f^*(\mathbf{x}') \right] \quad (3.11)$$

where $g(\mathbf{x}, \mathbf{a}(\mathbf{x}))$ denotes the cost rate incurred when the system is in state \mathbf{x} and action \mathbf{a} is taken. It is given as follows

$$g(\mathbf{x}, \mathbf{a}(\mathbf{x})) = h(\mathbf{x}) + \sum_{j=1}^2 \lambda_j L_j (1 - w_j) \quad (3.12)$$

By the weak accessibility condition, we know there exists an optimal average cost that is same for all initial states (Bertsekas, 2007).

3.2.3 Numerical example

In order to numerically determine the optimal policy, we use the average cost rate criterion as it does not depend on the initial state. Furthermore, we convert the dynamic programming optimization problem into a linear programming optimization one. Manne (1960) was the first to introduce such reformulation. We note that in general the value iteration and policy iteration algorithms (see Puterman 1994) are faster than linear programming. However, in addition to the optimal solution, linear programming provides the steady-state probability distribution of the system. Such distribution can be helpful in designing efficient heuristic policies as we show in the next section.

In an effort to simplify the notation and conform to a linear programming format, we first order the state space in the lexicographic order such that $(0,0,0)$, $(0,0,1)$, $(0,0,2), \dots$, $(0,1,0)$, $(0,1,1), \dots$ is mapped to state $1, 2, 3, \dots$. We further truncate the state space such that $x_i \in [0, n_T]$, and let $\bar{S} = \{1, 2, \dots, N\}$ be the truncated state space after lexicographic ordering and mapping, where $N = (n_T + 1)^3 = |\bar{S}|$. We further order the action space accordingly.

Based on (3.11), we can show that the optimal average cost rate, γ^* , is the solution of the following linear programming problem.

$$\begin{aligned}
& \text{Minimize } \sum_{s \in \bar{S}} \sum_{a \in A(s)} g(s,a) f(s,a) \\
& \text{s.t.} \\
& \sum_{a \in A(s)} f(s,a) = \sum_{s' \in \bar{S}} \sum_{a \in A(s')} p_{s',s}(a) f(s',a), \quad s \in \bar{S} \\
& \sum_{s \in \bar{S}} \sum_{a \in A(s)} f(s,a) = 1, \\
& f(s,a) \geq 0, \quad s \in \bar{S}.
\end{aligned} \tag{3.13}$$

Here, the decision variable $f(s,a)$ denotes the steady state probability of being in state s when action a is taken and $\gamma^* = \sum_{s \in \bar{S}} \sum_{a \in A(s)} g(s,a) f^*(s,a)$ where $f^*(s,a)$, $s \in \bar{S}$, $a \in A(s)$, is the optimal solution to LP (3.13).

Once LP (3.13) is solved, one can obtain several quantities of interest. First, the optimal policy, π^* , is determined as follows:

$$\pi^*(s) = \arg \max_{a \in A(s)} f^*(s,a), \quad s \in \bar{S} \tag{3.14}$$

The stationary probability of being in state s as a result of the optimal policy is given by:

$$f^*(s) = \sum_{a \in A(s)} f^*(s,a) \tag{3.15}$$

Here, we note that for all recurrent states, $f^*(s) > 0$, and for all transient states, $f^*(s) = 0$, for $s \in \bar{S}$.

We should note that in principle, LP (3.13) is solved multiple times starting with an initial truncation level n_T that is gradually increased until the average cost rate is no longer sensitive to the truncation level. However, in practice, choosing a relatively large truncation level n_T usually results in at most two runs of LP (3.13).

Using the LP (3.13), for a state space truncated at a sufficiently large n_T value, we computed the actions minimizing the average cost rate, for $\mu = (1.23, 3.20, 1.83)$, $\lambda = (1.16, 1.79)$, $h = (0.14, 0.06, 0.13)$ and $L = (35.56, 32.60)$ (we assumed linear holding cost rates, *i.e.*, $h_i(x_i) = h_i x_i$, $i \in \{0, 1, 2\}$). Figures 3.2 and 3.3 show the optimal inventory allocation policy for $x_0 = 1$ and $x_0 = 2$, respectively. A “●” indicates that it is optimal to satisfy Product 1 demand in state (x_0, x_1, x_2) . A “○” at coordinates (x_1, x_2) indicates that it is

optimal to satisfy Product 2 demand in state (x_0, x_1, x_2) . Delimiting black lines indicate the recurrent region where it is optimal to produce any component for states in the interior of the region and not produce components for states on the boundaries of the region. The blue (red) lines indicate the projection of the rationing threshold $r_1^1(x_0, x_2)$ ($r_2^2(x_0, x_1)$) onto the two-dimensional space (x_1, x_2) , respectively.

Figures 3.2 and 3.3 reveal that the inventory allocation or rationing policy is not simple and counter intuitive. Since $L_1 > L_2$, one would expect that demand for Product 1 would be always satisfied (as long as components are available) and that demand for Product 2 is rationed. However, this is not the case. Actually, the optimal rationing policy does not give absolute priority to neither product. In other words, for some states, Product 1 demand takes priority over Product 2 demand, and vice versa for other states. To see this, note that for low inventory levels for components 1 and 2, both product demands are satisfied. As the inventory level of Component j ($j \neq 0$) builds up, it is optimal to satisfy Product j demand and reject Product i ($i \neq j$) demand. In this case, the shift in priority can be explained by the fact that for sufficiently large inventory level of Component j , the only way to reduce it is by giving priority to Product j demand. In this case, the holding cost becomes significant enough to offset the lost sale penalty due to Product i 's demand. Rejecting Product i 's demand preserves the inventory level of Component 0 in order to satisfy the demand of Product j . Such a feature has not been observed in many integrated production/inventory problem settings where inventory allocation usually follows a fixed priority in satisfying demands. *i.e.*, classes or products are governed by ordered rationing levels (See Ha (1997a, 1997b), de Vericourt (2002) for example).

One would also expect the optimal rationing policy to give priority to Product 1 over Product 2 in the case $\lambda_1 L_1 > \lambda_2 L_2$, however in many numerical examples we found that it is not the case and the optimal rationing policy still exhibits behavior similar to the one shown in Figures 3.2 and 3.3. This indicates that in general ATO systems the product rationing policy is very complicated and does not lend itself to a simple rule. Furthermore, due to the state dependency of the allocation policy, determining a rule that indicates when Product 1 demand has strict priority over Product 2 demand, and when not, is very challenging. Actually, such

endeavor requires solving the Bellman equation (3.5) to a closed form, which is known to be an impossible task. However, we have noticed that when Product 1's holding and lost sale costs are significantly larger than those of Product 2, then Product 1's demand is always satisfied and Product 2's demand is rationed; and when the products' holding and lost sale costs are close to each other, both product demands maybe rationed. We have also noticed that the above result mostly occurs for small inventory levels of the common component. Of course, the remaining parameters of the system (production and demand rates) also do contribute to when such a result is observed or not. In any case, one has to be careful by taking into account the above result in order to design effective heuristic policies for more general settings.

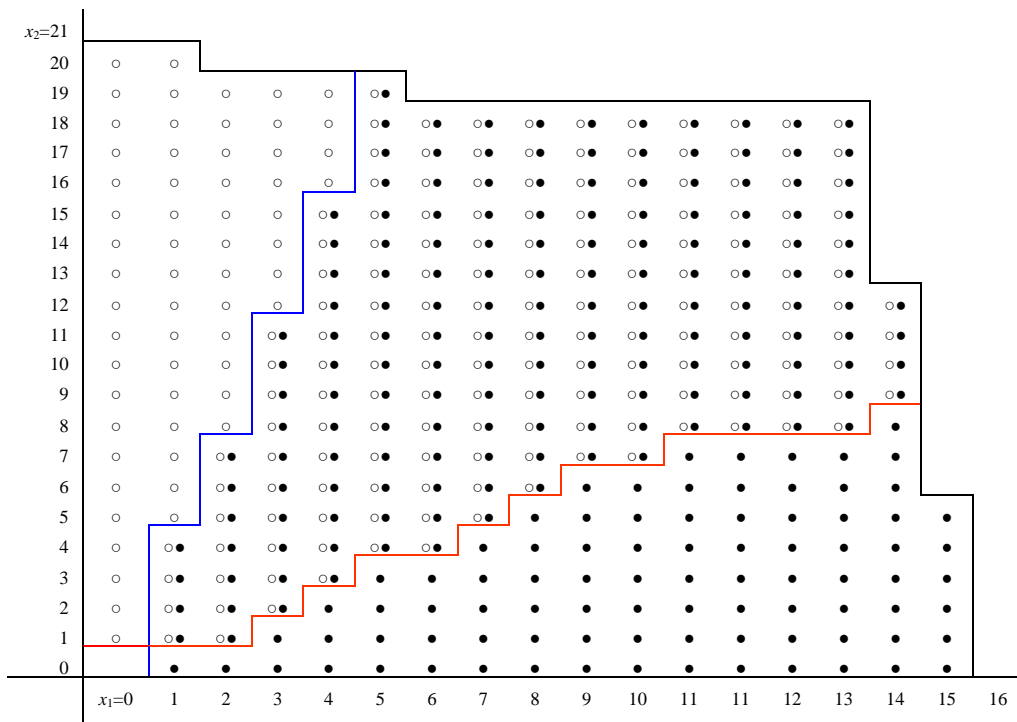


Figure 3.2: The optimal inventory allocation policy for $x_0=1$, $\mu = (1.23, 3.20, 1.83)$, $\lambda = (1.16, 1.79)$, $h = (0.14, 0.06, 0.13)$ and $L = (35.56, 32.60)$

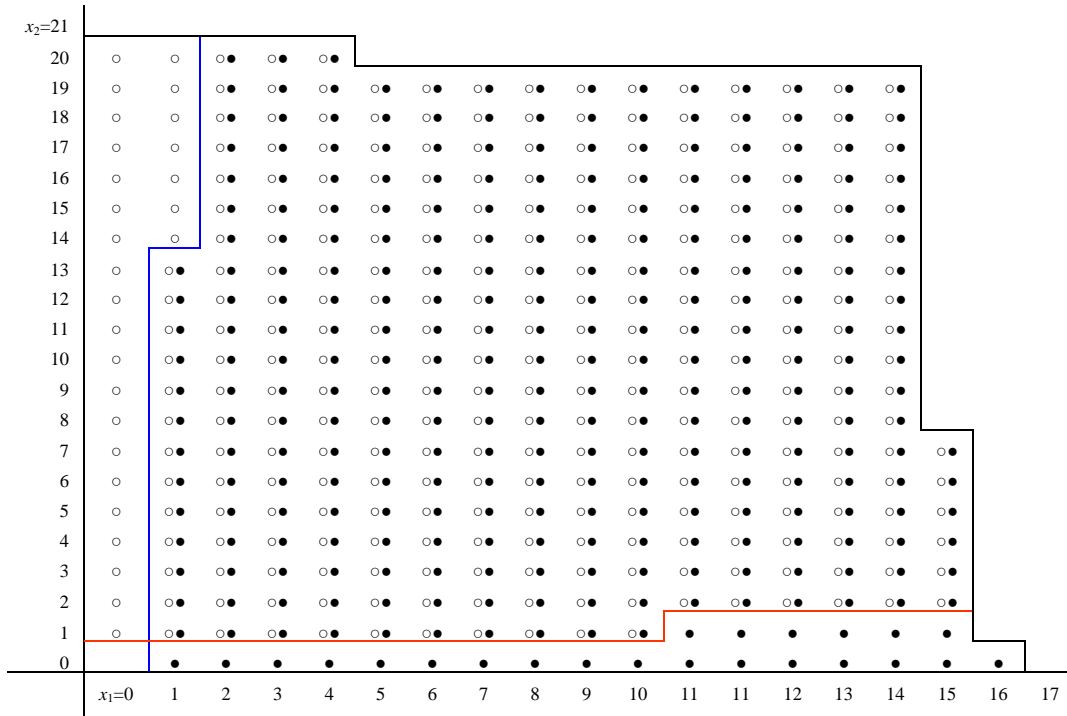


Figure 3.3: The optimal inventory allocation policy for $x_0 = 2$, $\mu = (1.23, 3.20, 1.83)$, $\lambda = (1.16, 1.79)$, $h = (0.14, 0.06, 0.13)$ and $L = (35.56, 32.60)$

3.3 Conclusion

In this chapter, we applied mathematical expressions to model 3 components, 2 end products W -configuration ATO systems. We used a linear programming formulation to solve the system and presented the limiting probability distribution in the recurrent region. The optimal policy revealed that the inventory allocation or rationing policy is not simple and is counter intuitive. Even though Product 1 is more expensive than Product 2 with regard to the lost sale cost, the optimal rationing policy does not give absolute priority to neither product. In addition, we showed that the rationing of the common component always happens at its low inventory level.

Chapter 4. Structure of optimal policy in W -configuration ATO systems with lost sales and its extensions

In this chapter, we identify a set of properties and show that the optimal value function satisfies these properties in the recurrent region, in W -configuration ATO systems under lost sale case. We use a propagation argument to show that these properties are preserved through the optimality equation. This will allow us to characterize the structure of optimal inventory and allocation policy. We then show that the optimal policy structure is preserved for systems with batch production, non-unitary compound demand, and Erlang distributed production times.

4.1 Structure of optimal policy

In this section, we provide a detailed characterization of the optimal production and inventory allocation policy. Similar to the literature on integrated production/inventory control, we use the approach developed by Porteus (1982), which consists in identifying a set of structural properties and then showing, using a propagation argument, that these properties are preserved. However, because in our case some of the properties do not hold for all system states, we adopt an approach that is different from the standard methodology used in the related literature: First, we characterize a compact sub-space of \mathbb{Z}_+^3 which we identify as the recurrent region. Then, we show that for any state within the recurrent region, the optimal cost value function satisfies a set of properties. This will allow us to infer the structure of the optimal component production and inventory allocation policy within the recurrent region. We note that some of the properties that hold within the recurrent region do not hold in the entire state space. However, we will show that under the optimal policy, it does not matter where the system starts, it always ends up in the recurrent region after a sufficient amount of time. Specifically, our approach is to first define the hypersurfaces (or base-stock thresholds) defining the boundaries of the recurrent region. Then, using certain properties, we show that these boundaries are unique and thus enclose a compact sub-space of \mathbb{Z}_+^3 .

In order to identify the recurrent region, we first define the following two-dimensional hypersurfaces (two-dimensional thresholds):

$$S_i(\tilde{\mathbf{x}}_i) = \min\{x_i \in \mathbb{Z}_+ \mid v(\mathbf{x} + \mathbf{e}_i) - v(\mathbf{x}) \geq 0\}, \quad (4.1)$$

where $\tilde{\mathbf{x}}_i$ is a two-dimensional vector obtained by dropping the i^{th} element from vector \mathbf{x} .

Practically, $S_i(\tilde{\mathbf{x}}_i)$ defines the level at which production of Component i should be stopped.

Since beyond this level, the cost of adding one unit of Component i , $v(\mathbf{x} + \mathbf{e}_i)$, becomes larger

than the cost of not producing the component, $v(\mathbf{x})$. $S_i(\tilde{\mathbf{x}}_i)$ is referred to as the base-stock

level of Component i . Let $\mathbf{S}(\mathbf{x}) = (S_1(\tilde{\mathbf{x}}_1), S_2(\tilde{\mathbf{x}}_2), S_3(\tilde{\mathbf{x}}_3))$ and define the sub-space $\mathfrak{S}_{\mathbf{a}}$ as

follows

$$\mathfrak{S}_{\mathbf{a}} = \{\mathbf{x} \in \mathbb{Z}_+^3 : \mathbf{x} \leq \mathbf{S}(\mathbf{x})\}. \quad (4.2)$$

Here, the inequality is taken vector component-wise. Using Optimality equation (3.6) and operators P_i , in particular, we note that it is always optimal to produce any of the components when the state of the system, \mathbf{x} , is in the interior of the sub-space $\mathfrak{S}_{\mathbf{a}}$.

Let \mathcal{V} be the set of real-valued functions v on $\mathfrak{S}_{\mathbf{a}}$ that satisfy the following properties:

Property 1: $\Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x}) \geq 0$, for $j \in \{1, 2\}$;

Property 2: $\Delta_j v(\mathbf{x} + \mathbf{e}_0) - \Delta_j v(\mathbf{x}) \leq 0$, for $j \in \{1, 2\}$;

Property 3: $\Delta_j v(\mathbf{x} + \mathbf{e}_i) - \Delta_j v(\mathbf{x}) \geq 0$, for $i \neq j$, and $i, j \in \{1, 2\}$;

Property 4: $\Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i) - \Delta_{\mathbf{E}_j} v(\mathbf{x}) \geq 0$, for $i \neq j$, and $i, j \in \{1, 2\}$;

Property 5: $\Delta_j v(\mathbf{x} + \mathbf{E}_i) - \Delta_j v(\mathbf{x}) \leq 0$, for $i \neq j$, and $i, j \in \{1, 2\}$.

Property 1 implies that the marginal cost of an ‘‘additional unit’’ of Product j (i.e. a one unit simultaneous increase of both Component 0 and Component j) is non-increasing in the stock level of Component j . Property 2 implies that the marginal cost of an additional unit of Component j is non-decreasing in the stock level of Component 0. Property 3 implies that the marginal cost of an additional unit of Component j is non-increasing in the stock level of Component i . Property 4 implies that the marginal cost of an additional unit of Product j is non-increasing in an increase in the ‘‘stock level’’ of Product i . Property 5 implies that the

marginal cost of an additional unit of Component j is non-decreasing in the “stock level” of Product i . Mathematically, Properties 1, 3, and 4 imply that the cost function v is supermodular in the directions $(\mathbf{E}_j, \mathbf{e}_j)$, $(\mathbf{e}_j, \mathbf{e}_i)$, and $(\mathbf{E}_j, \mathbf{E}_i)$, respectively. Properties 2 and 5 imply that the cost function v is submodular in the directions $(\mathbf{e}_j, \mathbf{e}_0)$ and $(\mathbf{e}_j, \mathbf{E}_i)$, respectively.

A real-valued function v is said to be convex in the direction \mathbf{e}_j (or point-wise convex in coordinate x_j) if and only if the following holds

$$\Delta_j v(\mathbf{x} + \mathbf{e}_j) - \Delta_j v(\mathbf{x}) \geq 0 \quad (4.3)$$

The following observations show that for any real-valued function v on $\mathfrak{S}_{\mathfrak{R}}$, v is point-wise convex in coordinate x_i , $i \in \{0, 1, 2\}$.

Observation 4.1: for any $\mathbf{x} \in \mathfrak{S}_{\mathfrak{R}}$ and $v \in \mathfrak{V}$, we have

$$\Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_0) - \Delta_{\mathbf{E}_j} v(\mathbf{x}) \geq 0, \text{ for } j \in \{1, 2\}.$$

Proof: By Properties 4 and 5, we have

$$\Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_0) \geq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_i) = \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i) \geq \Delta_{\mathbf{E}_j} v(\mathbf{x}), \text{ for } i \neq j, \text{ and } i, j \in \{1, 2\}. \quad \square$$

Observation 4.2: for any $\mathbf{x} \in \mathfrak{S}_{\mathfrak{R}}$ and $v \in \mathfrak{V}$, v is point-wise convex in x_i , $i \in \{0, 1, 2\}$.

Proof: Using Properties 1 and 2, we have

$$\Delta_j v(\mathbf{x} + \mathbf{e}_j) \geq \Delta_j v(\mathbf{x} + \mathbf{e}_j + \mathbf{e}_0) = \Delta_j v(\mathbf{x} + \mathbf{E}_j) \geq \Delta_j v(\mathbf{x}).$$

which establishes convexity in the direction \mathbf{e}_j for $j \in \{1, 2\}$. To establish convexity in the

direction \mathbf{e}_0 , using Observation 1 and Property 2, we have

$$\Delta_0 v(\mathbf{x}) \leq \Delta_0 v(\mathbf{x} + \mathbf{E}_j) = \Delta_0 v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j) \leq \Delta_0 v(\mathbf{x} + \mathbf{e}_0). \quad \square$$

Observation 4.3: $\Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x}) \leq 0$.

Proof:

$$\begin{aligned}
\Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x}) &= \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i) + \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i) - \Delta_{\mathbf{E}_j} v(\mathbf{x}) \\
&= \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{e}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i) + \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i) - \Delta_{\mathbf{E}_j} v(\mathbf{x}) \\
&\leq 0.
\end{aligned}$$

The last inequality is due to properties 1 and 5. \square

Lemma 4.1 establishes the structural properties of the optimal cost function.

Lemma 4.1: For any cost function $v \in \mathcal{V}$ and state $\mathbf{x} \in \mathfrak{S}_{\mathbf{g}}$, Tv is also an element of \mathcal{V} . In addition, the optimal cost function, v^* , defined in (3.6) is an element of \mathcal{V} .

Proof: In order to prove Lemma 4.1, we first show that Operator A_k and the holding cost function $h(\mathbf{x})$ both preserve properties 1-5. Then, since set \mathcal{V} is closed under multiplication by a scalar and under addition, we show that Operator Tv preserves properties 1-5 as well. Finally, to show that $v^* \in \mathcal{V}$, we use the fact that (1) $v^* = \lim_{n \rightarrow \infty} T^n v$ for any $v \in \mathcal{V}$ (see Proposition 3.1.5 and 3.1.6, Bertsekas (2001)), and (2) $T^n v \in \mathcal{V}$ since $Tv \in \mathcal{V}$.

Property 1

In order to prove Operator $A_k v(\mathbf{x})$ satisfies property 1, we need to show $\Delta_{\mathbf{E}_j} A_k v(\mathbf{x} + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} A_k v(\mathbf{x}) \geq 0$ for $j, k \in \{1, 2\}$. By the definition of allocation operator, we know

$$\begin{aligned}
\Delta_{\mathbf{E}_j} A_k v(\mathbf{x} + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} A_k v(\mathbf{x}) &= \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_k + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_k) + \min\{\Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k + \mathbf{E}_j + \mathbf{e}_j) + L_k, 0\} \\
&\quad - \min\{\Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k + \mathbf{e}_j) + L_k, 0\} - \min\{\Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k + \mathbf{E}_j) + L_k, 0\} \\
&\quad + \min\{\Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k) + L_k, 0\}
\end{aligned}$$

We discuss from the following two cases.

Case 1: $j = k$

In this case, we know

$$\begin{aligned}
\Delta_{\mathbf{E}_j} A_j v(\mathbf{x} + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} A_j v(\mathbf{x}) &= \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + \min\{\Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_j) + L_j, 0\} \\
&\quad - \min\{\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_j) + L_j, 0\} - \min\{\Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j, 0\} + \min\{\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j, 0\}
\end{aligned}$$

By property 1 and observation 4.1, we know

$\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_j) \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_j)$, which leads to the following five subcases.

$$1. \quad 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_j) + L_j$$

$$\Rightarrow \Delta_{\mathbf{E}_j} A_j v(\mathbf{x} + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} A_j v(\mathbf{x}) = \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) \geq 0$$

The above inequality is held by property 1.

$$2. \quad \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_j) + L_j$$

$$\Rightarrow \Delta_{\mathbf{E}_j} A_j v(\mathbf{x} + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} A_j v(\mathbf{x}) = \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j$$

$$= \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_j) + L_j \geq 0$$

The above inequality is held by the assumption.

$$3. \quad \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_j) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_j) + L_j$$

$$\Rightarrow \Delta_{\mathbf{E}_j} A_j v(\mathbf{x} + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} A_j v(\mathbf{x}) = \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_j) = 0$$

$$4. \quad \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_j) + L_j$$

$$\Rightarrow \Delta_{\mathbf{E}_j} A_j v(\mathbf{x} + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} A_j v(\mathbf{x}) = -(\Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j) \geq 0$$

The above inequality is held by the assumption.

$$5. \quad \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_j) + L_j \leq 0$$

$$\Rightarrow \Delta_{\mathbf{E}_j} A_j v(\mathbf{x} + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} A_j v(\mathbf{x}) = -\Delta_{\mathbf{E}_j} v(\mathbf{x}) + \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_j) \geq 0$$

The above inequality is held by property 1.

Case 2: $j \neq k$

By property 4 and 5, we know $\Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k + \mathbf{e}_j) \leq \Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k) \leq \Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k + \mathbf{E}_j)$ and $\Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k + \mathbf{e}_j) \leq \Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k + \mathbf{E}_j + \mathbf{e}_j) \leq \Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k + \mathbf{E}_j)$ holds, which leads to the following six subcases.

$$1. \quad 0 \leq \Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k + \mathbf{e}_j) + L_k \leq \Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k) + L_k \leq \Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k + \mathbf{E}_j) + L_k \quad \text{and}$$

$$\begin{aligned}
&\Rightarrow \Delta_{\mathbf{E}_j} A_k v(\mathbf{x} + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} A_k v(\mathbf{x}) = \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_k + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_k) - \Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k + \mathbf{e}_j) + \Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k) \\
&\quad + \Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k + \mathbf{E}_j + \mathbf{e}_j) - \Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k + \mathbf{E}_j) \\
&= \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x}) \geq 0
\end{aligned}$$

The above inequality is held by property 1.

Property 2

In order to show that Operator $A_k v(\mathbf{x})$ satisfies property 2, we must show

$$\Delta_j A_k v(\mathbf{x} + \mathbf{e}_0) - \Delta_j A_k v(\mathbf{x}) \leq 0 \quad \text{for } j, k = \{1, 2\}.$$

By the definition of allocation operator, we know

$$\begin{aligned}
\Delta_j A_k v(\mathbf{x} + \mathbf{e}_0) - \Delta_j A_k v(\mathbf{x}) &= \Delta_j v(\mathbf{x} - \mathbf{E}_k + \mathbf{e}_0) - \Delta_j v(\mathbf{x} - \mathbf{E}_k) + \min\{\Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k + \mathbf{e}_0 + \mathbf{e}_j) + L_k, 0\} \\
&\quad - \min\{\Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k + \mathbf{e}_0) + L_k, 0\} - \min\{\Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k + \mathbf{e}_j) + L_k, 0\} \\
&\quad + \min\{\Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k) + L_k, 0\}
\end{aligned}$$

We discuss from the following two cases.

Case 1: $j = k$

$$\begin{aligned}
\Delta_j A_j v(\mathbf{x} + \mathbf{e}_0) - \Delta_j A_j v(\mathbf{x}) &= \Delta_j v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_0) - \Delta_j v(\mathbf{x} - \mathbf{E}_j) + \min\{\Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j, 0\} - \min\{\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_j) + L_j, 0\} \\
&\quad - \min\{\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) + L_j, 0\} + \min\{\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j, 0\}
\end{aligned}$$

By property 1 and observation 4.1, we know $\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_j) \leq \Delta_{\mathbf{E}_j} v(\mathbf{x})$ and

$\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) \leq \Delta_{\mathbf{E}_j} v(\mathbf{x})$, which leads to the following six subcases.

$$1. \quad 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j \quad \text{and}$$

$$0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j$$

$$\Rightarrow \Delta_j A_j v(\mathbf{x} + \mathbf{e}_0) - \Delta_j A_j v(\mathbf{x}) = \Delta_j v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_0) - \Delta_j v(\mathbf{x} - \mathbf{E}_j) \leq 0$$

The above inequality is held by property 2.

$$2. \quad \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j \quad \text{and}$$

$$\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j$$

$$\Rightarrow \Delta_j A_j v(\mathbf{x} + \mathbf{e}_0) - \Delta_j A_j v(\mathbf{x}) = \Delta_j v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_0) - \Delta_j v(\mathbf{x} - \mathbf{E}_j) + \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq 0$$

The above inequality is held by property 2 and the assumption.

$$3. \quad \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_j) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j \quad \text{and}$$

$$\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j$$

$$\Rightarrow \Delta_j A_j v(\mathbf{x} + \mathbf{e}_0) - \Delta_j A_j v(\mathbf{x}) = \Delta_j v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_0) - \Delta_j v(\mathbf{x} - \mathbf{E}_j) + \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_j) \leq 0$$

The above inequality is held by property 2 and observation 4.1.

$$4. \quad \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j \quad \text{and}$$

$$\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j$$

$$\Rightarrow \Delta_j A_j v(\mathbf{x} + \mathbf{e}_0) - \Delta_j A_j v(\mathbf{x}) = \Delta_j v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_0) - \Delta_j v(\mathbf{x} - \mathbf{E}_j) + \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) \leq 0$$

The above inequality is held by property 1 and 2.

$$5. \quad \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_j) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j \quad \text{and}$$

$$\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j$$

$$\begin{aligned} \Rightarrow \Delta_j A_j v(\mathbf{x} + \mathbf{e}_0) - \Delta_j A_j v(\mathbf{x}) &= \Delta_j v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_0) - \Delta_j v(\mathbf{x} - \mathbf{E}_j) + \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) \\ &\quad - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_j) + \Delta_{\mathbf{E}_j} v(\mathbf{x}) - \Delta_{\mathbf{E}_j} v(\mathbf{x}) - L_j \\ &= \Delta_j v(\mathbf{x} + \mathbf{e}_0) - \Delta_j v(\mathbf{x}) - \left(\Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j \right) \leq 0 \end{aligned}$$

The above inequality is held by property 2 and the assumption.

$$6. \quad \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j \leq 0 \quad \text{and}$$

$$\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j \leq 0$$

$$\begin{aligned} \Rightarrow \Delta_j A_j v(\mathbf{x} + \mathbf{e}_0) - \Delta_j A_j v(\mathbf{x}) &= \Delta_j v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_0) - \Delta_j v(\mathbf{x} - \mathbf{E}_j) + \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) \\ &\quad - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_j) + \Delta_{\mathbf{E}_j} v(\mathbf{x}) \\ &= \Delta_j v(\mathbf{x} + \mathbf{e}_0) - \Delta_j v(\mathbf{x}) \leq 0 \end{aligned}$$

The above inequality is held by property 2.

Case 2: $j \neq k$

By property 4 and 5, we know

$$\Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k + \mathbf{e}_j) \leq \Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k) \leq \Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k + \mathbf{e}_0 + \mathbf{e}_j) \leq \Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k + \mathbf{e}_0), \text{ which leads to the}$$

In order to show that Operator $A_k v(\mathbf{x})$ satisfies property 3, we must show that

$$\Delta_j A_k v(\mathbf{x} + \mathbf{e}_i) - \Delta_j A_k v(\mathbf{x}) \geq 0 \text{ for } i \neq j \text{ and } i, j, k = \{1, 2\}.$$

By the definition of allocation operator, we know

$$\begin{aligned} \Delta_j A_k v(\mathbf{x} + \mathbf{e}_i) - \Delta_j A_k v(\mathbf{x}) &= \Delta_j v(\mathbf{x} - \mathbf{E}_k + \mathbf{e}_i) - \Delta_j v(\mathbf{x} - \mathbf{E}_k) + \min \left\{ \Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k + \mathbf{e}_i + \mathbf{e}_j) + L_k, 0 \right\} \\ &\quad - \min \left\{ \Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k + \mathbf{e}_i) + L_k, 0 \right\} - \min \left\{ \Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k + \mathbf{e}_j) + L_k, 0 \right\} \\ &\quad + \min \left\{ \Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k) + L_k, 0 \right\} \end{aligned}$$

We discuss from the following two cases.

Case 1: $j = k$

$$\begin{aligned} \Delta_j A_j v(\mathbf{x} + \mathbf{e}_i) - \Delta_j A_j v(\mathbf{x}) &= \Delta_j v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) - \Delta_j v(\mathbf{x} - \mathbf{E}_j) + \min \left\{ \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0 + \mathbf{e}_i) + L_j, 0 \right\} \\ &\quad - \min \left\{ \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) + L_j, 0 \right\} - \min \left\{ \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) + L_j, 0 \right\} \\ &\quad + \min \left\{ \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j, 0 \right\} \end{aligned}$$

By property 1 and 5, we know $\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0 + \mathbf{e}_i) \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0)$ and

$\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0)$, which leads to the following six subcases.

1. $0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0 + \mathbf{e}_i) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) + L_j$ and

$$\begin{aligned} 0 &\leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) + L_j \\ \Rightarrow \Delta_j A_j v(\mathbf{x} + \mathbf{e}_i) - \Delta_j A_j v(\mathbf{x}) &= \Delta_j v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) - \Delta_j v(\mathbf{x} - \mathbf{E}_j) \geq 0 \end{aligned}$$

The above inequality is held by property 3.

2. $\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0 + \mathbf{e}_i) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) + L_j$ and

$$\begin{aligned} \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) + L_j &\leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) + L_j \\ \Rightarrow \Delta_j A_j v(\mathbf{x} + \mathbf{e}_i) - \Delta_j A_j v(\mathbf{x}) &= \Delta_j v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) - \Delta_j v(\mathbf{x} - \mathbf{E}_j) - \left(\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) + L_j \right) \geq 0 \end{aligned}$$

The above inequality is held by property 3 and the assumption.

3. $\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0 + \mathbf{e}_i) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) + L_j$ and

$$\begin{aligned} \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) + L_j &\leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) + L_j \\ \Rightarrow \Delta_j A_j v(\mathbf{x} + \mathbf{e}_i) - \Delta_j A_j v(\mathbf{x}) &= \Delta_j v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) - \Delta_j v(\mathbf{x} - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) + \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0 + \mathbf{e}_i) \geq 0 \end{aligned}$$

The above inequality is held by property 1 and 3.

$$4. \quad \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0 + \mathbf{e}_i) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) + L_j \quad \text{and}$$

$$\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) + L_j$$

$$\Rightarrow \Delta_j A_j v(\mathbf{x} + \mathbf{e}_i) - \Delta_j A_j v(\mathbf{x}) = \Delta_j v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) - \Delta_j v(\mathbf{x} - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) + \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) \geq 0$$

The above inequality is held by property 3 and 5.

$$5. \quad \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0 + \mathbf{e}_i) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) + L_j \quad \text{and}$$

$$\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) + L_j$$

$$\begin{aligned} \Rightarrow \Delta_j A_j v(\mathbf{x} + \mathbf{e}_i) - \Delta_j A_j v(\mathbf{x}) &= \Delta_j v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) - \Delta_j v(\mathbf{x} - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) \\ &\quad + \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0 + \mathbf{e}_i) + \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \\ &= \Delta_j v(\mathbf{x} + \mathbf{e}_i) - \Delta_j v(\mathbf{x}) + \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) + L_j \\ &\geq 0 \end{aligned}$$

The above inequality is held by property 3 and the assumption.

$$6. \quad \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0 + \mathbf{e}_i) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) + L_j \leq 0 \quad \text{and}$$

$$\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) + L_j \leq 0$$

$$\begin{aligned} \Rightarrow \Delta_j A_j v(\mathbf{x} + \mathbf{e}_i) - \Delta_j A_j v(\mathbf{x}) &= \Delta_j v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) - \Delta_j v(\mathbf{x} - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j + \mathbf{e}_i) \\ &\quad + \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0 + \mathbf{e}_i) + \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{e}_0) \\ &= \Delta_j v(\mathbf{x} + \mathbf{e}_i) - \Delta_j v(\mathbf{x}) \\ &\geq 0 \end{aligned}$$

The above inequality is followed by property 3.

