Contributions to second order reflected backward stochastic differentials equations

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À ma' a Anna

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Résumé

Cette thèse traite des équations différentielles stochastiques rétrogrades(EDSR) du second ordre réfléchies dans une filtration générale . Dans un premier temps, nous considérons deux cas de réflexions, l'un qui avec un obstacle inferieur et l'autre avec un obstacle supérieur. Notre contribution consiste à démontrer l'existence et l'unicité de ses équations dans le cadre d'une filtration générale sous des hypothèses faibles sur le générateur, la condition terminal et l'obstacle. L'étape essentielle de l'existence étant la démonstration du principe de programmation dynamique, nous construisons une fonction valeur qui soit mesurable par rapport au temps, à l'espace et à la mesure de probabilité. Ainsi, nous avons recours au théorème de sélection mesurable pour prouver le principe de programmation dynamique. La non symétrie entre l'obstacle inferieur et l'obstacle supérieur dans le cadre des équations du second ordre est également mise en évidence. Ensuite nous considérons le problème d'approximation de la valeur initiale de la solution d'une EDSR réfléchie de second ordre. Ceci revient encore à donner une approximation d'un problème de contrôle sur les solutions d'EDSR standards réfléchies avec incertitude sur le modèle. Notre technique consiste à donner approximation en temps discret qui transfère l'incertitude sur le modèle en incertitude sur la volatilité et ensuite de montrer la convergence de cette approximation.

Mots clés: Equations différentielles stochastiques rétrogrades, équations différentielles stochastiques rétrogrades de second ordre réfléchies,approximation faible, contrôle stochastique robuste, modèle avec incertitude sur la volatilité, capacité, Ensemble analytique, problème avec obstacle, problème de Skorokhod, schéma numérique, équations différentielles stochastiques rétrogrades réfléchies.

Abstract

This thesis deals with second order reflected backward stochastic differential equations (2RBSDEs) in general filtration. At first, we consider two cases of reflection, one with a lower obstacle and the other with an upper obstacle. We prove existence and uniqueness of the solutions of these equations under weak assumptions about the generator, the terminal condition and the obstacle in the context of general filtration. The dynamic programming principle plays a key role in the proof of existence, we construct a value function that is measurable with respect to time, space and probability measure. Therefore, we use the measurable selection theorem to prove dynamic programming principle. The non-symmetry between the lower obstacle and the upper obstacle in the second-order framework is also highlighted. Then we consider the problem of approximation of the initial value of the solution of a 2RBSDE. This can be interpreted as an approximation of the value of stochastic control problem associated to standard reflected backward stochastic differentials equations solutions under model uncertainty. Our approach is based on the time discretization of the value of stochastic problem and the discretization of the model trough the discretization of the volatility process.

Key Words: Backward stochastic differential equations, second order reflected stochastic differential equations, weak approximation, robust stochastic control, Uncertainty Volatility Model, capacity, analytic set, obstacle problem, reflected backward stochastic differential equation, numerical scheme, skorokhod problem.

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Chapter 1

Introduction

1.1 Second order Reflected BSDEs

Second order Backward Stochastic Differential Equations (2BSDEs for short) were first introduced by Cheredito, Soner, Touzi and Victoir in [21] motivated by applications in financial mathematics and probabilistic numerical methods. Later, inspired by the quasi-sure stochastic analysis of Denis and Martini [28], Soner, Touzi and Zhang [91] gave the quasi-sure formulation of 2BSDEs. Given a filtered measurable space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T})$ generated by a d-dimensional canonical process B and a set of probability measures $\mathcal{P}_H^\kappa$ on $(\Omega, \mathcal{F})$, a solution to a one-dimensional 2BSDE consists in a triple of progressively measurable processes (Y, Z, K) taking values in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ such that

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s, \widehat{a}_s)ds - \int_t^T Z_s dB_s + K_T - K_t,\ t \in [0,T],\ \mathbb{P}\text{-a.s. } \forall \mathbb{P} \in \mathcal{P}_h.$$

were the generator f is a progressively measurable function and the terminal condition ξ is a $\mathcal{F}_T$-measurable random variable. Before we go into the details, we briefly recall the definition of Backward Stochastic Differential Equations (BSDEs for short) and their link with Partial Differential Equations(PDEs for short).

1.1.1 BSDEs and connection with semi-linear PDEs

Backward stochastic differential equations first appeared in Bismut [9] in the linear case but the theory of these equations was introduced by Pardoux and Peng [77]. Given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$ generated by a d-dimensional Brownian motion W , a solution to a one-dimensional BSDE consists in a couple of progressively measurable processes (Y, Z) taking values in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ such that

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds - \int_t^T Z_s dW_s,\ t \in [0,T],\ \mathbb{P}\text{-a.s.} \tag{1.1.1}$$

where the generator f is a progressively measurable function and the terminal condition is an $\mathcal{F}_T$-measurable random variable. Under the following conditions

- f is uniformly Lipschitz with respect to both y and z;
- ξ and $f_s(0,0)$ are square integrable,

Pardoux and Peng [77] proved the existence and uniqueness of the solution to the equation (1.1.1). These results were followed by many others in order to relax the condition on f among which we can mention the existence of a solution when

- f is uniformly Lipschitz w.r.t. z and has polynomial growth w.r.t. y (see Briand and Carmona [17]);
- f is only continuous in (y,z) with linear growth (see Lepeltier and San Martin [59]);
- f is quadratic growth w.r.t. z (see Kobylanski [55]);

Most of these results appear to be useful for problems in financial mathematics, stochastic control and differential games (see [37] and [43]), provide probabilistic interpretation for semi-linear partial differential equations etc. We also refer the reader to El Karoui, Hamadene and Matoussi [32] for some applications.

The connection between BSDEs and semi-linear PDEs can be established by considering Markovian BSDEs where the randomness of the generator and terminal condition comes from a diffusion process $(X_s, 0 \leq s \leq T)$ which is the strong solution of a standard Itô stochastic differential equation (SDE):

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s,\ 0 \leq t \leq T,\ \mathbb{P}\text{-a.s.}$$

Let consider the following semi-linear PDE

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) + \mathcal{L}u(t,x) + f(t,x,u(t,x),(\nabla u\sigma)(t,x)) = 0,\ \forall t \in [0,T],\ x \in \mathbb{R}^d, \\ &u(T,x) = \psi(x),\ \forall x \in \mathbb{R}^d. \end{aligned} \tag{1.1.2}$$

where $\mathcal{L}$ is the second order differential operator associated to X given by

$$\mathcal{L}u(t,x) := \frac{1}{2}\mathrm{Tr}\left[\sigma(t,x)\sigma'(t,x)\nabla^2 u(t,x)\right] + b(t,x)\cdot\nabla u(t,x).$$

Under suitable assumptions on f, ψ, b and σ the PDE (1.1.2) has a classic smooth solution and then processes $(Y,Z) = (Y_t, Z_t)_{t\in[0,t]} := (u(t,X_t), \nabla u(t,X_t)\sigma(t,X_t))_{t\in[0,t]}$ solves the following BSDEs

$$Y_t = \psi(X_T) + \int_t^T f_s(X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s,\ t \in [0,T],\ \mathbb{P}\text{-a.s.}$$

This link gives a probabilistic interpretation for solutions of the semi linear PDE (1.1.2) using the solution of the Markovian BSDE (1.1.1) and generalizes the Feynman-Kac formula to a semi-linear case. From that interpretation, one can use probabilistic methods for numerical simulations of solutions of semi-linear PDEs. Motivated by the applications of fully non-linear PDEs and probabilistic numerical methods for this class of PDEs in many fields especially in financial mathematics, Cheredito, Soner, Touzi and Victoir [21] defined the first formulation of 2BSDEs which are connected to fully non-linear PDEs.

1.1.2 Second order backward stochastic differential equations

We begin with the first formulation of 2BSDEs.

1.1.2.1 First formulation of 2BSDEs and connection with fully non-linear PDEs

Given a d-dimensional diffusion process X defined for every initial condition $(s,x) \in [0,T]\times\mathbb{R}^d$ by

$$\begin{aligned} dX_t^{s,x} &= b(X_t^{s,x})dt + \sigma(X_t^{s,x})dW_t,\ t \in (s,T],\ \mathbb{P}\text{-a.s.} \\ X_s^{s,x} &= x,\ \mathbb{P}\text{-a.s.} \end{aligned}$$

and two continuous functions h and ψ, Cheredito, Soner, Touzi and Victoir [21] introduced the following 2BSDE:

$$\begin{aligned} dY_t &= h(t, X_t^{s,x}, Y_t, Z_t, \Gamma_t)dt + Z_t' \circ dX_t^{s,x},\ t \in [s,T),\ \mathbb{P}\text{-a.s.} \\ dZ_t &= A_t dt + \Gamma_t dX_t^{s,x},\ t \in [s,T),\ \mathbb{P}\text{-a.s.} \\ Y_T &= \psi(X_T^{s,x}), \end{aligned} \tag{1.1.3}$$

where $Z_t' \circ dX_t^{s,x}$ denotes Fisk-Stratonovich integration, which is related to Iô integration by

$$Z_t' \circ dX_t^{s,x} = Z_t' dX_t^{s,x} + \frac{1}{2} d\langle Z, X^{s,x}\rangle_t.$$

A quadruple (Y, Z, Γ, A) of progressively measurable processes (w.r.t. the augmented filtration generated by $W^s := W_\cdot - W_s$) is called a solution of the 2BSDE associated to $(X^{s,x}, h, \psi)$ if Y, Z, Γ and A satisfy the relation (1.1.3).
The class of admissible strategies is $\mathcal{Z} := \bigcup_{m\geq 0} \mathcal{Z}_m$ where $\mathcal{Z}_m$ is the class of all processes of the form

$$Z_t = z + \int_0^t A_r dr + \int_0^t \Gamma_r dX_t^{s,x},\ t \in [s,T],$$

with Z, Γ and A such that

$$\max\{|Z_t|, |\Gamma_t|, |A_t|\} \leq m(1 + |X_t^{s,x}|^p),\ s \in [t,T]$$

and

$$|\Gamma_t| - |\Gamma_r| \leq m(1 + |X_t^{s,x}|^q + |X_r^{s,x}|^q)(|t-r| + |X_t^{s,x} - X_r^{s,x}|),\ r,t \in [s,T].$$

Let consider the following PDE

$$\begin{aligned} -\frac{\partial u}{\partial t}(t,x) + h(t,x,u(t,x), \nabla u(t,x), \nabla^2 u(t,x)) &= 0,\ (t,x) \in [0,T) \times \mathbb{R}^d \\ u(T,x) &= \psi(x), \forall x \in \mathbb{R}^d. \end{aligned} \tag{1.1.4}$$

and $\mathcal{L}$ the operator defined by

$$\mathcal{L}u(t,x) := \frac{\partial u}{\partial t}(t,x) + \frac{1}{2}\mathrm{Tr}\big[\nabla^2 u(t,x)\sigma(x)\sigma'(x)\big].$$

σ' is the transposed of σ. If u is a continuous function such that $\partial u/\partial t, \nabla u, \nabla^2 u$ and $\mathcal{L}\nabla u$ exists and are continuous and u solves the PDE (1.1.4), then it follows directly from Itô's formula that for each $(s,x) \in [0,T] \times \mathbb{R}^d$, the processes

$$\begin{aligned} Y_t &= u(t, X_t^{s,x}),\ t \in [s,T],\ \mathbb{P}\text{-a.s.}, \\ Z_t &= \nabla u(t, X_t^{s,x}),\ t \in [s,T],\ \mathbb{P}\text{-a.s.}, \\ \Gamma_t &= \nabla^2 u(t, X_t^{s,x}),\ t \in [s,T],\ \mathbb{P}\text{-a.s.}, \\ A_t &= \mathcal{L}\nabla u(t, X_t^{s,x}),\ t \in [s,T],\ \mathbb{P}\text{-a.s.} \end{aligned}$$

solve the 2BSDE (1.1.3) associated to $(X^{s,x}, h, \psi)$.

Uniqueness of the solution: The uniqueness of the solution (Y, Z, Γ, A) of 2BSDE(1.1.3) is proved only when the process Z belongs to the class of processes $\mathcal{Z}$ defined above. We notice that $\mathcal{Z}$ is limited and cannot be extended to a class of square integrable processes. In general there is no uniqueness result if the solution is allowed to be a general square integrable process.

Existence of the solution: Except for the trivial case where the PDE(1.1.4) has a sufficiently smooth solution, the existence problem was left open in [21].
The necessity to provide a complete theory of existence and uniqueness of the solution of 2BSDEs on a less restricted setting lead to the reformulation of 2BSDE.

1.1.2.2 New formulation of 2BSDEs

The key idea for the new formulation is inspired by the quasi-sure analysis of Denis and Martini [28] in the sense that the 2BSDE must hold $\mathbb{P}$-a.s. for every $\mathbb{P}$ in a non-dominated class of (eventually mutually singular) probability measures. Before recalling the new formulation of 2BSDE, It is worth pointing out some intuition which will help to understand this formulation.

Example 1: G-expectation. The G-expectation is defined by Peng using the following fully non-linear PDE

$$\begin{aligned}&\frac{\partial u}{\partial t}(t,x) + G(\nabla^2 u(t,x)) = 0,\ (t,x) \in [0,T) \times \mathbb{R}^d\\ &u(T,x) = \psi(x), \forall x \in \mathbb{R}^d.\end{aligned}$$

where the time maturity is taken to be T. Let denote by $\mathbb{S}_d^{>0}$ the set of all real valued positive definite $d \times d$ matrices. For $\overline{a}, \underline{a} \in \mathbb{S}_d^{>0}$, the nonlinearity h is defined by

$$h(t,x,y,z,\gamma) := G(\gamma) = \frac{1}{2} \sup_{a \in \mathcal{A}} \mathrm{Tr}[a\gamma].$$

where $\mathcal{A} := \{a \in \mathbb{S}_d^{>0}, \underline{a} \leq a \leq \overline{a}\}$ and the order $\leq$ in the set $\mathcal{A}$ is in the sense of symmetric matrices.
We suppose that the above PDE has a smooth solution. Let also consider $\alpha := (\alpha_s)_{s \in [0,T]}$ a process taking values in $[\underline{a}, \overline{a}]$ and define X_t^α by

$$X_t^\alpha = \int_0^t \alpha_s^{1/2} dW_s.$$

Then by using the link establish in [21] between fully nonlinear PDEs and 2BSDEs, we have that $Y_t = u(t, X_t^\alpha)$ and $Z_t = \nabla u(t, X_t^\alpha)$ satisfies the following 2BSDE

$$\begin{aligned}Y_t &= \psi(X_T^\alpha) + \int_t^T G(\nabla^2 u)(s, X_s^\alpha) ds - \int_t^T Z_t' \circ dX_s^\alpha;\\ &= \psi(X_T^\alpha) + \int_t^T [G(\nabla^2 u) - \frac{1}{2}\alpha_s \nabla^2 u](s, X_s^\alpha) ds - \int_t^T Z_t' dX_s^\alpha;\\ &= \psi(X_T^\alpha) - \int_t^T Z_t' dX_s^\alpha + K_T - K_t.\end{aligned}$$

where $K_t := \int_0^t [G(\nabla^2 u) - \frac{1}{2}\alpha_s \nabla^2 u](s, X_s^\alpha) ds$ is a non-decreasing process with $K_0 = 0$. The last equation highlights the presence of a non-drecreasing process K in the equation.

Example 2. Let h defined by

$$h(t,x,y,z,\gamma) := \sup_{a \in \mathbb{S}_d^{>0}} \left\{ \frac{1}{2}\mathrm{Tr}[a\gamma] - f(t,x,y,z,a) \right\}.$$

where f is a Fenchel-Legendre transform of h. Let consider the following fully non-linear PDE

$$\begin{aligned}&-\frac{\partial u}{\partial t}(t,x) + h(t,x,u(t,x), \nabla u(t,x), \nabla^2 u(t,x)) = 0,\ (t,x) \in [0,T) \times \mathbb{R}^d\\ &u(T,x) = \psi(x), \forall x \in \mathbb{R}^d.\end{aligned}$$

Then it is natural to expect that the solution of this PDE is given by

$$u(t,x) = \sup_{a \in \mathbb{S}_d^{>0}} u^a(t,x),$$

where u^a solves the following PDE

$$-\frac{\partial u^a}{\partial t}(t,x) + \frac{1}{2}\mathrm{Tr}[a\nabla^2 u^a(t,x)] - f(t,x,u^a(t,x),\nabla u^a(t,x),a) = 0,\ (t,x) \in [0,T) \times \mathbb{R}^d$$
$$u^a(T,x) = \psi(x), \forall x \in \mathbb{R}^d.$$

Since the above PDE is semi-linear, it corresponds to the BSDE

$$y_t^a = \psi(X_T^a) + \int_t^T f(s, X_s^a, y_s^a, z_s^a, a_s)ds - \int_t^T z_s^a dX_s^a,$$

with

$$X_t^a = \int_0^t a_s^{1/2} dW_s.$$

Consequently, using the link between fully non-linear PDEs and 2BSDEs in Markovian case given in [21], a first component of a solution to a 2BSDE will write as

$$Y_t = \sup_{a \in \mathbb{S}_d^{>0}} y_t^a.$$

Formulation. Let $\Omega := \left\{\omega \in C([0,T],\mathbb{R}^d) \ :\ \omega_0 = 0\right\}$ the canonical space equipped with the uniform norm $\|\omega\|_\infty := \sup_{t\in[0,T]} |\omega_t|$, $\mathbb{P}_0$ the Wiener measure and B the canonical process.

Given a map $h(t,\omega,y,z,\gamma) : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_h \longrightarrow \mathbb{R}$ we define its conjugate f with respect to γ by

$$f(t,\omega,y,z,a) := \sup_{\gamma \in D_h} \left\{\frac{1}{2}\mathrm{Tr}[a\gamma] - h(t,\omega,y,z,\gamma)\right\}.$$

We denote by $D_{f_t(y,z)} := \{a,\ f_t(\omega,y,z,a) < \infty\}$ the domain of f in a for fixed (t,ω,y,z).

Models space: The local martingale measures. We outline the space of models defined in [91]. A probability measure $\mathbb{P}$ is a local martingale measure if the canonical process B is a local martingale under $\mathbb{P}$. By Karandikar [53], we know that we can give pathwise definition of quadratic variation $\langle B\rangle$ and its density $\widehat{a}$ with

$$\langle B\rangle_t := B_t B_t' - 2\int_0^t B_s dB_s' \quad \text{and} \quad \widehat{a}_t := \overline{\lim_{\epsilon \downarrow 0}}\ \frac{1}{\epsilon}(\langle B\rangle_t - \langle B\rangle_{t-\epsilon}).$$

Clearly, $\langle B\rangle$ coincides with the $\mathbb{P}$-quadratic variation of B, $\mathbb{P}$-a.s. for all local martingale measures $\mathbb{P}$.

Let $\overline{\mathcal{P}}_W$ denote the set of all local martingale measures $\mathbb{P}$ such that

$$\langle B\rangle_t \text{ is absolutely continuous in } t \text{ and } \widehat{a} \text{ takes values in } \mathbb{S}_d^{>0},\ \mathbb{P}\text{-a.s.}$$

We note that, for different $\mathbb{P}_1, \mathbb{P}_2 \in \overline{\mathcal{P}}_W$, in general $\mathbb{P}_1$ and $\mathbb{P}_2$ are mutually singular. The subclass $\overline{\mathcal{P}}_S \subset \overline{\mathcal{P}}_W$ consists in all probability measures $\mathbb{P}^\alpha$ with

$$\mathbb{P}^\alpha = \mathbb{P}_0 \circ (X^\alpha)^{-1} \text{ where } X^\alpha := \int_0^t \alpha_s^{1/2} dB_s,\ t \in [0,T],\ \mathbb{P}\text{-a.s.} \tag{1.1.5}$$

for some progressively measurable process α taking values in $\mathbb{S}_d^{>0}$ with $\int_0^T |\alpha_t| dt < \infty$ We recall from [92] that every $\mathbb{P} \in \tilde{\mathcal{P}}_S$ satisfies the Blumenthal zero-one law and the martingale representation property. We concentrate on the subclass $\tilde{\mathcal{P}}_S \subset \overline{\mathcal{P}}_S$ defined by

$$\tilde{\mathcal{P}}_S := \left\{ \mathbb{P}^\alpha \in \overline{\mathcal{P}}_S,\ \underline{a} \leq \alpha \leq \bar{a},\ \mathbb{P}_0\text{-a.s.} \right\} \tag{1.1.6}$$

for fixed matrices $\underline{a}$ and $\bar{a}$ in $\mathbb{S}_d^{>0}$.
We fix a constant $\kappa \in (1, 2]$ and restrict the probability measures in the following subset $\mathcal{P}_h^\kappa$:

Definition 1.1.1. *$\mathcal{P}_h^\kappa$ denotes the collection of all those $\mathbb{P} \in \tilde{\mathcal{P}}_S$ such that*

$$\mathbb{E}^{\mathbb{P}} \left[\left(\int_0^T |f_s(0,0,\widehat{a}_t)|^\kappa dt \right)^{\frac{2}{\kappa}} \right] < \infty.$$

$\mathbb{E}^{\mathbb{P}}$ denotes the expectation under $\mathbb{P}$. It is clear that $\mathcal{P}_h^\kappa$ is decreasing in κ, and $\widehat{a}_t \in D_{f_t(0,0)}$, $dt \times d\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}_h^\kappa$.

Remark 1.1.1. *We emphasize the fact that the bounds $(\underline{a}, \bar{a})$ are not uniform with respect to the underlined probability measure. In fact unlike in Denis and Martini [28], this restriction on the set of measure are not essential, because $\mathcal{P}_h^\kappa$ does not need to be relatively weakly compact.*

Definition 1.1.2. *We say a property holds $\mathcal{P}_h^\kappa$-quasi-surely ($\mathcal{P}_h^\kappa$-q.s. for short) if it holds $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}_h^\kappa$.*

We shall consider the following 2BSDE:

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s, \widehat{a}_s) ds - \int_t^T Z_s dB_s + K_T - K_t,\ t \in [0,T],\ \mathcal{P}_h^\kappa\text{-q.s..} \tag{1.1.7}$$

Definition 1.1.3. *We say that (Y, Z) is a solution to 2BSDE (1.1.7) if:*

- *$Y_T = \xi$, $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}_h^\kappa$;*
- *For all $\mathbb{P} \in \mathcal{P}_h^\kappa$, the process $K^{\mathbb{P}}$ defined below has nondecreasing paths $\mathbb{P}$-a.s.*

$$K_t^{\mathbb{P}} := Y_0 - Y_t - \int_0^t f_s(Y_s, Z_s, \widehat{a}_s) ds + \int_t^T Z_s dB_s,\ t \in [0,T],\ \mathbb{P}\text{-a.s.} \tag{1.1.8}$$

- *The family $\left\{ K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_h^\kappa \right\}$ satisfies the minimun condition:*

$$K_t^{\mathbb{P}} = \operatorname*{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_h^\kappa(t^+, \mathbb{P})}{}^{\mathbb{P}} \mathbb{E}^{\mathbb{P}'} [K_T^{\mathbb{P}'} | \mathcal{F}_t],\ t \in [0,T],\ \mathbb{P}\text{-a.s.}\ \ \forall \mathbb{P} \in \mathcal{P}_h, \tag{1.1.9}$$

where $\mathcal{P}_h^\kappa(t^+, \mathbb{P})$ is the set of all probability measures in $\mathcal{P}_h^\kappa$ which coincides with $\mathbb{P}$ until t^+.

Moreover, if the family $\left\{ K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_h^\kappa \right\}$ can be aggregated into a universal process K, we call (Y, Z, K) the solution of 2BSDE (1.1.7)

The process K is added to keep the process Y above all the solutions of BSDEs defined $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}_h^\kappa$ with the same generator f and same terminal condition ξ. With this formulation there is no longer one fixed probability measure but a non-dominated class of eventually mutually singular probability measures.

Existence and Uniqueness of the solution: Under the following condition

- $\mathcal{P}_h^\kappa$ is non empty, the domain D_f of f is independent of (ω, y, z);
- For $t \in [0,T]$, $y_1, y_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}^d$, we have

$$|f_t(y_1, z_1, \widehat{a}_t) - f_t(y_2, z_2, \widehat{a}_t)| \leq C\big(|y_1 - y_2| + \|\widehat{a}^{1/2}(z_1 - z_2)\|\big),\ \mathcal{P}_h^\kappa\text{-q.s.}$$

- $f_s(0,0,\widehat{a}_s)$ and ξ verify some integrability conditions;
- f and ξ are uniformly continuous in ω under the uniform convergence norm.

Soner, Touzi and Zhang [91] proved the existence and uniqueness of the solution to 2BSDE (1.1.7). The uniqueness is provided by the following representation formula: For any $\mathbb{P} \in \mathcal{P}_h^\kappa$ and $0 \leq t_1 \leq t_2 \leq T$,

$$Y_{t_1} = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_h^\kappa(t_1^+, \mathbb{P})} y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}),\ \mathbb{P}\text{-a.s.} \tag{1.1.10}$$

where for any $\mathbb{F}_+$-stopping time τ, and $\mathcal{F}_{\tau+}$-measurable square integrable random variable ξ, $y^{\mathbb{P}}$ derives from $(y^{\mathbb{P}}, z^{\mathbb{P}}) := (y^{\mathbb{P}}(\tau,\xi), z^{\mathbb{P}}(\tau,\xi))$ the unique solution to the following BSDE under a fixed probability $\mathbb{P}$

$$y_t^{\mathbb{P}} = \xi + \int_t^\tau f_s(y_s^{\mathbb{P}}, z_s^{\mathbb{P}}, \widehat{a}_s)ds - \int_t^\tau z_s^{\mathbb{P}} dB_s,\ t \in [0,\tau],\ \mathbb{P}\text{-a.s.}$$

With this representation formula, the process K can be interpreted as allowing Y to remain above all the processes $y^{\mathbb{P}}$, $\mathbb{P} \in \mathcal{P}_h^\kappa$.
The assumptions of uniform continuity in ω for f and ξ are used to prove the existence of the solution using dynamic programming principle(with regular conditional probability measures). The construction of the solution is done in their accompanying paper [92]. In the markovian case, a stochastic representation result for fully nonlinear parabolic PDEs is provided.
These results were followed by many others in order to relax the condition on f among which we can mention the case where ,

- f has quadratic growth (see Possamai and Zhou[86]);
- f has linear growth (see Possamai [82]);
- f has quadratic growth (see Possamai [83]).

Connection with fully non-linear PDEs: The connection between 2BSDEs and fully non-linear PDEs is established by considering the Markovian 2BSDEs where the randomness of the generator and terminal condition comes from the canonical process B. Given a deterministic map $h(t, B_t(\omega), y, z, \gamma) : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times D_h \longrightarrow \mathbb{R}$, then the corresponding conjugate and bi-conjugate functions become

$$f(t,x,y,z,a) := \sup_{\gamma \in D_h} \left\{ \frac{1}{2}\mathrm{Tr}[a\gamma] - h(t,x,y,z,\gamma) \right\}\ a \in \mathbb{S}_d^{>0};$$
$$\widehat{h}(t,x,y,z,\gamma) := \sup_{a \in \mathbb{S}_d^{>0}} \left\{ \frac{1}{2}\mathrm{Tr}[a\gamma] - f(t,x,y,z,a) \right\},\ \gamma \in \mathbb{R}^{d\times d}.$$

Let consider the following PDE

$$-\frac{\partial u}{\partial t}(t,x)+\widehat{h}(t,x,u(t,x),\nabla u(t,x),\nabla^2 u(t,x))=0,\ (t,x)\in[0,T)\times\mathbb{R}^d$$
$$u(T,x)=\psi(x),\forall x\in\mathbb{R}^d. \tag{1.1.11}$$

Under suitable assumptions on $\mathcal{P}_h^\kappa, D_f, f$ and ψ the PDE (1.1.11) has a classic smooth solution and then

$$Y_t:=u(t,B_t),\ Z_t:=\nabla u(t,B_t),\ K_t:=\int_0^t k_s ds,$$
$$\text{with } k_s:=\widehat{h}(t,B_t,Y_t,Z_t,\Gamma_t)-\frac{1}{2}\mathrm{Tr}[\widehat{a}_t\Gamma_t]-f(t,B_t,Y_t,Z_t,\widehat{a}_t)\ \text{ and }\Gamma_t:=\nabla^2 u(t,x)$$

solves the following 2BSDE:

$$Y_t=\psi(B_T)+\int_t^T f(s,B_s,Y_s,Z_s,\widehat{a}_s)ds-\int_t^T Z_s dB_s+K_T-K_t,\ t\in[0,T],\ \mathcal{P}_h^\kappa\text{-q.s..}$$

1.1.2.3 Some applications of 2BSDEs

Robust utility maximization in non-dominated models.

Utility maximization and connection with quadratic BSDEs: The utility maximization consists in a problem of optimal investment faced by an economic agent who has the opportunity to invest in a financial market consisting of a risk-less asset and for simplicity one risky asset. Given a fixed investment horizon T, the aim of the agent is to find an optimal allocation between the two assets, so as to maximize his welfare at time T. Following the seminal work of Von Neumann and Morgenstern[96], where they assumed that the preference of the agent could be represented by a utility function U and a given probability measure P reflecting his views, the now classical formulation of the problem consists in solving the optimization

$$V(x)=\sup_{\pi\in\mathcal{A}}\mathbb{E}^{\mathbb{P}}\left[U(X_T^{x,\pi}-\xi)\right],$$

where $\mathcal{A}$ is the set of admissible strategies π for the agent, $X_T^{x,\pi}$ is his wealth at time T with initial capital x and a trading strategy π, and ξ is a terminal liability. This optimal problem was solved first by Merton [70] in the particular case where the risky asset follows a Black-Scholes model in complete market, where the utility function is of power type and where the liability is equal to 0. The same problem in complete market but with a a general utility function was solved by Pliska [81].

As the hypothesis of complete market is too much restrictive and unrealistic from the point of view of applications, a large trend of literature was developed. Among which one can mention Cvitanic and Karatzas [25] and Zariphopoulou [98] where convex duality and control techniques were used to solve the problem and Kramkov et Schachermayer [52] in a general semimartingale model.

The appearance of the theory of BSDEs and their link with optimal investment problem allowed El Karoui and Rouge [38] to connect the value function of the problem with the initial value of a BSDE where the generator has quadratic growth w.r.t. to z. They considered the problem of indifference pricing with an exponential utility in the case where the strategies are constrained to stay in a given closed and convex set. The generalization of this approach to the case where the utility function is of logarithmic and power types and strategies in a closed set was done by Hu, Imkeller and Muller [47].

Robust utility maximization in dominated models: The problem of robust utility maximization with dominated models was introduced by Gilboa and Schmeilder [40]. An example is the uncertainty in the drift. Based on the robust utility maximization penalized by a relative entropy term of the model uncertainty with respect to a given probability reference introduced by Anderson, Hansen and Sargent [4] and Hansen et al. [45], many results have been obtained in dominated models. Many authors introduced a dominated set of probability measures which are absolutely continuous with respect to a reference probability measure . A set $\mathcal{P}$ is said to be dominated if every probability measure $\mathbb{P} \in \mathcal{P}$ is absolutely continuous with respect to some reference probability measure in $\mathcal{P}$. Then the problem can be written as follows

$$V^{rob}(x) = \sup_{\pi \in \mathcal{A}} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[U(X_T^{x,\pi} - \xi) \right],$$

where $\mathcal{P}$ is a set of all possible probabilities measures. With reference to results linked to BSDEs, Muller in [72] study the robust problem in the case when the drift is unkown, Bordigoni, Matoussi and Schweizer [10] have solved the robust problem in a more general semimartingale framework by using stochastic control techniques and proved that the solution was related to a quadratic semimartingale BSDE. Also, Gundel [41] and Schied and Wu [88] studied the case of continuous filtrations.

Robust utility maximization in non-dominated models: To consider a fixed reference probability measure, means that the investor knows the historical probability $\mathbb{P}$ that describes the dynamics of the state process. In reality, the investor may have some uncertainty on this probability, which means that several objective probability measures may be considered. Now the set $\mathcal{P}$ is no longer dominated, which happens when introducing volatility uncertainty, in the sense that the volatility process is only assumed to lie between two given bounds.

The robust utility maximization in non-dominated models was first studied with duality theory by Denis and Kervarec [27]. They worked with bounded utility function and their approach takes into account the uncertainty under both the drift and the volatility. Under suitable assumptions on the utility functions, they showed that there is a least favourable probability measure and an optimal strategy. More recently Matoussi, Possamai and Zhou [67] proved for exponential, power and logarithmic utilities that the value function of the problem can be written as

$$V^{rob}(x) = U(X_T^{x,\pi} - Y_0),$$

where Y_0 is the initial value of a quadratic 2BSDE. In [67] volatility uncertainty is considered, in the sense that the volatility of the market is only assumed to lie between two given bounds. The intuition in [67] is that, exactly as the problem of utility maximization under constraints was linked to BSDEs with quadratic growth, the problem of robust utility maximization under volatility uncertainty should be linked to some kind of backward equations. They showed that in incomplete markets with volatility uncertainty, the solution of the robust utility maximization problem is related to the initial value of a particular 2BSDE with quadratic growth and prove existence of an optimal strategy.

1.1.3 Second order Reflected backward stochastic differential equations

Second order reflected backward stochastic equations (2RBSDEs for short) with lower obstacle were introduced by Matoussi, Possamai and Zhou [66] motivated by the problem of pricing of American options in a market with uncertain volatility. Following the theory of 2BSDEs

treated in [91], they defined the 2BSDEs reflected with a lower obstacle. The solution of these equations are represented by a triple (Y, Z, K) satisfies:

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s, \widehat{a}_s)ds - \int_t^T Z_s dB_s + K_T - K_t,\ t \in [0,T],\ \mathcal{P}_h\text{-q.s.} \tag{1.1.12}$$

where K is a non decreasing process nul at 0.

In order to better understand this notion of 2RBSDEs, let recall the notion of Reflected Backward stochastic differentials equations (RBSDEs for short) with a lower obstacle.

1.1.3.1 Reflected BSDEs with a lower obstacle and related problems

Reflected BSDEs were introduced by El Karoui, Kapoudjian, Pardoux, Peng and Quenez in [35]. The principal idea is to keep the solution of a backward equations above a given stochastic process, called the obstacle. An increasing process is introduced to push the solution upwards, so that it may remain above the obstacle. The formulation of this problem is the following: Given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T})$ generated by a d-dimensional Brownian motion W, given also an obstacle process L, a generator f (progressively measurable) and a terminal condition $\xi(\mathcal{F}_T$-measurable), a solution to a one dimensional RBSDE consists in a triple (Y, Z, K) of processes taking value in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ such that

$$\begin{cases} Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds - \int_t^T Z_s dW_s + K_T - K_t,\ t \in [0,T],\ \mathbb{P}\text{-a.s.} \\ Y_t \geq L_t,\ t \in [0,T],\ \mathbb{P}\text{-a.s.}, \\ \int_0^T (Y_s - L_s)dK_s = 0,\ \mathbb{P}\text{-a.s.} \end{cases} \tag{1.1.13}$$

The process K is non-decreasing and null at 0 and this process is added in order to keep the solution Y above the obstacle L. The last condition means that the process K only acts when Y reaches the obstacle L, and provides the uniqueness of the solution. The uniqueness and existence are both proved in [35] by a fixed point argument (and Snell envelope) and by approximation via penalization. Existence and uniqueness of the solution was proved under the following conditions

- f is uniformly Lipschitz with respect to both y and z;
- L is continuous and progressively measurable;
- $\forall\ (y,z) \in \mathbb{R} \times \mathbb{R}^d$ $f(\cdot, y, z)$ and ξ are square integrable;
- $E^{\mathbb{P}}\big(\sup_{0\leq t\leq T}(S_t^+)^2\big) < \infty$.

More information about obstacle. If the obstacle L is a general semimartingale of the form

$$L_t = L_0 + \int_0^t U_s ds + C_t + \int_0^t S_s dW_s,$$

where C is continuous process of integrable variation such that the measure dC_t is singular with respect to dt and which admits a decomposition

$$C_t = C_t^+ - C_t^-,$$

where C_t^+ and C_t^- are increasing processes. Also U and S are $\mathbb{R}$ and $\mathbb{R}^d$ progressively measurable and satisfying

$$\int_0^T (|U_t|dt + |S_t|^2 dt) + C_T^+ + C_T^- < \infty,\ \mathbb{P}\text{-a.s.}$$

Then under the above assumptions, the solution (Y, Z, K) satisfies

$$Z_t = S_t, d\mathbb{P} \times dt \text{ a.e. on the set } \{Y_t = L_t\}$$

and there exists a predictable process $\{\alpha_t, 0 \leq t \leq T\}$ such that $0 \leq \alpha_t \leq 1$ and

$$dK_t = \alpha_t 1_{\{Y_t = L_t\}}([f(t, L_t, S_t) + U_t]^- dt + dC_t^-).$$

These results was followed by many others in order to relax the condition on the obstacle, the generator, the filtration, high dimension ... (see Lepeltier and Xu [60] for the right continuous with left limit obstacle, Hamadene [42] for discontinuous obstacle, Matoussi [64] and Lepeltier, Matoussi and Xu [58] for the generator with arbitrary growth in y, [44] for L^p-solutions).

In [35] they also proved that the solution of RBSDE is a value function of an optimal stopping problem and provided the probabilistic interpretation for the viscosity solution of an obstacle problem associated with a non-linear PDE.

RBSDEs and non-linear PDEs. The connection between RBSDEs and non linear PDEs is established as in section 1.1.1 by considering Markovian RBSDEs where the randomness of the generator, the terminal condition and the obstacle comes from a diffusion process $(X_s, 0 \leq s \leq T)$. Let consider the following PDE

$$\begin{cases} \min\left(u(t,x) - h(t,x), -\frac{\partial u}{\partial t}(t,x) - \mathcal{L}u(t,x) - f(t,x,u(t,x),(\nabla u\sigma)(t,x))\right) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall (t,x) \in (0,T) \times \mathbb{R}^d \\ u(T,x) = \psi(x), \forall x \in \mathbb{R}^d. \end{cases} \tag{1.1.14}$$

where $\mathcal{L}$ have the same expression to section 1.1.1. Under suitable assumptions, the PDE (1.1.14) has at most one viscosity solution in the class of continuous functions which grow at most polynomially at infinity and a viscosity solution of (1.1.14) is given by $u(t,x) := Y_t, (t,x) \in [0,T] \times \mathbb{R}^d$ where (Y, Z, K) is the solution of the following RBSDE

$$\begin{cases} Y_t = \psi(X_T) + \int_t^T f_s(X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + K_T - K_t, \ t \in [0,T], \ \mathbb{P}\text{-a.s.} \\ Y_t \geq h(t, X_t), \ t \in [0,T], \ \mathbb{P}\text{-a.s.}, \\ \int_0^T (Y_s - h(s, X_s))dK_s = 0, \ \mathbb{P}\text{-a.s.} \end{cases}$$

One can also refer to Bally, Caballero, El Karoui and Fernandez [5] to the related problems for PDEs.

RBSDEs and optimal stopping time. For the relation with optimal stopping time control problems established in [35] , the first component of the solution Y can be represented as a Snell envelope

$$Y_t = \operatorname*{ess\,sup}_{v \in \mathcal{T}_t} \mathbb{E}^{\mathbb{P}}\Big[\int_t^v f_s(Y_s, Z_s)ds + L_v 1_{\{t<T\}} + \xi 1_{\{t=T\}} | \mathcal{F}_t\Big], \ t \in [0,T].$$

where $\mathcal{T}_t$ is the set of stopping times taking value in $[t,T]$. They generalized the above representation, first to the case where f is a linear function of (y,z) and write the solution Y as the optimal stopping time problem. Second, they considered f a concave or convex function of (y,z) and obtained that Y is the value function of a mixture of an optimal stopping time problem. This is resumed by following results (Theorem 7.2 in [35]).

Theorem 1.1.1. *For each* $(\beta, \gamma) \in \mathcal{A}$,

$$Y_t^{\beta,\gamma} = \operatorname*{ess\,sup}_{v \in \mathcal{T}_t} \mathbb{E}^{\mathbb{P}}\left[\Phi(t, v, \beta, \gamma) | \mathcal{F}_t\right].$$

Moreover,

$$\begin{aligned} Y_t &= \operatorname*{ess\,inf}_{(\beta,\gamma)\in\mathcal{A}} Y_t^{\beta,\gamma} \\ &= \operatorname*{ess\,inf}_{(\beta,\gamma)\in\mathcal{A}} \operatorname*{ess\,sup}_{v\in\mathcal{T}_t} \mathbb{E}^{\mathbb{P}}\left[\Phi(t, v, \beta, \gamma) | \mathcal{F}_t\right] \\ &= \operatorname*{ess\,sup}_{v\in\mathcal{T}_t} \operatorname*{ess\,inf}_{(\beta,\gamma)\in\mathcal{A}} \mathbb{E}^{\mathbb{P}}\left[\Phi(t, v, \beta, \gamma) | \mathcal{F}_t\right]. \end{aligned}$$

In other words, Y_t *is the value function of a minimax control problem and there exists an optimal triple* (β^*, γ^*, D_t)*, where* $D_t = \inf\{t \le s \le T\,; Y_s = L_s\}$.

For more details about the functions Φ, the process $Y^{\beta,\gamma}$ and the set $\mathcal{A}$, refer to section 7 of [35].

RBSDEs and american options. The idea of El Karoui, Pardoux and Quenez in [36] is to consider the strategy wealth portfolio (Y_t, π_t) as a pair of adapted processes which satisfy the following BSDE:

$$-dY_t = b(t, Y_t, \pi_t)dt - \pi_t' \sigma_t dW_t,$$

where b is a $\mathbb{R}$-valued convex and Lipschitz function with respect to (y, π). In addition, the volatility matrix of d risky assets is invertible and its inverse $(\sigma_t)^{-1}$ is bounded. The classical case correspond to $b(t, y, \pi) = -r_t y - \pi_t \cdot \sigma_t \theta_t$, where θ_t is the risk premium vector.

In complete market, we are concerned with the problem of pricing an American contingent at time t, which consists in selection of a stopping time $v \ge t$ and a payoff L_v on exercise if $v < T$ and ξ if $v = T$. In other words, the payoff $\tilde{L}$ is represented by

$$\tilde{L}_v = L_v 1_{\{v<T\}} + \xi 1_{\{v=T\}}.$$

Then the price of an American contingent claim $(\tilde{L}_s, 0 \le s \le T)$ at time t is given by

$$Y_t = \operatorname*{ess\,sup}_{v\in\mathcal{T}_t} Y_t(v, \tilde{L}_v).$$

Moreover the price $(Y_t, 0 \le t \le T)$ corresponds to the unique solution of the RBSDE associated with terminal condition ξ, generator b and obstacle L, there exists (π_t) and (K_t) an increasing continuous process with $K_0 = 0$ such that for all $t \in [0, T]$,

$$\begin{cases} Y_t = \xi + \int_t^T b(s, Y_s, \pi_s)ds - \int_t^T \pi_s \sigma_s dW_s + K_T - K_t,\ t \in [0, T],\ \mathbb{P}\text{-a.s.} \\ Y_t \ge L_t,\ t \in [0, T],\ \mathbb{P}\text{-a.s.}, \\ \int_0^T (Y_s - L_s)dK_s = 0,\ \mathbb{P}\text{-a.s.} \end{cases}$$

Furthermore, the stopping time $D_t = \inf\{s \ge t, Y_s = L_s\} \wedge T$ is optimal after t, that is

$$Y_t = Y_t(D_t, \tilde{L}_{D_t}).$$

1.1.3.2 Reflected BSDEs driven by G-Brownian Motion

On the base that the G-expectation theory shares many similarities with second order BSDEs, we will recall briefly the notion of reflected BSDEs driven by a G-Brownian motion. Recently, two independent studies of reflected G-BSDEs have appeared, the first by Li and Peng [61] and the second in the PhD thesis of Soumana Hima [93]. A reflected G-BSDE is in the following form

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s + A_T - A_t,\ t \in [0,T], \\ Y_t \geq L_t,\ t \in [0,T],\ \mathbb{P}\text{-a.s.} \end{cases}$$

in addition to the following conditions:
- *Peng and Li [61]:* The process A is a continuous, nondecreasing and $\{-\int_0^t (Y_s - L_s)dA_s\}_{t\in[0,T]}$ is a nonincreasing G-martingale.
- *Soumana Hima [93]:* The process A admits the decomposition $A := R - K$, where R is a continuous increasing process null at 0 verified the following Skorokhod condition

$$\int_0^T (Y_s - L_s)dR_s = 0,\ q.s.$$

and K is a decreasing G-martingale, which also means that -K satisfies the minimum condition (1.1.9) (see [90] for representation of G-martingale).
Existence of solution The existence of the solution of the triplet (Y, Z, A)(and quadruplet (Y, Z, R, K) [93]) of the above reflected G-BSDE have been proved through penalization method. The assumptions: the generator is Lipschitz in y, z , the terminal condition, the generator and the obstacle satisfies some integrability conditions (with respect to G-expectation) and the obstacle is a G-Itô process.
Uniqueness of solution In [93], the uniqueness of (Y, Z, A) is proved under assumptions, but the uniqueness of the decomposition $A := R - K$ is not provided, whereas Li and Peng [61] proved the uniqueness of (Y, Z, A) under an additional assumption, which is the upper boundedness of the obstacle. For the uniqueness, they don't need the obstacle as a G-Itô process.

1.1.3.3 Second order Reflected BSDEs and American options under volatility uncertainty

Formulation. The 2RBSDEs are formulated in [66, 68] with a similar framework to [91]. These equations are the kind of combination of 2BSDEs and RBSDEs. The goal is to define quasi-surely (w.r.t. $\mathcal{P}_h$) the equations and force the solution to stay above a given obstacle, then a non-decreasing process is introduced. A solution of a 2RBSDE associated to a terminal condition ξ, a generator f and an obstacle L is a triple of processes (Y, Z, K) which verifies (1.1.12). The formulation do not need to add another nondecreasing process, unlike in the classical case and do not need to impose a condition similar to the Skorokhod condition. The only change necessary is in the minimal condition that the non-decreasing process must satisfy.

For any $\mathbb{P} \in \mathcal{P}_h$, $\mathbb{F}$-stopping time τ, let $(y^{\mathbb{P}}, z^{\mathbb{P}}, k^{\mathbb{P}}) := (y^{\mathbb{P}}(\tau, \xi), z^{\mathbb{P}}(\tau, \xi), k^{\mathbb{P}}(\tau, \xi))$ de note the unique solution to the following RBSDE with the lower obstacle L, the generator f and the terminal condition ξ(existence and uniqueness have been proved under appropriate assumptions by Lepeltier and Xu [60]).

$$\begin{cases} y_t^{\mathbb{P}} = \xi + \int_t^\tau f_s(y_s^{\mathbb{P}}, z_s^{\mathbb{P}}, \widehat{a}_s)ds - \int_t^\tau z_s^{\mathbb{P}} dB_s + k_\tau^{\mathbb{P}} - k_t^{\mathbb{P}},\ 0 \leq t \leq \tau,\ \mathbb{P}\text{-a.s.} \\ y_t^{\mathbb{P}} \geq L_t,\ t \in [0,T],\ \mathbb{P}\text{-a.s.}, \\ \int_0^\tau (y_{s^-}^{\mathbb{P}} - L_{s^-})dk_s^{\mathbb{P}} = 0,\ \mathbb{P}\text{-a.s.},\ \forall t \in [0,T] \end{cases} \tag{1.1.15}$$

The following definition is formulated in [66].

Definition 1.1.4. *We say that (Y, Z) is a solution to 2RBSDE (1.1.12) if:*

- $Y_T = \xi$ *and* $Y_t \geq L_t$ $t \in [0, T]$, $\mathcal{P}_h^\kappa$*-q.s. ;*
- *For all* $\mathbb{P} \in \mathcal{P}_h^\kappa$*, the process* $K^{\mathbb{P}}$ *defined below has nondecreasing paths* $\mathbb{P}$*-a.s.*

$$K_t^{\mathbb{P}} := Y_0 - Y_t - \int_0^t f_s(Y_s, Z_s, \widehat{a}_s)ds + \int_t^T Z_s dB_s,\ t \in [0, T],\ \mathbb{P}\text{-a.s.} \tag{1.1.16}$$

- *The family* $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_h^\kappa\}$ *satisfies the minimum condition:*

$$\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_h^\kappa(t^+, \mathbb{P})} \mathbb{E}^{\mathbb{P}'}\Big[\int_t^T G_s^{t,\mathbb{P}'} d(K_s^{\mathbb{P}'} - k_s^{\mathbb{P}'}) \big| \mathcal{F}_{t+}^{\mathbb{P}}\Big] = 0, \quad 0 \leq t \leq T,\ \mathbb{P}\text{-a.s.}, \forall \mathbb{P} \in \mathcal{P}_h^\kappa. \tag{1.1.17}$$

where for any $t \in [0, T]$ *and for any* $\mathbb{P} \in \mathcal{P}_0$ *, the process* $G^{t,\mathbb{P}}$ *is defined by*

$$G_s^{t,\mathbb{P}} := \exp\Big(\int_t^s (\lambda_u^{\mathbb{P}} - \frac{1}{2}\|\eta_u^{\mathbb{P}}\|^2)(Y_u, y_u^{\mathbb{P}}, Z_u, z_u^{\mathbb{P}})du + \int_t^s \eta_u^{\mathbb{P}}(Y_u, y_u^{\mathbb{P}}, Z_u, z_u^{\mathbb{P}}) \cdot \widehat{a}_u^{-1/2} dB_u\Big).$$

The processes $\lambda^{\mathbb{P}}$ *and* $\eta^{\mathbb{P}}$ *are bounded and determined by the Lipschitz assumption under the genrator* f*. We recall that* $\mathcal{P}_h^\kappa(t^+, \mathbb{P})$ *is the set of all probability measures in* $\mathcal{P}_h^\kappa$ *which coincides with* $\mathbb{P}$ *until* t^+*.*

In this case, the nondecreasing process K plays two roles, the first to maintain the solution Y above the obstacle process L and secondly to maintain the solution Y above all the solutions $y^{\mathbb{P}}$ of RBSDEs defined $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}_H^\kappa$ in (1.1.15).

Existence and uniqueness Under the following assumptions

- $\mathcal{P}_h^\kappa$ is non empty, the domain D_f of f is independent of (ω, y, z);
- For $t \in [0, T]$, $y_1, y_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}^d$, we have

$$|f_t(y_1, z_1, \widehat{a}_t) - f_t(y_2, z_2, \widehat{a}_t)| \leq C\big(|y_1 - y_2| + \|\widehat{a}^{1/2}(z_1 - z_2)\|\big),\ \mathcal{P}_h\text{-q.s.}$$

- $f_s(0, 0, \widehat{a}_s)$, ξ and L verify some integrability conditions;
- f, ξ and L are uniformly continuous in ω under the uniform convergence norm.

Matoussi, Possamai and Zhou [66] proved the existence and uniqueness of the solution to 2RBSDE (1.1.12). Following [91], the uniqueness derives from the representation formula:
For any $\mathbb{P} \in \mathcal{P}_h^\kappa$ and $0 \leq t \leq T$,

$$Y_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_h^\kappa(t^+, \mathbb{P})} y_t^{\mathbb{P}'},\ \mathbb{P}\text{-a.s.} \tag{1.1.18}$$

where for any $\mathbb{P} \in \mathcal{P}_h^\kappa$, $\mathbb{F}$-stopping time τ, $y^{\mathbb{P}}$ comes from $(y^{\mathbb{P}}, z^{\mathbb{P}}, k^{\mathbb{P}})$ the unique solution to the RBSDE (1.1.15).
The existence is obtained using the dynamic programming principle. The value function is chosen pathwise as as the supremum over a shifted set of probability measures of a solution of RBSDEs. The strong assumptions about uniform continuity in ω for f, ξ and L allows to obtain the dynamic programming principle. After that, they obtained a semi-martingale decomposition of the value function which ensures the existence of the solution to the 2RBSDE.

Symmetry about upper and lower obstacle ? In [66] the authors highlight that unlike with classical RBSDEs, considering a lower obstacle was fundamentally different from considering the upper obstacle. Because, having a lower obstacle corresponds to add an increasing process in the definition, then they still end up with an increasing process. However, in the case of upper obstacle, they would have to add a decreasing process, therefore ending up with a finite variation process(in place of an increasing process). Consequently, there is no longer any symmetry between lower and upper obstacles in the second-order framework. This remark have been verified in [65], where Matoussi, Piozin and Possamai studied the case of doubling reflected 2BSDEs. As usual to prove the existence result for doubling reflected 2BSDEs, they assumed that the upper obstacle is a semimartingale. This is directly linked to the fact that this is one of the conditions under which existence and uniqueness of solutions to standard doubly reflected BSDEs are guaranteed. More precisely, this assumption will be also crucial in order to obtain a priori estimates for 2BSDEs with two obstacles. This assumption is the heart of their approach, and theirs proofs no longer work without it. They also notice however that such an assumption was not needed for the lower obstacles considered in [66].

More details about the action of $K^{\mathbb{P}}$ and $k^{\mathbb{P}}$.

Proposition 1.1.1. *Under assumptions listed above, let (Y, Z) be the solution to the 2RBSDE (1.1.12) and $(y^{\mathbb{P}}, z^{\mathbb{P}}, k^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_h^\kappa}$ be the solutions to the corresponding RBSDEs (1.1.15) starting at $t = 0$. Then we have the following result. For all $t \in [0, T]$,*

$$\int_0^t 1_{\{Y_{s^-}=L_{s^-}\}} dK_s^{\mathbb{P}} = \int_0^t 1_{\{Y_{s^-}=L_{s^-}\}} dk_s^{\mathbb{P}}.$$

The above result tells us that if Y becomes equal to the obstacle L, then it suffices to push it exactly as in the standard RBSDE case. This is in accordance to the following intuition: When Y reaches L, since the representation formula (1.1.18), then all the $y^{\mathbb{P}}$ are also on the obstacle and therefore there is no need additional efforts other than $k^{\mathbb{P}}$ to the second order effects. Moreover, we have the following decomposition of $K^{\mathbb{P}}$

$$\begin{aligned} K_t^{\mathbb{P}} &= \int_0^t 1_{\{Y_{s^-}=L_{s^-}\}} dK_s^{\mathbb{P}} + \int_0^t 1_{\{Y_{s^-}>L_{s^-}\}} dK_s^{\mathbb{P}}, \quad \mathbb{P}\text{-a.s.} \\ &= \int_0^t 1_{\{Y_{s^-}=L_{s^-}\}} dk_s^{\mathbb{P}} + \int_0^t 1_{\{Y_{s^-}>L_{s^-}\}} dK_s^{\mathbb{P}}, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

the last equality comes from the above proposition. The last above inequality leads naturally to think that one could decompose $K^{\mathbb{P}}$ into two nondecreasing processes $A^{\mathbb{P}}$ and $V^{\mathbb{P}}$ with

$$V_t^{\mathbb{P}} = \int_0^t 1_{\{Y_{s^-}=L_{s^-}\}} dk_s^{\mathbb{P}},$$

satisfies the usual Skorokhod condition for RBSDEs, that is

$$\int_0^T (Y_{s^-} - L_{s^-}) dV_s^{\mathbb{P}} = 0, \ \ t \in [0, T], \ \mathbb{P}\text{-a.s.} \ \ \forall \mathbb{P} \in \mathcal{P}_h^\kappa. \tag{1.1.19}$$

and

$$\begin{aligned} A_t^{\mathbb{P}} &= K_t^{\mathbb{P}} - \int_0^t 1_{\{Y_{s^-}=L_{s^-}\}} dk_s^{\mathbb{P}}, \\ &= \int_0^t 1_{\{Y_{s^-}>L_{s^-}\}} dK_s^{\mathbb{P}}. \end{aligned}$$

On $\{Y_{t^-} = L_{t^-}\} \cup \{y^{\mathbb{P}}_{t^-} > L_{t^-}\}$, the process $A^{\mathbb{P}}$ satisfies the minimum condition (1.1.9) of 2BSDEs on , i.e.

$$A^{\mathbb{P}}_t = \operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}^{\kappa}_h(t^+,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}[A^{\mathbb{P}'}_T | \mathcal{F}_t],\ \mathbb{P}\text{-a.s.}\ \ \forall \mathbb{P} \in \mathcal{P}^{\kappa}_h.$$

and otherwise (that means on $\{Y_{t^-} > L_{t^-}\} \cap \{y^{\mathbb{P}}_{t^-} = L_{t^-}\}$) we cannot say anything. Such a decomposition isolates the effects due to the obstacle by $V^{\mathbb{P}}$ and the ones due to the second order by $A^{\mathbb{P}}$. Of course, the choice $V^{\mathbb{P}} := k^{\mathbb{P}}$ would be natural, given the minimum condition (1.1.19). However, this is not necessary true unless on $\{Y_{s^-} = L_{s^-}\}$. The existence of such a decomposition $K^{\mathbb{P}} := A^{\mathbb{P}} + V^{\mathbb{P}}$ with $K^{\mathbb{P}}$ and $A^{\mathbb{P}}$ satisfies the above properties on the whole space Ω, is therefore still an open problem in [66].

If we have more information on the obstacle L, then we can give a more explicit representation for the process $K^{\mathbb{P}}$. The following result comes from Proposition 1.1.1 and the representation in the paragraph 1.1.3.1.

More information about obstacle. If the obstacle L is a general semimartingale of the form

$$L_t = L_0 + \int_0^t U_s ds + C_t + \int_0^t S_s dB_s,$$

where C is càdlàg process of integrable variation such that the measure dC_t is singular with respect to dt and which admits a decomposition

$$C_t = C^+_t - C^-_t,$$

where C^+_t and C^-_t are nondecreasing processes. Also U and V are $\mathbb{R}$ and $\mathbb{R}^d$ progressively measurable and satisfying

$$\int_0^T (|U_t| dt + |S_t|^2 dt) + C^+_T + C^-_T < \infty,\ \mathcal{P}^{\kappa}_h\text{-q.s.}$$

Then under the above assumptions, the solution (Y, Z) to the 2RBSDE(1.1.12) satisfies

$$Z_t = S_t, dt \times \mathcal{P}^{\kappa}_h \text{ q.s. on the set } \{Y_{t^-} = L_{t^-}\}$$

and there exists a predictable process $\{\alpha^{\mathbb{P}}_t, 0 \le t \le T\}$ such that $0 \le \alpha^{\mathbb{P}}_t \le 1$ and

$$1_{\{Y_{t^-} = L_{t^-}\}} dK^{\mathbb{P}}_t = \alpha^{\mathbb{P}}_t 1_{\{Y_{t^-} = L_{t^-}\}} ([f(t, L_t, S_t, \widehat{a}_t) + U_t]^- dt + dC^-_t).$$

Connection with optimal stopping problem: Let (Y, Z) be the solution to the 2RBSDE 1.1.12, then for each $t \in [0, T]$ and for all $\mathbb{P} \in \mathcal{P}_h$,

$$\begin{aligned} Y_t &= \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}^{\kappa}_h(t^+,\mathbb{P})} \operatorname*{ess\,sup}_{v \in \mathcal{T}_t} \mathbb{E}^{\mathbb{P}'}\Big[\int_t^v f_s(y^{\mathbb{P}'}_s, z^{\mathbb{P}'}_s, \widehat{a}_s) ds + L_v 1_{\{t<T\}} + \xi 1_{\{t=T\}} | \mathcal{F}_t\Big],\ \mathbb{P}\text{-a.s.} \\ &= \operatorname*{ess\,sup}_{v \in \mathcal{T}_t} \mathbb{E}^{\mathbb{P}}\Big[\int_t^v f_s(Y_s, Z_s, \widehat{a}_s) ds + A^{\mathbb{P}}_v - A^{\mathbb{P}}_t + L_v 1_{\{t<T\}} + \xi 1_{\{t=T\}} | \mathcal{F}_t\Big],\ \mathbb{P}\text{-a.s.} \end{aligned}$$

where $\mathcal{T}_t$ is the set of all stopping times taking valued in $[t, T]$ where $A^{\mathbb{P}}_t := \int_0^t 1_{Y_{s^-} > L_{s^-}} dK^{\mathbb{P}}_s$ is the part of $K^{\mathbb{P}}$ which only increases when $Y_{s^-} > L_{s^-}$.

Connection with American options under volatility uncertainty: In incomplete market, the problem consists in pricing an American contingent claims under uncertain volatility. In [66], they consider a superhedging price for the contingent claim given by

$$Y_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_h^\kappa(t^+,\mathbb{P})} Y_t^{\mathbb{P}'}, \ \mathbb{P}\text{-a.s.}, \ \forall \mathbb{P} \in \mathcal{P}_h^\kappa,$$

where $Y_t^{\mathbb{P}}$ is the price at time t of an American contingent claim a complete market with underlying probability measure $\mathbb{P}$. Then the link with 2RBSDE is given by the following result

Theorem 1.1.2. *There exist an $\mathbb{F}_+$-progressively measurable process π and a non-decreasing càdlàg process K such that for all $t \in [0,T]$ and for all $\mathbb{P} \in \mathcal{P}_h^\kappa$,*

$$\begin{cases} Y_t = \xi + \int_t^T b(s, Y_s, \pi_s)ds - \int_t^T \pi_s \sigma_s dW_s + K_T - K_t, \ t \in [0,T], \ \mathbb{P}\text{-a.s.} \\ Y_t \geq L_t, \ t \in [0,T], \ \mathbb{P}\text{-a.s.}, \\ \operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_h^\kappa(t^+,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}\Big[\int_t^T G_s^{t,\mathbb{P}'} d(K_s^{\mathbb{P}'} - k_s^{\mathbb{P}'}) \big| \mathcal{F}_{t+}^{\mathbb{P}}\Big] = 0, \ \mathbb{P}\text{-a.s.} \end{cases}$$

Furthermore, for all ϵ, the stopping time $D_t^\epsilon = \inf\{s \leq t, \ Y_s \leq L_s + \epsilon, \ \mathcal{P}_h^\kappa\text{-q.s.}\} \wedge T$ is ϵ-optimal after t. Besides, for all $\mathbb{P}$, if we consider the stopping times $D_t^{\mathbb{P},\epsilon} = \inf\{s \leq t, \ Y_s \leq L_s + \epsilon, \ \mathbb{P}\text{-a.s.}\} \wedge T$, which are ϵ-optimal for the American options under each $\mathbb{P}$, then for all $\mathbb{P}$,

$$D_t^\epsilon \geq D_t^{\mathbb{P},\epsilon}, \ \mathbb{P}\text{-a.s.}$$

where for any $t \in [0,T]$ and for any $\mathbb{P} \in \mathcal{P}_0$, the process $G^{t,\mathbb{P}}$ is defined by

$$G_s^{t,\mathbb{P}} := \exp\Big(\int_t^s (\lambda_u^{\mathbb{P}} - \frac{1}{2}\|\eta_u^{\mathbb{P}}\|^2)(Y_u, Y_u^{\mathbb{P}}, \pi_u\sigma_u, \pi_u^{\mathbb{P}}\sigma_u)du + \int_t^s \eta_u^{\mathbb{P}}(Y_u, Y_u^{\mathbb{P}}, \pi_u\sigma_u, \pi_u^{\mathbb{P}}\sigma_u)dW_u^{\mathbb{P}}\Big).$$

1.2 Contribution

Chapter 2 provides basis tools and preliminary results which will be useful to establish results in the following chapters. It includes especially the reminder of universal measurability, regular conditional probability distribution, dynamic programming principle, measurability with respect to a probability law, RBSDEs in a general filtration, the weak convergence of filtrations and some applications.

1.2.1 2RBSDEs under weak assumption: the lower case

In Chapter 3, we study 2RBSDEs with a lower obstacle under weak assumptions in general filtration. Our models space (more precisely the space of probability measures) on which we are working is much more general than the ones considered in [91] and [66]. As mentioned earlier about existence and uniqueness results in the above papers, the uniformly continuous conditions on the terminal condition, the generator (and the obstacle in [66]) are made to obtain the continuity of the value function and hence the measurability of this one. This very strong regularity assumptions allows to obtain the dynamic programming principle by avoiding completely the use of measurable selection theorem. In section 2.3 of Chapter 2 we recall that one can use measurable selection theorem together with the stability of controls to derive the dynamic programming principle. It is this last method together with the measurability with respect a probability law introduced by Neufeld and Nutz [73] which is used by Possamai,Tan and Zhou [85] to prove existence and uniqueness of 2BSDEs without any regularity conditions on the terminal condition and the generators.

We start from the wellposdeness result in [85] which extends the 2BSDE theory in a general filtration with no regularity conditions on the parameters. We extend 2RBSDEs introduced by Matoussi, Possamai and Zhou [66] to a large class of probability measures in general filtration with no regularity conditions on the parameters.
We briefly recall the context of our work. Let $\Omega = \{\omega \in C([0,T],\mathbb{R}^d) : \omega_0 = 0\}$ ($C([0,T],\mathbb{R}^d)$ denotes the space of continuous map from $[0,T]$ to $\mathbb{R}^d$) be the canonical space equipped with the uniform convergence norm, X the canonical process and $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ the canonical filtration. We summarize the space of models defined in [85].

Models space: the semimartingale measures. A probability measure $\mathbb{P}$ is a semimartingale measure if the canonical process X is a semimartingale under $\mathbb{P}$. By Karandikar [53], we know that we can give pathwise definition of quadratic variation $\langle X\rangle$ and its density $\widehat{a}$ with

$$\langle X\rangle_t := X_tX_t' - 2\int_0^t X_s dX_s' \quad \text{and} \quad \widehat{a}_t := \overline{\lim_{\epsilon\downarrow 0}}\ \frac{1}{\epsilon}(\langle X\rangle_t - \langle X\rangle_{t-\epsilon}).$$

Clearly, $\langle X\rangle$ coincides with the $\mathbb{P}$-quadratic variation of X, $\mathbb{P}$-a.s. for all local semimartingale measures $\mathbb{P}$. For every $t\in[0,T]$, Let $\mathcal{P}_t^W$ denotes the collection of all probability measures $\mathbb{P}$ on $(\Omega,\mathcal{F}_T)$ such that

$-$ $(X_s)_{s\in[t,T]}$ is a $(\mathbb{P},\mathbb{F})$-semimartingale admitting the canonical decomposition

$$X_s = \int_t^s b_r^{\mathbb{P}} dr + X_s^{c,\mathbb{P}},\ s\in[t,T],\ \mathbb{P}\text{-a.s.},$$

where $b^{\mathbb{P}}$ is a predictable $\mathbb{R}^d$-valued process, and $X^{c,\mathbb{P}}$ is the continuous local martingale part of X under $\mathbb{P}$.

$-$ $(\langle X\rangle_s)_{s\in[t,T]}$ is absolutely continuous in s with respect to the Lebesgue measure, and $\widehat{a}$ takes values in $\mathbb{S}_d^{\ge 0}$, $\mathbb{P}$-a.s.($\mathbb{S}_d^{\ge 0}$ denotes the set of all symmetric positive semi-definite $d\times d$ matrices).
In the sequel, we consider the generator

$$f : (t,\omega,y,z,a,b)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}_d^{\ge 0}\times\mathbb{R}^d \longrightarrow \mathbb{R}.$$

The additional dependency b comes from the fact that we work with semimartingale and b represents the finite-variation process of the semimartingale. We also consider a random variable $\xi:\Omega\longrightarrow\mathbb{R}$, a lower obstacle process $(L_t)_{t\in[0,T]}$ and $(\mathcal{P}(t,\omega))_{(t,\omega)\in[0,T]\times\Omega}$ a class of probability measure families where $\mathcal{P}(t,\omega)\subset\mathcal{P}_t^W$ for all $(t,\omega)\in[0,T]\times\Omega$. We define the graph of $(\mathcal{P}(t,\omega))_{(t,\omega)\in[0,T]\times\Omega}$ by

$$[[\mathcal{P}]] := \{(t,\omega,\mathbb{P})\ :\ (t,\omega)\in[0,T]\times\Omega,\ \mathbb{P}\in\mathcal{P}(t,\omega)\}.$$

The class $(\mathcal{P}(t,\omega))_{(t,\omega)\in[0,T]\times\Omega}$ of semimartingale measures replaces the class $\tilde{\mathcal{P}}_S$ of local martingale measure defined in (1.1.2.2). This class is larger than $\tilde{\mathcal{P}}_S$ introduced in [91] and takes over in [66] in the sense that the quadratic variation of X does not need to be bounded in $\mathbb{S}_d^{>0}$ and the probability measures does not need to be represented as in (1.1.5). We notice that for $t=0$, we have $\mathcal{P}_0 := \mathcal{P}(0,\omega)$ for any $\omega\in\Omega$.

Assumptions. We assume that
$-$ The r.v. ξ is $\mathcal{F}_T$-measurable.
$-$ $f_s(0,0,\widehat{a}_s,b_s^{\mathbb{P}})$, ξ and L verify the following integrability conditions.

$$\sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}[|\xi|^p]<\infty,\quad \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}[\sup_{0\le t\le T}|L_t|^p]<\infty,\quad \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\big[(\int_0^T|f_s(0,0,\widehat{a}_s,b_s^{\mathbb{P}})|ds)^p\big]<\infty.$$

− The generator function f is jointly Borel measurable, Lipschitz with respect to y and z and the map $(t,\omega) \mapsto f(t,\omega,y,z,a,b)$ is $\mathbb{F}$-progressively measurable .

− The graph $[[\mathcal{P}]]$ of $\mathcal{P}$ is upper semi-analytic in $[0,T] \times \Omega \times \mathbb{M}_1$ (see definition 2.1.2 (4)).

− $\mathcal{P}$ is stable under conditioning and concatenation (see Assumption 2.3.1 of Chapter 2).

Remark 1.2.1. *Notice that in the assumptions about integrability of ξ, f and L are a sketch of integrability conditions, we will come back in more details in chapter 3 (see Assumption 3.3.1 for L and f and space $\mathbb{L}_0^{p,\kappa}$ define in (3.2.4) for ξ).*

The first three assumptions excepted the jointly Borel measurability are quite standard in the RBSDE literature. The last two assumptions comes from [85](see also [39]) and are classic to establish the measurability of a value function of the stochastic control problem over a family of probability measures. The jointly measurable assumption about time, space and probability measure is introduced in [73](and reminded in section 2.4). Unlike in [66], we do not need regularity conditions under the terminal value, the generator and the obstacle. These conditions were necessary in [66] to establish the existence of the solution through the dynamic programming principle. In this work, we use measurable selection theorem to provide the measurability of the value function.
We consider the following 2RBSDE with the lower obstacle L:

$$\begin{cases} Y_t = \xi + \int_t^T f_s(Y_s, \widehat{a}_s^{1/2} Z_s, \widehat{a}_s, b_s^{\mathbb{P}})ds - \int_t^T Z_s \cdot dX_s^{c,\mathbb{P}} - \int_t^T dM_t^{\mathbb{P}} + K_T^{\mathbb{P}} - K_t^{\mathbb{P}}, \\ Y_t \geq L_t, \qquad\qquad 0 \leq t \leq T, \ \mathbb{P}\text{-a.s. } \forall \mathbb{P} \in \mathcal{P}_0 \end{cases}$$

where $M^{\mathbb{P}}$ is a martingale orthogonal to $X^{c,\mathbb{P}}$.
For any $\mathbb{P} \in \mathcal{P}_0$, $\mathcal{F}_T$-measurable random variable ξ, we consider $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$ the following RBSDE

$$\begin{cases} y_t^{\mathbb{P}} = \xi + \int_t^T f_s(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}}, \widehat{a}_s, b_s^{\mathbb{P}})ds - \int_t^T z_s^{\mathbb{P}} \cdot dX_s^{c,\mathbb{P}} - \int_t^T dm_s^{\mathbb{P}} + k_T^{\mathbb{P}} - k_t^{\mathbb{P}}, \ 0 \leq t \leq T, \ \mathbb{P}\text{-a.s.} \\ y_t^{\mathbb{P}} \geq L_t, \ t \in [0,T], \ \mathbb{P}\text{-a.s.}, \\ \int_0^T (y_{s^-}^{\mathbb{P}} - L_{s^-}) dk_s^{\mathbb{P}} = 0, \ \mathbb{P}\text{-a.s.} \end{cases}$$

For any, $\mathbb{P} \in \mathcal{P}_0$, The existence and uniqueness of these solutions under $\mathbb{P}$ is given by Theorem 2.5.1 and the solution is a quadruple $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$ where $m^{\mathbb{P}}$ is a martingale orthogonal to the canonical process X.

Definition 1.2.1. *We say that (Y,Z) is a solution to the 2RBSDE if :*

(i) $Y_T = \xi$, *and* $Y_t \geq L_t$, $t \in [0,T]$, $\mathcal{P}_0$*-q.s.;*

(ii) $\forall \mathbb{P} \in \mathcal{P}_0$, *the process $K^{\mathbb{P}}$ has non-decreasing paths $\mathbb{P}$-a.s.*

$$K_t^{\mathbb{P}} := Y_0 - Y_t - \int_0^t f_s(Y_s, \widehat{a}_s^{1/2} Z_s, \widehat{a}_s, b_s^{\mathbb{P}})ds + \int_s^T Z_s \cdot dX_s^{c,\mathbb{P}} + \int_s^T dM_s^{\mathbb{P}}, \ t \in [0,T].$$

(iii) We have the following minimality condition: $\forall \mathbb{P} \in \mathcal{P}_0$,

$$\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}\Big[\int_t^T G_s^{t,\mathbb{P}'} d(K_s^{\mathbb{P}'} - k_s^{\mathbb{P}'}) \big| \mathcal{F}_{t+}^{\mathbb{P}} \Big] = 0, \quad 0 \leq t \leq T, \ \mathbb{P}\text{-a.s..}$$

where for any $t \in [0,T]$ and for any $\mathbb{P} \in \mathcal{P}_0$, the process $G^{t,\mathbb{P}}$ is defined by

$$G_s^{t,\mathbb{P}} := \exp\Big(\int_t^s (\lambda_u^{\mathbb{P}} - \frac{1}{2} \|\eta_u^{\mathbb{P}}\|^2)(Y_u, y_u^{\mathbb{P}}, Z_u, z_u^{\mathbb{P}})du + \int_t^s \eta_u^{\mathbb{P}}(Y_u, y_u^{\mathbb{P}}, Z_u, z_u^{\mathbb{P}}) \cdot \widehat{a}_u^{-1/2} dX_u^{c,\mathbb{P}} \Big).$$

$\lambda^{\mathbb{P}}$ and $\eta^{\mathbb{P}}$ are bounded processes.

Remark 1.2.2. *Rigorously, the solution is* $(Y, Z, (M^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}, (K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$ *and through misuse of language, we denote* (Y, Z)*, given the dependence in* $\mathbb{P}$ *of* $K^{\mathbb{P}}$ *and* $M^{\mathbb{P}}$*.*

We prove existence and uniqueness of this 2RBSDE defined in a general filtration under the assumptions mentioned above.

1.2.2 2RBSDEs under weak assumption: the upper case

In Chapter 4, we study the case where we have an upper obstacle. Of course the space of models and the tools used for the lower one are still the same. But the added process K is no longer non decreasing, with an upper obstacle, K plays the role to maintain the solution at the same time below the obstacle and above standard RBSDEs associated. These two effects cannot be ensured by a non decreasing process, but rather by a finite variation process which can be seen as a difference of two non decreasing process. Our main goal is to establish existence and uniqueness of 2RBSDE with an upper obstacle without any regularity assumptions on the terminal condition, the generator and the obstacle. As in chapter 3, we work in a general filtration.
We consider the following generator

$$f : (t, \omega, y, z, a, b) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d^{\geq 0} \times \mathbb{R}^d \longrightarrow \mathbb{R}.$$

We also consider a random variable $\xi : \Omega \longrightarrow \mathbb{R}$, an upper obstacle process $(U_t)_{t\in[0,T]}$ and $(\mathcal{P}(t, \omega))_{(t,\omega)\in[0,T]\times\Omega}$ a class of probability measure families where $\mathcal{P}(t, \omega) \subset \mathcal{P}_t^W$ for all $(t, \omega) \in [0, T] \times \Omega$.
The assumptions are the same to the ones in section 1.2.1 excepted that U replaces L. Let consider the following 2RBSDE with the upper obstacle U:

$$\begin{cases} Y_t = \xi + \int_t^T f_s(Y_s, \widehat{a}_s^{1/2} Z_s, \widehat{a}_s, b_s^{\mathbb{P}})ds - \int_t^T Z_s \cdot dX_s^{c,\mathbb{P}} - \int_t^T dM_t^{\mathbb{P}} + K_T^{\mathbb{P}} - K_t^{\mathbb{P}}, \\ Y_t \leq U_t, \qquad\qquad 0 \leq t \leq T, \ \mathbb{P}\text{-a.s. } \forall \mathbb{P} \in \mathcal{P}_0 \end{cases}$$

where $M^{\mathbb{P}}$ is a martingale orthogonal to $X^{c,\mathbb{P}}$. For any $\mathbb{P} \in \mathcal{P}_0$, ξ is a $\mathcal{F}_T$-measurable random variable , we consider the following RBSDE

$$\begin{cases} y_t^{\mathbb{P}} = \xi + \int_t^T f_s(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}}, \widehat{a}_s, b_s^{\mathbb{P}})ds - \int_t^T z_s^{\mathbb{P}} \cdot dX_s^{c,\mathbb{P}} - \int_t^T dm_s^{\mathbb{P}} - k_T^{\mathbb{P}} + k_t^{\mathbb{P}}, \ 0 \leq t \leq T, \ \mathbb{P}\text{-a.s.} \\ y_t^{\mathbb{P}} \leq U_t, \ t \in [0, T], \ \mathbb{P}\text{-a.s.}, \\ \int_0^T (U_{s^-} - y_{s^-}^{\mathbb{P}}) dk_s^{\mathbb{P}} = 0, \ \mathbb{P}\text{-a.s.} \end{cases}$$

By the symmetry about lower and upper obstacles in the case of RBSDEs explained in section 2.5, we can deduce by Theorem 2.5.1, that under specified assumptions we have the existence and uniqueness of the solution to the above RBSDE under each $\mathbb{P}$. Moreover, the solution is represented by a quadruple $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$, where $m^{\mathbb{P}}$ is a martingale orthogonal to the canonical process X.

Definition 1.2.2. *We say that* (Y, Z) *is a solution to the 2RBSDE if :*

(*i*) $Y_T = \xi$*, and* $Y_t \leq U_t$, $t \in [0, T]$, $\mathcal{P}_0$*-q.s.;*

(*ii*) $\forall \mathbb{P} \in \mathcal{P}_0$*, the process* $K^{\mathbb{P}}$ *has paths of bounded variation* $\mathbb{P}$*-a.s.*

$$K_t^{\mathbb{P}} := Y_0 - Y_t - \int_0^t f_s(Y_s, \widehat{a}_s^{1/2} Z_s, \widehat{a}_s, b_s^{\mathbb{P}})ds + \int_s^T Z_s \cdot dX_s^{c,\mathbb{P}} + \int_s^T dM_s^{\mathbb{P}}, \ t \in [0, T].$$

(iii) *We have the following minimality condition:* $\forall \mathbb{P} \in \mathcal{P}_0$,

$$\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}\Big[\int_t^T G_s^{t,\mathbb{P}'} d(K_s^{\mathbb{P}'} + k_s^{\mathbb{P}'}) \big| \mathcal{F}_{t+}^{\mathbb{P}}\Big] = 0, \quad 0 \le t \le T, \ \mathbb{P}\text{-}a.s..$$

where for any $t \in [0,T]$ *and for any* $\mathbb{P} \in \mathcal{P}_0$ *, the process* $G^{t,\mathbb{P}}$ *is defined by*

$$G_s^{t,\mathbb{P}} := \exp\Big(\int_t^s (\lambda_u^{\mathbb{P}} - \frac{1}{2}\|\eta_u^{\mathbb{P}}\|^2)(Y_u, y_u^{\mathbb{P}}, Z_u, z_u^{\mathbb{P}})du + \int_t^s \eta_u^{\mathbb{P}}(Y_u, y_u^{\mathbb{P}}, Z_u, z_u^{\mathbb{P}}) \cdot \widehat{a}_u^{-1/2} dX_u^{c,\mathbb{P}}\Big).$$

$\lambda^{\mathbb{P}}$ *and* $\eta^{\mathbb{P}}$ *are bounded processes.*

Remark 1.2.3. *Here again, rigorously speaking, the solution is* $(Y, Z, (M^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}, (K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$ *and through misuse of language, we denote* (Y, Z)*, given the dependence in* $\mathbb{P}$ *of* $K^{\mathbb{P}}$ *and* $M^{\mathbb{P}}$*.*

We prove existence and uniqueness of this 2RBSDE defined in a general filtration under the assumptions mentioned above.

1.3 Approximation of second order Reflected BSDEs and convergence

In this section we state some approximation methods for BSDEs and a weak approximation of a class of 2BSDE introduced in [84]. We first recall the approximation of BSDEs by the Euler scheme, then approximation of the driving Brownian motion by a random walk. The following section is dedicated to a brief statement of weak approximation of 2BSDE. The we state our contribution.

1.3.1 Discrete-time approximation of BSDEs

Let consider the following BSDE defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s$$

W is a Brownian motion whose right-continuous and completed filtration assciated is $\mathbb{F} = (\mathcal{F}_t)_{\le t \le T}$. Let $\pi : t_0 = 0 < t_1 < \ldots < t_n = T$, be an equidistant partition of $[0,T]$ and let $h = T/n$ the step. We follow Bouchard, Elie and Touzi [12] to give the discretization. The above BSDEs can be discretized as by the following Euler scheme:

$$Y_{t_i}^\pi = Y_{t_{i+1}}^\pi + hf(t_i, Y_{t_i}^\pi, Z_{t_i}^\pi) - Z_{t_i}(W_{t_{i+1}} - W_{t_i}).$$

The solution of this scheme is obtained by applying the following operations:
- Taking conditional expectation with respect to $\mathcal{F}_{t_i}$, leads to

$$Y_{t_i}^\pi = \mathbb{E}[Y_{t_{i+1}}^\pi | \mathcal{F}_{t_i}] + hf(t_i, Y_{t_i}^\pi, Z_{t_i}^\pi).$$

- Pre-multiplying by $(W_{t_{i+1}} - W_{t_i})$ and then taking conditional provides

$$0 = \mathbb{E}[Y_{t_{i+1}}^\pi (W_{t_{i+1}} - W_{t_i}) | \mathcal{F}_{t_i}] - hZ_{t_i}^\pi$$

Then the BSDE can be approximated by the following backward Euler scheme

$$\begin{cases} Y_{t_n}^\pi &= \xi; \\ Y_{t_i}^\pi &= \mathbb{E}[Y_{t_{i+1}}^\pi | \mathcal{F}_{t_i}] + hf(t_i, Y_{t_i}^\pi, Z_{t_i}^\pi); \\ Z_{t_i}^\pi &= \frac{1}{h}\mathbb{E}[Y_{t_{i+1}}^\pi (W_{t_{i+1}} - W_{t_i}) | \mathcal{F}_{t_i}]. \end{cases}$$

For the estimation of error, the classical approach is to suppose that the randomness comes from a forward SDE process. Then this process is discretized on π and the terminal value is choose as a function of this process. Under suitable conditions on the drift and the volatility the following estimation of error is obtained

$$\max_{0\leq i\leq n}\mathbb{E}\Big[\sup_{t_i\leq t\leq t_{i+1}}|Y_t-Y^\pi_{t_i}|^2+\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}|Z_t-Z^\pi_{t_i}|^2dt\Big]\leq C|\pi|.$$

where $|\pi|:=\max_{1\leq i\leq n}|t_i-t_{i-1}|$ and C is a positive constant independent of n. This error only depends on the regularity of BSDEs, for the details about regularity we refer to Ma and Zhang [62] and for the details about this error, one can refer to the work of Zhang [99] and [100] and Bouchard and Touzi [15].

Remark 1.3.1. *The following explicit scheme can also be considered*

$$\begin{cases}\tilde{Y}^\pi_{t_n}=\xi;\\ \tilde{Y}^\pi_{t_i}=\mathbb{E}[\tilde{Y}^\pi_{t_{i+1}}|\mathcal{F}_{t_i}]+hf(t_i,\tilde{Y}^\pi_{t_{i+1}},\tilde{Z}^\pi_{t_i});\\ \tilde{Z}^\pi_{t_i}=\frac{1}{h}\mathbb{E}[\tilde{Y}^\pi_{t_{i+1}}(W_{t_{i+1}}-W_{t_i})|\mathcal{F}_{t_i}].\end{cases}$$

and the convergence rate is still the same.

Let consider the following RBSDE defined on $(\Omega,\mathcal{F},\mathbb{P})$

$$\begin{cases}Y_t=\xi+\int_t^T f(s,Y_s,Z_s)ds-\int_t^T Z_sdW_s+K_T-K_t,\\ Y_t\geq L_t,\ \ \int_0^T(Y_s-L_s)dK_s=0.\end{cases}$$

Following Bouchard and Touzi [15], we can discretize the above RBSDE using the following backward Euler scheme:

$$\begin{cases}Y^\pi_{t_n}=\xi;\\ Y^\pi_{t_i}=\max\{L_{t_i},\mathbb{E}[Y^\pi_{t_{i+1}}|\mathcal{F}_{t_i}]+hf(t_i,Y^\pi_{t_i},Z^\pi_{t_i})\};\\ Z^\pi_{t_i}=\frac{1}{h}\mathbb{E}[Y^\pi_{t_{i+1}}(W_{t_{i+1}}-W_{t_i})|\mathcal{F}_{t_i}].\end{cases}$$

Similarly to the case of BSDEs the estimates of the error is provided by the regularity of RBSDEs, for more details about regularity of RBSEs one can refer to Ma and Zhang [63]. Similarly to the case of BSDEs, in literature the randomness comes from a forward SDE process and therefore the terminal value and the obstacle are taken as functions of this forward SDE process. The estimate of the error is also linked to the discretization error of the forward process. Many results have been shown for the discrete time approximation of RBSDEs, among which we can cite Bouchard and Touzi in [15] for the case where the generator f does not depend on z, they obtained an estimation of error in order of $|\pi|^{\frac{1}{2}}$. The case where the generator depends on z have been studied in [11] and [12] with additional condition on the volatility of the forward process and the obstacle.

1.3.2 Convergence of approximation of BSDEs

The weak approximation of BSDEs or also called Donsker's type theorem for BSDEs by Briand, Delyon and Mémin in [19] is a method of approximation of BSDEs by discretization of BSDEs using an approximation of the Brownian motion by a sequence of i.i.d. random variables.
Approximation of a Brownian motion. The key idea is that a Brownian motion can be approximated by a sequence of scaled random walked as follows: Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability

space and a sequence of i.i.d. Bernoulli symmetric sequence $(\varepsilon_k^n)_{1\le k\le n}$. We define for $n \in \mathbb{N}^*$, the scaled random walked

$$W_t^n = \sqrt{h}\sum_{k=1}^{t/h} \varepsilon_k^n, \quad 0 \le t \le T, \quad h = \frac{T}{n}.$$

where $[x]$ is the integer part of x. We have by Donsker's theorem that

$$\sup_{0\le t\le T} |W_t^n - W_t| \to 0, \quad \text{as } n \to \infty,$$

in probability as well as in L^p, $1 \le p < \infty$, where $(W_t)_{0\le t\le T}$ is a Brownian motion.

Remark 1.3.2. *We notice that if W and the sequence (ε^n) are defined on two different probability spaces, we can use Skorokhod representation theorem to find a new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ carrying a Brownian motion $(\tilde{W}_t)_{0\le t\le T}$ and a Bernoulli symmetric sequence $(\tilde{\varepsilon}_k)_{1\le k\le n}$ of such that*

$$\tilde{W}_t^n = \sqrt{h}\sum_{k=1}^{[t/h]} \tilde{\varepsilon}_k^n, \quad 0 \le t \le T, \quad h = \frac{T}{n}.$$

and by Donsker's Theorem,

$$\sup_{0\le t\le T} |W_t^n - W_t| \to 0, \quad \textit{as } n \to \infty,$$

in probability as well as in L^p, $1 \le p < \infty$.

We begin by approximation by numeric schemes.

1.3.2.1 Approximation by numeric scheme

Approximation of a BSDEs. We consider the following BSDE defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying the Brownian motion $(W_t)_{0\le t\le T}$ and with respect to the Brownian filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ defined by $\mathcal{F}_t := \sigma(W_s, 0 \le s \le t)$.

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \tag{1.3.1}$$

For $n \ge 1$, we consider a probability space $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$ containing the sequence of i.i.d. Bernoulli symmetric sequence $(\varepsilon_k^n)_{1\le k\le n}$. Let denote $\mathbb{F}^n = (\mathcal{F}_k^n)_{1\le k\le n}$ er $\mathcal{F}_k^n = \sigma(\varepsilon_1^n, \ldots, \varepsilon_k^n)$. We define the discrete time BSDE on the small interval $[kh, (k+1)h]$

$$\begin{aligned} y_k^n &= y_{k+1}^n + hf(kh, y_k^n, z_k^n) - \sqrt{h}z_k^n\varepsilon_{k+1}^n, \quad k = n-1, \ldots, 0,\ y_n^n = \xi^n, \\ z_k^n &= \frac{1}{\sqrt{h}}\mathbb{E}[y_{k+1}^n\varepsilon_{k+1}^n|\mathcal{F}_k^n]. \end{aligned} \tag{1.3.2}$$

Then we define two continuous time processes by setting, for $0 \le t \le T$, $Y_t^n := y_{\lfloor t/h\rfloor}^n, Z_t^n := z_{\lfloor t/h\rfloor}^n$, where $\lfloor n\rfloor := (n-1)^+$ for all integer n and $\lfloor x\rfloor = [x]$ if x is not an integer.
Assumptions. (i) f is uniformly Lipschitz under y and z, that is, there is $K \ge 0$ such that

$$\forall (y, y', z, z') \quad |f(s, y, z) - f(s, y', z')| \le K(|y - y'| + |z - z'|);\ s \in [0, T].$$

(ii) ξ is $\mathcal{F}_T$-measurable and ξ^n is $\mathcal{F}_n^n$-measurable such that

$$\mathbb{E}[|\xi|^2] + \sup_n \mathbb{E}[|\xi^n|] < \infty;$$

(iii) ξ^n converges to ξ in L^1 as $n \to \infty$.

Briand, Delyon and Mémin in [19] have proved that under the above Assumptions, (Y^n, Z^n) converges to (Y, Z) in the following sense:

$$\sup_{0\le t\le T} |Y_t^n - Y_t|^2 + \int_0^T |Z_s^n - Z_s|^2 ds \to 0 \text{ as } n \to \infty \text{ in probability.}$$

where (Y, Z) is the solution of the BSDE (1.3.1) (existence and uniqueness are satisfied under assumptions). The proof of this result can be found in [19].
Peng and Xu [79](see also [97]) have established the same convergence when the implicit discretization (1.3.2) is replaced by the following explicit discretization on the small interval $[kh, (k+1)h]$

$$\begin{aligned}
&\bar{y}_k^n = \bar{y}_{k+1}^n + hf(kh, \mathbb{E}[\bar{y}_{k+1}^n|\mathcal{F}_k^n], \bar{z}_k^n) - \sqrt{h}\bar{z}_k^n\varepsilon_{k+1}^n, \quad k = n-1, \dots, 0,\ \bar{y}_n^n = \xi^n,\\
&\bar{z}_k^n = \frac{1}{\sqrt{h}}\mathbb{E}[\bar{y}_{k+1}^n\varepsilon_{k+1}^n|\mathcal{F}_k^n]
\end{aligned}$$

and then by setting, for $0 \le t \le T$, $\bar{Y}_t^n := \bar{y}_{\lfloor t/h\rfloor}^n, \bar{Z}_t^n := \bar{z}_{\lfloor t/h\rfloor}^n$

$$\sup_{0\le t\le T} |\bar{Y}_t^n - Y_t|^2 + \int_0^T |\bar{Z}_s^n - Z_s|^2 ds \to 0 \text{ as } n \to \infty \text{ in probability.}$$

where (Y, Z) is the solution of the BSDE (1.3.1). Moreover, in [79] the simulations of explicit and implicit approximation can also be found.
Approximation of a RBSDEs. Ménin, Peng and Xu [69] and Xu and Peng [79] have extended these convergence results to the case of reflected BSDEs. Let consider the following RBSDEs on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$

$$\begin{aligned}
&Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + K_T - K_t,\\
&Y_t \ge L_t,\ \int_0^T (Y_s - L_s)dK_s = 0.
\end{aligned} \tag{1.3.3}$$

In [69], the obstacle L is consider to be the null process and in [79], the obstacle L is consider to be an Itô process , i.e., $L_t = L_0 + \int_0^t l_s ds + \int_0^t \sigma_s dW_s$. For the existence and uniqueness of the above RBSDEs one can refer to section 1.1.3.1. In [69], the discrete time RBSDE is defined on the small interval $[kh, (k+1)h]$ by

$$\begin{aligned}
&y_k^n = y_{k+1}^n + hf(kh, y_k^n, z_k^n) - \sqrt{h}z_k^n\varepsilon_{k+1}^n + d_k^n, \quad k = n-1, \dots, 0,\ y_n^n = \xi^n,\\
&k_k^n = \sum_{j=1}^k d_k^n,\ y_k^n \ge L_k^n,\ d_k^n \ge 0\\
&(y_k^n - L_k^n)d_k^n = 0.
\end{aligned}$$

where $L_k^n = L_0 + h\sum_{j=0}^{k-1} l_{t_j} + \sqrt{h}\sum_{j=0}^{k-1}\sigma_{t_i}\varepsilon_{j+1}^n$. Under suitable assumption among them,

$$\lim_n \mathbb{E}[|\xi^n - \xi|^2] = 0..$$

By setting, for $0 \le t \le T$, $Y_t^n := y_{\lfloor t/h\rfloor}^n, Z_t^n := z_{\lfloor t/h\rfloor}^n$ we have the following convergence result

$$\mathbb{E}\Big[\sup_{0\le t\le T} |Y_t^n - Y_t|^2 + \int_0^T |Z_s^n - Z_s|^2 ds\Big] \to 0 \text{ as } n \to \infty.$$

Numeric scheme using penalization. We recall that a solution of RBSDE (1.3.3) is a limit of solution of the following penalized BSDEs: For $p \geq 1$,

$$Y_t^p = \xi + \int_t^T f(s, Y_s^p, Z_s^p)ds + p\int_t^T (Y_s^p - L_s)^- - \int_t^T Z_s^p dW_s, \quad K_t^p = p\int_0^t (Y_s^p - L_s)^-.$$

and there exists a positive constant c independent of p such that

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}|Y_t - Y_t^p|^2 + \int_0^T (Z_s - Z_s^p)^2 ds + \sup_{0\leq t\leq T}|K_t - K_t^p|^2\Big] \leq \frac{c}{\sqrt{p}}.$$

The penalized BSDEs can be discretized on the small interval $[kh, (k+1)h]$ as : for $p \geq 1$,

$$y_k^{n,p} = y_{k+1}^{n,p} + h[f(kh, y_k^{n,p}, z_k^{n,p}) + p(y_k^{n,p} - L_k^n)^-] - \sqrt{h} z_k^{n,p}\varepsilon_{k+1}^n, \quad k = n-1, \ldots, 0,\ y_n^{n,p} = \xi^n,$$

and $d_k^{p,n} = p((y_k^{n,p} - L_k^n)^-$. By setting, $Y_t^{n,p} := y_{\lfloor t/h\rfloor}^{n,p}$, $Z_t^{n,p} := z_{\lfloor t/h\rfloor}^{n,p}$ and $K_t^{n,p} := \sum_{j=0}^{\lfloor t/h\rfloor} d_j^{n,p}$, for $0 \leq t \leq T$, we have the following convergence

$$\lim_{p\to\infty}\lim_{n\to\infty}\mathbb{E}\Big[\sup_{0\leq t\leq T}|Y_t - Y_t^{n,p}|^2 + \int_0^T (Z_s - Z_s^{n,p})^2 ds\Big] = 0. \tag{1.3.4}$$

and $K_t^{n,p} \to K_t$ in L^2, for $0 \leq t \leq T$ as $n, p \to \infty$.
Similarly to the case of BSDEs, We can discretize the RBSDE using an explicit scheme. The convergence and simulations have been done in [79] and [69].

