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## Ordres sur les groupes de Garside

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► **To cite this version:**

Diego Arcis. Ordres sur les groupes de Garside. Théorie des groupes [math.GR]. Université Bourgogne Franche-Comté, 2017. Français. NNT : 2017UBFCK049 . tel-01816942

**HAL Id: tel-01816942**

**<https://theses.hal.science/tel-01816942>**

Submitted on 15 Jun 2018

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**THÈSE**  
**pour l'obtention du grade de**  
**Docteur de l'Université de Bourgogne**  
**en Mathématiques**

*présentée et soutenue publiquement par*

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le 29 septembre 2017

# **Ordering Garside groups**

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## Abstract

We introduce a condition on Garside groups that we call Dehornoy structure. An iteration of such a structure leads to a left order on the group. We show conditions for a Garside group to admit a Dehornoy structure, and we apply these criteria to prove that the Artin groups of type  $A$  and  $I_2(m)$ ,  $m \geq 4$ , have Dehornoy structures. We show that the left orders on the Artin groups of type  $A$  obtained from their Dehornoy structures are the Dehornoy orders. In the case of the Artin groups of type  $I_2(m)$ ,  $m \geq 4$ , we show that the left orders derived from their Dehornoy structures coincide with the orders obtained from embeddings of the groups into braid groups.



## Résumé

Nous présentons une condition sur les groupes de Garside que nous appelons la structure de Dehornoy. Une itération d'une telle structure conduit à un ordre à gauche sur le groupe. Nous montrons des conditions pour qu'un groupe de Garside admette une structure de Dehornoy, et nous appliquons ce critère pour prouver que les groupes d'Artin de type  $A$  et  $I_2(m)$ ,  $m \geq 4$ , ont des structures de Dehornoy. Nous montrons que les ordres à gauche sur les groupes d'Artin de type  $A$  obtenus à partir de leurs structures de Dehornoy sont les ordres de Dehornoy. Dans le cas des groupes d'Artin du type  $I_2(m)$ ,  $m \geq 4$ , nous montrons que les ordres à gauche dérivés de leurs structures de Dehornoy coïncident avec les ordres obtenus à partir des plongements de ces groupes dans les groupes de tresses.



## Résumé

Les *groupes de tresses* ont été pendant de nombreuses années un des objets d'étude de la topologie de basse dimension, en particulier de la théorie de noeuds, à travers les tresses fermées. Depuis quelques années cette théorie a étendu ses ramifications à d'autres domaines des mathématiques (théorie des groupes, géométrie algébrique, théorie de l'homotopie) allant jusqu'à des incursions en biologie et en cryptologie. Le *groupe de tresses à  $n$  brins*, noté  $\mathcal{B}_n$ , a été introduit par Artin [1, 2] en 1925. Dans son travail apparaît la présentation du groupe de tresse suivante comme un des aspects du groupe:

$$\mathcal{B}_n = \left\langle x_1, \dots, x_{n-1} \left| \begin{array}{ll} x_i x_j = x_j x_i & \text{si } |i - j| \geq 2 \\ x_i x_j x_i = x_j x_i x_j & \text{si } |i - j| = 1 \end{array} \right. \right\rangle.$$

Soit  $S$  un ensemble fini. Une *matrice de Coxeter* sur  $S$  est une matrice carrée  $M = (m_{s,t})_{s,t \in S}$ , indexée par les éléments de  $S$ , à coefficients dans  $\mathbb{N} \cup \{\infty\}$ , et vérifiant:

- (a)  $m_{s,s} = 1$  pour tout  $s \in S$ ;
- (b)  $m_{s,t} = m_{t,s} \geq 2$  pour tout  $s, t \in S, s \neq t$ .

Une matrice de Coxeter  $M$  comme ci-dessus se représente par un graphe pondéré,  $\Gamma$ , appelé *graphe de Coxeter* de  $M$ . Celui-ci est défini comme suit:

- (a) L'ensemble des sommets de  $\Gamma$  est  $S$ .
- (b) Deux sommets  $s, t \in S$  sont reliés par une arête si  $m_{s,t} \geq 3$ . Cette arête est pondérée par  $m_{s,t}$  si  $m_{s,t} \geq 4$ .

Si  $a, b$  sont deux lettres et  $m$  est un entier  $\geq 2$ , on pose:

$$\text{prod}(a, b : m) = \begin{cases} (ab)^{\frac{m}{2}} & \text{si } m \text{ est pair,} \\ (ab)^{\frac{m-1}{2}} a & \text{si } m \text{ est impair.} \end{cases}$$

En d'autres termes,  $\text{prod}(a, b : m)$  désigne le mot  $aba \dots$  de longueur  $m$ . Soit  $\Gamma$  un graphe de Coxeter comme ci-dessus. Le *groupe d'Artin* associé à  $\Gamma$  est le groupe  $A = A_\Gamma$  défini par la présentation:

$$A = \langle S \mid \text{prod}(s, t : m_{s,t}) = \text{prod}(t, s : m_{s,t}) \text{ pour tous } s, t \in S, s \neq t \text{ et } m_{s,t} \neq \infty \rangle.$$



Le monoïde  $A^+ = A_\Gamma^+$  ayant la présentation de monoïde suivante:

$$A^+ = \langle S \mid \text{prod}(s, t : m_{s,t}) = \text{prod}(t, s : m_{s,t}) \text{ pour tous } s, t \in S, s \neq t \text{ et } m_{s,t} \neq \infty \rangle^+$$

est appelé *monoïde d'Artin* associé à  $\Gamma$ . Par Paris [26], l'homomorphisme naturel  $A^+ \rightarrow A$  est injectif. Le *groupe de Coxeter* associé à  $\Gamma$ , noté  $W = W_\Gamma$ , est le quotient de  $A = A_\Gamma$  par les relations  $s^2 = 1, s \in S$ .

Les groupes d'Artin ont été introduits par Tits [30] comme extensions de groupes de Coxeter. Il existe une littérature étendue sur les groupes d'Artin, mais la plupart des résultats ne concernent que des sous-familles de groupes d'Artin. Une des familles de groupes d'Artin les plus étudiées est celle des groupes de type sphérique. Ce sont eux qui sont concernés par cette thèse. Rappelons qu'un groupe d'Artin  $A_\Gamma$  est dit *de type sphérique* si le groupe de Coxeter associé  $W = W_\Gamma$  est fini. Les groupes de tresses sont les exemples phare de groupes d'Artin de type sphérique.

La classification des groupes de Coxeter finis, et donc des groupes d'Artin de type sphérique, est un des résultats précurseurs du domaine. Elle est due à Coxeter [6]. Soit  $\Gamma$  un graphe de Coxeter. Soient  $\Gamma_1, \dots, \Gamma_\ell$  les composantes connexes de  $\Gamma$ . Il est clair que  $W_\Gamma = W_{\Gamma_1} \times \dots \times W_{\Gamma_\ell}$ , donc  $W_\Gamma$  est fini si et seulement si  $W_{\Gamma_i}$  est fini pour tout  $i \in \{1, \dots, \ell\}$ . Par ailleurs, selon Coxeter [6], les graphes de Coxeter connexes dont le groupe de Coxeter est finis sont ceux donnés dans la figure 1.

Un groupe  $G$  est dit *ordonnable à gauche* s'il existe un ordre total  $<$  sur  $G$  qui est invariant par multiplication à gauche, c'est-à-dire tel que  $g_1 g_2 < g_1 g_3$  si  $g_2 < g_3$ , pour tous  $g_1, g_2, g_3 \in G$ . Étant donné un tel ordre  $<$  sur  $G$ , on définit le *cône positif* (de  $G$  par rapport à  $<$ ) comme étant  $P = P_< = \{g \in G \mid 1 < g\}$ . Ce cône vérifie les propriétés suivantes:

- (1)  $PP \subset P$  (i.e.  $P$  est un sous-semi-groupe).
- (2)  $G = P \sqcup P^{-1} \sqcup \{1\}$ .

Réciproquement, une partie  $P$  de  $G$  qui vérifie (1) et (2) détermine un ordre à gauche  $<$  sur  $G$  défini par  $g_1 < g_2$  si  $g_1^{-1} g_2 \in P$ , et  $P$  est le cône positif pour  $<$ .

Le premier ordre à gauche explicite sur les groupes de tresses a été défini par Dehornoy [10]. Le fait que les groupes de tresses soient ordonnables est important, mais l'ordre de Dehornoy est aussi en lui-même intéressant. Par exemple, il intervient dans différentes démonstrations de

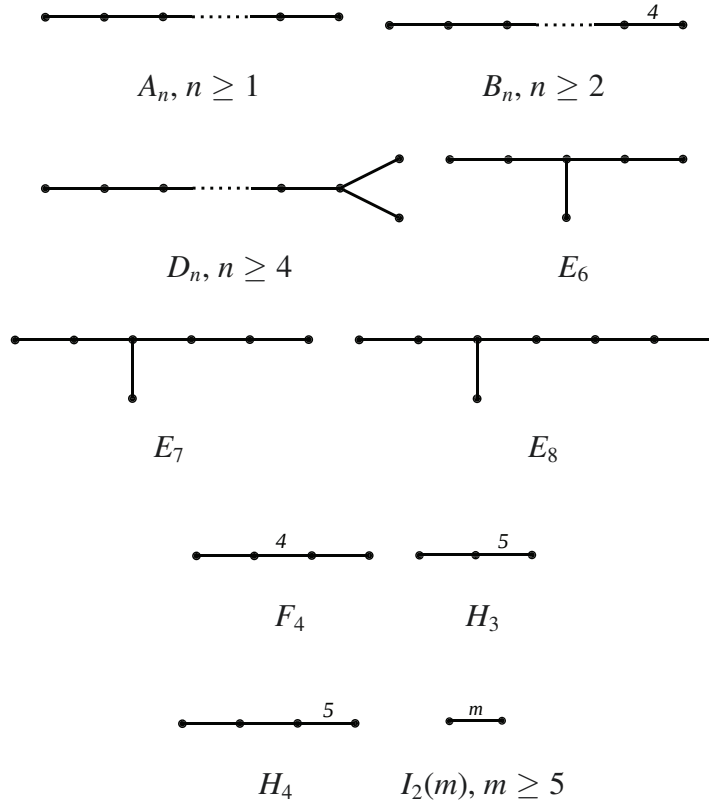


FIGURE 1. Graphes de Coxeter de type sphérique

la fidélité de certaines représentations des groupes de tresses (voir Shpilrain [28] et Crisp–Paris [9]). Une présentation assez complète des ordres sur les tresses et de l’ordre de Dehornoy est donnée dans Dehornoy–Dynnikov–Rolfsen–Wiest [14]. Sa définition repose sur la construction qui suit.

Soient  $G$  un groupe et  $X = \{x_1, \dots, x_k\}$  une famille génératrice de  $G$  que l’on suppose totalement ordonnée. Soit  $i \in \{1, \dots, k\}$ . On dit qu’un mot  $w \in (X \sqcup X^{-1})^*$  est  $x_i$ -positif (resp.  $x_i$ -négatif) si  $w \in \{x_i, x_{i+1}^{\pm 1}, \dots, x_k^{\pm 1}\}^*$  (resp.  $w \in \{x_i^{-1}, x_{i+1}^{\pm 1}, \dots, x_k^{\pm 1}\}^*$ ), mais  $w \notin \{x_{i+1}^{\pm 1}, \dots, x_k^{\pm 1}\}^*$ . Un élément  $g \in G$  est dit  $x_i$ -positif (resp.  $x_i$ -négatif) s’il admet un représentant  $x_i$ -positif (resp.  $x_i$ -négatif). On note  $G_i$  le sous-groupe de  $G$  engendré par  $\{x_i, x_{i+1}, \dots, x_k\}$ ,  $P_i^+$  l’ensemble des éléments  $x_i$ -positifs de  $G$ , et  $P_i^-$  l’ensemble des éléments  $x_i$ -négatifs. Le résultat clé dans la définition de l’ordre de Dehornoy est le suivant:

**Théorème 1** (Dehornoy [10]). *Supposons que  $G = \mathcal{B}_n$  est le groupe de tresses à  $n$  brins muni de sa famille génératrice standard,  $\{x_1, \dots, x_{n-1}\}$ . Pour tout  $i \in \{1, \dots, n-1\}$  on a la réunion disjointe  $G_i = P_i^+ \sqcup P_i^- \sqcup G_{i+1}$ , où  $G_n = \{1\}$ .*

Supposons encore que  $G = \mathcal{B}_n$  est le groupe de tresses à  $n$  brins, et posons  $P_D = P_1^+ \sqcup P_2^+ \sqcup \dots \sqcup P_{n-1}^+$ . Alors, par le Théorème 1,  $P_D$  est un cône positif pour un ordre à gauche sur  $G$ . Cet ordre est l'ordre de Dehornoy. Le lecteur attentif remarquera que le Théorème 1 ne fournit pas un ordre mais plusieurs ordres sur le groupe de tresses. Ainsi, pour  $\epsilon = (\epsilon_1, \dots, \epsilon_{n-1}) \in \{\pm 1\}^{n-1}$ , on pose:

$$P^\epsilon = P_1^{\epsilon_1} \sqcup P_2^{\epsilon_2} \sqcup \dots \sqcup P_{n-1}^{\epsilon_{n-1}}.$$

Alors, par le Théorème 1,  $P^\epsilon$  est le cône positif d'un ordre à gauche sur  $G = \mathcal{B}_n$ .

Comme les groupes de tresses sont un cas particulier de groupes d'Artin, une question naturelle est de savoir s'il existe une définition d'ordre sur les groupes d'Artin qui généralise celle de Dehornoy. Une première approche est de savoir si la construction antérieure donne aussi un ordre à gauche sur un groupe d'Artin qui ne soit pas de type  $A$ . la réponse à cette question a été donnée par Sibert [29]. Celui-ci a démontré, en utilisant des opérations de déploiement sur les graphes de Coxeter, que l'ordre de Dehornoy peut être étendu uniquement sur les groupes d'Artin qui sont produits directs de groupes de tresses.

En 1969 Garside [20] a donné une solution au problème de conjugaison sur les groupes de tresses de manière purement algébrique, en utilisant ce qui maintenant s'appelle la théorie de la divisibilité sur le monoïde des tresses positives  $\mathcal{B}_n^+$ . Il a aussi introduit un élément distingué,  $\Delta_n$ , appelé *élément fondamental* de  $\mathcal{B}_n$ , qui joue un rôle prépondérant dans la théorie. En 1972 la théorie développée par Garside a été étendue à tous les groupes d'Artin de type sphérique par Brieskorn–Saito [4]. Dans cette théorie chaque groupe  $G$  est un groupe de fractions d'un monoïde  $M$ , et on a un élément distingué  $\Delta$ , appelé *élément fondamental* de  $G$ . Par la suite, en 1999, Dehornoy–Paris [15] ont défini la notion de groupe de Garside en généralisant des propriétés et caractéristiques découvertes par Garside pour les groupes de tresses. Un groupe de Garside  $G$  est le groupe de fractions d'un monoïde  $M$ , appelé *monoïde de Garside*, et a un élément distingué, appelé *élément de Garside*, de façon à ce que tout élément de  $G$  s'écrit d'une manière unique sous la forme  $\Delta^m g$  avec  $(g, m) \in M \times \mathbb{Z}$ . Par Brieskorn–Saito [4] et Deligne

[16] tous les groupes d'Artin de type sphérique sont des groupes de Garside, et ses éléments fondamentaux sont des éléments de Garside. Ceux-ci sont toutefois un peu particulier dans la théorie car coïncident avec les plus petit multiples communs des générateurs standards. La théorie de Garside est maintenant un outil fondamental dans l'étude formelle des groupes de tresses, et plus généralement des groupes d'Artin de type sphérique.

De nombreuses propriétés connues pour les groupes d'Artin de type sphérique, ou plus particulièrement pour les groupes des tresses, ont été étendues aux groupes de Garside. Dans la même veine les sous-groupes paraboliques ont été étendus aux groupes de Garside par Godelle [22].

L'objectif de cette thèse est d'étendre l'ordre de Dehornoy à quelques groupes de Garside en satisfaisant certaines propriétés. Nous basons notre étude sur une définition de l'ordre de Dehornoy donnée dans Fromentin–Paris [19] qui utilise la longueur des formes alternées introduites par Dehornoy [12] et qui s'adapte facilement au contexte des groupes de Garside.

Soit  $M$  un monoïde. On dit que  $M$  est *atomique* s'il existe une application  $\nu : M \rightarrow \mathbb{N}$  telle que:

- (a)  $\nu(g) = 0$  si et seulement si  $g = 1$ ;
- (b)  $\nu(gg') \geq \nu(g) + \nu(g')$  pour tous  $g, g' \in M$ .

Une telle application  $\nu : M \rightarrow \mathbb{N}$  est appelée une *norme* sur  $M$ . Un élément  $g \in M$  est appelé un *atome* s'il est indécomposable dans le sens que, si  $g = g_1g_2$ , alors  $g_1 = 1$  ou  $g_2 = 1$ .

Supposons que  $M$  est atomique. Alors toute famille génératrice de  $M$  contient tous les atomes, et l'ensemble des atomes engendre  $M$ . En particulier,  $M$  est finiment engendré si et seulement s'il contient un nombre fini d'atomes. On peut définir des ordres partiels  $\leq_L$  et  $\leq_R$  sur  $M$  comme suit:

- On pose  $g_1 \leq_L g_2$  s'il existe  $g_3 \in M$  tel que  $g_1g_3 = g_2$ ;
- On pose  $g_1 \leq_R g_2$  s'il existe  $g_3 \in M$  tel que  $g_3g_1 = g_2$ .

Ces ordres s'appellent *ordre de divisibilité à gauche* et *ordre de divisibilité à droite*, respectivement. Pour  $g \in M$  on pose:

$$\text{Div}_L(g) = \{g' \in M \mid g' \leq_L g\} \text{ et } \text{Div}_R(g) = \{g' \in M \mid g' \leq_R g\}.$$

On dit que  $g$  est *balancé* si  $\text{Div}_L(g) = \text{Div}_R(g)$ .

Un *monoïde de Garside* est un monoïde  $M$  telle que:

- (a)  $M$  est atomique et finiment engendré;
- (b)  $M$  est *simplifiable*, c'est-à-dire  $g_1 g g_2 = g_1 g' g_2$  implique  $g = g'$  pour tous  $g, g', g_1, g_2 \in M$ ;
- (c)  $(M, \leq_L)$  et  $(M, \leq_R)$  sont des treillis;
- (d)  $M$  possède un *élément de Garside*, c'est-à-dire un élément balancé  $\Delta$  tel que  $\text{Div}_L(\Delta) = \text{Div}_R(\Delta)$  engendre  $M$ .

Les opérations de treillis dans  $(M, \leq_L)$  (resp.  $(M, \leq_R)$ ) sont notées  $\wedge_L$  et  $\vee_L$  (resp.  $\wedge_R$  et  $\vee_R$ ). Les éléments de  $\text{Div}(\Delta) = \text{Div}_L(\Delta) = \text{Div}_R(\Delta)$  sont appelés *éléments simples*.

Soit  $M$  un monoïde. Le *groupe enveloppant* de  $M$ , noté  $G(M)$ , est le groupe présenté par l'ensemble de générateurs  $M$  et les relations  $g_1 g_2 = g$  si l'on a  $g_1 g_2 = g$  dans  $M$ . Il existe un homomorphisme canonique évident  $M \rightarrow G(M)$  qui n'est pas injectif en général. Un *groupe de Garside* est un groupe enveloppant d'un monoïde de Garside.

Une paire  $(s, s')$  d'éléments simples est appelée *pondérée à droite* (resp. *pondérée à gauche*) si  $ss' \wedge_R \Delta = s'$  (resp.  $ss' \wedge_L \Delta = s$ ).

**Proposition 2** (Dehornoy–Paris [15]). *Tout élément  $g \in G$  admet des décompositions uniques  $g = s_p \cdots s_1 \Delta^d$  et  $g = \Delta^d s'_1 \cdots s'_p$  telles que:*

- (a)  $s_1, \dots, s_p, s'_1, \dots, s'_p \in \text{Div}(\Delta) \setminus \{1, \Delta\}$ ;
- (b)  $(s_{i+1}, s_i)$  est pondérée à droite pour tout  $i \in \{1, \dots, p-1\}$ ;
- (c)  $(s'_i, s'_{i+1})$  est pondérée à gauche pour tout  $i \in \{1, \dots, p-1\}$ .

Les expressions  $g = s_p \cdots s_1 \Delta^d$  et  $g = \Delta^d s'_1 \cdots s'_p$  de la Proposition 2 s'appellent la *forme normale avide à droite* et la *forme normale avide à gauche* de  $g$ , respectivement. L'entier  $d$  s'appelle l'*infimum* de  $g$  et se note  $\text{inf}(g)$ , l'entier  $p$  s'appelle la *longueur canonique* et se note  $\ell(g)$ , et l'entier  $d + p$  s'appelle le *supremum* et se note  $\text{sup}(g)$ . Dans cette thèse nous avons besoin d'un quatrième invariant que l'on appelle l'*infimum négatif* et qui est défini par:

$$\text{Ninf}(g) = \begin{cases} 0 & \text{si } d \geq 0, \\ -d & \text{si } d < 0. \end{cases}$$

Par ailleurs, on pose  $g_R = s_p \cdots s_1$  et  $g_L = s'_1 \cdots s'_p$ . Remarquez que  $g_L = g_R$  si  $\Delta$  est central.

Soit  $\delta$  un élément balancé. On note  $M_\delta$  (resp.  $G_\delta$ ) le sous-monoïde de  $M$  (resp. sous-groupe de  $G$ ) engendré par  $\text{Div}(\delta)$ . On dit que  $M_\delta$  (resp.  $G_\delta$ ) est un *sous-monoïde parabolique* de  $M$  (resp. un *sous-groupe parabolique* de  $G$ ) défini par  $\delta$  si  $\text{Div}(\delta) = \text{Div}(\Delta) \cap M_\delta$ .

**Proposition 3** (Godelle [22]). *Soit  $\delta \in M$  un élément balancé tel que  $M_\delta$  est un sous-monoïde parabolique de  $M$ .*

- (1)  $M_\delta$  est un monoïde de Garside,  $\delta$  est un élément de Garside pour  $M_\delta$  et  $G_\delta$  est le groupe enveloppant de  $M_\delta$ .
- (2) Notons  $\vee_{\delta,L}$  et  $\wedge_{\delta,L}$  (resp.  $\vee_{\delta,R}$  et  $\wedge_{\delta,R}$ ) les opérations de treillis de  $(M_\delta, \leq_L)$  (resp.  $(M_\delta, \leq_R)$ ). On a  $g_1 \vee_L g_2, g_1 \wedge_L g_2 \in M_\delta$ ,  $g_1 \vee_L g_2 = g_1 \vee_{\delta,L} g_2$ , et  $g_1 \wedge_L g_2 = g_1 \wedge_{\delta,L} g_2$ , pour tous  $g_1, g_2 \in M_\delta$ . De même,  $g_1 \vee_R g_2, g_1 \wedge_R g_2 \in M_\delta$ ,  $g_1 \vee_R g_2 = g_1 \vee_{\delta,R} g_2$ , et  $g_1 \wedge_R g_2 = g_1 \wedge_{\delta,R} g_2$  pour tous  $g_1, g_2 \in M_\delta$ .
- (3) Si  $g_1 \leq_L g_2$  (resp.  $g_1 \leq_R g_2$ ) et  $g_2 \in M_\delta$ , alors  $g_1 \in M_\delta$ , pour tous  $g_1, g_2 \in M$ .

Les groupes de tresses et, plus généralement, les groupes d'Artin de type sphérique sont les exemples phare de groupes de Garside. Plus précisément:

**Théorème 4** (Brieskorn–Saito [4], Deligne [16]). *Soient  $\Gamma$  un graphe de Coxeter de type sphérique,  $A^+$  son monoïde d'Artin et  $A$  son groupe d'Artin.*

- (1) Le monoïde  $A^+$  est un monoïde de Garside,  $\Delta = \vee_L(S) = \vee_R(S)$  est un élément de Garside de  $A^+$ , et  $A$  est le groupe enveloppant de  $A^+$ .
- (2) Soit  $X$  une partie de  $S$ . Alors  $A_X^+$  (resp.  $A_X$ ) est un monoïde d'Artin (resp. groupe d'Artin) associé au graphe  $\Gamma_X$ . Soit  $\Delta_X = \vee_L(X)$ . Alors  $\Delta_X = \vee_R(X)$ ,  $\Delta_X$  est balancé, et  $A_X^+$  est le sous-monoïde parabolique associé à  $\Delta_X$ .

Le résultat suivant servira à la définition de structure de Dehornoy, qui est l'objet central de cette thèse:

**Proposition 5** (Dehornoy [13]). *Soient  $M$  un monoïde de Garside,  $N$  un sous-monoïde parabolique, et  $g$  un élément de  $M$ . Alors il existe un unique élément  $R_N(g)$  dans  $N$  tel que  $\{h \in N \mid h \leq_R g\} = \{h \in N \mid h \leq_R R_N(g)\}$ .*

On se donne un monoïde de Garside  $M$  et deux sous-monoïdes paraboliques  $N_1$  et  $N_2$  tels que  $N_1 \cup N_2$  engendre  $M$ . Alors tout  $g \in M$  s'écrit de façon unique sous la forme  $g = g_p \cdots g_2 g_1$  où:

$$g_i = \begin{cases} R_{N_1}(g_p \cdots g_i) & \text{si } i \text{ est impair,} \\ R_{N_2}(g_p \cdots g_i) & \text{si } i \text{ est pair,} \end{cases}$$

et  $g_p \neq 1$ . Cette écriture s'appelle la *forme alternée* de  $g$  (par rapport à  $(N_1, N_2)$ ). Remarquez que l'on peut avoir  $g_1 = 1$ , mais on a  $g_i \neq 1$  pour tout  $i \in \{2, \dots, p\}$ . Le nombre  $p$  s'appelle le  $(N_1, N_2)$ -*breadth* de  $g$  et se note  $p = \text{bh}(g)$  (ou  $p = \text{bh}_{N_1, N_2}(g)$  s'il est nécessaire de préciser les deux sous-monoïdes  $N_1$  et  $N_2$ ).

Soient  $M$  un monoïde de Garside,  $G(M)$  le groupe enveloppant de  $M$  et  $\Delta$  un élément de Garside de  $M$ . On suppose que le flip automorphisme  $\Phi$  est triviale, c'est-à-dire que  $\Delta$  est central. On se donne deux éléments balancés  $\Delta_1, \Lambda \in \text{Div}(\Delta)$  et on suppose que  $M_1 = M_{\Delta_1}$  est un sous-monoïde parabolique associé à  $\Delta_1$  et  $N = M_\Lambda$  est un sous-monoïde parabolique associé à  $\Lambda$ . On pose  $G_1 = G(M_1)$  et  $H = G(N)$ . On suppose que  $M_1 \cup N$  engendre  $M$ ,  $\Delta_1$  est central dans  $G_1$  et  $\Lambda$  est central dans  $H$ . Une telle donnée s'appellera une *pré-structure de Dehornoy*. Par commodité, on dira souvent que la paire  $(G_1, H)$  est la pré-structure de Dehornoy sans préciser les autres données telles que  $\Delta_1$  et  $\Lambda$ .

Un élément  $g \in M$  est *inamovible* si  $\Delta \not\leq_R g$ . Remarquez que  $g$  est inamovible si et seulement si  $\Delta \not\leq_L g$ . De plus, pour tout  $g \in G$  il existe un unique  $d \in \mathbb{Z}$  et un unique élément inamovible  $g_0 \in M$  tels que  $g = g_0 \Delta^d$ . Dans ce cas on a aussi  $g = \Delta^k g_0$  car  $\Delta$  est central. Soit  $g \in M$  un élément inamovible. La *profondeur* de  $g$ , notée  $\text{dpt}(g)$ , est définie par:

$$\text{dpt}(g) = \begin{cases} \frac{\text{bh}(g)-1}{2} & \text{si } \text{bh}(g) \text{ est impair,} \\ \frac{\text{bh}(g)}{2} & \text{si } \text{bh}(g) \text{ est pair.} \end{cases}$$

En d'autres termes, si  $g = g_p \cdots g_2 g_1$  est la forme alternée de  $g$ , alors  $\text{dpt}(g) = |\{i \in \{1, \dots, p\} \mid g_i \in N\}|$ . Soit  $g \in G$  quelconque. On écrit  $g = \Delta^d g_0$  avec  $d \in \mathbb{Z}$  et  $g_0 \in M$  inamovible. Alors la *profondeur* de  $g$  est  $\text{dpt}(g) = \text{dpt}(g_0)$ .

Posons  $\theta = \Delta_1^{-1}\Delta = \Delta\Delta_1^{-1}$  et, pour  $k \geq 0$ ,  $q_k = \text{dpt}(\theta^k)$ . On dit qu'un élément  $g \in G$  est  $(G_1, H)$ -positif si  $\text{dpt}(g) \geq q_k$ , où  $k = \text{Ninf}(g)$ . On note  $P = P_{G_1, H}$  l'ensemble des éléments  $(G_1, H)$ -positifs. On dit que  $(G_1, H)$  est une *structure de Dehornoy* si  $P$  vérifie les propriétés suivantes:

- (a)  $P^2 \subset P$ .
- (b)  $G_1 P G_1 \subset P$ .
- (c) On a la réunion disjointe  $G = P \sqcup P^{-1} \sqcup G_1$ .

On suppose données deux suites de sous-groupes paraboliques  $G_0 = G > G_1 > \dots > G_n$  et  $G > H_1 > \dots > H_n$  telles que  $(G_i, H_i)$  soit une structure de Dehornoy de  $G_{i-1}$  pour tout  $i \in \{1, \dots, n\}$  et  $G_n \simeq \mathbb{Z}$ . Pour  $i \in \{1, \dots, n\}$  on note  $P_i$  l'ensemble des éléments positifs de  $G_{i-1}$ . Par ailleurs, on fixe un générateur  $g_n$  de  $G_n$  et on pose  $P_{n+1} = \{g_n^k \mid k \geq 1\}$ . Pour  $\epsilon = (\epsilon_1, \dots, \epsilon_n, \epsilon_{n+1}) \in \{\pm 1\}^{n+1}$  on pose:

$$P^\epsilon = P_1^{\epsilon_1} \cup \dots \cup P_n^{\epsilon_n} \cup P_{n+1}^{\epsilon_{n+1}}.$$

Le premier résultat de cette thèse est le suivant:

**Proposition 6.** *Sous les hypothèse ci-dessus l'ensemble  $P^\epsilon$  est le cône positif pour un ordre sur  $G$ .*

Soit  $(G_1, H)$  une pré-structure de Garside avec les définitions et notations ci-dessus. On dit que  $(G_1, H)$  vérifie la *condition A* s'il existe une constante  $c \in \mathbb{N}$ , telle que  $q_k = kc + 1$  pour tout  $k \geq 1$ .

Remarquer que  $q_0 = 0$ , donc l'égalité  $q_k = kc + 1$  ne peut être vraie pour  $k = 0$ . Comme dans notre étude nous avons parfois besoin de remplacer 0 par 1 lorsque  $k = 0$ , nous utilisons aussi la suite  $\tilde{q}_k$  définie comme suit à la place de  $q_k$ .

$$\tilde{q}_k = \begin{cases} 1 & \text{si } k = 0, \\ q_k & \text{si } k \geq 1. \end{cases}$$

On dit qu'un élément  $g \in M$  est un *theta élément* s'il existe  $g_0 \in M_1$  et  $k \geq 1$  tels que  $g = \theta^k g_0$ . On note  $\Theta = \Theta(G)$  l'ensemble des theta éléments.



Soient  $g, g' \in M$  deux éléments inamovibles tels que  $\{g, g'\} \not\subset M_1 \cup \Theta$ . On dit que la paire  $(g, g')$  vérifie la *condition B* s'il existe  $\epsilon \in \{0, 1\}$  tel que:

- (a)  $\text{dpt}(gg') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_k \geq 0$ , où  $k = \inf(gg')$ ;
- (b)  $\epsilon = 0$  si  $\{g, g'\} \cap \Theta \neq \emptyset$ .

On dit que  $(G_1, H)$  vérifie la *condition B* si, pour tous  $g, g' \in M$  inamovibles tels que  $\{g, g'\} \not\subset M_1 \cup \Theta$ , la paire  $(g, g')$  vérifie la condition B.

Le résultat principal de cette thèse est:

**Théorème 7.** *Si  $(G_1, H)$  vérifie les conditions A et B, alors  $(G_1, H)$  est une structure de Dehornoy.*

Le chapitre 4 de la thèse est dédié à l'étude du groupe de tresses ou groupe d'Artin de type  $A_n$ . Rappelons que le monoïde des tresses positives  $\mathcal{B}_{n+1}^+$  est défini par la présentation de monoïde avec ensemble de générateurs  $S = \{x_1, \dots, x_n\}$  et relations:

$$\begin{aligned} x_i x_{i+1} x_i &= x_{i+1} x_i x_{i+1} & \text{si } 1 \leq i \leq n-1, \\ x_i x_j &= x_j x_i & \text{si } |i-j| \geq 2. \end{aligned}$$

Le groupe de tresses  $G = \mathcal{B}_{n+1}$  est le groupe enveloppant de  $\mathcal{B}_{n+1}^+$ . Par Garside [20],  $\mathcal{B}_{n+1}^+$  est un monoïde de Garside et  $\Omega = x_1 \vee_R \dots \vee_R x_n$  est un élément de Garside de  $\mathcal{B}_{n+1}^+$ .

On a  $\Omega x_i = x_{n+1-i} \Omega$  pour tout  $i \in \{1, \dots, n\}$ . En particulier,  $\Omega$  n'est pas central, mais  $\Delta = \Omega^2$  est central. L'élément  $\Delta$  est un élément de Garside par Dehornoy [11], et on a  $\Delta = (x_1 \dots x_n)^{n+1}$  par Brieskorn–Saito [4].