Case 2: $j \neq k$

In this case, if $j \neq k$, we know $i = k$. By the definition of allocation operator, we know

$$\begin{aligned} \Delta_j A_i v(\mathbf{x} + \mathbf{e}_i) - \Delta_j A_i v(\mathbf{x}) &= \Delta_j v(\mathbf{x} - \mathbf{E}_i + \mathbf{e}_i) - \Delta_j v(\mathbf{x} - \mathbf{E}_i) + \min\{\Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{e}_0 + \mathbf{e}_j) + L_i, 0\} \\ &\quad - \min\{\Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{e}_0) + L_i, 0\} - \min\{\Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i + \mathbf{e}_j) + L_i, 0\} \\ &\quad + \min\{\Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) + L_i, 0\} \end{aligned}$$

By property 1 and 5, we know $\Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i + \mathbf{e}_j) \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{e}_0 + \mathbf{e}_j) \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{e}_0)$ and

$\Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i + \mathbf{e}_j) \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{e}_0)$, which leads to the following six subcases.

$$\begin{aligned}
\Rightarrow \Delta_j A_i v(\mathbf{x} + \mathbf{e}_i) - \Delta_j A_i v(\mathbf{x}) &= \Delta_j v(\mathbf{x} - \mathbf{E}_i + \mathbf{e}_i) - \Delta_j v(\mathbf{x} - \mathbf{E}_i) - \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i + \mathbf{e}_j) + \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) \\
&\quad + \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{e}_0 + \mathbf{e}_j) - \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{e}_0) \\
&= \Delta_j v(\mathbf{x} + \mathbf{e}_i) - \Delta_j v(\mathbf{x}) \geq 0
\end{aligned}$$

The above inequality is held by property 3.

Property A4

In order to show that Operator $A_k v(\mathbf{x})$ satisfies property 4, we must show that

$$\Delta_{\mathbf{E}_j} A_k v(\mathbf{x} + \mathbf{E}_i) - \Delta_{\mathbf{E}_j} A_k v(\mathbf{x}) \geq 0 \text{ for } i \neq j \text{ and } i, j, k = \{1, 2\}.$$

By the definition of allocation operator, we know

$$\begin{aligned}
\Delta_{\mathbf{E}_j} A_k v(\mathbf{x} + \mathbf{E}_i) - \Delta_{\mathbf{E}_j} A_k v(\mathbf{x}) &= \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i - \mathbf{E}_k) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_k) + \min\{\Delta_{\mathbf{E}_k} v(\mathbf{x} + \mathbf{E}_i + \mathbf{E}_j - \mathbf{E}_k) + L_k, 0\} \\
&\quad - \min\{\Delta_{\mathbf{E}_k} v(\mathbf{x} + \mathbf{E}_i - \mathbf{E}_k) + L_k, 0\} - \min\{\Delta_{\mathbf{E}_k} v(\mathbf{x} + \mathbf{E}_j - \mathbf{E}_k) + L_k, 0\} \\
&\quad + \min\{\Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k) + L_k, 0\}
\end{aligned}$$

We discuss from the following two cases.

Case 1: $j = k$

$$\begin{aligned}
\Delta_{\mathbf{E}_j} A_k v(\mathbf{x} + \mathbf{E}_i) - \Delta_{\mathbf{E}_j} A_k v(\mathbf{x}) &= \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + \min\{\Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i) + L_j, 0\} \\
&\quad - \min\{\Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i - \mathbf{E}_j) + L_j, 0\} - \min\{\Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j, 0\} \\
&\quad + \min\{\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j, 0\}
\end{aligned}$$

By property 4 and observation 4.3, we know

$$\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i - \mathbf{E}_j) \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i),$$

which leads to the following five subcases.

$$1. \quad 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i) + L_j$$

$$\Rightarrow \Delta_{\mathbf{E}_j} A_k v(\mathbf{x} + \mathbf{E}_i) - \Delta_{\mathbf{E}_j} A_k v(\mathbf{x}) = \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) \geq 0$$

The above inequality is held by property 4.

$$2. \quad \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i) + L_j$$

$$\begin{aligned}
&\Rightarrow \Delta_{\mathbf{E}_j} A_k v(\mathbf{x} + \mathbf{E}_i) - \Delta_{\mathbf{E}_j} A_k v(\mathbf{x}) = \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \\
&\quad = \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i - \mathbf{E}_j) + L_j \\
&\quad \geq 0
\end{aligned}$$

The above inequality is held by the assumption.

$$3. \quad \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i - \mathbf{E}_j) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i) + L_j$$

$$\Rightarrow \Delta_{\mathbf{E}_j} R_k v(\mathbf{x} + \mathbf{E}_i) - \Delta_{\mathbf{E}_j} R_k v(\mathbf{x}) = \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i - \mathbf{E}_j) = 0$$

$$4. \quad \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i) + L_j$$

$$\Rightarrow \Delta_{\mathbf{E}_j} A_k v(\mathbf{x} + \mathbf{E}_i) - \Delta_{\mathbf{E}_j} A_k v(\mathbf{x}) = -(\Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j) \geq 0$$

The above inequality is held by the assumption.

$$5. \quad \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i) + L_j \leq 0$$

$$\Rightarrow \Delta_{\mathbf{E}_j} A_k v(\mathbf{x} + \mathbf{E}_i) - \Delta_{\mathbf{E}_j} A_k v(\mathbf{x}) = -\Delta_{\mathbf{E}_j} v(\mathbf{x}) + \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{E}_i) \geq 0$$

The above inequality is held by property 4.

Case 2: $j \neq k$

In this case, we know $i = k$. By the definition of allocation operator, we know

$$\begin{aligned}
\Delta_{\mathbf{E}_j} A_i v(\mathbf{x} + \mathbf{E}_i) - \Delta_{\mathbf{E}_j} A_i v(\mathbf{x}) &= \Delta_{\mathbf{E}_j} v(\mathbf{x}) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_i) + \min\{\Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j) + L_i, 0\} \\
&\quad - \min\{\Delta_{\mathbf{E}_i} v(\mathbf{x}) + L_i, 0\} - \min\{\Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j - \mathbf{E}_i) + L_i, 0\} \\
&\quad + \min\{\Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) + L_i, 0\}
\end{aligned}$$

By property 4 and observation 4.3, we know

$$\Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j - \mathbf{E}_i) \leq \Delta_{\mathbf{E}_i} v(\mathbf{x}) \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j),$$

which leads to the following five subcases.

$$1. \quad 0 \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j - \mathbf{E}_i) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x}) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j) + L_i$$

$$\Rightarrow \Delta_{\mathbf{E}_j} A_i v(\mathbf{x} + \mathbf{E}_i) - \Delta_{\mathbf{E}_j} A_i v(\mathbf{x}) = \Delta_{\mathbf{E}_j} v(\mathbf{x}) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_i) \geq 0$$

The above equation is held by property 4.

$$2. \quad \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) + L_i \leq 0 \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j - \mathbf{E}_i) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x}) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j) + L_i$$

$$\Rightarrow \Delta_{\mathbf{E}_j} A_i v(\mathbf{x} + \mathbf{E}_i) - \Delta_{\mathbf{E}_j} A_i v(\mathbf{x}) = \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_i \geq 0$$

The above inequality is held by the assumption.

$$3. \quad \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j - \mathbf{E}_i) + L_i \leq 0 \leq \Delta_{\mathbf{E}_i} v(\mathbf{x}) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j) + L_i$$

$$\Rightarrow \Delta_{\mathbf{E}_j} A_i v(\mathbf{x} + \mathbf{E}_i) - \Delta_{\mathbf{E}_j} A_i v(\mathbf{x}) = \Delta_{\mathbf{E}_j} v(\mathbf{x}) - \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j - \mathbf{E}_i) \geq 0$$

The above inequality is held by observation 4.3.

$$4. \quad \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j - \mathbf{E}_i) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x}) + L_i \leq 0 \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j) + L_i$$

$$\Rightarrow \Delta_{\mathbf{E}_j} A_i v(\mathbf{x} + \mathbf{E}_i) - \Delta_{\mathbf{E}_j} A_i v(\mathbf{x}) = -\Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j - \mathbf{E}_i) - L_i \geq 0$$

The above inequality is held by the assumption.

$$5. \quad \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j - \mathbf{E}_i) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x}) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j) + L_i \leq 0$$

$$\Rightarrow \Delta_{\mathbf{E}_j} A_i v(\mathbf{x} + \mathbf{E}_i) - \Delta_{\mathbf{E}_j} A_i v(\mathbf{x}) = \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j) - \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j - \mathbf{E}_i) \geq 0$$

The above inequality is held by property 1.

Property A5

In order to show that Operator $A_k v(\mathbf{x})$ satisfies property 5, we must show that

$$\Delta_{\mathbf{E}_j} A_k v(\mathbf{x} + \mathbf{e}_i) - \Delta_{\mathbf{E}_j} A_k v(\mathbf{x}) \leq 0 \quad \text{for } i \neq j \text{ and } i, j, k = \{1, 2\}.$$

By the definition of allocation operator, we know

$$\begin{aligned} \Delta_{\mathbf{E}_j} A_k v(\mathbf{x} + \mathbf{e}_i) - \Delta_{\mathbf{E}_j} A_k v(\mathbf{x}) &= \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_k) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_k) + \min\{\Delta_{\mathbf{E}_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{E}_j - \mathbf{E}_k) + L_k, 0\} \\ &\quad - \min\{\Delta_{\mathbf{E}_k} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_k) + L_k, 0\} - \min\{\Delta_{\mathbf{E}_k} v(\mathbf{x} + \mathbf{E}_j - \mathbf{E}_k) + L_k, 0\} \\ &\quad + \min\{\Delta_{\mathbf{E}_k} v(\mathbf{x} - \mathbf{E}_k) + L_k, 0\} \end{aligned}$$

We discuss from the following two cases.

Case 1: $j = k$

$$\begin{aligned} \Delta_{\mathbf{E}_j} A_j v(\mathbf{x} + \mathbf{e}_i) - \Delta_{\mathbf{E}_j} A_j v(\mathbf{x}) &= \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + \min\{\Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i) + L_j, 0\} \\ &\quad - \min\{\Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_j) + L_j, 0\} - \min\{\Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j, 0\} \\ &\quad + \min\{\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j, 0\} \end{aligned}$$

By property 4 and 5, we know $\Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_j) \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i) \leq \Delta_{\mathbf{E}_j} v(\mathbf{x})$ and

$\Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_j) \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) \leq \Delta_{\mathbf{E}_j} v(\mathbf{x})$, which leads to the following six subcases.

1. $0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j$ and

$$0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j$$

$$\Rightarrow \Delta_{\mathbf{E}_j} A_j v(\mathbf{x} + \mathbf{e}_i) - \Delta_{\mathbf{E}_j} A_j v(\mathbf{x}) = \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) \leq 0$$

The above inequality is held by property 5.

2. $\Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_j) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j$ and

$$\Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_j) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j$$

$$\begin{aligned} \Rightarrow \Delta_{\mathbf{E}_j} A_j v(\mathbf{x} + \mathbf{e}_i) - \Delta_{\mathbf{E}_j} A_j v(\mathbf{x}) &= \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) - \left(\Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_j) + L_j \right) \\ &= -\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) - L_j \leq 0 \end{aligned}$$

The above inequality is held by the assumption.

3. $\Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j$ and

$$\Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_j) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j$$

$$\begin{aligned} \Rightarrow \Delta_{\mathbf{E}_j} A_j v(\mathbf{x} + \mathbf{e}_i) - \Delta_{\mathbf{E}_j} A_j v(\mathbf{x}) &= \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_j) + \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i) \\ &= -\left(\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \right) + \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i) + L_j \leq 0 \end{aligned}$$

The above inequality is held by the assumption.

4. $\Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_j) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j$ and

$$\Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j$$

$$\Rightarrow \Delta_{\mathbf{E}_j} A_j v(\mathbf{x} + \mathbf{e}_i) - \Delta_{\mathbf{E}_j} A_j v(\mathbf{x}) = \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) - \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_j) + \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) = 0$$

5. $\Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j$ and

$$\Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq 0 \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j$$

$$\Rightarrow \Delta_{\mathbf{E}_j} A_j v(\mathbf{x} + \mathbf{e}_i) - \Delta_{\mathbf{E}_j} A_j v(\mathbf{x}) = \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i) + L_j \leq 0$$

The above inequality is held by assumption.

$$6. \quad \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j \leq 0 \quad \text{and}$$

$$\Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j) + L_j \leq \Delta_{\mathbf{E}_j} v(\mathbf{x}) + L_j \leq 0$$

$$\Rightarrow \Delta_{\mathbf{E}_j} A_j v(\mathbf{x} + \mathbf{e}_i) - \Delta_{\mathbf{E}_j} A_j v(\mathbf{x}) = \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i) - \Delta_{\mathbf{E}_j} v(\mathbf{x}) \leq 0$$

The above inequality is held by property 5.

Case 2: $j \neq k$

In this case, we know $i = k$. By the definition of allocation operator, we know

$$\begin{aligned} \Delta_{\mathbf{E}_j} A_i v(\mathbf{x} + \mathbf{e}_i) - \Delta_{\mathbf{E}_j} A_i v(\mathbf{x}) &= \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_i) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_i) + \min\{\Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i + \mathbf{E}_j - \mathbf{E}_i) + L_i, 0\} \\ &\quad - \min\{\Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_i) + L_i, 0\} - \min\{\Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j - \mathbf{E}_i) + L_i, 0\} \\ &\quad + \min\{\Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) + L_i, 0\} \end{aligned}$$

By property 1 and 4, we know $\Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j - \mathbf{E}_i) \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i + \mathbf{E}_j - \mathbf{E}_i)$ and

$\Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_i) \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i + \mathbf{E}_j - \mathbf{E}_i)$, which leads to the following six

subcases.

$$1. \quad 0 \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j - \mathbf{E}_i) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i + \mathbf{E}_j - \mathbf{E}_i) + L_i \quad \text{and}$$

$$0 \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_i) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i + \mathbf{E}_j - \mathbf{E}_i) + L_i$$

$$\Rightarrow \Delta_{\mathbf{E}_j} A_i v(\mathbf{x} + \mathbf{e}_i) - \Delta_{\mathbf{E}_j} A_i v(\mathbf{x}) = \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_i) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_i) \leq 0$$

The above inequality is held by property 5.

$$2. \quad \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) + L_i \leq 0 \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j - \mathbf{E}_i) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i + \mathbf{E}_j - \mathbf{E}_i) + L_i \quad \text{and}$$

$$\Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) + L_i \leq 0 \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_i) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i + \mathbf{E}_j - \mathbf{E}_i) + L_i$$

$$\Rightarrow \Delta_{\mathbf{E}_j} A_i v(\mathbf{x} + \mathbf{e}_i) - \Delta_{\mathbf{E}_j} A_i v(\mathbf{x}) = \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_i) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_i) + \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) + L_i \leq 0$$

The above inequality is held by property 5 and assumption.

$$3. \quad \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j - \mathbf{E}_i) + L_i \leq 0 \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i + \mathbf{E}_j - \mathbf{E}_i) + L_i \quad \text{and}$$

$$\Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) + L_i \leq 0 \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_i) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i + \mathbf{E}_j - \mathbf{E}_i) + L_i$$

$$\Rightarrow \Delta_{\mathbf{E}_j} A_i v(\mathbf{x} + \mathbf{e}_i) - \Delta_{\mathbf{E}_j} A_i v(\mathbf{x}) = \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_i) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_i) + \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) - \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j - \mathbf{E}_i) \leq 0$$

The above inequality is held by property 4 and 5.

$$4. \quad \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) + L_i \leq 0 \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j - \mathbf{E}_i) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i + \mathbf{E}_j - \mathbf{E}_i) + L_i \quad \text{and}$$

$$\Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_i) + L_i \leq 0 \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i + \mathbf{E}_j - \mathbf{E}_i) + L_i$$

$$\Rightarrow \Delta_{\mathbf{E}_j} A_i v(\mathbf{x} + \mathbf{e}_i) - \Delta_{\mathbf{E}_j} A_i v(\mathbf{x}) = \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_i) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_i) + \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) - \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_i) \leq 0$$

The above inequality is held by property 1 and 5.

$$5. \quad \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j - \mathbf{E}_i) + L_i \leq 0 \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i + \mathbf{E}_j - \mathbf{E}_i) + L_i \quad \text{and}$$

$$\Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_i) + L_i \leq 0 \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i + \mathbf{E}_j - \mathbf{E}_i) + L_i$$

$$\begin{aligned} \Rightarrow \Delta_{\mathbf{E}_j} A_i v(\mathbf{x} + \mathbf{e}_i) - \Delta_{\mathbf{E}_j} A_i v(\mathbf{x}) &= \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_i) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_i) + \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) - \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_i) \\ &\quad - \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j - \mathbf{E}_i) - L_i \\ &= \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i) - \Delta_{\mathbf{E}_j} v(\mathbf{x}) - \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i + \mathbf{E}_j - \mathbf{E}_i) - L_i \\ &\leq 0 \end{aligned}$$

The above inequality is held by property 5 and the assumption.

$$6. \quad \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j - \mathbf{E}_i) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i + \mathbf{E}_j - \mathbf{E}_i) + L_i \leq 0 \quad \text{and}$$

$$\Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_i) + L_i \leq \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i + \mathbf{E}_j - \mathbf{E}_i) + L_i \leq 0$$

$$\begin{aligned} \Rightarrow \Delta_{\mathbf{E}_j} A_i v(\mathbf{x} + \mathbf{e}_i) - \Delta_{\mathbf{E}_j} A_i v(\mathbf{x}) &= \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_i) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_i) + \Delta_{\mathbf{E}_i} v(\mathbf{x} - \mathbf{E}_i) - \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i - \mathbf{E}_i) \\ &\quad - \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{E}_j - \mathbf{E}_i) + \Delta_{\mathbf{E}_i} v(\mathbf{x} + \mathbf{e}_i + \mathbf{E}_j - \mathbf{E}_i) \\ &= \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_i) - \Delta_{\mathbf{E}_j} v(\mathbf{x}) \leq 0 \end{aligned}$$

The above inequality is held by property 5.

It is straightforward to show that $h(\mathbf{x})$ satisfies Property 1-5, since $h(\mathbf{x})$ is separable in the variables x_i and increasing convex in each x_i . Since \mathfrak{V} is closed under multiplication by a scalar and addition, Tv satisfies Property 1-5.

Hence, Operator $Tv \in \mathfrak{V}$. \square

Lemma 4.2 establishes the structural properties of the subspace $\mathfrak{S}_{\mathfrak{a}}$ and identifies it as the recurrent region under the optimal policy.

Lemma 4.2: for any state $\mathbf{x} \notin \mathfrak{S}_{\mathfrak{a}}$, if the cost function v satisfies $\Delta_j v(\mathbf{x}) \geq 0$, then Tv satisfies $\Delta_j Tv(\mathbf{x}) \geq 0$, for $j \in \{0,1,2\}$. Furthermore, for $\mathbf{x} \notin \mathfrak{S}_{\mathfrak{a}}$, the optimal cost function

$v^*(\mathbf{x})$ satisfies the property $\Delta_j v^*(\mathbf{x}) \geq 0$, for $j \in \{0,1,2\}$.

Proof: We need to prove that $\Delta_j T v(\mathbf{x}) \geq 0$, for $x_j > S_j(\tilde{\mathbf{x}}_j)$. Since for $x_j > S_j(\tilde{\mathbf{x}}_j)$, we have

$\Delta_j v(\mathbf{x}) = v(\mathbf{x} + \mathbf{e}_j) - v(\mathbf{x}) \geq 0$. $\Delta_j T v(\mathbf{x})$ can be written as follows:

$$\Delta_j T v(\mathbf{x}) = h_j(x_j + 1) - h_j(x_j) + \mu_j \Delta_j v(\mathbf{x}) + \sum_{i \neq j} \mu_i \Delta_j P_i v(\mathbf{x}) + \sum_{k=1}^2 \lambda_k \Delta_j A_k v(\mathbf{x}),$$

Where $j \neq i \in \{0,1,2\}$, $k \in \{1,2\}$.

Operator P_i :

$$\begin{aligned} \Delta_j P_i v(\mathbf{x}) &= P_i v(\mathbf{x} + \mathbf{e}_j) - P_i v(\mathbf{x}) \\ &= \min\{v(\mathbf{x} + \mathbf{e}_j + \mathbf{e}_i), v(\mathbf{x} + \mathbf{e}_j)\} - \min\{v(\mathbf{x} + \mathbf{e}_i), v(\mathbf{x})\} \\ &\geq \min\{v(\mathbf{x} + \mathbf{e}_j), v(\mathbf{x})\} - \min\{v(\mathbf{x} + \mathbf{e}_i), v(\mathbf{x})\} = 0. \end{aligned}$$

Operator A_k : We distinguish three cases:

$j = 0$:

$$\begin{aligned} \Delta_0 A_k v(\mathbf{x}) &= A_k v(\mathbf{x} + \mathbf{e}_0) - A_k v(\mathbf{x}) \\ &= \min\{v(\mathbf{x} - \mathbf{e}_k), v(\mathbf{x} + \mathbf{e}_0) + L_k\} - \min\{v(\mathbf{x} - \mathbf{E}_k), v(\mathbf{x}) + L_k\} \\ &\geq \min\{v(\mathbf{x} - \mathbf{E}_k), v(\mathbf{x}) + L_k\} - \min\{v(\mathbf{x} - \mathbf{E}_k), v(\mathbf{x}) + L_k\} = 0. \end{aligned}$$

$j = k$:

$$\begin{aligned} \Delta_j A_j v(\mathbf{x}) &= A_j v(\mathbf{x} + \mathbf{e}_j) - A_j v(\mathbf{x}) \\ &= \min\{v(\mathbf{x} - \mathbf{e}_0), v(\mathbf{x} + \mathbf{e}_j) + L_j\} - \min\{v(\mathbf{x} - \mathbf{E}_j), v(\mathbf{x}) + L_j\} \\ &\geq \min\{v(\mathbf{x} - \mathbf{E}_j), v(\mathbf{x}) + L_j\} - \min\{v(\mathbf{x} - \mathbf{E}_j), v(\mathbf{x}) + L_j\} = 0. \end{aligned}$$

$j \neq k \neq 0$:

$$\begin{aligned} \Delta_j A_k v(\mathbf{x}) &= A_k v(\mathbf{x} + \mathbf{e}_j) - A_k v(\mathbf{x}) \\ &= \min\{v(\mathbf{x} - \mathbf{E}_k + \mathbf{e}_j), v(\mathbf{x} + \mathbf{e}_j) + L_k\} - \min\{v(\mathbf{x} - \mathbf{E}_k), v(\mathbf{x}) + L_k\} \\ &\geq \min\{v(\mathbf{x} - \mathbf{E}_k), v(\mathbf{x}) + L_k\} - \min\{v(\mathbf{x} - \mathbf{E}_k), v(\mathbf{x}) + L_k\} = 0. \end{aligned}$$

Since, by assumption, we have $h_j(x_j + 1) - h_j(x_j) \geq 0$, it follows that $\Delta_j T v(\mathbf{x}) \geq 0$ for

$x_j > S_j(\tilde{\mathbf{x}}_j)$ \square

Theorem 4.1 uses the results of Lemmas 4.1 and 4.2 to identify the recurrent region under the optimal policy.

Theorem 4.1: Under the optimal policy, the sub-Space $\mathfrak{S}_{\mathfrak{a}}$ is unique, closed and is identical to the recurrent region, *i.e.*, the region with all states \mathbf{x} being recurrent states.

Proof: By Observation 4.1, we know that for $\mathbf{x} \in \mathfrak{S}_{\mathfrak{a}}$, $\Delta_i v^*(\mathbf{x} + \mathbf{e}_i) \geq \Delta_i v^*(\mathbf{x})$. Hence, $\Delta_i v^*(\mathbf{x})$ is increasing in the direction \mathbf{e}_i . Also, by Lemma 4.2, we know that for $\mathbf{x} \notin \mathfrak{S}_{\mathfrak{a}}$, $\Delta_i v^*(\mathbf{x}) \geq 0$. As a result, $\Delta_i v^*(\mathbf{x}) \leq 0$ for $\mathbf{x} \in \mathfrak{S}_{\mathfrak{a}}$. This implies that thresholds $S_i(\tilde{\mathbf{x}}_i)$, $i \in \{0, 1, 2\}$, are unique and together enclose a unique bounded region identified by the sub-space $\mathfrak{S}_{\mathfrak{a}}$. Furthermore, since for states $\mathbf{x} \in \mathfrak{S}_{\mathfrak{a}}$, $\Delta_i v^*(\mathbf{x}) \leq 0$ it follows that it is always optimal to produce components within the region $\mathfrak{S}_{\mathfrak{a}}$. Also, since for $x_i \geq S_i(\tilde{\mathbf{x}}_i)$, $\Delta_i v^*(\mathbf{x}) \geq 0$, it is not optimal to produce Component i . This means that the inventory level of Component i , x_i , can only decrease due to product demand, for $x_i \geq S_i(\tilde{\mathbf{x}}_i)$. Thus, sooner or later, the state of the system enters sub-space $\mathfrak{S}_{\mathfrak{a}}$. On the other hand, if $x_i < S_i(\tilde{\mathbf{x}}_i)$, and since it is optimal to produce Component i then its inventory will increase to a maximum level of $S_i(\tilde{\mathbf{x}}_i)$, at which level production of Component i is no longer optimal. In conclusion, once the state of the system enters sub-space $\mathfrak{S}_{\mathfrak{a}}$, it never leaves it thereafter. Hence, sub-space $\mathfrak{S}_{\mathfrak{a}}$ identifies the recurrent region. \square

In order to characterize the structure of the optimal policy within the recurrent region, we further define the following two-dimensional threshold functions:

$$r_j^k(\tilde{\mathbf{x}}_j) = \min\{x_k \in \mathbb{Z}_+ \mid v(\mathbf{x}) - v(\mathbf{x} - \mathbf{E}_j) \geq -L_j\}, \text{ for } j \in \{1, 2\}, k \in \{0, j\}. \quad (4.4)$$

We can easily verify that $r_j^k(\tilde{\mathbf{x}}_j) \leq S_k(\tilde{\mathbf{x}}_k)$ for $\mathbf{x} \in \mathfrak{S}_{\mathfrak{a}}$ and $j \in \{1, 2\}$, $k \in \{0, j\}$. We are now ready to describe the structure of the optimal policy.

Theorem 4.2: There exists an optimal control policy that is stationary and is specified, for any initial state \mathbf{x} , in the recurrent region $\mathfrak{S}_{\mathfrak{a}}$ as follows:

(i) It is always optimal to produce Component i for $x_i < S_i(\tilde{\mathbf{x}}_i)$ and not to produce it

otherwise.

(ii) The optimal inventory allocation policy for Product j is characterized by a two-dimensional vector of state-dependent rationing levels, $(r_j^0(\tilde{\mathbf{x}}_0), r_j^j(\tilde{\mathbf{x}}_j))$, such that it is optimal to fulfill Product j incoming demand if $(x_0, x_j) > (r_j^0(\tilde{\mathbf{x}}_0), r_j^j(\tilde{\mathbf{x}}_j))$ and not to fulfill it otherwise.

(iii) Component j 's base-stock level, $S_j(\tilde{\mathbf{x}}_j)$, $j \in \{1, 2\}$, is non-decreasing in the inventory level of Component 0. Component 0's base-stock level, $S_0(\tilde{\mathbf{x}}_0)$, is non-decreasing in the inventory level of Component j , $j \in \{1, 2\}$. Component j base-stock level, $S_j(\tilde{\mathbf{x}}_j)$, is non-increasing in the inventory level of Component i , $j \neq i \in \{1, 2\}$. Rationing level $r_j^j(\tilde{\mathbf{x}}_j)$ is non-increasing in the inventory level of Component j and non-decreasing in the inventory level of Component i , $j \neq i \in \{1, 2\}$. Rationing level $r_j^0(\tilde{\mathbf{x}}_0)$ is non-increasing in the inventory level of Component j , $j \in \{1, 2\}$.

Proof: Part (i) follows from the definition of the recurrent region, $\mathfrak{S}_{\mathfrak{a}}$, and Theorem 4.1.

Part (ii): by Observation 4.1, $\Delta_{\mathbf{E}_j} v(\mathbf{x} - \mathbf{E}_j)$, for $j \in \{1, 2\}$, is non-decreasing in x_0 . Thus,

there exists a unique value, $r_j^0(\tilde{\mathbf{x}}_0)$, such that

$\Delta_{\mathbf{E}_j} v(r_j^0(\tilde{\mathbf{x}}_0) - 2, x_1 - 1, x_2) < -L_j \leq \Delta_{\mathbf{E}_j} v(r_j^0(\tilde{\mathbf{x}}_0) - 1, x_1 - 1, x_2)$. Similarly, by Property 1,

$\Delta_{\mathbf{E}_1} v(\mathbf{x} - \mathbf{E}_1)$ ($\Delta_{\mathbf{E}_2} v(\mathbf{x} - \mathbf{E}_2)$) is non-decreasing in x_1 (x_2). Thus, there exists a unique value,

$r_1^1(\tilde{\mathbf{x}}_1)$ ($r_2^2(\tilde{\mathbf{x}}_2)$) such that $\Delta_{\mathbf{E}_1} v(x_0 - 1, r_1^1(\tilde{\mathbf{x}}_1) - 2, x_2) < -L_1 \leq \Delta_{\mathbf{E}_1} v(x_0 - 1, r_1^1(\tilde{\mathbf{x}}_1) - 1, x_2)$

$(\Delta_{\mathbf{E}_2} v(x_0 - 1, x_1, r_2^2(\tilde{\mathbf{x}}_2) - 2) < -L_2 \leq \Delta_{\mathbf{E}_2} v(x_0 - 1, x_1, r_2^2(\tilde{\mathbf{x}}_2) - 1))$.

Part (iii): $S_j(\tilde{\mathbf{x}}_j)$ is non-decreasing in x_0 is due to Property 2. $S_0(\tilde{\mathbf{x}}_0)$ is non-decreasing in

x_j $j \in \{1, 2\}$, is due to Property 2. $S_j(\tilde{\mathbf{x}}_j)$ is non-increasing in x_i , $j \neq i \in \{1, 2\}$, is due to

Property 3. $r_j^j(\tilde{\mathbf{x}}_j)$ is non-increasing in x_j , $j \in \{1, 2\}$, is due to Property 1. $r_j^j(\tilde{\mathbf{x}}_j)$ is

non-decreasing in x_i , $j \neq i \in \{1, 2\}$, is due to Property 5. $r_j^0(\tilde{\mathbf{x}}_0)$ is non-increasing in x_j ,

$j \in \{1, 2\}$, is due to observation 4.1. \square

Theorem 4.2 states that it is always optimal to produce all components once the state of the system \mathbf{x} enters the recurrent region $\mathfrak{S}_{\mathfrak{a}}$. It also states that component inventory allocation is governed by a rationing policy that depends on the other components' inventory level that specifies whether a demand from a particular product is fulfilled or not. In particular, a demand for a product is satisfied only if the inventory level of each component, used to assemble the product, is above its rationing level for that product. We note that the components inventory rationing levels, for a product, are in general non-identical.

Furthermore, Theorem 4.2 indicates that production of Component 0 increases the desirability of producing components 1 and 2. Production of Component 1 (Component 2) increases the desirability to produce Component 0 and decreases the desirability to produce Component 2 (Component 1). i.e., $S_j(\tilde{\mathbf{x}}_j + \mathbf{e}_0) \geq S_j(\tilde{\mathbf{x}}_j)$, $S_0(\tilde{\mathbf{x}}_0 + \mathbf{e}_j) \geq S_0(\tilde{\mathbf{x}}_0)$, and $S_j(\tilde{\mathbf{x}}_j + \mathbf{e}_i) \leq S_j(\tilde{\mathbf{x}}_j)$, for $i \neq j$, $j \in \{1, 2\}$. An increase in the inventory level of Component j could lead to a decrease in the rationing levels of Components j and 0. An increase in the inventory level of Component 0 could lead to a decrease in the rationing levels of the other components. An increase in the inventory level of Component j could lead to an increase in the rationing level of Component i , $i \neq j$.

4.1.1 Structure of optimal policy under average cost rate criterion

The main take away of this section is that the structural characteristics of the optimal policy described under the discounted cost criterion carry over to the average cost rate criterion.

Theorem 4.3: Under the average cost rate criterion, there exists an optimal stationary policy that retains all the properties of the optimal policy under the discounted cost criterion. Namely, under this optimal policy, the sub-Space $\mathfrak{S}_{\mathfrak{a}}$ is unique, closed and is identical to the recurrent region. For any state within the interior of the recurrent region, it is always optimal to produce any of the components; the optimal inventory allocation policy for a particular product is characterized by a two-dimensional vector of state-dependent rationing levels; and the base-stock and rationing levels satisfy (iii) in Theorem 4.2. Furthermore, the optimal average cost rate is finite and independent of the initial state, \mathbf{x} . in other words, there exists a finite

constant γ^* such that $v^*(\mathbf{x}) = \gamma^*$, for all states \mathbf{x} .

Proof: we take the limit as $\alpha \rightarrow 0$, in the discounted cost problem in order to show that, under the average cost rate criterion, an optimal policy exists with a finite average cost rate that is independent of the starting state. As such, two conditions must be satisfied (Cavazos-Cadena and Sennott 1992; Weber and Stidham, 1987): (1) there exists a stationary policy that induces an irreducible positive recurrent Markov chain with finite average cost, γ , and (2) the number of states for which the holding cost $h(\mathbf{x}) \leq \gamma$ is finite. For (1), one can consider the policy that does not produce any of the components and thus yields an average cost rate $\gamma = \lambda_1 L_1 + \lambda_2 L_2$. (2) holds since $h(\mathbf{x})$ is component-wise increasing convex in its arguments.

Hence, for any positive value δ , the number of states for which $h(\mathbf{x}) \leq \delta$ is always finite.

