Méthodes d'analyse de Fourier en hydrodynamique : des mascarets aux fluides avec capillarité

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THÈSE DE DOCTORAT DE l'UNIVERSITÉ PARIS-EST

Spécialité

Mathématiques appliquées

L'École Doctorale Mathématiques et Sciences et Technologies de l'Information et de la Communication

Présentée par

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Méthodes d'analyse de Fourier en hydrodynamique : des mascarets aux fluides avec capillarité

soutenue le 6 Juillet 2017

devant le jury composé de :

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Fourier analysis methods in hydrodynamics: from bores to capillary fluids

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Résumé

Dans la première partie de cette thèse on étudie les systèmes *abcd* qui ont été dérivés par J.L. Bona, M. Chen et J.-C. Saut en 2002. Ces systèmes sont des modèles approximant le problème d'ondes hydrodynamiques dans le régime de Boussinesq, à savoir, des vagues de faible amplitude et de grande longueur d'onde. Dans les deux premiers chapitres on considère le problème d'existence en temps long à savoir la construction de solutions pour les systèmes *abcd* qui ont leur temps d'existence minoré par $1/\varepsilon$ où ε est le rapport entre une amplitude typique de vague et la profondeur du canal. Dans un premier temps on considère des données initiales appartenant aux espaces de Sobolev qui sont inclus dans l'espace des fonctions continues qui s'annulent à l'infini. D'un point de vue physique cette situation correspond à des vagues localisées en espace. Le point clé est la construction d'une fonctionnelle non linéaire d'énergie qui contrôle certaines normes de Sobolev sur un intervalle de temps long. Pour y arriver, on travaille avec des équations localisées en fréquence. Cette approche nous permet d'obtenir des résultats d'existence en temps long en demandant moins de régularité sur les données initiales. Un deuxième avantage de notre méthode est que l'on peut traiter d'une manière unifiée presque tous les cas correspondant aux différentes valeurs des paramètres *abcd*. Dans le deuxième chapitre on montre des résultats d'existence en temps long pour le cas des données ayant un comportement non trivial à l'infini. Ce type des données est pertinent pour l'étude de la propagation des mascarets. L'idée qui est à la base de ces résultats est de considérer un découpage convenable de la donnée initiale en hautes et basses fréquences. Dans le troisième chapitre on emploie des schémas de volumes finis afin de construire des solutions numériques. On utilise ensuite nos schémas pour étudier l'interaction d'ondes progressives.

La deuxième partie de ce manuscrit est consacrée à l'étude des problèmes de régularité optimale pour le système de Navier-Stokes qui régit l'évolution d'un fluide incompressible, inhomogène et pour le système Navier-Stokes-Korteweg utilisé pour prendre en compte les effets de capillarité. Plus précisément, on montre que ces systèmes sont bien-posés dans leurs espaces critiques, à savoir, les espaces qui ont la même invariance par changement d'échelle que les systèmes eux-mêmes. Pour pouvoir démontrer ce type de résultats on a besoin d'établir de nouvelles estimations pour un problème de type Stokes avec des coefficients variables peu réguliers.

Mots clés systèmes de Boussinesq ; existence en temps long ; propagation de mascarets ; système de Navier-Stokes-Korteweg ; système de Navier-Stokes pour un fluide incompressible inhomogène ; regularité critique ; coordonnées lagrangiennes ;

MSC : 35Q35, 76B15, 35A01, 76B03, 35Q30, 76D05

Abstract

The first part of the present thesis deals with the so -called *abcd* systems which were derived by J.L. Bona, M. Chen and J.-C. Saut back in 2002. These systems are approximation models for the water-waves problem in the Boussinesq regime, that is, waves of small amplitude and long wavelength. In the first two chapters we address the long time existence problem which consists in constructing solutions for the Cauchy problem associated to the *abcd* systems and prove that the maximal time of existence is bounded from below by some physically relevant quantity. First, we consider the case of initial data belonging to some Sobolev spaces imbedded in the space of continuous functions which vanish at infinity. Physically, this corresponds to spatially localized waves. The key ingredient is to construct a nonlinear energy functional which controls appropriate Sobolev norms on the desired time scales. This is accomplished by working with spectrally localized equations. The two important features of our method is that we require lower regularity levels in order to develop a long time existence theory and we may treat in an unified manner most of the cases corresponding to the different values of the parameters. In the second chapter, we prove the long time existence results for the case of data that does not necessarily vanish at infinity. This is especially useful if one has in mind bore propagation. One of the key ideas of the proof is to consider a well-adapted high-low frequency decomposition of the initial data. In the third chapter, we propose finite-volume schemes in order to construct numerical solutions. We use these schemes in order to study traveling waves interaction.

The second part of this manuscript, is devoted to the study of optimal regularity issues for the incompressible inhomogeneous Navier-Stokes system and the Navier-Stokes-Korteweg system used in order to take in account capillarity effects. More precisely, we prove that these systems are well-posed in their truly critical spaces i.e. the spaces that have the same scale invariance as the system itself. In order to achieve this we derive new estimates for a Stoke-like problem with time independent variable coefficients.

Keywords Boussinesq systems; long time existence; bore propagation; Navier-Stokes-Korteweg system; Inhomogeneous Navier-Stokes system; critical regularity; Lagrangian coordinates;

MSC: 35Q35, 76B15, 35A01, 76B03, 35Q30, 76D05

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Introduction

0.1 Première partie : l'étude des systèmes *abcd*

0.1.1 Origine des modèles *abcd* et résultats connus

L'étude de la propagation des ondes non linéaires en hydrodynamique est très importante à cause des nombreuses applications dans les domaines de l'océanographie, de la météorologie, de l'ingénierie côtière, etc. En tant qu'exemples de phénomènes physiques concrets où ce comportement non linéaire joue un rôle très important, on pourra mentionner la propagation des tsunamis, le changement du fond de l'océan après un orage ou l'érosion des plages.

Dans la suite, on s'intéresse à l'écoulement potentiel d'un fluide homogène, incompressible et non visqueux dans un canal à fond plat de hauteur $h > 0$ représenté par le plan horizontal :

$$\{(x, y, z) : z = -h\}. \tag{0.1.1}$$

Supposons que la surface libre de l'eau qui résulte d'une perturbation initiale de l'état d'équilibre puisse être décrite par une fonction η au-dessus du fond plat. En toute généralité l'évolution de la surface libre η et du champ de vitesses de l'écoulement, $u = \nabla\phi$ est régie par les équations des ondes hydrodynamiques (*water waves* en anglais) :

$$\begin{cases} \Delta\phi + \partial_z^2\phi = 0 & \text{dans} \quad -h \le z \le \eta(t, x, y), \\ \partial_z\phi = 0 & \text{sur} \quad z = -h, \\ \partial_t\eta + \nabla\phi \cdot \nabla\eta - \partial_z\phi = 0 & \text{sur} \quad z = \eta(t, x, y), \\ \partial_t\phi + \frac{1}{2}\left(|\nabla\phi|^2 + |\partial_z\phi|^2\right) + gz = 0 & \text{sur} \quad z = \eta(t, x, y), \end{cases} \tag{$\mathcal{WW}$}$$

où g est l'accélération de la pesanteur. En dépit d'un grand nombre de progrès théoriques sur ce modèle, il reste encore très difficile à employer dans la pratique. Citons l'exemple de [37] que l'on trouve très éclairant : en cas d'un tsunami on aimerait avoir des modèles permettant de prédire son comportement dans un temps assez court, de l'ordre de dizaines de minutes, afin de pouvoir émettre des avertissements pertinents. Il va sans dire que pour être capable de faire de telles prédictions on aurait besoin de schémas numériques très précis. Cependant, le traitement numérique du système ($\mathcal{WW}$) reste très délicat. De plus, dans de nombreuses situations, il faut utiliser le système ($\mathcal{WW}$), couplé avec d'autres modèles.

Une idée, remontant à Boussinesq [25] pour surmonter les difficultés mathématiques et pratiques est d'approcher le système ($\mathcal{WW}$) selon les particularités du phénomène physique en question. On va détailler ci-dessous cette idée, en suivant la démarche formelle employée par Bona, Chen et Saut [15] et obtenir une nouvelle famille de systèmes permettant d'étudier *des ondes de surface de faible amplitude et de grande longueur d'onde*. Ce régime physique particulier des ondes à la surface de l'eau s'appelle *régime de Boussinesq*. Plus précisément, soit A une amplitude typique, respectivement l une longueur d'onde typique et considérons les quantités suivantes :

$$\alpha = \frac{A}{h}, \ \beta = \left(\frac{h}{l}\right)^2, \ S = \frac{\alpha}{\beta}. \tag{0.1.2}$$

L'aspect faible amplitude est modélisé en supposant que $\alpha \ll 1$, la petitesse est comprise par rapport à la profondeur du canal. Les vagues de grande longueur d'onde sont caractérisées par $\beta \ll 1$. De plus, on supposera qu'il y a une balance entre ces deux rapports, à savoir $S \approx 1$. En fait, pour alléger les notations on va prendre $S = 1$ et donc $\alpha = \beta = \varepsilon$, sans perdre de vue l'ordre formel des quantités impliquées.

Le premier pas vers la simplification du modèle $(\mathcal{WW})$ est de faire un changement de variables qui rend toutes les nouvelles inconnues d'ordre formel $O(1)$. Ainsi, si l'on considère :

$$x = l\tilde{x},\ y = l\tilde{y},\ z = h(\tilde{z}-1),\ \eta = A\tilde{\eta},\ t = \frac{\tilde{t}l}{\sqrt{gh}},\ \phi = gAl\frac{\tilde{\phi}}{\sqrt{gh}}$$

on obtient que

$$\begin{cases} \varepsilon\Delta\tilde{\phi} + \partial_z^2\tilde{\phi} = 0 & \text{dans} \quad 0 \le \tilde{z} \le 1 + \varepsilon\tilde{\eta}\left(\tilde{t},\tilde{x},\tilde{y}\right), \\ \partial_z\tilde{\phi} = 0 & \text{sur} \quad \tilde{z} = -1, \\ \partial_t\tilde{\eta} + \varepsilon\nabla\tilde{\phi}\cdot\nabla\tilde{\eta} - \frac{1}{\varepsilon}\partial_z\tilde{\phi} = 0 & \text{sur} \quad \tilde{z} = 1 + \varepsilon\tilde{\eta}\left(\tilde{t},\tilde{x},\tilde{y}\right), \\ \partial_t\tilde{\phi} + \frac{1}{2}\left(\varepsilon\left|\nabla\tilde{\phi}\right|^2 + \left|\partial_z\tilde{\phi}\right|^2\right) + \tilde{\eta} = 0 & \text{sur} \quad \tilde{z} = 1 + \varepsilon\tilde{\eta}\left(\tilde{t},\tilde{x},\tilde{y}\right). \end{cases} \qquad (\mathcal{WW}_\varepsilon)$$

Remarquons que dans les nouvelles variables, $\tilde{\eta}$ représente la déviation de la surface libre par rapport à l'état d'équilibre $z = 0$. Pour des soucis de clarté, dans la suite, on va renoncer à la notation tilde. On suppose qu'il est possible d'écrire le potentiel ϕ comme une série par rapport à la variable de hauteur :

$$\phi(t,x,y,z) = \sum_{k=0}^{\infty} f_k(t,x,y)\, z^k. \qquad (0.1.3)$$

En utilisant la première équation de $(\mathcal{WW}_\varepsilon)$ on en déduit que :

$$-\varepsilon\Delta f_k = (k+1)(k+2)\, f_{k+2}.$$

La deuxième équation de $(\mathcal{WW}_\varepsilon)$ implique que

$$f_1 = 0,$$

par conséquent, on a :

$$\phi = \sum_{k=0}^{\infty} \frac{(-1)^k \varepsilon^k}{(2k)!}\Delta^{(k)} f_0 z^{2k}. \qquad (0.1.4)$$

En remplaçant la formule (0.1.4) dans la troisième respectivement dans la quatrième équation de $(\mathcal{WW}_\varepsilon)$, on obtient

$$\begin{cases} \partial_t\eta + \operatorname{div} V_0 - \frac{\varepsilon}{6}\Delta \operatorname{div} V_0 + \varepsilon \operatorname{div}(\eta V_0) = O\left(\varepsilon^2\right), \\ \left(Id - \frac{\varepsilon}{2}\Delta\right)\partial_t V_0 + \nabla\eta + \frac{\varepsilon}{2}\nabla\left|V_0\right|^2 = O\left(\varepsilon^2\right), \end{cases} \qquad (0.1.5)$$

où $V_0 = \nabla f_0$. Grâce à la formule (0.1.3) on voit que $f_0 = \phi_{|z=0}$ et donc V_0 est une renormalisation du champ de vitesses au fond du canal. On peut introduire un degré de liberté dans le système (0.1.5) en exprimant V_0 à l'aide du champ de vitesses corespondant au plan situé à la hauteur $z = -(1-\theta)h$ où $\theta \in [0,1]$. Plus exactement, en vue du (0.1.4) on a :

$$V_\theta \overset{def.}{=} V_0 - \frac{\varepsilon\theta^2}{2}\Delta V_0 + O\left(\varepsilon^2\right),$$

formule que l'on peut formellement inverser afin d'obtenir :

$$V_0 = V_\theta + \frac{\varepsilon\theta^2}{2}\Delta V_\theta + O\left(\varepsilon^2\right). \qquad (0.1.6)$$

En remplaçant la formule (0.1.6) dans (0.1.5) on obtient

$$\left\{ \begin{array}{l} \partial_t \eta + \operatorname{div} V_\theta + \frac{\varepsilon}{2}\left(\theta^2 - \frac{1}{3}\right)\Delta \operatorname{div} V_\theta + \varepsilon \operatorname{div}(\eta V_\theta) = O\left(\varepsilon^2\right), \\ \left(Id - \frac{\varepsilon}{2}\left(\theta^2 - 1\right)\Delta\right)\partial_t V_\theta + \nabla \eta + \frac{\varepsilon}{2}\nabla\left|V_0\right|^2 = O\left(\varepsilon^2\right). \end{array} \right. \tag{0.1.7}$$

Ensuite, il est possible d'introduire deux autres degrés de liberté en observant que

$$\left\{ \begin{array}{l} \Delta \operatorname{div} V_\theta = -\Delta \partial_t \eta + O(\varepsilon), \\ \Delta \partial_t V_\theta = -\Delta \nabla \eta + O(\varepsilon), \end{array} \right.$$

donc on peut remplacer $\Delta \operatorname{div} V_\theta$ dans la première équation de (0.1.7) par $\lambda \Delta \operatorname{div} V_\theta - (1-\lambda)\Delta\partial_t \eta + O(\varepsilon)$ respectivement $\Delta \partial_t V_\theta$ dans la première équation de (0.1.7) par $\mu \Delta \partial_t V_\theta - (1-\mu)\Delta \partial_t \eta + O(\varepsilon)$ sans affecter l'ordre de l'approximation. En utilisant les notations

$$a = \left(\frac{\theta^2}{2} - \frac{1}{6}\right)\lambda,\ b = \left(\frac{\theta^2}{2} - \frac{1}{6}\right)(1-\lambda),$$
$$c = \frac{\left(1-\theta^2\right)}{2}\mu,\ d = \frac{\left(1-\theta^2\right)}{2}(1-\mu),$$

et en négligeant les termes d'ordre $O\left(\varepsilon^2\right)$ on obtient le système suivant :

$$\left\{ \begin{array}{l} (I - b\varepsilon\Delta)\partial_t \eta + \operatorname{div} V + a\varepsilon \operatorname{div} \Delta V + \varepsilon \operatorname{div}(\eta V) = 0, \\ (I - d\varepsilon\Delta)\partial_t V + \nabla \eta + c\varepsilon \nabla \Delta \eta + \frac{\varepsilon}{2}\nabla |V|^2 = 0. \end{array} \right. \tag{0.1.8}$$

Les systèmes (0.1.8) sont connus sous le nom de systèmes *abcd*. La famille des paramètres vérifie la contrainte :

$$a + b + c + d = \frac{1}{3}\cdot$$

Dans la littérature, on distingue certains membres de cette famille :

- le système "classique" de Boussinesq qui correspond au choix

$$a = b = c = 0, d = \frac{1}{3}\cdot \tag{0.1.9}$$

  Ceci est une version régularisée du système originellement dérivé par Boussinesq dans [25] ;

- la famille de Bona-Smith,

$$b = d > 0, c < 0, a = 0, \tag{0.1.10}$$

  dérivé pour la première fois dans [23] ;

- le cas générique :

$$b, d > 0,\ a, c < 0\ . \tag{0.1.11}$$

- le système BBM-BBM :

$$a = c = 0,\ b, d > 0; \tag{0.1.12}$$

- le système KdV-KdV

$$b = d = 0,\ a, c \leq 0 \text{ ou } a = c > 0. \tag{0.1.13}$$

Il faut déjà souligner deux simplifications majeures des nouveaux modèles. D'une part, on étudie les systèmes (0.1.8) sur un domaine fixe tandis que $(\mathcal{WW}_\varepsilon)$ est un problème à frontière libre. D'autre part on voit que pour décrire une situation physique en trois dimensions, il suffit de résoudre un système bidimensionnel. Signalons que Chazel [33] a obtenu des modèles qui approchent le problème $(\mathcal{WW}_\varepsilon)$ en prenant en compte des topographies plus compliquées que (0.1.1). Dans cette situation on doit distinguer deux autres régimes : de grandes, respectivement de petites, variations du fond. Des modèles qui autorisent des changements topographiques du fond du canal au cours du temps ont été obtenus par Chen en [36]. Pour un traitement systématique de modèles approchés pour le problème $(\mathcal{WW}_\varepsilon)$ dans différents régimes physiques ainsi que leur justification on pourra consulter l'ouvrage de Lannes [70].

D'un point de vue pratique, il est essentiel qu'une théorie d'existence des solutions soit valable sur des temps d'ordre ε^{-1} : on voit que les effets dispersifs et les effets non linéaires ont une contribution d'ordre $O(1)$ à l'évolution des vagues dès que $t \approx 1/\varepsilon$.

Formellement, les zéros du membre de droite du (0.1.8) peuvent être vus comme une erreur quadratique en ε que l'on obtient par le processus d'approximation présenté ci-dessus. Ainsi, on s'attend à ce que l'erreur entre les solutions de $(\mathcal{WW}_\varepsilon)$ et (0.1.8) s'accumule au cours du temps comme $O\left(\varepsilon^2 t\right)$. On voit que même sur des temps longs, à savoir de l'ordre ε^{-1}, l'erreur d'approximation doit rester petite, de l'ordre $O(\varepsilon)$. En fait, en continuant ce raisonnement, sur des échelles de temps de l'ordre ε^{-2} les systèmes (0.1.8) n'ont plus aucune raison de rester des approximations pertinentes du système de départ.

La justification mathématique rigoureuse de la validité du système (0.1.8) en tant qu'approximation du problème $(\mathcal{WW}_\varepsilon)$ se fait en trois étapes :

- montrer que le problème $(\mathcal{WW}_\varepsilon)$ est bien-posé dans un certain espace de fonctions et que l'on peut borner inférieurement le temps d'existence des solutions par une quantité qui est d'ordre $O\left(\varepsilon^{-1}\right)$. Ce pas fut accompli par Alvarez et Lannes en [5] ;

- montrer que le problème (0.1.8) est bien-posé dans le même espace que $(\mathcal{WW}_\varepsilon)$ et que l'on peut borner inférieurement le temps d'existence par une quantité d'ordre $O\left(\varepsilon^{-1}\right)$;

- finalement, trouver des estimations optimales d'erreur. Ce pas fut réalisé en [19] par Bona, Colin et Lannes. Plus précisément, si l'on considère la solution du système (0.1.8) on peut reconstruire une fonction tridimensionnelle en utilisant la formule (0.1.4). La différence entre cette nouvelle fonction et la solution de $(\mathcal{WW}_\varepsilon)$ au temps $t > 0$, mesurée dans une certaine norme de Sobolev est de l'ordre $O\left(\varepsilon^2 t\right)$.

Dans cette thèse on s'est intéressé d'abord au deuxième point exposé ci-dessus, parfois appelé le problème d'existence en temps long, à savoir, la construction des solutions pour les systèmes *abcd* sur des temps d'ordre $O\left(\varepsilon^{-1}\right)$. Citons, d'après J.-C. Saut, une façon plus abstraite de formuler ce problème : étant donné que ε représente un paramètre petit, on peut regarder les systèmes (0.1.8) comme des perturbations dispersives du système hyperbolique :

$$\begin{cases} \partial_t \eta + \operatorname{div} V + \varepsilon \operatorname{div}(\eta V) = 0, \\ \partial_t V + \nabla \eta + \frac{\varepsilon}{2} \nabla |V|^2 = 0. \end{cases} \tag{0.1.14}$$

Il est connu dans la théorie des systèmes hyperboliques que le temps d'existence des solutions régulières du système (0.1.14) est borné inférieurement par une quantité d'ordre $O\left(\varepsilon^{-1}\right)$. Donc le problème d'existence en temps long peut être reformulé de la façon suivante : si l'on considère des perturbations dispersives de (0.1.14), les solutions du nouveau système vont-elles hériter du temps d'existence hyperbolique, à savoir d'ordre $O\left(\varepsilon^{-1}\right)$? Une autre question qui se pose naturellement est alors de comprendre si l'on peut améliorer, au moins pour un certain choix de ces perturbations, la borne inférieure sur leur temps d'existence.

Même si tous les membres de la famille des systèmes (0.1.8) sont, d'un point de vue formel, équivalents, ils possèdent des propriétés mathématiques bien différentes. Pour mettre en évidence cet aspect, on pourra regarder la matrice de dispersion du linéarisé du système (0.1.8) en dimension 1 :

$$\begin{pmatrix} 0 & \frac{1-a\xi^2}{1+b\xi^2} \\ \frac{1-c\xi^2}{1+d\xi^2} & 0 \end{pmatrix}$$

qui admet les valeurs propre réelles

$$\pm\sqrt{\frac{(1-a\xi^2)(1-c\xi^2)}{(1+b\xi^2)(1+d\xi^2)}},$$

dans les deux cas

$$a \leq 0,\ c \leq 0,\ b \geq 0,\ d \geq 0 \tag{0.1.15}$$

$$\text{ou}\quad a = c \geq 0 \ \text{ et } \ b \geq 0,\ d \geq 0. \tag{0.1.16}$$

Passons en revue les résultats d'existence et unicité connus pour le système (0.1.8). Précisons d'abord qu'il y a deux types de résultats. Grâce à la présence des opérateurs de dispersion du (0.1.8), on arrive à montrer l'existence et l'unicité de solutions (pour certaines valeurs des paramètres) dans le cas de données initiales ayant un niveau de régularité plus bas que le niveau hyperbolique classique, à savoir $\frac{n}{2}+1$. Par contre, si on cherche à montrer l'existence en temps long, d'ordre ε^{-1}, on a besoin au moins de la régularité hyperbolique.

Dans [15], Bona, Chen et Saut ont étudié le linéarisé du système (0.1.8) en dimension 1. Les mêmes auteurs, [16], montrent que le système non linéaire est bien posé dans des espaces des Sobolev dans les deux cas (0.1.15) et (0.1.16). De plus, ils regardent des systèmes d'ordre plus grand.

Dans [53], Dougalis, Mitsotakis et Saut montrent que le système bidimensionnel est bien posé pour des données initiales dans $H^{s+\mathrm{sgn}(a)-\mathrm{sgn}(c)}\left(\mathbb{R}^2\right) \times [H^s\left(\mathbb{R}^2\right)]^2$ pour $s > 0$ avec $b, d > 0$, $a = c \geq 0$ ou $a, c \leq 0$. Ils montrent que le temps d'existence est minoré par une quantité d'ordre $O\left(\varepsilon^{\beta}\right)$ pour tout $\beta \in (0, 1/2)$. Les autres cas sont traités dans [7].

Linares, Pilod et Saut dans [71] ont traité le système KdV-KdV (0.1.13) en dimension 2, et ont montré qu'il est bien posé au sens de Hadamard dans $H^s\left(\mathbb{R}^2\right) \times [H^s\left(\mathbb{R}^2\right)]^2$ pour $s > 3/2$ et que le temps d'existence est d'ordre $O\left(\varepsilon^{-1/2}\right)$.

L'existence globale des solutions est connue seulement pour trois cas en dimension 1 d'espace. Le système classique de Boussinesq (0.1.9) a été étudié par Schonbek [83]. Plus précisément, pour des données initiales qui sont essentiellement de classe C^2, à support compact, et telles que

$$\inf_{x\in\mathbb{R}} \{1 + \varepsilon\eta_0(x)\} > 0, \tag{0.1.17}$$

elle construit des solutions faibles. L'hypothèse (0.1.17) est tout à fait raisonnable du point de vue physique : $1+\varepsilon\eta$ représente la hauteur totale de l'eau et donc sa positivité exprime qu'il n'y a pas de portions sèches dans le canal. L'idée clé, dans ce cas, est que le système possède la fonctionnelle d'entropie :

$$\int_{-\infty}^{\infty}\Big((1+\varepsilon\eta)\log(1+\varepsilon\eta) - \varepsilon\eta + \varepsilon^2 V^2 + \varepsilon^4 (V_x)^2\Big)dx,$$

qui fournit des estimations a priori dans un certain espace d'Orlicz pour η, et dans $H^1(\mathbb{R})$ pour la vitesse. L'unicité des solutions régulières a été étudiée par Amick [6].

Les deux autres cas pour lesquels l'existence globale des solutions est connue appartiennent à la famille des systèmes de Bona-Smith [23] et [16] avec $b = d > 0$, $a \leq 0$ et $c < 0$. A nouveau, en supposant que la donnée initiale vérifie (0.1.17) et en utilisant la structure hamiltonienne du système pour le cas $b = d$, à savoir :

$$\begin{cases} H(\eta, V) = \displaystyle\int_{-\infty}^{\infty}\Big(\eta^2 + (1+\varepsilon\eta)V^2 - \varepsilon c(\partial_x\eta)^2 - \varepsilon a(\partial_x V)^2\Big)dx, \\ \dfrac{d}{dt}H(\eta, V) = 0, \end{cases} \tag{0.1.18}$$

les auteurs arrivent à montrer que le système (0.1.8) est globalement bien posé pour des données $(\eta_0, V_0) \in H^{s-\mathrm{sgn}(a)}(\mathbb{R}) \times H^s(\mathbb{R})$ avec $s \geq 1$ si, de plus, $H(\eta_0, V_0)$ est suffisamment petite.

Le premier résultat d'existence en temps long fut obtenu par Ming, Saut et Zhang [74] pour BBM-BBM dans le cas générique (0.1.11). Leur preuve est basée sur un théorème de type Nash-Moser, et donc les solutions construites ont moins de régularité que la donnée initiale.

Ce défaut fut corrigé par Saut et Xu [82] respectivement Saut, Wang et Xu [81]. Donnons quelques détails sur leur approche. En considérant $U = (\bar{\eta}, \bar{v}_1, \bar{v}_2) = (\varepsilon\eta, \varepsilon V)$, ils mettent (0.1.8) sous la forme :

$$\begin{cases} (I - \varepsilon b\Delta)\, \partial_t U + M(U, D)\, U = 0, \\ U_{|t=0} = \varepsilon\, (\eta_0, V_0)\,, \end{cases} \tag{0.1.19}$$

où

$$M(U, D) = \begin{pmatrix} \varepsilon\bar{V} \cdot \nabla & (1 + \bar{\eta} + a\varepsilon\Delta)\, \partial_1 & (1 + \bar{\eta} + a\varepsilon\Delta)\, \partial_2 \\ g(D)\,(1 + c\varepsilon\Delta)\, \partial_1 & g(D)\,(\bar{v}_1\partial_1) & g(D)\,(\bar{v}_2\partial_1) \\ g(D)\,(1 + c\varepsilon\Delta)\, \partial_2 & g(D)\,(\bar{v}_1\partial_2) & g(D)\,(\bar{v}_2\partial_2) \end{pmatrix}, \tag{0.1.20}$$

respectivement

$$g(D) = (1 - b\varepsilon\Delta)\,(1 - d\varepsilon\Delta)^{-1}\,.$$

L'idée est que si ε est petit par rapport à $\|(\eta_0, V_0)\|_{H^s}$ alors le système (0.1.19) n'est pas loin "d'être hyperbolique" par conséquent on peut attaquer sa résolution dans cette optique. En fait, pour ε suffisamment petit les auteurs montrent que l'opérateur $M(U, D)$ peut être symétrisé en un certain sens à l'aide de

$$S_U(D) = \begin{pmatrix} 1 + c\varepsilon\Delta & \bar{v}_1 & \bar{v}_2 \\ \bar{v}_1 & 1 + \bar{\eta} + a\varepsilon\Delta & 0 \\ \bar{v}_2 & 0 & 1 + \bar{\eta} + a\varepsilon\Delta \end{pmatrix} \text{ si } b = d,$$

respectivement

$$S_U(D) = \begin{pmatrix} (1 + c\varepsilon\Delta)^2\, g(D) & g(D)\,(\bar{v}_1\,(1 + c\varepsilon\Delta)) & g(D)\,(\bar{v}_1\,(1 + c\varepsilon\Delta)) \\ g(D)\,(\bar{v}_1\,(1 + c\varepsilon\Delta)) & (1 + \bar{\eta} + a\varepsilon)\,(1 + c\varepsilon\Delta) & 0 \\ g(D)\,(\bar{v}_2\,(1 + c\varepsilon\Delta)) & 0 & (1 + \bar{\eta} + a\varepsilon)\,(1 + c\varepsilon\Delta) \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \bar{v}_1\bar{v}_1 & \bar{v}_1\bar{v}_2 \\ 0 & \bar{v}_1\bar{v}_2 & \bar{v}_2\bar{v}_2 \end{pmatrix} (g(D) - 1)\,, \text{ si } b \neq d$$

et que la fonctionnelle d'énergie

$$E_s(U) = \langle (I - b\varepsilon\Delta)\, \Lambda^s U, S_U(D)\, \Lambda^s U \rangle_{L^2}$$

se comporte comme une norme sur les espaces de la donnée initiale. Le cas exceptionnel

$$a = b = d = 0,\ c < 0,$$

ne peut pas être traité de cette manière. Des estimations d'énergie encore plus fines, incluant les dérivées temporelles de U, doivent être prises en compte. De plus, leurs résultats restent valables quand on considère de petites variations du fond du canal par rapport au plan $z = -h$. Pour des résultats prenant en compte des variations plus importantes on renvoie au travail de Mésognon-Gireau [73].

Les résultats d'existence en temps long mentionnés ci-dessus sont obtenus en supposant que les données initiales se trouvent dans des espaces de Sobolev H^s avec un exposant de régularité qui satisfait :

$$s > \frac{n}{2} + 1.$$

De ce fait, les données initiales doivent s'annuler à l'infini :

$$\lim_{|x|+|y| \to \infty} |\eta(x, y)| = \lim_{|x|+|y| \to \infty} |V(x, y)| = 0.$$

Cette restriction nous empêche de considérer des données modélisant la propagation des mascarets. Les mascarets sont des vagues formées à cause de l'interaction des fleuves avec le flux d'une marée. Physiquement, ils se présentent comme une surélévation brusque d'eau qui remonte le cours d'un fleuve. En laboratoire, ils peuvent être créés par le soulèvement brusque d'une porte séparant deux niveaux différents d'un fluide dans un bassin. Une des premières études en ce sens fut menée par Favre [54]. Au niveau de la modélisation mathématique les mascarets sont représentés à l'aide de fonctions qui manifestent un comportement non trivial à l'infini.

On retrouve des études numériques des mascarets dans le travail de Peregrine [77]. Dans ce papier, on considère l'évolution de données du type :

$$u = \frac{1}{2}u_0\left[1 - \tanh\left(\frac{x}{a}\right)\right], \ \eta = u + \frac{1}{4}u^2,$$

en utilisant l'équation BBM :

$$\left(I - \varepsilon\partial_{xx}^2\right)\partial_t u + \partial_x u + u\partial_x u = 0, \tag{0.1.21}$$

et le système classique de Boussinesq (0.1.9).

Des résultats théoriques sont obtenus par Benjamin, Bona, Mahony [12]. Ils montrent l'existence et unicité des solutions de l'équation (0.1.21) pour des données initiales :

$$h \in C^2(\mathbb{R}), \ \lim_{x\to\pm\infty} h(x) = C_\pm \text{ et } h' \in L^2(\mathbb{R}).$$

Ces résultats sont étendus par Bona, Rajopadhye et Schonbek [22] pour les équations BBM-Burgers, respectivement KdV-Burgers, à savoir

$$\left(I - \varepsilon\partial_{xx}^2\right)\partial_t u + \partial_x u + u\partial_x u = \nu\partial_{xx}^2 u,$$

respectivement

$$\partial_t u + \partial_x u + \partial_{xxx}^3 u + u\partial_x u = \nu\partial_{xx}^2 u$$

où $\nu > 0$. Iorio, Linares, Scialom [65] ont étudié l'équation de KdV et de Benjamin-Ono avec des données de type mascaret. Plus récemment, Bona, Colin et Guillopé [17] ont étudié le problème de Cauchy pour le système BBM-BBM en dimension 2, en considérant des données initiales qui sont seulement continues et bornées. Bien évidemment, cette classe contient des données de type mascaret. De plus, ils montrent que l'on peut transporter, au moins localement en temps, certaines structures présentes au temps initial dans l'asymptotique $|y| \to \infty$. Pendant la finalisation de ce manuscrit on a été informé de l'existence de l'article [18] où les auteurs continuent l'étude initiée dans [17] et ils montrent en utilisant une autre méthode l'existence en temps long de solutions pour le problème de Cauchy pour le système BBM-BBM avec des données initiales modÃĺlisant des mascarets.

Comme on a précisé avant, les systèmes *abcd* se montrent très utiles dans des situations pratiques. C'est pour cette raison qu'un nombre important d'études numériques ont été meneés parallèlement à des résultats théoriques.

On a déjà cité le travail de Peregrine où il fait des expériences numériques en considérant des données de type mascaret.

Bona et Chen [13] ont étudié le problème aux limites pour le système BBM-BBM en une dimension. En utilisant la formulation intégrale du système, ils construisent un algorithme d'approximation qui est précis à l'ordre 4 en temps et en espace. Ils emploient leur algorithme pour trouver des ondes progressives généralisées et étudient leur interaction.

Un grand nombre d'études ont été menées en utilisant des méthodes de type Galerkin pour la discrétisation spatiale combinée avec l'algorithme de Runge-Kutta pour la discrétisation temporelle. Bona, Dougalis et Mitsotakis [20], [21] ont traité *le problème périodique* pour le système KdV-KdV, $b = d = 0$, $a = c = 1/6$ et ont étudié l'interaction de solitons généralisés.

Dans [9], Antonopoulus, Dougalis et Mitsotakis ont étudié la semi-discrétisation en espace du *problème périodique* pour une plage assez large de paramètres. Ils ont étendu leurs résultats dans [9] pour la famille des systèmes de Bona-Smith (0.1.10) en considérant des *conditions aux limites de type Dirichlet non homogènes* ainsi que *des conditions de réflexion*. Dans [8], Antonopoulus et Dougalis ont considéré le problème aux limites avec conditions de Dirichlet pour le système classique de Boussinesq.

Pour le cas bidimensionnel, Dougalis, Mitsotakis et Saut [53] ont abordé le problème aux limites avec *conditions de Dirichlet*. Plus précisément ils regardent le système BBM-BBM et de Bona-Smith $a = 0$, $c < 0$, $b = d > 0$. Le système de BBM-BBM (0.1.12), pour le cas où le fond du canal n'est pas nécessairement plat est investigué en [37] par Chen en employant un algorithme Crank-Nicolson semi-implicite en tandem avec des méthodes spectrales.

Dans un article assez récent, Bona et Chen [14] ont réalisé quelques expériences numériques qui suggèrent que la solution du système BBM-BBM explose en temps fini pour certaines données initiales. L'explosion semble avoir lieu après la collision de deux ondes progressives du système.

0.1.2 Présentation des résultats obtenus

Notre premier but sera de raffiner les résultats théoriques connus sur les systèmes *abcd* :

$$\left\{\begin{array}{l} (I - b\varepsilon\Delta)\,\partial_t\eta + \operatorname{div} V + a\varepsilon \operatorname{div}\Delta V + \varepsilon \operatorname{div}(\eta V) = 0, \\ (I - d\varepsilon\Delta)\,\partial_t V + \nabla\eta + c\varepsilon\nabla\Delta\eta + \frac{\varepsilon}{2}\nabla\,|V|^2 = 0. \end{array}\right. \tag{0.1.22}$$

Plus précisément, on va construire des solutions pour le système (0.1.22) avec un temps de vie minoré par une quantité d'ordre ε^{-1}. Le point clé est la construction d'une fonctionnelle non linéaire d'énergie qui contrôle une certaine norme de Sobolev. Pour réaliser cela on va travailler avec des équations localisées en fréquence. Par rapport aux résultats présentés en [82] respectivement en [81], nos résultats ont deux grands avantages :

- on a besoin d'un niveau de régularité plus bas pour montrer l'existence et unicité des solutions en temps long ;
- notre méthode nous permet de traiter d'une manière uniforme la plupart des cas correspondant aux différents choix des paramètres a, b, c, d.

De plus, même si cet aspect n'est pas très important dans la pratique, pour une certaine plage d'indices on montre que l'hypothèse

$$1 + \varepsilon\eta_0(x) \geq \alpha > 0, \tag{0.1.23}$$

n'est pas cruciale. Comme on a déjà précisé, la condition (0.1.23) signifie d'un point de vue physique que le canal ne devient pas sec dans certaines régions.

Avant d'énoncer notre premier résultat, on va introduire quelques notations. Étant donnés $s \in \mathbb{R}$, $a, c \leq 0$, $b, d \geq 0$, on considère les indices suivants :

$$\left\{\begin{array}{l} s_{bc} = s + \operatorname{sgn}(b) - \operatorname{sgn}(c)\,, \\ s_{ad} = s + \operatorname{sgn}(d) - \operatorname{sgn}(a)\,, \end{array}\right. \tag{0.1.24}$$

où la fonction sgn est définie par :

$$\operatorname{sgn}(x) = \left\{\begin{array}{rll} 1 & \text{si} & x > 0, \\ 0 & \text{si} & x = 0, \\ -1 & \text{si} & x < 0. \end{array}\right.$$

De plus, soit $0 < \varepsilon \leq 1$ et $r \in [1, \infty)$. Pour tout $(\eta, V) \in H^{s_{bc}}(\mathbb{R}^n) \times (H^{s_{ad}}(\mathbb{R}^n))^n$, on considère :

$$\begin{aligned} U^2_{s,\varepsilon}(\eta, V) = {} & \|\eta\|^2_{H^s} + \varepsilon\,(b - c)\,\|\nabla\eta\|^2_{H^s} + \varepsilon^2\,(-c)\,b\,\left\|\nabla^2\eta\right\|^2_{H^s} \\ & + \|V\|^2_{H^s} + \varepsilon\,(d - a)\,\|\nabla V\|^2_{H^s} + \varepsilon^2\,(-a)\,d\,\left\|\nabla^2 V\right\|^2_{H^s}. \end{aligned} \tag{0.1.25}$$

Bien évidemment, l'espace $H^{s_{bc}}(\mathbb{R}^n) \times (H^{s_{ad}}(\mathbb{R}^n))^n$ muni de $U_{s,\varepsilon}(.,.)$ est un espace de Banach. Dans le premier chapitre, on montre le résultat suivant :

Théorème 1. *Soit $a, c \leq 0$ et $b, d \geq 0$ en excluant le cas $a = b = d = 0$, $c < 0$. Considérons un entier $n \geq 1$ et $s \in \mathbb{R}$ tel que :*

$$s > \frac{n}{2} + 2 - \operatorname{sgn}(b+d).$$

Soit $(\eta_0, V_0) \in H^{s_{bc}}(\mathbb{R}^n) \times (H^{s_{ad}}(\mathbb{R}^n))^n$ où s_{bc}, s_{ad} sont définis par (0.1.24). Alors, il existe deux constantes ε_0 et $T(\eta_0, V_0)$ qui dépendent de s, $U_{s,1}(\eta_0, V_0)$ et de la dimension n tels que pour tout $\varepsilon \in (0, \varepsilon_0)$ il existe une unique solution de (0.1.22) :

$$\left\{ \begin{array}{l} (\eta, V) \in \mathcal{C}\left(\left[0, \varepsilon^{-1} T(\eta_0, V_0)\right], H^{s_{bc}} \times (H^{s_{ad}})^n\right) \text{ avec} \\ (\partial_t \eta, \partial_t V) \in \mathcal{C}\left(\left[0, \varepsilon^{-1} T(\eta_0, V_0)\right], H^{s-2+\operatorname{sgn}(b+d)} \times \left(H^{s-2+\operatorname{sgn}(b+d)}\right)^n\right) \end{array} \right.$$

qui vérifie :

$$\inf_{x \in \mathbb{R}^n} \{1 + \varepsilon \eta(t, x)\} > 0. \tag{0.1.26}$$

De plus, on a :

$$\sup_{t \in [0, \varepsilon^{-1} T(\eta_0, V_0)]} \|(\eta(t), V(t))\|_{s,\varepsilon} \leq 4 \|(\eta_0, V_0)\|_{s,\varepsilon}.$$

Le résultat du Théorème 1 est une mise à jour de notre travail antérieur [27].

Le point de départ du deuxième chapitre de cette thèse est le travail de [17] où Bona, Colin et Guillopé ont abordé le problème de l'existence et unicité des solutions pour le système BBM-BBM en autorisant des données initiales qui peuvent manifester un comportement non trivial à l'infini.

On se propose d'enrichir leurs résultats avec une théorie d'existence en temps long pour des données de type mascaret. De plus, on va considérer les deux autres systèmes qui présentent un terme de dispersion de type BBM, à savoir :

$$\left\{ \begin{array}{l} (I - \varepsilon b \Delta) \partial_t \eta + \operatorname{div} V + \varepsilon \operatorname{div}(\eta V) = 0, \\ (I - \varepsilon d \Delta) \partial_t V + \nabla \eta + \varepsilon V \cdot \nabla V = 0, \end{array} \right. \tag{0.1.27}$$

où les paramètres b, d vérifient :

$$b, d \geq 0, \ \ b + d > 0.$$

On souhaite résoudre les systèmes (0.1.27) dans des espaces qui contiennent des fonctions continues η_0^{1D} telles que :

$$\lim_{x \to \pm\infty} \eta_0^{1D}(x) = \eta_{\pm}.$$

Dans le cadre bidimensionnel, le scénario envisagé est une perturbation de la situation précédente, à savoir on veut considérer des données

$$\left\{ \begin{array}{r} \eta_0^{2D}(x, y) = \eta_0^{1D}(x) + \phi(x, y), \\ V_0^{2D}(x, y) = \left(u_0^{1D}(x) + \psi_1(x, y), \psi_2(x, y)\right) \end{array} \right.$$

où on demande que

$$\lim_{|(x,y)| \to \infty} |\phi(x, y)| = 0 \text{ et } \lim_{|(x,y)| \to \infty} |\psi_i(x, y)| = 0 \text{ pour } i = 1, 2.$$

Donnons plus de détails. Pour tout $s \in \mathbb{R}$ soit

$$\left\{ \begin{array}{l} s_b = s + \operatorname{sgn}(b), \\ s_d = s + \operatorname{sgn}(d). \end{array} \right. \tag{0.1.28}$$

On considère l'espace

$$E^s_{b,d}(\mathbb{R}) = \left\{(\eta, V) \in L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R}) : (\partial_x \eta, \partial_x V) \in H^{s_b - 1}(\mathbb{R}) \times H^{s_d - 1}(\mathbb{R})\right\}$$

muni de la norme :

$$\|(\eta, V)\|_{E^{s,\varepsilon}_{b,d}} = \|(\eta, V)\|_{L^\infty} + U_{s,\varepsilon}\left(\partial_x \eta, \partial_x V\right).$$

où $0 < \varepsilon < 1$. Évidement, $E^s_{b,d}(\mathbb{R})$ est un espace de Banach. Un exemple de fonction qui appartient à cet espace est

$$\eta_0(x) = \tanh(-x) := \frac{e^{-x} - e^x}{e^x + e^{-x}}.$$

Pour η_0 ainsi définie, on a

$$\lim_{x\to\infty} \eta_0(x) = -1 \text{ et } \lim_{x\to-\infty} \eta_0(x) = 1.$$

On peut vérifier que la fonction η_0 est dans $\bigcap\limits_{s\geq 0} E^{s,1}_{b,d}$ mais qu'elle n'appartient à aucun espace H^s.

Les espaces de fonctions $f \in L^\infty(\mathbb{R})$ telle que $\partial_x f \in H^s(\mathbb{R})$ s'appellent espaces de Zhidkov et ils ont été utilisés dans un autre context pour etudier l'équation de Gross-Pitaevskii (voir [56])

Notre résultat principal pour les mascarets 1-dimensionnels est le suivant :

Théorème 2. *Soit $s > \frac{3}{2}$ et considérons $(\eta_0, u_0) \in E^s_{b,d}(\mathbb{R})$. Il existe deux réels positifs ε_0, C dependant de s, b, d, et de $\|(\eta_0, u_0)\|_{E^{s,1}_{b,d}}$ et une constante $\tilde{C}$ tels que pour tout $\varepsilon \leq \varepsilon_0$, le problème de Cauchy pour le système (0.1.27) avec la donnée initiale (η_0, u_0), admet une unique solution $(\bar{\eta}^\varepsilon, \bar{u}^\varepsilon) \in \mathcal{C}\left(\left[0, \frac{C}{\varepsilon}\right], E^s_{b,d}\right)$. De plus, on a l'estimation suivante :*

$$\sup_{t\in\left[0,\frac{C}{\varepsilon}\right]} \|(\bar{\eta}^\varepsilon(t), \bar{u}^\varepsilon(t))\|_{E^{s,\varepsilon}_{b,d}} + \sup_{t\in\left[0,\frac{C}{\varepsilon}\right]} \|\partial_t \bar{\eta}^\varepsilon(t)\|_{L^\infty} \leq \tilde{C}\, \|(\eta_0, u_0)\|_{E^{s,\varepsilon}_{b,d}}.$$

Le cas bidimensionnel est un peu plus technique. Soit $\sigma > \frac{1}{2}$ et $s > 1$ et considérons $M^{\sigma,s}_{b,d}$ l'espace des fonctions $(\eta, V) \in \mathcal{C}\left(\mathbb{R}^2\right) \times \left(\mathcal{C}\left(\mathbb{R}^2\right)\right)^2$ telles que

$$\left\{\begin{array}{c} \eta(x,y) = \eta^{1D}(x) + \eta^{2D}(x,y), \\ V(x,y) = \left(V^{1D}(x) + V^{2D}_1(x,y), V^{2D}_2(x,y)\right), \end{array}\right. \tag{0.1.29}$$

pour tout $(x,y) \in \mathbb{R}^2$ où $\left(\eta^{1D}, V^{1D}\right) \in E^\sigma_{b,d}(\mathbb{R})$ respectivement $\left(\eta^{2D}, V^{2D}\right) \in H^{s_b}\left(\mathbb{R}^2\right) \times \left(H^{s_b}\left(\mathbb{R}^2\right)\right)^2$. L'espace $X^s_{b,d} = H^{s_b}\left(\mathbb{R}^2\right) \times \left(H^{s_b}\left(\mathbb{R}^2\right)\right)^2$ sera muni de la famille de normes

$$\|(\phi, \psi)\|_{X^{s,\varepsilon}_{b,d}(\mathbb{R}^2)} := \|\phi\|_{H^s} + \varepsilon b\, \|\nabla\phi\|_{H^s} + \|\psi\|_{H^s} + \varepsilon d\, \|\nabla\psi\|_{H^s},$$

pour $\varepsilon \in (0,1)$. Soit $i : E^\sigma_{b,d}(\mathbb{R}) \to \mathcal{C}\left(\mathbb{R}^2\right) \times \left(\mathcal{C}\left(\mathbb{R}^2\right)\right)^2$ definie pour tout $(x,y) \in \mathbb{R}^2$ par

$$i((\eta, u))(x,y) = (\eta(x), (u(x), 0)).$$

Bien sûr, à cause de $i\left(E^\sigma_{b,d}(\mathbb{R})\right) \cap X^s_{b,d}\left(\mathbb{R}^2\right) = \{0\}$, les fonctions de la décomposition (0.1.29) sont uniques. Pour tout $\varepsilon > 0$, soit

$$\|(\eta, V)\|_{M^{\sigma,s,\varepsilon}_{b,d}} = \left\|\left(\eta^{1D}, V^{1D}\right)\right\|_{E^{\sigma,\varepsilon}_{b,d}(\mathbb{R})} + \left\|\left(\eta^{2D}, V^{2D}\right)\right\|_{X^{s,\varepsilon}_{b,d}(\mathbb{R}^2)}.$$

L'espace $\left(M^{\sigma,s}_{b,d}, \|\cdot\|_{M^{\sigma,s,\varepsilon}_{b,d}}\right)$ est un espace de Banach.

Maintenant, on est en mesure d'énoncer le résultat suivant :

Théorème 3. *Soit s et σ deux réels tels que*

$$s > 2,\ \sigma > s + \frac{3}{2} \text{ et } \sigma - \frac{1}{2} \in \mathbb{R}\backslash\mathbb{N}.$$

Considérons $(\eta_0, u_0) \in E^{\sigma}_{b,d}(\mathbb{R})$ *et* $(\phi, \psi) \in H^{s_b}(\mathbb{R}^2) \times \left(H^{s_b}(\mathbb{R}^2)\right)^2$*. Alors, il existe* ε_0*,* C *dépendant de* s, b, d *et de* $\|(\eta_0, u_0)\|_{E^{s,1}_{b,d}} + \|(\phi, \psi)\|_{X^{s,1}_{b,d}(\mathbb{R}^2)}$ *ainsi qu'une constante* $\tilde{C}$ *tels que pour tout* $\varepsilon \leq \varepsilon_0$*, le problème de Cauchy pour le système* $(0.1.27)$ *avec la donnée initiale*

$$\begin{cases} \bar{\eta}_0(x,y) = \eta_0(x) + \phi(x,y), \\ \bar{V}_0(x,y) = (u_0(x), 0) + \psi(x,y), \end{cases}$$

admet une solution unique $\left(\bar{\eta}^{\varepsilon}, \bar{V}^{\varepsilon}\right) \in \mathcal{C}\left(\left[0, \frac{C}{\varepsilon}\right], M^{\sigma,s}_{b,d}\right)$*. De plus, on a l'estimation suivante :*

$$\sup_{t\in\left[0,\frac{C}{\varepsilon}\right]} \left(\left\|\left(\bar{\eta}^{\varepsilon}(t), \bar{V}^{\varepsilon}(t)\right)\right\|_{M^{\sigma,s,\varepsilon}_{b,d}}\right) + \sup_{t\in\left[0,\frac{C}{\varepsilon}\right]} \left(\|\partial_t \bar{\eta}^{\varepsilon}(t)\|_{L^{\infty}}\right) \leq \tilde{C}\left(\|(\eta_0, u_0)\|_{E^{\sigma,\varepsilon}_{b,d}} + \|(\phi,\psi)\|_{X^{s,\varepsilon}_{b,d}}\right)$$

L'idée qui est à la base des deux résultats, Théorème 2 respectivement Théorème 3, est de considérer un découpage convenable de la donnée initiale en hautes et basses fréquences. Ainsi, on montre que l'on peut essentielllement se ramener à la résolution d'un système un peu plus général que (0.1.27) mais qui peut être traité dans l'esprit des résultats du premier chapitre. La démonstration que l'on présente dans le deuxième chapitre provient de notre article [26].

Dans le troisième chapitre on s'intéresse à l'analyse numérique des systèmes *abcd* en dimension 1. En utilisant des schémas de volumes finis, on montre la convergence et on établit des estimations de stabilité. Pour traiter numériquement les systèmes on va prendre $\varepsilon = 1$. Rappelons que dans ce cas les systèmes *abcd* se réduisent à :

$$\begin{cases} \left(I - b\partial^2_{xx}\right)\partial_t \eta + \left(I + a\partial^2_{xx}\right)\partial_x u + \partial_x(\eta u) = 0, \\ \left(I - d\partial^2_{xx}\right)\partial_t u + \left(I + c\partial^2_{xx}\right)\partial_x \eta + \frac{1}{2}\partial_x u^2 = 0, \\ \eta_{|t=0} = \eta_0,\ u_{|t=0} = u_0. \end{cases} \tag{S_{abcd}}$$

On suppose que les paramètres vérifient

$$a \leq 0, c \leq 0, b \geq 0, d \geq 0,$$

et on va exclure de notre analyse les cinq cas :

$$\begin{cases} a = b = 0,\ d < 0, c < 0;\ a = b = c = d = 0; \\ a = d = 0,\ b > 0, c < 0;\ a = b = d = 0, c < 0; \\ b = d = 0, c < 0, a < 0. \end{cases} \tag{0.1.30}$$

Considérons Δt et Δx deux nombres réels strictement positifs. Soit $t_n = n\Delta t$ et $x_j = j\Delta x$. On va munir l'espace $\ell^2(\mathbb{Z})$ du produit scalaire et de la norme suivante :

$$\langle v, w\rangle := \Delta x \sum_{j\in\mathbb{Z}} v_j w_j,\ \ \|v\|_{\ell^2_{\Delta}} := \langle v, v\rangle^{\frac{1}{2}}.$$

Pour tout $v = (v_j)_{j\in\mathbb{Z}} \in \ell^{\infty}(\mathbb{Z})$ on va considérer les opérateurs de décalage :

$$(S_{\pm}v)_j := v_{j\pm 1}.$$

Pour $v = (v_j)_{j\in\mathbb{Z}} \in \ell^{\infty}(\mathbb{Z})$ on introduit $D_+ v$, $D_- v$, Dv les opérateurs discrets de dérivation :

$$\begin{cases} D_+ v = \frac{1}{\Delta x}(S_+ v - v), \\ D_- v = \frac{1}{\Delta x}(v - S_- v), \\ Dv = \frac{1}{2}(D_+ v + D_- v) = \frac{1}{2\Delta x}(S_+ v - S_- v). \end{cases}$$

De plus, soit

$$\mathcal{E}(e,f) \stackrel{def.}{=} \|e\|^2_{\ell^2_{\Delta}} + (b - c)\|D_+ e\|^2_{\ell^2_{\Delta}} + b(-c)\|D_+ D_- e\|^2_{\ell^2_{\Delta}}$$

$$+ \|f\|_{\ell^2_\Delta}^2 + (d-a)\|D_+f\|_{\ell^2_\Delta}^2 + d(-a)\|D_+D_-f\|_{\ell^2_\Delta}^2. \tag{0.1.31}$$

On va considérer d'abord le cas où $b > 0$ et $d > 0$. De ce fait, on obtiendra un contrôle ℓ^2 sur les dérivées discrètes ce qui nous permettra d'imiter la démarche du cas continu afin de montrer des estimations d'énergie. De plus, le schéma que l'on propose sera inconditionnellement stable. Soit $(\eta_0, u_0) \in H^{s_{bc}}(\mathbb{R}) \times H^{s_{ad}}(\mathbb{R})$ où s_{bc},s_{ad} sont définis par (0.1.24) et s est un réel suffisamment grand. On considère :

$$\begin{cases} \frac{1}{\Delta t}(I - bD_+D_-)(\eta^{n+1} - \eta^n) + (I + aD_+D_-)D\left(\frac{u^n+u^{n+1}}{2}\right) + D(\eta^n u^n) = 0, \\ \frac{1}{\Delta t}(I - dD_+D_-)(u^{n+1} - u^n) + (I + cD_+D_-)D\left(\frac{\eta^n+\eta^{n+1}}{2}\right) + \frac{1}{2}D\left((u^n)^2\right) = 0, \end{cases} \tag{0.1.32}$$

pour tout $n \geq 0$, avec

$$\begin{cases} \eta_j^0 = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \eta_0(y_1)\,dy_1, \\ u_j^0 = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} u_0(y_1)\,dy_1. \end{cases} \tag{0.1.33}$$

Soit (η, u) la solution du système (S_{abcd}) et pour tout $n \geq 1$, soit $\eta_\Delta^n, u_\Delta^n \in \ell^2_\Delta(\mathbb{Z})$ donnés par :

$$\begin{cases} \eta_{\Delta j}^n = \frac{1}{\Delta t \Delta x}\int_{t_n}^{t_{n+1}}\int_{x_j}^{x_{j+1}} \eta(s_1, y_1)\,dy_1 ds_1, \\ u_{\Delta j}^n = \frac{1}{\Delta t \Delta x}\int_{t_n}^{t_{n+1}}\int_{x_j}^{x_{j+1}} u(s_1, y_1)\,dy_1 ds_1. \end{cases} \tag{0.1.34}$$

pour tout $j \in \mathbb{Z}$. De plus, on considère $\left(\eta_\Delta^0, u_\Delta^0\right) = \left(\eta^0, u^0\right)$. Pour tout $n \geq 0$, l'erreur de consistance $(\epsilon_1^n, \epsilon_2^n) \in \ell^2_\Delta$ est définie par :

$$\begin{cases} \frac{1}{\Delta t}(I - bD_+D_-)(\eta_\Delta^{n+1} - \eta_\Delta^n) + \frac{1}{2}(I + aD_+D_-)D\left(u_\Delta^n + u_\Delta^{n+1}\right) + D(\eta_\Delta^n u_\Delta^n) = \epsilon_1^n, \\ \frac{1}{\Delta t}(I - dD_+D_-)(u_\Delta^{n+1} - u_\Delta^n) + \frac{1}{2}(I + cD_+D_-)D\left(\eta_\Delta^n + \eta_\Delta^{n+1}\right) + \frac{1}{2}D\left((u_\Delta^n)^2\right) = \epsilon_2^n. \end{cases} \tag{0.1.35}$$

L'erreur de convergence est donnée par la relation suivante :

$$e^n = \eta^n - \eta_\Delta^n, \ f^n = u^n - u_\Delta^n. \tag{0.1.36}$$

Enonçons maintenant notre premier résultat :

Théorème 4. *Soit $a \leq 0$, $b > 0$, $c \leq 0$ et $d > 0$. Considérons $s > 6$, $(\eta_0, u_0) \in H^{s_{bc}}(\mathbb{R}) \times H^{s_{ad}}(\mathbb{R})$ et $(\eta, u) \in C_T(H^{s_{bc}} \times H^{s_{ad}})$ la solution du système (S_{abcd}). Soit $N \in \mathbb{N}$, $\Delta x > 0$ et $\Delta t = T/N$. On considère le schéma (0.1.32) couplé avec la donnée initiale (0.1.33). De plus, soit $(\eta_\Delta^n, u_\Delta^n)_{n\in\overline{1,N}}$ définie par (0.1.35).*

Il existe une constante ε_0 qui dépend de $\sup_{t\in[0,T]} \|(\eta(t), u(t))\|_{H^{s_{bc}}\times H^{s_{ad}}}$ telle que si l'on choisit le nombre de pas de temps N et le pas d'espace dx tels que $\Delta t = T/N \leq \varepsilon_0$ et $\Delta x \leq \varepsilon_0$ alors, le schéma numérique (0.1.32) est convergent à l'ordre un en espace et temps, à savoir, l'erreur de convergence definie par (0.1.36) verifie :

$$\sup_{n\in\overline{0,N}} \mathcal{E}(e^n, f^n) \leq C_{abcd}\left\{(\Delta t)^2 + (\Delta x)^2\right\}, \tag{0.1.37}$$

où C_{abcd} dépend des paramètres a,b,c,d et du $\sup_{t\in[0,T]} \|(\eta(t), u(t))\|_{H^{s_{bc}}\times H^{s_{ad}}}$.

Pour traiter le cas où $bd = 0$, on considère le schéma :

$$\begin{cases} \frac{1}{\Delta t}(I - bD_+D_-)(\eta^{n+1} - \eta^n) + (I + aD_+D_-)D\left(u^{n+1}\right) + D(\eta^n u^n) \\ \qquad = \frac{1}{2}(1 - \operatorname{sgn}(b))\tau_1^n \Delta x D_+D_-(\eta^n), \\ \frac{1}{\Delta t}(I - dD_+D_-)(u^{n+1} - u^n) + (I + cD_+D_-)D\left(\eta^{n+1}\right) + \frac{1}{2}D\left((u^n)^2\right) \\ \qquad = \frac{1}{2}(1 - \operatorname{sgn}(d))\tau_2^n \Delta x D_+D_-(u^n) \end{cases} \tag{0.1.38}$$

pour tout $n \geq 0$, où

$$\begin{cases} \eta_j^0 = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} \eta_0 (y_1)\, dy_1, \\ u_j^0 = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} u_0 (y_1)\, dy_1. \end{cases} \tag{0.1.39}$$

Soit (η, u) la solution du système (S_{abcd}) et pour tout $n \geq 1$, on considère $\eta_\Delta^n, u_\Delta^n \in \ell_\Delta^2 (\mathbb{Z})$ definis par (0.1.34). Comme avant, pour tout $n \geq 0$, l'erreur de consistance $(\epsilon_1^n, \epsilon_2^n) \in \ell_\Delta^2$ est définie par

$$\begin{cases} \frac{1}{\Delta t} (I - bD_+D_-) (\eta_\Delta^{n+1} - \eta_\Delta^n) + (I + aD_+D_-) D \left(u_\Delta^{n+1}\right) + D (\eta_\Delta^n u_\Delta^n) \\ \qquad\qquad\qquad = \epsilon_1^n + \frac{1}{2} (1 - \mathrm{sgn}(b)) \tau_1^n \Delta x D_+D_- (\eta_\Delta^n), \\ \frac{1}{\Delta t} (I - dD_+D_-) (u_\Delta^{n+1} - u_\Delta^n) + (I + cD_+D_-) D \left(\eta_\Delta^{n+1}\right) + \frac{1}{2} D \left((u_\Delta^n)^2\right) \\ \qquad\qquad\qquad = \epsilon_2^n + \frac{1}{2} (1 - \mathrm{sgn}(d)) \tau_2^n \Delta x D_+D_- (u^n). \end{cases} \tag{0.1.40}$$

L'erreur de convergence (e^n, f^n) est définie par (0.1.36). Notre deuxième résultat est :

Théorème 5. *Soit $a \leq 0$, $b \geq 0$, $c \leq 0$ et $d \geq 0$ où $bd = 0$, sans prendre en compte les cas (0.1.30). Considérons $s > 6$, $(\eta_0, u_0) \in H^{s_{bc}} (\mathbb{R}) \times H^{s_{ad}} (\mathbb{R})$ et $(\eta, u) \in C_T(H^{s_{bc}} \times H^{s_{ad}})$, l'unique solution du système (S_{abcd}). Soit $N \in \mathbb{N}$ et $\Delta x > 0$, $\Delta t = T/N$. On considère le schéma numérique (0.1.38) couplé avec la donnée initiale (0.1.39). De plus, on considère $(\eta_\Delta^n, u_\Delta^n)_{n \in \overline{1,N}}$ définie par (0.1.40). Soit $\alpha \in (0,1)$ et τ_1^n, τ_2^n tels que :*

$$\|u_\Delta^n\|_{\ell^\infty} + \alpha \leq \min\{\tau_1^n, \tau_2^n\}\ .$$

Il existe une constante ε_0 qui dépend de α, β, τ_1^n, τ_2^n et de $\sup_{t\in[0,T]} \|(\eta(t), u(t))\|_{H^{s_{bc}} \times H^{s_{ad}}}$ telle que si le nombre de pas de temps N, le pas de discrétisation spatiale et les viscosités numériques sont choisies tels que :

$$\Delta t \leq \varepsilon_0,\ \Delta x \leq \varepsilon_0$$

respectivement

$$\max\{(1 - \mathrm{sgn}(b))\, \tau_1^n, (1 - \mathrm{sgn}(d))\, \tau_2^n\}\Delta t \leq (1 - \beta)\, \Delta x,$$

alors le schéma numérique (0.1.38) est convergent à l'ordre un, à savoir, l'on a :

$$\sup_{n \in \overline{0,N}} \mathcal{E} (e^n, f^n) \leq C_{abcd} (\Delta x)^2 ,$$

où C_{abcd} dépend des paramètres a,b,c,d et de $\sup\limits_{t\in[0,T]} \|(\eta(t), u(t))\|_{H^{s_{bc}} \times H^{s_{ad}}}$.

Les résultats du Théorème 4 et Théorème 5 font partie d'un travail, en cours de finalisation, réalisé en collaboration avec C. Courtès.

Enfin, on utilise ces schémas pour simuler l'interaction d'ondes progressives. Les résultats sont présentés dans la Section 3.5 du troisième chapitre. En particulier, nos résultats semblent indiquer que l'on a explosion en temps fini pour le choix de paramètres $a = c = d = 0$, $b = 3/5$.

0.2 Deuxième partie : méthodes Lagrangiennes en mécanique des fluides incompressibles

0.2.1 Introduction

L'évolution de l'écoulement d'un fluide non homogène, incompressible est régi par le système de Navier-Stokes :

$$\begin{cases} \partial_t \rho + \mathrm{div} (\rho u) = 0, \\ \partial_t (\rho u) + \mathrm{div} (\rho u \otimes u) - \mathrm{div} (\mu (\rho) D (u)) + \nabla P = 0, \\ \mathrm{div}\, u = 0, \\ u_{|t=0} = u_0, \end{cases} \tag{0.2.1}$$

où $\rho > 0$ est la densité du fluide, $u \in \mathbb{R}^n$ représente le champ de vitesses tandis que P est la pression. La viscosité $\mu = \mu(\rho)$ est une fonction C^∞, strictement positive qui dépend de la densité et

$$D(u) = \nabla u + Du$$

représente le (double du) tenseur de déformation.

Ce système est utilisé pour étudier le mélange de deux ou plusieurs fluides de densités différentes comme par exemple des mélanges eau/vapeur ou eau salée/eau douce, etc. Il existe une très vaste littérature consacrée à l'étude de l'existence et unicité des solutions du système de Navier-Stokes (0.2.1). On essayera de résumer les grands résultats d'existence et unicité des solutions ci-desous dans le cas où le fluide remplit tout l'espace ambiant $\mathbb{R}^n$.

L'existence des solutions faibles avec énergie finie pour le cas où la viscosité est constante fut démontrée pour la première fois par Kazhikov en [67] (voir aussi [10]). Le cas à viscosité variable fut abordé par Lions [72]. Des solutions faibles plus régulières ont été construites par Desjardins en [52]. Récemment, des solutions faibles ont été étudiées par Huang, Paicu et Zhang [64]. La question de l'unicité faible-fort a été adressée en [39].

L'existence *et unicité* du problème (0.2.1) fut initiée par Ladyženskaja et Solonnikov [69]. Lorsque la viscosité est constante et Ω domaine borné, en considérant $u_0 \in W^{2-\frac{2}{p},p}(\Omega)$, où $p > 2$, est un champ solénoïdal dont la trace est 0 sur $\partial\Omega$ et $\rho_0 \in C^1(\Omega)$ strictement positive ils construisent une solution globale dans le cas bidimensionnel et une solution locale en dimension trois. De plus, si u_0 est petite dans $W^{2-\frac{2}{p},p}(\Omega)$ alors la solution est globale même en dimension trois.

Dans les dernières années, des efforts considérables ont été menés pour étudier le système (0.2.1) dans les espaces critiques, c'est-à-dire les espaces qui possèdent la même invariance d'échelle que le système lui-même :

$$\left\{ \begin{array}{r} (\rho_0(x), u_0(x)) \to (\rho_0(lx), lu_0(lx)), \\ (\rho(t,x), u(t,x), P(t,x)) \to \left(\rho\left(l^2t, lx\right), lu\left(l^2t, lx\right), l^2P\left(l^2t, lx\right)\right). \end{array} \right.$$

Pour plus des détails sur cette approche classique on renvoie à [42] ou [51]. Les espaces de Besov critiques (voir l'Appendice A) pour la donnée initiale sont

$$\rho_0 - \bar{\rho} \in \dot{B}^{\frac{n}{p_1}}_{p_1,r_1} \text{ respectivement } u_0 \in \dot{B}^{\frac{n}{p_2}-1}_{p_2,r_2}, \tag{0.2.2}$$

où $\bar{\rho}$ est une constante et n représente la dimension de l'espace.

Dans la suite on va passer en revue des résultats d'existence et unicité pour le cas des fluides presque homogènes, à savoir pour le cas où la densité est proche d'une constante quand on la mesure dans une norme adéquate. Lorsque la viscosité est constante, Danchin [42] montre que le système (0.2.1) est globalement bien posé pour

$$\rho_0 - \bar{\rho} \in L^\infty \cap \dot{B}^{\frac{n}{2}}_{2,\infty},\ u_0 \in \dot{B}^{\frac{n}{2}-1}_{2,1}$$

sous l'hypothèse que $\|\rho_0 - \bar{\rho}\|_{L^\infty \cap \dot{B}^{\frac{n}{2}}_{2,\infty}}$ soit petite. Abidi [1] généralise ces résultats en permettant un certain degré de liberté pour l'indice d'intégrabilité :

$$\rho_0 - \bar{\rho} \in \dot{B}^{\frac{n}{p}}_{p,1} \text{ et } u_0 \in \dot{B}^{\frac{n}{p}-1}_{p,1},\ p \in [1, 2n).$$

Néanmoins, dans ce papier l'unicité des solutions est assurée seulement pour $p \in [1, n)$. Ces résultats ont été encore raffinés par Abidi et Paicu [4] en observant que $\rho_0 - \bar{\rho}$ peut être choisie dans un espace de Besov plus grand. Dans [59], Haspot obtient des résultats dans le même esprit mais en autorisant des champs de vitesses non lipschitziens (cependant, ses résultats sont en dehors des vrais espaces critiques). Quand la viscosité est supposée constante, R. Danchin et P.B. Mucha obtiennent un résultat d'existence et d'unicité des solutions en considérant des densités très générales. Plus exactement, ils peuvent construire une solution locale si $\rho_0 - \bar{\rho} \in \mathcal{M}(\dot{B}_{p,1}^{\frac{n}{p}-1})$, $u_0 \in \dot{B}_{p,1}^{\frac{n}{p}-1}$, $p \in [1, 2n)$, sous la condition de petitesse :

$$\|\rho_0 - \bar{\rho}\|_{\mathcal{M}(\dot{B}_{p,1}^{\frac{n}{p}-1})} \ll 1, \tag{0.2.3}$$

où $\mathcal{M}(\dot{B}_{p,1}^{\frac{n}{p}-1})$ représente l'espace de multiplicateurs de l'espace $\dot{B}_{p,1}^{\frac{n}{p}-1}$. Cet espace contient en particulier des fonctions qui présentent des discontinuités. De plus, si la vitesse est petite alors la solution devient globale. Leur méthode repose sur la formulation lagrangienne du (0.2.1) ce qui leur permet de se débarrasser de l'équation de la densité. Ainsi, ils obtiennent l'unicité pour toute la plage d'indices $p \in [1, 2n)$ et ils peuvent montrer que le problème de Cauchy (0.2.1) est bien posé à la Hadamard. Dans les articles [75], [63], [62], [64] les auteurs ont amélioré la condition de petitesse (0.2.3) utilisée pour obtenir l'existence globale. Pour résumer, tous les résultats ci-dessus utilisent d'une manière fondamentale l'hypothèse de fluide "presque homogène", la densité étant proche d'un état constant.

Dans de nombreuses situations physiques on doit prendre en compte des variations arbitraires de la densité autour d'un état d'équilibre. L'analyse mathématique de cette situation s'avère être plus délicate. Pour que l'on puisse développer une théorie d'existence et unicité des solutions on doit considérer des données initiales plus régulières. De ce fait, pour le cas de la viscosité constante, Danchin [43] obtient des résultats locaux et globaux dans le cas bidimensionnel pour des données appartenant aux espaces de Sobolev non homogènes :

$$(\rho_0 - \bar{\rho}, u_0) \in H^{\frac{n}{2}+\alpha} \times H^{\frac{n}{2}-1+\beta} \text{ avec } \alpha, \beta > 0.$$

Ce résultat est généralisé par Abidi [1] pour des lois de viscosité générales. Le cas de champs de vitesses non lipschitziens a été étudié par Haspot [59]. Danchin et Mucha [50] construisent des solutions locales en autorisant des densités assez générales $\rho_0 \in L^\infty(\mathbb{R}^d)$ et des champs de vitesses $u_0 \in H^2\left(\mathbb{R}^d\right)$. À nouveau, en dimension 2, si la variation de la densité autour d'un état d'équilibre est supposée petite et la vitesse $u_0 \in B_{4,2}^1 \cap L^2$ alors la solution est globale. En dimension 3, la vitesse doit être considerée petite en $B_{q,p}^{2-\frac{2}{q}}$ où $p \in (1, \infty)$ respectivement $q \in (3, \infty)$. Leurs résultats ont été ensuite généralisés par Paicu, Zhang et Zhang [76] en autorisant des vitesses moins régulières ainsi que des variations arbitraires de la densité autour d'un état constant. Plus précisement, en $2D$ ils montrent l'existence et unicité d'une solution globale avec la densité initiale $\rho_0 \in L^\infty\left(\mathbb{R}^2\right)$ et la vitesse $u_0 \in H^s\left(\mathbb{R}^2\right)$ avec $s > 0$. Dans le cas 3-dimensionnel la vitesse doit être petite en $H^1\left(\mathbb{R}^3\right)$ pour que la solution soit globale.

Dans les vrais espaces critiques (0.2.2) du système de Navier-Stokes, il y a peu des résultats connus. Récemment, pour le cas bidimensionnel avec viscosité variable, Xu, Li et Zhai [86] construisent une solution locale de (0.2.1) pour des données initiales $\rho_0 - \bar{\rho} \in \dot{B}_{p,1}^{\frac{2}{p}}\left(\mathbb{R}^2\right)$ respectivement $u_0 \in \dot{B}_{p,1}^{\frac{2}{p}-1}\left(\mathbb{R}^2\right)$. De plus, si la viscosité est constante et $\rho_0 - \bar{\rho} \in L^p \cap \dot{B}_{p,1}^{\frac{2}{p}}\left(\mathbb{R}^2\right)$ alors la solution devient globale. Dans le cas 3-dimensionnel, les résultats les plus proches des espaces critiques sont ceux de [2] et [3] (pour un résultat similaire dans le cas périodique voir Poulon [78]). Plus précisément, en supposant que

$$\rho_0 - \bar{\rho} \in L^2 \cap \dot{B}_{2,1}^{\frac{3}{2}} \text{ et } u_0 \in \dot{B}_{2,1}^{\frac{1}{2}}$$

et en prenant la viscosité constante, Abidi, Gui et Zhang [2], montrent que le système (0.2.1) est localement bien posé. De plus, si la vitesse initiale est suffisamment petite alors la solution est globale. Dans l'article [3], ils montrent qu'il est possible de prendre la vitesse dans des espaces plus grandes, le prix à payer étant la régularité de la densité :

$$\rho_0 - \bar{\rho} \in L^\lambda \cap \dot{B}_{\lambda,1}^{\frac{3}{\lambda}} \text{ et } u_0 \in \dot{B}_{p,1}^{\frac{3}{p}-1}$$

où $\lambda \in [1, 2]$, $p \in [3, 4]$ sont tels que $\frac{1}{\lambda} + \frac{1}{p} > \frac{5}{6}$ et $\frac{1}{\lambda} - \frac{1}{p} \leq \frac{1}{3}$.

Un modèle classique pour l'étude de l'écoulement d'un fluide compressible en prenant en compte la capillarité est le système de Navier-Stokes Korteweg :

$$\left\{\begin{aligned} \partial_t \rho + \operatorname{div}(\rho u) &= 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mathcal{A}u + \nabla(P(\rho)) &= -\operatorname{div}\mathcal{K}, \\ u_{|t=0} &= u_0, \end{aligned}\right. \tag{0.2.4}$$

où

$$\mathcal{A}u = \operatorname{div}(\mu(\rho) D(u)) + \nabla(\lambda(\rho)\operatorname{div} u)$$

est le tenseur de diffusion, et

$$\mathcal{K} = \nabla\left(\rho k(\rho)\Delta\rho + \frac{1}{2}\left(k(\rho) + \rho k'(\rho)|\nabla\rho|^2\right)\right) - \operatorname{div}(k(\rho)\nabla\rho\otimes\nabla\rho),$$

est le tenseur de Korteweg.

Le cas des coefficients de viscosité et capillarité constants fut étudié par Danchin et Desjardins dans [47] où ils mettent en avant le fort couplage entre la partie potentielle du champ de vitesses et la densité, couplage qui fournit une régularisation parabolique de la densité. Plus exactement, ils montrent que si l'on prend

$$\rho_0 - 1 \in \dot{B}^{\frac{n}{2}}_{2,1},\ u_0 \in \dot{B}^{\frac{n}{2}-1}_{2,1}$$

alors le système (0.2.4) admet une unique solution locale (ρ, u) telle que :

$$\left\{\begin{aligned} \rho - 1 &\in C_T(\dot{B}^{\frac{n}{2}}_{2,1}) \cap L^1_T(\dot{B}^{\frac{n}{2}+2}_{2,1}), \\ u &\in C_T(\dot{B}^{\frac{n}{2}-1}_{2,1}) \cap L^1_T(\dot{B}^{\frac{n}{2}+1}_{2,1}). \end{aligned}\right.$$

De plus, si $P'(1) > 0$, les données initiales sont petites et la fluctuation de la densité est un peu plus régulière, alors la solution est globale. On pourra mentionner ici les généralisations dues à Haspot [58], [57], [60] dans la direction de la régularité minimale sur les données initiales et des coefficients de viscosité et de capillarité plus généraux.

La capillarité peut être modélisée par des termes non locaux comme cela a été suggéré par Van der Waals [85] ou plus récemment par Rhode [79], [80] :

$$\left\{\begin{aligned} \partial_t \rho + \operatorname{div}(\rho u) &= 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mathcal{A}u + \nabla(P(\rho)) &= k(\rho)\rho\nabla L[\rho], \\ u_{|t=0} &= u_0, \end{aligned}\right. \tag{0.2.5}$$

où l'opérateur L est donné par

$$L[\rho] = \phi * \rho - \rho,$$

ϕ étant une fonction paire, positive telle que

$$\int_{\mathbb{R}^n} \phi(x)\,dx = 1 \text{ et } \left(|x| + |x|^2\right)\phi(x) \in L^1(\mathbb{R}^n).$$

Des résultats d'existence et unicité des solutions ainsi que la convergence des solutions du système (0.2.5) vers la solution du modèle local ont été obtenus par Charve et Haspot en [30],[31],[55] et [32].

Dans sa version incompressible :

$$\left\{\begin{aligned} \partial_t \rho + \operatorname{div}(\rho u) &= 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu(\rho) D(u)) + \nabla P &= -\operatorname{div}\mathcal{K}, \\ \operatorname{div} u &= 0, \\ u_{|t=0} &= u_0, \end{aligned}\right. \tag{0.2.6}$$

le système Navier-Stokes-Korteweg est beaucoup moins étudié. On pourra citer les deux travaux [84] et [87] pour le cas de la viscosité et capillarité constantes. Dans [84], les auteurs montrent le caractère localement bien-posé du système (0.2.6) pour des données $(\rho_0, u_0) \in W^{2,p}(\Omega) \times W^{1,p}(\Omega)$ pour $p > 1$ et Ω un borné convexe. Dans [87], Yang, Yao et Zhu montrent que le problème est localement bien posé dans $H^4(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$ et ils étudient le problème de la viscosité et capillarité évanescentes.

0.2.2 Les résultats obtenus

Dans la deuxième partie de la thèse, on s'intéresse à des questions de régularité optimale pour le problème de Cauchy associé aux systèmes de Navier-Stokes et Navier-Stokes-Korteweg, où l'on autorise :

- la viscosité $\mu = \mu(\rho)$ à dépendre de la densité ;
- des variations arbitraires de la densité autour d'un état d'équilibre.

Les résultats principaux que l'on obtient sont :

- l'existence et unicité de solutions dans les vrais espaces critiques :
$$\begin{cases} p \in \left(\frac{6}{5}, 4\right) \text{ si } n = 3, \\ \rho_0 - \bar{\rho} \in \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n), \; u_0 \in \dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n), \end{cases}$$
pour le système (0.2.1)
- l'existence et unicité de solutions dans les espaces critiques
$$\begin{cases} p \in \left(\frac{6}{5}, 4\right) \text{ si } n = 3 \text{ ou } p \in (1,4) \text{ si } n = 2, \\ \rho_0 - \bar{\rho}, \nabla\rho_0 \in \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n), \; u_0 \in \dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n), \end{cases}$$
pour le système (0.2.6).