Le résultat principal du chapitre 4 est le suivant:

**Théorème 8.** *La paire  $(G_1, H)$  vérifie les conditions A et B.*

En appliquant le théorème 7 on obtient:

**Corollaire 9.** *La pair  $(G_1, H)$  est une structure de Dehornoy.*

Comme  $G_1$  est un groupe d'Artin de type  $A_{n-1}$ , en itérant le Corollaire 9 et en appliquant la Proposition 6 on obtient:

**Corollaire 10.** *Posons  $G_i = \langle x_{i+1}, \dots, x_n \rangle$ ,  $H_i = \langle x_i, \dots, x_{n-1} \rangle$  et  $\Delta_i = (x_{i+1} \cdots x_n)^{n+1-i}$  pour  $i \in \{1, \dots, n-1\}$ .*

- (1) *Le groupe  $G_i$  est un groupe de Garside et  $\Delta_i$  est un élément de Garside de  $G_i$ , pour tout  $i \in \{1, \dots, n-1\}$ .*
- (2) *La paire  $(G_i, H_i)$  est une structure de Dehornoy de  $G_{i-1}$  pour tout  $i \in \{1, \dots, n-1\}$ , où  $G_0 = G$ .*
- (3) *Pour  $i \in \{1, \dots, n-1\}$  on note  $P_i$  l'ensemble des éléments  $(G_i, H_i)$ -positifs de  $G_{i-1}$ . Par ailleurs, on pose  $P_n = \{x_n^k \mid k \geq 1\}$ . Pour  $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{\pm 1\}^n$ , on pose:*

$$P^\epsilon = P_1^{\epsilon_1} \sqcup \dots \sqcup P_n^{\epsilon_n} .$$

*Alors  $P^\epsilon$  est le cône positif pour un ordre sur  $G$ .*

Dans une seconde partie du chapitre 4 nous montrons que les ordres ainsi obtenus coïncident avec ceux obtenus à partir du Théorème 1. Plus précisément, nous démontrons le résultat suivant:

**Proposition 11.** *L'ensemble  $P_{(G_1, H)}$  est égal à l'ensemble des éléments  $x_1$ -positifs de  $G = \mathcal{B}_{n+1}$ .*

Les chapitres 5 et 6 de la thèse sont dédiés à l'étude des groupes d'Artin de type diédral. Soit  $G$  (resp.  $M$ ) le groupe d'Artin (resp. monoïde d'Artin) de type  $I_2(m)$ ,  $m \geq 3$  et  $m \neq \infty$ . Rappelons que  $M$  est défini par la présentation:

$$M = \langle x, y \mid \text{prod}(x, y : m) = \text{prod}(y, x : m) \rangle^+ .$$

Le groupe  $G$  est le groupe enveloppant de  $M$ . Par le Théorème 4,  $M$  est un monoïde de Garside et  $\Omega = x \vee_R y$  est un élément de Garside de  $M$ . Par Brieskorn–Saito [4],  $\Delta = \Omega$  est un élément de Garside central si  $m$  est pair, et  $\Delta = \Omega^2$  est un élément de Garside central si  $m$  est impair.

Soient  $G_1 = \langle y \rangle$  et  $H = \langle x \rangle$ . Par le Théorème 4,  $G_1$  et  $H$  sont des sous-groupes paraboliques de  $G$ . Le sous-groupe  $G_1$  est associé à  $\Delta_1 = y$  si  $m$  est pair et à  $\Delta_1 = y^2$  si  $m$  est impair. Le

sous-groupe  $H$  est associé à  $\Lambda = x$  si  $m$  est pair et à  $\Lambda = x^2$  si  $m$  est impair. Comme d'habitude, on pose  $M_1 = G_1 \cap M \simeq \mathbb{N}$  et  $N = H \cap M \simeq \mathbb{N}$ . Il est évident que  $M_1 \cup N$  engendre  $M$ , donc  $(G_1, H)$  est une pré-structure de Dehornoy.

Le résultat principal des chapitres 5 et 6 est le suivant:

**Théorème 12.** *La paire  $(G_1, H)$  vérifie les conditions A et B.*

En appliquant le Théorème 7 on obtient:

**Corollaire 13.** *La paire  $(G_1, H)$  est une structure de Dehornoy.*

On note  $P_1$  l'ensemble des éléments  $(G_1, H)$ -positifs de  $G$ , et on pose  $P_2 = \{y^n \mid n \geq 1\}$ . Pour  $\epsilon = (\epsilon_1, \epsilon_2) \in \{\pm 1\}^2$  on pose  $P^\epsilon = P_1^{\epsilon_1} \cup P_2^{\epsilon_2}$ . Alors, par la Proposition 6:

**Corollaire 14.**  *$P^\epsilon$  est le cône positif pour un ordre sur  $G$ .*

On note  $z_1, \dots, z_{m-1}$  les générateurs standards du groupe de tresses  $\mathcal{B}_m$ . Par Crisp [7], l'application  $\phi : S \rightarrow \mathcal{B}_m$  définie par:

$$\phi(x) = \prod_{i < m, i \text{ pair}} z_i, \quad \phi(y) = \prod_{i < m, i \text{ impair}} z_i,$$

induit un plongement  $\phi : G \hookrightarrow \mathcal{B}_m$ . Nous démontrons aussi comment les ordres sur  $G$  obtenus au Corollaire 14 peuvent se déduire à partir de  $\phi$ . Plus précisément, nous montrons le résultat suivant.

**Proposition 15.** *Soit  $g \in G$ . Alors  $g$  est  $(G_1, H)$ -positif si et seulement si  $\phi(g)$  est  $z_1$ -positif.*

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## CHAPTER 1

### Introduction

In this thesis the *braid group on  $n$  strands* is defined by the presentation:

$$\mathcal{B}_n = \left\langle x_1, \dots, x_{n-1} \left| \begin{array}{ll} x_i x_j = x_j x_i & \text{if } |i - j| \geq 2 \\ x_i x_j x_i = x_j x_i x_j & \text{if } |i - j| = 1 \end{array} \right. \right\rangle.$$

A group  $G$  is called *left orderable* if there exists a total order  $<$  on  $G$  which is invariant under left multiplication, that is, such that  $g_1 g_2 < g_1 g_3$  if  $g_2 < g_3$ , for all  $g_1, g_2, g_3 \in G$ . Given such an order  $<$  on  $G$ , we define the *positive cone* of  $<$  to be the set  $P = P_< = \{g \in G \mid 1 < g\}$ . This satisfies the following properties:

- (1)  $PP \subset P$  (i.e.  $P$  is a subsemigroup).
- (2)  $G = P \sqcup P^{-1} \sqcup \{1\}$ .

Conversely, a subset  $P$  of  $G$  which satisfies (1) and (2) determines a left order  $<$  on  $G$  defined by  $g_1 < g_2$  if  $g_1^{-1} g_2 \in P$ , and  $P$  is the positive cone of  $<$ . In the literature the left orders on groups are often defined by their positive cones, and they will be defined in this way in the present thesis.

The first explicit left order on the braid group  $\mathcal{B}_n$  was determined by Dehornoy [10]. The fact that the braid group is left orderable is an important result, but, also, the Dehornoy order is interesting by itself, and it is used for several purposes such as in the proof of the faithfulness of some representations of the braid groups (see Shpilrain [28] and Crisp–Paris [9], for example). A complete presentation on left orders on the braid group and, in particular, on the Dehornoy order, is given in Dehornoy–Dyannikov–Rolfsen–Wiest [14]. The definition of the Dehornoy order is based on the following construction:

Let  $G$  be a group, and let  $X = \{x_1, \dots, x_k\}$  be an ordered generating family for  $G$ . Let  $i \in \{1, \dots, k\}$ . We say that a word  $w \in (X \sqcup X^{-1})^*$  is  *$x_i$ -positive* (resp.  *$x_i$ -negative*) if it satisfies

the following condition:

$$w \in \{x_i, x_{i+1}^{\pm 1}, \dots, x_k^{\pm 1}\}^* \text{ (resp. } w \in \{x_i^{-1}, x_{i+1}^{\pm 1}, \dots, x_k^{\pm 1}\}^*), \text{ but } w \notin \{x_{i+1}^{\pm 1}, \dots, x_k^{\pm 1}\}^*.$$

An element  $g \in G$  is called  $x_i$ -positive (resp.  $x_i$ -negative) if it has an  $x_i$ -positive representative (resp.  $x_i$ -negative representative). We denote by  $G_i$  the subgroup of  $G$  generated by  $\{x_i, x_{i+1}, \dots, x_k\}$ , by  $P_i^+$  the set of  $x_i$ -positive elements of  $G$ , and by  $P_i^-$  the set of  $x_i$ -negative elements. The key of the definition of the Dehornoy order is the following:

**Theorem 1.1** (Dehornoy [10]). *Assume that  $G = \mathcal{B}_n$  is the braid group on  $n$  strands, and that  $\{x_1, \dots, x_{n-1}\}$  is its standard generating set. For all  $i \in \{1, \dots, n-1\}$ , we have the disjoint union  $G_i = P_i^+ \sqcup P_i^- \sqcup G_{i+1}$ , where  $G_n = \{1\}$ .*

Let  $G = \mathcal{B}_n$  be the braid group on  $n$  strands. Set  $P_D = P_1^+ \sqcup P_2^+ \sqcup \dots \sqcup P_{n-1}^+$ . Then, by Theorem 1.1,  $P_D$  is the positive cone of a left order on  $G$ . This order is the *Dehornoy order*. The order relation respective to  $P_D$  shall be denoted by  $<_D$ .

A careful reader may notice that Theorem 1.1 leads to more than one left order on the braid group. Indeed, for  $\epsilon = (\epsilon_1, \dots, \epsilon_{n-1}) \in \{\pm 1\}^{n-1}$ , we set:

$$P^\epsilon = P_1^{\epsilon_1} \sqcup P_2^{\epsilon_2} \sqcup \dots \sqcup P_{n-1}^{\epsilon_{n-1}}.$$

Then, by Theorem 1.1,  $P^\epsilon$  is the positive cone of a left order on  $G = \mathcal{B}_n$ . The case  $\epsilon = (1, -1, 1, -1, \dots)$  is particularly interesting because of the following:

**Theorem 1.2** (Dubrovina–Dubrovin [17]). *Assume that  $G = \mathcal{B}_n$  is the braid group on  $n$  strands, and that  $\{x_1, \dots, x_{n-1}\}$  is its standard generating set. Let  $\epsilon = (1, -1, 1, -1, \dots)$ . Then  $P^\epsilon$  is a finitely generated semigroup. In particular, the left order on  $\mathcal{B}_n$  defined by  $P^\epsilon$  is isolated in the space of left orders on  $\mathcal{B}_n$ .*

The goal of the present thesis is to extend the Dehornoy order to some Garside groups.

A first approach would be to adopt the same definition as above. Let  $G$  be a group, and let  $\{x_1, \dots, x_k\}$  be an ordered generating set for  $G$ . We say that  $\{x_1, \dots, x_k\}$  determines a *Dehornoy structure* (in *Ito's sense*) if we have the disjoint union  $G_i = P_i^+ \sqcup P_i^- \sqcup G_{i+1}$  for all  $i \in \{1, \dots, k\}$ , where  $G_{k+1} = \{1\}$ . In this case, as for the braid group, if  $\epsilon = (\epsilon_1, \dots, \epsilon_k) \in \{\pm 1\}^k$ , then

$$P^\epsilon = P_1^{\epsilon_1} \sqcup P_2^{\epsilon_2} \sqcup \dots \sqcup P_k^{\epsilon_k}$$

is the positive cone of a left order on  $G$ . This approach is actually used in Ito [27] to construct groups having isolated left orders in their spaces of left orders.

In this thesis we adopt another approach based on a definition of the Dehornoy order in terms of Garside groups (see Dehornoy [12], Fromentin [18], Fromentin–Paris [19]). In Chapter 3 we give a (new and different) definition of a *Dehornoy structure* on a Garside group, and we show a criterion for a Garside group to admit such a structure. We warn the reader that our new notion of a Dehornoy structure is different and, probably, non-equivalent to the one in Ito's sense given above (see Sibert [29]). In Chapter 4 we prove that the braid group admits a Dehornoy structure (in the new sense), and that this structure leads to the Dehornoy order. In Chapters 5 and 6 we prove that an Artin group of dihedral type admits a Dehornoy structure which leads to a left order on the group. Then, we compare the left order obtained in this way with the one obtained using some embedding of the group in a braid group defined by Crisp [7]. For technical reasons, the study of the Artin groups of dihedral type is divided into two chapters, depending on the parity of the length of the Garside element. In Chapter 2 we give some preliminaries on Garside groups and Artin groups.

Unfortunately, this thesis does not produce any new example of an Artin monoid that admits a left ordering (see Question 3.1 in [14]), but we hope that our method will be applied in the future to tackle this question.

This thesis has been supported by "Beca de Doctorado en el Extranjero" (Nro. 72130288) Becas Chile – CONICYT.





## CHAPTER 2

### Preliminaries

#### 1. Garside groups

The Garside groups are a generalization of the braid groups on which one can apply Garside's ideas [20]. They were introduced in Dehornoy–Paris [15]. We refer to Dehornoy et al. [13] for a complete presentation on this theory. Here, we will just give some basic definitions and properties that will be needed for our purpose.

Let  $M$  be a monoid. We say that  $M$  is *atomic* if there exists a map  $\nu : M \rightarrow \mathbb{N}$  such that:

- (a)  $\nu(g) = 0$  if and only if  $g = 1$ ;
- (b)  $\nu(gg') \geq \nu(g) + \nu(g')$  for all  $g, g' \in M$ .

Such a map  $\nu : M \rightarrow \mathbb{N}$  is called a *norm* on  $M$ . A non-trivial element  $g \in M$  is called an *atom* if it is indecomposable in the sense that, if  $g = g_1g_2$ , then either  $g_1 = 1$  or  $g_2 = 1$ . We denote by  $\mathcal{A} = \mathcal{A}_M$  the set of atoms.

Assume that  $M$  is atomic. Then any generating set of  $M$  contains  $\mathcal{A}$ , and  $\mathcal{A}$  generates  $M$ . In particular,  $M$  is finitely generated if and only if  $\mathcal{A}$  is finite. We can define two partial orders  $\leq_L$  and  $\leq_R$  on  $M$  as follows:

- We set  $g_1 \leq_L g_2$  if there exists  $g_3 \in M$  such that  $g_1g_3 = g_2$ .
- We set  $g_1 \leq_R g_2$  if there exists  $g_3 \in M$  such that  $g_3g_1 = g_2$ .

These orders are called *left-divisibility order* and *right-divisibility order*, respectively. For  $g \in M$  we set:

$$\text{Div}_L(g) = \{g' \in M \mid g' \leq_L g\}, \quad \text{Div}_R(g) = \{g' \in M \mid g' \leq_R g\}.$$

We say that  $g$  is *balanced* if  $\text{Div}_L(g) = \text{Div}_R(g)$ .

A *Garside monoid* is a monoid  $M$  such that:

- (a)  $M$  is atomic;
- (b)  $M$  is *cancellative*, that is,  $g_1gg_2 = g_1g'g_2$  implies  $g = g'$ , for all  $g, g', g_1, g_2 \in M$ ;

- (c)  $(M, \leq_L)$  and  $(M, \leq_R)$  are lattices;
- (d)  $M$  contains a *Garside element*, that is, a balanced element  $\Delta$  such that  $\text{Div}_L(\Delta) = \text{Div}_R(\Delta)$  is finite and generates  $M$ .

The lattice operations in  $(M, \leq_L)$  (resp.  $(M, \leq_R)$ ) are denoted by  $\wedge_L$  and  $\vee_L$  (resp.  $\wedge_R$  and  $\vee_R$ ). For every  $g, g' \in M$  we denote by  $g' \setminus g$  and  $g/g'$  the unique elements in  $M$  such that  $g \vee_L g' = g'(g' \setminus g)$  and  $g \vee_R g' = (g/g')g'$ .

**Proposition 2.1** (Dehornoy [13]). *For every  $g, g_1, g_2$  in a Garside monoid, we have:*

- (1)  $g \vee_R (g_2 g_1) = ((g/g_1) \vee_R g_2) g_1$ .
- (2)  $(g_1 g_2) \vee_L g = g_1 (g_2 \vee_L (g_1 \setminus g))$ .
- (3)  $(g_2 g_1)/g = (g_2/(g/g_1))(g_1/g)$  and  $g/(g_2 g_1) = (g/g_1)/g_2$ .
- (4)  $g \setminus (g_1 g_2) = (g \setminus g_1)((g_1 \setminus g) \setminus g_2)$  and  $(g_1 g_2) \setminus g = g_2 \setminus (g_1 \setminus g)$ .

Let  $M$  be a monoid. The *enveloping group* of  $M$ , denoted by  $G = G(M)$ , is the group presented by the generating set  $M$  and the relations  $g_1 g_2 = g$  if  $g_1 g_2 = g$  holds in  $M$ . There is a canonical homomorphism  $M \rightarrow G(M)$  which is not injective in general. A *Garside group* is the enveloping group of a Garside monoid.

**Remark** (1) A Garside monoid  $M$  satisfies the Öre conditions, hence the canonical homomorphism  $M \rightarrow G(M)$  is injective (see Dehornoy–Paris [15]). Furthermore, the orders  $\leq_L$  and  $\leq_R$  extend to lattice orders on  $G(M)$  with positive cone  $M$ . These are defined by  $g \leq_L g'$  (resp.  $g \leq_R g'$ ) if  $g^{-1}g' \in M$  (resp.  $g'g^{-1} \in M$ ).

(2) A Garside element is never unique. For example, if  $\Delta$  is a Garside element, then  $\Delta^k$  is a Garside element for all  $k \geq 1$  (see Dehornoy [11, Lemma 2.2]). We will talk about a *Garside system*  $(M, \Delta)$  whenever we will need to indicate the Garside element.

Let  $M$  be a Garside monoid, and let  $\Delta$  be a fixed Garside element. The elements of  $\text{Div}_L(\Delta) = \text{Div}_R(\Delta)$  are called the *simple elements* of  $M$ , and the set of simple elements is denoted by  $\text{Div}(\Delta)$ . By Dehornoy–Paris [15], there is an automorphism  $\Phi : M \rightarrow M$ , called the *flip automorphism*, such that  $\Delta g = \Phi(g) \Delta$  for all  $g \in M$ . On the other hand, there is a one-to-one correspondence  $\partial : \text{Div}(\Delta) \rightarrow \text{Div}(\Delta)$  such that  $\partial(s) s = \Delta$  for all  $s \in \text{Div}(\Delta)$ . It is easily seen that  $\partial^2(s) = \Phi(s)$  and  $s \partial^{-1}(s) = \Delta$  for all  $s \in \text{Div}(\Delta)$ .

A pair  $(s, s')$  of simple elements is called *right-weighted* (resp. *left-weighted*) if  $ss' \wedge_R \Delta = s'$  (resp.  $ss' \wedge_L \Delta = s$ ).

**Proposition 2.2** (Dehornoy–Paris [15]). *Every element  $g \in G$  admits unique decompositions  $g = s_p \cdots s_1 \Delta^d$  and  $g = \Delta^d s'_1 \cdots s'_p$  such that:*

- (a)  $s_1, \dots, s_p, s'_1, \dots, s'_p \in \text{Div}(\Delta) \setminus \{1, \Delta\}$ ;
- (b)  $(s_{i+1}, s_i)$  is right-weighted for all  $i \in \{1, \dots, p-1\}$ ;
- (c)  $(s'_i, s'_{i+1})$  is left-weighted for all  $i \in \{1, \dots, p-1\}$ .

The expressions  $g = s_p \cdots s_1 \Delta^d$  and  $g = \Delta^d s'_1 \cdots s'_p$  of Proposition 2.2 are called the *right greedy normal form* and the *left greedy normal form* of  $g$ , respectively. The integer  $d$  is called the *infimum* of  $g$ , and is denoted by  $\text{inf}(g)$ , the integer  $p$  is called the *canonical length*, and is denoted by  $\ell(g)$ , and the integer  $d + p$  is called the *supremum*, and is denoted by  $\text{sup}(g)$ . These invariants of the greedy normal forms are frequently used in the theory of Garside groups, especially in the study of the conjugacy problem (see Gebhardt–González-Meneses [21], for example). In this paper we will need a fourth invariant, that we call the *negative-infimum* of  $g$ , and which is defined by:

$$\text{Ninf}(g) = \begin{cases} 0 & \text{if } d \geq 0, \\ -d & \text{if } d < 0. \end{cases}$$

Moreover, we set  $g_R = s_p \cdots s_1$  and  $g_L = s'_1 \cdots s'_p$ . Note that  $g_L = g_R$  if the flip automorphism  $\Phi$  is trivial.

Note that  $M = \{g \in G(M) \mid \text{Ninf}(g) = 0\}$ .

**Proposition 2.3** (Gebhardt–González-Meneses [21]). *Let  $g \in G$ .*

- (1) *Then  $\text{inf}(g)$  is the greatest integer  $d \in \mathbb{Z}$  such that  $\Delta^d \leq_R g$ , and  $\text{sup}(g)$  is the least integer  $k \in \mathbb{Z}$  such that  $g \leq_R \Delta^k$ .*
- (2) *We have  $\text{inf}(g^{-1}) = -\text{sup}(g)$ ,  $\text{sup}(g^{-1}) = -\text{inf}(g)$ , and  $\ell(g^{-1}) = \ell(g)$ .*

We define the *left complement* (resp. the *right complement*) of an element  $g \in G$  to be  $\text{com}_L(g) = \Delta^k g^{-1}$  (resp.  $\text{com}_R(g) = g^{-1} \Delta^k$ ), where  $k = \text{sup}(g)$ . Note that  $\text{com}_L(g) \in M$  because  $\text{com}_L(g) = \Delta^k \Delta^{-d} g_R^{-1} = \Delta^p g_R$ , where  $d = \text{inf}(g)$  and  $p = \ell(g)$ . Similarly  $\text{com}_R(g) \in M$ .

Further, if  $s \in \text{Div}(\Delta) \setminus \{1, \Delta\}$ , then  $\text{com}_L(s) = \partial(s)$  and  $\text{com}_R(s) = \partial^{-1}(s)$ . Moreover, if  $\Phi = 1$ , then  $\text{com}_L(g) = \text{com}_R(g)$ . The proof of the following is left to the reader:

**Lemma 2.4.** *Let  $g \in G$ , let  $g = s_p \cdots s_1 \Delta^d$  be the right greedy normal form of  $g$ , and let  $g = \Delta^d s'_1 \cdots s'_p$  be its left greedy normal form. Then  $\inf(\text{com}_L(g)) = \inf(\text{com}_R(g)) = 0$ , and  $\ell(\text{com}_L(g)) = \ell(\text{com}_R(g)) = p = \ell(g)$ . Moreover,*

$$\begin{aligned} \text{com}_L(g) &= \Phi^{p-1}(\partial(s_1)) \cdots \Phi(\partial(s_{p-1})) \partial(s_p), \\ \text{com}_R(g) &= \partial^{-1}(s'_p) \Phi^{-1}(\partial^{-1}(s'_{p-1})) \cdots \Phi^{-p+1}(\partial^{-1}(s'_1)). \end{aligned}$$

Furthermore,  $\text{com}_L(g) = \Phi^{d+p}(\text{com}_R(g))$ .

For example, if  $g = \Delta^{-3} s_1 s_2$ , then  $\text{com}_R(g) = s_2^{-1} s_1^{-1} \Delta^3 \Delta^{-1} = s_2^{-1} \Delta \Phi^{-1}(s_1^{-1} \Delta) = \partial^{-1}(s_2) \Phi^{-1}(\partial^{-1}(s_2))$  because  $\text{sup}(g) = -1$ .

**Proposition 2.5** (Dehornoy–Paris [15]). *For every  $h \in G$  there exists a unique pair  $(g, g')$  (resp.  $(g', g)$ ) of elements in  $M$ , such that  $h = g^{-1} g'$  (resp.  $h = g' g^{-1}$ ) and  $g \wedge_L g' = 1$  (resp.  $g \wedge_R g' = 1$ ).*

The pair  $(g, g')$  (resp.  $(g', g)$ ) in Proposition 2.5 is called the *left orthogonal splitting* (resp. *right orthogonal splitting*) of  $h$ .

## 2. Parabolic subgroups

We keep the above hypothesis and notations, namely,  $M$  is a Garside monoid,  $\Delta$  is a fixed Garside element, and  $G$  is the enveloping group of  $M$ . Recall that an element  $\delta \in M$  is *balanced* if  $\text{Div}_L(\delta) = \text{Div}_R(\delta)$ . In this case, this set is denoted by  $\text{Div}(\delta)$ . Let  $\delta$  be a balanced element. We denote by  $\mathcal{S}(\delta)$  the set of atoms of  $M$  that belong to  $\text{Div}(\delta)$ , and by  $G_\delta$  the subgroup of  $G$  generated by  $\mathcal{S}(\delta)$ . We say that  $G_\delta$  is a *parabolic subgroup* of  $G$  (or of  $(G, \Delta)$ , when the Garside element needs to be specified) *associated with  $\delta$*  if  $\text{Div}(\delta) = \text{Div}(\Delta) \cap G_\delta$ . We list in the following proposition some results on parabolic subgroups that we will need, and refer to Godelle [22] for the proofs.

**Proposition 2.6** (Godelle [22]). *Let  $\delta \in M$  be a balanced element such that  $G_\delta$  is a parabolic subgroup of  $G$ .*

- (1)  $M_\delta = M \cap G_\delta$  is a Garside monoid, and  $\delta$  is a Garside element for  $M_\delta$ . Moreover,  $G_\delta$  is the enveloping group of  $M_\delta$ .
- (2) Let  $\vee_{\delta,L}$  and  $\wedge_{\delta,L}$  (resp.  $\vee_{\delta,R}$  and  $\wedge_{\delta,R}$ ) be the lattice operations on  $(M_\delta, \leq_L)$  (resp. on  $(M_\delta, \leq_R)$ ). We have  $g_1 \vee_L g_2, g_1 \wedge_L g_2 \in M_\delta$ ,  $g_1 \vee_L g_2 = g_1 \vee_{\delta,L} g_2$ , and  $g_1 \wedge_L g_2 = g_1 \wedge_{\delta,L} g_2$ , for all  $g_1, g_2 \in M_\delta$ . Similarly, we have  $g_1 \vee_R g_2, g_1 \wedge_R g_2 \in M_\delta$ ,  $g_1 \vee_R g_2 = g_1 \vee_{\delta,R} g_2$ , and  $g_1 \wedge_R g_2 = g_1 \wedge_{\delta,R} g_2$ , for all  $g_1, g_2 \in M_\delta$ .
- (3) We have  $\mathcal{A} \cap M_\delta = \mathcal{S}(\delta)$ , and this is the set of atoms of  $M_\delta$ .
- (4) If  $g_1 \leq_L g_2$  (resp.  $g_1 \leq_R g_2$ ) and  $g_2 \in M_\delta$ , then  $g_1 \in M_\delta$ , for all  $g_1, g_2 \in M$ .

The following lemmas in this section will be needed in our study:

**Lemma 2.7.** *Let  $\delta_1, \delta_2 \in M$  be two balanced elements such that  $G_{\delta_1}$  and  $G_{\delta_2}$  are parabolic subgroups, and  $\mathcal{S}(\delta_1) \subseteq \mathcal{S}(\delta_2)$ . Then  $\delta_1$  is a balanced element of  $M_{\delta_2}$ , and  $G_{\delta_1} = (G_{\delta_2})_{\delta_1}$  is a parabolic subgroup of  $G_{\delta_2}$ .*

PROOF. Clearly  $\delta_1$  is balanced in  $M_{\delta_2}$ . Since  $\mathcal{S}(\delta_1) \subseteq \mathcal{S}(\delta_2)$  then  $M_{\delta_1}$  is a submonoid of  $M_{\delta_2}$ . Hence  $\text{Div}(\delta_1) = \text{Div}(\Delta) \cap M_{\delta_1} \subseteq \text{Div}(\Delta) \cap M_{\delta_2} = \text{Div}(\delta_2)$ . Therefore:

$$\text{Div}(\delta_2) \cap M_{\delta_1} \subseteq \text{Div}(\Delta) \cap M_{\delta_1} = \text{Div}(\delta_1) \subseteq \text{Div}(\delta_2) \cap M_{\delta_1}.$$

Further  $\mathcal{S}(\Delta) \supseteq \mathcal{S}(\delta_2) \supseteq \mathcal{S}(\delta_1) = \mathcal{S}(\Delta) \cap \text{Div}(\delta_1) \subseteq \text{Div}(\delta_1)$ . Hence  $\mathcal{S}(\delta_1) = \mathcal{S}(\delta_2) \cap \text{Div}(\delta_1)$ . Therefore  $G_{\delta_1} = (G_{\delta_2})_{\delta_1}$  is a parabolic subgroup of  $G_{\delta_2}$ .  $\square$

Recall that, by Dehornoy [11],  $\Delta^k$  is a Garside element for all  $k \geq 1$ .

**Lemma 2.8.** *Let  $\delta$  be a balanced element such that  $G_\delta$  is a parabolic subgroup of  $(G, \Delta)$ . Let  $k \in \mathbb{N}$ ,  $k \geq 1$ . Then  $\delta^k$  is balanced,  $G_{\delta^k} = G_\delta$ , and  $G_{\delta^k}$  is a parabolic subgroup of  $(G, \Delta^k)$  associated with  $\delta^k$ .*

PROOF. Let  $s \in \text{Div}_L(\delta^k)$ . Then  $s \in M_\delta$  and  $\text{sup}(s) \leq k$ , i.e.  $s = s_1 \cdots s_t$ ,  $t \leq k$ , for some  $s_1, \dots, s_t \in \text{Div}(\delta)$ . Hence  $s \leq_R \delta^t \leq_R \delta^k$  because  $\delta$  is balanced. Analogously, we obtain that if  $s \in \text{Div}_R(\delta^k)$ , then  $s \leq_L \delta^k$ . Therefore  $\delta^k$  is balanced in  $M$ . Since divisors of  $\delta^k$  are products of divisors of  $\delta$ , then  $G_{\delta^k} = G_\delta$ . Clearly  $\text{Div}(\delta^k) \subseteq \text{Div}(\Delta^k) \cap G_{\delta^k}$ . Let  $s \in \text{Div}(\Delta^k) \cap G_{\delta^k}$ ,  $s \neq 1$ . Then  $s = s_1 \cdots s_t$  for some  $s_1, \dots, s_t \in \text{Div}(\Delta)$ ,  $t \leq k$ . If  $k = 1$ , then  $s \in \text{Div}(\Delta) \cap G_\delta = \text{Div}(\delta)$ . If  $k > 1$  we suppose that the statement is true for values

less than  $k$ . If  $t < k$ , i.e.  $s \in \text{Div}(\Delta^t) \cap G_{\delta^t}$ , then, the inductive hypothesis implies that  $s \in \text{Div}(\delta^t) \subseteq \text{Div}(\delta^k)$ . If  $t = k$ , Proposition 2.6.4 implies that  $s_1 \cdots s_{k-1} \in \text{Div}(\Delta^{k-1}) \cap G_{\delta^{k-1}}$  because  $s \in G_{\delta^k} = G_\delta$ . So, by the inductive hypothesis, we have  $s_1 \cdots s_{k-1} \in \text{Div}(\delta^{k-1})$ . Hence  $s_k = (s_1 \cdots s_{k-1})^{-1}s \in G_{\delta^k}$ , i.e.  $s_k \in \text{Div}(\Delta) \cap G_\delta = \text{Div}(\delta)$ , thus  $s \in \text{Div}(\delta^k)$ . Therefore  $\text{Div}(\Delta^k) \cap G_{\delta^k} = \text{Div}(\delta^k)$  which implies that  $G_{\delta^k}$  is a parabolic subgroup of  $(G, \Delta^k)$ .  $\square$

**Lemma 2.9.** *Let  $g, g' \in M$  such that  $\Delta \not\leq_R g, g'$  and  $\Delta^k \leq_R gg'$  for some  $k \geq 1$ . Then  $g = g_1h$  and  $g' = \text{com}_R(h)g'_1$  for some  $g_1, g'_1 \in A^+$  and  $h \in \text{Div}(\Delta^k)$  with  $\ell(h) = k$ .*

PROOF. Let  $g_0 \in M$  such that  $gg' = g_0\Delta^k$ . Note that  $\ell(g) \geq k$ . Otherwise  $\Delta \leq_L g'$  which is a contradiction. Similarly  $\ell(g') \geq k$ . Let  $s_1, \dots, s_k \in \text{Div}(\Delta)$  be the first  $k$  elements of the right greedy normal form of  $g$ . Then  $g = g_1h$  where  $h = s_k \cdots s_1$ . We have  $g_0 = g_1h\Delta^{-k}\Phi^k(g') = g_1\text{com}_L(h)^{-1}\Phi^k(g')$ . Further  $g_1 \wedge_R \text{com}_L(h) = 1$  because  $h = g \wedge_R \Delta^k$ . Therefore  $\text{com}_L(h) \leq_L \Phi^k(g')$ . So, by Lemma 2.4,  $\text{com}_R(h) = \Phi^{-k}(\text{com}_L(h)) \leq_L g'$ . Hence, there exists  $g'_1 \in M$  such that  $g' = \text{com}_R(h)g'_1$ .  $\square$

### 3. Artin groups

Let  $S$  be a finite set. A *Coxeter matrix* over  $S$  is a square matrix  $M = (m_{s,t})_{s,t \in S}$ , indexed by the elements of  $S$ , with coefficients in  $\mathbb{N} \cup \{\infty\}$ , and satisfying:

- (a)  $m_{s,s} = 1$  for all  $s \in S$ ;
- (b)  $m_{s,t} = m_{t,s} \geq 2$  for all  $s, t \in S, s \neq t$ .

A Coxeter matrix  $M$  as above is usually represented by a labeled graph,  $\Gamma$ , called *Coxeter graph*. This is defined as follows:

- (a) The set of vertices of  $\Gamma$  is  $S$ .
- (b) Two vertices  $s, t \in S$  are connected by an edge if  $m_{s,t} \geq 3$ . This edge is labeled by  $m_{s,t}$  if  $m_{s,t} \geq 4$ .

If  $a, b$  are two letters and  $m$  is an integer  $\geq 2$ , we set:

$$\text{prod}(a, b : m) = \begin{cases} (ab)^{\frac{m}{2}} & \text{if } m \text{ is even,} \\ (ab)^{\frac{m-1}{2}}a & \text{if } m \text{ is odd.} \end{cases}$$

In other words,  $\text{prod}(a, b : m)$  denotes the word  $aba \cdots$  of length  $m$ . Let  $\Gamma$  be a Coxeter graph as above. The *Artin group* associated with  $\Gamma$  is the group  $A = A_\Gamma$  defined by the presentation:

$$A = \langle S \mid \text{prod}(s, t : m_{s,t}) = \text{prod}(t, s : m_{s,t}) \text{ for all } s, t \in S, s \neq t \text{ and } m_{s,t} \neq \infty \rangle.$$

The monoid  $A^+ = A_\Gamma^+$  having the monoid presentation:

$$A^+ = \langle S \mid \text{prod}(s, t : m_{s,t}) = \text{prod}(t, s : m_{s,t}) \text{ for all } s, t \in S, s \neq t \text{ and } m_{s,t} \neq \infty \rangle^+$$

is called the *Artin monoid* associated with  $\Gamma$ . By Paris [26], the natural homomorphism  $A^+ \rightarrow A$  is injective. The *Coxeter group* associated with  $\Gamma$ , denoted by  $W = W_\Gamma$ , is the quotient of  $A = A_\Gamma$  by the relations  $s^2 = 1, s \in S$ .

