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POUR OBTENIR LE GRADE DE
DOCTEUR DE
L'UNIVERSITÉ DE BORDEAUX

ÉCOLE DOCTORALE de MATHÉMATIQUES
SPÉCIALITÉ

Par Stefano SCROBOGNA

On some models in geophysical fluids

Sous la direction de : Marius PAICU

Soutenue le 1er Juin 2017

Membres du jury :

| | | |
|-----------------------|-----------------------------------|------------|
| M. SUEUR, Franck | Université de Bordeaux | Président |
| M. BRESCH, Didier | Université Savoie Mont Blanc | rapporteur |
| M. CHARVE, Frédéric | Université Paris-Creteil | rapporteur |
| M. DEL SANTO, Daniele | Università degli Studi di Trieste | Examineur |
| M. LANNES, David | Université de Bordeaux | Examineur |

Titre: Sur quelques modèles des fluides géophysiques

Résumé : Dans cette thèse nous étudions trois modèles décrivant la dynamique de l'écoulement d'un fluide à densité variable, dans des échelles spatio-temporelles grandes. Dans ce cadre, le mouvement relatif induit par des forces extérieures, comme la force de Coriolis ou la poussée hydrostatique, s'avère être beaucoup plus important que le mouvement intrinsèque du fluide induit par le transport des particules. Une tel déséquilibre contraint ainsi le mouvement, induisant des structures persistantes dans l'écoulement du fluide.

D'un point de vue mathématique, l'une des difficultés consiste en l'étude des perturbations induites par les forces extérieures, qui se propagent à grande vitesse. Ce type d'analyse peut être effectué au moyen de plusieurs outils mathématiques ; on choisit ici d'employer des techniques caractéristiques de l'analyse de Fourier, comme l'analyse des propriétés dispersives des intégrales oscillantes.

Tout au long de cette thèse, on se restreint à considérer des domaines spatiaux sans frontière : c'est le cas de l'espace entier, ou encore de l'espace périodique. Les modèles considérés sont donc les suivants:

- Équations primitives dont les nombres de Froude et de Rossby sont comparables, et pour lesquelles la diffusion verticale est nulle,
- fluides stratifiés dans un régime à faible nombre de Froude,
- fluides faiblement compressibles et tournants dans un régime où les nombres de Mach et de Rossby sont comparables.

On prouve que ces systèmes propagent globalement dans le temps des données peu régulières. Nous n'imposons jamais de condition de petitesse sur les données initiales. Toutefois, on prendra en compte certaines hypothèses spécifiques de régularité, lorsque des raisons techniques l'imposent.

Mots clés : équations de Navier-Stokes, dynamique des fluides, fluides géophysiques, inégalités de Strichartz

Title : On some models in geophysical fluids

Abstract : In this thesis we discuss three models describing the dynamics of density-dependent fluids in long lifespans and on a planetary scale. In such setting the relative displacement induced by various external physical forces, such as the Coriolis force and the stratification buoyancy, is far more relevant than the intrinsic motion generated by the collision of particles of the fluid itself. Such disproportion of balance limits hence the motion, inducing persistent structures in the velocity flow.

On a mathematical level one of the main difficulties relies in giving a full description of the perturbations induced by the external forces, which propagate at high speed. This analysis can be performed by the aid of several tools, we chose here to adopt techniques characteristic of harmonic analysis, such as the analysis of the dispersive properties of highly oscillating integrals.

All along the thesis we consider boundary-free, three-dimensional domains, and in specific we study only the case in which the domain is either the whole space or the periodic space . The models we consider are the following ones:

- Primitive equations with comparable Froude and Rossby number and zero vertical diffusivity,
- density-dependent stratified fluids in low Froude number regime,
- Weakly compressible and fast rotating fluid in a regime in which Mach and Rossby number are comparable.

We prove that these systems propagate globally-in-time data with low-regularity. No smallness assumption is ever made, specific constructive hypothesis are assumed on the initial data when required.

Keywords: Navier-Stokes equations, fluid dynamics, geophysical fluids, Strichartz estimates

*Dedicated to my family,
without them nothing would have been possible.*

It's like asking why is Ludwig van
Beethoven's Ninth Symphony beautiful.
If you don't see why, someone can't tell
you. I know numbers are beautiful. If
they aren't beautiful, nothing is.

Paul Erdős

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Chapter 1

Introduction (version française).

Mathematics allows for no hypocrisy
and no vagueness.

Stendhal

L'objectif principal de cette thèse est l'étude de l'évolution de plusieurs phénomènes physiques pouvant être décrits par une famille d'équations aux dérivées partielles paramétrées par des grandeurs physiques. En particulier on s'intéresse aux régimes asymptotiques quand ces paramètres explosent.

Il existe une grande quantité de phénomènes physiques dans la dynamique est bien approximée, dans un premier temps, par la limite formelle de certains systèmes lorsque des paramètres tendent à l'infini. Pour cette raison il est intéressant d'étudier ce type d'équations

Pour une construction physique physique complète des systèmes traités dans cette thèse on renvoie le lecteur au Chapitre 2 et à la monographie [50].

L'étude des fluides géophysiques est une branche de la mécanique des fluides. Les modèles étudiés sont généralement obtenus grâce à une analyse multi-échelle sur des systèmes qui décrivent l'évolution d'un fluide hors d'un système de référence planétaire, dans des temps longs. Plusieurs simplifications mathématiques doivent être apportées à ce système pour être étudié mathématiquement.

Les fluides étudiés peuvent avoir une grande variété de propriétés physiques (compressibilité/incompressibilité, inhomogénéité, etc.), mais en général, les systèmes les décrivant à une échelle planétaire partagent une caractéristique particulière : il y a une force extérieure très forte (en magnitude) qui agit sur le fluide (force de Coriolis, poussée hydrostatique, perturbations acoustiques...). La présence d'une telle force extérieure comporte des propriétés de rigidité sur l'écoulement du fluide, et donc contraint le mouvement. Une telle propriété de rigidité peut donc être utilisée pour prouver que des modèles hydrodynamiques tridimensionnels sont globalement bien posés dans des espaces invariants par le changement

d'échelle

$$u(t, x) \mapsto \lambda u(\lambda^2 t, \lambda x).$$

1.1 Formulation mathématique du problème.

La méthodologie que l'on appliquera à l'étude de ce type de problème varie beaucoup selon la géométrie du domaine considéré. Dans cette thèse on considère seulement les cas de l'espace entier tridimensionnel \mathbb{R}^3 , et d'un espace périodique tridimensionnel \mathbb{T}^3 . On considère donc le problème à perturbation singulière générique suivante:

$$\begin{cases} \partial_t U^\varepsilon(t, x) + \mathcal{B}(U^\varepsilon(t, x), U^\varepsilon(t, x)) + A_2 U^\varepsilon(t, x) \\ \quad + \frac{1}{\varepsilon} \mathcal{S}(U^\varepsilon(t, x)) = 0, & (t, x) \in \mathbb{R}_+ \times \Omega \\ U^\varepsilon(t, x)|_{t=0} = U_0^\varepsilon(x), & x \in \Omega. \end{cases} \quad (\text{SPP}_\varepsilon)$$

Dans (SPP_ε) la forme bilinéaire \mathcal{B} est de la forme

$$\mathcal{B}(u, v) = \sum_{i,j=1}^3 q_{i,j}(D)(u \otimes v),$$

où les coefficients $q_{i,j}$ sont des multiplicateurs de Fourier d'ordre un symétriques par rapport à i et j . Par simplicité on peut toutefois identifier \mathcal{B} avec la forme de transport $\mathcal{B}(u, v) \sim u \cdot \nabla v$. L'opérateur A_2 est un opérateur différentiel elliptique d'ordre deux et \mathcal{S} est un perturbation linéaire et antisymétrique. On s'intéresse à l'analyse de solutions fortes de (SPP_ε) quand $\varepsilon \rightarrow 0$, le domaine Ω est toujours considéré comme \mathbb{R}^3 où \mathbb{T}^3 , mais il peut avoir des formes plus génériques. Quand le régime s'approche à celui de la limite $\varepsilon \rightarrow 0$ les solutions du (SPP_ε) se "divisent" en deux parties:

- \bar{U}^ε qui est la projection de U^ε sur $\ker \mathcal{S}$,
- $\tilde{U}^\varepsilon = U^\varepsilon - \bar{U}^\varepsilon$ qui est une partie fortement oscillante de la solution.

On peut donc étudier la décomposition $\bar{U}^\varepsilon, \tilde{U}^\varepsilon$ tel que $U^\varepsilon = \bar{U}^\varepsilon + \tilde{U}^\varepsilon$. Si on projette l'équation (SPP_ε) sur le noyau de l'opérateur pénalisé $\ker \mathcal{S}$ l'équation résultante, qui est satisfaite par \bar{U}^ε , peut changer sensiblement selon la structure même de l'espace $\ker \mathcal{S}$. Néanmoins il y a une caractéristique qui reste invariante: la projection de la perturbation linéaire singulière $\frac{1}{\varepsilon} \mathcal{S}$ est nulle, et donc l'équation satisfaite par \bar{U}^ε ne présente plus un comportement turbulent. L'écoulement pénalisé \bar{U}^ε est (généralement) un champ de vecteurs tridimensionnel, mais il partage parfois des propriétés caractéristiques des écoulements bidimensionnels. Par exemple, il est parfois possible définir un "tourbillon modifié" qui satisfait, soit une équation de transport, soit une équation de transport-diffusion (comme dans le cas d'Euler ou Navier-Stokes bidimensionnel). Cette propriété est la clé qui nous permettra (éventuellement) de prouver que \bar{U}^ε , la partie pénalisée de la solution, est globalement bien posée.

La partie à hautes oscillations \tilde{U}^ε doit être traitée différemment selon la géométrie de l'espace:

1. $\Omega = \mathbb{R}^3$. L'équation satisfaite par \tilde{U}^ε présente encore une perturbation singulière linéaire. Il est donc possible (pour cette partie seulement de la solution) d'appliquer des outils d'analyse harmonique, à savoir les estimations de Strichartz, pour prouver que cette partie converge fortement vers zéro.
2. $\Omega = \mathbb{T}^3$. Dans ce cas-ci il peut y avoir une interaction constructive des interactions fortement oscillantes (résonance). Une étude détaillée de l'ensemble de résonance détermine des conditions géométriques qui doivent être satisfaites afin de ne pas avoir d'interactions entre oscillations fortes, et donc simplifier l'équation satisfaite par U_{osc} .

1.2 Contributions de la thèse.

1.2.1 Équations primitives.

Les équations primitives décrivent l'évolution d'un fluide sous l'effet de la rotation de la terre et de la poussée hydrostatique dans de grandes échelles spatio-temporelles. L'atmosphère terrestre et les océans sont ici nos domaines de référence. Ces domaines rentrent dans la classification de *domaines minces*, (longueur caractéristique verticale ~ 10 km, longueur caractéristique horizontale $\sim 10^3$ km). Ce type de domaine est également déterminé par une échelle spatiale verticale caractéristique bien plus petite de celle horizontale. Cette particularité géométrique du domaine est à la base du phénomène suivant: les forces de frottement dans la direction verticale sont négligeables par rapport à celles horizontales (voir [126] pour une description plus détaillée) c'est à dire que l'on considère le système suivant:

$$\left\{ \begin{array}{l} \partial_t u^{1,\varepsilon} + u^\varepsilon \cdot \nabla u^{1,\varepsilon} - \nu_h \Delta_h u^{1,\varepsilon} - \frac{1}{\varepsilon} u^2 = -\frac{1}{\varepsilon} \partial_1 \Phi_\varepsilon \\ \partial_t u^{2,\varepsilon} + u^\varepsilon \cdot \nabla u^{2,\varepsilon} - \nu_h \Delta_h u^{2,\varepsilon} + \frac{1}{\varepsilon} u^{1,\varepsilon} = -\frac{1}{\varepsilon} \partial_2 \Phi_\varepsilon \\ \partial_t u^{3,\varepsilon} + u^\varepsilon \cdot \nabla u^{3,\varepsilon} - \nu_h \Delta_h u^{3,\varepsilon} + \frac{1}{F \varepsilon} \theta^\varepsilon = -\frac{1}{\varepsilon} \partial_3 \Phi_\varepsilon \\ \partial_t \theta^\varepsilon + u^\varepsilon \cdot \nabla \theta^\varepsilon - \nu'_h \Delta_h \theta^\varepsilon - \frac{1}{F \varepsilon} u^{3,\varepsilon} = 0 \\ \operatorname{div} u^\varepsilon = 0, \\ (u^\varepsilon, \theta^\varepsilon)|_{t=0} = (u_0, \theta_0) = V_0. \end{array} \right. \quad (\text{PEA}_\varepsilon)$$

Pour une interprétation physique du système (PEA_ε) on renvoie le lecteur au Chapitre 2.

On considère le suivante domaine périodique

$$\mathbb{T}^3 = \mathbb{R}^3 / \prod_{i=1}^3 a_i \mathbb{Z} = \prod_{i=1}^3 [0, 2\pi a_i].$$

Les paramètres a_i , $i = 1, 2, 3$ doivent satisfaire la condition suivante:

Definition 1.2.1. On dit que un tore \mathbb{T}^3 satisfait la condition (\mathcal{P}) si une des deux conditions suivantes est satisfaite:

1. \mathbb{T}^3 n'est pas résonant,
2. si \mathbb{T}^3 est résonante, le nombre de Froude F^2 est rationnel et
 - soit $a_3^2/a_1^2 \in \mathbb{Q}$ et a_3^2/a_2^2 n'est pas un nombre algébrique de degré plus petit ou égal à quatre,
 - soit $a_3^2/a_2^2 \in \mathbb{Q}$ et a_3^2/a_1^2 n'est pas un nombre algébrique de degré plus petit ou égal à quatre.

On renvoie à la Définition 3.1.6 pour une définition de domaine (périodique) résonante.

Le système (PEA $_\varepsilon$) est un système non-linéaire du type mixte parabolique-hyperbolique. Les espaces de Sobolev anisotropes suivantes sont bien adaptés pour étudier Ce type de problème:

$$H^{s,s'}(\mathbb{T}^3) = \left\{ u \in \mathcal{S} \left| \sum_{n \in \mathbb{Z}^3} (1 + |\check{n}_h|^2)^s (1 + |\check{n}_3|^2)^{s'} |\hat{u}_n|^2 < \infty \right. \right\},$$

avec $s, s' \in \mathbb{R}$ et $\check{n} = (n_1/a_1, n_2/a_2, n_3/a_3)$ avec $a_i, i = 1, 2, 3$ paramètres du tore. Notamment on étudie dans ce travail des espaces $H^{s,s'}(\mathbb{T}^3)$ avec s' suffisamment grand pour avoir une inclusion du type $H^{s'}(\mathbb{T}_v^1) \hookrightarrow L^\infty(\mathbb{T}_v^1)$.

Dans ces espaces il est possible prouver le théorème suivant

Theorem 1.2.2. *Soit $s > 1/2$ et $V_0 \in H^{0,s}$ un champ de vecteurs à divergence nulle. Il existe donc un $T = T_{V_0} > 0$ indépendant de ε et une unique solution V^ε du système (PEA $_\varepsilon$) qui appartient à l'espace*

$$U^\varepsilon \in \mathcal{C}([0, T]; H^{0,s}), \quad \nabla_h U^\varepsilon \in L^2([0, T]; H^{0,s}).$$

Si en plus $\|V_0\|_{H^{0,s}} \leq c \min\{\nu_h, \nu'_h\}$ alors $T = T_{V_0} = \infty$.

Le système (PEA $_\varepsilon$) est égal à

$$\begin{cases} \partial_t U^\varepsilon + u^\varepsilon \cdot \nabla U^\varepsilon - \mathbb{D}U^\varepsilon + \frac{1}{\varepsilon} \mathcal{A}U^\varepsilon = -\frac{1}{\varepsilon} \nabla \Phi_\varepsilon, \\ \operatorname{div} u^\varepsilon = 0, \\ U^\varepsilon|_{t=0} = (u_0, \theta_0), \end{cases}$$

avec

$$\mathbb{D} = \begin{pmatrix} \nu \Delta_h & 0 & 0 & 0 \\ 0 & \nu \Delta_h & 0 & 0 \\ 0 & 0 & \nu \Delta_h & 0 \\ 0 & 0 & 0 & \nu' \Delta_h \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{-1} \\ 0 & 0 & -F^{-1} & 0 \end{pmatrix}.$$

Définissons maintenant l'opérateur

$$\mathbb{P} = \left(\frac{1 - \Delta^{-1} \nabla \operatorname{div}}{0} \middle| \begin{array}{c} 0 \\ 1 \end{array} \right), \quad (1.2.1)$$

qui agit comme l'opérateur de Leray sur les premières trois composantes et comme l'identité sur la quatrième. On définit le groupe de transformations

$$\mathcal{L}(\tau) = e^{-\tau \mathbb{P}A},$$

et l'inconnue auxiliaire

$$V^\varepsilon = \mathcal{L}\left(\frac{t}{\varepsilon}\right) \mathbb{P} U^\varepsilon,$$

qui satisfait l'équation suivante

$$\begin{cases} \partial_t V^\varepsilon + \mathcal{Q}^\varepsilon(V^\varepsilon, V^\varepsilon) - \mathbb{D}^\varepsilon V^\varepsilon = 0, \\ V^\varepsilon|_{t=0} = (u_0, \theta_0), \end{cases} \quad (1.2.2)$$

où

$$\begin{aligned} \mathcal{Q}^\varepsilon(V^\varepsilon, V^\varepsilon) &= \mathcal{L}\left(-\frac{t}{\varepsilon}\right) \mathbb{P} \left[\mathcal{L}\left(\frac{t}{\varepsilon}\right) V^\varepsilon \cdot \nabla \mathcal{L}\left(\frac{t}{\varepsilon}\right) V^\varepsilon \right], \\ \mathbb{D}^\varepsilon V^\varepsilon &= \mathcal{L}\left(-\frac{t}{\varepsilon}\right) \mathbb{D} \mathcal{L}\left(\frac{t}{\varepsilon}\right) V^\varepsilon. \end{aligned}$$

Il est possible de prouver (comme dans [79], [72] ou [125], situations que nous envisagerons plus tard) que le système (1.2.2) admet une limite distributionnelle de la forme

$$\begin{cases} \partial_t V + \mathcal{Q}(V, V) - \overline{\mathbb{D}}V = 0, \\ V|_{t=0} = (u_0, \theta_0). \end{cases} \quad (1.2.3)$$

La formulation explicite des formes limites $\mathcal{Q}, \overline{\mathbb{D}}$ est omise pour le moment.

Une caractéristique intéressante du système (1.2.3) est qu'il admet des solutions à la *Leray*. Celle-ci n'est pas une conséquence de l'application du théorème de Leray au système filtré (1.2.2). L'opérateur \mathbb{D}^ε fait défaut au terme diffusif verticale et donc l'espace anisotrope de Sobolev $H^{1,0}$ n'est pas compactement inclus dans L^2 , donc aucun argument de compacité peut fonctionner dans ce cas.

Malgré ce manque d'inclusions compactes, on peut contourner la difficulté en remarquant que l'opérateur quadratique $\mathcal{Q}(V, V)$ agit localement dans l'espace de Fourier. En particulier les modes qui donnent une interaction bilinéaire sont les solutions d'une équation polynomiale du type

$$P(X) = 0,$$

avec $X \in \mathbb{Z}^9$. En définissant $X = (X_1, X')$ et en fixant $X' \in \mathbb{Z}^8$ on se retrouve avec une équation unidimensionnelle du type $P(X_1, X') = 0$ dont les solutions peuvent s'écrire de la forme $X_1(P, X')$, et naturellement elles sont en nombre fini en vertu du théorème fondamentale de l'algèbre. Il est donc possible d'appliquer le résultat prouvé dans [112] pour déduire que

$$|X_1(P, X')| \leq K_P(X'),$$

où K_P est une fonction polynômiale de la variable X' .

Cette localisation anisotrope des racines nous permet de transformer une dérivé verticale ∂_3 dans un multi-indice $(C_1 \partial_1^{N_1}, C_2 \partial_2^{N_2})$, avec, éventuellement C_1, C_2, N_1, N_2 grands. Cette observation clé nous permet donc de prouver le théorème suivant [137]

Theorem 1.2.3. Soit \mathbb{T}^3 un tore tridimensionnel, considérons $U_0 \in L^2(\mathbb{T}^3)$ un champ de vecteurs à divergence nulle tel que $\Omega_0 = -\partial_2 u_0^1 + \partial_1 u_0^2 - F\partial_3 \theta_0 \in L^2(\mathbb{T}^3)$. Il existe donc une solution distributionnelle V de (1.2.3) telle que

$$V \in L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^3)) \quad \nabla_h V \in L^2(\mathbb{R}_+; L^2(\mathbb{T}^3)),$$

et que satisfait, pour chaque $t > 0$, l'estimation

$$\|V(t)\|_{L^2(\mathbb{T}^3)}^2 + c \int_0^t \|\nabla_h V(s)\|_{L^2(\mathbb{T}^3)}^2 ds \leq C \|U_0\|_{L^2(\mathbb{T}^3)}^2,$$

avec $c = \min\{\nu_h, \nu'_h\} > 0$.

Il est possible d'étendre le résultat du Théorème 1.2.3 au cadre des solutions fortes si la donnée initiale est plus régulière (dans le sens de régularité anisotrope) et bien préparé.

Theorem 1.2.4. Soit \mathbb{T}^3 un tore tridimensionnel qui satisfait la condition (P) et $U_0 \in H^{0,s}$ un champ vecteur à divergence nulle et à moyenne horizontale nulle tel que $\Omega_0 \in H^{0,s}$, pour $s \geq 1$ et $F \neq 1$. Le système limite (1.2.3) admet une solution globale tel que

$$V \in L^\infty(\mathbb{R}_+; H^{0,s}) \quad \nabla_h V \in L^2(\mathbb{R}_+; H^{0,s}),$$

et qui satisfait l'estimation suivante

$$\|V(t)\|_{H^{0,s}}^2 + c \int_0^t \|\nabla_h V(s)\|_{H^{0,s}}^2 ds \leq \mathcal{E}(\|U_0\|_{H^{0,s}}^2),$$

où \mathcal{E} c'est une fonction réelle bornée sur les ensembles bornés. La solution U est unique dans la topologie de $L^\infty(\mathbb{R}_+; H^{0,\sigma}) \cap L^2(\mathbb{R}_+; H^{1,\sigma})$ pour $\sigma \in [-1/2, s)$.

À ce stade, on peut donc s'interroger sur la question du comportement de solutions du (PEA $_\varepsilon$) par rapport aux solutions (maintenant globales) de (1.2.3) quand ε est très petit.

Theorem 1.2.5. Soit $\mathbb{T}^3 \subset \mathbb{R}^3$ un tore qui satisfait la condition (P), $\Omega_0 = -\partial_2 u_0^1 + \partial_1 u_0^2 - F\partial_3 \theta_0 \in H^{0,s}$, $U_0 \in H^{0,s}$ de moyenne horizontale nulle, alors

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left(U^\varepsilon - \mathcal{L} \left(\frac{t}{\varepsilon} \right) V \right) &= 0 && \text{dans } \mathcal{C}(\mathbb{R}_+; H^{0,\sigma}) \\ \lim_{\varepsilon \rightarrow 0} \nabla_h \left(U^\varepsilon - \mathcal{L} \left(\frac{t}{\varepsilon} \right) V \right) &= 0 && \text{dans } L^2(\mathbb{R}_+; H^{0,\sigma}) \end{aligned}$$

pour $\sigma < s$ et V c'est la seule solution du système (1.2.3).

1.2.2 Dynamique à nombre de Froude petit.

Supposons qu'on néglige l'effet de la rotation terrestre dans les équations (PEA_ε). La poussée hydrostatique est donc la seule force qui agit sur le fluide. Etant donné qu'on s'intéresse aux fluides stratifiés sans rotation, cela n'a plus de sens de considérer des grandes échelles spatiales (le nombre d'Ekman devient donc proche de 1 et non plus négligeable) mais on considère toujours une dynamique à temps longs. Un tel fluide est décrit par les équations

$$\begin{cases} \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \nu \Delta u^\varepsilon - \frac{1}{\varepsilon} \theta^\varepsilon \vec{e}_3 &= -\frac{1}{\varepsilon} \nabla \Phi^\varepsilon, \\ \partial_t \theta^\varepsilon + u^\varepsilon \cdot \nabla \theta^\varepsilon - \nu' \Delta \theta^\varepsilon + \frac{1}{\varepsilon} u^{3,\varepsilon} &= 0, \\ \operatorname{div} u^\varepsilon &= 0, \\ (u^\varepsilon, \theta^\varepsilon)|_{t=0} &= U_0 = (u_0, \theta_0). \end{cases} \quad (\text{PBS}_\varepsilon)$$

Pour une description complète du système (PBS_ε) on renvoie le lecteur au Chapitre 5. Il est donc clair que (PBS_ε) peut s'écrire sous la forme

$$\begin{cases} \partial_t U^\varepsilon + U^\varepsilon \cdot \nabla U^\varepsilon - \mathbb{D}U^\varepsilon + \frac{1}{\varepsilon} \mathcal{A}U^\varepsilon &= -\frac{1}{\varepsilon} \begin{pmatrix} \nabla \Phi^\varepsilon \\ 0 \end{pmatrix}, \\ \operatorname{div} u^\varepsilon &= 0, \\ U^\varepsilon &= (u^\varepsilon, \theta^\varepsilon), \end{cases} \quad (\text{PBS}_\varepsilon)$$

avec

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \mathbb{D} = \begin{pmatrix} \nu \Delta & 0 & 0 & 0 \\ 0 & \nu \Delta & 0 & 0 \\ 0 & 0 & \nu \Delta & 0 \\ 0 & 0 & 0 & \nu' \Delta \end{pmatrix}.$$

Le cas de l'espace périodique.

Il est donc possible de définir l'opérateur \mathbb{P} comme dans (1.2.1) et la variable auxiliaire

$$V^\varepsilon = \mathcal{L} \left(\frac{t}{\varepsilon} \right) U^\varepsilon.$$

La fonction V^ε satisfait donc l'équation

$$\begin{cases} \partial_t V^\varepsilon + \mathcal{Q}^\varepsilon(V^\varepsilon, V^\varepsilon) - \mathbb{D}^\varepsilon V^\varepsilon &= 0, \\ V^\varepsilon|_{t=0} &= (u_0, \theta_0), \end{cases} \quad (1.2.4)$$

où

$$\begin{aligned} \mathcal{Q}^\varepsilon(V^\varepsilon, V^\varepsilon) &= \mathcal{L} \left(-\frac{t}{\varepsilon} \right) \mathbb{P} \left[\mathcal{L} \left(\frac{t}{\varepsilon} \right) V^\varepsilon \cdot \nabla \mathcal{L} \left(\frac{t}{\varepsilon} \right) V^\varepsilon \right], \\ \mathbb{D}^\varepsilon V^\varepsilon &= \mathcal{L} \left(-\frac{t}{\varepsilon} \right) \mathbb{D} \mathcal{L} \left(\frac{t}{\varepsilon} \right) V^\varepsilon. \end{aligned}$$

On peut alors prouver que le système (2.8.2) converge (dans un sens distributionnel) vers

$$\begin{cases} \partial_t V + \mathcal{Q}(V, V) - \mathbb{D}V = 0, \\ V|_{t=0} = (u_0, \theta_0). \end{cases} \quad (1.2.5)$$

La définition explicite des opérateurs \mathcal{Q} , \mathbb{D} est omise pour le moment.

Il est naturel se demander dans quel sens (1.2.4) converge vers (1.2.5). L'opérateur du filtrage \mathcal{L} transforme le système (PBS $_\varepsilon$) en (1.2.4) : le dernière ne présente plus un opérateur singulier, donc il est possible trouver des bornes uniformes sur la suite $(\partial_t V^\varepsilon)_{\varepsilon>0}$ dans l'espace $L^p_{\text{loc}}(\mathbb{R}_+; H^{-N})$ avec N suffisamment grand et $p \in [2, \infty]$. Des arguments standard de compacité comme le lemme de Aubin-Lions [4] prouvent qu'à extraction près il y a la convergence

$$V^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} V \quad \text{in } L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3)).$$

Malheureusement ce type de raisonnement ne prouve pas que l'interaction bilinéaire $\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon)$ converge vers une interaction limite $\mathcal{Q}(U, U)$. Ce résultat peut être prouvé au moyen d'une application du théorème de phase non-stationnaire. Pour plus de détails on renvoi au Chapitre 4.

On considère $U_0 = (u_0, \theta_0)$ de moyenne horizontale nulle, i.e.

$$\frac{1}{|\mathbb{T}_h^2|} \int_{\mathbb{T}_h^2} U_0(y_h, x_3) \, dy_h = 0.$$

La moyenne globale des champ de vecteurs est toujours considérée comme nulle.

Définissons

$$\bar{U}_0 = \Delta_h^{-1} \begin{pmatrix} -\partial_2 \\ \partial_1 \\ 0 \\ 0 \end{pmatrix} (-\partial_2 u_0^1 + \partial_1 u_0^2), \quad U_{\text{osc},0} = U_0 - \bar{U}_0,$$

le premier résultat qui est prouvé est le suivant :

Theorem 1.2.6. *Soit V la solution limite distributionnelle de (1.2.5), alors V peut être écrit comme*

$$V(x) = \bar{V}(x_h, x_3) + V_{\text{osc}}(x_h, x_3),$$

où \bar{V} , V_{osc} sont respectivement des solutions faibles des systèmes suivantes

$$\begin{cases} \partial_t \bar{v}^h + \bar{v}^h \cdot \nabla_h \bar{v}^h - \nu \Delta \bar{v}^h = -\nabla_h \bar{p}, \\ \bar{v}^h|_{t=0} = \bar{u}_0^h, \end{cases} \quad (1.2.6)$$

$$\begin{cases} \partial_t V_{\text{osc}} + 2\mathcal{Q}(\bar{V}, V_{\text{osc}}) - (\nu + \nu') \Delta V_{\text{osc}} = 0, \\ V_{\text{osc}}|_{t=0} = U_{\text{osc},0}, \end{cases} \quad (1.2.7)$$

avec \mathcal{Q} définie dans (4.3.2) pour presque tous $(a_1, a_2, a_3) \in \mathbb{R}^3$ paramètres du tore $\mathbb{T}^3 = \prod_i [0, 2\pi a_i]$.

La forme bilinéaire limite \mathcal{Q} dans (1.2.5) est bien définie seulement pour des interactions bilinéaires dont les deux premières composantes sont de moyenne horizontale nulle. Il est donc important de prouver qu'à la limite $\varepsilon \rightarrow 0$ la moyenne horizontale de $\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon)$ est nulle, au moins au sens de distributions. Nous prouvons donc le résultat suivant:

Lemma 1.2.7. *Au sens de distributions, la limite*

$$\lim_{\varepsilon \rightarrow 0} \left(\int_{\mathbb{T}_h^2} (\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon))^h dx_h, 0, 0 \right) = 0,$$

est vraie.

L'avantage d'étudier (1.2.5) comme superposition du systèmes (1.2.6)–(1.2.7) réside dans le fait que le système (1.2.5) présente les mêmes difficultés que les equations de Navier-Stokes tridimensionnelles pour la propagation de la régularité des interactions bilinéaires dans \mathcal{Q} . Les systèmes (1.2.6)–(1.2.7) sont respectivement un système bidimensionnel de Navier-Stokes et un système linéaire: ça nous permet de prouver le résultat suivante:

Theorem 1.2.8. *Soit $\bar{u}_0^h \in L_v^\infty(H_h^\sigma)$ et $\nabla_h \bar{u}_0^h \in L_v^\infty(H_h^\sigma)$ où $\sigma > 0$, alors \bar{V} , solution faible de (1.2.6) est globalement bien définie dans \mathbb{R}_+ , et appartient à l'espace*

$$\bar{v}^h \in \mathcal{C}(\mathbb{R}_+; H^s(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; H^{s+1}(\mathbb{T}^3)), \quad s > 1/2,$$

et pour chaque $t > 0$ l'inégalité suivante est vérifié:

$$\|\bar{v}^h(t)\|_{H^s(\mathbb{T}^3)}^2 + \nu \int_0^t \|\nabla \bar{v}^h(\tau)\|_{H^s(\mathbb{T}^3)}^2 d\tau \leq \mathcal{E}_1(U_0).$$

La fonction \mathcal{E}_1 est définie comme le membre de droit de (4.5.2).

Soit V_{osc} une solution faible de (1.2.7), alors V_{osc} est globalement bien posé et

$$V_{\text{osc}} \in \mathcal{C}(\mathbb{R}_+; H^s(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; H^{s+1}(\mathbb{T}^3)),$$

pour tout $s > 1/2$. Pour chaque $t > 0$ on a la borne suivante:

$$\|V_{\text{osc}}(t)\|_{H^s(\mathbb{T}^3)}^2 + \frac{\nu + \nu'}{2} \int_0^t \|\nabla V_{\text{osc}}(\tau)\|_{H^s(\mathbb{T}^3)}^2 d\tau \leq \mathcal{E}_2(U_0),$$

la fonction \mathcal{E}_2 est définie dans (4.5.19).

Le dernier théorème est un résultat de convergence forte pour les solutions de (PBS_ε) dans le régime $\varepsilon \rightarrow 0$ aux solutions de (1.2.5):

Theorem 1.2.9. *Soit U_0 dans $H^s(\mathbb{T}^3)$ pour $s > 1/2$, si $\varepsilon > 0$ suffisamment petit (PBS_ε) est globalement bien définie dans $\mathcal{C}(\mathbb{R}_+; H^s(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; H^{s+1}(\mathbb{T}^3))$, et, si V est la solution forte globale de (1.2.5), alors*

$$U^\varepsilon - \mathcal{L}\left(-\frac{t}{\varepsilon}\right)V = o(1),$$

dans $\mathcal{C}(\mathbb{R}_+; H^s(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; H^{s+1}(\mathbb{T}^3))$.

Le cas de l'espace entier.

Dans le cas de l'espace entier, motivé par le travail dans l'espace périodique, on considère le système suivant :

$$\begin{cases} \partial_t \bar{u}^h + \bar{u}^h \cdot \nabla_h \bar{u}^h - \nu \Delta \bar{u}^h = -\nabla_h \bar{p} \\ \operatorname{div}_h \bar{u}^h = 0, \\ \bar{u}^h|_{t=0} = \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix} \Delta_h^{-1} (-\partial_2 u_0^1 + \partial_1 u_0^2) = \bar{u}_0^h. \end{cases} \quad (1.2.8)$$

Le système (1.2.8) est un écoulement tridimensionnel (\bar{u}^h dépend de toutes les variables spatiales) qui présente des caractéristiques de fluides en dimension deux. Par exemple le "tourbillon horizontal"

$$\omega^h = -\partial_2 \bar{u}^1 + \partial_1 \bar{u}^2,$$

peut décrire le champ vitesse au travers de la relation de Biot-Savart bidimensionnelle

$$\bar{u}^h = \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix} \Delta_h^{-1} \omega^h,$$

et satisfait l'équation de transport-diffusion

$$\begin{cases} \partial_t \omega^h + \bar{u}^h \cdot \nabla_h \omega^h - \nu \Delta \omega^h = 0, \\ \omega^h|_{t=0} = -\partial_2 u_0^1 + \partial_1 u_0^2 = \omega_0^h. \end{cases} \quad (1.2.9)$$

C'est bien sûr cette structure particulière qui nous permet de prouver le résultat suivant d'existence globale dans des espaces à régularité faible:

Theorem 1.2.10. *Soit $\bar{u}_0^h \in L^2(\mathbb{R}^3) \cap \dot{H}^s(\mathbb{R}^3)$, $s > 0$ tel que $\omega_0^h \in L^2(\mathbb{R}^3)$, il existe donc une solution forte globale de l'équation (1.2.8) dans l'espace*

$$\bar{u}^h \in L^\infty(\mathbb{R}_+; \dot{H}^s(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+; \dot{H}^{s+1}(\mathbb{R}^3)),$$

telle que, pour chaque $t > 0$

$$\begin{aligned} & \|\bar{u}^h(t)\|_{\dot{H}^s(\mathbb{R}^3)}^2 + \nu \int_0^t \|\nabla_h \bar{u}^h(\tau)\|_{\dot{H}^s(\mathbb{R}^3)}^2 d\tau \\ & \leq C \|\bar{u}_0^h\|_{\dot{H}^s(\mathbb{R}^3)}^2 \exp \left\{ \frac{CK^2}{\nu} \left(\|\bar{u}_0^h\|_{L^2(\mathbb{R}^3)}^4 + \|\omega_0^h\|_{L^2(\mathbb{R}^3)}^4 \right) \right\}. \end{aligned} \quad (1.2.10)$$

Si $s \geq 1/2$ la solution \bar{u}^h est de plus unique dans $L^\infty(\mathbb{R}_+; \dot{H}^s(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+; \dot{H}^{s+1}(\mathbb{R}^3))$.

La question de convergence en régime de nombre de Froude petit se pose donc naturellement, en suivant les travaux [42], [38], [28] on s'attend à ce que

$$U^\varepsilon + (\text{terme perturbatif en } \varepsilon) \xrightarrow{\varepsilon \rightarrow 0} \bar{U} = (\bar{u}^h, 0, 0).$$

Theorem 1.2.11. Soit $U_0 \in L^2(\mathbb{R}^3) \cap \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, $\omega_0^h \in L^2(\mathbb{R}^3)$, alors

$$\begin{aligned} U^\varepsilon - W^\varepsilon - (\bar{u}^h, 0, 0)^\top &\xrightarrow{\varepsilon \rightarrow 0} 0, & \text{dans l'espace } L^\infty\left(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\right), \\ \nabla(U^\varepsilon - W^\varepsilon - (\bar{u}^h, 0, 0)^\top) &\xrightarrow{\varepsilon \rightarrow 0} 0, & \text{dans l'espace } L^2\left(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\right), \end{aligned}$$

avec U^ε solution forte locale unique de (PBS_ε) et W^ε, \bar{u}^h unique solution globale de

$$\begin{cases} \partial_t W^\varepsilon - \mathbb{D}W^\varepsilon + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} W^\varepsilon = \begin{pmatrix} 0 \\ 0 \\ -\partial_3 \bar{p} \\ 0 \end{pmatrix}, \\ \operatorname{div} w^\varepsilon = 0, \\ W^\varepsilon|_{t=0} = (\mathbb{P}_- + \mathbb{P}_+) U_0, \end{cases}$$

et (1.2.8).

Les opérateurs $\mathbb{P}_0, \mathbb{P}_\pm$ sont les projections sur les espaces $\mathbb{C}E_0, \mathbb{C}E_\pm$, où E_0, E_\pm sont les vecteurs propres à divergence nulle de $L_\varepsilon = \mathbb{P} \mathcal{A} - \varepsilon \mathbb{D}$. La solution forte U^ε de (PBS_ε) en particulier est globale et est dans l'espace

$$L^\infty\left(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\right) \cap L^2\left(\mathbb{R}_+; \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)\right).$$

1.2.3 Fluides faiblement compressibles et tournantes.

La description physique d'un modèle décrivant le mouvement d'un fluide faiblement compressible sujet à une force de rotation très intense est décrit dans la Partie 2.6. L'équation dérivée est

$$\begin{cases} \partial_t(\rho^{\varepsilon, \theta} u^{\varepsilon, \theta}) + \operatorname{div}(\rho^{\varepsilon, \theta} u^{\varepsilon, \theta} \otimes u^{\varepsilon, \theta}) + \frac{1}{\theta^2} \nabla P(\rho^{\varepsilon, \theta}) + \frac{1}{\varepsilon} e^3 \wedge (\rho^{\varepsilon, \theta} u^{\varepsilon, \theta}) = 0 \\ \partial_t \rho^{\varepsilon, \theta} + \operatorname{div}(\rho^{\varepsilon, \theta} u^{\varepsilon, \theta}) = 0 \\ (\rho^{\varepsilon, \theta}, u^{\varepsilon, \theta})|_{t=0} = (\rho_0^{\varepsilon, \theta}, u_0^{\varepsilon, \theta}), \end{cases}$$

où ε est le nombre de Rossby et θ celui de Mach. Considérons le régime particulier $\varepsilon = \theta$, et introduisons l'hypothèse de faible compressibilité

$$\rho^\varepsilon = 1 + \varepsilon b^\varepsilon.$$

La densité ρ^ε est donc une petite variation autour d'un état constant. C'est le cas par exemple dans les océans et dans la troposphère.

Considérons encore une pression barotrope de la forme

$$P(\rho^\varepsilon) = A \rho^\gamma, \quad A > 0, \quad \gamma > 1,$$

et définissons $\bar{\gamma} = (\gamma - 1)/2$. Avec la substitution

$$1 + \varepsilon b^\varepsilon = \frac{(4\gamma A)^{1/2}}{\gamma - 1} (\rho^\varepsilon)^{\bar{\gamma}},$$

le système considéré devient

$$\begin{cases} \partial_t u^\varepsilon + \frac{1}{\varepsilon} (\bar{\gamma} \nabla b^\varepsilon + e^3 \wedge u^\varepsilon) + u^\varepsilon \cdot \nabla u^\varepsilon + \bar{\gamma} b^\varepsilon \nabla b^\varepsilon = 0 \\ \partial_t b^\varepsilon + \frac{\bar{\gamma}}{\varepsilon} \operatorname{div} u^\varepsilon + u^\varepsilon \nabla b^\varepsilon + \bar{\gamma} b^\varepsilon \operatorname{div} u^\varepsilon = 0 \\ (b^\varepsilon, u^\varepsilon)|_{t=0} = (b_0, u_0). \end{cases} \quad (1.2.11)$$

Le système (1.2.11) peut s'écrire sous la forme compacte

$$\begin{cases} \partial_t \begin{pmatrix} u^\varepsilon \\ b^\varepsilon \end{pmatrix} - \frac{1}{\varepsilon} \mathcal{B} \begin{pmatrix} u^\varepsilon \\ b^\varepsilon \end{pmatrix} + \begin{pmatrix} u^\varepsilon \cdot \nabla u^\varepsilon + \bar{\gamma} b^\varepsilon \nabla b^\varepsilon \\ u^\varepsilon \cdot \nabla b^\varepsilon + \bar{\gamma} b^\varepsilon \operatorname{div} u^\varepsilon \end{pmatrix} = 0, \\ (u^\varepsilon, b^\varepsilon)|_{t=0} = (u_0, b_0), \end{cases} \quad (1.2.12)$$

avec \mathcal{B} défini comme

$$\mathcal{B} = \begin{pmatrix} 0 & 1 & 0 & -\bar{\gamma} \partial_1 \\ -1 & 0 & 0 & -\bar{\gamma} \partial_2 \\ 0 & 0 & 0 & -\bar{\gamma} \partial_3 \\ -\bar{\gamma} \partial_1 & -\bar{\gamma} \partial_2 & -\bar{\gamma} \partial_3 & 0 \end{pmatrix}. \quad (1.2.13)$$

On peut encore modifier le système (1.2.12) en écrivant la nonlinearité sous la forme

$$\begin{pmatrix} u^\varepsilon \cdot \nabla u^\varepsilon + \bar{\gamma} b^\varepsilon \nabla b^\varepsilon \\ u^\varepsilon \cdot \nabla b^\varepsilon + \bar{\gamma} b^\varepsilon \operatorname{div} u^\varepsilon \end{pmatrix} = \mathcal{A}(U, D) U = \begin{pmatrix} u^\varepsilon \cdot \nabla & 0 & 0 & \bar{\gamma} b^\varepsilon \partial_1 \\ 0 & u^\varepsilon \cdot \nabla & 0 & \bar{\gamma} b^\varepsilon \partial_2 \\ 0 & 0 & u^\varepsilon \cdot \nabla & \bar{\gamma} b^\varepsilon \partial_3 \\ \bar{\gamma} b^\varepsilon \partial_1 & \bar{\gamma} b^\varepsilon \partial_2 & \bar{\gamma} b^\varepsilon \partial_3 & u^\varepsilon \cdot \nabla \end{pmatrix} \begin{pmatrix} u^\varepsilon \\ b^\varepsilon \end{pmatrix}. \quad (1.2.14)$$

En se basant sur ces considérations, on peut donc se limiter à l'étude du système

$$\begin{cases} \partial_t U - \frac{1}{\varepsilon} \mathcal{B} U + \mathcal{A}(U, D) U = 0, \\ U|_{t=0} = U_0 = (u_0, b_0). \end{cases} \quad (1.2.15)$$

Le système sous la forme (1.2.15) est un système hyperbolique symétrique avec perturbation antisymétrique, il est donc avantageux de travailler avec un système dans cette forme.

La donnée initiale $U_0 \in H^s(\mathbb{R}^3)$, $s > 5/2$ est tridimensionnelle et donc, grâce à l'analyse faite dans [68] qui montre que les éléments du noyau de \mathcal{B} sont des champ de vecteurs bi-dimensionnels, on s'attend à un résultat de dispersion complète.

Pour prouver cet résultat-ci on va définir

$$U^\varepsilon = (u^\varepsilon, b^\varepsilon) = \bar{U}^\varepsilon + \tilde{U}^\varepsilon$$

où $\bar{U}^\varepsilon = (\bar{u}^\varepsilon, \bar{b}^\varepsilon)$ et $\tilde{U}^\varepsilon = (\tilde{u}^\varepsilon, \tilde{b}^\varepsilon)$ sont respectivement solutions des systèmes suivants

$$\begin{cases} \partial_t \bar{U}^\varepsilon - \frac{1}{\varepsilon} \mathcal{B} \bar{U}^\varepsilon = 0 \\ \bar{U}^\varepsilon|_{t=0} = \Psi_{r,R}(D)(u_0, b_0) \end{cases}, \quad (1.2.16)$$

$$\begin{cases} \partial_t \tilde{U}^\varepsilon - \frac{1}{\varepsilon} \mathcal{B} \tilde{U}^\varepsilon + \mathcal{A}(U, D)U = 0 \\ \tilde{U}^\varepsilon|_{t=0} = (1 - \Psi_{r,R}(D))(u_0, b_0). \end{cases} \quad (1.2.17)$$

L'opérateur $\Psi_{r,R}$ est un opérateur de troncature de fréquences, où les paramètres $0 < r < R$ seront choisis en fonction du paramètre ε . En particulier $\Psi_{r,R}$ est une fonction test (dans l'espace de Fourier) qui est supporté dans $\mathcal{C}_{\frac{\varepsilon}{2}, 2R}$ et qui est identiquement égale à un sur l'ensemble $\mathcal{C}_{r,R}$. On est forcé d'avoir une dépendance explicite de r et R en fonction de ε parce que le système (1.2.15) est du type hyperbolique. On ne peut pas donc absorber plusieurs termes comme dans le cas parabolique. Le système (1.2.16) est bien sûr linéaire, homogène et localisé en fréquences, il est donc possible prouver le suivant résultat dispersif:

Theorem 1.2.12. *Soit $q \in [2, +\infty]$ et $p \geq \frac{4q}{q-2}$. Pour tout $U_0 \in L^2(\mathbb{R}^3)$, le système (1.2.16) admet une solution globale \bar{U}^ε telle que,*

$$\|\bar{U}^\varepsilon\|_{L^p(\mathbb{R}_+; L^q(\mathbb{R}^3))} \leq CR^{\frac{3}{2} - \frac{3}{q} + \frac{4}{p}} r^{-\frac{2}{p}} \varepsilon^{\frac{1}{p}} \|U_0\|_{L^2(\mathbb{R}^3)}.$$

Au contraire le système (1.2.17) est fortement non-linéaire, mais on peut quand même utiliser la symétrie de l'opérateur \mathcal{A} et les propriétés dispersives de \bar{U}^ε pour déduire un résultat d'existence locale dans l'espace $L^\infty([0, T]; H^s)$ avec T indépendant de ε . Un fois que ce résultat est établi, on s'intéresse au contrôle du temps de vie maximale de \tilde{U}^ε .

Theorem 1.2.13. *Soient $s > 5/2$, $s_0 > 0$, $1 < p < 2$ et soit la donnée initiale*

$$U_0 \in Y_{s, s_0, p} = H^{s+s_0}(\mathbb{R}^3) \cap L^2(\mathbb{R}_h^2; L^p(\mathbb{R}_v^1)) \cap L^p(\mathbb{R}_h^2; L^2(\mathbb{R}_v^1)).$$

Il existe un temps $T_\varepsilon^ > 0$ et une unique solution $U^\varepsilon = (u^\varepsilon, b^\varepsilon)$ de (2.8.16) qui satisfait*

$$U^\varepsilon \in L^\infty([0, T_\varepsilon^*]; H^s(\mathbb{R}^3)) \cap C([0, T_\varepsilon^*]; H^s(\mathbb{R}^3)),$$

avec T_ε^ qui explose quand ε converge vers zéro. De façon plus détaillé, il existe un constante $\bar{C} > 0$ et $\alpha > 0$ tel que*

$$T_\varepsilon^* \geq \frac{\bar{C}}{\mathcal{C}(U_0) \varepsilon^\alpha},$$

et $\mathcal{C}(U_0)$ est une constante qui dépende de la donnée initiale seulement.

L'espace $Y_{s, s_0, p}$ est un espace de Banach avec la norme

$$\|u\|_{s, s_0, p} = \max \left\{ \|u\|_{H^{s+s_0}}, \|u\|_{L_h^2 L_v^p}, \|u\|_{L_h^p L_v^2} \right\}.$$

On peut donc maintenant définir la constante $\mathcal{C}(U_0)$, de la façon suivante:

$$\mathcal{C}(U_0) = \max \left\{ \|U_0\|_{s, s_0, p}, \|U_0\|_{s, s_0, p}^2 \right\}.$$

Chapter 2

Introduction.

But in my opinion, all things in nature occur mathematically.

René Descartes

The geophysical fluid dynamic is a discipline which studies the dynamic of naturally occurring flows on a large scale. The physical and mechanical theory on which the study of geophysical fluids builds its roots is general enough to describe motions in both liquid and gas phase, but it focuses on the dynamical properties which are characteristic of large-scale motions.

The present section aims to give a self-contained physical motivation of the study of geophysical fluid systems. The will is to provide a simple chapter explaining the most relevant physical features of geophysical fluids, yet detailed enough to develop a completely self-contained theory without any accessory knowledge of advanced mathematics. The reader which is interested in a more comprehensive discussion on physical motivations of geophysical fluids is referred to the beautiful monograph [50] and references therein, which was as well the main reference in writing this introduction.

The variety of physical phenomena which can be described in terms of geophysical fluids equations is vast and includes physical manifestations which may appear very far related; such as the motion in the Earth's interior responsible of the dynamo effect and the tendency of oceanic currents to move West-Eastwards. We shall in any case consider geophysical dynamics only in the restricted case of liquid and gases motion on a large scale.

Two main ingredients distinguish the discipline from traditional fluid mechanics: the effects of rotation and those of stratification. The predominant influence of one, the other, or both give rise to substantially different dynamics.

Notably the physical occurrence characterizing the motion of a geophysical fluid is the rotation of the Earth around its axis. Such rotation inevitably modifies the motion of any particle on the Earth, and fluids are not immune to such effect. This variation on the velocity flow is twofold: in one stance it forces a particle of a fluid to move toward the outer space, in a second stance it induces a variation of the velocity perpendicularly to the direction of the

motion at a time t . These accelerations can be interpreted, thanks to Newton's second law of dynamic, as forces. These are respectively the centrifugal force and the Coriolis force (for a detailed explanation on the Coriolis force, see [140]). Despite the fact that the centrifugal force is the more palpable of the two on a planetary scale its effect is irrelevant, on the other hand the Coriolis force is fundamental in determining the dynamics of such motions.

The Coriolis force has a rather *bizarre* effect on the overall dynamic of a fluid: if the magnitude of the force is sufficiently large and the fluid homogeneous it constraints the motion to be strictly columnar. This tendency of a rotating fluid to displace himself in vertical homogeneous columns is generally known as *Taylor-Proudman theorem*. Taylor-Proudman theorem was first derived by Sidney Samuel Hough (1870-1923), a mathematician at Cambridge in the work [83], but it was named after the works of G.I. Taylor [141] and Joseph Proudman [130].

The stratification instead is present only in *non-homogeneous fluids*. Gravity is the cause; it tends to lower regions of the fluid with higher density and raise regions with lower density. On a dynamical point of view this mechanism is easily explained: a naturally occurring motion, in absence of external forces, tends to stabilize in a configuration in which the potential of the forces acting on it is minimal, in such configuration the motion stops. It is hence natural to imagine that a fluid which is composed of horizontally stacked layers of decreasing density is not affected any more (dynamically) by gravity.

2.1 A gentle introduction to the main physical actors.

Let us give a first glimpse in the main physical forces which constrain the motion at a macroscopic scale: the rotation and the gravitational stratification.

2.1.1 The rotation.

As we mentioned above the rotation of the Earth around his axis (Coriolis force) is one of the most relevant constraint in the motion of a geophysical flow. This fact is completely counter-intuitive for an observer placed on the surface of the Earth: the relative location with respect to a certain reference system trick us to suppose that we live in an inertial reference system. However, in a sufficiently large lifespan (say a week), (almost) any relative motion performed on the surface of the Earth is negligible with respect to the relative motion induced by the rotation of the planet in the space.

A rather important question when it comes to study rotating fluids is to determinate a reasonable criterion in order to describe the influence of the rotation on the motion. To answer this question, we must first define the ambient rotation rate, which we denote by Ω and define as:

$$\Omega = \frac{2\pi \text{ radians}}{\text{time of revolution}}.$$

Let us take in consideration a time-scale T comparable or bigger to the characteristic time of rotation. We define the dimensionless quantity

$$\omega = \frac{\text{time of one revolution}}{\text{motion time-scale}} = \frac{2\pi/\Omega}{T} = \frac{2\pi}{\Omega T}.$$

We shall use implicitly the following criterion: if ω is comparable to the unity or smaller ($\omega \lesssim 1$) then the rotation should be taken in consideration. On the earth, loosely speaking, if the characteristic time-span T is bigger than 24 hours than the rotation is not any more a negligible effect in the relative motion of a fluid for an observer situated outside the planet.

The limit of the above criterion resides in the fact that it depends exclusively on a ratio between characteristic rotation time and observation time. If a motion occur in a short time-scale but it is bestowed with a large spatial imprint the rotation effects shall be non-negligible again. Whence we define a second more adapted criterion : we consider the velocity and length scales of the motion, and we denote them respectively as U and L . Naturally, if a particle traveling at the speed U covers the distance L in a time longer than or comparable to a rotation period, we expect the trajectory to be influenced by the ambient rotation, and so we write

$$\begin{aligned} \varepsilon &= \frac{\text{time of one revolution}}{\text{time taken by particle to cover distance } L \text{ at speed } U}, \\ &= \frac{2\pi/\Omega}{L/U} = \frac{2\pi U}{\Omega L}. \end{aligned} \quad (2.1.1)$$

Hence we can replace the criterion $\omega \lesssim 1$ above with the more general $\varepsilon \lesssim 1$ in order to determinate whether the rotation has a strong influence on the motion.

The one above is a very brief introduction which explains how, in some determinate, simplified models, the rotation may become a significant factor once large space- and time-scales are considered. Let us give now hence a quick glimpse to the mathematical consequences of the Coriolis force in some inertial¹ reference system.

To facilitate the mathematical developments, let us first investigate the two-dimensional case. Let the X - and Y -axes form the inertial framework of reference and the x - and y -axes be those of a framework with the same origin but rotating at the angular rate Ω (defined as positive in the trigonometric sense). The correspondent unit vectors will be denoted as (\mathbf{I}, \mathbf{J}) and (\mathbf{i}, \mathbf{j}) respectively. At any time t , the rotating x -axis makes an angle Ωt with the fixed X -axis. It follows that

$$\begin{aligned} \mathbf{i} &= +\mathbf{I} \cos(\Omega t) + \mathbf{J} \sin(\Omega t), & \mathbf{I} &= +\mathbf{i} \cos(\Omega t) - \mathbf{j} \sin(\Omega t), \\ \mathbf{j} &= -\mathbf{I} \sin(\Omega t) + \mathbf{J} \cos(\Omega t), & \mathbf{J} &= +\mathbf{i} \sin(\Omega t) + \mathbf{j} \cos(\Omega t). \end{aligned} \quad (2.1.2)$$

Let us define the *position vector* \mathbf{r} . In the two reference systems it is defined as

$$\begin{aligned} \mathbf{r} &= X\mathbf{I} + Y\mathbf{J}, \\ &= x\mathbf{i} + y\mathbf{j}. \end{aligned}$$

¹By inertial we mean a reference system which is *not* rotating with the planet, i.e. a reference system which is fixed with respect to the distant stars, and whose origin is fixed for practical purposes at the center of the earth.

The coordinate functions (X, Y) and (x, y) are related by

$$\begin{aligned} x &= +X \cos(\Omega t) + Y \sin(\Omega t), \\ y &= -X \sin(\Omega t) + Y \cos(\Omega t). \end{aligned}$$

It suffice hence to differentiate in time the above relations in order to deduce the velocity field $(\dot{x}, \dot{y}) = (u, v)$

$$\begin{aligned} u &= \dot{x} = +\dot{X} \cos(\Omega t) + \dot{Y} \sin(\Omega t) + \Omega y, \\ v &= \dot{y} = -\dot{X} \sin(\Omega t) + \dot{Y} \cos(\Omega t) - \Omega x, \end{aligned} \quad (2.1.3)$$

and indeed

$$\begin{aligned} \mathbf{u} &= \dot{x} \mathbf{i} + \dot{y} \mathbf{j}, \\ \mathbf{U} &= \dot{X} \mathbf{I} + \dot{Y} \mathbf{J}, \end{aligned}$$

whence using the relations (2.1.2) we can write the absolute velocity in terms of the rotating unit vectors

$$\begin{aligned} \mathbf{U} &= \left(+\dot{X} \cos(\Omega t) + \dot{Y} \sin(\Omega t) \right) \mathbf{i} + \left(-\dot{X} \sin(\Omega t) + \dot{Y} \cos(\Omega t) \right) \mathbf{j}, \\ &= U \mathbf{i} + V \mathbf{j}. \end{aligned} \quad (2.1.4)$$

Comparing hence (2.1.3) and (2.1.4) we easily deduce

$$U = u - \Omega y, \quad V = v + \Omega x. \quad (2.1.5)$$

In three-dimensions the procedure is very similar and we deduce that

$$\mathbf{U} = \mathbf{u} + \boldsymbol{\Omega} \wedge \mathbf{r}, \quad (2.1.6)$$

whence, considering that the rotation is homogeneous ($\boldsymbol{\Omega} = \Omega \mathbf{K}$ is a constant vector, in particular it is time-independent). This implies that taking a time derivative of a vector with respect to the inertial framework is equivalent to applying the operator

$$\frac{d}{dt} + \boldsymbol{\Omega} \wedge,$$

in the rotating framework of reference. From the equation above we hence deduce that

$$\frac{d}{dt} \mathbf{U} = \frac{d}{dt} \mathbf{u} + 2 \boldsymbol{\Omega} \wedge \mathbf{u} + \frac{1}{2} \nabla (|\boldsymbol{\Omega} \wedge \mathbf{r}|),$$

the term $2 \boldsymbol{\Omega} \wedge \mathbf{u}$ represents the *Coriolis force*.

2.1.2 The centrifugal force.

The centrifugal force is a concept of which everybody of us is familiar with thanks to direct experience. Unlike the Coriolis force, which is proportional to the velocity, the centrifugal force depends solely on the rotation rate and the distance of the particle to the rotation axis.

A particle subjected to the centrifugal force has the tendency to move radially, outward. Yet we do not experience in our everyday life a force which displaces bodies to the outer space: Newton first principle applies again, and gravity exerts an equal and contrary force on the body.

This is not to say that such force has really no effect; it flattens planets as well as any other rotating elastic body distorting the geometry: the equatorial radius is (slightly) larger than the polar one. On the earth, for example, the distortion is very slight, because gravity by far exceeds the centrifugal force; the terrestrial equatorial radius is 6378 km, slightly greater than its polar radius of 6357 km.

2.1.3 The stratification.

The next question concerns the condition under which stratification effects are expected to play an important dynamical role. Geophysical fluids are generally miscible fluids with varying density. Gravity acts on *masses* hence it affects strongly the region of a fluid with higher density: it tends to dispose a fluid in stacks whose density is decreasing in height. In a situation of perfect dynamical stability gravity tends to minimize the gravitational potential, hence a configuration in which the density profile is decreasing is a dynamical attractor. The intrinsic motion of a fluid however can disturb such configuration lowering hi-density regions or lifting low-density ones, increasing the potential energy of the fluid. Being the system conservative the increase of potential energy is performed at the expense of kinetic energy, slowing the flow. On occasions, obviously, the opposite happens.

If $\Delta\rho$ is the density variations in the fluid and H is its height scale, a prototypical perturbation to the stratification consists in raising a fluid element of density $\rho_0 + \Delta\rho$ over the height H and lowering a lighter fluid element of density ρ_0 over the same height. The corresponding change in potential energy, per unit volume, is

$$(\rho_0 + \Delta\rho)gH - \rho_0gH = \Delta\rho gH.$$

With a typical fluid velocity U , the kinetic energy available per unit volume is $\frac{1}{2}\rho_0U^2$. We construct the relative comparative energy ratio

$$\sigma = \frac{\frac{1}{2}\rho_0U^2}{\Delta\rho gH}.$$

Let us give a sense of the energy ratio σ .

- $\sigma \ll 1$: in this case the kinetic effects are insignificant respect to the stratification, and hence the variation of the velocity does not perturbs the stratification which constrains greatly the flow.
- $\sigma \approx 1$: a typical potential-energy increase necessary to perturb the stratification consumes a sizable portion of the available kinetic energy, thereby modifying the flow field substantially. Stratification is then important.
- $\sigma \gg 1$: potential-energy modifications occur at very little cost to the kinetic energy, and stratification hardly affects the flow.

In conclusion we can hence say that the gravitational stratification of a flow plays an important role in the evolution if $\sigma \lesssim 1$. The quantity σ is to stratification what the number ε defined in (2.1.1), is to rotation.

Indeed studying the motion of a fluid at a planetary scale it may happen that rotation and stratification effects are both relevant at the same time, i.e. the dimensionless quantities $\varepsilon \approx 1$ and $\sigma \approx 1$ and yields the following relations among the various scales:

$$L \approx \frac{U}{\Omega}, \quad U \approx \sqrt{\frac{\Delta\rho}{\rho_0} gH},$$

Elimination of the velocity U yields a fundamental length scale:

$$L \approx \frac{1}{\Omega} \sqrt{\frac{\Delta\rho}{\rho_0} gH}.$$

Experimental verifications give us the following characteristic scales

$$\begin{aligned} L_{\text{atmosphere}} &\approx 500 \text{ km}, & L_{\text{ocean}} &\approx 60 \text{ km}, \\ U_{\text{atmosphere}} &\approx 30 \text{ m/s}, & U_{\text{ocean}} &\approx 4 \text{ m/s}. \end{aligned} \tag{2.1.7}$$

2.2 A glimpse into continuum mechanics: budget laws.

Fluids, as well as any object in the universe, are subjected to *motions*. The concept of motion is very intuitive and well understood by anybody, but a mathematical formulation of such natural concept may not be straightforward. We shall adopt the following definition

Definition 2.2.1. Let $\Omega \subset \mathbb{R}^d$, a **motion** is a \mathcal{C}^3 map $\psi : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^d$ such that for every $t \in \mathbb{R}_+$, $x \in \Omega$ the map $\psi_t = \psi(t, \cdot)$ is an affine transformation. We define

- $\Omega_t = \psi(t, \Omega)$ as the set which the body occupies at time t ,
- $\mathcal{T} = \{(t, x) : t \in \mathbb{R}_+, x \in \Omega_t\}$ is the set of trajectories,
- the map $\Psi_t = \psi_t^{-1} : \Omega_t \rightarrow \Omega$ is called the reference map or back-to-label map,
- the velocity flow is defined as $\mathbf{u}(t, \psi(t, x)) = \frac{d}{dt}\psi_t = \partial_t\psi(t, x)$.

We shall many times refer to a generic motion ψ as a *flow*.

One of the most important properties of bodies is that they possess mass. We here consider bodies whose mass is distributed continuously. No matter how severely such a body is deformed, its mass is the integral of a density field. Let us make these concept formal, a *reference density* is a function

$$\rho_0 : \Omega \rightarrow \mathbb{R}_+,$$

and the *mass* of the body Ω is defined as

$$m(\Omega) = \int_{\Omega} \rho_0(x) \, dV(x).$$

Indeed the deformed body Ω_t possesses as well a mass, and we assume that, no matter the deformation ψ_t , the mass is an invariant of the motion, i.e.

$$m(\Omega_t) = m(\Omega), \quad \forall t > 0,$$

and since the mass can always be described as the integral of a density field

$$m(\Omega_t) = \int_{\Omega_t} \rho(t, X) \, dV(X).$$

We want to express such integral in terms of the space variable $x \in \Omega$, indeed

$$X = X_t = \psi(t, x),$$

hence

$$\begin{aligned} m(\Omega_t) &= \int_{\Omega} \rho(t, \psi(t, x)) \, dV(\psi(t, x)), \\ &= \int_{\Omega} \rho(t, \psi(t, x)) \det[\nabla_x \psi(t, x)] \, dV(x). \end{aligned}$$

Since Ω is arbitrary we can hence deduce the following relation which describes the density of the deformed body Ω_t in terms of the reference density ρ_0 :

$$\rho_0(x) = \rho(t, \psi(t, x)) \det[\nabla_x \psi(t, x)].$$

Being the mass constant we deduce hence that

$$\frac{d}{dt} \int_{\Omega_t} \rho(t, X) \, dV(X) = 0,$$

applying Reynold's transport theorem (see [81]) we deduce that

$$\int_{\Omega_t} \left(\frac{d}{dt} \rho(t, X) + \rho(t, X) \operatorname{div} \mathbf{u}(t, X) \right) dV(X) = 0,$$

but, since Ω was arbitrarily, the fact that

$$\frac{d}{dt} \rho(t, X) = \frac{d}{dt} \rho(t, \psi(t, x)) = \partial_t \rho(t, \psi(t, x)) + \mathbf{u}(t, \psi(t, x)) \cdot \nabla \rho(t, \psi(t, x)),$$

we finally deduce the *conservation of mass equation*

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0.$$

We say that a motion is *isochoric* if and only if $\operatorname{div} \mathbf{u} = 0$.

Next we focus on the forces acting on a determinate body. From first Newton's law from each action it corresponds a second, equal with opposite direction, hence we can assume that, for a moving body Ω_t , at each $t \geq 0$ there is a balance of forces, i.e.

$$\mathbf{F}_i = \mathbf{F}_e,$$

where \mathbf{F}_i are the internal forces and \mathbf{F}_e are the external forces acting on the body. One of the most important and far reaching axioms in continuum mechanics is *Cauchy's hypothesis* concerning the form of the contact (internal) forces. Cauchy assumed the existence of a surface force density $\sigma(\mathbf{n}, x, t)$ defined for each unit vector \mathbf{n} and every (x, t) in the trajectory \mathcal{T} of the motion. Let \mathcal{B} be an oriented surface in Ω_t with positive unit normal \mathbf{n} at x . Then $\sigma(\mathbf{n}, x, t)$ is the force, per unit area, exerted across \mathcal{B} upon the material on the negative side of \mathcal{B} by the material on the positive side. This is a kind of stress that the body (fluid) exerts on himself, and falls into the effects of \mathbf{F}_i . Translated in mathematical terms it means that for every $(x, t) \in \mathcal{T}$, $\mathbf{n} \in \mathbb{S}^{d-1}$ and for all $\omega_t \subset \Omega_t$ the force that $\Omega_t \setminus \omega_t$ exerts upon ω_t is

$$\mathbf{F}_\sigma(\Omega_t, \omega_t) = - \int_{\Omega_t \cap \partial\omega_t} \sigma(\mathbf{n}(x), x, t) dS(x),$$

where dS is assumed to be the $d - 1$ dimensional Hausdorff measure.

This stress and eventual external forces \mathbf{F}_e modify the velocity of the body, inducing an acceleration and with it producing a force (Newton's law), such force is quantified by the variation of the *linear momentum*, hence the balance becomes

$$\mathbf{F}_{LM} + \mathbf{F}_\sigma = \mathbf{F}_e.$$

The linear momentum is defined as

$$\mathbf{l}(t) = \int_{\Omega_t} \rho_t \mathbf{u} dV,$$

while, as explained, the stress force \mathbf{F}_σ is defined as

$$\mathbf{F}_\sigma = - \int_{\partial\Omega_t} \sigma dS,$$

while we can assume that $\mathbf{F}_e = \int_{\Omega_t} \mathbf{F} dV$. The balance equation becomes hence

$$\frac{d}{dt} \int_{\Omega_t} \rho_t \mathbf{u} dV - \int_{\partial\Omega_t} \sigma dS = \int_{\Omega_t} \mathbf{F} dV.$$

Gauss theorem and Reynold's transport theorem allow us hence to deduce the equation for the conservation of the linear momentum (here we abuse of the notation and we denote with \mathbf{F} the acceleration related to the point-wise external force)

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div}(\sigma) = \rho \mathbf{F}.$$

2.3 A closer look: models for incompressible fluids.

The previous section was devoted to prove that the equations describing the evolution of a fluid are derived from physical considerations, and are the *conservation of mass equation*

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{2.3.1}$$

where ρ is the density of the fluid and \mathbf{u} is its velocity field, and the *conservation of the linear momentum* $\mathbf{m} = \rho \mathbf{u}$ equation,

$$\partial_t (\rho \mathbf{u}) + \operatorname{div} (\rho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div} (\sigma) = \rho \mathbf{F}, \quad (2.3.2)$$

and σ is the *stress tensor* and \mathbf{F} is the ensemble of the external forces (rotation, stratification buoyancy, acoustic waves, gravity...) acting on the fluid. We refer to [11] and [101]. Although it is true that, a priori, the density function ρ depends from both time and space variable t, x the oceans and the troposphere (lower layers of the atmosphere) have generally a density that does not varies wildly, i.e.

$$\rho(t, x) = \rho_0 + d\rho(t, x),$$

where ρ_0 is a (positive) constant, which we shall call *characteristic density*, around which the density oscillates. This simplification is usually known as *Boussinesq approximation*². We want to deduce some relation of magnitude between the function $d\rho$ and the constant ρ , we shall perform this analysis for adiabatic perfect gases, although it remains true for different fluids as well.

The quantity M is a dimensionless scalar called *Mach number*. In the literature ([101], [109]...) the Mach number can be defined as follows. Let us consider a fluid moving in a domain \mathcal{U} of length-scale comparable to the ones given in (2.1.7), with a velocity $\mathbf{u}(t, x)$. The Mach number is the ratio of the speed of the gas, to the speed of sound in the gas. The speed of sound is equal to the speed of transmission of small, isentropic disturbances in the flow, formally it is defined as

$$c = \sqrt{\gamma RT},$$

where R is the specific gas constant and T is the absolute temperature. We can hence define the Mach number as the ratio

$$M = \frac{V}{c}, \quad (2.3.3)$$

where V is the local flow velocity with respect to the boundaries, and c is the speed of sound in the medium.

To determine the role of the Mach number on compressibility effects we begin with the conservation of momentum equation³:

$$\rho V dV = -dp,$$

where ρ is the fluid density, V is the velocity, and p is the pressure. Here dV and dp denote differential changes in the velocity and pressure. From our derivation of the conditions for

²The full description of Boussinesq approximation is, actually slightly more complex and can be applied to stratified fluids only, for a more detailed description we refer to Section 2.3.3. By an abuse of lexicon we denote this simplified case as Boussinesq approximation as well.

³Here we simplify the internal stresses of the fluid $\sigma = p \mathbf{1}_{\mathbb{R}^d}$, i.e. we consider the effects of the hydrostatic pressure only. Such hypothesis is physically relevant in the sense that for general fluids (and even more so for geophysical fluids) the main contributions in σ are effectively induced by the hydrostatic pressure.

isentropic flow, we know that:

$$\begin{aligned}\frac{dp}{p} &= \gamma \frac{d\rho}{\rho}, \\ dp &= \gamma RT d\rho, \\ &= c^2 d\rho,\end{aligned}$$

which hence leads to the relation

$$\rho V dV = -c^2 d\rho,$$

which gives, after some basic manipulations

$$-M^2 \frac{dV}{V} = \frac{d\rho}{\rho}. \quad (2.3.4)$$

Let us hence now denote the (dimensionless) function

$$\langle V \rangle (t, x) = \frac{dV}{V} (t, x),$$

the variation of the fluid velocity. As the following formal calculations show

$$\begin{aligned}\langle V \rangle (x) &\approx \frac{V(x) - V(x + dx)}{V(x)}, \\ &= 1 - \frac{V(x + dx)}{V(x)},\end{aligned}$$

is a function whose value is close to zero for fluids which are not very turbulent. Whence considering the relation $\rho = \rho_0 + d\rho$ and (2.3.4) we deduce

$$d\rho = \frac{-M^2 \rho_0}{1 + M^2 \langle V \rangle}.$$

Thanks to a Taylor expansion around zero with respect to the parameter M we deduce that

$$d\rho = \mathcal{O}(M^2),$$

hence it makes sense to deduce the following *low Mach number approximation*:

$$\rho(t, x) = \rho_0 + M^2 \tilde{\rho}(t, x), \quad (2.3.5)$$

The speed of sound c depends on the fluid considered, the temperature, pressure, salinity and many other physical variables. For the air at 15 degrees Celsius and the water at 20 degree Celsius it has approximately the following values

$$c_{\text{air}} \approx 340 \text{ m/s}, \quad c_{\text{water}} \approx 1481 \text{ m/s},$$

which compared to the characteristic scales given in (2.1.7) give the following values for the Mach number

$$M_{\text{atmosphere}} \approx 0.08823529411, \quad M_{\text{ocean}} \approx 0.00270087778,$$

hence the Mach number is generally very small if it is evaluated accordingly to the standard displacement of air and water at a planetary scale. This is very reasonable since the autonomous motion of a fluid is always subsonic ($M < 1$). For practical purposes, if $M < 0.3$, the fluid is considered to be incompressible, i.e. compression waves do not disturb the global motion of the fluid. With this consideration (2.3.1) becomes:

$$\operatorname{div} \mathbf{u} = \frac{1}{\rho_0} \mathcal{O}(M^2),$$

which can be substituted with the more familiar (and simple) incompressibility condition

$$\operatorname{div} \mathbf{u} = 0. \quad (2.3.6)$$

In this first part we shall give some physical derivation of systems which are supposed to satisfy the "zero Mach number" condition. In the following section we shall focus on the case in which the Mach number is small and not identically null.

What follows is a multi-scale analysis for rotating and stratified fluids, we underline the fact that there is no novelty in this section and the result presented are taken from [50] and [27].

2.3.1 The equations on the linear momentum.

We adopt the same notation as in Section 2.1.1, i.e. we consider the motion of a fluid from two different reference systems: the first one is the inertial one, the velocity flow with respect to this reference shall be denoted as \mathbf{U} , the latter is the rotating reference system, whose velocity flow is \mathbf{u} .

The orthonormal basis with respect to the inertial reference system shall be denoted as $(\mathbf{I}, \mathbf{J}, \mathbf{K})$. We recall that this reference system is fixed. In particular hence we recall that we can write the velocity flow \mathbf{U} and the position vector \mathbf{r} as

$$\begin{aligned} \mathbf{U} &= U^1 \mathbf{I} + U^2 \mathbf{J} + U^3 \mathbf{K}, \\ \mathbf{r} &= X^1 \mathbf{I} + X^2 \mathbf{J} + X^3 \mathbf{K}. \end{aligned}$$

In a very similar way we denote with $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ the basis of a reference system whose origin stays on a determinate point at latitude ϕ on the surface of the earth. The vector \mathbf{i} denotes the direction West-East, \mathbf{j} the direction South-North and \mathbf{k} has the same direction of the radius of the earth and it is pointed outward. With this notation we have

$$\begin{aligned} \mathbf{u} &= u^1 \mathbf{i} + u^2 \mathbf{j} + u^3 \mathbf{k}, \\ \mathbf{r} &= x^1 \mathbf{i} + x^2 \mathbf{j} + x^3 \mathbf{k}. \end{aligned}$$

Let us consider the equation of conservation of linear momentum (2.3.2) in the inertial reference system under the hypothesis

$$\mathbf{F} = -\nabla\Phi_g,$$

where $\Phi_g = gx_3 \mathbf{k}$, i.e. the external force is the gradient of the gravitational potential and that the stress tensor σ assumes the form

$$\sigma = -p \mathbf{1}_{\mathbb{R}^3} + \tau,$$

where p is a scalar function and τ is the *viscous tensor*. It reads as

$$\rho \left(\frac{D}{Dt} \right)_i \mathbf{U} = -\rho \nabla \Phi_g - \nabla p + \operatorname{div} (\tau). \quad (2.3.7)$$

The operator $\left(\frac{D}{Dt} \right)_i$ denotes the material derivative with respect to the inertial reference system and it assumes the surprisingly simply form of

$$\left(\frac{D}{Dt} \right)_i = \frac{D}{Dt} + \boldsymbol{\Omega} \wedge, \quad (2.3.8)$$

where indeed the material derivative $\frac{D}{Dt}$ is defined as

$$\frac{D}{Dt} = \partial_t + \mathbf{u} \cdot \nabla.$$

The (vectorial) angular velocity $\boldsymbol{\Omega}$ is defined as

$$\begin{aligned} \boldsymbol{\Omega} &= \Omega \mathbf{K}, \\ &= \Omega \cos(\phi) \mathbf{j} + \Omega \sin(\phi) \mathbf{k}, \end{aligned} \quad (2.3.9)$$

where Ω is indeed a scalar value which we consider to be constant in both space and time.

By the aid of the relations (2.1.6) and (2.3.8) we deduce that

$$\left(\frac{D}{Dt} \right)_i \mathbf{U} = \frac{D}{Dt} \mathbf{u} + 2\boldsymbol{\Omega} \wedge \mathbf{u} + \boldsymbol{\Omega} \wedge (\boldsymbol{\Omega} \wedge \mathbf{r}). \quad (2.3.10)$$

Let us analyze just for a moment the right hand side of (2.3.10):

- The term $\frac{D}{Dt} \mathbf{u} = \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}$ represents the *material derivative*, which describes the rate of change of the velocity flow subjected to transport effects due to the velocity flow itself.
- The *Coriolis force* $\boldsymbol{\Omega} \wedge \mathbf{u}$.
- The term $\boldsymbol{\Omega} \wedge (\boldsymbol{\Omega} \wedge \mathbf{r})$ is the *centrifugal acceleration*. Let us remark that

$$\boldsymbol{\Omega} \wedge (\boldsymbol{\Omega} \wedge \mathbf{r}) = \nabla \left(\frac{1}{2} |\boldsymbol{\Omega} \wedge \mathbf{r}|^2 \right),$$

hence it makes sense to define the *geopotential*

$$\Phi = \Phi_g - \frac{1}{2} |\boldsymbol{\Omega} \wedge \mathbf{r}|^2. \quad (2.3.11)$$

Let us consider a (vertical) displacement $d\mathbf{r}$ of a volume of a fluid. Let us denote $\Psi_c = \frac{1}{2} |\boldsymbol{\Omega} \wedge \mathbf{r}|^2$ the potential associated to the centrifugal acceleration. Let us denote as $d\Phi_g$ and $d\Psi_c$ respectively the variation of the gravitational and centrifugal potential caused by a vertical displacement $d\mathbf{r}$ of the fluid. It is clear that

$$\begin{aligned} d\Phi_g &= \mathcal{O}(d\mathbf{r}), \\ d\Psi_c &= \mathcal{O}((d\mathbf{r})^2). \end{aligned}$$

Since any vertical displacement which may happen in the atmosphere or in the oceans has magnitude irrelevant if compared to the radius of the earth it is reasonable to assume that $\Phi \approx \Phi_g = gx_3 \mathbf{k}$ as a first-order approximation.

The equations (2.3.11) and (2.3.10) applied to (2.3.7) allow to deduce the following evolution equation satisfied by the velocity flow in the rotating system $\mathbf{u} = (u^1, u^2, u^3)$:

$$\rho \frac{D}{Dt} \mathbf{u} + 2\rho (\boldsymbol{\Omega} \wedge \mathbf{u}) = -\rho \nabla \Phi - \nabla p + \operatorname{div}(\boldsymbol{\tau}).$$

Still we need to simplify more the above equation. Let us *suppose* that the viscous tensor $\boldsymbol{\tau}$ assumes the following form:

$$\boldsymbol{\tau} = \mu \nabla \mathbf{u},$$

where $\mu > 0$. Moreover as we explained above at a first-order approximation we can consider $\nabla \Phi \approx g \mathbf{k}$. This combined with (2.3.9) allows us to deduce

$$\begin{cases} \rho \left(\frac{D}{Dt} u^1 + f_* u^3 - f u^2 \right) &= -\partial_1 p + \mu \Delta u^1, \\ \rho \left(\frac{D}{Dt} u^2 + f u^1 \right) &= -\partial_2 p + \mu \Delta u^2, \\ \rho \left(\frac{D}{Dt} u^3 - f_* u^1 \right) &= -\partial_3 p + \mu \Delta u^3 - \rho g, \end{cases} \quad (2.3.12)$$

where $f = 2\Omega \sin(\phi)$ and $f_* = 2\Omega \cos(\phi)$. The preceding equations assume a Cartesian system of coordinates and thus hold only if the dimension of the domain under consideration is much shorter than the earth's radius. On Earth, a length scale not exceeding 1000 km is usually acceptable.

2.3.2 The energy budget.

Obviously a fluid has an (absolute) temperature. This physical quantity evolves indeed accordingly to the density and the velocity flow. From the first law of thermodynamics the internal energy gained by a parcel of matter is equal to the heat it receives minus the mechanical work it performs, which translates in the equation

$$\frac{D}{Dt} e = Q - W.$$

The internal energy is the amount of agitation of internal molecules and it is proportional to the temperature

$$e = C_v T,$$

C_v is the capacity at a constant volume.

At a first approximation we can consider that there is no internal heat source in the air or water, hence using the Fourier law of heat conduction, which states that the thermal energy transforms into an homogeneous, local, change of temperature

$$Q = \frac{k_T}{\rho} \Delta T,$$

while

$$W = -\frac{p}{\rho^2} \frac{d}{dt} \rho,$$

hence

$$C_v \frac{D}{Dt} T = \frac{k_T}{\rho} \Delta T - -\frac{p}{\rho^2} \frac{d}{dt} \rho,$$

we finally deduce

$$\rho C_v \frac{D}{Dt} T + p \operatorname{div} \mathbf{u} = k_T \Delta T. \quad (2.3.13)$$

Since we introduced the new equation T we need a new equation relating p and T to close the system

- If we assume the air a perfect gas

$$p = \rho R T,$$

- the water can be considered to be incompressible, hence the pressure does not affect the motion of the fluid and we can consider it constant. On the other hand the salinity S may have physical relevance, this indeed modifies the density by mean of the following equation of state

$$\rho = \rho_0 (1 - \alpha (T - T_0) + \beta (S - S_0)), \quad (2.3.14)$$

where S satisfied a diffusion law

$$\frac{D}{Dt} S = \kappa_S \Delta S. \quad (2.3.15)$$

2.3.3 The Boussinesq approximation.

Let us consider a fluid with variable density. Accordingly to the theory explained in the previous sections it evolves accordingly to the laws described by (2.3.12). We say that a stratified fluid satisfies the *Boussinesq approximation* if the variation of density is relevant only in the buoyancy term ρg while it can be approximated as a constant ρ_0 otherwise.

As we already mentioned various times we consider large perturbations around a stationary state for a stratified fluid, and we are interested to study the dynamics induced by the fluid when it strives to restore such equilibrium, whence we conceive the variation in density mainly along the vertical direction and we suppose that heavier layers lay under lighter ones (i.e. ρ is decreasing in the variable x_3), i.e.

$$\rho(t, x) = \rho_0 + \bar{\rho}(x_3) + \theta(t, x), \quad (2.3.16)$$

with

$$|\theta| \ll 1.$$

Let us denote

$$\rho' = \bar{\rho} + \theta,$$

with the Boussinesq approximation here explained, the incompressibility condition (2.3.6) the conservation of mass equation (2.3.1) becomes

$$\partial_t \rho' + \mathbf{u} \cdot \nabla \rho' = 0. \quad (2.3.17)$$

Combing the above equation (2.3.17) with the evolution equation (2.3.15), (2.3.13) and the equation of state (2.3.14), supposing $\kappa_T = k_t / (\rho_0 C_v) = \kappa_S = \kappa$ we finally deduce the equation

$$\frac{D}{Dt} \rho' = \kappa \Delta \rho'. \quad (2.3.18)$$

Thanks to the Boussinesq approximation $\rho \approx \rho_0$ we can rewrite (2.3.12) in the new form

$$\begin{cases} \frac{D}{Dt} u^1 + f_* u^3 - f u^2 &= -\frac{1}{\rho_0} \partial_1 p + \nu \Delta u^1, \\ \frac{D}{Dt} u^2 + f u^1 &= -\frac{1}{\rho_0} \partial_2 p + \nu \Delta u^2, \\ \frac{D}{Dt} u^3 - f_* u^1 &= -\frac{1}{\rho_0} \partial_3 p + \nu \Delta u^3 - \frac{\rho}{\rho_0} \mathbf{g}, \end{cases} \quad (2.3.19)$$

where $\nu = \mu / \rho_0$ is the kinematic diffusion. Combining (2.3.6), (2.3.19) and (2.3.18) we finally deduce the very generic model (here we denote $\rho = \rho'$)

$$\begin{cases} \partial_t u^1 + \mathbf{u} \cdot \nabla u^1 + f_* u^3 - f u^2 &= -\frac{1}{\rho_0} \partial_1 p + \nu \Delta u^1, \\ \partial_t u^2 + \mathbf{u} \cdot \nabla u^2 + f u^1 &= -\frac{1}{\rho_0} \partial_2 p + \nu \Delta u^2, \\ \partial_t u^3 + \mathbf{u} \cdot \nabla u^3 - f_* u^1 &= -\frac{1}{\rho_0} \partial_3 p + \nu \Delta u^3 - \frac{\rho}{\rho_0} \mathbf{g}, \\ \partial_t \rho + \mathbf{u} \cdot \nabla \rho &= \kappa \Delta \rho, \\ \operatorname{div} \mathbf{u} &= 0. \end{cases} \quad (2.3.20)$$

2.3.4 Two dimensional dynamics: the geostrophic equilibrium.

Until now we have been studying how the rotation modifies the structure of the evolution equations for a geophysical fluid, but we did not really understand until now if the rotation has a stabilizing effect on the motion of the fluid itself.

To deduce informally the limit system we shall perform a multi-scale analysis as it is done in [27] and [50]. Let us define as L the characteristic length in the horizontal directions, H in the vertical direction, T is the characteristic time, which we suppose to be bigger than

the rotation time T_Ω , U is the characteristic horizontal speed and W the vertical one, P the pressure and $\Delta\rho$ the variation of density. Indeed the following relations hold true

$$T \gg T_\Omega = \frac{1}{\Omega}, \quad \frac{U}{L} \lesssim \Omega, \quad H \ll L, \quad \Delta\rho \ll \rho.$$

This analysis of magnitudes will allow us to remarkably simplify the equations (2.3.20). With the incompressibility condition $\text{div } \mathbf{u} = 0$ we can deduce that $U/L \sim W/H$, but since $H \ll L$ we easily deduce that $W \ll U$. This implies that the term $f_* u^3$, whose order of magnitude is ΩW is negligible if compared to $-f u^2$ of magnitude ΩU , under the hypothesis that the latitude ϕ is far from zero. In the same way the term $(\partial_1^2 + \partial_2^2) u$ is small compared to $\partial_3^2 u$. In the same way the left-hand side of the equation governing the motion of u^3 is negligible compared to the right-hand side, with these simplification (2.3.20) becomes

$$\begin{cases} \partial_t u^1 + \mathbf{u} \cdot \nabla u^1 - f u^2 &= -\frac{1}{\rho_0} \partial_1 p + \nu \partial_3^2 u^1, \\ \partial_t u^2 + \mathbf{u} \cdot \nabla u^2 + f u^1 &= -\frac{1}{\rho_0} \partial_2 p + \nu \partial_3^2 u^2, \\ 0 &= -\frac{1}{\rho_0} \partial_3 p - \frac{\rho}{\rho_0} \mathbf{g}, \\ \partial_t \rho + \mathbf{u} \cdot \nabla \rho &= \kappa \partial_3^2 \rho, \\ \text{div } \mathbf{u} &= 0. \end{cases} \quad (2.3.21)$$

The scaling analysis just performed was developed in order to neglect some small terms. The remaining terms may, though, be of very different magnitude: this is the second shell of simplifications that we are going to perform.

The terms of the horizontal momentum equations in the form (2.3.21) scale as

$$\frac{U}{T}, \quad \frac{U^2}{L}, \quad \frac{U^2}{L}, \quad \frac{WU}{H}, \quad \Omega U, \quad \frac{P}{\rho_0 L}, \quad \frac{\nu U}{H^2}.$$

We are interested to study the motion as long as the rotation plays a significant role, for this reason we divide the above relations for ΩU obtaining the following order of magnitude

$$\frac{1}{\Omega T}, \quad \frac{U}{\Omega L}, \quad \frac{U}{\Omega L}, \quad \frac{W}{\Omega H}, \quad 1, \quad \frac{P}{\rho_0 L \Omega U}, \quad \frac{\nu}{\Omega H^2}.$$

- The ratio

$$\text{Ro}_T = \frac{1}{\Omega T},$$

is called the *temporal Rossby number*, it compares the local time rate of change of the velocity to the Coriolis force and is on the order of unity or less.

- The ratio

$$\text{Ro} = \frac{U}{\Omega L}, \quad (2.3.22)$$

which compares advection to Coriolis force, is called the *Rossby number* and is again of order smaller than one.

- The ratio

$$\frac{W}{\Omega H} = \frac{WL}{UH} \cdot \frac{U}{\Omega L},$$

is the product of the Rossby number by $\frac{WL}{UH}$ which is of order smaller than one.

- The ratio $\frac{P}{\rho_0 L \Omega U}$ is of order one.

- Finally the ratio

$$\text{Ek} = \frac{\nu}{\Omega H^2}, \quad (2.3.23)$$

is called the *Ekman number*, and for geophysical flows it assumes a very small value.

Neglecting the terms whose order is small compared to one we deduce hence the *geostrophic equilibrium*

$$-f u^2 = -\frac{1}{\rho_0} \partial_1 p, \quad (2.3.24)$$

$$f u^1 = -\frac{1}{\rho_0} \partial_2 p, \quad (2.3.25)$$

$$0 = -\partial_3 p, \quad (2.3.26)$$

$$\text{div } \mathbf{u} = 0. \quad (2.3.27)$$

The dynamic described by this set of equations is constraint and simplified: taking the operator ∂_3 onto the equation (2.3.24) and (2.3.25) with the information of equation (2.3.26) we deduce

$$\partial_3 u^1 = \partial_3 u^2 = 0,$$

moreover thanks to equations (2.3.24) and (2.3.25)

$$\partial_1 u^1 + \partial_2 u^2 = 0.$$

This result is known as *Taylor-Proudman theorem* (for more we refer to [50] and [126]). Physically it means that in a regime in which the rotation plays an important role the fluid moves horizontally and has no vertical displacement. Solving (2.3.24) and (2.3.25) with respect to u^1 and u^2 we deduce

$$u^1 = -\frac{1}{f \rho_0} \partial_2 p, \quad u^2 = +\frac{1}{f \rho_0} \partial_1 p.$$

The horizontal velocity flow moves hence *orthogonally* with respect to the gradient flow of the pressure. This means that the flow displaces along the directions in which there is no variation of pressure, these directions are known as *isobars*.

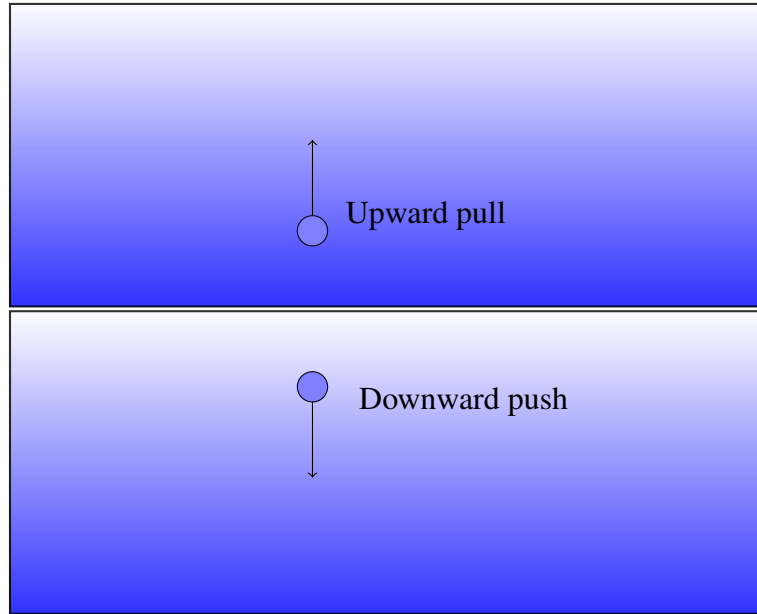


Figure 2.1 – Stratification buoyancy.

2.3.5 Stratification effects.

In the previous section we have seen how the rotation affects the motion of (2.3.21) deducing the geostrophic equilibrium (2.3.24)–(2.3.27). In this analysis we did not take in any consideration the effects induced by the stratification. In the present section our aim is to give description of the stratification effects for geophysical fluids.

Let us consider a fluid which is in a statical equilibrium, composed of layers of different densities. The gravity tends to stratify the fluid in a way that heavier layers lay below lighter ones. We are interested to study how large perturbations of equilibrium modifies the motion in large time and space scales. The idea is the following one, let us suppose that that a small volume dV of a fluid with a certain density is displaced (upwards) in a region of lower density. The gravity will tend to move the fluid volume downwards until it reaches a layer whose density is comparable to its own. Once the fluid reaches such situation its motion does not stops immediately, but, thanks to the inertia, it will have a tendency to move downwards still, entering in a region with higher density. Once it reaches such configuration Archimede's principle will provide a upward buoyancy, creating a pseudo-periodic motion around the layer of density comparable to the one of dV . We refer to Figure 2.1 as well for a graphical intuition of the effect. Let us verify such intuition, we consider hence a volume dV placed at a height x_3 with density $\rho(x_3)$, and let us move it to a height $x_3 + h$ where the fluid has density $\rho(x_3 + h)$, if the fluid can be considered to be incompressible the vertical displacement does not influence the volume dV , and accordingly to Archimede's principle it is subject to a buoyancy force which is

$$\mathbf{g}(\rho(x_3 + h) - \rho(x_3))dV.$$

Accordingly to Newton's principle $F = m a$ we deduce hence the following balance of forces

$$\rho(x_3) dV \ddot{h} = \mathbf{g}(\rho(x_3 + h) - \rho(x_3)) dV.$$

As we have already see in the case in which we consider regimes with low Mach number the Boussinesq approximation hold, hence we can say that

$$\rho(x_3 + h) - \rho(x_3) \approx \partial_3 \rho(x_3) h,$$

with this consideration we can hence deduce the following differential equation

$$\ddot{h} - \frac{\mathbf{g}}{\rho_0} \partial_3 \rho h = 0. \quad (2.3.28)$$

The above differential equation presents two different behaviors accordingly to the sign of $-\frac{\mathbf{g}}{\rho_0} \partial_3 \rho$. As we already stated we suppose $\partial_3 \rho < 0$, i.e. the density is decreasing with the altitude, in this case if we set

$$N^2 = -\frac{\mathbf{g}}{\rho_0} \partial_3 \rho,$$

and we consider N to be constant the solution h of (2.3.28) has an oscillatory character, with frequency N . The quantity N is called *stratification frequency* or Brunt-Väisälä frequency and quantifies the rate with which a fluid tends to stratify to a stable configuration.

Quantifying the stratification: the Froude number.

In Section 2.3.4 we performed a multi-scale analysis which allowed us to introduce the dimensionless Rossby number Ro : such number quantifies the effects of the Coriolis force on the motion of a fluid at planetary scale in time-spans bigger than a day. Conceptually what we want to derive here is the same, but considering the intrinsic buoyancy provided by gravity as explained in the previous section. By analogy we might expect that N and H shall play a similar role as Ω and L for the rotation. We consider a stratified fluid of thickness H and stratification frequency N moving horizontally at a speed U over an obstacle of length L and height Δz . The presence of such obstacle will force some heavy fluid layer to move upward, a configuration that generally does not occur. Hence there is a transformation of kinetic energy into gravitational energy due to a displacement from a situation of equilibrium, such transformation of energy provides a vertical velocity of magnitude comparable to

$$W = \frac{\Delta z}{T} = \frac{U \Delta z}{L}.$$

The vertical modification causes as well a density perturbation of the order of

$$\begin{aligned} \Delta \rho &= \left| \frac{d\bar{\rho}}{dz} \right| \Delta z, \\ &= \frac{\rho_0 N^2}{\mathbf{g}} \Delta z. \end{aligned}$$

In turn, this density variation gives rise to a pressure disturbance that scales, via the hydrostatic balance, as

$$\begin{aligned} \Delta P &= \mathbf{g} H \Delta \rho, \\ &= \rho_0 N^2 H \Delta z. \end{aligned}$$

Obviously hence a variation in the horizontal components of the pressure induces a variation of the velocity flow

$$\frac{U^2}{L} = \frac{\Delta P}{\rho_0 L} \implies U^2 = N^2 H \Delta z,$$

and since $\partial_1 u^1 + \partial_2 u^2 = -\partial_3 u^3$ we derive that

$$\frac{W/H}{U/L} = \frac{\Delta z}{H} = \frac{U^2}{N^2 H^2}.$$

The above equation implies that if U is smaller than NH then W/H has to be smaller than U/L : the variation in the vertical direction hence cannot fully meet the horizontal divergence. Supposing hence that $|\partial_3 u^3| < |\text{div}_h \mathbf{u}^h|$ some compensation has to be performed by the term $\text{div}_h \mathbf{u}^h$: The fluid cannot displace in a purely vertical way, and some of its motion is deflected horizontally (winds which tends to surround an obstacle).

We define hence at this point the *Froude number*

$$\text{Fr} = \frac{U}{N H},$$

which is a measure of the tendency of the fluid to dispose in horizontal stacks; a quantification of the stratification. Obviously the smaller Fr is the more relevant stratification effects are.

The quasi-geostrophic three-dimensional equilibrium.

As in Section 2.3.4 we derived the two-dimensional geostrophic equilibrium by use of the rotation of the Earth only in this section we want to deduce a simplified dynamics which takes in consideration both rotation and stratification; such dynamics is called quasi-geostrophic or semi-geostrophic equilibrium and it is way more complex and rich than the geostrophic equilibrium. We shall omit to give a full derivation and we refer the reader to [27] and [50]. Following the Boussinesq simplification (2.3.16) we can indeed suppose that the pressure p can be written as

$$p(t, x) = \bar{p}(x_3) + \pi(t, x),$$

we can hence define the *stream function*

$$\psi = \frac{\pi}{f_0 \rho_0},$$

where f_0 is the first order approximation of $2\Omega \sin \phi$ and the *potential vorticity*

$$q = -\partial_2 u^1 + \partial_1 u^2 - \partial_3 \left(\frac{f_0 \mathbf{g}}{N^2 \rho_0} \theta \right).$$

The quasi-geostrophic flow is hence defined as

$$\begin{cases} u^1 = -\partial_2 \psi, \\ u^2 = \partial_1 \psi, \\ u^3 = -\frac{f_0}{N^2} (\partial_t \partial_3 \psi + u^1 \partial_1 q + u^2 \partial_2 q), \\ \pi = \rho_0 f_0 \psi, \\ \theta = \frac{\rho_0 f_0}{g} \partial_3 \psi. \end{cases}$$

2.4 Final remarks on the simplification of the primitive equations (2.3.20).

Let us consider the primitive equations as in (2.3.20), neglecting the parameter f_* we deduce the following equations

$$\begin{cases} \partial_t u^1 + \mathbf{u} \cdot \nabla u^1 - f u^2 &= -\frac{1}{\rho_0} \partial_1 p + \nu \Delta u^1, \\ \partial_t u^2 + \mathbf{u} \cdot \nabla u^2 + f u^1 &= -\frac{1}{\rho_0} \partial_2 p + \nu \Delta u^2, \\ \partial_t u^3 + \mathbf{u} \cdot \nabla u^3 &= -\frac{1}{\rho_0} \partial_3 p + \nu \Delta u^3 - \frac{\rho}{\rho_0} \mathbf{g}, \\ \partial_t \rho + \mathbf{u} \cdot \nabla \rho &= \kappa \Delta \rho, \\ \operatorname{div} \mathbf{u} &= 0. \end{cases}$$

With the Boussinesq approximation (2.3.16), recalling that $\partial_3 \bar{\rho} = -\rho_0 N^2 / g$, neglecting the term $\kappa \partial_3^2 \bar{\rho}$ we obtain the following set of equations (here we denote θ appearing in (2.3.16) as ρ):

$$\begin{cases} \partial_t u^1 + \mathbf{u} \cdot \nabla u^1 - f u^2 &= -\frac{1}{\rho_0} \partial_1 p + \nu \Delta u^1, \\ \partial_t u^2 + \mathbf{u} \cdot \nabla u^2 + f u^1 &= -\frac{1}{\rho_0} \partial_2 p + \nu \Delta u^2, \\ \partial_t u^3 + \mathbf{u} \cdot \nabla u^3 + \frac{g \rho}{\rho_0} &= -\frac{1}{\rho_0} \partial_3 p + \nu \Delta u^3, \\ \partial_t \rho + \mathbf{u} \cdot \nabla \rho - \frac{\rho_0 N^2}{g} u^3 &= \kappa \Delta \rho, \\ \operatorname{div} \mathbf{u} &= 0. \end{cases}$$

We want to perform a dimensional re-scaling which nondimensionalises the above set of equations and magnifies the rotational and gravitational effects via suitably defined dimensionless numbers. Let us introduce hence the following characteristic scales

- L , characteristic horizontal length,
- U , characteristic horizontal speed,

- $T = L/U$, characteristic time,
- $T_R = f^{-1}$, rotation time,
- $T_N = N^{-1}$, stratification time,
- ρ_0 , average density,
- p_0 , average pressure.

With the following change of scales

$$\left\{ \begin{array}{l} x' = \frac{x}{L}, \\ t' = \frac{t}{T}, \\ v = \frac{v'}{U}, \\ \rho' = \frac{\rho}{\rho_0} - \frac{g}{\rho_0 N U} \rho, \\ p' = \frac{p}{p_0}, \end{array} \right.$$

and, defining the following quantities

$$\left\{ \begin{array}{l} \text{Ro} = \frac{T_R}{T} = \frac{U}{L f}, \\ \text{Fr} = \frac{T_N}{T} = \frac{U}{L N}, \\ \bar{p} = \frac{p_0}{\rho_0 U^2}, \end{array} \right.$$

and with the abuse of notation $\nu = \frac{\nu U}{L}$, $\nu' = \frac{\kappa U}{L}$ we can finally deduce the system

$$\left\{ \begin{array}{l} \partial_t u^1 + \mathbf{u} \cdot \nabla u^1 - \frac{1}{\text{Ro}} u^2 = -\frac{\bar{p}}{\rho_0} \partial_1 p + \nu \Delta u^1 \\ \partial_t u^2 + \mathbf{u} \cdot \nabla u^2 + \frac{1}{\text{Ro}} u^1 = -\frac{\bar{p}}{\rho_0} \partial_2 p + \nu \Delta u^2 \\ \partial_t u^3 + \mathbf{u} \cdot \nabla u^3 + \frac{1}{\text{Fr}} \rho = -\frac{\bar{p}}{\rho_0} \partial_3 p + \nu \Delta u^3 \\ \partial_t \rho + \mathbf{u} \cdot \nabla \rho - \frac{1}{\text{Fr}} u^3 = \nu' \Delta \rho \end{array} \right.$$

The Froude and Rossby number are small compared to one, and in particular we shall consider the following relations

$$\varepsilon = \text{Ro}, \quad \varepsilon F = \text{Fr},$$

defining $\Phi = \frac{\bar{p}}{\rho_0} p$ we deduce the system

$$\begin{cases} \partial_t u^{1,\varepsilon} + u^\varepsilon \cdot \nabla u^{1,\varepsilon} - \nu \Delta u^{1,\varepsilon} - \frac{1}{\varepsilon} u^2 & = -\frac{1}{\varepsilon} \partial_1 \Phi_\varepsilon \\ \partial_t u^{2,\varepsilon} + u^\varepsilon \cdot \nabla u^{2,\varepsilon} - \nu \Delta u^{2,\varepsilon} + \frac{1}{\varepsilon} u^{1,\varepsilon} & = -\frac{1}{\varepsilon} \partial_2 \Phi_\varepsilon \\ \partial_t u^{3,\varepsilon} + u^\varepsilon \cdot \nabla u^{3,\varepsilon} - \nu \Delta u^{3,\varepsilon} + \frac{1}{F \varepsilon} \theta^\varepsilon & = -\frac{1}{\varepsilon} \partial_3 \Phi_\varepsilon \\ \partial_t \theta^\varepsilon + u^\varepsilon \cdot \nabla \theta^\varepsilon - \nu' \Delta \theta^\varepsilon - \frac{1}{F \varepsilon} u^{3,\varepsilon} & = 0, \end{cases}$$

which can be reduced to the more compact form (here we denote $U^\varepsilon = (u^\varepsilon, \theta^\varepsilon)$)

$$\begin{cases} \partial_t U^\varepsilon + u^\varepsilon \cdot \nabla U^\varepsilon - \mathbb{D} U^\varepsilon + \frac{1}{\varepsilon} \mathcal{A} U^\varepsilon = -\frac{1}{\varepsilon} \nabla \Phi_\varepsilon, \\ U^\varepsilon|_{t=0} = (u_0, \theta_0), \end{cases} \quad (\text{PE}_\varepsilon)$$

where

$$\mathbb{D} = \begin{pmatrix} \nu \Delta & 0 & 0 & 0 \\ 0 & \nu \Delta & 0 & 0 \\ 0 & 0 & \nu \Delta & 0 \\ 0 & 0 & 0 & \nu' \Delta \end{pmatrix} \quad \mathcal{A} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{-1} \\ 0 & 0 & -F^{-1} & 0 \end{pmatrix}.$$

2.4.1 Informal derivation of the limit model in the regime $\varepsilon \rightarrow 0$.

As we mentioned we are interested in the study of the regime $\varepsilon \rightarrow 0$, in particular we would like to determinate whether U^ε converges to some limit flow as $\varepsilon \rightarrow 0$. This kind of question was already introduced in Section 2.3.5, whence we want to deduce a informally simplified dynamics for the limit flow. The following deduction have a mere informal motivation.

Let us *suppose* that $(U^\varepsilon, \Phi^\varepsilon) \rightarrow (U_{\text{QG}}, \Phi_{\text{QG}})$ in a sufficiently strong topology to justify the convergence $u^\varepsilon \cdot \nabla U^\varepsilon \rightarrow u_{\text{QG}} \cdot \nabla U_{\text{QG}}$. Letting $\varepsilon \rightarrow 0$ we deduce from (PE_ε)

$$\begin{aligned} -u_{\text{QG}}^2 &= -\partial_1 \Phi_{\text{QG}}, \\ +u_{\text{QG}}^1 &= -\partial_2 \Phi_{\text{QG}}, \\ +\theta_{\text{QG}} &= -F \partial_3 \Phi_{\text{QG}}, \\ +u_{\text{QG}}^3 &= 0. \end{aligned} \quad (2.4.1)$$

From (2.4.1) we immediately deduce

$$\text{div}_h u_{\text{QG}}^h = 0,$$

as it was already mentioned in Section 2.3.5. Let us define the operator

$$\Delta_F = \partial_1^2 + \partial_2^2 + F^2 \partial_3^2,$$

from the equations of the quasi-geostrophic balance (2.4.1) we deduce

$$\begin{aligned} \Omega &= -\partial_2 u_{\text{QG}}^1 + \partial_1 u_{\text{QG}}^2 + F \partial_3 \theta_{\text{QG}}, \\ &= +\Delta_F \Phi_{\text{QG}}. \end{aligned} \quad (2.4.2)$$

The quantity Ω is called *potential vorticity* and, supposing that in the domain considered the operator Δ_F is invertible the quasi-geostrophic balance (2.4.1) writes as

$$\begin{aligned} -u_{\text{QG}}^2 &= -\partial_1 \Delta_F^{-1} \Omega, \\ +u_{\text{QG}}^1 &= -\partial_2 \Delta_F^{-1} \Omega, \\ -\theta_{\text{QG}} &= -F \partial_3 \Delta_F^{-1} \Omega, \\ +u_{\text{QG}}^3 &= 0. \end{aligned} \tag{2.4.3}$$

Considering hence equations (PE $_\epsilon$) we can derive the equation satisfied by Ω which is (after a suitable number of algebraic simplifications):

$$\partial_t \Omega + u_{\text{QG}}^h \cdot \nabla_h \Omega + \Gamma(D) \Omega = 0, \tag{2.4.4}$$

where Γ is a suitable elliptic operator of order two which assumes the form

$$\Gamma = -\Delta \Delta_F^{-1} (\nu \partial_1^2 + \nu \partial_2^2 + \nu' F^2 \partial_3^2).$$

From (2.4.4) and (2.4.3) we can easily deduce the limit dynamic equations satisfied by the quasi-geostrophic flow U_{QG} :

$$\partial_t \begin{pmatrix} u_{\text{QG}}^1 \\ u_{\text{QG}}^2 \\ 0 \\ \theta_{\text{QG}} \end{pmatrix} + \Gamma \begin{pmatrix} u_{\text{QG}}^1 \\ u_{\text{QG}}^2 \\ 0 \\ \theta_{\text{QG}} \end{pmatrix} = - \begin{pmatrix} -\partial_2 \\ \partial_1 \\ 0 \\ -F \partial_3 \end{pmatrix} \Delta_F^{-1} (u_{\text{QG}}^h \cdot \nabla_h \Omega). \tag{2.4.5}$$

2.5 Non-dimensionalized Navier-Stokes inhomogeneous system.

We do consider in this section a density-dependent incompressible fluid whose only external force acting on it the gravity.

Let us suppose moreover that the fluid density ρ can be written as

$$\rho(t, x) = \rho_0 + \bar{\rho}(x_3) + \rho'(t, x), \tag{2.5.1}$$

and such that

$$|\rho'| \ll 1, \tag{2.5.2}$$

where $\rho \in \mathbb{R}_+$ is a constant state. This physical hypothesis is very reasonable and non-restrictive, we considered in fact the gravity as the only external force acting on the fluid, the downwards acceleration affects in a more accentuated way the regions of a fluid with a higher density. The variation of density in macroscopic fluids, such as the oceans, is given by the variation of some determined physical quantities. Let us analyze the case of the variation of salinity in the oceans, the salinity varies in a relatively simple way with respect to the depth and depends on the latitude. The variation is illustrated in Figure 2.2.

Some consideration on the salinity is due:

- the rate of variation of oceanic salinity *on an entire planetary scale* is about 10%,

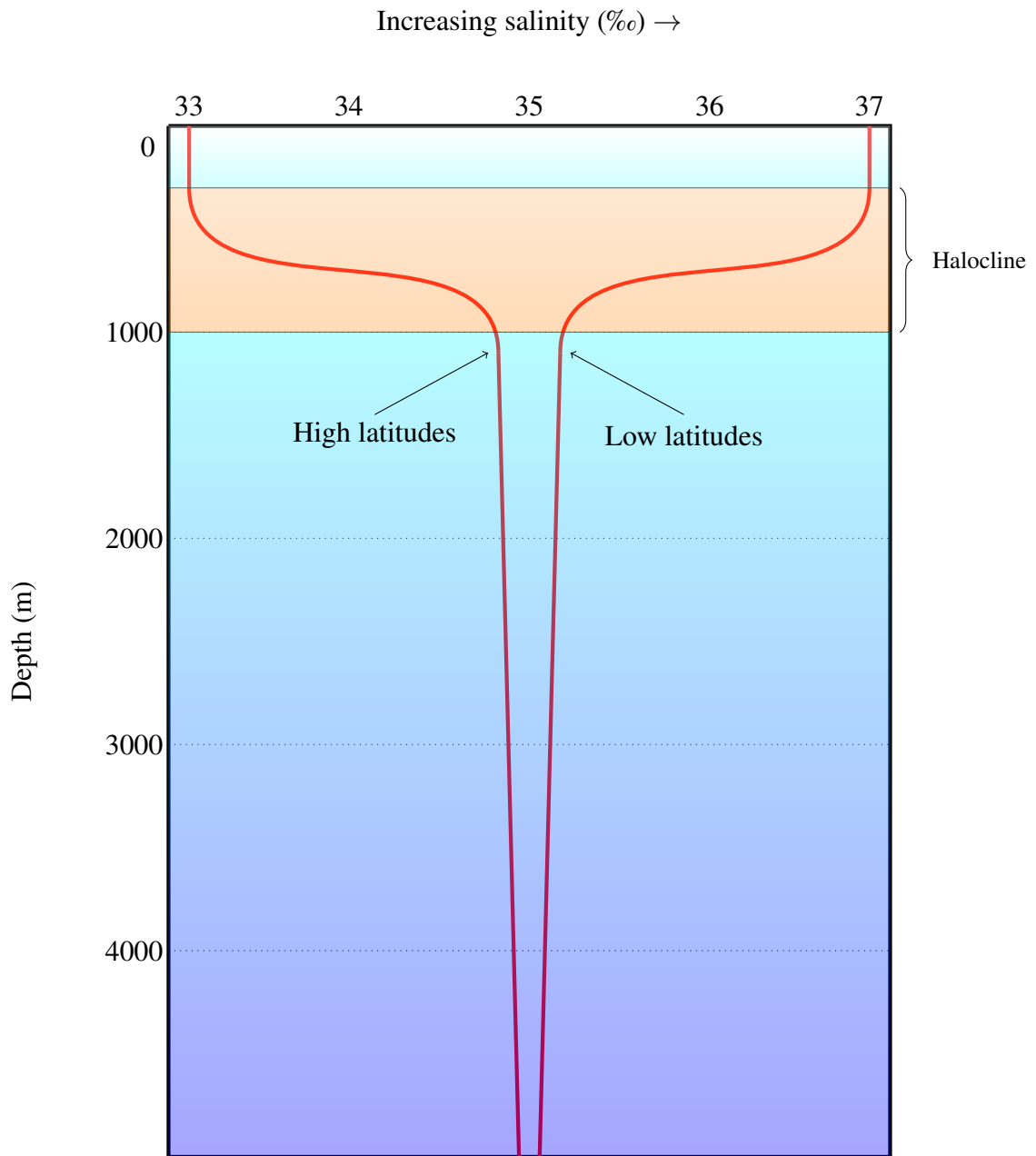


Figure 2.2 – Salinity density in relation with depth.

- on the upper layers and the lower layers of oceans the salinity is virtually constant, more than 90% of variation of salinity occurs in an intermediate region called *Halocline*. The depth of the Halocline varies accordingly to the latitude.

Since the variation of salinity can hence be generally considered to be "small" or "irrelevant" the hypothesis (2.5.1)–(2.5.2) is hence justified.

The Navier-Stokes incompressible equations with hypothesis (2.5.1)–(2.5.2) and Boussinesq approximation read hence as

$$\begin{cases} \partial_t u + u \cdot \nabla u & = -\frac{1}{\rho_0} \nabla p + \frac{1}{\rho_0} \nu \Delta u - \begin{pmatrix} 0 \\ 0 \\ \frac{\rho}{\rho_0} g \end{pmatrix}, \\ \partial_t \rho' + u \cdot \nabla \rho' + u^3 \partial_3 \bar{\rho} & = \kappa \Delta \rho' + \kappa \partial_3^2 \bar{\rho}, \\ \operatorname{div} u = 0. \end{cases} \quad (2.5.3)$$

If we define the Brunt-Väisälä stratification frequency as it was done in Section 2.3.5, i.e.

$$N^2 = -\frac{g}{\rho_0} \partial_3 \bar{\rho} \implies \partial_3^2 \bar{\rho} \approx 0,$$

equation (2.5.3) becomes (denoting ρ' as ρ)

$$\begin{cases} \partial_t u + u \cdot \nabla u & = -\frac{1}{\rho_0} \nabla p + \frac{1}{\rho_0} \nu \Delta u - \begin{pmatrix} 0 \\ 0 \\ \frac{\rho}{\rho_0} g \end{pmatrix}, \\ \partial_t \rho + u \cdot \nabla \rho & = \kappa \Delta \rho + \frac{\rho_0}{g} N^2 u^3, \\ \operatorname{div} u = 0. \end{cases} \quad (2.5.4)$$

Equation (2.5.4) describe hence the dynamics of a density-dependent fluid under the sole (physically relevant) hypothesis that the variation of density is small and the density increase with depth.

On a point of view of oceanography (2.5.4) is hence a good approximation of the local dynamic of sea water in non-tropical regions⁴, as well as a number of other physical manifestations.

Let us now nondimensionalize equations (2.5.4), this process is very often used by physi-

⁴Tropical regions exhibit the characteristic of having waters with higher salinity on the surface as shown in Figure 2.2. This imbalance is not motivated by any dynamical effect, but rather from a disproportion between evaporation and precipitation effects.

cists in order to compare magnitudes in different terms of an equation. Defining

$$\begin{cases} L = \text{standard length-scale of the system,} \\ U = \text{standard velocity of the flow,} \\ T_N = N^{-1}, \\ T = L/U, \\ \text{Fr} = T_N/T, \end{cases}$$

we can define the following adimensional unknowns:

$$\begin{cases} t^* = t/T, \\ x^* = x/L, \\ u^* = u/U, \\ p^* = \frac{p}{\rho U^2}, \\ \rho^* = \frac{g}{\rho_0 N U} \rho. \end{cases}$$

The dimensionless number Fr is the **Froude number** as is was introduced in Section 2.3.5. The equations (2.5.4) in nondimensional form become

$$\begin{cases} \partial_t u^* + u^* \cdot \nabla u^* - \nu^* \Delta u^* + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\text{Fr}} \rho^* \end{pmatrix} = -\nabla p^*, \\ \partial_t \rho^* + u^* \cdot \nabla \rho^* - \kappa^* \Delta \rho^* - \frac{1}{\text{Fr}} u^{3,*} = 0, \\ \text{div } u^* = 0, \end{cases}$$

where ν^* and κ^* are modified kinematic viscosities which depend on the Reynold number. Setting $\text{Fr} = \varepsilon$ we hence derived the system

$$\begin{cases} \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \nu \Delta u^\varepsilon - \frac{1}{\varepsilon} \rho^\varepsilon \vec{e}_3 = -\frac{1}{\varepsilon} \nabla \Phi^\varepsilon, \\ \partial_t \rho^\varepsilon + u^\varepsilon \cdot \nabla \rho^\varepsilon - \nu' \Delta \rho^\varepsilon + \frac{1}{\varepsilon} u^{3,\varepsilon} = 0, \\ \text{div } u^\varepsilon = 0, \end{cases} \quad (2.5.5)$$

which will be studied in Chapter 4 and 5.

2.6 The weakly compressible models.

In the previous section we derived via a careful scale-analysis the evolution equation governing the motion of a fluid under the incompressibility condition (2.3.6). Despite the fact that for many practical applications a fluid might be considered incompressible, in general, the propagation of acoustic (compression) waves is not a negligible phenomenon. Let us

give a motivation of a very reduced dynamics of geophysical compressible fluids. We adopt the same notation as in Section 2.3.1, i.e. capital letters shall denote the inertial reference and small letters shall denote the rotating reference system. The conservation of momentum equations read as

$$\rho \left(\frac{D}{Dt} \right)_i \mathbf{U} - \operatorname{div}(\sigma) = \rho \mathbf{F}.$$

Let us neglect the gravitational effects, i.e. $\mathbf{F} = 0$. Let us moreover consider the stress tensor σ to have the following reduced form

$$\sigma = p(\rho) \mathbf{1}_{\mathbb{R}^3},$$

with p scalar function depending on the unknown ρ . The equation above becomes hence

$$\rho \left(\frac{D}{Dt} \right)_i \mathbf{U} + \nabla p(\rho) = 0. \quad (2.6.1)$$

We perform the following change of variable

$$\rho^M(t, x) = \rho \left(\frac{t}{M}, x \right), \quad \mathbf{U}^M = \frac{1}{M} \mathbf{U} \left(\frac{t}{M}, x \right), \quad (2.6.2)$$

which hence transforms (2.6.1) into

$$\rho^M \left(\frac{D}{Dt} \right)_i \mathbf{U}^M + \frac{1}{M^2} \nabla p(\rho^M) = 0. \quad (2.6.3)$$

Now we know that

$$\left(\frac{D}{Dt} \right)_i \mathbf{U} = \frac{D}{Dt} \mathbf{u} + 2\boldsymbol{\Omega} \wedge \mathbf{u} + \boldsymbol{\Omega} \wedge (\boldsymbol{\Omega} \wedge \mathbf{r}).$$

Let us consider for a moment the term

$$\frac{D}{Dt} \mathbf{u} + 2\boldsymbol{\Omega} \wedge \mathbf{u},$$

we suppose that

$$\boldsymbol{\Omega} = \Omega \mathbf{k},$$

where Ω is the scalar magnitude of the vector associated. We know it is defined as

$$\Omega = \frac{1}{T_R} = \frac{T}{T_R} \cdot \frac{1}{T} = \frac{1}{\operatorname{Ro} T},$$

where T_R is the time of rotation and T is the timespan considered. We deduced hence:

$$\frac{D}{Dt} \mathbf{u} + 2\boldsymbol{\Omega} \wedge \mathbf{u} = \frac{D}{Dt} \mathbf{u} + \frac{2}{\operatorname{Ro} T} \mathbf{k} \wedge \mathbf{u}.$$

We have been finally able to make appear a direct dependence on the Rossby number Ro for the rotation. Unfortunately there is a scaling parameter T which messes up with our notation, our aim is to absorb such term via a suitable change of parameters. Some observation has to be done.

- The parameter T which we want to absorb is the reference time-scale of the motion. the material derivative

$$\frac{D}{Dt} \mathbf{u} = \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u},$$

involves a differentiation-in-space. Hence in order to avoid multiple change in scales we shall consider the propagation of \mathbf{u} along the flow (say ψ), and abusing the notation we shall denote $\mathbf{u}(t) = \mathbf{u}(t, \psi(t))$.

- We are interested to define a change of unknown which is sensible to changes in the velocity flow in a time-scale T (along the lines of the flow the space-scale is determinate by the variation of time), hence we define

$$\mathbf{u}^{\text{Ro}}(t) = \frac{1}{T} \mathbf{u} \left(\frac{t}{T} \right).$$

Whence considering the change of time-scale $\mathbf{u}(t) \mapsto \mathbf{u} \left(\frac{t}{T} \right)$ we deduce that

$$\frac{d}{dt} \mathbf{u} \left(\frac{t}{T} \right) + \frac{2}{\text{Ro} T} \mathbf{k} \wedge \mathbf{u} \left(\frac{t}{T} \right) = \frac{d}{dt} \mathbf{u}^{\text{Ro}}(t) + \frac{2}{\text{Ro}} \mathbf{k} \wedge \mathbf{u}^{\text{Ro}}(t).$$

Finally hence defining

$$\rho^{\text{Ro}}(t, x) = \rho \left(\frac{t}{T}, x \right),$$

we deduce from (2.6.3)

$$\rho^{\text{M,Ro}} \frac{D}{Dt} \mathbf{u}^{\text{M,Ro}} + \frac{2}{\text{Ro}} \mathbf{k} \wedge (\rho^{\text{M,Ro}} \mathbf{u}^{\text{M,Ro}}) + \frac{1}{\text{Ro}^2} \nabla (|x_h|^2) + \frac{1}{\text{M}^2} \nabla p(\rho^{\text{M,Ro}}) = 0 \quad (2.6.4)$$

There is a number of precisions that have to be done relatively on how we deduced the system (2.6.4):

1. we supposed $\Omega = \Omega \mathbf{k}$ when in reality $\Omega = \Omega \mathbf{K}$. This simplification is motivated by the fact that, thanks to a multi-scale analysis, we neglected the terms $f_* u^3$ and $-f_* u^1$ in equation (2.3.20),
2. there is two kind of perturbations which modify the dynamics of (2.6.4): the first ones are small isotropic perturbations, these are fast-traveling (of speed c) local waves. Their speed of propagation is indeed quantified by the Mach number M and they produce a rapid variation of the pressure (from here in fact the appearance of the term $\frac{1}{\text{M}^2} \nabla p(\rho^{\text{M,Ro}})$). The scaling (2.6.3) was in fact introduced in order to magnify the effect of such waves and, doing so, avoiding to neglect the effects of compressibility of the flow,
3. the final scale can be read as:

$$\begin{cases} \mathbf{u}^{\text{Ro,M}}(t) &= \frac{1}{\text{MT}} \mathbf{u} \left(\frac{t}{\text{MT}}, x \right), \\ \rho^{\text{Ro,M}}(t) &= \rho \left(\frac{t}{\text{MT}}, x \right), \end{cases}$$

this scale allowed us to put in evidence in (2.6.4) terms describing isentropic perturbations,

$$\frac{1}{M^2} \nabla p (\rho^{M, Ro}),$$

and terms describing geophysical perturbations

$$\frac{2}{Ro} \mathbf{k} \wedge (\rho^{M, Ro} \mathbf{u}^{M, Ro}) + \frac{1}{Ro^2} \nabla (|x_h|^2),$$

the magnitude of these terms is evaluated accordingly to the importance of the physical numbers Ro, M ,

4. if the Rossby number $Ro \ll 1$ the centrifugal acceleration

$$\frac{1}{Ro^2} \nabla (|x_h|^2),$$

has indeed a large impact on the dynamics of (2.6.4). As explained in Section 2.1.2, generally, centrifugal effects are negligible in magnitude due to a compensated balance with gravitational effects. In order to simplify the model we decided *not to take in consideration* the gravity, but in a real-life word such balance takes place, for this reason from now on we neglect the centrifugal effects.

