Zero-cycles and constant cycle subvarieties in Calabi-Yau and hyper-Kähler varieties

Ivan Bazhov

Université Pierre et Marie Curie
Ecole Doctorale Paris Centre

Thèse de Doctorat

Discipline : Mathématiques

presentée par Ivan Bazhov

Zero-cycles and constant cycle subvarieties in Calabi–Yau and hyper-Kähler varieties

dirigée par Claire Voisin

Soutenue publiquement le 17 Novembre 2017 devant le jury composé de :

M. Samuel Boissière	Université de Poitiers	rapporteur
M. Alessandro Chiodo	Université Paris–6	examinateur
M. Olivier Debarre	Université Paris–7 and ENS	examinateur
M. Laurent Manivel	Institut de Mathématiques de Marseille	examinateur
M. Nicolas Perrin	Université de Versailles	examinateur
Mme Claire Voisin	Collège de France	directrice de thése

Rapporteur absent lors de la soutenance :

M. Kieran O'Grady	Universita di Roma I

Université Pierre et Marie Curie
Ecole Doctorale Paris Centre

Thèse de Doctorat

Discipline : Mathématiques

presentée par Ivan Bazhov

Zero-cycles and constant cycle subvarieties in Calabi–Yau and hyper-Kähler varieties

---

dirigée par Claire Voisin

Soutenue publiquement le 17 Novembre 2017 devant le jury composé de :

M. Samuel Boissière	Université de Poitiers	rapporteur
M. Alessandro Chiodo	Université Paris–6	examinateur
M. Olivier Debarre	Université Paris–7 and ENS	examinateur
M. Laurent Manivel	Institut de Mathématiques de Marseille	examinateur
M. Nicolas Perrin	Université de Versailles	examinateur
Mme Claire Voisin	Collège de France	directrice de thése

Rapporteur absent lors de la soutenance :

M. Kieran O'Grady	Universita di Roma I

Institut de Mathématiques de Jussieu
4 Place Jussieu
75005 Paris

Acknowledgement

I would like to express my gratitude to my thesis advisor Claire Voisin. I would not be able to accomplish this work without her kind help, her ideas and suggestions she made to me, and without her patience she showed during my study. It is a great honour for me to do a mathematical research under her guidance.

I would like to thank Samuel Boissière and Kieran O'Grady for agreeing to be rapporteurs of this thesis. I also thank Samuel Boissière, Alessandro Chiodo, Olivier Debarre, Laurent Manivel, Nicolas Perrin for honoring me on the jury.

I would also like to thank Laurent Manivel for a careful reading of the manuscript, his remarks and suggestions were very fruitful for me.

I am grateful to Nicolas Perrin for the discussions and his suggestions on the second chapter of the present thesis. I am grateful to Lie Fu for his comments and ideas on the third chapter and for his invitation to give a talk in University of Lyon. I would like to thank Alessandro Chiodo for his support and encouragement in the beginning of my study in Paris, where the thesis is done.

I would like to thank the professors of my undergraduate school for developing my interests in mathematics. Especially, I am grateful to Sergey Loktev, who showed me the beauty of algebraic geometry, and to Ivan Arzhantsev, who was my diploma paper advisor.

I would next like to thank Hsueh-Yung Lin and René Mboro for our fruitful discussions on algebraic geometry and for the organisation of the interesting reading group. I thank all PhD-students in Jussieu and, especially, my officemates Alexander Adam, Mattia Galeotti, and Anne Giralt for creating a beautiful and productive atmosphere.

I thank my Russian friends, who motivate and encourage me continuously: Roman Bachev and Tanya Ivanova, Nikita Vaganov, Finski Sergey, Sasha Zubkova, Ivan Tsylin, Daria Smirnova. Special thanks to Yana Semyoshkina for her strong and permanent support.

Finally, and most importantly, I would like to thank my mother, Zoya Bazhova for everything she has given to me. I dedicate this work to her.

Résumé

Nous présentons trois résultats dans cette thèse.

Dans le chapitre 2 nous montrons l'existence d'un zéro-cycle c_x sur une hypersurface X de type Calabi–Yau dans une varieté homogène projective complexe. Plus précisement, nous montrons que l'intersection de n diviseurs sur X, où $n = \dim X$, est proportionnelle à la classe d'un point supporté sur une courbe rationnelle dans X.

Dans le chapitre 3 nous donnons une nouvelle preuve du théorème de Beauville et Voisin portant sur la décomposition de la petite diagonale d'une surface K3 notée S. La preuve que nous donnons est explicite et utilise le plongement de degré $2g - 2$ de S dans $\mathbb{P}^g$. Elle est différente de celle donnée par Beauville et Voisin, qui repose sur l'existence d'une famille à un paramètre de courbes elliptiques.

Le chapitre 4 est consacré à l'étude des similitudes entre la variété de Fano des droites d'une cubique de dimension 4, qui est une variété hyper-Kählerienne étudiée par Beauville et Donagi, et la variété hyper-Kählerienne de dimension 4 construite par Debarre et Voisin dans [11]. Nous introduisons un analogue de la notion de triangle pour ces variétés et prouvons que la variété des triangles, qui est de dimension 6, est une sous-variété Lagrangienne du cube de la variété hyper-Kählerienne construite par Debarre et Voisin.

Abstract

We present in this thesis three results.

In Chapter 2 we prove the existence of a canonical zero-cycle c_X on a Calabi–Yau hypersurfacee X in a complex projective homogeneous variety. Namely, we show that the intersection of any n divisors on X, $n = \dim X$ is proportional to the class of a point on a rational curve in X.

In Chapter 3 we give a new proof of the theorem of Beauville and Voisin about the decomposition of the small diagonal of a K3 surface S. Our proof is explicit and uses the degree $2g - 2$ embedding of S in $\mathbb{P}^g$. It is different from the one used by Beauville and Voisin, which employed the existence of one-parameters familie of elliptic curves.

Chapter 4 is devoted to the study of similarities between the Fano varieties of lines on a cubic fourfold, a hyper-Kähler fourfold studied by Beauville and Donagi, and the hyper-Kähler fourfold constructed by Debarre and Voisin in [11]. We exhibit an analog of the notion of "triangle" for these varieties and prove that the 6-dimensional variety of "triangles" is a Lagrangian subvariety in the cube of the constructed hyper-Kähler fourfold.

Contents

Chapter 1

Introduction

1.1 Chow groups

1 Basic definition

First of all we introduce the basic invariant that we will work with. This invariant, the Chow ring, is associated with a projective algebraic variety. All varieties will be defined over the field of the complex numbers $\mathbb{C}$.

Let X be an algebraic variety. A *cycle* of dimension i is a formal finite sum of irreducible subvarieties of X of dimension i. The free abelian group generated by the cycles is denoted by $\mathcal{Z}_i(X)$. We define $\mathcal{Z}_i(X)_{rat}$ as the abelian subgroup of $\mathcal{Z}_i(X)$ generated by elements of the form $W(0) - W(\infty)$, where W is a subvariety of $X \times \mathbb{P}^1$ (flat over $\mathbb{P}^1$) and $W(0)$ and $W(\infty)$ are fibers over 0 and ∞. Two cycles are called rationally equivalent if their difference is contained in the group $\mathcal{Z}_i(X)_{rat}$.

Definition 1.1. The *Chow group* of dimension i of the variety X

$$\mathrm{CH}_i(X) = \mathcal{Z}_i(X)/\mathcal{Z}_i(X)_{rat}$$

is by definition the group of algebraic cycles of dimension i modulo rational equivalence.

Assuming X is smooth and projective of dimension n, we can also define Chow groups of codimension i as:

$$\mathrm{CH}^i(X) = \mathrm{CH}_{n-i}(X).$$

The intersection product constructed in [14] defines a structure of a ring on the Chow groups:

$$\mathrm{CH}^*(X) = \bigoplus_{i=0}^{\dim X} \mathrm{CH}^i(X).$$

Example 1.2. If X is a smooth variety, then $\mathrm{CH}^1(X)$ is just the Picard group $\mathrm{Pic}(X)$. For instance, if X is a smooth projective connected curve, we have

$$0 \longrightarrow Jac(X) \longrightarrow \mathrm{CH}^1(X) \xrightarrow{deg} \mathbb{Z} \longrightarrow 0.$$

Cycle class map. For variety X defined over $\mathbb{C}$, one can compare the Chow ring $\mathrm{CH}^*(X)$ and the cohomology ring $H^*(X)$ of an algebraic variety X by considering the cycle class map. The cycle class map associates to every subvariety $Y \subset X$ the class which is Poincaré dual to the fundamental class $[Y]$ in the homology of X:

$$cl : \mathrm{CH}^k(X) \to H^{2k}(X, \mathbb{Z}).$$

Example 1.3. [32] The long exact sequence associated to the following short exact sequence of sheaves of holomorphic functions in the analytic topology

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{X^{an}} \xrightarrow{exp} \mathcal{O}^*_{X^{an}} \longrightarrow 0$$

gives a map

$$H^1(X^{an}, \mathcal{O}^*_{X^{an}}) \to H^2(X, \mathbb{Z}),$$

where

$$H^1(X^{an}, \mathcal{O}^*_{X^{an}}) = \mathrm{Pic}(X^{an}) = \mathrm{Pic}(X) = \mathrm{CH}^1(X),$$

where the second equality follows from Serre's GAGA principle. One can prove ([32, Chapter 4]) that the first Chern class coincides with the cycle class map:

$$c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z}).$$

Correspondences. Let us consider a morphism $f : X \to Y$ of smooth proper varieties. We have the push-forward $f_* : \mathrm{CH}_r(X) \to \mathrm{CH}_r(Y)$ and the pull-back $f^* : \mathrm{CH}^s(Y) \to \mathrm{CH}^s(X)$ maps. Treating the graph $\Gamma_f \subset X \times Y$ of f as a cycle of $X \times Y$ we can describe these two maps in another way:

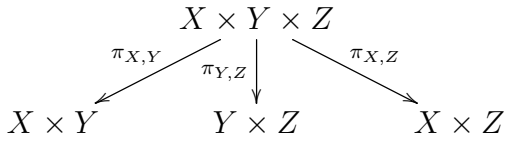
$$\alpha \in \mathrm{CH}_r(X), \;\; f_*(\alpha) = (\pi_Y)_*(\pi_X^*(\alpha) \cdot \Gamma_f), \tag{1.1}$$

$$\beta \in \mathrm{CH}^s(X), \;\; f^*(\beta) = (\pi_X)_*(\pi_Y^*(\beta) \cdot \Gamma_f), \tag{1.2}$$

where π_X and π_Y are the natural projections of $X \times Y$ to its factors. Substituting Γ_f with any cycle $\Gamma \in \mathrm{CH}_*(X \times Y)$ the construction leads to the following definition.

Definition 1.4. Let X and Y be smooth projective varieties. A *correspondence* from X to Y is an element of the Chow group $\mathrm{CH}_k(X \times Y)$.

A correspondence γ defines two maps γ_* and γ^* given by the formulas (1.1) and (1.2). We notice that correspondences as well as morphisms can be composed. Namely, for $\gamma_1 \in \mathrm{CH}_*(X \times Y)$ and $\gamma_2 \in \mathrm{CH}_*(Y \times Z)$ we have the natural projections:



the composition can be defined as

$$\gamma_1 \circ \gamma_2 = (\pi_{XZ})_*((\pi_{XY})^*(\gamma_2) \cdot (\pi_{YZ})^*(\gamma_1)) \in \mathrm{CH}_*(X \times Z).$$

It follows from the properties of the pull-backs and the push-forwards that this operation is associative. One can check that if $f : X \to Y$ and $g : Y \to Z$ are morphisms then $\Gamma_g \circ \Gamma_f = \Gamma_{g \circ f}$. In particular, on the group $\mathrm{CH}^*(X \times X)$ the composition defines a structure of a graded associative algebra with the class of the diagonal Δ_X as the unit.

2 Mumford's theorem, conjecture of Bloch and Beilinson

While $\mathrm{CH}^1(X)$ is relatively simple due to the exponential exact sequence, the higher codimensional Chow groups are less understood, which can be seen from the following theorem proved by Mumford in [24].

Theorem 1.5. *Let S be a projective smooth surface over the complex numbers. If $H^0(S, \Omega_S^2) \neq \{0\}$ then $\mathrm{CH}_0(S)$ has infinite dimension.*

Originally, this theorem was formulated in another way: the dimension of constant cycle subvarieties in $\mathcal{S}ym^n S$ is restricted above (cf. Remark 1.23). The theorem of Mumford was generalized for higher dimensional varieties by Roitman, see [30].

The following conjecture of Bloch can be viewed as a complement of a result of Mumford and Roitman.

Conjecture 1.6. *Let S be a surface. If $h^{2,0}(S) = 0$ then the Albanese map $\mathrm{CH}_0(S)_{hom} \to \mathrm{Alb}(S)$ is an isomorphism.*

This conjecture is a particular case of the most general conjecture of Bloch and Beilinson ([6], see also [33]).

Conjecture 1.7. *For any smooth irreducible projective variety X over the complex numbers there exists a filtration on the Chow groups with rational coefficients:*

$$F^{\cdot}\mathrm{CH}^k(X)_{\mathbb{Q}},$$

$0 \leq k \leq n$, $n = \dim X$, *such that*

a. $F^{\cdot}$ *is stable with respect to the action of correspondences, in particular, with respect to push-forwards and pull-backs,*

b. $F^0\mathrm{CH}^k(X)_{\mathbb{Q}} = \mathrm{CH}^k(X)_{\mathbb{Q}}$ *and* $F^1\mathrm{CH}^k(X)_{\mathbb{Q}} = \mathrm{CH}^k(X)_{\mathbb{Q},hom}$

c. $F^i\mathrm{CH}^*(X)_{\mathbb{Q}} \cdot F^j\mathrm{CH}^*(X)_{\mathbb{Q}} \subset F^{i+j}\mathrm{CH}^*(X)_{\mathbb{Q}}$

d. $\mathrm{Gr}_F^r\mathrm{CH}^k(X)_{\mathbb{Q}}$ *is controlled by the Hodge structure. Namely,*

$$\mathrm{Gr}_F^r\mathrm{CH}^k(X)_{\mathbb{Q}} = 0$$

if $H^{2k-r}(X)$ *is of coniveau greater than* $k - r$,

e. $F^{k+1}\mathrm{CH}^k(X) = 0$.

1.2 Varieties with trivial canonical bundle

In this section we make an introduction to the theory of manifolds (compact Kähler or complex projective manifolds) with trivial canonical bundle. These varieties are the main geometric context of our work. Although we are interested into their properties as algebraic varieties, their theory started more naturally in the context of compact Kähler geometry. So in this section X will denote a compact Kähler manifold.

1 The decomposition theorem

The starting point of the theory of Calabi–Yau manifolds is the theorem of Yau [40].

Theorem 1.8. *Let X be a compact Kähler manifold. If the topological first Chern class $c_1(X)$ is zero then in each Kähler class on X, there exists a unique Kähler form ω such that the Ricci curvature of the corresponding Kähler metric is zero:* $\mathrm{Ric}(X, \omega) = 0$.

Remark 1.9. For a Kähler manifold the Ricci curvature is just the curvature of the Chern connection of the induced metric on the canonical bundle. Thus an equivalent way of formulating the theorem is to say that the representation of the restricted holonomy group is in the special unitary group $\mathrm{SU}(n)$, where n is the dimension of the variety.

Using the classification of holomony groups, Beauville decomposes varieties with trivial canonical bundle in the product of irreducible factors.

Theorem 1.10. *([2]). Let X be a compact Kähler manifold such that $c_1(X) = 0$. There exists a finite étale cover $\tilde{X}$ of X such that $\tilde{X}$ is isomorphic to a product*

$$\tilde{X} \cong T \times \prod Y_i \times \prod Z_j,$$

where T is a complex torus, Y_i are simply connected Calabi–Yau manifolds and Z_j are compact hyper-Kähler manifolds.

Complex tori are simply the quotients V/Λ of a complex vector space V by a lattice Λ of maximal rank. We continue with the other types of varieties in the next two subsections.

2 Calabi–Yau manifolds

By definition, a compact Kähler manifold X is a Calabi–Yau manifold if it is simply connected, the canonical bundle K_X is trivial, and $H^0(X, \Omega_X^i) = 0$ for all $0 \leq i \leq n$, where $n = \dim X$. By Yau's theorem we have the following equivalent definition in terms of holonomy group.

Lemma 1.11. *([2]) Let X be a simply connected compact Kähler manifold of dimension n. Then X is a Calabi–Yau manifold if an only if it admits a Kähler metric with holonomy group equal to* $\mathrm{SU}(n)$.

Here are some examples of Calabi–Yau manifolds.

Example 1.12.

a. If X is a complete intersection of hypersurfaces of degrees $d_1, d_2, \ldots, d_k$ in a projective space $\mathbb{P}^n$ such that $d_1 + d_2 + \ldots + d_k = n + 1$ and $\dim X \geq 2$, then X is a Calabi–Yau manifold.

b. If X is a hypersurface in the anti-canonical linear system $|-K_Y|$ of a Fano variety Y, then X is a Calabi–Yau manifold.

3 Hyper-Kähler manifolds

By definition a compact Kähler manifold X is a hyper-Kähler manifold (or irreducible symplectic manifold) if X is simply connected and $H^0(X, \Omega_X^2)$ is of dimension 1, generated by a holomorphic 2-form σ, which is non-degenerated at any point of X. The 2-form σ is called the symplectic holomorphic form of X. It is defined up to a multiplicative constant.

There is an equivalent definition in terms of holonomy group.

Lemma 1.13. *[2] Let X be a simply connected compact Kähler manifold of dimension $2n$. Then X is a hyper-Kähler manifold if and only if it admits a Kähler metric whose holomony group is* $\mathrm{Sp}(n)$.