On commence par :

Théorème 1. *Soit $p \in \left(\frac{6}{5}, 4\right)$. Supposons qu'il existe des constantes positives $(\bar{\rho}, \rho_\star, \rho^\star)$ telles que $\rho_0 - \bar{\rho} \in \dot{B}^{\frac{3}{p}}_{p,1}(\mathbb{R}^3)$ et $0 < \rho_\star < \rho_0 < \rho^\star$. De plus, considérons u_0 un champ solénoïdal à coefficients dans $\dot{B}^{\frac{3}{p}-1}_{p,1}(\mathbb{R}^3)$. Alors, il existe un temps $T > 0$ et une unique solution $(\rho, u, \nabla P)$ du système (0.2.1) avec*
$$\rho - \bar{\rho} \in \mathcal{C}_T(\dot{B}^{\frac{3}{p}}_{p,1}(\mathbb{R}^3)), \; u \in \mathcal{C}_T(\dot{B}^{\frac{3}{p}-1}_{p,1}(\mathbb{R}^3)) \text{ et } \left(\partial_t u, \nabla^2 u, \nabla P\right) \in L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1}(\mathbb{R}^3)).$$

Le résultat analogue pour le système (0.2.6) est le suivant :

Théorème 2. *Considérons $p \in (1,4)$ si $n = 2$ ou $p \in \left(\frac{6}{5}, 4\right)$ si $n = 3$. De plus, considérons (ρ_0, u_0) tels que*
$$\operatorname{div} u_0 = 0, \; \inf_{x\in\mathbb{R}^n} \rho_0(x) > 0, \; u_0 \in \dot{B}^{\frac{n}{p}-1}_{p,1} \text{ et } \rho_0 - \bar{\rho}, \nabla\rho_0 \in \dot{B}^{\frac{n}{p}}_{p,1},$$
où $\bar{\rho}$ est une constante positive. Alors, il existe un temps $T > 0$ et une unique solution $(\rho, u, \nabla P)$ du système (0.2.6) telle que
$$\rho - \bar{\rho}, \nabla\rho \in \mathcal{C}_T(\dot{B}^{\frac{3}{p}}_{p,1}(\mathbb{R}^3)), \; u \in \mathcal{C}_T(\dot{B}^{\frac{3}{p}-1}_{p,1}(\mathbb{R}^3)) \text{ et } \left(\partial_t u, \nabla^2 u, \nabla P\right) \in L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1}(\mathbb{R}^3)).$$

En supposant que la densité est proche d'une constante, on peut récupérer une plage plus grande d'indices d'integrabilité, à savoir :

Théorème 3. *Considérons $p \in [1, 2n)$ et (ρ_0, u_0) tels que*
$$\operatorname{div} u_0 = 0, \; \inf_{x\in\mathbb{R}^n} \rho_0(x) > 0, \; u_0 \in \dot{B}^{\frac{n}{p}-1}_{p,1} \text{ et } \rho_0 - \bar{\rho}, \nabla\rho_0 \in \dot{B}^{\frac{n}{p}}_{p,1},$$
où $\bar{\rho}$ est une constante positive. Il existe une constante $c > 0$ tel que si
$$\|\rho_0 - 1\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \leq c,$$
alors il existe un temps $T > 0$ et une unique solution $(\rho, u, \nabla P)$ du système (0.2.6) tels que
$$\rho - \bar{\rho}, \nabla\rho \in \mathcal{C}_T(\dot{B}^{\frac{3}{p}}_{p,1}(\mathbb{R}^3)), \; u \in \mathcal{C}_T(\dot{B}^{\frac{3}{p}-1}_{p,1}(\mathbb{R}^3)) \text{ et } \left(\partial_t u, \nabla^2 u, \nabla P\right) \in L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1}(\mathbb{R}^3)).$$

Les preuves que l'on présentera ci-dessous sont empruntées de nos travaux [28] et [29]. Les résultats des Théorèmes 1 et 2 sont obtenus en étudiant les systèmes correspondants dans leur formulation lagrangienne. Dans le cadre des espaces de Besov, une première forme de cette méthode fut utilisée par Hmidi [61] pour étudier un problème de type transport et diffusion. Dans la mécanique des fluides non homogènes Danchin et Mucha [49] construisent une théorie rigoureuse de changement des coordonnées eulériennes en coordonnées lagrangiennes. À partir de là, on utilise parfois cette formulation pour montrer l'unicité des solutions, voir [38], [75], [76] etc. Le grand avantage mathématique de cette approche dans la mécanique des fluides incompressible est que dans ces nouvelles coordonnées la densité demeure indépendante du temps. Le prix à payer se voit au niveau des opérateurs de viscosité qui dépendent d'une façon non linéaire du champ de vitesses. Ainsi, on peut montrer que le problème se réduit à l'étude d'un système linéaire de type Stokes :

$$\begin{cases} \partial_t u - a\,\mathrm{div}\,(bD(u)) + a\nabla P = f, \\ \mathrm{div}\, u = \mathrm{div}\, R, \\ u_{|t=0} = u_0, \end{cases} \tag{0.2.7}$$

où les coefficients a, b sont variables. On va regrouper les résultats obtenus pour ce système dans le théorème suivant :

Théorème 4. *Considérons $n \in \{2,3\}$ et $p \in (1,4)$ si $n=2$ ou $p \in \left(\frac{6}{5},4\right)$ si $n=3$. On suppose qu'il existe des constantes positives $\left(a_\star, b_\star, a^\star, b^\star, \bar{a}, \bar{b}\right)$ telles que $a-\bar{a} \in \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)$, $b-\bar{b} \in \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)$ et*

$$\begin{aligned} 0 < a_\star \le a \le a^\star, \\ 0 < b_\star \le b \le b^\star. \end{aligned}$$

De plus, on considère les champs vectoriels u_0 et f à coefficients dans $\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)$ respectivement dans $L^1_{loc}(\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n))$. De plus, soit $R \in (\mathcal{S}'(\mathbb{R}^n))^n$ vérifiant[1] :

$$\mathcal{Q}R \in \mathcal{C}([0,\infty); \dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)),\ (\partial_t R, \nabla\,\mathrm{div}\, R) \in L^1_{loc}(\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n))$$

respectivement :

$$\mathrm{div}\, u_0 = \mathrm{div}\, R\,(0,\cdot)\,.$$

Alors, le système (0.2.7) admet une solution unique $(u, \nabla P)$ vérifiant :

$$u \in C([0,\infty), \dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)) \text{ et } \partial_t u, \nabla^2 u, \nabla P \in L^1_{loc}(\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)).$$

De plus, il existe une constante $C = C(a,b)$ telle que l'on ait :

$$\begin{aligned} &\|u\|_{L^\infty_t(\dot{B}^{\frac{n}{p}-1}_{p,1})} + \left\|\left(\partial_t u, \nabla^2 u, \nabla P\right)\right\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})} \\ &\le \left(\|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} + \|(f, \partial_t R, \nabla\,\mathrm{div}\, R)\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}\right)\exp\left(C(t+1)\right), \end{aligned} \tag{0.2.8}$$

pour tout $t \in [0,\infty)$.

Le résultat du Théorème 4 peut être vu comme une généralisation des résultats de Danchin et Mucha, [51] qui ont étudié le système (0.2.7) dans le cas des coefficients constants.

La grande difficulté par rapport au cas à coefficients constants est que le champ de vitesses et la pression sont fortement couplés. En effet, si l'on suppose que a et b sont des constantes (ou s'ils sont proches d'une constante), en appliquant l'opérateur $\mathcal{Q}$ dans la première équation de (0.2.7), on obtient la formule de la pression. Ensuite la vitesse vérifie une équation classique de la chaleur. Dans le cas

[1] $\mathcal{P}$ est le projecteur de Leray sur les champs à divergence nulle (ou solénoidaux) et $\mathcal{Q} = Id - \mathcal{P}$

à coefficients variables, la pression dépend de la vitesse et pour l'estimer on utilisera un découpage hautes-basses fréquences comme en [44] ainsi que des nouvelles estimations pour l'équation elliptique :

$$\operatorname{div}(a\nabla P) = \operatorname{div} f. \tag{0.2.9}$$

Pour estimer ∇P qui satisfait l'équation (0.2.9) on remarque que

$$\mathcal{Q}(a\nabla P) = \mathcal{Q}f.$$

La partie solénoïdale de $a\nabla P$ satisfait l'identité suivante :

$$\begin{aligned}\mathcal{P}(a\nabla P) &= \mathcal{P}((a-\bar{a})\nabla P) = \mathcal{P}((a-\bar{a})\nabla P) - (a-\bar{a})\mathcal{P}(\nabla P)\\ &:= [\mathcal{P}, a-\bar{a}]\nabla P\end{aligned}$$

qui nous permettra de montrer la continuité de l'operateur solution $f \mapsto \nabla P$ de (0.2.9) dans les espaces de Besov qui ont la même invariance par changement d'échelle que L^2. Un autre ingrédient qui interviendra d'une manière cruciale dans la construction des solutions de (0.2.7) sera une variante perturbative d'un résultat de Danchin et Mucha [51] pour le système à coefficients constants.

Chapter 1

Long time existence results for Sobolev initial data

1.1 The main result

In this chapter, we address the long time existence issue for the *abcd* Boussinesq systems:

$$\begin{cases} (I-\varepsilon b\Delta)\,\partial_t\eta+\operatorname{div}V+a\varepsilon\operatorname{div}\Delta V+\varepsilon\operatorname{div}(\eta V)=0,\\ (I-\varepsilon d\Delta)\,\partial_t V+\nabla\eta+c\varepsilon\nabla\Delta\eta+\varepsilon V\cdot\nabla V=0,\\ \eta_{|t=0}=\eta_0\ ,\ V_{|t=0}=V_0. \end{cases}\tag{1.1.1}$$

Our aim is to generalize most of the results presented in [82] and [81]. More precisely we wish to construct solutions for (1.1.1) for which the lifespan is bounded from below by a $O\left(\frac{1}{\varepsilon}\right)$-order quantity. The key ingredient is the construction of a nonlinear energy functional which controls appropriate Sobolev norms. This is accomplished by working with spectrally localized equations. The two most important features of our method are:

- we require lower regularity levels than the ones used to develop the long time existence theory in [82] and [81]
- as opposed to the previous works, our method allows us to treat in a unified manner most of the cases corresponding to the values of the a,b,c,d parameters.

Also, we mention that, although not relevant from a practical point of view, we may avoid using the non-cavitation condition

$$1+\varepsilon\eta_0(x)\geq\alpha>0.\tag{1.1.2}$$

Remark 1.1.1. *Let us observe that we consider the nonlinearity $\varepsilon V\cdot\nabla V$ instead of $\frac{\varepsilon}{2}\nabla|V|^2$ in the second equation of system (1.1.1).The two formulations are equivalent provided that* $\operatorname{curl}V=0$ *at any time, i.e. for all $l,k\in\overline{1,n}$:*

$$\partial_l V^k=\partial_k V^l.\tag{1.1.3}$$

First, let us establish some notations. The spaces $L^p(\mathbb{R}^n)$ with $p\in[1,\infty]$ will denote the classical Lebesgue spaces. Consider

$$a\leq 0,\ \ c\leq 0,\ \ b\geq 0,\ \ d\geq 0.\tag{1.1.4}$$

Given $s\in\mathbb{R}$ we will consider the following set of indices:

$$\begin{cases} s_{bc}=s+sgn(b)-sgn(c),\\ s_{ad}=s+sgn(d)-sgn(a), \end{cases}\tag{1.1.5}$$

where the sign function sgn is given by:

$$sgn\,(x)=\begin{cases}\ \ 1 & \text{if} \quad x>0,\\ \ \ 0 & \text{if} \quad x=0,\\ -1 & \text{if} \quad x<0,\end{cases}$$

We will denote by $H^s(\mathbb{R}^n)$ the classical Sobolev space of regularity index s with the norm

$$\|\eta\|_{H^s}^2=\int\limits_{\mathbb{R}^n}\left(1+|\xi|^2\right)^s|\hat{\eta}(\xi)|^2\,d\xi. \tag{1.1.6}$$

For any vector, matrix or 3-tensor field with components in $H^s(\mathbb{R}^n)$ we denote its Sobolev norm by the square root of the sum of the squares of the Sobolev norms of its components. For any pair $(\eta,V)\in H^{s_{bc}}(\mathbb{R}^n)\times(H^{s_{ad}}(\mathbb{R}^n))^n$ we will use the notation

$$\begin{aligned}\|(\eta,V)\|_{s,\varepsilon}^2=\|\eta\|_{H^s}^2+\varepsilon(b-c)\|\nabla\eta\|_{H^s}^2+\varepsilon^2(-c)b\left\|\nabla^2\eta\right\|_{H^s}^2+\\ +\|V\|_{H^s}^2+\varepsilon(d-a)\|\nabla V\|_{H^s}^2+\varepsilon^2(-a)d\left\|\nabla^2V\right\|_{H^s}^2,\end{aligned}$$

where $\nabla^2\eta=\left(\partial_{ij}^2\eta\right)_{i,j}$ and $\nabla^2V=\left(\partial_{ij}^2V^k\right)_{i,j,k}$. Clearly, $H^{s_{bc}}(\mathbb{R}^n)\times(H^{s_{ad}}(\mathbb{R}^n))^n$ endowed with $\|(\cdot,\cdot)\|_s$ is a Banach space which is continuously imbedded in $L^2(\mathbb{R}^n)\times\left(L^2(\mathbb{R}^n)\right)^n$.

We are now in the position of stating our long time existence result:

Theorem 1.1.1. *Consider $a,c\le 0$ and $b,d\ge 0$ excluding the case $a=b=d=0$, $c<0$. Also, consider an integer $n\ge 1$, s such that*

$$s>\frac{n}{2}+2-\operatorname{sgn}(b+d),$$

and s_{bc}, s_{ad} defined by (1.1.5). *Let us consider $(\eta_0,V_0)\in H^{s_{bc}}(\mathbb{R}^n)\times(H^{s_{ad}}(\mathbb{R}^n))^n$. Then, there exist two constants ε_0 and $T(\eta_0,V_0)$ depending on $\|(\eta_0,V_0)\|_{s,1}$, s and on the dimension n such that the following holds true. For any $\varepsilon\in(0,\varepsilon_0)$ there exists a unique solution*

$$(\eta,V)\in\mathcal{C}\left(\left[0,\varepsilon^{-1}T(\eta_0,V_0)\right],H^{s_{bc}}(\mathbb{R}^n)\times(H^{s_{ad}}(\mathbb{R}^n))^n\right)$$

of (1.1.1) *which satisfies*

$$\inf_{x\in\mathbb{R}^n}\{1+\varepsilon\eta(t,x)\}>0. \tag{1.1.7}$$

Moreover,

$$\sup_{t\in[0,\varepsilon^{-1}T(\eta_0,V_0)]}\|(\eta(t),V(t))\|_{s,\varepsilon}\le 4\,\|(\eta_0,V_0)\|_{s,\varepsilon}\,.$$

Remark 1.1.2. *In our proof, we use the non-cavitation condition (1.1.7) to show uniqueness for some values of the parameters. However, from a practical point of view this is not restrictive as all physical acceptable solutions will verify (1.1.7). If $c=0$ or $c<0$ and*

$$\operatorname{sgn}(b)+\operatorname{sgn}(d)+\operatorname{sgn}(-a)\ge 2$$

then we show that uniqueness holds even in the regions where (1.1.7) is not valid.

Remark 1.1.3. *Our proof also works for the "dispersionless" case $a=b=c=d=0$.*

Remark 1.1.4. *As it will be seen in the following, Theorem 1.1.1 can be formulated in the context of the more general Besov spaces.*

Our approach is based on an energy method applied to spectrally localized equations. We first derive a priori estimates and we establish local existence and uniqueness of solutions for the general *abcd* system. The rest of this chapter is organized as follows. In Section 1.2, considering a nonlinear energy functional, we establish a priori estimates. In section 1.4, for technical reasons, we are forced to consider a non-classical Friedrichs-type family of regularizations of the system 1.1.1. We prove that the solutions of this regularized systems uniformly satisfy the estimates established in Section 1.2 on time intervals that are of order ε^{-1}. Finally, we show that we can pass to the limit thus proving Theorem 1.1.1. We end the chapter with Section 1.4.3 where we discuss the possibility of applying our method to the *abcd* systems in the case of a more general bottom topography (see [33], Section 2).

Notations

Because our proof makes use of elementary tensor calculus let us introduce some basic notations. For any vector field $U : \mathbb{R}^n \to \mathbb{R}^n$ we denote by $\nabla U : \mathbb{R}^n \to \mathcal{M}_n(\mathbb{R})$ and by $\nabla^t U : \mathbb{R}^n \to \mathcal{M}_n(\mathbb{R})$ the $n \times n$ matrices defined by:

$$\begin{aligned} (\nabla U)_{ij} &= \partial_i U^j, \\ \left(\nabla^t U\right)_{ij} &= \partial_j U^i. \end{aligned}$$

In the same manner we define $\nabla^2 U : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ as:

$$\left(\nabla^2 U\right)_{ijk} = \partial^2_{ij} U^k.$$

We will suppose that all vectors appearing are column vectors and thus the (classical) product between a matrix field A and a vector field U will be the vector[1]:

$$(AU)^i = A_{ij} U^j.$$

For two vector fields U and V we define the following matrix operators

$$((\nabla U \nabla) \cdot V)_{ij} = \partial^2_{ik} U^k V^j,$$

and

$$(U \cdot \nabla (\nabla V))_{ij} = U^k \partial^2_{ki} V^j.$$

If $U, V : \mathbb{R}^n \to \mathbb{R}^n$ are two vector fields and $A, B : \mathbb{R}^n \to \mathcal{M}_n(\mathbb{R})$ two matrix fields we denote:

$$\begin{aligned} UV = U^i V^i, &\ \ A : B = A_{ij} B_{ij}, \\ \langle U, V \rangle_{L^2} = \int U^i V^i, &\ \ \langle A, B \rangle_{L^2} = \int A_{ij} B_{ij} \\ \|U\|^2_{L^2} = \langle U, U \rangle_{L^2}, &\ \ \|A\|^2_{L^2} = \langle A, A \rangle_{L^2} \\ \left\|\nabla^2 U\right\|^2_{L^2} = \int \nabla U : \nabla U &= \int \left(\partial_{ij} U^k\right)^2 \end{aligned}$$

Also, the tensorial product of two vector fields U, V is defined as the matrix field $U \otimes V : \mathbb{R}^n \to \mathcal{M}_n(\mathbb{R})$ given by:

$$(U \otimes V)_{ij} = U^i V^j.$$

We have the following derivation rule: if u is a scalar field, U, V are vector fields and $A : \mathbb{R}^n \to \mathcal{M}_n(\mathbb{R})$ then:

$$\nabla \operatorname{div}(uU) = \nabla^2 u U + \nabla u \operatorname{div} U + \nabla U \nabla u + u \nabla \operatorname{div} U$$

[1] From now on we will use the Einstein summation convention over repeted indices.

$$\begin{aligned}\nabla (UV) &= \nabla UV + \nabla VU \\ \nabla(AV) &= \nabla^2 A : V + \nabla V A^t\end{aligned}$$

Also, the following integration by parts identity holds true:

$$\begin{aligned}\langle U\cdot\nabla(\nabla V), \nabla V\rangle_{L^2} &= \int (U\cdot\nabla(\nabla V))_{ij}\,\partial_i V^j = \int U^k\partial^2_{ki}V^j\partial_i V^j \\ &= -\frac{1}{2}\int \partial_k U^k \left(\partial_i V^j\right)^2 = -\frac{1}{2}\int \operatorname{div} U(\nabla V:\nabla V).\end{aligned}\tag{1.1.8}$$

For all $m\in\mathbb{N}$, let us consider $\mathbb{E}_m$ the low frequency cut-off operator defined by:

$$\mathbb{E}_m f = \mathcal{F}^{-1}\left(\mathbf{1}_{B(0,2^{m+3}/3)}\hat{f}\right),\tag{1.1.9}$$

where, in general, $\mathbf{1}_A$ represents the characteristic function of the set A. We define the space

$$L^2_m = \left\{ f\in L^2 : \operatorname{Supp}\hat{f}\subset B\left(0,2^{m+3}/3\right)\right\}\tag{1.1.10}$$

which, endowed with the $\|\cdot\|_{L^2}$-norm is a Banach space. Let us observe that due to Bernstein's lemma, all nonhomogeneous Sobolev norms are equivalent on L^2_m.

Let $\mathcal{C}$ be the annulus $\{\xi\in\mathbb{R}^n : 3/4\leq|\xi|\leq 8/3\}$. Let us choose two radial functions $\chi\in\mathcal{D}(B(0,4/3))$ and $\varphi\in\mathcal{D}(\mathcal{C})$ valued in the interval $[0,1]$ and such that:

$$\forall\xi\in\mathbb{R}^n,\ \ \chi(\xi)+\sum_{j\geq 0}\varphi(2^{-j}\xi)=1.$$

Let us denote by $h=\mathcal{F}^{-1}\varphi$ and $\tilde{h}=\mathcal{F}^{-1}\chi$. For all $u\in\mathcal{S}'$, the nonhomogeneous dyadic blocks are defined as follows:

$$\left\{\begin{array}{c}\Delta_j u = 0 \ \text{ if } \ j\leq -2,\\ \Delta_{-1}u = \chi(D)\,u = \tilde{h}\star u,\\ \Delta_j u = \varphi\left(2^{-j}D\right)u = 2^{jd}\int_{\mathbb{R}^n} h\left(2^q y\right)u\left(x-y\right)dy \ \text{ if } \ j\geq 0.\end{array}\right.\tag{1.1.11}$$

The high frequency cut-off operator S_j is defined by

$$S_j u = \sum_{k\leq j-1}\Delta_k u.$$

Let us define now the nonhomogeneous Besov spaces.

Definition 1.1.1. *Let $s\in\mathbb{R}$, $r\in[1,\infty]$. The Besov space $B^s_{2,r}$ is the set of tempered distributions $u\in\mathcal{S}'$ such that:*

$$\|u\|_{B^s_{2,r}} := \left\|\left(2^{js}\left\|\Delta_j u\right\|_{L^2}\right)_{j\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})}<\infty.$$

Let us mention that $H^s(\mathbb{R}^n)=B^s_{2,2}(\mathbb{R}^n)$ and that we have the following continuous embedding

$$B^s_{2,1}\hookrightarrow H^s\hookrightarrow B^s_{2,\infty}\hookrightarrow H^{s'},$$

for all $s'<s$. Some basic properties about Besov spaces can be found in the Appendix. For more details and full proofs we refer to [11].

Let us consider $\varepsilon\leq 1$ and $s>0$, $r\in[1,\infty)$. For all $(\eta,V)\in B^{s_{bc}}_{2,r}(\mathbb{R}^n)\times\left(B^{s_{ad}}_{2,r}(\mathbb{R}^n)\right)^n$ we introduce the following quantities:

$$U_j^2 := U_j^2(\eta,V) = \int \eta_j^2 + \varepsilon\,(b-c)\left|\nabla\eta_j\right|^2 + \varepsilon^2(-c)b\left(\nabla^2\eta_j:\nabla^2\eta_j\right)\tag{1.1.12}$$

$$+\int V_j^2+\varepsilon\left(d-a\right)\left(\nabla V_j:\nabla V_j\right)+\varepsilon^2(-a)d\left(\nabla^2 V_j:\nabla^2 V_j\right),$$

and

$$U_s:=U_s\left(\eta,V\right)=\left\|\left(2^{js}U_j\right)_{j\in\mathbb{Z}}\right\|_{\ell^r},\tag{1.1.13}$$

where $(\eta_j,V_j):=(\Delta_j\eta,\Delta_j V)$ are the frequency-localized dyadic blocks defined by relation (1.1.11). It is easy to check that $U_s(\cdot,\cdot)$ is a norm on the space $B^{s_{bc}}_{2,r}(\mathbb{R}^n)\times\left(B^{s_{ad}}_{2,r}(\mathbb{R}^n)\right)^n$. Also, it transpires that:

$$\begin{aligned}U_s^2&\approx\|\eta\|^2_{B^s_{2,r}}+\varepsilon\left(b-c\right)\|\nabla\eta\|^2_{B^s_{2,r}}+\varepsilon^2\left(-c\right)b\left\|\nabla^2\eta\right\|^2_{B^s_{2,r}}\\&+\|V\|^2_{B^s_{2,r}}+\varepsilon\left(d-a\right)\|\nabla V\|^2_{B^s_{2,r}}+\varepsilon^2\left(-a\right)d\left\|\nabla^2 V\right\|^2_{B^s_{2,r}}.\end{aligned}$$

We will sometimes use the notation

$$\|(\eta,V)\|_s=U_s\left(\eta,V\right),$$

for all $(\eta,V)\in B^{s_{bc}}_{2,r}(\mathbb{R}^n)\times\left(B^{s_{ad}}_{2,r}(\mathbb{R}^n)\right)^n$. We observe that $\left(B^{s_{bc}}_{2,r}(\mathbb{R}^n)\times\left(B^{s_{ad}}_{2,r}(\mathbb{R}^n)\right)^n,\|(\cdot,\cdot)\|_s\right)$ is a Banach space. In order to ease the notations we will rather write $B^s_{2,r}$ instead of $B^s_{2,r}(\mathbb{R}^n)$. Another quantity that will play an important role in the following analysis is:

$$\begin{aligned}N_j^2=N_j^2\left(\eta,V\right)&=\int\eta_j^2+\varepsilon\left(b-c\right)\left|\nabla\eta_j\right|^2+\varepsilon^2(-c)b\left(\nabla^2\eta_j:\nabla^2\eta_j\right)\\&+\int\left(1+\varepsilon\eta\right)V_j^2+\varepsilon\left(d-a+d\varepsilon\eta\right)\left(\nabla V_j:\nabla V_j\right)+\varepsilon^2(-a)d\left(\nabla^2 V_j:\nabla^2 V_j\right),\end{aligned}\tag{1.1.14}$$

Denote

$$N_s=N_s\left(\eta,V\right)=\left\|\left(2^{js}N_j\left(\eta,V\right)\right)_{j\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})}.\tag{1.1.15}$$

1.2 Energy estimates

We begin by localizing equation (1.1.1) in Fourier space thus obtaining that:

$$\begin{cases}\left(I-\varepsilon b\Delta\right)\partial_t\eta_j+\operatorname{div}V_j+a\varepsilon\operatorname{div}\Delta V_j+\varepsilon V\nabla\eta_j+\varepsilon\eta\operatorname{div}V_j=\varepsilon R_{1j}\\\left(I-\varepsilon d\Delta\right)\partial_t V_j+\nabla\eta_j+c\varepsilon\nabla\Delta\eta_j+\varepsilon V\cdot\nabla V_j=\varepsilon R_{2j}\\\eta_{j|t=0}=\Delta_j\eta_0\ ,\ V_{j|t=0}=\Delta_j V_0\end{cases}\tag{1.2.1}$$

where the remainder terms are given by[2]:

$$\begin{cases}R_{1j}=\left[V,\Delta_j\right]\nabla\eta+\left[\eta,\Delta_j\right]\operatorname{div}V\\R_{2j}=\left[V,\Delta_j\right]\nabla V.\end{cases}\tag{1.2.2}$$

Let us establish our first useful identity. We multiply the first equation in (1.2.1) by η_j and the second one with V_j and by adding them up and integrating in space we get that[3]:

$$\frac{1}{2}\frac{d}{dt}\left(\|\eta_j\|^2_{L^2}+\varepsilon b\|\nabla\eta_j\|^2_{L^2}+\|V_j\|^2_{L^2}+\varepsilon d\|\nabla V_j\|^2_{L^2}\right)+a\varepsilon\int\eta_j\operatorname{div}\Delta V_j+c\varepsilon\int V_j\nabla\Delta\eta_j\tag{1.2.3}$$

[2]From now on, we agree that if A, B are two operator then the operator $[A,B]$ is given by:

$$[A,B]=AB-BA.$$

[3]Observe that here we use the fact that $\operatorname{curl}V=0$. Indeed, under (1.1.3) we have

$$\int\nabla V_jVV_j=\frac{1}{2}\int V\nabla\left|V_j\right|^2=-\frac{1}{2}\int\operatorname{div}V\left|V_j\right|^2.$$

$$+\varepsilon\int \eta\eta_j \operatorname{div} V_j = \frac{\varepsilon}{2}\int \operatorname{div} V\left(\eta_j^2 + V_j^2\right) + \varepsilon\int R_{1j}\eta_j + \varepsilon\int R_{2j}V_j.$$

Let us denote by T_1 the right hand side of the above identity:

$$T_1 = \frac{\varepsilon}{2}\int \operatorname{div} V\left(\eta_j^2 + V_j^2\right) + \varepsilon\int R_{1j}\eta_j + \varepsilon\int R_{2j}V_j. \tag{1.2.4}$$

Next, we wish to derive higher order estimates. In order to do so, let us observe that applying ∇ to the first equation in (1.1.1) gives us:

$$(I - \varepsilon b\Delta)\,\partial_t\nabla\eta + \nabla\operatorname{div} V + a\varepsilon\nabla\operatorname{div}\Delta V + \varepsilon\nabla\operatorname{div}(\eta V) = 0$$

and that by applying Δ_j we end up with:

$$\begin{gathered}(I - \varepsilon b\Delta)\,\partial_t\nabla\eta_j + \nabla\operatorname{div} V_j + a\varepsilon\nabla\operatorname{div}\Delta V_j + \varepsilon\nabla^2\eta_j V + \varepsilon\Delta_j\left(\eta\nabla\operatorname{div} V\right) \\ +\varepsilon\operatorname{div} V_j\nabla\eta + \varepsilon\nabla V_j\nabla\eta = \varepsilon R_{3j},\end{gathered} \tag{1.2.5}$$

where

$$R_{3j} = [V, \Delta_j]\,\nabla^2\eta + [\nabla\eta, \Delta_j]\operatorname{div} V + [\nabla\eta, \Delta_j]\,\nabla V.$$

We multiply (1.2.5) with $-c\varepsilon\nabla\eta_j$ and by integration we get:

$$\begin{gathered}\frac{-c\varepsilon}{2}\frac{d}{dt}\left(\int\left(|\nabla\eta_j|^2 + \varepsilon b\nabla^2\eta_j : \nabla^2\eta_j\right)\right) - c\varepsilon\int\nabla\operatorname{div} V_j\nabla\eta_j - ac\varepsilon^2\int\nabla\operatorname{div}\Delta V_j\nabla\eta_j \\ -c\varepsilon^2\int\Delta_j\left(\eta\nabla\operatorname{div} V\right)\nabla\eta_j = T_2,\end{gathered} \tag{1.2.6}$$

with T_2 given by

$$T_2 = c\varepsilon^2\int(\nabla^2\eta_j V)\nabla\eta_j + c\varepsilon^2\int\operatorname{div} V_j\nabla\eta\nabla\eta_j + c\varepsilon^2\int(\nabla V_j\nabla\eta)\nabla\eta_j - c\varepsilon^2\int R_{3j}\nabla\eta_j. \tag{1.2.7}$$

We proceed similarly with the second equation in (1.1.1) and we obtain:

$$(I - \varepsilon d\Delta)\,\partial_t\nabla V + \nabla^2\eta + \varepsilon c\nabla^2\Delta\eta + \varepsilon(\nabla V : \nabla)\cdot\nabla V + \varepsilon V\cdot\nabla(\nabla V) = 0.$$

We localize the last equation and we get that:

$$(I - \varepsilon d\Delta)\,\partial_t\nabla V_j + \nabla^2\eta_j + \varepsilon c\nabla^2\Delta\eta_j + \varepsilon V\cdot\nabla(\nabla V_j) = \varepsilon R_{4j}, \tag{1.2.8}$$

where:

$$R_{4j} = [V\cdot\nabla, \Delta_j]\,\nabla V + \Delta_j\left((\nabla V : \nabla)\cdot\nabla V\right).$$

We contract (1.2.8) with $-a\varepsilon\nabla V_j$ and by integration we get that:

$$\frac{-a\varepsilon}{2}\frac{d}{dt}\left(\int\nabla V_j : \nabla V_j + \varepsilon d\nabla^2 V_j : \nabla^2 V_j\right) - a\varepsilon\int\nabla^2\eta_j : \nabla V_j - ac\varepsilon^2\int\nabla^2\Delta\eta_j : \nabla V_j = T_3 \tag{1.2.9}$$

with T_3 given by:

$$T_3 = a\varepsilon^2\int\left(V\cdot\nabla(\nabla V_j)\right) : \nabla V_j - a\varepsilon^2\int R_{4j} : \nabla V_j. \tag{1.2.10}$$

Let us add up identities (1.2.6) and (1.2.9) to get that[4]:

$$\frac{d}{dt}\left(\int\left\{\frac{-c\varepsilon}{2}\left(|\nabla\eta_j|^2 + \varepsilon b\nabla^2\eta_j : \nabla^2\eta_j\right) + \frac{-a\varepsilon}{2}\left(\nabla V_j : \nabla V_j + \varepsilon d\nabla^2 V_j : \nabla^2 V_j\right)\right\}\right)$$

[4]Observe that by integration by parts we get

$$-ac\varepsilon^2\int\nabla\operatorname{div}\Delta V_j : \nabla\eta_j - ac\varepsilon^2\int\nabla^2\Delta\eta_j : \nabla V_j = 0.$$

$$-a\varepsilon\int\nabla^2\eta_j:\nabla V_j-c\varepsilon\int\nabla\operatorname{div}V_j\nabla\eta_j-c\varepsilon^2\int\Delta_j\left(\eta\nabla\operatorname{div}V\right)\nabla\eta_j=T_2+T_3.$$

Finally add up (1.2.3) to the last identity in order to obtain[5]:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\int\eta_j^2+\varepsilon\left(b-c\right)\left|\nabla\eta_j\right|^2+\varepsilon^2(-c)b\left(\nabla^2\eta_j:\nabla^2\eta_j\right)\right)\\&+\frac{1}{2}\frac{d}{dt}\left(\int\left|V_j\right|^2+\varepsilon\left(d-a\right)\left(\nabla V_j:\nabla V_j\right)+\varepsilon^2(-a)d\left(\nabla^2V_j:\nabla^2V_j\right)\right)\\&+\varepsilon\int\eta\eta_j\operatorname{div}V_j-c\varepsilon^2\int\Delta_j\left(\eta\nabla\operatorname{div}V\right)\nabla\eta_j=T_1+T_2+T_3.\end{aligned}\tag{1.2.11}$$

1.3 Refined estimates

The identity (1.2.11) cannot by itself provide the desired result. Indeed, we observe that the terms $\varepsilon\int\eta\eta_j\operatorname{div}V_j$, $-c\varepsilon^2\int\eta\nabla\operatorname{div}V_j\nabla\eta_j$ prevent us from directly applying a Gronwall type argument and establishing long time existence. In this section, we establish a refined energy-estimate which proves to be crucial for establishing Theorem 1.1.1.

Let us multiply the second equation of (1.2.1) with $\varepsilon\eta V_j$. We thus get:

$$\left\langle\varepsilon\left(I-d\Delta\right)\partial_tV_j,\eta V_j\right\rangle_{L^2}+\varepsilon\int\eta\nabla\eta_jV_j+c\varepsilon^2\int\eta\nabla\Delta\eta_jV_j=-\varepsilon^2\int\left(V\cdot\nabla V_j\right)\left(\eta V_j\right)+\varepsilon^2\int\eta R_{2j}V_j.$$

Observing that the first term can be put in the form:

$$\begin{aligned}\left\langle\varepsilon\left(I-d\Delta\right)\partial_tV_j,\eta V_j\right\rangle_{L^2}=&\frac{1}{2}\frac{d}{dt}\left(\int\varepsilon\eta\left|V_j\right|^2+\varepsilon^2d\eta\nabla V_j:\nabla V_j\right)-\frac{\varepsilon}{2}\int\partial_t\eta\left|V_j\right|^2\\&-\varepsilon^2\frac{d}{2}\int\partial_t\eta\nabla V_j:\nabla V_j+\varepsilon^2d\left\langle\partial_t\nabla V_j,\nabla\eta\otimes V_j\right\rangle_{L^2},\end{aligned}$$

we write that

$$\frac{1}{2}\frac{d}{dt}\left(\int\varepsilon\eta\left|V_j\right|^2+\varepsilon^2d\eta\nabla V_j:\nabla V_j\right)+\varepsilon\int\eta\nabla\eta_jV_j+c\varepsilon^2\int\eta\nabla\Delta\eta_jV_j=T_4\tag{1.3.1}$$

with T_4 given by:

$$\begin{aligned}T_4=&-\varepsilon^2\int\left(V\cdot\nabla V_j\right)\eta V_j+\varepsilon^2\int\eta R_{2j}V_j+\\&+\frac{\varepsilon}{2}\int\partial_t\eta\left|V_j\right|^2+\varepsilon^2\frac{d}{2}\int\partial_t\eta\nabla V_j:\nabla V_j-\varepsilon^2d\left\langle\partial_t\nabla V_j,\nabla\eta\otimes V_j\right\rangle_{L^2}.\end{aligned}\tag{1.3.2}$$

We add up (1.2.11) and (1.3.1) in order to obtain:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\int\eta_j^2+\varepsilon\left(b-c\right)\left|\nabla\eta_j\right|^2+\varepsilon^2(-c)b\left(\nabla^2\eta_j:\nabla^2\eta_j\right)\right)\\&+\frac{1}{2}\frac{d}{dt}\left(\int\left(1+\varepsilon\eta\right)\left|V_j\right|^2+\varepsilon\left(d-a+d\varepsilon\eta\right)\left(\nabla V_j:\nabla V_j\right)+\varepsilon^2(-a)d\left(\nabla^2V_j:\nabla^2V_j\right)\right)\\&=T_0+T_1+T_2+T_3+T_4,\end{aligned}\tag{1.3.3}$$

with

$$T_0=\varepsilon\int\nabla\eta\eta_jV_j-c\varepsilon^2\int\eta\nabla\Delta\eta_jV_j+c\varepsilon^2\int\Delta_j\left(\eta\nabla\operatorname{div}V\right)\nabla\eta_j.\tag{1.3.4}$$

---

[5]Observe that the terms $a\varepsilon\int\eta_j\operatorname{div}\Delta V_j-a\varepsilon\int\nabla^2\eta_j:\nabla V_j$ and $c\varepsilon\int V_j\nabla\Delta\eta_j-c\varepsilon\int\nabla\operatorname{div}V_j:\nabla\eta_j$ are both 0.

1.3.1 Estimates for the T_i's

Having established the energy identity (1.3.3), we proceed by conveniently bounding the right hand side term. We want to obtain a bound of the form:

$$T_0 + T_1 + T_2 + T_3 + T_4 \leq \varepsilon P(U_j, U_s)$$

with $P(x, y)$ some polynomial function with coefficients independant of ε. We suppose that

$$s > \frac{n}{2} + 1 \text{ or } s = \frac{n}{2} + 1 \text{ and } r = 1.$$

Moreover, $C > 0$ will denote a generic positive constant depending only on the dimension n and on s.

We claim that we can bound $T_0 + T_1 + T_2 + T_3 + T_4$ in such a way that the following estimate holds true:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\int \eta_j^2 + \varepsilon\,(b-c)\,|\nabla\eta_j|^2 + \varepsilon^2(-c)b\left(\nabla^2\eta_j : \nabla^2\eta_j\right)\right) \\ +&\frac{1}{2}\frac{d}{dt}\left(\int (1+\varepsilon\eta)\,V_j^2 + \varepsilon\,(d-a+d\varepsilon\eta)\,(\nabla V_j : \nabla V_j) + \varepsilon^2(-a)d\left(\nabla^2 V_j : \nabla^2 V_j\right)\right) \\ &\leq \varepsilon C_{abcd} U_j \left\{U_j\left(U_s + U_s^2\right) + c_j(t)(U_s^2 + U_s^3\right\}, \end{aligned} \tag{1.3.5}$$

where $C_{abcd} > 0$ only depends on a,b,c,d,n and s but not on ε, $(c_j\,(t))_j$ is a sequence with $\left(2^{js}c_j\,(t)\right)_j \in \ell^r(\mathbb{Z})$, having norm 1. Let us detail this below.

Regarding T_0, we begin by observing that:

$$\varepsilon \int \nabla\eta\eta_j V_j \leq \varepsilon \left\|\nabla\eta\right\|_{L^\infty} \left\|\eta_j\right\|_{L^2} \left\|V_j\right\|_{L^2} \leq \varepsilon U_j^2 U_s.$$

If $d - a > 0$, then, using Proposition 1.2.5, we write[6]:

$$\begin{aligned} &-c\varepsilon^2 \int \eta\Delta\nabla\eta_j V_j + c\varepsilon^2 \int \Delta_j\left(\eta\nabla \operatorname{div} V\right)\nabla\eta_j \\ &= -c\varepsilon^2 \int \left(\nabla^2\eta V_j\right)\nabla\eta_j - c\varepsilon^2 \int \left(\nabla V_j \nabla\eta\right)\nabla\eta_j - c\varepsilon^2 \int \nabla\eta\nabla\eta_j \operatorname{div} V_j \\ &- c\varepsilon^2 \int \eta\nabla \operatorname{div} V_j \nabla\eta_j + c\varepsilon^2 \int \Delta_j\left(\eta\nabla \operatorname{div} V\right)\nabla\eta_j \\ &\leq -c\varepsilon^2 \left(\left\|\nabla^2\eta\right\|_{L^\infty} \left\|V_j\right\|_{L^2} \left\|\nabla\eta_j\right\|_{L^2} + \left\|\nabla\eta\right\|_{L^\infty} \left\|\nabla V_j\right\|_{L^2} \left\|\nabla\eta_j\right\|_{L^2} \right. \\ &\qquad \left. + \left\|\nabla\eta_j\right\|_{L^2} \left\|\left[\Delta_j, \eta\right]\nabla \operatorname{div} V_j\right\|_{L^2}\right) \\ &\leq -c\varepsilon^2 \left(\left\|\nabla^2\eta\right\|_{L^\infty} \left\|V_j\right\|_{L^2} \left\|\nabla\eta_j\right\|_{L^2} + \left\|\nabla\eta\right\|_{L^\infty} \left\|\nabla V_j\right\|_{L^2} \left\|\nabla\eta_j\right\|_{L^2} \right. \\ &\qquad \left. + c_j\,(t) \left\|\nabla\eta_j\right\|_{L^2} \left\|\nabla\eta\right\|_{B^{s-1}_{2,r}} \left\|\operatorname{div} V\right\|_{B^s_{2,1}}\right) \\ &\leq \varepsilon C \max\left(\frac{-c}{b-c}, \frac{-c}{d-a}\right)\left(U_j^2 U_s + c_j\,(t)\,U_j U_s^2\right), \end{aligned}$$

where $\left(2^{js}c_j\,(t)\right)_{j\in\mathbb{Z}} \in \ell^r\,(\mathbb{Z})$. Next, if $a = d = 0$ then $b > 0$ and we get that:

$$\begin{aligned} &-c\varepsilon^2 \int \eta\Delta\nabla\eta_j V_j + c\varepsilon^2 \int \Delta_j\left(\eta\nabla \operatorname{div} V\right)\nabla\eta_j \\ &= c\varepsilon^2 \int \nabla\eta V_j \Delta\eta_j + c\varepsilon^2 \int \eta \operatorname{div} V_j \Delta\eta_j - c\varepsilon^2 \int \Delta_j\left(\nabla\eta \operatorname{div} V\right)\nabla\eta_j - c\varepsilon^2 \int \Delta_j\left(\eta \operatorname{div} V\right)\Delta\eta_j \end{aligned}$$

[6]From now on, we adopt the convention $\frac{0}{0} = 0$.

$$
\begin{aligned}
&= c\varepsilon^2 \int \nabla\eta V_j \Delta\eta_j + c\varepsilon^2 \int \eta \operatorname{div} V_j \Delta\eta_j + c\varepsilon^2 \int \Delta_j \left(\nabla^2 \eta V\right) \nabla\eta_j \\
&+ c\varepsilon^2 \int \nabla^2\eta_j \Delta_j \left(\nabla\eta V\right) - c\varepsilon^2 \int \Delta_j \left(\eta \operatorname{div} V\right) \Delta\eta_j \\
&= c\varepsilon^2 \int \nabla\eta V_j \Delta\eta_j + c\varepsilon^2 \int [\eta, \Delta_j] \operatorname{div} V \Delta\eta_j + c\varepsilon^2 \int [\Delta_j, V] \nabla^2\eta \nabla\eta_j + c\varepsilon^2 \int \nabla^2\eta_j V \nabla\eta_j \\
&+ c\varepsilon^2 \int \nabla^2\eta_j \nabla\eta_j V + c\varepsilon^2 \int \nabla^2\eta_j [\Delta_j, V] \nabla\eta \\
&\leq -c\varepsilon^2 \Big(\|\nabla\eta\|_{L^\infty} \|\Delta\eta_j\|_{L^2} \|V_j\|_{L^2} + \|[\Delta_j, \eta] \operatorname{div} V\|_{L^2} \|\Delta\eta_j\|_{L^2} \\
&\qquad\qquad + \|\operatorname{div} V\|_{L^\infty} \|\nabla\eta_j\|_{L^2}^2 + \left\|\nabla^2\eta_j\right\|_{L^2} \|[\Delta_j, V] \nabla\eta\|_{L^2} \Big) \\
&\leq \varepsilon C \max\left(\frac{-c}{b-c}, \sqrt{\frac{-c}{b}} \right) \left(U_j^2 U_s + c_j(t) U_j U_s^2 \right).
\end{aligned}
$$

where $\left(2^{js} c_j(t)\right)_{j\in\mathbb{Z}} \in \ell^r(\mathbb{Z})$. We choose

$$
C^1_{abcd} = \begin{cases} \max\left(\frac{-c}{b-c}, \sqrt{\frac{-c}{b}} \right) & \text{if } d-a=0, \\ \max\left(\frac{-c}{b-c}, \frac{-c}{d-a} \right) & \text{if } d-a>0. \end{cases} \tag{1.3.6}
$$

Consequently, we have:

$$
T_0 \leq \varepsilon C C^1_{abcd} \left(U_j^2 U_s + c_j(t) U_j U_s^2 \right). \tag{1.3.7}
$$

Next, we analyze T_1. According to Proposition 1.2.5 and Proposition 1.2.1 we get that:

$$
\|R_{1j}\|_{L^2} \leq C c_j^1(t) \left(\|\nabla V\|_{B^{s-1}_{2,r}} \|\eta\|_{B^s_{2,r}} + \|\nabla\eta\|_{B^{s-1}_{2,r}} \|V\|_{B^s_{2,r}} \right) \leq C c_j^1(t) U_s^2, \tag{1.3.8}
$$

$$
\|R_{2j}\|_{L^2} \leq C c_j^2(t) \|\nabla V\|_{B^{s-1}_{2,r}} \|V\|_{B^s_{2,r}} \leq C c_j^2(t) U_s^2, \tag{1.3.9}
$$

with $\left(2^{js} c_j^i(t)\right) \in \ell^r(\mathbb{Z})$, $i=1,2$, with norm 1. In order to avoid mentioning each time, from now on, for all natural number, $i \in \mathbb{N}$, $\left(c_j^i(t)\right)_{j\in\mathbb{Z}}$ will be a sequence such that $\left(2^{js} c_j^i(t)\right)_{j\in\mathbb{Z}} \in \ell^r(\mathbb{Z})$ with norm 1. We get that

$$
\begin{aligned}
T_1 &= \frac{\varepsilon}{2} \int \operatorname{div} V \left(\eta_j^2 + V_j^2 \right) + \varepsilon \int R_{1j}\eta_j + \varepsilon \int R_{2j} V_j \\
&\leq \varepsilon \|\operatorname{div} V\|_{L^\infty} \left(\|\eta_j\|_{L^2}^2 + \|V_j\|_{L^2}^2 \right) + C c_j^1(t) U_j U_s^2 + C c_j^2(t) U_j U_s^2 \\
&\leq C\varepsilon \left(U_j^2 U_s + c_j^3(t) U_j U_s^2 \right).
\end{aligned} \tag{1.3.10}
$$

We turn our attention towards T_2

$$
T_2 = c\varepsilon^2 \int (\nabla^2\eta_j V) \nabla\eta_j + c\varepsilon^2 \int \operatorname{div} V_j \nabla\eta \nabla\eta_j + c\varepsilon^2 \int \nabla V_j \nabla\eta \nabla\eta_j - c\varepsilon^2 \int R_{3j} \nabla\eta_j.
$$

According to Proposition 1.2.1 and Proposition 1.2.5 we get that

$$
\|R_{3j}\|_{L^2} \leq C c_j^4(t) \left(\|\nabla V\|_{B^{s-1}_{2,r}} \|\nabla\eta\|_{B^s_{2,r}} + \left\|\nabla^2\eta\right\|_{B^{s-1}_{2,r}} \|V\|_{B^s_{2,r}} \right). \tag{1.3.11}
$$

Let us observe that:

$$
\int (\nabla^2\eta_j V) \nabla\eta_j = -\frac{1}{2} \int |\nabla\eta_j|^2 \operatorname{div} V.
$$

If $d-a>0$ then:

$$
T_2 = c\varepsilon^2 \int (\nabla^2\eta_j V) \nabla\eta_j + c\varepsilon^2 \int \operatorname{div} V_j \nabla\eta \nabla\eta_j + c\varepsilon^2 \int \nabla V_j \nabla\eta \nabla\eta_j - c\varepsilon^2 \int R_{3j} \nabla\eta_j
$$

$$
\begin{aligned}
&\leq \frac{-c}{2}\varepsilon^2 \Big\{ \|\operatorname{div} V\|_{L^\infty} \|\nabla \eta_j\|_{L^2}^2 + 2\|\nabla \eta\|_{L^\infty} \|\nabla \eta_j\|_{L^2} \|\nabla V_j\|_{L^2} \\
&\qquad + C c_j^4(t) \|\nabla \eta_j\|_{L^2} \left(\|\nabla V\|_{B^{s-1}_{2,r}} \|\nabla \eta\|_{B^s_{2,r}} + \left\|\nabla^2 \eta\right\|_{B^{s-1}_{2,r}} \|V\|_{B^s_{2,r}} \right) \Big\} \\
&\leq \varepsilon C \max\left(\frac{-c}{b-c}, \frac{-c}{d-a} \right) \left(U_j^2 U_s + c_j^4(t) U_j U_s^2 \right). \qquad (1.3.12)
\end{aligned}
$$

If $d-a=0$, $c<0$ and $b>0$ then, we write that

$$
\begin{aligned}
T_2 &= c\varepsilon^2 \int (\nabla^2 \eta_j V) \nabla \eta_j + c\varepsilon^2 \int \operatorname{div} V_j \nabla \eta \nabla \eta_j + c\varepsilon^2 \int \nabla V_j \nabla \eta \nabla \eta_j - c\varepsilon^2 \int R_{3j} \nabla \eta_j \\
&= c\varepsilon^2 \int (\nabla^2 \eta_j V) \nabla \eta_j - c\varepsilon^2 \int V_j \nabla (\nabla \eta \nabla \eta_j) - c\varepsilon^2 \int V_j \operatorname{div}(\nabla \eta \otimes \nabla \eta_j) - c\varepsilon^2 \int R_{3j} \nabla \eta_j \qquad (1.3.13) \\
&\leq \varepsilon \max\left(\frac{-c}{b-c}, \frac{1}{b} \right) \left(U_j^2 U_s + c_j^4(t) U_j U_s^2 \right). \qquad (1.3.14)
\end{aligned}
$$

We let

$$
C^2_{abcd} = \begin{cases} \max\left(\frac{-c}{b-c}, \frac{-c}{d-a} \right) & \text{if } d-a>0, \\ \max\left(\frac{-c}{b-c}, \frac{1}{b} \right) & \text{otherwise.} \end{cases} \qquad (1.3.15)
$$

Let us estimate T_3. As above, owing to Proposition 1.2.5 we may bound R_{4j} in the following manner:

$$
\|R_{4j}\|_{L^2} \leq c_j^5(t) \|V\|_{B^{s+1}_{2,r}} \|\nabla V\|_{B^s_{2,r}}. \qquad (1.3.16)
$$

Then, using the integration by parts identity (1.1.8) and the last estimate, we get that:

$$
\begin{aligned}
T_3 &= a\varepsilon^2 \int (V \cdot \nabla(\nabla V_j)) : \nabla V_j + -a\varepsilon^2 \int R_{4j} : \nabla V_j \\
&\leq -a\varepsilon^2 \left(\frac{1}{2} \|\operatorname{div} V\|_{L^\infty} \|\nabla V_j\|_{L^2}^2 + c_j^5(t) \|V\|_{B^{s+1}_{2,r}} \|\nabla V\|_{B^s_{2,r}} \|\nabla V_j\|_{L^2} \right) \\
&\leq C\varepsilon \frac{-a}{d-a} \left(U_j^2 U_s + c_j^5(t) U_j U_s^2 \right), \qquad (1.3.17)
\end{aligned}
$$

and as before, let us denote by

$$
C^3_{abcd} = \frac{-a}{d-a}. \qquad (1.3.18)
$$

Finally let us turn our attention towards T_4. Let us observe that integration by parts and (1.3.9) gives us

$$
\begin{aligned}
&- \varepsilon^2 \int (\nabla V_j V) \eta V_j + \varepsilon^2 \int \eta R_{2j} V_j \\
&\leq \frac{\varepsilon^2}{2} \int \operatorname{div}(\eta V) |V_j|^2 + C\varepsilon^2 c_j^2(t) \|V_j\|_{L^2} \|\eta\|_{L^\infty} U_s^2 \\
&\leq C\varepsilon \left(\varepsilon U_j^2 U_s^2 + \varepsilon c_j^2(t) U_j U_s^3 \right). \qquad (1.3.19)
\end{aligned}
$$

Next, we write that

$$
\frac{\varepsilon}{2} \int \partial_t \eta |V_j|^2 \leq \varepsilon \|\partial_t \eta\|_{L^\infty} \|V_j\|_{L^2}^2 \leq \frac{\varepsilon}{2} U_j^2 \|\partial_t \eta\|_{L^\infty}.
$$

We observe that:

$$
\|\partial_t \eta\|_{L^\infty} = \left\| (I - b\varepsilon\Delta)^{-1} \left[(I + a\varepsilon\Delta) \operatorname{div} V + \varepsilon \operatorname{div}(\eta V) \right] \right\|_{L^\infty}
$$

$$
\begin{aligned}
&\leq \left\| (I - b\varepsilon\Delta)^{-1} (I + a\varepsilon\Delta) \operatorname{div} V \right\|_{L^\infty} + \varepsilon \left\| (I - b\varepsilon\Delta)^{-1} \operatorname{div} (\eta V)) \right\|_{L^\infty} \\
&\leq \left\| (I - b\varepsilon\Delta)^{-1} (I + a\varepsilon\Delta) \operatorname{div} V \right\|_{B^{\frac{n}{2}}_{2,1}} + \varepsilon \left\| \eta V \right\|_{B^s_{2,r}}.
\end{aligned}
\tag{1.3.20}
$$

If $b > 0$ or $a = b = 0$, then because the operator $(I - b\varepsilon\Delta)^{-1} (I + a\varepsilon\Delta)$ maps L^2 to L^2 and has its norm bounded by a quantity independent of ε, we get that:

$$
\|\partial_t \eta\|_{L^\infty} \leq C \max\left(1, \frac{-a}{b}\right) \left(U_s + U_s^2\right).
$$

If $b = 0$, $a < 0$ and $d > 0$ and we observe that

$$
-a\varepsilon \|\operatorname{div} \Delta V\|_{B^{\frac{n}{2}}_{2,1}} \leq -a\varepsilon \left\|\nabla^2 V\right\|_{B^{\frac{n}{2}+1}_{2,1}} \leq -a\varepsilon \left\|\nabla^2 V\right\|_{B^{s}_{2,1}} \leq \sqrt{\frac{-a}{d}} U_s. \tag{1.3.21}
$$

Finally, if $b = 0$, $a < 0$ and $d = 0$ then $s \geq n/2 + 2$ with equality if $r = 1$ and, consequently, we get:

$$
-a\varepsilon \|\operatorname{div} \Delta V\|_{B^{\frac{n}{2}}_{2,1}} \leq -a\varepsilon \|\nabla V\|_{B^{\frac{n}{2}+2}_{2,1}} \leq \sqrt{-a} U_s. \tag{1.3.22}
$$

We set

$$
C^4_{abcd} = \begin{cases} \max\left(1, \frac{-a}{b}\right) \text{ if } b > 0 \text{ or } a = b = 0, \\ \max\left(1, \sqrt{\frac{-a}{d}}\right) \text{ if } b = 0,\ d > 0\ a < 0, \\ \sqrt{-a} \text{ if } b = d = 0 \text{ and } a < 0. \end{cases} \tag{1.3.23}
$$

Thus, we get that

$$
\frac{\varepsilon}{2} \int \partial_t \eta \left|V_j\right|^2 \leq \varepsilon C C^4_{abcd} U_j^2 \left(U_s + U_s^2\right). \tag{1.3.24}
$$

In a similar fashion we can treat the second term of B_4 thus obtaining

$$
\varepsilon^2 \frac{d}{2} \int \partial_t \eta \nabla V_j : \nabla V_j \leq C \frac{d}{d-a} C^4_{abcd} U_j^2 \left(U_s + U_s^2\right), \tag{1.3.25}
$$

and we set

$$
C^5_{abcd} = \frac{d}{d-a} C^4_{abcd}. \tag{1.3.26}
$$

Finally, the last term of B_4 is estimated as follows:

$$
-\varepsilon^2 d \left\langle \partial_t \nabla V_j, \nabla \eta \otimes V_j \right\rangle_{L^2} \leq \varepsilon^2 d \|\nabla \eta\|_{L^\infty} \|\partial_t \nabla V_j\|_{L^2} \|V_j\|_{L^2},
$$

and using

$$
\partial_t \nabla V_j = -\left(I - \varepsilon d\Delta\right)^{-1} \nabla \left(\nabla \eta_j + c\varepsilon \nabla \Delta \eta_j + \varepsilon V \cdot \nabla V_j - \varepsilon R_{2j}\right)
$$

we observe that

$$
\begin{aligned}
\varepsilon d \|\partial_t \nabla V_j\|_{L^2} &\leq \sqrt{d} \|\eta_j\|_{L^2} - c\sqrt{d}\varepsilon \|\nabla \eta_j\|_{L^2} + \varepsilon d \|V\|_{L^\infty} \|\nabla V_j\|_{L^2} + \varepsilon d \|R_{2j}\|_{L^2} \\
&\leq C C^6_{abcd} \left(U_j \left(1 + \varepsilon U_s\right) + c_j^2(t) U_s^2\right),
\end{aligned}
$$

where

$$
C^6_{abcd} = \max\left(\sqrt{d}, \frac{-c\sqrt{d}}{sgn(d)\sqrt{b-c}}, \frac{d}{\sqrt{d-a}}\right), \tag{1.3.27}
$$

thus, we conclude that

$$
-\varepsilon^2 d \left\langle \partial_t \nabla V_j, \nabla \eta \otimes V_j \right\rangle_{L^2} \leq \varepsilon C C^6_{abcd} U_j \left(U_j \left(U_s + \varepsilon U_s^2\right) + c_j^2(t) U_s^3\right). \tag{1.3.28}
$$

Combining estimates (1.3.24), (1.3.25), (1.3.28) we obtain:

$$\frac{\varepsilon}{2}\int \partial_t\eta\left|V_j\right|^2+\varepsilon^2\frac{d}{2}\int \partial_t\eta\nabla V_j:\nabla V_j-\varepsilon^2 d\left\langle \partial_t\nabla V_j,\nabla\eta\otimes V_j\right\rangle_{L^2} \tag{1.3.29}$$

$$\leq \varepsilon C\tilde{C}_{abcd}U_j\left\{U_j(U_s+U_s^2)+c_j^2\left(t\right)U_s^3\right\}, \tag{1.3.30}$$

where

$$\tilde{C}_{abcd}=\max_{i=\overline{1,6}}C^i_{abcd} \tag{1.3.31}$$

is the maximum of all constants depending on a,b,c,d that appear in relations (1.3.6), (1.3.15), (1.3.18), (1.3.23), (1.3.26), (1.3.27).

Finally, after adding up the estimations (1.3.7), (1.3.10), (1.3.12), (1.3.17), (1.3.19), (1.3.29) we obtain that there exists a positive constant $C>0$ depending only on n and s but not on ε such that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\int \eta_j^2+\varepsilon\left(b-c\right)\left|\nabla\eta_j\right|^2+\varepsilon^2(-c)b\left(\nabla^2\eta_j:\nabla^2\eta_j\right)\right)\\
&+\frac{1}{2}\frac{d}{dt}\left(\int \left(1+\varepsilon\eta\right)V_j^2+\varepsilon\left(d-a+d\varepsilon\eta\right)\left(\nabla V_j:\nabla V_j\right)+\varepsilon^2(-a)d\left(\nabla^2V_j:\nabla^2V_j\right)\right)\\
&\leq \varepsilon C\tilde{C}_{abcd}U_j\left(U_j\left(U_s+U_s^2\right)+c_j(t)(U_s^2+U_s^3\right),
\end{aligned} \tag{1.3.32}$$

where $\left(c_j\left(t\right)\right)_j$ is a sequence with $\left(2^{js}c_j\left(t\right)\right)_j\in\ell^r(\mathbb{Z})$, having norm 1 and $\tilde{C}_{abcd}$ is defined in (1.3.31). Actually for the sake of simplicity, from now on we will not carry on the distinction between constants that depend on the parameters a,b,c,d and the ones depending on the dimension n and regularity index s.

1.3.2 Another useful estimation

In [27], we showed that the non-cavitation hypothesis:

$$1+\varepsilon\eta_0\left(x\right)\geq\alpha>0, \tag{1.3.33}$$

is not essential for some of the values of the parameters. (although, all physically relevant data will verify (1.3.33) as $1+\varepsilon\eta$ represents the total height of the water over the flat bottom). The idea is that we can consider "a positive version" of the energy functional $N_s\left(\eta,V\right)$. We will recall how to derive such a functional although we will not be using it in the following sections. Let us stress out that the estimates established in this section are only available for the parameters a, b, c, d verifying that whenever $c<0$, then

$$sgn\left(b\right)+sgn\left(d\right)-sgn\left(a\right)\geq 2.$$

Let us give some details for obtaining. We will investigate the following quantity:

$$\partial_t\left(\frac{\varepsilon}{2}\left\|\eta\right\|_{L^\infty}U_j^2\right)=I_1+I_2,$$

where

$$2I_1=\varepsilon\left\|\eta\right\|_{L^\infty}\partial_tU_j^2 \text{ and } 2I_2=\varepsilon U_j^2\partial_t\left\|\eta\right\|_{L^\infty}.$$

Owing to (1.2.11) we see that:

$$\begin{aligned}
I_1&=\frac{\varepsilon\left\|\eta\right\|_{L^\infty}}{2}\left(T_1+T_2+T_3\right)+\frac{\varepsilon\left\|\eta\right\|_{L^\infty}}{2}\left(-\varepsilon\int\eta\eta_j\operatorname{div}V_j+c\varepsilon^2\int\eta\nabla\operatorname{div}V_j\nabla\eta_j\right)\\
&\leq C\varepsilon^2\left(U_j^2U_s^2+c_j\left(t\right)U_jU_s^3\right)+C\varepsilon^2\left\|\eta\right\|^2_{L^\infty}\left\|\eta_j\right\|_{L^2}\left\|\operatorname{div}V_j\right\|_{L^2}\\
&+C\varepsilon^3\left\|\eta\right\|^2_{L^\infty}\left\|\nabla\operatorname{div}V_j\right\|_{L^2}\left\|\nabla\eta_j\right\|_{L^2}
\end{aligned}$$

$$\leq C\varepsilon^2\left(U_j^2U_s^2 + c_j(t)U_jU_s^3\right) + C\varepsilon^{\frac{3}{2}}U_j^2U_s^2 + C\varepsilon^{\frac{3}{2}}U_j^2U_s^2$$
$$\leq C\varepsilon^2\left(U_j^2U_s^2 + c_j(t)U_jU_s^3\right) \tag{1.3.34}$$

Remark 1.3.1. *Let us notice that the term $c\varepsilon^2\int\eta\nabla\operatorname{div}V_j\nabla\eta_j$ raises some important issues. When $c<0$, in order to successfully estimate it, we need the following restriction on the parameters:*

$$sgn(b) + sgn(d) - sgn(a) \geq 2.$$

This ensures that one of the unknown functions η, V has at least $B_{2,r}^{s+2}$-regularity level while the other one has $B_{2,r}^{s+1}$-regularity level. Thus, the following cases are excluded:

$$\begin{cases} a=d=0,\ b>0,\ c<0;\ a=d=0,\ b>0,\ c<0. \\ a=b=d=0,\ c<0,\ b=d=0,\ a<0,\ c<0. \end{cases}$$

In order to handle I_2 we use the fact that the function $t\to\|\eta(t)\|_{L^\infty}$ is locally Lipschitz we get that a.e. $\partial_t\|\eta(t)\|_{L^\infty}$ exists and besides, a.e. in time we have

$$\begin{aligned}|\partial_t\|\eta(t)\|_{L^\infty}| &= \left|\lim_{s\to t}\frac{\|\eta(t)\|_{L^\infty}-\|\eta(s)\|_{L^\infty}}{t-s}\right| \leq \lim_{s\to t}\left\|\frac{\eta(t)-\eta(s)}{t-s}\right\|_{L^\infty} \\ &\leq \|\partial_t\eta(t)\|_{L^\infty} \leq \|\partial_t\eta(t)\|_{B^{\frac{n}{2}}_{2,1}}.\end{aligned}$$

Thus, owing to (1.3.20), (1.3.21) we get that:

$$I_2 \leq \frac{1}{2}\varepsilon U_j^2\partial_t\|\eta\|_{L^\infty} \leq C\varepsilon U_j^2\left(U_s+U_s^2\right). \tag{1.3.35}$$

Combining (1.3.34) and (1.3.35) we get that

$$\partial_t\left(\frac{\varepsilon}{2}\|\eta\|_{L^\infty}U_j^2\right) \leq C\varepsilon\left(U_j^2\left(U_s+U_s^2\right) + c_j(t)U_jU_s^3\right). \tag{1.3.36}$$

Finally let us observe that by adding (1.3.5) with (1.3.36) we obtain that:

$$\frac{1}{2}\frac{d}{dt}\left(\int(1+\varepsilon\|\eta\|_{L^\infty})\left\{\eta_j^2+\varepsilon(b-c)|\nabla\eta_j|^2+\varepsilon^2(-c)b\left(\nabla^2\eta_j:\nabla^2\eta_j\right)\right\}\right) \tag{1.3.37}$$
$$+\frac{1}{2}\frac{d}{dt}\left(\int(1+\varepsilon\eta+\varepsilon\|\eta\|_{L^\infty})V_j^2+\varepsilon(d-a+d\varepsilon\eta+d\varepsilon\|\eta\|_{L^\infty})(\nabla V_j:\nabla V_j)\right) \tag{1.3.38}$$
$$+\frac{1}{2}\frac{d}{dt}\int\left(\varepsilon^2(-a)d(1+\varepsilon\|\eta\|_{L^\infty})\left(\nabla^2V_j:\nabla^2V_j\right)\right) \tag{1.3.39}$$
$$\leq\varepsilon C_{abcd}\left(U_j^2\left(U_s+U_s^2\right)+c_j(t)U_j(U_s^2+U_s^3)\right). \tag{1.3.40}$$

1.4 The proof of Theorem 1.1.1

Once the estimates (1.3.32) are in hand we proceed with the construction of the solution announced in Theorem 1.1.1. A classical way to proceed would be to consider a Friedrichs regularization, i.e. solve the system[7]

$$\begin{cases}(I-\varepsilon b\Delta)\partial_t\eta+\mathbb{E}_m\operatorname{div}V+a\varepsilon\mathbb{E}_m\operatorname{div}\Delta V+\varepsilon\mathbb{E}_m(\operatorname{div}(\eta V))=0,\\(I-\varepsilon d\Delta)\partial_tV+\mathbb{E}_m\nabla\eta+c\varepsilon\mathbb{E}_m\nabla\Delta\eta+\varepsilon\mathbb{E}_m(V\cdot\nabla V)=0,\\\eta_{|t=0}=\mathbb{E}_m\eta_0\ ,\ V_{|t=0}=\mathbb{E}_mV_0.\end{cases} \tag{1.4.1}$$

[7]see (1.1.9) and (1.1.10) for the definitions of the operator $\mathbb{E}_m$ and the space L^2_m.

and use the estimates previously obtained in order to obtain uniform bounds of the solution. There is a technical detail that prevents us from adapting directly this method namely the energy functional appearing in (1.3.32) is nonlinear and thus we do not obtain this estimate for solutions of (1.4.1) proceeding as in the previous section. In order to bypass this problem, we will prove first that the following system is well-posed:

$$\begin{cases} (I-\varepsilon b\Delta)\,\partial_t\eta + \mathbb{E}_m \operatorname{div} V + a\varepsilon\mathbb{E}_m \operatorname{div}\Delta V + \varepsilon\mathbb{E}_m\left(\operatorname{div}(\eta V)\right) = 0, \\ (I-\varepsilon d\Delta)\,\partial_t V + \nabla\eta + c\varepsilon\nabla\Delta\eta + \varepsilon V\cdot\nabla V = 0, \\ \eta_{|t=0} = \eta_0\ ,\ V_{|t=0} = V_0. \end{cases} \tag{1.4.2}$$

with $(\eta_0, V_0) \in L^2_m \times \left(B^{s_{ad}}_{2,r}\right)^n$. Owing to the fact that $\mathbb{E}_m^2 = \mathbb{E}_m$, solutions of system (1.4.2) do verify the estimates (1.3.32). In a second time we will study the limit when $m\to\infty$ for the solutions of (1.4.2).

1.4.1 Approximate solutions of the *abcd* system

The main result of this section is summarized in the following:

Proposition 1.4.1. *Let a,b,c,d be chosen as in* (1.1.4)*, $r\in[0,\infty)$ and $s\in\mathbb{R}$ such that:*

$$s > \frac{n}{2}+1 \text{ or } s = \frac{n}{2}+1 \text{ and } r=1. \tag{1.4.3}$$

Furthermore, let us consider s_{bc} and s_{ad} defined by relation (1.1.5) *and let us fix $m\in\mathbb{N}$. Then, for all $(\eta_0,V_0)\in L^2_m\times\left(B^{s_{ad}}_{2,r}\right)^n$ there exists a positive $T>0$ and an unique solution*

$$(\eta, V) \in \mathcal{C}\left([0,T], L^2_m \times \left(B^{s_{ad}}_{2,r}\right)^n\right) \cap \mathcal{C}^1\left([0,T], L^2_m \times \left(H^{2\operatorname{sgn}(d)}\right)^n\right),$$

of the system (1.4.2)*. Moreover, if we denote by $T(\eta_0,V_0)$ the maximal time of existence then, if $T(\eta_0,V_0)<\infty$, we have that:*

$$\limsup_{t\to T(\eta_0,V_0)} U_s(t) = \infty. \tag{1.4.4}$$

For all $m,p\in\mathbb{N}$ such that $p\geq m\geq 1$, we consider the following differential equation on $L^2_m\times\left(L^2_p\right)^n$:

$$\begin{cases} \partial_t\eta = F_m(\eta,V), \\ \partial_t V = G_p(\eta,V), \\ \eta_{|t=0} = \mathbb{E}_m\eta_0,\ V_{|t=0} = \mathbb{E}_p V_0, \end{cases} \tag{1.4.5}$$

where $(F_m,G_p): L^2_m\times\left(L^2_p\right)^n \to L^2_m\times\left(L^2_p\right)^n$ are defined by:

$$F_m(\eta,V) = -\mathbb{E}_m\left((I-\varepsilon b\Delta)^{-1}\left[(I+a\varepsilon\Delta)\operatorname{div}V + \varepsilon\operatorname{div}(\eta V)\right]\right), \tag{1.4.6}$$

$$G_p(\eta,V) = -\mathbb{E}_p\left((I-\varepsilon d\Delta)^{-1}\left[(I+c\varepsilon\Delta)\nabla\eta + \varepsilon V\cdot\nabla V\right]\right). \tag{1.4.7}$$

It transpires that due to the equivalence of the Sobolev norm, (F_m,G_p) is continuous and locally Lipschitz on $L^2_m\times\left(L^2_p\right)^n$. Thus, the classical Picard theorem ensures that there exists a nonnegative time $T_{m,p}>0$ and a unique solution $(\eta^{m,p},V^{m,p}): \mathcal{C}^1\left([0,T_{m,p}), L^2_m\times\left(L^2_p\right)^n\right)$. Moreover, if $T_{m,p}$ is finite, then:

$$\lim_{t\to T_{m,p}} \|(\eta^{m,p},V^{m,p})\|_{L^2} = \infty.$$

Another important aspect is that because of the property $\mathbb{E}_m^2=\mathbb{E}_m$ we get that the identity obtained in (1.2.11) still holds true for $(\eta^{m,p},V^{m,p})$, namely

$$\frac{1}{2}\frac{d}{dt}\left(\int \left(\eta_j^{m,p}\right)^2 + \varepsilon(b-c)\left|\nabla\eta_j^{m,p}\right|^2 + \varepsilon^2(-c)b\left(\nabla^2\eta_j^{m,p}:\nabla^2\eta_j^{m,p}\right)\right) \tag{1.4.8}$$

$$+\frac{1}{2}\frac{d}{dt}\left(\int\left|V_j^{m,p}\right|^2+\varepsilon\left(d-a\right)\left(\nabla V_j^{m,p}:\nabla V_j^{m,p}\right)+\varepsilon^2(-a)d\left(\nabla^2 V_j^{m,p}:\nabla^2 V_j^{m,p}\right)\right)$$
$$+\varepsilon\int\eta^{m,p}\eta_j^{m,p}\operatorname{div}V_j^{m,p}-c\varepsilon^2\int\eta^{m,p}\nabla\operatorname{div}V_j^{m,p}\nabla\eta_j^{m,p}=T_1+T_2+T_3$$

with T_1, T_2, T_3 defined as in relations (1.2.4), (1.2.7), (1.2.10) but with $(\eta^{m,p}, V^{m,p})$ instead of (η, V). Also, we let $U_j^{m,p}=U_j\left(\eta^{m,p},V^{m,p}\right)$ and $U_s^{m,p}=U_s\left(\eta^{m,p},V^{m,p}\right)$, see (1.1.12) and (1.1.13). Taking advantage of the estimates (1.3.10), (1.3.12) and (1.3.17) we gather that:

$$T_1+T_2+T_3\leq\varepsilon CU_j^{m,p}\left(U_j^{m,p}U_s^{m,p}+c_j\left(t\right)\left(U_s^{m,p}\right)^2\right).$$

Let us notice that an integration by parts combined with Bernstein's lemma 1.1.1 yields:

$$-\varepsilon\int\eta^{m,p}\eta_j^{m,p}\operatorname{div}V_j^{m,p}\leq\varepsilon 2^m C\left(U_j^{m,p}\right)^2U_s^{m,p}.$$

Next, observe that

$$c\varepsilon^2\int\eta^{m,p}\nabla\operatorname{div}V_j^{m,p}\nabla\eta_j^{m,p}\leq\left(-c\right)\varepsilon^2 2^{3m}C\left(U_j^{m,p}\right)^2U_s^{m,p}.$$

We thus gather that:

$$\frac{1}{2}\frac{d\left(U_j^{m,p}\right)^2}{dt}\leq\varepsilon 2^{3m}C\left(\left(U_j^{m,p}\right)^2U_s^{m,p}+c_j\left(t\right)U_j^{m,p}\left(U_s^{m,p}\right)^2\right)$$

which implies that for all $t\in[0,T_{m,p})$:

$$U_j^{m,p}\left(t\right)\leq U_j^{m,p}\left(0\right)+\varepsilon 2^{3m}C\int_0^t\left(U_j^{m,p}U_s^{m,p}+c_j\left(t\right)\left(U_s^{m,p}\right)^2\right)d\tau,\tag{1.4.9}$$

thus multiplying with 2^{js} and performing an $\ell^r(\mathbb{Z})$-summation, owing to Minkowski's theorem, we get that:

$$U_s^{m,p}\left(t\right)\leq U_s^{m,p}\left(0\right)+\varepsilon 2^{3m}C\int_0^t\left(U_s^{m,p}\left(\tau\right)\right)^2d\tau.\tag{1.4.10}$$

The Gronwall's lemma along with the blow-up criterion for ODEs gives us that:

$$\frac{1}{\varepsilon C2^{3m}U_s\left(\eta_0,V_0\right)}\leq T_{m,p},$$

Moreover, for each time $T>0$ such that

$$\varepsilon TC2^{3m}U_s\left(\eta_0,V_0\right)<1,\tag{1.4.11}$$

we obtain due to Gronwall's lemma that for all $t\in[0,T]$

$$U_s^{m,p}\left(t\right)\leq\frac{1}{\frac{1}{U_s(\eta_0,V_0)}-\varepsilon 2^{3m}CT},\tag{1.4.12}$$

and thus, the solution $\left(\eta^{m,p},V^{m,p}\right)_{p\geq m}$ is uniformly bounded with respect to p in $L_T^\infty\left(L_m^2\times(B_{2,r}^{s_{ad}})^n\right)$. Next, we prove that there exists a time $T_m^\star>0$ such that $\left(\eta^{m,p},V^{m,p}\right)_{p\geq m}$ is a Cauchy sequence in $C_{T_m^\star}(L_m^2\times\left(H^{\operatorname{sgn}(d)}\right)^n)$. First of all, let us fix T_m by

$$2\varepsilon T_mC2^{3m}U_s\left(\eta_0,V_0\right)=1,\tag{1.4.13}$$

which, of course, verifies (1.4.11). Then, the uniform estimates (1.4.12) simplifies into

$$U_s^{m,p}(t) \leq 2U_s(\eta_0, V_0).$$

We consider $p \geq q \geq m$ and let us denote by:

$$(\delta\eta, \delta V) = (\eta^{m,p} - \eta^{m,q}, V^{m,p} - V^{m,q}),$$

where, for the sake of simplicity, we drop the dependence on p and q in the notation. Observe that $(\delta\eta, \delta V)$ verify the equation

$$\begin{cases} (I - b\varepsilon\Delta)\,\partial_t\delta\eta + (I + a\varepsilon\Delta)\,\mathbb{E}_m \operatorname{div}\delta V + \varepsilon\mathbb{E}_m\left(\operatorname{div}(\delta\eta V^{m,p}) + \operatorname{div}(\eta^{m,q}\delta V)\right) = 0, \\ (I - d\varepsilon\Delta)\,\partial_t\delta V + (I + c\varepsilon\Delta)\,\nabla\delta\eta + \varepsilon\mathbb{E}_p\left(V^{m,p}\cdot\nabla\delta V + \delta V\cdot\nabla V^{m,q} + (I - \mathbb{E}_q)\left(V^{m,q}\cdot\nabla V^{m,q}\right)\right) = 0. \end{cases} \tag{1.4.14}$$

Multiplying the first equation with $\delta\eta$, the second with δV, integrating in space and adding up the results yields:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int\left\{(\delta\eta)^2 + b\,|\nabla\delta\eta|^2 + (\delta V)^2 + d\,|\nabla\delta V|^2\right\}dx \\ &\leq \varepsilon\,(-a-c)\,2^{3m}\,\|\delta\eta\|_{L^2}\,\|\delta V\|_{L^2} + \varepsilon C\,\|\operatorname{div}V^{m,p}\|_{L^\infty}\,\|\delta\eta\|_{L^2}^2 + \varepsilon 2^m C\,\|\eta^{m,q}\|_{L^\infty}\,\|\delta\eta\|_{L^2}\,\|\delta V\|_{L^2} \\ &+ \varepsilon C\,\|\operatorname{div}V^{m,p}\|_{L^\infty}\,\|\delta V\|_{L^2}^2 + \varepsilon C\,\|\nabla V^{m,q}\|_{L^\infty}\,\|\delta V\|_{L^2}^2 + \|(\mathbb{E}_p - \mathbb{E}_q)\left(V^{m,q}\cdot\nabla V^{m,q}\right)\|_{L^2}\,\|\delta V\|_{L^2}. \end{aligned} \tag{1.4.15}$$

where the constants $2^m, 2^{3m}$ appearing above come from applying Bernstein's lemma. Next, we see that

$$\begin{aligned} \|(\mathbb{E}_p - \mathbb{E}_q)\left(V^{m,q}\cdot\nabla V^{m,q}\right)\|_{L^2} &\leq \sum_{-2+q\leq j\leq p}\|\Delta_j V^{m,q}\cdot\nabla V^{m,q}\|_{L^2} \leq C2^{-\frac{qn}{2}}\,\|V^{m,q}\cdot\nabla V^{m,q}\|_{B^{\frac{n}{2}}_{2,1}} \\ &\leq C2^{-\frac{qn}{2}}\,\|V^{m,q}\|^2_{B^{\frac{n}{2}+1}_{2,1}}. \end{aligned}$$

The last relation along with (1.4.12), (1.4.13) and (1.4.15) imply that for all $t \in [0, T_m]$ we have:

$$\begin{aligned} Y(t) &\leq C2^{-\frac{qn}{2}}tU_s^2(\eta_0, V_0) + C(2^{3m} + 2^m U_s(\eta_0, V_0))\int_0^t Y(\tau)\,d\tau \\ &\leq 2^{-\frac{qn}{2}}\frac{CU_s(\eta_0, V_0)}{\varepsilon 2^{3m}} + C(2^{3m} + 2^m U_s(\eta_0, V_0))\int_0^t Y(\tau)\,d\tau, \end{aligned}$$

where

$$Y(t) = \sup_{\tau\in[0,t]}\left(\|\delta\eta(\tau)\|_{L^2}^2 + \|\delta V(\tau)\|_{L^2}^2\right)^{\frac{1}{2}}.$$

Let us denote

$$M_{m,s} = \max\left(\frac{CU_s(\eta_0, V_0)}{\varepsilon 2^{3m}}, C(2^{3m} + 2^m U_s(\eta_0, V_0))\right). \tag{1.4.16}$$

Observe that by taking

$$T_m^\star = \min\left\{\frac{1}{2M_{m,s}}\ln\left(1 + \frac{1}{M_{m,s}}\right), T_m\right\}, \tag{1.4.17}$$

this ensures us that

$$\left(1 + \frac{2^{\frac{qn}{2}}}{M_{m,s}}\right)e^{-T_m^\star M_{m,s}} - 1 > 0. \tag{1.4.18}$$

Gronwall's lemma implies that for any $t \in [0, T_m^\star]$, one has:

$$Y(t) \leq \frac{1}{\left(1 + \frac{2^{\frac{qn}{2}T_m^\star}}{M_{m,s}}\right)e^{-T_m^\star M_{m,s}} - 1}.$$

Obviously, this implies that $(\eta^{m,p}, V^{m,p})_{p\geq m}$ is a Cauchy sequence in the space $C_{T_m^\star}(L^2_m \times (H^{2\,\mathrm{sgn}(d)})^n)$. Moreover, using interpolation and the uniform bounds (1.4.12) we see the sequence is in fact Cauchy in $C_{T_m^\star}(L^2_m \times (B^{s_{ad}-\gamma}_{2,r})^n)$ for any $\gamma > 0$. Also, using the equations (1.4.14), we see that $(\partial_t \eta^{m,p}, \partial_t V^{m,p})$ is a Cauchy sequence in the space $C_{T_m^\star}(L^2_m \times (H^{2\,\mathrm{sgn}(d)})^n)$. Therefore, we conclude the existence of a pair $(\eta^m, V^m) \in \bigcap\limits_{1>\gamma>0} C_{T_m^\star}(L^2_m \times (B^{s_{ad}-\gamma}_{2,r})^n) \cap L^\infty_{T_m^\star}(L^2_m \times (B^{s_{ad}}_{2,r})^n)$ such that

$$\begin{cases} (\eta^m, V^m) = \lim_{p\to\infty} (\eta^{m,p}, V^{m,p}) \text{ in } C_{T_m^\star}(L^2_m \times (B^{s_{ad}-\gamma}_{2,r})^n) \text{ and} \\ (\partial_t\eta^m, \partial_t V^m) = \lim_{p\to\infty} (\partial_t\eta^{m,p}, \partial_t V^{m,p}) \text{ in } C_{T_m^\star}(L^2_m \times \left((H^{2\,\mathrm{sgn}(d)})^n\right)^n), \end{cases}$$

for all $1 > \gamma > 0$. This implies that (η^m, V^m) is a solution, in the sense of distributions, of the system (1.4.2).