The *reversed element* of  $g \in A^+$ , denoted by  $\text{rev}(g)$ , is the element represented by a word  $x_{i_k} \cdots x_{i_1} \in S^*$ , where  $x_{i_1} \cdots x_{i_k}$  is a word representative of  $g$ . Note that  $\text{rev}(g)$  is well defined because relations of Artin monoids are symmetrical. Further,  $\text{rev}^2(g) = g$  for all  $g \in M$ .

The Artin groups were introduced by Tits [30] as extensions of Coxeter groups. There is an extensive literature on these groups, but most of the results concern only special classes. One of the most popular classes is the one of spherical type Artin groups, which is the class that concerns the present paper. We say that a Coxeter graph  $\Gamma$  (or an Artin group  $A = A_\Gamma$ ) is of *spherical type* if the Coxeter group  $W = W_\Gamma$  is finite. A classification of these groups is given in the following theorem:

- Theorem 2.10** (Coxeter [6]). (1) *A Coxeter graph is of spherical type if and only if all its connected components are of spherical type.*
- (2) *A connected Coxeter graph is of spherical type if and only if it is isomorphic to one of the graphs depicted in Figure 2.1.*

Note that the braid group  $\mathcal{B}_{n+1}$  is the Artin group of type  $A_n$ .

Let  $\Gamma$  be a Coxeter graph, let  $S$  be its set of vertices, let  $A$  be its associated Artin group, and let  $A^+$  be its associated Artin monoid. For  $X \subset S$ , we denote by  $\Gamma_X$  the full subgraph of  $\Gamma$  spanned by  $X$ , by  $A_X$  the subgroup of  $A$  generated by  $X$ , and by  $A_X^+$  the submonoid of  $A^+$  generated by  $X$ . By Van der Lek [24], the group  $A_X$  (resp. the monoid  $A_X^+$ ) is the Artin group (resp. Artin monoid) associated with  $\Gamma_X$ . Now, Artin groups and Garside groups are related by the following:



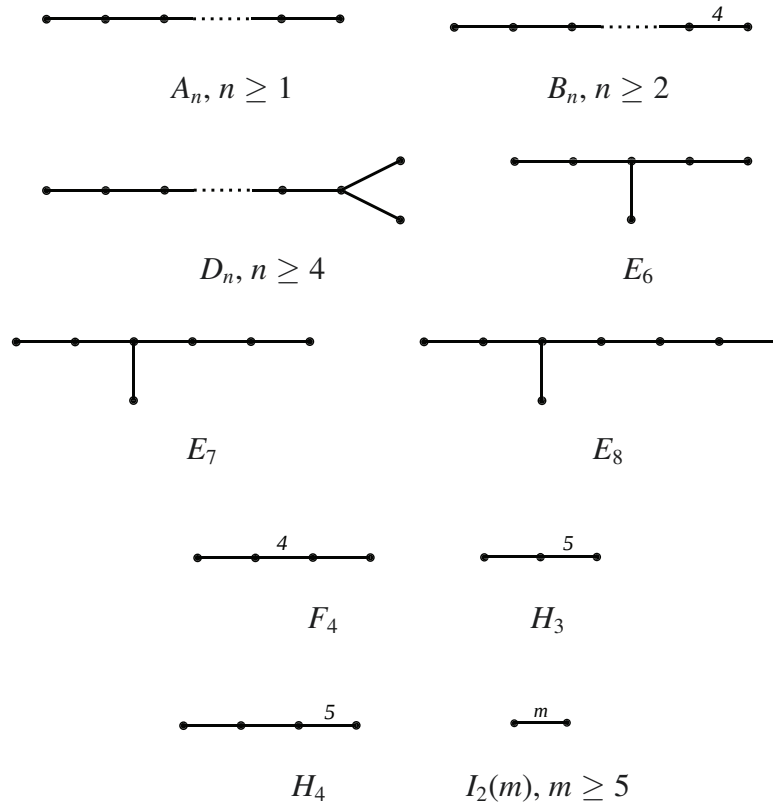


FIGURE 2.1. Spherical type Coxeter graphs.

**Theorem 2.11** (Brieskorn–Saito [4], Deligne [16]). *Assume that  $\Gamma$  is of spherical type.*

- (1) *The monoid  $A^+$  is a Garside monoid,  $\Delta = \vee_L(S) = \vee_R(S)$  is a Garside element of  $A^+$ , and  $A$  is the enveloping group of  $A^+$ .*
- (2) *Let  $X$  be a subset of  $S$ , and let  $\Delta_X = \vee_L(X)$ . Then  $\Delta_X = \vee_R(X)$ ,  $\Delta_X$  is balanced,  $A_X$  is a parabolic subgroup associated with  $\Delta_X$ , and  $A_X^+ = A^+ \cap A_X$ . In particular,  $A_X = A_{\Delta_X}$ .*
- (3) *We have  $|\Phi| \in \{1, 2\}$ , and  $Z(A)$  is the infinite cyclic group generated by  $\Delta^{|\Phi|}$ . In particular,  $\Delta^2$  is central. Furthermore  $\text{rev}(\Delta) = \Delta$ .*

In particular, by Garside [20] and Brieskorn–Saito [4],  $\mathcal{B}_{n+1}$  is a Garside group with the following Garside element:

$$\Omega = (x_n \cdots x_2 x_1) \underbrace{(x_n \cdots x_2) \cdots (x_n x_{n-1}) x_n}_{x_2 \vee_R \cdots \vee_R x_n} .$$

#### 4. Alternating forms and the Dehornoy order

In this section we recall some definitions and results on alternating forms from Dehornoy [12] and Fromentin–Paris [19], and how one can define the Dehornoy order from this alternating forms. This is the point of view that we intend to extend in the following chapters.

**Proposition 2.12** (Dehornoy [12]). *Let  $M$  be a Garside monoid, let  $N$  be a parabolic submonoid, and let  $g$  be an element of  $M$ . Then there exists a unique element  $R_N(g)$  in  $N$  such that  $\{h \in N \mid h \leq_R g\} = \{h \in N \mid h \leq_R R_N(g)\}$ .*

The element  $R_N(g)$  is called the  $N$ -tale of  $g$ .

Now, we suppose given a Garside monoid  $M$  and two parabolic submonoids  $N_1$  and  $N_2$  such that  $N_1 \cup N_2$  generates  $M$ . Then every non-trivial element  $g \in M$  can be uniquely written in the form  $g = g_p \cdots g_2 g_1$ , where:

$$g_i = \begin{cases} R_{N_1}(g_p \cdots g_i) & \text{if } i \text{ is odd,} \\ R_{N_2}(g_p \cdots g_i) & \text{if } i \text{ is even,} \end{cases}$$

and  $g_p \neq 1$ . This form is called the *alternating form* of  $g$  (with respect to  $(N_2, N_1)$ ). Note that we may have  $g_1 = 1$ , but we have  $g_i \neq 1$  for all  $i \in \{2, \dots, p\}$ . The number  $p$  is called the  $(N_2, N_1)$ -breadth of  $g$  and is denoted by  $p = \text{bh}(g)$  (or by  $p = \text{bh}_{N_2, N_1}(g)$  if one needs to specify the submonoids  $N_1$  and  $N_2$ ). Since  $1 \in N_1$  we set  $\text{bh}(1) = 1$ .

Assume that  $M = \mathcal{B}_{n+1}^+ = A_{A_n}^+$  is the positive braid monoid on  $(n + 1)$  strands, and that  $S = \{x_1, \dots, x_n\}$  is the standard generating set of  $M$ . Let  $N_1$  be the submonoid of  $M$  generated by  $\{x_2, \dots, x_n\}$ , and let  $N_2$  be the submonoid generated by  $\{x_1, \dots, x_{n-1}\}$ . Note that, by Theorem 2.11,  $N_1$  and  $N_2$  are both parabolic submonoids of  $M$ , and  $N_1 \simeq \mathcal{B}_n^+ \simeq N_2$ . On the other hand, it is easily seen that  $N_1 \cup N_2$  generates  $M$ . In this context we have the following results:

**Theorem 2.13** (Fromentin–Paris [19]). *Let  $g \in \mathcal{B}_{n+1}^+$ , and let  $k$  be a positive integer. Then  $\Omega^{-k}g$  is  $x_1$ -negative if and only if  $k \geq \max\{1, \text{bh}(g) - 1\}$ . In particular, if  $g \in \mathcal{B}_{n+1}$ , then  $g$  is  $x_1$ -negative if and only if  $\text{Ninf}(g) \geq \max\{1, \text{bh}(g_L) - 1\}$ .*

**Theorem 2.14** (Dehornoy [12], Burckel [5]). *Let  $g, g' \in M$ , and let  $(g_p, \dots, g_1), (g'_q, \dots, g'_1)$  be their respective alternating forms. We have  $g <_D g'$ , if and only if, either  $p < q$ , or  $p = q$  and  $\Phi^{k-1}(g_k) <_D \Phi^{k-1}(g'_k)$ , where  $k = \min\{i \mid g_i \neq g'_i\}$ .*



## CHAPTER 3

### Orders on Garside groups

Let  $M$  be a Garside monoid, let  $G(M)$  be the enveloping group of  $M$ , and let  $\Delta$  be a Garside element. We assume that the flip automorphism  $\Phi$  is trivial, that is,  $\Delta$  is central. We take two balanced elements  $\Delta_1, \Lambda \in \text{Div}(\Delta) \setminus \{1, \Delta\}$ , and we assume that  $G_1 = G_{\Delta_1}$  is a parabolic subgroup associated with  $\Delta_1$ ,  $H = G_{\Lambda}$  is a parabolic subgroup associated with  $\Lambda$ , and  $\Delta_1$  is central in  $G_1$ . We set  $M_1 = G_1 \cap M$  and  $N = H \cap M$ , and we assume that  $M_1 \cup N$  generates  $M$ . Such a data will be called a *Dehornoy pre-structure*. For convenience, we will often say that the pair  $(H, G_1)$  is the Dehornoy pre-structure, without specifying any other data such as  $\Delta_1$  and  $\Lambda$ .

We will consider alternating forms in  $M$  with respect to  $(N, M_1)$ . Note also that  $M_1$  will play a different role in our study from that of  $N$ .

An element  $g \in M$  is called *unmovable* if  $\Delta \not\leq_R g$ . Note that  $g$  is unmovable if and only if  $\Delta \not\leq_L g$ . Moreover, by Proposition 2.2, for all  $g \in G$ , there exists a unique  $d \in \mathbb{Z}$  and a unique unmovable element  $g_0 \in M$  such that  $g = g_0 \Delta^d$ . In that case we also have  $g = \Delta^d g_0$ , since  $\Delta$  is central. Let  $g \in M$  be an unmovable element. The *depth* of  $g$ , denoted by  $\text{dpt}(g)$ , is defined by

$$\text{dpt}(g) = \begin{cases} \frac{\text{bh}(g)-1}{2} & \text{if } \text{bh}(g) \text{ is odd,} \\ \frac{\text{bh}(g)}{2} & \text{if } \text{bh}(g) \text{ is even.} \end{cases}$$

In other words, if  $g = g_p \cdots g_2 g_1$  is the alternating form of  $g$ , then  $\text{dpt}(g) = |\{i \in \{1, \dots, p\} \mid g_i \in N\}|$ . Now, take any  $g \in G$ , and write  $g = \Delta^d g_0$ , where  $d \in \mathbb{Z}$  and  $g_0 \in M$  is unmovable. Then the *depth* of  $g$  is defined to be  $\text{dpt}(g) = \text{dpt}(g_0)$ .

Set  $\theta = \Delta_1^{-1} \Delta = \Delta \Delta_1^{-1}$ , and, for  $k \geq 0$ , set  $q_k = \text{dpt}(\theta^k)$ . We say that an element  $g \in G$  is  $(H, G_1)$ -*positive* if  $g \notin G_1$  and  $\text{dpt}(g) \geq q_k$ , where  $k = \text{Ninf}(g)$ . We denote by  $P = P_{H, G_1}$  the set of  $(H, G_1)$ -positive elements.

Note that  $M \setminus G_1 \subset P$  because  $\text{Ninf}(g) = 0$  for all  $g \in M$ . Hence, every non  $(H, G_1)$ -positive element has a positive negative-infimum.

We say that  $(H, G_1)$  is a *Dehornoy structure* if  $P$  satisfies the following conditions:

- (a)  $P^2 \subset P$ .
- (b)  $G_1 P G_1 \subset P$ .
- (c) We have the disjoint union  $G = P \sqcup P^{-1} \sqcup G_1$ .

Our goal in this chapter is to prove a condition on  $(H, G_1)$  that implies that it is a Dehornoy structure (see Theorem 3.2). We will apply in the next chapters this criterion to the Artin groups of type  $A$ ,  $B$ , and  $I_2(m)$  ( $m \geq 5$ ). But, firstly, we show how left orders appear in this way.

Assume given two sequences of parabolic subgroups  $G_0 = G, G_1, \dots, G_n$  and  $H_1, \dots, H_n$  such that  $G_i, H_i \subset G_{i-1}$  and  $(H_i, G_i)$  is a Dehornoy structure of  $G_{i-1}$ , for all  $i \in \{1, \dots, n\}$ , and  $G_n \simeq \mathbb{Z}$ . For  $i \in \{1, \dots, n\}$ , we denote by  $P_i$  the set of  $(H_i, G_i)$ -positive elements in  $G_{i-1}$ . Furthermore, we choose a generator  $g_n$  of  $G_n$  and we set  $P_{n+1} = \{g_n^k \mid k \geq 1\}$ . For  $i \in \{1, \dots, n+1\}$  we set  $P_i^{-1} = \{g^{-1} \mid g \in P_i\}$ , and for  $\epsilon = (\epsilon_1, \dots, \epsilon_n, \epsilon_{n+1}) \in \{\pm 1\}^{n+1}$  we set:

$$P^\epsilon = P_1^{\epsilon_1} \cup \dots \cup P_n^{\epsilon_n} \cup P_{n+1}^{\epsilon_{n+1}}.$$

**Proposition 3.1.** *Under the above hypothesis, the set  $P^\epsilon$  is the positive cone for a left order on  $G$ .*

PROOF. Let  $g, g' \in P^\epsilon$ . Then  $g \in P_i^{\epsilon_i}$  and  $g' \in P_j^{\epsilon_j}$  for some  $i, j \in \{1, \dots, n+1\}$ . If  $i = j$ , (a) implies that  $gg' \in P_i^{\epsilon_i} \subset P^\epsilon$ . If  $i \neq j$ , without loss of generality we suppose that  $i < j$ . Hence  $g' \in G_{i+1}$ . By (b),  $gg', g'g \in P_i^{\epsilon_i} \subset P^\epsilon$ . Therefore  $P^\epsilon$  is a subsemigroup of  $G$ . Since  $1 \in G_n$ , (c) implies that  $1 \notin P_i^{\pm 1}$  for all  $i \in \{1, \dots, n\}$ . Further  $1 \notin P_{n+1}^{\pm 1}$  because  $g_n \neq 1$  and  $G_n \simeq \mathbb{Z}$ . Hence  $G = (P^\epsilon \cup P^{-\epsilon}) \sqcup \{1\}$ . Since  $P_i \subset G_{i-1}$  for all  $i \in \{1, \dots, n+1\}$ , then, by (c),  $(P_i \cup P_i^{-1}) \cap (P_j \cup P_j^{-1}) = \emptyset$  for all  $i \neq j$ . Finally, (c) implies that  $P_i \cap P_i^{-1} = \emptyset$ . Thus  $G = P^\epsilon \sqcup (P^\epsilon)^{-1} \sqcup \{1\}$ . Therefore  $P^\epsilon$  is the positive cone for a left order on  $G$ .  $\square$

Let  $(H, G_1)$  be a Dehornoy pre-structure, with the above notations. We say that  $(H, G_1)$  satisfies *Condition A* if there exists a constant  $c \in \mathbb{N}$ ,  $c \geq 1$ , such that  $q_k = kc + 1$  for all  $k \geq 1$ .

Note that  $q_0 = 0$ , so the equality  $q_k = kc + 1$  is never true for  $k = 0$ . Since in our study we will need sometimes to replace 0 by  $1 = 0c + 1$  when  $k = 0$ , we may use the sequence  $\tilde{q}_k$

defined as follows in place of  $q_k$  :

$$\tilde{q}_k = \begin{cases} 1 & \text{if } k = 0, \\ q_k & \text{if } k \geq 1. \end{cases}$$

We say that an element  $g \in M$  is a *theta element* if there exist  $g_0 \in M_1$  and  $k \geq 1$  such that  $g = \theta^k g_0$ , where  $\theta = \Delta_1^{-1} \Delta = \Delta \Delta_1^{-1}$ . The set of theta elements is denoted by  $\Theta = \Theta(G)$ .

Let  $g, g' \in M$  be unmovable elements such that  $\{g, g'\} \not\subset M_1 \cup \Theta$ . We say that the pair  $(g, g')$  satisfies *Condition B* if there exists  $\epsilon \in \{0, 1\}$  such that:

- (a)  $\text{dpt}(gg') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_k \geq 0$ , where  $k = \inf(gg')$ ,
- (b)  $\epsilon = 0$  if  $\{g, g'\} \cap \Theta \neq \emptyset$ .

We say that  $(H, G_1)$  satisfies *Condition B* if, for all  $g, g' \in M$  unmovable such that  $\{g, g'\} \not\subset M_1 \cup \Theta$ , the pair  $(g, g')$  satisfies Condition B.

The purpose of this chapter is to prove the following:

**Theorem 3.2.** *Let  $(H, G_1)$  be a Dehornoy pre-structure. If  $(H, G_1)$  satisfies Conditions A and B, then  $(H, G_1)$  is a Dehornoy structure.*

The rest of the chapter is dedicated to the proof of Theorem 3.2.

**Lemma 3.3.** *We have  $\theta g = g\theta$  for all  $g \in G_1$ . Hence  $\Delta^k = \theta^k \Delta_1^k$  for all  $k \geq 1$ . Furthermore  $g \not\leq_L \theta$  and  $g \not\leq_R \theta$  for all  $g \in M_1 \setminus \{1\}$ .*

PROOF. Let  $g \in G_1$ . Then  $g = \Delta g \Delta^{-1} = \theta g \theta^{-1}$  because  $\Delta$  is central in  $G$ , and  $\Delta_1$  is central in  $G_1$ . Hence  $\theta g = g\theta$ . Let  $g \in M_1$  be a right divisor of  $\theta$ . Then  $g \Delta_1$  lies in  $\text{Div}(\Delta) \cap M_1 = \text{Div}(\Delta_1)$ , therefore  $g = 1$ . Similarly we prove that  $g = 1$  whenever it left divides  $\theta$ .  $\square$

**Corollary 3.4.** *Let  $g = \theta^k g_0$  be a theta element. Then  $R_{M_1}(g) = g_0$ . Furthermore, if  $g$  is unmovable, then  $\text{dpt}(g) = q_k$ .*

PROOF. Lemma 3.3 implies that  $g_1 \not\leq_R \theta^k$  for all  $g_1 \in M_1$ , thus  $R_{M_1}(g) = g_0$ . Hence  $\text{bh}(g) = \text{bh}(\theta^k)$ . Therefore  $\text{dpt}(g) = q_k$  if  $g$  is unmovable.  $\square$

In what follows of this chapter we assume that Condition A holds.

**Lemma 3.5.** *Let  $g = \theta^k g_0$  be an unmovable theta element of  $G$ , and let  $p = \ell(g_0)$ . Then  $\ell(g) = \max\{k, p\}$ ,  $\text{com}_L(g) = \theta^{\ell(g)-k} \Delta_1^{\ell(g)-p} g_1$ , and:*

$$\text{dpt}(g) + \text{dpt}(\text{com}_L(g)) = \begin{cases} q_k & \text{if } p \leq k \\ q_p + 1 & \text{if } p > k \end{cases}$$

where  $g_1 = \Delta_1^p g_0^{-1}$  is the left complement of  $g_0$  with respect to  $\Delta_1$ .

PROOF. Let  $s_p \cdots s_1$  be the right greedy normal form of  $g_0$ . Note that  $s_i \neq \Delta$  for all  $i$  because  $g$  is unmovable. If  $p \leq k$ , Lemma 3.3 implies that  $g = \theta \cdots \theta \theta_{s_p} \cdots \theta_{s_1}$ , where  $\theta \cdots \theta = \theta^{k-p}$ , is the right greedy normal form of  $g$ . Hence  $\ell(g) = k$ . Therefore  $\text{com}_L(g) = \Delta_1^{k-p} g_1$ , and  $\text{dpt}(g) + \text{dpt}(\text{com}_L(g)) = q_k$ . If  $p > k$ , Lemma 3.3 implies that  $g = s_p \cdots s_{k+1} \theta_{s_k} \cdots \theta_{s_1}$  is the right greedy normal form of  $g$ . Hence  $\ell(g) = p$ . Therefore  $\text{com}_L(g) = \theta^{p-k} g_1$ , and  $\text{dpt}(g) + \text{dpt}(\text{com}_L(g)) = q_k + q_{p-k} = q_p + 1$ .  $\square$

**Corollary 3.6.** *Recall that we have assumed that Condition A holds. Let  $g = \theta^k g_0$  be an unmovable theta element of  $G$ , and let  $t > 0$  be an integer. Then:*

$$\begin{cases} \Delta^{-t} g \in P \text{ and } (\Delta^{-t} g)^{-1} \notin P & \text{if } t < k \\ \Delta^{-t} g \in G_1 & \text{if } t = k \\ \Delta^{-t} g \notin P \text{ and } (\Delta^{-t} g)^{-1} \in P & \text{if } t > k \end{cases}$$

Furthermore  $\Delta^{t'} g \in P$  and  $(\Delta^{t'} g)^{-1} \notin P$  for all integer  $t' \geq 0$ .

PROOF. We have  $(\Delta^{-t} g)^{-1} = \Delta^t g^{-1} = \Delta^{t-d} \text{com}_L(g)$ , where  $d = \ell(g)$ . If  $t < k$ , then  $\text{dpt}(\theta^k g_0) = q_k > q_t$ . Hence  $\Delta^{-t} g \in P$ . Lemma 3.5 implies that  $d \geq k > t$ , and that  $\text{dpt}(\text{com}_L(g)) \leq q_d + 1 - \text{dpt}(g) = q_d - kc = q_{d-k} < q_{d-t}$ . Thus  $(\Delta^{-t} g)^{-1} \notin P$ . If  $t = k$ , then  $\Delta^{-t} g = \Delta_1^{-1} g_0 \in G_1$ . If  $t > k$ , then  $\text{dpt}(\theta^k g_0) = q_k < q_t$ . Therefore  $\Delta^{-t} g \notin P$ . If  $t \geq d$ , then  $(\Delta^{-t} g)^{-1} \in P$ . Otherwise, Lemma 3.5 implies that  $\text{dpt}(\text{com}_L(g)) \geq q_d - \text{dpt}(g) = (d-k)c \geq q_{d-t}$ . Hence  $(\Delta^{-t} g)^{-1} \in P$ . We have  $(\Delta^{t'} g)^{-1} = \Delta^{-t'-d} \text{com}_L(g)$ . Lemma 3.5 implies that  $\text{dpt}(\text{com}_L(g)) \leq q_d + 1 - \text{dpt}(g) = q_d - kc = q_{d-k} < q_{d+t'}$ . Therefore  $(\Delta^{t'} g)^{-1} \notin P$ .  $\square$

**Lemma 3.7.** *Suppose that  $\text{dpt}(g) + \text{dpt}(\text{com}_L(g)) = q_k$  for all unmovable element  $g \in M \setminus \Theta$ , where  $k = \ell(g)$ . Then  $G = P \sqcup P^{-1} \sqcup G_1$ .*

PROOF. By definition, no  $(H, G_1)$ -positive element belongs to  $G_1$ , hence  $(P \cup P^{-1}) \cap G_1 = \emptyset$ . We shall prove that  $P^{-1} = (G \setminus G_1) \setminus P$ . Let  $g^{-1} \in P^{-1}$  for some  $g = \Delta^d g_0 \in P$ , where  $d = \inf(g)$ , and let  $k = \ell(g_0)$ . Suppose that  $g_0 = \theta^t g_1 \in \Theta$  for some  $g_1 \in M_1$ . If  $d \geq 0$ , Corollary 3.6 implies that  $g^{-1} \notin P$ . If  $d < 0$ , then  $t > -d$  because  $g$  is  $(H, G_1)$ -positive, hence  $\text{dpt}(g) = q_t > q_{-d}$ . Then, by Corollary 3.6,  $g^{-1} \notin P$ . Suppose that  $g_0 \in M_1$ . Then  $d > 0$  because  $g$  is  $(H, G_1)$ -positive. Hence  $g^{-1} = \Delta^{-d-k} \text{com}_L(g_0)$ . By hypothesis, we have  $\text{dpt}(\text{com}_L(g_0)) = q_k < q_{d+k}$ . Therefore  $g^{-1} \notin P$ . Suppose that  $g_0 \in M \setminus (\Theta \cup M_1)$ . In particular  $\text{dpt}(g_0) \geq 1$ . If  $d \geq 0$ , then  $g^{-1} = \Delta^{-d-k} \text{com}_L(g_0)$ . Hence, by hypothesis,  $\text{dpt}(\text{com}_L(g_0)) = q_k - \text{dpt}(g_0) < q_k \leq q_{d+k}$ . In particular, we have proved that  $M^{-1} \cap P = \emptyset$ . If  $d < 0$ , i.e.  $d = -t$  for some  $t \geq 1$ , then  $g^{-1} = \Delta^{t-k} \text{com}_L(g_0)$ . We have  $t < k$ . Otherwise  $g^{-1} \in M$ , and so  $g \notin P$ , which is a contradiction. By hypothesis, we have  $\text{dpt}(\text{com}_L(g_0)) = q_k - \text{dpt}(g_0) \leq q_k - q_t < q_{k-t}$ . Therefore  $P^{-1} \subseteq (G \setminus G_1) \setminus P$ . Let  $g = \Delta^{-t} g_0 \in (G \setminus G_1) \setminus P$  with  $\inf(g) = -t$ , and  $\text{dpt}(g_0) < q_t$ . Note that  $\inf(g) < 0$  for all  $g \in (G \setminus G_1) \setminus P$ , because  $M \subseteq P$ . Suppose that  $g_0 = \theta^d g_1 \in \Theta$  for some  $g_1 \in M_1$ . Hence,  $d < t$  because  $q_d = \text{dpt}(g_0) < q_t$ . Corollary 3.6 implies that  $g^{-1} \in P$ . Suppose that  $g_0 \in M \setminus \Theta$ . We have  $g^{-1} = \Delta^{t-k} \text{com}_L(g_0)$ . If  $t \geq k$ , then  $g^{-1} \in M \subseteq P$ . If  $t < k$ , then, by hypothesis,  $\text{dpt}(\text{com}_L(g_0)) = q_k - \text{dpt}(g_0) \geq q_k - q_t + 1 = q_{k-t}$ . Therefore  $(G \setminus G_1) \setminus P \subseteq P^{-1}$ .  $\square$

**Lemma 3.8.** *Condition B implies that  $\text{dpt}(g) + \text{dpt}(\text{com}_L(g)) = q_k$  for all unmovable elements  $g \in M \setminus \Theta$ ,  $g \neq 1$ , where  $k = \ell(g)$ .*

PROOF. Let  $g \in M \setminus \Theta$  be an unmovable element, and let  $k = \ell(g)$ . Note that  $\text{com}_L(g) \notin M_1$  because  $g \notin \Theta$ . If  $g \in M_1$ , then  $\text{com}_L(g) = \theta^k (\Delta_1^k g^{-1})$ . Therefore  $\text{dpt}(g) + \text{dpt}(\text{com}_L(g)) = q_k$ . If  $g \in M \setminus M_1$ , Lemma 3.5 implies that  $\text{com}_L(g) \notin \Theta$ . So, by Condition B, we have  $0 = \text{dpt}(\Delta^k) = \text{dpt}(\text{com}_L(g)g) \geq \text{dpt}(\text{com}_L(g)) + \text{dpt}(g) - \tilde{q}_k \geq 0$ . Therefore  $\text{dpt}(\text{com}_L(g)) + \text{dpt}(g) = \tilde{q}_k = q_k$ , because  $k \geq 1$ .  $\square$

**Lemma 3.9.** *Let  $g \in G_1 \setminus M_1$ . Then  $g_L \in \Theta$  and  $\text{dpt}(g) = q_k$ , where  $k = \text{Ninf}(g)$ .*

PROOF. We have  $g = \Delta_1^{-k} g_0$  for some  $k \geq 1$  and  $g_0 \in M_1$  such that  $\Delta_1 \not\leq_R g_0$ . Note that  $g = \Delta_1^{-k} \theta^{-k} \theta^k g_0 = \Delta^{-k} \theta^k g_0$ . If  $\Delta \leq_L \theta^k g_0$ , Corollary 3.4 implies that  $\Delta_1 \leq_L g_0$ , which is a contradiction. Therefore  $g_L = \theta^k g_0 \in \Theta$  and  $\inf(g) = -k$ . By Corollary 3.4,  $\text{bh}(g_L) = \text{bh}(\theta^k)$ . Hence  $\text{dpt}(g) = q_k$ .  $\square$



Lemmas 3.7, 3.8 and 3.9 imply that, if Condition B holds, then:

$$P^{-1} = \{g \in G \mid \text{dpt}(g) < q_k \text{ where } k = \text{Ninf}(g)\}.$$

**Lemma 3.10.** *Condition B implies that  $P^{-1}$  is a subsemigroup of  $G$ .*

PROOF. Let  $g, g' \in P^{-1}$ , that is,  $g = \Delta^{-t}g_0$  and  $g' = \Delta^{-t'}g'_0$ , where  $t = \text{Ninf}(g)$  and  $t' = \text{Ninf}(g')$ , such that  $\text{dpt}(g_0) < q_t$  and  $\text{dpt}(g'_0) < q_{t'}$ . We have  $gg' = \Delta^{-t-t'+k}h$ , where  $k = \text{inf}(g_0g'_0)$ , and  $h = (g_0g'_0)_L$ , i.e.  $h = \Delta^{-k}g_0g'_0$ . If  $g_0, g'_0 \in M_1$ , then  $k = 0$ , and  $\text{dpt}(h) < q_{t+t'}$ . Hence  $gg' \in P^{-1}$ . If  $g_0, g'_0 \in \Theta$ , that is,  $g_0 = \theta^d g_1$  and  $g'_0 = \theta^{d'} g'_1$ . Then  $g_0g'_0 = \theta^{d+d'} g_1g'_1$  and  $h = \theta^{d+d'-k}h_0$  for some  $h_0 \in M_1$ . Notice that  $d < t$  and  $d' < t'$  because  $g, g' \in P^{-1}$ , therefore  $k \leq d + d' < t + t'$ . We have  $\text{dpt}(h) = q_{d+d'-k} \leq \tilde{q}_{d+d'-k} = \tilde{q}_d + \tilde{q}_{d'} - \tilde{q}_k < \tilde{q}_t + \tilde{q}_{t'} - \tilde{q}_k = \tilde{q}_{t+t'-k} = q_{t+t'-k}$ . Thus  $gg' \in P^{-1}$ . If  $g_0 = \theta^d g_1 \in \Theta$  and  $g'_0 \in M_1$ , then  $g_0g'_0 = \theta^d g_1g'_0 \in \Theta$  and  $h = \theta^{d-k}h_0$  for some  $h_0 \in M_1$ . Note that  $d < t$  because  $g \in P^{-1}$ , therefore  $k \leq d < t$ . We have  $\text{dpt}(h) = q_{d-k} < q_{t-k} \leq q_{t+t'-k}$ . Thus  $gg' \in P^{-1}$ . Similarly, we prove that  $gg' \in P^{-1}$  whenever  $g_0 \in M_1$  and  $g'_0 \in \Theta$ . If  $\{g_0, g'_0\} \not\subseteq M_1 \cup \Theta$ , Condition B implies that  $\tilde{q}_k \leq \text{dpt}(g_0) + \text{dpt}(g'_0) \leq q_t + q_{t'} - 2 = q_{t+t'} - 1 < q_{t+t'}$ , therefore  $k < t + t'$ . By Condition B, we have  $\text{dpt}(h) \leq \text{dpt}(g_0) + \text{dpt}(g'_0) - \tilde{q}_k + 1 \leq q_t + q_{t'} - \tilde{q}_k - 1 = q_{t+t'-k} - 1 < q_{t+t'-k}$ . Thus  $gg' \in P^{-1}$ . Therefore  $P^{-1}$  is a subsemigroup of  $G$ .  $\square$

Lemma 3.10 implies that, if Condition B holds, then  $P$  is a semigroup.

