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Sur la dynamique des fluides dans le domaine de communication extérieur d'un espace-temps de Schwarzschild

Shuyang Xiang

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HAL Id: tel-01832314

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**Thèse de Doctorat de
l'Université Pierre et Marie Curie**

Présentée et soutenue publiquement le 5 juillet 2017

pour l'obtention du grade de
Docteur de l'Université Pierre et Marie Curie
Spécialité : Mathématiques Appliquées

par

Shuyang XIANG

**Sur la dynamique des fluides
dans le domaine de communication extérieur
d'un espace-temps de Schwarzschild**

après avis des rapporteurs

François Dubois
Allen Tesdall

devant le jury composé de

François Dubois	Rapporteur
Philippe G. LeFloch	Directeur de Thèse
Sidi Mahmoud Kaber	Examineur
Carlos Parés	Examineur
Giovanni Russo	Examineur



Shuyang XIANG :

Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis
Lions, F-75005, Paris, France.

Email: xiang@ljl.math.upmc.fr

Remerciements

Je tiens tout à bord à exprimer ma profonde et infinie gratitude envers mon directeur de thèse, Philippe LeFloch, pour la confiance qu'il m'a accordé en acceptant d'encadrer ce travail doctoral, pour ses conseils géniaux et pour toutes les heures qu'il a consacré à m'écouter.

Je veux remercier Allen Tesdall et François Dubois d'avoir pris pris de temps de rapporter cette thèse, ainsi que Sidi Mahmoud Kaber, Carlos Parés et Giovanni Russo d'avoir accepté de faire partie de mon jury.

Je remercie tous les membre du Laboratoire Jaques-Louis Lions, pour le climat sympathique dans lequel ils m'ont permis de travailler. Merci donc à Catherine, Salima, Malika et Khashayar, pour toute l'aide.

Tout cela n'aurait pas été possible sans mes amis qui ont été présent durant cette thèse. Je pense en particulier à Stefan, Guillaume(s), Nicolas, Dena, Lucile, Cécile, Jean-François, Malik, Giacomo, Ludovick, Casimir, Jean-Paul, Carlo, Philippe, Olivier, Léa, Chaoyu, Jamin, Yangyang...

Merci beaucoup aussi à Lingling, Siyuan, Di, Qing (aussi le petit Clément), Vincent Galmiche et toute sa famille pour le support moral pendant ces années de theèse.

J'ai envie de dire un énorme merci à Florian, pour sa présence à mes côtés, pour son soutien, pour ses conseils, pour tout ce qu'il m'a donné, tout ce qu'il m'a apporté de ces années parfois compliquées. Merci beaucoup aussi à Alain, Fabienne et toute la famille de Florian.

Je tiens finalement à remercier ma famille. 最后要感谢的, 是我远在中国的爸爸妈妈。感谢你们一直以来对我的支持, 理解, 包容, 特别是我远在巴黎求学工作的时光。感谢你们与我分享这些年我所有的快乐, 悲伤, 焦虑, 孤独。或许这苍白的

文字不足以表达我心里全部的感受，但我依然想告诉你们，爸爸妈妈，我爱你们。
希望你们永远健康快乐。

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Introduction

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1 Objectif de ce travail

Dans cette thèse nous nous intéressons à des problèmes liés à la dynamique de fluides évoluant dans le domaine de communication extérieur d'un espace-temps de Schwarzschild. Ce travail est directement motivé par le programme initié par LeFloch sur les solutions faibles des équations d'Einstein-Euler [2, 3, 16, 26, 27, 28]. Cependant, nous nous concentrons ici sur les équations d'Euler posées sur un espace courbe et nous étudions la dynamique de fluides compressibles relativistes, en introduisant ici de nouvelles techniques mathématiques et numériques pour ce problème.

Comme une sous-discipline de la mécanique des fluides qui traite des liquides et des gaz, la dynamique des fluides offre une structure systématique et sa solution implique généralement le calcul des diverses propriétés du fluide. Déterminer les solutions aux équations classiques du mouvement d'un fluide reste un domaine de recherche très actif et ces équations deviennent encore plus complexes lorsque l'on considère le problème posé dans un espace-temps courbe, qui est, dans notre cas, l'espace-temps de Schwarzschild.

La métrique de Schwarzschild, une solution des équations d'Einstein de la relativité générale, permet de décrire la géométrie de l'espace-temps (sa courbure), et donc le champ gravitationnel à l'extérieur d'une masse sphérique. Cette solution fournit une approximation très pertinente pour décrire des objets astronomiques tournant lentement, comme par exemple la Terre ou le Soleil.

Dans les coordonnées dite "de Schwarzschild" (t, r, θ, φ) , la métrique de Schwarzschild a la forme :

$$g = -\left(1 - \frac{2M}{r}\right)c^2 dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (1.1)$$

où:

- t est la coordonnée de temps du point et r est la coordonnée radiale du point,
- $d\theta^2 + \sin^2 \theta d\varphi^2$ est la métrique canonique, avec $\theta \in [0, 2\pi)$, $\varphi \in [0, \pi)$,

- la constante $c \in [0, +\infty]$ est la vitesse de la lumière et la constante $M \in (0, +\infty)$ est la masse de l'objet.

Observons que le coefficient en dr^2 de la métrique tend vers l'infini lorsque $r = 2M$. Néanmoins, ce point limite n'est pas une vraie singularité de l'espace-temps de Schwarzschild car il s'agit d'une pathologie des coordonnées choisies. Dans les coordonnées de Lemaitre, par exemple, les coordonnées de Eddington-Finkelstein ou les coordonnées de Kruskal-Szekeres, la métrique devient régulière. L'intérieur de la solution Schwarzschild avec $0 < r \leq 2M$ est complètement séparé de l'extérieur par $r = 2M$ que nous nommons l'horizon du trou noir de Schwarzschild. D'ici, il est naturel d'étudier la dynamique des ondes nonlinéaires dans le domaine de communication extérieur d'un espace-temps de Schwarzschild $r > 2M$. Une remarque ici est que le cas où $r = 0$ est totalement différent. À ce point singulier $r = 0$, l'espace-temps lui-même n'est pas bien défini. Mais ce genre de singularité est aussi une caractéristique générique de la théorie de la relativité générale et son existence n'est pas un cas spécifique.

L'équation d'Einstein est en conformité avec la conservation locale de l'énergie et du moment, qui s'exprime

$$\nabla_\alpha (T^\alpha_\beta(\rho, u)) = 0, \quad (1.2)$$

où ∇ est la connection Levi-Civita associée avec la métrique (dans notre cas, la métrique de Schwarzschild (1.1)) et T est le tenseur d'énergie-impulsion qui représente la répartition de masse et d'énergie dans l'espace-temps. Dans le cas d'un fluide parfait, T a la forme:

$$T^\alpha_\beta(\rho, u) = \rho c^2 u^\alpha u_\beta + p(\rho)(g^\alpha_\beta + u^\alpha u_\beta), \quad (1.3)$$

dépendant de la densité de la masse-énergie des fluides $\rho \in (0, +\infty)$ et son champ de vitesse $u = (u^\alpha)$, normalisé et orienté vers le futur:

$$u^\alpha u_\alpha = -1, \quad u^0 > 0. \quad (1.4)$$

La pression p est une fonction $p = p(\rho)$ de la densité et, par souci de simplicité, on suppose que le fluide est isotherme, c'est à dire, $p(\rho) = k^2 \rho$ où $k \in (0, c)$ représente la vitesse du son. On utilise une notation standard pour la métrique $g = g_{\alpha\beta} x^\alpha x^\beta$ et son inverse ($g^{\alpha\beta}$) dans les coordonnées (x^α) et on monte ou descend les indices selon la convention $u^\alpha = g^{\alpha\beta} u_\beta$, $u_\alpha = g_{\alpha\beta} u^\beta$.

Les modèles que nous étudierons prennent la forme d'un système hyperbolique non-linéaire de loi de bilan. Ce genre de systèmes ont été d'abord étudiés par Dafermos et Hsiao [8], Liu [40] et, plus tard, [14, 17, 9] (et dans un contexte plus général [3, 15]).

Ce chapitre introductif décrit des modèles déduits des équations (1.1), (1.2) et présente les contributions principales apportées par cette thèse. Le chapitre est organisé de la façon suivante. Dans la Section 2, on traite le modèle d'Euler dans

l'espace-temps de Schwarzschild, en donnant la théorie d'existence dans la classe des solutions à variation totale bornée contenant éventuellement des ondes de choc. Les résultats du modèle de Burgers, le cas sans pression du modèle d'Euler sont présentés dans la Section 3, consacrée à l'étude de l'existence, l'unicité de la solution dont une version de variation totale modifiée par la géométrie reste bornée, ainsi que la stabilité nonlinéaire de la solution stationnaire. La Section 4 porte sur un travail numérique, motivé par des questions d'analyse des comportements asymptotiques des solutions avec des données initiales générales. Pour conclure, nous donnons une discussion sur les directions de recherche possibles.

2 Modèle d'Euler à l'extérieur d'un trou noir de Schwarzschild

2.1 Équation d'Euler

L'équation d'Euler pour un fluide isotherme à l'extérieur du trou noir de Schwarzschild est obtenue à partir de (1.1) et (1.2):

$$\begin{aligned} & \partial_t \left(r^2 \frac{1 + \epsilon^4 k^2 v^2}{1 - \epsilon^2 v^2} \rho \right) + \partial_r \left(r(r - 2M) \frac{1 + \epsilon^2 k^2}{1 - \epsilon^2 v^2} \rho v \right) = 0, \\ & \partial_t \left(r(r - 2M) \frac{1 + \epsilon^2 k^2}{1 - \epsilon^2 v^2} \rho v \right) + \partial_r \left((r - 2M)^2 \frac{v^2 + k^2}{1 - \epsilon^2 v^2} \rho \right) \\ & = \frac{3M}{r} (r - 2M) \frac{v^2 + k^2}{1 - \epsilon^2 v^2} \rho - \frac{M}{\epsilon^2 r} (r - 2M) \frac{1 + \epsilon^4 k^2 v^2}{1 - \epsilon^2 v^2} \rho + \frac{2}{r} (r - 2M)^2 k^2 \rho, \end{aligned} \quad (2.1)$$

pour tout $r > 2M$, où $\frac{1}{\epsilon} = c$ est la vitesse de la lumière et nous rappelons que M est la masse du trou noir et $k \in (0, 1/\epsilon)$ est la vitesse du son. Les inconnues sont la densité de la masse-énergie du fluide et la vitesse v donnée par le champ de vitesse:

$$v := \frac{1}{(1 - 2M/r)} \frac{u^1}{u^0} \in (-1/\epsilon, 1/\epsilon). \quad (2.2)$$

Remarquons que nous avons introduit ϵ au lieu d'utiliser c directement en vue d'étudier les régimes limites.

Le système est strictement hyperbolique et vraiment nonlinéaire dans tous les champs de caractéristiques quand la vitesse du son $k \in (0, 1/\epsilon)$. Ses invariants de Riemann a la forme:

$$w = \frac{1}{2\epsilon} \ln \left(\frac{1 - \epsilon v}{1 + \epsilon v} \right) + \frac{k}{1 + \epsilon^2 k^2} \ln \rho, \quad z = \frac{1}{2\epsilon} \ln \left(\frac{1 - \epsilon v}{1 + \epsilon v} \right) - \frac{k}{1 + \epsilon^2 k^2} \ln \rho, \quad (2.3)$$

et les valeurs propres correspondantes sont données par

$$\lambda(\rho, v) = \left(1 - \frac{2M}{r}\right) \frac{v - k}{1 - \epsilon^2 k v}, \quad \mu(\rho, v) = \left(1 - \frac{2M}{r}\right) \frac{v + k}{1 + \epsilon^2 k v}. \quad (2.4)$$

2.2 Fluides à l'équilibre

Nous considérons maintenant la solution stationnaire du modèle d'Euler dans l'espace-temps de Schwarzschild, c'est à dire, une solution qui satisfait (2.1) et ne dépend pas du temps. Cette solution est donnée par:

$$\begin{aligned} & \frac{d}{dr} \left(r(r - 2M) \frac{(1 + \epsilon^2 k^2)}{1 - \epsilon^2 v^2} \rho v \right) = 0, \\ & \frac{d}{dr} \left((r - 2M)^2 \frac{v^2 + k^2}{1 - \epsilon^2 v^2} \rho \right) \\ & = \frac{M}{r} \frac{(r - 2M)}{1 - \epsilon^2 v^2} \left(3\rho v^2 + 3k^2 \rho - \epsilon^{-2} \rho - \epsilon^2 k^2 \rho v^2 \right) + \frac{2k^2}{r} (r - 2M)^2 \rho, \end{aligned} \quad (2.5)$$

pour $r > 2M$. Par le système (2.5), on peut obtenir une relation algébrique entre le rayon r et la vitesse v :

$$G(r, v) := \ln \frac{1 - \epsilon^2 v^2}{1 - 2M/r} + \frac{2\epsilon^2 k^2}{1 - \epsilon^2 k^2} \ln (r^2 |v|) = \text{const.}, \quad r > 2M. \quad (2.6)$$

Pour avoir une solution stationnaire $v(r)$ sur $(0, 2M)$, on souhaite que pour tout $r > 2M$ fixé, (2.6) admette une racine v . Mais malheureusement, ce n'est pas toujours le cas. En effet, la constante à la droite de l'équation (2.6) détermine si une solution stationnaire régulière peut être définie dans toute la région hors du trou noir de Schwarzschild $r > 2M$. On trace la fonction $G(r, v)$ en fixant r dans la Figure 2.1 et on voit immédiatement que v ne peut pas être définie si la constante à la droite de l'équation (2.6) est plus petite que le minimum de G .

Plus précisément, on a

$$\frac{dv}{dr} = \frac{v}{\epsilon^2} \frac{(1 - \epsilon^2 v^2)(1 - \epsilon^2 k^2)}{r(r - 2M)} \left(\frac{2\epsilon^2 k^2}{1 - \epsilon^2 k^2} (r - 2M) - M \right) / (v^2 - k^2).$$

Donc, s'il existe un point r_* où la solution est sonique, c'est à dire, $v(r_*) = \pm k$ avec k la vitesse du son, la définition de solution ne peut pas continuer. On nomme ce point r_* le point sonique. L'existence du point sonique distingue deux régimes des solutions.

Théorème 2.1. *On se donne des valeurs de la vitesse de la lumière $1/\epsilon > 0$, de la vitesse du son $k \in (0, 1/\epsilon)$ et de la masse du trou noir $M > 0$, et on considère le modèle d'Euler statique dans l'espace-temps de Schwarzschild (2.5). Pour $r_0 > 2M$*

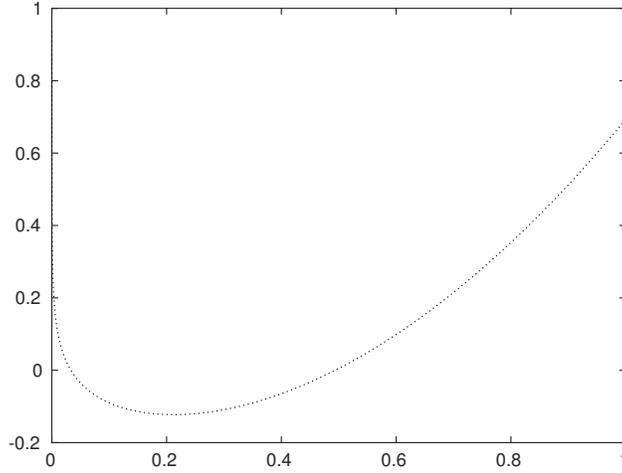


Figure 2.1: La fonction G .

arbitraire, et pour toute densité $\rho_0 > 0$ et vitesse $|v_0| \leq 1/\epsilon$, il existe une seule solution régulière:

$$\rho = \rho(r; r_0, \rho_0, v_0), \quad v = v(r; r_0, \rho_0, v_0),$$

satisfaisant (2.5) et la condition initiale $\rho(r_0) = \rho_0$, $v(r_0) = v_0$. De plus, le signe de v et $|v| - k$ ne change pas dans le domaine de définition. On a de plus deux régimes différents:

1. **Régime sans point sonique.** Si pour tout r dans le domaine de définition, $|v| \neq k$, la solution maximale est définie sur $(2M, +\infty)$.
2. **Régime avec un point sonique.** S'il existe un point r_* où $v(r_*; r_0, \rho_0, v_0) = \pm k$, la solution maximale est définie seulement sur $(2M, r_*)$ ou sur $(r_*, +\infty)$ et nous avons $\frac{d}{dr}v(r_*; r_0, \rho_0, v_0) = \infty$.

Pour construire une solution stationnaire globale en espace, nous avons recours à un choc stationnaire. La solution stationnaire que nous construisons satisfait le modèle d'Euler statique (2.5) au sens des distributions et est composée de deux solutions stationnaires régulières (ρ_L, v_L) , (ρ_R, v_R) connectées par un choc stationnaire au point unique r_0 tel que

$$v_R(r_0) = \frac{k^2}{v_L(r_0)}, \quad \rho_R(r_0) = \frac{1 - \epsilon^2 k^4 / v_L(r_0)^2}{1 - \epsilon^2 v_L(r_0)^2} \frac{v_L(r_0)^2}{k^2} \rho_L(r_0).$$

On annonce le théorème et réfère le lecteur à la Section 1.7 pour plus de détails.

Théorème 2.2 (Solution stationnaire globale en espace). *Pour tout rayon $r_0 > 2M$, toute densité $\rho_0 > 0$ et toute vitesse $|v_0| < 1/\epsilon$, le modèle d'Euler (2.5) avec $\rho(r_0) = \rho_0$, $v(r_0) = v_0$ admet une solution faible unique définie pour tout $r \in (2M, +\infty)$*

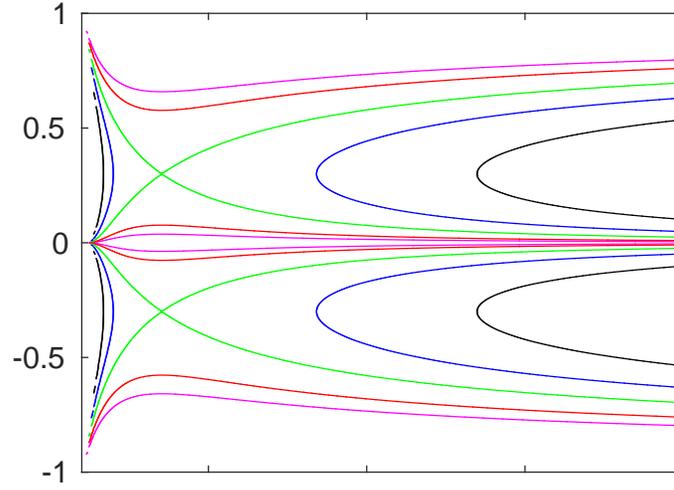


Figure 2.2: Solutions stationnaires régulières du système d'Euler

contenant au plus un choc statique. De plus, la famille des solutions stationnaires contenant éventuellement un choc dépendant de manière Lipschitzienne de ses arguments r_0, ρ_0, v_0 , lorsqu'ils varient dans le domaine admissible.

2.3 Existence pour le problème de Cauchy

Théorème principal

La solution stationnaire est sans doute une solution du modèle d'Euler à l'extérieur du trou noir de Schwarzschild. Pour une donnée générale, nous avons le résultat dans la classe des solutions à variation totale bornée:

Théorème 2.3 (Existence des solutions du modèle d'Euler). *Nous considérons le système d'Euler décrivant les fluides dans un espace-temps (2.1) posé sur $r > 2M$. Lorsque la donnée initiale $\rho_0 = \rho_0(r) > 0$, $|v_0| = |v_0(r)| < 1/\epsilon$ satisfait*

$$TV_{[2M+\delta, +\infty)}(\ln \rho_0) + TV_{[2M+\delta, +\infty)}\left(\ln \frac{1 - \epsilon v_0}{1 + \epsilon v_0}\right) < +\infty,$$

pour tout $\delta > 0$, il existe une solution faible $\rho = \rho(t, r)$, $v = v(t, r)$ définie sur $(0, T)$ avec $\rho(0, \cdot) = \rho_0$, $v(0, \cdot) = v_0$ où T est un instant fixé arbitraire (éventuellement infini) et pour un temps fini $t \in (0, T)$ et $\delta > 0$, nous avons

$$\sup_{t \in (0, T)} \left(TV_{[2M+\delta, +\infty)}(\ln \rho(t, \cdot)) + TV_{[2M+\delta, +\infty)}\left(\ln \frac{1 - \epsilon v(t, \cdot)}{1 + \epsilon v(t, \cdot)}\right) \right) < +\infty.$$

Nous ne donnons que l'idée de la preuve. Inspirée par Nishida [41], Smoller et Temple [44] qui ont traité les fluides dans un espace plat sans effet de la géométrie, la preuve du Théorème 2.3 est réalisée par une version de méthode de Glimm fondée sur le problème de Riemann généralisé.

Problème de Riemann généralisé

Le problème de Riemann généralisé est un problème de Cauchy avec une donnée initiale comprenant deux solutions stationnaires U_L, U_R séparées par une discontinuité de saut à un rayon fixé $r_0 > 2M$:

$$U_0 = \begin{cases} U_L(r) & 2M < r < r_0 \\ U_R(r) & r > r_0, \end{cases}$$

et on écrit $U_L(r_0) = U_L^0, U_R(r_0) = U_R^0$. On commence par étudier un système d'Euler homogène:

$$\begin{aligned} \partial_t \left(r^2 \frac{1 + \epsilon^4 k^2 v^2}{1 - \epsilon^2 v^2} \rho \right) + \partial_r \left(r(r - 2M) \frac{1 + \epsilon^2 k^2}{1 - \epsilon^2 v^2} \rho v \right) &= 0, \\ \partial_t \left(r(r - 2M) \frac{1 + \epsilon^2 k^2}{1 - \epsilon^2 v^2} \rho v \right) + \partial_r \left((r - 2M)^2 \frac{v^2 + k^2}{1 - \epsilon^2 v^2} \rho \right) &= 0, \end{aligned}$$

au point r_0 fixé. Nous considérons un problème de Riemann standard, qui est un problème de Cauchy de ce système sans terme source avec une donnée initiale formée par deux constantes U_L^0, U_R^0 avec une discontinuité de saut au point fixé r_0 . Ce problème est résolu par trois états constants y compris les deux états donnés et un autre état unique U_M^0 situé entre les deux précédents connectés par les ondes élémentaires (les chocs ou les raréfactions) qui sont les lignes droites. La solution ne dépend que de $\frac{r-r_0}{t}$.

Le modèle d'Euler dans l'espace-temps de Schwarzschild, par contre, a un terme source donnée par la géométrie qui a courbé les caractéristiques et nous pouvons avoir les formes approximatives des ondes élémentaires généralisées du modèle d'Euler (dans le sens distributionnel pour $t \rightarrow 0$). Nous avons donc donné une solution approximative du problème de Riemann généralisé $\tilde{U} = \tilde{U}(t, r)$ construite à partir de trois solutions stationnaires U_L, U_M, U_R parmi lesquelles U_M est déterminée uniquement par $U_M(r_0) = U_M^0$ et les trois états stationnaires sont connectés par les ondes élémentaires généralisées. Nous avons le lemme suivant pour estimer la différence entre cette solution construite \tilde{U} et la solution exacte.

Lemme 2.4. *Soit $\tilde{U} = \tilde{U}(t, r)$ la solution approximative du problème de Riemann généralisé du modèle d'Euler dans l'espace-temps de Schwarzschild (2.1). Pour $\Delta t, \Delta r > 0$ donnés tels que $\frac{\Delta r}{\Delta t} > \max(|\lambda|, |\mu|)$, et toutes les fonctions régulières ϕ définies sur*

$[0, +\infty) \times [r_0 - \Delta r, r_0 + \Delta r]$, nous avons

$$\begin{aligned} \Theta(\Delta t, \Delta r; \phi) &= \int_{r_0 - \Delta r}^{r_0 + \Delta r} \tilde{U}(t_0 + \Delta t, \cdot) \phi(t_0 + \Delta t, \cdot) dr - \int_{r_0 - \Delta r}^{r_0 + \Delta r} \tilde{U}(t_0, \cdot) \phi(t_0, \cdot) dr \\ &\quad + \int_{t_0}^{t_0 + \Delta t} F(r_0 + \Delta r, \tilde{U}(\cdot, r_0 + \Delta r)) \phi(\cdot, r_0 + \Delta r) dt \\ &\quad - \int_{t_0}^{t_0 + \Delta t} F(r_0 - \Delta r, \tilde{U}(\cdot, r_0 - \Delta r)) \phi(\cdot, r_0 - \Delta r) dt \\ &\quad + O(1) |U_R(r_0) - U_L(r_0)| \Delta t^2 \|\phi\|_{C^1}, \end{aligned} \tag{2.7}$$

où

$$\Theta(\Delta t, \Delta r; \phi) := \int_{t_0}^{t_0 + \Delta t} \int_{r_0 - \Delta r}^{r_0 + \Delta r} \left(\tilde{U} \partial_t \phi + F(r, \tilde{U}) \partial_r \phi + S(r, \tilde{U}) \phi \right) dr dt,$$

qui s'annule si et seulement si U est une solution exacte.

Méthode de Glimm

Nous pouvons alors introduire une version de méthode de Glimm qui nous produit une suite de solutions approchées dépendant d'un pas d'espace $\Delta r \rightarrow 0$. Elles sont les solutions du problème de Riemann généralisé par morceaux à chaque pas de temps et ses valeurs dans chaque cellule sont choisies par une valeur aléatoire. Une fois le pas d'espace Δr et le pas de temps Δt satisfait la condition de CFL, on peut continuer la construction. Plus précisément, on écrit

$$t_i = i\Delta t, \quad r_j = 2M + j\Delta r,$$

et

$$r_{i,j} = 2M + (w_i + j)\Delta r,$$

où $(w_i)_i$ est une suite equidistribuée dans $(-1, 1)$. Si la solution approximative U_Δ a déjà été construite pour tout $0 \leq t < t_i$, nous voulons prolonger la solution sur $t_i \leq t < t_{i+1}$:

1. Au temps $t = t_i$, on définit U_Δ comme une solution stationnaire par morceaux donnée par

$$\begin{aligned} \frac{d}{dr} F(r, U_\Delta(t_i, r)) &= S(r, U_\Delta(t_i, r)), \quad i + j \pmod{2} = 0, \quad r_j < r < r_{j+2}, \\ U_\Delta(t_i, r_{i,j+1}) &= U_\Delta(t_i, r_{i,j}). \end{aligned} \tag{2.8}$$

Remarquons que les solutions stationnaires peuvent contenir des chocs stationnaires.

2. On construit maintenant U_Δ sur $\{t_i < t < t_{i+1}, r_{j-1} < r < r_{j+1}\}$ (avec $i + j \bmod 2 = 0$):

$$U_\Delta(t, r) := \tilde{U}\left(t, r; t_i, r_j, \bar{U}_L, \bar{U}_R\right)$$

avec $\bar{U}_L(r) = U_\Delta(t_i, r)$, $r \in (r_{j-1}, r_j)$, $\bar{U}_R(r) = U_\Delta(t_i, r)$, $r \in (r_j, r_{j+1})$ deux morceaux des solutions stationnaires de $U_\Delta(t_i, r)$ et \tilde{U} la solutions du problème de Riemann avec les état \bar{U}_L, \bar{U}_R .

Ce schéma, tout d'abord, est capable de bien préserver une solution régulière ainsi qu'une solution stationnaire faible qui contient un choc stationnaire. Finalement, comme nous pouvons avoir toujours une borne de la variation totale de $\ln \rho$ sur $[2M + \delta, +\infty)$ où ρ est la densité des fluides, le Théorème d'Helly nous permet d'avoir une limite des solutions approximatives pour $\Delta r \rightarrow 0$ et cette fonction limite est exactement une solution faible du modèle d'Euler dans l'espace-temps de Schwarzschild pour $r > 2M$.

2.4 Etude des régimes limites

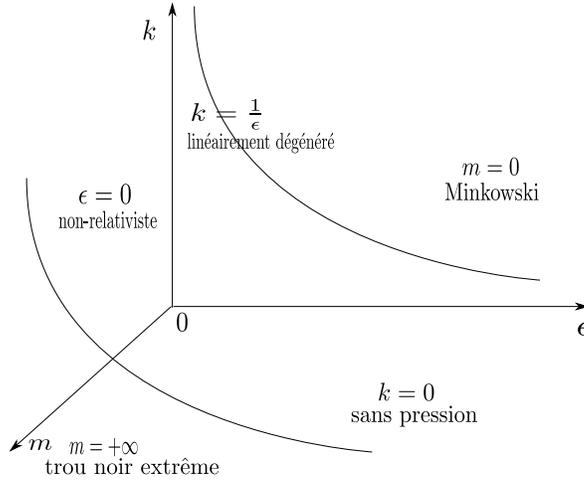
Enfin, on considère plusieurs cas limites, qui sont obtenus en faisant tendre les paramètres physiques vers leurs valeurs extrêmes. Il est nécessaire de mettre à l'échelle la masse M pour éviter l'explosion du terme source:

$$m := \frac{M}{\epsilon^2}. \quad (2.9)$$

Le modèle d'Euler a donc la forme $\mathcal{M}(\epsilon, k, m)$:

$$\begin{aligned} & \partial_t \left(r^2 \frac{1 + \epsilon^4 k^2 v^2}{1 - \epsilon^2 v^2} \rho \right) + \partial_r \left(r(r - 2\epsilon^2 m) \frac{1 + \epsilon^2 k^2}{1 - \epsilon^2 v^2} \rho v \right) = 0, \\ & \partial_t \left(r(r - 2\epsilon^2 m) \frac{1 + \epsilon^2 k^2}{1 - \epsilon^2 v^2} \rho v \right) + \partial_r \left((r - 2\epsilon^2 m)^2 \frac{v^2 + k^2}{1 - \epsilon^2 v^2} \rho \right) \\ & = \frac{3\epsilon^2 m}{r} (r - 2\epsilon^2 m) \frac{v^2 + k^2}{1 - \epsilon^2 v^2} \rho - \frac{m}{r} (r - 2\epsilon^2 m) \frac{1 + \epsilon^4 k^2 v^2}{1 - \epsilon^2 v^2} \rho + \frac{2}{r} (r - 2\epsilon^2 m)^2 k^2 \rho. \end{aligned} \quad (2.10)$$

Nous nous intéressons aux régimes limites déterminés par les valeurs des paramètres physiques, i.e. la masse du trou noir $m \in (0, +\infty)$, la vitesse de la lumière $\frac{1}{\epsilon} \in (0, +\infty)$, et la vitesse du son $k \in (0, 1/\epsilon)$. La Figure 2.3 fournit une illustration de tous les régimes.

Figure 2.3: Régimes du modèle $\mathcal{M}(\epsilon, k, m)$.

Fluide non-relativiste

En prenant $\epsilon \rightarrow 0$, on arrive au modèle Euler non-relativiste dans l'espace-temps de Schwarzschild $\mathcal{M}(0, k, m)$:

$$\begin{aligned} \partial_t(r^2\rho) + \partial_r(r^2\rho v) &= 0, \\ \partial_t(r^2\rho v) + \partial_r\left(r^2\rho(v^2 + k^2)\right) - 2k^2r\rho + m\rho &= 0, \quad t \geq 0, r > 0. \end{aligned} \quad (2.11)$$

Un phénomène intéressant est que le modèle est appliqué par le fluide non-relativiste mais il a quand même un terme de relaxation $m\rho$, qui est induit par la géométrie du trou noir de Schwarzschild. Ce modèle est étudié en Chapitre 4.

Fluide rigide

Le modèle pour décrire le fluide rigide dans l'espace-temps de Schwarzschild $\mathcal{M}(\epsilon, \frac{1}{\epsilon}, m)$ est obtenue quand on prend $k \rightarrow 1/\epsilon$:

$$\begin{aligned} \partial_t\left(r^2\frac{1 + \epsilon^2v^2}{1 - \epsilon^2v^2}\rho\right) + \partial_r\left(r(r - 2\epsilon^2m)\frac{2\rho v}{1 - \epsilon^2v^2}\right) &= 0, \\ \partial_t\left(r(r - 2\epsilon^2m)\frac{2\rho v}{1 - \epsilon^2v^2}\right) + \partial_r\left((r - 2\epsilon^2m)^2\frac{1 + \epsilon^2v^2}{\epsilon^2(1 - \epsilon^2v^2)}\rho\right) & \\ = 2\epsilon^2m\frac{r - 2\epsilon^2m}{\epsilon^2r}\frac{1 + \epsilon^2v^2}{1 - \epsilon^2v^2}\rho + 2\frac{(r - 2\epsilon^2m)^2}{\epsilon^2r}\rho. & \end{aligned} \quad (2.12)$$

Ce modèle a deux champs de caractéristiques linéairement dégénérés donc le problème de Riemann généralisé correspondant est résolu par trois états stationnaires liés par des discontinuités de contact. Suivant des étapes similaires au modèle $\mathcal{M}(\epsilon, k, m)$, nous avons l'existence pour le problème de Cauchy pour tout $t > 0$ fixé si les données initiales ρ_0, v_0 satisfont que pour tout $\delta > 0$ donné,

$$TV_{[2M+\delta, +\infty)}(\ln \rho_0) + TV_{[2M+\delta, +\infty)}\left(\ln \frac{1 - \epsilon v_0}{1 + \epsilon v_0}\right) < +\infty.$$

Une observation ici est que toutes nos estimations lorsque la vitesse du son est strictement inférieure à la vitesse de la lumière sont uniformes lorsque la vitesse du son s'approche de la vitesse de la lumière.

Fluide sans pression

Soit $k \rightarrow 0$, on obtient le modèle d'Euler sans pression $\mathcal{M}(\epsilon, 0, m)$:

$$\begin{aligned} \partial_t \left(r^2 \frac{\rho}{1 - \epsilon^2 v^2} \right) + \partial_r \left(r(r - 2\epsilon^2 m) \frac{\rho v}{1 - \epsilon^2 v^2} \right) &= 0, \\ \partial_t \left(r(r - 2\epsilon^2 m) \frac{\rho v}{1 - \epsilon^2 v^2} \right) + \partial_r \left((r - 2\epsilon^2 m)^2 \frac{\rho v^2}{1 - \epsilon^2 v^2} \right) &= \frac{m}{r} (3\epsilon^2 - 1)(r - 2\epsilon^2 m) \frac{\rho v^2}{1 - \epsilon^2 v^2}. \end{aligned} \quad (2.13)$$

La technique qu'on utilise n'est pas complètement pareille que celle du modèle d'Euler parce que (2.13) n'est pas strictement hyperbolique. Une analyse complète de ce cas sans pression est donnée comme une partie principale de cette thèse.