It remains to show that V^m is continuous in time with values in $B^{s_{ad}}_{2,r}$. As a consequence of the fact that $V^m \in \mathcal{C}_{T_m^\star}(B^{s_{ad}-\gamma}_{2,r})$ we obtain that $S_l V^m \in \mathcal{C}_{T_m^\star}(B^{s_{ad}}_{2,r})$ for all $l \in \mathbb{Z}$. The conclusion follows as the sequence of $B^{s_{ad}}_{2,r}$–valued functions $(S_j V^m)_{j\in\mathbb{Z}}$ tends uniformly to V^m. First, let us observe that proceeding as above (η^m, V^m) obeys the estimate (1.4.9):

$$U_j^m(t) \leq U_j^m(0) + \varepsilon 2^{3m} C \int_0^t \left(U_j^m U_s^m + c_j(t)(U_s^m)^2\right) d\tau, \tag{1.4.19}$$

and thus performing an $\ell^r(\mathbb{Z})$-summation for $j \geq l+2$, we obtain that

$$\begin{aligned} \left\|\left(2^{js} U_j^m(t)\,\mathbf{1}_{(j>l+2)}\right)\right\|_{\ell^r} &\leq \left\|\left(2^{js} U_j^m(0)\,\mathbf{1}_{(j>l+2)}\right)\right\|_{\ell^r} \\ +\varepsilon 2^{3m} C \int_0^t &\left(\left\|\left(2^{js} U_j^m(\tau)\,\mathbf{1}_{(j>l+2)}\right)\right\|_{\ell^r} U_s^m(\tau) + \left\|\left(2^{js} c_j(\tau)\,\mathbf{1}_{(j>l+2)}\right)\right\|_{\ell^r} (U_s^m(\tau))^2\right) d\tau. \end{aligned}$$

The result is obtained by combining the last inequality with the dominated convergence theorem and the following inequality:

$$\|V^m - S_l V^m\|_{B^{s_{ad}}_{2,r}} \leq \left\|\left(2^{js}\|\Delta_j V^m\|_{L^2}\,\mathbf{1}_{(j>l+2)}\right)\right\|_{\ell^r} \leq \left\|\left(2^{js} U_j^m(t)\,\mathbf{1}_{(j>l+2)}\right)\right\|_{\ell^r}.$$

Let us also mention that the estimate of (1.4.10) holds true for (η^m, V^m), i.e.:

$$U_s^m(t) \leq U_s^m(0) + \varepsilon 2^{3m} C \int_0^t (U_s^m(\tau))^2\, d\tau. \tag{1.4.20}$$

The uniqueness of such a solution follows from basic L^2-energy estimates.

Let us consider $T_m^{\max}$, the maximal time of existence of the solution (η^m, V^m) and let us assume that

$$T_m^{\max} < \infty. \tag{1.4.21}$$

We claim that

$$\limsup_{t\to T_m^{\max}} U_s(\eta^m(t), V^m(t)) = \infty. \tag{1.4.22}$$

The argument is standard, we assume by contradiction that there exists a constant $M > 0$ such that

$$U_s(\eta^m(t), V^m(t)) \leq M$$

for all $t \in [0, T_m^{\max})$. Of course, the above process of constructing a solution can be carried out with any $(\eta^m(t), V^m(t))$, $t \in [0, T_m^{\max})$ as initial data. We observe that the above uniform estimate enables us to find a positive lower bound which is independent of $t \in [0, T_m^{\max})$ and $U_s(\eta^m(t), V^m(t))$ for the times appearing in (1.4.13), (1.4.17). Thus, for t sufficiently close to $T_m^{\max}$ we can reiterate the above process in order to produce a solution having the required regularity on a time interval that is strictly larger than $T_m^{\max} - t$ which is in contradiction with the maximal character of $T_m^{\max}$. This concludes the proof of Proposition 1.4.1.

1.4.2 The proof of Theorem 1.1.1

Having established the existence of solutions for system (1.4.2) we proceed with the proof of Theorem 1.1.1. Thus, let us consider $(\eta_0, V_0) \in B^{s_{bc}}_{2,r} \times \left(B^{s_{ad}}_{2,r}\right)^n$. For any $m \in \mathbb{N}$, let us consider $(\eta_m, V_m) \in C_{T_m}\left(L^2_m \times \left(B^{s_{ad}}_{2,r}\right)^n\right)$ the solution of (1.4.2) with initial data $(\mathbb{E}_m\eta_0, V_0)$. Proceeding as in Section 1.2 we deduce that:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\int \left(\eta_j^m\right)^2 + \varepsilon\left(b-c\right)\left|\nabla\eta_j^m\right|^2 + \varepsilon^2(-c)b\left(\nabla^2\eta_j^m : \nabla^2\eta_j^m\right)\right)\\
&+\frac{1}{2}\frac{d}{dt}\left(\int (1+\varepsilon\eta^m)\left|V_j^m\right|^2 + \varepsilon\left(d-a+d\varepsilon\eta^m\right)\left(\nabla V_j^m : \nabla V_j^m\right) + \varepsilon^2(-a)d\left(\nabla^2 V_j^m : \nabla^2 V_j^m\right)\right)\\
&\leq \varepsilon C U_j^m\left(U_j^m\left(U_s^m + (U_s^m)^2\right) + c_j(t)((U_s^m)^2 + (U_s^m)^3\right). \qquad (1.4.23)
\end{aligned}$$

where C is a constant that depends on s, n, a, b, c, d but is independent of ε. Let us consider

$$T_m = \sup\left\{T \in (0, T_m^{\max}] : \sup_{\tau\in[0,T]} U_s^m(\tau) \leq \alpha U_s(0)\right\},$$

where α will be chosen in the following. Let us consider C_0 the constant of the embedding $B^{\frac{n}{2}}_{2,1} \hookrightarrow L^\infty$ which depends only on the dimension n. We may suppose that ε is sufficiently small such that

$$\varepsilon\alpha C_0 U_s(0) < K < 1,$$

for a certain K that will be specified in the following. Then, for all $t \in [0, T_m]$, we have

$$1 - K \leq 1 + \varepsilon\eta^m \leq 1 + K.$$

Using the notation introduced in (1.1.15), we get that

$$(1-K)^{\frac{1}{2}} U_s(\eta^m, V^m) \leq N_s(\eta^m, V^m) \leq (1+K)^{\frac{1}{2}} U_s(\eta^m, V^m), \qquad (1.4.24)$$

and

$$(1-K)^{\frac{1}{2}} U_j(\eta^m, V^m) \leq N_j(\eta^m, V^m) \leq (1+K)^{\frac{1}{2}} U_j(\eta^m, V^m). \qquad (1.4.25)$$

Using the last relation and (1.4.23) we get that

$$N_j^m(t) \leq N_j^m(0) + \varepsilon C(1+K)^{\frac{1}{2}}\int_0^t \left(U_j^m\left(U_s^m + (U_s^m)^2\right) + c_j(t)((U_s^m)^2 + (U_s^m)^3\right) d\tau,$$

which implies that

$$U_j^m(t) \leq \sqrt{\frac{1+K}{1-K}} U_j^m(0) + \varepsilon C\sqrt{\frac{1+K}{1-K}}\int_0^t \left(U_j^m\left(U_s^m + (U_s^m)^2\right) + c_j(t)((U_s^m)^2 + (U_s^m)^3\right) d\tau,$$

for all $t \in [0, T_m]$. Multiplying by 2^{js} and performing an $\ell^r(\mathbb{Z})$ summation, we obtain that:

$$U_s^m(t) \leq \sqrt{\frac{1+K}{1-K}} U_s(0) + \varepsilon C\sqrt{\frac{1+K}{1-K}}\int_0^t (U_s^m)^2 + (U_s^m)^3\, d\tau$$

$$\leq \sqrt{\frac{1+K}{1-K}}U_s(0) + \varepsilon\alpha C\sqrt{\frac{1+K}{1-K}}\left(U_s(0) + \alpha U_s^2(0)\right)\int_0^t U_s^m d\tau,$$

which implies, by Gronwall's lemma that

$$U_s^m(t) \leq \sqrt{\frac{1+K}{1-K}}U_s(0)\exp\left(\varepsilon\alpha T_m C\sqrt{\frac{1+K}{1-K}}\left(U_s(0) + \alpha U_s^2(0)\right)\right) \tag{1.4.26}$$

for all $t \in [0, T_m]$. If the right hand side of (1.4.26) is strictly smaller than $\alpha U_s(0)$ then, a continuity argument gives us

$$T_m = T_m^{\max}.$$

The fact that the solution is bounded on the maximal time of existence implies that $T_m^{\max} = \infty$. Let us suppose that this is not the case. Then, we have that

$$\varepsilon T_m \geq \frac{\ln\left(\alpha\sqrt{\frac{1-K}{1+K}}\right)}{\alpha C\sqrt{\frac{1+K}{1-K}}\left(U_s(0) + \alpha U_s^2(0)\right)}.$$

We may chose, for example $\alpha = 4$, and $K = \frac{16-e^2}{16+e^2}$ in order to transform the last inequality into

$$\varepsilon T_m \geq \frac{e}{16C\left(U_s(0) + \alpha U_s^2(0)\right)} :\overset{not.}{=} T(\eta_0, V_0).$$

This shows that the time of existence of solutions of (1.4.2) is bounded below by a quantity that is independent of m and that is of order ε^{-1}. In the following, we denote $T_\varepsilon = \varepsilon^{-1}T(\eta_0, V_0)$. The sequence of solutions $(\eta_m, V_m)_{m\in\mathbb{N}}$ is uniformly bounded in $C_{T_\varepsilon}\left(B_{2,r}^{s_{bc}} \times \left(B_{2,r}^{s_{ad}}\right)^n\right)$ i.e.

$$U_s(\eta^m(t), V^m(t)) \leq 4U_s(\eta_0, V_0) \tag{1.4.27}$$

for all $t \in [0, T_\varepsilon]$ Also, from the fact that (η^m, V^m) is a solution of we deduce that:

$$\begin{aligned}
&\int_0^t \langle \eta^m, (I - \varepsilon b\Delta)\partial_t\phi\rangle_{L^2} + \int_0^t \langle V^m, (I + a\varepsilon\Delta)\mathbb{E}_m\nabla\phi\rangle_{L^2} + \varepsilon\int_0^t \langle \eta^m V^m, \mathbb{E}_m\nabla\phi\rangle_{L^2}\\
&= \langle \eta^m(t), (I - \varepsilon b\Delta)\phi(t)\rangle_{L^2} - \langle \eta_0, (I - \varepsilon b\Delta)\phi(0)\rangle_{L^2},
\end{aligned} \tag{1.4.28}$$

respectively

$$\begin{aligned}
&\int_0^t \langle V^m, (I - \varepsilon d\Delta)\partial_t\psi\rangle_{L^2} + \int_0^t \langle \eta^m, (I + c\varepsilon\Delta)\operatorname{div}\psi\rangle_{L^2} + \varepsilon\int_0^t \langle V^m\cdot\nabla V^m, \operatorname{div}\psi\rangle_{L^2}\\
&= \langle V^m(t), (I - d\varepsilon\Delta)\psi(t)\rangle_{L^2} - \langle V_0, (I - d\varepsilon\Delta)\psi(0)\rangle_{L^2},
\end{aligned} \tag{1.4.29}$$

for all $(\phi, \psi) \in \mathcal{C}^1([0, T_\varepsilon], \mathcal{S}\times\mathcal{S}^n)$. We observe that $(\partial_t\eta^m)_{m\in\mathbb{N}}$ is uniformly bounded on $[0, T_\varepsilon]$ in $B_{2,1}^{\frac{n}{2}}$. Indeed, the first term is

$$(I - b\varepsilon\Delta)^{-1}(I + a\varepsilon\Delta)\operatorname{div}V^m \in B_{2,r}^{s_4}$$

with

$$s_4 = s + 2sgn(b) + sgn(d) + sgn(a) - 1 \geq \frac{n}{2},$$

with equality if $r = 1$. Using that $B_{2,1}^{\frac{n}{2}}$ is an algebra, the term $V^m\cdot\nabla V^m$ belongs to $B_{2,1}^{\frac{n}{2}}$. Thus, in view of the uniform estimates (1.4.27), we get that

$$\left(\|\partial_t\eta^m(t)\|_{B_{2,1}^{\frac{n}{2}}}\right)_{m\in\mathbb{N}}$$

is uniformly bounded on $[0, T_\varepsilon]$. Next, considering for all $p \in \mathbb{N}$ a smooth function ϕ_p such that:

$$\begin{cases} Supp\,(\phi_p) \subset B\,(0, p+1)\,, \\ \phi_p = 1 \text{ on } B\,(0,p)\,, \end{cases}$$

it follows, in view of Proposition 1.2.4 that for each $p \in \mathbb{N}$, the sequence $(\phi_p \eta^m)_{m\in\mathbb{N}}$ viewed as

$$\phi_p \eta^m : [0, T_\varepsilon] \to B^{\frac{n}{2}}_{2,1}$$

is uniformly equicontiniuous on $[0, T_\varepsilon]$ and that for all $t \in [0, T_\varepsilon]$, the set $\{\phi_p \eta^m(t) : m \in \mathbb{N}\}$ is relatively compact in $B^{\frac{n}{2}}_{2,1}$. Thus, the Ascoli-Arzela Theorem combined with Cantor's diagonal process provides us a subsequence of $(\eta^m)_{m\in\mathbb{N}}$ [8]and a tempered distribution $\eta \in \mathcal{C}\left([0, T_\varepsilon], \mathcal{S}'\right)$ such that for all $\phi \in \mathcal{D}\left(\mathbb{R}^n\right)$:

$$\phi\eta^m \to \phi\eta \text{ in } \mathcal{C}\left([0, T_\varepsilon], B^{\frac{n}{2}}_{2,1}\right).$$

Moreover, owing to Proposition 1.2.2 and (1.4.12) we get that $\eta \in L^\infty_{T_\varepsilon}\left(B^{s_{bc}}_{2,r}\right)$ and using interpolation we get that

$$\phi\eta^m \to \phi\eta \text{ in } \mathcal{C}\left([0, T_\varepsilon], B^{s_{bc}-\gamma}_{2,1}\right)$$

for all $1 > \gamma > 0$. Of course, by the same argument one can get $V \in L^\infty_{T_\varepsilon}\left(\left(B^{s_{ad}}_{2,r}\right)^n\right)$ such that for all $\psi \in \left(\mathcal{D}\left(\mathbb{R}^n\right)\right)^n$:

$$\psi V^m \to \phi V \text{ in } \mathcal{C}\left([0, T_\varepsilon], \left(B^{s_{ad}-\gamma}_{2,1}\right)^n\right)$$

for all $1 > \gamma > 0$. We claim that the properties enlisted above allow us to pass to the limit when $m \to \infty$ in the equations (1.4.28) and (1.4.29) verified by η^m and V^m. It remains only to verify that $(\eta, V) \in C_{T_\varepsilon}(B^{s_{bc}}_{2,r} \times \left(B^{s_{ad}}_{2,r}\right)^n)$. As in the case of Proposition 1.4.1, this a consequence of the fact that $(S_l\eta, S_l V) \in C_{T_\varepsilon}(B^{s_{bc}}_{2,r} \times \left(B^{s_{ad}}_{2,r}\right)^n)$ and

$$(S_l\eta, S_l V) \to (\eta, V) \text{ in } L^\infty_{T_\varepsilon}(B^{s_{bc}}_{2,r} \times \left(B^{s_{ad}}_{2,r}\right)^n).$$

This completes the proof of existence.

Uniqueness, is a consequence of the following stability estimate: let us consider two solutions of (1.1.1), $\left(\eta^1, V^1\right)$, $\left(\eta^2, V^2\right)$ and observe that the difference

$$(\delta\eta, \delta V) = \left(\eta^1 - \eta^2, V^1 - V^2\right)$$

satisfies the following system:

$$\begin{cases} (I - \varepsilon b\Delta)\,\partial_t \delta\eta + \operatorname{div}\delta V + a\varepsilon \operatorname{div}\Delta\delta V + \varepsilon \operatorname{div}\left(\delta\eta V^1\right) + \varepsilon \operatorname{div}\left(\eta^2 \delta V\right) = 0 \\ (I - \varepsilon d\Delta)\,\partial_t \delta V + \nabla\delta\eta + c\varepsilon\nabla\Delta\delta\eta + \varepsilon V^1 \cdot \nabla(\delta V) + \varepsilon\delta V \cdot \nabla V^2 = 0 \\ \delta\eta_{|t=0} = 0,\ \delta V_{|t=0} = 0. \end{cases} \tag{1.4.30}$$

We consider $U^1_s = U_s\left(\eta^1, V^1\right)$ and $U^2_s = U_s\left(\eta^2, V^2\right)$, see (1.1.13). For the sake of simplicity we will prove stability estimates in the classical Sobolev space $X = H^{r_{bc}} \times \left(H^{r_{ad}}\right)^n$ with:

$$\begin{cases} r_{bc} = sgn\,(b) - sgn\,(c)\,, \\ r_{ad} = sgn\,(d) - sgn\,(a)\,. \end{cases}$$

We endow X with the norm:

$$\|\eta, V\|^2_X = \|\eta\|^2_{L^2} + \varepsilon\,(b - c)\,\|\nabla\eta\|^2_{L^2} - \varepsilon^2 bc\left\|\nabla^2\eta\right\|^2_{L^2}$$

[8]Still denoted $(\eta^m)_{m\in\mathbb{N}}$ for the sake of simplicity.

$$+\|V\|_{L^2}^2+\varepsilon(d-a)\|\nabla V\|_{L^2}^2-\varepsilon^2 da\left\|\nabla^2 V\right\|_{L^2}^2$$

Observe that due to the fact that s_{bc}, s_{ad} are chosen so as to satisfy (1.1.5) with s chosen as in (1.4.3), we have that $H^{r_{bc}}\times\left(H^{r_{ad}}\right)^n$ is continuously embedded in $B^{s_{bc}}_{2,r}\times\left(B^{s_{ad}}_{2,r}\right)^n$. Multiplying the first equation in (1.4.30) with $(I+c\varepsilon\Delta)\,\delta\eta$, the second equation with $(I+a\varepsilon\Delta)\,\delta V$ and adding up the results we get that:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left(\|\delta\eta\|_{L^2}^2+\varepsilon(b-c)\|\nabla\delta\eta\|_{L^2}^2-\varepsilon^2 bc\left\|\nabla^2\delta\eta\right\|_{L^2}^2+\|\delta V\|_{L^2}^2+\varepsilon(d-a)\|\nabla\delta V\|_{L^2}^2-\varepsilon^2 da\left\|\nabla^2\delta V\right\|_{L^2}^2\right)\\&\le\varepsilon\int\operatorname{div}\left(\delta\eta V^1\right)(I+c\varepsilon\Delta)\,\delta\eta+\varepsilon\int\operatorname{div}\left(\eta^2\delta V\right)(I+c\varepsilon\Delta)\,\delta\eta\\&+\varepsilon\int V^1\cdot\nabla(\delta V)\,(I+a\varepsilon\Delta)\,\delta V+\varepsilon\int\delta V\cdot\nabla V^2\,(I+a\varepsilon\Delta)\,\delta V.\end{aligned}\tag{1.4.31}$$

The first term of the right hand side is treated as follows. Suppose first that $b>0$ and $c<0$. Then, we have:

$$\begin{aligned}\varepsilon\int\operatorname{div}\left(\delta\eta V^1\right)(I+c\varepsilon\Delta)\,\delta\eta&\le\varepsilon\left\|\operatorname{div}V^1\right\|_{L^\infty}\|\delta\eta\|_{L^2}^2-c\varepsilon^2\|\operatorname{div}(\delta\eta V)\|_{L^2}\|\Delta\delta\eta\|_{L^2}\\&\le C\left(\left\|V^1\right\|_{L^\infty}+\left\|\nabla V^1\right\|_{L^\infty}\right)\|\delta\eta,\delta V\|_X^2.\\&\le CU_s\left(\eta^1,V^1\right)\|\delta\eta,\delta V\|_X^2.\end{aligned}\tag{1.4.32}$$

If $b=0$ and $c<0$ then $d-a>0$ which implies that

$$s_{bc}\ge\frac{n}{2}+2,$$

with equality if $r=1$. Then, we write that

$$\begin{aligned}\int\operatorname{div}\left(\delta\eta V^1\right)\Delta\delta\eta&=-\left\langle\nabla\operatorname{div}\left(\delta\eta V^1\right),\nabla\delta\eta\right\rangle\\&=-\left\langle\nabla^2\delta\eta V^1+\nabla V^1\nabla\delta\eta+\nabla\delta\eta\operatorname{div}V^1+\delta\eta\nabla\operatorname{div}V^1,\nabla\delta\eta\right\rangle\\&=-\frac{1}{2}\left\langle\operatorname{div}V^1\nabla\delta\eta,\nabla\delta\eta\right\rangle-\left\langle\nabla V^1\nabla\delta\eta,\nabla\delta\eta\right\rangle-\left\langle\delta\eta\nabla\operatorname{div}V^1,\nabla\delta\eta\right\rangle.\end{aligned}$$

Finally, we observe that

$$\begin{aligned}\varepsilon\int\operatorname{div}\left(\delta\eta V^1\right)(I+c\varepsilon\Delta)\,\delta\eta&\le\left(\left\|\operatorname{div}V^1\right\|_{L^\infty}+\left\|\nabla\operatorname{div}V^1\right\|\right)\|\delta\eta,\delta V\|_X^2\\&\le CU_s\left(\eta^1,V^1\right)\|\delta\eta,\delta V\|_X^2.\end{aligned}\tag{1.4.33}$$

In the following we use that $a<0$ implies that

$$s_{bc}\ge\frac{n}{2}+2,$$

with equality if $r=1$. We begin with:

$$\begin{aligned}&\varepsilon\int V^1\cdot\nabla(\delta V)\,(I+a\varepsilon\Delta)\,\delta V\\&\le\frac{\varepsilon}{2}\left\|\nabla V^1\right\|_{L^\infty}\|\delta V\|_{L^2}^2+(-a)\varepsilon^2\left\|\nabla V^1\right\|_{L^\infty}\|\delta V\|_{L^2}\|\nabla\delta V\|_{L^2}+\frac{-a\varepsilon^2}{2}\left\|\operatorname{div}V^1\right\|_{L^\infty}\|\nabla\delta V\|_{L^2}^2\\&\le CU_s\left(\eta^1,V^1\right)\|\delta\eta,\delta V\|_X^2.\end{aligned}\tag{1.4.34}$$

Next, we compute

$$\varepsilon\int\delta V\cdot\nabla V^2\,(I+a\varepsilon\Delta)\,\delta V=\varepsilon\int\delta V\cdot\nabla V^2\delta V+a\varepsilon^2\int\delta V\cdot\nabla V^2\Delta\delta V$$

$$
\begin{aligned}
&= \varepsilon \int \delta V \cdot \nabla V^2 \delta V - a\varepsilon^2 \int \nabla \left(\delta V \cdot \nabla V^2\right) : \nabla \delta V \\
&\leq \left\| \nabla V^2 \right\|_{L^\infty} \|\delta V\|_{L^2}^2 - a\varepsilon^2 \left\| \nabla V^2 \right\|_{L^\infty} \|\nabla \delta V\|_{L^2}^2 \\
&- a\varepsilon^2 \left\| \nabla^2 V^2 \right\|_{L^\infty} \|\delta V\|_{L^2} \|\nabla \delta V\|_{L^2} \\
&\leq C U_s \left(\eta^2, V^2\right) \|\delta\eta, \delta V\|_X^2 .
\end{aligned} \tag{1.4.35}
$$

The remaining term is a bit more delicate to treat. First, it is easy to show that

$$
\varepsilon \int \operatorname{div}\left(\eta^2 \delta V\right) \delta\eta \leq C U_s \left(\eta^2, V^2\right) \|\delta\eta, \delta V\|_X^2 .
$$

Next, if $c < 0$, then supposing that

$$
\operatorname{sgn}(b) + \operatorname{sgn}(d) + \operatorname{sgn}(-a) \geq 2, \tag{1.4.36}
$$

eventually integrating by parts, we may prove that

$$
c\varepsilon^2 \int \operatorname{div}\left(\eta^2 \delta V\right) \Delta\delta\eta \leq C U_s \left(\eta^2, V^2\right) \|\delta\eta, \delta V\|_X^2 . \tag{1.4.37}
$$

The estimates (1.4.32), (1.4.33), (1.4.34) and (1.4.35) yield:

$$
\|(\delta\eta(t), \delta V(t))\|_X^2 \leq C \sup_{\tau \in [0,t]} \left\{ U_s\left(\eta^1(\tau), V^1(\tau)\right) + U_s\left(\eta^2(\tau), V^2(\tau)\right) \right\} \int_0^t \|\delta\eta(\tau), \delta V(\tau)\|_X^2 \, d\tau.
$$

Hence, Gronwall's lemma ensures the desired result.
If (1.4.36) is not true then, let us multiply the second equation of (1.4.30) with $\varepsilon\eta^2\delta V$, integrate in space and add the result to (1.4.31) in order to conclude that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\delta\eta\|_{L^2}^2 + \varepsilon(b-c)\|\nabla\delta\eta\|_{L^2}^2 - \varepsilon^2 bc \left\|\nabla^2\delta\eta\right\|_{L^2}^2 \right) \\
&+\frac{1}{2}\frac{d}{dt}\left(\int \left(\left(1+\varepsilon\eta^2\right)|\delta V|^2 + \varepsilon\left(d\left(1+\varepsilon\eta^2\right) - a\right)\nabla\delta V : \nabla\delta V - \varepsilon^2 da \nabla^2\delta V : \nabla^2\delta V \right) \right) \\
&\leq \varepsilon \int \operatorname{div}\left(\delta\eta V^1\right)(I + c\varepsilon\Delta)\delta\eta + \varepsilon \int V^1 \cdot \nabla(\delta V)\left(I + \varepsilon\eta^2 + a\varepsilon\Delta\right)\delta V \\
&+\varepsilon \int \delta V \cdot \nabla V^2 \left(I + \varepsilon\eta^2 + a\varepsilon\Delta\right)\delta V + \frac{\varepsilon}{2}\int \partial_t\eta^2 |\delta V|^2 \qquad (1.4.38) \\
&+\frac{\varepsilon d}{2}\int \partial_t\eta^2 \nabla\delta V : \nabla\delta V - \varepsilon d \int \partial_t \nabla\delta V : \nabla\eta^2 \otimes \delta V. \qquad (1.4.39)
\end{aligned}
$$

The first three terms of the right hand side of the above inequality are treated exactly as in (1.4.32), (1.4.33), (1.4.34) and (1.4.35). Next, we know thanks to relations (1.3.20)-(1.3.24) that

$$
\left\|\partial_t\eta^2\right\|_{L^\infty} \leq C\left(U_s\left(\eta^2, V^2\right) + \left[U_s\left(\eta^2, V^2\right)\right]^2 \right)
$$

and thus we get that

$$
\frac{\varepsilon}{2}\int \partial_t\eta^2 |\delta V|^2 + \frac{\varepsilon d}{2}\int \partial_t\eta^2 \nabla\delta V : \nabla\delta V \leq C\left(U_s\left(\eta^2, V^2\right) + \left[U_s\left(\eta^2, V^2\right)\right]^2 \right)\|\delta\eta, \delta V\|_X^2 . \tag{1.4.40}
$$

The last term of (1.4.38) is treated as follows:

$$
\varepsilon^2 d \int \partial_t \nabla\delta V : \nabla\eta^2 \otimes \delta V = -\varepsilon^2 d \int \partial_t \delta V : \nabla\left(\nabla\eta^2 \otimes \delta V\right)
$$

$$\leq \varepsilon^2 d \left\|\partial_t \delta V\right\|_{L^2} \left\|\nabla^2\eta^2\right\|_{L^\infty} \left\|\delta V\right\|_{L^2} + \varepsilon^2 d \left\|\partial_t \delta V\right\|_{L^2} \left\|\nabla\eta^2\right\|_{L^\infty} \left\|\nabla\delta V\right\|_{L^2}$$
$$\leq CU_s\left(\eta^2, V^2\right) \left\|\partial_t \delta V\right\|_{L^2} \left\|\delta\eta, \delta V\right\|_X .$$

The time derivative is treated in the following lines

$$\begin{aligned}\left\|\partial_t \delta V\right\|_{L^2} &= \left\|(I-\varepsilon d\Delta)^{-1}(I+c\varepsilon\Delta)\nabla\delta\eta + \varepsilon(I-\varepsilon d\Delta)^{-1}\left(V^1\cdot\nabla(\delta V)+\delta V\cdot\nabla V^2\right)\right\|_{L^2}\\ &\leq C\left\|\nabla\delta\eta\right\|_{L^2} + C\left\|V^1\right\|_{L^\infty}\left\|\nabla\delta V\right\|_{L^2} + C\left\|\nabla V^2\right\|_{L^\infty}\left\|\delta V\right\|_{L^2}\\ &\leq C\left(1+U_s\left(\eta^1,V^1\right)+U_s\left(\eta^2,V^2\right)\right)\left\|\delta\eta,\delta V\right\|_X .\end{aligned}$$

Combining the last two relations yields

$$\varepsilon^2 d\int \partial_t\nabla\delta V : \nabla\eta^2\otimes\delta V \leq \left(U_s\left(\eta^2,V^2\right)+U_s^2\left(\eta^1,V^1\right)+U_s^2\left(\eta^2,V^2\right)\right)\left\|\delta\eta,\delta V\right\|_X^2 . \tag{1.4.41}$$

The estimates (1.4.32), (1.4.33), (1.4.34), (1.4.35), (1.4.40) respectively (1.4.41) yield

$$\begin{gathered}\left\|\delta\eta\right\|_{L^2}^2 + \varepsilon(b-c)\left\|\nabla\delta\eta\right\|_{L^2}^2 - \varepsilon^2 bc\left\|\nabla^2\delta\eta\right\|_{L^2}^2 + \\ +\int\left(\left(1+\varepsilon\eta^2\right)|\delta V|^2 + \varepsilon\left(d\left(1+\varepsilon\eta^2\right)-a\right)\nabla\delta V:\nabla\delta V - \varepsilon^2 da\nabla^2\delta V:\nabla^2\delta V\right)\\ \leq C\sup_{\tau\in[0,t]}\left\{P\left(U_s\left(\eta^1(\tau),V^1(\tau)\right)\right)+P\left(U_s\left(\eta^2(\tau),V^2(\tau)\right)\right)\right\}\int_0^t\left\|\delta\eta(\tau),\delta V(\tau)\right\|_X^2 d\tau,\end{gathered}$$

where $P(x) = x + x^2$. This concludes the proof of the announced result.

1.4.3 The *abcd* systems with non flat topography

As we have mentioned in the introduction, approximate models taking in account general topographies of the bottom have been established in [33]. In the case of small variations with respect to a reference plane i.e. the bottom is given by the surface:

$$\{(x,y,z) : z = -h + \varepsilon S(x,y)\},$$

where S is smooth enough, these models are not very different from those we studied above:

$$\begin{cases}(I-\varepsilon b\Delta)\partial_t\eta + \operatorname{div} V + a\varepsilon\operatorname{div}\Delta V + \varepsilon\operatorname{div}((\eta-S)V) = 0,\\ (I-\varepsilon d\Delta)\partial_t V + \nabla\eta + c\varepsilon\nabla\Delta\eta + \varepsilon V\cdot\nabla V = 0,\\ \eta_{|t=0} = \eta_0,\ V_{|t=0} = V_0.\end{cases} \tag{1.4.42}$$

We claim that our method applies to this system with some minor modifications. Let us give a few details. By localizing the above equation in Fourier space and proceeding in the same manner as in Section 1.2 we can see that the terms preventing us from establishing long time existence are

$$\varepsilon\int(\eta-S)\eta_j\operatorname{div} V_j \text{ and } -c\varepsilon^2\int(\eta-S)\nabla\operatorname{div} V_j\nabla\eta_j .$$

In order to repair this inconvenience we multiply the localized equation of V with $(\eta - S)V_j$ rather than simply ηV_j. Thus, we can obtain an estimate similar to (1.3.32), namely:

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\left(\int\eta_j^2 + \varepsilon(b-c)|\nabla\eta_j|^2 + \varepsilon^2(-c)b\left(\nabla^2\eta_j:\nabla^2\eta_j\right)\right)\\ +\frac{1}{2}\frac{d}{dt}\left(\int(1+\varepsilon(\eta-S))V_j^2 + \varepsilon(d-a+d\varepsilon(\eta-S))(\nabla V_j:\nabla V_j) + \varepsilon^2(-a)d\left(\nabla^2 V_j:\nabla^2 V_j\right)\right)\\ \leq \varepsilon G(U_s)\end{gathered}$$

where G should be some polynomial function with coefficients not depending on ε. Afterwards proceeding like above, provided that $S \in B_{2,r}^{\tilde{s}}$ with some $\tilde{s}$ large enough, one can obtain long time existence results for (1.4.42) similar to those obtained in Theorem 1.1.1.

Chapter 2

Long time existence results for bore type initial data

2.1 The main results

Bores are water waves resulting from the interaction of a river with the flux of a tide. Physically, they appear as a sudden rise of the water level which is propagating against the river's current. In the case of the whole space $\mathbb{R}^n$, $n \in \{1, 2\}$, bores are modeled by functions that manifest a non trivial behavior at infinity. Recently, Bona, Colin and Guillopé, [17] studied the Cauchy problem associated to the BBM-BBM type system, $a = c = 0$ with initial data that can be used to model bores. More precisely, they establish well-posedness results for data that are continuous and bounded. They show that if there exists some discernible structure at infinity at the initial time:

$$\lim_{y\to\pm\infty} \eta_0(x, y) = \eta_\pm(x), \quad \lim_{y\to\pm\infty} V_0(x, y) = (v_\pm(x), 0),$$

then it can be transported in time, i.e. for all time $t > 0$ where the solution is defined, there exists a limit function

$$\lim_{y\to\pm\infty} \eta(t, x, y) = \eta_\pm(t, x), \quad \lim_{y\to\pm\infty} V(t, x, y) = (v_\pm(t, x), 0)$$

which turns out to be the solution of the one dimensional system.

In this chapter we wish to supplement the results of the previous cited paper with a long time existence theory for all the BBM-type systems. We recall that the long time existence problem has been studied for initial data belonging to Sobolev function spaces H^s with the index of regularity satisfying (at least):

$$s > \frac{n}{2} + 1.$$

This corresponds to initial perturbation of the rest state that are essentially localized in the space variables. Indeed, when s is chosen as above, H^s is embedded in $\mathcal{C}_0$ the class of continuous bounded functions which vanish at infinity. Of course, if we aim at studying bore propagation, we need to change the functional setting of the initial data. This is exactly this issue that we address in the following. As far as we know, these are the first l.t.e. results for data that are outside the Sobolev functional setting. More precisely, the methods that we employ here will allow us to construct solutions of the BBM-type systems, i.e. we take $a = c = 0$, which live on time intervals of order $O\left(\varepsilon^{-1}\right)$ where, for the 1-dimensional case, we might consider general disturbances modeled by continuous functions[1] η_0^{1D} such that:

$$\lim_{x\to\pm\infty} \eta_0^{1D}(x) = \eta_\pm.$$

[1] with some extra regularity on its derivative

In the two dimensional setting, the situation in view is that of a 2-dimensional perturbation of the essentially 1-dimensional situation, more precisely we consider:

$$\begin{cases} \eta_0^{2D}(x,y) = \eta_0^{1D}(x) + \phi(x,y), \\ V_0^{2D}(x,y) = \left(u_0^{1D}(x) + \psi_1(x,y), \psi_2(x,y)\right) \end{cases}$$

where we ask

$$\lim_{|(x,y)|\to\infty} |\phi(x,y)| = 0 \text{ and } \lim_{|(x,y)|\to\infty} |\psi_i(x,y)| = 0 \text{ for } i = 1,2.$$

Let us formulate all of the above in a more precise manner. In the present chapter we will focus our attention only on the so called BBM-type Boussinesq systems of equations:

$$\begin{cases} (I - \varepsilon b\Delta)\,\partial_t\bar{\eta} + \operatorname{div}\bar{V} + \varepsilon\operatorname{div}\left(\bar{\eta}\bar{V}\right) = 0, \\ (I - \varepsilon d\Delta)\,\partial_t\bar{V} + \nabla\bar{\eta} + \varepsilon\bar{V}\cdot\nabla\bar{V} = 0, \end{cases} \tag{2.1.1}$$

where the parameters b, d obey:

$$b, d \geq 0.$$

We address the long time existence problem for the general (2.1.1) system with initial data that can be used to model bore-type waves. Before stating our results, let us fix the functional framework where we are going to construct our solutions.

We fix two functions χ and φ satisfying:

$$\forall\xi \in \mathbb{R}^n, \;\; \chi(\xi) + \sum_{j\geq 0}\varphi(2^{-j}\xi) = 1,$$

(see Proposition 1.1.1 from the Appendix) and let us denote by h respectively $\tilde{h}$ their Fourier inverses. For all $u \in \mathcal{S}'$, we recall that the nonhomogeneous dyadic blocks are defined as follows:

$$\begin{cases} \Delta_j u = 0 \;\text{ if }\; j \leq -2, \\ \Delta_{-1}u = \chi(D)\,u = \tilde{h}\star u, \\ \Delta_j u = \varphi\left(2^{-j}D\right)u = 2^{jd}\int_{\mathbb{R}^n} h(2^q y)\,u(x-y)\,dy \;\text{ if }\; j \geq 0. \end{cases} \tag{2.1.2}$$

Let us define now the nonhomogeneous Besov spaces. In all that follows, unless otherwise mentioned, the j-subscript for a tempered distribution is reserved for denoting the frequency localized distribution i.e.:

$$u_j \overset{not.}{=} \Delta_j u. \tag{2.1.3}$$

The following remarks will be used in the following.

Remark 2.1.1. *Taking advantage of the Fourier-Plancherel theorem and using (1.1.9) one sees that the classical Sobolev spaces H^s coincide with $B^s_{2,2}$.*

Remark 2.1.2. *Let us suppose that $s \in \mathbb{R}^+\backslash\mathbb{N}$. The space $B^s_{\infty,\infty}$ coincides with the Hölder space $\mathcal{C}^{[s],s-[s]}$ of bounded functions $u \in L^\infty$ whose derivatives of order $|\alpha| \leq [s]$ are bounded and satisfy*

$$|\partial^\alpha u(x) - \partial^\alpha u(y)| \leq C\,|x-y|^{s-[s]} \;\textit{for}\; |x-y| \leq 1.$$

For all $s \in \mathbb{R}$, we define[2]:

$$\begin{cases} s_b = s + \operatorname{sgn}(b), \\ s_d = s + \operatorname{sgn}(d). \end{cases} \tag{2.1.4}$$

Let us denote by

$$X^s_{b,d,r}(\mathbb{R}^n) = B^{s_b}_{2,r}(\mathbb{R}^n) \times \left(B^{s_d}_{2,r}(\mathbb{R}^n)\right)^n.$$

[2]Here, we use the convention $\operatorname{sgn}(x) = \frac{x}{|x|}$ for $x \neq 0$ and $\operatorname{sgn}(0) = 0$.

For any $\varepsilon > 0$, we consider the norm:

$$\begin{cases} \|(\eta, V)\|_{X^{s,\varepsilon}_{b,d,r}} = \left\| \left(2^{js} U_j(\eta, V)\right)_{j\in\mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})} \text{ where} \\ U_j^2(\eta, V) = \int\limits_{\mathbb{R}^n} \left(|\eta_j|^2 + \varepsilon b \sum\limits_{k=\overline{1,n}} |\partial_k \eta_j|^2 + \sum\limits_{k=\overline{1,n}} \left|V_j^k\right|^2 + \varepsilon d \sum\limits_{k,l=\overline{1,n}} \left|\partial_l V_j^k\right|^2 \right). \end{cases}$$

The space

$$E^s_{b,d,r}(\mathbb{R}) = \left\{ (\eta, V) \in L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R}) : (\partial_x \eta, \partial_x V) \in X^{s-1}_{b,d,r} \right\},$$

endowed with the norm

$$\|(\eta, V)\|_{E^{s,\varepsilon}_{b,d,r}} = \|(\eta, V)\|_{L^\infty} + \|(\partial_x \eta, \partial_x V)\|_{X^{s-1,\varepsilon}_{b,d,r}},$$

is a Banach space. An important aspect is that the space $E^s_{b,d,r}(\mathbb{R})$ admits functions that manifest nontrivial behavior at infinity, see Remark 2.1.3. Let us point out that in some sense, the change of the functional setting in order to study bore propagation corresponds in solving a Neumann problem. Indeed, solving the BBM-BBM systems with an initial data in $\mathcal{C}_0$ amounts in solving a Dirichlet-type problem whereas solving it with initial data which is bounded with its gradient belonging to $\mathcal{C}_0$ amounts to solving a Neumann-type problem.

Our first result, pertaining to the 1-dimensional case is formulated in the following theorem.

Theorem 2.1.1. *Consider $r \in [1, \infty)$ and $s \in \mathbb{R}$ such that*

$$s > \frac{3}{2} \text{ or } s = \frac{3}{2} \text{ and } r = 1.$$

Consider $(\eta_0, u_0) \in E^s_{b,d,r}(\mathbb{R})$. Then, there exist two real numbers ε_0, C depending on s, b, d, and on $\|(\eta_0, u_0)\|_{E^{s,1}_{b,d,r}}$ and a numerical constant $\tilde{C}$ such that the following holds true. For any $\varepsilon \leq \varepsilon_0$, System (2.1.1) supplemented with the initial data (η_0, u_0), admits an unique solution

$$(\bar{\eta}^\varepsilon, \bar{u}^\varepsilon) \in \mathcal{C}\left(\left[0, \frac{C}{\varepsilon}\right], E^s_{b,d,r}\right).$$

Moreover, the following estimate holds true:

$$\sup_{t\in\left[0,\frac{C}{\varepsilon}\right]} \|(\bar{\eta}^\varepsilon(t), \bar{u}^\varepsilon(t))\|_{E^{s,\varepsilon}_{b,d,r}} + \sup_{t\in\left[0,\frac{C}{\varepsilon}\right]} \|\partial_t \bar{\eta}^\varepsilon(t)\|_{L^\infty} \leq \tilde{C} \|(\eta_0, u_0)\|_{E^{s,\varepsilon}_{b,d,r}}.$$

Remark 2.1.3. *Let us give an example of function that fits into our framework but is not covered by previous works dedicated to the long time existence problem. A reasonable initial data for modeling bores would be:*

$$\eta_0(x) = \tanh(-x) := \frac{e^{-x} - e^x}{e^x + e^{-x}}.$$

which is smooth and manifests nontrivial behavior at $\pm\infty$, i.e.

$$\lim_{x\to\infty} \eta_0(x) = -1 \text{ and } \lim_{x\to-\infty} \eta_0(x) = 1.$$

One can verify that $\tanh \in \bigcap\limits_{s\geq 0} E^{s,1}_{b,d,r}$ *but does not belong to any Sobolev space H^s.*

Let us consider $\sigma \geq \frac{1}{2}$, $s \geq 1$ with the convention that whenever there is equality in one of the previous relations then $r = 1$. We denote by $M^{\sigma,s}_{b,d,r}$ the space of $(\eta, V) \in \mathcal{C}(\mathbb{R}^2) \times \left(\mathcal{C}(\mathbb{R}^2)\right)^2$ such that there exists $\left(\eta^{1D}, V^{1D}\right) \in E^\sigma_{b,d,r}(\mathbb{R})$ and $\left(\eta^{2D}, V^{2D}\right) \in X^s_{b,d,r}(\mathbb{R}^2)$ for which

$$\begin{cases} \eta(x, y) = \eta^{1D}(x) + \eta^{2D}(x, y), \\ V(x, y) = \left(V^{1D}(x) + V_1^{2D}(x, y), V_2^{2D}(x, y)\right) \end{cases} \tag{2.1.5}$$

for all $(x,y)\in\mathbb{R}^2$. Let us introduce $i: E^{\sigma}_{b,d,r}(\mathbb{R})\to\mathcal{C}(\mathbb{R}^2)\times\left(\mathcal{C}(\mathbb{R}^2)\right)^2$ such that for all $(x,y)\in\mathbb{R}^2$ we have:

$$i((\eta,u))(x,y)=(\eta(x),(u(x),0)).$$

Of course, because of the fact $i\left(E^{\sigma}_{b,d,r}(\mathbb{R})\right)\cap X^{s}_{b,d,r}(\mathbb{R}^2)=\{0\}$, the functions appearing in the decomposition (2.1.5) are unique. For all $\varepsilon>0$, let us consider on $M^{\sigma,s}_{b,d,r}$, the norm

$$\|(\eta,V)\|_{M^{\sigma,s,\varepsilon}_{b,d,r}}=\left\|\left(\eta^{1D},V^{1D}\right)\right\|_{E^{\sigma,\varepsilon}_{b,d,r}(\mathbb{R})}+\left\|\left(\eta^{2D},V^{2D}\right)\right\|_{X^{s,\varepsilon}_{b,d,r}(\mathbb{R}^2)}.$$

It is easy to see that $\left(M^{\sigma,s}_{b,d,r},\|\cdot\|_{M^{\sigma,s,\varepsilon}_{b,d,r}}\right)$ is a Banach space. We can now formulate the result pertaining to the 2-dimensional case:

Theorem 2.1.2. *Consider $s,\sigma\in\mathbb{R}$, $r\in[1,\infty)$ such that*

$$s>2 \text{ or } s=2 \text{ and } r=1 \tag{2.1.6}$$

and

$$\sigma>s+\frac{3}{2} \text{ and } \sigma-\frac{1}{2}\in\mathbb{R}\backslash\mathbb{N}. \tag{2.1.7}$$

Also, consider $(\eta_0,u_0)\in E^{\sigma}_{b,d,r}(\mathbb{R})$ and $(\phi,\psi)\in X^{s}_{b,d,r}(\mathbb{R}^2)$. Then, there exist two real numbers ε_0, C depending on $s,b,d,$ and on $\|(\eta_0,u_0)\|_{E^{s,1}_{b,d,r}}+\|(\phi,\psi)\|_{X^{s,1}_{b,d,r}(\mathbb{R}^2)}$ and a numerical constant $\tilde{C}$ such that the following holds true. For any $\varepsilon\le\varepsilon_0$, system (2.1.1) supplemented with the initial data

$$\begin{cases}\bar{\eta}_0(x,y)=\eta_0(x)+\phi(x,y),\\ \bar{V}_0(x,y)=(u_0(x),0)+\psi(x,y),\end{cases}$$

admits an unique solution

$$(\bar{\eta}^{\varepsilon},\bar{V}^{\varepsilon})\in\mathcal{C}\left(\left[0,\frac{C}{\varepsilon}\right],M^{\sigma,s}_{b,d,r}\right).$$

Moreover, the following estimate holds true:

$$\sup_{t\in\left[0,\frac{C}{\varepsilon}\right]}\left(\left\|\left(\bar{\eta}^{\varepsilon}(t),\bar{V}^{\varepsilon}(t)\right)\right\|_{M^{\sigma,s,\varepsilon}_{b,d,r}}\right)+\sup_{t\in\left[0,\frac{C}{\varepsilon}\right]}\left(\|\partial_t\bar{\eta}^{\varepsilon}(t)\|_{L^{\infty}}\right)\le\tilde{C}\left(\|(\eta_0,u_0)\|_{E^{\sigma,\varepsilon}_{b,d,r}}+\|(\phi,\psi)\|_{X^{s,\varepsilon}_{b,d,r}}\right)$$

The results presented in Theorems 2.1.1, 2.1.2 are a by-product of a general result that we obtain later in this chapter (see Theorem 2.2.1). The method of proof consists of conveniently splitting the initial data into two parts and of performing energy estimates on a slightly more general system than (2.1.1). The estimates are a refined version of those obtained in [27] and they allow us to handle the fact that the bore type functions do not belong to L^2.

2.2 Some intermediate results

The method of proof of the main results naturally leads us to study the following system:

$$\begin{cases}(I-\varepsilon b\Delta)\partial_t\eta+\operatorname{div}V+\varepsilon\operatorname{div}(\eta W_1+hV+\beta\eta V)=\varepsilon f,\\ (I-\varepsilon d\Delta)\partial_t V+\nabla\eta+\varepsilon(W_2+\beta V)\cdot\nabla V+\varepsilon V\cdot\nabla W_3=\varepsilon g\\ \eta_{|t=0}=\eta_0,\ V_{|t=0}=V_0\end{cases} \tag{$\mathcal{S}_\varepsilon(\mathcal{D})$}$$

where $\varepsilon,\beta\in[0,1]$ and $\mathcal{D}=(\eta_0,V_0,f,g,h,W_1,W_2,W_3)$ are given. The above system captures a very general form of weakly dispersive quasilinear systems. Let us consider $s\in\mathbb{R}$ such that

$$s>\frac{n}{2}+1 \text{ or } s=\frac{n}{2}+1 \text{ and } r=1. \tag{2.2.1}$$

Let us fix the following notations:

$$\begin{cases} U_j^2(t) = \|(\eta_j(t), V_j(t))\|_{L^2}^2 + \varepsilon \left\| \left(\sqrt{b}\nabla\eta_j(t), \sqrt{d}\nabla V_j(t) \right) \right\|_{L^2}^2, \\ U_s(t) = \left\| \left(2^{js} U_j(t) \right)_{j\in\mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})}, U(t) = \|(\eta, \nabla\eta, V, \nabla V)\|_{L^\infty \cap L^{p_2}}, \ F_s(t) = \|(f(t), g(t))\|_{B^s_{2,r}}, \\ \mathcal{W}_s(t) = \|h(t)\|_{L^\infty} + \|\nabla h(t)\|_{B^s_{p_1,r}} + \|\partial_t h(t)\|_{L^\infty} + \sum_{i=1}^3 \left(\|(W_i(t))\|_{L^\infty} + \|\nabla W_i(t)\|_{B^s_{p_1,r}} \right). \end{cases} \tag{2.2.2}$$

In order to ease the reading we will rather skip denoting the time dependency in the computations that follow. We are now in the position of stating the following:

Theorem 2.2.1. *Consider $b, d \geq 0$, two real numbers with $b + d > 0$, $(p_1, r) \in [2, \infty] \times [1, \infty)$, p_2 such that*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$$

and $s, s_b, s_d > 0$ defined in (2.2.1) and (2.1.4). Let us also consider

$$\begin{cases} (f, g) \in \mathcal{C}\left([0, T], B^s_{2,r} \times \left(B^s_{2,r}\right)^n\right), \\ h \in \mathcal{C}([0, T], L^\infty) \ \textit{with} \ \nabla h \in \mathcal{C}\left([0, T], B^s_{p_1,r}\right), \partial_t h \in L^\infty([0, T], L^\infty) \end{cases}$$

and for $i = \overline{1,3}$ consider the vector fields

$$W_i \in \mathcal{C}([0, T], (L^\infty)^n) \ \textit{with} \ \nabla W_i \in \mathcal{C}\left([0, T], \left(B^s_{p_1,r}\right)^n\right).$$

Then, for all $(\eta_0, V_0) \in B^{s_b}_{2,r} \times \left(B^{s_d}_{2,r}\right)^n$, writing $\mathcal{D} = (\eta_0, V_0, f, g, h, W_1, W_2, W_3)$, there exists a $\bar{T} \in (0, T]$ such that $\mathcal{S}_\varepsilon(\mathcal{D})$ admits an unique solution

$$\begin{cases} (\eta, V) \in \mathcal{C}\left([0, \bar{T}], B^{s_b}_{2,r} \times \left(B^{s_d}_{2,r}\right)^n\right), \\ \textit{with} \ (\partial_t\eta, \partial_t V) \in \mathcal{C}([0, \bar{T}], B^{s-1+2\,\mathrm{sgn}\, b}_{2,r} \times (B^{s-1+2\,\mathrm{sgn}\, d}_{2,r})^n). \end{cases}$$

Moreover if $T^\star \in \mathbb{R}^+$ is such that

$$\int_0^{T^\star} U(\tau)\, d\tau < \infty, \tag{2.2.3}$$

then the solution (η, V) can be continued after $T^\star$.

Of course, in what the existence and uniqueness of solutions of $\mathcal{S}_\varepsilon(\mathcal{D})$ is concerned, the results presented in Theorem 2.2.1 are not optimal. However, the aspect that we wish to emphasize in this paper is the possibility of solving the above system on what we named long time scales and with regards to this matter the extra regularity is necessary in order to develop the forthcoming theory. The next result is the main ingredient in proving the l.t.e. results announced in Theorem 2.1.1 and Theorem 2.1.2.

Theorem 2.2.2. *Let us fix $(\eta_0, V_0) \in X^s_{b,d,r}$. For $\varepsilon \in (0, 1]$ we consider*

$$\begin{cases} (f^\varepsilon, g^\varepsilon) \in \mathcal{C}\left([0, T^\varepsilon], B^s_{2,r} \times \left(B^s_{2,r}\right)^n\right), \\ h^\varepsilon \in \mathcal{C}([0, T^\varepsilon], L^\infty) \ \textit{with} \ \nabla h^\varepsilon \in \mathcal{C}\left([0, T^\varepsilon], B^s_{p_1,r}\right), \partial_t h^\varepsilon \in L^\infty([0, T^\varepsilon], L^\infty) \end{cases}$$

and for $i = \overline{1,3}$ we consider the vector fields

$$W_i^\varepsilon \in \mathcal{C}([0, T^\varepsilon], (L^\infty)^n) \ \textit{with} \ \nabla W_i^\varepsilon \in \mathcal{C}\left([0, T^\varepsilon], \left(B^s_{p_1,r}\right)^n\right).$$

Assume that

$$\sup_{\varepsilon\in[0,1]} \sup_{\tau\in[0,T^\varepsilon]} \mathcal{W}_s^\varepsilon(\tau) < \infty, \quad \sup_{\varepsilon\in[0,1]} \sup_{\tau\in[0,T^\varepsilon]} F_s^\varepsilon(\tau) < \infty,$$

where

$$\mathcal{W}_s^{\varepsilon}(t) = \|h^{\varepsilon}(t)\|_{L^\infty} + \|\nabla h^{\varepsilon}(t)\|_{B^s_{p_1,r}} + \|\partial_t h^{\varepsilon}(t)\|_{L^\infty} + \sum_{i=1}^{3}\left(\|(W_i^{\varepsilon}(t))\|_{L^\infty} + \|\nabla W_i^{\varepsilon}(t)\|_{B^s_{p_1,r}}\right),$$

and

$$F_s^{\varepsilon}(t) = \|(f^{\varepsilon}(t), g^{\varepsilon}(t))\|_{B^s_{2,r}}.$$

We denote by $\mathcal{D}^{\varepsilon} = (\eta_0, V_0, f^{\varepsilon}, g^{\varepsilon}, h^{\varepsilon}, W_1^{\varepsilon}, W_2^{\varepsilon}, W_3^{\varepsilon})$*. Then, there exist two real numbers* ε_0*,* C *depending on* $s, n, b, d, \|(\eta_0, V_0)\|_{X^{s,1}_{b,d,r}}$*,* $\sup_{\varepsilon\in[0,1]}\sup_{\tau\in[0,T^{\varepsilon}]} \mathcal{W}_s^{\varepsilon}(\tau)$*,* $\sup_{\varepsilon\in[0,1]}\sup_{\tau\in[0,T^{\varepsilon}]} F_s^{\varepsilon}(\tau)$ *and a numerical constant* $\tilde{C} = \tilde{C}(n)$ *such that the following holds true. For any* $\varepsilon \le \varepsilon_0$*, the maximal time of existence* $T^{\varepsilon}_{\max}$ *of the unique solution* $(\eta^{\varepsilon}, V^{\varepsilon})$ *of system* $\mathcal{S}_{\varepsilon}(\mathcal{D}^{\varepsilon})$ *satisfies the following lower bound:*

$$T^{\varepsilon}_{\max} \ge T^{\varepsilon}_{\star} \stackrel{def.}{:=} \min\left\{T^{\varepsilon}, \frac{C}{\varepsilon}\right\}. \tag{2.2.4}$$

Moreover, we have that:

$$\sup_{t\in[0,T^{\varepsilon}_{\star}]} \|(\eta^{\varepsilon}(t), V^{\varepsilon}(t))\|_{X^{s,\varepsilon}_{b,d,r}} + \sup_{t\in[0,T^{\varepsilon}_{\star}]} \|\partial_t \eta^{\varepsilon}(t)\|_{L^\infty} \le \tilde{C}(n)\|(\eta_0, V_0)\|_{X^{s,\varepsilon}_{b,d,r}}. \tag{2.2.5}$$

Of course, when the data $\mathcal{D}^{\varepsilon}$ does not depend on ε we obtain the following:

Corollary 2.2.3. *Let us fix* $(\eta_0, V_0) \in X^s_{b,d,r}$ *and* s *and* p_1 *as above. Also, consider*

$$\begin{cases} (f,g) \in \mathcal{C}\left([0,T], B^s_{2,r} \times \left(B^s_{2,r}\right)^n\right), \\ h \in \mathcal{C}\left([0,T], L^\infty\right) \text{ with } \nabla h \in \mathcal{C}\left([0,T], B^s_{p_1,r}\right), \partial_t h \in L^\infty\left([0,T], L^\infty\right) \end{cases}$$

and for $i = \overline{1,3}$ *consider the vector fields*

$$W_i \in \mathcal{C}\left([0,T], (L^\infty)^n\right) \text{ with } \nabla W_i \in \mathcal{C}\left([0,T], \left(B^s_{p_1,r}\right)^n\right).$$

We denote by $\mathcal{D} = (\eta_0, V_0, f, g, h, W_1, W_2, W_3)$*. Then, there exist two real numbers* ε_0*,* C *both depending on* s*,*n*,*b*,*d*,*$\|(\eta_0, V_0)\|_{X^{s,1}_{b,d,r}}$*,* $\sup_{\tau\in[0,T]} \mathcal{W}_s(\tau)$*,* $\sup_{\tau\in[0,T]} F_s(\tau)$ *and a numerical constant* $\tilde{C} = \tilde{C}(n)$ *such that the following holds true. For any* $\varepsilon \le \varepsilon_0$*, the maximal time of existence* $T^{\varepsilon}_{\max}$ *of the unique solution* $(\eta^{\varepsilon}, V^{\varepsilon})$ *of the system* $\mathcal{S}_{\varepsilon}(\mathcal{D})$ *satisfies the following lower bound:*

$$T^{\varepsilon}_{\max} \ge T^{\varepsilon}_{\star} \stackrel{def.}{:=} \min\left\{T^{\varepsilon}, \frac{C}{\varepsilon}\right\}. \tag{2.2.6}$$

Moreover, we have that:

$$\sup_{t\in[0,T^{\varepsilon}_{\star}]} \|(\eta^{\varepsilon}(t), V^{\varepsilon}(t))\|_{X^{s,\varepsilon}_{b,d,r}} + \sup_{t\in[0,T^{\varepsilon}_{\star}]} \|\partial_t \eta^{\varepsilon}(t)\|_{L^\infty} \le \tilde{C}(n)\|(\eta_0, V_0)\|_{X^{s,\varepsilon}_{b,d,r}}. \tag{2.2.7}$$

Remark 2.2.1. *The choice of* s *according to relation (2.2.1) ensures that we have the following embedding:* $B^s_{2,r} \hookrightarrow L^{p_2}$ *and* $B^s_{2,r} \hookrightarrow L^\infty$*. In particular, we also have*

$$U(t) \le_n U_s(t),$$

a fact that will be systematically used in all that follows.

Remark 2.2.2. *The explosion criterion (2.2.3) implies that the life span of the solution does not depend on its possible extra regularity above the critical level* $B^{\frac{n}{2}+1}_{2,1}$*.*

The plan of the proof of Theorem 2.2.1 is the following. First, we derive a priori estimates using a spectral localization of the system $(\mathcal{S}_{\varepsilon}(\mathcal{D}))$. Then, we use the so called Friedrichs method in order to construct a sequence of functions that solves a family of ODE's which approximate system $(\mathcal{S}_{\varepsilon}(\mathcal{D}))$. Finally, using a compactness method we show that we can construct a solution of the system $(\mathcal{S}_{\varepsilon}(\mathcal{D}))$. Theorem 2.2.2 is obtained using a bootstrap argument.

2.2.1 A priori estimates

Firs of all we will derive a priori estimates. Thus, let us consider $(\eta, V) \in \mathcal{C}\left([0,\bar{T}], B^{s_b}_{2,r} \times \left(B^{s_d}_{2,r}\right)^n\right)$ a solution of $(\mathcal{S}_\varepsilon(\mathcal{D}))$. As announced, we proceed by localizing the system $(\mathcal{S}_\varepsilon(\mathcal{D}))$ in the frequency space such that we obtain:

$$\left\{\begin{array}{l} (I-\varepsilon b\Delta)\,\partial_t\eta_j + \operatorname{div} V_j + \varepsilon\nabla\eta_j\,(W_1+\beta V) + \varepsilon\,(h+\beta\eta)\operatorname{div} V_j = \varepsilon f_j + \varepsilon R_{1j}, \\ (I-\varepsilon d\Delta)\,\partial_t V_j + \nabla\eta_j + \varepsilon\,(W_2+\beta V)\cdot\nabla V_j = \varepsilon g_j + \varepsilon R_{2j}, \end{array}\right. \tag{2.2.8}$$

where

$$\left\{\begin{array}{l} R_{1j} = \beta\,[V,\Delta_j]\,\nabla\eta + [W_1,\Delta_j]\,\nabla\eta + \beta\,[\eta,\Delta_j]\operatorname{div} V + [h,\Delta_j]\operatorname{div} V - \Delta_j\left(\nabla h V\right) - \Delta_j\left(\eta\operatorname{div} W_1\right), \\ R_{2j} = [(W_2+\beta V)\cdot\nabla,\Delta_j]\,V - \Delta_j\left(V\cdot\nabla W_3\right), \end{array}\right. \tag{2.2.9}$$

We multiply the first equation of (2.2.8) with η_j and the second one with $1+\alpha\varepsilon\,(h+\beta\eta)\,V_j$, we add the results and by integrating over $\mathbb{R}^n$ we obtain that:

$$\frac{1}{2}\frac{d}{dt}\left[\int \eta_j^2 + \varepsilon b\,|\nabla\eta_j|^2 + (1+\alpha\varepsilon\,(h+\beta\eta))\left(V_j^2+\varepsilon d\nabla V_j:\nabla V_j\right)\right] = \sum_{i=1}^{7} T_i$$

where:

$$\begin{aligned} T_1 &= -\int (1+\varepsilon\,(h+\beta\eta))\,\eta_j\operatorname{div} V_j - \int (1+\alpha\varepsilon\,(h+\beta\eta))\,\nabla\eta_j V_j, \\ T_2 &= -\varepsilon\int \nabla\eta_j\,(W_1+\beta V)\,\eta_j, \\ T_3 &= -\varepsilon\,\langle (W_2+\beta V)\cdot\nabla V_j, (1+\alpha\varepsilon\,(h+\beta\eta))\,V_j\rangle_{L^2}, \\ T_4 &= \varepsilon\int f_j\eta_j + \varepsilon\int (1+\alpha\varepsilon\,(h+\beta\eta))\,g_j V_j, \\ T_5 &= \varepsilon\int R_{1j}\eta_j + \varepsilon\int (1+\alpha\varepsilon\,(h+\beta\eta))\,R_{2j}V_j, \\ T_6 &= -\frac{1}{2}\alpha\varepsilon\,\langle V_j, (\partial_t h+\beta\partial_t\eta)\,V_j\rangle_{L^2} - \frac{1}{2}\alpha d\varepsilon^2\,\langle \nabla V_j, (\partial_t h+\beta\partial_t\eta)\,\nabla V_j\rangle_{L^2}, \\ T_7 &= -\alpha d\varepsilon^2\,\langle \partial_t\nabla V_j, \nabla(h+\beta\eta)\otimes V_j\rangle_{L^2}. \end{aligned}$$

Here, we use α as a parameter in order to obtain the desired estimates from a single "strike". Indeed only the values $\alpha\in\{0,1\}$ will be of interest to us and, as we will see, $\alpha=0$ will give us the estimate necessary to develop the existence theory while $\alpha=1$ will lead to an estimate that is the key point in finding the lower bound on the time of existence.

Let us estimate the T_i's. Regarding the first term, we write that:

$$\begin{aligned} T_1 &= -\int (1+\varepsilon\,(h+\beta\eta))\,\eta_j\operatorname{div} V_j - \int (1+\alpha\varepsilon\,(h+\beta\eta))\,\nabla\eta_j V_j \\ &= \alpha\varepsilon\int (\nabla h+\beta\nabla\eta)\,\eta_j V_j - \varepsilon\,(1-\alpha)\int (\beta\eta+h)\,\eta_j\operatorname{div} V_j \\ &\leq \alpha\varepsilon C\left(\|\nabla h\|_{L^\infty} + \beta\,\|\nabla\eta\|_{L^\infty}\right)\|\eta_j\|_{L^2}\,\|V_j\|_{L^2} \\ &+ C\frac{(1-\alpha)\sqrt{\varepsilon}}{\max(\sqrt{b},\sqrt{d})}U_j^2\left(\mathcal{W}_s+\beta U\right) \\ &\leq \alpha\varepsilon C U_j^2\left(\mathcal{W}_s+\beta U\right) + C\frac{(1-\alpha)\sqrt{\varepsilon}}{\max(\sqrt{b},\sqrt{d})}U_j^2\left(\mathcal{W}_s+\beta U\right). \end{aligned} \tag{2.2.10}$$

Let us bound the second term:

$$T_2 = -\varepsilon\int (W_1+\beta V)\,\nabla\eta_j\eta_j \leq \frac{\varepsilon}{2}\int \left(\|\operatorname{div} W_1\|_{L^\infty} + \beta\,\|\operatorname{div} V\|_{L^\infty}\right)\eta_j^2$$

$$\leq \frac{\varepsilon}{2} U_j^2 \left(\mathcal{W}_s + \beta U\right). \tag{2.2.11}$$

Using the Einstein summation convention over repeated indices, the term T_3 is treated as follows:

$$\begin{aligned}
T_3 &= -\varepsilon \left\langle \left(W_2 + \beta V\right) \cdot \nabla V_j, \left(1 + \alpha\varepsilon \left(\beta\eta + h\right)\right) V_j \right\rangle_{L^2} \\
&= -\varepsilon \int \left(1 + \alpha\varepsilon(\beta\eta + h)\right) \left(\beta V^m + W_2^m\right) \partial_m V_j^k V_j^k \\
&= \frac{\varepsilon}{2} \int \partial_m \left(\left(1 + \alpha\varepsilon(\beta\eta + h)\right) \left(\beta V^m + W_2^m\right)\right) V_j^k V_j^k \\
&= \frac{\varepsilon}{2} \int \operatorname{div} \left(\left(1 + \alpha\varepsilon(\beta\eta + h)\right) \left(\beta V + W_2\right)\right) |V_j|^2 \\
&\leq \frac{\varepsilon}{2} U_j^2 \left(\|W_2\|_{L^\infty} + \beta \|V\|_{L^\infty} + \alpha\varepsilon \|(\beta\eta, \beta\nabla\eta, h, \nabla h)\|_{L^\infty} \|(\beta V, \beta\nabla V, W_2, \nabla W_2)\|_{L^\infty}\right) \\
&\leq \frac{\varepsilon}{2} U_j^2 \left(\mathcal{W}_s + \beta U + \alpha\varepsilon \left(\mathcal{W}_s + \beta U\right)^2\right).
\end{aligned} \tag{2.2.12}$$

Next, we bound the term T_4 :

$$\begin{aligned}
T_4 &= \varepsilon \int f_j \eta_j + \varepsilon \int \left(1 + \alpha\varepsilon \left(h + \beta\eta\right)\right) g_j V_j \\
&\leq \varepsilon \|f_j\|_{L^2} \|\eta_j\|_{L^2} + \varepsilon \left(1 + \alpha\varepsilon \|h\|_{L^\infty} + \alpha\beta\varepsilon \|\eta\|_{L^\infty}\right) \|g_j\|_{L^2} \|V_j\|_{L^2} \\
&\leq \varepsilon C 2^{-js} c_j^1(t) U_j F_s \left(1 + \alpha\varepsilon \left(\mathcal{W}_s + \beta U\right)\right).
\end{aligned} \tag{2.2.13}$$

Treating the fifth term is done in the following lines. First we write that:

$$\begin{aligned}
T_5 &= \varepsilon \int R_{1j} \eta_j + \varepsilon \int \left(1 + \alpha\varepsilon \left(h + \beta\eta\right)\right) R_{2j} V_j \\
&\leq \varepsilon C U_j \left(1 + \alpha\varepsilon \left(\mathcal{W}_s + \beta U\right)\right) \left(\|R_{1j}\|_{L^2} + \|R_{2j}\|_{L^2}\right).
\end{aligned} \tag{2.2.14}$$

Next, let us estimate the remainder terms R_{1j} and R_{2j}. In order to do so, we apply Proposition 1.2.5 and Proposition 1.2.1 from the Appendix in order to get:

$$\begin{aligned}
\|R_{1j}\|_{L^2} \leq C 2^{-js} c_j^2(t) \Big\{ &\beta \|\nabla V\|_{L^\infty} \|\nabla\eta\|_{B^{s-1}_{2,r}} + \beta \|\nabla\eta\|_{L^\infty} \|\nabla V\|_{B^{s-1}_{2,r}} + \\
&\|\nabla W_1\|_{L^\infty} \|\nabla\eta\|_{B^{s-1}_{2,r}} + \|\nabla\eta\|_{L^{p_2}} \|\nabla W_1\|_{B^{s-1}_{p_1,r}} + \\
&\beta \|\nabla\eta\|_{L^\infty} \|\nabla V\|_{B^{s-1}_{2,r}} + \beta \|\nabla V\|_{L^\infty} \|\nabla\eta\|_{B^{s-1}_{2,r}} + \\
&\|\nabla h\|_{L^\infty} \|\nabla V\|_{B^{s-1}_{2,r}} + \|\nabla V\|_{L^{p_2}} \|\nabla h\|_{B^{s-1}_{p_1,r}} + \\
&\|\nabla h\|_{L^\infty} \|\nabla V\|_{B^{s-1}_{2,r}} + \|V\|_{L^{p_2}} \|\nabla h\|_{B^{s}_{p_1,r}} + \\
&\|\operatorname{div} W_1\|_{L^\infty} \|\nabla\eta\|_{B^{s-1}_{2,r}} + \|\eta\|_{L^{p_2}} \|\operatorname{div} W_1\|_{B^{s}_{p_1,r}} \Big\}
\end{aligned} \tag{2.2.15}$$

and consequently:

$$\|R_{1j}\|_{L^2} \leq C 2^{-js} c_j^2(t) \left(\mathcal{W}_s U + U_s \left(\mathcal{W}_s + \beta U\right)\right).$$

Proceeding as above, we get a similar bound for R_{2j}. Thus, we get that:

$$T_5 \leq \varepsilon C 2^{-js} c_j^2(t) U_j \left(1 + \alpha\varepsilon \left(\mathcal{W}_s + \beta U\right)\right) \left(\mathcal{W}_s U + U_s \left(\mathcal{W}_s + \beta U\right)\right) \tag{2.2.16}$$

When $\alpha = 0$, $T_6 = T_7 = 0$ and we are in the position of obtaining the first estimate. Combining (2.2.10), (2.2.11), (2.2.12), (2.2.13) and (2.2.16) we get that:

$$\frac{d}{dt} U_j^2 \leq \sqrt{\varepsilon} C U_j^2 \left(\mathcal{W}_s + \beta U\right) + \varepsilon C 2^{-js} c_j^5 U_j \left(F_s + \mathcal{W}_s U + U_s \left(\mathcal{W}_s + \beta U\right)\right). \tag{2.2.17}$$

Time integration of (2.2.17) reveals the following:

$$
\begin{aligned}
U_j(t) \leq U_j(0) + \sqrt{\varepsilon} C \int_0^t U_j(\tau)\left(\mathcal{W}_s(\tau) + \beta U(\tau)\right) d\tau + \\
\sqrt{\varepsilon} C \int_0^t 2^{-j\sigma} c_j^5(\tau)\left(F_s(\tau) + \mathcal{W}_s(\tau) U(\tau) + U_s(\tau)\left(\mathcal{W}_s(\tau) + \beta U(\tau)\right)\right) d\tau.
\end{aligned}
$$

Multiplying the last inequality with 2^{js} and performing an $\ell^r(\mathbb{Z})$-summation we end up with:

$$
\begin{aligned}
U_s(t) &\leq U_s(0) + \sqrt{\varepsilon} C \int_0^t F_s(\tau) d\tau + \sqrt{\varepsilon} C \int_0^t \mathcal{W}_s(\tau) U(\tau) d\tau \\
&+ \sqrt{\varepsilon} C \int_0^t U_s(\tau)\left(\mathcal{W}_s(\tau) + \beta U(\tau)\right) d\tau. \\
&\leq U_s(0) + \sqrt{\varepsilon} C \int_0^t F_s(\tau) d\tau + \sqrt{\varepsilon} C \int_0^t U_s(\tau)\left(\mathcal{W}_s(\tau) + \beta U(\tau)\right) d\tau. \qquad (2.2.18)
\end{aligned}
$$

Obviously, relation (2.2.18) and Gronwall's lemma imply the explosion criteria.