1.3.2.2 Approximation by a sequence of BSDE

These above approximations are based on the discretization of BSDEs and since (Y, Z) is the solution of BSDE driven by the Brownian motion W, among whom conditions, we can represent the approximations Y^n, Z^n as the solutions of a BSDE driven by a martingale W^n. This is also called robustness of BSDEs and have been studied by Briand and al. [19] and [20].
In this section, we still work on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying a Brownian motion $(W_t)_{0\leq t\leq T}$ and $\mathbb{F} := (\mathcal{F}_t)$ still denote the filtration generated by the Brownian motion W. $(W^n)_{n\geq 1}$ is a sequence of càdlàg martingale with respect to the right continuous and complete filtration $\mathbb{F}^n := (\mathcal{F}_t^n)_{0\leq t\leq T}$. They consider the following assumptions.
Assumptions. (i) W^n is a square integrable martingale which converges to W in $\mathcal{S}^2$;
(ii) there exists $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ with $\rho(0^+) = 0$ and a deterministic sequence $(a_n)_n$ with $lim_{n\to\infty} a_n = 0$ such that, $\mathbb{P}$-a.s.,

$$\forall\, 0 \leq s \leq t \leq T,\ \langle W^n\rangle_t - \langle W^n\rangle_s \leq \rho(t-s) + a_n;$$

(iii) ξ is $\mathcal{F}_T$-measurable and for all n, ξ^n is $\mathcal{F}_T^n$-measurable such that ξ^n converges to ξ in L^2.
(iv) The function f^n is Lipschitz in (y, z) and converges to f(who is also Lipschitz in (y, z)) in $\mathcal{S}^2$;
Under these conditions, the following BSDEs has a unique solution (Y^n, Z^n, N^n) for n large enough.

$$Y_t^n = \xi^n + \int_t^T f^n(r, Y_{r^-}^n, Z_{r^-}^n)d\langle W^n\rangle_r - \int_t^T Z_r^n dW_r^n - (N_T^n - N_t^n).$$

and we have the following converge

$$\Big(Y^n, \int_0^\cdot Z_r^n dW_r^n, N^n\Big) \to (Y, \int_0^\cdot Z_r dW_r, 0),$$

and n tends to infinity in $\mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{S}^2$ and

$$\Big(\int_0^\cdot Z_r^n d\langle W^n \rangle_r, \int_0^\cdot |Z_r^n|^2 d\langle W^n \rangle_r \Big) \to \big(\int_0^\cdot Z_r dr, \int_0^\cdot |Z_r|^2 dr \big)$$

in $\mathcal{S}^2 \times \mathcal{S}^1$. where (Y, Z) solves the following BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s.$$

The proof of this result use the weak convergence of filtration $\mathbb{F}^n$ to $\mathbb{F}$ as an essential arguments.

1.3.3 Weak approximation of a class of 2BSDEs

We have already introduced the notion of second order BSDE in section 1.1.2.2. In [84] Possamai, Tan and Zhou provided the weak approximation of a class of 2BSDEs. We give a brief description of this class and state the result of weak convergence.
Let $d \in \mathbb{N}^*$, and consider a canonical space Ω of continuous paths on $[0, T]$ valued in $\mathbb{R}^d$ starting at 0. The canocical process is denoted by B, $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ is the canonical filtration and $\mathbb{F}_+ := (\mathcal{F}_{t+})_{0 \le t \le T}$ is the right continuous limit of $\mathbb{F}$. $\mathbb{P}_0$ is the Wiener measure on Ω under which B is a standard Brownian motion. Consider a compact convex set $A \subset \mathbb{S}_d^{>0}$($\mathbb{S}_d^{>0}$ is the set of positive, symmetric $d \times d$ matrices) such that for all $a \in A$, $a \ge \varepsilon_0 I_d$, where $\varepsilon_0 > 0$ is a fixed constant.
Class of models. Let define the two class $\mathcal{P}_W$ and $\mathcal{P}_S$ by

$$\mathcal{P}_W := \Big\{ \mathbb{P}, \text{ local martingale measure such that } \frac{d\langle B \rangle_t}{dt} \in A, \ d\mathbb{P} \times dt\text{-a.e.} \Big\}$$

and

$$\mathcal{P}_S := \Big\{ \mathbb{P}^\alpha \in \mathcal{P}_W, \mathbb{P}^\alpha := \mathbb{P}_0 \otimes (X^\alpha)^{-1}, \text{ where } X_t^\alpha := \int_0^t \alpha_s^{1/2} dB_s, \quad \mathbb{P}_0\text{-a.s.} \Big\}$$

The process α in the above set is a $\mathbb{F}$-progressively measurable process taking values in A. We consider the following 2BSDE

$$Y_t = \xi(B) + \int_t^T (g(s, B, Y_s, Z_s, \widehat{a}_s) : \widehat{a}_s) ds - \int_t^T Z_s dB_s + K_T - K_t, \mathbb{P}\text{-a.s.}$$

where $\widehat{a}$ denotes the density of the quadratic variation of B. The goal is to approximate the initial value Y_0, the value of the solution Y at time $t = 0$. But the definition of 2BSDE doesn't give more information about the process K except the minimum condition (1.1.9). Then the approximation of Y_0 by discretization of the equation is not very interesting, this leads to use the representation formula (1.1.10) of 2BSDE. For this purpose, the BSDEs associated to the above 2BSDE under the space of models $\mathcal{P}_W$ are defined as follows: For every $\mathbb{P} \in \mathcal{P}_W$,

$$y_t^{\mathbb{P}} = \xi(B) + \int_t^T (g(s, B, y_s^{\mathbb{P}}, z_s^{\mathbb{P}}, \widehat{a}_s) : \widehat{a}_s) ds - \int_t^T z_s^{\mathbb{P}} dB_s + n_T^{\mathbb{P}} - n_t^{\mathbb{P}}, \mathcal{P}_S\text{-q.s.}$$

where $n^{\mathbb{P}}$ is a martingale orthogonal to B. The presence of the additional orthogonal martingale is due to the fact every $\mathbb{P} \in \mathcal{P}_W$ doesn't a priori satisfies the predictable martingale representation property unlike every $\mathbb{P} \in \mathcal{P}_S$ which satisfies the predictable martingale representation and the Blumental 0-1 law. Then for $\mathbb{P} \in \mathcal{P}_S$, $n^{\mathbb{P}} = 0$. Using the representation formula of the 2BSDE (1.1.10), by the representation formula of 2BSDE, the approximation of the initial value Y_0 of the 2BSDE comes back to approximate the following control problem under uncertainty

$$Y_0 := \sup_{\mathbb{P} \in \mathcal{P}_S} y_0^{\mathbb{P}} = \sup_{\mathbb{P} \in \mathcal{P}_W} \mathbb{E}^{\mathbb{P}}[y_0^{\mathbb{P}}]. \tag{1.3.5}$$

The weak approximation of Y_0 will be a discrete time approximation, where the uncertainty about the space of models is translate to uncertainty about conditional variance of the increments. This discrete approximation method of a continuous problem have been used by Dolinsky, Nutz and Soner [30] to show a weak approximation of G-expectation. In this case the approximation is based on a control problem over the space of martingale law defined on a discrete time canonical space. Morover, Dolinsky [29] provided a numerical scheme for G- expectation, the key idea was to approximate a control problem under uncertainty on the space of models by a control problem on a space of discrete time martingale. Furthermore, the author provides a convergence rate of the approximation.
Since under every $\mathbb{P} \in \mathcal{P}_S$ the canonical process B is a martingale, the BSDEs associated to the 2BSDEs can be approximated by a sequence of BSDEs driven by a sequence of martingales which converge to the martingale B(see section 1.3.2.2 for the approximation of a BSDE by a sequence of BSDEs).

Martingale approximation. Let $\pi : t_0 = 0 < t_1 < \ldots < t_n = T$, be an equidistant partition of $[0, T]$ and let $h = T/n$ the step. For $n \geq 1$, let $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$ be a probability space containing n, independent random variables $U_1, \ldots, U_n$. Let define a family of functions $(H_k^n)_{1 \leq k \leq n, n \geq 1}$ such that every $H_k^n : A \times [0, 1]$ is continuous in a and for some $\delta > 0$, hence

$$\mathbb{E}[H_k^n(a, U_k)] = 0, \ \mathbb{E}[|H_k^n(a, U_k)|^2] = ah, \ \ \mathbb{E}[|H_k^n(a, U_k)|^{2+\delta}] \leq Ch^{1+\delta/2},$$

Define the filtration $\mathbb{F}^n := (\mathcal{F}_{t_k}^n)_{1 \leq k \leq n}$ with $\mathcal{F}_{t_k}^n := \sigma(U_1, \ldots, U_n)$ and denote by E_n the collection of all $\mathbb{F}^n$-predictable A-valued processes $e = (a_{t_1}^e, \ldots, a_{t_n}^e)$. Then for every $e \in E_n, M^e$ is defined by

$$M_{t_k}^e := \sum_{i=1}^{k} H_i^n(a_{t_i}^e, U_i)$$

Observe that

$$\langle M^e \rangle_{t_k} = h \sum_{i=1}^{k} a_{t_i}^e$$

In order to consider a BSDE driven by the martingale M^e, the continuous time version of M^e is defined by $M_t^e := M_{t_k}^e$ for all $t \in [t_k, t_{k+1})$ and by the same way the continuous version of the filtration $\mathbb{F}^n$ by $\mathcal{F}_t^n := \mathcal{F}_{t_k}^n$ for all $t \in [t_k, t_{k+1})$. Then for every $e \in E_n$, and $n \geq 1$, the BSDE driven by M^e is given by

$$y_t^e = \xi(\widehat{M}^e) + \int_t^T g_n(s, \widehat{M}^e, y_{s^-}^e, z_s^e, a_s^e) : d\langle M^e \rangle_s - \int_t^T z_s^e dM_s^e + n_T^e - n_t^e,$$

$\widehat{M}^e$ is the linear interpolation of M^e and g_n is defined by

$$g_n(t, x, y, z, a) := g(t_k, x, y, z, a), \ t \in [t_k, t_{k+1}).$$

For the existence and uniqueness result of this BSDE, see [84].

Weak approximation. For every $n \geq 1$, denote

$$Y_0^n = \sup_{e \in E_n} y_0^e.$$

Under specific assumptions under g and ξ the following approximations are established in [84].

Theorem 1.3.1.

$$\liminf_{n\to\infty} Y_0^n \geq Y_0,$$

and if g does not depend on z, then

$$\lim_{n\to\infty} Y_0^n = Y_0.$$

1.4 Contribution

Let Ω be the canonical space of continuous paths on $[0,T]$ which start at 0 equipped with the following norm $\|\omega\|_t := \sup_{0\leq s\leq t} |\omega_s|$, for any $\omega \in \Omega$ and $t \in [0,T]$ and X the canonical process.

The models space. We consider a nonempty compact convex subset $\mathbf{D} = [\underline{a}, \overline{a}]$, where $\underline{a} > 0$. We denote by $\mathcal{P}_{\mathbf{D}} \subset \mathcal{P}(\Omega)$ the collection all probability measures $\mathbb{P}$ satisfies:

- X is a $(\mathbb{P}, \mathbb{F})$-martingale;
- $\langle X \rangle$ is absolutely continuous $\mathbb{P}$-a.s.
- $d\langle X\rangle_t/dt \in \mathbf{D}, \quad \mathbb{P}\times dt$-a.e.

Formulation. We consider the following 2RBSDE where the terminal condition is defined with an $\mathbb{R}$-valued function Φ and the obstacle with an $\mathbb{R}$-valued function h satisfying $\Phi \geq h$.

$$\begin{cases} Y_t = \Phi(X_\cdot) + \int_t^T f(s, X_\cdot, Y_s, \widehat{a}_s^{1/2} Z_s, \widehat{a}_s) ds - \int_t^T Z_s dX_s - \int_t^T dM_s^{\mathbb{P}} + K_T^{\mathbb{P}} - K_t^{\mathbb{P}}, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \leq t \leq T, \ \mathbb{P}\text{-a.s.}, \forall \mathbb{P} \in \mathcal{P}_{\mathbf{D}}, \\ Y_t \geq h(X_t), 0 \leq t \leq T, \ \mathbb{P}\text{-a.s.}, \forall \mathbb{P} \in \mathcal{P}_{\mathbf{D}}. \end{cases} \tag{1.4.1}$$

and for each $\mathbb{P} \in \mathcal{P}_0$, we define the following RBSDE

$$\begin{cases} y_t^{\mathbb{P}} = \Phi(X_\cdot) + \int_t^T f(s, X_\cdot, y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}}, \widehat{a}_s) ds - \int_t^T z_s^{\mathbb{P}} dX_s - \int_t^T dm_s^{\mathbb{P}} + k_T^{\mathbb{P}} - k_t^{\mathbb{P}}, \\ y_t^{\mathbb{P}} \geq h(X_t), \ t \in [0,T] \\ \int_0^T (y_{s^-}^{\mathbb{P}} - h(X_{s^-})) dk_s^{\mathbb{P}} = 0. \end{cases} \tag{1.4.2}$$

We recall the definition of 2RBSDE

Definition 1.4.1. *We say that (Y,Z) is a solution to the 2RBSDE (4.2.1) if :*

(i) $Y_T = \Phi(X_\cdot)$, *and* $Y_t \geq h(X_t)$, $t \in [0,T]$, $\mathcal{P}_{\mathbf{D}}$*-q.s.;*

(ii) $\forall \mathbb{P} \in \mathcal{P}_{\mathbf{D}}$, *the process $K^{\mathbb{P}}$ defined below has non-decreasing paths $\mathbb{P}$-a.s.*

$$K_t^{\mathbb{P}} := Y_0 - Y_t - \int_0^t f(s, X_\cdot, Y_s, \widehat{a}_s^{1/2} Z_s, \widehat{a}_s) ds + \int_s^T Z_s \cdot dX_s + \int_s^T dM_s^{\mathbb{P}}, \ t \in [0,T], \tag{1.4.3}$$

(iii) We have the following minimality condition:

$$\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_{\mathbf{D}}(t,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}\Big[\int_t^T G_s^{t,\mathbb{P}'} d(K_s^{\mathbb{P}'} - k_s^{\mathbb{P}'}) \big| \mathcal{F}_{t+}^{\mathbb{P}} \Big] = 0, \quad 0 \leq t \leq T, \ \mathbb{P}\text{-a.s. } \forall \mathbb{P} \in \mathcal{P}_{\mathbf{D}}.$$

where for any $t \in [0,T]$ and for any $\mathbb{P} \in \mathcal{P}_0$, the process $G^{t,\mathbb{P}}$ is defined by

$$G_s^{t,\mathbb{P}} := \exp\Big(\int_t^s (\lambda_u^{\mathbb{P}} - \frac{1}{2} \|\eta_u^{\mathbb{P}}\|^2)(Y_u, y_u^{\mathbb{P}}, Z_u, z_u^{\mathbb{P}}) du + \int_t^s \eta_u^{\mathbb{P}}(Y_u, y_u^{\mathbb{P}}, Z_u, z_u^{\mathbb{P}}) \cdot \widehat{a}_u^{-1/2} dX_u^{c,\mathbb{P}} \Big).$$

$\lambda^{\mathbb{P}}$ and $\eta^{\mathbb{P}}$ are bounded processes.

Notation: $\mathcal{P}_{\mathbf{D}}(t,\mathbb{P},\mathbb{F}_+) := \{\mathbb{P}' \in \mathcal{P}_{\mathbf{D}} : \mathbb{P} = \mathbb{P}' \text{ on } \mathcal{F}_{t+}\}$.

Our goal is to give an approximation of the solution Y, the classical idea is the approximation of Y by a discretized backward scheme. But the definition of 2BSDE doesn't give more information about the process K except the minimum condition. Then the approximate of Y by discretization of the equation is not very interesting, this leads to use the representation formula (1.1.10) of 2BSDE. In other words, we look for an approximation of Y in particular for $t = 0$ represented by

$$Y_0 = \sup_{\mathbb{P}\in\mathcal{P}_{\mathbf{D}}} \mathbb{E}^{\mathbb{P}}[y_0^{\mathbb{P}}] \tag{1.4.4}$$

where $y^{\mathbb{P}}$ is the y-component of the solution of the above RBSDE.
Discrete time approximation. Given $n \geq 1$, we discretize $[0,T]$ in $n+1$ equidistant points $0 = t_0 < t_1 < \cdots < t_n = T$, with $h = T/n$ and set $\Omega^n = \mathbb{R}^{n+1}$ the canonical space of d-dimensional paths in discrete time $k = 0, 1, \ldots, n$. Let $U_1, \ldots, U_n$ be a sequence of n independent random variables . We denote $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$ a probability space carrying $U_1, \ldots, U_n$. Define the filtration $\mathbb{F}^n := (\mathcal{F}_k^n)_{1\leq k\leq n}$ with $\mathcal{F}_{t_k}^n := \sigma(U_1, \ldots, U_k)$.
Denote $\mathcal{A}_n$ the collection of all $\mathbb{F}^n$-predictable $\mathbf{D}$-valued process $a = (a_{t_1}, \ldots, a_{t_n})$. Then for every $a \in \mathcal{A}_n$, we define the discrete time martingale $M^a = (M^a_{t_0}, \ldots, M^a_{t_n})$ of the form, $M^a_{t_0} = 0$ and

$$M^a_{t_k} = \sum_{j=1}^{k} H_j^n(a_{t_j}, U_j),\ 1 \leq k \leq n.$$

Then for every $a \in \mathcal{A}_n$, we define the following RBSDE

$$\begin{cases} y_t^a = \Phi(\widehat{M}^a_\cdot) + \int_t^T f(s, \widehat{M}^a_\cdot, y^a_{s^-}, a_s^{1/2} z_s^a, a_s) dC_s^n - \int_t^T z_s^a dM_s^a - \int_t^T dm_s^a + k_T^a - k_t^a, \\ y_t^a \geq h(\widehat{M}_t^a),\ t \in [0,T] \\ \int_0^T (y^a_{s^-} - h(\widehat{M}^a_{s^-}))dk_s^a = 0. \end{cases} \tag{1.4.5}$$

where $C_s^n = [s/h]h$ and y^a_- denotes the càdlàg process associated to y^a. $a^{1/2}$ denotes the unique square root of a in $\mathbf{D}$ and a is given by

$$a_t := \sum_{k=0}^{n-1} a_{t_k} 1_{[t_k, t_{k+1}[}(t), \quad t \in [0,T].$$

We make the following assumptions.
Assumptions. There exists a constant $K > 0$ such that for any $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}, s, t \in [0,T], a_1, a_2 \in \mathbf{D}$,

(i)- Φ is positive continuous and $|\Phi(x_1) - \Phi(x_2)| \leq K(\|x_1 - x_2\|_T)$,

(ii)- h is positive continuous and $|h(y_1) - h(y_2)| \leq K(|y_1 - y_2|)$,

(iii)- $|f(t, x_1, y_1, z_1, a) - f(t, x_2, y_2, z_2, a)| \leq K(\|x_1 - x_2\|_t + |y_1 - y_2| + \|z_1 - z_2\| + |a_1 - a_2|)$,

(iv)- f is positive and the map $a \mapsto f(t, x, y, z, a)$ is concave and uniformly continuous for every $(t, x, y, z) \in [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$.

(v) - The process $t \mapsto f(t, x_\cdot, y_t, z_t, a_t)$ is progressively measurable given progressive processes (x, y, z, a) and uniformly continuous with modulus ρ in the sense that for every $s \leq t$ and x, y, z,

$$|f(t, x_{s\wedge\cdot}, y, z, a) - f(s, x_{s\wedge\cdot}, y, z, a)| \leq \rho(t - s).$$

(vi) - $\sup_{(t,x,\gamma)\in[0,T]\times\Omega\times\in\mathbf{D}} |f(t, x, 0, 0, \gamma)| + |\Phi(0)| + |h(0)| \leq K$.

(vii)- h is bounded, derivable and his derivative is Lipschitz (with Lipschitz constant K).
We proof the following result

Proposition 1.4.1. *Under the above assumptions, there is a unique solution* (y^a, z^a, m^a, k^a) *to the RBSDE (5.3.3) for n large enough.*

Given $n \geq 1$, let set

$$Y_0^n = \sup_{a \in \mathcal{A}_n} \mathbb{E}^n[y_0^a].$$

where $\mathbb{E}^n$ denotes the expectation with respect to the distribution of U_1.

1.4.1 Main result

Our main result is the following weak approximation

Theorem 1.4.1. *(i) Suppose that the above assumptions hold true. Then*

$$\liminf_{n\to\infty} Y_0^n \geq Y_0.$$

(ii)Suppose in addition that f *does not depend on* z *and for every* $a \in \mathcal{A}_n$,

$$K < \frac{1}{h} \quad \text{and} \quad K|H_k^n(a_{kh}, U_k)| < a_{kh}^{1/2}, \text{ for } k = 1, \cdots, n.$$

then

$$\lim_{n\to\infty} Y_0^n = Y_0.$$

1.4.2 Auxiliary results

As an auxiliary result, we extend the approximation method of a BSDE by a sequence of BSDEs (see section 1.3.2.2) to the case of RBSDEs defined in general filtration. Let set the following RBSDE

$$\begin{cases} y_t^n = \Phi(\widehat{M}_\cdot^n) + \int_t^T f(s, \widehat{M}_\cdot^n, y_{s^-}^n, (\tilde{a}_s^n)^{1/2} z_s^n, \tilde{a}_s^n) dC_s^n - \int_t^T z_s^n dM_s^n - \int_t^T dm_s^n + k_T^n - k_t^n, \\ y_t^n \geq h(\widehat{M}_t^n), \ t \in [0,T] \\ \int_0^T (y_{s^-}^n - h(\widehat{M}_{s^-}^n)) dk_s^n = 0. \end{cases}$$

where $\widehat{M^n}$ is the linear interpolation of M^n. M^n is a martingale which converge to the canonical process X under the Skorokhod topology $\mathbf{J}_1$. For n large enough, we establish existence and uniqueness the above RBSDE and estimates of the solution. We also establish the following convergence result.

Theorem 1.4.2. *Let the above assumptions holds and* (y^n, z^n, m^n, k^n) *be the solution to above the RBSDE. Denote* (y, z, m, k) *the solution of the following RBSDE .*

$$\begin{cases} y_t = \Phi(X_\cdot) + \int_t^T f(s, X_\cdot, y_s, \widehat{a}_s^{1/2} z_s, a_s) ds - \int_t^T z_s dX_s - \int_t^T dm_s + k_T - k_t, \\ y_t \geq h(X_t), \ t \in [0,T] \\ \int_0^T (y_{s^-} - h(X_{s^-})) dk_s = 0. \end{cases}$$

Then we have

$$\left(y^n, \int_0^\cdot z_r^n dM_r^n, \int_0^\cdot (\tilde{a}_s^n)^{1/2} z_r^n dC_r^n, m^n, k^n\right) \underset{n\to+\infty}{\to} \left(y^{\mathbb{P}}, \int_0^\cdot z_r^{\mathbb{P}} dX_r, \int_0^\cdot \widehat{a}_r^{1/2} z_r^{\mathbb{P}} dr, m^{\mathbb{P}}, k^{\mathbb{P}}\right),$$

in law for the topology of uniform convergence.

Chapter 2

Basics tools and preliminaries results

Contents

The aim of this chapter is to introduce the basics tools and some results which will be useful in the next chapters. The main themes concern the notions of universal measurability, regular conditional probability, dynamic programming principle, measurability with respect to a probability law and finally, RBSDEs in a general filtration and the convergence of filtrations and Snell envelope.

2.1 Analytic selection theorem

Before formulating the analytic measurable theorem, we need the following definitions.

Definition 2.1.1. *Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathcal{P}(\Omega)$ the set of all probability measures on $(\Omega, \mathcal{F})$. For all $\mathbb{P} \in \mathcal{P}(\Omega)$, denote by $\mathcal{F}^{\mathbb{P}}$ the completed σ-field of $\mathcal{F}$ under $\mathbb{P}$. The universal completion of $\mathcal{F}$ is the σ-field defined as the intersection of $\mathcal{F}^{\mathbb{P}}$ for all probability measures $\mathbb{P} \in \mathcal{P}(\Omega)$, i.e.*

$$\mathcal{F}^{U} := \bigcap_{\mathbb{P} \in \mathcal{P}(\Omega)} \mathcal{F}^{\mathbb{P}}.$$

Definition 2.1.2.

1. *A Borel set is any set in a topological space that can be formed from open sets (or, equivalently, from closed sets) through the operations of countable union, countable intersection, and relative complement.*

2. *A topological space is said to be a Borel space, if it is topologically homeomorphic to a Borel subset of a Polish space.*

3. *Let E be a Borel space, then a subset B is an analytic set in E if there is another Borel space F and a Borel subset $A \subseteq E \times F$ such that $B = \pi_E(A)$, where π_E is the projection operator map of subsets of $E \times F$ to subsets of F,*

$$\pi_E(A) := \{x : \exists y \in F \text{ such that } (x,y) \in A\}.$$

A subset $C \subseteq E$ is co-analytic if its complement C^c is analytic.

4. *A function $g : E \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is upper semianalytic (u.s.a. for short) if $\{x \in E : g(x) > c\}$ is analytic for every $c \in \mathbb{R}$.*

5. *Let E be a Borel set and $\mathcal{A}(E)$ denote the σ-field generated by all analytic subsets. A function $f : E \to F$, where F is a Borel set, is analytically measurable if $f^{-1}(C) \in \mathcal{A}(E)$ for every $C \in \mathcal{B}(F)$.*

In a Borel space E, every Borel set is analytic, every analytic set is universally measurable, i.e. $\mathcal{B}(E) \subset \mathcal{A}(E) \subset \mathcal{B}^U(E)$. It follows that every upper semianalytic function is universally measurable. However, the complement of an analytic set may not be an analytic set and the class of analytic sets is not a σ-field.

Theorem 2.1.1 (Analytic selection theorem). *Let E and F be Borel spaces, A be an analytic subset of $E \times F$, and $f : A \to \mathbb{R}$ be an upper semianalytic function. Define $g(x) := \sup\{f(x,y) : (x,y) \in A\}$.*

(i) The projection set $\pi_E(A)$ is an analytic subset in E.

(ii) There exists an analytically measurable function $\phi : \pi_E(A) \to F$ such that $(x, \phi(x)) \in A$ for every $x \in \pi_E(A)$.

(iii) The function $g : \pi_E(A) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is upper semianalytic.

(iv) For every $\epsilon > 0$, there is an analytically measurable function $\phi_\epsilon : \pi_E(A) \to F$ such that $f(x, \phi_\epsilon(x)) \geq g^\epsilon(x) := (g(x) - \epsilon)1_{\{g(x) < \infty\}} + \frac{1}{\epsilon}1_{\{g(x)=\infty\}}$ for every $x \in \pi_E(A)$.

This theorem comes from El Karoui and Tan[39] for the measurable selection theorem and we also refer to Chapter 7 of Bertsekas and Shreve [7], Dellacherie and Meyer [26] for the measurable selection theorem and the dynamic programming principle.
These results can be useful to prove the dynamic programming principle and present the advantage of not requiring any regularity conditions on the value functions. This will be applied in the chapters 3 and 4, to prove the dynamic programming principle of the value function, in order to construct the solution of 2RBSDEs under weak conditions.

2.2 Regular conditional probability distribution

Let $\mathbb{P}$ be a probability measure on a filtered space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t\in[0,T]})$ and τ a $\mathbb{F}$-stopping time τ taking values in $[0,T]$. Following the terminology of Stroock and Varadhan [94], there exists a family of regular conditional probability distribution (r.c.p.d. for short) $(\mathbb{P}^\tau_\omega)_{\omega\in\Omega}$ satisfying:

(i) For every $\omega \in \Omega$, $\mathbb{P}^\tau_\omega$ is a probability measure on $(\Omega, \mathcal{F}_T)$;

(ii) For every $E \in \mathcal{F}_T$, the mapping $\omega \mapsto \mathbb{P}^\tau_\omega(E)$ is $\mathcal{F}_\tau$-measurable;

(iii) The family $(\mathbb{P}^\tau_\omega)_{\omega\in\Omega}$ is a version of the conditional probability measure of $\mathbb{P}$ on $\mathcal{F}_\tau$, i.e., for every integrable $\mathcal{F}_T$-measurable random variable ξ we have $\mathbb{E}^{\mathbb{P}}[\xi|\mathcal{F}_\tau](\omega) = \mathbb{E}^{\mathbb{P}^\tau_\omega}[\xi]$, for $\mathbb{P}$-a.e. $\omega \in \Omega$.

(iv) For every $\omega \in \Omega$, $\mathbb{P}^{\tau}_{\omega}(\Omega^{\omega}_{\tau}) = 1$, where $\Omega^{\omega}_{\tau} := \{\omega' \in \Omega : \omega'(s) = \omega(s),\ 0 \leq s \leq \tau(\omega)\}$.

Moreover, given some $\mathbb{P}$ and a family $(\mathbb{Q}_{\omega})_{\omega\in\Omega}$ such that $\omega \mapsto \mathbb{Q}_{\omega}$ is $\mathcal{F}_{\tau}$-measurable and $\mathbb{Q}_{\omega}(\Omega^{\omega}_{\tau}) = 1$ for all $\omega \in \Omega$, one can then define a concatenated probability measure $\mathbb{P} \otimes_{\tau} \mathbb{Q}_{\cdot}$ by

$$\mathbb{P} \otimes_{\tau} \mathbb{Q}_{\cdot[A]} := \int_{\Omega} \mathbb{Q}_{\omega[A]}\, \mathbb{P}(d\omega),\ \forall A \in \mathcal{F}_T.$$

2.3 The dynamic programming principle

Dynamic programming is an optimization approach that transforms a complex problem into a sequence of simpler problems; its essential characteristic is the multistage nature of the optimization procedure. The dynamic programming principle (DPP for short) plays an essential role in studying the control problems and shows that a global optimization problem can be split into a serie of local optimization problems. Bertsekas and Shreve in [7] give the DPP for the discrete time case . In the literature, this principle is classically established under assumptions which ensure that the value function satisfies some regularity/measurability properties. For stochastic control problems to derive the DPP, it is classical to use a measurability selection argument together with the stability of controls with respect to conditioning and concatenation. The measurable selection theorem plays an important role and is generally used to prove the measurability of the associated value function. But there is another commonly used approach to derive the DPP without the measurable selection argument, for example by assuming additional assumptions to prove the regularity of the value function. The dynamic programming principle in continuous time mentioned below can be found in [39].

Let consider the following notations

- E a Polish space and Ω the canonical space of all E-valued càdlàg paths on $\mathbb{R}^+$;
- $\mathcal{P}(\Omega)$ the set of all probability measures on Ω. Due to the fact that Ω is a Polish space, $\mathcal{P}(\Omega)$ is a Polish space (one can refer to [73]);
- $\mathcal{A}_{usa}(\Omega)$ the collection of all upper semianalytic functions bounded from below defined on the Polish space Ω;
- $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ the filtration generated by the canonical process on Ω;
- $\mathcal{F}^{U}_{\tau}$ the universally completed σ-field of $\mathcal{F}_{\tau}$, where τ is a $\mathbb{F}$-stopping time;
- $\mathcal{A}^{U}_{\tau}(\omega)$ the collection of all $\mathcal{F}^{U}_{\tau}$-measurable functions in $\mathcal{A}_{usa}(\Omega)$;
- $\left(\mathcal{P}_{t,\omega}\right)_{(t,\omega)\in\mathbb{R}^+\times\Omega}$ a class of probability measures families.

We consider a family of nonlinear operators associated with the class $\left(\mathcal{P}_{t,\omega}\right)_{(t,\omega)\in\mathbb{R}^+\times\Omega}$:

$$\mathcal{E}_{\tau}[\xi](\omega) := \sup\left\{\mathbb{E}^{\mathbb{P}}[\xi] : \mathbb{P} \in \mathcal{P}_{\tau(\omega),\omega}\right\},\quad \forall \text{ where } \tau \text{ is a finite } \mathbb{F}\text{-stopping time .}$$

The family $\left(\mathcal{P}_{t,\omega}\right)_{(t,\omega)\in\mathbb{R}^+\times\Omega}$ can be considered as a family of section sets of subsets in $\mathbb{R}^+ \times \Omega \times \mathcal{P}(\Omega)$. Equivalently, its graph is given by

$$[[\mathcal{P}]] := \left\{(t,\omega,\mathbb{P}) :\ (t,\omega) \in \mathbb{R}^+ \times \Omega,\ \mathbb{P} \in \mathcal{P}_{t,\omega}\right\}.$$

Suppose that $[[\mathcal{P}]]$ is an analytic set in the Polish space $\mathbb{R}^+ \times \Omega \times \mathcal{P}(\Omega)$. Moreover, we assume the progressive measurability, i.e. for every $(t,\omega) \in \mathbb{R}^+ \times \Omega$, $\mathcal{P}_{t,\omega}$ is not empty and $\mathbb{F}$-adapted, for any $(t,\omega) \in \mathbb{R}^+ \times \Omega$,

$$\mathcal{P}_{t,\omega} = \mathcal{P}_{t,\omega_{t\wedge\cdot}} \text{ and } \mathbb{P}(\{w : w_{t\wedge\cdot} = \omega_{t\wedge\cdot}\})) = 1, \ \forall \mathbb{P} \in \mathcal{P}_{t,\omega} \tag{2.3.1}$$

The following hypotheses of conditioning and concatenation are commonly used to establish the DPP.

Assumption 2.3.1. *Let $(t_0,\omega_0) \in [0,T] \times \Omega$ be arbitrary, τ be an arbitrary stopping time taking value in $[t_0,T]$ and $\mathbb{P} \in \mathcal{P}_{t_0,\omega_0}$.*
Stability by conditioning: *There is a family of r.c.p.d. $(\mathbb{P}_\omega)_{\omega\in\Omega}$ of $\mathbb{P}$ w.r.t. $\mathbb{F}_\tau$ such that $\mathbb{P}_\omega \in \mathcal{P}_{\tau(\omega),\omega}$ for $\mathbb{P}$-almost every $\omega \in \Omega$.*
Stability by concatenation: *Let $(\mathbb{Q}_\omega)_{\omega\in\Omega}$ be such that $\omega \mapsto \mathbb{Q}_\omega$ is $\mathbb{F}_\tau$-measurable and $\mathbb{Q}_\omega \in \mathcal{P}_{\tau(\omega),\omega}$ for $\mathbb{P}$a.e. $\omega \in \Omega$, then $\mathbb{P} \otimes_\tau \mathbb{Q}_\cdot \in \mathcal{P}_{t_0,\omega_0}$.*

Since the DPP is mainly based on the regularity (measurability) of the value function, the following results which can be found in [39] give this regularity using the selection measurable theorem.

Lemma 2.3.1. *Let $\left(\mathcal{P}_{t,\omega}\right)_{(t,\omega)\in[0,T]\times\Omega}$ be satisfying the above hypotheses , τ a $\mathbb{F}$-stopping time and $\xi \in \mathcal{A}_{usa}(\Omega)$. Then $\mathcal{E}_\tau[\xi] \in \mathcal{A}^U_\tau(\Omega)$. In particular, $\mathcal{E}_\tau$ is an operator from $\mathcal{A}_{usa}(\Omega)$ to $\mathcal{A}^U_\tau(\Omega) \subset \mathcal{A}_{usa}(\Omega)$.*

Now we have the following DPP for $\mathcal{E}$.

Theorem 2.3.1. *Suppose that $[[\mathcal{P}]]$ is analytic in $[0,T] \times \Omega \times \mathcal{P}(\Omega)$, the condition (2.3.1) and Assumption 2.3.1 hold true. Then for every $\mathbb{F}$-stopping times $\tau \geq \sigma$, we have the following time consistency property:*

$$\mathcal{E}_\tau[\xi] = \mathcal{E}_\tau\left[\mathcal{E}_\sigma[\xi]\right], \quad \forall \xi \in \mathcal{A}_{usa}(\Omega).$$

2.4 Measurability with respect to the probability law.

Given a set of probability measures, Neufeld and Nutz in [73] provided a measurability of semi-martingales with respect to a probability measure considered as parameter. There are numerous applications of stochastic analysis and dynamic programming. It is well known that the DPP is delicate as soon as the regularity of the value function is not known a priori; this is often the case when the reward/cost function is discontinuous or in the presence of state constraints. In this situation, the measurability of the set of controls is crucial to establish the dynamic programming and the measurability of the value function;
This result is very interesting because we work with a large set $\mathcal{P}_0$ of semimartingale laws, often mutually singular. For instance, when considering a standard stochastic control problem based on a controlled BSDEs, it is useful to recast the problem on Skorokhod space by taking $\mathcal{P}_h^\kappa$ to be the set of all semimartingale laws. We will need the following results of [73] which are respectively Lemmas 3.1, 3.2 and 3.5, to construct a right-continuous jointly measurable version of the RBSDEs associated to a 2RBSDE (see the proof of Lemma 3.4.2).

We have the following notations
- $(\Omega,\mathcal{F})$ is a measurable space,

- $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration of sub-σ-fields of $\mathcal{F}$ and $\mathbb{F}_+ := (\mathcal{F}_{t+})_{t\geq 0}$ his right-continuous version,
- $\mathfrak{P}$ the set of all probability measure on $(\Omega, \mathcal{F})$ and $\mathcal{B}(\mathfrak{P})$ its Borel σ-field.
- $(\widehat{\Omega}, \widehat{\mathcal{F}}) := (\mathfrak{P} \times \Omega, \mathcal{B}(\mathfrak{P}) \otimes \mathcal{F})$.
- $\widehat{\mathbb{F}} = (\widehat{\mathcal{F}}_t)_{t\geq 0}$ a filtration , where $\widehat{\mathcal{F}}_t := \mathcal{B}(\mathfrak{P}) \otimes \mathcal{F}_t$ and $\widehat{\mathbb{F}}_+$ its right continuous filtration.

Lemma 2.4.1. *Let $t \geq 0$ and let $f : \widehat{\Omega} \to \overline{\mathbb{R}}$ be measurable. Then the function $\mathfrak{P} \to \overline{\mathbb{R}}$, $\mathbb{P} \mapsto \mathbb{E}^{\mathbb{P}}[f(\mathbb{P}, \cdot)]$ is measurable. Moreover, there exists a version of the conditional expectations $\mathbb{E}^{\mathbb{P}}[f(\mathbb{P}, \cdot)|\mathcal{F}_t]$ and $\mathbb{E}^{\mathbb{P}}[f(\mathbb{P}, \cdot)|\mathcal{F}_{t+}]$ such that*

$$\widehat{\Omega} \to \overline{\mathbb{R}}, \quad (\mathbb{P}, \omega) \mapsto \mathbb{E}^{\mathbb{P}}[f(\mathbb{P}, \cdot)|\mathcal{F}_t](\omega), \quad (\mathbb{P}, \omega) \mapsto \mathbb{E}^{\mathbb{P}}[f(\mathbb{P}, \cdot)|\mathcal{F}_{t+}](\omega)$$

are measurable with respect to $\widehat{\mathcal{F}}_t$ and $\widehat{\mathcal{F}}_{t+}$, respectively, while for fixed $\mathbb{P} \in \mathfrak{P}$,

$$\Omega \to \overline{\mathbb{R}}, \quad \omega \mapsto \mathbb{E}^{\mathbb{P}}[f(\mathbb{P}, \cdot)|\mathcal{F}_t](\omega), \quad \omega \mapsto \mathbb{E}^{\mathbb{P}}[f(\mathbb{P}, \cdot)|\mathcal{F}_{t+}](\omega)$$

are measurable with respect to $\mathcal{F}_t$ and $\mathcal{F}_{t+}$, respectively.

Lemma 2.4.2. *Let $f^n : \Omega \times \mathfrak{P} \to \overline{\mathbb{R}}^d$ be measurable functions such that $f^n(\mathbb{P}, \cdot)$ is a convergent sequence in $\mathbb{L}^1(\mathbb{P})$ for every $\mathbb{P} \in \mathfrak{P}$. There exists a measurable function $f : \Omega \times \mathfrak{P} \to \overline{\mathbb{R}}^d$ such that $f(\mathbb{P}, \cdot) = \lim_n f^n(\mathbb{P}, \cdot)$ in $\mathbb{L}^1(\mathbb{P})$ for every $\mathbb{P} \in \mathfrak{P}$. Moreover, there exists an increasing sequence $(n_k^{\mathbb{P}})_k \subseteq \mathbb{N}$ such that $\mathbb{P} \mapsto n_k^{\mathbb{P}}$ is measurable and $\lim_k f^{n_k^{\mathbb{P}}}(\mathbb{P}, \cdot) = f(\mathbb{P}, \cdot)$ $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathfrak{P}$.*

Lemma 2.4.3. *Let $f : \mathfrak{P} \times \Omega \times \mathbb{R}_+ \to \mathbb{R}$ be such that $f(\cdot, \cdot, t)$ is $\mathcal{B}(\mathfrak{P}) \otimes \mathcal{F}_{t+}$ measurable for all t. There exists a measurable function $\bar{f} : \mathfrak{P} \times \Omega \times \mathbb{R}_+ \to \mathbb{R}$ such that $\bar{f}$ is $\mathcal{B}(\mathfrak{P}) \times \mathbb{F}_+$-optional, $\bar{f}(\mathbb{P}, \omega, \cdot)$ is right-continuous for all $(\mathbb{P}, \omega)$, and for any $\mathbb{P} \in \mathfrak{P}$ such that $f(\mathbb{P}, \cdot, \cdot)$ is an $\mathbb{F}_+$-adapted $\mathbb{P}$ - $\mathbb{F}_+$ supermartingale with right-continuous expectation $t \mapsto \mathbb{E}^{\mathbb{P}}[f(\mathbb{P}, \cdot, t)]$, the process $\bar{f}(\mathbb{P}, \cdot, \cdot)$ is an $\mathbb{F}_+$-adapted $\mathbb{P}$-modification of $f(\mathbb{P}, \cdot, \cdot)$ and in particular a $\mathbb{P}$-$\mathbb{F}_+$-supermartingale.*

Since the 2BSDEs are defined $\mathbb{P}$-a.s. for every probability measure $\mathbb{P}$, the solution can be viewed as a function of probability measure $\mathbb{P}$ (in other word $\mathbb{P}$ plays the role of a measurable parameter). This result allows us to consider a jointly measurable version of the solutions processes in time, space and probability law. This is exactly the approach followed by Possamai,Tan and Zhou in [85] to provide wellposedness of 2BSDEs in a general filtration, existence and uniqueness of solution without any regularity condition on the generator and the terminal condition. In [91] and [66] they deduce the measurability of the value function by the uniform continuity of this one. To prove this uniform continuity, the generator and the terminal value (and the obstacle in [66]) should be uniformly continuous, this is why they need regularity conditions on the parameters. Using the Borel jointly measurability, Possamai, Tan and Zhou in [85] obtain the DPP of the value function without any regularity conditions on the terminal condition, the generator and proved existence and uniqueness of the 2BSDEs. We follow this approach and prove the existence and uniqueness of the 2RBSDEs in a general filtration without any uniform continuity condition on the terminal condition, the generator nor the obstacle.

2.5 RBSDEs in general filtration

Definition 2.5.1. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}\mathbb{P})$ a filtered probability space.*

1. *The filtration $(\mathcal{F}_t)_{t\geq 0}$ is complete if Ω is a complete space and all $\mathbb{P}$-negligible sets are in $\mathcal{F}_0$.*
2. *The filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous if $\mathcal{F}_t = \mathcal{F}_{t+}$ for all $t \geq 0$, with*

$$\mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s.$$

3. *The filtration $(\mathcal{F}_t)_{t\geq 0}$ is said to satisfy the usual conditions if it is complete and right-continuous.*

4. *The filtration $(\mathcal{F}_t)_{t\geq 0}$ is quasi-left continuous if $(\mathcal{F}_t)_{t\geq 0}$ satisfy usual conditions and $\mathcal{F}_T = \mathcal{F}_{T-}$ for all previsible time T, with*

$$\mathcal{F}_{t-} := \bigvee_{s<t} \mathcal{F}_s.$$

Given a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, \mathbb{P})$, equipped with a quasi left-continuous, right-continuous and complete filtration $(\mathcal{F}_t)$ (see Definition 2.5.1) and a d-dimensional càdlàg square integrable martingale $(M_t, t \in \mathbb{R}^+)$. We are interested in BSDE driven by a martingale M. Here the main difference is that the filtration $(\mathcal{F}_t)$ is no longer generated by a Brownian motion. Then the representation property of a local martingale is no more true and an additional orthogonal martingale term has to be introduced in the definition of the solution. A solution to one-dimensional BSDEs with respect to $(\mathcal{F}_t)$ becomes a triple of adapted processes (Y, Z, N) taking values in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ such that

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds - \int_t^T Z_s dM_s - \int_t^T dN_s, \quad \mathbb{P}\text{-a.s.} \tag{2.5.1}$$

where N is a càdlàg square integrable martingale orthogonal to M, f the generator is predictable and ξ the terminal condition.

These BSDEs were first introduced by El Karoui and Huang [33]. Under the following conditions,

- For any (y_1, z_1, y_2, z_2),

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq r_t|y_1 - y_2| + \theta_t|m_t||z_1 - z_2|, \ dt \otimes d\mathbb{P}\text{-a.s.}$$

  where m is the density of the quadratic variation of M, r and θ are two non-negative predictable processes.

- $f(t, 0, 0)/(r_t + \theta_t^2)$ and ξ are square integrable, for all $t \in [0, T]$.

they proved the existence and the uniqueness of the solution to the BSDE (2.5.2). The time horizon T may be a stopping time possibly with infinite values. Unlike in the classical results of existence and uniqueness related to a BSDE with respect to a Brownian filtration, the assumptions of uniformly Lipschitz condition on the driving parameter is relaxed.

Kruse and Popier [56] consider BSDEs with respect to filtration generated by a Brownian motion and a Poison random measure(notice that the filtration is still quasi left-continuous and verifies usual conditions) and establish existence and uniqueness of multimensional solutions. They work under monotonicity assumption on the driver and generalize to random time horizons given by a stopping time, they also proved comparison principle in the one-dimensional case.

Following (2.5.2), the solution of a RBSDE with a lower obstacle in general filtration consists in a quadruple (Y, Z, N, K) of processes which verifies

$$\begin{cases} Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds - \int_t^T Z_s dM_s - \int_t^T dN_s + K_T - K_t, \ \mathbb{P}\text{-a.s.} \\ Y_t \geq L_t, \ \mathbb{P}\text{-a.s.} \\ \int_0^T (Y_s - L_s)dK_s = 0, \ \mathbb{P}\text{-a.s.} \end{cases} \tag{2.5.2}$$

where K is non decreasing and null at 0 and N is a martingale orthogonal to M null at 0. Klimsiak [54] studied the problem of existence and uniqueness of solutions of this class of

RBSDEs in multiple dimension under monotonicity and weak integrability conditions with finite terminal condition. When $1 < p \leq 2$ the assumption of quasi-left continuous of the filtration is needed.In this section we choose to remind the wellposedness of RBSDEs in general filtration, which may not be quasi left-continuous introduced by Bouchard et al [14].

Definition 2.5.2. *Let L be a càdlàg process such that $L^+ := L \vee 0$ satisfies $\mathbb{E}^{\mathbb{P}}[\sup_{t\leq T} |L_t^+|^p] < \infty$. We say that (Y, Z, N, K) is a solution of the RBSDE with a lower obstacle L if*

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds - \int_t^T Z_s dM_s - \int_t^T dN_s + K_T - K_t$$

holds for any $t \in [0, T]$ $\mathbb{P}$-a.s. where K is a non decreasing process null at 0, N is a martingale orthogonal to M and if

$$\begin{cases} Y_t \geq L_t, t \in [0, T], \ \mathbb{P}\text{-a.s.} \\ \int_0^T (Y_{s^-} - L_{s^-})dK_s = 0, \ \mathbb{P}\text{-a.s.} \end{cases}$$

Symmetry about upper and lower obstacle. In the case of standard RBSDE the is a symmetry about upper and lower obstacle in the sense that for the upper one, the non decreasing process K null at 0 is adding by subtraction in the equation. This is due to the fact that for the upper obstacle K maintains the solution below the obstacle in opposite to the lower case where K maintains the solution above the obstacle. This symmetry is still verified in a general filtration. So, we have a similar definition for an upper obstacle.

Assumption 2.5.1. - *$\xi \in L^p$, $f(s, 0, 0)$ is square integrable and $(s, \omega) \mapsto f_s(\omega, y, z)$ is progressively measurable for all (y, z).*

- *There are two positive constants L_y and L_z such that*

$$|f_t(\omega, y, z) - f_t(\omega, y', z')| \leq L_y|y - y'| + L_z|z - z'|,$$

for all $(t, y, z, y', z') \in [0, T] \times \Omega \times (\mathbb{R} \times \mathbb{R}^d)^2$.

The following results have been proved in [14].

Theorem 2.5.1. *Let Assumption 2.5.1 holds true. Then, there is a unique solution (Y, Z, N, K) to the reflected BSDE introduced in Definition 2.5.2.*

Similar to the case of filtration generated by Brownian motion, we have the following result which state that the solution of RBSDE can be represented as a snell enveloppe.

Proposition 2.5.1. *Let (Y, Z, M, K) and L be as in Definition 2.5.2. Then*

$$Y_t = \operatorname*{ess\,sup}_{v\in\mathcal{T}_t} \mathbb{E}^{\mathbb{P}}\Big[\int_t^v f_s(Y_s, Z_s)ds + L_v 1_{\{v<T\}} + \xi 1_{\{v=T\}}|\mathcal{F}_t\Big], \ \text{for all } t \in [0, T].$$

where $\mathcal{T}_t$ is the set of all stopping times taking values in $[t, T]$.

And with an upper obstacle denoted U, we have the following:

Proposition 2.5.2. *Let (Y, Z, M, K) the solution of a RBSDE associated to the generator f, ξ and the obstacle U . Then*

$$Y_t = \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_t} \mathbb{E}^{\mathbb{P}}\Big[\int_t^\sigma f_s(Y_s, Z_s)ds + U_\sigma 1_{\{\sigma<T\}} + \xi 1_{\{\sigma=T\}}|\mathcal{F}_t\Big], \ \text{for all } t \in [0, T].$$

where $\mathcal{T}_t$ is the set of all stopping times taking values in $[t, T]$.

2.6 Convergence of filtrations

2.6.1 Skorokhod topology $\mathbf{J}_1$

For a fixed $T > 0$, we denote $D = D([0,T],\mathbb{R})$ the space of $\mathbb{R}$-valued càdlàg paths defined on $[0,T]$. In this space the uniform convergence topology is not the most appropriate one. When we consider the behaviour of a sequence of process $(x_n)_n$ in the neighbourhood of points of discontinuity of its limit process $x(t)$. The topology $\mathbf{J}_1$ suggests that the convergence be uniform also at points of discontinuity. It is much more natural to suppose that the functions we can obtain from each other by small deformation of the times scale lies close to each other. Following this observation, Skorokhod in [89] was brought to propose the convergence with respect to the topology $\mathbf{J}_1$ which we will see below. This topology has also been studied by Billingsley [8] for $T = 1$ and by Jacod and Shiryaev [50] for stochastic processes indexed by $[0,+\infty[$.

Before this we introduce Λ the set of all continuous functions $\lambda : [0,T] \to [0,T]$ that are strictly increasing with $\lambda(0) = 0$ and $\lambda(T) = T$. The following result is a part of Theorem 1.14 of [50]

Theorem 2.6.1. *A sequence $(x_n)_n$ is called $\mathbf{J}_1$-convergent to x if there exists a sequence $\{\lambda_n\} \subset \Lambda$ such that*

$$\sup_s |\lambda_n(s) - s| \to 0 \text{ and } \sup_{s \le N} |x_n \circ \lambda_n(s) - x(s)| \to 0 \text{ for all } N \in \mathbb{N}^\star.$$

Remark 2.6.1. ***1.*** *The topology $\mathbf{J}_1$ is weaker than the uniform topology.*

2. *If a sequence x^n is $\mathbf{J}_1$-convergent to x and x is continuous on the compact set $[0,T]$, then x^n converge to x with respect to the uniform topology.*

2.6.2 Convergence of σ-algebra and filtration

The notions of convergence of σ-algebras and filtrations was introduced by Hoover [46] in 1991. A few years later, Coquet, Mackevicius, Menin and Slominski in [23], [24] gave some results about the links between the convergence of processes and the convergence of the filtrations generated by these processes. The following results essentially comes from [46] for the convergence of σ-algebras and from [24] for the $\mathbf{J}_1$-convergence of filtrations.

Definition 2.6.1. *A sequence of σ-algebras $\mathcal{A}^n$ converges weakly to a σ-algebra $\mathcal{A}$ if and only if, for all $B \in \mathcal{A}$, the sequence of random variables $(\mathbb{E}[1_B|\mathcal{A}^n])$ converges in probability to 1_B.*

Definition 2.6.2. *A sequence of filtrations $\mathcal{F}^n$ converges weakly to the filtration $\mathcal{F}$ if and only if for all $B \in \mathcal{F}_T$, the sequence of càdlàg martingales $(\mathbb{E}[1_B|\mathcal{F}^n_\cdot])$ converges in probability under the Shorokhod $\mathbf{J}_1$-topology on D to the martingale $(\mathbb{E}[1_B|\mathcal{F}_\cdot])$*

In [24], there are many examples of weak convergence of filtrations essentially based on the behaviour of the sequence of processes which generated the sequence of filtrations. Among these, we have the following

Proposition 2.6.1. *Let $\{X_n\}$ be a sequence of càdlàg processes with independent increments (initial values are considered as increments). If $X^n \to X$ in probability under $\mathbf{J}_1$, then $\mathcal{F}^{X^n}$ converge weakly to $\mathcal{F}^X$.*

2.7 Convergence of Snell envelope

Let consider

$$J_t^n = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t^n} \mathbb{E}\left[f(X_\tau^n)|\mathcal{F}_t^{X^n}\right]$$

and

$$J_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}^n\left[f(X_\tau)|\mathcal{F}_t^{X}\right].$$

where f is a bounded continuous function defined on $\mathbb{R}$, $\mathcal{F}^X$(resp. $\mathcal{F}^{X^n}$) the filtration generated by X (resp. X^n) and $\mathcal{T}_t^n$(resp. $\mathcal{T}_t$) the set of $[t,T]$-valued $\mathcal{F}^X$-stopping times(resp. $\mathcal{F}^{X^n}$-stopping times).
In this section we only mention a result about stability for Snell envelope under convergence in distribution of the underlying processes. Mulinacci and Pratelli in [71] prove that if a sequence (X^n) of stochastic process converges in distribution for the Skorokhod topology (in particular $\mathbf{J}_1$ topology) to a process X and satisfies some additional hypotheses, the sequence of Snell envelope of X converges in distribution for the Meyer-Zheng topology to the Snell envelope of X. Also when the Snell envelope of the limit process is continuous, the convergence is in fact in the Skorokhod sense. Let $\{X^n\}$ be a sequence of positive stochastic processes.

Assumption 2.7.1. ***1.*** *The processes X^n are uniformly of class D (see section 2.6 for the definition of D), i.e. the random variable X_τ^n, for $n \in \mathbb{N}$ and τ stopping time for the filtration $\mathcal{F}^{X^n}$, are uniformly integrable;*

2. " Aldous tightness criterion " *For every $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ and $\delta > 0$ such that if $n \geq n_0$, $0 < s < \delta$ and τ is an $\mathcal{F}^{X^n}$-stopping time, we have*

$$\mathbb{E}^n\left[|X_{\tau+s}^n - X_\tau^n|\right] < \epsilon.$$

Remark 2.7.1. *Let us comment the above assumptions. The second condition introduced by Aldous in [1] and [3] are very useful in proofs of weak convergence. This condition is automatically verified by quasi-left-continuous processes and together with the first condition, this ensures the tightness of the sequence (X^n) (one can refer for example to [50]).*

Remark 2.7.2. *In the framework of chapter 5, X is the canonical process of the probability space, then the condition $(\mathscr{H})$ (introduced in [16] and then taken by Mulinacci and Pratelli in [71]) that must be verified by the pair $(X, \mathcal{F}^X)$ is obvious since the filtration $\mathcal{F}^X$ is generated by the canonical process X. This condition is very important, since the convergence result of [71] can't be obtained otherwise.*

Then the following result is a reformulation of Theorem 3.5 of [71] in our context.

Theorem 2.7.1. *Suppose that X^n is $\mathbf{J}_1$-convergent to X, that Assumption 2.7.1 holds true, then (X^n, J^n) is convergent to (X, J) with respect to the Meyer-Zheng topology.*

The original definition is not about $\mathbf{J}_1$ topology but $\mathbf{S}$. Since the $\mathbf{S}$ topology is coarser than $\mathbf{J}_1$ (one can refer to Jakubowski [51]), the theorem can be stated as above.

Chapter 3

Second order Reflected BSDEs under weak assumptions: the case of a lower obstacle

Contents

3.1 Introduction

Following Possamai, Tan and Zhou [85], we consider their wellposedness result for 2BSDEs in a general filtration which does not require any regularity assumption on terminal condition and generator together with 2RBSDEs with a lower obstacle defined in [66, 68]. Recall that in [91] and [66] , the dynamic programming principle was the key idea to show existence of the solution. To prove this dynamic programming, they imposed regularity condition on terminal condition, generator and obstacle to obtain the regularity of the value function.
Recently considering the optimization over a set of non-dominated probability measures of solutions of BSDEs, Possamai, Tan and Zhou [85] proved a dynamic programming principle for this stochastic control problem using analytic selection measurable through the Borel measurability argument.

Our goal is to prove existence and uniqueness of solution to 2RBSDEs in a general filtration without any regularity condition on generator, terminal condition and obstacle. The remainder of this chapter is organized as follows. First, we give some of the standard notations and the definition of 2RBSDE introduced in [66, 68], then the following section is dedicated to representation and uniqueness of the solution. The last section is devoted to the existence of the solution using DPP.

3.2 Preliminaries

3.2.1 Notation

Let us fix $T > 0$ and $d \in \mathbb{N}^*$. Let $\Omega = \{\omega \in C([0,T],\mathbb{R}^d) : \omega_0 = 0\}$ be the canonical space equipped with the uniform convergence norm $\|\omega\|_\infty = \sup_{0\le t\le T}\|\omega_t\|$, X the canonical process, i.e. $X_t(\omega) = \omega_t$ for all $\omega \in \Omega$ and $\mathbb{P}_0$ the Wiener measure on Ω under which X is a Brownian motion. We denote by $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ the canonical filtration, $\mathbb{F}_+ = (\mathcal{F}_{t+})_{0\le t\le T}$ the right limit of $\mathbb{F}$ with $\mathcal{F}_{t+} := \cap_{s>t}\mathcal{F}_s$ for all $t \in [0,T)$ and $\mathcal{F}_{T+} = \mathcal{F}_T$.

We shall need the following notations introduced in [85].

– $\mathbb{M}_1$ is the collection of all probability measures on $(\Omega, \mathcal{F}_T)$. Notice that $\mathbb{M}_1$ is a Polish space equipped with the weak convergence topology and set $\mathscr{B}$ its Borel σ-field. For any $\mathbb{P} \in \mathbb{M}_1$, denote by $\mathcal{F}_t^{\mathbb{P}}$ the completed σ-field of $\mathcal{F}_t$ under $\mathbb{P}$, $\mathbb{F}^{\mathbb{P}} = \left(\mathcal{F}_t^{\mathbb{P}}\right)_{0\le t\le T}$ the completed filtration and $\mathbb{F}_+^{\mathbb{P}}$ its right limit. Thus $\mathbb{F}_+^{\mathbb{P}}$ satisfies the usual conditions.

– For $\mathcal{P} \subset \mathbb{M}_1$, we have

$$\mathbb{F}^U := \left(\mathcal{F}_t^U\right)_{0\le t\le T},\ \mathbb{F}^{\mathcal{P}} := \left(\mathcal{F}_t^{\mathcal{P}}\right)_{0\le t\le T} \text{ and } \mathbb{F}_+^{\mathcal{P}} := \left(\mathcal{F}_{t+}^{\mathcal{P}}\right)_{0\le t\le T} \text{ such that}$$

$$\mathcal{F}_t^U := \bigcap_{\mathbb{P}\in\mathbb{M}_1} \mathcal{F}_t^{\mathbb{P}},\ \mathcal{F}_t^{\mathcal{P}} := \bigcap_{\mathbb{P}\in\mathcal{P}} \mathcal{F}_t^{\mathbb{P}},\ \text{and } \mathcal{F}_{t+}^{\mathcal{P}} := \cap_{s>t}\mathcal{F}_t^{\mathcal{P}},\ t \in [0,T],\ \text{and } \mathcal{F}_{T+}^{\mathcal{P}} := \mathcal{F}_T^{\mathcal{P}}.$$

3.2.2 The models space: the semimartingles measures

We call a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ a semimartingale measure if X is a semimartingale under $\mathbb{P}$. By Karandikar [53], there is some $\mathbb{F}$-progressively measurable non-decreasing process on Ω denoted by $\langle X\rangle = (\langle X\rangle_t)_{0\le t\le T}$ which coincides with the quadratic variation of X under each semi-martingale measure $\mathbb{P}$. In particular, this provides a pathwise definition of $\langle X\rangle$ and its density $\widehat{a}_t$,

$$\langle X\rangle_t := X_tX_t' - 2\int_0^t X_s dX_s' \quad \text{and} \quad \widehat{a}_t := \overline{\lim_{\epsilon\downarrow 0}}\ \frac{1}{\epsilon}(\langle X\rangle_t - \langle X\rangle_{t-\epsilon}).$$

where X' denotes the transposed of X, and the $\overline{\lim}$ is componentwise defined. For every $t \in [0,T]$, let $\mathcal{P}_t^W$ denote the collection of all probability measures $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ such that:

– $(X_s)_{s\in[t,T]}$ is a $(\mathbb{P},\mathbb{F})$-semimartingale admitting the canonical decomposition

$$X_s = \int_t^s b_r^{\mathbb{P}} dr + X_s^{c,\mathbb{P}},\ s \in [t,T],\ \mathbb{P}\text{-a.s.},$$

where $b^{\mathbb{P}}$ is a $\mathbb{F}$-predictable $\mathbb{R}^d$-valued process, and $X^{c,\mathbb{P}}$ is the continuous local martingale part of X under $\mathbb{P}$.

– $(\langle X\rangle_s)_{s\in[t,T]}$ is absolutely continuous in s with respect to the Lebesgue measure, and $\widehat{a}$ takes values in $\mathbb{S}_d^{\geq 0}$, $\mathbb{P}$-a.s.($\mathbb{S}_d^{\geq 0}$ denotes the set of all symmetric positive semi-definite $d\times d$ matrices).

Remark 3.2.1. *If $\widehat{a}_s$ is non-degenerate $\mathbb{P}$-a.s., for all $s\in[0,T]$, the we can construct a Brownian motion $W^{\mathbb{P}}$ on Ω by*

$$W_t^{\mathbb{P}} := \int_0^t \widehat{a}_s^{-1/2} dX_s^{c,\mathbb{P}},\ t\in[0,T],\ \mathbb{P}\text{-a.s.} \tag{3.2.1}$$

Otherwise we define the enlarged canonical space $\overline{\Omega} := \Omega\times\Omega'$, where Ω' is identical to Ω and set (X,B) its canonical process, i.e. $X_t(\overline{\omega}) := \omega_t$, $B_t(\overline{\omega}) := \omega'_t$ for all $\bar{\omega} := (\omega,\omega')\in\overline{\Omega}$. The extension from Ω to $\overline{\Omega}$ of a random variable or a process λ is defined by

$$\lambda(\overline{\omega}) := \lambda(\omega),\ \forall\overline{\omega} = (\omega,\omega')\in\overline{\Omega}. \tag{3.2.2}$$

In particular $\widehat{a}$ can be extended on $\overline{\Omega}$. For $\mathbb{P}\in\mathcal{P}_t^W$, a probability measure on the enlarged space $\overline{\Omega}$ is denote by $\overline{\mathbb{P}}$ with $\overline{\mathbb{P}} := \mathbb{P}\otimes\mathbb{P}_0$. We also consider like in [66], the canonical filtration $\overline{\mathbb{F}}$ generated by (X,B), the filtration $\overline{\mathbb{F}}^X$ generated by X, the right-continuous filtrations $\overline{\mathbb{F}}_+^X$ and $\overline{\mathbb{F}}_+$, and the augmented filtration $\overline{\mathbb{F}}_+^{X,\overline{\mathbb{P}}}$ and $\overline{\mathbb{F}}_+^{\overline{\mathbb{P}}}$ given a probability measure $\overline{\mathbb{P}}$ on $\overline{\Omega}$.
From the above it follows that X in $(\overline{\Omega},\overline{\mathcal{F}}_T,\overline{\mathbb{P}},\overline{\mathbb{F}})$ is a semimartingale measure with the same triplet of characteristics as X in $(\Omega,\mathcal{F}_T,\mathbb{P},\mathbb{F})$, B is a $\overline{\mathbb{F}}$-Brownian motion and X is independent of B. Then for every $\mathbb{P}\in\mathcal{P}_t^W$, there is some $\mathbb{R}^d$-valued $\overline{\mathbb{F}}$-Brownian motion $W^{\mathbb{P}}$ such that(see Theorem 4.5.2 of Stroock and Varadhan [94])

$$X_s = \int_t^s b_r^{\mathbb{P}} dr + \int_t^s \widehat{a}_r^{\frac{1}{2}} dW_r^{\mathbb{P}},\ s\in[t,T],\ \overline{\mathbb{P}}\text{-a.s.} \tag{3.2.3}$$

where the definition of $b^{\mathbb{P}}$ and $\widehat{a}$ are extended on $\overline{\Omega}$.

Remark 3.2.2. *The decomposition (3.2.3) of the canonical process X with a Brownian motion $W^{\mathbb{P}}$ is on the enlarged space $\overline{\Omega}$. The interest of this decomposition lies in the fact that in calculations, we can apply some known results related to Brownian Motion like Girsanov Theorem, linearisation arguments and others. To have the same decomposition on Ω, $\widehat{a}$ needs to be non-degenerate. Since we cannot ensure that this condition will be satisfied, we will extend some equivalence introduced in [85] in our work.*

Assumptions. We consider a random variable $\xi:\Omega\longrightarrow\mathbb{R}$ and a generator function

$$f:(t,\omega,y,z,a,b)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}_d^{\geq 0}\times\mathbb{R}^d\longrightarrow\mathbb{R}.$$

For simplicity of notation, we denote

$$\widehat{f}_s^{\mathbb{P}}(y,z) := f(s,X_{\cdot\wedge s},y,z,\widehat{a}_s,b_s^{\mathbb{P}}) \text{ and } \widehat{f}_s^{\mathbb{P},0} := f(s,X_{\cdot\wedge s},0,0,\widehat{a}_s,b_s^{\mathbb{P}}).$$

We denote by $(\mathcal{P}(t,\omega))_{(t,\omega)\in[0,T]\times\Omega}$ a class of probability measure families where $\mathcal{P}(t,\omega)\subset\mathcal{P}_t^W$ for all $(t,\omega)\in[0,T]\times\Omega$. Denote also $\mathcal{P}_t := \cup_{\omega\in\Omega}\mathcal{P}(t,\omega)$. The family $(\mathcal{P}(t,\omega))_{(t,\omega)\in[0,T]\times\Omega}$ can be considered as a family of section sets of a subset in $[0,T]\times\Omega\times\mathbb{M}_1$. Then, we define its graph

$$[[\mathcal{P}]] := \{(t,\omega,\mathbb{P})\ :\ (t,\omega)\in[0,T]\times\Omega,\ \mathbb{P}\in\mathcal{P}(t,\omega)\}.$$

Following [85], we make the following assumption.

Assumption 3.2.1. *(i) The random variable ξ is $\mathcal{F}_T$-measurable, the generator function f is jointly Borel measurable and uniformly Lipschitz in y and z such that for every $(t,\omega,y,y',z,z',a,b) \in [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}_d^{\geq 0}\times\mathbb{R}^d$,*

$$|f(t,\omega,y,z,a,b) - f(t,\omega,y',z',a,b)| \leq L_f(|y-y'| + \|z-z'\|),$$

L_f is a positive constant and for every fixed (y,z,a,b), the map $(t,\omega) \mapsto f(t,\omega,y,z,a,b)$ is $\mathbb{F}$-progressively measurable .

(ii) For every $(t,\omega) \in [0,T]\times\Omega$, one has $\mathcal{P}(t,\omega) = \mathcal{P}(t,\omega_{\cdot\wedge t})$ and $\mathbb{P}(\Omega_t^\omega) = 1$ whenever $\mathbb{P} \in \mathcal{P}(t,\omega)$, where $\Omega_t^\omega := \{\omega' \in \Omega : \omega'(s) = \omega(s),\ 0 \leq s \leq t\}$. The graph $[[\mathcal{P}]]$ of $\mathcal{P}$ is upper semi-analytic in $[0,T]\times\Omega\times\mathbb{M}_1$.

(iii) $\mathcal{P}$ is stable under conditioning, i.e. for every $(t,\omega) \in [0,T]\times\Omega$ and every $\mathbb{P} \in \mathcal{P}(t,\omega)$ together with an $\mathbb{F}$-stopping time τ taking values in $[t,T]$, there is a family of r.c.p.d. $(\mathbb{P}_\mathsf{w})_{\mathsf{w}\in\Omega}$ such that $\mathbb{P}_\mathsf{w} \in \mathcal{P}(\tau(\mathsf{w}),\mathsf{w})$ for $\mathbb{P}$-a.e. $\mathsf{w} \in \Omega$.

(iv) $\mathcal{P}$ is stable under concatenation, i.e. for every $(t,\omega) \in [0,T]\times\Omega$ and $\mathbb{P} \in \mathcal{P}(t,\omega)$ together with an $\mathbb{F}$-stopping time τ taking values in $[t,T]$, let $(\mathbb{Q}_\mathsf{w})_{\mathsf{w}\in\Omega}$ be a family of probability measures such that $\mathbb{Q}_\mathsf{w} \in \mathcal{P}(\tau(\mathsf{w}),\mathsf{w})$ for all $\mathsf{w} \in \Omega$ and $\mathsf{w} \mapsto \mathbb{Q}_\mathsf{w}$ is $\mathcal{F}_\tau$-measurable, then the concatenated probability measure $\mathbb{P}\otimes_\tau\mathbb{Q}_\cdot \in \mathcal{P}(t,\omega)$.

We notice that for $t = 0$, we have $\mathcal{P}_0 := \mathcal{P}(0,\omega)$ for any $\omega \in \Omega$.

Remark 3.2.3. *1. Let us comment the above assumptions. The first assumption except a jointly Borel measurability are quite standard in the classical RBSDE literature. The second one is classic assumption to derive time consistence property for a family of nonlinear operators(see [39], [85]). The last two assumptions are related to regular conditional probability and cumulate to the second assumption, this allow us to establish the measurability of a value function of the stochastic control problem over a family of probability measures. The jointly measurable assumption is introduced in [73] where a probability measure is consider as a parameter of measurability . Unlike in [66], we do not need regularity conditions under the terminal value, the generator and the obstacle. These conditions were necessary in [66] to establish the existence of the solution through the dynamic programming principle. In this work, we use measurable selection theorem to provide the measurability of the value function.*

2. Using the Lipschitz property of f in the first assumption, we can define bounded functions $\lambda : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}_d^{\geq 0}\times\mathbb{R}^d \longrightarrow \mathbb{R}$ and $\eta : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}_d^{\geq 0}\times\mathbb{R}^d \longrightarrow \mathbb{R}^d$ such that for any (t,ω,y,y',z,z',a,b)

$$f(t,\omega,y,z,a,b) - f(t,\omega,y',z',a,b) = \lambda_t(\omega,y,y',z,z',a,b)(y-y') + \eta_t(\omega,y,y',z,z',a,b)\cdot(z-z').$$

For simplicity of notation, we denote

$$\lambda_t^{\mathbb{P}}(y,y',z,z') := \lambda_t(X_{\cdot\wedge t},y,y',z,z',\widehat{a}_s,b_s^{\mathbb{P}}) \text{ and } \eta_t^{\mathbb{P}}(y,y',z,z') := \eta_t(X_{\cdot\wedge t},y,y',z,z',\widehat{a}_s,b_s^{\mathbb{P}}).$$

3.2.3 Spaces and norms

We follow [85] and define the spaces and norms which will be needed for the formulation of our problem. Fix some $t \in [0,T]$ and some $\omega \in \Omega$. In what follows, $\mathbb{X} := (\mathcal{X}_s)_{t\leq s\leq T}$ will denote an arbitrary filtration on $(\Omega,\mathcal{F}_T)$ and $\mathbb{P}$ and arbitrary element in $\mathcal{P}(t,\omega)$. Denote also by $\mathbb{X}_\mathbb{P}$ the $\mathbb{P}$-augmented filtration associated to $\mathbb{X}$.

For $p > 1$, $\mathbb{L}_{t,\omega}^p(\mathbb{X})$ (resp. $\mathbb{L}_{t,\omega}^p(\mathbb{X},\mathbb{P})$) denotes the space of all $\mathcal{X}_T$-measurable scalar random variable ξ with

$$\|\xi\|_{\mathbb{L}_{t,\omega}^p}^p := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}[|\xi|^p] < +\infty, \left(\text{resp. } \|\xi\|_{\mathbb{L}_{t,\omega}^p(\mathbb{P})}^p := \mathbb{E}^{\mathbb{P}}[|\xi|^p] < +\infty\right).$$

$\mathbb{H}^p_{t,\omega}(\mathbb{X})$(resp. $\mathbb{H}^p_{t,\omega}(\mathbb{X},\mathbb{P})$) denotes the space of all $\mathbb{X}$-predictable $\mathbb{R}^d$-valued processes Z, which are defined $\widehat{a}_s(\omega)ds$-a.e. on $[t,T]$ with

$$\|Z\|^p_{\mathbb{H}^p_{t,\omega}} := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\left[\left(\int_t^T \left\|\widehat{a}_s^{1/2}Z_s\right\|^2 ds\right)^{\frac{p}{2}}\right] < \infty,$$

$$\left(\text{resp. } \|Z\|^p_{\mathbb{H}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\left[\left(\int_t^T \left\|\widehat{a}_s^{1/2}Z_s\right\|^2 ds\right)^{\frac{p}{2}}\right] < \infty\right).$$

$\mathbb{M}^p_{t,\omega}(\mathbb{X},\mathbb{P})$ denotes the space of all $(\mathbb{X},\mathbb{P})$-optional martingales M with $\mathbb{P}$-a.s. càdlàg paths on $[t,T]$ with $M_t = 0$ $\mathbb{P}$-a.s., and

$$\|M\|^p_{\mathbb{M}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\left[[M]_T^{\frac{p}{2}}\right] < +\infty.$$

Furthermore, we will say that a family $(M^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(t,\omega)}$ belongs to $\mathbb{M}^p_{t,\omega}((\mathbb{X}_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(t,\omega)})$ if for any $\mathbb{P}\in\mathcal{P}(t,\omega)$, $M^{\mathbb{P}}\in\mathbb{M}^p_{t,\omega}(\mathbb{X}_{\mathbb{P}},\mathbb{P})$ and

$$\sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \left\|M^{\mathbb{P}}\right\|_{\mathbb{M}^p_{t,\omega}(\mathbb{P})} < +\infty.$$

$\mathbb{I}^p_{t,\omega}(\mathbb{X},\mathbb{P})$ denotes the set of all $\mathbb{X}$-predictable processes K with $\mathbb{P}$-a.s. càdlàg and non-decreasing paths on $[t,T]$ with $K_0 = 0$, $\mathbb{P}$-a.s. and

$$\|K\|^p_{\mathbb{I}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}[K_T^p] < +\infty.$$

We will say that a family $(K^{\mathbb{P}})_{\mathbb{P}\in\mathbb{P}(t,\omega)}$ belongs to $\mathbb{I}^p_{t,\omega}((\mathbb{X}_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(t,\omega)})$ if for any $\mathbb{P}\in\mathcal{P}(t,\omega)$, $K^{\mathbb{P}}\in\mathbb{I}^p_{t,\omega}(\mathbb{X},\mathbb{P})$ and

$$\sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \left\|K^{\mathbb{P}}\right\|_{\mathbb{I}^p_{t,\omega}(\mathbb{P})} < +\infty.$$

$\mathbb{D}^p_{t,\omega}(\mathbb{X})$ (resp. $\mathbb{D}^p_{t,\omega}(\mathbb{X},\mathbb{P})$) denotes the space of all $\mathbb{X}$-progressively measurable $\mathbb{R}$-valued processes Y with $\mathcal{P}(t,\omega)$-q.s.(resp. $\mathbb{P}$-a.s.) càdlàg paths on $[t,T]$ with

$$\|Y\|^p_{\mathbb{D}^p_{t,\omega}} := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\le s\le T}|Y_s|^p\Big] < +\infty, \quad \Big(\text{resp.}\|Y\|^p_{\mathbb{D}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\le s\le T}|Y_s|^p\Big] < +\infty\Big).$$

For each $\xi\in\mathbb{L}^1_0(\mathbb{F}_+^{\mathcal{P}_0})$ and $s\in[t,T]$ denote

$$\mathbb{E}_s^{\mathcal{P}_0,\mathbb{P}}[\xi] := \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_0(s,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}[\xi|\mathcal{F}_{s+}] \text{ where } \mathcal{P}_0(s,\mathbb{P},\mathbb{F}_+) := \{\mathbb{P}'\in\mathcal{P}_0,\ \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_{s+}\}.$$

Then we define for each $1\le\kappa<p$,

$$\mathbb{L}_0^{p,\kappa} := \Big\{\xi\in\mathbb{L}_0^p(\mathbb{F}_+),\ \|\xi\|_{\mathbb{L}_0^{p,\kappa}} < +\infty\Big\}, \tag{3.2.4}$$

where

$$\|\xi\|^p_{\mathbb{L}^{p,\kappa}_0} := \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\Big[\operatorname*{ess\,sup}^{\mathbb{P}}_{t\le s\le T}\Big(\mathbb{E}^{\mathcal{P}_0,\mathbb{P}}_s[|\xi|^\kappa]\Big)^{\frac{p}{\kappa}}\Big].$$

Similarly given a probability measure $\overline{\mathbb{P}}$ and a filtration $\overline{\mathbb{X}}$ on the enlarged canonical space $\overline{\Omega}$, we denote the corresponding spaces by $\mathbb{D}^p_{t,\omega}(\overline{\mathbb{X}},\overline{\mathbb{P}})$, $\mathbb{H}^p_{t,\omega}(\overline{\mathbb{X}},\overline{\mathbb{P}})$, $\mathbb{M}^p_{t,\omega}(\overline{\mathbb{X}},\overline{\mathbb{P}})$,... Furthermore when $t=0$, there is no longer any dependence on ω since $\omega_0=0$, so that we simplify the notations by suppressing the ω-dependence and write $\mathbb{H}^p_0(\overline{\mathbb{X}})$, $\mathbb{H}^p_0(\overline{\mathbb{X}},\overline{\mathbb{P}})$,... Similar notations are used on the enlarged canonical space.

3.2.4 Formulation of second order reflected BSDE

The following formulation is an extension of the one introduced by Matoussi, Possamai and Zhou [66] to the case of general filtration. Unlike in [66], we work with the filtration $\mathbb{F}^{\mathcal{P}_0}_+$. Since every $\mathbb{P}\in\mathcal{P}_0$ doesn't a priori satisfy the martingale representation property, then for every $\mathbb{P}\in\mathcal{P}_0$, we consider a 2RBSDE driven by the $\mathbb{P}$-martingale part $X^{c,\mathbb{P}}$ of X. Following the definition of RBSDEs in general filtration (see section 2.5) and the wellposedness of 2BSDE in [85], we formulate 2RBSDE with respect to filtration $\mathbb{F}^{\mathcal{P}_0}_+$. Our lower obstacle is represented by the process L. We will assume that $(L_t)_{t\in[0,T]}$ is càdlàg and $L\in\mathbb{D}^p_0(\mathbb{F}^{\mathcal{P}_0}_+)$.We now consider the following 2RBSDE with lower obstacle L:

$$\begin{cases} Y_t = \xi + \int_t^T \widehat{f}^{\mathbb{P}}_s(Y_s,\widehat{a}^{1/2}_s Z_s)ds - \int_t^T Z_s\cdot dX^{c,\mathbb{P}}_s - \int_t^T dM^{\mathbb{P}}_t + K^{\mathbb{P}}_T - K^{\mathbb{P}}_t, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0\le t\le T,\ \mathbb{P}\text{-a.s.},\ \forall\mathbb{P}\in\mathcal{P}_0, \\ Y_t \ge L_t, 0\le t\le T,\ \mathbb{P}\text{-a.s.},\ \forall\mathbb{P}\in\mathcal{P}_0, \end{cases} \tag{3.2.5}$$

where for every $\mathbb{P}\in\mathcal{P}_0$, $M^{\mathbb{P}}$ is a $\mathbb{P}$-martingale null at 0 orthogonal to $X^{c,\mathbb{P}}$ and $K^{\mathbb{P}}$ is a non decreasing process null at 0.

For any $\mathbb{P}\in\mathcal{P}_0$ and $\xi\in\mathbb{L}^p_0(\mathbb{P})$ a $\mathcal{F}_T$-measurable random variable , let $(y^{\mathbb{P}},z^{\mathbb{P}},m^{\mathbb{P}},k^{\mathbb{P}}) := (y^{\mathbb{P}}(T,\xi),z^{\mathbb{P}}(T,\xi),m^{\mathbb{P}}(T,\xi),k^{\mathbb{P}}(T,\xi))$ denote the solution to the following standard RBSDE with lower obstacle L:

$$\begin{cases} y^{\mathbb{P}}_t = \xi + \int_t^T \widehat{f}^{\mathbb{P}}_s(y^{\mathbb{P}}_s,\widehat{a}^{1/2}_s z^{\mathbb{P}}_s)ds - \int_t^T z^{\mathbb{P}}_s\cdot dX^{c,\mathbb{P}}_s - \int_t^T dm^{\mathbb{P}}_s + k^{\mathbb{P}}_T - k^{\mathbb{P}}_t,\ 0\le t\le T,\ \mathbb{P}\text{-a.s.} \\ y^{\mathbb{P}}_t \ge L_t,\ t\in[0,T],\ \mathbb{P}\text{-a.s.}, \\ \int_0^T (y^{\mathbb{P}}_{s^-} - L_{s^-})dk^{\mathbb{P}}_s = 0,\ \mathbb{P}\text{-a.s.} \end{cases} \tag{3.2.6}$$

where $m^{\mathbb{P}}$ is a martingale orthogonal to X under $\mathbb{P}$. Bouchard et al. in [14](see Theorem 3.1) have proved existence and uniqueness of a solution to the reflected BSDEs (3.2.6) with $(y^{\mathbb{P}}_t,z^{\mathbb{P}}_t,m^{\mathbb{P}}_t,k^{\mathbb{P}}_t)_{t\in[0,T]}\in\mathbb{D}^p_0(\mathbb{F}^{\mathbb{P}}_+,\mathbb{P})\times\mathbb{H}^p_0(\mathbb{F}^{\mathbb{P}}_+,\mathbb{P})\times\mathbb{M}^p_0(\mathbb{F}^{\mathbb{P}}_+,\mathbb{P})\times\mathbb{I}^p_0(\mathbb{F}^{\mathbb{P}}_+,\mathbb{P})$ satisfying equation (3.2.6) under $\mathbb{P}$. This equation gives the classical formulation to a RBSDE on Ω in a general filtration. We recall the definition of a 2RBSDE for fixed p in the following.

Definition 3.2.1. *We say that $(Y,Z)\in\mathbb{D}^p_0(\mathbb{F}^{\mathcal{P}_0}_+)\times\mathbb{H}^p_0(\mathbb{F}^{\mathcal{P}_0}_+)$ is a solution to the 2RBSDE (3.2.5)if :*

(i) $Y_T=\xi$, *and* $Y_t\ge L_t$, $t\in[0,T]$, $\mathcal{P}_0$*-q.s.;*

(ii) $\forall\mathbb{P}\in\mathcal{P}_0$, *the process $K^{\mathbb{P}}$ defined below has non-decreasing paths $\mathbb{P}$-a.s.*

$$K^{\mathbb{P}}_t := Y_0 - Y_t - \int_0^t \widehat{f}^{\mathbb{P}}_s(Y_s,\widehat{a}^{1/2}_s Z_s)ds + \int_0^t Z_s\cdot dX^{c,\mathbb{P}}_s + M^{\mathbb{P}}_t,\ t\in[0,T], \tag{3.2.7}$$

(*iii*) *We have the following minimality condition:*

$$\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}\Big[\int_t^T G_s^{t,\mathbb{P}'} d(K_s^{\mathbb{P}'} - k_s^{\mathbb{P}'})\big|\mathcal{F}_{t+}^{\mathbb{P}}\Big] = 0, \quad 0 \le t \le T,\ \mathbb{P}\text{-}a.s.,\ \forall \mathbb{P}\in\mathcal{P}_0. \quad (3.2.8)$$

where for any $t \in [0,T]$ and for any $\mathbb{P} \in \mathcal{P}_0$, the process $G^{t,\mathbb{P}}$ is defined by

$$G_s^{t,\mathbb{P}} := \exp\Big(\int_t^s (\lambda_u^{\mathbb{P}} - \frac{1}{2}\,\|\eta_u^{\mathbb{P}}\|^2)(Y_u, y_u^{\mathbb{P}}, Z_u, z_u^{\mathbb{P}})du + \int_t^s \eta_u^{\mathbb{P}}(Y_u, y_u^{\mathbb{P}}, Z_u, z_u^{\mathbb{P}})\cdot dW_u^{\mathbb{P}}\Big).$$

The processes $\lambda^{\mathbb{P}}$ and $\eta^{\mathbb{P}}$ are introduced in Remark 3.2.3(2).

Notation: $\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+) := \{\mathbb{P}' \in \mathcal{P}_0 : \mathbb{P} = \mathbb{P}' \text{ on } \mathcal{F}_{t+}\}$.

Remark 3.2.4. 1. *Rigorously, the solution is $(Y, Z, (M^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}, (K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}) \in \mathbb{D}_0^p(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{H}_0^p(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{M}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}) \times \mathbb{I}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$ and through misuse of language, we denote (Y,Z), given the dependence in $\mathbb{P}$ of $K^{\mathbb{P}}$ and $M^{\mathbb{P}}$.*

2. *Here we use the review minimality condition introduced by Matoussi, Possamai and Zhou [68]. It has been found that the condition introduce in [66] was wrong in the sense that in general $K^{\mathbb{P}} - k^{\mathbb{P}}$ is not non decreasing(one can refer to the counter-example of [68] in section A.1).*

3. *Using the above definition, the for any $\mathbb{P} \in \mathcal{P}_0$, $K^{\mathbb{P}} - M^{\mathbb{P}}$ is a semimartingale defined by*

$$K_t^{\mathbb{P}} - M_t^{\mathbb{P}} := Y_0 - Y_t - \int_0^t \widehat{f}_s^{\mathbb{P}}(Y_s, \widehat{a}_s^{1/2} Z_s)ds + \int_0^t Z_s \cdot dX_s^{c,\mathbb{P}},\ t \in [0,T],$$

Using recent results of Nutz [76], under additional assumptions (related to axiomatic set theory) the family of semimartingales $(K^{\mathbb{P}} - M^{\mathbb{P}})_{\mathbb{P}}$ can always be aggregated into a universal semimartingale $K - M$. Then by the uniqueness of decomposition of semimartingales, the processes $K^{\mathbb{P}}$ and $M^{\mathbb{P}}$ can be aggregated into processes K and M.

Throughout the rest of this chapter for the sake of simplicity , we consider the case where $\widehat{a}$ is non-degenerate and then there exists the Brownian motion $W^{\mathbb{P}}$ on Ω under $\mathbb{P}$ satisfying (3.2.1). Therefore the RBSDEs $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$ associated to the 2RBSDE (3.2.5) will be considered on $(\Omega, \mathcal{F}_T, \mathbb{P})$ w.r.t. the filtration $\mathbb{F}_+^{\mathbb{P}}$

$$\begin{cases} y_t^{\mathbb{P}} = \xi + \int_t^T \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}})ds - \int_t^T z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_t^T dm_s^{\mathbb{P}} + k_T^{\mathbb{P}} - k_t^{\mathbb{P}},\ 0 \le t \le T,\ \mathbb{P}\text{-a.s.} \\ y_t^{\mathbb{P}} \ge L_t,\ t \in [0,T],\ \mathbb{P}\text{-a.s.}, \\ \int_0^T (y_{t^-}^{\mathbb{P}} - L_{t^-}) dk_t^{\mathbb{P}} = 0,\ \mathbb{P}\text{-a.s.} \end{cases} \quad (3.2.9)$$

The case where $\widehat{a}$ is degenerate can be easily adapted by working in the enlarged space (see Remark 3.2.1) with equivalence of RBSDEs established in section A.2.

3.3 Uniqueness of the solution

Following [66] and [85] in addition to Assumption 3.2.1, we will always assume the following in order to prove uniqueness of the solution to the 2RBSDE (3.2.5).

Assumption 3.3.1. *For fixed $p \geq 2$, there is some $\kappa \in (1, p]$ such that the following integrability conditions are satisfied:*

$$\phi_f^{p,\kappa} := \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{0\leq t\leq T}^{\mathbb{P}}\left(\mathbb{E}_t^{\mathcal{P}_0,\mathbb{P}}\left[\int_0^T |\widehat{f}_s^{\mathbb{P},0}|^\kappa ds\right]\right)^{\frac{p}{\kappa}}\right] < +\infty, \tag{3.3.1}$$

$$\psi_L^{p,\kappa} := \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{0\leq t\leq T}^{\mathbb{P}}\left(\mathbb{E}_t^{\mathcal{P}_0,\mathbb{P}}\left[\sup_{0\leq s\leq T}|L_s|^\kappa\right]\right)^{\frac{p}{\kappa}}\right] < +\infty, \tag{3.3.2}$$

3.3.1 Representation and uniqueness of the solution

We have similarly as in Theorem 3.1 of [66], the following representation:

Theorem 3.3.1. *Let Assumptions 3.2.1 and 3.3.1 hold. Let $\xi \in \mathbb{L}_0^{p,\kappa}$ and (Y, Z) be a solution to the 2RBSDE (3.2.5). For any $\mathbb{P} \in \mathcal{P}_0$, let $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}}) \in \mathbb{D}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P}) \times \mathbb{H}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P}) \times \mathbb{M}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P}) \times \mathbb{I}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P})$ be the solutions to the corresponding RBSDEs (3.2.6). Then for any $\mathbb{P} \in \mathcal{P}_0$ and $0 \leq t \leq T$,*

$$Y_t = \operatorname*{ess\,sup}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}^{\mathbb{P}} y_t^{\mathbb{P}'},\ \mathbb{P}\text{-a.s.} \tag{3.3.3}$$

Thus, the 2RBSDE (3.2.5) has at most one solution in $\mathbb{D}_0^p(\mathbb{F}_+^{\mathbb{P}}) \times \mathbb{H}_0^p(\mathbb{F}_+^{\mathbb{P}})$.

Proof. The proof is divided in two parts. The first part provides the representation (3.3.3) and the last one consists in uniqueness of the solution.