Using (1) and (2), Weber and Stidham (1987) show that there exists a constant γ^* and a function $f(\mathbf{x})$ such that

$$\gamma^* = \min_{\mathbf{a}(\mathbf{x}) \in A(\mathbf{x})} \left[g(\mathbf{x}, \mathbf{a}(\mathbf{x})) + \sum_{\mathbf{x}'} p_{\mathbf{x}, \mathbf{x}'}(\mathbf{a}(\mathbf{x})) f(\mathbf{x}') - f(\mathbf{x}) \right],$$

Furthermore, the stationary policy that minimizes the right hand side of the above equation for each state \mathbf{x} is an optimal policy for the average cost criterion and yields a constant average cost rate γ^* . As a result, properties of the average cost optimal policy are the same as determined through function $f(\mathbf{x})$ in much the same way as were properties of the discounted cost optimal policy determined by $v(\mathbf{x})$. \square

4.2 Extension to batch production and non-unitary compound Poisson demand process

In this section, we extend the original model to the case where production occurs in batches and demand for the products is non-unitary and constitutes a compound Poisson process. We use the same notation as the previous section. Component 0 is the common component, Component 1 (Component 2) is the product specific component and is assembled with component 0 to an end product 1 (product 2). $\sum_{i=0}^2 h_i(x_i)$ denotes the total inventory holding

cost and L_k , $k = \{1, 2\}$ denotes the lost sale cost for each product if demands are not satisfied.

Assuming the demand process of each product follows by a compound Poisson distribution with rate λ_k , $k = \{1, 2\}$. The requested amount for each product is bounded above by quantity D_k , $k = \{1, 2\}$. The probability that the order size of product k 's demand will be d is $P\{\delta_k = d\} = p_k(d)$, $k = \{1, 2\}$ and $d = \{1, 2, \dots, D_k\}$. The production process of each component follows by an exponential rate μ_i , $i = \{0, 1, 2\}$ and the batch production size can be any value of b . Since the memoryless property still holds in this case, we can formulate the system as a MDP. Hence, the optimality equation can be written as Eq. (4.5). The Operator $H_k^d v(x)$ is the allocation operator and the Operator $P_i v(x)$ is the production operator.

$$v^*(x) = h(x) + \sum_{k=1}^2 \lambda_k \left(\sum_{d=1}^{D_k} p_k(d) H_k^d v^*(x) \right) + \sum_{i=0}^2 \mu_i P_i v^*(x) \quad (4.5)$$

Where $H_k^d v(x) = \min_{0 \leq q \leq \min\{\min(x_0, x_k), d\}} \{v(\mathbf{x} - qE_k) + (d - q)L_k\}$, $P_i v(x) = \min_{b \geq 0} \{v(x_i + b)\}$, $i = \{0, 1, 2\}$

and $k = \{1, 2\}$.

Note

$$\begin{aligned} H_k^d v(x) &= \min_{0 \leq q \leq \min\{\min(x_0, x_k), d\}} \{v(\mathbf{x} - qE_k) + (d - q)L_k\} \\ &= v(\mathbf{x} + E_k) + (d + 1)L_k - \max_{0 \leq q \leq \min\{\min(x_0, x_k), d\}} \left\{ \sum_{n=0}^q (\Delta_{E_k}(\mathbf{x} - nE_k) + L_k) \right\} \end{aligned}$$

Let $q_k^*(\mathbf{x}, d)$ denote the value that minimizes $v(\mathbf{x} - qE_k) + (d - q)L_k$. Then, $q_k^*(\mathbf{x}, d)$ is characterized by the Lemma 4.3.

$$\text{Let } Ov(x) = h(x) + \sum_{k=1}^2 \lambda_k \left(\sum_{d=1}^{D_k} p_k(d) H_k^d v(x) \right) + \sum_{i=0}^2 \mu_i P_i v(x).$$

Lemma 4.3: If $v \in V$, let $\min(x_0, x_k) = m(\mathbf{x})$, then we have

$$q_k^*(\mathbf{x}, d) = \begin{cases} 0 & \text{if } \Delta_{E_k}(\mathbf{x}) \leq -L_k \\ \max \left\{ q \mid \Delta_{E_k}(\mathbf{x} - qE_k) \geq -L_k, 0 \leq q \leq \min \{m(\mathbf{x}), d\} \right\} & \text{otherwise} \end{cases}$$

Proof: By property 1 and observation 4.1, we have $\Delta_{E_k}(\mathbf{x}) \geq \Delta_{E_k}(\mathbf{x} - E_k) \geq \dots \geq \Delta_{E_k}(\mathbf{x} - nE_k)$.

If $\Delta_{E_k}(\mathbf{x}) \leq -L_k$, which means it is not optimal to satisfy any value of the batch demand.

If $\Delta_{E_k}(\mathbf{x} - qE_k) \geq -L_k$, then we need to find the value $q_k^*(\mathbf{x}, d) \in [0, \min \{m(\mathbf{x}), d\}]$, which

lets $\Delta_{E_k}(\mathbf{x} - q_k^*(\mathbf{x}, d)E_k) \geq -L_k$ and $\Delta_{E_k}(\mathbf{x} - (q_k^*(\mathbf{x}, d) + 1)E_k) \leq -L_k$. \square

Lemma 4.4 and Lemma 4.5 determine the changes in the value $q_k^*(\mathbf{x}, d)$, with changes in components' inventory levels under different conditions.

Lemma 4.4: Let $\mathbf{e}' = \mathbf{e}_0 (\mathbf{e}_k, \mathbf{e}_0 + \mathbf{e}_k, \mathbf{e}_0 + \mathbf{e}_j)$ and $v \in V$, where $k = \{1, 2\}$ then we have

$$q_k^*(\mathbf{x} + \mathbf{e}', d) = \begin{cases} q_k^*(\mathbf{x}, d) = 0 \Leftrightarrow \Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - E_k) < -L_k \\ q_k^*(\mathbf{x}, d) = d \Leftrightarrow \min \{ \min(x_0, x_k), d \} = d \text{ and } \Delta_{E_k} v(\mathbf{x} - dE_k) \geq -L_k \\ q_k^*(\mathbf{x}, d) \Leftrightarrow 1 \leq q_k^*(\mathbf{x}, d) < m(\mathbf{x}) < d \text{ and } \Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - (q_k^*(\mathbf{x}, d) + 1)E_k) < -L_k \\ q_k^*(\mathbf{x} + \mathbf{e}', d) \Leftrightarrow q_k^*(\mathbf{x}, d) + 1 \Leftrightarrow 1 \leq q_k^*(\mathbf{x}, d) < m(\mathbf{x}) < d \text{ and } \Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - (q_k^*(\mathbf{x}, d) + 1)E_k) \geq -L_k \\ q_k^*(\mathbf{x}, d) \Leftrightarrow q_k^*(\mathbf{x}, d) = m(\mathbf{x}) = m(\mathbf{x} + \mathbf{e}') < d \\ q_k^*(\mathbf{x}, d) \Leftrightarrow q_k^*(\mathbf{x}, d) = m(\mathbf{x}) = m(\mathbf{x} + \mathbf{e}') - 1 < d \text{ and } \Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - m(\mathbf{x} + \mathbf{e}')E_k) < -L_k \\ q_k^*(\mathbf{x}, d) + 1 \Leftrightarrow q_k^*(\mathbf{x}, d) = m(\mathbf{x}) = m(\mathbf{x} + \mathbf{e}') - 1 < d \text{ and } \Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - m(\mathbf{x} + \mathbf{e}')E_k) \geq -L_k \end{cases}$$

Proof: we discuss each sub-equality by the following cases.

Case 1: By property 1, we know $\Delta_{E_k} v(\mathbf{x} - E_k) \leq \Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - E_k) < -L_k$ and by Lemma 4.3,

we have $q_k^*(\mathbf{x}, d) < 1$ and $q_k^*(\mathbf{x} + \mathbf{e}', d) < 1$. Hence, $q_k^*(\mathbf{x} + \mathbf{e}', d) = q_k^*(\mathbf{x}, d) = 0$.

Case 2: By property 1, we know $\Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - dE_k) \geq \Delta_{E_k} v(\mathbf{x} - dE_k) \geq -L_k$. In addition, as

$\min \{ \min(x_0, x_k), d \} = d$, $q_k^*(\mathbf{x} + \mathbf{e}', d) = q_k^*(\mathbf{x}, d) = d$.

Case 3: By property 1, we know $\Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - E_k) \geq \Delta_{E_k} v(\mathbf{x} - E_k)$, which means

$q_k^*(\mathbf{x} + \mathbf{e}', d) \geq q_k^*(\mathbf{x}, d)$. By Lemma 4.3, we have $q_k^*(\mathbf{x} + \mathbf{e}', d) < q_k^*(\mathbf{x}, d) + 1$. Hence,

$$q_k^*(\mathbf{x} + \mathbf{e}', d) = q_k^*(\mathbf{x}, d).$$

Case 4: By Lemma 4.3, we know $q_k^*(\mathbf{x} + \mathbf{e}', d) \geq q_k^*(\mathbf{x}, d) + 1$. In addition,

$$\Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - (q_k^*(\mathbf{x}, d) + 2)E_k) \leq \Delta_{E_k} v(\mathbf{x} + E_k - (q_k^*(\mathbf{x}, d) + 2)E_k) = \Delta_{E_k} v(\mathbf{x} - (q_k^*(\mathbf{x}, d) + 1)E_k) < L_k,$$

which means $q_k^*(\mathbf{x} + \mathbf{e}', d) < q_k^*(\mathbf{x}, d) + 2$. Hence, $q_k^*(\mathbf{x} + \mathbf{e}', d) = q_k^*(\mathbf{x}, d) + 1$.

Case 5: By Lemma 4.3, we know $q_k^*(\mathbf{x} + \mathbf{e}', d) < q_k^*(\mathbf{x}, d) + 1$. By property 1, we have

$$\Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - q_k^*(\mathbf{x}, d)E_k) \geq \Delta_{E_k} v(\mathbf{x} - q_k^*(\mathbf{x}, d)E_k) \geq -L_k, \text{ which means } q_k^*(\mathbf{x} + \mathbf{e}', d) \geq q_k^*(\mathbf{x}, d).$$

Hence, $q_k^*(\mathbf{x} + \mathbf{e}', d) = q_k^*(\mathbf{x}, d)$.

Case 6: By $\Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - (q_k^*(\mathbf{x}, d) + 1)E_k) < -L_k$, we know $q_k^*(\mathbf{x} + \mathbf{e}', d) < q_k^*(\mathbf{x}, d) + 1$. By

property 1, we know $\Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - q_k^*(\mathbf{x}, d)E_k) \geq \Delta_{E_k} v(\mathbf{x} - q_k^*(\mathbf{x}, d)E_k) \geq -L_k$, which means

$$q_k^*(\mathbf{x} + \mathbf{e}', d) \geq q_k^*(\mathbf{x}, d). \text{ Hence, } q_k^*(\mathbf{x} + \mathbf{e}', d) = q_k^*(\mathbf{x}, d).$$

Case 7: By $\Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - (q_k^*(\mathbf{x}, d) + 1)E_k) \geq -L_k$, we know $q_k^*(\mathbf{x} + \mathbf{e}', d) \geq q_k^*(\mathbf{x}, d) + 1$.

Besides,

$$\Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - (q_k^*(\mathbf{x}, d) + 2)E_k) \leq \Delta_{E_k} v(\mathbf{x} + E_k - (q_k^*(\mathbf{x}, d) + 2)E_k) = \Delta_{E_k} v(\mathbf{x} - (q_k^*(\mathbf{x}, d) + 1)E_k) < -L_k,$$

then $q_k^*(\mathbf{x} + \mathbf{e}', d) < q_k^*(\mathbf{x}, d) + 2$. Hence, $q_k^*(\mathbf{x} + \mathbf{e}', d) = q_k^*(\mathbf{x}, d) + 1$. \square

Lemma 4.5: Let $\mathbf{e}' = \mathbf{e}_j$ and $v \in V$, where $j \neq k = \{1, 2\}$, then we have

$$q_k^*(\mathbf{x} + \mathbf{e}', d) = \begin{cases} q_k^*(\mathbf{x}, d) = 0 \Leftrightarrow \Delta_{E_k} v(\mathbf{x} - E_k) < -L_k \\ q_k^*(\mathbf{x}, d) = d \Leftrightarrow \min\{\min(x_0, x_k), d\} = d \text{ and } \Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - dE_k) \geq -L_k \\ q_k^*(\mathbf{x}, d) - 1 \Leftrightarrow 1 \leq q_k^*(\mathbf{x}, d) < m(\mathbf{x}) < d \text{ and } \Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - q_k^*(\mathbf{x}, d)E_k) < -L_k \\ q_k^*(\mathbf{x}, d) \Leftrightarrow 1 \leq q_k^*(\mathbf{x}, d) < m(\mathbf{x}) < d \text{ and } \Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - q_k^*(\mathbf{x}, d)E_k) \geq -L_k \\ q_k^*(\mathbf{x}, d) \Leftrightarrow q_k^*(\mathbf{x}, d) = m(\mathbf{x}) = m(\mathbf{x} + \mathbf{e}') < d \text{ and } \Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - m(\mathbf{x})E_k) \geq -L_k \\ q_k^*(\mathbf{x}, d) - 1 \Leftrightarrow q_k^*(\mathbf{x}, d) = m(\mathbf{x}) = m(\mathbf{x} + \mathbf{e}') < d \text{ and } \Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - m(\mathbf{x})E_k) < -L_k \end{cases}$$

Proof: we discuss each sub-equality by the following cases.

Case 1: By property 5, we know $\Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - E_k) \leq \Delta_{E_k} v(\mathbf{x} - E_k) < -L_k$ and by Lemma 4.3,

we have $q_k^*(\mathbf{x}, d) < 1$ and $q_k^*(\mathbf{x} + \mathbf{e}', d) < 1$. Hence, $q_k^*(\mathbf{x} + \mathbf{e}', d) = q_k^*(\mathbf{x}, d) = 0$.

Case 2: By property 5, we know $-L_k \leq \Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - E_k) \leq \Delta_{E_k} v(\mathbf{x} - E_k)$. In addition, as

$$\min\{\min(x_0, x_k), d\} = d, \text{ hence, } q_k^*(\mathbf{x} + \mathbf{e}', d) = q_k^*(\mathbf{x}, d) = d.$$

Case 3: By $\Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - q_k^*(\mathbf{x}, d)E_k) < -L_k$, we know $q_k^*(\mathbf{x} + \mathbf{e}', d) < q_k^*(\mathbf{x}, d)$. Besides, we

have

$$\Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - (q_k^*(\mathbf{x}, d) - 1)E_k) \geq \Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - E - (q_k^*(\mathbf{x}, d) - 1)E_k) = \Delta_{E_k} v(\mathbf{x} - q_k^*(\mathbf{x}, d)E_k) \geq -L_k,$$

which means $q_k^*(\mathbf{x} + \mathbf{e}', d) \geq q_k^*(\mathbf{x}, d) - 1$. Hence, $q_k^*(\mathbf{x} + \mathbf{e}', d) = q_k^*(\mathbf{x}, d) - 1$.

Case 4: By $\Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - q_k^*(\mathbf{x}, d)E_k) \geq -L_k$, we know $q_k^*(\mathbf{x} + \mathbf{e}', d) \geq q_k^*(\mathbf{x}, d)$. Besides, by

property 3, we have $\Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - (q_k^*(\mathbf{x}, d) + 1)E_k) \leq \Delta_{E_k} v(\mathbf{x} - (q_k^*(\mathbf{x}, d) + 1)E_k) < -L_k$, which

means $q_k^*(\mathbf{x} + \mathbf{e}', d) < q_k^*(\mathbf{x}, d) + 1$. Hence, $q_k^*(\mathbf{x} + \mathbf{e}', d) = q_k^*(\mathbf{x}, d)$.

Case 5: By $\Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - q_k^*(\mathbf{x}, d)E_k) \geq -L_k$, we know $q_k^*(\mathbf{x} + \mathbf{e}', d) \geq q_k^*(\mathbf{x}, d)$. Besides, as

$$m(\mathbf{x} + \mathbf{e}') = m(\mathbf{x}), \text{ we have } q_k^*(\mathbf{x} + \mathbf{e}', d) < q_k^*(\mathbf{x}, d) + 1. \text{ Hence, } q_k^*(\mathbf{x} + \mathbf{e}', d) = q_k^*(\mathbf{x}, d).$$

Case 6: By $\Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - m(\mathbf{x})E_k) < -L_k$, we know $q_k^*(\mathbf{x} + \mathbf{e}', d) < q_k^*(\mathbf{x}, d)$. By property 5,

$$\Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - (q_k^*(\mathbf{x}, d) - 1)E_k) \geq \Delta_{E_k} v(\mathbf{x} + \mathbf{e}' - \mathbf{e} - (q_k^*(\mathbf{x}, d) - 1)E_k) = \Delta_{E_k} v(\mathbf{x} - q_k^*(\mathbf{x}, d)E_k) \geq -L_k$$

, we have $q_k^*(\mathbf{x} + \mathbf{e}', d) \geq q_k^*(\mathbf{x}, d) - 1$. Hence, $q_k^*(\mathbf{x} + \mathbf{e}', d) = q_k^*(\mathbf{x}, d) - 1$. \square

Lemma 4.6: The observations in Lemma 4.1 are preserved in the batch production and non-unitary compound Poisson demand case. That is, for any cost function $v \in \mathfrak{V}$ and state $\mathbf{x} \in \mathfrak{S}_\alpha$, Ov is also an element of \mathfrak{V} . In addition, the optimal cost function, v^* , defined in (4.5) is an element of \mathfrak{V} .

Proof: In order to prove Lemma 4.6, we first show that Operator $H_k^d v$, Operator $P_i v(x)$ and the holding cost function $h(\mathbf{x})$ preserve properties 1-5. Then, since set \mathfrak{V} is closed under multiplication by a scalar and under addition, we show that Operator Ov preserves properties 1-5 as well. Finally, to show that $v^* \in \mathfrak{V}$, we use the fact that (1) $v^* = \lim_{n \rightarrow \infty} O^n v$ for any $v \in \mathfrak{V}$ (see Proposition 3.1.5 and 3.1.6, Bertsekas (2001)), and (2) $O^n v \in \mathfrak{V}$ since $Ov \in \mathfrak{V}$.

Property 1

In order to prove operator $H_k^d v$ satisfies property 1, we need to show

$$\Delta_{\mathbf{E}_j} H_k^d v(\mathbf{x} + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} H_k^d v(\mathbf{x}) \geq 0 \quad \text{for } j, k \in \{1, 2\}.$$

By the definition of the allocation operator,

$$\begin{aligned} & \Delta_{\mathbf{E}_j} H_k^d v(\mathbf{x} + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} H_k^d v(\mathbf{x}) \\ &= H_k^d v(\mathbf{x} + \mathbf{e}_j + E_j) - H_k^d v(\mathbf{x} + \mathbf{e}_j) - H_k^d v(\mathbf{x} + E_j) + H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_j + E_j - q_k^*(\mathbf{x} + \mathbf{e}_j + E_j, d)E_k) - v(\mathbf{x} + \mathbf{e}_j - q_k^*(\mathbf{x} + \mathbf{e}_j, d)E_k) \\ &\quad - v(\mathbf{x} + E_j - q_k^*(\mathbf{x} + E_j, d)E_k) + v(\mathbf{x} - q_k^*(\mathbf{x}, d)E_k) \\ &\quad - (q_k^*(\mathbf{x} + \mathbf{e}_j + E_j, d) - q_k^*(\mathbf{x} + \mathbf{e}_j, d) - q_k^*(\mathbf{x} + E_j, d) + q_k^*(\mathbf{x}, d))L_k \end{aligned}$$

We discuss the following two cases.

Case 1: $j = k$

$$\begin{aligned} & \Delta_{\mathbf{E}_k} H_k^d v(\mathbf{x} + \mathbf{e}_k) - \Delta_{\mathbf{E}_k} H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_k + E_k - q_k^*(\mathbf{x} + \mathbf{e}_k + E_k, d)E_k) - v(\mathbf{x} + \mathbf{e}_k - q_k^*(\mathbf{x} + \mathbf{e}_k, d)E_k) \\ &\quad - v(\mathbf{x} + E_k - q_k^*(\mathbf{x} + E_k, d)E_k) + v(\mathbf{x} - q_k^*(\mathbf{x}, d)E_k) \\ &\quad - (q_k^*(\mathbf{x} + \mathbf{e}_k + E_k, d) - q_k^*(\mathbf{x} + \mathbf{e}_k, d) - q_k^*(\mathbf{x} + E_k, d) + q_k^*(\mathbf{x}, d))L_k \end{aligned}$$

Using the results of Lemma 4.4, we recognize that:

$$q_k^*(\mathbf{x}, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_k, d) \leq q_k^*(\mathbf{x}, d) + 1,$$

$$q_k^*(\mathbf{x}, d) \leq q_k^*(\mathbf{x} + E_k, d) \leq q_k^*(\mathbf{x}, d) + 1,$$

$$q_k^*(\mathbf{x} + \mathbf{e}_k, d) \leq q_k^*(\mathbf{x} + E_k, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_k, d) + 1 \quad \text{and}$$

$$q_k^*(\mathbf{x} + E_k, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_k + E_k, d) \leq q_k^*(\mathbf{x} + E_k, d) + 1.$$

We consider all combinations of these conditions and discuss the following subcases.

$$1. \quad q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q_k^*(\mathbf{x} + E_k, d) = q_k^*(\mathbf{x} + \mathbf{e}_k + E_k, d)$$

$$\begin{aligned} & \Delta_{\mathbf{E}_k} H_k^d v(\mathbf{x} + \mathbf{e}_k) - \Delta_{\mathbf{E}_k} H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_k + E_k - q_k^*(\mathbf{x}, d)E_k) - v(\mathbf{x} + \mathbf{e}_k - q_k^*(\mathbf{x}, d)E_k) - v(\mathbf{x} + E_k - q_k^*(\mathbf{x}, d)E_k) + v(\mathbf{x} - q_k^*(\mathbf{x}, d)E_k) \\ &= \Delta_{\mathbf{E}_k} v(\mathbf{x} + \mathbf{e}_k - q_k^*(\mathbf{x}, d)E_k) - \Delta_{\mathbf{E}_k} v(\mathbf{x} - q_k^*(\mathbf{x}, d)E_k) \geq 0 \end{aligned}$$

The inequality holds by property 1.

$$2. \quad q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q_k^*(\mathbf{x} + E_k, d) = q^* \quad \text{and} \quad q_k^*(\mathbf{x} + \mathbf{e}_k + E_k, d) = q^* + 1$$

$$\begin{aligned}
& \Delta_{\mathbf{E}_k} H_k^d v(\mathbf{x} + \mathbf{e}_k) - \Delta_{\mathbf{E}_k} H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_k + E_k - (q^* + 1)E_k) - v(\mathbf{x} + \mathbf{e}_k - q^* E_k) - v(\mathbf{x} + E_k - q^* E_k) + v(\mathbf{x} - q^* E_k) - L_k \\
&= \Delta_{\mathbf{E}_k} v(\mathbf{x} + \mathbf{e}_k - q^* E_k) - \Delta_{\mathbf{E}_k} v(\mathbf{x} - q^* E_k) - \Delta_{\mathbf{E}_k} v(\mathbf{x} + \mathbf{e}_k + E_k - (q^* + 1)E_k) - L_k \\
&\geq \Delta_{\mathbf{E}_k} v(\mathbf{x} + \mathbf{e}_k - q^* E_k) - \Delta_{\mathbf{E}_k} v(\mathbf{x} - q^* E_k) - \Delta_{\mathbf{E}_k} v(\mathbf{x} + \mathbf{e}_k + E_k - (q^* + 1)E_k) + \Delta_{\mathbf{E}_k} v(\mathbf{x} + E_k - (q^* + 1)E_k) = 0
\end{aligned}$$

The above inequality holds by Lemma 4.3, which shows

$$\Delta_{\mathbf{E}_k} v(\mathbf{x} + E_k - (q^* + 1)E_k) < -L_k \leq \Delta_{\mathbf{E}_k} v(\mathbf{x} + E_k - q^* E_k).$$

$$3. \quad q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q^* \text{ and } q_k^*(\mathbf{x} + E_k, d) = q_k^*(\mathbf{x} + \mathbf{e}_k + E_k, d) = q^* + 1$$

$$\begin{aligned}
& \Delta_{\mathbf{E}_k} H_k^d v(\mathbf{x} + \mathbf{e}_k) - \Delta_{\mathbf{E}_k} H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_k + E_k - (q^* + 1)E_k) - v(\mathbf{x} + \mathbf{e}_k - q^* E_k) - v(\mathbf{x} + E_k - (q^* + 1)E_k) + v(\mathbf{x} - q^* E_k) \\
&= \Delta_{\mathbf{E}_k} v(\mathbf{x} + \mathbf{e}_k - q^* E_k) - \Delta_{\mathbf{E}_k} v(\mathbf{x} - q^* E_k) - \Delta_{\mathbf{E}_k} v(\mathbf{x} + \mathbf{e}_k + E_k - (q^* + 1)E_k) + \Delta_{\mathbf{E}_k} v(\mathbf{x} + E_k - (q^* + 1)E_k) \\
&= 0
\end{aligned}$$

$$4. \quad q_k^*(\mathbf{x}, d) = q^* \text{ and } q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q_k^*(\mathbf{x} + E_k, d) = q_k^*(\mathbf{x} + \mathbf{e}_k + E_k, d) = q^* + 1$$

$$\begin{aligned}
& \Delta_{\mathbf{E}_k} H_k^d v(\mathbf{x} + \mathbf{e}_k) - \Delta_{\mathbf{E}_k} H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_k + E_k - (q^* + 1)E_k) - v(\mathbf{x} + \mathbf{e}_k - (q^* + 1)E_k) - v(\mathbf{x} + E_k - (q^* + 1)E_k) + v(\mathbf{x} - q^* E_k) + L_k \\
&= \Delta_{\mathbf{E}_k} v(\mathbf{x} + \mathbf{e}_k - (q^* + 1)E_k) + L_k \geq 0
\end{aligned}$$

The above inequality holds by Lemma 4.3 ($\Delta_{\mathbf{E}_k} v(\mathbf{x} + \mathbf{e}_k - (q^* + 1)E_k) \geq -L_k$).

$$5. \quad q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q^*, q_k^*(\mathbf{x} + E_k, d) = q^* + 1 \text{ and } q_k^*(\mathbf{x} + \mathbf{e}_k + E_k, d) = q^* + 2$$

$$\begin{aligned}
& \Delta_{\mathbf{E}_k} H_k^d v(\mathbf{x} + \mathbf{e}_k) - \Delta_{\mathbf{E}_k} H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_k + E_k - (q^* + 2)E_k) - v(\mathbf{x} + \mathbf{e}_k - q^* E_k) - v(\mathbf{x} + E_k - (q^* + 1)E_k) + v(\mathbf{x} - q^* E_k) - L_k \\
&= -\Delta_{\mathbf{E}_k} v(\mathbf{x} + \mathbf{e}_k - (q^* + 1)E_k) - L_k \geq 0
\end{aligned}$$

The above inequality holds by Lemma 4.3 ($\Delta_{\mathbf{E}_k} v(\mathbf{x} + \mathbf{e}_k - (q^* + 1)E_k) < -L_k$).

$$6. \quad q_k^*(\mathbf{x}, d) = q^*, q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q_k^*(\mathbf{x} + E_k, d) = q^* + 1 \text{ and } q_k^*(\mathbf{x} + \mathbf{e}_k + E_k, d) = q^* + 2$$

$$\begin{aligned}
& \Delta_{\mathbf{E}_k} H_k^d v(\mathbf{x} + \mathbf{e}_k) - \Delta_{\mathbf{E}_k} H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_k + E_k - (q^* + 2)E_k) - v(\mathbf{x} + \mathbf{e}_k - (q^* + 1)E_k) - v(\mathbf{x} + E_k - (q^* + 1)E_k) + v(\mathbf{x} - q^* E_k) = 0
\end{aligned}$$

Case 2: $j \neq k$

$$\begin{aligned}
& \Delta_{\mathbf{E}_j} H_k^d v(\mathbf{x} + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} H_k^d v(\mathbf{x}) \\
&= H_k^d v(\mathbf{x} + \mathbf{e}_j + E_j) - H_k^d v(\mathbf{x} + \mathbf{e}_j) - H_k^d v(\mathbf{x} + E_j) + H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_j + E_j - q_k^*(\mathbf{x} + \mathbf{e}_j + E_j, d)E_k) - v(\mathbf{x} + \mathbf{e}_j - q_k^*(\mathbf{x} + \mathbf{e}_j, d)E_k) \\
&\quad - v(\mathbf{x} + E_j - q_k^*(\mathbf{x} + E_j, d)E_k) + v(\mathbf{x} - q_k^*(\mathbf{x}, d)E_k) \\
&\quad - (q_k^*(\mathbf{x} + \mathbf{e}_j + E_j, d) - q_k^*(\mathbf{x} + \mathbf{e}_j, d) - q_k^*(\mathbf{x} + E_j, d) + q_k^*(\mathbf{x}, d))L_k
\end{aligned}$$

By using Lemma 4.4 and Lemma 4.5, we recognize that:

$$q_k^*(\mathbf{x}, d) - 1 \leq q_k^*(\mathbf{x} + \mathbf{e}_j, d) \leq q_k^*(\mathbf{x}, d) ,$$

$$q_k^*(\mathbf{x}, d) \leq q_k^*(\mathbf{x} + E_j, d) \leq q_k^*(\mathbf{x}, d) + 1 ,$$

$$q_k^*(\mathbf{x} + \mathbf{e}_j, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_j + E_j, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_j, d) + 1 \text{ and}$$

$$q_k^*(\mathbf{x} + E_j, d) - 1 \leq q_k^*(\mathbf{x} + \mathbf{e}_j + E_j, d) \leq q_k^*(\mathbf{x} + E_j, d) .$$

We consider all combinations of these conditions and discuss the following subcases.

$$1. \quad q_k^*(\mathbf{x} + \mathbf{e}_j, d) = q_k^*(\mathbf{x} + \mathbf{e}_j + E_j, d) = q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + E_j, d) = q^*$$

$$\begin{aligned}
& \Delta_{\mathbf{E}_j} H_k^d v(\mathbf{x} + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_j + E_j - q^*E_k) - v(\mathbf{x} + \mathbf{e}_j - q^*E_k) - v(\mathbf{x} + E_j - q^*E_k) + v(\mathbf{x} - q^*E_k) \\
&= \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_j - q^*E_k) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - q^*E_k) \geq 0
\end{aligned}$$

The last inequality holds by property 1.

$$2. \quad q_k^*(\mathbf{x} + \mathbf{e}_j, d) = q_k^*(\mathbf{x} + \mathbf{e}_j + E_j, d) = q_k^*(\mathbf{x}, d) = q^* \text{ and } q_k^*(\mathbf{x} + E_j, d) = q^* + 1$$

$$\begin{aligned}
& \Delta_{\mathbf{E}_j} H_k^d v(\mathbf{x} + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_j + E_j - q^*E_k) - v(\mathbf{x} + \mathbf{e}_j - q^*E_k) - v(\mathbf{x} + E_j - (q^* + 1)E_k) + v(\mathbf{x} - q^*E_k) + L_k \\
&= \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_j - q^*E_k) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - q^*E_k) + \Delta_{\mathbf{E}_k} v(\mathbf{x} + E_j - (q^* + 1)E_k) + L_k \geq 0
\end{aligned}$$

The last inequality holds by property 1 and Lemma 4.3.

$$3. \quad q_k^*(\mathbf{x} + \mathbf{e}_j, d) = q_k^*(\mathbf{x} + \mathbf{e}_j + E_j, d) = q^* \text{ and } q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + E_j, d) = q^* + 1$$

$$\begin{aligned}
& \Delta_{\mathbf{E}_j} H_k^d v(\mathbf{x} + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_j + E_j - q^*E_k) - v(\mathbf{x} + \mathbf{e}_j - q^*E_k) - v(\mathbf{x} + E_j - (q^* + 1)E_k) + v(\mathbf{x} - (q^* + 1)E_k) \\
&= \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_j - q^*E_k) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - q^*E_k) + \Delta_{\mathbf{E}_k} v(\mathbf{x} + E_j - (q^* + 1)E_k) - \Delta_{\mathbf{E}_k} v(\mathbf{x} - (q^* + 1)E_k) \\
&= \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_j - q^*E_k) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - (q^* + 1)E_k) \geq 0
\end{aligned}$$

The last inequality holds by property 1 and 4 similar to inequality below.

$$\begin{aligned}
& \Delta_{\mathbf{e}_j} v(\mathbf{x} + \mathbf{e}_j - q^* E_k) - \Delta_{\mathbf{e}_j} v(\mathbf{x} - (q^* + 1) E_k) \\
&= \Delta_{\mathbf{e}_j} v(\mathbf{x} + \mathbf{e}_j - q^* E_k) - \Delta_{\mathbf{e}_j} v(\mathbf{x} - q^* E_k) + \Delta_{\mathbf{e}_j} v(\mathbf{x} - q^* E_k) - \Delta_{\mathbf{e}_j} v(\mathbf{x} - (q^* + 1) E_k) \\
&= \Delta_{j, \mathbf{e}_j} v(\mathbf{x} - q^* E_k) + \Delta_{\mathbf{e}_j, \mathbf{e}_k} v(\mathbf{x} - (q^* + 1) E_k) \geq 0
\end{aligned}$$

4. $q_k^*(\mathbf{x} + \mathbf{e}_j, d) = q_k^*(\mathbf{x}, d) = q^*$ and $q_k^*(\mathbf{x} + \mathbf{e}_j + E_j, d) = q_k^*(\mathbf{x} + E_j, d) = q^* + 1$

$$\begin{aligned}
& \Delta_{\mathbf{e}_j} H_k^d v(\mathbf{x} + \mathbf{e}_j) - \Delta_{\mathbf{e}_j} H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_j + E_j - (q^* + 1) E_k) - v(\mathbf{x} + \mathbf{e}_j - q^* E_k) - v(\mathbf{x} + E_j - (q^* + 1) E_k) + v(\mathbf{x} - q^* E_k) \\
&= \Delta_{\mathbf{e}_j} v(\mathbf{x} + \mathbf{e}_j - q^* E_k) - \Delta_{\mathbf{e}_j} v(\mathbf{x} - q^* E_k) + \Delta_{\mathbf{e}_k} v(\mathbf{x} + E_j - (q^* + 1) E_k) - \Delta_{\mathbf{e}_k} v(\mathbf{x} + \mathbf{e}_j + E_j - (q^* + 1) E_k) \\
&= \Delta_j v(\mathbf{x} + E_j - (q^* + 1) E_k) - \Delta_j v(\mathbf{x} - q^* E_k) \geq 0
\end{aligned}$$

The last inequality holds by convexity property and property 3 similar to inequality below.

$$\begin{aligned}
& \Delta_j v(\mathbf{x} + E_j - (q^* + 1) E_k) - \Delta_j v(\mathbf{x} - q^* E_k) \\
&= \Delta_j v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}_k - q^* E_k) - \Delta_j v(\mathbf{x} - q^* E_k) \\
&= \Delta_j v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}_k - q^* E_k) - \Delta_j v(\mathbf{x} + \mathbf{e}_j - q^* E_k) + \Delta_j v(\mathbf{x} + \mathbf{e}_j - q^* E_k) - \Delta_j v(\mathbf{x} - q^* E_k) \\
&= -\Delta_{j, k} v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}_k - q^* E_k) + \Delta_{j, j} v(\mathbf{x} - q^* E_k) \geq 0
\end{aligned}$$

5. $q_k^*(\mathbf{x} + \mathbf{e}_j, d) = q^*$ and $q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + \mathbf{e}_j + E_j, d) = q_k^*(\mathbf{x} + E_j, d) = q^* + 1$

$$\begin{aligned}
& \Delta_{\mathbf{e}_j} H_k^d v(\mathbf{x} + \mathbf{e}_j) - \Delta_{\mathbf{e}_j} H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_j + E_j - (q^* + 1) E_k) - v(\mathbf{x} + \mathbf{e}_j - q^* E_k) - v(\mathbf{x} + E_j - (q^* + 1) E_k) + v(\mathbf{x} - (q^* + 1) E_k) - L_k \\
&= \Delta_{\mathbf{e}_j} v(\mathbf{x} + \mathbf{e}_j - (q^* + 1) E_k) - \Delta_{\mathbf{e}_j} v(\mathbf{x} - (q^* + 1) E_k) - \Delta_{\mathbf{e}_k} v(\mathbf{x} + \mathbf{e}_j - (q^* + 1) E_k) - L_k \geq 0
\end{aligned}$$

The last inequality holds by property 1 and Lemma 4.3 ($\Delta_{\mathbf{e}_k} v(\mathbf{x} + \mathbf{e}_j - (q^* + 1) E_k) < -L_k$).