Let us now bound the term T_6:

$$
\begin{aligned}
T_6 &= -\frac{1}{2}\alpha\varepsilon \left\langle V_j, \left(\partial_t h + \beta \partial_t \eta\right) V_j \right\rangle_{L^2} - \frac{1}{2}\alpha\varepsilon^2 d \left\langle \nabla V_j, \left(\partial_t h + \beta \partial_t \eta\right) \nabla V_j \right\rangle_{L^2} \\
&\leq \alpha\varepsilon U_j^2 \left(\|\partial_t h\|_{L^\infty} + \beta \|\partial_t \eta\|_{L^\infty}\right).
\end{aligned}
$$

Using the first equation of $(\mathcal{S}_\varepsilon(\mathcal{D}))$ we write:

$$
\partial_t \eta = \varepsilon (I - \varepsilon b \Delta)^{-1} f - (I - \varepsilon b \Delta)^{-1} \operatorname{div}\left(V + \varepsilon\left(\eta W_1 + hV + \beta \eta V\right)\right)
$$

and using again relation (2.2.1) combined with the fact that $(I - \varepsilon b\Delta)^{-1}$ has at most norm 1 when regarded as an L^2 to L^2 operator, we obtain that:

$$
\begin{aligned}
\|\partial_t \eta\|_{L^\infty} &\leq \left\| \varepsilon (I - \varepsilon b \Delta)^{-1} f - (I - \varepsilon b \Delta)^{-1} \operatorname{div}\left(V + \varepsilon\left(\eta W_1 + hV + \beta \eta V\right)\right) \right\|_{B^{\frac{n}{2}}_{2,1}} \\
&\leq \varepsilon \|f\|_{B^s_{2,r}} + \|V\|_{B^s_{2,r}} + \varepsilon \|\eta W_1 + hV + \beta \eta V\|_{B^s_{2,r}} \\
&\leq \varepsilon \|f\|_{B^s_{2,r}} + \|V\|_{B^s_{2,r}} + \varepsilon\beta \|(\eta, V)\|^2_{B^s_{2,r}} + \varepsilon \|(\eta, V)\|_{B^s_{2,r}} \|(h, W_1)\|_{L^\infty} \\
&+ \varepsilon \|(\eta, V)\|_{L^{p_2}} \|(\nabla h, \nabla W_1)\|_{B^{s-1}_{p_1,r}} \\
&\leq C(U_s + \varepsilon F_s + \varepsilon U_s\left(\mathcal{W}_s + \beta U_s\right)). \qquad (2.2.19)
\end{aligned}
$$

Thus, putting together the last estimates, we find that:

$$
T_6 \leq \alpha\varepsilon U_j^2 \left(\mathcal{W}_s + \beta U_s + \varepsilon\beta F_s + \varepsilon\left(\mathcal{W}_s + \beta U_s\right)^2\right). \qquad (2.2.20)
$$

Finally, let us estimate the last term:

$$
T_7 = -\alpha d \varepsilon^2 \left\langle \partial_t \nabla V_j, \nabla(\beta\eta + h) \otimes V_j \right\rangle_{L^2} \leq \alpha d \varepsilon^2 \|\partial_t \nabla V_j\|_{L^2} \|V_j\|_{L^2} \left(\beta \|\nabla \eta\|_{L^\infty} + \|\nabla h\|_{L^\infty}\right).
$$

Using the second equation of $(\mathcal{S}_\varepsilon(\mathcal{D}))$ we write that:

$$
\begin{aligned}
\partial_t \nabla V_j &= \varepsilon (I - \varepsilon d \Delta)^{-1} \nabla(g_j) - (I - \varepsilon\Delta)^{-1} \nabla^2 \eta_j - \varepsilon (I - \varepsilon\Delta)^{-1} \nabla \Delta_j \left[(W_2 + \beta V)\cdot \nabla V\right] \\
&- \varepsilon (I - \varepsilon\Delta)^{-1} \nabla \Delta_j \left(V \cdot \nabla W_3\right)
\end{aligned}
$$

and because $(\varepsilon d)^{\frac{1}{2}} (I-\varepsilon d\Delta)^{-1} \nabla$ and $\varepsilon d (I-\varepsilon d\Delta)^{-1} \nabla^2$ have $O(1)$-norms when regarded as operators from L^2 to L^2, we can write that

$$\begin{aligned}\varepsilon d \left\|\partial_t \nabla V_j\right\|_{L^2} &\leq \varepsilon^{\frac{3}{2}} C\left(\sqrt{d}\left\|g_j\right\|_{L^2}+\left\|\eta_j\right\|_{L^2}\right) \\ &\quad + C\left(\varepsilon^{\frac{3}{2}}\sqrt{d}\left\|\Delta_j\left[\left(W_2+\beta V\right)\cdot\nabla V\right]\right\|_{L^2}+\varepsilon^{\frac{3}{2}}\sqrt{d}\left\|\Delta_j\left(V\cdot\nabla W_3\right)\right\|_{L^2}\right) \\ &\leq C\max\left\{1,\sqrt{\varepsilon d}\right\}\left(U_j+2^{-js}c_j^4\left(\varepsilon F_s+\varepsilon\left(\mathcal{W}_s+\beta U_s\right)^2\right)\right).\end{aligned}$$

Thus, we get that:

$$T_7 \leq \alpha\varepsilon U_j^2\left(\mathcal{W}_s+\beta U\right)+\alpha\varepsilon C 2^{-js}c_j^4(t)\,U_j\left(\varepsilon F_s\left(\mathcal{W}_s+\beta U\right)+\varepsilon U_s\left(\mathcal{W}_s+\beta U_s\right)^2\right). \tag{2.2.21}$$

Let us consider $\alpha=1$. Putting together estimates (2.2.10)-(2.2.16) we get that:

$$\begin{aligned}&\frac{d}{dt}\left[\int \eta_j^2+\varepsilon b\left|\nabla\eta_j\right|^2+\left(1+\varepsilon(\eta+h)\right)\left(V_j^2+\varepsilon d\nabla V_j:\nabla V_j\right)\right] \\ &\leq \varepsilon C U_j^2\left(\varepsilon\beta F_s+\mathcal{W}_s+\beta U_s+\varepsilon\left(\mathcal{W}_s+\beta U_s\right)^2\right) && (2.2.22)\\ &+\varepsilon C 2^{-js}c_j(t)\,U_j\left(F_s\left(1+\varepsilon\left(\mathcal{W}_s+\beta U_s\right)\right)+U_s\left(\mathcal{W}_s+\beta U_s\right)\left(1+\varepsilon\left(\mathcal{W}_s+\beta U_s\right)\right)\right). && (2.2.23)\end{aligned}$$

2.2.2 Existence and uniqueness of solutions

We are now in the position to prove the existence and uniqueness of solutions for system $(\mathcal{S}_\varepsilon(\mathcal{D}))$. We will use the so called Friedrichs method. For all $m\in\mathbb{N}$, let us consider $\mathbb{E}_m$ the low frequency cut-off operator defined by:

$$\mathbb{E}_m f=\mathcal{F}^{-1}\left(\chi_{B(0,m)}\hat{f}\right).$$

We define the space

$$L_m^2=\left\{f\in L^2:\operatorname{Supp}\hat{f}\subset B(0,m)\right\}$$

which, endowed with the $\|\cdot\|_{L^2}$-norm is a Banach space. Let us observe that due to Bernstein's lemma, all Sobolev norms are equivalent on L_m^2. For all $m\in\mathbb{N}$, we consider the following differential equation on L_m^2:

$$\begin{cases}\partial_t\eta=F_m(\eta,V), \\ \partial V=G_m(\eta,V), \\ \eta_{|t=0}=\mathbb{E}_m\eta_0,\ V_{|t=0}=\mathbb{E}_m V_0,\end{cases} \tag{2.2.24}$$

where $(F_m,G_m):L_m^2\times\left(L_m^2\right)^n\to L_m^2\times\left(L_m^2\right)^n$ are defined by:

$$F_m(\eta,V)=-\mathbb{E}_m\left((I-\varepsilon b\Delta)^{-1}\left[\operatorname{div}V+\varepsilon\operatorname{div}\left(\eta W_1+hV+\beta\eta V\right)-\varepsilon f\right]\right), \tag{2.2.25}$$

$$G_m(\eta,V)=-\mathbb{E}_m\left((I-\varepsilon d\Delta)^{-1}\left[\nabla\eta+\varepsilon\left(W_2+\beta\right)\cdot\nabla V+\varepsilon V\cdot\nabla W_3-\varepsilon g\right]\right). \tag{2.2.26}$$

It transpires that due to the equivalence of the Sobolev norm, (F_m,G_m) is continuous and locally Lipschitz on $L_m^2\times\left(L_m^2\right)^n$. Thus, the classical Picard theorem ensures that there exists a nonnegative time $T_m>0$ and a unique solution $(\eta^m,V^m):\mathcal{C}^1\left([0,T_m],L_m^2\times\left(L_m^2\right)^n\right)$. Let us denote by

$$\begin{cases}U_j^m(t)=\left(\left\|\left(\Delta_j\eta^m(t),\Delta_j V^m(t)\right)\right\|_{L^2}^2+\varepsilon\left\|\left(\sqrt{b}\nabla\Delta_j\eta^m(t),\sqrt{d}\nabla\Delta_j V^m(t)\right)\right\|_{L^2}^2\right)^{\frac{1}{2}}, \\ U_s^m(t)=\left\|\left(2^{js}U_j^m(t)\right)_{j\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})},\ U^m(t)=\left\|\left(\eta^m,\nabla\eta^m,V^m,\nabla V^m\right)\right\|_{L^\infty\cap L^{p_2}}.\end{cases}$$

Thanks to the property $\mathbb{E}_m^2 = \mathbb{E}_m$, we get that the estimate obtained in (2.2.18) still holds true for (η^m, V^m), namely:

$$\begin{aligned} U_s^m(t) &\leq U_s^m(0) + \sqrt{\varepsilon} C \int_0^t F_s(\tau)\, d\tau + \sqrt{\varepsilon} C \int_0^t U_s^m(\tau)\left(U^m(\tau) + \mathcal{W}_s(\tau)\right) d\tau \\ &\leq U_s(0) + \sqrt{\varepsilon} C \int_0^t \left(F_s(\tau) + \mathcal{W}_s^2(\tau)\right) d\tau + \sqrt{\varepsilon} C \int_0^t \left(U_s^m(\tau)\right)^2 d\tau. \end{aligned}$$

We consider

$$T' = \sup\left\{T > 0 : \sqrt{\varepsilon} C \int_0^t \left(F_s(\tau) + \mathcal{W}^2(\tau)\right) d\tau \leq U_s(0)\right\}$$

and

$$\bar{T}_m = \sup\left\{T > 0 : U_s(\tau) \leq 3U_s(0)\right\}.$$

Gronwall's lemma ensures the existence of a constant $\bar{C} = \bar{C}(s, b, d)$ such that:

$$\bar{T}_m \geq \min\left\{\frac{\bar{C}}{\sqrt{\varepsilon} U_s(0)}, T'\right\} := \bar{T}$$

and thus, the sequence $(\eta^m, V^m) \in \mathcal{C}\left([0, T], B_{2,r}^{s_1} \times \left(B_{2,r}^{s_2}\right)^n\right)$ is uniformly bounded i.e.

$$\|(\eta^m, V^m)\|_E \leq 3U_s(0). \tag{2.2.27}$$

From the relation

$$\partial_t \eta^m = -\mathbb{E}_m\left((I - \varepsilon b \Delta)^{-1}\left[\operatorname{div} V^m + \varepsilon \operatorname{div}\left(\eta^m W_1 + h V^m + \beta \eta^m V^m\right) - \varepsilon f\right]\right)$$

and from (2.2.27) we get that the sequence $(\partial_t \eta^m)_{m \in \mathbb{N}}$ is uniformly bounded in $B_{2,r}^{s-1}$ on $\left[0, \bar{T}\right]$. Next, considering for all $p \in \mathbb{N}$ a smooth function ϕ_p such that:

$$\begin{cases} \operatorname{Supp} \phi_p \subset B(0, p+1), \\ \phi_p = 1 \text{ on } B(0, p) \end{cases}$$

it follows that for each $p \in \mathbb{N}$, the sequence $(\phi_p \eta^m)_{m \in \mathbb{N}}$ is uniformly equicontinuous on $\left[0, \bar{T}\right]$ and that for all $t \in \left[0, \bar{T}\right]$, the set $\{\phi_p \eta^m(t) : m \in \mathbb{N}\}$ is relatively compact in $B_{2,r}^{s-1}$. Thus, the Ascoli-Arzela Theorem combined with Proposition 1.2.4 and with Cantor's diagonal process provides us a subsequence of $(\eta^m)_{m \in \mathbb{N}}$ and a tempered distribution $\eta \in \mathcal{C}([0, T], S')$ such that for all $\phi \in \mathcal{D}(\mathbb{R}^n)$:

$$\phi \eta^m \to \phi \eta \text{ in } \mathcal{C}\left([0, T], B_{2,r}^{s-1}\right).$$

Moreover, owing to the Fatou property for Besov spaces, see Proposition 1.2.2, we get that $\eta \in L^\infty\left([0, T], B_{2,r}^{s_1}\right)$ and thus, using interpolation we get that:

$$\phi \eta^m \to \phi \eta \text{ in } \mathcal{C}\left([0, T], B_{2,r}^{s_b - \gamma}\right) \tag{2.2.28}$$

for any $\gamma > 0$. Of course, using the same argument we can construct a $V \in L^\infty\left(\left[0, \bar{T}\right], \left(B_{2,r}^{s_2}\right)^n\right)$ such that for any $\psi \in \left(\mathcal{D}(\mathbb{R}^n)\right)^n$:

$$\psi V^m \to \psi V \text{ in } \mathcal{C}\left([0, T], \left(B_{2,r}^{s_d - \gamma}\right)^n\right) \tag{2.2.29}$$

for any $\gamma > 0$. Also, by the Fatou property in Besov spaces we get that

$$(\eta, V) \in L^\infty\left(\left[0, \bar{T}\right], B_{2,r}^{s_b} \times \left(B_{2,r}^{s_d}\right)^n\right).$$

We claim that the properties enlisted above allow us to pass to the limit when $m \to \infty$ in the equation verified by η^m and V^m. Let us show on two examples, how this process is carried out in practice. Let $\phi \in \mathcal{D}(\mathbb{R}^n)$ and let us write that:

$$\begin{aligned}
&\left|\int \phi \operatorname{div}\left[(\eta^m - \eta) W_1\right]\right| \\
&= \left|\int (\eta^m - \eta) W_1 \nabla\phi\right| \le \left|\int (\eta^m - \eta)(S_q W_1)\nabla\phi\right| + \left|\int (\eta^m - \eta)((Id - S_q) W_1)\nabla\phi\right| \\
&\le \left|\int (\eta^m - \eta)(S_q W_1)\nabla\phi\right| + C\left\|(Id - S_q) W_1\right\|_{B^{s+1}_{p_1,r}} \left\|(\eta^m - \eta)\nabla\phi\right\|_{B^{-s-1}_{p_1',r'}} \\
&\le \left|\int (\eta^m - \eta)(S_q W_1)\nabla\phi\right| + C\left\|(Id - S_q)\nabla W_1\right\|_{B^{s}_{p_1,r}} \left\|(\eta^m - \eta)\nabla\phi\right\|_{B^{0}_{p_1',\infty}} \\
&\le \left|\int (\eta^m - \eta)(S_q W_1)\nabla\phi\right| + C\left\|(Id - S_q)\nabla W_1\right\|_{B^{s}_{p_1,r}} \left(\|\eta^m\|_{L^{p_2}} + \|\eta\|_{L^{p_2}}\right)\|\nabla\phi\|_{L^2} \\
&\le \left|\int (\eta^m - \eta)(S_q W_1)\nabla\phi\right| + CU_s(0)\left\|(Id - S_q)\nabla W_1\right\|_{B^{s}_{p_1,r}} \|\nabla\phi\|_{L^2}. \qquad (2.2.30)
\end{aligned}$$

The fact that the first term of (2.2.30) tends to zero as $m \to \infty$ is a consequence of (2.2.28). The second term tends to zero as $q \to \infty$ owing to the fact that $\nabla W_1 \in B^s_{p_1,r}$. Let us also show how to deal with the nonlinear terms. We write that

$$\begin{aligned}
\left|\int \phi \operatorname{div}(\eta^m V^m - \eta V)\right| &= \left|\int (\eta^m V^m - \eta V)\nabla\phi\right| \le \left|\int \eta^m (V^m - V)\nabla\phi\right| + \left|\int (\eta^m - \eta) V\nabla\phi\right| \\
&\le C\left\|V^m\nabla\phi - V\nabla\phi\right\|_{B^{s-1}_{2,r}} \|\eta^m\|_{B^{1-s}_{2,r'}} + \left|\int (\eta^m - \eta) V\nabla\phi\right| \\
&\le C\left\|V^m\nabla\phi - V\nabla\phi\right\|_{B^{s-1}_{2,r}} \|\eta^m\|_{B^{s}_{2,r}} + \left|\int (\eta^m - \eta) V\nabla\phi\right| \\
&\le CU_s(0)\left\|V^m\nabla\phi - V\nabla\phi\right\|_{B^{s-1}_{2,r}} + \left|\int (\eta^m - \eta) V\nabla\phi\right|. \qquad (2.2.31)
\end{aligned}$$

The first term of (2.2.31) tends to zero as $m \to \infty$, owing to relation (2.2.29). In order to show that the second term of (2.2.31) tends to zero as $m \to \infty$ one proceeds exactly like we did in (2.2.30).

Recovering the time regularity of (η, V) is again classical. One can show for example that for all $j \in \mathbb{Z}$ we have

$$S_j\eta \in \mathcal{C}\left([0,T], B^{s_1}_{2,r}\right)$$

and by using energy estimates:

$$\lim_{j\to\infty} \|\eta - S_j\eta\|_{L^\infty_T\left(B^{s_1}_{2,r}\right)} = 0.$$

As for the uniqueness of solutions let us consider $\left(\eta^1, V^1\right)$, $\left(\eta^2, V^2\right)$ two solutions of $(\mathcal{S}_\varepsilon(\mathcal{D}))$. The system verified by

$$(\delta\eta, \delta V) = \left(\eta^1 - \eta^2, V^1 - V^2\right)$$

is the following:

$$\begin{cases}
(I - \varepsilon b\Delta)\partial_t\delta\eta + \operatorname{div}\delta V + \varepsilon\operatorname{div}\left(\delta\eta\tilde{W}_1 + \tilde{h}\delta V\right) = 0, \\
(I - \varepsilon d\Delta)\partial_t\delta V + \nabla\delta\eta + \varepsilon\tilde{W}_2\cdot\nabla\delta V + \delta V\cdot\nabla\tilde{W}_3 = 0 \\
\eta_{|t=0} = 0,\ V_{|t=0} = 0,
\end{cases} \qquad (2.2.32)$$

with

$$\begin{cases}
\tilde{h} = h + \beta\eta^2, & \tilde{W}_1 = W_1 + \beta V^1, \\
\tilde{W}_2 = W_2 + \beta V^1, & \tilde{W}_3 = W_3 + \beta V^2.
\end{cases}$$

Let us multiply the firs equation of (2.2.32) with $\delta\eta$ and the second one with δV such that by repeated integration by parts, one obtains:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\delta U^2(t) &\leq C\left(\varepsilon\left\|\operatorname{div}\tilde{W}_1\right\|_{L^\infty}+\varepsilon\left\|\operatorname{div}\tilde{W}_2\right\|_{L^\infty}\right)\delta U^2(t)\\ &+C\left(\varepsilon\left\|\nabla\tilde{W}_3\right\|_{L^\infty}+\frac{\sqrt{\varepsilon}\left(\left\|\tilde{h}\right\|_{L^\infty}+\left\|\nabla\tilde{h}\right\|_{L^\infty}\right)}{\max\left\{\sqrt{b},\sqrt{d}\right\}}\right)\delta U^2(t)\end{aligned} \tag{2.2.33}$$

with

$$\delta U^2(t) \overset{not.}{:=} \int \|(\delta\eta,\delta V)\|_{L^2}^2+\|(b\nabla\delta\eta,d\nabla\delta V)\|_{L^2}^2.$$

As $\delta U(0)=0$, uniqueness follows by Gronwall's Lemma and thus the proof of Theorem 2.2.1 is achieved.

2.2.3 The lower bound on the time of existence

In this section we prove Theorem 2.2.2. Let us consider $(\eta_0,V_0)\in X^s_{b,d,r}$ and let us denote by

$$R_0^\varepsilon \overset{not.}{=} \|(\eta_0,V_0)\|_{X^{s,\varepsilon}_{b,d,r}}.$$

Then, according to Theorem 2.2.1, for all $\varepsilon>0$, there exists an unique maximal solution of $\mathcal{S}_\varepsilon(\mathcal{D}^\varepsilon)$ which we denote by $(\eta^\varepsilon(t),V^\varepsilon(t))\in X^s_{b,d,r}$. We introduce the following notations:

$$\begin{cases} U_j^2(\varepsilon,t)=\left\|\left(\eta_j^\varepsilon(t),V_j^\varepsilon(t)\right)\right\|_{L^2}^2+\varepsilon\left\|\left(\sqrt{b}\nabla\eta_j^\varepsilon(t),\sqrt{d}\nabla V_j^\varepsilon(t)\right)\right\|_{L^2}^2,\\ \|(\eta^\varepsilon(t),V^\varepsilon(t))\|_{X^{s,\varepsilon}_{b,d,r}}=U_s(\varepsilon,t)=\left\|\left(2^{js}U_j(\varepsilon,t)\right)_{j\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})}.\end{cases} \tag{2.2.34}$$

Let us consider

$$T_\star^\varepsilon=\sup\left\{T\in[0,T^\varepsilon]:\sup_{t\in[0,T]}\|(\eta^\varepsilon(t),V^\varepsilon(t))\|_{X^{s,\varepsilon}_{b,d,r}}\leq\left(1+e\sqrt{7}\right)R_0^\varepsilon\right\},$$

and

$$\varepsilon_0=\min\{\varepsilon_{01},\varepsilon_{02},\varepsilon_{03},\varepsilon_{04}\},$$

where

$$\begin{cases}\varepsilon_{01}=\frac{3}{4C_1\left(1+e\sqrt{7}\right)R_0^1+4\sup\limits_{\tilde{\varepsilon}\in[0,1]}\sup\limits_{\tau\in\left[0,T^{\tilde{\varepsilon}}\right]}\|h^{\tilde{\varepsilon}}(\tau)\|_{L^\infty}},\ \varepsilon_{02}=\frac{1}{2\left(1+e\sqrt{7}\right)R_0^1+2\sup\limits_{\tilde{\varepsilon}\in[0,1]}\sup\limits_{\tau\in\left[0,T^{\tilde{\varepsilon}}\right]}\mathcal{W}_s^{\tilde{\varepsilon}}(\tau)},\\ \varepsilon_{03}=\frac{\left(1+e\sqrt{7}\right)R_0^0+\sup\limits_{\tilde{\varepsilon}\in[0,1]}\sup\limits_{\tau\in\left[0,T^{\tilde{\varepsilon}}\right]}\mathcal{W}_s^{\tilde{\varepsilon}}(\tau)}{2\sup\limits_{\tilde{\varepsilon}\in[0,1]}\sup\limits_{\tau\in\left[0,T^{\tilde{\varepsilon}}\right]}F_s^{\tilde{\varepsilon}}(\tau)},\ \varepsilon_{04}=\frac{1}{2\left(1+e\sqrt{7}\right)R_0^1+2\sup\limits_{\tilde{\varepsilon}\in[0,1]}\sup\limits_{\tau\in\left[0,T^{\tilde{\varepsilon}}\right]}\mathcal{W}_s^{\tilde{\varepsilon}}(\tau)}.\end{cases}$$

where C_1 is the constant appearing in the embedding $B^{\frac{n}{2}}_{2,1}\hookrightarrow L^\infty$ i.e.

$$\|f\|_{L^\infty}\leq C_1\|f\|_{B^{\frac{n}{2}}_{2,1}}.$$

Let us put

$$\tilde{C}=\min\left\{\frac{R_0^0}{3eC\sup\limits_{\tilde{\varepsilon}\in[0,1]}\sup\limits_{\tau\in\left[0,T^{\tilde{\varepsilon}}\right]}F_s^{\tilde{\varepsilon}}(\tau)},\frac{1}{16C\left(1+e\sqrt{7}\right)R_0^1},\frac{1}{16C\sup\limits_{\tilde{\varepsilon}\in[0,1]}\sup\limits_{\tau\in\left[0,T^{\tilde{\varepsilon}}\right]}\mathcal{W}_s^{\tilde{\varepsilon}}(\tau)}\right\} \tag{2.2.35}$$

where C is the universal constant appearing in (2.2.23). We claim that for all $\varepsilon \leq \varepsilon_0$

$$T_\star^\varepsilon \geq \min\left\{\frac{\tilde{C}}{\varepsilon}, T^\varepsilon\right\}.$$

Let us suppose that this is not the case. Then there exists an $\varepsilon > 0$ such that

$$T_\star^\varepsilon < \min\left\{\frac{\tilde{C}}{\varepsilon}, T^\varepsilon\right\}. \tag{2.2.36}$$

We begin by writing that

$$\frac{1}{4} \leq 1 + \varepsilon\left(\eta^\varepsilon(t,x) + h^\varepsilon(t,x)\right) \leq \frac{7}{4},$$

for all $(t,x) \in [0, T_\star^\varepsilon] \times \mathbb{R}^n$, owing to the fact that

$$\varepsilon \leq \varepsilon_{01}. \tag{2.2.37}$$

Thus, denoting by

$$N_j^2(\varepsilon,t) := \int_{\mathbb{R}^n} \left(\eta_j^\varepsilon\right)^2(t) + \varepsilon b\left|\nabla\eta_j^\varepsilon(t)\right|^2 + \left(1+\varepsilon\left(\eta^\varepsilon(t) + h^\varepsilon(t)\right)\right)\left(\left|V_j^\varepsilon\right|^2(t) + \varepsilon d\nabla V_j^\varepsilon(t) : \nabla V_j^\varepsilon(t)\right)$$

we see that for all $t \in [0, T_\star^\varepsilon]$ we get that:

$$\frac{1}{2} U_j(\varepsilon,t) \leq N_j(\varepsilon,t) \leq \frac{\sqrt{7}}{2} U_j(\varepsilon,t). \tag{2.2.38}$$

Recall that according to (2.2.23) we have that:

$$\frac{d}{dt} N_j^2(\varepsilon,t) \leq \varepsilon C U_j^2(\varepsilon,t)\left(\varepsilon F_s^\varepsilon(t) + \mathcal{W}_s^\varepsilon(t) + U_s(\varepsilon,t) + \varepsilon\left(\mathcal{W}_s^\varepsilon(t) + U_s(\varepsilon,t)\right)^2\right) \tag{2.2.39}$$

$$+\varepsilon C 2^{-js} c_j^\varepsilon(t)\, U_j(\varepsilon,t)\, F_s^\varepsilon(t)\left\{1+\varepsilon\left(\mathcal{W}_s^\varepsilon(t) + U_s(\varepsilon,t)\right)\right\} \tag{2.2.40}$$

$$+\varepsilon C 2^{-js} c_j^\varepsilon(t)\, U_j(\varepsilon,t)\, U_s(\varepsilon,t)\left(\mathcal{W}_s^\varepsilon(t) + U_s(\varepsilon,t)\right)\left\{1+\varepsilon\left(\mathcal{W}_s^\varepsilon(t) + U_s(\varepsilon,t)\right)\right\} \tag{2.2.41}$$

and using (2.2.38) we get that:

$$U_j(\varepsilon,t) \leq \sqrt{7} U_j(\varepsilon,0) + 2\varepsilon C\int_0^t U_j(\varepsilon,\tau)\left(\varepsilon F_s^\varepsilon(\tau) + \mathcal{W}_s^\varepsilon(\tau) + U_s(\varepsilon,\tau)\right)$$

$$+ 2\varepsilon^2 C\int_0^t \left(\mathcal{W}_s^\varepsilon(\tau) + U_s(\varepsilon,\tau)\right)^2 \tag{2.2.42}$$

$$+ 2\varepsilon C 2^{-js}\int_0^t c_j F_s^\varepsilon(\tau)\left(1+\varepsilon\left(\mathcal{W}_s^\varepsilon(\tau) + U_s(\varepsilon,\tau)\right)\right) \tag{2.2.43}$$

$$+ 2\varepsilon C 2^{-js}\int_0^t U_s(\varepsilon,\tau)\left(\mathcal{W}_s^\varepsilon(\tau) + U_s(\varepsilon,\tau)\right)\left(1+\varepsilon\left(\mathcal{W}_s^\varepsilon(\tau) + U_s(\varepsilon,\tau)\right)\right). \tag{2.2.44}$$

Multiplying (2.2.43) with 2^{js} and performing an $\ell^r(\mathbb{Z})$-summation we end up with

$$\begin{aligned} U_s(\varepsilon,t) &\leq \sqrt{7} U_s(\varepsilon,0) + 2\varepsilon C\int_0^t F_s^\varepsilon(\tau)\left(1+\varepsilon\left(\mathcal{W}_s^\varepsilon(\tau) + U_s(\varepsilon,\tau)\right)\right) d\tau \\ &\quad + 4\varepsilon C\int_0^t U_s(\varepsilon,\tau)\left(\varepsilon F_s^\varepsilon(\tau) + \mathcal{W}_s^\varepsilon(\tau) + U_s(\varepsilon,\tau) + \varepsilon\left(\mathcal{W}_s^\varepsilon(\tau) + U_s(\varepsilon,\tau)\right)^2\right) d\tau \\ &\leq \sqrt{7} R_0^\varepsilon + 3\varepsilon C T_\star^\varepsilon \sup_{\tilde{\varepsilon}\in[0,1]} \sup_{\tau\in[0,T^{\tilde{\varepsilon}}]} F_s^{\tilde{\varepsilon}}(\tau) \end{aligned}$$

$$+4\varepsilon C\int_0^t U_s(\varepsilon,\tau)\left(\varepsilon F_s^{\varepsilon}(\tau)+\mathcal{W}_s^{\varepsilon}(\tau)+U_s(\varepsilon,\tau)+\varepsilon\left(\mathcal{W}_s^{\varepsilon}+U_s(\varepsilon,\tau)\right)^2\right)d\tau, \tag{2.2.45}$$

owing to the fact that ε has been chosen such that:

$$\varepsilon\leq\varepsilon_{02}. \tag{2.2.46}$$

Using the definition of $\tilde{C}$ we get that

$$T_\star^{\varepsilon}<\frac{R_0^0}{3\varepsilon eC\sup\limits_{\tilde{\varepsilon}\in[0,1]}\sup\limits_{\tau\in[0,T^{\tilde{\varepsilon}}]}F_s^{\tilde{\varepsilon}}(\tau)}<\frac{R_0^{\varepsilon}}{3\varepsilon eC\sup\limits_{\tilde{\varepsilon}\in[0,1]}\sup\limits_{\tau\in[0,T^{\tilde{\varepsilon}}]}F_s^{\tilde{\varepsilon}}(\tau)} \tag{2.2.47}$$

and consequently, we get that:

$$U_s(\varepsilon,t)\leq\left(e^{-1}+\sqrt{7}\right)R_0^{\varepsilon}+8\varepsilon C\left(\left(1+e\sqrt{7}\right)R_0^{\varepsilon}+\sup_{\tilde{\varepsilon}\in[0,1]}\sup_{\tau\in[0,T^{\tilde{\varepsilon}}]}\mathcal{W}_s^{\tilde{\varepsilon}}(\tau)\right)\int_0^t U_s(\varepsilon,\tau)\,d\tau, \tag{2.2.48}$$

where we have used that $\varepsilon\leq\varepsilon_{03}$ respectively $\varepsilon\leq\varepsilon_{04}$. The estimate (2.2.48) along with Gronwall's Lemma imply that:

$$U_s(\varepsilon,t)\leq\left(e^{-1}+\sqrt{7}\right)R_0^{\varepsilon}\exp\left(8\varepsilon CT_\star^{\varepsilon}\left(\left(1+e\sqrt{7}\right)R_0^{\varepsilon}+\sup_{\tilde{\varepsilon}\in[0,1]}\sup_{\tau\in[0,T^{\tilde{\varepsilon}}]}\mathcal{W}_s^{\tilde{\varepsilon}}(\tau)\right)\right).$$

Owing to

$$\begin{aligned}T_\star^{\varepsilon}&<\frac{1}{\varepsilon}\min\left\{\frac{1}{16C\left(1+e\sqrt{7}\right)R_0^1},\frac{1}{16C\sup\limits_{\tilde{\varepsilon}\in[0,1]}\sup\limits_{\tau\in[0,T^{\tilde{\varepsilon}}]}\mathcal{W}_s^{\tilde{\varepsilon}}(\tau)}\right\}\\&\leq\frac{1}{8\varepsilon C\left(\left(1+e\sqrt{7}\right)R_0^1+\sup\limits_{\tilde{\varepsilon}\in[0,1]}\sup\limits_{\tau\in[0,T^{\tilde{\varepsilon}}]}\mathcal{W}_s^{\tilde{\varepsilon}}(\tau)\right)}\end{aligned}$$

we obtain that

$$\sup_{t\in[0,T_\star^{\varepsilon}]}U_s(\varepsilon,t)<\left(1+e\sqrt{7}\right)R_0^{\varepsilon}$$

which in view of the time continuity of $(\eta^{\varepsilon},V^{\varepsilon})$ is a contradiction. Thus, our initial suppositions is false. Thus, if $\varepsilon\leq\varepsilon_0$ we get that

$$T_\star^{\varepsilon}\geq\min\left\{T^{\varepsilon},\frac{\tilde{C}}{\varepsilon}\right\}.$$

By the definition of $T_\star^{\varepsilon}$ we have that

$$\sup_{t\in[0,T_\star^{\varepsilon}]}\left\|\left(\eta^{\varepsilon}(t),V^{\varepsilon}(t)\right)\right\|_{X_{b,d,r}^{s,\varepsilon}}\leq\left(1+e\sqrt{7}\right)R_0^{\varepsilon}. \tag{2.2.49}$$

Observe that according to the uniform bounds of (2.2.49) and relation (2.2.19) we get that

$$\sup_{t\in[0,T_\star^{\varepsilon}]}\left\|\partial_t\eta^{\varepsilon}(t)\right\|_{L^\infty}\leq 2C'\left(1+e\sqrt{7}\right)R_0^{\varepsilon},$$

where C' is the constant appearing in relation (2.2.19). This ends the proof of Theorem 2.2.2.

2.3 Proofs of the main results

2.3.1 The 1d case: proof of Theorem 2.1.1

In the following lines we prove the 1-dimensional result announced in Theorem 2.1.1. For the reader's convenience, let us write below the system:

$$\begin{cases} \left(I-\varepsilon b\partial_{xx}^2\right)\partial_t\bar{\eta}+\partial_x\bar{u}+\varepsilon\partial_x\left(\bar{\eta}\bar{u}\right)=0,\\ \left(I-\varepsilon d\partial_{xx}^2\right)\partial_t\bar{u}+\partial_x\bar{\eta}+\varepsilon\bar{u}\partial_x\bar{u}=0,\\ \bar{\eta}_{|t=0}=\eta_0,\ \ \bar{u}_{|t=0}=u_0. \end{cases} \tag{2.3.1}$$

The strategy of the proof is the following. First, we split the initial data into low-high frequency parts i.e. :

$$\begin{aligned} (\eta_0,u_0)&=\left(\Delta_{-1}\eta_0,\Delta_{-1}u_0\right)+\left(\left(I-\Delta_{-1}\right)\eta_0,\left(I-\Delta_{-1}\right)u_0\right)\\ &\overset{not.}{=}\left(\eta_0^{low},u_0^{low}\right)+\left(\eta_0^{high},u_0^{high}\right). \end{aligned}$$

We solve the linear acoustic waves system with initial data $\left(\eta_0^{low},u_0^{low}\right)$. Searching for a solution $(\bar{\eta},\bar{u})$ of (2.3.1) in the form $(\eta+\eta_L,u+u_L)$ we observe that in fact (η,u) verifies a system of type $\left(\mathcal{S}_\varepsilon\left(\mathcal{D}\right)\right)$. In view of the uniform bounds (with respect to time) of (η_L,u_L) we obtain the existence of the pair (η,u) as a consequence of Corollary 2.2.3. Finally, we prove the uniqueness property by performing classical L^2-energy estimates with respect to the system of equations governing the difference of two solutions with the same initial data.

Owing to Bernstein's Lemma (1.1.1) we have:

$$\left\|\left(\eta_0^{low},u_0^{low}\right)\right\|_{L^\infty}\leq C_1\left\|(\eta_0,u_0)\right\|_{L^\infty}. \tag{2.3.2}$$

Moreover, it transpires that $\left(\eta_0^{high},u_0^{high}\right)\in X_{b,d,r}^s$ and that

$$\left\|\left(\eta_0^{high},u_0^{high}\right)\right\|_{X_{b,d,r}^{s,\varepsilon}}\leq C_2\left\|(\partial_x\eta_0,\partial_xu_0)\right\|_{X_{b,d,r}^{s-1,\varepsilon}}. \tag{2.3.3}$$

Let us consider $(\eta_L,u_L)\in\mathcal{C}\left([0,\infty),E_{b,d,r}^s\left(\mathbb{R}\right)\right)$ the unique solution of the linear acoustic waves system:

$$\begin{cases} \partial_t\eta_L+\partial_xu_L=0,\\ \partial_tu_L+\partial_x\eta_L=0,\\ \eta_{L|t=0}=\eta_0^{low}\ ,\ u_{L|t=0}=u_0^{low} \end{cases} \tag{2.3.4}$$

which is given explicitly by the following relation:

$$\begin{cases} 2\eta_L\left(t,x\right)=\left(\eta_0^{low}\left(x+t\right)+\eta_0^{low}\left(x-t\right)\right)+\left(u_0^{low}\left(x+t\right)-u_0^{low}\left(x-t\right)\right),\\ 2u_L\left(t,x\right)=\left(\eta_0^{low}\left(x+t\right)-\eta_0^{low}\left(x-t\right)\right)+\left(u_0^{low}\left(x+t\right)+u_0^{low}\left(x-t\right)\right), \end{cases} \tag{2.3.5}$$

for all $(t,x)\in\mathbb{R}^+\times\mathbb{R}$. From (2.3.5) we conclude that:

$$\begin{aligned} \left\|\left(\eta_L\left(t\right),u_L\left(t\right)\right)\right\|_{L^\infty}&\leq 2C_1\left\|(\eta_0,u_0)\right\|_{L^\infty},\\ \left\|\left(\partial_x\eta_L\left(t\right),\partial_xu_L\left(t\right)\right)\right\|_{B_{2,r}^\sigma}&\leq 2C_1\left\|(\partial_x\eta_0,\partial_xu_0)\right\|_{B_{2,\infty}^0} \end{aligned} \tag{2.3.6}$$

for all $\sigma\geq 0$. In particular, we also have

$$\left\|\left(\eta_L\left(t\right),u_L\left(t\right)\right)\right\|_{E_{b,d,r}^{s,\varepsilon}}\leq 2C_1\left\|(\eta_0,u_0)\right\|_{E_{b,d}^{s,\varepsilon}} \tag{2.3.7}$$

for all $t\geq 0$. Let us consider:

$$\begin{cases} f_L=-\eta_L\partial_xu_L-u_L\partial_x\eta_L-b\partial_{xxx}^3u_L,\\ g_L=-u_L\partial_xu_L-d\partial_{xxx}^3\eta_L. \end{cases}$$

Owing to the fact that $(\widehat{\eta_L}, \widehat{u_L})$ is supported in a ball centered at the origin, we obtain that

$$\begin{aligned}\|(f_L(t), g_L(t))\|_{B^s_{2,r}} &\leq C_s \Big\{ \|(\eta_L, u_L)\|_{L^\infty} \|(\partial_x \eta_L, \partial_x u_L)\|_{B^s_{2,r}} + \|(\partial_x \eta_L, \partial_x u_L)\|_{L^\infty} \|(\partial_x \eta_L, \partial_x u_L)\|_{B^{s-1}_{2,r}} \\ &\qquad + b \left\|\partial^3_{xxx} u_L\right\|_{B^s_{2,r}} + d \left\|\partial^3_{xxx} \eta_L\right\|_{B^s_{2,r}} \Big\} \\ &\leq C_s \|(\partial_x \eta_0, \partial_x u_0)\|_{B^0_{2,\infty}} \Big(\|(\eta_0, u_0)\|_{L^\infty} + \|(\partial_x \eta_0, \partial_x u_0)\|_{B^0_{2,\infty}} + b + d \Big). \quad (2.3.8)\end{aligned}$$

Owing to the uniform bounds of (η_L, u_L, f_L, g_L) announced in (2.3.6) and in (2.3.8) we can apply Corollary 2.2.3 with $p_1 = 2$, $p_2 = \infty$ in order to obtain the existence of two positive real numbers ε_0, C depending on s, b, d and the norm of the initial data such that for any $\varepsilon \leq \varepsilon_0$ we may consider $(\eta^\varepsilon, u^\varepsilon) \in \mathcal{C}\left([0, \frac{C}{\varepsilon}], X^s_{b,d,r}\right)$ the solution of

$$\begin{cases} \left(I - \varepsilon b \partial^2_{xx}\right) \partial_t \eta + \partial_x u + \varepsilon \partial_x (\eta u_L + \eta_L u + \eta u) = \varepsilon f_L, \\ \left(I - \varepsilon d \partial^2_{xx}\right) \partial_t u + \partial_x \eta + \varepsilon (u + u_L) \partial_x u + u \partial_x u_L = \varepsilon g_L, \\ \eta_{|t=0} = \eta_0^{high}, \ u_{|t=0} = u_0^{high} \end{cases} \quad (2.3.9)$$

which satisfies

$$\begin{aligned}\sup_{t \in \left[0, \frac{C}{\varepsilon}\right]} \Big(\|(\eta^\varepsilon(t), u^\varepsilon(t))\|_{X^{s,\varepsilon}_{b,d,r}} + \|\partial_t \eta^\varepsilon(t)\|_{L^\infty} \Big) &\leq \tilde{C} \left\| \left(\eta_0^{high}, u_0^{high}\right) \right\|_{X^{s,\varepsilon}_{b,d,r}} \\ &\leq \tilde{C} C_2 \|(\partial_x \eta_0, \partial_x u_0)\|_{X^{s-1,\varepsilon}_{b,d,r}}, \quad (2.3.10)\end{aligned}$$

for some numerical constant $\tilde{C}$. Considering

$$\bar{\eta}^\varepsilon = \eta^\varepsilon + \eta_L, \ \bar{u}^\varepsilon = u^\varepsilon + u_L$$

we see that $(\bar{\eta}^\varepsilon, \bar{u}^\varepsilon) \in \mathcal{C}\left([0, \infty), E^s_{b,d,r}(\mathbb{R})\right)$ and for all $t \in [0, \frac{C}{\varepsilon}]$ we have that:

$$\begin{aligned}&\|(\bar{\eta}^\varepsilon(t), \bar{u}(t))\|_{E^{s,\varepsilon}_{b,d}} + \|\partial_t \bar{\eta}^\varepsilon(t)\|_{L^\infty} \\ &\leq \|(\eta^\varepsilon(t), u^\varepsilon(t))\|_{E^{s,\varepsilon}_{b,d,r}} + \|\partial_t \eta^\varepsilon(t)\|_{L^\infty} + \|(\eta_L(t), u_L(t))\|_{E^{s,\varepsilon}_{b,d,r}} + \|\partial_t \eta_L(t)\|_{L^\infty} \\ &\leq 2C_1 \|(\eta_0, u_0)\|_{E^{s,\varepsilon}_{b,d,r}} + \|\partial_x u_L(t)\|_{L^\infty} + \tilde{C} C_2 \|(\partial_x \eta_0, \partial_x u_0)\|_{X^{s-1,\varepsilon}_{b,d,r}} \\ &\leq C_3 \|(\eta_0, u_0)\|_{E^{s,\varepsilon}_{b,d,r}},\end{aligned}$$

where we used (2.3.6), (2.3.7) and (2.3.10).

Proving the uniqueness of the solution is done in the following lines. Let us suppose that $\left(\bar{\eta}^1, \bar{u}^1\right)$, $\left(\bar{\eta}^2, \bar{u}^2\right) \in \mathcal{C}\left([0, T], E^s_{b,d,r}\right)$ are two solutions of (2.3.1). Then, we observe that:

$$\begin{cases} \left(I - \varepsilon b \partial^2_{xx}\right) \partial_t \delta\bar{\eta} + \partial_x \delta\bar{u} + \varepsilon \partial_x \left(\bar{u}^1 \delta\bar{\eta} + \bar{\eta}^2 \delta\bar{u}\right) = 0, \\ \left(I - \varepsilon d \partial^2_{xx}\right) \partial_t \delta\bar{u} + \partial_x \delta\bar{\eta} + \varepsilon \bar{u}^1 \partial_x \delta\bar{u} + \varepsilon \delta\bar{u} \partial_x \bar{u}^2 = 0, \\ \delta\bar{\eta}_{|t=0} = 0, \ \delta\bar{u}_{|t=0} = 0. \end{cases} \quad (2.3.11)$$

Writing the first equation of (2.3.11) in integral form we get that

$$\delta\bar{\eta}(t) = \int_0^t \left(I - \varepsilon b \partial^2_{xx}\right)^{-1} \left(\partial_x \delta\bar{u} + \delta\bar{\eta} \partial_x \bar{u}^1 + \bar{u}^1 \partial_x \delta\bar{\eta} + \bar{\eta}^2 \partial_x \delta\bar{u} + \delta\bar{u} \partial_x \bar{\eta}^2\right) d\tau$$

and because

$$\partial_x \delta\bar{u} + \delta\bar{\eta} \partial_x \bar{u}^1 + \bar{u}^1 \partial_x \delta\bar{\eta} + \bar{\eta}^2 \partial_x \delta\bar{u} + \delta\bar{u} \partial_x \bar{\eta}^2 \in L^2(\mathbb{R})$$

we get that

$$\delta\bar{\eta} \in H^{2\,\mathrm{sgn}(b)}(\mathbb{R}).$$

Similarly

$$\delta\bar{u} \in H^{2\,\mathrm{sgn}(d)}\left(\mathbb{R}\right).$$

Thus, multiplying the first equation of (2.3.11) with $\delta\bar{\eta}$ and the second one with $\delta\bar{u}$ and using Gronwall's lemma will lead us to the conclusion $(\delta\bar{\eta}, \delta\bar{u}) = (0,0)$ on $[0,T]$. This ends the proof of Theorem 2.1.1.

2.3.2 The 2d case: proof of Theorem 2.1.2

For the reader's convenience, let us rewrite below the problem that we wish to solve:

$$\begin{cases} (I - \varepsilon b\Delta)\,\partial_t\bar{\eta} + \operatorname{div}\bar{V} + \varepsilon\operatorname{div}\left(\bar{\eta}\bar{V}\right) = 0, \\ (I - \varepsilon d\Delta)\,\partial_t\bar{V} + \nabla\bar{\eta} + \varepsilon\bar{V}\cdot\nabla\bar{V} = 0, \\ \bar{\eta}_{|t=0}\left(x,y\right) = \eta_0\left(x\right) + \phi\left(x,y\right), \\ \bar{V}_{|t=0}\left(x,y\right) = \left(u_0\left(x\right),0\right) + \psi\left(x,y\right). \end{cases} \tag{2.3.12}$$

According to Theorem 2.1.1, there exist $\varepsilon_0, C^{1D} > 0$, depending on σ, b, d and the norm of the initial data (η_0, u_0) such that for any $\varepsilon \leq \varepsilon_0$ we may consider $\left(\eta^{\varepsilon,1D}, u^{\varepsilon,1D}\right) \in \mathcal{C}\left([0, \frac{C^{1D}}{\varepsilon}], E^{\sigma}_{b,d,r}\left(\mathbb{R}\right)\right)$ the unique solution of the 1-dimensional system (2.3.1) with initial data (η_0, u_0) and

$$\sup_{t\in[0,\frac{C^{1D}}{\varepsilon}]} \left\|\left(\eta^{\varepsilon,1D}\left(t\right), u^{\varepsilon,1D}\left(t\right)\right)\right\|_{E^{\sigma,\varepsilon}_{b,d,r}} + \sup_{t\in[0,\frac{C^{1D}}{\varepsilon}]} \left\|\partial_t\eta^{\varepsilon,1D}\left(t\right)\right\|_{L^\infty} \leq \tilde{C}_1\left\|(\eta_0, u_0)\right\|_{E^{\sigma,\varepsilon}_{b,d,r}}, \tag{2.3.13}$$

with $\tilde{C}_1$ some numerical constant. Let us observe that if we consider

$$\begin{cases} \eta_1^{\varepsilon}\left(t,x,y\right) = \eta^{1D}\left(t,x\right), \\ V_1^{\varepsilon}\left(t,x,y\right) = \left(u^{1D}\left(t,x\right),0\right), \end{cases}$$

for all $(t,x,y) \in [0, \frac{C^{1D}}{\varepsilon}] \times \mathbb{R}^2$, then:

$$\begin{cases} (I - \varepsilon b\Delta)\,\partial_t\eta_1^{\varepsilon} + \operatorname{div}V_1^{\varepsilon} + \varepsilon\operatorname{div}\left(\eta_1^{\varepsilon}V_1^{\varepsilon}\right) = 0, \\ (I - \varepsilon d\Delta)\,\partial_t V_1^{\varepsilon} + \nabla\eta_1^{\varepsilon} + \varepsilon V_1^{\varepsilon}\cdot\nabla V_1^{\varepsilon} = 0, \\ \eta^{\varepsilon}_{1|t=0}\left(x,y\right) = \eta_0\left(x\right),\ V^{\varepsilon}_{1|t=0} = \left(u_0\left(x\right),0\right). \end{cases}$$

Observe that

$$\partial_x\eta^{\varepsilon,1D}\left(t\right) \in B^{\sigma-1}_{2,r}\left(\mathbb{R}\right) \hookrightarrow B^{\sigma-\frac{3}{2}}_{\infty,\infty}\left(\mathbb{R}\right).$$

Let us observe that for all (x_1,y_1),$(x_2,y_2) \in \mathbb{R}^2$ such that

$$\left|x_1 - x_2\right|^2 + \left|y_1 - y_2\right|^2 \leq 1$$

the following holds true

$$\begin{aligned} \left|\nabla\eta_1^{\varepsilon}\left(t,x_1,y_1\right) - \nabla\eta_1^{\varepsilon}\left(t,x_2,y_2\right)\right| &= \left|\partial_x\eta^{\varepsilon,1D}\left(t,x_1\right) - \partial_x\eta^{\varepsilon,1D}\left(t,x_2\right)\right| \\ &\leq C\left\|\partial_x\eta^{\varepsilon,1D}\left(t\right)\right\|_{B^{\sigma-\frac{3}{2}}_{\infty,\infty}(\mathbb{R})}\left|x_1 - x_2\right|^{\left(\sigma-\frac{3}{2}\right)-\left[\sigma-\frac{3}{2}\right]}. \end{aligned}$$

Using the fact that $B^{\sigma-\frac{3}{2}}_{\infty,\infty}\left(\mathbb{R}^2\right) = \mathcal{C}^{\sigma-\frac{3}{2},\sigma-\frac{3}{2}-\left[\sigma-\frac{3}{2}\right]}\left(\mathbb{R}^2\right)$, see Remark 2.1.2 and relation (2.1.7), we get that $\nabla\eta_1^{\varepsilon}\left(t\right) = \left(\partial_x\eta^{\varepsilon,1D}\left(t\right),0\right) \in B^{s}_{\infty,r}\left(\mathbb{R}^2\right)$ with

$$\left\|\nabla\eta_1^{\varepsilon}\left(t\right)\right\|_{B^{s}_{\infty,r}(\mathbb{R}^2)} \leq C\left\|\partial_x\eta^{\varepsilon,1D}\left(t\right)\right\|_{B^{\sigma-1}_{2,r}(\mathbb{R})}. \tag{2.3.14}$$

Similarly, we get:

$$\left\|\nabla V_1^{\varepsilon}\right\|_{B^{s}_{\infty,r}(\mathbb{R}^2)} \leq C\left\|\partial_x u^{\varepsilon,1D}\right\|_{B^{\sigma-1}_{2,r}(\mathbb{R})}. \tag{2.3.15}$$

It is also clear that:

$$\|(\eta_1^\varepsilon, V_1^\varepsilon)\|_{L^\infty(\mathbb{R}^2)} = \left\|\left(\eta^{\varepsilon,1D}, u^{\varepsilon,1D}\right)\right\|_{L^\infty(\mathbb{R})} \text{ and } \|\partial_t\eta_1^\varepsilon\|_{L^\infty(\mathbb{R}^2)} = \left\|\partial_t\eta^{\varepsilon,1D}\right\|_{L^\infty(\mathbb{R})}, \tag{2.3.16}$$

and thus, using (2.3.14), (2.3.15), (2.3.16) respectively the uniform bounds of (2.3.13), we get that:

$$\sup_{\varepsilon\in[0,\varepsilon_0]} \sup_{t\in[0,\frac{C_{1D}}{\varepsilon}]} \left(\|(\eta_1^\varepsilon(t), V_1^\varepsilon(t))\|_{L^\infty(\mathbb{R}^2)} + \|(\nabla\eta_1^\varepsilon(t), \nabla V_1^\varepsilon(t))\|_{B^s_{\infty,r}(\mathbb{R}^2)}\right) + \sup_{t\in[0,\frac{C_{1D}}{\varepsilon}]} \|\partial_t\eta_1^\varepsilon(t)\|_{L^\infty(\mathbb{R}^2)}$$
$$\leq \tilde{C}_2 \|(\eta_0, u_0)\|_{E^{s,1}_{b,d,r}}. \tag{2.3.17}$$

Let us consider the system:

$$\begin{cases} (I - \varepsilon b\Delta)\,\partial_t\eta + \operatorname{div} V + \varepsilon \operatorname{div}(\eta V_1^\varepsilon + \eta_1^\varepsilon V + \eta V) = 0, \\ (I - \varepsilon d\Delta)\,\partial_t V + \nabla\eta + \varepsilon(V_1^\varepsilon + V)\cdot\nabla V + \varepsilon V\cdot\nabla V_1^\varepsilon = 0, \\ \eta_{|t=0} = \phi,\ V_{|t=0} = \psi. \end{cases}$$

Using the uniform bounds of $(\eta_1^\varepsilon, V_1^\varepsilon)_{\varepsilon\leq\varepsilon_0}$ and owing to Theorem 2.2.2, there exist two real numbers $C^{2D} \leq C^{1D}$, $\varepsilon_1 \leq \varepsilon_0$ depending on s, σ, b, d and on $\|(\eta_0, u_0)\|_{E^{s,1}_{b,d,r}} + \|(\phi,\psi)\|_{X^{s,1}_{b,d,r}}$ such that for any $\varepsilon \leq \varepsilon_1$ we can uniquely construct $\left(\eta^{\varepsilon,2D}, V^{\varepsilon,2D}\right) \in \mathcal{C}\left([0, \frac{C^{2D}}{\varepsilon}], X^s_{b,d,r}\right)$ and, moreover,

$$\sup_{t\in\left[0,\frac{C^{2D}}{\varepsilon}\right]} \left\|\left(\eta^{\varepsilon,2D}(t), V^{\varepsilon,2D}(t)\right)\right\|_{X^{s,\varepsilon}_{b,d,r}} \leq \tilde{C}_3 \left(\|(\eta_0,u_0)\|_{E^{s,\varepsilon}_{b,d,r}} + \|(\phi,\psi)\|_{X^{s,\varepsilon}_{b,d,r}}\right) \tag{2.3.18}$$

where $\tilde{C}_3$ is numerical constant. Then, we see that defining

$$\begin{cases} \bar{\eta}^\varepsilon(t,x,y) = \eta^{\varepsilon,1D}(t,x) + \eta^{\varepsilon,2D}(t,x,y), \\ \bar{V}^\varepsilon(t,x,y) = \left(u^{\varepsilon,1D}(t,x), 0\right) + V^{\varepsilon,2D}(t,x,y) \end{cases}$$

for all $(t,x,y) \in [0, \frac{C^{2D}}{\varepsilon}] \times \mathbb{R}^2$ is a solution of (2.1.1) which, by construction, belongs to $M^{\sigma,s}_{b,d,r}$. Moreover, owing to (2.3.13) and (2.3.18) there exists a constant $\tilde{C}_4$ such that:

$$\left\|\left(\bar{\eta}^\varepsilon, \bar{V}^\varepsilon\right)\right\|_{M^{\sigma,s,\varepsilon}_{b,d,r}} \leq \tilde{C}_4 \left(\|(\eta_0,u_0)\|_{E^{s,\varepsilon}_{b,d,r}} + \|(\phi,\psi)\|_{X^{s,\varepsilon}_{b,d,r}}\right).$$

We proceed by proving the uniqueness property. Consider two solutions $\left(\eta^i + \phi^i, \left(u^i + \psi_1^i, \psi_2^i\right)\right) \in \mathcal{C}\left([0,T]; M^{\sigma,s}_{b,d,r}\right)$ with the same initial data and we introduce the following notations:

$$\begin{cases} \delta\eta = \eta^1 - \eta^2, \delta u = u^1 - u^2, \\ \delta\phi = \phi^1 - \phi^2, \delta\psi_i = \psi_i^1 - \psi_i^2. \end{cases}$$

For any $\rho \in \mathcal{C}\left([0,T]; \mathcal{S}\left(\mathbb{R}^2\right)\right)$ we get that:

$$\int_0^t \int_{\mathbb{R}^2} (\delta\eta + \delta\phi)(I - \varepsilon b\Delta)\,\partial_t\rho + \int_0^t \int_{\mathbb{R}^2} \left(\varepsilon(\delta\eta + \delta\phi)\left(u^1 + \psi_1^1\right) + \left(1 + \varepsilon\left(\eta^2 + \phi^2\right)\right)(\delta u + \delta\psi_1)\right)\partial_x\rho$$
$$+ \int_0^t \int_{\mathbb{R}^2} \left(\varepsilon(\delta\eta + \delta\phi)\,\psi_2^1 + \left(1 + \varepsilon\left(\eta^2 + \phi^2\right)\right)\delta\psi_2\right)\partial_y\rho - \int_{\mathbb{R}^2} (\delta\eta(t) + \delta\phi(t))(I - \varepsilon b\Delta)\,\rho(t) = 0. \tag{2.3.19}$$

Taking for any $m \in \mathbb{N}^\star$

$$\rho^m(t,x,y) = \frac{1}{m}\tilde{\rho}(t,x)\,\chi\left(\frac{y}{m}\right)$$

with $\tilde{\rho} \in \mathcal{C}\left([0,T];\mathcal{S}\left(\mathbb{R}\right)\right)$ and $\chi \in \mathcal{S}\left(\mathbb{R}\right)$ with

$$\int_{\mathbb{R}} \chi dx = 1,$$

we find that

$$\int_0^t \int_{\mathbb{R}} \delta\eta \left(I - \varepsilon b \partial_{xx}^2\right) \partial_t \tilde{\rho} + \int_0^t \int_{\mathbb{R}} \left(\varepsilon \delta\eta u^1 + \left(1 + \varepsilon \eta^2\right) \delta u\right) \partial_x \tilde{\rho}$$
$$- \int_{\mathbb{R}} \delta\eta\left(t\right) \left(I - \varepsilon b \partial_{xx}^2\right) \tilde{\rho}\left(t\right) = o\left(m\right) \text{ when } m \to \infty.$$

Indeed, let us observe that the first term of (2.3.19) rewrites

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^2} \left(\delta\eta(x) + \delta\phi(x,y)\right) \left(I - \varepsilon b \Delta\right) \partial_t \rho^m\left(t,x,y\right) \\
&= \int_0^t \int_{\mathbb{R}} \delta\eta(x) \left(I - \varepsilon b \partial_{xx}^2\right) \partial_t \tilde{\rho}(t,x) + \int_0^t \int_{\mathbb{R}^2} \delta\phi(x,y) \left(I - \varepsilon b \partial_{xx}^2\right) \partial_t \tilde{\rho}(t,x) \frac{1}{m} \chi\left(\frac{y}{m}\right) dxdy \\
&+ \frac{1}{m^3} \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(\delta\eta(x) + \delta\phi(x,y)\right) \partial_t \tilde{\rho}(t,x) \chi''\left(\frac{y}{m}\right) dxdy \\
&= \int_0^t \int_{\mathbb{R}} \delta\eta(x) \left(I - \varepsilon b \partial_{xx}^2\right) \partial_t \tilde{\rho}(t,x) + \int_0^t \int_{\mathbb{R}^2} \delta\phi(x,my) \left(I - \varepsilon b \partial_{xx}^2\right) \partial_t \tilde{\rho}(t,x) \chi\left(y\right) dxdy \\
&+ \frac{1}{m^2} \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(\delta\eta(x) + \delta\phi(x,my)\right) \partial_t \tilde{\rho}(t,x) \chi''\left(y\right) dxdy.
\end{aligned}$$

However owing to $\delta\phi \in B_{2,r}^{s_b} \hookrightarrow B_{2,1}^{\frac{n}{2}}$ we have that

$$\lim_{|x|+|y| \to \infty} \delta\phi(x,y) = 0,$$

which combined with the dominated convergence theorem yields

$$\lim_{m \to \infty} \int_0^t \int_{\mathbb{R}^2} \delta\phi(x,my) \left(I - \varepsilon b \partial_{xx}^2\right) \partial_t \tilde{\rho}(t,x) \chi\left(y\right) dxdy = 0.$$

Owing to

$$\left|\left(\delta\eta(x) + \delta\phi(x,my)\right) \partial_t \tilde{\rho}(t,x) \chi''\left(y\right)\right| \leq \left(\|\delta\eta\|_{L^\infty} + \|\delta\phi\|_{L^\infty}\right) \left|\partial_t \tilde{\rho}(t,x) \chi''\left(y\right)\right|,$$

one gets that:

$$\lim_{m \to \infty} \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(\delta\eta(x) + \delta\phi(x,my)\right) \partial_t \tilde{\rho}(t,x) \chi''\left(y\right) dxdy.$$

Proceeding similarly we get that for any $\mu \in \mathcal{C}\left([0,T];\mathcal{S}\left(\mathbb{R}\right)\right)$

$$\int_0^t \int_{\mathbb{R}} \delta u \left(I - \varepsilon d \partial_{xx}^2\right) \partial_t \mu + \int_0^t \int_{\mathbb{R}} \left(\delta\eta + \varepsilon u^1 \delta u\right) \partial_x \mu + \int_0^t \int_{\mathbb{R}} \varepsilon \delta u \left(\partial_x u^1 - \partial_x u^2\right) \mu$$
$$- \int_{\mathbb{R}} \delta u\left(t\right) \left(I - \varepsilon d \partial_{xx}^2\right) \mu\left(t\right) = o\left(m\right) \text{ when } m \to \infty.$$

We thus recover that the pair $(\delta\eta, \delta u) \in \mathcal{C}\left([0,T];E_{b,d,r}^{\sigma}\left(\mathbb{R}\right)\right)$ is a solution of

$$\begin{cases} \left(I - \varepsilon b \partial_{xx}\right) \partial_t \delta\eta + \partial_x \delta u + \varepsilon \partial_x \left(\delta\eta u^1 + \eta^2 \delta u\right) = 0, \\ \left(I - \varepsilon d \partial_{xx}\right) \partial_t \delta u + \partial_x \eta + \varepsilon u^1 \partial_x \delta u + \varepsilon \delta u \partial_x u^2 = 0, \\ \delta\bar{\eta}_{|t=0} = 0, \ \delta\bar{u}_{|t=0} = 0. \end{cases} \tag{2.3.20}$$

According to Theorem 2.1.1 this implies that $(\delta\eta, \delta u) = (0, 0)$. It follows that

$$\left\{\begin{array}{l} (I - \varepsilon b\Delta)\, \partial_t \delta\phi + \operatorname{div} \delta\psi + \varepsilon \operatorname{div} \left(\delta\phi \psi^1 + \phi^2 \delta\psi\right) = 0, \\ (I - \varepsilon d\Delta)\, \partial_t \delta\psi + \nabla \delta\phi + \varepsilon \psi^1 \cdot \nabla \delta\psi + \varepsilon \delta\psi \cdot \nabla \psi^2 = 0, \\ \delta\phi_{|t=0} = 0,\ \delta\psi_{|t=0} = 0, \end{array}\right. \tag{2.3.21}$$

which by Theorem 2.2.1 implies that $(\delta\phi,\ \delta\psi) = (0, 0)$. Thus we obtain that the two solutions coincide.

Chapter 3

Numerical solutions for the *abcd* systems

3.1 Statement of the main results

In this chapter, attention is given to the numerical analysis for some of the *abcd* systems in the 1D case. We implement finite volume schemes, and we prove convergence and stability estimates, using a discrete variant of the "natural" energy functional associated to these systems. One practical advantage of our approach is that it allows us to treat explicitly the nonlinear terms.

Let us recall that the Cauchy problem associated with the one dimensional *abcd*-systems reads:

$$\begin{cases} \left(I - b\partial_{xx}^2\right)\partial_t\eta + \left(I + a\partial_{xx}^2\right)\partial_x u + \partial_x(\eta u) = 0, \\ \left(I - d\partial_{xx}^2\right)\partial_t u + \left(I + c\partial_{xx}^2\right)\partial_x\eta + \frac{1}{2}\partial_x u^2 = 0, \\ \eta_{|t=0} = \eta_0,\ u_{|t=0} = u_0. \end{cases} \tag{S_{abcd}}$$

We will treat the case where the parameters verify:

$$a \leq 0,\ c \leq 0,\ b \geq 0,\ d \geq 0, \tag{3.1.1}$$

excluding the five cases:

$$\begin{cases} a = b = 0,\ d < 0, c < 0;\ a = b = c = d = 0; \\ a = d = 0,\ b > 0, c < 0;\ a = b = d = 0, c < 0; \\ b = d = 0, c < 0, a < 0. \end{cases} \tag{3.1.2}$$

Given $s \in \mathbb{R}$ we will consider the set of indices defined by (1.1.5). The notation, $\mathcal{C}_T(H^{s_{bc}} \times H^{s_{ad}})$ stands for the space of continuous functions on $[0,T]$ with values in the space $H^{s_{bc}}(\mathbb{R}) \times H^{s_{ad}}(\mathbb{R})$. Let us recall in the following lines, the existence result of regular solution that can be found for instance in the first chapter:

Theorem 3.1.1. *Consider $a, c \leq 0$ and $b, d \geq 0$ excluding the cases (3.1.2). Also, consider an integer s such that*

$$s > \frac{5}{2} - \operatorname{sgn}(b+d),$$

and s_{bc}, s_{ad} defined by (1.1.5). *Let us consider $(\eta_0, u_0) \in H^{s_{bc}}(\mathbb{R}) \times H^{s_{ad}}(\mathbb{R})$. Then, there exists a positive time T and a unique solution*

$$(\eta, u) \in \mathcal{C}_T(H^{s_{bc}} \times H^{s_{ad}})$$

of (S_{abcd}).

Consider Δt and Δx two positive real numbers. In the following, $n\Delta t := t_n$ and $x_j = j\Delta x$. We endow $\ell^2(\mathbb{Z})$ with the following scalar product and norm

$$\langle v, w\rangle := \Delta x \sum_{j\in\mathbb{Z}} v_j w_j,\ \|v\|_{\ell^2_\Delta} := \langle v, v\rangle^{\frac{1}{2}}.$$

For all $v = (v_j)_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ we introduce the spatial shift operators:

$$(S_\pm v)_j := v_{j\pm 1},$$

If $v = (v_j)_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ we denote by D_+v, D_-v, Dv the discrete derivation operators:

$$\begin{cases} D_+v = \frac{1}{\Delta x}(S_+v - v), \\ D_-v = \frac{1}{\Delta x}(v - S_-v), \\ Dv = \frac{1}{2}(D_+v + D_-v) = \frac{1}{2\Delta x}(S_+v - S_-v). \end{cases}$$

Also, we consider the following discrete energy functional

$$\begin{aligned} \mathcal{E}(e,f) &\overset{def.}{=} \|e\|^2_{\ell^2_\Delta} + (b-c)\|D_+e\|^2_{\ell^2_\Delta} + b(-c)\|D_+D_-e\|^2_{\ell^2_\Delta} \\ &+ \|f\|^2_{\ell^2_\Delta} + (d-a)\|D_+f\|^2_{\ell^2_\Delta} + d(-a)\|D_+D_-f\|^2_{\ell^2_\Delta}. \end{aligned} \tag{3.1.3}$$

The case when $b > 0$ and $d > 0$: The fact that $b, d > 0$ ensures an ℓ^2-control on the discrete derivatives: this makes it possible to implement an energy method that mimics the one from the continuous case. Moreover, it allows us to close the estimates even without considering numerical viscosity. Moreover, no CFL-type condition is needed.

Consider $(\eta_0, u_0) \in H^{s_{bc}}(\mathbb{R}) \times H^{s_{ad}}(\mathbb{R})$ with s large enough. Also, consider the following numerical scheme:

$$\begin{cases} \frac{1}{\Delta t}(I - bD_+D_-)(\eta^{n+1} - \eta^n) + \frac{1}{2}(I + aD_+D_-)D\left(u^n + u^{n+1}\right) + D(\eta^n u^n) = 0, \\ \frac{1}{\Delta t}(I - dD_+D_-)(u^{n+1} - u^n) + \frac{1}{2}(I + cD_+D_-)D\left(\eta^n + \eta^{n+1}\right) + \frac{1}{2}D\left((u^n)^2\right) = 0. \end{cases} \tag{3.1.4}$$

for all $n \geq 0$, with

$$\begin{cases} \eta^0_j = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \eta_0(y_1)\, dy_1, \\ u^0_j = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} u_0(y_1)\, dy_1. \end{cases} \tag{3.1.5}$$

Let (η, u) be the solution of (S_{abcd}) and for $n \geq 1$, we consider $\eta^n_\Delta, u^n_\Delta \in \ell^2_\Delta(\mathbb{Z})$ given by:

$$\begin{cases} \eta^n_{\Delta j} = \frac{1}{\Delta t\Delta x}\int_{t_n}^{t_{n+1}}\int_{x_j}^{x_{j+1}} \eta(s_1, y_1)\, dy_1 ds_1, \\ u^n_{\Delta j} = \frac{1}{\Delta t\Delta x}\int_{t_n}^{t_{n+1}}\int_{x_j}^{x_{j+1}} u(s_1, y_1)\, dy_1 ds_1, \end{cases} \tag{3.1.6}$$

for all $j \in \mathbb{Z}$. Also, we let $\left(\eta^0_\Delta, u^0_\Delta\right) = \left(\eta^0, u^0\right)$. For all $n \geq 0$, we will consider $(\epsilon^n_1, \epsilon^n_2) \in \ell^2_\Delta$ the consistency error defined as:

$$\begin{cases} \frac{1}{\Delta t}(I - bD_+D_-)(\eta^{n+1}_\Delta - \eta^n_\Delta) + \frac{1}{2}(I + aD_+D_-)D\left(u^n_\Delta + u^{n+1}_\Delta\right) + D(\eta^n_\Delta u^n_\Delta) = \epsilon^n_1, \\ \frac{1}{\Delta t}(I - dD_+D_-)(u^{n+1}_\Delta - u^n_\Delta) + \frac{1}{2}(I + cD_+D_-)D\left(\eta^n_\Delta + \eta^{n+1}_\Delta\right) + \frac{1}{2}D\left((u^n_\Delta)^2\right) = \epsilon^n_2. \end{cases} \tag{3.1.7}$$

The convergence error is defined as:

$$e^n = \eta^n - \eta^n_\Delta,\ f^n = u^n - u^n_\Delta. \tag{3.1.8}$$

We are now in the position of stating our first result.

Theorem 3.1.2. *Let $a \leq 0$, $b > 0$, $c \leq 0$ and $d > 0$. Consider $s > 6$, $T > 0$, $(\eta_0, u_0) \in H^{s_{bc}}(\mathbb{R}) \times H^{s_{ad}}(\mathbb{R})$ and $(\eta, u) \in C_T(H^{s_{bc}} \times H^{s_{ad}})$ the solution of the system (S_{abcd}). Let $N \in \mathbb{N}$ and $\Delta x > 0$, $\Delta t = T/N$. We consider the numerical scheme (3.1.4) along with the initial data (3.1.5) as well as the approximation $(\eta_\Delta^n, u_\Delta^n)_{n \in \overline{1,N}}$ defined by (3.1.7). Then, there exists a constant ε_0 depending on $\sup_{t\in[0,T]} \|(\eta(t), u(t))\|_{H^{s_{bc}} \times H^{s_{ad}}}$ such that if the number of time steps N and the space discretization step are chosen such that*

$$\Delta t = T/N \leq \varepsilon_0 \text{ and } \Delta x \leq \varepsilon_0$$

then, the numerical scheme is first order convergent i.e. the convergence error defined in (3.1.8) satisfies:

$$\sup_{n\in\overline{0,N}} \mathcal{E}\left(e^n, f^n\right) \leq C_{abcd} \left\{(\Delta t)^2 + (\Delta x)^2\right\}, \tag{3.1.9}$$

where C_{abcd} depends on the parameters a,b,c,d and on $\sup_{t\in[0,T]} \|(\eta(t), u(t))\|_{H^{s_{bc}} \times H^{s_{ad}}}$.

The case when $bd = 0$ As above we consider$(\eta_0, u_0) \in H^{s_{bc}}(\mathbb{R}) \times H^{s_{ad}}(\mathbb{R})$ with s large enough along with the following numerical scheme:

$$\left\{\begin{array}{l} \frac{1}{\Delta t}\left(I - bD_+D_-\right)\left(\eta^{n+1} - \eta^n\right) + \left(I + aD_+D_-\right) D\left(u^{n+1}\right) + D\left(\eta^n u^n\right) \\ = \frac{1}{2}\left(1 - \operatorname{sgn}(b)\right) \tau_1^n \Delta x D_+ D_-\left(\eta^n\right), \\ \frac{1}{\Delta t}\left(I - dD_+D_-\right)\left(u^{n+1} - u^n\right) + \left(I + cD_+D_-\right) D\left(\eta^{n+1}\right) + \frac{1}{2} D\left(\left(u^n\right)^2\right) \\ = \frac{1}{2}\left(1 - \operatorname{sgn}(d)\right) \tau_2^n \Delta x D_+ D_-\left(u^n\right). \end{array}\right. \tag{3.1.10}$$

for all $n \geq 0$, with

$$\left\{\begin{array}{l} \eta_j^0 = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} \eta_0\left(y_1\right) dy_1, \\ u_j^0 = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} u_0\left(y_1\right) dy_1. \end{array}\right. \tag{3.1.11}$$

Let (η, u) be the solution of (S_{abcd}) and for $n \geq 1$, we consider $\eta_\Delta^n, u_\Delta^n \in \ell_\Delta^2(\mathbb{Z})$ as defined by the relation (3.1.6). Also, we let $\left(\eta_\Delta^0, u_\Delta^0\right) = \left(\eta^0, u^0\right)$. For all $n \geq 0$, we will consider $\left(\epsilon_1^n, \epsilon_2^n\right) \in \ell_\Delta^2$ the consistency error defined as:

$$\left\{\begin{array}{l} \frac{1}{\Delta t}\left(I - bD_+D_-\right)\left(\eta_\Delta^{n+1} - \eta_\Delta^n\right) + \left(I + aD_+D_-\right) D\left(u_\Delta^{n+1}\right) + D\left(\eta_\Delta^n u_\Delta^n\right) \\ = \epsilon_1^n + \frac{1}{2}\left(1 - \operatorname{sgn}(b)\right) \tau_1^n \Delta x D_+ D_-\left(\eta_\Delta^n\right), \\ \frac{1}{\Delta t}\left(I - dD_+D_-\right)\left(u_\Delta^{n+1} - u_\Delta^n\right) + \left(I + cD_+D_-\right) D\left(\eta_\Delta^{n+1}\right) + \frac{1}{2} D\left(\left(u_\Delta^n\right)^2\right) \\ = \epsilon_2^n + \frac{1}{2}\left(1 - \operatorname{sgn}(d)\right) \tau_2^n \Delta x D_+ D_-\left(u^n\right). \end{array}\right. \tag{3.1.12}$$

Of course, the convergence error is defined by (3.1.8). We are now in the position of stating our second main result.

Theorem 3.1.3. *Let $a \leq 0$, $b \geq 0$, $c \leq 0$ and $d \geq 0$ with $bd = 0$, excluding the cases (3.1.2). Consider $s > 8$, $T > 0$, $(\eta_0, u_0) \in H^{s_{bc}}(\mathbb{R}) \times H^{s_{ad}}(\mathbb{R})$ and $(\eta, u) \in C_T(H^{s_{bc}} \times H^{s_{ad}})$, the solution of the system (S_{abcd}). Let $N \in \mathbb{N}$ and $\Delta x > 0$, $\Delta t = T/N$. We consider the numerical scheme (3.1.10) along with the initial data (3.1.11) as well as the approximation $(\eta_\Delta^n, u_\Delta^n)_{n\in\overline{1,N}}$ defined by (3.1.12). Choose $\alpha > 0, \beta \in (0,1), \tau_1^n, \tau_2^n$ such that:*

$$\|u_\Delta^n\|_{\ell^\infty} + \alpha \leq \min\{\tau_1^n, \tau_2^n\}.$$

Then, there exists a constant ε_0 depending on $\alpha, \beta, \tau_1^n, \tau_2^n$ and on $\sup_{t\in[0,T]} \|(\eta(t), u(t))\|_{H^{s_{bc}} \times H^{s_{ad}}}$ such that if the number of time steps N, the space discretization step and the numerical viscosities are chosen in order to verify

$$\Delta t \leq \varepsilon_0, \ \Delta x \leq \varepsilon_0,$$

respectively

$$\max\{(1-\operatorname{sgn}(b))\,\tau_1^n,(1-\operatorname{sgn}(d))\,\tau_2^n\}\Delta t\leq(1-\beta)\,\Delta x,$$

then the numerical scheme is first order convergent i.e. the convergence error defined in (3.1.8) satisfies:

$$\sup_{n\in\overline{0,N}}\mathcal{E}\left(e^n,f^n\right)\leq C_{abcd}\left(\Delta x\right)^2,$$

where C_{abcd} depends on the parameters a,b,c,d and on $\sup_{t\in[0,T]}\|(\eta(t),u(t))\|_{H^{s_{bc}}\times H^{s_{ad}}}$.

Our results owes much to the technique developed by Courtès, Lagoutière and Rousset in [41]. Let us give some more details. For the Korteweg-de-Vries equation

$$\partial_t v+v\partial_x v+\partial^3_{xxx}v=0$$

they employ the following θ-scheme

$$\frac{1}{\Delta t}\left(v^{n+1}-v^n\right)+D_+D_+D_-((1-\theta)v^n+\theta v^{n+1})+D\left(\frac{(v^n)^2}{2}\right)=\frac{\tau^n\Delta x}{2}D_+D_-v^n.$$

With the aim to study the order of convergence, they consider the finite volume discrete operators:

$$\left(v^n_\Delta\right)_j=\frac{1}{\Delta t\Delta x}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}}v\left(s,y\right)dsdy$$

and the convergence error

$$e^n=v^n-v^n_\Delta,$$

which obeys the following equation:

$$e^{n+1}+\theta\Delta tD_+D_+D_-(e^{n+1})=\mathcal{A}e^n,$$

with

$$\begin{aligned}\mathcal{A}e^n&=e^n-(1-\theta)\,\Delta tD_+D_+D_-(e^{n+1})-\Delta tD\left(e^nv^n_\Delta\right)\\&-\Delta tD\left(\frac{(e^n)^2}{2}\right)+\frac{\tau^n\Delta x\Delta t}{2}D_+D_-e^n-\Delta t\epsilon^n,\end{aligned}$$

ϵ^n being the consistency error. In order to establish ℓ^2-estimates, they show that under the CFL condition

$$\begin{cases}\tau^n\Delta t<\Delta x,\ \tau^n<\|v^n_\Delta\|_{\ell^\infty}\ \text{and}\\(1-2\theta)\,\Delta t<\frac{(\Delta x)^3}{4},\end{cases}$$

the following holds true:

$$\|\mathcal{A}e^n\|^2_{\ell^2_\Delta}\lesssim\|\epsilon^n\|^2_{\ell^2_\Delta}+(1+C\Delta t)\,\|e^n+\theta\Delta tD_+D_+D_-(e^n)\|^2_{\ell^2_\Delta}\,,\tag{3.1.13}$$

provided that $\|e^n\|_{\ell^\infty}$ is sufficiently small. Thus, they are able to use the discrete Gronwall lemma and loop their estimates. The proof of (3.1.13) is rather technical and tricky. Essentially, using some clever identities when computing $\|\mathcal{A}e^n\|^2_{\ell^2_\Delta}$ they get several negative terms which, under the above CFL and smallness conditions, are used to balance the "bad" terms.

In Section 3.3.1 we study a discrete operator which appears in hyperbolic systems and we provide a bound on its ℓ^2_Δ norm, the proof being largely inspired from [41]. As a consequence, we establish a higher order estimate which proves to be crucial in the analysis of some of the *abcd* systems. Among the systems in view, we distinguish three situations:

- when $bd > 0$, establishing energy estimates for the convergence error can be done by imitating the approach from the continuous case. The structure of the equations provides enough control such that we do not need to impose numerical viscosity or CFL-type conditions.

- when at least one of the weakly dispersive operators does not appear, we have to work only with estimations established in Section 3.3.1 (see for instance the case $a < 0$, $b = c = d = 0$) either

- combine the techniques of Section 3.3.1 with "continuous-type" estimates like those established for the case when $bd > 0$ (see for instance the case $a = b = c = 0$, $d > 0$).

In order to validate the theoretical results regarding the order of convergence we compare the numerical solutions with the exact travelling wave solutions established by Chen in [34] and [35].

Finally, we use our results in order to study exact traveling waves interactions. We perform three such numerical experiments. Recently, in [14] Bona and Chen pointed out that finite time blow-up seems to occur at the head-on collision of the two exact solutions:

$$\begin{cases} \eta_{\pm,x0}\left(t,x\right) = \frac{15}{2}\operatorname{sech}^2\left(\frac{3}{\sqrt{10}}\left(x - x_0 \mp \frac{5}{2}t\right)\right) - \frac{45}{4}\operatorname{sech}^4\left(\frac{3}{\sqrt{10}}\left(x - x_0 \mp \frac{5}{2}t\right)\right), \\ u_{\pm,x_0}\left(t,x\right) = \pm\frac{15}{2}\operatorname{sech}^2\left(\frac{3}{\sqrt{10}}(x - x_0 \mp \frac{5}{2}t)\right). \end{cases}$$

More precisely, the numerical solution that they construct for the Cauchy problem with initial data

$$\begin{cases} \eta_0(x) = \eta_{+,x_0}\left(0,x\right) + \eta_{-,-x_0}\left(0,x\right), \\ u_0(x) = u_{+,x_0}\left(0,x\right) + u_{-,-x_0}\left(0,x\right) \end{cases}$$

sees its L^∞-norm growing arbitrarly large. In order to build confidence in our codes, we repeated their experiment and roughly obtained similar results. We performed a second experiment, for the BBM-type system $a = c = d = 0$, $b = 3/5$ which poses exact traveling waves of the above form. Our results point out that finite time blow-up also holds for this system. Finally, we perform an experiment witch shows that, as opposed to the "dispersionless" case $a = b = c = d = 0$, the total quantity $1 + \eta(t,x)$ which represents the water height at x at time t, could become negative even if at the initial time it is strictly positive.

The plan of the rest of this chapter is the following. In the next section we give a list of the main notations and identities that we use all along in this manuscript. Section 3.2 is devoted to the proof of Theorem 3.1.2. The proof of Theorem 3.1.3 is more involved as we cannot treat in a uniformly manner all the cases when $bd = 0$. First, we establish in Section 3.3.1 some estimates for discrete operators appearing in hyperbolic systems. The rest of Section 3.3 is dedicated to the proof corresponding to the different values of the *abcd* parameters appearing in Theorem 3.1.3. Finally, in Section 3.5, we present some simulations performed in Scilab using our schemes.