(i) *Representation of the solution.* Choose $\mathbb{P} \in \mathcal{P}_0$. For every $\mathbb{P}' \in \mathcal{P}_0(t, \mathbb{P}, \mathbb{F}_+)$ we set

$$\delta Y := Y - y^{\mathbb{P}'},\ \delta Z := Z - z^{\mathbb{P}'},\ \delta M := M^{\mathbb{P}'} - m^{\mathbb{P}'} \text{ and } \delta K := K^{\mathbb{P}'} - k^{\mathbb{P}'}.$$

By Remark 3.2.3(2), there exists bounded processes $\lambda^{\mathbb{P}'}$ and $\eta^{\mathbb{P}'}$ such that for all $t \in [0, T]$

$$\delta Y_t = \int_t^T (\lambda_s^{\mathbb{P}'}\delta Y_s + \eta_s^{\mathbb{P}'} \cdot \widehat{a}_s^{1/2}\delta Z_s)ds - \int_t^T \delta Z_s \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}'} - \int_t^T d(\delta M_s^{\mathbb{P}'} - \delta K_s^{\mathbb{P}'}),\ \mathbb{P}'\text{-a.s.}$$

By applying Itô's formula to $G^{t,\mathbb{P}'}\delta Y$ between t and T, we have :

$$\begin{aligned}
G_T^{t,\mathbb{P}'}\delta Y_T &= G_t^{t,\mathbb{P}'}\delta Y_t + \int_t^T \delta Y_{s^-} dG_s^{t,\mathbb{P}'} + \int_t^T G_{s^-}^{t,\mathbb{P}'} d\delta Y_s + \int_t^T d\langle G^{t,\mathbb{P}'}, \delta Y\rangle_s + \sum_{t<s\leq T} \Delta G_s^{t,\mathbb{P}'}\Delta\delta Y_s \\
&= G_t^{t,\mathbb{P}'}\delta Y_t + \int_t^T \delta Y_s G_s^{t,\mathbb{P}'}(\lambda_s^{\mathbb{P}'}ds + \eta_s^{\mathbb{P}'}dW_s^{\mathbb{P}'}) \\
&\quad + \int_t^T G_s^{t,\mathbb{P}'}(-\lambda_s^{\mathbb{P}'}\delta Y_s - \eta_s^{\mathbb{P}'} \cdot \widehat{a}_s^{1/2}\delta Z_s)ds + \int_t^T G_s^{t,\mathbb{P}'}\delta Z_s \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}'} \\
&\quad + \int_t^T G_s^{t,\mathbb{P}'} d\left(\delta M_s^{\mathbb{P}'} - \delta K_s^{\mathbb{P}'}\right) + \int_t^T G_s^{t,\mathbb{P}'}\eta_s^{\mathbb{P}'} \cdot \widehat{a}_s^{1/2}\delta Z_s ds \\
&= G_t^{t,\mathbb{P}'}\delta Y_t + \int_t^T G_s^{t,\mathbb{P}'}(\widehat{a}_s^{1/2}\delta Z_s + \eta_s^{\mathbb{P}'}\delta Y_s)dW_s^{\mathbb{P}'} + \int_t^T G_s^{t,\mathbb{P}'} d(\delta M_s^{\mathbb{P}'} - \delta K_s^{\mathbb{P}'}).
\end{aligned}$$

Since $\delta Y_T = 0$ and $G_t^{t,\mathbb{P}'} = 1$, then taking conditional expectation under $\mathbb{P}'$ with respect to $\mathcal{F}_{t+}^{\mathbb{P}'}$ on both side of equality we obtain

$$\delta Y_t = \mathbb{E}^{\mathbb{P}'}\Big[\int_t^T G_s^{t,\mathbb{P}'} d\delta K_s^{\mathbb{P}'} \big| \mathcal{F}_{t+}^{\mathbb{P}'}\Big]. \tag{3.3.4}$$

Because the conditional expectation of the martingale terms is equal to 0. Thus, we deduce that

$$\delta Y_t := Y_t - y_t^{\mathbb{P}'} = \mathbb{E}^{\mathbb{P}'}\Big[\int_t^T G_s^{t,\mathbb{P}'} d(K_s^{\mathbb{P}'} - k_s^{\mathbb{P}'}) \big| \mathcal{F}_{t+}^{\mathbb{P}'}\Big].$$

By arbitrariness of $\mathbb{P}'$ and thanks to the minimality condition (3.2.8) it follows that

$$Y_t - \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} y_t^{\mathbb{P}'} = 0,\ \mathbb{P}\text{-a.s.}$$

(ii) *Uniqueness of the solution.* The uniqueness of Y is deduce from the representation (3.3.3). Since quadratic covariation is uniquely define and

$$\big[Y,X\big]_t = \int_0^t \widehat{a}_s Z_s ds,\ \mathbb{P}\text{-a.s.},$$

Z is also uniquely defined, $\hat{a}_t dt \otimes \mathcal{P}_0$-q.s.
By definition of the solution of (3.2.5), it follows by the uniqueness of Y and Z that the processes $M^{\mathbb{P}} - K^{\mathbb{P}}$ are also uniquely defined for any $\mathbb{P} \in \mathcal{P}_0$. Since $K^{\mathbb{P}}$ is a nondecreasing process and $M^{\mathbb{P}}$ is a martingale, we observe that for any $\mathbb{P} \in \mathcal{P}_0$, $M^{\mathbb{P}} - K^{\mathbb{P}}$ is a $(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P})$-supermartingale. Furthermore, $(K_t^{\mathbb{P}}, M_t^{\mathbb{P}}) \in \mathbb{L}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P}) \times \mathbb{L}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P})$ for any $t \in [0,T]$, and $K^{\mathbb{P}}$ is $\mathbb{F}_+^{\mathbb{P}}$-predictable. Then by the uniqueness of the Doob-Meyer decomposition of supermartingales, we deduce the uniqueness of $M^{\mathbb{P}}$ and $K^{\mathbb{P}}$ for any $\mathbb{P} \in \mathcal{P}_0$. □

Remark 3.3.1. *The representation formula (3.3.3) also established in [68] thanks to the minimum condition (3.2.8) is more restrictive than the following classical representation formula ,*

$$Y_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} y_t^{\mathbb{P}'}(s, Y_s), \quad 0 \le t \le s \le T, \quad \mathbb{P}\text{-a.s.}$$

To get back this dynamic programming representation, it suffices to extend the minimality condition (3.2.8) to the following:

$$\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}\Big[\int_t^s G_u^{t,\mathbb{P}'} d(K_u^{\mathbb{P}'} - k_u^{\mathbb{P}'})\big|\mathcal{F}_{t+}^{\mathbb{P}}\Big] = 0, \quad 0 \le t \le s \le T,\ \mathbb{P}\text{-a.s.}, \forall \mathbb{P} \in \mathcal{P}_0.$$

3.3.2 Some properties of the solution

More details about the action of $K^{\mathbb{P}}$. In this section we recall the results obtained in [66] concerning the action of $K^{\mathbb{P}}$. The demonstrations are not mentioned because the slight difference is the additional martingale term $M^{\mathbb{P}}$ in the definition and this does not change much. This is an answer to the question of whether the action of $K^{\mathbb{P}}$ can be represented by two non decreasing process which act separately respectively to the obstacle (namely satisfies the Skorokhod condition) and to the space of probability measures (namely satisfy the minimum condition (1.1.9) of 2BSDEs).

Proposition 3.3.1. *Let Assumptions 3.2.1 and 3.3.1 hold. Assume $\xi \in \mathbb{L}_0^{p,\kappa}$ and $(Y,Z) \in \mathbb{D}_0^p(\mathbb{F}_+^{\mathbb{P}}) \times \mathbb{H}_0^p(\mathbb{F}_+^{\mathbb{P}})$ is a solution to the 2RBSDE(3.2.5). Let $\{(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}}), \mathbb{P} \in \mathcal{P}_0\}$ be the solutions of the corresponding RBSDEs (3.2.6). Then we have the following: for all $t \in [0,T]$ and for all $\mathbb{P} \in \mathcal{P}_0$*

$$\int_0^t 1_{\{Y_{s^-}=L_{s^-}\}} dK_s^{\mathbb{P}} = \int_0^t 1_{\{Y_{s^-}=L_{s^-}\}} dk_s^{\mathbb{P}}, \quad \mathbb{P}\text{-a.s.} \tag{3.3.5}$$

Remark 3.3.2. *The above result tells us that if Y becomes equal to the obstacle L, then it suffices to push it exactly as in the standard RBSDE case. Moreover, we have the following decomposition of $K^{\mathbb{P}}$*

$$\begin{aligned} K_t^{\mathbb{P}} &= \int_0^t 1_{\{Y_{s^-}=L_{s^-}\}} dK_s^{\mathbb{P}} + \int_0^t 1_{\{Y_{s^-}>L_{s^-}\}} dK_s^{\mathbb{P}}, \quad \mathbb{P}\text{-a.s.} \\ &= \int_0^t 1_{\{Y_{s^-}=L_{s^-}\}} dk_s^{\mathbb{P}} + V_s^{\mathbb{P}}, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

with

$$V_t^{\mathbb{P}} = \int_0^t 1_{\{Y_{s^-}=L_{s^-}\}} dk_s^{\mathbb{P}},$$

Then $K^{\mathbb{P}}$ can be decomposed as the sum of two non decreasing processes. On $\{Y_{t^-} = L_{t^-}\} \cup \{y_{t^-}^{\mathbb{P}} > L_{t^-}\}$, the process $A_t^{\mathbb{P}} := K_t^{\mathbb{P}} - \int_0^t 1_{\{Y_{s^-}=L_{s^-}\}} dk_s^{\mathbb{P}}$ satisfies the minimum condition (1.1.9) of 2BSDEs. Otherwise (that means on $\{Y_{t^-} > L_{t^-}\} \cap \{y_{t^-}^{\mathbb{P}} = L_{t^-}\}$) we cannot say anything and the question was left open in [66]. A decomposition which isolates the effects due to the obstacle and the ones due to the second order is therefore still an open problem .

Following Proposition 4.2 of [35] on the standard RBSDEs , the following result shows that if the obstacle is a general semimartingale, we can give a more explicit representation of $K^{\mathbb{P}}$.

Assumption 3.3.2. *L is a semimartingale with the following decomposition:*

$$L_t = L_0 + \int_0^t U_s^{\mathbb{P}} ds + B_t^{\mathbb{P}} + \int_0^t P_s \cdot dX_s^{c,\mathbb{P}} + N_t^{\mathbb{P}}, \ \mathbb{P}\text{-a.s.}, \quad \text{for all } \mathbb{P} \in \mathcal{P}_0.$$

where $N^{\mathbb{P}}$ is a $\mathbb{F}_+^{\mathbb{P}}$ - càdlàg martingale orthogonal to $X^{c,\mathbb{P}}$ such that

$$\mathbb{E}^{\mathbb{P}}\left[[N^{\mathbb{P}}]_T\right] < +\infty, \forall \mathbb{P} \in \mathcal{P}_0.$$

$B^{\mathbb{P}}$ is a càdlàg process of integrable variation such that the measure $dB_t^{\mathbb{P}}$ is singular with respect to the Lebesgue measure dt and which admits the following decomposition $B_t^{\mathbb{P}} = B_t^{\mathbb{P},+} - B_t^{\mathbb{P},-}$, where $B^{\mathbb{P},+}$ and $B^{\mathbb{P},-}$ are non decreasing processes. Also, $U^{\mathbb{P}}$ and P are respectively $\mathbb{R}$ and $\mathbb{R}^d$-valued $\mathcal{F}_{t+}^{\mathbb{P}}$ and $\mathcal{F}_{t+}^{\mathcal{P}_0}$ progressively measurable processes such that

$$\int_0^T (|U_t^{\mathbb{P}}|dt + \|\widehat{a}_t^{1/2} P_t\|^2)dt + B_T^{\mathbb{P},+} + B_T^{\mathbb{P},-} < \infty, \ \mathbb{P}\text{-a.s.} \ \forall \mathbb{P} \in \mathcal{P}_0.$$

Proposition 3.3.2. *Let Assumptions 3.2.1, 3.3.1 and 3.3.2 hold. Let (Y, Z) be the solution of the 2RBSDE (3.2.5), then for all $\mathbb{P} \in \mathcal{P}_0$*

$$Z_t = P_t \text{ and } M_t^{\mathbb{P}} = N_t^{\mathbb{P}}, \ dt \times \mathbb{P}\text{-a.s. on the set } \{Y_{t^-} = L_{t^-}\},$$

and there exists a progressively measurable process $(\alpha_t^{\mathbb{P}})_{0 \le t \le T}$ such that $0 \le \alpha^{\mathbb{P}} \le 1$ and

$$1_{\{Y_{t^-}=L_{t^-}\}} dK_t^{\mathbb{P}} = 1_{\{Y_{t^-}=L_{t^-}\}} dk_t^{\mathbb{P}} = \alpha_t^{\mathbb{P}} 1_{\{Y_{t^-}=L_{t^-}\}} \left([\widehat{f}_s^{\mathbb{P}}(L_t, \widehat{a}_t^{1/2} P_t) + U_t^{\mathbb{P}}]^- dt + dB_t^{\mathbb{P},-} \right).$$

Connection with optimal stopping problem. The following result gives the link between 2RBSDEs and optimal stopping problem.

Proposition 3.3.3. *Let (Y, Z) be the solution to the above 2RBSDE (3.2.5) and $\{(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}}), \mathbb{P} \in \mathcal{P}_0\}$ be the solutions of the corresponding RBSDEs (3.2.6). Then for each $t \in [0, T]$ and for all $\mathbb{P} \in \mathcal{P}_0$*

$$\begin{aligned} Y_t &= \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} \operatorname*{ess\,sup}_{v \in \mathcal{T}_t} \mathbb{E}^{\mathbb{P}'} \Big[\int_t^v \widehat{f}_s^{\mathbb{P}'}(y_s^{\mathbb{P}'}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}'}) ds + L_v 1_{\{v<T\}} + \xi 1_{\{v=T\}} \big| \mathcal{F}_{t+}^{\mathbb{P}'} \Big], \quad \mathbb{P}\text{-a.s.} \\ &= \operatorname*{ess\,sup}_{v \in \mathcal{T}_t} \mathbb{E}^{\mathbb{P}} \Big[\int_t^v \widehat{f}_s^{\mathbb{P}}(Y_s, \widehat{a}_s^{1/2} Z_s) ds + A_v^{\mathbb{P}} - A_t^{\mathbb{P}} + L_v 1_{\{v<T\}} + \xi 1_{\{v=T\}} \big| \mathcal{F}_{t+}^{\mathbb{P}} \Big], \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

where $A_t^{\mathbb{P}} := \int_0^t 1_{\{Y_{s^-}>L_{s^-}\}} dK_s^{\mathbb{P}}$.
Moreover, for each $\mathbb{P}$, the following stopping time is ϵ-optimal

$$D_t^{\mathbb{P},\epsilon} := \inf \left\{ v \ge t, \ y_v^{\mathbb{P}} \le L_v + \epsilon, \ \mathbb{P}\text{-a.s.} \right\} \wedge T.$$

Comparison result. Let us consider two triple (f,ξ,L) and (f',ξ',L') and assume the 2RBSDEs associated with this triples has a solution respectively (Y,Z) and (Y',Z'). We also denote respectively $\{(y^{\mathbb{P}},z^{\mathbb{P}},m^{\mathbb{P}},k^{\mathbb{P}}),\mathbb{P}\in\mathcal{P}_0\}$ and $\{(y'^{\mathbb{P}},z'^{\mathbb{P}},m'^{\mathbb{P}},k'^{\mathbb{P}}),\mathbb{P}\in\mathcal{P}_0\}$ the RBSDEs associated to the 2RBSDEs. The following result allows us to compare the components Y and Y' if we can compare the triples (f,ξ,L) and (f',ξ',L').

Theorem 3.3.2. *Assume that f satisfies the Lipschitz condition in Assumption 3.2.1(i) and that*

$$\sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[\int_0^T|\widehat{f}_s^{\mathbb{P}}(0,0)|^2ds\Big]<\infty\quad\text{and}\quad\sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[\int_0^T|\widehat{f}_s'^{\mathbb{P}}(y_s'^{\mathbb{P}},\widehat{a}_s^{1/2}z_s'^{\mathbb{P}})|^2ds\Big]<\infty.$$

Suppose in addition the following:

- $\xi\le\xi'$, $\mathcal{P}_0$-*q.s.*
- $\widehat{f}_t^{\mathbb{P}}(y_t'^{\mathbb{P}},\widehat{a}_t^{1/2}z_t'^{\mathbb{P}})\le\widehat{f}_t'^{\mathbb{P}}(y_t'^{\mathbb{P}},\widehat{a}_t^{1/2}z_t'^{\mathbb{P}})$, $d\mathbb{P}\times dt$ *a.e.*
- $L_t\le L_t'$, $0\le t\le T$, $\mathcal{P}_0$-*q.s.*

Then $Y\le Y'$, $\mathcal{P}_0$-q.s.

Proof. Using the comparison result of RBSDEs establish in Theorem 8.21 of [32], we deduce that for every $\mathbb{P}\in\mathcal{P}_0$,

$$y^{\mathbb{P}}\le y'^{\mathbb{P}},\quad\mathbb{P}-\text{a.e.}\tag{3.3.6}$$

and if in addition f' is Lipschitz with respect to y and z and $L\equiv L'$ then

$$k_t^{\mathbb{P}}-k_s^{\mathbb{P}}\ge k_t'^{\mathbb{P}}-k_s'^{\mathbb{P}},\ \mathbb{P}\text{-a.s. for any } s\le t.\tag{3.3.7}$$

In fact despite we are in general filtration, the additional martingale term does not create any else difficulty since it disappears by taking the expectation.
It follows directly by the representation formula (3.3.3) and (3.3.6) that for any $t\in[0,T]$, $Y_t\le Y_t'$, $\mathbb{P}$-a.s. for every $\mathbb{P}\in\mathcal{P}_0$. □

3.3.3 A priori estimates

We start by recalling some properties of the solution of the RBSDEs given in [14]. By Lipschitz condition in Assumption 3.2.1(i), there exists a $\mathbb{R}$-valued $\mathbb{F}$-progressively measurable process λ and a $\mathbb{R}^d$-valued, $\mathbb{F}$-predictable process η, with $|\lambda|\le L_f$ and $\|\eta\|\le L_f$ such that

$$\widehat{f}_t^{\mathbb{P}}(y_t^{\mathbb{P}},\widehat{a}_t^{1/2}z_t^{\mathbb{P}})=\widehat{f}_t^{\mathbb{P},0}+\lambda_t y_t^{\mathbb{P}}+\eta_t\cdot\widehat{a}_t^{1/2}z_t^{\mathbb{P}},\ t\in[0,T].$$

It should be noted that λ and η depend on $\mathbb{P}$, but for the sake of simplicity we skipped highlighting this dependence. We define

$$I_t:=e^{\int_0^t\lambda_s ds},\ \frac{d\mathbb{Q}_{\mathbb{P}}}{d\mathbb{P}}:=\mathcal{E}\Big(\int_0^{\cdot}\eta_t\cdot dW_t^{\mathbb{P}}\Big)_T\text{ and }\ W^{\mathbb{Q}_{\mathbb{P}}}:=W_{\cdot}^{\mathbb{P}}-\int_0^{\cdot}\eta_s ds,\tag{3.3.8}$$

where $\mathcal{E}$ denotes the Doléans-Dade exponential. Then, by Girsanov theorem, $W^{\mathbb{Q}_{\mathbb{P}}}$ is a $\mathbb{Q}_{\mathbb{P}}$-Brownian motion and $m^{\mathbb{P}}$ is still a $\mathbb{Q}_{\mathbb{P}}$-martingale orthogonal to $W^{\mathbb{Q}_{\mathbb{P}}}$. By Itô's formula applied to the product $Iy^{\mathbb{P}}$, we can re-write the solution of the RBSDE (3.2.9) as

$$\begin{aligned}
&I_t y_t^{\mathbb{P}}=I_T\xi+\int_t^T I_s\widehat{f}_s^{\mathbb{P},0}ds-\int_t^T I_s z_s^{\mathbb{P}}\cdot\widehat{a}_s^{1/2}dW_s^{\mathbb{Q}_{\mathbb{P}}}-\int_t^T I_s dm_s^{\mathbb{P}}+\int_t^T I_s k_s^{\mathbb{P}},\ \mathbb{P}\text{-a.s.}\\
&I_t y_t^{\mathbb{P}}\ge I_t L_t,\ t\in[0,T],\ \mathbb{P}\text{-a.s.},\\
&\int_0^T I_{s^-}(y_{s^-}^{\mathbb{P}}-L_{s^-})dk_s^{\mathbb{P}}=0,\ \mathbb{P}\text{-a.s.}
\end{aligned}$$

One can now use Proposition 3.1 of Lepeltier and Xu [60] to establish the link between RBSDEs with a càdlàg barrier and optimal stopping problems. In the following, we will denote $\mathbb{Q}$ instead of $\mathbb{Q}_{\mathbb{P}}$ for simplicity.

Proposition 3.3.4. *Let $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$ be a solution of the RBSDE (3.2.9), with the reflected lower barrier L, terminal condition ξ and generator f. Then for each $t \in [0,T]$,*

$$I_t y_t^{\mathbb{P}} = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}^{\mathbb{Q}}\Big[\int_t^{\tau} I_s \widehat{f}^{\mathbb{P},0}_s ds + I_\tau L_\tau 1_{\{\tau<T\}} + I_T \xi 1_{\{\tau=T\}} \big| \mathcal{F}_{t+} \Big]$$

and

$$y_t^{\mathbb{P}} = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}^{\mathbb{P}}\Big[\int_t^{\tau} \widehat{f}^{\mathbb{P}}_s(y^{\mathbb{P}}_s, \widehat{a}^{1/2}_s z^{\mathbb{P}}_s) ds + L_\tau 1_{\{\tau<T\}} + \xi 1_{\{\tau=T\}} \big| \mathcal{F}^{\mathbb{P}}_{t+} \Big].$$

where $\mathcal{T}_t$ is the set of all $\mathbb{F}$-stopping times valued in $[t,T]$.

This result will be useful to establish the a priori estimates of the first component $y^{\mathbb{P}}$ of the RBSDE (3.2.9) and the apriori estimates of the other components derived from the first estimates.

Proposition 3.3.5. *Let Assumptions 3.2.1 and 3.3.1 hold. Assume $\xi \in \mathbb{L}_0^{p,\kappa}$ and (Y,Z) is the solution to the 2RBSDE(3.2.5). Let $\{(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})\}_{\mathbb{P}\in\mathcal{P}_0}$ be the solutions of the corresponding RBSDEs(3.2.9). Then there exists a constant C depending only on κ, T, p and the Lipschitz constant L_f of f such that*

$$\sup_{\mathbb{P}\in\mathcal{P}_0} \|y^{\mathbb{P}}\|^p_{\mathbb{D}^p_0(\mathbb{P})} \le C\Big(\|\xi\|^p_{\mathbb{L}_0^{p,\kappa}} + \phi_f^{p,\kappa} + \psi_L^{p,\kappa} \Big) \text{ and } \|Y\|^p_{\mathbb{D}^p_0} \le C\Big(\|\xi\|^p_{\mathbb{L}_0^{p,\kappa}} + \phi_f^{p,\kappa} + \psi_L^{p,\kappa} \Big).$$

Proof. The proof falls naturally into two steps. We begin with the a priori estimates of $y^{\mathbb{P}}$ and finish with that of Y. Throughout this proof we will use the fact that by definition of the norms and Assumption 3.3.1, we have

$$\|\xi\|^p_{\mathbb{L}_0^p} \le \|\xi\|^p_{\mathbb{L}_0^{p,\kappa}}, \quad \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\Big[\big(\int_0^T |\widehat{f}^{\mathbb{P},0}_s| ds \big)^p \Big] \le \phi_f^{p,\kappa} \quad \text{and} \quad \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} |L_t|^p ds \Big] \le \psi_L^{p,\kappa}.$$

and also for any $l > 0$ and $(a_i)_{1\le i\le n} \subset (0,+\infty)$ we have the following inequalities (see Remark 2.1 of [14]),

$$(1 \wedge n^{l-1}) \sum_{i=1}^n a_i^l \le \Big(\sum_{i=1}^n a_i \Big)^l \le (1 \vee n^{l-1}) \sum_{i=1}^n a_i^l. \tag{3.3.9}$$

Step 1. We follow the proof of Proposition 3.2 of [14] to establish the a priori estimates of $y^{\mathbb{P}}$.

We deduce from Proposition 3.3.4 that

$$
\begin{aligned}
|y_t^{\mathbb{P}}| &\le e^{-L_f t}|I_t y_t^{\mathbb{P}}| \\
&\le \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} e^{-L_f t}\mathbb{E}^{\mathbb{Q}}\Big[\int_t^\tau I_s|\widehat{f}_s^{\mathbb{P},0}|ds + I_\tau|L_\tau|1_{\{\tau<T\}} + I_T|\xi|1_{\{\tau=T\}}\big|\mathcal{F}_{t+}\Big] \\
&\le e^{L_f(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[\int_0^T|\widehat{f}_s^{\mathbb{P},0}|ds + \sup_{s\in[0,T]}|L_s| + |\xi|\big|\mathcal{F}_{t+}\Big] \\
&\le e^{L_f(T-t)}\mathbb{E}^{\mathbb{P}}\Big[\exp\big(\int_t^T\eta_s\cdot dW_s^{\mathbb{P}} + \frac{1}{2}\int_t^T\|\eta_s\|^2ds\big)\big(\int_0^T|\widehat{f}_s^{\mathbb{P},0}|ds + \sup_{s\in[0,T]}|L_s| + |\xi|\big)\big|\mathcal{F}_{t+}\Big] \\
&\le e^{L_f(T-t)}\mathbb{E}^{\mathbb{P}}\Big[\exp\big(\frac{\kappa}{\kappa-1}\int_t^T\eta_s\cdot dW_s^{\mathbb{P}} + \frac{\kappa}{2(\kappa-1)}\int_t^T\|\eta_s\|^2ds\big)\big|\mathcal{F}_{t+}\Big]^{\frac{\kappa-1}{\kappa}} \\
&\qquad\times\mathbb{E}^{\mathbb{P}}\Big[\big(\int_0^T|\widehat{f}_s^{\mathbb{P},0}|ds + \sup_{s\in[0,T]}|L_s| + |\xi|\big)^\kappa\big|\mathcal{F}_{t+}\Big]^{\frac{1}{\kappa}} \\
&\le e^{L_f(T-t)-\frac{1}{2(\kappa-1)}L_f^2(T-t)}\mathbb{E}^{\mathbb{P}}\Big[\big(\int_0^T|\widehat{f}_s^{\mathbb{P},0}|ds + \sup_{s\in[0,T]}|L_s| + |\xi|\big)^\kappa\big|\mathcal{F}_{t+}\Big]^{\frac{1}{\kappa}} \\
&\le e^{L_f(T-t)-\frac{1}{2(\kappa-1)}L_f^2(T-t)}3^{\frac{\kappa-1}{\kappa}}T^{\kappa-1}\mathbb{E}^{\mathbb{P}}\Big[\int_0^T|\widehat{f}_s^{\mathbb{P},0}|^\kappa ds + \sup_{s\in[0,T]}|L_s|^\kappa + |\xi|^\kappa\big|\mathcal{F}_{t+}\Big]^{\frac{1}{\kappa}}.
\end{aligned}
$$

where we have used the boundedness of λ in the first and third inequalities, the fifth inequality follows from the Cauchy-Schwarz inequality. The boundedness of η and the fact that the expectation of the exponential martingale is equal to 1 lead to the sixth inequality and finally, the seventh one is given by the property (3.3.9).
Taking the supremum over $t\in[0,T]$,

$$
\sup_{t\in[0,T]}|y_t^{\mathbb{P}}| \le C\sup_{t\in[0,T]}\mathbb{E}^{\mathbb{P}}\Big[\int_0^T|\widehat{f}_s^{\mathbb{P},0}|^\kappa ds + \sup_{s\in[0,T]}|L_s|^\kappa + |\xi|^\kappa\big|\mathcal{F}_{t+}\Big]^{1/\kappa}.
$$

where C is a constant depending only on L_f, T, κ whose value can vary line to line. Hence, taking the expectation we deduce from Doob's inequality, Holder's inequality and relation (3.3.9) that

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|y_t^{\mathbb{P}}|^p\Big] &\le C\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}\Big(\mathbb{E}^{\mathbb{P}}\Big[\int_0^T|\widehat{f}_s^{\mathbb{P},0}|^\kappa ds + \sup_{s\in[0,T]}|L_s|^\kappa + |\xi|^\kappa\big|\mathcal{F}_{t+}\Big]\Big)^{p/\kappa}\Big] \\
&\le C\Big(\mathbb{E}^{\mathbb{P}}\Big[\int_0^T|\widehat{f}_s^{\mathbb{P},0}|^\kappa ds + \sup_{s\in[0,T]}|L_s|^\kappa + |\xi|^\kappa\Big]\Big)^{p/\kappa} \\
&\le C\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T|\widehat{f}_s^{\mathbb{P},0}|^\kappa ds + \sup_{s\in[0,T]}|L_s|^\kappa + |\xi|^\kappa\Big)^{p/\kappa}\Big] \\
&\le C\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T|\widehat{f}_s^{\mathbb{P},0}|ds\Big)^p + \sup_{s\in[0,T]}|L_s|^p + |\xi|^p\Big]
\end{aligned}
$$

Therefore taking the supremum over $\mathbb{P}\in\mathcal{P}_0$, we obtain

$$
\|y^{\mathbb{P}}\|^p_{\mathbb{D}_0^p(\mathbb{P})} \le C\Big(\|\xi\|^p_{\mathbb{L}_0^{p,\kappa}} + \phi_f^{p,\kappa} + \psi_f^{p,\kappa}\Big). \tag{3.3.10}
$$

Step 2. We now turn to the a priori estimates of Y. By the representation of Y in (3.3.3), it follows that

$$
Y_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} y_t^{\mathbb{P}'}(T,\xi),\ \mathbb{P}\text{-a.s.}
$$

We recall that from step 1 , there exists a constant C depending only on κ, T and L_f such that

$$|y_t^{\mathbb{P}}(T,\xi)| \leq C\mathbb{E}^{\mathbb{P}}\Big[\int_0^T |\widehat{f}_s^{\mathbb{P},0}|^\kappa ds + \sup_{s\in[0,T]}|L_s|^\kappa + |\xi|^\kappa|\mathcal{F}_{t+}\Big]^{\frac{1}{\kappa}}$$

Then by representation formula, we get

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|Y_t|^p\Big] &\leq \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}\Big(\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}|y_t^{\mathbb{P}'}(T,\xi)|\Big)^p\Big],\\
&\leq C\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}\Big(\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}\mathbb{E}^{\mathbb{P}'}\Big[\int_0^T|\widehat{f}_s^{\mathbb{P}',0}|^\kappa ds + \sup_{s\in[0,T]}|L_s|^\kappa + |\xi|^\kappa|\mathcal{F}_{t+}\Big]^{\frac{p}{\kappa}}\Big)\Big]\\
&\leq C\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}\Big(\mathbb{E}_t^{\mathcal{P}_0,\mathbb{P}}\Big[\int_0^T|\widehat{f}_s^{\mathbb{P},0}|^\kappa ds + \sup_{s\in[0,T]}|L_s|^\kappa + |\xi|^\kappa\Big]^{\frac{p}{\kappa}}\Big)\Big]\\
&\leq C\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}\Big(\mathbb{E}_t^{\mathcal{P}_0,\mathbb{P}}\Big[\int_0^T|\widehat{f}_s^{\mathbb{P},0}|^\kappa ds\Big]\Big)^{\frac{p}{\kappa}}\Big] + \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}\Big(\mathbb{E}_t^{\mathcal{P}_0,\mathbb{P}}\Big[\sup_{s\in[0,T]}|L_s|^\kappa\Big]\Big)^{\frac{p}{\kappa}}\Big]\\
&\qquad + C\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}\Big(\mathbb{E}_t^{\mathcal{P}_0,\mathbb{P}}\big[|\xi|^\kappa\big]\Big)^{\frac{p}{\kappa}}\Big].
\end{aligned}$$

Taking supremum over $\mathbb{P}\in\mathcal{P}_0$ we have that,

$$\|Y\|^p_{\mathbb{D}_0^p} \leq C\Big(\|\xi\|^p_{\mathbb{L}_0^{p,\kappa}} + \phi_f^{p,\kappa} + \psi_L^{p,\kappa}\Big). \tag{3.3.11}$$

□

Proposition 3.3.6. *We keep the notations and hypotheses of Proposition 3.3.5. For $p\geq 2$ there exists a constant C depending only on κ, T, p and the Lipschitz constant L_f of f such that*

$$\begin{aligned}
\sup_{\mathbb{P}\in\mathcal{P}_0}\Big\{\|z^{\mathbb{P}}\|^p_{\mathbb{H}_0^p(\mathbb{P})} + \mathbb{E}^{\mathbb{P}}\Big[[m^{\mathbb{P}}]_T^{\frac{p}{2}}\Big] + \mathbb{E}^{\mathbb{P}}\Big[(k_T^{\mathbb{P}})^p\Big]\Big\} &\leq C\Big(\|\xi\|^p_{\mathbb{L}_0^{p,\kappa}} + \phi_f^{p,\kappa} + \psi_L^{p,\kappa}\Big),\\
\|Z\|^p_{\mathbb{H}_0^p} + \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[[M^{\mathbb{P}}]_T^{\frac{p}{2}}\Big] + \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[(K_T^{\mathbb{P}})^p\Big] &\leq C\Big(\|\xi\|^p_{\mathbb{L}_0^{p,\kappa}} + \phi_f^{p,\kappa} + \psi_L^{p,\kappa}\Big).
\end{aligned}$$

Proof. Fix $\mathbb{P}\in\mathcal{P}_0$, the proof will be divided into three steps.
Step 1. We show the following

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T\|\widehat{a}_t^{1/2}z_t^{\mathbb{P}}\|^2dt\Big)^{p/2}\Big] + \mathbb{E}^{\mathbb{P}}\big[[m^{\mathbb{P}}]_T^{\frac{p}{2}}\big] \leq C\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|y_t^{\mathbb{P}}|^p + \big(\int_0^T|\widehat{f}_t^{\mathbb{P},0}|dt\big)^p\Big]. \tag{3.3.12}$$

Hamadène and Popier proved similar estimates for the $z^{\mathbb{P}}$ component in Lemma 1 of [44]. Their estimates were shown for the RBSDEs with respect to a Brownian filtration. We follow their proof to show estimates in our case where the RBSDEs are defined in a general filtration. Let us define for each integer n

$$\tau_n = \inf\{t\in[0,T], \int_0^t\|\widehat{a}_s^{1/2}z_s^{\mathbb{P}}\|^2ds + [m^{\mathbb{P}}]_T \geq n\}\wedge T.$$

Since $z^{\mathbb{P}}$ is in $\mathbb{H}_0^p(\mathbb{P})$ and $m^{\mathbb{P}}\in\mathbb{M}_0^p(\mathbb{P})$, then $\int_0^T\|\widehat{a}_s^{1/2}z_s^{\mathbb{P}}\|^2ds + [m^{\mathbb{P}}]_T < \infty$, $\mathbb{P}$-a.s. and it turns out that $(\tau_n)_{n\geq 0}$ is a stationary sequence. Now considering a real constant α and using Itô's

formula to $e^{\alpha t}|y_t^{\mathbb{P}}|^2$ between 0 and τ_n, we get

$$
\begin{aligned}
&|y_0^{\mathbb{P}}|^2 + \int_0^{\tau_n} e^{\alpha s}\|\widehat{a}_s^{1/2} z_s^{\mathbb{P}}\|^2 ds + \int_0^{\tau_n} e^{\alpha s} d[m^{\mathbb{P}}]_s \\
&= e^{\alpha\tau_n}|y_{\tau_n}^{\mathbb{P}}|^2 + \int_0^{\tau_n} e^{\alpha s} y_s^{\mathbb{P}} \big(2\widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}}) - \alpha y_s^{\mathbb{P}}\big) ds + 2\int_0^{\tau_n} e^{\alpha s} y_{s^-}^{\mathbb{P}} dk_s^{\mathbb{P}} \\
&\qquad - 2\int_0^{\tau_n} e^{\alpha s} y_{s^-}^{\mathbb{P}} dm_s^{\mathbb{P}} - 2\int_0^{\tau_n} e^{\alpha s} y_s^{\mathbb{P}} z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \sum_{0<s\le\tau_n} e^{\alpha s}|\Delta y_s^{\mathbb{P}}|^2 \\
&\le e^{\alpha\tau_n}|y_{\tau_n}^{\mathbb{P}}|^2 + \int_0^{\tau_n} e^{\alpha s} y_s^{\mathbb{P}} \big(2|\widehat{f}_s^{\mathbb{P},0}| + 2L_f|y_s^{\mathbb{P}}| + 2L_f\|\widehat{a}_s^{1/2} z_s^{\mathbb{P}}\| - \alpha y_s^{\mathbb{P}}\big) ds \\
&\qquad + 2\int_0^{\tau_n} e^{\alpha s} y_{s^-}^{\mathbb{P}} dk_s^{\mathbb{P}} - 2\int_0^{\tau_n} e^{\alpha s} y_{s^-}^{\mathbb{P}} dm_s^{\mathbb{P}} - 2\int_0^{\tau_n} e^{\alpha s} y_s^{\mathbb{P}} z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} \\
&\le e^{\alpha\tau_n}|y_{\tau_n}^{\mathbb{P}}|^2 + 2\sup_{s\le\tau_n} e^{\alpha s}|y_s^{\mathbb{P}}| \times \int_0^{\tau_n} |\widehat{f}_s^{\mathbb{P},0}| ds + (2L_f + L_f\epsilon^{-1} - \alpha)\int_0^{\tau_n} e^{\alpha s}|y_s^{\mathbb{P}}|^2 ds \\
&\qquad + \epsilon L_f \int_0^{\tau_n} \|\widehat{a}_s^{1/2} z_s^{\mathbb{P}}\|^2 ds + 2\int_0^{\tau_n} e^{\alpha s} y_{s^-}^{\mathbb{P}} dk_s^{\mathbb{P}} - 2\int_0^{\tau_n} e^{\alpha s} y_{s^-}^{\mathbb{P}} dm_s^{\mathbb{P}} - 2\int_0^{\tau_n} e^{\alpha s} y_s^{\mathbb{P}} z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}}.
\end{aligned}
$$

for any $\epsilon > 0$. The last inequality is obtained by the following Young's inequality

$$
ab \le \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}, \text{ for any } \epsilon > 0. \tag{3.3.13}
$$

Thus,

$$
\begin{aligned}
&|y_0^{\mathbb{P}}|^2 + (1 - \epsilon L_f)\int_0^{\tau_n} e^{\alpha s}\big\|\widehat{a}_s^{1/2} z_s^{\mathbb{P}}\big\|^2 ds + \int_0^{\tau_n} e^{\alpha s} d[m^{\mathbb{P}}]_s \\
&\qquad \le e^{\alpha\tau_n}|y_{\tau_n}^{\mathbb{P}}|^2 + \sup_{s\le\tau_n} e^{2\alpha s}|y_s^{\mathbb{P}}|^2 + \Big(\int_0^{\tau_n} |\widehat{f}_s^{\mathbb{P},0}| ds\Big)^2 + (2L_f + L_f\epsilon^{-1} - \alpha)\int_0^{\tau_n} e^{\alpha s}|y_s^{\mathbb{P}}|^2 ds \\
&\qquad + \frac{1}{\epsilon}\sup_{s\le\tau_n} e^{2\alpha s}|y_s^{\mathbb{P}}|^2 + \epsilon\big(k_{\tau_n}^{\mathbb{P}}\big)^2 - 2\int_0^{\tau_n} e^{\alpha s} y_{s^-}^{\mathbb{P}} dm_s^{\mathbb{P}} - 2\int_0^{\tau_n} e^{\alpha s} y_s^{\mathbb{P}} z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}}.
\end{aligned}
$$

Furthermore by (3.2.9), we have

$$
k_{\tau_n}^{\mathbb{P}} = y_0^{\mathbb{P}} - y_{\tau_n}^{\mathbb{P}} - \int_0^{\tau_n} \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}}) ds + \int_0^{\tau_n} z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} + m_{\tau_n}^{\mathbb{P}}.
$$

Then, there exists a constant C depending on the Lipschitz constant L_f such that

$$
\begin{aligned}
\big(k_{\tau_n}^{\mathbb{P}}\big)^2 \le C\Big(&|y_0^{\mathbb{P}}|^2 + |y_{\tau_n}^{\mathbb{P}}|^2 + \Big(\int_0^{\tau_n} |\widehat{f}_s^{\mathbb{P},0}| ds\Big)^2 + \int_0^{\tau_n} |y_s^{\mathbb{P}}|^2 ds + \int_0^{\tau_n} \|\widehat{a}_s^{1/2} z_s^{\mathbb{P}})\|^2 ds \\
&+ \Big|\int_0^{\tau_n} z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}}\Big|^2 + |m_{\tau_n}^{\mathbb{P}}|^2\Big).
\end{aligned}
$$

By taking into account the last inequality in the right-hand side of the previous one, we obtain

$$
\begin{aligned}
&(1 - \epsilon C)|y_0^{\mathbb{P}}|^2 + (1 - \epsilon L_f)\int_0^{\tau_n} e^{\alpha s}\big\|\widehat{a}_s^{1/2} z_s^{\mathbb{P}}\big\|^2 ds - \epsilon C\int_0^{\tau_n} \big\|\widehat{a}_s^{1/2} z_s^{\mathbb{P}}\big\|^2 ds \\
&\qquad + \int_0^{\tau_n} e^{\alpha s} d[m^{\mathbb{P}}]_s \\
&\le (\epsilon C + e^{\alpha\tau_n})|y_{\tau_n}^{\mathbb{P}}|^2 + \big(1 + \frac{1}{\epsilon}\big)\sup_{s\le\tau_n} e^{2\alpha s}|y_s^{\mathbb{P}}|^2 + (1 + \epsilon C)\Big(\int_0^{\tau_n} |\widehat{f}_s^{\mathbb{P},0}| ds\Big)^2 \\
&+ (2L_f + L_f\epsilon^{-1} - \alpha)\int_0^{\tau_n} e^{\alpha s}|y_s^{\mathbb{P}}|^2 ds + \epsilon C\Big|\int_0^{\tau_n} z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}}\Big|^2 + \epsilon C|m_{\tau_n}^{\mathbb{P}}|^2 \\
&+ 2\Big|\int_0^{\tau_n} e^{\alpha s} y_s^{\mathbb{P}} z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}}\Big| + 2\Big|\int_0^{\tau_n} e^{\alpha s} y_{s^-}^{\mathbb{P}} dm_s^{\mathbb{P}}\Big|
\end{aligned}
$$

Choosing now ϵ small enough, α such that $2L_f + L_f\epsilon^{-1} - \alpha < 0$ and using inequality (3.3.9), we get

$$
\begin{aligned}
&(1 \wedge 2^{\frac{p}{2}-1})\Big(\int_0^{\tau_n} e^{\alpha s}\|\widehat{a}_s^{1/2} z_s^{\mathbb{P}}\|^2 ds\Big)^{\frac{p}{2}} + (1 \wedge 2^{\frac{p}{2}-1})[m^{\mathbb{P}}]_{\tau_n}^{\frac{p}{2}} \\
&\qquad \le C(1 \vee 2^{\frac{p}{2}-1})\Big(\Big(\int_0^{\tau_n} |\widehat{f}_s^{\mathbb{P},0}| ds\Big)^p + \sup_{t\in[0,T]} |y_t^{\mathbb{P}}|^p \\
&\qquad\quad + \Big|\int_0^{\tau_n} e^{\alpha s} y_s^{\mathbb{P}} z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}}\Big|^{\frac{p}{2}} + \Big|\int_0^{\tau_n} e^{\alpha s} y_{s^-}^{\mathbb{P}} dm_s^{\mathbb{P}}\Big|^{\frac{p}{2}}\Big). \qquad (3.3.14)
\end{aligned}
$$

Otherwise, thanks to the BDG's inequality under probability $\mathbb{P}$ and the Young's inequality, we can can write

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\Big[\Big|\int_0^{\tau_n} e^{\alpha s} y_s^{\mathbb{P}} z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}}\Big|^{\frac{p}{2}}\Big] &\le C_p \mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^{\tau_n} |y_s^{\mathbb{P}}|^2 \|\widehat{a}_s^{1/2} z_s^{\mathbb{P}}\|^2 ds\Big)^{\frac{p}{4}}\Big] \\
&\le C_p \mathbb{E}^{\mathbb{P}}\Big[\Big(\sup_{t\in[0,T]} |y_s^{\mathbb{P}}|\Big)^{\frac{p}{2}} \Big(\int_0^{\tau_n} \|\widehat{a}_s^{1/2} z_s^{\mathbb{P}}\|^2 ds\Big)^{\frac{p}{4}}\Big] \\
&\le \frac{C_p^2}{\eta}\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} |y_s^{\mathbb{P}}|^p\Big] + \eta \mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^{\tau_n} \|\widehat{a}_s^{1/2} z_s^{\mathbb{P}}\|^2 ds\Big)^{\frac{p}{2}}\Big].
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\Big[\Big|\int_0^{\tau_n} e^{\alpha s} y_{s^-}^{\mathbb{P}} dm_s^{\mathbb{P}}\Big|^{\frac{p}{2}}\Big] &\le C_p' \mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^{\tau_n} |y_{s^-}^{\mathbb{P}}|^2 d[m^{\mathbb{P}}]_s\Big)^{\frac{p}{4}}\Big] \\
&\le C_p' \mathbb{E}^{\mathbb{P}}\Big[\Big(\sup_{t\in[0,T]} |y_s^{\mathbb{P}}|\Big)^{\frac{p}{2}} [m^{\mathbb{P}}]_{\tau_n}^{\frac{p}{4}}\Big] \\
&\le \frac{C_p'^2}{\eta'}\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} |y_s^{\mathbb{P}}|^p\Big] + \eta' \mathbb{E}^{\mathbb{P}}\Big[[m^{\mathbb{P}}]_{\tau_n}^{\frac{p}{2}}\Big].
\end{aligned}
$$

where C_p, C_p' depend on the BDG's constants and α and η, η' are positive constants. By taking expectation under $\mathbb{P}$ in (4.3.14), using the two previous estimates, choosing η and η' small enough and using Fatou's Lemma we deduce that (3.3.12) holds. Finally, combining (3.3.12) with (3.3.10), we obtain the required estimates of $z^{\mathbb{P}}$ and $m^{\mathbb{P}}$

Step 2. Estimates of $k^{\mathbb{P}}$. By the definition of our RBSDE (A.2.2),

$$
k_T^{\mathbb{P}} = y_0^{\mathbb{P}} - y_T^{\mathbb{P}} - \int_0^T \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}}) ds + \int_0^T z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} + m_T^{\mathbb{P}}.
$$

Using (3.3.9), and Lipschitz assumption on f, we easily have

$$
\begin{aligned}
(k^{\mathbb{P}})^p &\le 5^{p-1}\Big(|y_0^{\mathbb{P}}|^p + |\xi|^p + \Big(\int_0^T |\widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}})| ds\Big)^p + \Big|\int_0^T z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}}\Big|^p + |m_T^{\mathbb{P}}|^p\Big) \\
&\le 5^{p-1}\Big(|y_0^{\mathbb{P}}|^p + |\xi|^p + \Big(\int_0^T (|\widehat{f}_s^{\mathbb{P},0}| + L_f|y_s^{\mathbb{P}}| + L_f\|\widehat{a}_s^{1/2} z_s^{\mathbb{P}})\|) ds\Big)^p + \Big|\int_0^T z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}}\Big|^p \\
&\qquad + |m_T^{\mathbb{P}}|^p\Big) \\
&\le 5^{p-1}\Big(|y_0^{\mathbb{P}}|^p + |\xi|^p + 3^{p-1}\Big(\int_0^T |\widehat{f}_s^{\mathbb{P},0}| ds\Big)^p + 3^{p-1}L_f^p\Big(\int_0^T |y_s^{\mathbb{P}}| ds\Big)^p \\
&\qquad + 3^{p-1}L_f^p\Big(\int_0^T \|\widehat{a}_s^{1/2} z_s^{\mathbb{P}})\| ds\Big)^p + \Big|\int_0^T z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}}\Big|^p + |m_T^{\mathbb{P}}|^p\Big)
\end{aligned}
$$

Next,

$$(k^{\mathbb{P}})^p \leq C\Big(|\xi|^p + \sup_{t\in[0,T]}|y_t^{\mathbb{P}}|^p + \Big(\int_0^T |\widehat{f}_s^{\mathbb{P},0}|ds\Big)^p + \Big(\int_0^T \|\widehat{a}_s^{1/2} z_s^{\mathbb{P}})\|^2 ds\Big)^{\frac{p}{2}} + \Big|\int_0^T z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}}\Big|^p + |m_T^{\mathbb{P}}|^p\Big).$$

Taking expectation under $\mathbb{P}$ in the last inequality and using BDG's inequality, we obtain

$$\mathbb{E}^{\mathbb{P}}\big[(k_T^{\mathbb{P}})^p\big] \leq C\mathbb{E}^{\mathbb{P}}\Big[|\xi|^p + \sup_{t\in[0,T]}|y_t^{\mathbb{P}}|^p + \Big(\int_0^T |\widehat{f}_s^{\mathbb{P},0}|ds\Big)^p + \Big(\int_0^T \|\widehat{a}_s^{1/2} z_s^{\mathbb{P}})\|^2 ds\Big)^{\frac{p}{2}} + \big[m_T^{\mathbb{P}}\big]_T^{\frac{p}{2}}\Big].$$

where C depends only on p, T, L_f and the BDG's constants. The above estimates together with the estimates obtained in step 1 give that

$$\mathbb{E}^{\mathbb{P}}\big[(k_T^{\mathbb{P}})^p\big] \leq C\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|y_t^{\mathbb{P}}|^p + \Big(\int_0^T |\widehat{f}_s^{\mathbb{P},0}|ds\Big)^p\Big]. \tag{3.3.15}$$

Using Proposition 3.3.5 together with (3.3.10) and (3.3.12) we deduce the first assertion of the Proposition.

Step 3. We now turn to the case of $Z, (M^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}$ and $(K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}$. Since we have the a priori estimates of Y, we proceed analogously to step 1 with the solution of the 2RBSDE instead of the solution of the RBSDE and obtain

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T \|\widehat{a}_t^{1/2} Z_t\|^2 dt\Big)^{p/2}\Big] + \mathbb{E}^{\mathbb{P}}\big[[M^{\mathbb{P}}]_T^{\frac{p}{2}}\big] \leq C\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|Y_t^{\mathbb{P}}|^p + (\int_0^T |\widehat{f}_t^{\mathbb{P},0}|dt)^p\Big].$$

Doing the same as in step 2 with the solution of the 2RBSDE instead of the solution of the RBSDE , we also have

$$\mathbb{E}^{\mathbb{P}}\big[(K_T^{\mathbb{P}})^p\big] \leq C\mathbb{E}^{\mathbb{P}}\Big[|\xi|^p + \sup_{t\in[0,T]}|Y_t^{\mathbb{P}}|^p + \Big(\int_0^T |\widehat{f}_s^{\mathbb{P},0}|ds\Big)^p + \Big(\int_0^T \|\widehat{a}_s^{1/2} Z_s^{\mathbb{P}})\|^2 ds\Big)^{\frac{p}{2}} + \big[M_T^{\mathbb{P}}\big]_T^{\frac{p}{2}}\Big].$$

Combining these two above estimates with the estimates of Y in Proposition 3.3.5, we deduce the second claim of our proof and complete the proof. □

Theorem 3.3.3. *Let Assumptions 3.2.1 hold, and consider two generators f^1 , f^2 and two lower obstacles L^1, L^2 such that Assumption 3.3.1 holds. For $i = 1, 2$, let $((Y^i, Z^i)$ be a solution to the 2RBSDE (3.2.5) with the generator f^i, the terminal condition ξ^i and lower obstacle L^i. Define*

$$\phi_{f^1,f^2}^{p,\kappa} := \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[\operatorname{ess\,sup}_{0\leq t\leq T}^{\mathbb{P}}\Big(\mathbb{E}_t^{\mathcal{P}_0,\mathbb{P}}\Big[\int_0^T |\widehat{f}_s^{1,\mathbb{P}} - \widehat{f}_s^{2,\mathbb{P}}|^\kappa (y_s^{1,\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{1,\mathbb{P}}) ds\Big]\Big)^{\frac{p}{\kappa}}\Big] < +\infty,$$

$$\varphi_{f^1,f^2}^{p,\kappa} := \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T |\widehat{f}_s^{1,\mathbb{P}} - \widehat{f}_s^{2,\mathbb{P}}|(Y_s^1, \widehat{a}_s^{1/2} Z_s^1)\Big)^p\Big] + \infty,$$

$$\psi_{L^1,L^2}^{p,\kappa} := \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[\operatorname{ess\,sup}_{0\leq t\leq T}^{\mathbb{P}}\Big(\mathbb{E}_t^{\mathcal{P}_0,\mathbb{P}}\Big[\sup_{0\leq s\leq T}|L_s^1 - L_s^2|^\kappa\Big]\Big)^{\frac{p}{\kappa}}\Big] < +\infty.$$

Then, there exists a constant C depending only on κ, T and the Lipschitz constants of f^1 and f^2 such

that

$$\begin{aligned}
&\big\|Y^1-Y^2\big\|^p_{\mathbb{D}^p_0}\le C\Big(\big\|\xi^1-\xi^2\big\|^p_{\mathbb{L}^{p,\kappa}_0}+\psi^{p,\kappa}_{L^1,L^2}+\varphi^{p,\kappa}_{f^1,f^2}\Big),\\
&\big\|Z^1-Z^2\big\|^p_{\mathbb{H}^p_0}+\sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[\big[M^{1,\mathbb{P}}-M^{2,\mathbb{P}}\big]_T^{\frac{p}{2}}\Big]+\sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[(K_T^{1,\mathbb{P}}-K_T^{2,\mathbb{P}})^p\Big]\\
&\qquad\le C\Big(\big\|\xi^1-\xi^2\big\|^p_{\mathbb{L}^{p,\kappa}_0}+\phi^{p,\kappa}_{f^1,f^2}+\psi^{p,\kappa}_{L^1,L^2}\Big)\\
&+C\Big(\big\|\xi^1-\xi^2\big\|^{p/2}_{\mathbb{L}^{p,\kappa}_0}+\big(\phi^{p,\kappa}_{f^1,f^2}\big)^{p/2}+\big(\psi^{p,\kappa}_{L^1,L^2}\big)^{p/2}\Big)\times\Big(\big\|\xi^1\big\|^p_{\mathbb{L}^{p,\kappa}_0}+\big\|\xi^2\big\|^p_{\mathbb{L}^{p,\kappa}_0}+\phi^{p,\kappa}_{f^1}+\phi^{p,\kappa}_{f^2}\Big)^{1/2}.
\end{aligned}$$

Proof. (i) The representation formula (3.3.3) gives,

$$\big|Y^1_t-Y^2_t\big|\le\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}|y^{1,\mathbb{P}'}_t(T,\xi^1)-y^{2,\mathbb{P}'}_t(T,\xi^2)|,\ \mathbb{P}\text{-a.s. for all }\mathbb{P}\in\mathcal{P}_0,\ t\in[0,T].$$

where $y^{1,\mathbb{P}}$ and $y^{2,\mathbb{P}}$ are the RBSDEs associated to Y^1 and Y^2. Following the proof of Proposition A.4.1, there exists a constant C depending on κ, T and the Lipschitz constant of f^2 such that we have $\mathbb{P}$-a.s. for all $\mathbb{P}\in\mathcal{P}_0$ and $t\in[0,T]$

$$|\tilde y^{1,\mathbb{P}}_t-\tilde y^{2,\mathbb{P}}_t|\le C\mathbb{E}^{\mathbb{P}}\Big[\int_0^T|\widehat f^{1,\mathbb{P}}_s-\widehat f^{2,\mathbb{P}}_s|^\kappa(\tilde y^{1,\mathbb{P}}_s,\widehat a^{1/2}_s\tilde z^{1,\mathbb{P}}_s)ds+\sup_{s\in[0,T]}|L^1_s-L^2_s|^\kappa+|\xi^1-\xi^2|^\kappa\,\big|\mathcal{F}^{\mathbb{P}}_{t+}\Big]^{\frac1\kappa}.$$

Now using Cauchy-Schwarz inequality and (3.3.9), the above inequality provides $\mathbb{P}$-a.s.,

$$\begin{aligned}
|\tilde y^{1,\mathbb{P}}_t-\tilde y^{2,\mathbb{P}}_t|^p&\le C\mathbb{E}^{\mathbb{P}}\Big[\int_0^T|\widehat f^{1,\mathbb{P}}_s-\widehat f^{2,\mathbb{P}}_s|^\kappa(\tilde y^{1,\mathbb{P}}_s,\widehat a^{1/2}_s\tilde z^{1,\mathbb{P}}_s)ds+\sup_{s\in[0,T]}|L^1_s-L^2_s|^\kappa+|\xi^1-\xi^2|^\kappa\,\big|\mathcal{F}^{\mathbb{P}}_{t+}\Big]^{\frac p\kappa}\\
&\le C\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T|\widehat f^{1,\mathbb{P}}_s-\widehat f^{2,\mathbb{P}}_s|^\kappa(\tilde y^{1,\mathbb{P}}_s,\widehat a^{1/2}_s\tilde z^{1,\mathbb{P}}_s)ds\Big)^{\frac p\kappa}+\Big(\sup_{s\in[0,T]}|L^1_s-L^2_s|^\kappa\Big)^{\frac p\kappa}+|\xi^1-\xi^2|^p\,\big|\mathcal{F}^{\mathbb{P}}_{t+}\Big]^{\frac p\kappa}
\end{aligned}$$

By the definition of the norms, we deduce the first assertion of the theorem.
(ii) We consider the same notations used in the proof of Proposition A.4.1 and the following notations

$$\delta Y:=Y^1-Y^2,\ \delta Z:=Z^1-Z^2,\ \delta K^{\mathbb{P}}=K^{1,\mathbb{P}}-K^{2,\mathbb{P}},\ \delta M^{\mathbb{P}}:=M^{1,\mathbb{P}}-M^{2,\mathbb{P}}.$$

By Definition 3.2.1, we have

$$\begin{aligned}
\delta Y_t&=\delta\xi+\int_t^T(\widehat f^{1,\mathbb{P}}_s(Y^1_s,\widehat a^{1/2}_sZ^1_s)-\widehat f^{2,\mathbb{P}}_s(Y^2_s,\widehat a^{1/2}_sZ^2_s))ds-\int_t^T\delta Z_s\cdot dX^{c,\mathbb{P}}_s-\int_t^Td\delta M^{\mathbb{P}}_s+\int_t^Td\delta K^{\mathbb{P}}_s\\
&=\delta\xi+\int_t^T(\delta\widehat f^{\mathbb{P}}_s(Y^1_s,\widehat a^{1/2}_sZ^1_s)+\lambda_s\delta Y_s+\eta_s\cdot\widehat a^{1/2}_s\delta Z_s)ds-\int_t^T\delta Z_s\cdot dX^{c,\mathbb{P}}_s-\int_t^Td\delta M^{\mathbb{P}}_s+\int_t^Td\delta K^{\mathbb{P}}_s.
\end{aligned}$$

where λ and η are such that $|\lambda|\le L_{f^2}$ and $\|\eta\|\le L_{f^2}$(see section 3.3.3). Using Itô's formula to $|\delta Y|^2$ between 0 and T, we have

$$\begin{aligned}
|\delta Y_0|^2=|\delta\xi|^2&+2\int_0^T\delta Y_t\delta\widehat f^{\mathbb{P}}_t(Y^1_t,\widehat a^{1/2}_tZ^1_t)dt+2\int_0^T\lambda_t|\delta Y_t|^2dt+2\int_0^T\eta_t\widehat a^{1/2}_t\delta Y_t\delta Z_tdt\\
&-2\int_0^T\delta Y_t\delta Z_t\cdot dX^{c,\mathbb{P}}_t-2\int_0^T\delta Y_{t^-}d\delta M^{\mathbb{P}}_t+2\int_0^T\delta Y_{t^-}d\delta K^{\mathbb{P}}_t\\
&-\int_0^T\|\widehat a^{1/2}_t\delta Z_t\|^2dt-[\delta M^{\mathbb{P}}]_T-\sum_{0<t\le T}|\Delta\delta Y_t|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}\int_0^T \|\widehat{a}_t^{1/2}\delta Z_t\|^2 dt + [\delta M^{\mathbb{P}}]_T &\le |\delta\xi|^2 + 2\int_0^T |\delta Y_t||\delta \widehat{f}_t^{\mathbb{P}}(Y_t^1, \widehat{a}_t^{1/2}Z_t^1)|dt + 2\int_0^T \lambda_t |\delta Y_t|^2 dt \\ &\quad + 2\int_0^T \eta_t \widehat{a}_t^{1/2}\delta Y_t \delta Z_t dt - 2\int_0^T \delta Y_t \delta Z_t \cdot dX_t^{c,\mathbb{P}} \\ &\quad - 2\int_0^T \delta Y_{t^-} d\delta M_t^{\mathbb{P}} + 2\int_0^T \delta Y_{t^-} d\delta K_t^{\mathbb{P}} \\ &\le |\delta\xi|^2 + 2\sup_{t\in[0,T]}|\delta Y_t| \times \int_0^T |\delta \widehat{f}_t^{\mathbb{P}}(Y_t^1, \widehat{a}_t^{1/2}Z_t^1)|dt + \frac{1}{2}\int_0^T \|\widehat{a}_t^{1/2}\delta Z_t\|^2 dt \\ &\quad + (2L_{f^2} + 2L_{f^2}^2)T \sup_{t\in[0,T]}|\delta Y_t|^2 - 2\int_0^T \delta Y_t \delta Z_t \cdot dX_t^{c,\mathbb{P}} \\ &\quad - 2\int_0^T \delta Y_{t^-} d\delta M_t^{\mathbb{P}} + 2\sup_{t\in[0,T]}|\delta Y_t|(K_T^{1,\mathbb{P}} + K_T^{2,\mathbb{P}}) \\ &\le |\delta\xi|^2 + (2L_{f^2}T + 2L_{f^2}^2 T)\sup_{t\in[0,T]}|\delta Y_t|^2 + \frac{1}{2}\int_0^T \|\widehat{a}_t^{1/2}\delta Z_t\|^2 dt \\ &\quad + \Big(\int_0^T |\delta \widehat{f}_t^{\mathbb{P}}(Y_t^1, \widehat{a}_t^{1/2}Z_t^1)|dt\Big)^2 + 2\Big|\int_0^T \delta Y_t \delta Z_t \cdot dX_t^{c,\mathbb{P}}\Big| \\ &\quad + \Big|2\int_0^T \delta Y_{t^-} d\delta M_t^{\mathbb{P}}\Big| + \sup_{t\in[0,T]}|\delta Y_t|(K_T^{1,\mathbb{P}} + K_T^{2,\mathbb{P}})\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{2}\int_0^T \|\widehat{a}_t^{1/2}\delta Z_t\|^2 dt + [\delta M^{\mathbb{P}}]_T &\le C\Big(|\delta\xi|^2 + \sup_{t\in[0,T]}|\delta Y_t|^2 + \Big(\int_0^T |\delta \widehat{f}_t^{\mathbb{P}}(Y_t^1, \widehat{a}_t^{1/2}Z_t^1)|dt\Big)^2 \\ &\quad + \Big|\int_0^T \delta Y_t \delta Z_t \cdot dX_t^{c,\mathbb{P}}\Big| + \Big|\int_0^T \delta Y_{t^-} d\delta M_t^{\mathbb{P}}\Big| + \sup_{t\in[0,T]}|\delta Y_t|(K_T^{1,\mathbb{P}} + K_T^{2,\mathbb{P}})\Big).\end{aligned}$$

Therefore, by inequality (3.3.9) we get

$$\begin{aligned}&2^{\frac{p}{2}-1}\Big(\int_0^T \|\widehat{a}_t^{1/2}\delta Z_t\|^2 dt\Big)^{\frac{p}{2}} + (1 \wedge 2^{\frac{p}{2}-1})[\delta M^{\mathbb{P}}]_T^{\frac{p}{2}} \\ &\qquad \le 7^{\frac{p}{2}-1}C\Big(|\delta\xi|^p + \sup_{t\in[0,T]}|\delta Y_t|^p + \Big(\int_0^T |\delta \widehat{f}_t^{\mathbb{P}}(Y_t^1, \widehat{a}_t^{1/2}Z_t^1)|dt\Big)^p \\ &\qquad\quad + \Big|\int_0^T \delta Y_t \delta Z_t \cdot dX_t^{c,\mathbb{P}}\Big|^{\frac{p}{2}} + \Big|\int_0^T \delta Y_{t^-} d\delta M_t^{\mathbb{P}}\Big|^{\frac{p}{2}} + \sup_{t\in[0,T]}|\delta Y_t|^{\frac{p}{2}}(K_T^{1,\mathbb{P}} + K_T^{2,\mathbb{P}})^{\frac{p}{2}}\Big).\end{aligned}$$

Taking the expectation under $\mathbb{P}$ gives us

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T \|\widehat{a}_t^{1/2}\delta Z_t\|^2 dt\Big)^{\frac{p}{2}}\Big] + \mathbb{E}^{\mathbb{P}}\big[[\delta M^{\mathbb{P}}]_T^{\frac{p}{2}}\big] \\ &\qquad \le C\mathbb{E}^{\mathbb{P}}\Big[|\delta\xi|^p + \sup_{t\in[0,T]}|\delta Y_t|^p + \Big(\int_0^T |\delta \widehat{f}_t^{\mathbb{P}}(Y_t^1, \widehat{a}_t^{1/2}Z_t^1)|dt\Big)^p\Big] \\ &\qquad + \mathbb{E}^{\mathbb{P}}\Big[\Big|\int_0^T \delta Y_t \delta Z_t \cdot dX_t^{c,\mathbb{P}}\Big|^{\frac{p}{2}}\Big] + \mathbb{E}^{\mathbb{P}}\Big[\Big|\int_0^T \delta Y_{t^-} d\delta M_t^{\mathbb{P}}\Big|^{\frac{p}{2}}\Big] \\ &\qquad + \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^p\Big]^{1/2}\Big]\mathbb{E}^{\mathbb{P}}\Big[(K_T^{1,\mathbb{P}})^p + (K_T^{2,\mathbb{P}})^p\Big]^{1/2}.\end{aligned}$$

Now the Burkholder's inequality allows us to write

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\Big[\Big|\int_0^T \delta Y_{t^-} d\delta M_t^{\mathbb{P}}\Big|^{\frac{p}{2}}\Big] &\le C_p\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T |\delta Y_{t^-}|^2 d[\delta M^{\mathbb{P}}]_t\Big)^{\frac{p}{4}}\Big] \\ &\le C_p\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{\frac{p}{2}}[\delta M^{\mathbb{P}}]_T^{\frac{p}{4}}\Big] \\ &\le 2C_p^2\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^p\Big] + \frac{1}{8}\mathbb{E}^{\mathbb{P}}\Big[[\delta M^{\mathbb{P}}]_T^{\frac{p}{2}}\Big]\end{aligned}$$

and

$$\mathbb{E}^{\mathbb{P}}\Big[\Big|\int_0^T \delta Y_t \delta Z_t \cdot dX_t^{c,\mathbb{P}}\Big|^{\frac{p}{2}}\Big] \le C_p \mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T |\delta Y_t|^2 \|\widehat{a}_t^{1/2}\delta Z_t\|^2 dt\Big)^{\frac{p}{4}}\Big]$$
$$\le C_p \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} |\delta Y_t|^{\frac{p}{2}} \Big(\int_0^T \|\widehat{a}_t^{1/2}\delta Z_t\|^2 dt\Big)^{\frac{p}{4}}\Big]$$
$$\le 2C_p^2 \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} |\delta Y_t|^p\Big] + \frac{1}{8}\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T \|\widehat{a}_t^{1/2}\delta Z_t\|^2 dt\Big)^{\frac{p}{2}}\Big]$$

We deduce that there exists a constant C depending on T, p, L_f^1 and L_f^2 such that

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T \|\widehat{a}_t^{1/2}\delta Z_t\|^2 dt\Big)^{\frac{p}{2}}\Big] + \mathbb{E}^{\mathbb{P}}\big[[\delta M^{\mathbb{P}}]_T^{\frac{p}{2}}\big]$$
$$\le C\mathbb{E}^{\mathbb{P}}\Big[|\delta\xi|^p + \sup_{t\in[0,T]}|\delta Y_t|^p + \big(\int_0^T |\delta \widehat{f}_t^{\mathbb{P}}(Y_t^1, \widehat{a}_t^{1/2}Z_t^1)|dt\big)^p\Big]$$
$$+ \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^p\Big]^{1/2}\mathbb{E}^{\mathbb{P}}\Big[(K_T^{1,\mathbb{P}})^p + (K_T^{2,\mathbb{P}})^p\Big]^{1/2}.$$

By the same way as Step 2 of Proposition 3.3.6, we show that for $i = 1, 2$

$$\mathbb{E}^{\mathbb{P}}\big[(K_T^{i,\mathbb{P}})^p\big] \le C\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|Y_t^i|^p + \Big(\int_0^T |\widehat{f}_s^{i,\mathbb{P},0}|ds\Big)^p\Big].$$

Combining the two above estimates, we get

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T \|\widehat{a}_t^{1/2}\delta Z_t\|^2 dt\Big)^{\frac{p}{2}}\Big] + \mathbb{E}^{\mathbb{P}}\big[[\delta M^{\mathbb{P}}]_T^{\frac{p}{2}}\big]$$
$$\le C\mathbb{E}^{\mathbb{P}}\Big[|\delta\xi|^p + \sup_{t\in[0,T]}|\delta Y_t|^p + \big(\int_0^T |\delta \widehat{f}_t^{\mathbb{P}}(Y_t^1, \widehat{a}_t^{1/2}Z_t^1)|dt\big)^p$$
$$+ \Big(|\delta\xi|^{p/2} + \sup_{t\in[0,T]}|\delta Y_t|^{p/2} + \big(\int_0^T |\delta \widehat{f}_t^{\mathbb{P}}(Y_t^1, \widehat{a}_t^{1/2}Z_t^1)|dt\big)^{p/2}\Big)\times$$
$$\Big(\sup_{t\in[0,T]}|Y_t^1|^p + \sup_{t\in[0,T]}|Y_t^2|^p + \Big(\int_0^T |\widehat{f}_s^{1,\mathbb{P},0}|ds\Big)^p + \Big(\int_0^T |\widehat{f}_s^{2,\mathbb{P},0}|ds\Big)^p\Big)^{1/2}\Big].$$

Then taking the supremum over $\mathbb{P} \in \mathcal{P}_0$ in the above estimates, we have

$$\big\|Z^1 - Z^2\big\|_{\mathbb{H}_0^p}^p + \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[\big[M^{1,\mathbb{P}} - M^{2,\mathbb{P}}\big]_T^{\frac{p}{2}}\Big] \le C\Big(\big\|\xi^1 - \xi^2\big\|_{\mathbb{L}_0^{p,\kappa}}^p + \phi_{f^1,f^2}^{p,\kappa} + \psi_{L^1,L^2}^{p,\kappa}\Big) +$$
$$C\Big(\big\|\xi^1 - \xi^2\big\|_{\mathbb{L}_0^{p,\kappa}}^{p/2} + \big(\phi_{f^1,f^2}^{p,\kappa}\big)^{p/2} + \big(\psi_{L^1,L^2}^{p,\kappa}\big)^{p/2}\Big)\times$$
$$\Big(\big\|\xi^1\big\|_{\mathbb{L}_0^{p,\kappa}}^p + \big\|\xi^2\big\|_{\mathbb{L}_0^{p,\kappa}}^p + \phi_{f^1}^{p,\kappa} + \phi_{f^2}^{p,\kappa}\Big)^{1/2}.$$

It remains to prove the estimates of $\delta K^{\mathbb{P}}$. By definition, 3.2.1, we have

$$\delta K_T^{\mathbb{P}} = \delta Y_0 - \delta\xi - \int_0^T \big(\widehat{f}_s^{1,\mathbb{P}}(Y_s^1, \widehat{a}_s^{1/2}Z_s^1) - \widehat{f}_s^{2,\mathbb{P}}(Y_s^2, \widehat{a}_s^{1/2}Z_s^2)\big)ds + \int_0^T \delta Z_s \cdot dX_s^{c,\mathbb{P}} + \delta M_T^{\mathbb{P}}$$
$$= \delta Y_0 - \delta\xi - \int_0^T \big(\delta \widehat{f}_s^{\mathbb{P}}(Y_s^1, \widehat{a}_s^{1/2}Z_s^1) + \lambda_s \delta Y_s + \eta_s \cdot \widehat{a}_s^{1/2}\delta Z_s\big)ds + \int_0^T \delta Z_s \cdot dX_s^{c,\mathbb{P}} + \delta M_T^{\mathbb{P}}.$$

Next, there exists a constant C which can vary line to line and only depending on p, T and the Lipschitz constants of f^1 and f^2 such that

$$(\delta K_T^{\mathbb{P}})^p \leq C\Big[|\delta Y_0|^p + |\delta\xi|^p + \big(\int_0^T |\delta \widehat{f}_s^{\mathbb{P}}(Y_s^1, \widehat{a}_s^{1/2} Z_s^1)|ds\big)^p + \big(\int_0^T \lambda_s |\delta Y_s| ds\big)^p \\ + \big(\int_0^T |\eta_s \cdot \widehat{a}_s^{1/2} \delta Z_s| ds\big)^p ds + \big(\int_0^T \delta Z_s \cdot dX_s^{c,\mathbb{P}}\big)^p + \big(\delta M_T^{\mathbb{P}}\big)^p\Big].$$

After that, we take the expectation under $\mathbb{P}$ in the above and use the Burhkholder's and Young's inequalities to obtain

$$\mathbb{E}^{\mathbb{P}}\Big[(\delta K_T^{\mathbb{P}})^p\Big] \leq C\mathbb{E}^{\mathbb{P}}\Big[|\delta\xi|^p + \sup_{t\in[0,T]} |\delta Y_t|^p + \big(\int_0^T |\delta \widehat{f}_s^{\mathbb{P}}(Y_s^1, \widehat{a}_s^{1/2} Z_s^1)|ds\big)^p \\ + \Big(\int_0^T \|\widehat{a}_t^{1/2} \delta Z_t\|^2 dt\Big)^{\frac{p}{2}} + [\delta M^{\mathbb{P}}]_T^{\frac{p}{2}}\Big].$$

We finally get the expected estimate of $\delta K^{\mathbb{P}}$ by taking the supremum over $\mathbb{P} \in \mathcal{P}_0$ in the above inequality and using the estimates of $\delta Y, \delta Z$ and δM. □

3.4 Existence of the solution

The key idea to prove existence of a solution is the dynamic programming principle and the selection measurable Theorem. The value function is defined pathwise as a supremum of the conditional expectation of Picard iteration of solution to BSDEs over a set of probability measures. After proved the DPP for the value function, the solution is obtained from this one. Following the representation formula of 2RBSDEs (3.3.3), a natural candidate to the solution of 2RBSDEs could be: For every $(t, \omega) \in [0, T] \times \Omega$,

$$V_t(\omega) := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}[y_t^{\mathbb{P}}],$$

where $y^{\mathbb{P}}$ is the first component of the solution of RBSDE(3.2.9).
The proof of existence of a solution of 2RBSDE(3.2.5) will be divided in four steps:
Step one. In order to establish the dynamic programming principle for the above value function V, we need a jointly measurable (with respect to time, space and probability $\mathbb{P}$) version of $y^{\mathbb{P}}$ solution of RBSDEs. We recall that for every $\mathbb{P} \in \mathcal{P}_0$, these RBSDEs already have a unique solution then our goal is to construct a jointly measurable version of the solutions . Thereby, we use the Picard iteration of the solution to the penalized BSDEs and prove the converge of the iterations.
Step two. After the convergence of approximations, the resulting solutions $y_t^{\mathbb{P}}$ can be interpreted as a function of t, ω and $\mathbb{P}$. We now show that $y^{\mathbb{P}}$ is jointly measurable: that means $(t, \omega, \mathbb{P}) \mapsto y_t^{\mathbb{P}}$ is a measurable function.
Step three. This step is dedicated to establish the following dynamic programming principle for the value function.

$$V_t(\omega) = \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\big[y_t^{\mathbb{P}}(\tau, V_\tau)\big]$$

where τ is a stopping time taking value in $[t, T]$
Step four. This last step consists in path modification of the value function in order to obtain a càdlàg process and deduce the solutions by the Doob decomposition.

3.4.1 Construction of measurable version of solution

We recall the classical construction of the $y^{\mathbb{P}}$ part of the solution to the RBSDE (3.2.9) under some probability $\mathbb{P} \in \mathcal{P}_0$ using Penalization and Picard's iterations. These approximations are very closed to the definition of BSDEs in general filtration, for more details refer to section 2.5. The obstacle is represented by L which is a càdlàg $\mathbb{F}^{\mathcal{P}_0+}$-progressively measurable process.
For $n \in \mathbb{N}$, let $(y^{\mathbb{P},n}, z^{\mathbb{P},n}, m^{\mathbb{P},n}, k^{\mathbb{P},n})$ be the solution of the following penalized BSDE

$$y_t^{\mathbb{P},n} = \xi + \int_t^T \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P},n}, \widehat{a}_s^{1/2} z_s^{\mathbb{P},n}) ds - \int_t^T z_s^{\mathbb{P},n} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_t^T dm_s^{\mathbb{P},n} + n \int_t^T (y_s^{\mathbb{P},n} - L_s)^- ds. \tag{3.4.1}$$

with

$$k_t^{\mathbb{P},n} = n \int_0^t (y_s^{\mathbb{P},n} - L_s)^- ds.$$

We prove in section A.3.1 that the solution of the penalized BSDEs converge to the solution of RBSDEs. This result has been already proved in the literature, we can mention [35] for continuous obstacle and [60] for a càdlàg obstacle and those result concern a filtration generated by a Brownian motion. In section A.3.1, we use the reasoning of the above papers and show that the presence of a martingale m does not affect the result. More generally one can also refer to Klimsiak [54] for RBSDEs in general filtration.
Therefore $(y^{\mathbb{P},n}, z^{\mathbb{P},n}, m^{\mathbb{P},n}, k^{\mathbb{P},n})$ converge to $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$ the solution of the RBSDE (3.2.9).
Now for a fixed n, let us define $y_t^{\mathbb{P},n,0} \equiv 0$, $z_t^{\mathbb{P},n,0} \equiv 0$, $m_t^{\mathbb{P},n,0} \equiv 0$ and $k_t^{\mathbb{P},n,0} \equiv 0$, for all $t \in [0,T]$ and given a family of $\mathbb{F}_+$-progressively measurable processes $\left(y_t^{\mathbb{P},n,m}, z_t^{\mathbb{P},n,m}, m_t^{\mathbb{P},n,m}, k_t^{\mathbb{P},n,m}\right)_{t\in[0,T]}$,

$$y_t^{\mathbb{P},n,m+1} = \xi + \int_t^T \widehat{f}_s^{\mathbb{P},n}(y_s^{\mathbb{P},n,m}, \widehat{a}_s^{1/2} z_s^{\mathbb{P},n,m}) ds - \int_t^T z_s^{\mathbb{P},n,m+1} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_t^T dm_s^{\mathbb{P},n,m+1} \quad \mathbb{P}\text{-a.s.} \tag{3.4.2}$$

where $\widehat{f}_s^{\mathbb{P},n}(x,y) := \widehat{f}_s^{\mathbb{P}}(x,y) + n(y - L_s)^-$. By definition $\widehat{f}^{\mathbb{P},n}$ still verify the Lipschitz assumption in y and z.
We have that $y^{\mathbb{P},n,m+1}$ is a semi-martingale under $\mathbb{P}$. Let $\langle y^{\mathbb{P},n+1}, X\rangle^{\mathbb{P}}$ be the predictable quadratic covariation of the process $y^{\mathbb{P},n,m+1}$ and X under $\mathbb{P}$. Define

$$\widehat{a}_s^{1/2} z_s^{\mathbb{P},n,m+1} := \limsup_{\epsilon \in \mathbb{Q}, \epsilon \downarrow 0} \frac{\langle y^{\mathbb{P},n,m+1}, X\rangle_s^{\mathbb{P}} - \langle y^{\mathbb{P},n,m+1}, X\rangle_{s-\epsilon}^{\mathbb{P}}}{\epsilon}. \tag{3.4.3}$$

Convergence. We first show that the sequence $(y^{\mathbb{P},n,m}, z^{\mathbb{P},n,m})_{m\geq 0}$ is a Cauchy sequence for the norm

$$\|(y,z,m)\|_{\mathbb{P},\alpha} := \left(\mathbb{E}^{\mathbb{P}}\left[\int_0^T e^{\alpha s}|y_s|^2 ds + \int_0^T e^{\alpha s}\|\widehat{a}_s^{1/2} z_s\|^2 ds + \int_0^T e^{\alpha s} d[m]_s\right]\right)^{\frac{1}{2}}. \tag{3.4.4}$$

where α is positive well chosen. Let α be a positive real number

$$\begin{aligned} y_t^{\mathbb{P},n,m+1} - y_t^{\mathbb{P},n,m} &= \int_t^T \left(\widehat{f}_s^{\mathbb{P},n}(y_s^{\mathbb{P},n,m}, \widehat{a}_s^{1/2} z_s^{\mathbb{P},n,m}) - \widehat{f}_s^{\mathbb{P},n}(y_s^{\mathbb{P},n,m-1}, \widehat{a}_s^{1/2} z_s^{\mathbb{P},n,m-1})\right) ds \\ &\quad - \int_t^T (z_s^{\mathbb{P},n,m+1} - z_s^{\mathbb{P},n,m}) \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_t^T d(m_s^{\mathbb{P},n,m+1} - m_s^{\mathbb{P},n,m}). \end{aligned}$$

We apply Itô's formula to $e^{\alpha t}(y_t^{\mathbb{P},n,m+1} - y_t^{\mathbb{P},n,m})^2$ to obtain

$$\begin{aligned} &e^{\alpha T}(y_T^{\mathbb{P},n,m+1} - y_T^{\mathbb{P},n,m})^2 \\ &= e^{\alpha t}(y_t^{\mathbb{P},n,m+1} - y_t^{\mathbb{P},n,m})^2 + \alpha \int_t^T e^{\alpha s}(y_s^{\mathbb{P},n,m+1} - y_s^{\mathbb{P},n,m})^2 ds \\ &+ 2\int_t^T e^{\alpha s}(y_{s^-}^{\mathbb{P},n,m+1} - y_{s^-}^{\mathbb{P},n,m}) d(y_s^{\mathbb{P},n,m+1} - y_s^{\mathbb{P},n,m}) + \int_t^T e^{\alpha s} d\langle y^{\mathbb{P},n,m+1,c}, y^{\mathbb{P},n,m,c}\rangle_s \\ &+ \sum_{t<s\leq T} e^{\alpha s}\left\{(y_s^{\mathbb{P},n,m+1} - y_s^{\mathbb{P},n,m})^2 - (y_{s^-}^{\mathbb{P},n,m+1} - y_{s^-}^{\mathbb{P},n,m})^2 - 2(y_{s^-}^{\mathbb{P},n,m+1} - y_{s^-}^{\mathbb{P},n,m})\Delta(y_s^{\mathbb{P},n,m+1} - y_s^{\mathbb{P},n,m})\right\}. \end{aligned}$$

Since $y_T^{\mathbb{P},n,m+1} - y_T^{\mathbb{P},n,m} = 0$ and

$$\begin{aligned}(y_s^{\mathbb{P},n,m+1} - y_s^{\mathbb{P},n,m})^2 - (y_{s^-}^{\mathbb{P},n,m+1} - y_{s^-}^{\mathbb{P},n,m})^2 - 2(y_{s^-}^{\mathbb{P},n,m+1} - y_{s^-}^{\mathbb{P},n,m})\Delta(y_s^{\mathbb{P},n,m+1} - y_s^{\mathbb{P},n,m}) \\ = (\Delta(y_s^{\mathbb{P},n,m+1} - y_s^{\mathbb{P},n,m}))^2 \geq 0,\end{aligned}$$

then

$$\begin{aligned}&e^{\alpha t}(y_t^{\mathbb{P},n,m+1} - y_t^{\mathbb{P},n,m})^2 + \int_t^T e^{\alpha s}\widehat{a}_s\|z_s^{\mathbb{P},n,m+1} - z_s^{\mathbb{P},n,m}\|^2 ds + \int_t^T e^{\alpha s} d[m^{\mathbb{P},n,m+1} - m^{\mathbb{P},n,m}]_s \\ &\leq -\alpha\int_t^T e^{\alpha s}(y_s^{\mathbb{P},n,m+1} - y_s^{\mathbb{P},n,m})^2 ds - 2\int_t^T e^{\alpha s}(y_{s^-}^{\mathbb{P},n,m+1} - y_{s^-}^{\mathbb{P},n,m}) d(m_s^{\mathbb{P},n,m+1} - m_s^{\mathbb{P},n,m}) \\ &\quad + 2\int_t^T e^{\alpha s}(y_s^{\mathbb{P},n,m+1} - y_s^{\mathbb{P},n,m})(\widehat{f}_s^{\mathbb{P},n}(y_s^{\mathbb{P},n,m}, \widehat{a}_s^{1/2} z_s^{\mathbb{P},n,m}) - \widehat{f}_s^{\mathbb{P},n}(y_s^{\mathbb{P},n,m-1}, \widehat{a}_s^{1/2} z_s^{\mathbb{P},n,m-1}))ds \\ &\quad - 2\int_t^T e^{\alpha s}(y_s^{\mathbb{P},n,m+1} - y_s^{\mathbb{P},n,m})(z_s^{\mathbb{P},n,m+1} - z_s^{\mathbb{P},n,m}) \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} \\ &\leq -\alpha\int_t^T e^{\alpha s}(y_s^{\mathbb{P},n,m+1} - y_s^{\mathbb{P},n,m})^2 ds - 2\int_t^T e^{\alpha s}(y_{s^-}^{\mathbb{P},n,m+1} - y_{s^-}^{\mathbb{P},n,m}) d(m_s^{\mathbb{P},n,m+1} - m_s^{\mathbb{P},n,m}) \\ &\quad + 2C\int_t^T e^{\alpha s}|y_s^{\mathbb{P},n,m+1} - y_s^{\mathbb{P},n,m}|(|y_s^{\mathbb{P},n,m} - y_s^{\mathbb{P},n,m-1}| + \|\widehat{a}_s^{1/2}(z_s^{\mathbb{P},n,m} - z_s^{\mathbb{P},n,m-1})\|)ds \\ &\quad - 2\int_t^T e^{\alpha s}(y_s^{\mathbb{P},n,m+1} - y_s^{\mathbb{P},n,m})(z_s^{\mathbb{P},n,m+1} - z_s^{\mathbb{P},n,m}) \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}}\end{aligned}$$

Taking expectation under $\mathbb{P}$, we have

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\Big[e^{\alpha t}(y_t^{\mathbb{P},n,m+1} - y_t^{\mathbb{P},n,m})^2 + \int_t^T e^{\alpha s}\widehat{a}_s\|z_s^{\mathbb{P},n,m+1} - z_s^{\mathbb{P},n,m}\|^2 ds + \int_t^T e^{\alpha s} d[m^{\mathbb{P},n,m+1} - m^{\mathbb{P},n,m}]_s\Big] \\ &\leq \mathbb{E}^{\mathbb{P}}\Big[2C\int_t^T e^{\alpha s}|y_s^{\mathbb{P},n,m+1} - y_s^{\mathbb{P},n,m}|\big(|y_s^{\mathbb{P},n,m} - y_s^{\mathbb{P},n,m-1}| + \|\widehat{a}_s^{1/2}(z_s^{\mathbb{P},n,m} - z_s^{\mathbb{P},n,m-1})\|\big)ds \\ &\qquad - \alpha\int_t^T e^{\alpha s}(y_s^{\mathbb{P},n,m+1} - y_s^{\mathbb{P},n,m})^2 ds\Big].\end{aligned}$$

Then

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\Big[e^{\alpha t}(y_t^{\mathbb{P},n,m+1} - y_t^{\mathbb{P},n,m})^2 + \int_t^T e^{\alpha s}\widehat{a}_s\|z_s^{\mathbb{P},n,m+1} - z_s^{\mathbb{P},n,m}\|^2 ds + \int_t^T e^{\alpha s} d[m^{\mathbb{P},n,m+1} - m^{\mathbb{P},n,m}]_s\Big] \\ &\leq \frac{C^2}{\alpha}\mathbb{E}^{\mathbb{P}}\Big[\int_t^T e^{\alpha s}\Big(|y_s^{\mathbb{P},n,m} - y_s^{\mathbb{P},n,m-1}| + \|\widehat{a}_s^{1/2}(z_s^{\mathbb{P},n,m} - z_s^{\mathbb{P},n,m-1})\|\Big)^2\Big] \\ &\leq \frac{2C^2}{\alpha}\mathbb{E}^{\mathbb{P}}\Big[\int_t^T e^{\alpha s}\big(|y_s^{\mathbb{P},n,m} - y_s^{\mathbb{P},n,m-1}|^2 + \|\widehat{a}_s^{1/2}(z_s^{\mathbb{P},n,m} - z_s^{\mathbb{P},n,m-1})\|^2\big)\Big].\end{aligned} \tag{3.4.5}$$

the first inequality follows by using the following polarization identity

$$-\alpha a^2 + 2Cab = -\alpha\Big(a - \frac{C}{\alpha}b\Big)^2 + \frac{C^2}{\alpha}b^2 \leq \frac{C^2}{\alpha}b^2.$$

Therefore, in terms of the $(\mathbb{P},\alpha)$-norm defined in (3.4.7), we have

$$\|z^{\mathbb{P},n,m+1} - z^{\mathbb{P},n,m}\|_{\mathbb{P},\alpha}^2 \leq \frac{2C^2}{\alpha}\|(y^{\mathbb{P},n,m} - y^{\mathbb{P},n,m-1}, z^{\mathbb{P},n,m} - z^{\mathbb{P},n,m-1})\|_{\mathbb{P},\alpha}^2.$$

and

$$\|m^{\mathbb{P},n,m+1} - m^{\mathbb{P},n,m}\|_{\mathbb{P},\alpha}^2 \leq \frac{2C^2}{\alpha}\|(y^{\mathbb{P},n,m} - y^{\mathbb{P},n,m-1}, z^{\mathbb{P},n,m} - z^{\mathbb{P},n,m-1})\|_{\mathbb{P},\alpha}^2.$$

By (3.4.5), we also deduce that at any time t

$$e^{\alpha t}\mathbb{E}\Big[|y_t^{\mathbb{P},n,m+1} - y_t^{\mathbb{P},n,m}|^2\Big] \leq \frac{2C^2}{\alpha}\|(y^{\mathbb{P},n,m} - y^{\mathbb{P},n,m-1}, z^{\mathbb{P},n,m} - z^{\mathbb{P},n,m-1})\|_{\mathbb{P},\alpha}^2.$$

Moreover, by integrating between 0 and T both sides of this inequality, we obtain

$$\|y^{\mathbb{P},n,m+1} - y^{\mathbb{P},n,m}\|^2_{\mathbb{P},\alpha} \leq \frac{2C^2T}{\alpha}\|(y^{\mathbb{P},n,m} - y^{\mathbb{P},n,m-1}, z^{\mathbb{P},n,m} - z^{\mathbb{P},n,m-1})\|^2_{\mathbb{P},\alpha}.$$

Consequently, there exists $K > 0$ depending only on C and T such that

$$\begin{aligned}\|(y^{\mathbb{P},n,m+1} - y^{\mathbb{P},n,m}, z^{\mathbb{P},n,m+1} - z^{\mathbb{P},n,m}, m^{\mathbb{P},n,m+1} - m^{\mathbb{P},n,m})\|^2_{\mathbb{P},\alpha} \qquad (3.4.6)\\ \leq \frac{K}{\alpha}\|(y^{\mathbb{P},n,m} - y^{\mathbb{P},n,m-1}, z^{\mathbb{P},n,m} - z^{\mathbb{P},n,m-1})\|^2_{\mathbb{P},\alpha}\\ \leq \left(\frac{K}{\alpha}\right)^m\|(y^{\mathbb{P},n,1}, z^{\mathbb{P},n,1})\|^2_{\mathbb{P},\alpha}. \qquad (3.4.7)\end{aligned}$$

Let p be a positive integer, then

$$\begin{aligned}&\|(y^{\mathbb{P},n,m+p} - y^{\mathbb{P},n,m}, z^{\mathbb{P},n,m+p} - z^{\mathbb{P},n,m}, m^{\mathbb{P},n,m+p} - m^{\mathbb{P},n,m})\|^2_{\mathbb{P},\alpha}\\ &= \|(y^{\mathbb{P},n,m+p} - y^{\mathbb{P},n,m+p-1}) + (y^{\mathbb{P},n,m+p-1} - y^{\mathbb{P},n,m+p-2}) + \cdots + (y^{\mathbb{P},n,m+1} - y^{\mathbb{P},n,m}),\\ &\quad (z^{\mathbb{P},n,m+p} - z^{\mathbb{P},n,m+p-1}) + (z^{\mathbb{P},n,m+p-1} - z^{\mathbb{P},n,m+p-2}) + \cdots + (z^{\mathbb{P},n,m+1} - z^{\mathbb{P},n,m})\\ &\quad (m^{\mathbb{P},n,m+p} - m^{\mathbb{P},n,m+p-1}) + (m^{\mathbb{P},n,m+p-1} - m^{\mathbb{P},n,m+p-2}) + \cdots + (m^{\mathbb{P},n,m+1} - m^{\mathbb{P},n,m})\|^2_{\mathbb{P},\alpha}\\ &\leq \sum_{j=1}^{p}\|(y^{\mathbb{P},n,m+j} - y^{\mathbb{P},n,m+j-1}, z^{\mathbb{P},n,m+j} - z^{\mathbb{P},n,m+j-1})\|^2_{\mathbb{P},\alpha}\\ &\leq \sum_{j=1}^{p}\left(\frac{K}{\alpha}\right)^{m+j}\|(y^{\mathbb{P},n,1}, z^{\mathbb{P},n,1})\|^2_{\mathbb{P},\alpha}\\ &\leq \frac{(K/\alpha)^m}{1-K/\alpha}\|(y^{\mathbb{P},n,1}, z^{\mathbb{P},n,1})\|^2_{\mathbb{P},\alpha}\end{aligned}$$

for α large enough such that $K/\alpha < 1$, we have $\lim_{m\to+\infty}(K/\alpha)^m = 0$ and consequently, $(y^{\mathbb{P},n,m}, z^{\mathbb{P},n,m}, m^{\mathbb{P},n,m})_{m\geq 0}$ is a Cauchy sequence. Hence, by taking some suitable sub-sequence $\left(m_k^{\mathbb{P}}\right)_{k\geq 1}$, we can define the solution to the penalized BSDE(3.4.1) as

$$y_t^{\mathbb{P},n} := \limsup_{k\to\infty} y_t^{\mathbb{P},n,m_k^{\mathbb{P}}}, \quad z_t^{\mathbb{P},n} := \limsup_{k\to\infty} z_t^{\mathbb{P},n,m_k^{\mathbb{P}}} \quad \text{and} \quad m_t^{\mathbb{P},n} := \limsup_{k\to\infty} m_t^{\mathbb{P},n,m_k^{\mathbb{P}}}, \ \mathbb{P}\text{-a.s.} \qquad (3.4.8)$$

Since the solution to the penalized BSDE converges to solution of the reflected BSDE (3.2.9)(see A.3), then there exists $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$ the solution of (3.2.9) such that for $t \in [0,T]$

$$y_t^{\mathbb{P}} := \lim_{n\to\infty} y_t^{\mathbb{P},n}, \quad z_t^{\mathbb{P}} := \lim_{n\to\infty} z_t^{\mathbb{P},n} \quad m_t^{\mathbb{P}} := \lim_{n\to\infty} m_t^{\mathbb{P},n} \text{ and } k_t^{\mathbb{P}} := \lim_{n\to\infty} k_t^{\mathbb{P},n}.$$

3.4.2 Measurability of the constructed solution

In this paragraph, we justify the measurability with respect to a probability $\mathbb{P}$ of the construction in Section 3.4.1. The following result proved in [85](Lemma 2.3) provides the measurability of a quadratic variation of a semimartingale under a probability law $\mathbb{P}$ using results of [73].