Combining hence the momentum budget equation with the conservation of mass we deduce the following system describing the motion at a planetary scale of a fluid which is weakly compressible (the unknowns here obviously depend on the quantities M and Ro):

$$\begin{cases} \partial_t (\rho^{M, Ro} \mathbf{u}^{M, Ro}) + \operatorname{div} (\rho^{M, Ro} \mathbf{u}^{M, Ro} \otimes \mathbf{u}^{M, Ro}) \\ \quad + \frac{2}{Ro} \mathbf{k} \wedge (\rho^{M, Ro} \mathbf{u}^{M, Ro}) + \frac{1}{M^2} \nabla p (\rho^{M, Ro}) = 0, \\ \partial_t \rho^{M, Ro} + \operatorname{div} (\rho^{M, Ro} \mathbf{u}^{M, Ro}) = 0. \end{cases}$$

Supposing Rossby and Mach number are comparably small, i.e.

$$Ro = 2\varepsilon, \quad M = \varepsilon,$$

we can finally write the system in the form in which it will be used

$$\begin{cases} \partial_t (\rho^\varepsilon \mathbf{u}^\varepsilon) + \operatorname{div} (\rho^\varepsilon \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon) + \frac{1}{\varepsilon} \mathbf{k} \wedge (\rho^\varepsilon \mathbf{u}^\varepsilon) + \frac{1}{\varepsilon^2} \nabla p (\rho^\varepsilon) = 0, \\ \partial_t \rho^\varepsilon + \operatorname{div} (\rho^\varepsilon \mathbf{u}^\varepsilon) = 0. \end{cases} \quad (\text{CE}_\varepsilon)$$

2.7 (Some) Known results on Navier-Stokes equations and singular perturbation problems.

The aim of the present section is to give an overview of what is known about singular perturbation problems. The singular perturbation problems we are interested in arise from the study of the fluid dynamic, hence we (briefly) present here a (very) partial exposition on known results for Navier-Stokes equations. For a much deeper survey on incompressible Navier-Stokes equations we refer the reader to the two monographs of P.-G. Lemarie-Rieusset [102], [103] and references therein.

2.7.1 Navier-Stokes and Euler.

In 1755 L. Euler in [64] derived the equations for a general fluid, compressible or not, in the presence of arbitrary external forces. The *Euler equations*

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, \\ u|_{t=0} = u_0 \end{cases} \quad (\text{E})$$

use Newton's law when the fluid element is submitted only to the external forces and to the pressure exerted by the other elements.

Taking into account internal stresses which the fluid exerts onto itself led to the *Navier–Stokes equations*

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = 0, \\ u|_{t=0} = u_0, \end{cases} \quad (\text{NS})$$

which were introduced by Navier in [118] and then formalized by Stokes in [139]. Back in those days the tools of analysis were completely inappropriate in order to have an analytical understanding of solutions of equations (E) and (NS), but Oseen first in [122] proved the existence of local-in-time solutions for the Navier-Stokes equations (NS). The long-time existence of smooth solutions of (NS) is a famous open problem. It is in fact so famous that it is one of the few mathematical problems known in mainstream culture, thanks to the million-dollar prize awarded by the Clay institute. In his Ph.D. thesis [104] Leray proved that, as long as we can define derivatives in a weak sense, there exist global "solutions". Such solutions does not solve the equation (NS) in the classical sense, since they may even not be differentiable, these were consequently denoted as solutions *à la Leray* or weak solutions.

Nonetheless the modern study of strong solutions for Navier-Stokes equations had a new birth in the sixties, with the pioneering work of H. Fujita and T. Kato [70]. The conceptual novelty in this work has been to conceive (NS) as an equation of the form

$$\begin{cases} \partial_t u + \mathcal{L}(t) u = f(t, u), \\ u|_{t=0} = u_0, \end{cases} \quad (2.7.1)$$

where \mathcal{L} is a suitable linear operator and f a forcing term, and hence to apply semigroup theory to study existence and regularity of solutions of such problem. This methodology is now known as *mild solutions* approach, which hence allowed the authors to prove in [70] that, given $u_0 \in H^1$ and $f \in L^2([0, T]; L^2)$, there exists a positive time $T_0 \leq T$ such that and a solution u of (2.7.1) defined in $[0, T_0]$ and such that

$$u \in \mathcal{C}([0, T_0]; H^1) \cap L^2([0, T_0]; H^2).$$

Such methodology has been subsequently applied in a wide number of cases in order to prove that (NS) is solvable locally in suitable spaces, and globally if the initial data is small. We mention some work but the list is non-exhaustive. T. Kato proved in [94] that (NS) is locally well posed in L^p spaces, where p satisfies some compatibility condition. Cannone Meyer and Planchon proved in [18] that the same result holds true in the homogeneous Besov space $\dot{B}_{p,\infty}^{\frac{d}{p}-1}(\mathbb{R}^d)$, until finally arriving to the endpoint space BMO^{-1} introduced by H. Koch and D. Tataru in the work [97] which encompasses all the cases mentioned above and gives a final answer in the widest possible space for mild solutions of Navier-Stokes equations.

2.7.2 When ε goes to zero: singular perturbation problems.

Singular perturbation problems have a long history. The interest in studying this kind of systems is motivated by the fact that they describe the dynamics of many turbulent physical phenomena occurring in Nature. Harmonic analysis tools are very well adapted in order to comprehend the behavior of perturbations which propagates at a high speed, such as acoustic waves and geophysical waves. Since the literature concerning such topic is vast is not trivial to give a comprehensive and exhaustive introduction to such problems. We decide hence first to give a brief introduction on the general methodology used in such problem, and to introduce the functional setting which will be adopted all along the thesis. Next we consider three systems which fall within the category of singular perturbation problems:

1. Navier-Stokes-Coriolis equations,
2. weakly compressible fluids equations,
3. primitive equations,

and to explain the mathematical background concerning such systems.

2.7.3 Physical motivations.

The study of geophysical fluids is a discipline which falls in the wider category of Fluid Mechanics. The mathematical models used are generally derived from a physical multi-scale analysis describing the motion of a fluid outside a planetary reference system, in large time-spans. Several mathematical simplifications are generally performed in order to reduce the physical system in a suitable mathematical form.

The fluids studied may have a wide variety of physical properties (compressible/ incompressible, density-dependent etc.), but, generally, the systems describing them at a planetary scale share some characteristic feature: there is a strong (in the sense of singular) external force acting on them (Coriolis-force, stratification buoyancy, acoustic perturbation...). The presence of such external forcing, i.e. a moving action which does not takes origin from the internal collision of particles, induces some rigidity on the evolution of the fluid, constraining the motion. This rigidity may hence be used, in some model, to prove that three-dimensional geophysical fluids are in fact globally well-posed in suitable scale-invariant spaces with no particular smallness assumption on the initial data.

2.7.4 Mathematical formulation.

The methodology applied varies greatly depending on the domain considered, In this thesis we shall consider either the whole space \mathbb{R}^3 or a generic periodic-in-space domain \mathbb{T}^3 . Let us consider the generic singular perturbation problem

$$\begin{cases} \partial_t U^\varepsilon(t, x) + \mathcal{B}(U^\varepsilon(t, x), U^\varepsilon(t, x)) + A_2 U^\varepsilon(t, x) \\ \quad + \frac{1}{\varepsilon} \mathcal{S}(U^\varepsilon(t, x)) = 0, & (t, x) \in \mathbb{R}_+ \times \Omega \\ U^\varepsilon(t, x)|_{t=0} = U_0^\varepsilon(x), & x \in \Omega. \end{cases} \quad (\text{SPP}_\varepsilon)$$

2.7. (Some) Known results on Navier-Stokes equations and singular perturbation problems.

In (SPP_ε) \mathcal{B} is a bilinear (generally transport) form, A_2 is a second order elliptic operator and \mathcal{S} a skew-symmetric, *linear* perturbation. This analysis can be done for a wider category of singular perturbations. We are interested in the regime $\varepsilon \rightarrow 0$. The domain Ω shall be considered to be $\Omega = \mathbb{R}^3, \mathbb{T}^3$, but it can assume more general forms. In such state the solutions of (SPP_ε) tend to "split" in two parts:

- \bar{U}^ε which is the projection of U^ε onto $\ker \mathcal{S}$,
- $\tilde{U}^\varepsilon = U^\varepsilon - \bar{U}^\varepsilon$ which is a highly perturbative part of the solution.

We can hence study the decomposition $\bar{U}^\varepsilon, \tilde{U}^\varepsilon$, such that $U^\varepsilon = \bar{U}^\varepsilon + \tilde{U}^\varepsilon$ separately. At this stage the techniques adopted differ from the geometry of the domain Ω . Nonetheless we can proceed as follows: we project the equation (SPP_ε) onto $\ker \mathcal{S}$. The resulting equation satisfied by \bar{U}^ε , which is the projection of U^ε onto $\ker \mathcal{S}$, can noticeably differ, accordingly to the singular perturbation \mathcal{S} . In any case there is one feature which is invariant: being $\bar{U}^\varepsilon \in \ker \mathcal{S}$ the equation satisfied by \bar{U}^ε does not present any more the singular external force $\varepsilon^{-1}\mathcal{S}$, and with it a turbulent behavior. At this point hence the kernel flow \bar{U}^ε is (generally) a three-dimensional flow, but it sometimes shares some property which is characteristic of bi-dimensional flows, such as we can extract some invariant "modified vorticity" which solves a transport or transport-diffusion equation. This is the key property which, eventually, allows us to prove that the penalized part of the solution is globally well-posed in some suitable space of subcritical regularity.

The part \tilde{U}^ε has to be dealt differently accordingly to the geometry of the domain Ω . We shall briefly explain the cases $\Omega = \mathbb{R}^3, \mathbb{T}^3$:

1. $\Omega = \mathbb{R}^3$. The equation satisfied by \tilde{U}^ε presents still a singular linear perturbation. Dispersive tools such as Strichartz estimates can be used to prove that \tilde{U}^ε converges to zero, strongly, in some suitable space.
2. $\Omega = \mathbb{T}^3$. In this case there can be a constructive interaction of highly oscillating nodes (resonance). A careful study of the resonance sets leads hence to a suitable simplification of the equation satisfied by \tilde{U}^ε .

The above explanation is a first, very simplified, step of the analysis which is usually performed in dealing with such problems, nonetheless it give some interesting insight already on the approach which has to be used in order to obtain stronger results.

$\Omega = \mathbb{R}^3$: In this case the dispersive estimates suggest that the part $\tilde{U}^\varepsilon \rightarrow 0$ strongly in some suitable space. Hence we can already deduce that, at least in a distributional sense

$$U^\varepsilon - \bar{U}^\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\mathcal{D}'} 0.$$

In particular if we allow $U_0^\varepsilon = \bar{U}_0 \in \ker \mathcal{S}$ the derived solution $U^\varepsilon = \bar{U} \in \ker \mathcal{S}$ is stationary w.r.t. ε , in the sense that it does not presents a highly oscillating behavior. Once we consider a generic ill-prepared initial data U_0 (we omit the dependence of ε for the initial data for the sake of simplicity) indeed there cannot be a convergence of

the form $U^\varepsilon - \bar{U}^\varepsilon = U^\varepsilon - \bar{U} \rightarrow 0$. This is motivated by the fact that the initial data $U_0 = \bar{U}_0 + \tilde{U}_0$ is not small in any norm under the assumption that there is a nonzero oscillating part \tilde{U}_0 . To avoid such problem we define the *free-wave linear equation*

$$\begin{cases} \partial_t W_{r,R}^\varepsilon + \frac{1}{\varepsilon} \mathcal{S}(W_{r,R}^\varepsilon) = 0, \\ W_{r,R}^\varepsilon|_{t=0} = \Psi_{r,R}(D) \tilde{U}_0. \end{cases}$$

In the above equation $\Psi_{r,R}$ is a localization (in the Fourier space) on a set that we call $\mathcal{C}_{r,R}$. Such localization is generally chosen in a way that the eigenvalues of \mathcal{S} which span spaces which are orthogonal to $\ker \mathcal{S}$ are non-singular on $\mathcal{C}_{r,R}$. This choice implies that $\mathcal{G}(t/\varepsilon)$, the semigroup generated by $\varepsilon^{-1}\mathcal{S}$ has no stationary phase in his evolution. Under these assumptions hence the free-wave function $W_{r,R}^\varepsilon$ has very nice dispersive properties (see [10, Chapter 8]), whence we can finally deduce the equation satisfied by the auxiliary function

$$\delta_{r,R}^\varepsilon = U^\varepsilon - W_{r,R}^\varepsilon - \bar{U},$$

and to prove that, fixed $0 < r \ll 1 \ll R$, $\delta_{r,R}^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0$ in some suitable topology of strong type.

$\Omega = \mathbb{T}^3$: The spatially periodic setting has a more involved behavior, due to the fact that dispersive estimates cannot be applied in such setting. Nonetheless several authors (notably [133], [72], [5], [8], [79], [123], [13], [127], [128]) developed suitable techniques in order to handle such case. Let us define as above $\mathcal{G}(t/\varepsilon)$, the semigroup generated by $\varepsilon^{-1}\mathcal{S}$, whence we can define the auxiliary unknown

$$V^\varepsilon = \mathcal{G}(-t/\varepsilon)U^\varepsilon,$$

which satisfies an equation of the form

$$\begin{cases} \partial_t V^\varepsilon + \mathcal{Q}^\varepsilon(V^\varepsilon, V^\varepsilon) - \mathcal{A}_2^\varepsilon V^\varepsilon = 0, \\ V^\varepsilon|_{t=0} = U_0. \end{cases} \quad (2.7.2)$$

The operators $\mathcal{Q}^\varepsilon, \mathcal{A}_2^\varepsilon$ are of the form

$$\begin{aligned} \mathcal{Q}^\varepsilon(V^\varepsilon, V^\varepsilon) &= \mathcal{G}\left(-\frac{t}{\varepsilon}\right) \left[\mathcal{B}\left(\mathcal{G}\left(\frac{t}{\varepsilon}\right)V^\varepsilon, \mathcal{G}\left(\frac{t}{\varepsilon}\right)V^\varepsilon\right) \right], \\ \mathcal{A}_2^\varepsilon V^\varepsilon &= \mathcal{G}\left(-\frac{t}{\varepsilon}\right) A_2 \mathcal{G}\left(\frac{t}{\varepsilon}\right) V^\varepsilon. \end{aligned}$$

The advantage of considering the system in the form (2.7.2) is that the time-derivative $\partial_t V^\varepsilon$ is bounded in the distributional space $L^p(\mathbb{R}_+; H^{-N}(\mathbb{T}^3))$, $p \in [2, \infty]$ for N sufficiently large. Standard compactness arguments and an application of stationary phase theorem (see [133] and [72]) allow us to deduce that $V^\varepsilon \rightarrow V^*$ distributionally, such function solves the following Cauchy problem

$$\begin{cases} \partial_t V^* + \mathcal{Q}^*(V^*, V^*) - \mathcal{A}_2^* V^* = 0, \\ V^*|_{t=0} = U_0, \end{cases} \quad (2.7.3)$$

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where \mathcal{Q}^* and \mathcal{A}_2^* are suitable limit-operators. We avoid to give any detail about the explicit form of such operators, what has to be retained is the fact they have better smoothing properties than \mathcal{B} , \mathcal{Q}^ε , A_2 , $\mathcal{A}_2^\varepsilon$. This is motivated by the fact that the application of the stationary phase theorem cancels many interactions in the Fourier space, and hence what remain are operators whose action is localized on a very specific set of Fourier modes. With this in mind it is possible (sometimes, up to suitable hypothesis of various kind) to prove that (2.7.3) is globally well posed in some strong-type topology.

What remains to understand is the (eventual) convergence of solutions of (2.7.2) to solutions of (2.7.3). Defining the unknown $W^\varepsilon = V^\varepsilon - V^*$ and deducing the equations satisfied by the vector field W^ε it can be remarked that there appear some nonlinearity \mathcal{R}^ε which converges to zero only in the sense of distributions, hence a direct bootstrap procedure is impossible to apply. To avoid such problem I. Gallagher developed in [72] a technique based on the work [133] of S. Schochet which consists in a suitable change of unknown which cancels the problematic terms. This technique will be applied in the present thesis as well.

2.7.5 Functional framework.

As this thesis is focused to study hydrodynamical systems in the whole space and in the periodic setting Fourier analysis provides an extremely well-adapted tool to study the propagation of perturbations arising from nonlinear interactions. We shall use all along the manuscript the remarkable *Bony decomposition* tool, introduced by J.-M. Bony in [14]. Such technique is based on the following concept: let us define a sequence (indexed by q) of operators

$$\Delta_q f = \mathcal{F}^{-1} \left(\phi_q \left(\frac{\cdot}{2^q} \right) \hat{f}(\cdot) \right),$$

which basically localize (in the Fourier space) the function f on frequencies of size comparable to 2^q . The advantage of such choice is that derivation operators act on $\Delta_q f$ as multiplications of constants C_q which depend (exponentially) on q . Bony decomposition hence consists in studying how dyadic localizations Δ_q acts on products of functions (see as well [10, Chapter 2]).

This kind of decomposition can moreover be used in order to define in a very simple way an entire plethora of functional spaces, most notably Sobolev spaces and *Besov spaces*. Besov spaces can be understand as, somehow, a generalization of Sobolev spaces, in particular they are functional spaces which describe very well the rate of oscillations of a function. Despite being these spaces very well-adapted to study critical propagation of perturbations we shall restrict, in the present thesis, to the use of Sobolev spaces.

Let us start with the simplest possible case: let us consider a tempered distribution u defined on \mathbb{R}^d , we say that u belongs to the homogeneous Sobolev space $H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$ if

$$\|u\|_{H^s(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \right)^{1/2} < \infty.$$

It is hence immediate to deduce that there is natural counterpart for Sobolev spaces in periodic domains: let us consider a tempered distribution u defined on \mathbb{T}^d , we say that u belongs

to the homogeneous Sobolev space $H^s(\mathbb{T}^d)$, $s \in \mathbb{R}$ if

$$\|u\|_{H^s(\mathbb{T}^d)} = \left(\sum_{n \in \mathbb{Z}^d} (1 + |n|^2)^s |\hat{u}(n)|^2 \right)^{1/2} < \infty.$$

The notion of such spaces belongs to the mathematical folklore, hence we shall not give more detailed descriptions of them.

In the study of evolutionary nonlinear equations a very important concept is the propagation of regularity. This concept is very intuitive: given an initial data we want to understand how the evolution of the system can create eventual perturbations or waves. Mathematically speaking we are hence considering how a derivative of a certain degree evolves in time. We would like hence to define functional spaces who carry *less information* on a determinate function: ideally we want to understand if a derivative of a certain degree (even large) blows up in a finite or asymptotic time.

Let us hence consider the set of functions u whose Fourier transform belong to the space $L^1_{\text{loc}}(\mathbb{R}^d_\xi)$, we say that u belongs to the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$, $s \in \mathbb{R}$ if

$$\|u\|_{\dot{H}^s(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \right)^{1/2} < \infty.$$

The hypothesis $\hat{u} \in L^1_{\text{loc}}(\mathbb{R}^d_\xi)$ may seem unusual at a first sight, but it can be easily justified. The multiplier $|\xi|^s$ acts on u as a derivative of order s . Let us hence consider a polynomial $p = p(x)$ such that $\deg p \leq s$, then indeed $\mathcal{F}^{-1}(|\xi|^s \hat{p}(\xi)) = 0$, whence the above norm would be well defined modulo a polynomial in the case such hypothesis would be missing.

There is no periodic counterpart for homogeneous Sobolev spaces.

As explained in detail in Section 2.7.4, in the regime $\varepsilon \rightarrow 0$, solutions of (SPP_ε) converge (in some way) to elements which belong to $\ker \mathcal{S}$. Obviously such space ($\ker \mathcal{S}$) varies accordingly to the explicit formulation of the penalized operator \mathcal{S} . There is no generic classification of this kind of space, but very often it happens that solutions which belong to $\ker \mathcal{S}$ have a very different behavior along vertical and horizontal directions. This motivates to introduce functional spaces which are well adapted to such anisotropy, in the context of fluid dynamics these spaces have been introduced by D. Iftimie in [88], [85] and [86] and have been later refined by M. Paicu in [124] and [125]. We shall consider anisotropic spaces of Sobolev type for functions defined of three-dimensional domains only.

Let u be a tempered distribution, u belongs to the non-homogeneous anisotropic Sobolev space $H^{s,s'}(\mathbb{R}^3)$, $s, s' \in \mathbb{R}$ if

$$\|u\|_{H^{s,s'}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} (1 + |\xi_h|^2)^s (1 + |\xi_3|^2)^{s'} |\hat{u}(\xi)|^2 d\xi \right)^{1/2} < \infty.$$

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The periodic counterpart is indeed defined as the tempered distributions such that

$$\|u\|_{H^{s,s'}(\mathbb{T}^3)} = \left(\sum_{n \in \mathbb{Z}^3} (1 + |n_h|^2)^s (1 + |n_3|^2)^{s'} |\hat{u}(n)|^2 \right)^{1/2} < \infty.$$

2.7.6 The Navier-Stokes-Coriolis equations.

The Navier-Stokes-Coriolis equations describe the motion of a homogeneous fluid at a planetary scale, namely these equations describe the motion of a fluid, at a mid-latitude, under the effect of a strong (in magnitude) external rotating force orthogonal to the x_3 axis. These read as

$$\begin{cases} \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \nu \Delta u^\varepsilon + \frac{1}{\varepsilon} e^3 \wedge u^\varepsilon = -\frac{1}{\varepsilon} \nabla p^\varepsilon & \text{in } \mathbb{R}_+ \times \Omega, \\ u^\varepsilon|_{t=0} = u_0^\varepsilon & \text{in } \Omega. \end{cases} \quad (\mathbf{RF}_\varepsilon)$$

The vector e^3 in $(\mathbf{RF}_\varepsilon)$ stays for $e^3 = (0, 0, 1)$.

Let us consider at first the case in which the (three dimensional) spatial domain is periodic in all dimensions, i.e.

$$\Omega = \mathbb{T}^3 = \mathbb{R}^3 / \prod_{i=1}^3 a_i \mathbb{Z} = \prod_{i=1}^3 [0, 2\pi a_i].$$

We start with the work [79] of E. Grenier. In such paper the author studied the convergence of solutions of $(\mathbf{RF}_\varepsilon)$ to the solutions of the two-dimensional Navier-Stokes equations in the case in which the initial data $u_0^\varepsilon = u_0 = u_0(x_h)$ depends on the horizontal variables only. Such setting for the initial data is known as the *well-prepared case*. This name comes from the fact that vector fields depending on the horizontal variable only belong to the kernel of the operator $\mathbb{P}(e^3 \wedge \cdot)$, where \mathbb{P} is the Leray projector onto divergence-free vector fields, defined as

$$\mathbb{P} = 1_{\mathbb{R}^3} - \Delta^{-1} \nabla \operatorname{div}.$$

Generally, for singular perturbation problems, the definition of well-prepared initial data shall always denote a choice of an initial data which belongs to the kernel of the singular perturbation. The result is attained studying the perturbative waves thanks to a group of isometries known as the Poincaré group.

Next we mention the work [5] of A. Babin, A. Mahalov and B. Nikolaenko, which studied the system $(\mathbf{RF}_\varepsilon)$ in a periodic setting in the case in which the initial data depends on all three-dimensional variables, case known as *ill-prepared*. In this work the authors decide to impose certain geometric restrictions on the domain in way that bilinear interactions between highly oscillating perturbations does not occur. Such domain setting is known as *non-resonant domain*. Then the same authors studied in [8] the same problem but in generic domains. The result is attained via improved product algebraic rules deduced on the bilinear form.

In [91] the global stability of the Navier-Stokes-Coriolis equations is studied in a different functional setting

T. Colin and P. Fabrie studied in [48], [49] the periodic system $(\mathbf{RF}_\varepsilon)$ in the case in which the diffusivity assumes the form $(\nu_h, \varepsilon \nu_v)$, i.e. the vertical diffusivity goes to zero at the same rate as the Rossby number. For periodic boundary conditions and suitable, well-prepared

initial data they study the long-time existence of the system $(\mathbf{RF}_\varepsilon)$ and convergence to the two-dimensional Navier-Stokes equations.

Isabelle Gallagher studied in [72] singular perturbation periodic problems in a very general setting. In the case with ill-prepared initial data and non-resonant domains she proved global well posedness and convergence of solutions of $(\mathbf{RF}_\varepsilon)$ to the ones of the two-dimensional Navier-Stokes equations in the critical Besov-Sobolev space $\mathcal{B}^{0, \frac{1}{2}}(\mathbb{T}^3)$.

Marius Paicu studied in [123] the more generic case in which the viscosity in $(\mathbf{RF}_\varepsilon)$ assumes an anisotropic form (ν_h, ν_v) , where $\nu_h \geq c > 0$ and $\nu_v \geq 0$. He assumes all over his work hence that $\nu_v \equiv 0$ in order to take in account all cases possible. This kind of problem presents a mixed parabolic-hyperbolic behavior respectively in the horizontal and vertical variables. The initial data is considered to be ill-prepared, and the domain satisfies a suitable geometrical condition. Such condition (which we do not explicit here, as it is very tedious) is generic enough to include all non-resonant tori and a large class of resonant domains.

Next we focus on the case in which

$$\Omega = \mathbb{R}^3,$$

in [40] and [38] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier proved that the solutions of the system $(\mathbf{RF}_\varepsilon)$ with zero vertical diffusivity converge globally to the solutions of the two dimensional Navier-Stokes equations, and such solutions are globally well posed for ε sufficiently small. Then in [42] they proved that such result holds true in the case in which the viscosity is isotropic (i.e. spherically symmetric). These results are attained via a methodology which is completely different to the one adopted in the case of the periodic space. The main concept in this kind of approach is that the singular operator $\varepsilon^{-1}\mathbb{P}(e^3 \wedge \cdot)$ produces perturbations which propagates with a speed of order ε^{-1} , such a fact can be used, implementing what is called *dispersive Strichartz estimates*, to prove that there is a part of the solution which converges globally and strongly to zero in some $L^q(\mathbb{R}_+, L^p(\mathbb{R}^3))$ space. This fact can be hence exploited via a bootstrap procedure in order to prove that

$$w^\varepsilon - w^\varepsilon - \bar{u}^h \rightarrow 0$$

in some suitable, global, energy space. The function w^ε is called the *free-wave function*, and is solution to the linear problem associated to $(\mathbf{RF}_\varepsilon)$

$$\begin{cases} \partial_t w^\varepsilon - \nu \Delta w^\varepsilon + \frac{1}{\varepsilon} e^3 \wedge w^\varepsilon = 0, \\ w^\varepsilon|_{t=0} = w_0^\varepsilon. \end{cases}$$

The function \bar{u}^h is the solution of the two-dimensional Navier-Stokes equation. In [98] P. Konieczny and T. Yoneda prove that the Coriolis force has stabilizing effects even for flows which are not decaying at infinity.

V.-S. Ngo proved in [119] that this result remain true if the horizontal viscosity vanishes as well as a suitable positive power of the Rossby numer, i.e. $\nu_h = \mathcal{O}(\varepsilon^\alpha)$, and even when the vertical viscosity is null.

Indeed on a planetary scale the oceans and the atmosphere are thin layers, in the sense that their vertical dimension is very small if compared with the horizontal reference scale.

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For this reason it makes sense to consider a spatial domain of the form

$$\Omega = \mathbb{R}_h^2 \times [0, 1].$$

Close to the boundary of the domain, i.e. close to $x_3 = 0, 1$, there happens a physical effect which is worth to be taken in consideration, the *Ekman layer*. This is the layer in a fluid where there is a force balance between pressure gradient force, Coriolis force and turbulent drag. It was first described by Vagn Walfrid Ekman in [61], and it describes a phenomenon in which turbulence tends to vanish close to surface of the Earth, which translates, on a mathematical point of view, to consider Dirichlet boundary conditions on $\partial\Omega$. Moreover experimental verifications show that the vertical viscosity tends to be irrelevant when the rotation is sufficiently strong. All in all hence it makes sense to consider the following system

$$\begin{cases} \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - (\nu_h \Delta_h + \varepsilon \beta \partial_3^2) u^\varepsilon + \frac{1}{\varepsilon} e^3 \wedge u^\varepsilon = -\frac{1}{\varepsilon} \nabla p^\varepsilon, \\ u^\varepsilon|_{t=0} = u_0^\varepsilon, \\ u^\varepsilon|_{x_3=0,1} = 0. \end{cases} \quad (\text{Ek}_\varepsilon)$$

E. Grenier and N. Masmoudi studied in [80] the case in which the spatial domain is $\Omega_h \times [0, 1]$, with $\Omega_h = \mathbb{R}_h^2, \mathbb{T}_h^2$. The initial data was considered to be well prepared, in the sense that is independent on the variable x_3 . They proved the following result via a WKB analysis: it is possible to construct a function $u_{\text{BL}}^\varepsilon$, of the form

$$u_{\text{BL}}^\varepsilon = u_{\text{BL}}^\varepsilon \left(t, x_h, \frac{x_3}{\varepsilon} \right),$$

which is regular, such that, given u^ε a weak solution of (Ek_ε) then

$$u^\varepsilon - u_{\text{BL}}^\varepsilon \rightarrow \bar{u}^h,$$

weakly, where \bar{u}^h is a solution of the two-dimensional Navier-Stokes equation with damping. N. Masmoudi proved in [114] the same result but for generic initial data, and in [113] he proves that weak solution of Navier-Stokes equations with large Coriolis term converges to the Euler system with damping term under the conditions that viscosity and Rossby number go to zero at a suitable rate. J.-Y. Chemin et al. in [41] proved that the same result is true for the case of ill-prepared data in the space domain $\Omega = \mathbb{R}_h^2 \times [0, 1]$. In this case a boundary layer in time has to be taken as well in consideration.

Concerning the problem of Ekman boundary we mention that in the works [80], [114] and [41] were considered *horizontal* boundary layers only. The problem of nonflat Ekman boundary layers appears to be significantly more complex. In this setting we mention the work [16] in which the authors investigate the limit dynamic, when the Rossby number tends to zero, of the Navier-Stokes-Coriolis equations in a cylinder $C = B_1(0) \times [0, 1]$. In the work [51], instead, are studied the Navier-Stokes-Coriolis equations in a non-flat bottom setting.

2.7.7 Fluid dynamics with vanishing Mach number.

Classically a fluid in motion is submitted to two kinds of stresses corresponding to compression effects and viscous effects, then one writes

$$\sigma = -p\mathbf{1} + \tau,$$

where p is a scalar function, the *pressure* and τ is the viscous stress tensor. We denote by $\mathbf{1}$ the identity matrix (tensor).

Let us *postulate* that τ is a linear function of ∇u , invariant under translation and rotation, and that the fluid is isotropic. It is possible to deduce that necessarily

$$\tau = \lambda \operatorname{div} u + \mu (\nabla u + \nabla^t u),$$

and μ, λ are the so-called Lamé viscosity coefficients. These three assumptions on the stress tensor τ (linearity, invariance, isotropy) define what is known as a *newtonian fluid*. A priori the Lamé coefficients λ, μ may depend on the density ρ , but in a general mathematical framework they are considered to be constants such that $\mu > 0$ and $\nu = \lambda + 2\mu > 0$. Such conditions ensure that the operator $\mu\Delta + (\lambda + \mu)\nabla\operatorname{div}$ is elliptic. For most fluids experiments indicate that λ and μ are, in practice, very small.

As explained in Section 2.6 it is easy, thanks to the scaling (2.6.3), and without considering the influence of the rotation of the Earth, to deduce the system

$$\begin{cases} \partial_t(\rho^\varepsilon u^\varepsilon) + \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \mu\Delta u^\varepsilon - (\lambda + \mu)\nabla\operatorname{div} u^\varepsilon + \frac{1}{\varepsilon^2}\nabla P(\rho^\varepsilon) = \rho^\varepsilon f \\ \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0 \\ (\rho^\varepsilon, u^\varepsilon)|_{t=0} = (\rho_0^\varepsilon, u_0^\varepsilon), \end{cases} \quad (\text{WC}_\varepsilon)$$

where we implicitly assumed $M = \varepsilon$. Many results have been obtained concerning the system (WC_ε) in the case of *well prepared* initial data, *i.e.*

$$\rho_0^\varepsilon = 1 + \mathcal{O}(\varepsilon^2) \quad \text{and} \quad \operatorname{div} u_0^\varepsilon = \mathcal{O}(\varepsilon),$$

such hypothesis on the magnitude of Mach number is motivated by the physical analysis performed in Section 2.3. In this setting S. Klainerman and A. Majda in [95] develop a first general theory adapted to study existence and convergence of some quasi-linear hyperbolic system arising in physics when some physical parameter tends to infinity. This theory is broad enough to study a wide variety of singular limits in compressible fluid flow and magneto-fluid dynamics. Always in the setting of well-prepared initial data it is proved in [99] that solutions of (WC_ε) in the simplified form

$$\begin{cases} \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \nabla\Delta u^\varepsilon + \nabla p^\varepsilon = 0, \\ \partial_t p^\varepsilon + u^\varepsilon \cdot \nabla p^\varepsilon + \frac{1}{\varepsilon^2} \operatorname{div} u^\varepsilon = 0, \end{cases}$$

in a periodic setting admit the following asymptotic expansion

$$\begin{aligned} u^\varepsilon &= U + u_1^\varepsilon + \mathcal{O}(\varepsilon^4), \\ p^\varepsilon &= P + p_1^\varepsilon + \mathcal{O}(\varepsilon), \end{aligned}$$

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where (U, P) solve the incompressible Navier-Stokes equations and $(u_1^\varepsilon, p_1^\varepsilon)$ are highly oscillating in-time functions.

Always in the well prepared initial data setting it is proved in [105] and [82] that solutions of the compressible Navier-Stokes equation converge strongly to the solutions of the incompressible system,

In the case of *ill prepared* initial data, it is only assumed that

$$\rho_0^\varepsilon = 1 + \varepsilon b_0^\varepsilon$$

and $(b_0^\varepsilon, u_0^\varepsilon)$ are only bounded in some suitable spaces which does not necessarily belong to the kernel of the penalized operator. If $\mathbb{P}u_0^\varepsilon \rightarrow v_0$ when ε goes to zero, one expects that $u^\varepsilon \rightarrow v$ where v is the solution of the incompressible Navier-Stokes equations (NS). The expected convergence is however not easy to be rigorously justified. The main difficulty lies in the fact that one has to deal with the propagation of acoustic waves with speed of order ε^{-1} , a phenomenon which does not occur in the case of well prepared data.

In [110], P.-L. Lions proved the existence of global weak solutions of (WC_ε) for initial data with minimal regularity assumptions. The fluid is supposed to be isentropic and the pressure is of the form $P(\rho) = a\rho^\gamma$, with certain restrictions on γ depending on the space dimension d . In the same setting P.-L. Lions and N. Masmoudi in [111] proved that weak solutions of (WC_ε) converges weakly to weak solutions of (NS) in various boundary settings. This result is proved via some weak compactness methods (see also [77] and [66]). In the work of B. Desjardins, E. Grenier, P.-L. Lions and N. Masmoudi [57], considering (WC_ε) with $f \equiv 0$, in a bounded domain Ω with Dirichlet boundary conditions, the authors proved that as $\varepsilon \rightarrow 0$, the global weak solutions of (WC_ε) converge weakly in L^2 to a global weak solution of the incompressible Navier-Stokes equations (NS). In [56], using dispersive Strichartz-type estimates, Desjardins and Grenier proved that the gradient part of the velocity field (*i.e.* the gradient of the acoustic potential) of the system (WC_ε) converges strongly to zero. Finally, we want to mention the works of R. Danchin [52] and [54]. In [52], the author proved global existence of *strong* solutions for the system (WC_ε) for small initial data in some suitable, critical, scale-invariant (Besov) spaces, in the same spirit as in the work of Cannone, Planchon and Meyer [18] or the work of Fujita-Kato [70] for the incompressible model. In [54], the author addressed to the convergence of (WC_ε) to (NS) for ill-prepared initial data when the Mach number ε tends to zero. When the initial data are small, the author obtains global convergence and existence, while for large initial data with some further regularity assumptions, it is shown that the solution of (WC_ε) exists and converges to the solution of (NS) in the same time interval of existence of the solution of (NS). For compressible inviscid fluids in the non-rotating case, in A. Dutrifoy and T. Hmidi [60], the authors considered the system (WC_ε) in \mathbb{R}^2 with initial data not uniformly smooth (*i.e.* the C^1 norm is of order $\mathcal{O}(\varepsilon^{-\alpha})$, $\alpha > 0$). The convergence to strong, global solutions of 2D Euler equation is proved by mean of Strichartz estimates and the propagation of the minimal regularity.

We mention as well the work [73]; in such work it is proved that three-dimensional, periodic, weakly compressible fluids propagate (globally) smoothness when the Mach number is sufficiently small. It is assumed that the initial data is small with respect to a constant which only depends on the viscosity.

Let us now focus on fast rotating, compressible fluids. To the best of our knowledge, there is no result yet concerning the the inviscid system (CE_ε) . In the viscous fast rotating case,

in [68], E. Fereisl, I. Gallagher and A. Novotný studied the dynamics, when the Mach number equals the Rossby number, of weak solutions of the system system $(\mathbf{CE}_\varepsilon)$ with viscous tensor in $\mathbb{R}^2 \times \mathbb{T}^1$, with non-slip boundary conditions

$$u^{\varepsilon,3}|_{x_3=0,1} = 0 \quad \text{and} \quad (S_{2,3}, -S_{1,3}, 0)|_{x_3=0,1} = 0,$$

where \mathbb{S} is the stress viscous tensor

$$\mathbb{S}(\nabla u) = \mu \left(\nabla u + \nabla^\top u - \frac{2}{3} \operatorname{div} u I \right).$$

i.e. the system

$$\begin{cases} \partial_t (\rho^\varepsilon u^\varepsilon) + \operatorname{div} (\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \operatorname{div} \mathbb{S}(\nabla u) + \frac{1}{\varepsilon} e^3 \wedge (\rho^\varepsilon u^\varepsilon) + \frac{1}{\varepsilon^2} \nabla p(\rho^\varepsilon) = 0, \\ \partial_t \rho^\varepsilon + \operatorname{div} (\rho^\varepsilon \mathbf{u}^\varepsilon) = 0, \\ (u^\varepsilon, \rho^\varepsilon)|_{t=0} = (u_0, \rho_0). \end{cases} \quad (\mathbf{CNS}_\varepsilon)$$

Their result relies on the spectral analysis of the singular perturbation operator. Using RAGE theorem (see [132]), the authors proved the dispersion due to fast rotation and that weak solutions of $(\mathbf{CNS}_\varepsilon)$ converges to a 2D viscous quasi-geostrophic equation for the limit density. We refer to [68] for a detailed description of the limit system. In [67], Feireisl, Gallagher, Gérard-Varet and Novotný studied the system $(\mathbf{CNS}_\varepsilon)$ in the case where the effect of the centrifugal force was taken into account. Noticing that this term scales as ε^{-2} , they studied both the isotropic limit and the multi-scale limit: namely, they supposed the Mach-number to be proportional to ε^m , for $m \geq 1$. We want to point out that, in the analysis of the isotropic scaling ($m = 1$), the authors had to resort to compensated compactness arguments in order to pass to the limit: as a matter of fact, the singular perturbation operator had variable coefficients, and spectral analysis tools were no more available. Recently in [65], F. Fanelli proved a similar result as the one proved in [68], by adding to the system $(\mathbf{CNS}_\varepsilon)$ a capillarity term and studying various regimes depending on some positive parameter. We mention as well the work [93] in which A. Jüngel, C.-K. Lin and K.-C. Wu study the incompressible and vanishing capillarity limit for the two-dimensional, periodic, barotropic Navier-Stokes equations with constant (hence nonsingular) horizontal rotation effects.

We want to remark that all the compressible systems previously mentioned are isothermal. In the case of variable temperature, the generic system governing a heat conductive, compressible fluid is the following

$$\begin{cases} \partial_t \rho + \operatorname{div} (\rho u) = 0, \\ \partial_t (\rho u) + \operatorname{div} (\rho u \otimes u) - \operatorname{div} (\tau) + \nabla P = \rho f, \\ \partial_t \left(\rho \left(\frac{|u|^2}{2} + e \right) \right) + \operatorname{div} \left[u \left(\rho \left(\frac{|u|^2}{2} + e \right) + P \right) \right] \\ = \operatorname{div} (\tau \cdot u) - \operatorname{div} q + \rho f \cdot u, \end{cases} \quad (\mathbf{HCCNS})$$

which can be derived from the conservation of mass, linear momentum and energy. We refer the reader to [109] and references therein for more details. Here, the fluid is always supposed to be newtonian and $e = e(t, x)$ is the internal (thermal) energy per unit mass. The heat

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conduction q is given by $q = -k\nabla\mathcal{T}$, where k is positive and \mathcal{T} stands for the temperature. If e obeys Joule rule (*i.e.* e is a function of \mathcal{T} only), the initial data is smooth and the initial density is bounded and bounded away from zero, the existence and uniqueness of a local classical solution has already been known for a long time (see [117] or [90]). In [53], R. Danchin proved that (HCCNS) is locally well posed in the critical scale-invariant space $B_{p,1}^{\frac{d}{p}-1}(\mathbb{R}^d)$, $p \in [1, \infty)$.

Indeed the system (HCCNS) can be studied in the regime in which the Mach number goes to zero, we mention hence the works of G. Métivier and S. Schochet [115] and T. Alazard [2] in which is proved that, for generic initial data, the solutions of the Navier-Stokes equations exist and they are uniformly bounded for a time interval which is independent of the Mach, Reynolds and Peclet numbers.

2.7.8 Primitive equations.

Primitive equations (PE_ε) can be understood as a system describing the dynamics of a fluid at a planetary scale in long time-scales where the difference of density induced by gravitational stratification effect is taken in consideration. This was explained in great detail in Section 2.3 and 2.4, but for a more detailed physical discussion on the topic we refer the reader to the monographs [50] and [126].

As in the subsection concerning the Navier-Stokes-Coriolis equations we divide tentatively the present section accordingly to the geometry of the domain. As in the case of Navier-Stokes-Coriolis equations we analyze in detail the results present in the literature when the space domain Ω is either

$$\mathbb{T}^3, \quad \mathbb{R}^3, \quad \mathbb{R}^2 \times [0, 1].$$

Before starting to discuss such particular configurations it is (in our opinion) worth to mention the works [107] and [108] in which J.-L. Lions R. Temam and S. Wang derive the formulation of the primitive equations in a spherical reference system (hence the formulation is more complex than in (PE_ε)). They prove that such equations admit strong local-in-time and analytical in time solutions, and then they focus on the Hausdorff and fractal dimension of the attractors of the system.

For the space-periodic setting we mention the work of T. Beale and A. Bourgeois [15] which studied the inviscid primitive equations with a simplified equation for the density in a domain which is periodic in the horizontal direction and bounded in the vertical, with a regular initial data. Adopting a suitable change of variables they reduce themselves to a problem which is periodic in all directions, in such domain they prove local and then global well-posedness for the quasi-geostrophic system and convergence of solutions of primitive equations (PE_ε) in the case in which the initial data is H^3 and well-prepared, in the sense that it is in the quasi-geostrophic equilibrium.

Next we mention the works [62] and [63]. In these works the authors consider primitive equations (and even more generic systems) in different regimes in domain which are periodic-in-space. In particular the authors are interested to study the interactions between fast gravity

waves. They state and prove various weak convergence results, such as the (weak) convergence for well-prepared data of rotating Boussinesq and shallow-water equations to quasi-geostrophic flows in the regime in which the Rossby number vanishes.

In [6] Babin et al. study the inviscid primitive equations: they prove that the lifespan of the solutions tends to infinity as the rotation and the stratification buoyancy increase in magnitude. I. Gallagher proved always in [72] that the solutions of the primitive equations converge to the solutions of the quasi-geostrophic system in a periodic setting, for generic initial data and non-resonant domains. The result she proves is derived as a special case of convergence for singular parabolic systems with symmetric nonlinearity: being the limit equation globally defined she sets up an adapted bootstrap procedure which permits to prove convergence globally.

A. Babin, A. Mahalov and B. Nicolaenko continue in [7] the development of the theory started in [5] and [8]. In such work the limit dynamic of primitive equations in the periodic setting is described. It is proved that the flow describing the limit dynamic can be globally split in a geostrophic and ageostrophic part which represent the dynamical attractor of the system.

Next we mention the works [127], [92] and [128]. In [127] M. Petcu, R. Temam and D. Wirosoetisno study the two-dimensional primitive equations using a renormalization group technique introduced in [44], [45], [145], [116], [142] and [143]. The result they prove is an approximation of the exact solutions of the bi-dimensional primitive equations by mean of an asymptotic smooth solution which exists for all times, moreover the authors estimate the difference between approximate and exact solutions. In [92] the authors study the problem of a singularly perturbed quasigeostrophic equation when the viscosity tends to zero: this generates boundary layers. They hence construct suitable smooth correctors which compensate the perturbative effects generated by a vanishing viscosity. In the article [128] the authors address the same problem as in [127] but in dimension three.

In the case of the whole space \mathbb{R}^3 , J.-Y. Chemin proved in [36] that, if the Froude number is equal to the Rossby number, and both of them converge to zero at the same rate, the solutions of the primitive equations converge to the ones of the quasi-geostrophic system assuming that only a part of the initial data (called $U_{\text{osc},0}^\varepsilon$) is small (in fact converges to zero as $\varepsilon \rightarrow 0$) in a suitable norm. The convergence takes place in the homogeneous Sobolev $\dot{H}^s(\mathbb{R}^3)$ topology. The assumption that $F = 1$ (see (PE_ε)) alienates any hope of using dispersive tools.

The fact that $F = 1$ implies that, for algebraic reasons, the phase is no more oscillating, hence dispersive estimates cannot be applied. All along this thesis we shall instead always consider systems which present oscillating behaviors.

D. Iftimie studied in [87] the inviscid primitive equations in the case that $F \neq 1$. He proves the convergence of solutions of (PE_ε) to the ones of the quasi-geostrophic system in the $L_{\text{loc}}^\infty(\mathbb{R}_+; H^\sigma(\mathbb{R}^3))$ topology, with $\sigma < s, s > 5/2$. He assumes the initial data to be well prepared, in the sense that $U_0^\varepsilon = U_{\text{QG},0}^\varepsilon + U_{\text{osc},0}^\varepsilon$, and $U_{\text{osc},0}^\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$ in $L^2(\mathbb{R}^3)$. The methodology adopted is a propagation of $H^s(\mathbb{R}^3)$, $s > 5/2$ regularity in the spirit of the well known Beale-Kato-Majda criterion [12] for the potential vorticity Ω^ε .

We mention next the works of F. Charve, whose Ph.D. thesis [27] is entirely devoted to the primitive equations in the form (PE_ε) . He proved several results on such system: first of all he proved in [29] that, under the assumption $F \neq 1$ and minimal regularity assumption $U_0 \in$

2.7. (Some) Known results on Navier-Stokes equations and singular perturbation problems.

$L^2(\mathbb{R}^3)$, the weak solutions of (PE_ε) converge weakly to the ones of the quasi-geostrophic system. In [28] he proves that, if the initial data is in $\dot{H}^s(\mathbb{R}^3)$, $s \in [\frac{1}{2}, 1]$, then

$$U^\varepsilon - W^\varepsilon \rightarrow U_{\text{QG}},$$

in the space $L^\infty(\mathbb{R}_+; \dot{H}^s(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+; \dot{H}^{s+1}(\mathbb{R}^3))$, $s \in [1/2, 1]$. The function W^ε is a solution of a suitable linear system derived from the system solved by U_{QG} . We do not go into details. Next he improved the above results in [31] assuming that the initial data U_0^ε can explode as some suitable negative power $\varepsilon^{-\gamma}$ of the Rossby and Froude number and in [32] assuming less regular initial data. In the joint work [34] F. Charve and V.-S. Ngo proved that the primitive equations (PE_ε) are globally well posed even in the case that the vertical diffusivity $\nu_v, \nu'_v = 0$ and the horizontal diffusivities ν_h, ν'_h go to zero as a suitable power of the Rossby number in the case that the initial data is well-prepared, in the sense that the quasi-geostrophic part of the initial data is null. The works [33] and [26] are a generalization of [31]. In [31] the author assumes the viscosity and the thermal diffusivity to be equal, while in the works [33] and [26] it is not the case. In such setting the three-dimensional quasi-geostrophic system is a transport equation with nonlocal diffusive operator $-\Gamma$, such operator induces a number of additional technical difficulties in the study of the propagation of regularity for the quasi-geostrophic equations.

In the case in which the spatial domain is an infinite slab between two fixed planes i.e. $\mathbb{R}_h^2 \times [0, 1]$ with Dirichlet boundary conditions B. Desjardins and E. Grenier proved in [55] that, for well-prepared initial data, there is existence and weak convergence as the Rossby number tends to zero. In the same setting D. Niu proves in [121] a similar result adopting slightly different techniques; it is proved that, for well-prepared initial data, the smooth solutions of the primitive equations converge to smooth solutions of quasi-geostrophic equations as the Rossby number, the vertical viscosity and the vertical heat conductivity tend to zero. In [9] Babin et al. study the viscid primitive equations in an infinite slab and they focus on the convergence toward a suitable limit system.

We mention as well the work [131] in which M. Puel and A. Vasseur study a system in a semi-geostrophic balance. In such setting the quasi-geostrophic velocity flow can be fully recovered from a scalar function known as *stream function* Ψ . They hence prove that the evolution equations satisfied by Ψ admit a weak solution in the half space $\mathbb{R}_{(x_1, x_2)}^2 \times (\mathbb{R}_+)_{x_3}$.

In the (non exhaustive) series of works [23], [24], [25], [20], [19], [21] and [22], C. Cao, J. Li and E. Titi proved several results of global well posedness for the equations

$$\begin{cases} \partial_t u^h + u \cdot \nabla u^h + \nabla_h p + f e^3 \wedge u^h + L_1 u^h = 0, \\ u = (u^h, u^3), \\ \partial_3 p + T = 0, \\ \text{div } u = 0, \\ \partial_t T + u \cdot \nabla T + L_2 T = 0, \end{cases}$$

in which L_1 and L_2 are second order differential (isotropic or anisotropic) diffusion operators, in various domains and boundary settings.

At last we mention that the primitive equation (PE_ε) describe the dynamic of a stratified, rotating fluid in the very simplified setting in which gravity and rotation waves have comparable frequencies. In the series of works [77], [75], [76], [46], [47] and [74] I. Gallagher and L. Saint-Raymond with other authors study the dynamics of more physically relevant systems: gravity and rotation waves are considered to have a wider spectrum of frequencies and the superposition dynamic of such perturbation is hence way richer. We do not give a more detailed characterization of such works since the systems considered in the present thesis are considerable simplifications of the systems considered in the aforementioned papers.

2.8 Contributions of the thesis.

The main idea underlying the present thesis is to use the techniques developed in the works [38], [72], [133] and [123] (dispersion, Schochet's technique, control of resonance) and to apply them to a variety of problems which present strong (in magnitude) forces acting on them. These external forces may have very different physical motivations.

2.8.1 Primitive equations.

We re-write here for the convenience of the reader the primitive equations (PE_ε):

$$\left\{ \begin{array}{l} \partial_t u^{1,\varepsilon} + u^\varepsilon \cdot \nabla u^{1,\varepsilon} - \nu_h \Delta_h u^{1,\varepsilon} - \frac{1}{\varepsilon} u^2 = -\frac{1}{\varepsilon} \partial_1 \Phi_\varepsilon \\ \partial_t u^{2,\varepsilon} + u^\varepsilon \cdot \nabla u^{2,\varepsilon} - \nu_h \Delta_h u^{2,\varepsilon} + \frac{1}{\varepsilon} u^{1,\varepsilon} = -\frac{1}{\varepsilon} \partial_2 \Phi_\varepsilon \\ \partial_t u^{3,\varepsilon} + u^\varepsilon \cdot \nabla u^{3,\varepsilon} - \nu_h \Delta_h u^{3,\varepsilon} + \frac{1}{F \varepsilon} \theta^\varepsilon = -\frac{1}{\varepsilon} \partial_3 \Phi_\varepsilon \\ \partial_t \theta^\varepsilon + u^\varepsilon \cdot \nabla \theta^\varepsilon - \nu'_h \Delta_h \theta^\varepsilon - \frac{1}{F \varepsilon} u^{3,\varepsilon} = 0 \\ \operatorname{div} u^\varepsilon = 0, \\ (u^\varepsilon, \theta^\varepsilon)|_{t=0} = (u_0, \theta_0) = V_0. \end{array} \right. \quad (\text{PEA}_\varepsilon)$$

These equations, as explained in detail in Section 2.3 and 2.4 describe the long-time behavior of an incompressible, stratified, rotating fluid. There is a simplification between system (PE_ε) and (PEA_ε) which was made implicitly: we assume that the viscosity $\nu = (\nu_h, 0)$, $\nu' = (\nu'_h, 0)$ has no vertical component. This choice is motivated by the fact that experimental verifications show that the Ekman number (2.3.23) is very small in geophysical fluids, hence it makes sense to consider the vertical diffusivity null.

We place ourselves in a spatially-periodic domain

$$\mathbb{T}^3 = \mathbb{R}^3 / \prod_{i=1}^3 a_i \mathbb{Z} = \prod_{i=1}^3 [0, 2\pi a_i].$$

The parameters $(a_1, a_2, a_3) \in \mathbb{R}^3$ shall not be chosen randomly but they have to satisfy the following criterion:

Definition 2.8.1. We say that a torus $\mathbb{T}^3 \subset \mathbb{R}^3$ satisfies the condition (\mathcal{P}) if either one or the other of the following situation is satisfied:

1. \mathbb{T}^3 is non-resonant.
2. If \mathbb{T}^3 is resonant, the Froude number F^2 is rational, and either
 - $a_3^2/a_1^2 \in \mathbb{Q}$ and a_3^2/a_2^2 is not algebraic of degree smaller or equal than four.
 - $a_3^2/a_2^2 \in \mathbb{Q}$ and a_3^2/a_1^2 is not algebraic of degree smaller or equal than four.

We refer to Definition 3.1.6 for a formal definition of resonant domain.

The first result we address to is whether there exists a local strong solution of (PEA_ε) for any initial data in a suitable space of low-regularity:

Theorem 2.8.2. *Let $s > 1/2$ and $V_0 \in H^{0,s}(\mathbb{T}^3)$ a divergence-free vector field. Then there exists a time $T > 0$ independent of ε and a unique solution V^ε for the system (PEA_ε) which belongs to the space*

$$V^\varepsilon \in \mathcal{C}([0, T]; H^{0,s}), \quad \nabla_h V^\varepsilon \in L^2([0, T]; H^{0,s}).$$

The existence part of Theorem 2.8.2 has been proved in [38], while the uniqueness (in the same energy space) has been proved in [89].

Again as for (PE_ε) we can re-write (PEA_ε) in a more compact form:

$$\begin{cases} \partial_t U^\varepsilon + u^\varepsilon \cdot \nabla U^\varepsilon - \mathbb{D}U^\varepsilon + \frac{1}{\varepsilon} \mathcal{A}U^\varepsilon = -\frac{1}{\varepsilon} \nabla \Phi_\varepsilon, \\ U^\varepsilon|_{t=0} = (u_0, \theta_0), \end{cases}$$

where

$$\mathbb{D} = \begin{pmatrix} \nu_h \Delta_h & 0 & 0 & 0 \\ 0 & \nu_h \Delta_h & 0 & 0 \\ 0 & 0 & \nu_h \Delta_h & 0 \\ 0 & 0 & 0 & \nu'_h \Delta_h \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{-1} \\ 0 & 0 & -F^{-1} & 0 \end{pmatrix}.$$

Let us now define the following operator

$$\mathbb{P} = \left(\begin{array}{c|c} 1 - \Delta^{-1} \nabla \text{div} & 0 \\ \hline 0 & 1 \end{array} \right), \quad (2.8.1)$$

the operator \mathbb{P} acts as the Leray projector onto divergence-free vector fields in the first three components as the identity on the fourth. Let us define the propagator

$$\mathcal{L}(\tau) = e^{-\tau \mathbb{P} \mathcal{A}},$$

defining the auxiliary unknown

$$V^\varepsilon = \mathcal{L}\left(\frac{t}{\varepsilon}\right) U^\varepsilon,$$

we can deduce (following S. Schochet [133], E. Grenier [79], I. Gallagher [72] and M. Paicu [123]) that the function V^ε satisfies the following equation

$$\begin{cases} \partial_t V^\varepsilon + \mathcal{Q}^\varepsilon(V^\varepsilon, V^\varepsilon) - \mathbb{D}^\varepsilon V^\varepsilon = 0, \\ V^\varepsilon|_{t=0} = (u_0, \theta_0), \end{cases} \quad (2.8.2)$$

where

$$\begin{aligned} \mathcal{Q}^\varepsilon(V^\varepsilon, V^\varepsilon) &= \mathcal{L}\left(-\frac{t}{\varepsilon}\right) \mathbb{P}\left[\mathcal{L}\left(\frac{t}{\varepsilon}\right) V^\varepsilon \cdot \nabla \mathcal{L}\left(\frac{t}{\varepsilon}\right) V^\varepsilon\right], \\ \mathbb{D}^\varepsilon V^\varepsilon &= \mathcal{L}\left(-\frac{t}{\varepsilon}\right) \mathbb{D} \mathcal{L}\left(\frac{t}{\varepsilon}\right) V^\varepsilon. \end{aligned}$$

It can be proved that the system (2.8.2) admits a distributional limit

$$\begin{cases} \partial_t V + \mathcal{Q}(V, V) - \bar{\mathbb{D}}V = 0, \\ V|_{t=0} = (u_0, \theta_0), \end{cases} \quad (2.8.3)$$

where the explicit expression of the operators $\mathcal{Q}, \bar{\mathbb{D}}$ is now omitted.

We can moreover prove that system (2.8.3) admits global solutions of weak-type, such result is not a standard derivation of solutions *à la Leray*. Indeed the operators \mathbb{D}^ε and $\bar{\mathbb{D}}$ above are differential operators in the horizontal directions x_h only. In particular is hence missing a diffusive vertical term of the form $\nu_v \partial_3^2$. This implies that standard Galerkin approximation scheme cannot be applied, being the anisotropic Sobolev space $H^{1,0}$ not compactly embedded in L^2 . Despite such problem we are able to overcome such difficulty by noticing that the limit form \mathcal{Q} acts in very well defined localized (in the Fourier space) way. In particular the nodes which provide a bilinear interaction are solutions of a polynomial equation of the form

$$P(X) = 0,$$

with $X \in \mathbb{Z}^9$. Defining $X = (X_1, X')$ and fixing $X' \in \mathbb{Z}^8$ we can hence deduce that the equation $P(X_1, X') = 0$ has become a one-dimensional equation, whose solutions $X_1(P, X')$ are finite due to the very well know fundamental theorem of algebra. Moreover applying the result [112], which uses tools of complex analysis, we can deduce a bound of the form

$$|X_1(P, X')| \leq K(P, X'),$$

on the roots of the equation $P(X_1, X') = 0$. This anisotropic localization of nodes allows us to transform a vertical derivative ∂_3 to a multi-index $(C_1 \partial_1^{N_1}, C_2 \partial_2^{N_2})$, where eventually C_1, C_2, N_1, N_2 are large. This is the key observation which allows us to prove the following result

Theorem 2.8.3. *Let \mathbb{T}^3 be a 3-dimensional torus, for each divergence-free vector field $U_0 \in L^2(\mathbb{T}^3)$ and $\Omega_0 = -\partial_2 u_0^1 + \partial_1 u_0^2 - F \partial_3 \theta_0 \in L^2(\mathbb{T}^3)$ there exists a distributional solution to the system (2.8.3) in the space $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{T}^3)$ which moreover belongs to the space*

$$V \in L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^3)) \quad \nabla_h V \in L^2(\mathbb{R}_+; L^2(\mathbb{T}^3)),$$

and satisfies the following energy estimate for each $t > 0$

$$\|V(t)\|_{L^2(\mathbb{T}^3)}^2 + c \int_0^t \|\nabla_h V(s)\|_{L^2(\mathbb{T}^3)}^2 ds \leq C \|U_0\|_{L^2(\mathbb{T}^3)}^2,$$

where the constant $c = \min \{\nu_h, \nu'_h\} > 0$.

Despite this result has a theoretical interest since, due to the special properties of the limit system (2.8.3), we have to prove a result that generally does not holds true for Navier-Stokes anisotropic equations, we are still interested to prove propagation of data regularity in *sub-critical topologies*.

Nonetheless it is possible to prove that the solution of (2.8.3) is globally well-posed in some anisotropic super-critical Sobolev space, i.e. we prove

Theorem 2.8.4. *Let \mathbb{T}^3 satisfy the condition (P) and consider a vector field $U_0 \in H^{0,s}$ with zero horizontal average and $\Omega_0 \in H^{0,s}$, for $s \geq 1$ such that $\operatorname{div} u_0 = 0$, $F \neq 1$ the limit system (2.8.3) admits a unique global solution*

$$V \in L^\infty(\mathbb{R}_+; H^{0,s}) \quad \nabla_h V \in L^2(\mathbb{R}_+; H^{0,s}),$$

that satisfies the following energy bound

$$\|V(t)\|_{H^{0,s}}^2 + c \int_0^t \|\nabla_h V(s)\|_{H^{0,s}}^2 ds \leq \mathcal{E}(\|U_0\|_{H^{0,s}}^2),$$

where \mathcal{E} is a suitable (bounded on compact sets) function. Moreover the solution U is unique in the space $L^\infty(\mathbb{R}_+; H^{0,\sigma}) \cap L^2(\mathbb{R}_+, H^{1,\sigma})$ for $\sigma \in [-1/2, s)$.

The limit system (2.8.3) admits hence unique global strong solution. The next step is to understand if such state is an attractor for the local solution of (PEA $_\varepsilon$) identified in Theorem 2.8.2 in the regime $\varepsilon \rightarrow 0$ of strong rotation and stratification, and, if that is the case, which is the appropriate topology in which such convergence takes place.

Theorem 2.8.5. *Let $\mathbb{T}^3 \subset \mathbb{R}^3$ satisfy the condition (P), $\Omega_0 = -\partial_2 u_0^1 + \partial_1 u_0^2 - F \partial_3 \theta_0 \in H^{0,s}$, $U_0 \in H^{0,s}$ with zero horizontal average, then the following convergences take place*

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left(U^\varepsilon - \mathcal{L} \left(\frac{t}{\varepsilon} \right) V \right) &= 0 && \text{in } \mathcal{C}(\mathbb{R}_+; H^{0,\sigma}) \\ \lim_{\varepsilon \rightarrow 0} \nabla_h \left(U^\varepsilon - \mathcal{L} \left(\frac{t}{\varepsilon} \right) V \right) &= 0 && \text{in } L^2(\mathbb{R}_+; H^{0,\sigma}) \end{aligned}$$

for $\sigma < s$ and V is the unique solution of the limit system (2.8.3).

It is hence interesting to remark that the above theorem implies the fact that the local solutions of (PEA $_\varepsilon$) are in fact globally well-defined, improving the lifespan of Theorem 2.8.2 which can be deduced thanks to energy estimates alone.

2.8.2 Stratified fluids in low Froude number regime.

The primitive equations (PE_ε) describe stratified fluids under the influence of the rotation of the Earth. We can indeed neglect the effects of the rotation and study the motion of a fluid in a long time-scale under the effects of the stratification buoyancy. The system was derived from physical considerations in Section 2.5 and reads as:

$$\begin{cases} \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \nu \Delta u^\varepsilon - \frac{1}{\varepsilon} \theta^\varepsilon \vec{e}_3 &= -\frac{1}{\varepsilon} \nabla \Phi^\varepsilon, \\ \partial_t \theta^\varepsilon + u^\varepsilon \cdot \nabla \theta^\varepsilon - \nu' \Delta \theta^\varepsilon + \frac{1}{\varepsilon} u^{3,\varepsilon} &= 0, \\ \operatorname{div} u^\varepsilon &= 0, \\ (u^\varepsilon, \theta^\varepsilon)|_{t=0} &= U_0 = (u_0, \theta_0). \end{cases} \quad (\text{PBS}_\varepsilon)$$

Again we can rewrite the system (PBS_ε) into the following more compact form

$$\begin{cases} \partial_t U^\varepsilon + U^\varepsilon \cdot \nabla U^\varepsilon - \mathbb{D}U^\varepsilon + \frac{1}{\varepsilon} \mathcal{A}U^\varepsilon &= -\frac{1}{\varepsilon} \begin{pmatrix} \nabla \Phi^\varepsilon \\ 0 \end{pmatrix}, \\ U^\varepsilon &= (u^\varepsilon, \theta^\varepsilon), \\ \operatorname{div} u^\varepsilon &= 0, \end{cases} \quad (\text{PBS}_\varepsilon)$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \mathbb{D} = \begin{pmatrix} \nu \Delta & 0 & 0 & 0 \\ 0 & \nu \Delta & 0 & 0 \\ 0 & 0 & \nu \Delta & 0 \\ 0 & 0 & 0 & \nu' \Delta \end{pmatrix}.$$

The spatially-periodic setting.

Again we can define the operator \mathbb{P} as in (2.8.1) and defining the auxiliary unknown

$$V^\varepsilon = \mathcal{L} \left(\frac{t}{\varepsilon} \right) U^\varepsilon,$$

we can deduce that the function V^ε satisfies the following equation

$$\begin{cases} \partial_t V^\varepsilon + \mathcal{Q}^\varepsilon(V^\varepsilon, V^\varepsilon) - \mathbb{D}^\varepsilon V^\varepsilon &= 0, \\ V^\varepsilon|_{t=0} &= (u_0, \theta_0), \end{cases} \quad (2.8.4)$$

where

$$\begin{aligned} \mathcal{Q}^\varepsilon(V^\varepsilon, V^\varepsilon) &= \mathcal{L} \left(-\frac{t}{\varepsilon} \right) \mathbb{P} \left[\mathcal{L} \left(\frac{t}{\varepsilon} \right) V^\varepsilon \cdot \nabla \mathcal{L} \left(\frac{t}{\varepsilon} \right) V^\varepsilon \right], \\ \mathbb{D}^\varepsilon V^\varepsilon &= \mathcal{L} \left(-\frac{t}{\varepsilon} \right) \mathbb{D} \mathcal{L} \left(\frac{t}{\varepsilon} \right) V^\varepsilon. \end{aligned}$$

It can be proved (details omitted here) that the system (2.8.2) admits a distributional limit

$$\begin{cases} \partial_t V + \mathcal{Q}(V, V) - \mathbb{D}V = 0, \\ V|_{t=0} = (u_0, \theta_0), \end{cases} \quad (2.8.5)$$

where the explicit expression of the operators \mathcal{Q} , \mathbb{D} is now omitted.

Naturally one may wonder if, and in which sense (2.8.4) converges to (2.8.5). The filtering operator \mathcal{L} transforms the system (PBS $_\varepsilon$) in (2.8.4): the latter does not presents any more the singular perturbation, being this the case we can provide uniform bounds for the sequence $(\partial_t V^\varepsilon)_{\varepsilon>0}$ in the space $L^p_{\text{loc}}(\mathbb{R}_+; H^{-N})$ where N is sufficiently large and $p \in [2, \infty]$. Hence standard compactness arguments in functional spaces can be applied; applying Aubin-Lions lemma [4] we prove in fact that, up to subsequences, not relabeled

$$V^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} V \quad \text{in } L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3)).$$

The above argument does not explain how the bilinear interactions $\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon)$ converge to a limit bilinear interaction $\mathcal{Q}(U, U)$: this can be proved applying the non-stationary phase theorem. We omit to introduce the details of such result of convergence and we refer to the Chapter 4.

We consider $U_0 = (u_0, \theta_0)$ to be of *zero horizontal average*, i.e.

$$\frac{1}{|\mathbb{T}_h^2|} \int_{\mathbb{T}_h^2} U_0(y_h, x_3) dy_h = 0.$$

The global average of the vector fields instead will be always considered to be zero.

Let us define

$$\bar{U}_0 = \Delta_h^{-1} \begin{pmatrix} -\partial_2 \\ \partial_1 \\ 0 \\ 0 \end{pmatrix} (-\partial_2 u_0^1 + \partial_1 u_0^2), \quad U_{\text{osc},0} = U_0 - \bar{U}_0,$$

the first result proved is the following one:

Theorem 2.8.6. *Let V be the distributional limit solution of (2.8.5), then V can be written as*

$$V(x) = \bar{V}(x_h, x_3) + V_{\text{osc}}(x_h, x_3),$$

where \bar{V} , V_{osc} are respectively distributional solutions of the systems

$$\begin{cases} \partial_t \bar{v}^h + \bar{v}^h \cdot \nabla_h \bar{v}^h - \nu \Delta \bar{v}^h = -\nabla_h \bar{p}, \\ \bar{v}^h|_{t=0} = \bar{u}_0^h, \end{cases} \quad (2.8.6)$$

$$\begin{cases} \partial_t V_{\text{osc}} + 2\mathcal{Q}(\bar{V}, V_{\text{osc}}) - (\nu + \nu') \Delta V_{\text{osc}} = 0, \\ V_{\text{osc}}|_{t=0} = U_{\text{osc},0}, \end{cases} \quad (2.8.7)$$

where \mathcal{Q} , is the bilinear form defined in (4.3.2) for almost all $(a_1, a_2, a_3) \in \mathbb{R}^3$ parameters of the three-dimensional torus $\mathbb{T}^3 = \prod_i [0, 2\pi a_i]$.

Let us point out some technical difficulty characteristic of the system that we study in this part of the thesis. The limit bilinear form \mathcal{Q} in (2.8.5) is well defined only for bilinear interactions whose first two components have zero horizontal average. Whence it is important to prove that in the limit $\varepsilon \rightarrow 0$ the horizontal average of the horizontal components of the bilinear interaction $\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon)$ are zero, at least in a distributional sense. Whence we require to prove the following result:

Lemma 2.8.7. *The limit*

$$\lim_{\varepsilon \rightarrow 0} \left(\int_{\mathbb{T}_h^2} (\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon))^h dx_h, 0, 0 \right) = 0,$$

holds in a distributional sense.

The proof of the above lemma relies on a careful analysis on the horizontal average of the vertical oscillations: we exploit symmetric properties of the localization of the sum in order to deduce that suitable cancellation properties can be applied.

The main advantage in studying the limit system (2.8.5) as the superposition of the evolutions of the two systems (2.8.6)–(2.8.7) is that system (2.8.5) presents the same difficulties as the classic three-dimensional Navier-Stokes equation as far as concerns the propagation of Sobolev regularity due to the bilinear interaction \mathcal{Q} , while on the other hand (2.8.6)–(2.8.7) are respective a 2D-stratified Navier-Stokes system and a linear system. If we consider separately such equations we can prove that they are globally well posed as it is formalized in the following theorem

Theorem 2.8.8. *Let us assume $\bar{u}_0^h \in L_v^\infty(H_h^\sigma)$ and $\nabla_h \bar{u}_0^h \in L_v^\infty(H_h^\sigma)$ with $\sigma > 0$, then \bar{V} , distributional solution of (2.8.6) is globally well posed in \mathbb{R}_+ , and belongs to the space*

$$\bar{v}^h \in \mathcal{C}(\mathbb{R}_+; H^s(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; H^{s+1}(\mathbb{T}^3)), \quad s > 1/2,$$

and for each $t > 0$ the following estimate holds true

$$\|\bar{v}^h(t)\|_{H^s(\mathbb{T}^3)}^2 + \nu \int_0^t \|\nabla \bar{v}^h(\tau)\|_{H^s(\mathbb{T}^3)}^2 d\tau \leq \mathcal{E}_1(U_0).$$

Where the function \mathcal{E}_1 is defined as the right-hand-side of equation (4.5.2).

On the other hand, if we denote as V_{osc} as the distributional solution of (2.8.7), then, V_{osc} is globally defined and it belongs to the space

$$V_{\text{osc}} \in \mathcal{C}(\mathbb{R}_+; H^s(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; H^{s+1}(\mathbb{T}^3)),$$

for $s > 1/2$. For each $t > 0$ the following bound holds true

$$\|V_{\text{osc}}(t)\|_{H^s(\mathbb{T}^3)}^2 + \frac{\nu + \nu'}{2} \int_0^t \|\nabla V_{\text{osc}}(\tau)\|_{H^s(\mathbb{T}^3)}^2 d\tau \leq \mathcal{E}_2(U_0),$$

and the function \mathcal{E}_2 is defined as the right-hand-side of equation (4.5.19).

The last step is obviously to prove that solutions of (PBS_ε) converge as $\varepsilon \rightarrow 0$ in some appropriate topology to the solutions of (2.8.5):

Theorem 2.8.9. *Let U_0 in $H^s(\mathbb{T}^3)$ for $s > 1/2$ as above, for $\varepsilon > 0$ small enough (PBS_ε) is globally well posed in $\mathcal{C}(\mathbb{R}_+; H^s(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; H^{s+1}(\mathbb{T}^3))$, and, if V is the global strong solution of (2.8.5), then*

$$U^\varepsilon - \mathcal{L}\left(-\frac{t}{\varepsilon}\right)V = o(1),$$

in $\mathcal{C}(\mathbb{R}_+; H^s(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; H^{s+1}(\mathbb{T}^3))$.

The above theorem is proved applying a methodology known as *Schochet method*, which was formalized by I. Gallagher in [72] for parabolic nonlinear systems. Nonetheless this technique has to be adapted in order to control the decay of the bilinear vertical perturbations mentioned above.

The whole-space setting.

Let us now consider the system (PBS_ε) in the whole space \mathbb{R}^3 . The projection of (PBS_ε) onto $\ker \mathbb{P}\mathcal{A}$ is a two-dimensional, stratified, Navier-Stokes equation with full diffusion:

$$\begin{cases} \partial_t \bar{u}^h + \bar{u}^h \cdot \nabla_h \bar{u}^h - \nu \Delta \bar{u}^h = -\nabla_h \bar{p} \\ \operatorname{div}_h \bar{u}^h = 0, \\ \bar{u}^h|_{t=0} = \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix} \Delta_h^{-1} (-\partial_2 u_0^1 + \partial_1 u_0^2) = \bar{u}_0^h. \end{cases} \quad (2.8.8)$$

Such system shares many interesting features with the classical two-dimensional Navier-Stokes equations, in particular the "horizontal vorticity"

$$\omega^h = -\partial_2 \bar{u}^1 + \partial_1 \bar{u}^2,$$

can describe the velocity flow via a two-dimensional Biot-Savart law,

$$\bar{u}^h = \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix} \Delta_h^{-1} \omega^h,$$

and satisfies the following transport-diffusion equation

$$\begin{cases} \partial_t \omega^h + \bar{u}^h \cdot \nabla_h \omega^h - \nu \Delta \omega^h = 0, \\ \omega^h|_{t=0} = -\partial_2 u_0^1 + \partial_1 u_0^2 = \omega_0^h. \end{cases} \quad (2.8.9)$$

We are able to prove that, under suitable hypothesis, the system (2.8.8) is globally well posed in suitable homogeneous Sobolev space, in detail

Theorem 2.8.10. *Let $\bar{u}_0^h \in L^2(\mathbb{R}^3) \cap \dot{H}^s(\mathbb{R}^3)$, $s > 0$ and $\omega_0^h \in L^2(\mathbb{R}^3)$, then there exists a global strong solution of the system (2.8.8) which belongs to the space*

$$\bar{u}^h \in L^\infty\left(\mathbb{R}_+; L^2(\mathbb{R}^3) \cap \dot{H}^s(\mathbb{R}^3)\right) \cap L^2\left(\mathbb{R}_+; \dot{H}^1(\mathbb{R}^3) \cap \dot{H}^{s+1}(\mathbb{R}^3)\right),$$

and for each $t > 0$ the following bound holds

$$\begin{aligned} \|\bar{u}^h(t)\|_{\dot{H}^s(\mathbb{R}^3)}^2 + \nu \int_0^t \|\nabla_h \bar{u}^h(\tau)\|_{\dot{H}^s(\mathbb{R}^3)}^2 d\tau \\ \leq C \|\bar{u}_0^h\|_{\dot{H}^s(\mathbb{R}^3)}^2 \exp \left\{ \frac{CK^2}{\nu} \left(\|\bar{u}_0^h\|_{L^2(\mathbb{R}^3)}^4 + \|\omega_0^h\|_{L^2(\mathbb{R}^3)}^4 \right) \right\}. \end{aligned} \quad (2.8.10)$$

The question of convergence in low Froude number regime arise hence naturally, following the works [42], [38], [28] we expect that

$$U^\varepsilon + \text{perturbative term in } \varepsilon \xrightarrow{\varepsilon \rightarrow 0} \bar{U} = (\bar{u}^h, 0, 0).$$

The detailed statement of the result we prove is the following one:

Theorem 2.8.11. *Let $U_0 \in L^2(\mathbb{R}^3) \cap \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, $\omega_0^h \in L^2(\mathbb{R}^3)$, then*

$$\begin{aligned} U^\varepsilon - W^\varepsilon - (\bar{u}^h, 0, 0)^\top &\xrightarrow{\varepsilon \rightarrow 0} 0, & \text{in the space } L^\infty\left(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\right), \\ \nabla(U^\varepsilon - W^\varepsilon - (\bar{u}^h, 0, 0)^\top) &\xrightarrow{\varepsilon \rightarrow 0} 0, & \text{in the space } L^2\left(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\right), \end{aligned}$$

where U^ε is the local strong solution of (PBS $_\varepsilon$) and W^ε, \bar{u}^h are respectively the global solutions of

$$\begin{cases} \partial_t W^\varepsilon - \mathbb{D}W^\varepsilon + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} W^\varepsilon = \begin{pmatrix} 0 \\ 0 \\ -\partial_3 (-\Delta_h)^{-1} \operatorname{div}_h (\bar{u}^h \cdot \nabla_h \bar{u}^h) \\ 0 \end{pmatrix}, \\ W^\varepsilon|_{t=0} = (\mathbb{P}_{-, \varepsilon} + \mathbb{P}_{+, \varepsilon}) U_0, \end{cases}$$

and (2.8.8).

The operators $\mathbb{P}_0, \mathbb{P}_{\pm, \varepsilon}$ are the projections onto $\mathbb{C}E_0, \mathbb{C}E_\pm$, where E_0, E_\pm are the divergence-free eigenvectors of the operator $L_\varepsilon = \mathbb{P} \mathcal{A} - \varepsilon \mathbb{D}$. In particular hence the strong solution U^ε of (PBS $_\varepsilon$) is in fact global and belongs to the space

$$L^\infty\left(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\right) \cap L^2\left(\mathbb{R}_+; \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)\right)$$

The methodology adopted in order to prove such result is very similar to the one adopted in [42], [28] and [34]. We use the following dispersive result

Proposition 2.8.12. *Let us consider the linear system*

$$\begin{cases} \partial_t W_{r,R}^\varepsilon - \mathbb{D}W_{r,R}^\varepsilon + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} W_{r,R}^\varepsilon \\ = -\mathcal{P}_{r,R} (\mathbb{P}_{+, \varepsilon} + \mathbb{P}_{-, \varepsilon}) \begin{pmatrix} 0 \\ 0 \\ \partial_3 (-\Delta_h)^{-1} \operatorname{div}_h \operatorname{div}_h (\bar{u}^h \otimes \bar{u}^h) \\ 0 \end{pmatrix}, \\ \operatorname{div} w_{r,R}^\varepsilon = 0, \\ W_{r,R}^\varepsilon|_{t=0} = \mathcal{P}_{r,R} (\mathbb{P}_{+, \varepsilon} + \mathbb{P}_{-, \varepsilon}) U_0. \end{cases} \quad (2.8.11)$$

The operator $\mathcal{P}_{r,R}$ localizes tempered distribution onto the set

$$\mathcal{C}_{r,R} = \{\xi \in \mathbb{R}_\xi^3 : |\xi_h| > r, |\xi| < R\}.$$

The unique global solution of (2.8.11) is such that

$$\|W_{r,R}^\varepsilon\|_{L^p(\mathbb{R}_+; L^\infty(\mathbb{R}^3))} \leq C_{r,R} \left(1 + \frac{1}{\nu}\right) \varepsilon^{\frac{1}{4p}} \max \left\{ \|U_0\|_{L^2(\mathbb{R}^3)}, \|U_0\|_{L^2(\mathbb{R}^3)}^2 \right\},$$

for $p \in [1, \infty)$.

We want to point out that, differently as in [42], the dispersive result is proved on a linear non-homogeneous system associated to (PBS_ε) in the same fashion as in [56], [28] and [34]. The bulk force on the right hand side of (2.8.11) is introduced for technical reasons which we do not explain in detail at the moment.

2.8.3 Isentropic rotating fluids.

The derivation of a model describing the motion of a fluid which is compressible and rotating was explained in detail in Section 2.6. The equation derived is the following one

$$\begin{cases} \partial_t (\rho^{\varepsilon,\theta} u^{\varepsilon,\theta}) + \operatorname{div} (\rho^{\varepsilon,\theta} u^{\varepsilon,\theta} \otimes u^{\varepsilon,\theta}) + \frac{1}{\theta^2} \nabla P (\rho^{\varepsilon,\theta}) + \frac{1}{\varepsilon} e^3 \wedge (\rho^{\varepsilon,\theta} u^{\varepsilon,\theta}) = 0 \\ \partial_t \rho^{\varepsilon,\theta} + \operatorname{div} (\rho^{\varepsilon,\theta} u^{\varepsilon,\theta}) = 0 \\ (\rho^{\varepsilon,\theta}, u^{\varepsilon,\theta})|_{t=0} = (\rho_0^{\varepsilon,\theta}, u_0^{\varepsilon,\theta}), \end{cases}$$

where ε is the Rossby number and θ is the Mach number. Let us set $\varepsilon = \theta$, and let us consider

$$\rho = 1 + \varepsilon b^\varepsilon,$$

i.e. the density ρ is a small variation around a steady state which we normalized to one.

Let us consider the barotropic pressure to assume the form

$$P(\rho^\varepsilon) = A \rho^\gamma, \quad A > 0, \gamma > 1,$$

and let us define $\bar{\gamma} = (\gamma - 1)/2$. We consider the substitution

$$1 + \varepsilon b^\varepsilon = \frac{(4\gamma A)^{1/2}}{\gamma - 1} (\rho^\varepsilon)^{\bar{\gamma}}$$

hence the system becomes (after a few algebraic calculations)

$$\begin{cases} \partial_t u^\varepsilon + \frac{1}{\varepsilon} (\bar{\gamma} \nabla b^\varepsilon + e^3 \wedge u^\varepsilon) + u^\varepsilon \cdot \nabla u^\varepsilon + \bar{\gamma} b^\varepsilon \nabla b^\varepsilon = 0 \\ \partial_t b^\varepsilon + \frac{\bar{\gamma}}{\varepsilon} \operatorname{div} u^\varepsilon + u^\varepsilon \nabla b^\varepsilon + \bar{\gamma} b^\varepsilon \operatorname{div} u^\varepsilon = 0 \\ (b^\varepsilon, u^\varepsilon)|_{t=0} = (b_0, u_0), \end{cases} \quad (2.8.12)$$

or in a more compact form

$$\begin{cases} \partial_t \begin{pmatrix} u^\varepsilon \\ b^\varepsilon \end{pmatrix} - \frac{1}{\varepsilon} \mathcal{B} \begin{pmatrix} u^\varepsilon \\ b^\varepsilon \end{pmatrix} + \begin{pmatrix} u^\varepsilon \cdot \nabla u^\varepsilon + \bar{\gamma} b^\varepsilon \nabla b^\varepsilon \\ u^\varepsilon \cdot \nabla b^\varepsilon + \bar{\gamma} b^\varepsilon \operatorname{div} u^\varepsilon \end{pmatrix} = 0, \\ (u^\varepsilon, b^\varepsilon)|_{t=0} = (u_0, b_0), \end{cases} \quad (2.8.13)$$

where \mathcal{B} is the following operator

$$\mathcal{B} = \begin{pmatrix} 0 & 1 & 0 & -\bar{\gamma} \partial_1 \\ -1 & 0 & 0 & -\bar{\gamma} \partial_2 \\ 0 & 0 & 0 & -\bar{\gamma} \partial_3 \\ -\bar{\gamma} \partial_1 & -\bar{\gamma} \partial_2 & -\bar{\gamma} \partial_3 & 0 \end{pmatrix}, \quad (2.8.14)$$

and where ∂_i , for any $i \in \{1, 2, 3\}$ stands for the derivative with respect to the x_i variable. Moreover we can write the nonlinearity as follows

$$\begin{pmatrix} u^\varepsilon \cdot \nabla u^\varepsilon + \bar{\gamma} b^\varepsilon \nabla b^\varepsilon \\ u^\varepsilon \cdot \nabla b^\varepsilon + \bar{\gamma} b^\varepsilon \operatorname{div} u^\varepsilon \end{pmatrix} = \mathcal{A}(U, D) U = \begin{pmatrix} u^\varepsilon \cdot \nabla & 0 & 0 & \bar{\gamma} b^\varepsilon \partial_1 \\ 0 & u^\varepsilon \cdot \nabla & 0 & \bar{\gamma} b^\varepsilon \partial_2 \\ 0 & 0 & u^\varepsilon \cdot \nabla & \bar{\gamma} b^\varepsilon \partial_3 \\ \bar{\gamma} b^\varepsilon \partial_1 & \bar{\gamma} b^\varepsilon \partial_2 & \bar{\gamma} b^\varepsilon \partial_3 & u^\varepsilon \cdot \nabla \end{pmatrix} \begin{pmatrix} u^\varepsilon \\ b^\varepsilon \end{pmatrix}, \quad (2.8.15)$$

where U stays for $\begin{pmatrix} u^\varepsilon \\ b^\varepsilon \end{pmatrix}$. With all the above considerations, the system (2.8.13) can be rewritten as

$$\begin{cases} \partial_t U - \frac{1}{\varepsilon} \mathcal{B} U + \mathcal{A}(U, D) U = 0, \\ U|_{t=0} = U_0 = (u_0, b_0). \end{cases} \quad (2.8.16)$$

The advantage in in the above formulation is that we expressed the system as an hyperbolic symmetric system with singular perturbation.

The initial data considered $(u_0, b_0) \in H^s(\mathbb{R}^3)$, $s > 5/2$, whence it is fully three dimensional. The analysis performed in [68] showed that the elements belonging to $\ker \mathcal{B}$ are bi-dimensional, whence in this particular work we do not consider data which belong to the kernel of the penalized operator. As explained in Section 2.7.4 we do not expect hence a convergence to stationary flows, but rather a full dispersive result.

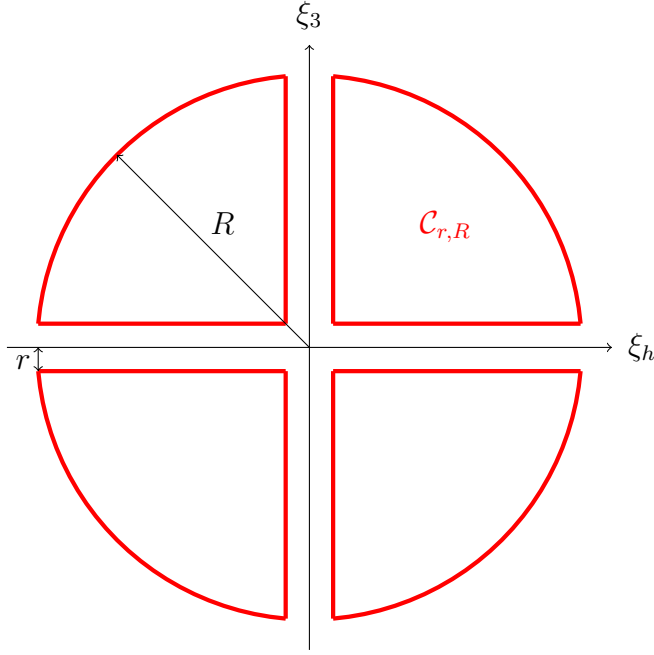
To achieve such result we proceed as follows: we set

$$\mathcal{C}_{r,R} = \{ \xi \in \mathbb{R}_\xi^3 \mid |\xi| \leq R, |\xi_h| \geq r, |\xi_3| \geq r \},$$

for a visual definition of $\mathcal{C}_{r,R}$ we refer to Figure 2.3.

Our strategy to study the system (2.8.16) consists in finding a solution of the form

$$U^\varepsilon = (u^\varepsilon, b^\varepsilon) = \bar{U}^\varepsilon + \tilde{U}^\varepsilon$$


 Figure 2.3 – The set $C_{r,R}$ in dimension two.

where $\bar{U}^\varepsilon = (\bar{u}^\varepsilon, \bar{b}^\varepsilon)$ and $\tilde{U}^\varepsilon = (\tilde{u}^\varepsilon, \tilde{b}^\varepsilon)$ are respectively solutions to the following systems

$$\begin{cases} \partial_t \bar{U}^\varepsilon - \frac{1}{\varepsilon} \mathcal{B} \bar{U}^\varepsilon = 0 \\ \bar{U}^\varepsilon|_{t=0} = \Psi_{r,R}(D)(u_0, b_0) \end{cases}, \quad (2.8.17)$$

$$\begin{cases} \partial_t \tilde{U}^\varepsilon - \frac{1}{\varepsilon} \mathcal{B} \tilde{U}^\varepsilon + \mathcal{A}(U, D)U = 0 \\ \tilde{U}^\varepsilon|_{t=0} = (1 - \Psi_{r,R}(D))(u_0, b_0) \end{cases}. \quad (2.8.18)$$

Here, the frequency cut-off radii $0 < r < R$ will be precisely chosen, depending on ε and $\Psi_{r,R}$ is a radial function supported in $C_{\frac{r}{2}, 2R}$ and is identically equal to 1 in $C_{r,R}$. The dependence on ε of the cut-off radii is motivated by the fact that the system (2.8.16) is of hyperbolic type, hence we cannot absorb many terms as we could usually do for parabolic systems, this implicated that we shall have to require a more delicate control of the dispersion in terms of powers of ε . The system (2.8.17) is indeed linear and localized, hence we can prove the following dispersive result:

Theorem 2.8.13. *Let $q \in [2, +\infty]$ and $p \geq \frac{4q}{q-2}$. For any $U_0 \in L^2(\mathbb{R}^3)$, the system (2.8.17) has a global solution \bar{U}^ε such that,*

$$\|\bar{U}^\varepsilon\|_{L^p(\mathbb{R}_+, L^q(\mathbb{R}^3))} \leq C R^{\frac{3}{2} - \frac{3}{q} + \frac{4}{p}} r^{-\frac{2}{p}} \varepsilon^{\frac{1}{p}} \|U_0\|_{L^2(\mathbb{R}^3)}.$$

The system (2.8.18) on the other hand is highly nonlinear. We can use the symmetry of the operator \mathcal{A} and the dispersive properties of \bar{U}^ε stated above to deduce a local-existence result in the space $L^\infty([0, T], H^s)$ where T is independent of ε . Once such result is established we can control the optimal lifespan of $\tilde{U}^\varepsilon = U^\varepsilon - \bar{U}^\varepsilon$, this is done in the following:

Theorem 2.8.14. *Let $s > 5/2$, $s_0 > 0$ be fixed constants, $1 < p < 2$ and the initial data*

$$U_0 \in Y_{s,s_0,p} = H^{s+s_0}(\mathbb{R}^3) \cap L^2(\mathbb{R}_h^2; L^p(\mathbb{R}_v^1)) \cap L^p(\mathbb{R}_h^2; L^2(\mathbb{R}_v^1)),$$

There exists a time $T_\varepsilon^ > 0$ and a unique solution $U^\varepsilon = (u^\varepsilon, b^\varepsilon)$ of system (2.8.16) satisfying*

$$U^\varepsilon \in L^\infty([0, T_\varepsilon^*]; H^s(\mathbb{R}^3)) \cap C([0, T_\varepsilon^*]; H^s(\mathbb{R}^3)),$$

where the maximal time T_ε^ tends to infinity as ε tends to zero, more precisely, there exist positive constants $\bar{C} > 0$ and $\alpha > 0$ such that*

$$T_\varepsilon^* \geq \frac{\bar{C}}{\mathcal{C}(U_0) \varepsilon^\alpha},$$

where $\mathcal{C}(U_0)$ is a constant depending on the initial data only.

The space $Y_{s,s_0,p}$ is a Banach space once it is endowed with the norm

$$\|u\|_{s,s_0,p} = \max \left\{ \|u\|_{H^{s+s_0}}, \|u\|_{L_h^2 L_v^p}, \|u\|_{L_h^p L_v^2} \right\}.$$

We can hence at this point to make sense of the constant $\mathcal{C}(U_0)$, given in fact a $U_0 \in Y_{s,s_0,p}$, we set

$$\mathcal{C}(U_0) = \max \left\{ \|U_0\|_{s,s_0,p}, \|U_0\|_{s,s_0,p}^2 \right\}.$$

Chapter 3

Primitive equations with null vertical diffusivity.

The essence of mathematics lies in its freedom.

Georg Cantor

3.1 Introduction

The primitive equations describe the hydro-dynamical flow in a large scale (of order of hundreds or thousands of kilometers) on the Earth, typically the ocean or the atmosphere, under the assumption that the vertical motion is much smaller than the horizontal one and that the fluid layer depth is small compared to the radius of the Earth. Concerning the difference between horizontal and vertical scale, it is observed that for geophysical fluids the vertical component of the diffusion term (viscosity or thermal diffusivity in the case of primitive equations) is much smaller than the horizontal components. In the case of rotating fluids between two planes (see [80] for the first work in which the initial data is well prepared, in the sense that it is a two-dimensional vector field and [114] and [41] for the generic case) the viscosity assumes the form $(-\nu_h \Delta_h - \varepsilon \beta \partial_3^2)$, with $\Delta_h = \partial_1^2 + \partial_2^2$, whence such geophysical motivation justifies the study of anisotropic (i.e. non-spherically symmetric) viscosities.

The present chapter was submitted for a peer-review under the name *Highly rotating fluids with vertical stratification for periodic data and vanishing vertical viscosity*, see [137].

The primitive system consists in the following equations

$$\left\{ \begin{array}{l} \partial_t v^{1,\varepsilon} + v^\varepsilon \cdot \nabla v^{1,\varepsilon} - \nu_h \Delta_h v^{1,\varepsilon} - \nu_v \partial_3^2 v^{1,\varepsilon} - \frac{1}{\varepsilon} v^{2,\varepsilon} = -\frac{1}{\varepsilon} \partial_1 \Phi_\varepsilon + f_1 \\ \partial_t v^{2,\varepsilon} + v^\varepsilon \cdot \nabla v^{2,\varepsilon} - \nu_h \Delta_h v^{2,\varepsilon} - \nu_v \partial_3^2 v^{2,\varepsilon} + \frac{1}{\varepsilon} v^{1,\varepsilon} = -\frac{1}{\varepsilon} \partial_2 \Phi_\varepsilon + f_2 \\ \partial_t v^{3,\varepsilon} + v^\varepsilon \cdot \nabla v^{3,\varepsilon} - \nu_h \Delta_h v^{3,\varepsilon} - \nu_v \partial_3^2 v^{3,\varepsilon} + \frac{1}{F\varepsilon} T^\varepsilon = -\frac{1}{\varepsilon} \partial_3 \Phi_\varepsilon + f_3 \\ \partial_t T^\varepsilon + v^\varepsilon \cdot \nabla T^\varepsilon - \nu'_h \Delta_h T^\varepsilon - \nu'_v \partial_3^2 T^\varepsilon - \frac{1}{F\varepsilon} v^{3,\varepsilon} = f_4 \\ \operatorname{div} v^\varepsilon = 0 \\ (v^\varepsilon, T^\varepsilon) \Big|_{t=0} = (v_0, T_0) = V_0, \end{array} \right. \quad (\text{PE}_\varepsilon)$$

in the unknown $v^\varepsilon = (v^{1,\varepsilon}, v^{2,\varepsilon}, v^{3,\varepsilon}), T^\varepsilon, \Phi_\varepsilon$. In what follows we write $V^\varepsilon = (v^\varepsilon, T^\varepsilon) = (V^{1,\varepsilon}, V^{2,\varepsilon}, V^{3,\varepsilon}, V^{4,\varepsilon})$. All the functions described depend on a couple $(x, t) \in \mathbb{T}^3 \times \mathbb{R}_+$ where \mathbb{T}^3 represents the torus

$$\mathbb{T}^3 = \mathbb{R}^3 / \prod_{i=1}^3 a_i \mathbb{Z} = \prod_{i=1}^3 [0, 2\pi a_i).$$

The only assumption which is made on the vertical viscosity is $\nu_v, \nu'_v \geq 0$, while the horizontal viscosities ν_h, ν'_h are strictly positive constants. The results obtained will be uniform with respect to the vertical viscosities (ν_v, ν'_v) and hence from now on we can suppose them zero without loss of generality. We refer to [124] for a result of well-posedness of the Navier-Stokes equation in critical spaces in the whole space with anisotropic viscosity and to [125] for the periodic case.

Under the assumption $\nu_v = \nu'_v = 0$ we can rewrite system (PE_ε) in the more compact form

$$\left\{ \begin{array}{l} \partial_t V^\varepsilon + v^\varepsilon \cdot \nabla V^\varepsilon - \mathbf{D} V^\varepsilon + \frac{1}{\varepsilon} \mathcal{A} V^\varepsilon = \frac{1}{\varepsilon} (-\nabla \Phi_\varepsilon, 0) + f \\ \operatorname{div} v^\varepsilon = 0 \\ V^\varepsilon \Big|_{t=0} = V_0 \end{array} \right. \quad (\text{PE}_\varepsilon)$$

where

$$\mathbf{D} = \begin{pmatrix} \nu_h \Delta_h & 0 & 0 & 0 \\ 0 & \nu_h \Delta_h & 0 & 0 \\ 0 & 0 & \nu_h \Delta_h & 0 \\ 0 & 0 & 0 & \nu'_h \Delta_h \end{pmatrix} \quad \mathcal{A} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{-1} \\ 0 & 0 & -F^{-1} & 0 \end{pmatrix} \quad (3.1.1)$$

with $\nu_h, \nu'_h > 0$ and $V^\varepsilon = (v^\varepsilon, T^\varepsilon)$.

This system is obtained by combining the effects of the Coriolis force and the vertical stratification induced by the Boussinesq approximation. We refer to [42], [126] or [50] for a discussion on the model and its derivations.

In the study of hydrodynamical flows on this scale two important phenomena have to be taken in consideration: the Earth rotation and the vertical stratification induced by the gravity. The Coriolis force induces a vertical rigidity on the fluid. Namely, in the asymptotic regime, the high rotation tends to stabilize the motion, which becomes constant in the direction parallel to the rotation axis: the fluid moves along vertical columns (the so called Taylor-Proudman columns), and the flow is purely horizontal.

Gravity forces the fluid masses to have a vertical structure: heavier layers lay under lighter ones. Internal movements of the fluid tend to destroy this structure and gravity tries to restore it, which gives a horizontal rigidity (to be opposed to the vertical rigidity induced by the rotation). In order to formally estimate the importance of this rigidity we also compare the typical time scale of the system with the Brunt-Väisälä frequency and introduce the Froude number εF . We shall not give more details in here, we refer to [126], [42], [50].

The primitive equations are obtained with moment, energy and mass conservation (see [62]). The coefficient $\varepsilon > 0$ denotes the Rossby number, which is defined as

$$\varepsilon = \frac{\text{displacement due to inertial forces}}{\text{displacement due to Coriolis force}}.$$

As the characteristic displacement of a particle in the ocean within a day is very small compared to the displacement caused by the rotation of the Earth (generally ε is of order 10^{-3} outside persistent currents such as the gulf stream), the Rossby number is supposed to be very small hence it is reasonable to study the behavior of the solutions to (PE_ε) in the limit regime as $\varepsilon \rightarrow 0$.

We denote the Froude number as εF . Assuming that the Brunt-Väisälä frequency is constant, in the whole space \mathbb{R}^3 , when $\varepsilon \rightarrow 0$, the formal limit of the system (PE_ε) , when the viscosity is isotropic, is the quasi-geostrophic system

$$\begin{cases} \partial_t V_{\text{QG}} + \Gamma(D) V_{\text{QG}} = - \begin{pmatrix} \nabla_h^\perp \\ 0 \\ -F\partial_3 \end{pmatrix} \Delta_F^{-1} (v_{\text{QG}}^h \cdot \nabla_h \Omega), \\ \operatorname{div} v_{\text{QG}} = 0 \\ V_{\text{QG}}|_{t=0} = V_{\text{QG},0}, \end{cases} \quad (\text{QG})$$

and $\Gamma(D)$ is the pseudo-differential operator given by the formula

$$\Gamma(D) u = \mathcal{F}^{-1} \left(\frac{|\xi|^2 (\nu |\xi_h|^2 + \nu' F^2 \xi_3^2)}{|\xi_h|^2 + F^2 \xi_3^2} \hat{u}(\xi) \right).$$

The differential operator Δ_F is defined as $\Delta_F = \partial_1^2 + \partial_2^2 + F^2 \partial_3^2$, while its inverse Δ_F^{-1} in L^2 is the Fourier multiplier

$$-\Delta_F^{-1} f = \mathcal{F}^{-1} \left(\frac{1}{\xi_1^2 + \xi_2^2 + F^2 \xi_3^2} \hat{f} \right).$$

The quantities V_{QG} and Ω are respectively called the *quasi-geostrophic flow* and the *potential vorticity*. We focus on the latter first, the potential vorticity is defined as

$$\Omega = -\partial_2 V_{\text{QG}}^1 + \partial_1 V_{\text{QG}}^2 - F \partial_3 V_{\text{QG}}^4,$$

and it is related to the quasi-geostrophic flow via the 2D-like Biot -Savart law

$$V_{\text{QG}} = \begin{pmatrix} -\partial_2 \\ \partial_1 \\ 0 \\ -F\partial_3 \end{pmatrix} \Delta_F^{-1} \Omega.$$

The vectors v_{QG}^h and v_{QG} represent respectively the first two and three components of the vector field V_{QG} . In the present setting, i.e. with periodic data, the limit system is more involved than the one mentioned above. In this case, as well as in many problems with singular perturbation, the idea is to decompose the unknown (in the case of the system (PE_ε) is V^ε) into two parts $V^\varepsilon = V_{\text{ker}}^\varepsilon + V_{\text{osc}}^\varepsilon$, where $V_{\text{ker}}^\varepsilon$ belongs to the kernel of the perturbation $\mathbb{P}\mathcal{A}$, where \mathbb{P} is the Leray projector in the first three components which leaves untouched the fourth one, and $V_{\text{osc}}^\varepsilon$ to its orthogonal complement. In the whole space it can be proved that the oscillating part, $V_{\text{osc}}^\varepsilon$, tends to zero *strongly* as $\varepsilon \rightarrow 0$. In the case of periodic data instead these perturbations interact constructively, as in [5], [8], [72] and [123], whence the limit system is different from the quasi-geostrophic system mentioned above (see (S)). We aim to study the behavior of strong solutions of (PE_ε) in the regime $\varepsilon \rightarrow 0$ in the periodic setting for a large class of tori (see Definition 3.1.8) which may as well present resonant effects. In particular we prove that the equation (S) is globally well posed in some suitable space of low-regularity, hence we prove the (global) convergence of solutions of (PE_ε) to solutions of (S).

We recall some results on primitive equations. We refer to J.-L. Lions, R. Temam and S. Wang ([107] and [108]) for the asymptotic expansion of the primitive equations with respect the Rossby number ε in a spherical and Cartesian geometry.

J.T. Beale and A. J. Bourgeois in [15] study the primitive equations (without viscosity, and with a simplified equation for the density) in a domain which is periodic in the horizontal direction and bounded in the vertical one. By the use of a change of variables they recover a purely periodic setting, on which they prove their result. They study as well the quasi-geostrophic system (first on short times, then globally) as well as the convergence of primitive equations for regular (i.e. H^3) and well prepared initial data.

In [62] P. Embid and A. Majda present a general formulation for the EVOLUTION of geophysical fluids in the periodic setting and derive the limit equation for the kernel part of the solution.

Let us now mention some known result of existence and convergence of solutions for the primitive equations when the spatial domain is \mathbb{R}^3 . In [36] J.-Y. Chemin proved that the solutions of the primitive equations converge toward those of the quasi geostrophic system in the case $F = 1$ for regular, well prepared data and under the assumption that $|\nu - \nu'|$ (the difference between the diffusion and the thermal diffusivity) is small.

When $F \neq 1$, F. Charve proved in [28] and [29], using dispersive Strichartz estimates, that the solutions of the primitive equations (PE_ε) converge globally toward a linear correction of the global solutions of the quasi-geostrophic system (QG).

For the inviscid case in the whole space, when $F = 1$, we mention the work of D. Iftimie [87] which proves that the potential vorticity Ω propagates $H^s(\mathbb{R}^3)$, $s > 5/2$ data under the hypothesis $U_{\text{osc},0}^\varepsilon = o_\varepsilon(1)$ in $L^2(\mathbb{R}^3)$. If $F \neq 1$, A. Dutrifoy proved in [58] the same result under much weaker assumptions, i.e. Ω_0 is a vortex patch and $\|U_0^\varepsilon\|_{H^s(\mathbb{R}^3)} = \mathcal{O}(\varepsilon^{-\gamma})$, $\gamma > 0$ and small. For the viscid case in the periodic setting I. Gallagher proves in [72] the global convergence of (PE_ε) toward (QG) using a technique introduced by S. Schochet in [133]. Such technique consists in a smart change of variables which cancels some nonlinear interactions which are otherwise impossible to control. We mention at last the work of F. Charve and V.-S. Ngo in [34] for the primitive equation in the whole space for $F \neq 1$ and anisotropic vanishing (horizontal) viscosity.

We recall that the primitive equations and the rotating fluid system

$$\partial_t v + v \cdot \nabla v - \nu \Delta v + \frac{e^3 \wedge v}{\varepsilon} = -\nabla p, \quad (\text{RF}_\varepsilon)$$

are intimately connected. In such system the rotation has a stabilizing effect on the solutions of (RF_ε) , inducing the fluid to have a strictly columnar dynamic. This was proved at first by E. Grenier in [79] and A. Babin et al. in [5] for the periodic case and by J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier [39] in \mathbb{R}^3 . We recall as well the results in [77] in which I. Gallagher and L. Saint-Raymond proved a weak convergence result for weak solutions for fast rotating fluids in which the rotation is inhomogeneous and given by $\frac{1}{\varepsilon} v \wedge b(x_h) e_3$.

3.1.1 Notation and results.

In this article we are interested to obtain a global-in-time result of existence and uniqueness for solutions of the system (PE_ε) and to some results of convergence when the Rossby and Froude number tend to zero at a comparable rate.

Before stating the results that we prove let us give a brief introduction about the spaces that we are going to use.

All the vector fields that we consider are real i.e we consider applications of the following form $V : \mathbb{T}^3 \rightarrow \mathbb{R}^4$. We will often associate to a vector field V the vector field $v : \mathbb{T}^3 \rightarrow \mathbb{R}^3$, which is simply the projection on the first three components of V . Moreover all the vector fields considered are periodic in all their components x_i , $i = 1, 2, 3$ and have zero global average, i.e. $\int_{\mathbb{T}^3} v dx = 0$, which is equivalent to assume that the first Fourier coefficient $\hat{V}_0 = 0$. We remark that this property is preserved for the Navier-Stokes equations as well as for the primitive equations (PE_ε) . We will always work with divergence-free vector fields.

3.1.2 Anisotropic spaces.

The anisotropy of the problem forces to introduce anisotropic spaces, i.e. spaces which behave differently in the horizontal and vertical directions. Let us recall that, in the periodic case, the non-homogeneous Sobolev anisotropic spaces are defined by the norm

$$\|u\|_{H^{s,s'}(\mathbb{T}^3)}^2 = \|u\|_{H^{s,s'}}^2 = \sum_{n=(n_h, n_3) \in \mathbb{Z}^3} (1 + |\check{n}_h|^2)^s (1 + |\check{n}_3|^2)^{s'} |\hat{u}_n|^2, \quad (3.1.2)$$

where we denoted $\check{n}_i = n_i/a_i$, $\check{n}_h = (\check{n}_1, \check{n}_2)$ and the Fourier coefficients \hat{u}_n are given by $u = \sum_n \hat{u}_n e^{2\pi i \check{n} \cdot x}$. In the whole text \mathcal{F} denotes the Fourier transform and \mathcal{F}^v the Fourier transform in the vertical variable.

We are interested to study the regularity of the product of two distributions (which is a priori not well defined), in the framework of Sobolev spaces it can be proved (see [72]) the following product rule

Lemma 3.1.1. *Let u, v be two distributions with zero average defined on $H^s(\mathbb{T}^d)$ and $H^t(\mathbb{T}^d)$ respectively, with $s + t > 0$, $s, t < d/2$, then*

$$\|u \cdot v\|_{H^{s+t-d/2}(\mathbb{T}^d)} \leq C_{s,t} \|u\|_{H^s(\mathbb{T}^d)} \|v\|_{H^t(\mathbb{T}^d)}.$$

As in classical isotropic spaces (see [1]) if $s > 1/2$ the space $H^s(\mathbb{T}_v^1)$ is a Banach algebra. Combining this fact with the above lemma we deduce the following result which we shall apply all along the paper

Lemma 3.1.2. *Let $u \in H^{s_1, s'}$, $v \in H^{s_2, s'}$ distributions with zero horizontal average with $s_1 + s_2 > 0$, $s_1, s_2 < 1$ and $s' > 1/2$, then $u \cdot v \in H^{s_1+s_2-1, s'}$ and the following bound holds true*

$$\|u \cdot v\|_{H^{s_1+s_2-1, s'}} \leq C \|u\|_{H^{s_1, s'}} \|v\|_{H^{s_2, s'}}.$$

Let us recall as well the definition of the anisotropic Lebesgue spaces, we denote with $L_h^p L_v^q$ the space $L^p(\mathbb{T}_h^2; L^q(\mathbb{T}_v^1))$, defined by the norm:

$$\|f\|_{L_h^p L_v^q} = \left\| \|f(x_h, \cdot)\|_{L^q(\mathbb{T}_v^1)} \right\|_{L^p(\mathbb{T}_h^2)} = \left(\int_{\mathbb{T}_h^2} \left(\int_{\mathbb{T}_v^1} |f(x_h, x_3)|^q dx_3 \right)^{\frac{p}{q}} dx_h \right)^{\frac{1}{p}},$$

in a similar way we define the space $L_v^q L_h^p$. It is well-known that the order of integration is important as it is described in the following lemma

Lemma 3.1.3. *Let $1 \leq p \leq q$ and $f : X_1 \times X_2 \rightarrow \mathbb{R}$ a function belonging to $L^p(X_1; L^q(X_2))$ where $(X_1; \mu_1), (X_2; \mu_2)$ are measurable spaces, then $f \in L^q(X_2; L^p(X_1))$ and we have the inequality*

$$\|f\|_{L^q(X_2; L^p(X_1))} \leq \|f\|_{L^p(X_1; L^q(X_2))}$$

In the anisotropic setting the Hölder inequality becomes;

$$\|fg\|_{L_h^p L_v^q} \leq \|f\|_{L_h^{p'} L_v^{q'}} \|g\|_{L_h^{p''} L_v^{q''}},$$

where $1/p = 1/p' + 1/p''$, $1/q = 1/q' + 1/q''$.

3.1.3 Results.

We recall at first a result of local existence and uniqueness of solutions for Navier-Stokes equations without vertical viscosity and periodic initial conditions.

Theorem 3.1.4. *Let $s > 1/2$ and $V_0 \in H^{0,s}(\mathbb{T}^3)$ a divergence-free vector field. Then there exists a time $T > 0$ independent of ε and a unique solution V^ε for the system (PE $_\varepsilon$) in the space*

$$V^\varepsilon \in \mathcal{C}([0, T]; H^{0,s}), \quad \nabla_h V^\varepsilon \in L^2([0, T]; H^{0,s}).$$

Moreover $(V^\varepsilon)_{\varepsilon>0}$ is uniformly bounded (in ε) in the space

$$V^\varepsilon \in L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^3)), \quad \nabla_h V^\varepsilon \in L^2(\mathbb{R}_+; L^2(\mathbb{T}^3)).$$

The existence part of Theorem 3.1.4 was proved in [39], while the uniqueness (in the same energy space) was proved in [89].

Remark 3.1.5. We want to point out that, as it was proved by M. Paicu in [125] (see Proposition 3.2.7) the maximal lifespan does not depend on the regularity of the initial data, as long as $V_0 \in H^{0,s}$, $s > 1/2$. \blacklozenge

Let $\mathcal{L}(\tau)$ be the semigroup generated by $\mathbb{P}\mathcal{A}$, where \mathbb{P} is the Leray projector on the divergence-free vector fields on the first three components, which leaves unchanged the fourth. In particular the Leray projector in three dimensions is given by the formula $\mathbb{P}^{(3)} = 1 - \mathcal{R}^{(3)} \otimes \mathcal{R}^{(3)}$, where $\mathcal{R}^{(3)}$ is the three dimensional Riesz transform

$$\mathcal{R}^{(3)} = \left(\frac{\partial_1}{\sqrt{-\Delta}}, \frac{\partial_2}{\sqrt{-\Delta}}, \frac{\partial_3}{\sqrt{-\Delta}} \right),$$

while \mathcal{A} is the matrix defined in (5.1.3). In the same way we define the operators $\Lambda = \sqrt{-\Delta}$, $\Lambda_h = \sqrt{-\Delta_h}$, $\Lambda_v = |\partial_3|$.

Let $\mathcal{L}(t)V_0$ be the unique global solution of

$$\begin{cases} \partial_t V_L + \mathbb{P}\mathcal{A} V_L = 0, \\ V_L|_{t=0} = V_0. \end{cases}$$

Let us further define $U^\varepsilon = \mathcal{L}\left(-\frac{t}{\varepsilon}\right)V^\varepsilon$. We will denote U^ε as the sequence of filtered solutions, we define

$$\mathcal{Q}^\varepsilon(U, V) = \mathcal{L}\left(-\frac{t}{\varepsilon}\right)\mathbb{P}\left[\mathcal{L}\left(\frac{t}{\varepsilon}\right)U \cdot \nabla \mathcal{L}\left(\frac{t}{\varepsilon}\right)V\right], \quad \mathbb{D}^\varepsilon U = \mathcal{L}\left(-\frac{t}{\varepsilon}\right)\mathbf{D}\mathcal{L}\left(\frac{t}{\varepsilon}\right)U,$$

where \mathbf{D} is defined in (5.1.3), and we consider their limits \mathcal{Q}, \mathbb{D} in \mathcal{D}' (we shall see that these limit exists). We can hence formally introduce the *limit system*

$$\begin{cases} \partial_t U + \mathcal{Q}(U, U) - \mathbb{D}U = 0, \\ \operatorname{div} u = 0, \\ U|_{t=0} = V_0, \end{cases} \quad (\text{S})$$

Since the space domain is periodic resonant effect may play an important role.

Definition 3.1.6. The resonant set \mathcal{K}^* is the set of frequencies such that

$$\begin{aligned} \mathcal{K}^* &= \{(k, m, n) \in \mathbb{Z}^9 \mid \omega^a(k) + \omega^b(m) = \omega^c(n) \text{ with } k + m = n, \ (a, b, c) \in \{-, +\}\}, \\ &= \{(k, n) \in \mathbb{Z}^6 \mid \omega^a(k) + \omega^b(n - k) = \omega^c(n), \ (a, b, c) \in \{-, +\}\}, \end{aligned}$$

where ω^j , $j = \pm$ are the eigenvalues of a suitable operator (see Section 3.3 for further details). Relatively to the present problem the explicit expression of the eigenvalues is

$$i \omega^\pm(n) = \pm \frac{i}{F} \frac{\sqrt{|\check{n}_h|^2 + F^2 \check{n}_3^2}}{|\check{n}|}.$$

We may as well associate a resonant space to a determinate frequency n , in this case we define

$$\mathcal{K}_n^* = \{(k, m) \in \mathbb{Z}^6 \mid \omega^a(k) + \omega^b(m) = \omega^c(n) \text{ with } k + m = n, \ (a, b, c) \in \{-, +\}\}.$$

Definition 3.1.7. We say that the torus \mathbb{T}^3 is non-resonant if $\mathcal{K}^* = \emptyset$.

Tori which are non-resonant, are, generally, a better choice since the oscillating part of the solution satisfies a linear equation (see [72]). Indeed though a generic torus may as well present resonant effects. For this reason we introduce the following definition:

Definition 3.1.8. We say that a torus $\mathbb{T}^3 \subset \mathbb{R}^3$ satisfies the condition (\mathcal{P}) if either one or the other of the following conditions is satisfied:

1. \mathbb{T}^3 is non-resonant.
2. If \mathbb{T}^3 is resonant, the Froude number F^2 is rational, and either
 - $a_3^2/a_1^2 \in \mathbb{Q}$ and a_3^2/a_2^2 is not algebraic of degree smaller or equal than four.
 - $a_3^2/a_2^2 \in \mathbb{Q}$ and a_3^1/a_2^2 is not algebraic of degree smaller or equal than four.

Remark 3.1.9. The above definition (Definition 3.1.8) is motivated in Section 3.5. Point 2 ensures that even with resonant effects we can propagate the horizontal average of the initial data, thing that, generally, is not true for three-dimensional Navier-Stokes equations. \blacklozenge

Although (S) is an hyperbolic system in the vertical variable we are able to prove that there exist weak (in the sense of distributions) global solutions. This was first remarked by M. Paicu in [123] and it is due to the fact that the limit bilinear form \mathcal{Q} has in fact better product rules than the standard bilinear transport form (see as well Lemma 3.8.4). The complete statement of the theorem is the following one.

Theorem 3.1.10. *Let \mathbb{T}^3 be a 3-dimensional torus in \mathbb{R}^3 and let $F \neq 1$, for each divergence-free vector field $V_0 \in L^2(\mathbb{T}^3)$ and $\Omega_0 = -\partial_2 v_0^1 + \partial_1 v_0^2 - F \partial_3 T_0 \in L^2(\mathbb{T}^3)$ there exists a distributional solution of the system*

$$\begin{cases} \partial_t U + \mathcal{Q}(U, U) - \mathbb{D}U = 0, \\ \operatorname{div} u = 0, \\ U|_{t=0} = V_0, \end{cases} \quad (\text{S})$$

in the space $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{T}^3)$ which moreover belongs to the space

$$U \in L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^3)), \quad \nabla_h U \in L^2(\mathbb{R}_+; L^2(\mathbb{T}^3)),$$

and satisfies the following energy estimate

$$\|U(t)\|_{L^2(\mathbb{T}^3)}^2 + 2c \int_0^t \|\nabla_h U(s)\|_{L^2(\mathbb{T}^3)}^2 ds \leq \|U_0\|_{L^2(\mathbb{T}^3)}^2,$$

where the constant $c = \min\{\nu_h, \nu'_h\} > 0$.

We remark that Theorem 3.1.10 holds for any three-dimensional torus. We do not require the condition (\mathcal{P}) to hold.

A natural question we address to is whether system (PE_ε) converges (even in a weak sense) to the limit system (S) as $\varepsilon \rightarrow 0$. This is the scope of the following theorem:

Theorem 3.1.11. *Let the initial data V_0 be as in Theorem 3.1.10, then defining the operator*

$$\mathcal{L}(\tau) = e^{-\tau \mathbb{P}^A},$$

and denoting as U the distributional solution of the limit system (S) identified in Theorem 3.1.10 the following convergence holds in the sense of distributions

$$V^\varepsilon - \mathcal{L}\left(\frac{t}{\varepsilon}\right) U \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Moreover U weak solution of the limit system (S) can be described as the superposition of the evolution of $U = U_{\text{QG}} + U_{\text{osc}} = V_{\text{QG}} + U_{\text{osc}}$ where V_{QG} solves

$$\begin{cases} \partial_t V_{\text{QG}} + a_{\text{QG}}(D_h) V_{\text{QG}} = - \begin{pmatrix} \nabla_h^\perp \\ 0 \\ -F \partial_3 \end{pmatrix} \Delta_F^{-1} (v_{\text{QG}}^h \cdot \nabla_h \Omega), \\ \operatorname{div}_h v_{\text{QG}}^h = \operatorname{div} v_{\text{QG}} = 0, \\ V_{\text{QG}}|_{t=0} = V_{\text{QG},0}, \end{cases}$$

and U_{osc} solves

$$\begin{cases} \partial_t U_{\text{osc}} + \mathcal{Q}(V_{\text{QG}}, U_{\text{osc}}) + \mathcal{Q}(U_{\text{osc}}, V_{\text{QG}}) + \mathcal{Q}(U_{\text{osc}}, U_{\text{osc}}) + a_{\text{osc}}(D_h) U_{\text{osc}} = 0, \\ \operatorname{div} u_{\text{osc}} = 0, \\ U_{\text{osc}}|_{t=0} = U_{\text{osc},0} = (V_0)_{\text{osc}}. \end{cases}$$

The operators a_{QG} and a_{osc} are elliptic in the horizontal variables, in the sense that there exists a positive constant $c > 0$ such that

$$(a_{\text{QG}}(D_h) u | u)_{L^2}, (a_{\text{osc}}(D_h) u | u)_{L^2} \geq c \|\nabla_h u\|_{L^2}^2,$$

and \mathcal{Q} is a bilinear form which shares many aspects with the more classical transport form, but has better properties as far as the regularity of the product is concerned.

Performing some a priori estimates on the limit system (S) we can improve of the above theorem, at the cost of having well prepared initial data and tori which satisfy Condition (P). We say that a data V_0 is well prepared if it has zero horizontal mean, i.e. $\int_{\mathbb{T}_h^2} V_0(x_h, x_3) dx_h = 0$. This property is conserved by the limit system (S) as long as the condition (P) is satisfied (see Lemma 3.5.6, 3.5.7). Moreover we ask as well that the potential vorticity, defined as

$$\Omega(t, x) = -\partial_2 U^1(t, x) + \partial_1 U^2(t, x) - F \partial_3 U^4(t, x), \quad (3.1.3)$$

belongs to $H^{0,s}$, $s \geq 0$ at time $t = 0$.

Theorem 3.1.12. *Let \mathbb{T}^3 satisfy the condition (P) and consider a divergence-free vector field $U_0 \in H^{0,s}$ with zero horizontal average. Let $\Omega_0 \in H^{0,s}$, for $s \geq 1$ and $F \neq 1$, the global weak solution of (S) is in fact strong and it belongs to the space of sub-critical regularity*

$$U \in L^\infty(\mathbb{R}_+; H^{0,s}), \quad \nabla_h U \in L^2(\mathbb{R}_+; H^{0,s}).$$

Moreover it satisfies, for each $t > 0$, the energy bound

$$\|U(t)\|_{H^{0,s}}^2 + c \int_0^t \|\nabla_h U(s)\|_{H^{0,s}}^2 \leq \mathcal{E}(\|U_0\|_{H^{0,s}}^2),$$

where \mathcal{E} is a suitable function which is bounded on compact sets. The solution U is unique in the space $L^\infty(\mathbb{R}_+; H^{0,\sigma}) \cap L^2(\mathbb{R}_+, H^{1,\sigma})$ for $\sigma \in [-1/2, s)$.

Remark 3.1.13. Compared to the work of M. Paicu [123] the author requires only $s > 1/2$. This discrepancy is due to the fact that in the present work the limit system is well-posed only for $s \geq 1$. Indeed we are able to propagate $H^{0,s}$, $s \geq 0$ norms for the potential vorticity Ω , and, as explained in Lemma 3.5.5, $\|V_{\text{QG}}\|_{H^{0,s+1}} \lesssim \|\Omega\|_{H^{0,s}}$.

The main idea in the propagation of regularity stated in Theorem 3.1.12 is that we can recover the missing viscosity in the vertical direction using the fact that the vector field u is divergence-free. We can in fact observe that in the nonlinear term the vertical derivative is always multiplied by the third component u^3 of the vector field considered (i.e. terms of the form $u^3 \partial_3$). We hence remark the fact that the term $\partial_3 u^3$ is more regular thanks to the relation $-\partial_3 u^3 = \text{div}_h u^h$, and due to the fact that the horizontal viscosity has a regularizing effect on the derivatives in the horizontal variable x_h .