Notations: In the following we present the main notations that will be used through the rest of the paper. Let us fix $\Delta x > 0$. For $v = (v_j)_{j\in\mathbb{Z}}$, $w = (w_j)_{j\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ we define the product operator:

$$vw = (v_j w_j)_{j\in\mathbb{Z}} \in \ell^1(\mathbb{Z}).$$

Also, we denote by

$$v^2 = vv.$$

We endow $\ell^2(\mathbb{Z})$ with the following scalar product and norm

$$\langle v, w\rangle := \Delta x \sum_{j\in\mathbb{Z}} v_j w_j,$$
$$\|v\|_{\ell^2_\Delta} := \langle v, v\rangle^{\frac{1}{2}}.$$

The following identity describes the derivation law of a product

$$D(vw) = vDw + \frac{1}{2}\left(S_+wD_+v + S_-wD_-v\right), \tag{3.1.14}$$

$$D(vw) = S_+vDw + DvS_-w, \tag{3.1.15}$$

$$D_+(vw) = S_+vD_+w + wD_+w. \tag{3.1.16}$$

We observe that we dispose of the following basic integration by parts rules:

$$\langle D_+v, w\rangle = -\langle v, D_-w\rangle,$$
$$\langle Dv, w\rangle = -\langle v, Dw\rangle.$$

In particular, we see that:

$$\langle Dv, v\rangle = 0,$$

and

$$\langle v, D_+v\rangle = -\frac{\Delta x}{2}||D_+v||^2_{\ell^2_\Delta}.$$

More elaborate integration by parts identities are the following

$$\langle v, D(vw)\rangle = \frac{1}{2}\langle D_+w, vS_+v\rangle, \tag{3.1.17}$$

respectively

$$\langle D_+D_-v, D(vw)\rangle = -\frac{1}{\Delta x^2}\langle D_+w, vS_+v\rangle + \frac{1}{\Delta x^2}\langle Dw, S_-vS_+v\rangle. \tag{3.1.18}$$

Also, it holds true that

$$\left\langle v, D\left(v^2\right)\right\rangle = -\frac{\Delta x}{3}\left\langle v, (D_+v)^2 - (D_-v)^2\right\rangle, \tag{3.1.19}$$

and

$$||D_+D_-u||^2_{\ell^2_\Delta} = \frac{4}{\Delta x^2}||D_+u||^2_{\ell^2_\Delta} - \frac{4}{\Delta x^2}||Du||^2_{\ell^2_\Delta}. \tag{3.1.20}$$

In the following we will frequently use the notations:

$$\begin{cases} e^n = \eta^n - \eta^n_\Delta,\ E^\pm(n) = e^{n+1} \pm e^n, \\ f^n = u^n - u^n_\Delta,\ F^\pm(n) = f^{n+1} \pm f^n. \end{cases} \tag{3.1.21}$$

We end this section with the following discrete version of Gronwall's lemma, a proof of which can be found in [40]:

Lemma 3.1.1. *Let $v = (v^n)_{n\in\mathbb{N}}$, $a = (a^n)_{n\in\mathbb{Z}}$, $b = (b^n)_{j\in\mathbb{N}}$ be sequences of real numbers with $b^n \geq 0$ for all $n \in \mathbb{N}$ which satisfy*

$$v^n \leq a^n + \sum_{j=0}^{n-1} b^j v^j,$$

for all $n \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$, we have:

$$v^n \leq \max_{k\in\overline{1,n}} a^k \prod_{j=0}^{n-1}\left(1 + b^j\right).$$

3.2 The proof of Theorem 3.1.2

Using the notations introduced in (3.1.8) and (3.1.21) we see that the equations governing the convergence error (e^n, f^n) are the following:

$$\begin{cases} (I - bD_+D_-)(E^-(n)) + \frac{\Delta t}{2}(I + aD_+D_-)D(F^+(n)) + \Delta t D(e^n u_\Delta^n) \\ \qquad + \Delta t D(\eta_\Delta^n f^n) + \Delta t D(e^n f^n) = -\Delta t \epsilon_1^n, \\ (I - dD_+D_-)(F^-(n)) + \frac{\Delta t}{2}(I + cD_+D_-)D(E^+(n)) + \Delta t D(f^n u_\Delta^n) \\ \qquad + \frac{\Delta t}{2} D\left((f^n)^2\right) = -\Delta t \epsilon_2^n. \end{cases} \tag{3.2.1}$$

Let us recall the energy functional:

$$\mathcal{E}(e,f) \overset{def.}{=} \|e\|_{\ell_\Delta^2}^2 + (b-c)\|D_+e\|_{\ell_\Delta^2}^2 + b(-c)\|D_+D_-e\|_{\ell_\Delta^2}^2 + \|f\|_{\ell_\Delta^2}^2 + (d-a)\|D_+f\|_{\ell_\Delta^2}^2 + d(-a)\|D_+D_-f\|_{\ell_\Delta^2}^2. \tag{3.2.2}$$

As we announced in the introduction, we will establish energy estimates imitating the approach from the continuous case. Let $n \in \overline{0, N-1}$ and observe that by multiplying the first equation by $(I + cD_+D_-)E^+(n)$, the second by $(I + aD_+D_-)F^+(n)$ and adding up the results, we find that:

$$\begin{aligned} &\mathcal{E}\left(e^{n+1}, f^{n+1}\right) - \mathcal{E}(e^n, f^n) = \\ &= -\Delta t \left\langle \epsilon_1^n, (I + cD_+D_-)E^+(n)\right\rangle - \Delta t \left\langle \epsilon_2^n, (I + aD_+D_-)F^+(n)\right\rangle \\ &- \Delta t \left\langle D(e^n u_\Delta^n) + D(\eta_\Delta^n f^n), (I + cD_+D_-)E^+(n)\right\rangle \\ &- \Delta t \left\langle D(e^n f^n), (I + cD_+D_-)E^+(n)\right\rangle \\ &- \Delta t \left\langle D(f^n u_\Delta^n), (I + aD_+D_-)F^+(n)\right\rangle \\ &- \frac{\Delta t}{2}\left\langle D\left((f^n)^2\right), (I + aD_+D_-)F^+(n)\right\rangle \overset{not.}{=} \sum_{i=1}^{5} T_i. \end{aligned} \tag{3.2.3}$$

Let us arbitrarily fix $\gamma \in (0, \frac{1}{2})$. Suppose that

$$\mathcal{E}(e^n, f^n) \le \left((\Delta x)^2 + (\Delta t)^2\right)^{1-\gamma}. \tag{3.2.4}$$

In particular, using the inequality:

$$\|e^n\|_{\ell^\infty} \le \|e^n\|_{\ell_\Delta^2}^{\frac{1}{2}} \|D_+e^n\|_{\ell_\Delta^2}^{\frac{1}{2}},$$

we get that for sufficiently small Δx and Δt one has:

$$\|e^n\|_{\ell^\infty} \le 1 \text{ and } \|f^n\|_{\ell^\infty} \le 1. \tag{3.2.5}$$

We begin by treating T_1. We denote

$$\|\epsilon^n\|_{\ell_\Delta^2} = \arg\max\left\{\|\epsilon_1^n\|_{\ell_\Delta^2}, \|\epsilon_2^n\|_{\ell_\Delta^2}\right\},$$

and we write that

$$\begin{aligned} &-\Delta t \left\langle \epsilon_1^n, (I + cD_+D_-)E^+(n)\right\rangle - \Delta t \left\langle \epsilon_2^n, (I + aD_+D_-)F^+(n)\right\rangle \\ &\le \Delta t \|\epsilon_1^n\|_{\ell_\Delta^2}\left(\left\|E^+(n)\right\|_{\ell_\Delta^2} - c\left\|D_+D_-E^+(n)\right\|_{\ell_\Delta^2}\right) \\ &+ \Delta t \|\epsilon_2^n\|_{\ell_\Delta^2}\left(\left\|F^+(n)\right\|_{\ell_\Delta^2} - a\left\|D_+D_-F^+(n)\right\|_{\ell_\Delta^2}\right) \end{aligned}$$

$$\leq \Delta t \left\|\epsilon\right\|_{\ell^2}^2 + 2\Delta t \left(\left\|e^n\right\|_{\ell^2}^2 + \left\|f^n\right\|_{\ell^2}^2 + c^2 ||D_+D_-e^n||_{\ell^2_\Delta}^2 + a^2 ||D_+D_-f^n||_{\ell^2_\Delta}^2\right)$$
$$+ 2\Delta t \left(\left\|e^{n+1}\right\|_{\ell^2}^2 + \left\|f^{n+1}\right\|_{\ell^2}^2 + c^2 ||D_+D_-e^{n+1}||_{\ell^2_\Delta}^2 + a^2 ||D_+D_-f^{n+1}||_{\ell^2_\Delta}^2\right) \tag{3.2.6}$$
$$\leq \Delta t \left\|\epsilon^n\right\|_{\ell^2}^2 + 2\Delta t \max\left\{1, \frac{-c}{b}, \frac{-a}{d}\right\} \mathcal{E}\left(e^n, f^n\right) + 2\Delta t \max\left\{1, \frac{-c}{b}, \frac{-a}{d}\right\} \mathcal{E}\left(e^{n+1}, f^{n+1}\right). \tag{3.2.7}$$

Let us treat T_2. Using (3.1.15), we first write

$$\begin{aligned}
&\Delta t \left\|D\left(e^n u_\Delta^n\right)\right\|_{\ell^2} \\
&= \Delta t \left\|S_- e^n D u_\Delta^n + S_+ u_\Delta^n D e^n\right\|_{\ell^2} \\
&\leq \Delta t \left\|D u_\Delta^n\right\|_{\ell^\infty} \left\|S_- e^n\right\|_{\ell^2} + \Delta t \left\|S_+ u_\Delta^n\right\|_{\ell^\infty} \left\|D e^n\right\|_{\ell^2} \\
&\leq \Delta t \max\left\{1, \frac{1}{\sqrt{b-c}}\right\} \left(\left\|D u_\Delta^n\right\|_{\ell^\infty} \left\|e^n\right\|_{\ell^2} + \left\|u_\Delta^n\right\|_{\ell^\infty} \sqrt{b-c} \left\|D e^n\right\|_{\ell^2}\right) \\
&\leq \Delta t \max\left\{\left\|u_\Delta^n\right\|_{\ell^\infty}, \left\|D u_\Delta^n\right\|_{\ell^\infty}\right\} \max\left\{1, \frac{1}{\sqrt{b-c}}\right\} \left(\left\|e^n\right\|_{\ell^2} + \sqrt{b-c} \left\|D e^n\right\|_{\ell^2}\right).
\end{aligned}$$

Proceeding in a similar fashion with the other terms we arrive at

$$\begin{aligned}
&- \Delta t \left\langle D\left(e^n u_\Delta^n\right) + D\left(\eta_\Delta^n f^n\right), \left(I + cD_+D_-\right) E^+(n)\right\rangle \\
&\leq \Delta t \left(\left\|D\left(e^n u_\Delta^n\right)\right\|_{\ell^2} + \left\|D\left(\eta_\Delta^n f^n\right)\right\|_{\ell^2}\right) \left(\left\|E^+(n)\right\|_{\ell^2} - c \left\|D_+D_-E^+(n)\right\|_{\ell^2}\right) \\
&\leq \Delta t C_{1,1} \mathcal{E}\left(e^n, f^n\right) + \Delta t C_{2,1} \mathcal{E}\left(e^{n+1}, f^{n+1}\right)
\end{aligned} \tag{3.2.8}$$

where $C_{1,1}$ and $C_{2,1}$ can be written, for example, as numerical constants multiplied with:

$$\max\left\{1, \sqrt{-c/b}\right\} \left\{\max\left\{\left\|u_\Delta^n\right\|_{\ell^\infty}, \left\|D u_\Delta^n\right\|_{\ell^\infty}\right\} \max\left\{1, 1/\sqrt{b-c}\right\}\right. \tag{3.2.9}$$
$$\left. + \max\left\{\left\|\eta_\Delta^n\right\|_{\ell^\infty}, \left\|D \eta_\Delta^n\right\|_{\ell^\infty}\right\} \max\left\{1, 1/\sqrt{d-a}\right\}\right\}. \tag{3.2.10}$$

We treat T_4 in same spirit as above in order to obtain that

$$-\Delta t \left\langle D\left(f^n u_\Delta^n\right), \left(I + aD_+D_-\right) F^+(n)\right\rangle \leq \Delta t C_{1,2} \mathcal{E}\left(e^n, f^n\right) + \Delta t C_{2,2} \mathcal{E}\left(e^{n+1}, f^{n+1}\right) \tag{3.2.11}$$

where $C_{1,2}$ and $C_{2,2}$ are multiples of :

$$\max\left\{\left\|u_\Delta^n\right\|_{\ell^\infty}, \left\|D u_\Delta^n\right\|_{\ell^\infty}\right\} \max\left\{1, 1/\sqrt{d-a}\right\} \max\left\{1, \sqrt{-a/d}\right\}. \tag{3.2.12}$$

In order to treat T_3 and T_5 we first observe that

$$\begin{aligned}
\left\|D\left(e^n f^n\right)\right\|_{\ell^2} &= \left\|S_- e^n D f^n + S_+ f^n D e^n\right\|_{\ell^2} \leq \left\|e^n\right\|_{\ell^\infty} \left\|D f^n\right\|_{\ell^2} + \left\|f^n\right\|_{\ell^\infty} \left\|D e^n\right\|_{\ell^2} \\
&\leq \max\left\{1/\sqrt{b-c}, 1/\sqrt{d-a}\right\} \left(\sqrt{d-a} \left\|D f^n\right\|_{\ell^2} + \sqrt{b-c} \left\|D e^n\right\|_{\ell^2}\right).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
T_3 &= -\Delta t \left\langle D\left(e^n f^n\right), \left(I + cD_+D_-\right) E^+(n)\right\rangle \\
&\leq \Delta t C_{1,3} \mathcal{E}\left(e^n, f^n\right) + \Delta t C_{2,3} \mathcal{E}\left(e^{n+1}, f^{n+1}\right)
\end{aligned} \tag{3.2.13}$$

where $C_{1,3}$ respectively $C_{2,3}$ are proportional with

$$\max\left\{1/(b-c), 1/(d-a)\right\} \max\left\{1, \sqrt{-c/b}\right\}. \tag{3.2.14}$$

The same holds for T_5, namely, we get that

$$-\frac{\Delta t}{2} \left\langle D\left(\left(f^n\right)^2\right), \left(I + aD_+D_-\right) F^+(n)\right\rangle \leq \Delta t C_{1,4} \mathcal{E}\left(e^n, f^n\right) + \Delta t C_{2,4} \mathcal{E}\left(e^{n+1}, f^{n+1}\right) \tag{3.2.15}$$

where $C_{1,4}$ respectively $C_{2,4}$ are proportional with

$$\frac{1}{\sqrt{d-a}} \max \left\{1, \sqrt{-a/d}\right\}. \tag{3.2.16}$$

Gathering the informations from (3.2.3), (3.2.7), (3.2.8), (3.2.11), (3.2.13) and (3.2.15) we obtain the existence of two constants C_1 and C_2 that depend on a, b, c, d, and the ℓ^∞-norm of $(\eta^n_\Delta, u^n_\Delta)$ and $(D\eta^n_\Delta, Du^n_\Delta)$ (dependence which we can track using relations (3.2.9), (3.2.12), (3.2.14) respectively (3.2.16)) such that

$$\mathcal{E}\left(e^{n+1}, f^{n+1}\right) - \mathcal{E}\left(e^n, f^n\right) \le \Delta t \left\|\epsilon^n\right\|^2_{\ell^2} + \Delta t C_1 \mathcal{E}\left(e^n, f^n\right) + \Delta t C_2 \mathcal{E}\left(e^{n+1}, f^{n+1}\right)$$

which rewrites as

$$\mathcal{E}\left(e^{n+1}, f^{n+1}\right) \le \frac{\Delta t}{1-\Delta t C_2} \left\|\epsilon^n\right\|^2_{\ell^2} + \left(1 + \frac{\Delta t\left(C_1 + C_2\right)}{1-\Delta t C_2}\right) \mathcal{E}\left(e^n, f^n\right). \tag{3.2.17}$$

Of course, if we suppose the stronger induction hypothesis

$$\mathcal{E}\left(e^k, f^k\right) \le \left((\Delta x)^2 + (\Delta t)^2\right)^{1-\gamma}, \tag{3.2.18}$$

for all $k \in \overline{0, n}$, then, inequality (3.2.17) is available for any all $k \in 1, n$. Thus, taking the sum of all these inequalities, we end up with

$$\mathcal{E}\left(e^{n+1}, f^{n+1}\right) \le \frac{(n+1)\Delta t}{1-\Delta t C_2} \max_{k \in \overline{0,n}} \left\|\epsilon^n\right\|^2_{\ell^2} + \frac{\Delta t\left(C_1 + C_2\right)}{1-\Delta t C_2} \sum_{k=1}^{n} \mathcal{E}\left(e^k, f^k\right).$$

Applying the discrete Grönwall lemma 3.1.1 and using the fact that the consistency error is order one accurate in time and space, see Section 3.4, we get

$$\begin{aligned}
\mathcal{E}\left(e^{n+1}, f^{n+1}\right) &\le \frac{(n+1)\Delta t}{1-\Delta t C_2} \exp\left(\frac{(n+1)\Delta t\left(C_1 + C_2\right)}{1-\Delta t C_2}\right) \max_{k \in \overline{0,n}} \left\|\epsilon^n\right\|^2_{\ell^2} \\
&\le \frac{TC\left(\eta_0, u_0\right)}{1-\Delta t C_2} \exp\left(\frac{T\left(C_1 + C_2\right)}{1-\Delta t C_2}\right) \left\{(\Delta x)^2 + (\Delta t)^2\right\},
\end{aligned}$$

where $C\left(\eta_0, u_0\right)$ is some constant depending on the initial data (η_0, u_0). Thus, for $\Delta x, \Delta t$ small enough such that

$$\frac{TC\left(\eta_0, u_0\right)}{1-\Delta t C_2} \exp\left(\frac{T\left(C_1 + C_2\right)}{1-\Delta t C_2}\right) \left((\Delta x)^2 + (\Delta t)^2\right)^\gamma \le 1,$$

we can assure that the inductive hypothesis (3.2.18) holds for all $k \in \overline{0, n+1}$. Obviously, this allows to close the estimates and provide the desired bound.

This concludes the proof of Theorem 3.1.2.

3.3 The proof of Theorem 3.1.3

As announced in the introduction, in this section we aim at providing a proof for our second main result. As opposed to the previous result, the proof of Theorem 3.1.3 is rather sensitive to the different values of the *abcd* parameters. First, we establish a technical result that interferes in a crucial manner in establishing the a priori estimates.

3.3.1 Burgers-type estimates

Let us state the first result of this section:

Proposition 3.3.1. *Let $u \in \ell^\infty(\mathbb{Z})$, $\lambda \geq 0$ respectively $\alpha > 0$, $\beta \in (0,1)$. Fix τ such that*

$$\|u\|_{\ell^\infty} + \alpha \leq \tau.$$

Then, there exists a sufficiently small positive numbers $\varepsilon_0, \varepsilon_1$ such that the following holds true. Consider two positive real numbers $\Delta t, \Delta x$ such that

$$\frac{\tau \Delta t}{\Delta x} \leq 1 - \beta,\ \Delta x \leq \varepsilon_0,\ \Delta t \left\| D_+ u \right\|_{\ell^\infty} \leq \varepsilon_0$$

and $a \in \ell^2(\mathbb{Z})$, such that

$$\lambda \left\| a \right\|_{\ell^\infty} \leq \varepsilon_1.$$

Then, there exists a positive constant C depending on the ℓ^∞-norm of $D_+(u)$ such that

$$\left\| a - \Delta t D\left((u + \lambda a) a \right) + \frac{\tau}{2} \Delta x \Delta t D_+ D_- a \right\|_{\ell^2_\Delta} \leq (1 + C\Delta t) \left\| a \right\|_{\ell^2_\Delta}.$$

Proof. We define

$$\mathcal{B}a = a - \Delta t D\left((u + \lambda a) a \right) + \frac{\tau}{2} \Delta x \Delta t D_+ D_- a.$$

We compute the square of the ℓ^2_Δ-norm of $\mathcal{B}a$:

$$\begin{aligned}
\left\| \mathcal{B}a \right\|^2_{\ell^2_\Delta} &= \left\| a \right\|^2_{\ell^2_\Delta} + (\Delta t)^2 \left\| D(au) \right\|^2_{\ell^2_\Delta} + \lambda^2 (\Delta t)^2 \left\| D\left(a^2 \right) \right\|^2_{\ell^2_\Delta} \\
&+ (\tau^2/4)(\Delta x)^2 (\Delta t)^2 \left\| D_+ D_- a \right\|^2_{\ell^2_\Delta} + 2\lambda (\Delta t)^2 \left\langle D(au), D(a^2) \right\rangle \\
&- 2\Delta t \left\langle a, D\left((u + \lambda a) a \right) \right\rangle + \tau \Delta x \Delta t \left\langle a, D_+ D_- a \right\rangle - \tau \Delta x (\Delta t)^2 \left\langle D\left((u + \lambda a) a \right), D_+ D_- a \right\rangle.
\end{aligned} \tag{3.3.1}$$

We use the derivation formula (3.1.15) in order to get:

$$\begin{aligned}
\Delta t^2 || D(au) ||^2_{\ell^2_\Delta} &= (\Delta t)^2 \left(\left\| S_+ u D a \right\|^2_{\ell^2_\Delta} + 2 \left\langle S_+ u D a, S_- a D u \right\rangle + \left\| S_- a D u \right\|^2_{\ell^2_\Delta} \right) \\
&\leq \Delta t^2 \left\{ ||u||^2_{\ell^\infty} + \Delta t ||u||^2_{\ell^\infty} \right\} ||Da||^2_{\ell^2_\Delta} + \Delta t \left\{ ||Du||^2_{\ell^\infty} + \Delta t ||Du||^2_{\ell^\infty} \right\} ||a||^2_{\ell^2_\Delta}.
\end{aligned} \tag{3.3.2}$$

Using again relation (3.1.15), we write

$$\lambda^2 (\Delta t)^2 \left\| D\left(a^2 \right) \right\|^2_{\ell^2_\Delta} \leq 4\lambda^2 (\Delta t)^2 \left\| a \right\|^2_{\ell^\infty} \left\| Da \right\|^2_{\ell^2_\Delta}. \tag{3.3.3}$$

Owing to (3.1.20), we have

$$\frac{\tau^2 \Delta t^2 \Delta x^2}{4} ||D_+ D_- a||^2_{\ell^2_\Delta} = \tau^2 \Delta t^2 ||D_+ a||^2_{\ell^2_\Delta} - \tau^2 \Delta t^2 ||Da||^2_{\ell^2_\Delta}. \tag{3.3.4}$$

We treat the next term as it follows:

$$\begin{aligned}
&2\lambda (\Delta t)^2 \left\langle D(au), D(a^2) \right\rangle \\
&= 2\lambda (\Delta t)^2 \left\langle S_+ u D a + S_- a D u, (S_+ a + S_- a) D a \right\rangle \\
&\leq 4\lambda (\Delta t)^2 \left\| u \right\|_{\ell^\infty} \left\| a \right\|_{\ell^\infty} \left\| Da \right\|^2_{\ell^2_\Delta} + 4\lambda (\Delta t)^2 \left\| a \right\|_{\ell^\infty} \left\| Du \right\|_{\ell^\infty} \left\| a \right\|_{\ell^2_\Delta} \left\| Da \right\|_{\ell^2_\Delta} \\
&\leq 4\lambda (\Delta t)^2 \left\| u \right\|_{\ell^\infty} \left\| a \right\|_{\ell^\infty} \left\| Da \right\|^2_{\ell^2_\Delta} + 4\lambda^2 (\Delta t)^2 \left\| a \right\|^2_{\ell^\infty} \left\| Da \right\|^2_{\ell^2_\Delta} + (\Delta t)^2 \left\| Du \right\|^2_{\ell^\infty} \left\| a \right\|^2_{\ell^2_\Delta}.
\end{aligned} \tag{3.3.5}$$

Next, observe that (3.1.17) and (3.1.19) imply that:

$$
\begin{aligned}
-2\Delta t \left\langle a, D\left(au+\lambda a^2\right)\right\rangle &= -\Delta t \left\langle D_+u, aS^+a\right\rangle + \frac{2\lambda\Delta x\Delta t}{3}\left\langle a,(D_+a)^2-(D_-a)^2\right\rangle \\
&\le \Delta t \left\|D_+u\right\|_{\ell^\infty}\left\|a\right\|^2_{\ell^2_\Delta} + \frac{4\lambda\Delta x\Delta t}{3}\left\|a\right\|_{\ell^\infty}\left\|D_+a\right\|^2_{\ell^2_\Delta}.
\end{aligned}
\tag{3.3.6}
$$

Obviously, we have

$$
\tau\Delta x\Delta t \left\langle a, D_+D_-a\right\rangle = -\tau\Delta t\Delta x||D_+a||^2_{\ell^2_\Delta}. \tag{3.3.7}
$$

Next, using (3.1.16) combined with (3.1.17), let us write that

$$
\begin{aligned}
&-\tau\Delta x\left(\Delta t\right)^2\left\langle D\left(au\right), D_+D_-a\right\rangle \\
&= \tau\Delta x\left(\Delta t\right)^2\left\langle D(S_+aD_+u+uD_+a), D_+a\right\rangle \\
&= \tau\Delta x\left(\Delta t\right)^2\left\langle D(S_+aD_+u), D_+a\right\rangle + \tau\Delta x\left(\Delta t\right)^2\left\langle D(uD_+a), D_+a\right\rangle \\
&= \tau\Delta x\left(\Delta t\right)^2\left\langle D_-(S_+aD_+u), Da\right\rangle + \frac{\tau\Delta x\left(\Delta t\right)^2}{2}\left\langle D_+u, D_+aS_+D_+a\right\rangle \\
&\le \tau\left(\Delta t\right)^2\left\|D_+u\right\|_{\ell^\infty}\left\|a\right\|_{\ell^2_\Delta}\left\|Da\right\|_{\ell^2_\Delta} + \frac{\tau\Delta x\left(\Delta t\right)^2}{2}\left\|D_+u\right\|_{\ell^\infty}\left\|D_+a\right\|^2_{\ell^2_\Delta} \\
&\le \frac{\tau^2\left(\Delta t\right)^3}{4}\left\|Da\right\|^2_{\ell^2_\Delta} + \Delta t\left\|D_+u\right\|^2_{\ell^\infty}\left\|a\right\|^2_{\ell^2_\Delta} + \frac{\tau\Delta x\left(\Delta t\right)^2}{2}\left\|D_+u\right\|_{\ell^\infty}\left\|D_+a\right\|^2_{\ell^2_\Delta}.
\end{aligned}
\tag{3.3.8}
$$

The last term of (3.3.1) is treated as follows:

$$
\begin{aligned}
-\tau\lambda\Delta x\left(\Delta t\right)^2\left\langle D\left(a^2\right), D_+D_-a\right\rangle &= \tau\lambda\Delta x\left(\Delta t\right)^2\left\langle D\left((a+S_+a)D_+a\right), D_+a\right\rangle \\
&= \frac{\tau\lambda\Delta x\left(\Delta t\right)^2}{2}\left\langle D_+a+S_+D_+a, S_+D_+aD_+a\right\rangle \\
&\le \tau\lambda\Delta x\left(\Delta t\right)^2\left\|D_+a\right\|_{\ell^\infty}\left\|D_+a\right\|^2_{\ell^2_\Delta} \\
&\le \tau\lambda\left(\Delta t\right)^2\left\|a\right\|_{\ell^\infty}\left\|D_+a\right\|^2_{\ell^2_\Delta}. \\
&\le \lambda\Delta t\Delta x\left\|a\right\|_{\ell^\infty}\left\|D_+a\right\|^2_{\ell^2_\Delta}.
\end{aligned}
\tag{3.3.9}
$$

Using the estimates (3.3.2)-(3.3.9) with (3.3.1) we get that

$$
\begin{aligned}
\left\|\mathcal{B}a\right\|^2_{\ell^2_\Delta} \le{}& \Delta t\left(2\left(1+\Delta t\right)\left\|D_+u\right\|^2_{\ell^\infty} + \left\|D_+u\right\|_{\ell^\infty}\right)\left\|a\right\|^2_{\ell^2_\Delta} \\
&+ \tau\Delta x\Delta t C_1\left\|D_+a\right\|^2_{\ell^2_\Delta} + C_2\left(\Delta t\right)^2\left\|Da\right\|^2_{\ell^2_\Delta}
\end{aligned}
$$

where

$$
C_1 = \frac{\tau\Delta t}{\Delta x} - 1 + \frac{7\lambda}{3\tau}\left\|a\right\|_{\ell^\infty} + \frac{\Delta t}{2}\left\|D_+u\right\|_{\ell^\infty},
$$

respectively

$$
C_2 = ||u||^2_{\ell^\infty} - \tau^2 + \Delta t||u||^2_{\ell^\infty} + 4\lambda\left\|u\right\|_{\ell^\infty}\left\|a\right\|_{\ell^\infty} + 8\lambda^2\left\|a\right\|^2_{\ell^\infty} + \frac{\tau^2\Delta t}{4}.
$$

We obtain that

$$
C_1 \le \frac{7\lambda}{3\tau}\varepsilon_1 + \frac{\Delta t}{2}\left\|D_+u\right\|_{\ell^\infty} - \beta
$$

and

$$
C_2 \le \Delta t||u||^2_{\ell^\infty} + 4\varepsilon_1\lambda\left\|u\right\|_{\ell^\infty} + 8\lambda^2\varepsilon_1^2 + \frac{\tau^2\Delta t}{4} - \alpha\left(||u||_{\ell^\infty} + \tau\right).
$$

Then, it becomes clear that we may chose Δt, Δx and ε_1 sufficiently small such that C_1 and C_2 are negative. This concludes the proof of Proposition 3.3.1. □

Proposition 3.3.2. *Consider $u \in \ell^\infty(\mathbb{Z})$, $\lambda \geq 0$ respectively $\alpha > 0$, $\beta \in (0,1)$. Fix τ such that*

$$\|u\|_{\ell^\infty} + \alpha \leq \tau.$$

Then, there exists sufficiently small positive numbers $\varepsilon_0, \varepsilon_1$ such that the following holds true. Consider two positive real numbers $\Delta t, \Delta x$ such that

$$\frac{\tau \Delta t}{\Delta x} \leq 1 - \beta,\ \Delta x \leq \varepsilon_0,\ \Delta t \|D_+ u\|_{\ell^\infty} \leq \varepsilon_0$$

and $a \in \ell^2(\mathbb{Z})$, $b \in \ell^\infty(\mathbb{Z})$ such that

$$\lambda \|a\|_{\ell^\infty}, \|b\|_{\ell^\infty} \leq \varepsilon_1 \text{ and } \|D_+ b\|_{\ell^\infty} \leq 1.$$

Then, there exists a positive constant C depending on the ℓ^∞-norm of u and $D_+(u)$ such that:

$$\left\| a - \Delta t D\left(a\left(u + b + \lambda a\right)\right) + \frac{\tau}{2} \Delta x \Delta t D_+ D_- a \right\|_{\ell^2_\Delta} \leq \left(1 + C\Delta t\right) \|a\|_{\ell^2_\Delta}. \tag{3.3.10}$$

Proof. Let us suppose that ε_1 is chosen small enough such that $\alpha - \varepsilon_1 > 0$. Then, for all $b \in \ell^\infty(\mathbb{Z})$ such that

$$\|b\|_{\ell^\infty} \leq \varepsilon_1$$

we have

$$\|u + b\|_{\ell^\infty} + \alpha - \varepsilon_1 \leq \tau.$$

Then, taking sufficiently small Δt and Δx, we may apply Proposition 3.3.1 in order to establish the existence of $\varepsilon_2 > 0$ such that if

$$\lambda \|a\|_{\ell^\infty} \leq \varepsilon_2$$

then the estimate (3.3.10) holds true. □

Proposition 3.3.3. *Consider $u \in \ell^\infty(\mathbb{Z})$, $\lambda \geq 0$ respectively $\alpha > 0$, $\beta \in (0,1)$. Fix τ such that*

$$\|u\|_{\ell^\infty} + \alpha \leq \tau.$$

Then, there exists a sufficiently small positive numbers $\varepsilon_0, \varepsilon_1$ such that the following holds true. Consider two positive real numbers $\Delta t, \Delta x$ such that

$$\frac{\tau \Delta t}{\Delta x} \leq 1 - \beta,\ \Delta x \leq \varepsilon_0,\ \Delta t \|D_+ u\|_{\ell^\infty} \leq \varepsilon_0$$

and $a \in \ell^2(\mathbb{Z})$, $b \in \ell^\infty(\mathbb{Z})$ such that

$$\lambda \|a\|_{\ell^\infty}, \lambda \|D_+ a\|_{\ell^\infty}, \|b\|_{\ell^\infty} \leq \varepsilon_1 \text{ and } \|D_+ b\|_{\ell^\infty}, \|D_+ D_-(b)\|_{\ell^\infty} \leq 1. \tag{3.3.11}$$

Then, there exist positive constants C_1,C_2 depending on the ℓ^∞-norm of u, $D_+(u)$ and $DD_+(u)$ such that:

$$\left\| D_+ a - \Delta t D_+ D\left(a\left(u + b + \lambda a\right)\right) + \frac{\tau}{2} \Delta x \Delta t D_+ D_+ D_- a \right\|_{\ell^2_\Delta} \leq C_1 \Delta t \|a\|_{\ell^2_\Delta} + \left(1 + C_2 \Delta t\right) \|D_+ a\|_{\ell^2_\Delta}.$$

Proof. Let us observe that

$$\begin{aligned}
&D_+ a - \Delta t D_+ D\left(a\left(u + b + \lambda a\right)\right) + \frac{\tau}{2} \Delta x \Delta t D_+ D_+ D_-(a) \\
&= D_+ a - \Delta t D\left(D_+(a)\left(u + b + \lambda a\right)\right) - \Delta t D\left(S_+(a) D_+\left(u + b + \lambda a\right)\right) \\
&+ \frac{\tau}{2} \Delta x \Delta t D_+ D_-\left(D_+(a)\right)
\end{aligned}$$

$$
\begin{aligned}
&= D_+ a - \Delta t D \left(D_+ (a) \left(u + b + \lambda S_+ (a) + \lambda a\right)\right) + \frac{\tau}{2} \Delta x \Delta t D_+ D_- \left(D_+ (a)\right) \\
&- \Delta t D \left(S_+ (a) D_+ (u)\right) - \Delta t D \left(S_+ (a) D_+ (b)\right)
\end{aligned}
$$

Owing to the hypothesis (3.3.11) and Proposition 3.3.2 with $b + \lambda S_+ a + \lambda a$ instead of b and $\lambda = 0$, we may choose ε_0 and ε_1 small enough which ensures the existence of a positive constant C_2 such that:

$$
\left\| D_+ a - \Delta t D \left(D_+ (a) \left(u + b + \lambda S_+ (a) + \lambda a\right)\right) + \frac{\tau}{2} \Delta x \Delta t D_+ D_- \left(D_+ (a)\right) \right\|_{\ell^2_\Delta} \leq \left(1 + C_2 \Delta t\right) \left\| D_+ (a) \right\|_{\ell^2_\Delta} \tag{3.3.12}
$$

Moreover, using the derivation formula (3.1.15) it transpires that

$$
\begin{aligned}
&\left\| -\Delta t D \left(S_+ (a) D_+ (u)\right) - \Delta t D \left(S_+ (a) D_+ (b)\right) \right\|_{\ell^2_\Delta} \\
&\leq \Delta t \left(\left\| D D_+ (u) \right\|_{\ell^\infty} + \left\| D D_+ (b) \right\|_{\ell^\infty} \right) \left\| a \right\|_{\ell^2_\Delta} + \Delta t \left(\left\| D_+ (u) \right\|_{\ell^\infty} + \left\| D_+ (b) \right\|_{\ell^\infty} \right) \left\| D_+ (a) \right\|_{\ell^2_\Delta}.
\end{aligned} \tag{3.3.13}
$$

The conclusion follows from estimates (3.3.12) and (3.3.13). □

3.3.2 The case $b = d = 0$

The case $a < 0$, $b = c = d = 0$ The convergence error satisfies:

$$
\begin{cases}
e^{n+1} + \Delta t D f^{n+1} + a \Delta t D_+ D_- D f^{n+1} \\
= e^n - \Delta t D \left(e^n u^n_\Delta\right) - \Delta t D \left(e^n f^n\right) - \Delta t D \left(\eta^n_\Delta f^n\right) + \frac{\tau^n_1}{2} \Delta t \Delta x D_+ D_- e^n - \Delta t \epsilon^n_1, \\
f^{n+1} + \Delta t D e^{n+1} = f^n - \Delta t D \left(f^n \left(u^n_\Delta + \frac{1}{2} f^n\right)\right) + \frac{\tau^n_2}{2} \Delta t \Delta x D_+ D_- f^n - \Delta t \epsilon^n_2,
\end{cases}
$$

where, we chose $\Delta t, \Delta x, \tau^n_1, \tau^n_2$ such that

$$
\begin{cases}
\frac{\max\{\tau^n_1, \tau^n_2\} \Delta t}{\Delta x} < 1, \\
\left\| u^n_\Delta \right\|_{\ell^\infty} < \min\left\{\tau^n_1, \tau^n_2\right\}.
\end{cases}
$$

We consider the energy functional

$$
\mathcal{E}(e, f) \overset{def.}{=} \|e\|^2_{\ell^2_\Delta} + \|f\|^2_{\ell^2_\Delta} + (-a) \|D_+ f\|^2_{\ell^2_\Delta}.
$$

Let us suppose that

$$
\mathcal{E}(e^n, f^n) \leq \Delta x^{2-2\gamma}
$$

which is sufficient in order to assure the hypothesis of Proposition 3.3.2 and Proposition 3.3.3. In order to recover a H^1-type control for f^n, let us apply $\sqrt{-a} D_+$ to the second equation. We get that:

$$
\begin{aligned}
&\sqrt{-a} D_+ f^{n+1} + \Delta t \sqrt{-a} D_+ D e^{n+1} \\
&= \sqrt{-a} D_+ f^n - \sqrt{-a} \Delta t D D_+ \left(f^n \left(u^n_\Delta + \frac{1}{2} f^n \right) \right) \\
&+ \sqrt{-a} \frac{\tau^n_2}{2} \Delta t \Delta x D_+ D_+ D_- f^n - \sqrt{-a} \Delta t D_+ \epsilon^n_2.
\end{aligned}
$$

Next, we compute

$$
\begin{aligned}
&\left\| \sqrt{-a} D_+ f^{n+1} + \Delta t \sqrt{-a} D_+ D e^{n+1} \right\|^2_{\ell^2_\Delta} \\
&= -a \left\| D_+ f^{n+1} \right\|^2_{\ell^2_\Delta} - a (\Delta t)^2 \left\| D_+ D e^{n+1} \right\|^2_{\ell^2_\Delta} + 2 a \Delta t \left\langle D_+ D_- f^{n+1}, D e^{n+1} \right\rangle.
\end{aligned}
$$

We observe that

$$
\left\| e^{n+1} + \Delta t \left(I + a D_+ D_-\right) D f^{n+1} \right\|^2_{\ell^2_\Delta} + \left\| f^{n+1} + \Delta t D e^{n+1} \right\|^2_{\ell^2_\Delta}
$$

$$
\begin{aligned}
&+ \left\|\sqrt{-a}D_+ f^{n+1} + \Delta t\sqrt{-a}D_+ De^{n+1}\right\|_{\ell^2_\Delta}^2 \\
&= \mathcal{E}\left(e^{n+1}, f^{n+1}\right) + (\Delta t)^2 \left\|\left(I + aD_+D_-\right) Df^{n+1}\right\|_{\ell^2_\Delta}^2 \\
&+ (\Delta t)^2 \left\|De^{n+1}\right\|_{\ell^2_\Delta}^2 - a\,(\Delta t)^2 \left\|D_+ De^{n+1}\right\|_{\ell^2_\Delta}^2 .
\end{aligned}
$$

On the other hand, we have that

$$
\begin{aligned}
&\left\|e^{n+1} + \Delta t\left(I + aD_+D_-\right) Df^{n+1}\right\|_{\ell^2_\Delta}^2 \\
&+ \left\|f^{n+1} + \Delta t De^{n+1}\right\|_{\ell^2_\Delta}^2 + \left\|\sqrt{-a}D_+ f^{n+1} + \Delta t\sqrt{-a}D_+ De^{n+1}\right\|_{\ell^2_\Delta}^2 \\
&\leq \left(\Delta t + (\Delta t)^2\right) \|\epsilon_1^n\|_{\ell^2_\Delta}^2 + \left(\Delta t + (\Delta t)^2\right) \|\epsilon_2^n\|_{\ell^2_\Delta}^2 + (-a)\left(\Delta t + (\Delta t)^2\right) \|D_+\epsilon_2^n\|_{\ell^2_\Delta}^2 \\
&+ (1+\Delta t)^2 \left\|e^n - \Delta t D\left(e^n u_\Delta^n\right) - \Delta t D\left(e^n f^n\right) + \frac{\tau_1^n}{2}\Delta t \Delta x D_+D_- e^n\right\|_{\ell^2_\Delta}^2 \\
&+ (1+\Delta t)\left\|f^n - \Delta t D\left(f^n\left(u_\Delta^n + \frac{1}{2}f^n\right)\right) + \frac{\tau_2^n}{2}\Delta t\Delta x D_+D_- f^n\right\|_{\ell^2_\Delta}^2 \\
&+ \left(\Delta t + (\Delta t)^2\right) \left\|D\left(\eta_\Delta^n f^n\right)\right\|_{\ell^2_\Delta}^2 \\
&+ (1+\Delta t)(-a)\left\|D_+ f^n - \Delta t D D_+\left(f^n\left(u_\Delta^n + \frac{1}{2}f^n\right)\right) + \frac{\tau_2^n}{2}\Delta t\Delta x D_+D_+D_- f^n\right\|_{\ell^2_\Delta}^2 .
\end{aligned} \tag{3.3.14}
$$

Owing to Proposition (3.3.1) and Proposition (3.3.2) we have that

$$
\left\|e^n - \Delta t D\left(e^n u_\Delta^n\right) - \Delta t D\left(e^n f^n\right) + \frac{\tau_1^n}{2}\Delta t\Delta x D_+D_- e^n\right\|_{\ell^2_\Delta}^2 \leq \left(1 + C_1\Delta t\right)\|e^n\|_{\ell^2_\Delta}^2 , \tag{3.3.15}
$$

respectively that

$$
\left\|f^n - \Delta t D\left(f^n\left(u_\Delta^n + \frac{1}{2}f^n\right)\right) + \frac{\tau_2^n}{2}\Delta t\Delta x D_+D_- f^n\right\|_{\ell^2_\Delta}^2 \leq \left(1 + C_2\Delta t\right)\|f^n\|_{\ell^2_\Delta}^2 . \tag{3.3.16}
$$

Proposition (3.3.3) ensures the existence of constants C_3, C_4 such that:

$$
\begin{aligned}
&(-a)\left\|D_+ f^n - \Delta t D_+ D\left(f^n\left(u_\Delta^n + \frac{f^n}{2}\right)\right) + \frac{\tau_2^n}{2}\Delta t\Delta x D_+D_+D_- f^n\right\|_{\ell^2_\Delta}^2 \\
&\leq \left(1 + C_3\Delta t\right)(-a)\left\|D_+ f\right\|_{\ell^2_\Delta}^2 + C_4(-a)\Delta t\left\|f^n\right\|_{\ell^2_\Delta}^2
\end{aligned} \tag{3.3.17}
$$

Finally, we have that

$$
\begin{aligned}
\left\|D\left(\eta_\Delta^n f^n\right)\right\|_{\ell^2_\Delta}^2 &\leq 2\left\|D\left(\eta_\Delta^n\right)\right\|_{\ell^\infty}^2 \|f^n\|_{\ell^2_\Delta}^2 + 2\left\|\eta_\Delta^n\right\|_{\ell^\infty}^2 \left\|D\left(f^n\right)\right\|_{\ell^2_\Delta}^2 \\
&\leq 2\max\left\{\left\|D\left(\eta_\Delta^n\right)\right\|_{\ell^\infty}^2 , \frac{\left\|\eta_\Delta^n\right\|_{\ell^\infty}^2}{-a}\right\}\mathcal{E}\left(e^n, f^n\right).
\end{aligned} \tag{3.3.18}
$$

Gathering relations (3.3.14), (3.3.15), (3.3.16), (3.3.17) and (3.3.18) yields

$$
\begin{aligned}
\mathcal{E}\left(e^{n+1}, f^{n+1}\right) \leq &\left(\Delta t + (\Delta t)^2\right)\left(\|\epsilon_1^n\|_{\ell^2_\Delta}^2 + \|\epsilon_2^n\|_{\ell^2_\Delta}^2 + (-a)\|D_+\epsilon_2^n\|_{\ell^2_\Delta}^2\right) \\
&+ \left(1 + \tilde{C}\Delta t\right)\mathcal{E}\left(e^n, f^n\right)
\end{aligned} \tag{3.3.19}
$$

From this point, one can argue as in Theorem 3.1.2 in order to conclude to the validity of Theorem 3.1.3 in the case $b = c = d = 0$ and $a < 0$.

Remark 3.3.1. *In fact, from now on, we show only how to derive estimates similar to (3.3.19) claiming that they are sufficient in order to prove Theorem 3.1.3.*

3.3.3 The case $b = 0, d > 0$

The case $a = b = c = 0, d > 0$. In this case, without any difficulties, we are able to prove a more general result. Indeed, we will show that the following general scheme:

$$\begin{cases} \frac{1}{\Delta t}(\eta^{n+1} - \eta^n) + D\left((1-\theta)u^n + \theta u^{n+1}\right) + D\left(\eta^n u^n\right) = \frac{\tau_1 \Delta x}{2} D_+ D_-\left(\eta^n\right), \\ \frac{1}{\Delta t}\left(I - dD_+D_-\right)\left(u^{n+1} - u^n\right) + D\left((1-\theta)\eta^n + \theta\eta^{n+1}\right) + \frac{1}{2}D\left(\left(u^n\right)^2\right) = 0. \end{cases} \tag{3.3.20}$$

is adapted for studying the classical Boussinesq system. The convergence error verifies:

$$\begin{cases} E^-(n) + \Delta t D\left((1-\theta) f^n + \theta f^{n+1}\right) + \Delta t D\left(e^n u_\Delta^n\right) + \Delta t D\left(\eta_\Delta^n f^n\right) \\ \qquad + \Delta t D\left(e^n f^n\right) = \frac{\tau_1^n}{2}\Delta t \Delta x D_+ D_-\left(e^n\right) - \Delta t \epsilon_1^n, \\ \left(I - dD_+D_-\right) F^-(n) + \Delta t D\left((1-\theta) e^n + \theta e^{n+1}\right) + \Delta t D\left(f^n u_\Delta^n\right) \\ \qquad + \frac{\Delta t}{2} D\left(\left(f^n\right)^2\right) = -\Delta t \epsilon_2^n. \end{cases} \tag{3.3.21}$$

Recall that in this case:

$$\mathcal{E}(e,f) = \|e\|_{\ell^2_\Delta}^2 + \|f\|_{\ell^2_\Delta}^2 + d\left\|D_+ f\right\|_{\ell^2_\Delta}^2. \tag{3.3.22}$$

and let us suppose that

$$\|e^n\|_{\ell^\infty}, \|D_+ f^n\|_{\ell^\infty}, \|f^n\|_{\ell^\infty} \le \Delta x^{\frac{1}{2}-\gamma}.$$

Multiply the second equation of (3.3.21) with $F^+(n)$ and proceeding as we did in Section 3.2, we obtain that there exists two constant $C_{1,1}$ and $C_{1,2}$ depending on d, $\|u_\Delta^n\|_{\ell^\infty}$, $\|Du_\Delta^n\|_{\ell^\infty}$ and θ, such that:

$$\begin{aligned} &\left\|f^{n+1}\right\|_{\ell^2_\Delta}^2 + d\left\|D_+ f^{n+1}\right\|_{\ell^2_\Delta}^2 - \left\|f^n\right\|_{\ell^2_\Delta}^2 - d\left\|D_+ f^n\right\|_{\ell^2_\Delta}^2 \\ &= -\left\langle F^+(n), \Delta t D\left((1-\theta) e^n + \theta e^{n+1}\right) + \Delta t D\left(f^n u_\Delta^n\right) + \frac{\Delta t}{2} D\left(\left(f^n\right)^2\right) - \Delta t \epsilon_2^n \right\rangle \\ &\le \Delta t \left\|\epsilon_2^n\right\|_{\ell^2_\Delta}^2 + \Delta t C_{1,1}\mathcal{E}\left(e^n, f^n\right) + \Delta t C_{1,2}\mathcal{E}\left(e^{n+1}, f^{n+1}\right). \end{aligned} \tag{3.3.23}$$

Next we rewrite the first equation of (3.3.21) as

$$\begin{aligned} e^{n+1} &= e^n - \Delta t D\left(e^n\left(u_\Delta^n + f^n\right)\right) + \frac{\tau_1^n}{2}\Delta t \Delta x D_+ D_-\left(e^n\right) \\ &\quad - \Delta t D\left((1-\theta) f^n + \theta f^{n+1}\right) - \Delta t D\left(\eta_\Delta^n f^n\right) - \Delta t \epsilon_1^n. \end{aligned}$$

Thus, using the Proposition (3.3.2) we get that

$$\begin{aligned} \left\|e^{n+1}\right\|_{\ell^2_\Delta} &\le \left\|e^n - \Delta t D\left(e^n\left(u_\Delta^n + f^n\right)\right) + \frac{\tau_1^n}{2}\Delta t \Delta x D_+ D_-\left(e^n\right)\right\|_{\ell^2_\Delta} \\ &\quad + \Delta t\left\|D\left((1-\theta) f^n + \theta f^{n+1}\right)\right\|_{\ell^2_\Delta} + \Delta t\left\|D\left(\eta_\Delta^n f^n\right)\right\|_{\ell^2_\Delta} + \Delta t\left\|\epsilon_1^n\right\|_{\ell^2_\Delta} \\ &\le \Delta t\left\|\epsilon_1^n\right\|_{\ell^2_\Delta} + (1 + C\Delta t)\left\|e^n\right\|_{\ell^2_\Delta} + \Delta t\left((1-\theta)\left\|D\left(f^n\right)\right\|_{\ell^2_\Delta} + \theta\left\|D\left(f^{n+1}\right)\right\|_{\ell^2_\Delta}\right) \\ &\quad + \Delta t \max\left\{\left\|\eta_\Delta^n\right\|_{\ell^\infty}, \left\|D\left(\eta_\Delta^n\right)\right\|_{\ell^\infty}\right\}\left(\left\|f^n\right\|_{\ell^2_\Delta} + \left\|D\left(f^n\right)\right\|_{\ell^2_\Delta}\right) \\ &\le \Delta t\left\|\epsilon_1^n\right\|_{\ell^2_\Delta} + (1 + C\Delta t)\left\|e^n\right\|_{\ell^2_\Delta} + \Delta t C_{1,2}\mathcal{E}^{\frac{1}{2}}\left(e^n, f^n\right) + C_{2,2}\Delta t \mathcal{E}^{\frac{1}{2}}\left(e^{n+1}, f^{n+1}\right). \end{aligned} \tag{3.3.24}$$

From (3.3.24) we deduce that

$$\begin{aligned} \left\|e^{n+1}\right\|_{\ell^2_\Delta}^2 &\le \left(\Delta t + (\Delta t)^2\right)\left\|\epsilon_1^n\right\|_{\ell^2_\Delta}^2 + \left(1 + C_{1,3}\Delta t\right)\left\|e^n\right\|_{\ell^2_\Delta}^2 \\ &\quad + \left(\Delta t + (\Delta t)^2\right) C_{1,3}\mathcal{E}\left(e^n, f^n\right) + \left(\Delta t + (\Delta t)^2\right) C_{2,3}\mathcal{E}\left(e^{n+1}, f^{n+1}\right). \end{aligned} \tag{3.3.25}$$

Adding up the estimates yields

$$\left(1 - \tilde{C}_1\Delta t\right)\mathcal{E}\left(e^{n+1}, f^{n+1}\right) \le \left(\Delta t + (\Delta t)^2\right)\left\|\epsilon_1^n\right\|_{\ell^2_\Delta}^2 + \Delta t\left\|\epsilon_2^n\right\|_{\ell^2_\Delta}^2 + \left(1 + \tilde{C}_2\Delta t\right)\mathcal{E}\left(e^n, f^n\right).$$

The case $a<0, b=0, c=0, d>0$ In this case, the convergence error satisfies:

$$\begin{cases} e^{n+1}+\Delta t D f^{n+1}+a\Delta t D_+D_-Df^{n+1} \\ = e^n - \Delta t D\left(e^n u^n_\Delta\right) - \Delta t D\left(e^n f^n\right) - \Delta t D\left(\eta^n_\Delta f^n\right) + \frac{\tau^n_1}{2}\Delta t \Delta x D_+D_- e^n - \Delta t \epsilon^n_1, \\ \left(I - dD_+D_-\right) f^{n+1} + \Delta t D e^{n+1} \\ = \left(I - dD_+D_-\right) f^n - \Delta t D\left(f^n\left(u^n_\Delta + \frac{1}{2}f^n\right)\right) - \Delta t \epsilon^n_2. \end{cases}$$

In this section, we will work in the following with the energy functional:

$$\mathcal{E}\left(e,f\right) = d\left\|e\right\|^2_{\ell^2_\Delta} + (-a)\left\|\left(I - dD_+D_-\right)f\right\|^2_{\ell^2_\Delta},$$

which is better adapted to the system corresponding to the particular values of the parameters in view here. Of course, $\mathcal{E}$ is equivalent to the energy from (3.2.2). As before, by induction, we may suppose that

$$\mathcal{E}\left(e^n, f^n\right) \leq \Delta x^{2-2\gamma}.$$

By summing up the square of the ℓ^2_Δ-norm of the first equation by d, the square of the ℓ^2_Δ norm of the second one by $(-a)$, we get that:

$$\begin{aligned} &\mathcal{E}\left(e^{n+1}, f^{n+1}\right) + 2(d+a)\Delta t\left\langle Df^{n+1}, e^{n+1}\right\rangle \\ &\leq (\Delta t + \Delta t^2)Cd\left\|\epsilon^n_1\right\|^2_{\ell^2_\Delta} + (\Delta t + \Delta t^2)C(-a)\left\|\epsilon^n_2\right\|^2_{\ell^2_\Delta} \\ &+ (1+\Delta t C)\,d\left\|e^n - \Delta t D\left(e^n\left(u^n_\Delta + f^n\right)\right) + \frac{\tau^n_1}{2}\Delta t\Delta x D_+D_-e^n\right\|^2_{\ell^2_\Delta} \\ &+ \left(\Delta t + \Delta t^2\right)d\left|\left|D\left(\eta^n_\Delta f^n\right)\right|\right|^2_{\ell^2_\Delta} \\ &+ (1+\Delta t C)(-a)\left\|\left(I - dD_+D_-\right)f^n\right\|^2_{\ell^2_\Delta} \\ &+ \left(\Delta t + \Delta t^2 C\right)(-a)\left\|D\left(f^n\left(u^n_\Delta + \frac{1}{2}f^n\right)\right)\right\|^2_{\ell^2_\Delta}. \end{aligned}$$

We notice that

$$\begin{aligned} 2\Delta t(d+a)\left\langle f^{n+1}, De^{n+1}\right\rangle &\geq -\Delta t(d-a)||Df^{n+1}||^2_{\ell^2_\Delta} - \Delta t(d-a)||e^{n+1}||^2_{\ell^2_\Delta} \\ &\geq -\Delta t C\mathcal{E}\left(e^{n+1}, f^{n+1}\right). \end{aligned}$$

Using Proposition 3.3.2 we get that

$$\begin{aligned} &(1+\Delta t C)\,d\left\|e^n - \Delta t D\left(e^n\left(u^n_\Delta + f^n\right)\right) + \frac{\tau^n_1}{2}\Delta t\Delta x D_+D_-e^n\right\|^2_{\ell^2_\Delta} \\ &\leq (1 + C\Delta t)\,\mathcal{E}\left(e^n, f^n\right). \end{aligned}$$

Also, we have:

$$||D\left(\eta^n_\Delta f^n\right)||^2_{\ell^2_\Delta} \leq ||D\left(\eta^n_\Delta\right)||^2_{\ell^\infty}||f^n||^2_{\ell^2_\Delta} + ||\eta^n_\Delta||^2_{\ell^\infty}||D\left(f^n\right)||^2_{\ell^2_\Delta} \leq C\mathcal{E}\left(e^n, f^n\right).$$

Proceeding as above, we get that:

$$\left\|D\left(f^n\left(u^n_\Delta + \frac{1}{2}f^n\right)\right)\right\|^2_{\ell^2_\Delta} \leq C\mathcal{E}\left(e^n, f^n\right).$$

Adding up the above estimate gives us:

$$\begin{aligned} (1 - C\Delta t)\,\mathcal{E}\left(e^{n+1}, f^{n+1}\right) \leq &\left(\Delta t + (\Delta t)^2\right)Cd\left\|\epsilon^n_1\right\|^2_{\ell^2_\Delta} + \left(\Delta t + (\Delta t)^2\right)C(-a)\left\|\epsilon^n_2\right\|^2_{\ell^2_\Delta} \\ &+ (1 + C\Delta t)\,\mathcal{E}\left(e^n, f^n\right). \end{aligned}$$

The case $a<0, b=0, c<0, d>0$ In this case, the convergence error satisfies:

$$\begin{cases} e^{n+1}+\Delta t D f^{n+1}+a \Delta t D_{+} D_{-} D f^{n+1} \\ =e^{n}-\Delta t D\left(e^{n} u_{\Delta}^{n}\right)-\Delta t D\left(e^{n} f^{n}\right)-\Delta t D\left(\eta_{\Delta}^{n} f^{n}\right)+\frac{\tau_{1}^{n}}{2} \Delta t \Delta x D_{+} D_{-} e^{n}-\Delta t \epsilon_{1}^{n}, \\ \left(I-d D_{+} D_{-}\right) f^{n+1}+\Delta t D e^{n+1}+c \Delta t D_{+} D_{-} D e^{n+1} \\ =\left(I-d D_{+} D_{-}\right) f^{n}-\Delta t D\left(f^{n}\left(u_{\Delta}^{n}+\frac{1}{2} f^{n}\right)\right)-\Delta t \epsilon_{2}^{n}. \end{cases}$$

Again, in order to close the estimates and prove the convergence of the scheme, we will use the following energy functional:

$$\mathcal{E}(e, f) \stackrel{def.}{=}\|e\|_{\ell_{\Delta}^{2}}^{2}+(-c d)\left\|D_{+} e\right\|_{\ell_{\Delta}^{2}}^{2}+(-a)\left\|\left(I-d D_{+} D_{-}\right) f\right\|_{\ell_{\Delta}^{2}}^{2}, \tag{3.3.26}$$

which is equivalent to that one from (3.2.2). In order to derive a H^1 control on e^{n+1}, let us apply $\sqrt{-cd}D_+$ in the first equation, to obtain

$$\begin{aligned} & \sqrt{-c d} D_{+} e^{n+1}+\Delta t D_{+} D f^{n+1}+\sqrt{-c d} a \Delta t D_{+} D_{+} D_{-} D f^{n+1} \\ & =\sqrt{-c d} D_{+} e^{n}-\Delta t \sqrt{-c d} D_{+} D\left(e^{n} u_{\Delta}^{n}\right)-\Delta t \sqrt{-c d} D_{+} D\left(e^{n} f^{n}\right) \\ & +\frac{\tau_{1}^{n}}{2} \sqrt{-c d} \Delta t \Delta x D_{+} D_{+} D_{-} e^{n}-\sqrt{-c d} \Delta t D_{+} D\left(\eta_{\Delta}^{n} f^{n}\right)-\sqrt{-c d} \Delta t D_{+} \epsilon_{1}^{n}. \end{aligned}$$

Let us observe that

$$\begin{aligned} I_{1} & =\left\|e^{n+1}+\Delta t D f^{n+1}+a \Delta t D_{+} D_{-} D f^{n+1}\right\|_{\ell_{\Delta}^{2}}^{2} \\ & =\left\|e^{n+1}\right\|_{\ell_{\Delta}^{2}}^{2}+(\Delta t)^{2}\left\|D f^{n+1}\right\|_{\ell_{\Delta}^{2}}^{2}+a^{2}(\Delta t)^{2}\left\|D_{+} D_{-} D f^{n+1}\right\|_{\ell_{\Delta}^{2}}^{2} \\ & -2 a(\Delta t)^{2}\left\|D_{+} D f^{n+1}\right\|_{\ell_{\Delta}^{2}}^{2}+2 \Delta t\left\langle e^{n+1}, D f^{n+1}\right\rangle+2 a \Delta t\left\langle e^{n+1}, D_{+} D_{-} D f^{n+1}\right\rangle \end{aligned} \tag{3.3.27}$$

along with

$$\begin{aligned} I_{2} & =(-c d)\left\|D_{+} e^{n+1}+\Delta t D_{+} D f^{n+1}+a \Delta t D_{+} D_{+} D_{-} D f^{n+1}\right\|_{\ell_{\Delta}^{2}}^{2} \\ & =(-c d)\left\|D_{+} e^{n+1}\right\|_{\ell_{\Delta}^{2}}^{2}+(-c d)(\Delta t)^{2}\left\|D_{+} D f^{n+1}\right\|_{\ell_{\Delta}^{2}}^{2}+(-c d) a^{2}(\Delta t)^{2}\left\|D_{+} D_{+} D_{-} D f^{n+1}\right\|_{\ell_{\Delta}^{2}} \\ & +2 a c d(\Delta t)^{2}\left\|D_{+} D_{-} D f^{n+1}\right\|_{\ell_{\Delta}^{2}}^{2}-2 c d \Delta t\left\langle D_{+} e^{n+1}, D_{+} D f^{n+1}\right\rangle \\ & -2 a c d \Delta t\left\langle D_{+} e^{n+1}, D_{+} D_{+} D_{-} D f^{n+1}\right\rangle, \end{aligned} \tag{3.3.28}$$

and

$$\begin{aligned} I_{3} & =(-a)\left\|\left(I-d D_{+} D_{-}\right) f^{n+1}+\Delta t D e^{n+1}+c \Delta t D_{+} D_{-} D e^{n+1}\right\|_{\ell_{\Delta}^{2}}^{2} \\ & =(-a)\left\|\left(I-d D_{+} D_{-}\right) f^{n+1}\right\|_{\ell_{\Delta}^{2}}^{2}+(-a) c^{2}(\Delta t)^{2}\left\|D_{+} D_{-} D e^{n+1}\right\|_{\ell_{\Delta}^{2}}^{2}+(\Delta t)^{2}\left\|D e^{n+1}\right\|_{\ell_{\Delta}^{2}} \\ & +2 a c(\Delta t)^{2}\left\|D_{+} D e^{n+1}\right\|_{\ell_{\Delta}^{2}}+2 \Delta t\left\langle\left(I-d D_{+} D_{-}\right) f^{n+1}, D e^{n+1}\right\rangle \\ & -2 a c \Delta t\left\langle f^{n+1}, D_{+} D_{-} D e^{n+1}\right\rangle+2 a c d \Delta t\left\langle D_{+} D_{-} f^{n+1}, D_{+} D_{-} D e^{n+1}\right\rangle. \end{aligned} \tag{3.3.29}$$

Observe that the last term from (3.3.28) cancels with the last term of (3.3.29) such that:

$$\begin{aligned} I_{1}+I_{2}+I_{3} \geq\; & \mathcal{E}\left(e^{n+1}, f^{n+1}\right)-2 a \Delta t\left\langle D e^{n+1}, D_{+} D_{-} f^{n+1}\right\rangle \\ & -2 c d \Delta t\left\langle D_{+} e^{n+1}, D_{+} D f^{n+1}\right\rangle-2 d \Delta t\left\langle D_{+} D_{-} f^{n+1}, D e^{n+1}\right\rangle \\ & -2 a c \Delta t\left\langle D_{+} D_{-} f^{n+1}, D e^{n+1}\right\rangle \\ \geq\; & (1-C \Delta t) \mathcal{E}\left(e^{n+1}, f^{n+1}\right). \end{aligned}$$

As before, we suppose that

$$\mathcal{E}\left(e^n, f^n\right) \leq \Delta x^{2-2\gamma}, \tag{3.3.30}$$

for some $\gamma \in \left(0, \frac{1}{2}\right)$. In view of the definition of our energy functional, (3.3.26), we claim that (3.3.30) implies that the hypothesis of Propositions 3.3.2 and 3.3.3 are verified such that we may perform the following manipulations. Using Proposition 3.3.2, we get that

$$\begin{aligned}\sqrt{I_1} \leq & \left\| e^n - \Delta t D\left(e^n u_\Delta^n\right) - \Delta t D\left(e^n f^n\right) + \frac{\tau_1^n}{2} \Delta t \Delta x D_+ D_- e^n \right\|_{\ell^2_\Delta} \\ & + \Delta t \left\| D\left(\eta_\Delta^n f^n\right)\right\|_{\ell^2_\Delta} + \Delta t \left\|\epsilon_1^n\right\|_{\ell^2_\Delta} \\ & \leq \left(1 + C\Delta t\right)\left\|e^n\right\|_{\ell^2} + \Delta t \left\| D\left(\eta_\Delta^n f^n\right)\right\|_{\ell^2_\Delta} + \Delta t \left\|\epsilon_1^n\right\|_{\ell^2_\Delta}\end{aligned} \tag{3.3.31}$$

Using the triangle inequality, we get that

$$\begin{aligned}\sqrt{I_3} &= \sqrt{(-a)} \left\| \left(I - dD_+D_-\right) f^{n+1} + \Delta t D e^{n+1} + c\Delta t D_+ D_- D e^{n+1}\right\|_{\ell^2_\Delta} \\ &\leq \sqrt{(-a)} \left\|\left(I - dD_+D_-\right) f^n\right\|_{\ell^2_\Delta} + \Delta t \sqrt{(-a)} \left\| D\left(f^n\left(u_\Delta^n + \frac{1}{2} f^n\right)\right)\right\|_{\ell^2_\Delta} + \Delta t\sqrt{-a}\left\|\epsilon_2^n\right\|_{\ell^2_\Delta} \\ &\leq \Delta t \sqrt{-a}\left\|\epsilon_2^n\right\|_{\ell^2_\Delta} + \left(1 + C\Delta t\right)\mathcal{E}^{\frac{1}{2}}\left(e^n, f^n\right).\end{aligned} \tag{3.3.32}$$

Using Proposition 3.3.3 and the triangle inequality, we get that

$$\begin{aligned}\sqrt{I_2} &= \sqrt{-cd}\left\| D_+ e^{n+1} + \Delta t D f^{n+1} + a\Delta t D_+ D_+ D_- D f^{n+1}\right\|_{\ell^2_\Delta} \\ &\leq \sqrt{-cd}\left\| D_+ e^n - \Delta t D_+ D\left(e^n\left(u_\Delta^n + f^n\right)\right) + \frac{\tau_1^n}{2}\Delta t \Delta x D_+ D_+ D_- e^n\right\|_{\ell^2_\Delta} \\ &+ \sqrt{-cd}\Delta t \left\| D_+ D\left(\eta_\Delta^n f^n\right)\right\|_{\ell^2_\Delta} + \sqrt{-cd}\Delta t\left\| D_+ \epsilon_1^n\right\|_{\ell^2_\Delta} \\ &\leq \sqrt{-cd}\Delta t\left\|D_+\epsilon_1^n\right\|_{\ell^2_\Delta} + \sqrt{-cd}\left(\Delta t C_1 \left\|e^n\right\|_{\ell^2_\Delta} + \left(1 + C_2\Delta t\right)\left\|D_+ e^n\right\|_{\ell^2_\Delta}\right) \\ &+ \sqrt{-cd}\Delta t\left\|D_+ D\left(\eta_\Delta^n f^n\right)\right\|_{\ell^2_\Delta} \\ &\leq \sqrt{-cd}\Delta t\left\|D_+\epsilon_1^n\right\|_{\ell^2_\Delta} + \left(1 + \Delta t C_3\right)\mathcal{E}\left(e^n, f^n\right)^{\frac{1}{2}}\end{aligned} \tag{3.3.33}$$

From (3.3.31)- (3.3.33) we get that

$$\begin{aligned}\mathcal{E}\left(e^{n+1}, f^{n+1}\right)\left(1 - \tilde{C}_1 \Delta t\right) \leq & \left(\Delta t + \Delta t^2\right)\left(\left\|\epsilon_1^n\right\|^2_{\ell^2_\Delta} + (-a)\left\|\epsilon_2^n\right\|^2_{\ell^2_\Delta} + (-c)d\left\|D_+\epsilon_1^n\right\|^2_{\ell^2_\Delta}\right) \\ & + \left(1 + \Delta t \tilde{C}_2\right)\mathcal{E}\left(e^n, f^n\right)\end{aligned}$$

3.3.4 The case $d = 0, b > 0$

The case $a = c = d = 0, b > 0$. In this case, as for the classical Boussinesq system we derive estimates for a more general scheme:

$$\begin{cases} \frac{1}{\Delta t}\left(\eta^{n+1} - \eta^n\right) + D\left((1-\theta)u^n + \theta u^{n+1}\right) + D\left(\eta^n u^n\right) = 0, \\ \frac{1}{\Delta t}\left(I - dD_+D_-\right)\left(u^{n+1} - u^n\right) + D\left((1-\theta)\eta^n + \theta\eta^{n+1}\right) + \frac{1}{2}D\left(\left(u^n\right)^2\right) = \frac{\tau_2 \Delta x}{2} D_+ D_-\left(\eta^n\right). \end{cases} \tag{3.3.34}$$

The convergence error verifies:

$$\begin{cases} E^-(n) + \Delta t D\left((1-\theta) f^n + \theta f^{n+1}\right) + \Delta t D\left(e^n u_\Delta^n\right) + \Delta t D\left(\eta_\Delta^n f^n\right) + \Delta t D\left(e^n f^n\right) \\ \qquad = \frac{\tau_1^n}{2}\Delta t \Delta x D_+ D_-\left(e^n\right) - \Delta t \epsilon_1^n, \\ \left(I - dD_+D_-\right) F^-(n) + \Delta t D\left((1-\theta)e^n + \theta e^{n+1}\right) + \Delta t D\left(f^n u_\Delta^n\right) + \frac{\Delta t}{2} D\left(\left(f^n\right)^2\right) \\ \qquad = -\Delta t \epsilon_2^n. \end{cases} \tag{3.3.35}$$

$$\begin{cases} (I - bD_+D_-)\left(E^-(n)\right) + \Delta t D\left((1-\theta) f^n + \theta f^{n+1}\right) \\ +\Delta t D\left(e^n u_\Delta^n\right) + \Delta t D\left(\eta_\Delta^n f^n\right) + \Delta t D\left(e^n f^n\right) = -\Delta t \epsilon_1^n, \\ F^-(n) + \Delta t D\left((1-\theta) e^n + \theta e^{n+1}\right) + \Delta t D\left(f^n u_\Delta^n\right) \\ +\frac{\Delta t}{2} D\left((f^n)^2\right) = \frac{\tau_2^n}{2}\Delta t \Delta x D_+D_-\left(f^n\right) - \Delta t \epsilon_2^n. \end{cases} \tag{3.3.36}$$

We work with

$$\mathcal{E}(e,f) = \|e\|_{\ell^2_\Delta}^2 + b\|D_+(e)\|_{\ell^2_\Delta}^2 + \|f\|_{\ell^2_\Delta}^2, \tag{3.3.37}$$

and we suppose that

$$\|e^n\|_{\ell^\infty}, \|f^n\|_{\ell^\infty} \le \Delta x^{1/2-\gamma}.$$

Multiply the first equation by $E^+(n)$ in order to get that

$$\begin{aligned} &\left(\left\|e^{n+1}\right\|_{\ell^2}^2 + b\left\|D_+e^{n+1}\right\|_{\ell^2}^2\right) - \left(\|e^n\|_{\ell^2}^2 + b\|D_+e^n\|_{\ell^2}^2\right) \\ &= -\Delta t\left\langle D\left((1-\theta) f^n + \theta f^{n+1}\right) + D\left(e^n u_\Delta^n\right) + D\left(\eta_\Delta^n f^n\right) + D\left(e^n f^n\right) + \epsilon_1^n, E^+(n)\right\rangle \\ &= -\Delta t\left\langle \epsilon_1^n, E^+(n)\right\rangle + \Delta t\left\langle (1-\theta) f^n + \theta f^{n+1} + e^n u_\Delta^n + \eta_\Delta^n f^n + e^n f^n, DE^+(n)\right\rangle \\ &\le \Delta t\|\epsilon_1^n\|_{\ell^2_\Delta}^2 + \Delta t C_1 \mathcal{E}\left(e^n, f^n\right) + \Delta t C_2 \mathcal{E}\left(e^{n+1}, f^{n+1}\right). \end{aligned} \tag{3.3.38}$$

Using the second equation of (3.3.36), the triangle inequality along with Proposition 3.3.1 we get that:

$$\begin{aligned} \left\|f^{n+1}\right\|_{\ell^2} \le &\left\|f^n - \Delta t D\left(f^n u_\Delta^n\right) - \frac{\Delta t}{2} D\left((f^n)^2\right) + \frac{\tau_2^n}{2}\Delta t \Delta x D_+D_-\left(f^n\right)\right\|_{\ell^2} \\ &+ \Delta t\|\epsilon_2^n\|_{\ell^2_\Delta} + \Delta t(1-\theta)\|D(e^n)\|_{\ell^2_\Delta} + \Delta t\theta\left\|D\left(e^{n+1}\right)\right\|_{\ell^2_\Delta} \\ &\le \Delta t\|\epsilon_2^n\|_{\ell^2_\Delta} + (1+C\Delta t)\|f^n\|_{\ell^2_\Delta} + \Delta t(1-\theta)\|D(e^n)\|_{\ell^2_\Delta} + \Delta t\theta\left\|D\left(e^{n+1}\right)\right\|_{\ell^2_\Delta}. \end{aligned} \tag{3.3.39}$$

Thus, by adding up estimate (3.3.38) with the square of the estimate (3.3.39) we get that

$$(1 - C\Delta t)\mathcal{E}\left(e^{n+1}, f^{n+1}\right) \le \Delta t\|\epsilon_1^n\|_{\ell^2_\Delta}^2 + \left(\Delta t + \Delta t^2\right)\|\epsilon_2^n\|_{\ell^2_\Delta}^2 + (1 + C\Delta t)\mathcal{E}\left(e^n, f^n\right).$$

The case $a<0, b>0, c=d=0$. The convergence error satisfies:

$$\begin{cases} (I - bD_+D_-) e^{n+1} + \Delta t D f^{n+1} + a\Delta t D_+D_-D f^{n+1} = (I - bD_+D_-) e^n \\ -\Delta t D\left(e^n u_\Delta^n\right) - \Delta t D\left(e^n f^n\right) - \Delta t D\left(\eta_\Delta^n f^n\right) - \Delta t \epsilon_1^n, \\ f^{n+1} + \Delta t D e^{n+1} = f^n - \Delta t D\left(f^n\left(u_\Delta^n + \frac{1}{2} f^n\right)\right) + \frac{\tau_2^n}{2}\Delta t \Delta x D_+D_-\left(f^n\right) - \Delta t \epsilon_2^n. \end{cases} \tag{3.3.40}$$

Consider the energy functional we will work with will be

$$\mathcal{E}(e,f) = \|e\|_{\ell^2_\Delta}^2 + b\|D_+e\|_{\ell^2_\Delta}^2 + \|f\|_{\ell^2_\Delta}^2 + (-a)\|D_+f\|_{\ell^2_\Delta}^2.$$

By induction, we may suppose that

$$\mathcal{E}\left(e^n, f^n\right) \le \Delta x^{2-2\gamma}.$$

Let us multiply the first equation with $2e^{n+1}$ to obtain:

$$\begin{aligned} &2\left\|e^{n+1}\right\|_{\ell^2_\Delta}^2 + 2b\left\|D_+e^{n+1}\right\|_{\ell^2_\Delta}^2 + 2\Delta t\left\langle Df^{n+1}, e^{n+1}\right\rangle + 2a\Delta t\left\langle D_+D_-Df^{n+1}, e^{n+1}\right\rangle \\ &= 2\left\langle (I - bD_+D_-) e^n, e^{n+1}\right\rangle - 2\Delta t\left\langle D\left(e^n u_\Delta^n\right) + D\left(e^n f^n\right) + D\left(\eta_\Delta^n f^n\right) + \epsilon_1^n, e^{n+1}\right\rangle \\ &\le \left\|e^{n+1}\right\|_{\ell^2_\Delta}^2 + b\left\|D_+e^{n+1}\right\|_{\ell^2_\Delta}^2 + \|e^n\|_{\ell^2_\Delta}^2 + b\|D_+e^n\|_{\ell^2_\Delta}^2 \\ &+ C\Delta t\left(\mathcal{E}^{\frac{1}{2}}\left(e^n, f^n\right) + \|\epsilon_1^n\|_{\ell^2_\Delta}\right)\left\|e^{n+1}\right\|_{\ell^2_\Delta} \end{aligned}$$

and thus

$$\left\|e^{n+1}\right\|_{\ell^2_\Delta}^2+b\left\|D_+e^{n+1}\right\|_{\ell^2_\Delta}^2+2\Delta t\left\langle Df^{n+1},e^{n+1}\right\rangle+2a\Delta t\left\langle D_+D_-Df^{n+1},e^{n+1}\right\rangle$$
$$\le\left\|e^n\right\|_{\ell^2_\Delta}^2+b\left\|D_+e^n\right\|_{\ell^2_\Delta}^2+C\Delta t\left(\mathcal{E}^{\frac{1}{2}}\left(e^n,f^n\right)+\left\|\epsilon_1^n\right\|_{\ell^2_\Delta}\right)\left\|e^{n+1}\right\|_{\ell^2_\Delta}. \tag{3.3.41}$$

Next, taking the square of the ℓ^2_Δ-norm of the second equation of (3.3.40) and using Proposition 3.3.2 we get that

$$\left\|f^{n+1}\right\|_{\ell^2_\Delta}^2+2\Delta t\left\langle f^{n+1},De^{n+1}\right\rangle+(\Delta t)^2\left\|De^{n+1}\right\|_{\ell^2_\Delta}^2$$
$$\le\left(\Delta t+(\Delta t)^2\right)\left\|\epsilon_2^n\right\|_{\ell^2_\Delta}^2+(1+\Delta t)\left\|f^n-\Delta tD\left(f^n\left(u_\Delta^n+\frac{1}{2}f^n\right)\right)+\frac{\tau_2^n}{2}\Delta t\Delta xD_+D_-\left(f^n\right)\right\|_{\ell^2_\Delta}^2$$
$$\le\left(\Delta t+(\Delta t)^2\right)\left\|\epsilon_2^n\right\|_{\ell^2_\Delta}^2+(1+C\Delta t)\left\|f^n\right\|_{\ell^2_\Delta}^2. \tag{3.3.42}$$

Applying $\sqrt{-a}D_+$ into the second equation of (3.3.40) and taking the square of the ℓ^2_Δ-norm of the resulting equation yields

$$(-a)\left\|D_+f^{n+1}\right\|_{\ell^2_\Delta}^2+2(-a)\Delta t\left\langle D_+f^{n+1},D_+De^{n+1}\right\rangle+(-a)(\Delta t)^2\left\|D_+De^{n+1}\right\|_{\ell^2_\Delta}^2$$
$$\le\left(\Delta t+(\Delta t)^2\right)(-a)\left\|D_+\epsilon_2^n\right\|_{\ell^2_\Delta}^2$$
$$+(1+\Delta t)(-a)\left\|D_+f^n-\Delta tD_+D\left(f^n\left(u_\Delta^n+\frac{1}{2}f^n\right)\right)+\frac{\tau_2^n}{2}\Delta t\Delta xD_+D_+D_-\left(f^n\right)\right\|_{\ell^2_\Delta}^2$$
$$\le\left(\Delta t+(\Delta t)^2\right)(-a)\left\|D_+\epsilon_2^n\right\|_{\ell^2_\Delta}^2+(1+C\Delta t)(-a)\left\|D_+f^n\right\|_{\ell^2_\Delta}^2+\Delta tC(-a)\left\|f^n\right\|_{\ell^2_\Delta}^2. \tag{3.3.43}$$

Adding up the estimates (3.3.41), (3.3.42) and (3.3.43) and noticing that leads to

$$\mathcal{E}\left(e^{n+1},f^{n+1}\right)\le\Delta t\left\|\epsilon_1^n\right\|_{\ell^2_\Delta}^2+\left(\Delta t+(\Delta t)^2\right)\left(\left\|\epsilon_2^n\right\|_{\ell^2_\Delta}^2+(-a)\left\|D_+\epsilon_2^n\right\|_{\ell^2_\Delta}^2\right) \tag{3.3.44}$$
$$+(1+C\Delta t)\,\mathcal{E}\left(e^n,f^n\right). \tag{3.3.45}$$

The case $a<0,b>0,c<0,d=0$. Finally, in this case, the convergence error satisfies:

$$\begin{cases}(I-bD_+D_-)\,e^{n+1}+\Delta tDf^{n+1}+a\Delta tD_+D_-Df^{n+1}=(I-bD_+D_-)\,e^n\\ -\Delta tD\left(e^nu_\Delta^n\right)-\Delta tD\left(e^nf^n\right)-\Delta tD\left(\eta_\Delta^nf^n\right)-\Delta t\epsilon_1^n,\\ f^{n+1}+\Delta tDe^{n+1}+c\Delta tD_+D_-De^{n+1}=f^n-\Delta tD\left(f^n\left(u_\Delta^n+\frac{1}{2}f^n\right)\right)\\ +\frac{\tau_2^n}{2}\Delta t\Delta xD_+D_-\left(f^n\right)-\Delta t\epsilon_2^n.\end{cases} \tag{3.3.46}$$

In this case, we will use the energy functional:

$$\mathcal{E}(e,f)=(-c)\left\|(I-bD_+D_-)\,e\right\|_{\ell^2_\Delta}^2+\left\|f\right\|_{\ell^2_\Delta}^2+(-a)\,b\left\|D_+f\right\|_{\ell^2_\Delta}^2.$$

By induction, we may suppose that

$$\mathcal{E}\left(e^n,f^n\right)\le\Delta x^{2-2\gamma},$$

which is sufficient in order to assure the hypothesis of Proposition 3.3.2 and Proposition 3.3.3. Taking the square of the first equations of (3.3.46) and multiplying the result with $(-c)$ yields

$$I_1=(-c)\left\|(I-bD_+D_-)\,e^{n+1}\right\|_{\ell^2_\Delta}^2+(-c)(\Delta t)^2\left\|Df^{n+1}\right\|_{\ell^2_\Delta}^2+(-c)\,a^2(\Delta t)^2\left\|D_+D_-Df^{n+1}\right\|_{\ell^2_\Delta}^2$$
$$+2ac(\Delta t)^2\left\|D_+Df^{n+1}\right\|_{\ell^2_\Delta}^2-2c\Delta t\left\langle(I-bD_+D_-)\,e^{n+1},Df^{n+1}\right\rangle$$
$$-2ac\Delta t\left\langle e^{n+1},D_+D_-Df^{n+1}\right\rangle+2acb\Delta t\left\langle D_+D_-e^{n+1},D_+D_-Df^{n+1}\right\rangle$$

$$\leq (-c)\left(\Delta t+(\Delta t)^2\right)\|\epsilon_1^n\|_{\ell^2_\Delta}^2+(1+\Delta t)(-c)\|(I-bD_+D_-)e^n\|_{\ell^2_\Delta}^2+C\Delta t\mathcal{E}(e^n,f^n). \tag{3.3.47}$$

Taking the square of the ℓ^2_Δ-norm of the second equation of (3.3.46) and applying Proposition 3.3.2, we get that

$$\begin{aligned}
I_2 &= \left\|f^{n+1}\right\|_{\ell^2_\Delta}^2+(\Delta t)^2\left\|De^{n+1}\right\|_{\ell^2_\Delta}^2+(c\Delta t)^2\left\|D_+D_-De^{n+1}\right\|_{\ell^2_\Delta}^2-2c(\Delta t)^2\left\|D_+De^{n+1}\right\|_{\ell^2_\Delta}^2\\
&2\Delta t\left\langle f^{n+1},De^{n+1}\right\rangle+2c\Delta t\left\langle f^{n+1},D_+D_-De^{n+1}\right\rangle\\
&\leq \left(\Delta t+(\Delta t)^2\right)\|\epsilon_2^n\|_{\ell^2_\Delta}^2+(1+\Delta t)\left\|f^n-\Delta tD\left(f^n\left(u_\Delta^n+\frac{1}{2}f^n\right)\right)+\frac{\tau_2^n}{2}\Delta t\Delta xD_+D_-(f^n)\right\|_{\ell^2_\Delta}^2\\
&\leq \left(\Delta t+(\Delta t)^2\right)\|\epsilon_2^n\|_{\ell^2_\Delta}^2+(1+C\Delta t)\|f^n\|_{\ell^2_\Delta}^2+\left(\Delta t+(\Delta t)^2\right)\|De^n\|_{\ell^2_\Delta}^2.
\end{aligned}\tag{3.3.48}$$

Let us observe that the terms

$$2\Delta t\left\langle f^{n+1},De^{n+1}\right\rangle,-2ac\Delta t\left\langle e^{n+1},D_+D_-Df^{n+1}\right\rangle \text{ and } 2c\Delta t\left\langle f^{n+1},D_+D_-De^{n+1}\right\rangle$$

can be controlled by the energy so that they do not raise any issues. In order to get rid of the term

$$2acb\Delta t\left\langle D_+D_-e^{n+1},D_+D_-Df^{n+1}\right\rangle \tag{3.3.49}$$

we will apply $\sqrt{(-a)\,b}D_+$ into the second equation of (3.3.46) and consider the square of the ℓ^2_Δ-norm of the result. First, let us compute:

$$\begin{aligned}
I_3 &= (-a)\,b\left\|D_+f^{n+1}+\Delta tD_+De^{n+1}+c\Delta tD_+D_+D_-De^{n+1}\right\|_{\ell^2_\Delta}^2\\
&=(-a)\,b\left\|D_+f^{n+1}\right\|_{\ell^2_\Delta}^2+(-a)\,b(\Delta t)^2\left\|D_+De^{n+1}\right\|_{\ell^2_\Delta}^2+(-a)\,b(c\Delta t)^2\left\|D_+D_+D_-De^{n+1}\right\|_{\ell^2_\Delta}^2\\
&+2(-a)\,b\Delta t\left\langle D_+f^{n+1},D_+De^{n+1}\right\rangle+2(-a)bc\Delta t\left\langle D_+f^{n+1},D_+D_+D_-De^{n+1}\right\rangle\\
&+2abc(\Delta t)^2\left\|D_+D_-De^{n+1}\right\|_{\ell^2_\Delta}^2.
\end{aligned}\tag{3.3.50}$$

Of course the problematic term from (3.3.49) will cancel with

$$-2abc\Delta t\left\langle D_+f^{n+1},D_+D_+D_-De^{n+1}\right\rangle$$

appearing in (3.3.50). Moreover, using Proposition (3.3.3) we get

$$\begin{aligned}
I_3 &\leq (-a)\,b\left(\Delta t+(\Delta t)^2\right)\|D_+\epsilon_2^n\|_{\ell^2_\Delta}^2+C\Delta t(-a)b\,\|f^n\|_{\ell^2_\Delta}^2\\
&+(1+C\Delta t)(-a)\,b\,\|D_+f^n\|_{\ell^2_\Delta}^2
\end{aligned}\tag{3.3.51}$$

Putting together the estimates (3.3.47), (3.3.48) and (3.3.51) we get that

$$\begin{aligned}
&(1-C\Delta t)\,\mathcal{E}\left(e^{n+1},f^{n+1}\right)\\
&\leq \left(\Delta t+\Delta t^2\right)(-c)\|\epsilon_1^n\|_{\ell^2_\Delta}^2+\left(\Delta t+(\Delta t)^2\right)\left(\|\epsilon_2^n\|_{\ell^2_\Delta}^2+(-ab)\|D_+\epsilon_2^n\|_{\ell^2_\Delta}^2\right)\\
&+(1+C\Delta t)\,\mathcal{E}(e^n,f^n).
\end{aligned}$$

3.4 Consistency error

In this part we aim at providing estimates for the consistency errors. For all functions f appearing below, we will denote by $(f)_\Delta^n$

$$(f_\Delta^n)_j=\frac{1}{\Delta t\Delta x}\int_{Q_j^n}f(t,x)\,dxdt$$

for all $j \in \mathbb{Z}$ where $Q_j^n = [n\Delta t, (n+1)\Delta t] \times [j\Delta x, (j+1)\Delta x]$. We recall that the consistency errors are defined as:

$$\begin{aligned}\epsilon_1^n &= \frac{1}{\Delta t}\left(I - bD_+D_-\right)\left(\eta_\Delta^{n+1} - \eta_\Delta^n\right) + \left(I + aD_+D_-\right)D\left(u_\Delta^{n+1}\right) \\ &\quad + D\left(\eta_\Delta^n u_\Delta^n\right) - \frac{\left(1 - \operatorname{sgn}(b)\right)\tau_1^n}{2}\Delta x D_+D_-\left(\eta_\Delta^n\right),\end{aligned}$$

respectively:

$$\begin{aligned}\epsilon_2^n &= \frac{1}{\Delta t}\left(I - dD_+D_-\right)\left(u_\Delta^{n+1} - u_\Delta^n\right) + \left(I + cD_+D_-\right)D\left(\eta_\Delta^{n+1}\right) \\ &\quad + \frac{1}{2}D\left(\left(u_\Delta^n\right)^2\right) - \frac{\left(1 - \operatorname{sgn}(d)\right)\tau_2^n}{2}\Delta x D_+D_-\left(u_\Delta^n\right),\end{aligned}$$

where $(\eta, u) : [0, T] \to H^s(\mathbb{R}) \times H^s(\mathbb{R})$ is a solution of (S_{abcd}).