Lemma 3.4.1. *Let $\mathcal{P}$ be a measurable set in $\mathbb{M}_1$, $(t,\omega,\mathbb{P}) \mapsto H_t^{\mathbb{P}}(\omega)$ be a measurable function such that for all $\mathbb{P} \in \mathcal{P}$, $H^{\mathbb{P}}$ is right continuous, $\mathbb{F}_+$-adapted and $(\mathbb{P},\mathbb{F}_+^{\mathbb{P}})$-semimartingale. Then there is a measurable function $(t,\omega,\mathbb{P}) \mapsto \langle H\rangle_t^{\mathbb{P}}(\omega)$ such that for all $\mathbb{P} \in \mathcal{P}$, $\langle H\rangle^{\mathbb{P}}$ is right continuous, $\mathbb{F}_+$-adapted and $\mathbb{F}_+^{\mathbb{P}}$-predictable, and*

$$\langle H\rangle_\cdot^{\mathbb{P}} \text{ is the predictable quadratic variation of the semimartingale } H^{\mathbb{P}} \text{ under } \mathbb{P}.$$

Proof. (i) For every $n \geq 1$, we define the following sequence of random times

$$\begin{cases} \tau_0^{\mathbb{P},n} & := \quad 0,\ \omega \in \Omega, \\ \tau_{i+1}^{\mathbb{P},n} & := \quad \inf\left\{ t \geq \tau_i^{\mathbb{P},n}(\omega), |H_t^{\mathbb{P}}(\omega) - H_{\tau_i^{\mathbb{P},n}}^{\mathbb{P}}(\omega)| \geq 2^{-n} | \right\} \wedge 1,\ \omega \in \Omega,\ i \geq 1. \end{cases} \tag{3.4.9}$$

We notice that $\tau_i^{\mathbb{P},n}$ are all $\mathbb{F}_+$-stopping times since the $H^{\mathbb{P}}$ are right continuous and $\mathbb{F}_+$-adapted. We then define

$$[H^{\mathbb{P}}].(\omega) := \limsup_{n \to +\infty} \sum_{i \geq 0} \left(H^{\mathbb{P}}_{\tau_{i+1}^{\mathbb{P},n} \wedge .}(\omega) - H^{\mathbb{P}}_{\tau_i^{\mathbb{P},n} \wedge .}(\omega) \right)^2. \tag{3.4.10}$$

It is clear that $(t, \omega, \mathbb{P}) \mapsto [H^{\mathbb{P}}]_t(\omega)$ is a measurable function, and for all $\mathbb{P} \in \mathcal{P}$, $[H^{\mathbb{P}}]$ is non-decreasing, $\mathbb{F}_+$-adapted and $\mathbb{F}_+$-optional. Then it follows by Karandikar[53] that $[H^{\mathbb{P}}]$ coincides with the quadratic variation of the semimartingale $H^{\mathbb{P}}$ under $\mathbb{P}$. Moreover, by taking its right limit over rational time instants, we can choose $[H^{\mathbb{P}}]$ to be right continuous.
(ii) Finally, using Proposition 5.1 of Neufeld and Nutz [73], we can then construct a process $\langle H \rangle_t^{\mathbb{P}}(\omega)$ satisfying the required conditions. □

Notice that the construction above can also be carried out for the predictable quadratic covariation $\langle H^{\mathbb{P}}, G^{\mathbb{P}} \rangle^{\mathbb{P}}$, by using the polarization identity

$$\langle H^{\mathbb{P}}, G^{\mathbb{P}} \rangle^{\mathbb{P}} := \frac{1}{4} \left(\langle H^{\mathbb{P}} + G^{\mathbb{P}} \rangle^{\mathbb{P}} - \langle H^{\mathbb{P}} - G^{\mathbb{P}} \rangle^{\mathbb{P}} \right) \tag{3.4.11}$$

The following result gives the property of the solution of the penalized BSDE (3.4.1).

Lemma 3.4.2. *For each $n \geq 1$, there exists a measurable map $(t, \omega, \mathbb{P}) \mapsto (y_t^{\mathbb{P},n}(\omega), z_t^{\mathbb{P},n}(\omega), m_t^{\mathbb{P},n}(\omega), k_t^{\mathbb{P},n}(\omega))$ such that for every $\mathbb{P} \in \mathcal{P}_0$, we have the following properties:*

(i) $y^{\mathbb{P},n}$ is right continuous, $\mathbb{F}_+$-adapted and $\mathbb{F}_+^{\mathbb{P}}$-optional;

(ii) $z^{\mathbb{P},n}$ is $\mathbb{F}_+$-adapted and $\mathbb{F}_+^{\mathbb{P}}$-predictable ;

(iii) $m^{\mathbb{P},n}$ is right continuous, $\mathbb{F}_+^{\mathbb{P}}$-martingale orthogonal to X under $\mathbb{P}$ such that $(y^{\mathbb{P},n}, z^{\mathbb{P},n}, m^{\mathbb{P},n})$ satisfies

$$y_t^{\mathbb{P},n} = \xi + \int_t^T \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P},n}, \widehat{a}_s^{1/2} z_s^{\mathbb{P},n}) ds - \int_t^T z_s^{\mathbb{P},n} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_t^T dm_s^{\mathbb{P},n} + n \int_t^T (y_s^{\mathbb{P},n} - L_s)^- ds, \mathbb{P}\text{-}a.s. \tag{3.4.12}$$

Proof. **(i)-** For every $t \in [0, T]$, (3.4.8) leads to

$$y_t^{\mathbb{P},n} = \lim_{m \to \infty} y_t^{\mathbb{P},n,m},$$

with

$$y_t^{\mathbb{P},n,m+1} = \xi + \int_t^T \widehat{f}_s^{\mathbb{P},n}(y_s^{\mathbb{P},n,m}, \widehat{a}_s^{1/2} z_s^{\mathbb{P},n,m}) ds - \int_t^T z_s^{\mathbb{P},n,m+1} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_t^T dm_s^{\mathbb{P},n,m+1} \ \mathbb{P}\text{-a.s.} \tag{3.4.13}$$

To prove the statement (i), we first prove by induction under m that $y^{\mathbb{P},n,m}$ is right continuous, $\mathbb{F}_+$-adapted and $\mathbb{F}_+^{\mathbb{P}}$-optional and then deduce the result by tending m to infinity.

(a)-Basis $m = 1$**:** By definition $y^{\mathbb{P},n,0} \equiv 0$, then by setting $m = 0$ and by taking conditional expectation under $\mathbb{P}$ with respect to $\mathcal{F}_{t+}$, we have

$$y_t^{\mathbb{P},n,1} = \mathbb{E}^{\mathbb{P}}\left[\xi + \int_t^T \widehat{f}_s^{\mathbb{P},n}(0,0)ds \,\middle|\mathcal{F}_{t+}\right]$$

Since f is jointly measurable, we can apply Lemma 2.4.1. This Lemma states that there is a version of conditional expectation $\mathbb{E}^{\mathbb{P}}\left[\xi + \int_t^T \widehat{f}_r^{\mathbb{P},n}(0,0)dr\middle|\mathcal{F}_{t+}\right]$ measurable with respect to $\mathbb{P}$ and ω, in other words, $(\mathbb{P},\omega) \mapsto y_t^{\mathbb{P},n,1}(\omega)$ is $\mathscr{B}\otimes\mathcal{F}_{t+}$-measurable. With this jointly measurability version and the fact that $y_t^{\mathbb{P},n,1}$ is $\mathcal{F}_{t+}$-measurable, we use Lemma 2.4.3 to choose for any $\mathbb{P}\in\mathcal{P}_0$ a $\mathbb{P}$-modification of $y^{\mathbb{P},n,1}$ which is right-continuous, $\mathbb{F}_+$-adapted and $\mathbb{F}_+^{\mathbb{P}}$-optional and then the property is verified for $m = 1$.

(b)- Inductive step: Assume that the property holds to the order m, then by definition we have

$$y_t^{\mathbb{P},n,m+1} := \mathbb{E}^{\mathbb{P}}\left[\xi + \int_t^T \widehat{f}_r^{\mathbb{P},n}(y_s^{\mathbb{P},n,m}, z_s^{\mathbb{P},n,m})dr\,\middle|\mathcal{F}_{t+}\right], \quad \mathbb{P}\text{-a.s.}$$

and the same reasoning to the above about measurability applies to $y^{\mathbb{P},n,m+1}$ implies that there exists a measurable map $(t,\omega,\mathbb{P}) \mapsto y_t^{\mathbb{P},n,m+1}(\omega)$ such that for every $\mathbb{P}\in\mathcal{P}_0$, $y^{\mathbb{P},n,m+1}$ is right-continuous, $\mathbb{F}_+$-adapted and $\mathbb{F}_+$-optional.
Thereby showing that $(t,\omega,\mathbb{P}) \mapsto y_t^{\mathbb{P},n,m+1}(\omega)$ holds to the order $n+1$. Since both the basis and the inductive step have been performed, by mathematical induction, for all natural numbers $m \geq 1$, $y^{\mathbb{P},n,m}$ is right continuous, $\mathbb{F}_+$-adapted and $\mathbb{F}_+^{\mathbb{P}}$-optional.

(c)- The passage to the limit: The sequence $(y^{\mathbb{P},n,m})_{m\geq 1}$ is the Picard iterations and we have seen in the above section that $(y^{\mathbb{P},n,m})_{m\geq 1}$ is Cauchy sequence under the $(\mathbb{P},\alpha)$-norm(for $\alpha > 0$ large enough) and that $(y^{\mathbb{P},n,m})_{m\geq 1}$ converge uniformly in t, $\mathbb{P}$-a.s. Therefore, $y^{\mathbb{P},n,m}$ converges (under the $(\mathbb{P},\alpha)$-norm) to some process $y^{\mathbb{P},n}$ as $m\to\infty$ which solves the penalized BSDE (3.4.1). Following the previous paragraph the iteration $y^{\mathbb{P},n,m}$ are jointly measurable with respect to time, space and probability law . Now we can use the Lemma 2.4.2 to find a family of subsequences $\left((m_k^{\mathbb{P}})_{k\geq 1}, \mathbb{P}\in\mathcal{P}_0\right)$ such that the limit of $y^{\mathbb{P},n,m_k^{\mathbb{P}}}$ is jointly measurable. And this jointly measurable limit process still the solution of the penalized BSDE (3.4.1).

(ii) We prove the second statement using (3.4.8), $z^{\mathbb{P},n}$ is the limit of the Picard iteration $(z^{\mathbb{P},n,m+1})_m$ also represented by

$$\widehat{a}_s^{1/2} z_s^{\mathbb{P},n,m+1} := \limsup_{\epsilon\in\mathbb{Q},\epsilon\downarrow 0} \frac{\langle y^{\mathbb{P},n,m+1}, X\rangle_s^{\mathbb{P}} - \langle y^{\mathbb{P},n,m+1}, X\rangle_{s-\epsilon}^{\mathbb{P}}}{\epsilon}.$$

The same reasoning applies to $z^{\mathbb{P},n,m}$ and a passage to the limit similar to above allows to deduce the result. First let show that by induction under m that $z^{\mathbb{P},n,m+1}$ is $\mathbb{F}_+$-adapted and $\mathbb{F}_+^{\mathbb{P}}$-predictable.

(a)-Basis $m = 1$**:** For every $t\in[0,T]$, we have

$$\widehat{a}_s^{1/2} z_s^{\mathbb{P},n,1} := \limsup_{\epsilon\in\mathbb{Q},\epsilon\downarrow 0} \frac{\langle y^{\mathbb{P},n,1}, X\rangle_s^{\mathbb{P}} - \langle y^{\mathbb{P},n,1}, X\rangle_{s-\epsilon}^{\mathbb{P}}}{\epsilon}.$$

The definition of quadratic covariation in (3.4.11), Lemma 3.4.1 and the first part of this proof(i) prove that there is a measurable version of the function $(t,\omega,\mathbb{P}) \longmapsto \langle y^{\mathbb{P},n,1}, X\rangle_t^{\mathbb{P}}(\omega)$, such that for every $\mathbb{P}\in\mathcal{P}_0$, $\langle y^{\mathbb{P},n,1}, X\rangle^{\mathbb{P}}$ is right-continuous, $\mathbb{F}_+$-adapted and coincides with the predictable quadratic covariation of $y^{\mathbb{P},n,1}$ and X under $\mathbb{P}$. With this version of $\langle y^{\mathbb{P},n,1}, X\rangle^{\mathbb{P}}$, it is

clear that the family $z^{\mathbb{P},n,1}$ defined above is measurable in $(t,\omega,\mathbb{P})$ and for every $\mathbb{P}\in\mathcal{P}_0$, $z^{\mathbb{P},n,1}$ is $\mathbb{F}_+$-adapted and $\mathbb{F}_+^{\mathbb{P}}$-predictable. Therefore assertion is verified for $m=1$.

(b)- Inductive step: Assume that the property holds to the order m,

$$\widehat{a}_s^{1/2} z_s^{\mathbb{P},n,m+1} := \limsup_{\epsilon\in\mathbb{Q},\epsilon\downarrow 0} \frac{\langle y^{\mathbb{P},n,m+1}, X\rangle_s^{\mathbb{P}} - \langle y^{\mathbb{P},n,m+1}, X\rangle_{s-\epsilon}^{\mathbb{P}}}{\epsilon}.$$

Similar arguments to the basis step (a) induce that there is a measurable version of the map $(t,\omega,\mathbb{P})\mapsto z_t^{\mathbb{P},n,m+1}$ such that for every $\mathbb{P}\in\mathcal{P}_0$, $z^{\mathbb{P},n,m+1}(\omega)$ is $\mathbb{F}_+$-adapted and $\mathbb{F}_+^{\mathbb{P}}$-predictable. Therefore we conclude by induction that $z^{\mathbb{P},n,m+1}$ is $\mathbb{F}_+$-adapted and $\mathbb{F}_+^{\mathbb{P}}$-predictable for all natural numbers $m\geq 1$.

(c)- The passage to the limit: $(z^{\mathbb{P},n,m})_{m\geq 1}$ is a Picard iteration and a Cauchy sequence under the $(\mathbb{P},\alpha)$-norm, then by (3.4.8) $(z^{\mathbb{P},n,m})_{m\geq 1}$ converges uniformly $\mathbb{P}$-a.s. to some process $z^{\mathbb{P},n}$ which is the z- composant of the solution to the penalized BSDE (3.4.1). By the same way, we can use the Lemma 2.4.2 to find a family of subsequences $\big((m_k^{\mathbb{P}})_{k\geq 1}, \mathbb{P}\in\mathcal{P}_0\big)$ such that the limit $z^{\mathbb{P},n}$ of $z_t^{\mathbb{P},n,m_k^{\mathbb{P}}}$ still jointly measurable and verifies (3.4.1). This conclude the proof of (ii).

(iii) For $n\geq 1$, on the one hand by (3.4.8) $m^{\mathbb{P},n}$ is a limit of Picard sequence $(m^{\mathbb{P},n,m})_{m\geq 0}$ which verifies 3.4.2, thus for a fixed $m\geq 0$, $(m^{\mathbb{P},n,m})_{t\geq 0}$ is a $\mathbb{F}_+^{\mathbb{P}}$-adapted right continuous martingale orthogonal to X. By letting m to infinity, the limit still a right continuous $\mathbb{F}_+^{\mathbb{P}}$-adapted right continuous martingale orthogonal to X. On the other hand we have already proved in the two above steps the jointly measurability of $y^{\mathbb{P},n}$ and $z^{\mathbb{P},n}$. Also Picard iterations converge to the solution of the penalized BSDE, thus by (3.4.1), $m^{\mathbb{P},n}$ verifies

$$m_t^{\mathbb{P},n} = y_t^{\mathbb{P},n} - y_0^{\mathbb{P},n} + \int_0^t \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P},n}, \widehat{a}_s^{1/2} z_s^{\mathbb{P},n})ds - \int_0^t z_s^{\mathbb{P},n}\cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} + n\int_0^t (y_s^{\mathbb{P},n} - L_s)^- ds.$$

We deduce that there exists a measurable map $(t,\omega,\mathbb{P})\mapsto m_t^{\mathbb{P},n}(\omega)$. Indeed f is jointly measurable by Assumption 3.2.1(i) then $m^{\mathbb{P},n}$ is a sum of measurable terms. □

Lemma 3.4.3. *There is a subsequence $(n_k^{\mathbb{P}}, k\geq 1)$ such that the sequence $(y^{\mathbb{P},n_k^{\mathbb{P}}})_{n_k^{\mathbb{P}}}$ of jointly measurable functions with respect to $\mathcal{B}[0,T]\times\mathcal{F}_T\times\mathscr{B}$ converges and the limit denoted $y^{\mathbb{P}}$ is also a jointly measurable function. Moreover, for every $t\in[0,T]$ and $\mathbb{P}\in\mathcal{P}_0$, the limit process $y_t^{\mathbb{P}}$ provides the solution to the RBSDE (3.2.9).*

Proof. The sequence $(y^{\mathbb{P},n}, z^{\mathbb{P},n}, m^{\mathbb{P},n}, k^{\mathbb{P},n})_{n\geq 0}$ is solution of Penalized BSDE and we have seen that $(y^{\mathbb{P},n}, z^{\mathbb{P},n}, m^{\mathbb{P},n})_{n\geq 0}$ converges to $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$ (see A.3.1). Moreover $(y^{\mathbb{P},n})_{n\geq 0}$ converge uniformly $\mathbb{P}$-a.s. Therefore, $y^{\mathbb{P},n}$ converges (under the norm defined on $\mathbb{D}_0^2(\mathbb{F}_+^{\mathbb{P}},\mathbb{P})$) to some process $y^{\mathbb{P}}$ as $n\to\infty$ which solves the RBSDE (3.2.9). Following the previous Lemma for each $n\geq 0$, $y^{\mathbb{P},n}$ are jointly measurable with respect to time, space and probability law . Now we can use the Lemma 2.4.2 to find a family of subsequence $\big((n_k^{\mathbb{P}})_{k\geq 0}, \mathbb{P}\in\mathcal{P}_0\big)$ such that the limit of $y_t^{\mathbb{P},n_k^{\mathbb{P}}}$ is jointly measurable. And this jointly measurable limit process still the solution of RBSDE (3.2.9). □

3.4.3 Dynamic programming principle

The dynamic programming principle here is principally based on universally selection measurable theorem seen in section 2.3. The following result extends Theorem 2.1 of [85]in the case of RBSDEs. Our value function V is given by: For every $(t,\omega)\in[0,T]\times\Omega$,

$$V_t(\omega) := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}[y_t^{\mathbb{P}}],$$

where $y^{\mathbb{P}}$ is the first component of the solution of RBSDE(3.2.9). We have the following dynamic programming principle on V.

Theorem 3.4.1. *Suppose that Assumption 3.2.1 holds true. Then for all $(t,\omega) \in [0,T] \times \Omega$, one has $V_t(\omega) = V_t(\omega_{\cdot \wedge t})$, and $(t,\omega) \to V_t(\omega)$ is $\mathcal{B}([0,T]) \otimes \mathcal{F}_T$-universally measurable. Moreover, for all $(t,\omega) \in [0,T] \times \Omega$ and $\mathbb{F}$-stopping time τ taking values in $[t,T]$, we have*

$$V_t(\omega) = \sup_{\mathbb{P} \in \mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}[y_t^{\mathbb{P}}(\tau, V_\tau)],$$

where $y_t^{\mathbb{P}}(\tau, V_\tau)$ is obtained from the solution to the following RBSDE with terminal time τ and terminal condition V_τ,

$$\begin{cases} y_t^{\mathbb{P}} = V_\tau + \int_t^\tau \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}})ds - \int_t^\tau z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_t^\tau dm_s^{\mathbb{P}} + k_\tau^{\mathbb{P}} - k_t^{\mathbb{P}}, \ \mathbb{P}\text{-a.s.} \\ y_t^{\mathbb{P}} \geq L_t, \ t \in [0,\tau], \ \mathbb{P}\text{-a.s.}, \\ \int_0^\tau (y_{t^-}^{\mathbb{P}} - L_{t^-}) dk_t^{\mathbb{P}} = 0, \ \mathbb{P}\text{-a.s.} \end{cases} \tag{3.4.14}$$

To prove this result, we proceed similarly to [85]. The first step of our prove is to establish the dynamic programming principle of our RBSDE associated to the 2RBSDE.

Lemma 3.4.4. *Let $t \in [0,T]$, $\mathbb{P} \in \mathcal{P}_0$, τ be an $\mathbb{F}$-stopping time taking values in $[t,T]$ and $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$ be a solution to the RBSDE (3.2.9) under $\mathbb{P}$. Then one has*

$$y_t^{\mathbb{P}}(T,\xi) = y_t^{\mathbb{P}}(\tau, y_\tau^{\mathbb{P}}) = y_t^{\mathbb{P}}(\tau, \mathbb{E}^{\mathbb{P}}[y_\tau^{\mathbb{P}} | \mathcal{F}_\tau^{\mathbb{P}}]), \ \mathbb{P}\text{-a.s.}$$

Proof. First, we consider a solution $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$ to the RBSDE (3.2.9) associated to (ξ, f, L) under $\mathbb{P}$ w.r.t. the filtration $\mathbb{F}_+^{\mathbb{P}} = (\mathcal{F}_{s+}^{\mathbb{P}})_{0 \leq s \leq T}$, then

$$y_t^{\mathbb{P}} = y_\tau^{\mathbb{P}} + \int_t^\tau \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}})ds - \int_t^\tau z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_t^\tau dm_s^{\mathbb{P}} + \int_t^\tau dk_s^{\mathbb{P}}, \ \mathbb{P}\text{-a.s.}$$

Taking conditional expectation w.r.t. $\mathcal{F}_\tau^{\mathbb{P}}$ under $\mathbb{P}$, we get $\mathbb{P}$-a.s. that

$$y_t^{\mathbb{P}} = \mathbb{E}^{\mathbb{P}}[y_\tau^{\mathbb{P}} | \mathcal{F}_\tau^{\mathbb{P}}] + \int_t^\tau \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}})ds - \int_t^\tau z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_t^\tau d\widehat{m}_s^{\mathbb{P}} + \int_t^\tau d\widehat{k}_s^{\mathbb{P}}, \ \mathbb{P}\text{-a.s.}$$

where the process $\widehat{k}^{\mathbb{P}}$ defined by $\widehat{k}_\tau^{\mathbb{P}} := \mathbb{E}^{\mathbb{P}}[k_\tau^{\mathbb{P}} | \mathcal{F}_\tau^{\mathbb{P}}]$, and $\widehat{k}_s^{\mathbb{P}} := k_s^{\mathbb{P}}$ is non-decreasing by the fact that $k^{\mathbb{P}}$ is non-decreasing. Since $dk_s^{\mathbb{P}} = 1_{\left\{y_{s^-}^{\mathbb{P}} \leq L_{s^-}\right\}} dk_s^{\mathbb{P}}$ for $s \leq \tau$, then

$$\begin{aligned} \int_0^\tau (y_{t^-}^{\mathbb{P}} - L_{t^-}) d\widehat{k}_t^{\mathbb{P}} &= \lim_{u \to \tau^-} \int_0^u (y_{t^-}^{\mathbb{P}} - L_{t^-}) d\widehat{k}_t^{\mathbb{P}} + (y_{\tau^-}^{\mathbb{P}} - L_{\tau^-}) \Delta \widehat{k}_\tau^{\mathbb{P}} \\ &= \lim_{u \to \tau^-} \int_0^u (y_{t^-}^{\mathbb{P}} - L_{t^-}) dk_t^{\mathbb{P}} + (y_{\tau^-}^{\mathbb{P}} - L_{\tau^-}) \mathbb{E}^{\mathbb{P}}[k_\tau^{\mathbb{P}} - k_{\tau^-}^{\mathbb{P}} | \mathcal{F}_\tau] \\ &= \mathbb{E}^{\mathbb{P}}[(y_{\tau^-}^{\mathbb{P}} - L_{\tau^-})(k_\tau^{\mathbb{P}} - k_{\tau^-}^{\mathbb{P}}) | \mathcal{F}_\tau] \\ &= 0. \end{aligned}$$

The last equality is provided by (3.4.14), if $(y_{\tau^-}^{\mathbb{P}} - L_{\tau^-}) > 0$, then $k_\tau^{\mathbb{P}} - k_{\tau^-}^{\mathbb{P}} = 0$. Therefore,

$$\int_0^\tau (y_{t^-}^{\mathbb{P}} - L_{t^-}) d\widehat{k}_t^{\mathbb{P}} = 0, \ \mathbb{P}\text{-a.s. and } y_t^{\mathbb{P}} \geq L_t, \ \forall t \in [0,\tau], \mathbb{P}\text{-a.s.}$$

We also have $\widehat{m}_\tau^{\mathbb{P}} := \mathbb{E}^{\mathbb{P}}[m_\tau^{\mathbb{P}} | \mathcal{F}_\tau^{\mathbb{P}}]$, and $\widehat{m}_s^{\mathbb{P}} := m_s^{\mathbb{P}}$ when $s < \tau$. It is apparent that $\widehat{m}^{\mathbb{P}} \in \mathbb{M}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P})$ and by identification, we deduce that

$$\widehat{m}_\tau^{\mathbb{P}} = m_\tau^{\mathbb{P}} + \mathbb{E}^{\mathbb{P}}[y_\tau^{\mathbb{P}} + k_\tau^{\mathbb{P}} | \mathcal{F}_\tau^{\mathbb{P}}] - (y_\tau^{\mathbb{P}} + k_\tau^{\mathbb{P}}).$$

and then $\widehat{m}^{\mathbb{P}}$ is orthogonal to the continuous martingale X under $\mathbb{P}$. Let consider a RBSDE associated to $(\mathbb{E}^{\mathbb{P}}[y_\tau^{\mathbb{P}}|\mathcal{F}_\tau^{\mathbb{P}}], f, L)$ on $[0,\tau]$, by uniqueness of this solution associated with the properties verifies by $\widehat{m}^{\mathbb{P}}$ and $\widehat{k}^{\mathbb{P}}$, it follows that

$$y_t^{\mathbb{P}}(\tau, y_\tau^{\mathbb{P}}) = y_t^{\mathbb{P}}(\tau, \mathbb{E}^{\mathbb{P}}[y_\tau^{\mathbb{P}}|\mathcal{F}_\tau^{\mathbb{P}}]),\ \mathbb{P}\text{-a.s.}$$

Finally, by definition of the RBSDE (3.2.9) it is clear that $y_t^{\mathbb{P}}(T,\xi) = y_t^{\mathbb{P}}(\tau, y_\tau^{\mathbb{P}})$. □

We now back to the proof of Theorem 3.4.1. The proof is exactly the same to Theorem 2.1 of [85] since we have proved the previous Lemma.

Proof of Theorem 3.4.1 (i) Assumption 3.2.1(iii) gives $\mathcal{P}(t,\omega) = \mathcal{P}(t,\omega_{\cdot\wedge t})$, and by definition of V we have $V_t(\omega) = V_t(\omega_{t\wedge\cdot})$. Furthermore, since $(t,\omega,\mathbb{P}) \longmapsto y_t^{\mathbb{P}}(\omega)$ is a Borel measurable map from $[0,T]\times\Omega\times\mathbb{M}_1$ by Lemma 3.4.3 and the graph $[[\mathcal{P}]]$ is also Borel measurable in $[0,T]\times\Omega\times\mathbb{M}_1$ by Assumption 3.2.1, it follows by the measurable selection theorem that $(t,\omega)\mapsto V_t(\omega)$ is $\mathcal{B}([0,T])\otimes\mathcal{F}_T$-universally measurable theorem 2.1.1 (or more precisely upper semi-analytic).
(ii) Now, using the measurable selection argument, the DPP is a direct consequence of the comparison principle and the stability of RBSDE (3.2.9). First, for every $\mathbb{P}\in\mathcal{P}_0$, we have

$$y_t^{\mathbb{P}}(T,\xi) = y_t^{\mathbb{P}}(\tau, y_\tau^{\mathbb{P}}) = y_t^{\mathbb{P}}(\tau, \mathbb{E}^{\mathbb{P}}[y_\tau^{\mathbb{P}}|\mathcal{F}_\tau^{\mathbb{P}}]),\ \mathbb{P}\text{-a.s.}$$

it follows by the comparison principle of the RBSDE (3.2.9) that

$$V_t(\omega) := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}[y_t^{\mathbb{P}}(T,\xi)] = \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\big[y_t^{\mathbb{P}}(\tau, \mathbb{E}^{\mathbb{P}}[y_\tau^{\mathbb{P}}|\mathcal{F}_\tau^{\mathbb{P}}])\big] \le \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}[y_t^{\mathbb{P}}(\tau, V_\tau)].$$

Next, for every $\mathbb{P}\in\mathcal{P}(t,\omega)$ and $\epsilon>0$, using the measurable selection theorem (see proposition 7.50 of [7]), one can choose a family of probability measures $(\mathbb{Q}_w^\epsilon)_{w\in\Omega}$ such that $w\mapsto\mathbb{Q}_w^\epsilon$ is $\mathcal{F}_\tau$-measurable, and for $\mathbb{P}$-a.e. $w\in\Omega$,

$$\mathbb{Q}_w^\epsilon \in \mathcal{P}(\tau(w), w) \ \text{ and } \ \mathbb{E}^{\mathbb{Q}_w^\epsilon}[y_{\tau(w)}^{\mathbb{Q}_w^\epsilon}(T,\xi)] \ge V_{\tau(w)}(w) - \epsilon.$$

Then $\mathbb{P}\otimes_\tau\mathbb{Q}_\cdot^\epsilon \in \mathcal{P}(t,\omega)$ by Assumption 3.2.1(v). Finally, using stability of the solution to RBSDE (3.2.9)(see Proposition 3.3 of Bouchard et al [14]), it follows that

$$\begin{aligned}
V_t(\omega) \ge \mathbb{E}^{\mathbb{P}\otimes_\tau\mathbb{Q}_\cdot^\epsilon}\big[y_t^{\mathbb{P}\otimes_\tau\mathbb{Q}_\cdot^\epsilon}\big] &= \mathbb{E}^{\mathbb{P}\otimes_\tau\mathbb{Q}_\cdot^\epsilon}\big[y_t^{\mathbb{P}\otimes_\tau\mathbb{Q}_\cdot^\epsilon}(\tau, y_\tau^{\mathbb{P}\otimes_\tau\mathbb{Q}_\cdot^\epsilon})\big] \\
&= \mathbb{E}^{\mathbb{P}\otimes_\tau\mathbb{Q}_\cdot^\epsilon}\big[y_t^{\mathbb{P}\otimes_\tau\mathbb{Q}_\cdot^\epsilon}(\tau, \mathbb{E}^{\mathbb{P}\otimes_\tau\mathbb{Q}_\cdot^\epsilon}[y_\tau^{\mathbb{P}\otimes_\tau\mathbb{Q}_\cdot^\epsilon}|\mathcal{F}_\tau])\big] \\
&= \mathbb{E}^{\mathbb{P}}\big[y_t^{\mathbb{P}}(\tau, \mathbb{E}^{\mathbb{P}\otimes_\tau\mathbb{Q}_\cdot^\epsilon}[y_\tau^{\mathbb{P}\otimes_\tau\mathbb{Q}_\cdot^\epsilon}|\mathcal{F}_\tau])\big] \\
&\ge \mathbb{E}^{\mathbb{P}}\big[y_t^{\mathbb{P}}(\tau, V_\tau)\big] - C\epsilon,
\end{aligned}$$

for some constant $C>0$ independent of ϵ. And hence the other inequality of the DPP holds true by the arbitrariness of $\epsilon>0$ and as well as that of $\mathbb{P}\in\mathcal{P}(t,\omega)$. □

3.4.4 Path regularization of the value function

After proving the DPP, we are interested in the right-continuity property that the first component of the solution of the 2RBSDE (3.2.5) should verify. The first step is to represent the right-continuity modification of V as a semi-martingale under any $\mathbb{P}\in\mathcal{P}_0$ and then give its decomposition. We define for all $(t,\omega)\in[0,T)\times\Omega$

$$V_t^+ := \overline{\lim_{r\in\mathbb{Q}\cap[0,T], r\downarrow t}} V_t, \ \text{ and } \ V_T^+ := V_T.$$

Our first objective is to show that V^+ admits right- and left- limits outside a $\mathcal{P}_0$-polar set. Since for all $t \in (0,T]$, V_t^+ is by definition $\mathcal{F}_t^{U+}$-measurable, we can deduce that V^+ is in fact $\mathbb{F}^{\mathcal{P}_0+}$-optionnal. The downcrossing inequalities below is proved in [66] for RBSDEs but not in a general filtration. We apply the same arguments to prove the result to the case of general filtration. Let $J := (\tau_n)_{n\in\mathbb{N}}$ be a countable family of $\mathbb{F}$-stopping times taking values in $[0,T]$ such that for any $(i,j) \in \mathbb{N}^2$, one has either $\tau_i \leq \tau_j$, or $\tau_i \geq \tau_j$, for every $\omega \in \Omega$. Let $a > b$ and $J_n \subset J$ be a finite subset ($J_n = \{0 \leq \tau_1 \leq \cdots \tau_n \leq T\}$). We denote by $D_a^b(V, J_n)$ the number of downcrossings of the process $(V_{\tau_k})_{1\leq k\leq n}$ from b to a. We then define

$$D_a^b(V,J) := \sup\left\{ D_a^b(V,J_n) : J_n \subset J, \text{ and } J_n \text{ is a finite set} \right\}.$$

The following lemma follows very closely the related result proved in Lemma $A.1$ of [13].

Lemma 3.4.5. *Fix some $\mathbb{P} \in \mathcal{P}_0$. Let Assumptions 3.2.1 and 3.3.1 hold. Denote by L_f the Lipschitz constant of the generator f. Then, for all $a < b$, there exists a probability measures $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that*

$$\mathbb{E}^{\mathbb{Q}}\left[D_a^b(V,J)\right] \leq \frac{e^{L_f T}}{b-a}\mathbb{E}^{\mathbb{Q}}\Big[e^{L_f T}(V_0 \wedge b - a) - e^{-L_f T}(V_T \wedge b - a)^+ + e^{L_f T}(V_T \wedge b - a)^- + e^{L_f T}\int_0^T |\widehat{f}^{\mathbb{P}}(a,0)|ds\Big]$$

Moreover, outside a $\mathcal{P}_0$-polar set, we have

$$\overline{\lim_{r\in\mathbb{Q}\cap(t,T],r\downarrow t}} V_t(\omega) := \lim_{r\in\mathbb{Q}\cap(t,T],r\downarrow t} V_t(\omega), \text{ and } \overline{\lim_{r\in\mathbb{Q}\cap(t,T],r\uparrow t}} V_t(\omega) := \lim_{r\in\mathbb{Q}\cap(t,T],r\uparrow t} V_t(\omega).$$

To prove the above result, we need to recall some properties verifed by V defined at $\mathbb{F}$-stopping times. For any $\mathbb{F}$-stopping times $\tau \geq \sigma$, we have from Theorem 3.4.1 that

$$V_{\sigma(\omega)}(\omega) = \sup_{\mathbb{P}\in\mathcal{P}(\sigma(\omega),\omega)} \mathbb{E}^{\mathbb{P}}\left[y^{\mathbb{P}}_{\sigma(\omega)}(\tau, V_\tau)\right], \tag{3.4.15}$$

We refer the reader to [22] for the precise details about the proof of this result.

Lemma 3.4.6. *For any $\mathbb{P} \in \mathcal{P}_0$, for any $\mathbb{F}$-stopping times $0 \leq \sigma \leq \tau \leq T$, we have*

$$\mathbb{E}^{\mathbb{P}^{\sigma(\omega)}_\omega}\left[y^{\mathbb{P}^{\sigma(\omega)}_\omega}_{\sigma(\omega)}(\tau, V_\tau)\right] = \mathbb{E}^{\mathbb{P}}\left[y^{\mathbb{P}}_\sigma(\tau, V_\tau)\Big|\mathcal{F}_\sigma\right](\omega), \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

The following inequality is the consequence of the above equation.

$$V_{\sigma(\omega)} \geq \mathbb{E}^{\mathbb{P}}\left[y^{\mathbb{P}}_{\sigma(\omega)}(\tau, V_\tau)\right], \text{ for any } \mathbb{P} \in \mathcal{P}(\sigma(\omega),\omega). \tag{3.4.16}$$

These inequalities allow one to prove Lemma 3.4.5.

Proof of Lemma 3.4.5. Without loss of generality, we suppose that $a = 0$. Let $J_n = \{\tau_0, \tau_1, \cdots, \tau_n\}$ with $0 = \tau_0 < \tau_1 < \cdots < \tau_n = T$. For any $i = 1, \ldots, n$, and $\omega \in \Omega$, let the following RBSDE under $\mathbb{P}^{\tau_{i-1}(\omega)}_\omega$ on $[\tau_{i-1}, \tau_i]$

$$\begin{cases} y_t^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega} := V_{\tau_i} + \int_t^{\tau_i}\left(\widehat{f}_s^{\mathbb{P}^{\tau_{i-1}(\omega)}_\omega,0} + \lambda_s^i y_s^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega} + \eta_s^i \cdot \widehat{a}_s^{1/2} z_s^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega}\right)ds \\ \qquad\qquad - \int_t^{\tau_i} z_s^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}^{\tau_{i-1}(\omega)}_\omega} - \int_t^{\tau_i} dm_s^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega} + \int_t^{\tau_i} dk_s^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega}, \ \mathbb{P}^{\tau_{i-1}(\omega)}_\omega\text{-a.s.}, \\ y_t^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega} \geq L_t, \ t \in [\tau_{i-1}, \tau_i], \ \mathbb{P}^{\tau_{i-1}(\omega)}_\omega\text{-a.s.}, \\ \int_{\tau_{i-1}}^{\tau_i}(y_{s^-}^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega} - L_{s^-})dk_s^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega} = 0, \ \mathbb{P}^{\tau_{i-1}(\omega)}_\omega\text{-a.s.} \end{cases}$$

where λ^i and η^i are two bounded processes (by the the Lipschitz constant L_f of f) appearing in the linearisation of f due to the Lipschitz property of f. Define the linear RBSDE,

$$\begin{cases} \bar{y}_t^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} := V_{\tau_i} + \int_t^{\tau_i} \big(-|\widehat{f}_s^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)},0}| + \lambda_s^i \bar{y}_s^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} + \eta_s^i \cdot \widehat{a}_s^{1/2} \bar{z}_s^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} \big) ds \\ \qquad\qquad - \int_t^{\tau_i} \bar{z}_s^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} - \int_t^{\tau_i} d\bar{m}_s^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} + \int_t^{\tau_i} d\bar{k}_s^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}}, \ \mathbb{P}_\omega^{\tau_{i-1}(\omega)}\text{-a.s.}, \\ \bar{y}_t^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} \geq L_t, \ t \in [\tau_{i-1}, \tau_i], \ \mathbb{P}_\omega^{\tau_{i-1}(\omega)}\text{-a.s.}, \\ \int_{\tau_{i-1}}^{\tau_i} (\bar{y}_{s^-}^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} - L_{s^-}) d\bar{k}_s^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} = 0, \ \mathbb{P}_\omega^{\tau_{i-1}(\omega)}\text{-a.s.} \end{cases}$$

By Assumption 3.2.1(iv), for $\mathbb{P}$-almost every where $\omega \in \Omega$, we have $\mathbb{P}_\omega^{\tau_{i-1}(\omega)} \in \mathcal{P}(\tau_{i-1}(\omega), \omega)$. Therefore on one side by comparison principle for supersolution of BSDEs and on another side by (3.4.15), we have

$$\bar{y}_{\tau_{i-1}}^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} \leq y_{\tau_{i-1}}^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} \leq V_{\tau_{i-1}}(\omega). \tag{3.4.17}$$

We consider $X_t^i = e^{\int_0^t \lambda_s^i ds}$. By Itô's formula applied to $X_t^i y_t^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}}$ between t and τ_i, we can rewrite the above RBSDE as

$$\begin{cases} X_t^i \bar{y}_t^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} = X_{\tau_i}^i V_{\tau_i} - \int_t^{\tau_i} X_s^i \Big| \widehat{f}_s^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)},0} \Big| ds - \int_t^{\tau_i} X_s^i \bar{z}_s^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} \cdot \widehat{a}_s^{1/2} (dW_s^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} - \eta_s^i ds) \\ \qquad\qquad - \int_t^{\tau_i} X_{s^-}^i d\bar{m}_s^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} + \int_t^{\tau_i} X_{s^-}^i d\bar{k}_s^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}}, \ \mathbb{P}_\omega^{\tau_{i-1}(\omega)}\text{-a.s.}, \\ X_t^i \bar{y}_t^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} \geq X_t^i L_t, \ t \in [\tau_{i-1}, \tau_i], \ \mathbb{P}_\omega^{\tau_{i-1}(\omega)}\text{-a.s.}, \\ \int_{\tau_{i-1}}^{\tau_i} X_{s^-}^i (\bar{y}_{s^-}^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} - L_{s^-}) d\bar{k}_s^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} = 0, \ \mathbb{P}_\omega^{\tau_{i-1}(\omega)}\text{-a.s.} \end{cases}$$

One can now use the link between the RBSDEs and optimal stopping problems to establish

$$\begin{aligned} \bar{y}_{\tau_{i-1}}^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} = \operatorname*{ess\,sup}_{\substack{\tau_{i-1} \leq \nu \leq \tau_i \\ \nu \ \mathbb{F}\text{-s.t.}}} \mathbb{E}^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} \Big[\mathcal{E}\Big(\int_{\tau_{i-1}}^{\tau_i} \eta_s^i \cdot dW^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} \Big) \Big(- \int_{\tau_{i-1}}^{\nu} e^{\int_{\tau_{i-1}}^s \lambda_s^i ds} \Big| \widehat{f}_s^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)},0} \Big| ds \\ + L_\nu e^{\int_{\tau_{i-1}}^\nu \lambda_s^i ds} 1_{\nu < \tau_i} + V_{\tau_i} e^{\int_{\tau_{i-1}}^{\tau_i} \lambda_s^i ds} 1_{\nu = \tau_i} \Big) \Big| \mathcal{F}_{\tau_{i-1}}^+ \Big]. \end{aligned}$$

where ν is a $\mathbb{F}$-stopping time. Hence by (3.4.17), it is clear that

$$\begin{aligned} \operatorname*{ess\,sup}_{\substack{\tau_{i-1} \leq \nu \leq \tau_i \\ \nu \ \mathbb{F}\text{-s.t.}}} \mathbb{E}^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} \Big[\mathcal{E}\Big(\int_{\tau_{i-1}}^{\tau_i} \eta_s^i \cdot dW^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} \Big) \Big(- \int_{\tau_{i-1}}^{\nu} e^{\int_{\tau_{i-1}}^s \lambda_s^i ds} \Big| \widehat{f}_s^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)},0} \Big| ds \\ + L_\nu e^{\int_{\tau_{i-1}}^\nu \lambda_s^i ds} 1_{\nu < \tau_i} + V_{\tau_i} e^{\int_{\tau_{i-1}}^{\tau_i} \lambda_s^i ds} 1_{\nu = \tau_i} \Big) \Big| \mathcal{F}_{\tau_{i-1}}^+ \Big] \leq V_{\tau_{i-1}}(\omega). \end{aligned}$$

Using the definition of the r.c.p.d., it follows that

$$\begin{aligned} \operatorname*{ess\,sup}_{\substack{\tau_{i-1} \leq \nu \leq \tau_i \\ \nu \ \mathbb{F}\text{-s.t.}}} \mathbb{E}^{\mathbb{Q}} \Big[- \int_{\tau_{i-1}}^{\nu} e^{\int_{\tau_{i-1}}^s \lambda_s^i ds} \Big| \widehat{f}_s^{\mathbb{P}_\cdot^{\tau_{i-1}(\cdot)},0} \Big| ds \\ + L_\nu e^{\int_{\tau_{i-1}}^\nu \lambda_s^i ds} 1_{\nu < \tau_i} + V_{\tau_i} e^{\int_{\tau_{i-1}}^{\tau_i} \lambda_s^i ds} 1_{\nu = \tau_i} \Big| \mathcal{F}_{\tau_{i-1}} \Big] \leq V_{\tau_{i-1}}, \ \mathbb{P}\text{-a.s..} \end{aligned}$$

where the probability measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \mathcal{E}\left(\int_{\tau_{i-1}}^t \eta_s^i \cdot dW_s^{\mathbb{P}} \right), \ t \in [\tau_{i-1}, \tau_i].$$

Hence, by choosing $\nu = \tau_i$ above, we get

$$\mathbb{E}^{\mathbb{Q}}\Big[- \int_{\tau_{i-1}}^{\tau_i} e^{\int_{\tau_{i-1}}^{s} \lambda_s^i ds} \Big| \widehat{f}_s^{\mathbb{P}^{\tau_{i-1}}(\cdot),0} \Big| ds + \widehat{Y}_{\tau_i} e^{\int_{\tau_{i-1}}^{\tau_i} \lambda_s^i ds} \Big| \mathcal{F}_{\tau_{i-1}} \Big] \leq V_{\tau_{i-1}}, \ \mathbb{P}\text{-a.s..}$$

Let $\lambda_s := \sum_{i=1}^n \lambda_s^i 1_{[\tau_{i-1},\tau_i)}(s)$, then one has that the discrete process $(U_{\tau_i})_{0\leq i\leq n}$ defined by

$$U_{\tau_i} := V_{\tau_i} e^{\int_0^{\tau_i} \lambda_s ds} - \int_0^{\tau_i} e^{\int_0^s \lambda_r dr} \big| \widehat{f}_s^{\mathbb{P},0} \big| ds,$$

is a $\mathbb{Q}$-supermartingale relative to $\mathbb{F}$. Define further

$$\bar{U}_{\tau_i} := U_{\tau_i} \wedge \Big(b e^{L_f T} - \int_0^{\tau_i} e^{\int_0^s \lambda_r dr} \big| \widehat{f}_s^{\mathbb{P},0} \big| ds \Big),$$

which is also a $\mathbb{Q}$-supermartingale relative to $\mathbb{F}$. Let

$$u_t := b e^{\int_0^t \lambda_r dr} - \int_0^t e^{\int_0^s \lambda_r dr} \big| \widehat{f}_s^{\mathbb{P},0} \big| ds,$$

and

$$l_t := - \int_0^t e^{\int_0^s \lambda_r dr} \big| \widehat{f}_s^{\mathbb{P},0} \big| ds.$$

It is clear that $D_l^u(U, J) = D_l^u(\bar{U}, J)$, in others words the number of downcrossings of the process U from l to u is equal to the number of downcrossings of the process $\bar{U}$ from l to u. Since l is decreasing in t, so that we can apply the classic downcrossing theorem from supermartingales (see Doob [31]) to $\bar{U}$ and obtain

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\big[D_0^b(V, J)\big] &\leq \mathbb{E}^{\mathbb{Q}}\big[D_l^u(\bar{U}, J)\big] \\
&\leq \frac{e^{L_f T}}{b} \mathbb{E}^{\mathbb{Q}}\big[(\bar{U}_0 - \bar{U}_T) - (u_T - \bar{U}_T) \wedge 0\big] \\
&\leq \frac{e^{L_f T}}{b} \mathbb{E}^{\mathbb{Q}}\Big[(U_0 \wedge b e^{L_f T}) - U_T \wedge \big(b e^{L_f T} - \int_0^T e^{\int_0^s \lambda_r dr} \big| \widehat{f}_s^{\mathbb{P},0} \big| ds \big) \\
&\qquad\qquad - (u_T - (b e^{L_f T} - \int_0^T e^{\int_0^s \lambda_r dr} \big| \widehat{f}_s^{\mathbb{P},0} \big| ds)) \wedge 0\Big] \\
&\leq \frac{e^{L_f T}}{b} \mathbb{E}^{\mathbb{Q}}\Big[e^{L_f T}(V_0 \wedge b) + e^{L_f T} \int_0^T \big| \widehat{f}_s^{\mathbb{P},0} \big| ds - (V_T e^{\int_0^T \lambda_s ds} \wedge b e^{L_f T})\Big] \\
&\leq \frac{e^{L_f T}}{b} \mathbb{E}^{\mathbb{Q}}\Big[e^{L_f T}(V_0 \wedge b) + e^{L_f T} \int_0^T \big| \widehat{f}_s^{\mathbb{P},0} \big| ds - (V_T e^{\int_0^T \lambda_s ds} \wedge b e^{L_f T})^+ \\
&\qquad\qquad + (V_T e^{\int_0^T \lambda_s ds} \wedge b e^{L_f T})^-\Big] \\
&\leq \frac{e^{L_f T}}{b} \mathbb{E}^{\mathbb{Q}}\Big[e^{L_f T}(V_0 \wedge b) + e^{L_f T} \int_0^T \big| \widehat{f}_s^{\mathbb{P},0} \big| ds - e^{-L_f T}(V_T \wedge b)^+ + e^{L_f T}(V_T \wedge b)^-\Big].
\end{aligned}$$

which proof the statement for $a = 0$.
Let set

$$\Sigma := \{\omega \in \Omega \text{ s.t. } V_\cdot(\omega) \text{ has no right- or left-limits along the rationals } \}.$$

We claim that Σ is a $\mathcal{P}_0$-polar set. Indeed suppose that there exists $\mathbb{P} \in \mathcal{P}_0$ satisfying $\mathbb{P}(\Sigma) > 0$. Then, Σ is non-empty and for any $\omega \in \Sigma$, the path $V_\cdot(\omega)$ has , e.g., no right-limit along the rational at some point $t \in [0, T]$. Hence we can find two rational numbers a, b such that

$$\underline{\lim}_{r\in\mathbb{Q}\cap(t,T], r\downarrow t} V_r(\omega) < a < b < \overline{\lim}_{r\in\mathbb{Q}\cap(t,T], r\downarrow t} V_r(\omega),$$

and the number of downcrossing $D_a^b(V,J)(\omega)$ of the path $V_\cdot(\omega)$ on the interval $[a,b]$ is equal to $+\infty$. However, the downcrossing inequality proved above shows that $D_a^b(V,J)$ is $\mathbb{Q}$-a.s. and thus $\mathbb{P}$-a.s. (see Lemma A.2.1), finite, for any pair (a,b). This implies a contradiction since we assumed that $\mathbb{P}(\Sigma) > 0$. Therefore, outside the $\mathcal{P}_0$-polar set Σ, V admits both right- and left-limits along the rationals. □

Using Lemma 3.4.5, we obtain

$$V_t^+ := \lim_{r\in\mathbb{Q}\cap[0,T], r\downarrow t} V_t, \text{ outside a } \mathcal{P}_0\text{-polar set},$$

and from this we deduced that V^+ is right-continuous outside a $\mathcal{P}_0$-polar set.

3.4.4.1 Representation formula

We begin by extend this inequality (3.4.16) to V^+.

Lemma 3.4.7. *For any $0 \le s \le t \le T$, for any $\mathbb{P} \in \mathcal{P}_0$, we have*

$$V_s^+ \ge y_s^{\mathbb{P}}(t, V_t^+),\ \mathbb{P}\text{-a.s.}$$

The proved of this Lemma is in [85] and the proof is the following.

Proof. Choose some $(s,t,\omega) \in [0,T]\times[s,T]\times\Omega$ and some $\mathbb{P}\in\mathcal{P}_0$. Let $r_n^1 \in \mathbb{Q}\cap(s,T], r_n^1 \downarrow s$ and $r_m^2 \in \mathbb{Q}\cap(t,T], r_m^2 \downarrow t$. For any $m,n \ge 1$, inequality (3.4.16) applies to r_n^1 and r_m^2 give that for $\tilde{\mathbb{P}} \in \mathcal{P}(r_n^1,\omega)$

$$V_{r_n^1} \ge \mathbb{E}^{\tilde{\mathbb{P}}}\left[y_{r_n^1}^{\tilde{\mathbb{P}}}(r_m^2, V_{r_m^2})\right].$$

In particular, the stability under conditioning of $\mathcal{P}$ (remember Assumption 3.2.1(iv)) yields for $\mathbb{P}$-a.e. $\omega\in\Omega$, we have

$$V_{r_n^1}(\omega) \ge \mathbb{E}^{\mathbb{P}_\omega^{r_n^1}}\left[y_{r_n^1}^{\mathbb{P}_\omega^{r_n^1}}(r_m^2, V_{r_m^2})\right] = \mathbb{E}^{\mathbb{P}}\left[y_{r_n^1}^{\mathbb{P}}(r_m^2, V_{r_m^2})\big|\mathcal{F}_{r_n^1}\right](\omega), \tag{3.4.18}$$

where we have used Lemma 3.4.6. We have by definition of V^+ that

$$\lim_{n\to+\infty} V_{r_n^1} = V_s^+, \mathbb{P}\text{-a.s.}$$

Our next goal is to show that

$$\mathbb{E}^{\mathbb{P}}\left[y_{r_n^1}^{\mathbb{P}}(r_m^2, V_{r_m^2})\big|\mathcal{F}_{r_n^1}\right] \underset{n\to+\infty}{\longrightarrow} y_s^{\mathbb{P}}(r_m^2, V_{r_m^2}), \text{ for the norm } \|\cdot\|_{\mathbb{L}^1_{s,\omega}}.$$

In fact, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\left|\mathbb{E}^{\mathbb{P}}\left[y_{r_n^1}^{\mathbb{P}}(r_m^2, V_{r_m^2})\big|\mathcal{F}_{r_n^1}\right] - y_s^{\mathbb{P}}(r_m^2, V_{r_m^2})\right|\right] &= \mathbb{E}^{\mathbb{P}}\left[\left|\mathbb{E}^{\mathbb{P}}\left[y_{r_n^1}^{\mathbb{P}}(r_m^2, V_{r_m^2}) - y_s^{\mathbb{P}}(r_m^2, V_{r_m^2})\big|\mathcal{F}_{r_n^1}\right]\right|\right] \\
&\le \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\left|y_{r_n^1}^{\mathbb{P}}(r_m^2, V_{r_m^2}) - y_s^{\mathbb{P}}(r_m^2, V_{r_m^2})\right|\big|\mathcal{F}_{r_n^1}\right]\right] \\
&\le \mathbb{E}^{\mathbb{P}}\left[\left|y_{r_n^1}^{\mathbb{P}}(r_m^2, V_{r_m^2}) - y_s^{\mathbb{P}}(r_m^2, V_{r_m^2})\right|\right].
\end{aligned}$$

Then since $y_{r_n^1}^{\mathbb{P}}(r_m^2, V_{r_m^2})$is càdlàg, we know that $y_{r_n^1}^{\mathbb{P}}(r_m^2, V_{r_m^2})$ goes, $\mathbb{P}$-a.s. to $y_s^{\mathbb{P}}(r_m^2, V_{r_m^2})$ as n goes to $+\infty$. Moreover, by the a priori estimates of the solution of RBSDEs(recall Theorem 3.3.5), the quantity $y_{r_n^1}^{\mathbb{P}}(r_m^2, \mathbb{V}_{r_m^2})$ is uniformly bounded in $L^p(\mathcal{F}_T^{\mathbb{P}^+}, \mathbb{P})$, and thus the sequence $(y_{r_n^1}^{\mathbb{P}}(r_m^2, V_{r_m^2}))_{n\ge1}$ is a uniformly integrable family by de la Vallée-Poussin criterion

(since $p > 1$). Therefore the dominated convergence theorem implies that the last term of the above inequalities go to 0 and then

$$\lim_{n\to+\infty} \mathbb{E}^{\mathbb{P}}\left[\left|\mathbb{E}^{\mathbb{P}}\left[y^{\mathbb{P}}_{r^1_n}(r^2_m, V_{r^2_m})\middle|\mathcal{F}_{r^1_n}\right] - y^{\mathbb{P}}_s(r^2_m, V_{r^2_m})\right|\right] = 0.$$

Hence taking a subsequence if necessary, we have that the right-hand side of (3.4.18) goes $\mathbb{P}$-a.s. to $y^{\mathbb{P}}_s(r^2_m, V_{r^2_m})$ as n goes to $+\infty$. Letting n goes to $+\infty$ in (3.4.18), we have

$$V^+_s \geq y^{\mathbb{P}}_s(r^2_m, V_{r^2_m}),\ \mathbb{P}\text{-a.s.}$$

Otherwise, we have by the dynamic programming for RBSDEs

$$\begin{aligned} y^{\mathbb{P}}_s(r^2_m, V_{r^2_m}) - y^{\mathbb{P}}_s(t, V^+_t) &= y^{\mathbb{P}}_s(r^2_m, V_{r^2_m}) - y^{\mathbb{P}}_s(r^2_m, V^+_t) + y^{\mathbb{P}}_s(r^2_m, V^+_t) - y^{\mathbb{P}}_s(t, V^+_t) \\ &= y^{\mathbb{P}}_s(r^2_m, V_{r^2_m}) - y^{\mathbb{P}}_s(r^2_m, V^+_t) + y^{\mathbb{P}}_s(t, y^{\mathbb{P}}_t(r^2_m, V^+_t)) - y^{\mathbb{P}}_s(t, V^+_t). \end{aligned}$$

The inequality $(A.4.3)$ yields the first difference on the last equality converges to 0, $\mathbb{P}$-a.s. Applying the same method that we have used for the estimates $(A.4.3)$, we control the second difference by

$$\mathbb{E}^{\mathbb{P}}\left[|V^+_t - y^{\mathbb{P}}_t(r^2_m, V^+_t)|^{\tilde{p}}\middle|\mathcal{F}_s\right]$$

for some $1 < \tilde{p} < p$. This term goes $\mathbb{P}$-a.s.(at least along a subsequence) to 0 when m goes $+\infty$(see Proposition A.4.1). Consequently,

$$\lim_{m\to+\infty}(y^{\mathbb{P}}_s(r^2_m, V_{r^2_m}) - y^{\mathbb{P}}_s(t, V^+_t)) = 0$$

which completes the proof. □

The next result is an extension of the previous result to stopping times.

Lemma 3.4.8. *For any $\mathbb{F}$-stopping times $0 \leq \sigma \leq \tau \leq T$, for any $\mathbb{P} \in \mathcal{P}_0$, we have*

$$V^+_\sigma \geq y^{\mathbb{P}}_\sigma(\tau, V^+_\tau),\ \mathbb{P}\text{-a.s.}$$

In particular V^+ is càdlàg, $\mathcal{P}_0$-q.s.

Proof. Assume first that σ takes a finite number of values $\{t_1, \cdots, t_n\}$ and that τ is deterministic. Then, we have for any $\mathbb{P} \in \mathcal{P}_0$

$$V^+_\sigma = \sum_{i=1}^n V^+_{t_i} 1_{\{\sigma=t_i\}} \geq \sum_{i=1}^n y^{\mathbb{P}}_{t_i}(\tau, V^+_\tau) 1_{\{\sigma=t_i\}} = y^{\mathbb{P}}_\sigma(\tau, V^+_\tau),\ \mathbb{P}\text{-a.s.}$$

Assume next that both τ and σ take a finite number of values $\{t_1, \cdots, t_n\}$. We have similarly

$$y^{\mathbb{P}}_\sigma(\tau, V^+_\tau) = \sum_{i=1}^n y^{\mathbb{P}}_\sigma(t_i, V^+_{t_i}) 1_{\{\tau=t_i\}} \leq \sum_{i=1}^n V^+_\sigma 1_{\{\tau=t_i\}} = V^+_\sigma,\ \mathbb{P}\text{-a.s.}$$

Then, if σ is general, we can always approach it from above by a decreasing sequence of $\mathbb{F}_+$-stopping times $(\sigma^n)_{n\geq 1}$ taking only a finite number of values. The above results imply directly that

$$V^+_{\sigma^n\wedge\tau} \geq y^{\mathbb{P}}_{\sigma^n\wedge\tau}(\tau, V^+_\tau),\ \mathbb{P}\text{-a.s.}$$

Then, we can use the right-continuity of V^+ and $y^{\mathbb{P}}(\tau, V_\tau^+$ to let n goes to $+\infty$ and obtain

$$V_\sigma^+ \geq y_\sigma^{\mathbb{P}}(\tau, V_\tau^+),\ \mathbb{P}\text{-a.s.}$$

Finally, let us take a general stopping time τ. We once more approach it by a decreasing sequence of $\mathbb{F}^+$-stopping times $(\tau^n)_{n\geq 1}$ taking only a finite number of values. We thus have

$$V_\sigma^+ \geq y_\sigma^{\mathbb{P}}(\tau^n, V_{\tau^n}^+),\ \mathbb{P}\text{-a.s.}$$

The term on the right-hand side converges (along a subsequence if necessary) $\mathbb{P}$-a.s. to $y_\sigma^{\mathbb{P}}(\tau, \widehat{Y}_\tau^+)$ by Lemma A.4.1.
It remains to justify that V^+ admits left-limits outside a $\mathcal{P}_0$-polar set. Fix some $\mathbb{P} \in \mathcal{P}_0$. Following the same arguments as in the proof of Lemma 3.4.5, we can show that for some probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ and some bounded process λ,

$$U_t := V_t e^{\int_0^t \lambda_s ds} + \int_0^t e^{\int_0^s \lambda_r dr} \left|\widehat{f}_s^{\mathbb{P},0}\right| ds,$$

is a right-continuous $(\mathbb{Q}, \mathbb{F}_+)$-supermatingale, which is in addition uniformly integrable under $\mathbb{Q}$ since V and $\widehat{f}^{\mathbb{P},0}$ are uniformly bounded in $\mathbb{L}^p(\widehat{\mathcal{F}}_T, \mathbb{P})$ and thus in $\mathbb{L}^{\tilde{p}}(\widehat{\mathcal{F}}_T, \mathbb{Q})$ for some $1 < \tilde{p} < p$. Therefore, for any increasing sequence of $\mathbb{F}_+$-stopping times $(\rho^n)_{n\geq 0}$ taking values in $[0, T]$, the sequence $(\mathbb{E}^{\mathbb{Q}}[V_{\rho^n}])_{n\geq 0}$ is non-increasing and admits a limit. By Theorem VI-48 and Remark VI-50(f) of [26], we deduce that V, and thus V^+, admit left-limits outside a $\mathbb{Q}$-negligible set. Moreover, the above implies that the set

$$\left\{\omega \in \Omega, V^+(\omega) \text{ admits left-limits } \right\},$$

is $\mathcal{P}_0$-polar, which ends the proof. □

Similarly to [66] and [85], we have the following representations.

Lemma 3.4.9. *For any $\mathbb{F}$-stopping times $0 \leq \sigma \leq \tau \leq T$, for any $0 \leq t \leq T$, for any $\mathbb{P} \in \mathcal{P}_0$, we have*

$$V_\sigma = \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_0(\sigma,\mathbb{P},\mathbb{F})}^{\mathbb{P}} \mathbb{E}^{\mathbb{P}'}\left[y_\sigma^{\mathbb{P}'}(\tau, y_\tau^{\mathbb{P}'})\middle|\mathcal{F}_\sigma\right],\ \mathbb{P}\text{-a.s. and } V_t^+ = \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}^{\mathbb{P}} y_t^{\mathbb{P}'}(T, \xi),\ \mathbb{P}\text{-a.s.}$$

where $\mathcal{P}_0(\sigma, \mathbb{P}, \mathbb{F})$ is defined in Section 3.2.3. In particular, if Assumption 3.3.1 holds, one has $V^+ \in \mathbb{D}_0^p(\mathbb{F}^{\mathcal{P}_0+})$.

Proof. We start with the first equality. By definition and Lemma 3.4.6, for any $\mathbb{P}' \in \mathcal{P}_0(\sigma, \mathbb{P}, \mathbb{F})$ we have

$$V_\sigma \geq \mathbb{E}^{\mathbb{P}'}\left[y_\sigma^{\mathbb{P}'}(\tau, y_\tau^{\mathbb{P}'})\middle|\mathcal{F}_\tau\right],\ \mathbb{P}'\text{-a.s.}$$

But since both sides of the inequality are $\mathcal{F}_\sigma^U$-measurable and $\mathbb{P}'$ coincides with $\mathbb{P}$ on $\mathcal{F}_\sigma$ (and thus on $\mathcal{F}_\sigma^U$, by uniqueness of universal completion) the above also holds $\mathbb{P}$-a.s. We deduce

$$V_\sigma \geq \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_0(\sigma,\mathbb{P},\mathbb{F})}^{\mathbb{P}} \mathbb{E}^{\mathbb{P}'}\left[y_\sigma^{\mathbb{P}'}(\tau, y_\tau^{\mathbb{P}'})\middle|\mathcal{F}_\tau\right],\ \mathbb{P}'\text{-a.s.}$$

Next, notice that by Lemmas 3.4.3 and 4.4.3, $(t, \omega, \mathbb{Q}) \mapsto \mathbb{E}^{\mathbb{Q}}\left[y_t^{\mathbb{Q}}(T, \xi)\right] = \mathbb{E}^{\mathbb{Q}}\left[y_t^{\mathbb{Q}}(\tau, y_\tau^{\mathbb{Q}})\right]$ is Borel measurable. As in the proof of Theorem 3.4.1, it follows by the measurable selection theorem(see e.g. Proposition 7.47 of [7]) that for every $\epsilon > 0$, there is a family of probability measure $(\mathbb{Q}_w^\epsilon)_{w\in\Omega}$ such that $w \mapsto \mathbb{Q}_w^\epsilon$ is $\mathcal{F}_\sigma$-measurable and for $\mathbb{P}$-a.e. $w \in \Omega$,

$$V_{\sigma(w)}(w) \leq \mathbb{E}^{\mathbb{Q}_w^\epsilon}\left[y_{\sigma(w)}^{\mathbb{Q}_w^\epsilon}\right] + \epsilon,\ \mathbb{P}\text{-a.s.}$$

Let us now define the concatenated probability $\mathbb{P}^\epsilon := \mathbb{P} \otimes \mathbb{Q}^\epsilon_\cdot$ so that $\mathbb{P}^\epsilon \in \mathcal{P}_0(\sigma, \mathbb{P}, \mathbb{F})$, it follows then by Lemma 3.4.6 that

$$V_\sigma \leq \mathbb{E}^{\mathbb{P}^\epsilon}\left[y^{\mathbb{P}^\epsilon}_\sigma(\tau, y^{\mathbb{P}^\epsilon}_\tau)\middle|\mathcal{F}_\tau\right] + \epsilon \leq \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_0(\sigma,\mathbb{P},\mathbb{F})} \mathbb{E}^{\mathbb{P}'}\left[y^{\mathbb{P}'}_\sigma(\tau, y^{\mathbb{P}'}_\tau)\middle|\mathcal{F}_\tau\right] + \epsilon, \ \mathbb{P}\text{-a.s.}$$

We hence finish the proof of the first equality by arbitrariness of $\epsilon > 0$.

Let us now prove the second equality. Let $r^1_n \in \mathbb{Q} \cap (t, T]$, $r^1_n \downarrow t$. By the first part of the proof, we have

$$V_{r^1_n} = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_0(r^1_n,\mathbb{P},\mathbb{F})} \mathbb{E}^{\mathbb{P}'}\left[y^{\mathbb{P}'}_{r^1_n}(T, \xi)\middle|\mathcal{F}_{r^1_n}\right], \ \mathbb{P}\text{-a.s.}$$

Since for every $n \in \mathbb{N}, \mathcal{P}_0(r^1_n, \mathbb{P}, \mathbb{F}) \subset \mathcal{P}_0(t, \mathbb{P}, \mathbb{F}_+)$, we deduce as above that for any $\mathbb{P}' \in \mathcal{P}_0(t, \mathbb{P}, \mathbb{F}_+)$ and for n large enough

$$V_{r^1_n} \geq \mathbb{E}^{\mathbb{P}'}\left[y^{\mathbb{P}'}_{r^1_n}(T, \xi)\middle|\mathcal{F}_{r^1_n}\right], \ \mathbb{P}\text{-a.s.}$$

Arguing exactly as in the proof of Lemma 3.4.7, we can let n go to $+\infty$ to obtain

$$V^+_t \geq y^{\mathbb{P}'}_t(T, \xi), \ \mathbb{P}\text{-a.s.},$$

which implies by arbitrariness of $\mathbb{P}'$

$$V^+_t \geq \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} y^{\mathbb{P}'}_t(T, \xi), \ \mathbb{P}\text{-a.s.}.$$

We claim next that for any $n \in \mathbb{N}$, the following family is upward directed

$$\left\{\mathbb{E}^{\mathbb{P}'}\left[y^{\mathbb{P}'}_{r^1_n}(T, \xi)\middle|\mathcal{F}_{r^1_n}\right], \ \mathbb{P}' \in \mathcal{P}_0(r^1_n, \mathbb{P}, \mathbb{F})\right\}.$$

Indeed this can be proved exactly as in Step 3 of Theorem 3.3.1. According to Neveu [74], we then know that there exists some sequence $(\mathbb{P}^m_n)_{m\geq 0} \subset \mathcal{P}_0(r^1_n, \mathbb{P}, \mathbb{F})$ such that

$$V_{r^1_n} = \lim_{m\to+\infty} \uparrow \mathbb{E}^{\mathbb{P}^m_n}\left[y^{\mathbb{P}^m_n}_{r^1_n}(T, \xi)\middle|\mathcal{F}_{r^1_n}\right], \ \mathbb{P}\text{-a.s.}$$

By dominated convergence(recall that the $y^{\mathbb{P}}$ are in $\mathbb{D}^p_0(\mathbb{F}^{\mathbb{P}+}, \mathbb{P})$, with a norm independent of $\mathbb{P}$, by 3.3.10), the above convergence also holds for the $\mathbb{L}^{\tilde{p}}_0(\mathbb{P})$-norm, for any $1 < \tilde{p} < p$. By the stability result of Proposition A.4.1 and the monotone convergence theorem, we deduce that

$$\begin{aligned}
y^{\mathbb{P}}_t(r^1_n, V_{r^1_n}) &= y^{\mathbb{P}}_t\left(r^1_n, \lim_{m\to+\infty} \uparrow \mathbb{E}^{\mathbb{P}^m_n}\left[y^{\mathbb{P}^m_n}_{r^1_n}(T, \xi)\middle|\mathcal{F}_{r^1_n}\right]\right), \ \mathbb{P}\text{-a.s.} \\
&= \lim_{m\to+\infty} y^{\mathbb{P}}_t\left(r^1_n, \mathbb{E}^{\mathbb{P}^m_n}\left[y^{\mathbb{P}^m_n}_{r^1_n}(T, \xi)\middle|\mathcal{F}_{r^1_n}\right]\right), \ \mathbb{P}\text{-a.s.} \\
&= \lim_{m\to+\infty} y^{\mathbb{P}^m_n}_t\left(r^1_n, \mathbb{E}^{\mathbb{P}^m_n}\left[y^{\mathbb{P}^m_n}_{r^1_n}(T, \xi)\middle|\mathcal{F}_{r^1_n}\right]\right), \ \mathbb{P}\text{-a.s.} \\
&= \lim_{m\to+\infty} y^{\mathbb{P}^m_n}_t\left(r^1_n, y^{\mathbb{P}^m_n}_{r^1_n}(T, \xi)\right), \ \mathbb{P}\text{-a.s.} \\
&= \lim_{m\to+\infty} y^{\mathbb{P}^m_n}_t(T, \xi), \ \mathbb{P}\text{-a.s.} \\
&\leq \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} y^{\mathbb{P}'}_t(T, \xi), \ \mathbb{P}\text{-a.s.}
\end{aligned}$$

where we have used in the third equality the fact that $\mathbb{P}^m_n$ coincides with $\mathbb{P}$ on $\mathcal{F}_{r^1_n}$ and that $y^{\mathbb{P}}_t$ is $\mathcal{F}^+_t$-measurable, Lemma 4.4.3 in the fourth equality, and the dynamic programming principle for RBSDEs in the fifth equality.

Finally, it remains to let n to to $+\infty$ and to use Lemma A.4.1 (together with Lemma A.2.2) to obtain the desired equality, from which we deduce exactly as in the proof of Proposition 3.3.5 that $V^+ \in \mathbb{D}_0^p(\mathbb{F}^{\mathcal{P}_0+})$. □

The next result shows that V^+ is actually a semi-martingale under any $\mathbb{P} \in \mathcal{P}_0$, gives its decomposition and deduce the existence of a solution to the 2RBSDE (3.2.5)

Lemma 3.4.10. *Let Assumptions 3.2.1 and 3.3.1 hold. For any $\mathbb{P} \in \mathcal{P}_0$, there exists $(Z^{\mathbb{P}}, M^{\mathbb{P}}, K^{\mathbb{P}}) \in \mathbb{H}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P}) \times \mathbb{M}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P}) \times \mathbb{I}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P})$ such that*

$$V_t^+ = \xi + \int_t^T \widehat{f}_s^{\mathbb{P}}(V_s^+, \widehat{a}^{1/2} Z_s^{\mathbb{P}})ds - \int_t^T Z_s^{\mathbb{P}} \cdot dX_s^{c,\mathbb{P}} - \int_t^T dM_s^{\mathbb{P}} + \int_t^T dK_s^{\mathbb{P}},\ t \in [0,T], \mathbb{P}\text{-a.s.}$$

Moreover, there is some $\mathbb{F}^{\mathcal{P}_0}$-predictable process Z which aggregates the family $(Z^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}$ and the quadruple $(V^+, Z, (M^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}, (K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$ is solution to the 2RBSDE (3.2.5).

Proof. The proof will be divided into three steps. The first step will be devote to the semi-martingale decomposition of V^+, in the second one, we will justify the aggregation of the family $(Z^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}$ and finally we show that the quadruple is solution to the 2RBSDE (3.2.5)

(i) Fix some $\mathbb{P} \in \mathcal{P}_0$. Consider the following reflected BSDE on the enlarged space. For $0 \leq t \leq T, \mathbb{P}$-a.s.

$$\begin{cases} \check{y}_t^{\mathbb{P}} = \xi + \int_t^T \widehat{f}_s^{\mathbb{P}}(\check{y}_s^{\mathbb{P}}, \widehat{a}_s^{1/2}\check{z}_s^{\mathbb{P}})ds - \int_t^T \check{z}_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_t^T d\check{m}_s^{\mathbb{P}} + \int_t^T d\check{k}_s^{\mathbb{P}} \\ \check{y}_t^{\mathbb{P}} \geq V_t^+, \\ \int_0^T (\check{y}_{t^-}^{\mathbb{P}} - V_{t^-}^+) d\check{k}_t^{\mathbb{P}} = 0. \end{cases}$$

By Theorem 3.1 in [14], this RBSDE is well posed and $\check{y}^{\mathbb{P}}$ is càdlàg. We claim that $\check{y}^{\mathbb{P}} = V^+$, $\mathbb{P} \otimes \mathbb{P}_0$-a.s. Indeed, we argue by contradiction, and assume without loss of generality that $\check{y}^{\mathbb{P}} > V_0^+$. For each $\epsilon > 0$, denote $\tau_\epsilon := \inf\{t : \check{y}_t^{\mathbb{P}} \leq V_t^+ + \epsilon\}$. Then τ_ϵ is an $\mathbb{F}_+$-stopping time and $\check{y}_{t^-}^{\mathbb{P}} \geq V_{t^-}^+ + \epsilon > V_{t^-}^+$ for all $t \leq \tau_\epsilon$. Thus $\check{k}_t^{\mathbb{P}} = \check{k}_{\tau_\epsilon}^{\mathbb{P}}$, $\mathbb{P}$-a.s. for $0 \leq t \leq \tau_\epsilon$ and thus

$$\check{y}_t^{\mathbb{P}} = \check{y}_{\tau_\epsilon}^{\mathbb{P}} + \int_t^{\tau_\epsilon} \widehat{f}_s^{\mathbb{P}}(\check{y}_s^{\mathbb{P}}, \widehat{a}_s^{1/2}\check{z}_s^{\mathbb{P}})ds - \int_t^{\tau_\epsilon} \check{z}_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_t^{\tau_\epsilon} d\check{m}_s^{\mathbb{P}},\ \mathbb{P} \otimes \mathbb{P}_0\text{-a.s.}$$

Therefore,

$$\begin{aligned} \check{y}_t^{\mathbb{P}} - y_t^{\mathbb{P}}(\tau_\epsilon, V_{\tau_\epsilon}^+) = \check{y}_{\tau_\epsilon}^{\mathbb{P}} - \widehat{Y}_{\tau_\epsilon}^+ + \int_t^{\tau_\epsilon} \{\widehat{f}_s^{\mathbb{P}}(\check{y}_s^{\mathbb{P}}, \widehat{a}_s^{1/2}\check{z}_s^{\mathbb{P}}) - \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}})\} ds \\ - \int_t^{\tau_\epsilon} (\check{z}_s^{\mathbb{P}} - z_s^{\mathbb{P}}) \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_t^{\tau_\epsilon} d(\check{m}_s^{\mathbb{P}} - m_s^{\mathbb{P}}) - (k_{\tau_\epsilon}^{\mathbb{P}} - k_t^{\mathbb{P}}). \end{aligned}$$

where $(y^{\mathbb{P}}, y^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$ is a solution to the RBSDE (3.2.9). Similar linearisation argument that we we used in Step 1 of proof of Proposition A.4.1 implies that there exists two processes λ and η such that

$$\widehat{f}_s^{\mathbb{P}}(\check{y}_s^{\mathbb{P}}, \widehat{a}_s^{1/2}\check{z}_s^{\mathbb{P}}) - \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}}) = \lambda_s(\check{y}_s^{\mathbb{P}} - y_s^{\mathbb{P}}) + \eta_s \cdot \widehat{a}_s^{1/2}(\check{z}_s^{\mathbb{P}} - z_s^{\mathbb{P}})$$

and

$$\begin{aligned} \check{y}_0^{\mathbb{P}} - y_0^{\mathbb{P}} = e^{\int_0^{\tau_\epsilon} \lambda_s ds}(\check{y}_{\tau_\epsilon}^{\mathbb{P}} - V_{\tau_\epsilon}^+) - \int_0^{\tau_\epsilon} e^{\int_0^s \lambda_s ds}(\check{z}_s^{\mathbb{P}} - z_s^{\mathbb{P}}) \cdot \widehat{a}_s^{1/2}\{dW_s^{\mathbb{P}} - \eta_s ds\} \\ - \int_0^{\tau_\epsilon} e^{\int_0^s \lambda_s ds} d(\check{m}_s^{\mathbb{P}} - m_s^{\mathbb{P}}) - \int_0^{\tau_\epsilon} e^{\int_0^s \lambda_s ds} dk_s^{\mathbb{P}}. \end{aligned}$$

Then, there exists a probability measure $\mathbb{Q}_{\mathbb{P}}$ equivalent to $\mathbb{P}$ such that

$$\begin{aligned}\check{y}_0^{\mathbb{P}} - y_0^{\mathbb{P}} &= \mathbb{E}^{\mathbb{Q}_{\mathbb{P}}}\Big[e^{\int_0^{\tau_\epsilon}\lambda_s ds}(\check{y}_{\tau_\epsilon}^{\mathbb{P}} - V_{\tau_\epsilon}^+) - \int_0^{\tau_\epsilon} e^{\int_0^s \lambda_s ds} d\tilde{k}_s^{\mathbb{P}}\Big] \\ &\leq \mathbb{E}^{\mathbb{Q}_{\mathbb{P}}}\big[e^{\int_0^{\tau_\epsilon}\lambda_s ds}(\check{y}_{\tau_\epsilon}^{\mathbb{P}} - V_{\tau_\epsilon}^+)\big] \\ &\leq C\mathbb{E}^{\mathbb{Q}_{\mathbb{P}}}\big[\check{y}_{\tau_\epsilon}^{\mathbb{P}} - V_{\tau_\epsilon}^+\big] \leq C\epsilon.\end{aligned}$$

where $C > 0$ is a constant only depending on the Lipschitz constant of f. Note that this equivalent to $\check{y}_0^{\mathbb{P}} \leq y_0^{\mathbb{P}}(\tau_\epsilon, V_{\tau_\epsilon}^+) + C\epsilon$. However by Lemma 3.4.7, we know that $y_0^{\mathbb{P}}(\tau_\epsilon, V_{\tau_\epsilon}^+) \leq V_0^+$, which contradicts the fact that $\check{y}_0^{\mathbb{P}} > V_0^+$.
For some $(Z^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0} \subset \mathbb{H}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P})$, and $(M^{\mathbb{P}}, K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0} \subset \mathbb{M}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P}) \times \mathbb{I}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P})$

$$V_t^+ = \xi + \int_t^T \widehat{f}_s^{\mathbb{P}}(V_s^+, \widehat{a}^{1/2} Z_s^{\mathbb{P}})ds - \int_t^T Z_s^{\mathbb{P}} \cdot dX_s^{c,\mathbb{P}} - \int_t^T dM_s^{\mathbb{P}} + \int_t^T dK_s^{\mathbb{P}},\ t \in [0,T], \mathbb{P}\text{-a.s.}$$

(ii) By Karandikar [53], since V^+ is a càdlàg semi-martingale, we can define a universal process denote by $\langle V^+, X\rangle$ which coincides with the quadratic co-variation of V^+ and X under each probability $\mathbb{P} \in \mathcal{P}_0$. In particular, the process $\langle V^+, X\rangle$ is $\mathcal{P}_0$-quasi-surely continuous and hence is $\mathbb{F}^{\mathcal{P}_0+}$-predictable (or equivalently $\mathbb{F}^{\mathcal{P}_0}$-predictable). Similar to the proof of Theorem 2.4 in [75], we can then define a universal $\mathbb{F}^{\mathcal{P}_0}$-predictable process Z by

$$Z_t := \widehat{a}_t^{\oplus} \frac{d\langle V^+, X\rangle_t}{dt},$$

where $\widehat{a}_t^{\oplus}$ represents the Moore-Penrose pseudoinverse of $\widehat{a}_t$. In particular, Z aggregates the family $(Z^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}$

(iii) Fix $t \in [0,T]$ and $\mathbb{P} \in \mathcal{P}_0$. According to the two previous steps, it remains to show that the families $(K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}$ and $(k^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}$ satisfies the minimality condition (3.2.8). Let us denote for any $\mathbb{P}' \in \mathcal{P}_0(t, \mathbb{P}, \mathbb{F}_+)$,

$$\delta V_t^+ := V_t^+ - y_t^{\mathbb{P}'} \quad \text{and} \quad \delta K_t^{\mathbb{P}'} := K_t^{\mathbb{P}'} - k_t^{\mathbb{P}'}.$$

By (3.3.4), we have

$$\delta V_t^+ = \mathbb{E}^{\mathbb{P}'}\Big[\int_t^T G_s^{t,\mathbb{P}'} d\delta K_s^{\mathbb{P}'} \big| \mathcal{F}_t^+\Big].$$

where $G^{t,\mathbb{P}'}$ is defined in Remark (3.2.3). Then taking the essential infimum over $\mathbb{P}' \in \mathcal{P}_0(t, \mathbb{P}, \mathbb{F}_+)$ in the above equality, we get

$$\operatorname*{ess\,inf}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}\Big[\int_t^T G_s^{t,\mathbb{P}'} d(K_s^{\mathbb{P}'} - k_s^{\mathbb{P}'}) \big| \mathcal{F}_t^+\Big] = V_t^+ - \operatorname*{ess\,sup}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} y_t^{\mathbb{P}'}$$

Finally the minimum condition is obtained by using the representation formula of V^+ established in Lemma 3.4.9 and this concludes the proof.

□

Chapter 4

Second order Reflected BSDEs under weak assumptions: the case of an upper obstacle

Contents

4.1 Introduction

In this chapter, we consider the 2RBSDEs with an upper obstacle, that is, we study the case where the solution is forced to stay below a given stochastic process. Following Soner, Touzi and Zhang [91], these solutions are closely linked to the standard RBSDEs with an upper obstacle.

Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ generated by a Brownian motion W, a solution to a RBSDE with an upper obstacle $(U_t)_{t\in[0,T]}$ associated to a generator f and a

terminal value ξ is a triple of progressively measurable processes (Y, Z, K) such that

$$
\begin{aligned}
&Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds - K_T + K_t - \int_t^T Z_s dW_s, \ \mathbb{P}\text{-a.s.} \\
&Y_t \leq U_t, \ t \in [0, T], \ \mathbb{P}\text{-a.s.} \\
&\int_0^T (U_s - Y_s)dK_s = 0, \ \mathbb{P}\text{-a.s.}
\end{aligned}
$$

The increasing process K plays the role to push the solution downwards, so that it may remain below the obstacle. The last condition is known as the Skorokhod condition and guarantees that the process K acts in a minimal way, that is to say only when the process Y reaches the obstacle U. In comparison with the definition of the RBSDEs with a lower obstacle given in section 1.1.3.1, one can observe that in the theory of standard BSDEs, the reflection problem with obstacle is symmetric. In other words, for the lower case, a non decreasing process is added by addition in order to maintain the solution above the obstacle and for the upper case, a non decreasing process is added by subtraction in order to maintain the solution below the obstacle.

Following the wellposedness of 2BSDEs of [91], Matoussi, Possamai and Zhou in [66, 68] introduced 2RBSDEs with a càdlàg lower obstacle. They represented a solution of a 2RBSDE as a supremum of standard RBSDEs with the same generator, terminal condition and lower obstacle, but written under the different probability measures. A non decreasing process K has been added by addition and this process plays two roles. The first is to maintain the solution of the 2RBSDE above the obstacle and the second is to maintain the solution of 2RBSDE above all the solutions of standard RBSDEs associated.
Similarly a solution of a 2RBSDE with an upper obstacle must be at the same time below the obstacle and above standard RBSDEs with the same upper obstacle, generator and terminal condition. These two effects start to counterbalance each other and the situation changes drastically. This is clearly explained in [66] that unlike with classical RBSDEs, considering a lower obstacle in the context of second order is fundamentally different from considering an upper obstacle. Therefore, in contrast to the case of lower obstacle, one needs to add by subtraction a non decreasing process to push the solution downwards and to add by addition a non decreasing process to ensure that this solution stays above the standard RBSDEs associated which at the end is equivalent to add by addition a finite variation process.

It is worth pointing out again that the existence and uniqueness result of [91] and [66] have been obtained under uniform continuity assumption on the terminal condition, the generator and the obstacle. Since the existence of the solution is essentially based on the dynamic programming principle which needs the measurability of the value function, this regularity condition allowed to establish the uniform continuity of the value function.
Recently considering the optimization over a set of non-dominated probability measures of solutions of BSDEs, Possamai, Tan and Zhou proved a dynamic programming principle for this stochastic control problem using selection measurable argument. The authors obtained wellposedness result for 2BSDEs which does not require any regularity assumption on the terminal condition and the generator. In the same way as Chapter 3, our goal is to prove existence and uniqueness to 2RBSDEs with an upper obstacle in a general filtration without any regularity condition on generator, terminal condition and obstacle.

This work is carried out in the same approach as chapter 3 and then we will keep same notations and results obtained in this chapter. The remainder of this chapter is organized as follows. First, we give the formulation and the definition of 2RBSDEs, then the following section is dedicated to representation and uniqueness of the solution. The last section is devoted to the existence of the solution using DPP.

4.2 Formulation

4.2.1 Notation, spaces and norms

We consider a fixed $T > 0$ and $d \in \mathbb{N}$. Let $\Omega = \{\omega \in C([0,T], \mathbb{R}^d) : \omega_0 = 0\}$ be the canonical space equipped with the uniform convergence norm $\|\omega\|_\infty = \sup_{0\le t\le T} \|\omega_t\|$, X the canonical process, i.e. $X_t(\omega) = \omega_t$ for all $\omega \in \Omega$ and $\mathbb{P}_0$ the Wiener measure on Ω under which X is a Brownian motion. We denote by $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ the canonical filtration, $\mathbb{F}_+ = (\mathcal{F}_t^+)_{0\le t\le T}$ the right limit of $\mathbb{F}$ with $\mathcal{F}_t^+ := \cap_{s>t}\mathcal{F}_s$ for all $t \in [0,T)$ and $\mathcal{F}_T^+ = \mathcal{F}_T$.

We keep all the notations, the space of models, assumptions and remarks, spaces and norms introduced from section 3.2.1 to section 3.2.3. In addition, for a fixed $t \in [0,T]$ and $\omega \in \Omega$ we define the following spaces where $\mathbb{X} := (\mathcal{X}_s)_{t\le s\le T}$ denotes an arbitrary filtration on $(\Omega, \mathcal{F}_T)$, $\mathbb{P}$ denote arbitrary element in $\mathcal{P}(t,\omega)$ and $\mathbb{X}_\mathbb{P}$ denote the $\mathbb{P}$-augmented filtration associated to $\mathbb{X}$:

– For $p > 1$, we introduce $\mathbb{V}^p_{t,\omega}(\mathbb{X},\mathbb{P})$ the set of all $\mathbb{X}$-progressively measurable processes K with $\mathbb{P}$-a.s. càdlàg and bounded variation paths on $[t,T]$ with $K_0 = 0$, $\mathbb{P}$-a.s. and

$$\|K\|^p_{\mathbb{V}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^\mathbb{P}[(\mathrm{Var}_{t,T}(K))^p] < +\infty.$$

In the above $\mathrm{Var}_{t,T}(K)$ denotes the total variation of K on $[t,T]$.

– We will say that a family $(K^\mathbb{P})_{\mathbb{P}\in\mathbb{P}(t,\omega)}$ belongs to $\mathbb{V}^p_{t,\omega}((\mathbb{X}_\mathbb{P})_{\mathbb{P}\in\mathcal{P}(t,\omega)})$ if for any $\mathbb{P} \in \mathcal{P}(t,\omega)$, $K^\mathbb{P} \in \mathbb{V}^p_{t,\omega}(\mathbb{X},\mathbb{P})$ and

$$\sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \left\|K^\mathbb{P}\right\|_{\mathbb{V}^p_{t,\omega}(\mathbb{P})} < +\infty.$$

4.2.2 Formulation

Our upper obstacle is represented by the process U. We will assume that $(U_t)_{t\in[0,T]}$ is càdlàg and $U \in \mathbb{D}^p_0(\mathbb{F}_+^{\mathcal{P}_0})$. The terminal condition ξ and the generator f are introduced in section 3.2.2. The following formulation is related to the universal filtration $\mathbb{F}_+^{\mathcal{P}_0}$ and remember that every $\mathbb{P} \in \mathcal{P}_0$ doesn't a priori satisfy the martingale representation property, then we follow [85] and for every $\mathbb{P} \in \mathcal{P}_0$, we consider a 2BSDE driven by the $\mathbb{P}$-martingale part $X^{c,\mathbb{P}}$ of X reflected to the upper obstacle U given by

$$\begin{cases} Y_t = \xi + \int_t^T \widehat{f}^\mathbb{P}_s(Y_s, \widehat{a}_s^{1/2} Z_s)ds - \int_t^T Z_s \cdot dX^{c,\mathbb{P}}_s - \int_t^T dM^\mathbb{P}_t + K^\mathbb{P}_T - K^\mathbb{P}_t, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le t \le T,\ \mathbb{P}\text{-a.s.},\ \forall \mathbb{P} \in \mathcal{P}_0, \\ Y_t \le U_t, 0 \le t \le T,\ \mathbb{P}\text{-a.s.},\ \forall \mathbb{P} \in \mathcal{P}_0, \end{cases} \tag{4.2.1}$$

where for every $\mathbb{P} \in \mathcal{P}_0$, $M^\mathbb{P}$ is a $\mathbb{P}$-martingale null at 0 orthogonal to $\mathbb{X}^{c,\mathbb{P}}$. This additional martingale comes from the fact that this equation are with respect to a general filtration, more details about the definition of RBSDEs in general filtration are given in section 2.5.

For any $\mathbb{P} \in \mathcal{P}_0$ and a $\mathcal{F}_T$-measurable random variable $\xi \in \mathbb{L}^p_0(\mathbb{F})$, let $(y^\mathbb{P}, z^\mathbb{P}, m^\mathbb{P}, k^\mathbb{P}) := (y^\mathbb{P}(T,\xi), z^\mathbb{P}(T,\xi), m^\mathbb{P}(T,\xi), k^\mathbb{P}(T,\xi))$ denote the solution to the following standard RBSDE with upper obstacle U:

$$\begin{cases} y^\mathbb{P}_t = \xi + \int_t^T \widehat{f}^\mathbb{P}_s(y^\mathbb{P}_s, \widehat{a}_s^{1/2} z^\mathbb{P}_s)ds - \int_t^T z^\mathbb{P}_s \cdot dX^{c,\mathbb{P}}_s - \int_t^T dm^\mathbb{P}_s - k^\mathbb{P}_T + k^\mathbb{P}_t,\ 0 \le t \le T,\ \mathbb{P}\text{-a.s.} \\ y^\mathbb{P}_t \le U_t,\ t \in [0,T],\ \mathbb{P}\text{-a.s.}, \\ \int_0^T (U_{s^-} - y^\mathbb{P}_{s^-}) dk^\mathbb{P}_s = 0,\ \mathbb{P}\text{-a.s.} \end{cases} \tag{4.2.2}$$

where $m^{\mathbb{P}}$ is a martingale orthogonal to X under $\mathbb{P}$. Bouchard et al in [14](see Theorem 3.1) have proved existence and uniqueness of a solution to the reflected BSDEs (4.2.2) with $(y_t^{\mathbb{P}}, z_t^{\mathbb{P}}, m_t^{\mathbb{P}}, k_t^{\mathbb{P}})_{t\in[0,T]} \in \mathbb{D}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P}) \times \mathbb{H}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P}) \times \mathbb{M}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P}) \times \mathbb{I}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P})$ satisfying equation(4.2.2) under $\mathbb{P}$.

Remark 4.2.1. *Bouchard et al in [14] proved existence and uniqueness of RBSDEs with lower obstacle in a general filtration. Since in the theory of standard BSDEs there is a symmetry between lower and upper obstacles, this is sufficient to deduce that this result is still true when the obstacle is an upper one. For more details about existence of these solutions (about upper and lower obstacles) via penalization, we refer the reader to Appendix section A.3.*

We define a 2RBSDE with the upper obstacle U with respect to the filtration $\mathbb{F}_+^{\mathcal{P}_0}$ as follow: for fixed $p > 1$,

Definition 4.2.1. *We say that $(Y, Z) \in \mathbb{D}_0^p(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{H}_0^p(\mathbb{F}_+^{\mathcal{P}_0})$ is a solution to the 2RBSDE (4.2.1) if :*

(i) $Y_T = \xi$, *and* $Y_t \leq U_t$, $t \in [0,T]$, $\mathcal{P}_0$*-q.s.;*

(ii) $\forall \mathbb{P} \in \mathcal{P}_0$, *the process $K^{\mathbb{P}}$ defined below has paths of bounded variation $\mathbb{P}$-a.s.*

$$K_t^{\mathbb{P}} := Y_0 - Y_t - \int_0^t \widehat{f}_s^{\mathbb{P}}(Y_s, \widehat{a}_s^{1/2} Z_s)ds + \int_0^t Z_s \cdot dX_s^{c,\mathbb{P}} + M_t^{\mathbb{P}},\ t \in [0,T], \tag{4.2.3}$$

(iii) We have the following minimality condition:

$$\operatorname*{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}{}^{\mathbb{P}} \mathbb{E}^{\mathbb{P}'}\Big[\int_t^T G_s^{t,\mathbb{P}'} d(K_s^{\mathbb{P}'} + k_s^{\mathbb{P}'}) \big| \mathcal{F}_{t+}^{\mathbb{P}}\Big] = 0, \quad 0 \leq t \leq T,\ \mathbb{P}\text{-a.s.}, \forall \mathbb{P} \in \mathcal{P}_0. \tag{4.2.4}$$

where for any $t \in [0,T]$ and for any $\mathbb{P} \in \mathcal{P}_0$, the process $G^{t,\mathbb{P}}$ is defined by

$$G_s^{t,\mathbb{P}} := \exp\Big(\int_t^s (\lambda_u^{\mathbb{P}} - \frac{1}{2}\|\eta_u^{\mathbb{P}}\|^2)(Y_u, y_u^{\mathbb{P}}, Z_u, z_u^{\mathbb{P}})du + \int_t^s \eta_u^{\mathbb{P}}(Y_u, y_u^{\mathbb{P}}, Z_u, z_u^{\mathbb{P}}) \cdot dW_u^{\mathbb{P}}\Big).$$

The processes $\lambda^{\mathbb{P}}$ and $\eta^{\mathbb{P}}$ are introduced in Remark 3.2.3(2)

Notation: $\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+) := \{\mathbb{P}' \in \mathcal{P}_0 : \mathbb{P} = \mathbb{P}'$ *on* $\mathcal{F}_{t+}\}$.