Fluide non-relativiste sans pression

On peut aussi prendre la limite $\epsilon \rightarrow 0$ dans (2.13) et définir le modèle d'Euler non-relativiste sans pression $\mathcal{M}(0, 0, m)$:

$$\partial_t(r^2 \rho) + \partial_r(r^2 \rho v) = 0, \quad \partial_t(r^2 \rho v) + \partial_r(r^2 \rho v^2) + m\rho = 0. \quad (2.14)$$

Mais l'effet du trou noir de Schwarzschild existe toujours, même dans ce modèle assez simple, exprimé en $m\rho$.

La masse de trou noir disparaissante

Quand la masse du trou noir s'annule, $m \rightarrow 0$, la métrique de Schwarzschild tend vers la métrique de Minkowski:

$$g = -c^2 dt^2 + dr^2 + r^2 g_{S^2}$$

et on a maintenant le modèle d'Euler dans l'espace-temps de Minkowski $\mathcal{M}(\epsilon, k, 0)$:

$$\begin{aligned} \partial_t \left(\frac{1 + \epsilon^4 k^2 v^2}{1 - \epsilon^2 v^2} \rho \right) + \partial_r \left(\frac{1 + \epsilon^2 k^2}{1 - \epsilon^2 v^2} \rho v \right) &= 0, \\ \partial_t \left(\frac{1 + \epsilon^2 k^2}{1 - \epsilon^2 v^2} \rho v \right) + \partial_r \left(\frac{v^2 + k^2}{1 - \epsilon^2 v^2} \rho \right) &= 0. \end{aligned} \tag{2.15}$$

Ce modèle a été étudié par Smoller et Temple [44] qui ont donné le résultat suivant: pour n'importe quelles données $\rho_0 = \rho_0(r) > 0$, $|v_0| = |v_0(r)| \leq 1/\epsilon$ telles que

$$TV(\ln \rho_0) + TV\left(\frac{1 - \epsilon v_0}{1 + \epsilon v_0}\right) < +\infty,$$

il existe une solution faible du modèle (2.15) $\rho = \rho(t, r)$ et $v = v(t, r)$, telle que pour tout $t > 0$, on a

$$\left(TV(\ln \rho(t, \cdot)) + TV\left(\ln \frac{1 - \epsilon v(t, \cdot)}{1 + \epsilon v(t, \cdot)}\right) \right) < +\infty.$$

3 Modèle de Burgers à l'extérieur d'un trou noir de Schwarzschild

3.1 Théorie d'existence

Théorème principal

Nous avons analysé le modèle d'Euler (2.1) lorsque la vitesse du son $0 < k < \frac{1}{\epsilon}$. Le régime limite $k \rightarrow 0$, ou le cas sans pression, par contre, est une autre histoire parce que le système n'est plus strictement hyperbolique. Nous considérons ici un modèle de Burgers qui peut être directement déduit du système d'Euler sans pression:

$$\partial_t(r^2 v) + \partial_r \left(r(r - 2M) \frac{v^2}{2} \right) = rv^2 - \frac{M}{\epsilon^2}, \quad r > 2M, \tag{3.1}$$

où nous rappelons que M est la masse du trou noir de Schwarzschild et $1/\epsilon$ est la vitesse de la lumière. Un calcul formel donne l'équation de Burgers dans la forme conservative:

$$\partial_t \left(\frac{v}{(1 - 2M/r)^2} \right) + \partial_r \left(\frac{v^2 - 1/\epsilon^2}{2(1 - 2M/r)} \right) = 0, \quad r > 2M. \quad (3.2)$$

Les solutions stationnaires de l'équation de Burgers sont données par

$$\partial_r \left(\frac{v^2 - 1/\epsilon^2}{2(1 - 2M/r)} \right) = 0, \quad r > 2M. \quad (3.3)$$

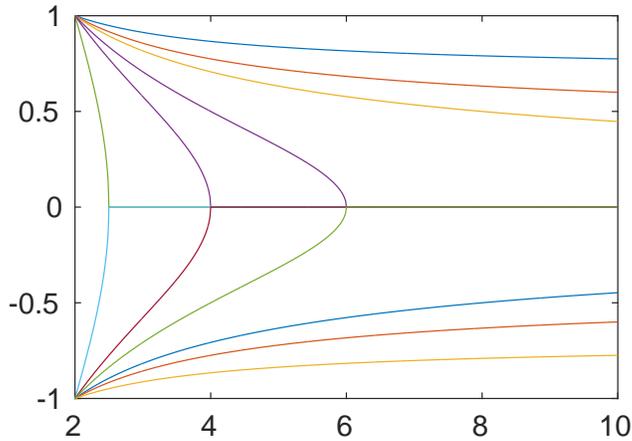


Figure 3.1: Solutions stationnaires de Burgers

Pour l'existence de la solution du problème de Cauchy, nous introduisons une classe de fonctions:

$$\mathcal{E} := \left\{ v \mid |v| \leq \frac{1}{\epsilon}, \widetilde{TV}(v) := \int_{2M}^{+\infty} \left| \partial_r \left(\left| \frac{v^2 - 1/\epsilon^2}{1 - 2M/r} + \frac{1}{\epsilon^2} \right|^{1/2} \text{sgn}(v) \right) \right| < \infty \right\}, \quad (3.4)$$

et nous travaillons maintenant sur les solutions dans la classe \mathcal{E} . En effet, nous avons le théorème suivant:

Théorème 3.1 (L'existence pour le modèle de Burgers). *Nous considérons l'équation de Burgers relativiste dans l'espace-temps de Schwarzschild (3.2) à l'extérieur du trou noir $r > 2M$ où M est la masse du trou noir de Schwarzschild. Pour toute vitesse $|v_0| = |v_0(r)| < 1/\epsilon$ où $1/\epsilon$ est la vitesse de la lumière telle que $\widetilde{TV}(v_0) < +\infty$ où \widetilde{TV}*

est donné par (3.4), il existe une solution faible de l'équation de Burgers relativiste dans l'espace-temps de Schwarzschild $v = v(t, r)$ définie sur $(0, +\infty) \times (2M, +\infty)$ telle que pour tout $t > 0$,

$$\widetilde{TV}(v(t, \cdot)) < +\infty. \quad (3.5)$$

De plus, \widetilde{TV} est décroissante en fonction du temps t :

$$\widetilde{TV}(v(s, \cdot)) \leq \widetilde{TV}(v(t, \cdot)), \quad \text{pour tout } 0 \leq t \leq s.$$

Nous sommes obligés de répondre aux trois questions suivantes avant de donner une preuve:

- Quels sont les comportements des solutions stationnaires satisfaisant (3.3)?
- Est-ce que le problème de Riemann généralisé dont la donnée initiale est composée par deux solutions stationnaires (différentes) admet une solution (globale en temps)?
- Comment construire les solutions approximatives par une méthode de Glimm raisonnable fondée sur les problèmes de Riemann généralisé et quelle variation totale utiliser?

Solutions stationnaires et problème de Riemann généralisé

Les deux premières questions sont traitées par les théorèmes suivants:

Théorème 3.2. *Nous considérons le modèle de Burgers statique (3.3) décrivant les fluides dans un l'espace-temps de Schwarzschild. Pour tout $r_0 > 2M$ et toute vitesse $v_0 \in [-\frac{1}{\epsilon}, \frac{1}{\epsilon}]$, il existe une solution stationnaire $v_* = v_*(r)$ avec $v_*(r_0) = v_0$ et $\text{sgn}(v_*) = \text{sgn}(v_0)$ dans le domaine de définition. De plus, nous avons:*

- Si $0 \leq \sqrt{\frac{1/\epsilon^2 - v_0^2}{1 - 2M/r_0}} \leq \frac{1}{\epsilon}$, la solution stationnaire maximale est définie sur toute la région à l'extérieur du trou noir $(2M, +\infty)$.
- Si $\sqrt{\frac{1/\epsilon^2 - v_0^2}{1 - 2M/r_0}} > \frac{1}{\epsilon}$, la solution stationnaire maximale est définie sur $(2M, r^\sharp)$ avec r^\sharp le rayon où la vitesse s'annule.

En fait, dans le domaine de définition, une solution stationnaire a toujours la forme:

$$v = \pm \frac{1}{\epsilon} \sqrt{1 - K^2 \left(1 - \frac{2M}{r}\right)}, \quad (3.6)$$

où $K \geq 0$ est une constante.

Notons que même si la solution stationnaire est probablement indéfinie sur $(2M, +\infty)$, nous pouvons toujours avoir la solution du problème de Riemann dont la donnée initiale est:

$$v_0(x) = \begin{cases} v_L(r) & 2M < r < r_0, \\ v_R(r) & r > r_0, \end{cases}$$

où v_L, v_R sont deux solutions stationnaires.

Théorème 3.3. *Il existe une solution unique du problème de Riemann généralisé définie pour tout $t > 0$, réalisée soit par un choc, soit par une raréfaction et la variation totale $\widetilde{TV}(v)$ est toujours constante. De plus, si la solution est un choc, nous avons, lorsque $t \rightarrow +\infty$:*

- *Le choc tend vers l'horizon du trou noir si et seulement si $v_L + v_R < 0$;*
- *Le choc tend vers l'infini si et seulement si $v_L + v_R > 0$;*
- *Le choc ne bouge pas avec le temps, si et seulement si $v_L + v_R = 0$.*

Si, par contre, la solution est une raréfaction, nous avons, lorsque $t \rightarrow +\infty$:

- *La borne inférieure (supérieure) de la raréfaction tend vers l'horizon du trou noir si et seulement si $v_L < 0$ ($v_R < 0$);*
- *La borne inférieure (supérieure) de la raréfaction tend vers l'infini si et seulement si $v_L > 0$ ($v_R > 0$);*
- *La borne inférieure (supérieure) de la raréfaction tend vers $r = r_L^h$ ($r = r_R^h$) si et seulement s'il existe un rayon r_L^h (r_R^h) tel que $v_L(r_L^h) = 0$ ($v_R(r_R^h) = 0$).*

Schéma de Glimm

Malheureusement, le Théorème 3.3 n'est pas suffisant pour construire le schéma de Glimm car la vitesse peut s'annuler et nous sommes obligés d'introduire un problème de Riemann multiple dont la donnée initiale est composée par trois états stationnaires séparées par deux rayons fixés. Nous annonçons qu'il existe une solution unique du problème de Riemann multiple dont la variation totale $\widetilde{TV}(v)$ est décroissante sans donner de détail. La méthode de Glimm pour l'équation de Burgers est donc un peu différente que celle pour le système d'Euler. En fait, remarquons que pour un r_0 donné, une solution stationnaire peut être toujours définie sur $(2M, r_0)$. Dans ce sens là, si la solutions stationnaire n'est pas définie dans une cellule entière, nous utilisons la valeur de v dans la cellule à sa droite pour l'approximer. Comme avant, nous écrivons

$$t_i = i\Delta t, \quad r_j = 2M + j\Delta r,$$

et

$$r_{i,j} = 2M + (w_i + j)\Delta r,$$

où $(w_i)_i$ est une suite equidistribuée dans $(-1, 1)$. Si la solution approximative v_Δ a été déjà construite pour tout $0 \leq t < t_i$, nous voulons prolonger la solution pour $t_i \leq t < t_{i+1}$:

1. Au temps $t = t_i$, on donne:

$$v_\Delta(t_i, r) = \begin{cases} v_{\Delta,i}^{j+1}(r) & i+j \text{ pair, } r_j < r < \min(r_{j+2}, r_{i,j+1}^\natural), \\ v_{\Delta,i}^{j+3}(r) & i+j \text{ pair, } \min(r_j, r_{i,j+1}^\natural) < r < r_{j+2}, \end{cases}$$

avec

$$\begin{aligned} v_{\Delta,i}^{j+1}(r) &= \operatorname{sgn}(v_\Delta(t_i^-, r_{i,j+1})) \sqrt{\frac{1}{\epsilon^2} - K_{i,j+1}^0 \left(1 - \frac{2M}{r}\right)}, & i+j \text{ pair,} \\ K_{i,j+1}^0 &= \frac{1}{1 - 2M/r_{i,j+1}} \left(\frac{1}{\epsilon^2} - v_\Delta(t_i^-, r_{i,j+1})^2\right), \\ r_{i,j+1}^\natural &= \sup\{r > 2M \mid v_{\Delta,i}^{j+1}(r) \neq 0\}. \end{aligned}$$

2. On définit la solution approximative sur $\{t_i < t < t_{i+1}, r_{j-1} < r < r_{j+1}\}$ ($i+j$ pair) par

$$v_\Delta(t, r) := \begin{cases} v_{\mathcal{R}}(t, r; t_i, v_\Delta(t_i, r)), & r_j > r_{i,j-1}^\natural, \\ v_{\mathcal{MR}}(t, r; t_i, v_\Delta(t_i, r)), & r_j \leq r_{i,j-1}^\natural, \end{cases}$$

où $v_{\mathcal{R}}$ est la solution du problème de Riemann généralisé et $v_{\mathcal{MR}}$ la solution du problème de Riemann multiple.

Comme la variation totale de $\left(\left|\frac{v^2-1/\epsilon^2}{1-2M/r} + \frac{1}{\epsilon^2}\right|^{1/2} \operatorname{sgn}(v)\right)$ est décroissante en fonction du temps, on peut prouver que la limite de la suite v_Δ pour $\Delta r \rightarrow 0$ est une solution du modèle de Burgers dans l'espace-temps de Schwarzschild $r > 2M$.

3.2 Unicité de la solution

L'unicité de la solution de l'équation de Burgers est donné par le théorème suivant.

Théorème 3.4. *Soit $v_{0,1} = v_{0,1}(r), v_{0,2} = v_{0,2}(r) \in \mathcal{E}$ deux vitesses initiales, les solutions correspondantes $v_1 = v_1(t, r), v_2 = v_2(t, r)$ du modèle de Burgers (3.2) avec $v_1(0, \cdot) = v_{0,1}$ et $v_2(0, \cdot) = v_{0,2}$ satisfont*

$$\int_{2M}^{+\infty} \frac{|v_2(t, r) - v_1(t, r)|}{(1 - 2M/r)^2} \leq e^{t/2\epsilon M} \int_{2M}^{+\infty} \frac{|v_{0,2} - v_{0,1}|}{(1 - 2M/r)^2}. \quad (3.7)$$

La preuve du Théorème 3.4 suit une méthode de viscosité. On introduit l'équation suivante:

$$\partial_t \left(\frac{v_\alpha}{(1 - 2M/r)^2} \right) + \partial_r \left(\frac{v_\alpha^2 - 1/\epsilon^2}{2(1 - 2M/r)} \right) = \alpha \partial_r (K(r) \partial_r v_\alpha), \quad r > 2M,$$

où $\alpha > 0$ est un paramètre et $K = K(r) > 0$ une fonction régulière dépendante d'une seule variable r . Par une régularisation standard et une intégration sur r entre $2M$ et $+\infty$, nous avons l'estimation dans L^1 .

En effet, cette méthode de viscosité nous permet de prouver une fois de plus que la variation totale de $\left(\left| \frac{v^2 - 1/\epsilon^2}{1 - 2M/r} + \frac{1}{\epsilon^2} \right|^{1/2} \operatorname{sgn}(v) \right)$ sur $(2M, +\infty)$ est décroissante en temps.

3.3 Stabilité des solutions stationnaires

Nous nous intéressons aussi à la stabilité des solutions stationnaires par morceaux. Nous voulons étudier le comportement d'une solution dont la donnée initiale est formée de deux solutions stationnaires v_*, v_{**} avec une perturbation.

Théorème 3.5. *Considérons une solution du modèle de Burgers (3.2) $v = v(t, r)$ dans l'espace-temps de Schwarzschild dont la donnée initiale est composée de deux solutions stationnaires v_*, v_{**} perturbée par une fonction à support compacte. Nous avons les résultats suivants:*

- Si $v_* > v_{**}$, nous avons:
 - Si de plus, $v_* > 0$, il existe un temps fini à partir duquel la solution est un choc généré par v_*, v_{**} .
 - Si de plus, $v_* < 0$, la solution tend vers un choc généré par v_*, v_{**} quand $t \rightarrow +\infty$.
- Si $v_* > v_{**}$, nous pouvons définir une "N-wave" $N = N(t, r)$ telle que: (i) $|v(t, r) - N(t, r)| = O(t^{-1})$ dans la région bornée par les bornes de N-wave; (ii) $|v(t, r) - N(t, r)| = O(t^{-1/2})$ dans la région entre les bornes de N-wave et les caractéristiques généralisées; (iii) Sinon, $v(t, r) = N(t, r)$.
- Si $v_* = v_{**}$, alors $\|v(t, r) - v_*(t, r)\|_{L^1(2M, +\infty)} = O(t^{-1/2})$.

Dans le Théorème 3.5, nous parlons des caractéristiques généralisées. En effet, une caractéristique généralisée associée à la solution $v = v(t, r)$ du modèle de Burgers (3.2) est une courbe intégrale de

$$\frac{d\xi}{dt} = \left(1 - \frac{2M}{\xi}\right)v(t, \xi),$$

au sens que

$$\frac{d\xi}{dt} \in \left[\left(1 - \frac{2M}{\xi}\right)v(t, \xi+), \left(1 - \frac{2M}{\xi}\right)v(t, \xi-) \right], \quad \text{a.e. on } t.$$

Nous observons qu'une caractéristique généralisée se propage soit avec la vitesse du choc, soit avec la vitesse caractéristique. Le comportement de la caractéristique généralisée nous permet d'avoir une preuve du Théorème 3.5. En effet, supposons que la donnée initiale est v_* sur $(2M, r_*)$ et v_{**} sur (r_{**}, ∞) et nous pouvons tracer deux caractéristiques généralisées depuis $(0, r_*)$, $(0, r_{**})$. Ces deux courbes peuvent s'approcher ou s'éloigner l'une de l'autre, en fonction des valeurs de v_* et v_{**} . Nous obtiendrons:

- un éventail de la raréfaction si les deux courbes sont plus en plus loins,
- un choc si les deux courbes sont plus en plus proches.

3.4 Comparaison avec le système avec pression

Nous terminons cette partie par une comparaison entre le modèle d'Euler et l'équation de Burgers dans l'espace-temps de Schwarzschild $r > 2M$. Comme expliqué avant, le modèle de Burgers est un cas limite du système d'Euler. Nous avons observé que les formes des solutions stationnaires sont plutôt similaires, présentées dans les Figures 2.2, 3.1.

Mais en même temps, nous avons aussi observé les résultats différents de ces deux cas, résumés dans le Tableau 1.

4 Étude numérique dans la géométrie de Schwarzschild

4.1 Schéma "équilibre"

Dans cette section, on présente les résultats numériques obtenus par plusieurs schémas différents. Ce travail a été motivé par les questions ouvertes dans le Tableau 1 pour la dynamique des fluides isothermes évoluant dans le domaine de la communication hors d'un trou noir de Schwarzschild. L'objectif est de construire des schémas "équilibres" qui sont formulés à partir de la géométrie de Schwarzschild pour bien préserver l'état stationnaire à l'extérieur du trou noir de Schwarzschild. Nous avons vu que tous les schémas proposés dans ce travail sont capables de préserver exacte-

<i>Modèle</i>	<i>de Burgers</i>	<i>d'Euler</i>
Solution stationnaire	L'existence et l'unicité d'une solution avec sa forme explicite obtenue	L'existence et l'unicité d'une solution sans forme explicite
Problème de Riemann généralisé	L'existence et l'unicité d'une solution avec sa forme explicite obtenue, globale en temps	L'existence d'une solution sans forme exacte, locale en temps
Problème de Cauchy	L'existence et l'unicité d'une solution, la variation totale de $\left(\left \frac{v^2 - 1/\epsilon^2}{1 - 2M/r} + \frac{1}{\epsilon^2} \right ^{1/2} \text{sgn}(v) \right)$ décroissante	L'existence d'une solution sans la variation totale de $\ln \rho$ bornée
Stabilité des solutions stationnaires	Les comportements exacts	Pas encore de conclusion

Table 1: Les résultats des modèles de Burgers et d'Euler

ment (y compris au voisinage du bord du trou noir) les équilibres numériques discrets, et de servir éventuellement à des données initiales adaptées à notre problème.

Schéma des volumes finis de l'équation de Burgers

Pour simplifier le problème, la vitesse de la lumière est normalisée à 1 dans toutes les discussions numériques qui suivent. Soit $\Delta t, \Delta r$ le pas de temps et de l'espace satisfaisant la condition CFL, pour éviter les interactions des ondes de deux problèmes de Riemann pendant un pas du temps. Soit $v_j^n = \int_{r_{j-1/2}}^{r_{j+1/2}} v(t_n, r) dr$, et on introduit le schéma des volumes finis pour l'équation de Burgers dans l'espace-temps de Schwarzschild (3.2):

$$v_j^{n+1} = v_j^n - \frac{\Delta t}{\Delta r} (F_{j+1/2} - F_{j-1/2}) - \Delta t \frac{2M}{r_j^2} (V_j^{n2} - 1),$$

où $F_{j+1/2}$ et $F_{j-1/2}$ sont les flux $F_{j+1/2} = \mathcal{F}(r_{j+1/2}, v_j^n, v_{j-1}^n)$

$$\mathcal{F}(r, v_L, v_R) = \left(1 - \frac{2M}{r}\right) \frac{q^2(v_L, v_R) - 1}{2}. \tag{4.1}$$

On donne $q(\cdot, \cdot)$ comme la solution d'un problème de Riemann standard, c'est à dire, la solution de l'équation de Burgers standard:

$$\partial_t v + \partial_x \frac{v^2}{2} = 0,$$

avec la donnée initiale:

$$v_0 = \begin{cases} v_L & r < r_0, \\ v_R & r > r_0, \end{cases}$$

où v_L et v_R sont deux constantes. On a fait ce choix q en démontrant que q est assez proche de la solution de problème de Riemann généralisé quand le temps Δt est suffisamment petit.

Nous pouvons développer le schéma à l'ordre deux en modifiant les valeurs utilisées dans les flux $\mathcal{F}(r, v_L, v_R)$. L'idée est de considérer v_j^n comme une fonction linéaire et prendre en compte la condition d'entropie en même temps. Pour plus de détails, nous renvoyons le lecteur à la Section 3.3.

Nous donnons ici un choc de l'équation de Burgers par le schéma des volumes finis d'ordre un et d'ordre deux .

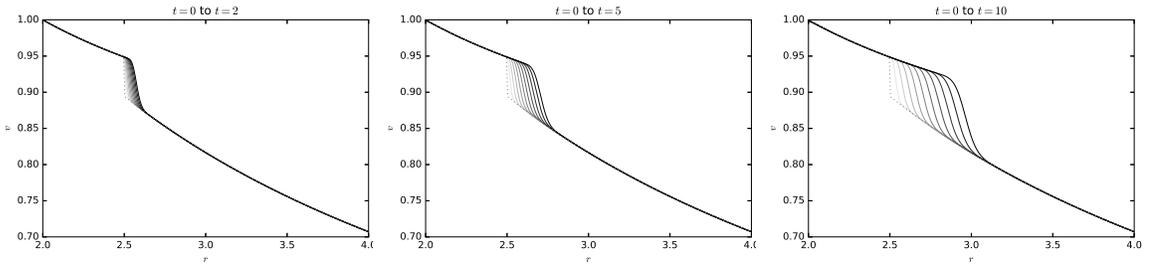


Figure 4.1: Un choc avec le schéma des volumes finis d'ordre un

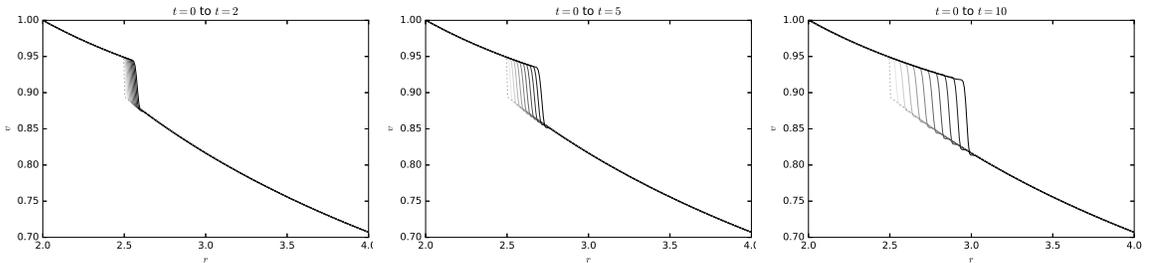


Figure 4.2: Un choc avec le schéma des volumes finis d'ordre deux

Schéma de Glimm pour l'équation de Burgers

L'une des contributions apportées par notre étude théorique de l'équation de Burgers est que la solution exacte du problème de Riemann généralisé est calculée explicitement. Cette forme nous permet de créer un schéma de Glimm qui considère la solution de Burgers comme une solution du problème de Riemann généralisé par

morceaux. On donne d'abord une suite equidistribuée dans $(-\frac{1}{2}, \frac{1}{2})$ et on écrit $r_{n,j} = 2M + (j + w_n)\Delta r$. Notre schéma de Glimm a la forme:

$$v_j^{n+1} = v_{\mathcal{R}}^{j,n}(t_{n+1}, r_{n,j}), \tag{4.2}$$

où $v_{\mathcal{R}}^{j,n} = v_{\mathcal{R}}^{j,n}(t, r)$ est la solution du problème de Riemann généralisé avec sa donnée initiale

$$v_0^{j,n} = \begin{cases} v_L^{j,n}(r), & r < r_{j+\text{sgn}(w_n)/2}, \\ v_R^{j,n}(r), & r > r_{j+\text{sgn}(w_n)/2}, \end{cases}$$

où les deux états $v_L^{j,n} = v_L^{j,n}(r)$ et $v_R^{j,n} = v_R^{j,n}(r)$ sont les solutions stationnaires de l'équation de Burgers telles que

$$\begin{cases} v_L^{j,n}(r_j) = v_j^n, & w_n \geq 0, \\ v_L^{j,n}(r_{j-1}) = v_{j-1}^n, & w_n < 0, \end{cases} \quad \begin{cases} v_R^{j,n}(r_j) = v_j^n, & w_n < 0, \\ v_R^{j,n}(r_{j+1}) = v_{j+1}^n, & w_n \geq 0. \end{cases}$$

Notons que nous avons choisi la valeur aléatoire une seule fois à chaque pas de temps au lieu de tous les points. Ce schéma a une forme un peu différente que celle de la partie théorique afin de mieux programmer. Le schéma de Glimm (4.2) a sans doute bien préservé toutes les solutions stationnaires. Il n'y presque pas de diffusion numérique parce qu'on la solution exacte du problème de Riemann. Nous donnons ici le même choc que celui dans les Figures 4.1, 4.2 tracé par le schéma de Glimm.

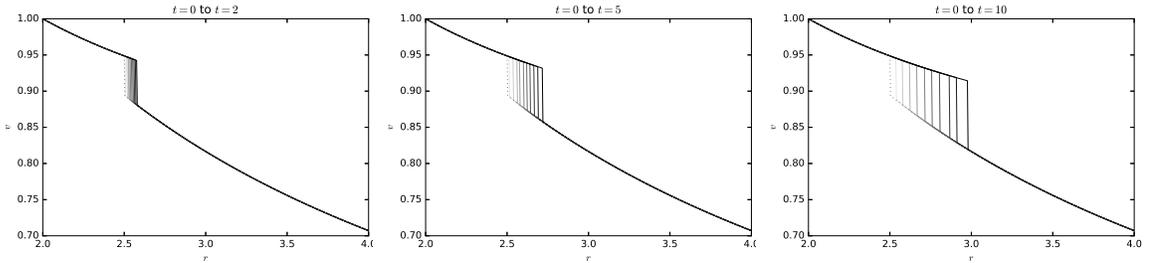


Figure 4.3: Un choc avec le schéma de Glimm

Schéma des volumes finis pour l'équation d'Euler

Pour commencer le schéma pour le modèle d'Euler, nous utilisons les notations

$$\partial_t U + \partial_r \left(\left(1 - \frac{2M}{r} \right) F(U) \right) = S(r, U),$$

avec

$$U = \begin{pmatrix} U^0 \\ U^1 \end{pmatrix} = \begin{pmatrix} \frac{1+k^2v^2}{1-v^2}\rho \\ \frac{1+k^2}{1-v^2}\rho v \end{pmatrix}, \quad F(U) = \begin{pmatrix} \frac{1+k^2}{1-v^2}\rho v \\ \frac{v^2+k^2}{1-v^2}\rho \end{pmatrix}, \quad (4.3)$$

et

$$S(r, U) = \begin{pmatrix} -\frac{2}{r}(1-2M/r)\frac{1+k^2}{1-v^2}\rho v \\ \frac{-2r+5M}{r^2}\frac{v^2+k^2}{1-v^2}\rho - \frac{M}{r^2}\frac{1+k^2v^2}{1-v^2}\rho + 2\frac{r-2M}{r^2}k^2\rho \end{pmatrix}.$$

Soit Δt , Δr le pas de temps et d'espace satisfaisant la condition CFL, nous donnons le schéma numérique pour le modèle d'Euler dans l'espace-temps de Schwarzschild:

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta r}(F_{j+1/2}^n - F_{j-1/2}^n) + \Delta t S_j^n, \quad (4.4)$$

où le flux est

$$F_{j-1/2}^n = \left(1 - \frac{2M}{r_{j-1/2}}\right) \mathcal{F}(U_{j-1/2-}^n, U_{j-1/2+}^n),$$

avec

$$\mathcal{F}(U_L, U_R) = \frac{F(U_L) + F(U_R)}{2} - \frac{1}{\lambda} \frac{U_R - U_L}{2},$$

où $\lambda = \Delta r / \Delta t$. Ici, F est le flux exact donné par (4.3). Les états $U_{j+1/2\pm}$, $U_{j-1/2\pm}$ et le terme source reflètent la géométrie de Schwarzschild:

$$\begin{aligned} (1 - v_{j+1/2-}^n)^2 v_{j+1/2-}^n \frac{2k^2}{1-k^2} r_{j+1/2}^{\frac{4k^2}{1-k^2}} / (1 - 2M/r_{j+1/2}) &= (1 - v_j^{n2}) v_j^n \frac{2k^2}{1-k^2} r_j^{\frac{4k^2}{1-k^2}} / (1 - 2M/r_j), \\ r_{j+1/2}(r_{j+1/2} - 2M) \rho_{j+1/2-}^n \frac{v_{j+1/2-}^n}{1 - v_{j+1/2-}^n} &= r_j(r_j - 2M) \rho_j^n \frac{v_j^n}{1 - v_j^{n2}}, \\ (1 - v_{j+1/2+}^n)^2 v_{j+1/2+}^n \frac{2k^2}{1-k^2} r_{j+1/2}^{\frac{4k^2}{1-k^2}} / (1 - 2M/r_{j+1/2}) &= (1 - v_{j+1}^n)^2 v_{j+1}^n \frac{2k^2}{1-k^2} r_{j+1}^{\frac{4k^2}{1-k^2}} / (1 - 2M/r_{j+1}), \\ r_{j+1/2}(r_{j+1/2} - 2M) \rho_{j+1/2+}^n \frac{v_{j+1/2+}^n}{1 - v_{j+1/2+}^n} &= r_j + 1(r_{j+1} - 2M) \rho_{j+1}^n \frac{v_{j+1}^n}{1 - v_{j+1}^n}, \end{aligned}$$

et

$$\begin{aligned} S_j^n &= \frac{1}{\Delta r} \int_{r_{j-1/2}}^{r_{j+1/2}} S(t_n, r) dr = \frac{1}{\Delta r} \int_{r_{j-1/2}}^{r_{j+1/2}} \partial_r \left((1 - 2M/r) F(U(t_n, r)) \right) dr \\ &= \frac{1}{\Delta r} \left((1 - 2M/r_{j+1/2}) F(U_{j+1/2-}^n) \right. \\ &\quad \left. - (1 - 2M/r_{j-1/2+}) F(U_{j-1/2+}^n) \right). \end{aligned}$$

La construction a garanti que le schéma préserve la solution stationnaire d'Euler et le schéma est d'ordre deux en r . On donne une solution du problème de Riemann généralisé résolu par notre schéma.

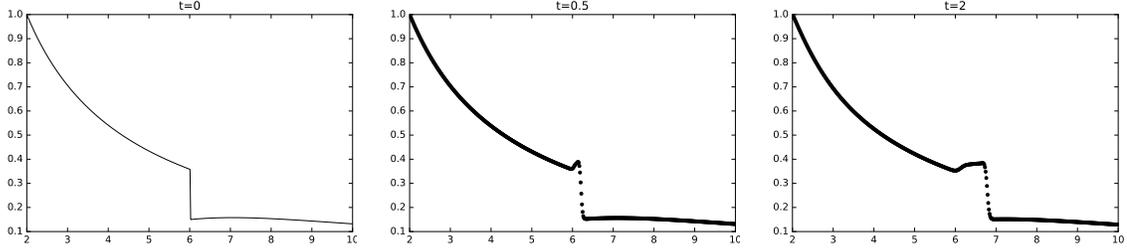


Figure 4.4: Une solution du problème de Riemann généralisé

4.2 Résultats numériques principaux

En vérifiant toutes les conclusions théoriques, nous avons présenté les expériences numériques pour obtenir les résultats sans preuve théorique rigoureuse, y compris les comportements des solutions stationnaires faibles (avec un choc) de tous les modèles introduits auparavant, la propagation des solutions du problème de Riemann généralisé du modèle d'Euler, etc. Les conclusions principales sont les suivantes.