One can then prove that $H^0(X, \Omega_X^{2i}) = \mathbb{C} \cdot \sigma^i$, for $0 \leq i \leq m = \dim X/2$.

Beauville in [2] provides two series of families of examples, for each even complex dimension: (a) the $n-$punctual Hilbert scheme $S^{[n]}$ of a K3 surface S and (b) the fiber at the origin of the Albanese map of the $(n+1)-$st punctual Hilbert scheme of an abelian surface. All of the irreducible hyper-Kähler manifolds constructed later on are deformation-equivalent to one of Beauville's examples, with two exceptions: O'Grady examples in dimension 6 and in dimension 10 (see [26, 25]).

We note that the varieties in Beauville's examples have Picard number two, while a general algebraic deformation of a hyper-Kähler manifold has Picard number one. There are not so many available explicit constructions of these general deformations with Picard number one. Only four such families, each of which is 20-dimensional and parametrizes general polarized deformations of the second punctual Hilbert scheme of a K3 surface, are known:

a. (Beauville and Donagi, [4]) The Fano variety of lines of a cubic fourfold. It was proven in [4] that the variety $F(X)$ of lines on a smooth cubic hypersurface $F \subset \mathbb{P}^5$ is an algebraic hyper-Kähler fourfold. It gives a 20-dimensional moduli space of fourfolds.

b. (Iliev and Ranestad, [17, 18]) The variety $V(X)$ of sum of powers of a general cubic $X \subset \mathbb{P}^5$. It was proven in [17, 18] that it is another algebraic hyper-Kähler fourfold, with 20 moduli.

c. (O'Grady, [27]) O'Grady constructed a 20-parameter family of hyper-Kähler algebraic fourfolds. They are quasi-étale double covers of certain singular sextic hypersurfaces constructed by Eisenbud, Popescu, and Walter.

d. (Debarre and Voisin, [11]) Using Grassmann geometry another 20-dimensional family of hyper-Kähler varieties which are deformations of $S^{[2]}$ for S of genus 12 is constructed in this paper.

The last example will be studied in Chapter 4.

There was found recently a new family of hyper-Kähler varieties of dimension 8, see [19].

1.3 Chow groups of varieties with trivial K_X

1 Around a result of Beauville and Voisin

The following theorem, which contrasts with the results of Mumford (Theorem 1.5), is the starting point of this work as well as many other researches ([3, 12, 13, 22, 28, 29, 35, 36, 37, 38]).

Theorem 1.14 ([5]). *Let X be a K3 surface.*

a. *All points of X which lie on some (possibly singular) rational curve have the same class c_X in $\mathrm{CH}_0(X)$.*

b. *The image of the intersection product*

$$\mathrm{Pic}(X) \otimes \mathrm{Pic}(X) \to \mathrm{CH}_0(X)$$

is contained in $\mathbb{Z}c_X$.

c. *The second Chern class $c_2(X) \in \mathrm{CH}_0(X)$ is equal to $24c_X$.*

Originally, the proof of (b) was done separately from the proof of (c). The proof of (b) used the existence of a rational curve in any ample class $L \in \mathrm{Pic}(S)$ (see [23]). It allows to represent the intersection of two divisors L_1 and L_2 as a cycle with support on rational curves. Taking into account that all points on any rational curve represent the same class c_X, the result (b) follows.

The proof of (c) involves the idea of the decomposition of the small diagonal ([5, Proposition 2.1]):

Proposition 1.15. *The following equality holds in $\mathrm{CH}_2(S \times S \times S)_{\mathbb{Q}}$*

$$\Delta_{123} = \Delta_{12} \times o_3 + \Delta_{23} \times o_1 + \Delta_{13} \times o_2 - S \times o \times o - o \times S \times o - o \times o \times S, \quad (1.3)$$

where o is any point representing the canonical zero cycle, Δ_{123} is the small diagonal in S^3, and the notation $\Delta_{ij} \times o_k$ stands for $\pi_{ij}^*(\Delta) \cdot \pi_k^* o$, $\Delta \in \mathrm{CH}_2(S \times S)$ being the class of the diagonal.

Considering the small diagonal as a correspondence between $X \times X$ and X, we can use (1.3) to control the intersection product on X. For example, applying this correspondence to $\pi_1^* L_1 \cdot \pi_2^* L_2$ we get the intersection product of L_1 and L_2 on one hand and something proportional to c_X on the other, which proves (b). Similarly (this is the argument in [5] to get (c)), applying

the correspondence to the diagonal $\Delta \subset X \times X$ we get $c_2(X)$ on one hand and something proportional to c_X on the other, which proves (c).

These two results led to extensive study of Chow groups for Calabi–Yau varieties and hyper-Kähler manifolds, and we will focus on them in the rest of the introduction. One of our results in this thesis is a new proof of Proposition 1.15, see Section "K3 surfaces and an invitation to Chapter 3" and Chapter 3.

2 Hyper-Kähler varieties

From now on we work with rational coefficients: $\mathrm{CH}^i(X) = \mathrm{CH}_{n-i}(X) \otimes \mathbb{Q}$. The Theorem 1.14 can be restated as saying that any polynomial relation

$$P(c_1(L)) = 0$$

in $H^*(S)$ already holds at the level of Chow groups. In [3] Beauville put this statement in a more general framework and proposed a conjectural explanation for hyper-Kähler manifolds: the class map $cl : \mathrm{CH}^i(X) \to H^{2i}(X)$ is injective on the subring generated by divisors.

Conjecture 1.16. *([3]) Let Y be an algebraic hyper-Kähler variety. Then any polynomial cohomological relation*

$$P([c_1(L_j)]) = 0$$

in $H^{2k}(Y, \mathbb{Q})$, where $L_j \in \mathrm{Pic}(Y)$, already holds at the level of Chow groups :

$$P([c_1(L_j)]) = 0$$

in $\mathrm{CH}^k(Y)$.

Beauville proved this conjecture in the case of the second and third punctual Hilbert scheme of an algebraic K3 surface. A stronger version, which involves Chern classes of tangent bundles, was formulated by Voisin:

Conjecture 1.17. *([35, Conjecture 1.3]) Let Y be an algebraic hyper-Kähler variety. Then any polynomial cohomological relation*

$$P([c_1(L_j)], [c_i(T_Y))]) = 0$$

in $H^{2k}(Y, \mathbb{Q})$, where $L_j \in \mathrm{Pic}(Y)$, already holds at the level of Chow groups:

$$P([c_1(L_j)], [c_i(T_Y))]) = 0$$

in $\mathrm{CH}^k(Y)$.

This conjecture was proven for (a) $Y = S^{[n]}$ with $n \leq 2b_2(S)_{tr}+4$ and any k and for (b) Y the Fano variety of lines of a cubic fourfold ([35, Theorem 0.4]).

The next result has been obtained by Fu for a generalized Kummer variety. We recall that a generalized Kummer variety K_n can be defined in the following way: if A is an abelian surface, $s : A^{[n+1]} \to A$ is the natural morphism defined by the composition of the Hilbert–Chow morphism $A^{[n+1]} \to A^{(n+1)}$ and the summation $A^{(n+1)} \to A$ using the group of law of A, then s is an isotrivial fibration and a generalized Kummer variety is a fibre $K_n := s^{-1}(O_A)$ (cf. [2] and [13, Example 1.3]). We note, that the following result is stated for numerical equivalence, i.e., in a slightly stronger version than the conjecture above.

Theorem 1.18. *([13]) Let $z \in \mathrm{CH}(K_n)_{\mathbb{Q}}$ be an algebraic cycle, which is a polynomial with rational coefficients of the first Chern classes of line bundles on K_n and the Chern classes of the tangent bundle of K_n, then z is numerically trivial if and only if z is (rationally equivalent to) zero.*

The reader can find some other results about hyper-Kähler manifolds in [22, 37, 38].

3 Calabi–Yau varieties and an invitation to Chapter 2

The following example from [3] shows that Conjecture 1.16 does not hold for Calabi–Yau varieties, namely, the class map may be not injective on the subring $DCH^*(X) \subset \mathrm{CH}^*(X)$ generated by divisors if X is a Calabi–Yau variety.

Example 1.19. ([3, Example 1.7]) Let X be the blow-up of $\mathbb{P}^3$ along a curve C of degree d and genus g. Let $\tilde{X}$ be the Calabi–Yau variety, obtained as a double cover of X ramified over an element of $|-2K_X|$. We have the following diagram:

$$\begin{array}{ccc} & & \tilde{X} \\ & & \downarrow{\scriptstyle 2:1}\ \downarrow{\scriptstyle \pi} \\ E & \xrightarrow{i} & X \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \varepsilon} \\ C & \xrightarrow{i_c} & \mathbb{P}^3 \end{array}$$

Let H be the class of a hyperplane section on $\mathbb{P}^3$ and put $H_C = i_c^* H$. The space $DCH^2(X) \subset \mathrm{CH}(X)_{\mathbb{Q}}$ is generated by

$$\varepsilon^* H^2,\ \varepsilon^* H \cdot [E] = i_* p^* H_C,\ [E]^2 = i_* p^* c_1(N) - \varepsilon^*[C],$$

where N is the normal bundle on C for the embedding $i_c : C \to \mathbb{P}^3$. Since $c_1(N) = 4H_C + K_C$, $DCH^2(X)$ contains the elements $i_*p^*H_C$ and $i_*p^*K_C$.

The map $i_*p^* : \mathrm{CH}^1(C)_{\mathbb{Q}} \to \mathrm{CH}^2(X)_{\mathbb{Q}}$ induces an isomorphism of the subspace of degree 0 divisor classes on C onto the subspace of homologically trivial classes in $\mathrm{CH}^2(X)_{\mathbb{Q}}$. If we choose C in such a way that H_C is non proportional to K_C in $\mathrm{CH}^1(C)_{\mathbb{Q}}$ then the class $Z = i_*p^*(dK_C - (2g-2)H_C)$ in $DCH^2(X)$ is homologically trivial, but not trivial. Finally, its pull-back $\pi^*(Z)$ is non-trivial in $DCH(\tilde{X})$ (because $\pi_*(\pi^*(Z)) = 2Z$ is not trivial in $\mathrm{CH}(X)_{\mathbb{Q}}$), but it is homologically trivial.

However, this example deals with cycles of strictly positive dimension and still there is a hope of existence partial results in this case. More precisely, results of Voisin [36] provide a decomposition of the small diagonal for a Calabi–Yau hypersurface X in $\mathbb{P}^n$. This result is similar to Proposition 1.15 (although it contains an extra term) and allows to control the intersection product on X.

Theorem 1.20. *([36, Theorem 3.1]) Let X be a Calabi–Yau hypersurface in $\mathbb{P}^n$ and let $F(X)$ be the variety of lines contained in X. We denote the class of a hyperplane section on X by H and put $o = H^{\dim X}/\deg(H^{\dim X})$. For $t \in F(X)$ we denote $\mathbb{P}^1_t \subset X \subset \mathbb{P}^n$ the corresponding line.*

The following relation holds in $\mathrm{CH}^{2n-2}(X \times X \times X)_{\mathbb{Q}}$:

$$\Delta = \Delta_{12} \cdot o_3 + (perm.) + Z + \Gamma',$$

where Z is the restriction to $X \times X \times X$ of a cycle on $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$ and Γ' is a multiple of the following effective cycle of dimension $n-1$:

$$\Gamma = \cup_{t \in F(X)} \mathbb{P}^1_t \times \mathbb{P}^1_t \times \mathbb{P}^1_t.$$

Corollary 1.21. *([36, Theorem 3.4]) Let X be as above and let Z_i and Z'_i be cycles of codimension greater than 0 on X such that $\mathrm{codim}\, Z_i + \mathrm{codim} Z'_i = n-1$. Then if we have a cohomological relation*

$$\sum_i n_i \deg(Z_i \cdot Z'_i) = 0,$$

this relation holds at the level of Chow groups:

$$\sum_i n_i (Z_i \cdot Z'_i) = 0$$

in $\mathrm{CH}_0(X)_{\mathbb{Q}}$.

This theorem and its corollary were generalized by Fu in [12] for a Calabi–Yau complete intersection in projective space which is general for a given degrees of hypersurfaces.

The results above suggest that despite the counterexample given by Example 1.19, it is true that the cycle map is injective on zero-cycles which are intersections of divisors or, equivalently, that the intersection of any $\dim(X)$ divisors on X is proportional to a canonical zero-cycle c_X in $\mathrm{CH}_0(X)$. The existence of such a cycle is trivial if $\mathrm{Pic}(X)$ is generated by one element H: we can put $c_X = H^{\dim X}/\deg\left(H^{\dim X}\right)$.

Our result of Chapter 2 provides evidence for the existence of such a canonical zero cycle c_X for some Calabi–Yau manifolds with higher Picard rank (see also [1]):

Theorem 1.22. *(= Theorem 2.1) Let Y be a complex homogeneous variety of dimension $n+1$ which is a Fano variety and let X be a general element of the anti-canonical system $|-K_Y|$ on Y.*

a. There exists a constant cycle subvariety (cf. Definition 1.25) of positive dimension $n-1$. All points of X which lie on such a constant cycle variety have the same class c_X in $\mathrm{CH}_0(X)$.

b. The image of the intersection product

$$\mathrm{Pic}(X)^{\otimes n} \to \mathrm{CH}_0(X)$$

is contained in $\mathbb{Z}c_X$.

c. The top Chern class $c_n(X) \in \mathrm{CH}_0(X)$ is proportional to c_X.

Result (a) shows that a constant cycle subvariety can be a divisor in a Calabi–Yau manifold. This is not possible in hyper-Kähler manifolds of dimension at least 4 by the following fact.

Remark 1.23. ([24, 38]) If X is a hyper-Kähler manifold and $Y \subset X$ is a constant cycle subvariety, then Y is Lagrangian for the holomorphic 2-form σ_X, hence $2\dim Y \leq \dim X$.

4 $K3$ surfaces and an invitation to Chapter 3

Chapter 3 is devoted to a new proof of the decomposition of the small diagonal, which is stated in Proposition 1.15 (cf. Theorem 3.1). The presented proof is short and there are some related ideas that are not involved and not reflected in the proof.

At first, we would like to emphasise that our proof is very different from the original proof. The original proof uses the existence of a one-parameter family of elliptic curves and the theorem of Bloch and Srinivas [7].

Our proof uses only projective geometry; it is similar in spirit to the proof of the decomposition of the small diagonal for a Calabi–Yau hypersurface (Theorem 1.20). Namely, we use the degree $2g-2$ embedding of a K3 surface S in projective space $\mathbb{P}^g$. It is based on the study of the set of pairs of points (x, y) in $S \times S$ such that two hyperplane sections intersect exactly at these two points with given multiplicities. Specifically, we choose the multiplicities $2g-3$ and 1.

Now we are going to explain the philosophy behind this approach, which formally is not involved in the proof.

Chow groups of $S^{[n]}$. Let S be a K3 surface, and $\mathcal{P}(n)$ be the set of all partitions of n. For any such a partition $\mu = (\mu_1, \dots, \mu_k)$, we denote as k_μ its length. Put $S^\mu = S^{k_\mu} = S \times S \times \ldots \times S$ (k_μ times). We also have a natural morphism to a symmetric product:

$$S^\mu \to S^{(n)}$$

$$(x_1, \dots, x_k) \mapsto \mu_1 x_1 + \ldots + \mu_k x_k.$$

Now define $E_\mu = \left(S^{[n]} \times_{S^{(n)}} S^\mu\right)$ to be the reduced incidence variety in $S^{[n]} \times S^\mu$. The variety E_μ can be viewed as a correspondence between $S^{[n]}$ to S^μ, it defines maps: $E_\mu^* : \mathrm{CH}(S^\mu) \to \mathrm{CH}(S^{[n]})$ and $E_{\mu *} : \mathrm{CH}(S^{[n]}) \to \mathrm{CH}(S^\mu)$. Let $\mu = (\mu_1, \dots, \mu_k) = 1^{a_1} 2^{a_2} \dots k^{a_n}$ be a partition of n, we put

$$m_\mu = (-1)^{n-k} \prod \mu_i, \quad c_\mu = \frac{1}{m_\mu a_1! \cdot \ldots \cdot a_n!}.$$

Theorem 1.24. *([10]) The group* $\mathrm{CH}(S^{[n]})$ *has the following decomposition:*

$$\mathrm{CH}(S^{[n]}) = \bigoplus_{\mu \in \mathcal{P}(n)} c_\mu E_\mu^* \circ E_{\mu *} \left(\mathrm{CH}(S^{[n]})\right).$$

This theorem allows us to relate cycles in $\mathrm{CH}(S^{[n]})$ and cycles in $\mathrm{CH}(S^\mu)$. In particular, we will consider the following construction. If S is embedded in $\mathbb{P}^g$ by a primitive linear system $|L|$, then the intersection of S with a general linear subspace $\mathbb{P}^{g-2} \subset \mathbb{P}^g$ is a zero-cycle of degree $2g-2$ on S. The set of all $P = \mathbb{P}^{g-2} \subset \mathbb{P}^g$ is the Grassmann variety $Gr(g-1, g+1)$ and we have a birational map:

$$Gr(g-1, g+1) \dashrightarrow S^{[2g-2]},$$

$$P \mapsto P \cap S.$$

Let Gr denote the class in $\mathrm{CH}(S^{[2g-2]})$ of the image of $Gr(g-1,g+1)$ in $S^{[2g-2]}$. Using the theorem above, we can relate Gr with its components in S^{μ}, and, particularly, if we take μ to be the partition in $2g-3$ and 1, we expect to get a surface in $S\times S$ as a component of the decomposition for Gr.