Theorem 3.1.14. *Let \mathbb{T}^3 satisfy the condition (P), $\Omega_0 = -\partial_2 v_0^1 + \partial_1 v_0^2 - F \partial_3 T_0 \in H^{0,s}$ and $V_0 \in H^{0,s}$ for $s > 1$ a divergence free vector field. Let V^ε be a local solution of (PE $_\varepsilon$) and U be the unique global solution of the limit system (S). Then the following convergences take place*

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left(V^\varepsilon - \mathcal{L} \left(\frac{t}{\varepsilon} \right) U \right) &= 0 && \text{in } \mathcal{C}(\mathbb{R}_+; H^{0,\sigma}), \\ \lim_{\varepsilon \rightarrow 0} \nabla_h \left(V^\varepsilon - \mathcal{L} \left(\frac{t}{\varepsilon} \right) U \right) &= 0 && \text{in } L^2(\mathbb{R}_+; H^{0,\sigma}), \end{aligned}$$

for $\sigma \in [1, s)$.

The paper is divided as follows

- In Section 3.2 we introduce some mathematical tools that will be useful in the development of the paper.
- Section 3.3 we provide a careful analysis of the spectral properties of the linear system whose evolution is determined by the operator $\mathbb{P}\mathcal{A}$. In Subsection 3.3.1 we state some results proved in [62], [63] and [7] which describe the behavior of the limit bilinear interaction $\mathcal{Q}(U, U)$ in (S) along the eigendirections spanned by the eigenvectors of $\mathbb{P}\mathcal{A}$.
- In Section 3.4 we prove Theorem 3.1.10. Such result is not a straightforward application of Leray Theorem since, due to the lack of the vertical diffusivity, the solutions are bounded in the space $L^2_{\text{loc}}(\mathbb{R}_+; H^{1,0})$ only. Such space is *not compactly embedded* in $L^2_{\text{loc}}(\mathbb{R}_+; L^2)$, this prevents us to use standard compactness theorems in functional spaces such as Aubin-Lions lemma (see [4]). Nonetheless using Fujiwara near-optimal bound (see [112]) we can transform a vertical derivative ∂_3 in a multi-index of the form $C(\partial_1^{N_1}, \partial_2^{N_2})$, where N_1, N_2 may as well be large. The system (PE_ε) has a non-zero diffusive effects in the horizontal directions, hence we can prove that bilinear interactions of weakly converging (in the sense that converge w.r.t. a Sobolev topology of negative index) sequences converge in the sense of distributions to some limit element.
- In Section 3.5 we prove Theorem 3.1.11. The approach is twofold:
 - Thanks to a topological argument we prove that the sequence $(V^\varepsilon)_{\varepsilon>0}$ is compact in some weak sense,
 - A careful analysis of the bilinear interactions in the limit $\varepsilon \rightarrow 0$ gives us the explicit form of the bilinear limit interactions.

Next in Subsection 3.5 we prove that, under some suitable geometric conditions (see Definition 3.1.8) the limit system (S) propagates globally-in-time the horizontal average of the initial data.

- In Section 3.6 we prove that the limit system propagates globally-in-time $H^{0,s}$ data, at the price of having well prepared (in the sense of zero-horizontal average) initial data and domains which satisfy the condition (P) given in Definition 3.1.8. Hence we prove Theorem 3.1.12.
- Lastly in Section 3.7 we prove Theorem 3.1.14, i.e. that we can approximate globally the solutions of (PE_ε) as $\varepsilon \rightarrow 0$ with the (global) solutions of (S) in some suitable subcritical topology.

3.2 Preliminaries.

This section is devoted to introduce the mathematical tools that will be used all along the paper and which are necessary to understand the contents described in the following pages.

3.2.1 Elements of Littlewood-Paley theory.

A tool that will be widely used all along the paper is the anisotropic theory of Littlewood-Paley, which consists in doing a dyadic cut-off of the vertical frequencies.

Let us define the (non-homogeneous) vertical truncation operators as follows:

$$\begin{aligned}\Delta_q^v u &= \sum_{n \in \mathbb{Z}^3} \hat{u}_n \varphi \left(\frac{|\check{n}_3|}{2^q} \right) e^{i\check{n} \cdot x} && \text{for } q \geq 0 \\ \Delta_{-1}^v u &= \sum_{n \in \mathbb{Z}^3} \hat{u}_n \chi (|\check{n}_3|) e^{i\check{n} \cdot x} \\ \Delta_q^v u &= 0 && \text{for } q \leq -2\end{aligned}$$

where $u \in \mathcal{D}'(\mathbb{T}^3)$ and \hat{u}_n are the Fourier coefficients of u . The functions φ and χ represent a partition of the unity in \mathbb{R} , which means that are smooth functions with compact support such that

$$\text{Supp } \chi \subset B \left(0, \frac{4}{3} \right), \quad \text{Supp } \varphi \subset \mathcal{C} \left(\frac{3}{4}, \frac{8}{3} \right),$$

Moreover for each $t \in \mathbb{R}$ the sequence $(\chi(\cdot), \varphi(2^{-q}\cdot))_{q \in \mathbb{N}}$ is a partition of the unity. Let us define further the vertical cut-off operator as $S_q^v u = \sum_{q' \leq q-1} \Delta_{q'}^v u$.

Anisotropic paradifferential calculus.

The dyadic decomposition turns out to be very useful also when it comes to study the product between two distributions. We can in fact, at least formally, write for two distributions u and v

$$u = \sum_{q \in \mathbb{Z}} \Delta_q^v u; \quad v = \sum_{q' \in \mathbb{Z}} \Delta_{q'}^v v; \quad uv = \sum_{\substack{q \in \mathbb{Z} \\ q' \in \mathbb{Z}}} \Delta_q^v u \Delta_{q'}^v v \quad (3.2.1)$$

We are going to perform a Bony decomposition in the vertical variable (see [10], [14], [37] for the isotropic case and [39], [86] for the anisotropic one).

Paradifferential calculus is a mathematical tool for splitting the above sum in three parts

$$uv = T_u^v v + T_v^v u + R^v(u, v),$$

where

$$\begin{aligned}T_u^v v &= \sum_q S_{q-1}^v u \Delta_q^v v, \\ T_v^v u &= \sum_{q'} S_{q'-1}^v v \Delta_{q'}^v u, \\ R^v(u, v) &= \sum_k \sum_{|\mu| \leq 1} \Delta_k^v u \Delta_{k+\mu}^v v.\end{aligned}$$

In particular the following almost orthogonality properties hold

$$\begin{aligned} \Delta_q^v (S_{q'-1}^v a \Delta_{q'}^v b) &= 0 && \text{if } |q - q'| \geq 5 \\ \Delta_q^v (\Delta_{q'}^v a \Delta_{q'+\mu}^v b) &= 0 && \text{if } q' < q - 4, |\mu| \leq 1 \end{aligned}$$

and hence we will often use the following relation

$$\Delta_q^v (uv) = \sum_{|q-q'|\leq 4} \Delta_q^v (S_{q'-1}^v v \Delta_{q'}^v u) + \sum_{|q-q'|\leq 4} \Delta_q^v (S_{q'-1}^v u \Delta_{q'}^v v) \quad (3.2.2)$$

$$\begin{aligned} &+ \sum_{q' \geq q-4} \sum_{|\mu| \leq 1} \Delta_q^v (\Delta_{q'}^v u \Delta_{q'+\mu}^v v) \\ &= \sum_{|q-q'|\leq 4} \Delta_q^v (S_{q'-1}^v v \Delta_{q'}^v u) + \sum_{q' > q-4} \Delta_q^v (S_{q'+2}^v u \Delta_{q'}^v v). \end{aligned} \quad (3.2.3)$$

In the paper [43] J.-Y. Chemin and N. Lerner introduced the following asymmetric decomposition, which was first used by J.-Y. Chemin et al. in [39] in its anisotropic version. This particular decomposition turns out to be very useful in our context

$$\begin{aligned} \Delta_q^v (uv) &= S_{q-1}^v u \Delta_q^v v \\ &+ \sum_{|q-q'|\leq 4} \{ [\Delta_q^v, S_{q'-1}^v u] \Delta_{q'}^v v + (S_q^v u - S_{q'-1}^v u) \Delta_q^v \Delta_{q'}^v v \} \\ &+ \sum_{q' > q-4} \Delta_q^v (S_{q'+2}^v v \Delta_{q'}^v u), \end{aligned} \quad (3.2.4)$$

where the commutator $[\Delta_q^v, a] b$ is defined as $[\Delta_q^v, a] b = \Delta_q^v (ab) - a \Delta_q^v b$.

All along the following we shall denote as $(b_q)_{q \geq -1}$ any sequence which is summable that may depend on different parameters such that $\sum_q b_q \leq 1$. In the same way we shall denote as $(c_q)_q \in \ell^2(\mathbb{Z})$ any sequence such that $\sum_q c_q^2 \leq 1$. As well C is a (large) positive constant independent of any parameter and c a small one, these two constants may differ implicitly from line to line. We remark that the regularity of a function can be rephrased in the following way: we say that $u \in H^{0,s}$ only if there exists a sequence $(c_q)_q$ depending on u such that

$$\| \Delta_q^v u \|_{L^2(\mathbb{T}^3)} \leq C c_q(u) 2^{-qs} \|u\|_{H^{0,s}}. \quad (3.2.5)$$

Dyadic blocks and commutators as convolution operators.

The dyadic blocks and the low-frequencies truncation operators can be seen as convolution operators, in particular if we denote as $h = \mathcal{F}^{-1}\varphi$ and $g = \mathcal{F}^{-1}\chi$ we have

$$\begin{aligned} \Delta_q^v u &= \varphi(2^{-q}D) u = 2^q \int_{\mathbb{T}} h(2^q y) u(x-y) dy, \\ S_q^v u &= \chi(2^{-q}D) u = 2^q \int_{\mathbb{T}} g(2^q y) u(x-y) dy. \end{aligned} \quad (3.2.6)$$

This is due to the fact that $\Delta_q^v u(x) = (\mathcal{F}^v)^{-1}(\varphi(\cdot) \hat{u}(\cdot))(x)$. We introduce this alternative way to consider commutators and truncations because we need it in Appendix 3.8. In particular we want to express a commutator as a convolution operator, since a commutator is defined as

$$[\Delta_q^v, a] b(x) = \Delta_q^v(ab)(x) - a(x) \Delta_q^v b(x),$$

and we apply to the right hand side of the above equation the relation in (3.2.6) we obtain in fact that

$$[\Delta_q^v, a] b(x) = 2^q \int_{\mathbb{T}} h(x_h, x_3 - y_3) (a(x_h, y_3) - a(x_h, x_3)) b(x_h, y_3) dy_3.$$

Thanks to Taylor expansion with reminder in Cauchy form we know that

$$a(x_h, y_3) - a(x_h, x_3) = \partial_3 a(x_h, x_3 + \tau(x_3 - y_3))(x_3 - y_3),$$

for some $\tau \in (0, 1)$, hence we can write the commutator as

$$[\Delta_q^v, a] b(x) = 2^q \int_{\mathbb{T}} (x_3 - y_3) h(x_h, x_3 - y_3) \partial_3 a(x_h, x_3 + \tau(x_3 - y_3)) b(x_h, y_3) dy_3. \quad (3.2.7)$$

Some basic estimates.

The interest in the use of the dyadic decomposition is that the derivative in the vertical direction of a function localized in vertical frequencies of size 2^q acts like the multiplication of a factor 2^q (up to a constant independent of q). In our setting (periodic case) a Bernstein type inequality holds. For a proof of the following lemma we refer to the work [86].

Lemma 3.2.1. *Let u be a function such that $\text{supp } \mathcal{F}^v u \subset \mathbb{T}_h^2 \times 2^q \mathcal{C}$, where \mathcal{F}^v denotes the Fourier transform in the vertical variable. For all integers $k, p \in [1, \infty]$, $1 \leq r' \leq r \leq \infty$, the following relations hold*

$$\begin{aligned} 2^{qk} C^{-k} \|u\|_{L_h^p L_v^r} &\leq \|\partial_{x_3}^k u\|_{L_h^p L_v^r} \leq 2^{qk} C^k \|u\|_{L_h^p L_v^r}, \\ 2^{qk} C^{-k} \|u\|_{L_v^r L_h^p} &\leq \|\partial_{x_3}^k u\|_{L_v^r L_h^p} \leq 2^{qk} C^k \|u\|_{L_v^r L_h^p}. \end{aligned}$$

Let now $\infty \geq r \geq r' \geq 1$ be real numbers. Let $\text{supp } \mathcal{F}^v u \subset \mathbb{T}_h^2 \times 2^q B$, then

$$\begin{aligned} \|u\|_{L_h^p L_v^r} &\leq C 2^{q(\frac{1}{r'} - \frac{1}{r})} \|u\|_{L_h^p L_v^{r'}} \\ \|u\|_{L_v^r L_h^p} &\leq C 2^{q(\frac{1}{r'} - \frac{1}{r})} \|u\|_{L_v^{r'} L_h^p} \end{aligned}$$

The following are inequalities of Gagliardo-Nirenberg type, we will avoid to give the proofs of such tools since they are already present in [123].

Lemma 3.2.2. *There exists a constant C such that for all periodic vector fields u on \mathbb{T}^3 with zero horizontal average ($\int_{\mathbb{T}_h^2} u(x_h, x_3) dx_h = 0$) we have*

$$\|u\|_{L_v^2 L_h^4} \leq C_1 \|u\|_{H^{1/2,0}} \leq C_2 \|u\|_{L^2(\mathbb{T}^3)}^{1/2} \|\nabla_h u\|_{L^2(\mathbb{T}^3)}^{1/2}. \quad (3.2.8)$$

From Lemma 4.1.8 and (3.2.8) we can deduce the following result

Corollary 3.2.3. *Let u be a periodic vector field such that $\text{Supp } \mathcal{F}^v u \subset \mathbb{T}_h^2 \times 2^q B$, then*

$$\|u\|_{L_v^\infty L_h^2} \leq C 2^{q/2} \|u\|_{L^2(\mathbb{T}^3)}, \quad (3.2.9)$$

moreover if u has zero horizontal average

$$\|u\|_{L_v^\infty L_h^4} \leq C 2^{q/2} \|u\|_{L^2(\mathbb{T}^3)}^{1/2} \|\nabla_h u\|_{L^2(\mathbb{T}^3)}^{1/2} \quad (3.2.10)$$

Lemma 3.2.4. *Let s be a real number and \mathbb{T}^3 a three dimensional torus. For all vector fields u with zero horizontal average, the following inequality holds*

$$\|u\|_{H^{1/2,s}} \leq C \|u\|_{H^{0,s}}^{1/2} \|\nabla_h u\|_{H^{0,s}}^{1/2} \quad (3.2.11)$$

Corollary 3.2.5. *Let $s > 1/2$. There exists a constant C such that the inequality*

$$\|u\|_{L_v^\infty L_h^2} \leq C \|u\|_{H^{0,s}}$$

holds. Moreover if u is of zero horizontal average we have

$$\|u\|_{L_v^\infty L_h^4} \leq C \|u\|_{H^{0,s}}^{1/2} \|\nabla_h u\|_{H^{0,s}}^{1/2}$$

Finally we state a lemma that shows that the commutator with the truncation operator in the vertical frequencies is a regularizing operator. The proof of such lemma can be found in [125].

Lemma 3.2.6. *Let \mathbb{T}^3 be a 3D torus and p, r, s real positive numbers such that $\infty \geq r', s', p, r, s \geq 1$, $\frac{1}{r'} + \frac{1}{s'} = \frac{1}{2}$ and $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$. There exists a constant C such that for all vector fields u and v on \mathbb{T}^3 we have the inequality*

$$\|[\Delta_q^v, u] v\|_{L_v^2 L_h^p} \leq C 2^{-q} \|\partial_3 u\|_{L_v^{r'} L_h^r} \|v\|_{L_v^{s'} L_h^s}$$

3.2.2 Preliminary results on the Navier-Stokes equations with zero vertical diffusivity.

A primary tool in the study of the convergence of the primitive equations (PE_ε) to the limit system (S) will be a careful study of the Navier-Stokes equation with only horizontal diffusion horizontal diffusion

$$\begin{cases} \partial_t v + v \cdot \nabla v - \nu_h \Delta_h v + \nabla p = 0 & \text{in } \mathbb{R}_+ \times \mathbb{T}^3 \\ \text{div } v = 0 \\ v|_{t=0} = v_0 \end{cases} \quad (\text{NS}_h)$$

This equation in the case of the periodic data on \mathbb{T}^3 has been carefully studied in [125], hence we will refer to this work as we go along.

Indeed the equation satisfied by U^ε , i.e. (FS_ε) is a Navier-Stokes equation with zero vertical diffusion and hence can be well described by the system (NS_h). Here we start giving the following energy estimate for three-dimensional anisotropic Navier-Stokes equations

Proposition 3.2.7. *Let $s \geq s_0 > \frac{1}{2}$ and v a solution of (NS_h) belonging to the space $\mathcal{C}([0, T]; H^{0,s})$ whose horizontal gradient $\nabla_h v \in L^2([0, T]; H^{0,s})$. Let us suppose moreover that $v = \underline{v} + \tilde{v}$ where \underline{v} is the horizontal average of v and \tilde{v} has zero horizontal mean. Suppose moreover that $\|\underline{v}(t)\|_{H^{s_0}} \leq ca_3^{-1}\nu_h$ in $[0, T]$, then for $t \in [0, T]$*

$$\begin{aligned} & \|v(t)\|_{H^{0,s}}^2 + \nu_h \int_0^t \|\nabla_h v(\tau)\|_{H^{0,s}}^2 d\tau \\ & \leq \|v_0\|_{H^{0,s}}^2 \exp\left(C \int_0^t \|\nabla_h v(\tau)\|_{H^{0,s_0}}^2 d\tau + C \int_0^t \|v(\tau)\|_{H^{0,s_0}}^2 \|\nabla_h v(\tau)\|_{H^{0,s_0}}^2 d\tau\right) \end{aligned}$$

Remark 3.2.8. Proposition 3.2.7 has been proved by M. Paicu for $s \geq s_0 > \frac{1}{2}$. Indeed in [123] the limit system was a coupling between a 2d Navier-Stokes system and the oscillating part. Indeed the 2d Navier-Stokes system is globally well posed if the initial data depends on x_h only and it is in $H^{0,s}$ for $s \geq 0$. The oscillating part instead is globally well posed in $H^{0,s}$ for $s > 1/2$. In our case though the limit flow is the sum of V_{QG} satisfying (3.5.11) and the oscillating part U_{osc} which are two three-dimensional vector fields. Now, U_{osc} is globally well posed in $H^{0,s}$ for $s > 1/2$ (see Proposition 3.6.5), but V_{QG} is globally well posed in $H^{0,s}$ for $s \geq 1$ (see Proposition 3.6.2 and Lemma 3.5.5). This is why in the following as long as we are required to apply Proposition 3.2.7 we shall use the index $s_0 \geq 1$ instead of $s_0 > 1/2$. \blacklozenge

For a proof of Proposition 3.2.7 we refer to the works [123, Proposition 3.1] and [125].

Given any vector field A we denote

$$\underline{A}(x_3) = \frac{1}{|\mathbb{T}_h^2|} \int_{\mathbb{T}_h^2} A(y_h, x_3) dy_h,$$

and

$$\tilde{A}(x_h, x_3) = A(x_h, x_3) - \underline{A}(x_3).$$

Proposition 3.2.9. *Let $s > \frac{1}{2}$ and \mathbb{T}^3 an arbitrary torus and $w \in \mathcal{C}([0, T]; H^{0,s})$, $\nabla_h w \in L^2([0, T]; H^{0,s})$ a solution of the problem*

$$\begin{cases} \partial_t w + w \cdot \nabla w + u \cdot \nabla w + w \cdot \nabla u - \nu_h \Delta_h w + \nabla p = f \\ \operatorname{div} w = 0 \\ w|_{t=0} = w_0, \end{cases} \quad (3.2.12)$$

where $u \in \mathcal{C}([0, T]; H^{0,s})$, $\nabla_h u \in L^2([0, T]; H^{0,s})$ a divergence-free vector field such that its horizontal average satisfies $\|\underline{u}(t)\|_{H^{s_0}} \leq ca_3^{-1}\nu_h$ for all $t \in [0, T]$ and $f = \underline{f} + \tilde{f}$ is such that

$$\begin{aligned} \underline{f} & \in L^1\left([0, T]; H_v^{-\frac{1}{2}}\right) \\ \tilde{f} & \in L^2\left([0, T]; H^{-1, -\frac{1}{2}}\right). \end{aligned}$$

Then there exists a constant $C > 0$ such that we have for all $t \in [0, T]$

$$\begin{aligned} & \|w(t)\|_{H^{0,-\frac{1}{2}}}^2 + \nu_h \int_0^t \|\nabla_h w(s)\|_{H^{0,-\frac{1}{2}}}^2 ds \\ & \leq C \left(\|w_0\|_{H^{0,-\frac{1}{2}}}^2 + \int_0^t \|\tilde{f}(s)\|_{H^{-1,-\frac{1}{2}}}^2 ds + \int_0^t \|\underline{f}(s)\|_{H_v^{-\frac{1}{2}}} ds \right) \times \\ & \exp \left\{ \int_0^t \|\underline{f}(s)\|_{H_v^{-\frac{1}{2}}} ds \right\} \exp \left\{ \int_0^t (1 + \|w(s)\|_{H^{0,s}}^2) \|\nabla_h w(s)\|_{H^{0,s}}^2 ds \right. \\ & \quad \left. + \int_0^t (1 + \|u(s)\|_{H^{0,s}}^2) \|\nabla_h u(s)\|_{H^{0,s}}^2 ds \right\} \end{aligned}$$

Proof. [123, Proposition 3.2, p. 182] □

Remark 3.2.10. Let us remark the fact that we impose two different kind of regularities on the exterior force. In order to obtain global results in time we shall apply this proposition for bulk forces which are $f \in L^1(\mathbb{R}_+, H^{-1,-1/2}) \cap L^2(\mathbb{R}_+, H^{-1,-1/2})$. ◆

3.3 Spectral analysis of the linear system an analysis of the Poincaré filtration $e^{-\tau\mathbb{P}\mathcal{A}}$.

Let us consider the following linear equation

$$\begin{cases} \partial_t V_L + \mathbb{P}\mathcal{A} V_L = 0 \\ V_L|_{t=0} = V_0 \end{cases}, \quad (3.3.1)$$

where \mathbb{P} is the Leray projection onto the divergence free vector fields, without changing V_L^4 . The Fourier multiplier associated to \mathbb{P} has the following form

$$\mathbb{P}_n = 1 - \frac{1}{|\tilde{n}|^2} \begin{pmatrix} \frac{n_1^2}{a_1^2} & \frac{n_1 n_2}{a_1 a_2} & \frac{n_1 n_3}{a_1 a_3} & 0 \\ \frac{n_2 n_1}{a_2 a_1} & \frac{n_2^2}{a_2^2} & \frac{n_2 n_3}{a_2 a_3} & 0 \\ \frac{n_3 n_1}{a_3 a_1} & \frac{n_3 n_2}{a_3 a_2} & \frac{n_3^2}{a_3^2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.3.2)$$

where $|\tilde{n}|^2 = \sum_j \frac{n_j^2}{a_j^2}$ and 1 is the identity matrix on \mathbb{C}^4 . The operator \mathcal{A} was defined in (3.1.1). The solution to the linear equation is indeed $V_L(\tau) = e^{-\tau\mathbb{P}\mathcal{A}} V_0$. We denote the propagator operator $e^{-\tau\mathbb{P}\mathcal{A}}$ as $\mathcal{L}(\tau)$. One can compute the matrix $\mathbb{P}_n \mathcal{A}$

$$\mathbb{P}_n \mathcal{A} = \begin{pmatrix} -\frac{\tilde{n}_1 \tilde{n}_2}{|\tilde{n}|^2} & -1 + \frac{\tilde{n}_1^2}{\tilde{n}^2} & 0 & -\frac{\tilde{n}_1 \tilde{n}_3}{F|\tilde{n}|^2} \\ 1 - \frac{\tilde{n}_2^2}{|\tilde{n}|^2} & \frac{\tilde{n}_1 \tilde{n}_2}{|\tilde{n}|^2} & 0 & -\frac{\tilde{n}_2 \tilde{n}_3}{F|\tilde{n}|^2} \\ -\frac{\tilde{n}_2 \tilde{n}_3}{|\tilde{n}|^2} & \frac{\tilde{n}_1 \tilde{n}_3}{|\tilde{n}|^2} & 0 & \frac{1}{F} \left(1 - \frac{\tilde{n}_3^2}{|\tilde{n}|^2} \right) \\ 0 & 0 & -\frac{1}{F} & 0 \end{pmatrix}, \quad (3.3.3)$$

whose eigenvalues are

$$\omega^0(n) = 0, \quad i \omega^\pm(n) = \pm \frac{i}{F} \frac{\sqrt{|\check{n}_h|^2 + F^2 \check{n}_3^2}}{|\check{n}|}, \quad (3.3.4)$$

where the eigenvalue ω^0 has multiplicity 2, and we can write $\omega^\pm = \pm\omega$. The associated normalized eigenvectors are

$$e^0(n) = \frac{1}{|\check{n}|_F} \begin{pmatrix} -\check{n}_2 \\ \check{n}_1 \\ 0 \\ -F\check{n}_3 \end{pmatrix}, \quad (3.3.5)$$

$$e^\pm(n) = \frac{1}{(1 + F^2 |\omega(n)|^2) |\check{n}_h|^2 |\check{n}|^2} \begin{pmatrix} -F\check{n}_3 (\check{n}_2 \mp i\check{n}_1\omega(n)) \\ F\check{n}_3 (\check{n}_1 \pm i\check{n}_2\omega(n)) \\ \mp iF\omega(n) |\check{n}_h|^2 \\ |\check{n}_h|^2 \end{pmatrix},$$

where $|\check{n}|_F = \sqrt{\check{n}_1^2 + \check{n}_2^2 + F^2 \check{n}_3^2}$.

A case of particular interest happens when $|n_h| = 0$, in such setting we recover the following matrix

$$\mathbb{P}_{(0,n_3)} \mathcal{A} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{F} & 0 \end{pmatrix},$$

while the oscillating eigenvalues become

$$i \omega^\pm(0, n_3) = \pm i,$$

and the oscillating and non-oscillating eigenvectors are given by

$$e^\pm(0, n_3) = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm i \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad e^0(0, n_3) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (3.3.6)$$

The eigenvalue ω^0 has algebraic multiplicity 2, but there is only one eigenvector related to it, namely e^0 . Indeed the matrix $\mathbb{P}\mathcal{A}$ has a nontrivial 2×2 Jordan block structure associated to the eigenvalue 0, hence the fourth is a generalized eigenvector \tilde{e}^0 . This though is not divergence-free, hence it shall play no role in the evolution of the system (\mathbf{PE}_ϵ) , for this reason it is omitted. For a more detailed discussion on the spectral properties of the linear system we refer the reader to the papers [62] and [63].

Once we have introduced the eigenvectors in (3.3.5) we can consider a generic divergence-free vector field V as direct sum of the elements belonging to $\mathbb{C}e^0$ and $\mathbb{C}e^- \oplus \mathbb{C}e^+$. We shall call the projection of V onto $\mathbb{C}e^0$ the quasi-geostrophic part, while the projection onto $\mathbb{C}e^- \oplus \mathbb{C}e^+$ the oscillating part. The projection can be explicitly defined as follows

$$V_{\text{QG}} = \mathcal{F}^{-1} \left(\left(\hat{V}_n \Big| e^0(n) \right)_{\mathbb{C}^4} e^0(n) \right),$$

$$V_{\text{osc}} = \sum_{i=\pm} \mathcal{F}^{-1} \left(\left(\hat{V}_n \Big| e^i(n) \right)_{\mathbb{C}^4} e^i(n) \right). \quad (3.3.7)$$

The element V_{osc} is called oscillating because is the only part of the initial vector field V_0 which is affected in the evolution of the system (3.3.1), V_{QG} stays still being in the kernel of $\mathbb{P}\mathcal{A}$.

We would like to point out the following relevant fact, the non-oscillating eigenspace $\mathbb{C}e^0$ is orthogonal to the oscillating eigenspace $\mathbb{C}e^- \oplus \mathbb{C}e^+$, whence in particular it is always true that $V_{\text{QG}} \perp V_{\text{osc}}$.

In the following we shall denote as $e^a(n)$ the eigenvector of $\mathbb{P}_n\mathcal{A}$ associated with the eigenvalue $i\omega^a$, i.e. $\mathbb{P}_n\mathcal{A}(e^{i\check{n}\cdot x}e^a(n)) = \exp\{i\check{n}\cdot x + i\tau\omega^a(n)\}e^a(n)$. Let us define $U^\varepsilon = \mathcal{L}\left(-\frac{t}{\varepsilon}\right)V^\varepsilon$, we want to reformulate (PE $_\varepsilon$) in terms of the new unknown U^ε . A straightforward computation shows that the vector field U^ε satisfies the following equation

$$\begin{cases} \partial_t U^\varepsilon + \mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon) - \mathbb{D}^\varepsilon U^\varepsilon = 0 \\ \operatorname{div} v^\varepsilon = 0 \\ U^\varepsilon|_{t=0} = V_0 \end{cases} \quad (\text{FS}_\varepsilon)$$

where

$$\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon) = \mathcal{L}\left(-\frac{t}{\varepsilon}\right) \mathbb{P} \left[\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right) U^\varepsilon \cdot \nabla \right) \mathcal{L}\left(\frac{t}{\varepsilon}\right) U^\varepsilon \right] \quad (3.3.8)$$

$$\mathbb{D}^\varepsilon U^\varepsilon = \mathcal{L}\left(-\frac{t}{\varepsilon}\right) \mathbf{D} \mathcal{L}\left(\frac{t}{\varepsilon}\right) U^\varepsilon. \quad (3.3.9)$$

We shall call the system (FS $_\varepsilon$) the filtered system.

Before using the above results to find the limit of (FS $_\varepsilon$) we introduce the "potential vorticity"

$$\Omega^\varepsilon = -\partial_2 U^{1,\varepsilon} + \partial_1 U^{2,\varepsilon} - F \partial_3 U^{4,\varepsilon}. \quad (3.3.10)$$

The potential vorticity has been introduced by J.-Y. Chemin in [36] and it is now a well-known tool in the study of primitive equation (see [30], [34], [72], [87]). The diagonalization explained in (3.3.7) can as well be obtained by writing $U^\varepsilon = U_{\text{QG}}^\varepsilon + U_{\text{osc}}^\varepsilon$, with

$$U_{\text{QG}}^\varepsilon = \left(-\partial_2 \Delta_F^{-1} \Omega^\varepsilon, \partial_1 \Delta_F^{-1} \Omega^\varepsilon, 0, -\partial_3 F \Delta_F^{-1} \Omega^\varepsilon \right), \quad (3.3.11)$$

where Δ_F^{-1} denotes the Fourier multiplier

$$-\Delta_F^{-1} u = \mathcal{F}^{-1} \left(\left(\frac{1}{\check{n}_1^2 + \check{n}_2^2 + F^2 \check{n}_3^2} \hat{u}_n \right)_n \right).$$

We remark the fact that since $U_{\text{QG}}^\varepsilon$ belongs to the kernel of $\mathbb{P}\mathcal{A}$ we obtain indeed that $U_{\text{QG}}^\varepsilon = V_{\text{QG}}^\varepsilon$.

One of the major problem is to understand exactly which is the limit for $\varepsilon \rightarrow 0$ of the forms $\mathcal{Q}^\varepsilon, \mathbb{D}^\varepsilon$ and, if possible, how to give a closed formulation for it. To do so we use the explicit formulation of $\mathcal{Q}^\varepsilon, \mathbb{D}^\varepsilon$ given in equation (3.3.8) and (3.3.9). Let us decompose divergence-free vector field U as:

$$\mathcal{F}U(n) = \sum_{a \in \{-, 0, +\}} U^a(n) = \sum_{a \in \{-, 0, +\}} (\mathcal{F}U(n)|_{e^a(n)})_{\mathbb{C}^4} e^a(n),$$

and after some computations we obtain that;

$$\begin{aligned} \mathcal{F}(\mathcal{Q}^\varepsilon(U, V))(n) &= \sum_{a,b,c \in \{-,0,+\}} e^{-i\frac{t}{\varepsilon}(\omega^a(k) + \omega^b(n-k) - \omega^c(n))} \\ &\quad \times \left(\sum_{j=1,2,3} (\tilde{n}_j - \check{k}_j) U^{a,j}(k) V^b(n-k) \Big|_{\mathbb{C}^4} e^c(n) \right) e^c(n). \end{aligned} \quad (3.3.12)$$

In the following we will write $\omega_{k,n-k,n}^{a,b,c} = \omega^a(k) + \omega^b(n-k) - \omega^c(n)$ for the sake of conciseness, as well as $\omega_n^{a,b} = \omega^a(n) + \omega^b(n)$. With $U^{a,j}$ we denote the j -th component of the vector $U^a = \left(\hat{U} \Big|_{\mathbb{C}^4} e^a \right) e^a$ for $a = 0, \pm$.

Similar calculations give us that

$$\mathbb{D}^\varepsilon U = \mathcal{F}^{-1} \left(\sum_{a,b \in \{-,0,+\}} e^{-i\frac{t}{\varepsilon}\omega_n^{a,b}} (\mathbf{D}(n)U^b(n) \Big|_{\mathbb{C}^4} e^a(n)) e^a(n) \right), \quad (3.3.13)$$

where $\mathbf{D}(n)$ is the Fourier symbol associated to the second-order differential operator \mathbf{D} , see (3.1.1).

Letting $\varepsilon \rightarrow 0$ we only have to use the non stationary phase theorem (see, for instance [3], [10], [138]) to obtain that, if U, V are smooth functions;

$$\mathcal{Q}(U, V) = \mathcal{F}^{-1} \left(\mathbb{P}_n \sum_{\omega_{k,n-k,n}^{a,b,c}=0} \left(\sum_{j=1}^3 (\tilde{n}_j - \check{k}_j) U^{a,j}(k) V^b(n-k) \Big|_{\mathbb{C}^4} e^c(n) \right) e^c(n) \right), \quad (3.3.14)$$

$$\mathbb{D}U = \mathcal{F}^{-1} \left(\sum_{\omega_n^{a,b}=0} (\mathbf{D}(n)U^b(n) \Big|_{\mathbb{C}^4} e^a(n)) e^a(n) \right). \quad (3.3.15)$$

Here we implicitly define as $\mathbf{D}(n)$ the Fourier symbol associated to the matrix \mathbf{D} defined in (3.1.1), while \mathbb{P}_n is the Fourier symbol associated to the operator (3.3.2).

3.3.1 The global splitting of the limit bilinear form \mathcal{Q} .

This section is aimed to explain how the bilinear interaction \mathcal{Q} defined in (3.3.14) behaves along non-oscillating and oscillating subspaces $\mathbb{C}e^0$ and $\mathbb{C}e^- \oplus \mathbb{C}e^+$. Such kind of result is very well known in the theory of singular perturbation problems in periodic domains, and the results that we present here have been already proved by several authors in [7], [62] and [63], for this reason we will not prove them but instead we will refer to the works mentioned and references therein.

The results presented in the present section derive from the geometrical properties of vector decomposed as in (3.3.7) and from the localization in the frequency space of the limit bilinear form \mathcal{Q} , localization which reads as

$$\left\{ (k, n) \in \mathbb{Z}^6 \mid \omega^a(k) + \omega^b(n-k) = \omega^c(n), a, b, c \in \{0, \pm\} \right\},$$

where the eigenvalues are defined in (4.2.3).

In this section we will always consider *smooth* vector fields, in particular given a smooth vector field W we define

$$\begin{aligned}\Omega_W &= -\partial_2 W^2 + \partial_1 W^2 - F\partial_3 W^4, \\ W_{\text{QG}} &= \begin{pmatrix} -\partial_2 \\ \partial_1 \\ 0 \\ -F\partial_3 \end{pmatrix} \Delta_F^{-1} \Omega_W, \\ &= (w_{\text{QG}}, W_{\text{QG}}^4), \\ W_{\text{osc}} &= W - W_{\text{QG}}.\end{aligned}$$

Obviously W_{QG} and W_{osc} are respectively the projections of W onto the non-oscillating and oscillating subspaces defined in (3.3.7).

Lemma 3.3.1. *The following identity holds true*

$$\mathcal{F}^{-1} \left((\mathcal{F} \mathcal{Q}(W, W) | |\check{n}|_F e^0)_{\mathbb{C}^4} \right) = w_{\text{QG}} \cdot \nabla \Omega_W,$$

where \mathcal{Q} is defined in (3.3.14) and e^0 is the non-oscillating eigenvector defined in (3.3.5).

Corollary 3.3.2. *The following identity holds true*

$$\mathcal{F}^{-1} \left((\mathcal{F} \mathcal{Q}(W, W) | e^0)_{\mathbb{C}^4} \right) = \begin{pmatrix} -\partial_2 \\ \partial_1 \\ 0 \\ -F\partial_3 \end{pmatrix} \Delta_F^{-1} (w_{\text{QG}} \cdot \nabla \Omega_W).$$

For a proof of Lemma 3.3.1 we refer the reader to [62] and [63]. What has to be retained is the facts that the projection of $\mathcal{Q}(W, W)$ onto the potential non-oscillating subspace does not presents interactions of the oscillating part of the vector field.

Lemma 3.3.3. *Let W be a smooth vector field, then the following identity holds true*

$$(\mathcal{Q}(W_{\text{QG}}, W_{\text{QG}}))_{\text{osc}} = 0.$$

Proof. Considering the explicit formulation of the limit bilinear form \mathcal{Q} we deduce

$$\begin{aligned} & (\mathcal{Q}(W_{\text{QG}}, W_{\text{QG}}))_{\text{osc}} \\ &= \mathcal{F}^{-1} \left(\sum_{\substack{k+m=n \\ \omega_{k,m,n}^{0,0,\pm}=0}} (\check{n} \cdot (W^0(k) \otimes W^0(m)) | e^\pm(n))_{\mathbb{C}^4} e^\pm(n) \right). \end{aligned} \quad (3.3.16)$$

Let us consider hence the equation $\omega_{k,m,n}^{0,0,\pm} = 0$, thanks to the explicit expression of the eigenvalues in (3.3.4) then it is equivalent to the equation

$$|\check{n}_h|^2 + F^2 \check{n}_3^2 = 0,$$

which is true only if $n = 0$, and in this case the contributions arising in (3.3.16) are null, concluding. \square

Corollary 3.3.4. *The projection of the limit bilinear form \mathcal{Q} onto the oscillating subspace can be written as*

$$(\mathcal{Q}(W, W))_{\text{osc}} = (\mathcal{Q}(W_{\text{QG}}, W_{\text{osc}}))_{\text{osc}} + (\mathcal{Q}(W_{\text{osc}}, W_{\text{QG}}))_{\text{osc}} + (\mathcal{Q}(W_{\text{osc}}, W_{\text{osc}}))_{\text{osc}},$$

thanks to the decomposition (3.3.7).

3.4 Proof of Theorem 3.1.10.

Remark 3.4.1. As the reader may have noted Theorem 3.1.10 states the existence of *à la* Leray-type solutions. This can seem to be unexpected since, generally, Leray solutions are constructed thanks to compactness methods. In system (S) we cannot apply any compactness method since we do not have any second-order vertical derivative ∂_3^2 and L^2 is not compactly embedded in $H^{1,0}$. Nonetheless the bilinear form \mathcal{Q} has better product rules than the standard bilinear form in the Navier-Stokes equations, this will allow us to make sense (distributionally) of the term $\mathcal{Q}(U, U)$. Moreover we require the initial potential vorticity Ω_0 to be $L^2(\mathbb{T}^3)$, which, roughly speaking, is "almost as" requiring the initial velocity field to be H^1 .

Proof of Theorem 3.1.10 : Before starting the proof we point out the following fact, Navier-Stokes equations preserve the global average of the unknown function. This happens as well for the system (PE $_\epsilon$), whence we can consider data with zero horizontal average. Thanks to this property homogeneous and non-homogeneous Sobolev spaces are equivalent, we shall use this constantly in the present proof. In particular they will be always non-homogeneous. This fact concerns the isotropic spaces $H^s(\mathbb{R}^3)$ only.

The proof is standard application of Galerkin's approximation. We define the truncation operator

$$J_N u = \sum_{\{k \in \mathbb{Z}^3 \mid |k| \leq N\}} \hat{u}_k e^{ik \cdot x},$$

and consequently the approximated system

$$\begin{cases} \partial_t U_N + J_N \mathcal{Q}(U_N, U_N) + \mathbb{D}U_N = 0 \\ \operatorname{div} u_N = 0 \\ U_N|_{t=0} = J_N U_0, \end{cases} \quad (3.4.1)$$

in the unknown U_N . We recall that for a fixed N , J_N maps continuously any H^k space to any H^{k+h} space for $h \geq 0$ thanks to Bernstein inequality. Thus (3.4.1) is a differential equation in the space

$$L_N^2(\mathbb{T}^3) = \{u \in L^2(\mathbb{T}^3) \mid \hat{u}_k = 0 \text{ if } |k| > N\}.$$

Since the support of the Fourier transform of $U_N \in L_N^2(\mathbb{T}^3)$ is included in the ball of center 0 and radius N and the support of $\mathcal{F}(U_N \otimes U_N)$ is included in $B_{2N}(0)$ we obtain easily that $J_N \mathcal{Q} \in \mathcal{C}(L_N^2(\mathbb{T}^3) \times L_N^2(\mathbb{T}^3); L_N^2(\mathbb{T}^3))$. Hence Cauchy-Lipschitz theorem gives the existence of a unique solution to (3.4.1) on a maximal interval of time $[0, T_N)$ taking values in $L_N^2(\mathbb{T}^3)$.

Moreover since

$$\mathcal{Q}(A, B) = \lim_{\varepsilon \rightarrow 0} \mathcal{L}\left(-\frac{t}{\varepsilon}\right) \mathbb{P}\left[\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right) A \cdot \nabla\right) \mathcal{L}\left(\frac{t}{\varepsilon}\right) B\right],$$

it is clear that $(J_N \mathcal{Q}(U_N, U_N)|U_N)_{L^2(\mathbb{T}^3)} = (\mathcal{Q}(U_N, U_N)|U_N)_{L^2(\mathbb{T}^3)} = 0$ since $\operatorname{div} u_N = 0$. Hence by a standard energy estimate on the parabolic-hyperbolic equation (3.4.1) we get

$$\frac{1}{2} \|U_N(t)\|_{L^2(\mathbb{T}^3)}^2 + c \int_0^t \|\nabla_h U_N(s)\|_{L^2(\mathbb{T}^3)}^2 ds \leq \frac{1}{2} \|U_0\|_{L^2(\mathbb{T}^3)}^2,$$

from which for all $t \in [0, T_N)$ we have $\|U_N(t)\|_{L^2(\mathbb{T}^3)}^2 \leq \|J_N U_0\|_{L^2(\mathbb{T}^3)}^2 \leq \|U_0\|_{L^2(\mathbb{T}^3)}^2$. We deduce that $T_N = \infty$ and for all $t > 0$ $U_N(t)$ satisfies

$$\|U_N(t)\|_{L^2(\mathbb{T}^3)}^2 + 2c \int_0^t \|\nabla_h U_N(s)\|_{L^2(\mathbb{T}^3)}^2 ds \leq \|U_0\|_2^2.$$

Consider the relation $\|U_N\|_{L^2((0,t); L^2(\mathbb{T}^3))} \leq \sqrt{t} \|U_N\|_{L^\infty((0,t); L^2(\mathbb{T}^3))} \leq \sqrt{t} \|U_0\|_{L^2(\mathbb{T}^3)}$ we can say that the sequence U_N is bounded in $L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^3)) \cap L_{\text{loc}}^2(\mathbb{R}_+; H^{1,0})$. By the structure of (3.4.1) we obtain easily that $\partial_t U_N$ is bounded in $L_{\text{loc}}^2(\mathbb{R}_+; H^{-N})$ for N sufficiently big (the proof of such fact is identical as the proof of Proposition 3.5.1), hence $(\partial_t U_N)_N$ is a sequence of uniformly bounded functions in $L_{\text{loc}}^2(\mathbb{R}_+; H^{-M})$. We can infer via Aubin-Lions lemma [4] obtaining that $U_N \rightarrow U$ in $L_{\text{loc}}^2(\mathbb{R}_+; H^{-\varepsilon}(\mathbb{T}^3))$ where $\varepsilon \in (0, M)$ up to (non-relabeled) subsequences.

Since the sequence $(U_N)_N$ converges in $L_{\text{loc}}^2(\mathbb{R}_+; H^{-\varepsilon}(\mathbb{T}^3))$ only, and products of $H^{-\varepsilon}$ functions are, a priori, not well defined we introduce a diagonalization method which allows us to split (3.4.1) in two systems which we will be able to handle.

We rely on a diagonalization method introduced by P. Embid and A. Majda in [62], in detail, we define

$$\Omega_N = -\partial_2 U_N^1 + \partial_1 U_N^2 - F \partial_3 U_N^4, \quad (3.4.2)$$

$$V_{\text{QG}, N} = U_{\text{QG}, N} = \begin{pmatrix} \nabla_h^\perp \\ 0 \\ -F \partial_3 \end{pmatrix} \Delta_F^{-1} \Omega_N, \quad (3.4.3)$$

$$U_{\text{osc}, N} = U_N - U_{\text{QG}, N}.$$

Applying Lemma 3.3.1 on the smooth vector field U_N we deduce that

$$(\mathcal{F} J_N \mathcal{Q}(U_N, U_N)| |\check{n}|_F e^0(n))_{\mathbb{C}^4} = \mathcal{F}(J_N(v_{\text{QG}, N}^h \cdot \nabla_h \Omega_N)).$$

Whence the projection of the element $J_N \mathcal{Q}(U_N, U_N)$ onto the potential space defined by the potential vorticity is the quasi-geostrophic transport $J_N (v_{\text{QG},N}^h \cdot \nabla_h \Omega_N)$. The proof of such result is omitted in the present work, but it relies on a careful analysis of the cancellation properties induced by the limit bilinear form \mathcal{Q} .

Applying Corollary 3.3.4 we deduce:

$$\begin{aligned} & (J_N \mathcal{Q}(U_N, U_N))_{\text{osc}} \\ &= (J_N \mathcal{Q}(V_{\text{QG},N}, U_{\text{osc},N}))_{\text{osc}} + (J_N \mathcal{Q}(U_{\text{osc},N}, V_{\text{QG},N}))_{\text{osc}} + (J_N \mathcal{Q}(U_{\text{osc},N}, U_{\text{osc},N}))_{\text{osc}}. \end{aligned}$$

Projecting hence (3.4.1) onto the oscillating subspace and the potential nonoscillating subspace we obtain the following global splitting for the first equation of (3.4.1):

$$\begin{aligned} & \partial_t \Omega_N + J_N (v_{\text{QG},N}^h \cdot \nabla_h \Omega_N) + a_{\text{QG}}(D_h) \Omega_N = 0, \\ & \partial_t U_{\text{osc},N} + (J_N \mathcal{Q}(V_{\text{QG},N}, U_{\text{osc},N}))_{\text{osc}} + (J_N \mathcal{Q}(U_{\text{osc},N}, V_{\text{QG},N}))_{\text{osc}} \\ & \quad + (J_N \mathcal{Q}(U_{\text{osc},N}, U_{\text{osc},N}))_{\text{osc}} + a_{\text{osc}}(D_h) U_{\text{osc},N} = 0. \end{aligned} \tag{3.4.4}$$

The operators a_{QG} and a_{osc} are nothing but the projection of the operator $-\mathbb{D}$ onto the potential space defined by Ω and the oscillating subspace. We avoid to give a detailed description of such operators now (see Section 3.5), what has to be retained is that they are symbols such that there exists a positive constant c such that $|a_{\text{QG}}(n)|, |a_{\text{osc}}(n)| \geq c |\check{n}_h|^2$.

On the splitting (3.4.4) we can apply the same procedure as above to obtain that $\Omega_N \rightarrow \Omega$ in

$L_{\text{loc}}^2(\mathbb{R}_+; H^{-\varepsilon}(\mathbb{T}^3))$, and defining $V_{\text{QG}} = \begin{pmatrix} \nabla_h^\perp \\ 0 \\ -F\partial_3 \end{pmatrix} \Delta_F^{-1} \Omega$ for Ω the limit of the sequence

$(\Omega_N)_N$, and since

$$\begin{pmatrix} \nabla_h^\perp \\ 0 \\ -F\partial_3 \end{pmatrix} \Delta_F^{-1} \in \mathcal{L}(H^\alpha, H^{\alpha+1}), \quad \alpha \in \mathbb{R}$$

we obtain as well that

$$V_{\text{QG},N} \rightarrow V_{\text{QG}} \text{ in } L_{\text{loc}}^2(\mathbb{R}_+; H^{1-\varepsilon}),$$

and $(V_{\text{QG},N})_n$ uniformly (in N) bounded in $L^\infty(\mathbb{R}_+, H^1)$.

Combining the definitions (3.4.2) and (3.4.3) we can hence rewrite $V_{\text{QG},N}$ as

$$V_{\text{QG},N} = \begin{pmatrix} -\partial_2 \\ \partial_1 \\ 0 \\ -F\partial_3 \end{pmatrix} \Delta_F^{-1} \begin{pmatrix} -\partial_2 & \partial_1 & 0 & -F\partial_3 \end{pmatrix} \cdot U_N = \Pi_{\text{QG}} U_N,$$

with Π_{QG} Fourier multiplier of order zero, hence $\Pi_{\text{QG}} \in \mathcal{L}(H^\alpha(\mathbb{T}^3))$ for each $\alpha \in \mathbb{R}$. This implies in particular that, defining $U_{\text{osc}} = U - V_{\text{QG}}$

$$\begin{aligned} \|U_{\text{osc},N} - U_{\text{osc}}\|_{H^{-\varepsilon}} &= \|(U_N - V_{\text{QG},N}) - (U - V_{\text{QG}})\| \\ &= \|(1 - \Pi_{\text{QG}})(U_N - U)\|_{H^{-\varepsilon}} \\ &\leq C \|U_N - U\|_{H^{-\varepsilon}}. \end{aligned}$$

This implies in particular that $U_{\text{osc},N} \rightarrow U_{\text{osc}}$ in $L_{\text{loc}}^2(\mathbb{R}_+; H^{-\varepsilon}(\mathbb{T}^3))$. The same idea can be applied to show that $(U_{\text{osc},N})_N$ is bounded in $L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^3))$ and $(\nabla_h U_{\text{osc},N})_N$ is bounded in $L^2(\mathbb{R}_+; L^2(\mathbb{T}^3))$.

At this point we can project $\mathcal{Q}(U_N, U_N)$ on the spaces $\mathbb{C}e^0, \mathbb{C}e^- \oplus \mathbb{C}e^+$ (see (3.3.5)) obtaining, thanks to the results of Corollary 3.3.2 and 3.3.4:

$$\begin{aligned} \mathcal{Q}(U_N, U_N) &= \mathcal{Q}(U_N, U_N)_{\text{QG}} + \mathcal{Q}(U_N, U_N)_{\text{osc}} \\ &= (-\partial_2, \partial_1, 0, -F\partial_3)^\top \Delta_F^{-1} (v_{\text{QG},N}^h \cdot \nabla_h \Omega_N) \\ &\quad + (\mathcal{Q}(V_{\text{QG},N}, U_{\text{osc},N}))_{\text{osc}} + (\mathcal{Q}(U_{\text{osc},N}, V_{\text{QG},N}))_{\text{osc}} + (\mathcal{Q}(U_{\text{osc},N}, U_{\text{osc},N}))_{\text{osc}}. \end{aligned}$$

It is matter of standard energy bounds with classical product rules in Sobolev spaces to prove that

$$\begin{aligned} (-\partial_2, \partial_1, 0, -F\partial_3)^\top \Delta_F^{-1} (v_{\text{QG},N}^h \cdot \nabla_h \Omega_N) &\rightarrow (-\partial_2, \partial_1, 0, -F\partial_3)^\top \Delta_F^{-1} (v_{\text{QG}}^h \cdot \nabla_h \Omega), \\ (\mathcal{Q}(V_{\text{QG},N}, U_{\text{osc},N}))_{\text{osc}} &\rightarrow (\mathcal{Q}(V_{\text{QG}}, U_{\text{osc}}))_{\text{osc}}, \\ (\mathcal{Q}(U_{\text{osc},N}, V_{\text{QG},N}))_{\text{osc}} &\rightarrow (\mathcal{Q}(U_{\text{osc}}, V_{\text{QG}}))_{\text{osc}}, \end{aligned}$$

in the sense of distributions as $N \rightarrow \infty$. The limit of the product of terms of the form $U_{\text{osc},N}$ is, in general, not well defined. Indeed system (S) lacks of vertical dissipation, hence the best we know is that $U_{\text{osc},N} \rightarrow U_{\text{osc}}$ in $L_{\text{loc}}^2(\mathbb{R}_+; H^{-\varepsilon})$, but generally a product between $H^{-\varepsilon}$ elements is not well-defined. We claim that

$$(\mathcal{Q}(U_{\text{osc},N}, U_{\text{osc},N}))_{\text{osc}} \xrightarrow[n \rightarrow \infty]{\mathcal{D}'(\mathbb{R}_+ \times \mathbb{T}^3)} (\mathcal{Q}(U_{\text{osc}}, U_{\text{osc}}))_{\text{osc}}, \quad (3.4.5)$$

The proof of (3.4.5) is postponed. Whence we finally proved that $\mathcal{Q}(U_N, U_N) \rightarrow \mathcal{Q}(U, U)$ in $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{T}^3)$, concluding. \square

Proof of (3.4.5).

As we already stated M. Paicu in [123] proved a similar result. We shall prove (3.4.5) using different techniques.

Defining $\mathcal{Q}(A, B) = \text{div } \tilde{\mathcal{Q}}(A, B)$, i.e.

$$\begin{aligned} \tilde{\mathcal{Q}}(A, B) &= \sum_{\mathcal{K}} \left(\hat{A}^a(k) \hat{B}^b(m) \Big| e^c(n) \right)_{\mathbb{C}^4} e^c(n), \\ &= \sum_{\mathcal{K}} \hat{A}^a(k) \hat{B}^{b,c}(m, n) \end{aligned}$$

where $\hat{A}^a(k) = \left(\hat{A}(k) \Big| e^a(k) \right) e^a(k)$, $\hat{B}^{b,c}(m, n) = \left(\hat{B}^b(m) \Big| e^c(n) \right) e^c(n)$. It suffices in fact to prove that

$$\tilde{\mathcal{Q}}(U_{\text{osc},j} - U_{\text{osc}}, U_{\text{osc},j} + U_{\text{osc}}) \rightarrow 0,$$

in $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{T}^3)$ as $j \rightarrow \infty$ to conclude. To do so we consider a $\phi \in \mathcal{D}$ and, by Plancherel theorem

$$\begin{aligned} &\int_{\mathbb{R}_+ \times \mathbb{T}^3} \phi(t, x) \tilde{\mathcal{Q}}(U_{\text{osc},j} - U_{\text{osc}}, U_{\text{osc},j} + U_{\text{osc}})(t, x) \, dx dt \\ &= \int_{\mathbb{R}_+} \sum_{n \in \mathbb{Z}^3} \sum_{\mathcal{K}_n^*} \hat{\phi}(t, n) (\widehat{U_{\text{osc},j} - U_{\text{osc}}})^a(t, k) (\widehat{U_{\text{osc},j} + U_{\text{osc}}})^{b,c}(t, m, n) \, dt \\ &= \int_{\mathbb{R}_+} \sum_{n, k_h, m_h} \hat{\phi}_n(t) \sum_{\{k_3: (k, (m_h, n_3 - k_3), n) \in \mathcal{K}^*\}} (\widehat{U_{\text{osc},j} - U_{\text{osc}}})^a(t, k) (\widehat{U_{\text{osc},j} + U_{\text{osc}}})^{b,c}(t, m_h, n_3 - k_3, n) \, dt. \end{aligned} \quad (3.4.6)$$

We make a couple of remarks in order to simplify the notation. Since we considered the eigenvectors as normalized all along the paper the following relations are easy to deduce

$$\left| \widehat{U}^{b,c}(m, n) \right| \lesssim \left| \widehat{U}^b(m) \right| \lesssim \left| \widehat{U}(m) \right|.$$

Hence from now on the terms $(\widehat{U_{\text{osc},j} - U_{\text{osc}}})^a(t, k)$ and $(\widehat{U_{\text{osc},j} + U_{\text{osc}}})^{b,c}(t, m_h, n_3 - k_3, n)$ shall be substituted respectively to $(U_{\text{osc},j} - U_{\text{osc}})_k$ and $(U_{\text{osc},j} + U_{\text{osc}})_{(m_h, n_3 - k_3)}$. Here we chose to make implicit the dependence on the variable t . We want to stress out the fact that this choice is made only to simplify the notation. Indeed we have that

$$\begin{aligned} & (\widehat{U_{\text{osc},j} - U_{\text{osc}}})_k (\widehat{U_{\text{osc},j} + U_{\text{osc}}})_{(m_h, n_3 - k_3)} \\ &= \check{k}^{-\varepsilon/2} (\widehat{U_{\text{osc},j} - U_{\text{osc}}})_k \check{k}^{\varepsilon/2} (\widehat{U_{\text{osc},j} + U_{\text{osc}}})_{(m_h, n_3 - k_3)}. \end{aligned} \quad (3.4.7)$$

The set $\{k_3 : (n, k) \in \mathcal{K}^*\}$ is indeed finite and, in particular, it is composed by the k_3 which satisfy the following equation

$$\begin{aligned} & (F^2 (\check{k}_3)^2 + (\check{k}_h)^2)^{1/2} ((\check{m}_h)^2 + (\check{n}_3 - \check{k}_3)^2)^{1/2} \\ &= ((\check{k}_h)^2 + (\check{k}_3)^2)^{1/2} ((\check{m}_h)^2 + (\check{n}_3 - \check{k}_3)^2)^{1/2} \\ &\quad - ((\check{k}_h)^2 + (\check{k}_3)^2)^{1/2} (F^2 (\check{n}_3 - \check{k}_3)^2 + (\check{m}_h)^2)^{1/2}. \end{aligned}$$

Expanding the above equation and collecting term by term in the powers of k_3 give us the following polynomial equation

$$\wp(\check{k}_3) = \sum_{i=0}^8 A_i (\check{k}_h, \check{m}_h, \check{n}) \check{k}_3^i = 0,$$

where the A_i take the following form

$$\begin{aligned} A_8 &= (1 - 4F^2) \\ A_7 &= 4(-1 + 4F^2)\check{n}_3 \\ A_6 &= -6(F^2\check{k}_h^2 + F^2\check{m}_h^2 + (-1 + 4F^2)\check{n}_3^2) \\ A_5 &= 4\check{n}_3(6F^2\check{k}_h^2 + 3F^2\check{m}_h^2 + (-1 + 4F^2)\check{n}_3^2) \\ A_4 &= -(F^2(-4 + F^2)\check{k}_h^4 + F^2(-4 + F^2)\check{m}_h^4 \\ &\quad - 6F^2\check{m}_h^2\check{n}_3^2 + (1 - 4F^2)\check{n}_3^4 - 2\check{k}_h^2((3 + 2F^2 + F^4)\check{m}_h^2 + 18F^2\check{n}_3^2)) \\ A_3 &= 4\check{k}_h^2\check{n}_3(-F^2(-4 + F^2)\check{k}_h^2 + (3 + 2F^2 + F^4)\check{m}_h^2 + 6F^2\check{n}_3^2) \\ A_2 &= -2\check{k}_h^2((2 + F^2)\check{m}_h^4 + (3 + 2F^2 + F^4)\check{m}_h^2\check{n}_3^2 \\ &\quad + 3F^2\check{n}_3^4 + \check{k}_h^2((2 + F^2)\check{m}_h^2 - 3F^2(-4 + F^2)\check{n}_3^2)) \\ A_1 &= 4\check{k}_h^4\check{n}_3((2 + F^2)\check{m}_h^2 - F^2(-4 + F^2)\check{n}_3^2) \\ A_0 &= -\check{k}_h^4(3\check{m}_h^4 + 2(2 + F^2)\check{m}_h^2\check{n}_3^2 - F^2(-4 + F^2)\check{n}_3^4). \end{aligned}$$

Although we have been giving the explicit expression of the A_i 's we outline the fact that the explicit expression by itself is irrelevant, the only thing that matters is that the A_i 's are polynomials in the variables $\check{k}_h, \check{m}_h, \check{n}$. We can hence apply the following result which bounds the modulus of a root of a complex polynomial in terms of its coefficients, the following proposition is known as Fujiwara near-optimal bound.

Proposition 3.4.2. *Let $P(z) = \sum_{k=0}^n a_n z^k$ a polynomial $P \in \mathbb{C}[z]$, let ζ be one of the n complex roots of P , then*

$$|\zeta| \leq 2 \max \left\{ \left| \frac{a_{n-1}}{a_n} \right|, \left| \frac{a_{n-2}}{a_n} \right|^{1/2}, \dots, \left| \frac{a_1}{a_n} \right|^{1/(n-1)}, \left| \frac{a_0}{a_n} \right|^{1/n} \right\}.$$

We shall omit to prove Proposition 3.4.2 and refer the reader to the work [112] instead. Proposition 3.4.2 applied on $\wp(k_3)$ tells us that

$$|\check{k}_3| \lesssim |\check{n}|^{\alpha_1} |\check{m}_h|^{\alpha_2} |\check{k}_h|^{\alpha_3},$$

where \check{k}_3 is any root of \wp , hence

$$|\check{k}|^{\varepsilon/2} \lesssim |\check{k}_h|^{\varepsilon/2} + (|\check{n}|^{\alpha_1} |\check{m}_h|^{\alpha_2} |\check{k}_h|^{\alpha_3})^{\varepsilon/2}.$$

by concavity on the function $h_\varepsilon(x) = x^{\varepsilon/2}$, with $\alpha_1 + \alpha_2 + \alpha_3 < N$ for some large and finite N . Coming back to (3.4.6) and (3.4.7) this means that

$$\begin{aligned} & \left| \int_{\mathbb{R}_+} \int_{\mathbb{T}^3} \phi(x) \tilde{Q}(U_{\text{osc},j} - U_{\text{osc}}, U_{\text{osc},j} + U_{\text{osc}})(x) dx dt \right| \\ & \lesssim \int_{\mathbb{R}_+} \sum_{n, k_h, m_h} |\hat{\phi}_n| \sum_{\{k_3: (k, (m_h, n_3 - k_3), n) \in \mathcal{K}^*\}} |\check{k}_h|^{\varepsilon/2} |\check{k}|^{-\varepsilon/2} \left| (U_{\text{osc},j} - U_{\text{osc}})_k \right| \\ & \quad \times \left| (U_{\text{osc},j} + U_{\text{osc}})_{(m_h, n_3 - k_3)} \right| dt \\ & + \int_{\mathbb{R}_+} \sum_{n, k_h, m_h} |\hat{\phi}_n| \sum_{\{k_3: (k, (m_h, n_3 - k_3), n) \in \mathcal{K}^*\}} |\check{k}|^{-\varepsilon/2} \left| (U_{\text{osc},j} - U_{\text{osc}})_k \right| \\ & \quad \times \left(|\check{n}|^{\alpha_1} |\check{m}_h|^{\alpha_2} |\check{k}_h|^{\alpha_3} \right)^{\varepsilon/2} \left| (U_{\text{osc},j} + U_{\text{osc}})_{(m_h, n_3 - k_3)} \right| dt \\ & = \int_{\mathbb{R}_+} \sum_{n, k_h, m_h} |\hat{\phi}_n| \sum_{\{k_3: (k, (m_h, n_3 - k_3), n) \in \mathcal{K}^*\}} |\check{k}_h|^{\varepsilon/2} |\check{k}|^{-\varepsilon/2} \left| (U_{\text{osc},j} - U_{\text{osc}})_k \right| \\ & \quad \times \left| (U_{\text{osc},j} + U_{\text{osc}})_{(m_h, n_3 - k_3)} \right| dt \\ & + \int_{\mathbb{R}_+} \sum_{n, k_h, m_h} |n|^{\frac{\alpha_1 \varepsilon}{2}} |\hat{\phi}_n| \sum_{\{k_3: (k, (m_h, n_3 - k_3), n) \in \mathcal{K}^*\}} |\check{k}_h|^{\frac{\alpha_3 \varepsilon}{2}} |\check{k}|^{-\varepsilon/2} \left| (U_{\text{osc},j} - U_{\text{osc}})_k \right| \\ & \quad \times |\check{m}_h|^{\frac{\alpha_2 \varepsilon}{2}} \left| (U_{\text{osc},j} + U_{\text{osc}})_{(m_h, n_3 - k_3)} \right| dt \\ & = I_{1,j} + I_{2,j}. \end{aligned}$$

We prove that $I_{2,j} \rightarrow 0$ as $j \rightarrow \infty$. In order to prove that $I_{1,j} \rightarrow 0$ the procedure is very similar (and actually simpler) to the one we are going to perform now, for this reason is omitted. We start remarking that

$$\begin{aligned} & |\check{k}_h|^{\frac{\alpha_3 \varepsilon}{2}} |\check{k}|^{-\varepsilon/2} \left| (U_{\text{osc},j} - U_{\text{osc}})_k \right| \\ & = \left(|\check{k}|^{-\varepsilon} \left| (U_{\text{osc},j} - U_{\text{osc}})_k \right| \right)^{1/2} \left(|\check{k}_h|^{\alpha_3 \varepsilon} \left| (U_{\text{osc},j} - U_{\text{osc}})_k \right| \right)^{1/2}, \end{aligned}$$

hence

$$\begin{aligned} I_{2,j} & \lesssim \int_{\mathbb{R}_+} \sum_{n, k_h, m_h} |\check{n}|^{\frac{\alpha_1 \varepsilon}{2}} |\hat{\phi}_n| \sum_{\{k_3: (k, (m_h, n_3 - k_3), n) \in \mathcal{K}^*\}} \left(|\check{k}|^{-\varepsilon} \left| (U_{\text{osc},j} - U_{\text{osc}})_k \right| \right)^{1/2} \\ & \quad \left(|\check{k}_h|^{\alpha_3 \varepsilon} \left| (U_{\text{osc},j} - U_{\text{osc}})_k \right| \right)^{1/2} |\check{m}_h|^{\frac{\alpha_2 \varepsilon}{2}} \left| (U_{\text{osc},j} + U_{\text{osc}})_{(m_h, n_3 - k_3)} \right| dt. \quad (3.4.8) \end{aligned}$$

Applying Lemma 3.8.4 we obtain

$$I_{2,j} \lesssim \|\phi\|_{L_{\text{loc}}^\infty(\mathbb{R}_+; H^{\frac{1}{2} + \frac{\alpha_1 \varepsilon}{2}})} \|U_{\text{osc},j} + U_{\text{osc}}\|_{L_{\text{loc}}^2(\mathbb{R}_+; H^{\frac{1}{2} + \frac{\alpha_2 \varepsilon}{2}, 0})} \\ \times \|U_{\text{osc},j} - U_{\text{osc}}\|_{L_{\text{loc}}^2(\mathbb{R}_+; H^{\alpha_3 \varepsilon, 0})}^{1/2} \|U_{\text{osc},j} - U_{\text{osc}}\|_{L_{\text{loc}}^\infty(\mathbb{R}_+; H^{-\varepsilon})}^{1/2}.$$

Both $U_{\text{osc},j}, U_{\text{osc}}$ belong to $L^\infty(\mathbb{R}_+; L^2)$ and $L^2(\mathbb{R}_+; \dot{H}^{1,0})$, and hence to $L_{\text{loc}}^2(\mathbb{R}_+; L^2)$ and by interpolation to $L_{\text{loc}}^2(\mathbb{R}_+; \dot{H}^{\sigma,0})$ for $\sigma \in (0, 1)$. This means that if ε is sufficiently small the quantities $\|U_{\text{osc},j} + U_{\text{osc}}\|_{L_{\text{loc}}^2(\mathbb{R}_+; H^{\frac{1}{2} + \frac{\alpha_2 \varepsilon}{2}, 0})}, \|U_{\text{osc},j} - U_{\text{osc}}\|_{L_{\text{loc}}^2(\mathbb{R}_+; H^{\alpha_3 \varepsilon, 0})}^2$ are bounded, while since

$$\|U_{\text{osc},j} - U_{\text{osc}}\|_{L_{\text{loc}}^\infty(\mathbb{R}_+; H^{-\varepsilon})}^2 \xrightarrow{j \rightarrow \infty} 0,$$

we proved that $I_{2,j} \rightarrow 0$ distributionally. This implies hence that $\mathcal{Q}(U_{\text{osc},j}, U_{\text{osc},j}) \rightarrow \mathcal{Q}(U_{\text{osc}}, U_{\text{osc}})$ in a distributional sense.

3.5 Weak convergence in the weak limit as $\varepsilon \rightarrow 0$.

Introducing the filtered system (FS_ε) allows us to deal with a system of equations which has a closer form to the classical Navier-Stokes system. In particular we can not have any uniform bound, in ε , for the norm $\|\partial_t V^\varepsilon\|_{H^s(\mathbb{T}^3)}$, but this is possible for the system (FS_ε) . We recall that we denoted $|\check{n}|_F = \sqrt{\check{n}_1^2 + \check{n}_2^2 + F^2 \check{n}_3^2}$.

It is natural to ask ourselves if in the limit $\varepsilon \rightarrow 0$ the filtered system (FS_ε) converges to the limit system (S) .

Proposition 3.5.1. *Let $U_0 \in H^{0,s}$ and U^ε be a local strong solution identified by Theorem 3.1.4 of (FS_ε) , then the sequence $(U^\varepsilon)_{\varepsilon > 0}$ has the following regularity uniformly in ε*

$$U^\varepsilon \in L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^3)), \quad \nabla_h U^\varepsilon \in L^2(\mathbb{R}_+; L^2(\mathbb{T}^3)), \quad (3.5.1)$$

and is compact in the space

$$L_{\text{loc}}^2(\mathbb{R}_+; H^{-\eta}(\mathbb{T}^3)),$$

for some $\eta > 0$ (possibly small).

Proof. The proof of (3.5.1) is merely an $L^2(\mathbb{T}^3)$ energy estimate on the filtered system (FS_ε) , hence is omitted.

We prove now that $(\partial_t U^\varepsilon)_\varepsilon$ is bounded, uniformly in ε , in $L_{\text{loc}}^2(\mathbb{R}_+; H^{-N})$ where N is large.

The only thing to prove is to control the bilinear interaction $\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon)$ in the $L_{\text{loc}}^2(\mathbb{R}_+; H^{-N})$ space. Since the propagator $\mathcal{L}(\tau)$ acts as an isometry in any Sobolev space we can safely assert that as far as concerns Sobolev estimates we can identify $\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon)$ with the transport

form $U^\varepsilon \cdot \nabla U^\varepsilon$. Indeed

$$\begin{aligned} U^\varepsilon \cdot \nabla U^\varepsilon &= u^{h,\varepsilon} \cdot \nabla_h U^\varepsilon + u^{3,\varepsilon} \partial_3 U^\varepsilon, \\ &= B_h^\varepsilon + B_v^\varepsilon. \end{aligned}$$

The term B_h^ε is easy to bound by the aid of the uniform estimates (3.5.1). The term B_v^ε is slightly more involved, but the methodology is the same. Let us consider a smooth, compactly supported function ϕ . Integrating by parts and applying Hölder inequality we deduce

$$\begin{aligned} \left| \int_{\mathbb{R}_+ \times \mathbb{T}^3} u^{3,\varepsilon} \partial_3 U^\varepsilon \phi \, dx dt \right| &= \left| \int_{\mathbb{R}_+ \times \mathbb{T}^3} U^\varepsilon (\operatorname{div}_h u^{h,\varepsilon} \phi + u^{3,\varepsilon} \partial_3 \phi) \, dx dt \right|, \\ &\leq \|U^\varepsilon\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^3))} \|\nabla_h U^\varepsilon\|_{L^2(\mathbb{R}_+; L^2(\mathbb{T}^3))} \|\phi\|_{L^2(\mathbb{R}_+; L^\infty(\mathbb{R}^3))} \\ &\quad + \|U^\varepsilon\|_{L^2_{\text{loc}}(\mathbb{R}_+; L^2_v L^4_h)} \|\partial_3 \phi\|_{L^2(\mathbb{R}_+; L^2_v L^{4/3})}. \end{aligned}$$

But indeed

$$\|U^\varepsilon\|_{L^2_{\text{loc}}(\mathbb{R}_+; L^2_v L^4_h)}^2 \leq C \|U^\varepsilon\|_{L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))} \|U^\varepsilon\|_{L^2_{\text{loc}}(\mathbb{R}_+; H^{1,0}(\mathbb{T}^3))}.$$

Indeed (3.5.1) assures us that $U^\varepsilon \in L^2_{\text{loc}}(\mathbb{R}_+; H^{1,0}(\mathbb{T}^3))$ uniformly in ε , whence, by density, we proved that $(\partial_t U^\varepsilon)_\varepsilon$ is bounded, uniformly in ε , in $L^2_{\text{loc}}(\mathbb{R}_+; H^{-N})$ where N is large. It suffices hence to apply Aubin-Lions lemma (see [4]) to deduce the claim. \square

Proposition 3.5.1 asserts hence that (up to subsequences, not relabeled):

$$U^\varepsilon = U + r^\varepsilon,$$

where r^ε is an $L^2_{\text{loc}}(\mathbb{R}_+; H^{-\eta}(\mathbb{T}^3))$ perturbation and U is a non-highly-oscillating state. In what follows we denote as V_{QG} the projection onto the non-oscillating space defined in (3.3.7) of the limit non-highly-oscillating state U , similarly U_{osc} is the projection of U onto the oscillating subspace. The element Ω is indeed defined as $\Omega = -\partial_2 U^1 + \partial_1 U^2 - F \partial_3 U^4$.

First of all we have to make sense of a convergence of the form

$$\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon) \rightarrow \mathcal{Q}(U, U),$$

where U is a weak solution of the limit system (S) of which we can say at best that it belongs to the space

$$U \in L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^3)), \quad \nabla_h U \in L^2(\mathbb{R}_+; L^2(\mathbb{T}^3)), \quad (3.5.2)$$

thanks to Theorem 3.1.10, and $(U^\varepsilon)_{\varepsilon>0}$ a (not relabeled) sequence of local strong solutions of (FS_ε) which satisfy (3.5.1) uniformly in ε and that converge to a limit element U in $L^2_{\text{loc}}(\mathbb{R}_+; H^{-\eta})$ for some $\eta > 0$. In fact in order to define \mathcal{Q} in (3.3.14) we applied the nonstationary phase theorem for smooth function. This is obviously not the case but by mollification we can deduce the same result.

Lemma 3.5.2. *Let $(U^\varepsilon)_{\varepsilon>0}$ a (not relabeled) sequence of local strong solutions of (FS_ε) which satisfy (3.5.1) uniformly in ε and that converges to a limit element U in $L^2_{\text{loc}}(\mathbb{R}_+; H^{-\eta})$ for some $\eta > 0$. Then the following limit holds in the sense of distributions*

$$\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon) \rightarrow \mathcal{Q}(U, U).$$

Proof. Let us define the mollifications

$$U_\alpha^\varepsilon = \mathcal{F}^{-1} \left(1_{\{|n| \leq \frac{1}{\alpha}\}} \hat{U}^\varepsilon \right), \quad U_\alpha = \mathcal{F}^{-1} \left(1_{\{|n| \leq \frac{1}{\alpha}\}} \hat{U} \right).$$

Indeed

$$\begin{aligned} \mathcal{Q}^\varepsilon (U^\varepsilon, U^\varepsilon) - \mathcal{Q} (U, U) &= \mathcal{Q}^\varepsilon (U^\varepsilon, U^\varepsilon) - \mathcal{Q}^\varepsilon (U_\alpha^\varepsilon, U_\alpha^\varepsilon) \\ &\quad + \mathcal{Q}^\varepsilon (U_\alpha^\varepsilon, U_\alpha^\varepsilon) - \mathcal{Q} (U_\alpha, U_\alpha) \\ &\quad + \mathcal{Q} (U_\alpha, U_\alpha) - \mathcal{Q} (U, U), \end{aligned} \quad (3.5.3)$$

and

$$\begin{aligned} \mathcal{Q}^\varepsilon (U^\varepsilon, U^\varepsilon) - \mathcal{Q}^\varepsilon (U_\alpha^\varepsilon, U_\alpha^\varepsilon) &\xrightarrow{\alpha \rightarrow 0} 0, \\ \mathcal{Q} (U_\alpha, U_\alpha) - \mathcal{Q} (U, U) &\xrightarrow{\alpha \rightarrow 0} 0, \end{aligned} \quad (3.5.4)$$

weakly since $U_\alpha^\varepsilon \xrightarrow{\alpha \rightarrow 0} U^\varepsilon$, $U_\alpha \xrightarrow{\alpha \rightarrow 0} U$ in $L_{\text{loc}}^\infty(\mathbb{R}_+; L^2(\mathbb{T}^3))$. As the space domain \mathbb{T}^3 is compact we do not require a passage to subsequences on the parameter α but the convergence holds true for the entire sequence. Next we can say that

$$\begin{aligned} \mathcal{Q}^\varepsilon (U_\alpha^\varepsilon, U_\alpha^\varepsilon) - \mathcal{Q} (U_\alpha, U_\alpha) &= (\mathcal{Q}^\varepsilon (U_\alpha^\varepsilon, U_\alpha^\varepsilon) - \mathcal{Q}^\varepsilon (U_\alpha, U_\alpha)) \\ &\quad + (\mathcal{Q}^\varepsilon (U_\alpha, U_\alpha) - \mathcal{Q} (U_\alpha, U_\alpha)), \end{aligned}$$

and again, for $\alpha > 0$ fixed

$$\mathcal{Q}^\varepsilon (U_\alpha^\varepsilon, U_\alpha^\varepsilon) - \mathcal{Q}^\varepsilon (U_\alpha, U_\alpha) \xrightarrow{\varepsilon \rightarrow 0} 0, \quad (3.5.5)$$

weakly since $U_\alpha^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} U_\alpha$ in $L_{\text{loc}}^2(\mathbb{R}_+; H^{-\eta})$ due to the topological argument performed in Proposition 3.5.1, while finally we can apply the nonstationary phase theorem on $\mathcal{Q}^\varepsilon (U_\alpha, U_\alpha) - \mathcal{Q} (U_\alpha, U_\alpha)$ deducing that

$$\mathcal{Q}^\varepsilon (U_\alpha, U_\alpha) - \mathcal{Q} (U_\alpha, U_\alpha) \xrightarrow{\varepsilon \rightarrow 0} 0, \quad (3.5.6)$$

in the sense of distributions for $\alpha > 0$ fixed. Whence (3.5.3)–(3.5.6) imply that, fixed a (possibly small) positive $\alpha > 0$, considering a $\phi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{T}^3)$, there exists a $c_\alpha = c_\alpha(\phi) > 0$ such that $c_\alpha \rightarrow 0$ as $\alpha \rightarrow 0$ and such that

$$\lim_{\varepsilon \rightarrow 0} \left| \int_{\mathbb{R}_+ \times \mathbb{T}^3} (\mathcal{Q}^\varepsilon (U^\varepsilon, U^\varepsilon) - \mathcal{Q} (U, U)) \cdot \phi \, dx \, dt \right| \leq c_\alpha. \quad (3.5.7)$$

The left-hand side of (3.5.7) is indeed independent from the parameter α , whence

$$\lim_{\varepsilon \rightarrow 0} \left| \int_{\mathbb{R}_+ \times \mathbb{T}^3} (\mathcal{Q}^\varepsilon (U^\varepsilon, U^\varepsilon) - \mathcal{Q} (U, U)) \cdot \phi \, dx \, dt \right| \leq \lim_{\alpha \rightarrow 0} c_\alpha = 0.$$

□

We underline the fact that the following calculations are an adaptation of the ones present in the work [72] to the case of anisotropic viscosity. For this reason many calculations shall not be carried out in detail, or we shall directly refer to the work [72] and references therein.

Once the convergence for the bilinear interactions is formalized we focus to understand how the global splitting introduced in Section 3.3.1 can be applied on bilinear interactions of elements which are not smooth.

P. Embid and A. Majda proved the following lemma in [62]:

Lemma 3.5.3. $\mathcal{F}^{-1} \left((\mathcal{F}\mathcal{Q}^\varepsilon (U^\varepsilon, U^\varepsilon) \|\check{n}\|_F e^0(n))_{\mathbb{C}^4} \right) \xrightarrow{\varepsilon \rightarrow 0} v_{\text{QG}} \cdot \nabla \Omega$. The limit holds in the sense of distributions.

Proof. Let us compute

$$\begin{aligned} & \mathcal{F}^{-1} \left((\mathcal{F}\mathcal{Q}^\varepsilon (U^\varepsilon, U^\varepsilon) \|\check{n}\|_F e^0(n))_{\mathbb{C}^4} \right) - v_{\text{QG}} \cdot \nabla \Omega \\ &= \mathcal{F}^{-1} \left((\mathcal{F}\mathcal{Q}^\varepsilon (U^\varepsilon, U^\varepsilon) \|\check{n}\|_F e^0(n))_{\mathbb{C}^4} \right) - \mathcal{F}^{-1} \left((\mathcal{F}\mathcal{Q}^\varepsilon (U_\alpha^\varepsilon, U_\alpha^\varepsilon) \|\check{n}\|_F e^0(n))_{\mathbb{C}^4} \right) \\ & \quad + \mathcal{F}^{-1} \left((\mathcal{F}\mathcal{Q}^\varepsilon (U_\alpha^\varepsilon, U_\alpha^\varepsilon) \|\check{n}\|_F e^0(n))_{\mathbb{C}^4} \right) - v_{\text{QG},\alpha} \cdot \nabla \Omega_\alpha \\ & \quad + v_{\text{QG},\alpha} \cdot \nabla \Omega_\alpha - v_{\text{QG}} \cdot \nabla \Omega. \end{aligned}$$

The element

$$\begin{aligned} & \mathcal{F}^{-1} \left((\mathcal{F}\mathcal{Q}^\varepsilon (U^\varepsilon, U^\varepsilon) \|\check{n}\|_F e^0(n))_{\mathbb{C}^4} \right) - \mathcal{F}^{-1} \left((\mathcal{F}\mathcal{Q}^\varepsilon (U_\alpha^\varepsilon, U_\alpha^\varepsilon) \|\check{n}\|_F e^0(n))_{\mathbb{C}^4} \right) \\ & \quad \xrightarrow[\alpha \rightarrow 0]{\mathcal{D}'(\mathbb{R}_+ \times \mathbb{T}^3)} 0, \end{aligned}$$

since $U_\alpha^\varepsilon \xrightarrow{\alpha \rightarrow 0} U^\varepsilon$ in $L_{\text{loc}}^\infty(\mathbb{R}_+; L^2)$. Next applying the nonstationary phase theorem and Lemma 3.3.1 we can say that

$$\mathcal{F}^{-1} \left((\mathcal{F}\mathcal{Q}^\varepsilon (U_\alpha^\varepsilon, U_\alpha^\varepsilon) \|\check{n}\|_F e^0(n))_{\mathbb{C}^4} \right) - v_{\text{QG},\alpha} \cdot \nabla \Omega_\alpha \rightarrow 0,$$

as $\varepsilon \rightarrow 0$ in the sense of distributions. Lastly again we can argue as above in order to state that

$$v_{\text{QG},\alpha} \cdot \nabla \Omega_\alpha - v_{\text{QG}} \cdot \nabla \Omega \xrightarrow[\alpha \rightarrow 0]{\mathcal{D}'(\mathbb{R}_+ \times \mathbb{T}^3)} 0,$$

since $v_{\text{QG},\alpha} \rightarrow v_{\text{QG}}$ and $\Omega_\alpha \rightarrow \Omega$ in $L_{\text{loc}}^\infty(\mathbb{R}_+; L^2)$, concluding. \square

We want to understand which are the projections of $\mathbb{D}^\varepsilon U$ on the oscillatory and non oscillatory space as $\varepsilon \rightarrow 0$. This is easily done if we consider the formulation of the limit form as it is given in (3.3.15). Let us consider the projection of the limit linear form onto the potential space defined by $\Omega = \mathcal{F}^{-1} \left((\mathcal{F}U \|\check{n}\|_F e^0)_{\mathbb{C}^4} \right)$,

$$(\mathcal{F}\mathbb{D}U \|\check{n}\|_F e^0)_{\mathbb{C}^4} = \sum_{\omega_n^{a,b}=0} (\mathbf{D}(n)U^b(n) | e^a(n))_{\mathbb{C}^4} (e^a(n) | \|\check{n}\|_F e^0)_{\mathbb{C}^4}.$$

As it has been pointed out above $e^0 \perp e^\pm$, hence $a = 0$. On the other hand if we consider the limit set $\omega_n^{a,b} = 0$ with the fact that $a = 0$ we easily obtain that $\omega^b(n) \equiv 0$, whence $b = 0$ as well, hence we obtained that

$$\begin{aligned} & \left(-\mathcal{L} \left(-\frac{t}{\varepsilon} \right) \mathbf{D}\mathcal{L} \left(\frac{t}{\varepsilon} \right) U^\varepsilon \left| \mathcal{F}^{-1} (\|\check{n}\|_F e^0) \right. \right)_{\mathbb{C}^4} \\ & \quad \xrightarrow{\varepsilon \rightarrow 0} a_{\text{QG}}(D_h) \Omega \\ & \quad = \mathcal{F}^{-1} \left(\frac{\nu (\check{n}_1^2 + \check{n}_2^2) + \nu' F^2 \check{n}_3^2}{\check{n}_1^2 + \check{n}_2^2 + F^2 \check{n}_3^2} (\check{n}_1^2 + \check{n}_2^2) \widehat{\Omega}_n \right). \end{aligned}$$

In the same way, defining $U^a = (\mathcal{F}U|e^a)e^a$

$$\begin{aligned} -\lim_{\varepsilon \rightarrow 0} \mathcal{L} \left(-\frac{t}{\varepsilon} \right) \mathbf{D}\mathcal{L} \left(\frac{t}{\varepsilon} \right) U_{\text{osc}}^\varepsilon &= a_{\text{osc}} (D_h) U_{\text{osc}}^\varepsilon \\ &= \mathcal{F}^{-1} \left(\sum_{\substack{\omega_n^{a,b}=0 \\ a,b=\pm}} (\mathbf{D}(n)U^b(n)|e^a(n))_{\mathbb{C}^4} e^a(n) \right). \end{aligned}$$

We want now to understand which form assumes the limit as $\varepsilon \rightarrow 0$ of the projection of $\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon)$ onto the oscillatory subspace $\mathbb{C}e^- \oplus \mathbb{C}e^+$. In particular the following result holds true:

Lemma 3.5.4. *For every three-dimensional torus \mathbb{T}^3 we have*

$$\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon)_{\text{osc}} \xrightarrow{\varepsilon \rightarrow 0} (\mathcal{Q}(V_{\text{QG}}, U_{\text{osc}}))_{\text{osc}} + (\mathcal{Q}(U_{\text{osc}}, V_{\text{QG}}))_{\text{osc}} + (\mathcal{Q}(U_{\text{osc}}, U_{\text{osc}}))_{\text{osc}}. \quad (3.5.8)$$

Proof. We avoid to give a detailed proof of such result since the proof is very similar to the one performed in Lemma 3.5.3 but using Corollary 3.3.4 instead of Lemma 3.3.1. \square

The above lemmas hence states that in the limit $\varepsilon \rightarrow 0$ there is no bilinear interaction of kernel elements in the equation describing the evolution of U_{osc} .

Whence the filtered system $(\mathbf{FS}_\varepsilon)$ can be described, as $\varepsilon \rightarrow 0$, thanks to the following two systems:

$$\begin{cases} \partial_t \Omega + v_{\text{QG}}^h \cdot \nabla_h \Omega + a_{\text{QG}}(D_h) \Omega = 0 \\ \text{div}_h v_{\text{QG}}^h = \text{div} v_{\text{QG}} = 0 \\ \Omega|_{t=0} = \Omega_0 \end{cases} \quad (3.5.9)$$

$$\begin{cases} \partial_t U_{\text{osc}} + (\mathcal{Q}(V_{\text{QG}}, U_{\text{osc}}))_{\text{osc}} + (\mathcal{Q}(U_{\text{osc}}, V_{\text{QG}}))_{\text{osc}} \\ \quad + (\mathcal{Q}(U_{\text{osc}}, U_{\text{osc}}))_{\text{osc}} + a_{\text{osc}}(D_h) U_{\text{osc}} = 0 \\ \text{div} u_{\text{osc}} = 0 \\ U_{\text{osc}}|_{t=0} = U_{\text{osc},0} = (V_0)_{\text{osc}}. \end{cases} \quad (3.5.10)$$

System (3.5.9) represents the projection of the limit system onto the non-oscillatory potential subspace defined by Ω , and (3.5.10) represents the projection onto $\mathbb{C}e^- \oplus \mathbb{C}e^+$.

It is easy to deduce from (3.5.9) that if $V_{\text{QG}} = \begin{pmatrix} \nabla_h^\perp \\ 0 \\ -F\partial_3 \end{pmatrix} \Delta_F^{-1} \Omega$ then

$$\begin{cases} \partial_t V_{\text{QG}} + a_{\text{QG}}(D_h) V_{\text{QG}} = - \begin{pmatrix} \nabla_h^\perp \\ 0 \\ -\partial_3 F \end{pmatrix} \Delta_F^{-1} (v_{\text{QG}}^h \cdot \nabla_h \Omega), \\ \text{div}_h v_{\text{QG}}^h = \text{div} v_{\text{QG}} = 0 \\ V_{\text{QG}}|_{t=0} = V_{\text{QG},0} = (\nabla_h^\perp, 0, -F\partial_3)^\top \Delta_F^{-1} \Omega_0. \end{cases} \quad (3.5.11)$$

We remark that in (3.5.10) the term $\mathcal{Q}(U_{\text{osc}}, U_{\text{osc}})$ represents a bilinear interaction between highly oscillating modes, i.e. we are taking into account some potentially resonant effect such as in [123].

The following lemma gives a connection in terms of regularity between the solutions of (3.5.9) and (3.5.11), and will result to be extremely useful in the energy estimates for the global well posedness of the limit system.

Lemma 3.5.5. *Let $\Lambda_h^s \Lambda_v^{s'} \Omega \in L^2(\mathbb{T}^3)$, with $V_{\text{QG}} = \begin{pmatrix} \nabla_h^\perp \\ 0 \\ -F\partial_3 \end{pmatrix} \Delta_F^{-1} \Omega$. Let $\sigma \in [0, 1]$, then there exists a uniformly finite (in σ) constant C_σ depending only on σ such that*

$$\left\| \Lambda_h^{s+\sigma} \Lambda_v^{s'+(1-\sigma)} v_{\text{QG}} \right\|_{L^2(\mathbb{T}^3)} \leq C_\sigma \left\| \Lambda_h^s \Lambda_v^{s'} \Omega \right\|_{L^2(\mathbb{T}^3)}.$$

Propagation of the horizontal average.

In the following lemmas we identify some conditions which suffices to guarantee that the horizontal average of $U = U_{\text{QG}} + U_{\text{osc}}$ solution of the limit system (3.5.10)-(3.5.11) is preserved for each time $t > 0$. This turns out to be very important since we are dealing with periodic functions, hence, generally we cannot use inequalities such as the one stated in (3.2.8) or Corollary 3.2.5 unless the horizontal mean of the function considered is zero. It is in this setting that the condition (P) shall play a fundamental role.

Lemma 3.5.6. *Let V_{QG} the solution of (3.5.11), if we define*

$$\underline{V_{\text{QG}}}(t, x_3) = \frac{1}{|\mathbb{T}_h^2|} \int_{\mathbb{T}_h^2} V_{\text{QG}}(t, y_h, x_3) \, dy_h,$$

then

$$\partial_t \underline{V_{\text{QG}}}(t, x_3) = 0.$$

Proof. It suffices to remark that

$$(-\partial_2, \partial_1, 0, -F\partial_3)^\top \Delta_F^{-1} (v_{\text{QG}}^h \cdot \nabla_h \Omega) = (-\partial_2, \partial_1, 0, -F\partial_3)^\top \Delta_F^{-1} \text{div}_h (v_{\text{QG}}^h \Omega).$$

□

Lemma 3.5.7. *Suppose that the limit system (3.5.9)–(3.5.11) is well posed. Then, setting $U = V_{\text{QG}} + U_{\text{osc}}$*

$$\partial_t \int_{\mathbb{T}_h^2} U(t, x_h, x_3) \, dx_h = 0,$$

for almost every torus $\mathbb{T} \subset \mathbb{R}^3$, and $F \neq 1$.

Proof. Taking in consideration the oscillatory part described by equation (3.5.10) it suffices to prove that

$$\int_{\mathbb{T}_h^2} \mathcal{Q}(V_{\text{QG}}, U_{\text{osc}}) \, dx_h = \int_{\mathbb{T}_h^2} \mathcal{Q}(U_{\text{osc}}, V_{\text{QG}}) \, dx_h = \int_{\mathbb{T}_h^2} \mathcal{Q}(U_{\text{osc}}, U_{\text{osc}}) \, dx_h = 0,$$

we consider at first the term $\int_{\mathbb{T}_h^2} \mathcal{Q}(V_{\text{QG}}, U_{\text{osc}}) dx_h$. To do so we consider

$$\mathcal{F} \int_{\mathbb{T}_h^2} \mathcal{Q}(V_{\text{QG}}, U_{\text{osc}}) dx_h = \sum_{\substack{\omega_{k,m,(0,n_3)}^{0,b,c} = 0 \\ b,c = \pm \\ k+m=(0,n_3)}} (n_3 \hat{v}_{\text{QG}}^3(k)) \hat{U}_{\text{osc}}(m).$$

If we look what the term $\hat{v}_{\text{QG}}^3(k)$ is we can easily deduce that $\hat{v}_{\text{QG}}^3(k) = \hat{V}_{\text{QG}}(k) \cdot e^0(k) e^{0,3}(k)$, where e^0 is defined in (3.3.4) and $e^{0,3}$ is the third component of e^0 . Looking at (3.3.4) we immediately notice that $e^{0,3} \equiv 0$, and hence the above value is null.

Next we consider the following term

$$\mathcal{F} \int_{\mathbb{T}_h^2} \mathcal{Q}(U_{\text{osc}}, V_{\text{QG}}) dx_h = \sum_{\substack{\omega_{k,m,(0,n_3)}^{a,0,c} = 0 \\ a,c = \pm \\ k+m=(0,n_3)}} \left(\left((\check{n}_3 \hat{u}_{\text{osc}}^3(k)) \hat{V}_{\text{QG}}(m) \right) \Big| e^c(n) \right)_{\mathbb{C}^4} e^c(n), \quad (3.5.12)$$

to show that the above quantity is zero we have to study the summation set. Recall that the eigenvalues are given by formula (3.3.4), the right hand side of the above equation has been evaluated explicitly thanks to the explicit formulation of the bilinear form \mathcal{Q} . The formulation of the summation set turns out to be quite simple thanks to the relation $n_h \equiv 0$, writing down in fact explicitly the relation $\omega_{k,(0,n_3)-k,(0,n_3)}^{a,0,c} = 0$ ¹ we deduce that we are considering the following modes:

$$\mathcal{K}_{\pm} = \{ k \in \mathbb{Z}^3 \mid \omega^{\pm}(k) = 1 \}.$$

The equation $\omega^{\pm}(k) = 1$ characterizing \mathcal{K}_{\pm} reads as

$$\frac{(F^2 \check{k}_3^2 + |\check{k}_h|^2)^{1/2}}{|\check{k}|} = \pm F,$$

which is equivalent to

$$(F^2 - 1) |\check{k}_h|^2 = 0.$$

It is trivial that this relation is satisfied only if $k_h \equiv 0$, but let us consider now in detail what the element $\hat{u}_{\text{osc}}^3(k)|_{k=(0,k_3)}$ appearing in (3.5.12) is. By definition $\hat{u}_{\text{osc}}^3(k) = (\mathcal{F}U(k)|e^{\pm,3}(k)) e^{\pm,3}(k)$, where $e^{\pm,3}(k)$ is the third component of the oscillating eigenvectors defined in (3.3.6), i.e. $e^{\pm,3}(k) \equiv 0$. Whence $\hat{u}_{\text{osc}}^3(0, k_3) \equiv 0$ and this implies that the contribution in (3.5.12) is zero.

Next we shall deal with the more complex term, namely the term

$$\int_{\mathbb{T}_h^2} (\mathcal{Q}(U_{\text{osc}}, U_{\text{osc}}))_{\text{osc}} dx_h,$$

being the deduction for the other ones appearing a matter of straightforward computations. In this term there are present interactions between perturbations which do not live in the

¹ $(0, n_3) - k = m$ and we recover the same summation set as in (3.5.12).

kernel of the penalized operator. In this context the resonance set defined in Definition 3.1.6 shall play a fundamental role. Let us consider the explicit expression of the above term

$$\begin{aligned} & \int_{\mathbb{T}_h^2} (\mathcal{Q}(U_{\text{osc}}, U_{\text{osc}}))_{\text{osc}} \, dx_h \\ &= \mathcal{F}^{-1} \left(\sum_{\mathcal{K}_{(0, \tilde{n}_3)}^*} \left(\sum_{j=1,2,3} U^{a,j}(k) m_j U^b(m) \Big|_{\mathbb{C}^4} e^c(0, n_3) \right) e^c(0, n_3) \right). \end{aligned}$$

We prove that the above quantity is zero by proving that $\mathcal{K}_{(0, \tilde{n}_3)}^* = \emptyset$. Since $\tilde{n}_h = 0$ and we have the convolution constraint $\check{k} + \check{m} = \check{n}$ we immediately understand $\check{k}_h + \check{m}_h = 0$, i.e. $|\check{k}_h| = |\check{m}_h| = \lambda$. Writing down the resonant equation we obtain the following equality

$$\frac{(F^2 \check{k}_3^2 + \lambda^2)^{1/2}}{(\lambda^2 + \check{k}_3^2)^{1/2}} \pm \frac{(F^2 \check{m}_3^2 + \lambda^2)^{1/2}}{(\lambda^2 + \check{m}_3^2)^{1/2}} = \pm 1.$$

Taking square (twice) and after some algebraic manipulation we obtain that the above equation is equivalent to

$$(\lambda^4 + F^2 \lambda^2 \check{m}_3^2 + \check{k}_3^2 (-(-2 + F^2) \lambda^2 + \check{m}_3^2))^2 = 4 (\lambda^2 + \check{k}_3^2)^2 (\lambda^2 + \check{m}_3^2) (\lambda^2 + F^2 \check{m}_3^2).$$

We multiply the above equation for a_3^8 , obtaining the new equality in the unknown $\mu^2 = \lambda^2 a_3^2$

$$\begin{aligned} & (\mu^4 + F^2 \mu^2 m_3^2 + k_3^2 (-(-2 + F^2) \mu^2 + m_3^2))^2 \\ &= 4 (\mu^2 + k_3^2)^2 (\mu^2 + m_3^2) (\mu^2 + F^2 m_3^2), \quad (3.5.13) \end{aligned}$$

and

$$\mu^2 = \lambda^2 a_3^2 = \left(\frac{a_3}{a_1} \right)^2 k_1^2 + \left(\frac{a_3}{a_2} \right)^2 k_2^2 = \mu_1 k_1^2 + \mu_2 k_2^2.$$

Since the torus satisfies the Condition (\mathcal{P}) we know that $F = r_1/r_2 \in \mathbb{Q}$, hence we can transform the expression in (3.5.13) into an equation of the form $P(\mu) = 0$, with $P \in \mathbb{Z}[\mu]$. Whence by the definition of Condition (\mathcal{P}) given in Definition 3.1.8 we argue that

- If $\mu_1 = a_3^2/a_1^2 \in \mathbb{Q}$ the (3.5.13) can be rewritten as $\tilde{P}(\mu_2) = 0$ where $\deg \tilde{P} = 4$, hence by hypothesis in Definition 3.1.8 we have that μ_2 is not algebraic of degree smaller or equal than four, this implies that the equation $\tilde{P}(\mu_2) = 0$ has no solution, concluding.
- If $\mu_2 = a_3^2/a_2^2 \in \mathbb{Q}$ the procedure is the same as above, but symmetric (see Definition 3.1.8).

□

We have hence identified some conditions under such we can say that the horizontal mean of the limit function $U = \lim_{\varepsilon \rightarrow 0} \mathcal{L} \left(-\frac{t}{\varepsilon} \right) V^\varepsilon$ is preserved. Hence if we consider initial data with zero horizontal average we can use freely (3.2.8) and moreover the following Poincaré inequality $\|U\|_{L^p(\mathbb{T}^3)} \leq C \|\nabla_h U\|_{L^p(\mathbb{T}^3)}$, holds.

3.6 Propagation of $H^{0,s}$ regularity.

3.6.1 The quasi-geostrophic part.

Section 3.4 ensures us that there exists a solution U for the limit system (S) which is

$$U \in L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^3)) \quad \nabla_h U \in L^2(\mathbb{R}_+; L^2(\mathbb{T}^3)).$$

The scope of the present and following section though is to prove if, under suitable initial conditions, the equations (3.5.9) and (3.5.10) propagate $H^{0,s}$ regularity.

Proposition 3.6.1. *Let Ω be a solution of (3.5.9). Then if $\Omega_0 \in L^2(\mathbb{T}^3)$, $\Omega \in L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^3))$, $\nabla_h \Omega \in L^2(\mathbb{R}_+; L^2(\mathbb{T}^3))$, and in particular for each $t > 0$ the following bound holds true*

$$\|\Omega(t)\|_{L^2(\mathbb{T}^3)}^2 + 2c \int_0^t \|\nabla_h \Omega(\tau)\|_{L^2(\mathbb{T}^3)}^2 d\tau \leq C \|\Omega_0\|_{L^2(\mathbb{T}^3)}^2.$$

This is a standard L^2 energy estimate on the parabolic equation (3.5.9) which has been already proved in Theorem 3.1.10.

Proposition 3.6.2. *Let Ω be the solution of (3.5.9) and let $\Omega_0 \in H^{0,s}$ for some $s > 0$. Then for all $t \in \mathbb{R}$ we have that $\Omega \in C(\mathbb{R}_+; H^{0,s})$ and $\nabla_h \Omega \in L^2(\mathbb{R}_+; H^{0,s})$, and in particular the following estimates hold:*

$$\begin{aligned} \|\Omega(t)\|_{H^{0,s}}^2 + c \int_0^t \|\nabla_h \Omega(\tau)\|_{H^{0,s}}^2 d\tau \\ \leq C \|\Omega_0\|_{H^{0,s}}^2 \exp \left\{ \frac{2C}{c} \left(1 + \|\Omega_0\|_{L^2(\mathbb{T}^3)}^2 \right) \|\Omega_0\|_{L^2(\mathbb{T}^3)}^2 \right\} \end{aligned} \quad (3.6.1)$$

Proof. Applying the vertical truncation Δ_q^v on both sides of equation (3.5.9), multiplying both sides for $\Delta_q^v \Omega$ and taking the scalar product in $L^2(\mathbb{T}^3)$ we obtain

$$\frac{1}{2} \frac{d}{dt} \|\Delta_q^v \Omega\|_{L^2(\mathbb{T}^3)}^2 + c \|\Delta_q^v \nabla_h \Omega\|_{L^2(\mathbb{T}^3)}^2 \leq \left| (\Delta_q^v (v_{\text{QG}}^h \cdot \nabla_h \Omega) | \Delta_q^v \Omega)_{L^2(\mathbb{T}^3)} \right|.$$

By use of Cauchy-Schwartz inequality and (3.8.1) we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta_q^v \Omega\|_{L^2(\mathbb{T}^3)}^2 + c \|\Delta_q^v \nabla_h \Omega\|_{L^2(\mathbb{T}^3)}^2 \\ \leq C 2^{-2qs} b_q(t) \left[\|\Omega\|_{L^2(\mathbb{T}^3)}^{1/2} \|\nabla_h \Omega\|_{L^2(\mathbb{T}^3)}^{1/2} \|\Omega\|_{H^{0,s}}^{1/2} \|\nabla_h \Omega\|_{H^{0,s}}^{3/2} \right. \\ \left. + \|\nabla_h \Omega\|_{L^2(\mathbb{T}^3)} \|\Omega\|_{H^{0,s}} \|\nabla_h \Omega\|_{H^{0,s}} \right] \end{aligned} \quad (3.6.2)$$

We recall that in (3.6.2) $(b_q)_q$ is a $\ell^1(\mathbb{Z})$ positive sequence which depends on Ω and such that $\sum_q b_q(t) \leq 1$. Multiplying equation (3.6.2) on both sides for 2^{2qs} , summing on $q \in \mathbb{Z}$ and using the convexity inequalities $2ab \leq a^2 + b^2$ and $ab \leq \frac{1}{4}a^4 + \frac{3}{4}b^{4/3}$ we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Omega\|_{H^{0,s}}^2 + c \|\nabla_h \Omega\|_{H^{0,s}}^2 \\ \leq \frac{c}{2} \|\nabla_h \Omega\|_{H^{0,s}}^2 + C \left(\left(1 + \|\Omega\|_{L^2(\mathbb{T}^3)}^2 \right) \|\nabla_h \Omega\|_{L^2(\mathbb{T}^3)}^2 \right) \|\Omega\|_{H^{0,s}}^2 \end{aligned} \quad (3.6.3)$$

whence, applying Gronwall inequality to (3.6.3) in $[0, t]$ we get the bound

$$\begin{aligned} \|\Omega(t)\|_{H^{0,s}}^2 + c \int_0^t \|\nabla_h \Omega(\tau)\|_{H^{0,s}}^2 d\tau \\ \leq C \|\Omega_0\|_{H^{0,s}}^2 \exp \left\{ 2C \int_0^t \left(1 + \|\Omega(s)\|_{L^2(\mathbb{T}^3)}^2 \right) \|\nabla_h \Omega(s)\|_{L^2(\mathbb{T}^3)}^2 ds \right\}. \end{aligned}$$

Hence, considering that Ω is bounded in $L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^3))$ and $\nabla_h \Omega$ is bounded in $L^2(\mathbb{R}_+; L^2(\mathbb{T}^3))$ we deduce the estimate (3.6.1). \square

Remark 3.6.3. In Proposition 3.6.2 we do not require the initial data to be of zero horizontal average in order to propagate $H^{0,s}$ norms. \blacklozenge

3.6.2 The oscillatory part.

We can now turn our attention on the oscillatory part U_{osc} solution of the equation (3.5.10). Indeed the terms $\mathcal{Q}(V_{\text{QG}}, U_{\text{osc}})$ and $\mathcal{Q}(U_{\text{osc}}, V_{\text{QG}})$ present in (3.5.10) should not present a problem in the propagation of regularity, being linear in U_{osc} . The term $\mathcal{Q}(U_{\text{osc}}, U_{\text{osc}})$ though is a bilinear term of the form $\sqrt{-\Delta}(U_{\text{osc}} \otimes U_{\text{osc}})$. Fortunately as pointed out in Lemma 3.8.4 the bilinear form \mathcal{Q} has better product rules than the standard Navier-Stokes bilinear form, this will allow us to recover the global well posedness result for (3.5.10) as well.

Lemma 3.6.4. *Let U be the weak solution defined in Theorem 3.1.10, then $U_{\text{osc}} = U - V_{\text{QG}}$ satisfies the energy bound*

$$\|U_{\text{osc}}(t)\|_{L^2(\mathbb{T}^3)}^2 + c \int_0^t \|\nabla_h U_{\text{osc}}(\tau)\|_{L^2(\mathbb{T}^3)}^2 d\tau \leq C \|U_0\|_{L^2(\mathbb{T}^3)}^2.$$

Proof. The proof stems from the fact that $U_{\text{osc}} = \Pi_{\text{osc}} U$ where $\Pi_{\text{osc}} = 1 - \Pi_{\text{QG}}$ is a pseudo-differential operator of order zero as it has been explained in the proof of Theorem 3.1.10. \square

Proposition 3.6.5. *Let U_{osc} be the solution of (3.5.10) and $V_{\text{QG},0}, U_{\text{osc},0} = 0$. Let \mathbb{T}^3 satisfy the condition ((P)) and $U_{\text{osc},0}, \Omega_0 \in H^{0,s}$ for $s > 1/2$, then $U_{\text{osc}} \in \mathcal{C}(\mathbb{R}_+; H^{0,s})$ and $\nabla_h U_{\text{osc}} \in L^2(\mathbb{R}_+; H^{0,s})$ and the following bound holds*

$$\begin{aligned} \|U_{\text{osc}}(t)\|_{H^{0,s}}^2 + c \int_0^t \|U_{\text{osc}}(\tau)\|_{H^{0,s}}^2 d\tau \\ \leq C \|U_{\text{osc},0}\|_{H^{0,s}}^2 \exp \left\{ \frac{2C}{c} \left[\|\Omega_0\|_{H^{0,s}} \exp \left\{ \frac{2C}{c} \left(1 + \|\Omega_0\|_{L^2(\mathbb{T}^3)}^2 \right) \|\Omega_0\|_{L^2(\mathbb{T}^3)}^2 \right\} \right. \right. \\ \left. \left. + \left(1 + \|U_0\|_{L^2(\mathbb{T}^3)}^2 \right) \|U_0\|_{L^2(\mathbb{T}^3)}^2 \right] \right\}. \end{aligned}$$

Proof. As in the proof of Proposition 3.6.2 apply the vertical truncation Δ_q^v on both sides of (3.5.10) and taking scalar product in $L^2(\mathbb{T}^3)$ we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta_q^v U_{\text{osc}}\|_{L^2(\mathbb{T}^3)}^2 + c \|\Delta_q^v \nabla_h U_{\text{osc}}\|_{L^2(\mathbb{T}^3)}^2 \leq |(\Delta_q^v \mathcal{Q}(V_{\text{QG}}, U_{\text{osc}}) | \Delta_q^v U_{\text{osc}})| \\ + |(\Delta_q^v \mathcal{Q}(U_{\text{osc}}, V_{\text{QG}}) | \Delta_q^v U_{\text{osc}})| + |(\Delta_q^v \mathcal{Q}(U_{\text{osc}}, U_{\text{osc}}) | \Delta_q^v U_{\text{osc}})|. \end{aligned}$$

Taking moreover in account the estimates (3.8.8) and (3.8.9) the above inequality turns into

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \left\| \Delta_q^v U_{\text{osc}} \right\|_{L^2(\mathbb{T}^3)}^2 + c \left\| \Delta_q^v \nabla_h U_{\text{osc}} \right\|_{L^2(\mathbb{T}^3)}^2 \\
 & \leq C b_q(t) 2^{-2qs} \left\| \nabla_h \Omega \right\|_{H^{0,s}} \left\| \nabla_h U_{\text{osc}} \right\|_{H^{0,s}} \left\| U_{\text{osc}} \right\|_{H^{0,s}} \\
 & \quad + C b_q(t) 2^{-2qs} \left\| \Omega \right\|_{H^{0,s}}^{1/2} \left\| \nabla_h \Omega \right\|_{H^{0,s}}^{1/2} \left\| U_{\text{osc}} \right\|_{H^{0,s}}^{1/2} \left\| \nabla_h U_{\text{osc}} \right\|_{H^{0,s}}^{3/2} \\
 & \quad + C b_q(t) 2^{-2qs} \left\| \nabla_h U_{\text{osc}} \right\|_{L^2(\mathbb{T}^3)} \left\| U_{\text{osc}} \right\|_{H^{0,s}} \left\| \nabla_h U_{\text{osc}} \right\|_{H^{0,s}} \\
 & \quad + C b_q(t) 2^{-2qs} \left\| U_{\text{osc}} \right\|_{L^2(\mathbb{T}^3)}^{1/2} \left\| \nabla_h U_{\text{osc}} \right\|_{L^2(\mathbb{T}^3)}^{1/2} \left\| U_{\text{osc}} \right\|_{H^{0,s}}^{1/2} \left\| \nabla_h U_{\text{osc}} \right\|_{H^{0,s}}^{3/2}.
 \end{aligned} \tag{3.6.4}$$

We recall that $(b_q)_q$ is a $\ell^1(\mathbb{Z})$ positive sequence which depends on Ω and U_{osc} and such that $\sum_q b_q(t) \leq 1$. Multiplying both sides of (3.6.4) for 2^{2qs} , summing over $q \in \mathbb{Z}$, and using the inequalities $2ab \leq a^2 + b^2$ and $ab \leq \frac{1}{4}a^4 + \frac{3}{4}b^{4/3}$ we obtain

$$\begin{aligned}
 & \frac{d}{dt} \left\| U_{\text{osc}} \right\|_{H^{0,s}}^2 + c \left\| \nabla_h U_{\text{osc}} \right\|_{L^2(\mathbb{T}^3)}^2 \\
 & \leq 2C \left((1 + \left\| \Omega \right\|_{H^{0,s}}^2) \left\| \nabla_h \Omega \right\|_{H^{0,s}}^2 + \left(1 + \left\| U_{\text{osc}} \right\|_{L^2(\mathbb{T}^3)}^2 \right) \left\| \nabla_h U_{\text{osc}} \right\|_{L^2(\mathbb{T}^3)}^2 \right) \left\| U_{\text{osc}} \right\|_{H^{0,s}}^2
 \end{aligned} \tag{3.6.5}$$

applying Gronwall inequality to (3.6.5) we obtain

$$\begin{aligned}
 & \left\| U_{\text{osc}}(t) \right\|_{H^{0,s}}^2 + c \int_0^t \left\| U_{\text{osc}}(\tau) \right\|_{H^{0,s}}^2 d\tau \\
 & \leq C \left\| U_{\text{osc},0} \right\|_{H^{0,s}}^2 \exp \left\{ 2C \int_0^t \left((1 + \left\| \Omega(\tau) \right\|_{H^{0,s}}^2) \left\| \nabla_h \Omega(\tau) \right\|_{H^{0,s}}^2 + \right. \right. \\
 & \quad \left. \left. (1 + \left\| U_{\text{osc}}(\tau) \right\|_{L^2(\mathbb{T}^3)}^2) \left\| \nabla_h U_{\text{osc}}(\tau) \right\|_{L^2(\mathbb{T}^3)}^2 \right) d\tau \right\},
 \end{aligned}$$

concluding. \square

3.6.3 Proof of Theorem 3.1.12

At this point it is very easy to prove Theorem 3.1.12 Let us consider a data $V_0 \in H^{0,s}$, $\Omega_0 \in H^{0,s}$, $s \geq 1$ and V_0 with zero horizontal average. Thanks to Proposition 3.6.2 we have that $\Omega \in \mathcal{C}(\mathbb{R}_+; H^{0,s-1}) \cap \mathcal{C}(\mathbb{R}_+; H^{0,s})$, $\nabla_h \Omega \in L^2(\mathbb{R}_+; H^{0,s-1}) \cap L^2(\mathbb{R}_+; H^{0,s})$, which in particular implies, thanks to Lemma 3.5.5 that $\Lambda_v^s V_{\text{QG}} \in \mathcal{C}(\mathbb{R}_+; L^2(\mathbb{T}^3))$, $\nabla_h \Lambda_v^s V_{\text{QG}} \in L^2(\mathbb{R}_+; L^2(\mathbb{T}^3))$. Since V_{QG} is defined as $V_{\text{QG}} = \Pi_{\text{QG}} U$ where Π_{QG} is a Fourier multiplier of order zero which maps continuously any $H^{s,s'}$ space to itself, this implies that $V_{\text{QG}} \in L^\infty(\mathbb{R}_+; L^2)$, $\nabla_h V_{\text{QG}} \in L^2(\mathbb{R}_+; L^2)$ since U is so thanks to Theorem 3.1.10, hence $V_{\text{QG}} \in \mathcal{C}(\mathbb{R}_+; H^{0,s})$, $\nabla_h V_{\text{QG}} \in L^2(\mathbb{R}_+; H^{0,s})$. For the oscillating part it suffices to apply Proposition 3.6.5 and the proof is complete.

We outline how to prove that solutions to the limit system are $H^{0,s'}$ -stable, for $s' \in [-1/2, s)$ globally with a continuous dependence of the initial data. To do so consider the

two solutions U_1, U_2 to the limit system

$$\begin{cases} \partial_t U_1 + \mathcal{Q}(U_1, U_1) - \mathbb{D}U_1 = 0 \\ \operatorname{div} u_1 = 0 \\ U_1|_{t=0} = U_{1,0} \end{cases} \quad (3.6.6)$$

$$\begin{cases} \partial_t U_2 + \mathcal{Q}(U_2, U_2) - \mathbb{D}U_2 = 0 \\ \operatorname{div} u_2 = 0 \\ U_2|_{t=0} = U_{2,0}. \end{cases} \quad (3.6.7)$$

Subtracting (3.6.7) from (3.6.6) and setting $U = U_1 - U_2$ we obtain the following system

$$\begin{cases} \partial_t U + \mathcal{Q}(U_1, U) + \mathcal{Q}(U, U_2) - \mathbb{D}U = 0 \\ \operatorname{div} u = 0 \\ U|_{t=0} = U_0 = U_{1,0} - U_{2,0}. \end{cases} \quad (3.6.8)$$

We apply now a stability result proved by M. Paicu in [125], namely Proposition 3.2.9, to the system (3.6.8). This gives the following estimate

$$\begin{aligned} & \|U\|_{H^{0,-\frac{1}{2}}}^2 + c \int_0^t \|\nabla_h U(\tau)\|_{H^{0,-\frac{1}{2}}}^2 \, d\tau \\ & \leq C \|U_0\|_{H^{0,-\frac{1}{2}}}^2 \exp \left\{ \int_0^t (1 + \|U(\tau)\|_{H^{0,s}}^2) \|\nabla_h U(\tau)\|_{H^{0,s}}^2 \, d\tau \right. \\ & \quad + \int_0^t (1 + \|U_1(\tau)\|_{H^{0,s}}^2) \|\nabla_h U_1(\tau)\|_{H^{0,s}}^2 \, d\tau \\ & \quad \left. + \int_0^t (1 + \|U_2(\tau)\|_{H^{0,s}}^2) \|\nabla_h U_2(\tau)\|_{H^{0,s}}^2 \, d\tau \right\}. \end{aligned}$$

The argument of the exponential is indeed uniformly bounded thanks to the estimates on the limit system performed above, whence if $\|U_0\|_{H^{0,-\frac{1}{2}}}^2$ is small the whole right hand side of the above equation is small. Since moreover

$$\|U\|_{H^{0,s}}^2 + c \int_0^t \|\nabla_h U(\tau)\|_{H^{0,s}}^2 \, d\tau \leq C (\|U_0\|_{H^{0,s}}^2),$$

uniformly in t by interpolation we prove the assertion stated above. \square

3.7 Convergence of the system as $\varepsilon \rightarrow 0$.

Remark 3.7.1. We point out the fact that Proposition 3.2.9 can be applied as well to systems with the form

$$\partial_t w + \mathcal{Q}^\varepsilon(w, w) + \mathcal{Q}^\varepsilon(u, w) - a_h(D)w = f, \quad \operatorname{div} w = 0.$$

◆

Remark 3.7.2. In the present section our aim is to use Proposition 3.2.7 and 3.2.9 to the systems (FS_ε) and (S) . Let us compare these two systems with (NS_h) : the only structural difference between these two is that in (FS_ε) and (S) the Poincaré semigroup couples velocity field and temperature $v^\varepsilon, T^\varepsilon$ in a new variable U^ε , but the structure itself of the equation is unchanged. For this reason Propositions 3.2.7 and 3.2.9 can be applied in the present case.

We shall require as well the following result

Lemma 3.7.3. *Let $f \in H^{s,s'}$, $s, s' \in \mathbb{R}$ such that the horizontal average $\underline{f} \in H_v^{s'}$. Then*

$$\|\underline{f}\|_{H_v^{s'}} \leq \|f\|_{H^{s,s'}}.$$

Proof. Since the element \underline{f} is the horizontal average of the function f we can indeed argue that

$$\underline{f}(x_3) = \mathcal{F}_v^{-1} \left(\left(\hat{f}(0, n_3) \right)_{n_3} \right),$$

at least in L^2 . Whence calculating explicitly the Sobolev norms

$$\|\underline{f}\|_{H_v^{s'}}^2 = \sum_{n_3 \in \mathbb{Z}} (1 + n_3^2)^{s'} \left| \hat{f}(0, n_3) \right|^2, \quad (3.7.1)$$

$$\|f\|_{H^{s,s'}}^2 = \sum_{n \in \mathbb{Z}^3} (1 + |n_h|^2)^s (1 + n_3^2)^{s'} \left| \hat{f}(n_h, n_3) \right|^2, \quad (3.7.2)$$

Comparing the expressions in (3.7.1) and (3.7.2) we remark that (3.7.1) is the restriction of (3.7.2) on the fiber $\{n_h = 0\}$, concluding. \square

Remark 3.7.4. Let us recall that Theorem 3.1.4 implies that for each $\varepsilon > 0$ fixed there exists a maximal time $T_\varepsilon^* \leq \infty$ such that for each $T^* < T_\varepsilon^*$ and $s > 1/2$ the function U^ε belongs to the space

$$U^\varepsilon \in L^\infty([0, T^*]; H^{0,s}), \quad \nabla_h U^\varepsilon \in L^2([0, T^*]; H^{0,s}).$$

◆

We prove that, given $V_0^\varepsilon \in H^{0,s}$, $s > 1$, the solution of our filtered system (FS_ε) converges to the solutions of the limit system (3.5.9), (3.5.10) in the sense that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left(V^\varepsilon - \mathcal{L} \left(\frac{t}{\varepsilon} \right) U \right) &= 0 && \text{in } \mathcal{C}(\mathbb{R}_+; H^{0,\sigma}) \\ \lim_{\varepsilon \rightarrow 0} \nabla_h \left(V^\varepsilon - \mathcal{L} \left(\frac{t}{\varepsilon} \right) U \right) &= 0 && \text{in } L^2(\mathbb{R}_+; H^{0,\sigma}) \end{aligned}$$

for $\sigma \in [1, s)$, where $U = U_{\text{osc}} + U_{\text{QG}}$ and $U_{\text{QG}} = V_{\text{QG}} = \begin{pmatrix} \nabla_h^\perp \\ 0 \\ -F\partial_3 \end{pmatrix} \Delta_F^{-1} \Omega$ with Ω solution

of (3.5.9). A suitable change of variable has to be performed so that the singular perturbations cancels among themselves. The same method has been studied in a wide generality by I. Gallagher in [72] in the generic context of parabolic (nonlinear) equations with singular,

linear, skew-symmetric perturbation. We mention as well the works [79] and [123] in which such technique has been used.

We want to underline a major difference between the application of Schochet method in the present work and in the work [123]. In [123] in fact the convergence takes place for the values of σ between $1/2$ and s . Indeed in our case $\sigma \in [1, s)$. This difference is motivated by the fact that our limit system is globally well posed in $H^{0,s}$, $s > 1$ only. This is due to the fact that we have been proving the propagation of $H^{0,s}$, $s > 0$ data for Ω in Proposition 3.6.2 and hence we have applied Lemma 3.5.5 to state that $H^{0,s}$, $s > 1$ data is propagated for V_{QG} .

Let us denote T_ε^* the maximal lifespan of U^ε solution of (FS_ε) in the space $H^{0,s}(\mathbb{T}^3)$ with $s > 1$, which exists thanks to the work [125]. Then there exists a time $T_\varepsilon^* \geq T > 0$ such that $U^\varepsilon \in \mathcal{C}([0, T]; H^{0,s})$ and $\nabla_h U^\varepsilon \in L^2([0, T]; H^{0,s})$ uniformly in ε small enough. Let us define $W^\varepsilon = U^\varepsilon - U$ defined on the interval $[0, T_\varepsilon^*]$ taking values in $H^{0,s}$. We obtain that W^ε satisfies the following equation

$$\begin{cases} \partial_t W^\varepsilon + \mathcal{Q}^\varepsilon(W^\varepsilon, W^\varepsilon) + \tilde{\mathcal{Q}}^\varepsilon(U, W^\varepsilon) - \mathbb{D}^\varepsilon W^\varepsilon \\ \quad = -(\mathbb{D}^\varepsilon - \mathbb{D})U - (\mathcal{Q}^\varepsilon(U, U) - \mathcal{Q}(U, U)), \\ \text{div } w^\varepsilon = 0, \\ W^\varepsilon|_{t=0} = 0, \end{cases} \quad (3.7.3)$$

where the form $\tilde{\mathcal{Q}}^\varepsilon$ is symmetric, bilinear and defined via

$$\tilde{\mathcal{Q}}^\varepsilon(A, B) = \mathcal{Q}^\varepsilon(A, B) + \mathcal{Q}^\varepsilon(B, A).$$

Let us define $R_{\text{osc}}^\varepsilon(U) = \mathcal{Q}^\varepsilon(U, U) - \mathcal{Q}(U, U)$, where $\mathcal{Q}^\varepsilon(A, B) \xrightarrow[\mathcal{D}'(\mathbb{R}_+ \times \mathbb{T}^3)]{\varepsilon \rightarrow 0} \mathcal{Q}(A, B)$.

It is a strongly oscillating in time function, given by the formula

$$R_{\text{osc}}^\varepsilon(U) = \mathcal{F}^{-1} \left(\sum_{\substack{\omega_{k,n-k,n}^{a,b,c} \neq 0 \\ 1 \leq j \leq 3}} e^{i \frac{t}{\varepsilon} \omega_{k,n-k,n}^{a,b,c}} (U^{a,j}(k) (n_j - k_j) U^b(n-k) | e^c(n))_{\mathbb{C}^4} e^c(n) \right),$$

where we have been using the notation $\omega_{k,n-k,n}^{a,b,c} = \omega^a(k) + \omega^b(n-k) - \omega^c(n)$, $a, b, c \in \{\pm\}$, $\omega^\pm(n)$ defined as in (3.3.4), $U^a(k) = \left(\hat{U}(k) | e^a(k) \right) e^a(k)$ and $U^{a,j}$ is the j -th component of U^a .