Conjecture 4.1. *Pour une donnée initiale $v_0 = v_0(r) \in [-1, 1]$ définie sur $[2M, +\infty)$, la solution $v = v(t, r)$ de l'équation de Burgers (3.2) satisfait:*

- Si $v_0(2M) = 1$, il existe un temps fini $t_0 > 0$ tel que pour tout $t > t_0$, la solution v est un seul choc avec l'état à la gauche 1 et l'état à la droite $-\sqrt{\frac{2M}{r}}$.
- Si $v_0(2M) < 1$ et $\lim_{r \rightarrow +\infty} v_0(r) > 0$, il existe un temps fini $t_0 > 0$ tel que pour tout $t > t_0$, la solution $v(t, r) = -\sqrt{\frac{2M}{r}}$.
- Si $v_0(2M) < 1$ et $\lim_{r \rightarrow +\infty} v_0(r) \leq 0$, il existe un temps fini $t_0 > 0$ tel que pour tout $t > t_0$,

$$v(t, r) = -\sqrt{1 - (1 - (v_0^\infty)^2)\left(1 - \frac{2M}{r}\right)}, \quad \lim_{r \rightarrow +\infty} v_0(r) =: v_0^\infty \leq 0.$$

Conjecture 4.2. *Soit $(\rho_*, v_*) = (\rho_*, v_*)(r)$, $r > 2M$ une solution stationnaire (qui peut contenir un choc stationnaire) du modèle d'Euler dans l'espace-temps de Schwarzschild et $(\rho_0, v_0) = (\rho_0, v_0)(r) = (\rho_*, v_*)(r) + (\delta_\rho, \delta_v)(r)$ où $(\delta_\rho, \delta_v) = (\delta_\rho, \delta_v)(r)$ est une fonction à support compact. La solution de l'équation d'Euler (2.1) $(\rho, v) = (\rho, v)(t, r)$ avec la donnée initiale (ρ_0, v_0) satisfait:*

- Si $|\int \delta_\rho(r)dr| + |\int \delta_v(r)dr| = 0$, il existe un temps fini $t_0 > 0$ tel que pour tout $t > t_0$, $(\rho, v)(t, r) = (\rho_*, v_*)(r)$.
- Si $|\int \delta_\rho(r)dr| + |\int \delta_v(r)dr| \neq 0$, il existe un temps fini $t_0 > 0$ tel que pour tout $t > t_0$, $(\rho, v)(t, r) = (\rho_{**}, v_{**})(r)$ et (ρ_{**}, v_{**}) est éventuellement une autre une solution stationnaire.

Nous invitons le lecteur à voir les illustrations de ces conjectures en Chapitre 3.

5 Conclusion et perspectives

En résumé, les questions suivantes ont été traitées dans cette thèse:

- L'existence pour le problème de Cauchy du modèle d'Euler relativiste/ non-relativiste dans le domaine de communication extérieur d'un espace-temps de Schwarzschild.
- L'existence, l'unicité pour le problème de Cauchy du modèle de Burgers relativiste dans le domaine de communication extérieur d'un espace-temps de Schwarzschild.
- La stabilité des solutions stationnaires par morceaux du modèle de Burgers dans le domaine de communication extérieur d'un espace-temps de Schwarzschild.
- La stabilité des solutions stationnaires régulières du modèle d'Euler dans le domaine de communication extérieur d'un espace-temps de Schwarzschild.
- Le comportement des solutions du modèle de Burgers dans le domaine de communication extérieur d'un espace-temps de Schwarzschild déterminé seulement par des valeurs au bord du trou noir et à l'infini.

Le travail réalisé sur la dynamique des fluides dans un espace-temps courbe est déjà assez complet, même si d'autres questions proches pourraient être aussi abordées avec les techniques mathématiques et numériques que nous avons proposées ici:

- Quel est le comportement d'une loi de conservation scalaire posée sur un autre espace-temps courbe autre que l'espace-temps de Schwarzschild?
- Quel est le comportement complet d'un système de loi de conservation sur l'espace-temps de Schwarzschild et les autres espace-temps courbes?
- Comment optimiser le schéma numérique pour étudier un système de flux des fluides avec une géométrie courbée arbitraire?

Fondé sur les questions ci-dessus, le résultat attendu est de pouvoir décrire en détail un modèle de la dynamique des fluides (plus d'une dimension en générale) posés sur l'espace-temps de Schwarzschild et sur l'autre espace-temps courbe qui devrait être plus compliqué, par exemple, la métrique de Kerr, dont la géométrie est influencée par un effet de rotation du corps de la masse. Numériquement, on attend un schéma d'ordre supérieur pour voir ce qui se passe exactement à l'horizon du trou noir et pour tester le changement sensible causé par la singularité attendue. Ces résultats donneront une meilleure compréhension de la mécanique des fluides dans des espace-temps courbes.

Part I

Fluid Dynamics on a Schwarzschild Spacetime Background: A Theoretical Study

Chapter 1

Weakly regular fluid flows with bounded variation on the domain of outer communication of a Schwarzschild spacetime¹

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¹ Réalisé en collaboration avec P.G. LeFloch et publié dans J. Math. Pures Appl. (2016), 1038-1090.

1.1 Introduction

We are interested in compressible fluids evolving on a curved background and, specifically, on the domain of outer communication of a Schwarzschild black hole spacetime. The fluid flows under consideration may contain shock waves and we must work within a class of weak solutions to the Euler equations. Our main result in this paper is a global-in-time existence theory for the initial value problem, when the fluid data are prescribed on a spacelike hypersurface. We also establish the nonlinear stability of equilibrium fluid solutions and investigate various limiting regimes when the light speed denoted by $c \in (0, +\infty)$, the (constant) sound speed denoted by $k \in [0, +\infty)$, and the mass of the black hole denoted by $M \in [0, +\infty)$ reach extremal values.

Recall that Schwarzschild spacetime is a spherically symmetric² solution to the vacuum Einstein equations of general relativity, and describes a massive body surrounded by a vacuum region. It is one of the simplest non-flat solution to the Einstein equations, but yet the analysis of (linear and) nonlinear waves propagating on this spacetime is very challenging and has attracted a lot of attention by mathematicians in recent years. The present paper is part of a program initiated by the first author on the Cauchy problem for the Einstein-Euler equations: see [2, 3, 16, 26, 27, 28], as well as the graduate course [22] on self-gravitating matter and weakly regular spacetimes.

In the so-called Schwarzschild coordinates $t \geq 0$ and $r \in (2M, +\infty)$, the domain of outer communication of Schwarzschild spacetime is described by the metric

$$g = -\left(1 - \frac{2M}{r}\right) c^2 dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 g_{S^2}, \quad r > 2M, \quad (1.1.1)$$

in which $g_{S^2} := d\theta^2 + (\sin\theta)^2 d\varphi^2$ is the canonical metric on the two-sphere S^2 , with $\theta \in [0, 2\pi)$ and $\varphi \in [0, \pi]$. Observe that the metric coefficients are singular as $r \rightarrow 2M$, but this boundary is not a genuine singularity of the spacetime and the coefficients would become regular at $r = 2M$ by suitably changing coordinates and the metric could be extended beyond this boundary. The boundary $r = 2M$ is the horizon of the black hole, and it is natural to study the dynamics of nonlinear waves outside the black hole region.

The Levi-Civita connection associated with (2.1.4) being denoted by ∇ , the Euler equations for a perfect compressible fluid on this spacetime read

$$\nabla_\alpha (T_\beta^\alpha(\rho, u)) = 0, \quad (1.1.2)$$

²that is, invariant under the group of rotations $SO(3)$

in which the energy-momentum tensor

$$T_{\beta}^{\alpha}(\rho, u) = \rho c^2 u^{\alpha} u_{\beta} + p(\rho) \left(g_{\beta}^{\alpha} + u^{\alpha} u_{\beta} \right) \quad (1.1.3)$$

(with $c > 0$ denoting the speed of light) depends on the mass-energy density of the fluid $\rho : M \mapsto (0, +\infty)$ and its velocity field $u = (u^{\alpha})$, normalized to be unit and future oriented:

$$u^{\alpha} u_{\alpha} = -1, \quad u^0 > 0. \quad (1.1.4)$$

The pressure p is prescribed as a function $p = p(\rho)$ of the mass energy density and, for the sake of simplicity, we assume that the fluid flow is isothermal, that is, $p(\rho) = k^2 \rho$ where $k \in (0, c)$ represents the speed of sound. We use here standard notation for the metric $g = (g_{\alpha\beta})$ and its inverse $g^{-1} = (g^{\alpha\beta})$ in an arbitrary local coordinate system $x = (x^{\alpha})$, where the Greek indices describe $0, 1, 2, 3$. We raise and lower indices by using this metric and, for instance, we write $u_{\alpha} = g_{\alpha\beta} u^{\beta}$ and we have $g_{\beta}^{\alpha} = \delta_{\beta}^{\alpha}$ (the Kronecker symbol).

The content of this paper is as follows. In Section 1.2, we formulate the Euler equations in our context and establish hyperbolicity and genuine nonlinearity properties. In Section 1.3, we formally derive several simpler models, arising when the light speed sound speed and/or black hole mass approach extremal values. Our model takes the form of a nonlinear hyperbolic system of balance laws. such systems were first investigated (for rather different applications) by Dafermos and Hsiao [8], Liu [40] and, later, [14, 17, 9]; see also Dafermos [9] the references cited therein. We also refer to [3?, 15] for the related problem of self-gravitating fluids in spherical symmetry.

A systematic study of the class of steady state solutions to the Euler model under consideration is one of the main contribution of the present paper. In Section 1.4, we first study the non-relativistic model, by taking into account the effect of the mass of the black hole. Next, in Section 1.5, we treat the full Euler model on a Schwarzschild background and, in particular, we establish that (smooth) steady state solutions are defined on intervals of the form $(2M, \underline{r}_*)$ or $(\bar{r}_*, +\infty)$.

Our next task is to study the Riemann problem which is solved in Section 1.6, while the generalized Riemann problem based on prescribing two steady state solutions (rather than constant states) separated by a jump discontinuity is investigated in Section 1.7.

In Section 1.8, we are then in a position to establish an existence theory for general flows of isothermal fluids evolving in the domain of outer communication (2.1.4). The technique developed earlier in Grubic and LeFloch [16] (in a different geometric setup) applies and provides us with the desired global-in-time result. Recall that, according to Nishida [41] and Smoller and Temple [44] who treated fluid flows in flat space, provided all curved geometrical effects are (formally) suppressed, a suitable notion of total variation is available and, specifically, the total variation of the log of the matter

density is non-increasing in time. For the fluids on a Schwarzschild background under consideration in the present paper, we also need to take geometrical terms into account and the total variation may grow, but yet is uniformly controlled on any compact interval of time. Furthermore, an analysis of the solutions near the horizon is also necessary and we observe that no boundary condition is required at $r = 2M$ and that solutions need not have finite bounded variation near the horizon, as is the case for some steady state solutions.

We also propose here a version of the random choice method which we design from piecewise equilibrium solutions and, in turn, preserves equilibria exactly. We then prove that equilibria are nonlinearly stable under small BV perturbations, and the proposed technique provides a possible approach in order to investigate the time-asymptotic behavior of weak solutions. Finally, in Section 1.9, we briefly consider the models obtained when the physical parameters take extremal values. Our total variation estimate is uniform with respect to these parameters, so that our main theorem has counterparts for these limiting systems.

1.2 The Euler equations on a Schwarzschild background

Derivation of the Euler equations

By using the subscripts $(0, 1, 2, 3)$ to denote the coordinates (t, r, θ, φ) , we can write

$$(g_{\alpha\beta}) = \begin{pmatrix} -(1 - 2M/r)c^2 & 0 & 0 & 0 \\ 0 & (1 - 2M/r)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2(\sin \theta)^2 \end{pmatrix}, \quad (1.2.1)$$

with inverse

$$(g^{\alpha\beta}) = \begin{pmatrix} -(1 - 2M/r)^{-1}c^{-2} & 0 & 0 & 0 \\ 0 & (1 - 2M/r) & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2} \sin^{-2} \theta \end{pmatrix} \quad (1.2.2)$$

and, by using $\Gamma_{\alpha\beta}^\gamma := \frac{1}{2}g^{\gamma\theta}(\partial_\alpha g_{\beta\theta} + \partial_\beta g_{\alpha\theta} - \partial_\theta g_{\alpha\beta})$, a tedious calculation shows that the non-vanishing Christoffel symbols are

$$\begin{aligned} \Gamma_{00}^1 &= \frac{c^2 M}{r^2}(r - 2M), & \Gamma_{11}^1 &= -\frac{M}{r(r - 2M)}, & \Gamma_{01}^0 &= \frac{M}{r(r - 2M)}, \\ \Gamma_{12}^2 &= \frac{1}{r}, & \Gamma_{22}^1 &= -(r - 2M), & \Gamma_{13}^3 &= \frac{1}{r}, \\ \Gamma_{33}^1 &= -(r - 2M)(\sin \theta)^2, & \Gamma_{33}^2 &= -\sin \theta \cos \theta, & \Gamma_{23}^3 &= \frac{\cos \theta}{\sin \theta}. \end{aligned} \tag{1.2.3}$$

On the other hand, we can express the Euler equations (1.1.2) in the form

$$\partial_0 T^{0\beta} + \partial_j T^{j\beta} + \Gamma_{00}^0 T^{0\beta} + \Gamma_{j0}^j T^{\beta 0} + \Gamma_{0j}^0 T^{j\beta} + \Gamma_{jk}^j T^{k\beta} + \Gamma_{00}^\beta T^{00} + 2\Gamma_{j0}^\beta T^{j0} + \Gamma_{jk}^\beta T^{jk} = 0$$

and, in view of (1.2.3), write the Euler equations on a Schwarzschild background as

$$\begin{aligned} \partial_0 \left(r(r - 2M)T^{00} \right) + \partial_1 \left(r(r - 2M)T^{01} \right) &= 0, \\ \partial_0 \left(r(r - 2M)T^{01} \right) + \partial_1 \left(r(r - 2M)T^{11} \right) &= \Omega_1, \\ \Omega_1 &:= 3MT^{11} - \frac{c^2 M}{r^2}(r - 2M)^2 T^{00} + r(r - 2M)^2 T^{22} + r(\sin \theta)^2 (r - 2M)^2 T^{33}. \end{aligned} \tag{1.2.4}$$

Here, we have assumed that not only the background geometry but also the fluid flows are *spherically symmetric*, so that the ‘‘transverse’’ components of the fluid velocity vanish: $T^{02} = T^{03} = 0$. Next, recalling the expression (2.2.2) of the energy-momentum tensor, we find (with $= p(\rho)$)

$$\begin{aligned} \partial_0 \left(r(r - 2M)(pu^1 u^1 + (1 - 2M/r)^2 c^4 \rho u^0 u^0) \right) + \partial_1 \left(r(r - 2M)(p + c^2 \rho)u^0 u^1 \right) &= 0, \\ \partial_0 \left(r(r - 2M)(p + c^2 \rho)u^0 u^1 \right) + \partial_1 \left(r(r - 2M)(pu^0 u^0 + (1 - 2M/r)^{-2} \rho u^1 u^1) \right) &= \Omega_1, \\ \Omega_1 &= 2r(r - 2M)^2 p + 3M \left(pu^1 u^1 + (1 - 2M/r)^2 c^4 \rho u^0 u^0 \right) \\ &\quad - \frac{c^2 M}{r^2} (r - 2M)^2 \left(pu^0 u^0 + (1 - 2M/r)^{-2} \rho u^1 u^1 \right). \end{aligned} \tag{1.2.5}$$

Observe that the ‘first’ Euler equation admits a ‘conservative form’, while the second one is a general ‘balance law’.

By definition, the velocity vector satisfies $(1 - 2M/r)c^2 u^0 u^0 - (1 - 2M/r)^{-1} u^1 u^1 = 1$ and $u^0 > 0$, and we find it convenient to introduce the **rescaled velocity vector**

$$v^0 := \frac{u^0}{\epsilon}, \quad v^1 := \frac{u^1}{\epsilon}, \quad \text{with } \epsilon := \frac{1}{c}. \tag{1.2.6}$$

Hence, the components of the energy-momentum tensor read

$$\begin{aligned} T^{00} &= (1 - 2M/r)^2 \frac{1}{\epsilon^2} \rho v^0 v^0 + \epsilon^2 p v^1 v^1, & T^{01} &= (\rho + \epsilon^2 p) v^0 v^1, \\ T^{11} &= \epsilon^2 p v^0 v^0 + (1 - 2M/r)^{-2} \epsilon^2 \rho v^1 v^1, & T^{22} &= T^{33} = p. \end{aligned}$$

and the system (1.2.5) takes the form:

$$\begin{aligned} \partial_0 \left(r(r-2M) \left((1-2M/r)^2 \rho v^0 v^0 + \epsilon^4 p v^1 v^1 \right) \right) + \partial_1 \left(r(r-2M) \epsilon^2 (\rho + \epsilon^2 p) v^0 v^1 \right) &= 0, \\ \partial_0 \left(r(r-2M) \left((\rho + \epsilon^2 p) v^0 v^1 \right) \right) + \partial_1 \left(r(r-2M) \left(\epsilon^2 (p v^0 v^0 + (1-2M/r)^{-2} \rho v^1 v^1) \right) \right) &= \tilde{\Omega}, \\ \tilde{\Omega} &:= \frac{3M}{\epsilon^2} \left(\epsilon^4 p v^1 v^1 + (1-2M/r)^2 \rho v^0 v^0 \right) \\ &\quad - \frac{M}{r^2} (r-2M)^2 \left(p v^0 v^0 + (1-2M/r)^{-2} \rho v^1 v^1 \right) + 2r(r-2M)^2 p, \end{aligned} \tag{1.2.7}$$

supplemented by the relation for the velocity vector

$$(1 - 2M/r) v^0 v^0 - \epsilon^2 (1 - 2M/r)^{-1} v^1 v^1 = 1, \quad v^0 > 0. \tag{1.2.8}$$

It is convenient also to introduce the **scalar velocity**

$$v := \frac{1}{(1 - 2M/r)} \frac{v^1}{v^0}, \tag{1.2.9}$$

leading us to

$$(v^0)^2 = \frac{1}{(1 - \epsilon^2 v^2)(1 - 2M/r)}, \quad (v^1)^2 = (1 - 2M/r) \frac{v^2}{1 - \epsilon^2 v^2}.$$

In summary, we have shown that the **Euler system on a Schwarzschild background** takes the form:

$$\begin{aligned} \partial_0 \left(r^2 \frac{\rho + \epsilon^4 p v^2}{1 - \epsilon^2 v^2} \right) + \partial_1 \left(r(r-2M) \frac{\rho + \epsilon^2 p}{1 - \epsilon^2 v^2} v \right) &= 0, \\ \partial_0 \left(r(r-2M) \frac{\rho + \epsilon^2 p}{1 - \epsilon^2 v^2} v \right) + \partial_1 \left((r-2M)^2 \frac{\rho v^2 + p}{1 - \epsilon^2 v^2} \right) &= 3M \left(1 - \frac{2M}{r} \right) \frac{\rho v^2 + p}{1 - \epsilon^2 v^2} - M \frac{r-2M}{\epsilon^2 r} \frac{\rho + \epsilon^4 p v^2}{1 - \epsilon^2 v^2} + 2 \frac{(r-2M)^2}{r} p. \end{aligned} \tag{1.2.10}$$

Remark 1.2.1. *1. In the limit $M \rightarrow 0$, the Schwarzschild metric converges to the Minkowski metric in radial coordinates*

$$g = -c^2 dt^2 + dr^2 + r^2 g_{S^2} \tag{1.2.11}$$

and from (3.1.3) we deduce the radially-symmetric Euler equations in Minkowski space:

$$\begin{aligned} \partial_0 \left(r^2 (\rho v^0 v^0 + \epsilon^4 p v^1 v^1) \right) + \partial_1 \left(r^2 \epsilon^2 (\rho + p \epsilon^2) v^0 v^1 \right) &= 0, \\ \partial_0 \left(r^2 (\rho + \epsilon^2 p) v^0 v^1 \right) + \partial_1 \left(r^2 \epsilon^2 (p v^0 v^0 + \rho v^1 v^1) \right) &= 2r p, \end{aligned} \quad (1.2.12)$$

with $v^0 v^0 - \epsilon^2 v^1 v^1 = 1$ and $v^0 > 0$ and $p = p(\rho)$.

2. In the singular limit $v \rightarrow \pm 1/\epsilon$, the (unit) velocity vector $v = (v^0, v^1)$ converges (after normalization!) to a null vector, namely:

$$(1 - \epsilon^2 v^2)^{1/2} (1 - 2M/r)^{1/2} (v^0, v^1) = (1, (1 - 2M/r)v) \rightarrow (1, \pm(1 - 2M/r)/\epsilon).$$

Hyperbolicity and genuine nonlinearity properties

Throughout the rest of this section, we regard (3.1.3) as a **system of nonlinear balance laws**, that is,

$$\partial_0 U + \partial_1 F(U, r) = S(U, r) \quad (1.2.13)$$

(with obvious notation) and we study the homogeneous part $\partial_0 U + \partial_1 F(U, r_0) = 0$, where the expressions F and S are evaluated at some fixed $r_0 > 2M$. We determine necessary and sufficient conditions ensuring the hyperbolicity and genuine nonlinearity properties for (3.1.3). We are going to rewrite the homogeneous part of (1.2.7) in the diagonal form (with the source-terms suppressed)

$$\partial_0 w + \lambda(w, z, r_0) \partial_1 w = 0, \quad \partial_0 z + \mu(w, z, r_0) \partial_1 z = 0 \quad (1.2.14)$$

for a suitable choice of functions $w = w(\rho, v)$ and $z = z(\rho, v)$, referred to as the **Riemann invariants**, and $\lambda = \lambda(\rho, v, r_0)$ and $\mu = \mu(\rho, v, r_0)$, referred to as the **wave speeds**.

Lemma 1.2.2. *For the Euler system on a Schwarzschild background (3.1.3), a choice of Riemann invariants is*

$$w = \frac{1}{2\epsilon} \ln \left(\frac{1 + \epsilon v}{1 - \epsilon v} \right) + \int_1^\rho \frac{\sqrt{p'(s)}}{s + \epsilon^2 p(s)} ds, \quad z = \frac{1}{2\epsilon} \ln \left(\frac{1 + \epsilon v}{1 - \epsilon v} \right) - \int_1^\rho \frac{\sqrt{p'(s)}}{s + \epsilon^2 p(s)} ds, \quad (1.2.15)$$

while the corresponding eigenvalues read

$$\lambda := \left(1 - \frac{2M}{r_0} \right) \frac{v - \sqrt{p'(\rho)}}{1 - \epsilon^2 \sqrt{p'(\rho)} v}, \quad \mu := \left(1 - \frac{2M}{r_0} \right) \frac{v + \sqrt{p'(\rho)}}{1 + \epsilon^2 \sqrt{p'(\rho)} v}. \quad (1.2.16)$$

Proof. 1. In order to determine the Riemann invariants, we may fix a time $t_0 \geq 0$

and search for solutions depending on the self-similar variable $y := \frac{r-r_0}{t-t_0}$ (further studied in Section 1.6 below), therefore satisfying $-y \frac{dw}{dy} + \lambda(w, z, r_0) \frac{dw}{dy} = 0$ and $-y \frac{dz}{dy} + \mu(w, z, r_0) \frac{dz}{dy} = 0$. Either w or z must thus be constant for such solutions. Moreover, by parametrizing such solutions by one of the unknown variables, say with the density ρ , we can regard the unknowns v^0 and v^1 as functions of ρ and, using a prime to denote the derivative with respect to ρ , we find

$$\begin{aligned} \left((1 - 2M/r_0)^2 \rho v^0 v^0 + \epsilon^4 p v^1 v^1 \right)' \partial_0 \rho + \left(\epsilon^2 (\rho + \epsilon^2 p) v^0 v^1 \right)' \partial_1 \rho &= 0, \\ \left((\rho + \epsilon^2 p) v^0 v^1 \right)' \partial_0 \rho + \left(\epsilon^2 (p v^0 v^0 + (1 - 2M/r_0)^{-2} \rho v^1 v^1) \right)' \partial_1 \rho &= 0, \end{aligned} \quad (1.2.17)$$

where we have neglected low-order, algebraic terms. By differentiating (1.2.8), we also have

$$(1 - 2M/r_0)(v^0)'v^0 - (1 - 2M/r_0)^{-1}\epsilon^2(v^1)'v^1 = 0. \quad (1.2.18)$$

By combining the two equations in (1.2.17) together, we obtain

$$\left((\epsilon^2 p + \rho) v^0 v^1 \right)' = \left(p v^0 v^0 + (1 - 2M/r_0)^{-2} \rho v^1 v^1 \right)' \left(\epsilon^4 p v^1 v^1 + (1 - 2M/r_0)^2 \rho v^0 v^0 \right)',$$

from which we deduce

$$\begin{aligned} p' \left((1 - 2M/r_0)(v^0)^2 - (1 - 2M/r_0)^{-1}(\epsilon v^1)^2 \right)^2 \\ = (\epsilon^2 p + \rho)^2 \left(((v^0)'v^1 + v^0(v^1)')^2 - 4v^0(v^0)'v^1(v^1)' \right). \end{aligned}$$

Using again (1.2.8), we find

$$\frac{(v^0)'\epsilon v^1 - c v^0(\epsilon v^1)'}{(1 - 2M/r_0)(v^0)^2 - (1 - 2M/r_0)^{-1}\epsilon^2(v^1)^2} \pm \frac{\epsilon \sqrt{p'}}{\epsilon^2 p + \rho} = 0. \quad (1.2.19)$$

After integration, we see that

$$\frac{1}{2} \ln \left(\frac{(1 - 2M/r_0)v^0 + \epsilon v^1}{(1 - 2M/r_0)v^0 - \epsilon v^1} \right) \pm \int_1^\rho \epsilon \frac{\sqrt{p'(s)}}{s + \epsilon^2 p(s)} ds \quad (1.2.20)$$

is a constant for the solutions under consideration. This calculation provides us with the Riemann invariants

$$\begin{aligned} w &= \frac{1}{2\epsilon} \ln \left(\frac{(1 - 2M/r_0)v^0 + \epsilon v^1}{(1 - 2M/r_0)v^0 - \epsilon v^1} \right) + \int_1^\rho \frac{\sqrt{p'(s)}}{s + \epsilon^2 p(s)} ds, \\ z &= \frac{1}{2\epsilon} \ln \left(\frac{(1 - 2M/r_0)v^0 + \epsilon v^1}{(1 - 2M/r_0)v^0 - \epsilon v^1} \right) - \int_1^\rho \frac{\sqrt{p'(s)}}{s + \epsilon^2 p(s)} ds, \end{aligned}$$

which take the form (1.2.15) by replacing v^0 and v^1 by their expression in terms of $v = \frac{1}{1-2M/r_0} \frac{v^1}{v^0}$.

2. We determine the eigenvalue λ from the first equation in the system $\partial_t U + \partial_r F(U, r_0) = 0$, that is,

$$\lambda = \frac{(1 - 2M/r_0)^2 \left((p\epsilon^2 + \rho)v^0v^1 \right)'}{\left(\epsilon^4 p v^1 v^1 + (1 - 2M/r_0)^2 \rho v^0 v^0 \right)'}. \quad (1.2.21)$$

In view of (1.2.18) and (1.2.19) (where we take the minus sign), we have

$$(v^0)' = -(1 - 2M/r_0)^{-1} \frac{\epsilon^2 \sqrt{p'}}{\epsilon^2 p + \rho} v^1, \quad (v^1)' = -(1 - 2M/r_0) \frac{\sqrt{p'}}{\epsilon^2 p + \rho} v^0. \quad (1.2.22)$$

Therefore, the ‘first’ eigenvalue reads

$$\begin{aligned} \lambda &= (1 - 2M/r_0)^2 \frac{(p'\epsilon^2 + 1)v^0v^1 - \sqrt{p'} \left((1 - 2M/r_0)(v^0)^2 + (1 - 2M/r_0)^{-1}(\epsilon v^1)^2 \right)}{\epsilon^4 p' v^1 v^1 + (1 - 2M/r_0)^2 v^0 v^0 - 2(1 - 2M/r_0)\epsilon^2 \sqrt{p'} v^0 v^1} \\ &= (1 - 2M/r_0)^2 \frac{\left((1 - 2M/r_0)v^0 - \epsilon^2 \sqrt{p'} v^1 \right) \left((1 - 2M/r_0)^{-1} v^1 - \sqrt{p'} v^0 \right)}{\left((1 - 2M/r_0)v^0 - \epsilon^2 \sqrt{p'} v^1 \right)^2} \\ &= (1 - 2M/r_0) \frac{(1 - 2M/r_0)^{-1} v^1 - \sqrt{p'} v^0}{v^0 - \epsilon^2 (1 - 2M/r_0)^{-1} \sqrt{p'} v^1}. \end{aligned}$$

Recalling that $v = \frac{1}{1-2M/r_0} \frac{v^1}{v^0}$, we obtain the desired expression for λ . The arguments for μ are entirely similar. \square

We arrive at the following result.

Proposition 1.2.3 (Necessary and sufficient conditions for hyperbolicity and genuine nonlinearity).

1. *The Euler system on a Schwarzschild background (3.1.3) (within the range $r > 2M$) is **strictly hyperbolic**, that is, admits two real and distinct wave speeds, if and only if the pressure satisfies the condition*

$$p'(\rho) > 0 \quad \text{for all } \rho > 0. \quad (1.2.23)$$

2. *This system is **genuinely nonlinear**, that is, the derivatives $\frac{\partial \lambda}{\partial w}$ and $\frac{\partial \mu}{\partial z}$ never vanish, if and only if the pressure satisfies the condition*

$$\rho p'' + 2p' + \epsilon^2 (p'' p - 2(p')^2) > 0 \quad \text{for all } \rho > 0 \quad (1.2.24)$$

and, therefore, is **linearly degenerate**, that is, the derivatives $\frac{\partial \lambda}{\partial w}$ and $\frac{\partial \mu}{\partial z}$ identically vanish, if and only if the pressure satisfies the condition

$$\rho p'' + 2p' + \epsilon^2(p''p - 2(p')^2) = 0 \quad \text{for all } \rho > 0. \quad (1.2.25)$$

When the sound speed is a constant k (which is the case of main interest in the present paper), that is, when $p = k^2\rho$ (with $0 < k < 1/\epsilon$), the eigenvalues read

$$\lambda = \left(1 - \frac{2M}{r}\right) \frac{v - k}{1 - \epsilon^2 k v}, \quad \mu = \left(1 - \frac{2M}{r}\right) \frac{v + k}{1 + \epsilon^2 k v}, \quad (1.2.26)$$

and the Riemann invariants take the form

$$w = \frac{1}{2\epsilon} \ln \left(\frac{1 + \epsilon v}{1 - \epsilon v} \right) + \frac{k}{1 + \epsilon^2 k^2} \ln \rho, \quad z = \frac{1}{2\epsilon} \ln \left(\frac{1 + \epsilon v}{1 - \epsilon v} \right) - \frac{k}{1 + \epsilon^2 k^2} \ln \rho. \quad (1.2.27)$$

The Euler system, therefore, is strictly hyperbolic and genuinely nonlinear in this case.

Proof. In view of Lemma 1.2.2, the condition $p' > 0$ is the necessary and sufficient condition for the eigenvalues to be real. Moreover, by definition, the first family λ (the second family μ , respectively) is genuinely nonlinear if and only if $\partial_w \lambda \neq 0$ (and $\partial_z \mu \neq 0$, resp.). We compute

$$\partial_w \lambda = \left(\frac{\partial \lambda}{\partial v^0} (v^0)' + \frac{\partial \lambda}{\partial v^1} (v^1)' \frac{\partial \lambda}{\partial v^0} + \frac{\partial \lambda}{\partial \rho} \right) \frac{\partial \rho}{\partial w},$$

following with the calculations:

$$\begin{aligned} \frac{\partial \lambda}{\partial v^0} &= -(1 - 2M/r) \frac{1 - (\epsilon^2 p'(\rho))v^1}{\left((1 - 2M/r)v^0 - \epsilon^2 \sqrt{p'(\rho)}v^1 \right)^2}, \\ \frac{\partial \lambda}{\partial v^1} &= (1 - 2M/r) \frac{(1 - \epsilon^2 p'(\rho))v^0}{\left((1 - 2M/r)v^0 - \epsilon^2 \sqrt{p'(\rho)}v^1 \right)^2}, \\ \frac{\partial \lambda}{\partial \rho} &= -(1 - 2M/r) \frac{p''(\rho)}{2\sqrt{p'(\rho)} \left((1 - 2M/r)v^0 - \epsilon^2 \sqrt{p'(\rho)}v^1 \right)^2}. \end{aligned}$$

Combining these formulas, we obtain

$$\partial_w \lambda = -(1 - 2M/r) \frac{\frac{p''}{2\sqrt{p'(\rho)}} + \frac{(1 - \epsilon^2 p)\sqrt{p'(\rho)}}{\epsilon^2 p + \rho}}{\left((1 - 2M/r)v^0 - \epsilon^2 \sqrt{p'(\rho)}v^1 \right)^2} \frac{\partial \rho}{\partial w}.$$

A similar calculation gives the result associated with the second eigenvalue:

$$\partial_z \mu = -(1 - 2M/r) \frac{\frac{p''}{2\sqrt{p'(\rho)}} + \frac{(1-\epsilon^2 p)\sqrt{p'(\rho)}}{\epsilon^2 p + \rho}}{\left((1 - 2M/r)v^0 + \epsilon^2 \sqrt{p'(\rho)}v^1\right)^2} \frac{\partial \rho}{\partial z}.$$

Therefore, the sufficient and necessary condition for genuine nonlinearity is (1.2.24). On the contrary, the system is linearly degenerate if and only if (1.2.25) holds. \square

Linearly degenerate equations of state

The following special case is of particular interest.

Proposition 1.2.4 (Linearly degenerate equations of state). *The Euler system (3.1.3) is linearly degenerate if and only if the pressure (which is defined up to a constant) takes one of the forms (for all $\rho > 0$)*

$$p(\rho) = 0 \quad \text{or} \quad p(\rho) = \frac{\rho}{\epsilon^2}, \quad \text{or} \quad p(\rho) = -\frac{A^2}{\rho + \epsilon^2 B}, \quad (1.2.28)$$

where $A, B > 0$ are arbitrary constants and only the latter two pressure-laws lead to a strictly hyperbolic model.