Voisin showed that every term $E^*_{\mu}(P)$ of the decomposition of a polynomial expression $P\in \mathrm{CH}(S^{[n]})$ in $c_i(\mathcal{O}^{[n]})$ and $c_j(T_n)$ is a polynomial expression in $\pi_i^*(o)$ and $\pi_{ij}(\Delta_S)$ (see [35, Propostion 2.4]): all these cycles as well as the decomposition of the diagonal are cycles defined globally over the moduli space. Since Gr can be represented as a polynomial in $c_i(L^{[2g-2]})$, the elements of its decomposition are expected to be polynomials in $\pi_i^*(o)$, $\pi_i^*(L)$ and $\pi_{ij}(\Delta_S)$. A careful consideration of relations between polynomials gives a relation which turned out to be the decomposition of the small diagonal up to multiplicative constant. The relation itself can be obtained in a simpler way, which does not involve Theorem 1.24 visibly.

1.4 Our main tool: constant cycle subvarieties

The notion of *constant cycle subvarieties* was introduced in [15] and used in [22, 38] to study $\mathrm{CH}_0(X)$ for hyper-Kähler manifolds. Constant cycle subvarieties will play a crucial role in our proofs.

Definition 1.25. A subvariety $Z\subset X$ of positive dimension is called *a constant cycle subvariety* if for any two points $z,z'\in Z$ one has $[z]=[z']$ in $\mathrm{CH}_0(X)$. If, moreover, $\mathrm{codim}_X(Z)=1$, we call Z a *constant cycle divisor* (or CCD for short). We will denote by $c_Z\in \mathrm{CH}_0(X)$ the common class of the points $z\in Z$.

Example 1.26. Let S be surface and let $C\subset S$ be a rational (possibly, singular) curve. Clearly, C is a constant cycle subvariety. More generally, any rationally connected subvariety Y of X is a constant cycle subvariety.

Example 1.27. Let $\pi: X\to \mathbb{P}^n$ be a ramified double cover. Clearly, the class in $\mathrm{CH}_0(X)$ of the degree two cycle $x_1+x_2=\pi^{-1}(p)$ does not depend on the choice of $p\in\mathbb{P}^n$, and if, moreover, x_1 and x_2 coincide, then the class $[x_1]=[x_2]$ is determined in $\mathrm{CH}_0(X)$ uniquely. The locus, where the two preimages x_1 and x_2 coincide is the divisor D of ramification. So, D is a constant cycle divisor on X.

Example 1.28 (The curve of hyperflexes, cf. [15] and [39])**.** Recall that for a quartic $X\subset\mathbb{P}^3$ a line $l\subset\mathbb{P}^3$ is called bitangent of X if at every point $x\in X\cap l$ the intersection multiplicity is at least two, a bitangent l is a

hyperflex if there is a unique point of an intersection. We can consider the universal family of bitangents:

$$\begin{array}{ccc} B & \xrightarrow{q} & X \subset \mathbb{P}^3 \\ \downarrow{\scriptstyle p} & & \\ F & & \end{array}$$

here $F \subset \mathrm{Gr}(\mathbb{P}^1, \mathbb{P}^3)$ is the variety of all bitangents and B is the variety of pairs (l, x), where x is a point of contact of l and X. The map $p : B \to F$ has degree two and let D be the ramification divisor. The curve $C_{hf} \subset X$ is defined as $q(p^{-1}(D))$.

The curve C_{hf} is a constant cycle curve in X because every point in C_{hf} has class $(l|_X)/4$ in $\mathrm{CH}_0(X)$ and there is no torsion in $\mathrm{CH}_0(X)$ by [30]. It is is a singular and irreducible curve in the linear system $|\mathcal{O}_X(20)|$ of geometric genus 201, in particular, C_{hf} is not rational.

Construction 2.2 in Chapter 2 generalizes Example 1.28 for Calabi–Yau hypersurfaces in Fano manifolds with trivial Chow groups.

1.5 Debarre–Voisin hyper-Kähler fourfolds and Lagrangian subvarieties

Remark 1.23 states that any constant cycle subvariety Y in a hyper-Kähler manifold X is Lagrangian for the holomorphic 2-form σ_X and hence $2 \dim Y \leq \dim X$.

Example 1.29. Let $S \subset \mathbb{P}^g$ be a K3 surface and $S^{[2g-2]}$ be the Hilbert scheme of $2g-2$ points on it. The image of the map

$$Gr(g-1, g+1) \dashrightarrow S^{[2g-2]},$$

which sends a subspace $\mathbb{P}^{g-2} \subset \mathbb{P}^g$ to the intersection $\mathbb{P}^{g-2} \cap S$, is a constant cycle subvariety. We note that the dimension of the Grassmanian is $2g-2$, i.e., half of the dimension of $S^{[2g-2]}$.

Example 1.30. ([35, Proof of Lemma 2.2]) Let $X \subset \mathbb{P}^5$ be a cubic fourfold and $F(X)$ be its Fano variety of lines. If $Y \subset X$ is a hyperplane section then the set $F(Y)$ of all lines contained in Y forms a Lagrangian subvariety in $F(X)$. If, moreover, Y has 5 nodal points then $F(Y)$ is birational to the projective plane $\mathbb{P}^2$ and, in particular, is a constant cycle surface.

In Chapter 4 we study a hyper-Kähler four-dimensional manifold F constructed by Debarre and Voisin from a hyperplane section X in $Gr(3, 10)$ (see [11]). Similarly to the case of Fano variety of lines, we introduce a notion of triangle on X and define a corresponding subvariety $I_3 \subset F \times F \times F$. The result of Chapter 4 is the following theorem.

Theorem 1.31. *The $6-$dimensional subvariety I_3 is Lagrangian for 2-form $\sum_{i=1}^{3} \pi_i^* \sigma_F$, where $\pi_i : F^3 \to F$ are the natural projections and σ_F is a holomorphic 2-form on F.*

Our proof uses the fact that the cycle $[W_1]+[W_2]+[W_3]$ on X corresponding to a point of I_3 is constant in $\mathrm{CH}_9(X)$ (i.e., does not depend on a choice of point in I_3). The similar result is obviously true for a cubic fourfold: any triangle is just a restriction of some plane to the cubic hypersurface.

However, we do not expect that I_3 is a constant cycle subvariety, because there is no similar result for a cubic fourfold (see [31, Theorem 20.5] for details).

Chapter 2

Result for some Calabi–Yau varieties

2.1 Introduction

Example 1.19 shows that Conjecture 1.16 does not hold for Calabi–Yau varieties, namely, the class map may be not injective on the ring generated by divisors if X is a Calabi–Yau variety. Fortunately, this example deals with cycles of strictly positive dimension and still there is a hope that the class map is injective on zero-cycles or, equivalently, that the intersection of any $\dim(X)$ divisors on X is proportional to a canonical zero-cycle c_X in $\mathrm{CH}_0(X)$. The existence of such a cycle is trivial if $\mathrm{Pic}(X)$ is generated by one element H: we can put $c_X = H^{\dim X}/\deg\left(H^{\dim X}\right)$. We aim to show the existence of c_X prove the following analog of the Theorem 1.14 at least for some Calabi–Yau varieties with higher Picard rank.

The natural candidate for Calabi–Yau variety X is a Calabi–Yau hypersurface in a Fano variety Y. In this case we can study $\mathrm{Pic}(X)$ using an existence of an identification between $\mathrm{Pic}(X)$ and $\mathrm{Pic}(Y)$ due to the Grothendieck–Leftschetz theorem. In the result below we concentrate on the case when Y is a homogeneous variety, because of the wonderful structure of its Picard group. We recall that variety Y is homogeneous if its automorphism group acts transitively on Y.

Theorem 2.1. *(= Theorem 1.22) Let Y be a complex homogeneous variety of dimension $n+1$ which is a Fano variety and let X be a general element of the anti-canonical system $|-K_Y|$ on Y.*

a. There exists a constant cycle subvariety (cf. Definition 1.25) of positive dimension. All points of X which lie on such constant cycle variety have the same class c_X in $\mathrm{CH}_0(X)$.

b. The image of the intersection product

$$\mathrm{Pic}(X)^{\otimes n} \to \mathrm{CH}_0(X)$$

is contained in $\mathbb{Z}c_X$.

c. The top Chern class $c_n(X) \in \mathrm{CH}_0(X)$ *is proportional to* c_X.

We have a hope that, using the construction and ideas presented in this chapter, it is possible to generalize Theorem 2.1 for wider classes of Fano varieties Y. Our principal requirement for Y, the triviality of Chow groups, will be stated in Construction 2.2 and fulfilled not only for rational homogeneous varieties but also for other classes such as smooth toric varieties or varieties admitting a stratification by affine spaces. We use few properties of homogeneous varieties in our proof and one can probably avoid using them to generalize the result.

The chapter is organized as follows. Firstly, we briefly recall basic facts about homogeneous spaces. Then, we construct a constant cycle divisor associated with an effective class of curves in Y. Using these divisors we finish the proof in the last section.

The main point of our proof is an existence of a *constant cycle divisor* in an ample class (in the situation of K3 surface, the analog is the existence of rational curves in an ample class). Our result (b) also shows that the restriction to our Calabi–Yau hypersurface of any curve in a homogeneous space is proportional to the canonical cycle c_X (cf. Proposition 2.8). Indeed, by the hard Lefschetz theorem, any curve class in Y is an intersection of divisor classes in Y and furthermore the cycle class $\mathrm{CH}^n(Y) \to H^{2n}(Y)$ is injective, so any element of $\mathrm{CH}^n(Y)$ is an intersection of divisors.

2.2 Homogeneous varieties

We recall that a projective variety Y is called *homogeneous* if there is an algebraic group acting transitively on Y. Due to a fundamental theorem of Chevalley, any projective homogeneous variety is isomorphic to a product of an abelian variety and a rational homogeneous variety. Examples of rational homogeneous varieties include Grassmann varieties, flag varieties and their products.

In the present work we deal with Y which is a homogeneous Fano variety, therefore Y is a rational homogeneous variety. For the sake of simplicity, the notion of a homogeneous variety will refer to a rational homogeneous variety.

We recall some basic facts about homogeneous varieties (see [8] and the references therein). For a projective homogeneous variety Y the cone of

effective divisors $\mathrm{Eff}(Y)$, which is the closure of the cone of ample divisors, is a polyhedral cone. Every rational point in it represents a globally generated line bundle, which is (very) ample if and only if the point belongs to the interior of $\mathrm{Eff}(Y)$. Since Y is Fano, its anti-canonical class $-K_Y$ belongs to the interior of $\mathrm{Eff}(Y)$.

For every face $\sigma \subset \mathrm{Eff}(Y)$ of codimension one, one defines the extremal contraction

$$Y \xrightarrow{f_\beta} Y_\beta,$$

of the primitive class $\beta \in \sigma^\perp$ of curves, where β is the unique integral positive generator of the rational line $\sigma^\perp$, and Y_β is a homogeneous variety of smaller dimension and f_β is a fibration. There is the natural identification

$$\beta^\perp = f_\beta^*(\mathrm{Pic}(Y_\beta)) \subset \mathrm{Pic}(Y).$$

Vice versa, for any extremal ray $\mathbb{R}_{>0}\beta$ there is a face $\sigma = \beta^\perp \cap \mathrm{Eff}(Y)$ of codimension one and we can define a contraction f_β. We denote the set of all primitive effective classes of curves on extremal rays by $\mathcal{R}$.

A general fiber Y_0 of f_β is *a generalized Grassmann variety* in the following sense: it is a homogeneous space G/P, where P is a maximal parabolic subgroup of G. We use only one fact: $\mathrm{Pic}(Y_0)$ is generated by the class of an ample bundle $\mathcal{O}_{Y_0}(1)$ and the unique primitive effective class $\beta \in \mathrm{CH}_1(Y_0)$ has degree 1 with respect to it (we call β the class of a line).

2.3 Constant cycle divisors

The notion of constant cycle subvarieties was introduced in [15] and used in [22] and [38]. The goal of this section is to construct constant cycle divisors on Calabi–Yau hypersurfaces expanding the idea of examples from the introduction: having a family of zero-cycles on X, all of the same class and degree, we are looking for cycles represented by one point (with the corresponding multiplicity).

Construction 2.2. *Let Y be a Fano variety of dimension $n+1$ and assume that Y has trivial $\mathrm{CH}_1(Y)$ group, which means the following.*

($\star$) The natural class map

$$\mathrm{cl} : \mathrm{CH}_1(Y)_{\mathbb{Q}} \longrightarrow H^{2n}(Y, \mathbb{Q})$$

is injective.

(Varieties with trivial Chow groups include projective homogeneous spaces, smooth toric varieties, and varieties admitting a stratification by affine spaces.)

Let β be any effective class in $\mathrm{CH}_1(Y)$ and M_β be the space of all (irreducible) rational curves $C \subset Y$ representing class β. Denoting by $\mathcal{C}_\beta$ the universal curve we have the natural diagram:

$$\begin{array}{ccc} \mathcal{C}_\beta & \xrightarrow{q} & Y \\ \downarrow{\scriptstyle p} & & \\ M_\beta & & \end{array} \tag{2.1}$$

where p and q are the natural projections.

For a given hypersurface $X \subset Y$ we can construct the following variety:

$$V_{X,\beta} = \{(C, x) \in \mathcal{C}_\beta : C \cap X = \deg(C \cap X) \cdot x\},$$

where the equality is an equality of zero-cycles on C (or even of subschemes of C if C is smooth). The subvariety $p(V_{X,\beta})$ in M_β describes all curves intersecting X in one point with the maximal multiplicity. Due to our assumption $(\star)$, the class of the zero-cycle $C|_X$ in $\mathrm{CH}_0(X)$ does not depend on the point C in M_β. By [30], a family of torsion cycles in a variety has to be constant, hence the subvariety $q(V_{X,\beta}) \subset X$ is a constant cycle subvariety.

If X is a smooth Calabi–Yau hypersurface in Y, and furthermore there is no obstruction for deformation of curves in Y, then denoting $d = \deg(C \cap X)$ we have

$$\begin{aligned} \dim V_{X,\beta} \geq \dim M_\beta - (d-1) &= (-K_Y \cdot \beta + (n+1) - 3) - (d-1) \\ &= (X \cdot C + \dim X - 2) - d + 1 = \dim X - 1, \end{aligned}$$

and we expect that $\overline{q(V_{X,\beta})}$ is a CCD on X.

Notation 2.3. Continuing with the settings of Construction 2.2, we put

$$H_\beta = \overline{q(V_{X,\beta})} \subset X.$$

Lemma 2.4. *Let Y be a projective homogeneous variety with Picard number one and dimension at least two and let β be the class of a line. If $X \in |-K_Y|$ is general, then H_β is a non-empty effective divisor on X.*

Proof. First of all, H_β cannot be the whole of X, because it would imply that all points in X are rationally equivalent. In order to show that H_β is a non-empty effective divisor, it thus suffices to show that its class is non-zero.

We denote the class of the standard polarization on Y by H_1 and notation as in (3.2), let $\tilde{H}_1 = q^*H_1$ be the corresponding divisor on the universal curve. We also put $H_2 := p_*(\tilde{H}_1^2) = p_*q^*H_1^2$, let $\tilde{H}_2 = p^*H_2$. Denote by K_{rel} the class of the relative cotangent bundle on the universal curve, it is easy to see that $K_{rel} = -2\tilde{H}_1 + \tilde{H}_2'$, where $\tilde{H}_2' = p^*H_2'$ for some divisor H_2' on M_β.

Let us show that $H_2 = H_2'$. By the Grothendieck–Riemann–Roch formula,

$$c_1\left(R^0p_*\mathcal{O}_{\mathcal{C}}(\tilde{H}_1)\right) = p_*\left(\tilde{H}_1^2 + \frac{5}{6}\left(\tilde{H}_1^2 - \tilde{H}_1\tilde{H}_2'\right) + \frac{1}{12}\tilde{H}_2^2\right),$$

but the left-hand side can be calculated geometrically: it is the locus of all curves C in M_β such that the restrictions of two general sections in $H^0(Y, \mathcal{O}_Y(H_1))$ to C coincide. Hence the left-hand side is equal to $p_*q^*(H_1^2) = p_*\left(\tilde{H}_1^2\right)$. Since $p_*\left(\tilde{H}_2^2\right) = 0$, we get

$$p_*\left(\frac{5}{6}\left(\tilde{H}_1^2 - \tilde{H}_1\tilde{H}_2'\right)\right) = 0.$$

Since $\tilde{H}_1$ has degree one on fibers of $p : \mathcal{C} \to M_\beta$, we get $p_*\left(\tilde{H}_1\tilde{H}_2'\right) = H_2'$ and

$$H_2' = p_*\left(\tilde{H}_1\tilde{H}_2'\right) = p_*\left(\tilde{H}_1^2\right) = H_2.$$

The subvariety $V_{X,\beta} \subset \mathcal{C}$ is the intersection of d divisors of classes ($0 \le r \le d-1$)

$$d\tilde{H}_1 + rK_{rel},$$

where the divisor number r is to say that the $r-$th derivative of the restriction $f_X|_C$ is zero at a point on the curve C, where f_X is the defining equation for X; all together they mean that $f_X|_C$ is one point with the maximal multiplicity. So, the degree of $H_\beta = q_*(V_{X,\beta})$ can be calculated as the degree of the following intersection on Y (we recall $\dim X = n$, $\dim Y = n+1$):

$$\begin{aligned}
H_1^{n-1} \cdot q_*\left[d\tilde{H}_1 \cdot \left(d\tilde{H}_1 + K_{rel}\right) \cdot \ldots \cdot \left(d\tilde{H}_1 + (d-1)K_{rel}\right)\right] \\
= q_*\left[\tilde{H}_1^{n-1} \cdot d\tilde{H}_1 \cdot \left(d\tilde{H}_1 + K_{rel}\right) \cdot \ldots \cdot \left(d\tilde{H}_1 + (d-1)K_{rel}\right)\right] \\
= q_*\left[d\tilde{H}_1^n \prod_{r=1}^{d-1}\left(d\tilde{H}_1 + r\left(-2\tilde{H}_1 + \tilde{H}_2\right)\right)\right] \\
= q_*\left[d!\left(\tilde{H}_1^n\tilde{H}_2^{d-1} + \sum_{r=1}^{d-1}\left(\frac{d-2r}{r}\right)\tilde{H}_1^{n+1}\tilde{H}_2^{d-2}\right)\right].
\end{aligned}$$

Clearly, points of $p_*(\tilde{H}_1^{n+1}) = p_*q^*(H_1^{n+1})$ in M_β correspond to curves which pass through a point of the intersection H_1^{n+1} in Y. Since H_2 is ample on M_β, the intersection $\tilde{H}_1^{n+1}\tilde{H}_2^{d-2}$ is not empty. To finish the proof, we note that the coefficient of $\tilde{H}_1^{n+1}\tilde{H}_2^{d-2}$, which is the sum $\sum_{r=1}^{d-1}(d/r-2)$, is strictly positive if $d > 2$. □

Lemma 2.5. *Let Y be a homogeneous variety and $X \in |-K_Y|$ be general. If $\beta \in \mathcal{R}$ such that $f_\beta : Y \to Y_\beta$ is not a $\mathbb{P}^1-$fibration, then H_β is a non-empty effective CCD on X.*

Proof. It is enough to show that $V_{X,\beta}$ is non-empty of the expected dimension and a general fiber of $q : V_{X,\beta} \to q(V_{X,\beta}) \subset H_\beta$ is finite.