6. $q_k^*(\mathbf{x} + \mathbf{e}_j, d) = q^* - 1$, $q_k^*(\mathbf{x} + \mathbf{e}_j + E_j, d) = q_k^*(\mathbf{x}, d) = q^*$ and $q_k^*(\mathbf{x} + E_j, d) = q^* + 1$

$$\begin{aligned}
& \Delta_{\mathbf{e}_j} H_k^d v(\mathbf{x} + \mathbf{e}_j) - \Delta_{\mathbf{e}_j} H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_j + E_j - q^* E_k) - v(\mathbf{x} + \mathbf{e}_j - (q^* - 1) E_k) - v(\mathbf{x} + E_j - (q^* + 1) E_k) + v(\mathbf{x} - q^* E_k) \\
&= v(\mathbf{x} + \mathbf{e}_j + E_j - q^* E_k) - v(\mathbf{x} + \mathbf{e}_j - q^* E_k) - v(\mathbf{x} + E_j - q^* E_k) + v(\mathbf{x} - q^* E_k) \\
&\quad + \Delta_{\mathbf{e}_k} v(\mathbf{x} + E_j - (q^* + 1) E_k) - \Delta_{\mathbf{e}_k} v(\mathbf{x} + \mathbf{e}_j - q^* E_k) \\
&= \Delta_{\mathbf{e}_j} v(\mathbf{x} + \mathbf{e}_j - q^* E_k) - \Delta_{\mathbf{e}_j} v(\mathbf{x} - q^* E_k) + \Delta_{\mathbf{e}_k} v(\mathbf{x} + E_j - (q^* + 1) E_k) - \Delta_{\mathbf{e}_k} v(\mathbf{x} + \mathbf{e}_j - q^* E_k) \geq 0
\end{aligned}$$

The last inequality holds by property 1 and Lemma 4.3 ($\Delta_{\mathbf{e}_k} v(\mathbf{x} + \mathbf{e}_j - q^* E_k) < -L_k \leq \Delta_{\mathbf{e}_k} v(\mathbf{x} + E_j - (q^* + 1) E_k)$).

$$7. \quad q_k^*(\mathbf{x} + \mathbf{e}_j, d) = q^* - 1, \quad q_k^*(\mathbf{x} + \mathbf{e}_j + E_j, d) = q^* \text{ and } q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + E_j, d) = q^* + 1$$

$$\begin{aligned} & \Delta_{\mathbf{E}_j} H_k^d v(\mathbf{x} + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_j + E_j - q^* E_k) - v(\mathbf{x} + \mathbf{e}_j - (q^* - 1)E_k) - v(\mathbf{x} + E_j - (q^* + 1)E_k) + v(\mathbf{x} - (q^* + 1)E_k) - L_k \\ &= \Delta_{\mathbf{E}_j} v(\mathbf{x} + \mathbf{e}_j - q^* E_k) - \Delta_{\mathbf{E}_j} v(\mathbf{x} - (q^* + 1)E_k) - \Delta_{\mathbf{E}_k} v(\mathbf{x} + \mathbf{e}_j - q^* E_k) - L_k \\ &= \Delta_{j, \mathbf{E}_j} v(\mathbf{x} - q^* E_k) + \Delta_{\mathbf{E}_j, \mathbf{E}_k} v(\mathbf{x} - (q^* + 1)E_k) - \Delta_{\mathbf{E}_k} v(\mathbf{x} + \mathbf{e}_j - q^* E_k) - L_k \geq 0 \end{aligned}$$

The last inequality holds by property 1, 4 and Lemma 4.3 ($\Delta_{\mathbf{E}_k} v(\mathbf{x} + \mathbf{e}_j - q^* E_k) < -L_k$).

$$8. \quad q_k^*(\mathbf{x} + \mathbf{e}_j, d) = q^* - 1, \quad q_k^*(\mathbf{x}, d) = q^* \text{ and } q_k^*(\mathbf{x} + \mathbf{e}_j + E_j, d) = q_k^*(\mathbf{x} + E_j, d) = q^* + 1$$

$$\begin{aligned} & \Delta_{\mathbf{E}_j} H_k^d v(\mathbf{x} + \mathbf{e}_j) - \Delta_{\mathbf{E}_j} H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_j + E_j - (q^* + 1)E_k) - v(\mathbf{x} + \mathbf{e}_j - (q^* - 1)E_k) - v(\mathbf{x} + E_j - (q^* + 1)E_k) + v(\mathbf{x} - q^* E_k) - L_k \\ &= \Delta_j v(\mathbf{x} + E_j - (q^* + 1)E_k) - \Delta_j v(\mathbf{x} - q^* E_k) - \Delta_{\mathbf{E}_k} v(\mathbf{x} + \mathbf{e}_j - q^* E_k) - L_k \\ &= \Delta_{j, j} v(\mathbf{x} - \mathbf{e}_k - q^* E_k) - \Delta_{j, k} v(\mathbf{x} - \mathbf{e}_k - q^* E_k) - \Delta_{\mathbf{E}_k} v(\mathbf{x} + \mathbf{e}_j - q^* E_k) - L_k \geq 0 \end{aligned}$$

The above inequality holds due to the convexity property, property 3 and Lemma 4.3 (

$$\Delta_{\mathbf{E}_k} v(\mathbf{x} + \mathbf{e}_j - q^* E_k) \leq -L_k).$$

Property 2

In order to prove operator $H_k^d v$ satisfy Property 2, we need to show

$\Delta_j H_k^d v(\mathbf{x} + \mathbf{e}_0) - \Delta_j H_k^d v(\mathbf{x}) \leq 0$ for $j \in \{1, 2\}$. By the definition of the allocation operator, we

know

$$\begin{aligned} & \Delta_j H_k^d v(\mathbf{x} + \mathbf{e}_0) - \Delta_j H_k^d v(\mathbf{x}) \\ &= H_k^d v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j) - H_k^d v(\mathbf{x} + \mathbf{e}_0) - H_k^d v(\mathbf{x} + \mathbf{e}_j) + H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j - q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j, d)E_k) - v(\mathbf{x} + \mathbf{e}_0 - q_k^*(\mathbf{x} + \mathbf{e}_0, d)E_k) \\ & \quad - v(\mathbf{x} + \mathbf{e}_j - q_k^*(\mathbf{x} + \mathbf{e}_j, d)E_k) + v(\mathbf{x} - q_k^*(\mathbf{x}, d)E_k) \\ & \quad - (q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j, d) - q_k^*(\mathbf{x} + \mathbf{e}_0, d) - q_k^*(\mathbf{x} + \mathbf{e}_j, d) + q_k^*(\mathbf{x}, d))L_k \end{aligned}$$

We discuss the following two cases.

Case 1: $j = k$

$$\begin{aligned}
& \Delta_k H_k^d v(\mathbf{x} + \mathbf{e}_0) - \Delta_k H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_k - q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_k, d) E_k) - v(\mathbf{x} + \mathbf{e}_0 - q_k^*(\mathbf{x} + \mathbf{e}_0, d) E_k) \\
&\quad - v(\mathbf{x} + \mathbf{e}_k - q_k^*(\mathbf{x} + \mathbf{e}_k, d) E_k) + v(\mathbf{x} - q_k^*(\mathbf{x}, d) E_k) \\
&\quad - (q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_k, d) - q_k^*(\mathbf{x} + \mathbf{e}_0, d) - q_k^*(\mathbf{x} + \mathbf{e}_k, d) + q_k^*(\mathbf{x}, d)) L_k
\end{aligned}$$

By Lemma 4.4, we recognize that:

$$q_k^*(\mathbf{x}, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_0, d) \leq q_k^*(\mathbf{x}, d) + 1;$$

$$q_k^*(\mathbf{x}, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_k, d) \leq q_k^*(\mathbf{x}, d) + 1;$$

$$q_k^*(\mathbf{x} + \mathbf{e}_0, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_k, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_0, d) + 1 \quad \text{and}$$

$$q_k^*(\mathbf{x} + \mathbf{e}_k, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_k, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_k, d) + 1.$$

We consider all combinations of these conditions and discuss the following subcases.

$$1. \quad q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + \mathbf{e}_0, d) = q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_k, d) = q^*$$

$$\begin{aligned}
& \Delta_k H_k^d v(\mathbf{x} + \mathbf{e}_0) - \Delta_k H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_k - q^* E_k) - v(\mathbf{x} + \mathbf{e}_0 - q^* E_k) - v(\mathbf{x} + \mathbf{e}_k - q^* E_k) + v(\mathbf{x} - q^* E_k) \\
&= \Delta_k v(\mathbf{x} + \mathbf{e}_0 - q^* E_k) - \Delta_k v(\mathbf{x} - q^* E_k) \leq 0
\end{aligned}$$

The above inequality holds by property 2.

$$2. \quad q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + \mathbf{e}_0, d) = q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q^* \quad \text{and} \quad q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_k, d) = q^* + 1$$

$$\begin{aligned}
& \Delta_k H_k^d v(\mathbf{x} + \mathbf{e}_0) - \Delta_k H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_k - (q^* + 1) E_k) - v(\mathbf{x} + \mathbf{e}_0 - q^* E_k) - v(\mathbf{x} + \mathbf{e}_k - q^* E_k) + v(\mathbf{x} - q^* E_k) - L_k \\
&= \Delta_k v(\mathbf{x} + \mathbf{e}_0 - q^* E_k) - \Delta_k v(\mathbf{x} - q^* E_k) - \Delta_{E_k} v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_k - (q^* + 1) E_k) - L_k \leq 0
\end{aligned}$$

The above inequality holds by property 2 and Lemma 4.3 (

$$\Delta_{E_k} v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_k - (q^* + 1) E_k) \geq -L_k).$$

$$3. \quad q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + \mathbf{e}_0, d) = q^* \quad \text{and} \quad q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_k, d) = q^* + 1$$

$$\begin{aligned}
& \Delta_k H_k^d v(\mathbf{x} + \mathbf{e}_0) - \Delta_k H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_k - (q^* + 1) E_k) - v(\mathbf{x} + \mathbf{e}_0 - q^* E_k) - v(\mathbf{x} + \mathbf{e}_k - (q^* + 1) E_k) + v(\mathbf{x} - q^* E_k) \\
&= \Delta_0 v(\mathbf{x} + \mathbf{e}_k - (q^* + 1) E_k) - \Delta_0 v(\mathbf{x} - q^* E_k) \\
&= -\Delta_{0,0} v(\mathbf{x} - \mathbf{e}_0 - q^* E_k) \leq 0
\end{aligned}$$

The above inequality holds by the convexity property.

$$4. \quad q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q^* \text{ and } q_k^*(\mathbf{x} + \mathbf{e}_0, d) = q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_k, d) = q^* + 1$$

$$\begin{aligned} & \Delta_k H_k^d v(\mathbf{x} + \mathbf{e}_0) - \Delta_k H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_k - (q^* + 1)E_k) - v(\mathbf{x} + \mathbf{e}_0 - (q^* + 1)E_k) - v(\mathbf{x} + \mathbf{e}_k - q^*E_k) + v(\mathbf{x} - q^*E_k) \\ &= \Delta_k v(\mathbf{x} + \mathbf{e}_0 - (q^* + 1)E_k) - \Delta_k v(\mathbf{x} - q^*E_k) \\ &= -\Delta_{k,k} v(\mathbf{x} - \mathbf{e}_k - q^*E_k) \leq 0 \end{aligned}$$

The above inequality holds by the convexity property.

$$5. \quad q_k^*(\mathbf{x}, d) = q^* \text{ and } q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q_k^*(\mathbf{x} + \mathbf{e}_0, d) = q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_k, d) = q^* + 1$$

$$\begin{aligned} & \Delta_k H_k^d v(\mathbf{x} + \mathbf{e}_0) - \Delta_k H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_k - (q^* + 1)E_k) - v(\mathbf{x} + \mathbf{e}_0 - (q^* + 1)E_k) - v(\mathbf{x} + \mathbf{e}_k - (q^* + 1)E_k) + v(\mathbf{x} - q^*E_k) + L_k \\ &= \Delta_k v(\mathbf{x} + \mathbf{e}_0 - (q^* + 1)E_k) - \Delta_k v(\mathbf{x} - (q^* + 1)E_k) + \Delta_{E_k} v(\mathbf{x} - (q^* + 1)E_k) + L_k \leq 0 \end{aligned}$$

The above inequality holds by property 2 and Lemma 4.3 ($\Delta_{E_k} v(\mathbf{x} - (q^* + 1)E_k) < -L_k$).

$$6. \quad q_k^*(\mathbf{x}, d) = q^*, q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q_k^*(\mathbf{x} + \mathbf{e}_0, d) = q^* + 1 \text{ and } q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_k, d) = q^* + 2$$

$$\begin{aligned} & \Delta_k H_k^d v(\mathbf{x} + \mathbf{e}_0) - \Delta_k H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_k - (q^* + 2)E_k) - v(\mathbf{x} + \mathbf{e}_0 - (q^* + 1)E_k) - v(\mathbf{x} + \mathbf{e}_k - (q^* + 1)E_k) + v(\mathbf{x} - q^*E_k) \\ &= -\Delta_0 v(\mathbf{x} - (q^* + 1)E_k) + \Delta_0 v(\mathbf{x} - \mathbf{e}_0 - q^*E_k) \\ &= \Delta_{0,k} v(\mathbf{x} - (q^* + 1)E_k) \leq 0 \end{aligned}$$

The above inequality holds by property 2.

$$7. \quad q_k^*(\mathbf{x}, d) = q^*, q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q^* + 1 \text{ and } q_k^*(\mathbf{x} + \mathbf{e}_0, d) = q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_k, d) = q^* + 2$$

$$\begin{aligned} & \Delta_k H_k^d v(\mathbf{x} + \mathbf{e}_0) - \Delta_k H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_k - (q^* + 2)E_k) - v(\mathbf{x} + \mathbf{e}_0 - (q^* + 2)E_k) - v(\mathbf{x} + \mathbf{e}_k - (q^* + 1)E_k) + v(\mathbf{x} - q^*E_k) + L_k \\ &= \Delta_k v(\mathbf{x} + \mathbf{e}_0 - (q^* + 2)E_k) - \Delta_k v(\mathbf{x} - (q^* + 1)E_k) + \Delta_{E_k} v(\mathbf{x} - (q^* + 1)E_k) + L_k \\ &= -\Delta_{k,k} v(\mathbf{x} + \mathbf{e}_0 - (q^* + 2)E_k) + \Delta_{E_k} v(\mathbf{x} - (q^* + 1)E_k) + L_k \leq 0 \end{aligned}$$

The above inequality holds by the convexity property and Lemma 4.3 ($\Delta_{E_k} v(\mathbf{x} - (q^* + 1)E_k) < -L_k$).

$$\Delta_{E_k} v(\mathbf{x} - (q^* + 1)E_k) < -L_k).$$

$$8. \quad q_k^*(\mathbf{x}, d) = q^*, q_k^*(\mathbf{x} + \mathbf{e}_0, d) = q^* + 1 \text{ and } q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_k, d) = q^* + 2$$

$$\begin{aligned}
& \Delta_k H_k^d v(\mathbf{x} + \mathbf{e}_0) - \Delta_k H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_k - (q^* + 2)E_k) - v(\mathbf{x} + \mathbf{e}_0 - (q^* + 1)E_k) - v(\mathbf{x} + \mathbf{e}_k - (q^* + 2)E_k) + v(\mathbf{x} - q^* E_k) + L_k \\
&= \Delta_0 v(\mathbf{x} + \mathbf{e}_k - (q^* + 2)E_k) - \Delta_0 v(\mathbf{x} - (q^* + 1)E_k) + \Delta_{E_k} v(\mathbf{x} - (q^* + 1)E_k) + L_k \\
&= -\Delta_{0,0} v(\mathbf{x} + \mathbf{e}_k - (q^* + 2)E_k) + \Delta_{E_k} v(\mathbf{x} - (q^* + 1)E_k) + L_k \leq 0
\end{aligned}$$

The above inequality holds by the convexity property and Lemma 4.3 (

$$\Delta_{E_k} v(\mathbf{x} - (q^* + 1)E_k) < -L_k).$$

Case 2: $j \neq k$

$$\begin{aligned}
& \Delta_j H_k^d v(\mathbf{x} + \mathbf{e}_0) - \Delta_j H_k^d v(\mathbf{x}) \\
&= H_k^d v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j) - H_k^d v(\mathbf{x} + \mathbf{e}_0) - H_k^d v(\mathbf{x} + \mathbf{e}_j) + H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j - q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j, d)E_k) - v(\mathbf{x} + \mathbf{e}_0 - q_k^*(\mathbf{x} + \mathbf{e}_0, d)E_k) \\
&\quad - v(\mathbf{x} + \mathbf{e}_j - q_k^*(\mathbf{x} + \mathbf{e}_j, d)E_k) + v(\mathbf{x} - q_k^*(\mathbf{x}, d)E_k) \\
&\quad - (q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j, d) - q_k^*(\mathbf{x} + \mathbf{e}_0, d) - q_k^*(\mathbf{x} + \mathbf{e}_j, d) + q_k^*(\mathbf{x}, d))L_k
\end{aligned}$$

By Lemma 4.5, we recognize that:

$$q_k^*(\mathbf{x} + \mathbf{e}_j, d) \leq q_k^*(\mathbf{x}, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_j, d) + 1;$$

$$q_k^*(\mathbf{x}, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j, d) \leq q_k^*(\mathbf{x}, d) + 1 \quad \text{and}$$

$$q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_0, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j, d) + 1.$$

We consider all combinations of these conditions and discuss the following subcases.

$$1. \quad q_k^*(\mathbf{x} + \mathbf{e}_j, d) = q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j, d) = q_k^*(\mathbf{x} + \mathbf{e}_0, d) = q^*$$

$$\begin{aligned}
& \Delta_j H_k^d v(\mathbf{x} + \mathbf{e}_0) - \Delta_j H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j - q^* E_k) - v(\mathbf{x} + \mathbf{e}_0 - q^* E_k) - v(\mathbf{x} + \mathbf{e}_j - q^* E_k) + v(\mathbf{x} - q^* E_k) \\
&= \Delta_j v(\mathbf{x} + \mathbf{e}_0 - q^* E_k) - \Delta_j v(\mathbf{x} - q^* E_k) \leq 0
\end{aligned}$$

The above inequality holds by property 2.

$$2. \quad q_k^*(\mathbf{x} + \mathbf{e}_j, d) = q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j, d) = q^* \quad \text{and} \quad q_k^*(\mathbf{x} + \mathbf{e}_0, d) = q^* + 1$$

$$\begin{aligned}
& \Delta_j H_k^d v(\mathbf{x} + \mathbf{e}_0) - \Delta_j H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j - q^* E_k) - v(\mathbf{x} + \mathbf{e}_0 - (q^* + 1) E_k) - v(\mathbf{x} + \mathbf{e}_j - q^* E_k) + v(\mathbf{x} - q^* E_k) + L_k \\
&\leq \Delta_j v(\mathbf{x} + \mathbf{e}_0 - q^* E_k) - \Delta_j v(\mathbf{x} - q^* E_k) + \Delta_{E_k} v(\mathbf{x} + \mathbf{e}_0 - (q^* + 1) E_k) - \Delta_{E_k} v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j - (q^* + 1) E_k) \\
&= -\Delta_j v(\mathbf{x} - q^* E_k) + \Delta_j v(\mathbf{x} + \mathbf{e}_0 - (q^* + 1) E_k) \\
&= -\Delta_{j,k} v(\mathbf{x} - \mathbf{e}_k - q^* E_k) \leq 0
\end{aligned}$$

The first inequality holds by Lemma 4.3 that states $\Delta_{E_k} v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j - (q^* + 1) E_k) < -L_k$.

The last inequality holds by property 3.

3. $q_k^*(\mathbf{x} + \mathbf{e}_j, d) = q_k^*(\mathbf{x}, d) = q^*$ and $q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j, d) = q_k^*(\mathbf{x} + \mathbf{e}_0, d) = q^* + 1$

$$\begin{aligned}
& \Delta_j H_k^d v(\mathbf{x} + \mathbf{e}_0) - \Delta_j H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j - (q^* + 1) E_k) - v(\mathbf{x} + \mathbf{e}_0 - (q^* + 1) E_k) - v(\mathbf{x} + \mathbf{e}_j - q^* E_k) + v(\mathbf{x} - q^* E_k) \\
&= \Delta_j v(\mathbf{x} + \mathbf{e}_0 - (q^* + 1) E_k) - \Delta_j v(\mathbf{x} - q^* E_k) \\
&= -\Delta_{j,k} v(\mathbf{x} - \mathbf{e}_k - q^* E_k) \leq 0
\end{aligned}$$

The above inequality holds by property 3.

4. $q_k^*(\mathbf{x} + \mathbf{e}_j, d) = q^*$ and $q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j, d) = q_k^*(\mathbf{x} + \mathbf{e}_0, d) = q^* + 1$

$$\begin{aligned}
& \Delta_j H_k^d v(\mathbf{x} + \mathbf{e}_0) - \Delta_j H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j - (q^* + 1) E_k) - v(\mathbf{x} + \mathbf{e}_0 - (q^* + 1) E_k) - v(\mathbf{x} + \mathbf{e}_j - q^* E_k) + v(\mathbf{x} - (q^* + 1) E_k) - L_k \\
&= \Delta_j v(\mathbf{x} + \mathbf{e}_0 - (q^* + 1) E_k) - \Delta_j v(\mathbf{x} - q^* E_k) - \Delta_{E_k} v(\mathbf{x} - (q^* + 1) E_k) - L_k \\
&= -\Delta_{j,k} v(\mathbf{x} - \mathbf{e}_k - q^* E_k) - \Delta_{E_k} v(\mathbf{x} - (q^* + 1) E_k) - L_k \leq 0
\end{aligned}$$

The above inequality holds by property 3 and Lemma 4.3 ($\Delta_{E_k} v(\mathbf{x} - (q^* + 1) E_k) \geq -L_k$).

5. $q_k^*(\mathbf{x} + \mathbf{e}_j, d) = q^*$, $q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j, d) = q^* + 1$ and $q_k^*(\mathbf{x} + \mathbf{e}_0, d) = q^* + 2$

$$\begin{aligned}
& \Delta_j H_k^d v(\mathbf{x} + \mathbf{e}_0) - \Delta_j H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j - (q^* + 1) E_k) - v(\mathbf{x} + \mathbf{e}_0 - (q^* + 2) E_k) - v(\mathbf{x} + \mathbf{e}_j - q^* E_k) + v(\mathbf{x} - (q^* + 1) E_k) \\
&= \Delta_j v(\mathbf{x} + \mathbf{e}_0 - (q^* + 1) E_k) - \Delta_j v(\mathbf{x} - (q^* + 1) E_k) + \Delta_{E_k} v(\mathbf{x} + \mathbf{e}_0 - (q^* + 2) E_k) - \Delta_{E_k} v(\mathbf{x} + \mathbf{e}_j - (q^* + 1) E_k) \\
&= \Delta_j v(\mathbf{x} + \mathbf{e}_0 - (q^* + 1) E_k) - \Delta_j v(\mathbf{x} - (q^* + 1) E_k) - \Delta_{k,E_k} v(\mathbf{x} - \mathbf{e}_k - (q^* + 1) E_k) - \Delta_{j,E_k} v(\mathbf{x} - (q^* + 1) E_k) \\
&= \Delta_j v(\mathbf{x} + \mathbf{e}_0 - (q^* + 1) E_k) - \Delta_j v(\mathbf{x} - q^* E_k) - \Delta_{k,E_k} v(\mathbf{x} - \mathbf{e}_k - (q^* + 1) E_k) \\
&= -\Delta_{j,k} v(\mathbf{x} - \mathbf{e}_k - q^* E_k) - \Delta_{k,E_k} v(\mathbf{x} - \mathbf{e}_k - (q^* + 1) E_k) \leq 0
\end{aligned}$$

The above inequality holds by property 1 and 3.

$$6. \quad q_k^*(\mathbf{x} + \mathbf{e}_j, d) = q^*, \quad q_k^*(\mathbf{x}, d) = q^* + 1 \text{ and } q_k^*(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j, d) = q_k^*(\mathbf{x} + \mathbf{e}_0, d) = q^* + 2$$

$$\begin{aligned} & \Delta_j H_k^d v(\mathbf{x} + \mathbf{e}_0) - \Delta_j H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j - (q^* + 2)E_k) - v(\mathbf{x} + \mathbf{e}_0 - (q^* + 2)E_k) - v(\mathbf{x} + \mathbf{e}_j - q^*E_k) + v(\mathbf{x} - (q^* + 1)E_k) - L_k \\ &= -\Delta_{j,k} v(\mathbf{x} - \mathbf{e}_k - q^*E_k) - \Delta_{k,E_k} v(\mathbf{x} - \mathbf{e}_k - (q^* + 1)E_k) - \Delta_{E_k} v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j - (q^* + 2)E_k) - L_k \\ &\leq 0 \end{aligned}$$

The above inequality holds by property 1, 3 and Lemma 4.3 (

$$\Delta_{E_k} v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_j - (q^* + 2)E_k) \geq -L_k).$$

Property A3

In order to prove operator $H_k^d v$ satisfy Property 3, we need to show

$$\Delta_j H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_j H_k^d v(\mathbf{x}) \geq 0 \quad \text{for } i \neq j \text{ and } i, j \in \{1, 2\}. \text{ By the definition of the allocation}$$

operator, we know

$$\begin{aligned} & \Delta_j H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_j H_k^d v(\mathbf{x}) \\ &= H_k^d v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) - H_k^d v(\mathbf{x} + \mathbf{e}_i) - H_k^d v(\mathbf{x} + \mathbf{e}_j) + H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j - q_k^*(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j, d)E_k) - v(\mathbf{x} + \mathbf{e}_i - q_k^*(\mathbf{x} + \mathbf{e}_i, d)E_k) \\ &\quad - v(\mathbf{x} + \mathbf{e}_j - q_k^*(\mathbf{x} + \mathbf{e}_j, d)E_k) + v(\mathbf{x} - q_k^*(\mathbf{x}, d)E_k) \\ &\quad - (q_k^*(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j, d) - q_k^*(\mathbf{x} + \mathbf{e}_i, d) - q_k^*(\mathbf{x} + \mathbf{e}_j, d) + q_k^*(\mathbf{x}, d))L_k \end{aligned}$$

We consider the following two cases.

Case 1: $j = k$

$$\begin{aligned} & \Delta_k H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_k H_k^d v(\mathbf{x}) \\ &= H_k^d v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_k) - H_k^d v(\mathbf{x} + \mathbf{e}_i) - H_k^d v(\mathbf{x} + \mathbf{e}_k) + H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_k - q_k^*(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_k, d)E_k) - v(\mathbf{x} + \mathbf{e}_i - q_k^*(\mathbf{x} + \mathbf{e}_i, d)E_k) \\ &\quad - v(\mathbf{x} + \mathbf{e}_k - q_k^*(\mathbf{x} + \mathbf{e}_k, d)E_k) + v(\mathbf{x} - q_k^*(\mathbf{x}, d)E_k) \\ &\quad - (q_k^*(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_k, d) - q_k^*(\mathbf{x} + \mathbf{e}_i, d) - q_k^*(\mathbf{x} + \mathbf{e}_k, d) + q_k^*(\mathbf{x}, d))L_k \end{aligned}$$

By Lemma 4.4, we recognize that:

$$q_k^*(\mathbf{x}, d) - 1 \leq q_k^*(\mathbf{x} + \mathbf{e}_i, d) \leq q_k^*(\mathbf{x}, d);$$

$$q_k^*(\mathbf{x}, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_k, d) \leq q_k^*(\mathbf{x}, d) + 1 \text{ and}$$

$$q_k^*(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_k, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_k, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_k, d) + 1.$$

We consider all combinations of these conditions and discuss the following subcases.

$$1. \quad q_k^*(\mathbf{x} + \mathbf{e}_i, d) = q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_k, d) = q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q^*$$

$$\begin{aligned} & \Delta_k H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_k H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_k - q^* E_k) - v(\mathbf{x} + \mathbf{e}_i - q^* E_k) - v(\mathbf{x} + \mathbf{e}_k - q^* E_k) + v(\mathbf{x} - q^* E_k) \\ &= \Delta_k v(\mathbf{x} + \mathbf{e}_i - q^* E_k) - \Delta_k v(\mathbf{x} - q^* E_k) \geq 0 \end{aligned}$$

The above inequality follows by property 3.

$$2. \quad q_k^*(\mathbf{x} + \mathbf{e}_i, d) = q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_k, d) = q^* \text{ and } q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q^* + 1$$

$$\begin{aligned} & \Delta_k H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_k H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_k - q^* E_k) - v(\mathbf{x} + \mathbf{e}_i - q^* E_k) - v(\mathbf{x} + \mathbf{e}_k - (q^* + 1) E_k) + v(\mathbf{x} - q^* E_k) + L_k \\ &= \Delta_k v(\mathbf{x} + \mathbf{e}_i - q^* E_k) - \Delta_k v(\mathbf{x} - q^* E_k) + \Delta_{E_k} v(\mathbf{x} + \mathbf{e}_k - (q^* + 1) E_k) + L_k \geq 0 \end{aligned}$$

The above inequality follows by property 3 and Lemma 4.3 ($\Delta_{E_k} v(\mathbf{x} + \mathbf{e}_k - (q^* + 1) E_k) \geq -L_k$).

$$3. \quad q_k^*(\mathbf{x} + \mathbf{e}_i, d) = q_k^*(\mathbf{x}, d) = q^* \text{ and } q_k^*(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_k, d) = q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q^* + 1$$

$$\begin{aligned} & \Delta_k H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_k H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_k - (q^* + 1) E_k) - v(\mathbf{x} + \mathbf{e}_i - q^* E_k) - v(\mathbf{x} + \mathbf{e}_k - (q^* + 1) E_k) + v(\mathbf{x} - q^* E_k) \\ &= \Delta_i v(\mathbf{x} + \mathbf{e}_k - (q^* + 1) E_k) - \Delta_i v(\mathbf{x} - q^* E_k) \\ &= -\Delta_{0,i} v(\mathbf{x} + \mathbf{e}_k - (q^* + 1) E_k) \geq 0 \end{aligned}$$

The above inequality holds by property 2.

$$4. \quad q_k^*(\mathbf{x} + \mathbf{e}_i, d) = q^* \text{ and } q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_k, d) = q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q^* + 1$$

$$\begin{aligned} & \Delta_k H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_k H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_k - (q^* + 1) E_k) - v(\mathbf{x} + \mathbf{e}_i - q^* E_k) - v(\mathbf{x} + \mathbf{e}_k - (q^* + 1) E_k) + v(\mathbf{x} - (q^* + 1) E_k) - L_k \\ &= \Delta_i v(\mathbf{x} + \mathbf{e}_k - (q^* + 1) E_k) - \Delta_i v(\mathbf{x} - q^* E_k) - \Delta_{E_k} v(\mathbf{x} - (q^* + 1) E_k) - L_k \\ &= -\Delta_{0,i} v(\mathbf{x} + \mathbf{e}_k - (q^* + 1) E_k) - \Delta_{E_k} v(\mathbf{x} - (q^* + 1) E_k) - L_k \geq 0 \end{aligned}$$

The above inequality holds by property 2 and Lemma 4.3 ($\Delta_{E_k} v(\mathbf{x} - (q^* + 1) E_k) < -L_k$).

$$5. \quad q_k^*(\mathbf{x} + \mathbf{e}_i, d) = q^* - 1, q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_k, d) = q^* \text{ and } q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q^* + 1$$

$$\begin{aligned}
& \Delta_k H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_k H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_k - q^* E_k) - v(\mathbf{x} + \mathbf{e}_i - (q^* - 1)E_k) - v(\mathbf{x} + \mathbf{e}_k - (q^* + 1)E_k) + v(\mathbf{x} - q^* E_k) \\
&= \Delta_k v(\mathbf{x} + \mathbf{e}_i - q^* E_k) - \Delta_k v(\mathbf{x} - q^* E_k) - \Delta_{E_k} v(\mathbf{x} + \mathbf{e}_i - q^* E_k) + \Delta_{E_k} v(\mathbf{x} + \mathbf{e}_k - (q^* + 1)E_k) \geq 0
\end{aligned}$$

The above inequality holds by property 3 and Lemma 4.3 (

$$\Delta_{E_k} v(\mathbf{x} + \mathbf{e}_i - q^* E_k) < -L_k \leq \Delta_{E_k} v(\mathbf{x} + \mathbf{e}_k - (q^* + 1)E_k)).$$

$$6. \quad q_k^*(\mathbf{x} + \mathbf{e}_i, d) = q^* - 1, q_k^*(\mathbf{x}, d) = q^* \text{ and } q_k^*(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_k, d) = q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q^* + 1$$

$$\begin{aligned}
& \Delta_k H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_k H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_k - (q^* + 1)E_k) - v(\mathbf{x} + \mathbf{e}_i - (q^* - 1)E_k) - v(\mathbf{x} + \mathbf{e}_k - (q^* + 1)E_k) + v(\mathbf{x} - q^* E_k) - L_k \\
&= \Delta_i v(\mathbf{x} + \mathbf{e}_k - (q^* + 1)E_k) - \Delta_i v(\mathbf{x} - q^* E_k) - \Delta_{E_i} v(\mathbf{x} + \mathbf{e}_i - q^* E_k) - L_k \\
&= -\Delta_{0,i} v(\mathbf{x} - \mathbf{e}_0 - q^* E_k) - \Delta_{E_i} v(\mathbf{x} + \mathbf{e}_i - q^* E_k) - L_k \geq 0
\end{aligned}$$

The above inequality holds by property 2 and Lemma 4.3 ($\Delta_{E_k} v(\mathbf{x} + \mathbf{e}_i - q^* E_k) < -L_k$).