**Lemma 3.11.** *Condition B implies that  $G_1P^{-1}G_1 \subseteq P^{-1}$ .*

PROOF. Let  $g = \Delta^{-t}g_0 \in P^{-1}$ , and let  $g' \in G_1$ . If  $g' \in M_1$ , then  $gg' = \Delta^{-t}g_0g' = \Delta^{-t+k}h$ , where  $k = \text{inf}(g_0g')$  and  $h = \Delta^{-k}g_0g'$ . If  $g_0 \in M_1$ , then  $k = 0$ . Hence  $gg' \in P^{-1}$  because  $\text{dpt}(h) = 0 < q_t$ . If  $g_0 = \theta^p g_1 \in \Theta$ , then  $h = \theta^{p-k}h'$ . Note that  $k \leq p < t$  because  $g$  is  $(H, G_1)$ -negative. Then  $\text{dpt}(h) = q_{p-k} \leq \tilde{q}_{p-k} < \tilde{q}_{t-k}$ . Hence  $gg' \in P^{-1}$ . If  $g_0 \in M \setminus (M_1 \cup \Theta)$ , Condition B implies that  $\tilde{q}_k \leq \text{dpt}(g_0) + \text{dpt}(g') < q_t$ . Hence  $k < t$ . Further, by Condition B, we have  $\text{dpt}(h) \leq \text{dpt}(g_0) + \text{dpt}(g') - \tilde{q}_k + 1 \leq q_t - \tilde{q}_k < q_{t-k}$ . Therefore  $gg' \in P^{-1}$ . Similarly, we can prove that  $g'g \in P^{-1}$ . If  $g' \in G_1 \setminus M_1$ , Lemma 3.9 implies that  $g' = \Delta_1^{-t'}g'_0 = \Delta^{-t'}\theta^{t'}g'_0$ , where  $\text{inf}(g') = -t'$ . Hence  $gg' = \Delta^{-t-t'}g_0\theta^{t'}g'_0 = \Delta^{-t-t'+k}h$ , where  $k = \text{inf}(g_0\theta^{t'}g'_0)$  and  $h = \Delta^{-k}g_0\theta^{t'}g'_0$ . If  $g_0 \in M_1$ , then  $h = \theta^{t'-k}h'$  for some  $h' \in M_1$ . Note that  $k \leq t' < t + t'$ . Further  $\text{dpt}(h) = q_{t'-k} \leq \tilde{q}_{t'-k} < q_{t+t'-k}$ . Hence  $gg' \in P^{-1}$ . If  $g_0 = \theta^p g_1 \in \Theta$ , then

$h = \theta^{p+t'-k}h'$  for some  $h' \in M_1$ . Note that  $p < t$  because  $g$  is  $(H, G_1)$ -negative. Hence  $k \leq p + t' < t + t'$ . Further  $\text{dpt}(h) = q_{p+t'-k} \leq \tilde{q}_{p+t'-k} < q_{t+t'-k}$ . Therefore  $gg' \in P^{-1}$ . If  $g_0 \in M \setminus (M_1 \cup \Theta)$ , Condition B implies that  $\tilde{q}_k \leq \text{dpt}(g_0) + q_{t'} \leq q_t + q_{t'} - 1 = q_{t+t'}$ . Hence  $k \leq t + t'$ . If  $k = t + t'$ , then  $gg' = h \in M$ . If  $h \in M \setminus M_1$ , Lemma 3.10 implies that  $g' = g^{-1}h \in P$  since  $g^{-1} \in P$ , which is a contradiction because  $g' \in G_1$ . If  $h \in M_1$ , then  $g_0\theta^{t'}g'_0 = \Delta^{t+t'}h = \theta^{t+t'}\Delta_1^{t+t'}h = \theta^{t+t'}g_1g'_0 = \theta^t g_1\theta^{t'}g'_0$  for some  $g_1 \in M_1$ . Then  $g_0 = \theta^t g_1$ , which is a contradiction because  $g_0 \notin \Theta$  and  $g$  is  $(H, G_1)$ -negative. Hence  $k < t + t'$ . By Condition B,  $\text{dpt}(h) \leq \text{dpt}(g_0) + q_{t'} - \tilde{q}_k \leq q_t + q_{t'} - \tilde{q}_k - 1 = q_{t+t'-k} - 1 < q_{t+t'-k}$  because  $g'_L \in \Theta$ . Thus  $gg' \in P^{-1}$ . Similarly, we have  $g'g \in P^{-1}$ . Hence  $P^{-1}G_1 \subset P^{-1}$  and  $G_1P^{-1} \subset P^{-1}$ . Therefore  $G_1P^{-1}G_1 \subseteq P^{-1}$ .  $\square$

Lemma 3.11 implies that, if Condition B holds, then  $G_1PG_1 \subseteq P$ .

**Remark** Assume that  $(H, G_1)$  is a Dehornoy pre-structure of  $G$ . Let  $g \in G$  with  $\text{inf}(g) < 0$ , and set  $t = -\text{inf}(g) \geq 1$ . Then  $g$  is  $(H, G_1)$ -positive, if and only if,  $\text{bh}(g) \geq 2\tilde{q}_t$ . Indeed, if  $\text{dpt}(g) \geq \tilde{q}_t$ , then  $\text{bh}(g) \geq 2\text{dpt}(g) \geq 2\tilde{q}_t$ . Suppose that  $\text{bh}(g) \geq 2\tilde{q}_t$ . If  $\text{bh}(g)$  is even, then  $\text{dpt}(g) \geq \tilde{q}_t$ . If  $\text{bh}(g)$  is odd, then  $\text{bh}(g) > 2\tilde{q}_t$ , hence  $\text{dpt}(g) \geq \tilde{q}_t$ .



## CHAPTER 4

### Artin groups of type A

In this chapter  $G$  and  $M$  denote the Artin group and the Artin monoid of type  $A_n$ , respectively, where  $n \geq 2$ . Recall that  $M$  is defined by the monoid presentation with generating set  $S = \{x_1, \dots, x_n\}$  and relations:

$$\begin{aligned} x_i x_{i+1} x_i &= x_{i+1} x_i x_{i+1} && \text{if } 1 \leq i \leq n-1, \\ x_i x_j &= x_j x_i && \text{if } |i-j| \geq 2. \end{aligned}$$

The group  $G$  is the enveloping group of  $M$ , and it is isomorphic to the braid group  $\mathcal{B}_{n+1}$  on  $n+1$  strands. By Theorem 2.11,  $M$  is a Garside monoid, and  $\Omega = x_1 \vee_R \cdots \vee_R x_n$  is a Garside element of  $M$ .

We have  $\Omega x_i = x_{n+1-i} \Omega$  for all  $i \in \{1, \dots, n\}$ . In particular,  $\Omega$  is not central, but  $\Delta = \Omega^2$  is central. Note that, by Dehornoy [11],  $\Delta$  is a Garside element, and, by Brieskorn–Saito [4],  $\Delta = (x_1 \cdots x_n)^{n+1}$ .

Let  $G_1$  be the subgroup of  $G$  generated by  $\{x_2, \dots, x_n\}$ , and let  $H$  be the subgroup generated by  $\{x_1, \dots, x_{n-1}\}$ . By Theorem 2.11 and Lemma 2.8,  $G_1$  and  $H$  are parabolic subgroups of  $(G, \Delta)$ . The subgroup  $G_1$  is associated with  $\Delta_1 = (x_2 \vee_R \cdots \vee_R x_n)^2 = (x_2 \cdots x_n)^n$ , and  $H$  is associated with  $\Lambda = (x_1 \vee_R \cdots \vee_R x_{n-1})^2 = (x_1 \cdots x_{n-1})^n$ . As ever, we set  $M_1 = G_1 \cap M$  and  $N = H \cap M$ . Obviously,  $M_1 \cup N$  generates  $M$ , hence  $(H, G_1)$  is a Dehornoy pre-structure. Note that  $\Delta$  is decomposed as follows:

$$\Delta = \underbrace{(x_1 x_2 \cdots x_{n-1} x_n^2 x_{n-1} \cdots x_2 x_1)}_{\theta = \Delta \Delta_1^{-1} = \Delta_1^{-1} \Delta} \underbrace{(x_2 \cdots x_{n-1} x_n^2 x_{n-1} \cdots x_2) \cdots (x_{n-1} x_n^2 x_{n-1}) x_n^2}_{\Delta_1 = (x_2 \vee_R \cdots \vee_R x_n)^2}.$$

The main result of this chapter is the following:

**Theorem 4.1.** *The pair  $(H, G_1)$  satisfies Conditions A and B.*

Applying Theorem 3.2 we get:

**Corollary 4.2.** *The pair  $(H, G_1)$  is a Dehornoy structure.*

Set:

$$G_i = \langle x_{i+1}, \dots, x_n \rangle, \quad H_i = \langle x_i, \dots, x_{n-1} \rangle, \quad \Delta_i = (x_{i+1} \cdots x_n)^{n+1-i}$$

for  $i \in \{1, \dots, n-1\}$ . Since each  $G_i$  and  $H_i$  are Artin groups of type A (see Theorem 2.11), iterating Corollary 4.2 and applying Proposition 3.1 we get:

**Corollary 4.3.**

- (1) The pair  $(H_i, G_i)$  is a Dehornoy structure of  $G_{i-1}$  for all  $i \in \{1, \dots, n-1\}$ , where  $G_0 = G$ .
- (2) For  $i \in \{1, \dots, n-1\}$ , we denote by  $P_i$  the set of  $(H_i, G_i)$ -positive elements of  $G_{i-1}$ . Furthermore, we set  $P_n = \{x_n^k \mid k \geq 1\}$ . Take  $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{\pm 1\}^n$ , and set:

$$P^\epsilon = P_1^{\epsilon_1} \sqcup \cdots \sqcup P_n^{\epsilon_n}.$$

Then  $P^\epsilon$  is the positive cone of a left order on  $G$ .

Before proving Theorem 4.1, we show that the left order obtained from it coincides with that obtained from Theorem 1.1. More precisely, we prove the following:

Recall that the definition of an  $x_1$ -positive element is given in the introduction.

**Proposition 4.4.** *The set  $P = P_{(H, G_1)}$  is the set of  $x_1$ -positive elements of  $G = \mathcal{B}_{n+1}$ .*

PROOF. Let  $g$  be a  $x_1$ -positive element of  $G$ . Then  $g = g_0 x_1^{k_1} g_1 \cdots g_{p-1} x_1^{k_p} g_p$  for some  $k_1, \dots, k_p \geq 1$  and  $g_0, \dots, g_p \in G_1$ . Note that  $x_1^{k_i} \in P$  for all  $i \in \{1, \dots, p\}$ . By Theorem 4.1,  $G_1 P G_1 \subseteq P$ . Therefore  $g \in P$ . Conversely, let  $g \in P$ . If  $g \in M$ , then  $g$  is  $x_1$ -positive. Otherwise  $g = \Delta^{-k} g_0$ , where  $k = \text{Ninf}(g)$ , such that  $\text{dpt}(g_0) \geq q_k = ck + 1 \geq k + 1$ . (Actually, as we will see soon, we have  $q_k = k + 1$ .) Hence  $\text{bh}(g_0) \geq 2\text{dpt}(g_0) \geq 2k + 2$ . Since  $g \notin G_1$  and  $g = \Omega^{-2k} g_0$ , Theorem 2.13 implies that  $g$  is  $x_1$ -positive.  $\square$

The rest of the chapter is dedicated to the proof of Theorem 4.1.

By Dehornoy [12, Lemma 4.19], the  $(N, M_1)$ -breadth of  $\Delta^k$  is  $2k + 2$ , for all  $k \geq 1$ . Lemma 3.3 implies that  $\text{bh}(\theta^k) = \text{bh}(\Delta^k)$  because  $\Delta_1^k \in M_1$ . Hence  $\tilde{q}_k = k + 1$  for all  $k \geq 0$ . Therefore  $(H, G_1)$  satisfies Condition A.

In what follows we will prove that Condition B holds:

**Lemma 4.5.** *Let  $g \in M \setminus M_1$ , and let  $k \geq 1$  be an integer. Then  $\text{bh}(g\theta^k) = \text{bh}(g) + 2k$ .*

PROOF. Let  $(g_p, \dots, g_1)$  be the alternating form of  $g$ . Note that  $gg_1^{-1} \neq 1$  because  $g \notin M_1$ . Hence  $x_1 \leq_R g_2$  and  $x_t \not\leq_R gg_1^{-1}$  for all  $t \geq 2$ . By Lemma 3.3,  $\theta g_1 = g_1\theta$ . Then  $(g_p, \dots, g_3, g_2x_1, x_2 \cdots x_{n-1}x_n^2, x_{n-1} \cdots x_1, g_1)$  is an  $(N, M_1)$ -sequence of  $g\theta$ . If  $x_t \leq_R g\theta g_1^{-1} = gg_1^{-1}\theta$  for some  $t \in \{2, \dots, n\}$ , Lemma 3.3 implies that  $x_t \leq_R gg_1^{-1}$  which is a contradiction. Hence  $g_1 = R_{M_1}(g\theta)$ . If  $x_{n-1} \leq_R gg_1^{-1}x_1 \cdots x_{n-1}x_n^2$ , then  $x_n x_{n-1} \leq_R gg_1^{-1}x_1 \cdots x_{n-1}x_n$ . In particular,  $x_{n-1} \leq_R gg_1^{-1}x_1 \cdots x_{n-1}x_n$ . Then  $x_n \leq_R gg_1^{-1}$  which is a contradiction. If  $x_k \leq_R gg_1^{-1}x_1 \cdots x_{n-1}x_n^2$  for some  $k \in \{1, \dots, n-2\}$ , then  $x_{k+1} \leq_R gg_1^{-1}$  which is a contradiction as well. Therefore  $\text{bh}(g\theta) = \text{bh}(g) + 2$ . Inductively, we obtain that  $\text{bh}(g\theta^k) = \text{bh}(g) + 2k$  for all  $k \geq 2$ .  $\square$

In general, if  $M$  is a monoid, and  $N_1, N_2$  are two submonoids of  $M$ . A  $(N_2, N_1)$ -sequence of an element  $g \in M$  is a tuple  $(g_p, \dots, g_1)$  such that,  $g = g_p \cdots g_1$ ,  $g_i \in N_1$  when  $i$  is odd, and  $g_i \in N_2$  when  $i$  is even. If  $M$  is Garside and it allows alternating decompositions with respect to  $(N_2, N_1)$ , we say that  $(N_2, N_1)$ -breadths are *minimal* if for all  $g \in M$ , its  $(N_2, N_1)$ -breadth is minimal in the set of lengths of all  $(N_2, N_1)$ -sequences of  $g$ .

For example,  $(x_2, x_2^3, x_1x_2, x_3x_2, x_1, 1)$  is a  $(\mathcal{B}_2^+, \Phi(\mathcal{B}_2^+))$ -sequence of length 6 in  $\mathcal{B}_3$ .

**Proposition 4.6** (Dehornoy [12], Burckel [5]).  *$(N, M_1)$ -breadths are minimal.*

In general, if  $(M, \Delta)$  is a Garside monoid,  $(H, G_1)$  is a Dehornoy pre-structure, and  $g \in M$ , then we denote by  $\text{dpt}^*(g)$  the depth of the element  $gR_{M_1}(g)^{-1}$ . More specifically, we have the following result:

**Lemma 4.7.** *Let  $(M, \Delta)$  be a Garside monoid, let  $(H, G_1)$  be a Dehornoy pre-structure, and let  $g \in M$ . Then:*

$$\text{dpt}^*(g) = \begin{cases} \frac{\text{bh}(g)-1}{2} & \text{if } \text{bh}(g) \text{ is odd,} \\ \frac{\text{bh}(g)}{2} & \text{if } \text{bh}(g) \text{ is even.} \end{cases}$$

PROOF. Let  $g' = gR_{M_1}(g)^{-1}$ . If  $\Delta \leq_R g'$ , then  $\Delta_1 \leq_R g'$  because  $\Delta_1 \in \text{Div}(\Delta) \setminus \{1\}$ . This is impossible because  $\Delta_1 \in M_1$  and  $h \not\leq_R g'$  for all  $h \in M_1 \setminus \{1\}$ . Therefore  $g'$  is unmovable. Since  $R_{M_1}(g) \in M_1$ , then  $\text{bh}(g) = \text{bh}(g')$ . This implies the equality.  $\square$

Note that  $\text{dpt}^*(g) = \text{dpt}(g)$  for all unmovable element  $g \in M$ .

We come back to the hypothesis given at the beginning of the chapter. In particular, now  $G$  denotes the Artin group of type  $A_n$ . Note that  $\text{rev}(g\Delta) = \text{rev}(g)\Delta$  for all  $g \in M$  because  $\text{rev}(\Delta) = \Delta$ . Thus  $\text{inf}(g) = \text{inf}(\text{rev}(g))$ .

**Lemma 4.8.** *Let  $g \in M \setminus M_1$ , and let  $k$  be its infimum with respect to  $\Delta$ . Let  $g_0 \in M_1$ , and let  $t \geq 1$  be an integer. Then:*

- (1)  $\text{dpt}^*(g_0\Delta^t) = \text{dpt}^*(g_0\theta^t) = \text{dpt}^*(\theta^t g_0) = \text{dpt}^*(\Delta^t g_0) = \tilde{q}_t$ .
- (2)  $\text{dpt}^*(g) = \text{dpt}(g) + k$ .
- (3)  $\text{dpt}^*(g) = \text{dpt}^*(\text{rev}(g))$ .
- (4)  $\text{dpt}^*(gg_0) = \text{dpt}^*(g_0g) = \text{dpt}^*(g)$ .
- (5)  $\text{dpt}^*(\Delta^t g) = \text{dpt}^*(\theta^t g) = \text{dpt}^*(g\theta^t) = \text{dpt}^*(g\Delta^t) = \text{dpt}^*(g) + \tilde{q}_t - 1$ .

PROOF. (1) It is a consequence of the fact that  $g_0, \Delta_1^t \in M_1$  and Lemma 3.3. (2) Lemma 4.5 implies that  $\text{bh}(g) = \text{bh}(g_R\Delta^k) = \text{bh}(g_R\theta^k) = \text{bh}(g_R) + 2k$ . Therefore  $\text{dpt}^*(g) = \text{dpt}^*(g_R) + k = \text{dpt}(g) + k$ . (3) Let  $(g_p, \dots, g_1)$  be the alternating form of  $g' = g\Delta^{-k}$ . If  $p$  is odd, then  $(\text{rev}(g_1), \dots, \text{rev}(g_p))$  is an  $(N, M_1)$ -sequence of  $\text{rev}(g')$ . Proposition 4.6 implies that  $\text{bh}(\text{rev}(g')) \leq p$ . Therefore  $\text{dpt}(\text{rev}(g')) \leq (p-1)/2 = \text{dpt}(g')$ . If  $p$  is even, then  $(\text{rev}(g_1), \dots, \text{rev}(g_p), 1)$  is an  $(N, M_1)$ -sequence of  $\text{rev}(g')$ . Proposition 4.6 implies that  $\text{bh}(\text{rev}(g')) \leq p + 1$ . Hence  $\text{dpt}(\text{rev}(g')) \leq p/2 = \text{dpt}(g')$ . Analogously we prove that  $\text{dpt}(g') \leq \text{dpt}(\text{rev}(g'))$  because  $\text{rev}^2(g') = g'$ . Hence, (2) implies that  $\text{dpt}^*(g) = \text{dpt}(g') + k = \text{dpt}(\text{rev}(g')) + k = \text{dpt}^*(\text{rev}(g))$ . (4) Since  $g_0 \in M_1$ , then  $\text{dpt}^*(gg_0) = \text{dpt}^*(g)$ . Further, by (3),  $\text{dpt}^*(g_0g) = \text{dpt}^*(\text{rev}(g_0g)) = \text{dpt}^*(\text{rev}(g)\text{rev}(g_0)) = \text{dpt}^*(\text{rev}(g)) = \text{dpt}^*(g)$ . (5) Lemma 4.5 implies that  $\text{dpt}^*(g\Delta^t) = \text{dpt}^*(g\theta^t) = \text{dpt}^*(g) + \tilde{q}_t - 1$  and  $\text{dpt}^*(\text{rev}(g)\theta^t) = \text{dpt}^*(\text{rev}(g)) + \tilde{q}_t - 1$ . By (4),  $\text{dpt}^*(\Delta^t g) = \text{dpt}^*(\theta^t g)$ . Finally, (3) implies that  $\text{dpt}^*(\theta^t g) = \text{dpt}^*(\text{rev}(g)\theta^t) = \text{dpt}^*(\text{rev}(g)) + \tilde{q}_t - 1 = \text{dpt}^*(g) + \tilde{q}_t - 1$ .  $\square$

**Lemma 4.9.** *Let  $g \in M_1 \cup \Theta$ , and let  $g' \in M$  such that either  $gg' \in \Theta$  or  $g'g \in \Theta$ . Then  $g' \in M_1 \cup \Theta$  as well.*

PROOF. Assume that  $gg' \in \Theta$ . Let  $k \geq 1$  be such that  $gg' = \theta^k h$  for some  $h \in M_1$ . If  $g \in M_1$ , Lemma 3.3 implies that  $g \leq_L h$  and  $g' = \theta^k(g^{-1}h) \in \Theta$ . If  $g \in \Theta$ , namely  $g = \theta^p g_0$

for some  $p \geq 1$  and  $g_0 \in M_1$ , Lemma 3.3 implies that  $g_0 \leq_L h$ . Further  $p \leq k$ . Hence  $g' = \theta^{k-p}(g_0^{-1}h) \in M_1 \cup \Theta$ . If  $g'g \in \Theta$ , then we show in the same way that  $g' \in M_1 \cup \Theta$ .  $\square$

**Lemma 4.10.** *Let  $g \in M \setminus (M_1 \cup \Theta)$  be an unmovable element, let  $g' \in \Theta$  be another unmovable element, and let  $g_0 \in M_1$ . Then:*

- (1)  $\text{dpt}(gg_0) - 1 = \text{dpt}(g) - \tilde{q}_k \geq 0$ , where  $k = \inf(gg_0)$ .
- (2)  $\text{dpt}(g_0g) - 1 = \text{dpt}(g) - \tilde{q}_k \geq 0$ , where  $k = \inf(g_0g)$ .
- (3)  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_k$ , where  $k = \inf(gg')$ .
- (4)  $\text{dpt}(g'g) = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_k$ , where  $k = \inf(g'g)$ .

PROOF. (1) We have  $gg_0 = \Delta^k h$  for some  $h \in M$  with  $\text{dpt}(gg_0) = \text{dpt}(h)$ . Lemma 4.9 implies that  $h \notin M_1$  because  $g \notin M_1 \cup \Theta$ . Hence  $\text{dpt}(h) \geq 1$ . Lemma 4.8.4 implies that  $\text{dpt}^*(gg_0) = \text{dpt}^*(g) = \text{dpt}(g)$ , and Lemma 4.8.5 implies that  $\text{dpt}^*(gg_0) = \text{dpt}^*(\Delta^k h) = \text{dpt}^*(h) + \tilde{q}_k - 1 = \text{dpt}(gg_0) + \tilde{q}_k - 1$ . Therefore  $\text{dpt}(gg_0) - 1 = \text{dpt}(g) - \tilde{q}_k$ . (2) By Lemma 4.8.2-3 and (1),  $\text{dpt}(g_0g) = \text{dpt}^*(g_0g) - k = \text{dpt}^*(\text{rev}(g_0g)) - k = \text{dpt}(\text{rev}(g_0g)) = \text{dpt}(\text{rev}(g)) - \tilde{q}_k + 1 = \text{dpt}(g) - \tilde{q}_k + 1$ . Further, Lemma 4.9 implies that  $\text{dpt}(g_0g) \geq 1$ . (3) Let  $p \geq 1$  be such that  $g' = \theta^p h'$  for some  $h' \in M_1$ , and let  $h \in M$  be an unmovable element such that  $gg' = \Delta^k h$ . By Lemma 4.9,  $h \notin M_1$  because  $g \notin M_1 \cup \Theta$ . Lemma 4.8.4-5 implies that  $\text{dpt}^*(gg') = \text{dpt}^*(gh'\theta^p) = \text{dpt}^*(gh') + \tilde{q}_p - 1 = \text{dpt}(g) + \tilde{q}_p - 1 = \text{dpt}(g) + \text{dpt}(g') - 1$  and  $\text{dpt}^*(gg') = \text{dpt}(h) + \tilde{q}_k - 1 = \text{dpt}(gg') + \tilde{q}_k - 1$ . Therefore  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_k$ . (4) We have  $\text{rev}(\Theta) \subseteq \Theta$  because  $\text{rev}(\theta) = \theta$ . Lemma 4.8.2-3 and (3) imply that  $\text{dpt}(g'g) = \text{dpt}^*(g'g) - k = \text{dpt}^*(\text{rev}(g'g)) - k = \text{dpt}(\text{rev}(g'g)) = \text{dpt}(\text{rev}(g)) + \text{dpt}(\text{rev}(g')) - \tilde{q}_k = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_k$ .  $\square$

**Lemma 4.11.** *Let  $g, g' \in M \setminus M_1$  be two unmovable elements such that  $gg' = \Delta^k$  for some integer  $k \geq 1$ . If  $\text{dpt}(g) + \text{dpt}(g') = k + 2$ . Then  $g, g' \in \Theta$ .*

PROOF. Let  $p = \text{dpt}(g) - 1$ , and let  $q = \text{dpt}(g') - 1$ . Then  $1 = (g^{-1}\Delta^p)(\Delta^q g'^{-1})$  because  $gg' = \Delta^{p+q}$ . Note that  $g^{-1}\Delta^p = \Omega^{-2q}g'$  and  $\Delta^q g'^{-1} = \Omega^{-2p}g$ . Further,  $2q < 2\text{dpt}(g') - 1 \leq \text{bh}(g') - 1$  and  $2p < 2\text{dpt}(g) - 1 \leq \text{bh}(g) - 1$ . So, Theorem 2.13 implies that  $g^{-1}\Delta^p$  (resp.  $\Delta^q g'^{-1}$ ) is either  $x_1$ -positive or belongs to  $G_1$ . Note that, if at least one of them is  $x_1$ -positive, then 1 is  $x_1$ -positive, which is a contradiction. Therefore  $g^{-1}\Delta^p = h$  and  $\Delta^q g'^{-1} = h'$  for some  $h, h' \in G_1$ . Thus  $g = \theta^p(\Delta_1^p h^{-1})$  and  $g' = (h'^{-1}\Delta_1^q)\theta^q$ . Let  $(h_0, h'_0)$  be the left orthogonal



splitting of  $\Delta_1^p h^{-1}$ , and let  $(h'_1, h_1)$  be the right orthogonal splitting of  $h'^{-1} \Delta_1^q$ . Then  $g = \theta^p h_0^{-1} h'_0$  and  $g' = h'_1 h_1^{-1} \theta^q$ . Since  $h_0 \wedge_L h'_0 = 1$  and  $h_1 \wedge_R h'_1 = 1$ , then  $h_0 \leq_R \theta^p$  and  $h_1 \leq_L \theta^q$ , because  $g, g' \in M$ . Lemma 3.3 implies that  $h_0 = h_1 = 1$ . Hence  $g = \theta^p h'_0$  and  $g' = h'_1 \theta^q = \theta^q h'_1$ . Note that  $\Delta_1 \not\leq_R h'_0, h'_1$  because  $\Delta \not\leq_R g, g'$ . Therefore  $g, g' \in \Theta$ .  $\square$

**Lemma 4.12.** *Let  $h, h' \in M \setminus M_1$  be two unmovable elements, and let  $k = \inf(hh')$ . If  $\Delta^{-k} hh' \in M_1$  and  $\text{dpt}(h) + \text{dpt}(h') = k + 2$ . Then  $h, h' \in \Theta$ .*

PROOF. By Lemma 2.9, there exist  $g, g', g_0, g'_0 \in M$  such that  $h = g_0 g$ ,  $h' = g' g'_0$ , and  $g g' = \Delta^k$ . Then  $\Delta^{-k} hh' = g_0 g'_0$  because  $\Delta$  is central, hence  $g_0, g'_0 \in M_1$ . We have  $\text{dpt}(g') = \text{dpt}(h')$ , and, by Lemma 4.8.4,  $\text{dpt}(g) = \text{dpt}(h)$ . Therefore  $\text{dpt}(g) + \text{dpt}(g') = k + 2$ . Lemma 4.11 implies that  $g, g' \in \Theta$ , i.e.  $g = \theta^p g_1$  and  $g' = \theta^q g'_1$  for some  $p, q \geq 1$  and  $g_1, g'_1 \in M_1$ . Hence  $h = \theta^p (g_0 g_1)$  and  $h' = \theta^q (g'_1 g'_0)$ . Note that  $\Delta_1 \not\leq_R g_0 g_1, g'_1 g'_0$  because  $\Delta \not\leq_R h, h'$ . Therefore  $h, h' \in \Theta$ .  $\square$

**Lemma 4.13.** *Let  $g, g' \in M \setminus (M_1 \cup \Theta)$  be two unmovable elements, and let  $k = \inf(gg')$ . Then  $\text{dpt}(gg') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_k \geq 0$  for some  $\epsilon \in \{0, 1\}$ .*

PROOF. Let  $p, q \geq 1$  be the respective depths of  $g, g'$ . Then  $\text{bh}(g) \in \{2p, 2p + 1\}$  and  $\text{bh}(g') \in \{2q, 2q + 1\}$ . Theorem 2.13 implies that  $\Omega^{-2p} g$  and  $\Omega^{-2q} g'$  are  $x_1$ -negative, and that  $\Omega^{-2p+2} g$  and  $\Omega^{-2q+2} g'$  are either  $x_1$ -positive or belong to  $G_1$ . Then  $\Omega^{-2p-2q} gg' = \Delta^{-p-q+k} g_0$  is  $x_1$ -negative, and  $\Omega^{-2p-2q+4} gg' = \Delta^{-p-q+2+k} g_0$  is either  $x_1$ -positive or belongs to  $G_1$ , where  $g_0 = \Delta^{-k} gg'$ . Note that  $k < p + q$  because  $\Delta^{-p-q+k} g_0 \notin M$ . Hence  $p + q - \tilde{q}_k \geq 0$ . Theorem 2.13 implies that  $2p + 2q - 2 \leq \text{bh}(gg') \leq 2p + 2q + 1$ . Therefore  $p + q - 1 \leq \text{dpt}^*(gg') \leq p + q$ . Assume that  $g_0 \in M_1$ , then  $\text{dpt}^*(gg') = q_k$ , and  $k \geq 1$  because  $g, g' \notin M_1$ . If  $q_k = p + q - 1$ , i.e.  $\text{dpt}(g) + \text{dpt}(g') = k + 2$ , Lemma 4.12 implies that  $g, g' \in \Theta$  which contradicts the enunciate. Therefore  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - q_k$ . Assume that  $g_0 \notin M_1$ . Then, by Lemma 4.8.2,  $p + q - \tilde{q}_k = p + q - 1 - k \leq \text{dpt}(gg') \leq p + q - k = p + q - \tilde{q}_k + 1$ .  $\square$

Thus, Lemmas 4.10 and 4.13 have proved that Condition B holds.

## CHAPTER 5

### Artin groups of dihedral type, the even case

In this chapter  $G$  and  $M$  denote the Artin group and the Artin monoid of type  $I_2(m)$ , respectively, where  $m = 2k$  is even and  $k \geq 2$ . Recall that  $M$  is defined by the monoid presentation:

$$M = \langle x, y \mid (xy)^k = (yx)^k \rangle^+.$$

The group  $G$  is the enveloping group of  $M$ . By Theorem 2.11,  $M$  is a Garside monoid and  $\Delta = x \vee_R y$  is a Garside element of  $M$ . By Brieskorn–Saito [4],  $\Delta$  is central, and  $\Delta = (xy)^k = (yx)^k$ .

Let  $G_1 = \langle y \rangle \simeq \mathbb{Z}$  and  $H = \langle x \rangle \simeq \mathbb{Z}$ . By Theorem 2.11,  $G_1$  and  $H$  are parabolic subgroups of  $(G, \Delta)$ . The subgroup  $G_1$  is associated with  $\Delta_1 = y$ , and  $H$  is associated with  $\Lambda = x$ . Once again, we set  $M_1 = G_1 \cap M \simeq \mathbb{N}$  and  $N = H \cap M \simeq \mathbb{N}$ . Obviously,  $M_1 \cup N$  generates  $M$ , hence  $(H, G_1)$  is a Dehornoy pre-structure.

The main result of the chapter is the following:

**Theorem 5.1.** *The pair  $(H, G_1)$  satisfies Conditions A and B.*

By Theorem 3.2, this implies the following:

**Corollary 5.2.** *The pair  $(H, G_1)$  is a Dehornoy structure.*

We denote by  $P_1$  the set of  $(H, G_1)$ -positive elements of  $G$ , and we set  $P_2 = \{y^n \mid n \geq 1\}$ . For  $\epsilon = (\epsilon_1, \epsilon_2) \in \{\pm 1\}^2$  we set  $P^\epsilon = P_1^{\epsilon_1} \cup P_2^{\epsilon_2}$ . Then, by Proposition 3.1:

**Corollary 5.3.**  *$P^\epsilon$  is the positive cone of a left order on  $G$ .*

We denote by  $z_1, \dots, z_{2k-1}$  the standard generators of the braid group  $\mathcal{B}_{2k}$ . By Crisp [7], the map  $\phi : S \rightarrow \mathcal{B}_{2k}$  defined by:

$$\phi(x) = \prod_{i=0}^{k-1} z_{2i+1}, \quad \phi(y) = \prod_{i=1}^{k-1} z_{2i},$$

induces an embedding  $\phi : G \hookrightarrow \mathcal{B}_{2k}$  satisfying  $\phi(M) \subseteq \mathcal{B}_{2k}^+$ . In a second part of the chapter we will show how the left orders on  $G$  obtained in Corollary 5.3 can be deduced from  $\phi$ . More precisely, we prove the following:

**Proposition 5.4.** *Let  $g \in G$ . Then  $g$  is  $(H, G_1)$ -positive if and only if  $\phi(g)$  is  $z_1$ -positive.*

The rest of the chapter is dedicated to the proofs of Theorem 5.1 and Proposition 5.4. Theorem 5.1 is proved in Section 1, and Proposition 5.4 is proved in Section 2.

### 1. Proof of Theorem 5.1

Note that, if  $g \in M$  is unmovable, then it has a unique word representative in  $\{x, y\}^*$ .

**Lemma 5.5.** *Let  $g, g' \in M$  be two unmovable elements such that  $\inf(gg') = 0$ . Then:*

$$\text{dpt}(gg') = \begin{cases} \text{dpt}(g) + \text{dpt}(g') - 1 & \text{if } x \leq_R g \text{ and } x \leq_L g' \\ \text{dpt}(g) + \text{dpt}(g') & \text{otherwise} \end{cases}$$

PROOF. If  $x \not\leq_R g$  or  $x \not\leq_L g'$  it is obvious that  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g')$ . If  $x \leq_R g$  and  $x \leq_L g'$ , let  $k$  be the greatest integer such that  $x^k \leq_L g'$ . Then  $g' = x^k g'_0$  for some  $g'_0 \in M$  unmovable where  $x \not\leq_L g'_0$ . We have  $\text{dpt}(gx^k) = \text{dpt}(g)$ , and  $\text{dpt}(g'_0) = \text{dpt}(g') - 1$ . Hence  $\text{dpt}(gg') = \text{dpt}(gx^k) + \text{dpt}(g'_0) = \text{dpt}(g) + \text{dpt}(g') - 1$ .  $\square$

Note that  $\theta = (xy)^{k-1}x = x(yx)^{k-1}$  because  $\Delta_1 = y$ .

**Lemma 5.6.** *Let  $g_0 \in M_1$ , and let  $g \in M \setminus M_1$ . Then:*

- (1)  $\text{bh}(g_0\theta) = m$ .
- (2)  $\text{bh}(g\theta) = \text{bh}(g) + m - 2$ .
- (3)  $\text{bh}(\theta^t) = (m - 2)t + 2$  for all  $t \geq 1$ .