A general fiber Y_0 of $f_\beta : Y \to Y_\beta$ is a Schubert variety in Y, hence the restriction map $H^0(Y, \mathcal{O}_Y(X)) \to H^0(Y_0, \mathcal{O}_Y(X)|_{Y_0})$ is surjective ([9, Section 3] or [8, Theorem 2.3.1]). Applying the previous lemma to Y_0 and $X|_{Y_0}$ we see that $V_{X,\beta}$ is non-empty and the projection q has some finite fibers, hence a general fiber is finite. The dimension of $V_{X,\beta}$ is expected because the dimension of M_β is expected. □

Example 2.6. Let $Y = \mathbb{P}^2 \times S$, where S is a Fano variety of positive dimension and let β be equal to $[l \times \text{point}]$, where l is a line in $\mathbb{P}^2$. If X is a general Calabi–Yau hypersurface in Y then the restriction of H_β to a fiber of $\mathbb{P}^2 \times S \to S$ is the union of nine Weierstrass points on a plane elliptic curve. In particular, H_β is non-empty and effective.

Example 2.7. Let Y be as in Construction 2.2 and assume that

$$f_\beta : Y \longrightarrow Y_\beta$$

is a $\mathbb{P}^1-$fibration (again β is the class of contracted curves, i.e, the class of a fiber). If X is a general Calabi–Yau hypersurface then H_β has class $(-K_Y + K_{Y/Y_\beta})|_X = f_\beta^*(-K_{Y_\beta})|_X$ in $\mathrm{Pic}(X)$ and H_β is trivial on fibers of f_β (in contrast to Lemma 2.5 and Example 2.6).

Example 1.28 from Introduction is another example of our construction.

2.4 The proof of Theorem 2.1

Let us first prove the following.

Proposition 2.8. *In the setting of Theorem 2.1, there exists a class c_X in $\mathrm{CH}_0(X)$ such that the restriction $C|_X$ of any curve in Y is proportional to c_X in $\mathrm{CH}_0(X)$.*

Proof. Because of the linearity of the intersection, it is enough to prove the statement only for curves with class $\beta \in \mathcal{R}$. By construction of H_β, the corresponding zero-cycles can be represented by points (with multiplicities) on H_β for $\beta \in \mathcal{R}$. Let us show that the class c_{H_β} does not depend on the choice of $\beta \in \mathcal{R}$.

Case 1: $\dim Y_\beta < n$ *for any* $\beta \in \mathcal{R}$. We may assume that the dimension of X is at least three and thus by the Grothendieck–Leftschetz theorem, we can identify $\mathrm{Pic}(X)$ and $\mathrm{Pic}(Y)$ by the restriction map. Let H_β be the restriction of a divisor $\tilde{H}_\beta$ on Y. We claim that the divisor

$$\tilde{H} := \sum_{\beta \in \mathcal{R}} \tilde{H}_\beta$$

is ample on Y. This is equivalent to $\tilde{H} \cdot \beta > 0$ for all $\beta \in \mathcal{R}$. To see this, it suffices to prove the inequalities

$$\tilde{H}_\beta|_{\beta'} \geq 0 \quad \text{and} \quad \tilde{H}_\beta|_\beta > 0$$

for any $\beta, \beta' \in \mathcal{R}$. Since the fibers of $f_{\beta'} : X \to Y_{\beta'}$ have positive dimension, any curve C contained in a fiber $F \subset X$ of $X \to Y_{\beta'}$ has its class in Y a non-zero multiple $N\beta'$. For smooth F, the curve C can be chosen to be movable in F and thus we get

$$N\tilde{H}_\beta \cdot \beta' = H_\beta \cdot C = H_\beta|_F \cdot C \geq 0,$$

because $H_\beta|_F$ is effective in F. The second inequality follows from Lemma 2.4. Thus the claim is proved.

The divisor $H := \tilde{H}|_X$ is thus ample on X, hence it is connected and any point on the divisors H_β represents the same class $c_X = c_{H_\beta}$ in $\mathrm{CH}_0(X)$.

Case 2: $\dim Y_\beta = n$ *for some* $\beta \in \mathcal{R}$. In this case, $f_\beta|_X : X \to Y_\beta$ is a double cover (out of some codimension two subvariety, where X contains a whole fiber of f_β). Due to Example 2.7, the divisor $H := H_\beta$ is a pull-back of an ample divisor on Y_β, so H meets $H_{\beta'}$ for any β'. It thus follows that $c_{H_{\beta'}} = c_{H_\beta}$ for any β' and we can take c_X as the class of a point on H. □

Note that in a situation where Y is not homogeneous, but still satisfies assumption $(\star)$ the following alternative lemma could be used.

Lemma 2.9. *If there exist an ample divisor $\tilde{H}$ on Y such that $H = \tilde{H} \cap X$ is a CCD on X with associated class c_H, then for any curve C on Y, $C|_X$ is proportional to c_H and, in particular, the intersection of any n divisors is proportional to c_H.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} \tilde{H}^{n-1} : \mathrm{CH}^1(Y) & \longrightarrow & \mathrm{CH}^n(Y) \\ \downarrow & & \downarrow \\ H^2(Y) & \longrightarrow & H^{2n}(Y) \end{array}$$

The vertical arrows are isomorphisms due to condition $(\star)$, the bottom arrow is also an isomorphism by the hard Lefschetz theorem, hence the top arrow is an isomorphism. Therefore the restriction $C|_X$ is equal to $(L \cdot \tilde{H}^{n-1})|_X$ for some divisor L on Y. But the last restriction can be represented by a zero-cycle with the support on H, hence is proportional to c_H. □

Proof of Theorem 2.1. The divisor H constructed in the proof of Lemma 2.8 does not intersect a constant cycle subvariety Z only if Z is a rational curve contracted by f_β and $\dim Y_\beta = n$. But in this case the class of a point on Z is proportional to $\beta|_X$ and hence is proportional to c_X. Let $\tilde{D}_i$ be a divisor on Y such that $\tilde{D}_i|_X = D_i$ in $\mathrm{Pic}(X)$; we then have in $\mathrm{CH}_0(X)$

$$D_1 \cdot \ldots \cdot D_n = \left(\tilde{D}_1 \cdot \ldots \cdot \tilde{D}_n\right)\Big|_X .$$

The intersection in the right-hand side is a (reducible) curve in Y, we proved that the restriction of any curve in Y to X is proportional to c_X in Proposition 2.8.

The top Chern class of X can be represented by a combination of Chern classes of Y and Chern classes of line bundle $\mathcal{O}_Y(X)$. This is again a restriction of some (reducible) curve on Y. □

Remark 2.10. There is another strategy to prove Theorem 2.1 b. If $\dim Y_\beta < n$ for some β then

$$D_1 \cdot \ldots \cdot D_n = 0$$

for any divisors D_i on Y_β. The same is true for divisors $f_\beta^*(D_i)|_X$ on X. Since H_β and $f_\beta^*(\mathrm{Pic}(Y_\beta))|_X$ generate $\mathrm{Pic}(X) \otimes \mathbb{Q}$, any intersection of n divisors on X is proportional to a zero-cycle with support on H_β and hence is proportional to c_X.

Chapter 3

A new proof of decomposition of a small diagonal for K3 surface

3.1 Introduction

The canonical zero cycle on a K3 surface S is defined in [5] as the rational equivalence class of any point lying on a rational curve $C \subset S$. The paper [5] shows that the intersection of any two divisors in S is proportional to the canonical cycle in $\mathrm{CH}_0(S)$. It is also shown that the second Chern class $c_2(S)$ is proportional to this canonical zero cycle o. Both results can be obtained as consequences of the following theorem.

Theorem 3.1. *(= Proposition 1.15, [5, Proposition 4.2]) Let S be a K3 surface. In $\mathrm{CH}_2(S^3)_{\mathbb{Q}}$ there is a decomposition*

$$\Delta_{123} = \Delta_{12} \times o_3 + \Delta_{23} \times o_1 + \Delta_{13} \times o_2 \\ - S \times o \times o - o \times S \times o - o \times o \times S, \quad (3.1)$$

where o is the canonical zero cycle, Δ_{123} is a small diagonal in S^3, and the notation $\Delta_{ij} \times o_k$ stands for $\pi_{ij}^*(\Delta) \cdot \pi_k^* o$.

The goal of this paper is to give a new proof of Theorem 3.1 for a K3 surface S with $\mathrm{Pic}(S) = \mathbb{Z}[L]$ with $L^2 = 2g-2$. The result can be generalised for K3 surfaces with higher Picard rank using the specialisation argument (cf. Remark 3.7). Our proof is very explicit using the embedding of S in $\mathbb{P}^g$. It is based on the study of the set of pairs of points (x, y) in $S \times S$ such that two curves in the linear system $|L|$ intersect exactly at these two points with given multiplicities. Specifically, we choose the multiplicities $2g-3$ and 1. In other words, we are studying the surface Σ parameterising complete intersections

subschemes of S consisting in the union of two points, one of them with multiplicity $2g-3$. We will prove that this is a surface and will establish two relations (3.6) and (3.7), from which we obtain the relation (3.1) up to some multiplicative factor μ, which is non-zero if the surface $\Sigma \subset S \times S$ dominates factors. The second part of the paper is then devoted to the proof that $\mu \neq 0$. In order to prove this non-vanishing we will interpret the surface Σ in a slightly different way: as $\mathrm{Pic}(S) = \mathbb{Z}[L]$, the curves in $|L|$ are irreducible hence the intersection of any two different curves in the linear system $|L|$ is a zero-dimensional subscheme of S of length $\deg(L)$, so we have a morphism $Gr(2, H^0(S,L)) \to S^{[2g-2]}$ and we let Gr denote the image. Using techniques from [10, 20, 21, 35] to work with cohomology groups of the Hilbert scheme $S^{[2g-2]}$, one can define the pieces $E^*_M(Gr) \in \mathrm{CH}(S^m)$ of the decomposition of the class of Gr in $\mathrm{CH}(S^{[2g-2]})$, where M is a partition of $\{1, \ldots, 2g-2\}$ and $m = |M|$. The proof that $\mu \neq 0$ involves the study of this class $E^*_M(Gr)$ in the case where M is a partition into two integers.

3.2 The proof of Theorem 3.1

1 Surface Σ.

Let us recall that a K3 surface with a very ample linear system L of degree $2g-2 \geq 4$ generating $\mathrm{Pic}(S)$ can be embedded in $\mathbb{P}^g$ by $|L|$ and the intersection of S with any linear subspace $\mathbb{P}^{g-2} \subset \mathbb{P}^g$ is a zero-cycle of degree $2g-2$ on S (cf. [16, Chapter 2]). The set of all $P = \mathbb{P}^{g-2} \subset \mathbb{P}^g$ is the Grassmann variety $Gr(g-1, g+1)$ and as already mentioned this provides a morphism

$$Gr(g-1, g+1) \to S^{[2g-2]},$$

which maps $[P]$ to $P \cap S$ for $[P] \in Gr(g-1, g+1)$. We denote by Gr the image of this map. We introduce the incidence scheme $\Xi \subset S \times S^{[2g-2]}$:

$$\begin{array}{ccc} \Xi & \xrightarrow{\pi_2} & S^{[2g-2]} \\ \downarrow{\scriptstyle \pi_1} & & \\ S & & \end{array} \tag{3.2}$$

Let $\mathcal{L}^{[2g-2]}$ denote the vector bundle $(\pi_2)_* \pi_1^*(\mathcal{O}_S(L))$ on $S^{[2g-2]}$. The image Gr is the locus in $S^{[2g-2]}$, where the rank of the map

$$H^0(S,L) \otimes \mathcal{O}_{S^{[2g-2]}} \to \mathcal{L}^{[2g-2]}$$

is at most $g-1$ and it follows by [14] that its class is given by

$$[Gr] = c^2_{g-1}(\mathcal{L}^{[2g-2]}) - c_{g-2}(\mathcal{L}^{[2g-2]}) c_g(\mathcal{L}^{[2g-2]})). \tag{3.3}$$

Let us also define a subset $G_0' \subset Gr(g-1, g+1)$ in the following way:

$$G_0' = \{[P] \in Gr(g-1, g+1) : P \cap S = (2g-3)p_1 + p_2 \text{ as a cycle}\}.$$

We also define the variety $\Sigma_0 \subset S \times S$ as

$$\Sigma_0 := \{(p_1, p_2) \in S \times S : (2g-3)p_1 + p_2 = S \cap P \\ \text{for some } [P] \in Gr(g-1, g+1)\}.$$

If we construct the following diagram

$$\begin{array}{ccc} G_0'' \subset E_{2g-3,1} & \xrightarrow{q} & S^{[2g-2]} \\ \Big\downarrow p & & \\ \Sigma_0 \subset S \times S & & \end{array} \tag{3.4}$$

where $E_{2g-3,1}$ represents schemes of the form $(2g-3)p_1+p_2$ on S, we then have $G_0'' = q^{-1}(Gr)$ and $\Sigma_0 = p(q^{-1}(Gr))$. We expect that the natural morphism $G_0' \to G_0''$ is an isomorphism, but since we will not use it, we omit a proof. Let us prove the following facts about the geometry of Σ_0.

Lemma 3.2. *The following holds:*

a. Σ_0 *is a non-empty surface, possibly reducible,*

b. there is a component $\Sigma \subset \Sigma_0$ *such that* Σ *dominates both factors of* $S \times S$.

Proof. Let us first prove that $\dim \Sigma_0 \leq 2$. Indeed, from the equality

$$p_1 + (2g-3)p_2 = L^2,$$

valid in $\mathrm{CH}^2(S)$ for any pair $(p_1, p_2) \in \Sigma_0$, and the theorem of Mumford [24], it follows that

$$\pi_1^* \sigma_S + (2g-3)\pi_2^* \sigma_S = 0 \tag{3.5}$$

on Σ_0, where σ_S is a non-vanishing 2-form on S. Therefore $\dim \Sigma_0 \leq 2$.

Notice that equation (3.5) characterises K-correspondences in the terminology of [34]. This equation implies that for any irreducible component Σ of Σ_0 the morphism $\pi_1|_\Sigma$ is dominant if and only if the morphism $\pi_2|_\Sigma$ is dominant. Indeed, these conditions are respectively equivalent to the generic non-vanishing of $\pi_i^* \sigma_S$. This argument also shows that (a) and (b) are implied by the fact that the first projection $\pi_1|_{\Sigma_0}$ is dominant. In order to prove this last statement we observe that the cycle $\Omega = p_* q^*(Gr)$ has for support

the surface Σ_0 although this cycle could be non-effective due to the fact that even if Σ_0 has the right dimension the scheme $G_0'' = q^{-1}(Gr)$ could be of a higher dimension leading to excess formulas in the computation of the cycle Ω. Nevertheless we can argue that if Ω can not be represented by a cycle supported on the union of divisors of the form $D \times S$, then one component of the support $\operatorname{supp}\Omega$ has to dominate S by the first projection, that is, one component of Σ_0 dominates S by the first projection. The next section is devoted to the proof that the class Ω can not be supported on the union of $D \times S$, see Proposition 3.10. □

Remark 3.3. Let us note that we expect $\dim G_0' = 2$, Σ_0 is irreducible and the projection $p : G_0' \to \Sigma_0$ is a one-to-one correspondence. In this case we get $[\Sigma_0] = \Omega$ in $H^4(S \times S)$ (actually, we can consider the equality even in $\mathrm{CH}^2(S \times S)$). Unfortunately, the author does not know how to prove these facts and we avoid them in our proof by introducing below a surface G' as a substitute of G_0'.