Case 2: $j \neq k$

In this case, if $j \neq k$, we know $i = k$. Hence, by symmetry, this case is equivalent to the case $j = k$. Thus, we omit the corresponding proof.

Property A4

In order to show that operator $H_k^d v$ satisfies Property 4, we must show that

$$\Delta_{E_j} H_k^d v(\mathbf{x} + E_i) - \Delta_{E_j} H_k^d v(\mathbf{x}) \geq 0 \text{ for } i \neq j \text{ and } i, j, k \in \{1, 2\}.$$

By the definition of allocation operator, we know

$$\begin{aligned}
& \Delta_{E_j} H_k^d v(\mathbf{x} + E_i) - \Delta_{E_j} H_k^d v(\mathbf{x}) \\
&= H_k^d v(\mathbf{x} + E_i + E_j) - H_k^d v(\mathbf{x} + E_i) - H_k^d v(\mathbf{x} + E_j) + H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + E_i + E_j - q_k^*(\mathbf{x} + E_i + E_j, d)E_k) - v(\mathbf{x} + E_i - q_k^*(\mathbf{x} + E_i, d)E_k) \\
&\quad - v(\mathbf{x} + E_j - q_k^*(\mathbf{x} + E_j, d)E_k) + v(\mathbf{x} - q_k^*(\mathbf{x}, d)E_k) \\
&\quad - (q_k^*(\mathbf{x} + E_i + E_j, d) - q_k^*(\mathbf{x} + E_i, d) - q_k^*(\mathbf{x} + E_j, d) + q_k^*(\mathbf{x}, d))L_k
\end{aligned}$$

We consider the following cases.

Case 1: $j = k$

$$\begin{aligned}
& \Delta_{E_k} H_k^d v(\mathbf{x} + E_i) - \Delta_{E_k} H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + E_i + E_k - q_k^*(\mathbf{x} + E_i + E_k, d)E_k) - v(\mathbf{x} + E_i - q_k^*(\mathbf{x} + E_i, d)E_k) \\
&\quad - v(\mathbf{x} + E_k - q_k^*(\mathbf{x} + E_k, d)E_k) + v(\mathbf{x} - q_k^*(\mathbf{x}, d)E_k) \\
&\quad - (q_k^*(\mathbf{x} + E_i + E_k, d) - q_k^*(\mathbf{x} + E_i, d) - q_k^*(\mathbf{x} + E_k, d) + q_k^*(\mathbf{x}, d))L_k
\end{aligned}$$

By Lemma 4.4, we recognize that:

$$q_k^*(\mathbf{x}, d) \leq q_k^*(\mathbf{x} + E_i, d) \leq q_k^*(\mathbf{x}, d) + 1;$$

$$q_k^*(\mathbf{x} + E_i, d) \leq q_k^*(\mathbf{x} + E_k, d) \leq q_k^*(\mathbf{x} + E_i, d) + 1 \quad \text{and}$$

$$q_k^*(\mathbf{x} + E_k, d) \leq q_k^*(\mathbf{x} + E_i + E_k, d) \leq q_k^*(\mathbf{x} + E_k, d) + 1.$$

We consider all combinations of these conditions and discuss the following subcases.

$$1. \quad q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + E_k, d) = q_k^*(\mathbf{x} + E_i, d) = q_k^*(\mathbf{x} + E_i + E_k, d) = q^*$$

$$\begin{aligned}
& \Delta_{E_k} H_k^d v(\mathbf{x} + E_i) - \Delta_{E_k} H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + E_i + E_k - q^*E_k) - v(\mathbf{x} + E_i - q^*E_k) - v(\mathbf{x} + E_k - q^*E_k) + v(\mathbf{x} - q^*E_k) \\
&= \Delta_{E_k} v(\mathbf{x} + E_i - q^*E_k) - \Delta_{E_k} v(\mathbf{x} - q^*E_k) \geq 0
\end{aligned}$$

The above inequality holds by property 4.

$$2. \quad q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + E_k, d) = q_k^*(\mathbf{x} + E_i, d) = q^* \quad \text{and} \quad q_k^*(\mathbf{x} + E_i + E_k, d) = q^* + 1$$

$$\begin{aligned}
& \Delta_{E_k} H_k^d v(\mathbf{x} + E_i) - \Delta_{E_k} H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + E_i + E_k - (q^* + 1)E_k) - v(\mathbf{x} + E_i - q^*E_k) - v(\mathbf{x} + E_k - q^*E_k) + v(\mathbf{x} - q^*E_k) - L_k \\
&= -\Delta_{E_k} v(\mathbf{x} + E_k - (q^* + 1)E_k) - L_k \geq 0
\end{aligned}$$

The above inequality holds by Lemma 4.3 ($\Delta_{E_k} v(\mathbf{x} + E_k - (q^* + 1)E_k) < -L_k$).

$$3. \quad q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + E_i, d) = q^* \quad \text{and} \quad q_k^*(\mathbf{x} + E_k, d) = q_k^*(\mathbf{x} + E_i + E_k, d) = q^* + 1$$

$$\begin{aligned}
& \Delta_{E_k} H_k^d v(\mathbf{x} + E_i) - \Delta_{E_k} H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + E_i + E_k - (q^* + 1)E_k) - v(\mathbf{x} + E_i - q^*E_k) - v(\mathbf{x} + E_k - (q^* + 1)E_k) + v(\mathbf{x} - q^*E_k) = 0
\end{aligned}$$

$$4. \quad q_k^*(\mathbf{x}, d) = q^* \quad \text{and} \quad q_k^*(\mathbf{x} + E_i, d) = q_k^*(\mathbf{x} + E_k, d) = q_k^*(\mathbf{x} + E_i + E_k, d) = q^* + 1$$

$$\begin{aligned}
& \Delta_{E_k} H_k^d v(\mathbf{x} + E_i) - \Delta_{E_k} H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + E_i + E_k - (q^* + 1)E_k) - v(\mathbf{x} + E_i - (q^* + 1)E_k) - v(\mathbf{x} + E_k - (q^* + 1)E_k) + v(\mathbf{x} - q^*E_k) + L_k \\
&= \Delta_{E_k} v(\mathbf{x} + E_i - (q^* + 1)E_k) + L_k \geq 0
\end{aligned}$$

The above inequality follows from Lemma 4.3 ($\Delta_{E_k} v(\mathbf{x} + E_i - (q^* + 1)E_k) \geq -L_k$).

$$5. \quad q_k^*(\mathbf{x}, d) = q^* - 1, \quad q_k^*(\mathbf{x} + E_i, d) = q_k^*(\mathbf{x} + E_k, d) = q^* \text{ and } q_k^*(\mathbf{x} + E_i + E_k, d) = q^* + 1$$

$$\begin{aligned} & \Delta_{E_k} H_k^d v(\mathbf{x} + E_i) - \Delta_{E_k} H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + E_i + E_k - (q^* + 1)E_k) - v(\mathbf{x} + E_i - q^*E_k) - v(\mathbf{x} + E_k - q^*E_k) + v(\mathbf{x} - (q^* - 1)E_k) = 0 \end{aligned}$$

$$6. \quad q_k^*(\mathbf{x}, d) = q^* - 1, \quad q_k^*(\mathbf{x} + E_i, d) = q^* \text{ and } q_k^*(\mathbf{x} + E_k, d) = q_k^*(\mathbf{x} + E_i + E_k, d) = q^* + 1$$

$$\begin{aligned} & \Delta_{E_k} H_k^d v(\mathbf{x} + E_i) - \Delta_{E_k} H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + E_i + E_k - (q^* + 1)E_k) - v(\mathbf{x} + E_i - q^*E_k) - v(\mathbf{x} + E_k - (q^* + 1)E_k) + v(\mathbf{x} - (q^* - 1)E_k) + L_k \\ &= \Delta_{E_k} v(\mathbf{x} + E_k - (q^* + 1)E_k) + L_k \geq 0 \end{aligned}$$

The above inequality follows by Lemma 4.3 ($\Delta_{E_k} v(\mathbf{x} + E_k - (q^* + 1)E_k) \geq -L_k$).

Case 2: $j \neq k$

In this case, if $j \neq k$, we know $i = k$. Hence, by symmetry, this case is equivalent to the case $j = k$. Thus, we omit the corresponding proof.

Property A5

In order to show that operator $H_k^d v$ satisfies Property 5, we must show that

$$\Delta_{E_j} H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_{E_j} H_k^d v(\mathbf{x}) \leq 0 \text{ for } i \neq j \text{ and } i, j, k \in \{1, 2\}.$$

By the definition of allocation operator, we know

$$\begin{aligned} & \Delta_{E_j} H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_{E_j} H_k^d v(\mathbf{x}) \\ &= H_k^d v(\mathbf{x} + \mathbf{e}_i + E_j) - H_k^d v(\mathbf{x} + \mathbf{e}_i) - H_k^d v(\mathbf{x} + E_j) + H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_i + E_j - q_k^*(\mathbf{x} + \mathbf{e}_i + E_j, d)E_k) - v(\mathbf{x} + \mathbf{e}_i - q_k^*(\mathbf{x} + \mathbf{e}_i, d)E_k) \\ & \quad - v(\mathbf{x} + E_j - q_k^*(\mathbf{x} + E_j, d)E_k) + v(\mathbf{x} - q_k^*(\mathbf{x}, d)E_k) \\ & \quad - (q_k^*(\mathbf{x} + \mathbf{e}_i + E_j, d) - q_k^*(\mathbf{x} + \mathbf{e}_i, d) - q_k^*(\mathbf{x} + E_j, d) + q_k^*(\mathbf{x}, d))L_k \end{aligned}$$

We consider the following two cases.

Case 1: $j = k$

$$\begin{aligned}
& \Delta_{E_k} H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_{E_k} H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_i + E_k - q_k^*(\mathbf{x} + \mathbf{e}_i + E_k, d)E_k) - v(\mathbf{x} + \mathbf{e}_i - q_k^*(\mathbf{x} + \mathbf{e}_i, d)E_k) \\
&\quad - v(\mathbf{x} + E_k - q_k^*(\mathbf{x} + E_k, d)E_k) + v(\mathbf{x} - q_k^*(\mathbf{x}, d)E_k) \\
&\quad - (q_k^*(\mathbf{x} + \mathbf{e}_i + E_k, d) - q_k^*(\mathbf{x} + \mathbf{e}_i, d) - q_k^*(\mathbf{x} + E_k, d) + q_k^*(\mathbf{x}, d))L_k
\end{aligned}$$

By Lemma 4.4 and Lemma 4.5, we recognize that:

$$q_k^*(\mathbf{x} + \mathbf{e}_i, d) \leq q_k^*(\mathbf{x}, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_i, d) + 1,$$

$$q_k^*(\mathbf{x}, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_i + E_k, d) \leq q_k^*(\mathbf{x}, d) + 1 \quad \text{and}$$

$$q_k^*(\mathbf{x} + \mathbf{e}_i + E_k, d) \leq q_k^*(\mathbf{x} + E_k, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_i + E_k, d) + 1.$$

We consider all combinations of these conditions and discuss the following subcases.

$$1. \quad q_k^*(\mathbf{x} + \mathbf{e}_i, d) = q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + \mathbf{e}_i + E_k, d) = q_k^*(\mathbf{x} + E_k, d) = q^*$$

$$\begin{aligned}
& \Delta_{E_k} H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_{E_k} H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_i + E_k - q^*E_k) - v(\mathbf{x} + \mathbf{e}_i - q^*E_k) - v(\mathbf{x} + E_k - q^*E_k) + v(\mathbf{x} - q^*E_k) \\
&= \Delta_{E_k} v(\mathbf{x} + \mathbf{e}_i - q^*E_k) - \Delta_{E_k} v(\mathbf{x} - q^*E_k) \leq 0
\end{aligned}$$

The above inequality follows by property 5.

$$2. \quad q_k^*(\mathbf{x} + \mathbf{e}_i, d) = q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + \mathbf{e}_i + E_k, d) = q^* \quad \text{and} \quad q_k^*(\mathbf{x} + E_k, d) = q^* + 1$$

$$\begin{aligned}
& \Delta_{E_k} H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_{E_k} H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_i + E_k - q^*E_k) - v(\mathbf{x} + \mathbf{e}_i - q^*E_k) - v(\mathbf{x} + E_k - (q^* + 1)E_k) + v(\mathbf{x} - q^*E_k) + L_k \\
&= \Delta_{E_k} v(\mathbf{x} + \mathbf{e}_i + E_k - (q^* + 1)E_k) + L_k \leq 0
\end{aligned}$$

The above inequality follows by Lemma 4.3 ($\Delta_{E_k} v(\mathbf{x} + \mathbf{e}_i + E_k - (q^* + 1)E_k) < -L_k$).

$$3. \quad q_k^*(\mathbf{x} + \mathbf{e}_i, d) = q_k^*(\mathbf{x}, d) = q^* \quad \text{and} \quad q_k^*(\mathbf{x} + \mathbf{e}_i + E_k, d) = q_k^*(\mathbf{x} + E_k, d) = q^* + 1$$

$$\begin{aligned}
& \Delta_{E_k} H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_{E_k} H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_i + E_k - (q^* + 1)E_k) - v(\mathbf{x} + \mathbf{e}_i - q^*E_k) - v(\mathbf{x} + E_k - (q^* + 1)E_k) + v(\mathbf{x} - q^*E_k) = 0
\end{aligned}$$

$$4. \quad q_k^*(\mathbf{x} + \mathbf{e}_i, d) = q^* \quad \text{and} \quad q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + \mathbf{e}_i + E_k, d) = q_k^*(\mathbf{x} + E_k, d) = q^* + 1$$

$$\begin{aligned}
& \Delta_{E_k} H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_{E_k} H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_i + E_k - (q^* + 1)E_k) - v(\mathbf{x} + \mathbf{e}_i - q^*E_k) - v(\mathbf{x} + E_k - (q^* + 1)E_k) + v(\mathbf{x} - (q^* + 1)E_k) - L_k \\
&= -\Delta_{E_k} v(\mathbf{x} - (q^* + 1)E_k) - L_k \leq 0
\end{aligned}$$

The above inequality follows by Lemma 4.3 ($\Delta_{E_k} v(\mathbf{x} - (q^* + 1)E_k) \geq -L_k$).

$$5. \quad q_k^*(\mathbf{x} + \mathbf{e}_i, d) = q^* - 1, \quad q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + \mathbf{e}_i + E_k, d) = q^* \text{ and } q_k^*(\mathbf{x} + E_k, d) = q^* + 1$$

$$\begin{aligned} & \Delta_{E_k} H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_{E_k} H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_i + E_k - q^* E_k) - v(\mathbf{x} + \mathbf{e}_i - (q^* - 1)E_k) - v(\mathbf{x} + E_k - (q^* + 1)E_k) + v(\mathbf{x} - q^* E_k) = 0 \end{aligned}$$

$$6. \quad q_k^*(\mathbf{x} + \mathbf{e}_i, d) = q^* - 1, \quad q_k^*(\mathbf{x}, d) = q^* \text{ and } q_k^*(\mathbf{x} + \mathbf{e}_i + E_k, d) = q_k^*(\mathbf{x} + E_k, d) = q^* + 1$$

$$\begin{aligned} & \Delta_{E_k} H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_{E_k} H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_i + E_k - (q^* + 1)E_k) - v(\mathbf{x} + \mathbf{e}_i - (q^* - 1)E_k) - v(\mathbf{x} + E_k - (q^* + 1)E_k) + v(\mathbf{x} - q^* E_k) - L_k \\ &= -\Delta_{E_k} v(\mathbf{x} + \mathbf{e}_i + E_k - (q^* + 1)E_k) - L_k \leq 0 \end{aligned}$$

The above inequality follows by Lemma 4.3 ($\Delta_{E_k} v(\mathbf{x} + \mathbf{e}_i + E_k - (q^* + 1)E_k) \geq -L_k$).

Case 2: $j \neq k$

In this case, we know $i = k$.

$$\begin{aligned} & \Delta_{E_j} H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_{E_j} H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_k + E_j - q_k^*(\mathbf{x} + \mathbf{e}_k + E_j, d)E_k) - v(\mathbf{x} + \mathbf{e}_k - q_k^*(\mathbf{x} + \mathbf{e}_k, d)E_k) \\ & \quad - v(\mathbf{x} + E_j - q_k^*(\mathbf{x} + E_j, d)E_k) + v(\mathbf{x} - q_k^*(\mathbf{x}, d)E_k) \\ & \quad - (q_k^*(\mathbf{x} + \mathbf{e}_k + E_j, d) - q_k^*(\mathbf{x} + \mathbf{e}_k, d) - q_k^*(\mathbf{x} + E_j, d) + q_k^*(\mathbf{x}, d))L_k \end{aligned}$$

By Lemma 4.4 and Lemma 4.5, we recognize that:

$$q_k^*(\mathbf{x}, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_k, d) \leq q_k^*(\mathbf{x}, d) + 1;$$

$$q_k^*(\mathbf{x}, d) \leq q_k^*(\mathbf{x} + E_j, d) \leq q_k^*(\mathbf{x}, d) + 1;$$

$$q_k^*(\mathbf{x} + \mathbf{e}_k, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_k + E_j, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_k, d) + 1 \text{ and}$$

$$q_k^*(\mathbf{x} + E_j, d) \leq q_k^*(\mathbf{x} + \mathbf{e}_k + E_j, d) \leq q_k^*(\mathbf{x} + E_j, d) + 1.$$

We consider all combinations of these conditions and discuss the following subcases.

$$1. \quad q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q_k^*(\mathbf{x} + E_j, d) = q_k^*(\mathbf{x} + \mathbf{e}_k + E_j, d) = q^*$$

$$\begin{aligned} & \Delta_{E_j} H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_{E_j} H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_k + E_j - q^* E_k) - v(\mathbf{x} + \mathbf{e}_k - q^* E_k) - v(\mathbf{x} + E_j - q^* E_k) + v(\mathbf{x} - q^* E_k) \\ &= \Delta_{E_j} v(\mathbf{x} + \mathbf{e}_k - q^* E_k) - \Delta_{E_j} v(\mathbf{x} - q^* E_k) \leq 0 \end{aligned}$$

The above inequality follows by property 5.

$$2. \quad q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q_k^*(\mathbf{x} + E_j, d) = q^* \text{ and } q_k^*(\mathbf{x} + \mathbf{e}_k + E_j, d) = q^* + 1$$

$$\begin{aligned} & \Delta_{E_j} H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_{E_j} H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_k + E_j - (q^* + 1)E_k) - v(\mathbf{x} + \mathbf{e}_k - q^* E_k) - v(\mathbf{x} + E_j - q^* E_k) + v(\mathbf{x} - q^* E_k) - L_k \\ &= \Delta_{E_j} v(\mathbf{x} + \mathbf{e}_k - q^* E_k) - \Delta_{E_j} v(\mathbf{x} - q^* E_k) - \Delta_{E_k} v(\mathbf{x} + \mathbf{e}_k + E_j - (q^* + 1)E_k) - L_k \leq 0 \end{aligned}$$

The above inequality follows by property 5 and Lemma 4.3 (

$$\Delta_{E_k} v(\mathbf{x} + \mathbf{e}_k + E_j - (q^* + 1)E_k) \geq -L_k).$$

$$3. \quad q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q^* \text{ and } q_k^*(\mathbf{x} + E_j, d) = q_k^*(\mathbf{x} + \mathbf{e}_k + E_j, d) = q^* + 1$$

$$\begin{aligned} & \Delta_{E_j} H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_{E_j} H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_k + E_j - (q^* + 1)E_k) - v(\mathbf{x} + \mathbf{e}_k - q^* E_k) - v(\mathbf{x} + E_j - (q^* + 1)E_k) + v(\mathbf{x} - q^* E_k) \\ &= \Delta_k v(\mathbf{x} + E_j - (q^* + 1)E_k) - \Delta_k v(\mathbf{x} - q^* E_k) \\ &= \Delta_{k, E_j} v(\mathbf{x} - q^* E_k) - \Delta_{k, E_k} v(\mathbf{x} + E_j - (q^* + 1)E_k) \leq 0 \end{aligned}$$

The above inequality follows by property 1 and 5.

$$4. \quad q_k^*(\mathbf{x}, d) = q_k^*(\mathbf{x} + E_j, d) = q^* \text{ and } q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q_k^*(\mathbf{x} + \mathbf{e}_k + E_j, d) = q^* + 1$$

$$\begin{aligned} & \Delta_{E_j} H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_{E_j} H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_k + E_j - (q^* + 1)E_k) - v(\mathbf{x} + \mathbf{e}_k - (q^* + 1)E_k) - v(\mathbf{x} + E_j - q^* E_k) + v(\mathbf{x} - q^* E_k) \\ &= \Delta_{E_j} v(\mathbf{x} + \mathbf{e}_k - (q^* + 1)E_k) - \Delta_{E_j} v(\mathbf{x} - q^* E_k) \\ &= -\Delta_{0, E_j} v(\mathbf{x} + \mathbf{e}_k - (q^* + 1)E_k) \leq 0 \end{aligned}$$

The above inequality follows by property 1.

$$5. \quad q_k^*(\mathbf{x}, d) = q^* \text{ and } q_k^*(\mathbf{x} + E_j, d) = q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q_k^*(\mathbf{x} + \mathbf{e}_k + E_j, d) = q^* + 1$$

$$\begin{aligned} & \Delta_{E_j} H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_{E_j} H_k^d v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_k + E_j - (q^* + 1)E_k) - v(\mathbf{x} + \mathbf{e}_k - (q^* + 1)E_k) - v(\mathbf{x} + E_j - (q^* + 1)E_k) + v(\mathbf{x} - q^* E_k) + L_k \\ &= \Delta_{E_j} v(\mathbf{x} + \mathbf{e}_k - (q^* + 1)E_k) - \Delta_{E_j} v(\mathbf{x} - (q^* + 1)E_k) + \Delta_{E_k} v(\mathbf{x} - (q^* + 1)E_k) + L_k \leq 0 \end{aligned}$$

The above inequality follows by property 5 and Lemma 4.3 ($\Delta_{E_k} v(\mathbf{x} - (q^* + 1)E_k) < -L_k$).

$$6. \quad q_k^*(\mathbf{x}, d) = q^* - 1, \quad q_k^*(\mathbf{x} + E_j, d) = q_k^*(\mathbf{x} + \mathbf{e}_k, d) = q^* \text{ and } q_k^*(\mathbf{x} + \mathbf{e}_k + E_j, d) = q^* + 1$$

$$\begin{aligned}
& \Delta_{E_j} H_k^d v(\mathbf{x} + \mathbf{e}_i) - \Delta_{E_j} H_k^d v(\mathbf{x}) \\
&= v(\mathbf{x} + \mathbf{e}_k + E_j - (q^* + 1)E_k) - v(\mathbf{x} + \mathbf{e}_k - q^*E_k) - v(\mathbf{x} + E_j - q^*E_k) + v(\mathbf{x} - (q^* - 1)E_k) \\
&= \Delta_{E_j} v(\mathbf{x} + \mathbf{e}_k - q^*E_k) - \Delta_{E_j} v(\mathbf{x} - q^*E_k) - \Delta_{E_k} v(\mathbf{x} + \mathbf{e}_k + E_j - (q^* + 1)E_k) + \Delta_{E_k} v(\mathbf{x} - q^*E_k) \\
&= \Delta_{k, E_j} v(\mathbf{x} - q^*E_k) - \Delta_{0, E_k} v(\mathbf{x} - q^*E_k) \leq 0
\end{aligned}$$

The above inequality follows by property 1 and 5.

For the production operator $P_i v(x)$ and the holding cost function $h(\mathbf{x})$, we can easily verify that they satisfy properties 1-5. Since \mathfrak{V} is closed under multiplication by a scalar and addition, Ov satisfies properties 1-5.

Hence, Operator $Ov \in \mathfrak{V}$. \square

Adopting a similar approach to that used in the previous section, we can show that the recurrent region, the base-stock, and rationing thresholds retain the same definition. The Optimal policy has a similar structure to the one described in Theorem 4.2: For any state, \mathbf{x} , in the recurrent region it is optimal to produce $S_i(\tilde{\mathbf{x}}_i) - \mathbf{x}_i$ units of Component i when $x_i < S_i(\tilde{\mathbf{x}}_i)$, and not to produce when $x_i = S_i(\tilde{\mathbf{x}}_i)$; For an incoming demand of size d units, it is optimal to fulfill q_j^* units of Product j , where $q_j^* = \min\{d, \max(0, x_0 - r_j^0(\tilde{\mathbf{x}}_0)), \max(0, x_j - r_j^j(\tilde{\mathbf{x}}_j))\}$. Furthermore, $S_i(\tilde{\mathbf{x}}_i)$, $i \in \{0, 1, 2\}$, and $r_j^k(\tilde{\mathbf{x}}_j)$, $k \in \{0, j\}$, $j \in \{1, 2\}$, retain all the properties described in (iii) in Theorem 4.2.

4.3 Extension to K-Erlang production process

In this case, we assume that the time required to produce Component i , $i \in \{0, 1, 2\}$, follows an Erlang distribution with shape parameter K_i and scale parameter μ . That is, the probability density function of the lead time is $f(t; K_i, \mu) = \mu^{K_i} t^{K_i-1} e^{-\mu} / (K_i - 1)!$. The Erlang distribution allows modelling a variety of production time distributions with varying scales and shapes. It is

also useful in modeling settings where production takes place over several stages. Thus, in order to simplify the analysis, we take this view and refer to K_i as the number of stages in the production process, with stage 1 being the starting stage. As a result, each production stage takes a production time that is exponentially distributed with mean $1/K_i\mu_i$. Let y_i denote the current production stage. Then, $y_i = 0$ means that the production facility is currently idle (not producing), and $y_i = K_i$ means that Component i is at its last production stage after which a unit is transferred to the stock of Component i , x_i . Thus the optimality equation in (3.6) can be modified as follows

$$v^*(\mathbf{x}, \mathbf{y}) = h(\mathbf{x}, \mathbf{y}) + \sum_{j=1}^2 \lambda_j A_j v^*(\mathbf{x}, \mathbf{y}) + \sum_{i=0}^2 K_i \mu_i P_i v^*(\mathbf{x}, \mathbf{y}) \quad (4.6)$$

where for any real-valued function, $v(\mathbf{x}, \mathbf{y})$, defined on \mathbb{Z}_+^3 , Operator A_j is defined as

$$A_j v(\mathbf{x}, \mathbf{y}) = \begin{cases} v(\mathbf{x}, \mathbf{y}) + L_j & \text{if } x_0 x_j = 0, \\ \min\{v(\mathbf{x} - \mathbf{e}_j, \mathbf{y}), v(\mathbf{x}, \mathbf{y}) + L_j\} & \text{otherwise,} \end{cases} \quad (4.7)$$

and Operator P_i is defined as

$$P_i v(\mathbf{x}, \mathbf{y}) = \begin{cases} \min\{v(\mathbf{x}, \mathbf{y}), v(\mathbf{x}, \mathbf{y} + \mathbf{e}_i)\} & \text{if } y_i = 0, \\ v(\mathbf{x}, \mathbf{y} + \mathbf{e}_i) & \text{if } 0 < y_i < K_i, \\ v(\mathbf{x} + \mathbf{e}_i, \mathbf{y} - y_i \mathbf{e}_i) & \text{otherwise,} \end{cases} \quad (4.8)$$

Let Operator $Tv(\mathbf{x}, \mathbf{y}) = h(\mathbf{x}, \mathbf{y}) + \sum_{j=1}^2 \lambda_j A_j v(\mathbf{x}, \mathbf{y}) + \sum_{i=0}^2 K_i \mu_i P_i v(\mathbf{x}, \mathbf{y})$.

We redefine the production base-stock level of component i as follows,

$$S_i(\tilde{\mathbf{x}}_i, \tilde{\mathbf{y}}_i) = \min\{x_i \in \mathbb{Z}_+, y_i = 0 \mid v(\mathbf{x} + \mathbf{e}_i, \mathbf{y}) - v(\mathbf{x}, \mathbf{y}) \geq 0\} \quad (4.9)$$

and letting $\mathbf{S}(\mathbf{x}, \mathbf{y}) = (S_1(\tilde{\mathbf{x}}_1, \tilde{\mathbf{y}}_1), S_2(\tilde{\mathbf{x}}_2, \tilde{\mathbf{y}}_2), S_3(\tilde{\mathbf{x}}_3, \tilde{\mathbf{y}}_3))$. We redefine the recurrent region, $\mathfrak{S}_{\mathbf{a}}$,

as $\mathfrak{S}_{\mathbf{a}} = \{\mathbf{x} \in \mathbb{Z}_+^3, 0 \leq y_i \leq K_i, i \in \{0, 1, 2\} : \mathbf{x} \leq \mathbf{S}(\mathbf{x}, \mathbf{y})\}$.

Then, for any state $\mathbf{x} \in \mathfrak{S}_{\mathbf{a}}$, we can rewrite $P_i v(\mathbf{x}, \mathbf{y})$ as

$$P_i v(\mathbf{x}, \mathbf{y}) = \begin{cases} v(\mathbf{x}, \mathbf{y} + \mathbf{e}_i) & \text{if } 0 \leq y_i < K_i, \\ v(\mathbf{x} + \mathbf{e}_i, \mathbf{y} - y_i \mathbf{e}_i) & \text{otherwise.} \end{cases} \quad (4.10)$$

As a result, all the properties we defined for the original model still hold. The optimal policy in

this case has the exact structure as in the original model.

Lemma 4.6 identifies the characterizations shown in Lemma 4.2 still preserve under the K-Erlang production process case.

Lemma 4.6: for any state $\mathbf{x} \notin \mathfrak{S}_{\mathfrak{a}}$, if the cost function v satisfies $\Delta_j v(\mathbf{x}) \geq 0$, then Tv satisfies $\Delta_j Tv(\mathbf{x}) \geq 0$, for $j \in \{0,1,2\}$. Furthermore, for $\mathbf{x} \in \mathfrak{S}_{\mathfrak{a}}$, the optimal cost function v^* satisfies the property $\Delta_j v^*(\mathbf{x}) \geq 0$, for $j \in \{0,1,2\}$.

Proof: From Operator $Tv(\mathbf{x})$, we know

$$\Delta_j Tv(\mathbf{x}, \mathbf{y}) = \sum_{i=0}^2 \Delta_j h(\mathbf{x}, \mathbf{y}) + \sum_{k=1}^2 \lambda_k \Delta_j A_k v(\mathbf{x}, \mathbf{y}) + \sum_{i=0}^2 K_i \mu_i \Delta_j P_i v(\mathbf{x}, \mathbf{y}), \text{ where } i, j = \{0,1,2\}, k = \{1,2\}$$

In order to prove $\Delta_j Tv(\mathbf{x}) \geq 0$, for $x_i > s_i(\tilde{\mathbf{x}}_i)$. We discuss each part of the optimality equation separately.

1. Operator $P_i v(\mathbf{x}, \mathbf{y})$

Case 1: $i = j$

$$\Delta_j P_j v(\mathbf{x}, \mathbf{y}) = P_j v(\mathbf{x} + \mathbf{e}_j, \mathbf{y}) - P_j v(\mathbf{x}, \mathbf{y}) = v(\mathbf{x} + \mathbf{e}_j, \mathbf{y}) - v(\mathbf{x}, \mathbf{y}) = \Delta_j v(\mathbf{x}, \mathbf{y}) \geq 0$$

Case 2: $i \neq j$

$$\begin{aligned} \Delta_j P_i v(\mathbf{x}, \mathbf{y}) &= P_i v(\mathbf{x} + \mathbf{e}_j, \mathbf{y}) - P_i v(\mathbf{x}, \mathbf{y}) \\ &= \min \{v(\mathbf{x} + \mathbf{e}_j, \mathbf{y} + \mathbf{e}_i), v(\mathbf{x} + \mathbf{e}_j, \mathbf{y})\} - \min \{v(\mathbf{x}, \mathbf{y} + \mathbf{e}_i), v(\mathbf{x}, \mathbf{y})\} \geq 0 \end{aligned}$$

The above inequality is held by the observations that $v(\mathbf{x} + \mathbf{e}_j, \mathbf{y} + \mathbf{e}_i) \geq v(\mathbf{x}, \mathbf{y} + \mathbf{e}_i)$ and $v(\mathbf{x} + \mathbf{e}_j, \mathbf{y}) \geq v(\mathbf{x}, \mathbf{y})$.

2. Operator $A_k v(\mathbf{x}, \mathbf{y})$

In order to show that operator $A_k v(\mathbf{x}, \mathbf{y})$ satisfies Lemma 4.6, we must show that

$$\Delta_j A_k v(\mathbf{x}, \mathbf{y}) \geq 0 \text{ in the transient region for all } j = \{0,1,2\} \text{ and } k = \{1,2\}.$$

$$\begin{aligned}\Delta_j R_k v(\mathbf{x}, \mathbf{y}) &= R_k v(\mathbf{x} + \mathbf{e}_j, \mathbf{y}) - R_k v(\mathbf{x}, \mathbf{y}) \\ &= \min\{v(\mathbf{x} + \mathbf{e}_j - E_k, \mathbf{y}), v(\mathbf{x} + \mathbf{e}_j, \mathbf{y}) + L_k\} - \min\{v(\mathbf{x} - E_k, \mathbf{y}), v(\mathbf{x}, \mathbf{y}) + L_k\} \geq 0\end{aligned}$$

The above inequality holds by the observations that $v(\mathbf{x} + \mathbf{e}_j - E_k, \mathbf{y}) \geq v(\mathbf{x} - E_k, \mathbf{y})$ and $v(\mathbf{x} + \mathbf{e}_j, \mathbf{y}) + L_k \geq v(\mathbf{x}, \mathbf{y}) + L_k$.