Remark 4.2.2.

1. *Rigorously, the solution is $(Y, Z, (M^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}, (K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}) \in \mathbb{D}_0^p(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{H}_0^p(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{M}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}) \times \mathbb{V}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$ and through misuse of language, we denote (Y, Z), given the dependence in $\mathbb{P}$ of $K^{\mathbb{P}}$ and $M^{\mathbb{P}}$.*

2. *For the same reasons as in Chapter 3, we use the review minimality condition introduced by Matoussi, Possamai and Zhou [68].*

3. *Using the above definition, the for any $\mathbb{P} \in \mathcal{P}_0$, $K^{\mathbb{P}} - M^{\mathbb{P}}$ is a semimartingale defined by*

$$K_t^{\mathbb{P}} - M_t^{\mathbb{P}} := Y_0 - Y_t - \int_0^t \widehat{f}_s^{\mathbb{P}}(Y_s, \widehat{a}_s^{1/2} Z_s)ds + \int_0^t Z_s \cdot dX_s^{c,\mathbb{P}},\ t \in [0,T],$$

Using recent results of Nutz [76], under additional assumptions (related to axiomatic set theory) the family of semimartingales $(K^{\mathbb{P}} - M^{\mathbb{P}})_{\mathbb{P}}$ can always be aggregated into a universal semimartingale $K - M$. Then by the uniqueness of decomposition of semimartingales, the processes $K^{\mathbb{P}}$ and $M^{\mathbb{P}}$ can be aggregated into processes K and M.

For the same reasons mentioned in chapter 3, it is more advantageous for us to work with a Brownian motion, then throughout the rest of this chapter, we consider the following RBSDE defined using $W^{\mathbb{P}}$ a $\mathbb{P}$-Brownian motion on Ω, which is equivalent to the RBSDE 4.2.2,

$$\begin{cases} y_t^{\mathbb{P}} = \xi + \int_t^T \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}})ds - \int_t^T z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_t^T dm_s^{\mathbb{P}} - k_T^{\mathbb{P}} + k_t^{\mathbb{P}},\ 0 \le t \le T,\ \mathbb{P}\text{-a.s.} \\ y_t^{\mathbb{P}} \le U_t,\ t \in [0,T],\ \mathbb{P}\text{-a.s.}, \\ \int_0^T (U_{t^-} - y_{t^-}^{\mathbb{P}}) dk_t^{\mathbb{P}} = 0,\ \mathbb{P}\text{-a.s.} \end{cases} \tag{4.2.5}$$

Remark 4.2.3. *The definition 4.2.1 differs from the definition 3.2.1 in the meaning that the process $K^{\mathbb{P}}$ has bounded variation. Therefore, $K^{\mathbb{P}}$ is the difference of two non decreasing processes. The first non decreasing process exactly as with 2BSDE framework forces the process Y to stay above all the $y^{\mathbb{P}}$ and the second one pushes the process Y to stay below the obstacle U. We notice that as shown in Example **1** in Introduction section 1.1.2.2, 2BSDE are a natural generalization of the G-expectation introduced by Peng [78] which is an example of sublinear expectation. One can also refer the reader to the paper by Pham and Zhang [80], whose problematic are strongly connected to 2RBSDEs. They study some norm estimates for semimartingales in the context of linear and sublinear expectations, and point out that there is a fundamental difference between non-linear submartingales and supermartingales. Translated in our framework, and using the intuition from the classical RBSDE theory, when the generator is equal to 0, a 2RBSDE with a lower obstacle should be a non-linear supermartingale, while a 2RBSDE with an upper obstacle should be a non-linear submartingale.*

4.3 Uniqueness of the solution and others properties

Following [66] and [85] in addition to Assumption 3.2.1, we will always assume the following in order to prove uniqueness of the solution to the 2RBSDE (4.2.1).

Assumption 4.3.1. *For fixed $p > 1$, there is some $\kappa \in (1, p]$ such that the following integrability conditions are statisfied:*

$$\phi_f^{p,\kappa} := \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{0\le t\le T}{}^{\mathbb{P}}\left(\mathbb{E}_t^{\mathcal{P}_0,\mathbb{P}}\left[\int_0^T |\widehat{f}_s^{\mathbb{P},0}|^\kappa ds\right]\right)^{\frac{p}{\kappa}}\right] < +\infty, \tag{4.3.1}$$

$$\psi_U^{p,\kappa} := \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{0\le t\le T}{}^{\mathbb{P}}\left(\mathbb{E}_t^{\mathcal{P}_0,\mathbb{P}}\left[\sup_{0\le s\le T} |U_s|^\kappa\right]\right)^{\frac{p}{\kappa}}\right] < +\infty, \tag{4.3.2}$$

4.3.1 Representation and uniqueness of the solution

We have similarly as in Theorem 3.3.1 of Chapter 3 , the following representation:

Theorem 4.3.1. *Let Assumptions 3.2.1 and 4.3.1 hold. Let $\xi \in \mathbb{L}_0^{p,\kappa}$ and (Y,Z) be a solution to the 2RBSDE (4.2.1). For any $\mathbb{P} \in \mathcal{P}_0$, let $(y_s^{\mathbb{P}}, z_s^{\mathbb{P}}, m_s^{\mathbb{P}}, k^{\mathbb{P}})_{s\in[t,T]} \in \mathbb{D}_0^p(\mathbb{F}_+^{\mathbb{P}},\mathbb{P}) \times \mathbb{H}_0^p(\mathbb{F}_+^{\mathbb{P}},\mathbb{P}) \times \mathbb{M}_0^p(\mathbb{F}_+^{\mathbb{P}},\mathbb{P}) \times \mathbb{I}_0^p(\mathbb{F}_+^{\mathbb{P}},\mathbb{P})$ be the solutions to the corresponding RBSDEs (4.2.2). Then for any $\mathbb{P} \in \mathcal{P}_0$ and $0 \le t \le T$,*

$$Y_t = \operatorname*{ess\,sup}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}{}^{\mathbb{P}} y_t^{\mathbb{P}'},\ \mathbb{P}\text{-a.s.} \tag{4.3.3}$$

Thus, the 2RBSDE (4.2.1) has at most one solution in $\mathbb{D}_0^p(\mathbb{F}_+^{\mathbb{P}}) \times \mathbb{H}_0^p(\mathbb{F}_+^{\mathbb{P}})$.

Proof. The proof is the same as the proof of Theorem 3.3.1 of Chapter 3, so we only mention the slight difference due to the fact that the minimum condition is about $K^{\mathbb{P}} + k^{\mathbb{P}}$. The rest of the proof is the same with $\delta K = K^{\mathbb{P}} + k^{\mathbb{P}}$ instead of $\delta K = K^{\mathbb{P}} - k^{\mathbb{P}}$. □

Remark 4.3.1. *To get the following dynamic programming representation,*

$$Y_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} y_t^{\mathbb{P}'}(s, Y_s), \quad 0 \le t \le s \le T, \quad \mathbb{P}\text{-}a.s.$$

it suffices to extend the minimality condition (4.2.4) to the following:

$$\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}\Big[\int_t^s G_u^{t,\mathbb{P}'} d(K_u^{\mathbb{P}'} + k_u^{\mathbb{P}'}) \big| \mathcal{F}_{t+}^{\mathbb{P}}\Big] = 0, \quad 0 \le t \le s \le T, \ \mathbb{P}\text{-}a.s., \forall \mathbb{P} \in \mathcal{P}_0.$$

4.3.2 More details about the action of $K^{\mathbb{P}}$

In this section we study the action of $K^{\mathbb{P}}$ under $\mathbb{P}$ relatively to the position of the solution $y^{\mathbb{P}}$ of the RBSDE with respect to the obstacle.

Proposition 4.3.1. *Let Assumptions 3.2.1 and 4.3.1 hold. Assume $\xi \in \mathbb{L}_0^{p,\kappa}$ and $(Y, Z) \in \mathbb{D}_0^p(\mathbb{F}_+^{\mathbb{P}}) \times \mathbb{H}_0^p(\mathbb{F}_+^{\mathbb{P}})$ is a solution to the 2RBSDE(4.2.1). Let $\{(y^{\mathbb{P}}, z^{\mathbb{P}}, k^{\mathbb{P}}), \mathbb{P} \in \mathcal{P}_0\}$ be the solutions of the corresponding RBSDEs (4.2.2). Then we have the following decomposition for all $t \in [0, T]$ and for all $\mathbb{P} \in \mathcal{P}_0$*

$$K_t^{\mathbb{P}} = \overline{K}_t^{\mathbb{P}} - k_t^{\mathbb{P}}, \quad \mathbb{P}\text{-}a.s. \tag{4.3.4}$$

where $\overline{K}_t^{\mathbb{P}} = \int_0^t 1_{\{y_{s^-}^{\mathbb{P}} < U_{s^-}\}} dK_s^{\mathbb{P}}$, $\mathbb{P}$-a.s. is a non decreasing process satisfying the following minimum condition

$$\overline{K}_t^{\mathbb{P}} = \operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}\big[\overline{K}_T^{\mathbb{P}'} | \mathcal{F}_{t^+}^{\mathbb{P}}\big], \ \mathbb{P}\text{-}a.s. \ \forall \mathbb{P} \in \mathcal{P}_0. \tag{4.3.5}$$

Proof. Consider $\mathbb{P} \in \mathcal{P}_0$, Choose τ_1 and τ_2 two stopping times such that for all $t \in [\tau_1, \tau_2)$, $y_{t^-}^{\mathbb{P}} < U_{t^-}$, $\mathbb{P}$-a.s. On the one hand we know from the skorokhod condition that $k^{\mathbb{P}}$ does not increase between τ_1 and τ_2, this can be rewrite as

$$k_u^{\mathbb{P}} - k_t^{\mathbb{P}} = 0, \quad \mathbb{P}\text{-a.s.} \quad \text{for } \tau_1 \le t \le u < \tau_2. \tag{4.3.6}$$

Using this result, we have $\mathbb{P}$-a.s. for $\tau_1 \le t \le u < \tau_2$,

$$\begin{aligned} K_u^{\mathbb{P}} - K_t^{\mathbb{P}} = \int_t^u dK_s^{\mathbb{P}} &= \int_t^u 1_{\{y_{s^-}^{\mathbb{P}} < U_{s^-}\}} dK_s^{\mathbb{P}} + \int_t^u 1_{\{y_{s^-}^{\mathbb{P}} = U_{s^-}\}} dK_s^{\mathbb{P}} \\ &= \int_t^u 1_{\{y_{s^-}^{\mathbb{P}} < U_{s^-}\}} dK_s^{\mathbb{P}} \\ &= \int_t^u 1_{\{y_{s^-}^{\mathbb{P}} < U_{s^-}\}} dK_s^{\mathbb{P}} - (k_u^{\mathbb{P}} - k_t^{\mathbb{P}}) \\ &= \overline{K}_u^{\mathbb{P}} - \overline{K}_t^{\mathbb{P}} - (k_u^{\mathbb{P}} - k_t^{\mathbb{P}}). \end{aligned}$$

which gives the decomposition (4.3.4). On the other hand we showed in the proof of Theorem 4.3.1 that $K^{\mathbb{P}} + k^{\mathbb{P}}$ is non decreasing. Therefore, on $[\tau_1, \tau_2)$, the process $K^{\mathbb{P}}$ is non decreasing due to (4.3.6). Since $\overline{K}^{\mathbb{P}} = K^{\mathbb{P}} + k^{\mathbb{P}}$, we get that $\overline{K}^{\mathbb{P}}$ is a non decreasing process.
Now choose τ_1 and τ_2 two stopping times such that for all $t \in [\tau_1, \tau_2)$, $y_{t^-}^{\mathbb{P}} = U_{t^-}$, $\mathbb{P}$-a.s. By the definition 4.2.1, the process Y is below the obstacle U, hence $Y_t \le U_t$, $\mathbb{P}$-a.s. for all $t \in [0, T]$. The representation formula (4.3.3) shows that $Y_t \ge y_t^{\mathbb{P}}$, $\mathbb{P}$-a.s. for all $t \in [0, T]$. Combining these observations together with the fact that these processes are càdlàg yields

$$Y_{t^-} = y_{t^-}^{\mathbb{P}} = U_{t^-}, \quad t \in [\tau_1, \tau_2), \ \mathbb{P}\text{-a.s.} \tag{4.3.7}$$

This implies that for $t \in [\tau_1, \tau_2)$, $\Delta Y_t = Y_t - Y_{t^-} = Y_t - U_{t^-}$ and $\Delta y_t^{\mathbb{P}} = y_t^{\mathbb{P}} - y_{t^-}^{\mathbb{P}} = y_t^{\mathbb{P}} - U_{t^-}$. Since Y and $y^{\mathbb{P}}$ solve respectively the equations (4.2.1) and (4.2.2), we can obtain the following $\mathbb{P}$-a.s. for $\tau_1 \leq t \leq u < \tau_1$,

$$
\begin{aligned}
U_{t^-} + \Delta Y_t &= Y_u + \int_t^u \widehat{f}_s^{\mathbb{P}}(Y_s, \widehat{a}_s^{1/2} Z_s) ds - \int_t^u Z_s \cdot dX_s^{c,\mathbb{P}} - \int_t^u dM_t^{\mathbb{P}} + K_u^{\mathbb{P}} - K_t^{\mathbb{P}}, \\
U_{t^-} + \Delta y_t^{\mathbb{P}} &= y_u^{\mathbb{P}} + \int_t^u \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}}) ds - \int_t^u z_s^{\mathbb{P}} \cdot dX_s^{c,\mathbb{P}} - \int_t^u dm_s^{\mathbb{P}} - k_u^{\mathbb{P}} + k_t^{\mathbb{P}}.
\end{aligned}
$$

By identifying the martingale parts above and the fact that both $m^{\mathbb{P}}$ and $M^{\mathbb{P}}$ are orthogonal to $X^{c,\mathbb{P}}$, we get that $Z_s = z_s^{\mathbb{P}}$ and $m_s^{\mathbb{P}} = M_s^{\mathbb{P}}$, $ds \times \mathbb{P}$-a.e. Applying the same identification to the finite variation parts, we have

$$
Y_u - \Delta Y_t + \int_t^u \widehat{f}_s^{\mathbb{P}}(Y_s, \widehat{a}_s^{1/2} Z_s) ds + K_u^{\mathbb{P}} - K_t^{\mathbb{P}} = y_u^{\mathbb{P}} - \Delta y_t^{\mathbb{P}} + \int_t^u \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}}) ds - k_u^{\mathbb{P}} + k_t^{\mathbb{P}}. \tag{4.3.8}
$$

Using (4.3.7) and the fact that for all $s \in [t, u]$, $Z_s = z_s^{\mathbb{P}}$, $ds \times \mathbb{P}$-a.e, we clearly have

$$
\int_t^u \widehat{f}_s^{\mathbb{P}}(Y_s, \widehat{a}_s^{1/2} Z_s) ds = \int_t^u \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}}) ds,
$$

Moreover, since $Y_{t^-} = y_{t^-}^{\mathbb{P}} = U_{t^-}$ for all $s \in [t, u]$ and since all the processes are càdlàg, the jumps of Y and $y^{\mathbb{P}}$ are equal to the jumps of U. Therefore, we can rewrite (4.3.8) as

$$
\begin{aligned}
K_u^{\mathbb{P}} - K_t^{\mathbb{P}} &= -(k_u^{\mathbb{P}} - k_t^{\mathbb{P}}) \\
&= \int_t^u 1_{\left\{y_{s^-}^{\mathbb{P}} < U_{s^-}\right\}} dK_s^{\mathbb{P}} - (k_u^{\mathbb{P}} - k_t^{\mathbb{P}}) \\
&= \overline{K}_u^{\mathbb{P}} - \overline{K}_t^{\mathbb{P}} - (k_u^{\mathbb{P}} - k_t^{\mathbb{P}}).
\end{aligned}
$$

which gives the decomposition (4.3.4). So we have shown that in all possible case, $K^{\mathbb{P}}$ admits the decomposition (4.3.4), where $\overline{K}^{\mathbb{P}}$ is a non decreasing process. The minimum condition (4.3.5) is an immediate consequence of (4.2.4). □

Remark 4.3.2. *It is important to point out the fact that it is enough that one $y^{\mathbb{P}}$ hits U so that Y hits the obstacle U. This means that in contrast to the 2RBSDEs with a lower obstacle, at a fixed time t it is enough to find one $\mathbb{P}$ such that $y_t^{\mathbb{P}}$ coincides with U_t to conclude that Y_t coincides with U_t.*
The above result gives the decomposition of $K^{\mathbb{P}}$ as the difference of two non decreasing processes where one acts only when Y hits U, which corresponds to a standard reflection with a Skorokhod condition and the other non decreasing process only acts to push Y below the standard RBSDEs associated and this process verifies the minimum condition introduced in [91]. Hence for a fixed $\mathbb{P}$, the act of reflection on the obstacle is clearly independent to the act on the set of probabilities.
Matoussi, Possamai and Zhou [66](see also Remark 3.3.2 for more details) showed for a 2RBSDE with a lower obstacle that outside the set $\{Y_{s^-} > L_{s^-}\} \cap \{y_{s^-}^{\mathbb{P}} = L_{s^-}\}$ one can clearly decompose K as the sum of two non decreasing processes where the actions of these processes are clearly identified. Namely, one of these processes acts only when Y hits the obstacle and then verified the usual skorokhod condition for RBSDEs. The other acts enough to push Y to maintain it above all the RBSDEs associated. However, the existence of a decomposition like (4.3.4) on the whole space Ω is still an open problem.

4.3.3 Some properties of the solution

We first give the connection with optimal stopping problem.

Proposition 4.3.2. *Let (Y, Z) be a solution to the above 2RBSDE (4.2.1) and $\{(y^{\mathbb{P}}, z^{\mathbb{P}}, k^{\mathbb{P}}), \mathbb{P} \in \mathcal{P}_0\}$ be the solutions of the corresponding RBSDEs (4.2.2). Then for each $t \in [0,T]$ and for all $\mathbb{P} \in \mathcal{P}_0$,*

$$Y_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} \operatorname*{ess\,inf}_{\tau \in \mathcal{T}_t} \mathbb{E}^{\mathbb{P}'}\Big[\int_t^\tau \widehat{f}^{\mathbb{P}}_s(y^{\mathbb{P}'}_s, \widehat{a}^{1/2}_s z^{\mathbb{P}'}_s)ds + U_\tau 1_{\{\tau<T\}} + \xi 1_{\{\tau=T\}} \big| \mathcal{F}^{\mathbb{P}'}_{t^+}\Big], \quad \mathbb{P}\text{-}a.s. \tag{4.3.9}$$

$$= \operatorname*{ess\,inf}_{\tau \in \mathcal{T}_t} \mathbb{E}^{\mathbb{P}'}\Big[\int_t^\tau \widehat{f}^{\mathbb{P}}_s(Y_s, \widehat{a}^{1/2}_s Z_s)ds + \overline{K}^{\mathbb{P}}_\tau - \overline{K}^{\mathbb{P}}_t + U_\tau 1_{\{\tau<T\}} + \xi 1_{\{\tau=T\}} \big| \mathcal{F}^{\mathbb{P}}_{t^+}\Big], \quad \mathbb{P}\text{-}a.s. \tag{4.3.10}$$

Moreover, for each $\mathbb{P}$, the following stopping time is ϵ-optimal

$$D^{\mathbb{P},\epsilon}_t := \inf\Big\{v \geq t,\ y^{\mathbb{P}}_v \geq U_v - \epsilon,\ \mathbb{P}\text{-}a.s.\Big\} \wedge T.$$

Proof. Proposition 2.5.1 shows that RBSDEs can be represented as a Snell envelope. Applying this proposition to the case of upper obstacle RBSDE(4.2.2) , we get for all $\mathbb{P} \in \mathcal{P}_0$,

$$y^{\mathbb{P}}_t = \operatorname*{ess\,inf}_{v \in \mathcal{T}_t} \mathbb{E}^{\mathbb{P}}\Big[\int_t^v \widehat{f}^{\mathbb{P}}_s(y^{\mathbb{P}}_s, \widehat{a}^{1/2}_s z^{\mathbb{P}}_s)ds + U_v 1_{\{v<T\}} + \xi 1_{\{v=T\}} | \mathcal{F}^{\mathbb{P}}_{t^+}\Big], \text{ for all } t \in [0,T].$$

Then the first equality is a simple consequence of the representation formula (4.3.3). To prove the second equality, we follow the proof of Proposition 3.1 of [60] about RBSDEs with a càdlàg obstacle and optimal stopping time. Fix some $\mathbb{P} \in \mathcal{P}_0$ and some $t \in [0,T]$. Let $v \in \mathcal{T}_t$, taking conditional expectation under $\mathbb{P}$ in the first line of (4.2.1) between t and v,

$$Y_t = \mathbb{E}^{\mathbb{P}}\Big[Y_v + \int_t^\tau \widehat{f}^{\mathbb{P}}_s(Y_s, \widehat{a}^{1/2}_s Z_s)ds + K^{\mathbb{P}}_v - K^{\mathbb{P}}_t \big| \mathcal{F}^{\mathbb{P}}_{t^+}\Big]$$

Using the decomposition (4.3.4), we have

$$\begin{aligned} Y_t &= \mathbb{E}^{\mathbb{P}}\Big[Y_v + \int_t^\tau \widehat{f}^{\mathbb{P}}_s(Y_s, \widehat{a}^{1/2}_s Z_s)ds + \overline{K}^{\mathbb{P}}_v - \overline{K}^{\mathbb{P}}_t - (k^{\mathbb{P}}_v - k^{\mathbb{P}}_t) \big| \mathcal{F}^{\mathbb{P}}_{t^+}\Big] \\ &\leq \mathbb{E}^{\mathbb{P}}\Big[\int_t^\tau \widehat{f}^{\mathbb{P}}_s(Y_s, \widehat{a}^{1/2}_s Z_s)ds + U_v 1_{\{v<T\}} + \xi 1_{\{v=T\}} + \overline{K}^{\mathbb{P}}_v - \overline{K}^{\mathbb{P}}_t \big| \mathcal{F}^{\mathbb{P}}_{t^+}\Big] \end{aligned}$$

which establishes the first the inequality.
Now fix $\epsilon > 0$ and consider $D^{\mathbb{P},\epsilon}_t$ defined in proposition. Firstly, on the set $\Big\{D^{\mathbb{P},\epsilon}_t < T\Big\}$, we have by definition $y^{\mathbb{P}}_{D^{\mathbb{P},\epsilon}_t} \geq U_{D^{\mathbb{P},\epsilon}_t} - \epsilon$ and secondly on $\Big\{D^{\mathbb{P},\epsilon}_t = T\Big\}$ we know that $y^{\mathbb{P}}_s < U_s - \epsilon$, for $t \leq s \leq T$. Then for all $s \in [t, D^{\mathbb{P},\epsilon}_t]$, $y^{\mathbb{P}}_{s^-} < U_{s^-}$. From (4.3.4), we have that

$$K^{\mathbb{P}}_{D^{\mathbb{P},\epsilon}_t} - K^{\mathbb{P}}_t = \overline{K}^{\mathbb{P}}_{D^{\mathbb{P},\epsilon}_t} - \overline{K}^{\mathbb{P}}_t.$$

Therefore, we have the following inequalities

$$\begin{aligned} Y_t &= \mathbb{E}^{\mathbb{P}}\Big[Y_{D^{\mathbb{P},\epsilon}_t} + \int_t^{D^{\mathbb{P},\epsilon}_t} \widehat{f}^{\mathbb{P}}_s(Y_s, \widehat{a}^{1/2}_s Z_s)ds + K^{\mathbb{P}}_{D^{\mathbb{P},\epsilon}_t} - K^{\mathbb{P}}_t \big| \mathcal{F}^{\mathbb{P}}_{t^+}\Big] \\ &= \mathbb{E}^{\mathbb{P}}\Big[Y_{D^{\mathbb{P},\epsilon}_t} + \int_t^{D^{\mathbb{P},\epsilon}_t} \widehat{f}^{\mathbb{P}}_s(Y_s, \widehat{a}^{1/2}_s Z_s)ds + \overline{K}^{\mathbb{P}}_{D^{\mathbb{P},\epsilon}_t} - \overline{K}^{\mathbb{P}}_t \big| \mathcal{F}^{\mathbb{P}}_{t^+}\Big] \\ &\geq \mathbb{E}^{\mathbb{P}}\Big[y^{\mathbb{P}}_{D^{\mathbb{P},\epsilon}_t} + \int_t^{D^{\mathbb{P},\epsilon}_t} \widehat{f}^{\mathbb{P}}_s(Y_s, \widehat{a}^{1/2}_s Z_s)ds + \overline{K}^{\mathbb{P}}_{D^{\mathbb{P},\epsilon}_t} - \overline{K}^{\mathbb{P}}_t \big| \mathcal{F}^{\mathbb{P}}_{t^+}\Big] \\ &\geq \mathbb{E}^{\mathbb{P}}\Big[\int_t^{D^{\mathbb{P},\epsilon}_t} \widehat{f}^{\mathbb{P}}_s(Y_s, \widehat{a}^{1/2}_s Z_s)ds + U_{D^{\mathbb{P},\epsilon}_t} 1_{\Big\{D^{\mathbb{P},\epsilon}_t < T\Big\}} + \xi 1_{\Big\{D^{\mathbb{P},\epsilon}_t = T\Big\}} + \overline{K}^{\mathbb{P}}_{D^{\mathbb{P},\epsilon}_t} - \overline{K}^{\mathbb{P}}_t \big| \mathcal{F}^{\mathbb{P}}_{t^+}\Big] - \epsilon \end{aligned}$$

for all ϵ. By arbitrariness of ϵ, we deduce the desired result. □

Following Proposition 4.2 of [35] on the standard RBSDEs , we are going to show below that if the obstacle is a general semimartingale, we can give a more explicit representation of $K^{\mathbb{P}}$. Within the framework of second order BSDEs, this result has been already stated in [66] for 2RBSDEs with a lower obstacle and [65] for 2RBSDEs with two obstacles.

Assumption 4.3.2. *U is a semimartingale with the following decomposition:*

$$U_t = U_0 + \int_0^t A_s^{\mathbb{P}} ds + B_t^{\mathbb{P}} + \int_0^t P_s \cdot dX_s^{c,\mathbb{P}} + N_t^{\mathbb{P}}, \ \mathbb{P}\text{-a.s.}, \quad \text{for all } \mathbb{P} \in \mathcal{P}_0.$$

where $N^{\mathbb{P}}$ is a $\mathbb{F}_+^{\mathbb{P}}$ - càdlàg martingale orthogonal to $X^{c,\mathbb{P}}$ such that

$$\mathbb{E}^{\mathbb{P}}\left[[N^{\mathbb{P}}]_T\right] < +\infty, \forall \mathbb{P} \in \mathcal{P}_0.$$

$B^{\mathbb{P}}$ is a càdlàg process of integrable variation such that the measure $dB_t^{\mathbb{P}}$ is singular with respect to the Lebesgue measure dt and which admits the following decomposition $B_t^{\mathbb{P}} = B_t^{\mathbb{P},+} - B_t^{\mathbb{P},-}$, where $B^{\mathbb{P},+}$ and $B^{\mathbb{P},-}$ are non decreasing processes. Also, $A^{\mathbb{P}}$ and P are respectively $\mathbb{R}$ and $\mathbb{R}^d$-valued $\mathcal{F}_{t+}^{\mathbb{P}}$ and $\mathcal{F}_{t+}^{\mathcal{P}_0}$ progressively measurable processes such that

$$\int_0^T (|A_t^{\mathbb{P}}|dt + \|\widehat{a}_t^{1/2} P_t\|^2)dt + B_T^{\mathbb{P},+} + B_T^{\mathbb{P},-} < \infty, \ \mathbb{P}\text{-a.s.} \ \forall \mathbb{P} \in \mathcal{P}_0.$$

Proposition 4.3.3. *Let Assumptions 3.2.1, 4.3.1 and 4.3.2 hold. Let (Y, Z) be the solution of the 2RBSDE (4.2.1), then for all $\mathbb{P} \in \mathcal{P}_0$*

$$Z_t = P_t \text{ and } M_t^{\mathbb{P}} = N_t^{\mathbb{P}}, \ dt \times \mathbb{P}\text{-a.s. on the set } \{Y_{t^-} = U_{t^-}\},$$

and there exists a progressively measurable process $(\alpha_t^{\mathbb{P}})_{0 \le t \le T}$ such that $0 \le \alpha^{\mathbb{P}} \le 1$ and

$$-1_{\{y_{t^-}^{\mathbb{P}} = U_{t^-}\}} dK_t^{\mathbb{P}} = 1_{\{y_{t^-}^{\mathbb{P}} = U_{t^-}\}} dk_t^{\mathbb{P}} = \alpha_t^{\mathbb{P}} 1_{\{y_{t^-}^{\mathbb{P}} = U_{t^-}\}} \left([\widehat{f}_s^{\mathbb{P}}(U_t, \widehat{a}_t^{1/2} P_t) + A_t^{\mathbb{P}}]^+ dt + dB_t^{\mathbb{P},+} \right).$$

Proof. Following the decomposition of U and (4.2.1), for all $\mathbb{P} \in \mathcal{P}_0$, the following holds $\mathbb{P}$-a.s.

$$\begin{aligned} U_t - Y_t = U_0 - Y_0 + \int_0^t (\widehat{f}_s^{\mathbb{P}}(Y_s, \widehat{a}_s^{1/2} Z_s) + A_s^{\mathbb{P}}) ds - \int_0^t (Z_s - P_s) dX_s^{c,\mathbb{P}} - (M_t^{\mathbb{P}} - N_t^{\mathbb{P}}) \\ + K_t^{\mathbb{P}} + B_t^{\mathbb{P},+} - B_t^{\mathbb{P},-} \end{aligned}$$

Now if we denote L_t the local time at 0 of $U_t - Y_t$, then by Itô-Tanaka formula under $\mathbb{P}$

$$\begin{aligned} (U_t - Y_t)^+ = (U_0 - Y_0)^+ + \int_0^t 1_{\{Y_{s^-} < U_{s^-}\}} (\widehat{f}_s^{\mathbb{P}}(Y_s, \widehat{a}_s^{1/2} Z_s) + A_s^{\mathbb{P}}) ds \\ - \int_0^t 1_{\{Y_{s^-} < U_{s^-}\}} (Z_s - P_s) dX_s^{c,\mathbb{P}} - \int_0^t 1_{\{Y_{s^-} < U_{s^-}\}} d(M_s^{\mathbb{P}} - N_s^{\mathbb{P}}) \\ + \int_0^t 1_{\{Y_{s^-} < U_{s^-}\}} d(K_t^{\mathbb{P}} + B_t^{\mathbb{P},+} - B_t^{\mathbb{P},-}) + \frac{1}{2} L_t \\ + \sum_{0 < s \le t} \{(U_s - Y_s)^+ - (U_{s^-} - Y_{s^-})^+ - 1_{\{Y_{s^-} < U_{s^-}\}} \Delta(U_s - Y_s)\}. \end{aligned}$$

But from (4.2.1), $(U_t - Y_t)^+ = U_t - Y_t$, hence the two above differentials coincide and so the martingale and the bounded variation parts coincide. By identification of the martingale parts, we obtain

$$1_{\{Y_{t^-} = U_{t^-}\}} (Z_t - P_t) dX_t^{c,\mathbb{P}} = 0 \quad \text{and} \quad 1_{\{Y_{t^-} = U_{t^-}\}} d(M_t^{\mathbb{P}} - N_t^{\mathbb{P}}), \ \mathbb{P}\text{-a.s. } \forall \mathbb{P} \in \mathcal{P}_0.$$

Hence, we get $Z_t = P_t$ and $M_t^{\mathbb{P}} = N_t^{\mathbb{P}}$, $dt \times \mathbb{P}$-a.s. on the set $\{Y_{t^-} = U_{t^-}\}$.
By identification of the bounded variation parts, we have

$$1_{\{Y_{t^-}=U_{t^-}\}} dK_t^{\mathbb{P}} + \frac{1}{2} L_t = -1_{\{Y_{t^-}=U_{t^-}\}} \Big([\widehat{f}_t^{\mathbb{P}}(Y_t, \widehat{a}_s^{1/2} Z_t) + A_t^{\mathbb{P}}] dt + dB_t^{\mathbb{P},+} - dB_t^{\mathbb{P},-} \Big) - \big\{ (U_t - Y_t)^+ - (U_{t^-} - Y_{t^-})^+ - 1_{\{Y_{t^-}<U_{t^-}\}} \Delta(U_t - Y_t) \big\}.$$

Since $(U_t - Y_t)^+ = U_t - Y_t$, we deduce that

$$\big\{ (U_t - Y_t)^+ - (U_{t^-} - Y_{t^-})^+ - 1_{\{Y_{t^-}<U_{t^-}\}} \Delta(U_t - Y_t) \big\} \leq U_t - Y_t.$$

Also, by the representation (4.3.3), it is obvious that $\{y_{t^-}^{\mathbb{P}} = U_{t^-}\} \subseteq \{Y_{t^-} = U_{t^-}\}$, $\mathbb{P}$-a.s. for every $\mathbb{P} \in \mathcal{P}_0$. Then we deduce that

$$\begin{aligned}
1_{\{y_{t^-}^{\mathbb{P}}=U_{t^-}\}} dK_t^{\mathbb{P}} &\leq 1_{\{Y_{t^-}=U_{t^-}\}} dK_t^{\mathbb{P}} \\
&\leq -1_{\{Y_{t^-}=U_{t^-}\}} \Big([\widehat{f}_t^{\mathbb{P}}(Y_t, \widehat{a}_s^{1/2} Z_t) + A_t^{\mathbb{P}}] dt + dB_t^{\mathbb{P},+} - dB_t^{\mathbb{P},-} \Big) \\
&\leq -1_{\{Y_{t^-}=U_{t^-}\}} \Big([\widehat{f}_t^{\mathbb{P}}(Y_t, \widehat{a}_s^{1/2} Z_t) + A_t^{\mathbb{P}}]^+ dt + dB_t^{\mathbb{P},+} \Big) \\
&\quad + 1_{\{Y_{t^-}=U_{t^-}\}} \Big([\widehat{f}_t^{\mathbb{P}}(Y_t, \widehat{a}_s^{1/2} Z_t) + A_t^{\mathbb{P}}]^- dt + dB_t^{\mathbb{P},-} \Big) \\
&\leq -1_{\{y_{t^-}^{\mathbb{P}}=U_{t^-}\}} \Big([\widehat{f}_t^{\mathbb{P}}(Y_t, \widehat{a}_s^{1/2} Z_t) + A_t^{\mathbb{P}}]^+ dt + dB_t^{\mathbb{P},+} \Big) \\
&\quad + 1_{\{Y_{t^-}=U_{t^-}\}} \Big([\widehat{f}_t^{\mathbb{P}}(Y_t, \widehat{a}_s^{1/2} Z_t) + A_t^{\mathbb{P}}]^- dt + dB_t^{\mathbb{P},-} \Big)
\end{aligned}$$

Proposition 4.3.1 yields

$$1_{\{y_{t^-}^{\mathbb{P}}=U_{t^-}\}} dK_t^{\mathbb{P}} = -1_{\{y_{t^-}^{\mathbb{P}}=U_{t^-}\}} dk_t^{\mathbb{P}}$$

By the fact that $k^{\mathbb{P}}$ is non decreasing, it follows by the two above inequalities that there exists a predictable process $(\alpha_t^{\mathbb{P}})_{0 \leq t \leq T}$ such that $0 \leq \alpha \leq 1$ and

$$-1_{\{y_{t^-}^{\mathbb{P}}=U_{t^-}\}} dK_t^{\mathbb{P}} = 1_{\{y_{t^-}^{\mathbb{P}}=U_{t^-}\}} dk_t^{\mathbb{P}} = \alpha_t^{\mathbb{P}} 1_{\{y_{t^-}^{\mathbb{P}}=U_{t^-}\}} \Big([\widehat{f}_s^{\mathbb{P}}(U_t, \widehat{a}_t^{1/2} P_t) + A_t^{\mathbb{P}}]^+ dt + dB_t^{\mathbb{P},+} \Big)$$

□

4.3.4 A priori estimates

Lemma 4.3.1. *Let $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$ be the solution of the RBSDE (4.2.2) and (Y, Z) be a solution to (4.2.1). There exists a real constant C which only depends on T, p and the Lipschitz constant L_f of f such that*

$$\mathbb{E}^{\mathbb{P}} \Big[\Big(\int_0^T \|\widehat{a}_t^{1/2} z_t^{\mathbb{P}}\|^2 dt \Big)^{p/2} \Big] + \mathbb{E}^{\mathbb{P}} \big[[m^{\mathbb{P}}]_T^{\frac{p}{2}} \big] \leq C \mathbb{E}^{\mathbb{P}} \Big[\sup_{t \in [0,T]} |y_t^{\mathbb{P}}|^p + \big(\int_0^T |\widehat{f}_t^{\mathbb{P},0}| dt \big)^p \Big]. \tag{4.3.11}$$

$$\mathbb{E}^{\mathbb{P}} \Big[\Big(\int_0^T \|\widehat{a}_t^{1/2} Z_t\|^2 dt \Big)^{p/2} \Big] + \mathbb{E}^{\mathbb{P}} \big[[M^{\mathbb{P}}]_T^{\frac{p}{2}} \big] \leq C \mathbb{E}^{\mathbb{P}} \Big[\sup_{t \in [0,T]} |Y_t|^p + \big(\int_0^T |\widehat{f}_t^{\mathbb{P},0}| dt \big)^p \Big]. \tag{4.3.12}$$

Proof. This first result is proved in step 1 of the proof of 3.3.6. The slight difference is the sign before $k^{\mathbb{P}}$, but this does not affect the result. We know focus on the second one. The method are the similar to step 1 of the proof of Proposition 3.3.6.

$$\tau_n = \inf\{t \in [0,T], \int_0^t \|\widehat{a}_s^{1/2} Z_s\|^2 ds + [M^{\mathbb{P}}]_T \geq n\} \wedge T.$$

Since Z is in $\mathbb{H}_0^p(\mathbb{P})$ and $M^{\mathbb{P}} \in \mathbb{M}_0^p(\mathbb{P})$, then $(\tau_n)_{n\geq 0}$ is a stationary sequence. Now considering a real constant α and using Itô's formula to $e^{\alpha t}|Y_t|^2$ between 0 and τ_n, we get

$$
\begin{aligned}
&|Y_0|^2 + \int_0^{\tau_n} e^{\alpha s}\|\widehat{a}_s^{1/2}Z_s\|^2 ds + \int_0^{\tau_n} e^{\alpha s} d[M^{\mathbb{P}}]_s \\
&\leq e^{\alpha\tau_n}|Y_{\tau_n}|^2 + \int_0^{\tau_n} e^{\alpha s} Y_s\big(2|\widehat{f}_s^{\mathbb{P},0}| + 2L_f|Y_s| + 2L_f\|\widehat{a}_s^{1/2}Z_s\| - \alpha Y_s\big) ds \\
&\qquad + 2\int_0^{\tau_n} e^{\alpha s} Y_{s^-} dK_s^{\mathbb{P}} - 2\int_0^{\tau_n} e^{\alpha s} Y_{s^-} dm_s^{\mathbb{P}} - 2\int_0^{\tau_n} e^{\alpha s} Y_s Z_s dX_s^{c,\mathbb{P}} \\
&\leq e^{\alpha\tau_n}|Y_{\tau_n}|^2 + 2\sup_{s\leq\tau_n} e^{\alpha s}|Y_s| \times \int_0^{\tau_n} |\widehat{f}_s^{\mathbb{P},0}| ds + (2L_f + L_f\epsilon^{-1} - \alpha)\int_0^{\tau_n} e^{\alpha s}|Y_s|^2 ds \\
&\qquad + \epsilon L_f \int_0^{\tau_n} \|\widehat{a}_s^{1/2}Z_s\|^2 ds + 2\int_0^{\tau_n} e^{\alpha s} Y_{s^-} d(\overline{K}_s^{\mathbb{P}} - k_s^{\mathbb{P}}) - 2\int_0^{\tau_n} e^{\alpha s} Y_{s^-} dM_s^{\mathbb{P}} \\
&\qquad - 2\int_0^{\tau_n} e^{\alpha s} Y_s Z_s dX_s^{c,\mathbb{P}}.
\end{aligned}
$$

for any $\epsilon > 0$. Thus,

$$
\begin{aligned}
&|Y_0|^2 + (1 - \epsilon L_f)\int_0^{\tau_n} e^{\alpha s}\big\|\widehat{a}_s^{1/2}Z_s\big\|^2 ds + \int_0^{\tau_n} e^{\alpha s} d[M^{\mathbb{P}}]_s \\
&\qquad \leq e^{\alpha\tau_n}|Y_{\tau_n}|^2 + \sup_{s\leq\tau_n} e^{2\alpha s}|Y_s|^2 + \Big(\int_0^{\tau_n} |\widehat{f}_s^{\mathbb{P},0}| ds\Big)^2 + (2L_f + L_f\epsilon^{-1} - \alpha)\int_0^{\tau_n} e^{\alpha s}|Y_s|^2 ds \\
&\qquad + \frac{1}{\epsilon}\sup_{s\leq\tau_n} e^{2\alpha s}|Y_s|^2 + \epsilon\big(\overline{K}_{\tau_n}^{\mathbb{P}}\big)^2 + \epsilon\big(k_{\tau_n}^{\mathbb{P}}\big)^2 - 2\int_0^{\tau_n} e^{\alpha s} Y_{s^-} dM_s^{\mathbb{P}} - 2\int_0^{\tau_n} e^{\alpha s} Y_s Z_s dX_s^{c,\mathbb{P}}.
\end{aligned}
$$

Furthermore by (4.2.5), we have

$$
\begin{aligned}
k_{\tau_n}^{\mathbb{P}} &= y_{\tau_n}^{\mathbb{P}} - y_0^{\mathbb{P}} + \int_0^{\tau_n} \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}}) ds - \int_0^{\tau_n} z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - m_{\tau_n}^{\mathbb{P}} \\
\overline{K}_{\tau_n}^{\mathbb{P}} &= Y_0 - Y_{\tau_n} - \int_0^{\tau_n} \widehat{f}_s^{\mathbb{P}}(Y_s, \widehat{a}_s^{1/2} Z_s) ds + \int_0^{\tau_n} Z_s dX_s^{c,\mathbb{P}} + M_{\tau_n}^{\mathbb{P}} + k_{\tau_n}^{\mathbb{P}}.
\end{aligned}
$$

Then, there exists a constant C depending on the Lipschitz constant L_f such that

$$
\begin{aligned}
\big(\overline{K}_{\tau_n}^{\mathbb{P}}\big)^2 + \big(k_{\tau_n}^{\mathbb{P}}\big)^2 &\leq C\Big(|y_0^{\mathbb{P}}|^2 + |Y_0|^2 + |y_{\tau_n}^{\mathbb{P}}|^2 + |Y_{\tau_n}|^2 + \Big(\int_0^{\tau_n} |\widehat{f}_s^{\mathbb{P},0}| ds\Big)^2 \\
&\qquad + \int_0^{\tau_n} |y_s^{\mathbb{P}}|^2 ds + \int_0^{\tau_n} |Y_s|^2 ds + \int_0^{\tau_n} \|\widehat{a}_s^{1/2} z_s^{\mathbb{P}})\|^2 ds + \int_0^{\tau_n} \|\widehat{a}_s^{1/2} Z_s)\|^2 ds \\
&\qquad + \Big|\int_0^{\tau_n} z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dX_s^{c,\mathbb{P}}\Big|^2 + \Big|\int_0^{\tau_n} Z_s dX_s^{c,\mathbb{P}}\Big|^2 + |m_{\tau_n}^{\mathbb{P}}|^2 + |M_{\tau_n}^{\mathbb{P}}|^2\Big) \\
&\leq C\Big(|Y_0|^2 + |Y_{\tau_n}|^2 + \Big(\int_0^{\tau_n} |\widehat{f}_s^{\mathbb{P},0}| ds\Big)^2 + \int_0^{\tau_n} |Y_s|^2 ds + \int_0^{\tau_n} \|\widehat{a}_s^{1/2} Z_s)\|^2 ds \\
&\qquad + \int_0^{\tau_n} \|\widehat{a}_s^{1/2} z_s^{\mathbb{P}})\|^2 ds + \Big|\int_0^{\tau_n} z_s^{\mathbb{P}} dX_s^{c,\mathbb{P}}\Big|^2 + \Big|\int_0^{\tau_n} Z_s dX_s^{c,\mathbb{P}}\Big|^2 \\
&\qquad + |m_{\tau_n}^{\mathbb{P}}|^2 + |M_{\tau_n}^{\mathbb{P}}|^2\Big)
\end{aligned} \tag{4.3.13}
$$

where we have used the representation formula(4.3.3) which implies $y_t^{\mathbb{P}} \leq Y_t$. By taking into

account the last inequality in the right-hand side of the previous one, we obtain

$$
\begin{aligned}
&(1-\epsilon C)|Y_0|^2+(1-\epsilon L_f)\int_0^{\tau_n} e^{\alpha s}\|\widehat{a}_s^{1/2}Z_s\|^2ds-\epsilon C\int_0^{\tau_n}\|\widehat{a}_s^{1/2}Z_s\|^2ds+\int_0^{\tau_n}e^{\alpha s}d[M^{\mathbb{P}}]_s\\
&\leq(\epsilon C+e^{\alpha\tau_n})|Y_{\tau_n}|^2+\big(1+\frac{1}{\epsilon}\big)\sup_{s\leq\tau_n}e^{2\alpha s}|Y_s|^2+(1+\epsilon C)\Big(\int_0^{\tau_n}|\widehat{f}_s^{\mathbb{P},0}|ds\Big)^2\\
&+(2L_f+L_f\epsilon^{-1}-\alpha)\int_0^{\tau_n}e^{\alpha s}|Y_s|^2ds+\epsilon C\Big|\int_0^{\tau_n}Z_s\cdot dX_s^{c,\mathbb{P}}\Big|^2+\epsilon C|M_{\tau_n}^{\mathbb{P}}|^2+\epsilon C|m_{\tau_n}^{\mathbb{P}}|^2\\
&\epsilon C\int_0^{\tau_n}\|\widehat{a}_s^{1/2}z_s^{\mathbb{P}}\|^2ds+2\Big|\int_0^{\tau_n}e^{\alpha s}Y_sZ_sdX_s^{c,\mathbb{P}}\Big|+2\Big|\int_0^{\tau_n}e^{\alpha s}Y_{s^-}dM_s^{\mathbb{P}}\Big|+\epsilon C\Big|\int_0^{\tau_n}z_s^{\mathbb{P}}dX_s^{c,\mathbb{P}}\Big|^2.
\end{aligned}
$$

Choosing now ϵ small enough, α such that $2L_f+L_f\epsilon^{-1}-\alpha<0$ and using inequality (3.3.9), we get

$$
\begin{aligned}
&(1\wedge 2^{\frac{p}{2}-1})\Big(\int_0^{\tau_n}e^{\alpha s}\|\widehat{a}_s^{1/2}Z_s\|^2ds\Big)^{\frac{p}{2}}+(1\wedge 2^{\frac{p}{2}-1})[M^{\mathbb{P}}]_{\tau_n}^{\frac{p}{2}}\\
&\leq C(1\vee 2^{\frac{p}{2}-1})\Big(\Big(\int_0^{\tau_n}|\widehat{f}_s^{\mathbb{P},0}|ds\Big)^p+\sup_{t\in[0,T]}|Y_t|^p\\
&\quad+\Big|\int_0^{\tau_n}e^{\alpha s}Y_sZ_sdX_s^{c,\mathbb{P}}\Big|^{\frac{p}{2}}+\Big|\int_0^{\tau_n}e^{\alpha s}Y_{s^-}dM_s^{\mathbb{P}}\Big|^{\frac{p}{2}}\Big).
\end{aligned}
\tag{4.3.14}
$$

Otherwise, thanks to the BDG's inequality under probability $\mathbb{P}$ and the Young's inequality, we can can write

$$
\mathbb{E}^{\mathbb{P}}\Big[\Big|\int_0^{\tau_n}e^{\alpha s}Y_sZ_sdX_s^{c,\mathbb{P}}\Big|^{\frac{p}{2}}\Big]\leq\frac{C_p^2}{\eta}\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|Y_s|^p\Big]+\eta\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^{\tau_n}\|\widehat{a}_s^{1/2}Z_s\|^2ds\Big)^{\frac{p}{2}}\Big].
$$

and

$$
\mathbb{E}^{\mathbb{P}}\Big[\Big|\int_0^{\tau_n}e^{\alpha s}Y_{s^-}dM_s^{\mathbb{P}}\Big|^{\frac{p}{2}}\Big]\leq\frac{C_p'^2}{\eta'}\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|Y_s|^p\Big]+\eta'\mathbb{E}^{\mathbb{P}}\Big[[M^{\mathbb{P}}]_{\tau_n}^{\frac{p}{2}}\Big].
$$

where C_p, C_p' depend on the BDG's constants and α and η,η' are positive constants. By taking expectation under $\mathbb{P}$ in (4.3.14), using the two previous estimates, choosing η and η' small enough and using Fatou's Lemma we deduce that (4.3.12) holds. □

Proposition 4.3.4. *Let $(y^{\mathbb{P}},z^{\mathbb{P}},m^{\mathbb{P}},k^{\mathbb{P}})$ be the solution of the RBSDE (4.2.2) and (Y,Z) be a solution to (4.2.1). There exists a real constant C which only depends on T, p and the Lipschitz constant L_f of f such that*

$$
\mathbb{E}^{\mathbb{P}}\Big[\sup_{0\leq t\leq T}|y_t^{\mathbb{P}}|^p\Big]\leq C\mathbb{E}^{\mathbb{P}}\Big[|\xi|^p+\Big(\int_0^T|\widehat{f}_t^{\mathbb{P},0}|dt\Big)^p+\sup_{t\in[0,T]}|U_t|^p\Big]
\tag{4.3.15}
$$

$$
\mathbb{E}^{\mathbb{P}}\Big[\sup_{0\leq t\leq T}|Y_t|^p\Big]\leq C\sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[|\xi|^p+\Big(\int_0^T|\widehat{f}_t^{\mathbb{P},0}|dt\Big)^p+\sup_{t\in[0,T]}|U_t|^p\Big].
\tag{4.3.16}
$$

We need the following result to prove the above proposition.

Lemma 4.3.2. *Fix $\mathbb{P}\in\mathcal{P}_0$. Assume that $(y^{\mathbb{P}},z^{\mathbb{P}},m^{\mathbb{P}},k^{\mathbb{P}})$ is the solution of the RBSDEs (3.2.9). Then for any $0\leq t\leq u\leq T$ we have*

$$
\begin{aligned}
&|y_t^{\mathbb{P}}|^p+c(p)\int_t^u|y_s^{\mathbb{P}}|^{p-2}1_{y_s^{\mathbb{P}}\neq0}\|\widehat{a}_s^{1/2}z_s^{\mathbb{P}}\|^2ds+c(p)\int_t^u|y_s^{\mathbb{P}}|^{p-2}1_{y_s^{\mathbb{P}}\neq0}d[m^{\mathbb{P}}]_s\\
&\qquad+\sum_{t<s\leq u}\{|y_s^{\mathbb{P}}|^p-|y_{s^-}^{\mathbb{P}}|^p-p|y_{s^-}^{\mathbb{P}}|^{p-1}\mathrm{sgn}(y_{s^-}^{\mathbb{P}})\Delta y_s^{\mathbb{P}}\}\\
&\leq|y_u^{\mathbb{P}}|^p+p\int_t^u|y_s^{\mathbb{P}}|^{p-1}\mathrm{sgn}(y_s^{\mathbb{P}})\widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}},\widehat{a}_s^{1/2}z_s^{\mathbb{P}})ds-p\int_t^u|y_s^{\mathbb{P}}|^{p-1}\mathrm{sgn}(y_s^{\mathbb{P}})z_s^{\mathbb{P}}\cdot dX_s^{c,\mathbb{P}}\\
&\qquad-p\int_t^u|y_{s^-}^{\mathbb{P}}|^{p-1}\mathrm{sgn}(y_{s^-}^{\mathbb{P}})dm_s^{\mathbb{P}}-p\int_t^u|y_{s^-}^{\mathbb{P}}|^{p-1}\mathrm{sgn}(y_{s^-}^{\mathbb{P}})dk_s^{\mathbb{P}}.
\end{aligned}
$$

with $c(p) = p(p-1)/2$ *and* $\operatorname{sgn}(x) = \frac{x}{|x|}1_{x\neq 0}$.

Proof. This follows directly by Lemma A.4.2. □

Proof of Proposition 4.3.4. We first prove the estimates for RBSDE and then deduce the estimates for the 2RBSDE. Let α be a real constant, Applying Itô formula to $e^{\alpha pt}|y_t^{\mathbb{P}}|^p$ and using the above Lemma yields

$$\begin{aligned}
&e^{\alpha pt}|y_t^{\mathbb{P}}|^p + \frac{1}{2}p(p-1)\int_t^T e^{\alpha ps}|y_s^{\mathbb{P}}|^{p-2}1_{y_s^{\mathbb{P}}\neq 0}(\|\widehat{a}_s^{1/2}z_s^{\mathbb{P}}\|^2 ds + d[m^{\mathbb{P}}]_s)\\
&\quad + \alpha p\int_t^T e^{\alpha ps}|y_s^{\mathbb{P}}|^p ds + \sum_{t<s\leq T} e^{\alpha ps}\{|y_s^{\mathbb{P}}|^p - |y_{s^-}^{\mathbb{P}}|^p - p|y_{s^-}^{\mathbb{P}}|^{p-1}\operatorname{sgn}(y_{s^-}^{\mathbb{P}})\Delta y_s^{\mathbb{P}}\}\\
&\leq e^{\alpha pT}|\xi|^p + p\int_t^T e^{\alpha ps}|y_s^{\mathbb{P}}|^{p-1}\operatorname{sgn}(y_s^{\mathbb{P}})\widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2}z_s^{\mathbb{P}})ds\\
&\quad - p\int_t^T e^{\alpha ps}|y_s^{\mathbb{P}}|^{p-1}\operatorname{sgn}(y_s^{\mathbb{P}})z_s^{\mathbb{P}}\cdot dX_s^{c,\mathbb{P}} - p\int_t^T e^{\alpha ps}|y_{s^-}^{\mathbb{P}}|^{p-1}\operatorname{sgn}(y_{s^-}^{\mathbb{P}})dm_s^{\mathbb{P}}\\
&\quad - p\int_t^T e^{\alpha ps}|y_{s^-}^{\mathbb{P}}|^{p-1}\operatorname{sgn}(y_{s^-}^{\mathbb{P}})dk_s^{\mathbb{P}}\\
&\leq e^{\alpha pT}|\xi|^p + p\int_t^T e^{\alpha ps}|y_s^{\mathbb{P}}|^{p-1}|\widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2}z_s^{\mathbb{P}})|ds - p\int_t^T e^{\alpha ps}|y_{s^-}^{\mathbb{P}}|^{p-1}\operatorname{sgn}(y_{s^-}^{\mathbb{P}})dk_s^{\mathbb{P}}\\
&\quad - p\int_t^T e^{\alpha ps}|y_s^{\mathbb{P}}|^{p-1}\operatorname{sgn}(y_s^{\mathbb{P}})z_s^{\mathbb{P}}\cdot dX_s^{c,\mathbb{P}} - p\int_t^T e^{\alpha ps}|y_{s^-}^{\mathbb{P}}|^{p-1}\operatorname{sgn}(y_{s^-}^{\mathbb{P}})dm_s^{\mathbb{P}}\\
&\leq e^{\alpha pT}|\xi|^p + p\int_t^T e^{\alpha ps}|y_s^{\mathbb{P}}|^{p-1}|\widehat{f}_s^{\mathbb{P},0}|ds + pL_f\int_t^T e^{\alpha ps}|y_s^{\mathbb{P}}|^p ds\\
&\quad + pL_f\int_t^T e^{\alpha ps}|y_s^{\mathbb{P}}|^{p-1}\|\widehat{a}_s^{1/2}z_s^{\mathbb{P}}\|ds - p\int_t^T e^{\alpha ps}|y_s^{\mathbb{P}}|^{p-1}\operatorname{sgn}(y_s^{\mathbb{P}})z_s^{\mathbb{P}}\cdot dX_s^{c,\mathbb{P}}\\
&\quad - p\int_t^T e^{\alpha ps}|y_{s^-}^{\mathbb{P}}|^{p-1}\operatorname{sgn}(y_{s^-}^{\mathbb{P}})dm_s^{\mathbb{P}} - p\int_t^T e^{\alpha ps}|y_{s^-}^{\mathbb{P}}|^{p-1}\operatorname{sgn}(y_{s^-}^{\mathbb{P}})dk_s^{\mathbb{P}}.
\end{aligned}$$

The Skorokhod condition implies that $dk_s^{\mathbb{P}} = 1_{\{y_{s^-}^{\mathbb{P}}\geq U_{s^-}\}}dk_s^{\mathbb{P}}$, also the function $x\mapsto |x|^{p-1}\operatorname{sgn}(x)$ is non decreasing then,

$$\begin{aligned}
\int_t^T e^{\alpha ps}|y_{s^-}^{\mathbb{P}}|^{p-1}\operatorname{sgn}(y_{s^-}^{\mathbb{P}})dk_s^{\mathbb{P}} &= \int_t^T e^{\alpha ps}|y_{s^-}^{\mathbb{P}}|^{p-1}\operatorname{sgn}(y_{s^-}^{\mathbb{P}})1_{\{y_{s^-}^{\mathbb{P}}\geq U_{s^-}\}}dk_s^{\mathbb{P}}\\
&\geq \int_t^T e^{\alpha ps}|U_{s^-}|^{p-1}\operatorname{sgn}(U_{s^-})1_{\{y_{s^-}^{\mathbb{P}}\geq U_{s^-}\}}dk_s^{\mathbb{P}}
\end{aligned}$$

By Young's inequality, we have

$$pL_f|y_s^{\mathbb{P}}|^{p-1}\|\widehat{a}_s^{1/2}z_s^{\mathbb{P}}\| \leq \frac{pL_f^2}{p-1}|y_s^{\mathbb{P}}|^p + \frac{1}{4}p(p-1)|y_s^{\mathbb{P}}|^{p-2}\|\widehat{a}_s^{1/2}z_s^{\mathbb{P}}\|^2 1_{y_s^{\mathbb{P}}\neq 0}.$$

and for any $\beta > 0$,

$$\begin{aligned}
p\int_t^T e^{\alpha ps}|y_s^{\mathbb{P}}|^{p-1}|\widehat{f}_s^{\mathbb{P},0}|ds &\leq p\beta\sup_{0\leq t\leq T}|y_t^{\mathbb{P}}|^{p-1}\int_t^T \beta^{-1}e^{\alpha ps}|\widehat{f}_s^{\mathbb{P},0}|ds\\
&\leq (p-1)\beta^{\frac{p}{p-1}}\sup_{0\leq t\leq T}|y_t^{\mathbb{P}}|^p + \beta^{-p}\Big(\int_t^T e^{\alpha ps}|\widehat{f}_s^{\mathbb{P},0}|ds\Big)^p
\end{aligned}$$

Hence combining these estimates, we obtain

$$
\begin{aligned}
&e^{\alpha p t}|y_t^{\mathbb{P}}|^p + \frac{1}{4}p(p-1)\int_t^T e^{\alpha p s}|y_s^{\mathbb{P}}|^{p-2}1_{y_s^{\mathbb{P}}\neq 0}\big(\|\widehat{a}_s^{1/2}z_s^{\mathbb{P}}\|^2 ds + d[m^{\mathbb{P}}]_s\big)\\
&\leq e^{\alpha p T}|\xi|^p + (p-1)\beta^{\frac{p}{p-1}}\sup_{0\leq t\leq T}|y_t^{\mathbb{P}}|^p + \beta^{-p}\Big(\int_t^T e^{\alpha p s}|\widehat{f}_s^{\mathbb{P},0}|ds\Big)^p\\
&\quad + p\big(L_f + \frac{L_f^2}{p-1} - \alpha\big)\int_t^T e^{\alpha p s}|y_s^{\mathbb{P}}|^p ds - p\int_t^T e^{\alpha p s}|y_s^{\mathbb{P}}|^{p-1}\mathrm{sgn}(y_s^{\mathbb{P}})z_s^{\mathbb{P}}\cdot dX_s^{c,\mathbb{P}}\\
&\quad - p\int_t^T e^{\alpha p s}|y_{s^-}^{\mathbb{P}}|^{p-1}\mathrm{sgn}(y_{s^-}^{\mathbb{P}})dm_s^{\mathbb{P}} - p\int_t^T e^{\alpha p s}|U_{s^-}|^{p-1}\mathrm{sgn}(U_{s^-})dk_s^{\mathbb{P}}\\
&\leq e^{\alpha p T}|\xi|^p + (p-1)\beta^{\frac{p}{p-1}}\sup_{0\leq t\leq T}|y_t^{\mathbb{P}}|^p + \beta^{-p}\Big(\int_t^T e^{\alpha p s}|\widehat{f}_s^{\mathbb{P},0}|ds\Big)^p\\
&\quad + p\big(L_f + \frac{L_f^2}{p-1} - \alpha\big)\int_t^T e^{\alpha p s}|y_s^{\mathbb{P}}|^p ds - p\int_t^T e^{\alpha p s}|y_s^{\mathbb{P}}|^{p-1}\mathrm{sgn}(y_s^{\mathbb{P}})z_s^{\mathbb{P}}\cdot dX_s^{c,\mathbb{P}}\\
&\quad - p\int_t^T e^{\alpha p s}|y_{s^-}^{\mathbb{P}}|^{p-1}\mathrm{sgn}(y_{s^-}^{\mathbb{P}})dm_s^{\mathbb{P}} + p\sup_{0\leq t\leq T}|U_t|^{p-1}\int_t^T e^{\alpha p s}dk_s^{\mathbb{P}}.
\end{aligned}
$$

By choosing α such that $L_f + Lf/(p-1) \leq \alpha$, we obtain for any $\epsilon > 0$,

$$
\begin{aligned}
&e^{\alpha p t}|y_t^{\mathbb{P}}|^p + \frac{1}{4}p(p-1)\int_t^T e^{\alpha p s}|y_s^{\mathbb{P}}|^{p-2}1_{y_s^{\mathbb{P}}\neq 0}\big(\|\widehat{a}_s^{1/2}z_s^{\mathbb{P}}\|^2 ds + d[m^{\mathbb{P}}]_s\big)\\
&\leq e^{\alpha p T}|\xi|^p + (p-1)\beta^{\frac{p}{p-1}}\sup_{0\leq t\leq T}|y_t^{\mathbb{P}}|^p + \beta^{-p}\Big(\int_t^T e^{\alpha p s}|\widehat{f}_s^{\mathbb{P},0}|ds\Big)^p\\
&\quad - p\int_t^T e^{\alpha p s}|y_s^{\mathbb{P}}|^{p-1}\mathrm{sgn}(y_s^{\mathbb{P}})z_s^{\mathbb{P}}\cdot dX_s^{c,\mathbb{P}} - p\int_t^T e^{\alpha p s}|y_{s^-}^{\mathbb{P}}|^{p-1}\mathrm{sgn}(y_{s^-}^{\mathbb{P}})dm_s^{\mathbb{P}}\\
&\quad + (p-1)\epsilon^{\frac{p}{1-p}}\sup_{0\leq t\leq T}|U_t|^p + \epsilon^p\Big(\int_t^T e^{\alpha p s}dk_s^{\mathbb{P}}\Big)^p. \qquad (4.3.17)
\end{aligned}
$$

Furthermore

$$
k_T^{\mathbb{P}} = \xi - y_0^{\mathbb{P}} + \int_0^T \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2}z_s^{\mathbb{P}})ds - \int_0^T z_s^{\mathbb{P}}dX_s^{c,\mathbb{P}} - m_T^{\mathbb{P}}
$$

Then

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_t^T e^{\alpha p s}dk_s^{\mathbb{P}}\Big)^p\Big] &\leq C\mathbb{E}^{\mathbb{P}}[(k_T^{\mathbb{P}})^p]\\
&\leq C\mathbb{E}^{\mathbb{P}}\Big[|\xi|^p + \Big(\int_0^T \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2}z_s^{\mathbb{P}})ds\Big)^p + \Big(\int_0^T z_s^{\mathbb{P}}\cdot\widehat{a}_s^{1/2}dW_s^{\mathbb{P}}\Big)^p + (m_T^{\mathbb{P}})^p\Big]\\
&\leq C\mathbb{E}^{\mathbb{P}}\Big[|\xi|^p + \sup_{0\leq t\leq T}|y_t^{\mathbb{P}}|^p + \Big(\int_0^T |\widehat{f}_s^{\mathbb{P},0}|ds\Big)^p + \Big(\int_0^T \|\widehat{a}_s^{1/2}z_s^{\mathbb{P}}\|^2 ds\Big)^{p/2} + \big[m^{\mathbb{P}}\big]_T^{\frac{p}{2}}\Big]\\
&\leq C\mathbb{E}^{\mathbb{P}}\Big[\sup_{0\leq t\leq T}|y_t^{\mathbb{P}}|^p + \Big(\int_0^T |\widehat{f}_s^{\mathbb{P},0}|ds\Big)^p\Big]
\end{aligned}
$$

where we have used Lemma 4.3.1 and where C is a constant depending on T, p, α and L_f which vary line to line. Then taking expectation under $\mathbb{P}$ in (4.3.17), we obtain

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Big[e^{\alpha p t}|y_t^{\mathbb{P}}|^p + \frac{1}{4}p(p-1)\int_t^T e^{\alpha p s}|y_s^{\mathbb{P}}|^{p-2}1_{y_s^{\mathbb{P}}\neq 0}\big(\|\widehat{a}_s^{1/2}z_s^{\mathbb{P}}\|^2 ds + d[m^{\mathbb{P}}]_s\big)\Big]\\
&\leq \mathbb{E}^{\mathbb{P}}\Big[e^{\alpha p T}\|\xi\|^p + \big((p-1)\beta^{\frac{p}{p-1}} + \epsilon^p C\big)\sup_{0\leq t\leq T}|y_t^{\mathbb{P}}|^p + \big(\beta^{-p} + \epsilon^p\big)C\Big(\int_t^T |\widehat{f}_s^{\mathbb{P},0}|ds\Big)^p\\
&\quad + (p-1)\epsilon^{\frac{p}{1-p}}\sup_{0\leq t\leq T}|U_t|^p\Big]. \qquad (4.3.18)
\end{aligned}
$$

Going back to (4.3.17), taking the supremum over t and then the expectation under $\mathbb{P}$, we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} e^{\alpha p t}|y_t^{\mathbb{P}}|^p\Big] \\
&\le \mathbb{E}^{\mathbb{P}}\Big[e^{\alpha p T}|\xi|^p + (p-1)\beta^{\frac{p}{p-1}}\sup_{0\le t\le T}|y_t^{\mathbb{P}}|^p + \beta^{-p}\Big(\int_0^T e^{\alpha p s}|\widehat{f}_s^{\mathbb{P},0}|ds\Big)^p \\
&+ p\sup_{t\in[0,T]}\Big|\int_t^T e^{\alpha p s}|y_s^{\mathbb{P}}|^{p-1}\mathrm{sgn}(y_s^{\mathbb{P}})z_s^{\mathbb{P}}\cdot dX_s^{c,\mathbb{P}}\Big| + (p-1)\epsilon^{\frac{p}{1-p}}\sup_{0\le t\le T}|U_t|^p \\
&\quad + p\sup_{t\in[0,T]}\Big|\int_t^T e^{\alpha p s}|y_{s^-}^{\mathbb{P}}|^{p-1}\mathrm{sgn}(y_{s^-}^{\mathbb{P}})dm_s^{\mathbb{P}}\Big| + \epsilon^p\sup_{t\in[0,T]}\Big(\int_t^T e^{\alpha p s}dk_s^{\mathbb{P}}\Big)^p\Big] \qquad (4.3.19)
\end{aligned}$$

By the Burkholder inequality, there exists positive constant C_p such that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}\Big|\int_t^T e^{\alpha p s}|y_s^{\mathbb{P}}|^{p-1}\mathrm{sgn}(y_s^{\mathbb{P}})z_s^{\mathbb{P}}\cdot dX_s^{c,\mathbb{P}}\Big|\Big] \\
&\le C_p\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T |y_s^{\mathbb{P}}|^{2p-2}1_{\{y_s^{\mathbb{P}}\neq 0\}}\|\widehat{a}_s^{1/2}z_s^{\mathbb{P}}\|^2 ds\Big)^{1/2}\Big] \\
&\le C_p\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|y_t^{\mathbb{P}}|^{p/2}\Big(\int_0^T |y_s^{\mathbb{P}}|^{p-2}1_{\{y_s^{\mathbb{P}}\neq 0\}}\|\widehat{a}_s^{1/2}z_s^{\mathbb{P}}\|^2 ds\Big)^{1/2}\Big] \\
&\le \frac{1}{4p}\mathbb{E}^{\mathbb{P}}\big[\sup_{t\in[0,T]}|y_t^{\mathbb{P}}|^p\big] + pC_p^2\mathbb{E}^{\mathbb{P}}\Big[\int_0^T |y_s^{\mathbb{P}}|^{p-2}1_{\{y_s^{\mathbb{P}}\neq 0\}}\|\widehat{a}_s^{1/2}z_s^{\mathbb{P}}\|^2 ds\Big].
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}\Big|\int_t^T e^{\alpha p s}|y_{s^-}^{\mathbb{P}}|^{p-1}\mathrm{sgn}(y_{s^-}^{\mathbb{P}})dm_s^{\mathbb{P}}\Big|\Big] \\
&\le C_p\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|y_t^{\mathbb{P}}|^{p/2}\Big(\int_0^T |y_s^{\mathbb{P}}|^{p-2}1_{\{y_s^{\mathbb{P}}\neq 0\}}d[m]_s\Big)^{1/2}\Big] \\
&\le C_p\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T |y_s^{\mathbb{P}}|^{2p-2}1_{\{y_s^{\mathbb{P}}\neq 0\}}d[m]_s\Big)^{1/2}\Big] \\
&\le \frac{1}{4p}\mathbb{E}^{\mathbb{P}}\big[\sup_{t\in[0,T]}|y_t^{\mathbb{P}}|^p\big] + pC_p^2\mathbb{E}^{\mathbb{P}}\Big[\int_0^T |y_s^{\mathbb{P}}|^{p-2}1_{\{y_s^{\mathbb{P}}\neq 0\}}d[m]_s\Big].
\end{aligned}$$

Combining the two above inequalities together with (4.3.19) and (4.3.18), taking β and ϵ small enough it follows that

$$\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|y_t^{\mathbb{P}}|^p\Big] \le C\mathbb{E}^{\mathbb{P}}\Big[|\xi|^p + \Big(\int_0^T |\widehat{f}_s^{\mathbb{P},0}|ds\Big)^p + \sup_{t\in[0,T]}|U_s|^p\Big]. \qquad (4.3.20)$$

Therefore, we have proved the first assertion of the proposition. The second concerning Y is deduced by the representation formula (4.3.3).

$$\sup_{t\in[0,T]}\big|Y_t\big|^p \le \sup_{t\in[0,T]}\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}|y_t^{\mathbb{P}'}(T,\xi)|^p,\ \mathbb{P}\text{-a.s.}$$

Taking expectation under $\mathbb{P}$, we have

$$\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}\big|Y_t\big|^p\Big] \le \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}\Big(\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}\mathbb{E}^{\mathbb{P}'}\Big[\sup_{0\le s\le T}|y_s^{\mathbb{P}'}|\Big|\mathcal{F}_{t+}\Big]\Big)^p\Big].$$

Taking the supremum over $\mathbb{P} \in \mathcal{P}_0$, it follows that

$$\sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} |Y_t|^p\Big] \le \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} \Big(\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}\Big[\sup_{0\le s\le T} |y_s^{\mathbb{P}'}| \Big| \mathcal{F}_{t+}\Big]\Big)^p\Big].$$

Moreover, using Lemma $A.2$ of [90] and Holder's inequality, we have

$$\begin{aligned}\sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} \Big(\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}\Big[\sup_{0\le s\le T} |y_s^{\mathbb{P}'}| \Big| \mathcal{F}_{t+}\Big]\Big)^p\Big] &\le \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} |y_s^{\mathbb{P}}|^{\frac{p}{\kappa}}\Big]^{\kappa} \\ &\le \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} |y_s^{\mathbb{P}}|^{\kappa}\Big]^{\frac{p}{\kappa}}\end{aligned}$$

Then (4.3.20) and Holder's inequality yield

$$\begin{aligned}\sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} |Y_t|^p\Big] &\le \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} |y_s^{\mathbb{P}}|^{\kappa}\Big]^{\frac{p}{\kappa}} \\ &\le \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\Big[|\xi|^{\kappa} + \Big(\int_0^T |\widehat{f}_s^{\mathbb{P},0}|ds\Big)^{\kappa} + \sup_{t\in[0,T]} |U_s|^{\kappa}\Big]^{\frac{p}{\kappa}} \\ &\le \sup_{\mathbb{P}\in\mathcal{P}_0} \Big(\mathbb{E}^{\mathbb{P}}\Big[|\xi|^{\kappa}\Big]^{\frac{p}{\kappa}} + \mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T |\widehat{f}_s^{\mathbb{P},0}|ds\Big)^{\kappa}\Big]^{\frac{p}{\kappa}} + \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} |U_s|^{\kappa}\Big]^{\frac{p}{\kappa}}\Big) \\ &\le \sup_{\mathbb{P}\in\mathcal{P}_0} \Big(\mathbb{E}^{\mathbb{P}}\Big[|\xi|^{p}\Big] + \mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T |\widehat{f}_s^{\mathbb{P},0}|ds\Big)^{p}\Big] + \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} |U_s|^{p}\Big]\Big).\end{aligned}$$

□

Proposition 4.3.5. *Let Assumptions 3.2.1 and 4.3.1 hold. Assume $\xi \in \mathbb{L}_0^{p,\kappa}$ and (Y,Z) is the solution to the 2RBSDE(4.2.1). Let $\{(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})\}_{\mathbb{P}\in\mathcal{P}_0}$ be the solutions of the corresponding RBSDEs(4.2.2). Then there exists a constant C depending only on κ, T, p and the Lipschitz constant L_f of f such that*

$$\sup_{\mathbb{P}\in\mathcal{P}_0}\Big\{\|y^{\mathbb{P}}\|^p_{\mathbb{D}_0^p(\mathbb{P})} + \|z^{\mathbb{P}}\|^p_{\mathbb{H}_0^p(\mathbb{P})} + \mathbb{E}^{\mathbb{P}}\Big[[m^{\mathbb{P}}]_T^{\frac{p}{2}}\Big] + \mathbb{E}^{\mathbb{P}}\Big[(k_T^{\mathbb{P}})^p\Big]\Big\} \le C\Big(\|\xi\|^p_{\mathbb{L}_0^{p,\kappa}} + \phi_f^{p,\kappa} + \psi_U^{p,\kappa}\Big),$$

$$\|Y\|^p_{\mathbb{D}_0^p} + \|Z\|^p_{\mathbb{H}_0^p} + \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[[M^{\mathbb{P}}]_T^{\frac{p}{2}}\Big] + \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[\mathrm{Var}_{0,T}(K^{\mathbb{P}})^p\Big] \le C\Big(\|\xi\|^p_{\mathbb{L}_0^{p,\kappa}} + \phi_f^{p,\kappa} + \psi_U^{p,\kappa}\Big).$$

Proof. Throughout this proof we will use the fact that by definition of the norms and Assumptions 3.2.1 and 4.3.1, we have

$$\|\xi\|^p_{\mathbb{L}_0^p} \le \|\xi\|^p_{\mathbb{L}_0^{p,\kappa}}, \quad \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T |\widehat{f}_s^{\mathbb{P},0}|ds\Big)^p\Big] \le \phi_f^{p,\kappa} \quad \text{and} \quad \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} |U_t|^p ds\Big] \le \psi_U^{p,\kappa}.$$

By Lemma 4.3.1 and Proposition 4.3.4, it follows that

$$\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} |y_t^{\mathbb{P}}|^p + \Big(\int_0^T \|\widehat{a}_s^{1/2} z_s^{\mathbb{P}}\|^2\Big)^{p/2} + [m^{\mathbb{P}}]_T^{\frac{p}{2}}\Big] \le C\mathbb{E}^{\mathbb{P}}\Big[|\xi|^p + \Big(\int_0^T |\widehat{f}_s^{\mathbb{P},0}|ds\Big)^p + \sup_{t\in[0,T]} |U_t|^p\Big]$$

Then taking the supremum under $\mathbb{P}$, we obtain

$$\begin{aligned}\sup_{\mathbb{P}\in\mathcal{P}_0}\Big\{\|y^{\mathbb{P}}\|^p_{\mathbb{D}_0^p(\mathbb{P})} + \|z^{\mathbb{P}}\|^p_{\mathbb{H}_0^p(\mathbb{P})} + \mathbb{E}^{\mathbb{P}}\Big[[m^{\mathbb{P}}]_T^{\frac{p}{2}}\Big]\Big\} &\le C \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[|\xi|^p + \Big(\int_0^T |\widehat{f}_s^{\mathbb{P},0}|ds\Big)^p + \sup_{t\in[0,T]} |U_t|^p\Big] \\ &\le C\Big(\|\xi\|_{\mathbb{L}_0^p} + \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T |\widehat{f}_s^{\mathbb{P},0}|ds\Big)^p + \sup_{t\in[0,T]} |U_t|^p\Big]\Big) \\ &\le C\Big(\|\xi\|^p_{\mathbb{L}_0^{p,\kappa}} + \phi_f^{p,\kappa} + \psi_U^{p,\kappa}\Big).\end{aligned}$$

Similarly,

$$\|Y\|^p_{\mathbb{D}^p_0} + \|Z\|^p_{\mathbb{H}^p_0} + \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\Big[[M^{\mathbb{P}}]_T^{\frac{p}{2}}\Big] \leq C \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\Big[|\xi|^p + \Big(\int_0^T |\widehat{f}^{\mathbb{P},0}_s|ds\Big)^p + \sup_{t\in[0,T]} |U_t|^p\Big]$$
$$\leq C\Big(\|\xi\|^p_{\mathbb{L}^{p,\kappa}_0} + \phi_f^{p,\kappa} + \psi_U^{p,\kappa}\Big).$$

It remains to prove the estimates of $k^{\mathbb{P}}$ and $K^{\mathbb{P}}$, we first focus on $k^{\mathbb{P}}$

$$k^{\mathbb{P}}_T = y^{\mathbb{P}}_T - y^{\mathbb{P}}_0 + \int_0^T \widehat{f}^{\mathbb{P}}_s(y^{\mathbb{P}}_s, \widehat{a}^{1/2}_s z^{\mathbb{P}}_s)ds - \int_0^T z^{\mathbb{P}}_s \cdot \widehat{a}^{1/2}_s dW^{\mathbb{P}}_s - m^{\mathbb{P}}_T$$

Then using the Lipschitz property of f, we get

$$(k^{\mathbb{P}}_T)^p \leq 5^{p-1}\Big(|\xi|^p + |y^{\mathbb{P}}_0|^p + \big(\int_0^T |\widehat{f}^{\mathbb{P}}_s(y^{\mathbb{P}}_s, \widehat{a}^{1/2}_s z^{\mathbb{P}}_s)ds|\big)^p + \big(\int_0^T z^{\mathbb{P}}_s dX^{c,\mathbb{P}}_s\big)^p + |m^{\mathbb{P}}_T|^p\Big)$$
$$\leq C\Big(|\xi|^p + |y^{\mathbb{P}}_0|^p + \big(\int_0^T |\widehat{f}^{\mathbb{P},0}_s|ds\big)^p + \big(\int_0^T |y^{\mathbb{P}}|_s ds\big)^p + \big(\int_0^T \|\widehat{a}^{1/2}_s z^{\mathbb{P}}_s\|ds\big)^p$$
$$+ \big(\int_0^T z^{\mathbb{P}}_s dX^{c,\mathbb{P}}_s\big)^p + |m^{\mathbb{P}}_T|^p\Big)$$

with C a constant only depending on p, T and L_f which varies line to line. Then taking the expectation under $\mathbb{P}$ and using Burkholder inequality, we obtain

$$\mathbb{E}^{\mathbb{P}}\Big[(k^{\mathbb{P}}_T)^p\Big] \leq C\mathbb{E}^{\mathbb{P}}\Big[|\xi|^p + \sup_{t\in[0,T]} |y^{\mathbb{P}}_t|^p + \big(\int_0^T |\widehat{f}^{\mathbb{P},0}_s|ds\big)^p + \big(\int_0^T \|\widehat{a}^{1/2}_s z^{\mathbb{P}}_s\|ds\big)^p + \big(\int_0^T z^{\mathbb{P}}_s dX^{c,\mathbb{P}}_s\big)^p + |m^{\mathbb{P}}_T|^p\Big]$$
$$\leq C\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} |y^{\mathbb{P}}_t|^p + \big(\int_0^T |\widehat{f}^{\mathbb{P},0}_s|ds\big)^p + \big(\int_0^T \|\widehat{a}^{1/2}_s z^{\mathbb{P}}_s\|^2 ds\big)^{p/2} + [m^{\mathbb{P}}]_T^{\frac{p}{2}}\Big]$$
$$\leq C\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} |y^{\mathbb{P}}_t|^p + \big(\int_0^T |\widehat{f}^{\mathbb{P},0}_s|ds\big)^p\Big].$$

Taking the supremum under $\mathbb{P}$, the estimates of $k^{\mathbb{P}}$ follows directly. Finally, we have by Proposition 4.3.1, the following decomposition

$$K^{\mathbb{P}}_t = \overline{K}^{\mathbb{P}}_t - k^{\mathbb{P}}_t,$$

where $\overline{K}^{\mathbb{P}}$ is a non decreasing process. Hence we have by definition of total variation and the fact that $\overline{K}^{\mathbb{P}}$ and $k^{\mathbb{P}}$ are non decreasing

$$\mathbb{E}^{\mathbb{P}}\Big[\big(\mathrm{Var}_{0,T}(K^{\mathbb{P}})\big)^p\Big] = \mathbb{E}^{\mathbb{P}}\Big[\big(\mathrm{Var}_{0,T}(\overline{K}^{\mathbb{P}} - k^{\mathbb{P}})\big)^p\Big]$$
$$\leq \mathbb{E}^{\mathbb{P}}\Big[\big(\mathrm{Var}_{0,T}(\overline{K}^{\mathbb{P}}) + \mathrm{Var}_{0,T}(k^{\mathbb{P}})\big)^p\Big]$$
$$\leq 2^{p-1}\mathbb{E}^{\mathbb{P}}\Big[\big(\mathrm{Var}_{0,T}(\overline{K}^{\mathbb{P}})\big)^p + \big(\mathrm{Var}_{0,T}(k^{\mathbb{P}})\big)^p\Big]$$
$$\leq 2^{p-1}\mathbb{E}^{\mathbb{P}}\Big[\big(\overline{K}^{\mathbb{P}}_T\big)^p + \big(k^{\mathbb{P}}_T\big)^p\Big]$$

We can now use (4.3.13) with power p ,

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\Big[(\overline{K}_T^{\mathbb{P}})^p + (k_T^{\mathbb{P}})^p\Big] &\leq C\mathbb{E}^{\mathbb{P}}\Big[|Y_0|^p + |\xi|^p + \Big(\int_0^T |\widehat{f}_s^{\mathbb{P},0}|ds\Big)^p + \int_0^T |Y_s|^p ds + \Big|\int_0^T z_s^{\mathbb{P}} dX_s^{c,\mathbb{P}}\Big|^p \\
&\quad + \Big(\int_0^T \|\widehat{a}_s^{1/2} Z_s)\|^2 ds\Big)^{p/2} + \Big(\int_0^T \|\widehat{a}_s^{1/2} z_s^{\mathbb{P}})\|^2 ds\Big)^{p/2} + \Big|\int_0^T Z_s dX_s^{c,\mathbb{P}}\Big|^p \\
&\quad + |m_T^{\mathbb{P}}|^p + |M_T^{\mathbb{P}}|^p\Big] \\
&\leq C\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} |Y_t|^p + \Big(\int_0^T |\widehat{f}_s^{\mathbb{P},0}|ds\Big)^p + \Big(\int_0^T \|\widehat{a}_s^{1/2} Z_s)\|^2 ds\Big)^{p/2} \\
&\quad + \Big(\int_0^T \|\widehat{a}_s^{1/2} z_s^{\mathbb{P}})\|^2 ds\Big)^{p/2} + [m^{\mathbb{P}}]_T^{\frac{p}{2}} + [M^{\mathbb{P}}]_T^{\frac{p}{2}}\Big] \\
&\leq C\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} |Y_t|^p + \Big(\int_0^T |\widehat{f}_s^{\mathbb{P},0}|ds\Big)^p\Big]
\end{aligned}$$

Then taking the supremum under $\mathbb{P}$,

$$\begin{aligned}
\sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\Big[\big(\mathrm{Var}_{0,T}(K^{\mathbb{P}})\big)^p\Big] &\leq C \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} |Y_t|^p + \Big(\int_0^T |\widehat{f}_s^{\mathbb{P},0}|ds\Big)^p + \sup_{t\in[0,T]} |U_t|^p\Big] \\
&\leq C\Big(\|\xi\|^p_{\mathbb{L}_0^{p,\kappa}} + \phi_f^{p,\kappa} + \psi_U^{p,\kappa}\Big).
\end{aligned}$$

this concludes the proof. □

Theorem 4.3.2. *Let Assumption 3.2.1 hold, and consider two generators f^1 , f^2 and two lower obstacles U^1, U^2 such that Assumption 4.3.1 holds. For $i = 1, 2$, let $((Y^i, Z^i)$ be a solution to the 2RBSDE (4.2.1) with the generator f^i, the terminal condition ξ^i and lower obstacle U^i. Define*

$$\begin{aligned}
\phi^{p,\kappa}_{f^1,f^2} &:= \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\Big[\operatorname*{ess\,sup}_{0\leq t\leq T}{}^{\mathbb{P}} \mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T |\widehat{f}_s^{1,\mathbb{P}} - \widehat{f}_s^{2,\mathbb{P}}|^\kappa (y_s^{1,\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{1,\mathbb{P}}) ds\Big)^{\frac{p}{\kappa}} \big| \mathcal{F}_{t+}^{\mathbb{P}}\Big]\Big] < +\infty, \\
\varphi^{p,\kappa}_{f^1,f^2} &:= \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T |\widehat{f}_s^{1,\mathbb{P}} - \widehat{f}_s^{2,\mathbb{P}}|(Y_s^1, \widehat{a}_s^{1/2} Z_s^1)\Big)^p\Big] + \infty, \\
\psi^{p,\kappa}_{U^1,U^2} &:= \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\Big[\operatorname*{ess\,sup}_{0\leq t\leq T}{}^{\mathbb{P}} \Big(\mathbb{E}^{\mathbb{P}}\Big[\sup_{0\leq s\leq T} |U_s^1 - U_s^2|^\kappa \big| \mathcal{F}_{t+}^{\mathbb{P}}\Big]\Big)^{\frac{p}{\kappa}}\Big] < +\infty.
\end{aligned}$$

Then, there exists a constant C depending only on κ, T and the Lipschitz constant of f^1 and f^2 such that

$$\begin{aligned}
&\|Y^1 - Y^2\|^p_{\mathbb{D}_0^p} \leq C\Big(\|\xi^1 - \xi^2\|^p_{\mathbb{L}_0^{p,\kappa}} + \psi^{p,\kappa}_{U^1,U^2} + \varphi^{p,\kappa}_{f^1,f^2}\Big), \\
&\|Z^1 - Z^2\|^p_{\mathbb{H}_0^p} + \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\Big[[M^{1,\mathbb{P}} - M^{2,\mathbb{P}}]_T^{\frac{p}{2}}\Big] + \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\Big[\mathrm{Var}_{0,T}(K^{1,\mathbb{P}} - K^{2,\mathbb{P}})^p\Big] \\
&\qquad \leq C\Big(\|\xi^1 - \xi^2\|^p_{\mathbb{L}_0^{p,\kappa}} + \phi^{p,\kappa}_{f^1,f^2} + \psi^{p,\kappa}_{U^1,U^2}\Big) \\
&+ C\Big(\|\xi^1 - \xi^2\|^{p/2}_{\mathbb{L}_0^{p,\kappa}} + (\phi^{p,\kappa}_{f^1,f^2})^{p/2} + (\psi^{p,\kappa}_{U^1,U^2})^{p/2}\Big) \times \Big(\|\xi^1\|^p_{\mathbb{L}_0^{p,\kappa}} + \|\xi^2\|^p_{\mathbb{L}_0^{p,\kappa}} + \phi^{p,\kappa}_{f^1} + \phi^{p,\kappa}_{f^2}\Big)^{1/2}.
\end{aligned}$$

Proof. (i) The representation formula (4.3.3) gives,

$$|Y_t^1 - Y_t^2| \leq \operatorname*{ess\,sup}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}{}^{\mathbb{P}} |y_t^{1,\mathbb{P}'}(T, \xi^1) - y_t^{2,\mathbb{P}'}(T, \xi^2)|, \ \mathbb{P}\text{-a.s. for all } \mathbb{P}\in\mathcal{P}_0,\ t\in[0,T].$$

where $y^{1,\mathbb{P}}$ and $y^{2,\mathbb{P}}$ are the RBSDEs associated to Y^1 and Y^2. Following the proof of Proposition A.4.2, there exists a constant C depending on κ, T and the Lipschitz constant of f^2 such that we

have $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}_0$ and $t \in [0,T]$

$$|\tilde{y}_t^{1,\mathbb{P}} - \tilde{y}_t^{2,\mathbb{P}}| \leq C\mathbb{E}^{\mathbb{P}}\Big[\int_0^T |\widehat{f}_s^{1,\mathbb{P}} - \widehat{f}_s^{2,\mathbb{P}}|^\kappa(\tilde{y}_s^{1,\mathbb{P}}, \widehat{a}_s^{1/2}\tilde{z}_s^{1,\mathbb{P}})ds + \sup_{s\in[0,T]}|U_s^1 - U_s^2|^\kappa + |\xi^1 - \xi^2|^\kappa \,|\mathcal{F}_{t+}^{\mathbb{P}}\Big]^{\frac{1}{\kappa}}.$$

Now using Cauchy-Schwarz inequality and (3.3.9), the above inequality provides $\mathbb{P}$-a.s.,

$$\begin{aligned}|\tilde{y}_t^{1,\mathbb{P}} - \tilde{y}_t^{2,\mathbb{P}}|^p &\leq C\mathbb{E}^{\mathbb{P}}\Big[\int_0^T |\widehat{f}_s^{1,\mathbb{P}} - \widehat{f}_s^{2,\mathbb{P}}|^\kappa(\tilde{y}_s^{1,\mathbb{P}}, \widehat{a}_s^{1/2}\tilde{z}_s^{1,\mathbb{P}})ds + \sup_{s\in[0,T]}|U_s^1 - U_s^2|^\kappa + |\xi^1 - \xi^2|^\kappa \,|\mathcal{F}_t^+\Big]^{\frac{p}{\kappa}} \\ &\leq C\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T |\widehat{f}_s^{1,\mathbb{P}} - \widehat{f}_s^{2,\mathbb{P}}|^\kappa(\tilde{y}_s^{1,\mathbb{P}}, \widehat{a}_s^{1/2}\tilde{z}_s^{1,\mathbb{P}})ds\Big)^{\frac{p}{\kappa}} + \Big(\sup_{s\in[0,T]}|U_s^1 - U_s^2|^\kappa\Big)^{\frac{p}{\kappa}} + |\xi^1 - \xi^2|^p \,|\mathcal{F}_t^+\Big]^{\frac{p}{\kappa}}\end{aligned}$$

By the definition of the norms, we deduce the first assertion of the theorem.