Let Σ be a surface as in the last lemma and let $G' \subset G_0'$ be any surface dominating Σ after the projection $\pi_1 : G_0' \to \Sigma_0$. We can consider G' as a subvariety of $Gr(g-1, g+1)$ and define $\Pi_2 \subset \mathbb{P}^g \times \mathbb{P}^g$ and $\Pi_3 \subset \mathbb{P}^g \times \mathbb{P}^g \times \mathbb{P}^g$ as the universal varieties:

$$\Pi_2 := \{(x_1, x_2) \in \mathbb{P}^g \times \mathbb{P}^g : x_1, x_2 \in P \text{ for some } [P] \in G'\},$$

$$\Pi_3 := \{(x_1, x_2, x_3) \in \mathbb{P}^g \times \mathbb{P}^g \times \mathbb{P}^g : x_1, x_2, x_3 \in P \text{ for some } [P] \in G'\}.$$

We have $\dim \Pi_2 = \dim G' + 2(g-2) = 2g-2$ and $\dim \Pi_3 = \dim G' + 3(g-2) = 3g-4$. Clearly, $\Sigma \subset \Pi_2 \cap (S \times S)$.

2 Key lemma

A key observation for our proof is the following lemma.

Lemma 3.4. *a. There is a decomposition in* $\mathrm{CH}_2(S \times S)$ *which, in fact, is an equality of effective cycles*

$$\Pi_2|_{S\times S} = \alpha\Delta + \beta\left(\Sigma + \Sigma^T\right), \tag{3.6}$$

where Δ *is the diagonal in* $S \times S$.

b. There is a decomposition in $\mathrm{CH}_2(S \times S \times S)$

$$\Pi_3|_{S\times S\times S} = \gamma\Delta_{123} + \varepsilon(\delta_{12*} + \delta_{23*} + \delta_{31*})(\Sigma + \Sigma^T), \tag{3.7}$$

where Δ_{123} *is the small diagonal in* S^3, *and* $\delta_{12}(x, y) = (x, x, y)$, $\delta_{23}(x, y) = (y, x, x)$, $\delta_{31}(x, y) = (x, y, x)$.

Proof. The proof of (3.6) follows from the facts that $\Pi_2|_{S\times S}$ is symmetric and supported on the union of the diagonal, Σ and Σ^T, and that Σ and Σ^T are chosen to be irreducible. The proof of (3.7) is similar. □

Lemma 3.5. *We have:*

a. *the denominators of the ratios* $\frac{\alpha}{\beta}$ *and* $\frac{\gamma}{\varepsilon}$ *are non-zero and both ratios are non-negative,*

b. *the following relation holds:*

$$\frac{\gamma}{\varepsilon} - \frac{3\alpha}{\beta} = -\left(\frac{\alpha}{\beta} + a + b\right), \tag{3.8}$$

where a *and* b *are the degrees of the projections of* $\Sigma \subset S \times S$ *to its factors.*

Proof. a. As numbers $\alpha, \beta, \gamma, \varepsilon$ are non-negative, we need only to show that $\beta, \varepsilon \neq 0$. We will use the fact that S has some transcendental cohomology. Let us consider the actions on $H^*(S)$ of $\Pi_2|_{S\times S}$ and of Δ viewed as correspondences between S and S. Since $\Pi_2|_{S\times S}$ can be supported on divisors, it acts as zero on the transcendental cohomology of S. Since Δ always acts as identity, it is not proportional to $\Pi_2|_{S\times S}$. It implies $\beta \neq 0$. Using a projection $S \times S \times S \to S \times S$, the fact $\varepsilon > 0$ can be proved in a similar way.

b. Projecting (3.7) to $S \times S$ and taking cohomology classes, we easily conclude that $\frac{\gamma}{\varepsilon} = \frac{2\alpha}{\beta} - a - b$, which is equivalent to (3.8).

□

Proof of Theorem 3.1. We chose a surface Σ as in Lemma 3.2. Due to [5, Proposition 2.6], $\delta_{ij*}(\Pi_2|_{S\times S})$ can be represented by a sum of $Z'|_{S^3}$ and $o_k \times \Delta_{ij}$. We also recall that $\delta_{ij*}(\Delta) = \Delta_{123}$. So, putting (3.6) and (3.7) together, we get a decomposition of the small diagonal:

$$\left(\frac{\gamma}{\varepsilon} - \frac{3\alpha}{\beta}\right)\Delta_{123} = \alpha_1\Delta_{12} \times o_3 + \alpha_2\Delta_{23} \times o_1 + \alpha_3\Delta_{13} \times o_2 + Z|_{S^3}, \tag{3.9}$$

where $Z \subset \mathbb{P}^g \times \mathbb{P}^g \times \mathbb{P}^g$.

Projecting to $S \times S$ and taking the cohomology classes, we easily conclude that $\alpha_1 = \alpha_2 = \alpha_3 = \gamma/\varepsilon - 3\alpha/\beta$, and by previous lemma, $\alpha_1 = -(\alpha/\beta + a + b)$. Since the decomposition of the small diagonal holds in cohomology (due to [5] and more generally [29]), we can deal with the term $Z|_{S^3}$ as follows:

this term is a polynomial in L_1, L_2, L_3, where $L_i := pr_i^* L$ and on the other hand it is cohomologous to

$$-\alpha_1(S \times o \times o + o \times S \times o + o \times o \times S).$$

By [5] it is thus rationally equivalent to $-\alpha_1(S \times o \times o + o \times S \times o + o \times o \times S)$. Since $a, b > 0$ by choice of Σ and α/β is non-negative, we can divide the equation by $-\alpha/\beta - a - b$ to get the result. The theorem is proved. □

Remark 3.6. We would like to emphasise that this proof is very different from the one used by Beauville and Voisin, which uses the existence of one-parameters families of elliptic curves. It is much more along the lines of the method used by Voisin in the Calabi–Yau hypersurface case, and Fu in the Calabi-Yau complete intersection case (see [36, Theorem 3.1], [12]). Namely, to study the case of Calabi-Yau varieties, one needs to replace G' by the set of lines intersecting the hypersurface in two points, the result [5, Proposition 2.6] used in our proof, corresponds to [36, Lemma 3.3].

Remark 3.7. The Picard number one condition can be removed using the following specialization argument. For a given K3 surface S, consider a family of projective K3 surfaces such that the general member is of Picard number one. The expression of decomposition of the small diagonal provides a (universal) relative 2-cycle of this family and by the proof above we know that it vanishes on general fibers. Since the specialization of the diagonal (resp. small diagonal) is the diagonal (resp. small diagonal), we can conclude that the decomposition holds in the special fiber.

Remark 3.8. As proved in [5], the decomposition of the small diagonal immediately gives the fact that $c_2(S)$ is proportional to the canonical cycle o. From our proof we can easily get another more direct proof of this fact, using only (3.6). Indeed, let us apply $\Delta^* : \mathrm{CH}_2(S \times S) \to \mathrm{CH}_0(S)$ to the decomposition (3.6). Since $\Delta^*(\Delta) = c_2(S)$, we get that $\alpha c_2(S)$ is a combination of a canonical zero cycle (corresponding to the term $\Delta^*(\Pi_2|_{S\times S})$) and zero cycle $\Delta^*(\Sigma + \Sigma^T)$ which is supported on $\Sigma \cap \Delta$ and $\Sigma^T \cap \Delta$. But clearly the points on $\Sigma \cap \Delta$ are rationally equivalent to o. This proves the statement concerning c_2, once we prove that $\alpha \neq 0$, which can be derived from the following remark or from the proofs of Lemma 3.2 and of Proposition 3.10 in the next section.

Remark 3.9. Let us present a relation between α/β, a, and b. We see from the definition of Σ that for any $(x, y) \in \Sigma$, we have the equality

$$(2g - 3)x + y = L^2 \text{ in } \mathrm{CH}_0(S).$$

It follows that we have for any $x \in S$

$$\Sigma_*(x) = -a(2g-3)x + C \text{ in } \mathrm{CH}_0(S),$$

where C is a constant multiple of L^2. Similarly

$$\Sigma_*^T(x) = -\frac{b}{2g-3}x + C' \text{ in } \mathrm{CH}_0(S).$$

Applying (3.6) and the fact that Π_2 is restricted from $\mathbb{P}^g \times \mathbb{P}^g$, we thus conclude that for any $x \in S$

$$C'' = -a(2g-3)x - \frac{b}{2g-3}x + \frac{\alpha}{\beta}x \text{ in } \mathrm{CH}_0(S),$$

where C'' is a constant multiple of L^2. It follows that

$$\alpha/\beta = a(2g-3) + \frac{b}{2g-3}.$$

3 Proof of the fact that $\mathrm{supp}\,(\Omega)$ dominates factors

The goal of this section is to prove the following lemma.

Proposition 3.10. *The class $\Omega = E^*_{2g-3,1}(Gr)$ in $H^*(S \times S)$ cannot be represented by a cycle supported on the union of divisors of the form $D_i \times S$ and $S \times D_j$, and hence its support has non-trivial projections to both factors of $S \times S$.*

To prove Proposition 3.10 we study $H^*(S^{[2g-2]})$ and introduce the following notation. Let

$$M = (m_1, m_2, \ldots, m_k)$$

be a partition of $\{1, \ldots, 2g-2\}$. Such a partition determines a partial diagonal

$$S_M \cong S^k \subset S^{2g-2},$$

defined by the conditions

$$x = (x_1, \ldots, x_{2g-2}) \in S_M \iff x_i = x_j \text{ if } i, j \in m_l, \text{for some } l.$$

Consider the quotient map

$$q_M : S^k \cong S_M \to S^{(2g-2)},$$

and denote by E_M the following fibered product:

$$E_M := S_M \times_{S^{(2g-2)}} S^{[2g-2]} \subset S^k \times S^{[2g-2]}.$$

We view E_M as a correspondence between S^k and $S^{[2g-2]}$ and we will denote by $E_M^* : \mathrm{CH}(S^{[2g-2]}) \to \mathrm{CH}(S^m)$ the map

$$\alpha \to \pi_{1*}(\pi_2^*(\alpha) \cdot E_M).$$

The main point of the proofs is considering $E_M^*(Gr)$ for the partition $M = (\{1, \ldots, 2g-3\}, \{2g-2\})$ and the intersection

$$\pi_{2*}(E_M) \cdot (c_{g-1}^2(\mathcal{L}^{[2g-2]}) - c_g(\mathcal{L}^{[2g-2]}) c_{g-2}(\mathcal{L}^{[2g-2]}))$$

We now turn our attention to the cup product on the Hilbert scheme $S^{[2g-2]}$.

4 Cup product on $S^{[n]}$

The paper [21] gives a description on the ring structure on $H^*(S^{[n]})$; the following theorem holds (cf. [21, Theorem 3.2]):

Theorem 3.11. *Let S be a smooth projective surface with numerically trivial canonical class. Then there is a canonical isomorphism of graded rings*

$$(H^*(S; \mathbb{Q})[2])^{[n]} \to H^*(S^{[n]}; \mathbb{Q})[2n].$$

In the theorem above we define $A^{[n]}$ as

$$A^{[n]} := (A\{S_n\})^{S_n}.$$

It is the subspace of invariants of the ring $A\{S_n\}$, which has the following grading by permutations in S_n

$$A\{S_n\} := \oplus_{\pi \in S_n} A^{\otimes(\pi)\backslash[n]} \cdot \pi,$$

where $(\pi)\backslash[n]$ denotes the orbit space for the action on $\{1, 2, \ldots, n\}$ of the group generated by permutation π. The action of σ on $\{1, 2, \ldots, n\}$ induces a bijection:

$$\sigma : (\pi)\backslash[n] \to (\sigma\pi\sigma^{-1})\backslash[n], \ x \mapsto \sigma x$$

for each π and hence an isomorphism of $A\{S_n\}$:

$$\sigma : A\{S_n\} \to A\{S_n\}, \ a\pi \mapsto \sigma^*(a)\sigma\pi\sigma^{-1}.$$

To describe $c_i(\mathcal{O}^{[2g-2]})$ in these terms, let us introduce the following notation. If $\sigma \in S_n$ is a permutation, then let $c(\sigma)$ be the number of cycles in

σ and $l(\sigma) = n - c(\sigma)$. The number $l(\sigma)$ is the minimal number of transpositions needed to generate σ.

The statement [21, Proposition 4.3] (see also [20]) gives

$$c_i(\mathcal{O}^{[2g-2]}) = \epsilon_i, \text{ where } \epsilon_i := (-1)^i \sum_{l(\sigma)=i} \sigma \in H^*(S)^{[2g-2]}. \tag{3.10}$$

The class of E_M for $M = (\{1, \dots, 2g-3\}, \{2g-2\})$ is proportional to the sum of all permutations which contains one cycle of length $2g-3$ (cf. [21, 2.10]).

5 Two technical lemmas and end of the proof

Before we start the proof of Proposition 3.10, we would like to state two lemmas about transpositions. Let us enumerate all transpositions in S_{2g-3} by $s_1, s_2, \dots s_{(2g-3)(g-2)}$ in such a way that

$$l(s_1 \cdot \ldots \cdot s_{2g-4}) = 2g - 4$$

and define $A(k)$ as the set of all permutations $\sigma \in S_{2g-3}$ such that $l(\sigma s_i) > l(\sigma)$ for any $i \leq k$. We note that for $\sigma \in A(k)$ one has $\sigma \in A(k+1)$ or $\sigma = \sigma' s_{k+1}$ for some $\sigma' \in A(k+1)$ with $l(\sigma') = l(\sigma) - 1$.

Let us define set of pairs $\mathcal{F}_k(i, j, \tau)$:

$$\{(\sigma_1, \sigma_2) \in A(k) \times A(k) : l(\sigma_1) = i, l(\sigma_2) = j, l(\sigma_1\sigma_2\tau) = i + j + l(\tau)\}.$$

And let $F_k(i, j, \tau)$ be the number of elements in $\mathcal{F}_k(i, j, \tau)$.

Lemma 3.12. *If $i + 1 < j$, one has*

$$F_k(i, j, \tau) \leq F_k(i + 1, j - 1, \tau). \tag{3.11}$$

and the inequality is strict in the case $i = g - 3$, $j = g - 1$, $k = 0$, $\tau = id$.

Proof. The set $\mathcal{F}_k(i, j, \tau)$ can be divided in four subsets:

a. pairs (σ_1, σ_2) such that $\sigma_1, \sigma_2 \in A(k+1)$. This subset coincides with $\mathcal{F}_{k+1}(i, j, \tau)$.

b. pairs (σ_1, σ_2) such that $\sigma_1 \in A(k+1)$ and $\sigma_2 \notin A(k+1)$, so $\sigma_2 = \sigma_2' s_{k+1}$ with $l(\sigma_2') = j - 1$. This subset is in bijection with pairs (σ_1, σ_2') of $\mathcal{F}_{k+1}(i, j-1, s_{k+1}\tau)$.

c. pairs (σ_1, σ_2) such that $\sigma_1 \notin A(k+1)$ and $\sigma_2 \in A(k+1)$, so $\sigma_1 = \sigma_1' s_{k+1}$ with $l(\sigma_1') = i-1$. This subset is in bijection with pairs $(\sigma_1', s_{k+1}\sigma_2 s_{k+1})$ of $\mathcal{F}_{k+1}(i-1, j, s_{k+1}\tau)$.

d. pairs (σ_1, σ_2) such that $\sigma_1, \sigma_2 \notin A(k+1)$, so $\sigma_1 = \sigma_1' s_{k+1}$ and $\sigma_2 = s_{k+1}\sigma_2'$ for some $\sigma_1', \sigma_2' \in A(k+1)$. Since

$$l(\sigma_1\sigma_2\tau) = l(\sigma_1' s_{k+1}^2 \sigma_2'\tau) = l(\sigma_1'\sigma_2'\tau) \leq l(\sigma_1) - 1 + l(\sigma_2) - 1 + l(\tau),$$

this subset is empty.

So we have

$$F_k(i,j,\tau) = F_{k+1}(i,j,\tau) + F_{k+1}(i-1,j,s_{k+1}\tau) + F_{k+1}(i,j-1,s_{k+1}\tau). \quad (3.12)$$

The proof easily follows by induction if we prove that

$$F_k(0,j,\tau) \leq F_k(1,j-1,\tau)$$

for all j,k and all τ. Inequality follows from the fact that any permutation σ_2, where $(id,\sigma_2) \in \mathcal{F}_k(0,j,\tau)$, has (more than one) decomposition $\sigma_1'\sigma_2'$, where $(\sigma_1',\sigma_2') \in \mathcal{F}_k(1,j-1,\tau)$. Different σ_2 have different decompositions. We note that for

$$\sigma_2 = s_{g-1}\dots s_{2g-4},\ \tau = (s_1\dots s_{g-2})^{-1}$$

the pair (id,σ_2) will be an element of $F_{g-2}(0,g-2,\tau)$ and therefore the inequality is strict. □

Let $\mathcal{G}(i,j)$ be the set of all pairs $(\sigma_1,\sigma_2) \in S_{2g-2} \times S_{2g-2}$ such that the following three conditions hold:

a. $l(\sigma_1) + l(\sigma_2) - 2 = l(\sigma_1\sigma_2)$,

b. $\sigma_1\sigma_2$ preserves the point $2g-2$: $(\sigma_1\sigma_2)(2g-2) = 2g-2$,

c. σ_1 and σ_2 do not preserve the point $2g-2$: $\sigma_{1,2}(2g-2) \neq 2g-2$.

Let $G(i,j)$ be the number of elements in $\mathcal{G}(i,j)$.

Lemma 3.13. *One has*

$$G(g-1,g-1) > G(g-2,g).$$

Proof. Let us define a map of sets $f : S_n \to S_{n-1}$ in the following way: if $\sigma' \in S_n$ has a cycle $(\dots,k,n,m,\dots)$ we replace it by a cycles $(\dots,k,m,\dots)$. Clearly, if $\sigma'(n) \neq n$ than $l(\sigma) = l(\sigma') - 1$ for $\sigma = f(\sigma')$.