3. Holding cost

Since the holding cost is always non-decreasing in its inventory level and work-in process inventory level, the optimality equation of K-Erlang production process satisfies this property.

□

4.4 Conclusion

In this chapter, we identified the recurrent region. Through numerical experiments, we showed that a set of properties in the recurrent region are satisfied by the optimal value function. We used a propagation argument to show that these properties are preserved by the optimal cost function. These properties allowed us to infer the structure of the optimal production and allocation policies of the continuous-time W -configuration ATO system with lost sales. Specifically, we showed that it is always optimal to produce all components once the state of the system enters the recurrent region and component inventory allocation is governed by a rationing policy that depends on the other components' inventory level that specifies whether a demand from a particular product is fulfilled or not. Finally, we extended the results to the cases of batch production, batch compound Poisson demand process and K-Erlang production times.

Chapter 5. Heuristic methods

In this chapter, we propose several heuristic methods that will provide effective approximate policies to ease the computational burden of the optimal policy. Then, we run a numerical experiment to compare the effectiveness of these heuristic policies under randomly generated system parameters. In addition, we show that the utilization of the heuristics as startup policies in the value/policy iteration algorithms leads to a faster convergence of these exact algorithms. At last, we test one of the heuristics in some larger ATO systems.

5.1 Introduction

As described in the previous chapter, the optimal policy structure is fairly complex since both components' production and allocation decisions are governed by state-dependent thresholds. As such, for a general W -configuration ATO system with N components and $N-1$ products, each threshold represents a switching surface embedded in an $(N-1)$ -dimensional hyperspace. For systems with a large number of components and products, the number of states increases exponentially thus making it prohibitive to compute the optimal policy in a reasonable amount of time. Though the optimal policy can be characterized with relatively simple mathematical expressions, it is difficult to implement in practice due to its complexity. In this chapter, we propose three heuristic methods and compare their effectiveness regarding to the optimal policy. Then, we apply one of the heuristics to some larger ATO systems.

The organization of this chapter is given below. In the second section, we propose a method that is based on a decomposition of the W -configuration ATO systems into two N -configuration sub-systems. We solve each sub-system separately and construct an approximate policy by combining the optimal policies of two sub-systems. In the third section, we propose two state-independent heuristic methods that provide fixed base-stock level and fixed rationing level. In the fourth section, we run a numerical experiment that comprises 100 randomly generated system parameters instances to compare the effectiveness of these heuristic methods. In addition, we show that the heuristic policies can be used as startup

policies in the value iteration and policy iteration algorithms, which results in a much faster converge of these algorithms. In the fifth section, we apply the decomposition method to three larger ATO systems and present its effectiveness.

5.2 Decomposition method

Due to the ‘curse of dimensionality’ in dynamic programming, it is prohibitive to apply it to high-dimensional problems. However, motivated by the relative ease of solving ATO systems with limited number of components and products, we can decompose a general ATO system into several sub-systems and construct a control policy by combining the optimal policy of all sub-systems. As the constructed policy derives the components’ production and allocation policy from the optimal policy of its sub-systems, the resulting heuristic control policy has state-dependent thresholds.

As illustrated in the properties of W -configuration ATO systems with lost sales, the inventory level of component 0 is non-decreasing in the inventory level of Component 1 (2), whereas the inventory level of Component 1(2) is non-increasing in the inventory level of Component 2(1) and non-decreasing in the inventory level of ‘Product’ 2(1). This means that Component 0 has a higher influence on the inventory level of Component 1(2) than Component 2(1). This relationship motivates us to decompose the general ATO system into single-product ATO systems. For instance, a general N -component, $(N-1)$ -product W -configuration ATO system, can be decomposed into $N-1$ sub-systems, where each sub-system contains a product specific component and a common component that is used in another sub-system. By solving each sub-system, we obtain the production policy of the two components and the allocation policy of the end product. For the production policy of product specific component, we map the optimal production policy generated in sub-system directly to the general system by adding the state variable of the rest of the components. For the production policy of common components, we first construct it by minimizing two optimal production policies generated from two sub-systems in order to take advantage of inventory pooling. Then, we map it to the general system by adding the state variable of the rest of the components. For product allocation policies, we take the optimal allocation policy generated

from each sub-system and map it to the general system.

For example, for a 3-component, 2-product, W -configuration ATO system, it can be decomposed into two 2-component ATO sub-systems (cf. Figure 5.1). Each system comprises a common component and a product specific component. The demands arise for the end product as well as for the common component. The demand rate and lost sale cost of the common component is the same as that of the other product. All other parameters remain the same as in the original system. By solving Sub-system 1, we obtain the optimal production policy of Component 0 and Component 1, together with the optimal allocation policy of Product 1, while sub-system 2 gives the optimal production policy of Component 0 and Component 2, together with the optimal allocation policy of Product 2. For the production policy of Component 1, we take the optimal production policy of Component 1 obtained from Sub-system 1 and map it from state (x_1, x_0) to state (x_1, x_0, x_2) . For Component 2, the production policy can be obtained in a similar way. For the production policy of Component 0, as Sub-system 1 and Sub-system 2 both provide an optimal production policy, we take the minimum of these two policies and map it to the original system. For the allocation policy of Product 1(2), we take the optimal allocation policy obtained from Sub-system 1(2) and map it to the original system.

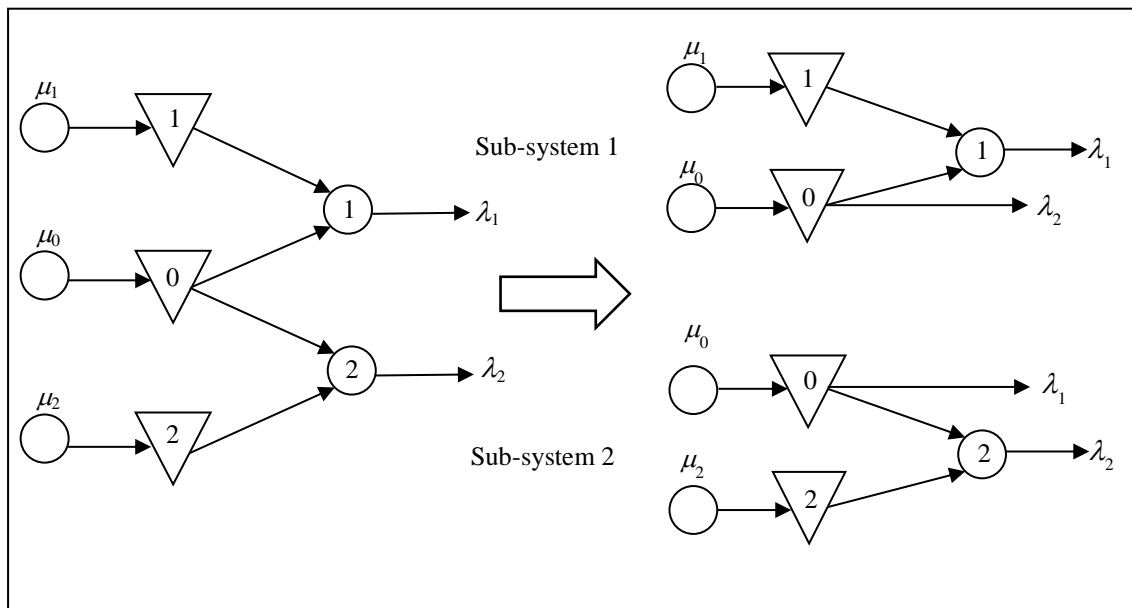


Figure 5.1: Decomposing the W -configuration ATO system into two N -configuration systems

By constructing the heuristic control policy in this way, we retain much of the characteristics of the optimal component production and rationing policy structure in the original system. For instance, the base stock level of Component 1(2) is state-dependent threshold, which is non-decreasing in the inventory level of Component 0, non-decreasing in the inventory level of ‘Product’ 2(1) and non-increasing in the inventory level of ‘Product’ 1(2). This decomposition method only neglects the influence between the product specific components. Thus, we expect this decomposition method to work well.

5.3 (S, R) method and Exhaustive search method

5.3.1 (S, R) method (Heuristic I)

Inspired by the structure of the optimal policy and noticing that the rationing of the inventory of Components 1 and 2 occurs only for low values of the inventory of Component 0, we will only ration the inventory of Component 0 in favor of Product 1. In other words, Product 1 demand is satisfied as long as components 0 and 1 are available. Let r_0^I denote the rationing level below which Product 2 demand is rejected. Thus Product 2 demand is only satisfied when Component 2 is available and the inventory level of Component 0 is above r_0^I . We also denote by s_j^I , $j \in \{0,1,2\}$, the base stock level of Component j under Heuristic I. As such, the inventory of Component 0 behaves as a two-class $M/M/1/s_0^I$ queue, while the inventory of Components 1 and 2, each behaves as a single-class $M/M/1/s_j^I$ queue. The average cost rate, γ^I , induced by Heuristic I can then be determined using the following dynamic programming equation:

$$v(\mathbf{x}) + \gamma^I = \frac{1}{\beta} \left(\sum_{i=0}^2 h_i x_i + \sum_{j=1}^2 \lambda_j A_j^I v(\mathbf{x}) + \sum_{i=0}^2 \mu_i P_i^I v(\mathbf{x}) \right) \quad (5.1)$$

Where Operator A_j^I , $j \in \{1,2\}$, is defined as

$$A_1^I v(\mathbf{x}) = \begin{cases} v(\mathbf{x}) + L_1 & \text{if } x_0, x_1 = 0 \\ v(\mathbf{x} - \mathbf{e}_0 - \mathbf{e}_1) & \text{otherwise} \end{cases} \quad (5.2)$$

$$A_2^I v(\mathbf{x}) = \begin{cases} v(\mathbf{x}) + L_2 & \text{if } x_0 < r_0^I \text{ or } x_2 = 0 \\ v(\mathbf{x} - \mathbf{e}_0 - \mathbf{e}_2) & \text{otherwise} \end{cases} \quad (5.3)$$

Operator P_i^I , $i \in \{0,1,2\}$, is defined as

$$P_i^I v(\mathbf{x}) = \begin{cases} v(\mathbf{x} + \mathbf{e}_i) & x_i < s_i^I \\ v(\mathbf{x}) & x_i = s_i^I \end{cases} \quad (5.4)$$

In order to determine the parameters r_0^I and s_j^I , $j \in \{0,1,2\}$, we treat the inventory level behavior of each component independently. This allows us to obtain the steady state probability distribution of the inventory level of each component which in turn allows us to determine the average holding and lost sales cost under such setting. Finally, minimizing this average cost yields the value of parameters r_0^I and s_j^I under Heuristic **I**.

Letting $p_j(l)$, $0 \leq l \leq s_j$, $j \in \{0,1,2\}$, denote the steady state probability distribution of the inventory level of Component i , we have

$$p_j(l) = \frac{(1 - \rho_j)}{1 - \rho_j^{s_j+1}}, \quad l = 0, 1, \dots, s_j, \text{ for } j \in \{1,2\} \quad (5.5)$$

$$p_0(l) = \begin{cases} \frac{(1 - \eta)(1 - \eta_1)\eta_1^{r_0-1}\eta^{s_0-r_0+1}}{(1 - \eta)(1 - \eta_1^{r_0-1})\eta^{s_0-r_0+1} + (1 - \eta_1)(1 - \eta^{s_0-r_0+1})} & l = 0 \\ \frac{(1 - \eta)(1 - \eta_1)\eta_1^{r_0-l-1}\eta^{s_0-r_0+1}}{(1 - \eta)(1 - \eta_1^{r_0-1})\eta^{s_0-r_0+1} + (1 - \eta_1)(1 - \eta^{s_0-r_0+1})} & l = 1, \dots, r_0 - 1 \\ \frac{(1 - \eta)(1 - \eta_1)\eta^{s_0-l}}{(1 - \eta)(1 - \eta_1^{r_0-1})\eta^{s_0-r_0+1} + (1 - \eta_1)(1 - \eta^{s_0-r_0+1})} & l = r_0, \dots, s_0 \end{cases} \quad (5.6)$$

Where $\rho_j = \lambda_j / \mu_j$, $j \in \{1,2\}$, is the utilization rate of Component 1 and Component 2. $\eta = (\lambda_1 + \lambda_2) / \mu_2$ is the utilization rate of Component 0 by both products, $\eta_1 = \lambda_1 / \mu_2$ is the utilization rate of Component 0 by Product 1.

Let $p_{j,0}$ denote the probability of not satisfying an arriving demand of Product j , $j \in \{1,2\}$. Then,

$$p_{j,0} = p_j(0) + p_0(0) - p_j(0)p_0(0) \quad (5.7)$$

The average cost is then given by

$$\bar{c}(r_0, s_0, s_1, s_2) = \sum_{i=0}^2 h_i \sum_{l=0}^{s_i} (l^* p_i(l)) + \sum_{j=1}^2 \lambda_j L_j p_{j,0} \quad (5.8)$$

and $(r_0^I, s_0^I, s_1^I, s_2^I) = \arg \min_{r_0, s_0, s_1, s_2} \bar{c}(r_0, s_0, s_1, s_2)$.

Through extensive numerical experiments, the cost function in (5.8) is convex in its arguments.

Thus parameters $r_0^I, s_0^I, s_1^I, s_2^I$ can be obtained using classical nonlinear minimization algorithms (see Bazara and Shetty 1993).

5.3.2 Exhaustive search method (Heuristic II)

The exhaustive search heuristic method has the same structure as the (S,R) heuristic method.

However, the parameters $(r_0^II, s_0^II, s_1^II, s_2^II)$ are determined via an exhaustive search on the spaces $[0, r_0^{\max}] \times [0, s_0^{\max}] \times [0, s_1^{\max}] \times [0, s_2^{\max}]$, making it much more computationally

demanding compared to (S,R) heuristic. Here, s_i^{\max} and r_0^{\max} are suitably chosen as noted

below. We propose exhaustive search method for two reasons: First, even though it uses fixed thresholds, in a similar fashion as the (S,R) heuristic method, the exhaustive search method preserves the correlation among component production and demands, which results in a better performance compared to (S,R) heuristic method. Second, as Table 5.1 shows, the intensive computational effort required by the exhaustive search method does not justify its use as it

does not offer much improvement over the (S,R) heuristic method, which is computationally much more efficient. In order to speed up the exhaustive search method, we found it efficient to restrict the search space of parameters $(r_0^II, s_0^II, s_1^II, s_2^II)$ to

$[r_0^I - m_1, r_0^I + m_1] \times [s_0^I - m_2, s_0^I + m_2] \times [s_1^I - m_3, s_1^I + m_3] \times [s_2^I - m_4, s_2^I + m_4]$, where m_k ,

$k = 1, \dots, 4$, have relatively small values.

5.4 Numerical experiments

Table 5.1 shows 100 randomly generated problems each is identified by its set of parameters.

The parameters, shown in the Table, are drawn from a uniform distribution with a relatively

large variation so that extreme cases could be generated. The table also shows the relative average cost gap between the heuristic and the optimal policy which is calculated as:

$$Gap\% = 100(\gamma^* - \gamma^H) / \gamma^* \quad (5.9)$$

Where γ^* and γ^H represent the average cost of the optimal and the heuristic policy, respectively. For ease of exposition, we let Gap I, Gap II and Gap III represent the gap between the cost of (S,R) heuristic, the cost of exhaustive search heuristic and the cost of decomposition heuristic (Heuristic III) policy compared to the optimal cost, respectively.

As Table 5.1 illustrates, all three heuristics perform very well compared to the optimal policy. In particular, for the (S,R) heuristic, the average gap over the 100 problem instances is 1.60% with a minimum gap of 0.04% and a maximum gap of 8.39%; for the exhaustive search heuristic, the average gap over the 100 problem instances is 0.69% with a minimum gap of 0.03% and a maximum gap of 2.35%; for the decomposition method, the average gap over the 100 problem instances is 0.19% with a minimum gap of 0.00% and a maximum gap of 2.57%. As expected, the decomposition method outperforms the other two heuristics, as it retains the same features of the optimal policy. Namely, a state-dependent base stock level and a state-dependent rationing level. For the state-independent heuristics, the exhaustive search method outperforms the (S,R) heuristic. However, this outperformance is not very significant and does not justify using the exhaustive search heuristic over the (S,R) heuristic given the extra computational effort required by the exhaustive search method.

We note that for all heuristics the relatively high % gaps occur for systems where the production rate of Component 0 cannot keep up with the demand rates of the two products. Since in these cases, the inventory level of Component 0 remains low most of the time and recalling, as we mentioned above, that the rationing of Components 1 and 2 occurs at low levels of the inventory of Component 0, given that both the (S,R) and the exhaustive heuristics ignore the rationing of Components 1 and 2, this explains the relatively larger % gaps in those cases. In addition, as the decomposition heuristic preserves part of this rationing structure, though the cost gap is larger than other cases, it is much lower than the previous two heuristics. We also note that the heuristics give the lowest % gaps for systems where the production rates are much larger than the demand rates. In these cases, the product demands are served in a

FCFS fashion and there is no inventory rationing. The cases with high % gap correspond to systems where the production rate of at least one component is below the demand rate of the product using that component. For example, in case 8, the production rate of Component 1 is smaller than the demand rate of Product 1, which resulted in a gap of 4.99%; in case 21 the production rate of Component 0 is smaller than the sum of the demand rates of the products, which resulted in a gap of 7.2%; in case 53, the production rate of Component 2 is smaller than the demand rate of product 2, which resulted in a gap of 8.39%. However, in practice these cases are not common.

The performance of the heuristics can also be partly explained through the optimal steady-state probability distribution of the system states, which is obtained by solving Linear Program (3.13). For example, in the case where the gap is 0.04%, we noted that the significant probabilities are for states where inventory of components is high. However, looking at figures 3.2 and 3.3, we note that within this region the base stock levels are less sensitive to the other components inventories thus can well be approximated by straight lines. Therefore, the % gap is expected to be relatively low in this case. On the other hand, looking at the case where the gap is 7.2%, we noted that the significant probabilities are for states where inventory of components is relatively low. In this case, also as shown in figures 3.2 and 3.3, the base stock levels are very sensitive to the other components inventory levels. Thus, approximating them with fixed thresholds results in a relatively higher gap.

The optimal policy for MDP problems is frequently obtained via dynamic programming based methods such as Value Iteration and Policy Iteration (Puterman, 1994) as they are usually relatively faster than other methods (e.g., Linear Programming method). Nevertheless, these methods still suffer from the curse of dimensionality and generally are very slow even for low-dimension problems. As Table 5.1 shows, the (S,R) heuristic performs quite well against the optimal policy with the advantage of being much faster to obtain. This inspired us to use it as a starting policy for the value and policy iteration methods. This allows us to quantify the gain in computational time by using the (S,R) heuristic as a starting policy in one hand and allows us to extrapolate this gain for systems with larger number of components and products, in the other hand.

Table 5.2 shows the CPU time (in seconds) and the % gain in CPU time using the (S,R)

heuristic as a startup policy for the same systems of Table 5.1. The % gain in CPU time is calculated as follows

$$\% \text{ Gain} = 100(\text{CPU Time} - \text{CPU Time}_1) / \text{CPU Time}_1 \quad (5.10)$$

where CPU Time indicates the CPU Time without using the Heuristic I and CPU Time_1 indicates the CPU Time using the Heuristic I as a startup policy.

Computations were carried on a DELL computer equipped with an Intel Core i5 650 3.2 GHz processor. We note that, without the heuristic, on average, Value Iteration is faster than Policy Iteration with an average CPU time of 38.92 seconds compared to 112.08 seconds. With the (S,R) heuristic, on average, Value Iteration is faster than Policy Iteration with an average CPU time of 16.18 seconds compared to 28.32 seconds. However, Policy Iteration benefit from using heuristic is more than that of Value iteration with an average % CPU time gain of 608.02% compared to 181.89%. The minimum and maximum % CPU time gain for Value Iteration (Policy Iteration) is 101.51% (237.81%) and 10624.19% (1146.75%), respectively.

As one can see from the above results, Using the (S,R) heuristic as a startup policy tremendously reduces the computational effort to obtain the optimal policy. This result is very encouraging as one can use the Heuristic I (in addition of the possibility of using it as an approximate policy to the optimal one as indicated above) to determine a starting policy for larger systems in order to obtain their optimal policy in relatively short amount of time.

Table 5.1: Performance of Heuristic methods

	System Parameters										Optimal cost	% Gap		
	λ_1	λ_2	μ_0	μ_1	μ_2	h_0	h_1	h_2	L_1	L_2		I	II	III
1	1.841	1.254	3.131	2.849	1.186	0.10	0.09	0.08	30.73	26.81	68,39	3.54	1.22	0.15
2	1.352	1.831	3.739	1.629	1.743	0.09	0.16	0.16	40.21	35.33	94,36	2.47	1.02	0.17
3	1.076	1.054	2.602	1.112	0.994	0.07	0.14	0.12	27.39	19.22	86,83	2.75	1.06	0.44
4	1.162	1.794	4.305	1.422	2.994	0.14	0.15	0.09	50.42	38.81	41,75	0.44	0.10	0.00
5	1.451	1.084	3.976	1.384	1.833	0.17	0.20	0.13	53.85	32.84	98,41	1.07	0.70	0.06
6	1.107	1.962	6.103	1.147	1.981	0.18	0.11	0.06	52.25	30.61	55,68	0.94	0.58	0.14
7	1.431	1.911	5.487	2.174	3.253	0.07	0.18	0.14	38.87	23.74	26,12	0.04	0.04	0.00
8	1.417	1.050	2.368	1.328	1.321	0.12	0.19	0.10	42.23	24.90	146,06	4.99	1.59	1.91
9	1.780	1.390	4.915	2.398	2.491	0.07	0.19	0.19	41.46	27.67	35,67	0.30	0.06	0.02
10	1.235	1.353	2.607	2.425	2.573	0.08	0.16	0.15	38.74	32.31	53,42	1.96	1.50	0.05
11	1.547	1.296	3.003	2.522	1.421	0.08	0.14	0.11	39.41	19.76	52,98	1.37	0.60	0.07

12	1.929	1.776	4.678	2.534	2.312	0.10	0.13	0.13	40.45	39.89	40,60	0.32	0.14	0.04
13	1.644	1.379	3.063	2.006	1.941	0.19	0.18	0.13	60.40	51.32	111,32	2.04	1.00	0.16
14	1.208	1.301	3.206	1.892	1.293	0.08	0.08	0.08	23.42	19.01	42,72	1.56	1.01	0.07
15	1.311	1.923	4.266	2.146	1.844	0.20	0.12	0.07	44.06	33.17	64,85	3.54	1.52	0.05
16	1.595	1.262	3.316	1.721	1.994	0.07	0.10	0.09	24.94	23.09	42,85	0.68	0.46	0.05
17	1.086	1.262	2.394	2.097	1.194	0.16	0.14	0.12	43.23	35.00	109,91	3.73	1.24	0.11
18	1.963	1.547	4.319	3.072	1.950	0.14	0.15	0.11	58.74	34.59	46,25	0.43	0.20	0.02
19	1.183	1.240	2.348	2.287	1.562	0.08	0.20	0.16	40.81	34.14	84,64	5.10	2.35	0.11
20	1.060	1.682	5.218	1.952	2.069	0.06	0.17	0.17	40.86	27.27	34,56	0.18	0.10	0.01
21	1.660	1.519	2.933	1.866	1.549	0.12	0.17	0.11	33.07	25.24	118,81	7.20	1.99	0.43
22	1.292	1.432	5.348	1.185	2.385	0.07	0.11	0.08	25.59	19.51	43,57	0.55	0.49	0.04
23	1.952	1.920	7.284	2.070	2.903	0.11	0.19	0.13	60.45	34.82	46,59	0.33	0.19	0.02
24	1.301	1.701	3.337	1.580	1.851	0.15	0.08	0.07	45.32	25.67	61,71	1.05	0.45	0.09
25	1.033	1.561	2.520	1.145	2.542	0.11	0.20	0.12	56.14	25.96	97,07	3.04	1.16	0.65
26	1.645	1.376	4.916	2.173	1.744	0.07	0.14	0.08	32.69	21.05	30,51	0.17	0.17	0.01
27	1.252	1.290	2.921	1.899	1.297	0.20	0.16	0.10	56.54	33.12	89,42	1.50	0.89	0.07
28	1.906	1.880	3.822	2.904	2.194	0.05	0.11	0.10	19.70	17.46	37,59	2.21	1.37	0.17
29	1.423	1.094	2.930	1.818	1.193	0.15	0.15	0.06	39.69	22.45	66,16	0.86	0.41	0.12
30	1.531	1.654	4.278	1.543	1.777	0.20	0.13	0.10	52.36	32.51	83,51	1.84	0.84	0.33
31	1.779	1.423	5.775	2.696	2.404	0.09	0.13	0.12	41.49	30.34	24,16	0.09	0.09	0.00
32	1.255	1.224	2.752	1.247	1.732	0.17	0.15	0.05	51.01	30.24	99,35	2.36	1.18	0.76
33	1.916	1.001	3.752	2.539	1.289	0.17	0.17	0.10	49.03	27.33	56,24	0.50	0.18	0.08
34	1.176	1.722	3.696	1.988	2.443	0.14	0.16	0.08	57.88	27.33	41,99	0.47	0.20	0.04
35	1.269	1.766	4.949	1.887	3.183	0.14	0.15	0.13	47.51	38.27	32,52	0.23	0.03	0.03
36	1.459	1.662	3.243	2.054	1.852	0.11	0.18	0.17	46.58	36.11	83,68	1.56	0.74	0.12
37	1.582	1.541	3.056	2.401	2.230	0.07	0.19	0.15	42.43	31.77	69,44	3.94	2.02	0.33
38	1.545	1.647	3.863	1.656	2.025	0.20	0.08	0.07	31.28	28.18	56,49	0.94	0.55	0.05
39	1.405	1.448	3.965	1.466	1.652	0.17	0.20	0.19	43.12	40.51	91,40	1.06	0.52	0.11
40	1.348	1.150	2.933	2.050	2.184	0.16	0.12	0.09	47.19	33.93	50,87	0.83	0.25	0.02
41	1.736	1.395	3.441	1.882	1.822	0.05	0.11	0.10	21.16	18.26	44,16	1.04	0.44	0.13
42	1.822	1.430	3.149	2.480	1.487	0.11	0.17	0.16	38.68	33.17	100,46	4.75	1.88	0.26
43	1.790	1.949	5.369	1.983	2.554	0.18	0.17	0.08	67.73	46.57	54,76	0.50	0.19	0.02
44	1.514	1.884	3.985	2.554	3.040	0.11	0.17	0.16	50.99	36.04	42,99	0.32	0.32	0.04
45	1.534	1.090	4.628	2.637	1.201	0.12	0.12	0.08	28.52	21.39	31,62	0.60	0.23	0.03
46	1.851	1.561	3.225	2.016	1.837	0.17	0.20	0.18	69.14	35.43	150,61	4.19	1.72	0.48
47	1.613	1.990	4.412	2.047	2.029	0.08	0.19	0.12	49.69	32.89	64,20	1.18	0.55	0.09
48	1.739	1.586	5.130	1.932	2.883	0.14	0.16	0.15	60.13	55.41	50,13	0.29	0.20	0.01
49	1.552	1.630	6.129	1.787	2.272	0.06	0.12	0.08	21.80	15.31	24,25	0.33	0.21	0.02
50	1.147	1.189	4.444	1.301	1.777	0.13	0.15	0.12	43.78	36.95	50,07	0.15	0.15	0.01
51	1.124	1.490	2.584	1.097	2.251	0.08	0.15	0.13	32.21	26.05	87,92	3.52	1.35	0.74
52	1.948	1.082	5.378	3.329	1.804	0.14	0.14	0.06	53.92	34.74	25,02	0.12	0.12	0.00
53	1.738	1.063	2.756	1.639	0.975	0.18	0.17	0.13	48.49	36.02	190,36	8.39	1.95	2.57
54	1.134	1.031	2.036	1.666	1.522	0.10	0.15	0.12	45.53	22.56	92,70	3.33	1.78	0.22

55	1.559	1.854	4.8160	2.031	3.482	0.08	0.15	0.10	42.90	19.70	28.26	0.19	0.19	0.11
56	1.178	1.360	4.752	1.449	1.938	0.08	0.19	0.08	43.92	23.16	38.07	0.13	0.13	0.01
57	1.912	1.104	3.184	2.030	1.319	0.08	0.14	0.09	26.34	19.58	63.85	1.97	0.69	0.34
58	1.895	1.072	4.596	3.560	1.401	0.05	0.18	0.08	30.93	14.38	23.13	0.11	0.11	0.03
59	1.684	1.132	3.016	2.974	1.984	0.15	0.15	0.10	51.46	38.86	57.60	0.80	0.55	0.01
60	1.631	1.090	4.961	1.688	1.045	0.13	0.17	0.07	40.67	25.42	71.46	1.38	0.75	0.23
61	1.746	1.010	5.210	1.939	1.172	0.13	0.16	0.16	51.37	36.71	60.81	0.37	0.37	0.04
62	1.693	1.557	4.403	3.152	1.608	0.10	0.16	0.14	29.53	26.72	44.81	0.73	0.47	0.03
63	1.789	1.368	5.062	3.242	1.420	0.08	0.13	0.11	35.08	23.24	33.17	0.41	0.41	0.01
64	1.485	1.152	2.720	2.649	1.703	0.09	0.13	0.06	30.25	16.50	42.10	1.17	0.90	0.02
65	1.112	1.784	4.2920	1.290	1.654	0.11	0.16	0.15	46.12	38.55	104.39	1.54	0.98	0.12
66	1.110	1.934	4.969	1.682	1.976	0.12	0.17	0.11	36.72	24.12	47.52	0.80	0.60	0.03
67	1.673	1.430	4.024	1.932	2.669	0.10	0.17	0.15	29.76	28.34	44.06	0.62	0.26	0.02
68	1.092	1.008	2.7860	1.223	1.079	0.13	0.14	0.07	31.13	22.08	69.38	1.03	0.68	0.16
69	1.099	1.142	3.729	1.778	1.654	0.10	0.09	0.08	35.04	30.68	29.26	0.12	0.12	0.01
70	1.556	1.184	4.369	2.847	1.130	0.16	0.13	0.10	47.00	29.501	52.74	1.33	0.90	0.03
71	1.988	1.170	4.824	2.693	2.150	0.15	0.20	0.11	56.73	36.87	41.73	0.15	0.15	0.01
72	1.154	1.381	4.250	1.209	1.351	0.10	0.15	0.09	46.81	30.11	78.50	0.94	0.64	0.18
73	1.598	1.335	4.3160	2.071	1.772	0.10	0.16	0.13	37.92	33.86	45.09	0.64	0.13	0.02
74	1.125	1.024	3.188	1.629	1.148	0.19	0.19	0.12	67.72	38.73	76.98	0.76	0.54	0.06
75	1.759	1.741	3.699	3.122	1.915	0.12	0.08	0.06	36.71	21.65	44.17	0.94	0.50	0.04
76	1.164	1.666	2.730	1.437	1.808	0.07	0.19	0.13	44.69	21.16	89.35	3.31	1.67	0.35
77	1.809	1.749	6.219	2.220	2.514	0.13	0.11	0.11	30.66	28.55	29.41	0.10	0.10	0.00
78	1.021	1.924	3.300	0.963	3.216	0.19	0.17	0.14	51.46	40.84	97.02	1.98	1.21	0.07
79	1.752	1.229	5.535	1.824	1.361	0.16	0.15	0.11	55.14	37.57	66.41	0.56	0.48	0.06
80	1.317	1.815	3.217	1.303	2.259	0.15	0.19	0.12	63.09	27.77	115.14	2.39	1.10	0.82
81	1.631	1.355	2.719	2.571	1.520	0.14	0.11	0.07	35.36	21.74	95.85	4.98	1.41	0.12
82	1.184	1.726	4.029	1.178	1.835	0.14	0.19	0.08	63.37	26.85	90.26	1.14	0.78	0.57
83	1.627	1.022	2.532	1.660	1.078	0.17	0.14	0.11	49.56	42.76	166.00	4.92	1.40	1.10
84	1.275	1.249	3.273	2.003	1.271	0.20	0.13	0.05	59.16	27.44	62.22	1.62	0.76	0.05
85	1.224	1.269	2.759	1.557	1.452	0.09	0.17	0.08	50.28	28.63	69.35	1.01	0.33	0.12
86	1.108	1.182	4.093	1.396	1.919	0.18	0.06	0.06	44.18	37.53	33.82	0.06	0.06	0.00
87	1.312	1.179	3.541	2.096	1.463	0.19	0.14	0.07	45.94	26.49	46.90	0.38	0.24	0.02
88	1.501	1.432	2.670	1.521	1.809	0.18	0.11	0.07	56.41	48.86	198.09	6.42	1.29	0.42
89	1.714	1.618	4.720	1.614	2.815	0.16	0.17	0.15	58.54	42.87	93.33	1.28	0.85	0.02
90	1.835	1.323	3.798	1.688	1.594	0.10	0.14	0.10	42.59	28.80	105.83	3.29	1.17	1.24
91	1.492	1.695	2.941	2.142	1.690	0.16	0.19	0.05	58.86	29.26	127.54	6.49	0.99	0.12
92	1.282	1.230	2.711	1.465	1.440	0.15	0.10	0.06	36.48	25.59	73.00	1.26	0.57	0.26
93	1.715	1.856	5.339	1.827	3.188	0.18	0.14	0.07	56.69	33.65	46.73	0.41	0.41	0.01
94	1.504	1.490	2.917	2.112	1.935	0.19	0.20	0.06	65.10	29.83	101.59	1.88	0.73	0.08
95	1.586	1.675	4.551	1.819	1.698	0.05	0.20	0.06	41.13	14.22	48.54	0.82	0.43	0.11
96	1.404	1.122	3.820	2.144	1.605	0.07	0.10	0.07	32.98	15.44	25.74	0.57	0.13	0.11
97	1.930	1.399	6.300	2.736	1.486	0.17	0.15	0.13	61.00	31.74	44.77	0.43	0.43	0.02

98	1.304	1.046	3.807	1.399	1.121	0.18	0.14	0.06	61.34	43.62	77.57	0.66	0.64	0.08
99	1.944	1.549	3.728	3.005	1.539	0.12	0.15	0.13	51.13	33.64	57.90	0.77	0.42	0.10
100	1.504	1.647	4.602	2.579	2.097	0.10	0.17	0.17	45.48	30.77	39.86	0.20	0.19	0.02

** The system parameters are drawn from uniform distribution, where $\lambda_k \sim U(1,2)$, $\rho_i \sim U(0.5,1.1)$, $\mu_k = \lambda_k / \rho_i$, $\mu_0 = (\lambda_1 + \lambda_2) / \rho_0$, $h_i \sim U(1,10)$ and $L_k \sim U(100,200) \times (h_0 + h_k)$, $i=(0,1,2)$; $k=(1,2)$. For ease of disposition, we let $h = h/100$ and $L = L/100$.