We thus have only two strictly hyperbolic and linearly degenerate models:

- **Case** $p = \frac{1}{\epsilon^2}\rho$. The system is well-defined within the full range $\rho > 0$ and $|v| < 1/\epsilon$. The eigenvalues

$$-\lambda = \mu = (1 - 2M/r)/\epsilon$$

are independent of the dependent variables, while the Riemann invariants read

$$w = \frac{1}{2\epsilon} \ln \left(\frac{1 + \epsilon v}{1 - \epsilon v} \right) + \frac{1}{2\epsilon} \ln \rho, \quad z = \frac{1}{2\epsilon} \ln \left(\frac{1 + \epsilon v}{1 - \epsilon v} \right) - \frac{1}{2\epsilon} \ln \rho.$$

- When $p = -\frac{A^2}{\rho + \epsilon^2 B}$, the system is well-defined in *limited range of ρ , only*. For instance, when $p = -\frac{A^2}{\rho + \epsilon^2 B}$, the eigenvalues read

$$\lambda = (1 - 2M/r) \frac{\rho v - A}{\rho - A\epsilon^2 v}, \quad \mu = (1 - 2M/r) \frac{\rho v + A}{\rho + A\epsilon^2 v}$$

while the Riemann invariants are

$$w = \frac{1}{2\epsilon} \ln \left(\frac{1 + \epsilon v}{1 - \epsilon v} \right) + \frac{1}{2\epsilon} \ln \left(\frac{\rho - \epsilon A}{\rho + \epsilon A} \right), \quad z = \frac{1}{2\epsilon} \ln \left(\frac{1 + \epsilon v}{1 - \epsilon v} \right) - \frac{1}{2\epsilon} \ln \left(\frac{\rho - \epsilon A}{\rho + \epsilon A} \right).$$

This model can be considered within the range $|\rho| < \epsilon A$ and $|v| < 1/\epsilon$ (even with negative density values).

Proof. From Proposition 1.2.3, we recall the condition $\rho p'' + 2p' + \epsilon^2(p''p - 2p'^2) = 0$. If we set $q := \epsilon^2 p + \rho$, we thus need to solve the ordinary differential equation

$$q''q - 2(q')^2 + 6q' - 4 = 0. \quad (1.2.29)$$

We treat q as an *independent variable* and set $\frac{dq(\rho)}{d\rho} =: \nu(q)$, hence

$$\frac{d^2q}{d\rho^2} = \frac{d\nu}{dq} \frac{dq}{d\rho} = \nu(q) \frac{d\nu}{dq}.$$

We see that (1.2.29) transforms into a *separable* equation for the function $\nu = \nu(q)$, that is, provided $(\nu - 1)(\nu - 2)$ does not vanish

$$\frac{\nu}{(\nu - 1)(\nu - 2)} \frac{d\nu}{dq} = \frac{2}{q} \quad (1.2.30)$$

or else $\nu \equiv 1$ or $\nu \equiv 2$. Solutions satisfying $\frac{dq}{d\rho} \equiv 1$ correspond a constant pressure function, since $\frac{dq}{d\rho} := \epsilon^2 p' + 1 = 1$ implies that p is a constant. The condition $\frac{dq(\rho)}{d\rho} = 2$ generates the solutions of the form $p(\rho) = \frac{\rho}{\epsilon^2} + C$. Finally, by integrating (1.2.30), we find the third class of solutions. \square

1.3 Formal derivation of simplified models

Fluid flows with constant sound speed

In this section, we formally analyze the structure of the Euler equation in a Schwarzschild background. We focus our attention on the Euler system (3.1.3) when the sound speed is assumed to be a constant $k \in [0, 1/\epsilon]$, that is, with the pressure

law $p(\rho) = k^2\rho$, (3.1.3) becomes

$$\begin{aligned} & \partial_t \left(r^2 \frac{1 + \epsilon^4 k^2 v^2}{1 - \epsilon^2 v^2} \rho \right) + \partial_r \left(r(r - 2M) \frac{1 + \epsilon^2 k^2}{1 - \epsilon^2 v^2} \rho v \right) = 0, \\ & \partial_t \left(r(r - 2M) \frac{1 + \epsilon^2 k^2}{1 - \epsilon^2 v^2} \rho v \right) + \partial_r \left((r - 2M)^2 \frac{v^2 + k^2}{1 - \epsilon^2 v^2} \rho \right) \\ & = 3M \left(1 - \frac{2M}{r} \right) \frac{v^2 + k^2}{1 - \epsilon^2 v^2} \rho - M \frac{r - 2M}{\epsilon^2 r} \frac{1 + \epsilon^4 k^2 v^2}{1 - \epsilon^2 v^2} \rho + 2 \frac{(r - 2M)^2}{r} k^2 \rho. \end{aligned} \tag{1.3.1}$$

It will be necessary to rescale the mass M and we thus set

$$m := \frac{M}{\epsilon^2} \tag{1.3.2}$$

and refer to the following system as **the family of Euler models** $\mathcal{M}(\epsilon, k, m)$

$$\begin{aligned} & \partial_t \left(r^2 \frac{1 + \epsilon^4 k^2 v^2}{1 - \epsilon^2 v^2} \rho \right) + \partial_r \left(r(r - 2\epsilon^2 m) \frac{1 + \epsilon^2 k^2}{1 - \epsilon^2 v^2} \rho v \right) = 0, \\ & \partial_t \left(r(r - 2\epsilon^2 m) \frac{1 + \epsilon^2 k^2}{1 - \epsilon^2 v^2} \rho v \right) + \partial_r \left((r - 2\epsilon^2 m)^2 \frac{v^2 + k^2}{1 - \epsilon^2 v^2} \rho \right) \\ & = \frac{3\epsilon^2 m}{r} (r - 2\epsilon^2 m) \frac{v^2 + k^2}{1 - \epsilon^2 v^2} \rho - \frac{m}{r} (r - 2\epsilon^2 m) \frac{1 + \epsilon^4 k^2 v^2}{1 - \epsilon^2 v^2} \rho + \frac{2}{r} (r - 2\epsilon^2 m)^2 k^2 \rho. \end{aligned} \tag{1.3.3}$$

Here the main unknowns are the mass-energy density $\rho > 0$ and the scalar velocity $|v| < 1/\epsilon$, and are defined for $r > 2\epsilon^2 m$. We are interested in investigating limiting regimes determined by extremal values of the physical parameters, i.e. the mass of the black hole $m \in (0, +\infty)$, the light speed $\frac{1}{\epsilon} \in (0, +\infty)$, and the sound speed $k \in (0, 1/\epsilon)$. Figure ?? provides an illustration of this family of models. Let us also summarize, for this family of models, our conclusions in the previous section.

Proposition 1.3.1. *Consider the Euler equation (1.3.3), take the pressure p as a linear function of the density $\rho > 0$, that is, $p(\rho) = k^2\rho$ where the sound speed k is a positive constant. When $0 < k < 1/\epsilon$, (1.3.3) is strictly hyperbolic and genuinely nonlinear. When $k = 0$, it is non-strictly hyperbolic and linearly degenerate; when $k = \frac{1}{\epsilon}$, it is strictly hyperbolic but linearly degenerate.*

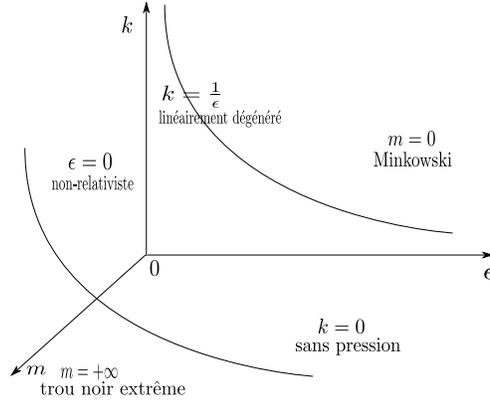


Figure 1.3.1: Limit regimes of model $\mathcal{M}(\epsilon, k, m)$.

Formal limits on the light speed and sound speed

Non-relativistic fluid flows

First of all, when $\epsilon \rightarrow 0$, the light speed goes to infinity and in order to avoid a blow-up of the source term, $M \frac{r-2M}{\epsilon^2 r} \frac{1+\epsilon^4 k^2 v^2}{1-\epsilon^2 v^2} \rho$ in the ‘second’ Euler equations in (1.3.1), we keep the ratio $m = \frac{M}{\epsilon^2}$ constant. Letting $\epsilon \rightarrow 0$, we arrive at the **Euler model for non-relativistic fluid flows on a Schwarzschild background**, denoted by $\mathcal{M}(0, k, m)$:

$$\begin{aligned} \partial_t(r^2 \rho) + \partial_r(r^2 \rho v) &= 0, \\ \partial_t(r^2 \rho v) + \partial_r(r^2 \rho (v^2 + k^2)) - 2k^2 r \rho + m \rho &= 0, \quad t \geq 0, r > 0. \end{aligned} \quad (1.3.4)$$

Interestingly, this model applies to non-relativistic flows but yet contains a “relaxation term”, that is $m\rho$, which is induced by the black hole geometry. Provided $k > 0$, this model is strictly hyperbolic (for $\rho > 0$ and $v \in \mathbb{R}$) and admits two genuinely nonlinear characteristic fields. In Section 1.4, we will first study the family of steady state solutions and, for the Cauchy problem in Section 1.9, we will establish a global-in-time theory of weak solutions.

Stiff fluid flows

Returning to the regime of finite light speed, we now consider limiting regimes for the sound speed $k \in (0, 1/\epsilon)$. By definition, a stiff fluid is governed by the equation $p = \epsilon^{-2} \rho$ for which the sound speed coincides with the light speed. Letting therefore $k \rightarrow 1/\epsilon$, we define the **Euler model for stiff fluid flows on a Schwarzschild**

background $\mathcal{M}(\epsilon, \frac{1}{\epsilon}, m)$ as

$$\begin{aligned} \partial_t \left(r^2 \frac{1 + \epsilon^2 v^2}{1 - \epsilon^2 v^2} \rho \right) + \partial_r \left(r(r - 2\epsilon^2 m) \frac{2\rho v}{1 - \epsilon^2 v^2} \right) &= 0, \\ \partial_t \left(r(r - 2\epsilon^2 m) \frac{2\rho v}{1 - \epsilon^2 v^2} \right) + \partial_r \left((r - 2\epsilon^2 m)^2 \frac{1 + \epsilon^2 v^2}{\epsilon^2 (1 - \epsilon^2 v^2)} \rho \right) & \quad (1.3.5) \\ &= 2\epsilon^2 m \frac{r - 2\epsilon^2 m}{\epsilon^2 r} \frac{1 + \epsilon^2 v^2}{1 - \epsilon^2 v^2} \rho + 2 \frac{(r - 2\epsilon^2 m)^2}{\epsilon^2 r} \rho. \end{aligned}$$

According to Proposition 1.2.3, this model has two linearly degenerate characteristic fields. The Cauchy problem for this system will be studied in Section 1.9, below.

Pressureless fluid flows

Letting now the sound speed $k \rightarrow 0$, we obtain a regime where the pressure vanishes identically and we can introduce the **Euler model of pressureless fluid flow** $\mathcal{M}(\epsilon, 0, m)$:

$$\begin{aligned} \partial_t \left(r^2 \frac{\rho}{1 - \epsilon^2 v^2} \right) + \partial_r \left(r(r - 2\epsilon^2 m) \frac{\rho v}{1 - \epsilon^2 v^2} \right) &= 0, \\ \partial_t \left(r(r - 2\epsilon^2 m) \frac{\rho v}{1 - \epsilon^2 v^2} \right) + \partial_r \left((r - 2\epsilon^2 m)^2 \frac{\rho v^2}{1 - \epsilon^2 v^2} \right) &= \frac{m}{r} (3\epsilon^2 - 1) (r - 2\epsilon^2 m) \frac{\rho v^2}{1 - \epsilon^2 v^2}. \end{aligned} \quad (1.3.6)$$

Observe that this system is not hyperbolic, since it admits only one eigenvalue: $\lambda = \mu = \left(1 - \frac{2M}{r}\right)v$. Note also $p \equiv 0$ obviously satisfies (1.2.25), so that (1.3.6) admits one linearly degenerate characteristic field, while it can be checked that the other field is genuinely nonlinear. This model can not be handled by the techniques in the present paper, and we postpone its analysis to a follow-up work.

Non-relativistic pressureless regime

In addition to having $k \rightarrow 0$, we can also take the limit $\epsilon \rightarrow 0$ in (1.3.6) and thus define the **Euler model for pressureless non-relativistic flows** $\mathcal{M}(0, 0, m)$:

$$\partial_t(r^2 \rho) + \partial_r(r^2 \rho v) = 0, \quad \partial_t(r^2 \rho v) + \partial_r(r^2 \rho v^2) + m\rho = 0. \quad (1.3.7)$$

Vanishing black hole mass

Relativistic regime

When the black hole mass is taken to vanish, that is, $m \rightarrow 0$, the Schwarzschild metric approaches the Minkowski metric (1.2.1), and we arrive at the **Euler model for radially symmetric fluid flows in Minkowski space** denoted by $\mathcal{M}(\epsilon, k, 0)$:

$$\begin{aligned} \partial_t \left(\frac{1 + \epsilon^4 k^2 v^2}{1 - \epsilon^2 v^2} \rho \right) + \partial_r \left(\frac{1 + \epsilon^2 k^2}{1 - \epsilon^2 v^2} \rho v \right) &= 0, \\ \partial_t \left(\frac{1 + \epsilon^2 k^2}{1 - \epsilon^2 v^2} \rho v \right) + \partial_r \left(\frac{v^2 + k^2}{1 - \epsilon^2 v^2} \rho \right) &= 0. \end{aligned} \tag{1.3.8}$$

Relativistic pressureless regime

If in addition we let the sound speed $k \rightarrow 0$ in (1.3.8), we have the **Euler model for radially symmetric, pressureless flows in Minkowski space** $\mathcal{M}(\epsilon, 0, 0)$:

$$\partial_t \left(\frac{\rho}{1 - \epsilon^2 v^2} \right) + \partial_r \left(\frac{\rho v}{1 - \epsilon^2 v^2} \right) = 0, \quad \partial_t \left(\frac{\rho v}{1 - \epsilon^2 v^2} \right) + \partial_r \left(\frac{\rho v^2}{1 - \epsilon^2 v^2} \right) = 0. \tag{1.3.9}$$

Observe that, for sufficiently regular solutions, these equations are equivalent to

$$\partial_t v + \partial_r \left(\frac{v^2}{2} \right) = 0, \quad \partial_t \left(\frac{\rho}{1 - \epsilon^2 v^2} \right) + \partial_r \left(\frac{\rho v}{1 - \epsilon^2 v^2} \right) = 0,$$

from which we see that the velocity component satisfies Burgers' equation.

Non-relativistic regime

Finally, letting both $m \rightarrow 0$ and $\epsilon \rightarrow 0$, we obtain the **Euler model for radially symmetric, non-relativistic fluid flows** $\mathcal{M}(0, k, 0)$:

$$\partial_t(r^2 \rho) + \partial_r(r^2 \rho v) = 0, \quad \partial_t(r^2 \rho v) + \partial_r(r^2 \rho(v^2 + k^2)) = 2k^2 \rho r, \tag{1.3.10}$$

and its pressureless version

$$\partial_t(r^2 \rho) + \partial_r(r^2 \rho v) = 0, \quad \partial_t(r^2 \rho v) + \partial_r(r^2 \rho v^2) = 0. \tag{1.3.11}$$

Fluid flows in a black hole background with extreme mass

Another limit of interest is obtained when $M \rightarrow +\infty$. In order to analyze this regime, we fix $\epsilon > 0$ and $k \in (0, 1/\epsilon)$ and we define a rescaled variable $\tilde{r} := \frac{r}{2M} \in (1, +\infty)$. We can rewrite (1.3.3) in the form

$$\begin{aligned} \partial_t \left(\tilde{r}^2 \frac{1 + \epsilon^4 k^2 v^2}{1 - \epsilon^2 v^2} \rho \right) + \frac{1}{2M} \partial_{\tilde{r}} \left(\tilde{r}(\tilde{r} - 1) \frac{1 + \epsilon^2 k^2}{1 - \epsilon^2 v^2} \rho v \right) &= 0, \\ \partial_t \left(\tilde{r}(\tilde{r} - 1) \frac{1 + \epsilon^2 k^2}{1 - \epsilon^2 v^2} \rho v \right) + \frac{1}{2M} \partial_{\tilde{r}} \left((\tilde{r} - 1)^2 \frac{v^2 + k^2}{1 - \epsilon^2 v^2} \rho \right) &= \tilde{\Omega}, \\ \tilde{\Omega} &:= \frac{3}{4M} \frac{\tilde{r} - 1}{\tilde{r}} \frac{v^2 + k^2}{1 - \epsilon^2 v^2} \rho - \frac{1}{4M} \frac{\tilde{r} - 1}{\epsilon^2 \tilde{r}} \frac{1 + \epsilon^4 k^2 v^2}{1 - \epsilon^2 v^2} \rho + \frac{(\tilde{r} - 1)^2}{M \tilde{r}} k^2 \rho, \end{aligned} \quad (1.3.12)$$

and we now formally investigate the singular limit $M \rightarrow +\infty$.

Lemma 1.3.2. *For solutions to (1.3.12) expanded in the form (for $t \geq 0$ and $\tilde{r} > 1$)*

$$\rho(t, \tilde{r}) = \sum_{j=0}^{+\infty} \frac{1}{M^j} \rho^{(j)}(t, \tilde{r}), \quad v(t, \tilde{r}) = \sum_{j=0}^{+\infty} \frac{1}{M^j} v^{(j)}(t, \tilde{r}),$$

it follows that the functions $\rho^{(0)}, v^{(0)}$ must be independent of the time variable t , while $\rho^{(j)}, v^{(j)}$ satisfy a coupled system of ordinary differential equations in the time variable:

$$\begin{aligned} \partial_t \rho^{(j)}(t, \cdot) &= \sum_{i=0}^{j-1} \left(A_3^{j,i} \rho^{(i)}(t, \cdot) + A_4^{j,i} v^{(i)}(t, \cdot) \right), \\ \partial_t v^{(j)}(t, \cdot) &= \sum_{i=0}^{j-1} \left(B_3^{j,i} \rho^{(i)}(t, \cdot) + B_4^{j,i} v^{(i)}(t, \cdot) \right), \end{aligned} \quad (1.3.13)$$

in which the coefficients are constants depending upon ϵ and k only.

Proof. Keeping only the terms of zero-order in $\frac{1}{M}$, we easily find the ordinary differential system

$$\partial_t \left(\frac{1 + \epsilon^4 k^2 (v^{(0)})^2}{1 - \epsilon^2 (v^{(0)})^2} \rho^{(0)} \right) = 0, \quad \partial_t \left(\frac{1 + \epsilon^2 k^2}{1 - \epsilon^2 (v^{(0)})^2} \rho^{(0)} v^{(0)} \right) = 0. \quad (1.3.14)$$

which is equivalent to saying that $\partial_t \rho^{(0)} = \partial_t v^{(0)} = 0$, so that $\rho^{(0)} = \rho^{(0)}(\tilde{r})$ and $v^{(0)} = v^{(0)}(\tilde{r})$ depend on the spatial variable only. Next, keeping the terms of the

first-order in $\frac{1}{M}$, we find the following system of equations:

$$\begin{aligned}
& \tilde{r}^2 \partial_t \left(\rho^{(1)} \frac{1 + \epsilon^4 k^2 (v^{(0)})^2}{1 - \epsilon^2 (v^{(0)})^2} + \rho^{(0)} v^{(0)} v^{(1)} \left(\frac{2\epsilon^4 k^2}{1 - \epsilon^2 (v^{(0)})^2} + 2\epsilon(1 + \epsilon^4 k^2 v^{(0)^2}) \right) \right) \\
& \quad + \frac{1}{2} \partial_{\tilde{r}} \left(\tilde{r}(\tilde{r} - 1) \frac{1 + \epsilon^2 k^2}{1 - \epsilon^2 v^{(0)^2} } \rho^{(0)} v^{(0)} \right) = 0, \\
& \tilde{r}(\tilde{r} - 1) \partial_t \left(\frac{1 + \epsilon^2 k^2}{1 - \epsilon^2 v^{(0)^2} } (\rho^{(1)} v^{(0)} + \rho^{(0)} v^{(1)}) + 2\epsilon^2 (1 + \epsilon^2 k^2) \rho^{(0)} v^{(0)^2} v^{(1)} \right) \\
& \quad + \frac{1}{2} \partial_{\tilde{r}} \left((\tilde{r} - 1)^2 \frac{v^{(0)^2} + k^2}{1 - \epsilon^2 v^{(0)^2} } \rho^{(0)} \right) \\
& = \frac{3\tilde{r} - 1}{4} \frac{v^{(0)^2} + k^2}{\tilde{r} (1 - \epsilon^2 v^{(0)^2})} \rho^{(0)} - \frac{1}{4} \frac{\tilde{r} - 1}{\epsilon^2 \tilde{r}} \frac{1 + \epsilon^4 k^2 v^{(0)^2}}{1 - \epsilon^2 v^{(0)^2} } \rho^{(0)} + \frac{(\tilde{r} - 1)^2}{\tilde{r}} k^2 \rho^{(0)}.
\end{aligned} \tag{1.3.15}$$

The functions $\rho^{(0)}, v^{(0)}$ being already fixed, we see that (1.3.15) is a differential system in the time variable t , which has the general form (higher-order terms $\rho^{(j)}, v^{(j)}$ (with $j > 2$) being determined similarly):

$$\begin{aligned}
A_1^j \partial_t \rho^{(j)}(t, \cdot) + A_2^j \partial_t v^{(j)}(t, \cdot) &= \sum_{i=0}^{j-1} \left(A_3^{j,i} \rho^{(i)}(t, \cdot) + A_4^{j,i} v^{(i)}(t, \cdot) \right), \\
B_1^j \partial_t \rho^{(j)}(t, \cdot) + B_2^j \partial_t v^{(j)}(t, \cdot) &= \sum_{i=0}^{j-1} \left(B_3^{j,i} \rho^{(i)}(t, \cdot) + B_4^{j,i} v^{(i)}(t, \cdot) \right),
\end{aligned}$$

in which the coefficients are constants depending upon ϵ and k only. This system is non-degenerate in the sense that it can be expressed as an ordinary differential system in t for the functions $\partial_t \rho^{(j)}(t, \cdot)$ and $\partial_t v^{(j)}(t, \cdot)$. Changing the notation, we thus arrive at (1.3.13). \square

In view of Lemma 1.3.2, in the extreme mass regime $M \rightarrow +\infty$, the leading-order behavior of solutions only depends on the space variable \tilde{r} , that is,

$$\rho(t, \tilde{r}) = \rho^{(0)}(\tilde{r}), \quad v(t, \tilde{r}) = v^{(0)}(\tilde{r}).$$

Proceeding at a formal level, the following result is now immediate. It would be interesting to rigorously justify the expansion below, but this is outside the scope of the present paper.

Proposition 1.3.3 (Asymptotic solutions for black holes with extreme mass). *Consider the Euler model (1.3.12) with initial data prescribed at $t = 0$:*

$$\rho(0, \tilde{r}) = \rho^{(0)}(\tilde{r}), \quad v(0, \tilde{r}) = v^{(0)}(\tilde{r}), \quad \tilde{r} > 0.$$

1. If the data $\rho^{(0)}, v^{(0)}$ belong to C^l for some $l \geq 1$, then there exists an approximate

solution, i.e.

$$\tilde{\rho}(t, \tilde{r}) := \rho^{(0)}(\tilde{r}) + \sum_{j=1}^l \frac{1}{M^j} \rho^{(j)}(t, \tilde{r}), \quad \tilde{v}(t, \tilde{r}) := v^{(0)}(\tilde{r}) + \sum_{j=1}^l \frac{1}{M^j} v^{(j)}(t, \tilde{r}),$$

which satisfies (1.3.12) up to an error $\mathcal{O}(1/M^{l+1})$.

2. If $\rho^{(0)}, v^{(0)}$ has C^∞ regularity, then exists a formal series defined at all order.

1.4 Non-relativistic equilibria on a Schwarzschild background

We now turn our attention to the main model of interest in this section, that is, the Euler model for non-relativistic flows on a Schwarzschild background (1.3.4), which we have denoted by $\mathcal{M}(0, k, m)$. We begin by considering general pressure-laws, that is,

$$\begin{aligned} \partial_t(r^2 \rho) + \partial_r(r^2 \rho v) &= 0, \\ \partial_t(r^2 \rho v) + \partial_r(r^2(\rho v^2 + p)) - 2pr + m\rho &= 0, \end{aligned} \tag{1.4.1}$$

for solutions defined on $r \in (0, +\infty)$. We search for steady state solutions $\rho = \rho(r)$ and $v = v(r)$, which satisfy the differential system:

$$\begin{aligned} \frac{d}{dr}(r^2 \rho v) &= 0, \\ \frac{d}{dr}(r^2(\rho v^2 + p)) - 2pr + m\rho &= 0 \end{aligned} \tag{1.4.2}$$

with initial condition $\rho_0, v_0 > 0$ prescribed at some given radius $r = r_0 > 0$,

$$\rho(r_0) = \rho_0 > 0, \quad v(r_0) = v_0. \tag{1.4.3}$$

It is straightforward to check the following statement.

Lemma 1.4.1. *All solutions (2.3.3)–(1.4.3) satisfy*

$$r^2 \rho(r) v(r) = r_0^2 \rho_0 v_0, \tag{1.4.4}$$

$$\frac{1}{2} v(r)^2 + h(\rho(r)) - m \frac{1}{r} = \frac{1}{2} v_0^2 + h(\rho_0) - m \frac{1}{r_0}, \tag{1.4.5}$$

where $h(\rho) := \int^\rho \frac{p'(s)}{s} ds$.

In view of (1.4.4), we see that the solution v has the sign of the initial condition

v_0 , and without loss of generality, we now assume that $v_0 \geq 0$. We are especially interested in a constant sound speed, that is, $p = k^2 \rho$ with $k > 0$, hence

$$\begin{aligned} \frac{d}{dr}(r^2 \rho v) &= 0, \\ \frac{d}{dr}\left(r^2 \rho (v^2 + k^2)\right) - 2k^2 r \rho + m \rho &= 0. \end{aligned} \tag{1.4.6}$$

According to Lemma 1.4.1, we must solve the system

$$\begin{aligned} r^2 \rho v &= r_0^2 \rho_0 v_0, \\ \frac{1}{2} v^2 + k^2 \ln \rho - m \frac{1}{r} &= \frac{1}{2} v_0^2 + k^2 \ln \rho_0 - m \frac{1}{r_0}. \end{aligned} \tag{1.4.7}$$

After eliminating ρ , we find an algebraic equation for the velocity, i.e.

$$\frac{1}{2} v^2 + k^2 \ln \frac{r_0^2 v_0}{r^2 v} - m \frac{1}{r} = \frac{1}{2} v_0^2 - m \frac{1}{r_0} \tag{1.4.8}$$

and we now focus on this equation.

Let us introduce the function

$$G(r, v; r_0, v_0) := \frac{1}{2}(v^2 - v_0^2) + k^2 \ln \frac{r_0^2 v_0}{r^2 v} - \frac{m}{r} + \frac{m}{r_0}. \tag{1.4.9}$$

By definition, if $v = v(r)$ is a steady state solution, then $G(r, v(r); r_0, v_0) \equiv 0$ and, in addition, $v(r_0) = v_0$. Clearly, we have $G(r_0, v_0; r_0, v_0) = 0$. Differentiating G with respect to v and r , we obtain

$$\partial_v G(r, v; r_0, v_0) = v - \frac{k^2}{v}, \quad \partial_r G(r, v; r_0, v_0) = \frac{1}{r^2}(m - 2k^2 r).$$

Hence, the function G is decreasing with respect to v when $v < k$, and is increasing when $v > k$ (that is, a non-sonic velocity). Also, the derivative of a solution $v = v(r)$ is found to be

$$\frac{dv}{dr} = \frac{v}{r^2} \frac{2k^2 r - m}{v^2 - k^2}. \tag{1.4.10}$$

Since $\partial_v G(r_0, v_0; r_0, v_0) \neq 0$ when $v_0 \neq k$, it is immediate to apply the implicit function theorem for the function $v = v(r)$ and then recover the density $\rho = \rho(r)$ by (1.4.7). We thus have the following local existence statement.

Lemma 1.4.2 (Locally-defined steady state solutions). *Given any values $r_0 > 0$, $\rho_0 > 0$, $v_0 \geq 0$ with $v_0 \neq k$, the system (1.4.6) with initial condition (1.4.3) at $r = r_0$ admits a unique smooth solution $\rho = \rho(r)$ and $v = v(r)$ defined in a neighborhood \mathcal{U}_0*

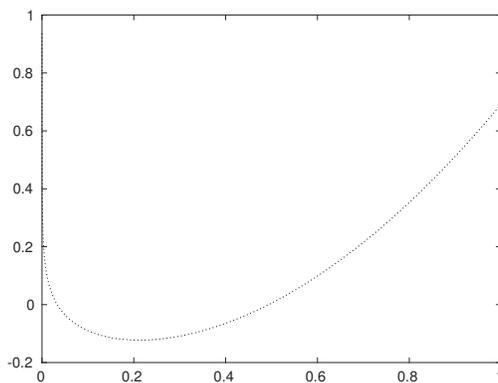


Figure 1.4.1: Plot of the map $v \mapsto G(v) = G(r, v; r_0, v_0)$.

of r_0 and denoted by

$$\rho = \rho(r; r_0, \rho_0, v_0), \quad v = v(r; r_0, \rho_0, v_0).$$

According to (4.3.6), the derivative of a solution $v = v(r)$ may blow up if at some radius r_* the velocity $v(r_*) = k$ reaches the sonic value. We will use the following notation.

Definition 1.4.3. A radius $r_* > 0$ is called a **sonic point** if it is a root of the equation

$$\frac{1}{2}(k^2 - v_0^2) + k^2 \ln \frac{v_0}{k} - k^2 \ln \frac{r^2}{r_0^2} - \frac{m}{r} + \frac{m}{r_0} = 0. \quad (1.4.11)$$

If such a radius r_* exists, then the derivative $\frac{dv}{dr}$ tends to infinity when $r \rightarrow r_*$ and the velocity loses its regularity.

Lemma 1.4.4. One can distinguish between two alternatives:

1. Either $\frac{3}{2} + \ln \frac{m^2}{4k^3 r_0^2 v_0} + \frac{1}{2k^2}(v_0^2 - \frac{2m}{r_0}) > 0$ and there exists no sonic point.
2. Or $\frac{3}{2} + \ln \frac{m^2}{4k^3 r_0^2 v_0} + \frac{1}{2k^2}(v_0^2 - \frac{2m}{r_0}) \leq 0$, there exist two (possibly coinciding) sonic points, denoted by $\underline{r}_* \leq \bar{r}_*$. Moreover, in this case, one has:
 - When $r_0 \geq \frac{m}{2k^2}$, the roots satisfy $\underline{r}_* \leq \bar{r}_* \leq r_0$.
 - When $r_0 < \frac{m}{2k^2}$, the roots satisfy $r_0 < \underline{r}_* \leq \bar{r}_*$.

Proof. We introduce the functions $f(x) := \frac{1}{2} - \frac{1}{2}x^2 + \ln x$ and $g(r) := \ln \frac{r^2}{r_0^2} + \frac{m}{k^2}(\frac{1}{r} - \frac{1}{r_0})$, so that a sonic point r_* is characterized by the condition $f(\frac{v_0}{k}) = g(r_*)$. Since $f'(x) = -x + \frac{1}{x}$, we see that f reaches its maximum at $x = 1$, with $f(1) = 0$. Since we assume $v_0 \neq k$, we have $-\infty < f(\frac{v_0}{k}) < 0$. Turning our attention to the

function $g = g(r)$, we have $g'(r) = \frac{1}{r^2}(2r - \frac{m}{k^2})$. Therefore, the minimum of $g = g(r)$ is obtained at $r = \frac{m}{2k^2}$, with $g(\frac{m}{2k^2}) = \ln \frac{m^2}{4k^4 r_0^2} + 2 - \frac{m}{k^2 r_0}$. We now set $\tilde{g}(x) := \ln \frac{x^2}{4} + 2 - x$. According to our definition, $\tilde{g}(\frac{m}{r_0 k^2}) = g(\frac{m}{2k^2})$. We have $\tilde{g}'(x) = \frac{2}{x} - 1$, so that $-\infty < \tilde{g}(x) \leq \tilde{g}(2) = 0$. Therefore, (1.4.11) admits no solution if and only if $f(\frac{v_0}{k}) < g(\frac{m}{2k^2})$. This yields us the condition $\frac{1}{2} - \frac{1}{2} \left(\frac{v_0}{k}\right)^2 + \ln \frac{v_0}{k} < \ln \frac{m^2}{4k^4 r_0^2} + 2 - \frac{m}{k^2 r_0}$, as announced. If the opposite inequality holds, then, (1.4.11) admits two solutions (which may coincide). Furthermore, since $g(r_0) = 0$, we have either $\underline{r}_* \leq \bar{r}_* \leq r_0$ or $r_0 \leq \underline{r}_* \leq \bar{r}_*$. If $\underline{r}_* \leq \bar{r}_* \leq r_0$, we must have $g'(r_0) \geq 0$, which gives $r_0 \geq \frac{m}{2k^2}$. \square

We now define a function P which only depends upon the initial radius r_0 and the initial velocity v_0 :

$$P(r_0, v_0) := \frac{3}{2} + \ln \frac{m^2}{4k^3 r_0^2 v_0} + \frac{1}{2k^2} \left(v_0^2 - \frac{2m}{r_0} \right). \quad (1.4.12)$$

According to Lemma 1.4.4, the existence/non-existence of sonic points is determined by the sign of P . We will now distinguish between several cases and introduce a general notation:

$$\begin{aligned} \mathfrak{A} : P(r_0, v_0) > 0, & \quad \mathfrak{B} : P(r_0, v_0) \leq 0, \\ \mathfrak{1} : v_0 < k, & \quad \mathfrak{2} : v_0 > k, \\ \mathfrak{i} : r_0 \geq \frac{m}{2k^2}, & \quad \mathfrak{ii} : r_0 < \frac{m}{2k^2}. \end{aligned} \quad (1.4.13)$$

Hence, the symbol $\mathfrak{A}\mathfrak{1}$ refers to the case where both conditions $\frac{3}{2} + \ln \frac{m^2}{4k^3 r_0^2 v_0} + \frac{1}{2k^2} (v_0^2 - \frac{2m}{r_0}) > 0$ and $v_0 < k$ hold.