We claim that $f(\mathcal{G}(i,j))$ is $(2g-3)-$fold covering of $\mathcal{F}_0(i-1,j-1,id)$, where $\mathcal{F}_0(i-1,j-1,id)$ was defined before Lemma 3.12. Indeed, let $(\sigma_1,\sigma_2) \in \mathcal{F}_0(i-1,j-1,id)$ and $1 \leq k \leq 2g-3$. If σ_1 contains a cycles $(...,k,m_1,...)$, then let σ_1' be the same permutation, but with the cycle $(...k,2g-2,m_1,...)$. Similarly, if σ_2 contains cycles $(...,m_2,k,...)$ we put σ_2' the permutation containing the cycle $(...m_2,2g-2,k,...)$. Particularly, $f(\sigma_1')=\sigma_1$ and $f(\sigma_2')=\sigma_2$. Now condition $l(\sigma_1\sigma_2)=l(\sigma_1)+l(\sigma_2)$ provides that $l(\sigma_1')+l(\sigma_2')-2=l(\sigma_1'\sigma_2')$ and so $(\sigma_1',\sigma_2') \in \mathcal{G}(i,j)$. Any element of $\mathcal{G}(i,j)$ can be obtained in this way. Due to Lemma 3.12, we get the result. □

Proof of Proposition 3.10. Let us recall, that we are going to understand the product

$$E_M \cdot (c_{g-1}^2(\mathcal{L}^{[2g-2]}) - c_{g-2}(\mathcal{L}^{[2g-2]})c_g(\mathcal{L}^{[2g-2]})),$$

which is a polynomial in the ring $H^*(S,\mathbb{Q})[2]^{[2g-2]}$. Some terms of this polynomial are divisible by L_i, and eventually, they will correspond to classes supported on divisors of $S \times S$. Therefore we are interested only in the remaining terms. Since $c(\mathcal{L}^{[2g-2]})$ has a uniform representation for all L (cf. [20, Theorem 4.6]), we can conclude that the part without L_i coincides with $c(\mathcal{O}^{[2g-2]})$. The classes $c_i(\mathcal{O}^{[2g-2]})$ are given by (3.10). Then, the class of E_M is proportional to the sum of all permutations which contains one cycle of length $2g-3$, i.e., all permutations conjugated to $\sigma_M = (1,2,...,2g-3)$. For simplicity, we study the part involving σ_M only (the total sum would be greater in $(2g-4)! \cdot (2g-2)$ times):

$$\sum_{l(\sigma_1)=g-1, l(\sigma_2)=g-1} \sigma_M\sigma_1\sigma_2 - \sum_{l(\sigma_1)=g-2, l(\sigma_2)=g} \sigma_M\sigma_1\sigma_2.$$

Every summand in the sum above correspond to a class of $c_2(S) \times S$ (or $S \times c_2(S)$) or Δ in $S \times S$. To distinguish the classes of the diagonal, we need to require that the factor $e^{g(\sigma_M,\sigma_1,\sigma_2)}$ appearing in the triple intersection and corresponding to $c_2(S)$ should be equal to one (cf. [21, Proof of Lemma 2.13]). It implies that the three following conditions must hold

a. $l(\sigma_1\sigma_2) = l(\sigma_M)$, in particular, $l(\sigma_1\sigma_2) = l(\sigma_1) + l(\sigma_2) - 2$, because $l(\sigma_M) = 2g-4$ and $l(\sigma_1)+l(\sigma_2) = 2g-2$,

b. $(\sigma_1\sigma_2)(2g-2) = 2g-2$,

c. $\sigma_1(2g-2) \neq 2g-2$ and $\sigma_2(2g-2) \neq 2g-2$.

The pairs (σ_1,σ_2) with $l(\sigma_1)=i$, $l(\sigma_2)=j$ satisfying these requirements are precisely the set $\mathcal{G}(i,j)$ defined previously. Since $G(g-1,g-1) >$

$G(g-2,g)$ by Lemma 3.13, we get that the representation of $\Omega = E_M^*(Gr)$ as a sum of Δ, $c_2(S) \times S$, $S \times c_2(S)$, and $L \times L$ contains the diagonal with a non-zero coefficient. Therefore Ω viewed as a self-correspondence of S does not act trivially on $H^{2,0}(S)$ and can not be supported on divisors of the form $D_i \times S$ and $S \times D_j$. $\square$

Chapter 4

Results for hyper-Kähler varieties

1 Introduction

In this section we study a 20-dimensional family of hyper-Kähler fourfolds described in [11]. An irreducible hyper-Kähler manifold is a simply connected compact Kähler manifold whose space of holomorphic 2-forms is generated by an everywhere non-degenerated form. In each dimension $2n$, Beauville constructed in [2] two families of such varieties: (a) the $n-$punctual Hilbert scheme $S^{[n]}$ of a K3 surface S and (b) the fiber at the origin of the Albanese map of the $(n+1)-$st punctual Hilbert scheme of an abelian surface. All of the irreducible hyper-Kähler manifolds constructed later on are deformation-equivalent to one of Beauville's examples, with two exceptions: O'Grady's examples in dimension 6 and in dimension 10 (see [26, 25]).

We noted in the introduction that the varieties constructed by Beauville have Picard number two, while a general algebraic deformation of a hyper-Kähler manifold has Picard number one. We also listed four families of hyper-Kähler manifold with Picard rank 1.

Two of them were the Fano variety of lines of a cubic fourfold and the hyper-Kähler manifold constructed by Debarre and Voisin in [11]. These two varieties have very similar constructions and we start in this section the study of their common properties. We hope that this start will allow to attack the Beauville-Voisin conjecture for the hyper-Kähler manifolds constructed by Debarre and Voisin, which is already proved for Fano varieties of lines of cubic fourfold [35].

Let us recall the construction of [11]. Let $G(3, V_{10})$ be the Grassmann variety of 3-dimensional vector subspaces in a 10-dimensional vector space V_{10} and let X be a hyperplane section in $Gr(3, V_{10})$. The variety X is defined by a 3-form

$$\alpha_X = \sum \alpha_{ijk} e_i^* \wedge e_j^* \wedge e_k^* \in \Lambda^3 V_{10}^*$$

where (e_i^*) is a basis of the dual vector space V_{10}^*.

The variety $F(X)$ is then defined as the subvariety of $Gr(6, V_{10})$ of all 6-dimensional spaces $V_6 \subset V_{10}$ such that the form $i_{V_6}^* \alpha_X \in \Lambda^3 V_6^*$ is zero, where $i_{V_6}^* : \Lambda^3 V_{10}^* \to \Lambda^3 V_6^*$ is that natural map. Equivalently, for any 3-dimensional $V_3 \subset V_6$ the restriction $i_{V_3}^* \alpha_X$ is zero and hence $Gr(3, V_6) \subset X$. We thus have a natural universal diagram:

$$\begin{array}{ccc} U & \xrightarrow{\quad q \quad} & X \subset Gr(3, V_{10}) \,, \\ \downarrow p & & \\ F(X) \subset Gr(6, V_{10}) & & \end{array} \tag{4.1}$$

where U is the universal variety consisting of pairs (V_3, V_6) such that $V_3 \subset V_6$ and $i_{V_6}^* \alpha_X$ is zero. For $[V_6] \in F(X)$ we will denote $Z_{V_6} := Gr(3, V_6) \subset X$, a nine-dimensional subvariety of X whose class is $U_*[V_6]$.

Theorem 4.1 ([11]). *For α_X general, the variety $F(X)$ is an irreducible hyper-Kähler manifold of dimension four. More precisely, endowed with the Plücker line bundle, it is deformation-equivalent to the second punctual Hilbert scheme $S^{[2]}$ of a K3 surface S of genus 12, endowed with the line bundle whose pull-back to $\widetilde{S \times S}$ is $(\mathcal{O}_S(1) \boxtimes \mathcal{O}_S(1))^{10}(-33\widetilde{E})$.*

In this theorem $\widetilde{S \times S} \to S \times S$ is the blow-up of the diagonal, $\widetilde{E}$ is the exceptional divisor, and the pull-back is via the canonical double cover $\widetilde{S \times S} \to S^{[2]}$.

The goal of this section is to study the variety $F(X)$ and its similarities with the variety of lines of a cubic fourfold, which has the following similar construction. Let $Y \subset \mathbb{P}^5 = Gr(2, 6)$ be a smooth hypersurface of degree 3, and $F(Y) \subset Gr(2, 6)$ be the variety of lines contained in Y:

$$\begin{array}{ccc} U & \xrightarrow{\quad q \quad} & Y \subset \mathbb{P}^5 \,, \\ \downarrow p & & \\ F(Y) \subset Gr(2, 6) & & \end{array}$$

here U is the universal variety consisting of pairs $(x, [l])$, where $x \in X$, the line l is contained in Y, and $x \in l$.

The notation $F(X)$ for the fourfold constructed by Debarre and Voisin and $F(Y)$ for the Fano variety of lines may look confusing, but the author decided to use them in order to emphasise the similarity between two varieties. The Fano variety of lines does not appear below, so $F(X)$ will unequivocally refer to the fourfold constructed by Debarre and Voisin.

2 Statements of main results

In this subsection we announce Theorems 4.2 and 4.3, which provide evidences of similarities between the Fano varieties of lines on a cubic fourfold and the hyper-Kähler fourfold constructed by Debarre and Voisin. The rest of the chapter will be devoted to the proofs of these theorems.

Following the ideas used in [35] for the Fano variety of lines in a cubic fourfold, we are going to consider the incidence variety I of pairs $([W_1],[W_2]) \in F(X)\times F(X)$ such that the corresponding subvarieties Z_{W_1} and Z_{W_2} on X have a common point. This common point, represented be a $3-$dimensional space, can be viewed as a point on the diagonal Δ_X in $X\times X$, and since we expect that $(p,p)(q,q)^{-1}\Delta_X$ is reducible and contains a diagonal as a component, we then define I in the following way:

$$I \subset (p,p)(q,q)^{-1}\Delta_X,$$

$$I = \overline{((p,p)(q,q)^{-1}\Delta_X) \setminus \Delta_{F(X)}},$$

where p and q were defined by the diagram (4.1). The variety I has a stratification:

$$I = I^{(3)} \cup I^{(4)} \cup I^{(5)} \cup I^{(6)},$$

where $I^{(i)}$ is the subvarity of I consisting of pairs (W_1,W_2) with i-dimensional intersection. For general α_X, calculations give

$$\dim I^{(3)} = \dim I = 6,\ \dim I^{(4)} = 4,\ \dim I^{(5)} = 3,$$

and away from $\Delta_{F(X)}$ which could be contained in I, we have $\dim I^{(6)} \leq 3$.

We define the variety of "triangles" as the closure of I_3^o:

$$I_3^o = \{([W_1],[W_2],[W_3]) | \dim(W_i \cap W_j) \geq 3\ \forall i,j \\ \text{and } \dim(W_1\cap W_2\cap W_3) = 0\} \subset F(X)\times F(X)\times F(X),$$

$$I_3 = \overline{I_3^o} \subset F(X)\times F(X)\times F(X).$$

In Lemma 4.7 below we will show that the natural projection $\pi_{12} : I_3 \to I$ has degree one. One can also consider a bigger variety defined as

$$I_3' = \{([W_1],[W_2],[W_3]) | \dim(W_i \cap W_j) \geq 3\ \forall i,j\} \subset F(X)\times F(X)\times F(X).$$

and show that I_3 is an irreducible component of I_3'.

Theorem 4.2. *a. There exists a cycle $\gamma \in \mathrm{CH}_{10}(Gr(3,10))$ such that for any $([W_1],[W_2],[W_3]) \in I_3$, the sum $Z_{W_1}+Z_{W_2}+Z_{W_3} \in \mathrm{CH}_9(X)$ is the restriction to X of γ.*

b. The $6-$*dimensional subvariety* $I_3 \subset F(X) \times F(X) \times F(X)$ *is a Lagrangian subvariety for the* $(2,0)-$*form* $pr_1^*\sigma_{F(X)} + pr_2^*\sigma_{F(X)} + pr_3^*\sigma_{F(X)}$*, where* $\sigma_{F(X)}$ *denotes a* $(2,0)-$*form on* $F(X)$ *generating* $H^{2,0}(F(X))$*.*

The proof of Theorem 4.2 starts in the next subsection and goes until the end of the chapter.

Another similarity between Fano varieties of lines and Debarre-Voisin fourfolds is given in the next theorem (see [35, Proposition 3.3] for the corresponding results for the Fano variety of lines).

Theorem 4.3. *There is a quadratic relation in* $\mathrm{CH}^4(F(X) \times F(X))$ *of the form*

$$I^2 = \alpha\Delta_{F(X)} + \beta I^{(4)} + I \cdot \Gamma_1 + \Gamma_2 + q^* p_* \Gamma_3,$$

where $\Gamma_1 \in \mathrm{CH}^2(Gr(6,10)^2)|_{F(X)\times F(X)}$*,* $\Gamma_2 \in \mathrm{CH}^4(Gr(6,10)^2)|_{F(X)\times F(X)}$ *and* Γ_3 *is proportional to* $\Delta_{X*}c_2(Gr(3,10)) \in \mathrm{CH}(X \times X)$*.*

We prove Theorem 4.3 in the next subsection.

3 Proofs of main results

We start with the proof of Theorem 4.2.

Proof of Theorem 4.2. Item (a) will be proved in Proposition 4.5 below.

From Lemma 4.6 and Lemma 4.7 below it is follows that I_3 is a $6-$dimensional subvariety. Let us show that (a) implies (b).

Let $\tilde{I}_3$ be a desingularisation of I_3. Let T be the natural correspondence between $\tilde{I}_3$ and $F(X)$. In particular, we have a map $T_* : \mathrm{CH}_0(\tilde{I}_3) \to \mathrm{CH}_0(F(X))$. Note that, by (a), the image of the composition

$$U_* \circ T_* : \mathrm{CH}_0(\tilde{I}_3) \to \mathrm{CH}_0(F(X)) \to \mathrm{CH}_9(X)$$

is $\mathbb{Z}$. Therefore, by the generalisation of Mumford's theorem [24], the map

$$(U \circ T)^* : H^{11,9}(X) \to H^{2,0}(\tilde{I}_3)$$

is zero. But $U^*H^{11,9}(X) = H^{2,0}(F(X))$ by [11], so $\sigma_{F(X)} = U^*\eta_X$ for some $\eta_X \in H^{11,9}(X)$ and thus $\left(pr_1^*\sigma_{F(X)} + pr_2^*\sigma_{F(X)} + pr_3^*\sigma_{F(X)}\right)|_{\tilde{I}_3} = (U\circ T)^*\eta_X = 0$, which proves that the subvariety I_3 is Lagrangian. □

Proof of Theorem 4.3. We follow the line of the proof of the similar statement for the Fano variety of lines (see [35, Proposition 3.3]).

We are going to establish a relation of the following form:

$$I_o^2 = I_o \cdot \Gamma_1 + \Gamma_2 + \Gamma_3,$$

in $\mathrm{CH}^4(F \times F \setminus \left(\Delta_{F(X)} \cup I^{(4)}\right))$, where I_o is the restriction of I to $(F \times F \setminus \left(\Delta_{F(X)} \cup I^{(4)}\right))$. The result will follow by the localisation exact sequence.

We recall that I is the image in $F \times F$ of $\tilde{I} = (q,q)^{-1}\Delta_X$ under the projection $(p,p) : \tilde{I} \to I$. Since (p,p) is an isomorphism away from $\Delta_{F(X)} \cup I^{(4)}$, we have a local isomorphism between I_o and $\tilde{I}_o = (p,p)^{-1}(I_o)$.

We denote some Chern classes in the following short way:

$$c_i^j(3) = pr_j^* c_i(Gr(3,10)) \in \mathrm{CH}_i(Gr(3,10)),$$

$$c_i^j(6) = pr_j^* c_i(Gr(6,10)) \in \mathrm{CH}_i(Gr(6,10)).$$

When we speak about $\mathrm{CH}_*(U \times U)$, where U is the universal variety in the diagram (4.1), to keep notation simple we will denote $(p,p)^*(c_i^j(6))$ as $c_i^j(6)$ and $(q,q)^*(c_i^j(3))$ as $c_i^j(3)$.

We have the normal sequence:

$$0 \to T_{U\times U/F\times F|_{\tilde{I}_0}} \to N_{\tilde{I}_0/U\times U} \to (p,p)^* N_{I_0/F\times F} \to 0,$$

therefore $(p,p)^* N_{I_o/F\times F}$ can be expressed as a polynomial in the Chern classes of the normal bundle $N_{\tilde{I}_o/U\times U}$ and in Chern classes of $T_{U\times U/F\times F|_{\tilde{I}_o}}$. The later ones are polynomial in $c_i^j(6)$ and $c_i^j(3)$. Next, we see that $\tilde{I} = (q,q)^{-1}(\Delta_X)$, therefore

$$c_i(N_{\tilde{I}_o/U\times U}) = (q,q)^* c_i(T_X),$$

but $c_i(T_X)$ are polynomial in $c_j(Gr(3,V_{10}))$. So we have that

$$I_o^2 = (p,p)_*(P \cdot \tilde{I}),$$

where P is a quadratic polynomial in $c_i^j(6)$ and $c_i^j(3)$. The polynomial P can be divided in three parts:

a. the part containing only $c_i^j(6)$. Since all these terms have from $(p,p)^*(c)$ for some $c \in \mathrm{CH}_2(F(X)\times F(X))$, the intersection with $\tilde{I}$ and projection $(p,p)_*$ gives the term $\Gamma_1 \cdot I_o$.

b. the part divisible by $c_1^j(3)$. The term $c_1^j(3)$ has the from $(q,q)^*(c)$, for some $c \in \mathrm{CH}_2(X \times X)$, and its intersection with $\tilde{I}$ can be represented as $(q,q)^*(c \cdot \Delta_X)$. Since $c_1(Gr(3,10)) \cdot \Delta_X$ is proportional to $\Delta_{Gr(3,10)}|_{X\times X}$, it is a cycle coming from $\mathrm{CH}(Gr(3,10) \times Gr(3,10))$. Therefore this part gives the term Γ_2 in the final relation.

c. the part proportional to $c_2^j(3)$. It will lead to the term

$$(p,p)_*(q,q)^*\Gamma_3 = (p,p)_*(q,q)^*(c_2(Gr(3,10)))$$

in the final relation.