Table 5.2: CPU Time Performance of Value and Policy Iteration Methods

	Value Iteration			Policy Iteration		
	Without Heuristic I (s)	With Heuristic I (s)	% gain	Without Heuristic I (s)	With Heuristic I (s)	% gain
1	1789,35	28,99	6071,31	299,98	56,03	435,41
2	279,91	62,40	348,56	124,67	26,49	370,65
3	39,96	4,72	747,30	63,10	12,02	425,15
4	10,13	3,50	189,06	6,90	0,97	612,29
5	761,51	25,80	2851,50	16,25	2,30	604,96
6	1132,44	70,66	1502,65	90,63	13,22	585,75
7	2,01	0,71	181,46	2,76	0,24	1039,86
8	150,68	20,19	646,21	434,42	62,44	595,78
9	3,72	1,19	212,20	4,42	0,49	793,68
10	2005,43	18,70	10624,19	16,07	1,30	1134,70
11	130,98	11,32	1057,58	34,20	5,13	566,85
12	10,78	4,04	166,75	19,04	3,19	496,99
13	365,02	31,82	1047,10	71,24	11,93	497,21
14	85,18	15,91	435,30	23,13	3,37	587,17
15	581,30	121,53	378,34	76,31	11,23	579,66
16	25,48	9,52	167,63	27,77	3,39	718,47
17	724,89	34,89	1977,65	89,37	10,76	730,26
18	20,27	3,89	421,29	16,19	2,34	591,03
19	4130,09	57,07	7137,49	55,31	7,20	667,84
20	8,95	1,05	748,76	3,07	0,35	771,76
21	3763,50	42,57	8741,10	500,81	95,13	426,44
22	1262,61	25,89	4776,99	13,02	1,04	1146,75
23	210,49	13,13	1503,73	11,72	1,76	564,98
24	87,83	13,18	566,43	82,86	12,36	570,21
25	834,43	37,64	2117,11	72,20	11,59	523,12
26	53,74	2,09	2466,49	7,85	0,95	728,91
27	84,39	17,56	380,51	33,21	5,08	553,13
28	2185,76	25,53	8462,72	48,46	6,74	618,51
29	63,79	7,84	713,24	21,88	3,67	496,69
30	969,50	16,27	5857,60	99,17	19,32	413,37

31	6,73	0,57	1084,70	2,78	0,27	938,03
32	211,41	10,78	1860,88	106,91	11,26	849,16
33	20,61	2,67	671,73	10,51	1,19	786,10
34	6,13	1,02	502,43	6,37	0,72	783,45
35	2,28	0,71	221,28	2,55	0,28	799,91
36	229,98	15,55	1378,77	40,55	7,24	460,37
37	4883,77	66,49	7245,49	63,36	10,25	518,23
38	58,33	14,13	312,87	32,45	6,43	404,76
39	43,14	9,36	360,90	22,85	3,79	502,37
40	9,56	1,65	478,45	5,76	0,61	847,51
41	34,84	6,95	401,33	34,12	5,08	571,09
42	1864,16	63,93	2816,16	141,38	28,02	404,65
43	59,74	13,04	357,99	28,70	5,60	412,00
44	20,21	2,31	773,91	6,14	0,71	759,71
45	9,55	3,22	196,83	4,14	0,38	987,59
46	1611,21	86,05	1772,50	313,67	57,11	449,25
47	113,62	7,64	1387,83	64,94	12,43	422,23
48	35,79	6,66	437,35	8,74	1,31	566,94
49	19,64	3,87	407,54	9,86	0,98	906,69
50	25,81	5,09	406,86	6,38	0,74	766,11
51	316,61	40,98	672,56	127,25	20,99	506,33
52	2,70	0,89	204,36	5,13	0,45	1027,80
53	127,32	50,67	151,28	1122,20	315,07	256,17
54	28,59	10,73	166,56	15,22	1,83	732,23
55	2,60	0,84	208,99	2,87	0,30	845,12
56	2,89	0,97	196,99	2,46	0,31	693,68
57	6,79	1,65	312,11	45,64	7,01	551,50
58	1,65	0,56	197,10	1,63	0,17	841,65
59	8,32	2,92	184,40	4,91	0,58	740,01
60	84,88	33,64	152,30	37,00	6,20	496,45
61	10,22	3,26	213,04	7,80	1,12	593,63
62	11,70	4,29	172,86	5,02	0,72	592,82
63	11,56	3,73	209,54	3,97	0,48	733,54
64	9,60	2,99	221,26	6,39	0,76	734,83
65	78,40	32,50	141,19	34,84	6,77	414,67
66	12,89	4,55	183,26	5,54	0,70	692,64
67	3,71	1,33	179,80	3,54	0,42	733,07
68	9,28	3,60	157,55	9,64	1,51	537,92
69	1,18	0,41	189,39	1,69	0,17	869,44
70	33,29	11,83	181,30	8,25	0,82	907,09
71	1,65	0,57	191,76	2,05	0,28	622,72
72	47,09	19,80	137,82	25,67	4,61	457,39
73	0,93	0,20	370,21	5,26	0,69	663,09

74	9.07	2.12	327.16	6.19	0.77	703.41
75	13.89	5.38	158.39	22.94	3.13	632.40
76	67.07	26.73	150.88	72.55	12.04	502.66
77	3.56	1.24	186.33	3.44	0.38	801.33
78	53.04	21.76	143.74	28.76	3.74	668.75
79	29.59	11.70	152.95	16.88	2.62	545.00
80	72.92	36.19	101.51	100.28	16.05	524.63
81	59.36	23.88	148.52	60.81	11.37	434.77
82	55.00	24.23	127.02	45.11	7.49	502.58
83	65.89	21.15	211.47	510.75	101.30	404.19
84	30.08	12.15	147.56	13.64	2.53	438.46
85	9.98	4.11	142.77	23.27	3.98	484.13
86	3.50	1.21	189.64	2.78	0.28	880.73
87	2.94	1.05	179.54	3.22	0.31	950.90
88	281.31	135.20	108.07	1835.85	543.46	237.81
89	94.02	37.78	148.86	23.33	3.39	587.43
90	152.36	71.62	112.72	153.23	27.20	463.42
91	147.61	68.41	115.77	334.76	87.87	280.97
92	4.50	1.12	303.18	27.92	4.65	499.97
93	20.28	7.65	165.20	8.03	1.30	517.14
94	12.24	3.03	304.10	33.52	5.32	530.54
95	24.07	9.91	142.97	19.83	4.43	347.85
96	1.15	0.40	186.91	1.87	0.27	597.59
97	11.60	4.25	173.02	5.82	0.70	734.86
98	31.89	13.45	137.15	22.05	4.16	430.66
99	114.60	39.29	157.37	648.93	159.88	881.35
100	1.98	0.68	189.07	2.37	0.24	891.97
Ave.	38.92	16.18	181.89	112.08	28.32	608.02

** System parameter is same as Table 5.1.

5.5 Decomposition method for larger systems

In the previous section, we showed that the average cost obtained by the decomposition method is very close to the optimal cost, with an average gap of 0.19%. In several cases, the cost gap even is zero. Furthermore, the computational burden is much less than the exhaustive search heuristic. Thus, both in performance and computational time, the decomposition method is a very effective heuristic method in *W*-configuration ATO systems. In order to show that the decomposition method is still effective for other ATO systems, we apply the decomposition

method to three larger ATO systems, which have 4 components and 2, 3, and 4 end products, respectively, with distinct configurations.

5.5.1 Decomposition method for system 1

System 1 has 4 components and 2 end products, where Product 1 is assembled from the first three components (Component 1, Component 2 and Component 3) and Product 2 is assembled from the last three components (Component 2, Component 3 and Component 4). The production rates of these components are μ_1 , μ_2 , μ_3 and μ_4 , respectively. The demand rates of the end products are λ_1 and λ_2 , respectively. If demand cannot be fulfilled immediately, it is lost and incurs a lost sales cost of L_1 for Product 1 and L_2 for Product 2 (cf. Figure 5.2). This system can be regarded as a general *W*-configuration ATO system, where each product is assembled from a product specific component and two common components.

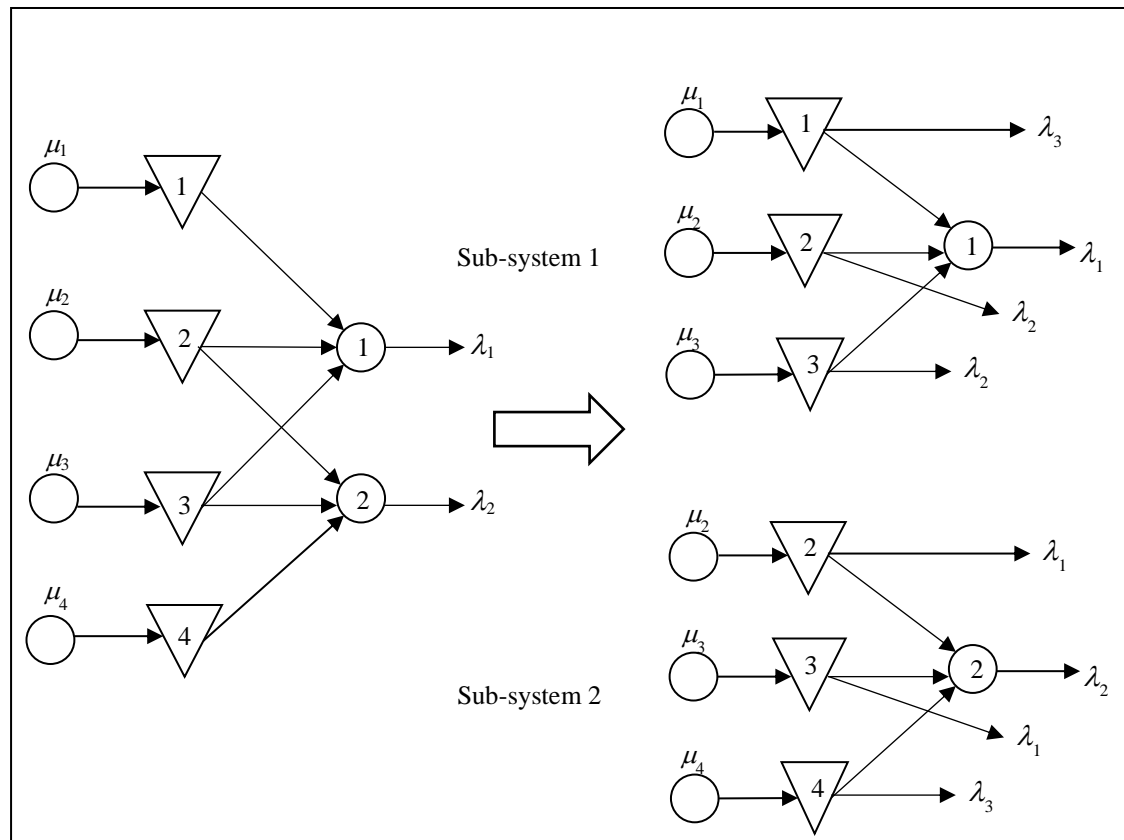


Figure 5.2: Decomposing map of 4 components, 2 end products systems

Applying the decomposition method, System 1 can be decomposed into two sub-systems, where each sub-system comprises the corresponding components for each product. For instance, Sub-system 1 includes Component 1, Component 2 and Component 3 with a demand of Product 1. As in the original system, Component 2 and Component 3 are also used in Product 2. We add a corresponding demand rate and lost sale cost L_2 for both Component 2 and Component 3. In addition, we add an artificial demand rate $\lambda_3 = 0$ for Component 1 (cf. Figure 5.2). This Sub-system constitutes a three components M -configuration ATO system, where demands arise for both end product and components. For Sub-system 2, it can be illustrated in the same way.

The optimal policy of M -configuration ATO systems can be difficult to obtain if they have large number of components or have special system parameters. In order to facilitate the process of obtaining the production and allocation policy for each component, we further adjust the Sub-systems. For instance, in Sub-system 1, to obtain the heuristic production and rationing policy of Component 1, we combine Component 2 and Component 3 together and regard them as a single component. For the combined component, the production rate and holding cost equal to the sum of that of the two components, that is, $\mu = \mu_2 + \mu_3$ and $h = h_2 + h_3$. For the demand rate of the combined component, as both Component 2 and Component 3 demand inter arrival times follow an exponential distribution, we treat it as the mean of maximum of two independent exponential distributed demand variables λ_2 and λ_3 . Though the inter arrival time between demands for components 2 and 3 no longer follows an exponential distribution, we use the rate of the maximum distribution as an approximate rate for an exponentially distributed inter arrival time. The mean of the maximum of n independent random variables (Z_1, Z_2, \dots, Z_n) where each random variable Z_i is exponentially distribution with mean $1/\lambda_i$ is given as follows.

$$E[\max(Z_1, Z_2, \dots, Z_n)] = \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \dots + \frac{1}{\lambda_n} \right) - \left(\frac{1}{\lambda_1 + \lambda_2} + \frac{1}{\lambda_1 + \lambda_3} + \dots + \frac{1}{\lambda_i + \lambda_j} + \dots \right) + \left(\frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{1}{\lambda_1 + \lambda_2 + \lambda_4} + \dots + \frac{1}{\lambda_i + \lambda_j + \lambda_k} + \dots \right) - \dots + (-1)^{n+1} \frac{1}{\sum_{i=1}^n \lambda_i} \quad (5.11)$$

The lost sale cost of the combined component is the sum of the lost sale costs of the components in the original sub-system. The other parameters remain the same as in Sub-system 1 (production rate, demand rate and holding cost of Component 1, and demand rate and lost sale cost of the Product 1). Thus, the modified Sub-system 1 is a two component, one end product M -configuration ATO system, whose optimal policy can be obtained easily. By solving this modified Sub-system 1, we can obtain the production policy and rationing policy of Component 1. To obtain the production policy and allocation policy of Component 2 and Component 3 in Sub-system 1, we combine Component 1 & Component 3 and Component 1 & Component 2 and solve the relative systems, respectively. As a result of the modified sub-system, the obtained production and allocation policy is two dimensional. Hence, we need to map the production and allocation policy to the three-dimensional system of Sub-system 1. From the structure of the optimal policy which is characterized in the previous chapter, we know that the production decision for a certain component is strongly influenced by the component with the lowest inventory level. However, numerical experiments show the rationing decision for a certain component is strongly influenced by the component with highest inventory level. Such behavior is intuitive, as the aim is to keep the inventory level balanced. Thus, for component i 's production policy, we map the original three-dimensional state to state $(x_i, \min\{x_j, j=1,2,3, j \neq i\})$. For component i 's allocation policy, we map the original policy to state $(x_i, \max\{x_j, j=1,2,3, j \neq i\})$.

We handle Sub-system 2 in the same manner as in Sub-system 1 to obtain the production and rationing policy of Component 2, Component 3 and Component 4. Then, we construct the heuristic policy as follows. First, for all components, we map their production and rationing policies directly to the 4 components system by adding the state of the rest component. For example, for Component 1's production policy, we map the policy in state (x_1, x_2, x_3) to state (x_1, x_2, x_3, x_4) by adding the state of Component 4. Second, for the production policy of common components, as both sub-systems provide a corresponding policy, we obtain it by multiplying the two production policies that were obtained from the two distinct Sub-systems in order to take advantage of inventory pooling property. Last, we note that the demand for a product can be satisfied only if the inventory of all its components is above their corresponding

rationing level. As such, we multiply the allocation policy of Component 1, Component 2 and Component 3 obtained by solving Sub-system 1 to obtain the allocation policy of Product 1 and we multiply the allocation policy of Component 2, Component 3 and Component 4 obtained by solving Sub-system 2 to obtain the allocation policy of Product 2.

In order to examine the effectiveness of our decomposition method, we carry out a numerical experiment consisting of 25 problems with randomly generated parameter settings (cf. Table 5.3). The results of the numerical experiment show that the decomposition method works very well in this system, despite being more involved than the 3 components *W*-configuration ATO system. The average gap over the 25 problems is 0.55%, with a minimum of 0.14% and a maximum of 1.10%.

Table 5.3: Parameter settings and decomposition results for System 1

	System Parameters												Optimal	Gap
	λ_1	λ_2	μ_1	μ_2	μ_3	μ_4	h_1	h_2	h_3	h_4	L_1	L_2	Cost	%
1	1,044	1,687	1,450	4,327	4,448	2,125	0,11	0,12	0,07	0,10	39,34	54,65	34,29	0,30
2	1,247	1,311	2,003	3,590	4,710	1,721	0,06	0,12	0,05	0,16	40,17	40,91	32,87	0,36
3	1,431	1,703	2,166	5,472	3,720	2,321	0,11	0,08	0,08	0,13	39,01	51,27	33,59	0,52
4	1,070	1,847	1,387	5,261	3,460	3,185	0,14	0,13	0,07	0,05	61,45	45,29	34,13	0,52
5	1,421	1,057	1,935	4,350	3,131	1,481	0,09	0,19	0,16	0,18	46,95	63,24	56,69	0,49
6	1,568	1,888	1,873	4,020	3,947	2,285	0,05	0,05	0,06	0,09	16,75	28,87	30,92	1,10
7	1,341	1,541	1,541	4,654	4,538	1,827	0,10	0,07	0,13	0,18	41,25	63,65	47,50	0,56
8	1,468	1,861	2,138	4,761	4,790	3,144	0,06	0,06	0,18	0,09	57,00	56,20	28,75	0,20
9	1,725	1,607	2,346	4,949	5,576	2,394	0,05	0,14	0,19	0,06	40,03	75,32	30,32	0,18
10	1,247	1,009	1,510	4,056	2,648	1,875	0,10	0,14	0,14	0,15	62,57	61,20	57,97	0,71
11	1,140	1,752	1,910	3,804	3,430	3,282	0,20	0,05	0,18	0,18	44,71	62,67	49,03	0,57
12	1,943	1,322	2,362	4,408	4,002	1,612	0,06	0,09	0,15	0,12	58,78	63,64	49,62	0,80
13	1,568	1,299	2,603	3,355	4,224	1,572	0,06	0,18	0,05	0,18	56,59	63,70	54,87	0,63
14	1,120	1,178	1,432	2,760	4,471	1,467	0,19	0,10	0,15	0,10	53,61	62,27	59,21	0,84
15	1,723	1,279	2,351	4,491	5,592	2,509	0,12	0,09	0,10	0,09	37,04	39,99	24,86	0,33
16	1,698	1,204	2,215	4,284	4,309	2,111	0,08	0,14	0,09	0,13	60,64	62,88	36,13	0,29
17	1,679	1,959	2,073	4,897	4,139	3,740	0,09	0,20	0,17	0,12	76,34	68,74	55,12	0,84
18	1,380	1,213	2,113	5,067	3,764	1,916	0,20	0,13	0,18	0,11	74,07	76,97	44,87	0,14
19	1,991	1,524	2,288	4,418	4,835	1,717	0,17	0,19	0,15	0,11	75,98	85,62	72,91	0,90
20	1,015	1,157	1,474	3,028	4,145	1,516	0,18	0,07	0,12	0,09	72,57	44,10	43,44	0,59
21	1,865	1,058	2,730	3,705	4,599	1,602	0,13	0,18	0,17	0,06	61,37	56,73	47,01	0,41
22	1,865	1,420	3,129	4,444	4,748	1,652	0,19	0,17	0,16	0,16	98,64	52,25	55,92	0,69

23	1,336	1,004	1,607	3,329	3,620	1,700	0,13	0,09	0,06	0,06	30,33	30,51	33,90	0,61
24	1,123	1,443	1,307	4,001	4,683	1,983	0,18	0,10	0,06	0,07	36,59	40,02	35,80	0,33
25	1,862	1,084	2,932	4,956	4,697	1,213	0,13	0,16	0,18	0,08	89,62	45,72	42,08	0,77

** The system parameters are drawn from uniform distribution, where $\lambda_k \sim U(1,2)$; $\rho_i \sim U(0.5,0.9)$; $\mu_k = \lambda_k / \rho_k$, ($k=1,4$); $\mu = (\lambda_1 + \lambda_2) / \rho_k$, ($k=2,3$); $h_i \sim U(1,10)$ and $L_k \sim U(100,200) \times (h_2 + h_3 + h_k)$, $i=1,2,3,4$; $k=1,2$. For ease of disposition, we let $h = h/100$ and $L = L/100$.

5.5.2 Decomposition method for system 2

System 2 has 4 components and 3 end products, where Product 1 is assembled from Component 1 and Component 2, Product 2 is assembled from the Component 2 and Component 3 and Product 3 is assembled from Component 2, Component 3 and Component 4. The production rate and holding cost of each component is μ_i and h_i , ($i = 1, 2, 3, 4$). The demand rate of each product is λ_k ($j = 1, 2, 3$). If demand cannot be fulfilled immediately, it is lost and incurs a lost sales cost of L_k for Product k (cf. Figure 5.3).

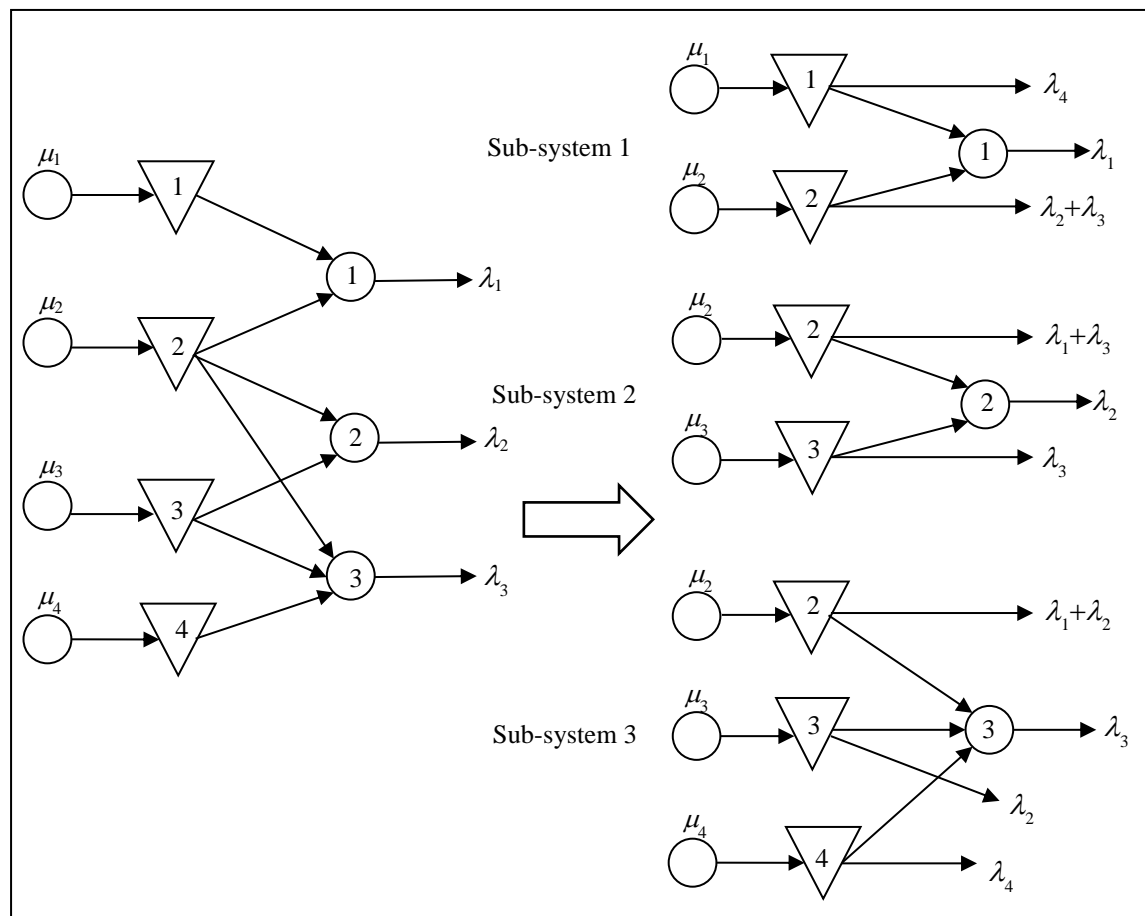


Figure 5.3: Decomposing map of 4 components, 3 end products systems

System 2 can be decomposed into three sub-systems, where each sub-system comprises the corresponding components for each product (cf. Figure 5.3). For instance, Sub-system 1 includes Component 1 and Component 2 with a demand for Product 1. As in the original system, Component 2 is also needed by Product 2 and Product 3. We add a corresponding demand rate $(\lambda_2 + \lambda_3)$ and lost sale cost $(L_2 + L_3)$ for Component 2. In addition, we add an artificial demand rate $\lambda_4 = 0$ for Component 1. This Sub-system represents a two components *M*-configuration ATO system, where demands arise for both end product and components. Sub-system 2 comprises Component 2 and Component 3, which can be illustrated in the same way. Sub-system 3 uses Component 2, Component 3 and Component 4 to assemble into Product 3. In addition, Component 2 is demanded by Product 1 and Product 2 and faces a demand rate of $\lambda_1 + \lambda_2$; Component 3 is demanded by Product 2, which faces a demand rate of λ_2 . Sub-system 3 is modelled as a 3 components *M*-configuration ATO system by adding an artificial demand rate $\lambda_4 = 0$ to Component 4.

By solving each Sub-system, we obtain the production and rationing policy of each component. After mapping the policy into the original system, we construct the heuristic control policy as in System 1. For example, as Component 2 is present in all three Sub-systems, we obtain the production policy of Component 2 by multiplying together the production policies obtained by solving the three Sub-systems. The other components' production policy can be obtained in a similar way. For the allocation policy of the end product we note that the demand for a product can be satisfied only if the inventory of all its components is above their corresponding rationing level. As such, we multiply the allocation policy of Component 2, Component 3 and Component 4 obtained by solving Sub-system 3 to obtain the allocation policy of Product 3. The other Products' allocation policy can be obtained in a similar way. To illustrate the effectiveness of the decomposition method, we run a numerical experiment that comprises 25 problems with distinct system parameters. Table 5.4 gives the system parameters of each problem together with the optimal cost and heuristic gap. Through Table 5.4, the average gap of the 25 problems is 1.52%, with a minimum of 0.64% and a maximum of 3.03%. The results of the numerical experiment show that the decomposition method still works well

in system 2.

Table 5.4: Parameter settings and decomposition results for System 2

	System Parameters														Optim.	Gap
	λ_1	λ_2	λ_3	μ_1	μ_2	μ_3	μ_4	h_1	h_2	h_3	h_4	L_1	L_2	L_3	Cost	%
1	1,57	1,70	1,74	1,96	7,65	5,13	1,97	0,14	0,18	0,09	0,14	49,78	52,80	44,77	38,94	2,06
2	1,48	1,34	1,80	1,66	8,20	5,27	2,30	0,11	0,20	0,20	0,15	56,25	54,94	87,87	41,74	1,13
3	1,99	1,56	1,93	2,52	7,90	4,62	2,26	0,08	0,11	0,20	0,11	31,37	58,59	83,50	38,67	1,10
4	1,65	1,11	1,04	2,21	5,22	2,42	1,30	0,15	0,13	0,09	0,19	42,80	22,41	69,85	55,78	1,65
5	1,52	1,06	1,89	2,40	7,55	5,41	3,03	0,08	0,15	0,06	0,09	29,75	39,05	43,19	18,11	1,06
6	1,76	1,60	1,78	3,22	5,77	4,03	3,43	0,12	0,10	0,14	0,08	34,56	39,02	52,46	35,21	2,79
7	1,45	1,04	1,51	2,18	7,36	3,73	2,23	0,13	0,17	0,16	0,18	31,93	39,75	73,97	36,50	1,10
8	1,96	1,79	1,45	3,09	9,93	4,07	2,07	0,08	0,11	0,08	0,16	26,56	36,77	35,82	21,11	0,88
9	1,83	1,63	1,54	2,41	6,32	5,89	1,81	0,05	0,09	0,08	0,19	15,63	27,06	58,95	30,11	1,61
10	1,65	1,86	1,06	2,00	6,43	3,76	1,81	0,13	0,16	0,19	0,12	31,14	36,88	75,85	44,76	1,43
11	1,80	1,69	1,35	2,04	6,83	3,44	2,54	0,08	0,17	0,19	0,17	32,04	40,70	96,23	49,83	1,81
12	1,28	1,68	1,90	1,48	6,08	5,92	2,45	0,07	0,07	0,08	0,07	21,92	15,80	39,91	22,03	1,77
13	1,92	1,14	1,33	2,23	6,26	3,31	1,82	0,15	0,05	0,13	0,05	38,21	24,60	44,02	34,26	0,95
14	1,25	1,58	1,94	2,40	9,14	6,92	2,51	0,14	0,07	0,17	0,14	22,14	25,29	43,08	19,97	0,88
15	1,79	1,09	1,24	2,99	7,60	2,76	1,59	0,16	0,15	0,13	0,10	51,12	30,74	42,93	35,18	0,72
16	1,02	1,96	1,97	1,86	7,21	5,16	3,20	0,16	0,13	0,11	0,09	52,07	47,06	58,42	28,46	1,71
17	1,50	1,65	1,80	2,53	6,68	6,32	2,64	0,18	0,19	0,12	0,09	60,12	59,98	61,07	31,88	3,03
18	1,97	1,20	1,11	3,19	6,50	3,45	1,78	0,15	0,06	0,11	0,09	28,46	19,25	42,77	23,42	1,11
19	1,28	1,16	1,00	2,09	4,77	2,54	1,94	0,19	0,07	0,18	0,17	48,97	27,83	62,41	47,37	0,88
20	1,41	1,17	1,58	1,89	7,09	3,88	1,76	0,12	0,15	0,11	0,06	35,90	47,91	43,24	32,75	1,32
21	1,08	1,51	1,37	1,36	5,58	3,50	1,65	0,08	0,07	0,17	0,15	14,93	45,83	58,70	46,28	1,04
22	1,54	1,61	1,76	1,83	7,52	6,31	2,22	0,10	0,18	0,11	0,17	32,44	31,82	85,71	34,89	1,77
23	1,07	1,54	1,28	1,54	5,03	4,84	1,72	0,10	0,18	0,07	0,07	55,06	29,07	36,59	33,37	2,98
24	1,16	1,09	1,45	1,51	4,44	3,12	1,85	0,12	0,16	0,19	0,13	36,17	32,64	58,84	67,01	2,67
25	1,23	1,46	1,62	1,65	7,87	5,61	2,64	0,16	0,11	0,17	0,19	38,06	30,10	74,84	30,12	0,64

** The system parameters are drawn from uniform distribution, where $\lambda_k \sim U(1,2)$; $\rho_i \sim U(0.5,0.9)$;

$$\mu_i = \left(\frac{\lambda_1}{\rho_1}, \frac{(\lambda_1 + \lambda_2 + \lambda_3)}{\rho_2}, \frac{(\lambda_2 + \lambda_3)}{\rho_3}, \frac{\lambda_4}{\rho_4} \right) \quad \text{and} \quad L_k \sim U(100,200) \times \left(\sum_{i=1}^2 h_i, \sum_{i=2}^3 h_i, \sum_{i=2}^4 h_i \right),$$

$i \in \{1,2,3,4\}$, $k \in \{1,2,3\}$. For ease of disposition, we let $h = h/100$ and $L = L/100$.

5.5.3 Decomposition method for system 3

System 3 is a 4-components nested ATO system, where Product 1 is assembled from Component 1, Product 2 is assembled from Component 1 and Component 2, Product 3 is assembled from Component 1, Component 2 and Component 3 and Product 4 is assembled from Component 1, Component 2 and Component 3 and Component 4. The production rate and holding cost of each component is μ_i and h_i , ($i = 1, 2, 3, 4$). The demand rate of each product is λ_k ($j = 1, 2, 3, 4$). If demand cannot be fulfilled immediately, it is lost and incurs a lost sales cost with L_k for Product k (cf. Figure 5.4).

System 3 can be decomposed into four sub-systems, where each sub-system comprises the corresponding components for each product. For instance, Sub-system 1 is a single component system facing two demand classes with different demand rates and lost sale penalties. Sub-system 4 uses Component 1, Component 2, Component 3 and Component 4 to assemble into Product 4. In addition, Component 1 is demanded by Product 1, Product 2 and Product 3, Component 2 is demanded by Product 2 and Product 3, Component 3 is demanded by Product 3. We add the corresponding demand rate for each component. Sub-systems 4 is modelled by a 4 components M -configuration ATO system by adding an artificial demand rate $\lambda_5 = 0$ to Component 4.

By solving each Sub-system, we obtain the production and rationing policy of each component. We construct the heuristic policy as in System 1 and System 2. To illustrate the effectiveness of the decomposition method, we run a numerical experiment that comprises 25 problems with distinct system parameters. Table 5.5 gives the system parameters of each problem together with the optimal result and heuristic gap.

The average gap of the 25 problems is 3.87%, with a minimum of 1.74% and a maximum of 5.65%. The results of the numerical experiment show that the decomposition method is still appropriate in System 3 with an acceptable gap. However, the heuristic result is not as good as that of System 1 and System 2. There are two reasons that lead to this result. First, our decomposition method breaks down the original system into its end products, thus ignores the influence of other components belonging to the other products. However, because of the state dependency structure of the optimal policy (see Chapter 4), the state of components that do not

belong to same end product still influence those in other sub-systems. For instance, with respect to the allocation policy, the demand of an end product can be satisfied in the sub-system, but may be denied in the original system as other components may have a higher inventory level, so that the common component is allocated to other products in order to decrease the inventory holding cost. In addition, since we decompose each sub-system as a 2-Component M-configuration ATO system, the demand rate of each component is equal to the sum of the demand rates of the other products. Furthermore, a demand that is satisfied in the sub-system may not be satisfied in the original system if other components that belong to the same product are not available. Also, a demand of an end product that is denied in the sub-system may be satisfied in the original system. As System 3 is more unbalanced in terms of components per product compared to system 1 and 2, this effect is more accentuated. Last, the demand rate of each component is the rate obtained from the distribution of the maximum of several exponential distributions which itself is not exponentially distributed. This will further deteriorate the performance as systems become larger.