(ii) We consider the same notations as in the proof of Proposition A.4.1 and the following notations

$$\delta Y := Y^1 - Y^2,\ \delta Z := Z^1 - Z^2,\ \delta K^{\mathbb{P}} = K^{1,\mathbb{P}} - K^{2,\mathbb{P}},\ \delta M^{\mathbb{P}} := M^{1,\mathbb{P}} - M^{2,\mathbb{P}}.$$

By Definition 3.2.1, we have

$$\begin{aligned}\delta Y_t &= \delta\xi + \int_t^T \big(\widehat{f}_s^{1,\mathbb{P}}(Y_s^1, \widehat{a}_s^{1/2}Z_s^1) - \widehat{f}_s^{2,\mathbb{P}}(Y_s^2, \widehat{a}_s^{1/2}Z_s^2)\big)ds - \int_t^T \delta Z_s \cdot dX_s^{c,\mathbb{P}} - \int_t^T d\delta M_s^{\mathbb{P}} + \int_t^T d\delta K_s^{\mathbb{P}} \\ &= \delta\xi + \int_t^T \big(\delta\widehat{f}_s^{\mathbb{P}}(Y_s^1, \widehat{a}_s^{1/2}Z_s^1) + \lambda_s\delta Y_s + \eta_s \cdot \widehat{a}_s^{1/2}\delta Z_s\big)ds - \int_t^T \delta Z_s \cdot dX_s^{c,\mathbb{P}} - \int_t^T d\delta M_s^{\mathbb{P}} + \int_t^T d\delta K_s^{\mathbb{P}}.\end{aligned}$$

where λ and η are such that $|\lambda| \leq L_{f^2}$ and $\|\eta\| \leq L_{f^2}$(see section 3.3.3). Using Itô's formula to $|\delta Y|^2$ between t and T, we have

$$\begin{aligned}|\delta Y_0|^2 = |\delta\xi|^2 &+ 2\int_0^T \delta Y_t\delta\widehat{f}_t^{\mathbb{P}}(Y_t^1, \widehat{a}_t^{1/2}Z_t^1)dt + 2\int_0^T \lambda_t|\delta Y_t|^2dt + 2\int_0^T \eta_t\widehat{a}_t^{1/2}\delta Y_t\delta Z_t dt \\ &- 2\int_0^T \delta Y_t\delta Z_t \cdot dX_t^{c,\mathbb{P}} - 2\int_0^T \delta Y_{t^-}d\delta M_t^{\mathbb{P}} + 2\int_0^T \delta Y_{t^-}d\delta K_t^{\mathbb{P}} \\ &- \int_0^T \|\widehat{a}_t^{1/2}\delta Z_t\|^2dt - [\delta M^{\mathbb{P}}]_T - \sum_{0<t\leq T}|\Delta\delta Y_t|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\int_0^T \|\widehat{a}_t^{1/2}\delta Z_t\|^2dt + [\delta M^{\mathbb{P}}]_T &\leq |\delta\xi|^2 + 2\int_0^T |\delta Y_t||\delta\widehat{f}_t^{\mathbb{P}}(Y_t^1, \widehat{a}_t^{1/2}Z_t^1)|dt + 2\int_0^T \lambda_t|\delta Y_t|^2dt \\ &\quad + 2\int_0^T \eta_t\widehat{a}_t^{1/2}\delta Y_t\delta Z_t dt - 2\int_0^T \delta Y_t\delta Z_t \cdot dX_t^{c,\mathbb{P}} \\ &\quad - 2\int_0^T \delta Y_{t^-}d\delta M_t^{\mathbb{P}} + 2\int_0^T \delta Y_{t^-}d\delta K_t^{\mathbb{P}} \\ &\leq |\delta\xi|^2 + 2\sup_{t\in[0,T]}|\delta Y_t| \times \int_0^T |\delta\widehat{f}_t^{\mathbb{P}}(Y_t^1, \widehat{a}_t^{1/2}Z_t^1)|dt + \frac{1}{2}\int_0^T \|\widehat{a}_t^{1/2}\delta Z_t\|^2dt \\ &\quad + (2L_{f^2} + 2L_{f^2}^2)T\sup_{t\in[0,T]}|\delta Y_t|^2 - 2\int_0^T \delta Y_t\delta Z_t \cdot dX_t^{c,\mathbb{P}} \\ &\quad - 2\int_0^T \delta Y_{t^-}d\delta M_t^{\mathbb{P}} + 2\sup_{t\in[0,T]}|\delta Y_t|(\mathrm{Var}_{0,T}(K^{1,\mathbb{P}}) + \mathrm{Var}_{0,T}(K^{2,\mathbb{P}})) \\ &\leq |\delta\xi|^2 + (2L_{f^2}T + 2L_{f^2}^2T)\sup_{t\in[0,T]}|\delta Y_t|^2 + \frac{1}{2}\int_0^T \|\widehat{a}_t^{1/2}\delta Z_t\|^2dt \\ &\quad + \Big(\int_0^T |\delta\widehat{f}_t^{\mathbb{P}}(Y_t^1, \widehat{a}_t^{1/2}Z_t^1)|dt\Big)^2 + 2\Big|\int_0^T \delta Y_t\delta Z_t \cdot dX_t^{c,\mathbb{P}}\Big| \\ &\quad + \Big|2\int_0^T \delta Y_{t^-}d\delta M_t^{\mathbb{P}}\Big| + \sup_{t\in[0,T]}|\delta Y_t|(\mathrm{Var}_{0,T}(K^{1,\mathbb{P}}) + \mathrm{Var}_{0,T}(K^{2,\mathbb{P}}))\end{aligned}$$

Hence,

$$\frac{1}{2}\int_0^T \|\widehat{a}_t^{1/2}\delta Z_t\|^2 dt + [\delta M^{\mathbb{P}}]_T \le C\Big(|\delta\xi|^2 + \sup_{t\in[0,T]}|\delta Y_t|^2 + \Big(\int_0^T |\delta \widehat{f}_t^{\mathbb{P}}(Y_t^1, \widehat{a}_t^{1/2}Z_t^1)|dt\Big)^2 + \Big|\int_0^T \delta Y_t \delta Z_t \cdot dX_t^{c,\mathbb{P}}\Big| + \Big|\int_0^T \delta Y_{t^-} d\delta M_t^{\mathbb{P}}\Big| + \sup_{t\in[0,T]}|\delta Y_t|(\mathrm{Var}_{0,T}(K^{1,\mathbb{P}}) + \mathrm{Var}_{0,T}(K^{2,\mathbb{P}}))\Big).$$

Therefore, by inequality (3.3.9) we get

$$2^{\frac{p}{2}-1}\Big(\int_0^T \|\widehat{a}_t^{1/2}\delta Z_t\|^2 dt\Big)^{\frac{p}{2}} + (1\wedge 2^{\frac{p}{2}-1})[\delta M^{\mathbb{P}}]_T^{\frac{p}{2}} \le 7^{\frac{p}{2}-1}C\Big(|\delta\xi|^p + \sup_{t\in[0,T]}|\delta Y_t|^p + \Big(\int_0^T |\delta \widehat{f}_t^{\mathbb{P}}(Y_t^1, \widehat{a}_t^{1/2}Z_t^1)|dt\Big)^p + \Big|\int_0^T \delta Y_t \delta Z_t \cdot dX_t^{c,\mathbb{P}}\Big|^{\frac{p}{2}} + \Big|\int_0^T \delta Y_{t^-} d\delta M_t^{\mathbb{P}}\Big|^{\frac{p}{2}} + \sup_{t\in[0,T]}|\delta Y_t|^{\frac{p}{2}}(\mathrm{Var}_{0,T}(K^{1,\mathbb{P}}) + \mathrm{Var}_{0,T}(K^{2,\mathbb{P}}))^{\frac{p}{2}}\Big).$$

Taking the expectation under $\mathbb{P}$ gives us

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T \|\widehat{a}_t^{1/2}\delta Z_t\|^2 dt\Big)^{\frac{p}{2}}\Big] + \mathbb{E}^{\mathbb{P}}\big[[\delta M^{\mathbb{P}}]_T^{\frac{p}{2}}\big] \le C\mathbb{E}^{\mathbb{P}}\Big[|\delta\xi|^p + \sup_{t\in[0,T]}|\delta Y_t|^p + \Big(\int_0^T |\delta \widehat{f}_t^{\mathbb{P}}(Y_t^1, \widehat{a}_t^{1/2}Z_t^1)|dt\Big)^p\Big] + \mathbb{E}^{\mathbb{P}}\Big[\Big|\int_0^T \delta Y_t \delta Z_t \cdot dX_t^{c,\mathbb{P}}\Big|^{\frac{p}{2}}\Big] + \mathbb{E}^{\mathbb{P}}\Big[\Big|\int_0^T \delta Y_{t^-} d\delta M_t^{\mathbb{P}}\Big|^{\frac{p}{2}}\Big] + \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^p\Big]^{1/2}\Big]\mathbb{E}^{\mathbb{P}}\Big[(\mathrm{Var}_{0,T}(K^{1,\mathbb{P}})^p + \mathrm{Var}_{0,T}(K^{2,\mathbb{P}})^p\Big]^{1/2}.$$

Now the Burkholder's inequality allows us to write

$$\mathbb{E}^{\mathbb{P}}\Big[\Big|\int_0^T \delta Y_{t^-} d\delta M_t^{\mathbb{P}}\Big|^{\frac{p}{2}}\Big] \le C_p\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T |\delta Y_{t^-}|^2 d[\delta M^{\mathbb{P}}]_t\Big)^{\frac{p}{4}}\Big] \le C_p\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{\frac{p}{2}}[\delta M^{\mathbb{P}}]_T^{\frac{p}{4}}\Big] \le 2C_p^2\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^p\Big] + \frac{1}{8}\mathbb{E}^{\mathbb{P}}\Big[[\delta M^{\mathbb{P}}]_T^{\frac{p}{2}}\Big]$$

and

$$\mathbb{E}^{\mathbb{P}}\Big[\Big|\int_0^T \delta Y_t \delta Z_t \cdot dX_t^{c,\mathbb{P}}\Big|^{\frac{p}{2}}\Big] \le C_p\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T |\delta Y_t|^2 \|\widehat{a}_t^{1/2}\delta Z_t\|^2 dt\Big)^{\frac{p}{4}}\Big] \le C_p\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{\frac{p}{2}}\Big(\int_0^T \|\widehat{a}_t^{1/2}\delta Z_t\|^2 dt\Big)^{\frac{p}{4}}\Big] \le 2C_p^2\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^p\Big] + \frac{1}{8}\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T \|\widehat{a}_t^{1/2}\delta Z_t\|^2 dt\Big)^{\frac{p}{2}}\Big]$$

We deduce that there exists a constant C depending on T, p, L_f^1 and L_f^2 such that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T \|\widehat{a}_t^{1/2}\delta Z_t\|^2 dt\Big)^{\frac{p}{2}}\Big] + \mathbb{E}^{\mathbb{P}}\big[[\delta M^{\mathbb{P}}]_T^{\frac{p}{2}}\big] \\
&\quad \le C\mathbb{E}^{\mathbb{P}}\Big[|\delta\xi|^p + \sup_{t\in[0,T]}|\delta Y_t|^p + \Big(\int_0^T |\delta \widehat{f}_t^{\mathbb{P}}(Y_t^1, \widehat{a}_t^{1/2}Z_t^1)|dt\Big)^p\Big] \\
&\quad + \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^p\Big]^{1/2}\mathbb{E}^{\mathbb{P}}\Big[(\mathrm{Var}_{0,T}(K^{1,\mathbb{P}})^p + \mathrm{Var}_{0,T}(K^{2,\mathbb{P}})^p\Big]^{1/2}.
\end{aligned}$$

By the same arguments as in the proof of Proposition 4.3.5, we show that for $i = 1, 2$

$$\mathbb{E}^{\mathbb{P}}\big[(\mathrm{Var}_{0,T}(K^{i,\mathbb{P}}))^p\big] \le C\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|Y_t^i|^p + \Big(\int_0^T |\widehat{f}_s^{i,\mathbb{P},0}|ds\Big)^p\Big].$$

Combining the two above estimates, we get

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T \|\widehat{a}_t^{1/2}\delta Z_t\|^2 dt\Big)^{\frac{p}{2}}\Big] + \mathbb{E}^{\mathbb{P}}\big[[\delta M^{\mathbb{P}}]_T^{\frac{p}{2}}\big] \\
&\le C\mathbb{E}^{\mathbb{P}}\Big[|\delta\xi|^p + \sup_{t\in[0,T]}|\delta Y_t|^p + \Big(\int_0^T |\delta \widehat{f}_t^{\mathbb{P}}(Y_t^1, \widehat{a}_t^{1/2}Z_t^1)|dt\Big)^p \\
&+ \Big(|\delta\xi|^{p/2} + \sup_{t\in[0,T]}|\delta Y_t|^{p/2} + \Big(\int_0^T |\delta \widehat{f}_t^{\mathbb{P}}(Y_t^1, \widehat{a}_t^{1/2}Z_t^1)|dt\Big)^{p/2}\Big)\times \\
&\Big(\sup_{t\in[0,T]}|Y_t^1|^p + \sup_{t\in[0,T]}|Y_t^2|^p + \Big(\int_0^T |\widehat{f}_s^{1,\mathbb{P},0}|ds\Big)^p + \Big(\int_0^T |\widehat{f}_s^{2,\mathbb{P},0}|ds\Big)^p\Big)^{1/2}\Big].
\end{aligned}$$

Then taking the supremum over $\mathbb{P} \in \mathcal{P}_0$ in the above estimates, we have

$$\begin{aligned}
\big\|Z^1 - Z^2\big\|_{\mathbb{H}_0^p}^p + \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[\big[M^{1,\mathbb{P}} - M^{2,\mathbb{P}}\big]_T^{\frac{p}{2}}\Big] &\le C\Big(\big\|\xi^1 - \xi^2\big\|_{\mathbb{L}_0^{p,\kappa}}^p + \phi_{f^1,f^2}^{p,\kappa} + \psi_{L^1,L^2}^{p,\kappa} + \\
&\Big(\big\|\xi^1 - \xi^2\big\|_{\mathbb{L}_0^{p,\kappa}}^{p/2} + \big(\phi_{f^1,f^2}^{p,\kappa}\big)^{p/2} + \big(\psi_{L^1,L^2}^{p,\kappa}\big)^{p/2}\Big)\times \\
&\Big(+\big\|\xi^1\big\|_{\mathbb{L}_0^{p,\kappa}}^p + \big\|\xi^2\big\|_{\mathbb{L}_0^{p,\kappa}}^p + \phi_{f^1}^{p,\kappa} + \phi_{f^2}^{p,\kappa}\Big)^{1/2}.
\end{aligned}$$

It remains to prove the estimates of $\delta K^{\mathbb{P}}$. By definition, By Definition 3.2.1, we have

$$\begin{aligned}
\delta K_T^{\mathbb{P}} &= \delta Y_0 - \delta\xi - \int_0^T \big(\widehat{f}_s^{1,\mathbb{P}}(Y_s^1, \widehat{a}_s^{1/2}Z_s^1) - \widehat{f}_s^{2,\mathbb{P}}(Y_s^2, \widehat{a}_s^{1/2}Z_s^2)\big)ds + \int_0^T \delta Z_s \cdot dX_s^{c,\mathbb{P}} + \delta M_T^{\mathbb{P}} \\
&= \delta Y_0 - \delta\xi - \int_0^T \big(\delta \widehat{f}_s^{\mathbb{P}}(Y_s^1, \widehat{a}_s^{1/2}Z_s^1) + \lambda_s\delta Y_s + \eta_s \cdot \widehat{a}_s^{1/2}\delta Z_s\big)ds + \int_0^T \delta Z_s \cdot dX_s^{c,\mathbb{P}} + \delta M_T^{\mathbb{P}}.
\end{aligned}$$

Next, there exists a constant C which may vary line to line and only depending on p, T and the Lipschitz constants of f^1 and f^2 such that

$$\begin{aligned}
(\delta K_T^{\mathbb{P}})^p \le C\Big[&|\delta Y_0|^p + |\delta\xi|^p + \Big(\int_0^T |\delta \widehat{f}_s^{\mathbb{P}}(Y_s^1, \widehat{a}_s^{1/2}Z_s^1)|ds\Big)^p + \Big(\int_0^T \lambda_s|\delta Y_s|ds\Big)^p \\
&+ \Big(\int_0^T |\eta_s \cdot \widehat{a}_s^{1/2}\delta Z_s|ds\Big)^p ds + \Big(\int_0^T \delta Z_s \cdot dX_s^{c,\mathbb{P}}\Big)^p + \big(\delta M_T^{\mathbb{P}}\big)^p\Big].
\end{aligned}$$

After that, we take the expectation under $\mathbb{P}$ in the above and use the Burkholder's and Young's inequalities to obtain

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\Big[(\delta K_T^{\mathbb{P}})^p\Big] \le \mathbb{E}^{\mathbb{P}}\Big[&|\delta\xi|^p + \sup_{t\in[0,T]}|\delta Y_t|^p + \Big(\int_0^T |\delta \widehat{f}_s^{\mathbb{P}}(Y_s^1, \widehat{a}_s^{1/2}Z_s^1)|ds\Big)^p \\
&+ \Big(\int_0^T \|\widehat{a}_t^{1/2}\delta Z_t\|^2 dt\Big)^{\frac{p}{2}} + [\delta M^{\mathbb{P}}]_T^{\frac{p}{2}}\Big].
\end{aligned}$$

Moreover,

$$\mathbb{E}^{\mathbb{P}}\left[\left(\mathrm{Var}_{0,T}(K^{\mathbb{P}})\right)^p\right] = \mathbb{E}^{\mathbb{P}}\left[\left(\mathrm{Var}_{0,T}(\delta\overline{K}^{\mathbb{P}} - \delta k^{\mathbb{P}})\right)^p\right] \leq 2^{p-1}\mathbb{E}^{\mathbb{P}}\left[\left(\delta\overline{K}_T^{\mathbb{P}}\right)^p + \left(\delta k_T^{\mathbb{P}}\right)^p\right].$$

Since we know the estimates of $\delta k^{\mathbb{P}}$, we replace into the above estimates and get the expected estimate of $\delta K^{\mathbb{P}}$ by taking the supremum over $\mathbb{P} \in \mathcal{P}_0$ in the above inequality and using the estimates of $\delta Y, \delta Z$ and δM. □

4.4 Existence of the solution

The proof of existence of the solution follows the same scheme that in Chapter 3. Following the representation formula of 2RBSDEs (4.3.3), a natural candidate to the solution of 2RBSDEs is: For every $(t,\omega) \in [0,T] \times \Omega$,

$$V_t(\omega) := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}[y_t^{\mathbb{P}}],$$

where $y^{\mathbb{P}}$ is the first component of the solution of RBSDE(4.2.5).
The proof of existence of a solution of 2RBSDE(4.2.1) will be divided into the following steps:
Step one. In order to establish the dynamic programming principle for the above value function V, we need a jointly measurable (with respect to time, space and probability $\mathbb{P}$) version of $y^{\mathbb{P}}$ solution of RBSDEs. We recall that for every $\mathbb{P} \in \mathcal{P}_0$, these RBSDEs already have a unique solution then our goal is to construct a jointly measurable version of the solutions . Thereby, we use the Picard iteration of the solution to the penalized BSDEs and prove the converge of the iterations.
Step two. After the convergence of approximations, the resulting solutions $y_t^{\mathbb{P}}$ can be interpreted as a function of t, ω and $\mathbb{P}$. We now show that $y^{\mathbb{P}}$ is jointly measurable: that means $(t,\omega,\mathbb{P}) \mapsto y_t^{\mathbb{P}}$ is a measurable function.
Step three. This step is dedicated to establish the following dynamic programming principle for the value function.

$$V_t(\omega) = \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\left[y_t^{\mathbb{P}}(\tau, V_\tau)\right]$$

where τ is a stopping time taking value in $[t,T]$
Step four. This last step consists in path modification of the value function in order to obtain a càdlàg process and deduce the solutions by the Doob decomposition.

4.4.1 Construction of measurable version of solution

We recall the classical construction of the $y^{\mathbb{P}}$ part of the solution to the RBSDE (4.2.5) under some probability $\mathbb{P} \in \mathcal{P}_0$ using Penalization and Picard's iterations. These approximations are very closed to the definition of BSDEs in general filtration, for more details let refer to section 2.5. The obstacle is represented by U which is a càdlàg $\mathbb{F}^{\mathcal{P}_0+}$-progressively measurable process. For $n \in \mathbb{N}$, let $(y^{\mathbb{P},n}, z^{\mathbb{P},n}, m^{\mathbb{P},n}, k^{\mathbb{P},n})$ be the solution of the following penalized BSDE

$$y_t^{\mathbb{P},n} = \xi + \int_t^T \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P},n}, \widehat{a}_s^{1/2} z_s^{\mathbb{P},n})ds - \int_t^T z_s^{\mathbb{P},n} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_t^T dm_s^{\mathbb{P},n} - n\int_t^T (U_s - y_s^{\mathbb{P},n})^- ds. \tag{4.4.1}$$

with

$$k_t^{\mathbb{P},n} = n\int_0^t (U_s - y_s^{\mathbb{P},n})^- ds.$$

We prove in section A.3.2 that the solution of the penalized BSDEs to an upper obstacle converges to the solution of RBSDEs. Since for the case of standard BSDEs there is a symmetry between lower an upper obstacles, the results in the literature are for the most part related to the lower obstacle. We can mention [35] for continuous obstacle and [60] for a càdlàg obstacle and those result concerned a filtration generated by a Brownian motion. In section A.3.2, we take the reasoning to the above papers and show that the presence of a martingale m does not affect the result. More generally one can also refer to Klimsiak [54] for RBSDEs in general filtration.

Therefore $(y^{\mathbb{P},n}, z^{\mathbb{P},n}, m^{\mathbb{P},n}, k^{\mathbb{P},n})$ converge to $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$ the solution of the RBSDE (4.2.5). Now for a fixed n, let us define $y_t^{\mathbb{P},n,0} \equiv 0$, $z_t^{\mathbb{P},n,0} \equiv 0$, $m_t^{\mathbb{P},n,0} \equiv 0$ and $k_t^{\mathbb{P},n,0} \equiv 0$, for all $t \in [0,T]$ and given a family of $\mathbb{F}_+$-progressively measurable processes $\left(y_t^{\mathbb{P},n,m}, z_t^{\mathbb{P},n,m}, m_t^{\mathbb{P},n,m}, k_t^{\mathbb{P},n,m}\right)_{t\in[0,T]}$,

$$y_t^{\mathbb{P},n,m+1} = \xi + \int_t^T \widehat{f}_s^{\mathbb{P},n}(y_s^{\mathbb{P},n,m}, \widehat{a}_s^{1/2} z_s^{\mathbb{P},n,m})ds - \int_t^T z_s^{\mathbb{P},n,m+1} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_t^T dm_s^{\mathbb{P},n,m+1} \quad \mathbb{P}\text{-a.s.} \tag{4.4.2}$$

where $\widehat{f}_s^{\mathbb{P},n}(x,y) := \widehat{f}_s^{\mathbb{P}}(x,y) - n(U_s - y)^-$. By definition $\widehat{f}^{\mathbb{P},n}$ still verifies the Lipschitz assumption in y and z.
We have that $y^{\mathbb{P},n,m+1}$ is a semi-martingale under $\mathbb{P}$. Let $\langle y^{\mathbb{P},n+1}, X\rangle^{\mathbb{P}}$ be the predictable quadratic covariation of the process $y^{\mathbb{P},n,m+1}$ and X under $\mathbb{P}$. Define

$$\widehat{a}_s^{1/2} z_s^{\mathbb{P},n,m+1} := \limsup_{\epsilon\in\mathbb{Q},\epsilon\downarrow 0} \frac{\langle y^{\mathbb{P},n,m+1}, X\rangle_s^{\mathbb{P}} - \langle y^{\mathbb{P},n,m+1}, X\rangle_{s-\epsilon}^{\mathbb{P}}}{\epsilon}. \tag{4.4.3}$$

Convergence. Using the same arguments as in the case of lower obstacle, we show that the sequence $(y^{\mathbb{P},n,m}, z^{\mathbb{P},n,m})_{m\geq 0}$ is a Cauchy sequence for the norm

$$\|(y,z,m)\|_{\mathbb{P},\alpha} := \left(\mathbb{E}^{\mathbb{P}}\Big[\int_0^T e^{\alpha s}|y_s|^2 ds + \int_0^T e^{\alpha s}\|\widehat{a}_s^{1/2} z_s\|^2 ds + \int_0^T e^{\alpha s} d[m]_s\Big]\right)^{\frac{1}{2}}. \tag{4.4.4}$$

for α large enough.

$$\begin{aligned} y_t^{\mathbb{P},n,m+1} - y_t^{\mathbb{P},n,m} = &\int_t^T \left(\widehat{f}_s^{\mathbb{P},n}(y_s^{\mathbb{P},n,m}, \widehat{a}_s^{1/2} z_s^{\mathbb{P},n,m}) - \widehat{f}_s^{\mathbb{P},n}(y_s^{\mathbb{P},n,m-1}, \widehat{a}_s^{1/2} z_s^{\mathbb{P},n,m-1})\right) ds \\ &- \int_t^T (z_s^{\mathbb{P},n,m+1} - z_s^{\mathbb{P},n,m}) \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_t^T d(m_s^{\mathbb{P},n,m+1} - m_s^{\mathbb{P},n,m}). \end{aligned}$$

In much the same way as for the lower obstacle case in section 3.4.1, it follows that $(y^{\mathbb{P},n,m}, z^{\mathbb{P},n,m}, m^{\mathbb{P},n,m})_{m\geq 0}$ is a Cauchy sequence, which converge to $y^{\mathbb{P},n}, z^{\mathbb{P},n}, m^{\mathbb{P},n})_{n\geq 0}$ the solution of the penalized BSDE 4.4.1. In other words, by taking some suitable sub-sequence $\left(m_k^{\mathbb{P}}\right)_{k\geq 1}$ if necessary, we can define the solution to the penalized BSDE(4.4.1) as

$$y_t^{\mathbb{P},n} := \limsup_{k\to\infty} y_t^{\mathbb{P},n,m_k^{\mathbb{P}}}, \quad z_t^{\mathbb{P},n} := \limsup_{k\to\infty} z_t^{\mathbb{P},n,m_k^{\mathbb{P}}} \quad \text{and} \quad m_t^{\mathbb{P},n} := \limsup_{k\to\infty} m_t^{\mathbb{P},n,m_k^{\mathbb{P}}}. \tag{4.4.5}$$

Since the solution to the penalized BSDE (4.4.1) converge to the solution of the reflected BSDE (4.2.5), then there exists $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$ the solution of (4.2.5) such that for $t \in [0,T]$

$$y_t^{\mathbb{P}} := \lim_{n\to\infty} y_t^{\mathbb{P},n}, \quad z_t^{\mathbb{P}} := \lim_{n\to\infty} z_t^{\mathbb{P},n} \quad m_t^{\mathbb{P}} := \lim_{n\to\infty} m_t^{\mathbb{P},n} \text{ and } k_t^{\mathbb{P}} := \lim_{n\to\infty} k_t^{\mathbb{P},n}.$$

4.4.2 Measurability of the constructed solution

In this paragragh, we justify the measurability with respect to probability $\mathbb{P}$ of the construction in Section 3.4.1. The following result extends Lemma 3.4.2 of Chapter 3 in the upper case.

Lemma 4.4.1. *For each $n \geq 1$, there exists a measurable map $(t,\omega,\mathbb{P}) \mapsto (y_t^{\mathbb{P},n}(\omega), z_t^{\mathbb{P},n}(\omega), m_t^{\mathbb{P},n}(\omega), k_t^{\mathbb{P},n}(\omega))$ such that for every $\mathbb{P} \in \mathcal{P}_0$, we have the following properties:*

(i) $y^{\mathbb{P},n}$ is right continuous, $\mathbb{F}_+$-adapted and $\mathbb{F}_+^{\mathbb{P}}$-optional;

(ii) $z^{\mathbb{P},n}$ is $\mathbb{F}_+$-adapted and $\mathbb{F}_+^{\mathbb{P}}$-predictable ;

(iii) $m^{\mathbb{P},n}$ is right continuous, $\mathbb{F}_+^{\mathbb{P}}$-martingale orthogonal to X under $\mathbb{P}$ such that $(y^{\mathbb{P},n}, z^{\mathbb{P},n}, m^{\mathbb{P},n})$ satisfies

$$y_t^{\mathbb{P},n} = \xi + \int_t^T \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P},n}, \widehat{a}_s^{1/2} z_s^{\mathbb{P},n})ds - \int_t^T z_s^{\mathbb{P},n} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_t^T dm_s^{\mathbb{P},n} - n\int_t^T (U_s - y_s^{\mathbb{P},n})^- ds. \tag{4.4.6}$$

Proof. **(i)-** For every $t \in [0,T]$, (4.4.5) leads to

$$y_t^{\mathbb{P},n} = \lim_{m\to\infty} y_t^{\mathbb{P},n,m},$$

with

$$y_t^{\mathbb{P},n,m+1} = \xi + \int_t^T \widehat{f}_s^{\mathbb{P},n}(y_s^{\mathbb{P},n,m}, \widehat{a}_s^{1/2} z_s^{\mathbb{P},n,m})ds - \int_t^T z_s^{\mathbb{P},n,m+1} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_t^T dm_s^{\mathbb{P},n,m+1} \quad \mathbb{P}\text{-a.s.} \tag{4.4.7}$$

To prove the statement (i), we first prove by induction under m that $y^{\mathbb{P},n,m}$ is right continuous, $\mathbb{F}_+$-adapted and $\mathbb{F}_+^{\mathbb{P}}$-optional and then deduce the result by tending m to infinity.

(a)-Basis $m = 1$: By definition $y^{\mathbb{P},n,0} \equiv 0$, then by setting $m = 0$ and by taking conditional expectation under $\mathbb{P}$ with respect to $\mathcal{F}_{t+}$, we have

$$y_t^{\mathbb{P},n,1} = \mathbb{E}^{\mathbb{P}}\left[\xi + \int_t^T \widehat{f}_s^{\mathbb{P},n}(0,0)ds \,\middle|\mathcal{F}_{t+}\right]$$

Since f is jointly measurable, we can apply Lemma 2.4.1. This Lemma states that there is a version of conditional expectation $\mathbb{E}^{\mathbb{P}}\left[\xi + \int_t^T \widehat{f}_r^{\mathbb{P},n}(0,0)dr \middle|\mathcal{F}_{t+}\right]$ measurable with respect to $\mathbb{P}$ and ω, in other words, $(\mathbb{P},\omega) \mapsto y_t^{\mathbb{P},n,1}(\omega)$ is $\mathscr{B}\otimes\mathcal{F}_{t+}$-measurable. With this jointly measurability version and the fact that $y_t^{\mathbb{P},n,1}$ is $\mathcal{F}_{t+}$-measurable, we use Lemma 2.4.3 to choose for any $\mathbb{P} \in \mathcal{P}_0$ a $\mathbb{P}$-modification of $y^{\mathbb{P},n,1}$ which is right-continuous, $\mathbb{F}_+$-adapted and $\mathbb{F}_+^{\mathbb{P}}$-optional and then the property is verified for $m = 1$.

(b)- Inductive step: Assume that assertion the property holds to the order m, then by definition we have

$$y_t^{\mathbb{P},n,m+1} := \mathbb{E}^{\mathbb{P}}\left[\xi + \int_t^T \widehat{f}_r^{\mathbb{P},n}(y_s^{\mathbb{P},n,m}, z_s^{\mathbb{P},n,m})dr \,\middle|\mathcal{F}_{t+}\right], \quad \mathbb{P}\text{-a.s.}$$

and the same reasoning to the above about measurability applies to $y^{\mathbb{P},n,m+1}$ implies that there exists a measurable map $(t,\omega,\mathbb{P}) \mapsto y_t^{\mathbb{P},n,m+1}(\omega)$ such that for every $\mathbb{P} \in \mathcal{P}_0$, $y^{\mathbb{P},n,m+1}$ is right-continuous, $\mathbb{F}_+$-adapted and $\mathbb{F}_+$-optional.
thereby showing that indeed $(t,\omega,\mathbb{P}) \mapsto y_t^{\mathbb{P},n,m+1}(\omega)$ holds to the order $n+1$, Since both the basis and the inductive step have been performed, by mathematical induction, for all natural numbers $m \geq 1$, $y^{\mathbb{P},n,m}$ is right continuous, $\mathbb{F}_+$-adapted and $\mathbb{F}_+^{\mathbb{P}}$-optional.

(c)- The passage to the limit: The sequence $(y^{\mathbb{P},n,m})_{m\geq 1}$ is the Picard iterations and we have seen in the above section that $(y^{\mathbb{P},n,m})_{m\geq 1}$ is Cauchy sequence under the $(\mathbb{P},\alpha)$-norm(for $\alpha > 0$ large enough) and that $(y^{\mathbb{P},n,m})_{m\geq 1}$ converge uniformly $\mathbb{P}$-a.s. Therefore, $y^{\mathbb{P},n,m}$ converges (under the $(\mathbb{P},\alpha)$-norm) to some process $y^{\mathbb{P},n}$ as $m \to \infty$ which solves the penalized

BSDE (4.4.1). Following the previous paragraph the iteration $y^{\mathbb{P},n,m}$ are jointly measurable with respect to time, space and probability law . Now we can use the Lemma 2.4.2 to find a family of subsequence $\big((m_k^{\mathbb{P}})_{k\geq 1}, \mathbb{P}\in\mathcal{P}_0\big)$ such that the limit of $y_t^{\mathbb{P},n,m_k^{\mathbb{P}}}$ is jointly measurable. And this jointly measurable limit process still the solution of the penalized BSDE (4.4.1).

(ii) We prove the second statement using (4.4.5), $z^{\mathbb{P},n}$ is the limit of the Picard iteration $(z^{\mathbb{P},n,m+1})_m$ also represented by

$$\widehat{a}_s^{1/2} z_s^{\mathbb{P},n,m+1} := \limsup_{\epsilon\in\mathbb{Q},\epsilon\downarrow 0} \frac{\langle y^{\mathbb{P},n,m+1}, X\rangle_s^{\mathbb{P}} - \langle y^{\mathbb{P},n,m+1}, X\rangle_{s-\epsilon}^{\mathbb{P}}}{\epsilon}.$$

The same reasoning applies to $z^{\mathbb{P},n,m}$ and a passage to the limit similar to above allows to deduce the result. First let show that by induction under m that $z^{\mathbb{P},n,m+1}$ is $\mathbb{F}_+$-adapted and $\mathbb{F}_+^{\mathbb{P}}$-predictable.

(a)-Basis $m=1$**:** For every $t\in[0,T]$, we have

$$\widehat{a}_s^{1/2} z_s^{\mathbb{P},n,1} := \limsup_{\epsilon\in\mathbb{Q},\epsilon\downarrow 0} \frac{\langle y^{\mathbb{P},n,1}, X\rangle_s^{\mathbb{P}} - \langle y^{\mathbb{P},n,1}, X\rangle_{s-\epsilon}^{\mathbb{P}}}{\epsilon}.$$

The definition of quadratic covariation in (3.4.11), Lemma 3.4.1 and the first part of this proof(i) prove that there is a measurable version of the function $(t,\omega,\mathbb{P})\longmapsto \langle y^{\mathbb{P},n,1}, X\rangle_t^{\mathbb{P}}(\omega)$, such that for every $\mathbb{P}\in\mathcal{P}_0$, $\langle y^{\mathbb{P},n,1}, X\rangle^{\mathbb{P}}$ is right-continuous, $\mathbb{F}_+$-adapted and coincides with the predictable quadratic covariation of $y^{\mathbb{P},n,1}$ and X under $\mathbb{P}$. With this version of $\langle y^{\mathbb{P},n,1}, X\rangle^{\mathbb{P}}$, it is clear that the family $z^{\mathbb{P},n,1}$ defined above is measurable in $(t,\omega,\mathbb{P})$ and for every $\mathbb{P}\in\mathcal{P}_0, z^{\mathbb{P},n,1}$ is $\mathbb{F}_+$-adapted and $\mathbb{F}_+^{\mathbb{P}}$-predictable. Therefore assertion is verified for $m=1$.

(b)- Inductive step: Assume that the property holds to the order m,

$$\widehat{a}_s^{1/2} z_s^{\mathbb{P},n,m+1} := \limsup_{\epsilon\in\mathbb{Q},\epsilon\downarrow 0} \frac{\langle y^{\mathbb{P},n,m+1}, X\rangle_s^{\mathbb{P}} - \langle y^{\mathbb{P},n,m+1}, X\rangle_{s-\epsilon}^{\mathbb{P}}}{\epsilon}.$$

Similar arguments to the basis step (a) induce that there is a measurable version of the map $(t,\omega,\mathbb{P})\mapsto z_t^{\mathbb{P},n,m+1}$ such that for every $\mathbb{P}\in\mathcal{P}_0, z^{\mathbb{P},n,m+1}(\omega)$ is $\mathbb{F}_+$-adapted and $\mathbb{F}_+^{\mathbb{P}}$-predictable. Therefore we conclude by induction that $z^{\mathbb{P},n,m+1}$ is $\mathbb{F}_+$-adapted and $\mathbb{F}_+^{\mathbb{P}}$-predictable for all natural numbers $m\geq 1$.

(c)- The passage to the limit: $(z^{\mathbb{P},n,m})_{m\geq 1}$ is a Picard iteration and a Cauchy sequence under the $(\mathbb{P},\alpha)$-norm, then by (3.4.8) $(z^{\mathbb{P},n,m})_{m\geq 1}$ converge uniformly $\mathbb{P}$-a.s. to some process $z^{\mathbb{P},n}$ which is the z- component of the solution to the penalized BSDE (4.4.1). By the same way, we can use the Lemma 2.4.2 to find a family of subsequence $\big((m_k^{\mathbb{P}})_{k\geq 1}, \mathbb{P}\in\mathcal{P}_0\big)$ such that the limit $z^{\mathbb{P},n}$ of $z_t^{\mathbb{P},n,m_k^{\mathbb{P}}}$ still jointly measurable and verifies (4.4.1). This conclude the proof of (ii).

(iii) For $n\geq 1$, on the one hand by (4.4.5) $m^{\mathbb{P},n}$ is a limit of Picard sequence $(m^{\mathbb{P},n,m})_{m\geq 0}$ which verifies 4.4.2, thus for a fixed $m\geq 0$, $(m^{\mathbb{P},n,m})_{t\geq 0}$ is a $\mathbb{F}_+^{\mathbb{P}}$-adapted right continuous martingale orthogonal to X. By letting m to infinity, the limit still a right continuous $\mathbb{F}_+^{\mathbb{P}}$-adapted right continuous martingale orthogonal to X. On the other hand we have already proved in the two above steps the jointly measurability of $y^{\mathbb{P},n}$ and $z^{\mathbb{P},n}$. Also Picard iterations converge to the solution of the penalized BSDE, thus by (4.4.1), $m^{\mathbb{P},n}$ verifies

$$m_t^{\mathbb{P},n} = y_t^{\mathbb{P},n} - y_0^{\mathbb{P},n} + \int_0^t \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P},n}, \widehat{a}_s^{1/2} z_s^{\mathbb{P},n})ds - \int_0^t z_s^{\mathbb{P},n}\cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - n\int_0^t (U_s - y_s^{\mathbb{P},n})^- ds.$$

We deduce that there exists a measurable map $(t,\omega,\mathbb{P})\mapsto m_t^{\mathbb{P},n}(\omega)$. Indeed f is jointly measurable by Assumption 3.2.1(i) then $m^{\mathbb{P},n}$ is a sum of measurable terms. □

Lemma 4.4.2. *There is family of subsequence $(n_k^{\mathbb{P}}, k \geq 1)$ such that the sequence $(y^{\mathbb{P},n_k^{\mathbb{P}}})_{n_k^{\mathbb{P}}}$ of measurable jointly measurable functions with respect to $\mathcal{B}[0,T] \times \mathcal{F}_T \times \mathscr{B}$ converges and the limit denoted $y^{\mathbb{P}}$ is also a jointly measurable function. Moreover, for every $t \in [0,T]$ and $\mathbb{P} \in \mathcal{P}_0$, the limit process $y_t^{\mathbb{P}}$ provides the solution to the RBSDE (4.2.5).*

The result is proved in the same way as Lemma 3.4.3.

4.4.3 Dynamic programming principle

The dynamic programming principle here is principally based on universally selection measurable theorem seen in section 2.3. The following result extends Theorem 3.4.1 of Chapter 3 in the case of upper obstacle.

Theorem 4.4.1. *Suppose that Assumption 3.2.1 holds true. Then for all $(t,\omega) \in [0,T] \times \Omega$, one has $V_t(\omega) = V_t(\omega_{\cdot \wedge t})$, and $(t,\omega) \to V_t(\omega)$ is $\mathcal{B}([0,T]) \otimes \mathcal{F}_T$-universally measurable. Moreover, for all $(t,\omega) \in [0,T] \times \Omega$ and $\mathbb{F}$-stopping time τ taking values in $[t,T]$, we have*

$$V_t(\omega) = \sup_{\mathbb{P} \in \mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}[y_t^{\mathbb{P}}(\tau, V_\tau)],$$

where $y_t^{\mathbb{P}}(\tau, V_\tau)$ is obtained from the solution to the following RBSDE with terminal time τ and terminal condition V_τ,

$$\begin{cases} y_t^{\mathbb{P}} = V_\tau + \int_t^\tau \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}}) ds - \int_t^\tau z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_t^\tau dm_s^{\mathbb{P}} - k_\tau^{\mathbb{P}} + k_t^{\mathbb{P}}, \ \mathbb{P}\text{-a.s.} \\ y_t^{\mathbb{P}} \leq U_t, \ t \in [0,\tau], \ \mathbb{P}\text{-a.s.}, \\ \int_0^\tau (U_{t^-} - y_{t^-}^{\mathbb{P}}) dk_t^{\mathbb{P}} = 0, \ \mathbb{P}\text{-a.s.} \end{cases} \tag{4.4.8}$$

The proof follows the same scheme as the proof of Theorem 3.4.1. The first step of our proof is to establish the dynamic programming principle of our RBSDE associated to the 2RBSDE.

Lemma 4.4.3. *Let $t \in [0,T]$, $\mathbb{P} \in \mathcal{P}_0$, τ be an $\mathbb{F}$-stopping time taking values in $[t,T]$ and $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$ be a solution to the RBSDE (4.2.5) under $\mathbb{P}$. Then one has*

$$y_t^{\mathbb{P}}(T,\xi) = y_t^{\mathbb{P}}(\tau, y_\tau^{\mathbb{P}}) = y_t^{\mathbb{P}}(\tau, \mathbb{E}^{\mathbb{P}}[y_\tau^{\mathbb{P}} | \mathcal{F}_\tau^{\mathbb{P}}]), \ \mathbb{P}\text{-a.s.}$$

Proof. First, we consider a solution $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$ to the RBSDE (4.2.5) associated to (ξ, f, L) under $\mathbb{P}$ w.r.t. the filtration $\mathbb{F}_+^{\mathbb{P}} = (\mathcal{F}_{s+}^{\mathbb{P}})_{0 \leq s \leq T}$, then

$$y_t^{\mathbb{P}} = y_\tau^{\mathbb{P}} + \int_t^\tau \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}}) ds - \int_t^\tau z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_t^\tau dm_s^{\mathbb{P}} - \int_t^\tau dk_s^{\mathbb{P}}, \ \mathbb{P}\text{-a.s.}$$

Taking conditional expectation w.r.t. $\mathcal{F}_\tau^{\mathbb{P}}$ under $\mathbb{P}$, we get $\mathbb{P}$-a.s. that

$$y_t^{\mathbb{P}} = \mathbb{E}^{\mathbb{P}}[y_\tau^{\mathbb{P}} | \mathcal{F}_\tau^{\mathbb{P}}] + \int_t^\tau \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}}) ds - \int_t^\tau z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_t^\tau d\widehat{m}_s^{\mathbb{P}} - \int_t^\tau d\widehat{k}_s^{\mathbb{P}}, \ \mathbb{P}\text{-a.s.}$$

where the process $\widehat{k}^{\mathbb{P}}$ defined by $\widehat{k}_\tau^{\mathbb{P}} := \mathbb{E}^{\mathbb{P}}[k_\tau^{\mathbb{P}} | \mathcal{F}_\tau^{\mathbb{P}}]$, and $\widehat{k}_s^{\mathbb{P}} := k_s^{\mathbb{P}}$ is non-decreasing by the fact that $k^{\mathbb{P}}$ is non-decreasing. Since $d\widehat{k}_s^{\mathbb{P}} = 1_{\{y_{s^-}^{\mathbb{P}} \geq U_{s^-}\}} dk_s^{\mathbb{P}}$ for $s \leq \tau$, then

$$\begin{aligned} \int_0^\tau (U_{t^-} - y_{t^-}^{\mathbb{P}}) d\widehat{k}_t^{\mathbb{P}} &= \lim_{u \to \tau^-} \int_0^u (U_{t^-} - y_{t^-}^{\mathbb{P}}) d\widehat{k}_t^{\mathbb{P}} + (U_{\tau^-} - y_{\tau^-}^{\mathbb{P}}) \Delta \widehat{k}_\tau^{\mathbb{P}} \\ &= \lim_{u \to \tau^-} \int_0^u (U_{t^-} - y_{t^-}^{\mathbb{P}}) dk_t^{\mathbb{P}} + (U_{\tau^-} - y_{\tau^-}^{\mathbb{P}}) \mathbb{E}^{\mathbb{P}}[k_\tau^{\mathbb{P}} - k_{\tau^-}^{\mathbb{P}} | \mathcal{F}_\tau] \\ &= \mathbb{E}^{\mathbb{P}}[(U_{\tau^-} - y_{\tau^-}^{\mathbb{P}})(k_\tau^{\mathbb{P}} - k_{\tau^-}^{\mathbb{P}}) | \mathcal{F}_\tau] \\ &= 0. \end{aligned}$$

The last equality is provided by (4.4.8), if $(U_{\tau^-} - y^{\mathbb{P}}_{\tau^-}) > 0$, then $k^{\mathbb{P}}_{\tau} - k^{\mathbb{P}}_{\tau^-} = 0$. Therefore,

$$\int_0^{\tau} (U_{t^-} - y^{\mathbb{P}}_{t^-}) d\widehat{k}^{\mathbb{P}}_t = 0, \ \mathbb{P}\text{-a.s. and } y^{\mathbb{P}}_t \leq U_t, \ \forall t \in [0, \tau], \mathbb{P}\text{-a.s.}$$

We also have $\widehat{m}^{\mathbb{P}}_{\tau} := \mathbb{E}^{\mathbb{P}}[m^{\mathbb{P}}_{\tau} | \mathcal{F}^{\mathbb{P}}_{\tau}]$, and $\widehat{m}^{\mathbb{P}}_s := m^{\mathbb{P}}_s$ when $s < \tau$. It is apparent that $\widehat{m}^{\mathbb{P}} \in \mathbb{M}^p_0(\mathbb{F}^{\mathbb{P}}_+, \mathbb{P})$ and by identification, we deduce that

$$\widehat{m}^{\mathbb{P}}_{\tau} = m^{\mathbb{P}}_{\tau} + \mathbb{E}^{\mathbb{P}}[y^{\mathbb{P}}_{\tau} - k^{\mathbb{P}}_{\tau} | \mathcal{F}^{\mathbb{P}}_{\tau}] - (y^{\mathbb{P}}_{\tau} - k^{\mathbb{P}}_{\tau}).$$

and then $\widehat{m}^{\mathbb{P}}$ is orthogonal to the continuous martingale X under $\mathbb{P}$. Let consider a RBSDE associated to $(\mathbb{E}^{\mathbb{P}}[y^{\mathbb{P}}_{\tau} | \mathcal{F}^{\mathbb{P}}_{\tau}], f, L)$ on $[0, \tau]$, by uniqueness of this solution associated with the properties verifies by $\widehat{m}^{\mathbb{P}}$ and $\widehat{k}^{\mathbb{P}}$, it follows that

$$y^{\mathbb{P}}_t(\tau, y^{\mathbb{P}}_{\tau}) = y^{\mathbb{P}}_t(\tau, \mathbb{E}^{\mathbb{P}}[y^{\mathbb{P}}_{\tau} | \mathcal{F}^{\mathbb{P}}_{\tau}]), \ \mathbb{P}\text{-a.s.}$$

Finally, by definition of the RBSDE (4.2.5) it is clear that $y^{\mathbb{P}}_t(T, \xi) = y^{\mathbb{P}}_t(\tau, y^{\mathbb{P}}_{\tau})$. □

We now back to the proof of Theorem 4.4.1. The proof is exactly the same to Theorem 3.4.1 in Chapter 3 and use the previous Lemma.

4.4.4 Path regularization of the value function

After proving the DPP, we are interested in the right-continuity property that the first component of the solution of the 2RBSDE (4.2.1) should verify. The first step is to represent the right-continuity modification of V as a semi-martingale under any $\mathbb{P} \in \mathcal{P}_0$ and then give its decomposition. We define for all $(t, \omega) \in [0, T) \times \Omega$

$$V^+_t := \overline{\lim_{r \in \mathbb{Q} \cap [0,T], r \downarrow t}} V_t, \text{ and } V^+_T := V_T.$$

Our first objective is to show that V^+ admits right- and left- limits outside a $\mathcal{P}_0$-polar set. Since for all $t \in (0, T]$, V^+_t is by definition $\mathcal{F}^{U+}_t$-measurable, we can deduce that V^+ is in fact $\mathbb{F}^{\mathcal{P}_0+}$-optionnal. The downcrossing inequalities below is prove in section 5 of [85] to the case of BSDEs. Similar arguments apply to the case of RBSDEs give the same result.

Let $J := (\tau_n)_{n \in \mathbb{N}}$ be a countable family of $\mathbb{F}$-stopping times taking values in $[0, T]$ such that for any $(i, j) \in \mathbb{N}^2$, on has either $\tau_i \leq \tau_j$, or $\tau_i \geq \tau_j$, for every $\omega \in \Omega$. Let $a > b$ and $J_n \subset J$ be a finite subset ($J_n = \{0 \leq \tau_1 \leq \cdots \tau_n \leq T\}$). We denote by $D^b_a(V, J_n)$ the number of downcrossings of the process $(V_{\tau_k})_{1 \leq k \leq n}$ from b to a. We then define

$$D^b_a(V, J) := \sup \left\{ D^b_a(V, J_n) : J_n \subset J, \text{ and } J_n \text{ is a finite set} \right\}$$

The following lemma follows very closely the related result proved in Lemma $A.1$ of [13].

Lemma 4.4.4. *Fix some $\mathbb{P} \in \mathcal{P}_0$. Let Assumptions 3.2.1 and 4.3.1 hold. Denote by L the Lipschitz constant of the generator f. Then, for all $a < b$, there exists a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that*

$$\mathbb{E}^{\mathbb{Q}}\left[D^b_a(V, J)\right] \leq \frac{e^{LT}}{b-a} \mathbb{E}^{\mathbb{Q}}\Big[e^{LT}(V_0 \wedge b - a) - e^{-LT}(V_T \wedge b - a)^+ + e^{LT}(V_T \wedge b - a)^- + e^{LT} \int_0^T |\widehat{f}^{\mathbb{P}}(a, 0)| ds\Big]$$

Moreover, outside a $\mathcal{P}_0$-polar set, we have

$$\overline{\lim_{r \in \mathbb{Q} \cap (t,T], r \downarrow t}} V_t(\omega) := \lim_{r \in \mathbb{Q} \cap (t,T], r \downarrow t} V_t(\omega), \text{ and } \overline{\lim_{r \in \mathbb{Q} \cap (t,T], r \uparrow t}} V_t(\omega) := \lim_{r \in \mathbb{Q} \cap (t,T], r \uparrow t} V_t(\omega).$$

To prove the above result, we need to recall some properties verifed by V defined at $\mathbb{F}$-stopping times. For any stopping $\mathbb{F}$-stopping times $\tau \geq \sigma$, we have from Theorem 4.4.1 that

$$V_{\sigma(\omega)}(\omega) = \sup_{\mathbb{P}\in\mathcal{P}(\sigma(\omega),\omega)} \mathbb{E}^{\mathbb{P}}\left[y^{\mathbb{P}}_{\sigma(\omega)}(\tau, V_\tau)\right], \tag{4.4.9}$$

Formally, the following result can be obtained by simply taking conditional expectations of the corresponding RBSDEs. However, this raises subtle problems about negligible sets and conditional probability measures. We refer the reader to [22] for the precise details.

Lemma 4.4.5. *For any $\mathbb{P} \in \mathcal{P}_0$, for any $\mathbb{F}$-stopping times $0 \leq \sigma \leq \tau \leq T$, we have*

$$\mathbb{E}^{\mathbb{P}^{\sigma(\omega)}_\omega}\left[y^{\mathbb{P}^{\sigma(\omega)}_\omega}_{\sigma(\omega)}(\tau, V_\tau)\right] = \mathbb{E}^{\mathbb{P}}\left[y^{\mathbb{P}}_{\sigma}(\tau, V_\tau)\Big|\mathcal{F}_\tau\right](\omega), \ \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

The following inequality is the consequence of the above equation.

$$V_{\sigma(\omega)} \geq \mathbb{E}^{\mathbb{P}}\left[y^{\mathbb{P}}_{\sigma(\omega)}(\tau, V_\tau)\right], \ \text{for any } \mathbb{P} \in \mathcal{P}(\sigma(\omega), \omega). \tag{4.4.10}$$

These inequalities allow one to prove Lemma 4.4.4.

Proof of Lemma 4.4.4. Without loss of generality, we suppose that $a = 0$. Let $J_n = \{\tau_0, \tau_1, \cdots, \tau_n\}$ with $0 = \tau_0 < \tau_1 < \cdots < \tau_n = T$. For any $i = 1, \ldots, n$, and $\omega \in \Omega$, let the following RBSDE under $\mathbb{P}^{\tau_{i-1}(\omega)}_\omega$ on $[\tau_{i-1}, \tau_i]$

$$\begin{cases} y^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega}_t := V_{\tau_i} + \int_t^{\tau_i}\Big(\widehat{f}^{\mathbb{P}^{\tau_{i-1}(\omega)}_\omega,0}_s + \lambda^i_s y^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega}_s + \eta^i_s \cdot \widehat{a}^{1/2}_s z^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega}_s\Big)ds \\ \qquad\qquad - \int_t^{\tau_i} z^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega}_s \cdot \widehat{a}^{1/2}_s dW^{\mathbb{P}^{\tau_{i-1}(\omega)}_\omega}_s - \int_t^{\tau_i} dm^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega}_s - \int_t^{\tau_i} dk^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega}_s, \ \mathbb{P}^{\tau_{i-1}(\omega)}_\omega\text{-a.s.}, \\ y^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega}_t \leq U_t, \ t \in [\tau_{i-1}, \tau_i], \ \mathbb{P}^{\tau_{i-1}(\omega)}_\omega\text{-a.s.}, \\ \int_{\tau_{i-1}}^{\tau_i}(U_{s^-} - y^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega}_{s^-})dk^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega}_s = 0, \ \mathbb{P}^{\tau_{i-1}(\omega)}_\omega\text{-a.s.} \end{cases}$$

where λ^i and η^i are two bounded processes (by the the Lipschitz constant L_f of f) appearing in the linearisation of f due to the Lipschitz property of f. This RBSDE can be also write as

$$\begin{cases} -y^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega}_t := V_{\tau_i} + \int_t^{\tau_i}\Big(-\widehat{f}^{\mathbb{P}^{\tau_{i-1}(\omega)}_\omega,0}_s - \lambda^i_s y^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega}_s - \eta^i_s \cdot \widehat{a}^{1/2}_s z^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega}_s\Big)ds \\ \qquad\qquad - \int_t^{\tau_i}(-z^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega}_s) \cdot \widehat{a}^{1/2}_s dW^{\mathbb{P}^{\tau_{i-1}(\omega)}_\omega}_s - \int_t^{\tau_i} d(-m^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega}_s) + \int_t^{\tau_i} dk^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega}_s, \ \mathbb{P}^{\tau_{i-1}(\omega)}_\omega\text{-a.s.}, \\ -y^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega}_t \geq -U_t, \ t \in [\tau_{i-1}, \tau_i], \ \mathbb{P}^{\tau_{i-1}(\omega)}_\omega\text{-a.s.}, \\ \int_{\tau_{i-1}}^{\tau_i}(-y^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega}_{s^-} - (-U_{s^-}))dk^{i,\mathbb{P}^{\tau_{i-1}(\omega)}_\omega}_s = 0, \ \mathbb{P}^{\tau_{i-1}(\omega)}_\omega\text{-a.s.} \end{cases}$$

and then we obtain a RBSDE with the lower obstacle $-U$, since we have already proved the downcrossing inequality in Lemma 3.4.5 of Chapter 3 when the BSDE are reflected with a lower obstacle, therefore using the following symmetry between lower and upper obstacle, we can follow step by step the proof of Lemma 3.4.5 of Chapter 3 to proof the our result with the RBSDE define above. □

Using Lemma 3.4.5, we obtain

$$V^+_t := \lim_{r\in\mathbb{Q}\cap[0,T], r\downarrow t} V_t, \ \text{outside a } \mathcal{P}_0\text{-polar set},$$

and from this we deduced that V^+ is right-continuous outside a $\mathcal{P}_0$-polar set.

4.4.4.1 Representation formula

In this section we don't give the proof of result since the arguments of proof are the same to the case of lower obstacle in Chapter 3. We begin by extending the inequality (3.4.16) to V^+.

Lemma 4.4.6. *For any $0 \leq s \leq t \leq T$, for any $\mathbb{P} \in \mathcal{P}_0$, we have*

$$V_s^+ \geq y_s^{\mathbb{P}}(t, V_t^+), \ \mathbb{P}\text{-a.s.}$$

The proof of this Lemma is the same as the proof of Lemma 3.4.7.

The next result is an extension of the previous result to stopping times and the proof is similar for the lower case.

Lemma 4.4.7. *For any $\mathbb{F}$-stopping times $0 \leq \sigma \leq \tau \leq T$, for any $\mathbb{P} \in \mathcal{P}_0$, we have*

$$V_\sigma^+ \geq y_\sigma^{\mathbb{P}}(\tau, V_\tau^+), \ \mathbb{P}\text{-a.s.}$$

In particular V^+ is càdlàg, $\mathcal{P}_0$-q.s.

Lemma 4.4.8. *For any $\mathbb{F}$-stopping times $0 \leq \sigma \leq \tau \leq T$, for any $0 \leq t \leq T$, for any $\mathbb{P} \in \mathcal{P}_0$, we have*

$$V_\sigma = \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_0(\sigma,\mathbb{P},\mathbb{F})}^{\mathbb{P}} \mathbb{E}^{\mathbb{P}'}\left[y_\sigma^{\mathbb{P}'}(\tau, y_\tau^{\mathbb{P}'}) \middle| \mathcal{F}_\sigma\right], \ \mathbb{P}\text{-a.s. and } \ V_t^+ = \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}^{\mathbb{P}} y_t^{\mathbb{P}'}(T, \xi), \ \mathbb{P}\text{-a.s.}$$

where $\mathcal{P}_0(\sigma, \mathbb{P}, \mathbb{F})$ is defined in Section 3.2.3. In particular, if Assumption 3.3.1 holds, one has $V^+ \in \mathbb{D}_0^p(\mathbb{F}^{\mathcal{P}_0+})$.

The next result shows that V^+ is actually a semi-martingale under any $\mathbb{P} \in \mathcal{P}_0$, gives its decomposition and deduce the existence of a solution to the 2RBSDE (4.2.1) and the proof is deduce by the same arguments to the case of lower obstacle.

Lemma 4.4.9. *Let Assumptions 3.2.1 and 4.3.1 hold. For any $\mathbb{P} \in \mathcal{P}_0$, there exists $(Z^{\mathbb{P}}, M^{\mathbb{P}}, K^{\mathbb{P}}) \in \mathbb{H}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P}) \times \mathbb{M}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P}) \times \mathbb{I}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P})$ such that*

$$V_t^+ = \xi + \int_t^T \widehat{f}_s^{\mathbb{P}}(V_s^+, \widehat{a}^{1/2} Z_s^{\mathbb{P}}) ds - \int_t^T Z_s^{\mathbb{P}} \cdot dX_s^{c,\mathbb{P}} - \int_t^T dM_s^{\mathbb{P}} + \int_t^T dK_s^{\mathbb{P}}, \ t \in [0, T], \mathbb{P}\text{-a.s.}$$

Moreover, there is some $\mathbb{F}^{\mathcal{P}_0}$-predictable process Z which aggregates the family $(Z^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}_0}$ and the quadruple $(V^+, Z, (M^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}_0}, (K^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}_0})$ is solution to the 2RBSDE (4.2.1).

Chapter 5

Weak approximation of second order reflected BSDEs

Contents

5.1 Introduction

In this chapter, we aim at approximating a solution of a second order reflected BSDEs studied in Chapter 3, in particular in terms of weak approximation. A second order reflected BSDE (2RBSDE) can be represented as an equation defined as follows on the canonical space $\Omega := C([0,T],\mathbb{R}^d)$

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, \widehat{a}_s^{1/2} Z_s, \widehat{a}_s) ds - \int_t^T Z_s dX_s - \int_t^T dM_s^{\mathbb{P}} + K_T^{\mathbb{P}} - K_t^{\mathbb{P}}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le t \le T,\ \mathbb{P}\text{-a.s.}, \forall \mathbb{P} \in \mathcal{P}, \\ Y_t \ge L_t, 0 \le t \le T,\ \mathbb{P}\text{-a.s.}, \forall \mathbb{P} \in \mathcal{P} \end{cases}$$

where $\mathcal{P}$ is the set of probability eventually non dominated such that under $\mathbb{P} \in \mathcal{P}$, the canonical process X is a local martingale (or more generally semimartingale see Chapter 3) measure with the density of its quadratic variation (also called volatility) $\widehat{a}$ taking value the set $\mathbb{S}_d^{\ge 0}$ of symmetric positive semi-definite $d \times d$ matrices. With this stochastic differential equation, it is natural to have recourse to the discretization of the continuous solution by numerical methods (among which we can mention Euler's scheme, Milstein's scheme ...), this turns out to be complicated for in the present case since the process $K^{\mathbb{P}}$ in the equation is not defined explicitly. The information about $K^{\mathbb{P}}$ is given by the minimum condition introduced by Soner, Touzi and Zhang [91] for the second order BSDEs and thereafter adapted for reflected second

order BSDEs by Matoussi, Possamaï and Zhou [66]. Thus it appears easier to make use of the representation formula through standard reflected BSDEs, and then our problem becomes to approximate a supremum of a family of nonlinear expectations on the canonical space of continuous trajectories.

The G-expectation introduced by Peng [78] is a particular case of nonlinear expectation. In [30] Dolinsky, Nutz and Soner studied the weak approximation of G-expectation, they introduced a notion volatility in discrete time and define an approximation of the G-expectation in discrete time. They proved the converge of the discrete time approximation to the continuous time G-expectation. And with this weak convergence approach, there is no estimation error of convergence rate. To overcome this disadvantage, Dolinsky [29] considers a different approximation with explicit discrete time martingale and then proves the convergence of approximation and provides the error estimates.

Possamaï and Zhou [84] proved weak approximation of a class of second order BSDE non reflected. Their approximation consists in a construction of a sequence of discrete martingale convergent to X and a sequence of BSDEs driven by these martingales. Then under specific conditions, they proved that the supremum under volatilities uncertainty in discrete time of this family of BSDEs converge to the solution of 2BSDE. In order to do this, they call on robustness properties of BSDEs introduced by Briand , Delyon and Mémin [19] and [20] (which is a weak approximation of BSDEs driven by a Brownian motion) and tightness of sequence of approximations.

Our main contribution is to extend weak approximation of [85] to the case of second order reflected BSDE, using an approximation of a sequence of discrete time martingale convergent to the canonical process. An auxiliary result is to extend the stability results for the case of reflected BSDEs driven by a martingale in a general filtration.

The rest of the paper is organized as follows: in section 5.2, we provides a framework to our class of second order BSDEs. Section 5.3 is devoted to an approximation in discrete time of martingale as random walk and the statement of the main result using a family of reflected BSDEs driven by the approximation. Finally the last section contains proofs of existence, uniqueness and stability of RBSDEs driven by discrete time approximation of martingale.

5.2 Framework and formulation of the problem

Framework. We consider $\Omega := \{\omega \in C_0([0,T],\mathbb{R}) : \omega_0 = 0\}$ the canonical space of continuous paths on $[0,T]$ which start at 0 equipped with the following norm $\|\omega\|_t := \sup_{0\le s\le t} |\omega_s|$, for any $\omega \in \Omega$ and $t \in [0,T]$ and X the canonical process. We denote by $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ the canonical filtration and $\mathbb{F}_+ = (\mathcal{F}_t^+)_{0\le t\le T}$ the right continuous filtration.

Let $\mathcal{P}(\Omega)$ be the collection of all probability measure on $(\Omega, \mathcal{F}_T)$, $\mathbb{P}_0$ the Wiener measure on Ω under which X is a standard Brownian motion. A probability measure $\mathbb{P}$ in $\mathcal{P}(\Omega)$ will be called a martingale measure if X is a martingale under $\mathbb{P}$ and $X_0 = 0$, $\mathbb{P}$-a.s. For $\mathbb{P} \in \mathcal{P}(\Omega)$, we define $\mathbb{F}^{\mathbb{P}} = (\mathcal{F}_t^{\mathbb{P}})_{0\le t\le T}$ to be the completed filtration with respect to $\mathbb{P}$ and $\mathbb{F}_+^{\mathbb{P}}$ its right limit. In this way, for $\mathcal{P} \subset \mathcal{P}(\Omega)$, we define the augmented filtration with respect to $\mathcal{P}$ by

$$\mathbb{F}^{\mathcal{P}} = (\mathcal{F}_t^{\mathcal{P}})_{0\le t\le T}, \quad \mathcal{F}_t^{\mathcal{P}} = \bigcap_{\mathbb{P}\in\mathcal{P}} \mathcal{F}_t^{\mathbb{P}}$$

and $\mathbb{F}_+^{\mathcal{P}}$ his right limit.

By Karandikar [53], there is some $\mathbb{F}$-progressively measurable non-decreasing process on Ω denoted by $\langle X\rangle = (\langle X\rangle_t)_{0\le t\le T}$ which coincides with the quadratic variation of X under each

semi-martingale measure $\mathbb{P}$ with

$$\langle X\rangle_t := X_t X_t' - 2\int_0^t X_s dX_s' \quad \text{and} \quad \widehat{a}_t := \overline{\lim_{\epsilon\downarrow 0}}\ \frac{1}{\epsilon}(\langle X\rangle_t - \langle X\rangle_{t-\epsilon}).$$

where X' denotes the transposed of X, and the $\overline{\lim}$ is componentwise defined.

The models space. We consider a nonempty compact convex subset $\mathbf{D} = [\underline{a}, \overline{a}]$, where $\underline{a} > 0$. We denote by $\mathcal{P}_{\mathbf{D}} \subset \mathcal{P}(\Omega)$ the collection all probability measures $\mathbb{P}$ satisfies:

- X is a $(\mathbb{P}, \mathbb{F})$-martingale
- $\langle X\rangle$ is absolutely continuous $\mathbb{P}$-a.s.
- $d\langle X\rangle_t/dt \in \mathbf{D}, \quad \mathbb{P}\times dt$-a.e.

Formulation of second order Reflected BSDE We consider the following 2RBSDE

$$\begin{cases} Y_t = \Phi(X_\cdot) + \int_t^T f(s, X_\cdot, Y_s, \widehat{a}_s^{1/2} Z_s, \widehat{a}_s)ds - \int_t^T Z_s dX_s - \int_t^T dM_s^{\mathbb{P}} + K_T^{\mathbb{P}} - K_t^{\mathbb{P}}, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \le t \le T,\ \mathbb{P}\text{-a.s.}, \forall \mathbb{P} \in \mathcal{P}_{\mathbf{D}}, \\ Y_t \ge h(X_t), 0 \le t \le T,\ \mathbb{P}\text{-a.s.}, \forall \mathbb{P} \in \mathcal{P}_{\mathbf{D}} \\ \operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_{\mathbf{D}}(t,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}\Big[\int_t^T G_s^{t,\mathbb{P}'} d(K_s^{\mathbb{P}'} - k_s^{\mathbb{P}'})\big|\mathcal{F}_{t+}^{\mathbb{P}}\Big] = 0, \quad 0 \le t \le T,\ \mathbb{P}\text{-a.s. } \forall \mathbb{P} \in \mathcal{P}_{\mathbf{D}}. \end{cases} \tag{5.2.1}$$

where for any $t \in [0,T]$ and for any $\mathbb{P} \in \mathcal{P}_0$, the process $G^{t,\mathbb{P}}$ is defined by

$$G_s^{t,\mathbb{P}} := \exp\Big(\int_t^s (\lambda_u^{\mathbb{P}} - \frac{1}{2}\,\|\eta_u^{\mathbb{P}}\|^2)(Y_u, y_u^{\mathbb{P}}, Z_u, z_u^{\mathbb{P}})du + \int_t^s \eta_u^{\mathbb{P}}(Y_u, y_u^{\mathbb{P}}, Z_u, z_u^{\mathbb{P}}) \cdot \widehat{a}_u^{-1/2} dX_u^{c,\mathbb{P}}\Big).$$

$\lambda^{\mathbb{P}}$ and $\eta^{\mathbb{P}}$ are bounded processes, f, Φ and h are three functions respectively on $[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbf{D}$ and $\mathbb{R}$ with values in $\mathbb{R}$ satisfying $\Phi \ge h$.
Notation: $\mathcal{P}_{\mathbf{D}}(t,\mathbb{P},\mathbb{F}_+) := \{\mathbb{P}' \in \mathcal{P}_{\mathbf{D}} : \mathbb{P} = \mathbb{P}' \text{ on } \mathcal{F}_{t+}\}$.
We proved existence and uniqueness of solution of the above equation in Chapter 3. Furthermore, the solution $(Y, Z, (M^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_{\mathbf{D}}}, (K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_{\mathbf{D}}})$ of this 2RBSDE is such that Y is $\mathbb{F}_+$-progressively measurable, Z is $\mathbb{F}_+$-predictable, $K^{\mathbb{P}}$ is $\mathbb{F}_+^{\mathbb{P}}$-predictable and $M^{\mathbb{P}}$ is an $\mathbb{F}_+^{\mathbb{P}}$-optional martingale orthogonal to X under $\mathbb{P}$.

Remark 5.2.1. *In this work, we have a slightly different formulation of 2RBSDE in the sense that in chapter* 3*, the generator depends on $\widehat{a}$ only by the product with z, while in the above expression the dependence with respect to $\widehat{a}$ is more general. Nevertheless, the existence and uniqueness of the 2RBSDE 5.2.1 is obtained under Assumptions 5.2.1 by the same arguments as in Chapter* 3.

We also provided in (3.3.3) of Chapter 3 the following representation formula: for $0 \le t \le T$,

$$Y_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_{\mathbf{D}}(t,\mathbb{P},\mathbb{F}_+)} y_t^{\mathbb{P}'}, \quad \mathbb{P}\text{-a.s.} \tag{5.2.2}$$

where $y^{\mathbb{P}}$ is the first component of the solution of the following RBSDE:

$$\begin{cases} y_t^{\mathbb{P}} = \Phi(X_\cdot) + \int_t^T f(s, X_\cdot, y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}}, \widehat{a})ds - \int_t^T z_s^{\mathbb{P}} dX_s - \int_t^T dm_s^{\mathbb{P}} + k_T^{\mathbb{P}} - k_t^{\mathbb{P}}, \\ y_t^{\mathbb{P}} \ge h(X_t),\ t \in [0,T] \\ \int_0^T (y_{s^-}^{\mathbb{P}} - h(X_{s^-}))dk_s^{\mathbb{P}} = 0. \end{cases} \tag{5.2.3}$$

The existence and uniqueness of solution to the above RBSDEs under each $\mathbb{P} \in \mathcal{P}_{\mathbf{D}}$ is proved in section 4 and the arguments used to prove this result are classical to the ones used in [14] for example. For every $\mathbb{P}$ the solution is represented by a quadruple $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$ of $\mathbb{F}_+$-measurable processes where $m^{\mathbb{P}}$ is a $(\mathbb{P}, \mathbb{F}_+)$-martingale orthogonal to X under $\mathbb{P}$. For every $\mathbb{P} \in \mathcal{P}_{\mathbf{D}}$, we have the following estimates

$$\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|y_t^{\mathbb{P}}|^2 + \int_0^T |\widehat{a}_s^{1/2} z_s^{\mathbb{P}}|^2 + [m^{\mathbb{P}}]_T + (k_T^{\mathbb{P}})^2\Big] \leq C. \tag{5.2.4}$$

where C is a constant independent of $\mathbb{P}$.
For every $\mathbb{P} \in \mathcal{P}_{\mathbf{D}}$, each RBSDE (5.2.3) are linked to an optimal stopping problem by the following relation

$$y_t^{\mathbb{P}} = \operatorname*{ess\,sup}_{v\in\mathcal{T}_t} \mathbb{E}^{\mathbb{P}}\Big[\int_t^v f(s, X_\cdot, y_s^{\mathbb{P}}, z_s^{\mathbb{P}}, \widehat{a}_s)ds + h(X_v)1_{\{v<T\}} + \Phi(X_\cdot)1_{\{v=T\}}\big|\mathcal{F}_{t+}\Big], \tag{5.2.5}$$

where $\mathcal{T}_t$ is the set of all $[t, T]$-valued $\mathbb{F}$-stopping times.
Formulation of the problem. Thanks to the representation formula (5.2.2), our goal is to give an approximation of Y_0 defined by

$$Y_0 = \sup_{\mathbb{P}\in\mathcal{P}_{\mathbf{D}}} \mathbb{E}^{\mathbb{P}}[y_0^{\mathbb{P}}]. \tag{5.2.6}$$

We make the following assumptions.

Assumption 5.2.1. *There exists a constant $K > 0$ such that for any $(\mathrm{x}_1, y_1, z_1), (\mathrm{x}_2, y_2, z_2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}, s, t \in [0, T], a_1, a_2 \in \boldsymbol{D}$,*

(i) Φ is positive continuous and $|\Phi(\mathrm{x}_1) - \Phi(\mathrm{x}_2)| \leq K(\|\mathrm{x}_1 - \mathrm{x}_2\|_T)$,

(ii) h is positive continuous and $|h(y_1) - h(y_2)| \leq K(|y_1 - y_2|)$,

(iii) $|f(t, \mathrm{x}_1, y_1, z_1, a) - f(t, \mathrm{x}_2, y_2, z_2, a)| \leq K(\|\mathrm{x}_1 - \mathrm{x}_2\|_t + |y_1 - y_2| + \|z_1 - z_2\| + |a_1 - a_2|)$,

(iv) f is positive and the map $a \mapsto f(t, \mathrm{x}, y, z, a)$ is concave and uniformly continuous for every $(t, \mathrm{x}, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

(v) The process $t \mapsto f(t, \mathrm{x}_\cdot, y_t, z_t, a_t)$ is progressively measurable given progressive processes (x, y, z, a) and uniformly continuous with modulus ρ in the sense that for every $s \leq t$ and x, y, z,

$$|f(t, \mathrm{x}_{s\wedge\cdot}, y, z, a) - f(s, \mathrm{x}_{s\wedge\cdot}, y, z, a)| \leq \rho(t - s).$$

(vi) $\sup_{(t,\mathrm{x},\gamma)\in[0,T]\times\Omega\times\boldsymbol{D}} |f(t, \mathrm{x}, 0, 0, \gamma)| + |\Phi(0)| + |h(0)| \leq K$.

(vii) h is convex, bounded, derivable and his derivative is Lipschitz (with Lipschitz constant K),

Remark 5.2.2. *Assumptions (i), (ii), (iii), (v) and (vi) are classical in the theory of BSDEs and the numerical approximation of RBSDEs and ensures the convergence of the numerical scheme to the solution. Assumption (iv) will be useful in section to prove the existence of an opimal function value and finally Assumption (vii) comes from the approximation of RBSDEs (we can see Bouchard and Chassagneux [11]). Since we don't give a bound on the convergence rate, the last one is only need to prove Lemma 5.3.1 to establish an estimate for a scheme of RBSDEs.*

5.3 Discrete time martingale and formulation of approximation

We begin with a discrete time analogue of martingale, we follow the approximation of [84] and then proved existence and uniqueness of solution to RBSDE driven by the approximation.

Discretisation. Given $n \geq 1$, we discretize $[0,T]$ in $n+1$ equidistant points $0 = t_0 < t_1 < \cdots < t_n = T$, with $h = T/n$ and for every $n \geq 1$, we set $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$ a probability space carrying a sequence of n independent random variables $U_1, \ldots, U_n$. Define the filtration $\mathbb{F}^n := (\mathcal{F}^n_k)_{1\leq k\leq n}$ with $\mathcal{F}^n_{t_k} := \sigma(U_1, \ldots, U_k)$.
We consider a family of functions $\{H^n_k, 1 \leq k \leq n\}$ such that $H^n_k : \mathbf{D} \times \mathbb{R} \to \mathbb{R}$ is a continuous in a and for some $\delta > 0$, we have for any γ,

$$\mathbb{E}[H^n_k(\gamma, U_k)] = 0, \quad \mathbb{E}[|H^n_k(\gamma, U_k)|^2] = \gamma h, \quad \mathbb{E}[|H^n_k(\gamma, U_k)|^{2+\delta}] = Kh^{1+\delta/2}. \tag{5.3.1}$$

where the expectation is with respect to the probability of $\mathbb{P}^n$.

Remark 5.3.1. *Observe that if $U_1, \ldots, U_n$ are i.i.d. and under $\mathbb{P}^n$ the have a standard normal distribution, the sequence of functions $H^n_k(a,u) := au$ h satisfiy the above conditions.*

Space of volatility uncertainty. Denote $\mathcal{A}_n$ the collection of all $\mathbb{F}^n$-predictable $\mathbf{D}$-valued process $a = (a_{t_1}, \ldots, a_{t_n})$. Then for every $a \in \mathcal{A}_n$, we define the discrete time martingale $M^a = (M^a_{t_0}, \ldots, M^a_{t_n})$ of the form, $M^a_{t_0} = 0$ and

$$M^a_{t_k} = \sum_{j=1}^{k} H^n_j(a_{t_j}, U_j),\ 1 \leq k \leq n. \tag{5.3.2}$$

together with its predictable variation process $\{\langle M^a\rangle_{t_k}, k = 0, \ldots, n\}$ which is given by $\langle M^a\rangle_{t_0} = 0$ and

$$\langle M^a\rangle_{t_k} = \sum_{j=1}^{k} \Delta\langle M^a\rangle_{t_j} = \sum_{j=1}^{k} \mathbb{E}\left[(\Delta\langle M^a\rangle_{t_j})^2 \middle| \mathcal{F}^n_{t_{j-1}}\right] = h\sum_{j=1}^{k} a_{t_j}.$$

where $\Delta X_{t_j} := X_{t_{j+1}} - X_{t_j}$.
We extend discrete paths in continuous paths. By abuse of notations, we define the continuous time filtration $\mathbb{F}^n = (\mathcal{F}^n_t)_{0\leq t\leq T}$, with $\mathcal{F}^n_t := \mathcal{F}^n_{t_k}$, for all $t \in [t_k, t_{k+1}[$ and continuous time martingales $M^a_t := M^a_{t_k}$ for all $t \in [t_k, t_{k+1}[$. We consider the complete filtration $\mathbb{G}^n := \overline{\mathbb{F}^n}^{\mathbb{P}^n}$. It is easily to see that $\mathbb{G}^n$ is right-continuous and completed under $\mathbb{P}^n$, and by definition for every $a \in \mathcal{A}_n$, M^a is a right-continuous, piecewise constant in time, $\mathbb{G}^n$-martingale.
Moreover, we also extend M of $(\mathbb{R}^d)^{n+1}$ to a continuous path by linear interpolation. We define the interpolation operator $\widehat{\cdot}$ by

$$\begin{aligned}\widehat{\cdot}&: (\mathbb{R}^d)^{n+1} \to \Omega,\ M = (M_0, \ldots, M_n) \mapsto \widehat{M} = (\widehat{M}_t)_{0\leq t\leq T},\\ \widehat{M}_t &:= ([t/h] + 1 - t/h)M_{[t/h]h} + (t/h - [t/h])M_{([t/h]+1)h},\end{aligned}$$

and $[z]$ denotes the integer part of z.

Reflected BSDEs driven by M^a. Then for every $a \in \mathcal{A}_n$, we define the following RBSDE

$$\begin{cases} y^a_t = \Phi(\widehat{M^a}_\cdot) + \int_t^T f(s, \widehat{M}^a_\cdot, y^a_{s^-}, a_s^{1/2} z^a_s, a_s) dC^n_s - \int_t^T z^a_s dM^a_s - \int_t^T dm^a_s + k^a_T - k^a_t,\\ y^a_t \geq h(\widehat{M}^a_t),\ t \in [0,T]\\ \int_0^T (y^a_{s^-} - h(\widehat{M}^a_{s^-}))dk^a_s = 0. \end{cases} \tag{5.3.3}$$

where $C^n_s = [s/h]h$ and y^a_- denotes the càdlàg process associated to y^a. $a^{1/2}$ denotes the unique square root of a in $\mathbf{D}$ and a is given by

$$a_t := \sum_{k=0}^{n-1} a_{t_k} 1_{[t_k, t_{k+1}[}(t), \quad t \in [0,T].$$

A solution of the above RBSDE is a quadruple (y^a, z^a, m^a, k^a) of $\mathbb{G}^n$-measurable processes such that m^a is a martingale orthogonal to M^a(existence and uniqueness result is provided in section 5.4.

Numeric scheme of RBSDE (5.3.3). Following Bouchard and Touzi [15], by discretization of the RBSDE (5.3.3), we have the following discretized backward dynamic program scheme:

$$\begin{cases} y^a_{nh} = \Phi(\widehat{M}^a_{\cdot}), \\ y^a_{kh} = \max\{h(M^a_{kh}), \mathbb{E}[y^a_{(k+1)h}|\mathcal{F}^n_{t_k}] + f(kh, \widehat{M}^a_{\cdot}, y^a_{kh}, a^{1/2}_{kh} z^a_{kh}, a_{kh})h\}, \\ z^a_{kh} = \frac{1}{h}\mathbb{E}\left[y^a_{(k+1)h}\, a^{-1}_{kh} \Delta M^a_{kh} | \mathcal{F}^n_{t_k}\right]. \end{cases} \tag{5.3.4}$$

We are interested by approximation of the y^a-component then we are not required to give a discretization of the martingale orthogonal.
The following assumption is useful and provided a comparison result for reflected discrete time BSDEs.

Assumption 5.3.1. *For every $n \geq 1$ and $a \in \mathcal{A}_n$, the following relations hold:*

$$K < \frac{1}{h} \quad \text{and} \quad K|H^n_k(a_{kh}, U_k)| < a^{1/2}_{kh}, \text{ for } k = 1, \cdots, n.$$

where K is the Lipschitz coefficient K introduced in Assumption 5.2.1 and h is the step of discretization.

These conditions guarantee that the following classical statement is satisfied: Let $n \geq 1$, the backward scheme (5.3.4) is monotone. In others words let $(y^{a,1}, z^{a,1}), (y^{a,2}, z^{a,2})$ be two solutions of (5.3.4), then

$$y^{a,1}_{(k+1)h} \leq y^{a,2}_{(k+1)h} \quad \Rightarrow \quad y^{a,1}_{kh} \leq y^{a,2}_{kh} \qquad \forall k = 0, \ldots, n-1.$$

Indeed,

$$\begin{aligned} y^{a,1}_{kh} - y^{a,2}_{kh} &= \max\{h(M^a_{kh}), \mathbb{E}[y^{a,1}_{(k+1)h}|\mathcal{F}^n_{t_k}] + f(kh, \widehat{M}^a_{\cdot}, y^{a,1}_{kh}, a^{1/2}_{kh} z^{a,1}_{kh}, a_{kh})h\} \\ &\quad - \max\{h(M^a_{kh}), \mathbb{E}[y^{a,2}_{(k+1)h}|\mathcal{F}^n_{t_k}] + f(kh, \widehat{M}^a_{\cdot}, y^{a,2}_{kh}, a^{1/2}_{kh} z^{a,2}_{kh}, a_{kh})h\} \\ &\leq \max\{0, \mathbb{E}[y^{a,1}_{(k+1)h} - y^{a,2}_{(k+1)h}|\mathcal{F}^n_{t_k}] \\ &\quad + f(kh, \widehat{M}^a_{\cdot}, y^{a,1}_{kh}, a^{1/2}_{kh} z^{a,1}_{kh}, a_{kh})h - f(kh, \widehat{M}^a_{\cdot}, y^{a,2}_{kh}, a^{1/2}_{kh} z^{a,2}_{kh}, a_{kh})h\} \\ &\leq \max\{0, \mathbb{E}[y^{a,1}_{(k+1)h} - y^{a,2}_{(k+1)h}|\mathcal{F}^n_{t_k}] + K(y^{a,1}_{kh} - y^{a,2}_{kh})h + Kha^{1/2}_{kh}(z^{a,1}_{kh} - z^{a,2}_{kh})\}. \end{aligned}$$

Furthermore,

$$\begin{aligned} z^{a,1}_{kh} - z^{a,2}_{kh} &= \frac{1}{h}\mathbb{E}\left[(y^{a,1}_{(k+1)h} - y^{a,2}_{(k+1)h})\, a^{-1}_{kh} \Delta M^a_{kh} | \mathcal{F}^n_{t_k}\right] \\ &= \frac{1}{h}\mathbb{E}\left[(y^{a,1}_{(k+1)h} - y^{a,2}_{(k+1)h})\, a^{-1}_{kh} H^n_{k+1}(a_{kh}, U_{k+1}) | \mathcal{F}^n_{t_k}\right]. \end{aligned}$$

Then

$$(1 - Kh)(y^{a,1}_{kh} - y^{a,2}_{kh}) \leq \max\{0, \mathbb{E}[\left(1 + Ka^{-1/2}_{kh} H^n_{k+1}(a_{kh}, U_{k+1})\right)(y^{a,1}_{(k+1)h} - y^{a,2}_{(k+1)h})|\mathcal{F}^n_{t_k}]\}.$$

Discrete time approximation of the problem. Given $n \geq 1$, let set

$$Y^n_0 = \sup_{a \in \mathcal{A}_n} \mathbb{E}[y^a_0]. \tag{5.3.5}$$

where the expectation is defined with respect to the probability $\mathbb{P}^n$.

Assumption 5.3.2. *$f(t,x,y,z,a) = f(t,x,y,a)$, in other words f does not depend on z.*

Our main result is the following weak approximation

Theorem 5.3.1. *(i) Suppose that Assumption 5.2.1 holds true. Then*

$$\liminf_{n\to\infty} Y_0^n \geq Y_0.$$

(ii)Suppose in addition that Assumptions 5.3.1 and 5.3.2 hold, then

$$\lim_{n\to\infty} Y_0^n = Y_0.$$

Proof. This proof is divided in two steps, the first step concerns the first assertion of the Theorem and the second one is devoted to the proof of the second assertion.
Step 1 *:Proof of the first assertion of the Theorem.* Let $\mathbb{P} \in \mathcal{P}_{\mathbf{D}}$ and consider $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$ the solution of the following RBSDE under $\mathbb{P}$.

$$\begin{cases} y_t^{\mathbb{P}} = \Phi(X_\cdot) + \int_t^T f(s, X_\cdot, y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}}, a_s)ds - \int_t^T z_s^{\mathbb{P}} dX_s - \int_t^T dm_s^{\mathbb{P}} + k_T^{\mathbb{P}} - k_t^{\mathbb{P}}, \quad \mathbb{P}\text{-a.s.} \\ y_t^{\mathbb{P}} \geq h(X_t),\ t \in [0,T] \quad \mathbb{P}\text{-a.s.} \\ \int_0^T (y_{s^-}^{\mathbb{P}} - h(X_{s^-}))dk_s^{\mathbb{P}} = 0 \quad \mathbb{P}\text{-a.s.} \end{cases}$$

with respect to the martingale X.
Let denote for $n \geq 1$, M^n the piecewise approximation of the martingale X defined by

$$M_{t_k}^n = \sum_{j=1}^{k} H_j^n(\tilde{a}_{t_j}^n, U_j),\ 1 \leq k \leq n.$$

where $\tilde{a}^n$ is a process which belongs to $\mathcal{A}_n$ and is defined from the density $\widehat{a}$ of the quadratic variation of X. We give all details about the construction of $\tilde{a}^n$ and the convergence result of M^n in section 5.4.2. Using this approximation, we consider (y^n, z^n, m^n, k^n) the solution of the following RBSDE

$$\begin{cases} y_t^n = \Phi(\widehat{M}_\cdot^n) + \int_t^T f(s, \widehat{M}_\cdot^n, y_{s^-}^n, (\tilde{a}_s^n)^{1/2} z_s^n, \tilde{a}_s^n)dC_s^n - \int_t^T z_s^n dM_s^n - \int_t^T dm_s^n + k_T^n - k_t^n, \quad \mathbb{P}^n\text{-a.s.} \\ y_t^n \geq h(\widehat{M}_t^n),\ t \in [0,T] \quad \mathbb{P}^n\text{-a.s.} \\ \int_0^T (y_{s^-}^n - h(\widehat{M}_{s^-}^n))dk_s^n = 0, \quad \mathbb{P}^n\text{-a.s.} \end{cases}$$

driven by $(M^n)_{n\geq 1}$, where $\widehat{M^n}$ is the linear interpolation of M^n.
For n large enough, we establish the existence and uniqueness of the above RBSDE in section 5.4. Moreover, we stated in Theorem 5.4.1 the following convergence

$$y^n \to y^{\mathbb{P}}, \quad \text{as} \quad n \to \infty \tag{5.3.6}$$

in law for the topology of uniform convergence. Moreover, by definition of Y_0^n, we have

$$\mathbb{E}[y_0^n] \leq Y_0^n.$$

Then taking the infimum limit into the following expression and using refnumcongyn we obtain

$$\liminf_{n\to\infty} Y_0^n \geq \liminf_{n\to\infty} \mathbb{E}[y_0^n] \to \mathbb{E}^{\mathbb{P}}[y_0^{\mathbb{P}}].$$

Now taking the supremum over $\mathbb{P}$, we have the first part of the Theorem.
We stand for the second assertion.

Step 2:*Proof of the second assertion of the Theorem* An essential argument for the proof of the second part of the Theorem is the existence of an optimal control for the approximation problem (5.3.5). Then we reformulated the problem (5.3.5) using a backward scheme as a control problem over the compact set $\mathbf{D}$ which is very useful to prove the existence of this optimal control.
This formulation is based on backward scheme of the RBSDE.Let define

$$\Lambda^n := \bigcup_{0\le k\le n} \{t_k\} \times \mathbb{R}^{k+1}.$$

For every $n \geq 1$, $(t_k, \mathrm{x}) \in \Lambda^n$ and $\gamma \in \mathbf{D}$, we define $M^{t_k,\mathrm{x},\gamma} \in \mathbb{R}^{k+2}$ by

$$\begin{cases} M^{t_k,\mathrm{x},\gamma}_{t_i} := \mathrm{x}_i, \quad \text{for every } i \le k \\ M^{t_k,\mathrm{x},\gamma}_{t_{k+1}} = M^{t_k,\mathrm{x},\gamma}_{t_k} + H^n_{k+1}(\gamma, U_{k+1}). \end{cases}$$

We then define $u^n : \Lambda^n \to \mathbb{R}$ by the following backward iteration. The terminal condition is given by

$$u^n(t_n, \mathrm{x}) := \Phi(\mathrm{x}) \quad \forall \mathrm{x} \in \mathbb{R}^{n+1},$$

and the backward iteration u^n is given by, for all $\mathrm{x} \in \mathbb{R}^{k+1}$,

$$\begin{cases} u^n(t_k, \mathrm{x}) = \sup_{\gamma\in\mathbf{D}} u^n_\gamma(t_k, \mathrm{x}), \\ u^n_\gamma(t_k, \mathrm{x}) = \max\{h(M^{t_k,\mathrm{x},\gamma}_{t_k}), \mathbb{E}[u^n_\gamma(t_{k+1}, M^{t_k,\mathrm{x},\gamma})] + f(t_k, \widehat{\mathrm{x}}, u^n_\gamma(t_k, \mathrm{x}), \gamma^{1/2} v^n_\gamma(t_k, \mathrm{x}), \gamma)h\}, \\ v^n_\gamma(t_k, \mathrm{x}) := \frac{1}{h}\mathbb{E}\Big[u^n_\gamma(t_{k+1}, M^{t_k,\mathrm{x},\gamma})\gamma^{-1} H^n_{k+1}(\gamma, U_{k+1})\Big], \end{cases} \tag{5.3.7}$$

Proposition 5.3.1. *Let u^n defined in 5.3.7, then*

$$u^n(0, \boldsymbol{0}) = Y^n_0.$$

In the rest of this section, we suppose that f does not depend on z and we consider the approximation u^n defined in 5.3.7 (see Proposition 5.3.1). Therefore, we have the following regularity property.

Lemma 5.3.1. *Let Assumption 5.3.1 holds. Consider u^n the solution to the backward iteration (5.3.7), then there exists a constant C independent of n such that*

$$|u^n(t_k, \mathrm{x}^1) - u^n(t_{k+1}, \mathrm{x}^{2,k})| \le C\big(\|\mathrm{x}^1 - \mathrm{x}^2\| + \sqrt{h}\big).$$

where $\mathrm{x}^1, \mathrm{x}^2 \in \mathbb{R}^{k+1}$ and $\mathrm{x}^{2,k} = (\mathrm{x}^2, \mathrm{x}^2_{k+1} + H^n_{k+1}(\cdot, U_{k+1})) \in \mathbb{R}^{k+2}$.

The following result states the existence of an optimal control to the problem defined in 5.3.7.

Proposition 5.3.2. *For every $n \geq 1$, there is $a^{n,*} \in \mathcal{A}^n$ such that the solution $(y^{n,*}, z^{n,*}, m^{n,*}, k^{n,*}) = (y^{a^{n,*}}, z^{a^{n,*}}, m^{a^{n,*}}, k^{a^{n,*}})$ of (5.3.3) satisfies $y^{n,*}_{t_k} = u^n(t_k, M^{n,*})$, $\mathbb{P}^n$-a.s. with $M^{n,*} = M^{a^{n,*}}$. Furthermore, the sequence $(y^{n,*})_{n\ge1}$ is tight.*

Hence with the above result, the proof of the second assertion can be resume by proving the following inequality

$$\limsup_{n\to\infty} \mathbb{E}[y^{n,*}_0] \le Y_0 = \sup_{\mathbb{P}\in\mathbb{P}_{\mathbf{D}}} \mathbb{E}^{\mathbb{P}}[y^{\mathbb{P}}_0]. \tag{5.3.8}$$

where $y^{n,*}$ is the process of Proposition 5.3.2. To prove (5.3.8), we will deal simultaneous with the convergence of controls over the set $\mathcal{A}_n$ and the convergence of processes. In order to guarantee the existence of limits, it will be necessary to work with a space of controls that have the appropriate closure property and which yield the same value function for the optimization problem we seek to approximate. This will done through deterministic relaxed controls.

Relaxed Controls. The notions related to relaxed controls used in the following can be found in the work El Karoui, Huu Nguyen and Jeanblanc-Picqué[34] and Kusnher and Dupuis [57]. Given the compact set $\mathbf{D}$, we define a Borel measure q on the σ-algebras $\mathcal{B}\big([0,T]\times\mathbf{D}\big)$ (the collection of Borel sets of $[0,T]\times\mathbf{D}$) such that $q([0,t]\times\mathbf{D}) = t$ for all $t\in[0,T]$. Denote $\mathbb{Q}$ the following set

$$\mathbb{Q} := \{q \text{ measure on } [0,T]\times\mathbf{D} :\ q(dt,d\gamma) = q(t,d\gamma)dt\}.$$

For $q\in\mathbb{Q}$, we can defined a derivative $q_t(\cdot)$ such that

$$q(B) = \int_{[0,T]\times\mathbf{D}} 1_{\{(t,\gamma)\in B\}} q_t(d\gamma)dt,$$

for all $B\in\mathcal{B}\big([0,T]\times\mathbf{D}\big)$, i.e. $q(dt,d\gamma) = q_t(d\gamma)dt$. For every t, q_t is a probability measure on $\mathbf{D}$. Let us introduce the canonical space of continuous paths , càdlàg paths and measures

$$\overline{\Omega} := C([0,T],\mathbb{R})\times D([0,T],\mathbb{R})\times\mathbb{Q}.$$

with the canonical process (X,U,q) and the canonical filtration $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{0\le t\le T}$ defined by

$$\overline{\mathcal{F}}_t := \sigma\{X_s, U_s, q_s(\phi),\ \phi\in C_b([[0,T]\times\mathbf{D}),\ s\le t\},$$

where $C_b([0,T]\times\mathbf{D})$ is the set of all bounded continuous function on $[0,T]\times\mathbf{D}$, and

$$q_s(\phi) := \int_0^s\int_{\mathbf{D}} \phi(s,\gamma)q(ds,d\gamma).$$

We denote $\overline{\mathbb{F}}_+ = (\overline{\mathcal{F}}_{t+})_{0\le t\le T}$ the right limit of $\overline{\mathbb{F}}$.