Proposition 3.4.1. *There exists a sufficiently large $s > 0$ such that the following holds true. There exists a positive constant C depending on $\|(\eta, u)\|_{L_T^\infty(H^s)}$ such that*

$$\|(\epsilon_1^n, \epsilon_2^n)\|_{\ell_\Delta^2} + \|(cD_+\epsilon_1^n, aD_+\epsilon_2^n)\|_{\ell_\Delta^2} \le C\left(\Delta t + \Delta x\right).$$

The proof of Proposition 3.4.1 is rather standard and in the following we will only sketch a proof. Obviously, we have:

$$\begin{aligned}\|\epsilon_1^n\|_{\ell_\Delta^2} &\le \left\|\left(I - bD_+D_-\right)\left(\eta_\Delta^{n+1} - \eta_\Delta^n\right) - \left(\left(I - b\partial_{xx}^2\right)\partial_t\eta\right)_\Delta^n\right\|_{\ell_\Delta^2} \\ &\quad + \left\|\left(I + aD_+D_-\right)D\left(u_\Delta^n + u_\Delta^{n+1}\right) - \left(\left(I + a\partial_{xx}^2\right)\partial_x u\right)_\Delta^n\right\|_{\ell_\Delta^2} \\ &\quad + \Delta x\left\|D_+D_-\left(\eta^n\right)\right\|_{\ell_\Delta^2} + \left\|D\left(\eta_\Delta^n u_\Delta^n\right) - \left(\partial_x\left(\eta u\right)_\Delta^n\right)\right\|_{\ell_\Delta^2}.\end{aligned}$$

In the following, we detail how to treat the term containing the nonlinearity. We begin with:

Proposition 3.4.2. *Let us consider the sequence defined by*

$$a_j = \frac{1}{\Delta t\Delta x}\int_{Q_j^n}\left(\frac{u(t, x + \Delta x) - u\left(t, x - \Delta x\right)}{2\Delta x} - \partial_x u(t, x)\right)v(t, x)dtdx,$$

for all $j \in \mathbb{Z}$. Then

$$\|a\|_{\ell_\Delta^2} \le \|v\|_{L_T^\infty(L^\infty)}\left\|\partial_{xxx}^3 u\right\|_{L_T^\infty(L^2)}(\Delta x)^2.$$

In view of Taylor's formula with integral remainder, we get that

$$\begin{aligned}&\frac{u(t, x + \Delta x) - u\left(t, x - \Delta x\right)}{2\Delta x} - \partial_x u(t, x) \\ &= \frac{\left(\Delta x\right)^2}{2}\int_0^1\left(\partial_{xxx}^3 u\left(t, x + \tau\Delta x\right) + \partial_{xxx}^3 u\left(t, x - \tau\Delta x\right)\right)\left(1 - \tau\right)^2 d\tau \overset{not.}{=} \frac{\left(\Delta x\right)^2}{2}\tilde{u}\left(t, x\right).\end{aligned}$$

Thus, we have that

$$a_j = \frac{\Delta x}{\Delta t}\int_{Q_j^n}\tilde{u}\left(t, x\right)v(t, x)dtdx,$$

such that using Cauchy's inequality yields

$$a_j^2\Delta x = \frac{(\Delta x)^3}{(\Delta t)^2}\left(\int_{Q_j^n}\tilde{u}\left(t, x\right)v(t, x)dtdx\right)^2$$

$$
\begin{aligned}
&\leq \frac{(\Delta x)^3}{(\Delta t)^2} \int_{Q_j^n} dtdx \int_{Q_j^n} \tilde{u}^2(t,x)\, v^2(t,x) dtdx \\
&\leq \frac{(\Delta x)^4}{\Delta t} \|v\|^2_{L^\infty_T(L^\infty)} \int_{Q_j^n} \tilde{u}^2(t,x)\, dtdx.
\end{aligned}
$$

Taking the sum over all $j \in \mathbb{Z}$, yields

$$
\|a\|^2_{\ell^2_\Delta} \leq \frac{(\Delta x)^4}{\Delta t} \|v\|^2_{L^\infty_T(L^\infty)} \int_{t^n}^{t^{n+1}} \|\tilde{u}(t,\cdot)\|^2_{L^2} dt \leq (\Delta x)^4 \|v\|^2_{L^\infty_T(L^\infty)} \|\tilde{u}\|^2_{L^\infty_T(L^2)}.
$$

Next, we observe that

$$
\begin{aligned}
\|\tilde{u}(t,\cdot)\|^2_{L^2} &= \int_{\mathbb{R}} \tilde{u}^2(t,x)\, dx \\
&\leq \frac{1}{2} \int_{\mathbb{R}} \int_0^1 \left[\partial^3_{xxx} u(t, x+\tau\Delta x)\right]^2 d\tau dx + \frac{1}{2} \int_{\mathbb{R}} \int_0^1 \left[\partial^3_{xxx} u(t, x-\tau\Delta x)\right]^2 d\tau dx \\
&= \frac{1}{2} \int_0^1 \int_{\mathbb{R}} \left[\partial^3_{xxx} u(t, x+\tau\Delta x)\right]^2 dx d\tau + \frac{1}{2} \int_0^1 \int_{\mathbb{R}} \left[\partial^3_{xxx} u(t, x-\tau\Delta x)\right]^2 dx d\tau \\
&= \left\|\partial^3_{xxx} u(t,\cdot)\right\|^2_{L^2}.
\end{aligned}
$$

Using the last two estimates, we get

$$
\|a\|_{\ell^2_\Delta} \leq \|v\|_{L^\infty_T(L^\infty)} \left\|\partial^3_{xxx} u\right\|_{L^\infty_T(L^2)} (\Delta x)^2.
$$

This concludes the proof of the proposition.

Proposition 3.4.3. *The following estimate holds true*

$$
\begin{aligned}
&\|D(\eta^n_\Delta u^n_\Delta) - (\partial_x(\eta u)^n_\Delta)\|_{\ell^2_\Delta} \\
&\lesssim (\Delta x)^2 \|\eta\|_{L^\infty_T(L^\infty)} \left\|\partial^3_{xxx} u\right\|_{L^\infty_T(L^2)} + (\Delta x)^2 \|u\|_{L^\infty_T(L^\infty)} \|\partial_{xxx}\eta\|_{L^\infty_T(L^2)} \\
&+ (\Delta t + \Delta x) \Big(\|(\partial_t \eta, \partial_t u)\|_{L^\infty_T(L^\infty)} + \|(\partial_x \eta, \partial_x u)\|_{L^\infty_T(L^\infty)}\Big) \|(\partial_x \eta, \partial_x u)\|_{L^\infty_T(L^2)}.
\end{aligned} \tag{3.4.1}
$$

First, we notice that

$$
\begin{aligned}
&\|D(\eta^n_\Delta u^n_\Delta) - (\partial_x(\eta u))^n_\Delta\|_{\ell^2_\Delta} \\
&\leq \|S_+\eta^n_\Delta D u^n_\Delta - (\eta\partial_x u)^n_\Delta\|_{\ell^2_\Delta} + \|S_- u^n_\Delta D\eta^n_\Delta - (u\partial_x \eta)^n_\Delta\|_{\ell^2_\Delta} \\
&\leq \|S_+\eta^n_\Delta D u^n_\Delta - S_+\eta^n_\Delta (\partial_x u)^n_\Delta\|_{\ell^2_\Delta} + \|S_+\eta^n_\Delta (\partial_x u)^n_\Delta - (\eta\partial_x u)^n_\Delta\|_{\ell^2_\Delta} \\
&+ \|S_- u^n_\Delta D\eta^n_\Delta - S_- u^n_\Delta (\partial_x \eta)^n_\Delta\|_{\ell^2_\Delta} + \|S_- u^n_\Delta (\partial_x \eta)^n_\Delta - (u\partial_x \eta)^n_\Delta\|_{\ell^2_\Delta}.
\end{aligned}
$$

Using Proposition 3.4.2, let us write that

$$
\begin{aligned}
&\|S_+\eta^n_\Delta D u^n_\Delta - S_+\eta^n_\Delta (\partial_x u)^n_\Delta\|_{\ell^2_\Delta} + \|S_- u^n_\Delta D\eta^n_\Delta - S_- u^n_\Delta (\partial_x \eta)^n_\Delta\|_{\ell^2_\Delta} \\
&\leq (\Delta x)^2 \|\eta\|^2_{L^\infty_T(L^\infty)} \left\|\partial^3_{xxx} u\right\|^2_{L^\infty_T(L^2)} + (\Delta x)^2 \|u\|^2_{L^\infty_T(L^\infty)} \|\partial_{xxx}\eta\|^2_{L^\infty_T(L^2)}.
\end{aligned} \tag{3.4.2}
$$

In order to treat the remaining two terms, we write that:

$$
(S_+\eta^n_\Delta (\partial_x u)^n_\Delta - (\eta\partial_x u)^n_\Delta)_j
$$

$$
\begin{aligned}
&= \frac{1}{(\Delta t \Delta x)^2} \int_{Q_j^n} \partial_x u\,(t,x) \left(\int_{Q_j^n} \left(\eta\,(t', x'+\Delta x) - \eta\,(t,x)\right) dt'dx' \right) dtdx \\
&\lesssim \frac{\Delta t + \Delta x}{\Delta t \Delta x} \int_{Q_j^n} \left|\partial_x u\,(t,x)\right| \left(\|\partial_t \eta\|_{L_T^\infty(L^\infty)} + \|\partial_x \eta\|_{L_T^\infty(L^\infty)} \right) dtdx.
\end{aligned}
$$

The last relation along with Cauchy's inequality implies that

$$
\begin{aligned}
&\Delta x \left(S_+ \eta_\Delta^n \left(\partial_x u\right)_\Delta^n - \left(\eta \partial_x u\right)_\Delta^n \right)_j^2 \\
&\lesssim \frac{(\Delta t)^2 + (\Delta x)^2}{\Delta t} \left(\|\partial_t \eta\|_{L_T^\infty(L^\infty)}^2 + \|\partial_x \eta\|_{L_T^\infty(L^\infty)}^2 \right) \int_{Q_j^n} \left|\partial_x u\,(t,x)\right|^2 dtdx
\end{aligned}
$$

We perform a summation over $\mathbb{Z}$ in order to get that

$$
\begin{aligned}
&\|S_+ \eta_\Delta^n \left(\partial_x u\right)_\Delta^n - \left(\eta \partial_x u\right)_\Delta^n\|_{\ell_\Delta^2}^2 \\
&\lesssim \frac{(\Delta t)^2 + (\Delta x)^2}{\Delta t} \left(\|\partial_t \eta\|_{L_T^\infty(L^\infty)}^2 + \|\partial_x \eta\|_{L_T^\infty(L^\infty)}^2 \right) \int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} \left|\partial_x u\,(t,x)\right|^2 dtdx \\
&\lesssim \left((\Delta t)^2 + (\Delta x)^2 \right) \left(\|\partial_t \eta\|_{L_T^\infty(L^\infty)}^2 + \|\partial_x \eta\|_{L_T^\infty(L^\infty)}^2 \right) \|\partial_x u\|_{L_T^\infty(L^2)}^2 . \qquad (3.4.3)
\end{aligned}
$$

Of course, proceeding in a similar manner we get that

$$
\|S_- u_\Delta^n \left(\partial_x \eta\right)_\Delta^n - \left(u \partial_x \eta\right)_\Delta^n\|_{\ell_\Delta^2} \lesssim (\Delta t + \Delta x) \left(\|\partial_t u\|_{L_T^\infty(L^\infty)} + \|\partial_x u\|_{L_T^\infty(L^\infty)} \right) \|\partial_x \eta\|_{L_T^\infty(L^2)}. \qquad (3.4.4)
$$

Combining the estimates (3.4.2), (3.4.3) and (3.4.4) we get the validity of (3.4.1).

3.5 Numerical simulations

3.5.1 Accuracy tests

Throughout this section we present the results of some numerical tests we performed in order to numerically verify the accuracy of our schemes. Unless mentioned otherwise, we use the computational domain $[-20, 20]$ with the time discretization step $\Delta t = 0.0001$ respectively the space discretization step $\Delta x = 10 \times 2^j$ with $j \in \overline{4, 8}$. We perform computations up to the time $T = 2$. We gather the errors between exact solutions for the *abcd* systems (obtained for instance in [35] and [34]) and the solutions build with our numerical schemes in order to estimate the order of convergence. To our surprise, in many situations, the order of convergence that we obtain is 2.

The case of proportional η and u Assuming the conditions

$$
a - b + 2d = 0,\ a = c,\ d > 0,
$$

it has been showed in [35] and [34] that the family

$$
\left\{
\begin{array}{l}
\eta\,(t,x) = \eta_0 \operatorname{sech}^2(\lambda(x + x_0 - C_s^\pm t)), \\
u_\pm\,(t,x) = \pm \eta_0 \sqrt{\frac{3}{\eta_0 + 3}} \operatorname{sech}^2(\lambda(x + x_0 - C_s^\pm t)),
\end{array}
\right. \qquad (3.5.1)
$$

where

$$
C_s^\pm = \pm \frac{3 + 2\eta_0}{\sqrt{3(3 + \eta_0)}},\ \lambda = \frac{1}{2} \sqrt{\frac{2\eta_0}{3(a-b) + 2b(\eta_0 + 3)}}
$$

$x_0 \in \mathbb{R}$ and $\eta_0 > -3$ with

$$
(\eta_0 - 1)\,(\eta_0 - b/d) > 0.
$$

Below we will take $x_0 = 0$.

We perform a first test choosing:

$$a = c = 0, b = 2d = 1/9,$$

with

$$\eta_0 = 0.5.$$

The results are gathered in the table bellow.

Δx	ℓ^2_Δ-error	$\mathcal{E}$-error	ℓ^2_Δ-order	$\mathcal{E}$-order
$2^{-2} = 0.25$	0.0285	0.0325		
$2^{-3} = 0.125$	0.0073	0.0083	1.9649	1.9692
$2^{-4} = 0.0625$	0.0018	0.0021	2.0198	1.9827
$2^{-5} = 0.03125$	0.00045	0.00052	2.0000	2.0138
$2^{-6} = 0.015625$	0.00011	0.00013	2.0324	2.0000

The next case we treat is

$$a = c = -1/6,\ b = d = 1/6,$$

with

$$\eta_0 = -0.5.$$

We obtain the following results:

Δx	ℓ^2_Δ-error	$\mathcal{E}$-error	ℓ^2_Δ-order	$\mathcal{E}$-order
$2^{-2} = 0.25$	0.0799	0.2554		
$2^{-3} = 0.125$	0.0211	0.0735	1.9209	1.7969
$2^{-4} = 0.0625$	0.0053	0.0179	1.9931	2.0377
$2^{-5} = 0.03125$	0.0013	0.0044	2.0274	2.0243
$2^{-6} = 0.015625$	0.00032	0.0011	2.0223	2.0000

In this case, $b = d$ and thus, the continuous system poses the conserved quantities

$$\begin{cases} I(\eta, u) = \int \eta u + \frac{1}{6}\partial_x\eta\partial_x u, \\ H(\eta, u) = \int (1+\eta)\, u^2 + \eta^2 + \frac{1}{6}(\partial_x\eta)^2 + \frac{1}{6}(\partial_x u)^2. \end{cases}$$

In figure (3.1) we represent the time evolution of the discrete quantities

$$\begin{cases} I^n = \langle \eta^n, u^n\rangle_{\ell^2_\Delta} + \langle D_+\eta^n, D_+u^n\rangle_{\ell^2_\Delta}, \\ H^n = \langle (1+\eta^n)\, u^n, u^n\rangle_{\ell^2_\Delta} + \|\eta^n\|^2_{\ell^2_\Delta} + \frac{1}{6}\|D_+\eta^n\|^2_{\ell^2_\Delta} + \frac{1}{6}\|D_+u^n\|^2_{\ell^2_\Delta}. \end{cases}$$

Next, we present the results obtained for

$$a = c = -\frac{1}{3},\ d = \frac{1}{6}, b = 0$$

and

$$\eta_0 = -0.5.$$

At every time step we set the artificial viscosity coefficient

$$\tau_1^n = 2\|u^n\|_{\ell^\infty}. \tag{3.5.2}$$

The results are gathered below

Δx	ℓ^2_Δ-error	$\mathcal{E}$-error	ℓ^2_Δ-order	$\mathcal{E}$-order
$2^{-2} = 0.25$	0.0467	0.057		
$2^{-3} = 0.125$	0.0239	0.0293	0.9664	0.9600
$2^{-4} = 0.0625$	0.0122	0.015	0.9701	0.9659
$2^{-5} = 0.03125$	0.0062	0.0076	0.9765	0.9808
$2^{-6} = 0.015625$	0.0031	0.0039	1.0000	0.9625

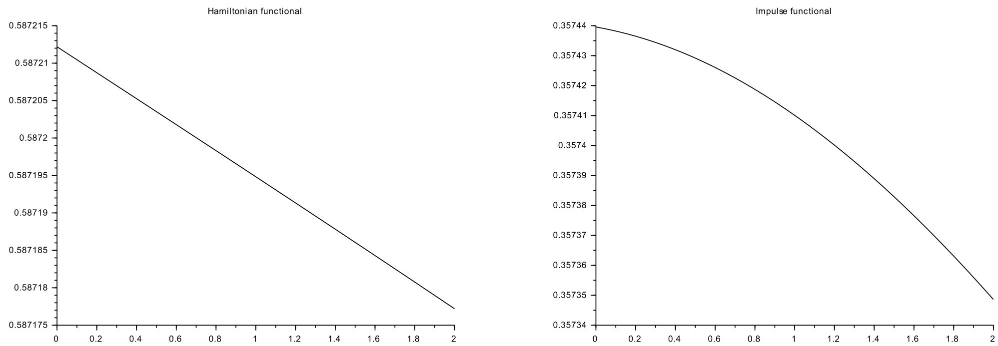

Figure 3.1: Time variation of the energy (left) and impulse (right) functionals defined in (3.5.1), computed with $dx = 0.125$

The classical Boussinesq system The couple

$$\begin{cases} \eta(t,x) = -1, \\ u(t,x) = 3\,\mathrm{sech}^2\left(\frac{3}{2}(x-t)\right), \end{cases}$$

is an exact solution for the *abcd* system with

$$a = b = c = 0, d = 1/9.$$

We chose the numerical viscosity coefficient τ_1^n as in (3.5.2). The errors between the numerical solution and the discretization of the exact solution, at time $T = 2$, are gathered below.

Δx	ℓ^2_Δ-error	$\mathcal{E}$-error	ℓ^2_Δ-order	$\mathcal{E}$-order
$2^{-2} = 0.25$	0.338	0.4394		
$2^{-3} = 0.125$	0.088	0.1174	1.9414	1.9041
$2^{-4} = 0.0625$	0.0221	0.0296	1.9934	1.9877
$2^{-5} = 0.03125$	0.0056	0.0073	1.9805	2.0196

The $a < 0$, $b = c = d = 0$ In this case, an exact solution is given in [35]:

$$\begin{cases} \eta(t,x) = -1 - \frac{a\rho}{4} + \frac{a\rho}{2}\,\mathrm{sech}^2\left(\frac{\sqrt{\rho}}{2}(x - x_0 - C_s t)\right), \\ u(t,x) = C_s + \mathrm{sech}\left(\frac{\sqrt{\rho}}{2}(x - x_0 - C_s t)\right). \end{cases}$$

We perform a comparison with the numerical solution, using

$$a = -0.25,\ \rho = 16.$$

At every time step we set the artificial viscosity coefficients

$$\tau_1^n = \tau_2^n = 1.5 \times \|u^n\|_{\ell^\infty}.$$

We obtain the results gathered in the next table:

Δx	ℓ^2_Δ-error	$\mathcal{E}$-error	ℓ^2_Δ-order	$\mathcal{E}$-order
$2^{-2} = 0.25$	0.2675	0.323		
$2^{-3} = 0.125$	0.0622	0.0765	2.1045	2.0780
$2^{-4} = 0.0625$	0.0134	0.0165	2.2146	2.2129
$2^{-5} = 0.03125$	0.0032	0.0039	2.0660	2.0809

An exact solution of the BBM-BBM system:

$$a = c = 0,\ b = d = \frac{1}{6}$$

is given by

$$\begin{cases} \eta_\pm(t,x) = \frac{15}{2}\operatorname{sech}^2\left(\frac{3}{\sqrt{10}}\left(x - x_0 \mp \frac{5}{2}t\right)\right) - \frac{45}{4}\operatorname{sech}^4\left(\frac{3}{\sqrt{10}}\left(x - x_0 \mp \frac{5}{2}t\right)\right), \\ u_\pm(t,x) = \pm\frac{15}{2}\operatorname{sech}^2\left(\frac{3}{\sqrt{10}}(x - x_0 \mp \frac{5}{2}t)\right). \end{cases} \tag{3.5.3}$$

The following table gathers the ℓ^2_Δ-error and the energy error between the numerical solution and the exact solution $(\eta_\pm, u_\pm)$ where $\Delta t = 0.0001$.

Δx	ℓ^2_Δ-error	$\mathcal{E}$-error	ℓ^2_Δ-order	$\mathcal{E}$-order
$2^{-2} = 0.25$	1.5846	2.1113		
$2^{-3} = 0.125$	0.3928	0.5345	2.0122	1.9818
$2^{-4} = 0.0625$	0.0734	0.0993	2.4199	2.4283
$2^{-5} = 0.03125$	0.0255	0.0311	1.5252	1.6748
$2^{-6} = 0.015625$	0.0397	0.0519	-0.6386	-0.7588

The last estimations of the order of convergence may seem strange. The explanation is the following: as the order seems to be 2, the two last values are explained by the value of Δt which is not sufficiently small in order to capture the quadratic order for $\Delta x = 2^{-6}$. Refining the time step to $\Delta t = 0.00001$ we obtain the following:

Δx	ℓ^2_Δ-error	$\mathcal{E}$-error	ℓ^2_Δ-order	$\mathcal{E}$-order
$2^{-2} = 0.25$	1.6111	2.1467		
$2^{-3} = 0.125$	0.4254	0.5791	1.9211	1.8902
$2^{-4} = 0.0625$	0.105	0.1437	2.0184	2.0907
$2^{-5} = 0.03125$	0.0235	0.0322	2.1596	2.5579
$2^{-6} = 0.015625$	0.0037	0.0048	2.6670	2.7459

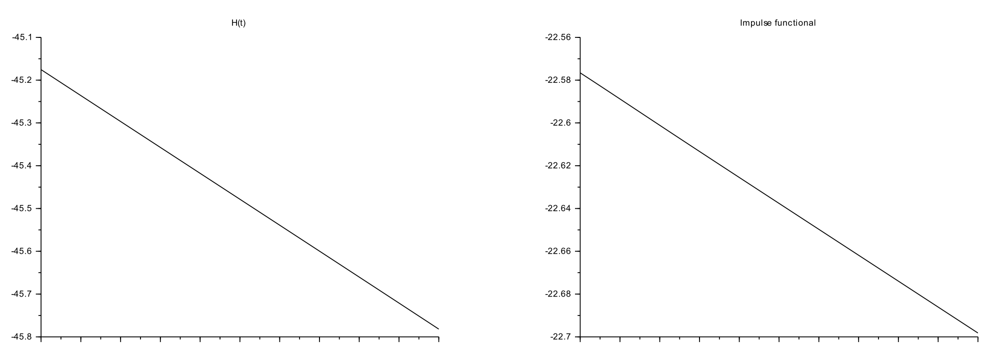

Figure 3.2: Time variation of the energy and impulse functionals defined in (3.5.1) and computed with $dx = 0.0625$

When

$$a = c = d = 0,\ b = \frac{3}{5}$$

the exact solution reads:

$$\begin{cases} \eta_{\pm}(t,x) = \frac{15}{2}\operatorname{sech}^2\left(\frac{1}{2}\left(x - x_0 \mp \frac{\sqrt{10}}{2}t\right)\right) - \frac{45}{4}\operatorname{sech}^4\left(\frac{1}{2}\left(x - x_0 \mp \frac{\sqrt{10}}{2}t\right)\right), \\ u_{\pm}(t,x) = \pm\frac{3\sqrt{10}}{2}\operatorname{sech}^2\left(\frac{1}{2}\left(x - x_0 \mp \frac{\sqrt{10}}{2}t\right)\right). \end{cases} \tag{3.5.4}$$

Taking in account that $d = 0$, we set the artificial viscosity coefficient to be

$$\tau_2^n = 1.5 \times \|u^n\|_{\ell^\infty}$$

We obtain the following results:

Δx	ℓ^2_Δ-error	$\mathcal{E}$-error	ℓ^2_Δ-order	$\mathcal{E}$-order
$2^{-4} = 0.0625$	0.9067	1.1202		
$2^{-5} = 0.03125$	0.4864	0.6023	0.8984	0.8952
$2^{-6} = 0.015625$	0.2587	0.3188	0.9108	0.9174
$2^{-7} = 0.0078125$	0.1364	0.1665	0.9234	0.9371
$2^{-8} = 0.00390625$	0.0711	0.0858	0.9399	0.9564

3.5.2 Traveling wave collision

The case $a = c = 0, b = d = 1/6$

Recently, J.B. Bona and M. Chen, [14], studied the interaction between the traveling waves (3.5.3) of the BBM-BBM system with $a = c = 0$ and $b = d = 1/6$. Their results point out that finite time blow-up might hold up. More precisely, considering the spatial domain $[-14, 14]$, they construct the numerical solution corresponding to the initial data:

$$(\eta_+(0, x - x_+), u_+(0, x - x_+)) + (\eta_-(0, x - x_-), u_-(0, x - x_-)) \tag{3.5.5}$$

where $x_\pm = \pm 7$. We repeated their experiment using our codes: we perform the computations on the domain $[-14, 14]$ starting with the initial data (3.5.5). The time and space steps are $\Delta t = 0.0001$ and $\Delta x = 0.001$. The program stops at $T = 3.71$ the first time when

$$\|D_+\eta^n\|^2_{\ell^2_\Delta} + \|D_+u^n\|^2_{\ell^2_\Delta} > 40000.$$

The results we obtained are gathered in the next figures:

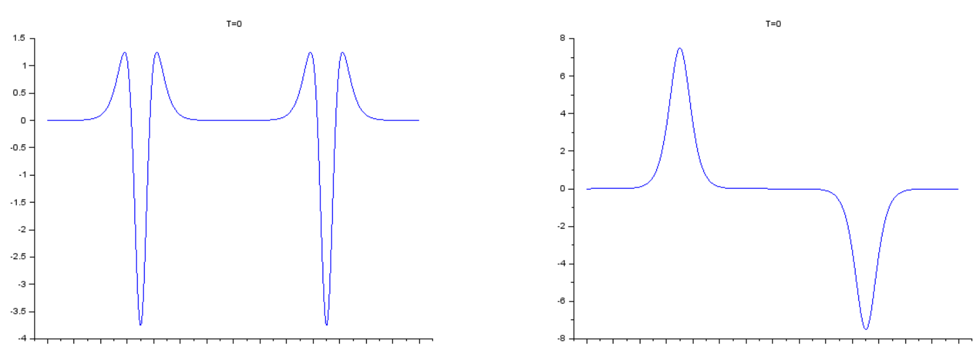

Figure 3.3: η (left) and u (right) at time $T = 0$

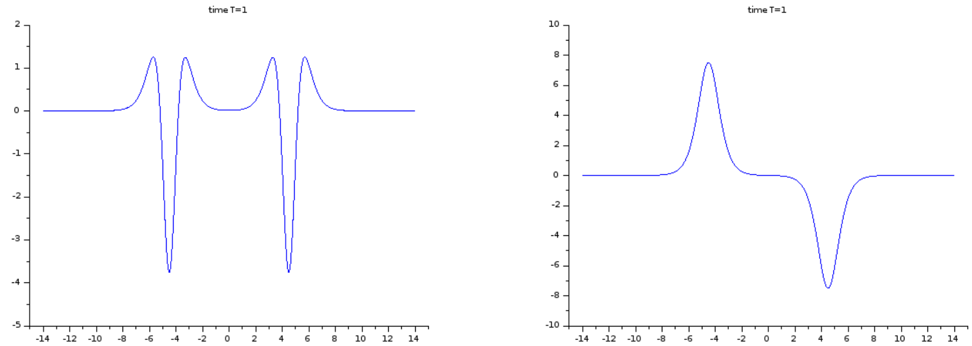

Figure 3.4: η (left) and u (right) at time $T = 1$

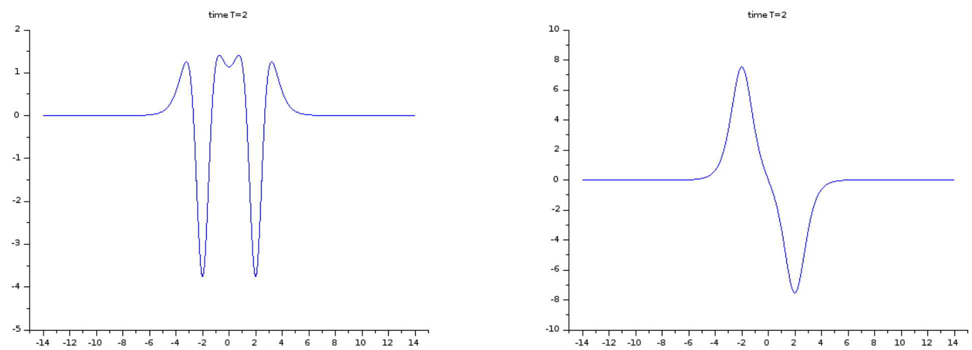

Figure 3.5: η (left) and u (right) at time $T = 2$

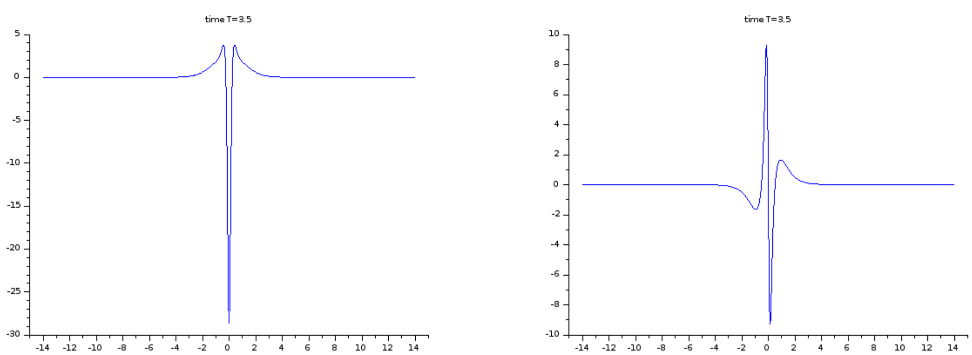

Figure 3.6: η (left) and u (right) at time $T = 3.5$

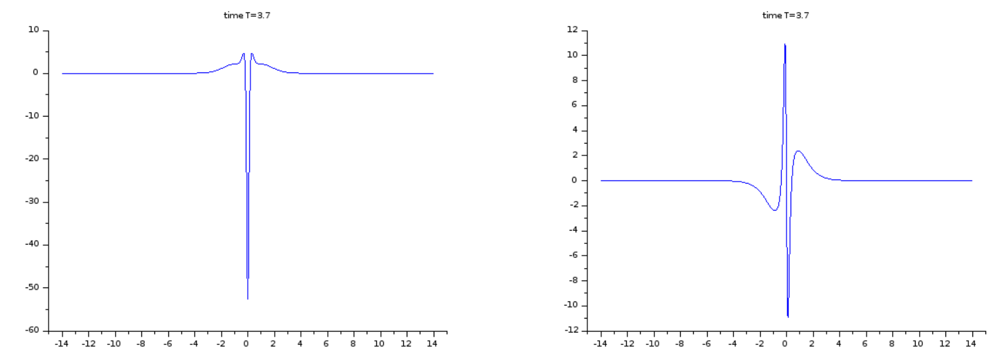

Figure 3.7: η (left) and u (right) at time $T = 3.7$

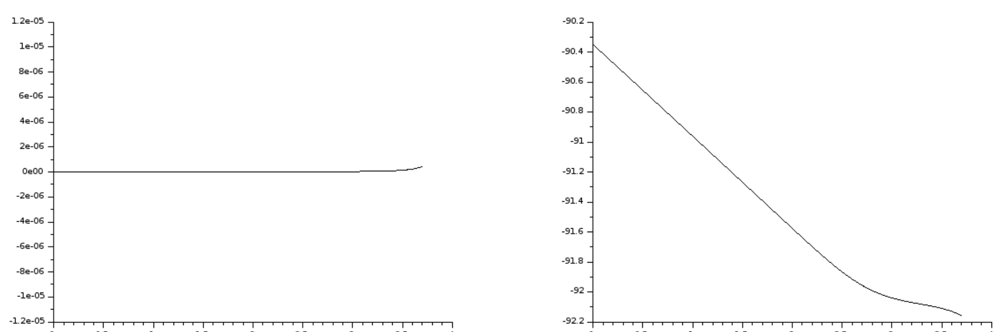

Figure 3.8: time variation of the Impulse functional (left) and the hamiltonian (right)

The case $a = c = d = 0,\ b = \frac{3}{5}$

As we have seen earlier, in the case $a = c = d = 0$, $b = \frac{3}{5}$ traveling waves of the same form as for the previous case are available i.e. (3.5.4). As in the later case, the computations are performed on the domain $[-14, 14]$ starting with the initial data (3.5.5) where, this time, $(\eta_{\pm}, u_{\pm})$ are given by (3.5.4) and $x_{\pm} = \pm 5$. The time and space steps are $\Delta t = 0.0001$ and $\Delta x = 0.001$. The program stops at $T = 4.83$ the first time when

$$\|D_+\eta\|^2_{\ell^2_\Delta} + \|D_+u\|^2_{\ell^2_\Delta} > 100.000.$$

Our results are illustrated in the following figures:

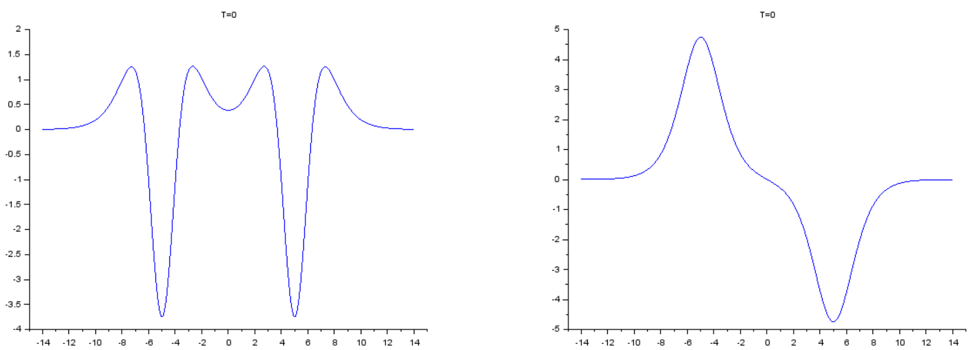

Figure 3.9: η (left) and u (right) at time $T = 0$

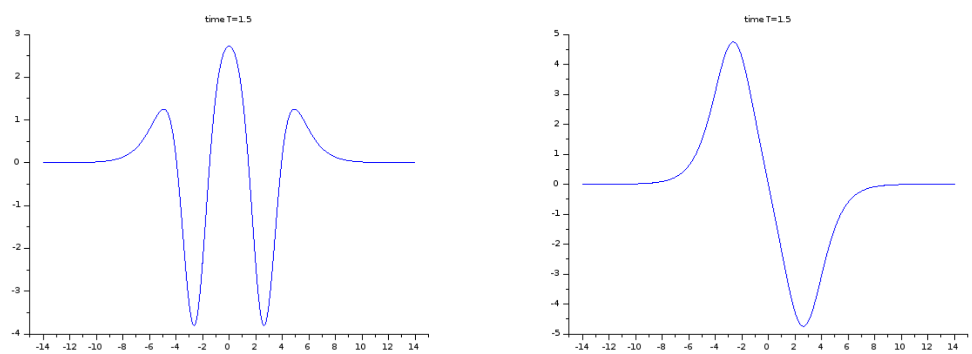

Figure 3.10: η (left) and u (right) at time $T = 1.5$

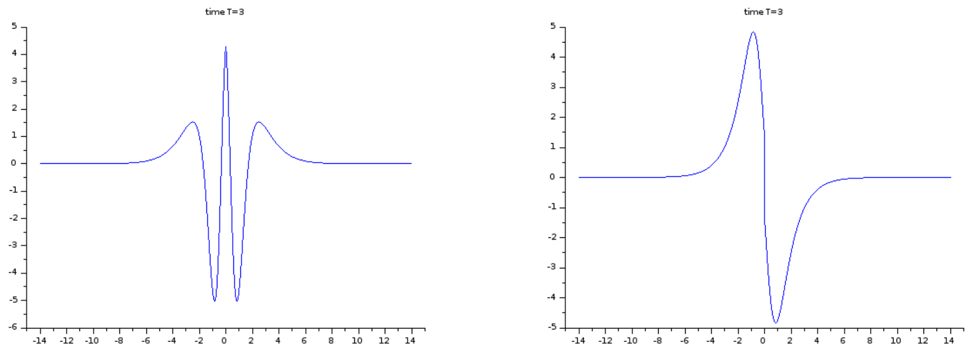

Figure 3.11: η (left) and u (right) at time $T = 3$

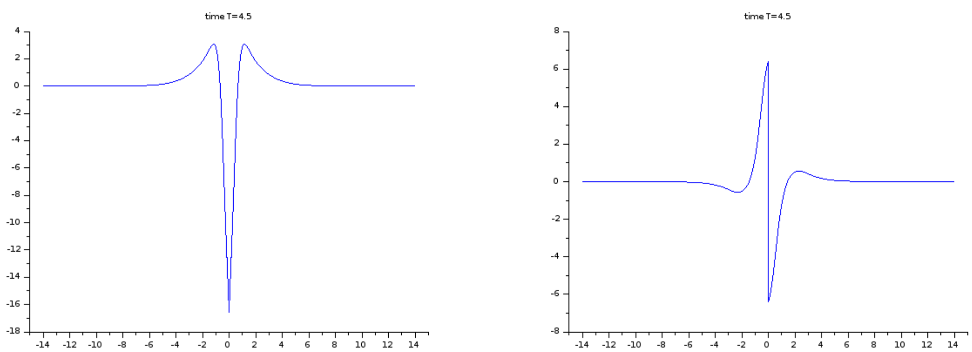

Figure 3.12: η (left) and u (right) at time $T = 4.5$

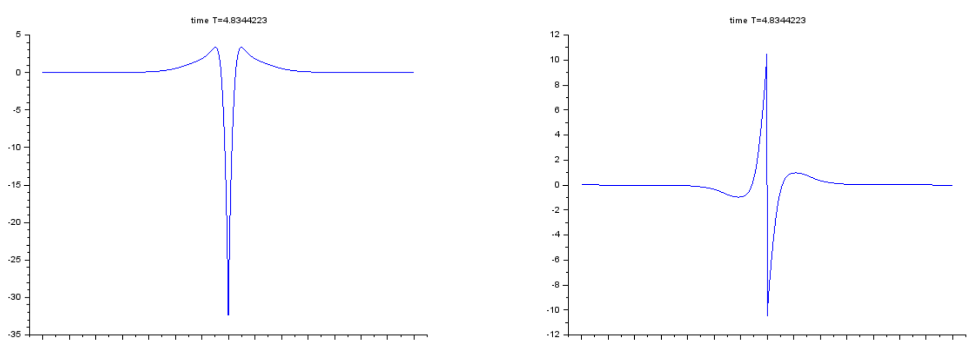

Figure 3.13: η(left) and u (right) at time $T = 4.83$

The case $a = c = -1/6,\ b = d = 1/6$

The last test that we performed is the traveling wave interaction when the value of the parameters are $a = c = -1/6$, $b = d = 1/6$. As before, we will consider the initial data in the form (3.5.5) where $(\eta_{\pm}, u_{\pm})$ are given by (3.5.1) and $\eta_0 = -0.7$ and $x_{\pm} = \pm 5$. As in the other case, we consider the spatial domain $[-14, 14]$ and we chose $\Delta t = 0.001$ and $\Delta x = 0.001$. We let the program run up until time $T = 15$. What we want to emphasize in this case is that, as opposed to the "dispersionless" case $a = b = c = d = 0$, the quantity $1 + \eta$ (which in physical variables should represent the total height of water above the flat bottom) although positive at the initial time, it becomes negative when the two depression waves interact. The results are gathered in the following figures.

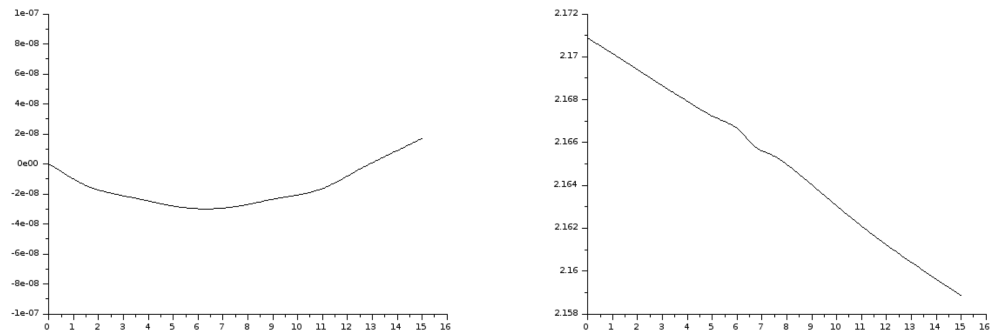

Figure 3.14: time variation of the impulse(left) and energy functional(right)

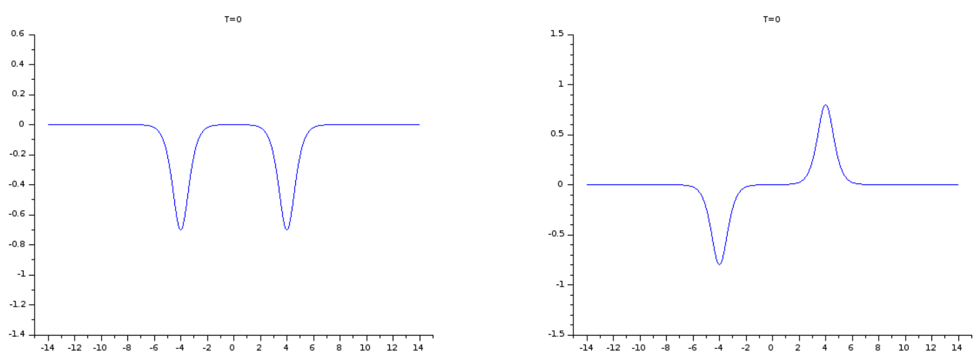

Figure 3.15: η (left) and u (right) at time $T = 0$

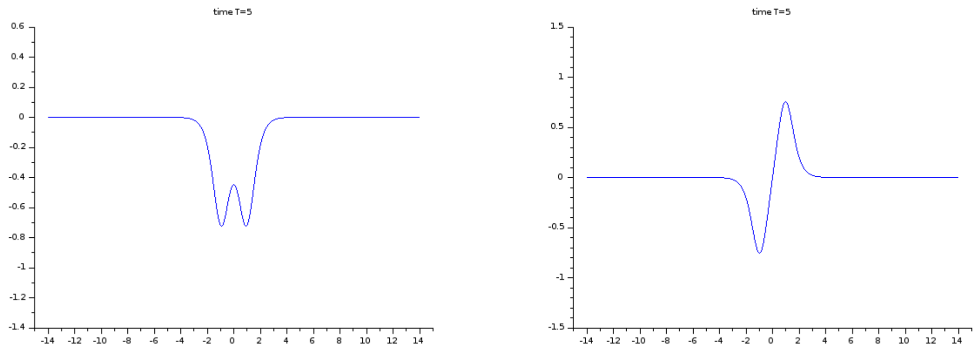

Figure 3.16: η (left) and u (right) at time $T = 5$

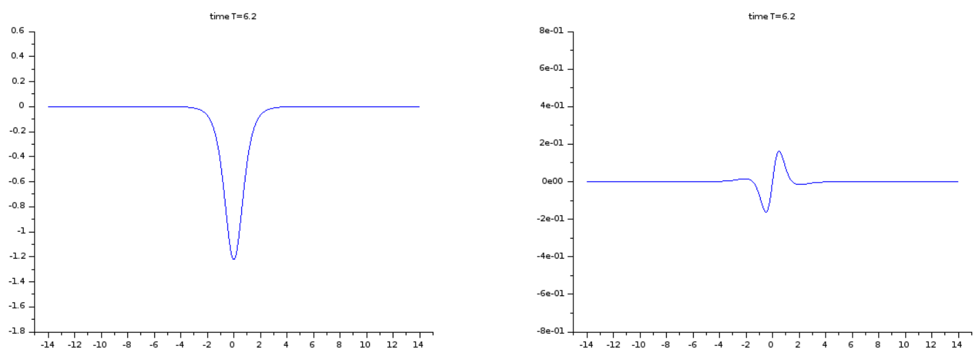

Figure 3.17: η (left) and u (right) at time $T = 6.2$

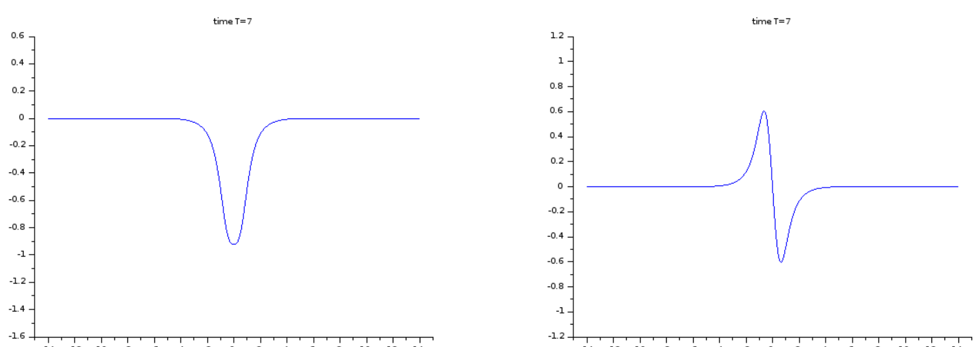

Figure 3.18: η (left) and u (right) at time $T = 7$

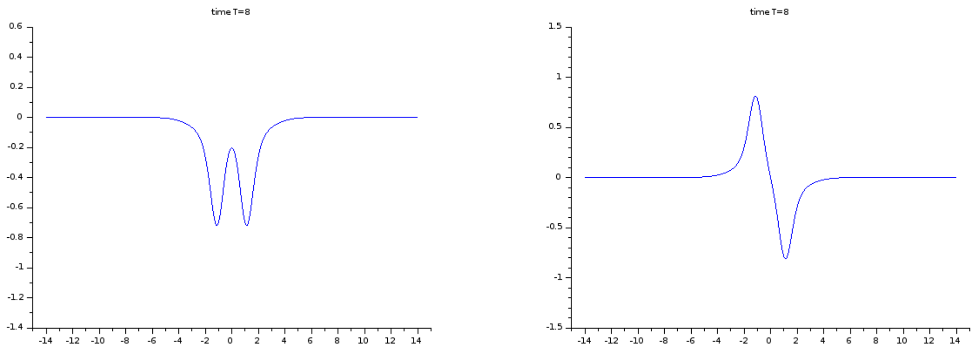

Figure 3.19: η (left) and u (right) at time $T = 8$

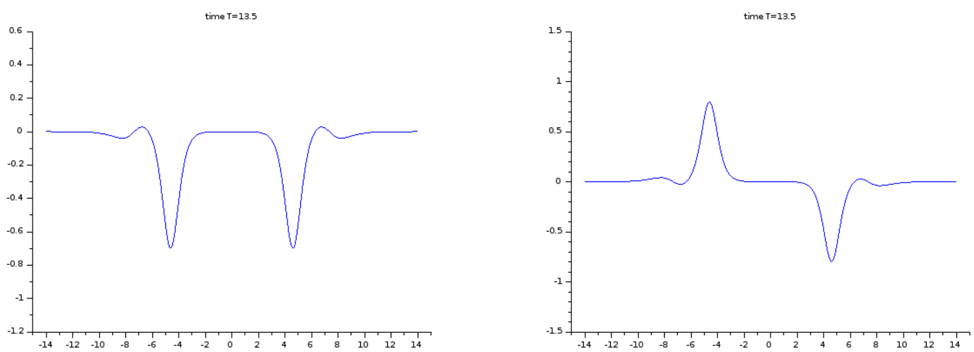

Figure 3.20: η (left) and u (right) at time $T = 13.5$

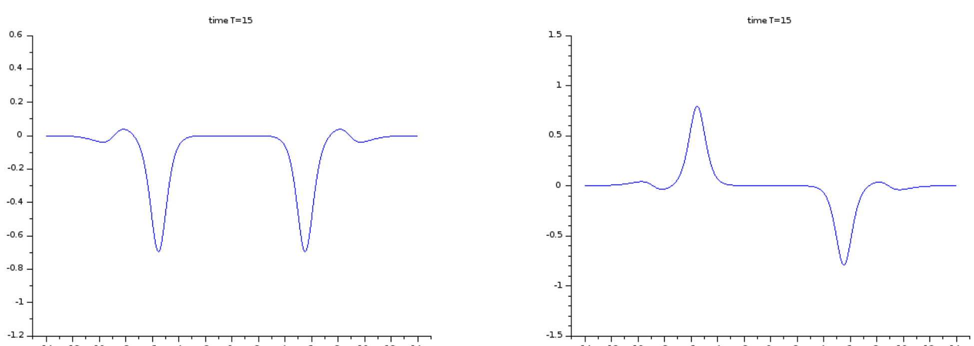

Figure 3.21: η (left) and u (right) at time $T = 15$

Chapter 4

Lagrangian methods for incompressible fluid mechanics

In this chapter, we gather the results obtained in [29] and [28]. We aim at providing well-posedness results under optimal regularity assumptions for some mathematical models describing the dynamics of inhomogeneous, incompressible viscous fluids. The plan of the chapter is the following: in Section 4.1 we study a linear Stokes-type system with variable coefficients which naturally arises when studying the Lagrangian formulation of different models from fluid mechanics. These new estimates prove to be the key ingredient when studying the Navier-Stokes and the Navier-Stokes Korteweg systems. More precisely, in the second part of the chapter, we prove that these systems are well-posed in the so-called critical spaces (see Introduction).

4.1 The Stokes system with nonconstant coefficients

4.1.1 The main result

In this section consideration is given to the following Stokes problem with time independent, nonconstant coefficients:

$$\begin{cases} \partial_t u - a\operatorname{div}(bD(u)) + a\nabla P = f, \\ \operatorname{div} u = \operatorname{div} R, \\ u_{|t=0} = u_0. \end{cases} \tag{4.1.1}$$

This system is encountered in the study of the Lagrangian formulation of systems of partial differential equations issued from fluid mechanics. We establish global well-posedness results for System (4.1.1). This can be viewed as a first step towards generalizing the results of Danchin and Mucha obtained in [51], Chapter 4, for the case of general viscosity and without assuming that the density is close to a constant state. The main result reads:

Theorem 4.1.1. *Consider $n\in\{2,3\}$ and $p\in(1,4)$ if $n=2$ or $p\in\left(\frac{6}{5},4\right)$ if $n=3$. Assume there exist positive constants $\left(a_\star, b_\star, a^\star, b^\star, \bar{a}, \bar{b}\right)$ such that $a-\bar{a}\in \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)$, $b-\bar{b}\in \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)$ and*

$$0 < a_\star \le a \le a^\star,$$
$$0 < b_\star \le b \le b^\star.$$

Furthermore, consider the vector fields $u_0 \in \dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)$ respectively $f\in L^1_{loc}(\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n))$. Also, consider the vector field $R\in(\mathcal{S}'(\mathbb{R}^n))^n$ with[1]

$$\mathcal{Q}R\in\mathcal{C}([0,\infty);\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)) \text{ and } (\partial_t R,\nabla\operatorname{div} R)\in L^1_{loc}(\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n))$$

[1] $\mathcal{P}$ is the Leray projector over divergence free vector fields, $\mathcal{Q} = Id - \mathcal{P}$

such that:

$$\operatorname{div} u_0 = \operatorname{div} R\,(0,\cdot)\,.$$

Then, system (4.1.1) has a unique global solution $(u, \nabla P)$ *with:*

$$u \in C([0,\infty), \dot{B}_{p,1}^{\frac{n}{p}-1}(\mathbb{R}^n)) \text{ and } \partial_t u, \nabla^2 u, \nabla P \in L^1_{loc}(\dot{B}_{p,1}^{\frac{n}{p}-1}(\mathbb{R}^n)).$$

Moreover, there exists a constant $C = C\,(a,b)$ *such that:*

$$\begin{aligned}
&\|u\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}-1})} + \left\|\left(\partial_t u, \nabla^2 u, \nabla P\right)\right\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}-1})} \\
&\leq e^{C(t+1)}(\|u_0\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}-1})}),
\end{aligned} \tag{4.1.2}$$

for all $t \in [0,\infty)$.

The difficulty in establishing such a result comes from the fact that the pressure and velocity are "strongly" coupled as opposed to the case where ρ is close to a constant see Remark 4.1.3 below. The key idea is to use the high-low frequency splitting technique first introduced in [44] combined with the special structure of the "incompressible" part of $a\nabla P$ i.e.

$$\begin{aligned}
\mathcal{P}(a\nabla P) &= \mathcal{P}((a-\bar{a})\nabla P) = \mathcal{P}((a-\bar{a})\nabla P) - (a-\bar{a})\mathcal{P}(\nabla P) \\
&:= [\mathcal{P}, a-\bar{a}]\nabla P.
\end{aligned}$$

which is, loosely speaking, more regular than ∇P. Let us mention that a similar principle holds for u which is divergence free[2]: whenever we estimate some term of the form $\mathcal{Q}\,(bM\,(D)\,u)$ where b lies in an appropriate Besov space and $M\,(D)$ is some pseudo-differential operator then we may write it

$$\mathcal{Q}\,(bM\,(D)\,u) = [\mathcal{Q}, b]M\,(D)\,u$$

and use the fact that the later expression in more regular than $M\,(D)\,u$, see Proposition 1.3.14.

The rest of this section is devoted to the proof of Theorem 4.1.1. The strategy is the following. First we prove some new estimates regarding the elliptic equation $\operatorname{div}(a\nabla P) = \operatorname{div} f$. These estimates are used in order to prove a weaker variant of 4.1.1 by demanding an extra low-frequency information on the initial data. Finally, using a perturbative version of Danchin and Mucha's results of [51] we arrive at constructing a solution with the optimal regularity. Uniqueness is obtained by a duality method.

4.1.2 Pressure estimates

Before handling System (4.1.1) we shall study the following elliptic equation:

$$\operatorname{div}(a\nabla P) = \operatorname{div} f. \tag{4.1.3}$$

For the reader's convenience let us cite the following classical result, a proof of which can be found, for instance in [45]:

Proposition 4.1.1. *Let us consider* $a \in L^\infty(\mathbb{R}^n)$ *and a constant* $a_\star$ *such that*

$$a \geq a_\star > 0$$

For all vector field f *with coefficients in* $L^2(\mathbb{R}^n)$*, there exists a tempered distribution* P *unique up to constant functions such that* $\nabla P \in L^2(\mathbb{R}^n)$ *and Equation (4.1.3) is satisfied. In addition, we have:*

$$a_\star \|\nabla P\|_{L^2} \leq \|\mathcal{Q}f\|_{L^2}\,.$$

[2] and thus $\mathcal{Q}u = 0$.

Recently, in [86], regarding the $2D$ case, H. Xu, Y. Li and X. Zhai studied the elliptic equation (4.1.3) with the data $(a-\bar{a}, f)$ in Besov spaces. Using a different approach, we obtain estimates in both two dimensional and three dimensional situations. Let us also mention that our method allows to obtain a wider range of indices than the one of Proposition 3.1. $i)$ of [86].

We choose to focus on the $3D$ case. We aim at establishing the following result:

Proposition 4.1.2. *Let us consider $p \in \left(\frac{6}{5}, 2\right)$ and $q \in [1, \infty)$ such that $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{2}$. Assume that there exist positive constants $(\bar{a}, a_\star, a^\star)$ such that $a - \bar{a} \in \dot{B}^{\frac{3}{q}}_{q,1}\left(\mathbb{R}^3\right)$ and $0 < a_\star \leq a \leq a^\star$. Furthermore, consider $f \in \dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}\left(\mathbb{R}^3\right)$. Then there exists a tempered distribution P unique up to constant functions such that $\nabla P \in \dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}\left(\mathbb{R}^3\right)$ and Equation (4.1.3) is satisfied. Moreover, the following estimate holds true:*

$$\|\nabla P\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} \lesssim (1/\bar{a} + \|1/a - 1/\bar{a}\|_{\dot{B}^{\frac{3}{q}}_{q,1}})(1 + (1/a_\star)\|a-\bar{a}\|_{\dot{B}^{\frac{3}{q}}_{q,1}})\|\mathcal{Q}f\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}. \tag{4.1.4}$$

Remark 4.1.1. *Working in Besov spaces with third index $r = 2$ is enough in view of the applications that we have in mind. However similar estimates do hold true when the third index is chosen in the interval $[1, 2]$.*

Proof. Because $p < 2$, Proposition 1.3.4 ensures that $\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2} \hookrightarrow L^2 = \dot{B}^0_{2,2}$ and owing to Proposition 4.1.1, we get the existence of $P \in \mathcal{S}'\left(\mathbb{R}^3\right)$ with $\nabla P \in L^2$ and

$$a_\star \|\nabla P\|_{L^2} \leq \|\mathcal{Q}f\|_{L^2}. \tag{4.1.5}$$

Moreover, as $\mathcal{Q}$ is a continuous operator on L^2 we deduce from (4.1.3) that

$$\mathcal{Q}(a\nabla P) = \mathcal{Q}f\ . \tag{4.1.6}$$

Using the Bony decomposition (see Definition 1.3.1 and the remark that follows) and the fact that $\mathcal{P}(\nabla P) = 0$ we write that:

$$\mathcal{P}(a\nabla P) = \mathcal{P}\left(\dot{T}'_{\nabla P}(a-\bar{a})\right) + \left[\mathcal{P}, \dot{T}_{a-\bar{a}}\right]\nabla P.$$

Using Proposition 1.3.10 along with Proposition 1.3.4 and relation (4.1.5), we get that

$$\left\|\mathcal{P}\left(\dot{T}'_{\nabla P}(a-\bar{a})\right)\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} \lesssim \|\nabla P\|_{L^2}\|a-\bar{a}\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p^\star,2}} \lesssim \frac{1}{a_\star}\|\mathcal{Q}f\|_{L^2}\|a-\bar{a}\|_{\dot{B}^{\frac{3}{q}}_{q,1}} \tag{4.1.7}$$

where

$$\frac{1}{p} = \frac{1}{2} + \frac{1}{p^\star}.$$

Next, proceeding as in Proposition 1.3.13 we get that:

$$\left\|\left[\mathcal{P}, \dot{T}_{a-\bar{a}}\right]\nabla P\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} \lesssim \|\nabla a\|_{\dot{B}^{\frac{3}{p}-\frac{5}{2}}_{p^\star,2}}\|\nabla P\|_{L^2} \lesssim \frac{1}{a_\star}\|\mathcal{Q}f\|_{L^2}\|a-\bar{a}\|_{\dot{B}^{\frac{3}{q}}_{q,1}}. \tag{4.1.8}$$

Putting together relations (4.1.7) and (4.1.8) we get

$$\|\mathcal{P}(a\nabla P)\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} \lesssim \frac{1}{a_\star}\|\mathcal{Q}f\|_{L^2}\|a-\bar{a}\|_{\dot{B}^{\frac{3}{q}}_{q,1}}.$$

Combining this with (4.1.6) and Proposition 1.3.4, we find that:

$$\|a\nabla P\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} \lesssim \left(1 + \frac{1}{a_\star}\|a-\bar{a}\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\right)\|\mathcal{Q}f\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}.$$

Of course, writing

$$\nabla P = \frac{1}{a} a \nabla P$$

using product rules one gets that:

$$\|\nabla P\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} \lesssim (1/\bar{a} + \|1/a - 1/\bar{a}\|_{\dot{B}^{\frac{3}{q}}_{q,1}})(1 + (1/a_\star)\|a - \bar{a}\|_{\dot{B}^{\frac{3}{q}}_{q,1}})\|\mathcal{Q}f\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} \tag{4.1.9}$$

This concludes the proof of Proposition 4.1.2. □

Applying the same technique as above leads to the 2 dimensional estimate:

Proposition 4.1.3. *Let us consider $p \in (1,2)$ and $q \in [1,\infty)$ such that $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{2}$. Assume that there exists positive constants $(\bar{a}, a_\star, a^\star)$ such that $a - \bar{a} \in \dot{B}^{\frac{2}{q}}_{q,1}(\mathbb{R}^2)$ and $0 < a_\star \leq a \leq a^\star$. Furthermore, consider $f \in \dot{B}^{\frac{2}{p}-1}_{p,2}(\mathbb{R}^2)$. Then there exists a tempered distribution P unique up to constant functions such that $\nabla P \in \dot{B}^{\frac{2}{p}-1}_{p,2}(\mathbb{R}^2)$ and Equation (4.1.3) is satisfied. Moreover, the following estimate holds true:*

$$\|\nabla P\|_{\dot{B}^{\frac{2}{p}-1}_{p,2}} \lesssim (1/\bar{a} + \|1/a - 1/\bar{a}\|_{\dot{B}^{\frac{2}{q}}_{q,1}})(1 + (1/a_\star)\|a - \bar{a}\|_{\dot{B}^{\frac{2}{q}}_{q,1}})\|\mathcal{Q}f\|_{\dot{B}^{\frac{2}{p}-1}_{p,2}}. \tag{4.1.10}$$

Let us point out that the restriction $p > \frac{6}{5}$ comes from the fact that we need $\frac{3}{p} - \frac{5}{2} < 0$ in relation (4.1.8). In $2D$ instead of $\frac{3}{p} - \frac{5}{2}$ we will have $\frac{2}{p} - 2$ which is negative provided that $p > 1$.

The next result covers the range of integrability indices larger than 2 :

Proposition 4.1.4. *Let us consider $p \in (2,6)$ and $q \in [1,\infty)$ such that $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{2}$. Assume that there exists positive constants $(\bar{a}, a_\star, a^\star)$ such that $a - \bar{a} \in \dot{B}^{\frac{3}{q}}_{q,1}(\mathbb{R}^3)$ and $0 < a_\star \leq a \leq a^\star$. Furthermore, consider $f \in \dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}(\mathbb{R}^3)$ and a tempered distribution P with $\nabla P \in \dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}(\mathbb{R}^3)$ such that Equation (4.1.3) is satisfied. Then, the following estimate holds true:*

$$\|\nabla P\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} \lesssim (1/\bar{a} + \|1/a - 1/\bar{a}\|_{\dot{B}^{\frac{3}{q}}_{q,1}})(1 + (1/a_\star)\|a - \bar{a}\|_{\dot{B}^{\frac{3}{q}}_{q,1}})\|\mathcal{Q}f\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}. \tag{4.1.11}$$

Proof. Let us notice that p' the conjugate Lebesgue exponent of p satisfies $p' \in \left(\frac{6}{5}, 2\right)$ and $\frac{1}{p'} - \frac{1}{q} \leq \frac{1}{2}$. Thus, according to Proposition 4.1.2, for any g belonging to the unit ball of $\mathcal{S} \cap \dot{B}^{\frac{3}{p'}-\frac{3}{2}}_{p',2}$ there exists a $P_g \in \mathcal{S}'(\mathbb{R}^3)$ with $\nabla P_g \in \mathcal{S} \cap \dot{B}^{\frac{3}{p'}-\frac{3}{2}}_{p',2}$ such that

$$\operatorname{div}(a \nabla P_g) = \operatorname{div} g$$

and

$$\|\nabla P_g\|_{\dot{B}^{\frac{3}{p'}-\frac{3}{2}}_{p',2}} \lesssim (1/\bar{a} + \|1/a - 1/\bar{a}\|_{\dot{B}^{\frac{3}{q}}_{q,1}})(1 + (1/a_\star)\|a - \bar{a}\|_{\dot{B}^{\frac{3}{q}}_{q,1}}).$$

We write that

$$\begin{aligned} \langle \nabla P, g \rangle &= -\langle P, \operatorname{div} g \rangle = -\langle P, \operatorname{div}(a \nabla P_g) \rangle \\ &= -\langle \operatorname{div} \mathcal{Q}f, P_g \rangle = \langle \mathcal{Q}f, \nabla P_g \rangle, \end{aligned}$$

and consequently

$$\begin{aligned} |\langle \nabla P, g \rangle| &\lesssim \|\mathcal{Q}f\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} \|\nabla P_g\|_{\dot{B}^{\frac{3}{p'}-\frac{3}{2}}_{p',2}} \\ &\lesssim (1/\bar{a} + \|1/a - 1/\bar{a}\|_{\dot{B}^{\frac{3}{q}}_{q,1}})(1 + (1/a_\star)\|a - \bar{a}\|_{\dot{B}^{\frac{3}{q}}_{q,1}})\|\mathcal{Q}f\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}. \end{aligned}$$

Using Proposition 1.3.5 we get that relation (4.1.11) holds true. □

As in the previous situation, by applying the same technique we get a similar result in $2D$:

Proposition 4.1.5. *Let us consider $p \in (2, \infty)$ and $q \in [1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{2}$. Assume that there exists positive constants $(\bar{a}, a_\star, a^\star)$ such that $a - \bar{a} \in \dot{B}^{\frac{2}{q}}_{q,1}(\mathbb{R}^2)$ and $0 < a_\star \leq a \leq a^\star$. Furthermore, consider $f \in \dot{B}^{\frac{2}{p}-1}_{p,2}(\mathbb{R}^2)$ and a tempered distribution P with $\nabla P \in \dot{B}^{\frac{2}{p}-1}_{p,2}(\mathbb{R}^2)$ such that Equation (4.1.3) is satisfied. Then, following estimate holds true:*

$$\|\nabla P\|_{\dot{B}^{\frac{2}{p}-1}_{p,2}} \lesssim (1/\bar{a} + \|1/a - 1/\bar{a}\|_{\dot{B}^{\frac{3}{q}}_{q,1}})(1 + (1/a_\star)\|a - \bar{a}\|_{\dot{B}^{\frac{3}{q}}_{q,1}})\|\mathcal{Q}f\|_{\dot{B}^{\frac{2}{p}-1}_{p,2}}. \tag{4.1.12}$$

4.1.3 Some preliminary results

In this section we derive estimates for a Stokes-like problem with time independent, nonconstant coefficients. Before proceeding to the actual proof, for the reader's convenience, let us cite the following results which were established by Danchin and Mucha in [48], [51]. These results correspond to the case where a and b constants:

Proposition 4.1.6. *Consider $u_0 \in \dot{B}^{\frac{n}{p}-1}_{p,1}$ and $(f, \partial_t R, \nabla \operatorname{div} R) \in L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})$ with $\mathcal{Q}R \in C_T(\dot{B}^{\frac{n}{p}-1}_{p,1})$ such that*

$$\operatorname{div} u_0 = \operatorname{div} R(0, \cdot).$$

Then, the system

$$\begin{cases} \partial_t u - \bar{a}\bar{b}\Delta u + \bar{a}\nabla P = f, \\ \operatorname{div} u = \operatorname{div} R, \\ u_{|t=0} = u_0, \end{cases}$$

has a unique solution $(u, \nabla P)$ with:

$$u \in \mathcal{C}([0, T); \dot{B}^{\frac{n}{p}-1}_{p,1}) \text{ and } \partial_t u, \nabla^2 u, \nabla P \in L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})$$

and the following estimate is valid:

$$\|u\|_{L^\infty_T(\dot{B}^{\frac{n}{p}-1}_{p,1})} + \left\|\left(\partial_t u, \bar{a}\bar{b}\nabla^2 u, \bar{a}\nabla P\right)\right\|_{L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})} \lesssim \|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} + \left\|\left(f, \partial_t R, \bar{a}\bar{b}\nabla \operatorname{div} R\right)\right\|_{L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})}.$$

As a consequence of the previous result, one can establish via a perturbation argument:

Proposition 4.1.7. *Consider $u_0 \in \dot{B}^{\frac{n}{p}-1}_{p,1}$ and $(f, \partial_t R, \nabla \operatorname{div} R) \in L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})$ with $\mathcal{Q}R \in C_T(\dot{B}^{\frac{n}{p}-1}_{p,1})$ such that*

$$\operatorname{div} u_0 = \operatorname{div} R(0, \cdot).$$

Then, there exists a $\eta = \eta(\bar{a})$ small enough such that for all $c \in \dot{B}^{\frac{n}{p}}_{p,1}$ with

$$\|c\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \leq \eta\ ,$$

the system

$$\begin{cases} \partial_t u - \bar{a}\bar{b}\Delta u + (\bar{a} + c)\nabla P = f, \\ \operatorname{div} u = \operatorname{div} R, \\ u_{|t=0} = u_0, \end{cases}$$

has a unique solution $(u, \nabla P)$ with:

$$u \in \mathcal{C}([0, T); \dot{B}^{\frac{n}{p}-1}_{p,1}) \text{ and } \partial_t u, \nabla^2 u, \nabla P \in L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})$$

and the following estimate is valid:

$$\|u\|_{L^\infty_T(\dot{B}^{\frac{n}{p}-1}_{p,1})} + \left\|\left(\partial_t u, \bar{a}\bar{b}\nabla^2 u, \bar{a}\nabla P\right)\right\|_{L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})} \lesssim \|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} + \left\|\left(f, \partial_t R, \bar{a}\bar{b}\nabla \operatorname{div} R\right)\right\|_{L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})}.$$

In all what follows we denote by E_{loc} the space of $(u, \nabla P)$ such that:

$$u \in \mathcal{C}([0,\infty); \dot{B}^{\frac{n}{p}-1}_{p,1}) \text{ and } \left(\nabla^2 u, \nabla P\right) \in L^1_{loc}(\dot{B}^{\frac{n}{p}-1}_{p,1}) \times L^1_{loc}(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2} \cap \dot{B}^{\frac{n}{p}-1}_{p,1}).$$

Also, let us introduce the space E_T of $(u, \nabla P)$ with

$$u \in \mathcal{C}_T(\dot{B}^{\frac{n}{p}-1}_{p,1}),\ \nabla^2 u \in L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1}) \text{ and } \nabla P \in L^1_T(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2} \cap \dot{B}^{\frac{n}{p}-1}_{p,1})$$

which we endow with the norm:

$$\|(u, \nabla P)\|_{E_T} = \|u\|_{L^\infty_T(\dot{B}^{\frac{n}{p}-1}_{p,1})} + \left\|\nabla^2 u\right\|_{L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})} + \|\nabla P\|_{L^1_T(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2} \cap \dot{B}^{\frac{n}{p}-1}_{p,1})} < \infty.$$

The first ingredient in proving Theorem 4.1.1 is the following:

Proposition 4.1.8. *Consider $n \in \{2,3\}$ and $p \in (1,4)$ if $n=2$ or $p \in \left(\frac{6}{5}, 4\right)$ if $n=3$. Assume there exists positive constants $\left(a_\star, b_\star, a^\star, b^\star, \bar{a}, \bar{b}\right)$ such that $a - \bar{a} \in \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)$, $b - \bar{b} \in \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)$ and*

$$0 < a_\star \leq a \leq a^\star,$$
$$0 < b_\star \leq b \leq b^\star.$$

Furthermore, consider $u_0 \in \dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)$, $f \in L^1_{loc}(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}(\mathbb{R}^n) \cap \dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n))$ respectively $R \in (\mathcal{S}'(\mathbb{R}^n))^n$ with $(\partial_t R, \nabla \operatorname{div} R) \in L^1_{loc}(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}(\mathbb{R}^n) \cap \dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n))$ and $\mathcal{Q}R \in \mathcal{C}([0,\infty); \dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n))$ such that

$$\operatorname{div} u_0 = \operatorname{div} R(0,\cdot).$$

Then, there exists a constant C_{ab} depending on a and b such that any solution $(u, \nabla P) \in E_T$ of the Stokes system (4.1.1) will satisfy:

$$\begin{aligned} &\|u\|_{L^\infty_t(\dot{B}^{\frac{n}{p}-1}_{p,1})} + \left\|\nabla^2 u\right\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})} + \|\nabla P\|_{L^1_t(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2} \cap \dot{B}^{\frac{n}{p}-1}_{p,1})} \\ &\leq e^{C_{ab}(t+1)}(\|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L^1_t(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2} \cap \dot{B}^{\frac{n}{p}-1}_{p,1})}), \end{aligned} \tag{4.1.13}$$

for all $t \in (0,T]$.

Before proceeding with the proof, a few remarks are in order:

Remark 4.1.2. *Proposition 4.1.8 is different from Theorem 4.1.1 when $n=3$. Indeed, in the 3 dimensional case the theory is more subtle and thus, as a first step we construct a unique solution for the case of more regular initial data.*

Remark 4.1.3. *The difficulty when dealing with the Stokes system with non constant coefficients lies in the fact that the pressure and the velocity u are coupled. Indeed, in the constant coefficients case, in view of*

$$\operatorname{div} u = \operatorname{div} R,$$

one can apply the divergence operator in the first equation of (4.1.1) in order to obtain the following elliptic equation verified by the pressure:

$$a\Delta P = \operatorname{div}\left(f - \partial_t R + 2ab\nabla \operatorname{div} R\right). \tag{4.1.14}$$

From (4.1.14) we can construct the pressure. Having built the pressure, the velocity satisfies a classical heat equation. In the non constant coefficient case, proceeding as above we find that:

$$\operatorname{div}\left(a\nabla P\right) = \operatorname{div}\left(f - \partial_t R + a \operatorname{div}(bD(u))\right). \tag{4.1.15}$$

therefore the strategy used in the previous case is not well-adapted. We will establish a priori estimate and use a continuity argument like in [46]. In order to be able to close the estimates on u, we have to bound $\|a\nabla P\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}$ in terms of $\|u\|^{\beta}_{L^\infty_t(\dot{B}^{\frac{n}{p}-1}_{p,1})} \left\|\nabla^2 u\right\|^{1-\beta}_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}$ for some $\beta \in (0,1)$. Thus, the difficulty is to find estimates for the pressure which do not feature the time derivative of the velocity.

In view of Proposition 4.1.6, let us consider $(u_L, \nabla P_L)$ the unique solution of the system

$$\begin{cases} \partial_t u - \bar{a}\,\mathrm{div}(\bar{b}D(u)) + \bar{a}\nabla P = f, \\ \mathrm{div}\, u = \mathrm{div}\, R, \\ u_{|t=0} = u_0, \end{cases} \tag{4.1.16}$$

with

$$u_L \in \mathcal{C}([0,\infty); \dot{B}^{\frac{n}{p}-1}_{p,1}) \text{ and } (\partial_t u_L, \nabla^2 u_L, \nabla P_L) \in L^1_{loc}(\dot{B}^{\frac{n}{p}-1}_{p,1}).$$

Recall that for any $t \in [0,\infty)$ we have

$$\|u_L\|_{L^\infty_t(\dot{B}^{\frac{n}{p}-1}_{p,1})} + \left\|\left(\partial_t u_L, \bar{a}\bar{b}\nabla^2 u_L, \bar{a}\nabla P_L\right)\right\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})} \leq C(\|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} + \left\|\left(f, \partial_t R, \bar{a}\bar{b}\nabla\,\mathrm{div}\, R\right)\right\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}). \tag{4.1.17}$$

In what follows, we will use the notation:

$$\tilde{u} = u - u_L,\ \nabla\tilde{P} = \nabla P - \nabla P_L. \tag{4.1.18}$$

Obviously, we have

$$\mathrm{div}\,\tilde{u} = 0. \tag{4.1.19}$$

Thus, the system (4.1.1) is recast into

$$\begin{cases} \partial_t \tilde{u} - a\,\mathrm{div}\,(bD(\tilde{u})) + a\nabla\tilde{P} = \tilde{f}, \\ \mathrm{div}\,\tilde{u} = 0, \\ \tilde{u}_{|t=0} = 0, \end{cases} \tag{4.1.20}$$

where

$$\tilde{f} = a\,\mathrm{div}(bD(u_L)) - \bar{a}\,\mathrm{div}(\bar{b}D(u_L)) - (a-\bar{a})\nabla P_L.$$

Using the last equality along with Proposition 1.3.11, we infer that:

$$\begin{aligned} \left\|\tilde{f}\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} &\leq \left\|a\,\mathrm{div}(bD(u_L)) - \bar{a}\,\mathrm{div}(\bar{b}D(u_L))\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} + \|(a-\bar{a})\nabla P_L\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \\ &\lesssim (\bar{a} + \|a-\bar{a}\|_{\dot{B}^{\frac{n}{p}}_{p,1}})(\bar{b} + \left\|b-\bar{b}\right\|_{\dot{B}^{\frac{n}{p}}_{p,1}})\,\|\nabla u_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}} + \|a-\bar{a}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\,\|\nabla P_L\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}. \end{aligned} \tag{4.1.21}$$

Let us estimate the pressure $a\nabla\tilde{P}$. First, we write that

$$\left\|a\nabla\tilde{P}\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \leq \left\|\mathcal{Q}\left(a\nabla\tilde{P}\right)\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} + \left\|\mathcal{P}\left(a\nabla\tilde{P}\right)\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}.$$

Applying the $\mathcal{Q}$ operator in the first equation of (4.1.20) we get that

$$\mathcal{Q}\left(a\nabla\tilde{P}\right) = \mathcal{Q}\tilde{f} + \mathcal{Q}(a\,\mathrm{div}\,(bD(\tilde{u}))).$$

Thus, we get that:

$$\left\|\mathcal{Q}\left(a\nabla\tilde{P}\right)\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \leq \left\|\mathcal{Q}\tilde{f}\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} + \|\mathcal{Q}\,(a\,\mathrm{div}\,(bD(\tilde{u})))\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}. \tag{4.1.22}$$

Let write that:

$$\mathcal{Q}\,(a\,\mathrm{div}\,(bD(\tilde{u}))) = \mathcal{Q}\left(D(\tilde{u})\dot{S}_m\,(a\nabla b)\right) + \mathcal{Q}\left(\dot{S}_m\,(ab - \bar{a}\bar{b})\,\Delta\tilde{u}\right) \tag{4.1.23}$$

$$+\,\mathcal{Q}\left(D(\tilde{u})\left(Id - \dot{S}_m\right)(a\nabla b)\right) \tag{4.1.24}$$

$$+\,\mathcal{Q}\left(\left(Id - \dot{S}_m\right)(ab - \bar{a}\bar{b})\,\Delta\tilde{u}\right), \tag{4.1.25}$$

where $m \in \mathbb{N}$ will be chosen later. According to Proposition 1.3.11 we have:

$$\left\|\mathcal{Q}\left(D\left(\tilde{u}\right)\dot{S}_m\left(a\nabla b\right)\right)\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \lesssim \left\|\dot{S}_m\left(a\nabla b\right)\right\|_{\dot{B}^{\frac{n}{p}-\frac{1}{2}}_{p,1}} \left\|\nabla\tilde{u}\right\|_{\dot{B}^{\frac{n}{p}-\frac{1}{2}}_{p,1}}. \tag{4.1.26}$$

Owing to the fact that $\tilde{u}$ is divergence free we can write that

$$\mathcal{Q}\left(\dot{S}_m\left(ab-\bar{a}\bar{b}\right)\Delta\tilde{u}\right) = [\mathcal{Q}, \dot{S}_m\left(ab-\bar{a}\bar{b}\right)]\Delta\tilde{u}, \tag{4.1.27}$$

such that applying Proposition 1.3.14 we get that

$$\begin{aligned}\left\|\mathcal{Q}\left(\dot{S}_m\left(ab-\bar{a}\bar{b}\right)\Delta\tilde{u}\right)\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} &\lesssim \left\|\left(\dot{S}_m\left(a\nabla b\right), \dot{S}_m\left(b\nabla a\right)\right)\right\|_{\dot{B}^{\frac{n}{p}-\frac{1}{2}}_{p,1}} \left\|\Delta\tilde{u}\right\|_{\dot{B}^{\frac{n}{p}-\frac{3}{2}}_{p,1}} \\ &\lesssim \left\|\left(\dot{S}_m\left(a\nabla b\right), \dot{S}_m\left(b\nabla a\right)\right)\right\|_{\dot{B}^{\frac{n}{p}-\frac{1}{2}}_{p,1}} \left\|\nabla\tilde{u}\right\|_{\dot{B}^{\frac{n}{p}-\frac{1}{2}}_{p,1}}. \end{aligned} \tag{4.1.28}$$

The last two terms of (4.1.23)-(4.1.25) are estimated as follows:

$$\left\|\mathcal{Q}\left(\left(Id-\dot{S}_m\right)\left(a\nabla b\right)D\left(\tilde{u}\right)\right)+\mathcal{Q}\left(\left(Id-\dot{S}_m\right)\left(ab-\bar{a}\bar{b}\right)\Delta\tilde{u}\right)\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \tag{4.1.29}$$

$$\lesssim \left(\left\|\left(Id-\dot{S}_m\right)\left(a\nabla b\right)\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} + \left\|\left(Id-\dot{S}_m\right)\left(ab-\bar{a}\bar{b}\right)\right\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\right)\left\|\nabla\tilde{u}\right\|_{\dot{B}^{\frac{n}{p}}_{p,1}}. \tag{4.1.30}$$

Thus, putting together relations (4.1.22)-(4.1.30) we get that:

$$\begin{aligned}\left\|\mathcal{Q}\left(a\nabla\tilde{P}\right)\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} &\lesssim \left\|\mathcal{Q}\tilde{f}\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} + \left\|\left(\dot{S}_m\left(a\nabla b\right), \dot{S}_m\left(b\nabla a\right)\right)\right\|_{\dot{B}^{\frac{n}{p}-\frac{1}{2}}_{p,1}} \left\|\nabla\tilde{u}\right\|_{\dot{B}^{\frac{n}{p}-\frac{1}{2}}_{p,1}} \\ &+ \left\|\nabla\tilde{u}\right\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\left(\left\|\left(Id-\dot{S}_m\right)\left(a\nabla b, b\nabla a\right)\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} + \left\|\left(Id-\dot{S}_m\right)\left(ab-\bar{a}\bar{b}\right)\right\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\right). \end{aligned} \tag{4.1.31}$$

Next, we turn our attention towards $\mathcal{P}\left(a\nabla\tilde{P}\right)$. The $2D$ case respectively the $3D$ case have to be treated differently.