Lemma 1.4.5 (Extension of solutions without sonic point). *Consider the local solution $\rho = \rho(r; r_0, \rho_0, v_0)$ and $v = v(r; r_0, \rho_0, v_0)$ given by Lemma 1.4.2.*

1. **Case $\mathfrak{A}\mathfrak{1}$.** *The solution can be extended to the whole domain $(0, +\infty)$ and globally satisfies $v < k$, with*

$$\lim_{r \rightarrow 0} v(r; r_0, \rho_0, v_0) = \lim_{r \rightarrow +\infty} v(r; r_0, \rho_0, v_0) = 0.$$

One has the following monotonicity property: the velocity v is increasing with respect to r on the interval $(0, \frac{m}{2k^2})$, while it is decreasing on the interval $(\frac{m}{2k^2}, +\infty)$.

2. **Case $\mathfrak{A}\mathfrak{2}$.** *The solution can be extended to the whole domain $(0, +\infty)$ and globally satisfies $v > k$, with*

$$\lim_{r \rightarrow 0} v(r; r_0, \rho_0, v_0) = \lim_{r \rightarrow +\infty} v(r; r_0, \rho_0, v_0) = +\infty.$$

One has the following monotonicity property: the velocity v is decreasing with respect to r on the interval $(0, \frac{m}{2k^2})$, while it is increasing on the interval $(\frac{m}{2k^2}, +\infty)$.

Proof. The two cases are completely similar and we treat the case $\mathfrak{A}1$. Since we have sonic point, the velocity $v = v(r)$ never reaches the sound speed k and, by the implicit function theorem, the solution can be continued and extended to the whole interval $(0, \infty)$. Its derivative, given by (4.3.6), remains finite. From the definition of the function G in (4.3.4), we obtain

$$\frac{1}{2}(v^2 - v_0^2) + k^2 \ln \frac{v_0}{v} = k^2 \ln \frac{r^2}{r_0^2} + \frac{m}{r} - \frac{m}{r_0}.$$

When $r \rightarrow 0$ or $r \rightarrow +\infty$, the left-hand side of this identity goes to infinity. Such a limit is reached if and only if v goes to 0 or infinity. Since $v < k$ always holds in this case, we obtain the asymptotic behavior limits, as stated in the lemma. Furthermore, the expression (4.3.6) of $\frac{dv}{dr}$ determines the monotonicity properties: $\frac{dv}{dr}$ has the sign of $\frac{m}{2k^2} - r$. \square

Lemma 1.4.6 (Extension of solutions with sonic points). *Consider the local solutions $\rho = \rho(r; r_0, \rho_0, v_0)$ and $v = v(r; r_0, \rho_0, v_0)$ given by Lemma 1.4.2.*

1. **Case $\mathfrak{B}1i$.** *The solution can be extended to the interval $(\bar{r}_*, +\infty)$ and satisfies $v \leq k$, with*

$$\lim_{r \rightarrow +\infty} v(r; r_0, \rho_0, v_0) = 0, \quad \lim_{r \rightarrow \bar{r}_*} v(r; r_0, \rho_0, v_0) = k.$$

Moreover, v is decreasing with respect to r on $(\bar{r}_, +\infty)$.*

2. **Case $\mathfrak{B}2i$.** *The solution can be extended to the interval $(\bar{r}_*, +\infty)$ and satisfies $v \geq k$, with*

$$\lim_{r \rightarrow +\infty} v(r; r_0, \rho_0, v_0) = +\infty, \quad \lim_{r \rightarrow \bar{r}_*} v(r; r_0, \rho_0, v_0) = k.$$

Moreover, v is increasing with respect to r on $(\bar{r}_, +\infty)$.*

3. **Case $\mathfrak{B}1ii$.** *The solution can be extended to the interval $(0, \underline{r}_*)$ and satisfies $v \leq k$, with*

$$\lim_{r \rightarrow 0} v(r; r_0, \rho_0, v_0) = 0, \quad \lim_{r \rightarrow \underline{r}_*} v(r; r_0, \rho_0, v_0) = k.$$

Moreover, v is increasing with respect to r on $(0, \underline{r}_)$.*

4. **Case $\mathfrak{B}2ii$.** *The solution can be extended to the interval $(0, \underline{r}_*)$ and satisfies $v \geq k$, with*

$$\lim_{r \rightarrow 0} v(r; r_0, \rho_0, v_0) = +\infty, \quad \lim_{r \rightarrow \underline{r}_*} v(r; r_0, \rho_0, v_0) = k.$$

Moreover, v is decreasing with respect to r on $(0, \underline{r}_)$.*

Proof. Consider the case $\mathfrak{B}1i$ (while the case $\mathfrak{B}2i$ is completely similar). According to Lemma 1.4.4, there exist two sonic points $\underline{r}_* \leq \bar{r}_* \leq r_0$, so that by continuation the solution can be extended to the whole interval $(\bar{r}_*, +\infty)$ and the limits $r \rightarrow +\infty$ and $r \rightarrow \underline{r}_*$ are easily computed. Moreover, since in this case $r \geq \bar{r}_* \geq \frac{m}{2k^2}$ and $v \leq k$, the function $v = v(r)$ is decreasing in r .

In the case $\mathfrak{B}1ii$ (while the case $\mathfrak{B}2ii$ can be treated similarly), Lemma 1.4.4 shows that there exist two sonic points $r_0 < \underline{r}_* < \bar{r}_*$. In this case, the solution can be extended to $(0, \underline{r}_*)$ and the limits $r \rightarrow 0$ and $r \rightarrow \underline{r}_*$ are easily computed. The condition $r < \bar{r}_* < \frac{m}{2k^2}$ gives the monotonicity property. \square

Observe also that no solution can be defined on the interval $r \in (\underline{r}_*, \bar{r}_*)$. Indeed, since G reaches its minimum at $v = k$, we deduce that, for any radius $r \in (\underline{r}_*, \bar{r}_*)$, the inequality

$$G(r, v, r_0, v_0) > G(r, k, r_0, v_0) > 0$$

holds, that is, G cannot admit roots between the two sonic points. Therefore, a solution cannot be further extended when it reaches a sonic point. We summarize our conclusions in this section in the following theorem.

Theorem 1.4.7 (Non-relativistic steady flows on a Schwarzschild background). For any sound speed $k > 0$ and black hole mass $m > 0$, consider the Euler model $\mathcal{M}(0, k, m)$ given in (1.3.4), describing non-relativistic flows on a Schwarzschild background. Then, given any radius $r_0 > 0$, density $\rho_0 > 0$, and velocity $v_0 \geq 0$, there exists a unique steady state solution denoted by

$$\rho = \rho(r; r_0, \rho_0, v_0), \quad v = v(r; r_0, \rho_0, v_0),$$

satisfying the system (1.4.6) together with the initial condition $\rho(r_0) = \rho_0$ and $v(r_0) = v_0$. Moreover, the velocity component satisfies $\text{sgn}(v(r) - k) = \text{sgn}(v_0 - k)$ for all relevant values r , and in order to specify the range of the independent variable r where this solution is defined, we distinguish between two alternatives:

1. **Regime without sonic point:** $P(r_0, v_0) > 0$ (with P defined in (1.4.12)). Then, the solution is defined on the whole interval $(0, +\infty)$.
2. **Regime with sonic points:** $P(r_0, v_0) \leq 0$. The solution is defined on the interval $\Xi \subsetneq (0, +\infty)$, defined by

$$\Xi := \begin{cases} (0, \underline{r}_*), & r_0 \leq \frac{m}{2k^2}, \\ (\bar{r}_*, +\infty), & r_0 > \frac{m}{2k^2}. \end{cases} \quad (1.4.14)$$

Moreover, the velocity $v(r) \rightarrow k$ when r approaches the sonic point.

These solutions will be used to design a method of approximation of general weak solutions to the Cauchy problem. In fact (cf. Section 1.7), we will need to introduce

discontinuous solutions in order to construct globally-defined steady state solution (defined for all r). This will be achieved with solutions containing a jump discontinuity connecting two smooth steady state solutions.

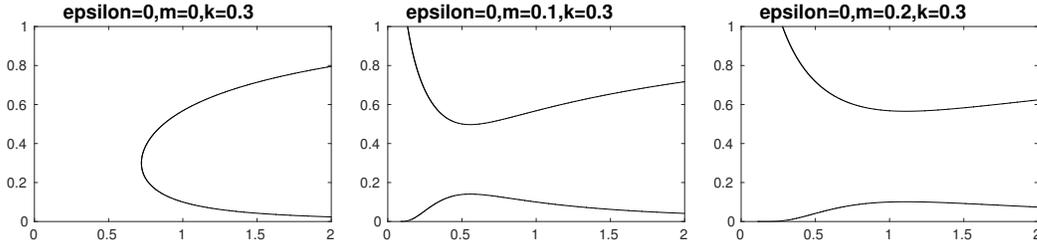


Figure 1.4.2: Plots of $v = v(r)$ with sound speed $k = 0.3$ and different masses.

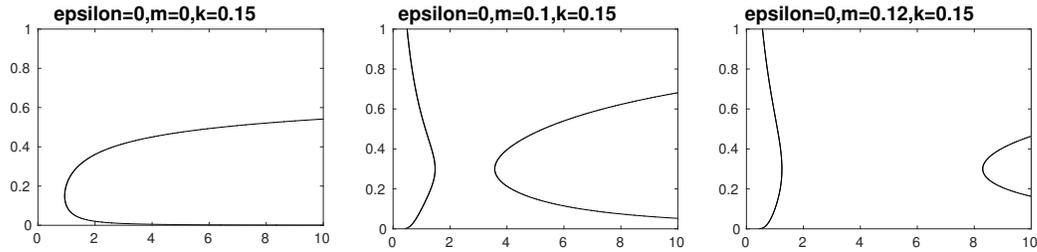


Figure 1.4.3: Plots of $v = v(r)$ with sound speed $k = 0.15$ and different masses.

1.5 Fluid equilibria on a Schwarzschild background

Local existence result

This section is devoted to the analysis of (smooth) steady state solutions to the Euler system on a Schwarzschild background, i.e. the general model (3.1.3). Such solutions must satisfy the following two coupled ordinary differential equations with unknowns $\rho = \rho(r)$ and $v = v(r)$ (defined over $r > 2M$)

$$\frac{d}{dr} \left(r(r-2M) \frac{(\rho + \epsilon^2 p)v}{1 - \epsilon^2 v^2} \right) = 0, \quad (1.5.1a)$$

$$\frac{d}{dr} \left((r-2M)^2 \frac{\rho v^2 + p}{1 - \epsilon^2 v^2} \right) = \frac{M}{r} \frac{(r-2M)}{1 - \epsilon^2 v^2} \left(3\rho v^2 + 3p - \epsilon^{-2} \rho - \epsilon^2 p v^2 \right) + \frac{2}{r} (r-2M) \frac{(\rho + \epsilon^2 p)v}{1 - \epsilon^2 v^2}, \quad (1.5.1b)$$

formulated here for a general pressure-law $p = p(\rho)$. We are interested in solving the associated initial value problem for a given radius $r_0 > 2M$ with data ρ_0, v_0 prescribed at $r = r_0$:

$$\rho(r_0) = \rho_0 > 0, \quad v(r_0) = v_0. \quad (1.5.2)$$

Lemma 1.5.1. *If $\rho = \rho(r)$ and $v = v(r)$ is a solution to (1.5.1)–(1.5.2), then one has*

$$\begin{aligned} r(r-2M) \frac{(\rho + \epsilon^2 p(\rho))v}{1 - \epsilon^2 v^2} &= D_0, \\ -\frac{1}{2\epsilon^2} \ln(1 - \epsilon^2 v^2) + l(\rho) + \frac{1}{2\epsilon^2} \ln \left(1 - \frac{2M}{r} \right) &= C_0, \end{aligned} \quad (1.5.3)$$

where the function $l = l(\rho)$ is defined by $l'(\rho) := \frac{p'(\rho)}{\rho + \epsilon^2 p(\rho)}$, and the constants above are determined by the initial conditions, that is,

$$\begin{aligned} D_0 &:= r_0(r_0 - 2M) \frac{(\rho_0 + \epsilon^2 p(\rho_0))v_0}{1 - \epsilon^2 v_0^2}, \\ C_0 &:= -\frac{1}{2\epsilon^2} \ln(1 - \epsilon^2 v_0^2) + l(\rho_0) + \frac{1}{2\epsilon^2} \ln \left(1 - \frac{2M}{r_0} \right). \end{aligned}$$

Observe that by letting $\epsilon \rightarrow 0$ in (1.5.3), we recover our earlier formulas (1.4.4), (1.4.5) for non-relativistic flows.

Proof. The equation (1.5.1a) leads us immediately to the first equation in (1.5.3). Next, by multiplying (1.5.1b) by $\frac{r}{r-2M}$, we find

$$\frac{d}{dr} \left(r(r-2M) \frac{\rho v^2 + p}{1 - \epsilon^2 v^2} \right) = M \frac{\rho v^2 + p}{1 - \epsilon^2 v^2} - \frac{M}{\epsilon^2} \frac{\rho + \epsilon^4 p v^2}{1 - \epsilon^2 v^2} + 2(r-2M)p,$$

which is equivalent to $\frac{\rho + \epsilon^2 p}{1 - \epsilon^2 v^2} v \frac{dv}{dr} + \frac{dp}{dr} + \frac{M}{r(r-2M)} (\epsilon^{-2} \rho + p) = 0$. Multiplying this equation by $\frac{1}{\rho + \epsilon^2 p}$, we thus find $\frac{v}{1 - \epsilon^2 v^2} \frac{dv}{dr} + \frac{1}{\rho + \epsilon^2 p} \frac{dp}{dr} + \frac{M}{\epsilon^2 r(r-2M)} = 0$, which, by integration, yields the second equation in (1.5.3). \square

By now assuming the linear pressure law $p(\rho) = k^2 \rho$ with (constant) sound speed $0 < k < 1/\epsilon$, we thus consider the differential system

$$\begin{aligned} \frac{d}{dr} \left(r(r-2M) \frac{(1 + \epsilon^2 k^2)}{1 - \epsilon^2 v^2} \rho v \right) &= 0, \\ \frac{d}{dr} \left((r-2M)^2 \frac{v^2 + k^2}{1 - \epsilon^2 v^2} \rho \right) & \\ = \frac{M}{r} \frac{(r-2M)}{1 - \epsilon^2 v^2} \left(3\rho v^2 + 3k^2 \rho - \epsilon^{-2} \rho - \epsilon^2 k^2 \rho v^2 \right) + \frac{2k^2}{r} (r-2M)^2 \rho. & \end{aligned} \quad (1.5.4)$$

By elementary algebra, in view of (1.5.3) and $l(\rho) = \frac{k^2}{1+\epsilon^2 k^2} \log \rho$, we obtain

$$\begin{aligned} r^2 \rho^{\frac{1-\epsilon^2 k^2}{1+\epsilon^2 k^2}} v &= r_0^2 \rho_0^{\frac{1-\epsilon^2 k^2}{1+\epsilon^2 k^2}} v_0, \\ \left(1 - \frac{2M}{r}\right) \frac{1}{1 - \epsilon^2 v^2} \rho^{\frac{2\epsilon^2 k^2}{1+\epsilon^2 k^2}} &= \left(1 - \frac{2M}{r_0}\right) \frac{1}{1 - \epsilon^2 v_0^2} \rho_0^{\frac{2\epsilon^2 k^2}{1+\epsilon^2 k^2}}. \end{aligned}$$

Consequently, by introducing the notation

$$\kappa := \frac{1 - \epsilon^2 k^2}{1 + \epsilon^2 k^2} \in (0, 1), \quad 1 - \kappa = \frac{2\epsilon^2 k^2}{1 + \epsilon^2 k^2}, \quad (1.5.5)$$

we find

$$\begin{aligned} r^2 \rho^\kappa v &= r_0^2 \rho_0^\kappa v_0, \\ \left(1 - \frac{2M}{r}\right) \frac{\rho^{1-\kappa}}{1 - \epsilon^2 v^2} &= \left(1 - \frac{2M}{r_0}\right) \frac{\rho_0^{1-\kappa}}{1 - \epsilon^2 v_0^2}. \end{aligned} \quad (1.5.6)$$

Clearly, the component v has a constant sign and, for definiteness, we can now assume that $v_0 \geq 0$. By eliminating the density variable ρ , we arrive at an algebraic equation of the velocity v , i.e.

$$\ln \frac{1 - \epsilon^2 v_0^2}{1 - \epsilon^2 v^2} + \frac{1 - \kappa}{\kappa} \ln \frac{v_0}{v} = \frac{1 - \kappa}{\kappa} \ln \frac{r^2}{r_0^2} + \ln \frac{r(r_0 - 2M)}{r_0(r - 2M)}. \quad (1.5.7)$$

Let us define a function G_ϵ of the variables r, v (depending also upon the data r_0, v_0) by

$$G_\epsilon(r, v; r_0, v_0) := \ln \frac{1 - \epsilon^2 v_0^2}{1 - \epsilon^2 v^2} + \frac{1 - \kappa}{\kappa} \ln \frac{r_0^2 v_0}{r^2 v} - \ln \frac{r(r_0 - 2M)}{r_0(r - 2M)}. \quad (1.5.8)$$

(See Figure 1.5.1 for an illustration.) Note that, in the limit $\epsilon \rightarrow 0$ we recover the non-relativistic expression (4.3.4.) By definition, a function $v = v(r)$ is a solution to the problem (1.5.4) with initial data (1.5.2) if and only if $G_\epsilon(r, v(r); r_0, v_0) \equiv 0$ and $v(r_0) = v_0$. We differentiate G_ϵ with respect to v and r and obtain

$$\partial_v G_\epsilon = \frac{v - k^2/v}{1 - \epsilon^2 v^2}, \quad \partial_r G_\epsilon = -\frac{1}{\epsilon^2 r} \left(\frac{1 - \kappa}{\kappa} + \frac{M}{r - 2M} \right) < 0.$$

Observe that $\partial_v G_\epsilon = 0$ if and only if $v = k$. Moreover, G_ϵ is decreasing with respect to v when $v < k$ and increasing when $v > k$. The derivative of a steady state solution is given by

$$\frac{dv}{dr} = \frac{v}{r(r - 2M)} \frac{\frac{1-\kappa}{\kappa}(r - 2M) - M}{\frac{\epsilon^2 v^2}{1 - \epsilon^2 v^2} - \frac{1-\kappa}{2\kappa}} \quad (1.5.9)$$

and changes sign once, at $r = \frac{2-\kappa}{1-\kappa} M \in (2M, +\infty)$.

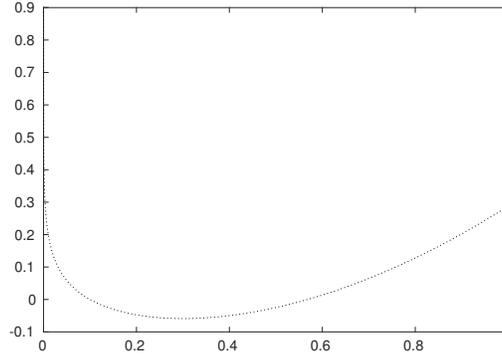


Figure 1.5.1: Plot of the function $v \mapsto G_\epsilon(v) = G_\epsilon(r, v; r_0, v_0)$ with $\epsilon = 0.01$.

Since $G_\epsilon(r_0, v_0; r_0, v_0) = 0$ and $\partial_v G_\epsilon(r_0, v_0; r_0, v_0) \neq 0$ provided $v_0 \neq k$, we can apply the implicit function theorem to a non-sonic velocity v_0 .

Lemma 1.5.2 (The family of locally-defined steady states). *Given any radius $r_0 > 2M$ and any initial data $\rho_0 > 0$ and $v_0 \geq 0$ satisfying the non-sonic condition $v_0 \neq k$, the initial value problem defined in (1.5.2) and (1.5.4) admits a solution $\rho = \rho(r)$ and $v = v(r)$ denoted by*

$$\rho = \rho(r; r_0, \rho_0, v_0), \quad v = v(r; r_0, \rho_0, v_0),$$

and defined in some neighborhood \mathcal{U}_0^ϵ of $r_0 \in \mathbb{R}$ (at least).

Global existence theory

We now analyze the possible extension of the (smooth) solutions above to their maximum domain of existence. Since $\partial_v G_\epsilon(r_0, v_0; r_0, v_0) = 0$ if and only if $v = k$, a solution can always be continued, unless the velocity component v reaches the sonic speed.

Definition 1.5.3. *A radius $r = r_* > 2M$ is called a **sonic point** for the problem (1.5.2) and (1.5.4) if it is a root of the following algebraic equation:*

$$\ln \left(\frac{1 - \epsilon^2 v_0^2}{1 - \epsilon^2 k^2} \right) + \frac{1 - \kappa}{\kappa} \ln \left(\frac{v_0}{k} \right) = \frac{1 - \kappa}{\kappa} \ln \left(\frac{r^2}{r_0^2} + \ln \frac{r(r_0 - 2M)}{r_0(r - 2M)} \right). \quad (1.5.10)$$

From (1.5.9), it follows that the derivative $\frac{dv}{dr}$ of a steady state solution blows-up when one approaches a sonic value. In the following, it will be useful to observe that

$$\epsilon^2 k^2 = \frac{1 - \kappa}{1 + \kappa}, \quad 1 + 3\epsilon^2 k^2 = \frac{2(2 - \kappa)}{1 + \kappa}, \quad \frac{1 + 3\epsilon^2 k^2}{2\epsilon^2 k^2} = \frac{2 - \kappa}{1 - \kappa}. \quad (1.5.11)$$

In order to simplify the notation, we introduce the following function P_ϵ of the radius r_0 and velocity v_0 :

$$P_\epsilon(r_0, v_0) := \ln \left(\frac{(2 - \kappa)^2 M^2 k}{(1 - \kappa)^2 r_0^2 v_0} \right) + \frac{\kappa}{1 - \kappa} \ln \left(\frac{2(2 - \kappa)}{1 + \kappa} \frac{(r_0 - 2M)}{r_0(1 - \epsilon^2 v_0^2)} \right). \quad (1.5.12)$$

The importance of the sign of $P_\epsilon(r_0, v_0)$ is identified in the following lemma.

Lemma 1.5.4 (Existence/non-existence of sonic points). *Consider a solution $v = v(r)$ associated with a positive and non-sonic velocity $v_0 > 0$ with $v_0 \neq k$:*

1. *If $P_\epsilon(r_0, v_0) > 0$, there exists no sonic point.*
2. *If $P_\epsilon(r_0, v_0) \leq 0$, there exist two sonic points $\underline{r}_* \leq \bar{r}_*$. Moreover, one has:*
 - *If $r_0 \geq \frac{2-\kappa}{1-\kappa}M$, the roots satisfy $2M < \underline{r}_* \leq \bar{r}_* \leq r_0$.*
 - *If $r_0 < \frac{2-\kappa}{1-\kappa}M$, the roots satisfy $2M < r_0 < \underline{r}_* \leq \bar{r}_*$.*

Proof. Introduce the following function of the velocity variable $v_0 > 0$:

$$L_\epsilon(v_0) := \frac{\kappa}{1 - \kappa} \ln \left(\frac{1 - \epsilon^2 v_0^2}{1 - \epsilon^2 k^2} \right) + \ln \frac{v_0}{k},$$

which satisfies $L'_\epsilon(v_0) = \frac{1}{v_0} \left(1 - \frac{v_0^2}{k^2} \frac{1 - \epsilon^2 k^2}{1 - \epsilon^2 v_0^2} \right)$. Thus, $L'_\epsilon(v_0) = 0$ if and only if $v_0 = k$. Hence, L_ϵ achieves its maximum at k , that is, $L_\epsilon(v_0) \leq L_\epsilon(k) = 0$. Therefore for all non-sonic v_0 , we have $-\infty < L_\epsilon(v_0) < 0$.

Now, consider the following function of the spatial variable

$$R_\epsilon(r) := \ln \frac{r^2}{r_0^2} + \frac{\kappa}{1 - \kappa} \left(\ln \frac{r}{r - 2M} - \ln \frac{r_0}{r_0 - 2M} \right),$$

which satisfies $R'_\epsilon(r) = \frac{2}{r(r-2M)} \left((r - 2M) - \frac{\kappa}{1-\kappa} M \right)$. Therefore, the function R_ϵ reaches its minimum at $r_{min} := \frac{2-\kappa}{1-\kappa}M$ and

$$R_\epsilon(r_{min}) = \ln \left(\frac{(2 - \kappa) M^2}{(1 - \kappa) r_0^2} \right) + \frac{\kappa}{1 - \kappa} \ln \left(\frac{2 - \kappa}{\kappa} \left(1 - \frac{2M}{r_0} \right) \right).$$

Observe also that the minimum value $R_\epsilon(r_{min})$ reaches its maximum value 0 when $r_0 = r_{min}$. Therefore, if and only if $R_\epsilon(r_{min}) - L_\epsilon(v_0) > 0$, no sonic point can be found; otherwise, we have two sonic points. The positions of r_0 and r_{min} determine the location of the sonic points $\underline{r}_* \leq \bar{r}_*$. Furthermore, since $R_\epsilon(2M) = +\infty$, we have the lower bound $2M < \underline{r}_*$. \square

We need now to distinguish between several cases and the following notation will be useful:

$$\begin{aligned} \widetilde{\mathfrak{A}} : P_\epsilon(r_0, v_0) &> 0, & \widetilde{\mathfrak{B}} : P_\epsilon(r_0, v_0) &\leq 0, \\ \widetilde{\mathfrak{i}} : v_0 &< k, & \widetilde{\mathfrak{z}} : v_0 &> k, \\ \widetilde{\mathfrak{i}} : r_0 &\geq \frac{2-\kappa}{1-\kappa}M, & \widetilde{\mathfrak{ii}} : r_0 &< \frac{2-\kappa}{1-\kappa}M. \end{aligned} \tag{1.5.13}$$

We are now ready to continue the local solutions in Lemma 1.5.2 beyond the neighborhood \mathcal{U}_0^ϵ . There are two main regimes, which we now discuss.

Lemma 1.5.5 (Extension of steady state solutions without sonic point). *Given a radius $r_0 > 2M$, a density $\rho_0 > 0$, and a non-sonic velocity $0 \leq v_0 < 1/\epsilon$ (satisfying $v_0 \neq k$), the local solution $\rho = \rho(r; r_0, \rho_0, v_0)$ and $v = v(r; r_0, \rho_0, v_0)$ given in Lemma 1.5.2 satisfies the following properties:*

1. **Case $\widetilde{\mathfrak{A}}\mathfrak{1}$.** *The solution can be extended to $(2M, +\infty)$ satisfying $v < k$ with*

$$\lim_{r \rightarrow 2M} v(r; r_0, \rho_0, v_0) = \lim_{r \rightarrow +\infty} v(r; r_0, \rho_0, v_0) = 0.$$

The solution satisfies the monotonicity that v is increasing with respect to r on the interval $(2M, \frac{2-\kappa}{1-\kappa}M)$ while it is decreasing on $(\frac{2-\kappa}{1-\kappa}M, +\infty)$.

2. **Case $\widetilde{\mathfrak{A}}\mathfrak{2}$.** *The solution can be extended to $(2M, +\infty)$ satisfying $v > k$ with*

$$\lim_{r \rightarrow 2M} v(r; r_0, \rho_0, v_0) = \lim_{r \rightarrow +\infty} v(r; r_0, \rho_0, v_0) = \frac{1}{\epsilon}.$$

The following monotonicity property holds: v is decreasing with respect to r on the interval $(2M, \frac{2-\kappa}{1-\kappa}M)$ while it is increasing on $(\frac{2-\kappa}{1-\kappa}M, +\infty)$.

Lemma 1.5.6 (Extension of steady state solutions with sonic points). *Given a radius $r_0 > 2M$, a density $\rho_0 > 0$, and a non-sonic velocity $0 \leq v_0 < 1/\epsilon$ (satisfying $v_0 \neq k$), the local solution $\rho = \rho(r; r_0, \rho_0, v_0)$ and $v = v(r; r_0, \rho_0, v_0)$ given in Lemma 1.5.2 satisfies the following properties, in which $\bar{r}_* \leq \underline{r}_*$ denotes the sonic points given by Lemma 1.5.4:*

1. **Case $\widetilde{\mathfrak{B}}\mathfrak{1i}$.** *The solution $v = v(r)$ can be extended to $(\bar{r}_*, +\infty)$ and satisfies $v \leq k$, with*

$$\lim_{r \rightarrow +\infty} v(r; r_0, \rho_0, v_0) = 0, \quad \lim_{r \rightarrow \bar{r}_*} v(r; r_0, \rho_0, v_0) = k.$$

Moreover, v is decreasing with respect to r on $(\bar{r}_, +\infty)$.*

2. **Case $\widetilde{\mathfrak{B}}\mathfrak{2i}$.** *The solution $v = v(r)$ can be extended to $(\bar{r}_*, +\infty)$ and satisfies*

$v \geq k$, with

$$\lim_{r \rightarrow +\infty} v(r; r_0, \rho_0, v_0) = \frac{1}{\epsilon}, \quad \lim_{r \rightarrow \bar{r}_*} v(r; r_0, \rho_0, v_0) = k.$$

Moreover, v is increasing with respect to r on $(\bar{r}_*, +\infty)$.

3. **Case $\widetilde{\mathfrak{B}1ii}$.** The solution $v = v(r)$ can be extended to $(0, \underline{r}_*)$ and satisfies $v \leq k$, with

$$\lim_{r \rightarrow 2M} v(r; r_0, \rho_0, v_0) = 0, \quad \lim_{r \rightarrow \underline{r}_*} v(r; r_0, \rho_0, v_0) = k.$$

Moreover, v is increasing with respect to r on $(0, \underline{r}_*)$.

4. **Case $\widetilde{\mathfrak{B}2ii}$.** The solution $v = v(r)$ can be extended to $(0, \underline{r}_*)$ and satisfies $v \geq k$, with

$$\lim_{r \rightarrow 2M} v(r; r_0, \rho_0, v_0) = \frac{1}{\epsilon}, \quad \lim_{r \rightarrow \underline{r}_*} v(r; r_0, \rho_0, v_0) = k.$$

Moreover, v is decreasing with respect to r on $(0, \underline{r}_*)$.

The proof of Lemmas 1.5.5 and 1.5.6 follows the same lines as the ones of Lemmas 1.4.5 and 1.4.6, respectively. Note that since G_ϵ has its minimum at $v = k$, we have $G_\epsilon(r, v; r_0, v_0) > G_\epsilon(r, k; r_0, v_0) > 0$ for all $r \in (\underline{r}_*, \bar{r}_*)$, and we see that no solution can be defined on the interval $(\underline{r}_*, \bar{r}_*)$ limited by the two roots.

Main conclusion for this section

We can now summarize the properties of steady state solutions. We refer to Figures 1.5.2 to 1.5.4 for an illustration for several values of the physical parameters ϵ, k, m .

Theorem 1.5.7 (Steady flows on a Schwarzschild background). Given some values of the light speed $\epsilon > 0$, sound speed $k \in (0, 1/\epsilon)$, and black hole mass $M > 0$, consider the Euler model $\mathcal{M}(\epsilon, k, m = M/\epsilon^2)$ in (1.3.3) describing fluid flows on a Schwarzschild background. Then, for any given any radius $r_0 > 2M$, density $\rho_0 > 0$, and velocity $v_0 \geq 0$ with $v_0 \neq k$, there exists a unique steady state solution denoted by

$$\rho = \rho(r; r_0, \rho_0, v_0), \quad v = v(r; r_0, \rho_0, v_0),$$

satisfying the steady state equations (1.5.4) together with the initial condition $\rho(r_0) = \rho_0$ and $v(r_0) = v_0$. Moreover, the velocity component satisfies $\text{sgn}(v(r) - k) = \text{sgn}(v_0 - k)$ for all relevant values r , and in order to specify the range of the independent variable r where this solution is defined, one distinguishes between two alternatives:

1. **Regime without sonic point.** If $P_\epsilon(r_0, v_0) > 0$ (this function being introduced in (1.5.12)), the solution is defined on the whole interval $(2M, +\infty)$.

2. Regime with sonic points. If $P_\epsilon(r_0, v_0) \leq 0$, the solution is defined on the interval $\Pi \subsetneq (2M, +\infty)$ defined by

$$\Pi := \begin{cases} (0, \underline{r}_*), & r_0 < \frac{2-\kappa}{1-\kappa}M, \\ (\bar{r}_*, +\infty), & r_0 \geq \frac{2-\kappa}{1-\kappa}M. \end{cases} \quad (1.5.14)$$

Moreover, the velocity $v(r)$ tends to the sonic velocity k when r approaches the sonic radius (\underline{r}_* or \bar{r}_* , introduced in Lemma 1.5.4).

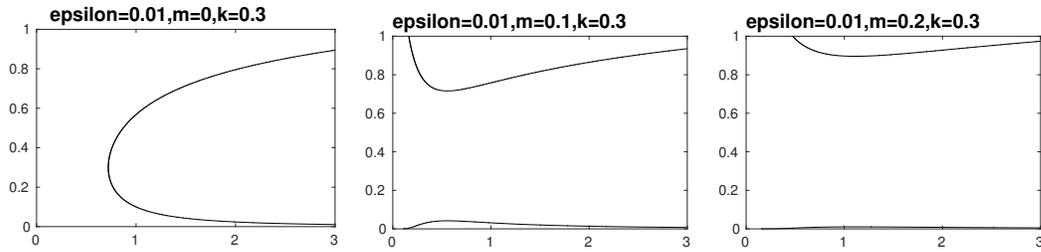


Figure 1.5.2: Solution $v = v(r)$ for $\epsilon = 0.01$, $k = 0.3$ and several values $m = M/\epsilon^2$.