As well the function $S_{\text{osc}}^\varepsilon = (\mathbb{D}^\varepsilon - \mathbb{D})U$ is a highly oscillating function given by the following formula

$$S_{\text{osc}}^\varepsilon(U) = \mathcal{F}^{-1} \left(\sum_{\omega_n^{a,b} \neq 0} e^{i \frac{t}{\varepsilon} \omega_n^{a,b}} (\mathbf{D}(n) U^b(n) | e^a(n))_{\mathbb{C}^4} e^a(n) \right),$$

and as well as $R_{\text{osc}}^\varepsilon$ even $S_{\text{osc}}^\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$ only in \mathcal{D}' . For the rest of the section when we write the scalar product $(\cdot | \cdot)$ we implicitly mean $(\cdot | \cdot)_{\mathbb{C}^4}$.

We decompose $R_{\text{osc}}^\varepsilon$ and $S_{\text{osc}}^\varepsilon$ in high and low frequencies, i.e.

$$R_{\text{osc,LF}}^{\varepsilon,N}(U) = \mathcal{F}^{-1} \left(1_{\{|n| \leq N\}} \sum_{\substack{\omega_{k,n-k,n}^{a,b,c} \neq 0 \\ 1 \leq j \leq 3}} e^{i \frac{t}{\varepsilon} \omega_{k,n-k,n}^{a,b,c}} 1_{\{|k| \leq N\}} (U^{a,j}(k)(n_j - k_j) U^b(n-k) | e^c(n)) e^c(n) \right),$$

$$S_{\text{osc,LF}}^{\varepsilon,N}(U) = \mathcal{F}^{-1} \left(1_{\{|n| \leq N\}} \sum_{\omega_n^{a,b} \neq 0} e^{i \frac{t}{\varepsilon} \omega_n^{a,b}} (\mathbf{D}(n) U^b(n) | e^a(n)) e^a(n) \right),$$

and

$$R_{\text{osc,HF}}^{\varepsilon,N}(U) = R_{\text{osc}}^\varepsilon(U) - R_{\text{osc,LF}}^{\varepsilon,N}(U)$$

$$S_{\text{osc,HF}}^{\varepsilon,N}(U) = S_{\text{osc}}^\varepsilon(U) - S_{\text{osc,LF}}^{\varepsilon,N}(U).$$

Indeed the subscript f_{HF} stands for high frequencies and the subscript f_{LF} stands for low frequencies.

Concerning the high frequencies terms the following lemma hold

Lemma 3.7.5. *If $N \rightarrow \infty$ the terms $R_{\text{osc,HF}}^{\varepsilon,N}(U)$, $S_{\text{osc,HF}}^{\varepsilon,N}(U)$ tend uniformly to 0 in ε respectively in the space $L^p([0, T]; H^{-1, -1/2})$ and $L^p([0, T]; H^{-1, s})$ for all $1 \leq p \leq 2$, $s > 1$.*

The proof of Lemma 3.7.5 is postponed to the end of the section for the sake of clarity.

The term $R_{\text{osc,LF}}^{\varepsilon,N}(U)$ tends only weakly to zero. In order to absorb it in the following computations we introduce the following notation

$$\tilde{R}_{\text{osc,LF}}^{\varepsilon,N}(U) = \mathcal{F}^{-1} \left(1_{\{|n| \leq N\}} \sum_{\substack{\omega_{k,n-k,n}^{a,b,c} \neq 0 \\ 1 \leq j \leq 3}} \frac{e^{i \frac{t}{\varepsilon} \omega_{k,n-k,n}^{a,b,c}}}{\omega_{k,n-k,n}^{a,b,c}} 1_{\{|n| \leq N\}} (U^{a,j}(t, k)(n_j - k_j) U^b(t, n-k) | e^c(n)) e^c(n) \right)$$

$$\tilde{S}_{\text{osc,LF}}^{\varepsilon,N}(U) = \mathcal{F}^{-1} \left(1_{\{|n| \leq N\}} \sum_{\omega_n^{a,b} \neq 0} \frac{e^{i \frac{t}{\varepsilon} \omega_n^{a,b}}}{i \omega_n^{a,b}} (\mathbf{D}(n) U^b(n) | e^a(n)) e^a(n) \right).$$

We do as well the following change of unknown

$$\Psi_{\text{LF}}^{\varepsilon,N} = W^\varepsilon + \varepsilon \left(\tilde{R}_{\text{osc,LF}}^{\varepsilon,N}(U) + \tilde{S}_{\text{osc,LF}}^{\varepsilon,N}(U) \right). \quad (3.7.4)$$

Considering the substitution defined in (3.7.4) into (3.7.3), and after some algebraic manipulation we obtain that $\Psi_{\text{LF}}^{\varepsilon,N}$ satisfies the following equation

$$\partial_t \Psi_{\text{LF}}^{\varepsilon,N} + \frac{1}{2} \tilde{\mathcal{Q}} \left(\Psi_{\text{LF}}^{\varepsilon,N}, \Psi_{\text{LF}}^{\varepsilon,N} - 2\varepsilon \left(\tilde{R}_{\text{osc,LF}}^{\varepsilon,N}(U) + \tilde{S}_{\text{osc,LF}}^{\varepsilon,N}(U) \right) + 2U \right) - \mathbb{D}^\varepsilon \Psi_{\text{LF}}^{\varepsilon,N} = \Gamma^{\varepsilon,N}(U), \quad (3.7.5)$$

where

$$\Gamma^{\varepsilon,N} = R_{\text{osc,HF}}^{\varepsilon,N} + S_{\text{osc,HF}}^{\varepsilon,N} + \varepsilon \Gamma_N^\varepsilon,$$

and

$$\begin{aligned} \Gamma_N^\varepsilon &= \mathbb{D}^\varepsilon \left(\tilde{R}_{\text{osc,LF}}^{\varepsilon,N}(U) + \tilde{S}_{\text{osc,LF}}^{\varepsilon,N}(U) \right) \\ &\quad + \frac{1}{2} \tilde{\mathcal{Q}} \left(\left(\tilde{R}_{\text{osc,LF}}^{\varepsilon,N}(U) + \tilde{S}_{\text{osc,LF}}^{\varepsilon,N}(U) \right), \varepsilon \left(\tilde{R}_{\text{osc,LF}}^{\varepsilon,N}(U) + \tilde{S}_{\text{osc,LF}}^{\varepsilon,N}(U) \right) - 2U \right) + \\ &\quad \left(\tilde{R}_{\text{osc,LF}}^{\varepsilon,N,t}(U) + \tilde{S}_{\text{osc,LF}}^{\varepsilon,N,t}(U) \right) \end{aligned} \quad (3.7.6)$$

and respectively

$$\begin{aligned} \tilde{R}_{\text{osc,LF}}^{\varepsilon,N,t} &= \mathcal{F}^{-1} \left(1_{\{|n| \leq N\}} \sum_{\substack{\omega_{k,n-k,n}^{a,b,c} \neq 0 \\ 1 \leq j \leq 3}} \frac{e^{i \frac{t}{\varepsilon} \omega_{k,n-k,n}^{a,b,c}}}{\omega_{k,n-k,n}^{a,b,c}} 1_{\{|n| \leq N\}} \partial_t [(U^{a,j}(t,k)(n_j - k_j) U^b(t,n-k) | e^c(n)) e^c(n)] \right) \\ \tilde{S}_{\text{osc,LF}}^{\varepsilon,N,t} &= \mathcal{F}^{-1} \left(1_{\{|n| \leq N\}} \sum_{\omega_n^{a,b} \neq 0} \frac{e^{i \frac{t}{\varepsilon} \omega_n^{a,b}}}{i \omega_n^{a,b}} \partial_t [(\mathbf{D}(n) U^b(t,n) | e^a(n)) e^a(n)] \right). \end{aligned}$$

Lemma 3.7.6. *The term Γ_N^ε given by the relation (3.7.6) is bounded uniformly in ε by a constant $C(N)$ which depend solely on N in the spaces $L^p([0, T]; H^{-1, -1/2})$ for $1 \leq p \leq 2$.*

Proof. The result is due to the fact that we are considering functions localized in a ball of radius N in the frequency space, hence we can gain all the regularity that we want at the price of a constant which behaves like a power of N , and, in particular if $\omega_n^{a,b}, \omega_{k,n-k,n}^{a,b,c} \neq 0$ implies that

$$\frac{1}{|\omega_n^{a,b}|}, \frac{1}{|\omega_{k,n-k,n}^{a,b,c}|} \leq C(N).$$

Whence we easily obtain that Γ_N^ε belongs to the space $L^p(\mathbb{R}_+, H^{-1, -1/2})$ and that is uniformly bounded by a constant $C(N)$. \square

We remark that for ε sufficiently small the term $U - \varepsilon \left(\tilde{R}_{\text{osc,LF}}^{\varepsilon,N}(U) + \tilde{S}_{\text{osc,LF}}^{\varepsilon,N}(U) \right)$ has a small horizontal mean in H_v^s , whence we can apply Proposition 3.2.9 to equation (3.7.5) in order to obtain, for all $t \in [0, T_\varepsilon^*]$ the following bound

$$\begin{aligned} &\left\| \Psi_{\text{LF}}^{\varepsilon,N}(t) \right\|_{H^{0,-1/2}}^2 + c \int_0^t \left\| \nabla_h \Psi_{\text{LF}}^{\varepsilon,N}(\tau) \right\|_{H^{0,-1/2}}^2 d\tau \leq C (\|U_0\|_{H^{0,s_0}}) \\ &\quad \times \left(\left\| \Psi_{\text{LF}}^{\varepsilon,N}(0) \right\|_{H^{0,-1/2}}^2 + \int_0^t \left\| \Gamma^{\varepsilon,N}(\tau) \right\|_{H^{-1,-1/2}} d\tau + \int_0^t \left\| \Gamma^{\varepsilon,N}(\tau) \right\|_{H^{-1,-1/2}}^2 d\tau \right) \\ &\quad \times \exp \left\{ \int_0^t \left\| \Gamma^{\varepsilon,N}(\tau) \right\|_{H^{-1,-1/2}} d\tau \right. \\ &\quad \left. + \int_0^t \left(1 + \left\| \Psi_{\text{LF}}^{\varepsilon,N}(\tau) \right\|_{H^{0,s_0}}^2 \right) \left\| \nabla_h \Psi_{\text{LF}}^{\varepsilon,N}(\tau) \right\|_{H^{0,s_0}}^2 d\tau \right\}. \end{aligned} \quad (3.7.7)$$

Since we want to obtain global in time solutions it is important to have $\Gamma^{\varepsilon,N}$ at the same time in both spaces $L^1(\mathbb{R}_+; H^{-1, -1/2})$ and $L^2(\mathbb{R}_+; H^{-1, -1/2})$.

- We remark the fact that writing the estimate (3.7.7) we have been using implicitly the bound

$$\int_0^t (1 + \|U(\tau)\|_{H^{0,s_0}}^2) \|\nabla_h U(\tau)\|_{H^{0,s_0}}^2 d\tau \leq \frac{C}{c} \tilde{\mathcal{C}}(\|U_0\|_{H^{0,s_0}}),$$

for $s_0 > 1$, and we denoted $\mathcal{C}(\|U_0\|_{H^{0,s_0}}) = \exp\left\{\frac{C}{c} \tilde{\mathcal{C}}(\|U_0\|_{H^{0,s_0}})\right\}$.

- We used Lemma 3.7.3 to deduce the inequality

$$\|\underline{\Gamma}^{\varepsilon,N}(\tau)\|_{H_v^{-1/2}} \leq \|\Gamma^{\varepsilon,N}(\tau)\|_{H^{-1,-1/2}},$$

which has consequently be applied in order to deduce (3.7.7).

Considering Lemma 3.7.5 we can say that for each $\eta > 0$ there exists a large enough N such that, setting $\mathcal{X} = L^1(\mathbb{R}_+; H^{-1,-1/2}) \cap L^2(\mathbb{R}_+; H^{-1,-1/2})$,

$$\left\| R_{\text{osc,HF}}^{\varepsilon,N} + S_{\text{osc,HF}}^{\varepsilon,N} \right\|_{\mathcal{X}} \leq \frac{\eta}{2},$$

and thanks to Lemma 3.7.6 for ε sufficiently small

$$\varepsilon \|\Gamma_N^\varepsilon\|_{\mathcal{X}} \leq \varepsilon C(N) \leq \frac{\eta}{2},$$

whence we obtain that

$$\|\Gamma^{\varepsilon,N}\|_{\mathcal{X}} \leq \eta.$$

Thanks to the definition (3.7.4) we can argue that for each $\eta > 0$ and $t < T$ time of local existence of the solutions, there exists a $\varepsilon_1 = \varepsilon_1(\eta, T)$ such that for each $\varepsilon \in (0, \varepsilon_1)$:

$$\left\| \Psi_{\text{LF}}^{\varepsilon,N}(t) - W^\varepsilon(t) \right\|_{H^{0,-1/2}}^2 + c \int_0^t \left\| \nabla_h \Psi_{\text{LF}}^{\varepsilon,N}(\tau) - \nabla_h W^\varepsilon(\tau) \right\|_{H^{0,-1/2}}^2 d\tau \leq \varepsilon C(N) \leq \frac{\eta}{2},$$

in the same way we can write

$$\left\| \Psi_{\text{LF}}^{\varepsilon,N}(0) \right\|_{H^{0,-1/2}} = \varepsilon \left\| \tilde{R}_{\text{osc,LF}}^{\varepsilon,N}(U_0) + \tilde{S}_{\text{osc,LF}}^{\varepsilon,N}(U_0) \right\|_{H^{0,-1/2}} \leq \varepsilon C(N) \|U_0\|_{H^{0,s}}^2 \leq \frac{\eta}{2}.$$

Whence for ε sufficiently small and $t \in [0, T_\varepsilon^*)$ we have

$$\begin{aligned} & \left\| \Psi_{\text{LF}}^{\varepsilon,N}(t) \right\|_{H^{0,-1/2}}^2 + c \int_0^t \left\| \nabla_h \Psi_{\text{LF}}^{\varepsilon,N}(\tau) \right\|_{H^{0,-1/2}}^2 d\tau \\ & \leq C\eta \left(1 + \exp \left\{ \int_0^t \left\| \nabla_h \Psi_{\text{LF}}^{\varepsilon,N}(\tau) \right\|_{H^{0,s_0}}^2 d\tau \right\} \left(1 + \left\| \Psi_{\text{LF}}^{\varepsilon,N}(\tau) \right\|_{H^{0,s_0}}^2 \right) \right). \end{aligned} \quad (3.7.8)$$

We want to use now the definition of $\Psi_{\text{LF}}^{\varepsilon,N}$ given in (3.7.4), in particular this implies that $\left\| \Psi_{\text{LF}}^{\varepsilon,N} \right\| = \|W^\varepsilon\| + \mathcal{O}_N(\varepsilon)$ for N fixed. This means that $\Psi_{\text{LF}}^{\varepsilon,N}$ and W^ε have the same norm

up to an error which is comparable to ε which is, anyway, considered to be small. Whence (3.7.8) gives us that

$$\begin{aligned} & \|W^\varepsilon(t)\|_{H^{0,-1/2}}^2 + c \int_0^t \|\nabla_h W^\varepsilon(\tau)\|_{H^{0,-1/2}}^2 d\tau \\ & \leq C\eta \left(1 + \exp \left\{ \int_0^t \|\nabla_h W^\varepsilon(\tau)\|_{H^{0,s_0}}^2 (1 + \|W^\varepsilon(\tau)\|_{H^{0,s_0}}^2) d\tau \right\} \right). \end{aligned} \quad (3.7.9)$$

For the real numbers $s' \in [-1/2, s]$ we introduce the following continuous function

$$f_{\varepsilon,s'}(t) = \|W^\varepsilon(t)\|_{H^{0,s'}}^2 + \int_0^t (1 + \|W^\varepsilon(\tau)\|_{H^{0,s'}}^2) \|\nabla_h W^\varepsilon(\tau)\|_{H^{0,s'}}^2 d\tau.$$

The function $\|W^\varepsilon(t)\|_{H^{0,s'}}$ is defined on the interval $[0, T_\varepsilon^*)$, by use of (3.7.9) we get

$$f_{\varepsilon,-1/2}(t) \leq C\eta, \quad (3.7.10)$$

for each $t \in [0, T_\varepsilon^*)$.

We consider now an $s_0 > 1$ and the maximal time

$$T_\varepsilon^{s_0} = \sup \left\{ 0 < t < T_\varepsilon^* \mid f_{\varepsilon,s_0}(t) \leq 1, \text{ for each } 0 \leq t \leq T_\varepsilon^{s_0} \right\}.$$

Interpolating between $H^{0,-1/2}$ and H^{0,s_0} we get

$$f_{\varepsilon,\sigma}(t) = \mathcal{O}(\eta^{\vartheta(s_0,\sigma)}) \leq 1, \quad t \in [0, T_\varepsilon^{s_0}), \quad (3.7.11)$$

where $0 < \vartheta(s_0, \sigma) \xrightarrow{\sigma \rightarrow s_0} 0$ and $0 < \sigma \in [-1/2, s_0)$.

We consider at this point $U^\varepsilon = W^\varepsilon + U$, since U has zero horizontal mean we can easily point out that

$$\underline{U}^\varepsilon(t) = \underline{W}^\varepsilon(t).$$

Whence using Lemma 3.7.3, the definition of the function $f_{\varepsilon,\sigma}$ given in (3.7), and the smallness property on $f_{\varepsilon,\sigma}$ given in (3.7.11) we deduce:

$$\begin{aligned} \|\underline{W}^\varepsilon(t)\|_{H^\sigma} & \leq \|W^\varepsilon(t)\|_{H^{0,\sigma}}, \\ & \leq C \sqrt{f_{\varepsilon,\sigma}(t)}, \\ & \leq C\eta^{\vartheta/2} \ll 1. \end{aligned}$$

Since the horizontal average of $\underline{U}^\varepsilon$ is small we can infer via Proposition 3.2.7 obtaining, for $\sigma \in (1, s_0)$;

$$\begin{aligned} & \|U^\varepsilon(t)\|_{H^{0,s}}^2 + c \int_0^t \|\nabla_h U^\varepsilon(\tau)\|_{H^{0,s}}^2 d\tau \\ & \leq C \|V_0\|_{H^{0,s}} \exp \left(\int_0^t (1 + \|U^\varepsilon(\tau)\|_{H^{0,\sigma}}^2) \|\nabla_h U^\varepsilon(\tau)\|_{H^{0,\sigma}}^2 d\tau \right), \end{aligned} \quad (3.7.12)$$

on the other hand $0 \leq t < T_\varepsilon^{s_0}$, and since $U^\varepsilon = W^\varepsilon + U$ we get

$$\begin{aligned} & \int_0^t (1 + \|U^\varepsilon(\tau)\|_{H^{0,\sigma}}^2) \|\nabla_h U^\varepsilon(\tau)\|_{H^{0,\sigma}}^2 \, ds \\ & \leq f_{\varepsilon,\sigma}(t) + \int_0^t (1 + \|U(\tau)\|_{H^{0,\sigma}}^2) \|\nabla_h U(\tau)\|_{H^{0,\sigma}}^2 \, d\tau + F_\sigma(t). \end{aligned} \quad (3.7.13)$$

$F_\sigma(t)$ in particular is defined as

$$\begin{aligned} F_\sigma(t) &= \int_0^t (1 + \|W^\varepsilon(\tau)\|_{H^{0,\sigma}}^2) \|\nabla_h U(\tau)\|_{H^{0,\sigma}}^2 \, d\tau + \int_0^t (1 + \|U(\tau)\|_{H^{0,\sigma}}^2) \|\nabla_h W^\varepsilon(\tau)\|_{H^{0,\sigma}}^2 \, d\tau \\ &\lesssim \int_0^t (1 + \|W^\varepsilon(\tau)\|_{H^{0,\sigma}}^2) \|\nabla_h U(\tau)\|_{H^{0,\sigma}}^2 \, d\tau + (1 + \|U\|_{L^\infty(\mathbb{R}_+; H^{0,\sigma})}^2) f_{\varepsilon,\sigma}(t) \\ &\lesssim \left(\sup_{[0,t]} f_{\varepsilon,\sigma} \right) \|\nabla_h U\|_{L^2(\mathbb{R}_+; H^{0,\sigma})} + (1 + \|U\|_{L^\infty(\mathbb{R}_+; H^{0,\sigma})}^2) f_{\varepsilon,\sigma}(t), \end{aligned}$$

which in turn implies that, considering the above estimate in (3.7.13),

$$\begin{aligned} & \int_0^t (1 + \|U^\varepsilon(\tau)\|_{H^{0,\sigma}}^2) \|\nabla_h U^\varepsilon(\tau)\|_{H^{0,\sigma}}^2 \, ds \\ & \leq f_{\varepsilon,\sigma}(t) + \int_0^t (1 + \|U(\tau)\|_{H^{0,\sigma}}^2) \|\nabla_h U(\tau)\|_{H^{0,\sigma}}^2 \, d\tau \\ & \quad + \left(\sup_{[0,t]} f_{\varepsilon,\sigma} \right) \|\nabla_h U\|_{L^2(\mathbb{R}_+; H^{0,\sigma})} + (1 + \|U\|_{L^\infty(\mathbb{R}_+; H^{0,\sigma})}^2) f_{\varepsilon,\sigma}(t). \end{aligned}$$

We have seen though that in $[0, T_\varepsilon^{s_0})$ that $f_{\varepsilon,\sigma}(t) \leq 1$ for $\sigma \in (-1/2, s_0)$, and since $U \in L^\infty(\mathbb{R}_+, H^{0,\sigma})$ and $\nabla_h U \in L^2(\mathbb{R}_+, H^{0,\sigma})$ for $\sigma \in (1, s_0]$ (this is simply Proposition 3.6.2 combined with Lemma 3.5.5), we obtained that

$$\int_0^t (1 + \|U^\varepsilon(\tau)\|_{H^{0,\sigma}}^2) \|\nabla_h U^\varepsilon(\tau)\|_{H^{0,\sigma}}^2 \, ds \leq C.$$

If we consider the above bound in (3.7.12) we have hence obtained that

$$\|U^\varepsilon(t)\|_{H^{0,s}}^2 + c \int_0^t \|\nabla_h U^\varepsilon(s)\|_{H^{0,s}}^2 \, ds \leq C,$$

for all times $t \in [0, T_\varepsilon^{s_0})$ and $s > 1$. We deduce that $T_\varepsilon^{s_0} = T_\varepsilon^*$ and since the constant C is independent of the time t , this implies that $U^\varepsilon(t)$ can be extended in $H^{0,s}$ beyond T_ε^* and hence we obtain that $T_\varepsilon^* = \infty$ as long as ε is sufficiently small. Recalling that $\|W^\varepsilon\| = o(1)$ in $[0, T_\varepsilon^*)$ we deduce that $U^\varepsilon \rightarrow U$ globally in time in $H^{0,\sigma}$ for $-1/2 \leq \sigma < s$.

Proof of Lemma 3.7.5: In the following the index s addressing to the anisotropic Sobolev space $H^{0,s}$ is always considered to be $s > 1$. An interesting feature is that if $s > 1/2$ then H_v^s is a Banach algebra. We shall use this property all along the proof. We perform at first the estimates for the term $R_{\text{osc,HF}}^{\varepsilon,N}$. Since $U(t)$ is of zero horizontal average for all $t > 0$ and $\nabla_h U \in L^2(\mathbb{R}_+; H^{0,s})$ we obtain that $U \in L^2(\mathbb{R}_+; H^{0,s})$. Consequently $U \in$

$\mathcal{C}(\mathbb{R}_+; H^{0,s}) \cap L^2(\mathbb{R}_+; H^{0,s})$, and, interpolating $U \in L^{p'}(\mathbb{R}_+, H^{0,s})$ for each $p' \in [2, \infty]$. Let us observe that the term $R_{\text{osc,HF}}^{\varepsilon,N}$ can be decomposed as

$$R_{\text{osc,HF}}^{\varepsilon,N} = R_{\text{osc,1}}^{\varepsilon,N} + R_{\text{osc,2}}^{\varepsilon,N},$$

where we denoted

$$R_{\text{osc,1}}^{\varepsilon,N} = \mathcal{F}^{-1} \left(1_{\{|n| \geq N\}} R_{\text{osc,HF}}^{\varepsilon,N} \right),$$

and

$$R_{\text{osc,2}}^{\varepsilon,N}(U) = \mathcal{F}^{-1} \left(1_{\{|n| \leq N\}} \sum_{\substack{\omega_{k,n-k,n}^{a,b,c} \neq 0 \\ 1 \leq j \leq 3}} e^{i \frac{t}{\varepsilon} \omega_{k,n-k,n}^{a,b,c}} 1_{\{|k| \geq N\}} (U^{a,j}(k) (n_j - k_j) U^b(n-k) | e^c(n)) e^c(n) \right)$$

For the first term we use the fact that we are on the high frequencies and of an element in $L^p(\mathbb{R}_+, H^{-1,-1/2})$ which tends uniformly at zero as long as $\varepsilon \rightarrow 0$ thanks to Lebesgue theorem and Sobolev embeddings. In fact

$$\begin{aligned} \|R_{\text{osc}}^{\varepsilon}(U)\|_{H^{-1,-1/2}} &\leq \left\| \mathcal{F}^{-1} \left(\sum_{k+m=n} (U^a(k) \otimes U^b(m) | e^c(n))_{\mathbb{Z}^4} e^c(n) \right) \right\|_{H^{0,1/2}} \\ &= \|U \otimes U\|_{H^{0,1/2}} \\ &\lesssim \|U\|_{H^{1/2,s}}^2. \end{aligned}$$

Now, since U has null horizontal average we can apply Lemma 3.2.4 to obtain finally that

$$\|R_{\text{osc}}^{\varepsilon}(U)\|_{H^{-1,-1/2}} \lesssim \|U\|_{H^{0,s}} \|\nabla_h U\|_{H^{0,s}}.$$

Since $L^2([0, T]) \subset L^{p'}([0, T])$ for $p' \in [1, 2)$ if we prove that $\|R_{\text{osc}}^{\varepsilon}(U)\|_{L^2([0, T]; H^{-1,-1/2})} < \infty$ we can apply Lebesgue theorem and conclude that $\|R_{\text{osc,1}}^{\varepsilon,N}\|_{L^{p'}([0, T]; H^{-1,-1/2})} \rightarrow 0$ as $N \rightarrow \infty$. But this is in fact true since

$$\begin{aligned} &\| \|U\|_{H^{0,s}} \|\nabla_h U\|_{H^{0,s}} \|_{L_t^2}^2 \\ &= \int_0^t \|U(\tau)\|_{H^{0,s}}^2 \|\nabla_h U(\tau)\|_{H^{0,s}}^2 d\tau \leq \|U\|_{L^\infty(\mathbb{R}_+; H^{0,s})}^2 \|\nabla_h U\|_{L^2(\mathbb{R}_+; H^{0,s})}^2. \end{aligned} \quad (3.7.14)$$

For the second term we argue as follows

$$\begin{aligned} \|R_{\text{osc,2}}^{\varepsilon,N}\|_{H^{-1,-1/2}} &\leq \left\| \mathcal{F}^{-1} \left(\sum_{k+m=n} 1_{\{|k| \geq N\}} ((U^a(k) \otimes U^b(m)) | e^c(n)) e^c(n) \right) \right\|_{H^{0,1/2}} \\ &\leq \left\| \mathcal{F}^{-1} \left(\sum_{k+m=n} 1_{\{|k| \geq N\}} (\hat{U}(k) | e^a(k)) e^a(k) \right) \right\|_{H^{1/2,s}} \|u\|_{H^{1/2,s}}, \end{aligned}$$

and, using (3.2.11) we obtain the following bound

$$\begin{aligned} &\|R_{\text{osc,2}}^{\varepsilon,N}\|_{H^{-1,-1/2}} \\ &\leq \left\| \mathcal{F}^{-1} (1_{\{|k| \geq N\}} U^a(k)) \right\|_{H^{0,s}}^{1/2} \left\| \mathcal{F}^{-1} (1_{\{|k| \geq N\}} (\nabla_h U)^a(k)) \right\|_{H^{0,s}}^{1/2} \|U\|_{H^{0,s}}^{1/2} \|\nabla_h U\|_{H^{0,s}}^{1/2}, \end{aligned}$$

which evidently tends to zero thanks to Lebesgue theorem.

For the term $S_{\text{osc, HF}}^{\varepsilon, N}$ it comes straightforward since

$$\begin{aligned} \left\| S_{\text{osc, HF}}^{\varepsilon, N} \right\|_{H^{-1, s}} &= \left\| \mathcal{F}^{-1} \left(1_{\{|n| \geq N\}} \sum_{\omega_n^{a, b} \neq 0} e^{i \frac{t}{\varepsilon} \omega_n^{a, b}} (\mathbf{D}(n) U^b(n) | e^a(n)) e^a(n) \right) \right\|_{H^{-1, s}} \\ &\leq C \|\nabla_h U\|_{H^{0, s}}. \end{aligned} \quad (3.7.15)$$

□

3.8 The energy estimates

In this appendix we refer to V_{QG} and U_{osc} respectively as the solution of equation (3.5.11) and (3.5.10). Moreover $v_{\text{QG}}, u_{\text{osc}}$ represents the projection of the first three components of V_{QG} and U_{osc} .

The aim of this section is essentially to give an energy bound for the bilinear term appearing in equation (3.5.10).

Given a generic vector field u we refer to \underline{u} as the horizontal average of u . This gives the natural decomposition $u = \underline{u} + \tilde{u}$. Since \tilde{u} has zero horizontal average the results given in the Subsection 4.1.5 can be applied.

3.8.1 Estimates for the global well-posedness of the limit system.

Proposition 3.8.1. *Let $V_{\text{QG}} = \begin{pmatrix} \nabla_h^\perp \\ 0 \\ -F \partial_3 \end{pmatrix} \Delta_F^{-1} \Omega$ where Ω is the potential vorticity defined in (3.3.10), then*

$$\begin{aligned} (\Delta_q^v (v_{\text{QG}}^h \cdot \nabla_h \Omega) | \Delta_q^v \Omega) &\leq C 2^{-2qs} b_q(t) \\ &\times \left[\|\Omega\|_{L^2(\mathbb{T}^3)}^{1/2} \|\nabla_h \Omega\|_{L^2(\mathbb{T}^3)}^{1/2} \|\Omega\|_{H^{0, s}}^{1/2} \|\nabla_h \Omega\|_{H^{0, s}}^{3/2} + \|\nabla_h \Omega\|_{L^2(\mathbb{T}^3)} \|\Omega\|_{H^{0, s}} \|\nabla_h \Omega\|_{H^{0, s}} \right], \end{aligned} \quad (3.8.1)$$

where $(b_q)_q$ is a $\ell^1(\mathbb{Z})$ positive sequence which depends on Ω and such that $\sum_q b_q(t) \leq 1$.

Proof. Thanks to Bony decomposition (4.1.11) we can write

$$\begin{aligned} \Delta_q^v (v_{\text{QG}}^h \cdot \nabla_h \Omega) &= S_{q-1}^v v_{\text{QG}}^h \Delta_q^v \nabla_h \Omega + \\ &\sum_{|q-q'| \leq 4} ([\Delta_q^v; S_{q'-1}^v v_{\text{QG}}^h] \Delta_{q'}^v \nabla_h \Omega + (S_{q'-1}^v v_{\text{QG}}^h - S_{q-1}^v v_{\text{QG}}^h) \Delta_q^v \Delta_{q'}^v \nabla_h \Omega) \\ &\quad + \sum_{q' > q-4} \Delta_q^v (S_{q'+2}^v \nabla_h \Omega \Delta_{q'}^v v_{\text{QG}}^h), \end{aligned} \quad (3.8.2)$$

and hence we can decompose $(\Delta_q^v (v_{\text{QG}}^h \cdot \nabla_h \Omega) | \Delta_q^v \Omega) = \sum_{k=1}^4 I_h^k(q)$.

First of all, since $\operatorname{div}_h v_{\text{QG}}^h = 0$ we have $I_h^1 = 0$. We remark that we proved in Lemma 3.5.6 that $\underline{v}_{\text{QG}}^h = 0$. Whence $v_{\text{QG}}^h = \tilde{v}_{\text{QG}}^h$. Moreover $\nabla_h \Omega = \nabla_h \tilde{\Omega}$, hence

$$\begin{aligned} I_h^2(q) &= \sum_{|q-q'|\leq 4} \left([\Delta_q^v; S_{q'-1}^v v_{\text{QG}}^h] \Delta_{q'}^v \nabla_h \Omega \Big| \Delta_q^v \Omega \right) \\ &= \sum_{|q-q'|\leq 4} \left([\Delta_q^v; S_{q'-1}^v \tilde{v}_{\text{QG}}^h] \Delta_{q'}^v \nabla_h \tilde{\Omega} \Big| \Delta_q^v \tilde{\Omega} \right) + \left([\Delta_q^v; S_{q'-1}^v \tilde{v}_{\text{QG}}^h] \Delta_{q'}^v \nabla_h \tilde{\Omega} \Big| \Delta_q^v \underline{\Omega} \right) \\ &= I_h^{2,1}(q) + I_h^{2,2}(q). \end{aligned}$$

We consider first the term $I_h^{2,1}$. By Hölder inequality and Lemma 4.1.10 we can deduce

$$I_h^{2,1}(q) \lesssim \sum_{|q-q'|\leq 4} 2^{-q} \|S_{q'-1}^v \partial_3 \tilde{v}_{\text{QG}}^h\|_{L_v^\infty L_h^4} \left\| \Delta_{q'}^v \nabla_h \tilde{\Omega} \right\|_{L^2(\mathbb{T}^3)} \left\| \Delta_q^v \tilde{\Omega} \right\|_{L_v^2 L_h^4}$$

we can hence apply (3.2.10) to the term $\|S_{q'-1}^v \partial_3 \tilde{v}_{\text{QG}}^h\|_{L_v^\infty L_h^4}$ and (3.2.8) to $\left\| \Delta_q^v \tilde{\Omega} \right\|_{L_v^2 L_h^4}$, and then (3.2.5) and Lemma 3.5.5 in order to deduce

$$\begin{aligned} I_h^{2,1}(q) &\lesssim \sum_{|q-q'|\leq 4} 2^{-q+q'/2} \|\partial_3 \tilde{v}_{\text{QG}}^h\|_{L^2(\mathbb{T}^3)}^{1/2} \|\partial_3 \nabla_h \tilde{v}^h\|_{L^2(\mathbb{T}^3)}^{1/2} \left\| \Delta_{q'}^v \nabla_h \tilde{\Omega} \right\|_{L^2(\mathbb{T}^3)} \\ &\quad \times \left\| \Delta_q^v \tilde{\Omega} \right\|_{L^2(\mathbb{T}^3)}^{1/2} \left\| \Delta_q^v \nabla_h \tilde{\Omega} \right\|_{L^2(\mathbb{T}^3)}^{1/2} \\ &\lesssim b_q(t) 2^{-q/2-2qs} \|\Omega\|_{L^2(\mathbb{T}^3)}^{1/2} \|\nabla_h \Omega\|_{L^2(\mathbb{T}^3)}^{1/2} \|\Omega\|_{H^{0,s}}^{1/2} \|\nabla_h \Omega\|_{H^{0,s}}^{3/2}. \end{aligned} \quad (3.8.3)$$

For the following terms the tools used are the same as for the term $I_h^{2,1}(q)$, hence, we shall not explain the procedure in details. For the term $I_h^{2,2}(q)$

$$\begin{aligned} I_h^{2,2}(q) &\lesssim 2^{-q} \|S_{q'-1}^v \partial_3 \tilde{v}_{\text{QG}}^h\|_{L_v^\infty L_h^2} \left\| \Delta_{q'}^v \nabla_h \tilde{\Omega} \right\|_{L^2(\mathbb{T}^3)} \left\| \Delta_q^v \underline{\Omega} \right\|_{L^2} \\ &\lesssim b_q(t) 2^{-2qs-q'/2} \|\nabla_h \partial_3 v_{\text{QG}}\|_{L^2(\mathbb{T}^3)} \|\Omega\|_{H^{0,s}} \|\nabla_h \Omega\|_{H^{0,s}} \\ &\lesssim b_q(t) 2^{-2qs-q'/2} \|\nabla_h \Omega\|_{L^2(\mathbb{T}^3)} \|\Omega\|_{H^{0,s}} \|\nabla_h \Omega\|_{H^{0,s}}, \end{aligned} \quad (3.8.4)$$

where in the first inequality we have used (3.2.9) and by Poincaré inequality in the horizontal variable to obtain

$$\|S_{q'-1}^v \partial_3 \tilde{v}_{\text{QG}}^h\|_{L_v^\infty L_h^2} \lesssim 2^{q'/2} \|S_{q'-1}^v \nabla_h \partial_3 \tilde{v}_{\text{QG}}^h\|_{L^2(\mathbb{T}^3)}$$

Next, we consider the term

$$\begin{aligned} I_h^3(q) &= \sum_{|q-q'|\leq 4} \left((S_{q'-1}^v v_{\text{QG}}^h - S_{q-1}^v v_{\text{QG}}^h) \Delta_q^v \Delta_{q'}^v \nabla_h \Omega \Big| \Delta_q^v \Omega \right) \\ &= \sum_{|q-q'|\leq 4} \left((S_{q'-1}^v \tilde{v}_{\text{QG}}^h - S_{q-1}^v \tilde{v}_{\text{QG}}^h) \Delta_q^v \Delta_{q'}^v \nabla_h \tilde{\Omega} \Big| \Delta_q^v \tilde{\Omega} \right) \\ &\quad + \left((S_{q'-1}^v \tilde{v}_{\text{QG}}^h - S_{q-1}^v \tilde{v}_{\text{QG}}^h) \Delta_q^v \Delta_{q'}^v \nabla_h \tilde{\Omega} \Big| \Delta_q^v \underline{\Omega} \right) = I_h^{3,1}(q) + I_h^{3,2}(q). \end{aligned}$$

With calculations similar and since $\text{Supp } \mathcal{F} (S_{q'-1}^v v_{\text{QG}}^h - S_{q-1}^v v_{\text{QG}}^h) \subset \bigcup_{|q-q'|\leq 4} 2^q \mathcal{C}$, and hence localized from above and below in the frequency space, using respectively in the first inequality (3.2.10), Bernstein inequality, (3.2.8), (3.2.5) and Lemma 3.5.5

$$\begin{aligned} I_h^{3,1}(q) &\leq \sum_{|q-q'|\leq 4} \|S_{q'-1}^v \tilde{v}_{\text{QG}}^h - S_{q-1}^v \tilde{v}_{\text{QG}}^h\|_{L_v^\infty L_h^4} \left\| \Delta_q^v \Delta_{q'}^v \nabla_h \tilde{\Omega} \right\|_{L^2(\mathbb{T}^3)} \left\| \Delta_q^v \tilde{\Omega} \right\|_{L_v^2 L_h^4} \\ &\lesssim b_q(t) 2^{-q/2-2qs} \|\partial_3 V_{\text{QG}}\|_{L^2(\mathbb{T}^3)}^{1/2} \|\nabla_h \partial_3 V_{\text{QG}}\|_{L^2(\mathbb{T}^3)}^{1/2} \|\Omega\|_{H^{0,s}}^{1/2} \|\nabla_h \Omega\|_{H^{0,s}}^{3/2} \\ &\lesssim b_q(t) 2^{-q/2-2qs} \|\Omega\|_{L^2(\mathbb{T}^3)}^{1/2} \|\nabla_h \Omega\|_{L^2(\mathbb{T}^3)}^{1/2} \|\Omega\|_{H^{0,s}}^{1/2} \|\nabla_h \Omega\|_{H^{0,s}}^{3/2} \end{aligned} \quad (3.8.5)$$

The procedure for the term $I_h^{3,2}(q)$ is almost the same as the one for the term $I_h^{3,1}(q)$, except that we do not use (3.2.8) and we use Poincaré inequality in the horizontal variables

$$I_h^{3,2}(q) \leq b_q(t) 2^{-q/2-2qs} \|\nabla_h \Omega\|_{L^2(\mathbb{T}^3)} \|\Omega\|_{H^{0,s}} \|\nabla_h \Omega\|_{H^{0,s}}. \quad (3.8.6)$$

The last term

$$\begin{aligned} I_h^4(q) &= \sum_{q'>q-1} (\Delta_q^v (S_{q'+2}^v \nabla_h \Omega \Delta_{q'}^v v_{\text{QG}}^h) | \Delta_q^v \Omega) \\ &= \sum_{q'>q-1} (\Delta_q^v (S_{q'+2}^v \nabla_h \Omega \Delta_{q'}^v v_{\text{QG}}^h) | \Delta_q^v \tilde{\Omega}) + \sum_{q'>q-1} (\Delta_q^v (S_{q'+2}^v \nabla_h \Omega \Delta_{q'}^v v_{\text{QG}}^h) | \Delta_q^v \underline{\Omega}) \\ &= I_h^{4,1}(q) + I_h^{4,2}(q). \end{aligned}$$

Let us deal with the term $I_h^{4,1}(q)$. Applying Hölder inequality we deduce

$$I_h^{4,1}(q) \leq \sum_{q'>q-1} \|\Delta_{q'}^v v_{\text{QG}}^h\|_{L_v^\infty L_h^4} \|S_{q'+2}^v \nabla_h \Omega\|_{L^2(\mathbb{T}^3)} \left\| \Delta_q^v \tilde{\Omega} \right\|_{L_v^2 L_h^4}.$$

Using Bernstein inequality twice, (3.2.8), Lemma 3.5.5 and lastly (3.2.5) we deduce

$$\begin{aligned} \|\Delta_{q'}^v v_{\text{QG}}^h\|_{L_v^\infty L_h^4} &\lesssim 2^{q'/2} \|\Delta_{q'}^v v_{\text{QG}}^h\|_{L_v^2 L_h^4} \\ &\lesssim 2^{-q'/2} \|\partial_3 \Delta_{q'}^v v_{\text{QG}}^h\|_{L_v^2 L_h^4} \\ &\lesssim 2^{-q'/2} \|\partial_3 \Delta_{q'}^v v_{\text{QG}}^h\|_{L^2(\mathbb{T}^3)}^{1/2} \|\partial_3 \nabla_h \Delta_{q'}^v v_{\text{QG}}^h\|_{L^2(\mathbb{T}^3)}^{1/2} \\ &\lesssim 2^{-q'/2} \|\Delta_{q'}^v \Omega\|_{L^2(\mathbb{T}^3)}^{1/2} \|\nabla_h \Delta_{q'}^v \Omega\|_{L^2(\mathbb{T}^3)}^{1/2} \\ &\lesssim c_{q'}(\Omega, t) 2^{-q'/2-q's} \|\Omega\|_{H^{0,s}}^{1/2} \|\nabla_h \Omega\|_{H^{0,s}}^{1/2}. \end{aligned}$$

An application of (3.2.8) and (3.2.5) gives instead

$$\left\| \Delta_q^v \tilde{\Omega} \right\|_{L_v^2 L_h^4} \lesssim c_q(\Omega, t) 2^{-qs} \|\Omega\|_{H^{0,s}}^{1/2} \|\nabla_h \Omega\|_{H^{0,s}}^{1/2}, \quad (3.8.7)$$

whence we deduce the bound

$$I_h^{4,1}(q) \leq C 2^{-2qs-q/2} b_q(\Omega, t) \|\nabla_h \Omega\|_{L^2(\mathbb{T}^3)} \|\Omega\|_{H^{0,s}} \|\nabla_h \Omega\|_{H^{0,s}}.$$

To bound the term $I_h^{4,2}(q)$ is a similar procedure and hence is omitted. Whence collecting estimates (3.8.3)–(3.8.7) we deduce the bound (3.8.1). \square

Proposition 3.8.2. *Let V_{QG} and U_{osc} respectively be the solution of equation (3.5.11) and (3.5.10), then if the horizontal mean of V_{QG} and U_{osc} is zero (see Lemmas 3.5.6 and 3.5.7) the following estimates hold*

$$\begin{aligned} & (\Delta_q^v \mathcal{Q}(V_{\text{QG}}, U_{\text{osc}}) | \Delta_q^v U_{\text{osc}})_{L^2(\mathbb{T}^3)} + (\Delta_q^v \mathcal{Q}(U_{\text{osc}}, V_{\text{QG}}) | \Delta_q^v U_{\text{osc}})_{L^2(\mathbb{T}^3)} \\ & \leq C 2^{-2qs} b_q(t) \|\nabla_h \Omega\|_{H^{0,s}} \|\nabla_h U_{\text{osc}}\|_{H^{0,s}} \|U_{\text{osc}}\|_{H^{0,s}} \\ & \quad + C 2^{-2qs} b_q(t) \|\Omega\|_{H^{0,s}}^{1/2} \|\nabla_h \Omega\|_{H^{0,s}}^{1/2} \|U_{\text{osc}}\|_{H^{0,s}}^{1/2} \|\nabla_h U_{\text{osc}}\|_{H^{0,s}}^{3/2} \end{aligned} \quad (3.8.8)$$

$$\begin{aligned} & (\Delta_q^v \mathcal{Q}(U_{\text{osc}}, U_{\text{osc}}) | \Delta_q^v U_{\text{osc}})_{L^2(\mathbb{T}^3)} \\ & \leq C b_q(t) 2^{-2qs} \|\nabla_h U_{\text{osc}}\|_{L^2(\mathbb{T}^3)} \|U_{\text{osc}}\|_{H^{0,s}} \|\nabla_h U_{\text{osc}}\|_{H^{0,s}} \\ & \quad + C b_q(t) 2^{-2qs} \|U_{\text{osc}}\|_{L^2(\mathbb{T}^3)}^{1/2} \|\nabla_h U_{\text{osc}}\|_{L^2(\mathbb{T}^3)}^{1/2} \|U_{\text{osc}}\|_{H^{0,s}}^{1/2} \|\nabla_h U_{\text{osc}}\|_{H^{0,s}}^{3/2}. \end{aligned} \quad (3.8.9)$$

The sequence $(b_q)_q$ is a $\ell^1(\mathbb{Z})$ positive sequence which depends on Ω, U_{osc} and such that $\sum_q b_q(t) \leq 1$.

Remark 3.8.3. From now on $(\cdot | \cdot) = (\cdot | \cdot)_{L^2(\mathbb{T}^3)}$ ◆

Proof. We shall divide the proof of the above proposition in two parts, namely one part for each estimate.

In the following we always consider $s > 1/2$, hence in particular the embedding $H_v^s \hookrightarrow L_v^\infty$ holds true. Moreover we underline the fact that $V_{\text{QG}}(t)$ and $U_{\text{osc}}(t)$ have zero horizontal average for each $t > 0$ is the initial data has zero horizontal average thanks to the results of Lemma 3.5.6 and Lemma 3.5.7, whence the estimates (3.2.8) and (3.2.11) can be applied in this context as well as Lemma 3.1.2.

Proof of (3.8.8): in order to prove the estimate (3.8.8) we shall substitute the bilinear form \mathcal{Q} with the transport bilinear form. This choice is done only in order to simplify the notation.

Indeed we have

$$\begin{aligned} & |(\Delta_q^v (v_{\text{QG}} \cdot \nabla U_{\text{osc}}) | \Delta_q^v U_{\text{osc}})| = |(\Delta_q^v (v_{\text{QG}}^h \cdot \nabla_h U_{\text{osc}}) | \Delta_q^v U_{\text{osc}})|, \\ & \quad = |(\text{div}_h \Delta_q^v (v_{\text{QG}}^h \otimes U_{\text{osc}}) | \Delta_q^v U_{\text{osc}})|, \\ & |(\Delta_q^v (u_{\text{osc}} \cdot \nabla V_{\text{QG}}) | \Delta_q^v U_{\text{osc}})| \leq |(\text{div}_h \Delta_q^v (u_{\text{osc}}^h \otimes V_{\text{QG}}) | \Delta_q^v U_{\text{osc}})| \\ & \quad + |(\partial_3 \Delta_q^v (u_{\text{osc}}^3 V_{\text{QG}}) | \Delta_q^v U_{\text{osc}})|, \end{aligned}$$

and indeed

$$\begin{aligned} & |(\text{div}_h \Delta_q^v (v_{\text{QG}}^h \otimes U_{\text{osc}}) | \Delta_q^v U_{\text{osc}})| + |(\text{div}_h \Delta_q^v (u_{\text{osc}}^h \otimes V_{\text{QG}}) | \Delta_q^v U_{\text{osc}})| \\ & \leq 2 |(\Delta_q^v (U_{\text{osc}} \otimes V_{\text{QG}}) | \Delta_q^v \nabla_h U_{\text{osc}})|, \end{aligned}$$

whence

$$\begin{aligned} & |(\Delta_q^v (v_{\text{QG}} \cdot \nabla U_{\text{osc}}) | \Delta_q^v U_{\text{osc}})| + |(\Delta_q^v (u_{\text{osc}} \cdot \nabla V_{\text{QG}}) | \Delta_q^v U_{\text{osc}})| \\ & \leq 2 |(\Delta_q^v (U_{\text{osc}} \otimes V_{\text{QG}}) | \Delta_q^v \nabla_h U_{\text{osc}})| + |(\partial_3 \Delta_q^v (u_{\text{osc}}^3 V_{\text{QG}}) | \Delta_q^v U_{\text{osc}})| \\ & \quad = B_h(q) + B_v(q). \end{aligned}$$

Thanks to (3.2.5) and Lemma 3.1.2 we deduce

$$\begin{aligned} B_h(q) &\lesssim 2^{-2qs} b_q(t) \|U_{\text{osc}} \otimes V_{\text{QG}}\|_{H^{0,s}} \|\nabla_h U_{\text{osc}}\|_{H^{0,s}}, \\ &\lesssim 2^{-2qs} b_q(t) \|V_{\text{QG}}\|_{H^{1/2,s}} \|U_{\text{osc}}\|_{H^{1/2,s}} \|\nabla_h U_{\text{osc}}\|_{H^{0,s}}. \end{aligned} \quad (3.8.10)$$

An application of Poincaré inequality and (3.2.11) allow us to deduce that

$$\begin{aligned} \|V_{\text{QG}}\|_{H^{1/2,s}} &\lesssim \|\nabla_h V_{\text{QG}}\|_{H^{1/2,s}}, \\ &\lesssim \|\nabla_h V_{\text{QG}}\|_{H^{0,s}}^{1/2} \|\nabla_h^2 V_{\text{QG}}\|_{H^{0,s}}^{1/2}. \end{aligned}$$

An application of Lemma 3.5.5 leads to

$$\|\nabla_h V_{\text{QG}}\|_{H^{0,s}}^{1/2} \|\nabla_h^2 V_{\text{QG}}\|_{H^{0,s}}^{1/2} \lesssim \|\Omega\|_{H^{0,s}}^{1/2} \|\nabla_h \Omega\|_{H^{0,s}}^{1/2},$$

whence with use of (3.2.11) we deduce the bound

$$B_h(q) \lesssim 2^{-2qs} b_q(t) \|\Omega\|_{H^{0,s}}^{1/2} \|\nabla_h \Omega\|_{H^{0,s}}^{1/2} \|U_{\text{osc}}\|_{H^{0,s}}^{1/2} \|\nabla_h U_{\text{osc}}\|_{H^{0,s}}^{3/2}. \quad (3.8.11)$$

The term B_v can instead be written as

$$B_v(q) = |(\Delta_q^v(\text{div}_h u_{\text{osc}}^h V_{\text{QG}}) | \Delta_q^v U_{\text{osc}})| + |(\Delta_q^v(u_{\text{osc}}^3 \partial_3 V_{\text{QG}}) | \Delta_q^v U_{\text{osc}})| = B_v^1(q) + B_v^2(q).$$

For the term $B_v^1(q)$, applying (3.2.5) and Lemma 3.1.2

$$\begin{aligned} B_v^1(q) &\lesssim 2^{-2qs} b_q(t) \|\text{div}_h u_{\text{osc}}^h V_{\text{QG}}\|_{H^{-1/2,s}} \|U_{\text{osc}}\|_{H^{1/2,s}} \\ &\lesssim 2^{-2qs} b_q(t) \|V_{\text{QG}}\|_{H^{1/2,s}} \|U_{\text{osc}}\|_{H^{1/2,s}} \|\nabla_h U_{\text{osc}}\|_{H^{0,s}}, \end{aligned}$$

which is the same estimate as (3.8.10) and whence we can deduce the same bound as for $B_h(q)$. i.e. (3.8.11).

The term $B_v^2(q)$ is indeed less regular due to the presence of the vertical derivative. Similarly as before we can apply (3.2.5) and Lemma 3.1.2 to deduce

$$B_v^2(q) \lesssim 2^{-2qs} b_q(t) \|\partial_3 V_{\text{QG}}\|_{H^{0,s}} \|U_{\text{osc}}\|_{H^{1/2,s}}^2.$$

Poincaré inequality and Lemma 3.5.5 imply

$$\|\partial_3 V_{\text{QG}}\|_{H^{0,s}} \lesssim \|\partial_3 \nabla_h V_{\text{QG}}\|_{H^{0,s}} \lesssim \|\nabla_h \Omega\|_{H^{0,s}},$$

while using (3.2.11) we can conclude with the following bound

$$B_v^2(q) \lesssim 2^{-2qs} b_q(t) \|\nabla_h \Omega\|_{H^{0,s}} \|U_{\text{osc}}\|_{H^{0,s}} \|\nabla_h U_{\text{osc}}\|_{H^{0,s}}. \quad (3.8.12)$$

Whence (3.8.11) and (3.8.12) prove (3.8.8).

Proof of (3.8.9): Lastly we consider the term

$$\begin{aligned} (\Delta_q^v \mathcal{Q}(U_{\text{osc}}, U_{\text{osc}}) | \Delta_q^v U_{\text{osc}}) &= (\Delta_q^v \mathcal{Q}^h(U_{\text{osc}}, U_{\text{osc}}) | \Delta_q^v U_{\text{osc}}) + (\Delta_q^v \mathcal{Q}^3(U_{\text{osc}}, U_{\text{osc}}) | \Delta_q^v U_{\text{osc}}) \\ &= C^h(q) + C^v(q), \end{aligned}$$

where \mathcal{Q}^h and \mathcal{Q}^3 are respectively defined as

$$\mathcal{Q}^h(U_{\text{osc}}, U_{\text{osc}}) = \lim_{\varepsilon \rightarrow 0} \mathcal{L}\left(-\frac{t}{\varepsilon}\right) \left[\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right) U_{\text{osc}}^\varepsilon \right)^h \cdot \nabla_h \mathcal{L}\left(\frac{t}{\varepsilon}\right) U_{\text{osc}} \right], \quad (3.8.13)$$

$$\mathcal{Q}^3(U_{\text{osc}}, U_{\text{osc}}) = \lim_{\varepsilon \rightarrow 0} \mathcal{L}\left(-\frac{t}{\varepsilon}\right) \left[\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right) U_{\text{osc}}^\varepsilon \right)^3 \partial_3 \mathcal{L}\left(\frac{t}{\varepsilon}\right) U_{\text{osc}} \right] \quad (3.8.14)$$

By aid of Bony decomposition as in (4.1.10) we can say that

$$\begin{aligned} C^h(q) &= \sum_{|q-q'|\leq 4} (\Delta_q^v \mathcal{Q}^h(S_{q'-1}^v U_{\text{osc}}, \Delta_{q'}^v U_{\text{osc}}) | \Delta_q^v U_{\text{osc}}) \\ &\quad + \sum_{q'>q-4} (\Delta_q^v \mathcal{Q}^h(\Delta_{q'}^v U_{\text{osc}}, S_{q'+2}^v U_{\text{osc}}) | \Delta_q^v U_{\text{osc}}) = C_1^h(q) + C_2^h(q). \end{aligned}$$

By use of Lemma 3.8.4

$$C_1^h(q) \lesssim \sum_{|q-q'|\leq 4} \|S_{q'-1}^v U_{\text{osc}}\|_{H^{1/2,0}} \|\Delta_{q'}^v \nabla_h U_{\text{osc}}\|_{L^2(\mathbb{T}^3)} \|\Delta_q^v U_{\text{osc}}\|_{H^{1/2,0}},$$

moreover since U_{osc} is a vector field with zero horizontal average we can apply (3.2.8)

$$\begin{aligned} \|S_{q'-1}^v U_{\text{osc}}\|_{H^{1/2,0}} &\lesssim \|U_{\text{osc}}\|_{L^2(\mathbb{T}^3)}^{1/2} \|\nabla_h U_{\text{osc}}\|_{L^2(\mathbb{T}^3)}^{1/2}, \\ \|\Delta_q^v U_{\text{osc}}\|_{H^{1/2,0}} &\lesssim \|\Delta_q^v U_{\text{osc}}\|_{L^2(\mathbb{T}^3)}^{1/2} \|\nabla_h \Delta_q^v U_{\text{osc}}\|_{L^2(\mathbb{T}^3)}^{1/2}, \end{aligned}$$

whence thanks to (3.2.5) and the fact that we are summing on a finite set of q'

$$C_1^h(q) \lesssim b_q(t) 2^{-2qs} \|U_{\text{osc}}\|_{L^2(\mathbb{T}^3)}^{1/2} \|\nabla_h U_{\text{osc}}\|_{L^2(\mathbb{T}^3)}^{1/2} \|U_{\text{osc}}\|_{H^{0,s}}^{1/2} \|\nabla_h U_{\text{osc}}\|_{H^{0,s}}^{3/2}. \quad (3.8.15)$$

Similar computations give us the result for C_2^h , here we sketch the procedure. Respectively using (3.8.21), (3.2.5) and summing on the summation set

$$\begin{aligned} C_2^h(q) &= \sum_{q'>q-4} (\Delta_q^v \mathcal{Q}^h(\Delta_{q'}^v U_{\text{osc}}, S_{q'+2}^v U_{\text{osc}}) | \Delta_q^v U_{\text{osc}}) \\ &\lesssim b_q(t) 2^{-2qs} \|\nabla_h U_{\text{osc}}\|_{L^2(\mathbb{T}^3)} \|U_{\text{osc}}\|_{H^{0,s}} \|\nabla_h U_{\text{osc}}\|_{H^{0,s}}. \end{aligned} \quad (3.8.16)$$

On the term C^v we apply instead Bony decomposition as in (3.8.2) obtaining

$$\begin{aligned} C^v(q) &= (\mathcal{Q}^3(S_{q-1}^v U_{\text{osc}}, \Delta_q^v U_{\text{osc}}) | \Delta_q^v U_{\text{osc}}) \\ &\quad + \sum_{|q-q'|\leq 4} (\mathcal{Q}^3(S_{q-1}^v U_{\text{osc}} - S_{q'-1}^v U_{\text{osc}}, \Delta_q^v \Delta_{q'}^v U_{\text{osc}}) | \Delta_q^v U_{\text{osc}}) \\ &\quad + \sum_{|q-q'|\leq 4} \lim_{\varepsilon \rightarrow 0} \left(\left[\Delta_q^v, S_{q'-1}^v \left(\mathcal{L}\left(\frac{t}{\varepsilon}\right) U_{\text{osc}} \right)^3 \right] \Delta_{q'}^v \partial_3 \mathcal{L}\left(\frac{t}{\varepsilon}\right) U_{\text{osc}} \middle| \Delta_q^v \mathcal{L}\left(\frac{t}{\varepsilon}\right) U_{\text{osc}} \right) \\ &\quad + \sum_{q'>q-4} (\Delta_q^v \mathcal{Q}^3(\Delta_{q'}^v U_{\text{osc}}, S_{q'+2}^v U_{\text{osc}}) | \Delta_q^v U_{\text{osc}}) = \sum_{k=1}^4 C_k^v(q), \end{aligned}$$

where \mathcal{Q}^3 is defined in (3.8.14). Let us consider the term $C_1^v(q)$ first. Integration by parts and the fact that we are considering divergence-free vector fields gives us

$$\begin{aligned} C_1^v(q) &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{T}^3} S_{q-1}^v \left(\mathcal{L} \left(\frac{t}{\varepsilon} \right) U_{\text{osc}} \right)^3 \partial_3 \mathcal{L} \left(\frac{t}{\varepsilon} \right) \Delta_q^v U_{\text{osc}} \mathcal{L} \left(\frac{t}{\varepsilon} \right) \Delta_q^v U_{\text{osc}} dx \\ &= -\frac{1}{2} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{T}^3} S_{q-1}^v \operatorname{div}_h \left(\mathcal{L} \left(\frac{t}{\varepsilon} \right) U_{\text{osc}} \right)^h \left| \mathcal{L} \left(\frac{t}{\varepsilon} \right) \Delta_q^v U_{\text{osc}} \right|^2 dx. \end{aligned}$$

Moreover using the fact that $\mathcal{L} \left(\frac{t}{\varepsilon} \right)$ is an isometry on Sobolev spaces, and (3.8.21) we deduce

$$\begin{aligned} C_1^v(q) &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{T}^3} S_{q-1}^v \operatorname{div}_h \left(\mathcal{L} \left(\frac{t}{\varepsilon} \right) U_{\text{osc}} \right)^h \left(\mathcal{L} \left(\frac{t}{\varepsilon} \right) \Delta_q^v U_{\text{osc}} \right)^2 dx \\ &\lesssim b_q(t) 2^{-2qs} \|\nabla_h U_{\text{osc}}\|_{L^2(\mathbb{T}^3)} \|U_{\text{osc}}\|_{H^{0,s}} \|\nabla_h U_{\text{osc}}\|_{H^{0,s}}. \end{aligned} \quad (3.8.17)$$

Let us consider the term C_2^v which is defined as

$$\begin{aligned} C_2^v(q) &= \sum_{|q-q'| \leq 4} \left(\mathcal{Q}^3 (S_{q-1}^v U_{\text{osc}} - S_{q'-1}^v U_{\text{osc}}, \Delta_q^v \Delta_{q'}^v U_{\text{osc}}) \middle| \Delta_q^v U_{\text{osc}} \right) \\ &= \sum_{|q-q'| \leq 4} \sum_{\substack{(k,m,n) \in \mathcal{K}^* \\ a,b,c,d = \pm}} \left(\widehat{S_{q-1}^v U}^{a,3}(k) - \widehat{S_{q'-1}^v U}^{a,3}(k) \right) \check{m}_3 \widehat{\Delta_q^v \Delta_{q'}^v U}^{b,c}(m,n) \widehat{\Delta_q^v U}^d(n), \end{aligned}$$

where $\widehat{U}^{b,c}(m,n) = \left(\widehat{U}^b(m) \middle| e^c(n) \right) e^c(n)$. Since the eigenvectors e^c can always be considered normalized to norm one we deduce $\left| \widehat{U}^{b,c}(m,n) \right| \lesssim |U^b(m)|$. At this point we can use Lemma 3.8.4 to obtain the bound

$$C_2^v(q) \lesssim \sum_{|q-q'| \leq 4} \sum_{a=\pm} \|S_{q-1}^v U^{a,3} - S_{q'-1}^v U^{a,3}\|_{L^2(\mathbb{T}^3)} \|\Delta_q^v \Delta_{q'}^v \partial_3 U_{\text{osc}}\|_{H^{1/2,0}} \|\Delta_q^v U_{\text{osc}}\|_{H^{1/2,0}}.$$

We remark that the term U^a is in fact divergence-free.

Thanks to Lemma 4.1.8

$$\begin{aligned} \|S_{q-1}^v U^{a,3} - S_{q'-1}^v U^{a,3}\|_{L^2(\mathbb{T}^3)} &\lesssim 2^{-q} \|(S_{q-1}^v - S_{q'-1}^v) \partial_3 U^{a,3}\|_{L^2(\mathbb{T}^3)} \\ &\lesssim 2^{-q} \|(S_{q-1}^v - S_{q'-1}^v) \nabla_h U_{\text{osc}}\|_{L^2(\mathbb{T}^3)} \\ \|\Delta_q^v \Delta_{q'}^v \partial_3 U_{\text{osc}}\|_{H^{1/2,0}} &\lesssim 2^q \|\Delta_q^v \Delta_{q'}^v U_{\text{osc}}\|_{H^{1/2,0}} \end{aligned}$$

Hence using first (3.2.5) and then (3.2.11)

$$C_2^v(q) \lesssim b_q(t) 2^{-2qs} \|\nabla_h U_{\text{osc}}\|_{L^2(\mathbb{T}^3)} \|U_{\text{osc}}\|_{H^{0,s}} \|\nabla_h U_{\text{osc}}\|_{H^{0,s}}. \quad (3.8.18)$$

The term $C_3^v(q)$ will be handled in a different way. First of all, writing $f_\varepsilon = \mathcal{L} \left(\frac{t}{\varepsilon} \right) f$ and considering that commutators can be expressed as convolutions (as it has been expressed in detail in the Section 3.2.1, see equation (3.2.6)) we can write $C_3^v(q)$ as

$$\begin{aligned} C_3^v(q) &= \lim_{\varepsilon \rightarrow 0} \sum_{|q-q'| \leq 4} \int_{\mathbb{T}^3} \int_{\mathbb{T}_b^1 \times [0,1]} \tilde{h}(2^q y_3) (S_{q'-1}^v \partial_3 U_{\text{osc},\varepsilon}^3)(x_h, x_3 + \tau(x_3 - y_3)) \\ &\quad \times \partial_3 \Delta_{q'}^v U_{\text{osc},\varepsilon}(x_h, x_3 - y_3) \Delta_q^v U_{\text{osc},\varepsilon}(x) dy_3 d\tau dx_h dx_3, \end{aligned}$$

with $\tilde{h}(z) = zh(z)$ and $h = \mathcal{F}^{-1}\varphi$. Taking the limit as $\varepsilon \rightarrow 0$, using the divergence free-property we obtain the following bound

$$|C_3^v(q)| \leq \sum_{|q-q'|\leq 4} \int_{\mathbb{T}_v^1 \times [0,1]} \sum_{(k,n) \in \mathcal{K}^*} \tilde{h}(2^q y_3) \left| \mathcal{F} \left((S_{q'-1}^v \nabla_h U_{\text{osc}}) (x_h, x_3 + \tau (x_3 - y_3)) \right) (k) \right| \\ \times \left| \mathcal{F} \left(\partial_3 \Delta_{q'}^v U_{\text{osc}} (x_h, x_3 - y_3) \right) (n-k) \widehat{\Delta_q^v U_{\text{osc}}}(n) \right| dy_3 d\tau,$$

applying Lemma 3.8.4

$$C_3^v(q) \lesssim \sum_{|q-q'|\leq 4} \int_{\mathbb{T}_v^1 \times [0,1]} \tilde{h}(2^q y_3) \left\| S_{q'-1}^v \nabla_h U_{\text{osc}} (x_h, x_3 + \tau (x_3 - y_3)) \right\|_{L^2(\mathbb{T}^3)} \\ \times \left\| \partial_3 \Delta_{q'}^v U_{\text{osc}} (x_h, x_3 - y_3) \right\|_{H^{1/2,0}} \left\| \Delta_q^v U_{\text{osc}} \right\|_{H^{1/2,0}} dy_3 d\tau,$$

by standard calculations, localization of the term $\partial_3 \Delta_{q'}^v U_{\text{osc}}$ and (3.2.8) we obtain

$$C_3^v(q) \lesssim b_q(t) 2^{-2qs} \left\| \nabla_h U_{\text{osc}} \right\|_{L^2(\mathbb{T}^3)} \left\| U_{\text{osc}} \right\|_{H^{0,s}} \left\| \nabla_h U_{\text{osc}} \right\|_{H^{0,s}}. \quad (3.8.19)$$

Lastly, for the reminder term $C_4^v(q)$, if we apply Lemma 3.8.4 and Lemma 4.1.8 as for the term $C_2^v(q)$ we get

$$C_4^v(q) \lesssim \sum_{q'>q-4} \left\| S_{q'+2}^v U_{\text{osc}} \right\|_{H^{1/2,0}} \left\| \Delta_{q'}^v \nabla_h U_{\text{osc}} \right\|_{L^2(\mathbb{T}^3)} \left\| \Delta_q^v U_{\text{osc}} \right\|_{H^{1/2,0}},$$

hence by localization and the interpolation (3.2.11) we obtain

$$C_4^v(q) \lesssim b_q(t) 2^{-2qs} \left\| U_{\text{osc}} \right\|_{L^2(\mathbb{T}^3)}^{1/2} \left\| \nabla_h U_{\text{osc}} \right\|_{L^2(\mathbb{T}^3)}^{1/2} \left\| U_{\text{osc}} \right\|_{H^{0,s}}^{1/2} \left\| \nabla_h U_{\text{osc}} \right\|_{H^{0,s}}^{3/2}. \quad (3.8.20)$$

The estimates (3.8.15)–(3.8.20) prove hence (3.8.9). \square

3.8.2 The bilinear form \mathcal{Q} .

In this section we state some particular property of the quadratic limit form defined in (3.3.14). In particular we state a product rule which can be applied thanks to the particular structure of the resonance set $\mathcal{K}^* = \bigcup_{n \in \mathbb{Z}^3} \mathcal{K}_n^*$, which is a crucial feature in the energy

estimates for the limit system.

The following property has been remarked at first by A. Babin et al. in [8], but was first explicitly proved by M. Paicu in [123]. The proof is based on the fact that, fixed (k_h, n) , the fiber $\mathcal{J}(k_h, n) = \{k_3 : (k, n) \in \mathcal{K}^*\}$ is of finite cardinality.

Lemma 3.8.4. *Let $a, b \in H^{1/2,0}(\mathbb{T}^3)$, $c \in L^2(\mathbb{T}^3)$ vector fields of zero horizontal average on \mathbb{T}_h^2 . Then there exists a constant C which depends only of a_1/a_2 such that*

$$\left| \sum_{(k,n) \in \mathcal{K}^*} \hat{a}(k) \hat{b}(n-k) \hat{c}(n) \right| \leq \frac{C}{a_3} \|a\|_{H^{1/2,0}(\mathbb{T}^3)} \|b\|_{H^{1/2,0}(\mathbb{T}^3)} \|c\|_{L^2(\mathbb{T}^3)} \quad (3.8.21)$$

The following proof can be found [42, Lemma 6.6, p. 150] or [123, Lemma 6.4, p. 222].

Proof. We shall give the proof on the torus $[0, 2\pi]^3$, whence

$$\begin{aligned} I_{\mathcal{K}^*} &= \left| \sum_{(k,n) \in \mathcal{K}^*} \hat{a}_k \hat{b}_{n-k} \hat{c}_n \right| \leq \sum_{(k_h, n) \in \mathbb{Z}^2 \times \mathbb{Z}^3} \sum_{\{k_3: (k,n) \in \mathcal{K}^*\}} \left| \hat{a}_k \hat{b}_{n-k} \hat{c}_n \right| \\ &\leq \sum_{(k_h, n) \in \mathbb{Z}^2 \times \mathbb{Z}^3} |\hat{c}_n| \sum_{\{k_3: (k,n) \in \mathcal{K}^*\}} |\hat{a}_k| |\hat{b}_{n-k}|, \end{aligned} \quad (3.8.22)$$

by Cauchy-Schwarz inequality

$$\sum_{\{k_3: (k,n) \in \mathcal{K}^*\}} |\hat{a}_k| |\hat{b}_{n-k}| \leq \left(\sum_{\{k_3: (k,n) \in \mathcal{K}^*\}} |\hat{a}_k|^2 |\hat{b}_{n-k}|^2 \right)^{1/2} \left(\sum_{\{k_3: (k,n) \in \mathcal{K}^*\}} 1 \right)^{1/2},$$

now, fixing $(k_h, n) \in \mathbb{Z}^2 \times \mathbb{Z}^3$ there exists only a finite number (8) or resonant modes k_3 , i.e. $\#(\{k_3 : (k, n) \in \mathcal{K}^*\}) \leq 8$. Let us briefly explain why this is true. We write explicitly the resonant condition $\omega_{k, n-k, n}^{+,+,+} = 0$ (the same procedure holds for the generic case $\omega_{k, n-k, n}^{a,b,c} = 0$, $a, b, c \neq 0$), this reads as

$$\left(\frac{|Fk_3|^2 + |k_h|^2}{|k_3|^2 + |k_h|^2} \right)^{1/2} + \left(\frac{(F|n_3 - k_3|)^2 + |n_h - k_h|^2}{|n_3 - k_3|^2 + |n_h - k_h|^2} \right)^{1/2} = \left(\frac{|Fn_3|^2 + |n_h|^2}{|n_3|^2 + |n_h|^2} \right)^{1/2}.$$

Taking squares several times on both sides of the above equation give us an expression which is free of square roots. Moreover putting everything to common factor and recalling that n, k_h are fixed we transformed the above equation in the form $R(k_3) = 0$, $R \in \mathbb{R}[x]$, hence thanks to fundamental theorem of algebra it has a finite number of roots.

From this we deduce

$$\sum_{\{k_3: (k,n) \in \mathcal{K}^*\}} |\hat{a}_k| |\hat{b}_{n-k}| \leq \sqrt{8} \left(\sum_{\{k_3: (k,n) \in \mathcal{K}^*\}} |\hat{a}_k|^2 |\hat{b}_{n-k}|^2 \right)^{1/2},$$

which considered into inequality (3.8.22) gives

$$I_{\mathcal{K}^*} \leq \sqrt{8} \sum_{k_h, n_h} \sum_{n_3} |\hat{c}_n| \left(\sum_{k_3} |\hat{a}_k|^2 |\hat{b}_{n-k}|^2 \right)^{1/2}.$$

Moreover

$$\sum_{n_3} |\hat{c}_n| \left(\sum_{k_3} |\hat{a}_k|^2 |\hat{b}_{n-k}|^2 \right)^{1/2} \leq \left(\sum_{n_3} |\hat{c}_n|^2 \right)^{1/2} \left(\sum_{n_3, k_3} |\hat{a}_k|^2 |\hat{b}_{n-k}|^2 \right)^{1/2},$$

and hence

$$I_{\mathcal{K}^*} \leq \sqrt{8} \sum_{(k_h, n) \in \mathbb{Z}^2 \times \mathbb{Z}^3} \left(\sum_{n_3} |\hat{c}_n|^2 \right)^{1/2} \left(\sum_{p_3} |\hat{b}_{n_h - k_h, p_3}|^2 \right)^{1/2} \left(\sum_{k_3} |\hat{a}_k|^2 \right)^{1/2}. \quad (3.8.23)$$

Let us denote at this point

$$\tilde{a}_{n_h} = \left(\sum_{n_3} |\hat{a}_n|^2 \right)^{1/2}, \quad \tilde{b}_{n_h} = \left(\sum_{n_3} |\hat{b}_n|^2 \right)^{1/2}, \quad \tilde{c}_{n_h} = \left(\sum_{n_3} |\hat{c}_n|^2 \right)^{1/2},$$

and the following distributions

$$\tilde{a}(x_h) = \mathcal{F}_h^{-1}(\tilde{a}_{n_h}) \quad \tilde{b}(x_h) = \mathcal{F}_h^{-1}(\tilde{b}_{n_h}) \quad \tilde{c}(x_h) = \mathcal{F}_h^{-1}(\tilde{c}_{n_h}).$$

Whence the inequality (3.8.23) can be read, applying Plancherel theorem and the product rules for Sobolev spaces, as

$$\begin{aligned} I_{\mathcal{K}^*} &\leq \left(\tilde{a} \tilde{b} \middle| \tilde{c} \right)_{L^2(\mathbb{T}_h^2)} \\ &\leq \left\| \tilde{a} \tilde{b} \right\|_{L^2(\mathbb{T}_h^2)} \left\| \tilde{c} \right\|_{L^2(\mathbb{T}_h^2)} \\ &\leq \left\| \tilde{a} \right\|_{H^{1/2}(\mathbb{T}_h^2)} \left\| \tilde{b} \right\|_{H^{1/2}(\mathbb{T}_h^2)} \left\| \tilde{c} \right\|_{L^2(\mathbb{T}_h^2)} \\ &= \left\| a \right\|_{H^{1/2,0}(\mathbb{T}^3)} \left\| b \right\|_{H^{1/2,0}(\mathbb{T}^3)} \left\| c \right\|_{L^2(\mathbb{T}^3)}. \end{aligned}$$

To lift this argument to a generic torus $\prod_{i=1}^3 [0, 2\pi a_i)$ it suffices to use the transform

$$\tilde{v}(x_1, x_2, x_3) = v(a_1 x_1, a_2 x_2, a_3 x_3),$$

and the identity

$$\left\| \tilde{v} \right\|_{L^2([0, 2\pi]^3)} = (a_1 a_2 a_3)^{-1/2} \left\| v \right\|_{L^2(\prod_{i=1}^3 [0, 2\pi a_i])}.$$

□

Chapter 4

Dynamic of stratified fluids in low Froude number regime in space-periodic domains.

If we knew what it was we were doing,
it would not be called research, would
it?

Albert Einstein

4.1 Introduction.

In the present article we study the behavior of strong solutions of the following modified Boussinesq system

$$\begin{cases} \partial_t v^\varepsilon + v^\varepsilon \cdot \nabla v^\varepsilon - \nu \Delta v^\varepsilon - \frac{1}{\varepsilon} \theta^\varepsilon \vec{e}_3 &= -\frac{1}{\varepsilon} \nabla \Phi^\varepsilon, \\ \partial_t \theta^\varepsilon + v^\varepsilon \cdot \nabla \theta^\varepsilon - \nu' \Delta \theta^\varepsilon + \frac{1}{\varepsilon} v^{3,\varepsilon} &= 0, \\ \operatorname{div} v^\varepsilon &= 0, \\ (v^\varepsilon, \theta^\varepsilon)|_{t=0} &= (v_0, \theta_0), \end{cases} \quad (\text{PBS}_\varepsilon)$$

for data which are periodic-in-space in the regime $\varepsilon \rightarrow 0$. The space variable x shall be many times considered separately with respect to the horizontal and vertical components, i.e. $x = (x_h, x_3) = (x_1, x_2, x_3)$. We denote $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ the standard laplacian, $\Delta_h = \partial_1^2 + \partial_2^2$ is the laplacian in the horizontal directions. The symbol ∇ represents the gradient in all space directions $\nabla = (\partial_1, \partial_2, \partial_3)^\top$, while we denote $\nabla_h = (\partial_1, \partial_2)^\top$, $\nabla_h^\perp = (-\partial_2, \partial_1)^\top$ respectively the horizontal gradient and the "orthogonal" horizontal gradient. Considered a

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vector field w we denote $\operatorname{div} w = \partial_1 w^1 + \partial_2 w^2 + \partial_3 w^3$. Given two three-components vector fields w, z the notation $w \cdot \nabla z$ indicates the operator

$$w \cdot \nabla z = \sum_{i=1}^3 w^i \partial_i z.$$

Generally for any two-components vector field $u = (u^1, u^2)$ we shall denote as $u^\perp = (-u^2, u^1)$. The viscosity ν, ν' above are strictly positive constants $\nu, \nu' \geq c > 0$.

As we already mentioned the goal of the present paper is to study the behavior of (strong) solutions of $(\mathbf{PBS}_\varepsilon)$ in the regime $\varepsilon \rightarrow 0$ for periodic-in-space data, i.e. given $a_i > 0, i = 1, 2, 3$ we consider the domain

$$\mathbb{T}^3 = \prod_{i=1}^3 [0, 2\pi a_i],$$

and we look for a divergence-free vector field $v^\varepsilon : \mathbb{R}_+ \times \mathbb{T}^3 \rightarrow \mathbb{R}^3$ and a scalar function $\theta^\varepsilon : \mathbb{R}_+ \times \mathbb{T}^3 \rightarrow \mathbb{R}$ such that $(v^\varepsilon, \theta^\varepsilon)$ solves $(\mathbf{PBS}_\varepsilon)$. The functions $(v^\varepsilon, \theta^\varepsilon)$ depend on $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^3$. The system $(\mathbf{PBS}_\varepsilon)$ belongs to a much wider family of problems which may be written in the following general form

$$\begin{cases} \partial_t V^\varepsilon + v^\varepsilon \cdot \nabla V^\varepsilon + A_2(D) V^\varepsilon + \frac{1}{\varepsilon} \mathcal{S}(V^\varepsilon) = 0, \\ V(0, x) = V_0(x), \end{cases} \quad (4.1.1)$$

where A_2 is an elliptic operator and \mathcal{S} is skew-symmetric.

The problem of systems with skew symmetric singular perturbation is not at all new in the literature. S. Klainerman and A. Majda in [96] develop a first generic theory whose aim is to study a number of problems arising in physics, when certain physical magnitudes blow-up, which can be described by the aid of quasilinear symmetric hyperbolic systems. They use this theory in order to study the following system describing the motion of a compressible fluid

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon v^\varepsilon) = 0, \\ \partial_t v^\varepsilon + v^\varepsilon \cdot \nabla v^\varepsilon = -\frac{1}{\varepsilon^2} \frac{1}{\rho^\varepsilon} \nabla p(\rho^\varepsilon), \\ v^\varepsilon|_{t=0} = v_0(x), \quad \rho^\varepsilon|_{t=0} = \rho^0, \\ \operatorname{div} v_0 = 0, \end{cases} \quad (\mathbf{WCE}_\varepsilon)$$

when $\varepsilon \rightarrow 0$. The parameter ε is called the *Mach number* and describes the rate of compressibility of a fluid. Let us underline the fact that the initial data is of "incompressible type". The choice of the initial data is relevant since "compressible initial data" generate fast oscillating perturbations which propagate at speed $\mathcal{O}(\varepsilon^{-1})$ which are not easy to handle mathematically. The authors prove that solutions of $(\mathbf{WCE}_\varepsilon)$ converge locally (in time) and strongly to local solutions of the incompressible Navier-Stokes system.

Another system which falls in the family of singular perturbations problems is the Navier-Stokes -Coriolis equations

$$\begin{cases} \partial_t v_{\text{RF}}^\varepsilon + v_{\text{RF}}^\varepsilon \cdot \nabla v_{\text{RF}}^\varepsilon - \nu \Delta v_{\text{RF}}^\varepsilon + \frac{1}{\varepsilon} e^3 \wedge v_{\text{RF}}^\varepsilon = -\frac{1}{\varepsilon} \nabla p_{\text{RF}}^\varepsilon, \\ \operatorname{div} v_{\text{RF}}^\varepsilon = 0, \\ v_{\text{RF}}^\varepsilon(0, x) = v_{\text{RF},0}^\varepsilon(x). \end{cases} \quad (\mathbf{RF}_\varepsilon)$$

E. Grenier in [79] proved that, as long as the initial data is a bidimensional flow (case which we refer as *well prepared initial data*) the solutions of $(\mathbf{RF}_\varepsilon)$ in a periodic setting converge strongly, after a suitable renormalization, to those of a two-dimensional Navier-Stokes system.

A. Babin A. Mahalov and B. Nicolaenko studied at first the equation $(\mathbf{RF}_\varepsilon)$ in the periodic setting in a series of work (such as [5], [8] , this list is non exhaustive) when the initial data is considered to be generic or *ill-prepared*, in the sense that it is not a bidimensional flow. Purely three-dimensional perturbations hence can interact constructively between each other, such as in standard Navier-Stokes equations. This problem is overcome in a twofold way: at first in [5] a geometric hypotheses on the domain is done so that no bilinear interaction can occur. Such domains are said to be *non-resonant*, we shall adopt this kind of approach in the present work. In [8] instead the domain is considered to be generic, but the authors manage to prove that three-dimensional bilinear interactions are localized in a very specific way in the frequency space. This observation allows hence to deduce an improved product rule which hence can be used to prove that the limit system, despite being three-dimensional and nonlinear, is well posed. This is the key observation which allows them to prove a result of strong, global convergence to a two-dimensional Navier-Stokes system after renormalization.

Finally in [72] I. Gallagher studied systems in an even more generic form than (4.1.1), giving a generic theory for the convergence of parabolic systems with singular perturbation in periodic domains. This allowed her to obtain some global strong convergence results for rapidly rotating fluids $(\mathbf{RF}_\varepsilon)$ and for a system describing density dependent fluids under the effects of rotation and gravitational stratification called the *primitive equations* (see $(\mathbf{PE}_\varepsilon)$). The convergence theory developed by the author is based on a theory developed by S. Schochet in [133] in the setting of quasilinear hyperbolic systems and hence adapted to the parabolic case. Such technique consists in determining a "smart" change of variable, which cancels interactions which converge to zero only in a distributional sense. The resulting new unknown is an $\mathcal{O}(\varepsilon)$ perturbation of the original unknown, but the equation satisfied by the new variable has a simpler spectrum of nonlinear interactions, making hence possible to prove that this new unknown is globally well posed and deducing the result for the initial functions. This technique shall be adopted in the present work as well.

As mentioned many times already we are interested in the dynamics of the system $(\mathbf{PBS}_\varepsilon)$ in the limit as $\varepsilon \rightarrow 0$ in the periodic case. Recently K. Widmayer in [144] proved that, in the whole space and for the inviscid case, the limit system solves a two dimensional incompressible stratified Euler equation

$$\begin{cases} \partial_t \bar{u}_E^h + \bar{u}_E^h \cdot \nabla_h \bar{u}_E^h + \nabla_h \bar{p}_E = 0, \\ \operatorname{div}_h \bar{u}_E^h = 0. \end{cases} \quad (\text{E-2D})$$

His proof relied on the fact that, in the whole space \mathbb{R}^3 , the highly perturbative part of the solution decay at infinity as $\varepsilon \rightarrow 0$. Recently S. Ibrahim and T. Yoneda studied in [84] the system $(\mathbf{PBS}_\varepsilon)$ in a quasi-periodic setting proving a stability result for arbitrarily, finite, timespans $[0, T]$.

Let us rewrite the system (PBS_ε) into the following more compact form

$$\begin{cases} \partial_t V^\varepsilon + v^\varepsilon \cdot \nabla V^\varepsilon - \mathbb{D}V^\varepsilon + \frac{1}{\varepsilon} \mathcal{A}V^\varepsilon = -\frac{1}{\varepsilon} \begin{pmatrix} \nabla \Phi^\varepsilon \\ 0 \end{pmatrix}, \\ V^\varepsilon = (v^\varepsilon, \theta^\varepsilon), \\ \operatorname{div} v^\varepsilon = 0, \end{cases} \quad (\text{PBS}_\varepsilon)$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \mathbb{D} = \begin{pmatrix} \nu \Delta & 0 & 0 & 0 \\ 0 & \nu \Delta & 0 & 0 \\ 0 & 0 & \nu \Delta & 0 \\ 0 & 0 & 0 & \nu' \Delta \end{pmatrix}. \quad (4.1.2)$$

4.1.1 A survey on the notation adopted.

All along this note we consider real valued vector fields, i.e. applications $V : \mathbb{R}_+ \times \mathbb{T}^3 \rightarrow \mathbb{R}^4$. We will often associate to a vector field V the vector field v which shall be simply the projection on the first three components of V . The vector fields considered are periodic in all their directions and they have zero global average $\int_{\mathbb{T}^3} V dx = 0$, which is equivalent to assume that the first Fourier coefficient $\hat{V}(0) = 0$. We remark that the zero average propriety stated above is preserved in time t for both Navier-Stokes equations as well as for the system (PBS_ε).

Let us define the Sobolev space $H^s(\mathbb{T}^3)$, which consists in all the tempered distributions u such that

$$\|u\|_{H^s(\mathbb{T}^3)} = \left(\sum_{n \in \mathbb{Z}^3} (1 + |\tilde{n}|^2)^s |\hat{u}_n|^2 \right)^{1/2} < \infty. \quad (4.1.3)$$

Since we shall consider always vector fields whose average is null the Sobolev norm defined above in particular is equivalent to the following semi-norm

$$\left\| (-\Delta)^{s/2} u \right\|_{L^2(\mathbb{T}^3)} \sim \|u\|_{H^s(\mathbb{T}^3)}, \quad s \in \mathbb{R},$$

which appears naturally in parabolic problems.

Let us define the operator \mathbb{P} as the three dimensional Leray operator $\mathbb{P}^{(3)}$ which leaves untouched the fourth component, i.e.

$$\mathbb{P} = \left(\frac{\delta_{i,j} - \Delta^{-1} \partial_i \partial_j}{0} \middle| \frac{0}{1} \right)_{i,j=1,2,3} = \left(\frac{\mathbb{P}^{(3)}}{0} \middle| \frac{0}{1} \right).$$

The operator \mathbb{P} is a pseudo-differential operator, in the Fourier space its symbol is

$$\mathbb{P}_n = \left(\frac{\delta_{i,j} - \frac{\tilde{n}_i \tilde{n}_j}{|\tilde{n}|^2}}{0} \middle| \frac{0}{1} \right)_{i,j=1,2,3}, \quad (4.1.4)$$

where $\delta_{i,j}$ is Kronecker's delta and $\tilde{n}_i = n_i/a_i$, $|\tilde{n}|^2 = \sum_i \tilde{n}_i^2$.

4.1.2 Anisotropic spaces.

The problem presents a singular perturbation \mathcal{A} which does not acts symmetrically on the two-dimensional unit-sphere \mathbb{S}^2 , namely there is a relevant external force acting along the vertical direction. This asymmetry in the balance of forces induces the solutions of (PBS $_{\varepsilon}$) to behave differently along the horizontal and vertical directions. For this reason we are forced to introduce anisotropic spaces, which means spaces which behaves differently in the horizontal or vertical direction. Let us recall that, in the periodic case, the non-homogeneous Sobolev anisotropic spaces are defined by the norm

$$\|u\|_{H^{s,s'}(\mathbb{T}^3)}^2 = \sum_{n=(n_h, n_3) \in \mathbb{Z}^3} (1 + |\check{n}_h|^2)^s (1 + |\check{n}_3|^2)^{s'} |\hat{u}_n|^2,$$

where we denoted $\check{n}_i = n_i/a_i$, $n_h = (n_1, n_2)$ and the Fourier coefficients \hat{u}_n are given by $u = \sum_n \hat{u}_n e^{2\pi i \check{n} \cdot x}$. In the whole text \mathcal{F} denotes the Fourier transform. In particular our notation will be

$$\mathcal{F}u(n) = \hat{u}(n) = \hat{u}_n = \int_{\mathbb{T}^3} u(x) e^{2\pi i \check{n} \cdot x} dx.$$

Let's recall as well the definition of the anisotropic Lebesgue spaces, we denote with $L_h^p L_v^q$ the space $L^p(\mathbb{T}_h^2; L^q(\mathbb{T}_v^1))$, defined by the norm:

$$\|f\|_{L_h^p L_v^q} = \left\| \|f(x_h, \cdot)\|_{L^q(\mathbb{T}_v^1)} \right\|_{L^p(\mathbb{T}_h^2)} = \left(\int_{\mathbb{T}_h^2} \left(\int_{\mathbb{T}_v^1} |f(x_h, x_3)|^q dx_3 \right)^{\frac{p}{q}} dx_h \right)^{\frac{1}{p}},$$

in a similar way we denote the space $L_v^q L_h^p$. It is well-known that the order of integration is important as it is described in the following lemma

Lemma 4.1.1. *Let $1 \leq p \leq q$ and $f : X_1 \times X_2 \rightarrow \mathbb{R}$ a function belonging to $L^p(X_1; L^q(X_2))$ where $(X_1; \mu_1), (X_2; \mu_2)$ are measurable spaces, then $f \in L^q(X_2; L^p(X_1))$ and we have the inequality*

$$\|f\|_{L^q(X_2; L^p(X_1))} \leq \|f\|_{L^p(X_1; L^q(X_2))}.$$

In the anisotropic setting the Cauchy-Schwarz inequality becomes;

$$\|fg\|_{L_h^p L_v^q} \leq \|f\|_{L_h^{p'} L_v^{q'}} \|g\|_{L_h^{p''} L_v^{q''}},$$

where $1/p = 1/p' + 1/p''$, $1/q = 1/q' + 1/q''$.

We shall need as well to define spaces which are of mixed Lebesgue-Sobolev type. Namely, we define the space

$$L_v^p(H_h^\sigma) = L^p(\mathbb{T}_v^1; H^\sigma(\mathbb{T}_h^2)), \quad p \in [1, \infty),$$

as the closure of the tempered distributions with respect to the norm

$$\|u\|_{L_v^p(H_h^\sigma)} = \left(\int_{\mathbb{T}_v^1} \left| \sum_{n_h \in \mathbb{Z}^2} (1 + |\check{n}_h|^2)^\sigma |\mathcal{F}_h u(n_h, x_3)|^2 \right|^{p/2} dx_3 \right)^{1/p}.$$

We define the space $L_v^\infty(H_h^\sigma) = L^\infty(\mathbb{T}_v^1; H^\sigma(\mathbb{T}_h^2))$ thanks to the norm

$$\|u\|_{L_v^\infty(H_h^\sigma)} = \sup_{x_3 \in \mathbb{T}_v^1} \|u(\cdot, x_3)\|_{H_h^\sigma}.$$

4.1.3 Results.

Theorem 4.1.7 is the main result proved in the present work. Unfortunately in order to understand in detail the statement of such theorem some notational explanation (notably Section 4.1.1) was introduced. The first part of the present section instead focuses in introducing some result which is classical in the theory of Navier-Stokes equations and which is of the utmost importance in order to develop the theory in the present work.

We recall at first the celebrated Leray and Fujita-Kato theorems. The first is a result of existence of distributional solutions for Navier-Stokes equations, while the second is a result of (local) well-posedness in Sobolev spaces for Navier-Stokes equations. The proof of such results is considered to be nowadays somehow classical and can be found in many texts, we refer to [69] and [71] or [42].

Theorem (LERAY). *Let us consider the following system describing the evolution of an incompressible viscid fluid in the d -dimensional periodic space \mathbb{T}^d*

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \nu \Delta u = -\nabla p, \\ \nabla \cdot u = 0, \\ u(0) = u_0. \end{cases} \quad (\text{NS})$$

Let u_0 be a divergence-free vector field in $L^2(\mathbb{T}^d)$, then (NS) has a weak solution u such that

$$u \in L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^d)), \quad \nabla u \in L^2(\mathbb{R}_+; L^2(\mathbb{T}^d)).$$

Theorem (FUJITA-KATO). *Let $u_0 \in H^{\frac{d}{2}-1}(\mathbb{T}^d)$, then there exists a positive time T^* such that (NS) has a unique solution $u \in L^4([0, T^*]; H^{\frac{d-1}{2}}(\mathbb{T}^d))$ which also belongs to*

$$\mathcal{C}([0, T^*]; H^{\frac{d}{2}-1}(\mathbb{T}^d)) \cap L^2([0, T^*]; H^{\frac{d}{2}}(\mathbb{T}^d)).$$

Let us denote $T_{u_0}^*$ be the maximal lifespan of the solution of (NS) with initial datum u_0 , then there exists a constant $c > 0$ such that if

$$\|u_0\|_{H^{\frac{d}{2}-1}} \leq c\nu \implies T_{u_0}^* = \infty.$$

Since the perturbation appearing in (PBS_ε) is skew symmetric we know that the bulk force $\mathcal{A}V^\varepsilon$ does not apport any energy in any $H^s(\mathbb{T}^3)$ space, whence Leray and Fujita-Kato theorem can be applied *mutatis mutandis* to the system (PBS_ε) , and in particular this is the formulation which we shall use:

Theorem 4.1.2. *Let $V_0 = (v_0, \theta_0) \in L^2(\mathbb{T}^3)$ and such that $\text{div } v_0 = 0$. Then for each $\varepsilon > 0$ there exists a weak solution V^ε of (PBS_ε) which belongs to the energy space*

$$V^\varepsilon \in L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^3)), \quad \nabla V^\varepsilon \in L^2(\mathbb{R}_+; L^2(\mathbb{T}^3)).$$

Theorem 4.1.3. *If $V_0 \in H^s(\mathbb{T}^3)$ with $s \geq 1/2$ there exists a positive time T^* independent of $\varepsilon > 0$ and a unique strong solution V^ε of (4.1.1) in the space $L^4([0, T]; H^{s+\frac{1}{2}}(\mathbb{T}^3))$ which also belongs to the space $\mathcal{C}([0, T]; H^s(\mathbb{T}^3)) \cap L^2([0, T]; H^{s+1}(\mathbb{T}^3))$ for each $T \in [0, T^*]$. In particular if $\|V_0^\varepsilon\|_{H^{s+\frac{1}{2}}(\mathbb{T}^3)} \leq c\nu$ for some positive and small constant c then the solution is global in \mathbb{R}_+ .*

In the framework of d -dimensional Navier-Stokes equations the propagation of $H^{\frac{d}{2}-1}$ Sobolev regularity is usually referred as propagation of *critical regularity*. It is hence a generally accepted choice of lexicon to denote the regularity H^s , $s > d/2 - 1$ as *subcritical* and H^s , $s < d/2 - 1$ as *supercritical*.

The dynamics of (PBS_ε) varies accordingly to the real parameter ε . The asymptotic regime $\varepsilon \rightarrow 0$ is of particular interest since it describes long-time dynamics of stratified fluids (for a more detailed physical discussion we refer to Section 4.1.4), it is hence relevant to prove that (PBS_ε) admits a limit when $\varepsilon \rightarrow 0$. The limit system may be written as follows:

$$\begin{cases} \partial_t U + \mathcal{Q}(U, U) - \mathbb{D}U = 0, \\ U|_{t=0} = V_0. \end{cases} \quad (\mathcal{S}_0)$$

The sense in which system (PBS_ε) converges to (\mathcal{S}_0) shall be explained in detail in Section 4.2 and 4.3. Section 4.3 is entirely devoted to explain in detail in what consists the limit form \mathcal{Q} and \mathbb{D} .

As it is proven in [72] any system in the generic form (4.1.1) converges to a limit system of the form (\mathcal{S}_0) in the sense of distributions. In the Section 4.6 we extend this convergence to a strong setting. Such technique has been introduced by S. Schochet in [133] in the framework of quasilinear symmetric hyperbolic systems, but the theory in the parabolic setting was developed by I. Gallagher in [72, Theorem 1]. The statement of [72, Theorem 1] is the following:

Theorem (GALLAGHER). *Let $U_0 \in H^s(\mathbb{T}^d)$ with $s \geq \frac{d}{2} - 1$. Let T^* be as in the statement of Theorem 4.1.3. Let $T \in [0, T^*)$, and U be the local, strong solution of (\mathcal{S}_0) determined by Theorem 4.1.3 satisfy*

$$U \in \mathcal{C}([0, T]; H^s(\mathbb{T}^d)) \cap L^2([0, T]; H^{s+1}(\mathbb{T}^d))$$

then, for $\varepsilon > 0$ small enough the associate solution V^ε of (4.1.1) is also defined on $[0, T]$ and

$$V^\varepsilon - \mathcal{L}\left(\frac{t}{\varepsilon}\right)U = o(1),$$

in $\mathcal{C}([0, T]; H^s(\mathbb{T}^d)) \cap L^2([0, T]; H^{s+1}(\mathbb{T}^d))$.

The operator $\mathcal{L}(\tau)$ appearing in the above theorem is nothing but the backward propagator $e^{\tau\mathbb{P}^A}$, we refer to Section 4.2 and 4.3 for a more detailed introduction.

The result we prove has rather long and technical statement, but it simply addresses a stability result of (PBS_ε) to a simplified 3-dimensional nonlinear model, and it is divided in four parts:

1. as $\varepsilon \rightarrow 0$ the system (PBS_ε) converges, in the sense of distributions, to a limit system,
2. the limit system can be simplified, in particular it can be written as the sum of two systems. The first one is similar to a 2D-Navier-Stokes system, the second is a linear system,

3. the aforementioned systems are, individually, globally well posed. Hence the limit system is globally well-posed,
4. (PBS_ε) converges (now strongly) to the limit system which now we know to be globally well-posed. We deduce the convergence to be global.

We would like to spend a couple of words more on the result (4) of the list here above. The convergence procedure gives an additional result which is crucial: we proved in the point (3) that the limit system solved by U is globally well posed in some Sobolev space: V^ε solution of (PBS_ε) converges *globally* to a renormalization of U , hence V^ε is globally well posed as well if ε is sufficiently small.

The first result we prove is the following compactness result concerning the solutions *à la Leray* of the system (PBS_ε) :

Theorem 4.1.4. *Let $\mathcal{L}(\tau) = e^{\tau\mathbb{P}\mathcal{A}}$ where \mathcal{A} and \mathbb{P} are defined respectively in (4.1.2) and (4.1.4), and let $V_0 \in L^2(\mathbb{T}^3)$ such that $\text{div } v_0 = 0$. The sequence $(\mathcal{L}(\frac{t}{\varepsilon})V^\varepsilon)_{\varepsilon>0}$ which V^ε energy solution determined in Theorem 4.1.2 is weakly compact in the $L^2_{\text{loc}}(\mathbb{R}_+ \times \mathbb{T}^3)$ topology and each element U of the topological closure of $(\mathcal{L}(\frac{t}{\varepsilon})V^\varepsilon)_{\varepsilon>0}$ w.r.t. the $L^2_{\text{loc}}(\mathbb{R}_+ \times \mathbb{T}^3)$ topology solves (\mathcal{S}_0) in the sense of distributions and belongs to the energy space*

$$L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; H^1(\mathbb{T}^3)).$$

The second result we prove is the following simplification of the limit system in the abstract form (\mathcal{S}_0) :

Theorem 4.1.5. *Let us define*

$$\begin{aligned} \omega_0^h &= \text{curl}_h V_0^h, & \bar{U}_0 &= (\nabla_h^\perp \Delta_h^{-1} \omega_0^h, 0, 0), & U_{\text{osc},0} &= V_0 - \bar{U}_0. \\ &= -\partial_2 V_0^1 + \partial_1 V_0^2 & &= (\bar{u}_0^h, 0, 0). \end{aligned}$$

The projection of (\mathcal{S}_0) onto $\ker \mathbb{P}\mathcal{A}$ is the following 2d-Navier-Stokes stratified system with full diffusion

$$\begin{cases} \partial_t \bar{u}^h(t, x_h, x_3) + \bar{u}^h(t, x_h, x_3) \cdot \nabla_h \bar{u}^h(t, x_h, x_3) - \nu \Delta \bar{u}^h(t, x_h, x_3) = -\nabla_h \bar{p}(t, x_h, x_3) \\ \bar{u}^h(t, x_h, x_3)|_{t=0} = \bar{u}_0^h(x_h, x_3). \end{cases} \quad (4.1.5)$$

While the projection of (\mathcal{S}_0) onto $(\ker \mathbb{P}\mathcal{A})^\perp$ satisfies, for almost all $(a_1, a_2, a_3) \in \mathbb{R}^3$ parameters of the three-dimensional torus $\mathbb{T}^3 = \prod_i [0, 2\pi a_i]$, the following linear system in the unknown U_{osc}

$$\begin{cases} \partial_t U_{\text{osc}} + 2\mathcal{Q}(\bar{U}, U_{\text{osc}}) - (\nu + \nu') \Delta U_{\text{osc}} = 0, \\ U_{\text{osc}}|_{t=0} = U_{\text{osc},0} = V_0 - \bar{U}_0. \end{cases} \quad (4.1.6)$$

Theorem 4.1.5 hinges to a rather important deduction: the limit system in the abstract form (\mathcal{S}_0) is hence the superposition of (4.1.5) and (4.1.6). General theory of 2D Navier-Stokes systems and of linear parabolic equations gives hence the tools to prove a global well-posedness result which reads as follows:

Theorem 4.1.6. *Let us consider a vector field $V_0 = (v_0, V_0^4) = (V_0^1, V_0^2, V_0^3, V_0^4) \in H^s(\mathbb{T}^3)$, $s > 1/2$. Let V_0 be of global zero average and of horizontal zero average, i.e.*

$$\int_{\mathbb{T}^3} V_0(y) dy = 0, \quad \int_{\mathbb{T}_h^2} V_0(y_h, x_3) dy_h = 0. \quad (4.1.7)$$

Let us assume $\bar{u}_0^h \in L_v^\infty(H_h^\sigma)$ and $\nabla_h \bar{u}_0^h \in L_v^\infty(H_h^\sigma)$ with $\sigma > 0$, then \bar{u}^h solution of (4.1.5), is globally well posed in \mathbb{R}_+ , and belongs to the space

$$\bar{u}^h \in \mathcal{C}(\mathbb{R}_+; H^s(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; H^{s+1}(\mathbb{R}^3)), \quad s > 1/2,$$

and for each $t > 0$ the following estimate holds true

$$\|\bar{u}^h(t)\|_{H^s(\mathbb{T}^3)}^2 + \nu \int_0^t \|\nabla \bar{u}^h(\tau)\|_{H^s(\mathbb{T}^3)}^2 d\tau \leq \mathcal{E}_1(U_0).$$

Where the function \mathcal{E}_1 is defined as the right-hand-side of equation (4.5.2).

Let U_{osc} be the solution of the linear system (4.1.6). It is globally defined and it belongs to the space

$$U_{\text{osc}} \in \mathcal{C}(\mathbb{R}_+; H^s(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; H^{s+1}(\mathbb{R}^3)),$$

for $s > 1/2$. For each $t > 0$ the following bound holds true

$$\|U_{\text{osc}}(t)\|_{H^s(\mathbb{T}^3)}^2 + \frac{\nu + \nu'}{2} \int_0^t \|\nabla U_{\text{osc}}(\tau)\|_{H^s(\mathbb{T}^3)}^2 d\tau \leq \mathcal{E}_2(U_0),$$

and the function \mathcal{E}_2 is defined as the right-hand-side of equation (4.5.19).

The assumption of zero horizontal average is important in this context: in what follows we will use operators¹ of the form $(-\Delta_h)^{-1}$, such operators are not well defined when we consider vector fields with non-zero horizontal average.

The last question we address to is the stability of the dynamics of (PBS_ε) in the limit regime $\varepsilon \rightarrow 0$. As mentioned above this is done with a methodology introduced by I. Gallagher in [72] and already outlined in the introduction:

Theorem 4.1.7. *Let V_0 be in $H^s(\mathbb{T}^3)$ for $s > 1/2$ as in Theorem 4.1.6. For $\varepsilon > 0$ small enough (PBS_ε) is globally well posed in $\mathcal{C}(\mathbb{R}_+; H^s(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; H^{s+1}(\mathbb{T}^3))$, and, if U is the solution of (\mathcal{S}_0) , then*

$$V^\varepsilon - \mathcal{L}\left(-\frac{t}{\varepsilon}\right)U = o(1),$$

in $\mathcal{C}(\mathbb{R}_+; H^s(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; H^{s+1}(\mathbb{T}^3))$.

Let us, now, outline the structure of the paper:

¹In the sense of Fourier multipliers.

- In Section 4.2 we shall study the linear problem associated to the singular perturbation $\varepsilon^{-1}\mathbb{P}\mathcal{A}$ characterizing the system (PBS $_{\varepsilon}$). By mean of a careful spectral analysis of the penalized operator $\mathbb{P}\mathcal{A}$ we define what shall be called the *non-oscillating* and *oscillating* subspace. The first is the subspace in Fourier variables defined by the divergence-free elements belonging to the kernel of $\mathbb{P}\mathcal{A}$. Being in the kernel of such operator the evolution of such elements shall not be influenced by the highly external force $\varepsilon^{-1}\mathbb{P}\mathcal{A}$ and hence it shall not exhibit any oscillating behavior. On the other hand the element belonging to the oscillating subspace, which is the orthogonal complement of $\ker \mathbb{P}\mathcal{A}$ will present an oscillating behavior which depends (inversely) on the parameter ε .
- In Section 4.3 we prove Theorem 4.1.4. We apply the Poicaré semigroup

$$\mathcal{L}(\tau) = e^{\tau\mathbb{P}\mathcal{A}},$$

to the system (PBS $_{\varepsilon}$). The new variable $U^{\varepsilon} = \mathcal{L}\left(\frac{t}{\varepsilon}\right) V^{\varepsilon}$ satisfies an equation which is very close to the three-dimensional periodic Navier-Stokes equation which we denote as *the filtered system*. We avoid to give a detailed description of the filtered system now, but the reader which is already familiar with this kind of mathematical tools is referred to ($\mathcal{S}_{\varepsilon}$). What has to be retained is the fact that it is possible to construct from (PBS $_{\varepsilon}$) another family of systems, indexed by ε , which is somehow better suited for the study of the problem. Thanks to this similarity we can deduce that the weak solutions $(U^{\varepsilon})_{\varepsilon}$ are in fact uniformly bounded in some suitable space, and thanks to standard compactness arguments we deduce that

$$U^{\varepsilon} \rightarrow U,$$

weakly. In particular U satisfies a three-dimensional Navier-Stokes-like equation, whose bilinear interaction (defined in (4.3.2)) has better product rules than the standard transport-form. Lastly we deduce that V^{ε} can in fact be written as

$$V^{\varepsilon} = \text{stationary state} + \text{high oscillation} + \text{remainder}.$$

- in Section 4.4 we prove Theorem 4.1.5 via a study of the limit (in the sense of distributions) of the filtered system as $\varepsilon \rightarrow 0$. In particular such limit has two qualitatively different behaviors once we consider its projection onto $(\ker \mathbb{P}\mathcal{A})$ and $(\ker \mathbb{P}\mathcal{A})^{\perp}$:
 - The projection of the limit system (\mathcal{S}_0) onto $(\ker \mathbb{P}\mathcal{A})$ presents, as a bilinear interaction, bilinear interactions of elements of $(\ker \mathbb{P}\mathcal{A})$ only, and in particular it is represented by a two-dimensional, stratified, Navier-Stokes equation with additional vertical diffusion.
 - The projection of the limit system (\mathcal{S}_0) onto $(\ker \mathbb{P}\mathcal{A})^{\perp}$ is, for almost all three-dimensional tori, a linear equation of the unknown U_{osc} . Such deduction is a result of a geometrical analysis on the domain, we denote the domains which satisfy such properties as *non-resonant domains*.
- The Section 4.5 is devoted to the proof of Theorem 4.1.6. As well as in Section 4.4 we divide the proof in two sub-parts, considering the projection of the solutions onto $(\ker \mathbb{P}\mathcal{A})$ and $(\ker \mathbb{P}\mathcal{A})^{\perp}$ respectively

- The kernel part, as already stated, is a two-dimensional stratified Navier-Stokes equation. We take advantage of the fact that, along the vertical direction, the equation is purely diffusive without transport term. This allows us to prove that in fact, for some suitable anisotropic strong norms, the solution decay exponentially-in-time, and hence the global-in-time result.
- For the oscillating subspace we exploit the fact that the solution satisfied by U_{osc} is linear to achieve the global result.
- Lastly, in Section 4.6, we prove Theorem 4.1.7 using a smart change of variable, to prove that

$$V^\varepsilon - \mathcal{L}\left(-\frac{t}{\varepsilon}\right)U \rightarrow 0 \quad \text{in } L^\infty(\mathbb{R}_+, H^s(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+, H^{s+1}(\mathbb{R}^3)), \quad s > 1/2.$$

4.1.4 Physical derivation of the system (PBS_ε) and previous works on similar systems.

In the present section we linger for a while on the physical motivations which induce us to study the system (PBS_ε) and we continue to (briefly) expose some relevant result concerning various system related to (PBS_ε) .

In the following $v = (v^1, v^2, v^3)$ represents the velocity flow of the fluid, and Ro, Fr are positive constants which have a physical relevance. They will be defined precisely in what follows. The fluid is considered to be density-dependent, and the density is considered to be a slight oscillation from a stationary state for a strongly stratified fluid (such as the oceans or the troposphere). We consider the full fluid density $\rho = \rho(t, x) = \rho(t, x_h, x_3)$ as

$$\rho(t, x_h, x_3) = \rho_0 + \bar{\rho}(x_3) + \theta(t, x_h, x_3),$$

and

$$|\theta| \ll 1,$$

i.e. the fluid density ρ is slight oscillation around the constant state ρ_0 whose main variation $\bar{\rho}$ is along the x_3 axis due to gravitational stratification effects and on which acts a different variation of density θ which can be arbitrary.

We start considering the equations governing the motion of viscous ($\nu, \nu' \geq c > 0$), density-dependent fluid at a planetary scale (primitive equations)

$$\left\{ \begin{array}{l} \partial_t v^1 + v \cdot \nabla v^1 - \frac{1}{\text{Ro}} v^2 = -p_0 \frac{1}{\rho_0} \partial_1 \phi + \nu \Delta v^1, \\ \partial_t v^2 + v \cdot \nabla v^2 + \frac{1}{\text{Ro}} v^1 = -p_0 \frac{1}{\rho_0} \partial_2 \phi + \nu \Delta v^2, \\ \partial_t v^3 + v \cdot \nabla v^3 + \frac{1}{\text{Fr}} \theta = -p_0 \frac{1}{\rho_0} \partial_3 \phi + \nu \Delta v^3, \\ \partial_t \theta + v \cdot \nabla \theta - \frac{1}{\text{Fr}} v^3 = \nu' \Delta \theta, \\ \text{div } v = 0. \end{array} \right. \quad (\text{P-I})$$

The term p_0 appearing in (P-I) is called the *reference pressure*. For static gases the *reference density* and the *reference absolute temperature* ρ_0, T_0 are given physical quantities. For adiabatic gases the equation of state and the conservation law read as

$$\begin{cases} p = R\rho T, \\ \frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma, \end{cases}$$

for $\gamma > 1$, hence the reference pressure is defined as $p_0 = R\rho_0 T_0$ with R physical constant depending on the gas. The above laws hence give us the variation of the absolute temperature with respect to the density, i.e.

$$T = \frac{T_0}{\rho_0^{\gamma-1}} \rho^{\gamma-1}.$$

The derivation of (P-I) is not completely trivial, for this reason we refer the reader to [50, Chapter 11, 15]. We point out though that there are two intrinsic features which characterize a motion described by a system of the form (P-I)

- The skew-symmetric perturbation $(v^1, v^2) \mapsto \text{Ro}^{-1}(-v^2, v^1)$ acting on the first two components of (P-I). This linear force is a rotational force acting along an axis which is parallel to the x_3 axis. It describes relatively well the effect of the Coriolis force locally on mid-latitude regions. Ro is called the *Rossby number*, and it is a physical magnitude describing the importance of the earth rotation on the system. We shall define it in detail in what follows.
- The second one is the linear application $(v^3, \theta) \mapsto \text{Fr}^{-1}(\theta, -v^3)$. This application describes the force which is applied due to stratification of the fluid. Namely, let us consider a (static) fluid whose density is decreasing along the vertical direction. This hypothesis is completely natural since layer with higher density are heavier, and the gravity tends to minimize the gravitational potential, moving them below the layers which are lighter. Suppose to move a volume of heavier fluid in a region of low density. Indeed the gravitation will tend to restore the stratification, hence the "high density volume" will tend to move down. Once it reaches the layers whose density are comparable to his he will not stop immediately, but by inertial force he will tend to go below it, until Archimedes' principle will provide sufficient buoyancy to invert such motion.

The quantity Fr is said the *Froude number* and measures the stratification of a fluid. We shall define it in detail in what follows.

As we said before the system (P-I) describes the motion of a fluid on which are present rotational and stratification effects. The rotating motion which induces the Coriolis force is indeed bestowed of a certain (angular) frequency, which we shall denote as Ω . On the other hand once a stratification equilibrium is broken we induce a vertical pulsating motion of frequency, say, N (during all this paper we shall consider N to be constant). This is known in the literature as the Brunt-Väisälä frequency.

We can indeed associate to Ω, N the following quantities

$$T_\Omega = \frac{1}{\Omega}, \quad T_N = \frac{1}{N},$$

which are respectively the *period of rotation* and the *period of stratification*. The system (P-I) describes hence how the rotation and gravity tend to restore this horizontal-vertical equilibrium in a timescale T , where Ro and Fr are respectively defined as

$$\text{Ro} = \frac{T_\Omega}{T}, \quad \text{Fr} = \frac{T_N}{T}.$$

The short introduction given above explains that in fact (P-I) is a family of systems, whose asymptotic varies accordingly to the ratio Ro/Fr . In the present paper we are interested to study the regime

$$T_N \ll T \ll T_\Omega,$$

hence we *suppose* that the stratification effects have a frequency $N \gg \Omega$ and we focus in a timescale T which is significantly bigger than T_N but still irrelevant respect to T_Ω , with these hypotheses hence the physical numbers Fr, Ro becomes

$$\text{Ro} \gg 1, \quad \text{Fr} \approx 0,$$

setting $\text{Fr} = \varepsilon$, and, omitting the term $(-\text{Ro}^{-1}v^2, \text{Ro}^{-1}v^1)$ since $1/\text{Ro} \approx 0$ we simplify the primitive equations to the following form

$$\left\{ \begin{array}{l} \partial_t v^1 + v \cdot \nabla v^1 = -p_0 \frac{1}{\rho_0} \partial_1 \phi + \nu \Delta v^1, \\ \partial_t v^2 + v \cdot \nabla v^1 = -p_0 \frac{1}{\rho_0} \partial_2 \phi + \nu \Delta v^2, \\ \partial_t v^3 + v \cdot \nabla v^3 + \frac{1}{\varepsilon} \theta = -p_0 \frac{1}{\rho_0} \partial_3 \phi + \nu \Delta v^3, \\ \partial_t \theta + v \cdot \nabla \theta - \frac{1}{\varepsilon} v^3 = \nu' \Delta \theta, \\ \text{div } v = 0. \end{array} \right. \quad (\text{P-II})$$

Indeed the solutions of (P-II) depend on the Froude number ε , for this reason from now on we denote

$$(v, \theta) = (v^\varepsilon, \theta^\varepsilon),$$

and

$$\frac{p_0}{\rho_0} \phi = \Phi^\varepsilon.$$

It may be interesting to notice that, as mentioned above, varying the ratio Ro/Fr we can deduce many different systems which are already well known.

Let us suppose, for instance, that the fluid considered is viscid, and *completely homogeneous*, i.e. its density is a constant value ρ_0 . Being absent any variation of density it lacks the physical mean to have stratification effects, i.e. the gravitational potential is considered to be constant (or linearized around a minimum). This in turn implies that $T_N = \infty$ and $\text{Fr} = \infty$. It is easy to deduce hence that, in a timescale $T \gg T_\Omega$ (hence $\text{Ro} = \varepsilon \approx 0$) the flow shall evolve accordingly to the following law (Navier-Stokes-Coriolis equations)

$$\left\{ \begin{array}{l} \partial_t v_{\text{RF}}^\varepsilon + v_{\text{RF}}^\varepsilon \cdot \nabla v_{\text{RF}}^\varepsilon - \nu \Delta v_{\text{RF}}^\varepsilon + \frac{1}{\varepsilon} e^3 \wedge v_{\text{RF}}^\varepsilon = -\frac{1}{\varepsilon} \nabla p_{\text{RF}}^\varepsilon, \\ v_{\text{RF}}^\varepsilon(0, x) = v_{\text{RF},0}^\varepsilon(x). \end{array} \right. \quad (\text{RF}_\varepsilon)$$

It is relevant to mention that Chemin et al. in [38] proved global strong convergence of solutions of $(\mathbf{RF}_\varepsilon)$ with only horizontal viscosity $-\nu_h \Delta_h$ instead of the full viscosity $-\nu \Delta$ to a purely 2D Navier-Stokes system in the case in which $\varepsilon \rightarrow 0$ and the space domain is \mathbb{R}^3 . This result is attained with methodologies which are very different with respect to the ones mentioned for the periodic setting. Using dispersive estimates the authors prove that the global solutions of the linear system associated to $(\mathbf{RF}_\varepsilon)$

$$\begin{cases} \partial_t w_{\mathbf{RF},r,R}^\varepsilon + \left(\frac{1}{\varepsilon} \mathbb{P} (e^3 \wedge \cdot) - \nu_h \Delta_h \right) w_{\mathbf{RF},r,R}^\varepsilon = 0, \\ \operatorname{div} w_{\mathbf{RF},r,R}^\varepsilon = 0, \\ w_{\mathbf{RF},r,R}^\varepsilon|_{t=0} = w_{\mathbf{RF},r,R,0}^\varepsilon, \end{cases}$$

has vanishing norm in some $L^q(\mathbb{R}_+; L^p(\mathbb{T}^3))$ space. The detailed hypothesis on the initial data is here omitted since it is pretty technical and does not represent an interest for the present introduction. Roughly speaking hence we can say that the perturbations induced by the external force are somehow negligible in a suitable topology, with this in mind a bootstrap argument can hence be put in place to prove that the solutions exist globally and converge to those of a 2D Navier-Stokes system.

Another interesting system which can be derived from a multi-scale analysis of the ratio $\operatorname{Ro}/\operatorname{Fr}$ is the case

$$\frac{\operatorname{Ro}}{\operatorname{Fr}} = \frac{1}{F}, \quad F \in \mathbb{R}.$$

Hence in this case stratification and rotation have comparable frequencies, it is easy to deduce that the system derived in this case is the following one

$$\begin{cases} \partial_t V_{\mathcal{P}}^\varepsilon + v_{\mathcal{P}}^\varepsilon \cdot \nabla V_{\mathcal{P}}^\varepsilon - \mathbb{D} V_{\mathcal{P}}^\varepsilon + \frac{1}{\varepsilon} \left(-V_{\mathcal{P}}^{2,\varepsilon}, V_{\mathcal{P}}^{1,\varepsilon}, \frac{1}{F} V_{\mathcal{P}}^{4,\varepsilon}, -\frac{1}{F} V_{\mathcal{P}}^{3,\varepsilon} \right)^\top \\ \hspace{15em} = -\frac{1}{\varepsilon} \left(\nabla \Phi_{\mathcal{P}}^\varepsilon, 0 \right)^\top, \quad (\mathbf{PE}_\varepsilon) \\ V_{\mathcal{P}}^\varepsilon(0, x) = V_{\mathcal{P},0}^\varepsilon(x), \end{cases}$$

where \mathbb{D} is defined in (4.1.2). J.-Y. Chemin studied the system $(\mathbf{PE}_\varepsilon)$ in the case $F \equiv 1$ in [36] obtaining a global existence result under a smallness condition made only on a part of the initial data. In the series of works [27], [28] and [29], F. Charve proved that, setting $\Omega^\varepsilon = -\partial_2 v_{\mathcal{P}}^{1,\varepsilon} + \partial_1 v_{\mathcal{P}}^{2,\varepsilon} - F \partial_3 v_{\mathcal{P}}^{4,\varepsilon}$, the system $(\mathbf{PE}_\varepsilon)$ converges to what it is known as the quasi-geostrophicsystem, i.e.

$$\partial_t V_{\text{QG}} - \Gamma V_{\text{QG}} + \begin{pmatrix} -\partial_2 \\ \partial_1 \\ 0 \\ -F \partial_3 \end{pmatrix} \Delta^{-1} (V_{\text{QG}}^h \cdot \nabla_h \Omega^\varepsilon) = 0, \quad (\mathbf{QG})$$

where $V_{\text{QG}} = (V_{\text{QG}}^h, V_{\text{QG}}^3, V_{\text{QG}}^4) = (-\partial_2 \Delta^{-1} \Omega, \partial_1 \Delta^{-1} \Omega, 0, -F \partial_3 \Delta^{-1} \Omega)$ and Γ is an elliptic operator of order two defined as

$$\Gamma = \frac{\Delta (\nu \Delta_h + \nu' \partial_3^2)}{\Delta_h + F^2 \partial_3^2}.$$

I. Gallagher proved strong convergence of solutions of (PE_ε) to a limit system in the form (\mathcal{S}_0) in the periodic case always in the work [72], F. Charve proved first weak convergence of solutions of (PE_ε) to solution of (QG) in [29], and strong convergence in [28]. The case in which the system (PE_ε) presents only horizontal diffusion, hence it is a mixed parabolic hyperbolic type has been studied by Charve and Ngo in [34] in the whole space in the case $\nu_h = \nu_h^\varepsilon = \mathcal{O}(\varepsilon^\alpha)$, $\alpha > 0$ and by the author in [137] in the periodic case when the horizontal viscosity is strictly positive.

We mention as well work of D. Bresch, D. Gérard-Varet and E. Grenier [17]. In this work the authors consider the primitive equations in the form

$$\begin{cases} \partial_t v_P^\varepsilon + u_P^\varepsilon \cdot \nabla v_P^\varepsilon - \nu \Delta v_P^\varepsilon + \frac{1}{\varepsilon} e^3 \wedge v_P^\varepsilon = -\frac{1}{\varepsilon} \nabla_h \phi_P^\varepsilon, \\ \partial_3 \phi_P^\varepsilon = \theta_P^\varepsilon \\ \operatorname{div}_h v_P^\varepsilon = -\partial_3 w_P^\varepsilon, \\ \partial_t \theta_P^\varepsilon + u_P^\varepsilon \cdot \nabla \theta_P^\varepsilon - \nu' \Delta \theta_P^\varepsilon + w_P^\varepsilon = Q, \\ u_P^\varepsilon = (v_P^\varepsilon, w_P^\varepsilon). \end{cases} \quad (\widetilde{\text{PE}}_\varepsilon)$$

The methodology used in [17] although is completely different respect to the other works mentioned. The penalization in particular is *not skew-symmetric*, this prevents the authors to apply energy methods as in the other works mentioned.

4.1.5 Elements of Littlewood-Paley theory.

A tool that will be widely used all along the paper is the theory of Littlewood–Paley, which consists in doing a dyadic cut-off of the frequencies.