Remark 5.3.2. *If $(\overline{\mathbb{P}}$ is a probability measure on $(\overline{\Omega},\overline{\mathcal{F}}_T)$ then By definition 2.4 of [34], for every $s\in[0,T]$, $g\in C_b^2(\mathbb{R})$,the process $C(g,q)$ defined by*

$$C_t(g,q) := g(X_t) - g(X_s) - \int_s^t\int_{\boldsymbol{D}} \frac{1}{2}\gamma\frac{d^2g}{dx^2}(X_s)q(ds,d\gamma)$$

is a $(\overline{\mathbb{P}},\overline{\mathbb{F}}_+)$-martingale.

We consider $(M^{n,*},y^{n,*},a^{n,*})$ the processes of Proposition 5.3.2 and define

$$q^{n,*}(dt,d\gamma) := \sum_{k=0}^{n-1}\delta_{a^{n,*}_{t_k}}(d\gamma)dt1_{[t_k,t_{k+1})}(t).$$

Define $\overline{\mathbb{P}}^n$ the law on $\overline{\Omega}$ induced by $(\widehat{M}^{n,*},y^{n,*},q^{n,*})$ in probability space $(\Omega^n,\mathcal{F}^n,\mathbb{P}^n)$. Since $(\widehat{M}^{n,*},y^{n,*})$ is tight (by Proposition 5.3.2) and $\mathbf{D}$ is a compact set, then $(\overline{\mathbb{P}}^n)_{n\ge1}$ is relatively compact. Let $\overline{\mathbb{P}}^\infty$ be a limit probability measures. We have the following result:

Proposition 5.3.3. *Let Assumptions 5.2.1 and 5.3.1 hold. There exists some probability space $(\Omega^*,\mathcal{F}^*,\mathbb{P}^*)$ containing (M^*,y^*,q^*) whose distribution in $\overline{\mathbb{P}}^\infty$ such that*

$$\limsup_{n\to\infty}\mathbb{E}^{\mathbb{P}^n}[y_0^{n,*}] = \mathbb{E}^{\mathbb{P}^*}[y_0^*].$$

Moreover if $\mathbb{F}^*_+$ denote the right limit of the filtration generated by (M^*, y^*, q^*) and completed under $\mathbb{P}^*$, then there exists $\mathbb{F}^*_+$-progressively measurable processes z^*, m^*, k^* and a^* such that m^* is a martingale orthogonal to M^* with $m_0 = 0$, k^* is a predictable nondecreasing process null at 0, a^* is the density of the quadratic variation of M^* and (y^*, z^*, m^*, k^*) solve the following constraint BSDEs

$$y^*_t = \Phi(M^*) + \int_t^T f(s, M^*_\cdot, y^*_s, a^*_s) - \int_t^T z^*_s dM^*_s - \int_t^T dm^*_s + \int_t^T dk^*_s, \quad \mathbb{P}^*\text{-a.s.}$$

such that for all $t \in [0,T]$, $y^*_t \geq h(M^*_t)$, $\mathbb{P}^*$-a.s. and

$$\int_0^T (y^*_{s^-} - h(M^*_{s^-}))dk^*_s = 0, \quad \mathbb{P}^*\text{-a.s.} \tag{5.3.9}$$

Remark 5.3.3. *We conjecture that the Skorokhod condition (5.3.9) is satisty. Unfortunately we are not able until now to improve this assertion.*

Proof. This proof will be divided in two steps, the first step consists in the existence of (y^*, z^*, m^*, k^*) and the second one is devoted reflected condition.

Step 1. The sequence $(\overline{\mathbb{P}}^n)_{n\geq 1}$ of law of $(M^{n,*}, y^{n,*}, q^{n,*})_{n\geq 1}$ is relatively compact, then there exists a probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ containing the limit (M^*, y^*, q^*) of $(M^{n,*}, y^{n,*}, q^{n,*})_{n\geq 1}$ whose distribution is $\overline{\mathbb{P}}^\infty$. That is the sequence $y^{n,*}$ converge in law to y^* and this leads to

$$\mathbb{E}^{\mathbb{P}^n}[y^{n,*}_0] \to \mathbb{E}^{\mathbb{P}^*}[y^*_0] \quad as\ n \to \infty.$$

Furthermore, since the process $(y^{n,*}, z^{n,*}, m^{n,*}, k^{n,*})$ solves the RBSDE (5.4.10), we can used the link between RBSDEs and optimal stopping problem to deduce that $y^{n,*}_\cdot + \int_0^\cdot f(s, M^{n,*}_\cdot, y^{n,*}_s, a^{n,*}_s)ds$ is a $(\mathbb{P}^n, \mathbb{G}^n)$-supermartingale then using relaxed controls and the canonical process (X, U, q), we get back that the process

$$D_\cdot(X, U, q) = U_\cdot + \int_0^\cdot \int_{\mathbf{D}} f(s, X_\cdot, U_s, \gamma) q(ds, d\gamma)$$

is a $(\overline{\mathbb{P}}^n, \overline{\mathbb{F}}_+)$-supermartingale. Then for $0 \leq s \leq t \leq T$ and $E \in \overline{F}_{s+}$, we have

$$\mathbb{E}^{\overline{\mathbb{P}}^n}\left[1_E(D_t(X, U, q) - D_s(X, U, q))\right] \leq 0.$$

Since $D_\cdot(X, U, q)$ is bounded and continuous (by Assumption 5.2.1) we let n goes to ∞ and get

$$\mathbb{E}^{\overline{\mathbb{P}}^\infty}\left[1_E(D_t(X, U, q) - D_s(X, U, q))\right] \leq 0.$$

$D_\cdot(X, U, q)$ becomes a $(\overline{\mathbb{P}}^\infty, \overline{\mathbb{F}}_+)$-supermartingale. Now came back to the probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$, we obtain that and then $y^*_\cdot + \int_0^\cdot \int_{\mathbf{D}} f(s, M^*_\cdot, y^*_s, \gamma_s) q^*(d\gamma)ds$ is a $\mathbb{F}^*_+$-supermartingale. Denote

$$a^*_s := \int_{\mathbf{D}} \gamma q^*_s(d\gamma)ds.$$

By the convexity assumption on $\mathbf{D}$, a^* takes values in $\mathbf{D}$. In addition, using the fact that q^* is the limit of $q^{n,*}$ it appears that a^* is the density of the quadratic variation of M^*. Using the fact that $\gamma \mapsto f(t, \mathbf{x}, y, \gamma)$ is a concave function, we have

$$\int_{\mathbf{D}} f(t, \mathbf{x}, y, \gamma) q^*_t(d\gamma) \leq f(t, \mathbf{x}, y, \int_{\mathbf{D}} \gamma q^*_t(d\gamma)).$$

Then $y^*_\cdot + \int_0^\cdot \int_{\mathbf{D}} f(s, M^*_\cdot, y^*_s, \gamma_s) q^*(d\gamma)ds$ still a $\mathbb{F}^*_+$-supermartingale.

There exists some probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ containing (M^*, y^*, q^*) We know apply the Doob-Meyer decomposition and the orthogonal decomposition with respect to M^* of the supermartingale $y^*_\cdot + \int_0^\cdot f(s, M^*_\cdot, y^*_s, a^*_s)ds$ to obtain the existence of z^*, m^* and k^* such that m^* is a $\mathbb{P}^*$-martingale orthogonal to M^*, k^* is a nondecreasing process with $k_0 = 0$ and

$$y^*_t + \int_0^t f(s, M^*_\cdot, y^*_s, a^*)ds = \int_0^t z^*_s dM^*_s + m^*_t - k^*_t.$$

Step 2. Since $(y^{n,*}, z^{n,*}, m^{n,*}, k^{n,*})$ solves the RBSDE (5.4.10) then we have by (5.4.11) that

$$\sup_{n\geq 1}\mathbb{E}^{\mathbb{P}^n}\Big[\sup_{0\leq t\leq T}|y^{n,*}_t|^2 + \int_0^T |(a^{n,*}_s)^{1/2}z^{n,*}_s|^2 dC^n_s + \langle m^{n,*}\rangle_T + (k^{n,*}_T)^2\Big] < +\infty. \tag{5.3.10}$$

Then $y^{n,*}$ and $M^{n,*}$ (see Lemma 5.4.2) are uniformly bounded and convergent in L^2(weakly for $y^{n,*}$ and strongly for $M^{n,*}$), so that a sequence of convex combinations $(\bar{M}^{n,*})$, $(\bar{y}^{n,*})$ and $(\bar{k}^{n,*})$ of the form

$$\bar{M}^{n,*} = \sum_{i=1}^{N_n}\alpha^n_k M^{n_i,*}, \quad \bar{y}^{n,*} = \sum_{i=1}^{N_n}\alpha^n_i y^{n_i,*} \quad \text{and} \quad \text{with} \quad \sum_{i=1}^{N_n}\alpha^n_i = 1,$$

converges strongly respectively to M^*, y^* and k^*, then using the fact that $y^{n_i,*}_t \geq h(M^{n_i,*}_t), \mathbb{P}^n$-a.s for any $t \in [0,T]$, we have by using the convexity of h that

$$\begin{aligned} y^*_t = \lim_{n\to+\infty}\bar{y}^{n,*}_t = \lim_{n\to+\infty}\sum_{i=1}^{N_n}\alpha^n_i y^{n_i,*}_t &\geq \lim_{n\to+\infty}\sum_{i=1}^{N_n}\alpha^n_i h(M^{n_i,*}_t) \\ &\geq \lim_{n\to+\infty}\sum_{i=1}^{N_n} h(\alpha^n_i M^{n_i,*}_t) \\ &\geq \lim_{n\to+\infty} h(\bar{M}^{n,*}_t) \\ &\geq h(M^*_t). \end{aligned}$$

and then $y^*_t \geq h(M^*_t), \mathbb{P}^*$-a.s for any $t \in [0,T]$. Hence since k^* is a non decreasing process null at 0, we deduce that

$$\int_0^T (y^*_{s^-} - h(M^*_{s^-}))dk^*_s \geq 0, \quad \mathbb{P}^*\text{-a.s.}$$

□

The proof of the second assumption follows directly from Proposition 5.3.3. In fact,

$$\limsup_{n\to\infty}\mathbb{E}[y^{n,*}_0] = \mathbb{E}^{\mathbb{P}^*}[y^*_0] \leq \sup_{\mathbb{P}\in\mathbb{P}_{\mathbf{D}}}\mathbb{E}^{\mathbb{P}}[y^{\mathbb{P}}_0] = Y_0. \tag{5.3.11}$$

□

Proof of Lemma 5.3.1. We have

$$\begin{aligned} |u^n(t_k, \mathrm{x}^1) - u^n(t_{k+1}, \mathrm{x}^{2,k})| &\leq |u^n(t_k, \mathrm{x}^1) - u^n(t_k, \mathrm{x}^2)| + |u^n(t_k, \mathrm{x}^2) - u^n(t_{k+1}, \mathrm{x}^{2,k})| \\ &\leq \sup_{\gamma\in\mathbf{D}}|u^n_\gamma(t_k, \mathrm{x}^1) - u^n_\gamma(t_k, \mathrm{x}^2)| + \sup_{\gamma\in\mathbf{D}}|u^n_\gamma(t_k, \mathrm{x}^2) - u^n_\gamma(t_{k+1}, \mathrm{x}^{2,k})|. \end{aligned}$$

Let $\gamma \in \mathbf{D}$ and $n \geq 1$, we claim that $u^n_\gamma(t_k, \mathrm{x})$ is Lipschitz in x for all t_k, $k = 0, \cdots, n$.

Let $\mathrm{x} \in \mathbb{R}^{n+1}, u_\gamma^n(t_n, x) = \Phi(\mathrm{x})$ is Lipschitz in x. Suppose that for $\mathrm{x} \in \mathbb{R}^{k+2}, u_\gamma^n(t_{k+1}, \mathrm{x})$ is Lipschitz in x and let $\mathrm{x}^1, \mathrm{x}^2 \in \mathbb{R}^{k+1}$,

$$\begin{aligned}
|u_\gamma^n(t_k, \mathrm{x}^1) - u_\gamma^n(t_k, \mathrm{x}^2)| &= |\max\{h(\mathrm{x}_{k+1}^1), \mathbb{E}[u_\gamma^n(t_{k+1}, M^{t_k,\mathrm{x}^1,\gamma})] + f(t_k, \widehat{\mathrm{x}}^1, u_\gamma^n(t_k, \mathrm{x}^1), \gamma)h\} \\
&\quad - \max\{h(\mathrm{x}_{k+1}^2), \mathbb{E}[u_\gamma^n(t_{k+1}, M^{t_k,\mathrm{x}^2,\gamma})] + f(t_k, \widehat{\mathrm{x}^2}, u_\gamma^n(t_k, \mathrm{x}^2), \gamma)h\}| \\
&\leq \max\{\mathbb{E}[|u_\gamma^n(t_{k+1}, M^{t_k,\mathrm{x}^1,\gamma}) - u_\gamma^n(t_{k+1}, M^{t_k,\mathrm{x}^2,\gamma})|] \\
&\quad + |f(t_k, \widehat{\mathrm{x}}^1, u_\gamma^n(t_k, \mathrm{x}^1), \gamma) - f(t_k, \widehat{\mathrm{x}^2}, u_\gamma^n(t_k, \mathrm{x}^2), \gamma)|h, |h(\mathrm{x}_{k+1}^1) - h(\mathrm{x}_{k+1}^2)|\} \\
&\leq K \max\{|\mathrm{x}_{k+1}^1 - \mathrm{x}_{k+1}^2|, \mathbb{E}[\|M^{t_k,\mathrm{x}^1,\gamma} - M^{t_k,\mathrm{x}^2,\gamma}\|] + \|\widehat{\mathrm{x}}^1 - \widehat{\mathrm{x}}^2\| \\
&\quad + |u_\gamma^n(t_k, \mathrm{x}^1) - u_\gamma^n(t_k, \mathrm{x}^2)|\}.
\end{aligned}$$

We recall that $M^{t_k,\mathrm{x}^1,\gamma} = (\mathrm{x}^1, \mathrm{x}_{k+1}^1 + H_{k+1}^n(\gamma, U_{k+1}))$ hence $\|M^{t_k,\mathrm{x}^1,\gamma} - M^{t_k,\mathrm{x}^2,\gamma}\|. = \|\mathrm{x}^1 - \mathrm{x}^2\|$. Moreover, $\|\widehat{\mathrm{x}}^1 - \widehat{\mathrm{x}}^2\| \leq 2\|\mathrm{x}^1 - \mathrm{x}^2\|$. Therefore, we obtain

$$|u_\gamma^n(t_k, \mathrm{x}^1) - u_\gamma^n(t_k, \mathrm{x}^2)| \leq \frac{3K}{1 - Kh}\|\mathrm{x}^1 - \mathrm{x}^2\|. \tag{5.3.12}$$

It remains to give an estimates to $u_\gamma^n(t_k, \mathrm{x}^2) - u_\gamma^n(t_{k+1}, \mathrm{x}^{2,k})$. If we use the approximation of [11] and denote $\tilde{u}_\gamma^n$ a kind of approximation of BSDE before reflection, we can write

$$\begin{aligned}
u_\gamma^n(t_k, \mathrm{x}^2) &= \max\{h(\mathrm{x}_{k+1}^2), \tilde{u}_\gamma^n(t_k, \mathrm{x}^2)\}; \\
\tilde{u}_\gamma^n(t_k, \mathrm{x}^2) &= \mathbb{E}[u_\gamma^n(t_{k+1}, M^{t_k,\mathrm{x}^2,\gamma})] + f(t_k, \widehat{\mathrm{x}}^2, \tilde{u}_\gamma^n(t_k, \mathrm{x}^1), \gamma)h
\end{aligned}$$

and then

$$|u_\gamma^n(t_k, \mathrm{x}^2) - u_\gamma^n(t_{k+1}, \mathrm{x}^{2,k})| \leq |\tilde{u}_\gamma^n(t_k, \mathrm{x}^2) - \tilde{u}_\gamma^n(t_{k+1}, \mathrm{x}^{2,k})| + |h(\mathrm{x}_{k+1}^2) - h(\mathrm{x}_{k+1}^{2,k})|.$$

Since $\mathrm{x}_{k+1}^2 = \mathrm{x}_{k+1}^{2,k}$, the last term is null and by approximation about BSDEs we obtain

$$|u_\gamma^n(t_k, \mathrm{x}^2) - u_\gamma^n(t_{k+1}, \mathrm{x}^{2,k})| \leq |\tilde{u}_\gamma^n(t_k, \mathrm{x}^2) - \tilde{u}_\gamma^n(t_{k+1}, \mathrm{x}^{2,k})| \leq C\sqrt{h}. \tag{5.3.13}$$

where $C > 0$ is a constant depending only on T and K, but independent of n. Combining the estimates (5.3.12) and (5.3.13) the proof is complete. □

Proof of Proposition 5.3.1. For a process $a \in \mathcal{A}_n$, we have

$$\begin{aligned}
y_{t_k}^a &= \max\left\{h(M_{t_k}^a), \mathbb{E}[y_{t_{k+1}}^a | \mathcal{F}_{t_k}^n] + f(t_k, \widehat{M^a}_\cdot, y_{t_k}^a, a_{t_k}^{1/2} z_{t_k}^a, a_{t_k})h\right\}, \\
z_{t_k}^a &= \frac{1}{h}\mathbb{E}\left[y_{t_{k+1}}^a \, a_{t_k}^{-1} H_{k+1}^n(a_{t_{k+1}}, U_{k+1}) \middle| \mathcal{F}_{t_k}^n\right].
\end{aligned}$$

where $\mathcal{F}_{t_k}^n = \sigma(U_1, \ldots, U_k)$ and $M_{t_k}^a = \sum_{j=1}^k H_j^n(a_{t_j}, U_j)$. In the same way, for $\gamma \in \mathbf{D}$, and $x \in \mathbb{R}^{k+1}$ we have

$$\begin{aligned}
u_\gamma^n(t_k, \mathrm{x}) &= \max\{h(M_{t_k}^{t_k,\mathrm{x},\gamma}), \mathbb{E}[u_\gamma^n(t_{k+1}, M^{t_k,\mathrm{x},\gamma})] + f(t_k, \widehat{\mathrm{x}}, u_\gamma^n(t_k, \mathrm{x}), \gamma^{1/2} v_\gamma^n(t_k, \mathrm{x}), \gamma)h\}, \\
&= \max\{h(M_{t_k}^{t_k,\mathrm{x},\gamma}), \mathbb{E}[u_\gamma^n(t_{k+1}, M^{t_k,\mathrm{x},\gamma}) | \mathcal{F}_{t_k}^n] + f(t_k, \widehat{\mathrm{x}}, u_\gamma^n(t_k, \mathrm{x}), \gamma^{1/2} v_\gamma^n(t_k, \mathrm{x}), \gamma)h\}
\end{aligned}$$

and

$$\begin{aligned}
v_\gamma^n(t_k, \mathrm{x}) &:= \frac{1}{h}\mathbb{E}\left[u_\gamma^n(t_{k+1}, M^{t_k,\mathrm{x},\gamma})\gamma^{-1}\Delta M_{t_{k+1}}^{t_k,\mathrm{x},\gamma}\right] \\
&:= \frac{1}{h}\mathbb{E}\left[u_\gamma^n(t_{k+1}, M^{t_k,\mathrm{x},\gamma})\gamma^{-1}\Delta M_{t_{k+1}}^{t_k,\mathrm{x},\gamma} \middle| \mathcal{F}_{t_k}^n\right].
\end{aligned}$$

This is due to the fact that $M^{t_k,x,\gamma}$ and $\Delta M^{t_k,x,\gamma}_{t_{k+1}}$ only depends on U_{k+1} by definition and therefore independent of $\mathcal{F}^n_{t_k}$. Comparison of the above equations shows that

$$\sup_{\gamma\in\mathbf{D}} u^n_\gamma(0,\mathbf{0}) \leq \sup_{a\in\mathcal{A}_n} \mathbb{E}[y^a_0],$$

and then we obtain one side inequality. Its remains to show the reverse inequality. Let $0 \leq k \leq n$

$$\sup_{a\in\mathcal{A}_n} \mathbb{E}[y^a_{t_k}] = \sup_{a\in\mathcal{A}_n} \max\left\{\mathbb{E}[h(M^a_{t_k})], \mathbb{E}[y^a_{t_{k+1}}] + \mathbb{E}\big[f(t_k, \widehat{M^a}_\cdot, y^a_{t_k}, a^{1/2}_{t_k} z^a_{t_k}, a_{t_k})h\big]\right\}.$$

Then using the selection measurable theorem we have that for any $\epsilon > 0$ there exists a measurable process $\gamma^\epsilon \in \mathcal{A}_n$ with M^{γ^ϵ} (defined using H^n, γ^ϵ and $U_1, \ldots, U_n$) such that

$$\begin{aligned}\sup_{a\in\mathcal{A}_n} \mathbb{E}[y^a_{t_k}] &\leq \max\left\{\mathbb{E}[h(M^{\gamma^\epsilon}_{t_k})], \mathbb{E}[y^{\gamma^\epsilon}_{t_{k+1}}] + \mathbb{E}\big[f(t_k, \widehat{M^{\gamma^\epsilon}}_\cdot, y^{\gamma^\epsilon}_{t_k}, (\gamma^\epsilon_{t_k})^{1/2} z^{\gamma^\epsilon}_{t_k}, \gamma^\epsilon_{t_k})h\big]\right\} + \epsilon \\ &\leq u^n(t_k, M^{\gamma^\epsilon}_{t_k}) + \epsilon.\end{aligned}$$

Therefore by taking $k = 0$ and by the arbitrariness of $\epsilon > 0$, we deduce that

$$Y^n_0 = \sup_{a\in\mathcal{A}_n} \mathbb{E}[y^a_0] \leq u^n(0,\mathbf{0}).$$

This concludes the proof. □

Proof of Proposition 5.3.2. This proof use the same arguments as in the proof of Proposistion 4.3 in [84]. Let $n \geq 1$, using the continuity of H^n_k in γ and the dominated convergence theorem, u^n_γ defined in (5.3.7) is continuous in γ. By the compactness of $\mathbf{D}$, any fixed t_k, there exists an optimal denoted by $\gamma^*(t_k)$ for the maximization problem (5.3.7). Then by the measurable selection theorem we can construct $a^{n,*}$ such that $a^{n,*} = (\gamma^*_{t_1}, \ldots, \gamma^*_{t_n})$ which verified

$$u^n(t_k, M^{n,*}) = u^n_{\gamma^*_{t_k}}(t_k, M^{n,*}) = y^{n,*}_{t_k}.$$

Moreover using Lemma 5.3.1, we have

$$\begin{aligned}\mathbb{E}[(\Delta y^{n,*}_{t_k})^2|\mathcal{F}^n_{t_k}] &= \mathbb{E}[|y^{n,*}_{t_{k+1}} - y^{n,*}_{t_k}|^2|\mathcal{F}^n_{t_k}] \\ &= \mathbb{E}[|u^n(t_{k+1}, M^{n,*,k}) - u^n(t_k, M^{n,*})|^2|\mathcal{F}^n_{t_k}] \\ &\leq C\mathbb{E}[||H^n_{k+1}(a^{n,*}_{t_{k+1}}, U_{k+1})| + \sqrt{h}|^2|\mathcal{F}^n_{t_k}] \\ &\leq C\big(\mathbb{E}[|H^n_{k+1}(a^{n,*}_{t_{k+1}}, U_{k+1})|^2] + h\big) \\ &\leq C(\bar{a}+1)h.\end{aligned}$$

where the constant C may vary line to line and does not depend of n. Since $y^{n,*}$ is a pure jump process, this implies that

$$\langle y^{n,*}\rangle_{t_k} \leq C(\bar{a}+1)t_k, \quad t_{k-1} \leq t \leq t_k.$$

Let defined G^n as follows

$$G^n_s := C(\bar{a}+1)\sum_{k=0}^{n-1} t_{k+1} 1_{[t_k,t_{k+1})}(s).$$

$(G^n)_{n\geq 1}$ is a sequence of non decreasing processes weakly converging to the deterministic process $s \mapsto C(\bar{a}+1)s$ as $n \to \infty$. Then by applying Theorem 2.3 of Jacod, Mémin and Métivier [49], where their condition $C1$ holds for the sequence of non decreasing processes (G^n), we deduce the tightness of $(y^{n,*})_{n\geq 1}$. □

5.4 Stability results for solutions of RBSDE in a general filtration

In this section, we prove existence, uniqueness and stability of the solution to RBSDE (5.3.3). The arguments of proof are closed to the one used by Briand, Delyon and Mémin [20] in their work on stability results for standard BSDEs. We also point out the fact we don't use Itô's formula here to prove estimates and for stability results because the processes M^a and X are not martingales with respect to a common filtration. Unless otherwise stated the expectation $\mathbb{E}$ will be understand with respect to the probability measure $\mathbb{P}^n$.

5.4.1 Existence and uniqueness

We prove existence by using fixed point arguments and Picard approximation and then use conventional methods and steps to proof our result. We begin with a priori estimates of RBSDEs where the generator does not depend on y and z.

Lemma 5.4.1. *Let $(U_t)_{0\leq t\leq T}$ be a progressively measurable process and $(V_t)_{0\leq t\leq T}$ be a predictable process with respect to $\mathbb{G}^n = \{\mathcal{G}_t^n, 0 \leq t \leq T\}$ such that*

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}|U_t|^2\Big] < +\infty \quad \textit{and} \quad \mathbb{E}\Big[\int_0^T |a_s^{1/2}V_s|^2 dC_s^n\Big] < +\infty.$$

Then the RBSDE

$$\begin{aligned} &y_t = \Phi(\widehat{M}^a_\cdot) + \int_t^T f(s, \widehat{M}^a_\cdot, U_{s^-}, a_s^{1/2}V_s, a_s)dC_s^n - \int_t^T z_s dM_s^a - \int_t^T dm_s + k_T - k_t, \\ &y_t \geq h(\widehat{M}_t^a),\ t \in [0,T] \\ &\int_0^T (y_{s^-} - h(\widehat{M}^a_{s^-}))dk_s = 0. \end{aligned} \tag{5.4.1}$$

where m is a $\mathbb{G}^n$-martingale orthogonal to M^a has a unique solution and

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}|y_t|^2 + \int_0^T |a_s^{1/2}z_s|^2 dC_s^n + \langle m\rangle_T + (k_T)^2\Big] < +\infty.$$

Proof. Using the link between RBSDEs and optimal stopping times, y set out in Lemma can be written as

$$y_t = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbb{E}\Big[\int_t^\tau f(s, \widehat{M}^a_\cdot, U_{s^-}, a_s^{1/2}V_s, a_s)dC_s^n + h(\widehat{M}^a_\tau)1_{\{\tau<T\}} + \Phi(\widehat{M}^a_\cdot)1_{\{\tau=T\}}\big|\mathcal{G}_t^n\Big], \quad 0 \leq t \leq T.$$

Moreover, we have that

$$|y_t| \leq \mathbb{E}\Big[\int_0^T f(s, \widehat{M}^a_\cdot, U_{s^-}, a_s^{1/2}V_s, a_s)dC_s^n + \sup_{0\leq t\leq T}|h(\widehat{M}_t^a)| + |\Phi(\widehat{M}^a_\cdot)|\big|\mathcal{G}_t^n\Big].$$

By Doob's inequality,

$$\begin{aligned} \mathbb{E}\Big[\sup_{0\leq t\leq T}|y_t|^2\Big] &\leq 4\mathbb{E}\Big[\Big(\int_0^T f(s, \widehat{M}^a_\cdot, U_{s^-}, a_s^{1/2}V_s, a_s)dC_s^n + \sup_{0\leq t\leq T}|h(\widehat{M}_t^a)| + |\Phi(\widehat{M}^a_\cdot)|\Big)^2\Big] \\ &\leq 12\mathbb{E}\Big[\Big(\int_0^T f(s, \widehat{M}^a_\cdot, U_{s^-}, a_s^{1/2}V_s, a_s)dC_s^n\Big)^2 + \sup_{0\leq t\leq T}|h(\widehat{M}_t^a)|^2 + |\Phi(\widehat{M}^a_\cdot)|^2\Big]. \end{aligned} \tag{5.4.2}$$

Furthermore, for $0 \leq s \leq t \leq T$,

$$\begin{aligned} C_t^n - C_s^n &= [t/h]h - [s/h]h = [(t-s)/h + s/h]h - [s/h]h \\ &\leq ([(t-s)/h] + [s/h] + 1)h - [s/h]h = [(t-s)/h]h + h \\ &\leq (t-s) + h. \end{aligned} \tag{5.4.3}$$

Then $\sup_n C_T^n$ is finite and therefore the right hand side of the inequality (5.4.2) is finite in view of Assumptions 5.2.1 (i),(ii),(iii) and (vi). We deduce that

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} |y_t|^2\Big] < +\infty.$$

We apply the Doob-Meyer decomposition and the orthogonal decomposition with respect to M^a of the supermartingale $y_t + \int_0^t f(s, \widehat{M}_\cdot^a, U_{s^-}, a_s^{1/2} V_s, a_s) dC_s^n$ to obtain the existence of z, m and k such that

$$y_t + \int_0^t f(s, \widehat{M}_\cdot^a, U_{s^-}, a_s^{1/2} V_s, a_s) dC_s^n = \int_0^t z_s dM_s^a + m_t - k_t. \tag{5.4.4}$$

k is an increasing process such that $k_0 = 0$ and m is a $\mathbb{G}^n$-martingale orthogonal to M^a with $m_0 = 0$. On can refer to for example Jacod [48], Theorem 1.53 for Doob decomposition of supermartingales and Theorem 4.27 for orthogonal decomposition of martingales. It is well known that for all $\epsilon > 0$, the stopping time D_t^ϵ defined by

$$D_t^\epsilon = \inf\{s \geq t, \;\; y_s \leq h(\widehat{M}_s^a) + \epsilon\} \wedge T$$

is ϵ-optimal, and $k_{D_t^\epsilon} = k_t$ (see proof of Proposition 3.1 of [60]). Then

$$\int_0^T (y_{s^-} - h(\widehat{M}_{s^-}^a)) dk_s = 0.$$

Since it is obvious that (y, z, m, k) solves the RBSDE, it remains to prove integrability results of z, m and k. Using (5.4.2), we have

$$\begin{aligned} \mathbb{E}[(k_T)^2] &= 2\mathbb{E}\Big[\int_0^T (k_T - k_s) dk_s\Big] \\ &= 2\mathbb{E}\Big[\int_0^T \mathbb{E}^n[k_T - k_s | \mathcal{G}_s^n] dk_s\Big] \\ &= 2\mathbb{E}\Big[\int_0^T \mathbb{E}^n\big[y_s - y_T - \int_0^t f(s, \widehat{M}_\cdot^a, U_{s^-}, a_s^{1/2} V_s, a_s) dC_s^n | \mathcal{G}_s^n\big] dk_s\Big] \\ &\leq 2\mathbb{E}\Big[\Big(2 \sup_{0\leq t\leq T} |y_s| + \int_0^T |f(s, \widehat{M}_\cdot^a, U_{s^-}, a_s^{1/2} V_s, a_s)| dC_s^n\Big) k_T\Big] \\ &\leq 2\mathbb{E}\Big[\Big(2 \sup_{0\leq t\leq T} |y_s| + \int_0^T |f(s, \widehat{M}_\cdot^a, U_{s^-}, a_s^{1/2} V_s, a_s)| dC_s^n\Big)^2\Big] + \frac{1}{2}\mathbb{E}[(k_T)^2]. \end{aligned}$$

Then

$$\mathbb{E}[(k_T)^2] \leq 8\mathbb{E}\Big[\sup_{0\leq t\leq T} |y_t|^2 + \Big(\int_0^T |f(s, \widehat{M}_\cdot^a, U_{s^-}, a_s^{1/2} V_s, a_s)| dC_s^n\Big)^2\Big] < +\infty.$$

Since M^a and m are orthogonal and using (5.4.2), we get

$$\begin{aligned} \mathbb{E}\Big[\int_0^T |a_s^{1/2} z_s|^2 dC_s^n + \langle m \rangle_T\Big] &\leq \mathbb{E}\Big[\Big(\int_0^T z_s dM_s^a + m_T\Big)^2\Big] \\ &\leq \mathbb{E}\Big[\Big(|\Phi(\widehat{M}^a_\cdot)| + \int_0^T |f(s, \widehat{M}_\cdot^a, U_{s^-}, a_s^{1/2} V_s, a_s)| dC_s^n + k_T\Big)^2\Big] \\ &\leq \mathbb{E}\Big[|\Phi(\widehat{M}^a_\cdot)|^2 + \Big(\int_0^T |f(s, \widehat{M}_\cdot^a], U_{s^-}, a_s^{1/2} V_s, a_s)| dC_s^n\Big)^2 + (k_T)^2\Big]. \end{aligned}$$

By the Lipschitz property of F, we have

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^T |f(s,\widehat{M}^a_\cdot,U_{s^-},a_s^{1/2}V_s,a_s)|dC^n_s\Big)^2\Big]\\
&\quad\le \mathbb{E}\Big[\Big(\int_0^T |f(s,\widehat{M}^a_\cdot,0,0,a_s)|dC^n_s + K\int_0^T |U_{s^-}|dC^n_s + K\int_0^T |a_s^{1/2}V_s|dC^n_s\Big)^2\Big]\\
&\quad\le 3\mathbb{E}\Big[\Big(\int_0^T |f(s,\widehat{M}^a_\cdot,0,0,a_s)|dC^n_s\Big)^2 + K^2\Big(\int_0^T |U_{s^-}|dC^n_s\Big)^2 + K^2\Big(\int_0^T |a_s^{1/2}V_s|dC^n_s\Big)^2\Big]\\
&\quad\le 3\mathbb{E}\Big[\sup_{0\le t\le T}|f(t,\widehat{M}^a_\cdot,0,0,a_s)|^2(C^n_T)^2 + K^2\sup_{0\le t\le T}|U_t|^2(C^n_T)^2 + K^2\Big(\int_0^T |a_s^{1/2}V_s|dC^n_s\Big)^2\Big]\\
&\quad\le 3\mathbb{E}\Big[\sup_{0\le t\le T}|f(t,\widehat{M}^a_\cdot,0,0,a_s)|^2T^2 + K^2\sup_{0\le t\le T}|U_t|^2T^2 + K^2\Big(\int_0^T |a_s^{1/2}V_s|dC^n_s\Big)^2\Big] < +\infty.
\end{aligned}$$

□

A stability result is given by:

Proposition 5.4.1. *Let (y,z,m,k) (resp. (y',z',m',k')) be the solution to the RBSDE (5.4.1) associated to (U,V) (resp. (U',V')) where U,U' (resp. V,V') satisty the same property as U(resp. V) in Lemma 5.4.1. For each $0\le\sigma\le\tau\le T$,*

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{\sigma\le t\le\tau}|\delta y_t|^2 + \int_\sigma^\tau |a_s^{1/2}\delta z_t|^2dC^n_s + \langle\delta m\rangle_\tau - \langle\delta m\rangle_\sigma + (\delta k_\tau - \delta k_\sigma)^2\Big]\\
&\qquad\le 1025\mathbb{E}[|\delta y_\tau|^2] + C(\tau,\sigma,h,K)\mathbb{E}\Big[\sup_{\sigma\le t\le\tau}|\delta U_t|^2 + \int_\sigma^\tau |a_s^{1/2}\delta V_t|^2dC^n_s\Big].
\end{aligned}$$

where $C(\tau,\sigma,h,K) = 1185K^2\max\{(\sigma-\tau)+h,((\sigma-\tau)+h)^2\}$ and δy stands for $y-y'$ and so on.

Proof. By definition we have for $t\ge\sigma$,

$$\begin{aligned}
\delta y_t = \delta y_\tau &+ \int_t^\tau (f(s,\widehat{M}^a_\cdot,U_{s^-},a_s^{1/2}V_s,a_s) - f(s,\widehat{M}^a_\cdot,U'_{s^-},a_s^{1/2}V'_s,a_s))dC^n_s - \int_t^\tau \delta z_s dM^a_s\\
&- \int_t^\tau d\delta m_s + \delta k_\tau - \delta k_t,\quad 0\le t\le\tau. \qquad (5.4.5)
\end{aligned}$$

Taking the conditional expectation of the above expression gives

$$\delta y_t = \mathbb{E}\Big[\delta y_\tau + \int_t^\tau (f(s,\widehat{M}^a_\cdot,U_{s^-},a_s^{1/2}V_s) - f(s,\widehat{M}^a_\cdot,U'_{s^-},a_s^{1/2}V'_s,a_s))dC^n_s + \delta k_\tau - \delta k_t\big|\mathcal{G}^n_t\Big].$$

Fix some $\epsilon>0$ and define the stopping time $D^\epsilon_t := \inf\{s\ge t : y_s\le h(\widehat{M}^a_s)+\epsilon\}\wedge\tau$. By definition of D^ϵ_t we have on one side, $y_{D^\epsilon_t}\le h(\widehat{M}^a_{D^\epsilon_t})+\epsilon$ on the set $\{D^\epsilon_t<T\}$ and then

$$\delta y_{D^\epsilon_t} = y_{D^\epsilon_t} - y'_{D^\epsilon_t} \le h(\widehat{M}^a_{D^\epsilon_t}) - y'_{D^\epsilon_t} + \epsilon \le \epsilon \quad \text{ on the set } \quad \{D^\epsilon_t<\tau\}.$$

and on the other side $y'_{s^-} > h(\widehat{M}^a_{s^-})$ for all $s\in[t,D^\epsilon_t]$ and $k_{D^\epsilon_t}-k_t=0$, by the Skorokhod condition. These observations permitted us to get

$$\begin{aligned}
\delta y_t &= \mathbb{E}\Big[\delta y_{D^\epsilon_t} + \int_t^{D^\epsilon_t}(f(s,\widehat{M}^a_\cdot,U_{s^-},a_s^{1/2}V_s,a_s) - f(s,\widehat{M}^a_\cdot,U'_{s^-},a_s^{1/2}V'_s))dC^n_s + k_{D^\epsilon_t} - k_t - (k'_{D^\epsilon_t} - k'_t)\big|\mathcal{G}^n_t\Big]\\
&\le \mathbb{E}\Big[(y_{D^\epsilon_t} - y'_{D^\epsilon_t}) + \int_t^{D^\epsilon_t}(f(s,\widehat{M}^a_\cdot,U_{s^-},a_s^{1/2}V_s,a_s) - f(s,\widehat{M}^a_\cdot,U'_{s^-},a_s^{1/2}V'_s,a_s))dC^n_s - (k'_{D^\epsilon_t} - k'_t)\big|\mathcal{G}^n_t\Big]\\
&\le \mathbb{E}\Big[(y_\tau - y'_\tau)1_{\{D^\epsilon_t=\tau\}} + (y_{D^\epsilon_t} - y'_{D^\epsilon_t})1_{\{D^\epsilon_t<\tau\}}\\
&\qquad\qquad + \int_t^{D^\epsilon_t}(f(s,\widehat{M}^a_\cdot,U_{s^-},a_s^{1/2}V_s,a_s) - f(s,\widehat{M}^a_\cdot,U'_{s^-},a_s^{1/2}V'_s,a_s))dC^n_s\big|\mathcal{G}^n_t\Big]\\
&\le \operatorname*{ess\,sup}_{\nu\in[\sigma,\tau]\cap\mathcal{T}_t}\mathbb{E}\Big[\delta y_\tau 1_{\{\nu=\tau\}} + \int_t^\nu (f(s,\widehat{M}^a_\cdot,U_{s^-},a_s^{1/2}V_s,a_s) - f(s,\widehat{M}^a_\cdot,U'_{s^-},a_s^{1/2}V'_s,a_s))dC^n_s\big|\mathcal{G}^n_t\Big] + \epsilon.
\end{aligned}$$

We know applying this reasoning again, with $-\delta y_t = y'_t - y_t$ and $D^\epsilon_t := \inf\{s \geq t : y'_s \leq h(\widehat{M}^a_s) + \epsilon\} \wedge \tau$ and deduce that

$$-\delta y_t \leq \operatorname*{ess\,sup}_{\nu \in [\sigma,\tau]\cap\mathcal{T}_t} \mathbb{E}\Big[-\delta y_\tau 1_{\{\nu=\tau\}} - \int_t^\nu (f(s,\widehat{M}^a_\cdot, U_{s^-}, a_s^{1/2}V_s, a_s) - f(s,\widehat{M}^a_\cdot, U'_{s^-}, a_s^{1/2}V'_s, a_s))dC^n_s \big| \mathcal{G}^n_t\Big] + \epsilon.$$

Let ϵ go to 0, we get by the Lipschitz property of f

$$\begin{aligned}
|\delta y_t| &\leq \operatorname*{ess\,sup}_{\nu \in [\sigma,\tau]\cap\mathcal{T}_t} \mathbb{E}\Big[|\delta y_\tau| 1_{\{\nu=\tau\}} + \int_t^\nu |f(s,\widehat{M}^a_\cdot, U_{s^-}, a_s^{1/2}V_s, a_s) - f(s,\widehat{M}^a_\cdot, U'_{s^-}, a_s^{1/2}V'_s, a_s)| dC^n_s \big| \mathcal{G}^n_t\Big] \\
&\leq \operatorname*{ess\,sup}_{\nu \in [\sigma,\tau]\cap\mathcal{T}_t} \mathbb{E}\Big[|\delta y_\tau| 1_{\{\nu=\tau\}} + K\int_t^\nu (|\delta U_{s^-}| + |a_s^{1/2}\delta V_s|) dC^n_s \big| \mathcal{G}^n_t\Big] \\
&\leq \mathbb{E}^n\Big[|\delta y_\tau| + K\int_\sigma^\tau (|\delta U_{s^-}| + |a_s^{1/2}\delta V_s|) dC^n_s \big| \mathcal{G}^n_t\Big].
\end{aligned}$$

Taking the supremum over the set $[\sigma,\tau]$ and using Doob's inequality yields

$$\begin{aligned}
\mathbb{E}\Big[\sup_{\sigma\leq t\leq\tau} |\delta y_t|^2\Big] &\leq 4\mathbb{E}\Big[\Big(|\delta y_\tau| + K\int_\sigma^\tau (|\delta U_{s^-}| + |a_s^{1/2}\delta V_s|) dC^n_s\Big)^2\Big] \\
&\leq 12\mathbb{E}\Big[|\delta y_\tau|^2 + K^2\Big(\int_\sigma^\tau |\delta U_{s^-}| dC^n_s\Big)^2 + K^2\Big(\int_\sigma^\tau |a_s^{1/2}\delta V_s| dC^n_s\Big)^2\Big]. \qquad (5.4.6)
\end{aligned}$$

Moreover, for all $t \geq \sigma$,

$$\begin{aligned}
\mathbb{E}[(\delta k_\tau - \delta k_t)^2] &= 2\mathbb{E}\Big[\int_t^\tau (\delta k_\tau - \delta k_s) d\delta k_s\Big] \\
&= 2\mathbb{E}\Big[\int_t^\tau \mathbb{E}[\delta k_\tau - \delta k_s | \mathcal{G}^n_s] d\delta k_s\Big] \\
&= 2\mathbb{E}\Big[\int_t^\tau \mathbb{E}^n\big[\delta y_s - \delta y_\tau - \int_t^\tau \{f(s,\widehat{M}^a_\cdot, U_{s^-}, a_s^{1/2}V_s, a_s) \\
&\qquad\qquad - f(s,\widehat{M}^a_\cdot, U'_{s^-}, a_s^{1/2}V'_s, a_s)\} dC^n_s | \mathcal{G}^n_s\big] d\delta k_s\Big] \\
&\leq 2\mathbb{E}\Big[\Big(\sup_{\sigma\leq t\leq\tau} |\delta y_t| + |\delta y_\tau| + K\int_t^\tau \big(|\delta U_{s^-}| + |a_s^{1/2}\delta V_s|\big) dC^n_s\Big)(\delta k_\tau - \delta k_t)\Big] \\
&\leq 2\mathbb{E}\Big[\Big(\sup_{\sigma\leq t\leq\tau} |\delta y_t| + |\delta y_\tau| + K\int_t^\tau \big(|\delta U_{s^-}| + |a_s^{1/2}\delta V_s|\big) dC^n_s\Big)^2\Big] \\
&\qquad + \frac{1}{2}\mathbb{E}[(\delta k_\tau - \delta k_t)^2].
\end{aligned}$$

Therefore, $\forall t \geq \sigma$, we have

$$\mathbb{E}[(\delta k_\tau - \delta k_t)^2] \leq 16\mathbb{E}\Big[\sup_{\sigma\leq t\leq\tau} |\delta y_t|^2 + |\delta y_\tau|^2 + K^2\Big(\int_t^\tau |\delta U_{s^-}| dC^n_s\Big)^2 + K^2\Big(\int_t^\tau |a_s^{1/2}\delta V_s| dC^n_s\Big)^2\Big].$$

Using (5.4.6), we get

$$\mathbb{E}[(\delta k_\tau - \delta k_t)^2] \leq 192\mathbb{E}[|\delta y_\tau|^2] + 224K^2\mathbb{E}\Big[\Big(\int_t^\tau |\delta U_{s^-}| dC^n_s\Big)^2 + \Big(\int_t^\tau |a_s^{1/2}\delta V_s| dC^n_s\Big)^2\Big].$$

On the other hand , using (5.4.5), we have

$$\begin{aligned}
&\mathbb{E}\Big[\int_\sigma^\tau |a_s^{1/2}\delta z_t|^2 dC_s^n + \langle m\rangle_\tau - \langle m\rangle_\sigma\Big] = \mathbb{E}^n\Big[\Big(\int_t^\tau \delta z_s dM_s^a - \int_t^\tau d\delta m_s\Big)^2\Big]\\
&= \mathbb{E}\Big[\Big(\delta y_\tau - \delta y_\sigma + \int_\sigma^\tau (f(s,\widehat{M}_\cdot^a, U_{s^-}, a_s^{1/2}V_s, a_s) - f(s,\widehat{M}_\cdot^a, U'_{s^-}, a_s^{1/2}V'_s, a_s))dC_s^n + \int_\sigma^\tau d\delta k_s\Big)^2\Big]\\
&\le \mathbb{E}\Big[\Big(|\delta y_\tau| + |\delta y_\sigma| + \int_\sigma^\tau |f(s,\widehat{M}_\cdot^a, U_{s^-}, a_s^{1/2}V_s, a_s) - f(s,\widehat{M}_\cdot^a, U'_{s^-}, a_s^{1/2}V'_s, a_s)|dC_s^n + \int_\sigma^\tau d\delta k_s\Big)^2\Big]\\
&\le \mathbb{E}\Big[\Big(|\delta y_\tau| + \sup_{\sigma\le t\le\tau}|\delta y_t| + K\int_\sigma^\tau (|\delta U_{s^-}| + |a_s^{1/2}\delta V_s|)dC_s^n + \delta k_\tau - \delta k_\sigma\Big)^2\Big]\\
&\le 5\mathbb{E}\Big[|\delta y_\tau|^2 + \sup_{\sigma\le t\le\tau}|\delta y_t|^2 + K^2\Big(\int_\sigma^\tau |\delta U_{s^-}|dC_s^n\Big)^2 + K^2\Big(\int_\sigma^\tau |a_s^{1/2}\delta V_s|dC_s^n\Big)^2 + (\delta k_\tau - \delta k_\sigma)^2\Big]\\
&\le 1025\mathbb{E}[|\delta y_\tau|^2] + 1185K^2\mathbb{E}^n\Big[\Big(\int_\sigma^\tau |\delta U_{s^-}|dC_s^n\Big)^2 + \Big(\int_\sigma^\tau |a_s^{1/2}\delta V_s|dC_s^n\Big)^2\Big].
\end{aligned}$$

Hölder's inequality leads to

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_\sigma^\tau |\delta U_{s^-}|dC_s^n\Big)^2 + \Big(\int_\sigma^\tau |a_s^{1/2}\delta V_s|dC_s^n\Big)^2\Big]\\
&\qquad\le \mathbb{E}\Big[\sup_{\sigma\le t\le\tau}|\delta U_t|^2(C_\tau^n - C_\sigma^n)^2 + (C_\tau^n - C_\sigma^n)\int_\sigma^\tau |a_s^{1/2}\delta V_s|^2 dC_s^n\Big].
\end{aligned}$$

Finally, using (5.4.3), we get

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_\sigma^\tau |\delta U_{s^-}|dC_s^n\Big)^2 + \Big(\int_\sigma^\tau |a_s^{1/2}\delta V_s|dC_s^n\Big)^2\Big]\\
&\qquad\le \mathbb{E}\Big[((\tau-\sigma)+h)^2\sup_{\sigma\le t\le\tau}|\delta U_t|^2 + ((\tau-\sigma)+h)\int_\sigma^\tau |a_s^{1/2}\delta V_s|^2 dC_s^n\Big].
\end{aligned}$$

This concludes the proof. □

Now we prove that when the generator is deterministic (does not depend on a and z), the solution is obtained as a fixed point of a contracting function. We introduce the following norm

$$\begin{aligned}
\|(y,z,m,k)\|_{\alpha,n} = \Big(\sum_{j=0}^{n-1}\alpha^j\mathbb{E}\Big[&\sup_{t\in[jh,(j+1)h]}|y_t|^2 + \int_{jh}^{(j+1)h}|a_s^{1/2}z_s|^2 dC_s^n\\
&+ \int_{jh}^{(j+1)h} d\langle m\rangle_s + (k_{(j+1)h})^2 - (k_{kh})^2\Big]\Big)^{1/2}.
\end{aligned}$$

This norm is equivalent to the following

$$\|(y,z,m,k)\| = \Big(\mathbb{E}\Big[\sup_{t\in[0,(T]}|y_t|^2 + \int_0^T |a_s^{1/2}z_s|^2 dC_s^n + \langle m\rangle_T + (k_T)^2\Big]\Big)^{1/2}.$$

Then if we consider (y,z,m,k), (y',z',m',k'), U, U', V and V' with the same hypothesis defined

in Proposition 5.4.1, we obtain

$$\begin{aligned}\|(\delta y,\delta z,\delta m,\delta k)\|^2_{\alpha,n} &= \Big(\sum_{j=0}^{n-1}\alpha^j\mathbb{E}\Big[\sup_{t\in[jh,(j+1)h]}|\delta y_t|^2+\int_{jh}^{(j+1)h}|a_s^{1/2}\delta z_s|^2dC_s^n\\ &\qquad\qquad+\int_{jh}^{(j+1)h}d\langle\delta m\rangle_s+(k_{(j+1)h})^2-(k_{jh})^2\Big]\Big)^{1/2}\\ &\le\sum_{j=0}^{n-1}\alpha^j\Big\{1025\mathbb{E}[|\delta y_{(j+1)h}|^2]\\ &\qquad\qquad+2\times1185K^2h\mathbb{E}\Big[\sup_{t\in[jh,(j+1)h]}|\delta U_t|^2+\int_{jh}^{(j+1)h}|a_s^{1/2}\delta z_s|^2dC_s^n\Big]\Big\}\\ &\le\frac{1025}{\alpha}\|(\delta y,\delta z,\delta m,\delta k)\|^2_\alpha+2\times1185K^2h\|(\delta U,\delta V,0,0)\|^2_\alpha\end{aligned}$$

Then since $h=T/n\to0$ when n goes to infinity, there exists n_0 such that for all $n\ge n_0$ and for α large enough we have

$$\|(\delta y,\delta z,\delta m,\delta k)\|^2_{\alpha,n}\le\beta\|(\delta U,\delta V,0,0)\|^2_\alpha.$$

where $\beta\in]0,1[$. Then using a fixed point argument, we deduce that the RBSDE (5.4.1) has unique solution for n large enough.

Picard iteration for general Lipschitz generator. We prove that our RBSDE admits a unique which is the limit of Picard iteration.

Proposition 5.4.2. *Let Assumption 5.2.1 holds true. Then there is a unique solution (y^a,z^a,m^a,k^a) to the RBSDE (5.3.3) for n large enough.*

Proof. The proof is based on the Picard iteration sequence $(y^{a,p},z^{a,p},m^{a,p},k^{a,p})$ which is recursively defined by: $(y^{a,0},z^{a,0},m^{a,0},k^{a,0})=(0,0,0,0)$, and

$$\begin{cases}y_t^{a,p+1}=\Phi(\widehat{M}^a_\cdot)+\int_t^T f(s,\widehat{M}^a_\cdot,y^{a,p}_{s^-},a_s^{1/2}z_s^{a,p},a_s)dC_s^n-\int_t^T z_s^{a,p+1}dM_s^a-\int_t^T dm_s^{a,p+1}+\int_t^T dk_s^{a,p+1},\\ y_t^{a,p+1}\ge h(\widehat{M}^a_t),\ t\in[0,T]\\ \int_0^T(y^{a,p+1}_{s^-}-h(\widehat{M}^a_{s^-}))dk_s^{a,p+1}=0.\end{cases}$$

For all $s\le t$; we have

$$\begin{aligned}y_s^{a,p+1}-y_s^{a,p}&=y_t^{a,p+1}-y_t^{a,p}+\int_s^t\{f(r,\widehat{M}^a_\cdot,y^{a,p}_{r^-},a_r^{1/2}z_r^{a,p},a_r)-f(r,\widehat{M}^a_\cdot,y^{a,p-1}_{r^-},a_r^{1/2}z_r^{a,p-1},a_r)\}dC_r^n\\ &\quad-\int_s^t(z_r^{a,p+1}-z_r^{a,p})dM_r^a-\int_s^t d(m_r^{a,p+1}-m_r^{a,p})+\int_s^t d(k_r^{a,p+1}-k_r^{a,p}).\end{aligned}$$

We apply similar arguments used in the proof of Proposition 5.4.1 to deduce that

$$\begin{aligned}&\mathbb{E}\Big[\sup_{s\le r\le t}|y_r^{a,p+1}-y_r^{a,p}|^2+\int_s^t|a_s^{1/2}(y_r^{a,p+1}-y_r^{a,p})|^2dC_s^n+\int_s^t d\langle m^{a,p+1}-m^{a,p}\rangle_r\\ &\qquad+\Big(\int_s^t d(k_s^{a,p+1}-k_s^{a,p})\Big)^2\Big]\\ &\le1025\mathbb{E}[|y_t^{a,p+1}-y_t^{a,p}|^2]+C(s,t,h,K)\mathbb{E}\Big[\sup_{s\le r\le t}|y_r^{a,p}-y_r^{a,p-1}|^2+\int_s^t|a_s^{1/2}(z_r^{a,p}-z_r^{a,p-1})|^2dC_s^n\Big].\end{aligned}$$

Therefore there exists n_0, such that by choosing α large enough, we get for $n\ge n_0$,

$$\begin{aligned}&\|(y^{a,p+1}-y^{a,p},z^{a,p+1}-z^{a,p},m^{a,p+1}-m^{a,p},k^{a,p+1}-k^{a,p})\|^2_{\alpha,n}\\ &\qquad\le\beta\|(y^{a,p}-y^{a,p-1},z^{a,p}-y^{a,p-1},0,0)\|^2_{\alpha,n}\\ &\qquad\le\beta^p\|(y^{a,1},z^{a,1},0,0)\|^2_{\alpha,n},\end{aligned}$$

$\beta \in]0,1[$. Since we have

$$\begin{aligned}\|(y^{a,1}, z^{a,1}, 0, 0)\|^2 &= \mathbb{E}\Big[\sup_{0\le t\le T}|y_t^{a,1}|^2 + \int_0^T |a_s^{1/2} z_s^{a,1}|^2 dC_s^n\Big] \\ &\le \mathbb{E}\Big[|\Phi(\widehat{M}_\cdot^a)| + \sup_{0\le t\le T}|f(t, \widehat{M}_\cdot^a, 0, 0, a_t)|^2 (C_T^n)^2\Big] \\ &\le \mathbb{E}\Big[|\Phi(\widehat{M}_\cdot^a)| + \sup_{0\le t\le T}|f(t, \widehat{M}_\cdot^a, 0, 0, a_t)|^2 T^2\Big] < +\infty\end{aligned}$$

and the norms $\|\cdot\|_{\alpha,n}$ and $\|\cdot\|$ are equivalent, we have

$$\begin{aligned}&\|(y^{a,p+1} - y^{a,p}, z^{a,p+1} - z^{a,p}, m^{a,p+1} - m^{a,p}, k^{a,p+1} - k^{a,p})\|_{\alpha,n}^2 \\ &\qquad \le \beta^p \mathbb{E}\Big[|\Phi(\widehat{M}_\cdot^a)| + \sup_{0\le t\le T}|f(t, \widehat{M}_\cdot^a, 0, 0, a_t)|^2 T^2\Big]. \end{aligned} \tag{5.4.7}$$

From there, one can easily proof that $(y^{a,p}, z^{a,p}, m^{a,p}, k^{a,p})$ is a Cauchy sequence and thus converges to the solution of the RBSDE (5.4.1). □

Since our RBSDEs admits a unique solution, we have the following estimates.

Proposition 5.4.3. *Let (y^a, z^a, m^a, k^a) be as in (5.3.3). Then,*

$$\mathbb{E}\Big[\sup_{0\le t\le T}|y_t^a|^2 + \int_0^T |a_s^{1/2} z_s^a|^2 dC_s^n + \langle m^a\rangle_T + (k_T^a)^2\Big] < +\infty.$$

5.4.2 Stability result for Reflected BSDE

In this section we consider the for a fixed $\mathbb{P}$ the following RBSDEs

$$\begin{cases} y_t^{\mathbb{P}} = \Phi(X_\cdot) + \int_t^T f(s, X_\cdot, y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}}, a_s) ds - \int_t^T z_s^{\mathbb{P}} dX_s - \int_t^T dm_s^{\mathbb{P}} + k_T^{\mathbb{P}} - k_t^{\mathbb{P}}, \quad \mathbb{P}\text{-a.s.} \\ y_t^{\mathbb{P}} \ge h(X_t),\ t \in [0,T] \quad \mathbb{P}\text{-a.s.} \\ \int_0^T (y_{s^-}^{\mathbb{P}} - h(X_{s^-}))dk_s^{\mathbb{P}} = 0 \quad \mathbb{P}\text{-a.s.} \end{cases} \tag{5.4.8}$$

Since X is a martingale under $\mathbb{P}$, the following process denoted by $W^{\mathbb{P}}$ is a $\mathbb{P}$-Brownian motion

$$W_t^{\mathbb{P}} = \int_0^t \widehat{a}_s^{-1/2} dX_s.$$

We approximate the martingale X by a sequence of martingale M^n (identical to the construction of M^a) and we prove that the solution of the RBSDE driven by M^n converges to the above RBSDE (5.4.8) driven by X. This robustness result is established by Briand et al. [19] and [20] for the classical BSDEs with respect to a Brownian filtration. We deal with the case of RBSDE in a general filtration.

5.4.3 Approximation of martingale and convergence

Approximation of X. Given $n \ge 1$, we introduce a^n an $\mathbf{D}$-valued piecewise constant $\mathbb{F}^{\mathbb{P}}$-predictable process defined by

$$a_t^n = \Pi_{\mathbf{D}}\Big[\Big(\frac{n}{T}\int_{(k-1)h}^{kh} \widehat{a}_s^{1/2} ds\Big)^2\Big], \quad t \in (kh, (k+1)h],$$

for $k = 1, \dots, n-1$, where $\Pi_{\mathbf{D}} : \mathbb{R}_+^\star \to \mathbf{D}$ is the Euclidean projection. On $[0,h]$, we take, $a_t^n := \Gamma$, where Γ is an invertible element of $\mathbf{D}$.

Consider $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$ a probability space containing n i.i.d. random variables $U_1, \dots, U_n$, with the filtration $\mathbb{F}^n$ and the completed filtration $\mathbb{G}^n = (\mathcal{G}_t^n)_{t\in[0,T]}$ introduced in section 5.3. Consider also the following discretized process

$$W_0^n := 0, \quad W_{(k+1)h}^n := W_{kh}^n + (a_{kh}^n)^{-1/2} H_k^n(a_{kh}^n, U_k),$$

By using the properties satisfied by H^n in (5.3.1), one can see W^n as a discrete time version of the Brownian motion $W^{\mathbb{P}}$. We know use the existence of regular conditional probability distributions, to construct functions $\Theta_k : D([0,kh];\mathbb{R}) \times \mathbb{R} \times \cdots \times \mathbb{R} \to \mathbf{D}$ such that the random variables $\tilde{a}_{kh}^n := \Theta_k(W^n\,|_{[0,kh]}, U_1, \dots, U_k)$ satisfy

$$\{W^n, \tilde{a}_0^n, \dots, \tilde{a}_{(n-1)h}^n\} = \{W^{\mathbb{P}}, a_0^n, \dots, a_{(n-1)h}^n\} \quad \text{in law}. \tag{5.4.9}$$

Then we obtained the discrete time volatility $\tilde{a}^n = (\tilde{a}_0^n, \dots, \tilde{a}_{nh}^n)$ such that $\tilde{a}_{kh}^n := \tilde{a}^n(W^n, U_1, \dots, U_k)$ is measurable with respect to $\sigma(U_1, \dots, U_k)$, hence $\tilde{a}^n$ belongs to $\mathcal{A}_n$.
Given $n \geq 1$, we define the discrete time martingale M^n by

$$M_0^n := 0, \quad M_{(k+1)h}^n := M_{kh}^n + H_k^n(\tilde{a}_{kh}^n, U_k).$$

By using similar notations as in section 5.3, we define the piecewise continuous processes $W^n, M^n, a^n, \tilde{a}^n$ of the processes define above. We also denote by $\widehat{M}^n$ the linear interpolation of M^n.

Remark 5.4.1. *We notice that W^n and M^n are defined in $(\Omega^n, \mathbb{F}^n, \mathbb{P}^n)$ while $W^{\mathbb{P}}$ and X are defined in $(\Omega, \mathcal{F}_T, \mathbb{P})$. The fact that the processes are in two different probability space does not pose any problem since we work with a convergence in law instead of in probability. By Donsker's theorem and Skorokhod representation theorem, there exists a probability space (where we denote by $\tilde{\mathbb{P}}$ the probability on this space), with a Brownian motion $W^{\mathbb{P}}$ and a sequence of i.i.d. variables $(U^n)_n = (U_1^n, \dots, U_n^n)_n$ such that the processes*

$$\tilde{W}_0^n := 0, \quad \tilde{W}_{(k+1)h}^n := \tilde{W}_{kh}^n + (a_{kh}^n)^{-1/2} H_k^n(a_{kh}^n, U_k^n), \quad 0 \leq t \leq T.$$

satisfy

$$\sup_{0\leq t\leq T} |\tilde{W}_t^n - W_t^{\mathbb{P}}| \longrightarrow 0, \quad \text{as } n \to \infty,$$

in probability $\tilde{\mathbb{P}}$ as well as in L^2 since H^n satisfied (5.3.1).
For simplicity of notation,throughout the remainder of the document we will work with the random variables $U_1, \dots, U_n$ instead of the sequence $(U^n)_n$ when necessary. It will be the same for the probability space, that is to say that we will consider that the processes are in the same probability space the probability measure $\tilde{\mathbb{P}}$.

Convergence of the approximation M^n and others.

Lemma 5.4.2. *We have*

$$\lim_{n\to\infty} \mathbb{E}^{\tilde{\mathbb{P}}}\Big[\sup_{0\leq t\leq T} |W_t^n - W_t^{\mathbb{P}}|^2\Big] = 0 \quad \text{and} \quad \lim_{n\to\infty} \mathbb{E}^{\tilde{\mathbb{P}}}\Big[\sup_{0\leq t\leq T} |M_t^n - X_t|^2\Big] = 0.$$

Moreover, suppose that Assumption 5.2.1 holds true. Then,

$$\lim_{n\to\infty} \mathbb{E}^{\tilde{\mathbb{P}}}\Big[|\Phi(\widehat{M}_\cdot^n) - \Phi(X_\cdot)|^2\Big] = 0$$

and for every $t \in [0,T]$, $(y,z) \in \mathbb{R}\times\mathbb{R}$

$$\lim_{n\to\infty} \mathbb{E}^{\tilde{\mathbb{P}}}\Big[|h(\widehat{M}_t^n) - h(X_t)|^2\Big] = 0$$

and

$$\lim_{n\to\infty} \mathbb{E}^{\tilde{\mathbb{P}}}\Big[\int_0^T |f(t, \widehat{M}_\cdot^n, y, (\tilde{a}_t^n)^{1/2} z, \tilde{a}_t^n) - f(t, X_\cdot, y, \widehat{a}_t^{1/2} z, \widehat{a}_t)|^2 dt\Big] = 0.$$

Proof. The convergence of W^n follows from Donsker's theorem and Remark 5.4.1. We use the Doob's inequality together with the fact that a^n is a piecewise continuous version of the projection of $\widehat{a}$ and (5.4.9) linked to the law of $\tilde{a}^n$ and a^n to establish the following

$$\begin{aligned}\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\sup_{0\le t\le T}|M_t^n - X_t|^2\Big] &\le 4\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\langle M^n - X\rangle_T\Big]\\ &\le 4\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\int_0^T |(\tilde{a}_t^n)^{1/2} - \widehat{a}_t^{1/2}|^2 dt\Big]\\ &= 4\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\int_0^T |(a_t^n)^{1/2} - \widehat{a}_t^{1/2}|^2 dt\Big] \longrightarrow 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

Now we use this result to derive the convergence of the linear interpolation of $\widehat{M}^n$

$$\begin{aligned}&\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\sup_{0\le t\le T}|X_t - \widehat{M}_t^n|^2\Big]\\ &\quad\le \mathbb{E}^{\tilde{\mathbb{P}}}\Big[\sup_{0\le t\le T}\Big|\Big(\Big[\frac{t}{h}\Big] + 1 - \frac{t}{h}\Big)(X_t - M^n_{[t/h]h}) + \Big(\frac{t}{h} - \Big[\frac{t}{h}\Big]\Big)(X_t - M^n_{([t/h]+1)h})\Big|^2\Big]\\ &\quad\le 4\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\sup_{0\le t\le T}\big|X_t - M^n_{[t/h]h}\big|^2\Big]\\ &\quad\le 16\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\langle X - M^n\rangle_T\Big]\\ &\quad\le 16\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\int_0^T |(a_t^n)^{1/2} - \widehat{a}_t^{1/2}|^2 dt\Big] \longrightarrow 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

The convergence about Φ and h follows directly from the above convergence of $X - \widehat{M}^n$.

$$\begin{aligned}&\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\int_0^T |f(t,\widehat{M}^n_\cdot, y, (\tilde{a}_t^n)^{1/2}z, \tilde{a}_t^n) - f(t, X_\cdot, y, \widehat{a}_t^{1/2}z, \widehat{a}_t)|^2 dt\Big]\\ &\qquad\le K^2\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\int_0^T \big(\|\widehat{M}^n_\cdot - X_\cdot\|_t + |((\tilde{a}_t^n)^{1/2} - \widehat{a}_t^{1/2})z| + |\tilde{a}_t^n - \widehat{a}_t|\big)^2 dt\Big]\\ &\qquad\le K^2\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\int_0^T \big(\sup_{0\le s\le t}|\widehat{M}_s^n - X_s| + |((\tilde{a}_t^n)^{1/2} - \widehat{a}_t^{1/2})z| + |\tilde{a}_t^n - \widehat{a}_t|\big)^2 dt\Big]\\ &\qquad\le K^2\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\sup_{0\le s\le T}|\widehat{M}_s^n - X_s|^2 + \int_0^T |((\tilde{a}_t^n)^{1/2} - \widehat{a}_t^{1/2})z|^2 dt + \int_0^T |\tilde{a}_t^n - \widehat{a}_t|\Big]\\ &\longrightarrow 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

□

Now since we have the convergence of M^n, to establish the robustness result we need to have the weakly convergence of filtrations and also the extended convergence.

5.4.4 Weak convergence of filtrations

In this section, we justify why the right continuous and completed filtration $\mathbb{G}^n$ generated by the càdlàg process M^n weakly converge to the right-continuous ans completed filtration $\mathbb{F}_+^{\mathbb{P}}$ generated by the canonical process X. We use the results of section 2.6 of chapter 2 to justify the following weak convergence.

Proposition 5.4.4. *The filtration $\mathbb{G}^n = (\mathbb{G}_t^n)_{a\le t\le T}$ weakly converge to the the canonical filtration $\mathbb{F}_+^{\mathbb{P}}$.*

Proof. Under the probability $\tilde{\mathbb{P}}$ introduced in 5.4.1, the martingale M^n converge uniformly in probability (and hence in law) to X. By the continuity of X, we deduce the convergence of M^n to X under the Skorokhod topology $\mathbf{J}_1$(see Proposition 1.2.1 of [87]). Hence we have

as an immediate consequence the following result. By construction of M^n and the fact that $U_1, \dots, U_n$ are i.i.d., we get that the right-continuous version of M^n is independent increments. This means (M^n) is a sequence of càdlàg processes with independent increments $\mathbf{J}_1$-convergent to X, therefore the desired result is obtained by applying Proposition 2.6.1. □

The above weakly convergence result of filtrations leads us to establish the following extended convergence.

Lemma 5.4.3. $(M^n, \mathbb{G}^n) \to (X, \mathbb{F}_+^{\mathbb{P}})$ *in probability* $\tilde{\mathbb{P}}$.

Proof. This result follows immediately by Proposition 5.4.4, the convergence of M^n to X under $\mathbf{J}_1$ (see proof of proposition 5.4.4 for justification) and Proposition 7 of [24] associated to the continuity of X. □

5.4.5 Convergence of Reflected BSDE

We introduce the following RBSDE under $\mathbb{P}^n$

$$\begin{cases} y_t^n = \Phi(\widehat{M}_\cdot^n) + \int_t^T f(s, \widehat{M}_\cdot^n, y_{s^-}^n, (\tilde{a}_s^n)^{1/2} z_s^n, \tilde{a}_s^n) dC_s^n - \int_t^T z_s^n dM_s^n - \int_t^T dm_s^n + k_T^n - k_t^n, \quad \mathbb{P}^n\text{-a.s.} \\ y_t^n \geq h(\widehat{M}_t^n), \ t \in [0,T] \quad \mathbb{P}^n\text{-a.s.} \\ \int_0^T (y_{s^-}^n - h(\widehat{M}_{s^-}^n)) dk_s^n = 0, \quad \mathbb{P}^n\text{-a.s.} \end{cases} \tag{5.4.10}$$

where $\widehat{M^n}$ is the linear interpolation of M^n, in this equation we take the pieces continuous version of M^n and $\tilde{a}^n$. For n large enough, the above RBSDE under admits a unique solution (y^n, z^n, m^n, k^n) of $\mathbb{G}^n$- progressive measurable processes where m^n is a martingale orthogonal to M^n (The existence and uniqueness result is the similar to the existence and uniqueness result for the RBSDE (5.3.3), see Proposition 5.4.2 for this result). We have the following estimates

$$\sup_{n\geq 1} \mathbb{E}\Big[\sup_{0\leq t\leq T} |y_t^n|^2 + \int_0^T |(\tilde{a}_s^n)^{1/2} z_s^n|^2 dC_s^n + \langle m^n \rangle_T + (k_T^n)^2 \Big] < +\infty. \tag{5.4.11}$$

We now establish the following robustness result. We show that the above RBSDE driven by M^n converges to the solution of the RBSDE 5.4.8. Let first denoted the space $\mathcal{S}^p$ where $1 \leq p < \infty$ the space of càdlàg process $U = (U_t)_{0\leq t\leq T}$ with values in $\mathbb{R}$ with

$$\|U\|_{\mathcal{S}^p}^p = \mathbb{E}^{\tilde{\mathbb{P}}}\Big[\sup_{t\in[0,T]} |U_t|^p \Big] < \infty.$$

Theorem 5.4.1. *Let Assumption 5.2.1 holds. Let (y^n, z^n, m^n, k^n) be the solution to the RBSDE (5.4.10) and $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$ the solution of the RBSDE (5.4.8). Then we have*

$$\Big(y^n, \int_0^\cdot z_r^n dM_r^n, \int_0^\cdot (\tilde{a}_s^n)^{1/2} z_r^n dC_r^n, m^n, k^n \Big) \to \Big(y^{\mathbb{P}}, \int_0^\cdot z_r^{\mathbb{P}} dX_r, \int_0^\cdot \hat{a}_r^{1/2} z_r^{\mathbb{P}} dr, m^{\mathbb{P}}, k^{\mathbb{P}} \Big),$$

as n tends to infinity in law for the topology of uniform convergence.

Proof. We consider the following Picard approximation $(y^{n,p}, z^{n,p}, m^{n,p}, k^{n,p})$ and $(y^{\infty,p}, z^{n,p}, m^{\infty,p}, k^{\infty,p})$ defined by

$$(y^{n,0}, z^{n,0}, m^{n,0}, k^{n,0}) = (0,0,0,0), \quad (y^{\infty,0}, z^{n,0}, m^{\infty,0}, k^{\infty,0}) = (0,0,0,0)$$

and

$$\begin{aligned} & y_t^{n,p+1} = \Phi(\widehat{M^n}_\cdot) + \int_t^T f(s, \widehat{M}_\cdot^n, y_{s^-}^{n,p}, (\tilde{a}_s^n)^{1/2} z_s^{n,p}, \tilde{a}_s^n) dC_s^n - \int_t^T z_s^{n,p+1} dM_s^n - \int_t^T dm_s^{n,p+1} + \int_t^T dk_s^{n,p+1}, \\ & y_t^{n,p+1} \geq h(\widehat{M}_t^n), \ t \in [0,T] \\ & \int_0^T (y_{s^-}^{n,p+1} - h(\widehat{M}_{s^-}^n)) dk_s^{n,p+1} = 0. \end{aligned}$$

and

$$y_t^{\infty,p+1} = \Phi(X_\cdot) + \int_t^T f(s, X_\cdot, y_s^{\infty,p}, \widehat{a}_s^{1/2} z_s^{\infty,p}, \widehat{a}_s) ds - \int_t^T z_s^{\infty,p+1} dX_s - \int_t^T dm_s^{\infty,p+1} + \int_t^T dk_s^{\infty,p+1},$$
$$y_t^{\infty,p+1} \geq h(X_t),\ t \in [0,T]$$
$$\int_0^T (y_{s^-}^{\infty,p+1} - h(X_{s^-})) dk_s^{\infty,p+1} = 0.$$

The same methods used in proof of Proposition 5.4.2 and estimates (5.4.7) applied to $(y^{n,p}, z^{n,p}, m^{n,p}, k^{n,p})$ show the convergence of $(y^{n,p}, z^{n,p}, m^{n,p}, k^{n,p})$ to (y^n, z^n, m^n, k^n) for n large enough. Then there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{n \geq n_0} \mathbb{E}^{\tilde{\mathbb{P}}} \Big[\sup_{0 \leq t \leq T} |y_t^n - y_t^{n,p}|^2 + \int_0^T |(\tilde{a}_s^n)^{1/2} (z_t^n - z_t^{n,p})|^2 dC_s^n + \langle m^n - m^{n,p} \rangle_T + (k_T^n - k_T^{n,p})^2 \Big] \leq C\beta^p,$$

where C is a positive constant and $\beta \in]0,1[$.

Similarly, the convergence of the Picard iteration of the solution $(y^{\infty,p}, z^{\infty,p}, m^{\infty,p}, k^{\infty,p})$ converge to the solution $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$ of the RBSDE (5.4.8) with

$$\mathbb{E}^{\tilde{\mathbb{P}}} \Big[\sup_{0 \leq t \leq T} |y_t^{\mathbb{P}} - y_t^{\infty,p}|^2 + \int_0^T |\widehat{a}_s^{1/2} (z_t^{\mathbb{P}} - z_t^{\infty,p})|^2 ds + \langle m^{\mathbb{P}} - m^{\infty,p} \rangle_T + (k_T^{\mathbb{P}} - k_T^{\infty,p})^2 \Big] \to \quad \text{as } p \to \infty.$$

Using the link between RBSDEs and optimal stopping times, we can write

$$y_t^{n,p+1} = \operatorname*{ess\,sup}_{\nu \in \mathcal{T}_t^n} \mathbb{E}^{\tilde{\mathbb{P}}} \Big[\int_t^\nu f(s, \widehat{M}_\cdot^n, y_{s^-}^{n,p}, (\tilde{a}_s^n)^{1/2} z_s^{n,p}, \tilde{a}_s^n) dC_s^n + \Phi(\widehat{M^n}_\cdot) 1_{\{\nu = T\}} + h(\widehat{M}_\nu^n) 1_{\{\nu < T\}} \big| \mathcal{G}_t^n \Big] \tag{5.4.12}$$

and

$$y_t^{\infty,p+1} = \operatorname*{ess\,sup}_{\nu \in \mathcal{T}_t} \mathbb{E}^{\tilde{\mathbb{P}}} \Big[\int_t^\nu f(s, X_\cdot, y_s^{\infty,p}, \widehat{a}_s^{1/2} z_s^{\infty,p}, \widehat{a}_s) ds + \Phi(X_\cdot) 1_{\{\nu = T\}} + h(X_\nu) 1_{\{\nu < T\}} \big| \mathcal{F}_{t+} \Big] \tag{5.4.13}$$

where $\mathcal{T}_t$ is the set of $\mathbb{F}$-stopping times valued in $[t,T]$ and $\mathcal{T}_t^n$ is the set of $\mathbb{G}^n$-stopping times valued in $[t,T]$.
We prove that for each p, the sequence $(y^{n,p}, \int_0^\cdot z_r^{n,p} dM_r^n, m^{n,p}, k^{n,p})_n$ converge to $(y^{\infty,p}, \int_0^\cdot z_r^{\infty,p} dX_r, m^{\infty,p}, k^{\infty,p})$ as n goes to ∞ as described in the statement using induction under p.
Basis: $p = 0$, $(y^{n,0}, z^{n,0}, m^{n,0}, k^{n,0}) = (y^{\infty,0}, z^{n,0}, m^{\infty,0}, k^{\infty,0}) = (0,0,0,0)$ therefore the assertion is verified for $p = 0$.
Inductive step: assume that the property holds to the order p, that is $(y^{n,p}, \int_0^\cdot z_r^{n,p} dM_r^n, m^{n,p}, k^{n,p})_n$ converge to $(y^{\infty,p}, \int_0^\cdot z^{\infty,p} dX_r, m^{\infty,p}, k^{\infty,p})$ as n goes to ∞. The process, defined by

$$x_t^{n,p+1} = y_t^{n,p+1} + \int_0^t f(s, \widehat{M}_\cdot^n, y_{s^-}^{n,p}, (\tilde{a}_s^n)^{1/2} z_s^{n,p}, \tilde{a}_s^n) dC_s^n, \quad 0 \leq t \leq T. \tag{5.4.14}$$

satisfies

$$x_t^{n,p+1} = \tilde{x}_0^{n,p+1} + \int_0^t z_s^{n,p+1} dM_s^n + m_t^{n,p+1} - k_t^{n,p+1}.$$

Therefore $x^{n,p+1}$ is an $\mathbb{G}^n$-supermartingale. In the same vein, the process $x^{\infty,p+1}$ defined below is an $\mathbb{F}_+^{\mathbb{P}}$-supermartingale

$$x_t^{\infty,p+1} = y_t^{\infty,p+1} + \int_0^t f(s, X_\cdot, y_s^{\infty,p}, \widehat{a}_s^{1/2} z_s^{\infty,p}, \widehat{a}_s) ds, \quad 0 \le t \le T. \tag{5.4.15}$$

$$= x_0^{\infty,p+1} + \int_0^t z_s^{\infty,p+1} dX_s + m_t^{\infty,p+1} - k_t^{\infty,p+1}. \tag{5.4.16}$$

The rest of the proof is divided in three steps.
Step 1: *$\int_0^t f(s, \widehat{M}_\cdot^n, y_{s^-}^{n,p}, (\tilde{a}_s^n)^{1/2} z_s^{n,p}, \tilde{a}_s^n) dC_s^n$ converges to $\int_0^t f(s, X_\cdot, y_s^{\infty,p}, \widehat{a}_s^{1/2} z_s^{\infty,p}, \widehat{a}_s) ds$* . Let $t \in [0,T]$, we have

$$\begin{aligned}
&\int_0^t f(s, \widehat{M}_\cdot^n, y_{s^-}^{n,p}, (\tilde{a}_s^n)^{1/2} z_s^{n,p}, \tilde{a}_s^n) dC_s^n - \int_0^t f(s, X_\cdot, y_s^{\infty,p}, \widehat{a}_s^{1/2} z_s^{\infty,p}, \widehat{a}_s) ds \\
&= \int_0^t \{ f(s, \widehat{M}_\cdot^n, y_{s^-}^{n,p}, (\tilde{a}_s^n)^{1/2} z_s^{n,p}, \tilde{a}_s^n) dC_s^n - f(s, X_\cdot, y_s^{\infty,p}, \widehat{a}_s^{1/2} z_s^{\infty,p}, \widehat{a}_s) \} dC_s^n \\
&\quad + \int_0^t f(s, X_\cdot, y_s^{\infty,p}, \widehat{a}_s^{1/2} z_s^{\infty,p}, \widehat{a}_s)(dC_s^n - ds).
\end{aligned} \tag{5.4.17}$$

However for all $s \in [0,T]$,

$$C_s^n = \left[\frac{s}{h}\right] h = \left[\frac{nt}{T}\right] \frac{T}{n} \text{ and } \lim_{n\to\infty} C_s^n = s, \text{ then } dC_s^n \to ds \text{ as } n \to \infty.$$

It follows that the last term in (5.4.17) tends to 0. Indeed,

$$|f(s, X_\cdot, y_s^{\infty,p}, \widehat{a}_s^{1/2} z_s^{\infty,p}, \widehat{a}_s)| \le |f(s, X_\cdot, 0, 0, \widehat{a}_s)| + K|y_s^{\infty,p}| + K|\widehat{a}_s^{1/2} z_s^{\infty,p}|,$$

and on the right hand side of the above inequality, the first term is bounded by Assumption 5.2.1 (vi) and the two last terms are also bounded in the sense of the estimates in (5.2.4). The induction assumption gives among others,

$$\sup_{t\in[0,T]} |y_t^{n,p} - y_t^{\infty,p}| \to 0 \quad \text{and} \quad \sup_{t\in[0,T]} \left| \int_0^t (\tilde{a}_s^n)^{1/2} z_s^{n,p} dC_s^n - \int_0^t \widehat{a}_s^{1/2} y_s^{\infty,p} ds \right| \to 0.$$

as n tends to ∞. From this convergence we deduce by using the Lipschitz property of f that

$$\begin{aligned}
&\sup_{t\in[0,T]} \left| \int_0^t \{ f(s, \widehat{M}_\cdot^n, y_{s^-}^{n,p}, (\tilde{a}_s^n)^{1/2} z_s^{n,p}, \tilde{a}_s^n) - f(s, X_\cdot, y_s^{\infty,p}, \widehat{a}_s^{1/2} z_s^{\infty,p}, \widehat{a}_s) \} dC_s^n \right| \\
&\le K \sup_{t\in[0,T]} \left| \int_0^t \{ \|\widehat{M}^n - X\|_t + |y_{s^-}^{n,p} - y_s^{\infty,p}| + |(\tilde{a}_s^n)^{1/2} z_s^{n,p}) - \widehat{a}_s^{1/2} z_s^{\infty,p}| + |\tilde{a}_s^n - \widehat{a}_s| \} dC_s^n \right| \\
&\le 3K \Big(\sup_{t\in[0,T]} |\widehat{M}_t^n - X_t\| + \sup_{t\in[0,T]} |y_t^{n,p} - y_t^{\infty,p}| + Big| \int_0^t |\tilde{a}_s^n - \widehat{a}_s| dC_s^n + \left| \int_0^t |(\tilde{a}_s^n)^{1/2} z_s^{n,p}) - \widehat{a}_s^{1/2} z_s^{\infty,p}| dC_s^n \right| \Big) \\
&\qquad \longrightarrow \quad \text{as} \quad n \to \infty.
\end{aligned}$$

which end the first step.
Step 2: *$y^{n,p+1} \to y^{\infty,p+1}$ for the Meyer-Zheng topology and the convergence also holds in L^2.* Using (5.4.12), $(y^{n,p+1})$ is a sequence of Snell envelopes of $g(\cdot, M^n)$ with

$$g(\cdot, M_\cdot^n) = \int_t^\cdot f(s, \widehat{M}_\cdot^n, y_{s^-}^{n,p}, (\tilde{a}_s^n)^{1/2} z_s^{n,p}, \tilde{a}_s^n) dC_s^n + \Phi(\widehat{M^n}_\cdot) 1_{\{\cdot = T\}} + h(\widehat{M}_\cdot^n) 1_{\{\cdot < T\}}.$$

We have by Lemma 5.4.3 the extend convergence of filtration $\mathbb{G}^n$ and the sequence (M^n). Using the continuity (more precisely the quasi-left-continuity) of X, we deduce by Proposition 1.1.39 of [95](who is a result of Aldous [2]) that the sequence (X^n) verifies the Aldous tightness criterion of Assumption 2.7.1. Using Assumption 5.2.1 the functions g is Lipschitz and then the sequence $(g(\cdot, M_\cdot^n))$ also verifies the verifies the Aldous tightness criterion.

$(g(\cdot, M^n_\cdot))$ is also uniformly of class D in the sense of Assumption 2.7.1. In fact, (M^n) is uniformly of class D and by Assumption 5.2.1 f, Φ ang h are bounded and Lipschitz. Also using *step one* and Lemma 5.4.2, we have the following convergence for Skrokhod topology as n tends to ∞

$$g(\cdot, M^n_\cdot) \to g(\cdot, X_\cdot) = \int_t^\cdot f(s, X_\cdot, y^{\infty,p}_{s^-}, \widehat{a}_s^{1/2} z^{\infty,p}_s, \widehat{a}_s) ds + \Phi(X_\cdot) 1_{\{\cdot = T\}} + h(X_\cdot) 1_{\{\cdot < T\}}.$$

Thus by Theorem 2.7.1, the sequence of Snell envelopes $(y^{n,p+1})$ converges to the Snell envelope $y^{\infty,p+1}$ with respect to the Meyer-Zheng topology. Its remains to prove that this convergence also holds in L^2 sense. We have

$$y^{n,p+1}_t = \operatorname*{ess\,sup}_{\nu \in \mathcal{T}_t} \mathbb{E}^{\tilde{\mathbb{P}}} \Big[\int_t^\nu f(s, \widehat{M}^n_\cdot, y^{n,p}_s, (\tilde{a}^n_s)^{1/2} z^{n,p}_s, \tilde{a}^n_s) dC^n s + \Phi(\widehat{M}^n_\cdot) 1_{\{\nu = T\}} + h(\widehat{M}^n_\nu) 1_{\{\nu < T\}} \big| \mathcal{F}_{t+} \Big].$$

Using the same arguments as in proof of Lemma 5.4.1, we get that

$$\mathbb{E}^{\tilde{\mathbb{P}}} \Big[\sup_{0 \le t \le T} |y^{n,p+1}_t|^2 \Big] \le C_1 \Big[\Big(\int_0^T f(s, \widehat{M}^n_\cdot, y^{n,p}_s, (\tilde{a}^n_s)^{1/2} z^{n,p}_s, \tilde{a}^n_s) dC^n_s \Big)^2 + |\Phi(\widehat{M}^n_\cdot)|^2 + \sup_{0 \le t \le T} |h(\widehat{M}^n_t)|^2 \Big] \le C.$$

with C_1 and C two positives constants depending on the Lipschitz coefficient K defined in Assumption **??** and T. We have also used the fact M^n and $y^{n,p}$ are uniformly bounded and $z^{n,p}$ are bounded in L^2. By Fatou's Lemma(or by using the estimates like above), we deduce that

$$\mathbb{E}^{\tilde{\mathbb{P}}} \Big[\sup_{0 \le t \le T} |y^{\infty,p+1}_t|^2 \Big] \le C.$$

Then we can now get the convergence in L^2 of $y^{n,p+1}$ to $y^{\infty,p+1}$ by using the dominated convergence under probability $\tilde{\mathbb{P}}$.

Step 3 *: others convergence.* We deduce by the two firsts steps that the sequence $(x^{n,p+1})$ defined in (5.4.14) is a sequence of positive supermartingales convergent to the positive supermatingale $x^{\infty,p+1}$ defined in (5.4.15) in L^1. Similar arguments of proof of Lemma 5.4.1 applied to $k^{n,p+1}$ yields

$$\mathbb{E}^{\tilde{P}} \Big[(k^{n,p+1}_T)^2 \Big] \le 8 \mathbb{E}^n \Big[\sup_{0 \le t \le T} |y^{n,p+1}_t|^2 + \Big(\int_0^T |f(s, \widehat{M}^n_\cdot, y^{n,p}_{s^-}, (\tilde{a}^n_s)^{1/2} z^{n,p}_s, \tilde{a}^n_s)| dC^n_s \Big)^2 \Big] < C.$$

Then using Corollary 2 of Protter and Barlow [6], we have that the predictable part $k^{n,p+1}$ of $x^{n,p+1}$ converge to the predictable part $k^{\infty,p+1}$ of $x^{\infty,p}$ in L^1 and we have the following convergence of the martingale part in $\mathcal{H}^1$,

$$\int_0^\cdot z^{n,p+1}_s dM^n_s + m^{n,p+1} \longrightarrow \int_0^\cdot z^{\infty,p+1}_s dX_s + m^{\infty,p+1}.$$

Since the $m^{n,p+1}$ and M^n are orthogonal, we deduce that

$$\int_0^\cdot z^{n,p+1}_s dM^n_s \longrightarrow \int_0^\cdot z^{\infty,p+1}_s dX_s \text{ and } m^{n,p+1} \longrightarrow m^{\infty,p+1}$$

as n tends to ∞. Furthermore, Corollary 2 of [6] and the Burkholder inequality allows us to deduce the following convergence

$$\mathbb{E}^{\tilde{\mathbb{P}}} \Big[\sup_{t \in [0,T]} \Big| \int_0^t z^{n,p+1}_s dM^n_s - \int_0^t z^{\infty,p+1}_s dX_s \Big| \Big] \le \mathbb{E}^{\tilde{\mathbb{P}}} \Big[\Big| \int_0^T (\tilde{a}^n_s)^{1/2} z^{n,p+1}_s dC^n_s - \int_0^T \widehat{a}^{1/2}_s z^{\infty,p+1}_s ds \Big| \Big] \longrightarrow 0,$$

and

$$\mathbb{E}^{\tilde{\mathbb{P}}} \Big[\sup_{t \in [0,T]} |m^{n,p+1}_t - m^{\infty,p+1}_t| \Big] = \mathbb{E}^{\tilde{\mathbb{P}}} \Big[|\langle m^{n,p+1} \rangle^{1/2}_T - \langle m^{\infty,p+1} \rangle^{1/2}_T| \Big] \longrightarrow 0,$$

as n tends to ∞. which concludes the proof □

Appendix A

Appendix

A.1 Counter example

Fix $T = 2$ and take as a lower obstacle a process L satisfying the required assumptions in [66] as well as

$$L_t := 2(1-t),\ 0 \leq t \leq 1, \text{ and } L_t \leq 2,\ 1 \leq t \leq 2.$$

Furthermore, take the generator f of the 2RBSDE to be 0, and the terminal condition to be L_2. In this case, the solution to the 2RBSDE being necessarily the supremum of the solutions to the associated RBSDEs, we will have automatically the representations

$$Y_t = \operatorname*{essup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} \operatorname*{essup}^{\mathbb{P}}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\mathbb{P}'}[L_\tau | \mathcal{F}_T],\ y_t^{\mathbb{P}} = \operatorname*{essup}^{\mathbb{P}}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\mathbb{P}'}[L_\tau | \mathcal{F}_T].$$

Furthermore, in this case since $f = 0$, $K^{\mathbb{P}'} - k^{\mathbb{P}'}$ being a $\mathbb{P}'$-submartingale is equivalent to $Y - y^{\mathbb{P}'}$ being a $\mathbb{P}'$-supermartingale, which would imply in particular that

$$Y_0 - y_0^{\mathbb{P}'} \geq \mathbb{E}^{\mathbb{P}'}\left[Y_1 - y_1^{\mathbb{P}'}\right]. \tag{A.1.1}$$

However, it is clear by definition of L that $Y_0 = y_0^{\mathbb{P}'} = 2$. However, there is absolutely no reason why in general one could not have, for some $\mathbb{P}'$, and for an appropriate choice of S, $Y_1 > y_1^{\mathbb{P}'}$ (recall that we always have $Y_1 \geq y_1^{\mathbb{P}'}$), at least with strictly positive $\mathbb{P}'$-probability, which then contradicts (A.1.1). So this is a counterexample to a possible definition that $K^P - k^P$ being a submartingale.

A.2 Equivalent formulation to the RBSDE on enlarged canonical space

Recall that $\overline{\Omega} := \Omega \times \Omega'$ and for any probability measure $\mathbb{P}$ on Ω, we define $\overline{\mathbb{P}} := \mathbb{P} \otimes \mathbb{P}_0$ a probability measure on $\overline{\Omega}$. Therefore, if we consider a $\mathbb{P}$-null set on Ω, it still a $\overline{\mathbb{P}}$-null set on $\overline{\Omega}$ if it is considered in the enlarged space.
Let $\pi : \Omega \times \Omega' \longrightarrow \Omega$ be the projection operator defined by $\pi(\omega, \omega') := \omega$, for any $(\omega, \omega') \in \overline{\Omega}$. The following result is proved in [85](Lemma 2.1).

Lemma A.2.1. *Let $A \subseteq \Omega$ be a subset in Ω. Then saying that A is a $\mathbb{P}$-null set is equivalent to saying that $\{\overline{\omega} : \pi(\overline{\omega}) \in A\}$ is a $\overline{\mathbb{P}} := \mathbb{P} \otimes \mathbb{P}_0$-null set.*

We now consider two RBSDEs on the enlarged space, w.r.t. two different filtrations.