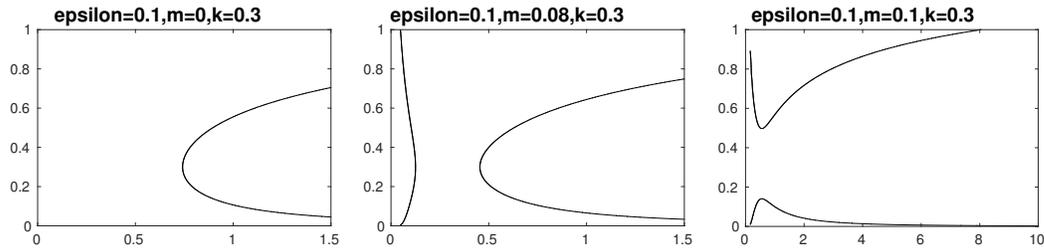


Figure 1.5.3: Solution $v = v(r)$ for $\epsilon = 0.1$, $k = 0.3$ and several values $m = M/\epsilon^2$.

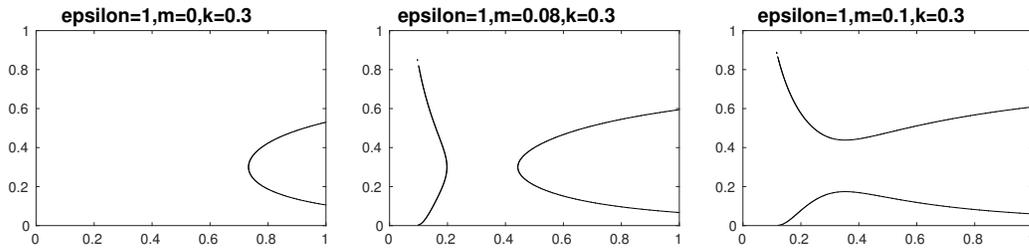


Figure 1.5.4: Solution $v = v(r)$ for $\epsilon = 1$, $k = 0.3$ and several values $m = M/\epsilon^2$.

1.6 The Riemann problem for the Euler equations

Preliminaries

In this section, we consider the solution of the Riemann problem for our general Euler model in a Schwarzschild background (1.3.3), which has the form of a nonlinear hyperbolic system of balance laws:

$$\partial_t U + \partial_r F(U, r) = S(U, r), \quad (1.6.1)$$

where the “conservative variables” and “flux variables” are

$$U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} r^2 \frac{1+\epsilon^4 k^2 v^2}{1-\epsilon^2 v^2} \rho \\ r(r-2M) \frac{1+\epsilon^2 k^2}{1-\epsilon^2 v^2} \rho v \end{pmatrix}, \quad (1.6.2)$$

$$F(U, r) = \begin{pmatrix} F_1(U, r) \\ F_2(U, r) \end{pmatrix} = \begin{pmatrix} r(r-2M) \frac{1+\epsilon^2 k^2}{1-\epsilon^2 v^2} \rho v \\ (r-2M)^2 \frac{v^2+k^2}{1-\epsilon^2 v^2} \rho \end{pmatrix}, \quad (1.6.3)$$

respectively, while the “source term” reads

$$S(U, r) = \begin{pmatrix} S_1(U, r) \\ S_2(U, r) \end{pmatrix} = \begin{pmatrix} 0 \\ 3M \left(1 - \frac{2M}{r}\right) \frac{v^2+k^2}{1-\epsilon^2 v^2} \rho - M \frac{r-2M}{\epsilon^2 r} \frac{1+\epsilon^4 k^2 v^2}{1-\epsilon^2 v^2} \rho + 2 \frac{(r-2M)^2}{r} k^2 \rho \end{pmatrix}. \quad (1.6.4)$$

By definition, the **Riemann problem** for (1.6.1) is the initial value problem associated with an initial data U_0 consisting of a left-hand constant state $U_L = (\rho_L, v_L)$ and a right-hand constant state $U_R = (\rho_R, v_R)$, separated by a jump discontinuity at some point $r = r_0$ (with $r_0 > 2M$). In other words, we set

$$U_0(r) = \begin{cases} U_L, & r < r_0, \\ U_R & r > r_0. \end{cases} \quad (1.6.5)$$

In Proposition 1.2.3, we have seen that both eigenvalues of (1.6.1) are genuinely nonlinear, when the sound speed k is a constant satisfying $0 < k < 1/\epsilon$, which we now assume throughout. We are going to solve the Riemann problem first for the **homogeneous system**

$$\partial_t U + \partial_r F(U, r_0) = 0 \quad (1.6.6)$$

for a given $r_0 > 2M$. in the class of self-similar functions (depending only on the variable $y := \frac{r-r_0}{t}$) consisting of constant states, separated by either shock waves or rarefaction waves. Furthermore, it is convenient to introduce the **fluid constant** χ and the **scaled velocity** defined by

$$\chi := \frac{2\epsilon k}{1 + \epsilon^2 k^2} \in (0, 1), \quad \nu := \frac{1}{2\epsilon} \frac{1 + \epsilon v}{1 - \epsilon v} \in (-\infty, +\infty). \quad (1.6.7)$$

Rarefaction curves

We begin by searching for smooth solutions to the Euler system depending only upon the self-similar variable. The partial differential system (1.6.1) then reduces to an ordinary differential system for functions $\rho = \rho(y)$ and $\nu = \nu(y)$ and, according to the discussion in the proof of Lemma 1.2.2, we know that one of the Riemann invariants w, z must remain constant throughout. Hence, we are led to the notion of **rarefaction curves**: given any state U_L , the 1-rarefaction curve $R_1^\rightarrow(U_L)$ is the curve passing through U_L along which the Riemann invariant w remains constant and, in addition, the first eigenvalue λ is *increasing*. The definition of the (backward) curve $R_2^\leftarrow(U_R)$ for a given right-hand state U_R is similar: the Riemann invariant z remains constant and, in addition, the second eigenvalue μ is *decreasing*. We thus have

$$\begin{aligned} R_1^\rightarrow(U_L) &= \left\{ w(\rho, \nu) = w(\rho_L, \nu_L), \quad z(\rho, \nu) < z(\rho_L, \nu_L) \right\}, \\ R_2^\leftarrow(U_R) &= \left\{ z(\rho, \nu) = z(\rho_R, \nu_R), \quad w(\rho, \nu) > w(\rho_R, \nu_R) \right\}. \end{aligned}$$

By observing that

$$w = \frac{1}{2\epsilon} \ln(2\epsilon\nu\rho^x), \quad z = \frac{1}{2\epsilon} \ln(2\epsilon\nu\rho^{-x}), \quad (1.6.8)$$

the following statement is immediate.

Lemma 1.6.1. *The two rarefaction curves associated with constant states U_L and U_R , respectively, are given by*

$$R_1^\rightarrow(U_L) = \left\{ \frac{\nu}{\nu_L} = \left(\frac{\rho}{\rho_L} \right)^{-x}, \quad \rho > \rho_L \right\}, \quad R_2^\leftarrow(U_R) = \left\{ \frac{\nu}{\nu_R} = \left(\frac{\rho}{\rho_R} \right)^x, \quad \rho < \rho_R \right\}. \quad (1.6.9)$$

Shock curves

We next search for solutions consisting of two constant states separating a single jump discontinuity satisfying the Euler system (1.6.1). Along a shock curve we impose the Rankine-Hugoniot relations (see below) as well as Lax entropy inequalities (see (1.6.14), below), which can be stated as follows: the characteristic speed λ must be *decreasing* when moving away from the left-hand state U_L on the 1-shock curve $S_1^\rightarrow(U_L)$, while μ is *increasing* as one moves away from the right-hand state U_R on the *backward* 2-shock curve $S_2^\leftarrow(U_R)$.

Lemma 1.6.2. *The 1-shock curve and the 2-shock curve issuing from given constant*

states, denoted by $U_L = (\rho_L, v_L)$ and $U_R = (\rho_R, v_R)$, respectively, are given by

$$\begin{aligned} S_1^\rightarrow(U_L) &= \left\{ \sqrt{\frac{\nu}{\nu_L}} - \sqrt{\frac{\nu_L}{\nu}} = -\chi \left(\sqrt{\frac{\rho}{\rho_L}} - \sqrt{\frac{\rho_L}{\rho}} \right), \quad \rho > \rho_L \right\}, \\ S_2^\leftarrow(U_R) &= \left\{ \sqrt{\frac{\nu}{\nu_R}} - \sqrt{\frac{\nu_R}{\nu}} = \chi \left(\sqrt{\frac{\rho}{\rho_R}} - \sqrt{\frac{\rho_R}{\rho}} \right), \quad \rho < \rho_R \right\}. \end{aligned} \quad (1.6.10)$$

The speed $s_1(U_L, U)$ along the 1-shock curve and the speed $s_2(U, U_R)$ along the 2-shock curve are given by

$$\begin{aligned} \epsilon s_1(U_L, U) &= - \left(1 - \frac{2M}{r} \right) \left(\frac{\rho}{\rho - \rho_L} \frac{\epsilon^2 v^2}{1 - \epsilon^2 v^2} + \frac{\epsilon^2 k^2}{1 + \epsilon^2 k^2} \right)^{1/2} \left(\frac{\rho}{\rho - \rho_L} \frac{\epsilon^2 v^2}{1 - \epsilon^2 v^2} + \frac{1}{1 + \epsilon^2 k^2} \right)^{-1/2}, \\ \epsilon s_2(U, U_R) &= \left(1 - \frac{2M}{r} \right) \left(\frac{\rho}{\rho - \rho_R} \frac{\epsilon^2 v^2}{1 - \epsilon^2 v^2} + \frac{\epsilon^2 k^2}{1 + \epsilon^2 k^2} \right)^{1/2} \left(\frac{\rho}{\rho - \rho_R} \frac{\epsilon^2 v^2}{1 - \epsilon^2 v^2} + \frac{1}{1 + \epsilon^2 k^2} \right)^{-1/2}. \end{aligned} \quad (1.6.11)$$

Proof. 1. We use here the notation $U_i = (\rho_i, v_i)$ and $U = (\rho, v)$ for the two states on each side of a jump discontinuity, which must satisfy the **Rankine-Hugoniot relations** associated with (1.6.1). To simplify the calculation, we use the tensor components v^0, v^1 rather than the scalar velocity v . Denoting the shock speed by s , we see that the Rankine-Hugoniot jump conditions read

$$\begin{aligned} s \left[r(r - 2M) \left((1 - 2M/r)^2 \rho v^0 v^0 + \epsilon^4 k^2 \rho v^1 v^1 \right) \right] &= \left[r(r - 2M) \epsilon^2 \left((1 + \epsilon^2 k^2) \rho v^0 v^1 \right) \right], \\ s \left[r(r - 2M) \left((1 + \epsilon^2 k^2) \rho v^0 v^1 \right) \right] &= \left[r(r - 2M) \epsilon^2 \left(k^2 \rho v^0 v^0 + (1 - 2M/r)^{-2} \rho v^1 v^1 \right) \right]. \end{aligned}$$

where, in our notation, the bracket $[\Phi] := \Phi - \Phi_i$ denotes the jump a quantity Φ . Eliminating s , we find

$$\left[(k^2 \epsilon^2 + 1) \rho v^0 v^1 \right]^2 = \left[(k^2 \epsilon^4 \rho v^1 v^1 + (1 - 2M/r)^2 \rho v^0 v^0) \right] \left[k^2 \rho v^0 v^0 + \frac{\rho v^1 v^1}{(1 - 2M/r)^2} \right].$$

On the other hand, a straightforward calculation gives

$$\begin{aligned} 0 &= k^2 \rho^2 \left((1 - 2M/r) v^0 v^0 - (1 - 2M/r)^{-1} \epsilon^2 v^1 v^1 \right)^2 \\ &\quad + k^2 \rho_i^2 \left((1 - 2M/r) v^0 v^0 - (1 - 2M/r)^{-1} \epsilon^2 v^1 v^1 \right)^2 \\ &\quad - 2k^2 \rho \rho_i \left((1 - 2M/r) v^0 v^0 - (1 - 2M/r)^{-1} \epsilon^2 v^1 v^1 \right)^2 + 2(k^2 \epsilon^2 + 1)^2 \rho \rho_i v_i^0 v_i^0 v_i^1 v_i^1 \\ &\quad - k^4 \epsilon^4 \rho \rho_i v_i^0 v_i^0 v_i^1 v_i^1 - 2\epsilon^2 k^2 \rho \rho_i v_i^1 v_i^1 v_i^1 v_i^1 - \rho \rho_i v_i^0 v_i^0 v_i^1 v_i^1 \\ &\quad - k^4 \epsilon^4 \rho \rho_i v_i^0 v_i^0 v_i^1 v_i^1 - 2\epsilon^2 k^2 \rho \rho_i v_i^0 v_i^0 v_i^0 v_i^0 - \rho \rho_i v_i^0 v_i^0 v_i^1 v_i^1. \end{aligned}$$

Using the fact that the velocity vector is unit, that is, $(1 - 2M/r) v^0 v^0 - (1 -$

$2M/r)^{-1}\epsilon^2v^1v^1 = 1$, we find

$$0 = k^2(\rho^2 + \rho_i^2) - 2k^2\rho\rho_i - (1 + \epsilon^2k^2)^2\rho\rho_i(v_i^0v^1 - v^0v_i^1)^2.$$

Thus, we have arrived at an equation for the density ratio $\frac{\rho}{\rho_i}$:

$$0 = \left(\frac{\rho}{\rho_i}\right)^2 - \left(2 + \frac{(1 + \epsilon^2k^2)^2(v_i^0v^1 - v^0v_i^1)^2}{k^2}\right)\frac{\rho}{\rho_i} + 1. \quad (1.6.12)$$

Furthermore, we obtain $\frac{1-2\epsilon\nu}{1+2\epsilon\nu} = -\epsilon(1 - 2M/r)^{-1}\frac{v^1}{v^0}$ in view of the definition of the velocity variable ν in (1.6.7) and, therefore,

$$\begin{aligned} (v_i^0v^1 - v^0v_i^1)^2 &= \frac{(1 - 2M/r)^{-2}\left(\frac{v^1}{v^0} - \frac{v_i^1}{v_i^0}\right)^2}{\left(1 - (1 - 2M/r)^{-2}\left(\frac{\epsilon v^1}{v^0}\right)^2\right)\left(1 - (1 - 2M/r)^{-2}\left(\frac{\epsilon v_i^1}{v_i^0}\right)^2\right)} \\ &= \frac{\left(\frac{1-2\epsilon\nu}{1+2\epsilon\nu} - \frac{1-2\epsilon\nu_i}{1+2\epsilon\nu_i}\right)^2}{\epsilon^2\left(1 - \left(\frac{1-2\epsilon\nu}{1+2\epsilon\nu}\right)^2\right)\left(1 - \left(\frac{1-2\epsilon\nu_i}{1+2\epsilon\nu_i}\right)^2\right)} = \frac{1}{4\epsilon^2}\left(\sqrt{\frac{\nu}{\nu_i}} - \sqrt{\frac{\nu_i}{\nu}}\right)^2. \end{aligned} \quad (1.6.13)$$

Now, plugging (1.6.13) into (1.6.12), we find

$$\frac{\rho}{\rho_i} = 1 + \frac{1}{2\chi^2}\left(\sqrt{\frac{\nu}{\nu_i}} - \sqrt{\frac{\nu_i}{\nu}}\right)^2 \pm \sqrt{\frac{1}{4(\chi)^4}\left(\sqrt{\frac{\nu}{\nu_i}} - \sqrt{\frac{\nu_i}{\nu}}\right)^4 + \frac{1}{\chi^2}\left(\sqrt{\frac{\nu}{\nu_i}} - \sqrt{\frac{\nu_i}{\nu}}\right)^2},$$

or

$$\left(\sqrt{\frac{\rho}{\rho_i}} - \sqrt{\frac{\rho_i}{\rho}}\right)^2 = \frac{1}{\chi^2}\left(\sqrt{\frac{\nu}{\nu_i}} - \sqrt{\frac{\nu_i}{\nu}}\right)^2.$$

We have $v < v_L$ for 1-shock, so we take the minus sign to guarantee that $\nu < \nu_L$. The analysis of the 2-shock curve is similar.

2. With the Rankine-Hugoniot relations, we obtain

$$s^2 = \left(1 - \frac{2M}{r}\right)^2 \left(\frac{\rho}{\rho - \rho_i} \frac{v^2}{1 - \epsilon^2v^2} + \frac{k^2}{1 + \epsilon^2k^2}\right) / \left(\frac{\rho}{\rho - \rho_i} \frac{\epsilon^2v^2}{1 - \epsilon^2v^2} + \frac{\epsilon^2k^2}{1 + \epsilon^2k^2}\right).$$

Lax's shock inequalities require that

$$\lambda(U_L) > s_1 > \lambda(U), \quad \mu(U) > s_2 > \mu(U_R), \quad (1.6.14)$$

which provide us with the relevant signs for both characteristic families. \square

Wave curves and wave interaction estimates

Combining shock waves and rarefaction waves together, we are able to construct the solution to the Riemann problem, as follows. Given a left-hand state U_L and a right-hand state U_R , by concatenating the two types of curves above, we define the **1-wave curve** and the **2-wave curve**, respectively, by

$$W_1^{\rightarrow}(U_L) := S_1^{\rightarrow}(U_L) \cup R_1^{\rightarrow}(U_L), \quad W_2^{\leftarrow}(U_R) := S_2^{\leftarrow}(U_R) \cup R_2^{\leftarrow}(U_R). \quad (1.6.15)$$

The following observation are in order:

- Observe that in the special case that the initial states satisfy $U_R \in R_1^{\rightarrow}(U_L)$ or $U_L \in R_2^{\leftarrow}(U_R)$, then the Riemann problem can be solved by a single rarefaction wave. In this case, each state U in the solution lie between U_L and U_R along the corresponding rarefaction curve and the associated propagation speed is $\lambda(U)$ and $\mu(U)$, respectively.
- Similarly, in the special case that $U_R \in S_1^{\rightarrow}(U_L)$ or $U_L \in S_2^{\leftarrow}(U_R)$, the Riemann solution consists of a single shock propagating at the speed given by (1.6.11).
- Moreover, it can be checked that the curves S_1 and S_2 are tangent up to second-order derivatives with the corresponding integral curves. Consequently, the wave curves $W_1^{\rightarrow}(U_L)$ and $W_2^{\leftarrow}(U_R)$ are of class C^2 .

Furthermore, according to (1.6.9) and (1.6.10), the density component ρ is increasing (from $-\infty$ to $+\infty$) along the wave curve $W_1^{\rightarrow}(U_L)$, while it is decreasing (from $+\infty$ to $-\infty$) along the wave curve $W_2^{\leftarrow}(U_R)$. This implies that the velocity component ν is increasing along $W_1^{\rightarrow}(U_L)$, and is decreasing along $W_2^{\leftarrow}(U_R)$.

To proceed, we need the following technical lemma.

Lemma 1.6.3. *The 1-shock curve $S_1^{\rightarrow}(U_L)$ satisfies*

$$0 \leq \frac{dz}{dw} = -\frac{\sqrt{2\beta + \chi^2\beta^2} - \sqrt{2\beta + \beta^2}}{-\sqrt{2\beta + \chi^2\beta^2} - \sqrt{2\beta + \beta^2}} < 1, \quad (1.6.16)$$

while the 2-shock curve $S_2^{\leftarrow}(U_R)$ satisfies

$$\frac{dz}{dw} = -\frac{-\sqrt{2\beta + \chi^2\beta^2} - \sqrt{2\beta + \beta^2}}{\sqrt{2\beta + \chi^2\beta^2} - \sqrt{2\beta + \beta^2}} > 1, \quad (1.6.17)$$

with the notation

$$\beta = \beta(U, U_i) := \frac{1}{2\chi^2} \left(\sqrt{\frac{\nu}{\nu_i}} - \sqrt{\frac{\nu_i}{\nu}} \right)^2.$$

Proof. The Riemann invariants read

$$\begin{aligned} w &= \frac{1}{2\epsilon} \ln \left(\frac{1 + \epsilon v}{1 - \epsilon v} \right) - \frac{k}{1 + \epsilon^2 k^2} \ln \rho, \\ z &= \frac{1}{2\epsilon} \ln \left(\frac{1 + \epsilon v}{1 - \epsilon v} \right) + \frac{k}{1 + \epsilon^2 k^2} \ln \rho. \end{aligned} \quad (1.6.18)$$

By introducing the functions $g_{\pm}(\beta) = 1 + \beta \left(1 \pm \sqrt{1 + \frac{2}{\beta}} \right)$, it is straightforward to see that

$$\frac{\rho}{\rho_i} = g_{\pm}(\beta). \quad (1.6.19)$$

Moreover, we can check that $g_+(\beta)g_-(\beta) = 1$. By Lemma 1.6.2, we have $2\beta\chi^2 = \left(\sqrt{\frac{\nu}{\nu_i}} - \sqrt{\frac{\nu_i}{\nu}} \right)^2$ and, therefore,

$$\frac{\nu}{\nu_i} = 1 + \frac{\chi^2 \beta}{2} \left(1 \pm \sqrt{1 + \frac{4}{\chi^2 \beta}} \right) = g_{\pm}(\chi^2 \beta / 2). \quad (1.6.20)$$

For definiteness, we consider 1-shocks. The tangent to the shock curve $S_1^{\rightarrow}(U_L)$ in the (w, z) -plane satisfies

$$\frac{dz}{dw} = \frac{d(z - z_L)}{d(w - w_L)} = \frac{d(z - z_L)}{d\beta} \frac{d\beta}{d(w - w_L)}.$$

Plugging (1.6.19) and (1.6.20) into the expression of the Riemann invariants (1.6.18), we obtain (for 1-shocks)

$$\begin{aligned} w - w_L &= \frac{1}{2\epsilon} \left(\ln g_+(\beta) + \chi \ln g_+(\chi^2 \beta / 2) \right), \\ z - z_L &= -\frac{1}{2\epsilon} \left(\ln g_+(\beta) - \chi \ln g_+(\chi^2 \beta / 2) \right), \end{aligned} \quad (1.6.21)$$

and, thus,

$$\frac{dz}{dw} = \frac{d(z - z_L)}{d(w - w_L)} = -\frac{\sqrt{2\beta + \chi^2 \beta^2} - \sqrt{2\beta + \beta^2}}{-\sqrt{2\beta + \chi^2 \beta^2} - \sqrt{2\beta + \beta^2}}.$$

Since $\chi = \frac{2\epsilon k}{1 + \epsilon^2 k^2} < 1$, we have $0 \leq \frac{dz}{dw} < 1$. \square

We have arrived at the main result of the present section.

Proposition 1.6.4 (The Riemann problem for fluid flows). *The homogeneous Euler system (1.6.6) supplemented with Riemann initial data (1.6.5) admits an entropy weak solution for arbitrary initial data $r_0 > 2M$, $U_L = (\rho_L, v_L)$, and $U_R = (\rho_R, v_R)$. This solution depends on the self-similarity variable r/t , only, and is piecewise smooth: it consists of two (shock or rarefaction) waves separated by constant states.*

Proof. Consider the intersection of the two curves state $W_1^{\rightarrow}(U_L) \cap W_2^{\leftarrow}(U_R)$. Thanks to Lemma 1.6.3 and our analysis above, the family of 1-curves and 2-curves covers the whole region in such a way that, for any given data U_L, U_R , the curves $W_1^{\rightarrow}(U_L)$ and $W_2^{\leftarrow}(U_R)$ admit precisely one intersection point U_M . The Riemann solution is then solved by a 1-wave connecting from U_L to U_M , followed by a 2-wave connecting from U_M to U_R . \square

Next, we define the **total wave strength** of the Riemann solution connecting three states U_L, U_M, U_R by

$$\mathcal{E}(U_L, U_R) := |\ln \rho_L - \ln \rho_M| + |\ln \rho_R - \ln \rho_M|, \quad (1.6.22)$$

where, by definition, $U_M = (\rho_M, v_M)$ is the intermediate state

$$\{U_M\} = W_1^{\rightarrow}(U_L) \cap W_2^{\leftarrow}(U_R).$$

The following observation is the key in order to establish the global existence theory for the Cauchy problem.

Proposition 1.6.5 (Diminishing total variation property). *Given three constant states U_L, U_* , and U_R , consider the associated Riemann problems (U_L, U_*) , (U_*, U_R) , and (U_L, U_R) . Then, the total wave strengths satisfy the inequality*

$$\mathcal{E}(U_L, U_R) \leq \mathcal{E}(U_L, U_*) + \mathcal{E}(U_*, U_R). \quad (1.6.23)$$

Proof. We consider the wave curves in the (w, z) -plane of the Riemann invariants. Recall that, in this plane, rarefaction curves are straightlines, while shock curves are described by explicit formulas. Importantly, the shock curves have the same geometric shape independently of the base point U_L or U_R and are described by the functions $g_{\pm}(\beta)$. Moreover, by observing the remarkable algebraic property $g_+(\beta)g_-(\beta) = 1$, we see that the 2-shock curve is the symmetric of the 1-shock curve with respect to the straightline $z = w$ (in the (w, z) -plane). Note that the strength \mathcal{E} does not change at interactions involving two rarefaction waves of the same family, only. Since the wave strengths, by definition, are measured in $w - z \sim \ln \rho$, these symmetry properties imply that the wave strengths are non-decreasing at interactions. \square

1.7 The generalized Riemann problem

Discontinuous steady states

Our strategy is now to solve the Riemann problem for the full Euler model, by replacing the two initial constant states by two equilibrium solutions. We refer to

this problem as the *generalized Riemann problem*. In order to proceed, we need first to revisit our analysis in (cf. Section 1.5 and to introduce first global equilibrium solutions, defined for all radius $r \in (2M, +\infty)$ and possibly containing a jump discontinuity. This is necessary since some (smooth) steady state solutions are defined on a sub-interval of $r > 2M$, only; this happens when the velocity component may reach the sonic value $\pm k$.

Recall that, according to Theorem 1.5.7, two possible behavior may arise, which are determined by the sign of the function $P_\epsilon(r_0, v_0)$ defined in (1.5.12), that is,

$$\begin{aligned} P_\epsilon(r_0, v_0) &= \ln \left(\frac{(2 - \kappa)^2 M^2 k}{(1 - \kappa)^2 r_0^2 v_0} \right) + \frac{\kappa}{1 - \kappa} \ln \left(\frac{2(2 - \kappa) (r_0 - 2M)}{1 + \kappa r_0 (1 - \epsilon^2 v_0^2)} \right) \\ &= \ln \left(\frac{(1 + 3\epsilon^2 k^2)^2 M^2}{4\epsilon^4 k^3 r_0^2 v_0} \right) + \frac{\kappa}{1 - \kappa} \ln \left(\frac{(1 + 3\epsilon^2 k^2)(r_0 - 2M)}{r_0 (1 - \epsilon^2 v_0^2)} \right). \end{aligned}$$

The function P_ϵ beign regular, the existence of a sonic point depends also continuously upon the data r_0, v_0 . Recall also that in the special case that the data satisfy $P_\epsilon(r_0, v_0) = 0$, then the associated two sonic points \bar{r}_* and \underline{r}_* are both equal to (cf. our notation (1.5.5) and (1.5.11))

$$r_* := \frac{2 - \kappa}{1 - \kappa} M, \quad (1.7.1)$$

which we refer to as the **critical sonic point**. We now consider this limiting case, which was excluded in our earlier analysis.

Proposition 1.7.1 (The global construction for sonic initial data). *When the initial data $r_0 > 2M$, $\rho_0 > 0$, and $0 \leq v_0 < 1/\epsilon$ satisfy the **sonic condition***

$$P_\epsilon(r_0, v_0) = 0, \quad (1.7.2)$$

*then the steady Euler system (1.5.4) admits a **global steady state solution** $\rho = \rho(r)$ and $v = v(r)$ satisfying the initial conditions $\rho(r_0) = \rho_0$ and $v(r_0) = v_0$ and such that $v(r) - k$ changes sign precisely once.*

Proof. Without loss of generality, we assume that $r_0 \geq r_*$. According to Theorem 1.5.7, there exists a smooth steady state solution defined on the interval $(r_*, +\infty)$. At any radius $r \in (2M, r_*)$, we have

$$G(r, v; r_0, v_0) < G(r, k; r_0, v_0) < G(r_*, k; r_0, v_0) = 0.$$

Therefore, for a given $r \in (2M, r_*)$, the equation $G(r, v; r_0, v_0) = 0$ admits two roots: $v^b(r) < k$ and $v^\sharp(r) > k$. Moreover, $v^b(r_*-) = v^\sharp(r_*-) = k$, and these solutions are continuous at the sonic point $r = r_*$. We could in principle define two continuous steady state solutions, but we want to make a unique selection. At the sonic point, the

derivative of the solution blows-up to infinity, and it is natural to keep the sign of the derivative. Hence, for the initial data under consideration satisfying $P_\epsilon(r_0, v_0) = 0$, we define a continuous global steady state solution by setting

$$\widehat{v}(r; r_0, \rho_0, v_0) = \begin{cases} v(r; r_0, \rho_0, v_0), & r \in \Pi, \\ v^\sharp(r; r_0, \rho_0, v_0), & r \notin \Pi, \end{cases} \quad (1.7.3)$$

in which we have selected $v^\sharp = v^b$ if $v_0 > k$ while $v^\sharp = v^\#$ if $v_0 < k$. Hence, the function $v(r) - k$ changes sign when we reach the sonic point. \square

We now turn our attention to general data, when two sonic points $r_* < \bar{r}_*$ are available. We can no longer “cross” the sonic velocity value, while by remaining within the class of continuous solutions. Instead, we must consider solutions with one shock, as we now explain it.

Lemma 1.7.2 (Jump conditions for steady state solutions). *A steady state discontinuity associated with left/right-hand limits (ρ, v) and (ρ_i, v_i) must satisfy*

$$\frac{\rho}{\rho_i} = \frac{1 - \epsilon^2 k^4 / v_i^2}{1 - \epsilon^2 v_i^2} \frac{v_i^2}{k^2}, \quad v v_i = k^2. \quad (1.7.4)$$

Proof. From the Rankine-Hugoniot relations $\left[\frac{1 + \epsilon^2 k^2}{1 - \epsilon^2 v^2} \rho v \right] = 0$ and $\left[\frac{v^2 + k^2}{1 - \epsilon^2 v^2} \rho \right] = 0$, we deduce that

$$\frac{1}{1 - \epsilon^2 v^2} \rho v = \frac{1}{1 - \epsilon^2 v_i^2} \rho_i v_i, \quad \frac{v^2 + k^2}{1 - \epsilon^2 v^2} \rho = \frac{v_i^2 + k^2}{1 - \epsilon^2 v_i^2} \rho_i,$$

which we solve for ρ and v . \square

In view of Lemma 4.3.4, there exist infinitely many discontinuous steady state solutions containing a shock discontinuity at some radius $r_1 \in \Pi$. At such a point, we have

$$\rho(r_1 \pm) := \frac{1 - \epsilon^2 k^4 / v(r_1 \pm)^2}{1 - \epsilon^2 v(r_1 \pm)^2} \frac{v(r_1 \pm)^2}{k^2} \rho(r_1 \pm), \quad v(r_1 \pm) := \frac{k^2}{v(r_1 \pm)}. \quad (1.7.5)$$

Of course, by introducing a shock within a steady state solution, we must guarantee that the new branch of solution allows us to make a global continuation in the sense that we are not limited again by a sonic point. In fact, in order to have also a unique construction, we propose to select the jump point so that the new branch of solution has the “sonic property” discussed above. The following lemma provides us with the key observation.

Lemma 1.7.3 (Existence and uniqueness of the critical jump radius). *Consider a smooth steady state solution $\rho = \rho(r; r_0, \rho_0, v_0)$ and $v = v(r; r_0, \rho_0, v_0)$, which is defined on the interval Π and satisfying the initial condition $\rho(r_0) = \rho_0$ and $v(r_0) = v_0$. The, if this solution admits a sonic point (which is denoted by \underline{r}_* or \bar{r}_*), then there exists a unique radius, referred to as the **critical jump radius** and denoted by $r_1^* \in \Pi$, such that*

$$P_\epsilon\left(r_1^*, \frac{k^2}{v_1^*}\right) = 0 \quad \text{with } v_1^* = v(r_1^*; r_0, \rho_0, v_0). \quad (1.7.6)$$

Moreover, r_1^* lies in the interval limited by r_0 and the sonic point.

Proof. First of all, it is straightforward to check that $P_\epsilon(r_1, v_1)$ is increasing in r_1 on $\left(2M, \frac{(1+3\epsilon^2 k^2)M}{2\epsilon^2 k^2}\right)$ and is decreasing on $\left(\frac{(1+3\epsilon^2 k^2)M}{2\epsilon^2 k^2}, +\infty\right)$. It is also decreasing in v_1 on $(0, k)$ and decreasing on $(k, 1/\epsilon)$.

Let us first establish the existence of a radius satisfying the condition (1.7.6). In the regime under consideration, we have two sonic points and $P_\epsilon(\bar{r}_*, k) < 0$ and $P_\epsilon(\underline{r}_*, k) < 0$. For definiteness in the discussion, we treat the following case

$$\frac{2-\kappa}{1-\kappa}M < \bar{r}_* < r_0, \quad k < v_0.$$

Thanks to the above monotonicity property of P_ϵ with respect to v , we can find a neighborhood of \bar{r}_* , denoted by U_* , such that the inequality $P_\epsilon\left(r_1, \frac{k^2}{v_1}\right) < 0$ holds for all $r_1 \in U_*$. For every (r_1, v_1) along the steady solution curve starting from (r_0, v_0) , the condition $P_\epsilon(r_1, v_1) < 0$ holds. By introducing $\widetilde{M}(r) = \frac{(1+3\epsilon^2 k^2)M}{2\epsilon^2 k^2 r} = \frac{2-\kappa}{1-\kappa} \frac{M}{r}$, we can rewrite the condition $P_\epsilon(r_1, v_1) < 0$ as

$$2 \ln \widetilde{M}(r_1) + \ln \left(\frac{k}{v_1} \right) - \frac{\kappa}{1-\kappa} \ln (1 - \epsilon^2 v_1^2) + \frac{\kappa}{1-\kappa} \ln \left(\frac{2(2-\kappa)}{1+\kappa} - 4\epsilon^2 k^2 \widetilde{M}(r_1) \right) \leq 0.$$

We need to show that there exists some point r_1 such that $P_\epsilon(r_1, \frac{k^2}{v_1}) > 0$. Indeed, by

setting $\overline{M}_* = \frac{2-\kappa}{1-\kappa} \frac{M}{\overline{r}_*}$ and $\underline{M}_* = \frac{2-\kappa}{1-\kappa} \frac{M}{\underline{r}_*}$, we have

$$\begin{aligned}
 P_\epsilon\left(r_1, \frac{k^2}{v_1}\right) &\geq P_\epsilon\left(r_1, \frac{k^2}{v_1}\right) + P_\epsilon(r_1, v_1) \\
 &\geq 4 \ln \widetilde{M}(r_1) - \frac{\kappa}{1-\kappa} \ln\left(\left(1 - \epsilon^2 k^4/v_1^2\right)\left(1 - \epsilon^2 v_1^2\right)\right) + \frac{2\kappa}{1-\kappa} \ln\left(\frac{2(2-\kappa)}{1+\kappa} - \frac{2(1-\kappa)}{1+\kappa} \widetilde{M}(r_1)\right) \\
 &\geq -\frac{\kappa}{1-\kappa} \ln\left(\left(1 - \epsilon^2 k^4/v_1^2\right)\left(1 - \epsilon^2 v_1^2\right)\right) \\
 &+ \max\left(4 \ln \underline{M}_*, \frac{2\kappa}{1-\kappa} \ln\left(\frac{2(2-\kappa)}{1+\kappa} - \frac{2(1-\kappa)}{1+\kappa} \overline{M}_*\right)\right).
 \end{aligned}$$

Since $-\frac{\kappa}{1-\kappa} \ln\left(\left(1 - \epsilon^2 k^4/v_1^2\right)\left(1 - \epsilon^2 v_1^2\right)\right) \in \left(-\frac{\kappa}{1-\kappa} \ln(1 - \epsilon^4 k^4), +\infty\right)$, we can find an interval of v_1 where $P_\epsilon\left(r_1, \frac{k^2}{v_1}\right) > 0$. By continuity, we conclude that there exists a radius r_1^* such that $P_\epsilon\left(r_1^*, \frac{k^2}{v_1^*}\right) = 0$.