□

4 Technical lemmas: open part

We start with the study of the local geometry of I_3. Let $([W_1], [W_2], [W_3])$ be a general point of I_3. By definition of I_3, the three spaces

$$K_1 = W_2 \cap W_3,\ K_2 = W_3 \cap W_1,\ K_3 = W_1 \cap W_2$$

are pairwise transversal. In particular, we have decompositions

$$W_1 = K_2 \oplus K_3, W_2 = K_3 \oplus K_1, W_3 = K_1 \oplus K_2, V_9 = K_1 \oplus K_2 \oplus K_3,$$

and since α_X vanishes on W_1, W_2, W_3, the restriction $\alpha' = i^*_{V_9}\alpha_X$ belongs to $K_1^* \otimes K_2^* \otimes K_3^*$. We note that α' defines a hypersurface X' in $Gr(3, V_9)$ and X' contains Z_{W_1}, Z_{W_2}, and Z_{W_3}, where the notation Z_{W_i} was introduced above.

Let O be an open chart of $Gr(3, K_1 \oplus K_2 \oplus K_3)$ defined in the following way:

$$O = \{V : \dim \pi_{K_1}(V) = 3\}.$$

We note that O is naturally isomorphic to the affine space $\mathrm{Hom}(K_1, W_1)$ or to $\mathrm{Hom}(K_1, K_2) \oplus \mathrm{Hom}(K_1, K_3)$. The following lemma shows that $\alpha'|_O$ is a quadratic form and hence $X'|_O$ is a quadratic hypersurface in O.

Lemma 4.4. *Let $\alpha' \in K_1^* \otimes K_2^* \otimes K_3^*$. Its restriction $\alpha'|_O$ defines a pairing between $\mathrm{Hom}(K_1, K_2)$ and $\mathrm{Hom}(K_1, K_3)$, which in some basis can be represented by a $9 \times 9-$matrix*

$$\begin{pmatrix} 0 & -Q_3 & -Q_2 \\ Q_3 & 0 & -Q_1 \\ Q_2 & Q_1 & 0 \end{pmatrix},$$

where Q_1, Q_2, Q_3 are 3×3-matrices. Moreover, this pairing is non-degenerate for a general choice of α'.

Proof. Let (e_1, e_2, e_3) be a basis of K_1, (e_4, e_5, e_6) be a basis of K_2, and (e_7, e_8, e_9) be a basis of K_3. Let a point $p \in \mathrm{Hom}(K_1, W_1)$ be given by the matrix (4.2). We evaluate α' on the trivector

$$\begin{aligned}(e_1 + (n_1e_4 + n_2e_5 + n_3e_6) + (m_1e_7 + m_2e_8 + m_3e_9)) \\ \wedge (e_2 + (n_4e_4 + n_5e_5 + n_6e_6) + (m_4e_7 + m_5e_8 + m_6e_9)) \\ \wedge (e_3 + (n_7e_4 + n_8e_5 + n_9e_6) + (m_7e_7 + m_8e_8 + m_9e_9)).\end{aligned}$$

As $\alpha' \in K_1^* \otimes K_2^* \otimes K_3^*$, we can as well evaluate α' on

$$\begin{aligned} e_1 \wedge (n_4e_4 + n_5e_5 + n_6e_6) \wedge (m_7e_7 + m_8e_8 + m_9e_9) \\ - e_1 \wedge (n_7e_4 + n_8e_5 + n_9e_6) \wedge (m_4e_7 + m_5e_8 + m_6e_9) \\ + e_2 \wedge (n_7e_4 + n_8e_5 + n_9e_6) \wedge (m_1e_7 + m_2e_8 + m_3e_9) \\ - e_2 \wedge (n_1e_4 + n_2e_5 + n_3e_6) \wedge (m_7e_7 + m_8e_8 + m_9e_9) \\ + e_3 \wedge (n_1e_4 + n_2e_5 + n_3e_6) \wedge (m_4e_7 + m_5e_8 + m_6e_9) \\ - e_3 \wedge (n_4e_4 + n_5e_5 + n_6e_6) \wedge (m_1e_7 + m_2e_8 + m_3e_9). \end{aligned}$$

On the other hand, α' can be written as $e_1^* \wedge Q_1 + e_2^* \wedge Q_2 + e_3^* \wedge Q_3$, where $Q_i \in K_2^* \otimes K_3^*$. This gives the desired matrix presentation.

Assuming that Q_3 is non-degenerate and using operations on lines, we can transform the matrix to

$$\begin{pmatrix} 0 & 1 & Q_3^{-1}Q_2 \\ 1 & 0 & -Q_3^{-1}Q_1 \\ 0 & 0 & -Q_2Q_3^{-1}Q_1 + Q_1Q_3^{-1}Q_2 \end{pmatrix}.$$

Now we see that the pairing is non-generate if and only if $-Q_2Q_1^{-1}Q_3 + Q_3Q_1^{-1}Q_2$ is non-degenerate. This condition is an open condition and it is true for the following choice:

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, Q_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, Q_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence it is true for a general choice of Q_1, Q_2, Q_3. □

Proposition 4.5. *a. The 9-dimensional cycle $Z_{W_1}+Z_{W_2}+Z_{W_3}$ in $\mathrm{CH}_9(X')$ is the restriction of a cycle Z of $Gr(3, K_1 \oplus K_2 \oplus K_3)$.*

b. The 9-dimensional cycle $Z_{W_1} + Z_{W_2} + Z_{W_3}$ in $\mathrm{CH}_9(X)$ is the restriction of a cycle Z' of $Gr(3, 10)$. In particular, $Z_{W_1} + Z_{W_2} + Z_{W_3}$ is constant in $\mathrm{CH}_9(X)$, i.e., it does not depend on the choice of $([W_1], [W_2], [W_3]) \in I_3$.

Proof. The statement (b) is an immediate consequence of (a). We prove (a).

Let us represent $Gr(3, K_1 \oplus K_2 \oplus K_3)$ as the union of the chart O (as above), and subvarieties D_1, D_2, D_3 where

$$O = \{V : \dim \pi_{K_1}(V) = 3\},$$

$$D_k = \{V : \dim \pi_{K_1}(V) = 3 - k\}.$$

Due to Lemma 4.4, the form $\alpha'|_O$ defines a non-degenerate quadratic hypersurface $Q = X'|_O$. Using this fact we are going to represent cycle $(Z_{W_2} + Z_{W_3})|_O$ as the sum

$$(B_1 - B_2 + B_3 - \ldots + B_9)|_Q,$$

where B_i are $10-$dimensional subspaces of O. We can take

$$B_i = \langle v_i, v_{i+1}, \ldots, v_9, v_1^*, \ldots, v_i^* \rangle ,$$

where $(v_1, v_2, \ldots, v_9)$ is an arbitrary basis of K_2^3 and $(v_1^*, v_2^*, \ldots, v_9^*)$ is the dual basis in K_3^3. Now if $A_1 = K_2^3$, $A_{10} = K_3^3$, and

$$A_i = \left\langle v_i, v_{i+1}, \ldots, v_9, v_1^*, \ldots, v_{i-1}^* \right\rangle$$

for $2 \leq i \leq 9$, it is easy to see that A_i are $9-$dimensional affine spaces contained in hypersurface Q. We also have $A_1 = Z_{W_2}|_O$ and $A_{10} = Z_{W_3}|_O$. Moreover, the restriction $B_i|_Q$ is $A_i \cup A_{i+1}$ for each i. Therefore $(B_1 - B_2 + B_3 - \ldots + B_9)|_Q = (Z_{W_2} + Z_{W_3})|_O$.

Let $\overline{B}_i \subset Gr(3, V_9)$ be the Zariski closure of $B_i \subset O$. We are going to take the cycle $(\overline{B}_1 - \overline{B}_2 + \overline{B}_3 - \ldots + \overline{B}_9)$ as the desired cycle in $Gr(3, V_9)$. To finish the proof we need to investigate the boundaries $B'_{ik} = \overline{B}_i \cap D_k$. For $k = 1, 2$, we will prove in Lemma 4.8 that the intersection of B'_{ik} and $X' \cap D_k$ has dimension at most 8 for a general choice of the basis $(v_1, \ldots, v_9)$. For $k = 3$, we note that Z_{W_2} identifies to $\mathrm{Hom}(K_1, K_3)$, via the isomorphism $W_2 = K_1 \oplus K_3$, therefore the cycle Z_{W_1} is contained in the complement of O and in fact equal to D_3. We have an inclusion:

$$\overline{B}_i \cap X' \subset \left(\overline{A}_i \cup \overline{A}_{i+1} \cup Z_{W_1}\right) \cup (\text{lower dimensional terms}) .$$

Therefore the restriction to X' of the closure of $B_1 - B_2 + B_3 - \ldots + B_9$ defines a cycle of the form $dZ_{W_1} + Z_{W_2} + Z_{W_3}$ for some $d \geq 0$. Permuting the W_i and adding up, we conclude that $(d+2)(Z_{W_1} + Z_{W_2} + Z_{W_3})$ is the restriction of a cycle of $Gr(3, V_9)$, which concludes the proof. □

In the end of this subsection, we are going to present a lemma about the relation of I and I_3.

Lemma 4.6. *For a general X the expected dimension of I is 6.*

Proof. The codimension of Δ_X is 20, therefore the codimension of $(q, q)^{-1}\Delta_X$ is also 20. Since $U \times U$ has dimension 26, the dimension of $(q, q)^{-1}\Delta_X$ as well as the dimension of $(p, p)(q, q)^{-1}\Delta_X$ should be 6. □

Lemma 4.7. *For a general choice of α_X, the natural projection $\pi_{12} : I_3^o \to I$ has degree one. As a consequence, there exists a birational map:*

$$\phi = \pi_3 \circ \pi_{12}^{-1} : I \dashrightarrow F(X).$$

Proof. We are going to understand the fiber $\pi_{12}^{-1}(p)$ for a general point $p = ([W_1], [W_2]) \in I$. Let $K_3 = W_1 \cap W_2$, $K_2 \subset W_1$ such that $K_2 \cap K_3 = 0$, and $K_1 \subset W_2$ such that $K_1 \cap K_2 = 0$, $V_9 = K_1 \oplus K_2 \oplus K_3$.

Any point of $I_3^o \cap \pi_{12}^{-1}(p)$ belongs to the following open char of $Gr(6, V_9)$:

$$\{V : \dim \pi_{K_1 \oplus K_2}(V) = 6\} \subset Gr(6, V_9).$$

This chart can be identify with $\mathrm{Hom}\,(K_1 \oplus K_2, K_3)$, and its point $(\phi \oplus \psi)$ belongs to I_3^o, if and only if the following equations hold:

$$\alpha'((v_1 + \phi(v_1)) \wedge (v_2 + \phi(v_2)) \wedge (w_1 + \psi(w_1))) = 0,$$

$$\alpha'((v_1 + \phi(v_1)) \wedge (w_1 + \psi(w_1)) \wedge (w_2 + \psi(w_2))) = 0$$

for any $v_1, v_2 \in K_1$ and any $w_1, w_2 \in K_2$. Since, $i_{W_1}^* \alpha' = i_{W_2}^* \alpha' = 0$, these equations are equivalent to

$$\alpha'(v_1 \wedge \phi(v_2) \wedge w_1) + \alpha'(\phi(v_1) \wedge v_2 \wedge w_1) = -\alpha'(v_1 \wedge v_2 \wedge w_1).$$

$$\alpha'(v_1 \wedge \psi(w_1) \wedge w_2) + \alpha'(v_1 \wedge w_1 \wedge \psi(w_2)) = -\alpha'(v_1 \wedge w_1 \wedge w_2).$$

The equations define two linear systems: for ϕ and for ψ. Since the coefficients in the left hand sides of equations are defined only by the part of α' which belongs to $K_1^* \otimes K_2^* \otimes K_3^*$, we may apply Lemma 4.4 to see that the matrix of coefficients in the linear system for $\phi \in \mathrm{Hom}\,(K_1, K_3)$ is a non-degenerate $9 \times 9-$matrix. Similarly, the matrix for ψ is non-degenerate. Therefore there is a unique solution and the natural projection $\pi_{12} : I_3^o \to I$ has degree one. □

5 Technical lemmas: boundary

The goal of this subsection is to prove the following lemma. We continue with the notation from the previous subsection.

Lemma 4.8. *For a general choice of basis (v_j) of K_2^3 and for any choice of integer i between 1 and 9, let $B \subset O$ be the 10-dimensional vector space*

$$\langle v_i, v_{i+1}, \ldots, v_9, v_1^*, \ldots, v_i^* \rangle,$$

where v_j^* denotes the dual basis of K_3^* with respect to $\alpha'|_O$. Then the intersection $\overline{B} \cap D_k \cap X'$ has dimension at most 8 for $k = 1, 2$.

The proof of Lemma 4.8 will rest on Lemma 4.9 for $k = 1$ and Lemma 4.11 for $k = 2$. Before the proof we introduce local coordinates on O and relate them to the local coordinates on D_1 and on D_2. In particular, we show that any point on D_k is the limit point of some affine line in O. Finally, we show that the intersection $\overline{B} \cap D_k \cap X'$ has dimension at most 8 by proving that any 9−dimensional component of $\overline{B} \cap D_k$ is not contained in X'.

We recall that a point in $Gr(3, 9)$ can be represented by three independent vectors, i.e., 3×9 matrix (both up to $GL(3)$−action). Fixing basis of K_1, K_2, and K_3, we have in chart O a representation

$$p_O = \begin{pmatrix} 1 & 0 & 0 & n_1 & n_2 & n_3 & m_1 & m_2 & m_3 \\ 0 & 1 & 0 & n_4 & n_5 & n_6 & m_4 & m_5 & m_6 \\ 0 & 0 & 1 & n_7 & n_8 & n_9 & m_7 & m_8 & m_9 \end{pmatrix}. \tag{4.2}$$

In this notation, n_i correspond to an element of $\mathrm{Hom}(K_1, K_2)$ and m_i correspond to an element of $\mathrm{Hom}(K_1, K_3)$. Unfortunately, it is not possible to relate coordinates n_j and m_j with the basis (v_j, v_j^*) in a simple way, because the quadratic form $\alpha'|_O$ is not a general quadratic form on O (its form was explained in Lemma 4.4).

Now we are going to study the boundaries D_1 and D_2. According to the definition of D_k, a point $p \in D_k$ can be represented by a 3×9−matrix, whose rank of the first three columns is equal to $3 - k$.

For $k = 1$, we need another chart O' where the rank of the first three columns may be two. Without loss of generality, we may assume that the columns number $1, 2$, and 4 are linearly independent in O'. On $O' \cap O$ we have

$$\begin{pmatrix} 1 & 0 & n_1' & 0 & n_2' & n_3' & m_1' & m_2' & m_3' \\ 0 & 1 & n_4' & 0 & n_5' & n_6' & m_4' & m_5' & m_6' \\ 0 & 0 & n_7' & 1 & n_8' & n_9' & m_7' & m_8' & m_9' \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & -\frac{n_1}{n_7} & 0 & \frac{n_2 n_7 - n_1 n_8}{n_7} & \frac{n_3 n_7 - n_1 n_9}{n_7} & \frac{m_1 n_7 - n_1 m_7}{n_7} & \frac{m_2 n_7 - n_1 m_8}{n_7} & \frac{m_3 n_7 - n_1 m_8}{n_7} \\ 0 & 1 & -\frac{n_4}{n_7} & 0 & \frac{n_5 n_7 - n_4 n_8}{n_7} & \frac{n_6 n_7 - n_4 n_9}{n_7} & \frac{m_4 n_7 - n_4 m_7}{n_7} & \frac{m_5 n_7 - n_4 m_8}{n_7} & \frac{m_6 n_7 - n_4 m_8}{n_7} \\ 0 & 0 & \frac{1}{n_7} & 1 & \frac{n_8}{n_7} & \frac{n_9}{n_7} & \frac{m_7}{n_7} & \frac{m_8}{n_7} & \frac{m_9}{n_7} \end{pmatrix}.$$

The intersection $D_1' = O' \cap D_1$ is the 17−dimensional affine space defined by $n_7' = 0$ in O'. Any point $p \in D_1'$ can be represented as the $(t \to \infty)$-limit of the affine line

$$l = t\,(0|N_0|M_0) + (id|N_1|M_1)$$

in O, where the 3×3−matrices N_0, N_1, M_0, M_1 are defined by the coordinates n_i' and m_i' of p. Moreover, since the coordinates n_2', n_3', n_5', n_6' and $m_1', \ldots, m_6'$ are well-defined as $t \to \infty$, the direction $(N_0|M_0)$ must satisfy the condition

$rk(N_0|M_0) = 1$. Conversely, a choice of $(N_0|M_0)$ with $rk(N_0|M_0) = 1$ and a point $(N_1|M_1) \in O$ defines a line, whose limit point p belongs to D_1. The coordinates

$$n'_1, n'_4, n'_7, n'_8, n'_9, m'_7, m'_8, m'_9$$

of the limit point $p \in D'_1$ are defined only by the choice of $(N_0|M_0)$ and the remaining coordinates

$$n'_2, n'_3, n'_5, n'_6, m'_1, \ldots, m'_6$$

are defined by $(N_1|M_1)$, while $(N_0|M_0)$ is fixed. The projection of D'_1 along the coordinates $n'_2, n'_3, n'_5, n'_6, m'_1, \ldots, m'_6$ defines the structure of a fibration on D'_1, which can be extended to D_1.