The first one is the following reflected BSDE on $(\overline{\Omega}, \overline{\mathcal{F}}_T^X, \overline{\mathbb{P}})$ w.r.t the filtration $\overline{\mathbb{F}}^{X,\overline{\mathbb{P}}}$:

$$\begin{aligned}
&\overline{y}_t^{\overline{\mathbb{P}}} = \xi + \int_t^T \widehat{f}_r^{\mathbb{P}}(\overline{y}_r^{\overline{\mathbb{P}}}, \widehat{a}_r^{1/2}\overline{z}_r^{\overline{\mathbb{P}}})dr - \int_t^T \overline{z}_r^{\overline{\mathbb{P}}} \cdot dX_r^{c,\mathbb{P}} - \int_t^T d\overline{m}_r^{\overline{\mathbb{P}}} + \overline{k}_T^{\overline{\mathbb{P}}} - \overline{k}_t^{\overline{\mathbb{P}}},\ 0 \le t \le T,\ \overline{\mathbb{P}}\text{-a.s.} \\
&\overline{y}_t^{\overline{\mathbb{P}}} \ge L_t,\ t \in [0,T],\ \overline{\mathbb{P}}\text{-a.s.}, \qquad (A.2.1) \\
&\int_0^T (\overline{y}_{t^-}^{\overline{\mathbb{P}}} - L_{t^-}) d\overline{k}_t^{\overline{\mathbb{P}}} = 0,\ \overline{\mathbb{P}}\text{-a.s.}
\end{aligned}$$

where a solution is a triple $(\overline{y}^{\overline{\mathbb{P}}}, \overline{z}^{\overline{\mathbb{P}}}, \overline{m}^{\overline{\mathbb{P}}}, \overline{k}^{\overline{\mathbb{P}}}) \in \mathbb{D}_0^p(\overline{\mathbb{F}}_+^{X,\overline{\mathbb{P}}}, \overline{\mathbb{P}}) \times \mathbb{H}_0^p(\overline{\mathbb{F}}_+^{X,\overline{\mathbb{P}}}, \overline{\mathbb{P}}) \times \mathbb{M}_0^p(\overline{\mathbb{F}}_+^{X,\overline{\mathbb{P}}}, \overline{\mathbb{P}}) \times \mathbb{I}_0^p(\overline{\mathbb{F}}_+^{X,\overline{\mathbb{P}}}, \overline{\mathbb{P}})$ satisfying (A.2.1).

The second reflected BSDE on the enlarged space $(\overline{\Omega}, \overline{\mathcal{F}}_T, \overline{\mathbb{P}})$, w.r.t. the filtration $\overline{\mathbb{F}}$ is the following

$$\begin{aligned}
&\tilde{y}_t^{\overline{\mathbb{P}}} = \xi + \int_t^T \widehat{f}_r^{\mathbb{P}}(\tilde{y}_r^{\overline{\mathbb{P}}}, \widehat{a}_r^{1/2}\tilde{z}_r^{\overline{\mathbb{P}}})dr - \int_t^T \tilde{z}_r^{\overline{\mathbb{P}}} \cdot \widehat{a}_r^{1/2} dW_r^{\mathbb{P}} - \int_t^T d\tilde{m}_r^{\overline{\mathbb{P}}} + \tilde{k}_T^{\overline{\mathbb{P}}} - \tilde{k}_t^{\overline{\mathbb{P}}},\ 0 \le t \le T,\ \overline{\mathbb{P}}\text{-a.s.} \\
&\tilde{y}_t^{\overline{\mathbb{P}}} \ge L_t,\ t \in [0,T],\ \overline{\mathbb{P}}\text{-a.s.}, \qquad (A.2.2) \\
&\int_0^T (\tilde{y}_{t^-}^{\overline{\mathbb{P}}} - L_{t^-}) d\tilde{k}_t^{\overline{\mathbb{P}}} = 0,\ \overline{\mathbb{P}}\text{-a.s.}
\end{aligned}$$

where a solution is a triple $(\tilde{y}^{\overline{\mathbb{P}}}, \tilde{z}^{\overline{\mathbb{P}}}, \tilde{m}^{\overline{\mathbb{P}}}, \tilde{k}^{\overline{\mathbb{P}}}) \in \mathbb{D}_0^p(\overline{\mathbb{F}}_+^{\overline{\mathbb{P}}}, \overline{\mathbb{P}}) \times \mathbb{H}_0^p(\overline{\mathbb{F}}_+^{\overline{\mathbb{P}}}, \overline{\mathbb{P}}) \times \mathbb{M}_0^p(\overline{\mathbb{F}}_+^{\overline{\mathbb{P}}}, \overline{\mathbb{P}}) \times \mathbb{I}_0^p(\overline{\mathbb{F}}_+^{\overline{\mathbb{P}}}, \overline{\mathbb{P}})$ satisfying (A.2.2).

The following result which is very closed to Lemma 2.2 of [85], gives the equivalence between the three RBSDEs in (3.2.6), (A.2.1) and (A.2.2).

Lemma A.2.2. *Let $\mathbb{P} \in \mathcal{P}_0$ and $\overline{\mathbb{P}} := \mathbb{P} \otimes \mathbb{P}_0$, then each of the three RBSDEs (3.2.6), (A.2.1) and (A.2.2) has a unique solution, denoted respectively by $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$, $(\overline{y}^{\overline{\mathbb{P}}}, \overline{z}^{\overline{\mathbb{P}}}, \overline{m}^{\overline{\mathbb{P}}}, \overline{k}^{\overline{\mathbb{P}}})$ and $(\tilde{y}^{\overline{\mathbb{P}}}, \tilde{z}^{\overline{\mathbb{P}}}, \tilde{m}^{\overline{\mathbb{P}}}, \tilde{k}^{\overline{\mathbb{P}}})$. Moreover, their solution coincide in the sense that there is some functional*

$$\Psi := (\Psi^y, \Psi^z, \Psi^m, \Psi^k) : [0,T] \times \Omega \longrightarrow \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R},$$

such that Ψ^y, Ψ^m and Ψ^k are $\mathbb{F}_+$-progressively measurable, $\mathbb{P}$-a.s. càdlàg, Ψ^z is $\mathbb{F}$-predictable,

$$y_t^{\mathbb{P}} = \Psi_t^y,\ z_t^{\mathbb{P}} = \Psi_t^z,\ k_t^{\mathbb{P}} = \Psi_t^{\mathbb{P}}, \widehat{a}_s ds\text{-a.e.},\ m_t^{\mathbb{P}} = \Psi_t^m \text{ for all } t \in [0,T],\ \mathbb{P}\text{-a.s.},$$

$$\overline{y}_t^{\overline{\mathbb{P}}} = \tilde{y}_t^{\overline{\mathbb{P}}} = \Psi_t^y(X.),\ \overline{z}_t^{\overline{\mathbb{P}}} = \tilde{z}_t^{\overline{\mathbb{P}}} = \Psi_t^z(X.),\ \overline{k}_t^{\overline{\mathbb{P}}} = \tilde{k}_t^{\overline{\mathbb{P}}} = \Psi_t^k(X.), \widehat{a}_s ds\text{-a.e. and } \overline{m}_t^{\overline{\mathbb{P}}} = \tilde{m}_t^{\overline{\mathbb{P}}} = \Psi_t^m(X.)$$
for all $t \in [0,T]$, $\overline{\mathbb{P}}$-a.s.,

Proof. (i) Since the existence and uniqueness of (3.2.6) have been proved in Theroem 3.1 of [14], then it remains to show that the three RBSDEs share the same solution.

(ii) In this paragraph we prove that (A.2.1) and (A.2.2) have the same solution in $(\overline{\Omega}, \overline{\mathcal{F}}_T^{\overline{\mathbb{P}}}, \overline{\mathbb{P}})$. By the decomposition (3.2.3) of the canonical process X under $\overline{\mathbb{P}}$, it is apparent that a solution to (A.2.1) is a solution solution to (A.2.2). To complete this paragragh, it remains to show that a solution to (A.2.2) is a solution to (A.2.1).

Let $\theta : \overline{\Omega} \longrightarrow \mathbb{R}$ be a $\overline{\mathcal{F}}_T^{X,\overline{\mathbb{P}}}$-measurable random variable, which admits the following unique martingale representation

$$\theta = \mathbb{E}^{\overline{\mathbb{P}}}[\theta] + \int_0^T \overline{z}_s^{\theta} \cdot dX_s^{c,\mathbb{P}} + \int_0^T d\overline{m}_s^{\theta},$$

w.r.t. the filtration $\overline{\mathbb{F}}_+^{X,\overline{\mathbb{P}}}$. We know that B is independent of X in the enlarged space, and X admits the same semi-martingale triplet of characteristics in both space(Ω and $\overline{\Omega}$), the above martingale representation of θ w.r.t. $\overline{\mathbb{F}}_+^{X,\overline{\mathbb{P}}}$ is the same as the one w.r.t. $\overline{\mathbb{F}}_+^{\overline{\mathbb{P}}}$, which are all unique up to a $\overline{\mathbb{P}}$-evanescent set.
Since the solution of RBSDE (A.2.2) is constructed as an iteration of the above martingale representation (see e.g. Section 3.4.1), then a solution to (A.2.2) is clearly a solution to (A.2.1).

(iii) We now show that a solution to (A.2.1) induces a solution to (3.2.6). Recall that $\overline{y}^{\overline{\mathbb{P}}}, \overline{m}^{\overline{\mathbb{P}}}, \overline{k}^{\overline{\mathbb{P}}}$ are $\overline{\mathbb{F}}_+^{X,\overline{\mathbb{P}}}$-optional, and $\overline{z}^{\overline{\mathbb{P}}}$ is $\overline{\mathbb{F}}_+^{X,\overline{\mathbb{P}}}$-predictable, then (see e.g. Lemma 2.4 of [91] and Theorem IV.78 of [26]) there exists a functional $(\overline{\Psi}^y, \overline{\Psi}^z, \overline{\Psi}^m, \overline{\Psi}^k) : [0,T] \times \overline{\Omega} \longrightarrow \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$ such that $\overline{\Psi}^y, \overline{\Psi}^m$ and $\overline{\Psi}^k$ are $\overline{\mathbb{F}}_+^X$-progressively measurable and $\overline{\mathbb{P}}$-a.s. càdlàg, $\overline{\Psi}^z$ is $\overline{\mathbb{F}}^X$-predictable, and $\overline{y}_t^{\overline{\mathbb{P}}} = \overline{\Psi}_t^y, \overline{z}_t^{\overline{\mathbb{P}}} = \overline{\Psi}_t^z, \overline{m}_t^{\overline{\mathbb{P}}} = \overline{\Psi}_t^m$ and $\overline{k}_t^{\overline{\mathbb{P}}} = \overline{\Psi}_t^k$,, for all $t \in [0,T]$, $\mathbb{P}$-a.s. Define

$$(\overline{\Psi}^{y,0}(\omega), \overline{\Psi}^{z,0}(\omega), \overline{\Psi}^{m,0}(\omega), \overline{\Psi}^{k,0}(\omega)) := (\overline{\Psi}^y(\omega,\mathbf{0}), \overline{\Psi}^z(\omega,\mathbf{0}), \overline{\Psi}^m(\omega,\mathbf{0}), \overline{\Psi}^k(\omega,\mathbf{0})),$$

where $\mathbf{0}$ denotes the path taking value 0 for all $t \in [0,T]$.
Since $(\overline{\Psi}^y, \overline{\Psi}^z, \overline{\Psi}^m, \overline{\Psi}^k)$ are $\overline{\mathbb{F}}^X$-progressively measurable, the functions $(\overline{\Psi}^{y,0}, \overline{\Psi}^{z,0}, \overline{\Psi}^{m,0}, \overline{\Psi}^{k,0})$ are $\overline{\mathbb{F}}$-progressively measurable, and it is easy to see that they provide a version of a solution to (3.2.6) in $(\Omega, \mathcal{F}_T^{\mathbb{P}}, \mathbb{P})$.

(iv) Finally, let $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$ be a solution to (3.2.6), then there exists a function $(\Psi^y, \Psi^z, \Psi^m, \Psi^k) : [0,T] \times \Omega \longrightarrow \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$ such that Ψ^y, Ψ^m and Ψ^k are $\mathbb{F}_+$-measurable and $\mathbb{P}$-a.s. càdlàg, Ψ^z is $\mathbb{F}$-predictable, and $y_t^{\mathbb{P}} = \Psi_t^y, z_t^{\mathbb{P}} = \Psi_t^z, m_t^{\mathbb{P}} = \Psi_t^m$ and $k_t^{\mathbb{P}} = \Psi_t^k$, for all $t \in [0,T]$, $\mathbb{P}$-a.s. Since $\overline{\mathbb{P}} := \mathbb{P} \otimes \mathbb{P}_0$, it is easy to see that $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$ is the required functional in the lemma.

A.3 Penalization method for RBSDEs in general filtration

In this section we follow [35] to show that the penalized BSDEs converge to the solution of the RBSDEs, to a slight difference that we have in addition a martingale M^n since backwards are relative to a general filtration.

A.3.1 Lower obstacle

For each $n \in \mathbb{N}$, we consider the following penalized BSDE

$$Y_t^n = \xi + \int_t^T f_s(Y_s^n, \widehat{a}_s^{1/2} Z_s^n) ds + n \int_t^T (Y_s^n - L_s)^- ds - \int_t^T Z_s^n \cdot \widehat{a}_s^{1/2} dW_s - \int_t^T dM_s^n,$$

where M^n is a martingale orthogonal to W. We define

$$K_t^n = n \int_0^t (Y_s^n - L_s)^- ds.$$

The generator $f : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ verify the classical Lipschitz condition with respect to y and z, ξ, $f_s(0,0)$ and the obstacle satisfy the following integrability conditions:

$$\mathbb{E}\big[|\xi|^2 + \int_0^T |f_s(0,0)|^2 ds + \sup_{0 \le t \le T} |L_t|^2\big] < \infty.$$

By El Karoui and Huang [33], for each $n \ge 0$, the penalized BSDEs has a unique solution.

A.3.1.1 Estimates

Proposition A.3.1. *For each $n \geq 0$, let (Y^n, Z^n, M^n) be the solution of the above penalized BSDE. There exists a constant $C \geq 0$ such that*

$$\forall t \leq T,\ \mathbb{E}\big[|Y_t^n|^2 + \int_0^T \|\widehat{a}_s^{1/2} Z_s^n\|^2 ds + [M^n]_T + (K_T^n)^2\big] < \infty.$$

Proof. Applying Itô's formula to $(Y^n)^2$ between t and T, we have

$$\begin{aligned}|Y_t^n|^2 = |\xi|^2 &+ 2\int_t^T Y_s^n f_s(Y_s^n, \widehat{a}_s^{1/2} Z_s^n) ds - 2\int_t^T Y_s^n Z_s^n \cdot \widehat{a}_s^{1/2} dW_s - 2\int_t^T Y_{s^-}^n dM_s^n \\ &+ 2\int_t^T Y_{s^-}^n dK_s^n - \int_t^T \|\widehat{a}_s^{1/2} Z_s^n\|^2 ds - [M^n]_T - \sum_{t<s\leq T} |\Delta Y_s^n|^2.\end{aligned}$$

Then

$$\begin{aligned}|Y_t^n|^2 + \int_t^T \|\widehat{a}_s^{1/2} Z_s^n\|^2 ds + [M^n]_T \leq |\xi|^2 &+ 2\int_t^T Y_s^n f_s(Y_s^n, \widehat{a}_s^{1/2} Z_s^n) ds - 2\int_t^T Y_s^n Z_s^n \cdot \widehat{a}_s^{1/2} dW_s \\ &- 2\int_t^T Y_{s^-}^n dM_s^n + 2n\int_t^T Y_{s^-}^n (Y_s^n - L_s)^- ds. \qquad \text{(A.3.1)}\end{aligned}$$

Taking the expectation yields

$$\begin{aligned}\mathbb{E}\Big[|Y_t^n|^2 + \int_t^T \|\widehat{a}_s^{1/2} Z_s^n\|^2 ds + [M^n]_T\Big] \leq \mathbb{E}\Big[|\xi|^2 &+ 2\int_t^T Y_s^n f_s(Y_s^n, \widehat{a}_s^{1/2} Z_s^n) ds \\ &+ 2\int_t^T L_{s^-} n (Y_s^n - L_s)^- ds\Big].\end{aligned}$$

By Young inequality,

$$ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}, \quad \forall \varepsilon > 0,$$

and the Lipschitz property of f, we get

$$\begin{aligned}\mathbb{E}\Big[|Y_t^n|^2 + \int_t^T \|\widehat{a}_s^{1/2} Z_s^n\|^2 ds + [M^n]_T\Big] &\leq \mathbb{E}\Big[|\xi|^2 + 2\int_t^T |Y_s^n||f_s(0,0)| ds + 2L_f \int_t^T |Y_s^n|^2 ds \\ &\qquad + 2L_f \int_t^T |Y_s^n| \|\widehat{a}_s^{1/2} Z_s^n\| ds + 2\int_t^T L_{s^-} dK_s^n\Big] \\ &\leq \mathbb{E}\Big[|\xi|^2 + \int_t^T |f_s(0,0)|^2 ds + C\int_t^T |Y_s^n|^2 ds \\ &\qquad + \frac{1}{2}\int_t^T \|\widehat{a}_s^{1/2} Z_s^n\|^2 ds + 2\sup_{t\in[0,T]} |L_t| \int_t^T dK_s^n\Big] \\ &\leq \mathbb{E}\Big[|\xi|^2 + \int_t^T |f_s(0,0)|^2 ds + C\int_t^T |Y_s^n|^2 ds \\ &\qquad + \frac{1}{2}\int_t^T \|\widehat{a}_s^{1/2} Z_s^n\|^2 ds + \frac{1}{\varepsilon}\sup_{t\in[0,T]} |L_t|^2 + \varepsilon (K_T^n - K_t^n)^2\Big]\end{aligned}$$

where C is a constant only depending on T and the Lipschitz coefficient of f and which can vary line to line. But, for any $t \leq T$, we have by definition

$$K_T^n - K_t^n = Y_t^n - \xi - \int_t^T f_s(Y_s^n, \widehat{a}_s^{1/2} Z_s^n) ds + \int_t^T Z_s^n \cdot \widehat{a}_s^{1/2} dW_s + \int_t^T dM_s^n$$

Thus,

$$\begin{aligned}\mathbb{E}\big[(K_T^n - K_t^n)^2\big] &\leq C\mathbb{E}\Big[|\xi|^2 + |Y_t^n|^2 + \big(\int_t^T |f_s(Y_s^n, \widehat{a}_s^{1/2} Z_s^n)|ds\big)^2 \\ &\qquad + \big(\int_t^T Z_s^n \cdot \widehat{a}_s^{1/2} dW_s\big)^2 + \big(\int_t^T dM_s^n\big)^2\Big] \\ &\leq C\mathbb{E}\Big[|\xi|^2 + |Y_t^n|^2 + \int_t^T |Y_s^n|^2 ds + \int_t^T |f_s(0,0)|^2 ds \\ &\qquad + \int_t^T \|\widehat{a}_s^{1/2} Z_s^n\|^2 ds + [M^n]_T - [M^n]_t\Big] \\ &\leq C\mathbb{E}\Big[|\xi|^2 + |Y_t^n|^2 + \int_t^T |Y_s^n|^2 ds + \int_t^T |f_s(0,0)|^2 ds \\ &\qquad + \int_t^T \|\widehat{a}_s^{1/2} Z_s^n\|^2 ds + [M^n]_T\Big].\end{aligned}$$

where the second inequality comes from the Lipschitz property of f, the Cauchy-Schwarz inequality and the Burkholder-Davis Gundy inequalities. Now plug this inequality into the previous one and choose $\varepsilon C_2 = 1/4$ yields

$$\begin{aligned}\mathbb{E}\Big[|Y_t^n|^2 + \int_t^T \|\widehat{a}_s^{1/2} Z_s^n\|^2 ds + [M^n]_T\Big] &\leq C\mathbb{E}\Big[|\xi|^2 + \int_t^T |f_s(0,0)|^2 ds + \sup_{t\in[0,T]} |L_t|^2 + \int_t^T |Y_s^n|^2 ds\Big] \\ &\leq C\mathbb{E}\Big[1 + \int_t^T |Y_s^n|^2 ds\Big].\end{aligned}$$

From there, we can write

$$\mathbb{E}\Big[|Y_t^n|^2\Big] \leq C\mathbb{E}\Big[1 + \int_t^T |Y_s^n|^2 ds\Big], \quad t \leq T$$

and Gronwall's inequality leads to

$$\mathbb{E}\Big[|Y_t^n|^2\Big] < \infty, \quad t \leq T$$

After that, the estimates related of Z^n and M^n come from

$$\begin{aligned}\mathbb{E}\Big[\int_t^T \|\widehat{a}_s^{1/2} Z_s^n\|^2 ds + [M^n]_T\Big] &\leq \mathbb{E}\Big[|Y_t^n|^2 + \int_t^T \|\widehat{a}_s^{1/2} Z_s^n\|^2 ds + [M^n]_T\Big] \\ &\leq C\mathbb{E}\Big[1 + \int_t^T |Y_s^n|^2 ds\Big] < \infty, \quad t \leq T.\end{aligned}$$

Finally, taking $t = 0$ in (A.3.2), we get

$$\mathbb{E}\big[(K_T^n)^2\big] \leq C\mathbb{E}\Big[|\xi|^2 + |Y_t^n|^2 + \int_t^T |Y_s^n|^2 ds + \int_t^T |f_s(0,0)|^2 ds + \int_t^T \|\widehat{a}_s^{1/2} Z_s^n\|^2 ds + [M^n]_T\Big] < \infty.$$

□

A.3.1.2 Convergence

Convergence of Y^n: Firstly, using comparison principle for BSDEs, we have for all $t \in [0,T]$, almost surely

$$Y_t^n \leq Y_t^{n+1}, \text{ a.s.}$$

Combining this monotony to the estimates of Y^n in the previous section, we obtain that for a fixed $t \in [0,T]$, $(Y_t^n)_{n\geq 0}$ converge and we denote his limit Y_t. On the other hand, taking the

supremun over $t \in [0,T]$ in (A.3.1) and the Burkholder -Davis-Gundy inequality, we deduce also that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|Y_t^n|^2\Big] < \infty.$$

According to the above and Fatou's Lemma,

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|Y_t|^2\Big] < \infty.$$

It follows by dominated convergence that the sequence $(Y^n)_{n\geq 0}$ converge to $(Y_t)_{t\in[0,T]}$ in L^2, in other words

$$\mathbb{E}\Big[\int_0^T (Y_t - Y_t^n)^2 dt\Big] \to \quad \text{as} \quad n \to \infty.$$

Convergence of Z^n, M^n and K^n: For $p \geq n \geq 0$ and $t \leq T$, Itô's formula applied to $(Y^n - Y^p)^2$ between t and T, leads to

$$\begin{aligned}
|Y_t^n - Y_t^p|^2 = {} & 2\int_t^T (Y_s^n - Y_s^p)(f_s(Y_s^n, \widehat{a}_s^{1/2} Z_s^n) - f_s(Y_s^p, \widehat{a}_s^{1/2} Z_s^p))ds \\
& - 2\int_t^T (Y_s^n - Y_s^p)(Z_s^n - Z_s^p)\cdot \widehat{a}_s^{1/2} dW_s - 2\int_t^T (Y_{s^-}^n - Y_{s^-}^p) d(M_s^n - M_s^p) \\
& + 2\int_t^T (Y_{s^-}^n - Y_{s^-}^p) d(K_s^n - K_s^p) - \int_t^T \|\widehat{a}_s^{1/2}(Z_s^n - Z_s^p)\|^2 ds \\
& - [M^n - M^p]_T - \sum_{t<s\leq T} |\Delta(Y_s^n - Y_s^p)|^2.
\end{aligned}$$

Similar arguments to the previous section implies that

$$\begin{aligned}
\mathbb{E}\Big[|Y_t^n - Y_t^p|^2 + \int_t^T \|\widehat{a}_s^{1/2}(Z_s^n - Z_s^p)\|^2 ds + [M^n - M^p]_T\Big] \\
\leq C\mathbb{E}\Big[\int_t^T |Y_s^n - Y_s^p|^2 ds + \int_t^T (Y_{s^-}^n - Y_{s^-}^p) d(K_s^n - K_s^p)\Big].
\end{aligned}$$

Since $p \geq n$, then

$$\int_t^T (Y_{s^-}^n - Y_{s^-}^p) d(K_s^n - K_s^p) \leq \sup_{t\in[0,T]} (Y_s^n - L_s)^- (K_T^p + K_T^n)$$

Then taking expectation we have

$$\begin{aligned}
\mathbb{E}\Big[|Y_t^n - Y_t^p|^2 + \int_t^T \|\widehat{a}_s^{1/2}(Z_s^n - Z_s^p)\|^2 ds + [M^n - M^p]_T\Big] \\
\leq C\mathbb{E}\Big[\int_t^T |Y_s^n - Y_s^p|^2 ds + \sup_{t\in[0,T]} (Y_s^n - L_s)^- (K_T^p + K_T^n)\Big]
\end{aligned}$$

The following property have been proved in [35] and [32] and still verify despite the fact that there is in addition M^n.

$$\lim_{n\to\infty} \mathbb{E}\Big[\sup_{t\in[0,T]} |(Y_s^n - L_s)^-|^2\Big] = 0.$$

The classical reasoning for penalization in [35] applies to this case shows that

$$\mathbb{E}\Big[\int_t^T \|\widehat{a}_s^{1/2}(Z_s^n - Z_s^p)\|^2 ds + [M^n - M^p]_T\Big] \to 0 \quad \text{as} \quad n \to \infty.$$

Which ensures the convergence of $(Z^n)_{n\geq 0}$ and $(M^n)_{n\geq 0}$ to respectively $(Z_t)_{t\in[0,T]}$ and $(M_t)_{t\in[0,T]}$. Also it is obvious to see that limit process M still orthogonal to W. The convergence of K^n to a process $(K_t)_{t\in[0,T]}$ comes from

$$K_t^n = Y_0^n - Y_t^n - \int_t^T f_s(Y_s^n, \widehat{a}_s^{1/2} Z_s^n)ds + \int_t^T Z_s^n \cdot \widehat{a}_s^{1/2} dW_s + M_t^n$$

As in [35] the limit process K satisfies the Skorokhod condition and (Y, Z, M, K) is the solution of the following RBSDE

$$\begin{aligned}
&Y_t = \xi + \int_t^T f_s(Y_s, \widehat{a}_s^{1/2} Z_s)ds - \int_t^T Z_s \cdot \widehat{a}_s^{1/2} dW_s - \int_t^T dM_s + K_T - K_t, \text{ a.s.} \\
&Y_t \geq L_t, \ t \in [0,T], \text{ a.s.} \\
&\int_0^T (Y_{s^-} - L_{s^-}) dK_s = 0, \text{ a.s..}
\end{aligned}$$

A.3.2 Upper obstacle

For each $n \in \mathbb{N}$, we consider the following penalized BSDE

$$Y_t^n = \xi + \int_t^T f_s(Y_s^n, \widehat{a}_s^{1/2} Z_s^n)ds - n\int_t^T (U_s - Y_s^n)^- ds - \int_t^T Z_s^n \cdot \widehat{a}_s^{1/2} dW_s - \int_t^T dM_s^n$$

where M^n is a martingale orthogonal to W. We define

$$K_t^n = n\int_0^t (U_s - Y_s^n)^- ds.$$

The penalized BSDEs related to an upper obstacle is slightly different for

The generator $f : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ verify the classical Lipschitz condition with respect to y and z, ξ, $f_s(0,0)$ and the obstacle satisfy the following integrability conditions:

$$\mathbb{E}\big[|\xi|^2 + \int_0^T |f_s(0,0)|^2 ds + \sup_{0\leq t\leq T} |U_t|^2\big] < \infty.$$

A.3.2.1 Estimates

Proposition A.3.2. *For each $n \geq 0$, let (Y^n, Z^n, M^n) be the solution of the above penalized BSDE. There exists a constant $C \geq 0$ such that*

$$\forall t \leq T, \ \mathbb{E}\big[|Y_t^n|^2 + \int_0^T \|\widehat{a}_s^{1/2} Z_s^n\|^2 ds + [M^n]_T + (K_T^n)^2\big] < \infty.$$

Proof. Applying Itô's formula to $(Y^n)^2$ between t and T, we have

$$\begin{aligned}
|Y_t^n|^2 = |\xi|^2 &+ 2\int_t^T Y_s^n f_s(Y_s^n, \widehat{a}_s^{1/2} Z_s^n)ds - 2\int_t^T Y_s^n Z_s^n \cdot \widehat{a}_s^{1/2} dW_s - 2\int_t^T Y_{s^-}^n dM_s^n \\
&- 2\int_t^T Y_{s^-}^n dK_s^n - \int_t^T \|\widehat{a}_s^{1/2} Z_s^n\|^2 ds - [M^n]_T - \sum_{t<s\leq T} |\Delta Y_s^n|^2.
\end{aligned}$$

Then

$$\begin{aligned}
|Y_t^n|^2 + \int_t^T \|\widehat{a}_s^{1/2} Z_s^n\|^2 ds + [M^n]_T &\leq |\xi|^2 + 2\int_t^T Y_s^n f_s(Y_s^n, \widehat{a}_s^{1/2} Z_s^n)ds - 2\int_t^T Y_s^n Z_s^n \cdot \widehat{a}_s^{1/2} dW_s \\
&\quad - 2\int_t^T Y_{s^-}^n dM_s^n - 2n\int_t^T Y_s^n (U_s - Y_s^n)^- ds \\
&\leq |\xi|^2 + 2\int_t^T Y_s^n f_s(Y_s^n, \widehat{a}_s^{1/2} Z_s^n)ds - 2\int_t^T Y_s^n Z_s^n \cdot \widehat{a}_s^{1/2} dW_s \\
&\quad - 2\int_t^T Y_{s^-}^n dM_s^n + 2n\int_t^T |U_s|(U_s - Y_s^n)^- ds. \qquad \text{(A.3.2)}
\end{aligned}$$

Taking the expectation yields and using Young inequality, we have for a fixed $\varepsilon > 0$

$$\mathbb{E}\Big[|Y_t^n|^2 + \int_t^T \|\widehat{a}_s^{1/2} Z_s^n\|^2 ds + [M^n]_T\Big] \leq \mathbb{E}\Big[|\xi|^2 + 2\int_t^T Y_s^n f_s(Y_s^n, \widehat{a}_s^{1/2} Z_s^n) ds + \frac{1}{\varepsilon} \sup_{t\in[0,T]} |U_t|^2 + \varepsilon (K_T^n - K_t^n)^2\Big].$$

Then by the Lipschitz property of f and by the same operations as lower obstacle, we obtain

$$\mathbb{E}\Big[|Y_t^n|^2 + \int_t^T \|\widehat{a}_s^{1/2} Z_s^n\|^2 ds + [M^n]_T\Big] \leq \mathbb{E}\Big[|\xi|^2 + \int_t^T |f_s(0,0)|^2 ds + C\int_t^T |Y_s^n|^2 ds + \frac{1}{2}\int_t^T \|\widehat{a}_s^{1/2} Z_s^n\|^2 ds + \frac{1}{\varepsilon} \sup_{t\in[0,T]} |U_t|^2 + \varepsilon (K_T^n - K_t^n)^2\Big]$$

where C is a constant only depending on T and the Lipschitz coefficient of f and which can vary line to line in the following. By definition

$$K_T^n - K_t^n = \xi - Y_t^n + \int_t^T f_s(Y_s^n, \widehat{a}_s^{1/2} Z_s^n) ds - \int_t^T Z_s^n \cdot \widehat{a}_s^{1/2} dW_s - \int_t^T dM_s^n$$

Same method using in section A.3.1 gives ,

$$\mathbb{E}\big[(K_T^n - K_t^n)^2\big] \leq C\mathbb{E}\Big[|\xi|^2 + |Y_t^n|^2 + \int_t^T |Y_s^n|^2 ds + \int_t^T |f_s(0,0)|^2 ds + \int_t^T \|\widehat{a}_s^{1/2} Z_s^n\|^2 ds + [M^n]_T\Big].$$

Therefore, the estimates follow directly by the same argument to the case of lower obstacle (see section A.3.1). □

A.3.2.2 Convergence

Convergence of Y^n: Firstly, using comparison principle for BSDEs, we have for all $t \in [0,T]$, almost surely

$$Y_t^n \geq Y_t^{n+1}, \text{ a.s.}$$

Then for a fixed $t \in (0,T]$, $(Y_t^n)_{n\geq 0}$ is a non increasing bounded sequence almost surely (see also the estimates of Y^n). Hence this sequence converge almost surely and we denote his limit Y_t. Therefore we follow section A.3.1 and the same reasoning applies to the case of upper obstacle shows that $(Y^n)_{n\geq 0}$ converge uniformly to $(Y_t)_{t\in[0,T]}$, $(Z^n)_{n\geq 0}$, $(M^n)_{n\geq 0}$ and $(K^n)_{n\geq 0}$ converge respectively to $(Z_t)_{t\in[0,T]}$, $(M_t)_{t\in[0,T]}$ and $(K_t)_{t\in[0,T]}$ and the limit processes Y, Z, M and K solve the following RBSDE

$$\begin{aligned}
&Y_t = \xi + \int_t^T f_s(Y_s, \widehat{a}_s^{1/2} Z_s) ds - \int_t^T Z_s \cdot \widehat{a}_s^{1/2} dW_s - \int_t^T dM_s - K_T + K_t, \text{ a.s.}\\
&Y_t \leq U_t, \ t \in [0,T], \text{ a.s.}\\
&\int_0^T (U_{s^-} - Y_{s^-}) dK_s = 0, \text{ a.s..}
\end{aligned}$$

A.4 Some results and estimates for RBSDEs

The following result gives us the stability of RBSDEs in a general filtration.

Proposition A.4.1. *Let Assumption 3.2.1 holds, and consider two generators f^1 and f^2 such that Assumptions 3.3.1 holds. For $i = 1,2$, let $(y^{i,\mathbb{P}}, z^{i,\mathbb{P}}, m^{i,\mathbb{P}}, k^{i,\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}$ be the solutions to the RBSDE (3.2.6) with terminal condition ξ^i and lower obstacle L^i. Define*

$$\phi_{f^1,f^2}^{p,\kappa} := \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{0\leq t\leq T}{}^{\mathbb{P}} \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T |\widehat{f}_s^{1,\mathbb{P}} - \widehat{f}_s^{2,\mathbb{P}}|^\kappa (y_s^{1,\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{1,\mathbb{P}}) ds\right)^{\frac{p}{\kappa}} \middle| \mathcal{F}_t^+\right]\right] < +\infty,$$

$$\psi_{L^1,L^2}^{p,\kappa} := \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{0\leq t\leq T}{}^{\mathbb{P}} \left(\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq s\leq T} |L_s^1 - L_s^2|^\kappa \middle| \mathcal{F}_t^+\right]\right)^{\frac{p}{\kappa}}\right] < +\infty.$$

Then, there exists a constant C depending only on κ, T and the Lipschitz constant of f^1 and f^2 such that

$$\|y^{2,\mathbb{P}} - y^{2,\mathbb{P}}\|^p_{\mathbb{D}^p_0(\mathbb{P})} \le C\Big(\|\xi^1 - \xi^2\|^p_{\mathbb{L}^{p,\kappa}_0} + \psi^{p,\kappa}_{L^1,L^2} + \varphi^{p,\kappa}_{f^1,f^2}\Big),$$
$$\|z^{1,\mathbb{P}} - z^{2,\mathbb{P}}\|^p_{\mathbb{H}^p_0(\mathbb{P})} + \|m^{1,\mathbb{P}} - m^{2,\mathbb{P}}\|^p_{\mathbb{M}^p_0(\mathbb{P})} + \|k^{1,\mathbb{P}}_T - k^{2,\mathbb{P}}_T\|^p_{\mathbb{I}^p_0(\mathbb{P})} \le C\Big(\|\xi^1 - \xi^2\|^p_{\mathbb{L}^{p,\kappa}_0} + \phi^{p,\kappa}_{f^1,f^2} + \psi^{p,\kappa}_{L^1,L^2}\Big).$$

Proof. In the proof, we use the fact that the RBSDEs defined in (3.2.6) are equivalent to the RBSDEs defined in (*A*.2.2) (see Lemma A.2.2). Since it is easier to make calculations with the Brownian motion, we show the result by using RBSDEs defined in (*A*.2.2). For simplicity of notations, we write $(\tilde{y}^{\mathbb{P}}, \tilde{z}^{\mathbb{P}}, \tilde{m}^{\mathbb{P}}, \tilde{k}^{\mathbb{P}})$ instead of $(\tilde{y}^{\overline{\mathbb{P}}}, \tilde{z}^{\overline{\mathbb{P}}}, \tilde{m}^{\overline{\mathbb{P}}}, \tilde{k}^{\overline{\mathbb{P}}})$.
Throughout the proof, we also use the following notations

$$\delta\tilde{y}^{\mathbb{P}} := \tilde{y}^{1,\mathbb{P}} - \tilde{y}^{2,\mathbb{P}},\ \delta\tilde{z}^{\mathbb{P}} := \tilde{z}^{1,\mathbb{P}} - \tilde{z}^{2,\mathbb{P}},\ \delta\tilde{m}^{\mathbb{P}} := \tilde{m}^{1,\mathbb{P}} - \tilde{m}^{2,\mathbb{P}},\ \delta\tilde{k}^{\mathbb{P}} := \tilde{k}^{1,\mathbb{P}} - \tilde{k}^{2,\mathbb{P}},$$
$$\delta\widehat{f}^{\mathbb{P}} := f^{1,\mathbb{P}} - f^{2,\mathbb{P}},\ \delta L := L^1 - L^2,\ \delta\xi := \xi^1 - \xi^2$$

(i) [14] We begin to prove the estimates of the first component $\delta\tilde{y}^{\mathbb{P}}$ which has been proved in Proposition 3.2 of [14]. By (*A*.2.2), we have

$$\delta\tilde{y}^{\mathbb{P}}_t = \delta\xi + \int_t^T \big(\widehat{f}^{1,\mathbb{P}}_s(\tilde{y}^{1,\mathbb{P}}_s, \widehat{a}^{1/2}_s\tilde{z}^{1,\mathbb{P}}_s) - \widehat{f}^{2,\mathbb{P}}_s(\tilde{y}^{2,\mathbb{P}}_s, \widehat{a}^{1/2}_s\tilde{z}^{2,\mathbb{P}}_s)\big)ds - \int_t^T \delta\tilde{z}^{\mathbb{P}}_s \cdot \widehat{a}^{1/2}_s dW^{\mathbb{P}}_s$$
$$- \int_t^T d\delta\tilde{m}^{\mathbb{P}}_s + \int_t^T d\delta\tilde{k}^{\mathbb{P}}_s. \tag{A.4.1}$$

Using the same argument as in section 3.3.3, there exist a $\mathbb{R}$-valued $\overline{\mathbb{F}}$-progressively measurable process $\tilde{\lambda}$ and a $\mathbb{R}^d$-valued, $\overline{\mathbb{F}}$-predictable process $\tilde{\eta}$, with $|\tilde{\lambda}| \le L_{f^2}$ and $\|\tilde{\eta}\| \le L_{f^2}$ such that for $s \in [0,T]$, we have $\mathbb{P}$-a.s.

$$\begin{aligned}\widehat{f}^{1,\mathbb{P}}_s(\tilde{y}^{1,\mathbb{P}}_s, \widehat{a}^{1/2}_s\tilde{z}^{1,\mathbb{P}}_s) - \widehat{f}^{2,\mathbb{P}}_s(\tilde{y}^{2,\mathbb{P}}_s, \widehat{a}^{1/2}_s\tilde{z}^{2,\mathbb{P}}_s) &= \widehat{f}^{1,\mathbb{P}}_s(\tilde{y}^{1,\mathbb{P}}_s, \widehat{a}^{1/2}_s\tilde{z}^{1,\mathbb{P}}_s) - \widehat{f}^{2,\mathbb{P}}_s(\tilde{y}^{1,\mathbb{P}}_s, \widehat{a}^{1/2}_s\tilde{z}^{1,\mathbb{P}}_s)\\ &\quad + \widehat{f}^{2,\mathbb{P}}_s(\tilde{y}^{1,\mathbb{P}}_s, \widehat{a}^{1/2}_s\tilde{z}^{1,\mathbb{P}}_s) - \widehat{f}^{2,\mathbb{P}}_s(\tilde{y}^{2,\mathbb{P}}_s, \widehat{a}^{1/2}_s\tilde{z}^{2,\mathbb{P}}_s)\\ &= \delta\widehat{f}^{\mathbb{P}}_s(\tilde{y}^{1,\mathbb{P}}_s, \widehat{a}^{1/2}_s\tilde{z}^{1,\mathbb{P}}_s) + \tilde{\lambda}_s\delta\tilde{y}^{\mathbb{P}}_s + \tilde{\eta}_s \cdot \widehat{a}^{1/2}_s\delta\tilde{z}^{\mathbb{P}}_s\end{aligned}$$

Therefore, we can define $\tilde{\mathbb{Q}}_{\mathbb{P}} \sim \mathbb{P}$ and a bounded positive process $\tilde{I}$ as in (3.3.8)

$$\tilde{I}_t := e^{\int_0^t \tilde{\lambda}_s ds}, \text{ and } \frac{d\tilde{\mathbb{Q}}_{\mathbb{P}}}{d\mathbb{P}} := \mathcal{E}\Big(\int_0^\cdot \tilde{\eta}_t \cdot dW^{\mathbb{P}}_t\Big)_T,\ W^{\tilde{\mathbb{Q}}_{\mathbb{P}}} := W^{\mathbb{P}}_\cdot - \int_0^\cdot \tilde{\eta}_s ds, \tag{A.4.2}$$

Apply Itô's formula to $\tilde{I}\tilde{y}^{\mathbb{P}}$, for all stopping time $\tau \ge t$,

$$\tilde{I}_t\delta\tilde{y}^{\mathbb{P}}_t = \tilde{I}_\tau\delta\tilde{y}^{\mathbb{P}}_\tau + \int_t^\tau \tilde{I}_s\delta\widehat{f}^{\mathbb{P}}_s(\tilde{y}^{1,\mathbb{P}}_s, \widehat{a}^{1/2}_s\tilde{z}^{1,\mathbb{P}}_s)ds - \int_t^\tau \tilde{I}_s\delta\tilde{z}^{\mathbb{P}}_s \cdot \widehat{a}^{1/2}_s dW^{\tilde{\mathbb{Q}}_{\mathbb{P}}}_s - \int_t^\tau \tilde{I}_s d\delta\tilde{m}^{\mathbb{P}}_s + \int_t^\tau \tilde{I}_s d\delta\tilde{k}^{\mathbb{P}}_s.$$

Taking conditional expectation under $\tilde{\mathbb{Q}}_{\mathbb{P}}$ w.r.t $\mathcal{F}^+_t$, gives

$$\tilde{I}_t\delta\tilde{y}^{\mathbb{P}}_t = \mathbb{E}^{\tilde{\mathbb{Q}}_{\mathbb{P}}}\Big[\tilde{I}_\tau\delta\tilde{y}^{\mathbb{P}}_\tau + \int_t^\tau \tilde{I}_s\delta\widehat{f}^{\mathbb{P}}_s(\tilde{y}^{1,\mathbb{P}}_s, \widehat{a}^{1/2}_s\tilde{z}^{1,\mathbb{P}}_s)ds + \int_t^\tau \tilde{I}_s d\delta\tilde{k}^{\mathbb{P}}_s\big|\mathcal{F}^+_t\Big]$$

Fix some $\epsilon > 0$ and define the stopping time $D^{\mathbb{P},\epsilon}_t := \inf\Big\{u \ge t : \tilde{y}^{1,\mathbb{P}}_u \le L^1_u + \epsilon,\ \mathbb{P}\text{-a.s.}\Big\} \wedge T$. By definition we have on one side, $\tilde{y}^{1,\mathbb{P}}_{D^{\mathbb{P},\epsilon}_t} \le L^1_{D^{\mathbb{P},\epsilon}_t} + \epsilon$ on the set $\Big\{D^{\mathbb{P},\epsilon}_t < T\Big\}$ and then

$$\tilde{y}^{1,\mathbb{P}}_{D^{\mathbb{P},\epsilon}_t} - \tilde{y}^{2,\mathbb{P}}_{D^{\mathbb{P},\epsilon}_t} \le L^1_{D^{\mathbb{P},\epsilon}_t} - \tilde{y}^{2,\mathbb{P}}_{D^{\mathbb{P},\epsilon}_t} + \epsilon \le L^1_{D^{\mathbb{P},\epsilon}_t} - L^2_{D^{\mathbb{P},\epsilon}_t} + \epsilon \text{ on the set } \Big\{D^{\mathbb{P},\epsilon}_t < T\Big\}.$$

and on the other side $\tilde{y}^{1,\mathbb{P}}_{s^-} > L^1_{s-}$ for all $s \in [t, D^{\mathbb{P},\epsilon}_t]$ and $\tilde{k}^{1,\mathbb{P}}_{D^{\mathbb{P},\epsilon}_t} - \tilde{k}^{1,\mathbb{P}}_t = 0$, by the Skorokhod condition. These observations enabled us to have

$$\begin{aligned}
\tilde{I}_t\delta\tilde{y}^{\mathbb{P}}_t &= \mathbb{E}^{\tilde{\mathbb{Q}}_{\mathbb{P}}}\Big[\tilde{I}_{D^{\mathbb{P},\epsilon}_t}\delta\tilde{y}^{\mathbb{P}}_{D^{\mathbb{P},\epsilon}_t} + \int_t^{D^{\mathbb{P},\epsilon}_t}\tilde{I}_s\delta\widehat{f}^{\mathbb{P}}_s(\tilde{y}^{1,\mathbb{P}}_s, \widehat{a}^{1/2}_s\tilde{z}^{1,\mathbb{P}}_s)ds + \int_t^{D^{\mathbb{P},\epsilon}_t}\tilde{I}_s d\tilde{k}^{1,\mathbb{P}}_s - \int_t^{D^{\mathbb{P},\epsilon}_t}\tilde{I}_s d\tilde{k}^{2,\mathbb{P}}_s\big|\mathcal{F}^+_t\Big] \\
&\leq \mathbb{E}^{\tilde{\mathbb{Q}}_{\mathbb{P}}}\Big[\tilde{I}_{D^{\mathbb{P},\epsilon}_t}(\tilde{y}^{1,\mathbb{P}}_{D^{\mathbb{P},\epsilon}_t} - \tilde{y}^{2,\mathbb{P}}_{D^{\mathbb{P},\epsilon}_t}) + \int_t^{D^{\mathbb{P},\epsilon}_t}\tilde{I}_s\delta\widehat{f}^{\mathbb{P}}_s(\tilde{y}^{1,\mathbb{P}}_s, \widehat{a}^{1/2}_s\tilde{z}^{1,\mathbb{P}}_s)ds - \int_t^{D^{\mathbb{P},\epsilon}_t}\tilde{I}_s d\tilde{k}^{2,\mathbb{P}}_s\big|\mathcal{F}^+_t\Big] \\
&\leq \mathbb{E}^{\tilde{\mathbb{Q}}_{\mathbb{P}}}\Big[\tilde{I}_{D^{\mathbb{P},\epsilon}_t}(\xi^1 - \xi^2)1_{\left\{D^{\mathbb{P},\epsilon}_t = T\right\}} + \tilde{I}_{D^{\mathbb{P},\epsilon}_t}(L^1_{D^{\mathbb{P},\epsilon}_t} - L^2_{D^{\mathbb{P},\epsilon}_t})1_{\left\{D^{\mathbb{P},\epsilon}_t < T\right\}} + \epsilon\tilde{I}_{D^{\mathbb{P},\epsilon}_t} \\
&\qquad\qquad + \int_t^{D^{\mathbb{P},\epsilon}_t}\tilde{I}_s\delta\widehat{f}^{\mathbb{P}}_s(\tilde{y}^{1,\mathbb{P}}_s, \widehat{a}^{1/2}_s\tilde{z}^{1,\mathbb{P}}_s)ds\big|\mathcal{F}^+_t\Big] \\
&\leq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}}\mathbb{E}^{\tilde{\mathbb{Q}}_{\mathbb{P}}}\Big[\int_t^\tau \tilde{I}_s\big|\delta\widehat{f}^{\mathbb{P}}_s(\tilde{y}^{1,\mathbb{P}}_s, \widehat{a}^{1/2}_s\tilde{z}^{1,\mathbb{P}}_s)\big|ds + \tilde{I}_\tau|\delta\xi|1_{\{\tau=T\}} + \tilde{I}_\tau|\delta L_\tau|1_{\{\tau<T\}} + \epsilon\tilde{I}_\tau\big|\mathcal{F}^+_t\Big].
\end{aligned}$$

We now apply this reasoning again, with $\delta y^{\mathbb{P}}_t$ replace by $y^{2,\mathbb{P}}_t - y^{1,\mathbb{P}}_t = -\delta y^{\mathbb{P}}_t$ and $D^{\mathbb{P},\epsilon}_t := \inf\left\{u \geq t : \tilde{y}^{2,\mathbb{P}}_u \leq L^2_u + \epsilon,\ \mathbb{P}\text{-a.s.}\right\} \wedge T$ and let ϵ go to 0 to obtain,

$$\tilde{I}_t|\delta\tilde{y}^{\mathbb{P}}_t| \leq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}}\mathbb{E}^{\tilde{\mathbb{Q}}_{\mathbb{P}}}\Big[\int_t^\tau \tilde{I}_s\big|\delta\widehat{f}^{\mathbb{P}}_s(\tilde{y}^{1,\mathbb{P}}_s, \widehat{a}^{1/2}_s\tilde{z}^{1,\mathbb{P}}_s)\big|ds + \tilde{I}_\tau|\delta\xi|1_{\{\tau=T\}} + \tilde{I}_\tau|\delta L_\tau|1_{\{\tau<T\}} + \epsilon\tilde{I}_\tau\big|\mathcal{F}^+_t\Big]$$

Repeated step 1 of the proof of Proposition 3.3.5 enables us to write

$$|\tilde{y}^{1,\mathbb{P}}_t - \tilde{y}^{2,\mathbb{P}}_t| \leq C\mathbb{E}^{\mathbb{P}}\Big[\int_0^T |\widehat{f}^{1,\mathbb{P}}_s - \widehat{f}^{2,\mathbb{P}}_s|^\kappa(\tilde{y}^{1,\mathbb{P}}_s, \widehat{a}^{1/2}_s\tilde{z}^{1,\mathbb{P}}_s)ds + \sup_{s\in[0,T]}|L^1_s - L^2_s|^\kappa + |\xi^1 - \xi^2|^\kappa\big|\mathcal{F}^+_t\Big]^{1/\kappa}. \tag{A.4.3}$$

Taking the supremum over $t \in [0,T]$, and using Doob's inequality, we prove the first assertion of the Proposition.

(ii) We now turn to the second estimates. We can rewrite $(A.4.1)$ as

$$\begin{aligned}
\delta\tilde{y}^{\mathbb{P}}_t = \delta\xi + \int_t^T \big(\delta\widehat{f}^{\mathbb{P}}_s(\tilde{y}^{1,\mathbb{P}}_s, \widehat{a}^{1/2}_s\tilde{z}^{1,\mathbb{P}}_s) + \tilde{\lambda}_s\delta\tilde{y}^{\mathbb{P}}_s + \tilde{\eta}_s\cdot\widehat{a}^{1/2}_s\delta\tilde{z}^{\mathbb{P}}_s\big)ds - \int_t^T \delta\tilde{z}^{\mathbb{P}}_s\cdot\widehat{a}^{1/2}_s dW^{\mathbb{P}}_s \\
- \int_t^T d\delta\tilde{m}^{\mathbb{P}}_s + \int_t^T d\delta\tilde{k}^{\mathbb{P}}_s.
\end{aligned}$$

Applying Itô's formula to $|\delta\tilde{y}^{\mathbb{P}}|^2$, we have

$$\begin{aligned}
|\delta\tilde{y}^{\mathbb{P}}_0|^2 = |\delta\xi|^2 + 2\int_0^T \delta y^{\mathbb{P}}_t\delta\widehat{f}^{\mathbb{P}}_t(\tilde{y}^{1,\mathbb{P}}_t, \widehat{a}^{1/2}_t\tilde{z}^{1,\mathbb{P}}_t)dt + 2\int_0^T \tilde{\lambda}_s|\delta\tilde{y}^{\mathbb{P}}_t|^2dt + 2\int_0^T \tilde{\eta}_s\widehat{a}^{1/2}_t\delta\tilde{y}^{\mathbb{P}}_t\delta\tilde{z}^{\mathbb{P}}_t dt \\
- 2\int_0^T \delta\tilde{y}^{\mathbb{P}}_t\delta\tilde{z}^{\mathbb{P}}_t\cdot\widehat{a}^{1/2}_t dW^{\mathbb{P}}_t - 2\int_0^T \delta\tilde{y}^{\mathbb{P}}_{t^-}d\delta\tilde{m}^{\mathbb{P}}_t + 2\int_0^T \delta\tilde{y}^{\mathbb{P}}_{t^-}d\delta\tilde{k}^{\mathbb{P}}_t \\
- \int_0^T \|\widehat{a}^{1/2}_t\delta\tilde{z}^{\mathbb{P}}_t\|^2dt - [\delta\tilde{m}^{\mathbb{P}}]_T - \sum_{0<t\leq T}\{|\delta\tilde{y}^{\mathbb{P}}_t|^2 - |\delta\tilde{y}^{\mathbb{P}}_{t^-}|^2 - 2|\delta\tilde{y}^{\mathbb{P}}_{t^-}|\Delta\delta\tilde{y}^{\mathbb{P}}_t\}.
\end{aligned}$$

Since $|\delta\tilde{y}^{\mathbb{P}}_t|^2 - |\delta\tilde{y}^{\mathbb{P}}_{t^-}|^2 - 2|\delta\tilde{y}^{\mathbb{P}}_{t^-}|\Delta\delta\tilde{y}^{\mathbb{P}}_t = |\Delta\delta\tilde{y}^{\mathbb{P}}_t|^2 \geq 0$, then

$$\begin{aligned}\int_0^T \|\widehat{a}_t^{1/2}\delta\tilde{z}_t^{\mathbb{P}}\|^2dt + [\delta\tilde{m}^{\mathbb{P}}]_T &\leq |\delta\xi|^2 + 2\int_0^T |\delta y_t^{\mathbb{P}}||\delta\widehat{f}_t^{\mathbb{P}}(\tilde{y}_t^{1,\mathbb{P}},\widehat{a}_t^{1/2}\tilde{z}_t^{1,\mathbb{P}})|dt + 2\int_0^T \tilde{\lambda}_s|\delta\tilde{y}_t^{\mathbb{P}}|^2dt \\ &\quad + 2\int_0^T \tilde{\eta}_s\widehat{a}_t^{1/2}\delta\tilde{y}_t^{\mathbb{P}}\delta\tilde{z}_t^{\mathbb{P}}dt - 2\int_0^T \delta\tilde{y}_t^{\mathbb{P}}\delta\tilde{z}_t^{\mathbb{P}}\cdot\widehat{a}_t^{1/2}dW_t^{\mathbb{P}} \\ &\quad - 2\int_0^T \delta\tilde{y}_{t^-}^{\mathbb{P}}d\delta\tilde{m}_t^{\mathbb{P}} + 2\int_0^T \delta\tilde{y}_{t^-}^{\mathbb{P}}d\delta\tilde{k}_t^{\mathbb{P}} \\ &\leq |\delta\xi|^2 + 2\sup_{t\in[0,T]}|\delta y_t^{\mathbb{P}}| \times \int_0^T |\delta\widehat{f}_t^{\mathbb{P}}(\tilde{y}_t^{1,\mathbb{P}},\widehat{a}_t^{1/2}\tilde{z}_t^{1,\mathbb{P}})|dt + \frac{1}{2}\int_0^T \|\widehat{a}_t^{1/2}\delta\tilde{z}_t^{\mathbb{P}}\|^2dt \\ &\quad + (2L_{f^2} + 2L_{f^2}^2)T\sup_{t\in[0,T]}|\delta\tilde{y}_t^{\mathbb{P}}|^2 - 2\int_0^T \delta\tilde{y}_t^{\mathbb{P}}\delta\tilde{z}_t^{\mathbb{P}}\cdot\widehat{a}_t^{1/2}dW_t^{\mathbb{P}} \\ &\quad - 2\int_0^T \delta\tilde{y}_{t^-}^{\mathbb{P}}d\delta\tilde{m}_t^{\mathbb{P}} + 2\int_0^T \delta\tilde{y}_{t^-}^{\mathbb{P}}d\delta\tilde{k}_t^{\mathbb{P}} \\ &\leq |\delta\xi|^2 + (2L_{f^2}T + 2L_{f^2}^2T + 1)\sup_{t\in[0,T]}|\delta\tilde{y}_t^{\mathbb{P}}|^2 + \frac{1}{2}\int_0^T \|\widehat{a}_t^{1/2}\delta\tilde{z}_t^{\mathbb{P}}\|^2dt \\ &\quad + \Big(\int_0^T |\delta\widehat{f}_t^{\mathbb{P}}(\tilde{y}_t^{1,\mathbb{P}},\widehat{a}_t^{1/2}\tilde{z}_t^{1,\mathbb{P}})|dt\Big)^2 - 2\int_0^T \delta\tilde{y}_t^{\mathbb{P}}\delta\tilde{z}_t^{\mathbb{P}}\cdot\widehat{a}_t^{1/2}dW_t^{\mathbb{P}} \\ &\quad - 2\int_0^T \delta\tilde{y}_{t^-}^{\mathbb{P}}d\delta\tilde{m}_t^{\mathbb{P}} + \epsilon^{-1}\sup_{t\in[0,T]}|\delta\tilde{y}_t^{\mathbb{P}}|^2 + \epsilon\big(\delta\tilde{k}_T^{\mathbb{P}}\big)^2.\end{aligned}$$

Moreover, by $(A.4.1)$, there exists $C > 0$ such that

$$\begin{aligned}\big(\delta\tilde{k}_T^{\mathbb{P}}\big)^2 \leq C\Big(|\delta\tilde{y}_0^{\mathbb{P}}|^2 + |\delta\xi|^2 + \Big(\int_0^T |\delta\widehat{f}_s^{\mathbb{P}}(\tilde{y}_s^{1,\mathbb{P}},\widehat{a}_s^{1/2}\tilde{z}_s^{1,\mathbb{P}})|ds\Big)^2 + \int_0^T |\delta\tilde{y}_s^{\mathbb{P}}|^2ds + \int_0^T \|\widehat{a}_s^{1/2}\delta\tilde{z}_s^{\mathbb{P}}\|^2ds \\ + \Big|\int_0^T \delta\tilde{z}_s^{\mathbb{P}}\cdot\widehat{a}_s^{1/2}dW_s^{\mathbb{P}}\Big|^2 + |\delta\tilde{m}_T^{\mathbb{P}}|^2\Big).\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{2}\int_0^T \|\widehat{a}_t^{1/2}\delta\tilde{z}_t^{\mathbb{P}}\|^2dt + (1-\epsilon C)[\delta\tilde{m}^{\mathbb{P}}]_T &\leq C\Big(|\delta\xi|^2 + \sup_{t\in[0,T]}|\delta\tilde{y}_t^{\mathbb{P}}|^2 + \big(\int_0^T |\delta\widehat{f}_t^{\mathbb{P}}(\tilde{y}_t^{1,\mathbb{P}},\widehat{a}_t^{1/2}\tilde{z}_t^{1,\mathbb{P}})|dt\big)^2\Big) \\ &\quad + 2\Big|\int_0^T \delta\tilde{y}_{t^-}^{\mathbb{P}}d\delta\tilde{m}_t^{\mathbb{P}}\Big|^2 + 2\Big|\int_0^T \delta\tilde{z}_s^{\mathbb{P}}\cdot\widehat{a}_s^{1/2}dW_s^{\mathbb{P}}\Big|^2\end{aligned}$$

By choosing ϵ small enough, using the same reasoning as in step 2 of Proof of Theorem 3.4.1, we come to

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T \|\widehat{a}_t^{1/2}\delta\tilde{z}_t^{\mathbb{P}}\|^2dt\Big)^{p/2}\Big] + \mathbb{E}^{\mathbb{P}}\big[[\delta\tilde{m}^{\mathbb{P}}]_T^{\frac{p}{2}}\big] \leq C\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|\delta\tilde{y}_t^{\mathbb{P}}|^p + \Big(\int_0^T |\delta\widehat{f}_t^{\mathbb{P}}(\tilde{y}_t^{1,\mathbb{P}},\widehat{a}_t^{1/2}\tilde{z}_t^{1,\mathbb{P}})|dt\Big)^p\Big].$$

We conclude from the first part of the proof that the announced estimates of $\delta\tilde{z}^{\mathbb{P}}$ and $\delta\tilde{k}^{\mathbb{P}}$ are verified and hence that the estimates of $\delta\tilde{k}^{\mathbb{P}}$ is a consequence of the previous estimates (see proof of Theorem 3.3.6 Step (iii)). □

Proposition A.4.2. *Let Assumption 3.2.1 holds, and consider two generators f^1 and f^2 such that Assumptions 3.3.1 holds. For $i = 1, 2$, let $(y^{i,\mathbb{P}}, z^{i,\mathbb{P}}, m^{i,\mathbb{P}}, k^{i,\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}$ be the solutions to the RBSDE (4.2.2) with terminal condition ξ^i and lower obstacle U^i. Define*

$$\begin{aligned}\phi_{f^1,f^2}^{p,\kappa} &:= \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\left[\operatorname{ess\,sup}_{0\leq t\leq T}^{\mathbb{P}}\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T |\widehat{f}_s^{1,\mathbb{P}} - \widehat{f}_s^{2,\mathbb{P}}|^\kappa(y_s^{1,\mathbb{P}},\widehat{a}_s^{1/2}z_s^{1,\mathbb{P}})ds\right)^{\frac{p}{\kappa}}\middle|\mathcal{F}_t^+\right]\right] < +\infty, \\ \psi_{U^1,U^2}^{p,\kappa} &:= \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\left[\operatorname{ess\,sup}_{0\leq t\leq T}^{\mathbb{P}}\left(\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq s\leq T}|U_s^1 - U_s^2|^\kappa\middle|\mathcal{F}_t^+\right]\right)^{\frac{p}{\kappa}}\right] < +\infty.\end{aligned}$$

Then, there exists a constant C depending only on κ, T and the Lipschitz constant of f^1 and f^2 such that

$$\begin{aligned}&\|y^{2,\mathbb{P}} - y^{2,\mathbb{P}}\|_{\mathbb{D}_0^p(\mathbb{P})}^p \leq C\Big(\|\xi^1 - \xi^2\|_{\mathbb{L}_0^{p,\kappa}}^p + \psi_{L^1,L^2}^{p,\kappa} + \varphi_{f^1,f^2}^{p,\kappa}\Big), \\ &\|z^{1,\mathbb{P}} - z^{2,\mathbb{P}}\|_{\mathbb{H}_0^p(\mathbb{P})}^p + \|m^{1,\mathbb{P}} - m^{2,\mathbb{P}}\|_{\mathbb{M}_0^p(\mathbb{P})}^p + \|k_T^{1,\mathbb{P}} - k_T^{2,\mathbb{P}}\|_{\mathbb{I}_0^p(\mathbb{P})}^p \leq C\Big(\|\xi^1 - \xi^2\|_{\mathbb{L}_0^{p,\kappa}}^p + \phi_{f^1,f^2}^{p,\kappa} + \psi_{U^1,U^2}^{p,\kappa}\Big).\end{aligned}$$

Proof. he arguments of this proof are essentially of the ones used to proof Proposition A.4.1. Then we retained the same notations to the proof of Proposition A.4.1 and just take $\delta U = U^1 - U^2$ instead of δL. By same arguments, we have the following

$$\tilde{I}_t \delta \tilde{y}_t^{\mathbb{P}} = \mathbb{E}^{\tilde{\mathbb{Q}}_{\mathbb{P}}}\Big[\tilde{I}_\tau \delta \tilde{y}_\tau^{\mathbb{P}} + \int_t^\tau \tilde{I}_s \delta \widehat{f}_s^{\mathbb{P}}(\tilde{y}_s^{1,\mathbb{P}}, \widehat{a}_s^{1/2}\tilde{z}_s^{1,\mathbb{P}})ds - \int_t^\tau \tilde{I}_s d\delta \tilde{k}_s^{\mathbb{P}} \big| \mathcal{F}_t^+\Big]$$

where $\tilde{\mathbb{Q}}_{\mathbb{P}} \sim \mathbb{P}$ and $\tilde{I}$ is a bounded positive process. Fix some $\epsilon > 0$ and define the stopping time $D_t^{\mathbb{P},\epsilon} := \inf\Big\{u \geq t : \tilde{y}_u^{2,\mathbb{P}} \geq U_u^2 - \epsilon,\ \mathbb{P}\text{-a.s.}\Big\} \wedge T$. By definition we have on one side, $\tilde{y}^{2,\mathbb{P}}_{D_t^{\mathbb{P},\epsilon}} \geq U^2_{D_t^{\mathbb{P},\epsilon}} - \epsilon$ on the set $\Big\{D_t^{\mathbb{P},\epsilon} < T\Big\}$ and then

$$\tilde{y}^{1,\mathbb{P}}_{D_t^{\mathbb{P},\epsilon}} - \tilde{y}^{2,\mathbb{P}}_{D_t^{\mathbb{P},\epsilon}} \leq \tilde{y}^{1,\mathbb{P}}_{D_t^{\mathbb{P},\epsilon}} - U^2_{D_t^{\mathbb{P},\epsilon}} + \epsilon \leq U^1_{D_t^{\mathbb{P},\epsilon}} - U^2_{D_t^{\mathbb{P},\epsilon}} + \epsilon \text{ on the set } \Big\{D_t^{\mathbb{P},\epsilon} < T\Big\}.$$

and on the other side $\tilde{y}_{s^-}^{1,\mathbb{P}} < U_{s^-}^1$ for all $s \in [t, D_t^{\mathbb{P},\epsilon}]$ and $\tilde{k}^{2,\mathbb{P}}_{D_t^{\mathbb{P},\epsilon}} - \tilde{k}_t^{2,\mathbb{P}} = 0$, by the Skorokhod condition. These observations enabled us to have

$$\begin{aligned}
\tilde{I}_t \delta \tilde{y}_t^{\mathbb{P}} &= \mathbb{E}^{\tilde{\mathbb{Q}}_{\mathbb{P}}}\Big[\tilde{I}_{D_t^{\mathbb{P},\epsilon}} \delta \tilde{y}^{\mathbb{P}}_{D_t^{\mathbb{P},\epsilon}} + \int_t^{D_t^{\mathbb{P},\epsilon}} \tilde{I}_s \delta \widehat{f}_s^{\mathbb{P}}(\tilde{y}_s^{1,\mathbb{P}}, \widehat{a}_s^{1/2}\tilde{z}_s^{1,\mathbb{P}})ds - \int_t^{D_t^{\mathbb{P},\epsilon}} \tilde{I}_s d\tilde{k}_s^{1,\mathbb{P}} + \int_t^{D_t^{\mathbb{P},\epsilon}} \tilde{I}_s d\tilde{k}_s^{2,\mathbb{P}} \big| \mathcal{F}_t^+\Big] \\
&\leq \mathbb{E}^{\tilde{\mathbb{Q}}_{\mathbb{P}}}\Big[\tilde{I}_{D_t^{\mathbb{P},\epsilon}}(\tilde{y}^{1,\mathbb{P}}_{D_t^{\mathbb{P},\epsilon}} - \tilde{y}^{2,\mathbb{P}}_{D_t^{\mathbb{P},\epsilon}}) + \int_t^{D_t^{\mathbb{P},\epsilon}} \tilde{I}_s \delta \widehat{f}_s^{\mathbb{P}}(\tilde{y}_s^{1,\mathbb{P}}, \widehat{a}_s^{1/2}\tilde{z}_s^{1,\mathbb{P}})ds - \int_t^{D_t^{\mathbb{P},\epsilon}} \tilde{I}_s d\tilde{k}_s^{1,\mathbb{P}} \big| \mathcal{F}_t^+\Big] \\
&\leq \mathbb{E}^{\tilde{\mathbb{Q}}_{\mathbb{P}}}\Big[\tilde{I}_{D_t^{\mathbb{P},\epsilon}}(\xi^1 - \xi^2)1_{\Big\{D_t^{\mathbb{P},\epsilon} = T\Big\}} + \tilde{I}_{D_t^{\mathbb{P},\epsilon}}(U^1_{D_t^{\mathbb{P},\epsilon}} - U^2_{D_t^{\mathbb{P},\epsilon}})1_{\Big\{D_t^{\mathbb{P},\epsilon} < T\Big\}} + \epsilon \tilde{I}_{D_t^{\mathbb{P},\epsilon}} \\
&\qquad\qquad + \int_t^{D_t^{\mathbb{P},\epsilon}} \tilde{I}_s \delta \widehat{f}_s^{\mathbb{P}}(\tilde{y}_s^{1,\mathbb{P}}, \widehat{a}_s^{1/2}\tilde{z}_s^{1,\mathbb{P}})ds \big| \mathcal{F}_t^+\Big] \\
&\leq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\tilde{\mathbb{Q}}_{\mathbb{P}}}\Big[\int_t^\tau \tilde{I}_s |\delta \widehat{f}_s^{\mathbb{P}}(\tilde{y}_s^{1,\mathbb{P}}, \widehat{a}_s^{1/2}\tilde{z}_s^{1,\mathbb{P}})|ds + \tilde{I}_\tau |\delta\xi| 1_{\{\tau = T\}} + \tilde{I}_\tau |\delta U_\tau| 1_{\{\tau < T\}} + \epsilon \tilde{I}_\tau \big| \mathcal{F}_t^+\Big].
\end{aligned}$$

We now apply this reasoning again, with $\delta y_t^{\mathbb{P}}$ replace by $y_t^{2,\mathbb{P}} - y_t^{1,\mathbb{P}} = -\delta y_t^{\mathbb{P}}$ and $D_t^{\mathbb{P},\epsilon} := \inf\Big\{u \geq t : \tilde{y}_u^{1,\mathbb{P}} \geq U_u^1 - \epsilon,\ \mathbb{P}\text{-a.s.}\Big\} \wedge T$ and let ϵ go to 0 to obtain,

$$\tilde{I}_t |\delta \tilde{y}_t^{\mathbb{P}}| \leq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\tilde{\mathbb{Q}}_{\mathbb{P}}}\Big[\int_t^\tau \tilde{I}_s |\delta \widehat{f}_s^{\mathbb{P}}(\tilde{y}_s^{1,\mathbb{P}}, \widehat{a}_s^{1/2}\tilde{z}_s^{1,\mathbb{P}})|ds + \tilde{I}_\tau |\delta\xi| 1_{\{\tau = T\}} + \tilde{I}_\tau |\delta U_\tau| 1_{\{\tau < T\}} + \epsilon \tilde{I}_\tau \big| \mathcal{F}_t^+\Big]$$

Repeated step 1 of the proof of Theorem 3.3.5 enables us to write

$$|\tilde{y}_t^{1,\mathbb{P}} - \tilde{y}_t^{2,\mathbb{P}}| \leq C\mathbb{E}^{\mathbb{P}}\Big[\int_0^T |\widehat{f}_s^{1,\mathbb{P}} - \widehat{f}_s^{2,\mathbb{P}}|^\kappa(\tilde{y}_s^{1,\mathbb{P}}, \widehat{a}_s^{1/2}\tilde{z}_s^{1,\mathbb{P}})ds + \sup_{s\in[0,T]} |U_s^1 - U_s^2|^\kappa + |\xi^1 - \xi^2|^\kappa \big| \mathcal{F}_t^+\Big]^{1/\kappa}. \tag{A.4.4}$$

Taking the supremum over $t \in [0,T]$, and using Doob's inequality, we prove the first assertion of the Proposition. The others estimates are deduce exactly as in the proof of proof Proposition A.4.1. □

Fix a map $g : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ which is $\mathbb{F}_+$-progressively measurable and uniformly Lipschitz in (y,z) satisfying for $\mathbb{P} \in \mathcal{P}_0$,

$$\mathbb{E}^{\mathbb{P}}\Big[\int_0^T |g_s(0,0)|^p ds\Big] < +\infty.$$

Let $(L_t)_{t\in[0,T]}$ a càdlàg process such that $L \in \mathbb{D}_0^p(\mathbb{F}^{\mathcal{P}_0+})$. To Given a RBSDE (defined on enlarged space $\overline{\Omega}$) when terminal time is a term of a decreasing sequence of stopping times and terminal value is a function of this term, the following result gives a convergence of such RBSDEs.

Lemma A.4.1. *Let $\mathbb{P} \in \mathcal{P}_0$. For any $\mathbb{F}$-stopping times $0 \leq \sigma \leq \tau \leq T$, any decreasing sequence of $\mathbb{F}$-stopping times $(\tau_n)_{n\geq 1}$ converging $\mathbb{P}$-a.s. to τ, any $\mathbb{F}^+$- progressively measurable and right-continuous process $V \in \mathbb{D}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P})$, if $y(\cdot, V_\cdot)$ denotes the first component of the solution to the following RBSDE*

$$\begin{cases} y_t = V_\cdot + \int_t^\cdot g_s(y_s, \widehat{a}_s^{1/2} z_s)ds - \int_t^\cdot z_s \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_t^\cdot dm_s + \int_t^\cdot dk_s, \mathbb{P} \otimes \mathbb{P}_0\text{-a.s.} \\ y_t \geq L_t, \forall t \in [0, \cdot], \ \mathbb{P} \otimes \mathbb{P}_0\text{-a.s.} \\ \int_0^\cdot (y_{s^-} - L_{s^-})dk_s = 0, \ \mathbb{P} \otimes \mathbb{P}_0\text{-a.s.} \end{cases} \tag{A.4.5}$$

Then

$$\mathbb{E}^{\mathbb{P}\otimes\mathbb{P}_0}\left[|y_\sigma(\tau, V_\tau) - y_\sigma(\tau_n, V_{\tau_n})|\right] \underset{n\to+\infty}{\longrightarrow} 0$$

Proof. We begin by recalling that Lemma 4.4.3 implies

$$y_\sigma(\tau, V_\tau) - y_\sigma = (\tau_n, V_{\tau_n}) = y_\sigma(\tau, V_\tau) - y_\sigma(\tau, y_\tau(\tau_n, V_{\tau_n}))$$

By $(A.4.3)$, we have for any $\kappa \in (1, p]$

$$\mathbb{E}^{\mathbb{P}\otimes\mathbb{P}_0}\left[|y_\sigma(\tau, V_\tau) - y_\sigma(\tau, y_\tau(\tau_n, V_{\tau_n}))|\right] \leq C\mathbb{E}^{\mathbb{P}\otimes\mathbb{P}_0}\left[|V_\tau - y_\tau(\tau_n, V_{\tau_n})|^\kappa\right]^{\frac{1}{\kappa}}.$$

Applying the same reasoning used in step (i) of Proof of Proposition $A.4.1$, we get

$$y_\tau(\tau_n, V_{\tau_n}) = \mathbb{E}^{\mathbb{P}\otimes\mathbb{P}_0}\left[\mathcal{E}\left(\int_\tau^{\tau_n} \eta_s \cdot dW_s^{\mathbb{P}}\right)\left(e^{\int_\tau^{\tau_n}\lambda_s ds}V_{\tau_n} - \int_\tau^{\tau_n} e^{\int_\tau^s \lambda_r dr} g_s(0,0)ds + \int_\tau^{\tau_n} e^{\int_\tau^s \lambda_r dr} dk_s\right)|\mathcal{F}_\tau^+\right]$$

Then,

$$\begin{aligned} V_\tau - y_\tau(\tau_n, V_{\tau_n}) = \mathbb{E}^{\mathbb{P}\otimes\mathbb{P}_0}\Big[& V_\tau - \mathcal{E}\Big(\int_\tau^{\tau_n} \eta_s \cdot dW_s^{\mathbb{P}}\Big)e^{\int_\tau^{\tau_n}\lambda_s ds}V_\tau \\ & + \mathcal{E}\Big(\int_\tau^{\tau_n} \eta_s \cdot dW_s^{\mathbb{P}}\Big)e^{\int_\tau^{\tau_n}\lambda_s ds}(V_\tau - V_{\tau_n}) \\ & - \mathcal{E}\Big(\int_\tau^{\tau_n} \eta_s \cdot dW_s^{\mathbb{P}}\Big)\int_\tau^{\tau_n} e^{\int_\tau^s \lambda_r dr} g_s(0,0)ds \\ & - \mathcal{E}\Big(\int_\tau^{\tau_n} \eta_s \cdot dW_s^{\mathbb{P}}\Big)\int_\tau^{\tau_n} e^{\int_\tau^s \lambda_r dr} dk_s|\mathcal{F}_\tau^+\Big] \end{aligned}$$

Therefore

$$\begin{aligned} |V_\tau - y_\tau(\tau_n, V_{\tau_n})|^\kappa \leq 4^{\kappa-1}\mathbb{E}^{\mathbb{P}\otimes\mathbb{P}_0}\Big[& \Big|1 - \mathcal{E}\Big(\int_\tau^{\tau_n} \eta_s \cdot dW_s^{\mathbb{P}}\Big)e^{\int_\tau^{\tau_n}\lambda_s ds}\Big|^\kappa |V_\tau|^\kappa \\ & + \mathcal{E}\Big(\int_\tau^{\tau_n} \eta_s \cdot dW_s^{\mathbb{P}}\Big)^\kappa e^{\kappa\int_\tau^{\tau_n}\lambda_s ds}|V_{\tau_n} - V_\tau|^\kappa \\ & + \mathcal{E}\Big(\int_\tau^{\tau_n} \eta_s \cdot dW_s^{\mathbb{P}}\Big)^\kappa \Big(\int_\tau^{\tau_n} e^{\int_\tau^s \lambda_r dr}|g_s(0,0)|ds\Big)^\kappa \\ & + \mathcal{E}\Big(\int_\tau^{\tau_n} \eta_s \cdot dW_s^{\mathbb{P}}\Big)^\kappa \Big(\int_\tau^{\tau_n} e^{\int_\tau^s \lambda_r dr} dk_s\Big)^\kappa |\mathcal{F}_\tau^+\Big] \end{aligned}$$

Using Hölder inequality and since λ and η are bounded (this leads to the Doléeans-Dade ex-

ponential above has finite moments of any order), we have for $\kappa < \tilde{p} < p$

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}\otimes\mathbb{P}_0}\big[|y_\sigma(\tau,V_\tau)-y_\sigma(\tau,y_\tau(\tau_n,V_{\tau_n}))|\big]\\
&\quad\le C\mathbb{E}^{\mathbb{P}\otimes\mathbb{P}_0}\big[|V_\tau|^p\big]^{\frac{\kappa}{p}}\mathbb{E}^{\mathbb{P}\otimes\mathbb{P}_0}\Big[\Big|1-\mathcal{E}\Big(\int_\tau^{\tau_n}\eta_s\cdot dW_s^{\mathbb{P}}\Big)e^{\int_\tau^{\tau_n}\lambda_s ds}\Big|^{\frac{p}{p-\kappa}}\Big]^{\frac{p-\kappa}{p}}\\
&\qquad+C\mathbb{E}^{\mathbb{P}\otimes\mathbb{P}_0}\big[|V_{\tau_n}-V_\tau|^{\tilde{p}}\big]^{\frac{\kappa}{\tilde{p}}}\mathbb{E}^{\mathbb{P}\otimes\mathbb{P}_0}\Big[\mathcal{E}\Big(\int_\tau^{\tau_n}\eta_s\cdot dW_s^{\mathbb{P}}\Big)^{\frac{\tilde{p}}{\tilde{p}-\kappa}}e^{\frac{\tilde{p}}{\tilde{p}-\kappa}\int_\tau^{\tau_n}\lambda_s ds}\Big]^{\frac{\tilde{p}-\kappa}{\tilde{p}}}\\
&\qquad+C\mathbb{E}^{\mathbb{P}\otimes\mathbb{P}_0}\Big[\Big(\int_\tau^{\tau_n}e^{\kappa\int_\tau^s\lambda_r dr}|g_s(0,0)|^\kappa ds\Big)^{\frac{\tilde{p}}{\kappa}}\Big]^{\frac{\kappa}{\tilde{p}}}\mathbb{E}^{\mathbb{P}}\Big[\mathcal{E}\Big(\int_\tau^{\tau_n}\eta_s\cdot dW_s^{\mathbb{P}}\Big)^{\frac{\tilde{p}}{\tilde{p}-\kappa}}\Big]^{\frac{\tilde{p}-\kappa}{\tilde{p}}}\\
&\qquad+C\mathbb{E}^{\mathbb{P}\otimes\mathbb{P}_0}\Big[\Big(\int_\tau^{\tau_n}e^{\int_\tau^s\lambda_r dr}dk_s\Big)^{\tilde{p}}\Big]^{\frac{\kappa}{\tilde{p}}}\mathbb{E}^{\mathbb{P}\otimes\mathbb{P}_0}\Big[\mathcal{E}\Big(\int_\tau^{\tau_n}\eta_s\cdot dW_s^{\mathbb{P}}\Big)^{\frac{\tilde{p}}{\tilde{p}-\kappa}}\Big]^{\frac{\tilde{p}-\kappa}{\tilde{p}}}\\
&\quad\le C\mathbb{E}^{\mathbb{P}\otimes\mathbb{P}_0}\Big[\Big|1-\mathcal{E}\Big(\int_\tau^{\tau_n}\eta_s\cdot dW_s^{\mathbb{P}}\Big)e^{\int_\tau^{\tau_n}\lambda_s ds}\Big|^{\frac{p}{p-\kappa}}\Big]^{\frac{p-\kappa}{p}}+C\mathbb{E}^{\mathbb{P}\otimes\mathbb{P}_0}\big[|V_{\tau_n}-V_\tau|^{\tilde{p}}\big]^{\frac{\kappa}{\tilde{p}}}\\
&\qquad+C\mathbb{E}^{\mathbb{P}\otimes\mathbb{P}_0}\Big[\int_\tau^{\tau_n}e^{\int_\tau^s\tilde{p}\lambda_r dr}|g_s(0,0)|^{\tilde{p}}ds\Big]+C\mathbb{E}^{\mathbb{P}\otimes\mathbb{P}_0}\big[(k_{\tau_n}-k_\tau)^{\tilde{p}}\big]^{\frac{\kappa}{\tilde{p}}}
\end{aligned}
$$

Since the terms inside the expectations on the right-hand side all converge in probability to 0, and are clearly uniformly integrable by de la Vallée-Poussin criterion (since $V\in\mathbb{D}_0^p(\mathbb{F}_+^{\mathbb{P}},\mathbb{P})$, $k\in\mathbb{I}_0^p(\mathbb{F}_+^{\mathbb{P}},\mathbb{P})$ and k is non-decreasing) and $\tilde{p}<p$, we can apply the dominated convergence theorem and get the result. □

Remark A.4.1. *The above result still verified if we take a RBSDEs with an upper barrier instead of a lower barrier. Because the slight difference is the sign in front of the càdlàg process k and when the absolute value is taken the rest of arguments is the same.*

The following results have been proved several times in different contexts, among others Briand et al. [18] for a semimartingale with respect to a filtration generated by a Brownian motion, Klimsiak [54] for a general filtration, Kruse and Popier [56] a filtration that supports a Brownian motion and a Poison random measure.

Lemma A.4.2. *Fix $\mathbb{P}$ a probability measure in $\mathcal{P}_0$ and let $\{H_t\}_{t\in[0,T]}$, $\{K_t\}_{t\in[0,T]}$ be two progressively measurable processes with values in $\mathbb{R}$, $\{Z_t\}_{t\in[0,T]}$ a predictable process with values in $\mathbb{R}^d$, $\{M_t\}_{t\in[0,T]}$ and $\{N_t\}_{t\in[0,T]}$ two càdlàg local martingales under $\mathbb{P}$ with Morthogonal to N such that $\mathbb{P}$-a.s*

$$\int_0^T(|H_t|+\|Z_t\|^2)dt<\infty.$$

We consider the $\mathbb{R}$-valued semimartingale $\{S_t\}_{t\in[0,T]}$ under $\mathbb{P}$ defined by

$$S_t=S_0+\int_0^t H_s ds+\int_0^t Z_s dN_s+M_t+K_t,\ \mathbb{P}\text{-a.s.}$$

Then for any $p\in[1,2)$, we have

$$
\begin{aligned}
|S_t|^p\ge|S_s|^p&+p\int_s^t|S_r|^{p-1}\hat{S}_r H_r dr+p\int_s^t|S_r|^{p-1}\hat{S}_r Z_r dN_r\\
&+p\int_s^t|S_{r^-}|^{p-1}\hat{S}_{r^-}dM_r+p\int_s^t|S_{r^-}|^{p-1}\hat{S}_{r^-}dK_r+\sum_{s<r\le t}\{|S_r|^p-|S_{r^-}|^p-p|S_r|^{p-1}\Delta S_r\}\\
&+\frac{1}{2}p(p-1)\int_s^t|S_r|^{p-2}1_{S_r\neq0}|Z_r|^2d[N]_r^c+\frac{1}{2}p(p-1)\int_s^t|S_r|^{p-2}1_{S_r\neq0}d[M]_r^c.
\end{aligned}
$$

where $\hat{x}=|x|^{-1}x1_{x\neq0}$.

Proof. **Case** $p \in [1,2)$: In this case the function $x \mapsto |x|$ is not smooth enough, to apply Itô's formula we use an approximation. Let $\varepsilon > 0$ and let us consider the function $u_\varepsilon(x) = (|x|^2 + \varepsilon^2)^{1/2},\ x \in \mathbb{R}$. It is a smooth function and we have

$$\frac{\partial u_\varepsilon^p}{\partial x}(x) = p x u_\varepsilon^{p-2}(x),\ \frac{\partial^2 u_\varepsilon^p}{\partial x^2}(x) = p u_\varepsilon^{p-2}(x) + p(p-2)x^2 u_\varepsilon^{p-4}(x).$$

We now apply Itô's formula and obtain

$$\begin{aligned}
u_\varepsilon^p(S_t) &= u_\varepsilon^p(S_s) + \int_s^t \frac{\partial u_\varepsilon^p}{\partial x}(S_{r^-})dS_r + \frac{1}{2}\int_s^t \frac{\partial^2 u_\varepsilon^p}{\partial x^2}(S_r)d[S,S]_r^c \\
&\quad + \sum_{s<r\le t}\{u_\varepsilon^p(S_r) - u_\varepsilon^p(S_{r^-}) - \frac{\partial u_\varepsilon^p}{\partial x}(S_{r^-})\Delta S_r\} \\
&= u_\varepsilon^p(S_s) + \int_s^t p S_{r^-} u_\varepsilon^{p-2}(S_{r^-})dS_r + \frac{1}{2}\int_s^t [p u_\varepsilon^{p-2}(S_r) + p(p-2)S_r^2 u_\varepsilon^{p-4}(S_r)]d[S,S]_r^c \\
&\quad + \sum_{s<r\le t}\{u_\varepsilon^p(S_r) - u_\varepsilon^p(S_{r^-}) - p S_{r^-} u_\varepsilon^{p-2}(S_r)\Delta S_r\}
\end{aligned}$$

Since N are orthogonal to M, $d[S,S]_r^c = \|Z_r\|^2 d[N]_r^c + d[M]_r^c$ and then

$$\begin{aligned}
u_\varepsilon^p(S_t) &= u_\varepsilon^p(S_s) + p\int_s^t S_r H_r u_\varepsilon^{p-2}(S_r)dr + p\int_s^t S_{r^-} Z_r u_\varepsilon^{p-2}(S_{r^-})dN_r \\
&\quad + p\int_s^t S_{r^-} u_\varepsilon^{p-2}(S_{r^-})dM_r + p\int_s^t S_{r^-} u_\varepsilon^{p-2}(S_{r^-})dK_r \\
&\quad + \frac{1}{2}p\int_s^t [u_\varepsilon^{p-2}(S_r) + (p-2)S_r^2 u_\varepsilon^{p-4}(S_r)]\|Z_r\|^2 d[N]_r^c \\
&\quad + \frac{1}{2}p\int_s^t [u_\varepsilon^{p-2}(S_r) + (p-2)S_r^2 u_\varepsilon^{p-4}(S_r)]d[M]_r^c \\
&\quad + \sum_{s<r\le t}\{u_\varepsilon^p(S_r) - u_\varepsilon^p(S_{r^-}) - p S_{r^-} u_\varepsilon^{p-2}(S_r)\Delta S_r\}.
\end{aligned} \tag{A.4.6}$$

Let us remark that $p u_\varepsilon^{p-2}(x)x \to p|x|^{p-1}\hat{x}$. Now we pass to the limit when $\varepsilon \to 0$ and by dominated convergence theorem, we have $\mathbb{P}$-a.s.

$$\begin{aligned}
p\int_s^t S_r H_r u_\varepsilon^{p-2}(S_r)dr &\longrightarrow p\int_s^t |S_r|^{p-1}\hat{S}_r H_r dr \\
p\int_s^t S_{r^-} Z_r u_\varepsilon^{p-2}(S_{r^-})dN_r &\longrightarrow p\int_s^t |S_{r^-}|^{p-1}\hat{S}_{r^-} Z_r dN_r \\
p\int_s^t S_{r^-} u_\varepsilon^{p-2}(S_{r^-})dM_r &\longrightarrow p\int_s^t |S_{r^-}|^{p-1}\hat{S}_{r^-} dM_r \\
p\int_s^t S_{r^-} u_\varepsilon^{p-2}(S_{r^-})dK_r &\longrightarrow p\int_s^t |S_{r^-}|^{p-1}\hat{S}_{r^-} dK_r.
\end{aligned} \tag{A.4.7}$$

By convexity of $u_\varepsilon^p,\ u_\varepsilon^p(S_s) - u_\varepsilon^p(S_{s^-}) - p S_{s^-} u_\varepsilon^{p-2}(S_{s^-})\Delta S_s \ge 0,\ s \in [0,T]$. Thus, by Fatou's lemma we have

$$\begin{aligned}
&\liminf_{\varepsilon\to 0^+} \sum_{s<r\le t}\{u_\varepsilon^p(S_r) - u_\varepsilon^p(S_{r^-}) - p S_{r^-} u_\varepsilon^{p-2}(S_r)\Delta S_r\} \\
&\qquad \ge \sum_{s<r\le t}\{|S_r|^p - |S_{r^-}|^p - p S_{r^-}|S_r|^{p-1}\Delta S_r\}
\end{aligned} \tag{A.4.8}$$

Furthermore, we have

$$\begin{aligned}
p u_\varepsilon^{p-2}(x) + p(p-2)x^2 u_\varepsilon^{p-4}(x) &= p\varepsilon^2 u_\varepsilon^{p-4}(x) + p(p-1)x^2 u_\varepsilon^{p-4}(x) \\
&= p(p-1)\Big(\frac{|x|}{u_\varepsilon(x)}\Big)^{4-p}|x|^{p-2}1_{x\neq 0} + p\varepsilon^2 u_\varepsilon^{p-4}(x).
\end{aligned}$$

and $\dfrac{|x|}{u_\varepsilon(x)} \nearrow 1_{x\neq 0}$ as $\varepsilon \searrow 0$. Hence by monotone convergence under $\mathbb{P}$, as $\varepsilon \to 0$,

$$\begin{aligned}\int_s^t \Big(\frac{|S_r|}{u_\varepsilon(S_r)}\Big)^{4-p} |S_r|^{p-2} 1_{S_r\neq 0} \|Z_r\|^2 d[N]^c_r \nearrow \int_s^t |S_r|^{p-2} 1_{S_r\neq 0} \|Z_r\|^2 d[N]^c_r,\\ \int_s^t \Big(\frac{|S_r|}{u_\varepsilon(S_r)}\Big)^{4-p} |S_r|^{p-2} 1_{S_r\neq 0} d[M]^c_r \nearrow \int_s^t |S_r|^{p-2} 1_{S_r\neq 0} d[M]^c_r.\end{aligned} \tag{A.4.9}$$

By taking the limit when ε goes to 0 in (A.4.6), we deduce from (A.4.7)-(A.4.9) the desired conclusion.

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UNIVERSITE BRETAGNE LOIRE

Le Mans Université

Thèse de Doctorat

Fanny Larissa NOUBIAGAIN CHOMCHIE

Contributions to second order reflected backward stochastic differentials equations

Résumé

Cette thèse traite des équations différentielles stochastiques rétrogrades réfléchies du second ordre dans une filtration générale. Le cas de réflexion à une barrière inférieure est d'abord traité, puis le résultat obtenu est étendu dans le cas d'une barrière supérieure. La contribution de ce travail consiste à démontrer l'existence et l'unicité de la solution de ces équations dans le cadre d'une filtration générale sous des hypothèses faibles. La régularité uniforme sur la barrière, la condition terminale et le générateur est remplacée par une régularité de type Borel sur le générateur. Le principe de programmation dynamique pour le problème de contrôle stochastique robuste est donc démontré sous les hypothèses faibles c'est à dire sans régularité sur le générateur, la condition terminale et la barrière. Dans le cadre des Equations Différentielles Stochastiques Rétrogrades (EDSRs) standard, les problèmes de réflexions à barrières inferieures et supérieures sont symétriques. Par contre dans le cadre des EDSRs de second ordre, cette symétrie n'est plus valable à cause de la non linéarité de l'espérance sous laquelle est définie notre problème de contrôle stochastique robuste non dominé.
Ensuite un schéma d'approximation numérique d'une classe d'EDSR de second ordre réfléchies est proposé ainsi que la convergence de schéma.

Mots clés: Equations différentielles stochastiques rétrogrades, Equations différentielles stochastiques rétrogrades de second ordre réfléchies, approximation faible, Contrôle stochastique robuste, modèle avec incertitude sur la volatilité, Capacité, Ensemble analytique, Problème avec obstacle, Problème de Skorokhod, Schéma numérique, Equations différentielles stochastiques rétrogrades réfléchies.

Abstract

This thesis deals with second order reflected backward stochastic differential equations (2RBSDEs) in general filtration. At first, we consider two cases of reflection, one with a lower obstacle and the other with an upper obstacle. We prove existence and uniqueness of these equations under weak assumptions about the generator, the terminal condition and the obstacle in the context of general filtration. The dynamic programming principle plays a key role in the proof of existence , we construct a value function that is measurable with respect to time, space and probability measure . Therefore, we use the measurable selection theorem to prove dynamic programming principle. The non-symmetry between the lower obstacle and the upper obstacle in the second-order framework is also highlighted. Then we consider the problem of approximation of the initial value of the solution of a 2RBSDE . This can be interpreted as an approximation of the value of stochastic control problem associated to standard reflected backward stochastic differentials equations solutions under model uncertainty. Our approach is based on the time discretization of the value of stochastic problem and the discretization of the model trough the discretization of the volatility process.

Key Words: Backward stochastic differential equations, Second order reflected stochastic differential equations, Weak approximation, Robust stochastic control, Uncertainty Volatility Model, Capacity, Analytic set, Obstacle problem, Reflected backward stochastic differential equation, Numerical scheme, Skorokhod problem..