Now, we turn to the uniqueness of r_1^* . We want to show that $P_\epsilon\left(r_1, \frac{k^2}{v_1}\right)$ changes its sign only once along the steady state curve. Recall that we assume (for definiteness) that $r_0 > \frac{2-\kappa}{1-\kappa} M$, so that the smooth solution is defined on $(\overline{r}_*, +\infty)$. Let r_1 be a point such that $P_\epsilon\left(r_1, \frac{k^2}{r_1}\right) > 0$. For $r > r_1$, according to the monotonicity properties of steady state solutions, we have $|k - v(r)| > |k - v_1|$, therefore, $P_\epsilon\left(r, \frac{k^2}{v(r)}\right) > 0$ always holds. Then let r_2 be a point such that $P_\epsilon\left(r_2, \frac{k^2}{v_2}\right) < 0$ holds. For $r \in (\overline{r}_*, r_2)$, according to the monotonicity properties of P_ϵ , we have

$$\begin{aligned}
 P_\epsilon\left(r, \frac{k^2}{v}\right) &= 2 \ln \widetilde{M}(r) + \ln\left(\frac{v(r)}{k}\right) - \frac{\kappa}{1-\kappa} \ln\left(1 - \epsilon^2 k^2/v(r)^2\right) \\
 &+ \frac{\kappa}{1-\kappa} \ln\left(1 + 3\epsilon^2 k^2 - 4\epsilon^2 k^2 \widetilde{M}(r)\right) < P_\epsilon\left(r_2, \frac{k^2}{v_2}\right) < 0.
 \end{aligned}$$

We have thus established that P_ϵ changes sign only once.

Moreover, let us emphasize that r_1^* lies in the interval limited by r_0 and the sonic point. Again, we treat the case $r_0 > \frac{2-\kappa}{1-\kappa} M$. If the desired property would not hold, then we would have $P_\epsilon(r_0, v_0) < 0$ and $P_\epsilon\left(r_0, \frac{k^2}{v_0}\right) < 0$ simultaneously, but this would

contradict

$$P_\epsilon(r_0, v_0) + P_\epsilon(r_0, \frac{k^2}{v_0}) > -\frac{\kappa}{1-\kappa} \ln \left((1 - \epsilon^2 k^4 / v_1^2)(1 - \epsilon^2 v_1^2) \right) \\ + \frac{\kappa}{1-\kappa} \ln \left(\frac{4(2-\kappa)^2}{(1+\kappa)^2} \right) > 0.$$

Therefore, we have $\bar{r}_* < r_1^* < r_0$ in the case under consideration. \square

From the family of smooth steady states $\rho = \rho(r; r_0, \rho_0, v_0)$ and $v = v(r; r_0, \rho_0, v_0)$ in the regime where they admit a sonic point, we are now in a position to define solutions on the whole interval $r \in (2M, +\infty)$. We introduce the domains

$$\Lambda_s := \begin{cases} [r_1^*, +\infty), & r_1^* \geq \frac{2-\kappa}{1-\kappa}M, \\ (2M, r_1^*), & r_1^* < \frac{2-\kappa}{1-\kappa}M, \end{cases} \quad \Lambda_d := \begin{cases} (2M, r_1^*], & r_1^* \geq \frac{2-\kappa}{1-\kappa}M, \\ (r_1^*, +\infty), & r_1^* < \frac{2-\kappa}{1-\kappa}M. \end{cases}$$

We arrive at our main conclusion in this section.

Theorem 1.7.4 (Globally-defined steady state solutions). Consider the family of smooth steady state solutions to the Euler system posed on a Schwarzschild background with black hole mass M . Given any radius $r_0 > 2M$, initial density $\rho_0 > 0$, and initial velocity $|v_0| < 1/\epsilon$, the initial value problem for the steady Euler system (1.5.4) with initial condition $\rho(r_0) = \rho_0$ and $v(r_0) = v_0$ admits a unique weak solution which is globally defined for all $r \in (2M, +\infty)$ and contains at most one shock (satisfying the Rankine-Hugoniot relations and Lax's shock inequalities), and such that the velocity component $|v| - k$ changes sign at most once. Furthermore, the family of steady state solutions with possibly one shock depends Lipschitz continuously upon its arguments r_0, ρ_0, v_0 when they vary within the whole range of solutions, encompassing smooth solutions with no sonic point, continuous solutions with exactly one sonic point, and discontinuous solutions containing exactly one continuous sonic point and one shock crossing a sonic point.

A precise statement of the continuity property above is as follows: in the case of a solution containing a shock, it is meant that the location of the shock and its left- and right-hand limit vary continuously; moreover, in the transition from a solution of one of three types to a solution of another type, the values taken by the solution vary continuously.

We have derived all the ingredients in order to establish the theorem above. First of all, for the case without sonic point, smooth solutions defined for all $r \in (2M, +\infty)$ were already constructed in Section 1.5, so that to shock is required when a branch of solution never reaches a sonic point.

Now consider the case with sonic points. The critical case where the two sonic points coincide is already dealt with in Proposition 1.7.1. So, it remains to discuss

the case $r_* < \bar{r}_*$. Let $\rho_1^* = \rho(r_1^*; r_0, \rho_0, v_0)$, $v_1^* = v(r_1^*; r_0, \rho_0, v_0)$ be values achieved by the smooth solution at the critical jump point r_1^* provided by Lemma 1.7.3. In view of Lemmas 4.3.4 and 1.7.3, we can now introduce the (discontinuous) **global steady state solution** as

$$v(r; r_0, \rho_0, v_0) := \begin{cases} v(r; r_0, \rho_0, v_0), & r \in \Lambda_s, \\ \widehat{v}\left(r; r_1^*, \frac{1-\epsilon^2 k^4/v_1^{*2}}{1-\epsilon^2 v_1^{*2}} \frac{v_1^{*2}}{k^2} \rho_1^*, \frac{k^2}{v_1^*}\right), & r \in \Lambda_d, \end{cases} \quad (1.7.7)$$

where \widehat{v} is the corresponding steady state solution containing a (unique) sonic point (cf. (1.7.3)). According to Lemma 4.3.4, the Rankine-Hugoniot relations hold along the discontinuity so that, for any smooth function with compact support $\phi = \phi(r)$,

$$\begin{aligned} & \int_{2M}^{+\infty} (F(r, U(r)) \frac{d}{dr} \phi(r) + S(r, U(r)) \phi(r)) dr \\ &= \left(\int_{2M}^{r_1^*} + \int_{r_1^*}^{+\infty} \right) \left(-\frac{d}{dr} F(r, U) + S(r, U) \right) \phi(r) dr \\ & - \left(F(r_1^*, U(r_1^*+)) - F(r_1^*, U(r_1^*-)) \right) \phi(r_1^*) = 0. \end{aligned}$$

Therefore, (1.7.7) defines a weak solution to the Euler equations in the distributional sense. Moreover, Lax's shock inequalities are satisfied by construction. Indeed, without loss of generality, suppose that $r_0 > \frac{2-\kappa}{1-\kappa} M$ and let us use the notation U_L, U_R , where U_R is the smooth steady flow. We have either a 1-shock wave if $v_R > k$ or a 2-shock wave if $v_R < k$. We treat, for instance, the case $v_R < k$. The two eigenvalues read (after using the jump relations (1.7.4))

$$\mu(U_R) = \left(1 - \frac{2M}{r}\right) \frac{v_R + k}{1 + \epsilon^2 k v_R}, \quad \mu(U_L) = \left(1 - \frac{2M}{r}\right) \frac{k^2/v_R + k}{1 + \epsilon^2 k^3/v_R},$$

while the shock speed is

$$\begin{aligned} s(U_L, U_R) &= \left(1 - \frac{2M}{r}\right) \left(\frac{k^2}{v_R^2/k^2 - \epsilon^2 k^2 - 1 + \epsilon^2 v_R^2} + \frac{k^2}{1 + \epsilon^2 k^2} \right)^{1/2} \\ & \cdot \left(\frac{\epsilon^2 k^2}{v_R^2/k^2 - \epsilon^2 k^2 - 1 + \epsilon^2 v_R^2} + \frac{1}{1 + \epsilon^2 k^2} \right)^{-1/2}. \end{aligned}$$

In fact, there is no new calculation to do here since, by construction, we have chosen $v_L > v_R$ for a 2 shock and, consequently as observed in our study of general 2-shock curves, Lax's shock inequalities $\mu(U_L) > s(U_L, U_R) > \mu(U_R)$ hold. For 1-shock waves, a similar argument gives $\lambda(U_L) > s(U_L, U_R) > \lambda(U_R)$.

For the continuity property, we observe that the regularity is obvious for the

family of smooth solutions and we only need to consider the continuous solutions and the discontinuous solutions, as well as the transitions from one case to another. Let us consider first continuous solutions that, by construction, cross the sonic value k at the critical radius. We claim that such solutions $r \mapsto v(r)$ are *Lipschitz continuous* everywhere (except at $r = 2M$ where they may blow-up, but later on we will first exclude a neighborhood of the horizon). Namely, we only need to check this property at the critical sonic point: from (1.5.9), we can compute the derivative at the point $r_* = \frac{2-\kappa}{1-\kappa}M$ and obtain

$$\begin{aligned} \frac{dv}{dr}(r_*) &\simeq \frac{k}{r_*(r_* - 2M)} \lim_{r \rightarrow r_*} \frac{\frac{1-\kappa}{\kappa}(r - 2M) - M}{\frac{\epsilon^2 v^2}{1-\epsilon^2 v^2} - \frac{1-\kappa}{2\kappa}} \\ &\simeq \frac{k}{r_*(r_* - 2M)} \frac{1 - \epsilon^2 k^2}{2\epsilon^2 k} \frac{1 - \kappa}{\kappa} / \frac{dv}{dr}(r_*) \simeq \frac{k}{r_*(r_* - 2M)} k / \frac{dv}{dr}(r_*) \end{aligned}$$

and, consequently, the derivative is finite and is given by

$$\frac{dv}{dr}(r_*) = \pm \frac{k}{(r_*(r_* - 2M))^{1/2}}, \quad (1.7.8)$$

the sign depending upon the choice of the branch. This shows that the continuous branch is Lipschitz continuous. The same calculation is valid to deal with discontinuous solutions and shows that, way from the jump discontinuity, the solution depends Lipschitz continuously. In the transition from discontinuous to continuous solutions, the strength of the jump discontinuity shrinks to zero, while the base point r_0 approaches the critical point r_* . All derivatives remain finite in this limit. Finally the transition from a continuous to a smooth solution is regular away from the sonic point (located at r_*), while at the critical point r_* we have a jump of the derivative which is a non-vanishing constant (for continuous solutions) and which vanishes for smooth solutions. Yet, the derivative remains bounded, and we still have the Lipschitz continuity property. This completes the proof of Theorem 1.7.4.

A generalized Riemann solver

The Riemann solver defined in Section 1.6 for the homogeneous system is now extended to the full Euler model (1.6.1) with source term $S(U, r)$. The Riemann solution no longer depends solely on $\frac{r-r_0}{t-t_0}$ and is no longer given by a closed formula. In particular, wave trajectories are no longer straightlines. We are going to construct an *approximate solver*, which will have sufficient accuracy in order to establish our existence theory. Precisely, we consider the **generalized Riemann problem** which, by definition, is based on two steady state solutions separated by a jump discontinuity, that is,

$$\partial_t U + \partial_r F(U, r) = S(U, r), \quad t > t_0, \quad (1.7.9)$$

$$U(t_0, r) = U_0(r) := \begin{cases} U_L(r), & r < r_0, \\ U_R(r), & r > r_0, \end{cases} \quad (1.7.10)$$

posed at the point $t_0 \geq 0$ and $r_0 > 2M$, in which the functions $U_L = U_L(r)$ and $U_R = U_R(r)$ are two (global) steady state solutions, that is, weak solutions to the ordinary differential system

$$\frac{d}{dr}F(U, r) = S(U, r), \quad (1.7.11)$$

constructed in Section 6. The exact solution to the generalized Riemann problem, denoted here by $U = U(t, r)$, cannot be determined explicitly, and we thus seek for an *approximate solution*, which we will denote by $\tilde{U} = \tilde{U}(t, r)$.

First of all, we can follow the discussion in Section 1.6 and we solve the (classical) Riemann problem posed at the point (t_0, r_0) for the homogeneous Euler system, that is, by denoting this solution by $U^c(t, r; t_0, r_0)$, we have

$$\partial_t U^c + \partial_r F(r_0, U^c) = 0, \quad t \geq t_0, \quad (1.7.12)$$

$$U_0^c(r) = \begin{cases} U_L^0 := U_L(r_0), & r < r_0, \\ U_R^0 := U_R(r_0), & r > r_0. \end{cases} \quad (1.7.13)$$

We know that the solution U^c depends upon $\xi := \frac{r-r_0}{t-t_0}$, only, and consists of three constant states U_L^0, U_M^0, U_R^0 , separated by shock waves or rarefaction waves. For all sufficiently small times $t > t_0$, the solution to the generalized Riemann problem is expected to remain sufficiently close to the solution of the classical Riemann problem.

Next, let us introduce the (possibly discontinuous) steady state solution $U_M = U_M(r)$ determined in Theorem 1.7.4 from the initial condition $U_M(r_0) =: U_M^0$ at r_0 . For the following discussion, it is convenient to set

$$U_0 := U_L^0, \quad U_1 := U_M^0, \quad U_2 := U_R^0. \quad (1.7.14)$$

We also set $s_j^- = \lambda(U_{j-1})$ and $\mu(U_{j-1})$, and $s_j^+ = \lambda(U_j)$ or $\mu(U_j)$ (for $j = 1, 2$) be the lower and upper bounds of the speeds in the j -rarefactions. If the j -wave is a shock, then $s_j^- = s_j^+ = s_j$ denotes the j -shock speed (given by (1.6.11)).

We are now ready to define the **approximate generalized Riemann solver** by

setting

$$\tilde{U}(t, r) := \begin{cases} U_L(r), & r - r_0 < s_1^-(t - t_0), \\ V_1(t, \eta_1(t, r)), & s_1^-(t - t_0) < r - r_0 < s_1^+(t - t_0), \\ U_M(r), & s_1^+(t - t_0) < r - r_0 < s_2^-(t - t_0), \\ V_2(t, \eta_2(t, r)), & s_2^-(t - t_0) < r - r_0 < s_2^+(t - t_0), \\ U_R(r), & r - r_0 > s_2^+(t - t_0), \end{cases} \quad (1.7.15)$$

in which we have also introduced (in the case that the classical Riemann problem admits rarefactions) the functions $V_j = V_j(t, \eta_j)$ and the change of variable $(t, r) \mapsto (t, \eta_j)$ given by the following integro-differential problem. Following Liu [40], we take into account the time-evolution of the generalized Riemann solution within a rarefaction fan and define “approximate rarefaction fans”, as follows. We first seek for $V_j = V_j(t, \eta_j)$ and $r^\sharp = r^\sharp(t, \eta_j)$ as functions of the time variable t together with a new variable denoted by η_j , satisfying

$$\begin{aligned} \partial_{\eta_j} r^\sharp \partial_t V_j + \left(\partial_U F(V_j) - \lambda_j(V_j) \right) \partial_{\eta_j} V_j &= S(V_j) \partial_{\eta_j} r^\sharp, \\ \partial_t r^\sharp &= \lambda_j(V_j), \end{aligned} \quad (1.7.16)$$

with the following boundary and initial conditions (with $\eta_j^0 = \lambda_j(U_{j-1}(r_0))$)

$$\begin{aligned} V_j(t, \eta_j^0) &= U_{j-1}(r^\sharp(t, \eta_j^0)), & V_j(t_0, \eta_j) &= h_j(\eta_j), \\ \partial_t r^\sharp(t, \eta_j^0) &= \lambda_j(U_{j-1}(r^\sharp)), & r^\sharp(t_0, \eta_j) &= r_0, \end{aligned} \quad (1.7.17)$$

where the function h_j is defined by inverting the eigenvalue functions along the rarefaction curves, i.e.

$$\lambda_j(h_j(\xi)) = \xi = \frac{r - r_0}{t - t_0}. \quad (1.7.18)$$

(As usual, $\lambda_1 = \lambda$ and $\lambda_2 = \mu$). Next, we recover the “standard” radial variable r by setting

$$r = r^\sharp(t, \eta_j),$$

and, therefore, expressing η_j as a function of (t, r) . We now check that the conditions above define a unique function.

Lemma 1.7.5. *For sufficiently small times Δt , there exists a unique smooth solution of the problem (1.7.16) defined within the time interval $t_0 < t < t_0 + \Delta t$, such that*

$$\partial_t V_j = O(1)G, \quad \partial_{\eta_j} V_j = h'_j(\eta_j) + O(1)G\Delta t,$$

where G is a constant independent of t and r .

Proof. Let us, for instance, treat the rarefaction waves of the first family $j = 1$ and

derive first an integral formulation of the problem. Denoting by l_1, l_2 two independent left-eigenvectors of the Jacobian matrix of the Euler system, we have

$$\begin{aligned}\mathcal{D}\tilde{V}_2 &= \frac{\partial_{\eta_2} r^\#}{\mu - \lambda} l_2 \cdot S + \mathcal{D}l_2 \cdot V_1, \\ \partial_t \tilde{V}_1 &= l_2 \cdot S + \partial_t l_2 \cdot V_1,\end{aligned}\tag{1.7.19}$$

where we write $\tilde{V}_1 = l_1 \cdot V_1$ and $\tilde{V}_2 = l_2 \cdot V_1$, and we have also introduced the differential operator $\mathcal{D} := \frac{\partial_{\eta_2} r^\#}{\mu - \lambda} \partial_t + \partial_{\eta_2}$, whose integral curves starting from $(s, \lambda(U_0))$ are denoted by \mathcal{L} . By integrating (1.7.19), we thus obtain

$$\begin{aligned}\tilde{V}_2(t, \eta_1) &= \tilde{V}_2(s, \lambda(U_0)) + \int_{\mathcal{L}} \left(\frac{\partial_{\eta_2} r^\#}{\mu - \lambda} l_2 \cdot S + \mathcal{D}l_2 \cdot V_1 \right) d\eta_1, \\ \tilde{V}_1(t, \eta_1) &= \tilde{V}_1(t_0, \xi) + \int_{t_0}^t \left(l_2 \cdot S + \partial_t l_2 \cdot V_1 \right) d\eta_1.\end{aligned}\tag{1.7.20}$$

Now we define an operator T to provide the right-hand side of (4.4.10) and we take an arbitrary function V_1^0 such that $V_1^0(t, \eta_1^0) = V_1(t, \eta_1^0)$ and $V_1^0(t_0, \eta_1) = V_1(t_0, \eta_1)$. We then study the iteration scheme $V_1^{(l)} := T^{(l)} V_1^0$. For all sufficiently small Δt , the operator T is contractive in the sup-norm of V_1^0 and their first-order derivatives, by a standard fixed point argument we deduce that there exists a unique solution V_1 to (4.4.10). Moreover, by integration, we can estimate the first-order derivatives of V_1 , as stated in the lemma. \square

We define the **wave trajectories** as

$$r_j^\pm(t) := s_j^\pm(t - t_0) + r_0\tag{1.7.21}$$

and, in particular, if the j -wave is a shock, we have $r_j(t) := r_j^\pm(t) = s_j(t - t_0) + r_0$.

Lemma 1.7.6 (Control of the error associated with the generalized Riemann solver). *Let \tilde{U} be the approximate generalized Riemann solver defined by (1.7.15). Then, for all $t_0 \leq t < t_0 + \Delta t$, one has:*

1. When (U_{j-1}, U_j) is a j -shock wave, one has

$$\begin{aligned}& s_j \left(\tilde{U}(t, r_j(t)+) - \tilde{U}(t, r_j(t)-) \right) \\ &= F(r_j(t), \tilde{U}(t, r_j(t)+)) - F(r_j(t), \tilde{U}(t, r_j(t)-)) \\ &+ O(1) |U_j - U_{j-1}| \Delta t.\end{aligned}\tag{1.7.22}$$

2. When (U_{j-1}, U_j) is a j -rarefaction wave, one has

$$\tilde{U}(t, r_j^+(t)) - \tilde{U}(t, r_j^-(t)) = O(1)|U_j - U_{j-1}|\Delta t. \quad (1.7.23)$$

Consequently, when there is no jump at r_0 , that is, $U_L(r) = U_R(r)$, then the term $|U_j - U_{j-1}|$ vanishes, and the approximate solution is, in fact, exact.

Proof. By our construction, if (U_{j-1}, U_j) is a shock wave, then we simply connect $U_{j-1}(r)$, and $U_j(r)$ by a jump discontinuity. A Taylor's expansion yields us

$$\tilde{U}(t, r_j^+(t)) - \tilde{U}(t, r_j^-(t)) = U_j - U_{j-1} + O(1)|U_j - U_{j-1}|\Delta t,$$

and, thanks to the Rankine-Hugoniot relations $s_j(U_j - U_{j-1}) = F(r, U_j) - F(r, U_{j-1})$, we arrive at

$$\begin{aligned} & s_j \left(\tilde{U}(t, r_j(t+)) - \tilde{U}(t, r_j(t-)) \right) \\ & F(r_j(t), \tilde{U}(t, r_j(t+))) - F(r_j(t), \tilde{U}(t, r_j(t-))) + O(1)|U_j - U_{j-1}|\Delta t. \end{aligned}$$

If, now, (U_{j-1}, U_j) is a rarefaction wave, it follows from our construction that

$$\begin{aligned} U_j(r_j^+(t)) - U_{j-1}(r_j^+(t)) &= U_j - U_{j-1} + O(1)|U_j - U_{j-1}|\Delta t. \\ U_j(r_j^-(t)) - U_{j-1}(r_j^-(t)) &= U_j - U_{j-1} + O(1)|U_j - U_{j-1}|\Delta t. \end{aligned}$$

Moreover, we have $r_j^+(t) - r_{j-1}^-(t) = O(1)|U_j - U_{j-1}|\Delta t$ and a Taylor's expansion yields us

$$U_{j-1}(r_j^+(t)) - U_{j-1}(r_j^-(t)) = O(1)|U_j - U_{j-1}|\Delta t.$$

Hence, we can compute

$$\begin{aligned} \tilde{U}(t, r_j^+(t)) - \tilde{U}(t, r_j^-(t)) &= U_j(r_j^+(t)) - U_{j-1}(r_j^-(t)) \\ &= U_j(r_j^+(t)) - U_{j-1}(r_j^+(t)) - (U_j(r_j^-(t)) - U_{j-1}(r_j^-(t))) + U_{j-1}(r_j^+(t)) - U_{j-1}(r_j^-(t)) \\ &= O(1)|U_j - U_{j-1}|\Delta t. \quad \square \end{aligned}$$

In order to estimate whether the function \tilde{U} is an “accurate” approximate solution, we consider any smooth function $\phi = \phi(t, r)$ and study the integral expression

$$\Theta(\Delta t, \Delta r; \phi) := \int_{t_0}^{t_0 + \Delta t} \int_{r_0 - \Delta r}^{r_0 + \Delta r} \left(\tilde{U} \partial_t \phi + F(r, \tilde{U}) \partial_r \phi + S(r, \tilde{U}) \phi \right) dr dt \quad (1.7.24)$$

for any $\Delta t, \Delta r > 0$ with $r_0 - \Delta r > 2M$. Observe that Θ would vanish if we would take the exact Riemann solution $U = U(t, r)$ in (1.7.24) and we would assume that the function is compactly supported in the slab under consideration. The expression (1.7.24) provides a measure of the discrepancy between the exact and the approximate

solutions, and can be expressed from integrals on the boundary of the slab, modulo an error term, as we now show it.

Proposition 1.7.7. *For given $\Delta t, \Delta r > 0$ with $r_0 - \Delta r > 2M$ satisfying the stability condition*

$$\frac{\Delta r}{\Delta t} > \max(-\lambda, \mu), \quad (1.7.25)$$

and for every smooth function ϕ defined on $[t_0, +\infty) \times [r_0 - \Delta r, r_0 + \Delta r]$, the integral expression in (1.7.24) satisfies

$$\begin{aligned} \Theta(\Delta t, \Delta r; \phi) &= \int_{r_0 - \Delta r}^{r_0 + \Delta r} \tilde{U}(t_0 + \Delta t, \cdot) \phi(t_0 + \Delta t, \cdot) dr - \int_{r_0 - \Delta r}^{r_0 + \Delta r} \tilde{U}(t_0, \cdot) \phi(t_0, \cdot) dr \\ &\quad + \int_{t_0}^{t_0 + \Delta t} F(r_0 + \Delta r, \tilde{U}(\cdot, r_0 + \Delta r)) \phi(\cdot, r_0 + \Delta r) dt \\ &\quad - \int_{t_0}^{t_0 + \Delta t} F(r_0 - \Delta r, \tilde{U}(\cdot, r_0 - \Delta r)) \phi(\cdot, r_0 - \Delta r) dt \\ &\quad + O(1) |U_R(r_0) - U_L(r_0)| \Delta t^2 \|\phi\|_{C^1}. \end{aligned} \quad (1.7.26)$$

Proof. We decompose the sum under consideration as

$$\Theta(\Delta t, \Delta r; \phi) = \sum_j \int_{t_0}^{t_0 + \Delta t} \int_{D_j^1} \theta(t, r) dr dt + \sum_j \int_{t_0}^{t_0 + \Delta t} \int_{D_j^2} \theta(t, r) dr dt,$$

where

$$D_0^1 := (r_0 - \Delta r, r_1^-(t)), \quad D_1^1 = (r_1^+(t), r_2^-(t)), \quad D_0^2 = (r_2^+(t), r_0 + \Delta r), \quad D_j^2 = (r_j^-(t), r_j^+(t)),$$

for $j = 1, 2$ which is used to denote the rarefaction regions. We first consider the interval D_j^1 where the approximate solution \tilde{U} is a steady state solution. Therefore, we have $\partial_t \tilde{U} + \partial_r \tilde{F}(r, \tilde{U}) - S(r, \tilde{U}) = 0$ in D_j^1 . Multiplying the equation by the test-function ϕ and integrating by parts, we obtain

$$\begin{aligned} \int_{t_0}^{t_0 + \Delta t} \int_{D_0^1} \theta(t, r) dr dt &= \int_{r_0 - \Delta r}^{r_1^-(t)} \tilde{U}(t_0 + \Delta t, r) \phi(t_0 + \Delta t, r) dr - \int_{r_0 - \Delta r}^{r_0} \tilde{U}(t_0, r) \phi(t_0, r) dr \\ &\quad + \int_{t_0}^{t_0 + \Delta t} \left(F(r_1^-(t), \tilde{U}(t, r_1^-(t)-)) - s_1^- \tilde{U}(t, r_1^-(t)-) \right) \phi(t, r_1^-(t)) dt \\ &\quad - \int_{t_0}^{t_0 + \Delta t} F(r_0 - \Delta r, \tilde{U}(t, r_0 - \Delta r)) \phi(t, r_0 - \Delta r) dt \end{aligned}$$

and

$$\begin{aligned} \int_{t_0}^{t_0+\Delta t} \int_{D_2^1} \theta(t, r) dr dt &= \int_{r_2^+(t)}^{r_0+\Delta r} \tilde{U}(t_0 + \Delta t, r) \phi(t_0 + \Delta t, r) dr - \int_{r_0}^{r_0+\Delta r} \tilde{U}(t_0, r) \phi(t_0, r) dr \\ &+ \int_{t_0}^{t_0+\Delta t} F(r_0 + \Delta r, \tilde{U}(t, r_0 - \Delta r)) \phi(t, r_0 + \Delta r) dt \\ &- \int_{t_0}^{t_0+\Delta t} \left(F(r_2^+(t), \tilde{U}(t, r_2^+(t)+)) - s_2^+ \tilde{U}(t, r_2^+(t)+) \right) \phi(t, r_2^+(t)) dt. \end{aligned}$$

A similar calculation for the integration in D_1^1 gives us:

$$\begin{aligned} \int_{t_0}^{t_0+\Delta t} \int_{D_1^1} \theta(t, r) dr dt &= \int_{r_1^+(t)}^{r_2^-(t)} \tilde{U}(t_0 + \Delta t, r) \phi(t_0 + \Delta t, r) dr \\ &+ \int_{t_0}^{t_0+\Delta t} \left(F(r_2^-(t), \tilde{U}(t, r_2^-(t)-)) - s_2^- \tilde{U}(t, r_2^-(t)-) \right) \phi(t, r_2^-(t)) dt \\ &- \int_{t_0}^{t_0+\Delta t} \left(F(r_1^+(t), \tilde{U}(t, r_1^+(t)+)) - s_1^+ \tilde{U}(t, r_1^+(t)+) \right) \phi(t, r_1^+(t)) dt. \end{aligned}$$

Next, consider the rarefaction region D_j^2 . According to the construction in (1.7.15) and (1.7.16), we have $U(t, r) = V_j(t, \eta_j)$ in D_j^2 . Performing the change the variable $(t, r) \rightarrow (t, \eta_j)$, we have (with the notation $\lambda_1 = \lambda$ and $\lambda_2 = \mu$)

$$\begin{aligned} \partial_t U + \partial_r F(U, r) - S(U, r) &= \partial_t V_j - \lambda_j(V_j) \partial_{\eta_j} V_j \partial_r \eta_j + \partial_{\eta_j} F \partial_r \eta_j - S(V_j) \\ &= \partial_r \eta_j (\partial_t V_j \partial_{\eta_j} r + (\partial_U F - \lambda_j) \partial_{\eta_j} V_j - S \partial_{\eta_j} r) = 0. \end{aligned}$$

Multiply the equation by the test function ϕ , then for the rarefaction region, we have

$$\begin{aligned} \int_{t_0}^{t_0+\Delta t} \int_{D_j^2} \theta(t, r) dr dt &= \int_{D_j^2} \tilde{U}(t_0 + \Delta t, r) \phi(t_0 + \Delta t, r) dr \\ &+ \int_{t_0}^{t_0+\Delta t} \left(F(r_j^+(t), \tilde{U}(t, r_j^+(t)-)) - s_j^+ \tilde{U}(t, r_j^+(t)-) \right) \phi(t, r_j^+(t)) dt \\ &- \int_{t_0}^{t_0+\Delta t} \left(F(r_j^-(t), \tilde{U}(t, r_j^-(t)+)) - s_j^- \tilde{U}(t, r_j^-(t)+) \right) \phi(t, r_j^-(t)) dt. \end{aligned}$$

According our construction of the generalized Riemann problem, if (U_{j-1}, U_j) is a shock, (1.7.22) gives

$$\begin{aligned} &s_j \left(\tilde{U}(t, r_j(t)+) - \tilde{U}(t, r_j(t)-) \right) \\ &= F(r_j(t), \tilde{U}(t, r_j(t)+)) - F(r_j(t), \tilde{U}(t, r_j(t)-)) + O(1) |U_j - U_{j-1}| \Delta t. \end{aligned}$$

Hence, we have

$$\begin{aligned} & \int_{t_0}^{t_0+\Delta t} \left(F(r_j(t), \tilde{U}(t, r_j(t)+)) - s_j \tilde{U}(t, r_j(t)+) \right) \phi(t, r_j(t)) dt \\ & - \int_{t_0}^{t_0+\Delta t} \left(F(r_j(t), \tilde{U}(t, r_j(t)-)) - s_j \tilde{U}(t, r_j(t)-) \right) \phi(t, r_j(t)) dt = O(1) |U_R - U_L| \Delta t^2 \|\phi\|_{C^0}. \end{aligned}$$

According to (1.7.23), if (U_{j-1}, U_j) is a rarefaction wave, we have

$$\tilde{U}(t, r_j^+(t)+) - \tilde{U}(t, r_j^+(t)-) = O(1) |U_j - U_{j-1}| \Delta t,$$

from which we obtain

$$\begin{aligned} & \int_{t_0}^{t_0+\Delta t} \left(F(r_j(t), \tilde{U}(t, r_j^+(t)+)) - s_j \tilde{U}(t, r_j^+(t)+) \right) \phi(t, r_j(t)) dt \\ & - \int_{t_0}^{t_0+\Delta t} \left(F(r_j(t), \tilde{U}(t, r_j^+(t)-)) - s_j \tilde{U}(t, r_j^+(t)-) \right) \phi(t, r_j(t)) dt \\ & = O(1) (U(t, r_j^+(t)+) - U(t, r_j^+(t)-)) \Delta t \|\phi\|_{C^0} = O(1) |U_R - U_L| \Delta t^2 \|\phi\|_{C^0}. \end{aligned}$$

Adding all the terms together, we thus estimate the discrepancy as

$$\begin{aligned} & \Theta(\Delta t, \Delta r; \phi) \\ & = \int_{r_0-\Delta r}^{r_0+\Delta r} \tilde{U}(t_0 + \Delta t, \cdot) \phi(t_0 + \Delta t, \cdot) dr - \int_{r_0-\Delta r}^{r_0+\Delta r} \tilde{U}(t_0, \cdot) \phi(t_0, \cdot) dr \\ & + \int_{t_0}^{t_0+\Delta t} F(r_0 + \Delta r, \tilde{U}(\cdot, r_0 + \Delta r)) \phi(\cdot, r_0 + \Delta r) dt \\ & - \int_{t_0}^{t_0+\Delta t} F(r_0 - \Delta r, \tilde{U}(\cdot, r_0 - \Delta r)) \phi(\cdot, r_0 - \Delta r) dt + O(1) |U_R(r_0) - U_L(r_0)| \Delta t^2 \|\phi\|_{C^1}. \quad \square \end{aligned}$$

1.8 The initial value problem

The global existence theory

We now consider the **initial value problem** for the Euler system on a Schwarzschild background, that is, (1.6.1)–(1.6.4), with some initial condition at $t_0 \geq 0$

$$(\rho, u)(t_0, \cdot) = (\rho_0, v_0) \tag{1.8.1}$$

for some prescribed data $\rho_0 : (2M, +\infty) \rightarrow (0, +\infty)$ and $v_0 : (2M, +\infty) \rightarrow (-1/\epsilon, +1/\epsilon)$. Before we introduce our method based on steady states, we first observe that the technique already developed by Grubic and LeFloch [16] (in a different geometric setup)

applies, which is based on a piecewise constant approximation and an ODE solver. This method applies to general initial data and solutions.