Lemma 4.9. *Let F be the intersection of $\overline{B}$ with a fiber of D_1. Then there exists a point $p \in F$, such that $p \notin X'$.*

Proof. A choice of fiber is equivalent to the choice of $(N_0|M_0)$ such that $rk(N_0|M_0) = 1$. We note that $\alpha'(id|N_0|M_0) = 0$, therefore $(N_0|M_0)$ belongs to $B \cap X'$, which is the union of two subspaces

$$\left\langle v_i, v_{i+1}, \ldots, v_9, v_1^*, \ldots v_{i-1}^* \right\rangle \text{ and } \left\langle v_{i+1}, \ldots, v_9, v_1^*, \ldots v_i^* \right\rangle.$$

Without loss of generality, we may assume that $(N_0|M_0)$ belongs to the first subspace. In this case v_i^* is not related with $(N_0|M_0)$ and can be chosen arbitrarily (more precisely, there is a choice of basis (v_j) providing the given choice of v_i^*). It allows to take a point $(N_1|M_1)$ in B corresponding to v_i^*. In this case $N_1 = 0$ and M_1 is arbitrary, which makes the coordinates $m'_1, \ldots, m'_6$ arbitrary (while $n'_1, n'_4, n'_7, n'_8, n'_9, m'_7, m'_8, m'_9$ are fixed by the choice of $(N_0|M_0)$). Representing α' as $Q'_1 \wedge e_7^* + Q'_2 \wedge e_8^* + Q'_3 \wedge e_9^*$, where $Q'_i \in K_1^* \otimes K_2^*$, we see that

$$\alpha'(p) = m_1 Q'_1((0,1,n'_4),(1,n'_8,n'_9)) + m_2 Q'_2((0,1,n'_4),(1,n'_8,n'_9)) + \\ m_3 Q'_3((0,1,n'_4),(1,n'_8,n'_9)) + \text{other terms.}$$

Since $(0,1,n'_4)$ can not be orthogonal to $(1,n'_8,n'_9)$ with respect to all three forms Q'_i, at least one of the first three terms is not zero. Therefore a general choice of $m'_1, \ldots, m'_6$ provides a point p with $\alpha(p) \neq 0$ and thus $p \notin X'$. $\square$

For $k = 2$, we need a chart O', where the rank of the first three columns may be one. Without loss of generality, we may assume that the columns number $1, 4$, and 5 are linearly independent in O'. On O' we have coordinates

$$\begin{pmatrix} 1 & n'_1 & n'_2 & 0 & 0 & n'_3 & m'_1 & m'_2 & m'_3 \\ 0 & n'_4 & n'_5 & 1 & 0 & n'_6 & m'_4 & m'_5 & m'_6 \\ 0 & n'_7 & n'_8 & 0 & 1 & n'_9 & m'_7 & m'_8 & m'_9 \end{pmatrix}.$$

The intersection $D_2' = D_2 \cap O'$ is the $14-$dimensional affine subspace in O', defined by $n_4' = n_5' = n_7' = n_8' = 0$. Again, any point on D_2' can be represented as the limit of the affine line

$$l = t\,(0|N_0|M_0) + (id|N_1|M_1)$$

in O and, similarly to the case $k = 1$, we have $rk(N_0|M_0) = 2$. Conversely, an affine line $t\,(0|N_0|M_0) + (id|N_1|M_1)$ with $rk(N_0|M_0) = 2$ defines a limit point $p \in D_2$. We also have the structure of a fibration on D_2: we can fix $(N_0|M_0)$ and then vary $(N_1|M_1)$. A fiber has the following matrix presentation in O':

$$\begin{pmatrix} 1 & n_1' & n_2' & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 1 & 0 & n_6' & m_4' & m_5' & m_6' \\ 0 & 0 & 0 & 0 & 1 & n_9' & m_7' & m_8' & m_9' \end{pmatrix},$$

where n_j' and m_j' are fixed. We note that such a fiber of D_2 is entirely contained in X' or has an empty intersection with X'. It makes sense to consider a projection

$$\pi : O' \to P,$$

along the coordinates n_3', m_1', m_2', m_3', where $P \subset O'$ is the linear subspace defined by $n_3' = m_1' = m_2' = m_3' = 0$. Then, any point $p(t)$ on the line

$$t\,(0|N_0|M_0) + (id|0|0)\,,$$

passing through the origin of O with $rk(N_0|M_0) = 2$, can be represented as

$$\begin{pmatrix} 1 & n_1' & n_2' & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & tn_4 & tn_5 & tn_6 & tm_4 & tm_5 & tm_6 \\ 0 & 0 & 1 & tn_7 & tn_8 & tn_9 & tm_7 & tm_8 & tm_9 \end{pmatrix}$$

or, for $t \neq 0$,

$$\begin{pmatrix} 1 & n_1' & n_2' & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/t & 0 & n_4 & n_5 & n_6 & m_4 & m_5 & m_6 \\ 0 & 0 & 1/t & n_7 & n_8 & n_9 & m_7 & m_8 & m_9 \end{pmatrix}.$$

In this notation the value $\alpha'(p(t))$ does not depend on t. Therefore the limit point of such a line belongs to $X' \cap D_2$ if and only if the line is contained in X'.

We are going to show that the intersection $\overline{B} \cap D_2$ is not contained in X'. We need the following auxiliary lemma.

Lemma 4.10. *Let* $1 \leq i \leq 9$ *and let*

$$A = \left\langle v_i, \ldots, v_9, v_1^*, \ldots, v_{i-1}^* \right\rangle .$$

Then $\pi(\overline{A} \cap D_2')$ *has dimension at most four.*

Proof. The dimension of $\pi(\overline{A} \cap D_2')$ depends only on the variety of "directions" in A:

$$\mathcal{M} = \{(N_0|M_0) \in A : rk(N_0|M_0) = 2\}.$$

Let $\mathcal{M}_0$ be an irreducible component of $\mathcal{M}$. There are two possibilities:

a. $M_0 = 0$ for general $(N_0|M_0) \in \mathcal{M}_0$. In this case we have $m_1' = m_2' = \ldots = m_6' = 0$ for the corresponding component of $\pi(\overline{A} \cap D_2')$. Hence its dimension is at most 4.

b. $M_0 \neq 0$ for general $(N_0|M_0) \in \mathcal{M}_0$. Let $\mathcal{M}_{0N}$ be the image of the projection of $\mathcal{M}_0$ to K_2^3, i.e., $\mathcal{M}_{0N}$ is the variety of N_0 in pairs $(N_0|M_0) \in \mathcal{M}_0$. We note that $\mathcal{M}_{0N}$ is contained in the cubic hypersurface defined in $\langle v_i, \ldots, v_9 \rangle$ by $det(N_0) = 0$. In particular, $\dim \mathcal{M}_{0N} \leq 9 - i$. Similarly, we can define $\mathcal{M}_{0M}$ as the image of the projection of $\mathcal{M}$ to K_3^3. We also have $\dim \mathcal{M}_{0M} \leq i - 2$. There are two possibilities:

   (a) $\dim \mathcal{M}_{0N} = 9 - i$ or $\dim \mathcal{M}_{0M} = i - 2$. Without loss of generality, we may assume that $\dim \mathcal{M}_{0N} = 9 - i$. We are going to show that the fiber of $\mathcal{M}_0 \to \mathcal{M}_{0N}$ over a general point of $\mathcal{M}_{0N}$ has dimension at most $i - 4$. This will imply that $\dim \mathcal{M}_0 \leq 5$ and therefore $\dim \pi(\overline{A} \cap D_2) \leq 4$.

   Let N_0 be a general point of $\mathcal{M}_{0N} \subset \langle v_i, \ldots, v_9 \rangle$. The fiber of $\mathcal{M}_0$ over N_0 is the subvariety of

$$A \cap K_3^3 = \langle v_1^*, \ldots, v_i^* \rangle$$

   consisting of all matrices M_0 such that $rk(N_0|M_0) = 2$. It can be defined by 3 linear equations (depending on N_0) and hence it can be represented as the subspace orthogonal with respect to the quadratic form $\alpha'|_O$ to some 3−dimensional subspace

$$\langle f_1, f_2, f_3 \rangle \subset K_2^3 = \langle v_1, \ldots, v_9 \rangle ,$$

   where f_1, f_2, f_3 depend on N_0. It can be also seen as the subspace of

$$K_3^3 = \langle v_1^*, \ldots, v_9^* \rangle$$

orthogonal to

$$\langle f_1, f_2, f_3, v_i, v_{i+1}, \ldots, v_9 \rangle .$$

For a general choice of α and a general choice of N_0, the vectors

$$N_0, f_1, f_2, f_3$$

are linearly independent in K_2^3. Moreover, we claim that all $13-i$ vectors

$$f_1, f_2, f_3, v_i, \ldots, v_9$$

are linearly independent. Indeed, this condition is an open condition, and it is enough to show the result for some choice of v_j. We can fix N_0 in K_3^3, take $v_i = N_0$, then define v_j for $j \neq i$ in such a way that the desired $13-i$ vectors are linearly independent. It is possible, while the dimension of $\langle f_1, f_2, f_3, v_i, \ldots, v_9 \rangle$ is less than 9. The assumption of the existence of non-zero M_0 in the fiber, provides that $\dim \langle f_1, f_2, f_3, v_i, \ldots, v_9 \rangle \leq 8$.

The independence of $f_1, f_2, f_3, v_i, \ldots, v_9$ will imply that the dimension of a fiber over N_0 is at most $i-4$. Hence $\dim \mathcal{M}_0 \leq 5$ and therefore $\dim \pi(\overline{A} \cap D_2') \leq 4$.

(b) $\dim \mathcal{M}_{0N} \leq 8-i$ and $\dim \mathcal{M}_{0M} \leq i-3$. In this case, $\dim \mathcal{M}_0 \leq 5$ and therefore $\dim \pi(\overline{A} \cap D_2') \leq 4$.

□

Lemma 4.11. *No $9-$dimensional component of $\overline{B} \cap D_2$ is contained in X'.*

Proof. We prove the lemma by contradiction. Assume that there is a component B' of $\overline{B} \cap D_2$ contained in X'. Hence $\pi(B')$ is contained in X' and therefore each point of $\pi(B')$ is the limit of a line contained in $B \cap X'$. Since $B \cap X' = A \cup A'$, where

$$A = \left\langle v_i, \ldots v_9, v_1^*, \ldots, v_{i-1}^* \right\rangle, A' = \langle v_{i+1}, \ldots v_9, v_1^*, \ldots, v_i^* \rangle .$$

We have that $\pi(B')$ is contained in $\pi(\overline{A} \cup \overline{A}')$. Due to Lemma 4.10, $\dim \pi(\overline{A} \cup \overline{A}') \leq 4$, but $\dim \pi(B') \geq 5$. We have a contradiction, which proves the lemma. □

Proof of Lemma 4.8. For $k = 1$ the result follows from Lemma 4.9. The intersection F of $\overline{B}$ with a fiber of D_1 is irreducible and therefore it is contained in one irreducible component of $\overline{B} \cap D_1$. Conversely, any component contains such a fiber. Since each fiber contains a point which does not belong to X', each irreducible component of $\overline{B} \cap D_1$ is not contained in X'.

For $k = 2$ the result follows from Lemma 4.11. □

Bibliography

[1] I. Bazhov. On the chow group of zero-cycles of calabi-yau hypersurfaces. *Manuscripta Math.*, 152(1), 2016.

[2] A. Beauville. Variétés kählériennes dont la premiere class de chern est nulle. *J. Differential Geom.*, 18:755–782, 1984.

[3] A. Beauville. On the splitting of the Bloch–Beilinson filtration. *Algebraic cycles and motives (vol. 2)*, 344:38–53, 2007.

[4] A. Beauville and R. Donagi. La variété des droits d'une hypersurface qubique de dimension 4. *C. R. Acad. Sci. Paris Sér. I Math*, 301(14):703–706, 1985.

[5] A. Beauville and C. Voisin. On the Chow ring of a K3 surface. *J. Alg. Geom*, 13:417–426, 2004.

[6] A. Beilinson. *Height pairing between algebraic cycles*, volume 1289 of *Lecture Notes in Math.* K-theory, arithmetic and geometry, 1987.

[7] S. Bloch and V. Srinivas. Remarks on correspondences and algebraic cycles. *Amer. J. Math.*, 105:1235–1253, 1983.

[8] M. Brion. Lectures on the geometry of flag varieties. 2004.

[9] M. Brion and S. Kumar. *Frobenius Splitting Methods in Geometry and Representation Theory*, volume 231 of *Progress in Mathematics*. Birkhäuser Basel, 2005.

[10] M. de Cataldo and J. Migliorini. The Chow groups and the motive of the Hilbert scheme of points on a surface. *J. Algebra*, 2(251):824–848, 2002.

[11] O. Debarre and C. Voisin. Hyper-Kaehler fourfolds and grassmann geometry. *Journal für die reine und angewandte Mathematik*, 649, 2010.

[12] L. Fu. Decomposition of small diagonals and Chow rings of hypersurfaces and Calabi–Yau complete intersections. *Advances in Mathematics*, 244:894–924, 2013.

[13] L. Fu. Beauville–Voisin conjecture for generalized Kummer varieties. *International Mathematics Research Notices*, 2014.

[14] W. Fulton. *Intersection Theory*. Springer-Verlag New York, 1998.

[15] D. Huybrechts. Curves and cycles on K3 surface. *Algebraic Geometry*, 1:69–106, 2014.

[16] D. Huybrechts. *Lectures on K3 surfaces*. Cambridge University Press., 2016.

[17] A. Iliev and K. Ranestad. K3 surface of genus 8 and varieties of sums of powers of cubic fourfolds. *Trans. Amer. Math. Soc.*, 353(4):1455–1458, 2001.

[18] A. Iliev and K. Ranestad. Addendum to "K3 surface of genus 8 and varieties of sums of powers of cubic fourfolds". *C.R. Acad. Bulgare Sci.*, 60(12):1265–1270, 2007.

[19] C. Lehn, M. Lehn, C. Sorger, and D. van Straten. Twisted cubics on cubic fourfolds. arXiv:1305.0178.

[20] M. Lehn. Chern classes of tautological sheaves on Hilbert schemes of points on surface. *Inventiones mathematicae*, 136(1):157–207, 1999.

[21] M. Lehn and C. Sorger. The cup product on the Hilbert Scheme for K3 surfaces. *Inventiones mathematicae*, 152(2):305–329, 2003.

[22] H.-Y. Lin. On the Chow group of zero-cycles of a generalized Kummer variety. arXiv:1507.05155v1, 2015.

[23] S. Mori and S. Mukai. *The uniruledness of the moduli space of curves of genus 11*, volume 1016 of *Lectures Notes in Math.* 1982.

[24] D. Mumford. Rational equivalence of zero-cycles on surfaces. *J. Math. Kyoto Univ.*, (9):195–204, 1968.

[25] K. O'Grady. Desingularized moduli spaces of sheaves on a K3. *J. reine angew. Math*, 512:49–117, 1999.

[26] K. O'Grady. A new six-dimensional irreducible symplectic variety. *J. Alg. Geom*, 12:435–505, 2003.

[27] K. O'Grady. Irreducible symplectic 4-folds and Eisenbud–Popescu–Walter sextics. *Duke Math. J.*, 134(1):99–137, 2006.

[28] K. O'Grady. Moduli of sheaves and the chow group of k3 surfaces. *Journal de Mathématiques Pures et Appliquées*, 100(5):701–718, 2013.

[29] K. O'Grady. Computations with modified diagonals. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur.*, 25(3):249–274, 2014.

[30] A. Roitman. The torsion of the group of zero-cycles modulo rational equivalence. *Ann. Math*, 111:553–569, 1980.

[31] M. Shen and C. Vial. The Fourier Transform for Certain Hyper-Kaehler Fourfolds. *Memoirs of the AMS*, 240(1139), 2016.

[32] C. Voisin. *Hodge theory and complex algebraic geometry. I*, volume 76 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2003.

[33] C. Voisin. *Hodge theory and complex algebraic geometry. II*, volume 77 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2003.

[34] C. Voisin. Intrinsic pseudovolume forms and K-correspondences. In M. M. A. Collino, A. Conte, editor, *The Fano Conference*, pages 761–792, 2004.

[35] C. Voisin. On the Chow ring of certain algebraic hyper-Kähler manifolds. *Pure Appl. Math. Q.*, 4(3):613–649, 2008.

[36] C. Voisin. Chow rings and decomposition theorems for families of K3 surfaces and Calabi–Yau hypersurfaces. *Geometry and topology*, 16:433–473, 2012.

[37] C. Voisin. Rational equivalence of zero-cycles on K3 surfaces and conjectures of Huybrechts and O'Grady. *To appear in "Recent Advances in Algebraic Geometry, a conference in honor of Rob Lazarsfeld's 60th birthday"*, 2012.

[38] C. Voisin. Remarks and questions on coisotropic subvarieties and zero-cycles of hyper-Kähler varieties. arXiv:1501.02984, 2015.

[39] G. Welters. Abel–Jacobi isogenies for certain types of Fano threefolds. *Mathematical Centre Tracts*, 141, 1981.

[40] S. Yau. the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampere quation. I. *Comm.Pure. Appl. Math.*, 31(3):339–411, 1978.