PROOF. (1) Note that  $(x, y, \dots, x, 1)$  is the alternating form of  $\theta$ , and  $\text{bh}(\theta) = m$  because  $\text{lg}_S(\theta) = m - 1$ . Lemma 3.3 implies that  $g_0\theta = \theta g_0$ . Hence  $\text{bh}(g_0\theta) = \text{bh}(\theta) = m$ . (2) Let  $g_1$  be the  $M_1$ -tale of  $g$ , and let  $g' = gg_1^{-1}$ . Note that  $g' \neq 1$  because  $g \notin M_1$ . By Lemma 3.3,  $g\theta = g'\theta g_1$ , and  $R_{M_1}(g\theta) = g_1$  because  $y \not\leq_R g'$ . Moreover, if  $h \leq_L \theta$ , and  $a \leq_R h$  for some  $a \in \{x, y\}$ , then  $b \not\leq_R g'h$  because  $y \not\leq_R g'$ , where  $b \in \{x, y\} \setminus \{a\}$ . Therefore  $(g_p, \dots, g_3, g_2x, y, \dots, x, g_1)$  is the alternating form of  $g\theta$ , where  $g' = g_p \cdots g_2$  and  $(g_p, \dots, g_1)$  is the alternating form of

$g$ . So,  $\text{bh}(g\theta) = \text{bh}(g) - 2 + m$ . (3) By (1),  $\text{bh}(\theta) = (m - 2) + 2$ . If  $t > 1$ , we suppose that  $\text{bh}(\theta^{t-1}) = (m - 2)(t - 1) + 2$ . Hence, (2) implies that  $\text{bh}(\theta^t) = \text{bh}(\theta^{t-1}) + m - 2 = (m - 2)(t - 1) + 2 + m - 2 = (m - 2)t + 2$ .  $\square$

Lemma 5.6.3 implies that  $q_t = (k - 1)t + 1$  for all  $k \geq 1$ , because  $\text{bh}(\theta^t)$  is even. Therefore, Condition A holds, where  $\tilde{q}_t = tc + 1$  with  $c = k - 1$  for all  $t \geq 1$ .

Note that, since  $\Delta = (xy)^k$ , then  $\text{dpt}(s) + \text{dpt}(\text{com}_R(s)) = q_1$  for all  $s \in \text{Div}(\Delta) \setminus \{\Delta\}$ .

Since relations of Artin monoids preserve lengths, all words representing an element  $g$  of an Artin monoid  $A$  have the same length. Thus, we may consider the *length* of  $g$ , denoted by  $\text{lg}_S(g)$ , defined as the length of its word representatives in  $S^*$ , where  $S$  is the standard generating set of  $A$ .

**Lemma 5.7.** *Let  $g, g' \in M$  be two unmovable elements, and let  $h_1, h_2$  be two simple elements in  $\text{Div}(\Delta) \setminus \{1, \Delta\}$  such that  $h_2 h_1 \leq_R g$  and  $\text{com}_R(h_1) \text{com}_R(h_2) \leq_L g'$ . If  $a \leq_L h_1$  and  $b \leq_R \text{com}_R(h_1)$ , where  $\{a, b\} = \{x, y\}$ , then:*

$$(1) \ a \leq_R h_2 \text{ and } b \leq_L \text{com}_R(h_2).$$

$$(2) \ \text{dpt}(gh_1^{-1}) + \text{dpt}(\text{com}_R(h_1)^{-1}g') = \text{dpt}(g) + \text{dpt}(g') - q_1 + 1.$$

PROOF. (1) Since  $h_2 \text{com}_R(h_2) = \Delta$ , if  $b \leq_R h_2$ , then  $a \leq_L \text{com}_R(h_2)$ . Similarly, if  $a \leq_R \text{com}_R(h_2)$ , then  $b \leq_R h_2$ . Hence  $h_2 h_1$  and  $\text{com}_R(h_1) \text{com}_R(h_2)$  are simple elements. Note that  $\text{lg}_S(h_2 h_1) < m$  and  $\text{lg}_S(\text{com}_R(h_1) \text{com}_R(h_2)) < m$  because  $g, g'$  are unmovable. Thus,  $\text{lg}_S(\Delta^2) = \text{lg}_S(h_2 h_1 \text{com}_R(h_1) \text{com}_R(h_2)) < 2m$ , which is a contradiction because  $\text{lg}_S(\Delta^2) = 2m$ . Therefore  $a \leq_R h_2$  and  $b \leq_L \text{com}_R(h_2)$ . (2) By Lemma 5.5, we have the following:

$$\text{dpt}(gh_1^{-1}) = \begin{cases} \text{dpt}(g) - \text{dpt}(h_1) & \text{if } a = y \\ \text{dpt}(g) - \text{dpt}(h_1) + 1 & \text{if } a = x \end{cases}$$

$$\text{dpt}(\text{com}_R(h_1)^{-1}g') = \begin{cases} \text{dpt}(g') - \text{dpt}(\text{com}_R(h_1)) + 1 & b = x \\ \text{dpt}(g') - \text{dpt}(\text{com}_R(h_1)) & b = y \end{cases}$$

Therefore  $\text{dpt}(gh_1^{-1}) + \text{dpt}(\text{com}_R(h_1)^{-1}g') = \text{dpt}(g) + \text{dpt}(g') - q_1 + 1$ .  $\square$

**Lemma 5.8.** *Let  $g, g' \in M$  be two unmovable elements such that  $\text{inf}(gg') = 1$ . Then  $\text{dpt}(gg') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - q_1 \geq 0$  for some  $\epsilon \in \{0, 1\}$ .*

PROOF. Let  $h \in M$  be a simple element, such that  $h \leq_R g$  and  $\text{com}_R(h) \leq_L g'$ , let  $g_1 = gh^{-1}$ , and let  $g'_1 = \text{com}_R(h)^{-1}g'$ . By Lemma 5.5, if  $g_1, g'_1 \neq 1$ , we have:

$$\text{dpt}(gg') = \begin{cases} \text{dpt}(g_1) + \text{dpt}(g'_1) - 1 & \text{if } x \leq_R g_1 \text{ and } x \leq_L g'_1 \\ \text{dpt}(g_1) + \text{dpt}(g'_1) & \text{otherwise} \end{cases}$$

and:

$$\text{dpt}(g_1) = \begin{cases} \text{dpt}(g) - \text{dpt}(h) + 1 & \text{if } x \leq_R g_1 \text{ and } x \leq_L h \\ \text{dpt}(g) - \text{dpt}(h) & \text{otherwise} \end{cases}$$

$$\text{dpt}(g'_1) = \begin{cases} \text{dpt}(g') - \text{dpt}(\text{com}_R(h)) + 1 & \text{if } x \leq_R \text{com}_R(h) \text{ and } x \leq_L g'_1 \\ \text{dpt}(g') - \text{dpt}(\text{com}_R(h)) & \text{otherwise} \end{cases}$$

If  $x \leq_L h$  and  $g_1, g'_1 \neq 1$  we obtain the following:

$$\text{dpt}(gg') = \begin{cases} \text{dpt}(g) + \text{dpt}(g') - q_1 + 1 & \text{if } x \leq_R g_1 \text{ and } y \leq_L g'_1 \\ \text{dpt}(g) + \text{dpt}(g') - q_1 & \text{otherwise} \end{cases}$$

If  $y \leq_L h$  and  $g_1, g'_1 \neq 1$  we obtain the following:

$$\text{dpt}(gg') = \begin{cases} \text{dpt}(g) + \text{dpt}(g') - q_1 + 1 & \text{if } y \leq_R g_1 \text{ and } x \leq_L g'_1 \\ \text{dpt}(g) + \text{dpt}(g') - q_1 & \text{otherwise} \end{cases}$$

If  $x \leq_L h$  and  $1 \in \{g_1, g'_1\}$  we obtain the following:

$$\text{dpt}(gg') = \begin{cases} \text{dpt}(g) + \text{dpt}(g') - q_1 + 1 & \text{if } x \leq_R g_1 \\ \text{dpt}(g) + \text{dpt}(g') - q_1 & \text{if } x \not\leq_R g_1 \end{cases}$$

If  $y \leq_L h$  and  $1 \in \{g_1, g'_1\}$  we obtain the following:

$$\text{dpt}(gg') = \begin{cases} \text{dpt}(g) + \text{dpt}(g') - q_1 + 1 & \text{if } x \leq_L g'_1 \\ \text{dpt}(g) + \text{dpt}(g') - q_1 & \text{if } x \not\leq_L g'_1 \end{cases}$$

Therefore  $\text{dpt}(gg') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - q_1$  for some  $\epsilon \in \{0, 1\}$ . If  $\Delta^{-1}gg' \notin M_1$ , then  $\text{dpt}(gg') \geq 1$ . Hence  $\text{dpt}(g) + \text{dpt}(g') - q_1 \geq 0$ . If  $\Delta^{-1}gg' \in M_1$ , then  $\text{dpt}(g) + \text{dpt}(g') = \text{dpt}(h) + \text{dpt}(\text{com}_R(h)) = q_1$ . Thus  $\text{dpt}(g) + \text{dpt}(g') - q_1 \geq 0$ .  $\square$

The following lemma proves Part (a) of Condition B:

**Lemma 5.9.** *Let  $g, g' \in M$  be two unmovable elements with  $t = \inf(gg')$ , such that  $\{g, g'\} \not\subseteq M_1 \cup \Theta$ . Then  $\text{dpt}(gg') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_t \geq 0$  for some  $\epsilon \in \{0, 1\}$ .*

PROOF. If  $t = 0$ , Lemma 5.5 implies that  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_0 + \epsilon$  for some  $\epsilon \in \{0, 1\}$ , where  $\epsilon = 0$  if and only if  $x \leq_R g$  and  $x \leq_L g'$ . Note that  $\text{dpt}(gg') = 0$  if and only if  $g, g' \in M_1$ . Therefore  $\text{dpt}(g) + \text{dpt}(g') - \tilde{q}_0 \geq 0$ . If  $t = 1$ , the result is clear by Lemma 5.8. If  $t \geq 2$ , there exist simple elements  $h_1, \dots, h_t$  such that  $g = g_1 h_t \cdots h_1$  and  $g' = \text{com}_R(h_1) \cdots \text{com}_R(h_t) g'_1$ , for some  $g_1, g'_1 \in M$  such that  $\text{inf}(g_1 g'_1) = 0$ . Lemma 5.7 implies that  $\text{dpt}(g_1 h_t) + \text{dpt}(\text{com}_R(h_t) g'_1) = \text{dpt}(g) + \text{dpt}(g') - q_{t-1} + 1$ . So, by Lemma 5.8,  $\text{dpt}(gg') - \epsilon = \text{dpt}(g_1 g'_1) - \epsilon = \text{dpt}(g_1 h_t) + \text{dpt}(\text{com}_R(h_t) g'_1) - q_1 = \text{dpt}(g) + \text{dpt}(g') - q_t$  for some  $\epsilon \in \{0, 1\}$  with  $\text{dpt}(g) + \text{dpt}(g') - q_t \geq 0$ .  $\square$

The following lemma proves Part (b) of Condition B:

**Lemma 5.10.** *Let  $g, g' \in M$  be two unmovable elements with  $t = \text{inf}(gg')$ , such that  $\{g, g'\} \not\subseteq M_1 \cup \Theta$  and  $\{g, g'\} \cap \Theta \neq \emptyset$ . Then  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_t$ .*

PROOF. Suppose that  $g \in \Theta$ , i.e.  $g = g_0 \theta^s$  for some  $s \geq 1$  and  $g_0 \in M_1$ . Note that  $g_0 = 1$  because  $g$  is unmovable. Hence  $g = \theta^s$ . If  $s < t$ , then  $y^s \Delta^{t-s} \leq_L g'$ , which is impossible because  $g'$  is unmovable. Then  $s \geq t$ , and  $y^t \leq_L g'$ . If  $s = t$ , then  $\text{dpt}(gg') = \text{dpt}(g') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_t$ . If  $s > t$ , Lemma 5.5 implies that  $\text{dpt}(gg') = \text{dpt}(\theta^{s-t}) + \text{dpt}(g') - 1 = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_t$  because  $y \not\leq_L y^{-t} g'$ .  $\square$

## 2. Proof of Proposition 5.4

We denote by  $\Omega$  the standard Garside element of  $\mathcal{B}_{2k}$ , by  $A_1$  the submonoid of  $\mathcal{B}_{2k}^+$  generated by  $z_1, \dots, z_{2k-2}$ , and by  $A_2$  the submonoid of  $\mathcal{B}_{2k}^+$  generated by  $z_2, \dots, z_{2k-1}$ . Note that  $\phi(G_1) \subseteq A_1 \cap A_2$ .

**Proposition 5.11** (Crisp [7]). *Let  $g, g' \in M$ , and let  $(g_p, \dots, g_1)$  be the greedy normal form of  $g$ . Then:*

- (1)  $g \leq_R g'$  if and only if  $\phi(g) \leq_R \phi(g')$ .
- (2)  $\phi(g \wedge_R g') = \phi(g) \wedge_R \phi(g')$  and  $\phi(g \vee_R g') = \phi(g) \vee_R \phi(g')$ .
- (3)  $(\phi(g_p), \dots, \phi(g_1))$  is the greedy normal form of  $\phi(g)$ .

**Lemma 5.12.** *We have  $\phi(g) \in \mathcal{B}_{2k} \setminus \langle z_2, \dots, z_{2k-1} \rangle$  for all  $g \in G \setminus G_1$ .*

PROOF. Let  $h \in \mathcal{B}_{2k}$ , and let  $(h_2, h_1)$  be its right orthogonal splitting, that is  $h = h_1 h_2^{-1}$  and  $h_1 \wedge_R h_2 = 1$ . Since  $G(A_2)$  is a parabolic subgroup, we have  $h \in G(A_2)$  if and only if  $h_1, h_2 \in A_2$ . Let  $g \in G$ , and let  $(g_2, g_1)$  be its right orthogonal splitting. By Proposition 5.11, we have  $\phi(g) = \phi(g_1)\phi(g_2)^{-1}$  and  $\phi(g_1) \wedge_R \phi(g_2) = 1$ . It follows that  $\phi(g) \in G(A_2)$  if and only if  $\phi(g_1), \phi(g_2) \in A_2$ . Furthermore, we have  $g \in G_1$  if and only if  $g_1, g_2 \in M_1$ . So, we only need to prove that  $\phi(g) \in \mathcal{B}_{2k}^+ \setminus A_2$  for all  $g \in M \setminus M_1$ . Let  $h \in \mathcal{B}_{2k}^+$ , and let  $(h_p, \dots, h_1)$  be its right greedy normal form. Since  $G(A_2)$  is a parabolic subgroup, we have  $h \in A_2$  if and only if  $h_i \in A_2$  for all  $i \in \{1, \dots, p\}$ . Let  $g \in M$ , and let  $(g_p, \dots, g_1)$  be its right greedy normal form. By Proposition 5.11,  $\phi(g) = \phi(g_p) \cdots \phi(g_1)$  is the right greedy normal form of  $\phi(g)$ . It follows that  $\phi(g) \in A_2$  if and only if  $\phi(g_i) \in A_2$  for all  $i \in \{1, \dots, p\}$ . Furthermore, we have  $g \in M_1$  if and only if  $g_i \in M_1$  for all  $i \in \{1, \dots, p\}$ . So, we only need to show that  $\phi(g) \in \text{Div}(\Omega) \setminus A_2$  for  $g \in \text{Div}(\Delta) \setminus M_1 = \text{Div}(\Delta) \setminus \text{Div}(x)$ . Let  $h \in \text{Div}(\Omega)$ , and let  $i_1, \dots, i_l \in \{1, \dots, 2k-1\}$  such that  $h = z_{i_1} \cdots z_{i_l}$ . By Bourbaki [3], Chapter IV, Section 1.8, we have  $h \in A_2$  if and only if  $\{z_{i_1}, \dots, z_{i_l}\} \subset \{z_2, \dots, z_{2k-1}\}$ . This implies that, for  $g \in \text{Div}(\Delta)$ , we have  $\phi(g) \in A_2$  if and only if  $g \in \{1, y\} = \text{Div}(y)$ .  $\square$

For each  $i, j \in \{0, \dots, k-1\}$  with  $j \neq 0$ , and for every  $t \geq 1$ , we define:

$$x_{2i+1}^t = \prod_{l=i}^{k-1} z_{2l+1}^t, \quad y_{2j}^t = \prod_{l=j}^{k-1} z_{2l}^t.$$

In particular  $x_1^t = \phi(x^t) = \phi(x)^t$  and  $y_2^t = \phi(y^t) = \phi(y)^t$ . Note that  $y_{2j}^t \in A_1 \cap A_2$ .

**Lemma 5.13.** *Let  $g \in M \setminus M_1$  be an unmovable element such that  $\text{dpt}(g) \leq k-1$ . Then  $\phi(g)$  has a  $(A_1, A_2)$ -sequence and a  $(A_2, A_1)$ -sequence of length 2.*

PROOF. Let  $d = \text{dpt}(g)$ . Then  $g = y^{t_d} x^{s_d} \cdots y^{t_1} x^{s_1} y^{t_0}$ , where  $s_1, \dots, s_d \geq 1$ ,  $t_0, t_d \geq 0$ , and  $t_1, \dots, t_{d-1} \geq 1$  if  $d \geq 2$ . Since  $d \leq k - 1$ , then:

$$\begin{aligned}
\phi(g) &= y_2^{t_d} x_1^{s_d} \cdots x_1^{s_2} y_2^{t_1} x_1^{s_1} \cdot y_2^{t_0} \\
&= y_2^{t_d} x_1^{s_d} \cdots x_1^{s_2} y_2^{t_1} z_1^{s_1} \cdot x_3^{s_1} y_2^{t_0} \\
&= y_2^{t_d} x_1^{s_d} \cdots x_1^{s_2} z_2^{t_1} z_1^{s_1} \cdot y_4^{t_1} x_3^{s_1} y_2^{t_0} \\
&= y_2^{t_d} x_1^{s_d} \cdots z_3^{s_2} z_1^{s_2} z_2^{t_1} z_1^{s_1} \cdot x_5^{s_2} y_4^{t_1} x_3^{s_1} y_2^{t_0} \\
&\quad \dots \\
&= y_2^{t_d} z_{2d-1}^{s_d} \cdots z_3^{s_d} z_1^{s_d} \cdots z_3^{s_2} z_1^{s_2} z_2^{t_1} z_1^{s_1} \cdot x_{2d+1}^{s_d} \cdots x_5^{s_2} y_4^{t_1} x_3^{s_1} y_2^{t_0}
\end{aligned}$$

Let  $h' = y_2^{t_d} z_{2d-1}^{s_d} \cdots z_3^{s_d} z_1^{s_d} \cdots z_3^{s_2} z_1^{s_2} z_2^{t_1} z_1^{s_1}$ , and let  $h = x_{2d+1}^{s_d} \cdots x_5^{s_2} y_4^{t_1} x_3^{s_1} y_2^{t_0}$ . Note that  $h \in A_2$  and  $h' \in A_1$ . Therefore  $(h', h)$  is a  $(A_1, A_2)$ -sequence of  $\phi(g)$ . Let  $\Phi$  be the flip automorphism of  $\mathcal{B}_{2k}$ . Then  $\phi(g) = \Phi(\phi(g)) = \Phi(h')\Phi(h)$  because  $\phi(\Delta)$  is the Garside element of  $\mathcal{B}_{2k}$ . Note that  $\Phi(h') \in A_2$  and  $\Phi(h) \in A_1$ . Therefore  $(\Phi(h'), \Phi(h))$  is a  $(A_2, A_1)$ -sequence of  $\phi(g)$ . Both sequences are of length 2.  $\square$

The *limit* of an unmovable element  $g \in M \setminus M_1$ , denoted by  $\lim(g)$ , is the unique integer  $t \geq 1$  such that  $\tilde{q}_{t-1} < \text{dpt}(g) + 1 \leq \tilde{q}_t$ . Since  $\tilde{q}_{t-1} - 1 < \text{dpt}(g)$ , then:

$$\lim(g) = \frac{2\tilde{q}_{t-1} + m - 4}{m - 2} < \frac{2\text{dpt}(g) + m - 2}{m - 2}. \quad (5.1)$$

We set  $\lim(g_0) = 0$  for all  $g_0 \in M_1$ . Thus  $\text{bh}_{A_1, A_2}(\phi(g_0)) = \lim(g_0) + 1$ .

**Lemma 5.14.** *Let  $g \in M \setminus M_1$  be an unmovable element. Then  $\phi(g)$  has a  $(A_1, A_2)$ -sequence of length  $\lim(g) + 1$ .*

PROOF. We have  $g = y^{t_d} x^{s_d} \cdots y^{t_1} x^{s_1} y^{t_0}$  with  $d = \text{dpt}(g)$ , where  $s_1, \dots, s_d \geq 1$ ,  $t_0, t_d \geq 0$ , and  $t_1, \dots, t_{d-1} \geq 1$  if  $d \geq 2$ . Let  $t = \lim(g)$ . For every  $i \in \{1, \dots, t\}$ , we define an element  $g_i \in M$ , as follows:

$$g_i = \begin{cases} y^{t^{i(k-1)}} x^{s^{i(k-1)}} \cdots y^{t^{(i-1)(k-1)+1}} x^{s^{(i-1)(k-1)+1}} & \text{if } i < t \\ y^{t_d} x^{s_d} \cdots y^{t^{(t-1)(k-1)+1}} x^{s^{(t-1)(k-1)+1}} & \text{if } i = t \end{cases}$$

Note that  $\text{dpt}(g_i) = k - 1$  for all  $i < t$ , and  $\text{dpt}(g_t) \in \{1, \dots, k - 1\}$ . Lemma 5.13 implies that there are  $h_1, \dots, h_t \in A_2$  and  $h'_1, \dots, h'_t \in A_1$ , such that  $(h'_i, h_i)$  is a  $(A_1, A_2)$ -sequence of



$g_i$  if  $i$  is odd, and  $(h_i, h'_i)$  is a  $(A_2, A_1)$ -sequence of  $g_i$  if  $i$  is even. Therefore, the following is a  $(A_1, A_2)$ -sequence of  $\phi(g)$ :

$$\begin{cases} (h'_t, h_t h_{t-1}, \dots, h_3 h_2, h'_2 h'_1, h_1 y^{t_0}) & \text{if } t \text{ is odd} \\ (h_t, h'_t h'_{t-1}, \dots, h_3 h_2, h'_2 h'_1, h_1 y^{t_0}) & \text{if } t \text{ is even} \end{cases}$$

Note that, the length of this sequence is  $t + 1$ .  $\square$

Proposition 4.6 and Lemma 5.14 imply that  $\text{bh}_{A_1, A_2}(\phi(g)) \leq \text{lim}(g) + 1$  for all unmovable element  $g \in M \setminus M_1$ .

**Lemma 5.15.** *Let  $g' \in G$  such that  $\phi(g')$  is  $z_1$ -positive. Then  $g'$  is  $(H, G_1)$ -positive.*

PROOF. Let  $m = \text{inf}(g')$  such that  $g' = \Delta^m g$  for some unmovable element  $g \in M$ . Note that  $g' \notin G_1$  because  $\phi(g')$  is  $z_1$ -positive. We have  $\phi(g') = \phi(\Delta)^m \phi(g)$ . If  $m \geq 0$ , then  $g' \in M \setminus M_1$ . Therefore  $g'$  is  $(H, G_1)$ -positive. If  $m < 0$ , namely  $m = -t$  for some  $t \geq 1$ , Theorem 2.13 and Equation (5.1) imply that  $t + 2 \leq \text{bh}_{A_1, A_2}(\phi(g)) \leq \text{lim}(g) + 1 < (2\text{dpt}(g) + 2(m - 2))/(m - 2)$ . Hence  $(k - 1)t < \text{dpt}(g)$ , i.e.  $\text{dpt}(g) \geq q_t$ . Therefore  $g'$  is  $(H, G_1)$ -positive.  $\square$

If  $g \in G$  is  $(H, G_1)$ -positive and  $\phi(g)$  is  $z_1$ -negative. Then  $\phi(g)^{-1}$  is  $z_1$ -positive. So, Lemma 5.15 implies that  $g^{-1}$  is  $(H, G_1)$ -positive as well, which contradicts Corollary 5.2. This proves Proposition 5.4.

We finish this section with the following consequence:

**Corollary 5.16.** *We have  $\text{bh}_{A_1, A_2}(\phi(g)) = \text{lim}(g) + 1$  for all unmovable  $g \in M \setminus M_1$ .*

PROOF. Let  $g_0 \in M_1$  be the  $M_1$ -tale of  $g$ , and let  $g' = g g_0^{-1}$ . By definition,  $\tilde{q}_{t-1} < \text{dpt}(g) = \text{dpt}(g')$  where  $t = \text{lim}(g)$ . Then  $\Delta^{-t+1} g'$  is  $(H, G_1)$ -positive. So, Proposition 5.4 implies that  $\phi(\Delta^{-t+1} g')$  is  $z_1$ -positive. By Theorem 2.13 and Lemma 5.14,  $t + 1 \leq \text{bh}_{A_1, A_2}(\phi(g')) \leq t + 1$ . Therefore  $\text{bh}_{A_1, A_2}(\phi(g)) = \text{lim}(g) + 1$ .  $\square$

## CHAPTER 6

### Artin groups of dihedral type, the odd case

In this chapter  $G$  and  $M$  denote the Artin group and the Artin monoid of type  $I_2(m)$ , respectively, where  $m = 2k + 1$  is odd and  $k \geq 2$ . Recall that  $M$  is defined by the monoid presentation

$$M = \langle x, y \mid (xy)^k x = (yx)^k y \rangle^+.$$

The group  $G$  is the enveloping group of  $M$ . By Theorem 2.11,  $M$  is a Garside monoid and  $\Omega = x \vee_R y$  is a Garside element of  $M$ .

What makes the difference between the case  $m$  even, treated in Chapter 5, and the case  $m$  odd, treated in the present chapter, is that, now, the Garside element  $\Omega = (xy)^k x = (yx)^k y$  is not central. It satisfies  $\Omega x = y \Omega$  and  $\Omega y = x \Omega$ , hence  $\Delta = \Omega^2 = (xy)^m$  is central. In order to apply our criteria, we will use the Garside element  $\Delta$  and not  $\Omega$ , making the proofs more complicated than in Chapter 5, although the statements are identical.

Let  $G_1 = \langle y \rangle \simeq \mathbb{Z}$ , and let  $H = \langle x \rangle \simeq \mathbb{Z}$ . By Theorem 2.11,  $G_1$  and  $H$  are parabolic subgroups of  $(G, \Delta)$ . The subgroup  $G_1$  is associated with  $\Delta_1 = y^2$ , and  $H$  is associated with  $\Lambda = x^2$ . Once again, we set  $M_1 = G_1 \cap M \simeq \mathbb{N}$  and  $N = H \cap M \simeq \mathbb{N}$ . Obviously,  $M_1 \cup N$  generates  $M$ , hence  $(H, G_1)$  is a Dehornoy pre-structure.

The main result of the chapter is the following:

**Theorem 6.1.** *The pair  $(H, G_1)$  satisfies Conditions A and B.*

Then, applying Theorem 3.2 we obtain:

**Corollary 6.2.** *The pair  $(H, G_1)$  is a Dehornoy structure.*

We denote by  $P_1$  the set of  $(H, G_1)$ -positive elements of  $G$ , and we set  $P_2 = \{y^n \mid n \geq 1\}$ . For  $\epsilon = (\epsilon_1, \epsilon_2) \in \{\pm 1\}^2$  we set  $P^\epsilon = P_1^{\epsilon_1} \cup P_2^{\epsilon_2}$ . Then, by Proposition 3.1:

**Corollary 6.3.** *The set  $P^\epsilon$  is the positive cone of a left order on  $G$ .*

We denote by  $z_1, \dots, z_{2k}$  the standard generators of the braid group  $\mathcal{B}_{2k+1}$ . By Crisp [7], the map  $\phi : S \rightarrow \mathcal{B}_{2k+1}$  defined by

$$\phi(x) = \prod_{i=0}^{k-1} z_{2i+1}, \quad \phi(y) = \prod_{i=1}^k z_{2i},$$

induces an embedding  $\phi : G \hookrightarrow \mathcal{B}_{2k+1}$ . As in the case  $m$  even, we will show how the left orders on  $G$  obtained in Corollary 6.3 can be deduced from  $\phi$ . More precisely, we will prove the following:

**Proposition 6.4.** *Let  $g \in G$ . Then  $g$  is  $(H, G_1)$ -positive if and only if  $\phi(g)$  is  $z_1$ -positive.*

The rest of the chapter is dedicated to the proofs of Theorem 6.1 and Proposition 6.4. Theorem 6.1 is proved in Section 1, and Proposition 6.4 is proved in Section 2.

### 1. Proof of Theorem 6.1

Note that, if  $g \in M$  and  $\Omega \not\leq_R g$ , then  $g$  does not contain  $x \vee_L y$  as a subword. Therefore  $g$  has a unique word representative in  $\{x, y\}^*$ . We will denote by  $\Phi$  the flip automorphism of  $G$  respective to  $\Omega$ . Note that, if  $g \in M \setminus \{1\}$  such that  $\Omega \not\leq_R g$ , then  $\text{bh}(\Phi(g)) = \text{bh}(g) + 1$  when  $y \leq_R g$ , and  $\text{bh}(\Phi(g)) = \text{bh}(g) - 1$  when  $x \leq_R g$ .

**Lemma 6.5.** *Let  $g, g' \in M$  such that  $\Omega \not\leq_R g'$ ,  $\Omega \not\leq_R g$ , and  $\Omega \not\leq_R gg'$ . Then:*

$$\text{bh}(gg') = \begin{cases} \text{bh}(g) + \text{bh}(g') & \text{if } y \leq_R g \text{ and } x \leq_L g' \\ \text{bh}(g) + \text{bh}(g') - 2 & \text{if } x \leq_R g \text{ and } x \leq_L g' \\ \text{bh}(g) + \text{bh}(g') - 1 & \text{otherwise} \end{cases}$$

Thus,  $\text{bh}(gg') \geq \text{bh}(g), \text{bh}(g')$ . Consequently:

$$\text{bh}(\Phi(g)g') = \begin{cases} \text{bh}(g) + \text{bh}(g') & \text{if } y \leq_R g \text{ and } y \leq_L g' \\ \text{bh}(g) + \text{bh}(g') - 2 & \text{if } x \leq_R g \text{ and } y \leq_L g' \\ \text{bh}(g) + \text{bh}(g') - 1 & \text{otherwise} \end{cases}$$

PROOF. If  $y \leq_R g$  and  $x \leq_L g'$ , then  $\text{bh}(gg') = \text{bh}(g) + \text{bh}(g')$ . If  $y \leq_L g'$ , then  $\text{bh}(gg') = \text{bh}(gy^k) + \text{bh}(y^{-k}g') = \text{bh}(g) + \text{bh}(g') - 1$ , where  $k \geq 1$  is the maximal integer such that  $y^k \leq_L g'$ . If  $x \leq_R g$  and  $x \leq_L g'$ , then  $\text{bh}(gg') = \text{bh}(gx^{-k}) + \text{bh}(x^k g') = \text{bh}(g) - 2 + \text{bh}(g')$ , where  $k \geq 1$  is the maximal integer such that  $x^k \leq_R g$ .  $\square$

Let  $\vartheta = \Omega y^{-1} = (yx)^k$ . Note that  $\theta = (xy)^k(yx)^k = \Phi(\vartheta)\vartheta$  because  $\Delta_1 = y^2$ .

**Lemma 6.6.** *Let  $g_0 \in M_1$ , and let  $g \in M \setminus M_1$ . Then:*

- (1)  $\text{bh}(g_0\theta) = 2(m-1)$ .
- (2)  $\text{bh}(g\theta) = \text{bh}(g) + 2(m-2)$ .
- (3)  $\text{bh}(\theta^t) = 2(m-2)t + 2$  for all  $t \geq 1$ .

PROOF. (1) Lemma 6.5 implies that  $\text{bh}(\theta) = \text{bh}(\Phi(\vartheta)) + \text{bh}(\vartheta) - 1 = 2(m-1) = 4k$ . So,  $\text{bh}(g_0\theta) = 2(m-1)$  because  $\theta g_0 = g_0\theta$ . (2) Let  $k \geq 0$  be the maximal integer such that  $y^k \leq_R g$ . Since  $y \not\leq_R gy^{-k}$  then  $\Omega \not\leq_R gy^{-k}$ . By Lemma 3.3,  $y \not\leq_R gy^{-k}\theta$ . Then  $\Omega \not\leq_R gy^{-k}\theta$  as well. So, Lemma 6.5 implies that  $\text{bh}(g\theta) = \text{bh}(gy^{-k}\theta) = \text{bh}(g) + 2(m-1) - 2 = \text{bh}(g) + 2(m-2)$ . (3) By (1),  $\text{bh}(\theta) = 2(m-2) + 2$ . If  $t > 1$ , we suppose that  $\text{bh}(\theta^{t-1}) = 2(m-2)(t-1) + 2$ . Hence, (2) implies that  $\text{bh}(\theta^t) = \text{bh}(\theta^{t-1}) + 2(m-2) = 2(m-2)(t-1) + 2 + 2(m-2) = 2(m-2)t + 2$ .  $\square$

Lemma 6.6.3 implies that  $q_t = (m-2)t + 1$  for all  $k \geq 1$ , because  $\text{bh}(\theta^t)$  is even. Therefore, Condition A holds, where  $\tilde{q}_t = tc + 1$  with  $c = m-2$  for all  $t \geq 0$ .

**Lemma 6.7.** *Let  $g \in M \setminus \{1\}$  such that  $\Omega \not\leq_R g$ . Then:*

- (1)  $\text{bh}(g\Omega) = \text{bh}(g) + (m-1)$  if  $y \leq_R g$ .
- (2)  $\text{bh}(g\Omega) = \text{bh}(g) + (m-2)$  if  $g \in N$ .
- (3)  $\text{bh}(g\Omega) = \text{bh}(g) + (m-3)$  if  $x \leq_R g$  and  $g \notin N$ .
- (4)  $\text{bh}(\Omega g) = \text{bh}(g) + (m-2)$  if  $x \leq_R g$ .
- (5)  $\text{bh}(\Omega g) = \text{bh}(g) + (m-1)$  if  $g \in M_1$ .
- (6)  $\text{bh}(\Omega g) = \text{bh}(g) + (m-2)$  if  $y \leq_R g$  and  $g \notin M_1$ .