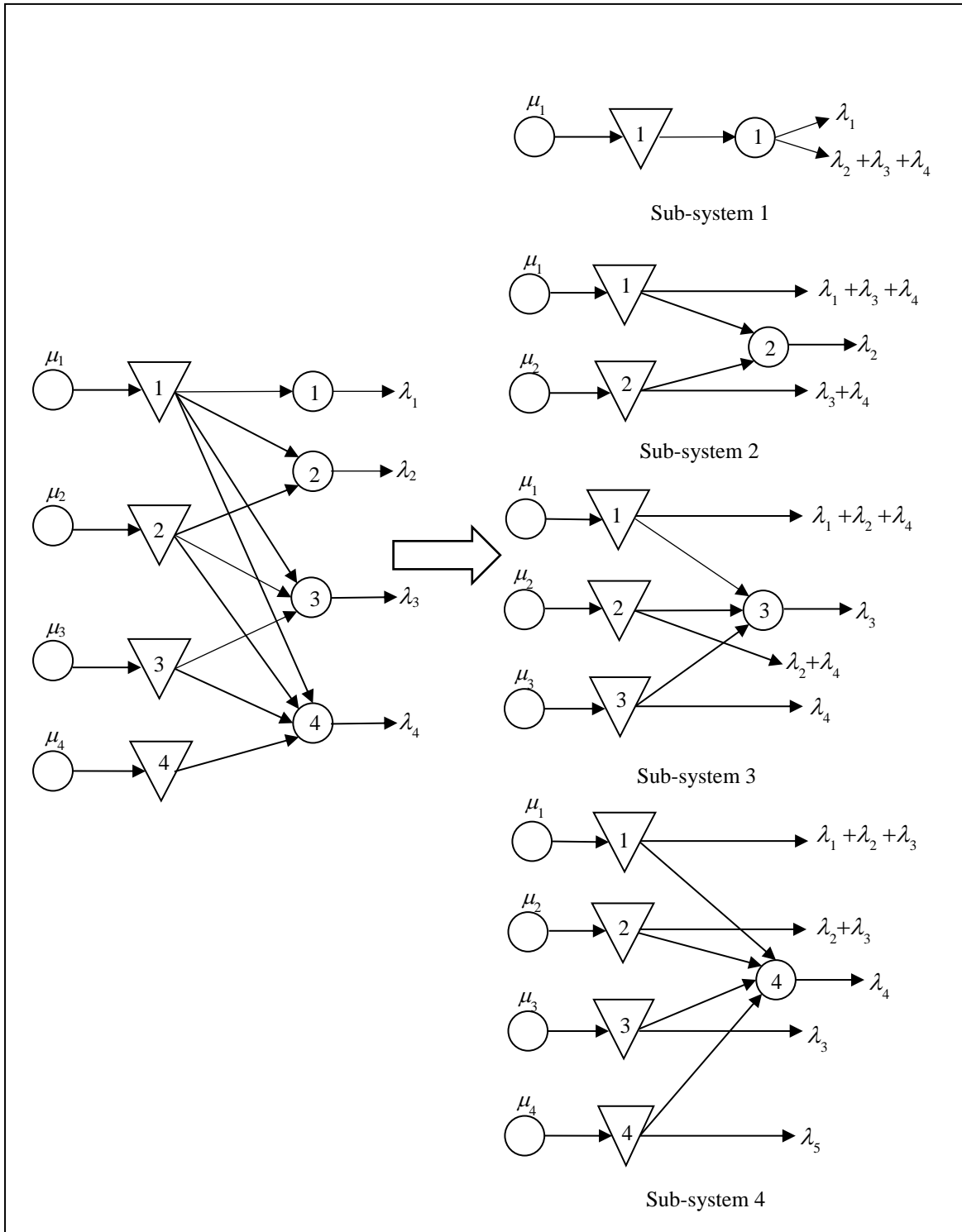


Figure 5.4: Decomposing map of 4 components, 4 end products nested systems

Table 5.5: Parameter settings and decomposition results for System 3

	System Parameters																Optim.	Gap
	λ_1	λ_2	λ_3	λ_4	μ_1	μ_2	μ_3	μ_4	h_1	h_2	h_3	h_4	L_1	L_2	L_3	L_4	Cost	%
1	1,8	1,6	1,4	1,5	9,6	6,5	4,0	2,6	0,1	0,2	0,1	0,2	7,5	26,3	65,9	96,1	23,8	2,8
2	1,8	1,5	1,6	1,2	8,7	7,8	5,3	1,6	0,1	0,1	0,1	0,1	22,0	25,3	39,2	48,0	19,4	3,7
3	1,4	1,5	1,2	1,7	11,3	8,0	3,4	2,0	0,1	0,1	0,2	0,1	10,0	32,4	58,3	44,6	22,2	2,3
4	1,4	1,9	1,7	1,4	8,8	8,5	4,4	1,6	0,2	0,1	0,2	0,1	35,3	29,3	52,8	95,3	28,2	4,3
5	1,1	1,5	1,8	1,7	11,2	7,8	5,7	2,6	0,1	0,1	0,2	0,1	16,1	32,9	76,4	96,6	20,5	1,9
6	1,7	1,5	1,3	1,6	10,2	7,0	3,5	1,9	0,2	0,2	0,1	0,1	27,5	68,5	72,4	78,1	29,1	4,4
7	1,3	1,7	1,5	1,8	11,3	7,2	6,4	2,4	0,2	0,2	0,1	0,2	22,9	55,5	53,6	64,0	22,9	4,5
8	1,9	1,1	1,4	1,6	7,5	6,3	3,8	2,9	0,1	0,2	0,1	0,2	15,6	47,9	44,7	64,1	26,3	5,5
9	1,5	1,1	1,9	1,3	9,9	6,3	5,0	2,1	0,1	0,2	0,1	0,1	17,9	48,8	49,5	107,8	25,0	4,2
10	1,6	1,2	1,5	1,1	7,1	6,0	3,9	1,6	0,2	0,1	0,1	0,1	15,9	44,8	33,2	85,6	25,6	5,2
11	1,7	1,9	1,5	1,2	11,1	6,4	3,5	2,0	0,2	0,1	0,1	0,1	28,2	28,9	40,4	70,2	22,4	3,7
12	1,6	1,1	1,5	1,7	7,9	7,2	5,0	2,2	0,2	0,1	0,2	0,1	19,8	24,3	55,7	71,7	25,1	4,7
13	1,3	1,3	1,2	1,9	7,6	7,2	4,9	2,9	0,1	0,1	0,2	0,1	23,0	22,1	54,9	85,7	24,3	4,0
14	1,6	1,2	2,0	1,5	11,0	6,9	4,5	2,0	0,1	0,1	0,1	0,1	21,5	27,0	45,5	55,0	18,9	3,3
15	1,7	1,9	1,1	1,0	8,4	5,9	3,3	2,1	0,2	0,2	0,2	0,1	20,8	42,9	82,1	95,7	30,1	3,5
16	1,3	1,8	1,8	1,8	11,0	10,4	5,0	2,2	0,1	0,1	0,2	0,1	13,8	25,6	39,0	56,2	17,7	1,7
17	1,1	1,6	1,8	1,5	11,7	6,1	5,1	2,5	0,1	0,1	0,1	0,2	8,8	29,8	40,7	96,0	22,6	2,8
18	1,9	1,5	1,8	1,5	8,0	7,2	4,1	1,6	0,1	0,1	0,1	0,1	14,3	33,4	43,1	56,7	29,4	5,6
19	1,8	1,5	1,5	1,8	12,1	6,4	5,2	2,8	0,1	0,2	0,1	0,2	25,9	38,0	51,4	81,8	22,1	3,4
20	1,9	1,5	1,3	1,5	8,5	6,5	5,6	2,0	0,1	0,1	0,1	0,1	15,6	28,1	53,1	55,3	20,0	4,4
21	1,7	1,9	1,7	1,7	10,8	9,1	4,9	3,0	0,2	0,1	0,2	0,2	25,9	42,7	84,1	132,6	26,1	3,1
22	1,8	1,8	1,7	1,6	8,6	7,3	3,9	3,0	0,2	0,1	0,1	0,1	21,6	44,8	63,3	89,5	30,7	5,3
23	1,3	1,8	1,5	1,5	7,4	5,5	5,7	1,9	0,1	0,1	0,1	0,2	17,0	32,2	48,6	88,9	32,9	5,0
24	1,7	1,6	1,4	1,6	9,7	6,9	4,7	2,4	0,1	0,2	0,1	0,1	19,8	46,4	49,6	78,0	22,3	3,8
25	1,9	1,1	1,4	1,1	7,5	5,0	3,1	1,8	0,1	0,1	0,2	0,1	18,7	34,6	54,7	114,3	38,2	3,8

** The system parameters are drawn from uniform distribution, where $\lambda_k \sim U(1,2)$; $\rho_i \sim U(0.5,0.9)$;

$$\mu_i = \left(\frac{(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)}{\rho_1}, \frac{(\lambda_2 + \lambda_3 + \lambda_4)}{\rho_2}, \frac{(\lambda_3 + \lambda_4)}{\rho_3}, \frac{\lambda_4}{\rho_4} \right) \quad \text{and} \quad L_k \sim U(100,200) \times \left(h_1, \sum_{i=1}^2 h_i, \sum_{i=1}^3 h_i, \sum_{i=1}^4 h_i \right),$$

$i \in \{1,2,3,4\}$, $k \in \{1,2,3,4\}$. For ease of disposition, we let $h = h/100$ and $L = L/100$; we keep one decimal point in this case.

5.6 Conclusion

In this chapter, we conducted a numerical experiment for a W -configuration ATO system. The numerical results showed that the decomposition method works best in this system, followed by the exhaustive search method and the (S,R) heuristic. The difference of heuristic results between exhaustive search method and the (S,R) heuristic is small, considering exhaustive search method requires intensive computational effort. Furthermore, we showed that the computation is much faster if we utilize the results obtained by the (S,R) heuristic in value the iteration/ policy iteration compared to just applying them without the startup policies obtained through the (S,R) heuristic. Finally, we applied the decomposition method to three larger ATO systems and showed that the heuristic continues to perform well.

Chapter 6. Conclusion and Future research perspectives

It is well recognized that ATO systems are difficult to analyze as they comprise both an assembly system and a distribution system. This dissertation significantly broadens the current literature in ATO systems. In particular, the characterization of the structure of the optimal policy within a MDP framework.

In this dissertation, we studied a *W*-configuration ATO system. While the vast majority of the literature dealing with integrated production and inventory control tries to characterize the structure of the optimal policy over the entire state space, we used a novel approach that characterizes the structure of the optimal policy within a sub space which we identified as the recurrent region within which the state remains in the long run. The advantage of such approach is that it allows us to identify a set of structural properties that are satisfied by the cost function (otherwise may not be satisfied outside this region). This approach greatly simplifies the analysis and can be applied to several similar integrated production and inventory control problems. In particular, within this region, the operator associated with the production decision, in the optimality equation, automatically satisfies the set of identified structural properties. This, in addition to greatly simplifying the proof of the structure of the optimal policy, avoids the challenge of having the cost function not satisfying certain properties in the region other than the recurrent. Furthermore, in addition to integrated production and inventory control systems, our approach can be applied to problems where the optimal policy yields both a recurrent region and a transient region. Specifically, we showed that this region is bounded by three state-dependent thresholds or base-stock levels, each corresponding to a component. We showed that it is optimal to produce/replenish a component if its inventory level is strictly below its corresponding base-stock level and not to produce it otherwise. We also showed that, within the recurrent region, the optimal inventory allocation policy for each product is characterized by two state-dependent thresholds corresponding to the components used by the product. Our analysis revealed that the optimal allocation policy does not follow the same structure usually encountered in the integrated production/inventory

literature. In particular, one would expect that the demand of the product with the higher lost sale cost or higher lost sale cost rate (i.e., demand rate times the lost sale cost) would always be satisfied and would take priority over that of the other product. However, we showed examples of the optimal allocation policy where the demand of the product with the lower lost sale and even lower lost sale cost rate takes priority over the product with higher values of the two parameters. This indicates that the optimal inventory allocation policy may not be simple to implement and thus has to be carefully designed when developing heuristic procedures. We have also showed that the optimal policy structure is preserved for systems with batch production, Erlang distributed production times, and non-unitary product demands.

Based on the structure of the optimal policy, we proposed a decomposition method that decomposes the W -configuration ATO system into two N -configuration systems. We compared it with two other state independent heuristic policies, which are obtained by the (S,R) heuristic and the exhaustive search method. Extensive numerical experiments showed that the three heuristics are efficient compared to the optimal policy. In particular, the decomposition method works best both in terms of cost performance computational time, whereas the (S,R) heuristic has the advantage of being very efficient in terms of computational time compared to the exhaustive search method. Taking advantage of this fact, we also used the heuristic as a startup policy to the dynamic programming algorithms used to obtain the optimal policy. We showed that doing so reduced the CPU time, required to obtain the optimal policy, by several orders of magnitude. In addition, to validate the effectiveness of the decomposition method, we applied it to three additional larger systems. For each system, we conducted a numerical experiment, which showed that the decomposition method continues to perform very well.

This work can be extended in several directions. One extension would be to consider the case where demand of a product is backordered if it cannot be fulfilled immediately. In this case, one needs to keep track not only of the inventory of components but also of the number of backorders of each product, adding further complexity to the model. Another extension is to consider the general W -configuration ATO system which consists of several products sharing multiple components. We expect the component production and inventory allocation policies to continue to have the same structure. However, it would be very challenging to obtain the optimal policies, numerically, in such a case. Finally, another direction would be to extend the

problem to the case of phase-type distributions for both component production times and demand inter-arrival times. We expect the structure of the optimal policy to remain the same in these cases.

References

1. Akcay, Y., & Xu, S. H. (2004). Joint Inventory Replenishment and Component Allocation Optimization in an Assemble-to-Order System. *Management Science*, 50(1):99–116.
2. Atan Z., Ahmadi T., Stegehuis C., de Kok T., & Adan I. (2017). Assemble-to-Order systems: A Review. *European Journal of Operation Research*.
3. Bazaraa M. S., & Shetty C. M. (1982) Nonlinear Programming. Theory and Algorithms [Russian translation], *Mir, Moscow*.
4. Benjaafar S., & ElHafsi M. (2006). “Production and Inventory Control of a Single Product Assemble-to-Order System with Multiple Customer Classes.” *Management Science*, 52 (12): 1896–1912.
5. Benjaafar, S., Gayon, J. P. & Tepe, S. (2010). Optimal control of a production–inventory system with customer impatience. *Operations Research Letters*, 38(4), 267–272.
6. Benjaafar, S. & Elhafsi, M. (2012). A production-inventory system with both patient and impatient demand classes. *IEEE Transactions on Automation Science and Engineering*, 9(1), 148–159.
7. Bernstein, F., DeCroix, G. A., & Wang, Y. (2007). Incentives and Commonality in a Decentralized Multiproduct Assembly System. *Operations Research*, 55(4):630–646.
8. Bernstein, F., DeCroix, G. A., & Wang, Y. (2011). The Impact of Demand Aggregation Through Delayed Component Allocation in an Assemble-to-Order System. *Management Science*, 57(6):1154–1171.
9. Bertsekas, D. (2001). Dynamic programming and optimal control (Vol. II). *Belmont: Athena Scientific*.
10. Basic, A., Vliegen, I., & Scheller-wolf, A. (2012). Comparing Markov chains: Aggregation and precedence relations applied to sets of states, with applications to assemble-to-order systems. *Mathematics of Operations Research*, 37(November 2014):259–287.
11. Buzacott, J. A., & Shanthikumar, J. G. (1993). Stochastic models of manufacturing systems (Vol. 4). *Englewood Cliffs, NJ: Prentice Hall*.
12. Carr, S., & Duenyas, I. (2000). Optimal admission control and sequencing in a make-to-stock/make-to-order production system. *Operations research*, 48(5), 709–720.
13. Cavazos-Cadena, R., & Sennott, L. I. (1992). Comparing recent assumptions for the existence of average optimal stationary policies. *Operations research letters*, 11(1), 33–37.
14. Ceryan, O., Duenyas, I., & Koren, Y. (2012). Optimal control of an assembly system with demand for the end-product and intermediate components. *IIE Transactions*, 44(5), 386–403.
15. Chen, J., & Dong, M. (2014). Available-to-promise-based flexible order allocation in ATO supply chains. *International Journal of Production Research*, 52(22):6717–6738.
16. Cheng, T., Gao, C., & Shen, H. (2011). Production planning and inventory allocation of a single-product assemble-to-order system with failure-prone machines. *International Journal of Production Economics*, 131(2):604–617.
17. Chod, J., Pyke, D., & Rudi, N. (2010). The Value of Flexibility in Make-to-Order Systems: The Effect of Demand Correlation. *Operations Research*, 58(4):834–848.
18. Dayanik, S., Song, J.-S., & Hu, S. (2003). The effectiveness of several performance bounds for capacitated production, partial-order-service, assemble-to-order systems. *Manufacturing & Service Operations Management*, 5(3):230–251.
19. De Vericourt, F., Karaesmen, F. & Dallery, Y. (2002). Optimal stock allocation for a capacitated supply system. *Management Science*, 48(11), 1486–1501.
20. Dogru, M. K., Reiman, M. I., & Wang, Q. (2010). A Stochastic Programming Based Inventory Policy for Assemble-to-Order Systems with Application to the W Model. *Operations Research*, 58(4):849–864.

21. Elhafsi, M. (2009). Optimal integrated production and inventory control of an assemble-to-order system with multiple non-unitary demand classes. *European Journal of Operational Research*, 194(1):127–142.
22. Elhafsi, M., & Hamouda, E. (2015). Managing an assemble-to-order system with after sales market for components. *European Journal of Operational Research*, 242(3):828–841.
23. Elhafsi, M., Zhi, L., Camus, H., & Craye, E. (2015). An assemble-to-order system with product and components demand with lost sales. *International Journal of Production Research*, 53(3), 718-735.
24. Eynan, A., & Rosenblatt, M. J. (2007). The impact of component commonality on composite assembly policies. *Naval Research Logistics*, 54(6):615–622.
25. Fang, X., So, K. C., & Wang, Y. (2008). Component Procurement Strategies in Decentralized Assemble-to-Order Systems with Time-Dependent Pricing. *Management Science*, 54(12):1997–2011.
26. Feng, Y., Ou, J., & Pang, Z. (2008). Optimal control of price and production in an assemble-to-order system. *Operations Research Letters*, 36(4):506–512.
27. Fong, D. K., Fu, H., & Li, Z. (2004). Efficiency in shortage reduction when using a more expensive common component. *Computers & Operations Research*, 31(1):123–138.
28. Fu K., Hsu V. N., & Lee C.-Y. (2006). Inventory and production decisions for an assemble-to-order system with uncertain demand and limited assembly capacity. *Operation Research*, 54(6): 1137-1150.
29. Fu, K., Hsu, V. N., & Lee, C.-Y. (2009). Note —Optimal Component Acquisition for a Single Product, Single-Demand Assemble-to-Order Problem with Expediting. *Manufacturing & Service Operations Management*, 11(2):229–236.
30. Gallien, J., & Wein, L. M. (2001). A simple and effective component procurement policy for stochastic assembly systems. *Queueing Systems*, 38(2), 221-248.
31. Gao, C., Shen, H., & Cheng, T. (2010). Order-fulfillment performance analysis of an assemble-to-order system with unreliable machines. *International Journal of Production Economics*, 126(2):341–349.
32. Glasserman, P., Wang, Y. (1998). Leadtime-inventory trade-offs in assemble-to-order systems. *Operations Research*, 46(6), 858-871.
33. Gunasekaran A., Ngai E.W.T. (2005). Build-to-order supply chain management: a literature review and framework for development. *Journal of Operations Management*, 23: 423-451.
34. Gühlich, H., Fleischmann, M., & Stoltetz, R. (2015). Revenue management approach to due date quoting and scheduling in an assemble-to-order production system. *OR Spectrum*, 37:951–982.
35. Ha, A. Y. (1997a). Inventory rationing in a make-to-stock production system with several demand classes and lost sales. *Management Science*, 43(8), 1093-1103.
36. Ha, A. Y. (1997b). Stock rationing policy for a make-to-stock production system with two priority classes and backordering. *Naval Research Logistics (NRL)*, 44(5), 457-472.
37. Ha, A. Y. (2000). Stock rationing in an M/Ek/1 make-to-stock queue. *Management Science*, 46(1), 77-87.
38. Feng, J., Liu, L., & Wan, Y.-W. (2012). Irreducibility of joint inventory positions in an assemble-to-order system under (r, nQ) policies. *Naval Research Logistics*, 59(1):18–25.
39. Fu, K., Hsu, V. N., & Lee, C.-Y. (2011). Approximation methods for the analysis of a multicomponent, multiproduct assemble-to-order system. *Naval Research Logistics*, 58(7):685–704.
40. Huang, K. (2014). Benchmarking non-first-come-first-served component allocation in an assemble-to-order system. *Annals of Operations Research*, 223(1), 217-237.
41. Huang, H., Jin, Y., Huang, B., & Qiu, H.-G. (2014). Mixed Replenishment Policy for ATO Supply Chain Based on Hybrid Genetic Simulated Annealing Algorithm. *Mathematical Problems in Engineering*, 2014:1–9.
42. Huang, K., & de Kok, T. (2015). Optimal FCFS allocation rules for periodic-review assemble-to-order systems. *Naval Research Logistics (NRL)*, 62(2):158–169.

43. Horng, S.-C., & Yang, F.-Y. (2012). Optimal base-stock policy of the assemble-to-order systems. *Artificial Life and Robotics*, 17(1):47–52.
44. Hsu, V. N., Lee, C. Y., & So, K. C. (2006). Optimal Component Stocking Policy for Assemble-to-Order Systems with Lead-Time-Dependent Component and Product Pricing. *Management Science*, 52(3):337–351.
45. Hsu, V. N., Lee, C. Y., & So, K. C. (2007). Managing components for assemble-to-order products with lead-time-dependent pricing: The full-shipment model. *Naval Research Logistics*, 54(5):510–523.
46. Inman, R. R., & Schmeling, D. M. (2003). Algorithm for agile assembling-to-order in the automotive industry. *International Journal of Production Research*, 41(16):3831–3848.
47. Iravani, S., Luangkesorn, K., & Simchi-Levi, D. (2004). A general decomposition algorithm for parallel queues with correlated arrivals. *Queueing Systems*, 47(4):313–344.
48. Joseph Pine II B. (1993). *Mass Customization: the new frontier in business competition*. Harvard Business School Press, Boston, Massachusetts.
49. Kapuscinski R., Zhang R. Q., Carbonneau P., Moore R., & Reeves B. (2004). Inventory decisions in Dell's supply chain. *Interfaces*, 34(3): 191-205.
50. Karaarslan, A., Kiesmüller, G., & de Kok, A. (2013). Analysis of an assemble-to-order system with different review periods. *International Journal of Production Economics*, 143(2):335–341.
51. Keblis, M. F. & Feng, Y. (2012). Optimal Pricing and Production Control in an Assembly System With a General Stockout Cost. *IEEE Transactions on Automatic Control*, 57(7):1821–1826.
52. Lee, H. L., Padmanabhan, V., & Whang, S. (1997). The bullwhip effect in supply chains. *Sloan management review*, 38(3), 93.
53. Lippman, S. A. (1975). Applying a new device in the optimization of exponential queuing systems. *Operations Research*, 23(4), 687-710.
54. Lu, Y., & Song, J.-S. (2005). Order-based cost optimization in assemble-to-order systems. *Operations Research*, 53(1):151–169.
55. Lu, Y. (2008). Performance analysis for assemble-to-order systems with general renewal arrivals and random batch demands. *European Journal of Operational Research*, 185(2):635–647.
56. Lu, Y., Song, J.-S., & Zhao, Y. (2010). No-Holdback Allocation Rules for Continuous-Time Assemble-to-Order Systems. *Operations Research*, 58(3):691–705.
57. Lu, L., Song, J.-S., & Zhang, H. (2015). Optimal and asymptotically optimal policies for assemble-to-order N-and W-systems. *Naval Research Logistics (NRL)*, 62(8):617–645.
58. Manne, A. S. (1960). Linear programming and sequential decisions. *Management Science*, 6(3), 259-267.
59. Mohebbi, E., & Choobineh, F. (2005). The impact of component commonality in an assemble-to-order environment under supply and demand uncertainty. *Omega*, 33(6):472–482.
60. Nadar, E., Akan, M., & Scheller-wolf, A. (2014). Technical Note-Optimal Structural Results for Assemble-to-Order Generalized M -Systems. *Operations Research*, 62(October):571–579.
61. Nadar, E., Akan, M., & Scheller-wolf, A. (2016). Experimental Results Indicating Lattice Dependent Policies May Be Optimal for General Assemble-to-Order Systems. *Production and Operations Management*, 25(4):647–661.
62. Oh, S., Sourirajan, K., & Ettl, M. (2014). Joint Pricing and Production Decisions in an Assemble-to-Order System. *Manufacturing & Service Operations Management*, 16(4):529–543.
63. Pang, Z., Chen, F. Y., & Feng, Y. (2012). Technical Note-A Note on the Structure of Joint Inventory-Pricing Control with Leadtimes. *Operations Research*, 60(3):581–587.
64. Pang, Z. (2015). Optimal control of a single-product assemble-to-order system with multiple demand classes and backordering. *IEEE Transactions on Automatic Control*, 60(2), 480-484.

65. Plambeck, E. L., & Ward, A. R. (2006). Optimal Control of a High-Volume Assemble-to-Order System. *Mathematics of Operations Research*, 31(3):453–477.
66. Plambeck, E. L., & Ward, A. R. (2007). Note: A Separation Principle for a Class of Assemble-to-Order Systems with Expediting. *Operations Research*, 55(3):603–609.
67. Porteus, E. (1982). Conditions for characterizing the structure of optimal strategies in infinite-horizon dynamic programs. *Journal of Optimization Theory and Applications*, 36(3), 419-432.
68. Puterman, M. L. (1994). Markov decision processes. *Jhon Wiley & Sons, New Jersey*.
69. Regattieri, A., Giazzi, A., Gamberi, M., & Gamberini, R. (2015). An innovative method to optimize the maintenance policies in an aircraft: General framework and case study. *Journal of air transport management*, 44, 8-20.
70. Reiman, M. I., & Wang, Q. (2015). Asymptotically optimal inventory control for assemble-to-order systems with identical lead times. *Operations Research*, 63(3), 716-732.
71. Rosling, K. (1989). Optimal inventory policies for assembly systems under random demands. *Operations Research*, 37(4):565–579.
72. Rudberg M., & Wikner J. (2004). Mass customization in terms of the customer order decoupling point, *Production Planning & Control*, 15(4): 445-458, DOI: 10.1080/0953728042000238764.
73. Song, J. S. (2000). A note on assemble-to-order systems with batch ordering. *Management Science*, 46(5), 739-743.
74. Song, J. S. (2002). Order-based backorders and their implications in multi-item inventory systems. *Management Science*, 48(4), 499-516.
75. Song, J. S., Xu, S. H., & Liu, B. (1999). Order-fulfillment performance measures in an assemble-to-order system with stochastic leadtimes. *Operations Research*, 47(1), 131-149.
76. Song J.-S. and Zipkin P. (2003). “Supply Chain Operations: Assemble-to-Order Systems.” In Handbooks in Operations Research and Management Science, 11:561-96. Supply Chain Management. *North-Holland*.
77. Song, J. S., Yao, D. D. (2002). Performance analysis and optimization of assemble-to-order systems with random lead times. *Operations Research*, 50(5), 889-903.
78. Teunter, R. H., & Haneveld, W. K. K. (2008). Dynamic inventory rationing strategies for inventory systems with two demand classes, Poisson demand and backordering. *European Journal of Operational Research*, 190(1), 156-178.
79. Tsai, K.-M., & Wang, S.-C. (2009). Multi-site available-to-promise modeling for assemble-to-order manufacturing: An illustration on TFT-LCD manufacturing. *International Journal of Production Economics*, 117(1):174–184.
80. Tsai, C.-Y., Chen, C.-J., & Lo, Y.-T. (2013). A cost-based module mining method for the assemble-to-order strategy. *Journal of Intelligent Manufacturing*, 25(6):1377–1392.
81. van Jaarsveld W., Dollevoet T., & Dekker R. (2015). Improving spare parts inventory control at a repair shop. *Omega: The International Journal of Management Science*, 57(B): 217-229.
82. van Jaarsveld W., & A. Scheller-Wolf. (2015). Optimization of industrial-scale assemble-to-order systems. *INFORMS Journal on Computing*, 27(3): 544-560.
83. van Mieghem, J. A. (2004). Note-commonality strategies: Value drivers and equivalence with flexible capacity and inventory substitution. *Management Science*, 50(3):419–424.
84. Veatch, M. H., & Wein, L. M. (1996). Scheduling a make-to-stock queue: Index policies and hedging points. *Operations Research*, 44(4), 634-647.
85. Vliegen, I. M. H. (2009). Integrated planning for service tools and spare parts for capital goods. *Eindhoven University of Technology, Eindhoven, The Netherlands*.

86. Weber, R. R., & Stidham, S. (1987). Optimal control of service rates in networks of queues. *Advances in applied probability*, 19(1), 202-218.
87. Williams K., Haslam C., & Williams J. (1992). Ford versus 'Fordism': the beginning of mass production? *Work, Employment & Society*, 6(4):517-555.
88. Xiao, Y., Chen, J., & Lee, C.-Y. (2010a). Optimal decisions for assemble-to-order systems with uncertain assembly capacity. *International Journal of Production Economics*, 123(1):155-165.
89. Xiao Y., Chen J., & Lee C.-Y. (2010b). Optimal decisions for assemble-to-order systems with uncertain assembly capacity. *Int. J. Production Economics*, 123:155-165.
90. Xu, P. J., Graves, S. C., & Allgor, R. (2006). The benefits of reevaluating the real-time fulfillment decisions. *Manufacturing and Service Operations Management*, 8(1), 104-117.
91. Xu, S. H., & Li, Z. (2007). Managing a Single-Product Assemble-to-Order System with Technology Innovations. *Management Science*, 53(9):1467-1485.
92. Yao Z., Lee L. H., Chew E. P., Hsu V. N., & Jaruphongs W. (2013). Dual-channel component replenishment problem in an assemble-to-order system. *IIE Transactions*, 45:229-243.
93. Zhang, X., Ou, J., & Gilbert, S. M. (2008). Coordination of stocking decisions in an assemble-to-order environment. *European Journal of Operational Research*, 189(2):540-558.
94. Zhao, Y. (2009). Analysis and evaluation of an assemble-to-order system with batch ordering policy and compound Poisson demand. *European Journal of Operational Research*, 198(3):800-809.
95. Zhao, Y. (2009). Analysis and evaluation of an Assemble-to-Order system with batch ordering policy and compound Poisson demand. *European Journal of Operational Research*, 198(3), 800-809.
96. Zhou, W., & Chao, X. (2012). Stein–chen approximation and error bounds for order fill rates in assemble-to-order systems. *Naval Research Logistics*, 59(8):643-655.
97. Zipkin, P. H. (2000). Foundations of inventory management. *McGraw-Hill, Boston*.
98. Zinn W. (1990). "Should You Assemble Products before an Order Is Received?" *Business Horizons*, March-April edition.
99. Zipkin, P. (2016). Some Specially Structured Assemble-to-Order Systems. *Operation Research Letters*, 44:136-142.

Analyse et algorithmes de résolution de systèmes ATO (Assemble-To-Order) : Applications aux systèmes du type W

Résumé: Les systèmes ATO qui comprennent un système d'assemblage et un système de distribution sont difficiles à analyser par sa nature. Dans cette thèse, nous analysons un type W de système ATO avec des délais de livraison aléatoires, l'arrivée aléatoire de la demande et des ventes perdues, en temps continu. Nous formulons le problème en tant que processus de décision Markov à l'horizon infini. Nous nous éloignons de l'approche standard en caractérisant une région de l'espace d'état où toutes les propriétés de la fonction de coût tiennent. Nous caractérisons la politique optimale dans cette région. En particulier, nous montrons que, dans l'intérieur de la région récurrente, les composants sont toujours produits. Nous caractérisons également la politique d'allocation de composants optimale qui spécifie si une demande de produit arrivant devrait être remplie. Notre analyse révèle que la politique d'allocation optimale est contre-intuitive. Par exemple, même lorsqu'un produit domine l'autre, en termes de coût/taux de vente perdue, sa demande peut ne pas avoir une priorité absolue par rapport à la demande de l'autre produit. Une telle caractéristique n'a pas été observée dans de nombreux paramètres intégrés de production/inventaire où l'allocation d'inventaire suit une priorité fixe pour satisfaire les exigences. Nous montrons également que la structure de la politique optimale reste la même pour les systèmes à production par lots, les temps de production répartis par Erlang et la demande de produits non unitaire. Enfin, nous proposons des heuristiques efficaces qui peuvent être utilisées comme substitut à la politique optimale ou peuvent être utilisées comme une politique de départ pour les algorithmes communs utilisés pour obtenir une politique optimale dans le but de réduire leur temps de calcul.

Mots-clefs: Assemble-To-Order, Processus de décision Markov, Programmation dynamique, Contrôle de la production et des stocks, Gestion de la demande

Analysis and Computational Algorithms for Assemble-To-Order systems: Application to W -configuration systems

Abstract: The ATO systems that comprise an assembly system and a distribution system are difficult to analyze by its nature. In this thesis, we analyze a W -configuration ATO system with random lead times, random arrival of demand, and lost sales, in continuous time. We formulate the problem as an infinite-horizon Markov decision process. We deviate from the standard approach by first characterizing a region (the recurrent region) of the state space where all properties of the cost function hold. We then characterize the optimal policy within this region. In particular, we show that within the interior of the recurrent region components are always produced. We also characterize the optimal component allocation policy which specifies whether an arriving product demand should be fulfilled. Our analysis reveals that the optimal allocation policy is counter-intuitive. For instance, even when one product dominates the other, in terms of lost sale cost and lost sale cost rate (i.e., demand rate times the lost sale cost), its demand may not have absolute priority over the other product's demand. Such a feature has not been observed in many integrated production/inventory settings where inventory allocation follows a fixed priority in satisfying demands. We also show that the structure of the optimal policy remains the same for systems with batch production, Erlang distributed production times, and non-unitary product demand. Finally, we propose efficient heuristics that can be either used as a substitute for the optimal policy or can be used as a starting policy for the common algorithms that are used to obtain the optimal policy in an effort to reduce their computational time.

Keywords: Assemble-To-Order, Markov decision process, Dynamic programming, Production and inventory control, Demand management