The $3D$ case

Noticing that

$$\mathcal{P}\left(a\nabla\tilde{P}\right) = \mathcal{P}\left(\left(Id-\dot{S}_m\right)\left(a-\bar{a}\right)\nabla\tilde{P}\right) + [\mathcal{P}, \dot{S}_m\left(a-\bar{a}\right)]\nabla\tilde{P},$$

and using again Proposition 1.3.14 combined with Proposition 4.1.2 and Proposition 4.1.4 we get that:

$$\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} + \left\|\mathcal{P}\left(a\nabla\tilde{P}\right)\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \tag{4.1.32}$$

$$\lesssim \left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} + \left\|\mathcal{P}((Id-\dot{S}_m)(a-\bar{a})\nabla\tilde{P})\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + \left\|[\mathcal{P}, \dot{S}_m(a-\bar{a})]\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \tag{4.1.33}$$

$$\lesssim \left\|(Id-\dot{S}_m)(a-\bar{a})\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + (1+\left\|\dot{S}_m\nabla a\right\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,2}})\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} \tag{4.1.34}$$

$$\lesssim \left\|(Id-\dot{S}_m)(a-\bar{a})\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}} (1/\bar{a}+\left\|1/a-1/\bar{a}\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}})\left\|a\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \tag{4.1.35}$$

$$+\tilde{C}\left(a\right)(1+\left\|\dot{S}_m\nabla a\right\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,2}})(\left\|\tilde{f}\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} + \left\|a\,\mathrm{div}\left(bD(\tilde{u})\right)\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}), \tag{4.1.36}$$

where

$$\tilde{C}(a)=(1/\bar{a}+\|1/a-1/\bar{a}\|_{\dot{B}^{\frac{3}{p}}_{p,1}})(1+1/a_\star\|a-\bar{a}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}).$$

We observe that

$$\|a\operatorname{div}(bD(\tilde{u}))\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}\lesssim(\bar{a}+\|a-\bar{a}\|_{\dot{B}^{\frac{3}{p}}_{p,1}})(\bar{b}+\left\|b-\bar{b}\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}})\|\nabla\tilde{u}\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,1}}. \tag{4.1.37}$$

Putting together (4.1.32)-(4.1.36) along with (4.1.37) we get that

$$\begin{aligned}\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}+\left\|\mathcal{P}(a\nabla\tilde{P})\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}&\lesssim\left\|(Id-\dot{S}_m)(a-\bar{a})\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}}(1/\bar{a}+\|1/a-1/\bar{a}\|_{\dot{B}^{\frac{3}{p}}_{p,1}})\left\|a\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\\&+\tilde{C}(a)(1+\left\|\dot{S}_m\nabla a\right\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,2}})(\left\|\tilde{f}\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}+(\bar{a}+\|a-\bar{a}\|_{\dot{B}^{\frac{3}{p}}_{p,1}})(\bar{b}+\left\|b-\bar{b}\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}})\|\nabla\tilde{u}\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,1}}).\end{aligned} \tag{4.1.38}$$

Combining (4.1.31) with (4.1.38) yields:

$$\begin{aligned}\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}+\left\|a\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\lesssim T^1_m(a,b)\left\|a\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}+T^2_m(a,b)\left\|\tilde{f}\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}\cap\dot{B}^{\frac{3}{p}-1}_{p,1}}\\+T^3_m(a,b)\|\nabla\tilde{u}\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,1}}+T^4_m(a,b)\|\nabla\tilde{u}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\end{aligned}$$

where

$$\begin{aligned}T^1_m(a,b)&=\left\|(Id-\dot{S}_m)(a-\bar{a})\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}}(1/\bar{a}+\|1/a-1/\bar{a}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}),\\T^2_m(a,b)&=\tilde{C}(a)(1+\left\|\dot{S}_m\nabla a\right\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,2}}),\\T^3_m(a,b)&=\left\|(\dot{S}_m(a\nabla b),\dot{S}_m(b\nabla a))\right\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,1}}\\&\quad+\tilde{C}(a)(1+\left\|\dot{S}_m\nabla a\right\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,2}})(\bar{a}+\|a-\bar{a}\|_{\dot{B}^{\frac{3}{p}}_{p,1}})(\bar{b}+\left\|b-\bar{b}\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}}),\\T^4_m(a,b)&=\left\|(Id-\dot{S}_m)(a\nabla b,b\nabla a)\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}+\left\|(Id-\dot{S}_m)(ab-\bar{a}\bar{b})\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}}.\end{aligned}$$

Observe that m could be chosen large enough such that $T^1_m(a,b)$ and $T^4_m(a,b)$ can be made arbitrarily small. Thus, there exists a constant C_{ab} depending on a and b such that:

$$\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}\cap\dot{B}^{\frac{3}{p}-1}_{p,1}}\leq C_{ab}(\left\|\tilde{f}\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}\cap\dot{B}^{\frac{3}{p}-1}_{p,1}}+\|\nabla\tilde{u}\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,1}})+\eta\|\nabla\tilde{u}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}, \tag{4.1.39}$$

where η can be made arbitrarily small (of course, with the price of increasing the constant C_{ab}). Let us take a look at the $\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}$-norm of $\tilde{f}$; we get that:

$$\begin{aligned}\left\|\tilde{f}\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}&\leq\left\|a\operatorname{div}(bD(u_L))-\bar{a}\operatorname{div}(\bar{b}D(u_L))\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}+\|(a-\bar{a})\nabla P_L\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}\\&\lesssim(\bar{a}+\|a-\bar{a}\|_{\dot{B}^{\frac{3}{p}}_{p,1}})(\bar{b}+\left\|b-\bar{b}\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}})\|\nabla u_L\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,1}}+\|a-\bar{a}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\|\nabla P_L\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}.\end{aligned} \tag{4.1.40}$$

As $u_L\in C([0,\infty),\dot{B}^{\frac{3}{p}-1}_{p,1})\cap L^1([0,\infty),\dot{B}^{\frac{3}{p}+1}_{p,1})$ and $\mathcal{Q}$ is a continuous operator on homogeneous Besov spaces from

$$\operatorname{div}(u_L-R)=0,$$

we deduce that

$$\mathcal{P}(u_L-R)=u_L-R,$$

which implies

$$\mathcal{Q}u_L = \mathcal{Q}R.$$

By applying the operator $\mathcal{Q}$ in the first equation of System (4.1.16) we get that:

$$\begin{aligned}\bar{a}\nabla P_L &= \mathcal{Q}f - \mathcal{Q}\partial_t u_L + \bar{a}\bar{b}\mathcal{Q}\Delta u_L + \bar{a}\bar{b}\nabla \operatorname{div} R\\ &= \mathcal{Q}f - \mathcal{Q}\partial_t R + 2\bar{a}\bar{b}\nabla \operatorname{div} R\end{aligned}$$

and thus

$$\|\nabla P_L\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} \leq \frac{1}{\bar{a}}\|\mathcal{Q}f\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} + \frac{1}{\bar{a}}\|\partial_t \mathcal{Q}R\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} + 2\bar{b}\|\nabla \operatorname{div} R\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}.$$

In view of (4.1.39), (4.1.21), (4.1.40) and interpolation we gather that there exists a constant C_{ab} such that:

$$\begin{aligned}\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}\cap\dot{B}^{\frac{3}{p}-1}_{p,1}} &\leq C_{ab}(\|\nabla u_L\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,1}} + \|\nabla P_L\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} + \left\|(\nabla^2 u_L, \nabla P_L)\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}) && (4.1.41)\\ &\quad + C_{ab}\|\nabla\tilde{u}\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,1}} + \eta\|\nabla\tilde{u}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}. && (4.1.42)\\ &\leq C_{ab}\|(\mathcal{Q}f, \partial_t\mathcal{Q}R, \nabla \operatorname{div} R)\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} + C_{ab}\|u_L\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} && (4.1.43)\\ &\quad + C_{ab}\left\|(\nabla^2 u_L, \nabla P_L)\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + C_{ab}\|\tilde{u}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + 2\eta\|\nabla\tilde{u}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}. && (4.1.44)\end{aligned}$$

where, again, at the price of increasing C_{ab}, η can be made arbitrarily small.

The $2D$ case

In this case, using again Proposition 1.3.14 combined with Proposition with Proposition 4.1.3 and Proposition 4.1.5 we get that:

$$\begin{aligned}&\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,2}} + \left\|\mathcal{P}(a\nabla\tilde{P})\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}\\ &\lesssim \left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,2}} + \left\|(Id-\dot{S}_m)(a-\bar{a})\right\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} + \left\|[\mathcal{P}, \dot{S}_m(a-\bar{a})]\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}\\ &\lesssim \left\|(Id-\dot{S}_m)(a-\bar{a})\right\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} + (1+\left\|\nabla\dot{S}_m a\right\|_{\dot{B}^{\frac{2}{p}}_{p,2}})\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,2}}\\ &\lesssim \left\|(Id-\dot{S}_m)(a-\bar{a})\right\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}\\ &+\tilde{C}(a)(1+\left\|\nabla\dot{S}_m a\right\|_{\dot{B}^{\frac{2}{p}}_{p,2}})(\left\|\tilde{f}\right\|_{\dot{B}^{\frac{2}{p}-1}_{2,2}} + \|\mathcal{Q}(a\operatorname{div}(bD(\tilde{u})))\|_{\dot{B}^{\frac{2}{p}-1}_{p,2}})\end{aligned}$$

where, as before

$$\tilde{C}(a) = (1/\bar{a} + \|1/a - 1/\bar{a}\|_{\dot{B}^{\frac{2}{p}}_{p,1}})(1 + (1/a_\star)\|a-\bar{a}\|_{\dot{B}^{\frac{2}{p}}_{p,1}}).$$

As we have already estimated $\|\mathcal{Q}(a\operatorname{div}(bD(\tilde{u})))\|_{\dot{B}^{\frac{2}{p}-1}_{p,2}}$ in (4.1.31), we gather that:

$$\begin{aligned}\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,2}} + \left\|a\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}}_{p,1}} &\lesssim T^1_m(a,b)\left\|a\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} + T^2_m(a,b)\left\|\tilde{f}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}\\ &\quad + T^3_{m,M}(a,b)\|\nabla\tilde{u}\|_{\dot{B}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} + T^4_{m,M}(a,b)\|\nabla\tilde{u}\|_{\dot{B}^{\frac{2}{p}}_{p,1}}, \qquad (4.1.45)\end{aligned}$$

where

$$T^1_m(a,b) = \left\|(Id-\dot{S}_m)(a-\bar{a})\right\|_{\dot{B}^{\frac{2}{p}}_{p,1}}(1/\bar{a} + \|1/a - 1/\bar{a}\|_{\dot{B}^{\frac{2}{p}}_{p,1}}),$$

$$
\begin{aligned}
T_m^2(a,b) &= \tilde{C}(a)\,(1+\left\|\nabla \dot{S}_m a\right\|_{\dot{B}^{\frac{2}{p}}_{p,2}}),\\
T_{m,M}^3(a,b) &= \left\|(\dot{S}_m(a\nabla b), \dot{S}_m(b\nabla a))\right\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\\
&\quad +\tilde{C}(a)(1+\left\|\nabla \dot{S}_m a\right\|_{\dot{B}^{\frac{2}{p}}_{p,2}})\left\|(\dot{S}_M(a\nabla b), \dot{S}_M(b\nabla a))\right\|_{\dot{B}^{\frac{2}{p}}_{p,1}},\\
T_{m,M}^4(a,b) &= \left\|(Id-\dot{S}_m)(a\nabla b)\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} + \left\|(Id-\dot{S}_m)\left(ab-\bar{a}\bar{b}\right)\right\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\\
&\qquad + \tilde{C}(a)\,(1+\left\|\nabla \dot{S}_m a\right\|_{\dot{B}^{\frac{2}{p}}_{p,2}})(\left\|(Id-\dot{S}_M)(a\nabla b)\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} + \left\|(Id-\dot{S}_M)\left(ab-\bar{a}\bar{b}\right)\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}).
\end{aligned}
$$

First, we fix an $\eta>0$. Let us fix an $m\in\mathbb{N}$ such that $T_m^1(a,b)\left\|a\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}$ can be "absorbed" by the left hand side of (4.1.45) and that

$$
\left\|(Id-\dot{S}_m)(a\nabla b)\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} + \left\|(Id-\dot{S}_m)\left(ab-\bar{a}\bar{b}\right)\right\|_{\dot{B}^{\frac{2}{p}}_{p,1}} \leq \eta/2.
$$

Next, we see that by choosing M large enough we have

$$
T_{m,M}^4(a,b)\leq\eta.
$$

Thus, using interpolation we can write that:

$$
\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,2}} + \left\|a\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} \leq C_{ab}(\left\|(\nabla^2 u_L,\nabla P_L)\right\|_{\dot{B}^{\frac{2}{p}-1}_{2,1}} + \left\|\tilde{u}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}) + 2\eta\left\|\nabla^2\tilde{u}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}. \tag{4.1.46}
$$

End of the proof of Proposition 4.1.8

Obviously, combining the two estimates (4.1.41)-(4.1.44) and (4.1.46) we can continue in a unified manner the rest of the proof of Proposition 4.1.8. First, choose $m\in\mathbb{N}$ large enough such that

$$
\bar{a}\bar{b}+\dot{S}_m(ab-\bar{a}\bar{b})\geq\frac{a_\star b_\star}{2}.
$$

We apply $\dot{\Delta}_j$ to (4.1.20) and we write that:

$$
\begin{aligned}
\partial_t\tilde{u}_j - \operatorname{div}\left((\bar{a}\bar{b}+\dot{S}_m(ab-\bar{a}\bar{b}))\nabla\tilde{u}_j\right) &= \tilde{f}_j - \dot{\Delta}_j\left(a\nabla\tilde{P}\right)\\
&\quad + \dot{\Delta}_j\operatorname{div}\left((Id-\dot{S}_m)(ab-\bar{a}\bar{b})\nabla\tilde{u}\right) + \operatorname{div}\left[\dot{\Delta}_j,\dot{S}_m(ab-\bar{a}\bar{b})\right]\nabla\tilde{u}\\
&\quad + \dot{\Delta}_j\left(D\tilde{u}\dot{S}_m(b\nabla a)\right) + \dot{\Delta}_j\left(D\tilde{u}(Id-\dot{S}_m)(b\nabla a)\right)\\
&\quad + \dot{\Delta}_j\left(\nabla\tilde{u}\dot{S}_m(a\nabla b)\right) + \dot{\Delta}_j\left(\nabla\tilde{u}(Id-\dot{S}_m)(a\nabla b)\right).
\end{aligned}
$$

Multiplying the last relation by $|\tilde{u}_j|^{p-1}\operatorname{sgn}\tilde{u}_j$, integrating and using Lemma 8 from the Appendix B of [45], we get that:

$$
\begin{aligned}
&\left\|\tilde{u}_j\right\|_{L^p} + a_\star b_\star 2^{2j}C\int_0^t\left\|\tilde{u}_j\right\|_{L^p}\\
&\lesssim \int_0^t\left\|\tilde{f}_j\right\|_{L^p} + \int_0^t\left\|\dot{\Delta}_j\left(a\nabla\tilde{P}\right)\right\|_{L^p}\\
&+\int_0^t\left\|\operatorname{div}\left[\dot{\Delta}_j,\dot{S}_m(ab-\bar{a}\bar{b})\right]\nabla\tilde{u}\right\|_{L^p} + \int_0^t\left\|\dot{\Delta}_j\operatorname{div}\left((Id-\dot{S}_m)(ab-\bar{a}\bar{b})\nabla\tilde{u}\right)\right\|_{L^p}
\end{aligned}
$$

$$+\int_0^t \left\|\dot{\Delta}_j\left(D\tilde{u}\dot{S}_m(b\nabla a)\right)\right\|_{L^p}+\int_0^t \left\|\dot{\Delta}_j\left(D\tilde{u}(Id-\dot{S}_m)(b\nabla a)\right)\right\|_{L^p}$$
$$+\int_0^t \left\|\dot{\Delta}_j\left(\nabla\tilde{u}\dot{S}_m(a\nabla b)\right)\right\|_{L^p}+\int_0^t \left\|\dot{\Delta}_j\left(\nabla\tilde{u}(Id-\dot{S}_m)(a\nabla b)\right)\right\|_{L^p}.$$

Multiplying the last relation by $2^{j(\frac{n}{p}-1)}$, performing an $\ell^1(\mathbb{Z})$-summation and using Proposition 1.3.12 to deal with $\left\|\operatorname{div}\left[\dot{\Delta}_j,\dot{S}_m(ab-\bar{a}\bar{b})\right]\nabla\tilde{u}\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}$ along with (4.1.41)-(4.1.44) and (4.1.45) to deal with the pressure, we get that:

$$\begin{aligned}
&\|\tilde{u}\|_{L^\infty_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}+a_\star b_\star C\left\|\nabla^2\tilde{u}\right\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}\\
&\lesssim \left\|\tilde{f}\right\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}+C\int_0^t\left\|a\nabla\tilde{P}\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\\
&+\int_0^t\left\|\left(\dot{S}_m(b\nabla a),\dot{S}_m(a\nabla b)\right)\right\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\nabla\tilde{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+T_m(a,b)\left\|\nabla^2\tilde{u}\right\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}\\
&\le C_{ab}(1+t)\left(\|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\|(f,\partial_tR,\nabla\operatorname{div}R)\|_{L^1_t(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}\right) &(4.1.47)\\
&\quad+C_{ab}\int_0^t\|\tilde{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+(T_m(a,b)+\eta)\left\|\nabla^2\tilde{u}\right\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}. &(4.1.48)
\end{aligned}$$

where

$$\begin{aligned}
T_m(a,b)=&\left\|(Id-\dot{S}_m)(b\nabla a)\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\left\|(Id-\dot{S}_m)(a\nabla b)\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\\
&+\left\|(Id-\dot{S}_m)(ab-\bar{a}\bar{b})\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}.
\end{aligned}$$

Assuming that m is large enough and η is small enough, we can "absorb" $(T_m(a,b)+\eta)\left\|\nabla^2u\right\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}$ in the left hand side of (4.1.47). Thus, we end up with

$$\begin{aligned}
&\|\tilde{u}\|_{L^\infty_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}+a_\star b_\star\frac{C}{2}\left\|\nabla^2\tilde{u}\right\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}\\
&\le C_{ab}(1+t)\left(\|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\|(f,\partial_tR,\nabla\operatorname{div}R)\|_{L^1_t(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}\right)\\
&+C_{ab}\int_0^t\|\tilde{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}
\end{aligned}$$

such that using Gronwall's lemma, (4.1.17) and the classical inequality

$$1+t^\alpha\le C_\alpha\exp(t)$$

yields:

$$\begin{aligned}
&\|\tilde{u}\|_{L^\infty_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}+a_\star b_\star\frac{C}{2}\left\|\nabla^2\tilde{u}\right\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})} &(4.1.49)\\
&\le C_{ab}\exp(C_{ab}t)\left(\|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\|(f,\partial_tR,\nabla\operatorname{div}R)\|_{L^1_t(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}\right). &(4.1.50)
\end{aligned}$$

Using the fact that $u=u_L+\tilde{u}$ along with (4.1.17) and (4.1.49) gives us:

$$\|u\|_{L^\infty_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}+a_\star b_\star\frac{C}{2}\left\|\nabla^2u\right\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})} \qquad (4.1.51)$$

$$\leq C_{ab}\exp\left(C_{ab}t\right)\left(\|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\|(f,\partial_t R,\nabla\operatorname{div}R)\|_{L^1_t(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}\right). \tag{4.1.52}$$

Next, using (4.1.41)-(4.1.44) and (4.1.46) combined with (4.1.17), we infer that:

$$\|\nabla P\|_{L^1_t((\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}\leq C_a\|a\nabla P_L\|_{L^1_t(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}+C_a\left\|a\nabla\tilde{P}\right\|_{L^1_t(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})} \tag{4.1.53}$$

$$\leq C_{ab}\exp\left(C_{ab}t\right)\left(\|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\|(f,\partial_t R,\nabla\operatorname{div}R)\|_{L^1_t(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}\right). \tag{4.1.54}$$

Combing (4.1.54) with (4.1.51) we finally get that:

$$\begin{aligned}&\|u\|_{L^\infty_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}+\left\|\nabla^2u\right\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}+\|\nabla P\|_{L^1_t((\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}\\&\leq e^{C_{ab}(t+1)}\left(\|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\|(f,\partial_t R,\nabla\operatorname{div}R)\|_{L^1_t(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}\right).\end{aligned} \tag{4.1.55}$$

Obviously, by obtaining the last estimate we conclude the proof of Proposition 4.1.8.

Next, let us deal with the existence part of the Stokes problem with the coefficients having regularity as in Proposition 4.1.8. More precisely, we have:

Proposition 4.1.9. *Let us consider (a,b,u_0,f,R) as in the statement of Proposition 4.1.8. Then, there exists a unique solution $(u,\nabla P)\in E_{loc}$ of the Stokes system (4.1.1). Furthermore, there exists a constant C_{ab} depending on a and b such that:*

$$\begin{aligned}&\|u\|_{L^\infty_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}+\left\|\nabla^2u\right\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}+\|\nabla P\|_{L^1_t((\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}\\&\leq\left(\|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\|(f,\partial_t R,\nabla\operatorname{div}R)\|_{L^1_t(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}\right)\exp\left(C_{ab}\left(t+1\right)\right).\end{aligned} \tag{4.1.56}$$

for all $t>0$.

The uniqueness property is a direct consequence of the estimates of Proposition 4.1.8. The proof of existence relies on Proposition 4.1.8 combined with a continuity argument as used in [46], see also [68]. Let us introduce

$$(a_\theta,b_\theta)=(1-\theta)\left(\bar{a},\bar{b}\right)+\theta\left(a,b\right)$$

and let us consider the following Stokes systems

$$\begin{cases}\partial_t u-a_\theta(\operatorname{div}(b_\theta D(u))-\nabla P)=f,\\ \operatorname{div}u=\operatorname{div}R,\\ u_{|t=0}=u_0.\end{cases} \tag{$\mathcal{S}_\theta$}$$

First of all, a more detailed analysis of the estimates established in Proposition 4.1.8 enables us to conclude that the constant $C_{a_\theta b_\theta}$ appearing in (4.1.55) is uniformly bounded with respect to $\theta\in[0,1]$ by a constant $c=c_{ab}$. Indeed, when repeating the estimation process carried out in Proposition 4.1.8 with (a_θ,b_θ) instead of (a,b) amounts in replacing $(a-\bar{a})$ and $\left(b-\bar{b}\right)$ with $\theta\left(a-\bar{a}\right)$ and $\theta\left(b-\bar{b}\right)$. Taking in account Proposition 1.3.9 and the remark that follows we get that there exists

$$c:=\sup_{\theta\in[0,1]}C_{a_\theta b_\theta}<\infty.$$

Let us take $T>0$ and let us consider $\mathcal{E}_T$ the set of those $\theta\in[0,1]$ such that for any (u_0,f,R) as in the statement of Proposition 4.1.8 Problem $(\mathcal{S}_\theta)$ admits a unique solution $(u,\nabla P)\in E_T$ which satisfies

$$\|u\|_{L^\infty_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}+\left\|\nabla^2u\right\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}+\|\nabla P\|_{L^1_t((\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}$$

$$\leq (\|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L^1_t(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2} \cap \dot{B}^{\frac{n}{p}-1}_{p,1})})(\exp(c(t+1)), \tag{4.1.57}$$

for all $t \in [0,T]$. According to Proposition 4.1.6, $0 \in \mathcal{E}_T$.

Let us suppose that $\theta \in \mathcal{E}_T$. First of all, we denote by $(u_\theta, \nabla P_\theta) \in E_T$ the unique solution of $(\mathcal{S}_\theta)$. We consider the space

$$E_{T,\mathrm{div}} = \{\left(\tilde{w}, \nabla\tilde{Q}\right) \in E_T : \operatorname{div} \tilde{w} = 0\}$$

and let $S_{\theta\theta'}$ be the operator which associates to $\left(\tilde{w}, \nabla\tilde{Q}\right) \in E_{T,\mathrm{div}}$, $\left(\tilde{u}, \nabla\tilde{P}\right)$ the unique solution of

$$\begin{cases} \partial_t \tilde{u} - a_\theta(\operatorname{div}(b_\theta D(\tilde{u})) - \nabla\tilde{P}) = g_{\theta\theta'}(u_\theta, \nabla P_\theta) + g_{\theta\theta'}\left(\tilde{w}, \nabla\tilde{Q}\right), \\ \qquad \operatorname{div} u = 0, \\ \qquad u_{|t=0} = 0. \end{cases} \tag{4.1.58}$$

where

$$g_{\theta\theta'}(u, \nabla P) = (a_\theta - a_{\theta'})\nabla P + a_{\theta'} \operatorname{div}(b_{\theta'} D(u)) - a_\theta \operatorname{div}(b_\theta D(u)). \tag{4.1.59}$$

Obviously, $S_{\theta\theta'}$ maps $E_{T,\mathrm{div}}$ into $E_{T,\mathrm{div}}$. We claim that there exists a positive quantity $\varepsilon = \varepsilon(T) > 0$ such that if $|\theta - \theta'| \leq \varepsilon(T)$ then $S_{\theta\theta'}$ has a fixed point $\left(\tilde{u}^\star, \nabla\tilde{P}^\star\right)$ in a suitable ball centered at the origin of the space $E_{T,\mathrm{div}}$. Obviously,

$$\left(\tilde{u}^\star + u_\theta, \nabla\tilde{P}^\star + \nabla P_\theta\right)$$

will solve $(\mathcal{S}_{\theta'})$ in E_T.

First, we note that, as a consequence of Proposition 4.1.8, we have that:

$$\left\|\left(\tilde{u}, \nabla\tilde{P}\right)\right\|_{E_T} \leq (\|g_{\theta\theta'}(u_\theta, \nabla P_\theta)\|_{L^1_T(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2} \cap \dot{B}^{\frac{n}{p}-1}_{p,1})} + \left\|g_{\theta\theta'}\left(\tilde{w}, \nabla\tilde{Q}\right)\right\|_{L^1_T(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2} \cap \dot{B}^{\frac{n}{p}-1}_{p,1})})e^{c(T+1)}. \tag{4.1.60}$$

Let us observe that

$$\|(a_\theta - a_{\theta'})\nabla P\|_{L^1_T(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2} \cap \dot{B}^{\frac{n}{p}-1}_{p,1})} \leq |\theta - \theta'| \|a - \bar{a}\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \|\nabla P\|_{L^1_T(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2} \cap \dot{B}^{\frac{n}{p}-1}_{p,1})}. \tag{4.1.61}$$

Next, we write that:

$$a_{\theta'} \operatorname{div}(b_{\theta'} D(u)) - a_\theta \operatorname{div}(b_\theta D(u)) = (a_{\theta'} - a_\theta)\operatorname{div}(b_{\theta'} D(u)) + a_\theta \operatorname{div}((b_{\theta'} - b_\theta) D(u)).$$

The first term of the last identity is estimated as follows:

$$\|(a_{\theta'} - a_\theta)\operatorname{div}(b_{\theta'} D(u))\|_{L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})} \leq |\theta - \theta'| \|a - \bar{a}\|_{\dot{B}^{\frac{n}{p}}_{p,1}} (\bar{b} + \left\|b - \bar{b}\right\|_{\dot{B}^{\frac{n}{p}}_{p,1}}) \|D(u)\|_{L^1_T(\dot{B}^{\frac{n}{p}}_{p,1})}.$$

Regarding the second term, we have that:

$$\|a_\theta \operatorname{div}((b_{\theta'} - b_\theta) D(u))\|_{L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})} \leq |\theta - \theta'| \left\|b - \bar{b}\right\|_{\dot{B}^{\frac{n}{p}}_{p,1}} (\bar{a} + \|a - \bar{a}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}) \|D(u)\|_{L^1_T(\dot{B}^{\frac{n}{p}}_{p,1})}$$

and thus:

$$\begin{aligned} &\|a_{\theta'} \operatorname{div}(b_{\theta'} D(u)) - a_\theta \operatorname{div}(b_\theta D(u))\|_{L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})} \\ &\leq |\theta - \theta'| (\bar{a} + \|a - \bar{a}\|_{\dot{B}^{\frac{n}{p}}_{p,1}})(\bar{b} + \left\|b - \bar{b}\right\|_{\dot{B}^{\frac{n}{p}}_{p,1}}) \|Du\|_{L^1_T(\dot{B}^{\frac{n}{p}}_{p,1})}. \end{aligned} \tag{4.1.62}$$

The only thing left is to treat the $L^1_t(\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2})$-norm of $a_{\theta'}\operatorname{div}(b_{\theta'}D(u)) - a_\theta \operatorname{div}(b_\theta D(u))$ in the case where $n=3$. Using the fact that $\nabla u \in L^{\frac{4}{3}}_T(\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,1})$ write that:

$$\begin{aligned}
&\|(a_{\theta'}-a_\theta)\operatorname{div}(b_{\theta'}D(u))\|_{L^1_T(\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2})}\\
&\leq |\theta-\theta'|\,\|a-\bar{a}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\,\|\operatorname{div}(b_{\theta'}D(u))\|_{L^1_T(\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,1})}\\
&\leq |\theta-\theta'|\,\|a-\bar{a}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\,(\bar{b}+\left\|b-\bar{b}\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}})\,\|Du\|_{L^1_T(\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,1})}\\
&\leq |\theta-\theta'|\,\|a-\bar{a}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\,(\bar{b}+\left\|b-\bar{b}\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}})T^{\frac{1}{4}}\,\|u\|^{\frac{1}{4}}_{L^\infty_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}\,\|u\|^{\frac{3}{4}}_{L^1_T(\dot{B}^{\frac{3}{p}+1}_{p,1})} &(4.1.63)\\
&\leq |\theta-\theta'|\,C(T,a,b)\,(\|u\|_{L^\infty_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}+\left\|\nabla^2 u\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}) &(4.1.64)
\end{aligned}$$

and, proceeding in a similar manner we can estimate $\|a_\theta \operatorname{div}((b_{\theta'}-b_\theta)D(u))\|_{L^1_T(\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2})}$.

Combining (4.1.61), (4.1.62) along with (4.1.64) we get that:

$$\|g_{\theta\theta'}(u,\nabla P)\|_{L^1_T(\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}\cap\dot{B}^{\frac{3}{p}-1}_{p,1})} \leq |\theta-\theta'|\,C_{T,a,b}(\|u\|_{L^\infty_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}+\left\|\nabla^2 u\right\|_{L^1_T(\dot{B}^{\frac{3}{p}+1}_{p,1})}+\|\nabla P\|_{L^1_T(\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}\cap\dot{B}^{\frac{3}{p}-1}_{p,1})}). \tag{4.1.65}$$

Let us replace this into (4.1.60) to get that

$$\left\|\left(\tilde{u},\nabla\tilde{P}\right)\right\|_{E_T} \leq |\theta-\theta'|\,C(T,a,b)\,(\|(u_\theta,\nabla P_\theta)\|_{E_T}+\left\|\left(\tilde{w},\nabla\tilde{Q}\right)\right\|_{E_T})$$

and by linearity

$$\left\|\left(\tilde{u}^1-\tilde{u}^2,\nabla\tilde{P}^1-\nabla\tilde{P}^2\right)\right\|_{E_T} \leq |\theta-\theta'|\,C(T,a,b)\left\|\left(\tilde{w}^1-\tilde{w}^2,\nabla\tilde{Q}^1-\nabla\tilde{Q}^2\right)\right\|_{E_T}$$

where for $k=1,2$:

$$\left(\tilde{u}^i,\nabla\tilde{P}^i\right) = S_{\theta\theta'}\left(\left(\tilde{w}^i,\nabla\tilde{Q}^i\right)\right)$$

Thus one can choose $\varepsilon(T)$ small enough such that $|\theta-\theta'|\leq\varepsilon(T)$ gives us a fixed point of the solution operator $S_{\theta\theta'}$ in $B_{E_{T,\operatorname{div}}}\left(0,2\,\|(u_\theta,\nabla P_\theta)\|_{E_T}\right)$.

Thus, for all $T>0$, $E_T=[0,1]$ and owing to the uniqueness property and to Proposition 4.1.8, we can construct a unique solution $(u,\nabla P)\in E_{loc}$ to (4.1.1) such that for all $t>0$ the estimate (4.1.13) is valid. This ends the proof of Proposition 4.1.9.

4.1.4 The proof of Theorem 4.1.1 in the case $n=3$

As it was discussed earlier, in dimension $n=3$, Proposition 4.1.8 is weaker than Theorem 4.1.1 as one requires additional low frequency informations on the data $(f,\partial_t R,\nabla\operatorname{div} R)\in L^1_t(\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2})$. Thus, we have to bring an extra argument in order to conclude the validity of Theorem 4.1.1. This is the object of interest of this section.

The existence part

We begin by taking $m\in\mathbb{N}$ large enough and owing to Proposition 4.1.7 we can consider $\left(u^1,\nabla P^1\right)$ the unique solution with $u^1\in\mathcal{C}(\mathbb{R}^+;\dot{B}^{\frac{3}{p}-1}_{p,1})$ and $\left(\partial_t u^1,\nabla^2 u^1,\nabla P^1\right)\in L^1_{loc}(\dot{B}^{\frac{3}{p}-1}_{p,1})$ of the system

$$\begin{cases}
\partial_t u - \bar{a}\bar{b}\operatorname{div} D(u) + \left(\bar{a}+\dot{S}_{-m}(a-\bar{a})\right)\nabla P = f,\\
\operatorname{div} u = \operatorname{div} R,\\
u_{|t=0} = u_0,
\end{cases}$$

which also satisfies:

$$\left\|u^1\right\|_{L^\infty_T(\dot B^{\frac3p-1}_{p,1})}+\left\|\left(\partial_t u^1,\bar a\bar b\nabla^2 u^1,\bar a\nabla P^1\right)\right\|_{L^1_T(\dot B^{\frac3p-1}_{p,1})}\le C(\|u_0\|_{\dot B^{\frac3p-1}_{p,1}}+\left\|\left(f,\partial_t R,\bar a\bar b\nabla\operatorname{div}R\right)\right\|_{L^1_T(\dot B^{\frac3p-1}_{p,1})}),$$

for all $T>0$. Let us consider

$$G\left(u^1,\nabla P^1\right)=a\operatorname{div}\left(bD(u^1)\right)-\bar a\operatorname{div}(\bar bD(u^1))-((Id-\dot S_{-m})(a-\bar a))\nabla P^1.$$

We claim that $G\left(u^1,\nabla P^1\right)\in L^1_{loc}(\dot B^{\frac3p-\frac32}_{p,2}\cap\dot B^{\frac3p-1}_{p,1})$. Indeed

$$a\operatorname{div}\left(bD(u^1)\right)-\bar a\operatorname{div}(\bar bD(u^1))=(a-\bar a)\operatorname{div}\left(bD(u^1)\right)+\bar a\operatorname{div}\left(\left(b-\bar b\right)D\left(u^1\right)\right)$$

and proceeding as in (4.1.62) and (4.1.63) we get that

$$\begin{aligned}&\left\|a\operatorname{div}\left(bD(u^1)\right)-\bar a\operatorname{div}(\bar bD(u^1))\right\|_{L^1_t(\dot B^{\frac3p-\frac32}_{p,2}\cap\dot B^{\frac3p-1}_{p,1})}\\&\le C_{ab}(1+t^{\frac14})(\left\|u^1\right\|_{L^\infty_t(\dot B^{\frac3p-1}_{p,1})}+\left\|u^1\right\|_{L^1_t(\dot B^{\frac3p+1}_{p,1})})\\&\le e^{C_{ab}(t+1)}(\|u_0\|_{\dot B^{\frac3p-1}_{p,1}}+\|(f,\partial_tR,\nabla\operatorname{div}R)\|_{L^1_T(\dot B^{\frac3p-1}_{p,1})}).\end{aligned}\tag{4.1.66}$$

Next, we obviously have

$$\left\|((Id-\dot S_{-m})(a-\bar a))\nabla P^1\right\|_{L^1_t(\dot B^{\frac3p-1}_{p,1})}\le C\left\|(a-\bar a)\right\|_{\dot B^{\frac3p}_{p,1}}\left\|\nabla P^1\right\|_{L^1_t(\dot B^{\frac3p-1}_{p,1})}.\tag{4.1.67}$$

Using the fact that the product maps $\dot B^{\frac3p-\frac12}_{p,1}\times\dot B^{\frac3p-1}_{p,1}\to\dot B^{\frac3p-\frac32}_{p,2}$ we get that:

$$\left\|((Id-\dot S_{-m})(a-\bar a))\nabla P^1\right\|_{L^1_t(\dot B^{\frac3p-\frac32}_{p,2})}\le C\left\|(Id-\dot S_{-m})(a-\bar a)\right\|_{\dot B^{\frac3p-\frac12}_{p,1}}\left\|\nabla P^1\right\|_{L^1_t(\dot B^{\frac3p-1}_{p,1})}.\tag{4.1.68}$$

Of course

$$\begin{aligned}\left\|(Id-\dot S_{-m})(a-\bar a)\right\|_{\dot B^{\frac3p-\frac12}_{p,1}}&\le C\sum_{j\ge-m}2^{j\left(\frac3p-\frac12\right)}\left\|\dot\Delta_j(a-\bar a)\right\|_{L^2}\le C2^{\frac m2}\sum_{j\ge-m}2^{\frac3pj}\left\|\dot\Delta_j(a-\bar a)\right\|_{L^2}.\\&\le C2^{\frac m2}\left\|a-\bar a\right\|_{\dot B^{\frac3p}_{p,1}}\end{aligned}$$

so that the first term in the right hand side of (4.1.68) is finite. We thus gather from (4.1.66), (4.1.67) and (4.1.68) that $G\left(u^1,\nabla P^1\right)\in L^1_{loc}(\dot B^{\frac3p-\frac32}_{p,2}\cap\dot B^{\frac3p-1}_{p,1})$ and that for all $t>0$ there exists a constant C_{ab} such that

$$\left\|G\left(u^1,\nabla P^1\right)\right\|_{L^1_t(\dot B^{\frac3p-\frac32}_{p,2}\cap\dot B^{\frac3p-1}_{p,1})}\le e^{C_{ab}(t+1)}(\|u_0\|_{\dot B^{\frac3p-1}_{p,1}}+\|(f,\partial_tR,\nabla\operatorname{div}R)\|_{L^1_t(\dot B^{\frac3p-1}_{p,1})}).$$

According to Proposition 4.1.9, there exists a unique solution $\left(u^2,\nabla P^2\right)\in E_{loc}$ of the system:

$$\begin{cases}\partial_tu-a\operatorname{div}(bD(u))+a\nabla P=G\left(u^1,\nabla P^1\right),\\ \operatorname{div}u=0,\\ u_{|t=0}=0,\end{cases}$$

which satisfies the following estimate

$$\left\|u^2\right\|_{L^\infty_t(\dot B^{\frac3p-1}_{p,1})}+\left\|\left(\nabla^2u^2,\nabla P^2\right)\right\|_{L^1_t(\dot B^{\frac3p-1}_{p,1})}$$

$$
\begin{aligned}
&\leq \left\|G\left(u^1, \nabla P^1\right)\right\|_{L_t^1(\dot{B}_{p,2}^{\frac{3}{p}-\frac{3}{2}} \cap \dot{B}_{p,1}^{\frac{3}{p}-1})} \exp\left(C_{ab}\left(t+1\right)\right)\\
&\leq (\|u_0\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L_t^1(\dot{B}_{p,1}^{\frac{3}{p}-1})}) \exp\left(C_{ab}\left(t+1\right)\right).
\end{aligned}
$$

We observe that

$$
(u, \nabla P) := \left(u^1 + u^2, \nabla P^1 + \nabla P^2\right)
$$

is a solution of (4.1.1) which satisfies

$$
\|u\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{3}{p}-1})} + \left\|\left(\nabla^2 u, \nabla P\right)\right\|_{L_t^1(\dot{B}_{p,1}^{\frac{3}{p}-1})} \leq e^{C_{ab}(t+1)}(\|u_0\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L_t^1(\dot{B}_{p,1}^{\frac{3}{p}-1})}). \tag{4.1.69}
$$

Of course, using again the first equation of (4.1.1) we get that

$$
\|\partial_t u\|_{L_t^1(\dot{B}_{p,1}^{\frac{3}{p}-1})} \leq C_{ab}\left\|\left(f, \nabla^2 u, \nabla P\right)\right\|_{L_t^1(\dot{B}_{p,1}^{\frac{3}{p}-1})}
$$

and thus, we get the estimate

$$
\|u\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{3}{p}-1})} + \left\|\left(\partial_t u, \nabla^2 u, \nabla P\right)\right\|_{L_t^1(\dot{B}_{p,1}^{\frac{3}{p}-1})} \leq e^{C_{ab}(t+1)}(\|u_0\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L_t^1(\dot{B}_{p,1}^{\frac{3}{p}-1})}). \tag{4.1.70}
$$

Uniqueness

Next, let us prove the uniqueness property. Let us suppose that there exists a $T>0$ and a pair $(u, \nabla P)$ that solves

$$
\begin{cases}
\partial_t u - a \operatorname{div}(bD(u)) + a\nabla P = 0,\\
\operatorname{div} u = 0,\\
u_{|t=0} = 0,
\end{cases} \tag{4.1.71}
$$

with

$$
u \in C_T(\dot{B}_{p,1}^{\frac{3}{p}-1}) \text{ and } \left(\partial_t u, \nabla^2 u, \nabla P\right) \in L_T^1(\dot{B}_{p,1}^{\frac{3}{p}-1}).
$$

Observe that we cannot directly conclude to the uniqueness property by appealing to Proposition 4.1.9 because the pressure does not belong (a priori) to $L_T^1(\dot{B}_{p,2}^{\frac{3}{p}-\frac{3}{2}})$. Recovering this low frequency information is done in the following lines. Let us suppose that $3<p<4$. Applying the operator $\mathcal{Q}$ in the first equation of (4.1.71) we write that:

$$
\mathcal{Q}((\bar{a} + \dot{S}_{-m}(a-\bar{a}))\nabla P) = Q(a \operatorname{div}(bD(u))) - \mathcal{Q}((Id - \dot{S}_{-m})\left(a-\bar{a}\right)\nabla P)
$$

where $m \in \mathbb{N}$ will be fixed later. We observe that:

$$
\begin{aligned}
&\left\|\mathcal{Q}(\bar{a} + \dot{S}_{-m}(a-\bar{a}))\nabla P)\right\|_{L_T^1(\dot{B}_{p,1}^{\frac{3}{p}-\frac{3}{2}})}\\
&\lesssim \|\mathcal{Q}(a \operatorname{div}(bD(u)))\|_{L_T^1(\dot{B}_{p,1}^{\frac{3}{p}-\frac{3}{2}})} + \left\|\mathcal{Q}((Id - \dot{S}_{-m})(a-\bar{a})\nabla P)\right\|_{L_T^1(\dot{B}_{p,1}^{\frac{3}{p}-\frac{3}{2}})}\\
&\lesssim T^{\frac{1}{4}}(\bar{a} + \|a-\bar{a}\|_{\dot{B}_{p,1}^{\frac{3}{p}}})(\bar{b} + \left\|b-\bar{b}\right\|_{\dot{B}_{p,1}^{\frac{3}{p}}})\left\|\nabla u\right\|_{L_T^{\frac{4}{3}}(\dot{B}_{p,1}^{\frac{3}{p}-\frac{1}{2}})}\\
&+ \left\|\left(Id - \dot{S}_{-m}\right)(a-\bar{a})\right\|_{\dot{B}_{p,1}^{\frac{3}{p}-\frac{1}{2}}} \|\nabla P\|_{L_T^1(\dot{B}_{p,1}^{\frac{3}{p}-1})}.
\end{aligned}
$$

Consequently, we get that

$$
\mathcal{Q}((\bar{a} + \dot{S}_{-m}(a-\bar{a}))\nabla P) \in L_T^1(\dot{B}_{p,1}^{\frac{3}{p}-\frac{3}{2}}). \tag{4.1.72}
$$

Let us observe that the condition $p \in (3,4)$ ensures that $\dot{B}_{p,1}^{\frac{3}{p}}$ is contained in the multiplier space of $\dot{B}_{p',2}^{-\frac{3}{p}+1} = \dot{B}_{p',2}^{\frac{3}{p'}-2}$. More precisely, we get

Proposition 4.1.10. *Let us consider $p\in(3,4)$ and $(u,v)\in \dot{B}^{\frac{3}{p}}_{p,1}\times \dot{B}^{-\frac{3}{p}+1}_{p',2}$. Then $uv\in \dot{B}^{-\frac{3}{p}+1}_{p',2}$ and*

$$\|uv\|_{\dot{B}^{-\frac{3}{p}+1}_{p',2}}\lesssim \|u\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\|v\|_{\dot{B}^{-\frac{3}{p}+1}_{p',2}}.$$

Proof. Indeed, considering $(u,v)\in \dot{B}^{\frac{3}{p}}_{p,1}\times \dot{B}^{-\frac{3}{p}+1}_{p',2}$ and using the Bony decomposition we get that

$$\left\|\dot{T}_u v\right\|_{\dot{B}^{-\frac{3}{p}+1}_{p',2}}\lesssim \|u\|_{L^\infty}\|v\|_{\dot{B}^{-\frac{3}{p}+1}_{p',2}}.$$

Next, considering

$$\frac{1}{p'}=\frac{1}{2}+\frac{1}{p^\star}.$$

we see that

$$\begin{aligned}2^{j\left(-\frac{3}{p}+1\right)}\left\|\dot{\Delta}_j\dot{T}'_v u\right\|_{L^{p'}} &\lesssim \sum_{\ell\geq j-3}2^{\left(-\frac{3}{p}+1\right)(j-\ell)}2^{\left(-\frac{3}{p}+1\right)\ell}\|S_{\ell+1}v\|_{L^2}\left\|\dot{\Delta}_\ell u\right\|_{L^{p^\star}}\\ &=\sum_{\ell\geq j-3}2^{\left(-\frac{3}{p}+1\right)(j-\ell)}2^{-\frac{1}{2}\ell}\|S_{\ell+1}v\|_{L^2}\,2^{\frac{3}{p^\star}\ell}\left\|\dot{\Delta}_\ell u\right\|_{L^{p^\star}},\end{aligned}$$

such that, with the help of Proposition 1.3.7, we get

$$\left\|\dot{T}'_v u\right\|_{\dot{B}^{-\frac{3}{p}+1}_{p',2}}\lesssim \|v\|_{\dot{H}^{-\frac{1}{2}}}\|u\|_{\dot{B}^{\frac{3}{p^\star}}_{p^\star,1}}\lesssim \|v\|_{\dot{B}^{-\frac{3}{p}+1}_{p',2}}\|u\|_{\dot{B}^{\frac{3}{p}}_{p,1}}.$$

□

Proposition 4.1.11. *Let us consider $p\in(3,4)$. Furthermore, consider a constant $\bar{c}>0$ and $c\in\dot{B}^{\frac{3}{p}}_{p,1}$. Then there exists an universal constant $\eta>0$ such that if*

$$\|c\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\leq\eta,$$

then for any $\psi\in \dot{B}^{\frac{3}{p'}-\frac{3}{2}}_{p',2}\cap\dot{B}^{\frac{3}{p'}-2}_{p',2}$ there exists a unique solution $\nabla P\in \dot{B}^{\frac{3}{p'}-\frac{3}{2}}_{p',2}\cap\dot{B}^{\frac{3}{p'}-2}_{p',2}$ of the elliptic equation

$$\operatorname{div}\left((\bar{c}+c)\nabla P\right)=\operatorname{div}\psi.$$

Moreover, the following estimate holds true

$$\|\nabla P\|_{\dot{B}^{\frac{3}{p'}-\sigma}_{p',2}}\lesssim\|\mathcal{Q}\psi\|_{\dot{B}^{\frac{3}{p'}-\sigma}_{p',2}},$$

where $\sigma\in\left\{\frac{3}{2},2\right\}$.

Proof. The proof is standard. Under some smallness condition on $c\in\dot{B}^{\frac{3}{p}}_{p,1}$ the operator

$$\nabla R\to\nabla P=\frac{1}{\bar{c}}\mathcal{Q}\left(\psi-c\nabla R\right)$$

has a fixed point in a suitable chosen ball of the space $\dot{B}^{\frac{3}{p'}-\frac{3}{2}}_{p',2}\cap\dot{B}^{\frac{3}{p'}-2}_{p',2}$. □

We choose $m \in \mathbb{N}$ such that $\left\|\dot{S}_{-m}(a-\bar{a})\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}}$ is small enough such that we can apply Proposition 4.1.11 with $\bar{a}$ and $\dot{S}_{-m}(a-\bar{a})$ instead of $\bar{c}$ and c. Let us consider ψ a vector field with coefficients in $\mathcal{S}$. As the Schwartz class is included in $\dot{B}^{\frac{3}{p'}-\frac{3}{2}}_{p',2} \cap \dot{B}^{\frac{3}{p'}-2}_{p',2}$, let us consider $\nabla P_\psi \in \dot{B}^{\frac{3}{p'}-\frac{3}{2}}_{p',2} \cap \dot{B}^{\frac{3}{p'}-2}_{p',2}$ the solution of the equation

$$\operatorname{div}\left(\left(\bar{a}+\dot{S}_{-m}(a-\bar{a})\right)\nabla P_\psi\right) = \operatorname{div}\psi,$$

the existence of which is granted by Proposition 4.1.11. Then, using Proposition 1.3.5 and Proposition 1.3.6, we write that[3]:

$$\begin{aligned}
\langle \nabla P, \psi\rangle_{\mathcal{S}'\times\mathcal{S}} &= \sum_j \left\langle \dot{\Delta}_j \nabla P, \tilde{\Delta}_j \psi\right\rangle = \sum_j -\left\langle \dot{\Delta}_j P, \tilde{\Delta}_j \operatorname{div}\psi\right\rangle && (4.1.73)\\
&= \sum_j -\left\langle \dot{\Delta}_j P, \tilde{\Delta}_j \operatorname{div}\left(\left(\bar{a}+\dot{S}_{-m}(a-\bar{a})\right)\nabla P_\psi\right)\right\rangle && (4.1.74)\\
&= \sum_j \left\langle \dot{\Delta}_j \nabla P, \tilde{\Delta}_j \left(\left(\bar{a}+\dot{S}_{-m}(a-\bar{a})\right)\nabla P_\psi\right)\right\rangle && (4.1.75)\\
&= \sum_j \left\langle \dot{\Delta}_j \left(\bar{a}+\dot{S}_{-m}(a-\bar{a})\right)\nabla P, \tilde{\Delta}_j \nabla P_\psi\right\rangle && (4.1.76)\\
&= \sum_j \left\langle \dot{\Delta}_j \mathcal{Q}\left(\left(\bar{a}+\dot{S}_{-m}(a-\bar{a})\right)\nabla P\right), \tilde{\Delta}_j \nabla P_\psi\right\rangle && (4.1.77)\\
&\lesssim \left\|\mathcal{Q}\left(\left(\bar{a}+\dot{S}_{-m}(a-\bar{a})\right)\nabla P\right)\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} \|\nabla P_\psi\|_{\dot{B}^{\frac{3}{p'}-\frac{3}{2}}_{p',1}} && (4.1.78)\\
&\lesssim \left\|\mathcal{Q}\left(\left(\bar{a}+\dot{S}_{-m}(a-\bar{a})\right)\nabla P\right)\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} \|\psi\|_{\dot{B}^{\frac{3}{p'}-\frac{3}{2}}_{p',1}}. && (4.1.79)
\end{aligned}$$

Taking the supremum over all $\psi \in \mathcal{S}$ with $\|\psi\|_{\dot{B}^{\frac{3}{p'}-\frac{3}{2}}_{p',2}} \leq 1$, owing to (4.1.72) and Proposition 1.3.5, it follows that $\nabla P \in L^1_T(\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2})$ and that

$$\|\nabla P\|_{L^1_T(\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2})} \lesssim \left\|\mathcal{Q}(\bar{a}+\dot{S}_{-m}(a-\bar{a}))\nabla P)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2})}.$$

According to the uniqueness property of Proposition 4.1.9 we conclude that $(u, \nabla P) = (0,0)$.

Let us observe that in the case $p \in \left(\frac{6}{5}, 3\right]$, owing to the fact that $\dot{B}^{\frac{3}{p}-1}_{p,1} \hookrightarrow \dot{B}^{\frac{3}{q}-1}_{q,1}$ for any $q \in (3,4)$ and $u \in C_T(\dot{B}^{\frac{3}{p}-1}_{p,1})$ along with $\left(\partial_t u, \nabla^2 u, \nabla P\right) \in L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})$ we get that $u \in C_T(\dot{B}^{\frac{3}{q}-1}_{q,1})$ along with $\left(\partial_t u, \nabla^2 u, \nabla P\right) \in L^1_T(\dot{B}^{\frac{3}{q}-1}_{q,1})$. Thus, owing to the uniqueness property for the case $q \in (3,4)$ we conclude that $(u, \nabla P)$ is identically null for $p \in \left(\frac{6}{5}, 3\right]$.

4.2 Applications of Theorem 4.2.1

In this section we will apply the result established in the previous section in the study of the well-posedness of the inhomogeneous, incompressible Navier-Stokes-Korteweg system:

$$\left\{\begin{aligned}
\partial_t \rho + \operatorname{div}(\rho u) = 0,\\
\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu(\rho) D(u)) + \nabla P + \gamma \operatorname{div}\mathcal{K} = 0,\\
\operatorname{div} u = 0,\\
u_{|t=0} = u_0.
\end{aligned}\right. \tag{4.2.1}$$

[3] We denote $\tilde{\Delta}_j := \dot{\Delta}_{j-1} + \dot{\Delta}_j + \dot{\Delta}_{j+1}$.

where, $\mathcal{K}$ is the Korteweg tensor:

$$\mathcal{K} = -\operatorname{div}\left(k\left(\rho\right)\nabla\rho\otimes\nabla\rho\right) \tag{4.2.2}$$

and $\gamma\in\{0,1\}$. Of course, this simplified form of $\mathcal{K}$ comes from the incompressibility assumption which allows us to absorb into the pressure the potential part. In the above, $\rho>0$ stands for the density of the fluid, $u\in\mathbb{R}^n$ is the fluid's velocity field while P is the pressure. The viscosity coefficient $\mu=\mu\left(\rho\right)$ is supposed to be a smooth, strictly positive function of the density. The capillarity coefficient $k=k\left(\rho\right)$ is a smooth function of the density while

$$D\left(u\right)=\nabla u+Du$$

is the deformation tensor. This system is used to study fluids obtained as a mixture of two (or more) incompressible fluids that have different densities: fluids containing a melted substance, polluted air/water etc.

We aim at proving local well-posedness with initial data:

$$\begin{cases} \rho_0-\bar{\rho}\in\dot{B}^{\frac{n}{p}+\gamma}_{p,1}\left(\mathbb{R}^n\right),\, u_0\in\dot{B}^{\frac{n}{p}-1}_{p,1}\left(\mathbb{R}^n\right), \\ p\in\left(1,4\right) \text{ if } n=2, \\ p\in\left(\frac{6}{5},4\right) \text{ if } n=3. \end{cases}$$

for system (4.2.1)

- with general smooth variable viscosity and capillarity laws $\mu=\mu\left(\rho\right)>0$, $k=k\left(\rho\right)$,
- without any smallness assumption on the density i.e. we authorize arbitrarily large variations of the density around a constant state,

When $\gamma=0$, then system (4.2.1) is the classical Navier-Stokes system for an incompressible, inhomogeneous viscous fluid. In dimension 2, the well-posedness result in critical spaces was first proved in [86] by H. Xu, Y. Li and X. Zhai. In the 3 dimensional situation, this result in proved in [28] thus achieving the critical regularity (see also [88] for a different approach).

As was mentioned in the introduction, when $\gamma=1$, this system is less studied than its compressible counterpart. The only well-posedness results that we are aware of are those presented in [84] respectively in [87] which treat the case of constant viscosity and capillarity. In [84], the authors prove that (4.2.1) is locally well-posed for $\left(\rho_0,u_0\right)\in W^{2,p}\left(\Omega\right)\times W^{1,p}\left(\Omega\right)$ where $p>1$ and $\Omega\subset\mathbb{R}^3$ is bounded and convex. In [87], Yang, Yao and Zhu construct a local unique solution with initial data drawn from $H^4\left(\mathbb{R}^3\right)\times H^3\left(\mathbb{R}^3\right)$ and they study the limit of vanishing viscosity and capillarity.

Our result is the following:

Theorem 4.2.1. *Consider $p\in(1,4)$ if $n=2$ or $p\in\left(\frac{6}{5},4\right)$ if $n=3$. Assume there exist positive constants $\left(\bar{\rho},\rho_\star,\rho^\star\right)$ such that $\rho_0-\bar{\rho}\in\dot{B}^{\frac{n}{p}+\gamma}_{p,1}\left(\mathbb{R}^n\right)$ and $0<\rho_\star<\rho_0<\rho^\star$. Furthermore, consider u_0 a divergence free vector field with coefficients in $\dot{B}^{\frac{n}{p}-1}_{p,1}\left(\mathbb{R}^n\right)$. Then, there exists a time $T>0$ and a unique solution $\left(\rho,u,\nabla P\right)$ of system (4.2.1) with*

$$\rho-\bar{\rho}\in\mathcal{C}_T(\dot{B}^{\frac{n}{p}+\gamma}_{p,1}(\mathbb{R}^n)),\ u\in\mathcal{C}_T(\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n))$$
$$\text{and }\ \left(\partial_t u,\nabla^2 u,\nabla P\right)\in L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)).$$

As in [49] we will not work directly with system (4.2.1) instead we will rather use its Lagrangian formulation. One salutary feature of working in the Lagrangian coordinates is that the density becomes independent of time. More precisely, considering $\left(\rho,u,\nabla P\right)$ a solution of (4.2.1) and denoting by X the flow associated to the vector field u:

$$X\left(t,y\right)=y+\int_0^t u\left(\tau,X(\tau,y)\right)dy \tag{4.2.3}$$

we introduce the new Lagrangian variables:

$$\bar{\rho}(t,y)=\rho(t,X(t,y)),\ \bar{u}(t,y)=u(t,X(t,y)) \text{ and } \bar{P}(t,y)=P(t,X(t,y)). \tag{4.2.4}$$

Then, using the chain rule and Proposition 1.4.2 we gather that $\bar{\rho}(t,\cdot)=\rho_0$ and

$$\begin{cases} \rho_0\partial_t\bar{u}-\operatorname{div}\left(\mu(\rho_0)A_{\bar{u}}D_{A_{\bar{u}}}(\bar{u})\right)+A_{\bar{u}}^T\nabla\bar{P}+\gamma\operatorname{div}\left(k(\rho_0)A_{\bar{u}}\left(A_{\bar{u}}^T\nabla\rho_0\right)\otimes\left(A_{\bar{u}}^T\nabla\rho_0\right)\right)=0,\\ \operatorname{div}(A_{\bar{u}}\bar{u})=0,\\ \bar{u}_{|t=0}=u_0, \end{cases} \tag{4.2.5}$$

where $A_{\bar{u}}$ is the inverse of the differential of X, and

$$D_{A_{\bar{u}}}(\bar{u})=D\bar{u}A_{\bar{u}}+A_{\bar{u}}^T\nabla\bar{u}.$$

Note that we can give a meaning to (4.2.5) independently of the Eulerian formulation by stating:

$$X(t,y)=y+\int_0^t\bar{u}(\tau,y)\,d\tau.$$

Let us note that in the Lagrangian formulation (4.2.5) the density appears just like a *known* variable coefficient. Of course, the price to pay is that the dissipative term depends now in a nonlinear manner on the velocity. However, for a sufficiently small amount of time, we do have

$$\operatorname{div}\left(\mu(\rho_0)A_{\bar{u}}D_{A_{\bar{u}}}(\bar{u})\right)\approx\operatorname{div}\left(\mu(\rho_0)D(\bar{u})\right).$$

Using this idea, which was developed in [49] (see also [61] in another context), it turns out to be possible to solve (4.2.5). More precisely, we get

Theorem 4.2.2. *Consider $p\in(1,4)$ if $n=2$ or $p\in\left(\frac{6}{5},4\right)$ if $n=3$. Assume there exist positive constants $(\bar{\rho},\rho_\star,\rho^\star)$ such that $\rho_0-\bar{\rho}\in\dot{B}_{p,1}^{\frac{n}{p}+\gamma}(\mathbb{R}^n)$ and $0<\rho_\star<\rho_0<\rho^\star$. Furthermore, consider u_0 a divergence free vector field with coefficients in $\dot{B}_{p,1}^{\frac{n}{p}-1}(\mathbb{R}^n)$. Then, there exists a time $T>0$ and a unique solution $\left(\bar{u},\nabla\bar{P}\right)$ of system (4.2.5) with*

$$\bar{u}\in\mathcal{C}_T(\dot{B}_{p,1}^{\frac{n}{p}-1}(\mathbb{R}^n)) \text{ and } \left(\partial_t\bar{u},\nabla^2\bar{u},\nabla\bar{P}\right)\in L_T^1(\dot{B}_{p,1}^{\frac{n}{p}-1}(\mathbb{R}^n)).$$

Moreover, there exists a positive constant $C=C(\rho_0)$ such that:

$$\|u\|_{L_T^\infty(\dot{B}_{p,1}^{\frac{n}{p}-1})}+\left\|\left(\nabla^2u,\nabla P\right)\right\|_{L_T^1(\dot{B}_{p,1}^{\frac{n}{p}-1})}\leq e^{C(T+1)}\|u_0\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}}.$$

Of course, the estimates of Theorem 4.1.1 are the key element of the proof of Theorem 4.2.2 which in turn implies Theorems 4.2.1 by returning in the Eulerian formulation.

We end this paragraph by remarking that opposed to the case when ρ is a small perturbation of a constant state, when considering the linearized system of the Lagrangian formulation i.e. system (4.1.1) we obtain the estimates (4.1.2) which have a time-dependent right hand side term. This in particular prevents us from adapting the arguments from [49] to our situation and obtaining a global solution for system (4.2.5) and consequently for the system (4.2.1). In fact, even if we were able to construct such a solution for system (4.2.5) it is not clear how we could go back into the original formulation as passing from the Eulerian formulation to the Lagrangian one needs some smallness condition on the $\|\cdot\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}})}$-norm of the velocity.

4.2.1 The proof of Theorem 4.2.2

In this section we aim at proving Theorem 4.2.2. In order to fix the ideas we will work in the 3 dimensional framework.

Let us fix some notations. The space $\tilde{F}_T$ consists of $\left(\tilde{w}, \nabla\tilde{Q}\right)$ with

$$\tilde{w} \in \mathcal{C}_T(\dot{B}^{\frac{3}{p}-1}_{p,1}) \text{ and } \left(\partial_t\tilde{w}, \nabla^2\tilde{w}, \nabla\tilde{Q}\right) \in L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1}),$$

which we endow with the norm

$$\left\|\left(\tilde{w}, \nabla\tilde{Q}\right)\right\|_{\tilde{F}_T} = \|\tilde{w}\|_{L^\infty_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \left\|\left(\partial_t\tilde{w}, \nabla^2\tilde{w}, \nabla\tilde{Q}\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}. \tag{4.2.6}$$

For any time dependent vector field $\bar{v}$ we denote

$$X_{\bar{v}}(t,x) = x + \int_0^t \bar{v}(\tau, x)\,d\tau,$$

and $A_{\bar{v}} = (DX_{\bar{v}})^{-1}$. Also, let us denote by $\operatorname{adj}(DX_{\bar{v}})$ the adjugate matrix (i.e. the transpose of the cofactor matrix) of $DX_{\bar{v}}$ and $J_{\bar{v}} = \det(DX_{\bar{v}})$.

Before attacking the well-posedness of (4.2.5), we first have to solve the following linear system:

$$\begin{cases} \rho_0\partial_t\bar{u} - \operatorname{div}\left(\mu(\rho_0) A_{\bar{v}} D_{A_{\bar{v}}}(\bar{u})\right) + A^T_{\bar{v}}\nabla\bar{P} = \gamma\operatorname{div}\left(k(\rho_0) A_{\bar{v}}((A^t_{\bar{v}}\nabla\rho_0)\otimes(A^t_{\bar{v}}\nabla\rho_0))\right), \\ \operatorname{div}\left(\operatorname{adj}(DX_{\bar{v}})\,\bar{u}\right) = 0, \\ \bar{u}_{|t=0} = u_0. \end{cases} \tag{4.2.7}$$

Proposition 4.2.1. *Consider $\bar{v} \in \mathcal{C}_T(\dot{B}^{\frac{3}{p}-1}_{p,1})$ with $\nabla\bar{v} \in L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})$. Assume there exist positive constants $(\bar{\rho}, \rho_\star, \rho^\star)$ such that $\rho_0 - \bar{\rho} \in \dot{B}^{\frac{3}{p}+\gamma}_{p,1}(\mathbb{R}^3)$ and $0 < \rho_\star < \rho_0 < \rho^\star$. Furthermore, consider u_0 a divergence free vector field with coefficients in $\dot{B}^{\frac{3}{p}-1}_{p,1}(\mathbb{R}^3)$. Then, there exists a constant α small enough such that if*

$$\|\nabla\bar{v}\|_{L^2_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \|\nabla\bar{v}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} \leq 2\alpha \tag{4.2.8}$$

then the following holds true. There exists $T > 0$ and a unique solution $\left(\bar{u}, \nabla\bar{P}\right) \in \tilde{F}_T$ of system (4.2.7).

Proof. Let us observe that (4.2.7) can be written in the form

$$\begin{cases} \partial_t\bar{u} - \frac{1}{\rho_0}\operatorname{div}\left(\mu(\rho_0) D(\bar{u})\right) + \frac{1}{\rho_0}\nabla\bar{P} = \frac{1}{\rho_0}F_{\bar{v}}\left(\bar{u}, \nabla\bar{P}\right), \\ \operatorname{div}\bar{u} = \operatorname{div}\left((Id - \operatorname{adj}(DX_{\bar{v}}))\,\bar{u}\right), \\ \bar{u}_{t=0} = u_0. \end{cases}$$

with

$$\begin{aligned} F_{\bar{v}}\left(\bar{w}, \nabla\bar{Q}\right) &:= \operatorname{div}\left(\mu(\rho_0) A_{\bar{v}} D_{A_{\bar{v}}}(\bar{w}) - \mu(\rho_0) D(\bar{w})\right) + \left(Id - A^T_{\bar{v}}\right)\nabla\bar{Q} &(4.2.9)\\ &\quad - \gamma\operatorname{div}\left(k(\rho_0) A_{\bar{v}}\left(A^t_{\bar{v}}\nabla\rho_0\right)\otimes\left(A^t_{\bar{v}}\nabla\rho_0\right)\right). &(4.2.10)\end{aligned}$$

Owing to Theorem 4.1.1 let us consider $(u_L, \nabla P_L)$ with $u_L \in \mathcal{C}(\mathbb{R}^+, \dot{B}^{\frac{3}{p}-1}_{p,1})$ and $\left(\partial_t u_L, \nabla^2 u_L, \nabla P_L\right) \in L^1_{loc}(\dot{B}^{\frac{3}{p}-1}_{p,1})$ the unique solution of

$$\begin{cases} \partial_t u_L - \frac{1}{\rho_0}\operatorname{div}\left(\mu(\rho_0) D(u_L)\right) + \frac{1}{\rho_0}\nabla P_L = 0, \\ \operatorname{div} u_L = 0, \\ u_{L|t=0} = u_0, \end{cases} \tag{4.2.11}$$

for which we know that:

$$\|(u_L, \nabla P_L)\|_{E_T} \leq \|u_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \exp\left(C_{\rho_0}\left(T+1\right)\right).$$

Moreover, for any α we see that T can be chosen small enough such that

$$\|\nabla u_L\|_{L^2_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \left\|\left(\partial_t u_L, \nabla^2 u_L, \nabla P_L\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} \leq \alpha. \tag{4.2.12}$$

Following the idea in [49], and owing to Theorem 4.1.1, we consider the operator

$$\Phi\left(\tilde{w}, \nabla\tilde{Q}\right) = \left(\tilde{u}, \nabla\tilde{P}\right) \tag{4.2.13}$$

which associates to $\left(\tilde{w}, \nabla\tilde{Q}\right) \in \tilde{F}_T$, the unique solution $\left(\tilde{u}, \nabla\tilde{P}\right) \in \tilde{F}_T$ of:

$$\begin{cases} \partial_t \tilde{u} - \frac{1}{\rho_0}\operatorname{div}\left(\mu\left(\rho_0\right) D\left(\tilde{u}\right)\right) + \frac{1}{\rho_0}\nabla\tilde{P} = \frac{1}{\rho_0} F_{\bar{v}}\left(u_L + \tilde{w}, \nabla P_L + \nabla\tilde{Q}\right), \\ \hfill \operatorname{div}\tilde{u} = \operatorname{div}\left(\left(Id - \operatorname{adj}\left(DX_{\bar{v}}\right)\right)\left(u_L + \tilde{w}\right)\right), \\ \hfill \tilde{u}_{|t=0} = 0. \end{cases}$$

We will show in the following that for any $R > 0$ there exists a sufficiently small $T > 0$ such that there exists a fixed point for Φ in the ball of radius R centered at the origin of $\tilde{F}_T$. More precisely, according to Theorem 4.1.1 we get that

$$\begin{aligned} \left\|\Phi\left(\tilde{w}, \nabla\tilde{Q}\right)\right\|_{\tilde{F}_T} \leq \left\|(1/\rho_0) F_{\bar{v}}\left(u_L + \tilde{w}, \nabla P_L + \nabla\tilde{Q}\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \\ \left\|\partial_t\left(Id - \operatorname{adj}\left(DX_{\bar{v}}\right)\right)\left(u_L + \tilde{w}\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \left\|\nabla\operatorname{div}\left(\left(Id - \operatorname{adj}\left(DX_{\bar{v}}\right)\right)\left(u_L + \tilde{w}\right)\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}. \end{aligned} \tag{4.2.14}$$

We begin by treating the first term:

$$\left\|(1/\rho_0) F_{\bar{v}}(u_L + \tilde{w}, \nabla P_L + \nabla\tilde{Q})\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} \tag{4.2.15}$$

$$\lesssim \left(1/\bar{\rho} + \|1/\rho_0 - 1/\bar{\rho}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\right)\left\|F_{\bar{v}}(u_L + \tilde{w}, \nabla P_L + \nabla\tilde{Q})\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}. \tag{4.2.16}$$

We write that

$$\begin{aligned} T_1 &= \operatorname{div}\left(\mu\left(\rho_0\right) A_{\bar{v}} D_{A_{\bar{v}}}\left(u_L + \tilde{w}\right)\right) - \operatorname{div}\left(\mu\left(\rho_0\right) D\left(u_L + \tilde{w}\right)\right) \\ &= \operatorname{div}\left(\mu\left(\rho_0\right)\left(A_{\bar{v}} - Id\right) D_{A_{\bar{v}}}\left(u_L + \tilde{w}\right)\right) + \operatorname{div}\left(\mu\left(\rho_0\right) D_{A_{\bar{v}} - Id}\left(u_L + \tilde{w}\right)\right) \\ &= \operatorname{div}\left(\mu\left(\rho_0\right)\left(A_{\bar{v}} - Id\right) D_{A_{\bar{v}} - Id}\left(u_L + \tilde{w}\right)\right) + \operatorname{div}\left(\mu\left(\rho_0\right)\left(A_{\bar{v}} - Id\right) D\left(u_L + \tilde{w}\right)\right) \\ &\quad + \operatorname{div}\left(\mu\left(\rho_0\right) D_{A_{\bar{v}} - Id}\left(u_L + \tilde{w}\right)\right). \end{aligned}$$

Thus, using (1.4.11) of Proposition 1.4.4 along with product laws in Besov spaces (see Proposition 1.3.11) we get the following bound for T_1:

$$\begin{aligned} \|T_1\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} &\lesssim C_{\rho_0}\|A_{\bar{v}} - Id\|_{L^\infty_T(\dot{B}^{\frac{3}{p}}_{p,1})}\left(1 + \|A_{\bar{v}} - Id\|_{L^\infty_T(\dot{B}^{\frac{3}{p}}_{p,1})}\right)\left(\|\nabla u_L\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} + \|\nabla\tilde{w}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}\right) \\ &\lesssim C_{\rho_0}\|\nabla\bar{v}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}\left(1 + \|\nabla\bar{v}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}\right)\left(\|\nabla u_L\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} + \|\nabla\tilde{w}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}\right) \\ &\lesssim C_{\rho_0}\alpha\left(\alpha + \left\|(\tilde{w}, \nabla\tilde{Q})\right\|_{F_T}\right). \end{aligned} \tag{4.2.17}$$

The second term is estimated as follows:

$$\left\|\left(Id - A^T_{\bar{v}}\right)\left(\nabla P_L + \nabla\tilde{Q}\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}$$

$$\lesssim \|Id - A_{\bar{v}}\|_{L^\infty_T(\dot{B}^{\frac{3}{p}}_{p,1})} (\|\nabla P_L\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \left\|\nabla \tilde{Q}\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}) \tag{4.2.18}$$

$$\lesssim \|\nabla \bar{v}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} (\|\nabla P_L\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \left\|\nabla \tilde{Q}\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}) \tag{4.2.19}$$

$$\lesssim \alpha(\alpha + \left\|(\tilde{w}, \nabla \tilde{Q})\right\|_{F_T}). \tag{4.2.20}$$

The last term of (4.2.9) is treated as follows. First, we write:

$$\begin{aligned}
&\left(A^T_{\bar{v}} \nabla \rho_0\right) \otimes (A^T_{\bar{v}} \nabla \rho_0) \\
&= (A^T_{\bar{v}} - Id)\nabla \rho_0 \otimes (A^T_{\bar{v}} - Id)\nabla \rho_0 + \nabla \rho_0 \otimes (A^T_{\bar{v}} - Id)\nabla \rho_0 \\
&+ \nabla \rho_0 \otimes (A^T_{\bar{v}} - Id)\nabla \rho_0 + \nabla \rho_0 \otimes \nabla \rho_0
\end{aligned}$$

such that using the last relation along with Proposition 1.4.4, estimate (1.4.11), we get that:

$$\begin{aligned}
\left\|\left(A^T_{\bar{v}} \nabla \rho_0\right) \otimes (A^T_{\bar{v}} \nabla \rho_0)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} &\leq CT(\left\|(A^T_{\bar{v}} - Id)\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}} + 1)^2 \|\nabla \rho_0\|^2_{\dot{B}^{\frac{3}{p}}_{p,1}} \\
&\leq CT(\alpha + 1)^2 \|\nabla \rho_0\|^2_{\dot{B}^{\frac{3}{p}}_{p,1}} \leq CT \|\nabla \rho_0\|^2_{\dot{B}^{\frac{3}{p}}_{p,1}}.
\end{aligned} \tag{4.2.21}$$

Next

$$\begin{aligned}
&\operatorname{div}\left(k\left(\rho_0\right) A_{\bar{v}} \left(A^T_{\bar{v}} \nabla \rho_0\right) \otimes \left(A^T_{\bar{v}} \nabla \rho_0\right)\right) \\
&= \operatorname{div}\left(\left(k\left(\rho_0\right) - k\left(\bar{\rho}\right)\right) A_{\bar{v}} \left(A^T_{\bar{v}} \nabla \rho_0\right) \otimes \left(A^T_{\bar{v}} \nabla \rho_0\right)\right) \\
&+ k\left(\bar{\rho}\right) \operatorname{div}\left(\left(A_{\bar{v}} - Id\right) \left(A^T_{\bar{v}} \nabla \rho_0\right) \otimes \left(A^T_{\bar{v}} \nabla \rho_0\right)\right),
\end{aligned}$$

thus, with the help of Proposition 1.3.9 and estimate (4.2.21) we get

$$\begin{aligned}
&\left\|\operatorname{div}\left(k\left(\rho_0\right) A_{\bar{v}} \left(A^T_{\bar{v}} \nabla \rho_0\right) \otimes \left(A^T_{\bar{v}} \nabla \rho_0\right)\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} \\
&\leq (\|k\left(\rho_0\right) - k\left(\bar{\rho}\right)\|_{\dot{B}^{\frac{3}{p}}_{p,1}} + k\left(\bar{\rho}\right))(\|A_{\bar{v}} - Id\|_{\dot{B}^{\frac{3}{p}}_{p,1}} + 1) \left\|\left(A^T_{\bar{v}} \nabla \rho_0\right) \otimes \left(A^T_{\bar{v}} \nabla \rho_0\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} \\
&\leq C_{\rho_0} T \|\nabla \rho_0\|^2_{\dot{B}^{\frac{3}{p}}_{p,1}}.
\end{aligned} \tag{4.2.22}$$

such that combining (4.2.15), (4.2.17), (4.2.20) and (4.2.22)we get that:

$$\left\|(1/\rho_0) F_{\bar{v}} \left(u_L + \tilde{w}, \nabla P_L + \nabla \tilde{Q}\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} \lesssim C_{\rho_0} \alpha(\alpha + \left\|(\tilde{w}, \nabla \tilde{Q})\right\|_{F_T}) + \gamma C_{\rho_0} T \|\nabla \rho_0\|^2_{\dot{B}^{\frac{3}{p}}_{p,1}}. \tag{4.2.23}$$

In order to treat the second term of (4.2.14) we use the estimates (1.4.12), (1.4.13) of Proposition 1.4.4 along with Hölder's inequality in order to obtain:

$$\begin{aligned}
&\|\partial_t \left(Id - \operatorname{adj}\left(DX_{\bar{v}}\right)\right) \left(u_L + \tilde{w}\right)\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} \\
&\lesssim \|(u_L + \tilde{w}) \partial_t \operatorname{adj}\left(DX_{\bar{v}}\right)\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \|\left(Id - \operatorname{adj}\left(DX_{\bar{v}}\right)\right) \left(\partial_t u_L + \partial_t \tilde{w}\right)\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} \\
&\lesssim \|\partial_t \operatorname{adj}\left(DX_{\bar{v}}\right)\|_{L^2_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} \|u_L + \tilde{w}\|_{L^2_T(\dot{B}^{\frac{3}{p}}_{p,1})} + \|Id - \operatorname{adj}\left(DX_{\bar{v}}\right)\|_{L^\infty_T(\dot{B}^{\frac{3}{p}}_{p,1})} \|\partial_t u_L + \partial_t \tilde{w}\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} \\
&\lesssim \|\nabla \bar{v}\|_{L^2_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} (\alpha + \left\|(\tilde{w}, \nabla \tilde{Q})\right\|_{F_T}) + \alpha(\alpha + \left\|(\tilde{w}, \nabla \tilde{Q})\right\|_{F_T}) \\
&\lesssim \alpha(\alpha + \left\|(\tilde{w}, \nabla \tilde{Q})\right\|_{F_T}).
\end{aligned} \tag{4.2.24}$$

Treating the next term of (4.2.14) is done with the aid of Corollary 1.4.1:

$$\begin{aligned}\operatorname{div}\left(\left(Id-\operatorname{adj}\left(DX_{\bar{v}}\right)\right)\left(u_L+\tilde{w}\right)\right) &= \left(Du_L+D\tilde{w}\right):\left(Id-J_{\bar{v}}A_{\bar{v}}\right)\\ &= J_{\bar{v}}\left(Du_L+D\tilde{w}\right):\left(Id-A_{\bar{v}}\right)+\left(1-J_{\bar{v}}\right)\left(\operatorname{div}u_L+\operatorname{div}\tilde{w}\right).\end{aligned}$$

Thus, using the estimates (1.4.11), (1.4.15) of Proposition 1.4.4, we may write:

$$\begin{aligned}&\left\|\nabla\operatorname{div}\left(\left(Id-\operatorname{adj}\left(DX_{\bar{v}}\right)\right)\left(u_L+\tilde{w}\right)\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}\\ &\lesssim \left\|J_{\bar{v}}\left(Du_L+D\tilde{w}\right):\left(Id-A_{\bar{v}}\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}+\left\|\left(1-J_{\bar{v}}\right)\left(\operatorname{div}u_L+\operatorname{div}\tilde{w}\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}\\ &\lesssim \left(1+\left\|J_{\bar{v}}-1\right\|_{L^\infty_T(\dot{B}^{\frac{3}{p}}_{p,1})}\right)\left\|Id-A_{\bar{v}}\right\|_{L^\infty_T(\dot{B}^{\frac{3}{p}}_{p,1})}\left\|Du_L+D\tilde{w}\right\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}\\ &+\left\|\left(J_{\bar{v}}-1\right)\right\|_{L^\infty_T(\dot{B}^{\frac{3}{p}}_{p,1})}\left\|\operatorname{div}u_L+\operatorname{div}\tilde{w}\right\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}\\ &\lesssim \alpha\left(1+\alpha\right)\left(\alpha+\left\|(\tilde{w},\nabla\tilde{Q})\right\|_{F_T}\right).\end{aligned}\tag{4.2.25}$$

Combining the estimates (4.2.23), (4.2.24), and (4.2.25) we get that:

$$\left\|\Phi(\tilde{w},\nabla\tilde{Q})\right\|_{\tilde{F}_T}\lesssim\alpha(\alpha+\left\|(\tilde{w},\nabla\tilde{Q})\right\|_{F_T})+\gamma C_{\rho_0}T\left\|\nabla\rho_0\right\|^2_{\dot{B}^{\frac{3}{p}}_{p,1}}.\tag{4.2.26}$$

Thus, for a suitably small α the operator Φ maps the ball of radius R centered at the origin of $\tilde{F}_T$ into itself. Due to the linearity of Φ one can repeat the above arguments in order to show that for small values of α, Φ is a contraction. This concludes the existence of a fixed point of Φ, say $\left(\tilde{u}^\star,\nabla\tilde{P}^\star\right)\in\tilde{F}_T$. Of course,

$$(\bar{u},\nabla\bar{P})=\left(\tilde{u}^\star,\nabla\tilde{P}^\star\right)+(u_L,\nabla P_L)$$

is a solution of (4.2.7). This concludes the proof of the Proposition 4.2.1. □

We are now in the position of proving the results announced in the beginning of Section 4.2

Proof of Theorem 4.2.2: We consider the closed set:

$$\tilde{F}_T(\alpha,R)=\left\{\left(\tilde{v},\nabla\tilde{Q}\right)\in F_T:\tilde{v}_{|t=0}=0,\ \left\|\left(\tilde{v},\nabla\tilde{Q}\right)\right\|_{F_T}\leq R\alpha\right\}\tag{4.2.27}$$

with R sufficiently small such that:

$$\left\|\nabla\tilde{v}\right\|_{L^2_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}+\left\|\nabla\tilde{v}\right\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}\leq\alpha.\tag{4.2.28}$$

Consider T small enough such that $(u_L,\nabla P_L)$ the solution of (4.2.11) satisfies

$$\left\|\nabla u_L\right\|_{L^2_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}+\left\|\nabla u_L\right\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}\leq\alpha.\tag{4.2.29}$$

Using relations (4.2.28), (4.2.29) along with Proposition 4.2.1, let us consider the operator S which associates to $\left(\tilde{v},\nabla\tilde{Q}\right)\in\tilde{F}_T(\alpha,R)$, the solution of:

$$\begin{cases}\partial_t\tilde{u}-\frac{1}{\rho_0}\operatorname{div}\left(\mu\left(\rho_0\right)D\left(\tilde{u}\right)\right)+\frac{1}{\rho_0}\nabla\tilde{P}=\frac{1}{\rho_0}F_{(u_L+\tilde{v})}\left(u_L+\tilde{u},\nabla P_L+\nabla\tilde{P}\right),\\ \operatorname{div}\left(\operatorname{adj}\left(DX_{u_L+\tilde{v}}\right)\left(u_L+\tilde{u}\right)\right)=0,\\ \tilde{u}_{|t=0}=0.\end{cases}$$

We will show that that for suitably small T and α, the operator S maps the closed set $\tilde{F}_T(\alpha, R)$ into itself and that S is a contraction. First of all, recalling that $\left(\tilde{u}, \nabla\tilde{P}\right)$ is in fact the fixed point of the operator Φ defined in (4.2.13) and using the estimates established in the last section we conclude that

$$\left\|\left(\tilde{u}, \nabla\tilde{P}\right)\right\|_{\tilde{F}_T} = \left\|S\left(\tilde{v}, \nabla\tilde{Q}\right)\right\|_{F_T} \leq R\alpha \tag{4.2.30}$$

for some small enough T.