PROOF. Note that  $\text{bh}(\vartheta) = m$ . (1) By Lemma 6.5, we have  $\text{bh}(g\Omega) = \text{bh}(g\vartheta) = \text{bh}(g) + m - 1$ . (2) We have  $\text{bh}(g\Omega) = \text{bh}(\Omega) = m = \text{bh}(g) + m - 2$  because  $\text{bh}(g) = 2$ . (3) Let  $k \geq 1$  be the maximal integer such that  $x^k \leq_R g$ . Lemma 6.5 implies that  $\text{bh}(g\Omega) = \text{bh}(gx^{-k}\Omega y^k) = \text{bh}(gx^{-k}\vartheta) = \text{bh}(gx^{-k}) + m - 1 = \text{bh}(g) + m - 3$ . (4) Since  $y \leq_R \Phi(g)$ , (1) implies that  $\text{bh}(\Omega g) = \text{bh}(\Phi(g)\Omega) = \text{bh}(\Phi(g)) + m - 1 = \text{bh}(g) + m - 2$ . (5) We have  $\Phi(g) \in N$ . So, by (2),  $\text{bh}(\Omega g) = \text{bh}(\Phi(g)\Omega) = \text{bh}(\Phi(g)) + m - 2 = \text{bh}(g) + m - 1$ . (6) Since  $x \leq_R \Phi(g)$  and  $\Phi(g) \notin N$ , (3) implies that  $\text{bh}(\Omega g) = \text{bh}(\Phi(g)\Omega) = \text{bh}(\Phi(g)) + m - 3 = \text{bh}(g) + m - 2$ .  $\square$

Thus, we obtain the depths of the unmovable elements in  $M$  which are divisible by  $\Omega$ :

**Corollary 6.8.** *Let  $g \in M \setminus \{1\}$  such that  $\Omega \not\leq_R g$ . Then:*

- (1)  $\text{dpt}(g\Omega) = \text{dpt}(g) + (m - 1)/2$  if  $y \leq_R g$ .
- (2)  $\text{dpt}(g\Omega) = \text{dpt}(g) + (m - 3)/2$  if  $x \leq_R g$ .
- (3)  $\text{dpt}(\Omega g) = \text{dpt}(g) + (m - 1)/2$  if  $y \leq_L g$ .
- (4)  $\text{dpt}(\Omega g) = \text{dpt}(g) + (m - 3)/2$  if  $x \leq_L g$ .

PROOF. (1) and (2) are direct consequences of Lemma 6.7.1-3 because  $m - 1, m - 3$  are even and  $\text{bh}(g) = 2$  for all  $g \in N$ . (3) Note that  $\text{bh}(g)$  is odd because  $y \leq_L g$ . If  $g \notin M_1$ , Lemma 6.7.4,6 implies that  $\text{dpt}(\Omega g) = (\text{bh}(g) - 1 + m - 1)/2 = \text{dpt}(g) + (m - 1)/2$ . If  $g \in M_1$ , Lemma 6.7.5 implies that  $\text{dpt}(\Omega g) = (m - 1)/2 = \text{dpt}(g) + (m - 1)/2$  because  $\text{bh}(g) = 1$ . (4) Note that  $\text{bh}(g)$  is even because  $x \leq_L g$ . Finally, by Lemma 6.7.4,6,  $\text{dpt}(\Omega g) = (\text{bh}(g) + m - 3)/2 = \text{dpt}(g) + (m - 3)/2$ .  $\square$

More generally, we have the following lemma:

**Lemma 6.9.** *Let  $g_0 \in M_1$ , let  $g \in M \setminus M_1$  such that  $\Omega \not\leq_R g$ , and let  $t \geq 2$ . Then:*

- (1)  $\text{bh}(g_0\Omega^t) = \text{bh}(\Omega^t g_0) = \text{bh}(g_0) + (m - 2)t + 1$ .
- (2)  $\text{bh}(g\Omega^t) = \text{bh}(\Omega^t g) = \text{bh}(g) + (m - 2)t$  when  $t$  is even.
- (3)  $\text{bh}(g\Omega^t) = \text{bh}(g) + (m - 2)t + 1$  when  $t$  is odd and  $y \leq_R g$ .
- (4)  $\text{bh}(g\Omega^t) = \text{bh}(g) + (m - 2)t$  when  $t$  is odd and  $g \in N$ .
- (5)  $\text{bh}(g\Omega^t) = \text{bh}(g) + (m - 2)t - 1$  when  $t$  is odd,  $x \leq_R g$  and  $g \notin N$ .
- (6)  $\text{bh}(\Omega^t g) = \text{bh}(g) + (m - 2)t$  when  $t$  is odd.

In particular  $\text{bh}(\Omega^r) = (m - 2)r + 2$  for all  $r \geq 1$ .

PROOF. (1) If  $t$  is even, Lemma 6.6.1,3 implies that  $\text{bh}(g_0\Omega^t) = \text{bh}(g_0\theta^{t/2}) = (m - 2)t + 2$ . If  $t$  is odd, Lemmas 6.6.3 and 6.7.1 imply that  $\text{bh}(g_0\Omega^t) = \text{bh}(g_0\Omega\theta^{(t-1)/2}) = \text{bh}(g_0\Omega) + (m - 2)(t - 1) = \text{bh}(g_0) + (m - 1) + (m - 2)(t - 1)$ . Lemmas 6.6.3 and 6.7.5 imply that  $\text{bh}(\Omega^t g_0) = \text{bh}(\Omega g_0\Omega^{t-1}) = \text{bh}(\Omega g_0\theta^{(t-1)/2}) = \text{bh}(\Omega g_0) + (m - 2)(t - 1) = \text{bh}(g_0) + (m - 1) + (m - 2)(t - 1)$ . (2) Lemma 6.6.2 implies that  $\text{bh}(g\Omega^t) = \text{bh}(g\theta^{t/2}) = \text{bh}(g) + (m - 2)t$ . (3) Lemmas 6.6.2 and 6.7.1 imply that  $\text{bh}(g\Omega^t) = \text{bh}(g\Omega\theta^{(t-1)/2}) = \text{bh}(g) + (m - 1) + (m - 2)(t - 1)$ . (4) Lemmas 6.6.2 and 6.7.2 imply that  $\text{bh}(g\Omega^t) = \text{bh}(g\Omega\theta^{(t-1)/2}) = \text{bh}(g) + (m - 2)t$ . (5) By Lemmas 6.6.2

and 6.7.3,  $\text{bh}(g\Omega^t) = \text{bh}(g\Omega\theta^{(t-1)/2}) = \text{bh}(g) + (m-3) + (m-2)(t-1)$ . (6) Lemmas 6.6.2 and 6.7.4,6 imply that  $\text{bh}(\Omega^t g) = \text{bh}(\Omega g\theta^{(t-1)/2}) = \text{bh}(g) + (m-2) + (m-2)(t-1)$ .  $\square$

Note that, if  $s, s' \in \text{Div}(\Omega) \setminus \{1, \Omega\}$  such that  $ss' = \Omega$ , then:

$$\text{bh}(s) + \text{bh}(s') = \begin{cases} m & \text{if } y \leq_L s \text{ and } y \leq_R s \\ m+2 & \text{if } x \leq_L s \text{ and } x \leq_R s \\ m+1 & \text{otherwise} \end{cases} \quad (6.1)$$

If  $s, s' \in \text{Div}(\Omega) \setminus \{1, \Omega\}$  such that  $\Phi(s)s' = \Omega$ , Equation (6.1) implies that:

$$\text{bh}(\Phi(s)) + \text{bh}(s') = \begin{cases} m & \text{if } x \leq_L s \text{ and } x \leq_R s \\ m+2 & \text{if } y \leq_L s \text{ and } y \leq_R s \\ m+1 & \text{otherwise} \end{cases}$$

Thus:

$$\text{bh}(s) + \text{bh}(s') = \begin{cases} m & \text{if } x \leq_L s \text{ and } y \leq_R s \\ m+2 & \text{if } y \leq_L s \text{ and } x \leq_R s \\ m+1 & \text{otherwise} \end{cases} \quad (6.2)$$

Let  $g, g' \in M$  such that  $\Omega \not\leq_R g, g'$  and  $gg' \neq \Omega$ , and let  $s, s' \in \text{Div}(\Omega) \setminus \{1, \Omega\}$  such that  $s \leq_R g, s' \leq_L g'$ , and  $ss' = \Omega$ . Note that,  $x \leq_L s$  if and only if  $x \leq_R s'$ . Further,  $x \leq_R s$  if and only if  $y \leq_L s'$ . In Lemmas 6.10 to 6.13 we will denote by  $T$  the sum  $\text{bh}(gs^{-1}) + \text{bh}(s'^{-1}g')$  and by  $T'$  the sum  $\text{bh}(\Phi(gs^{-1})) + \text{bh}(s'^{-1}g')$ .

**Lemma 6.10.** *If  $y \leq_L s$  and  $y \leq_R s$ , then:*

$$T = \begin{cases} \text{bh}(g) + \text{bh}(g') - m + 1 & \text{if } x \leq_L s'^{-1}g' \\ \text{bh}(g) + \text{bh}(g') - m + 2 & \text{if } x \not\leq_L s'^{-1}g' \end{cases}$$

Consequently:

$$T' = \begin{cases} \text{bh}(g) + \text{bh}(g') - m & \text{if } x \leq_L s'^{-1}g' \text{ and } x \leq_R gs^{-1} \\ \text{bh}(g) + \text{bh}(g') - m + 2 & \text{if } x \leq_L s'^{-1}g' \text{ and } y \leq_R gs^{-1} \\ \text{bh}(g) + \text{bh}(g') - m + 1 & \text{if } x \leq_L s'^{-1}g' \text{ and } g = s \\ \text{bh}(g) + \text{bh}(g') - m + 1 & \text{if } x \not\leq_L s'^{-1}g' \text{ and } x \leq_R gs^{-1} \\ \text{bh}(g) + \text{bh}(g') - m + 3 & \text{if } x \not\leq_L s'^{-1}g' \text{ and } y \leq_R gs^{-1} \\ \text{bh}(g) + \text{bh}(g') - m + 2 & \text{if } x \not\leq_L s'^{-1}g' \text{ and } g = s \end{cases}$$

PROOF. Equation (6.1) implies that  $\text{bh}(s) + \text{bh}(s') = m$ . Since  $y \leq_L s$ , Lemma 6.5 implies that  $\text{bh}(gs^{-1}) = \text{bh}(g) - \text{bh}(s) + 1$ . If  $y \leq_L s'^{-1}g'$ , Lemma 6.5 implies that  $\text{bh}(s'^{-1}g') = \text{bh}(g') - \text{bh}(s') + 1$ . Hence  $T = \text{bh}(g) + \text{bh}(g') - m + 2$ . If  $x \leq_L s'^{-1}g'$ , Lemma 6.5 implies that  $\text{bh}(s'^{-1}g') = \text{bh}(g') - \text{bh}(s')$ . Therefore  $T = \text{bh}(g) + \text{bh}(g') - m + 1$ . If  $g' = s'$ , then  $T = \text{bh}(g) + \text{bh}(g') - m + 2$  because  $\text{bh}(s'^{-1}g') = 1$ . We obtain  $T'$  by replacing  $\text{bh}(gs^{-1}) = \text{bh}(\Phi(gs^{-1})) \pm 1$  for each case.  $\square$

**Lemma 6.11.** *If  $y \leq_L s$  and  $x \leq_R s$ , then:*

$$T = \begin{cases} \text{bh}(g) + \text{bh}(g') - m & \text{if } x \leq_L s'^{-1}g' \\ \text{bh}(g) + \text{bh}(g') - m + 1 & \text{if } x \not\leq_L s'^{-1}g' \end{cases}$$

*Consequently:*

$$T' = \begin{cases} \text{bh}(g) + \text{bh}(g') - m - 1 & \text{if } x \leq_L s'^{-1}g' \text{ and } x \leq_R gs^{-1} \\ \text{bh}(g) + \text{bh}(g') - m + 1 & \text{if } x \leq_L s'^{-1}g' \text{ and } y \leq_R gs^{-1} \\ \text{bh}(g) + \text{bh}(g') - m & \text{if } x \leq_L s'^{-1}g' \text{ and } g = s \\ \text{bh}(g) + \text{bh}(g') - m & \text{if } x \not\leq_L s'^{-1}g' \text{ and } x \leq_R gs^{-1} \\ \text{bh}(g) + \text{bh}(g') - m + 2 & \text{if } x \not\leq_L s'^{-1}g' \text{ and } y \leq_R gs^{-1} \\ \text{bh}(g) + \text{bh}(g') - m + 1 & \text{if } x \not\leq_L s'^{-1}g' \text{ and } g = s \end{cases}$$

PROOF. Equation (6.1) implies that  $\text{bh}(s) + \text{bh}(s') = m + 1$ . Since  $y \leq_L s$ , Lemma 6.5 implies that  $\text{bh}(gs^{-1}) = \text{bh}(g) - \text{bh}(s) + 1$ . If  $y \leq_L s'^{-1}g'$ , Lemma 6.5 implies that  $\text{bh}(s'^{-1}g') = \text{bh}(g') - \text{bh}(s') + 1$ . Hence  $T = \text{bh}(g) + \text{bh}(g') - m + 1$ . If  $x \leq_L s'^{-1}g'$ , Lemma 6.5 implies that  $\text{bh}(s'^{-1}g') = \text{bh}(g') - \text{bh}(s')$ . Therefore  $T = \text{bh}(g) + \text{bh}(g') - m$ . If  $g' = s'$ , then  $T = \text{bh}(g) + \text{bh}(g') - m + 1$  because  $\text{bh}(s'^{-1}g') = 1$ . We obtain  $T'$  by replacing  $\text{bh}(gs^{-1}) = \text{bh}(\Phi(gs^{-1})) \pm 1$  for each case.  $\square$

**Lemma 6.12.** *If  $x \leq_L s$  and  $y \leq_R s$ , then:*

$$T = \begin{cases} \text{bh}(g) + \text{bh}(g') - m + 1 & \text{if } y \leq_R gs^{-1} \text{ and } x \leq_L s'^{-1}g' \\ \text{bh}(g) + \text{bh}(g') - m & \text{if } y \leq_R gs^{-1} \text{ and } x \not\leq_L s'^{-1}g' \\ \text{bh}(g) + \text{bh}(g') - m + 3 & \text{if } x \leq_R gs^{-1} \text{ and } x \leq_L s'^{-1}g' \\ \text{bh}(g) + \text{bh}(g') - m + 2 & \text{if } x \leq_R gs^{-1} \text{ and } x \not\leq_L s'^{-1}g' \\ \text{bh}(g) + \text{bh}(g') - m + 2 & \text{if } g = s \text{ and } x \leq_L s'^{-1}g' \\ \text{bh}(g) + \text{bh}(g') - m + 1 & \text{if } g = s \text{ and } y \leq_L s'^{-1}g' \end{cases}$$

*Consequently:*

$$T' = \begin{cases} \text{bh}(g) + \text{bh}(g') - m + 2 & \text{if } x \leq_L s'^{-1}g' \\ \text{bh}(g) + \text{bh}(g') - m + 1 & \text{if } x \not\leq_L s'^{-1}g' \end{cases}$$

PROOF. Equation (6.1) implies that  $\text{bh}(s) + \text{bh}(s') = m + 1$ . Assume that  $y \leq_R gs^{-1}$ , then  $\text{bh}(gs^{-1}) = \text{bh}(g) - \text{bh}(s)$ . If  $x \leq_L s'^{-1}g'$ , Lemma 6.5 implies that  $\text{bh}(s'^{-1}g') = \text{bh}(g') - \text{bh}(s') + 2$ . Therefore  $T = \text{bh}(g) + \text{bh}(g') - m + 1$ . If  $x \not\leq_L s'^{-1}g'$ , Lemma 6.5 implies that  $\text{bh}(s'^{-1}g') = \text{bh}(g') - \text{bh}(s') + 1$ . Hence  $T = \text{bh}(g) + \text{bh}(g') - m$ . Assume that  $x \leq_R gs^{-1}$ , then  $\text{bh}(gs^{-1}) = \text{bh}(g) - \text{bh}(s) + 2$ . If  $x \leq_L s'^{-1}g'$ , Lemma 6.5 implies that  $\text{bh}(s'^{-1}g') = \text{bh}(g') - \text{bh}(s') + 2$ . Thus  $T = \text{bh}(g) + \text{bh}(g') - m + 3$ . If  $x \not\leq_L s'^{-1}g'$ , Lemma 6.5 implies that  $\text{bh}(s'^{-1}g') = \text{bh}(g') - \text{bh}(s') + 1$ . Therefore  $T = \text{bh}(g) + \text{bh}(g') - m + 2$ . Assume that  $g = s$ , then  $\text{bh}(gs^{-1}) = \text{bh}(g) - \text{bh}(s) + 1$ . If  $x \leq_L s'^{-1}g'$ , Lemma 6.5 implies that  $\text{bh}(s'^{-1}g') = \text{bh}(g') - \text{bh}(s') + 2$ . Hence  $T = \text{bh}(g) + \text{bh}(g') - m + 2$ . If  $y \leq_L s'^{-1}g'$ , Lemma 6.5 implies that  $\text{bh}(s'^{-1}g') = \text{bh}(g') - \text{bh}(s') + 1$ . Therefore  $T = \text{bh}(g) + \text{bh}(g') - m + 1$ . We obtain  $T'$  by replacing  $\text{bh}(gs^{-1}) = \text{bh}(\Phi(gs^{-1})) \pm 1$  for each case.  $\square$

**Lemma 6.13.** *If  $x \leq_L s$  and  $x \leq_R s$ , then:*

$$T = \begin{cases} \text{bh}(g) + \text{bh}(g') - m & \text{if } y \leq_R gs^{-1} \text{ and } x \leq_L s'^{-1}g' \\ \text{bh}(g) + \text{bh}(g') - m - 1 & \text{if } y \leq_R gs^{-1} \text{ and } x \not\leq_L s'^{-1}g' \\ \text{bh}(g) + \text{bh}(g') - m + 2 & \text{if } x \leq_R gs^{-1} \text{ and } x \leq_L s'^{-1}g' \\ \text{bh}(g) + \text{bh}(g') - m + 1 & \text{if } x \leq_R gs^{-1} \text{ and } x \not\leq_L s'^{-1}g' \\ \text{bh}(g) + \text{bh}(g') - m + 1 & \text{if } g = s \text{ and } x \leq_L s'^{-1}g' \\ \text{bh}(g) + \text{bh}(g') - m & \text{if } g = s \text{ and } y \leq_L s'^{-1}g' \end{cases}$$



Consequently:

$$T' = \begin{cases} \text{bh}(g) + \text{bh}(g') - m + 1 & \text{if } x \leq_L s'^{-1}g' \\ \text{bh}(g) + \text{bh}(g') - m & \text{if } x \not\leq_L s'^{-1}g' \end{cases}$$

PROOF. Equation (6.1) implies that  $\text{bh}(s) + \text{bh}(s') = m + 2$ . Assume that  $y \leq_R gs^{-1}$ , then  $\text{bh}(gs^{-1}) = \text{bh}(g) - \text{bh}(s)$ . If  $x \leq_L s'^{-1}g'$ , Lemma 6.5 implies that  $\text{bh}(s'^{-1}g') = \text{bh}(g') - \text{bh}(s') + 2$ . Therefore  $T = \text{bh}(g) + \text{bh}(g') - m$ . If  $x \not\leq_L s'^{-1}g'$ , Lemma 6.5 implies that  $\text{bh}(s'^{-1}g') = \text{bh}(g') - \text{bh}(s') + 1$ . Hence  $T = \text{bh}(g) + \text{bh}(g') - m - 1$ . Assume that  $x \leq_R gs^{-1}$ , then  $\text{bh}(gs^{-1}) = \text{bh}(g) - \text{bh}(s) + 2$ . If  $x \leq_L s'^{-1}g'$ , Lemma 6.5 implies that  $\text{bh}(s'^{-1}g') = \text{bh}(g') - \text{bh}(s') + 2$ . Hence  $T = \text{bh}(g) + \text{bh}(g') - m + 2$ . If  $x \not\leq_L s'^{-1}g'$ , Lemma 6.5 implies that  $\text{bh}(s'^{-1}g') = \text{bh}(g') - \text{bh}(s') + 1$ . Therefore  $T = \text{bh}(g) + \text{bh}(g') - m + 1$ . Assume that  $g = s$ , then  $\text{bh}(gs^{-1}) = \text{bh}(g) - \text{bh}(s) + 1$ . If  $x \leq_L s'^{-1}g'$ , Lemma 6.5 implies that  $\text{bh}(s'^{-1}g') = \text{bh}(g') - \text{bh}(s') + 2$ . Hence  $T = \text{bh}(g) + \text{bh}(g') - m + 1$ . If  $y \leq_L s'^{-1}g'$ , Lemma 6.5 implies that  $\text{bh}(s'^{-1}g') = \text{bh}(g') - \text{bh}(s') + 1$ . Therefore  $T = \text{bh}(g) + \text{bh}(g') - m$ . We obtain  $T'$  by replacing  $\text{bh}(gs^{-1}) = \text{bh}(\Phi(gs^{-1})) \pm 1$  for each case.  $\square$

**Lemma 6.14.** *Let  $g, g' \in M$  with  $\Omega \not\leq_R g, g'$ , and let  $s_1, s_2, s'_1, s'_2 \in \text{Div}(\Omega) \setminus \{1, \Omega\}$  such that  $s_2s_1 \leq_R g$ ,  $s'_1s'_2 \leq_L g'$ , and  $s_1s'_1 = \Phi(s_2)s'_2 = \Omega$ . Then, there is a unique  $z \in \{x, y\}$  satisfying  $z \leq_L s_1, s'_2$  and  $z \leq_R s'_1, s_2$ .*

PROOF. Let  $z$  be the unique element in  $\{x, y\}$  such that  $z \leq_L s_1$  and  $z \leq_R s'_1$ , and let  $r \in \{x, y\}$  such that  $r \leq_L s'_2$ . Since  $\Phi(s_2)s'_2 = \Omega$ , then  $r \leq_R s_2$ . Suppose that  $r \neq z$ , then  $\text{lg}_S(s_2s_1) < m$  and  $\text{lg}_S(s'_1s'_2) < m$  because  $\Omega \not\leq_R g, g'$ . Hence,  $\text{lg}_S(\Delta) = \text{lg}_S(s_2s_1s'_1s'_2) < 2m$ , which is a contradiction. Therefore  $r = z$ .  $\square$

**Remark** Let  $g, g' \in M$  with  $\Omega \not\leq_R g, g'$ , let  $s_1, \dots, s_{t+1} \in \text{Div}(\Omega) \setminus \{1, \Omega\}$ , and let  $s'_1, \dots, s'_{t+1} \in \text{Div}(\Omega) \setminus \{1, \Omega\}$  for some  $t \geq 1$ , such that  $s_{t+1} \cdots s_1 \leq_R g$ ,  $s'_1 \cdots s'_{t+1} \leq_L g'$ , and  $\Phi^{i-1}(s_i)s'_i = \Omega$  for all  $i \in \{1, \dots, t+1\}$ . Hence  $\Omega^t \leq_R gg'$ . Let  $g_0, g'_0 \in M$  such that  $g = g_0s_{t+1} \cdots s_1$  and  $g' = s'_1 \cdots s'_{t+1}g'_0$ . Then  $\Omega^{t-i+1} \leq_R \Phi^{i-1}(g_0s_{t+1} \cdots s_{i+1})\Phi^{i-1}(s_i)s'_i \cdots s'_{t+1}g'_0$  for all  $i \in \{1, \dots, t\}$ . Lemma 6.14 implies that  $z \leq_R \Phi^{i-1}(s_{i+1}), s'_i$  and  $z \leq_L \Phi^{i-1}(s_i), s'_{i+1}$  for some  $z \in \{x, y\}$ . Then, there are  $a, b \in \{x, y\}$  such that  $a \leq_R s_{i+1}, a \leq_L s_i, b \leq_R s'_i$ , and  $b \leq_L s'_{i+1}$ , where  $a = b$  if and only if  $i$  is odd. Note that  $b = z$ .

**Lemma 6.15.** *Let  $g, g' \in M$  with  $\Omega \not\leq_R g, g'$ , and let  $s, s' \in \text{Div}(\Omega) \setminus \{1, \Omega\}$  such that  $s \leq_R g$ ,  $s' \leq_L g'$ , and  $ss' = \Omega$ . Let  $z \in \{x, y\}$  such that  $z \leq_L s$  and  $z \leq_R s'$ . Assume that  $z \leq_R gs^{-1}$ ,  $z \leq_L s'^{-1}g'$ , and  $\Omega \not\leq_R \Phi(gs^{-1})s'^{-1}g'$ . Then:*

$$\text{bh}(\Phi(gs^{-1})s'^{-1}g') = \begin{cases} \text{bh}(g) + \text{bh}(g') - m + 2 & \text{if } y \leq_R s \\ \text{bh}(\Phi(g)) + \text{bh}(g') - m + 2 & \text{if } x \leq_R s \end{cases}$$

PROOF. Let  $T' = \text{bh}(\Phi(gs^{-1})) + \text{bh}(s'^{-1}g')$ . So, Lemmas 6.10 to 6.13 imply that:

$$T' = \begin{cases} \text{bh}(g) + \text{bh}(g') - m + 3 & \text{if } y \leq_L s \text{ and } y \leq_R s \\ \text{bh}(g) + \text{bh}(g') - m + 1 & \text{if } x \leq_L s \text{ and } x \leq_R s \\ \text{bh}(g) + \text{bh}(g') - m + 2 & \text{otherwise} \end{cases}$$

Since  $\Omega \not\leq_R \Phi(gs^{-1})s'^{-1}g'$ , Lemma 6.5 implies that  $\text{bh}(\Phi(gs^{-1})s'^{-1}g') = T' - 1$  when  $y \leq_L s$ , and  $\text{bh}(\Phi(gs^{-1})s'^{-1}g') = T'$  when  $x \leq_L s$ . Hence:

$$\text{bh}(\Phi(gs^{-1})s'^{-1}g') = \begin{cases} \text{bh}(g) + \text{bh}(g') - m + 2 & \text{if } y \leq_R s \\ \text{bh}(g) + \text{bh}(g') - m + 1 & \text{if } x \leq_R s \end{cases}$$

We obtain the result by replacing  $\text{bh}(g) = \text{bh}(\Phi(g)) + 1$  when  $x \leq_R s$ .  $\square$

Let  $g, g' \in M$  such that  $\Omega \not\leq_R g, g'$ , and let  $s_1, s'_1, \dots, s_t, s'_t \in \text{Div}(\Omega) \setminus \{1, \Omega\}$  such that  $g = g_0 s_t \cdots s_1$  and  $g' = s'_1 \cdots s'_t g'_0$  for some  $g_0, g'_0 \in M$ . Assume that  $\Omega \not\leq_R \Phi^t(g_0)g'_0$  and  $\Phi^{i-1}(s_i)s'_i = \Omega$  for all  $i \in \{1, \dots, t\}$ . For every  $i \in \{1, \dots, t\}$ , let  $u_i(g, g') = \Phi^{i-1}(g_0 s_t \cdots s_i) a b s'_i \cdots s'_t g'_0$ , where  $a \leq_R \Phi^{i-1}(s_i)$  and  $b \leq_L s'_i$ . Note that  $\Omega \not\leq_R u_i(g, g')$  and  $a \neq b$  for all  $i \in \{1, \dots, t\}$ . Since  $\text{bh}(\Phi^{i-1}(g_0 s_t \cdots s_i) a) = \text{bh}(\Phi^{i-1}(g_0 s_t \cdots s_i))$  and  $\text{bh}(b s'_i \cdots s'_t g'_0) = \text{bh}(s'_i \cdots s'_t g'_0)$ , Lemma 6.5 implies the following:

$$\text{bh}(u_i(g, g')) = \begin{cases} \text{bh}(\Phi^{i-1}(g_0 s_t \cdots s_i)) + \text{bh}(s'_i \cdots s'_t g'_0) & \text{if } a = y \\ \text{bh}(\Phi^{i-1}(g_0 s_t \cdots s_i)) + \text{bh}(s'_i \cdots s'_t g'_0) - 1 & \text{if } a = x \end{cases} \quad (6.3)$$

In particular:

$$\text{bh}(u_1(g, g')) = \begin{cases} \text{bh}(g) + \text{bh}(g') & \text{if } y \leq_R g \\ \text{bh}(\Phi(g)) + \text{bh}(g') & \text{if } x \leq_R g \end{cases}$$

For every  $i \in \{1, \dots, t\}$  we denote by  $\widehat{u}_i(g, g')$  the element obtained by removing  $a$  and  $b$  from  $u_i(g, g')$ . Hence  $\widehat{u}_i(g, g') = \Omega^{t-i+1} \Phi^t(g_0)g'_0$ . Note that  $\widehat{u}_1(g, g') = gg'$ .

**Remark** Let  $g \in M$ , and let  $(g_p, \dots, g_1)$  be its alternating form. For every  $i \in \{1, \dots, p\}$ , let  $a_i, b_i \in \{1, x, y\}$  with  $a_2, \dots, a_p, b_2, \dots, b_{p-1} \neq 1$  such that  $a_i \leq_R g_i$  and  $b_i \leq_L g_i$ . Then  $\Omega \not\leq_R b_p g_p a_p \cdots b_1 g_1 a_1$  and  $\text{bh}(b_p g_p a_p \cdots b_1 g_1 a_1) = \text{bh}(g)$ .

**Lemma 6.16.** *Let  $g, g' \in M$  such that  $\Omega \not\leq_R g, g'$  and  $\Omega^{t+1} \leq_R gg'$  for some  $t \geq 1$ . Then  $\text{bh}(u_{t+1}(g, g')) = \text{bh}(u_1(g, g')) - \text{bh}(\Omega^t) + 2$ , that is:*

$$\text{bh}(u_{t+1}(g, g')) = \begin{cases} \text{bh}(g) + \text{bh}(g') - \text{bh}(\Omega^t) + 2 & \text{if } y \leq_R g \\ \text{bh}(\Phi(g)) + \text{bh}(g') - \text{bh}(\Omega^t) + 2 & \text{if } x \leq_R g \end{cases}$$

PROOF. Let  $s_1, s'_1, \dots, s_t, s'_t \in \text{Div}(\Omega) \setminus \{1, \Omega\}$  with  $\Phi^{i-1}(s_i)s'_i = \Omega$  for all  $i \in \{1, \dots, t\}$  such that  $g = g_0 s_t \cdots s_1$  and  $g' = s'_1 \cdots s'_t g'_0$  for some  $g_0, g'_0 \in M$ . Hence  $u_1(g, g') = g_0 s_t \cdots s_1 a b s'_1 \cdots s'_t g'_0$ . If  $t = 1$ , then  $u_2(g, g') = \Phi(g_0 s_2) c d s'_2 g'_0 = \Omega^{-1} g_0 s_2 d s_1 s'_1 d s'_2 g'_0$  for some  $c, d \in \{x, y\}$ . So, Lemma 6.14 implies that  $d \leq_L s_1$  and  $d \leq_R s_2$ . Thus, by Lemma 6.15, we obtain the following:

$$\text{bh}(u_2(g, g')) = \begin{cases} \text{bh}(g) + \text{bh}(g') - m + 2 & \text{if } y \leq_R g \\ \text{bh}(\Phi(g)) + \text{bh}(g') - m + 2 & \text{if } x \leq_R g \end{cases}$$

Equation (6.3) implies that  $\text{bh}(u_2(g, g')) = \text{bh}(u_1(g, g')) - \text{bh}(\Omega) + 2$  because  $m = \text{bh}(\Omega)$ . If  $t \geq 2$  we suppose the statement is true for  $t - 1$ , then:

$$\text{bh}(u_t(g, g')) = \begin{cases} \text{bh}(g) + \text{bh}(g') - \text{bh}(\Omega^{t-1}) + 2 & \text{if } y \leq_R g \\ \text{bh}(\Phi(g)) + \text{bh}(g') - \text{bh}(\Omega^{t-1}) + 2 & \text{if } x \leq_R g \end{cases}$$

Further, Equation (6.3) implies the following:

$$\text{bh}(u_t(g, g')) = \begin{cases} \text{bh}(\Phi^{t-1}(g_0 s_{t+1} s_t)) + \text{bh}(s'_t s'_{t+1} g'_0) & \text{if } y \leq_R \Phi^{t-1}(g_0 s_{t+1} s_t) \\ \text{bh}(\Phi^{t-1}(g_0 s_{t+1} s_t)) + \text{bh}(s'_t s'_{t+1} g'_0) - 1 & \text{if } x \leq_R \Phi^{t-1}(g_0 s_{t+1} s_t) \end{cases}$$

We have  $u_{t+1}(g, g') = \Phi^t(g_0 s_{t+1}) c d s'_{t+1} g'_0 = \Omega^{-1} \Phi^{t-1}(g_0 s_{t+1}) d s_t s'_t d s'_{t+1} g'_0$ . Lemma 6.14 implies  $d \leq_L s_t$  and  $d \leq_R s'_{t+1}$ . So, by Lemma 6.15, we have:

$$\text{bh}(u_{t+1}(g, g')) = \begin{cases} \text{bh}(\Phi^{t-1}(g_0 s_{t+1} s_t)) + \text{bh}(s'_t s'_{t+1} g'_0) - m + 2 & \text{if } y \leq_R \Phi^{t-1}(g_0 s_{t+1} s_t) \\ \text{bh}(\Phi^{t-1}(g_0 s_{t+1} s_t)) + \text{bh}(s'_t s'_{t+1} g'_0) - m + 1 & \text{if } x \leq_R \Phi^{t-1}(g_0 s_{t+1} s_t) \end{cases}$$

Therefore  $\text{bh}(u_{t+1}(g, g')) = \text{bh}(u_t(g, g')) - m + 2$ . Hence:

$$\text{bh}(u_t(g, g')) = \begin{cases} \text{bh}(g) + \text{bh}(g') - \text{bh}(\Omega^{t-1}) - m + 4 & \text{if } y \leq_R g \\ \text{bh}(\Phi(g)) + \text{bh}(g') - \text{bh}(\Omega^{t-1}) - m + 4 & \text{if } x \leq_R g \end{cases}$$

Finally, Lemma 6.9 implies that  $\text{bh}(\Omega^{t-1}) + m - 4 = \text{bh}(\Omega^t) - 2$ .  $\square$

Let  $g, g' \in M$  such that  $\Omega \not\leq_R \Phi(g)g'$ , and let  $s, s' \in \text{Div}(\Omega) \setminus \{1, \Omega\}$  such that  $ss' = \Omega$  and  $\Omega \not\leq_R gs, s'g'$ . In Lemmas 6.17 and 6.18 we will denote by  $a, b$  the unique maximal elements in  $\{1, x, y\}$  satisfying  $a \leq_R g$  and  $b \leq_L g'$ .