Let us define the (non-homogeneous) truncation operators as follows:

$$\begin{aligned} \Delta_q u &= \sum_{n \in \mathbb{Z}^3} \hat{u}_n \varphi \left(\frac{|\check{n}|}{2^q} \right) e^{i\check{n} \cdot x}, & \text{for } q \geq 0, \\ \Delta_{-1} u &= \sum_{n \in \mathbb{Z}^3} \hat{u}_n \chi(|\check{n}|) e^{i\check{n} \cdot x}, \\ \Delta_q u &= 0, & \text{for } q \leq -2, \end{aligned}$$

where $u \in \mathcal{D}'(\mathbb{T}^3)$ and \hat{u}_n are the Fourier coefficients of u . The functions φ and χ represent a partition of the unity in \mathbb{R} , which means that are smooth functions with compact support such that

$$\operatorname{supp} \chi \subset B \left(0, \frac{4}{3} \right), \quad \operatorname{supp} \varphi \subset \mathcal{C} \left(\frac{3}{4}, \frac{8}{3} \right),$$

and such that for all $t \in \mathbb{R}$,

$$\chi(t) + \sum_{q \geq 0} \varphi(2^{-q}t) = 1.$$

Let us define further the low frequencies cut-off operator

$$S_q u = \sum_{q' \leq q-1} \Delta_{q'} u.$$

Anisotropic paradifferential calculus.

The dyadic decomposition turns out to be very useful also when it comes to study the product between two distributions. We can in fact, at least formally, write for two distributions u and v

$$u = \sum_{q \in \mathbb{Z}} \Delta_q u; \quad v = \sum_{q' \in \mathbb{Z}} \Delta_{q'} v; \quad u \cdot v = \sum_{\substack{q \in \mathbb{Z} \\ q' \in \mathbb{Z}}} \Delta_q u \cdot \Delta_{q'} v. \quad (4.1.8)$$

We are going to perform a Bony decomposition (see [10], [14], [37] for the isotropic case and [39], [86] for the anisotropic one).

Paradifferential calculus is a mathematical tool for splitting the above sum in three parts

$$u \cdot v = T_u v + T_v u + R(u, v),$$

where

$$\begin{aligned} T_u v &= \sum_q S_{q-1} u \Delta_q v, \\ T_v u &= \sum_{q'} S_{q'-1} v \Delta_{q'} u, \\ R(u, v) &= \sum_k \sum_{|\nu| \leq 1} \Delta_k u \Delta_{k+\nu} v. \end{aligned}$$

The following almost orthogonality properties hold

$$\begin{aligned} \Delta_q (S_q a \Delta_{q'} b) &= 0, & \text{if } |q - q'| \geq 5, \\ \Delta_q (\Delta_{q'} a \Delta_{q'+\nu} b) &= 0, & \text{if } q' < q - 4, |\nu| \leq 1, \end{aligned}$$

and hence we will often use the following relation

$$\Delta_q (u \cdot v) = \sum_{|q-q'| \leq 4} \Delta_q (S_{q'-1} v \Delta_{q'} u) + \sum_{|q-q'| \leq 4} \Delta_q (S_{q'-1} u \Delta_{q'} v) \quad (4.1.9)$$

$$\begin{aligned} &+ \sum_{q' \geq q-4} \sum_{|\nu| \leq 1} \Delta_q (\Delta_{q'} a \Delta_{q'+\nu} b), \\ &= \sum_{|q-q'| \leq 4} \Delta_q (S_{q'-1} v \Delta_{q'} u) + \sum_{q' > q-4} \Delta_q (S_{q'+2} u \Delta_{q'} v). \end{aligned} \quad (4.1.10)$$

In the paper [43] J.-Y. Chemin and N. Lerner introduced the following decomposition, which will be used by Chemin et al. in [39] in its anisotropic version. This particular decomposition turns out to be very useful in our context

$$\begin{aligned} \Delta_q (uv) &= S_{q-1} u \Delta_q v + \sum_{|q-q'| \leq 4} \{[\Delta_q, S_q u] \Delta_{q'} v + (S_{q'} u - S_q u) \Delta_q \Delta_{q'} v\} \\ &\quad + \sum_{q' > q-4} \Delta_q (S_{q'+2} v \Delta_{q'} u), \end{aligned} \quad (4.1.11)$$

where the commutator $[\Delta_q, a]b$ is defined as

$$[\Delta_q, a]b = \Delta_q(ab) - a\Delta_q b.$$

There is an interesting relation of regularity between dyadic blocks and full function in the Sobolev spaces, i.e.

$$\|\Delta_q f\|_{L^2(\mathbb{T}^3)} \leq C c_q 2^{-qs} \|f\|_{H^s(\mathbb{T}^3)}, \quad (4.1.12)$$

with $\left\| \{c_q\}_{q \in \mathbb{Z}} \right\|_{\ell^2(\mathbb{Z})} \equiv 1$. In the same way we denote as b_q a sequence in $\ell^1(\mathbb{Z})$ such that $\sum_q |b_q| \leq 1$.

In particular in Section 4.5 we shall need paradifferential calculus in the horizontal variables, everything is the same as in the isotropic case except that we shall take the Fourier transform only on the horizontal components, i.e.

$$\mathcal{F}_h f(n_h, x_3) = \int_{\mathbb{T}_h^2} f(x_h, x_3) e^{-2\pi i x_h \cdot n_h} dx_h,$$

and we can define hence the horizontal truncation operators (as well as the low frequencies cut off, reminders, etc...) Δ_q^h in the same way as we did for Δ_q except that we act only on the horizontal variables. This difference shall be denoted by the fact that that we will always put an index h when it comes to the horizontal anisotropic paradifferential calculus.

Some basic estimates.

The interest in the use of the dyadic decomposition is that the derivative of a function localized in frequencies of size 2^q acts like the multiplication with the factor 2^q (up to a constant independent of q). In our setting (periodic case) a Bernstein type inequality holds. For a proof of the following lemma in the anisotropic (hence as well isotropic) setting we refer to the work [86]. For the sake of self-completeness we state the result in both isotropic and anisotropic setting.

Lemma 4.1.8. *Let u be a function such that $\mathcal{F}u$ is supported in $2^q \mathcal{C}$, where \mathcal{F} denotes the Fourier transform. For all integers k the following relation holds*

$$2^{qk} C^{-k} \|u\|_{L^p(\mathbb{T}^3)} \leq \left\| (-\Delta)^{k/2} u \right\|_{L^p(\mathbb{T}^3)} \leq 2^{qk} C^k \|u\|_{L^p(\mathbb{T}^3)}.$$

Let now $r \geq r' \geq 1$ be real numbers. Let $\text{supp } \mathcal{F}u \subset 2^q B$, then

$$\|u\|_{L^r} \leq C \cdot 2^{3q\left(\frac{1}{r'} - \frac{1}{r}\right)} \|u\|_{L^{r'}}.$$

Let us consider now a function u such that $\mathcal{F}u$ is supported in $2^q \mathcal{C}_h \times 2^q \mathcal{C}_v$. Let us define $D_h = (-\Delta_h)^{1/2}$, $D_3 = |\partial_3|$, then

$$C^{-q-q'} 2^{qs+q's'} \|u\|_{L^p(\mathbb{T}^3)} \leq \left\| D_h^s D_3^{s'} u \right\|_{L^p(\mathbb{T}^3)} \leq C^{q+q'} 2^{qs+q's'} \|u\|_{L^p(\mathbb{T}^3)},$$

and given $1 \leq p' \leq p \leq \infty$, $1 \leq r' \leq r \leq \infty$, then

$$\begin{aligned} \|u\|_{L_h^p L_v^{r'}} &\leq C^{q+q'} 2^{2q(\frac{1}{p'} - \frac{1}{p}) + q'(\frac{1}{r'} - \frac{1}{r})} \|u\|_{L_h^{p'} L_v^r}, \\ \|u\|_{L_v^r L_h^p} &\leq C^{q+q'} 2^{2q(\frac{1}{p'} - \frac{1}{p}) + q'(\frac{1}{r'} - \frac{1}{r})} \|u\|_{L_v^{r'} L_h^{p'}}. \end{aligned}$$

The following are inequalities of Gagliardo-Nirenberg type, which combined with the anisotropic version of the Bernstein lemma will give us some information that we will use continuously all along the paper. We will avoid to give the proofs of such tools since they are already present in [123].

Lemma 4.1.9. *There exists a constant C such that for all periodic vector fields u on \mathbb{T}^3 with zero horizontal average ($\int_{\mathbb{T}_h^2} u(x_h, x_3) dx_h = 0$) we have*

$$\|u\|_{L_v^2 L_h^4} \leq C \cdot \|u\|_{L^2(\mathbb{T}^3)}^{1/2} \|\nabla_h u\|_{L^2(\mathbb{T}^3)}^{1/2}. \quad (4.1.13)$$

Finally we state a lemma that shows that the commutator with the truncation operator is a regularizing operator.

Lemma 4.1.10. *Let \mathbb{T}^3 be a 3D torus and p, r, s real positive numbers such that $r', s', p, r, s \geq 1$, $\frac{1}{r'} + \frac{1}{s'} = \frac{1}{2}$ and $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$. There exists a constant C such that for all vector fields u and v on \mathbb{T}^3 we have the inequality*

$$\|[\Delta_q, u]v\|_{L_v^2 L_h^p} \leq C \cdot 2^{-q} \|\nabla u\|_{L_v^{r'} L_h^r} \|v\|_{L_v^{s'} L_h^s},$$

indeed there exists an isotropic counterpart of such Lemma (see [125], [137]).

4.2 The linear problem.

Let us introduce at first some notation. Let us consider the generic linear problem associated with the linear operator $\mathbb{P}\mathcal{A}$:

$$\begin{cases} \partial_\tau W + \mathbb{P}\mathcal{A} W = 0, \\ \operatorname{div} w = 0, \\ W = (w, W^4), \\ W|_{\tau=0} = W_0. \end{cases} \quad (4.2.1)$$

\mathcal{A} is the skew symmetric penalized matrix defined in (4.1.2). \mathbb{P} is the Leray projection onto the divergence free vector fields, without changing V^4 which is defined in (4.1.4). In the present section (and everywhere) the Fourier modes are considered to be

$$\tilde{n} = (n_1/a_1, n_2/a_2, n_3/a_3)$$

where $n_i \in \mathbb{Z}$ and the a_i 's are the parameters of the three-dimensional torus. We shall generally ignore the check notation (unless differently specified) in order to simplify the overall notation.

To the sake of completeness we give here the action of the matrix $\mathbb{P}\mathcal{A}$ in the Fourier space, which is

$$\mathcal{F}(\mathbb{P}\mathcal{A}u) = \begin{pmatrix} 0 & 0 & 0 & -\frac{n_1 n_2}{|n|^2} \\ 0 & 0 & 0 & -\frac{n_2 n_3}{|n|^2} \\ 0 & 0 & 0 & 1 - \frac{n_3^2}{|n|^2} \\ 0 & 0 & -1 & 0 \end{pmatrix} \hat{u}_n.$$

Such kind of equation has been thoroughly first studied by Poincaré in [129]. The study of the linear equation (4.2.1) is essential in the study of the nonlinear problem (PBS_ε). The solution of (4.2.1) is obviously

$$W(\tau) = e^{-\tau\mathbb{P}\mathcal{A}}W_0 = \mathcal{L}(-\tau)W_0. \quad (4.2.2)$$

We want to give an explicit sense to the propagator $e^{-\tau\mathbb{P}\mathcal{A}}$. To do so we perform a spectral analysis of the operator $\mathbb{P}\mathcal{A}$. After some calculations we obtain that the matrix $\mathbb{P}\mathcal{A}$ admits an eigenvalue $\omega^0(n) \equiv 0$ with multiplicity 2 and other two eigenvalues

$$\omega^\pm(n) = \pm i \frac{|n_h|}{|n|} = \pm i\omega(n). \quad (4.2.3)$$

The matrix $(\widehat{\mathbb{P}\mathcal{A}})_n$ admits a basis of normal (in the sense that they have norm one) eigenvectors. In particular the basis is the following one

$$\tilde{e}_1^0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{e}_2^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad e^\pm(n) = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm i \frac{n_1 n_3}{|n_h| |n|} \\ \pm i \frac{n_2 n_3}{|n_h| |n|} \\ \mp i \frac{|n_h|}{|n|} \\ 1 \end{pmatrix}.$$

We imposed that the solutions of (4.2.1) are divergence-free, in the sense that they are orthogonal, in the Fourier space, to the vector $(n_1, n_2, n_3, 0)$. Now, not all the subspace $\mathbb{C}\tilde{e}_1^0 \oplus \mathbb{C}\tilde{e}_2^0$, which is the kernel of the operator $\mathbb{P}\mathcal{A}$, satisfies this property. In any case there exist a subspace of $\mathbb{C}\tilde{e}_1^0 \oplus \mathbb{C}\tilde{e}_2^0$ which is divergence free. This space is the space generated by

$$e^0(n) = \frac{1}{|n_h|} \begin{pmatrix} -n_2 \\ n_1 \\ 0 \\ 0 \end{pmatrix},$$

we underline again that $\mathbb{C}e^0 \subset \mathbb{C}\tilde{e}_1^0 \oplus \mathbb{C}\tilde{e}_2^0$.

We have hence identified a basis of divergence-free, orthogonal eigenvectors associated to the linear problem (4.2.1), which is

$$e^0(n) = \frac{1}{|n_h|} \begin{pmatrix} -n_2 \\ n_1 \\ 0 \\ 0 \end{pmatrix}, \quad e^\pm(n) = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm i \frac{n_1 n_3}{|n_h| |n|} \\ \pm i \frac{n_2 n_3}{|n_h| |n|} \\ \mp i \frac{|n_h|}{|n|} \\ 1 \end{pmatrix}. \quad (4.2.4)$$

A case of particular interest which shall be crucial in Section 4.4 is the subspace $\{n_h = 0\}$ of the frequency space. Performing the required computation we prove that the only eigenvalue is $\omega(n) \equiv 0$ with multiplicity four. The Fourier multiplier $(\widehat{\mathbb{P}\mathcal{A}})(n)$ associated to $\mathbb{P}\mathcal{A}$ in this case is

$$(\widehat{\mathbb{P}\mathcal{A}})(0, n_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

which admits three eigenvalues related to ω :

$$\tilde{e}_1^0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{e}_2^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{e}_3^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (4.2.5)$$

We underline the fact that $\tilde{e}_i^0, i = 1, 2, 3$ are divergence-free on the restriction $\{n_h = 0\}$ of the Fourier space. We remark the fact that the hypothesis (4.1.7) automatically excludes the case which the initial data V_0 is a function depending on x_3 only. In fact

$$\int_{\mathbb{T}_h^2} V_0(x_3) dy_h = 0 \implies V(x_3) = 0 \text{ for each } x_3 \in \mathbb{T}_v^1.$$

This case is hence *not considered* in the present work.

The eigenvectors in (4.2.4) are orthogonal with respect the standard \mathbb{C}^4 scalar product, whence the generic solution given in (4.2.2) can be expressed in the following form

$$W = \bar{W} + W_{\text{osc}}.$$

We denoted as \bar{W} the orthogonal projection of W onto the space $\mathbb{C}e^0$, i.e. onto the divergence-free part of the kernel. This projection takes the form (in the Fourier variables)

$$\mathcal{F} \bar{W}(n) = (\mathcal{F} W(n) | e^0(n))_{\mathbb{C}^4} e^0(n). \quad (4.2.6)$$

In the same way W_{osc} is defined as

$$\mathcal{F} W_{\text{osc}}(n) = (\mathcal{F} W(n) | e^+(n))_{\mathbb{C}^4} e^+(n) + (\mathcal{F} W(n) | e^-(n))_{\mathbb{C}^4} e^-(n), \quad (4.2.7)$$

i.e. the decomposition of the (Fourier) data along the directions of e^+, e^- .

We shall denote these two parts of the solution respectively as the *oscillating* and the *non-oscillating* part of the solution. This choice of the names has an easy mathematical justification. Let us in fact consider \bar{W}_0 and let us consider the evolution imposed by the laws of the system (4.2.1) on such vector field. By mean of the explicit solution given in (4.2.2) we obtain (recall that \bar{W} belongs to $\ker \mathbb{P}\mathcal{A}$)

$$\bar{W}(t) = \bar{W}_0.$$

Hence the non-oscillating part of the solution \bar{W} is in fact a stationary (in the sense that is not time-dependent) flow. This is reasonable since once we consider the linear system (4.2) restricted on $\ker \mathbb{P}\mathcal{A}$ there is no external force at all acting on it.

On the other hand along the direction of (say) e^+ the evolution at time τ of the solution has direction (in the Fourier space)

$$\left(e^{-\tau(\widehat{\mathbb{P}\mathcal{A}})_n} \right) e^+(n) = e^{-i\tau\omega(n)} e^+(n),$$

hence it spins with a angular speed $\omega = \omega(n) = \frac{|n_h|}{|n|}$.

Introducing hence a small parameter ε we act, on a physical point of view, on the system in very well-defined way: the spinning linear force grows and with it the turbulent behavior of the solution.

Indeed as long as we consider a generic time-dependent nonlinearity the problem does not behave in such a rigid and well-defined way. Let us consider hence a nonlinear problem associated to (4.2.1)

$$\begin{cases} \partial_\tau W_N + \mathbb{P}\mathcal{A} W_N = \mathcal{N}(\tau), \\ \operatorname{div} w_N = 0, \\ W_N = (w_N, W_N^4), \\ W_N|_{\tau=0} = W_{N,0}. \end{cases} \quad (4.2.8)$$

The nonlinearity $\mathcal{N}(\tau)$ is very generic and only time-dependent, but this is not restrictive, since we want to give a qualitative analysis of the behavior of the solutions.

By mean of the Duhamel formula the solution is expressed as

$$W_N(\tau) = e^{-\tau\mathbb{P}\mathcal{A}} W_{N,0} + \int_0^\tau e^{-(\tau-\sigma)\mathbb{P}\mathcal{A}} \mathcal{N}(\sigma) d\sigma.$$

This generic formulation does not say much, but we can still extract interesting information. Let us denote $\bar{\mathcal{N}}$ the projection of \mathcal{N} onto the nonoscillatory space, i.e.

$$\mathcal{F} \bar{\mathcal{N}} = (\mathcal{F} \mathcal{N} | e^0) e^0.$$

The projection of W_N onto the nonoscillatory space shall hence be described by the law

$$\bar{W}_N(\tau) = \bar{W}_{N,0} + \int_0^\tau \bar{\mathcal{N}}(\sigma) d\sigma.$$

The influence of the propagator (spinning behavior) is not any more a direct consequence of the application of Duhamel formula, but it can still be present as long as $\bar{\mathcal{N}}$ is depending on the spinning eigenvectors e^\pm . We shall see that this will be a major problem in the comprehension of the limiting process as $\varepsilon \rightarrow 0$.

4.3 The filtered limit.

Our strategy shall be to "filter out" the system (PBS_ε) by mean of the propagator \mathcal{L} defined above. Such technique is classic in singular problems on the torus ([72], [79], [123], [137]).

Let us apply from the left the operator $\mathcal{L}\left(\frac{t}{\varepsilon}\right)$ to the equation (PBS_ε) . Setting $U^\varepsilon = \mathcal{L}\left(\frac{t}{\varepsilon}\right)V^\varepsilon$ we obtain that the vector field U^ε satisfies the following evolution equation

$$\begin{cases} \partial_t U^\varepsilon + \mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon) - \mathbb{D}^\varepsilon U^\varepsilon = 0, \\ \operatorname{div} v^\varepsilon = 0, \\ U^\varepsilon|_{t=0} = V_0, \end{cases} \quad (\mathcal{S}_\varepsilon)$$

where

$$\begin{aligned} \mathcal{Q}^\varepsilon(A, B) &= \frac{1}{2} \mathcal{L}\left(\frac{t}{\varepsilon}\right) \mathbb{P} \left[\mathcal{L}\left(-\frac{t}{\varepsilon}\right) A \cdot \nabla \mathcal{L}\left(-\frac{t}{\varepsilon}\right) B + \mathcal{L}\left(-\frac{t}{\varepsilon}\right) B \cdot \nabla \mathcal{L}\left(-\frac{t}{\varepsilon}\right) A \right], \\ \mathbb{D}^\varepsilon A &= \mathcal{L}\left(\frac{t}{\varepsilon}\right) \mathbb{D} \mathcal{L}\left(-\frac{t}{\varepsilon}\right) A. \end{aligned}$$

It is interesting to notice that the application of the Poincaré semigroup \mathcal{L} allowed us to deduce an equation (namely $(\mathcal{S}_\varepsilon)$) on which we can obtain uniform bounds for the sequence $(\partial_t U^\varepsilon)_\varepsilon$. It results in fact that $(\partial_t U^\varepsilon)_\varepsilon$ is uniformly bound in $L^p(\mathbb{R}_+; H^{-N})$, $p \in [2, \infty]$ for N large. This shall result to be fundamental in order to obtain some compactness result in the same fashion as it is done for solutions à la Leray of Navier-Stokes equations.

4.3.1 Uniform bounds of the weak solutions and formal identification of the limit system.

In this section we prove Theorem 4.1.4; whose extended claim is here proposed:

Lemma 4.3.1. *The sequence $(U^\varepsilon)_{\varepsilon>0}$ of distributional solutions of (PBS_ε) identified in Theorem 4.1.2 is uniformly bounded in*

$$L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; H^1(\mathbb{T}^3)),$$

and sequentially compact in $L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))$. Every U belonging to the topological closure of $(U^\varepsilon)_{\varepsilon>0}$ w.r.t. the $L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))$ topology belongs to the energy space $L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; H^1(\mathbb{T}^3))$ and is a distributional solution of the limit system (\mathcal{S}_0) .

Lemma 4.3.1 is composed of two parts:

- Topological convergence of a (not relabeled) subsequence $(U^\varepsilon)_{\varepsilon>0}$ to an element U as $\varepsilon \rightarrow 0$ in some suitable topology and,
- Determination of the limit system to whom $(\mathcal{S}_\varepsilon)$ converges.

More in specific the second point above proves that there exists a bilinear form \mathcal{Q} such that

$$\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} \mathcal{Q}(U, U),$$

in a weak sense.

The first point above will be proved in Lemma 4.3.2, while the second will be studied in detail in Lemma 4.3.4.

Lemma 4.3.2. *Let U^ε be a Leray solution of (PBS $_\varepsilon$). Let the initial data V_0 be bounded in $L^2(\mathbb{T}^3)$. The sequence $(U^\varepsilon)_\varepsilon$ is compact in the space $L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))$ and converges (up to a subsequence) to an element U which belongs to the space*

$$L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; H^1(\mathbb{T}^3)),$$

and the following uniform bound holds

$$\|U^\varepsilon(t)\|_{L^2(\mathbb{T}^3)}^2 + 2c \int_0^t \|\nabla U^\varepsilon(\tau)\|_{L^2(\mathbb{T}^3)}^2 d\tau \leq \|V_0\|_{L^2(\mathbb{T}^3)}^2, \quad (4.3.1)$$

where $c = \min\{\nu, \nu'\}$.

Proof. A standard $L^2(\mathbb{T}^3)$ estimate on the equation (\mathcal{S}_ε) shows that

$$\|U^\varepsilon(t)\|_{L^2(\mathbb{T}^3)}^2 + 2c \int_0^t \|\nabla U^\varepsilon(\tau)\|_{L^2(\mathbb{T}^3)}^2 d\tau \leq \|V_0\|_{L^2(\mathbb{T}^3)}^2.$$

Let us prove that $(\partial_t U^\varepsilon)_\varepsilon$ is uniformly bounded in $L^2_{\text{loc}}(\mathbb{R}_+; H^{-\frac{3}{2}}(\mathbb{T}^3))$ in terms of the $L^2(\mathbb{T}^3)$ norm of the initial data V_0 . Since $\mathcal{L}(\tau)$ is unitary as an application between any Sobolev space H^σ , $\sigma \in \mathbb{R}$ we can safely say that as long as concerns energy estimates in Sobolev spaces we can identify $\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon) \sim U^\varepsilon \cdot \nabla U^\varepsilon$. We can use the product rules in Sobolev spaces to deduce that

$$\begin{aligned} \|U^\varepsilon \cdot \nabla U^\varepsilon\|_{L^2_{\text{loc}}(\mathbb{R}_+; H^{-3/2})} &\leq C \|U^\varepsilon \otimes U^\varepsilon\|_{L^2_{\text{loc}}(\mathbb{R}_+; H^{-1/2})}, \\ &\leq C \|U^\varepsilon\|_{L^2_{\text{loc}}(\mathbb{R}_+; H^1)} \|U^\varepsilon\|_{L^\infty(\mathbb{R}_+; L^2)}. \end{aligned}$$

It is very easy moreover to deduce that $\|-\mathbb{D}^\varepsilon U^\varepsilon\|_{L^2_{\text{loc}}(\mathbb{R}_+; H^{-1})} \lesssim \|U^\varepsilon\|_{L^2_{\text{loc}}(\mathbb{R}_+; H^1)}$. Whence since $\partial_t U^\varepsilon = -\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon) + \mathbb{D}^\varepsilon U^\varepsilon$ we conclude. Since $L^2(\mathbb{T}^3)$ is compactly embedded in $H^{-3/2}(\mathbb{T}^3)$ and that $H^1(\mathbb{T}^3)$ is continuously embedded in $L^2(\mathbb{T}^3)$ it is sufficient to apply Aubin-Lions lemma [4] to deduce the claim. \square

The convergence of U^ε to the element U does not give any qualitative information of the (eventual) solution which is satisfied by U . Especially the bilinear limit involving $\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon)$ is a priori not well-defined.

The result we prove now is needed in order to prove that, in the limit as $\varepsilon \rightarrow 0$, the limit $\lim_{\varepsilon \rightarrow 0} \mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon)$ belongs to the space spanned by the eigenvectors e^0, e^\pm .

Lemma 4.3.3. *The limit*

$$\lim_{\varepsilon \rightarrow 0} \left(\int_{\mathbb{T}_h^2} (\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon))^h dx_h, 0, 0 \right) = 0,$$

holds in a distributional sense.

Since the proof of Lemma 4.3.3 is rather long and technical is postponed at the end of the present section.

By mean of stationary phase theorem we prove the following lemma

Lemma 4.3.4. *Let U^ε be a Leray solution of $(\mathcal{S}_\varepsilon)$ and let U be the limit of one of the converging subsequences of $(U^\varepsilon)_\varepsilon$ identified in Lemma 4.3.2. Then the following limits hold (in the sense of distributions, subsequence not relabeled)*

$$\begin{aligned}\lim_{\varepsilon \rightarrow 0} \mathcal{Q}^\varepsilon (U^\varepsilon, U^\varepsilon) &= \mathcal{Q} (U, U), \\ \lim_{\varepsilon \rightarrow 0} \mathbb{D}^\varepsilon U^\varepsilon &= \mathbb{D} U,\end{aligned}$$

where \mathcal{Q} and \mathbb{D} has the following form

$$\mathcal{F} \mathcal{Q} (U, U) = \mathbb{P}_n \sum_{\substack{\omega_{k,m,n}^{a,b,c}=0 \\ k+m=n \\ a,b,c \in \{0, \pm\}}} \left(\left(\sum_{j=1,2,3} U^{a,j} (k) m_j \right) U^b (m) \Big| e^c (n) \right)_{\mathbb{C}^4} e^c (n) \quad (4.3.2)$$

$$\mathcal{F} \mathbb{D} U = \sum_{\omega_n^{a,b}=0} \left(\mathbb{D} (n) U^a (n) \Big| e^b (n) \right)_{\mathbb{C}^4} e^b (n), \quad (4.3.3)$$

where $\omega_{k,m,n}^{a,b,c} = \omega^a (k) + \omega^b (m) - \omega^c (n)$ and $\omega_n^{a,b} = \omega^a (n) - \omega^b (n)$, the Fourier multiplier $\mathbb{D} (n)$ in noting but the Fourier multiplier associated to the matrix \mathbb{D} defined in (4.1.2), the eigenvalues ω^i are defined in (4.2.3) and the operator \mathbb{P} is defined in (4.1.4).

Remark 4.3.5. Lemma 4.3.3 proves that only the first two components of the horizontal average of $\mathcal{Q}^\varepsilon (U^\varepsilon, U^\varepsilon)$ converge (weakly) to zero. We need to prove such result since the eigenvectors defined in (4.2.4) present in their first two components a Fourier symbol of the form $|n_h|^{-1}$, and such operator is well-defined only for vector fields with zero horizontal average. These are hence applied on the bilinear interaction as it is shown in (4.3.2). \blacklozenge

Proof. We start proving (4.3.3) since it is easier. We claim that

$$\mathbb{D}^\varepsilon U^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \mathbb{D} U,$$

in \mathcal{D}' . Indeed via standard manipulations we can express the difference $\mathbb{D}^\varepsilon U^\varepsilon - \mathbb{D} U$ as

$$\mathbb{D}^\varepsilon U^\varepsilon - \mathbb{D} U = \mathbb{D} (U^\varepsilon - U) + (\mathbb{D}^\varepsilon - \mathbb{D}) U^\varepsilon.$$

The element $\mathbb{D} (U^\varepsilon - U) \xrightarrow{\varepsilon \rightarrow 0} 0$ in \mathcal{D}' since $U^\varepsilon \rightarrow U$ w.r.t. the $L^2_{\text{loc}} (\mathbb{R}_+; L^2 (\mathbb{T}^3))$ topology as it is proved in Lemma 4.3.1.

Hence all we have to prove is that

$$\mathcal{F} ((\mathbb{D}^\varepsilon - \mathbb{D}) U^\varepsilon) = \sum_{\omega_n^{a,b} \neq 0} e^{i \frac{t}{\varepsilon} \omega_n^{a,b}} \left(\mathbb{D} (n) U^{a,\varepsilon} (n) \Big| e^b (n) \right)_{\mathbb{C}^4} e^b (n) \rightarrow 0,$$

as $\varepsilon \rightarrow 0$ in the sense of distributions. To do so we consider $\phi \in \mathcal{D} (\mathbb{R}_+ \times \mathbb{T}^3)$. Since for $s, t < 3/2, s + t > 0$ the map

$$\begin{aligned}H^s (\mathbb{T}^3) \times H^t (\mathbb{T}^3) &\rightarrow H^{s+t-\frac{3}{2}} (\mathbb{T}^3), \\ (u, v) &\mapsto u \otimes v,\end{aligned}$$

is continuous we deduce that

$$(\partial_t U^\varepsilon)_\varepsilon \quad \text{uniformly bounded in } L^2(\mathbb{R}_+; H^{-3/2}(\mathbb{T}^3)).$$

We want hence to prove that

$$S_1^\varepsilon = \sum_n \sum_{\omega_n^{a,b} \neq 0} \int e^{i\frac{t}{\varepsilon} \omega_n^{a,b}} (\mathbb{D}(n) U^{a,\varepsilon}(t, n) |e^b(n))_{\mathbb{C}^4} e^b(n) \hat{\phi}(t, n) dt \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (4.3.4)$$

Indeed the sum on the left-hand-side of (4.3.4) is well defined (in the sense that the sum is smaller than infinity). We can decompose it as

$$\begin{aligned} S_1^\varepsilon &= S_{1,N}^\varepsilon + S_1^{N,\varepsilon}, \\ S_{1,N}^\varepsilon &= \sum_{|n| \leq N} \sum_{\omega_n^{a,b} \neq 0} \int e^{i\frac{t}{\varepsilon} \omega_n^{a,b}} (\mathbb{D}(n) U^{a,\varepsilon}(t, n) |e^b(n))_{\mathbb{C}^4} e^b(n) \hat{\phi}(t, n) dt, \\ S_1^{N,\varepsilon} &= \sum_{|n| > N} \sum_{\omega_n^{a,b} \neq 0} \int e^{i\frac{t}{\varepsilon} \omega_n^{a,b}} (\mathbb{D}(n) U^{a,\varepsilon}(t, n) |e^b(n))_{\mathbb{C}^4} e^b(n) \hat{\phi}(t, n) dt. \end{aligned}$$

The term $S_1^{N,\varepsilon}$ is indeed an $o_N(1)$ function, considering in fact that the symbol $\mathbb{D}(n)$ can be bounded as $|\mathbb{D}(n)| \leq C|n|^2$ we deduce

$$\begin{aligned} S_1^{N,\varepsilon} &\leq \frac{1}{N} \sum_{|n| > N} \sum_{\omega_n^{a,b} \neq 0} \int |\mathbb{D}(n) U^{a,\varepsilon}(t, n)| |n| |\hat{\phi}(t, n)| dt, \\ &\leq \frac{C}{N} \sum_{|n| > N} \sum_{\omega_n^{a,b} \neq 0} \int |n| |U^{a,\varepsilon}(t, n)| |n|^2 |\hat{\phi}(t, n)| dt \\ &\leq \frac{C}{N} \|U^\varepsilon\|_{L^2(\mathbb{R}_+; H^1)} \|\phi\|_{L^2(\mathbb{R}_+; H^2)}, \end{aligned}$$

which indeed tends to zero as $N \rightarrow \infty$ thanks to the uniform bound (4.3.1).

For the term $S_{1,N}^\varepsilon$ We exploit the fact that

$$e^{i\frac{t}{\varepsilon} \omega_n^{a,b}} = -\frac{i\varepsilon}{\omega_n^{a,b}} \partial_t \left(e^{i\frac{t}{\varepsilon} \omega_n^{a,b}} \right), \quad (4.3.5)$$

and the fact that $|\omega_n^{a,b}| \geq c = c_N > 0$ in the set $|n| \leq N$. Using (4.3.5) on $S_{1,N}^\varepsilon$ and integrating by parts we obtain that

$$\begin{aligned} S_{1,N}^\varepsilon &= \sum_{|n| \leq N} \sum_{\omega_n^{a,b} \neq 0} \frac{i\varepsilon}{\omega_n^{a,b}} \int e^{i\frac{t}{\varepsilon} \omega_n^{a,b}} (\mathbb{D}(n) \partial_t U^{a,\varepsilon}(t, n) |e^b(n))_{\mathbb{C}^4} e^b(n) \hat{\phi}(t, n) dt \\ &\quad + \sum_{|n| \leq N} \sum_{\omega_n^{a,b} \neq 0} \frac{i\varepsilon}{\omega_n^{a,b}} \int e^{i\frac{t}{\varepsilon} \omega_n^{a,b}} (\mathbb{D}(n) U^{a,\varepsilon}(t, n) |e^b(n))_{\mathbb{C}^4} e^b(n) \partial_t \hat{\phi}(t, n) dt. \end{aligned}$$

It is obvious that the term

$$\sum_{|n| \leq N} \sum_{\omega_n^{a,b} \neq 0} \frac{i\varepsilon}{\omega_n^{a,b}} \int e^{i\frac{t}{\varepsilon} \omega_n^{a,b}} (\mathbb{D}(n) U^{a,\varepsilon}(t, n) |e^b(n))_{\mathbb{C}^4} e^b(n) \partial_t \hat{\phi}(t, n) dt \xrightarrow{\varepsilon \rightarrow 0} 0,$$

hence we shall focus on the other one. Since $|\omega_n^{a,b}| \geq c = c_N > 0$, on the set $|n| \leq N$ and $|e^b| \equiv 1$ and the fact that the symbol $|\mathbb{D}(n)| \leq C |n|^2$ we can deduce

$$\begin{aligned} & \left| \sum_{|n| \leq N} \sum_{\omega_n^{a,b} \neq 0} \frac{i \varepsilon}{\omega_n^{a,b}} \int e^{i \frac{t}{\varepsilon} \omega_n^{a,b}} (\mathbb{D}(n) \partial_t U^{a,\varepsilon}(t, n) |e^b(n)_{\mathbb{C}^4}) e^b(n) \hat{\phi}(t, n) dt \right| \\ & \lesssim \sum_{|n| \leq N} \sum_{\omega_n^{a,b} \neq 0} \varepsilon \left| \int \partial_t U^{a,\varepsilon}(t, n) |n|^2 \hat{\phi}(t, n) dt \right| \\ & \leq C \varepsilon \|\partial_t U^\varepsilon\|_{L^2(\mathbb{R}_+; H^{-3/2})} \|\phi\|_{L^2(\mathbb{R}_+; H^{7/2})} \rightarrow 0. \end{aligned}$$

This concludes the proof of (4.3.3).

The proof of (4.3.2) is more delicate.

At first: if we consider the equation of the filtered system (\mathcal{S}_ε) it is easy to deduce that

$$\begin{aligned} (\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon))_\varepsilon & \text{ bounded in } L^4(\mathbb{R}_+; H^{-1/2}) \cap L^2(\mathbb{R}_+; H^{-3/2}), \\ (-\mathbb{D}^\varepsilon U^\varepsilon)_\varepsilon & \text{ bounded in } L^2(\mathbb{R}_+; H^{-1}), \end{aligned}$$

uniformly in ε . From this we deduce that

$$(\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon))_\varepsilon \text{ bounded in } L_{\text{loc}}^2(\mathbb{R}_+; H^{-1/2}) \cap L_{\text{loc}}^2(\mathbb{R}_+; H^{-3/2}),$$

since $L_{\text{loc}}^4(\mathbb{R}_+) \hookrightarrow L_{\text{loc}}^2(\mathbb{R}_+)$. Hence by interpolation in Sobolev spaces

$$(\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon))_\varepsilon \text{ bounded in } L_{\text{loc}}^2(\mathbb{R}_+; H^{-1}),$$

uniformly in ε .

This implies hence that

$$(\partial_t U^\varepsilon)_\varepsilon = (-\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon) + \mathbb{D}^\varepsilon U^\varepsilon)_\varepsilon \text{ bounded in } L_{\text{loc}}^2(\mathbb{R}_+; H^{-1}),$$

uniformly in ε .

We can finally focus on the proof of (4.3.2). As is done for the linear part standard algebraic manipulations on the bilinear form allow us to deduce that

$$\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon) - \mathcal{Q}(U, U) = (\mathcal{Q}^\varepsilon - \mathcal{Q})(U^\varepsilon, U^\varepsilon) + \mathcal{Q}(U^\varepsilon, U^\varepsilon - U) + \mathcal{Q}(U^\varepsilon - U, U).$$

Again we can assert that

$$\mathcal{Q}(U^\varepsilon, U^\varepsilon - U) + \mathcal{Q}(U^\varepsilon - U, U) \xrightarrow{\varepsilon \rightarrow 0} 0,$$

in \mathcal{D}' to the convergence of U^ε to U in $L_{\text{loc}}^2(\mathbb{R}_+; L^2(\mathbb{T}^3))$ proved in Lemma 4.3.1. What it remain hence to be proved is that

$$(\mathcal{Q}^\varepsilon - \mathcal{Q})(U^\varepsilon, U^\varepsilon) = \sum_{\substack{k+m=n \\ \omega_{k,m,n}^{a,b,c} \neq 0 \\ a,b,c=0,\pm}} e^{i \frac{t}{\varepsilon} \omega_{k,m,n}^{a,b,c}} \left(\sum_{j=1,2,3} U^{a,j}(k) m_j U^b(m) \Big| e^c(n) \right)_{\mathbb{C}^4} e^c(n) \xrightarrow{\varepsilon \rightarrow 0} 0, \quad (4.3.6)$$

in some weak sense.

To prove (4.3.6) is equivalent, thanks to the orthogonality of the eigenvectors e^i defined in (4.2.4), to prove that, for each $\phi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{T}^3)$

$$S_2^\varepsilon = \sum_n \sum_{\substack{\omega_{k,m,n}^{a,b,c} \neq 0 \\ a,b,c=0,\pm \\ n=k+m}} \int e^{i\frac{t}{\varepsilon} \omega_{k,m,n}^{a,b,c}} U^{a,\varepsilon}(t, k) \otimes U^{b,\varepsilon}(t, m) \hat{\phi}(t, n) dt \rightarrow 0,$$

as $\varepsilon \rightarrow 0$.

As it has been done above for the term S_1^ε we can decompose S_2^ε into

$$S_{2,N}^\varepsilon = \sum_{\substack{|n| \leq N \\ |k| \leq N}} \sum_{\substack{\omega_{k,m,n}^{a,b,c} \neq 0 \\ a,b,c=0,\pm \\ n=k+m}} \int e^{i\frac{t}{\varepsilon} \omega_{k,m,n}^{a,b,c}} U^{a,\varepsilon}(t, k) \otimes U^{b,\varepsilon}(t, m) \hat{\phi}(t, n) dt$$

$$S_2^{N,\varepsilon} = S_2^\varepsilon - S_{2,N}^\varepsilon.$$

The term $S_2^{N,\varepsilon} \rightarrow 0$ as $N \rightarrow \infty$ as for the term $S_1^{N,\varepsilon}$ above. Using the fact that

$$e^{i\frac{t}{\varepsilon} \omega_{k,m,n}^{a,b,c}} = -\frac{i\varepsilon}{\omega_{k,m,n}^{a,b,c}} \partial_t \left(e^{i\frac{t}{\varepsilon} \omega_{k,m,n}^{a,b,c}} \right),$$

and the fact that $\left| \omega_{k,m,n}^{a,b,c} \right| \geq c = c_N > 0$ uniformly in k, m, n in the frequency set $\{|n|, |k| \leq N\}$ to deduce that

$$S_{2,N}^\varepsilon = \sum_{\substack{|n| \leq N \\ |k| \leq N}} \sum_{\substack{\omega_{k,m,n}^{a,b,c} \neq 0 \\ a,b,c=0,\pm \\ n=k+m}} \frac{i\varepsilon}{\omega_{k,m,n}^{a,b,c}} \int e^{i\frac{t}{\varepsilon} \omega_{k,m,n}^{a,b,c}} \partial_t U^{a,\varepsilon}(t, k) \otimes U^{b,\varepsilon}(t, m) \hat{\phi}(t, n) dt$$

$$+ \sum_{\substack{|n| \leq N \\ |k| \leq N}} \sum_{\substack{\omega_{k,m,n}^{a,b,c} \neq 0 \\ a,b,c=0,\pm \\ n=k+m}} \frac{i\varepsilon}{\omega_{k,m,n}^{a,b,c}} \int e^{i\frac{t}{\varepsilon} \omega_{k,m,n}^{a,b,c}} U^{a,\varepsilon}(t, k) \otimes \partial_t U^{b,\varepsilon}(t, m) \hat{\phi}(t, n) dt$$

$$+ \sum_{\substack{|n| \leq N \\ |k| \leq N}} \sum_{\substack{\omega_{k,m,n}^{a,b,c} \neq 0 \\ a,b,c=0,\pm \\ n=k+m}} \frac{i\varepsilon}{\omega_{k,m,n}^{a,b,c}} \int e^{i\frac{t}{\varepsilon} \omega_{k,m,n}^{a,b,c}} U^{a,\varepsilon}(t, k) \otimes U^{b,\varepsilon}(t, m) \partial_t \hat{\phi}(t, n) dt.$$

Is obvious that the term

$$\sum_{\substack{|n| \leq N \\ |k| \leq N}} \sum_{\substack{\omega_{k,m,n}^{a,b,c} \neq 0 \\ a,b,c=0,\pm \\ n=k+m}} \frac{i\varepsilon}{\omega_{k,m,n}^{a,b,c}} \int e^{i\frac{t}{\varepsilon} \omega_{k,m,n}^{a,b,c}} U^{a,\varepsilon}(t, k) \otimes U^{b,\varepsilon}(t, m) \partial_t \hat{\phi}(t, n) dt \rightarrow 0,$$

while for the first two term on the right-hand-side of the equation above the procedure is the

same, hence we focus on the first one only. It is indeed true that

$$\begin{aligned}
 & \left| \sum_{\substack{|n| \leq N \\ |k| \leq N}} \sum_{\substack{\omega_{k,m,n}^{a,b,c} \neq 0 \\ a,b,c=0,\pm, \\ n=k+m}} \frac{i \varepsilon}{\omega_{k,m,n}^{a,b,c}} \int e^{i \frac{t}{\varepsilon} \omega_{k,m,n}^{a,b,c}} \partial_t U^{a,\varepsilon}(t, k) \otimes U^{b,\varepsilon}(t, m) \hat{\phi}(t, n) dt \right| \\
 & \leq \sum_{\substack{|n| \leq N \\ |k| \leq N}} \sum_{\substack{\omega_{k,m,n}^{a,b,c} \neq 0 \\ a,b,c=0,\pm, \\ n=k+m}} C \varepsilon \int |\partial_t U^{a,\varepsilon}(t, k)| |U^{b,\varepsilon}(t, m) \hat{\phi}(t, n)| dt \\
 & \leq C \varepsilon \|\partial_t U^\varepsilon\|_{L^2_{\text{loc}}(\mathbb{R}_+; H^{-1})} \|U^\varepsilon \phi\|_{L^2_{\text{loc}}(\mathbb{R}_+; H^1)},
 \end{aligned}$$

which uniformly tends to zero w.r.t. ε thanks to the uniform bounds given above, concluding the proof. \square

4.3.2 Oscillating behavior of V^ε .

The above uniform bounds give a shady determination of the limit function U^ε . By definition of $U^\varepsilon = \mathcal{L}\left(\frac{t}{\varepsilon}\right) V^\varepsilon$ with V^ε distributional solution of (PBS_ε) we want to determinate some qualitative connection between the oscillating behavior of the filtered system $(\mathcal{S}_\varepsilon)$ and the initial system (PBS_ε) .

Thanks to Lemma 4.3.1 we can say that

$$U^\varepsilon(t, x) = U(t, x) + r^\varepsilon(t, x),$$

where $r^\varepsilon \rightarrow 0$, in $L^2_{\text{loc}}(\mathbb{R}_+, L^2(\mathbb{T}^3))$. Hence r^ε is a perturbative term in the $L^2_{\text{loc}}(\mathbb{R}_+, L^2(\mathbb{T}^3))$ topology. As it has been explained in detail in Section 4.2 we can decompose the (weak) limit U projecting it onto the non-oscillating and oscillating space $U = \bar{U} + U_{\text{osc}}$, where the orthogonal projection is defined in (4.2.6)-(4.2.7). Since $U^\varepsilon = \mathcal{L}\left(\frac{t}{\varepsilon}\right) V^\varepsilon$ we can hence deduce

$$V^\varepsilon(t, x) = \mathcal{L}\left(-\frac{t}{\varepsilon}\right) \bar{U}(t, x) + \mathcal{L}\left(-\frac{t}{\varepsilon}\right) U_{\text{osc}}(t, x) + \mathcal{L}\left(-\frac{t}{\varepsilon}\right) r^\varepsilon(t, x).$$

By the definition itself of \bar{U} we know that \bar{U} belongs to the kernel of the penalized operator $\mathbb{P}\mathcal{A}$, hence

$$\mathcal{L}\left(-\frac{t}{\varepsilon}\right) \bar{U} = \bar{U}.$$

Moreover the operator $\mathcal{L}(\tau)$, $\tau \in \mathbb{R}$ is unitary in $L^2(\mathbb{T}^3)$, whence the function

$$R^\varepsilon(t) = \mathcal{L}\left(-\frac{t}{\varepsilon}\right) r^\varepsilon(t),$$

is still an $o_\varepsilon(1)$ function in the $L^2_{\text{loc}}(\mathbb{R}_+, L^2(\mathbb{T}^3))$ topology. Hence

$$V^\varepsilon(t, x) = \bar{U}(t, x) + \mathcal{L}\left(-\frac{t}{\varepsilon}\right) U_{\text{osc}}(t, x) + R^\varepsilon(t, x),$$

i.e. V^ε is a (high) oscillation around a stationary state \bar{U} modulated by a $L^2_{\text{loc}}(\mathbb{R}_+, L^2(\mathbb{T}^3))$ perturbation which tends to zero as $\varepsilon \rightarrow 0$.

4.3.3 Proof of Lemma 4.3.3.

Lemma 4.3.6. *The following limits hold, in the sense of distributions*

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left(\int_{\mathbb{T}_h^2} (\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon))^h dx_h, 0, 0 \right) &= \mathcal{F}_v^{-1} \left(\sum_{(a,b) \in \{0, \pm\}^2} \sum_{\mathcal{I}_{a,b}(n_3)} n_3 \left(U^{a,3}(k) \hat{U}^{b,h}(m) \right) \right), \\ &= \underline{\mathcal{Q}}(U, U), \end{aligned} \quad (4.3.7)$$

where, fixed $(a, b) \in \{0, \pm\}^2$ the summation set $\mathcal{I}_{a,b}$ is defined as

$$\mathcal{I}_{a,b}(n_3) = \{(k, m) \in \mathbb{Z}^6 \mid k + m = (0, n_3), \omega^a(k) + \omega^b(m) = 0\} \quad (4.3.8)$$

Proof. Let us hence study the distributional limit for $\varepsilon \rightarrow 0$ of $\left(\int_{\mathbb{T}_h^2} (\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon))^h dx_h, 0, 0 \right)$. Let us consider a function $\phi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{T}_v^1)$ of the form $\phi = (\phi_1, \phi_2, 0, 0)$, and evaluate

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\mathbb{T}_v^1} \left(\int_{\mathbb{T}_h^2} (\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon)) dx_h(t, x_3) \right) \cdot \phi(t, x_3) dx_3 dt \\ &= \int_{\mathbb{R}_+} \int_{\mathbb{T}_v^1} \left(\int_{\mathbb{T}_h^2} \mathcal{L}\left(\frac{t}{\varepsilon}\right) \left[\mathcal{L}\left(-\frac{t}{\varepsilon}\right) U^\varepsilon \cdot \nabla \mathcal{L}\left(-\frac{t}{\varepsilon}\right) U^\varepsilon \right] dx_h \right) (t, x_3) \cdot \phi(t, x_3) dx_3 dt \\ &= \int_{\mathbb{R}_+} \int_{\mathbb{T}_v^1} \left(\int_{\mathbb{T}_h^2} \left[\mathcal{L}\left(-\frac{t}{\varepsilon}\right) U^\varepsilon \cdot \nabla \mathcal{L}\left(-\frac{t}{\varepsilon}\right) U^\varepsilon \right] dx_h \right) (t, x_3) \cdot \phi(t, x_3) dx_3 dt. \end{aligned}$$

In the above equality (and for the rest of the proof) $A \cdot B$ is the standard scalar product in \mathbb{C}^4 . We underline the fact that, being $\phi = (\phi_1, \phi_2, 0, 0)$ an element of the form $A \cdot \phi$ has only the horizontal components which give a non-null contribution to the scalar product. The last equality is justified by the fact that the adjoint of $\mathcal{L}\left(\frac{t}{\varepsilon}\right)$ is $\mathcal{L}\left(-\frac{t}{\varepsilon}\right)$ and $\mathcal{L}\left(-\frac{t}{\varepsilon}\right) \phi(t, x_3) = \phi(t, x_3)$. By use of Placherel theorem we can hence deduce

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\mathbb{T}_v^1} \left(\int_{\mathbb{T}_h^2} (\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon)) dx_h(t, x_3) \right) \cdot \phi(t, x_3) dx_3 dt \\ &= \int_{\mathbb{R}_+} \sum_{\substack{n_3 \in \mathbb{Z} \\ k+m=(0, n_3) \\ a, b}} e^{i \frac{t}{\varepsilon} \omega_{k,m}^{a,b}} n_3 \left(U^{a,3,\varepsilon}(t, k) U^{b,\varepsilon}(t, m) \right) \cdot \hat{\phi}(t, n_3) dt. \end{aligned}$$

In this case $\omega_{k,m}^{a,b} = \omega^a(k) + \omega^b(m)$. Indeed an application of stationary phase theorem a

allows us to deduce that

$$\int_{\mathbb{R}_+} \sum_{\substack{n_3 \in \mathbb{Z} \\ k+m=(0,n_3) \\ a,b}} e^{i \frac{t}{\varepsilon} \omega_{k,m}^{a,b}} n_3 (U^{a,3,\varepsilon}(t,k) U^{b,\varepsilon}(t,m)) \cdot \hat{\phi}(t, n_3) dt$$

$$\xrightarrow{\varepsilon \rightarrow 0} \int_{\mathbb{R}_+} \sum_{\substack{n_3 \in \mathbb{Z} \\ k+m=(0,n_3) \\ \omega^a(k)+\omega^b(m)=0}} n_3 (U^{a,3}(t,k) U^b(t,m)) \cdot \hat{\phi}(t, n_3) dt,$$

which is exactly the result stated. □

Lemma 4.3.7. Let $\underline{Q}(U, U)$ be defined as in (4.3.7), then

$$\underline{Q}(U, U) = 0.$$

Proof. Let us recall that

$$\mathcal{F}_v \underline{Q}(U, U) = \sum_{(a,b) \in \{0, \pm\}^2} \sum_{\mathcal{I}_{a,b}(n_3)} n_3 (U^{a,3}(k) \hat{U}^{b,h}(m)),$$

hence we shall prove that for any $(a, b) \in \{0, \pm\}^2$ the quantity $\sum_{n_3, \mathcal{I}_{a,b}(n_3)} n_3 (U^{a,3}(k) \hat{U}^{b,h}(m))$ is null. The summation set $\mathcal{I}_{a,b}(n_3)$ is defined in (4.3.8).

- We consider at first the case in which $(a, b) = (0, 0)$, then the contributions of $\mathcal{F}_v \underline{Q}$ restricted on the set $(a, b) = (0, 0)$ are

$$\sum_{k+m=(0,n_3)} n_3 (U^{0,3}(k) \hat{U}^{0,h}(m)),$$

but $U^{0,3} \equiv 0$ (see (4.2.4) and (4.2.5)) hence this contribution is null.

- Let us suppose $(a, b) = (\pm, 0)$, the contributions of (4.3.7) restricted on such set are

$$\sum_{\substack{k+m=(0,n_3) \\ \omega^\pm(k)=0}} n_3 (U^{\pm,3}(k) \hat{U}^{0,h}(m)).$$

The condition $\omega^\pm(k) = 0$ implies that $k_h \equiv 0$, while the condition $k + m = (0, n_3)$ implies that $m_h \equiv 0$, but $U^{a,3}(0, k_3) \equiv 0$ (see (4.2.5)), whence such term gives a null contribution. The same approach can be applied for the case $(a, b) = (0, \pm)$.

- We consider now the case in which $(a, b) = (\pm, \pm)$, the contributions are hence

$$\sum_{\substack{k+m=(0,n_3) \\ \omega^\pm(k)+\omega^\pm(m)=0}} U^{\pm,3}(k) m_3 \hat{U}^\pm(m).$$

Since $k + m = (0, n_3)$ then $|k_h| = |m_h| = \lambda$. Taking in consideration the constraint $\omega^\pm(k) + \omega^\pm(m) = 0$, which reads as (thanks to the explicit formulation of the eigenvalues in (4.2.3))

$$\frac{\lambda}{\sqrt{\lambda^2 + k_3^2}} + \frac{\lambda}{\sqrt{\lambda^2 + m_3^2}} = 0,$$

which implies that $\lambda \equiv 0$. Then we can argue as in the two points above to deduce that such contribution is null.

- Next we handle the more delicate case in which $(a, b) = (\pm, \mp)$. In this case the contributions are given by

$$\sum_{\substack{k+m=(0, n_3) \\ \omega^\pm(k)=\omega^\pm(m)}} n_3 \left(U^{\pm,3}(k) \hat{U}^{\mp,h}(m) \right), \quad (4.3.9)$$

where we used implicitly the divergence-free property of the vector U^\pm . The conditions $k + m = (0, n_3)$, $\omega^\pm(k) = \omega^\pm(m)$ imply now that $k_h = -m_h$ and

$$\frac{\lambda}{\sqrt{\lambda^2 + k_3^2}} = \frac{\lambda}{\sqrt{\lambda^2 + m_3^2}},$$

which implies that $m_3 = \pm k_3$.

- If $m_3 = -k_3$ the convolution constraint $k_3 + m_3 = n_3$ in (4.3.9) implies that $n_3 \equiv 0$, and hence the contributions in (4.3.9) arising from this case are nil.
- In this case $k_h = -m_h$ and $k_3 = m_3 = \frac{n_3}{2}$. Hence we are dealing with an interaction of the form

$$B_{n_3}^{\pm, \mp} = \sum_{m_h \in \mathbb{Z}^2} n_3 \left(U^{\pm,3} \left(-m_h, \frac{n_3}{2} \right) \hat{U}^{\mp,h} \left(m_h, \frac{n_3}{2} \right) \right), \quad n_3 \in 2\mathbb{Z}.$$

We shall now reformulate the infinite sum $B_{n_3}^{+,-} + B_{n_3}^{-,+}$ in way in which its symmetric properties are explicit.

If we consider the element

$$\begin{aligned} \beta(m_h, n_3) = & \frac{n_3}{2} \left[U^{+,3} \left(-m_h, \frac{n_3}{2} \right) \hat{U}^{-,h} \left(m_h, \frac{n_3}{2} \right) \right. \\ & + U^{-,3} \left(-m_h, \frac{n_3}{2} \right) \hat{U}^{+,h} \left(m_h, \frac{n_3}{2} \right) + U^{+,3} \left(m_h, \frac{n_3}{2} \right) \hat{U}^{-,h} \left(-m_h, \frac{n_3}{2} \right) \\ & \left. + U^{-,3} \left(m_h, \frac{n_3}{2} \right) \hat{U}^{+,h} \left(-m_h, \frac{n_3}{2} \right) \right], \quad (4.3.10) \end{aligned}$$

we can indeed say that

$$B_{n_3}^{+,-} + B_{n_3}^{-,+} = \sum_{m_h \in \mathbb{Z}^2} \beta(m_h, n_3).$$

We claim that

$$\beta(m_h, n_3) = 0, \quad \forall m_h, n_3, \quad (4.3.11)$$

the proof of (4.3.11) is postponed, this implies that $B_{n_3}^{+,-} + B_{n_3}^{-,+} = 0$, and we finally conclude the proof.

□

Proof of (4.3.11).: Let us write the elements $\beta(m_h, n_3)$ as

$$\beta(m_h, n_3) = \beta^\pm(m_h, n_3) + \beta^\mp(m_h, n_3),$$

where

$$\beta^\pm(m_h, n_3) = \frac{n_3}{2} \left[U^{+,3} \left(-m_h, \frac{n_3}{2} \right) \hat{U}^{-,h} \left(m_h, \frac{n_3}{2} \right) + U^{-,3} \left(m_h, \frac{n_3}{2} \right) \hat{U}^{+,h} \left(-m_h, \frac{n_3}{2} \right) \right], \quad (4.3.12)$$

$$\beta^\mp(m_h, n_3) = \frac{n_3}{2} \left[U^{-,3} \left(-m_h, \frac{n_3}{2} \right) \hat{U}^{+,h} \left(m_h, \frac{n_3}{2} \right) + U^{+,3} \left(m_h, \frac{n_3}{2} \right) \hat{U}^{-,h} \left(-m_h, \frac{n_3}{2} \right) \right].$$

We shall prove only $\beta^\pm(m_h, n_3) \equiv 0$ being the proof of $\beta^\mp(m_h, n_3) \equiv 0$ identical. By definition itself of such elements we know that

$$\begin{aligned} U^{+,3} \left(-m_h, \frac{n_3}{2} \right) \hat{U}^{-,h} \left(m_h, \frac{n_3}{2} \right) &= \left(\hat{U} \left(-m_h, \frac{n_3}{2} \right) \middle| e^+ \left(-m_h, \frac{n_3}{2} \right) \right)_{\mathbb{C}^4} e^{+,3} \left(-m_h, \frac{n_3}{2} \right) \\ &\quad \times \left(\hat{U} \left(m_h, \frac{n_3}{2} \right) \middle| e^- \left(m_h, \frac{n_3}{2} \right) \right)_{\mathbb{C}^4} e^{-,h} \left(m_h, \frac{n_3}{2} \right), \\ U^{-,3} \left(m_h, \frac{n_3}{2} \right) \hat{U}^{+,h} \left(-m_h, \frac{n_3}{2} \right) &= \left(\hat{U} \left(m_h, \frac{n_3}{2} \right) \middle| e^- \left(m_h, \frac{n_3}{2} \right) \right)_{\mathbb{C}^4} e^{-,3} \left(m_h, \frac{n_3}{2} \right) \\ &\quad \times \left(\hat{U} \left(-m_h, \frac{n_3}{2} \right) \middle| e^+ \left(-m_h, \frac{n_3}{2} \right) \right)_{\mathbb{C}^4} e^{+,h} \left(-m_h, \frac{n_3}{2} \right). \end{aligned}$$

By the aid of the explicit definition of the eigenvectors given in (4.2.4) we can argue that

$$\begin{aligned} e^{-,h} \left(m_h, \frac{n_3}{2} \right) &= e^{+,h} \left(-m_h, \frac{n_3}{2} \right) = A_{m_h, n_3}^h, \\ e^{+,3} \left(-m_h, \frac{n_3}{2} \right) &= -e^{-,3} \left(m_h, \frac{n_3}{2} \right) = A_{m_h, n_3}^3. \end{aligned}$$

Whence setting

$$C_{m_h, n_3} = \left(\hat{U} \left(-m_h, \frac{n_3}{2} \right) \middle| e^+ \left(-m_h, \frac{n_3}{2} \right) \right)_{\mathbb{C}^4} \left(\hat{U} \left(m_h, \frac{n_3}{2} \right) \middle| e^- \left(m_h, \frac{n_3}{2} \right) \right)_{\mathbb{C}^4},$$

by the aid of the above definitions we hence deduced that

$$\begin{aligned} U^{+,3} \left(-m_h, \frac{n_3}{2} \right) \hat{U}^{-,h} \left(m_h, \frac{n_3}{2} \right) &= -C_{m_h, n_3} A_{m_h, n_3}^h A_{m_h, n_3}^3, \\ U^{-,3} \left(m_h, \frac{n_3}{2} \right) \hat{U}^{+,h} \left(-m_h, \frac{n_3}{2} \right) &= C_{m_h, n_3} A_{m_h, n_3}^h A_{m_h, n_3}^3. \end{aligned}$$

Inserting such relations in the definition of β^\pm (4.3.12) we deduce hence that $\beta^\pm(m_h, n_3) \equiv 0$, concluding.

4.4 The limit.

The aim of this section is to give a full description of the limit system written in the generic form (\mathcal{S}_0) in the case in which the penalized system is given by (PBS_ε), i.e. in the present

section we prove Theorem 4.1.5.

In the present work we aim to prove large time propagation of strong solutions with no particular assumption on the smallness of the initial data for the system (PBS_ε) . To achieve this goal we shall divide the limit system (\mathcal{S}_0) in two parts, the first one represents the evolution of U onto the kernel of the penalized operator $\mathbb{P}\mathcal{A}$. Such projection is defined by the element \bar{U} in (4.2.6). The second part will represent the projection of U onto the space $(\ker \mathbb{P}\mathcal{A})^\perp$, this projection is denoted by the element U_{osc} defined in (4.2.7).

The result we prove is Theorem 4.1.5, in order to prove it we proceed as follows: we consider separately the evolution of U distributional solution of (\mathcal{S}_0) onto the non-oscillating and oscillating subspace and prove that respectively \bar{U} solves (4.1.5) and U_{osc} solves (4.1.6). These results are codified respectively in Proposition 4.4.1 and Proposition 4.4.5. The structure of the following two subsections is very similar, namely we shall proceed in the following manner

- We project the equation (\mathcal{S}_0) onto the non-oscillatory or oscillatory subspace.
- We study the bilinear interaction $\mathcal{Q}(U, U)$ once it is projected onto these spaces. We deduce that, under some suitable assumptions, such bilinear interaction once it is restricted on the subspaces mentioned above assumes a "suitable" form. We refer to the statements of Proposition 4.4.1 and 4.4.5 for further details of what "suitable" means.
- We study the form of the second-order linear elliptic operator \mathbb{D} defined in (4.1.2) once it is projected on the non-oscillatory or oscillatory subspace.

4.4.1 Derivation of the equation for \bar{U} .

The procedure we adopt to derive the limit system is pretty straightforward, we mention the works [72] and [123] where the authors adopted the same techniques.

As we already mentioned in this section we want to deduce the equations satisfied by the projection of U onto the non-oscillating space $\mathbb{C}e^0$. Such projection will be denoted as $\bar{U} = (\bar{u}^h, 0, 0)$ as it is already mentioned in (4.2.6). The result we want to prove is codified in the following proposition

Proposition 4.4.1. *Let $\bar{U}_0 = (\bar{u}_0^h, 0, 0) = (\nabla_h^\perp \Delta_h^{-1} \omega^h, 0, 0)$ be in $L^2(\mathbb{T}^3)$. The projection of U distributional solution of (\mathcal{S}_0) onto the non-oscillating space $\mathbb{C}e^0$ (see (4.2.3)) defined as*

$$\bar{U} = \mathcal{F}^{-1} \left((\mathcal{F}U|_{e^0})_{\mathbb{C}^4} e^0 \right),$$

satisfies the following two-dimensional stratified Navier-Stokes equations with vertical diffusion (in the sense of distributions)

$$\begin{cases} \partial_t \bar{u}^h + \bar{u}^h \cdot \nabla_h \bar{u}^h - \nu \Delta \bar{u}^h = -\nabla_h \bar{p}, \\ \bar{u}^h|_{t=0} = \bar{u}_0^h. \end{cases}$$

We divide the proof in steps:

Step 1 We project the equation (\mathcal{S}_0) onto the non-oscillatory space generated by the vector e^0 defined in (4.2.4). We recall again that such projection is defined as follows (see (4.2.6) as well): given a vector field W the orthogonal projection is defined as

$$\mathcal{F} \bar{W} = \left(\hat{W} \Big|_{\mathbb{C}^4} e^0 \right),$$

with this projection we can derive the evolution equation for the limit flow \bar{U} , i.e.

$$\begin{cases} \partial_t \bar{U} + \overline{\mathcal{Q}(U, U)} - \overline{\mathbb{D}U} = 0, \\ \bar{U}_0|_{t=0} = \bar{U}_0 = \bar{V}_0, \end{cases} \quad (4.4.1)$$

Step 2 We prove by mean of a careful analysis that the projection of $\mathcal{Q}(U, U)$ onto the non-oscillating subspace $\mathbb{C}e^0$, i.e. the element $\overline{\mathcal{Q}(U, U)}$ is in fact

$$\overline{\mathcal{Q}(U, U)} = \mathcal{B}(\bar{U}, \bar{U}),$$

for a suitable bilinear form \mathcal{B} . Hence the projection onto the kernel of the penalized operator $\mathbb{P}\mathcal{A}$ of all the bilinear interactions is a suitable bilinear interaction of elements of the kernel.

Step 3 The last step of this section is to prove that

$$-\overline{\mathbb{D}U} = -\nu \Delta \bar{U}$$

We have explained the structure of the present session.

To prove Proposition 4.4.1 it is sufficient to prove Step 1 – Step 3 mentioned above.

To understand the limit of the system (PBS_ε) means to diagonalize the system (PBS_ε) in terms of the oscillating and non-oscillating modes introduced in Section 4.2. To do so we introduce the following quantities

$$\begin{aligned} \omega^{h,\varepsilon} &= -\partial_2 u^{1,\varepsilon} + \partial_1 u^{2,\varepsilon}; & \bar{u}^{h,\varepsilon} &= \nabla_h^\perp \Delta_h^{-1} \omega^{h,\varepsilon}; \\ \psi^\varepsilon &= \Delta_h^{-1} \omega^{h,\varepsilon}; & \tilde{\psi}^\varepsilon &= \Delta_h^{-1/2} \omega^{h,\varepsilon}. \end{aligned} \quad (4.4.2)$$

Step 1 is only a constructive consideration, hence there is nothing to prove.

Proof of Step 2.

The proof of the Step 2 is codified in the following lemma:

Lemma 4.4.2. *Let $U^\varepsilon \rightarrow U$ in $L_{\text{loc}}^2(\mathbb{R}_+; L^2(\mathbb{T}^3))$ as proved in Lemma 4.3.1, the limit of $(\mathcal{F}\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon)|_{e^0(n)})$ as $\varepsilon \rightarrow 0$ is $\mathcal{F}(\Delta_h^{-1/2}(\bar{u}^h \cdot \nabla_h \omega^h))$, in the sense of distributions.*

Proof. Let us recall that explicit expression of $\mathcal{FQ}^\varepsilon(U^\varepsilon, U^\varepsilon)$ is given in (4.3.2). As explained before in Lemma 4.3.4 thanks to the stationary phase theorem in the limit as $\varepsilon \rightarrow 0$ the only contributions remaining in (4.3.2) is

$$\mathcal{FQ}(U, U)(n) = \mathbb{P}_n \sum_{\substack{\omega_{k,m,n}^{a,b,c}=0 \\ k+m=n \\ j=1,2,3}} (U^{a,j}(k) m_j U^b(m) | e^c(n))_{\mathbb{C}^4} e^c(n). \quad (4.4.3)$$

Now, since e^0 is orthogonal to e^\pm as is evident from the definition of the eigenvectors given in (4.2.4) we obtain that, projecting on the non-oscillatory potential subspace $(\mathcal{FQ}(U, U)| e^0)_{\mathbb{C}^4}$

$$(\mathcal{FQ}(U, U)(n) | e^0(n))_{\mathbb{C}^4} = \mathbb{P}_n \sum_{\substack{\omega_{k,m,n}^{a,b,0}=0 \\ k+m=n \\ j=1,2,3}} (U^{a,j}(k) m_j U^b(m) | e^0(n))_{\mathbb{C}^4} |e^0(n)|^2,$$

whence we can reduce to the case $c = 0$.

Reading in the Fourier space the projection of the bilinear form it is clear that not all Fourier modes contribute to the bilinear interaction. In the special case that we are considering now in fact the set of bilinear interactions is

$$\left\{ (k, m, n) \in \mathbb{Z}^9 : \omega_{k,m,n}^{a,b,0} = 0, a, b = 0, \pm, k + m = n \right\} = \mathcal{R} = \bigcup_{i=0}^3 \mathcal{R}_i,$$

where

$$\begin{aligned} \mathcal{R}_0 &= \{ (k, m, n) \in \mathbb{Z}^9 : \omega_{k,m,n}^{0,0,0} = 0, k + m = n \}, \\ \mathcal{R}_1 &= \{ (k, m, n) \in \mathbb{Z}^9 : \omega_{k,m,n}^{\pm,\pm,0} = 0, k + m = n \}, \\ &= \left\{ (k, m, n) \in \mathbb{Z}^9 : \pm \frac{|k_h|}{|k|} \pm \frac{|m_h|}{|m|} = 0, k + m = n \right\}, \\ \mathcal{R}_2 &= \{ (k, m, n) \in \mathbb{Z}^9 : (\omega_{k,m,n}^{\pm,0,0} = 0) \vee (\omega_{k,m,n}^{0,\pm,0} = 0), k + m = n \}, \\ &= \left\{ (k, m, n) \in \mathbb{Z}^9 : \left(\pm \frac{|k_h|}{|k|} = 0 \right) \vee \left(\pm \frac{|m_h|}{|m|} = 0 \right), k + m = n \right\}, \\ \mathcal{R}_3 &= \{ (k, m, n) \in \mathbb{Z}^9 : \omega_{k,m,n}^{\pm,\mp,0} = 0, k + m = n \}, \\ &= \left\{ (k, m, n) \in \mathbb{Z}^9 : \pm \frac{|k_h|}{|k|} \mp \frac{|m_h|}{|m|} = 0, k + m = n \right\} \end{aligned}$$

Thanks to the above decomposition of the set of bilinear interactions we can assert that $(\mathcal{FQ}(U, U)| e^0) = \sum_{i=0}^3 \mathcal{B}_i$, where

$$\mathcal{B}_i(n) = \mathbb{P}_n \sum_{\substack{(k,m,n) \in \mathcal{R}_i \\ j=1,2,3}} (U^{a,j}(k) m_j U^{b,j'}(m) | e^0(n)) |e^0(n)|^2.$$

We start at this point to study the resonance effect on the expression (4.4.3). Indeed the triple $(a, b, c) = (0, 0, 0)$ is admissible which determinate the bilinear interaction set \mathcal{R}_0 .

Namely \mathcal{R}_0 describes the set of bilinear interactions between element of $\ker \mathbb{P}\mathcal{A}$. The term \mathcal{B}_0 gives hence a non-null contribution, we want to show that the contributions coming from the other \mathcal{B}_i 's are null. At first let us suppose that $a = b \neq 0$, i.e. we are considering the contributions coming from the term \mathcal{B}_1 which is defined by the resonant set \mathcal{R}_1 . Let us say $a = b = +$. Whence the resonance condition $\omega_{k,m,n}^{+,+,0} = 0$ reads as $|k_h| = |m_h| = 0$. As it was proved in Section 4.2 in the case in which $n_h = 0$ the eigenvalues collapse all to zero, and hence we obtain that

$$\{(k, m, n) \in \mathbb{Z}^9 : \omega_{k,m,n}^{+,+,0}, k + m = n\} \subset \mathcal{R}_0.$$

The very same analysis can be done for the triplets $(-, -, 0)$, $(\pm, 0, 0)$, $(0, \pm, 0)$, and hence to prove that $\mathcal{B}_1 = \mathcal{B}_2 = 0$.

What is left hence at this point is to prove that the triplets $(\pm, \mp, 0)$ do not produce any bilinear interaction, or, alternatively, to prove that the contribution coming from \mathcal{B}_3 is zero. To do so let us set

$$\begin{aligned} \hat{U}_n^a &= \left(\hat{U}(n) \middle| e^a(n) \right), \\ C_{k,m,n}^{a,b,c} &= \sum_{j=1}^3 e^{a,j}(k) m_j \left(e^b(m) \middle| e^c(n) \right), \end{aligned}$$

in particular with this notation the limit form (4.4.3) can be written as

$$\mathcal{FQ}(U, U) = \mathbb{P}_n \sum_{\substack{\omega_{k,m,n}^{a,b,c}=0 \\ k+m=n}} C_{k,m,n}^{a,b,c} \hat{U}_k^a \hat{U}_m^b e^c(n).$$

Let us consider at this point the resonant condition $\omega^\pm(k) + \omega^\mp(m) = 0$, it is equivalent, after some algebraic manipulation, considering the explicit expression of the eigenvalues given in (4.2.3) to

$$k_3^2 |m_h|^2 = m_3^2 |k_h|^2. \quad (4.4.4)$$

Some straightforward computations, using the explicit expression of the eigenvectors given in (4.2.4) gives us that

$$C_{k,m,n}^{-,+,0} = C_{k,m,n}^{+,-,0} \stackrel{\text{def}}{=} \frac{1}{2} C_{k,m,n}^{\pm,0} \in \mathbb{R}. \quad (4.4.5)$$

Moreover $\hat{U}_n^\pm = \pm i c(n) + d(n)$, c and d are complex-valued and assume the following form

$$\begin{aligned} c(n) &= \frac{n_1 n_3}{|n_h| |n|} \hat{U}^1 + \frac{n_2 n_3}{|n_h| |n|} \hat{U}^2 - \frac{|n_h|}{|n|} \hat{U}^3, \\ d(n) &= \hat{U}^4. \end{aligned}$$

The \hat{U}^i above is the i -th component of the Fourier transform of U . Hence we can write

$$\begin{aligned} \hat{U}_k^\mp \hat{U}_m^\pm &= C(k, m) \pm iD(k, m), \\ C(k, m) &= c(k)c(m) + d(k)d(m), \\ D(k, m) &= c(k)d(m) - c(m)d(k), \end{aligned}$$

with C symmetric and D skew-symmetric with respect to k and m . With these considerations hence

$$\begin{aligned} \sum_{\substack{\omega_{k,m,n}^{\pm,\mp,0}=0 \\ k+m=n}} C_{k,m,n}^{\pm,\mp,0} \hat{U}_k^{\pm} \hat{U}_m^{\mp} &= \sum_{\substack{k_3^2 |m_h|^2 = m_3^2 |k_h|^2 \\ k+m=n}} \left(C_{k,m,n}^{-,+0} \hat{U}_k^- \hat{U}_m^+ + C_{k,m,n}^{+,-0} \hat{U}_k^+ \hat{U}_m^- \right), \\ &= \sum_{\substack{k_3^2 |m_h|^2 = m_3^2 |k_h|^2 \\ k+m=n}} C_{k,m,n}^{\pm,0} C(k, m). \end{aligned} \quad (4.4.6)$$

We rely now on the following lemma whose proof is postponed at the end of the present section.

Lemma 4.4.3. *Under the convolution constraint $k + m = n$ the element $C_{k,m,k+m}^{\pm,0}$ defined in (4.4.5), is skew symmetric with respect to the variables k, m .*

Using at this point Lemma 4.4.3 it is easy to conclude. If we consider the expression in (4.4.6), and we remark that the summation set, given by the relation (4.4.4), is symmetric with respect to k and m , since C is symmetric and $C_{k,m,k+m}^{\pm,0}$ is skew symmetric we obtain that the sum in (4.4.6) is zero, hence the only admissible triple is $(0, 0, 0)$.

At this point hence all that remains is to fully describe what is the sum

$$(\mathcal{FQ}(U, U) | e^0) = \left(\mathbb{P}_n \sum_{\substack{k+m=n \\ j=1,2,3}} \hat{U}_k^0 \hat{U}_m^0 e^{0,j}(k) m_j (e^0(m) | e^0(n)) \left. e^0(n) \right|_{\mathbb{C}^4} e^0(n) \right).$$

The matrix \mathbb{P}_n is symmetric and purely real, hence selfadjoint, and the vector e^0 is divergence-free, this implies that

$$(\mathcal{FQ}(U, U) | e^0) = \sum_{\substack{k+m=n \\ j=1,2,3}} \hat{U}_k^0 \hat{U}_m^0 e^{0,j}(k) m_j (e^0(m) | e^0(n))_{\mathbb{C}^4} |e^0(n)|^2,$$

by our choice of e^0 (see (4.2.4)) we have that $|e^0(n)|^2 \equiv 1$ and a straightforward computation gives us that, considering the relations defined in (4.4.2),

$$\sum_{j=1}^3 \hat{U}_k^0 \hat{U}_m^0 e^{0,j}(\bar{k}) \bar{m}_j = \mathcal{F} \left(\bar{u}^h \cdot \nabla_h \tilde{\psi} \right) (\bar{n}),$$

where $\bar{k} + \bar{m} = \bar{n}$, whence evaluating what $(e^0(m) | e^0(n))$ is, under the convolution condition $k + m = n$ we obtain

$$\begin{aligned} (e^0(m) | e^0(n)) &= \frac{1}{|n_h| |m_h|} (n_1 m_1 + n_2 m_2), \\ &= \frac{1}{|n_h| |m_h|} [(m_1^2 + m_2^2) + (k_1 m_1 + k_2 m_2)]. \end{aligned}$$

At this point we first apply the operator defined by the symbol $\frac{1}{|n_h| |m_h|} (m_1^2 + m_2^2)$ to the element evaluated above $\mathcal{F} \left(\bar{u}^h \cdot \nabla_h \tilde{\psi} \right) (n)$, this gives

$$\mathcal{F} \left(\Delta_h^{-1/2} (\bar{u}^h \cdot \nabla_h \omega^h) \right) (n),$$

while computing

$$\begin{aligned}
 & \frac{1}{|n_h| |m_h|} (k_1 m_1 + k_2 m_2) \mathcal{F} \left(\bar{u}^1 \partial_1 \tilde{\psi} + \bar{u}^2 \partial_2 \tilde{\psi} \right) \\
 &= \frac{1}{|n_h|} (k_1 m_1 + k_2 m_2) \mathcal{F} \left(\bar{u}^1 \partial_1 \psi + \bar{u}^2 \partial_2 \psi \right), \\
 &= \frac{1}{|n_h|} \mathcal{F} \left(\partial_1 \bar{u}^1 \partial_1^2 \psi + \partial_2 \bar{u}^2 \partial_2^2 \psi + \partial_2 \bar{u}^1 \partial_{1,2}^2 \psi + \partial_1 \bar{u}^2 \partial_{1,2}^2 \psi \right), \\
 &= 0,
 \end{aligned}$$

where in the last equality we used the relation $\bar{u}^h = \begin{pmatrix} -\partial_2 \psi \\ \partial_1 \psi \end{pmatrix}$ already defined in (4.4.2). Putting together all the results we hence obtained that

$$(\mathcal{FQ}(U, U)|e^0) = \mathcal{F} \left(\Delta_h^{-1/2} (\bar{u}^h \cdot \nabla_h \omega^h) \right),$$

which concludes the proof of the lemma. \square

This concludes the proof of the Step 2, the bilinear interactions of the kernel part are the same as the ones present in the evolution equation for 2d Euler equations in vorticity form.

Proof of Lemma 4.4.3. We recall that

$$\frac{1}{2} C_{k,m,n}^{\pm,0} = \sum_{j=1}^3 e^{\pm,j}(k) m_j (e^{\pm}(m)|e^0(n)),$$

whence in particular thanks to the explicit expressions of the eigenvectors e^{\pm} given in (4.2.4) and the convolution constrain $k + m = n$ we obtain that

$$\begin{aligned}
 -\frac{1}{2} C_{k,m,k+m}^{\pm,0} &= (k_1 k_3 m_1 + k_2 k_3 m_2 - |k_h|^2 m_3) (m_2 m_3 (k_1 + m_1) - m_1 m_3 (k_2 + m_2)) \\
 &= (m_1 m_2 m_3 k_1^2 k_3 - k_1 k_2 k_3 m_1^2 m_3) + (k_1 k_2 k_3 m_2^2 m_3 - m_1 m_2 m_3 k_2^2 k_3),
 \end{aligned}$$

which is indeed skew-symmetric. \square

Proof of Step 3.

It remains to understand how the projection onto the non-oscillating space $\mathbb{C}e^0$ affects the second-order linear operator \mathbb{D} defined in (4.1.2). I.e. we want to prove the Step 3 of the list above. We study the limit as $\varepsilon \rightarrow 0$ of the second order linear part. The result we prove is the following one

Lemma 4.4.4. *The following limit holds in the sense of distributions*

$$\lim_{\varepsilon \rightarrow 0} \mathcal{F}^{-1} \left((-\mathcal{F}(\mathbb{D}^\varepsilon U^\varepsilon)|_{|n_h|} e^0(n))_{\mathbb{C}^4} \right) = -\nu \Delta \omega^h.$$

Proof. Let us write explicitly what $\lim_{\varepsilon \rightarrow 0} (-\mathcal{F}(\mathbb{D}^\varepsilon U^\varepsilon)_n |n_h| e^0(n))_{\mathbb{C}^4}$ is. By the aid of the limit formulation for the second order linear differential operator given in (4.3.3) and some computations which can be performed explicitly thanks to the exact formulation of the eigenvector e^0 given in (4.2.4) (and recalling that the eigenvectors (4.2.4) are orthonormal) we deduce

$$\lim_{\varepsilon \rightarrow 0} (-\mathcal{F}(\mathbb{D}^\varepsilon U^\varepsilon)_n |n_h| e^0(n))_{\mathbb{C}^4} = \sum_{\omega_n^{0,b}=0} \nu |n|^2 (-n_2 U^{b,1} + n_1 U^{b,2}). \quad (4.4.7)$$

$\omega_n^{a,b} = \omega^a(n) - \omega^b(n)$. Let us consider hence what the interaction condition $\omega_n^{0,b} = 0$ means. If $b = \pm$ then indeed $\omega_n^{0,b} = 0$ is equivalent to $n_h = 0$ since the equation we derive it the following one

$$\frac{|n_h|}{|n|} = 0.$$

As it has been explained in Section 4.2 as long as $n_h = 0$ the eigenvalue corresponding, i.e. ω^b , it collapses to zero, and hence it belongs to the kernel of the penalized operator. This implies that in (4.4.7) the only nonzero contributions are given if $b = 0$, proving Step 3. \square

With the proof of Step 1–Step 3 above we hence proved that, given an initial ω_0^h , the element

$$\omega^h = \text{curl}_h \bar{u}^h,$$

solves in the sense of distribution the following Navier-Stokes system in vorticity form

$$\begin{cases} \partial_t \omega^h + \bar{u}^h \cdot \nabla_h \omega^h - \nu \Delta \omega^h = 0, \\ \omega^h|_{t=0} = \omega_0^h. \end{cases} \quad (4.4.8)$$

We hence apply the 2d-Biot-Savart law $\bar{u}^h = \nabla_h^\perp \Delta_h^{-1} \omega^h$, to the system (4.4.8) to deduce the claim of Proposition 4.4.1.

4.4.2 Derivation of the equation for U_{osc} .

The result we want to prove in the present section is the following one

Proposition 4.4.5. *Let be $U_{\text{osc},0} = V_0 - \bar{U}_0 \in L^2(\mathbb{T}^3)$. Then the projection of U distributional solution of (\mathcal{S}_0) onto the oscillating space defined $\mathbb{C}e^- \oplus \mathbb{C}e^+$ defined as*

$$U_{\text{osc}} = \mathcal{F}^{-1} \left((\mathcal{F}U|e^-)_{\mathbb{C}^4} e^- + (\mathcal{F}U|e^+)_{\mathbb{C}^4} e^+ \right)$$

satisfies, for almost all $(a_1, a_2, a_3) \in \mathbb{R}^3$ parameters defining the three-dimensional periodic domain \mathbb{T}^3 the linear equation

$$\begin{cases} \partial_t U_{\text{osc}} + 2\mathcal{Q}(\bar{U}, U_{\text{osc}}) - (\nu + \nu') \Delta U_{\text{osc}} = 0, \\ U_{\text{osc}}|_{t=0} = U_{\text{osc},0}, \end{cases}$$

where \mathcal{Q} is defined (4.3.2).

Step 1 We project the equation (\mathcal{S}_0) onto the oscillatory space generated by the vectors e^- , e^+ defined in (4.2.4). We recall again that such projection is defined as follows (see (4.2.7) as well): given a vector field W the orthogonal projection onto the oscillating subspace is defined as

$$\mathcal{F} W_{\text{osc}} = \left(\hat{W} \Big| e^- \right)_{\mathbb{C}^4} e^- + \left(\hat{W} \Big| e^+ \right)_{\mathbb{C}^4} e^+,$$

with this decomposition we can derive the evolution equation for the limit flow U , i.e.

$$\begin{cases} \partial_t U_{\text{osc}} + (\mathcal{Q}(U, U))_{\text{osc}} - (\mathbb{D} U)_{\text{osc}} = 0, \\ U_{\text{osc}}|_{t=0} = U_{\text{osc},0} = V_{\text{osc},0}. \end{cases} \quad (4.4.9)$$

Step 2 Next we turn our attention to the oscillating part of the bilinear interaction $(\mathcal{Q}(U, U))_{\text{osc}}$. We prove that for almost all tori

$$(\mathcal{Q}(U, U))_{\text{osc}} = 2\mathcal{Q}(\bar{U}, U_{\text{osc}}).$$

This result is not a free-deduction and it can be attained only thanks to some geometrical hypothesis on the domain. We say in fact in this case that we consider *non-resonant* domain.

A direct consequence is that U_{osc} satisfies hence a *linear equation*, hence it is globally well posed if the perturbation \bar{U} acting on his evolution system is globally well posed as well.

Step 3 The last step of this section is to prove that

$$-(\mathbb{D} U)_{\text{osc}} = -(\nu + \nu') \Delta U_{\text{osc}}.$$

As well as in the previous section in order to prove Proposition 4.4.5 it is to prove Step 1–Step 3 above.

As well as above the Step 1 consists of constructive considerations only, hence there is nothing really to prove.

Proof of Step 2.

Our goal is to study the interaction of the kind $(\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon))_{\text{osc}}$, hence to prove the Step 2. These are bilinear interactions between highly oscillating modes, which create a bilinear interaction of the same form of the classical three-dimensional Navier-Stokes equations. We want to prove that in the limit as $\varepsilon \rightarrow 0$, for almost each torus \mathbb{T}^3 , interactions between highly oscillating modes vanishes, leaving linear interactions between U_{osc} and \bar{U} only. Since $U^\varepsilon = \bar{U}^\varepsilon + U_{\text{osc}}^\varepsilon$ it shall hence suffice to prove that

$$\lim_{\varepsilon \rightarrow 0} (\mathcal{Q}^\varepsilon(U_{\text{osc}}^\varepsilon, U_{\text{osc}}^\varepsilon))_{\text{osc}} = 0, \quad (4.4.10)$$

$$\lim_{\varepsilon \rightarrow 0} (\mathcal{Q}^\varepsilon(\bar{U}^\varepsilon, \bar{U}^\varepsilon))_{\text{osc}} = 0. \quad (4.4.11)$$

We prove (4.4.10) and (4.4.11) respectively in Lemma 4.4.6 and Lemma 4.4.7.

Lemma 4.4.6. *For almost each torus \mathbb{T}^3 the following limit holds in the sense of distributions*

$$\lim_{\varepsilon \rightarrow 0} (\mathcal{Q}^\varepsilon (U_{\text{osc}}^\varepsilon, U_{\text{osc}}^\varepsilon))_{\text{osc}} = 0.$$

Proof. In the proof of this lemma we shall see how the resonant effects play a fundamental role in the limit of the projection of the bilinear form onto the oscillatory space. In this proof only we will use again the check notation on the Fourier modes since the structure of the torus itself shall play a significant role. First of all we recall that

$$(\mathcal{F}\mathcal{Q}^\varepsilon (U^\varepsilon, U^\varepsilon))_{\text{osc}} = \left(\mathbb{P}_n \sum_{\substack{a,b,c \in \{0,\pm\} \\ k+m=n}} e^{i\frac{t}{\varepsilon} \omega_{k,m,n}^{a,b,c}} \left(\sum_{j=1,2,3} U^{a,\varepsilon,j} (k) m_j U^{b,\varepsilon} (m) \middle| e^c (n) \right) e^c (n) \right)_{\text{osc}},$$

whence, since $F_{\text{osc}} = (F|e^\pm)e^\pm$ and $e^0 \perp e^\pm$ we easily deduce that $c = \pm$. Letting $\varepsilon \rightarrow 0$ by stationary phase theorem all that remain are interactions of the form,

$$\mathcal{F}\mathcal{Q}(U, U) = \mathbb{P}_n \sum_{\substack{\omega_{k,m,n}^{a,b,\pm} = 0 \\ k+m=n \\ j=1,2,3}} (U^{a,j} (k) \check{m}_j U^b (m) | e^\pm (n)) e^\pm (n),$$

and in particular we focus on the ones which have purely highly oscillating modes interacting, i.e. when $a = \pm, b = \pm$ (but they may be different the one from the other) and the frequency set of bilinear interaction satisfies the relation

$$\frac{|\check{k}_h|}{|\check{k}|} + \varepsilon_1 \frac{|\check{m}_h|}{|\check{m}|} = \varepsilon_2 \frac{|\check{n}_h|}{|\check{n}|} \quad \varepsilon_1, \varepsilon_2 = \pm 1. \quad (4.4.12)$$

We want to prove, specifically, that the bilinear interaction restricted on these modes gives a zero contribution for almost all tori.

The above relation can be expressed as a polynomial in the variables $(\check{k}, \check{m}, \check{n})$ at the cost of long and tedious computations. In particular we shall use the following expansion

$$\begin{aligned} & 2 |\check{k}_h|^2 |\check{m}_h|^2 \left(|\check{k}_h|^2 + |\check{k}_3|^2 \right) \left(|\check{m}_h|^2 + |\check{m}_3|^2 \right) \left(|\check{n}_h|^4 + |\check{n}_3|^4 + 2 |\check{n}_h|^2 \check{n}_3^2 \right) = \\ & |\check{k}_h|^4 \left(|\check{m}_h|^4 + \check{m}_3^4 + 2 |\check{m}_h|^2 \check{m}_3^2 \right) \left(|\check{n}_h|^4 + \check{n}_3^4 + 2 |\check{n}_h|^2 \check{n}_3^2 \right) \\ & + |\check{m}_h|^4 \left(|\check{k}_h|^4 + \check{k}_3^4 + 2 |\check{k}_h|^2 \check{k}_3^2 \right) \left(|\check{n}_h|^4 + \check{n}_3^4 + 2 |\check{n}_h|^2 \check{n}_3^2 \right) \\ & + |\check{n}_h|^4 \left(|\check{k}_h|^4 + \check{k}_3^4 + 2 |\check{k}_h|^2 \check{k}_3^2 \right) \left(|\check{m}_h|^4 + \check{m}_3^4 + 2 |\check{m}_h|^2 \check{m}_3^2 \right) \\ & - 2 |\check{k}_h|^2 |\check{n}_h|^2 \left(|\check{k}_h|^2 + \check{k}_3^2 \right) \left(|\check{n}_h|^2 + \check{n}_3^2 \right) \left(|\check{m}_h|^4 + \check{m}_3^4 + 2 |\check{m}_h|^2 \check{m}_3^2 \right) \\ & - 2 |\check{m}_h|^2 |\check{n}_h|^2 \left(|\check{m}_h|^2 + \check{m}_3^2 \right) \left(|\check{n}_h|^2 + \check{n}_3^2 \right) \left(|\check{k}_h|^4 + \check{k}_3^4 + 2 |\check{k}_h|^2 \check{k}_3^2 \right). \end{aligned} \quad (4.4.13)$$

We underline the fact that (4.4.12) and (4.4.13) are equivalent. The expression in (4.4.13) could be further expanded and refined, but for our purposes the form in (4.4.13) shall be sufficient.

We take the expression in (4.4.13) and we evaluate the sum of monomials in the leading order in the variables $\check{k}_h, \check{m}_h, \check{n}_h$, which is

$$\check{P}_0(\check{k}_h, \check{m}_h) = -3 |\check{k}_h|^4 |\check{m}_h|^4 |\check{n}_h|^4,$$

while the sum of monomial in the leading order for the variables $\check{k}_3, \check{m}_3, \check{n}_3$ is

$$\begin{aligned} \check{P}_8(\check{k}, \check{m}) &= \check{m}_3^4 \check{n}_3^4 |\check{k}_h|^4 + \check{k}_3^4 \check{n}_3^4 |\check{m}_h|^4 + \check{k}_3^4 \check{m}_3^4 |\check{n}_h|^4 \\ &\quad - 2 |\check{k}_h|^2 |\check{m}_h|^2 \check{k}_3^2 \check{m}_3^2 \check{n}_3^4 - 2 |\check{k}_h|^2 |\check{n}_h|^2 \check{k}_3^2 \check{n}_3^2 \check{m}_3^4 - 2 |\check{m}_h|^2 |\check{n}_h|^2 \check{m}_3^2 \check{n}_3^2 \check{k}_3^4. \end{aligned}$$

We point out the \check{P}_8 is homogeneous of degree 8 in the variables $\check{k}_3, \check{m}_3, \check{n}_3$ while \check{P}_0 is homogeneous of degree zero.

Since $\check{P}_8(\check{k}, \check{m})$ is homogeneous of degree 8 we can rewrite is as

$$\check{P}_8(\check{k}, \check{m}) = \check{P}_8\left(\frac{k}{a}, \frac{m}{a}\right) = a_3^{-8} P_8(k, m, a_h).$$

Since a_1, a_2, a_3 are parameters of a torus we can indeed consider them different from zero. Moreover

$$P_0(k, m, a_h) = -3 (a_1 a_2)^{-12} (a_2^2 k_1^2 + a_1^2 k_2^2)^2 (a_2^2 m_1^2 + a_1^2 m_2^2)^2 (a_2^2 n_1^2 + a_1^2 n_2^2)^2 = 0,$$

if and only of k_h or m_h or $n_h = k_h + m_h$ is equal to zero. Let us suppose hence that one of these three conditions is satisfied. We have seen in Section 4.2 that once we consider (say) $n_h = 0$ all the eigenvalues collapse to the degenerate case of $\omega = 0$ with multiplicity four, whence there is no triple interaction of highly oscillating modes and we can consider $k_h, m_h, n_h \neq 0$.

As explained under this condition hence $P_0(k, m, a_h) \neq 0$, hence we can rewrite the resonant condition (4.4.13) in the abstract form

$$P_0(k, m, a_h) a_3^8 + \sum_{\alpha=1}^8 P_\alpha(k, m, a_h) a_3^{8-\alpha} = 0, \quad (4.4.14)$$

where we made sure that $P_0(k, m, a_h) \neq 0$. Whence fixing $(k, m, a_h) \in \mathbb{Z}^6 \times (\mathbb{R}_+)^2$ we can state that there exists a finite $a_3(k, m, a_h)$ solving (4.4.14). These elements are finite and unique once we fix a 8-tuple (k, m, a_h) . At this point hence it is obvious that

$$a_3(\mathbb{Z}^6, a_h) = \bigcup_{(k,m) \in \mathbb{Z}^6} a_3(k, m, a_h),$$

has zero measure in \mathbb{R} . Whence we proved that outside a null measure set in \mathbb{R}^3 there is not bilinear interaction of highly oscillating modes, proving the lemma. \square

We turn now our attention to study the limit dynamic as $\varepsilon \rightarrow 0$ of the projection of the bilinear interactions of elements in the kernel onto the oscillating subspace, i.e. we prove (4.4.11).

Lemma 4.4.7. *The following limit*

$$\left(\mathcal{Q}^\varepsilon (\bar{U}^\varepsilon, \bar{U}^\varepsilon) \right)_{\text{osc}} \xrightarrow{\varepsilon \rightarrow 0} 0,$$

holds in the sense of distributions.

Lemma 4.4.7 states that, on the oscillatory subspace in the limit $\varepsilon \rightarrow 0$ there is no bilinear interaction of elements of the kernel.

Proof. The element $\left(\mathcal{Q}^\varepsilon (\bar{U}^\varepsilon, \bar{U}^\varepsilon) \right)_{\text{osc}}$ reads as, in the Fourier space

$$\mathcal{F} \left(\mathcal{Q}^\varepsilon (\bar{U}^\varepsilon, \bar{U}^\varepsilon) \right)_{\text{osc}} = \sum_{\substack{k+m=n \\ j=1,2,3}} e^{i \frac{t}{\varepsilon} \omega^\pm(n)} \left(\mathbb{P}_n U^{0,j,\varepsilon}(k) m_j U^{0,\varepsilon}(m) \Big| e^\pm(n) \right)_{\mathbb{C}^4} e^\pm(n),$$

letting $\varepsilon \rightarrow 0$ and applying the stationary phase theorem the limit results to be

$$\lim_{\varepsilon \rightarrow 0} \mathcal{F} \left(\mathcal{Q}^\varepsilon (\bar{U}^\varepsilon, \bar{U}^\varepsilon) \right)_{\text{osc}} = \sum_{\substack{k+m=n \\ j=1,2,3 \\ \omega^\pm(n)=0}} \left(\mathbb{P}_n U^{0,j}(k) m_j U^0(m) \Big| e^\pm(n) \right)_{\mathbb{C}^4} e^\pm(n).$$

The condition

$$\omega^\pm(n) = \frac{|n_h|}{|n|} = 0 \Rightarrow n_h = 0,$$

combined with the convolution condition $k + m = (0, n_3)$ imply that $m_h = -k_h$. Under this assumption

$$\begin{aligned} \sum_{j=1,2,3} U^{0,j}(k) m_j U^0(m) &= k_3 U^{0,3}(k) U^0(m) + U^{0,3}(k) m_3 U^0(m), \\ &= n_3 U^{0,3}(k) U^0(m). \end{aligned}$$

We deduced hence that

$$\lim_{\varepsilon \rightarrow 0} \mathcal{F} \left(\mathcal{Q}^\varepsilon (\bar{U}^\varepsilon, \bar{U}^\varepsilon) \right)_{\text{osc}} = \sum_{\substack{k+m=n \\ j=1,2,3 \\ n_h=0}} \left(n_3 U^{0,3}(k) U^0(m) \Big| e^\pm(0, n_3) \right)_{\mathbb{C}^4} e^\pm(0, n_3). \quad (4.4.15)$$

The term $U^0(m) = \left(\hat{U}(m) \Big| e^0(m) \right)_{\mathbb{C}^4} e^0(m)$ and e^0 has the first two components only which are nonzero (see (4.2.4)), while $e^\pm(0, n_3) = (0, 0, 0, 1)$ as it is given in (4.2.5), hence the contribution in (4.4.15) is null, concluding. \square

Proof of Step 3.

It remains hence only to prove the Step 3 above, i.e. to understand the (distributional) limit as $\varepsilon \rightarrow 0$ of the interaction generated by the second-order elliptic operator \mathbb{D} defined in (4.1.2). This is done in the following lemma:

Lemma 4.4.8. *The following limit holds in the sense of distributions*

$$\lim_{\varepsilon \rightarrow 0} \left(-\mathbb{D}^\varepsilon U^\varepsilon | e^- + e^+ \right) = -(\nu + \nu') \Delta U_{\text{osc}}.$$

Proof. We proceed as follows. By definition of the projection onto the oscillatory space (see (4.2.7)) $(-\mathbb{D}^\varepsilon U^\varepsilon)_{\text{osc}}$ is given by the formula (4.3.3)

$$\begin{aligned} \mathcal{F}(-\mathbb{D}^\varepsilon U^\varepsilon)_{\text{osc}}(n) &= \sum_{\omega_n^{a,\pm}=0} \left(-\mathbb{D}^\varepsilon(n) \hat{U}^\varepsilon(n) \Big| e^\pm(n) \right)_{\mathbb{C}^4} e^\pm(n), \\ &= \sum_{\omega_n^{+,+}=0} \left(-\mathbb{D}^\varepsilon(n) \hat{U}^{\varepsilon,+}(n) e^+(n) \Big| e^+(n) \right)_{\mathbb{C}^4} e^+(n) \\ &\quad + \sum_{\omega_n^{-,+}=0} \left(-\mathbb{D}^\varepsilon(n) \hat{U}^{\varepsilon,-}(n) e^-(n) \Big| e^+(n) \right)_{\mathbb{C}^4} e^+(n) \quad (4.4.16) \\ &\quad + \sum_{\omega_n^{+,-}=0} \left(-\mathbb{D}^\varepsilon(n) \hat{U}^{\varepsilon,+}(n) e^+(n) \Big| e^-(n) \right)_{\mathbb{C}^4} e^-(n) \\ &\quad + \sum_{\omega_n^{-,-}=0} \left(-\mathbb{D}^\varepsilon(n) \hat{U}^{\varepsilon,-}(n) e^-(n) \Big| e^-(n) \right)_{\mathbb{C}^4} e^-(n), \end{aligned}$$

where for the second equality we used the decomposition $\hat{U}^\varepsilon = \sum_{a=0,\pm} \hat{U}^{a,\varepsilon} e^a$ and the fact that the eigenvectors are orthogonal.

Computing the explicit expression of $(-\mathbb{D}^\varepsilon(n) e^+(n) | e^+(n))_{\mathbb{C}^4}$ we deduce

$$\begin{aligned} (-\mathbb{D}^\varepsilon(n) e^+(n) | e^+(n))_{\mathbb{C}^4} &= \left(-\mathbb{D}(n) e^{-i\frac{t}{\varepsilon}\omega(n)} e^+(n) \Big| e^{-i\frac{t}{\varepsilon}\omega(n)} e^+(n) \right)_{\mathbb{C}^4}, \\ &= (-\mathbb{D}(n) e^+(n) | e^+(n))_{\mathbb{C}^4}, \quad (4.4.17) \\ &= (\nu + \nu') |n|^2. \end{aligned}$$

While for the element $(-\mathbb{D}^\varepsilon(n) e^-(n) | e^+(n))_{\mathbb{C}^4}$

$$(-\mathbb{D}^\varepsilon(n) e^-(n) | e^+(n))_{\mathbb{C}^4} = e^{2i\frac{t}{\varepsilon}\omega(n)} (-\mathbb{D}(n) e^-(n) | e^+(n))_{\mathbb{C}^4} \rightarrow 0, \quad (4.4.18)$$

in the sense of distributions thanks to the stationary phase theorem. In this case we automatically excluded the case $\omega(n) = 0$ since, as explained in Section 4.2, saying $\omega(n) = 0$ is equivalent to say that $n_h = 0$ and hence, in this case, all eigenvectors belong to the kernel of the penalized operator and hence $\mathcal{Q}(U, U)_{\text{osc}}|_{n_h=0} = 0$.

The same ideas can be applied to deduce

$$(-\mathbb{D}^\varepsilon(n) e^-(n) | e^-(n))_{\mathbb{C}^4} = (\nu + \nu') |n|^2, \quad (4.4.19)$$

$$(-\mathbb{D}^\varepsilon(n) e^+(n) | e^-(n))_{\mathbb{C}^4} \rightarrow 0. \quad (4.4.20)$$

The limit (4.4.20) has to be understood in the sense of distributions. Inserting (4.4.17)–(4.4.20) into (4.4.16) we deduce the claim, proving the Step 3. \square

4.5 Global existence of the limit system.

In Section 4.4.1 and 4.4.2 we performed a careful analysis whose goal was to understand which equations are solved (in the sense of distributions) by the functions \bar{U} and U_{osc} which were defined as the projection respectively onto the non-oscillating subspace $\mathbb{C}e^0$ and the oscillating space $\mathbb{C}e^- \oplus \mathbb{C}e^+$ of U , distributional solution of (\mathcal{S}_0) . The present section is devoted to study the propagation of strong (Sobolev) norms under the assumption that the initial data is sufficiently regular.

In particular we are interested to understand if the system (\mathcal{S}_0) propagates (isotropic) Sobolev data $H^s(\mathbb{T}^3)$, $s > 1/2$ and, if so, under which conditions on the initial data. Our expectation is that such system can propagate sub-critical Sobolev regularity globally-in-time without any particular smallness assumption on the initial data. The result we prove is the following one:

Proposition 4.5.1. *Let $U_0 \in H^s(\mathbb{T}^3) \cap L^\infty(\mathbb{T}_v; H^\sigma(\mathbb{T}_h^2))$, and $\nabla_h U \in L^\infty(\mathbb{T}_v; H^\sigma(\mathbb{T}_h^2))$ for $s > 1/2, \sigma > 0$, and let U be of zero horizontal average, i.e.*

$$\int_{\mathbb{T}_h^2} U_0(x_h, x_3) dx_h = 0 \text{ for each } x_3 \in \mathbb{T}_v^1,$$

then the weak solution of

$$\begin{cases} \partial_t \bar{u}^h + \bar{u}^h \cdot \nabla_h \bar{u}^h - \nu \Delta \bar{u}^h = 0, \\ \operatorname{div}_h \bar{u}^h = -\nabla_h \bar{p}, \\ U(0, x) = U_0 \end{cases} \quad (4.5.1)$$

is in fact strong, and has the following regularity:

$$\bar{u}^h \in \mathcal{C}(\mathbb{R}_+; H^s(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; H^{s+1}(\mathbb{T}^3)).$$

Moreover for each $t > 0$ the following estimate holds true

$$\begin{aligned} \|\bar{u}^h(t)\|_{H^s(\mathbb{T}^3)}^2 + \nu \int_0^t \|\bar{u}^h(\tau)\|_{H^{s+1}(\mathbb{T}^3)}^2 d\tau \\ \leq C \|\bar{u}_0^h\|_{H^s(\mathbb{T}^3)}^2 \exp \left\{ \frac{CK}{c\nu} \Phi(U_0) \|\nabla_h \bar{u}_0^h\|_{L_v^p(H_h^\sigma)} \right\}. \end{aligned} \quad (4.5.2)$$

where

$$\begin{aligned} \Phi(U_0) = \\ \exp \left\{ \frac{CK^2 \|\nabla_h \bar{u}_0^h\|_{L_v^\infty(L_h^2)}^2}{c\nu} \exp \left\{ \frac{K}{c\nu} \left(1 + \|\bar{u}_0^h\|_{L_v^\infty(L_h^2)}^2 \right) \|\nabla_h \bar{u}_0^h\|_{L_v^\infty(L_h^2)}^2 \right\} \right\}. \end{aligned} \quad (4.5.3)$$

Moreover if $U_{\text{osc},0} \in H^s(\mathbb{T}^3)$, $s > 1/2$ then U_{osc} , weak solution of

$$\begin{cases} \partial_t U_{\text{osc}} + 2\mathcal{Q}(\bar{U}, U_{\text{osc}}) - (\nu + \nu') \Delta U_{\text{osc}} = 0, \\ U_{\text{osc}}|_{t=0} = U_{\text{osc},0}. \end{cases}$$

is global-in-time and belongs to the space

$$U_{\text{osc}} \in \mathcal{C}(\mathbb{R}_+; H^s(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; H^{s+1}(\mathbb{T}^3)),$$

for $s > 1/2$. For each $t > 0$ the following bound holds true

$$\begin{aligned} & \|U_{\text{osc}}(t)\|_{H^s(\mathbb{T}^3)}^2 + \frac{\nu + \nu'}{2} \int_0^t \|U_{\text{osc}}(\tau)\|_{H^{s+1}(\mathbb{T}^3)}^2 d\tau \\ & \leq C \|U_{\text{osc},0}\|_{H^s(\mathbb{T}^3)}^2 \exp \left\{ \frac{C}{\nu} \|\bar{u}_0^h\|_{H^s(\mathbb{T}^3)}^2 \exp \left\{ \frac{CK}{c\nu} \Phi(U_0) \|\nabla_h \bar{u}_0^h\|_{L_v^p(H_h^\sigma)} \right\} \right\}. \end{aligned}$$

Thanks to the bounds above we can hence claim that, if $U_0 \in H^s(\mathbb{T}^3) \cap L^\infty(\mathbb{T}_v; H^s(\mathbb{T}_h^2))$, and $\nabla_h U \in L^\infty(\mathbb{T}_v; H^s(\mathbb{T}_h^2))$ for $s > 1/2$, and let U be of zero horizontal average, then

$$U = \bar{U} + U_{\text{osc}},$$

distributional solution of (\mathcal{S}_0) is in fact global-in-time and belongs to the space

$$U \in \mathcal{C}(\mathbb{R}_+; H^s(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; H^{s+1}(\mathbb{T}^3)),$$

for $s > 1/2$.

The statement of Proposition 4.5.1 is divided in two macro areas:

1. Control of strong Sobolev norms for the kernel-part of the solution \bar{U} .
2. Control of strong Sobolev norms for the orthogonal of the kernel-part of the solution U_{osc} .

The derivation of the regularity of $U = \bar{U} + U_{\text{osc}}$ is hence a simple deduction due to the orthogonality of \bar{U} and U_{osc} .

We shall hence divide the proof of Proposition 4.5.1 in the next two sub-sections: at first we prove the propagation of Sobolev norms for \bar{U} , then we prove the same result for U_{osc} .

4.5.1 The kernel part: propagation of $H^s(\mathbb{T}^3)$, $s > 1/2$ data.

The equation (4.4.8) is the vorticity equation associated to \bar{u}^h , this comes from the fact that $\nabla_h^\perp \cdot \bar{u}^h = \omega^h$ and hence $\nabla_h^\perp \cdot (u - \bar{u}^h) = 0$. Whence \bar{u}^h satisfies the following equation

$$\begin{cases} \partial_t \bar{u}^h + \bar{u}^h \cdot \nabla_h \bar{u}^h - \nu \Delta \bar{u}^h = -\nabla_h \bar{p}^h, \\ \text{div}_h \bar{u}^h = 0, \\ \bar{U}(0, x) = \bar{U}_0(x). \end{cases} \quad (4.5.4)$$

This tells us that \bar{u}^h satisfies a stratified 2D Navier-Stokes equation with full diffusion. In particular the 2D Biot-Savart $\bar{u}^h = \nabla_h^\perp \Delta_h^{-1} \omega^h$ law holds.

Lemma 4.5.2. *Let $\bar{u}_0^h \in H^s(\mathbb{T}^3)$, $s > 1/2$ and of zero horizontal average i.e.*

$$\int_{\mathbb{T}_h^2} \bar{u}_0^h(y_h, x_3) dy_h = 0.$$

The function \bar{u}^h local solution of (4.5.4) defined in the space

$$\bar{u}^h \in \mathcal{C}([0, T^*]; H^s(\mathbb{T}^3)) \cap L^2([0, T^*]; H^{s+1}(\mathbb{T}^3)),$$

for some $T^ > 0$ is of zero horizontal average in its lifespan, i.e.*

$$\int_{\mathbb{T}_h^2} \bar{u}^h(t, y_h, x_3) dy_h = 0,$$

for each $0 < t < T^$.*

Remark 4.5.3. The above lemma in particular implies that, for local solutions of equation (4.5.4), the horizontal homogeneous and nonhomogeneous Sobolev spaces are equivalent i.e.

$$\left\| (-\Delta_h)^{s/2} u(\cdot, x_3) \right\|_{L_h^2} \sim \|u(\cdot, x_3)\|_{H_h^s}.$$

For this reason, from now on, we shall always use the nonhomogeneous Sobolev space (although, as explained, for equation (4.5.4) they are equivalent) since the embedding

$$H^{1+\varepsilon}(\mathbb{T}_h^2) \hookrightarrow L^\infty(\mathbb{T}_h^2), \varepsilon > 0,$$

holds true (which is not the case with homogeneous spaces, generally) and we do not leave any place to ambiguity. \blacklozenge

The main result of the present section shall be the first part of the claim of Proposition 4.5.1, namely

Proposition 4.5.4. *Let $U_0 \in H^s(\mathbb{T}^3) \cap L^\infty(\mathbb{T}_v; H^\sigma(\mathbb{T}_h^2))$, and $\nabla_h U \in L^\infty(\mathbb{T}_v; H^\sigma(\mathbb{T}_h^2))$ for $s > 1/2, \sigma > 0$, and let U be of zero horizontal average, i.e.*

$$\int_{\mathbb{T}_h^2} U_0(x_h, x_3) dx_h = 0 \text{ for each } x_3 \in \mathbb{T}_v^1,$$

then the weak solution of

$$\begin{cases} \partial_t \bar{u}^h + \bar{u}^h \cdot \nabla_h \bar{u}^h - \nu \Delta \bar{u}^h = -\nabla_h \bar{p}, \\ \operatorname{div}_h \bar{u}^h = 0, \\ U(0, x) = U_0 \end{cases} \quad (4.5.5)$$

is in fact strong, and has the following regularity:

$$\bar{u}^h \in \mathcal{C}(\mathbb{R}_+; H^s(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; H^{s+1}(\mathbb{T}^3)).$$

Moreover for each $t > 0$ the following estimate holds true

$$\begin{aligned} & \|\bar{u}^h(t)\|_{H^s(\mathbb{T}^3)}^2 + \nu \int_0^t \|\bar{u}^h(\tau)\|_{H^{s+1}(\mathbb{T}^3)}^2 d\tau \\ & \leq C \|\bar{u}_0^h\|_{H^s(\mathbb{T}^3)}^2 \exp \left\{ \frac{CK}{c\nu} \Phi(U_0) \|\nabla_h \bar{u}_0^h\|_{L_v^p(H_h^\sigma)} \right\}, \end{aligned} \quad (4.5.6)$$

where $\Phi(U_0)$ is defined in (4.5.3).

The following two sections exist to this scope

The kernel part : smoothing effects of the heat flow.

This first subsection is aimed to prove some global-in-time integrability results for some suitable norms for (weak) solutions of the limit system (4.5.4). The result we present here are a consequence of the fact that (4.5.4) is a transport-diffusion equation in the horizontal variables, but a purely diffusion equation in the vertical one, in the sense that there is no vertical transport contribution.

The final result we want to prove is the following one

Proposition 4.5.5. *Let \bar{u}^h be a weak solution of (4.5.4), and assume that $\bar{u}_0^h, \nabla_h \bar{u}_0^h \in L_v^\infty(H_h^\sigma)$. Let the initial data be of zero horizontal average, i.e.*

$$\int_{\mathbb{T}_h^2} \bar{u}_0^h(y_h, x_3) dy_h = 0,$$

for each $x_3 \in \mathbb{T}_v^1$. Then the solution \bar{u}^h belongs to the space

$$\bar{u}^h \in L^2(\mathbb{R}_+; L^\infty(\mathbb{T}^3)),$$

and in particular

$$\|\bar{u}^h\|_{L^2(\mathbb{R}_+; L^\infty(\mathbb{T}^3))} \leq \frac{CK}{c\nu} \Phi(U_0) \|\nabla_h \bar{u}_0^h\|_{L_v^p(H_h^\sigma)},$$

where c, C, K are constants which do not depend on any parameter of the problem and $\Phi(U_0)$ is defined in (4.5.3).

The tools required in order to prove Proposition 4.5.5 are rather easy, but the procedure adopted is slightly involved, for this reason we decide to outline the structure of the proof in the following lines:

1. Using the fact that the transport effects in (4.5.4) are horizontal only we perform an L^2 -energy estimate in the horizontal direction. Next, on the vertical direction we exploit the fact that (4.5.4) is purely diffusive and linear equation: this fact allows us to use the smoothing effects of the heat kernel (at least along the x_3 -direction) in order to prove that

$$\begin{aligned} \bar{u}^h &\in L^2(\mathbb{R}_+; L_v^\infty(L_h^2)), \\ \nabla_h \bar{u}^h &\in L^2(\mathbb{R}_+; L_v^\infty(L_h^2)). \end{aligned}$$

2. We use the results of the point 1 in order improve the regularity result to the following statement (at a cost of having smoother initial data):

$$\begin{aligned} \bar{u}^h &\in L^2(\mathbb{R}_+; L_v^\infty(H_h^\sigma)), \\ \nabla_h \bar{u}^h &\in L^2(\mathbb{R}_+; L_v^\infty(H_h^\sigma)). \end{aligned}$$

for $\sigma > 0$.

3. Since the equation (4.5.4) propagates the horizontal average we exploit the embedding $L_v^\infty (H_h^{1+\sigma}) \hookrightarrow L^\infty (\mathbb{T}^3)$ to deduce the result.

The following Poincaré inequality shall be crucial in the proof of time-smoothing effects we want to prove

Lemma 4.5.6. *Let $f \in W^{1,2} ([0, 2\pi a_1] \times [0, 2\pi a_2])$ and such that it zero average, i.e.*

$$\int_0^{2\pi a_1} \int_0^{2\pi a_2} f(x_1, x_2) dx_2 dx_1 = 0.$$

Then the following inequality holds true

$$\|f\|_{L^2([0, 2\pi a_1] \times [0, 2\pi a_2])} \leq C \|\nabla f\|_{L^2([0, 2\pi a_1] \times [0, 2\pi a_2])},$$

where in particular the constant C is independent of the parameters a_1, a_2 characterizing the torus $[0, 2\pi a_1] \times [0, 2\pi a_2]$.

The following lemma is a key step for the rest of the results presented in the present paper

Lemma 4.5.7. *Let \bar{u}^h be a (weak) solution of the equation (4.5.4). Let us suppose moreover that $u_0, \nabla_h u_0 \in L_v^p (L_h^2)$ for some $p \in [2, \infty]$. Let us assume as well that*

$$\int_{\mathbb{T}_h^2} \bar{u}_0^h(y_h, x_3) dy_h = 0,$$

for each $x_3 \in \mathbb{T}_v^1$. Then

$$\bar{u}^h \in L^q (\mathbb{R}_+; L_v^p (L_h^2)), \quad \text{for } q \in [1, \infty], p \in [2, \infty], \quad (4.5.7)$$

$$\nabla_h \bar{u}^h \in L^q (\mathbb{R}_+; L_v^p (L_h^2)), \quad \text{for } q \in [1, \infty], p \in [2, \infty]. \quad (4.5.8)$$

In particular the time-decay rate is exponential, i.e.

$$\begin{aligned} \|\bar{u}^h(t)\|_{L_v^p(L_h^2)} &\leq e^{-\nu c t} \|\bar{u}_0^h\|_{L_v^p(L_h^2)}, \\ \|\nabla_h \bar{u}^h(t)\|_{L_v^p(L_h^2)} &\leq K e^{-\nu c t} \|\nabla_h \bar{u}_0^h\|_{L_v^p(L_h^2)}, \end{aligned}$$

where c, K are strictly positive constants which depend on the dimension of the horizontal domain only (in this case two).

Proof. Let us multiply the equation (4.5.4) for \bar{u}^h and let us take L_h^2 scalar product. Since the vector field \bar{u}^h is horizontal-divergence-free, i.e. $\partial_1 \bar{u}^1(x_1, x_2, x_3) + \partial_2 \bar{u}^2(x_1, x_2, x_3) = 0$ for each $x \in \mathbb{T}^3$ we deduce the following normed equality

$$\frac{1}{2} \frac{d}{dt} \|\bar{u}^h(x_3)\|_{L_h^2}^2 + \nu \|\nabla_h \bar{u}^h(x_3)\|_{L_h^2}^2 + \nu \|\partial_3 \bar{u}^h(x_3)\|_{L_h^2}^2 - \nu \partial_3^2 \|\bar{u}^h(x_3)\|_{L_h^2}^2 = 0.$$

The term $\nu \|\partial_3 \bar{u}^h(x_3)\|_{L_h^2}^2$ has indeed a positive contribution, hence we deduce the following inequality

$$\frac{1}{2} \frac{d}{dt} \|\bar{u}^h(x_3)\|_{L_h^2}^2 + \nu \|\nabla_h \bar{u}^h(x_3)\|_{L_h^2}^2 - \nu \partial_3^2 \|\bar{u}^h(x_3)\|_{L_h^2}^2 \leq 0.$$

At the same time we can use the Poincaré inequality as stated in Lemma 4.5.6 to argue that

$$\nu \|\nabla_h \bar{u}^h(x_3)\|_{L_h^2}^2 \geq c\nu \|\bar{u}^h(x_3)\|_{L_h^2}^2,$$

where $c = C^{-1}$ appearing in Lemma 4.5.6. Whence we deduced the inequality

$$\frac{1}{2} \frac{d}{dt} \|\bar{u}^h(x_3)\|_{L_h^2}^2 + c\nu \|\bar{u}^h(x_3)\|_{L_h^2}^2 - \nu \partial_3^2 \|\bar{u}^h(x_3)\|_{L_h^2}^2 \leq 0. \quad (4.5.9)$$

Let us now consider a $p \in [2, \infty)$, and let us multiply (4.5.9) by

$$\|\bar{u}^h(x_3)\|_{L_h^2}^{(p-2)} = \left(\int_{\mathbb{T}_h^2} \bar{u}^h(y_h, x_3)^2 dy_h \right)^{\frac{p-2}{2}},$$

and hence integrate the resulting inequality with respect to $x_3 \in \mathbb{T}_v^1$. The resulting inequality we deduce is

$$\frac{1}{p} \frac{d}{dt} \|\bar{u}^h\|_{L_v^p(L_h^2)}^p + c\nu \|\bar{u}^h\|_{L_v^p(L_h^2)}^p + \underbrace{\frac{8(p-2)}{p^2} \int_{\mathbb{T}_v^1} \left[\partial_3 \left(\int_{\mathbb{T}_h^2} u^2 dx_h \right)^{\frac{p}{4}} \right]^2 dx_3}_{=I_p(u)} \leq 0,$$

and since $I_p(u) \geq 0$ for each p we deduce the following inequality neglecting it

$$\frac{d}{dt} \left(\left(e^{c\nu t} \|\bar{u}^h\|_{L_v^p(L_h^2)} \right)^p \right) \leq 0.$$

Integrating in-time the above equation we deduce hence that.

$$\|\bar{u}^h(t)\|_{L_v^p(L_h^2)} \leq e^{-c\nu t} \|\bar{u}_0^h\|_{L_v^p(L_h^2)}, \quad (4.5.10)$$

and hence \bar{u}^h is L^q -in-time for each $p \in [2, \infty)$. In order to lift the result when $p = \infty$ it suffice to recall that, given a finite measure space $(\mathcal{X}, \Sigma, \mu)$ and a $\phi \in L^p(\mathcal{X}, \Sigma, \mu)$ for each $p \in [1, \infty]$, the application $p \mapsto |\mathcal{X}|^{-1} \|\phi\|_{L^p(\mathcal{X}, \Sigma, \mu)}$ is continuous, increasing in p and converges to $\|\phi\|_{L^\infty(\mathcal{X})}$ as $p \rightarrow \infty$, hence it suffice to consider the limit for $p \rightarrow \infty$ in (4.5.10).

To prove the statement for $\nabla_h \bar{u}^h$ let us consider the equation satisfied by $\omega^h = \text{curl}_h \bar{u}^h$. The equation is the following one

$$\begin{cases} \partial_t \omega^h + \bar{u}^h \cdot \nabla_h \omega^h - \nu \Delta \omega^h = 0, \\ \omega^h|_{t=0} = \omega_0^h = \text{curl}_h \bar{u}_0^h. \end{cases}$$

We can perform the exactly same procedure as it has been done with \bar{u}^h , obtaining hence that

$$\|\omega^h(t)\|_{L_v^p(L_h^2)} \leq e^{-c\nu t} \|\omega_0^h\|_{L_v^p(L_h^2)}.$$

Since the application $\omega^h \mapsto \nabla_h \bar{u}^h$ is a Calderón-Zygmund operator it maps continuously L_h^2 to itself and has operator norm K we deduce that

$$\|\nabla_h \bar{u}^h(t)\|_{L_v^p(L_h^2)} \leq K e^{-c\nu t} \|\nabla_h \bar{u}_0^h\|_{L_v^p(L_h^2)},$$

for each $p \in [2, \infty]$. □

Lemma 4.5.7 deals hence with the propagation of some anisotropic $L_v^p(L_h^2)$ regularity for (weak) solutions of equation (4.5.4). In our context we are particularly interested to study the propagation of the anisotropic $L_v^\infty(L_h^2)$ norm.

Similarly we are interested to understand how (4.5.4) propagates data which are bounded in the horizontal variables. Standard theory of two-dimensional Navier-Stokes and Euler equations suggests that, if the data is sufficiently regular in terms of Sobolev regularity, the propagation of horizontal norms should not be problematic.