Next, we will deal with the stability estimates. For $i = 1, 2$, let us consider $\left(\tilde{v}_i, \nabla\tilde{Q}_i\right) \in \tilde{F}_T(\alpha, R)$ and $\left(\tilde{u}_i, \nabla\tilde{P}_i\right) = S\left(\tilde{v}_i, \nabla\tilde{Q}_i\right) \in \tilde{F}_T(\alpha, R)$. Denoting by

$$\begin{aligned}
\left(\delta\tilde{v}, \nabla\delta\tilde{Q}\right) &= \left(\tilde{v}_1 - \tilde{v}_2, \nabla\tilde{Q}_1 - \nabla\tilde{Q}_2\right), \\
\left(\delta\tilde{u}, \nabla\delta\tilde{P}\right) &= \left(\tilde{u}_1 - \tilde{u}_2, \nabla\tilde{P}_1 - \nabla\tilde{P}_2\right),
\end{aligned}$$

we see that:

$$\begin{cases}
\partial_t\delta\tilde{u} - \frac{1}{\rho_0}\operatorname{div}\left(\mu\left(\rho_0\right)D\left(\delta\tilde{u}\right)\right) + \frac{1}{\rho_0}\nabla\delta\tilde{P} = \frac{1}{\rho_0}\tilde{F}, \\
\hfill \operatorname{div}\delta\tilde{u} = \operatorname{div}\tilde{G}, \\
\hfill \delta\tilde{u}_{|t=0} = 0,
\end{cases}$$

where

$$\begin{cases}
\tilde{F} = F_1(\delta\tilde{v}, u_L + \tilde{u}_1) + F_1(u_L + \tilde{v}_2, \delta\tilde{u}) + F_2\left(\delta\tilde{v}, \nabla P_L + \nabla\tilde{P}_1\right) \\
\qquad + F_2(u_L + \tilde{v}_2, \nabla\delta\tilde{P}) + \gamma F_3\left(\tilde{v}_1, \tilde{v}_2\right), \\
\tilde{G} = -\left(\operatorname{adj}\left(DX_{(u_L + \tilde{v}_1)}\right) - Id\right)\delta\tilde{u} \\
\qquad - \left(\operatorname{adj}\left(DX_{(u_L + \tilde{v}_1)}\right) - \operatorname{adj}\left(DX_{(u_L + \tilde{v}_2)}\right)\right)\left(u_L + \tilde{u}_2\right) := \tilde{G}_1 + \tilde{G}_2
\end{cases}$$

and

$$\begin{cases}
F_1(\bar{v}, \bar{w}) = \operatorname{div}\left(\mu\left(\rho_0\right)A_{\bar{v}}D_{A_{\bar{v}}}\left(\bar{w}\right) - \mu\left(\rho_0\right)D\left(\bar{w}\right)\right), \\
F_2(\bar{v}, \nabla\bar{Q}) = \left(Id - A_{\bar{v}}^T\right)\nabla\bar{Q}, \\
F_3\left(\tilde{v}_1, \tilde{v}_2\right) = \operatorname{div}\left(k\left(\rho_0\right)A_{u_L + \tilde{v}_1}\left(A_{u_L + \tilde{v}_1}^T\nabla\rho_0\right) \otimes \left(A_{u_L + \tilde{v}_1}^T\nabla\rho_0\right)\right) \\
\qquad - \operatorname{div}\left(k\left(\rho_0\right)A_{u_L + \tilde{v}_2}\left(A_{u_L + \tilde{v}_2}^T\nabla\rho_0\right) \otimes \left(A_{u_L + \tilde{v}_2}^T\nabla\rho_0\right)\right).
\end{cases}$$

According to Theorem 4.1.1 we get that

$$\left\|\left(\delta\tilde{u}, \nabla\delta\tilde{P}\right)\right\|_{\tilde{F}_T} \lesssim C_{\rho_0}\left(\left\|\tilde{F}\right\|_{L_T^1(\dot{B}_{p,1}^{\frac{3}{p}-1})} + \left\|\nabla\operatorname{div}\tilde{G}\right\|_{L_T^1(\dot{B}_{p,1}^{\frac{3}{p}-1})} + \left\|\partial_t\tilde{G}\right\|_{L_T^1(\dot{B}_{p,1}^{\frac{3}{p}-1})}\right). \tag{4.2.31}$$

We begin with

$$\begin{aligned}
F_3\left(\tilde{v}_1, \tilde{v}_2\right) = {}& \operatorname{div}\left(k\left(\rho_0\right)\left(A_{u_L + \tilde{v}_1} - A_{u_L + \tilde{v}_2}\right)\left(A_{u_L + \tilde{v}_1}^T\nabla\rho_0\right) \otimes \left(A_{u_L + \tilde{v}_1}^T\nabla\rho_0\right)\right) \\
&+ \operatorname{div}\left(k\left(\rho_0\right)A_{u_L + \tilde{v}_2}\left(\left(A_{u_L + \tilde{v}_1}^T - A_{u_L + \tilde{v}_2}^T\right)\nabla\rho_0\right) \otimes \left(A_{u_L + \tilde{v}_1}^T\nabla\rho_0\right)\right) \\
&+ \operatorname{div}\left(k\left(\rho_0\right)A_{u_L + \tilde{v}_2}\left(A_{u_L + \tilde{v}_2}^T\nabla\rho_0\right) \otimes \left(\left(A_{u_L + \tilde{v}_1}^T - A_{u_L + \tilde{v}_2}^T\right)\nabla\rho_0\right)\right).
\end{aligned} \tag{4.2.32}$$

Using the last relation along with Proposition 1.3.9, the estimates (1.4.11) and (1.4.16) yields:

$$\begin{aligned}
&\left\|\operatorname{div}\left(k\left(\rho_0\right)\left(A_{u_L + \tilde{v}_1} - A_{u_L + \tilde{v}_2}\right)\left(A_{u_L + \tilde{v}_1}^T\nabla\rho_0\right) \otimes \left(A_{u_L + \tilde{v}_1}^T\nabla\rho_0\right)\right)\right\|_{L_T^1(\dot{B}_{p,1}^{\frac{3}{p}-1})} \\
&\lesssim \left\|A_{u_L + \tilde{v}_1} - A_{u_L + \tilde{v}_2}\right\|_{L_T^\infty(\dot{B}_{p,1}^{\frac{3}{p}})}\left\|\left(A_{u_L + \tilde{v}_1}^T\nabla\rho_0\right) \otimes \left(A_{u_L + \tilde{v}_1}^T\nabla\rho_0\right)\right\|_{L_T^1(\dot{B}_{p,1}^{\frac{3}{p}})} \\
&\lesssim \left\|\nabla\delta\tilde{v}\right\|_{L_T^1(\dot{B}_{p,1}^{\frac{3}{p}})}\left\|A_{u_L + \tilde{v}_1}^T\nabla\rho_0\right\|_{L_T^1(\dot{B}_{p,1}^{\frac{3}{p}})}\left\|A_{u_L + \tilde{v}_1}^T\nabla\rho_0\right\|_{L_T^\infty(\dot{B}_{p,1}^{\frac{3}{p}})} \\
&\lesssim T\left\|\nabla\delta\tilde{v}\right\|_{L_T^1(\dot{B}_{p,1}^{\frac{3}{p}})}\left(\left\|A_{u_L + \tilde{v}_1} - Id\right\|_{L_T^\infty(\dot{B}_{p,1}^{\frac{3}{p}})} + 1\right)^2\left\|\nabla\rho_0\right\|_{\dot{B}_{p,1}^{\frac{3}{p}}}^2
\end{aligned}$$

$$\lesssim C_{\rho_0} T \|\nabla \delta \tilde{v}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}$$

Proceeding in a similar manner with the rest of the terms of $F_3(\tilde{v}_1, \tilde{v}_2)$ form relation (4.2.32), we get that

$$\|F_3\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} \lesssim C_{\rho_0} T \|\nabla \delta \tilde{v}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}.$$

Next, using the last estimate and proceeding as in relations (4.2.15) and (4.2.17) we get that

$$\begin{aligned}
&\left\|\tilde{F}\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} \\
&\lesssim \|\nabla \delta \tilde{v}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} \|\nabla u_L + \nabla \tilde{u}_1\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} && (4.2.33)\\
&+ \|\nabla u_L + \nabla \tilde{v}_2\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} \|\nabla \delta \tilde{u}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} + \|\nabla \delta \tilde{v}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} \left\|\nabla P_L + \nabla \tilde{P}_1\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} \\
&+ \|\nabla u_L + \nabla \tilde{v}_2\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} \left\|\nabla \delta \tilde{P}\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \gamma C_{\rho_0} T^2 \|\nabla \delta \tilde{v}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} \\
&\lesssim (C_{\rho_0} \gamma T \|\nabla \delta \tilde{v}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} + \alpha) \left\|\left(\nabla \delta \tilde{v}, \nabla \delta \tilde{Q}\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} + \alpha \left\|\left(\delta \tilde{u}, \nabla \delta \tilde{P}\right)\right\|_{\tilde{F}_T}. && (4.2.34)
\end{aligned}$$

Of course, we will use the smallness of α to absorb $\alpha \left\|\left(\nabla \delta \tilde{u}, \nabla \delta \tilde{P}\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}$ into the left hand side of (4.2.31).

Next, we treat $\left\|\nabla \operatorname{div} \tilde{G}\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}$. Using Corollary 1.4.2, we write that

$$\begin{aligned}
&- \operatorname{div} \tilde{G}_2 \\
&= \operatorname{div}\left(\left(\operatorname{adj}\left(DX_{(u_L+\tilde{v}_1)}\right) - \operatorname{adj}\left(DX_{(u_L+\tilde{v}_2)}\right)\right)(u_L + \tilde{u}_2)\right) \\
&= \operatorname{div}\left(\operatorname{adj}\left(DX_{(u_L+\tilde{v}_1)}\right)(u_L + \tilde{u}_2)\right) - \operatorname{div}\left(\operatorname{adj}\left(DX_{(u_L+\tilde{v}_2)}\right)(u_L + \tilde{u}_2)\right) \\
&= J_{u_L+\tilde{v}_1} D(u_L + \tilde{u}_2) : A_{(u_L+\tilde{v}_1)} - J_{(u_L+\tilde{v}_2)} D(u_L + \tilde{u}_2) : A_{(u_L+\tilde{v}_2)} \\
&= \left(J_{u_L+\tilde{v}_1} - J_{(u_L+\tilde{v}_2)}\right) D(u_L + \tilde{u}_2) : A_{(u_L+\tilde{v}_1)} + J_{(u_L+\tilde{v}_2)} D(u_L + \tilde{u}_2) : \left(A_{(u_L+\tilde{v}_1)} - A_{(u_L+\tilde{v}_2)}\right)
\end{aligned}$$

and thus, using Proposition 1.4.4 and Proposition 1.4.5 we get that

$$\begin{aligned}
&\left\|\nabla \operatorname{div} \tilde{G}_2\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} \\
&\lesssim \left\|\left(J_{u_L+\tilde{v}_1} - J_{(u_L+\tilde{v}_2)}\right)\right\|_{L^\infty_T(\dot{B}^{\frac{3}{p}}_{p,1})} \|(Du_L, D\tilde{u}_2)\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} (1 + \left\|Id - A_{(u_L+\tilde{v}_1)}\right\|_{L^\infty_T(\dot{B}^{\frac{3}{p}}_{p,1})}) && (4.2.35)\\
&+ (1 + \|(J_{u_L+\tilde{v}_2} - 1)\|_{L^\infty_T(\dot{B}^{\frac{3}{p}}_{p,1})}) \|(Du_L, D\tilde{u}_2)\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} \left\|A_{(u_L+\tilde{v}_1)} - A_{(u_L+\tilde{v}_2)}\right\|_{L^\infty_T(\dot{B}^{\frac{3}{p}}_{p,1})} \\
&\lesssim \alpha \|\nabla \delta \tilde{v}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}. && (4.2.36)
\end{aligned}$$

Next, using again Corollary 1.4.2 we see that

$$\begin{aligned}
- \operatorname{div} \tilde{G}_1 &= \operatorname{div}\left(\left(\operatorname{adj}\left(DX_{(u_L+\tilde{v}_1)}\right) - Id\right) \delta \tilde{u}\right) = D\delta \tilde{u} : \left(J_{u_L+\tilde{v}_1} A_{(u_L+\tilde{v}_1)} - Id\right) \\
&= J_{u_L+\tilde{v}_1} D\delta \tilde{u} : \left(A_{(u_L+\tilde{v}_1)} - Id\right) + (J_{u_L+\tilde{v}_1} - 1) \operatorname{div} \delta \tilde{u}
\end{aligned}$$

and consequently

$$\begin{aligned}
&\left\|\nabla \operatorname{div} \tilde{G}_1\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} \\
&\lesssim \left\|J_{u_L+\tilde{v}_1} D\delta \tilde{u} : \left(A_{(u_L+\tilde{v}_1)} - Id\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} + \|(J_{u_L+\tilde{v}_1} - 1) \operatorname{div} \delta \tilde{u}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} && (4.2.37)
\end{aligned}$$

$$
\begin{aligned}
&\lesssim (1+\|J_{u_L+\tilde v_1}-1\|_{L^1_T(\dot B^{\frac{3}{p}}_{p,1})})\left\|D\delta\tilde u:\left(A_{(u_L+\tilde v_1)}-Id\right)\right\|_{L^1_T(\dot B^{\frac{3}{p}}_{p,1})}\\
&+\|J_{u_L+\tilde v_1}-1\|_{L^1_T(\dot B^{\frac{3}{p}}_{p,1})}\|\operatorname{div}\delta\tilde u\|_{L^1_T(\dot B^{\frac{3}{p}}_{p,1})} \qquad (4.2.38)\\
&\lesssim \alpha\,\|(\delta\tilde u,\nabla\delta P)\|_{\tilde F_T} \qquad (4.2.39)
\end{aligned}
$$

Combining (4.2.36) with (4.2.39) yields

$$
\left\|\nabla\operatorname{div}\tilde G\right\|_{L^1_T(\dot B^{\frac{3}{p}-1}_{p,1})}\lesssim \alpha\,\|\nabla\delta\tilde v\|_{L^1_T(\dot B^{\frac{3}{p}}_{p,1})}+\alpha\,\|(\delta\tilde u,\nabla\delta P)\|_{\tilde F_T}. \qquad (4.2.40)
$$

Again, we will use the smallness of α to absorb $\alpha\left\|\left(\nabla\delta\tilde u,\nabla\delta\tilde P\right)\right\|_{L^1_T(\dot B^{\frac{3}{p}}_{p,1})}$ into the left hand side of (4.2.40).

Finally, we write that

$$
\begin{aligned}
&\partial_t\left[\left(\operatorname{adj}\left(DX_{(u_L+\tilde v_1)}\right)-\operatorname{adj}\left(DX_{(u_L+\tilde v_2)}\right)\right)(u_L+\tilde u_2)\right]\\
&=\left(\partial_t\operatorname{adj}\left(DX_{(u_L+\tilde v_1)}\right)-\partial_t\operatorname{adj}\left(DX_{(u_L+\tilde v_2)}\right)\right)(u_L+\tilde u_2)\\
&+\operatorname{adj}\left(DX_{(u_L+\tilde v_1)}\right)-\operatorname{adj}\left(DX_{(u_L+\tilde v_2)}\right)(\partial_t u_L+\partial_t\tilde u_2).
\end{aligned}
$$

Using Proposition 1.4.5 gives us

$$
\begin{aligned}
&\left\|\partial_t\left[\left(\operatorname{adj}\left(DX_{(u_L+\tilde v_1)}\right)-\operatorname{adj}\left(DX_{(u_L+\tilde v_2)}\right)\right)(u_L+\tilde u_2)\right]\right\|_{L^1_T(\dot B^{\frac{3}{p}-1}_{p,1})}\\
&\lesssim\left\|\partial_t\operatorname{adj}\left(DX_{(u_L+\tilde v_1)}\right)-\partial_t\operatorname{adj}\left(DX_{(u_L+\tilde v_2)}\right)\right\|_{L^2_T(\dot B^{\frac{3}{p}-1}_{p,1})}\|u_L\|_{L^2_T(\dot B^{\frac{3}{p}}_{p,1})}\\
&+\left\|\partial_t\operatorname{adj}\left(DX_{(u_L+\tilde v_1)}\right)-\partial_t\operatorname{adj}\left(DX_{(u_L+\tilde v_2)}\right)\right\|_{L^1_T(\dot B^{\frac{3}{p}}_{p,1})}\|\tilde u_2\|_{L^\infty_T(\dot B^{\frac{3}{p}-1}_{p,1})}\\
&+\left\|\operatorname{adj}\left(DX_{(u_L+\tilde v_1)}\right)-\operatorname{adj}\left(DX_{(u_L+\tilde v_2)}\right)\right\|_{L^\infty_T(\dot B^{\frac{3}{p}}_{p,1})}\|\partial_t u_L+\partial_t\tilde u_2\|_{L^1_T(\dot B^{\frac{3}{p}-1}_{p,1})}\\
&\lesssim\alpha\,\|\delta\tilde v\|_{L^2_T(\dot B^{\frac{3}{p}}_{p,1})}.
\end{aligned}
$$

The conclusion is that

$$
\left\|\partial_t\tilde G\right\|_{L^1_T(\dot B^{\frac{3}{p}-1}_{p,1})}\lesssim\alpha\,\|\delta\tilde v\|_{L^2_T(\dot B^{\frac{3}{p}}_{p,1})}. \qquad (4.2.41)
$$

Gathering the information of (4.2.34), (4.2.40) and (4.2.41) we get that if α is chosen sufficiently small then

$$
\left\|\left(\left(\delta\tilde u,\nabla\delta\tilde P\right)\right)\right\|_{\tilde F_T}\le\frac{1}{2}\left\|\left(\left(\delta\tilde v,\nabla\delta\tilde Q\right)\right)\right\|_{F_T} \qquad (4.2.42)
$$

the operator S is also a contraction over $\tilde F_T(\alpha,R)$. Thus, according to Banach's theorem there exists a fixed point $\left(\bar u^\star,\nabla\bar P^\star\right)$ of S. Obviously,

$$
\left(\bar u,\nabla\bar P\right)=(u_L,\nabla P_L)+\left(\bar u^\star,\nabla\bar P^\star\right)
$$

is a solution of

$$
\begin{cases}
\rho_0\partial_t\bar u-\operatorname{div}\left(\mu(\rho_0)A_{\bar u}D_{A_{\bar u}}(\bar u)\right)+A^T_{\bar u}\nabla\bar P=0,\\
\operatorname{div}\left(\operatorname{adj}\left(DX_{\bar u}\right)\bar u\right)=0,\\
\bar u_{|t=0}=u_0.
\end{cases} \qquad (4.2.43)
$$

In view of Proposition 1.4.3 we also get that $J_{\bar u}=1$. Thus, the second equation of (4.2.43) becomes

$$
\operatorname{div}\left(A_{\bar u}\bar u\right)=0.
$$

The only thing left to prove is the uniqueness property. Let us consider $\left(\bar{u}^1, \nabla \bar{P}^1\right)$, $\left(\bar{u}^2, \nabla \bar{P}^2\right) \in F_T$, two solutions of (4.2.43) with the same initial data $u_0 \in \dot{B}^{\frac{3}{p}-1}_{p,1}$. With $(u_L, \nabla P_L)$ defined above we let

$$\left(\tilde{u}^i, \nabla \tilde{P}^i\right) = \left(\bar{u}^i, \nabla \bar{P}^i\right) - (u_L, \nabla P_L) \text{ for } i = 1, 2$$

such that the system verified by $\left(\tilde{u}^i, \nabla \tilde{P}^i\right)$ is

$$\begin{cases} \partial_t \tilde{u}^i - \frac{1}{\rho_0} \operatorname{div}\left(\mu\left(\rho_0\right) D\left(\tilde{u}^i\right)\right) + \frac{1}{\rho_0} \nabla \tilde{P}^i = \frac{1}{\rho_0} F_{(u_L + \tilde{u}^i)}\left(u_L + \tilde{u}^i, \nabla P_L + \nabla \tilde{P}^i\right), \\ \hfill \operatorname{div}\left(A_{(u_L + \tilde{u}^i)}(u_L + \tilde{u}^i)\right) = 0, \\ \hfill \tilde{u}_{|t=0} = 0. \end{cases}$$

We are now in the position of performing exactly the same computations as above such that we obtain a time $T^{'}$ sufficiently small such that:

$$\left(\bar{u}^1, \nabla \bar{P}^1\right) = \left(\bar{u}^2, \nabla \bar{P}^2\right) \text{ on } [0, T'].$$

It is classical that the above local uniqueness property extends to all $[0, T]$.

This concludes the proof of Theorem 4.2.2

Return to the Eulerian coordinates and proof of Theorem 4.2.1: Finally, we are in the position of proving the result announced in Theorem 4.2.1. Considering $(\rho_0, u_0) \in \dot{B}^{\frac{3}{p}+\gamma}_{p,1} \times \dot{B}^{\frac{3}{p}-1}_{p,1}$ and applying Theorem 4.2.2, there exists a positive $T > 0$ such that we may construct a solution $\left(\bar{u}, \nabla \bar{P}\right)$ to the system (4.2.5) in F_T. Then, working with a smaller T if needed and considering $X_{\bar{u}}$, the "flow" of $\bar{u}$ defined by (1.4.6), by using Proposition 1.4.3 from the Appendix, one obtains that for all $t \in [0, T]$, $X_{\bar{u}}$ is a measure preserving C^1-diffeomorphism over $\mathbb{R}^3$. Thus we may introduce the Eulerian variable:

$$\rho(t, x) = \rho_0\left(X_{\bar{u}}^{-1}(t, x)\right), \ u(t, x) = \bar{u}\left(t, X_{\bar{u}}^{-1}(t, x)\right) \text{ and } P(t, x) = \bar{P}\left(t, X_{\bar{u}}^{-1}(t, x)\right).$$

Then, Proposition 1.4.2 ensures that $(\rho, u, \nabla P)$ is a solution of (4.2.1). As $DX_{\bar{u}} - Id$ belongs to $\dot{B}^{\frac{3}{p}}_{p,1}$, using Proposition 1.4.1, we may conclude that $(\rho, u, \nabla P)$ has the announced regularity.

The uniqueness property comes from the fact that considering two solutions $\left(\rho^i, u^i, \nabla P^i\right)$ of (4.2.1), $i = 1, 2$, and considering Y_{u^i} the flow of u^i we find that $\left(u^i\left(t, Y_{u^i}(t, y)\right), \nabla P^i\left(t, Y_{u^i}(t, y)\right)\right)$ are solutions of the system (4.2.5) with the same data. Thus, they are equal according to the uniqueness property announced in Theorem 4.2.2. Thus, on some nontrivial interval $[0, T'] \subset [0, T]$, (chosen such as the condition (1.4.8) holds), the solutions $\left(\rho^i, u^i, \nabla P^i\right)$ are equal. This local uniqueness property obviously entails uniqueness on all $[0, T]$.

This concludes the proof of Theorem 4.2.1

4.2.2 Extending the range of integrability indices for the case $\gamma = 1$

Let us mention that if we consider almost homogeneous fluids, we may apply the maximal regularity results for the Stokes problem established by Danchin and Mucha, [48], [51], in order to obtain a larger range of integrability coefficients. These approach produces new well-posedness results for the Navier-Stokes-Korteweg system i.e. $\gamma = 1$. More precisely:

Theorem 4.2.3. *Consider $p \in [1, 2n)$ and (ρ_0, u_0) such that*

$$\operatorname{div} u_0 = 0, \ \inf_{x \in \mathbb{R}^n} \rho_0(x) > 0, \ u_0 \in \dot{B}^{\frac{n}{p}-1}_{p,1} \text{ and } \rho_0 - \bar{\rho} \in \dot{B}^{\frac{n}{p}+1}_{p,1},$$

where $\bar{\rho}$ is a positive constant. Then, there exists a constant $c > 0$ such that if

$$\|\rho_0 - 1\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \leq c,$$

then the following holds true. There exists a time $T > 0$ and an unique solution $(\rho, u, \nabla P)$ of the system (4.2.1) with $\gamma = 1$ such that

$$\Big\{\rho - \bar{\rho} \in \mathcal{C}_T(\dot{B}^{\frac{3}{p}+1}_{p,1}(\mathbb{R}^3)),\ u \in \mathcal{C}_T(\dot{B}^{\frac{3}{p}-1}_{p,1}(\mathbb{R}^3)) \text{ and } \big(\partial_t u, \nabla^2 u, \nabla P\big) \in L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1}(\mathbb{R}^3)).$$

As the proof is very similar to the one given in the last section we do not give details of the computations. Indeed, in order to take advantage of the results of Proposition 4.1.6 we observe that system (4.2.5) can be written in the form

$$\begin{cases} \partial_t \bar{u} - \mu\left(\bar{\rho}\right) \operatorname{div} D(\bar{u}) + \nabla \bar{P} = F_{\bar{v}}\left(\bar{u}, \nabla \bar{P}\right), \\ \operatorname{div}\left(\bar{u}\right) = \operatorname{div}\left(\left(Id - \operatorname{adj}\left(DX_{\bar{v}}\right)\right) \bar{u}\right), \\ \bar{u}_{|t=0} = u_0, \end{cases}$$

where

$$\begin{aligned} F_{\bar{v}}\left(\bar{w}, \nabla \bar{Q}\right) := {} & \left(\rho_0 - 1\right) \partial_t \bar{w} + \operatorname{div}\left(\mu\left(\rho_0\right) A_{\bar{v}} D_{A_{\bar{v}}}\left(\bar{w}\right)\right) - \mu\left(1\right) \operatorname{div} D(\bar{w}) \\ & + \left(Id - A^T_{\bar{v}}\right) \nabla \bar{Q} + \operatorname{div}\left(k\left(\rho_0\right) A_{\bar{v}}\left(A^t_{\bar{v}} \nabla \rho_0\right) \otimes \left(A^t_{\bar{v}} \nabla \rho_0\right)\right). \end{aligned}$$

As above, considering

$$\begin{cases} \partial_t u_L - \mu\left(1\right) \operatorname{div} D(u_L) + \nabla \bar{P}_L = 0, \\ \operatorname{div}\left(u_L\right) = 0, \\ u_{L|t=0} = u_0. \end{cases}$$

we first solve the system

$$\begin{cases} \partial_t \tilde{u} - \mu\left(\bar{\rho}\right) \operatorname{div} D(\tilde{u}) + \nabla \tilde{P} = F_{\bar{v}}\left(u_L + \tilde{u}, \nabla P_L + \nabla \tilde{P}\right), \\ \operatorname{div}\left(\bar{u}\right) = \operatorname{div}\left(\left(Id - \operatorname{adj}\left(DX_{\bar{v}}\right)\right) \bar{u}\right), \\ \bar{u}_{|t=0} = 0, \end{cases}$$

by considering the operator $\Phi : \tilde{F}_T \to \tilde{F}_T$, defined as $\Phi\left(\tilde{w}, \nabla \tilde{Q}\right) = \left(\tilde{u}, \nabla \tilde{P}\right)$ the unique solution of

$$\begin{cases} \partial_t \tilde{u} - \mu\left(1\right) \operatorname{div} D(\tilde{u}) + \nabla \tilde{P} = F_{\bar{v}}\left(u_L + \tilde{w}, \nabla P_L + \nabla \tilde{Q}\right), \\ \operatorname{div}\left(\bar{u}\right) = \operatorname{div}\left(\left(Id - \operatorname{adj}\left(DX_{\bar{v}}\right)\right) \tilde{u}\right), \\ \bar{u}_{|t=0} = u_0, \end{cases}$$

the existence and estimation of which is provided by R. Danchin and P.B. Mucha's result, Proposition 4.1.6. The rest of the proof follows closely the one presented in the previous section.

Appendix A

Appendix: Littlewood-Paley theory

We present here a few results of Fourier analysis used through the text. The full proofs along with other complementary results can be found in [11]. In the following if $\Omega \subset \mathbb{R}^n$ is a domain then $\mathcal{D}(\Omega)$ will denote the set of smooth functions on Ω with compact support and $\mathcal{S}$ will denote the Schwartz class of functions defined on $\mathbb{R}^n$. Also, we consider $\mathcal{S}'$ the set of tempered distributions on $\mathbb{R}^n$.

1.1 The dyadic partition of unity

Let us begin by recalling the so called Bernstein lemma:

Lemma 1.1.1. *Let $\mathcal{C}$ be a given annulus and B a ball of $\mathbb{R}^n$. Let us also consider any nonnegative integer k, a couple $p, q \in [1, \infty]^2$ with $p \leq q$ and any functions $u, v \in L^p$ such that* $\mathrm{Supp}(\hat{u}) \subset \lambda B$ *and* $\mathrm{Supp}(\hat{v}) \subset \lambda\mathcal{C}$. *Then, there exists a constant C such that the following inequalities hold true:*

$$\sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1} \lambda^{k+n\left(\frac{1}{p}-\frac{1}{q}\right)} \|u\|_{L^p}, \tag{1.1.1}$$

respectively

$$C^{-k-1} \lambda^k \|v\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^\alpha v\|_{L^p} \leq C^{k+1} \lambda^k \|v\|_{L^p}. \tag{1.1.2}$$

Next, let us introduce the dyadic partition of the space:

Proposition 1.1.1. *Let $\mathcal{C}$ be the annulus $\{\xi \in \mathbb{R}^n : 3/4 \leq |\xi| \leq 8/3\}$. There exist two radial functions $\chi \in \mathcal{D}(B(0, 4/3))$ and $\varphi \in \mathcal{D}(\mathcal{C})$ valued in the interval $[0, 1]$ and such that:*

$$\forall \xi \in \mathbb{R}^n, \quad \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1, \tag{1.1.3}$$

$$\forall \xi \in \mathbb{R}^n \backslash \{0\}, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \tag{1.1.4}$$

$$2 \leq |j - j'| \Rightarrow \mathrm{Supp}(\varphi(2^{-j}\cdot)) \cap \mathrm{Supp}(\varphi(2^{-j'}\cdot)) = \emptyset \tag{1.1.5}$$

$$j \geq 1 \Rightarrow \mathrm{Supp}(\chi) \cap \mathrm{Supp}(\varphi(2^{-j}\cdot)) = \emptyset \tag{1.1.6}$$

the set $\tilde{\mathcal{C}} = \mathcal{B}(0, 2/3) + \mathcal{C}$ is an annulus and we have

$$|j - j'| \geq 5 \Rightarrow 2^j \mathcal{C} \cap 2^{j'} \tilde{\mathcal{C}} = \emptyset. \tag{1.1.7}$$

Also the following inequalities hold true:

$$\forall \xi \in \mathbb{R}^n, \quad \frac{1}{2} \leq \chi^2(\xi) + \sum_{j \geq 0} \varphi^2(2^{-j}\xi) \leq 1, \tag{1.1.8}$$

$$\forall \xi \in \mathbb{R}^n \backslash \{0\}, \quad \frac{1}{2} \leq \sum_{j \in \mathbb{Z}} \varphi^2(2^{-j}\xi) \leq 1. \tag{1.1.9}$$

From now on we fix two functions χ and φ satisfying the assertions of Proposition 1.1.1 and let us denote by h respectively $\tilde{h}$ their Fourier inverses. For all $u \in \mathcal{S}'$, the nonhomogeneous dyadic blocks are defined as follows:

$$\begin{cases} \Delta_j u = 0 \;\text{ if }\; j \leq -2, \\ \Delta_{-1} u = \chi(D)\, u = \tilde{h} \star u, \\ \Delta_j u = \varphi\left(2^{-j} D\right) u = 2^{jd} \int_{\mathbb{R}^n} h\left(2^q y\right) u\left(x-y\right) dy \;\text{ if }\; j \geq 0. \end{cases} \tag{1.1.10}$$

The high frequency cut-off operator S_j is defined by

$$S_j u = \sum_{k \leq j-1} \Delta_k u.$$

The following lemma shows how to reconstruct a tempered distribution once we know its nonhomogeneous dyadic blocks:

Lemma 1.1.2. *For any $u \in \mathcal{S}'$ we have:*

$$u = \sum_{j \in \mathbb{Z}} \Delta_j u \quad \textit{in} \quad \mathcal{S}'\left(\mathbb{R}^n\right).$$

Let us define now the nonhomogeneous Besov spaces.

Definition 1.1.1. *Let $s \in \mathbb{R}$, $r \in [1, \infty]$. The Besov space $B^s_{2,r}$ is the set of tempered distributions $u \in \mathcal{S}'$ such that:*

$$\|u\|_{B^s_{2,r}} := \left\| \left(2^{js} \|\Delta_j u\|_{L^2}\right)_{j \in \mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})} < \infty.$$

The homogeneous dyadic blocks $\dot{\Delta}_j$ and the homogeneous low frequency cutt-off operators $\dot{S}_j$ are defined below:

$$\dot{\Delta}_j u = \varphi\left(2^{-j} D\right) u = 2^{jn} \int_{\mathbb{R}^n} h\left(2^j y\right) u\left(x-y\right) dy$$
$$\dot{S}_j u = \chi\left(2^{-j} D\right) u = 2^{jn} \int_{\mathbb{R}^n} \tilde{h}\left(2^j y\right) u\left(x-y\right) dy$$

for all $j \in \mathbb{Z}$.

Definition 1.1.2. *We denote by $\mathcal{S}'_h$ the space of tempered distributions such that:*

$$\lim_{j \to -\infty} \left\| \dot{S}_j u \right\|_{L^\infty} = 0.$$

Let us now define the homogeneous Besov spaces:

Definition 1.1.3. *Let s be a real number and $(p, r) \in [1, \infty]$. The homogenous Besov space $\dot{B}^s_{p,r}$ is the subset of tempered distributions $u \in S'_h$ such that:*

$$\|u\|_{\dot{B}^s_{p,r}} := \left\| \left(2^{js} \left\| \dot{\Delta}_j u \right\|_{L^p}\right)_{j \in \mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})} < \infty.$$

1.2 Properties of nonhomogeneous Besov spaces

The following propositions list important basic properties of Besov spaces that are used through the paper.

Proposition 1.2.1. *A tempered distribution $u \in S'$ belongs to $B^s_{p,r}$ if and only if there exists a sequence $(c_j)_j$ such that $\left(2^{js}c_j\right)_j \in \ell^r(\mathbb{Z})$ with norm 1 and a constant $C(u) > 0$ such that for any $j \in \mathbb{Z}$ we have*

$$\|\Delta_j u\|_{L^p} \leq C(u)\, c_j.$$

Proposition 1.2.2. *Let $s, \tilde{s} \in \mathbb{R}$ and $r, \tilde{r} \in [1, \infty]$.*

- *$B^s_{p,r}$ is a Banach space which is continuously embedded in $\mathcal{S}'$.*
- *The inclusion $B^s_{p,r} \subset B^{\tilde{s}}_{p,\tilde{r}}$ is continuous whenever $\tilde{s} < s$ or $s = \tilde{s}$ and $\tilde{r} > r$.*
- *We have the following continuous inclusion $B^{\frac{n}{p}}_{p,1} \subset \mathcal{C}_0$[1] $(\subset L^\infty)$.*
- *(Fatou property) If $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence of $B^s_{p,r}$ which tends to u in $\mathcal{S}'$ then $u \in B^s_{p,r}$ and*
$$\|u\|_{B^s_{p,r}} \leq \liminf_n \|u_n\|_{B^s_{p,r}}.$$
- *If $r < \infty$ then*
$$\lim_{j\to\infty} \|u - S_j u\|_{B^s_{p,r}} = 0.$$

Proposition 1.2.3. *Let us consider $m \in \mathbb{R}$ and a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ such that for all multi-index α, there exists a constant C_α such that:*

$$\forall \xi \in \mathbb{R}^n \quad |\partial^\alpha f(\xi)| \leq C_\alpha \left(1 + |\xi|\right)^{m-|\alpha|}.$$

Then the operator $f(D)$ is continuous from $B^s_{p,r}$ to $B^{s-m}_{p,r}$.

Proposition 1.2.4. *Let $1 \leq r \leq \infty$, $s \in \mathbb{R}$ and $\varepsilon > 0$. For all $\phi \in \mathcal{S}$, the map $u \to \phi u$ is compact from $B^s_{2,r}$ to $B^{s-\varepsilon}_{2,r}$.*

The next result deals with product estimates in nonhomogeneous Besov spaces.

Theorem 1.2.1. *A constant C exists such that the following holds true. Consider a real number $s > 0$ and any $(p, p_1, p_2, p_3, p_4, r) \in [1, \infty]^6$ such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

For any $(u, v) \in L^{p_2} \times \left(L^{p_4} \cap B^s_{p_1,r}\right)$ such that $\nabla u \in B^{s-1}_{p_3,r}$ we have:

$$\|uv\|_{B^s_{p,r}} \leq \|u\|_{L^{p_2}} \|v\|_{B^s_{p_1,r}} + \|\nabla u\|_{B^{s-1}_{p_3,r}} \|v\|_{L^{p_4}}.$$

All the above statements are given or immediate consequences of the results presented in [11], pages $107 - 108$.

[1] $\mathcal{C}_0$ is the space of continuous bounded functions which decay at infinity.

1.2.1 Commutator estimates

This section is devoted to establish commutator-type estimates both in homogeneous and in nonhomogeneous Besov spaces. We begin by stating the following basic, yet very useful lemma:

Lemma 1.2.1. *Let us consider θ a $\mathcal{C}^1$ function on $\mathbb{R}^n$ such that $(1+|\cdot|)\,\hat{\theta}\in L^1$. Let us also consider $p,q\in[1,\infty]$ such that:*

$$\frac{1}{r}:=\frac{1}{p}+\frac{1}{q}\leq 1.$$

Then, there exists a constant C such that for any Lipschitz function a with gradient in L^p, any function $b\in L^q$ and any positive λ:

$$\left\|\left[\theta\left(\lambda^{-1}D\right),a\right]b\right\|_{L^r}\leq C\lambda^{-1}\left\|\nabla a\right\|_{L^p}\left\|b\right\|_{L^q}.$$

In particular, when $\theta=\varphi$ and $\lambda=2^j$ we get that:

$$\left\|\left[\Delta_j,a\right]b\right\|_{L^r}\leq C2^{-j}\left\|\nabla a\right\|_{L^p}\left\|b\right\|_{L^q}.$$

Proposition 1.2.5. *Let us consider $s>0$ and $(p,p_1,r)\in[1,\infty]^3$ such that $p\leq p_1$. Also, let us consider two tempered distributions (u,v) such that $u\in B^M_{\infty,\infty}(\mathbb{R}^n)$, $\nabla u\in L^\infty(\mathbb{R}^n)\cap B^{s-1}_{p_1,r}(\mathbb{R}^n)$ and $\nabla v\in L^{p_2}(\mathbb{R}^n)\cap B^s_{p,r}(\mathbb{R}^n)$ where M is some strictly negative real number and $\frac{1}{p_2}=\frac{1}{p}-\frac{1}{p_1}$. We denote by*

$$R_j=[\Delta_j,u]\partial^\alpha v=\Delta_j\left(u\partial^\alpha v\right)-u\Delta_j\partial^\alpha v,$$

where $\alpha\in\overline{1,n}$. Then, the following estimate holds true:

$$\left\|\left(2^{js}\left\|R_j\right\|_{L^p}\right)\right\|_{\ell^r(\mathbb{Z})}\lesssim_s\left\|\nabla u\right\|_{L^\infty}\left\|\nabla v\right\|_{B^{s-1}_{p,r}}+\left\|\nabla v\right\|_{L^{p_2}}\left\|\nabla u\right\|_{B^{s-1}_{p_1,r}}.$$

Proof. The main ingredients of the proof are Lemma 1.2.1 and the Bony decomposition of the product into the paraproduct and remainder. For the definitions and some properties we refer to [11] pages 106-108. We begin by considering

$$\tilde{u}=u-\dot{S}_{-1}u$$

and we write that

$$R_j=\sum_{i=1,6}R^i_j$$

where

$$\begin{cases}R^1_j=\Delta_j\left(T_{\tilde{u}}\partial^\alpha v\right)-T_{\tilde{u}}\Delta_j\partial^\alpha v, & R^2_j=\Delta_j\left(T_{\partial^\alpha v}\tilde{u}\right),\\ R^3_j=T_{\Delta_j\partial^\alpha v}\tilde{u}, & R^4_j=\Delta_jR\left(\tilde{u},\partial^\alpha v\right),\\ R^5_j=R\left(\tilde{u},\Delta_j\partial^\alpha v\right). & R^6_j=\left[\Delta_j,S_{-1}u\right]\partial^\alpha v.\end{cases}$$

We begin with R^1_j. We notice that $\left(2^{lM}\left\|S_{l-1}\tilde{u}\Delta_l\partial^\alpha v\right\|_{L^\infty}\right)_{l\geq-1}\in\ell^\infty$ and

$$\left\|\left(2^{lM}\left\|S_{l-1}\tilde{u}\Delta_l\partial^\alpha v\right\|_{L^\infty}\right)_{l\geq-1}\right\|_{\ell^\infty}\leq C\left\|u\right\|_{B^M_{\infty,\infty}}\left\|\nabla v\right\|_{L^\infty}$$

such that the series $\sum_{l\geq-1}S_{l-1}u\Delta_l\partial^\alpha v$ is convergent. Thus, owing to Lemma 1.2.1 we may write that

$$2^{js}\left\|R^1_j\right\|_{L^p}\leq\sum_{|j-l|\leq4}2^{(j-l)s}2^{ls}\left\|\left[\Delta_j,S_{l-1}\tilde{u}\right]\Delta_l\partial^\alpha v\right\|_{L^p}\leq C\left\|\nabla u\right\|_{L^\infty}\sum_{|j-l|\leq4}2^{(j-l)(s-1)}2^{l(s-1)}\left\|\Delta_l\nabla v\right\|_{L^p}.$$

From Young's inequality we conclude that

$$\left\|\left(2^{js}\left\|R^1_j\right\|_{L^p}\right)\right\|_{\ell^r(\mathbb{Z})}\lesssim_s\left\|\nabla u\right\|_{L^\infty}\left\|\nabla v\right\|_{B^{s-1}_{p,r}}.$$

Let us pass to R_j^2. We have that:

$$2^{js}\left\|R_j^2\right\|_{L^p} \leq \sum_{|j-l|\leq 4} 2^{(j-l)s}2^{ls}\left\|S_{l-1}\partial^\alpha v\Delta_l\tilde{u}\right\|_{L^p} \leq C\left\|\nabla v\right\|_{L^{p_2}} \sum_{|j-l|\leq 4} 2^{(j-l)s}2^{ls}\left\|\Delta_l\tilde{u}\right\|_{L^{p_1}}$$
$$\leq C\left\|\nabla v\right\|_{L^{p_2}} \sum_{|j-l|\leq 4} 2^{(j-l)s}2^{l(s-1)}\left\|\Delta_l\nabla\tilde{u}\right\|_{L^{p_1}}.$$

From Young's inequality we conclude that

$$\left\|\left(2^{js}\left\|R_j^2\right\|_{L^p}\right)\right\|_{\ell^r(\mathbb{Z})} \lesssim_s \left\|\nabla v\right\|_{L^{p_2}}\left\|\nabla u\right\|_{B^{s-1}_{p_1,r}}.$$

The third term is treated similarly:

$$2^{js}\left\|R_j^3\right\|_{L^p} \leq \sum_{j-l\leq 5} 2^{(j-l)s}2^{ls}\left\|S_{l-1}\partial^\alpha v\Delta_l\tilde{u}\right\|_{L^p} \leq C\left\|\nabla v\right\|_{L^{p_2}} \sum_{j-l\leq 5} 2^{(j-l)s}2^{l(s-1)}\left\|\Delta_l\nabla\tilde{u}\right\|_{L^{p_1}}.$$

Because $s>0$ we can apply Young's inequality and obtain that

$$\left\|\left(2^{js}\left\|R_j^3\right\|_{L^p}\right)\right\|_{\ell^r(\mathbb{Z})} \lesssim_s \left\|\nabla v\right\|_{L^{p_2}}\left\|\nabla u\right\|_{B^{s-1}_{p_1,r}}.$$

As for the fourth term

$$2^{js}\left\|R_j^4\right\|_{L^p} \leq \sum_{j-l\leq 5} 2^{(j-l)s}2^{ls}\left\|\Delta_l\tilde{u}\right\|_{L^\infty}\left\|\tilde{\Delta}_l\partial^\alpha v\right\|_{L^p} \leq C\sum_{j-l\leq 5} 2^{(j-l)s}2^{l(s-1)}\left\|\Delta_l\nabla\tilde{u}\right\|_{L^\infty}\left\|\tilde{\Delta}_l\partial^\alpha v\right\|_{L^p}.$$

Because $s>0$ we can apply Young's inequality and obtain that:

$$\left\|\left(2^{js}\left\|R_j^4\right\|_{L^p}\right)\right\|_{\ell^r(\mathbb{Z})} \lesssim_s \left\|\nabla u\right\|_{L^\infty}\left\|\nabla v\right\|_{B^{s-1}_{p,r}}.$$

In a similar manner we get that:

$$\left\|\left(2^{js}\left\|R_j^5\right\|_{L^p}\right)\right\|_{\ell^r(\mathbb{Z})} \lesssim_s \left\|\nabla u\right\|_{L^\infty}\left\|\nabla v\right\|_{B^{s-1}_{p,r}}.$$

Finally, the sixth term is treated as it follows. First we write that there exists a integer N_0 such that:

$$\left[\Delta_j,\dot{S}_{-1}u\right]\partial^\alpha v = \sum_{l\geq -1}\left[\Delta_j,\dot{S}_{-1}u\right]\Delta_l\partial^\alpha v = \sum_{|l-j|\leq N_0}\left[\Delta_j,\dot{S}_{-1}u\right]\Delta_l\partial^\alpha v.$$

This comes from:

$$\operatorname{Supp}\left(\dot{\Delta}_j\left(\dot{S}_{-1}u\Delta_l\partial^\alpha v\right)\right) \subset 2^j\mathcal{C}\cap\left(B\left(0,\frac{2}{3}\right)+2^l\mathcal{C}\right) \subset 2^j\mathcal{C}\cap 2^l\left(B\left(0,\frac{2}{3}\right)+\mathcal{C}\right)$$

and from the fact that $B\left(0,\frac{2}{3}\right)+\mathcal{C}$ is an annulus. Thus from Lemma 1.2.1 we get that :

$$2^{js}\left\|R_j^6\right\|_{L^p} \leq \sum 2^{(j-l)(s-1)}2^{l(s-1)}\left\|\nabla\dot{S}_{-1}u\right\|_{L^\infty}\left\|\Delta_l\partial^\alpha v\right\|_{L^p}$$

and consequently

$$\left\|\left(2^{js}\left\|R_j^6\right\|_{L^p}\right)\right\|_{\ell^r(\mathbb{Z})} \lesssim_s \left\|\nabla u\right\|_{L^\infty}\left\|\nabla v\right\|_{B^{s-1}_{p,r}}.$$

By putting together the estimates for R_j^1,R_j^2,R_j^3,R_j^4,R_j^5,R_j^6, we end the proof of Proposition 1.2.5. $\square$

1.3 Properties of the homogeneous Besov spaces

The next propositions gather some basic properties of Besov spaces.

Proposition 1.3.1. *Let us consider $s \in \mathbb{R}$ and $p, r \in [1, \infty]$ such that*

$$s < \frac{n}{p} \text{ or } s = \frac{n}{p} \text{ and } r = 1. \tag{1.3.1}$$

Then $\left(\dot{B}^s_{p,r}, \|\cdot\|_{\dot{B}^s_{p,r}}\right)$ is a Banach space.

Proposition 1.3.2. *A tempered distribution $u \in S'_h$ belongs to $\dot{B}^s_{p,r}(\mathbb{R}^n)$ if and only if there exists a sequence $(c_j)_j$ such that $\left(2^{js} c_j\right)_j \in \ell^r(\mathbb{Z})$ with norm 1 and a constant $C = C(u) > 0$ such that for any $j \in \mathbb{Z}$ we have*

$$\left\|\dot{\Delta}_j u\right\|_{L^p} \le C c_j.$$

Proposition 1.3.3. *Let us consider s_1 and s_2 two real numbers such that $s_1 < s_2$ and $\theta \in (0,1)$. Then, there exists a constant $C > 0$ such that for all $r \in [1,\infty]$ we have:*

$$\|u\|_{\dot{B}^{\theta s_1 + (1-\theta)s_2}_{p,r}} \le \|u\|^{\theta}_{\dot{B}^{s_1}_{p,r}} \|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,r}} \text{ and}$$
$$\|u\|_{\dot{B}^{\theta s_1 + (1-\theta)s_2}_{p,1}} \le \frac{C}{s_2 - s_1}\left(\frac{1}{\theta} + \frac{1}{1-\theta}\right) \|u\|^{\theta}_{\dot{B}^{s_1}_{p,\infty}} \|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,\infty}}$$

Proposition 1.3.4. *1) Let $1 \le p_1 \le p_2 \le \infty$ and $1 \le r_1 \le r_2 \le \infty$. Then, for any real number s, the space $\dot{B}^s_{p_1,r_1}$ is continuously embedded in $\dot{B}^{s-n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_2,r_2}$.*

2) Let $1 \le p < \infty$. Then, $\dot{B}^{\frac{n}{p}}_{p,1}$ is continuously embedded in $\left(C_0(\mathbb{R}^n), \|\cdot\|_{L^\infty}\right)$ the space of continuous functions vanishing at infinity.

Proposition 1.3.5. *For all $1 \le p, r \le \infty$ and $s \in \mathbb{R}$,*

$$\begin{cases} \dot{B}^s_{p,r} \times \dot{B}^{-s}_{p',r'} \to \mathbb{R}, \\ (u,v) \to \sum_j \left\langle \dot{\Delta}_j u, \tilde{\Delta}_j v \right\rangle, \end{cases} \tag{1.3.2}$$

where $\tilde{\Delta}_j := \dot{\Delta}_{j-1} + \dot{\Delta}_j + \dot{\Delta}_{j+1}$, defines a continuous bilinear functional on $\dot{B}^s_{p,r} \times \dot{B}^{-s}_{p',r'}$. Denote by $Q^{-s}_{p',r'}$ the set of functions $\phi \in \mathcal{S} \cap \dot{B}^{-s}_{p',r'}$ such that $\|\phi\|_{\dot{B}^{-s}_{p',r'}} \le 1$. If $u \in \mathcal{S}'_h$, then we have

$$\|u\|_{\dot{B}^s_{p,r}} \lesssim \sup_{\phi \in Q^{-s}_{p',r'}} \langle u, \phi \rangle_{\mathcal{S}' \times \mathcal{S}}.$$

Proposition 1.3.6. *Let us consider $1 < p, r < \infty$ and $s \in \mathbb{R}$. Furthermore, let $u \in \dot{B}^s_{p,r}$, $v \in \dot{B}^{-s}_{p',r'}$ and $\rho \in L^\infty \cap \mathcal{M}\left(\dot{B}^s_{p,r}\right) \cap \mathcal{M}\left(\dot{B}^{-s}_{p',r'}\right)$. Then, we have that*

$$(\rho u, v) = \sum_j \left\langle \dot{\Delta}_j(\rho u), \tilde{\Delta}_j v \right\rangle = \sum_j \left\langle \dot{\Delta}_j u, \tilde{\Delta}_j(\rho v) \right\rangle = (u, \rho v). \tag{1.3.3}$$

The proof of Proposition 1.3.6 follows from a density argument. Relation (1.3.3) clearly holds for functions from the Schwartz class: then we may write

$$\int_{\mathbb{R}^n} \rho u v = (\rho u, v) = (u, \rho v).$$

The condition $1 < p, r < \infty$ and $s \in \mathbb{R}$ ensures that u and v may be approximated by Schwartz functions.

An important feature of Besov spaces with negative index of regularity is the following:

Proposition 1.3.7. *Let $s<0$ and $1\le p,r\le\infty$. Let u be a distribution in S'_h. Then, u belongs to $\dot{B}^s_{p,r}$ if and only if*

$$\left(2^{js}\left\|\dot{S}_j u\right\|_{L^p}\right)_{j\in\mathbb{Z}}\in\ell^r(\mathbb{Z}).$$

Moreover, there exists a constant C depending only on the dimension n such that:

$$C^{-|s|+1}\|u\|_{\dot{B}^s_{p,r}}\le\left\|\left(2^{js}\left\|\dot{S}_j u\right\|_{L^p}\right)_{j\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})}\le C\left(1+\frac{1}{|s|}\right)\|u\|_{\dot{B}^s_{p,r}}.$$

The next proposition tells us how certain multipliers act on Besov spaces.

Proposition 1.3.8. *Let us consider A a smooth function on $\mathbb{R}^n\backslash\{0\}$ which is homogeneous of degree m. Then, for any $(s,p,r)\in\mathbb{R}\times[1,\infty]^2$ such that*

$$s-m<\frac{n}{p}\text{ or }s-m=\frac{n}{p}\text{ and }r=1$$

the operator[2] $A(D)$ maps $\dot{B}^s_{p,r}$ continuously into $\dot{B}^{s-m}_{p,r}$.

The next proposition describes how smooth functions act on homogeneous Besov spaces.

Proposition 1.3.9. *Let f be a smooth function on $\mathbb{R}$ which vanishes at 0. Let us consider $(s,p,r)\in\mathbb{R}\times[1,\infty]^2$ such that*

$$0<s<\frac{n}{p}\text{ or }s=\frac{n}{p}\text{ and }r=1.$$

Then for any real-valued function $u\in\dot{B}^s_{p,r}\cap L^\infty$, the function $f\circ u\in\dot{B}^s_{p,r}\cap L^\infty$ and we have

$$\|f\circ u\|_{\dot{B}^s_{p,r}}\le C\left(f',\|u\|_{L^\infty}\right)\|u\|_{\dot{B}^s_{p,r}}.$$

Remark 1.3.1. *The constant $C\left(f',\|u\|_{L^\infty}\right)$ appearing above can be taken to be*

$$\sup_{i\in\overline{1,[s]+1}}\left\|f^{(i)}\right\|_{L^\infty\left(|[-M\|u\|_{L^\infty},-M\|u\|_{L^\infty}]\right)}$$

where M is a constant depending only on the dimension n.

1.3.1 Commutator and product estimates

Next, we want to see how the product acts in Besov spaces. The Bony decomposition, introduced in [24] offers a mathematical framework to obtain estimates of the product of two distributions, when the later is defined.

Definition 1.3.1. *Given two tempered distributions $u,v\in S'_h$, the homogeneous paraproduct of v by u is defined as:*

$$\dot{T}_u v=\sum_{j\in\mathbb{Z}}\dot{S}_{j-1}u\dot{\Delta}_j v. \tag{1.3.4}$$

The homogeneous remainder of u and v is defined by:

$$\dot{R}(u,v)=\sum_{j\in\mathbb{Z}}\dot{\Delta}_j u\dot{\Delta}'_j v \tag{1.3.5}$$

where

$$\dot{\Delta}'_j=\dot{\Delta}_{j-1}+\dot{\Delta}_j+\dot{\Delta}_{j+1}.$$

[2] $A(D)w=\mathcal{F}^{-1}(A\mathcal{F}w)$

Remark 1.3.2. *Notice that at a formal level, one has the following decomposition of the product of two (sufficiently well-behaved) distributions:*

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}\,(u,v) = \dot{T}_u v + \dot{T}'_v u.$$

The next result describes how the paraproduct and remainder behave.

Proposition 1.3.10. *1) Assume that* $(s,p,p_1,p_2,r) \in \mathbb{R}\times [1,\infty]^4$ *such that:*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2},\ s < \frac{n}{p} \text{ or } s = \frac{n}{p} \text{ and } r = 1.$$

Then, the paraproduct maps $L^{p_1} \times \dot{B}^s_{p_2,r}$ *into* $\dot{B}^s_{p,r}$ *and the following estimates hold true:*

$$\left\|\dot{T}_f g\right\|_{\dot{B}^s_{p,r}} \lesssim \|f\|_{L^{p_1}} \|g\|_{\dot{B}^s_{p_2,r}}.$$

2) Assume that $(s,p,p_1,p_2,r,r_1,r_2) \in \mathbb{R}\times [1,\infty]^6$ *and* $\nu > 0$ *such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2},\ \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$$

and

$$s < \frac{n}{p} - \nu \text{ or } s = \frac{n}{p} - \nu \text{ and } r = 1.$$

Then, the paraproduct maps $\dot{B}^{-\nu}_{p_1,r_1} \times \dot{B}^{s+\nu}_{p_2,r_2}$ *into* $\dot{B}^s_{p,r}$ *and the following estimate holds true:*

$$\left\|\dot{T}_f g\right\|_{\dot{B}^s_{p,r}} \lesssim \|f\|_{\dot{B}^{-\nu}_{p_1,r_1}} \|g\|_{\dot{B}^{s+\nu}_{p_2,r_2}}.$$

3) Let us consider $(s_1,s_2,p,p_1,p_2,r,r_1,r_2) \in \mathbb{R}^2 \times [1,\infty]^6$ *such that*

$$0 < s_1 + s_2 < \frac{n}{p} \text{ or } s_1 + s_2 = \frac{n}{p} \text{ and } r = 1.$$

Then, the remainder maps $\dot{B}^{s_1}_{p_1,r_1} \times \dot{B}^{s_2}_{p_2,r_2}$ *into* $\dot{B}^{s_1+s_2}_{p,r}$ *and*

$$\left\|\dot{R}\,(f,g)\right\|_{\dot{B}^{s_1+s_2}_{p,r}} \leq \|f\|_{\dot{B}^{s_1}_{p_1,r_1}} \|g\|_{\dot{B}^{s_2}_{p_2,r_2}}.$$

As a consequence we obtain the following product rules in Besov space:

Proposition 1.3.11. *Consider* $p \in [1,\infty]$ *and the real numbers* $\nu_1 \geq 0$ *and* $\nu_2 \geq 0$

$$\nu_1 + \nu_2 < \frac{n}{p} + \min\left\{\frac{n}{p}, \frac{n}{p'}\right\}.$$

Then, the following estimate holds true:

$$\|fg\|_{\dot{B}^{\frac{n}{p}-\nu_1-\nu_2}_{p,1}} \lesssim \|f\|_{\dot{B}^{\frac{n}{p}-\nu_1}_{p,1}} \|g\|_{\dot{B}^{\frac{n}{p}-\nu_2}_{p,1}}.$$

Proposition 1.3.12. *Assume that* s, ν *and* $p \in [1,\infty]$ *are such that*

$$0 \leq \nu \leq \frac{n}{p} \text{ and } -1 - \min\left\{\frac{n}{p}, \frac{n}{p'}\right\} < s \leq \frac{n}{p} - \nu.$$

Then, there exists a constant C *depending only on* s, ν, p *and* n *such that for all* $l \in \overline{1,n}$ *we have for some sequence* $(c_j)_{j\in\mathbb{Z}}$ *with* $\left\|(c_j)_{j\in\mathbb{Z}}\right\|_{\ell^1(\mathbb{Z})} = 1$*:*

$$\left\|\partial_l \left[a, \dot{\Delta}_j\right] w\right\|_{L^p} \leq C c_j 2^{-js} \|\nabla a\|_{\dot{B}^{\frac{n}{p}-\nu}_{p,1}} \|w\|_{\dot{B}^{s+\nu}_{p,1}}$$

for all $j \in \mathbb{Z}$.

For a proof of the above results we refer the reader to the Appendix of [46], Lemma $A.5.$ and Lemma $A.6.$

Proposition 1.3.13. *Consider a homogeneous function $A : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ of degree 0. Let us consider $s \in \mathbb{R}$, $0 < \nu \leq 1$ and $p, r, r_1, r_2 \in [1, \infty]$ such that*

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$$

and

$$s < \frac{n}{p} - \nu \text{ or } s = \frac{n}{p} - \nu \text{ and } r_2 = 1. \tag{1.3.6}$$

Moreover, assume that $w \in \dot{B}^{s+\nu}_{p,r_2}$ and that $a \in L^\infty$ with $\nabla a \in \dot{B}^{-\nu}_{\infty,r_1}$. Then, the following estimate holds true:

$$\left\| [A(D), \dot{T}_a] w \right\|_{\dot{B}^{s+1}_{p,r}} \lesssim \|\nabla a\|_{\dot{B}^{-\nu}_{\infty,r_1}} \|w\|_{\dot{B}^{s+\nu}_{p,r_2}}. \tag{1.3.7}$$

As this result is of great importance in the analysis of the pressure term, we present a sketched proof below (see also [11], Chapter 2, Lemma 2.99)

Proof. The fact that $a \in L^\infty$ along with relation (1.3.6) guarantees that $A(D)w \in \dot{B}^{s+\nu}_{p,r}$ and that the paraproducts $\dot{T}_a w$ and $\dot{T}_a A(D) w$ are well-defined. We observe that there exists a function $\tilde{\varphi}$ supported in some annulus which equals 1 on the support of φ such that one may write (of course it is here that we use the homogeneity of A):

$$[A(D), \dot{T}_a] w = \sum_j \left[(A\tilde{\varphi})(2^{-j}D), \dot{S}_{j-1} a \right] \dot{\Delta}_j w.$$

But according to Lemma 1.2.1 we have

$$2^{j(s+1)} \left\| \left[(A\tilde{\varphi})(2^{-j}D), \dot{S}_{j-1} a \right] \dot{\Delta}_j w \right\|_{L^p} \lesssim 2^{-j\nu} \left\| \nabla \dot{S}_{j-1} a \right\|_{L^\infty} 2^{j(s+\nu)} \left\| \dot{\Delta}_j w \right\|_{L^p}.$$

The last relation obviously implies (1.3.7). □

As a consequence of the above proposition and Proposition 1.3.10 we get the following:

Proposition 1.3.14. *Let us consider a homogeneous function $A : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ of degree 0, $s \in \mathbb{R}$, $0 < \nu \leq 1$ and $p, r, r_1, r_2 \in [1, \infty]$ such that*

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$$

and

$$-1 - \min\left\{ \frac{n}{p}, \frac{n}{p'} \right\} < s < \frac{n}{p} - \nu \text{ or } s = \frac{n}{p} - \nu \text{ and } r = r_2 = 1. \tag{1.3.8}$$

assume that $w \in \dot{B}^{s+\nu}_{p,r_2}$ and that $a \in L^\infty$ with $\nabla a \in \dot{B}^{\frac{n}{p}-\nu}_{p,r_1}$. Then, the following estimate hold true:

$$\|[A(D), a] w\|_{\dot{B}^{s+1}_{p,r}} \lesssim \|\nabla a\|_{\dot{B}^{\frac{n}{p}-\nu}_{p,r_1}} \|w\|_{\dot{B}^{s+\nu}_{p,r_2}}.$$

1.4 Properties of Lagrangian coordinates

The following results are gathered from papers [46] respectively [49]. More precisely, a proof of Propositions 1.4.1, 1.4.2, the estimate (1.4.15) of Proposition 1.4.4 respectively the estimate (1.4.20) of Proposition 1.4.5, can be found in [46], pages 782 – 786. Propositions 1.4.4 and 1.4.5 can be found in the Appendix of [49]. Proposition 1.4.3 is inspired from [49].

Proposition 1.4.1. *Let X be a globally defined bi-lipschitz diffeomorphism of $\mathbb{R}^3$ and $-\frac{3}{p'} < s \leq \frac{3}{p}$. Then $a \to a \circ X$ is a self map over $\dot{B}^s_{p,1}$ whenever*

$$1)\ s \in (0,1)\,;$$

$$2)\ s \geq 1 \text{ and } (DX - Id) \in \dot{B}^{\frac{3}{p}}_{p,1}.$$

The following result interferes in a crucial manner in the proof of the well-posedness result for the inhomogeneous incompressible Navier-Stokes system.

Proposition 1.4.2. *Let m be a C^1 scalar function over $\mathbb{R}^3$ and $u \in \mathbb{R}^3$ a C^1 vector field. Let X be a C^1 diffeomorphism and we denote by $J := \det(DX)$. Suppose that $J > 0$. Then, the following relations hold true:*

$$(\nabla m) \circ X = J^{-1} \operatorname{div}\left(\operatorname{adj}(DX) m \circ X\right), \tag{1.4.1}$$

$$(\operatorname{div} u) \circ X = J^{-1} \operatorname{div}\left(\operatorname{adj}(DX) u \circ X\right). \tag{1.4.2}$$

Corollary 1.4.1. *Let m be a C^1 scalar function over $\mathbb{R}^3$ and $u \in \mathbb{R}^3$ be a C^1 vector field. Let X be a C^1 diffeomorphism and $J := \det(DX)$. Suppose that $J > 0$. Then, we have that*

$$J^{-1} \operatorname{div}\left(\operatorname{adj}(DX) u\right) = Du : (DX)^{-1}, \tag{1.4.3}$$

$$J^{-1} \operatorname{div}\left(\operatorname{adj}(DX) m\right) = \left[(DX)^{-1}\right]^T \nabla m. \tag{1.4.4}$$

Proof. In order to ease reading we denote by $F_{|x} := F(x)$. Writing u as $u \circ X \circ X^{-1}$, using the chain rule and Einstein convention over repeated index we write that

$$\begin{aligned}
(\operatorname{div} u)_{|x} &= \partial_k (u^i \circ X)_{|X^{-1}(x)} \partial_i \left(X^{-1}\right)^k_{|x} \\
&= D\left(u \circ X\right)_{|X^{-1}(x)} : D\left(X^{-1}\right)_{|x} \\
&= D\left(u \circ X\right)_{|X^{-1}(x)} : (DX)^{-1}_{|X^{-1}(x)}
\end{aligned}$$

and thus, we get that

$$(\operatorname{div} u) \circ X = D\left(u \circ X\right) : (DX)^{-1}. \tag{1.4.5}$$

Thus, using (1.4.2) and (1.4.5) we get that:

$$\begin{aligned}
J^{-1} \operatorname{div}\left(\operatorname{adj}(DX) u\right) &= J^{-1} \operatorname{div}\left(\operatorname{adj}(DX) u \circ X^{-1} \circ X\right) = \left(\operatorname{div} u \circ X^{-1}\right) \circ X \\
&= D\left(u \circ X^{-1} \circ X\right) : (DX)^{-1} = Du : (DX)^{-1}.
\end{aligned}$$

In a similar manner we prove (1.4.4). □

For any $\bar{v}$ a time dependent vector field we set:

$$X_{\bar{v}}(t,x) = x + \int_0^t \bar{v}(\tau, x)\, d\tau \tag{1.4.6}$$

and we denote

$$A_{\bar{v}} = \left(DX_{\bar{v}}\right)^{-1}. \tag{1.4.7}$$

It is crucial to know (in order to pass back in Eulerian coordinates, for instance) when $X_{\bar{v}}$ is a global diffeomorphism. In order to achieve this, we will use the following theorem due to Hadamard

Theorem 1.4.2 (Hadamard)**.** *Let $X : \mathbb{R}^n \to \mathbb{R}^n$ a function of class C^1. Then, the following affirmation are equivalent:*

1) *X is a local diffeomorphism and $\lim_{|x|\to\infty} |X(x)| = \infty$.*
2) *X is a global C^1-diffeomorphism over $\mathbb{R}^n$.*

For a proof of this result one can consult for instance [66].

Proposition 1.4.3. *Let us consider $\bar{v} \in C_b([0,T], \dot{B}^{\frac{3}{p}-1}_{p,1})$ with $\partial_t \bar{v}$, $\nabla^2 \bar{v} \in L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})$. Then, there exists a positive α such that if*

$$\|\nabla \bar{v}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} \leq \alpha, \tag{1.4.8}$$

then, for any $t \in [0,T]$, $X_{\bar{v}}(t,\cdot)$ introduced in (1.4.6) is a global C^1-diffeomorphism over $\mathbb{R}^3$ and $\det(DX_{\bar{v}}) > 0$. *Moreover, if*

$$\operatorname{div}(\operatorname{adj}(DX_{\bar{v}})\bar{v}) = 0 \tag{1.4.9}$$

then, $X_{\bar{v}}$ is measure preserving i.e.

$$\det DX_{\bar{v}} = 1. \tag{1.4.10}$$

Proof. Differentiating $X_{\bar{v}}$, we obtain

$$DX_{\bar{v}}(t,\cdot) = Id + \int_0^t D\bar{v}(\tau,\cdot)\, d\tau$$

and because of the embedding of $\dot{B}^{\frac{3}{p}}_{p,1}$ into the space of continuous functions, see Proposition 1.3.4, we conclude that $X_{\bar{v}} \in C^1\left([0,T] \times \mathbb{R}^3\right)$. We observe that

$$\begin{aligned}\|DX_{\bar{v}}(t,\cdot) - Id\|_{L^\infty(\mathbb{R}^3)} &\leq \int_0^t \|D\bar{v}(\tau,\cdot)\|_{L^\infty}\, d\tau \\ &\leq C\|\nabla \bar{v}\|_{L^1_t(\dot{B}^{\frac{3}{p}}_{p,1})} \leq \alpha C\end{aligned}$$

thus choosing α sufficiently small ensures that $X_{\bar{v}}(t,\cdot)$ is a local C^1-diffeomorphism over $\mathbb{R}^3$. The second condition of Hadamard's Theorem is verified in the following lines. Using the triangle inequality we get that:

$$\begin{aligned}|X_{\bar{v}}(t,x)| \geq |x| - \int_0^t |\bar{v}(\tau,x)|\, d\tau &\geq |x| - \int_0^t \|\bar{v}(\tau,\cdot)\|_{L^\infty}\, d\tau \\ &\geq |x| - C\int_0^t \|\bar{v}(\tau,\cdot)\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\, d\tau \\ &\geq |x| - C\sqrt{t}\,\|\bar{v}\|_{L^2_t(\dot{B}^{\frac{3}{p}}_{p,1})}.\end{aligned}$$

The conclusion is that $X_{\bar{v}}(t,\cdot)$ is a global C^1-diffeomorphism over $\mathbb{R}^3$. Let us denote $J_{\bar{v}} := \det DX_{\bar{v}} \neq 0$. Using Jacobi's formula we get that

$$J_{\bar{v}}(t,x) = 1 + \int_0^t \operatorname{tr}\left(D\bar{v}(\tau,x)\operatorname{adj}(DX_{\bar{v}})(\tau,x)\right) d\tau.$$

Recall that according to [49], Lemma A.4. we may write

$$Id - \operatorname{adj}(DX_{\bar{v}}) = \int_0^t (D\bar{v} - \operatorname{div}\bar{v}\, Id)\, d\tau + P_2\left(\int_0^t D\bar{v}\, d\tau\right)$$

where the coefficients of the matrix $P_2 : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ are at least quadratic polynomial functions of degree $n-1$. Using this identity combined with the embedding $L^\infty \hookrightarrow \dot{B}^{\frac{3}{p}}_{p,1}$ and the smallness condition (1.4.8) we get that $J_{\bar{v}} > 0$. In order to prove the second part of the proposition, let us denote by

$$v\left(t,x\right) \stackrel{not.}{:=} \bar{v}\left(t, X_{\bar{v}}^{-1}\left(t,x\right)\right).$$

Using relation (1.4.9) combined with (1.4.2) we get that

$$0 = J_{\bar{v}}^{-1} \operatorname{div}\left(\operatorname{adj}(DX_{\bar{v}})\bar{v}\right) = \operatorname{div}\left(\bar{v} \circ X_{\bar{v}}^{-1}\right) \circ X_{\bar{v}}$$

which implies that

$$\operatorname{div} v = \operatorname{div}\left(\bar{v} \circ X_{\bar{v}}^{-1}\right) = 0.$$

Since $X_{\bar{v}}$ can be viewed as being the flow of v, using Jacobi's formula we can conclude the validity of (1.4.10). Indeed, we have that

$$\begin{aligned} X_{\bar{v}}\left(t,x\right) &= x + \int_0^t \bar{v}\left(\tau, x\right) d\tau = x + \int_0^t \bar{v}\left(\tau, X_{\bar{v}}^{-1}\left(\tau, X_{\bar{v}}\left(\tau,x\right)\right)\right) d\tau \\ &= x + \int_0^t v\left(\tau, X_{\bar{v}}\left(\tau,x\right)\right) d\tau. \end{aligned}$$

Then, Jacobi's formula implies that

$$\det\left(DX_{\bar{v}}\right)\left(t,x\right) = \exp\left(\int_0^t \left(\operatorname{div} v\right)\left(\tau, X_{\bar{v}}\left(\tau,x\right)\right)\right) = 1.$$

□

Proposition 1.4.4. *Let us consider $\bar{v} \in C_b([0,T], \dot{B}_{p,1}^{\frac{3}{p}-1})$ with $\partial_t \bar{v}$, $\nabla^2 \bar{v} \in L_T^1(\dot{B}_{p,1}^{\frac{3}{p}-1})$ satisfying the smallness condition (4.2.8). Let $X_{\bar{v}}$ be defined by (1.4.6) and $J_{\bar{v}} = \det DX_{\bar{v}}$. Then for all $t \in [0,T]$:*

$$\|Id - A_{\bar{v}}\left(t\right)\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \lesssim \|\nabla \bar{v}\|_{L_t^1(\dot{B}_{p,1}^{\frac{3}{p}})}, \tag{1.4.11}$$

$$\|Id - \operatorname{adj}(DX_{\bar{v}})\left(t\right)\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \lesssim \|\nabla \bar{v}\|_{L_t^1(\dot{B}_{p,1}^{\frac{3}{p}})}, \tag{1.4.12}$$

$$\|\partial_t \operatorname{adj}(DX_{\bar{v}})\left(t\right)\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}} \lesssim \|\nabla \bar{v}(t)\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}}, \text{ if } p < 6, \tag{1.4.13}$$

$$\|\partial_t \operatorname{adj}(DX_{\bar{v}})\left(t\right)\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \lesssim \|\nabla \bar{v}(t)\|_{\dot{B}_{p,1}^{\frac{3}{p}}}, \tag{1.4.14}$$

$$\left\|J_{\bar{v}}^{\pm}\left(t\right) - 1\right\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \lesssim \|\nabla \bar{v}\|_{L_t^1(\dot{B}_{p,1}^{\frac{3}{p}})}. \tag{1.4.15}$$

In order to establish stability estimates we use the following

Proposition 1.4.5. *Let $\bar{v}_1, \bar{v}_2 \in C_b([0,T], \dot{B}_{p,1}^{\frac{3}{p}-1})$ with $\partial_t \bar{v}_1, \partial_t \bar{v}_2$, $\nabla^2 \bar{v}_1, \nabla^2 \bar{v}_2 \in L_T^1(\dot{B}_{p,1}^{\frac{3}{p}-1})$, both satisfying the smallness condition (1.4.8) and $\delta v = \bar{v}_2 - \bar{v}_1$. Then we have:*

$$\|A_{\bar{v}_1} - A_{\bar{v}_2}\|_{L_T^\infty(\dot{B}_{p,1}^{\frac{3}{p}})} \lesssim \|\nabla \delta v\|_{L_T^1(\dot{B}_{p,1}^{\frac{3}{p}})}, \tag{1.4.16}$$

$$\|\operatorname{adj}(DX_{\bar{v}_1}) - \operatorname{adj}(DX_{\bar{v}_2})\|_{L_T^\infty(\dot{B}_{p,1}^{\frac{3}{p}})} \lesssim \|\nabla \delta v\|_{L_T^1(\dot{B}_{p,1}^{\frac{3}{p}})}, \tag{1.4.17}$$

$$\|\partial_t \operatorname{adj}(DX_{\bar{v}_1}) - \partial_t \operatorname{adj}(DX_{\bar{v}_2})\|_{L_T^1(\dot{B}_{p,1}^{\frac{3}{p}})} \lesssim \|\nabla \delta v\|_{L_T^1(\dot{B}_{p,1}^{\frac{3}{p}})}, \tag{1.4.18}$$

$$\|\partial_t \operatorname{adj}(DX_{\bar{v}_1}) - \partial_t \operatorname{adj}(DX_{\bar{v}_2})\|_{L_T^2(\dot{B}_{p,1}^{\frac{3}{p}-1})} \lesssim \|\nabla \delta v\|_{L_T^2(\dot{B}_{p,1}^{\frac{3}{p}-1})}, \text{ if } p < 6, \tag{1.4.19}$$

$$\left\|J_{\bar{v}_1}^{\pm}\left(t\right) - J_{\bar{v}_2}^{\pm}\left(t\right)\right\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \lesssim \|\nabla \delta v\|_{L_t^1(\dot{B}_{p,1}^{\frac{3}{p}})}. \tag{1.4.20}$$

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