**Lemma 6.17.** *Assume that  $y \leq_L s$ ,  $(a, b) \neq (1, 1)$  and  $(a, b) \neq (y, y)$ . Then:*

$$\text{bh}(\Phi(g)g') = \begin{cases} \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a, b \neq 1 \text{ and } y \leq_R s \\ \text{bh}(\Phi(gs)) + \text{bh}(s'g') - m & \text{if } a, b \neq 1 \text{ and } x \leq_R s \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } \{a, b\} = \{1, x\} \text{ and } y \leq_R s \\ \text{bh}(\Phi(gs)) + \text{bh}(s'g') - m & \text{if } \{a, b\} = \{1, x\} \text{ and } x \leq_R s \\ \text{bh}(gs) + \text{bh}(s'g') - m + 2 & \text{if } a = y \text{ and } b = 1 \text{ and } y \leq_R s \\ \text{bh}(\Phi(gs)) + \text{bh}(s'g') - m + 2 & \text{if } a = y \text{ and } b = 1 \text{ and } x \leq_R s \\ \text{bh}(gs) + \text{bh}(s'g') - m + 1 & \text{if } a = 1 \text{ and } b = y \text{ and } y \leq_R s \\ \text{bh}(\Phi(gs)) + \text{bh}(s'g') - m + 1 & \text{if } a = 1 \text{ and } b = y \text{ and } x \leq_R s \end{cases}$$

In particular  $\text{bh}(\Phi(g)g') = \text{bh}(u_1(gs, s'g')) - m$  when  $a, b \neq 1$ .

PROOF. Let  $R = \text{bh}(\Phi(g)) + \text{bh}(g')$ . If  $y \leq_R s$ , Lemmas 6.10 implies the following:

$$R = \begin{cases} \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = x \text{ and } b = x \\ \text{bh}(gs) + \text{bh}(s'g') - m + 1 & \text{if } a = x \text{ and } b = y \\ \text{bh}(gs) + \text{bh}(s'g') - m + 2 & \text{if } a = y \text{ and } b = x \\ \text{bh}(gs) + \text{bh}(s'g') - m + 1 & \text{if } a = x \text{ and } b = 1 \\ \text{bh}(gs) + \text{bh}(s'g') - m + 3 & \text{if } a = y \text{ and } b = 1 \\ \text{bh}(gs) + \text{bh}(s'g') - m + 1 & \text{if } a = 1 \text{ and } b = x \\ \text{bh}(gs) + \text{bh}(s'g') - m + 2 & \text{if } a = 1 \text{ and } b = y \end{cases}$$

So, by Lemma 6.5, we have:

$$\text{bh}(\Phi(g)g') = \begin{cases} \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = x \text{ and } b = x \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = x \text{ and } b = y \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = y \text{ and } b = x \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = x \text{ and } b = 1 \\ \text{bh}(gs) + \text{bh}(s'g') - m + 2 & \text{if } a = y \text{ and } b = 1 \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = 1 \text{ and } b = x \\ \text{bh}(gs) + \text{bh}(s'g') - m + 1 & \text{if } a = 1 \text{ and } b = y \end{cases}$$

Similarly, if  $x \leq_R s$ , Lemma 6.11 implies the following:

$$R = \begin{cases} \text{bh}(gs) + \text{bh}(s'g') - m - 1 & \text{if } a = x \text{ and } b = x \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = x \text{ and } b = y \\ \text{bh}(gs) + \text{bh}(s'g') - m + 1 & \text{if } a = y \text{ and } b = x \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = x \text{ and } b = 1 \\ \text{bh}(gs) + \text{bh}(s'g') - m + 2 & \text{if } a = y \text{ and } b = 1 \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = 1 \text{ and } b = x \\ \text{bh}(gs) + \text{bh}(s'g') - m + 1 & \text{if } a = 1 \text{ and } b = y \end{cases}$$

So, by Lemma 6.5, we have:

$$\text{bh}(\Phi(g)g') = \begin{cases} \text{bh}(gs) + \text{bh}(s'g') - m - 1 & \text{if } a = x \text{ and } b = x \\ \text{bh}(gs) + \text{bh}(s'g') - m - 1 & \text{if } a = x \text{ and } b = y \\ \text{bh}(gs) + \text{bh}(s'g') - m - 1 & \text{if } a = y \text{ and } b = x \\ \text{bh}(gs) + \text{bh}(s'g') - m - 1 & \text{if } a = x \text{ and } b = 1 \\ \text{bh}(gs) + \text{bh}(s'g') - m + 1 & \text{if } a = y \text{ and } b = 1 \\ \text{bh}(gs) + \text{bh}(s'g') - m - 1 & \text{if } a = 1 \text{ and } b = x \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = 1 \text{ and } b = y \end{cases}$$

Therefore:

$$\text{bh}(\Phi(g)g') = \begin{cases} \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a, b \neq 1 \text{ and } y \leq_R s \\ \text{bh}(gs) + \text{bh}(s'g') - m - 1 & \text{if } a, b \neq 1 \text{ and } x \leq_R s \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } \{a, b\} = \{1, x\} \text{ and } y \leq_R s \\ \text{bh}(gs) + \text{bh}(s'g') - m + 2 & \text{if } a = y \text{ and } b = 1 \text{ and } y \leq_R s \\ \text{bh}(gs) + \text{bh}(s'g') - m + 1 & \text{if } a = 1 \text{ and } b = y \text{ and } y \leq_R s \\ \text{bh}(gs) + \text{bh}(s'g') - m - 1 & \text{if } \{a, b\} = \{1, x\} \text{ and } x \leq_R s \\ \text{bh}(gs) + \text{bh}(s'g') - m + 1 & \text{if } a = y \text{ and } b = 1 \text{ and } x \leq_R s \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = 1 \text{ and } b = y \text{ and } x \leq_R s \end{cases}$$

We obtain the result by replacing  $\text{bh}(gs) = \text{bh}(\Phi(gs)) + 1$  when  $x \leq_R s$ .  $\square$

**Lemma 6.18.** *Assume that  $x \leq_L s$ ,  $(a, b) \neq (1, 1)$  and  $(a, b) \neq (x, x)$ . Then:*

$$\text{bh}(\Phi(g)g') = \begin{cases} \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a, b \neq 1 \text{ and } y \leq_R s \\ \text{bh}(\Phi(gs)) + \text{bh}(s'g') - m & \text{if } a, b \neq 1 \text{ and } x \leq_R s \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } b = 1 \text{ and } y \leq_R s \\ \text{bh}(\Phi(gs)) + \text{bh}(s'g') - m & \text{if } b = 1 \text{ and } x \leq_R s \\ \text{bh}(gs) + \text{bh}(s'g') - m + 1 & \text{if } a = 1 \text{ and } b = x \text{ and } y \leq_R s \\ \text{bh}(\Phi(gs)) + \text{bh}(s'g') - m + 1 & \text{if } a = 1 \text{ and } b = x \text{ and } x \leq_R s \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = 1 \text{ and } b = y \text{ and } y \leq_R s \\ \text{bh}(\Phi(gs)) + \text{bh}(s'g') - m & \text{if } a = 1 \text{ and } b = y \text{ and } x \leq_R s \end{cases}$$

In particular  $\text{bh}(\Phi(g)g') = \text{bh}(u_1(gs, s'g')) - m$  when  $a, b \neq 1$ .

PROOF. Let  $R = \text{bh}(\Phi(g)) + \text{bh}(g')$ . If  $y \leq_R s$ , Lemmas 6.12 implies the following:

$$R = \begin{cases} \text{bh}(gs) + \text{bh}(s'g') - m + 1 & \text{if } a = y \text{ and } b = y \\ \text{bh}(gs) + \text{bh}(s'g') - m + 1 & \text{if } a = x \text{ and } b = y \\ \text{bh}(gs) + \text{bh}(s'g') - m + 2 & \text{if } a = y \text{ and } b = x \\ \text{bh}(gs) + \text{bh}(s'g') - m + 1 & \text{if } a = x \text{ and } b = 1 \\ \text{bh}(gs) + \text{bh}(s'g') - m + 1 & \text{if } a = y \text{ and } b = 1 \\ \text{bh}(gs) + \text{bh}(s'g') - m + 2 & \text{if } a = 1 \text{ and } b = x \\ \text{bh}(gs) + \text{bh}(s'g') - m + 1 & \text{if } a = 1 \text{ and } b = y \end{cases}$$

So, by Lemma 6.5, we have:

$$\text{bh}(\Phi(g)g') = \begin{cases} \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = y \text{ and } b = y \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = x \text{ and } b = y \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = y \text{ and } b = x \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = x \text{ and } b = 1 \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = y \text{ and } b = 1 \\ \text{bh}(gs) + \text{bh}(s'g') - m + 1 & \text{if } a = 1 \text{ and } b = x \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = 1 \text{ and } b = y \end{cases}$$

Similarly, if  $x \leq_R s$ , Lemma 6.13 implies the following:

$$R = \begin{cases} \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = y \text{ and } b = y \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = x \text{ and } b = y \\ \text{bh}(gs) + \text{bh}(s'g') - m + 1 & \text{if } a = y \text{ and } b = x \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = x \text{ and } b = 1 \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = y \text{ and } b = 1 \\ \text{bh}(gs) + \text{bh}(s'g') - m + 1 & \text{if } a = 1 \text{ and } b = x \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = 1 \text{ and } b = y \end{cases}$$

So, by Lemma 6.5, we have:

$$\text{bh}(\Phi(g)g') = \begin{cases} \text{bh}(gs) + \text{bh}(s'g') - m - 1 & \text{if } a = y \text{ and } b = y \\ \text{bh}(gs) + \text{bh}(s'g') - m - 1 & \text{if } a = x \text{ and } b = y \\ \text{bh}(gs) + \text{bh}(s'g') - m - 1 & \text{if } a = y \text{ and } b = x \\ \text{bh}(gs) + \text{bh}(s'g') - m - 1 & \text{if } a = x \text{ and } b = 1 \\ \text{bh}(gs) + \text{bh}(s'g') - m - 1 & \text{if } a = y \text{ and } b = 1 \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = 1 \text{ and } b = x \\ \text{bh}(gs) + \text{bh}(s'g') - m - 1 & \text{if } a = 1 \text{ and } b = y \end{cases}$$

Therefore:

$$\text{bh}(\Phi(g)g') = \begin{cases} \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a, b \neq 1 \text{ and } y \leq_R s \\ \text{bh}(gs) + \text{bh}(s'g') - m - 1 & \text{if } a, b \neq 1 \text{ and } x \leq_R s \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } b = 1 \text{ and } y \leq_R s \\ \text{bh}(gs) + \text{bh}(s'g') - m - 1 & \text{if } b = 1 \text{ and } x \leq_R s \\ \text{bh}(gs) + \text{bh}(s'g') - m + 1 & \text{if } a = 1 \text{ and } b = x \text{ and } y \leq_R s \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = 1 \text{ and } b = x \text{ and } x \leq_R s \\ \text{bh}(gs) + \text{bh}(s'g') - m & \text{if } a = 1 \text{ and } b = y \text{ and } y \leq_R s \\ \text{bh}(gs) + \text{bh}(s'g') - m - 1 & \text{if } a = 1 \text{ and } b = y \text{ and } x \leq_R s \end{cases}$$

We obtain the result by replacing  $\text{bh}(gs) = \text{bh}(\Phi(gs)) + 1$  when  $x \leq_R s$ .  $\square$

Let  $g, g' \in M$  such that  $\Omega \not\leq_R g, g'$  and  $\Delta \not\leq_R gg'$ , and let  $s, s' \in \text{Div}(\Omega) \setminus \{1, \Omega\}$  such that  $s \leq_R g, s' \leq_L g'$ , and  $ss' = \Omega$ . Equation (6.1) implies that:

$$\text{bh}(\Omega^{-1}ss') = \begin{cases} \text{bh}(s) + \text{bh}(s') - m + 1 & \text{if } y \leq_L s \text{ and } y \leq_R s \\ \text{bh}(s) + \text{bh}(s') - m - 1 & \text{if } x \leq_L s \text{ and } x \leq_R s \\ \text{bh}(s) + \text{bh}(s') - m & \text{otherwise} \end{cases}$$

Equivalently:

$$\text{bh}(\Omega^{-1}ss') = \begin{cases} \text{bh}(s) + \text{bh}(s') - m + 1 & \text{if } y \leq_L s \text{ and } y \leq_R s \\ \text{bh}(\Phi(s)) + \text{bh}(s') - m + 1 & \text{if } y \leq_L s \text{ and } x \leq_R s \\ \text{bh}(s) + \text{bh}(s') - m & \text{if } x \leq_L s \text{ and } y \leq_R s \\ \text{bh}(\Phi(s)) + \text{bh}(s') - m & \text{if } x \leq_L s \text{ and } x \leq_R s \end{cases} \quad (6.4)$$

Let  $a, b, c, d$  be the maximal elements in  $\{1, x, y\}$  satisfying  $a \leq_R gs^{-1}, b \leq_L s, c \leq_R s',$  and  $d \leq_L s'^{-1}g'$ . So, by Lemmas 6.15, 6.17, 6.18 and Equation (6.4), we have  $\text{bh}(\Omega^{-1}gg') = \text{bh}(u_1(g, g')) - m + \epsilon'$ , where  $\epsilon'$  is given as follows:

$\epsilon'$	2	0	0	0	0	2	0	0	1	0	0	1	2	0	0	0	1	0
$a$	y	x	y	x	y	x	y	x	1	1	1	1	y	x	y	x	1	1
$b$	y	y	y	y	x	x	x	x	y	y	x	x	y	y	x	x	y	x
$c$	y	y	y	y	x	x	x	x	y	y	x	x	y	y	x	x	y	x
$d$	y	x	x	y	y	x	x	y	y	x	y	x	1	1	1	1	1	1



In what follows, the tuple  $(a, b, c, d)$  will be called a *feature* of the pair  $(g, g')$ .

**Lemma 6.19.** *Let  $h, h' \in M$  such that  $\Omega \not\leq_R h, h'$  and  $hh' = \Omega^t$  for some  $t \geq 1$ . Then:*

$$\text{bh}(h) + \text{bh}(h') = \begin{cases} (m-2)t + 2 & \text{if } y \leq_R h \text{ and } y \leq_R h' \\ (m-2)t + 4 & \text{if } x \leq_R h \text{ and } x \leq_R h' \\ (m-2)t + 3 & \text{otherwise} \end{cases}$$

PROOF. Let  $s_1, s'_1, \dots, s_t, s'_t \in \text{Div}(\Omega) \setminus \{1, \Omega\}$  such that  $h = s_1 \cdots s_t$  and  $h' = s'_1 \cdots s'_t$ , and  $\Phi^{i-1}(s_i)s'_i = \Omega$  for all  $i \in \{1, \dots, t\}$ . Let  $s, s' \in \text{Div}(\Omega) \setminus \{1, \Omega\}$  such that  $ss' = \Omega$ . Then, Equation (6.1) implies the following:

$$\text{bh}(s) + \text{bh}(s') = \begin{cases} (m-2) + 2 & \text{if } y \leq_R s \text{ and } y \leq_R s' \\ (m-2) + 4 & \text{if } x \leq_R s \text{ and } x \leq_R s' \\ (m-2) + 3 & \text{otherwise} \end{cases}$$

Furthermore:

$$\text{bh}(\Phi(s)) + \text{bh}(s') = \begin{cases} (m-2) + 2 & \text{if } y \leq_L s \text{ and } x \leq_R s \\ (m-2) + 4 & \text{if } x \leq_L s \text{ and } y \leq_R s \\ (m-2) + 3 & \text{otherwise} \end{cases}$$

So, by Lemmas 6.5 and 6.14, and Equation (6.2), we have:

$$\text{bh}(s_2 s_1) + \text{bh}(s'_1 s'_2) = \begin{cases} 2(m-2) + 2 & \text{if } y \leq_R s_1 \text{ and } y \leq_R s'_2 \\ 2(m-2) + 4 & \text{if } x \leq_R s_1 \text{ and } x \leq_R s'_2 \\ 2(m-2) + 3 & \text{otherwise} \end{cases}$$

If  $t \geq 3$  we suppose the statement is true for  $t-1$ , that is:

$$\text{bh}(s_i^{-1} h) + \text{bh}(h' s'_i{}^{-1}) = \begin{cases} (m-2)(t-1) + 2 & \text{if } y \leq_R s_1 \text{ and } y \leq_R s'_{t-1} \\ (m-2)(t-1) + 4 & \text{if } x \leq_R s_1 \text{ and } x \leq_R s'_{t-1} \\ (m-2)(t-1) + 3 & \text{otherwise} \end{cases}$$

So, Lemmas 6.5 and 6.14, and Equations (6.1) and (6.2) imply the result.  $\square$

**Corollary 6.20.** *Let  $g, g', h, h' \in M$  such that  $\Omega \not\leq_R g, g'$ ,  $h \leq_L g$ ,  $h' \leq_L g'$ , an  $hh' = \Omega^t$  for some  $t \geq 1$ . Then  $\text{bh}(u_1(g, g')) \geq \text{bh}(\Omega^t)$ .*

PROOF. By Lemmas 6.19 and 6.5, we have:

$$\text{bh}(g) + \text{bh}(g') \geq \begin{cases} (m-2)t + 2 & \text{if } y \leq_R h \text{ and } y \leq_R h' \\ (m-2)t + 4 & \text{if } x \leq_R h \text{ and } x \leq_R h' \\ (m-2)t + 3 & \text{otherwise} \end{cases}$$

Lemma 6.5 and Equation (6.3) imply that:

$$\text{bh}(u_1(g, g')) \geq \begin{cases} (m-2)t + 2 & \text{if } y \leq_R h' \\ (m-2)t + 3 & \text{if } x \leq_R h' \end{cases}$$

So, by Lemma 6.9,  $\text{bh}(u_1(g, g')) \geq \text{bh}(\Omega^t)$ . □

**Lemma 6.21.** *Let  $g, g' \in M$  such that  $\Omega \not\leq_R g, g'$  and  $\Omega \leq_R gg'$ , and let  $t \geq 1$  be the maximal integer satisfying  $\Omega^t \leq_R gg'$ . Then, there exists  $\epsilon' \in \{0, 1, 2\}$  such that:*

$$\text{bh}(\Omega^{-t}gg') - \epsilon' = \text{bh}(u_1(g, g')) - \text{bh}(\Omega^t) \geq 0.$$

PROOF. By assumption, there are  $s_1, s'_1, \dots, s_t, s'_t \in \text{Div}(\Omega) \setminus \{1, \Omega\}$  and  $h, h' \in M$  such that  $g = hs_t \cdots s_1$ ,  $g' = s'_1 \cdots s'_t h'$ , and  $\Phi^{i-1}(s_i)s'_i = \Omega$  for all  $i \in \{1, \dots, t\}$ . Let  $a, b$  be the maximal elements in  $\{1, x, y\}$  such that  $a \leq_R h$  and  $b \leq_L h'$ . We will denote by  $c$  the unique atom satisfying  $c \leq_R s'_t$ . If  $t = 1$ , then  $c \leq_L s_t$ . So, Lemmas 6.15, 6.17 and 6.18, and Equation (6.4) imply that:

$$\text{bh}(\Omega^{-1}gg') = \begin{cases} \text{bh}(u_1(g, g')) - \text{bh}(\Omega) + 2 & \text{if } a, b \neq 1 \text{ and } a = b = c \\ \text{bh}(u_1(g, g')) - \text{bh}(\Omega) + 2 & \text{if } (a, b, c) = (y, 1, y) \\ \text{bh}(u_1(g, g')) - \text{bh}(\Omega) + 1 & \text{if } (a, b, c) = (1, 1, y) \\ \text{bh}(u_1(g, g')) - \text{bh}(\Omega) + 1 & \text{if } a = 1 \text{ and } b = c \\ \text{bh}(u_1(g, g')) - \text{bh}(\Omega) & \text{otherwise} \end{cases} \quad (6.5)$$

Therefore  $\text{bh}(\Omega^{-1}gg') - \epsilon' = \text{bh}(u_1(g, g')) - \text{bh}(\Omega)$  for some  $\epsilon' \in \{0, 1, 2\}$ . If  $t \geq 2$ , Lemma 6.16 implies that  $\text{bh}(u_t(g, g')) = \text{bh}(u_1(g, g')) - \text{bh}(\Omega^{t-1}) + 2$ . Further  $u_t(g, g') = u_1(\Phi^{t-1}(hs_t), s'_t h')$ .

So, Equation (6.5) implies that:

$$\text{bh}(\Omega^{-t}gg') = \begin{cases} \text{bh}(u_t(g, g')) - \text{bh}(\Omega) + 2 & \text{if } a, b \neq 1 \text{ and } \Phi^{t-1}(a) = b = c \\ \text{bh}(u_t(g, g')) - \text{bh}(\Omega) + 2 & \text{if } (\Phi^{t-1}(a), b, c) = (y, 1, y) \\ \text{bh}(u_t(g, g')) - \text{bh}(\Omega) + 1 & \text{if } (a, b, c) = (1, 1, y) \\ \text{bh}(u_t(g, g')) - \text{bh}(\Omega) + 1 & \text{if } a = 1 \text{ and } b = c \\ \text{bh}(u_t(g, g')) - \text{bh}(\Omega) & \text{otherwise} \end{cases} \quad (6.6)$$

Then  $\text{bh}(\Omega^{-t}gg') = \text{bh}(u_1(g, g')) - \text{bh}(\Omega^{t-1}) + 2 - \text{bh}(\Omega) + \epsilon'$ . Lemma 6.9.1 and Corollary 6.20 imply that  $\text{bh}(\Omega^{-t}gg') - \epsilon' = \text{bh}(u_1(g, g')) - \text{bh}(\Omega^t) \geq 0$ .  $\square$

Let  $g, g' \in M$  such that  $\Omega \not\leq_R g, g'$ , and let  $h, h' \in M$  such that  $h \leq_R g, h' \leq_L g'$ , and  $hh' = \Omega^t$  for some  $t \geq 1$  maximal. Let  $(a, b, c, d)$  be a feature of  $(g, g')$ . Lemma 6.21 implies that  $\text{bh}(\Omega^{-t}gg') - \epsilon' = \text{bh}(u_1(g, g')) - \text{bh}(\Omega^t)$  for some  $\epsilon' \in \{0, 1, 2\}$ . More specifically, if  $t$  is odd, Equation (6.6) implies that  $\epsilon'$  is given by:

$\epsilon'$	2	0	0	0	0	2	0	0	1	0	0	1	2	0	0	0	1	0
$a$	$y$	$x$	$y$	$x$	$y$	$x$	$y$	$x$	1	1	1	1	$y$	$x$	$y$	$x$	1	1
$b$	$y$	$y$	$y$	$y$	$x$	$x$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$x$
$c$	$y$	$y$	$y$	$y$	$x$	$x$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$x$
$d$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	$y$	$x$	1	1	1	1	1	1

(6.7)

and, if  $t$  is even, Equation (6.6) implies that  $\epsilon'$  is given as follows:

$\epsilon'$	2	0	0	0	0	2	0	0	1	0	0	1	2	0	0	0	1	0
$a$	$x$	$y$	$x$	$y$	$x$	$y$	$x$	$y$	1	1	1	1	$x$	$y$	$x$	$y$	1	1
$b$	$x$	$x$	$x$	$x$	$y$	$y$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	$y$
$c$	$y$	$y$	$y$	$y$	$x$	$x$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$x$
$d$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	$y$	$x$	1	1	1	1	1	1

(6.8)

Recall that, for every  $g \in M$ ,  $\text{dpt}^*(g) = \text{dpt}(gR_{M_1}(g)^{-1})$ .

**Lemma 6.22.** *Let  $g, g' \in M$  such that  $\Omega \not\leq_R g, g'$ , and let  $t \geq 1$  satisfying  $\Omega^t \leq_R gg'$ . Assume that  $\text{bh}(\Omega^{-t}gg') - \epsilon' = \text{bh}(u_1(g, g')) - \text{bh}(\Omega^t) \geq 0$  for some  $\epsilon' \in \{0, 1, 2\}$ . Then  $\text{dpt}(\Omega^{-t}gg') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - \text{dpt}^*(\Omega^t) \geq 0$  for some  $\epsilon \in \{-1, 0, 1\}$ . More specifically,*

if  $t$  is odd, we have:

$$\epsilon = \begin{cases} -1 & \text{if } \epsilon' = 0 \text{ and } \text{bh}(g) \text{ is even} \\ 1 & \text{if } \epsilon' = 2 \text{ and } \text{bh}(g) \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \quad (6.9)$$

Similarly, if  $t$  is even, we have:

$$\epsilon = \begin{cases} 0 & \text{if either } \epsilon' = 0, \text{ or } \epsilon' = 1 \text{ and } \text{bh}(g) \text{ is even} \\ 1 & \text{if either } \epsilon' = 1 \text{ and } \text{bh}(g) \text{ is odd, or } \epsilon' = 2 \end{cases} \quad (6.10)$$

PROOF. Note that  $\text{bh}(u_1(g, g')) = \text{bh}(g) + \overline{\text{bh}}(g')$ , where  $\overline{\text{bh}}(g') = \text{bh}(g')$  if  $y \leq_R g$ , and  $\overline{\text{bh}}(g') = \text{bh}(g') - 1$  if  $x \leq_R g$ . Since  $t \geq 1$  then  $\overline{\text{bh}}(g')$  is even and  $\text{dpt}(g') = \overline{\text{bh}}(g')/2$ . Assume that  $t$  is even, then  $\text{bh}(\Omega^t)$  is even. If  $\epsilon' \in \{0, 2\}$ , then  $\text{bh}(\Omega^{-t}gg')$  and  $\text{bh}(g)$  have the same parity. Therefore  $\text{dpt}(\Omega^{-t}gg') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - \text{dpt}^*(\Omega^t) \geq 0$ , where  $\epsilon = \epsilon'/2$ . If  $\epsilon' = 1$  and  $\text{bh}(g)$  is even, then  $\text{bh}(\Omega^{-t}gg')$  is odd. Hence  $\text{dpt}(\Omega^{-t}gg') = \text{dpt}(g) + \text{dpt}(g') - \text{dpt}^*(\Omega^t)$ . If  $\epsilon' = 1$  and  $\text{bh}(g)$  is odd, then  $\text{bh}(\Omega^{-t}gg')$  is even. Hence  $\text{dpt}(\Omega^{-t}gg') = \text{dpt}(g) + \text{dpt}(g') - \text{dpt}^*(\Omega^t) + 1$ . Assume that  $t$  is odd, then  $\text{bh}(\Omega^t)$  is odd. If  $\epsilon' = 0$  and  $\text{bh}(g)$  is even, then  $\text{bh}(\Omega^{-t}gg')$  is odd. Hence  $\text{dpt}(\Omega^{-t}gg') = \text{dpt}(g) + \text{dpt}(g') - \text{dpt}^*(\Omega^t) - 1$ . If  $\epsilon' = 0$  and  $\text{bh}(g)$  is odd, then  $\text{bh}(\Omega^{-t}gg')$  is even. Hence  $\text{dpt}(\Omega^{-t}gg') = \text{dpt}(g) + \text{dpt}(g') - \text{dpt}^*(\Omega^t)$ . If  $\epsilon' = 1$ , then  $\text{bh}(\Omega^{-t}gg')$  and  $\text{bh}(g)$  have the same parity. Therefore  $\text{dpt}(\Omega^{-t}gg') = \text{dpt}(g) + \text{dpt}(g') - \text{dpt}^*(\Omega^t)$ . If  $\epsilon' = 2$  and  $\text{bh}(g)$  is even, then  $\text{bh}(\Omega^{-t}gg')$  is odd. Hence  $\text{dpt}(\Omega^{-t}gg') = \text{dpt}(g) + \text{dpt}(g') - \text{dpt}^*(\Omega^t)$ . If  $\epsilon' = 2$  and  $\text{bh}(g)$  is odd, then  $\text{bh}(\Omega^{-t}gg')$  is even. Hence  $\text{dpt}(\Omega^{-t}gg') = \text{dpt}(g) + \text{dpt}(g') - \text{dpt}^*(\Omega^t) + 1$ .  $\square$

Let  $g, g' \in M$  such that  $\Omega \not\leq_R g, g'$ , and let  $h, h' \in M$  such that  $h \leq_R g, h' \leq_L g'$ , and  $hh' = \Omega^t$  for  $t \geq 1$  maximal. Let  $(a, b, c, d)$  be a feature of  $(g, g')$ . Lemma 6.21 implies that  $\text{bh}(\Omega^{-t}gg') - \epsilon' = \text{bh}(u_1(g, g')) - \text{bh}(\Omega^t) \geq 0$ . So, Lemma 6.22 implies that  $\text{dpt}(\Omega^{-t}gg') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - \text{dpt}(\Omega^t)$  for some  $\epsilon \in \{-1, 0, 1\}$  obtained as in Equations (6.9) and (6.10). More specifically, if  $t$  is odd and  $y \leq_L g$ , Equation (6.8) implies that  $\epsilon$  is:

$\epsilon$	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
$a$	$y$	$x$	$y$	$x$	$y$	$x$	$y$	$x$	1	1	$y$	$x$	$y$	$x$	1
$b$	$y$	$y$	$y$	$y$	$x$	$x$	$x$	$x$	$y$	$y$	$y$	$y$	$x$	$x$	$y$
$c$	$y$	$y$	$y$	$y$	$x$	$x$	$x$	$x$	$y$	$y$	$y$	$y$	$x$	$x$	$y$
$d$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	1	1	1	1	1

(6.11)

Similarly, if  $t$  is odd and  $x \leq_L g$ , Equation (6.8) implies that  $\epsilon$  is:

$\epsilon$	0	-1	-1	-1	-1	0	-1	-1	-1	0	0	-1	-1	-1	-1
$a$	$y$	$x$	$y$	$x$	$y$	$x$	$y$	$x$	1	1	$y$	$x$	$y$	$x$	1
$b$	$y$	$y$	$y$	$y$	$x$	$x$	$x$	$x$	$x$	$x$	$y$	$y$	$x$	$x$	$x$
$c$	$y$	$y$	$y$	$y$	$x$	$x$	$x$	$x$	$x$	$x$	$y$	$y$	$x$	$x$	$x$
$d$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	1	1	1	1	1

(6.12)

Further, if  $t$  is even, Equation (6.7) implies that  $\epsilon$  is:

$\epsilon$	1	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0
$a$	$x$	$y$	$x$	$y$	$x$	$y$	$x$	$y$	1	1	1	1	$x$	$y$	$x$	$y$	1	1
$b$	$x$	$x$	$x$	$x$	$y$	$y$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	$y$
$c$	$y$	$y$	$y$	$y$	$x$	$x$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$x$
$d$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	$y$	$x$	1	1	1	1	1	1

(6.13)

**Lemma 6.23.** *Let  $g, g' \in M$  such that  $\Omega \not\leq_R g, g'$  and  $t = \inf(gg') \geq 0$ . Then  $\text{dpt}(gg') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_t \geq 0$  for some  $\epsilon \in \{0, 1\}$ .*

PROOF. Assume that  $\Omega \not\leq_R gg'$ . Lemma 6.5 implies that  $\text{bh}(gg') = \text{bh}(g) + \text{bh}(g') - 2\tilde{q}_0 + \epsilon'$ , where  $\epsilon' = 2$  if  $y \leq_R g$  and  $x \leq_L g'$ ,  $\epsilon' = 0$  if  $x \leq_R g$  and  $x \leq_L g'$ , and  $\epsilon' = 1$  otherwise. If  $1 \in \{g, g'\}$  it is clear that  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_0 + 1$  because  $\tilde{q}_0 = 1$ . If  $g, g' \neq 1$  and  $x \leq_L g'$ , then  $\epsilon'$  and  $\text{bh}(g')$  are even. So,  $\text{bh}(gg')$  and  $\text{bh}(g)$  have the same parity. Thus  $\text{dpt}(gg') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_t \geq 0$  with  $\epsilon = \epsilon'/2$ . If  $g, g' \neq 1$  and  $y \leq_L g'$ , then  $\epsilon' = 1$  and  $\text{bh}(g')$  is odd. So,  $\text{bh}(gg')$  and  $\text{bh}(g)$  have the same parity. Therefore  $\text{dpt}(gg') - 1 = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_0 \geq 0$ . Hence:

$$\epsilon = 0 \quad \text{if and only if} \quad x \leq_R g \quad \text{and} \quad x \leq_L g'. \quad (6.14)$$

Assume that  $\Omega \leq_R gg'$ , and let  $t' \geq 1$  be the maximal integer satisfying  $\Omega^{t'} \leq_R gg'$ . Lemma 6.22 implies that  $\text{dpt}(\Omega^{-t'} gg') = \text{dpt}(g) + \text{dpt}(g') - \text{dpt}^*(\Omega^{t'}) + \epsilon'$  for some  $\epsilon' \in \{-1, 0, 1\}$ . If  $t' = 2t$  we are done because  $\epsilon' \in \{0, 1\}$  and  $\text{dpt}^*(\Omega^{t'}) = \tilde{q}_t$ . Hence  $\epsilon = \epsilon'$ . More specifically, if

$(a, b, c, d)$  is a feature of  $(g, g')$ , Equation (6.13) implies:

$\epsilon$	1	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0
$a$	$x$	$y$	$x$	$y$	$x$	$y$	$x$	$y$	1	1	1	1	$x$	$y$	$x$	$y$	1	1
$b$	$x$	$x$	$x$	$x$	$y$	$y$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	$y$
$c$	$y$	$y$	$y$	$y$	$x$	$x$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$x$
$d$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	$y$	$x$	1	1	1	1	1	1

(6.15)

If  $t' = 2t + 1$ , then  $\text{dpt}^*(\Omega^{t'}) = \tilde{q}_t + (m - 3)/2$ . By Corollary 6.8, we have:

$$\text{dpt}(gg') = \begin{cases} \text{dpt}(\Omega^{-t'} gg') + (m - 3)/2 & \text{if } x \leq_L \Omega^{-t'} gg' \\ \text{dpt}(\Omega^{-t'} gg') + (m - 1)/2 & \text{if } x \not\leq_L \Omega^{-t'} gg' \end{cases}$$

Hence  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_t + \epsilon' + \epsilon''$  where  $\epsilon'' = 0$  if  $x \leq_L \Omega^{-t'} gg'$ , and  $\epsilon'' = 1$  if  $x \not\leq_L \Omega^{-t'} gg'$ . Then  $\epsilon = \epsilon' + \epsilon''$ . More specifically, if  $y \leq_L g$ , Equation (6.11) implies the following:

$\epsilon$	1	0	0	0	0	1	0	0	1	0	1	0	0	0	0	1
$a$	$y$	$x$	$y$	$x$	$y$	$x$	$y$	$x$	1	1	$y$	$x$	$y$	$x$	1	
$b$	$y$	$y$	$y$	$y$	$x$	$x$	$x$	$x$	$y$	$y$	$y$	$y$	$x$	$x$	$y$	
$c$	$y$	$y$	$y$	$y$	$x$	$x$	$x$	$x$	$y$	$y$	$y$	$y$	$x$	$x$	$y$	
$d$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	1	1	1	1	1	

(6.16)

Similarly, if  $x \leq_L g$ , Equation (6.12) implies that:

$\epsilon$	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
$a$	$y$	$x$	$y$	$x$	$y$	$x$	$y$	$x$	1	1	$y$	$x$	$y$	$x$	1
$b$	$y$	$y$	$y$	$y$	$x$	$x$	$x$	$x$	$x$	$x$	$y$	$y$	$x$	$x$	$x$
$c$	$y$	$y$	$y$	$y$	$x$	$x$	$x$	$x$	$x$	$x$	$y$	$y$	$x$	$x$	$x$
$d$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	1	1	1	1	1