The regularity statements proved until now are not sufficient to perform our analysis, for this reason we require the following lemma

Lemma 4.5.8. *Let us consider \bar{u}^h a (weak) solution of (4.5.4), with initial data $\bar{u}_0^h, \nabla_h \bar{u}_0^h \in L_v^\infty(H_h^\sigma)$, $\sigma \geq 0$ and assume \bar{u}_0^h has zero horizontal average, then*

$$\begin{aligned} \bar{u}^h &\in L^q(\mathbb{R}_+; L_v^p(H_h^\sigma)), & \text{for } q \in [1, \infty], p \in [2, \infty], \\ \nabla_h \bar{u}^h &\in L^q(\mathbb{R}_+; L_v^p(H_h^\sigma)), & \text{for } q \in [1, \infty], p \in [2, \infty]. \end{aligned}$$

Moreover the decay rate of the $L_v^p(H_h^\sigma)$ norms is exponential-in-time, in particular the following bounds hold

$$\begin{aligned} &\|\bar{u}^h(t)\|_{L_v^p(H_h^\sigma)} \\ &\leq C \exp\left\{\frac{K}{c\nu}\left(1 + \|\bar{u}_0^h\|_{L_v^\infty(L_h^2)}^2\right)\|\nabla_h \bar{u}_0^h\|_{L_v^\infty(L_h^2)}^2\right\} e^{-\frac{c\nu}{2}t} \|\bar{u}_0^h\|_{L_v^p(H_h^\sigma)}, \end{aligned} \quad (4.5.11)$$

$$\|\nabla_h \bar{u}^h(t)\|_{L_v^p(H_h^\sigma)} \leq CK \Phi(U_0) e^{-\frac{c\nu}{2}t} \|\nabla_h \bar{u}_0^h\|_{L_v^p(H_h^\sigma)}, \quad (4.5.12)$$

where $\Phi(U_0)$ is defined in (4.5.3).

Proof. We prove at first (4.5.11).

Let us recall the bound

$$\begin{aligned} &(\bar{u}^h(\cdot, x_3) \cdot \nabla_h \bar{u}^h(\cdot, x_3) | \bar{u}^h(\cdot, x_3))_{H_h^\sigma} \\ &\leq C \left(1 + \|\bar{u}^h(\cdot, x_3)\|_{L_h^2}\right) \|\nabla_h \bar{u}^h(\cdot, x_3)\|_{L_h^2} \|\bar{u}^h(\cdot, x_3)\|_{H_h^\sigma} \|\nabla_h \bar{u}^h(\cdot, x_3)\|_{H_h^\sigma}, \\ &\leq C \left(1 + \|\bar{u}^h(\cdot, x_3)\|_{L_h^2}^2\right) \|\nabla_h \bar{u}^h(\cdot, x_3)\|_{L_h^2}^2 \|\bar{u}^h(\cdot, x_3)\|_{H_h^\sigma}^2 + \frac{\nu}{2} \|\nabla_h \bar{u}^h(\cdot, x_3)\|_{H_h^\sigma}^2, \\ &= C f(t, x_3) \|\bar{u}^h(\cdot, x_3)\|_{H_h^\sigma}^2 + \frac{\nu}{2} \|\nabla_h \bar{u}^h(\cdot, x_3)\|_{H_h^\sigma}^2, \end{aligned} \quad (4.5.13)$$

Performing an H_h^σ energy estimate onto (4.5.4) with the bound (4.5.13) we deduce that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\bar{u}^h(\cdot, x_3)\|_{H_h^\sigma}^2 + \frac{\nu}{2} \|\nabla_h \bar{u}^h(\cdot, x_3)\|_{H_h^\sigma}^2 + \nu \|\partial_3 \bar{u}^h(\cdot, x_3)\|_{H_h^\sigma}^2 - \nu \partial_3^2 \|\bar{u}^h(\cdot, x_3)\|_{H_h^\sigma}^2 \\ - C f(t, x_3) \|\bar{u}^h(\cdot, x_3)\|_{H_h^\sigma}^2 \leq 0. \end{aligned}$$

By use of Lemma 4.5.6 and the fact that $\nu \|\partial_3 \bar{u}^h(\cdot, x_3)\|_{H_h^\sigma}^2 \geq 0$ we deduce

$$\frac{1}{2} \frac{d}{dt} \|\bar{u}^h(\cdot, x_3)\|_{H_h^\sigma}^2 + \left(\frac{c\nu}{2} - \nu \partial_3^2 - C f(t, x_3)\right) \|\bar{u}^h(\cdot, x_3)\|_{H_h^\sigma}^2 \leq 0. \quad (4.5.14)$$

Let us define

$$F(t, x_3) = C \int_0^t f(t', x_3) dt',$$

The function F is bounded in L_v^∞ thanks to the results in Lemma 4.5.7, in particular

$$e^{\|F\|_{L^\infty}} \leq C \exp \left\{ \frac{K}{c\nu} \left(1 + \|\bar{u}_0^h\|_{L_v^\infty(L_h^2)}^2 \right) \|\nabla_h \bar{u}_0^h\|_{L_v^\infty(L_h^2)}^2 \right\}$$

hence again as it was done in equation (4.5.9) we multiply (4.5.14) for

$$\|\bar{u}^h(x_3)\|_{H_h^\sigma}^{p-2} = \left(\int_{\mathbb{T}_h^2} (1 - \Delta_h)^\sigma \bar{u}^h(y_h, x_3)^2 dy_h \right)^{\frac{p-2}{2}},$$

where $p > 2$, $\sigma > 0$ and we integrate in x_3 to deduce

$$\|\bar{u}^h(t)\|_{L_v^p(H_h^\sigma)} \leq C \exp \left\{ \frac{K}{c\nu} \left(1 + \|\bar{u}_0^h\|_{L_v^\infty(L_h^2)}^2 \right) \|\nabla_h \bar{u}_0^h\|_{L_v^\infty(L_h^2)}^2 \right\} e^{-\frac{c\nu}{2} t} \|\bar{u}^h(t)\|_{L_v^p(L_h^2)},$$

in the same fashion as it was done in (4.5.10) for any $p \in [2, \infty]$. The bound (4.5.11) is then proved.

For the inequality (4.5.12) the procedure is the same but slightly more involved. We recall that the following bound holds true for zero-horizontal average vector fields:

$$\begin{aligned} (\bar{u}^h(\cdot, x_3) \cdot \nabla_h \omega^h(\cdot, x_3)) \omega^h(\cdot, x_3) \Big|_{H_h^\sigma} &\leq \frac{\nu}{2} \|\nabla_h \omega^h(\cdot, x_3)\|_{H_h^\sigma}^2 \\ &+ C K^2 \|\bar{u}^h(\cdot, x_3)\|_{H_h^\sigma}^2 \|\omega^h(\cdot, x_3)\|_{L_h^2}^2 \|\omega^h(\cdot, x_3)\|_{H_h^\sigma}^2. \end{aligned} \quad (4.5.15)$$

We postpone the proof of (4.5.15).

We set

$$\begin{aligned} g(t, x_3) &= \|\bar{u}^h(t, \cdot, x_3)\|_{H_h^\sigma}^2 \|\omega^h(t, \cdot, x_3)\|_{L_h^2}^2, \\ G(t, x_3) &= CK^2 \int_0^t g(t', x_3) dt', \end{aligned}$$

where K denotes again the norm of $\omega^h \mapsto \nabla_h \bar{u}^h$ as a Calderon-Zygmung application in L_h^2 .

Performing an H_h^σ energy estimate onto the equation satisfied by ω^h with the bound (4.5.15) we deduce the inequality

$$\frac{1}{2} \frac{d}{dt} \|\omega^h(x_3)\|_{H_h^\sigma}^2 + \left(\frac{c\nu}{2} - CK^2 g(t, x_3) - \nu \partial_3^2 \right) \|\omega^h(x_3)\|_{H_h^\sigma}^2 \leq 0.$$

Net, we multiply the above inequality for $\|\omega^h(x_3)\|_{H_h^\sigma}^{p-2}$, $p \geq 2$ in order to deduce as it was done for \bar{u}^h that

$$\|\omega^h(t)\|_{L_v^p(H_h^\sigma)} \leq C e^{-\frac{c\nu}{2} t} \|e^G\|_{L^\infty} \|\omega_0^h\|_{L_v^p(H_h^\sigma)}.$$

The function $e^G \in L^\infty(\mathbb{T}_v^1)$ thanks to the results in Lemma 4.5.7 and the estimate (4.5.11), hence we deduce the bound

$$e^{\|G\|_{L_v^\infty}} \leq C \exp \left\{ \frac{CK^2 \|\omega_0^h\|_{L_v^\infty(L_h^2)}^2}{c\nu} \exp \left\{ \frac{K}{c\nu} \left(1 + \|\bar{u}_0^h\|_{L_v^\infty(L_h^2)}^2 \right) \|\nabla_h \bar{u}_0^h\|_{L_v^\infty(L_h^2)}^2 \right\} \right\},$$

which lead to the final bound

$$\begin{aligned} & \|\omega^h(t)\|_{L_v^p(H_h^\sigma)} \\ & \leq C \exp \left\{ \frac{CK^2 \|\omega_0^h\|_{L_v^\infty(L_h^2)}^2}{c\nu} \exp \left\{ \frac{K}{c\nu} \left(1 + \|\bar{u}_0^h\|_{L_v^\infty(L_h^2)}^2 \right) \|\nabla_h \bar{u}_0^h\|_{L_v^\infty(L_h^2)}^2 \right\} \right\} \\ & \quad \times e^{-\frac{c\nu}{2}t} \|\omega^h\|_{L_v^p(H_h^\sigma)}, \end{aligned}$$

for $p \in [2, \infty]$.

Since the application $\omega^h \mapsto \nabla_h \bar{u}^h$ is a Calderón-Zygmung application we can conclude with the following estimate

$$\begin{aligned} & \|\nabla_h \bar{u}^h\|_{L_v^p(H_h^\sigma)} \\ & \leq CK \exp \left\{ \frac{CK^2 \|\nabla_h \bar{u}_0^h\|_{L_v^\infty(L_h^2)}^2}{c\nu} \exp \left\{ \frac{K}{c\nu} \left(1 + \|\bar{u}_0^h\|_{L_v^\infty(L_h^2)}^2 \right) \|\nabla_h \bar{u}_0^h\|_{L_v^\infty(L_h^2)}^2 \right\} \right\} \\ & \quad \times e^{-\frac{c\nu}{2}t} \|\nabla_h \bar{u}_0^h\|_{L_v^p(H_h^\sigma)}, \end{aligned}$$

which proves (4.5.12). □

Proof of Proposition 4.5.5 At this point the proof of Proposition 4.5.5 is direct corollary of Lemma 4.5.8. Since the vector field \bar{u}^h has zero horizontal average the following equivalence of norms hold true

$$\|\nabla_h \bar{u}^h(\cdot, x_3)\|_{H_h^\sigma} \sim \|\bar{u}^h(\cdot, x_3)\|_{H_h^{\sigma+1}}$$

It is sufficient in fact to remark now that, for vector fields with zero horizontal average, the embedding $H^{1+\sigma}(\mathbb{R}_h^2) \hookrightarrow L^\infty(\mathbb{R}_h^2)$, $\sigma > 0$ holds true. I.e.

$$\|\bar{u}^h(\cdot, x_3)\|_{L_h^\infty} \leq C \|\bar{u}^h(\cdot, x_3)\|_{H_h^{\sigma+1}}.$$

These considerations together with the inequality (4.5.12) (setting $p = \infty$) lead us to the following estimate

$$\|\bar{u}^h\|_{L^\infty(\mathbb{T}^3)} \leq CK \Phi(U_0) e^{-\frac{c\nu}{2}t} \|\nabla_h \bar{u}_0^h\|_{L_v^p(H_h^\sigma)}.$$

An integration-in-time completes hence the proof of Proposition 4.5.5. □

Proof of (4.5.15). This is the only part of the present paper in which we use the anisotropic (horizontal) paradiifferential calculus introduced at Section 4.1.5. We recall that, given two functions $f, g \in H_h^\sigma$

$$(f|g)_{H_h^\sigma} \sim \sum_{q \in \mathbb{Z}} 2^{2q\sigma} (\Delta_q^h f | \Delta_q^h g)_{L_h^2}.$$

This deduction is a consequence of the almost-orthogonality property of dyadic blocks. Whence it is sufficient to prove bounds for terms of the form

$$\begin{aligned} A_q &= (\Delta_q^h (\bar{u}^h(\cdot, x_3) \cdot \nabla_h \omega^h(\cdot, x_3)) | \Delta_q^h \omega^h(\cdot, x_3))_{L_h^2}, \\ &= (\Delta_q^h (\bar{u}^h(\cdot, x_3) \omega^h(\cdot, x_3)) | \Delta_q^h \nabla_h \omega^h(\cdot, x_3))_{L_h^2}. \end{aligned}$$

Using the (horizontal) Bony decomposition (4.1.10) we decompose A_q into the following infinite sum

$$\begin{aligned} A_q &= (\Delta_q^h (\bar{u}^h(\cdot, x_3) \omega^h(\cdot, x_3)) | \Delta_q^h \nabla_h \omega^h(\cdot, x_3))_{L_h^2}, \\ &= \sum_{|q-q'| \leq 4} (\Delta_q^h (S_{q'-1}^h \bar{u}^h(\cdot, x_3) \Delta_{q'}^h \omega^h(\cdot, x_3)) | \Delta_q^h \nabla_h \omega^h(\cdot, x_3))_{L_h^2}, \\ &\quad + \sum_{q' > q-4} (\Delta_q^h (\Delta_{q'}^h \bar{u}^h(\cdot, x_3) S_{q'+1}^h \omega^h(\cdot, x_3)) | \Delta_q^h \nabla_h \omega^h(\cdot, x_3))_{L_h^2}, \\ &= A_q^1 + A_q^2. \end{aligned}$$

We start bounding the term A_q^1 . We recall that thanks to the Bernstein inequality the operator Δ_q^h maps continuously any H_h^σ space to itself.

Using Hölder inequality (twice)

$$|A_q^1| \leq \sum_{|q-q'| \leq 4} \|\bar{u}^h(\cdot, x_3)\|_{L_h^4} \|\Delta_{q'}^h \omega^h(\cdot, x_3)\|_{L_h^4} \|\Delta_q^h \nabla_h \omega^h(\cdot, x_3)\|_{L_h^2}.$$

Thanks to the Remark 4.5.3 we know that \bar{u}^h and ω^h are vector fields with zero horizontal average for each x_3 . Hence we can use the Gagliardo-Nirenberg-type inequality (4.1.13), to deduce

$$\begin{aligned} \|\bar{u}^h(\cdot, x_3)\|_{L_h^4} &\leq C \|\bar{u}^h(\cdot, x_3)\|_{L_h^2}^{1/2} \|\nabla_h \bar{u}^h(\cdot, x_3)\|_{L_h^2}^{1/2}, \\ \|\Delta_{q'}^h \omega^h(\cdot, x_3)\|_{L_h^4} &\leq C \|\Delta_{q'}^h \omega^h(\cdot, x_3)\|_{L_h^2}^{1/2} \|\Delta_{q'}^h \nabla_h \omega^h(\cdot, x_3)\|_{L_h^2}^{1/2}. \end{aligned}$$

Since the application $\omega^h \mapsto \nabla_h \bar{u}^h$ is a Calderon-Zygmung application we can say that, there exists a K constant independent of any parameter of the problem such that

$$\|\nabla_h \bar{u}^h(\cdot, x_3)\|_{L_h^2}^{1/2} \leq K^{1/2} \|\omega^h(\cdot, x_3)\|_{L_h^2}^{1/2}.$$

Moreover for vector fields whose horizontal average is zero the embedding $H_h^\sigma \hookrightarrow L_h^2$, $\sigma > 0$ holds true, hence

$$\|\bar{u}^h(\cdot, x_3)\|_{L_h^2}^{1/2} \leq C \|\bar{u}^h(\cdot, x_3)\|_{H_h^\sigma}^{1/2}.$$

Since there exists a $\ell^2(\mathbb{Z})$ sequence such that

$$\begin{aligned} \|\Delta_{q'}^h \omega^h(\cdot, x_3)\|_{L_h^2}^{1/2} \|\Delta_{q'}^h \nabla_h \omega^h(\cdot, x_3)\|_{L_h^2}^{1/2} &\leq C c_{q'} 2^{-q'\sigma} \|\omega^h(\cdot, x_3)\|_{H_h^\sigma}^{1/2} \|\nabla_h \omega^h(\cdot, x_3)\|_{H_h^\sigma}^{1/2}, \\ \|\Delta_q^h \nabla_h \omega^h(\cdot, x_3)\|_{L_h^2} &\leq C c_q 2^{-q\sigma} \|\nabla_h \omega^h(\cdot, x_3)\|_{H_h^\sigma}, \end{aligned}$$

we formally deduced the bound

$$|A_q^1| \leq CK^{1/2} c_q 2^{-2q\sigma} \|\bar{u}^h(\cdot, x_3)\|_{H_h^\sigma}^{1/2} \|\omega^h(\cdot, x_3)\|_{L_h^2}^{1/2} \|\omega^h(\cdot, x_3)\|_{H_h^\sigma}^{1/2} \|\nabla_h \omega^h(\cdot, x_3)\|_{H_h^\sigma}^{3/2}.$$

It remains to prove the same kind of bound for the term A_q^2 . Again, using Hölder inequality

$$|A_q^2| \leq \sum_{q' > q-4} \|\Delta_{q'}^h \bar{u}^h(\cdot, x_3)\|_{L_h^4} \|S_{q'+2}^h \omega^h(\cdot, x_3)\|_{L_h^4} \|\Delta_q^h \nabla_h \omega^h(\cdot, x_3)\|_{L_h^2}.$$

Since the vector fields have zero horizontal average we can apply (4.1.13), the fact that $\omega^h \mapsto \nabla_h \bar{u}^h$ is a Calderon-Zygmund operator and (4.1.12) to deduce

$$\begin{aligned} \|\Delta_{q'}^h \bar{u}^h(\cdot, x_3)\|_{L_h^4} &\leq C \|\Delta_{q'}^h \bar{u}^h(\cdot, x_3)\|_{L_h^2}^{1/2} \|\Delta_{q'}^h \nabla_h \bar{u}^h(\cdot, x_3)\|_{L_h^2}^{1/2}, \\ &\leq CK^{1/2} \|\Delta_{q'}^h \bar{u}^h(\cdot, x_3)\|_{L_h^2}^{1/2} \|\Delta_{q'}^h \omega^h(\cdot, x_3)\|_{L_h^2}^{1/2}, \\ &\leq CK^{1/2} c_q 2^{-q'\sigma} \|\bar{u}^h(\cdot, x_3)\|_{H_h^\sigma}^{1/2} \|\omega^h(\cdot, x_3)\|_{H_h^\sigma}^{1/2}. \end{aligned}$$

Using (4.1.13) and the embedding $H_h^\sigma \hookrightarrow L_h^2$ which holds for vector fields with zero horizontal average

$$\begin{aligned} \|S_{q'+2}^h \omega^h(\cdot, x_3)\|_{L_h^4} &\leq C \|\omega^h(\cdot, x_3)\|_{L_h^2}^{1/2} \|\nabla_h \omega^h(\cdot, x_3)\|_{L_h^2}^{1/2}, \\ &\leq C \|\omega^h(\cdot, x_3)\|_{H_h^\sigma}^{1/2} \|\nabla_h \omega^h(\cdot, x_3)\|_{L_h^2}^{1/2}. \end{aligned}$$

Hence with the aid of (4.1.12)

$$\|\Delta_q^h \nabla_h \omega^h(\cdot, x_3)\|_{L_h^2} \leq C c_q 2^{-q\sigma} \|\nabla_h \omega^h(\cdot, x_3)\|_{H_h^\sigma}.$$

We hence deduced that

$$|A_q^2| \leq CK^{1/2} c_q 2^{-2q\sigma} \|\bar{u}^h(\cdot, x_3)\|_{H_h^\sigma}^{1/2} \|\omega^h(\cdot, x_3)\|_{L_h^2}^{1/2} \|\omega^h(\cdot, x_3)\|_{H_h^\sigma}^{1/2} \|\nabla_h \omega^h(\cdot, x_3)\|_{H_h^\sigma}^{3/2}.$$

With these bounds we hence proved that

$$\begin{aligned} &(\bar{u}^h(\cdot, x_3) \cdot \nabla_h \omega^h(\cdot, x_3) | \omega^h(\cdot, x_3))_{H_h^\sigma} \\ &\leq CK^{1/2} \|\bar{u}^h(\cdot, x_3)\|_{H_h^\sigma}^{1/2} \|\omega^h(\cdot, x_3)\|_{L_h^2}^{1/2} \|\omega^h(\cdot, x_3)\|_{H_h^\sigma}^{1/2} \|\nabla_h \omega^h(\cdot, x_3)\|_{H_h^\sigma}^{3/2}. \end{aligned}$$

To deduce the estimate (4.5.15) it is sufficient hence to apply the convexity inequality $ab \leq \frac{C^4}{4} a^4 + \frac{3}{4C^{4/3}} b^{4/3}$ to the above estimate. \square

Propagation of isotropic Sobolev regularity.

We apply in this Section the result proved in the previous one in order to conclude the proof of Proposition 4.5.4.

Proof of Proposition 4.5.4. Let us apply the operator Δ_q to both sides of (4.5.5) and let us multiply what we obtain with $\Delta_q \bar{u}^h$ and let us take scalar product in $L^2(\mathbb{T}^3)$, we obtain in particular

$$\frac{1}{2} \frac{d}{dt} \|\Delta_q \bar{u}^h\|_{L^2(\mathbb{T}^3)}^2 + \nu \|\Delta_q \nabla \bar{u}^h\|_{L^2(\mathbb{T}^3)}^2 \leq \left| (\Delta_q (\bar{u}^h \cdot \nabla_h \bar{u}^h) | \Delta_q \bar{u}^h)_{L^2(\mathbb{T}^3)} \right|, \quad (4.5.16)$$

whence to obtain the claim everything reduces to bound the term $\left| (\Delta_q (\bar{u}^h \cdot \nabla_h \bar{u}^h) | \Delta_q \bar{u}^h)_{L^2(\mathbb{T}^3)} \right|$. By Bony decomposition (4.1.11) we know that

$$\begin{aligned} \left| (\Delta_q (\bar{u}^h \cdot \nabla_h \bar{u}^h) | \Delta_q \bar{u}^h)_{L^2(\mathbb{T}^3)} \right| &\leq \left| (S_{q-1} \bar{u}^h \Delta_q \nabla_h \bar{u}^h | \Delta_q \bar{u}^h)_{L^2(\mathbb{T}^3)} \right| \\ &+ \left\{ \sum_{|q-q'|\leq 4} \left| ([\Delta_q, S_{q'-1} \bar{u}^h] \Delta_{q'} \nabla_h \bar{u}^h | \Delta_q \bar{u}^h)_{L^2(\mathbb{T}^3)} \right| \right. \\ &+ \left. \sum_{|q-q'|\leq 4} \left| ((S_q - S_{q'-1}) \bar{u}^h \Delta_q \Delta_{q'} \nabla_h \bar{u}^h | \Delta_q \bar{u}^h)_{L^2(\mathbb{T}^3)} \right| \right\} \\ &+ \sum_{q'>q-4} \left| (\Delta_q (S_{q'+2} \nabla_h \bar{u}^h \Delta_{q'} \bar{u}^h) | \Delta_q \bar{u}^h)_{L^2(\mathbb{T}^3)} \right| = \sum_{k=1}^4 I_k. \end{aligned}$$

Since $\operatorname{div}_h \bar{u}^h = 0$ we immediately obtain that $I_1 \equiv 0$, whence if we consider the second term, thanks to Hölder inequality and Lemma 4.1.10 we can argue that

$$I_2 \leq C \sum_{|q-q'|\leq 4} 2^{-q} \|S_{q'-1} \nabla \bar{u}^h\|_{L^\infty(\mathbb{T}^3)} \|\Delta_{q'} \nabla_h \bar{u}^h\|_{L^2(\mathbb{T}^3)} \|\Delta_q \bar{u}^h\|_{L^2(\mathbb{T}^3)}.$$

Accordingly to Bernstein inequality we have that

$$\|S_{q'-1} \nabla \bar{u}^h\|_{L^\infty(\mathbb{T}^3)} \leq C 2^{q'} \|S_{q'-1} \bar{u}^h\|_{L^\infty(\mathbb{T}^3)},$$

and hence, since $\|\Delta_q f\|_{L^2(\mathbb{T}^3)} \leq C c_q(t) 2^{-qs} \|f\|_{H^s(\mathbb{T}^3)}$ we obtain that

$$I_2 \leq C c_q 2^{-2qs} \|\bar{u}^h\|_{L^\infty(\mathbb{T}^3)} \|\bar{u}^h\|_{H^s(\mathbb{T}^3)} \|\nabla_h \bar{u}^h\|_{H^s(\mathbb{T}^3)}.$$

Similar calculations lead to the same bound for I_3 , i.e.

$$I_3 \leq C c_q 2^{-2qs} \|\bar{u}^h\|_{L^\infty(\mathbb{T}^3)} \|\bar{u}^h\|_{H^s(\mathbb{T}^3)} \|\nabla_h \bar{u}^h\|_{H^s(\mathbb{T}^3)}.$$

Finally from the reminder term I_4 the following computations hold

$$\begin{aligned} I_4 &= \sum_{q'>q-4} \left| (\Delta_q (S_{q'+2} \nabla_h \bar{u}^h \Delta_{q'} \bar{u}^h) | \Delta_q \bar{u}^h)_{L^2(\mathbb{T}^3)} \right| \\ &\leq \sum_{q'>q-4} \|S_{q'+2} \nabla_h \bar{u}^h\|_{L^\infty(\mathbb{T}^3)} \|\Delta_{q'} \bar{u}^h\|_{L^2(\mathbb{T}^3)} \|\Delta_q \bar{u}^h\|_{L^2(\mathbb{T}^3)}, \end{aligned}$$

but, since we are dealing with localized functions

$$\begin{aligned} \|S_{q'+2}\nabla_h\bar{u}^h\|_{L^\infty(\mathbb{T}^3)} &\leq 2^{q'} \|S_{q'+2}\bar{u}^h\|_{L^\infty(\mathbb{T}^3)}, \\ \|\Delta_{q'}\bar{u}^h\|_{L^2(\mathbb{T}^3)} &\leq c_{q'} 2^{-q's-q'} \|\nabla\bar{u}^h\|_{H^s(\mathbb{T}^3)}. \\ \|\Delta_q\bar{u}^h\|_{L^2(\mathbb{T}^3)} &\leq c_q 2^{-qs} \|\bar{u}^h\|_{H^s(\mathbb{T}^3)}, \end{aligned}$$

whence we obtain

$$I_4 \leq C c_q 2^{-2qs} \|\bar{u}^h\|_{L^\infty(\mathbb{T}^3)} \|\bar{u}^h\|_{H^s(\mathbb{T}^3)} \|\nabla\bar{u}^h\|_{H^s(\mathbb{T}^3)},$$

which in particular implies that

$$\left| (\Delta_q(\bar{u}^h \cdot \nabla_h \bar{u}^h) | \Delta_q \bar{u}^h)_{L^2(\mathbb{T}^3)} \right| \leq C c_q 2^{-2qs} \|\bar{u}^h\|_{L^\infty(\mathbb{T}^3)} \|\bar{u}^h\|_{H^s(\mathbb{T}^3)} \|\nabla_h \bar{u}^h\|_{H^s(\mathbb{T}^3)}. \quad (4.5.17)$$

Whence inserting (4.5.17) into (4.5.16), multiplying both sides for 2^{2qs} and summing over q we obtain that

$$\frac{1}{2} \frac{d}{dt} \|\bar{u}^h\|_{H^s(\mathbb{T}^3)}^2 + \nu \|\bar{u}^h\|_{H^{s+1}(\mathbb{R}^3)}^2 \leq C \|\bar{u}^h\|_{L^\infty(\mathbb{T}^3)} \|\bar{u}^h\|_{H^s(\mathbb{T}^3)} \|\bar{u}^h\|_{H^{s+1}(\mathbb{R}^3)}, \quad (4.5.18)$$

whence by Young inequality

$$\|\bar{u}^h\|_{L^\infty(\mathbb{T}^3)} \|\bar{u}^h\|_{H^s(\mathbb{T}^3)} \|\bar{u}^h\|_{H^{s+1}(\mathbb{R}^3)} \leq \frac{\nu}{2} \|\bar{u}^h\|_{H^{s+1}(\mathbb{R}^3)}^2 + C \|\bar{u}^h\|_{L^\infty(\mathbb{T}^3)}^2 \|\bar{u}^h\|_{H^s(\mathbb{T}^3)}^2,$$

which, together with (4.5.18) and a Gronwall argument lead to the following estimate

$$\|\bar{u}^h(t)\|_{H^s(\mathbb{T}^3)}^2 + \nu \int_0^t \|\bar{u}^h(\tau)\|_{H^{s+1}(\mathbb{T}^3)}^2 d\tau \leq C \|\bar{u}_0^h\|_{H^s(\mathbb{T}^3)}^2 \exp \left\{ \int_0^t \|\bar{u}^h(\tau)\|_{L^\infty(\mathbb{T}^3)}^2 d\tau \right\}.$$

We can hence apply on the above inequality Proposition 4.5.5 to deduce the bound (4.5.6). \square

4.5.2 The oscillating part: propagation of $H^s(\mathbb{T}^3)$, $s > 1/2$ data.

In this section we consider the propagation of Sobolev regularity for the equation (4.1.6) satisfied (distributionally) by the function U_{osc} . We proved in Section 4.4.2 that for almost all three-dimensional tori \mathbb{T}^3 the equation for U_{osc} is linear. The result claimed is hence the following:

Proposition 4.5.9. *Let $U_{\text{osc},0} \in H^s(\mathbb{T}^3)$, $s > 1/2$, suppose $\bar{U} = (\bar{u}^h, 0, 0)$ solution of (4.5.4) is globally defined and $\bar{u}^h \in L^2(\mathbb{R}_+; H^{s+1})$, hence the solution U_{osc} of the system (4.1.6) satisfies*

$$U_{\text{osc}} \in C(\mathbb{R}_+; H^s(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; H^{s+1}(\mathbb{T}^3)),$$

and the following estimates hold for each $t > 0$

$$\begin{aligned} \|U_{\text{osc}}(t)\|_{H^s(\mathbb{T}^3)}^2 + \frac{\nu + \nu'}{2} \int_0^t \|U_{\text{osc}}(\tau)\|_{H^{s+1}(\mathbb{T}^3)}^2 d\tau \\ \leq C \|U_{\text{osc},0}\|_{H^s(\mathbb{T}^3)}^2 \exp \left\{ C \|\nabla\bar{u}^h\|_{L^2(\mathbb{R}_+; H^s(\mathbb{T}^3))}^2 \right\}. \end{aligned} \quad (4.5.19)$$

For a proof for Proposition 4.5.9 we refer to [72, Appendix B].

4.6 Convergence for $\varepsilon \rightarrow 0$ and proof of Theorem 4.1.7

Remark 4.6.1. Given an $N \in \mathbb{N}$ (generally large) in the present section we denote with K_N and k_N two constant such that $K_N \rightarrow \infty$ and $k_N \rightarrow 0$ respectively as $N \rightarrow \infty$. These constant depend on N only, and their value may vary from line to line.

In the present proof for the convergence we shall reduce ourselves to the simplified case $\nu = \nu'$. It is a simple procedure to lift such result when the diffusivity is different. We chose to make such simplification in order not to make an already very complex notation even heavier. \blacklozenge

The previous section has been devoted to the study of the global-well-posedness of the limit system (\mathcal{S}_0) in some sub-critical $H^s(\mathbb{T}^3)$, $s > 1/2$ Sobolev space. The present section shall use this result to prove that, for $0 < \varepsilon \leq \varepsilon_0$ sufficiently small the (local, strong) solutions of (PBS_ε) converge (globally) in the space

$$\mathcal{C}(\mathbb{R}_+; H^s(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; H^{s+1}(\mathbb{T}^3)),$$

to the now global and strong solution U of (\mathcal{S}_0) . This shall imply that as long as ε is sufficiently close to zero the strong solutions of (PBS_ε) are in fact global.

The method we are going to explain reduces to a smart choice of variable substitution that cancels some problematic term appearing in the equations. This technique has been introduced by S. Schochet in [133] in the context of hyperbolic systems with singular perturbation. I. Gallagher in [72] adapted the method to parabolic systems. We mention as well the work of M. Paicu [123] and E. Grenier [79].

Let us subtract (\mathcal{S}_0) from $(\mathcal{S}_\varepsilon)$, and we denote the difference unknown by $W^\varepsilon = U^\varepsilon - U$. After some basic algebra we reduced hence ourselves to the following difference system

$$\begin{cases} \partial_t W^\varepsilon + \mathcal{Q}^\varepsilon(W^\varepsilon, W^\varepsilon + 2U) - \nu \Delta W^\varepsilon = -(\mathcal{Q}^\varepsilon(U, U) - \mathcal{Q}(U, U)), \\ \text{div } W^\varepsilon = 0, \\ W^\varepsilon|_{t=0} = 0. \end{cases} \quad (4.6.1)$$

We define $\mathcal{R}_{\text{osc}}^\varepsilon = \mathcal{Q}^\varepsilon(U, U) - \mathcal{Q}(U, U)$. We remark that $\mathcal{R}_{\text{osc}}^\varepsilon \rightarrow 0$ only in \mathcal{D}' , since it is defined as $\mathcal{R}_{\text{osc}}^\varepsilon = \mathcal{R}_{\text{osc,I}}^\varepsilon + \mathcal{R}_{\text{osc,II}}^\varepsilon$ where

$$\mathcal{R}_{\text{osc,I}}^\varepsilon = \mathcal{F}^{-1} \left(\sum_{\substack{\omega_{k,n-k,n}^{a,b,c} \neq 0 \\ j=1,2,3}} e^{i \frac{t}{\varepsilon} \omega_{k,n-k,n}^{a,b,c}} (U^{a,j}(k) (n_j - k_j) U^b(n-k) | e^c(n)) e^c(n) \right), \quad (4.6.2)$$

$$\mathcal{R}_{\text{osc,II}}^\varepsilon = \mathcal{F}_v^{-1} \left(\sum_{\substack{k+m=(0,n_3) \\ \tilde{\omega}_{k,m}^{a,b} \neq 0}} e^{i \frac{t}{\varepsilon} \tilde{\omega}_{k,m}^{a,b}} n_3 (U^{a,3}(k) U^{b,h}(m), 0, 0)^\top \right), \quad (4.6.3)$$

where $\tilde{\omega}_{k,m}^{a,b} = \omega^a(k) + \omega^b(m)$. The term $\mathcal{R}_{\text{osc,II}}^\varepsilon$ represents the high-frequency vertical perturbations induced by the horizontal average $\left(\int_{\mathbb{T}_h^2} (\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon))^h dx_h, 0, 0 \right)$ which converges to

zero only weakly as explained in Lemma 4.3.3. Hence we divide it in high-low frequencies in the following way, for the low-frequency part

$$\begin{aligned}\mathcal{R}_{\text{osc,I},N}^\varepsilon &= \mathcal{F}^{-1} \left(1_{\{|n| \leq N\} \cap \{|k| \leq N\}} \mathcal{F} \mathcal{R}_{\text{osc,I}}^\varepsilon \right), \\ \mathcal{R}_{\text{osc,II},N}^\varepsilon &= \mathcal{F}^{-1} \left(1_{\{|n_3| \leq N\} \cap \{|k| \leq N\}} \mathcal{F} \mathcal{R}_{\text{osc,II}}^\varepsilon \right), \\ \mathcal{R}_{\text{osc},N}^\varepsilon &= \mathcal{R}_{\text{osc,I},N}^\varepsilon + \mathcal{R}_{\text{osc,II},N}^\varepsilon,\end{aligned}$$

while the high-frequency part is defined as

$$\mathcal{R}_{\text{osc}}^{\varepsilon,N} = \mathcal{R}_{\text{osc}}^\varepsilon - \mathcal{R}_{\text{osc},N}^\varepsilon. \quad (4.6.4)$$

Lemma 4.6.2. *Let $\mathcal{R}_{\text{osc}}^{\varepsilon,N}$ be defined as in (4.6.4). $\mathcal{R}_{\text{osc}}^{\varepsilon,N} \xrightarrow{N \rightarrow \infty} 0$ in $L^2(\mathbb{R}_+; H^{s-1})$ uniformly in ε , and the following bound holds*

$$\|\mathcal{R}_{\text{osc}}^{\varepsilon,N}\|_{L^2(\mathbb{R}_+; H^{s-1})} \leq k_N \xrightarrow{N \rightarrow \infty} 0. \quad (4.6.5)$$

The proof of Lemma 4.6.2 is postponed to the end of the present section, at Subsection 4.6.1.

Let us now perform the following change of unknown

$$\psi_N^\varepsilon = W^\varepsilon + \varepsilon \tilde{\mathcal{R}}_{\text{osc},N}^\varepsilon, \quad (4.6.6)$$

where, in particular, $\tilde{\mathcal{R}}_{\text{osc},N}^\varepsilon$ is defined as $\tilde{\mathcal{R}}_{\text{osc},N}^\varepsilon = \tilde{\mathcal{R}}_{\text{osc,I},N}^\varepsilon + \tilde{\mathcal{R}}_{\text{osc,II},N}^\varepsilon$ where

$$\begin{aligned}\tilde{\mathcal{R}}_{\text{osc,I},N}^\varepsilon &= \mathcal{F}^{-1} \left(\sum_{\substack{\omega_{k,n-k,n}^{a,b,c} \neq 0 \\ j=1,2,3}} 1_{\{|n|,|k| \leq N\}} \frac{e^{i \frac{t}{\varepsilon} \omega_{k,n-k,n}^{a,b,c}}}{i \omega_{k,n-k,n}^{a,b,c}} (U^{a,j}(k) (n_j - k_j) U^b(n-k) | e^c(n) e^c(n)) \right), \\ \tilde{\mathcal{R}}_{\text{osc,II},N}^\varepsilon &= \mathcal{F}_v^{-1} \left(1_{\{|n_3| \leq N\}} \sum_{\substack{k+m=(0,n_3) \\ \tilde{\omega}_{k,m}^{a,b} \neq 0}} 1_{\{|k| \leq N\}} \frac{e^{i \frac{t}{\varepsilon} \tilde{\omega}_{k,m}^{a,b}}}{i \tilde{\omega}_{k,m}^{a,b}} n_3 (U^{a,3}(t,k) U^{b,h}(t,m), 0, 0)^\top \right)\end{aligned}$$

in particular we remark that

$$\partial_t \left(\varepsilon \tilde{\mathcal{R}}_{\text{osc},N}^\varepsilon \right) = \mathcal{R}_{\text{osc},N}^\varepsilon + \varepsilon \tilde{\mathcal{R}}_{\text{osc},N}^{\varepsilon,t}, \quad (4.6.7)$$

with $\tilde{\mathcal{R}}_{\text{osc},N}^{\varepsilon,t}$ is defined as $\tilde{\mathcal{R}}_{\text{osc},N}^{\varepsilon,t} = \tilde{\mathcal{R}}_{\text{osc,I},N}^{\varepsilon,t} + \tilde{\mathcal{R}}_{\text{osc,II},N}^{\varepsilon,t}$ where

$$\begin{aligned}\tilde{\mathcal{R}}_{\text{osc,I},N}^{\varepsilon,t} &= \mathcal{F}^{-1} \left(\sum_{\substack{\omega_{k,n-k,n}^{a,b,c} \neq 0 \\ j=1,2,3}} 1_{\{|n|,|k| \leq N\}} \frac{e^{i \frac{t}{\varepsilon} \omega_{k,n-k,n}^{a,b,c}}}{i \omega_{k,n-k,n}^{a,b,c}} \partial_t (U^{a,j}(t,k) (n_j - k_j) U^b(t,n-k) | e^c(n) e^c(n)) \right), \\ \tilde{\mathcal{R}}_{\text{osc,II},N}^{\varepsilon,t} &= \mathcal{F}_v^{-1} \left(1_{\{|n_3| \leq N\}} \sum_{\substack{k+m=(0,n_3) \\ \tilde{\omega}_{k,m}^{a,b} \neq 0}} 1_{\{|k| \leq N\}} \frac{e^{i \frac{t}{\varepsilon} \tilde{\omega}_{k,m}^{a,b}}}{i \tilde{\omega}_{k,m}^{a,b}} \partial_t n_3 (U^{a,3}(t,k) U^{b,h}(t,m), 0, 0)^\top \right).\end{aligned}$$

We underline the fact that the term $\mathcal{R}_{\text{osc},N}^\varepsilon$ in (4.6.7) is what we require in order to cancel the low frequencies of $\mathcal{R}_{\text{osc}}^\varepsilon$ which otherwise converge to zero only weakly due to stationary phase theorem. This here is the key observation and most important idea on which Schochet method is based: despite the fact that the difference system presents nonlinearities which does not converge strongly to zero we can define an alternative unknown ψ_N^ε which is an $\mathcal{O}(\varepsilon)$ -corrector of the difference W^ε which solves an equation in which this problematic nonlinear interaction vanishes.

Tanks to definition (4.6.6) and system (4.6.1) we can deduce the equation satisfied by ψ_N^ε after some elementary algebraic manipulation, which is

$$\begin{cases} \partial_t \psi_N^\varepsilon + \mathcal{Q}^\varepsilon \left(\psi_N^\varepsilon, \psi_N^\varepsilon - 2\varepsilon \tilde{\mathcal{R}}_{\text{osc},N}^\varepsilon + 2U \right) - \nu \Delta \psi_N^\varepsilon = -\mathcal{R}_{\text{osc}}^{\varepsilon,N} - \varepsilon \Gamma_N^\varepsilon, \\ \operatorname{div} \psi_N^\varepsilon = 0, \\ \psi_N^\varepsilon|_{t=0} = \psi_{N,0}^\varepsilon = \varepsilon \tilde{\mathcal{R}}_{\text{osc},N}^\varepsilon|_{t=0}, \end{cases} \quad (4.6.8)$$

with Γ_N^ε defined as

$$\Gamma_N^\varepsilon = \nu \Delta \tilde{\mathcal{R}}_{\text{osc},N}^\varepsilon + \mathcal{Q}^\varepsilon \left(\tilde{\mathcal{R}}_{\text{osc},N}^\varepsilon, \varepsilon \tilde{\mathcal{R}}_{\text{osc},N}^\varepsilon + 2U \right) + \tilde{\mathcal{R}}_{\text{osc},N}^{\varepsilon,t}.$$

We outline that ψ_N^ε is divergence-free since it is a linear combination of the eigenvectors e^0, e^\pm defined in (4.2.4) which are all divergence-free.

Now we claim that

Lemma 4.6.3. Γ_N^ε is bounded in $L^2(\mathbb{R}_+; H^{s-1})$ by a constant K_N which depends on N solely.

This is usually referred as *small divisor estimate* in the literature. The proof is due to the fact that all the elements composing $\Gamma_N^\varepsilon = \Gamma_N^\varepsilon(U)$ are localized in the frequency space, hence they have all the regularity we want them to have at the cost of some power of N . We omit a detailed proof only for the sake of brevity, but this can be deduced thanks to the energy estimates performed on U in the previous section.

Let us, at this point, perform an $H^s(\mathbb{T}^3)$ energy estimate on equation (4.6.8), we obtain that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\psi_N^\varepsilon\|_{H^s(\mathbb{T}^3)}^2 + \nu \|\psi_N^\varepsilon\|_{H^{s+1}(\mathbb{R}^3)}^2 \\ & \leq \left| \left(\mathcal{R}_{\text{osc}}^{\varepsilon,N} + \varepsilon \Gamma_N^\varepsilon + \mathcal{Q}^\varepsilon \left(\psi_N^\varepsilon, \psi_N^\varepsilon - 2\varepsilon \tilde{\mathcal{R}}_{\text{osc},N}^\varepsilon + 2U \right) \right) \psi_N^\varepsilon \right|_{H^s(\mathbb{T}^3)}. \end{aligned} \quad (4.6.9)$$

Now, if we consider two four component vector fields A, B such that their first three components are divergence-free it is indeed true that $\|\mathcal{Q}^\varepsilon(A, B)\|_{H^s(\mathbb{T}^3)} \leq C \|A \otimes B\|_{H^{s+1}(\mathbb{R}^3)}$,

we shall use repeatedly this property in what follows. We shall use as well the fact that $H^{s+1}(\mathbb{R}^3)$, $s > 1/2$ is a Banach algebra. Whence

$$\begin{aligned}
 (\mathcal{Q}^\varepsilon(\psi_N^\varepsilon, \psi_N^\varepsilon) | \psi_N^\varepsilon)_{H^s(\mathbb{T}^3)} &\leq C \|\psi_N^\varepsilon \otimes \psi_N^\varepsilon\|_{H^{s+1}(\mathbb{R}^3)} \|\psi_N^\varepsilon\|_{H^s(\mathbb{T}^3)}, \\
 &\leq C \|\psi_N^\varepsilon\|_{H^s(\mathbb{T}^3)} \|\psi_N^\varepsilon\|_{H^{s+1}(\mathbb{R}^3)}^2, \\
 (\mathcal{Q}^\varepsilon(\psi_N^\varepsilon, 2U) | \psi_N^\varepsilon)_{H^s(\mathbb{T}^3)} &\leq C \|\psi_N^\varepsilon\|_{H^s(\mathbb{T}^3)} \|\psi_N^\varepsilon\|_{H^{s+1}(\mathbb{R}^3)} \|U\|_{H^{s+1}(\mathbb{R}^3)}, \\
 (\mathcal{Q}^\varepsilon(\psi_N^\varepsilon, 2\varepsilon \tilde{\mathcal{R}}_{\text{osc},N}^\varepsilon) | \psi_N^\varepsilon)_{H^s(\mathbb{T}^3)} &\leq C\varepsilon \|\psi_N^\varepsilon\|_{H^s(\mathbb{T}^3)} \|\psi_N^\varepsilon\|_{H^{s+1}(\mathbb{R}^3)} \|\tilde{\mathcal{R}}_{\text{osc},N}^\varepsilon\|_{H^{s+1}(\mathbb{R}^3)}.
 \end{aligned} \tag{4.6.10}$$

Using the estimates in (4.6.10) into (4.6.9) and using repeatedly Young inequality $ab \leq \frac{\eta}{2}a^2 + \frac{1}{2\eta}b^2$ we obtain

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \|\psi_N^\varepsilon\|_{H^s(\mathbb{T}^3)}^2 + \left(\frac{\nu}{2} - C \|\psi_N^\varepsilon\|_{H^s(\mathbb{T}^3)} \right) \|\psi_N^\varepsilon\|_{H^{s+1}(\mathbb{R}^3)}^2 \\
 \leq C \left(\|U\|_{H^{s+1}(\mathbb{R}^3)}^2 + \varepsilon \|\tilde{\mathcal{R}}_{\text{osc},N}^\varepsilon\|_{H^{s+1}(\mathbb{R}^3)}^2 \right) \|\psi_N^\varepsilon\|_{H^s(\mathbb{T}^3)}^2 \\
 + C \|\mathcal{R}_{\text{osc}}^{\varepsilon,N} + \varepsilon \Gamma_N^\varepsilon\|_{H^{s-1}}^2.
 \end{aligned} \tag{4.6.11}$$

Whence let us define

$$\frac{1}{2} \Theta_{\varepsilon,N}(t) = C \left(\|U\|_{H^{s+1}(\mathbb{R}^3)}^2 + \varepsilon \|\tilde{\mathcal{R}}_{\text{osc},N}^\varepsilon\|_{H^{s+1}(\mathbb{R}^3)}^2 \right), \tag{4.6.12}$$

by variation of constant method we transform (4.6.11) into

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \left(\|\psi_N^\varepsilon\|_{H^s(\mathbb{T}^3)}^2 e^{-\int_0^t \Theta_{\varepsilon,N}(s) ds} \right) + \left(\frac{\nu}{2} - C \|\psi_N^\varepsilon\|_{H^s(\mathbb{T}^3)} \right) \|\psi_N^\varepsilon\|_{H^{s+1}(\mathbb{R}^3)}^2 e^{-\int_0^t \Theta_{\varepsilon,N}(s) ds} \\
 \leq C \|\mathcal{R}_{\text{osc}}^{\varepsilon,N} + \varepsilon \Gamma_N^\varepsilon\|_{H^{s-1}}^2 e^{-\int_0^t \Theta_{\varepsilon,N}(s) ds}.
 \end{aligned} \tag{4.6.13}$$

Now we claim the following

Lemma 4.6.4. *The function $\Theta_{\varepsilon,N}$ defined in (4.6.12) is an $L^1(\mathbb{R}_+)$ function uniformly in ε , moreover we can write the $L^1(\mathbb{R}_+)$ -bound as*

$$\|\Theta_{\varepsilon,N}\|_{L^1(\mathbb{R}_+)} \leq C + \varepsilon K_N. \tag{4.6.14}$$

We do not give a detailed proof of Lemma 4.6.4. What it has to be retained is that it is possible to bound the term $\tilde{\mathcal{R}}_{\text{osc},N}^\varepsilon$ since it is localized on the low frequencies, at the cost of making appear a (large in N) constant K_N depending on N only.

Lemma 4.6.4 in particular asserts, that fixing an (eventually large) $N > 0$ there exists an $\varepsilon = \varepsilon_N > 0$ such that there exist two constants $0 < c_1(\varepsilon, N) \leq c_2(\varepsilon, N)$ such that

$$c_1(\varepsilon, N) \leq \left| e^{-\int_0^t \Theta_{\varepsilon,N}(s) ds} \right| \leq c_2(\varepsilon, N),$$

independently of $t \in \mathbb{R}_+$.

We fix now an $\eta > 0$ (which we can suppose to be small) and we select two quantities $N = N_\eta$ and $\varepsilon = \varepsilon_N = \varepsilon_{N_\eta}$ such that

$$\|\psi_{N,0}^\varepsilon\|_{H^s(\mathbb{T}^3)} \leq \frac{\nu}{8C}, \quad e^{C+\varepsilon K_N} \|\psi_{N,0}^\varepsilon\|_{H^s(\mathbb{T}^3)} \leq \frac{\eta}{2}, \quad C c_2(\varepsilon, N) (k_N + \varepsilon K_N) \leq \frac{\eta}{2}. \quad (4.6.15)$$

The first and second inequality in (4.6.15) holds true thanks to the following procedure: we consider the definition of ψ_N^ε given in equation (4.6.6) we immediately deduce that $\psi_{N,0}^\varepsilon = \varepsilon \tilde{\mathcal{R}}_{\text{osc},N}^\varepsilon|_{t=0}$, but in particular $\|\tilde{\mathcal{R}}_{\text{osc},N}^\varepsilon|_{t=0}\|_{H^s(\mathbb{T}^3)} \leq K_N$ thanks to an argument similar to the one which proves Lemma 4.6.3, i.e. we exploit the fact that $\tilde{\mathcal{R}}_{\text{osc},N}^\varepsilon$ is supported in a ball of radius N in the frequency space and hence we can gain all the integrability we want at the price of some power of N . The constants C and K_N in particular are considered to be the ones appearing in (4.6.14).

We integrate now (4.6.13) in time, using the above consideration combined with Lemma 4.6.3 and (4.6.5) we transform (4.6.13) into

$$\begin{aligned} \|\psi_N^\varepsilon(t)\|_{H^s(\mathbb{T}^3)}^2 + \int_0^t \left(\nu - 2C \|\psi_N^\varepsilon(s)\|_{H^s(\mathbb{T}^3)} \right) \|\psi_N^\varepsilon(s)\|_{H^{s+1}(\mathbb{R}^3)}^2 e^{\int_s^t \Theta_{\varepsilon,N}(s') ds'} ds \\ \leq C c_2(k_N + \varepsilon K_N) + \|\psi_{N,0}^\varepsilon\|_{H^s(\mathbb{T}^3)}^2 e^{\int_0^t \Theta_{\varepsilon,N}(s) ds}, \end{aligned} \quad (4.6.16)$$

where we used the following notation $\psi_{N,0}^\varepsilon = \psi_N^\varepsilon|_{t=0}$. Whence considering the hypothesis (4.6.15) that we set for the bootstrap procedure we deduce

$$\|\psi_{N,0}^\varepsilon\|_{H^s(\mathbb{T}^3)}^2 e^{\int_0^t \Theta_{\varepsilon,N}(s) ds} \leq \frac{\eta}{2}, \quad C c_2(k_N + \varepsilon K_N) \leq \frac{\eta}{2}. \quad (4.6.17)$$

Thanks to the existence theorem given in Theorem 4.1.3 we can assert that the application $t \mapsto \|\psi_N^\varepsilon(t)\|_{H^s(\mathbb{T}^3)}$ is continuous, hence, since we considered $\|\psi_{N,0}^\varepsilon\|_{H^s(\mathbb{T}^3)}$ small in (4.6.15) it makes sense to define the time

$$\tilde{T}_\varepsilon^* = \sup \left\{ 0 \leq t \leq T^* \mid \|\psi_N^\varepsilon(t)\|_{H^s(\mathbb{T}^3)} \leq \frac{\nu}{4C} \right\}. \quad (4.6.18)$$

The definition of \tilde{T}_ε^* implies that $\nu - 2C \|\psi_N^\varepsilon(s)\|_{H^s(\mathbb{T}^3)} \leq \nu/2$ in $[0, \tilde{T}_\varepsilon^*]$, and moreover, since $\left| e^{\int_s^t \Theta_{\varepsilon,N}(s') ds'} \right| \geq 1$ and estimates (4.6.17) transform (4.6.16) in the following differential inequality

$$\|\psi_N^\varepsilon(t)\|_{H^s(\mathbb{T}^3)}^2 + \frac{\nu}{2} \int_0^t \|\psi_N^\varepsilon(s)\|_{H^{s+1}(\mathbb{R}^3)}^2 ds \leq \eta. \quad (4.6.19)$$

Now the bound on the right hand side of (4.6.19) is independent of t and arbitrary small, whence selecting $\eta < \frac{\nu^2}{16C^2}$ the condition (4.6.18) defining \tilde{T}_ε^* is always satisfied, whence we can assert that $\tilde{T}_\varepsilon^* = \infty$ (bootstrap) and hence we obtained the following result

Proposition 4.6.5. *Let be $\eta > 0$, there exists an $\varepsilon_\eta > 0$ and $N_\eta \in \mathbb{N}^*$ such that for each $\varepsilon \in (0, \varepsilon_\eta)$, $N > N_\eta$ the function ψ_N^ε defined as in (4.6.6) solves globally (4.6.8) and for each $t > 0$ the following bound holds true*

$$\|\psi_N^\varepsilon(t)\|_{H^s(\mathbb{T}^3)}^2 + \frac{\nu}{2} \int_0^t \|\nabla \psi_N^\varepsilon(s)\|_{H^s(\mathbb{T}^3)}^2 ds \leq \eta.$$

To prove the end of Theorem 4.1.7 is now a corollary of Proposition 4.6.5. Let us set

$$\mathcal{E}^s = L^\infty(\mathbb{R}_+; H^s(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+; H^{s+1}(\mathbb{T}^3)).$$

Thanks to the same procedure as always ($\tilde{\mathcal{R}}_{\text{osc},N}^\varepsilon$ is localized in the frequency set) we can safely assert that

$$\|W^\varepsilon\|_{\mathcal{E}^s} \leq \|\psi_N^\varepsilon\|_{\mathcal{E}^s} + \varepsilon K_N < \infty, \quad (4.6.20)$$

which accidentally implied that W^ε belongs to \mathcal{E}^s . Let us remind that $W^\varepsilon = U^\varepsilon - U$, and that U belongs to \mathcal{E}^s thanks to the results proved in Proposition 4.5.4 and 4.5.9, hence $U^\varepsilon \in \mathcal{E}^s$ if ε is sufficiently small.

From (4.6.20) we deduce that

$$\limsup_{\varepsilon \rightarrow 0} \|W^\varepsilon\|_{\mathcal{E}^s} \leq 2\eta,$$

for any $\eta > 0$, whence we finally deduced that $U^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} U$ in \mathcal{E}^s . \square

4.6.1 Proofs of technical lemmas.

Proof of Lemma 4.6.2 : The proof of Lemma 4.6.2 consists in an application of Lebesgue dominated convergence theorem. Since every time that Schochet method is applied (notably we refer to [72]) an estimate of this form on the high frequencies has to be performed we shall outline the proof of Lemma 4.6.2.

The element $\mathcal{R}_{\text{osc}}^{\varepsilon,N}$ converges point-wise (in the frequency space) to zero when $N \rightarrow \infty$ (computations omitted), and it is indeed true that

$$\left\| |n|^{s-1} |\mathcal{F} \mathcal{R}_{\text{osc}}^{\varepsilon,N}(t, n)| \right\|^2 \leq \left\| |n|^s |\mathcal{F}(U \otimes U)(t, n)| \right\|^2 = \mathcal{G}_s(t, n).$$

By Plancherel theorem the $L^1(\mathbb{R}_+ \times \mathbb{Z}^3, dt \times d\#)$ norm of \mathcal{G}_s is indeed the square of the $L^2(\mathbb{R}_+; H^s)$ norm of $U \otimes U$ (here we denote with $\#$ the discrete homogeneous measure on \mathbb{Z}^3). The function \mathcal{G}_s will be the dominating function. We apply the following product rule (for a proof of which we refer to [10, Corollary 2.86, p. 104])

$$\|U \otimes U\|_{H^s(\mathbb{T}^3)} \lesssim \|U\|_{L^\infty(\mathbb{T}^3)} \|U\|_{H^s(\mathbb{T}^3)},$$

while thanks to the embedding $H^{s+1}(\mathbb{T}^3) \hookrightarrow L^\infty(\mathbb{T}^3)$ for $s > 1/2$ we can finally state that

$$\|U \otimes U\|_{L^2(\mathbb{R}_+; H^s)} \lesssim \|U\|_{L^2(\mathbb{R}_+; H^{s+1}(\mathbb{R}^3))} \|U\|_{L^\infty(\mathbb{R}_+; H^s(\mathbb{T}^3))} < \infty.$$

Since $\mathcal{R}_{\text{osc}}^{\varepsilon,N}$ converges point-wise to zero in the Fourier space as $N \rightarrow \infty$ we can hence deduce (4.6.5). \square

Chapter 5

Low Froude number dynamic in the whole space.

Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country.

David Hilbert

5.1 Introduction.

In the present article we study the behavior of strong solutions of the following modified Boussinesq system

$$\begin{cases} \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \nu \Delta u^\varepsilon - \frac{1}{\varepsilon} \rho^\varepsilon \vec{e}_3 &= -\frac{1}{\varepsilon} \nabla \Phi^\varepsilon, \\ \partial_t \rho^\varepsilon + u^\varepsilon \cdot \nabla \rho^\varepsilon - \nu' \Delta \rho^\varepsilon + \frac{1}{\varepsilon} u^{3,\varepsilon} &= 0, \\ \operatorname{div} u^\varepsilon &= 0, \\ (u^\varepsilon, \rho^\varepsilon)|_{t=0} &= (u_0, \rho_0), \end{cases} \quad (\text{PBS}_\varepsilon)$$

where the functions $(u^\varepsilon, \rho^\varepsilon) = U^\varepsilon$ depend on the variables $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$, in the regime $\varepsilon \rightarrow 0$. The space variable x shall be many times considered separately with respect to the horizontal and vertical components, i.e. $x = (x_h, x_3) = (x_1, x_2, x_3)$. In the present paper we denote $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ the standard laplacian, $\Delta_h = \partial_1^2 + \partial_2^2$ is the laplacian in the horizontal directions, as well as $\nabla_h = (\partial_1, \partial_2)^\top$, $\nabla_h^\perp = (-\partial_2, \partial_1)^\top$. In the same way the symbol ∇ represents the gradient in all space directions $\nabla = (\partial_1, \partial_2, \partial_3)$. Considered a

This chapter is available as a preprint, see [136].

vector field w we denote $\operatorname{div} w = \partial_1 w^1 + \partial_2 w^2 + \partial_3 w^3$. Given two three-components vector fields w, z the notation $w \cdot \nabla z$ indicates the operator

$$w \cdot \nabla z = \sum_{i=1}^3 w^i \partial_i z.$$

Generally for any two-components vector field $u = (u^1, u^2)$ we shall denote as $u^\perp = (-u^2, u^1)$. The viscosity ν, ν' above are strictly positive constants $\nu, \nu' \geq c > 0$. We give in what follows a short physical justification of the system (PBS_ε) . The system describing the motion of a fluid with variable density under the effects of (external) gravitational force is (see [50]):

$$\left\{ \begin{array}{l} \partial_t u^1 + u \cdot \nabla u^1 \\ \partial_t u^2 + u \cdot \nabla u^2 \\ \partial_t u^3 + u \cdot \nabla u^3 + \frac{\mathbf{g} \cdot \rho}{\rho_0} \\ \partial_t \rho + u \cdot \nabla \rho - \frac{\rho_0 N^2}{\mathbf{g}} u^3 \\ \operatorname{div} u = 0. \end{array} \right. = \begin{array}{l} -\frac{1}{\rho_0} \partial_1 p + \nu \Delta u^1, \\ -\frac{1}{\rho_0} \partial_2 p + \nu \Delta u^2, \\ -\frac{1}{\rho_0} \partial_3 p + \nu \Delta u^3, \\ \kappa \Delta \rho, \end{array} \quad (5.1.1)$$

The system (5.1.1) already presents many physical simplifications which were made implicitly, such as the incompressibility condition $\operatorname{div} u = 0$ and the *Boussinesq approximation*:

$$\text{Fluid density} = \rho_0 + \bar{\rho}(x_3) + \rho(t, x), \quad |\rho| \ll |\bar{\rho}|.$$

For the sake of simplicity we renormalize the constants ρ_0, \mathbf{g} to one. The system (PBS_ε) describes how a stratified fluid reacts in a long time-scale T to big perturbations around a state of dynamical equilibrium. To understand in what consist such perturbation let us consider a fluid which is perfectly stratified: gravity tends to minimize the gravitational potential and hence to dispose heavier layers under lighter ones. An equilibrium state is hence a configuration in which the fluid density is a function depending on the vertical variable x_3 only and it is decreasing in x_3 . Let us suppose now to displace a certain volume of fluid with high density in a higher region with lower density (perturbation of equilibrium). Gravity will induce downward motion and Archimede's principle will provide upward buoyancy. This process induces a periodic motion of frequency N appearing in the third equation of (5.1.1). The value N appearing in the equation for ρ is called Brunt-Väisälä frequency, and describes the oscillatory behavior induced by the buoyancy which is caused by the stratification in decreasing-density stacks. We suppose N to be constant, and indeed $N = T_N^{-1}$ where T_N is the *characteristic time of stratification*. We define the *Froude number* as

$$\text{Fr} = \frac{T_N}{T} \ll 1.$$

The Froude number Fr quantifies the stratification effects on the dynamics of the fluid; the smaller it is the more relevant such effects are. In fact $\text{Fr} = T_N/T$ is a ratio which involves time-scales only; the characteristic time of stratification T_N is an intrinsic magnitude of the system which is determinate by the stratification frequency only, while T can be chosen as

large as the observer desires. It is reasonable hence to think that in very large time-scales T the periodic motion caused by an induced equilibrium disturbance will somehow disperse, and the fluid will once again recover a configuration of equilibrium.

Defining the following change of reference scale

$$u^{\text{Fr}}(t, x) = \frac{1}{T}u\left(\frac{t}{T}, x\right), \quad \rho^{\text{Fr}}(t, x) = \frac{T_N}{T}\rho\left(\frac{t}{T}, x\right), \quad p^{\text{Fr}}(t, x) = \frac{T_N}{T}p\left(\frac{t}{T}, x\right),$$

the system (5.1.1) becomes

$$\begin{cases} \partial_t u^{\text{Fr}} + u^{\text{Fr}} \cdot \nabla u^{\text{Fr}} - \nu \Delta u^{\text{Fr}} - \frac{1}{\text{Fr}} \rho^{\text{Fr}} \vec{e}_3 = -\frac{1}{\text{Fr}} \nabla p^{\text{Fr}}, \\ \partial_t \rho^{\text{Fr}} + u^{\text{Fr}} \cdot \nabla \rho^{\text{Fr}} - \nu' \Delta \rho^{\text{Fr}} + \frac{1}{\text{Fr}} u^{3, \text{Fr}} = 0, \\ \text{div } u^{\text{Fr}} = 0, \\ (u^{\text{Fr}}, \rho^{\text{Fr}})|_{t=0} = (u_0, \rho_0). \end{cases} \quad (5.1.2)$$

Setting $\text{Fr} = \varepsilon$ we deduce hence the system (PBS_ε) . Let us rewrite the system (PBS_ε) into the following more compact form

$$\begin{cases} \partial_t U^\varepsilon + u^\varepsilon \cdot \nabla U^\varepsilon - \mathbb{D}U^\varepsilon + \frac{1}{\varepsilon} \mathcal{A}U^\varepsilon = -\frac{1}{\varepsilon} \begin{pmatrix} \nabla \Phi^\varepsilon \\ 0 \end{pmatrix}, \\ U^\varepsilon = (u^\varepsilon, \rho^\varepsilon), \\ \text{div } u^\varepsilon = 0, \end{cases} \quad (\text{PBS}_\varepsilon)$$

where $U^\varepsilon = (u^\varepsilon, \rho^\varepsilon)$ and

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \mathbb{D} = \begin{pmatrix} \nu \Delta & 0 & 0 & 0 \\ 0 & \nu \Delta & 0 & 0 \\ 0 & 0 & \nu \Delta & 0 \\ 0 & 0 & 0 & \nu' \Delta \end{pmatrix}. \quad (5.1.3)$$

The above form for the system is the one we shall always adopt. We shall use as well the following differential operator

$$\mathbb{P} = \left(\begin{array}{c|c} \delta_{i,j} - \Delta^{-1} \partial_i \partial_j & 0 \\ \hline 0 & 1 \end{array} \right)_{i,j=1,2,3}. \quad (5.1.4)$$

The operator \mathbb{P} acts in the following way: given a four component vector field $V = V(x) = (V^1, V^2, V^3, V^4) = (V', V^4)$ it maps V' onto a divergence-free vector field and leaves untouched V^4 , i.e.

$$\mathbb{P}V = (V' - \Delta^{-1} \nabla \text{div } V', V^4).$$

We underline that \mathbb{P} and \mathbb{D} commute. We shall use this property (even implicitly) repeatedly all along the present work.

The system (PBS_ε) falls into a wider category of mathematical problems known as *singular perturbation problems*. The idea behind this kind of problems is that, once we have an external, linear, force acting on a system such as in (PBS_ε) , such force with great magnitude

will constraint the motion of the system described. This kind of rigidity can hence be used in order to prove that suitable three-dimensional hydrodynamical flows are globally well-posed without any smallness assumption on the initial data.

Another system which falls in the category of singular perturbation problems is the system describing the motion of a flow under the effect of a strong horizontal rotation, namely what is known as the Navier-Stokes-Coriolis equations:

$$\begin{cases} \partial_t v_{\text{RF}}^\varepsilon + v_{\text{RF}}^\varepsilon \cdot \nabla v_{\text{RF}}^\varepsilon - \nu \Delta v_{\text{RF}}^\varepsilon + \frac{e^3 \wedge v_{\text{RF}}^\varepsilon}{\varepsilon} = -\frac{1}{\varepsilon} \nabla p_{\text{RF}}^\varepsilon, \\ v_{\text{RF}}^\varepsilon|_{t=0} = v_{\text{RF},0}. \end{cases} \quad (\text{NSC}_\varepsilon)$$

In the case in which the spatial domain is the full three-dimensional space \mathbb{R}^3 , if the initial data is of the form

$$v_{\text{RF},0} = \bar{u}_{2\text{D},0}^h + \tilde{u}_{3\text{D},0},$$

where $\bar{u}_{2\text{D},0}^h$ is a two-dimensional vector field it is proved in [42] that

$$\begin{aligned} v_{\text{RF}}^\varepsilon - w^\varepsilon - \bar{u}_{2\text{D}}^h &\rightarrow 0, & \text{in } L^\infty \left(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \right), \\ \nabla (v_{\text{RF}}^\varepsilon - w^\varepsilon - \bar{u}_{2\text{D}}^h) &\rightarrow 0, & \text{in } L^2 \left(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \right), \end{aligned}$$

where w^ε is the global solution of the linear homogeneous equation associated to (NSC_ε) and $\bar{u}_{2\text{D}}^h$ is the global solution of the two-dimensional Navier-Stokes equations.

Many more are the works on global existence and convergence for the equation (NSC_ε) : in the whole space it was proved in [38] a result of global existence and convergence in Sobolev spaces of anisotropic type in the case in which the vertical diffusivity is null. Such result is physically significant since experimental proof suggests that for fluids at a planetary scale the vertical diffusivity (Ekman number) tends to be very small, see [50] and [126]. We mention as well [80], [114] and [41] for works describing rotating fluids between two parallel rigid layers with Dirichlet boundary conditions, [119] for rotating fluids with zero vertical diffusivity and vanishing horizontal diffusivity and [59] for propagation of tangential regularity in rotating inviscid fluids.

To the best of our knowledge there are not many works concerning the system (PBS_ε) . In [63] P. Embid and A. Majda study the distributional limit of the primitive equations

$$\begin{cases} \partial_t u^{\text{Ro,Fr}} + u^{\text{Ro,Fr}} \cdot \nabla u^{\text{Ro,Fr}} - \nu \Delta u^{\text{Ro,Fr}} \\ \quad + \frac{1}{\text{Ro}} u^{\text{Ro,Fr}} \wedge \vec{e}_3 - \frac{1}{\text{Fr}} \rho^{\text{Ro,Fr}} \vec{e}_3 = -\nabla P^{\text{Ro,Fr}}, \\ \partial_t \rho^{\text{Ro,Fr}} + u^{\text{Ro,Fr}} \cdot \nabla \rho^{\text{Ro,Fr}} - \nu' \Delta \rho^{\text{Ro,Fr}} + \frac{1}{\text{Fr}} u^{3,\text{Ro,Fr}} = 0, \\ \text{div } u^{\text{Ro,Fr}} = 0, \\ (u^{\text{Ro,Fr}}, \rho^{\text{Ro,Fr}})|_{t=0} = (u_0, \rho_0). \end{cases} \quad (\text{PE})$$

In the regimes $\text{Ro, Fr} = \mathcal{O}(\varepsilon)$ and $\text{Ro} \gg \text{Fr} = \mathcal{O}(\varepsilon)$ in the case in which the domain is periodic-in-space. The value Ro is called the *Rossby number* and quantifies the influence of the rotation on the motion of a fluid in the same way as the Froude number quantifies

the stratification effects. The systems (PE) and (PBS_ε) are intimately related, for a formal derivation of (PE) in the case Ro, Fr = O(ε) we refer to the beautiful introduction of [27]. Concerning always the equations (PE) in the regime Ro, Fr = O(ε) in the whole space we refer to the pioneering work [36] in which J.-Y. Chemin proves that (PE) is globally well posed in the case in which Fr = Ro = ε, only a certain part of the initial datum is small and the difference |ν - ν'| is small. Moreover in [28] and [29] F. Charve uses dispersive tools (Strichartz estimates) proves that (PE) in the regime Ro, Fr = O(ε), Ro ≠ Fr are globally well posed and converge to a suitable limit system known as the quasi-geostrophic system without any smallness assumption on the initial data.

The system (PBS_ε) is very close to the system (PE) in the regime Ro ≫ Fr = O(ε), we refer to [63] and references therein for a justification of such fact. Recently K. Widmayer proved in [144] that the inviscid counterpart of (PBS_ε) converges locally in L²(ℝ³) to a stratified two-dimensional Euler system. In Chapter 4 it is proved that, in a domain which is three-dimensional, periodic-in-space and non-resonant, the solutions of (PBS_ε) converge globally and strongly to the global solutions of a suitable limit system.

5.1.1 The functional setting.

In this section we introduce the functional spaces we shall adopt all along the paper. We define the homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^d)$, $s \in \mathbb{R}$ as the space of tempered distributions u on \mathbb{R}^d whose Fourier transform $\hat{u} \in L^1_{\text{loc}}(\mathbb{R}^d)$ and such that

$$\|u\|_{\dot{H}^s(\mathbb{R}^d)}^2 = \left\| (-\Delta)^{s/2} u \right\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi < \infty.$$

Since we intend to study the behavior of solutions of nonlinear partial differential equations we are interested to understand the regularity of a product of distributions. Generally a product of distributions is not well defined as it was first proved in [134]. This is no longer true if the distributions considered belong to some suitable homogeneous Sobolev space;

Lemma 5.1.1. *Let $u \in \dot{H}^{s_1}(\mathbb{R}^d)$, $v \in \dot{H}^{s_2}(\mathbb{R}^d)$ where $s_1, s_2 < d/2$ and $s_1 + s_2 > 0$. Then*

$$\|uv\|_{\dot{H}^{s_1+s_2-\frac{d}{2}}(\mathbb{R}^d)} \leq C_{s_1, s_2} \|u\|_{\dot{H}^{s_1}(\mathbb{R}^d)} \|v\|_{\dot{H}^{s_2}(\mathbb{R}^d)},$$

or, equivalently, the point-wise multiplication maps continuously $\dot{H}^{s_1}(\mathbb{R}^d) \times \dot{H}^{s_2}(\mathbb{R}^d)$ to $\dot{H}^{s_1+s_2-\frac{d}{2}}(\mathbb{R}^d)$.

Homogeneous Sobolev are Hilbert spaces if and only if $s < d/2$, in this case the scalar product of two elements of $\dot{H}^s(\mathbb{R}^d)$ is defined as

$$\begin{aligned} (u|v)_{\dot{H}^s(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \left[(-\Delta)^{s/2} u(x) \right] \cdot \left[(-\Delta)^{s/2} v(x) \right] dx, \\ &= \int_{\mathbb{R}^d} (|\xi|^s \hat{u}(\xi)) \cdot (|\xi|^s \hat{v}(\xi)) d\xi. \end{aligned} \tag{5.1.5}$$

We refer to [10, p. 26] for a counterexample in the case in which $s \geq d/2$.

The norm of $\dot{H}^s(\mathbb{R}^3)$ does not take in account the behavior of u in a frequency set close to zero, the non-homogeneous Sobolev space $H^s(\mathbb{R}^3)$ defined as

$$H^s(\mathbb{R}^d) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d) \mid \left\| (1 - \Delta)^{s/2} u \right\|_{L^2(\mathbb{R}^d)} < \infty \right\},$$

gives a deeper description of the tempered distribution u and a mean to control the low-frequencies as well. Nonetheless we shall work constantly with homogeneous Sobolev spaces since the propagation of a critical homogeneous Sobolev regularity suffices to deduce smoothness of solutions, we shall briefly explain such fact. It is moreover interesting to notice that

$$H^s(\mathbb{R}^3) = L^2(\mathbb{R}^3) \cap \dot{H}^s(\mathbb{R}^3), \text{ if } s \geq 0.$$

We shall see that the solutions of (PBS $_\varepsilon$), if ε is sufficiently close to zero, develop a behavior which is radically different along the horizontal direction x_h and the vertical x_3 . This motivates the introduction of the following *anisotropic Lebesgue spaces*, which are spaces of function whose integrability differs along horizontal and vertical directions. The anisotropic Lebesgue spaces $L_h^p(L_v^q)$ with $p, q \geq 1$ are defined as

$$\begin{aligned} L_h^p(L_v^q(\mathbb{R}^3)) &= L^p(\mathbb{R}_h^2; L^q(\mathbb{R}_v)) \\ &= \left\{ u \in \mathcal{S}' \mid \|u\|_{L_h^p L_v^q} = \left[\int_{\mathbb{R}_h^2} \left| \int_{\mathbb{R}_v} |u(x_h, x_3)|^q dx_3 \right|^{\frac{p}{q}} dx_h \right]^{\frac{1}{p}} < +\infty \right\}. \end{aligned}$$

Here, the order of integration is important. Indeed, if $1 \leq p \leq q$ and if $u : X_1 \times X_2 \rightarrow \mathbb{R}$ is a function in $L^p(X_1; L^q(X_2))$, where $(X_1, d\mu_1), (X_2, d\mu_2)$ are measurable spaces, then $u \in L^q(X_2; L^p(X_1))$ and

$$\|u\|_{L^q(X_2; L^p(X_1))} \leq \|u\|_{L^p(X_1; L^q(X_2))}. \quad (5.1.6)$$

Obviously we can define in a symmetric way the space $L_v^p(L_h^q) = L^p(\mathbb{R}_v; L^q(\mathbb{R}_h^2))$. We shall be interested to study spaces of the kind $L_v^\infty(L_h^q)$, $q \in [1, \infty]$, they are indeed defined as the tempered distributions such that

$$\|u\|_{L_v^\infty(L_h^q)} = \operatorname{ess\,sup}_{x_3 \in \mathbb{R}_v} \|u(\cdot, x_3)\|_{L^q(\mathbb{R}_h^2)} < \infty.$$

In a similar way we can define $L_h^p(L_v^\infty)$ spaces via the norm

$$\|u\|_{L_h^p(L_v^\infty)} = \left(\int_{\mathbb{R}_h^2} \operatorname{ess\,sup}_{x_3 \in \mathbb{R}_v} |u(x_h, x_3)|^p dx_h \right)^{\frac{1}{p}}.$$

Parabolic equations such as Navier-Stokes equations develop a classical integrability regularity of the form $L^\infty(\mathbb{R}_+; L^2) \cap L^2(\mathbb{R}_+; \dot{H}^1)$. It is well known (see [35], for instance) that as long as Navier-Stokes equations can propagate $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ data the solutions are in fact regular, for this reason it makes sense to define the following space for $s > 0$:

$$\dot{\mathcal{E}}_T^s(\mathbb{R}^d) = \mathcal{C}([0, T]; \dot{H}^s(\mathbb{R}^d)) \cap L^2([0, T]; \dot{H}^{s+1}(\mathbb{R}^d)),$$

and since we are interested in global-in-time regularity we define hence the space

$$\dot{\mathcal{E}}^s(\mathbb{R}^d) = \dot{\mathcal{E}}_\infty^s(\mathbb{R}^d).$$

5.2 Statement of the main result and preliminaries.

Before stating the main result let us mention that, as \mathcal{A} defined in (5.1.3) appear in (PBS_ε) is skew-symmetric it does not bring any energy in suitable energy spaces which are calibrated on L^2 , such as homogeneous and non-homogeneous Sobolev spaces $\dot{H}^s, H^s, s \in \mathbb{R}$ and Besov spaces $B_{2,r}^s, s \in \mathbb{R}, r \geq 1$. We refer to [10] for a detailed definition and a deep description of Besov spaces in the whole space. This implies in particular that Fujita-Kato and Leray theorem [10], [104] can be applied on system (PBS_ε) . At first we state the following theorem *à la Leray*:

Theorem 5.2.1. *Let U_0 be in $L^2(\mathbb{R}^3)$, for each $\varepsilon > 0$ there exist a sequence $(U^\varepsilon)_{\varepsilon>0}$ such that, for each $\varepsilon > 0$, the function U^ε is a distributional solution of (PBS_ε) with initial data U_0 . Moreover the sequence $(U^\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $\dot{\mathcal{E}}^0(\mathbb{R}^3)$.*

Let us now state a result of existence in the homogeneous Sobolev setting, for a proof we refer to [10],

Theorem 5.2.2. *Let us suppose $U_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, then there exists a time $T = T_{U_0}$ independent of ε such that there exists a unique solution U^ε of (PBS_ε) in $L^4([0, T]; \dot{H}^1(\mathbb{R}^3))$ which also belongs to the space $\dot{\mathcal{E}}_T^{1/2}(\mathbb{R}^3)$.*

- *If the initial data is small in the space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, i.e. if $\|U_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \leq \tilde{c} \min\{\nu, \nu'\}$, then $T_{U_0} = \infty$.*
- *If T_{U_0} is finite then*

$$\int_0^{T_{U_0}} \|U^\varepsilon(\tau)\|_{\dot{H}^1(\mathbb{R}^3)}^4 d\tau = \infty. \quad (5.2.1)$$

Theorem 5.2.2 states that there exist always local, strong solutions for the system (PBS_ε) , moreover if the $\dot{H}^{1/2}$ initial data is small with respect to the viscosities characterizing the system (namely if the constant \tilde{c} in Theorem 5.2.2 is small) the solution is global.

The blow-up condition (5.2.1) gives already an insight on how to connect the critical homogeneous Sobolev regularity $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ with its non-homogeneous counterpart. Indeed by interpolation of Sobolev spaces we can argue that

$$\|U^\varepsilon(\tau)\|_{\dot{H}^1(\mathbb{R}^3)}^4 \leq \|U^\varepsilon(\tau)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \|\nabla U^\varepsilon(\tau)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2,$$

whence to control the $\dot{\mathcal{E}}^{1/2}(\mathbb{R}^3)$ norm impose a control on the blow-up condition (5.2.1). The following result stems in a relatively simple way from Theorem 5.2.2:

Corollary 5.2.3. *Let $U_0 \in H^s(\mathbb{R}^3)$, $s \geq 1/2$, then the unique solution U^ε identified by Theorem 5.2.2 satisfies the following inequality, for each $t \in (0, T_{U_0})$:*

$$\begin{aligned} \|U^\varepsilon(t)\|_{H^s(\mathbb{R}^3)}^2 + c \int_0^t \|\nabla U^\varepsilon(\tau)\|_{H^s(\mathbb{R}^3)}^2 d\tau \\ \leq C \|U_0\|_{H^s(\mathbb{R}^3)}^2 \exp \left\{ \int_0^t \|U^\varepsilon(\tau)\|_{\dot{H}^1(\mathbb{R}^3)}^4 d\tau \right\}. \end{aligned}$$

The proof of Corollary 5.2.3 is rather simple, but it involves the tools of paradifferential calculus and Bony decomposition, for this reason is postponed to Section 5.2.2.

Whence the blow-up condition (5.2.1) which is calibrated in the critical homogeneous Sobolev setting suffices to determinate non-homogeneous subcritical regularity, and the maximal lifespan in Corollary 5.2.3 is hence the same one as in Theorem 5.2.2. In what follows it suffices hence that we focus on the propagation of homogeneous critical Sobolev regularity.

A question of great importance in the study of hydro-dynamical systems is whether three-dimensional hydro-dynamical flows admit classical solutions which are globally well-defined. For two-dimensional systems the answer is affirmative and it is known since the classical works [100] and [106]. In dimension three the question of global solvability for generic large data remain unsolved. Nonetheless there exist many three-dimensional systems which admit global-in-time solution of strong type for arbitrary data, notably geophysical fluids [42] belong to such category due to the constraining effects of the rotation of the Earth. The system (PBS_ε) can be studied with the methodologies characterizing such discipline and we prove the following result:

Theorem 5.2.4. *Let $U_0 \in H^{\frac{1}{2}}(\mathbb{R}^3)$, such that $\omega^h = -\partial_2 u_0^1 + \partial_1 u_0^2 \in L^2(\mathbb{R}^3)$, there exists a $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ the unique local solution U^ε of (PBS_ε) is in fact global and belongs to the space*

$$U^\varepsilon \in L^\infty\left(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\right) \cap L^2\left(\mathbb{R}_+; \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)\right) = \dot{\mathcal{E}}^{1/2}(\mathbb{R}^3).$$

Moreover as $\varepsilon \rightarrow 0$ the following convergence takes place,

$$U^\varepsilon - W^\varepsilon - (\bar{u}^h, 0, 0)^\top \xrightarrow{\varepsilon \rightarrow 0} 0, \quad \text{in the space } \dot{\mathcal{E}}^{1/2}(\mathbb{R}^3),$$

where U^ε is the strong solution identified by Theorem 5.2.2 and W^ε, \bar{u}^h are respectively the unique global solutions (in the space $\dot{\mathcal{E}}^{1/2}(\mathbb{R}^3)$) of

$$\begin{cases} \partial_t W^\varepsilon - \mathbb{D}W^\varepsilon + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} W^\varepsilon = \begin{pmatrix} 0 \\ 0 \\ -\partial_3 (-\Delta_h)^{-1} \operatorname{div}_h (\bar{u}^h \cdot \nabla_h \bar{u}^h) \\ 0 \end{pmatrix}, \\ W^\varepsilon|_{t=0} = (\mathbb{P}_{-, \varepsilon} + \mathbb{P}_{+, \varepsilon}) U_0, \end{cases} \quad (5.2.2)$$

and (5.4.3). The operators $\mathbb{P}_{\pm, \varepsilon}$ are defined in (5.3.7).

Remark 5.2.5. We point out that Theorem 5.2.4 is composed of two main statements:

1. global well-posedness in the energy space $\dot{\mathcal{E}}^{1/2}(\mathbb{R}^3)$ for positive, small ε ,
2. convergence as $\varepsilon \rightarrow 0$ in the space $\dot{\mathcal{E}}^{1/2}(\mathbb{R}^3)$ to the solutions of a suitable limit system,

we prove at first the global well-posedness, and subsequently, thanks to the theory developed in order to prove such result, we prove the convergence result. \blacklozenge

To prove Theorem 5.2.4 we shall proceed as follows:

- In Section 5.3 we perform a careful spectral analysis of the linear operator $L_\varepsilon = \mathbb{P}\mathcal{A} - \varepsilon\mathbb{D}$, where \mathbb{P} is the Leray projector onto the first three-components and the identity on the fourth and \mathcal{A} , \mathbb{D} are defined in (5.1.3). Such analysis will be of great relevance in the Sections 5.5 and 5.4.
- In Section 5.4 we prove that the system (5.4.3) is globally well posed in $\dot{\mathcal{E}}^s(\mathbb{R}^3)$, $s \geq 0$. The system (5.4.3) is the system to whom (PBS_ε) approaches as $\varepsilon \rightarrow 0$.
- In Section 5.5 we study the linear system (5.5.3). The initial data of (5.5.3) is considered to be in what we denote as the oscillating subspace of L_ε , which is introduced at the end of Section 5.3, and moreover is localized in a set $\mathcal{C}_{r,R}$ (see (5.3.4)) of the frequency space which makes his evolution to be described by an oscillating integral with no stationary phase. This observation is hence the key observation which allows us to prove some adapted dispersive estimates on the solutions of (5.5.3).
- In Section 5.6 we prove the global well-posedness part of Theorem 5.2.4, i.e. we prove that for ε sufficiently small the solution U^ε of (PBS_ε) belongs to the space $\dot{\mathcal{E}}^{1/2}(\mathbb{R}^3)$. To do so we perform a bootstrap argument on the function $\delta_{r,R}^\varepsilon = U^\varepsilon - W_{r,R}^\varepsilon - \bar{U}$ which requires the use of the dispersive estimates performed in Section 5.5.
- Finally in Section 5.7 we prove the convergence part of the statement of Theorem 5.2.4.

Remark 5.2.6. All along the paper we shall denote with C a generic positive constant, independent by any parameter. Such value may differ from line to line. The positive constant $C_{r,R}$ depends instead from the parameter $0 < r < R$, and

$$C_{r,R} \leq C \left(1 + \frac{R^N}{r^N} \right),$$

for some positive and finite $N \in \mathbb{N}$. ◆

5.2.1 Dyadic decomposition

We recall that in \mathbb{R}^d , with $d \in \mathbb{N}^*$, for $R > 0$, the ball $\mathcal{B}_d(0, R)$ is the set

$$\mathcal{B}_d(0, R) = \{ \xi \in \mathbb{R}^d : |\xi| \leq R \}.$$

For $0 < r_1 < r_2$, we defined the annulus

$$\mathcal{A}_d(r_1, r_2) \stackrel{\text{def}}{=} \{ \xi \in \mathbb{R}^d : r_1 \leq |\xi| \leq r_2 \}.$$

Next, we recall the following Bernstein-type lemma, which states that Fourier multipliers act almost as homotheties on distributions whose Fourier transforms are supported in a ball or an annulus. We refer the reader to [37, Lemma 2.1.1] or [10, Lemma 2.1] for a proof of this lemma.

Lemma 5.2.7. *Let $k \in \mathbb{N}$, $d \in \mathbb{N}^*$ and $R, r_1, r_2 \in \mathbb{R}$ satisfy $0 < r_1 < r_2$ and $R > 0$. There exists a constant $C > 0$ such that, for any $a, b \in \mathbb{R}$, $1 \leq a \leq b \leq +\infty$, for any $\lambda > 0$ and for any $u \in L^a(\mathbb{R}^d)$, we have*

$$\text{supp } (\widehat{u}) \subset \mathcal{B}_d(0, \lambda R) \implies \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^k \lambda^{k+d(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a}, \quad (5.2.3)$$

$$\text{supp } (\widehat{u}) \subset \mathcal{A}_d(\lambda r_1, \lambda r_2) \implies C^{-k} \lambda^k \|u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a} \leq C^k \lambda^k \|u\|_{L^a}. \quad (5.2.4)$$

In order to define the (non-homogeneous) dyadic partition of unity, we also recall the following proposition, the proof of which can be found in [37, Proposition 2.1.1] or [10, Proposition 2.10].

Proposition 5.2.8. *Let $d \in \mathbb{N}^*$. There exist smooth radial function χ and φ from \mathbb{R}^d to $[0, 1]$, such that*

$$\text{supp } \chi \in \mathcal{B}_d\left(0, \frac{4}{3}\right), \quad \text{supp } \varphi \in \mathcal{A}_d\left(\frac{3}{4}, \frac{8}{3}\right), \quad (5.2.5)$$

$$\forall \xi \in \mathbb{R}^3, \quad \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1, \quad (5.2.6)$$

$$|j - j'| \geq 2 \implies \text{supp } \varphi(2^{-j}\cdot) \cap \text{supp } \varphi(2^{-j'}\cdot) = \emptyset, \quad (5.2.7)$$

$$j \geq 1 \implies \text{supp } \chi \cap \text{supp } \varphi(2^{-j}\cdot) = \emptyset. \quad (5.2.8)$$

Moreover, for any $\xi \in \mathbb{R}^d$, we have

$$\frac{1}{2} \leq \chi^2(\xi) + \sum_{j \geq 0} \varphi^2(2^{-j}\xi) \leq 1. \quad (5.2.9)$$

The dyadic blocks are defined as follows

Definition 5.2.9. *For any $d \in \mathbb{N}^*$ and for any tempered distribution $u \in \mathcal{S}'(\mathbb{R}^d)$, we set*

$$\begin{aligned} \Delta_q u &= \mathcal{F}^{-1}(\varphi(2^{-q}|\xi|)\widehat{u}(\xi)), & \forall q \geq 0, \\ \Delta_{-1} u &= \mathcal{F}^{-1}(\chi(2^{-1}|\xi|)\widehat{u}(\xi)), \\ \Delta_q u &\equiv 0, & \forall q \leq -2, \\ S_q u &= \sum_{q' \leq q-1} \Delta_{q'} u, & \forall q \in \mathbb{Z}. \end{aligned}$$

Using the properties of ψ and φ , for any tempered distribution $u \in \mathcal{S}'(\mathbb{R}^d)$, one can formally write

$$u = \sum_q \Delta_q u \quad \text{in,}$$

and the homogeneous Sobolev spaces $H^s(\mathbb{R}^d)$, with $s \in \mathbb{R}$, can be characterized as follows

Proposition 5.2.10. *Let $d \in \mathbb{N}^*$, $s \in \mathbb{R}$ and $u \in H^s(\mathbb{R}^d)$. Then,*

$$\|u\|_{H^s} \sim \left(\sum_q 2^{2qs} \|\Delta_q u\|_{L^2}^2 \right)^{\frac{1}{2}} = \left\| (2^{qs} \|\Delta_q u\|_{L^2})_{q \in \mathbb{Z}} \right\|_{\ell^2}.$$

Moreover, there exists a square-summable sequence of positive numbers $(c_q)_q$ with $\sum_q c_q^2 = 1$, such that

$$\|\Delta_q u\|_{L^2} \leq c_q(u) 2^{-qs} \|u\|_{H^s}. \quad (5.2.10)$$

5.2.2 Paradifferential calculus.

The decomposition into dyadic blocks allows, at least formally, to write, for any tempered distributions u and v ,

$$u v = \sum_{\substack{q \in \mathbb{Z} \\ q' \in \mathbb{Z}}} \Delta_q u \Delta_{q'} v \quad (5.2.11)$$

The Bony decomposition (see for instance [14], [37] or [10] for more details) consists in splitting the above sum in three parts. The first corresponds to the low frequencies of u multiplied by the high frequencies of v , the second is the symmetric counterpart of the first, and the third part concerns the indices q and q' which are comparable. Then,

$$uv = T_u v + T_v u + R(u, v),$$

where

$$T_u v = \sum_q S_{q-1} u \Delta_q v, \quad T_v u = \sum_{q'} S_{q'-1} v \Delta_{q'} u, \quad R(u, v) = \sum_{|q-q'| \leq 1} \Delta_q u \Delta_{q'} v.$$

Using the quasi-orthogonality given in (5.2.7) and (5.2.8), we get the following relations.

Lemma 5.2.11. *For any tempered distributions u and v , we have*

$$\begin{aligned} \Delta_q (S_{q'-1} u \Delta_{q'} v) &= 0 && \text{if } |q - q'| \geq 5 \\ \Delta_q (S_{q'+1} u \Delta_{q'} v) &= 0 && \text{if } q' \leq q - 4. \end{aligned}$$

Lemma 5.2.11 implies the following decomposition, which we will widely use in this paper

$$\Delta_q (uv) = \sum_{|q'-q| \leq 4} \Delta_q (S_{q'-1} v \Delta_{q'} u) + \sum_{q' > q-4} \Delta_q (S_{q'+2} u \Delta_{q'} v). \quad (5.2.12)$$

Proof of Corollary 5.2.3 : Let us consider the three dimensional Navier-Stokes equations

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u = -\nabla p, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0. \end{cases}$$

Let us apply the dyadic block on such equation and let us multiply the resulting equation for $\Delta_q u$ and let us integrate in space; we deduce the following differential inequality:

$$\frac{1}{2} \frac{d}{dt} \|\Delta_q u\|_{L^2}^2 + \nu \|\Delta_q \nabla u\|_{L^2}^2 \leq |(\Delta_q (u \otimes u) | \Delta_q \nabla u)_{L^2}|. \quad (5.2.13)$$

With Bony decomposition we deduce

$$\begin{aligned} & |(\Delta_q (u \otimes u) | \Delta_q \nabla u)_{L^2}| \\ & \leq \sum_{|q-q'| \leq 4} |(S_{q'-1} u \otimes \Delta_{q'} u | \Delta_q \nabla u)_{L^2}| + \sum_{q' > q-4} |(S_{q'+2} u \otimes \Delta_{q'} u | \Delta_q \nabla u)_{L^2}| \\ & = I_{1,q} + I_{2,q}. \end{aligned}$$

Since $I_{1,q}$ and $I_{2,q}$ are symmetric we bound the term $I_{2,q}$ which involves an infinite sum and it is hence more difficult. Applying Hölder inequality we deduce

$$\begin{aligned} I_{2,q} &= \sum_{q' > q-4} |(S_{q'+2}u \otimes \Delta_{q'}u | \Delta_q \nabla u)_{L^2}| \\ &\leq \|u\|_{L^6} \|\Delta_q \nabla u\|_{L^2} \sum_{q' > q-4} \|\Delta_{q'}u\|_{L^3}. \end{aligned}$$

We use now the embedding $\dot{H}^1 \hookrightarrow L^6$ and $\dot{H}^{1/2} \hookrightarrow L^3$ and the property

$$\|\Delta_q u\|_{L^2} \sim 2^{-qs} c_q(u) \|u\|_{\dot{H}^s}, \quad (5.2.14)$$

where $(c_q)_q \in \ell^2$ to deduce that

$$I_{2,q} \lesssim 2^{-2qs} c_q \|u\|_{\dot{H}^1} \|u\|_{\dot{H}^{s+1/2}} \|u\|_{\dot{H}^{s+1}} \sum_{q' > q-4} 2^{(q-q')s} c_{q'}.$$

We notice that

$$\left(\sum_{q' > q-4} 2^{(q-q')s} c_{q'} \right)_q = ((2^{sp} 1_{p < 4}) \star c_p)_q \in \ell^1,$$

whence the sequence $(b_q)_q$ defined as

$$b_q = c_q \sum_{q' > q-4} 2^{(q-q')s} c_{q'} \in \ell^1.$$

Sobolev interpolation, Young inequality, a multiplication for 2^{2qs} , the use of (5.2.14), a parabolic absorption and Gronwall inequality transform (5.2.13) into

$$\|u\|_{\dot{H}^s}^2 + \nu \int_0^t \|\nabla u\|_{\dot{H}^s}^2 d\tau \lesssim \|u_0\|_{\dot{H}^s}^2 \exp \left\{ \frac{C}{\nu^3} \int_0^t \|u\|_{\dot{H}^1}^4 d\tau \right\}. \quad (5.2.15)$$

A standard L^2 estimate on the Navier-Stokes equations gives us the bound

$$\|u\|_{L^2}^2 + \nu \int_0^t \|\nabla u\|_{L^2}^2 d\tau \leq \|u_0\|_{L^2}^2. \quad (5.2.16)$$

Summing (5.2.15) and (5.2.16) and since for $s > 0$ the non-homogeneous Sobolev space is continuously embedded in both \dot{H}^s and L^2 , we deduce hence the inequality

$$\|u\|_{H^s}^2 + \nu \int_0^t \|\nabla u\|_{H^s}^2 d\tau \leq C \|u_0\|_{H^s}^2 \left[\exp \left\{ \frac{C}{\nu^3} \int_0^t \|u\|_{\dot{H}^1}^4 d\tau \right\} + 1 \right],$$

concluding. □

5.3 Spectral analysis of the linear operator.

In the context of singular perturbation problem a important role is determined by the dynamical effects induced by the singular operator $\mathbb{P}\mathcal{A}$, where \mathbb{P} is defined in (5.1.4) and \mathcal{A} in (5.1.3). In particular we are interested to study the effects of the perturbation induced by such operator. This is generally done with tools of Fourier analysis such as dispersive estimates on highly oscillating integrals ([138]). To perform such analysis is hence very important to understand the explicit structure of the eigenvalues of the linear operator $\varepsilon^{-1}\mathbb{P}\mathcal{A} - \mathbb{D}$, this is the scope of the present section.

We consider the linear operator

$$L_\varepsilon = \mathbb{P}\mathcal{A} - \varepsilon\mathbb{D}, \quad (5.3.1)$$

whose Fourier symbol is

$$\hat{L}_\varepsilon = \begin{pmatrix} \varepsilon\nu |\xi|^2 & 0 & 0 & -\frac{\xi_3 \xi_1}{|\xi|^2} \\ 0 & \varepsilon\nu |\xi|^2 & 0 & -\frac{\xi_3 \xi_2}{|\xi|^2} \\ 0 & 0 & \varepsilon\nu |\xi|^2 & \frac{|\xi_h|^2}{|\xi|^2} \\ 0 & 0 & -1 & \varepsilon\nu' |\xi|^2 \end{pmatrix}.$$

We study the parabolic operator L_ε instead of the hyperbolic $\mathbb{P}\mathcal{A}$ since we want to take in account the regularizing effects induced by the second-order elliptic operator $-\mathbb{D}$. This choice will become clear in Section 5.5.

The characteristic polynomial associated to \hat{L}_ε is

$$P_{\hat{L}_\varepsilon}(\lambda) = (\varepsilon\nu |\xi|^2 - \lambda)^2 \left(\lambda^2 - \lambda \varepsilon |\xi|^2 (\nu + \nu') + \frac{|\xi_h|^2}{|\xi|^2} + \varepsilon^2 \nu \nu' |\xi|^4 \right).$$

which admits four roots,

$$\lambda_0^\varepsilon(\xi) = \varepsilon\nu |\xi|^2, \quad (5.3.2)$$

which has multiplicity two and

$$\lambda_\pm^\varepsilon(\xi) = \frac{1}{2} \varepsilon (\nu + \nu') |\xi|^2 \pm i \frac{|\xi_h|}{|\xi|} S_\varepsilon(\xi), \quad (5.3.3)$$

where

$$S_\varepsilon(\xi) = \sqrt{1 - \varepsilon^2 \frac{(\nu - \nu')^2 |\xi|^6}{4 |\xi_h|^2}}.$$

Let us restrict ourselves on the localization

$$\mathcal{C}_{r,R} = \{\xi \in \mathbb{R}^3 : |\xi_h| > r, |\xi| < R\}, \quad (5.3.4)$$

we choose such localization since in $\mathcal{C}_{r,R}$ the eigenvalues $\lambda_0^\varepsilon, \lambda_\pm^\varepsilon$ are well-defined. Moreover $|\lambda_\pm^\varepsilon(\xi)| \geq \frac{r}{2R}$ (if ε is sufficiently small) for any $\xi \in \mathcal{C}_{r,R}$, hence the oscillating eigenvalues are never null in such set. It is clear that, for ε sufficiently small, on $\mathcal{C}_{r,R}$

$$|S_\varepsilon(\xi) - 1| \leq C_{r,R} \varepsilon,$$

hence from now on we shall consider implicitly $S_\varepsilon \approx 1$.

Let us evaluate the eigenvectors related to the eigenvalues λ_i^ε , relatively to the eigenvalue λ_0^ε , which has multiplicity two, we have two eigenvectors

$$e_1 = (1 \ 0 \ 0 \ 0)^\top, \quad e_2 = (0 \ 1 \ 0 \ 0)^\top.$$

These eigenvectors are not divergence-free, hence, a priori, they do not describe the evolution of solutions to equation (PBS $_\varepsilon$). In any case there is a subspace of the space $\mathbb{C}e_1 \oplus \mathbb{C}e_2$ which is composed by divergence-free vector fields, namely the space spanned by the vector

$$E_0(\xi) = \frac{1}{|\xi_h|} (-\xi_2 \ \xi_1 \ 0 \ 0)^\top. \quad (5.3.5)$$

Relatively to the eigenvalues λ_\pm^ε the following eigenvectors can be computed

$$E_\pm^\varepsilon(\xi) = \begin{pmatrix} \pm i \frac{\xi_3 \xi_1}{|\xi| |\xi_h|} \mathcal{S}_\varepsilon^\pm(\xi) \\ \pm i \frac{\xi_3 \xi_2}{|\xi| |\xi_h|} \mathcal{S}_\varepsilon^\pm(\xi) \\ \mp i \frac{\xi_h}{|\xi|} \mathcal{S}_\varepsilon^\pm(\xi) \\ 1 \end{pmatrix}, \quad (5.3.6)$$

where

$$\mathcal{S}_\varepsilon^\pm(\xi) = S_\varepsilon(\xi) \pm \frac{i}{2} \varepsilon \frac{(\nu - \nu') |\xi|^3}{|\xi_h|},$$

hence if $\xi \in \mathcal{C}_{r,R}$ and ε small $\mathcal{S}_\varepsilon^\pm \approx 1$.

An important feature of the spectral analysis of the operator L_ε is that the eigenvectors *are not orthogonal*. We will in Section 5.5 require to analyze the regularity of the propagation of some vector field along the eigendirections spanned by E_\pm^ε . this cannot hence be done by a standard application of the triangular inequality since, as we will see below, the projections onto the eigenspaces are defined by suitable Fourier multipliers, hence a more thorough analysis is required.

Let us now consider a solenoidal vector field $V = (V^1, V^2, V^3, V^4)$ which belongs to the space

$$\mathcal{X} = \bigoplus_{i=0,\pm} \mathbb{C} E_i^\varepsilon,$$

indeed

$$V = \mathcal{F}^{-1} \left(\sum_{i=0,\pm} k_{i,\varepsilon}(V) E_i^\varepsilon \right),$$

where the elements $k_{i,\varepsilon}$, $i = 0, \pm$ are suitable forms which act on the space of solenoidal vector fields and they describe the magnitude of the projection of \hat{V} onto the eigenspace $\mathbb{C} E_i^\varepsilon$. We can hence define the projections of a divergence-free vector field in \mathcal{X} onto the eigenspace spanned by E_i^ε as

$$\mathbb{P}_{i,\varepsilon}(V) = \mathcal{F}^{-1} (k_{i,\varepsilon}(V) E_i^\varepsilon), \quad i = 0, \pm. \quad (5.3.7)$$

The definition of $\mathbb{P}_{i,\varepsilon}$ does not give any insight of the regularity of the element $\mathbb{P}_{i,\varepsilon}(V)$ w.r.t. the regularity of V . We expect that the form k_i acts as a Fourier multiplier of a suitable degree. We prove in fact that, as long as we restrict ourselves in the set $\mathcal{C}_{r,R}$, the map $\hat{V} \mapsto k_{i,\varepsilon}(V) E_i^\varepsilon$ acts as a multiplication for a constant in terms of L^2 regularity:

Lemma 5.3.1. *Let $V \in \mathcal{X}$ a solenoidal vector field such that $\text{supp}(\hat{V}) \subset \mathcal{C}_{r,R}$, then for $i = 0, \pm$*

$$\|\mathbb{P}_{i,\varepsilon}(V)\|_{L^2(\mathbb{R}^3)} \leq C_{r,R} \|V\|_{L^2(\mathbb{R}^3)}.$$

Proof. This is a problem of linear algebra. Let us consider the following basis of \mathbb{C}^4 (in the Fourier space)

$$\mathcal{B} = \{e_1, E_0, E_+^\varepsilon, E_-^\varepsilon\},$$

and the canonical basis

$$\mathcal{B}_{\text{can}} = \{e_j\}_{j=1}^4.$$

The matrix \hat{L}_ε is indeed diagonalizable, hence there exists an invertible matrix Q such that

$$Q \hat{L}_\varepsilon(\xi) Q^{-1} = \text{diag} \{ \lambda_0^\varepsilon(\xi), \lambda_0^\varepsilon(\xi), \lambda_+^\varepsilon(\xi), \lambda_-^\varepsilon(\xi) \},$$

the matrix Q is the change of base matrix from the base \mathcal{B}_{can} to the base \mathcal{B} and, given the explicit expression of the eigenvectors in (5.3.5), (5.3.6) it assumes the form

$$Q = \begin{pmatrix} 1 & \frac{-\xi_2}{|\xi_h|} & i \frac{\xi_3 \xi_1}{|\xi| |\xi_h|} \mathcal{S}_\varepsilon^\pm(\xi) & -i \frac{\xi_3 \xi_1}{|\xi| |\xi_h|} \mathcal{S}_\varepsilon^\pm(\xi) \\ 0 & \frac{\xi_1}{|\xi_h|} & i \frac{\xi_3 \xi_2}{|\xi| |\xi_h|} \mathcal{S}_\varepsilon^\pm(\xi) & -i \frac{\xi_3 \xi_2}{|\xi| |\xi_h|} \mathcal{S}_\varepsilon^\pm(\xi) \\ 0 & 0 & -i \frac{|\xi_h|}{|\xi|} \mathcal{S}_\varepsilon^\pm(\xi) & i \frac{|\xi_h|}{|\xi|} \mathcal{S}_\varepsilon^\pm(\xi) \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Let us note that the first column of Q is $(1, 0, 0, 0)^\top$, this is motivated by the fact that we completed the basis \mathcal{B} with the vector e_1 in order to obtain a complete basis of \mathbb{C}^4 .

The matrix Q performs the following transformation,

$$Q \begin{pmatrix} 0 \\ k_{0,\varepsilon} \\ k_{+,\varepsilon} \\ k_{-,\varepsilon} \end{pmatrix} = \begin{pmatrix} \hat{V}^1 \\ \hat{V}^2 \\ \hat{V}^3 \\ \hat{V}^4 \end{pmatrix},$$

we deduce hence that the element

$$\begin{pmatrix} 0 \\ k_{0,\varepsilon} \\ k_{+,\varepsilon} \\ k_{-,\varepsilon} \end{pmatrix} = Q^{-1} \begin{pmatrix} \hat{V}^1 \\ \hat{V}^2 \\ \hat{V}^3 \\ \hat{V}^4 \end{pmatrix}, \quad (5.3.8)$$

gives the expression of the k_i 's in terms of the variables \hat{V}_i 's multiplied by suitable Fourier multipliers determined by the inverse matrix Q^{-1} . Whence it suffices to compute the explicit

expression of the matrix Q^{-1} to solve the linear system above. The matrix Q^{-1} assumes the form

$$Q^{-1} = \begin{pmatrix} 1 & \frac{\xi_2}{\xi_1} & \frac{\xi_3}{\xi_1} & 0 \\ 0 & \frac{|\xi_h|}{\xi_1} & \frac{\xi_2 \xi_3}{|\xi_h| \xi_1} & 0 \\ 0 & 0 & -i \frac{|\xi|}{2S_\varepsilon^\pm(\xi)} & \frac{1}{2} \\ 0 & 0 & +i \frac{|\xi|}{2S_\varepsilon^\pm(\xi)} & \frac{1}{2} \end{pmatrix}, \quad (5.3.9)$$

whence it is clear that, since \hat{V} is supported in $C_{r,R}$:

$$\begin{aligned} \left| \begin{pmatrix} 0 \\ k_{0,\varepsilon} \\ k_{+,\varepsilon} \\ k_{-,\varepsilon} \end{pmatrix} \right| &\leq |Q^{-1}| |\hat{V}|, \\ &\leq C_{r,R} |\hat{V}|. \end{aligned}$$

The claim follows applying Plancherel theorem. \square

Lemma 5.3.1 gives hence a complete answer regarding the regularity of the projectors $\mathbb{P}_{i,\varepsilon}$, nonetheless we did not compute explicitly their form. Regarding the first two equations of the system (5.3.8) we can deduce the following explicit equations thanks to the explicit expression of Q^{-1} given in (5.3.9):

$$\begin{aligned} 0 &= \xi_1 \hat{V}^1 + \xi_2 \hat{V}^2 + \xi_3 \hat{V}^3, \\ k_0 &= \frac{1}{|\xi_h|} \left(-\xi_2 \hat{V}^1 + \xi_1 \hat{V}^2 \right), \end{aligned}$$

hence we can compute explicit expression of the projector $\mathbb{P}_0 V = \mathcal{F}^{-1} (k_0(V) E_0)$, which in particular assumes the form (in the Fourier variables):

$$\begin{aligned} \mathcal{F}(\mathbb{P}_{0,\varepsilon} V) &= k_{0,\varepsilon}(V) E_0, \\ &= \frac{1}{|\xi_h|^2} \begin{pmatrix} -\xi_2 \\ +\xi_1 \\ 0 \\ 0 \end{pmatrix} \left(-\xi_2 \hat{V}^1 + \xi_1 \hat{V}^2 \right). \end{aligned}$$

Whence we can define the projector \mathbb{P}_0 (which does not depend any more on the parameter ε) which maps a solenoidal vector field V onto $\mathbb{C}E_0$ via the following pseudo-differential operator of order zero

$$\begin{aligned} \mathbb{P}_0 V &= \begin{pmatrix} -(-\Delta_h)^{-1} \partial_2 \operatorname{curl}_h V \\ +(-\Delta_h)^{-1} \partial_1 \operatorname{curl}_h V \\ 0 \\ 0 \end{pmatrix} \\ &= (-\Delta_h)^{-1} \begin{pmatrix} \partial_2^2 & -\partial_1 \partial_2 & 0 & 0 \\ -\partial_1 \partial_2 & \partial_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} V, \end{aligned} \quad (5.3.10)$$

where the operator curl_h is defined as $\text{curl}_h V = -\partial_2 V^1 + \partial_1 V^2$.

The space $\mathbb{C}E_0$ shall be denoted as *non-oscillating* subspace, whereas the space $\mathbb{C}E_+^\varepsilon \oplus \mathbb{C}E_-^\varepsilon$ shall be denoted as *oscillating* subspace. This choice of lexicon can easily be justified: let us consider the following linear system,

$$\begin{cases} \partial_t W_L + \frac{1}{\varepsilon} L_\varepsilon W_L = 0, \\ W_L|_{t=0} = W_{L,0}. \end{cases}$$

The unique solution of such system can be written as

$$W_L(t) = e^{\frac{t}{\varepsilon} L_\varepsilon} W_{L,0}.$$

Respectively hence the projection of W_L onto the subspaces $\mathbb{C}E_0, \mathbb{C}E_\pm^\varepsilon$ is

$$\begin{aligned} \mathbb{P}_0 W_L(t) &= \mathcal{F}^{-1} \left(e^{-\nu t |\xi|^2} \widehat{\mathbb{P}_0 W_{L,0}}(\xi) \right), \\ \mathbb{P}_{\pm, \varepsilon} W_L(t) &= \mathcal{F}^{-1} \left(e^{-i \frac{t}{\varepsilon} \lambda_\varepsilon^\pm(\xi)} \widehat{\mathbb{P}_{\pm, \varepsilon} W_{L,0}}(\xi) \right). \end{aligned}$$

We can immediately see hence that the elements $\mathbb{P}_0 W_L$ and $\mathbb{P}_{\pm, \varepsilon} W_L$ have two qualitatively very different behaviors: the former has a purely parabolic decay-in-time, while the latter is described by an oscillating integral.

5.4 Global well posedness of the limit system.

A consistent part of Theorem 5.2.4 deals with the convergence of solutions of (PBS_ε) in the regime $\varepsilon \rightarrow 0$ to a certain limit function.

We expect hence that once we restrict ourselves on $\mathbb{C}E^0$, no dispersive effect occur due to the absence of the singular perturbation, determining hence a candidate for the limit model we look for.

5.4.1 Formal derivation of the limit system.

An important step as long as concerns singular perturbation problems is to deduce formally a limit system to whom (PBS_ε) converges. Several works on geophysical fluids such as [42], [28] or [68] suggest that the solutions of (PBS_ε) converge (in a sense which we do not specify at the moment) to an element belonging to the nonoscillatory space $\mathbb{C}E^0$.

The next result is a direct deduction of Theorem 5.2.1 (see for instance [77, Corollary 2.1]):

Lemma 5.4.1. *Let U_0 be in $L^2(\mathbb{R}^3)$, and let U^ε be a weak solution of (PBS_ε) , there exists a $U^* \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+; \dot{H}^1(\mathbb{R}^3))$ and a subsequence $\varepsilon_j \xrightarrow{j \rightarrow \infty} 0$ such that*

$$U^{\varepsilon_j} \rightharpoonup U^* \text{ weakly in } L_{\text{loc}}^2(\mathbb{R}_+; L_{\text{loc}}^2(\mathbb{R}^3)) \text{ as } j \rightarrow \infty.$$

Taking a formal limit for $\varepsilon \rightarrow 0$ in (PBS_ε) and supposing that $(U^\varepsilon, \Phi^\varepsilon) \rightarrow (U^*, \Phi^*)$ the following balance

$$\begin{aligned} u^{3,*} &= 0, \\ \rho^* &= \partial_3 \Phi^*, \end{aligned} \tag{5.4.1}$$

has to take place by simple comparison of magnitude in (PBS_ε) in the limit $\varepsilon \rightarrow 0$.

Let us consider now the subsequence $(\varepsilon_j)_j$ identified in Lemma 5.4.1. With a standard argument of cancellation of the pressure on (PBS_ε) we can deduce that

$$-\Delta \Phi^{\varepsilon_j} = -\partial_3 \rho^{\varepsilon_j} + \varepsilon_j \operatorname{div} \operatorname{div} (u^{\varepsilon_j} \otimes u^{\varepsilon_j}).$$

Since

$$\|\operatorname{div} \operatorname{div} (u^\varepsilon \otimes u^\varepsilon)\|_{L^2_{\text{loc}}(\mathbb{R}_+; H^{-3})} \leq \|u^\varepsilon\|_{L^\infty_{\text{loc}}(\mathbb{R}_+; L^2)} \|u^\varepsilon\|_{L^2_{\text{loc}}(\mathbb{R}_+; H^1)} < \infty, \quad \forall \varepsilon > 0,$$

we deduce that $\varepsilon \operatorname{div} \operatorname{div} (u^\varepsilon \otimes u^\varepsilon)$ is an $\mathcal{O}(\varepsilon)$ function in the $L^2_{\text{loc}}(\mathbb{R}_+; H^{-3})$ topology, hence since $\rho^{\varepsilon_j} \rightharpoonup \rho^*$ in $L^2_{\text{loc}}(\mathbb{R}_+; L^2)$ for the same subsequence $(\varepsilon_j)_j$ we deduce

$$-\Delta \Phi^{\varepsilon_j} \rightarrow -\Delta \Phi^* = -\partial_3 \rho^*,$$

in the sense of distributions. The above relation together with (5.4.1) imply that

$$-\Delta \rho^* = -\partial_3^2 \rho^* \Rightarrow -\Delta_h \rho^* = 0.$$

But $-\Delta_h \rho^* = 0$ in the whole space implies that $\rho^* = \rho^*(x_3)$, and hence Lemma 5.4.1 allows us to state that $\rho^* \equiv 0$ in L^2 .

We hence deduced (formally) until now that

$$(u^{h,\varepsilon}, u^{3,\varepsilon}, \rho^\varepsilon, \Phi^\varepsilon) \rightharpoonup (u^{h,*}, 0, 0, \Phi^*),$$

we want to understand (heuristically) which equation is satisfied by the limit function $u^{h,*}$.

Next let us consider some very specific test functions $\phi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^3)$ such that $\phi = (\phi_1, \phi_2, 0, 0)$ and

$$\phi_1 = -\partial_2 \Delta_h^{-1} \Psi, \quad \phi_2 = \partial_1 \Delta_h^{-1} \Psi,$$

for some potential Ψ . This in particular implies that $\operatorname{div}_h \phi_h = 0$, these hypothesis have been imposed so that

$$\hat{\phi}(t) \in \mathbb{C}E_0, \quad \forall t > 0.$$

Let us suppose moreover that the weak convergence sated in Lemma 5.4.1 is strong enough so that

$$u^{\varepsilon_j} \otimes u^{\varepsilon_j} \rightharpoonup u^* \otimes u^*. \tag{5.4.2}$$

Obviously this is not the case, but an educated guess which motivated the development of the present work.

Testing the equation (PBS_ε) against functions of such form we deduce that (here we denote as $u^{h,\varepsilon}$ the horizontal components of U^ε)

$$\langle u^{h,\varepsilon_j} | \partial_t \phi_h \rangle - \langle u_0^h | \psi(0) \rangle + \langle u^{h,\varepsilon_j} \otimes u^{h,\varepsilon_j} | \nabla_h \phi_h \rangle + \langle u^{3,\varepsilon_j} u^{h,\varepsilon_j} | \partial_3 \phi_h \rangle + \langle u^{h,\varepsilon_j} | \Delta \phi_h \rangle = 0,$$

Let us take now formally the limit as $\varepsilon_j \rightarrow 0$, justified by Lemma 5.4.1. First of all we remark, thanks to the balance deduced in (5.4.1), and the hypothesis (5.4.2):

$$u^{3,\varepsilon_j} \rightharpoonup 0 \Rightarrow \langle u^{3,\varepsilon_j} u^{h,\varepsilon_j} | \partial_3 \phi_h \rangle \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Whence we deduce that, at least in this restricted distributional sense, the limit function describing the evolution of the horizontal components shall satisfy the system

$$\begin{cases} \partial_t u^{h,*} + u^{h,*} \cdot \nabla_h u^{h,*} - \nu \Delta u^{h,*} = -\nabla_h \Phi^*, \\ \operatorname{div}_h u^{h,*} = 0. \end{cases}$$

5.4.2 Detailed study of the limit system.

Section 5.4.1 motivates hence the study of the 2-dimensional, incompressible, stratified Navier-Stokes system

$$\begin{cases} \partial_t \bar{u}^h(x_h, x_3) + \bar{u}^h(x_h, x_3) \cdot \nabla_h \bar{u}^h(x_h, x_3) - \nu \Delta \bar{u}^h(x_h, x_3) = -\nabla_h \bar{p}(x_h, x_3) \\ \operatorname{div}_h \bar{u}^h(x_h, x_3) = 0, \\ \bar{u}^h(x_h, x_3)|_{t=0} = \mathbb{P}_0 U_0(x_h, x_3) = \bar{u}_0^h(x_h, x_3). \end{cases} \quad (5.4.3)$$

The operator \mathbb{P}_0 is defined in (5.3.10). The velocity field \bar{u}^h is endowed with a 2d-like vorticity

$$\omega^h(x_h, x_3) = -\partial_2 \bar{u}^{h,1}(x_h, x_3) + \partial_1 \bar{u}^{h,2}(x_h, x_3),$$

which, as well as for the two-dimensional Navier-Stokes equation satisfies the transport-diffusion equation

$$\begin{cases} \partial_t \omega^h(x_h, x_3) + \bar{u}^h(x_h, x_3) \cdot \nabla_h \omega^h(x_h, x_3) - \nu \Delta \omega^h(x_h, x_3) = 0 \\ \omega^h(x_h, x_3)|_{t=0} = \omega_0^h(x_h, x_3). \end{cases} \quad (5.4.4)$$

We can recover \bar{u}^h from ω^h via a 2D-like Biot-Savart law

$$\bar{u}^h(x_h, x_3) = \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix} \Delta_h^{-1} \omega^h(x_h, x_3),$$

as it was already outlined and justified in the previous section deducing the explicit expression of the projector \mathbb{P}_0 in (5.3.10).

Let us make a couple of remarks on the system (5.4.3), the unknown \bar{u}^h of (5.4.3) depends on all three space variables and is time-dependent, i.e. $\bar{u}^h = \bar{u}^h(t, x) = \bar{u}^h(t, x_h, x_3)$. The equations (5.4.3) represents hence a Navier-Stokes system in the horizontal directions x_h , while it is a diffusive equation along the vertical direction x_3 .

The results stated in the following lemmas are classical, hence the proof is omitted.