(6.17)

Therefore  $\text{dpt}(gg') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_t \geq 0$  for some  $\epsilon \in \{0, 1\}$ . □

**Corollary 6.24.** *Let  $g = \theta^t$  for some  $t \geq 1$ , and  $g' \in M \setminus M_1$  such that  $\Omega \not\leq_R g'$ . Then  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_r$  where  $r = \inf(gg')$ .*

PROOF. Let  $e$  be a feature of  $(g, g')$ . Let  $t'$  be the maximal integer satisfying  $\Omega^{t'} \leq_R gg'$ . If  $t' = 0$ , then  $y \not\leq_L g'$ . Hence  $x \leq_R g$  and  $x \leq_L g'$ . So, Equation (6.14) implies that

$\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_0$ . In what follows  $t' \geq 1$ . Assume that  $r = t$ . Then  $t' = 2t$  and  $y^{2t} \leq_L g'$ . Since  $g' \notin M_1$ , then  $e = (1, x, y, d)$  for some  $d \in \{x, y\}$ . Assume that  $r < t$ . If  $t' = 2r$ , then  $y^{2r}x \leq_L g'$  because  $g' \notin M_1$ . Hence  $e = (x, x, y, x)$ . If  $t' = 2r + 1$ , then  $y^{2r+1}x \leq_L g'$  because  $g' \notin M_1$ . Hence  $e = (y, y, y, x)$ . By Equations (6.15) to (6.17),  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_r$ .  $\square$

**Lemma 6.25.** *Let  $g, g' \in M$  be unmovable elements such that  $\Omega \leq_R g, g'$  and  $t = \inf(gg') \geq 1$ . Then  $\text{dpt}(gg') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_t \geq 0$  for some  $\epsilon \in \{0, 1\}$ .*

PROOF. Let  $g_0, g'_0 \in M$  such that  $g = \Omega g_0$  and  $g' = \Omega g'_0$ . Then  $gg' = \Omega^2 \Phi(g_0)g'_0$  and  $\inf(\Phi(g_0)g'_0) = t - 1$ . By Lemma 6.23,  $\text{dpt}(\Phi(g_0)g'_0) - \epsilon = \text{dpt}(\Phi(g_0)) + \text{dpt}(g'_0) - \tilde{q}_{t-1} \geq 0$  for some  $\epsilon \in \{0, 1\}$ . Assume that  $\Omega \not\leq_R \Phi(g_0)g'_0$ . Equation (6.14) implies that  $\epsilon = 0$  if and only if  $y \leq_R g_0$  and  $x \leq_L g'_0$ . Then, by Corollary 6.8, we have:

$$\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_1 + \begin{cases} 0 & \text{if } y \not\leq_R g_0 \text{ and } x \not\leq_L g'_0 \\ 1 & \text{otherwise} \end{cases} \quad (6.18)$$

Assume that  $\Omega \leq_R \Phi(g_0)g'_0$ . Then, either  $y \leq_R g_0$  and  $y \leq_L g'_0$ , or  $x \leq_R g_0$  and  $x \leq_L g'_0$ . So, Lemma 6.23 and Corollary 6.8 imply the result.  $\square$

**Corollary 6.26.** *Let  $g = \theta^t y$  for some  $t \geq 1$ , and let  $g' \in M$  be an unmovable element such that  $\Omega \leq_R g'$ . Then  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_r$  where  $r = \inf(gg')$ .*

PROOF. We have  $g = \Omega g_0$  where  $g_0 = (yx)^{(m-1)/2} \theta^{t-1}$ , and  $g' = \Omega g'_0$  for some  $g'_0 \in M$  with  $\Omega \not\leq_R g'_0$ . Then  $gg' = \Omega^2 \Phi(g_0)g'_0$  where  $\Phi(g_0) = \theta^{t-1} (xy)^{(m-1)/2}$ . Let  $t' \geq 0$  be the maximal integer satisfying  $\Omega^{t'} \leq_R \Phi(g_0)g'_0$ . If  $x \not\leq_L g'_0$ , then  $t' = 0$ . Equation (6.18) implies that  $\text{dpt}(gg') = \text{dpt}(\Phi(g_0)g'_0) = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_1$ . Assume that  $x \leq_L g'_0$ . In particular,  $t' \geq 1$  and  $\Omega \leq_R \Phi(g_0)g'_0$ . Let  $e$  be a feature of  $(\Phi(g_0), g'_0)$ . If  $t' = 2t - 1$ , then  $e = (1, x, x, d)$  for some  $d \in \{1, x, y\}$ . Assume that  $t' < 2t - 1$ . If  $t'$  is odd, then  $e = (x, x, x, d)$  for some  $d \in \{1, y\}$  because  $t'$  is maximal. If  $t'$  is even, then  $e = (y, y, x, d)$  for some  $d \in \{1, y\}$  because  $t'$  is maximal. Note that  $x \leq_L \Phi(g_0)$ . So, Equations (6.15) and (6.17) imply that  $\text{dpt}(\Phi(g_0)g'_0) = \text{dpt}(\Phi(g_0)) + \text{dpt}(g'_0) - \tilde{q}_{r-1}$ . Now, it is shown in the proof of Lemma 6.25 that, if  $\Omega \leq_R \Phi(g_0)g'_0$  and  $\text{dpt}(\Phi(g_0)g'_0) - \epsilon = \text{dpt}(\Phi(g_0)) + \text{dpt}(g'_0) - \tilde{q}_{r-1}$ , then  $\text{dpt}(gg') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_r$ . Hence  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_r$ .  $\square$

**Lemma 6.27.** *Let  $g, g' \in M$  be unmovable elements such that  $\Omega \leq_R g$ ,  $\Omega \not\leq_R g'$ , and  $t = \inf(gg') \geq 0$ . Then  $\text{dpt}(gg') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_t \geq 0$  for some  $\epsilon \in \{0, 1\}$ .*

PROOF. We have  $g = \Omega g_0$  for some  $g_0 \in M$  such that  $\Omega \not\leq_R g_0$ . Then  $gg' = \Omega g_0 g'$ . Let  $t' \geq 0$  be the maximal integer satisfying  $\Omega^{t'} \leq_R g_0 g'$ . Assume that  $\Omega \not\leq_R g_0 g'$ . Lemma 6.23 implies that  $\text{dpt}(g_0 g') = \text{dpt}(g_0) + \text{dpt}(g') - \tilde{q}_0 + \epsilon$  for some  $\epsilon \in \{0, 1\}$ . So, Corollary 6.8 implies that  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_0 + \epsilon$ . Assume that  $\Omega \leq_R g_0 g'$ . By Lemma 6.22,  $\text{dpt}(\Omega^{-t'} g_0 g') = \text{dpt}(g_0) + \text{dpt}(g') - \text{dpt}^*(\Omega^{t'}) + \epsilon'$  for some  $\epsilon' \in \{-1, 0, 1\}$ . Let  $(a, b, c, d)$  be a feature of  $(g_0, g')$ . Assume that  $t' = 2t$ . By Corollary 6.8 and Equation (6.13),  $\text{dpt}(gg') - \epsilon = \text{dpt}(\Omega^{1-t'} g_0 g') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_t \geq 0$ , where  $\epsilon$  is given as follows:

$\epsilon$	1	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0
$a$	$x$	$y$	$x$	$y$	$x$	$y$	$x$	$y$	1	1	1	1	$x$	$y$	$x$	$y$	1	1
$b$	$x$	$x$	$x$	$x$	$y$	$y$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	$y$
$c$	$y$	$y$	$y$	$y$	$x$	$x$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$x$
$d$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	$y$	$x$	1	1	1	1	1	1

(6.19)

Assume that  $t' = 2t - 1$ . By Corollary 6.8 and Equations (6.11) and (6.12),  $\text{dpt}(gg') - \epsilon = \text{dpt}(\Omega^{-t'} g_0 g') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_t \geq 0$ , where, if  $y \leq_L g_0$ , we have:

$\epsilon$	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0
$a$	$y$	$x$	$y$	$x$	$y$	$x$	$y$	$x$	1	1	$y$	$x$	$y$	$x$	1	1	1	1
$b$	$y$	$y$	$y$	$y$	$x$	$x$	$x$	$x$	$y$	$y$	$y$	$y$	$x$	$x$	$y$	1	1	1
$c$	$y$	$y$	$y$	$y$	$x$	$x$	$x$	$x$	$y$	$y$	$y$	$y$	$x$	$x$	$y$	1	1	1
$d$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	1	1	1	1	1	1	1	1

(6.20)

and, if  $x \leq_L g_0$ , we have:

$\epsilon$	1	0	0	0	0	1	0	0	0	1	1	0	0	0	0	0	0	0
$a$	$y$	$x$	$y$	$x$	$y$	$x$	$y$	$x$	1	1	$y$	$x$	$y$	$x$	1	1	1	1
$b$	$y$	$y$	$y$	$y$	$x$	$x$	$x$	$x$	$x$	$x$	$y$	$y$	$x$	$x$	$x$	1	1	1
$c$	$y$	$y$	$y$	$y$	$x$	$x$	$x$	$x$	$x$	$x$	$y$	$y$	$x$	$x$	$x$	1	1	1
$d$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	1	1	1	1	1	1	1	1

(6.21)

Therefore  $\text{dpt}(gg') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_t \geq 0$  for some  $\epsilon \in \{0, 1\}$ . □



**Corollary 6.28.** *Let  $g = \theta^t y$  for some  $t \geq 1$ , and let  $g' \in M \setminus M_1$  such  $\Omega \not\leq_R g'$ . Then  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_r$  where  $r = \inf(gg')$ .*

PROOF. We have  $g = \Omega g_0$  where  $g_0 = (yx)^{(m-1)/2} \theta^{t-1}$ . Let  $t' \geq 0$  be the maximal integer satisfying  $\Omega^{t'} \leq_R g_0 g'$ . If  $y \not\leq_L g'$ , then  $t' = 0$  and  $x \leq_L g'$  because  $g' \notin M_1$ . Equation (6.14) implies that  $\text{dpt}(g_0 g') = \text{dpt}(g_0) + \text{dpt}(g') - \tilde{q}_0$ . So, Corollary 6.8 implies that  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_0$ . Let  $e$  be a feature of  $(g_0, g')$ . Assume that  $y \leq_L g'$ . Then  $t' \geq 1$ . If  $t'$  is even, then  $e = (x, x, y, x)$  because  $t'$  is maximal and  $g' \notin M_1$ . If  $t'$  is odd and  $t' = 2t - 1$ , then  $e = (1, y, y, d)$  with  $d \in \{x, y\}$ , because  $g' \notin M_1$ . If  $t'$  is odd and  $t' < 2t - 1$ , then  $e = (y, y, y, x)$  because  $g' \notin M_1$ . So, Equations (6.19) and (6.20) imply that  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_r$ .  $\square$

**Lemma 6.29.** *Let  $g, g' \in M \setminus \{1\}$  be unmovable elements such that  $\Omega \not\leq_R g$ ,  $\Omega \leq_R g'$ , and  $t = \inf(gg') \geq 0$ . Then  $\text{dpt}(gg') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_t \geq 0$  for some  $\epsilon \in \{0, 1\}$ .*

PROOF. We have  $g' = \Omega g'_0$  for some  $g'_0 \in M$  such that  $\Omega \not\leq_R g'_0$ . Then  $gg' = \Omega \Phi(g) g'_0$ . Let  $t' \geq 0$  be the maximal integer satisfying  $\Omega^{t'} \leq_R \Phi(g) g'_0$ . Assume that  $\Omega \not\leq_R \Phi(g) g'_0$ . Lemma 6.23 implies that  $\text{dpt}(\Phi(g) g'_0) = \text{dpt}(\Phi(g)) + \text{dpt}(g'_0) - \tilde{q}_0 + \epsilon'$  for some  $\epsilon' \in \{0, 1\}$ . So, Corollary 6.8 implies that  $\text{dpt}(gg') = \text{dpt}(g\Omega) + \text{dpt}(g'_0) - \tilde{q}_0 + \epsilon'$ . Recall that, by Equation (6.14),  $\epsilon' = 0$  if and only if  $y \leq_R g$  and  $x \leq_L g'_0$ . Then, by Corollary 6.8, we have the following:

$$\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_0 + \begin{cases} 0 & \text{if } x \leq_R g \text{ and } x \leq_L g'_0 \\ 1 & \text{otherwise} \end{cases}$$

Assume that  $\Omega \leq_R \Phi(g) g'_0$ . Then, either  $x \leq_R g$  and  $x \leq_L g'_0$ , or  $y \leq_R g$  and  $y \leq_L g'_0$ . So, by Corollary 6.8,  $\text{dpt}(g\Omega) + \text{dpt}(g'_0) = \text{dpt}(g) + \text{dpt}(g')$ . By Lemma 6.22,  $\text{dpt}(\Omega^{-t'} \Phi(g) g'_0) = \text{dpt}(\Phi(g)) + \text{dpt}(g'_0) - \text{dpt}^*(\Omega^{t'}) + \epsilon'$  for some  $\epsilon' \in \{-1, 0, 1\}$ . Let  $(a, b, c, d)$  be a feature of  $(\Phi(g), g'_0)$ . Assume that  $t' = 2t$ . By Corollary 6.8 and Equation (6.13),  $\text{dpt}(gg') - \epsilon = \text{dpt}(\Omega^{1-t'} \Phi(g) g'_0) - \epsilon = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_t \geq 0$ , where  $\epsilon$  is given as follows:

$\epsilon$	1	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0
$a$	$x$	$y$	$x$	$y$	$x$	$y$	$x$	$y$	1	1	1	1	$x$	$y$	$x$	$y$	1	1
$b$	$x$	$x$	$x$	$x$	$y$	$y$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	$y$
$c$	$y$	$y$	$y$	$y$	$x$	$x$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$x$
$d$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	$y$	$x$	1	1	1	1	1	1

(6.22)

Assume that  $t' = 2t - 1$ . By Corollary 6.8 and Equations (6.11) and (6.12),  $\text{dpt}(gg') - \epsilon = \text{dpt}(\Omega^{-t'}\Phi(g)g'_0) - \epsilon = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_t \geq 0$ , where, if  $y \leq_L \Phi(g)$ , we have:

$\epsilon$	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
$a$	$y$	$x$	$y$	$x$	$y$	$x$	$y$	$x$	1	1	$y$	$x$	$y$	$x$	1
$b$	$y$	$y$	$y$	$y$	$x$	$x$	$x$	$x$	$y$	$y$	$y$	$y$	$x$	$x$	$y$
$c$	$y$	$y$	$y$	$y$	$x$	$x$	$x$	$x$	$y$	$y$	$y$	$y$	$x$	$x$	$y$
$d$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	1	1	1	1	1

(6.23)

Similarly, if  $x \leq_L \Phi(g)$ , we have:

$\epsilon$	1	0	0	0	0	1	0	0	0	1	1	0	0	0	0
$a$	$y$	$x$	$y$	$x$	$y$	$x$	$y$	$x$	1	1	$y$	$x$	$y$	$x$	1
$b$	$y$	$y$	$y$	$y$	$x$	$x$	$x$	$x$	$x$	$x$	$y$	$y$	$x$	$x$	$x$
$c$	$y$	$y$	$y$	$y$	$x$	$x$	$x$	$x$	$x$	$x$	$y$	$y$	$x$	$x$	$x$
$d$	$y$	$x$	$x$	$y$	$y$	$x$	$x$	$y$	$y$	$x$	1	1	1	1	1

(6.24)

Therefore  $\text{dpt}(gg') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_t \geq 0$  for some  $\epsilon \in \{0, 1\}$ . □

Lemmas 6.23 to 6.29 have proved Part (a) of Condition B.

**Corollary 6.30.** *Let  $g = \theta^t$  for some  $t \geq 1$ , and let  $g' \in M$  be an unmovable element such that  $\Omega \leq_R g'$ . Then  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_r$  where  $r = \inf(gg')$ .*

PROOF. We have  $g' = \Omega g'_0$  for some  $g'_0 \in M$  such that  $\Omega \not\leq_R g'_0$ . Then  $gg' = \Omega(yx)^{(m-1)/2}\theta^{t-1}(xy)^{(m-1)/2}g'_0$ . Let  $t' \geq 0$  be the maximal integer satisfying  $\Omega^{t'} \leq_R \Phi(g)g'_0$ . If  $x \not\leq_L g'_0$ , then  $t' = 0$ . Equation (6.14) implies that  $\text{dpt}(\Phi(g)g'_0) = \text{dpt}(\Phi(g)) + \text{dpt}(g'_0) - \tilde{q}_0 + 1$ . So, Corollary 6.8 implies that  $\text{dpt}(gg') = \text{dpt}(\Phi(g)) + \text{dpt}(g') - \tilde{q}_0 + 1 = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_0$ . Assume that  $x \leq_L g'_0$ . Then  $t' \geq 1$ . Let  $e$  be a feature of  $(\Phi(g), g'_0)$ . If  $t' = 2t$ , then  $e = (1, y, x, d)$  for some  $d \in \{1, x, y\}$ . If  $t' < 2t$  and  $t'$  is odd, then  $e = (x, x, x, d)$  for some  $d \in \{1, y\}$  because  $t'$  is maximal. If  $t' < 2t$  and  $t'$  is even, then  $e = (y, y, x, d)$  for some  $d \in \{1, y\}$ . So, Equations (6.22) and (6.23) imply that  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_r$ . □

**Lemma 6.31.** *Let  $g \in M \setminus M_1$ , and let  $g' \in \Theta$  unmovable. Then  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_r$  where  $r = \inf(gg')$ .*

PROOF. We have  $g' = \theta^t y^e$  for some  $t \geq 1$  and  $e \in \{0, 1\}$ . Let  $t' \geq 0$  be the maximal integer satisfying  $y^{t'} \leq_R g$ . Then  $g = g_0 y^{t'}$  for some  $g_0 \in M$  satisfying  $y \not\leq_R g_0$  and  $x \leq_R g_0$  because  $g \notin M_1$ . If  $r = t$ , then  $\text{dpt}(gg') = \text{dpt}(g) = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_r$ . Assume that  $r < t$ . If  $t' = 2r$ , then  $gg' = \Delta^r g_0 \theta^{t-r} y^e$ . If  $t' = 2r + 1$  and  $e = 0$ , then  $gg' = \Delta^r g_0 \theta^{t-r} y$ , and, if  $t' = 2r - 1$  and  $e = 1$ , then  $gg' = \Delta^r g_0 \theta^{t-r}$ . So, Lemma 6.6.2 implies that  $\text{dpt}(gg') = \text{dpt}(g_0) + (m - 2)(t - r) = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_r$ .  $\square$

Corollaries 6.24 to 6.30 and Lemma 6.31 have proved Part (b) of Condition B.

## 2. Proof of Proposition 6.4

In this section, we denote by  $\Omega_B$  the standard Garside element of  $\mathcal{B}_{2k+1}$ , by  $A_1$  the submonoid of  $\mathcal{B}_{2k+1}$  generated by  $z_1, \dots, z_{2k-1}$ , and by  $A_2$  the submonoid of  $\mathcal{B}_{2k+1}$  generated by  $z_2, \dots, z_{2k}$ . Note that  $\phi(H) \subseteq A_1 \setminus A_2$  and  $\phi(G_1) \subseteq A_2 \setminus A_1$ . Further, we denote by  $\Omega$  the standard Garside element of  $G$ .

**Proposition 6.32** (Crisp [7]). *Let  $g, g' \in M$ , and let  $(g_p, \dots, g_1)$  be the greedy normal form of  $g$ . Then:*

- (1)  $g \leq_R g'$  if and only if  $\phi(g) \leq_R \phi(g')$ .
- (2)  $\phi(g \wedge_R g') = \phi(g) \wedge_R \phi(g')$  and  $\phi(g \vee_R g') = \phi(g) \vee_R \phi(g')$ .
- (3)  $(\phi(g_p), \dots, \phi(g_1))$  is the greedy normal form of  $\phi(g)$ .

**Lemma 6.33.** *We have  $\phi(g) \in \mathcal{B}_{2k+1} \setminus \langle z_2, \dots, z_{2k} \rangle$  for all  $g \in G \setminus G_1$ .*

PROOF. Let  $h \in \mathcal{B}_{2k+1}$ , and let  $(h_2, h_1)$  be its right orthogonal splitting that is  $h = h_1 h_2^{-1}$  and  $h_1 \wedge_R h_2 = 1$ . Since  $G(A_2)$  is a parabolic subgroup, we have  $h \in G(A_2)$  if and only if  $h_1, h_2 \in A_2$ . Let  $g \in G$ , and let  $(g_2, g_1)$  be its right orthogonal splitting. By Proposition 6.32, we have  $\phi(g) = \phi(g_1) \phi(g_2)^{-1}$  and  $\phi(g_1) \wedge_R \phi(g_2) = 1$ . It follows that  $\phi(g) \in G(A_2)$  if and only if  $\phi(g_1), \phi(g_2) \in A_2$ . Furthermore, we have  $g \in G_1$  if and only if  $g_1, g_2 \in M_1$ . So, we only need to prove that  $\phi(g) \in \mathcal{B}_{2k+1}^+ \setminus A_2$  for all  $g \in M \setminus M_1$ . Let  $h \in \mathcal{B}_{2k+1}^+$ , and let  $(h_p, \dots, h_1)$  be its greedy normal form. Since  $G(A_2)$  is a parabolic subgroup, we have  $h \in A_2$  if and only if  $h_i \in A_2$  for all  $i \in \{1, \dots, p\}$ . Let  $g \in M$ , and let  $(g_p, \dots, g_1)$  be its right greedy normal form. By Proposition 6.32,  $\phi(g) = \phi(g_p) \cdots \phi(g_1)$  is the right greedy normal form of  $\phi(g)$ . It follows that  $\phi(g) \in A_2$  if

and only if  $\phi(g_i) \in A_2$  for all  $i \in \{1, \dots, p\}$ . Furthermore, we have  $g \in M_1$  if and only if  $g_i \in M_1$  for all  $i \in \{1, \dots, p\}$ . So, we only need to show that  $\phi(g) \in \text{Div}(\Omega_B) \setminus A_2$  if  $g \in \text{Div}(\Omega) \setminus M_1$ . Let  $h \in \text{Div}(\Omega_B)$ , and let  $i_1, \dots, i_l \in \{1, \dots, 2k\}$  such that  $h = z_{i_1} \cdots z_{i_l}$ . By Bourbaki [3], Chapter IV, Section 1.8, we have  $h \in A_2$  if and only if  $\{z_{i_1}, \dots, z_{i_l}\} \subset \{z_2, \dots, z_{2k}\}$ . This implies that, for  $g \in \text{Div}(\Omega)$ , we have  $\phi(g) \in A_2$  if and only if  $g \in \{1, y\} = \text{Div}(y)$ .  $\square$

For each  $i, j \in \{1, \dots, k\}$  with  $j \neq k$ , and for every  $t \geq 1$ , we define:

$$x_{2i-1}^t = \prod_{l=i}^k z_{2l-1}^t, \quad y_{2j}^t = \prod_{l=j}^k z_{2l}^t.$$

In particular  $x_1^t = \phi(x^t) = \phi(x)^t$  and  $y_2^t = \phi(y^t) = \phi(y)^t$ .

**Lemma 6.34.** *Let  $g \in M \setminus M_1$  be an unmovable element such that  $\text{bh}(g) \leq 2k$ . Then  $\phi(g)$  has a  $(A_1, A_2)$ -sequence of length 2. Moreover, if  $\text{bh}(g) \neq 2k$ , then  $\phi(g)$  has a  $(A_2, A_1)$ -sequence of length 2.*

PROOF. Let  $b = \text{bh}(g)$ . Then  $g = a^{t_b} \cdots y^{t_3} x^{t_2} y^{t_1}$ , where  $t_1 \geq 0$ ,  $t_2, \dots, t_b \geq 1$ ,  $a = x$  if  $b$  is even, and  $a = y$  if  $b$  is odd. Since  $b \leq 2k$ , then:

$$\begin{aligned} \phi(g) &= \phi(a)^{t_b} \cdots x_1^{t_4} y_2^{t_3} x_1^{t_2} \cdot y_2^{t_1} \\ &= \phi(a)^{t_b} \cdots x_1^{t_4} y_2^{t_3} z_1^{t_2} \cdot x_3^{t_2} y_2^{t_1} \\ &= \phi(a)^{t_b} \cdots x_1^{t_4} z_2^{t_3} z_1^{t_2} \cdot y_4^{t_3} x_3^{t_2} y_2^{t_1} \\ &= \phi(a)^{t_b} \cdots z_3^{t_4} z_1^{t_4} z_2^{t_3} z_1^{t_2} \cdot x_5^{t_4} y_4^{t_3} x_3^{t_2} y_2^{t_1} \\ &\dots \\ &= (z_{b-1}^{t_b} \cdots z_i^{t_b}) \cdots z_3^{t_4} z_1^{t_4} z_2^{t_3} z_1^{t_2} \cdot a^{t_b} \cdots x_5^{t_4} y_4^{t_3} x_3^{t_2} y_2^{t_1} \end{aligned}$$

where  $i = 1$  if  $a = x$ , and  $i = 2$  if  $a = y$ . Let  $h' = (z_{b-1}^{t_b} \cdots z_i^{t_b}) \cdots z_3^{t_4} z_1^{t_4} z_2^{t_3} z_1^{t_2}$ , and let  $h = a^{t_b} \cdots x_5^{t_4} y_4^{t_3} x_3^{t_2} y_2^{t_1}$ . Note that  $h \in A_2$  and  $h' \in A_1$ . Therefore  $(h', h)$  is a  $(A_1, A_2)$ -sequence of  $\phi(g)$ . If  $\text{bh}(g) \neq 2k$ , then  $\text{bh}(\Phi(g)) \leq 2k$ . As above,  $\Omega_B \phi(g) \Omega_B^{-1}$  has a  $(A_1, A_2)$ -sequence  $(h', h)$ . Then  $(\Omega_B h' \Omega_B^{-1}, \Omega_B h \Omega_B^{-1})$  is a  $(A_2, A_1)$ -sequence of  $g$ .  $\square$

The *limit* of an unmovable element  $g \in M \setminus M_1$ , denoted by  $\text{lim}(g)$ , is the unique integer  $t \geq 1$  such that  $\tilde{q}_{t-1} < \text{bh}(g) \leq \tilde{q}_t$ . Since  $\tilde{q}_{t-1} < \text{bh}(g)$ , then:

$$\text{lim}(g) = \frac{\tilde{q}_{t-1} + m - 3}{m - 2} < \frac{\text{bh}(g) + m - 3}{m - 2}. \quad (6.25)$$

We set  $\lim(g_0) = 0$  for all  $g_0 \in M_1$ . Thus  $\text{bh}_{A_1, A_2}(\phi(g_0)) = \lim(g_0) + 1$ .

**Lemma 6.35.** *Let  $g \in M \setminus M_1$  be an unmovable element. Then  $\phi(g)$  has a  $(A_1, A_2)$ -sequence of length  $\lim(g) + 1$ .*

PROOF. Let  $(g'_p, \dots, g'_0)$  be the alternating form of  $g$ , and let  $t = \lim(g)$ . Let  $g_0 = g'_0$ , for each  $i \in \{1, \dots, t-1\}$  let  $g_i = \prod_{j=1}^{m-2} g'_{(m-2)(i-1)+j}$ , and let  $g_t = g'_p \cdots g'_{(m-2)(t-1)+1}$ . Let  $i \in \{1, \dots, t-1\}$ . If  $i$  is odd, then  $R_{M_1}(g_i) = 1$  and  $\text{bh}(g_i) = m-1$ . Similarly, if  $i$  is even, then  $R_N(g_i) = 1$  and  $\text{bh}(g_i) = m-2$ . Furthermore,  $R_{M_1}(g_t) = 1$  and  $\text{bh}(g_t) \leq m-1$  if  $t$  is odd, and  $R_N(g_t) = 1$  and  $\text{bh}(g_t) \leq m-2$  if  $t$  is even. Lemma 6.34 implies that there are  $h_1, \dots, h_t \in A_2$  and  $h'_1, \dots, h'_t \in A_1$ , such that  $(h'_i, h_i)$  is a  $(A_1, A_2)$ -sequence of  $\phi(g_i)$  if  $i$  is odd, and  $(h_i, h'_i)$  is a  $(A_2, A_1)$ -sequence of  $\phi(g_i)$  if  $i$  is even. Therefore, the following is a  $(A_1, A_2)$ -sequence of  $\phi(g)$ :

$$\begin{cases} (h'_t, h_t h_{t-1}, \dots, h_3 h_2, h'_2 h'_1, h_1 g_0) & \text{if } t \text{ is odd} \\ (h_t, h'_t h'_{t-1}, \dots, h_3 h_2, h'_2 h'_1, h_1 g_0) & \text{if } t \text{ is even} \end{cases}$$

Note that the length of this sequence is  $t+1$ . □

**Lemma 6.36.** *Let  $g \in G$ , and let  $t, t' \geq 1$  such that  $g = \Delta^{-t} g_0 = \Omega^{-t'} g'_0$  for some  $g_0, g'_0 \in M$ , where  $\Delta \not\leq_R g_0$  and  $\Omega \not\leq_R g'_0$ . Then  $\text{bh}(g_0) \geq 2\tilde{q}_t$  if and only if  $\text{bh}(g'_0) \geq \text{bh}(\Omega^{t'})$ .*

PROOF. If  $\Omega \not\leq_R g_0$  we are done because  $g_0 = g'_0$  and  $t' = 2t$ . Otherwise  $g_0 = \Omega g'_0$  and  $t' = 2t-1$ . Lemma 6.7.4-8 implies that  $\text{bh}(g_0) = \text{bh}(g'_0) + (m-2)$  when  $g'_0 \notin M_1$ , and  $\text{bh}(g_0) = \text{bh}(g'_0) + (m-1)$  when  $g'_0 \in M_1$ . If  $\text{bh}(g_0) \geq 2\tilde{q}_t = 2(m-2)t + 2$ , then  $\text{bh}(g'_0) \geq (m-2) + 1 \geq 2$ , i.e.  $g'_0 \notin M_1$ . Hence  $\text{bh}(g_0) = \text{bh}(g'_0) + (m-2)$ . So  $\text{bh}(g'_0) \geq (m-2)(2t-1) + 2 = \text{bh}(\Omega^{t'})$ . Conversely, if  $\text{bh}(g'_0) \geq \text{bh}(\Omega^{t'})$ , then  $g'_0 \notin M_1$ . Therefore  $\text{bh}(g_0) = \text{bh}(g'_0) + (m-2) \geq \text{bh}(\Omega^{t'}) + (m-2) = 2\tilde{q}_t$ . □

**Lemma 6.37.** *Let  $g' \in G$  such that  $\phi(g')$  is  $z_1$ -positive. Then  $g'$  is  $(H, G_1)$ -positive.*

PROOF. Let  $m \in \mathbb{Z}$  such that  $g' = \Omega^m g$  for some  $g \in M$  with  $\Omega \not\leq_R g$ . Note that  $g' \notin G_1$  because  $\phi(g')$  is  $z_1$ -positive. We have  $\phi(g') = \Omega^m_B \phi(g)$ . If  $m \geq 0$ , then  $g' \in M \setminus M_1$ . Therefore  $g'$  is  $(H, G_1)$ -positive. If  $m < 0$ , namely  $m = -t$  for some  $t \geq 1$ , Theorem 2.13, Lemma 6.35 and Equation (6.25) imply that  $t+2 \leq \text{bh}_{A_1, A_2}(\phi(g)) \leq \lim(g) + 1 < (\text{bh}(g) + 2m - 5)/(m-2)$ . Hence  $\text{bh}(\Omega^t) \leq \text{bh}(g)$ . So, Lemma 6.36 implies that  $g'$  is  $(H, G_1)$ -positive. □

If  $g \in G$  is  $(H, G_1)$ -positive and  $\phi(g)$  is  $z_1$ -negative. Then  $\phi(g)^{-1}$  is  $z_1$ -positive. So, Lemma 6.36 implies that  $g^{-1}$  is  $(H, G_1)$ -positive as well, which contradicts Corollary 6.2. This proves Proposition 6.4.

**Corollary 6.38.** *We have  $\text{bh}_{A_1, A_2}(\phi(g)) = \text{lim}(g) + 1$  for all unmovable  $g \in M \setminus M_1$ .*

PROOF. Let  $g_0 \in M_1$  be the  $M_1$ -tale of  $g$ , and let  $g' = gg_0^{-1}$ . By definition,  $\tilde{q}_{t-1} < \text{bh}(g) = \text{bh}(g')$  where  $t = \text{lim}(g)$ . Lemma 6.36 implies that  $\Omega^{-t+1}g'$  is  $(H, G_1)$ -positive. Proposition 6.4 implies that  $\Omega_{\mathcal{B}}^{-t+1}\phi(g')$  is  $z_1$ -positive. By Theorem 2.13 and Lemma 6.35,  $t + 1 \leq \text{bh}_{A_1, A_2}(\phi(g')) \leq t + 1$ . Therefore  $\text{bh}_{A_1, A_2}(\phi(g)) = \text{lim}(g) + 1$ .  $\square$



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## Abstract

We introduce a condition on Garside groups that we call Dehornoy structure. An iteration of such a structure leads to a left order on the group. We show conditions for a Garside group to admit a Dehornoy structure, and we apply these criteria to prove that the Artin groups of type  $A$  and  $I_2(m)$ ,  $m \geq 4$ , have Dehornoy structures. We show that the left orders on the Artin groups of type  $A$  obtained from their Dehornoy structures are the Dehornoy orders. In the case of the Artin groups of type  $I_2(m)$ ,  $m \geq 4$ , we show that the left orders derived from their Dehornoy structures coincide with the orders obtained from embeddings of the groups into braid groups.

## Résumé

Nous présentons une condition sur les groupes de Garside que nous appelons la structure de Dehornoy. Une itération d'une telle structure conduit à un ordre à gauche sur le groupe. Nous montrons des conditions pour qu'un groupe de Garside admette une structure de Dehornoy, et nous appliquons ce critère pour prouver que les groupes d'Artin de type  $A$  et  $I_2(m)$ ,  $m \geq 4$ , ont des structures de Dehornoy. Nous montrons que les ordres à gauche sur les groupes d'Artin de type  $A$  obtenus à partir de leurs structures de Dehornoy sont les ordres de Dehornoy. Dans le cas des groupes d'Artin du type  $I_2(m)$ ,  $m \geq 4$ , nous montrons que les ordres à gauche dérivés de leurs structures de Dehornoy coïncident avec les ordres obtenus à partir des plongements de ces groupes dans les groupes de tresses.