Lemma 5.4.2. *Let $\bar{u}_0^h \in L^2(\mathbb{R}^3)$ and $\omega_0^h \in L^2(\mathbb{R}^3)$. Then there exists respectively a weak solution \bar{u}^h, ω^h of (5.4.3) and (5.4.4) such that*

$$\bar{u}^h, \omega^h \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+; \dot{H}^1(\mathbb{R}^3)),$$

and, for each $t > 0$, the following bounds hold

$$\begin{aligned} \|\bar{u}^h(t)\|_{L^2(\mathbb{R}^3)}^2 + 2\nu \int_0^t \|\nabla \bar{u}^h(\tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau &\leq \|\bar{u}_0^h\|_{L^2(\mathbb{R}^3)}^2, \\ \|\omega^h(t)\|_{L^2(\mathbb{R}^3)}^2 + 2\nu \int_0^t \|\nabla \omega^h(\tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau &\leq \|\omega_0^h\|_{L^2(\mathbb{R}^3)}^2. \end{aligned}$$

Lemma 5.4.3. *Let $U = U(x)$ be in $L^2(\mathbb{R}) \cap \dot{H}^1(\mathbb{R})$, then $U \in L^\infty(\mathbb{R})$ and*

$$\|U\|_{L^\infty(\mathbb{R})} \leq C \|U\|_{L^2(\mathbb{R})}^{1/2} \|U'\|_{L^2(\mathbb{R})}^{1/2}.$$

The results in Lemma 5.4.2 and 5.4.3 are classical results hence we do not prove them here. This is all we require in order to prove the following lemma, which is the main result which will allow us subsequently to prove that (5.4.3) is globally well posed in $\dot{H}^s(\mathbb{R}^3)$, $s \geq 0$:

Lemma 5.4.4. *Let \bar{u}_0^h, ω_0^h satisfy the hypotheses of Lemma 5.4.2, then*

$$\begin{aligned} \bar{u}^h &\in L^4(\mathbb{R}_+; L^4(\mathbb{R}_h^2; L^\infty(\mathbb{R}_v))) = L^4(\mathbb{R}_+; L_h^2(L_v^\infty)), \\ \bar{u}^h &\in L^4(\mathbb{R}_+; L^\infty(\mathbb{R}_v; L^4(\mathbb{R}_h^2))) = L^4(\mathbb{R}_+; L_v^\infty(L_h^4)), \end{aligned}$$

and for each $t > 0$ the following bounds hold

$$\int_0^t \|\bar{u}^h(\tau)\|_{L_v^\infty(L_h^4)}^4 d\tau \leq \int_0^t \|\bar{u}^h(\tau)\|_{L_h^4(L_v^\infty)}^4 d\tau \leq \frac{CK^2}{\nu} \left(\|\bar{u}_0^h\|_{L^2(\mathbb{R}^3)}^4 + \|\omega_0^h\|_{L^2(\mathbb{R}^3)}^4 \right).$$

Proof. Let us start considering the value $\|\bar{u}^h\|_{L_h^4(L_v^\infty)}^4$, applying Lemma 5.4.3 we deduce

$$\|\bar{u}^h\|_{L_h^4(L_v^\infty)}^4 \leq C \|\bar{u}^h\|_{L_h^4(L_v^2)}^2 \|\partial_3 \bar{u}^h\|_{L_h^4(L_v^2)}^2.$$

By use of (5.1.6) and a Gagliardo-Nirenberg interpolation inequality we deduce

$$\begin{aligned} \|\bar{u}^h\|_{L_h^4(L_v^2)}^2 &\leq \|\bar{u}^h\|_{L_v^2(L_h^4)}^2 \\ &\leq C \|\bar{u}^h\|_{L^2(\mathbb{R}^3)} \|\nabla_h \bar{u}^h\|_{L^2(\mathbb{R}^3)}, \\ \|\partial_3 \bar{u}^h\|_{L_h^4(L_v^2)}^2 &\leq \|\partial_3 \bar{u}^h\|_{L_v^2(L_h^4)}^2 \\ &\leq C \|\partial_3 \bar{u}^h\|_{L^2(\mathbb{R}^3)} \|\partial_3 \nabla_h \bar{u}^h\|_{L^2(\mathbb{R}^3)}, \end{aligned}$$

whence we deduce

$$\begin{aligned} \|\bar{u}^h\|_{L_h^4(L_v^\infty)}^4 &\leq C \|\bar{u}^h\|_{L^2(\mathbb{R}^3)} \|\nabla_h \bar{u}^h\|_{L^2(\mathbb{R}^3)} \|\partial_3 \bar{u}^h\|_{L^2(\mathbb{R}^3)} \|\partial_3 \nabla_h \bar{u}^h\|_{L^2(\mathbb{R}^3)}, \\ &\leq CK^2 \|\bar{u}^h\|_{L^2(\mathbb{R}^3)} \|\omega^h\|_{L^2(\mathbb{R}^3)} \|\partial_3 \bar{u}^h\|_{L^2(\mathbb{R}^3)} \|\partial_3 \omega^h\|_{L^2(\mathbb{R}^3)}, \\ &\leq CK^2 \|\bar{u}^h\|_{L^2(\mathbb{R}^3)} \|\omega^h\|_{L^2(\mathbb{R}^3)} \|\nabla \bar{u}^h\|_{L^2(\mathbb{R}^3)} \|\nabla \omega^h\|_{L^2(\mathbb{R}^3)} \end{aligned} \tag{5.4.5}$$

where in the second inequality we used the fact that the map $\omega^h \mapsto \nabla_h \bar{u}^h$ is a Calderon-Zygmund application of norm K in $L^2(\mathbb{R}_h^2)$. Integrating in time (5.4.5) using Young inequality and the results of Lemma 5.4.2 we deduce the inequality

$$\int_0^t \|\bar{u}^h(\tau)\|_{L_h^4(L_v^\infty)}^4 d\tau \leq \frac{CK^2}{\nu} \left(\|\bar{u}_0^h\|_{L^2(\mathbb{R}^3)}^4 + \|\omega_0^h\|_{L^2(\mathbb{R}^3)}^4 \right).$$

To complete the proof it suffices hence to apply (5.1.6). \square

Lemma 5.4.4 is the cornerstone of the proof of the propagation of the isotropic Sobolev regularity, which is formalized in the following proposition

Proposition 5.4.5. *Let $\bar{u}_0^h \in H^s(\mathbb{R}^3)$, $s > 0$ and $\omega_0^h \in L^2(\mathbb{R}^3)$, then the weak solution \bar{u}^h of (5.4.3) which exists thanks to Lemma 5.4.2 belongs to the space*

$$\bar{u}^h \in L^\infty(\mathbb{R}_+; H^s(\mathbb{R}^3)), \quad \nabla \bar{u}^h \in L^2(\mathbb{R}_+; H^s(\mathbb{R}^3)),$$

and for each $t > 0$ the following bound holds

$$\begin{aligned} \|\bar{u}^h(t)\|_{H^s(\mathbb{R}^3)}^2 + \nu \int_0^t \|\nabla \bar{u}^h(\tau)\|_{H^s(\mathbb{R}^3)}^2 d\tau \\ \leq C \|\bar{u}_0^h\|_{H^s(\mathbb{R}^3)}^2 \exp \left\{ \frac{CK^2}{\nu} \left(\|\bar{u}_0^h\|_{L^2(\mathbb{R}^3)}^4 + \|\omega_0^h\|_{L^2(\mathbb{R}^3)}^4 \right) \right\}. \end{aligned} \quad (5.4.6)$$

Proof. Let us apply the operator Δ_q to the equation (5.4.3) and multiply it by $\Delta_q \bar{u}^h$ and integrate in space, we deduce

$$\frac{1}{2} \frac{d}{dt} \|\Delta_q \bar{u}^h\|_{L^2(\mathbb{R}^3)}^2 + \nu \|\Delta_q \nabla \bar{u}^h\|_{L^2(\mathbb{R}^3)}^2 \leq \left| (\Delta_q (\bar{u}^h \cdot \nabla_h \bar{u}^h) | \Delta_q \bar{u}^h)_{L^2(\mathbb{R}^3)} \right|. \quad (5.4.7)$$

Indeed, since $\operatorname{div}_h \bar{u}^h = 0$ and integrating by parts,

$$\left| (\Delta_q (\bar{u}^h \cdot \nabla_h \bar{u}^h) | \Delta_q \bar{u}^h)_{L^2(\mathbb{R}^3)} \right| = \left| (\Delta_q (\bar{u}^h \otimes \bar{u}^h) | \Delta_q \nabla_h \bar{u}^h)_{L^2(\mathbb{R}^3)} \right|.$$

Applying Bony decomposition we deduce

$$\begin{aligned} \left| (\Delta_q (\bar{u}^h \otimes \bar{u}^h) | \Delta_q \nabla_h \bar{u}^h)_{L^2(\mathbb{R}^3)} \right| \\ \leq \sum_{|q-q'|\leq 4} \left| (\Delta_q (S_{q'-1} \bar{u}^h \otimes \Delta_{q'} \bar{u}^h) | \Delta_q \nabla_h \bar{u}^h)_{L^2(\mathbb{R}^3)} \right| \\ + \sum_{q'>q-4} \left| (\Delta_q (\Delta_{q'} \bar{u}^h \otimes S_{q'+2} \bar{u}^h) | \Delta_q \nabla_h \bar{u}^h)_{L^2(\mathbb{R}^3)} \right| = I_{1,q} + I_{2,q}. \end{aligned}$$

Since the operators Δ_q, S_q map continuously any $L^p(\mathbb{R}^3)$ space to itself and by Hölder inequality we deduce

$$\begin{aligned} I_{1,q} &\leq C \|\bar{u}^h\|_{L_v^\infty(L_h^4)} \sum_{|q-q'|\leq 4} \|\Delta_{q'} \bar{u}^h\|_{L_v^2(L_h^4)} \|\Delta_q \nabla_h \bar{u}^h\|_{L^2(\mathbb{R}^3)}, \\ &\leq C \|\bar{u}^h\|_{L_v^\infty(L_h^4)} \sum_{|q-q'|\leq 4} \|\Delta_{q'} \bar{u}^h\|_{L^2(\mathbb{R}^3)}^{1/2} \|\Delta_q \nabla_h \bar{u}^h\|_{L^2(\mathbb{R}^3)}^{3/2}, \\ &\leq C b_q 2^{-2qs} \|\bar{u}^h\|_{L_v^\infty(L_h^4)} \|\bar{u}^h\|_{H^s(\mathbb{R}^3)}^{1/2} \|\nabla_h \bar{u}^h\|_{H^s(\mathbb{R}^3)}^{3/2}. \end{aligned} \quad (5.4.8)$$

In the second inequality we used a Gagliardo-Nirenberg inequality and in the third one the regularity properties of dyadic blocks. The sequence $(b_q)_q \in \ell^1(\mathbb{Z})$. For the term $I_{2,q}$ we can apply the very same procedure to deduce the same bound

$$I_{2,q} \leq C b_q 2^{-2qs} \|\bar{u}^h\|_{L_v^\infty(L_h^4)} \|\bar{u}^h\|_{H^s(\mathbb{R}^3)}^{1/2} \|\nabla_h \bar{u}^h\|_{H^s(\mathbb{R}^3)}^{3/2}, \quad (5.4.9)$$

but in this case the sequence $(b_q)_q$, which is ℓ^1 , assumes the convolution form

$$b_q = c_q \sum_{q' > q-4} 2^{-(q'-q)s} c_{q'}.$$

Thanks to (5.4.8), (5.4.9) we hence deduced that

$$\left| (\Delta_q (\bar{u}^h \cdot \nabla_h \bar{u}^h) | \Delta_q \bar{u}^h)_{L^2(\mathbb{R}^3)} \right| \leq C b_q 2^{-2qs} \|\bar{u}^h\|_{L_v^\infty(L_h^4)} \|\bar{u}^h\|_{H^s(\mathbb{R}^3)}^{1/2} \|\nabla_h \bar{u}^h\|_{H^s(\mathbb{R}^3)}^{3/2}. \quad (5.4.10)$$

With the bound (5.4.10) applied to (5.4.7) we deduce

$$\frac{1}{2} \frac{d}{dt} \|\Delta_q \bar{u}^h\|_{L^2(\mathbb{R}^3)}^2 + \nu \|\Delta_q \nabla \bar{u}^h\|_{L^2(\mathbb{R}^3)}^2 \leq C b_q 2^{-2qs} \|\bar{u}^h\|_{L_v^\infty(L_h^4)} \|\bar{u}^h\|_{H^s(\mathbb{R}^3)}^{1/2} \|\nabla_h \bar{u}^h\|_{H^s(\mathbb{R}^3)}^{3/2}, \quad (5.4.11)$$

hence, multiplying (5.4.11) by 2^{2qs} , summing on $q \in \mathbb{Z}$ and using the convexity inequality $ab \leq Ca^4 + \frac{\nu}{2}b^{4/3}$ we deduce the bound

$$\frac{d}{dt} \|\bar{u}^h\|_{H^s(\mathbb{R}^3)}^2 + \nu \|\nabla \bar{u}^h\|_{H^s(\mathbb{R}^3)}^2 \leq \|\bar{u}^h\|_{L_v^\infty(L_h^4)}^4 \|\bar{u}^h\|_{H^s(\mathbb{R}^3)}^2. \quad (5.4.12)$$

it suffices hence to apply Gronwall inequality on (5.4.12) and consider the result of Lemma 5.4.4 to deduce the bound (5.4.6). \square

The following result is a direct deduction of the above proposition:

Corollary 5.4.6. *The solutions of (5.4.3) are $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ -stable if the initial data belong to the space $L^2(\mathbb{R}^3) \cap \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$.*

Proof. Let us consider $\bar{u}_{1,0}^h, \bar{u}_{2,0}^h \in L^2(\mathbb{R}^3) \cap \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, and let us set \bar{u}_1^h, \bar{u}_2^h respectively the solutions of

$$\begin{cases} \partial_t \bar{u}_1^h + \bar{u}_1^h \cdot \nabla_h \bar{u}_1^h - \nu \Delta \bar{u}_1^h = -\nabla_h \bar{p}_1, \\ \operatorname{div}_h \bar{u}_1^h = 0, \\ \bar{u}_1^h|_{t=0} = \bar{u}_{1,0}^h, \\ \partial_t \bar{u}_2^h + \bar{u}_2^h \cdot \nabla_h \bar{u}_2^h - \nu \Delta \bar{u}_2^h = -\nabla_h \bar{p}_2, \\ \operatorname{div}_h \bar{u}_2^h = 0, \\ \bar{u}_2^h|_{t=0} = \bar{u}_{2,0}^h. \end{cases}$$

Let us set now

$$\delta \bar{u}^h = \bar{u}_1^h - \bar{u}_2^h, \quad \delta \bar{p} = \bar{p}_1 - \bar{p}_2,$$

the function $\delta\bar{u}^h$ solves the equation

$$\begin{cases} \partial_t \delta\bar{u}^h - \nu \Delta \delta\bar{u}^h = -\nabla_h \delta\bar{p} - 2\operatorname{div}_h (\delta\bar{u}^h \otimes (\bar{u}_1^h + \bar{u}_2^h)), \\ \operatorname{div}_h \delta\bar{u}^h = 0, \\ \delta\bar{u}^h|_{t=0} = \bar{u}_{1,0}^h - \bar{u}_{2,0}^h. \end{cases} \quad (5.4.13)$$

Performing a $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ estimate on the system (5.4.13) we deduce the following differential inequality:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\delta\bar{u}^h(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + \nu \|\nabla \delta\bar{u}^h(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \\ \leq C \left| (\operatorname{div}_h (\delta\bar{u}^h(t) \otimes (\bar{u}_1^h(t) + \bar{u}_2^h(t))) | \delta\bar{u}^h(t) \right|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}. \end{aligned}$$

Applying an integration by parts, Hölder inequality, Sobolev product rules and an interpolation of Sobolev norms we can deduce:

$$\begin{aligned} \left| (\operatorname{div}_h (\delta\bar{u}^h \otimes (\bar{u}_1^h + \bar{u}_2^h)) | \delta\bar{u}^h \right|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \\ \leq C \left(\|\bar{u}_1^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{1/2} \|\nabla \bar{u}_1^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{1/2} + \|\bar{u}_2^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{1/2} \|\nabla \bar{u}_2^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{1/2} \right) \\ \times \|\delta\bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{1/2} \|\nabla \delta\bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{3/2}, \end{aligned}$$

whence applying the convexity inequality $ab \leq \frac{\nu}{2} a^{4/3} + \frac{C}{\nu^3} b^4$, a parabolic absorption and a Gronwall inequality we deduce

$$\begin{aligned} \|\delta\bar{u}^h(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + \nu \int_0^t \|\nabla \delta\bar{u}^h(\tau)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 d\tau \\ \leq \|\delta\bar{u}_0^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \exp \left\{ \frac{C}{\nu^3} \int_0^t \|\bar{u}_1^h(\tau)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \|\nabla \bar{u}_1^h(\tau)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 d\tau \right\} \\ \times \exp \left\{ \frac{C}{\nu^3} \int_0^t \|\bar{u}_2^h(\tau)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \|\nabla \bar{u}_2^h(\tau)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 d\tau \right\}, \end{aligned}$$

concluding. □

5.5 Dispersive properties.

We recall that, for $0 < r < R$, in (5.3.4), we defined

$$\mathcal{C}_{r,R} = \{\xi \in \mathbb{R}_\xi^3 : |\xi_h| > r, |\xi| < R\}.$$

Let ψ a C^∞ -function from \mathbb{R}^3 to \mathbb{R} such that

$$\chi(\xi) = \begin{cases} 1 & \text{if } 0 \leq |\xi| \leq 1 \\ 0 & \text{if } |\xi| \geq 2 \end{cases}$$

and $\Psi_{r,R} : \mathbb{R}^3 \rightarrow \mathbb{R}$ the following frequency cut-off function

$$\Psi_{r,R}(\xi) = \chi\left(\frac{|\xi|}{R}\right) \left[1 - \chi\left(\frac{|\xi_h|}{r}\right)\right]. \quad (5.5.1)$$

Then, we have $\Psi_{r,R} \in \mathcal{D}(\mathbb{R}^3)$, $\text{supp } \Psi_{r,R} \subset \mathcal{C}_{\frac{r}{2}, 2R}$ and $\Psi_{r,R} \equiv 1$ on $\mathcal{C}_{r,R}$. Indeed the operator $\Psi_{r,R}$ maps any tempered distribution f to

$$\Psi_{r,R}(D)f = \mathcal{F}^{-1} \left(\Psi_{r,R}(\xi) \widehat{f}(\xi) \right), \quad (5.5.2)$$

with this in mind we want to study the following linear system

$$\begin{cases} \partial_t W_{r,R}^\varepsilon + \frac{1}{\varepsilon} L_\varepsilon W_{r,R}^\varepsilon = -\Psi_{r,R}(D) (\mathbb{P}_{+,\varepsilon} + \mathbb{P}_{-,\varepsilon}) \Lambda(\bar{u}^h), \\ \text{div } w_{r,R}^\varepsilon = 0, \\ W_{r,R}^\varepsilon|_{t=0} = \Psi_{r,R}(D) (\mathbb{P}_{+,\varepsilon} + \mathbb{P}_{-,\varepsilon}) U_0, \end{cases} \quad (5.5.3)$$

where $\mathbb{P}_{\pm,\varepsilon}$ is the projection respectively onto the space $\mathbb{C}E_\pm^\varepsilon$ defined in (5.3.7) and L_ε is defined in (5.3.1). We stress out the fact that Lemma 5.3.1 implies that the maps $\mathbb{P}_{i,\varepsilon}$ are bounded operators onto L^2 as long as we consider functions localized on the set $\mathcal{C}_{r,R}$.

The forcing term Λ appearing on the right-hand-side of (5.5.3) is defined as

$$\Lambda(\bar{u}^h) = \begin{pmatrix} 0 \\ 0 \\ \partial_3 \bar{p} \\ 0 \end{pmatrix}, \quad (5.5.4)$$

where the scalar function \bar{p} the limit pressure of the limit system (5.4.3). We expressed the nonlinearity Λ as depending on the velocity flow \bar{u}^h , but in the above definition the dependence on \bar{p} is made explicit. Indeed we can express \bar{p} in term of \bar{u}^h as

$$\begin{aligned} \bar{p} &= (-\Delta_h)^{-1} \text{div}_h (\bar{u}^h \cdot \nabla_h \bar{u}^h), \\ &= (-\Delta_h)^{-1} \text{div}_h \text{div}_h (\bar{u}^h \otimes \bar{u}^h), \end{aligned} \quad (5.5.5)$$

and this justifies the above observation.

Let us remark that the operator $(-\Delta_h)^{-1} \text{div}_h \text{div}_h$ is a Fourier multiplier of degree zero. It is known hence that such operators map continuously $L^p(\mathbb{R}^3)$ onto itself if $p \in (1, \infty)$. Sometimes hence we shall implicitly simplify $\bar{p} \sim \bar{u}^h \otimes \bar{u}^h$ when we perform energy estimates in some $L^p(\mathbb{R}^3)$, $p \in (1, \infty)$ norm.

The forcing term Λ presents an interesting property

Lemma 5.5.1. *Let \mathbb{P}_0 be the projector onto the non-oscillating subspace defined in (5.3.10), then*

$$\mathbb{P}_0 \Lambda = 0.$$

Proof. It suffices to remark that the only non-zero component of Λ is the third one and that the projector \mathbb{P}_0 defined in (5.3.10) maps the third component to zero. \square

Lemma 5.5.1 implies in particular that

$$\Lambda = (\mathbb{P}_{+, \varepsilon} + \mathbb{P}_{-, \varepsilon}) \Lambda, \quad (5.5.6)$$

and hence we shall use (5.5.6) repeatedly along this work.

The presence of the external forcing term $-\Psi_{r,R}(D) \Lambda$ is motivated by technical needs which will be explained in detail in Section 5.6.

5.5.1 Study of the linear system (5.5.3).

In this small section we prove some existence and regularity result concerning the free-wave system (5.5.3). Let us define the space

$$H_{r,R}^{1/2} = \left\{ g \mid g = \Psi_{r,R}(D) f, f \in L^2(\mathbb{R}^3) \cap \dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \right\},$$

it is indeed trivial to deduce that $H_{r,R}^{1/2} \subset H^{\frac{1}{2}}(\mathbb{R}^3) = L^2(\mathbb{R}^3) \cap \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. The space $H_{r,R}^{1/2}$ endowed with the $H^{\frac{1}{2}}(\mathbb{R}^3)$ norm is a Banach space.

Lemma 5.5.2. *Let $U_0 \in H^{\frac{1}{2}}(\mathbb{R}^3)$ such that $\omega_0^h \in L^2(\mathbb{R}^3)$, for each $\varepsilon > 0$ and $0 < r < R$ there exist a solution $W_{r,R}^\varepsilon$ of (5.5.3) in the space*

$$W_{r,R}^\varepsilon \in \mathcal{C}^1(\mathbb{R}_+; H_{r,R}^{1/2}),$$

moreover the sequence $(W_{r,R}^\varepsilon)_{\varepsilon > 0}$ is bounded in the space $\dot{\mathcal{E}}^{1/2}(\mathbb{R}^3)$ and for each $t > 0$ and $\varepsilon > 0$, $0 < r < R$ the following bound holds true:

$$\begin{aligned} & \|W_{r,R}^\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + c \int_0^t \|\nabla W_{r,R}^\varepsilon(s)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 ds \leq C_{r,R} \|U_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \\ & + \frac{C}{\nu} \|U_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^4 \exp \left\{ \frac{CK^2}{\nu} \left(\|\bar{u}_0^h\|_{L^2(\mathbb{R}^3)}^4 + \|\omega_0^h\|_{L^2(\mathbb{R}^3)}^4 \right) \right\}, \end{aligned} \quad (5.5.7)$$

where $c = \min\{\nu, \nu'\}$.

In order to prove Lemma 5.5.2 it suffices apply Cauchy-Lipschitz theorem (which was already implicitly used in the proof of Corollary 5.2.3).

Lemma 5.5.3. *Let us consider the ordinary differential equation*

$$\begin{cases} \dot{u} = F(u, t) \\ u|_{t=0} = u_0 \in \omega \end{cases}, \quad (\text{ODE})$$

where ω is an open subset of a Banach space X . Let

$$\begin{aligned} F : \omega \times \mathbb{R}_+ & \rightarrow X \\ (u, t) & \mapsto F(u, t) \end{aligned}$$

be such that, for each $u_1, u_2 \in \omega$ there exists a function $L \in L^1_{\text{loc}}(\mathbb{R}_+)$ such that

$$\|F(u_1, t) - F(u_2, t)\|_X \leq L(t) \|u_1 - u_2\|_X. \quad (5.5.8)$$

Let us suppose moreover that

$$\|F(u, t)\|_X \leq \beta(t) M(\|u\|_X),$$

where $M \in L^\infty_{\text{loc}}(\mathbb{R}_+)$, $\beta \in L^1_{\text{loc}}(\mathbb{R}_+)$. Then there exists a unique maximal solution u in the space $\mathcal{C}^1([0, t^*]; X)$ of (ODE), such that, if $t^* < \infty$,

$$\limsup_{t \nearrow t^*} \|u(t)\|_X = \infty.$$

Proof. See [10, Proposition 3.11, p. 131]. □

Proof of Lemma 5.5.2 : It suffices to consider (5.5.3) in the form

$$\partial_t W_{r,R}^\varepsilon = F_\varepsilon(t, W_{r,R}^\varepsilon),$$

where (using as well (5.5.6)):

$$F_\varepsilon(t, W_{r,R}^\varepsilon) = -\frac{1}{\varepsilon} L_\varepsilon W_{r,R}^\varepsilon - \Psi_{r,R}(D) \Lambda(\bar{u}^h(t)).$$

It is easy to prove that F_ε satisfies (5.5.8) with a locally L^1 function which depends on ε, r and R . We aim to prove that, for each $r, R, \varepsilon > 0$ the function $W_{r,R}^\varepsilon$ belongs to the space $\mathcal{C}^1(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))$: accordingly to Lemma 5.5.3 it suffices hence to prove that

$$\sup_{t \geq 0} \|W_{r,R}^\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} < \infty.$$

Let us now multiply (5.5.3) for $W_{r,R}^\varepsilon$ and let us take the $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ scalar product of it, we deduce hence that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|W_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + c \|\nabla W_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \\ \leq \left| (\Psi_{r,R}(D) \partial_3 (-\Delta_h)^{-1} \operatorname{div}_h \operatorname{div}_h (\bar{u}^h \otimes \bar{u}^h) | W_{r,R}^\varepsilon)_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \right|, \end{aligned}$$

where $c = \min\{\nu, \nu'\}$. Integration by parts, Cauchy-Schwartz inequality, and the fact that the operator $\Psi_{r,R}(D) (-\Delta_h)^{-1} \operatorname{div}_h \operatorname{div}_h$ maps continuously any $\dot{H}^s(\mathbb{R}^3)$ space to itself with norm independent of r and R allow us to deduce

$$\begin{aligned} \left| (\Psi_{r,R}(D) \partial_3 (-\Delta_h)^{-1} \operatorname{div}_h \operatorname{div}_h (\bar{u}^h \otimes \bar{u}^h) | W_{r,R}^\varepsilon)_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \right| \\ \leq C \|\bar{u}^h \otimes \bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \|\nabla W_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}, \end{aligned}$$

and since

$$\begin{aligned} \|\bar{u}^h \otimes \bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} &\leq C \|\bar{u}^h\|_{\dot{H}^1(\mathbb{R}^3)}^2, \\ &\leq C \|\bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \|\nabla \bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}, \end{aligned}$$

whence applying Young inequality we obtain the estimate

$$\frac{1}{2} \frac{d}{dt} \|W_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + \frac{c}{2} \|\nabla W_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \leq C \|\bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \|\nabla \bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2.$$

Integrating in-time the above equation and to using the estimate (5.4.6) we hence conclude the proof. □

5.5.2 Dispersive properties of (5.5.3).

In the previous section we made sure that (5.5.3) is solvable (locally) in the classical sense and that the solutions of (5.5.3) belong to the space $\mathcal{E}^{1/2}(\mathbb{R}^3)$ uniformly w.r.t. the parameters ε, r, R . In the present section we are hence interested to study the perturbation induced by the operator $\varepsilon^{-1}L_\varepsilon$, and to prove that such perturbations induce some dispersive effect on $W_{r,R}^\varepsilon$.

The result we want to prove in this section is the following one

Theorem 5.5.4. *Let $U_0 \in H^{\frac{1}{2}}(\mathbb{R}^3)$, $0 < r < R$ and $\varepsilon > 0$. Then $W_{r,R}^\varepsilon$ solution of (5.5.3) belongs to the space $L^p(\mathbb{R}_+; L^\infty(\mathbb{R}^3))$, $p \in [1, \infty)$ and if $\varepsilon > 0$ is sufficiently small*

$$\|W_{r,R}^\varepsilon\|_{L^p(\mathbb{R}_+; L^\infty(\mathbb{R}^3))} \leq C_{r,R} \left(1 + \frac{1}{\nu}\right) \varepsilon^{\frac{1}{4p}} \max \left\{ \|U_0\|_{L^2(\mathbb{R}^3)}, \|U_0\|_{L^2(\mathbb{R}^3)}^2 \right\}, \quad (5.5.9)$$

for $p \in [1, \infty)$.

Some preparation is indeed required in order to prove Theorem 5.5.4.

By use of Duhamel formula we can write (at least formally) the solution of (5.5.3) as

$$W_{r,R}^\varepsilon(t) = e^{-\frac{t}{\varepsilon} L_\varepsilon} W_{r,R,0}^\varepsilon - \int_0^t e^{-\frac{t-s}{\varepsilon} L_\varepsilon} \Psi_{r,R}(D) (\mathbb{P}_{+,\varepsilon} + \mathbb{P}_{-,\varepsilon}) \Lambda(\bar{u}^h(s)) ds,$$

whence along the eigendirection E_\pm^ε the evolution of (5.5.3) assumes the value

$$\begin{aligned} & \mathbb{P}_{\pm,\varepsilon} (W_{r,R}^\varepsilon)(t, x) \\ &= \mathcal{F}^{-1} (k_\pm (W_{r,R}^\varepsilon(t)) E_\pm^\varepsilon)(x), \\ &= \int_{\mathbb{R}_y^3 \times \mathbb{R}_\xi^3} e^{\pm i \frac{t}{\varepsilon} \lambda_\pm^\varepsilon(\xi) + i \xi(x-y)} \Psi_{r,R}(\xi) \mathbb{P}_{\pm,\varepsilon}(U_0)(y) dy d\xi \\ &\quad - \int_0^t \int_{\mathbb{R}_y^3 \times \mathbb{R}_\xi^3} e^{\pm i \frac{t-s}{\varepsilon} \lambda_\pm^\varepsilon(\xi) + i \xi(x-y)} \Psi_{r,R}(\xi) \mathbb{P}_{\pm,\varepsilon} \Lambda(\bar{u}^h(s, y)) dy d\xi ds, \\ &= \mathcal{K}_{\pm,r,R} \left(t, \frac{t}{\varepsilon}, \cdot \right) \star \mathbb{P}_{\pm,\varepsilon}(U_0)(x) \\ &\quad - \int_0^t \mathcal{K}_{\pm,r,R} \left(t-s, \frac{t-s}{\varepsilon}, \cdot \right) \star \mathbb{P}_{\pm,\varepsilon} \Lambda(\bar{u}^h(s, \cdot)) ds, \\ &= \mathcal{G}_{\pm,r,R}^\varepsilon \left(\frac{t}{\varepsilon} \right) U_0(x) - \int_0^t \mathcal{G}_{\pm,r,R}^\varepsilon \left(\frac{t-s}{\varepsilon} \right) \Lambda(\bar{u}^h(s)) (x) ds. \end{aligned} \quad (5.5.10)$$

where λ_\pm^ε is defined in (5.3.3). The convolution kernels $\mathcal{K}_{\pm,r,R}$ assume the form

$$\mathcal{K}_{\pm,r,R}(t, \tau, z) = \int_{\mathbb{R}_\xi^3} e^{\pm i \tau \frac{|\xi_h|}{|\xi|}} S_\varepsilon(\xi) e^{-\frac{1}{2}(\nu+\nu')|\xi|^2 t + i \xi \cdot z} \Psi_{r,R}(\xi) d\xi. \quad (5.5.11)$$

The convolution kernel $\mathcal{K}_{\pm,r,R}$ is hence a highly oscillating integral. It is well known that integrals with such a behavior are $L^\infty(\mathbb{R}^3)$ functions whose $L^\infty(\mathbb{R}^3)$ norm decays in time (see [3], [10], [42], [138]...), we shall apply the methodology of [42] in order to prove the following result

Lemma 5.5.5. *For any r, R such that $0 < r < R$ there exists a constant $C_{r,R}$ such that for each $z \in \mathbb{R}^3$*

$$|\mathcal{K}_{\pm,r,R}(t, \tau, z)| \leq C_{r,R} \min\{1, \tau^{-1/2}\} e^{-\frac{1}{4}(\nu+\nu')r^2t}. \quad (5.5.12)$$

Proof. Taking the modulus of both sides of (5.5.11) and integrating, considering that $\Psi_{r,R}$ is supported in $\mathcal{C}_{r,R}$, it is sufficient to prove that

$$|\mathcal{K}_{\pm,r,R}(t, \tau, z)| \leq C_{r,R} e^{-\frac{1}{2}(\nu+\nu')r^2t},$$

for each $t, \tau \in \mathbb{R}_+$ and $z \in \mathbb{R}^3$. This holds hence in particular if $\tau \in [0, 1]$.

The rest of the proof is devoted to improve the above estimate in the case $\tau \geq 1$.

Let us fix some notation first, we denote as $\phi(\xi) = \frac{|\xi_h|}{|\xi|} S_\varepsilon(\xi)$ and thanks to Fubini's theorem

$$\begin{aligned} |\mathcal{K}_{\pm,r,R}(t, \tau, z)| &= \int_{\mathbb{R}_\xi^3} e^{\pm i\tau\phi(\xi) - \frac{1}{2}(\nu+\nu')|\xi|^2t + i\xi \cdot z} \Psi_{r,R}(\xi) \, d\xi \\ &= \int_{\mathbb{R}_{\xi_h}^2} e^{\xi_h \cdot z_h} \left(\int_{\mathbb{R}_{\xi_3}^1} e^{\pm i\tau\phi(\xi) - \frac{1}{2}(\nu+\nu')t|\xi|^2 + i\xi_3 \cdot z_3} \Psi_{r,R}(\xi) \, d\xi_3 \right) d\xi_h, \\ &= \int_{\mathbb{R}_{\xi_h}^2} e^{\xi_h \cdot z_h} \mathcal{I}_{\pm,r,R}(t, \tau, \xi_h, z_3) \, d\xi_h. \end{aligned}$$

Indeed since \mathcal{I}_\pm is supported, relatively to the variable ξ_h , in the set $\{\xi_h : r \leq |\xi_h| \leq R\}$, we deduce

$$|\mathcal{K}_{\pm,r,R}(t, \tau, z)| \leq C_{r,R} |\mathcal{I}_{\pm,r,R}(t, \tau, \xi_h, z_3)|,$$

hence it shall suffice to prove an L^∞ bound for the function \mathcal{I}_\pm . Let us remark that \mathcal{I}_\pm are even functions w.r.t. the variable z_3 , hence we can restrict ourselves to the case $z_3 \geq 0$.

We are interested to study the L^∞ norm of the elements \mathcal{I}_\pm , these norms are invariant under dilation, in particular hence we consider the transformation $z_3 \mapsto \tau z_3$, $\tau > 1$, with these

$$\mathcal{I}_{\pm,r,R}(t, \tau, \xi_h, \tau z_3) = \int_{\mathbb{R}_{\xi_3}^1} e^{i\tau(\pm\phi(\xi) + \xi_3 z_3) - \frac{1}{2}(\nu+\nu')t|\xi|^2} \Psi_{r,R}(\xi) \, d\xi_3.$$

Let us fix some notation, we define

$$\begin{aligned} \Phi(\xi) &= \partial_{\xi_3} \phi(\xi) \\ &= \left(\frac{|\xi_h|}{|\xi|} S_\varepsilon(\xi) - \varepsilon^2 (\nu - \nu') \frac{|\xi| |\xi_h|}{S_\varepsilon(\xi)} \right) \xi_3, \\ \theta_\pm(\xi, z_3) &= \pm \phi(\xi) + \xi_3 z_3, \\ \Theta_\pm(\xi, z_3) &= \partial_{\xi_3} \theta_\pm(\xi, z_3), \\ &= \left(\frac{|\xi_h|}{|\xi|} S_\varepsilon(\xi) - \varepsilon^2 (\nu - \nu') \frac{|\xi| |\xi_h|}{S_\varepsilon(\xi)} \right) \xi_3 + z_3. \end{aligned}$$

With this notation indeed

$$\mathcal{I}_{\pm, r, R}(t, \tau, \xi_h, \tau z_3) = \int_{\mathbb{R}_{\xi_3}} e^{i\tau\theta_{\pm}(\xi, z_3) - \frac{1}{2}(\nu+\nu')t|\xi|^2} \Psi_{r, R}(\xi) d\xi_3.$$

Let us define the differential operator

$$\mathcal{L}_{\pm} := \frac{1}{1 + \tau \Theta_{\pm}^2(\xi, z_3)} (1 + i \Theta_{\pm}(\xi, z_3) \partial_{\xi_3}),$$

in particular there exists a positive constant C independent by any parameter of the problem such that, being $\xi \in \mathcal{C}_{r, R}$ defined in (5.3.4),

$$\frac{r^2}{CR} \xi_3 + z_3 \leq |\Theta_{\pm}(\xi, z_3)| \leq \frac{CR^2}{r} \xi_3 + z_3. \quad (5.5.13)$$

Indeed $\mathcal{L}_{\pm}(e^{i\tau\theta_{\pm}}) = e^{i\tau\theta_{\pm}}$, hence integration by parts yields

$$\mathcal{I}_{\pm}(t, \tau, \xi_h, \tau z_3) = \int_{\mathbb{R}_{\xi_3}^1} e^{i\tau\theta_{\pm}(\xi, z_3)} \mathcal{L}_{\pm}^{\top} \left(\Psi_{r, R}(\xi) e^{-\frac{1}{2}(\nu+\nu')t|\xi|^2} \right) d\xi_3, \quad (5.5.14)$$

where

$$\begin{aligned} & \mathcal{L}_{\pm}^{\top} \left(\Psi_{r, R}(\xi) e^{-\frac{1}{2}(\nu+\nu')t|\xi|^2} \right) \\ &= \left(\frac{1}{1 + \tau \Theta_{\pm}^2} - i (\partial_{\xi_3} \Theta_{\pm}) \frac{1 - \tau \Theta_{\pm}^2}{(1 + \tau \Theta_{\pm}^2)^2} \right) \Psi_{r, R}(\xi) e^{\frac{1}{2}(\nu+\nu')t|\xi|^2} \\ & \quad - \frac{i \Theta_{\pm}}{1 + \tau \Theta_{\pm}^2} \partial_{\xi_3} \left(\Psi_{r, R}(\xi) e^{-\frac{1}{2}(\nu+\nu')t|\xi|^2} \right). \end{aligned}$$

Since $\xi \in \mathcal{C}_{r, R}$ and thanks to the estimate (5.5.13) we can deduce easily that (here we use the fact that $\left| \frac{1 - \tau \Theta_{\pm}^2}{(1 + \tau \Theta_{\pm}^2)^2} \right| \leq \left| \frac{1}{1 + \tau \Theta_{\pm}^2} \right|$)

$$\frac{1}{1 + \tau \Theta_{\pm}^2} \leq \frac{C_{r, R}}{1 + \tau \xi_3^2}.$$

Moreover

$$\begin{aligned} \frac{|\Theta_{\pm}|}{1 + \tau |\Theta_{\pm}|^2} &\leq C_{r, R} \frac{1 + z_3}{1 + \tau |z_3 + \xi_3|^2}, \\ &\leq C_{r, R} \frac{1 + z_3}{1 + \tau z_3^2 + \tau \xi_3^2}, \\ &\leq C_{r, R} \frac{1 + z_3}{(1 + \sqrt{\tau} z_3)^2} \frac{1}{1 + \tau \xi_3^2} \leq C_{r, R} \frac{1}{1 + \tau \xi_3^2}. \end{aligned}$$

The last inequality is true since $\tau > 1$. As is ξ localized in $\mathcal{C}_{r, R}$ it is a matter of straightforward computations to prove that

$$|\partial_{\xi_3} \Theta_{\pm}| \leq C_{r, R},$$

moreover, being $\Psi_{r,R} \in \mathcal{D}$,

$$\left| \partial_{\xi_3} \left(\Psi_{r,R}(\xi) e^{-\frac{1}{2}(\nu+\nu')t |\xi|^2} \right) \right| \leq C_{r,R} e^{-\frac{1}{4}(\nu+\nu')r^2 t},$$

whence we finally deduced that

$$\left| \mathcal{L}_{\pm}^T \left(\Psi_{r,R}(\xi) e^{-\frac{1}{2}(\nu+\nu')t |\xi|^2} \right) \right| \leq \frac{C_{r,R}}{1 + \tau \xi_3^2} e^{-\frac{1}{4}(\nu+\nu')r^2 t}.$$

With the above bound and (5.5.14) we deduce hence

$$\begin{aligned} |\mathcal{I}_{\pm,r,R}(t, \tau, \xi_h, \tau z_3)| &\leq C_{r,R} e^{-\frac{1}{4}(\nu+\nu')r^2 t} \int_{\mathbb{R}_{\xi_3}^1} \frac{d\xi_3}{1 + \tau \xi_3^2}, \\ &\leq C_{r,R} \tau^{-1/2} e^{-\frac{1}{4}(\nu+\nu')r^2 t} \end{aligned}$$

which concludes the proof. \square

Proposition 5.5.6. *Let us consider a vector field $U_0 \in L^2(\mathbb{R}^3)$ and the functions $\mathcal{G}_{\pm,r,R}^\varepsilon U_0$ of the variables (t, x) defined in (5.5.10). Then*

$$\left\| \mathcal{G}_{\pm,r,R}^\varepsilon U_0 \right\|_{L^p(\mathbb{R}_+; L^\infty(\mathbb{R}^3))} \leq C_{r,R} \varepsilon^{\frac{1}{4p}} \|U_0\|_{L^2(\mathbb{R}^3)}, \quad (5.5.15)$$

for each $p \in [1, \infty)$.

Proof. Indeed $\mathcal{G}_{\pm,r,R}^\varepsilon U_0$ can be written as a convolution operator as explained in equation (5.5.10), in particular

$$\mathcal{G}_{\pm,r,R}^\varepsilon \left(\frac{t}{\varepsilon} \right) U_0(x) = \mathcal{K}_{\pm,r,R} \left(t, \frac{t}{\varepsilon}, \cdot \right) \star \mathbb{P}_{\pm,\varepsilon}(U_0)(x),$$

where $\mathbb{P}_{\pm,\varepsilon}$ are the projections onto the eigenspaces generated by E_{\pm}^ε defined in (5.3.7), and the convolution kernels $\mathcal{K}_{\pm,r,R}$ are defined in (5.5.11). Considering the dispersive estimate (5.5.12) given in Lemma 5.5.5 we can apply what is known as TT^* argument (see [10, Chapter 8]) in the exact same way as it is done in [42], [40], [38], [28], [34] to deduce that

$$\left\| \mathcal{K}_{\pm,r,R} \star \mathbb{P}_{\pm,\varepsilon}(U_0) \right\|_{L^1(\mathbb{R}_+; L^\infty(\mathbb{R}^3))} \leq C_{r,R} \varepsilon^{1/4} \left\| \mathbb{P}_{\pm,\varepsilon}(U_0) \right\|_{L^2(\mathbb{R}^3)}.$$

We can hence apply Lemma 5.3.1 obtaining

$$\left\| \mathcal{G}_{\pm,r,R}^\varepsilon U_0 \right\|_{L^1(\mathbb{R}_+; L^\infty(\mathbb{R}^3))} \leq C_{r,R} \varepsilon^{1/4} \|U_0\|_{L^2(\mathbb{R}^3)}. \quad (5.5.16)$$

The element $\mathcal{G}_{\pm,r,R}^\varepsilon \left(\frac{t}{\varepsilon} \right) U_0$ has the following properties:

- $\mathcal{G}_{\pm,r,R}^\varepsilon \left(\frac{t}{\varepsilon} \right) U_0$ is localized in the frequency space,
- $\left\| \mathcal{G}_{\pm,r,R}^\varepsilon U_0 \right\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^3))} \leq \|U_0\|_{L^2(\mathbb{R}^3)},$

whence an application of Bernstein inequality allow us to deduce that

$$\left\| \mathcal{G}_{\pm,r,R}^\varepsilon U_0 \right\|_{L^\infty(\mathbb{R}_+; L^\infty(\mathbb{R}^3))} \leq C_{r,R} \|U_0\|_{L^2(\mathbb{R}^3)}. \quad (5.5.17)$$

An interpolation between (5.5.16) and (5.5.17) gives finally (5.5.15). \square

The oscillating behavior of the propagator allow us to deduce the following dispersive result on the external forcing $-\Psi_{r,R}(D)\Lambda$ as it is done, for instance, in [78], [56] or [28].

Proposition 5.5.7. *There exists a constant $C_{r,R}$ depending on the localization (5.3.4) such that, for ε small*

$$\left\| \int_0^t \mathcal{G}_{\pm,r,R}^\varepsilon \left(\frac{\cdot - s}{\varepsilon} \right) \Psi_{r,R}(D) \Lambda(s) \, ds \right\|_{L^p(\mathbb{R}_+; L^\infty(\mathbb{R}^3))} \leq C_{r,R} \varepsilon^{\frac{1}{4p}} \|\Psi_{r,R}(D) \Lambda\|_{L^1(\mathbb{R}_+; L^2(\mathbb{R}^3))}, \quad (5.5.18)$$

for each real $p \geq 1$.

Proof. For this proof only we write $\mathcal{G}_{\pm,r,R}^\varepsilon = \mathcal{G}$, $\Psi_{r,R} = \Psi$ in order to simplify the notation,

$$\left\| \int_0^t \mathcal{G} \left(\frac{\cdot - s}{\varepsilon} \right) \Psi(D) \Lambda(s) \, ds \right\|_{L^1(\mathbb{R}_+; L^\infty(\mathbb{R}^3))} \leq \int_0^\infty \int_0^t \left\| \mathcal{G} \left(\frac{t-s}{\varepsilon} \right) \Psi(D) \Lambda(s) \right\|_{L^\infty} \, ds \, dt,$$

applying Fubini theorem and performing the change of variable $\tau = t - s$ we deduce

$$\begin{aligned} \left\| \int_0^t \mathcal{G} \left(\frac{\cdot - s}{\varepsilon} \right) \Psi(D) \Lambda(s) \, ds \right\|_{L^1(\mathbb{R}_+; L^\infty(\mathbb{R}^3))} &\leq \int_0^\infty \int_0^\infty \left\| \mathcal{G} \left(\frac{\tau}{\varepsilon} \right) \Psi(D) \Lambda(s) \right\|_{L^\infty(\mathbb{R}^3)} \, d\tau \, ds, \\ &= \int_0^\infty \|\mathcal{G} \Psi(D) \Lambda(s)\|_{L^1(\mathbb{R}_+, \tau L^\infty(\mathbb{R}^3))} \, ds, \end{aligned}$$

whence applying (5.5.15) we deduce that

$$\|\mathcal{G} \Psi(D) \Lambda(s)\|_{L^1(\mathbb{R}_+, \tau L^\infty(\mathbb{R}^3))} \leq C_{r,R} \varepsilon^{1/4} \|\Psi(D) \Lambda(s)\|_{L^2(\mathbb{R}^3)},$$

which in turn implies the claim for $p = 1$.

To lift up the argument to a generic p it suffices to notice that, being $\Psi(D)\Lambda$ localized in $\mathcal{C}_{r,R}$, there exist a constant C_R depending on the magnitude of the localization $\mathcal{C}_{r,R}$ such that

$$\left\| \int_0^t \mathcal{G}_{\pm,r,R}^\varepsilon \left(\frac{\cdot - s}{\varepsilon} \right) \Psi_{r,R}(D) \Lambda(s) \, ds \right\|_{L^\infty(\mathbb{R}_+; L^\infty(\mathbb{R}^3))} \leq C_R \|\Psi(D) \Lambda\|_{L^1(\mathbb{R}_+; L^2(\mathbb{R}^3))},$$

hence (5.5.18) follows by interpolation. \square

Proof of Theorem 5.5.4: to prove Theorem 5.5.4 it suffices to collect all the results proved in the present section. By superposition we obviously have that

$$W_{r,R}^\varepsilon = \mathbb{P}_{-, \varepsilon} W_{r,R}^\varepsilon + \mathbb{P}_{+, \varepsilon} W_{r,R}^\varepsilon,$$

and applying (5.5.10)

$$\mathbb{P}_{\pm, \varepsilon} W_{r,R}^\varepsilon = \mathcal{G}_{\pm,r,R}^\varepsilon U_0 - \int_0^t \mathcal{G}_{\pm,r,R}^\varepsilon \left(\frac{\cdot - s}{\varepsilon} \right) \Psi_{r,R}(D) \Lambda(\bar{u}^h(s)) \, ds,$$

whence it suffices to apply (5.5.15) and (5.5.18) to deduce

$$\|W_{r,R}^\varepsilon\|_{L^p(\mathbb{R}_+; L^\infty(\mathbb{R}^3))} \leq C_{r,R} \varepsilon^{\frac{1}{4p}} \left(\|U_0\|_{L^2(\mathbb{R}^3)} + \|\Psi_{r,R}(D) \Lambda(\bar{u}^h)\|_{L^1(\mathbb{R}_+; L^2(\mathbb{R}^3))} \right),$$

hence since

$$\left\| \Psi_{r,R}(D) \Lambda(\bar{u}^h) \right\|_{L^1(\mathbb{R}_+; L^2(\mathbb{R}^3))} \leq R^{1/2} \left\| \bar{u}^h \right\|_{L^2(\mathbb{R}_+; \dot{H}^1(\mathbb{R}^3))}^2,$$

and thanks to the results of Lemma (5.4.2) we can hence argue that

$$\left\| \Psi_{r,R}(D) \Lambda(\bar{u}^h) \right\|_{L^1(\mathbb{R}_+; L^2(\mathbb{R}^3))} \leq \frac{C_{r,R}}{\nu} \|U_0\|_{L^2(\mathbb{R}^3)}^2,$$

which implies in turn that

$$\left\| W_{r,R}^\varepsilon \right\|_{L^p(\mathbb{R}_+; L^\infty(\mathbb{R}^3))} \leq C_{r,R} \left(1 + \frac{1}{\nu} \right) \varepsilon^{\frac{1}{4p}} \max \left\{ \|U_0\|_{L^2(\mathbb{R}^3)}, \|U_0\|_{L^2(\mathbb{R}^3)}^2 \right\},$$

concluding. \square

5.6 Long time behavior: the bootstrap procedure.

This section is devoted to deduce the maximal lifespan of the function

$$\delta_{r,R}^\varepsilon = U^\varepsilon - W_{r,R}^\varepsilon - \bar{U}, \quad (5.6.1)$$

where U^ε is the local solution of (PBS $_\varepsilon$) identified in the Theorem 5.2.2, $W_{r,R}^\varepsilon$ is the global solution of the free-wave system (5.5.3) and \bar{U} is the global solution of the limit system identified in Section 5.4.1, i.e. the system (5.4.3). By the definition itself of $\delta_{r,R}^\varepsilon$ we understand that, being \bar{U} and $W_{r,R}^\varepsilon$ globally well-posed, U^ε and $\delta_{r,R}^\varepsilon$ have the same lifespan in the space $\dot{\mathcal{E}}^{1/2}(\mathbb{R}^3)$.

This first regularity result is a very rough bound on the $\dot{\mathcal{E}}^0$ norm of $\delta_{r,R}^\varepsilon$:

Lemma 5.6.1. *Let $U_0 \in H^{\frac{1}{2}}(\mathbb{R}^3)$ such that $\omega_0^h \in L^2(\mathbb{R}^3)$, the function $\delta_{r,R}^\varepsilon$ defined as in (5.6.1) belongs uniformly in $\varepsilon > 0$ to the space $\dot{\mathcal{E}}^0(\mathbb{R}^3)$ and*

$$\begin{aligned} \left\| \delta_{r,R}^\varepsilon \right\|_{\dot{\mathcal{E}}^0(\mathbb{R}^3)}^2 &\leq C_{r,R} \left(1 + \frac{1}{c} \right) \|U_0\|_{L^2(\mathbb{R}^3)}^2 \\ &+ C \left(1 + \frac{1}{c^2} \right) \|U_0\|_{L^2(\mathbb{R}^3)}^2 \|U_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \exp \left\{ \frac{CK^2}{\nu} \left(\|U_0\|_{L^2(\mathbb{R}^3)}^4 + \|\omega_0^h\|_{L^2(\mathbb{R}^3)}^4 \right) \right\}, \end{aligned}$$

where $c = \min \{\nu, \nu'\}$.

Proof. Theorem 5.2.1 implies that $U^\varepsilon \in \dot{\mathcal{E}}^0$ as well as Lemma 5.4.2 implies that $\bar{U} \in \dot{\mathcal{E}}^0$ and moreover

$$\left\| U^\varepsilon \right\|_{\dot{\mathcal{E}}^0(\mathbb{R}^3)}^2 + \left\| \bar{U} \right\|_{\dot{\mathcal{E}}^0(\mathbb{R}^3)}^2 \leq C \left(1 + \frac{1}{c} \right) \|U_0\|_{L^2(\mathbb{R}^3)}^2,$$

where $c = \min \{\nu, \nu'\}$. For $W_{r,R}^\varepsilon$ the procedure is similar: let us multiply (5.5.3) for $W_{r,R}^\varepsilon$ and let us integrate in space. Recalling that $\bar{p} = (-\Delta_h)^{-1} \operatorname{div} \operatorname{div} (\bar{u}^h \otimes \bar{u}^h) = p_0(D) (\bar{u}^h \otimes \bar{u}^h)$ it suffices to prove a suitable energy bound on the element

$$\left| \left(\partial_3 (\bar{u}^h \otimes \bar{u}^h) \mid W_{r,R}^\varepsilon \right)_{L^2(\mathbb{R}^3)} \right|.$$

Integration by parts and Young inequality allow us to deduce that

$$\left| (\partial_3 (\bar{u}^h \otimes \bar{u}^h) | W_{r,R}^\varepsilon)_{L^2(\mathbb{R}^3)} \right| \leq \frac{c}{2} \|\nabla W_{r,R}^\varepsilon\|_{L^2}^2 + C \|\bar{u}^h \otimes \bar{u}^h\|_{L^2(\mathbb{R}^3)}^2.$$

Product rules in Sobolev spaces imply

$$\|\bar{u}^h \otimes \bar{u}^h\|_{L^2(\mathbb{R}^3)}^2 \leq \|\bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \|\bar{u}^h\|_{\dot{H}^1(\mathbb{R}^3)}^2,$$

whence an integration in time

$$\begin{aligned} \|W_{r,R}^\varepsilon(t)\|_{L^2(\mathbb{R}^3)}^2 + c \int_0^t \|\nabla W_{r,R}^\varepsilon(\tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau \\ \leq C_{r,R} \|U_0\|_{L^2(\mathbb{R}^3)}^2 + \|\bar{u}^h\|_{L^\infty(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))}^2 \|\bar{u}^h\|_{L^2(\mathbb{R}_+; \dot{H}^1(\mathbb{R}^3))}^2. \end{aligned}$$

it suffices hence to use the bounds in Lemma 5.4.2 and Proposition 5.4.5 to deduce the claim. \square

Lemma 5.6.1 provides a first rough bound on $\delta_{r,R}^\varepsilon$ under some rather strong regularity assumptions on the initial data ($U_0 \in H^{1/2}$ and $\text{curl}_h U_0 \in L^2(\mathbb{R}^3)$). Nonetheless such bound shall be required in the proof of Lemma 5.6.6 (see function $g_3^{r,R}$), which is an important step in the proof of Proposition 5.6.4, the main result of the present section. Let us remark moreover that the hypothesis on the initial data of Lemma 5.6.1 are the same as the ones of Proposition 5.6.4.

The following procedure is standard in singular perturbation problems (see [38], [42] and [28]). In particular, being the diffusion isotropic we shall follow closely the methodology in [42], proving that $\delta_{r,R}^\varepsilon$ is globally well posed in $\dot{\mathcal{E}}^{1/2}(\mathbb{R}^3)$. If we prove this, as mentioned above, we prove as well that U^ε is globally well-posed in the space $\dot{\mathcal{E}}^{1/2}(\mathbb{R}^3)$, and hence we prove the global-well-posedness part in Theorem 5.2.4.

Let us at first deduce the equation satisfied by the function $\delta_{r,R}^\varepsilon$. This is a matter of careful algebraic computations, which lead us to deduce the following equations

$$\begin{cases} \partial_t \delta_{r,R}^\varepsilon - \mathbb{D} \delta_{r,R}^\varepsilon + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} \delta_{r,R}^\varepsilon = -\frac{1}{\varepsilon} \nabla \tilde{p}^\varepsilon - (F_{r,R}^\varepsilon + G_{r,R}^\varepsilon) - (1 - \Psi_{r,R}(D)) \Lambda(\bar{u}^h), \\ \text{div} \delta_{r,R}^\varepsilon = 0, \\ \delta_{r,R}^\varepsilon|_{t=0} = [1 - \Psi_{r,R}(D)] (\mathbb{P}_{+, \varepsilon} + \mathbb{P}_{-, \varepsilon}) U_0. \end{cases} \quad (5.6.2)$$

Where the modified pressure $\tilde{p}^\varepsilon = \Phi^\varepsilon - \varepsilon \bar{p}$ and the nonlinearity is defined as

$$\begin{aligned} F_{r,R}^\varepsilon &= \delta_{r,R}^\varepsilon \cdot \nabla \delta_{r,R}^\varepsilon + \delta_{r,R}^\varepsilon \cdot \nabla \bar{u}^h + \delta_{r,R}^\varepsilon \cdot \nabla W_{r,R}^\varepsilon + \bar{u}^h \cdot \nabla_h \delta_{r,R}^\varepsilon + w_{r,R}^\varepsilon \cdot \nabla \delta_{r,R}^\varepsilon, \\ G_{r,R}^\varepsilon &= \bar{u}^h \cdot \nabla_h w_{r,R}^\varepsilon + w_{r,R}^\varepsilon \cdot \nabla \bar{u}^h + w_{r,R}^\varepsilon \cdot \nabla w_{r,R}^\varepsilon. \end{aligned}$$

We can now explain why in the equation (5.5.3) we introduced artificially the external forcing $-\Psi_{r,R}(D) \Lambda$ where Λ is defined in (5.5.4). The pressure \bar{p} appears with an horizontal gradient only in the equation (5.4.3), whence the difference

$$\nabla \Phi^\varepsilon - \varepsilon \nabla_h \bar{p},$$

arising when we compute the difference equation of $U^\varepsilon - \bar{U}$ is *not the gradient of a scalar function*, being \bar{p} dependent on the variable x_3 as it is clear from its expression in terms of the velocity flow \bar{u}^h given in (5.5.5).

The forcing term $-\Psi_{r,R}(D)\Lambda = -\Psi_{r,R}(D)(\mathbb{P}_{+,\varepsilon} + \mathbb{P}_{-,\varepsilon})\Lambda$ on the right-hand-side of (5.5.3) is hence a corrector term: it adds the intermediate frequencies of $\partial_3\bar{p}$ in order to later obtain a full gradient function in the system (5.6.2) describing the evolution of $\delta_{r,R}^\varepsilon$. Obviously we require an additional corrector which covers the very low and very high frequencies of Λ , for this reason it is present in equation (5.6.2) the term $-(1 - \Psi_{r,R}(D))\Lambda$. We had as well to use the property (5.5.6) in such process.

Let us define $\eta = \eta_{r,R}$ any positive function depending on the parameters r, R which determinate the localization $\mathcal{C}_{r,R}$ defined in (5.3.4) such that

$$\lim_{\substack{r \rightarrow 0 \\ R \rightarrow \infty}} \eta_{r,R} = 0. \quad (5.6.3)$$

We shall in fact require more than one η function. Let us point out that, given two functions η_1, η_2 which satisfy the above hypothesis if we define $\tilde{\eta} = \max\{\eta_1, \eta_2\}$ it still satisfy (5.6.3), hence from now on we shall write simply $\eta_{r,R}$ knowing that this process can be applied when required.

Lemma 5.6.2. *Let $U_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and P_0 be a Fourier multiplier of order 0, then the following bound holds true*

$$\|(1 - \Psi_{r,R}(D))P_0(D)(\bar{u}^h \otimes \bar{u}^h)\|_{L^2(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))} \leq \frac{\eta_{r,R}}{3C}.$$

Proof. The proof is an application of Lebesgue dominated convergence theorem. Indeed the function

$$|1 - \Psi_{r,R}(\xi)|^2 |\xi| |P_0(\xi)|^2 |\mathcal{F}(\bar{u}^h \otimes \bar{u}^h)(\xi)|^2,$$

converges point-wise to zero when $r \rightarrow 0, R \rightarrow \infty$, hence it suffices to prove that

$$|\xi| |P_0(\xi)|^2 |\mathcal{F}(\bar{u}^h \otimes \bar{u}^h)(\xi)|^2 \in L^1(\mathbb{R}_+; L^1).$$

By Plancherel theorem and product rules in Sobolev spaces we deduce

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^3} |\xi| P_0(\xi)^2 \mathcal{F}(\bar{u}^h \otimes \bar{u}^h)^2(t, \xi) d\xi dt &\leq C \|\bar{u}^h \otimes \bar{u}^h\|_{L^2(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))}^2 \\ &\leq C \left\| \|\bar{u}^h\|_{\dot{H}^1}^2 \right\|_{L^2(\mathbb{R}_+)}^2 \\ &\leq C \|\bar{u}^h\|_{L^\infty(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))}^2 \|\nabla \bar{u}^h\|_{L^2(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))}^2 \\ &< \infty, \end{aligned}$$

thanks to the results in Proposition 5.4.5, concluding. \square

Let us analyze now the initial data of the system (5.6.2), it is defined as

$$\delta_{r,R,0} = [1 - \Psi_{r,R}(D) (\mathbb{P}_{+,\varepsilon} + \mathbb{P}_{-,\varepsilon}) - \mathbb{P}_0] U_0, \quad (5.6.4)$$

where the projectors $\mathbb{P}_{i,\varepsilon}$ defined in (5.3.7), are the projections onto the eigendirections E_i^ε , $i = 0, \pm$ defined in (5.3.5) and (5.3.6). The initial data is localized onto the very high and low frequencies along the eigendirections of the eigenvectors E^\pm defined in (5.3.6). Unfortunately the projectors $\mathbb{P}_{\pm,\varepsilon}$ are *not bounded* on such set of frequencies, hence we cannot deduce directly the regularity of $\delta_{r,R,0}$ in terms of the regularity of U_0 . Nonetheless we can prove the following result

Lemma 5.6.3. *Let $\delta_{r,R,0}$ be the initial data of (5.6.2) be defined as in (5.6.4). For any $s \in \mathbb{R}$ if $U_0 \in \dot{H}^s(\mathbb{R}^3)$ there exists a constant C which does not depend on the parameters r, R of the localization $\mathcal{C}_{r,R}$ such that*

$$\|\delta_{r,R,0}\|_{\dot{H}^s(\mathbb{R}^3)} \leq C \|U_0\|_{\dot{H}^s(\mathbb{R}^3)}.$$

Proof. We can prove Lemma 5.6.3 in a simple way. Let us remark that $1 = \mathbb{P}_0 + \mathbb{P}_{+,\varepsilon} + \mathbb{P}_{-,\varepsilon}$, this in turn implies that

$$1 - \Psi_{r,R}(D) (\mathbb{P}_{+,\varepsilon} + \mathbb{P}_{-,\varepsilon}) - \mathbb{P}_0 = (1 - \Psi_{r,R}(D)) (1 - \mathbb{P}_0),$$

whence

$$\delta_{r,R,0} = (1 - \Psi_{r,R}(D)) (1 - \mathbb{P}_0) U_0.$$

The projector \mathbb{P}_0 has been evaluated in detail in (5.3.10), and in particular it is a Fourier multiplier of order zero. This implies that the operator $(1 - \Psi_{r,R}(D)) (1 - \mathbb{P}_0)$ is as well a Fourier multiplier of order zero, such operators map continuously any $\dot{H}^s(\mathbb{R}^3)$, $s \in \mathbb{R}$ space to itself, whence we deduce the claim. \square

Given $U_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ Lemma 5.6.3 and a dominated convergence argument allow us hence to choose some $0 < r < R$ such that

$$\|\delta_{r,R,0}\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} < \frac{\eta_{r,R}}{3C}, \quad (5.6.5)$$

with $\eta_{r,R}$ real and sufficiently small such that

$$\eta_{r,R} < \frac{c}{4C},$$

where c, C are fixed positive constant.

The result we prove in this section is the following one:

Proposition 5.6.4. *Let the initial data $U_0 \in H^{\frac{1}{2}}(\mathbb{R}^3)$ such that $\omega_0^h = -\partial_2 U_0^1 + \partial_1 U_0^2 \in L^2(\mathbb{R}^3)$. Let us set $0 < r \ll 1 \ll R$ such that $\delta_{r,R,0}$ defined in (5.6.4) satisfies (5.6.5). Let $(\delta_{r,R}^\varepsilon)_{\varepsilon>0}$ be a sequence indexed by ε of solutions of (5.6.2), there exists a $\varepsilon_0 = \varepsilon_0(r, R) > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ and $t > 0$*

$$\|\delta_{r,R}^\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + c \int_0^t \|\nabla \delta_{r,R}^\varepsilon(\tau)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 d\tau \leq \eta_{r,R}^2, \quad (5.6.6)$$

where $c = \min\{\nu, \nu'\}$ and $\eta_{r,R}$ satisfies (5.6.3) and it is independent of the time-variable. In particular hence for each $\varepsilon \in (0, \varepsilon_0)$, being $\delta_{r,R}^\varepsilon$ defined as in (5.6.1), the solution U^ε of (PBS_ε) is in fact global and belongs to the space $\dot{\mathcal{E}}^{1/2}(\mathbb{R}^3)$.

The proof of the above proposition consists in a bootstrap argument as it is done in [42]. The main step in order to prove such bootstrap argument is an energy bound on the nonlinearity $F_{r,R}^\varepsilon + G_{r,R}^\varepsilon$. This is formalized in the following lemma:

Lemma 5.6.5. *The following bounds hold true*

$$\begin{aligned}
 \left| (\delta_{r,R}^\varepsilon \cdot \nabla \delta_{r,R}^\varepsilon | \delta_{r,R}^\varepsilon)_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \right| &\leq C \|\delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \|\nabla \delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2, \\
 \left| (\operatorname{div} (\delta_{r,R}^\varepsilon \otimes (\bar{u}^h + W_{r,R}^\varepsilon)) | \delta_{r,R}^\varepsilon)_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \right| &\leq C \left(\|\bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{1/2} \|\nabla \bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{1/2} \right. \\
 &\quad \left. + \|W_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{1/2} \|\nabla W_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{1/2} \right) \\
 &\quad \times \|\delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{1/2} \|\nabla \delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{3/2}, \\
 \left| (\bar{u}^h \cdot \nabla_h W_{r,R}^\varepsilon | \delta_{r,R}^\varepsilon)_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \right| &\leq C_{r,R} \|\nabla \delta_{r,R}^\varepsilon\|_{L^2(\mathbb{R}^3)} \|W_{r,R}^\varepsilon\|_{L^\infty(\mathbb{R}^3)} \|\bar{u}^h\|_{L^2(\mathbb{R}^3)}, \\
 \left| (w_{r,R}^\varepsilon \cdot \nabla \bar{u}^h | \delta_{r,R}^\varepsilon)_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \right| &\leq C \|W_{r,R}^\varepsilon\|_{L^\infty(\mathbb{R}^3)} \|\bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{1/2} \|\nabla \bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{1/2} \\
 &\quad \times \|\delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{1/2} \|\nabla \delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{1/2}, \\
 \left| (w_{r,R}^\varepsilon \cdot \nabla W_{r,R}^\varepsilon | \delta_{r,R}^\varepsilon)_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \right| &\leq C_{r,R} \|W_{r,R}^\varepsilon\|_{L^\infty(\mathbb{R}^3)} \|W_{r,R}^\varepsilon\|_{L^2(\mathbb{R}^3)} \|\delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}.
 \end{aligned}$$

Thanks to the above bounds we can deduce the following bounds for the nonlinearity $F_{r,R}^\varepsilon + G_{r,R}^\varepsilon$, which shall be the ones that we will use in the proof of the bootstrap argument

Lemma 5.6.6. *The following bounds hold true*

$$\left| (F_{r,R}^\varepsilon | \delta_{r,R}^\varepsilon)_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \right| \leq \left(\frac{C}{16} + C \|\delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \right) \|\nabla \delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + f_{r,R} \|\delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2, \tag{5.6.7}$$

$$\begin{aligned}
 \left| (G_{r,R}^\varepsilon | \delta_{r,R}^\varepsilon)_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \right| &\leq \frac{C}{16} \|\nabla \delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + g_{1,\varepsilon}^{r,R} \|\delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \\
 &\quad + \left(g_2^{r,R} + g_3^{r,R} \right) \|W_{r,R}^\varepsilon\|_{L^\infty(\mathbb{R}^3)} + g_4 \|W_{r,R}^\varepsilon\|_{L^\infty(\mathbb{R}^3)}^2,
 \end{aligned} \tag{5.6.8}$$

where

$$\begin{aligned}
 f_{r,R}(t) &= C \left(\|\bar{u}^h(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \|\nabla \bar{u}^h(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \right. \\
 &\quad \left. + \|W_{r,R}^\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \|\nabla W_{r,R}^\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \right), && \in L^1(\mathbb{R}_+), \\
 g_{1,\varepsilon}^{r,R}(t) &= C \|\nabla \bar{u}^h(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + C_{r,R} \|W_{r,R}^\varepsilon\|_{L^2(\mathbb{R}^3)} \|W_{r,R}^\varepsilon\|_{L^\infty(\mathbb{R}^3)}, && \in L^1(\mathbb{R}_+), \\
 g_2^{r,R}(t) &= C_{r,R} \|W_{r,R}^\varepsilon\|_{L^2(\mathbb{R}^3)}, && \in L^\infty(\mathbb{R}_+), \\
 g_3^{r,R}(t) &= C_{r,R} \|\bar{u}^h\|_{L^2(\mathbb{R}^3)} \|\nabla \delta_{r,R}^\varepsilon\|_{L^2(\mathbb{R}^3)} && \in L^2(\mathbb{R}_+), \\
 g_4(t) &= C \|\bar{u}^h(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}, && \in L^\infty(\mathbb{R}_+).
 \end{aligned}$$

Moreover if $U_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ then $f = f_{r,R} \in L^1(\mathbb{R}_+)$ uniformly with respect to the variables r, R and for $0 < \varepsilon < \varepsilon_0(r, R)$ the function $g_1 = g_{1,\varepsilon}^{r,R}$ belongs to $L^1(\mathbb{R}_+)$ uniformly with respect to the variables r, R .

The proofs of Lemmas 5.6.5 and 5.6.6 are postponed.

We can finally prove the convergence result.

Proof of Proposition 5.6.4: Let us perform an $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ energy estimate onto the system (5.6.2), we indeed deduce that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\delta_{r,R}^\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + c \|\nabla \delta_{r,R}^\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \\ \leq \left| (F_{r,R}^\varepsilon(t) | \delta_{r,R}^\varepsilon(t))_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \right| + \left| (G_{r,R}^\varepsilon(t) | \delta_{r,R}^\varepsilon(t))_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \right| \\ + \left| ((1 - \Psi_{r,R}(D)) \Lambda(\bar{u}^h) | \delta_{r,R}^\varepsilon)_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \right|. \end{aligned} \quad (5.6.9)$$

Thanks to the explicit definition of Λ given in (5.5.4) an integration by parts and young inequality we deduce

$$\begin{aligned} \left| ((1 - \Psi_{r,R}(D)) \Lambda(\bar{u}^h) | \delta_{r,R}^\varepsilon)_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \right| \\ \leq \left| ((1 - \Psi_{r,R}(D)) (-\Delta_h)^{-1} \operatorname{div} \operatorname{div}(\bar{u}^h \otimes \bar{u}^h) | \partial_3 \delta_{r,R}^\varepsilon)_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \right|, \\ \leq \frac{c}{16} \|\nabla \delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + \|(1 - \Psi_{r,R}) P_0(\bar{u}^h \otimes \bar{u}^h)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2, \end{aligned} \quad (5.6.10)$$

where we denoted $P_0 = (-\Delta_h)^{-1} \operatorname{div} \operatorname{div}$.

With the bounds (5.6.7), (5.6.8) and (5.6.10) the equation (5.6.9) becomes

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\delta_{r,R}^\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + \left(\frac{3c}{4} - C \|\delta_{r,R}^\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \right) \|\nabla \delta_{r,R}^\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \\ \leq (f(t) + g_1(t)) \|\delta_{r,R}^\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + \left(g_2^{r,R}(t) + g_3^{r,R}(t) \right) \|W_{r,R}^\varepsilon(t)\|_{L^\infty(\mathbb{R}^3)} \\ + g_4(t) \|W_{r,R}^\varepsilon(t)\|_{L^\infty(\mathbb{R}^3)}^2 + g_5^{r,R}(t), \end{aligned} \quad (5.6.11)$$

where

$$g_5^{r,R} = \|(1 - \Psi_{r,R}) P_0(\bar{u}^h \otimes \bar{u}^h)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2. \quad (5.6.12)$$

We omit the dependence of f and g_1 on the parameters r, R, ε in light of the results of Lemma 5.6.6.

Let us define at this point the time

$$T^* = \sup \left\{ 0 < t \leq T \mid \|\delta_{r,R}^\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} < \frac{c}{4C} \right\},$$

where T is the lifespan defined in Theorem 5.2.2.

For each $t \in [0, T^*)$, thanks of the definition of T^* , we can deduce that

$$\frac{3c}{4} - C \|\delta_{r,R}^\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \geq \frac{c}{2},$$

from which, combined with (5.6.11) we can deduce:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\delta_{r,R}^\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + \frac{c}{2} \|\nabla \delta_{r,R}^\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \\ & \leq (f(t) + g_1(t)) \|\delta_{r,R}^\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + \left(g_2^{r,R}(t) + g_3^{r,R}(t) \right) \|W_{r,R}^\varepsilon(t)\|_{L^\infty(\mathbb{R}^3)} \\ & \quad + g_4(t) \|W_{r,R}^\varepsilon(t)\|_{L^\infty(\mathbb{R}^3)}^2 + g_5^{r,R}(t). \end{aligned} \quad (5.6.13)$$

Let us set

$$\Xi(t) = -2 \int_0^t (f(\tau) + g_1(\tau)) d\tau,$$

and let us remark that since $f, g_1 \in L^1(\mathbb{R}_+)$ then $\Xi, e^{\pm\Xi} \in L^\infty(\mathbb{R}_+)$, and moreover

$$e^{-\Xi(t)} \geq e^{-\|\Xi\|_{L^\infty(\mathbb{R}_+)}} , \quad e^{\Xi(t)} \leq e^{\|\Xi\|_{L^\infty(\mathbb{R}_+)}} . \quad (5.6.14)$$

Standard calculation on (5.6.13) and integration-in-time imply that

$$\begin{aligned} & \|\delta_{r,R}^\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + c \int_0^t e^{-(\Xi(t)-\Xi(\tau))} \|\nabla \delta_{r,R}^\varepsilon(\tau)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 d\tau \\ & \leq e^{-\Xi(t)} \|\delta_{r,R,0}\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + C \int_0^t e^{-(\Xi(t)-\Xi(\tau))} \left(\left(g_2^{r,R}(\tau) + g_3^{r,R}(\tau) \right) \|W_{r,R}^\varepsilon(\tau)\|_{L^\infty(\mathbb{R}^3)} \right. \\ & \quad \left. + g_4(\tau) \|W_{r,R}^\varepsilon(\tau)\|_{L^\infty(\mathbb{R}^3)}^2 + g_5^{r,R}(\tau) \right) d\tau, \end{aligned}$$

whence by the use of (5.6.14) we deduce

$$\begin{aligned} & \|\delta_{r,R}^\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + c \int_0^t \|\nabla \delta_{r,R}^\varepsilon(\tau)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 d\tau \\ & \leq C \|\delta_{r,R,0}\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + C \int_0^t \left(\left(g_2^{r,R}(\tau) + g_3^{r,R}(\tau) \right) \|W_{r,R}^\varepsilon(\tau)\|_{L^\infty(\mathbb{R}^3)} \right. \\ & \quad \left. + g_4(\tau) \|W_{r,R}^\varepsilon(\tau)\|_{L^\infty(\mathbb{R}^3)}^2 + g_5^{r,R}(\tau) \right) d\tau. \end{aligned} \quad (5.6.15)$$

Moreover since $g_2^{r,R}, g_4 \in L^\infty(\mathbb{R}_+)$, $g_3^{r,R} \in L^2(\mathbb{R}_+)$ and thanks to the estimates (5.5.9) we deduce

$$\begin{aligned} & C \int_0^t \left(\left(g_2^{r,R}(\tau) + g_3^{r,R}(\tau) \right) \|W_{r,R}^\varepsilon(\tau)\|_{L^\infty(\mathbb{R}^3)} + g_4(\tau) \|W_{r,R}^\varepsilon(\tau)\|_{L^\infty(\mathbb{R}^3)}^2 \right) d\tau \\ & \leq C \left(\|W_{r,R}^\varepsilon\|_{L^1(\mathbb{R}_+; L^\infty(\mathbb{R}^3))} + \|W_{r,R}^\varepsilon\|_{L^2(\mathbb{R}_+; L^\infty(\mathbb{R}^3))} + \|W_{r,R}^\varepsilon\|_{L^2(\mathbb{R}_+; L^\infty(\mathbb{R}^3))}^2 \right), \\ & \leq C_{r,R} (\varepsilon^{1/4} + \varepsilon^{1/8}), \end{aligned} \quad (5.6.16)$$

for $\varepsilon < \varepsilon_0$ positive and sufficiently small. In light of the definition of $g_5^{r,R}$ given in (5.6.12) and Lemma 5.6.2 we deduce

$$C \int_0^\infty g_5^{r,R}(\tau) d\tau \leq \frac{\eta_{r,R}^2}{3}, \quad (5.6.17)$$

The bound (5.6.16), (5.6.17) and (5.6.5) transform (5.6.15) into

$$\|\delta_{r,R}^\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + c \int_0^t \|\nabla \delta_{r,R}^\varepsilon(\tau)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 d\tau \leq \frac{2}{3} \eta_{r,R}^2 + C_{r,R} (\varepsilon^{1/4} + \varepsilon^{1/8}).$$

Moreover if ε is sufficiently small

$$\frac{2}{3} \eta_{r,R}^2 + C_{r,R} (\varepsilon^{1/4} + \varepsilon^{1/8}) < \eta_{r,R}^2,$$

and hence the bound is independent from the time variable. Applying Lemma 5.5.3 we deduce that $T^* = \infty$. Moreover, thanks to (5.6.3) and the above results:

$$\lim_{\substack{r \rightarrow 0 \\ R \rightarrow \infty}} \limsup_{\varepsilon \rightarrow 0} \left(\|\delta_{r,R}^\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + c \int_0^t \|\nabla \delta_{r,R}^\varepsilon(\tau)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 d\tau \right) = 0,$$

for each $t > 0$. □

5.6.1 Proof of Lemma 5.6.5

The first bound is a simple application of the definition (5.1.5) and of Lemma 5.1.1

$$\begin{aligned} \left| (\delta_{r,R}^\varepsilon \cdot \nabla \delta_{r,R}^\varepsilon | \delta_{r,R}^\varepsilon)_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \right| &= \|\delta_{r,R}^\varepsilon \otimes \delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \|\nabla \delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}, \\ &\leq C \|\delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \|\nabla \delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}. \end{aligned}$$

The estimate is derived by interpolation of Sobolev spaces.

For the second estimate

$$\begin{aligned} \left| (\operatorname{div} (\delta_{r,R}^\varepsilon \otimes (\bar{u}^h + W_{r,R}^\varepsilon)) | \delta_{r,R}^\varepsilon)_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \right| \\ \leq C \|\delta_{r,R}^\varepsilon \otimes (\bar{u}^h + W_{r,R}^\varepsilon)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \|\nabla \delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}, \\ \leq C \|\delta_{r,R}^\varepsilon\|_{\dot{H}^1(\mathbb{R}^3)} \|(\bar{u}^h + W_{r,R}^\varepsilon)\|_{\dot{H}^1(\mathbb{R}^3)} \|\nabla \delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}, \end{aligned}$$

an interpolation of Sobolev spaces and triangular inequality conclude the second estimate.

For the next term

$$\begin{aligned} \left| (\bar{u}^h \cdot \nabla_h W_{r,R}^\varepsilon | \delta_{r,R}^\varepsilon)_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \right| &\leq \|\bar{u}^h \cdot \nabla W_{r,R}^\varepsilon\|_{L^2(\mathbb{R}^3)} \|\nabla \delta_{r,R}^\varepsilon\|_{L^2(\mathbb{R}^3)}, \\ &\leq C_{r,R} \|\bar{u}^h\|_{L^2(\mathbb{R}^3)} \|W_{r,R}^\varepsilon\|_{L^\infty(\mathbb{R}^3)} \|\nabla \delta_{r,R}^\varepsilon\|_{L^2(\mathbb{R}^3)}, \end{aligned}$$

where in the last inequality we applied Hölder inequality and Bernstein inequality. For the last term it suffices to remark that the function $w_{r,R}^\varepsilon \cdot \nabla W_{r,R}^\varepsilon$ is well-defined and still localized in the Fourier space, hence apply Hölder and Bernstein inequalities.

5.6.2 Proof of Lemma 5.6.6

To deduce the bound (5.6.7) and (5.6.8) it suffices to apply repeatedly Young inequality to the bounds of Lemma 5.6.5, in detail:

applying the convexity inequality $\alpha \beta \leq \frac{c}{16} \alpha^4 + C \beta^4$ we deduce

$$\begin{aligned} & \left| (\operatorname{div} (\delta_{r,R}^\varepsilon \otimes (\bar{u}^h + W_{r,R}^\varepsilon)) | \delta_{r,R}^\varepsilon)_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \right| \\ & \leq C \left(\|\bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{1/2} \|\nabla \bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{1/2} + \|W_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{1/2} \|\nabla W_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{1/2} \right) \\ & \quad \times \|\delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{1/2} \|\nabla \delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{3/2} \\ & \leq \frac{c}{16} \|\nabla \delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \\ & \quad + C \left(\|\bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \|\nabla \bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + \|W_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \|\nabla W_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \right) \|\delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2, \end{aligned}$$

and hence we set

$$f_{r,R} = C \left(\|\bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \|\nabla \bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + \|W_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \|\nabla W_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \right),$$

obtaining the bound (5.6.7).

Next we prove (5.6.8). In the third inequality of Lemma 5.6.6 we proceed as follows

$$\begin{aligned} & \left| (\bar{u}^h \cdot \nabla_h W_{r,R}^\varepsilon | \delta_{r,R}^\varepsilon)_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \right| \leq C_{r,R} \|\nabla \delta_{r,R}^\varepsilon\|_{L^2(\mathbb{R}^3)} \|\bar{u}^h\|_{L^2(\mathbb{R}^3)} \|W_{r,R}^\varepsilon\|_{L^\infty(\mathbb{R}^3)} \\ & = g_3^{r,R} \|W_{r,R}^\varepsilon\|_{L^\infty(\mathbb{R}^3)}. \end{aligned}$$

Next, in the fourth inequality of Lemma 5.6.6 we apply the inequality

$$\alpha \beta \gamma \leq \frac{c}{64} \alpha^4 + C \beta^4 + C \gamma^2$$

in order to deduce the following inequality

$$\begin{aligned} & \left| (w_{r,R}^\varepsilon \cdot \nabla \bar{u}^h | \delta_{r,R}^\varepsilon)_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \right| \\ & \leq C \|W_{r,R}^\varepsilon\|_{L^\infty(\mathbb{R}^3)} \|\bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{1/2} \|\nabla \bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{1/2} \|\delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{1/2} \|\nabla \delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{1/2} \\ & \leq \frac{c}{64} \|\nabla \delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + C \|\nabla \bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \|\delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \\ & \quad + C \|\bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \|\nabla W_{r,R}^\varepsilon\|_{L^\infty(\mathbb{R}^3)}^2, \end{aligned}$$

hence we set

$$\begin{aligned} g_4 &= C \|\bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}, \\ g_{1,I}^{r,R} &= C \|\nabla \bar{u}^h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2. \end{aligned}$$

For the last inequality it suffices to remark that

$$\begin{aligned} \left| (w_{r,R}^\varepsilon \cdot \nabla W_{r,R}^\varepsilon | \delta_{r,R}^\varepsilon)_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \right| &\leq C_{r,R} \|W_{r,R}^\varepsilon\|_{L^\infty(\mathbb{R}^3)} \|W_{r,R}^\varepsilon\|_{L^2(\mathbb{R}^3)} \|\delta_{r,R}^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \\ &\quad + C_{r,R} \|W_{r,R}^\varepsilon\|_{L^\infty(\mathbb{R}^3)} \|W_{r,R}^\varepsilon\|_{L^2(\mathbb{R}^3)}, \end{aligned}$$

whence we set

$$\begin{aligned} g_{1,\text{II},\varepsilon}^{r,R} &= C_{r,R} \|W_{r,R}^\varepsilon\|_{L^\infty(\mathbb{R}^3)} \|W_{r,R}^\varepsilon\|_{L^2(\mathbb{R}^3)}, \\ g_2^{r,R} &= C_{r,R} \|W_{r,R}^\varepsilon\|_{L^2(\mathbb{R}^3)}. \end{aligned}$$

Lastly we finally define

$$g_{1,\varepsilon}^{r,R} = g_{1,\text{I}}^{r,R} + g_{1,\text{II},\varepsilon}^{r,R},$$

and we deduce the bound (5.6.8).

The function $f_{r,R}$ belongs indeed to $L^1(\mathbb{R}_+)$ uniformly with respect to r, R thanks to the result in Proposition 5.4.5 and Lemma 5.5.2.

For the function $g_{1,\varepsilon}^{r,R}$ it suffices to integrate in time and to use the result in Proposition 5.4.5 and (5.5.9) to obtain

$$\begin{aligned} \left\| g_{1,\varepsilon}^{r,R} \right\|_{L^1(\mathbb{R}_+)} &\leq C \left\| \nabla \bar{u}^h \right\|_{L^2(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))}^2 + C_{r,R} \|W_{r,R}^\varepsilon\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^3))} \|W_{r,R}^\varepsilon\|_{L^1(\mathbb{R}_+; L^\infty(\mathbb{R}^3))} \\ &\leq C + C_{r,R} \varepsilon^{1/4}, \\ &< \infty, \end{aligned}$$

if ε is sufficiently small.

5.7 Proof of the main result.

Section 5.6 gives us all the ingredients required in order to prove the main result of the present paper, namely Theorem 5.2.4. Remarkably the statement in Theorem 5.2.4 and Proposition 5.6.4 are very similar: the difference is that W^ε solution of (5.2.2) does not depends on the parameters r, R as $W_{r,R}^\varepsilon$ solution of (5.5.3). Let us hence define

$$\delta^\varepsilon = U^\varepsilon - W^\varepsilon - \bar{U}.$$

In Section 5.5 we focused on existence, regularity and dispersive results for $W_{r,R}^\varepsilon$, but no result was proved for W^ε . Remarkably the initial data of the system (5.2.2), which is solved by W^ε , is not any more localized in the frequency space. The estimate (5.5.7) hence does not hold true any more, in particular the bound

$$\left\| W_{r,R}^\varepsilon|_{t=0} \right\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \leq C_{r,R} \|U_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)},$$

is false for initial data which are not localized as for W^ε . Fortunately we can extend the result of Lemma 5.5.2 to the system (5.2.2) with an argument very similar to the one given

in the proof of Lemma 5.6.3. We omit a detailed proof here, but it suffices to remark that the operator $\mathbb{P}_{+, \varepsilon} + \mathbb{P}_{-, \varepsilon} = 1 - \mathbb{P}_0$, and that the operator $1 - \mathbb{P}_0$ is continuous in any $\dot{H}^s(\mathbb{R}^3)$ space. We hence showed that, if $U_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$,

$$W^\varepsilon \in \dot{\mathcal{E}}^{1/2}(\mathbb{R}^3),$$

and we shall use this property continuously in what follows

Let us fix $0 < r \ll 1 \ll R$ such that (5.6.3) is satisfied, indeed we have that

$$\delta^\varepsilon = \delta_{r,R}^\varepsilon - (W^\varepsilon - W_{r,R}^\varepsilon).$$

In the last equation we hence introduce artificially a dependence on r, R . From this we derive that

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \|\delta^\varepsilon\|_{\dot{\mathcal{E}}^{1/2}(\mathbb{R}^3)} &\leq \limsup_{\varepsilon \rightarrow 0} \left(\|\delta_{r,R}^\varepsilon\|_{\dot{\mathcal{E}}^{1/2}(\mathbb{R}^3)} + \|W^\varepsilon - W_{r,R}^\varepsilon\|_{\dot{\mathcal{E}}^{1/2}(\mathbb{R}^3)} \right), \\ &\leq \eta_{r,R} + \limsup_{\varepsilon \rightarrow 0} \|W^\varepsilon - W_{r,R}^\varepsilon\|_{\dot{\mathcal{E}}^{1/2}(\mathbb{R}^3)}, \end{aligned}$$

where we used (5.6.6). The left-hand-side of the above equation is independent from the parameters r, R , hence

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \|\delta^\varepsilon\|_{\dot{\mathcal{E}}^{1/2}(\mathbb{R}^3)} &\leq \lim_{\substack{r \rightarrow 0 \\ R \rightarrow \infty}} \left(\eta_{r,R} + \limsup_{\varepsilon \rightarrow 0} \|W^\varepsilon - W_{r,R}^\varepsilon\|_{\dot{\mathcal{E}}^{1/2}(\mathbb{R}^3)} \right), \\ &= 0, \end{aligned}$$

where in the last equality we used (5.6.3) and a dominated convergence argument to deduce that $\lim_{\substack{r \rightarrow 0 \\ R \rightarrow \infty}} \|W^\varepsilon - W_{r,R}^\varepsilon\|_{\dot{\mathcal{E}}^{1/2}(\mathbb{R}^3)} = 0$ for each $\varepsilon > 0$.

Chapter 6

Weakly compressible and fast rotating fluids.

Life stands before me like an eternal
spring with new and brilliant clothes.

Carl Friederich Gauss

The present chapter is a joint work with V.-S. Ngo, maître de conférences at the Laboratoire de Mathématiques Raphaël Salem, département of the Université de Rouen.

6.1 Introduction

In this paper, we consider the following system of weakly compressible, fast rotating fluids in the whole space \mathbb{R}^3

$$\begin{cases} \partial_t (\rho^{\varepsilon,\theta} u^{\varepsilon,\theta}) + \operatorname{div} (\rho^{\varepsilon,\theta} u^{\varepsilon,\theta} \otimes u^{\varepsilon,\theta}) + \frac{1}{\theta^2} \nabla P (\rho^{\varepsilon,\theta}) + \frac{1}{\varepsilon} e^3 \wedge (\rho^{\varepsilon,\theta} u^{\varepsilon,\theta}) = 0 \\ \partial_t \rho^{\varepsilon,\theta} + \operatorname{div} (\rho^{\varepsilon,\theta} u^{\varepsilon,\theta}) = 0 \\ (\rho^{\varepsilon,\theta}, u^{\varepsilon,\theta})|_{t=0} = (\rho_0^{\varepsilon,\theta}, u_0^{\varepsilon,\theta}). \end{cases} \quad (\text{CRE}_{\varepsilon,\theta})$$

Here, the Rossby number ε represents the ratio of the displacement due to inertia to the displacement due to Coriolis force. On a planetary scale, the displacement due by inertial forces, *i.e.* the collision of air molecules (in the case of the atmosphere) or water molecules (in the case of oceans) is generally much smaller than the relative displacement due to the rotation of the Earth around his own axis. Away from persistent streams such as the Gulf stream, the value of Rossby number is around 10^{-3} . On the other hand, the Mach number is a dimensionless number representing the ration between the local flow velocity and the speed

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of sound in the medium. For geophysical fluids appearing in meteorology for exemple, the Mach number θ is also very small.

We want to have a few words about the low Mach-number regime and the fast rotation limit. For the weak compressible limit, the fluid is expected to have an incompressible behaviour. In the fast rotation limit, the Coriolis force becomes dominant and plays a very important role. Indeed, the fast rotating fluid have tendency to stabilize and to move in vertical columns (the so-called ‘‘Taylor-Proudman’’ columns). This phenomenon can be observed in many geophysical fluids (such as oceanic currents in the western North Atlantic) and is well known in fluid mechanics as the Taylor-Proudman theorem (see [126] for more details).

We remark that if $\varepsilon \ll \theta$ or $\varepsilon \gg \theta$, then either the high rotation or the weak compressibility dominates the other, and one can separately take the high rotation limit and the weak compressible limit. In this paper, we are interested in the case where these two numbers are very small and where the high rotation and weak compressibility limits occur at the same scale, *i.e.* $\theta = \varepsilon \rightarrow 0$. Moreover, in our study, we suppose that the fluid is inviscid and isentropic, which means that it has no viscosity and the pressure satisfies

$$P = P(\rho) = A\rho^\gamma,$$

where $A > 0$ and $\gamma > 1$ are given. We refer the reader to [126] and the references therein for further physical explanations, and to [42] for a brief physical introduction of fast rotating hydrodynamic systems with strong emphasis on the problem under the mathematical point of view.

6.1.1 Formulation of the system

Let us give a brief explanation of the formulation of our system. In general, the motion of a compressible fluid with a homogeneous temperature can be derived from the laws of conservation of mass and of linear momentum (see [11], [101] or [109] for instance), and is described by the following system

$$\begin{cases} \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\sigma) = \rho f \\ \partial_t \rho + \operatorname{div}(\rho u) = 0. \end{cases}$$

Here, σ is the *stress tensor* and f represents the external body forces acting on the fluid (gravity, Coriolis, electromagnetic forces, etc. . .). For an isotropic *newtonian fluid*, the stress tensor is supposed to be linearly dependent on the strain rate tensor $D = \frac{1}{2}(\nabla u + {}^T\nabla u)$, and writes

$$\sigma = -p\mathbf{1} + \lambda \operatorname{div} u + \mu (\nabla u + {}^T\nabla u),$$

where the scalar function p stands for the *pressure*, $\mathbf{1}$ is the identity matrix (tensor) and $\mu, \lambda \geq 0$ are the Lamé viscosity coefficients (which may depend on the density ρ). In fluid mechanics, μ is referred to as the dynamic viscosity of the fluid and in a case of a barotropic fluid, p is a function of the density ρ only. These considerations lead to the following system describing the motion of a compressible newtonian barotropic fluid

$$\begin{cases} \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\lambda \operatorname{div} u + \mu (\nabla u + {}^T\nabla u)) + \nabla p = \rho f \\ \partial_t \rho + \operatorname{div}(\rho u) = 0. \end{cases} \quad (\text{CNS})$$

for which, we refer to the works [95], [99], [105] or [82]. In the case of *ill prepared* initial data, it is only assumed that

$$\rho_0^\theta = 1 + \theta b_0^\theta$$

and (b_0^θ, u_0^θ) are only bounded in some suitable spaces which does not necessarily belong to the kernel of the penalized operator. If $\mathbb{P}u_0^\theta \rightarrow v_0$ when θ goes to zero¹, one expects that $u^\theta \rightarrow v$ where v is the solution of the incompressible Navier-Stokes equations

$$\begin{cases} \partial_t v + v \cdot \nabla v - \mu \Delta v + \nabla \Pi = 0, \\ \operatorname{div} v = 0 \\ v|_{t=0} = v_0. \end{cases} \quad (\text{INS})$$

The expected convergence is however not easy to be rigorously justified. The main difficulty lies in the fact that one has to deal with the propagation of acoustic waves with speed of order θ^{-1} , a phenomenon which does not occur in the case of well prepared data.

In [110], P.-L. Lions proved the existence of global weak solutions of (CNS_θ) for initial data with minimal regularity assumptions. The fluid is supposed to be isentropic and the pressure is of the form $P(\rho) = a\rho^\gamma$, with certain restrictions on γ depending on the space dimension d . In the same setting P.-L. Lions and N. Masmoudi in [111] proved that weak solutions of (CNS_θ) converges weakly to weak solutions of (INS) in various boundary settings. This result is proved via some weak compactness methods (see also [77] and [66]). In the work of B. Desjardins, E. Grenier, P.-L. Lions and N. Masmoudi [57], considering (CNS_θ) with $f \equiv 0$, in a bounded domain Ω with Dirichlet boundary conditions, the authors proved that as $\theta \rightarrow 0$, the global weak solutions of (CNS_θ) converge weakly in L^2 to a global weak solution of the incompressible Navier-Stokes equations (INS) . In [56], using dispersive Strichartz-type estimates, Desjardins and Grenier proved that the gradient part of the velocity field (*i.e.* the gradient of the acoustic potential) of the system (CNS_θ) converges strongly to zero. Finally, we want to mention the works of R. Danchin [52] and [54]. In [52], the author proved global existence of *strong* solutions for the system (CNS_θ) for small initial data in some suitable, critical, scale-invariant (Besov) spaces, in the same spirit as in the work of Cannone, Planchon and Meyer [18] or the work of Fujita-Kato [70] for the incompressible model. In [54], the author addressed to the convergence of (CNS_θ) to (INS) for ill-prepared initial data when the Mach number θ tends to zero. When the initial data are small, the author obtains global convergence and existence, while for large initial data with some further regularity assumptions, it is shown that the solution of (CNS_θ) exists and converges to the solution of (INS) in the same time interval of existence of the solution of (INS) . For compressible inviscid fluids in the non-rotating case, in [60], A. Dutrifoy and T. Hmidi considered the system (CE_θ) in \mathbb{R}^2 with initial data not uniformly smooth (*i.e.* the \mathcal{C}^1 norm is of order $\mathcal{O}(\theta^{-\alpha})$, $\alpha > 0$). The convergence to strong, global solutions of 2D Euler equation is proved by mean of Strichartz estimates and the propagation of the minimal regularity.

In the case of incompressible fast rotating fluids, we first recall the works of J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier [40] and [38] for incompressible viscous rotating fluids, with initial data of the form

$$u_0 = \bar{u}_0 + \tilde{u}_0,$$

¹Here \mathbb{P} is the Leray projector on the space of solenoidal vector fields defined as $\mathbb{P} = I - \Delta^{-1} \nabla \operatorname{div}$

where the 2D part \bar{u}_0 only depends on (x_1, x_2) and the 3D part \tilde{u}_0 belongs to the anisotropic Sobolev spaces $H^{0,s}$, with $s > \frac{1}{2}$. It is proved that the 2D part is governed by a 2D incompressible Navier-Stokes system, while the 3D part converges to zero as the Rossby number $\varepsilon \rightarrow 0$, using Strichartz estimates obtained for the associated linear free-wave system. As a consequence, if the rotation is fast enough, the solution of the 3D incompressible viscous rotating fluids exists globally in time and converges to the solution of the 2D incompressible Navier-Stokes system. In the case of incompressible, inviscid fluids, however, we cannot get the global existence of strong solutions when the rotation is fast, due to the lack of smoothing effect given by the viscous term. It is proved in A. Dutrifoy [59] that if the rotation is fast enough ($\varepsilon \rightarrow 0$), the solution of a incompressible inviscid rotating fluids exists almost global in time, with the lifespan is at least equivalent to $\ln \ln \varepsilon^{-1}$. However, in the case where the viscosity is not zero, but very small (of order ε^α , for α in some interval $[0, \alpha_0[$), when the rotation is fast enough, the global existence on strong solutions can still be proven in the case of pure 3D initial data (see [119]).

Let us now focus on fast rotating, compressible fluids. To the best of our knowledge, there is no result yet concerning the the inviscid system (CRE $_{\varepsilon,\theta}$). In the viscous fast rotating case, in [68], E. Feireisl, I. Gallagher and A. Novotný studied the dynamics, when $\theta = \varepsilon \rightarrow 0$, of weak solutions of the system (CRNS $_{\varepsilon,\theta}$) in $\mathbb{R}^2 \times \mathbb{T}^1$, with non-slip boundary conditions

$$u^{\varepsilon,3}|_{x_3=0,1} = 0 \quad \text{and} \quad (S_{2,3}, -S_{1,3}, 0)|_{x_3=0,1} = 0,$$

where \mathbb{S} is the stress viscous tensor

$$\mathbb{S}(\nabla u) = \mu \left(\nabla u + {}^T \nabla u - \frac{2}{3} \operatorname{div} u I \right).$$

Their result relies on the spectral analysis of the singular perturbation operator. Using RAGE theorem (see [132]), the authors proved the dispersion due to fast rotation and that weak solutions of (CRNS $_{\varepsilon,\theta}$) converges to a 2D viscous quasi-geostrophic equation for the limit density. We refer to [68] for a detailed description of the limit system. In [67], Feireisl, Gallagher, Gérard-Varet and Novotný studied the system (CRNS $_{\varepsilon,\theta}$) in the case where the effect of the centrifugal force was taken into account. Noticing that this term scales as ε^{-2} , they studied both the isotropic limit and the multi-scale limit: namely, they supposed the Mach-number to be proportional to ε^m , for $m \geq 1$. We want to point out that, in the analysis of the isotropic scaling ($m = 1$), the authors had to resort to compensated compactness arguments in order to pass to the limit: as a matter of fact, the singular perturbation operator had variable coefficients, and spectral analysis tools were no more available. Recently in [65], F. Fanelli proved a similar result as the one proved in [68] and [93], by adding to the system (CRNS $_{\varepsilon,\theta}$) a capillarity term and studying various regimes depending on some positive parameter.

To complete our brief survey of known results, we want to remark that all the compressible systems previously mentioned are isothermal. In the case of variable temperature, the

generic system governing a heat conductive, compressible fluid is the following

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\tau) + \nabla P = \rho f, \\ \partial_t \left(\rho \left(\frac{|u|^2}{2} + e \right) \right) + \operatorname{div} \left[u \left(\rho \left(\frac{|u|^2}{2} + e \right) + P \right) \right] \\ = \operatorname{div}(\tau \cdot u) - \operatorname{div} q + \rho f \cdot u, \end{cases} \quad (\text{HCCNS})$$

which can be derived from the conservation of mass, linear momentum and energy. We refer the reader to [109] and references therein for more details. Here, the fluid is always supposed to be newtonian and $e = e(t, x)$ is the internal (thermal) energy per unit mass. The heat conduction q is given by $q = -k \nabla \mathcal{T}$, where k is positive and \mathcal{T} stands for the temperature. If e obeys Joule rule (*i.e.* e is a function of \mathcal{T} only), the initial data is smooth and the initial density is bounded and bounded away from zero, the existence and uniqueness of a local classical solution has already been known for a long time (see [117] or [90]). In [53], R. Danchin proved that (HCCNS) is locally well posed in the critical scale-invariant space $B_{p,1}^{\frac{N}{p}-1}(\mathbb{R}^N)$, $p \in [1, \infty[$.

6.1.3 Main result and structure of the paper.

The aim of this paper is to study the behavior of *strong* solutions of the system (CRE $_{\varepsilon,\theta}$) in the limit $\theta = \varepsilon \rightarrow 0$ and in the case of ill-prepared initial data in the whole space \mathbb{R}^3 , say

$$\rho_0 = 1 + \varepsilon b_0.$$

Let $\bar{\gamma} = (\gamma - 1)/2$. We consider the substitution

$$1 + \varepsilon b^\varepsilon = \frac{(4\gamma A)^{1/2}}{\gamma - 1} (\rho^\varepsilon)^{\bar{\gamma}}$$

and (CRE $_{\varepsilon,\theta}$) becomes (after a few algebraic calculations)

$$\begin{cases} \partial_t u^\varepsilon + \frac{1}{\varepsilon} (\bar{\gamma} \nabla b^\varepsilon + e^3 \wedge u^\varepsilon) + u^\varepsilon \cdot \nabla u^\varepsilon + \bar{\gamma} b^\varepsilon \nabla b^\varepsilon = 0 \\ \partial_t b^\varepsilon + \frac{\bar{\gamma}}{\varepsilon} \operatorname{div} u^\varepsilon + u^\varepsilon \nabla b^\varepsilon + \bar{\gamma} b^\varepsilon \operatorname{div} u^\varepsilon = 0 \\ (b^\varepsilon, u^\varepsilon)|_{t=0} = (b_0, u_0). \end{cases} \quad (6.1.1)$$

From now on we shall always consider the system (CRE $_{\varepsilon,\theta}$) in the form (6.1.1) or in a more compact form

$$\begin{cases} \partial_t \begin{pmatrix} u^\varepsilon \\ b^\varepsilon \end{pmatrix} - \frac{1}{\varepsilon} \mathcal{B} \begin{pmatrix} u^\varepsilon \\ b^\varepsilon \end{pmatrix} + \begin{pmatrix} u^\varepsilon \cdot \nabla u^\varepsilon + \bar{\gamma} b^\varepsilon \nabla b^\varepsilon \\ u^\varepsilon \cdot \nabla b^\varepsilon + \bar{\gamma} b^\varepsilon \operatorname{div} u^\varepsilon \end{pmatrix} = 0, \\ (u^\varepsilon, b^\varepsilon)|_{t=0} = (u_0, b_0), \end{cases} \quad (6.1.2)$$

where \mathcal{B} is the following operator

$$\mathcal{B} = \begin{pmatrix} 0 & 1 & 0 & -\bar{\gamma} \partial_1 \\ -1 & 0 & 0 & -\bar{\gamma} \partial_2 \\ 0 & 0 & 0 & -\bar{\gamma} \partial_3 \\ -\bar{\gamma} \partial_1 & -\bar{\gamma} \partial_2 & -\bar{\gamma} \partial_3 & 0 \end{pmatrix}, \quad (6.1.3)$$

and where ∂_i , for any $i \in \{1, 2, 3\}$ stands for the derivative with respect to x_i variable. Moreover we can write the nonlinearity as follows

$$\begin{aligned} \begin{pmatrix} u^\varepsilon \cdot \nabla u^\varepsilon + \bar{\gamma} b^\varepsilon \nabla b^\varepsilon \\ u^\varepsilon \cdot \nabla b^\varepsilon + \bar{\gamma} b^\varepsilon \operatorname{div} u^\varepsilon \end{pmatrix} &= \mathcal{A}(U, D) U \\ &= \begin{pmatrix} u^\varepsilon \cdot \nabla & 0 & 0 & \bar{\gamma} b^\varepsilon \partial_1 \\ 0 & u^\varepsilon \cdot \nabla & 0 & \bar{\gamma} b^\varepsilon \partial_2 \\ 0 & 0 & u^\varepsilon \cdot \nabla & \bar{\gamma} b^\varepsilon \partial_3 \\ \bar{\gamma} b^\varepsilon \partial_1 & \bar{\gamma} b^\varepsilon \partial_2 & \bar{\gamma} b^\varepsilon \partial_3 & u^\varepsilon \cdot \nabla \end{pmatrix} \begin{pmatrix} u^\varepsilon \\ b^\varepsilon \end{pmatrix}, \end{aligned} \quad (6.1.4)$$

where U stays for $\begin{pmatrix} u^\varepsilon \\ b^\varepsilon \end{pmatrix}$. With all the above considerations, the system (6.1.2) can be rewritten as

$$\begin{cases} \partial_t U - \frac{1}{\varepsilon} \mathcal{B}U + \mathcal{A}(U, D)U = 0, \\ U|_{t=0} = U_0 = (u_0, b_0). \end{cases} \quad (6.1.5)$$

Remark 6.1.1. We would like to underline that, given a $L^2(\mathbb{R}^3)$ vector field F , we have

$$(\mathcal{B}F|F)_{L^2(\mathbb{R}^3)} = \left(\widehat{\mathcal{B}F} \middle| \widehat{F} \right)_{L^2(\mathbb{R}^3)} = 0.$$

In order to state our result, we recall the definitions of the functional spaces we will use in our paper. We use the index ‘‘h’’ to refer to the horizontal variable, and the index ‘‘v’’ or ‘‘3’’ to refer to the vertical one. Thus, $x_h = (x_1, x_2)$ and $\xi_h = (\xi_1, \xi_2)$. The anisotropic Lebesgue spaces $L_h^p L_v^q$ with $p, q \geq 1$ are defined as

$$\begin{aligned} L_h^p L_v^q(\mathbb{R}^3) &= L^p(\mathbb{R}_h^2; L_v^q(\mathbb{R})) \\ &= \left\{ u \in \mathcal{S}' : \|u\|_{L_h^p L_v^q} = \left[\int_{\mathbb{R}_h^2} \left| \int_{\mathbb{R}_v} |u(x_h, x_3)|^q dx_3 \right|^{\frac{p}{q}} dx_h \right]^{\frac{1}{p}} < +\infty \right\}. \end{aligned}$$

Here, the order of integration is important. Indeed, if $1 \leq p \leq q$ and if $u : X_1 \times X_2 \rightarrow \mathbb{R}$ is a function in $L^p(X_1; L^q(X_2))$, where $(X_1, d\mu_1)$, $(X_2, d\mu_2)$ are measurable spaces, then $u \in L^q(X_2; L^p(X_1))$ and

$$\|u\|_{L^q(X_2; L^p(X_1))} \leq \|u\|_{L^p(X_1; L^q(X_2))}.$$

We recall that the non-homogeneous Sobolev spaces $H^s(\mathbb{R}^3)$, with $s \in \mathbb{R}$, are defined as the closure of the set of smooth functions under the norm

$$\|u\|_{H^s} \stackrel{\text{def}}{=} \left(\int_{\mathbb{R}^3} (1 + |\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

For any $s > 5/2$, $s_0 > 0$, $1 < p < 2$, we define the spaces

$$Y_{s, s_0, p} = H^{s+s_0}(\mathbb{R}^3)^4 \cap L_h^2 L_v^p(\mathbb{R}^3)^4 \cap L_h^p L_v^2(\mathbb{R}^3)^4, \quad (6.1.6)$$

endowed with the norm

$$\|u\|_{s, s_0, p} = \max \left\{ \|u\|_{H^{s+s_0}}, \|u\|_{L_h^2 L_v^p}, \|u\|_{L_h^p L_v^2} \right\}. \quad (6.1.7)$$

From now on, for any initial data $U_0 \in Y_{s,s_0,p}$, we set

$$\mathcal{C}(U_0) = \max \left\{ \|U_0\|_{s,s_0,p}, \|U_0\|_{s,s_0,p}^2 \right\}. \quad (6.1.8)$$

The main result of this paper is the following theorem.

Theorem 6.1.2. *Let $s > 5/2$, $s_0 > 0$ be fixed constants, $1 < p < 2$ and the initial data $U_0 \in Y_{s,s_0,p}$. There exists a time $T_\varepsilon^* > 0$ and a unique solution $U^\varepsilon = (u^\varepsilon, b^\varepsilon)$ of system (6.1.1) satisfying*

$$U^\varepsilon \in L^\infty([0, T_\varepsilon^*]; H^s(\mathbb{R}^3)) \cap C([0, T_\varepsilon^*]; H^s(\mathbb{R}^3)),$$

where the maximal time T_ε^* tends to infinity as ε tends to zero, more precisely, there exist positive constants $\bar{C} > 0$ and $\alpha > 0$ such that

$$T_\varepsilon^* \geq \frac{\bar{C}}{\mathcal{C}(U_0) \varepsilon^\alpha},$$

where $\mathcal{C}(U_0)$ is defined in (6.1.8).

Remark 6.1.3.

1. The estimate of the lifespan T_ε^* of U^ε is much better than in [59] (for incompressible fast rotating fluids). The reason is that we only consider 3D initial data, which is of finite energy in \mathbb{R}^3 . As a consequence, the limit system is zero, since the only vector field of finite energy in \mathbb{R}^3 which belongs to the kernel of the penalized operator \mathcal{B} is zero. In the more general case where the initial data is the sum of a 2D part (which belongs to the kernel of the penalization operator \mathcal{B}) and a 3D part (of finite energy in \mathbb{R}^3), the limit system is not zero but some 2D nonlinear hyperbolic system. Thus, in the case of general data, we can only hope for a similar lifespan as in [59]. This general case will be dealt in a forthcoming paper.
2. If U_0 is small, then the lifespan is inversely proportionnal to the $Y_{s,s_0,p}$ -norm of U_0 , which is somehow expected for this type of hyperbolic system with small initial data.
3. The initial data can be chosen not only to be large but to blow up as $\varepsilon \rightarrow 0$. Indeed, for data $U_0 \sim \varepsilon^{-\omega}$, with $0 < \omega < \frac{\alpha}{2}$, the maximal lifetime of the solution still goes to ∞ as

$$T_\varepsilon^* \gtrsim \varepsilon^{-(\alpha-2\omega)} \rightarrow \infty.$$

Throughout this paper, we set

$$\mathcal{C}_{r,R} = \left\{ \xi \in \mathbb{R}_\xi^3 \mid |\xi| \leq R, |\xi_h| \geq r, |\xi_3| \geq r \right\}. \quad (6.1.9)$$

Our strategy to study the system (6.1.1) consists in finding a solution of to (6.1.2) of the form

$$U^\varepsilon = (u^\varepsilon, b^\varepsilon) = \bar{U}^\varepsilon + \tilde{U}^\varepsilon$$

where $\bar{U}^\varepsilon = (\bar{u}^\varepsilon, \bar{b}^\varepsilon)$ and $\tilde{U}^\varepsilon = (\tilde{u}^\varepsilon, \tilde{b}^\varepsilon)$ are respectively solutions to the following systems

$$\begin{cases} \partial_t \bar{U}^\varepsilon - \frac{1}{\varepsilon} \mathcal{B} \bar{U}^\varepsilon = 0 \\ \bar{U}^\varepsilon|_{t=0} = \Psi_{r,R}(D)(u_0, b_0) \end{cases}, \quad \begin{cases} \partial_t \tilde{U}^\varepsilon - \frac{1}{\varepsilon} \mathcal{B} \tilde{U}^\varepsilon + \mathcal{A}(U, D)U = 0 \\ \tilde{U}^\varepsilon|_{t=0} = (1 - \Psi_{r,R}(D))(u_0, b_0) \end{cases}.$$

Here, the frequency cut-off radii $0 < r < R$ will be precisely chosen, depending on ε and $\Psi_{r,R}$ is a radial function supported in $\mathcal{C}_{\frac{r}{2}, 2R}$ and is identically equal to 1 in $\mathcal{C}_{r,R}$. The precise definition of $\Psi_{r,R}$ will be given in (6.3.1) in Section 6.3. We will also prove in Section 6.3 that, if R is sufficiently large, the system describing \tilde{U}^ε can be considered as a 3D hydrodynamical system with small initial data, which is known to be globally well posed in critical spaces ([70], [18], [97], [52]). For the linear part \bar{U}^ε which describes the evolution of 3D free waves, we will prove that it goes to zero in some appropriate topology using similar Strichartz-type estimates as in [40], [38], [59] or [119]. We want to emphasize that, unlike the RAGE theorem using in [68], Strichartz estimates give very precise quantitative estimates of the rate of decay to zero of \bar{U}^ε , as $\varepsilon \rightarrow 0$.

This paper will be organized as follows. In Section 6.2 we introduce the notation and a detailed description of the critical spaces that we are going to use all along the work. Moreover, we introduce some elements of the Littlewood-Paley and the paradifferential calculus, which is primordial to the study of critical behavior of nonlinearities. In Section 6.3, we study a specific decomposition of the initial data in two parts, one only containing medium Fourier frequencies and the other very high or very low frequencies and we provide a precise control of the latter. Section 6.4 is devoted to the study of the cut-off linear free-wave system associate to (6.1.1). Using the spectral properties of the penalized operator \mathcal{B} defined in (6.1.3), we prove some Strichartz-type estimates for this system, which show that its solutions vanish in some appropriate $L^p(\mathbb{R}_+; L^q(\mathbb{R}^3))$ spaces as $\varepsilon \rightarrow 0$. The nonlinear problem is finally dealt in Section 6.5, where, combining with the results of Section 6.4, we prove an existence result for the system (6.1.1). Performing a bootstrap procedure, we also prove that the solution of (6.1.1) is almost global when the rotation is fast enough.

6.2 Preliminary

The aim of this section is to briefly recall some elements of the Littlewood-Paley theory, which are the main technique used all along the paper.

6.2.1 Dyadic decomposition

We recall that in \mathbb{R}^d , with $d \in \mathbb{N}^*$, for $R > 0$, the ball $\mathcal{B}_d(0, R)$ is the set

$$\mathcal{B}_d(0, R) = \{\xi \in \mathbb{R}^d : |\xi| \leq R\}.$$

For $0 < r_1 < r_2$, we defined the annulus

$$\mathcal{A}_d(r_1, r_2) \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^d : r_1 \leq |\xi| \leq r_2\}.$$

Next, we recall the following Bernstein-type lemma, which states that Fourier multipliers act almost as homotheties on distributions whose Fourier transforms are supported in a ball or an annulus. We refer the reader to [37, Lemma 2.1.1] or [10, Lemma 2.1] for a proof of this lemma.

Lemma 6.2.1. *Let $k \in \mathbb{N}$, $d \in \mathbb{N}^*$ and $R, r_1, r_2 \in \mathbb{R}$ satisfy $0 < r_1 < r_2$ and $R > 0$. There exists a constant $C > 0$ such that, for any $a, b \in \mathbb{R}$, $1 \leq a \leq b \leq +\infty$, for any $\lambda > 0$ and for any $u \in L^a(\mathbb{R}^d)$, we have*

$$\text{supp } (\widehat{u}) \subset \mathcal{B}_d(0, \lambda R) \implies \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^k \lambda^{k+d(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a}, \quad (6.2.1)$$

$$\text{supp } (\widehat{u}) \subset \mathcal{A}_d(\lambda r_1, \lambda r_2) \implies C^{-k} \lambda^k \|u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a} \leq C^k \lambda^k \|u\|_{L^a}. \quad (6.2.2)$$

In order to define the dyadic partition of unity, we also recall the following proposition, the proof of which can be found in [37, Proposition 2.1.1] or [10, Proposition 2.10].

Proposition 6.2.2. *Let $d \in \mathbb{N}^*$. There exist smooth radial function χ and φ from \mathbb{R}^d to $[0, 1]$, such that*

$$\text{supp } \chi \in \mathcal{B}_d\left(0, \frac{4}{3}\right), \quad \text{supp } \varphi \in \mathcal{A}_d\left(\frac{3}{4}, \frac{8}{3}\right), \quad (6.2.3)$$

$$\forall \xi \in \mathbb{R}^3, \quad \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1, \quad (6.2.4)$$

$$|j - j'| \geq 2 \implies \text{supp } \varphi(2^{-j}\cdot) \cap \text{supp } \varphi(2^{-j'}\cdot) = \emptyset, \quad (6.2.5)$$

$$j \geq 1 \implies \text{supp } \chi \cap \text{supp } \varphi(2^{-j}\cdot) = \emptyset. \quad (6.2.6)$$

Moreover, for any $\xi \in \mathbb{R}^d$, we have

$$\frac{1}{2} \leq \chi^2(\xi) + \sum_{j \geq 0} \varphi^2(2^{-j}\xi) \leq 1. \quad (6.2.7)$$

The dyadic blocks are defined as follows

Definition 6.2.3. *For any $d \in \mathbb{N}^*$ and for any tempered distribution $u \in \mathcal{S}'(\mathbb{R}^d)$, we set*

$$\begin{aligned} \Delta_q u &= \mathcal{F}^{-1}(\varphi(2^{-q}|\xi|)\widehat{u}(\xi)), & \forall q \in \mathbb{N}, \\ \Delta_{-1} u &= \mathcal{F}^{-1}(\psi(|\xi|)\widehat{u}(\xi)), \\ \Delta_q u &= 0, & \forall q \leq -2, \\ S_q u &= \sum_{q' \leq q-1} \Delta_{q'} u, & \forall q \geq 1. \end{aligned}$$

Using the properties of ψ and φ , for any tempered distribution $u \in \mathcal{S}'(\mathbb{R}^d)$, one can formally write

$$u = \sum_{q \geq -1} \Delta_q u \quad \text{in,}$$

and the non-homogeneous Sobolev spaces $H^s(\mathbb{R}^d)$, with $s \in \mathbb{R}$, can be characterized as follows

Proposition 6.2.4. *Let $d \in \mathbb{N}^*$, $s \in \mathbb{R}$ and $u \in H^s(\mathbb{R}^d)$. Then,*

$$\|u\|_{H^s} := \left(\int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \sim \left(\sum_{q \geq -1} 2^{2qs} \|\Delta_q u\|_{L^2}^2 \right)^{\frac{1}{2}}$$

Moreover, there exists a square-summable sequence of positive numbers $\{c_q(u)\}_q$ with $\sum_q c_q(u)^2 = 1$, such that

$$\|\Delta_q u\|_{L^2} \leq c_q(u) 2^{-qs} \|u\|_{H^s}. \quad (6.2.8)$$

6.2.2 Paradifferential calculus.

The decomposition into dyadic blocks allows, at least formally, to write, for any tempered distributions u and v ,

$$uv = \sum_{\substack{q \in \mathbb{Z} \\ q' \in \mathbb{Z}}} \Delta_q u \Delta_{q'} v \quad (6.2.9)$$

The Bony decomposition (see for instance [14], [37] or [10] for more details) consists in splitting the above sum in three parts. The first corresponds to the low frequencies of u multiplied by the high frequencies of v , the second is the symmetric counterpart of the first, and the third part concerns the indices q and q' which are comparable. Then,

$$uv = T_u v + T_v u + R(u, v),$$

where

$$\begin{aligned} T_u v &= \sum_q S_{q-1} u \Delta_q v \\ T_v u &= \sum_{q'} S_{q'-1} v \Delta_{q'} u \\ R(u, v) &= \sum_{|q-q'| \leq 1} \Delta_q u \Delta_{q'} v. \end{aligned}$$

Using the quasi-orthogonality given in (6.2.5) and (6.2.6), we get the following relations.

Lemma 6.2.5. *For any tempered distributions u and v , we have*

$$\begin{aligned} \Delta_q (S_{q'-1} u \Delta_{q'} v) &= 0 && \text{if } |q - q'| \geq 5 \\ \Delta_q (S_{q'+1} u \Delta_{q'} v) &= 0 && \text{if } q' \leq q - 4. \end{aligned}$$

Lemma 6.2.5 implies the following decomposition, which we will widely use in this paper

$$\Delta_q(uv) = \sum_{|q'-q| \leq 4} \Delta_q (S_{q'-1} v \Delta_{q'} u) + \sum_{q' > q-4} \Delta_q (S_{q'+2} u \Delta_{q'} v). \quad (6.2.10)$$

As in J.-Y. Chemin and N. Lerner [43] we will also use the following decomposition of the first term on the right hand side of (6.2.10)

$$\begin{aligned} & \sum_{|q'-q|\leq 4} \Delta_q (S_{q'-1} v \Delta_{q'} u) \\ &= S_q u \Delta_q v + \sum_{|q'-q|\leq 4} [\Delta_q, S_{q'-1} u] \Delta_{q'} v + \sum_{|q'-q|\leq 4} (S_q - S_{q'-1}) u \Delta_{q'} v, \end{aligned} \quad (6.2.11)$$

where the commutator $[\Delta_q, a] b$ is defined as

$$[\Delta_q, a] b = \Delta_q (ab) - a \Delta_q b.$$

We also recall the following lemma concerning the commutators. One can find a proof of this lemma in [10, p. 110].

Lemma 6.2.6. *Let be $p, q, r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $f \in W^{1,p}(\mathbb{R}^3)$, $g \in L^q(\mathbb{R}^3)$. Then*

$$\|[\Delta_q, f] g\|_{L^r} \leq C 2^{-q} \|\nabla f\|_{L^p} \|g\|_{L^q}. \quad (6.2.12)$$

6.2.3 Auxiliary estimates

We first recall the following classical product rule in $H^s(\mathbb{R}^3)$ spaces.

Lemma 6.2.7. *For any $s > 0$, there exists a constant C such that, for any u, v in $H^s(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, we have*

$$\|uv\|_{H^s} \leq \frac{C^{s+1}}{s} (\|u\|_{L^\infty} \|v\|_{H^s} + \|v\|_{L^\infty} \|u\|_{H^s}). \quad (6.2.13)$$

To prove Lemma 6.2.7 it suffice to decompose the data uv using the decomposition (6.2.10) and apply repeatedly Hölder inequality.

In this paper, in order to perform a bootstrap argument in Section 6.5, for $t > 0$, we define the spaces $\tilde{L}^p([0, t], H^s(\mathbb{R}^3))$, with $p \geq 2$, as the closure of the set of smooth vector-fields under the norms

$$\|u\|_{\tilde{L}^p([0, t], H^s)} = \left(\sum_q 2^{2qs} \|\Delta_q u\|_{L^p([0, t], L^2)}^2 \right)^{\frac{1}{2}}.$$

From the above definition, it is easy to see that, for any $p \geq 2$, $\tilde{L}^p([0, t], H^s(\mathbb{R}^3))$ is smoother than $L^p([0, t], H^s(\mathbb{R}^3))$. From the above definition, we can prove the following lemma which gives similar estimates as (6.2.8).

Lemma 6.2.8. *Suppose that u belongs to $\tilde{L}^p([0, t], H^s(\mathbb{R}^3))$, with $s > 0$, then there exists a square-summable sequence of positive numbers $\{c_q(u)\}_{q \geq -1}$, with $\sum_q c_q(u)^2 = 1$, such that*

$$\|\Delta_q u\|_{L^p([0, t], L^2)} \leq c_q(u) 2^{-qs} \|u\|_{\tilde{L}^p([0, t], H^s)}.$$

For functions in $\tilde{L}^p([0, t], H^s(\mathbb{R}^3))$, we can prove similar estimates as in (6.2.13).

Lemma 6.2.9. *Let $T > 0$. For any $s > 0$, there exists a constant $C(s)$ depending on s such that, for any u, v in $\tilde{L}^\infty([0, T], H^s(\mathbb{R}^3)) \cap L^\infty([0, T], L^\infty(\mathbb{R}^3))$, we have*

$$\begin{aligned} & \|uv\|_{\tilde{L}^\infty([0, T], H^s)} \\ & \leq C(s) \left(\|u\|_{L^\infty([0, T], L^\infty)} \|v\|_{\tilde{L}^\infty([0, T], H^s)} + \|u\|_{\tilde{L}^\infty([0, T], H^s)} \|v\|_{L^\infty([0, T], L^\infty)} \right). \end{aligned}$$

Finally, we recall the definition of the weak- L^p spaces and a refined version of Young's inequality that we need in Section 6.4 (see [10] for a proof, for instance).

Definition 6.2.10. *For $1 < p < \infty$ and for any measurable function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, we define the space*

$$L^{p, \infty}(\mathbb{R}^d) \stackrel{\text{def}}{=} \left\{ f : \mathbb{R}^d \rightarrow \mathbb{R} \text{ measurable} : \|f\|_{L^{p, \infty}} < +\infty \right\},$$

where the quasinorm

$$\|f\|_{L^{p, \infty}} \stackrel{\text{def}}{=} \sup_{\lambda > 0} \lambda \mu(\{x \in \mathbb{R}^d : |f(x)| > \lambda\})^{\frac{1}{p}},$$

and where μ is the usual Lebesgue measure on \mathbb{R}^d .

Theorem 6.2.11. *Let $p, q, r \in]1, \infty[$ satisfying*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

*Then, a constant $C > 0$ exists such that, for any $f \in L^{p, \infty}(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, the convolution product $f * g$ belongs to $L^r(\mathbb{R}^d)$ and we have*

$$\|f * g\|_{L^r} \leq C \|f\|_{L^{p, \infty}} \|g\|_{L^q}. \quad (6.2.14)$$

6.3 Decomposition of the initial data

We recall that, for $0 < r < R$, in (6.1.9), we defined

$$\mathcal{C}_{r, R} = \left\{ \xi \in \mathbb{R}_\xi^3 \mid |\xi| \leq R, |\xi_h| \geq r, |\xi_3| \geq r \right\}.$$

Let ψ a C^∞ -function from \mathbb{R}^3 to \mathbb{R} such that

$$\psi(\xi) = \begin{cases} 1 & \text{if } 0 \leq |\xi| \leq 1 \\ 0 & \text{if } |\xi| \geq 2 \end{cases}$$

and $\Psi_{r, R} : \mathbb{R}^3 \rightarrow \mathbb{R}$ the following frequency cut-off function

$$\Psi_{r, R}(\xi) = \psi\left(\frac{|\xi|}{R}\right) \left[1 - \psi\left(\frac{|\xi_h|}{r}\right)\right] \left[1 - \psi\left(\frac{|\xi_3|}{r}\right)\right]. \quad (6.3.1)$$

Then, we have $\Psi_{r,R} \in \mathcal{D}(\mathbb{R}^3)$, $\text{supp } \Psi_{r,R} \subset \mathcal{C}_{\frac{r}{2}, 2R}$ and $\Psi_{r,R} \equiv 1$ on $\mathcal{C}_{r,R}$. We will decompose U_0 in the following way

$$U_0 = \bar{U}_0 + \tilde{U}_0,$$

where

$$\bar{U}_0 = \mathcal{P}_{r,R} U_0 = \Psi_{r,R}(D)U_0 = \mathcal{F}^{-1} \left(\Psi_{r,R}(\xi) \widehat{U}_0(\xi) \right).$$

Our goal is to get precise controls of the $H^s(\mathbb{R}^3)$ -norms of \tilde{U}_0 with respect to the frequency cut-off radii r and R .

Lemma 6.3.1. *Let $s > \frac{5}{2}$, $s_0 > 0$, $p \in]1, 2[$ and the initial data $U_0 \in Y_{s,s_0,p}$, where $Y_{s,s_0,p}$ is defined as in (6.1.6) and (6.1.7). There exists $\delta > 0$ such that, for $R > 0$ large enough and $r = R^{-\delta}$,*

$$\left\| \tilde{U}_0 \right\|_{H^s} \leq C \mathcal{C}(U_0) R^{-s_0},$$

where $\mathcal{C}(U_0)$ is defined in (6.1.8).

Proof. By the definition of $H^s(\mathbb{R}^3)$ -norm, we have

$$\begin{aligned} \left\| \tilde{U}_0 \right\|_{H^s}^2 &\leq \int_{\substack{|\xi_3| < r \\ |\xi_h| < R}} (1 + |\xi|^2)^s \left| \widehat{U}_0(\xi) \right|^2 d\xi_3 d\xi_h + \int_{\substack{|\xi_3| < R \\ |\xi_h| < r}} (1 + |\xi|^2)^s \left| \widehat{U}_0(\xi) \right|^2 d\xi_3 d\xi_h \\ &\quad + \int_{|\xi| > R} (1 + |\xi|^2)^s \left| \widehat{U}_0(\xi) \right|^2 d\xi \\ &= I_1 + I_2 + I_3. \end{aligned}$$

In what follows, we denote as \mathcal{F}_h and \mathcal{F}_v respectively the horizontal and vertical Fourier transforms. Let q, p' be positive numbers such that $q = \frac{p}{p-1}$, $q' = \frac{q}{2}$ and $p' = \frac{p}{2-p}$. Thus $1 \leq p < 2 < q$, and $p' \in [1, \infty]$ and the following relations hold

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{p'} + \frac{1}{q'} = 1.$$

For the first integral, we write

$$\begin{aligned} I_1 &= \int_{\substack{|\xi_3| < r \\ |\xi_h| < R}} \left(\frac{1 + |\xi_h|^2 + \xi_3^2}{1 + \xi_3^2} \right)^s (1 + \xi_3^2)^s \left| \widehat{U}_0(\xi) \right|^2 d\xi_3 d\xi_h \\ &\leq CR^{2s} \int_{|\xi_3| < r} \int_{|\xi_h| < R} (1 + \xi_3^2)^s \left| \widehat{U}_0(\xi) \right|^2 d\xi_h d\xi_3 \\ &\leq CR^{2s} \int_{|\xi_3| < r} \int_{\mathbb{R}_{\xi_h}^2} (1 + \xi_3^2)^s \left| \widehat{U}_0(\xi) \right|^2 d\xi_h d\xi_3. \end{aligned}$$

Plancherel theorem in the horizontal variable yields

$$\begin{aligned} I_1 &\leq CR^{2s} \int_{|\xi_3| < r} \int_{\mathbb{R}_{\xi_h}^2} (1 + \xi_3^2)^s \left| \widehat{U}_0(\xi) \right|^2 d\xi_h d\xi_3 \\ &= CR^{2s} \int_{|\xi_3| < r} \int_{\mathbb{R}_{x_h}^2} (1 + \xi_3^2)^s |\mathcal{F}_v U_0(x_h, \xi_3)|^2 dx_h d\xi_3. \end{aligned}$$

Applying Fubini theorem and Hölder inequality in the vertical direction, we get

$$\begin{aligned} I_1 &\leq CR^{2s} \left(\int_{|\xi_3| < r} (1 + \xi_3^2)^{p's} \right)^{\frac{1}{p'}} \int_{\mathbb{R}_{x_h}^2} \left(\int_{\mathbb{R}_{\xi_3}^2} |\mathcal{F}_v U_0(x_h, \xi_3)|^{2q'} \xi_3 \right)^{\frac{1}{q'}} dx_h \\ &\leq CR^{2s} r^{\frac{1}{p'}} \int_{\mathbb{R}_{x_h}^2} \left(\int_{\mathbb{R}_{\xi_3}^2} |\mathcal{F}_v U_0(x_h, \xi_3)|^q \xi_3 \right)^{\frac{2}{q}} dx_h, \end{aligned}$$

Finally, we use Hausdorff-Young inequality in the vertical direction, taking into account the relation $r \sim R^{-\delta}$, to obtain

$$I_1 = \int_{\substack{|\xi_3| < r \\ |\xi_h| < R}} (1 + |\xi_h|^2 + \xi_3^2)^s \left| \widehat{U}_0(\xi) \right|^2 d\xi_3 d\xi_h \leq CR^{2s - \frac{\delta}{p'}} \|U_0\|_{L_h^2 L_v^p}^2. \quad (6.3.2)$$

Similar calculations lead to the following estimate for the second integral

$$I_2 = \int_{\substack{|\xi_3| < R \\ |\xi_h| < r}} (1 + |\xi_h|^2 + \xi_3^2)^s \left| \widehat{U}_0(\xi) \right|^2 d\xi_3 d\xi_h \leq CR^{2s - \frac{\delta}{p'}} \|U_0\|_{L_h^2 L_v^p}^2. \quad (6.3.3)$$

The third term contains only the very high frequencies, hence is much simpler to control

$$I_3 = \int_{|\xi| > R} (1 + |\xi|^2)^{-s_0} (1 + |\xi|^2)^{s+s_0} \left| \widehat{U}_0(\xi) \right|^2 d\xi \leq R^{-2s_0} \|U_0\|_{H^{s+s_0}}^2. \quad (6.3.4)$$

We choose the free parameter δ such that

$$\frac{\delta}{p'} = 2(s + s_0).$$

Combining the estimates (6.3.2) to (6.3.4), we can conclude the proof. \square

6.4 Strichartz-type estimates for the linear system

We recall that the projector $\mathcal{P}_{r,R}$ associates any tempered distribution f to

$$\mathcal{P}_{r,R} f = \Psi_{r,R}(D) f = \mathcal{F}^{-1} \left(\Psi_{r,R}(\xi) \widehat{f}(\xi) \right), \quad (6.4.1)$$

where the function $\Psi_{r,R}$ is defined in (6.3.1). In this section, we consider the following frequency cut-off free-wave system

$$\begin{cases} \partial_t \bar{U}^\varepsilon = \frac{1}{\varepsilon} \mathcal{B} \bar{U}^\varepsilon \\ \bar{U}^\varepsilon|_{t=0} = \mathcal{P}_{r,R} U_0. \end{cases} \quad (6.4.2)$$

where the linear hyperbolic operator \mathcal{B} is defined in (6.1.3). Since the system (6.4.2) is linear and the Fourier transform of the initial data are supported in $\mathcal{C}_{\frac{r}{2}, 2R}$, the Fourier transform of the solution $\bar{U}^\varepsilon(t)$ is also supported in $\mathcal{C}_{\frac{r}{2}, 2R}$ for any $t > 0$. The aim of the present section is to analyze the dispersive properties of system (6.4.2) as $\varepsilon \rightarrow 0$, i.e to prove the following theorem

Theorem 6.4.1. *Let $q \in [2, +\infty]$ and $p \geq \frac{4q}{q-2}$. For any $U_0 \in L^2(\mathbb{R}^3)$, the system (6.4.2) has a global solution \bar{U}^ε such that,*

$$\|\bar{U}^\varepsilon\|_{L^p(\mathbb{R}_+; L^q(\mathbb{R}^3))} \leq CR^{\frac{3}{2}-\frac{3}{q}+\frac{4}{p}} r^{-\frac{2}{p}} \varepsilon^{\frac{1}{p}} \|U_0\|_{L^2(\mathbb{R}^3)}. \quad (6.4.3)$$

Writing this system in Fourier frequency variable, we get

$$\begin{cases} \partial_t \widehat{U}^\varepsilon = \frac{1}{\varepsilon} \widehat{\mathcal{B}} \widehat{U}^\varepsilon \\ \widehat{U}^\varepsilon|_{t=0} = \Psi_{r,R}(D) \widehat{U}_0, \end{cases} \quad (6.4.4)$$

where

$$\widehat{\mathcal{B}}(\xi) = \begin{pmatrix} 0 & 1 & 0 & -i\bar{\gamma}\xi_1 \\ -1 & 0 & 0 & -i\bar{\gamma}\xi_2 \\ 0 & 0 & 0 & -i\bar{\gamma}\xi_3 \\ -i\bar{\gamma}\xi_1 & -i\bar{\gamma}\xi_2 & -i\bar{\gamma}\xi_3 & 0 \end{pmatrix}.$$

The characteristic polynomial of $\widehat{\mathcal{B}}(\xi)$ writes

$$\mathcal{P}_{\widehat{\mathcal{B}}(\xi)}(\lambda) = \det(\widehat{\mathcal{B}}(\xi) - \lambda \mathbb{I}_{\mathbb{R}^4}) = \lambda^4 + (1 + \bar{\gamma}^2 |\xi|^2) \lambda^2 + \bar{\gamma}^2 \xi_3^2. \quad (6.4.5)$$

So, straightforward calculations shows that the eigenvalues of $\widehat{\mathcal{B}}(\xi)$ are

$$\lambda_{\varepsilon_1, \varepsilon_2}(\xi) = \varepsilon_1 i \sqrt{\frac{1}{2} \left((1 + \bar{\gamma}^2 |\xi|^2) + \varepsilon_2 \sqrt{(1 + \bar{\gamma}^2 |\xi|^2)^2 - 4\bar{\gamma}^2 \xi_3^2} \right)}.$$

where $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$. We recall that, for any $A, B \in \mathbb{R}$, we have

$$\sqrt{A \pm \sqrt{B}} = \sqrt{\frac{A + \sqrt{A^2 - B}}{2}} \pm \sqrt{\frac{A - \sqrt{A^2 - B}}{2}}.$$

Then, setting

$$\begin{aligned} A &= 1 + \bar{\gamma}^2 |\xi|^2 \\ B &= (1 + \bar{\gamma}^2 |\xi|^2)^2 - 4\bar{\gamma}^2 \xi_3^2, \end{aligned}$$

we can rewrite the eigenvalues as

$$\lambda_{\varepsilon_1, \varepsilon_2}(\xi) = \varepsilon_1 \frac{i}{2} \left(\sqrt{1 + \bar{\gamma}^2 |\xi|^2 + 2\bar{\gamma}\xi_3} + \varepsilon_2 \sqrt{1 + \bar{\gamma}^2 |\xi|^2 - 2\bar{\gamma}\xi_3} \right). \quad (6.4.6)$$

We remark that a similar spectral analysis has already been performed in the work [68] with the difference that the domain considered in [68] was of the form $\mathbb{R}_h^2 \times \mathbb{T}_v^1$ instead of \mathbb{R}^3 .

Now, in order to understand the behavior of the solutions to (6.4.2) we define the following operators

$$G_\lambda(t) f(x) = \mathcal{F}^{-1} \left(e^{\frac{t}{\varepsilon} \lambda(\xi)} \widehat{f}(\xi) \right) (x) = \int_{\mathbb{R}_\xi^3 \times \mathbb{R}_y^3} f(y) e^{\frac{t}{\varepsilon} \lambda(\xi) + i(x-y) \cdot \xi} d\xi dy,$$

where the eigenvalues $\lambda(\xi)$ are given in (6.4.6)

$$\lambda(\xi) = \pm \frac{i}{2} \left(\sqrt{1 + \bar{\gamma}^2 |\xi|^2 + 2\bar{\gamma}\xi_3} + \pm \sqrt{1 + \bar{\gamma}^2 |\xi|^2 - 2\bar{\gamma}\xi_3} \right).$$

Lemma 6.4.2. For any $f \in L^1(\mathbb{R}^3)$ and for any $t > 0$, we have

$$\|\Delta_j^h \Delta_k^v G_\lambda(t) f\|_{L^\infty(\mathbb{R}^3)} \leq C \max\left\{2^{\frac{5j}{2}-k}, 2^{5j-k}\right\} \left(\frac{\varepsilon}{t}\right)^{\frac{1}{2}} \|f\|_{L^1(\mathbb{R}^3)} \quad (6.4.7)$$

Remark 6.4.3. The estimates in Lemma 6.4.2 are not optimal for $t \leq \varepsilon$. Indeed, for $t \leq \varepsilon$, using Bernstein lemma 6.2.1, we can simply bound

$$\begin{aligned} \|\Delta_j^h \Delta_k^v G_\lambda(t) f\|_{L^\infty(\mathbb{R}^3)} &\leq C 2^j 2^{\frac{k}{2}} \|\Delta_j^h \Delta_k^v G_\lambda(t) f\|_{L^2(\mathbb{R}^3)} \\ &\leq C 2^j 2^{\frac{k}{2}} \|\Delta_j^h \Delta_k^v f\|_{L^2(\mathbb{R}^3)} \\ &\leq C 2^{2j} 2^k \|\Delta_j^h \Delta_k^v f\|_{L^1(\mathbb{R}^3)} \\ &\leq C 2^{2j} 2^k \|f\|_{L^1(\mathbb{R}^3)} \end{aligned} \quad (6.4.8)$$

To prove Lemma 6.4.2, we write

$$\begin{aligned} \Delta_j^h \Delta_k^v G_\lambda(t) f(x) &= \int_{\mathbb{R}_y^3} f(y) \int_{\mathbb{R}_\xi^3} e^{\frac{t}{\varepsilon} \lambda(\xi) + i(x-y) \cdot \xi} \varphi(2^{-j} |\xi_h|) \varphi(2^{-k} |\xi_3|) d\xi dy \\ &= K_{j,k}^\lambda \left(\frac{t}{\varepsilon}, \cdot \right) * f(x), \end{aligned}$$

where

$$K_{j,k}^\lambda(\tau, x) = \int_{\mathbb{R}_\xi^3} e^{\tau \lambda(\xi) + i x \cdot \xi} \varphi(2^{-j} |\xi_h|) \varphi(2^{-k} |\xi_3|) d\xi. \quad (6.4.9)$$

The key point to prove Lemma 6.4.2 is to estimate the kernel function $K_{j,k}^\lambda$ using the method of [40] and [38]. For that purpose, we perform the change of variables

$$z = 2^j x \quad \text{and} \quad \zeta = 2^{-j} \xi.$$

Then, we have

$$K_{j,k}^\lambda(\tau, x) = 2^{3j} \tilde{K}_{j,k}^\lambda(\tau, z),$$

where

$$\tilde{K}_{j,k}^\lambda(\tau, z) = \int_{\mathbb{R}_\zeta^3} e^{\tau \lambda(2^j \zeta) + i z \cdot \zeta} \varphi(|\zeta_h|) \varphi(2^{j-k} |\zeta_3|) d\zeta. \quad (6.4.10)$$

We remark that the invariance of $\tilde{K}_{j,k}^\lambda$ by rotation in the plane $\mathbb{R}_{\zeta_h}^2$ allows to restrict the study to the case $z_2 = 0$. Indeed, if $z_2 \neq 0$, we can perform a rotation of angle θ , with $\cot \theta = \frac{z_1}{z_2}$ to suppress the second component of z . Following the ideas of [40] and [38], we will apply an integration by parts to $\tilde{K}_{j,k}^\lambda(\tau, z)$. Let

$$\begin{aligned} A_j(\zeta) &= \sqrt{1 + 2^{2j} \bar{\gamma}^2 |\zeta|^2 + 2^{j+1} \bar{\gamma} \zeta_3} \\ B_j(\zeta) &= \sqrt{1 + 2^{2j} \bar{\gamma}^2 |\zeta|^2 - 2^{j+1} \bar{\gamma} \zeta_3} \\ \lambda_j(\zeta) &= \lambda(2^j \zeta) = \pm \frac{i}{2} (A_j(\zeta) \pm B_j(\zeta)) \\ a_j(\zeta) &= \partial_{\zeta_2} \lambda_j(\zeta) = \pm \frac{i 2^{2j} \bar{\gamma}^2 \zeta_2}{2} \left(\frac{1}{A_j(\zeta)} \pm \frac{1}{B_j(\zeta)} \right). \end{aligned}$$

We consider the operator

$$\mathcal{L}_\lambda \stackrel{\text{def}}{=} \frac{1}{1 + \tau a_j^2} (\text{Id} - i a_j \partial_{\zeta_2}). \quad (6.4.11)$$

Direct calculations give

$$\mathcal{L}_\lambda (e^{\tau \lambda_j(\zeta) + iz \cdot \zeta} \varphi (2^{j-k} |\zeta_3|)) = e^{\tau \lambda_j(\zeta) + iz \cdot \zeta} \varphi (2^{j-k} |\zeta_3|),$$

thus,

$$\tilde{K}_{j,k}^\lambda(\tau, z) = \int_{\mathbb{R}_\zeta^3} e^{\tau \lambda(2^j \zeta) + iz \cdot \zeta} \varphi (2^{j-k} |\zeta_3|) {}^T \mathcal{L}_\lambda (\varphi (|\zeta_h|)) d\zeta,$$

where

$${}^T \mathcal{L}_\lambda (\varphi (|\zeta_h|)) = \left[\frac{1}{1 + \tau a_j^2} + i (\partial_{\zeta_2} a_j) \frac{1 - \tau a_j^2}{(1 + \tau a_j^2)} \right] \varphi (|\zeta_h|) + \frac{i a_j}{1 + \tau a_j^2} \partial_{\zeta_2} \varphi (|\zeta_h|). \quad (6.4.12)$$

Lemma 6.4.4. *There exists a constant $C > 0$ such that*

$$|{}^T \mathcal{L}_\lambda (\varphi (|\zeta_h|))| \leq \frac{C (1 + 2^j)}{1 + \min \{1, 2^{3j}\} \tau \zeta_2^2 \zeta_3^2}.$$

Proof. By definition of φ , to estimate $\tilde{K}_{j,k}^\lambda$, we can consider

$$\frac{3}{4} \leq |\zeta_h|, |\zeta_3| \leq \frac{8}{3}.$$

Then, there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned} C_1 2^j &\leq A_j(\zeta) = \sqrt{2^{2j} \bar{\gamma}^2 |\zeta_h|^2 + (1 + 2^j \bar{\gamma} \zeta_3)^2} \leq C_2 \max \{1, 2^j\} \\ C_1 2^j &\leq B_j(\zeta) = \sqrt{2^{2j} \bar{\gamma}^2 |\zeta_h|^2 + (1 - 2^j \bar{\gamma} \zeta_3)^2} \leq C_2 \max \{1, 2^j\}, \end{aligned}$$

and

$$|A_j(\zeta) - B_j(\zeta)| = \frac{|A_j(\zeta)^2 - B_j(\zeta)^2|}{A_j(\zeta) + B_j(\zeta)} = \frac{2^{j+2} \bar{\gamma} |\zeta_3|}{A_j(\zeta) + B_j(\zeta)} > C_1 \min \{1, 2^j\} |\zeta_3|.$$

As a consequence, we have

$$C_1 \min \{1, 2^{3j}\} |\zeta_2 \zeta_3| < |a_j(\zeta)| < C_2 2^j. \quad (6.4.13)$$

Now, differentiating a_j with respect to ζ_2 , we get

$$\partial_{\zeta_2} a_j(\zeta) = \pm i 2^{2j-1} \bar{\gamma}^2 \left(\frac{1}{A_j(\zeta)} \pm \frac{1}{B_j(\zeta)} \right) - i 2^{4j-1} \bar{\gamma}^4 \zeta_2^2 \left(\frac{1}{A_j(\zeta)^3} \pm \frac{1}{B_j(\zeta)^3} \right)$$

Then, we can choose C_2 such that

$$|\partial_{\zeta_2} a_j(\zeta)| < C_2 2^j. \quad (6.4.14)$$

Using Estimates (6.4.13) and (6.4.14), we obtain

$$\begin{aligned} \left| \frac{\varphi(|\zeta_h|)}{1 + \tau a_j^2} \right| &\leq \frac{C}{1 + \min\{1, 2^{3j}\} \tau \zeta_2^2 \zeta_3^2} \\ \left| i(\partial_{\zeta_2} a_j) \frac{1 - \tau a_j^2}{(1 + \tau a_j^2)^2} \varphi(|\zeta_h|) \right| &\leq \frac{C 2^j}{1 + \min\{1, 2^{3j}\} \tau \zeta_2^2 \zeta_3^2} \\ \left| \frac{i a_j}{1 + \tau a_j^2} \partial_{\zeta_2} \varphi(|\zeta_h|) \right| &\leq \frac{C 2^j}{1 + \min\{1, 2^{3j}\} \tau \zeta_2^2 \zeta_3^2}, \end{aligned}$$

which imply

$$\left| {}^T \mathcal{L}_\lambda(\varphi(|\zeta_h|)) \right| \leq \frac{C(1 + 2^j)}{1 + \min\{1, 2^{3j}\} \tau \zeta_2^2 \zeta_3^2}.$$

□

Lemma 6.4.5. For any $\tau > 0$, $j, k \in \mathbb{N}$,

$$\left\| \tilde{K}_{j,k}^\lambda(\tau, \cdot) \right\|_{L_x^\infty} \leq C(1 + 2^j) \max\left\{1, 2^{-\frac{3j}{2}}\right\} \tau^{-\frac{1}{2}} \left\| \varphi(2^{j-k} \zeta_3) \zeta_3^{-1} \right\|_{L_\zeta^\infty}.$$

Proof. We recall that

$$\tilde{K}_{j,k}^\lambda(\tau, z) = \int_{\mathbb{R}_\zeta^3} e^{\tau \lambda(2^j \zeta) + iz \cdot \zeta} \varphi(2^{j-k} |\zeta_3|) {}^T \mathcal{L}_\lambda(\varphi(|\zeta_h|)) d\zeta,$$

Then, using Lemma 6.4.4 and the definition of φ , there exist positive constants $c_1, c_2 > 0$ such that

$$\begin{aligned} \left\| \tilde{K}_{j,k}^\lambda(\tau, \cdot) \right\|_{L_x^\infty} &\leq C(1 + 2^j) \left\| \varphi(2^{j-k} \zeta_3) \int_{c_1}^{c_2} \frac{d\zeta_2}{1 + \min\{1, 2^{3j}\} \tau \zeta_2^2 \zeta_3^2} \right\|_{L_{\zeta_3}^\infty} \\ &\leq C(1 + 2^j) \max\left\{1, 2^{-\frac{3j}{2}}\right\} \tau^{-\frac{1}{2}} \left\| \varphi(2^{j-k} \zeta_3) \zeta_3^{-1} \right\|_{L_\zeta^\infty}. \end{aligned}$$

□

From Lemma 6.4.5, we deduce the following immediate corollary

Corollary 6.4.6. For any $\tau > 0$, $j, k \in \mathbb{N}$,

$$\left\| K_{j,k}^\lambda(\tau, \cdot) \right\|_{L_x^\infty} \leq C(2^{4j-k} + 2^{5j-k}) \max\left\{1, 2^{-\frac{3j}{2}}\right\} \tau^{-\frac{1}{2}}.$$

Proof of Lemma 6.4.2. We recall that

$$\Delta_j^h \Delta_k^v G_\lambda(t) f(x) = K_{j,k}^\lambda\left(\frac{t}{\varepsilon}, \cdot\right) * f(x).$$

Using Young's inequality, we obtain

$$\begin{aligned} \left\| \Delta_j^h \Delta_k^v G_\lambda(t) f \right\|_{L_x^\infty} &\leq C \left\| K_{j,k}^\lambda\left(\frac{t}{\varepsilon}, \cdot\right) \right\|_{L_x^\infty}, \\ \|f\|_{L_x^1} &\leq C \max\left\{2^{\frac{5j}{2}-k}, 2^{5j-k}\right\} \left(\frac{\varepsilon}{t}\right)^{\frac{1}{2}} \|f\|_{L_x^1}. \end{aligned}$$

□

Lemma 6.4.7. *Let $0 < r \ll 1 \ll R$ and recall that*

$$\mathcal{C}_{r,R} = \{ \xi \in \mathbb{R}^3 \mid |\xi_h|, |\xi_3| \geq r, |\xi| \leq R \}.$$

Let $\Psi_{r,R} : \mathbb{R}^3 \rightarrow \mathbb{R}$, $\text{supp } \Psi_{r,R} \subset \mathcal{C}_{\frac{r}{2}, 2R}$ and $\Psi_{r,R}|_{\mathcal{C}_{r,R}} \equiv 1$. Then,

$$\| \Psi_{r,R}(D)G_\lambda(t)f \|_{L^\infty(\mathbb{R}^3)} \leq Cr^{-1}R^4 \left(\frac{\varepsilon}{t} \right)^{\frac{1}{2}} \|f\|_{L^1(\mathbb{R}^3)}.$$

Proof. We choose $m_1 \in \mathbb{Z}_-$ and $m_2 \in \mathbb{Z}_+$ such that

$$\frac{3}{4} \leq r2^{-m_1} \leq \frac{8}{3} \quad \text{and} \quad \frac{3}{4} \leq R2^{-m_2} \leq \frac{8}{3}.$$

Then,

$$\| \Psi_{r,R}(D)G_\lambda(t)f \|_{L^\infty(\mathbb{R}^3)} \leq \sum_{j=m_1}^{m_2} \sum_{k=m_1}^{m_2} \| \Delta_j^h \Delta_k^v G_\lambda(t)f \|_{L^\infty(\mathbb{R}^3)}.$$

It remains to apply Lemma 6.4.2 to obtain Lemma 6.4.7. □

Lemma 6.4.8. *For $t \leq \varepsilon$, we have*

$$\| \Psi_{r,R}(D)G_\lambda(t)f \|_{L^\infty(\mathbb{R}^3)} \leq CR^3 \|f\|_{L^1(\mathbb{R}^3)}.$$

Proof. We use the same estimates as in Remark 6.4.3. □

Lemma 6.4.9. *For any $q \in [2, +\infty]$ and $\bar{q} \in \mathbb{R}$ such that $\frac{1}{q} + \frac{1}{\bar{q}} = 1$, we have*

$$\| \Psi_{r,R}(D)G_\lambda(t)f \|_{L^q(\mathbb{R}^3)} \leq C \left[R^3 \min \left\{ 1, R^2 r^{-1} \left(\frac{\varepsilon}{t} \right)^{\frac{1}{2}} \right\} \right]^{1-\frac{2}{q}} \|f\|_{L^{\bar{q}}(\mathbb{R}^3)}.$$

Proof. We already proved that

$$\| \Psi_{r,R}(D)G_\lambda(t)f \|_{L^\infty(\mathbb{R}^3)} \leq CR^3 \min \left\{ 1, R^2 r^{-1} \left(\frac{\varepsilon}{t} \right)^{\frac{1}{2}} \right\} \|f\|_{L^1(\mathbb{R}^3)}$$

The definition of $\Psi_{r,R}(D)G_\lambda(t)$ implies that

$$\| \Psi_{r,R}(D)G_\lambda(t)f \|_{L^2(\mathbb{R}^3)} \leq C \|f\|_{L^2(\mathbb{R}^3)}.$$

Since the point $\left(\frac{1}{q}, \frac{1}{\bar{q}} \right)$ belongs to the line segment $\left[(0, 1), \left(\frac{1}{2}, \frac{1}{2} \right) \right]$, the Riesz-Thorin theorem yields

$$\| \Psi_{r,R}(D)G_\lambda(t)f \|_{L^q(\mathbb{R}^3)} \leq C \left[R^3 \min \left\{ 1, R^2 r^{-1} \left(\frac{\varepsilon}{t} \right)^{\frac{1}{2}} \right\} \right]^{1-\frac{2}{q}} \|f\|_{L^{\bar{q}}(\mathbb{R}^3)}.$$

□

The following theorem gives Strichartz estimates of \bar{U}^ε in the direction of each eigenvector of the operator $\bar{\mathcal{B}}$.

Theorem 6.4.10. *Let $q \in [2, +\infty]$ and $p \geq \frac{4q}{q-2}$. Then,*

$$\|\Psi_{r,R}(D)G_\lambda(t)f\|_{L_t^p(L^q(\mathbb{R}^3))} \leq CR^{\frac{3}{2}-\frac{3}{q}+\frac{4}{p}}r^{-\frac{2}{p}}\varepsilon^{\frac{1}{p}}\|f\|_{L^2(\mathbb{R}^3)}.$$

Proof. Following the ideas of [40] and [38], we will apply the so-called TT^* method, which consist in an argument of duality. Let \bar{p} and \bar{q} such that

$$\frac{1}{p} + \frac{1}{\bar{p}} = \frac{1}{q} + \frac{1}{\bar{q}} = 1,$$

and

$$\mathcal{B} = \left\{ \varphi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^3) \mid \|\varphi\|_{L_t^{\bar{p}}(L_x^{\bar{q}})} \leq 1 \right\}.$$

Then, considering $\Phi = \Psi_{r,R}(D)\varphi$ and using Plancherel theorem and Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|\Psi_{r,R}(D)G_\lambda(t)f\|_{L_t^p(L^q(\mathbb{R}^3))} &= \sup_{\varphi \in \mathcal{B}} \int_{\mathbb{R}_+} \langle \Psi_{r,R}(D)G_\lambda(t)f, \varphi \rangle_{L_x^2} dt \\ &= (2\pi)^{-3} \sup_{\varphi \in \mathcal{B}} \int_{\mathbb{R}_+ \times \mathbb{R}_\xi^3} \widehat{f}(t, \xi) \widehat{\Phi}(t, \xi) e^{\frac{t}{\varepsilon}\lambda(\xi)} dt d\xi \\ &\leq (2\pi)^{-3} \sup_{\varphi \in \mathcal{B}} \|f\|_{L^2} \left\| \int_{\mathbb{R}_+} \widehat{\Phi}(t, \xi) e^{\frac{t}{\varepsilon}\lambda(\xi)} dt \right\|_{L_\xi^2}. \end{aligned}$$

It remains to estimate

$$I = \left\| \int_{\mathbb{R}_+} \widehat{\Phi}(t, \xi) e^{\frac{t}{\varepsilon}\lambda(\xi)} dt \right\|_{L_\xi^2}.$$

Recalling that $\lambda(\xi)$ is an imaginary number, using several times Fubini's theorem, Plancherel theorem and Hölder's inequality, we have

$$\begin{aligned} I^2 &= \left\langle \int_{\mathbb{R}_+} \widehat{\Phi}(t, \xi) e^{\frac{t}{\varepsilon}\lambda(\xi)} dt, \int_{\mathbb{R}_+} \overline{\widehat{\Phi}(s, \xi) e^{\frac{s}{\varepsilon}\lambda(\xi)}} ds \right\rangle_{L^2} \\ &= \int_{\mathbb{R}_\xi^3} \left(\int_{\mathbb{R}_+} \widehat{\Phi}(t, \xi) e^{\frac{t}{\varepsilon}\lambda(\xi)} dt \right) \left(\int_{\mathbb{R}_+} \overline{\widehat{\Phi}(s, \xi) e^{\frac{s}{\varepsilon}\lambda(\xi)}} ds \right) d\xi \\ &= \int_{\mathbb{R}_\xi^3} \left(\int_{(\mathbb{R}_+)^2} \widehat{\Phi}(t, \xi) \overline{\widehat{\Phi}(s, \xi)} e^{-\frac{t-s}{\varepsilon}\lambda(\xi)} ds dt \right) d\xi \\ &= \int_{(\mathbb{R}_+)^2} \int_{\mathbb{R}_\xi^3} (\Psi_{r,R}(D)\varphi(t, -x)) (\Psi_{r,R}(D)G_\lambda(t-s)\varphi(t, x)) dx dt ds \\ &\leq C \int_{(\mathbb{R}_+)^2} \|\varphi(s)\|_{L_x^{\bar{q}}} \|\Psi_{r,R}(D)G_\lambda(t-s)\varphi(t)\|_{L_x^q} dt ds. \end{aligned}$$

Next, using Lemma 6.4.9, Hölder's inequality, we get

$$\begin{aligned}
 I^2 &\leq C \int_{(\mathbb{R}_+)^2} \|\varphi(s)\|_{L_x^{\bar{q}}} \|\varphi(t)\|_{L_x^{\bar{q}}} \left[R^3 \min \left\{ 1, \frac{R^2 r^{-1} \varepsilon^{-\frac{1}{2}}}{|t-s|^{\frac{1}{2}}} \right\} \right]^{1-\frac{2}{q}} ds dt \\
 &\leq C \|\varphi\|_{L_t^{\bar{p}}(L_x^{\bar{q}})} \left\| \int_{\mathbb{R}_+} \|\varphi(s)\|_{L_x^{\bar{q}}} \left[R^3 \min \left\{ 1, \frac{R^2 r^{-1} \varepsilon^{-\frac{1}{2}}}{|t-s|^{\frac{1}{2}}} \right\} \right]^{1-\frac{2}{q}} ds \right\|_{L_t^p} \\
 &\leq C \|\varphi\|_{L_t^{\bar{p}}(L_x^{\bar{q}})} R^{3(1-\frac{2}{q})} \|\varphi(\cdot)\|_{L_x^{\bar{q}}} *_t M(\cdot) \|_{L_t^p},
 \end{aligned}$$

where

$$M(t) = \left[\min \left\{ 1, \frac{R^2 r^{-1} \varepsilon^{-\frac{1}{2}}}{|t|^{\frac{1}{2}}} \right\} \right]^{1-\frac{2}{q}}.$$

If $(p, q) = (+\infty, 2)$, Theorem 6.4.10 is obvious from the definition of $\Psi_{r,R}(D)G_\lambda(t)$. In the case where $q > 2$, we study two different cases

- If $p > \frac{4q}{q-2}$ then $M \in L_t^{\frac{p}{2}}$. For any $q > 2$,

$$\begin{aligned}
 \|M\|_{L_t^{\frac{p}{2}}} &= \left(\int_0^{R^4 r^{-2} \varepsilon} dt + \int_{R^4 r^{-2} \varepsilon}^{+\infty} \left(\frac{R^4 r^{-2} \varepsilon}{t} \right)^{\frac{p}{2}(\frac{1}{2}-\frac{1}{q})} dt \right)^{\frac{2}{p}} \\
 &= (R^4 r^{-2} \varepsilon)^{\frac{2}{p}} \left(1 + \int_1^{+\infty} \left(\frac{1}{\tau} \right)^{\frac{p}{2}(\frac{1}{2}-\frac{1}{q})} d\tau \right)^{\frac{2}{p}} \leq C (R^4 r^{-2} \varepsilon)^{\frac{2}{p}}.
 \end{aligned}$$

Thus, using the classical Young's inequality, we obtain

$$I^2 \leq C R^{3(1-\frac{2}{q})} \|\varphi\|_{L_t^{\bar{p}}(L_x^{\bar{q}})}^2 \|M\|_{L_t^{\frac{p}{2}}} \leq C R^{3(1-\frac{2}{q})} (R^4 r^{-2} \varepsilon)^{\frac{2}{p}} \|\varphi\|_{L_t^{\bar{p}}(L_x^{\bar{q}})}^2. \quad (6.4.15)$$

- If $p = \frac{4q}{q-2}$ then Young's inequality does not work anymore because $M \notin L_t^{\frac{p}{2}}$. Since M belong to the space $L_t^{\frac{p}{2}, \infty}$ and

$$\|M\|_{L_t^{\frac{p}{2}, \infty}} \sim (R^4 r^{-2} \varepsilon)^{\frac{2}{p}},$$

applying Theorem 6.2.11, we also get

$$I^2 \leq C R^{3(1-\frac{2}{q})} \|\varphi\|_{L_t^{\bar{p}}(L_x^{\bar{q}})}^2 \|M\|_{L_t^{\frac{p}{2}, \infty}} \leq C R^{3(1-\frac{2}{q})} (R^4 r^{-2} \varepsilon)^{\frac{2}{p}} \|\varphi\|_{L_t^{\bar{p}}(L_x^{\bar{q}})}^2. \quad (6.4.16)$$

Since,

$$I^2 \leq C R^{3(1-\frac{2}{q})} \left(\frac{R^4}{r^2} \right)^{\frac{2}{p}} \varepsilon^{\frac{2}{p}} \|\varphi\|_{L_t^{\bar{p}}(L_x^{\bar{q}})}^2,$$

we immediately deduce that,

$$\|\Psi_{r,R}(D)G_\lambda(t)f\|_{L_t^p(L_x^q)} \leq C (2\pi)^{-3} R^{\frac{3}{2}-\frac{3}{q}+\frac{4}{p}} r^{-\frac{2}{p}} \varepsilon^{\frac{1}{p}} \|f\|_{L_x^2}.$$

□

Remark 6.4.11. We want to make some remarks about the dispersive result in Theorem 6.4.10.

1. Unlike the case of viscous fluids (see for instance [40], [38] or [119]), we cannot obtain dispersive estimates for $\Psi_{r,R}(D)G_\lambda(t)f$ in an $L_t^1(L_x^q)$ -norm, due to the fact that we do not have damping effect given by the viscosity terms. This is one of the main reasons why we can only obtain an almost global existence result.
2. The result of Theorem 6.4.10 is slightly better than the dispersive estimates obtained in [59] in the sense where we can treat the limit case $p = \frac{4q}{q-2}$, using Theorem 6.2.11 to get (6.4.16). In general cases, (6.4.16) is known as the Hardy-Littlewood-Sobolev inequality, where one uses the fact that the function $|x|^{-\frac{d}{p}}$ belongs to $L^{p,\infty}(\mathbb{R}^d)$ but not to $L^p(\mathbb{R}^d)$.

Proof of Theorem 6.4.1. We recall that in the Fourier variable, the system (6.4.2) writes as

$$\begin{cases} \partial_t \widehat{U}^\varepsilon = \frac{1}{\varepsilon} \widehat{\mathcal{B}} \widehat{U}^\varepsilon \\ \widehat{U}^\varepsilon|_{t=0} = \Psi_{r,R}(D) \widehat{U}_0, \end{cases} \quad (6.4.4)$$

where

$$\widehat{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & -i\bar{\gamma}\xi_1 \\ -1 & 0 & 0 & -i\bar{\gamma}\xi_2 \\ 0 & 0 & 0 & -i\bar{\gamma}\xi_3 \\ -i\bar{\gamma}\xi_1 & -i\bar{\gamma}\xi_2 & -i\bar{\gamma}\xi_3 & 0 \end{pmatrix}.$$

We also recall that the eigenvalues of $\widehat{\mathcal{B}}$ are

$$\lambda_{\varepsilon_1, \varepsilon_2}(\xi) = \varepsilon_1 \frac{i}{2} \left(\sqrt{1 + \bar{\gamma}^2 |\xi|^2 + 2\bar{\gamma}\xi_3} + \varepsilon_2 \sqrt{1 + \bar{\gamma}^2 |\xi|^2 - 2\bar{\gamma}\xi_3} \right),$$

with $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$. Since $\widehat{\mathcal{B}}$ is a skew-Hermitian matrix, the unit eigenvectors $\vec{V}_{\varepsilon_1, \varepsilon_2}(\xi)$ corresponding to the eigenvalues $\lambda_{\varepsilon_1, \varepsilon_2}(\xi)$ form an orthonormal basis of \mathbb{R}^4 . Decomposing

$$\widehat{U}_0(\xi) = \sum_{\varepsilon_1, \varepsilon_2 \in \{-1, 1\}} C_{\varepsilon_1, \varepsilon_2}(\xi) \vec{V}_{\varepsilon_1, \varepsilon_2}(\xi),$$

the solution of the system (6.4.4) write

$$\widehat{U}^\varepsilon(t, \xi) = \sum_{\varepsilon_1, \varepsilon_2 \in \{-1, 1\}} \Psi_{r,R}(\xi) e^{\frac{t}{\varepsilon} \lambda_{\varepsilon_1, \varepsilon_2}(\xi)} C_{\varepsilon_1, \varepsilon_2}(\xi) \vec{V}_{\varepsilon_1, \varepsilon_2}(\xi).$$

Using the orthogonality of $\{\vec{V}_{\epsilon_1, \epsilon_2}(\xi)\}$ and applying Theorem 6.4.10 and Plancherel theorem, we have

$$\begin{aligned}
 \|\bar{U}^\varepsilon\|_{L^p(\mathbb{R}_+; L^q(\mathbb{R}^3))} &\leq \sum_{\epsilon_1, \epsilon_2 \in \{-1, 1\}} \left\| \mathcal{F}^{-1} \left(\Psi_{r, R}(\xi) e^{\frac{t}{\varepsilon} \lambda_{\epsilon_1, \epsilon_2}(\xi)} C_{\epsilon_1, \epsilon_2}(\xi) \vec{V}_{\epsilon_1, \epsilon_2}(\xi) \right) \right\|_{L^p(\mathbb{R}_+; L^q(\mathbb{R}^3))} \\
 &\leq CR^{\frac{3}{2} - \frac{3}{q} + \frac{4}{p}} r^{-\frac{2}{p}} \varepsilon^{\frac{1}{p}} \sum_{\epsilon_1, \epsilon_2 \in \{-1, 1\}} \left\| C_{\epsilon_1, \epsilon_2}(\xi) \vec{V}_{\epsilon_1, \epsilon_2}(\xi) \right\|_{L_\xi^2} \\
 &\leq CR^{\frac{3}{2} - \frac{3}{q} + \frac{4}{p}} r^{-\frac{2}{p}} \varepsilon^{\frac{1}{p}} \left\| \sum_{\epsilon_1, \epsilon_2 \in \{-1, 1\}} C_{\epsilon_1, \epsilon_2}(\xi) \vec{V}_{\epsilon_1, \epsilon_2}(\xi) \right\|_{L_\xi^2} \\
 &\leq CR^{\frac{3}{2} - \frac{3}{q} + \frac{4}{p}} r^{-\frac{2}{p}} \varepsilon^{\frac{1}{p}} \|U_0\|_{L^2(\mathbb{R}^3)}.
 \end{aligned}$$

Theorem 6.4.1 is then proved. \square

6.5 The nonlinear part

In this section, we decompose the local solution $U^\varepsilon = \begin{pmatrix} u^\varepsilon \\ b^\varepsilon \end{pmatrix}$ of (6.1.1) into two parts

$$U^\varepsilon = \bar{U}^\varepsilon + \tilde{U}^\varepsilon,$$

where \bar{U}^ε is the global solution of (6.4.2) and \tilde{U}^ε solves (locally) the system

$$\begin{cases} \partial_t \tilde{U}^\varepsilon + \frac{1}{\varepsilon} \mathcal{B} \tilde{U}^\varepsilon = \begin{pmatrix} -u^\varepsilon \cdot \nabla u^\varepsilon - \bar{\gamma} b^\varepsilon \nabla b^\varepsilon \\ -u^\varepsilon \nabla b^\varepsilon - \bar{\gamma} b^\varepsilon \operatorname{div} u^\varepsilon \end{pmatrix} \\ \tilde{U}^\varepsilon|_{t=0} = \tilde{U}_0 = (1 - \mathcal{P}_{r, R}) U_0. \end{cases} \quad (6.5.1)$$

As proven in Section 6.4, the linear system (6.4.2) is globally well-posed in $L^\infty(\mathbb{R}_+, H^s(\mathbb{R}^3)^4)$ and its solution goes to zero as $\varepsilon \rightarrow 0$ in some $L^p(\mathbb{R}_+; L^q)$ -norm. On the contrary, the system (6.5.1) is a nonlinear hyperbolic system, the solutions of which can only be expected to exist almost globally in time, in the sense that, there exist $T^\varepsilon \rightarrow +\infty$ as $\varepsilon \rightarrow 0$, such that, $\tilde{U}^\varepsilon \in L^\infty([0, T^\varepsilon], H^s(\mathbb{R}^3)^4)$. The main goal of this section is to prove the following theorem.

Theorem 6.5.1. *Let $s > \frac{5}{2}$, $s_0 > 0$, $1 < p < 2$ and the initial data $U_0 \in Y_{s, s_0, p}$, where $Y_{s, s_0, p}$ is defined in (6.1.6) and (6.1.7). Then, for any $0 < \varepsilon < 1$, there exist $T_\varepsilon^* > 0$ and a unique solution \tilde{U}^ε to the system (6.5.1) satisfying*

$$\tilde{U}^\varepsilon \in L^\infty([0, T_\varepsilon^*], H^s(\mathbb{R}^3)^4) \cap C([0, T_\varepsilon^*], H^s(\mathbb{R}^3)^4).$$

Moreover, the lifespan T_ε^* of \tilde{U}^ε goes to ∞ as $\varepsilon \rightarrow 0$ and there exists constant $\bar{C} > 0$ and $\alpha > 0$ such that

$$T_\varepsilon^* \geq \frac{\bar{C}}{\mathcal{C}(U_0) \varepsilon^\alpha},$$

where $\mathcal{C}(U_0)$ is defined in (6.1.8). In addition, we can choose $\beta > 0$ such that the asymptotic behavior of \tilde{U}^ε when $\varepsilon \rightarrow 0$ is determined as follows

$$\left\| \tilde{U}^\varepsilon \right\|_{L^\infty([0, T_\varepsilon^*], H^s)} = \mathcal{O}(\varepsilon^{\beta s}).$$

The proof of Theorem 6.5.1 will be divided into two parts

- Part 1** In the first part, using an iterative scheme, we prove that, for ε_0 small enough (which will be precised later), and for any $\varepsilon \in]0, \varepsilon_0[$, there exists a unique strong solution \tilde{U}^ε of (6.5.1) in $L^\infty([0, T], H^s(\mathbb{R}^3))$, with the lifespan $T > 0$ independent of ε .
- Part 2** In the second part, we prove more refined estimates which, combining with a bootstrap argument, allow to prove that the maximal lifespan T_ε^* goes to ∞ as $\varepsilon \rightarrow 0$, despite the fact that (6.1.1) is a 3D nonlinear hyperbolic system.

6.5.1 Local-in-time existence result for the nonlinear part.

Throughout this part, we will always fix constants $\beta, \delta > 0$ and the Rossby number $\varepsilon > 0$. We also set the radii of the frequency cut-off to be

$$R = \varepsilon^{-\beta}, \quad r = R^{-\delta} = \varepsilon^{\beta\delta}. \quad (6.5.2)$$

Our choice of these parameters will be explained and precised during the proof, at the place where we need to ajust their values. The setting of r and R in (6.5.2) is to prepare for the bootstrap argument in the second part.

Our goal is to prove the existence of a unique, local strong solution of the system (6.5.1). To simplify the notations and the calculations, we rewrite (6.5.1) as follows

$$\begin{cases} \partial_t \tilde{U}^\varepsilon + \frac{1}{\varepsilon} \mathcal{B} \tilde{U}^\varepsilon = -u^\varepsilon \cdot \nabla U^\varepsilon - \bar{\gamma} \begin{pmatrix} b^\varepsilon \nabla b^\varepsilon \\ b^\varepsilon \operatorname{div} u^\varepsilon \end{pmatrix} \\ \tilde{U}^\varepsilon \Big|_{t=0} = (1 - \mathcal{P}_{r,R}) U_0 \end{cases} \quad (6.5.3)$$

where we set $U^\varepsilon = \bar{U}^\varepsilon + \tilde{U}^\varepsilon$ and where \bar{U}^ε is the solution of (6.4.2). Our approach can be resumed in the following steps

1. We introduce a sequence of linear systems, indexed by $n \in \mathbb{N}$, starting from (6.5.3) and by induction with respect to n , we construct a solution of the n -th system defined in $\tilde{L}^\infty([0, T_n], H^\sigma(\mathbb{R}^3)^4)$, for some given $\sigma \in]s, s + s_0[$ and for some $T_n > 0$.
2. We prove that we can choose $\varepsilon_0 > 0$ small enough such that, for any $\varepsilon \in]0, \varepsilon_0[$, the sequence of solutions of the previously introduced linear systems are uniformly bounded in $\tilde{L}^\infty([0, T_\varepsilon]; H^\sigma(\mathbb{R}^3)^4)$, for some $T_\varepsilon > 0$ independent of n .

3. We prove that the sequence of solutions is a Cauchy sequence in $\tilde{L}^\infty([0, T_\varepsilon]; H^s(\mathbb{R}^3)^4)$.

4. We check that the limit \tilde{U} satisfies (6.5.1).

The main technique result needed for our approach consists in the control of the bilinear terms. This control is given in the following lemma, which will be proven in the appendix.

Lemma 6.5.2. *The following estimates hold*

$$\begin{aligned} \int_0^t \left| \left\langle \Delta_q (u^\varepsilon(\tau) \cdot \nabla U^\varepsilon(\tau)) \mid \Delta_q \tilde{U}^\varepsilon(\tau) \right\rangle_{L^2} \right| d\tau \\ \leq C \mathcal{C}(U_0) b_q 2^{-2qs} R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} \\ + C \mathcal{C}(U_0) b_q 2^{-2qs} \left(R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} + t \left\| \tilde{U}^\varepsilon \right\|_{\tilde{L}^\infty([0,t], H^s)} \right) \left\| \tilde{U}^\varepsilon \right\|_{\tilde{L}^\infty([0,t], H^s)}^2, \end{aligned} \quad (6.5.4)$$

$$\begin{aligned} \int_0^t \left| \left\langle \Delta_q (b^\varepsilon(\tau) \nabla b^\varepsilon(\tau)) \mid \Delta_q \tilde{u}^\varepsilon(\tau) \right\rangle_{L^2} + \left\langle \Delta_q (b^\varepsilon(\tau) \operatorname{div} u^\varepsilon(\tau)) \mid \Delta_q \tilde{b}^\varepsilon(\tau) \right\rangle \right| d\tau \\ \leq C \mathcal{C}(U_0) b_q 2^{-2qs} R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} \\ + C \mathcal{C}(U_0) b_q 2^{-2qs} \left(R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} + t \left\| \tilde{U}^\varepsilon \right\|_{\tilde{L}^\infty([0,t], H^s)} \right) \left\| \tilde{U}^\varepsilon \right\|_{\tilde{L}^\infty([0,t], H^s)}^2, \end{aligned} \quad (6.5.5)$$

where b_q is a summable sequence such that $\sum_q b_q = 1$.

Step 1. We fix $\sigma \in]s, s + s_0[$, say $\sigma = (s + \frac{s_0}{2})$. For any $n \in \mathbb{N}^*$, we define the operator

$$J_n f = \mathcal{F}^{-1} \left(1_{\{|\xi| < n\}} \cap \{|\xi_h| > 1/n\}} \cap \{|\xi_3| > 1/n\}} \hat{f} \right)$$

which is continuous from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. Setting $\tilde{U}^0 = 0$, by induction, we define the following family of linear systems, related to (6.5.3)

$$\begin{cases} \partial_t \tilde{U}^{n+1} + \frac{1}{\varepsilon} \mathcal{B} \tilde{U}^{n+1} = -J_{n+1} (\mathcal{A}(U^n, D) J_{n+1} U^{n+1}) \\ \tilde{U}_{|t=0}^{n+1} = \tilde{U}_0^{n+1} = \mathcal{F}^{-1} (1_{B(0, n+1)} \mathcal{F} (1 - \mathcal{P}_{r, R}) U_0) \\ U^{n+1} = \bar{U}^\varepsilon + \tilde{U}^{n+1} \end{cases} \quad (6.5.3_n)$$

\mathcal{A} is defined in (6.1.4). We remark that since $\tilde{U}^0 \equiv 0$, \tilde{U}^1 is solution to the following linear system

$$\begin{cases} \partial_t \tilde{U}^1 + \frac{1}{\varepsilon} \mathcal{B} \tilde{U}^1 + J_1 (\mathcal{A}(\bar{U}^\varepsilon, D) J_1 \tilde{U}^1) = -J_1 (\mathcal{A}(\bar{U}^\varepsilon, D) J_1 \bar{U}^\varepsilon) \\ \tilde{U}_{|t=0}^1 = \tilde{U}_0^1 = \mathcal{F}^{-1} (1_{B(0,1)} \mathcal{F} (1 - \mathcal{P}_{r, R}) U_0) \end{cases} \quad (6.5.3_0)$$

We have $\bar{U}^\varepsilon \in L^\infty(\mathbb{R}_+, H^\alpha(\mathbb{R}^3)^4)$, for any $\alpha > 0$, because of its frequency localization property and Lemma 6.3.1 implies that $\tilde{U}_0^1 \in H^\sigma(\mathbb{R}^3)^4$. Then, we can easily construct \tilde{U}^1 , with the Fourier transform of which localized in $B(0, 1)$ using Hahn-Banach theorem.

Now, for any $n \in \mathbb{N}^*$, let $L_n^2(\mathbb{R}^3)$ be the space

$$L_n^2(\mathbb{R}^3) = \left\{ f \in L^2(\mathbb{R}^3) \mid \text{Supp} \widehat{f} \subset (\{|\xi| < n\} \cap \{|\xi_h| > 1/n\} \cap \{|\xi_3| > 1/n\}) \right\}.$$

Let $\eta > 0$ be a fix positive constant and we suppose that, for any $0 \leq k \leq n-1$, we can construct a unique maximal solution \widetilde{U}^{k+1} of (6.5.3_k) in

$$C^1\left([0, T_{k+1}], L_{k+1}^2(\mathbb{R}^3)^4\right) \cap \widetilde{L}^\infty\left([0, T_{k+1}], H^\sigma(\mathbb{R}^3)^4\right)$$

such that

$$\left\| \widetilde{U}^{k+1} \right\|_{\widetilde{L}^\infty([0, T_{k+1}]; H^\sigma)} \leq \eta.$$

Thanks to the embedding $H^\sigma(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, we have $U^n \in L^\infty\left([0, T_n], L^\infty(\mathbb{R}^3)^4\right)$, which implies that

$$J_{n+1}(\mathcal{A}(U^n(t), D) J_{n+1} U^{n+1}(t)) \in L^2(\mathbb{R}^3)^4,$$

and we can rewrite (6.5.3_n) as an ODE

$$\partial_t \widetilde{U}^{n+1} = \mathcal{L}_{n+1} \widetilde{U}^{n+1},$$

where the linear operator \mathcal{L}_{n+1} maps continuously $L^2(\mathbb{R}^3)^4$ to $L^2(\mathbb{R}^3)^4$. The Cauchy-Lipschitz theorem ensure the existence of a unique maximal solution to the system (6.5.3_n)

$$\widetilde{U}^{n+1} \in C^1\left([0, T_{n+1}]; L^2(\mathbb{R}^3)^4\right).$$

Moreover, since $J_{n+1}^2 = J_{n+1}$, applying J_{n+1} to (6.5.3_n), we obtain, by uniqueness, that

$$J_{n+1} \widetilde{U}^{n+1} = \widetilde{U}^{n+1}.$$

Hence, \widetilde{U}^{n+1} belongs not only to $L^2(\mathbb{R}^3)^4$ but to $L_{n+1}^2(\mathbb{R}^3)^4$, which conclude the first step by induction.

Step 2. We recall that throughout this paper, we use C to denote a generic positive constant which can change from line to line. In this step, we want to prove that, for previously chosen $\varepsilon > 0$ small enough, the sequence $\{T_n\}$ is bounded from below from zero, which means that there exists $T_\varepsilon > 0$ such that, for any $n \in \mathbb{N}$,

$$\left\| \widetilde{U}^n \right\|_{\widetilde{L}^\infty([0, T_\varepsilon]; H^\sigma)} \leq \eta. \quad (6.5.6)$$

We will prove (6.5.6) by induction. For $n = 0$, we have nothing to do. So we suppose that, for fix $T_\varepsilon > 0$ which will be precised later, (6.5.6) is true for any $0 \leq k \leq n$. Now, we want to estimate \widetilde{U}^{n+1} in $\widetilde{L}^\infty([0, T_\varepsilon], H^\sigma(\mathbb{R}^3))$ -norm. Applying Δ_q to (6.5.3_n), taking the

L^2 -scalar product of the obtained equation with $\Delta_q \tilde{U}^{n+1}$ and then integrating with respect to the time variable on $[0, t]$, we get

$$\begin{aligned} \left\| \Delta_q \tilde{U}^{n+1}(t) \right\|_{L^2}^2 &\leq \left\| \Delta_q \tilde{U}_0^{n+1} \right\|_{L^2}^2 + 2 \int_0^t \left| \langle J_{n+1} \Delta_q (u^n \cdot \nabla U^{n+1}) \rangle \mid \Delta_q \tilde{U}^{n+1} \right|(\tau) d\tau \\ &\quad + 2 \int_0^t \left| \langle J_{n+1} \Delta_q (b^n \nabla b^{n+1}) \mid \Delta_q \tilde{u}^{n+1} \rangle_{L^2} + \right. \\ &\quad \left. + \langle J_{n+1} \Delta_q (b^n \operatorname{div} u^{n+1}) \mid \Delta_q \tilde{b}^{n+1} \rangle \right|(\tau) d\tau \end{aligned} \quad (6.5.7)$$

Using the same method as in the proof of Lemma 6.5.2, we decompose the bilinear term on the right hand side of (6.5.7) into the following sums

$$\int_0^t \left| \langle J_{n+1} \Delta_q (u^n \cdot \nabla U^{n+1}) \rangle \mid \Delta_q \tilde{U}^{n+1} \right|(\tau) d\tau \leq B_1^{n+1} + B_2^{n+1} + B_3^{n+1} + B_4^{n+1},$$

and

$$\begin{aligned} \int_0^t \left| \langle J_{n+1} \Delta_q (b^n \nabla b^{n+1}) \mid \Delta_q \tilde{u}^{n+1} \rangle_{L^2} + \langle J_{n+1} \Delta_q (b^n \operatorname{div} u^{n+1}) \mid \Delta_q \tilde{b}^{n+1} \rangle \right|(\tau) d\tau \\ \leq C_1^{n+1} + C_2^{n+1} + C_3^{n+1} + C_4^{n+1}. \end{aligned}$$

where, applying Lemmas 6.6.1, 6.6.2 and 6.6.3 in the appendix, we have

$$\begin{aligned} B_1^{n+1} &= \int_0^t \left| \langle \Delta_q (\bar{u}^\varepsilon \cdot \nabla \bar{U}^\varepsilon) \rangle \mid \Delta_q \tilde{U}^{n+1} \right|(\tau) d\tau \\ &\leq C \|U_0\|_{H^\sigma} R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} b_q 2^{-2q\sigma} \|\bar{U}^\varepsilon\|_{\tilde{L}^\infty([0,t],H^\sigma)} \|\tilde{U}^{n+1}\|_{\tilde{L}^\infty([0,t],H^\sigma)} \\ &\leq C \mathcal{C}(U_0) t^{\frac{3}{4}} \varepsilon^{\frac{1}{4} - \frac{\beta(7+\delta)}{2}} b_q 2^{-2q\sigma} \left(1 + \|\tilde{U}^{n+1}\|_{\tilde{L}^\infty([0,t],H^\sigma)}^2 \right), \end{aligned} \quad (6.5.8)$$

$$\begin{aligned} B_2^{n+1} &= \int_0^t \left| \langle \Delta_q (\tilde{u}^n \cdot \nabla \bar{U}^\varepsilon) \rangle \mid \Delta_q \tilde{U}^{n+1} \right|(\tau) d\tau \\ &\leq C \mathcal{C}(U_0) R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} b_q 2^{-2q\sigma} \|\tilde{U}^n\|_{\tilde{L}^\infty([0,t],H^\sigma)} \|\tilde{U}^{n+1}\|_{\tilde{L}^\infty([0,t],H^\sigma)} \\ &\leq C \mathcal{C}(U_0) \eta t^{\frac{3}{4}} \varepsilon^{\frac{1}{4} - \frac{\beta(7+\delta)}{2}} b_q 2^{-2q\sigma} \left(1 + \|\tilde{U}^{n+1}\|_{\tilde{L}^\infty([0,t],H^\sigma)}^2 \right), \end{aligned} \quad (6.5.9)$$

$$\begin{aligned} B_3^{n+1} &= \int_0^t \left| \langle \Delta_q (\bar{u}^\varepsilon \cdot \nabla \tilde{U}^{n+1}) \rangle \mid \Delta_q \tilde{U}^{n+1} \right|(\tau) d\tau \\ &\leq C \mathcal{C}(U_0) R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} b_q 2^{-2q\sigma} \|\tilde{U}^{n+1}\|_{\tilde{L}^\infty([0,t],H^\sigma)}^2 \\ &\leq C \mathcal{C}(U_0) t^{\frac{3}{4}} \varepsilon^{\frac{1}{4} - \frac{\beta(7+\delta)}{2}} b_q 2^{-2q\sigma} \|\tilde{U}^{n+1}\|_{\tilde{L}^\infty([0,t],H^\sigma)}^2, \end{aligned} \quad (6.5.10)$$

$$\begin{aligned}
B_4^{n+1} &= \int_0^t \left| \left\langle \Delta_q \left(\tilde{u}^n \cdot \nabla \tilde{U}^{n+1} \right) \right\rangle \mid \Delta_q \tilde{U}^{n+1} \right| (\tau) d\tau \\
&\leq C b_q 2^{-2q\sigma} \left\| \tilde{U}^n \right\|_{\tilde{L}^\infty([0,t], H^\sigma)} \left\| \tilde{U}^{n+1} \right\|_{\tilde{L}^2([0,t], H^\sigma)}^2 \\
&\leq C \eta t b_q 2^{-2q\sigma} \left\| \tilde{U}^{n+1} \right\|_{\tilde{L}^\infty([0,t], H^\sigma)}^2,
\end{aligned} \tag{6.5.11}$$

and

$$\begin{aligned}
C_1^{n+1} &= \int_0^t \left| \left\langle \Delta_q \left(\bar{b}^n \nabla \bar{b}^{n+1} \right) \mid \Delta_q \tilde{u}^{n+1} \right\rangle \right| (\tau) d\tau \\
&\quad + \int_0^t \left| \left\langle \Delta_q \left(\bar{b}^n \operatorname{div} \bar{u}^{n+1} \right) \mid \Delta_q \tilde{b}^{n+1} \right\rangle \right| (\tau) d\tau \\
&\leq C \|U_0\|_{H^\sigma} R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} b_q 2^{-2q\sigma} \left\| \bar{U}^\varepsilon \right\|_{\tilde{L}^\infty([0,t], H^\sigma)} \left\| \tilde{U}^{n+1} \right\|_{\tilde{L}^\infty([0,t], H^\sigma)} \\
&\leq C \mathcal{C}(U_0) t^{\frac{3}{4}} \varepsilon^{\frac{1}{4} - \frac{\beta(7+\delta)}{2}} b_q 2^{-2q\sigma} \left(1 + \left\| \tilde{U}^{n+1} \right\|_{\tilde{L}^\infty([0,t], H^\sigma)}^2 \right),
\end{aligned}$$

$$\begin{aligned}
C_2^{n+1} &= \int_0^t \left| \left\langle \Delta_q \left(\tilde{b}^n \nabla \bar{b}^{n+1} \right) \mid \Delta_q \tilde{u}^{n+1} \right\rangle \right| (\tau) d\tau \\
&\quad + \int_0^t \left| \left\langle \Delta_q \left(\tilde{b}^n \operatorname{div} \bar{u}^{n+1} \right) \mid \Delta_q \tilde{b}^{n+1} \right\rangle \right| (\tau) d\tau \\
&\leq C \mathcal{C}(U_0) R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} b_q 2^{-2q\sigma} \left\| \tilde{U}^n \right\|_{\tilde{L}^\infty([0,t], H^\sigma)} \left\| \tilde{U}^{n+1} \right\|_{\tilde{L}^\infty([0,t], H^\sigma)} \\
&\leq C \mathcal{C}(U_0) \eta t^{\frac{3}{4}} \varepsilon^{\frac{1}{4} - \frac{\beta(7+\delta)}{2}} b_q 2^{-2q\sigma} \left(1 + \left\| \tilde{U}^{n+1} \right\|_{\tilde{L}^\infty([0,t], H^\sigma)}^2 \right),
\end{aligned}$$

$$\begin{aligned}
C_3^{n+1} &= \int_0^t \left| \left\langle \Delta_q \left(\bar{b}^n \nabla \tilde{b}^{n+1} \right) \mid \Delta_q \tilde{u}^{n+1} \right\rangle \right| (\tau) d\tau \\
&\quad + \int_0^t \left| \left\langle \Delta_q \left(\bar{b}^n \operatorname{div} \tilde{u}^{n+1} \right) \mid \Delta_q \tilde{b}^{n+1} \right\rangle \right| (\tau) d\tau \\
&\leq C \mathcal{C}(U_0) R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} b_q 2^{-2q\sigma} \left\| \tilde{U}^{n+1} \right\|_{\tilde{L}^\infty([0,t], H^\sigma)}^2 \\
&\leq C \mathcal{C}(U_0) t^{\frac{3}{4}} \varepsilon^{\frac{1}{4} - \frac{\beta(7+\delta)}{2}} b_q 2^{-2q\sigma} \left\| \tilde{U}^{n+1} \right\|_{\tilde{L}^\infty([0,t], H^\sigma)}^2,
\end{aligned}$$

$$\begin{aligned}
C_4^{n+1} &= \int_0^t \left| \left\langle \Delta_q \left(\tilde{b}^n \nabla \tilde{b}^{n+1} \right) \mid \Delta_q \tilde{u}^{n+1} \right\rangle \right| d\tau \\
&\quad + \int_0^t \left| \left\langle \Delta_q \left(\tilde{b}^n \operatorname{div} \tilde{u}^{n+1} \right) \mid \Delta_q \tilde{b}^{n+1} \right\rangle \right| (\tau) d\tau \\
&\leq C b_q 2^{-2q\sigma} \left\| \tilde{U}^n \right\|_{\tilde{L}^\infty([0,t], H^\sigma)} \left\| \tilde{U}^{n+1} \right\|_{\tilde{L}^2([0,t], H^\sigma)}^2 \\
&\leq C \eta t b_q 2^{-2q\sigma} \left\| \tilde{U}^{n+1} \right\|_{\tilde{L}^\infty([0,t], H^\sigma)}^2,
\end{aligned}$$

where $\mathcal{C}(U_0)$ is defined in (6.1.8).

Inserting Estimates (6.5.8) to (6.5.12) into (6.5.7), multiplying the obtained inequality by $2^{2q\sigma}$, then summing with respect to $q \geq -1$ and applying Lemma 6.3.1 leads to

$$\begin{aligned} \left\| \tilde{U}^{n+1} \right\|_{\tilde{L}^\infty([0,t],H^\sigma)}^2 &\leq C \mathcal{C}(U_0) \varepsilon^{2\beta s_0} + C \eta t \left\| \tilde{U}^{n+1} \right\|_{\tilde{L}^\infty([0,t],H^\sigma)}^2 \\ &\quad + C \mathcal{C}(U_0) t^{\frac{3}{4}} \varepsilon^{\frac{1}{4} - \frac{\beta(7+\delta)}{2}} \left(1 + \left\| \tilde{U}^{n+1} \right\|_{\tilde{L}^\infty([0,t],H^\sigma)}^2 \right). \end{aligned}$$

We recall that $\sigma \in]s, s + s_0[$ and $\eta > 0$ are fixed positive constants and $\delta > 0$ is given in Lemma 6.3.1. We choose $\varepsilon_0 > 0$, $T_\varepsilon > 0$ and $\beta > 0$ such that

$$\begin{cases} \beta(7 + \delta) < \frac{1}{2} \\ C\eta T_\varepsilon + C \mathcal{C}(U_0) T_\varepsilon^{\frac{3}{4}} \varepsilon_0^{\frac{1}{4} - \frac{\beta(7+\delta)}{2}} < \frac{1}{2} \\ C \mathcal{C}(U_0) \varepsilon_0^{2\beta s_0} + C \mathcal{C}(U_0) T_\varepsilon^{\frac{3}{4}} \varepsilon_0^{\frac{1}{4} - \frac{\beta(7+\delta)}{2}} \leq \frac{\eta^2}{2}. \end{cases} \quad (6.5.12)$$

Then, for any $\varepsilon \in [0, \varepsilon_0]$, we deduce that,

$$\left\| \tilde{U}^{n+1} \right\|_{\tilde{L}^\infty([0,T_\varepsilon],H^\sigma)} \leq \eta,$$

and Step 2 is concluded.

Remark 6.5.3. In fact, the time of existence $T_\varepsilon = T > 0$ depends only on $\varepsilon_0 > 0$ and thus is independent of ε , for $\varepsilon \in]0, \varepsilon_0[$.

Step 3. At first, we will prove that $\left\{ \tilde{U}^n \right\}_n$ is a Cauchy sequence in the space $L^\infty([0, T]; L^2(\mathbb{R}^3)^4)$. We define the auxiliary sequence $\left\{ \tilde{V}^n \right\}_n$ by

$$\tilde{V}^{n+1} = \tilde{U}^{n+1} - \tilde{U}^n, \quad \forall n \in \mathbb{N}.$$

For any $n \in \mathbb{N}$, \tilde{V}^{n+1} is solution of the system

$$\begin{cases} \partial_t \tilde{V}^{n+1} + \frac{1}{\varepsilon} \mathcal{B} \tilde{V}^{n+1} + \mathcal{A}(\tilde{U}^n, D) \tilde{V}^{n+1} + \mathcal{A}(\tilde{V}^n, D) \tilde{U}^n \\ \quad + \mathcal{A}(\tilde{V}^n, D) \bar{U}^\varepsilon + \mathcal{A}(\bar{U}^\varepsilon, D) \tilde{V}^{n+1} = 0 \\ \tilde{V}^{n+1}|_{t=0} = \tilde{U}_0^{n+1} - \tilde{U}_0^n. \end{cases} \quad (6.5.13)$$

We will need the following estimates, the proof of which is simple and direct.

Lemma 6.5.4. *The following estimates hold*

$$\begin{aligned} \left| \left\langle \mathcal{A}(\tilde{U}^n, D) \tilde{V}^{n+1} \mid \tilde{V}^{n+1} \right\rangle_{L^2} \right| &\leq C \left\| \nabla \tilde{U}^n \right\|_{L^\infty} \left\| \tilde{V}^{n+1} \right\|_{L^2}^2 \\ \left| \left\langle \mathcal{A}(\tilde{V}^n, D) \tilde{U}^n \mid \tilde{V}^{n+1} \right\rangle_{L^2} \right| &\leq C \left\| \nabla \tilde{U}^n \right\|_{L^\infty} \left\| \tilde{V}^n \right\|_{L^2} \left\| \tilde{V}^{n+1} \right\|_{L^2} \\ \left| \left\langle \mathcal{A}(\tilde{V}^n, D) \bar{U}^\varepsilon \mid \tilde{V}^{n+1} \right\rangle_{L^2} \right| &\leq C \left\| \nabla \bar{U}^\varepsilon \right\|_{L^\infty} \left\| \tilde{V}^n \right\|_{L^2} \left\| \tilde{V}^{n+1} \right\|_{L^2} \\ \left| \left\langle \mathcal{A}(\bar{U}^\varepsilon, D) \tilde{V}^{n+1} \mid \tilde{V}^{n+1} \right\rangle_{L^2} \right| &\leq C \left\| \nabla \bar{U}^\varepsilon \right\|_{L^\infty} \left\| \tilde{V}^{n+1} \right\|_{L^2}^2. \end{aligned}$$

Taking the L^2 scalar product of the first equation of (6.5.13) with $\left\| \tilde{V}^{n+1} \right\|_{L^2}^{-1} \tilde{V}^{n+1}$ and using Bernstein Lemma 6.2.1, Lemma 6.5.4 and the Sobolev inclusion $H^\sigma(\mathbb{R}^3) \hookrightarrow W^{1,\infty}(\mathbb{R}^3)$, we obtain

$$\frac{d}{dt} \left\| \tilde{V}^{n+1} \right\|_{L^2} \leq \left(\left\| \tilde{U}^n \right\|_{H^\sigma} + R \left\| \bar{U}^\varepsilon \right\|_{L^\infty} \right) \left(\left\| \tilde{V}^{n+1} \right\|_{L^2} + \left\| \tilde{V}^n \right\|_{L^2} \right). \quad (6.5.14)$$

We recall that, for any $n \in \mathbb{N}$,

$$\left\| \tilde{U}^n \right\|_{\tilde{L}^\infty([0,T];H^\sigma)} \leq \eta$$

and that Hölder inequality and Strichartz-type estimates (6.4.3) in Section 6.4 give, pour tout $0 < t \leq T$,

$$\int_0^t \left\| \bar{U}^\varepsilon \right\|_{L^\infty} ds \leq \left(\int_0^t ds \right)^{\frac{3}{4}} \left\| \bar{U}^\varepsilon \right\|_{L^4([0,t],L^\infty(\mathbb{R}^3))} \leq C \mathcal{C}(U_0) T^{\frac{3}{4}} R^{\frac{5+\delta}{2}} \varepsilon^{\frac{1}{4}},$$

where $\mathcal{C}(U_0)$ is defined in (6.1.8). Using Bernstein Lemma 6.2.1, we also have

$$\left\| \tilde{V}^{n+1}(0) \right\|_{L^2}^2 = \left\| \tilde{U}_0^{n+1} - \tilde{U}_0^n \right\|_{L^2}^2 \leq \int_{|\xi|>n} (1 + |\xi|^2)^{-s} (1 + |\xi|^2)^s \left| \widehat{U}_0 \right|^2 d\xi \leq C \mathcal{C}(U_0) n^{-2s}.$$

Integrating (6.5.14) with respect to the time variable and taking into account all the above inequalities and remarking that we already choose $R = \varepsilon^{-\beta}$, $\beta > 0$, we obtain

$$v_{n+1} \leq C \mathcal{C}(U_0) n^{-s} + \left(\eta T + C \mathcal{C}(U_0) T^{\frac{3}{4}} \varepsilon^{\frac{1}{4} - \frac{\beta(7+\delta)}{2}} \right) (v_{n+1} + v_n),$$

where for any $n \in \mathbb{N}$, we set

$$v_n = \left\| \tilde{V}^n \right\|_{L^\infty([0,T],L^2)}.$$

If we choose the parameters such that

$$\begin{cases} \beta(7 + \delta) < \frac{1}{2} \\ \eta T + C \mathcal{C}(U_0) T^{\frac{3}{4}} \varepsilon_0^{\frac{1}{4} - \frac{\beta(7+\delta)}{2}} < \frac{1}{3}, \end{cases} \quad (6.5.15)$$

then, for any $\varepsilon \in]0, \varepsilon_0[$, we have

$$v_{n+1} \leq C \mathcal{C}(U_0) n^{-s} + \frac{1}{2} v_n.$$

Since $s > \frac{5}{2}$, the series $\sum_{n \in \mathbb{N}} n^{-s}$ is convergent, which implies that the sequence $\{v^n\}_n$ is summable, which in turn implies that $\left\{ \tilde{U}^n \right\}_n$ is a Cauchy sequence in $L^\infty([0, T], L^2(\mathbb{R}^3)^4)$. Since $\left\{ \tilde{U}^n \right\}_n$ is a bounded sequence in $\tilde{L}^\infty([0, T], H^\sigma(\mathbb{R}^3)^4)$, for some $\sigma \in]s, s + s_0[$, by interpolation, we deduce that $\left\{ \tilde{U}^n \right\}_n$ is a Cauchy sequence in $\tilde{L}^\infty([0, T], H^s(\mathbb{R}^3)^4)$, and so, there exists \tilde{U}^ε in $\tilde{L}^\infty([0, T], H^s(\mathbb{R}^3)^4)$ such that

$$\tilde{U}^\varepsilon = \lim_{n \rightarrow +\infty} \tilde{U}^n.$$

Remark 6.5.5. Fixing $s > \frac{5}{2}$, $s_0 > 0$, $\sigma \in]s, s + s_0[$, $\eta > 0$, $\delta > 0$, the conditions (6.5.12) and (6.5.15) can easily be satisfied by choosing $\beta > 0$, $T > 0$ and $\varepsilon_0 > 0$ sufficiently small.

Step 4. It remains to verify if \tilde{U}^ε is a solution of (6.5.1). In fact, we only have to check if we can pass to the limit in the bilinear term. Since $s > \frac{5}{2}$, classical product laws in Sobolev spaces yield

$$\begin{aligned} & \left\| \mathcal{A}(\tilde{U}^\varepsilon, D) \tilde{U}^\varepsilon - \mathcal{A}(\tilde{U}^n, D) \tilde{U}^{n+1} \right\|_{H^{s-1}} \\ & \leq \left\| \mathcal{A}(\tilde{U}^n - \tilde{U}^\varepsilon, D) \tilde{U}^{n+1} \right\|_{H^{s-1}} + \left\| \mathcal{A}(\tilde{U}^\varepsilon, D) (\tilde{U}^{n+1} - \tilde{U}^\varepsilon) \right\|_{H^{s-1}} \\ & \leq \left\| \tilde{U}^n - \tilde{U}^\varepsilon \right\|_{H^{s-1}} \left\| \nabla \tilde{U}^{n+1} \right\|_{L^\infty} + \left\| \tilde{U}^n - \tilde{U}^\varepsilon \right\|_{L^\infty} \left\| \tilde{U}^{n+1} \right\|_{H^s} \\ & \quad + \left\| \tilde{U}^\varepsilon \right\|_{H^{s-1}} \left\| \nabla (\tilde{U}^{n+1} - \tilde{U}^\varepsilon) \right\|_{L^\infty} + \left\| \tilde{U}^\varepsilon \right\|_{L^\infty} \left\| \tilde{U}^{n+1} - \tilde{U}^\varepsilon \right\|_{H^s} \\ & \leq C \left\| \tilde{U}^\varepsilon \right\|_{H^s} \left\| \tilde{U}^{n+1} - \tilde{U}^\varepsilon \right\|_{H^s} + C \left\| \tilde{U}^{n+1} \right\|_{H^s} \left\| \tilde{U}^n - \tilde{U}^\varepsilon \right\|_{H^s} \end{aligned}$$

We recall that \tilde{U}^n and so \tilde{U}^ε are bounded in $L^\infty([0, T], H^s(\mathbb{R}^3)^4)$ by $\eta > 0$. Besides, we also prove in Step 3 that

$$\lim_{n \rightarrow +\infty} \left\| \tilde{U}^n - \tilde{U}^\varepsilon \right\|_{L^\infty([0, T], H^s)} = 0.$$

Thus, we obtain

$$\lim_{n \rightarrow +\infty} \left\| \mathcal{A}(\tilde{U}^\varepsilon, D) \tilde{U}^\varepsilon - \mathcal{A}(\tilde{U}^n, D) \tilde{U}^{n+1} \right\|_{L^\infty([0, T], H^{s-1})} = 0,$$

which allows to pass to the limit and conclude Step 4.

We remark that the continuity in $L^2(\mathbb{R}^3)^4$ and the boundedness in $H^\sigma(\mathbb{R}^3)^4$ implies the continuity of \tilde{U}^ε with respect to the time variable. To finish this part, we study the uniqueness and the continuity with respect to the initial data of the previously constructed solution. More precisely, we prove the following lemma.

Lemma 6.5.6. *Let $U_0 \in H^{s+s_0}$, $s > 5/2$, $s_0 > 0$. There exists a unique solution of the system (6.1.5) in $\tilde{L}^\infty([0, T]; H^s(\mathbb{R}^3)^4)$. Moreover, if Φ is the function which associates to $U_0 \in H^{s+s_0}$ the unique solution U of (6.1.5), then*

$$\Phi \in \mathcal{C} \left(H^s(\mathbb{R}^3)^4; L^\infty([0, T]; H^s(\mathbb{R}^3)^4) \right).$$

Proof. Let us consider two initial data $U_{i,0} \in H^{s+s_0}$, $s > 5/2$, $s_0 > 0$, $i = 1, 2$. These data generate two solutions U_i , $i = 1, 2$, to the system

$$\begin{cases} \partial_t U_i - \frac{1}{\varepsilon} \mathcal{B} U_i = -\mathcal{A}(U_i, D) U_i, \\ U_i|_{t=0} = U_{i,0}. \end{cases}$$

We remark that $\delta U = U_1 - U_2$ solves the system

$$\begin{cases} \partial_t \delta U + \frac{1}{\varepsilon} \mathcal{B} \delta U = -\mathcal{A}(\delta U, D) U_2 - \mathcal{A}(U_1, D) \delta U \\ \delta U|_{t=0} = U_{1,0} - U_{2,0}. \end{cases} \quad (6.5.16)$$

Taking $L^2(\mathbb{R}^3)$ scalar product of (6.5.16) with δU , and considering the following inequalities,

$$\begin{aligned} \left| \left\langle \mathcal{A}(\delta U, D) U_2 \mid \delta U \right\rangle_{L^2(\mathbb{R}^3)} \right| &\leq \|\nabla U_2\|_{L^\infty(\mathbb{R}^3)} \|\delta U\|_{L^2}^2 \\ \left| \left\langle \mathcal{A}(U_1, D) \delta U \mid \delta U \right\rangle_{L^2(\mathbb{R}^3)} \right| &\leq \|\nabla U_1\|_{L^\infty(\mathbb{R}^3)} \|\delta U\|_{L^2}^2, \end{aligned}$$

we deduce via Gronwall inequality, and the embedding $H^s(\mathbb{R}^3) \hookrightarrow W^{1,\infty}(\mathbb{R}^3)$ that

$$\frac{d}{dt} \|\delta U(t)\|_{L^2(\mathbb{R}^3)}^2 \leq \|\delta U_0\|_{L^2(\mathbb{R}^3)}^2 e^{2 \int_0^t (\|U_1(\tau)\|_{H^s(\mathbb{R}^3)} + \|U_2(\tau)\|_{H^s(\mathbb{R}^3)}) d\tau}.$$

From the construction of the solution, we have

$$\|U_i\|_{\tilde{L}^\infty([0,T]; H^s(\mathbb{R}^3)^4)} \leq \|U_{i,0}\|_{H^s(\mathbb{R}^3)} + \eta,$$

hence

$$e^{2 \int_0^t (\|U_1(\tau)\|_{H^s(\mathbb{R}^3)} + \|U_2(\tau)\|_{H^s(\mathbb{R}^3)}) d\tau} \leq e^{(4\eta+2\|U_{1,0}\|_{H^s(\mathbb{R}^3)}+2\|U_{2,0}\|_{H^s(\mathbb{R}^3)})T},$$

which implies the uniqueness and the continuity of the solution in the space $L^\infty([0, T]; L^2(\mathbb{R}^3)^4)$.

The uniqueness and the continuity in $L^\infty([0, T]; H^s(\mathbb{R}^3)^4)$ follows by interpolation. \square

6.5.2 Lifespan of the nonlinear part.

In this part, we will provide a control of the maximal lifespan T_ε^* of the solution previously constructed using a bootstrap argument. Applying Δ_q to (6.5.1), taking the L^2 -scalar product of the obtained equation with $\Delta_q \tilde{U}^\varepsilon$ and then integrating with respect to the time variable on $[0, t]$, we get

$$\begin{aligned} \left\| \Delta_q \tilde{U}^\varepsilon(t) \right\|_{L^2}^2 &\leq \left\| \Delta_q \tilde{U}_0 \right\|_{L^2}^2 + 2 \int_0^t \left| \left\langle \Delta_q (u^\varepsilon(\tau) \cdot \nabla U^\varepsilon(\tau)) \mid \Delta_q \tilde{U}^\varepsilon(\tau) \right\rangle \right| d\tau \\ &\quad + 2 \int_0^t \left| \left\langle \Delta_q \left(\begin{array}{c} b^\varepsilon \nabla b^\varepsilon \\ b^\varepsilon \operatorname{div} u^\varepsilon \end{array} \right) (\tau) \mid \Delta_q \tilde{U}^\varepsilon(\tau) \right\rangle \right|_{L^2} d\tau. \end{aligned} \quad (6.5.17)$$

We recall that $R = \varepsilon^{-\beta}$, $\beta > 0$. Then, Lemma 6.3.1 implies, for $\varepsilon > 0$ small enough,

$$\left\| \tilde{U}_0 \right\|_{H^s} \leq C \mathcal{C}(U_0) R^{-s_0} = C \mathcal{C}(U_0) \varepsilon^{\beta s_0} \leq \varepsilon^{\frac{\beta s_0}{2}},$$

where $\mathcal{C}(U_0)$ is defined in (6.1.8). Let

$$T_\varepsilon^* = \sup \left\{ T > 0 : \left\| \tilde{U}^\varepsilon(t) \right\|_{\tilde{L}^\infty([0,t], H^s)} \leq 2\varepsilon^{\frac{\beta s_0}{2}}, \forall t \in [0, T] \right\}. \quad (6.5.18)$$

The continuity of \tilde{U}^ε with respect to the time variable implies that $T_\varepsilon^* > 0$. Multiplying (6.5.17) by 2^{2qs} and summing with respect to $q \geq -1$, then using Lemma 6.5.2, for any $t \in [0, T_\varepsilon^*]$, we obtain

$$\begin{aligned} \left\| \tilde{U}^\varepsilon \right\|_{\tilde{L}^\infty([0,t],H^s)}^2 &\leq \left\| \tilde{U}_0 \right\|_{H^s}^2 + C \mathcal{C}(U_0) t^{\frac{3}{4}} \varepsilon^{\frac{1}{4} - \frac{\beta(7+\delta)}{2}} \\ &\quad + C \mathcal{C}(U_0) \left(t^{\frac{3}{4}} \varepsilon^{\frac{1}{4} - \frac{\beta(7+\delta)}{2}} + t \left\| \tilde{U}^\varepsilon \right\|_{\tilde{L}^\infty([0,t],H^s)} \right) \left\| \tilde{U}^\varepsilon \right\|_{\tilde{L}^\infty([0,t],H^s)}^2 \end{aligned}$$

which implies, for $0 < \varepsilon < 1$ small enough, and for any $t \in [0, T_\varepsilon^*]$,

$$\begin{aligned} \left\| \tilde{U}^\varepsilon \right\|_{\tilde{L}^\infty([0,t],H^s)}^2 &\leq \varepsilon^{\beta s_0} + C \mathcal{C}(U_0) t^{\frac{3}{4}} \varepsilon^{\frac{1}{4} - \frac{\beta(7+\delta)}{2}} \\ &\quad + C \mathcal{C}(U_0) \left(t^{\frac{3}{4}} \varepsilon^{\frac{1}{4} - \frac{\beta(7+\delta)}{2}} + t \varepsilon^{\beta s_0} \right) \left\| \tilde{U}^\varepsilon \right\|_{\tilde{L}^\infty([0,t],H^s)}^2. \end{aligned}$$

If

$$\beta(14 + 2\delta + 4s_0) < 1 \quad (6.5.19)$$

and if

$$C \mathcal{C}(U_0) \left((T_\varepsilon^*)^{\frac{3}{4}} \varepsilon^{\frac{1}{4} - \frac{\beta(7+\delta)}{2}} + T_\varepsilon^* \varepsilon^{\beta s_0} \right) \leq \frac{1}{2} \quad (6.5.20)$$

then for $\varepsilon > 0$ small enough, for any $0 < t < T_\varepsilon^*$, we have

$$\frac{1}{2} \left\| \tilde{U}^\varepsilon \right\|_{\tilde{L}^\infty([0,t],H^s)}^2 \leq \varepsilon^{\beta s_0} + C \mathcal{C}(U_0) t^{\frac{3}{4}} \varepsilon^{\frac{1}{4} - \frac{\beta(7+\delta)}{2}} < 2\varepsilon^{\beta s_0},$$

and so, for any $0 < t < T_\varepsilon^*$,

$$\left\| \tilde{U}^\varepsilon \right\|_{\tilde{L}^\infty([0,t],H^s)} < 2\varepsilon^{\frac{\beta s_0}{2}}.$$

Thus, the solution \tilde{U}^ε exists at least up to a time $T_\varepsilon^* > 0$ satisfies (6.5.20). From (6.5.19), if we set

$$\alpha = \min \left\{ \frac{1}{4} - \frac{\beta(7+\delta)}{2}, \frac{\beta s_0}{2} \right\} > 0 \quad \text{and} \quad \bar{C} = \frac{1}{4C},$$

then we have

$$T_\varepsilon^* \geq \bar{C} \mathcal{C}(U_0)^{-1} \varepsilon^{-\alpha}.$$

Theorem 6.5.1 is proved. \square

6.6 Estimates on the bilinear terms

In this appendix, we prove important estimates on the bilinear term, which allow to prove Lemma 6.5.2. First of all, we prove the following lemma

Lemma 6.6.1. *Let $i \in \{1, 2, 3\}$ and $\partial_i = \frac{\partial}{\partial x_i}$. For any $s > \frac{5}{2}$ and for any functions u, v and w in $H^s(\mathbb{R}^3)$, we have*

$$\begin{aligned} \int_0^t \left| \langle \Delta_q(w(\tau) \partial_i u(\tau)) \mid \Delta_q u(\tau) \rangle_{L^2} \right| d\tau \\ \leq C b_q 2^{-2qs} \|w\|_{\tilde{L}^\infty([0,t],H^s)} \|u\|_{\tilde{L}^2([0,t],H^s)}^2, \quad (6.6.1) \end{aligned}$$

and

$$\begin{aligned} \int_0^t & \left| \langle \Delta_q(w(\tau) \partial_i u(\tau)) \mid \Delta_q v(\tau) \rangle_{L^2} + \langle \Delta_q(w(\tau) \partial_i v(\tau)) \mid \Delta_q u(\tau) \rangle_{L^2} \right| d\tau \\ & \leq C b_q 2^{-2qs} \|w\|_{\tilde{L}^\infty([0,t],H^s)} \|u\|_{\tilde{L}^2([0,t],H^s)} \|v\|_{\tilde{L}^2([0,t],H^s)}, \end{aligned} \quad (6.6.2)$$

where b_q is a summable sequence such that $\sum_q b_q = 1$.

Proof. We will only prove Estimate (6.6.2). Estimate (6.6.1) can be obtained from (6.6.2) by choosing $u = v$. Applying the Bony decomposition as in (6.2.10) and (6.2.11) to the products $w\partial_i u$ and $w\partial_i v$, we can write

$$\int_0^t \left| \langle \Delta_q(w(\tau) \partial_i u(\tau)) \mid \Delta_q v(\tau) \rangle_{L^2} + \langle \Delta_q(w(\tau) \partial_i v(\tau)) \mid \Delta_q u(\tau) \rangle_{L^2} \right| d\tau \leq I_A + I_B, \quad (6.6.3)$$

where

$$\begin{aligned} I_A &= \int_0^t \left| \left\langle \Delta_q \sum_{q' > q-4} S_{q'+2}(\partial_i u) \Delta_{q'} w \mid \Delta_q v \right\rangle + \left\langle \Delta_q \sum_{q' > q-4} S_{q'+2}(\partial_i v) \Delta_{q'} w \mid \Delta_q u \right\rangle \right|(\tau) d\tau \\ I_B &= \int_0^t \left| \left\langle \Delta_q \sum_{|q'-q| \leq 4} S_{q'-1} w \partial_i \Delta_{q'} u \mid \Delta_q v \right\rangle + \left\langle \Delta_q \sum_{|q'-q| \leq 4} S_{q'-1} w \partial_i \Delta_{q'} v \mid \Delta_q u \right\rangle \right|(\tau) d\tau. \end{aligned}$$

Since $S_{q'+2}$ continuously maps $L^\infty(\mathbb{R}^3)$ to $L^\infty(\mathbb{R}^3)$, using Lemma 6.2.8, Hölder inequality and the Sobolev inclusion $H^s(\mathbb{R}^3) \hookrightarrow W^{1,\infty}(\mathbb{R}^3)$, we get

$$\begin{aligned} I_A &\leq \sum_{q' > q-4} \|S_{q'+2} \partial_i u\|_{L^2([0,t],L^\infty)} \|\Delta_{q'} w\|_{L^\infty([0,t],L^2)} \|\Delta_q v\|_{L^2([0,t],L^2)} \\ &\quad + \sum_{q' > q-4} \|S_{q'+2} \partial_i v\|_{L^2([0,t],L^\infty)} \|\Delta_{q'} w\|_{L^\infty([0,t],L^2)} \|\Delta_q u\|_{L^2([0,t],L^2)} \\ &\leq C b_q 2^{-2qs} \|w\|_{\tilde{L}^\infty([0,t],H^s)} \|u\|_{\tilde{L}^2([0,t],H^s)}^2 \end{aligned} \quad (6.6.4)$$

where

$$\{b_q\}_q = \left\{ \sum_{q' > q-4} 2^{-(q'-q)s} c_{q'}(w) (c_q(u) + c_q(v)) \right\}_q \in \ell^1,$$

using Young convolution inequality and the fact that $\{c_q(u)\}_q$, $\{c_q(v)\}_q$ and $\{c_{q'}(w)\}_{q'}$ are square-summable sequences.

To estimate the second term I_B of (6.6.3), we decompose it as follows

$$I_B \leq I_{B1} + I_{B2} + I_{B3},$$

where

$$\begin{aligned} I_{B1} &= \int_0^t \left| \langle S_q w \Delta_q \partial_i u \mid \Delta_q v \rangle + \langle S_q w \Delta_q \partial_i v \mid \Delta_q u \rangle \right|(\tau) d\tau \\ I_{B2} &= \int_0^t \sum_{|q-q'| \leq 4} \left| \langle (S_q - S_{q'-1}) w \Delta_{q'} \partial_i u \mid \Delta_q v \rangle + \langle (S_q - S_{q'-1}) w \Delta_{q'} \partial_i v \mid \Delta_q u \rangle \right|(\tau) d\tau \\ I_{B3} &= \int_0^t \sum_{|q-q'| \leq 4} \left| \langle [\Delta_q, S_{q'-1} w] \Delta_{q'} \partial_i u \mid \Delta_q v \rangle + \langle [\Delta_q, S_{q'-1} w] \Delta_{q'} \partial_i v \mid \Delta_q u \rangle \right|(\tau) d\tau. \end{aligned}$$

Recalling that S_q continuously maps $L^\infty(\mathbb{R}^3)$ to $L^\infty(\mathbb{R}^3)$, an integration by parts, Hölder inequality, Lemma 6.2.8 and the Sobolev inclusion $H^s(\mathbb{R}^3) \hookrightarrow W^{1,\infty}(\mathbb{R}^3)$ give

$$\begin{aligned} I_{B1} &= \int_0^t |S_q(\partial_i w(\tau)) \Delta_q u(\tau) \Delta_q v(\tau)| d\tau \\ &\leq \|S_q \partial_i w\|_{L^\infty([0,t],L^\infty)} \|\Delta_q u\|_{L^2([0,t],L^2)} \|\Delta_q v\|_{L^2([0,t],L^2)} \\ &\leq C b_q 2^{-2qs} \|w\|_{L^\infty([0,t],H^s)} \|u\|_{\tilde{L}^2([0,t],H^s)} \|v\|_{\tilde{L}^2([0,t],H^s)}, \end{aligned} \quad (6.6.5)$$

where $\{b_q\}_q = \{c_q(u)c_q(v)\}_q$ is a summable sequence. For I_{B2} , we remark that $S_q - S_{q'-1}$ does not contain low frequencies and continuously maps $L^\infty(\mathbb{R}^3)$ to $L^\infty(\mathbb{R}^3)$. Then, using Bernstein lemma 6.2.1 and Hölder inequality, we obtain the same estimates as in (6.6.5)

$$\begin{aligned} I_{B2} &\leq C \int_0^t \sum_{|q'-q|\leq 4} \|(S_q - S_{q'-1})w(\tau)\|_{L^\infty} 2^{q'} \|\Delta_{q'} u(\tau)\|_{L^2} \|\Delta_q v(\tau)\|_{L^2} d\tau \\ &\quad + C \int_0^t \sum_{|q'-q|\leq 4} \|(S_q - S_{q'-1})w(\tau)\|_{L^\infty} 2^{q'} \|\Delta_{q'} v(\tau)\|_{L^2} \|\Delta_q u(\tau)\|_{L^2} d\tau \\ &\leq C \sum_{|q'-q|\leq 4} \|(S_q - S_{q'-1})\partial_i w\|_{L^\infty([0,t],L^\infty)} \|\Delta_{q'} u\|_{L^2([0,t],L^2)} \|\Delta_q v\|_{L^2([0,t],L^2)} \\ &\quad + C \sum_{|q'-q|\leq 4} \|(S_q - S_{q'-1})\partial_i w\|_{L^\infty([0,t],L^\infty)} \|\Delta_{q'} v\|_{L^2([0,t],L^2)} \|\Delta_q u\|_{L^2([0,t],L^2)} \\ &\leq C b_q 2^{-2qs} \|w\|_{L^\infty([0,t],H^s)} \|u\|_{\tilde{L}^2([0,t],H^s)} \|v\|_{\tilde{L}^2([0,t],H^s)} \end{aligned} \quad (6.6.6)$$

where

$$\{b_q\}_q = \left\{ \sum_{|q'-q|\leq 4} 2^{-(q'-q)s} (c_q(v)c_{q'}(u) + c_q(u)c_{q'}(v)) \right\}_q \in \ell^1.$$

Finally, for the term I_{B3} , Hölder inequality and Lemma 6.2.6 yield

$$\begin{aligned} I_{B3} &\leq C \sum_{|q-q'|\leq 4} 2^{-q} \|S_{q'-1} \nabla w\|_{L^\infty([0,t],L^\infty)} \|\Delta_{q'} \partial_i u\|_{L^2([0,t],L^2)} \|\Delta_q v\|_{L^2([0,t],L^2)} \\ &\quad + C \sum_{|q-q'|\leq 4} 2^{-q} \|S_{q'-1} \nabla w\|_{L^\infty([0,t],L^\infty)} \|\Delta_{q'} \partial_i v\|_{L^2([0,t],L^2)} \|\Delta_q u\|_{L^2([0,t],L^2)}. \end{aligned}$$

Using the fact that S_q continuously maps $L^\infty(\mathbb{R}^3)$ to $L^\infty(\mathbb{R}^3)$, Bernstein lemma 6.2.1 and Estimate (6.2.8), we have

$$\begin{aligned} I_{B3} &\leq C \sum_{|q-q'|\leq 4} 2^{q'-q} \|S_{q'-1} \nabla w\|_{L^\infty([0,t],L^\infty)} \|\Delta_{q'} u\|_{L^2([0,t],L^2)} \|\Delta_q v\|_{L^2([0,t],L^2)} \\ &\quad + C \sum_{|q-q'|\leq 4} 2^{q'-q} \|S_{q'-1} \nabla w\|_{L^\infty([0,t],L^\infty)} \|\Delta_{q'} v\|_{L^2([0,t],L^2)} \|\Delta_q u\|_{L^2([0,t],L^2)} \\ &\leq C b_q 2^{-2qs} \|w\|_{L^\infty([0,t],H^s)} \|u\|_{\tilde{L}^2([0,t],H^s)} \|v\|_{\tilde{L}^2([0,t],H^s)}, \end{aligned} \quad (6.6.7)$$

where

$$\{b_q\}_q = \left\{ \sum_{|q'-q|\leq 4} 2^{-(q'-q)(s-1)} (c_{q'}(u)c_q(v) + c_{q'}(v)c_q(u)) \right\}_q \in \ell^1.$$

Inserting (6.6.4)–(6.6.7) into (6.6.3), we deduce Estimate (6.6.2). \square

In order to prove Lemma 6.5.2, we also need the following estimates when the bilinear term contains functions whose Fourier transform is localized in $\mathcal{C}_{r,R}$ (see (6.1.9) for the definition of $\mathcal{C}_{r,R}$).

Lemma 6.6.2. *Let $T > 0$, $i \in \{1, 2, 3\}$, $\partial_i = \frac{\partial}{\partial x_i}$ and \bar{U}^ε be the solution of the cut-off linear system (6.4.2). For any $s > \frac{5}{2}$, for any functions v and w in $\tilde{L}^\infty([0, T], H^s(\mathbb{R}^3))$, for any component \bar{u}^j of \bar{U}^ε , $j \in \{1, 2, 3, 4\}$, and for any $0 < t \leq T$, we have*

$$\begin{aligned} & \int_0^t |\langle \Delta_q (v(\tau) \partial_i \bar{u}^j(\tau)) \mid \Delta_q w(\tau) \rangle_{L^2}| d\tau \\ & \leq C \mathcal{C}(U_0) R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} b_q 2^{-2qs} \|v\|_{\tilde{L}^\infty([0,t], H^s)} \|w\|_{\tilde{L}^\infty([0,t], H^s)}, \end{aligned} \quad (6.6.8)$$

where $\mathcal{C}(U_0)$ is defined in (6.1.8) and b_q is a summable sequence such that $\sum_q b_q = 1$.

Lemma 6.6.3. *Let $T > 0$, $i \in \{1, 2, 3\}$, $\partial_i = \frac{\partial}{\partial x_i}$ and \bar{U}^ε be the solution of the cut-off linear system (6.4.2). For any $s > \frac{5}{2}$, for any functions v and w in $\tilde{L}^\infty([0, T], H^s(\mathbb{R}^3))$, for any component \bar{u}^j of \bar{U}^ε , $j \in \{1, 2, 3, 4\}$, and for any $0 < t \leq T$, we have*

$$\begin{aligned} & \int_0^t |\langle \Delta_q (\bar{u}^j(\tau) \partial_i v(\tau)) \mid \Delta_q v(\tau) \rangle_{L^2}| d\tau \\ & \leq C \mathcal{C}(U_0) R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} b_q 2^{-2qs} \|v\|_{\tilde{L}^\infty([0,t], H^s)}^2, \end{aligned} \quad (6.6.9)$$

and

$$\begin{aligned} & \int_0^t |\langle \Delta_q (\bar{u}^j(\tau) \partial_i v(\tau)) \mid \Delta_q w(\tau) \rangle_{L^2} + \langle \Delta_q (\bar{u}^j(\tau) \partial_i w(\tau)) \mid \Delta_q v(\tau) \rangle_{L^2}| d\tau \\ & \leq C \mathcal{C}(U_0) R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} b_q 2^{-2qs} \|v\|_{\tilde{L}^\infty([0,t], H^s)} \|w\|_{\tilde{L}^\infty([0,t], H^s)}, \end{aligned} \quad (6.6.10)$$

where $\mathcal{C}(U_0)$ is defined in (6.1.8) and b_q is a summable sequence such that $\sum_q b_q = 1$.

Proof of Lemma 6.6.2. We apply the same Bony decomposition into paraproducts and remainders as in (6.6.3) and we have

$$\int_0^t |\langle \Delta_q (v(\tau) \partial_i \bar{u}^j(\tau)) \mid \Delta_q w(\tau) \rangle_{L^2}| d\tau \leq J_A + J_B, \quad (6.6.11)$$

where

$$\begin{aligned} J_A &= \int_0^t \left| \left\langle \Delta_q \sum_{q' > q-4} S_{q'+2} (\partial_i \bar{u}^j(\tau)) \Delta_{q'} v(\tau) \mid \Delta_q w(\tau) \right\rangle_{L^2} \right| d\tau \\ J_B &= \int_0^t \left| \left\langle \Delta_q \sum_{|q'-q| \leq 4} S_{q'-1} v(\tau) \Delta_{q'} \partial_i \bar{u}^j(\tau) \mid \Delta_q w(\tau) \right\rangle_{L^2} \right| d\tau. \end{aligned}$$

For the term J_A , Lemma 6.2.8 and similar estimates as in (6.6.4) imply

$$\begin{aligned} J_A &\leq \sum_{q' > q-4} \|S_{q'+2} \partial_i \bar{w}^j\|_{L^1([0,t],L^\infty)} \|\Delta_{q'} v\|_{L^\infty([0,t],L^2)} \|\Delta_q v\|_{L^\infty([0,t],L^2)} \\ &\leq CRt^{\frac{3}{4}} \|\bar{U}^\varepsilon\|_{L^4([0,t],L^\infty)} 2^{-2qs} \left(\sum_{q' > q-4} 2^{-(q'-q)s} c_{q'}(v) c_q(w) \right) \|v\|_{\tilde{L}^\infty([0,t],H^s)} \|w\|_{\tilde{L}^\infty([0,t],H^s)} \end{aligned}$$

Using Strichartz-type estimates (6.4.3) and fixing $0 < r = R^\delta$, $\delta > 0$, we have

$$J_A \leq C \mathcal{C}(U_0) R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} b_q 2^{-2qs} \|v\|_{\tilde{L}^\infty([0,t],H^s)} \|w\|_{\tilde{L}^\infty([0,t],H^s)}, \quad (6.6.12)$$

where

$$\{b_q\}_q = \left\{ \sum_{q' > q-4} 2^{-(q'-q)s} c_{q'}(v) c_q(w) \right\}_q \in \ell^1.$$

The term J_B is a little more difficult to estimate. Using Hölder inequality and the fact that $S_{q'-1}$ continuously maps $H^s(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3)$, we have

$$\begin{aligned} J_B &\leq \sum_{|q'-q| \leq 4} \|S_{q'-1} v\|_{L^\infty([0,t],L^2)} \|\Delta_{q'} \partial_i \bar{w}^j\|_{L^1([0,t],L^\infty)} \|\Delta_q w\|_{L^\infty([0,t],L^2)} \\ &\leq CRt^{\frac{3}{4}} \sum_{|q'-q| \leq 4} \|\Delta_{q'} \bar{U}^\varepsilon\|_{L^4([0,t],L^\infty)} \|v\|_{L^\infty([0,t],H^s)} \|\Delta_q w\|_{L^\infty([0,t],L^2)}. \end{aligned}$$

Strichartz-type estimates (6.4.3) imply

$$\begin{aligned} J_B &\leq CR^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} \sum_{|q'-q| \leq 4} \|\Delta_{q'} \mathcal{P}_{r,R} U_0\|_{L^2} \|v\|_{L^\infty([0,t],H^s)} \|\Delta_q w\|_{L^\infty([0,t],L^2)} \quad (6.6.13) \\ &\leq CR^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} b_q 2^{-2qs} \|U_0\|_{H^s} \|v\|_{L^\infty([0,t],H^s)} \|w\|_{\tilde{L}^\infty([0,t],H^s)} \\ &\leq C \mathcal{C}(U_0) R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} b_q 2^{-2qs} \|v\|_{\tilde{L}^\infty([0,t],H^s)} \|w\|_{\tilde{L}^\infty([0,t],H^s)}, \end{aligned}$$

where

$$\{b_q\}_q = \left\{ \sum_{|q'-q| \leq 4} 2^{-(q'-q)s} c_{q'}(U_0) c_q(w) \right\}_q \in \ell^1.$$

Putting (6.6.12) and (6.6.13) into (6.6.11), we deduce Estimate (6.6.8). \square

Proof of Lemma 6.6.3. As in the proof of Lemma 6.6.1, we will only prove Estimate (6.6.10). Estimate (6.6.9) will follow if we choose $v = w$. Applying the Bony decomposition into paraproducts and remainders, we have

$$\int_0^t |\langle \Delta_q (\bar{w}^j(\tau) \partial_i v(\tau)) \mid \Delta_q w(\tau) \rangle_{L^2} + \langle \Delta_q (\bar{w}^j(\tau) \partial_i w(\tau)) \mid \Delta_q v(\tau) \rangle_{L^2}| d\tau \leq K_A + K_B, \quad (6.6.14)$$

where

$$\begin{aligned} K_A &= \int_0^t \left| \left\langle \Delta_q \sum_{q' > q-4} S_{q'+2} (\partial_i v) \Delta_{q'} \bar{w}^j \mid \Delta_q w \right\rangle + \left\langle \Delta_q \sum_{q' > q-4} S_{q'+2} (\partial_i w) \Delta_{q'} \bar{w}^j \mid \Delta_q v \right\rangle \right|(\tau) d\tau \\ K_B &= \int_0^t \left| \left\langle \Delta_q \sum_{|q'-q| \leq 4} S_{q'-1} \bar{w}^j \Delta_{q'} \partial_i v \mid \Delta_q w \right\rangle + \left\langle \Delta_q \sum_{|q'-q| \leq 4} S_{q'-1} \bar{w}^j \Delta_{q'} \partial_i w \mid \Delta_q v \right\rangle \right|(\tau) d\tau. \end{aligned}$$

The term K_A can be bounded by similar estimates as we did for J_B in (6.6.13)

$$\begin{aligned}
 K_A &\leq \sum_{q' > q-4} \|S_{q'+2}v\|_{L^\infty([0,t],L^2)} \|\Delta_{q'}\partial_i\bar{u}^j\|_{L^1([0,t],L^\infty)} \|\Delta_q w\|_{L^\infty([0,t],L^2)} \\
 &\quad + \sum_{q' > q-4} \|S_{q'+2}w\|_{L^\infty([0,t],L^2)} \|\Delta_{q'}\partial_i\bar{u}^j\|_{L^1([0,t],L^\infty)} \|\Delta_q v\|_{L^\infty([0,t],L^2)} \\
 &\leq CR^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} \sum_{q' > q-4} \|\Delta_{q'}\mathcal{P}_{r,R}U_0\|_{L^2} \|v\|_{L^\infty([0,t],H^s)} \|\Delta_q w\|_{L^\infty([0,t],L^2)} \\
 &\quad + CR^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} \sum_{q' > q-4} \|\Delta_{q'}\mathcal{P}_{r,R}U_0\|_{L^2} \|w\|_{L^\infty([0,t],H^s)} \|\Delta_q v\|_{L^\infty([0,t],L^2)} \\
 &\leq C \mathcal{C}(U_0) R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} b_q 2^{-2qs} \|v\|_{\tilde{L}^\infty([0,t],H^s)} \|w\|_{\tilde{L}^\infty([0,t],H^s)},
 \end{aligned} \tag{6.6.15}$$

where

$$\{b_q\}_q = \left\{ \sum_{q' > q-4} 2^{-(q'-q)s} c_{q'}(U_0) (c_q(v) + c_q(w)) \right\}_q \in \ell^1.$$

The term K_B is more difficult to estimate because we can not simply commute $S_{q'-1}$ and ∂_i . So, we use the same method as for the term I_B of (6.6.3) and we decompose

$$K_B \leq K_{B1} + K_{B2} + K_{B3},$$

where

$$\begin{aligned}
 K_{B1} &= \int_0^t |\langle S_q \bar{u}^j \Delta_q \partial_i v \mid \Delta_q w \rangle + \langle S_q \bar{u}^j \Delta_q \partial_i v \mid \Delta_q w \rangle|(\tau) d\tau \\
 K_{B2} &= \int_0^t \sum_{|q-q'| \leq 4} |\langle (S_q - S_{q'-1}) \bar{u}^j \Delta_{q'} \partial_i v \mid \Delta_q w \rangle + \langle (S_q - S_{q'-1}) \bar{u}^j \Delta_{q'} \partial_i w \mid \Delta_q v \rangle|(\tau) d\tau \\
 K_{B3} &= \int_0^t \sum_{|q-q'| \leq 4} |\langle [\Delta_q, S_{q'-1} \bar{u}^j] \Delta_{q'} \partial_i v \mid \Delta_q w \rangle + \langle [\Delta_q, S_{q'-1} \bar{u}^j] \Delta_{q'} \partial_i w \mid \Delta_q v \rangle|(\tau) d\tau.
 \end{aligned}$$

For K_{B1} , performing an integration by parts, we have

$$\begin{aligned}
 K_{B1} &= \int_0^t |S_q(\partial_i \bar{u}^j(\tau)) \Delta_q v(\tau) \Delta_q w(\tau)| d\tau \\
 &\leq \|S_q \partial_i \bar{U}^\varepsilon\|_{L^1([0,t],L^\infty)} \|\Delta_q v\|_{L^\infty([0,t],L^2)} \|\Delta_q w\|_{L^\infty([0,t],L^2)} \\
 &\leq C \mathcal{C}(U_0) R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} b_q 2^{-2qs} \|v\|_{\tilde{L}^\infty([0,t],H^s)} \|w\|_{\tilde{L}^\infty([0,t],H^s)},
 \end{aligned} \tag{6.6.16}$$

where $\{b_q\}_q = \{c_q(v)c_q(w)\}_q$ is a summable sequence. For K_{B2} , we have

$$\begin{aligned}
 K_{B2} &\leq C \int_0^t \sum_{|q'-q|\leq 4} \|(S_q - S_{q'-1})\bar{w}^j(\tau)\|_{L^\infty} 2^{q'} \|\Delta_{q'}v(\tau)\|_{L^2} \|\Delta_q w(\tau)\|_{L^2} d\tau \quad (6.6.17) \\
 &\quad + C \int_0^t \sum_{|q'-q|\leq 4} \|(S_q - S_{q'-1})\bar{w}^j(\tau)\|_{L^\infty} 2^{q'} \|\Delta_{q'}w(\tau)\|_{L^2} \|\Delta_q v(\tau)\|_{L^2} d\tau \\
 &\leq C \sum_{|q'-q|\leq 4} \|(S_q - S_{q'-1})\nabla\bar{U}^\varepsilon\|_{L^1([0,t],L^\infty)} \|\Delta_{q'}v\|_{L^\infty([0,t],L^2)} \|\Delta_q w\|_{L^\infty([0,t],L^2)} \\
 &\quad + C \sum_{|q'-q|\leq 4} \|(S_q - S_{q'-1})\nabla\bar{U}^\varepsilon\|_{L^1([0,t],L^\infty)} \|\Delta_{q'}w\|_{L^\infty([0,t],L^2)} \|\Delta_q v\|_{L^\infty([0,t],L^2)} \\
 &\leq C \mathcal{C}(U_0) R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} b_q 2^{-2qs} \|v\|_{\tilde{L}^\infty([0,t],H^s)} \|w\|_{\tilde{L}^\infty([0,t],H^s)},
 \end{aligned}$$

where

$$\{b_q\}_q = \left\{ \sum_{|q'-q|\leq 4} 2^{-(q'-q)s} (c_{q'}(v)c_q(w) + c_{q'}(w)c_q(v)) \right\}_q \in \ell^1.$$

Finally, for K_{B3} we can write

$$\begin{aligned}
 K_{B3} &\leq C \sum_{|q-q'|\leq 4} 2^{q'-q} \|S_{q'-1}\nabla\bar{w}^j\|_{L^1([0,t],L^\infty)} \|\Delta_{q'}v\|_{L^\infty([0,t],L^2)} \|\Delta_q w\|_{L^\infty([0,t],L^2)} \quad (6.6.18) \\
 &\quad + C \sum_{|q-q'|\leq 4} 2^{q'-q} \|S_{q'-1}\nabla\bar{w}^j\|_{L^1([0,t],L^\infty)} \|\Delta_{q'}w\|_{L^\infty([0,t],L^2)} \|\Delta_q v\|_{L^\infty([0,t],L^2)} \\
 &\leq C \mathcal{C}(U_0) R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} b_q 2^{-2qs} \|v\|_{\tilde{L}^\infty([0,t],H^s)} \|w\|_{\tilde{L}^\infty([0,t],H^s)},
 \end{aligned}$$

where

$$\{b_q\}_q = \left\{ \sum_{|q'-q|\leq 4} 2^{-(q'-q)(s-1)} (c_{q'}(v)c_q(w) + c_{q'}(w)c_q(v)) \right\}_q \in \ell^1.$$

Summing Estimates (6.6.15) to (6.6.18) and putting the obtained result into (6.6.14), we deduce Inequality (6.6.10) of Lemma 6.6.3. \square

Proof of Lemma 6.5.2 We recall the decomposition of U^ε as the sum

$$U^\varepsilon = \bar{U}^\varepsilon + \tilde{U}^\varepsilon,$$

then, we can write

$$\int_0^t \left| \left\langle \Delta_q (u^\varepsilon(\tau) \cdot \nabla U^\varepsilon(\tau)) \mid \Delta_q \tilde{U}^\varepsilon(\tau) \right\rangle_{L^2} \right| d\tau \leq A_1 + A_2 + A_3 + A_4,$$

where

$$\begin{aligned}
 A_1 &= \int_0^t \left| \left\langle \Delta_q (\bar{u}^\varepsilon(\tau) \cdot \nabla \bar{U}^\varepsilon(\tau)) \mid \Delta_q \tilde{U}^\varepsilon(\tau) \right\rangle_{L^2} \right| d\tau \\
 A_2 &= \int_0^t \left| \left\langle \Delta_q (\tilde{u}^\varepsilon(\tau) \cdot \nabla \bar{U}^\varepsilon(\tau)) \mid \Delta_q \tilde{U}^\varepsilon(\tau) \right\rangle_{L^2} \right| d\tau \\
 A_3 &= \int_0^t \left| \left\langle \Delta_q (\bar{u}^\varepsilon(\tau) \cdot \nabla \tilde{U}^\varepsilon(\tau)) \mid \Delta_q \tilde{U}^\varepsilon(\tau) \right\rangle_{L^2} \right| d\tau \\
 A_4 &= \int_0^t \left| \left\langle \Delta_q (\tilde{u}^\varepsilon(\tau) \cdot \nabla \tilde{U}^\varepsilon(\tau)) \mid \Delta_q \tilde{U}^\varepsilon(\tau) \right\rangle_{L^2} \right| d\tau.
 \end{aligned}$$

Using Lemma 6.6.2, we have

$$\begin{aligned}
 A_1 &\leq C \mathcal{C}(U_0) b_q 2^{-2qs} R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} \left\| \tilde{U}^\varepsilon \right\|_{\tilde{L}^\infty([0,t],H^s)} \\
 A_2 &\leq C \mathcal{C}(U_0) b_q 2^{-2qs} R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} \left\| \tilde{U}^\varepsilon \right\|_{\tilde{L}^\infty([0,t],H^s)}^2.
 \end{aligned}$$

For A_3 , using Lemma 6.6.3, we have

$$A_3 \leq C \mathcal{C}(U_0) b_q 2^{-2qs} R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} \left\| \tilde{U}^\varepsilon \right\|_{\tilde{L}^\infty([0,t],H^s)}^2.$$

Finally, for A_4 , Lemma 6.6.1 and the Sobolev embedding $H^s(\mathbb{R}^3) \hookrightarrow W^{1,\infty}(\mathbb{R}^3)$, with $s > \frac{5}{2}$, simply yield

$$A_4 \leq C b_q 2^{-2qs} \left\| \tilde{U}^\varepsilon \right\|_{\tilde{L}^\infty([0,t],H^s)} \left\| \tilde{U}^\varepsilon \right\|_{\tilde{L}^2([0,t],H^s)}^2 \leq C b_q 2^{-2qs} t \left\| \tilde{U}^\varepsilon \right\|_{\tilde{L}^\infty([0,t],H^s)} \left\| \tilde{U}^\varepsilon \right\|_{\tilde{L}^\infty([0,t],H^s)}^2.$$

Now, we can prove (6.5.5) exactly in the same way as we do to prove (6.5.4). We can decompose the term on the right hand side of (6.5.5) as

$$\begin{aligned}
 \int_0^t \left| \left\langle \Delta_q (b^\varepsilon(\tau) \nabla b^\varepsilon(\tau)) \mid \Delta_q \tilde{u}^\varepsilon(\tau) \right\rangle_{L^2} + \left\langle \Delta_q (b^\varepsilon(\tau) \operatorname{div} u^\varepsilon(\tau)) \mid \Delta_q \tilde{b}^\varepsilon(\tau) \right\rangle_{L^2} \right| d\tau \\
 \leq A'_1 + A'_2 + A'_3 + A'_4,
 \end{aligned}$$

where

$$\begin{aligned}
 A'_1 &= \int_0^t \left| \left\langle \Delta_q (\bar{b}^\varepsilon(\tau) \nabla \bar{b}^\varepsilon(\tau)) \mid \Delta_q \tilde{u}^\varepsilon(\tau) \right\rangle_{L^2} + \left\langle \Delta_q (\bar{b}^\varepsilon(\tau) \operatorname{div} \bar{u}^\varepsilon(\tau)) \mid \Delta_q \tilde{b}^\varepsilon(\tau) \right\rangle_{L^2} \right| d\tau \\
 A'_2 &= \int_0^t \left| \left\langle \Delta_q (\tilde{b}^\varepsilon(\tau) \nabla \bar{b}^\varepsilon(\tau)) \mid \Delta_q \tilde{u}^\varepsilon(\tau) \right\rangle_{L^2} + \left\langle \Delta_q (\tilde{b}^\varepsilon(\tau) \operatorname{div} \bar{u}^\varepsilon(\tau)) \mid \Delta_q \tilde{b}^\varepsilon(\tau) \right\rangle_{L^2} \right| d\tau \\
 A'_3 &= \int_0^t \left| \left\langle \Delta_q (\bar{b}^\varepsilon(\tau) \nabla \tilde{b}^\varepsilon(\tau)) \mid \Delta_q \tilde{u}^\varepsilon(\tau) \right\rangle_{L^2} + \left\langle \Delta_q (\bar{b}^\varepsilon(\tau) \operatorname{div} \tilde{u}^\varepsilon(\tau)) \mid \Delta_q \tilde{b}^\varepsilon(\tau) \right\rangle_{L^2} \right| d\tau \\
 A'_4 &= \int_0^t \left| \left\langle \Delta_q (\tilde{b}^\varepsilon(\tau) \nabla \tilde{b}^\varepsilon(\tau)) \mid \Delta_q \tilde{u}^\varepsilon(\tau) \right\rangle_{L^2} + \left\langle \Delta_q (\tilde{b}^\varepsilon(\tau) \operatorname{div} \tilde{u}^\varepsilon(\tau)) \mid \Delta_q \tilde{b}^\varepsilon(\tau) \right\rangle_{L^2} \right| d\tau.
 \end{aligned}$$

Using Lemma 6.6.2, we have

$$\begin{aligned} A'_1 &\leq C \mathcal{C}(U_0) b_q 2^{-2qs} R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} \left\| \tilde{U}^\varepsilon \right\|_{\tilde{L}^\infty([0,t],H^s)} \\ A'_2 &\leq C \mathcal{C}(U_0) b_q 2^{-2qs} R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} \left\| \tilde{U}^\varepsilon \right\|_{\tilde{L}^\infty([0,t],H^s)}^2. \end{aligned}$$

Next, Lemma 6.6.3 yields

$$A'_3 \leq C \mathcal{C}(U_0) b_q 2^{-2qs} R^{\frac{7+\delta}{2}} t^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} \left\| \tilde{U}^\varepsilon \right\|_{\tilde{L}^\infty([0,t],H^s)}^2.$$

Finally, Lemma 6.6.1 and the Sobolev embedding $H^s(\mathbb{R}^3) \hookrightarrow W^{1,\infty}(\mathbb{R}^3)$, with $s > \frac{5}{2}$, imply

$$A'_4 \leq C b_q 2^{-2qs} \left\| \tilde{u}^\varepsilon \right\|_{\tilde{L}^\infty([0,t],H^s)} \left\| \tilde{b}^\varepsilon \right\|_{\tilde{L}^2([0,t],H^s)}^2 \leq C b_q 2^{-2qs} t \left\| \tilde{U}^\varepsilon \right\|_{\tilde{L}^\infty([0,t],H^s)} \left\| \tilde{U}^\varepsilon \right\|_{\tilde{L}^\infty([0,t],H^s)}^2.$$

Lemma 6.5.2 is then proved. \square

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Résumé

Dans cette thèse nous étudions trois modèles décrivant la dynamique de l'écoulement d'un fluide à densité variable, dans des échelles spatio-temporelles grandes. Dans ce cadre, le mouvement relatif induit par des forces extérieures, comme la force de Coriolis ou la poussée hydrostatique, s'avère être beaucoup plus important que le mouvement intrinsèque du fluide induit par le transport des particules. Une tel déséquilibre contraint ainsi le mouvement, induisant des structures persistantes dans l'écoulement du fluide.

D'un point de vue mathématique, l'une des difficultés consiste en l'étude des perturbations induites par les forces extérieures, qui se propagent à grande vitesse. Ce type d'analyse peut être effectué au moyen de plusieurs outils mathématiques ; on choisit ici d'employer des techniques caractéristiques de l'analyse de Fourier, comme l'analyse des propriétés dispersives des intégrales oscillantes.

Tout au long de cette thèse, on se restreint à considérer des domaines spatiaux sans frontière : c'est le cas de l'espace entier \mathbb{R}^3 , ou encore de l'espace périodique \mathbb{T}^3 . Les modèles considérés sont donc les suivants:

- Équations primitives dont les nombres de Froude et de Rossby sont comparables, et pour lesquelles la diffusion verticale est nulle,
- fluides stratifiés dans un régime à faible nombre de Froude,
- fluides faiblement compressibles et tournants dans un régime où les nombres de Mach et de Rossby sont comparables.

On prouve que ces systèmes propagent globalement dans le temps des données peu régulières. Nous n'imposons jamais de condition de petitesse sur les données initiales. Toutefois, on prendra en compte certaines hypothèses spécifiques de régularité, lorsque des raisons techniques l'imposent.

Abstract

In this thesis we discuss three models describing the dynamics of density-dependent fluids in long lifespans and on a planetary scale. In such setting the relative displacement induced by various external physical forces, such as the Coriolis force and the stratification buoyancy, is far more relevant than the intrinsic motion generated by the collision of particles of the fluid itself. Such disproportion of balance limits hence the motion, inducing persistent structures in the velocity flow.

On a mathematical level one of the main difficulties relies in giving a full description of the perturbations induced by the external forces, which propagate at high speed. This analysis can be performed by the aid of several tools, we chose here to adopt techniques characteristic of harmonic analysis, such as the analysis of the dispersive properties of highly oscillating integrals.

All along the thesis we consider boundary-free, three-dimensional domains, and in specific we study only the case in which the domain is either the whole space \mathbb{R}^3 or the periodic space \mathbb{T}^3 . The models we consider are the following ones:

- Primitive equations with comparable Froude and Rossby number and zero vertical diffusivity,
- density-dependent stratified fluids in low Froude number regime,
- Weakly compressible and fast rotating fluid in a regime in which Mach and Rossby number are comparable.

We prove that these systems propagate globally-in-time data with low-regularity. No smallness assumption is ever made, specific constructive hypothesis are assumed on the initial data when required.