Theorem 1.8.1 (Global existence theory for fluid flows on a Schwarzschild background). Consider the Euler system describing fluid flows on a Schwarzschild background (1.3.3) posed in $r > 2M$. Given any initial density $\rho_0 = \rho_0(r) > 0$ and velocity $|v_0| = |v_0(r)| < 1/\epsilon$ satisfying, for any $\delta > 0$,

$$TV_{[2M+\delta,+\infty)}(\ln \rho_0) + TV_{[2M+\delta,+\infty)}\left(\ln \frac{1 - \epsilon v_0}{1 + \epsilon v_0}\right) < +\infty,$$

then there exists a weak solution $\rho = \rho(t, r)$ and $v = v(t, r)$ defined on $[t_0, +\infty)$ and satisfying the prescribed initial data at the time t_0 and such that, for all finite time $T \geq t_0$ and $\delta > 0$,

$$\sup_{t \in [t_0, T]} \left(TV_{[2M+\delta,+\infty)}(\ln \rho(t, \cdot)) + TV_{[2M+\delta,+\infty)}\left(\ln \frac{1 - \epsilon v(t, \cdot)}{1 + \epsilon v(t, \cdot)}\right) \right) < +\infty.$$

For the proof, we only need to observe that no boundary condition is required at $r = 2M$, since the wave speeds vanish on the horizon and that we can always “cut” an arbitrarily small neighborhood of the horizon and estimate the total variation outside this neighborhood, as explained in the following subsection. We omit the details.

Behavior near the horizon

In view of Lemma 1.2.2, the eigenvalues

$$\lambda = -\left(1 - \frac{2M}{r}\right) \frac{v - k}{1 - \epsilon^2 k v}, \quad \mu = \left(1 - \frac{2M}{r}\right) \frac{v + k}{1 + \epsilon^2 k v}$$

are distinct for all $r > 2M$ but both of them vanish on the horizon $r = 2M$. This indicates that no boundary condition should be required on the horizon. On the other hand, the Euler system (1.3.3) is not strictly hyperbolic at the horizon $r = 2M$. Yet, for any given $\delta > 0$, the system is strictly hyperbolic in the region $r \geq 2M + \delta$.

Furthermore, recall from Section 5 that steady state solutions may “blow-up” near the horizon, in the sense that the velocity component v may approach $\pm 1/\epsilon$, which does correspond to an algebraic singularity for the Euler system.

It follows that it is natural to study the Cauchy problem, first, away from the horizon within a domain of dependence where the solution is uniquely determined from the prescribed initial data. Observe that, according to Lemma 1.2.2, the eigenvalues

are uniformly bounded:

$$\begin{aligned}
 -\frac{1}{\epsilon} < -\frac{1}{\epsilon} \left(1 - \frac{2M}{r}\right) < \lambda < \mu < \frac{1}{\epsilon} \left(1 - \frac{2M}{r}\right) < \frac{1}{\epsilon} && \text{when } r > 2M, \\
 \lambda = \mu = 0 && \text{when } r = 2M.
 \end{aligned}$$

This provides us with a uniform a priori control on the wave speed, so that the stability condition required in the random choice method is automatically satisfied (without having to derive first a uniform sup-norm estimate).

We thus fix $\delta > 0$ and consider the curve $r = \bar{r}_\delta(t)$ characterized by

$$\frac{d\bar{r}}{dt}(t) = \frac{1}{\epsilon} \left(1 - \frac{2M}{\bar{r}(t)}\right), \quad \bar{r}(0) = 2M + \delta, \tag{1.8.2}$$

which, in the limit of vanishing δ , converges to the line $r = 2M$, in the sense that

$$\lim_{\delta \rightarrow 0} \bar{r}(t) = 2M \quad \text{uniformly for } t \text{ in a compact subset of } (2M, +\infty]. \tag{1.8.3}$$

In the following, we study the initial value problem with data prescribed at some time $t_0 \geq 0$, and we state first our BV estimate within the region $\Omega_\delta(T) = \{t_0 < t < T, r > \bar{r}(t)\}$. In turn, by letting $\delta \rightarrow 0$, we are able to control the total variation in every compact subset in (t, r) .

A random choice method based on equilibria

We are now ready to develop a theory based on steady state solutions as a building blocks, which has the advantage of preserving equilibria and allows to establish the nonlinear stability of equilibria. Our approach is based on the approximate solver of the generalized Riemann problem provided in Section 1.7. Use $\underline{U}(t, r; t_0, r_0, U_L(r), U_R(r))$ to denote the approximate solver of the generalized Riemann problem at (t_0, r_0) with initial steady states $U_L(r)$ and $U_R(r)$ at $t = t_0$ separated at $r = r_0$ provided in Section 1.6. Denote the mesh lengths in r and t by Δr and Δt respectively, and (t_i, r_j) the mesh point of the grid:

$$t_i = t_0 + i\Delta t, \quad r_j = 2M + j\Delta r.$$

Since $-\lambda, \mu < 1/\epsilon$, we assume $\frac{\Delta r}{\Delta t} > \frac{2}{\epsilon}$ to guarantee the stability condition (1.7.7). Interactions can thus be avoided within one step. First of all, we approximate the initial data U_0 by a piecewise steady state profile determined from the initial condition

at $r = r_{j+1}$:

$$\begin{aligned} \frac{d}{dr}F(r, U_\Delta(t_0, r)) &= S(r, U_\Delta(t_0, r)), & j \text{ even, } r_j < r < r_{j+2}, \\ U_\Delta(t_0, r_{j+1}) &= U_0(r_{j+1}). \end{aligned} \quad (1.8.4)$$

We set

$$r_{i,j} = 2M + (w_i + j)\Delta r,$$

where $(w_i)_i$ is a given random sequence in $(-1, 1)$. If the approximate solution U_Δ has been defined for all $t_{i-1} \leq t < t_i$, we define $U_\Delta(t, r)$ for all r and $t_i \leq t < t_{i+1}$, as follows:

1. At the time level $t = t_i$, we define U_Δ to be the piecewise smooth steady solution given by solving

$$\begin{aligned} \frac{d}{dr}F(r, U_\Delta(t_i, r)) &= S(r, U_\Delta(t_i, r)), & i + j \text{ even, } r_j < r < r_{j+2}, \\ U_\Delta(t_i, r_{i,j+1}) &= U_\Delta(t_i^-, r_{i,j+1}). \end{aligned} \quad (1.8.5)$$

2. Now define U_Δ on $t_i < t < t_{i+1}$:

For $j \geq 1$, define the solution on $\{t_i < t < t_{i+1}, r_{j-1} < r < r_{j+1}\}$ (with $i + j$ even) by

$$U_\Delta(t, r) := \tilde{U}\left(t, r; t_i, r_j, \bar{U}_L(r_j), \bar{U}_R(r_j)\right)$$

with $\bar{U}_L(r) = U_\Delta(t_i, r)$, $r \in (r_{j-1}, r_j)$ and $\bar{U}_R(r) = U_\Delta(t_i, r)$, $r \in (r_j, r_{j+1})$ the steady state components of $U_\Delta(t_i, r)$.

This completes the definition of the approximate solution U_Δ on $[t_0, +\infty) \times (2M, +\infty)$.

Wave interactions of the generalized Riemann problem

In Proposition 1.6.5, we studied wave interactions in the context of the classical Riemann problem and established a monotonicity property. For the generalized Riemann problem under consideration now, the initial data is no longer piecewise constant and we need to revisit this issue. Given a pattern consisting of three (possibly discontinuous) steady state solutions $U_L = U_L(r)$, $U_M = U_M(r)$, and $U_R = U_R(r)$, we are interested in the solution to the Euler system (1.7.9) with Cauchy data (with

$r_1 < r_0 < r_2$ given)

$$U_0(r) = \begin{cases} U_L(r), & r < r_1, \\ U_M(r), & r_1 < r < r_2, \\ U_R(r), & r > r_2, \end{cases} \quad (1.8.6)$$

and we want to compare it with the solution with Cauchy data (1.7.10), that is,

$$U_0(r) = \begin{cases} U_L(r), & r < r_0, \\ U_R(r), & r > r_0. \end{cases} \quad (1.8.7)$$

The following statement is a generalization of Proposition 1.6.5, which corresponds to the special case $r_1 = r_2 = r_0$. We restrict attention to continuous steady states. (A generalization to discontinuous steady states could possibly be established too, by including the strength of the steady shock.)

Proposition 1.8.2 (Diminishing total variation property for the generalized Riemann problem). *Suppose that all steady state under consideration are continuous. The wave strengths associated with radii $r_1 < r_0 < r_2$ and three steady state solutions $U_L = U_L(r)$, $U_M = U_M(r)$, and $U_R = U_R(r)$ to the Euler system (1.6.1). Then, one has*

$$\mathcal{E}(U_L(r_0-), U_R(r_0+)) \leq \left(\mathcal{E}(U_L(r_1-), U_M(r_1+)) + \mathcal{E}(U_M(r_2-), U_R(r_2+)) \right) (1 + O(1)(r_2 - r_1)). \quad (1.8.8)$$

Proof. Consider first smooth steady state solutions (which do not contain shocks). Since solutions to an ordinary differential system depend continuously upon their data, it is immediate that

$$U_L(r_1) - U_*(r_1) = U_L(r_0) - U_*(r_0) + O(1)(r_0 - r_1)|U_L(r_1) - U_*(r_1)|$$

and, since $|U_R - U_L| = O(1)\mathcal{E}(U_L, U_R)$, we obtain

$$\mathcal{E}(U_L(r_1), U_*(r_1)) = \mathcal{E}(U_L(r_0), U_*(r_0)) \left((1 + O(1)(r_0 - r_1)) \right).$$

With the same argument, we have

$$\mathcal{E}(U_*(r_2), U_R(r_2)) = \mathcal{E}(U_*(r_0), U_R(r_0)) \left(1 + O(1)(r_2 - r_0) \right)$$

and the conclusion follows for smooth equilibrium solutions. For steady state solutions which are only continuous, we recall the conclusion in Theorem 1.7.4, where we established a Lipschitz continuity property satisfied by global steady state solutions. \square

The existence theory based on equilibria

The existence property below is established under the restriction that only continuous steady states are involved in the scheme. Dealing with discontinuous steady states require a further investigation of the interaction between steady shocks and Riemann solutions (which is outside the scope of the present paper).

Theorem 1.8.3 (The generalized random method based on equilibria). Consider the Euler system describing fluid flows on a Schwarzschild background (1.3.3) posed in $r > 2M$. The generalized random choice scheme above has the following properties:

1. **Convergence to a weak solution.** Given any initial density $\rho_0 = \rho_0(r) > 0$ and velocity $|v_0| = |v_0(r)| < 1/\epsilon$ satisfying, for any $\delta > 0$,

$$TV_{[2M+\delta,+\infty)}(\ln \rho_0) + TV_{[2M+\delta,+\infty)}\left(\ln \frac{1 - \epsilon v_0}{1 + \epsilon v_0}\right) < +\infty,$$

and provided on some (possibly infinite) interval $[t_0, T) \subset [t_0, +\infty)$, the generalized Riemann solver involves continuous steady states, only, then there exists a weak solution $\rho = \rho(t, r)$ and $v = v(t, r)$ defined on $[t_0, T)$ and satisfying the prescribed initial data at the time t_0 and such that, for all finite $T' \in [t_0, T)$ and $\delta > 0$,

$$\sup_{t \in [t_0, T']} \left(TV_{[2M+\delta,+\infty)}(\ln \rho(t, \cdot)) + TV_{[2M+\delta,+\infty)}\left(\ln \frac{1 - \epsilon v(t, \cdot)}{1 + \epsilon v(t, \cdot)}\right) \right) < +\infty.$$

2. **The well-balanced property for smooth steady states.** When the initial density $\rho_0 = \rho_0(r) > 0$ and the initial velocity $|v_0| = |v_0(r)| < 1/\epsilon$ consist of a smooth steady state solution to (1.5.4), the corresponding approximate solution to the Euler system (1.3.3) constructed by the proposed generalized random choice method (in Section 1.7) coincides with the given solution, so that our method provides the exact solution in this special case.

3. **The well-balanced property for discontinuous steady states.** Consider an initial data $U_0 = (\rho_0(r), v_0(r))$ with $\rho_0(r) > 0$ and $|v_0(r)| < 1/\epsilon$ of the following form

$$U_0(r) = \begin{cases} U_L(r), & r \in (2M, r^\natural), \\ U_R(r), & r \in (r^\natural, +\infty), \end{cases} \quad (1.8.9)$$

where $r^\natural > 2M$ is a given radius, $U_L = U_L(r)$ and $U_R = U_R(r)$ are global smooth steady solutions such that the states $U_L(r^\natural)$ and $U_R(r^\natural)$ satisfy the equilibrium Rankine-Hugoniot relations (1.7.4). Then, the solution constructed by the generalized random choice method has, at each time, the same form (1.8.9), that is, a discontinuous steady state solution with possibly “shifted” location r^\natural .

Proof. Step 1a. Consistency of the method. With the proposed generalized random method, we obtain a sequence $\{U_\Delta(t, r)\}$. Once the uniform BV bound (established below) is known, it follows from Helly's theorem that there exists a subsequence of $\{U_\Delta(t, r)\}$ (still denoted by $\{U_\Delta(t, r)\}$) depending on the mesh length $\Delta r \rightarrow 0$ and a limit function $U = U(t, r)$ such that $U_\Delta \rightarrow U$ pointwise for all times t . To check that the limit function is a weak solution to the Euler system (1.3.3), we consider a compactly supported and smooth function $\phi : [t_0, +\infty) \times (2M, +\infty) \rightarrow \mathbb{R}$, and from the approximate solution U_Δ with mesh length $\Delta t, \Delta r$, we define

$$\Delta(U_\Delta, \phi) := \int_{t_0}^{+\infty} \int_{2M+\Delta r}^{+\infty} (U_\Delta \partial_t \phi + F(r, U_\Delta) \partial_r \phi + S(r, U_\Delta)) dr dt + \int_{2M+\Delta r}^{+\infty} U_0(r) \phi(t_0, r) dr. \quad (1.8.10)$$

By definition, U is a weak solution to the Euler system (1.3.3) with initial data $U(t_0, \cdot) = U_0$ if and only if $\Delta(U, \phi) = 0$. We write $\Delta(U_\Delta, \phi) = \sum_i \Delta_i^1(U_\Delta, \phi) + \Delta_i^2(U_\Delta, \phi)$ with

$$\begin{aligned} \Delta_i^1(U_\Delta, \phi) &= \int_{2M+\Delta r}^{+\infty} (U_\Delta(t_i+, r) - U_\Delta(t_i-, r)) \phi(t_i, r) dr, \\ \Delta_i^2(U_\Delta, \phi) &= \int_{t_i}^{t_{i+1}} \int_{2M+\Delta r}^{+\infty} (U_\Delta \partial_t \phi + F(r, U_\Delta) \partial_r \phi + S(r, U_\Delta)) dr dt + \int_{2M+\Delta r}^{+\infty} U_0(r) \phi(t_0, r) dr. \end{aligned}$$

According to Proposition 1.7.7, $\sum_i \Delta_i^2(U_\Delta, \phi) \rightarrow 0$ when $\Delta t \rightarrow 0$. Furthermore, it a standard matter that, since the sequence (w_i) is equidistributed and thanks to the approximation result in Lemma 1.7.6, we have $\sum_i \Delta_i^1(U_\Delta, \phi) \rightarrow 0$ when $\Delta t \rightarrow 0$, and therefore $\Delta(U_\Delta, \phi) \rightarrow 0$ when $\Delta t, \Delta r \rightarrow 0$.

Step 1b. Uniform total variation bound. Next, in order to study globally the total variation of the solution, we introduce the notion of *mesh curves* J , that is, polygonal curves connecting the points $(t_i, r_{i,j+1})$ (with $i + j$ even). Observe that J separates $[t_0, +\infty) \times [2M, +\infty)$ into two parts: the part including the initial time $t = t_0$ denoted by J_- and the other part J_+ . We call J_2 an immediate successor of J_1 if the every point of J_2 is either on J_1 or in the part J_1+ .

For the mesh point, set

$$U_\Delta(t_i, r_{j+1}) = U_{i,j+1}.$$

Denote by $\hat{U}_{i,j+1}$ as the solution of classical Riemann problem at the mesh point (t_i, r_{j+1}) . We define the total variation $L(J)$ of J as

$$L(J) = \sum \mathcal{E}(U_{i-1,j}, \hat{U}_{i,j-1}) + \mathcal{E}(U_{i-1,j}, \hat{U}_{i,j+1}). \quad (1.8.11)$$

Observe that we can divide the (t, r) plane as a set of diamonds $\diamond_{i,j}$ centered at (t_i, r_j) , $i + j$ even with vertices $(t_{i-1}, r_{i-1,j}), (t_i, r_{i,j-1}), (t_i, r_{i,j+1})$. In particular, for

$j = 1$, i odd, we only have a half diamond cut by the straightline $r = r_1$.

Now, consider a diamond $\diamond_{i,j}$, with $i + j$ even, and define

$$\mathcal{E}_1(\diamond_{i,j}) := \mathcal{E}(U_{i-1,j}, \hat{U}_{i,j-1}) + \mathcal{E}(U_{i-1,j}, \hat{U}_{i,j+1})$$

and

$$\mathcal{E}_2(\diamond_{i,j}) := \mathcal{E}(U_{i,j-1}, \hat{U}_{i+1,j}) + \mathcal{E}(U_{i,j+1}, \hat{U}_{i+1,j})$$

which represent the total strength of waves entering and leaving the diamond $\diamond_{i,j}$, respectively. We write $\triangleright_{i,1}$, with i odd, for the right-hand part of the diamond $\diamond_{i,1}$ cut by the straightline $r = 2M$. We define similarly

$$\mathcal{E}_1(\triangleright_{i,1}) := \mathcal{E}(U_{i-1,1}, \hat{U}_{i,2}), \quad \mathcal{E}_2(\triangleright_{i,1}) := \mathcal{E}(U_{i,2}, \hat{U}_{i+1,1})$$

which represent the total wave strength entering and leaving $\triangleright_{i,1}$, respectively. We now consider the total variation contribution “between” the mesh curve J_1 and its immediate successor J_2 .

We now claim that: Let J_1 and J_2 be two mesh curves such that J_2 is an immediate successor of J_1 . Then there exists a constant $C_1 > 0$ such that the total variation on the mesh curves satisfies

$$L(J_2) - L(J_1) \leq C_1(\Delta t + \Delta r)L(J_1).$$

Namely, suppose the mesh curve J_1 is sandwiched between the time levels t_{i-1} and t_i . In view of (1.8.11), we have

$$L(J_2) - L(J_1) = \mathcal{E}_2(\triangleright_{i,1}) - \mathcal{E}_1(\triangleright_{i,1}) + \sum_{i+j \text{ even}} \mathcal{E}_2(\diamond_{i,j}) - \mathcal{E}_1(\diamond_{i,j}).$$

Now consider the difference $\mathcal{E}_2(\diamond_{i,j}) - \mathcal{E}_1(\diamond_{i,j})$:

$$\begin{aligned} \mathcal{E}_2(\diamond_{i,j}) - \mathcal{E}_1(\diamond_{i,j}) &= \mathcal{E}(U_{i,j-1}, \hat{U}_{i+1,j}) + \mathcal{E}(U_{i,j+1}, \hat{U}_{i+1,j}) - \mathcal{E}(\hat{U}_{i,j-1}, \hat{U}_{i,j+1}) \\ &\quad + \mathcal{E}(\hat{U}_{i,j-1}, \hat{U}_{i,j+1}) - \mathcal{E}(U_{i-1,j}, \hat{U}_{i,j-1}) - \mathcal{E}(U_{i-1,j}, \hat{U}_{i,j+1}). \end{aligned}$$

According to Proposition 1.8.2, we have the inequality of the wave strength:

$$\mathcal{E}(U_{i-1,j}, \hat{U}_{i,j-1}) - \mathcal{E}(U_{i-1,j}, \hat{U}_{i,j+1}) - \mathcal{E}(\hat{U}_{i,j-1}, \hat{U}_{i,j+1}) \leq C_1 \Delta r \mathcal{E}(\hat{U}_{i,j-1}, \hat{U}_{i,j+1}).$$

Using Lemma 1.7.5, we have

$$\begin{aligned}
 & \mathcal{E}(U_{i,j-1}, \hat{U}_{i+1,j}) + \mathcal{E}(U_{i,j+1}, \hat{U}_{i+1,j}) - \mathcal{E}(\hat{U}_{i,j-1}, \hat{U}_{i,j+1}) \\
 & \leq C_1 \mathcal{E}(\hat{U}_{i,j-1}, \hat{U}_{i,j+1}) (|U_{i+1,j} - \hat{U}_{i+1,j}| + |U_{i-1,j} - \hat{U}_{i-1,j}|) \\
 & \quad + C_1 (|U_{i-1,j} - U_{i+1,j}| + |\hat{U}_{i-1,j} - \hat{U}_{i+1,j}|) \\
 & \leq C_1 \Delta t \left(\mathcal{E}(U_{i-1,j}, \hat{U}_{i,j-1}) + \mathcal{E}(U_{i,j-1}, \hat{U}_{i,j+1}) \right)
 \end{aligned}$$

for some constants C_1 which need not be the same at each occurrence. Therefore, we find

$$\mathcal{E}_2(\diamond_{i,j}) - \mathcal{E}_1(\diamond_{i,j}) \leq C_1 (\Delta t + \Delta r) \left(\mathcal{E}(U_{i-1,j}, \hat{U}_{i,j+1}) + \mathcal{E}(U_{i-1,j}, \hat{U}_{i+1,j}) \right),$$

and a similar analysis gives $\mathcal{E}_2(\triangleright_{i,1}) - \mathcal{E}_1(\triangleright_{i,1}) \leq C_1 (\Delta t + \Delta r) \mathcal{E}(\hat{U}_{i,2}, U_{i-1,1})$.

Step 1c. Convergence property. Let $T > t_0$ be given, and let J_0, J_T be the mesh curves lying below and above any other mesh curves between $t_0 \leq t \leq T$, respectively. Thanks to Step 2, there exist uniform constants $C_2, C_3 > 0$ such that

$$L(J_T) \leq C_3 e^{C_2(T-t_0)} L(J_0).$$

We now claim that for small Δr , the total variation of the approximate solver $\ln \rho_\Delta$ on the mesh curve J can be regarded equivalent as the total wave strength $L(J)$. In fact, according to construction, for the mesh curve between (t_i, t_{i+1}) ,

$$\begin{aligned}
 |TV(\ln \rho_\Delta(J)) - L(J)| &= \sum_{i+j \text{ even}} |TV_{(r_j+, r_{j+2}-)}(\ln \rho_\Delta(r))| \\
 &= O(1) \sum_{i+j \text{ even}} \Delta r |\ln \rho_\Delta(t_i, r_{j+2}-) - \ln \rho_\Delta(t_i, r_j+)| \leq O(1) \Delta r L(J).
 \end{aligned}$$

Letting $\Delta t, \Delta r \rightarrow 0$, we see that $TV_{[2M+\delta, L]}(\ln \rho(T, \cdot)) \leq C_3 TV_{[2M+\delta, L]}(\ln \rho_0) e^{C_2(T-t_0)}$ for any given $\delta > 0$ and $L > 0$. We have arrived at our main result stated in Theorem 4.6.1. \square

Step 2. The well-balanced property for smooth steady states. 2. We proceed by induction and assume that the numerical solution coincides with the steady state solution within the time interval $t_{i-1} \leq t < t_i$, and we consider the next interval $t_i \leq t < t_{i+1}$. In our method, the approximate solution is determined in two steps: (i) First of all, we must solve the steady state problem at the time $t = t_i$; (ii) Second, we must solve the generalized Riemann problem on the interval $t_i < t < t_{i+1}$. Since the initial data is a smooth steady state solution, it is clear that Step (i) is exact. On

the other hand, Lemma 1.7.6 provides us a control of the error associated with the generalized Riemann problem and implies that Step (ii) is also exact. This completes our argument.

Step 3. The well-balanced property for discontinuous steady states. We start from the initial data $U_\Delta(t_0, \cdot) = U_0$ at some time t_0 . Writing $U_L(r^\sharp) =: U_L^0$ and $U_R(r^\sharp) =: U_R^0$, we have either $U_R^0 \in S_1^\rightarrow(U_L^0)$ (if $|v_L^0| > k$) or $U_L^0 \in S_2^\leftarrow(U_R^0)$ (if $|v_R^0| > k$). For definiteness, we assume that $U_L^0 \in S_2^\leftarrow(U_R^0)$. Consider the solution for the time interval $t_0 < t < t_1$, and consider the unique even number j_0 such that $r^\sharp \in (r_{j_0-1}, r_{j_0+1}]$. We distinguish between two cases:

Case $r^\sharp \neq r_{j_0}$. The solution is a steady state solution with a shock at $r = r^\sharp$.

Case $r^\sharp = r_{j_0}$. We solve the generalized Riemann problem at $r = r^\sharp$. According to our construction, for all $t_0 < t < t_1$, the solution is defined by

$$U_\Delta(t, r) = \begin{cases} U_L(r), & r \in (2M, s_2^0(t - t_0) + r^\sharp), \\ U_R(r), & r \in (s_2^0(t - t_0) + r^\sharp, +\infty), \end{cases}$$

where

$$s_2^0 := \begin{cases} s_2(U_L^0, U_R^0), & r^\sharp = r_{j_0}, \\ 0, & r^\sharp \neq r_{j_0}. \end{cases}$$

To extend the construction, we solve the differential equation (1.5.4) and obtain the steady state solution at the time level $t = t_1$. We write $r_1^\sharp := s_2^0 \Delta t + r^\sharp$. Thanks to the stability condition (1.7.25), we have $r_1^\sharp \in [r_{j_0}, r_{j_0+1}]$. The definition of the approximate solution depends on the position of $r_{1,j_0} = r_{j_0} + w_i$. We have

$$U_\Delta(t_1, r) = \begin{cases} U_L(r), & r \in (2M, r_{j_1}), \\ U_R(r), & r \in (r_{j_1}, +\infty), \end{cases}$$

where $r_{j_1} = r_{j_0 - \text{sgn}(r_{1,j_0} - r_1^\sharp)}$. We then solve the generalized Riemann problem at r_{j_1} . By induction, we find the solution defined on the time interval $[t_i, t_{i+1})$:

$$U_\Delta(t, r) = \begin{cases} U_L(r), & r \in (2M, s_2^i(t - t_0) + r_{i+1}^\sharp), \\ U_R(r), & r \in (s_2^i(t - t_0) + r_{i+1}^\sharp, +\infty), \end{cases} \quad (1.8.12)$$

where s_2^i is (randomly) determined by the sequence (w_i) . This completes the proof.

1.9 Remarks on special models

Stiff fluids on a Schwarzschild background

Consider now the model corresponding to the pressure-law $p = \frac{1}{\epsilon^2}\rho$, so that the sound speed coincides with the light speed $\frac{1}{\epsilon}$. That is, consider the Euler model for stiff fluid flows on a Schwarzschild background $\mathcal{M}(\epsilon, \frac{1}{\epsilon}, m)$ presented in (1.3.5). Recall that it admits two real and distinct eigenvalues $\lambda = -(1 - 2M/r)/\epsilon$ and $\mu = (1 - 2M/r)\epsilon$. They satisfy $-\frac{1}{\epsilon} < \lambda < 0 < \mu < 1/\epsilon$. and, in the limit $r \rightarrow +\infty$, we have $\lambda, \mu \rightarrow \pm\frac{1}{\epsilon}$. According to Proposition 1.2.3, the two characteristics fields are both linearly degenerate. Denote by $D_1^{\rightarrow}(U_L)$ and $D_2^{\leftarrow}(U_R)$ the 1- and 2-contact discontinuities (that is, the notions of shock and rarefaction coincide in this case) corresponding to any given constant states U_L and U_R respectively.

Lemma 1.9.1 (Riemann problem for stiff fluids). *Consider the Euler model $\mathcal{M}(\epsilon, \frac{1}{\epsilon}, m)$ in (1.3.5). Given any constant states U_L, U_R , there exists a unique intermediate U_M , such that U_L can be connected to U_M by a contact discontinuity with the speed $-(1 - 2M/r)/\epsilon$, while U_M is connected to U_R by a contact discontinuity with speed $(1 - 2M/r)/\epsilon$.*

Unlike the case when the sound speed is strictly less than the light speed, in this linearly degenerate regime, steady state solutions are always defined globally. The system for steady state solutions reads

$$\begin{aligned} \frac{d}{dr} \left(r(r-2M) \frac{\rho v}{1-\epsilon^2 v^2} \right) &= 0, \\ \frac{d}{dr} \left((r-2M)^2 \frac{1+\epsilon^2 v^2}{1-\epsilon^2 v^2} \rho \right) &= 2M \frac{r-2M}{r} \frac{1+\epsilon^2 v^2}{1-\epsilon^2 v^2} \rho + 2 \frac{(r-2M)^2}{r} \rho. \end{aligned} \tag{1.9.1}$$

Lemma 1.9.2. *By imposing an initial condition $\rho(r_0) = \rho_0$ and $v(r_0) = v_0$, the system (1.9.1) has a unique global smooth solution given explicitly by*

$$\rho(r) = \left(1 - \frac{r_0^4 \epsilon^2 v_0^2}{r^4} \right) \frac{(r_0 - 2M)r}{r_0(r - 2M)(1 - \epsilon^2 v_0^2)} \rho_0, \quad v(r) = \frac{r_0^2}{r^2} v_0. \tag{1.9.2}$$

Proof. By taking $k = 1/\epsilon$ in (1.5.3), we obtain

$$\begin{aligned} r(r-2M) \frac{\rho v}{1-\epsilon^2 v^2} &= r_0(r_0-2M) \frac{\rho_0 v_0}{1-\epsilon^2 v_0^2}, \\ \left(1 - \frac{2M}{r} \right) \frac{\rho}{1-\epsilon^2 v^2} &= \frac{r_0-2M}{r_0} \frac{\rho_0}{1-\epsilon^2 v_0^2}, \end{aligned}$$

which we can solve explicitly for the density and velocity functions. □

In view of the classical Riemann solver and the explicit form of the steady state solutions, it is now straightforward to follow all the steps of the general proof and check the following result. Our main observation here is that all of our earlier estimates when the sound speed is strictly less than the light speed are *uniform* when the sound speed approaches the light speed.

Theorem 1.9.3 (Stiff fluid flows on a Schwarzschild background). Consider the Euler model $\mathcal{M}(\epsilon, \frac{1}{\epsilon}, m)$ for stiff fluids evolving on a Schwarzschild background, as presented in (1.3.5). Given any initial density $\rho_0 = \rho_0(r) > 0$ and velocity $v = v_0(r)$ defined for $r > 2M$ and satisfying (for all $\delta > L > 0$)

$$TV_{[2M+\delta, +\infty)}(\ln \rho_0) + TV_{[2M+\delta, +\infty)}\left(\ln \frac{1 - \epsilon v_0}{1 + \epsilon v_0}\right) < +\infty,$$

there exists a weak solution $\rho = \rho(t, r)$ and $v = v(t, r)$ satisfying the prescribed initial data at some given time t_0 , together with the following bound on every time interval $[t_0, T]$ and for all $\delta, L > 0$

$$\sup_{t \in [t_0, T]} \left(TV_{[2M+\delta, +\infty)}(\ln \rho(t, \cdot)) + TV_{[2M+\delta, +\infty)}\left(\ln \frac{1 - \epsilon v(t, \cdot)}{1 + \epsilon v(t, \cdot)}\right) \right) < +\infty.$$

Non-relativistic Euler equations on a Schwarzschild background

In this section, we state the existence theory for the non-relativistic Euler system (1.3.4):

$$\begin{aligned} \partial_t(r^2 \rho) + \partial_r(r^2 \rho v) &= 0, \\ \partial_t(r^2 \rho v) + \partial_r(r^2(v^2 + k^2)\rho) - 2k^2 \rho r + m\rho &= 0. \end{aligned}$$

For (1.3.4), we have the eigenvalues $\lambda = v - k$ and $\mu = v + k$ and a pair of Riemann invariants: $w = -v - k \ln \rho$ and $z = -v + k \ln \rho$. We can also give the form of the 1-shock and the 2-shock associated with the constant states U_L and U_R respectively:

$$\begin{aligned} S_1^\rightarrow(U_L) &= \left\{ v - v_L = -k \left(\sqrt{\frac{\rho}{\rho_L}} - \sqrt{\frac{\rho_L}{\rho}} \right), \quad \rho > \rho_L \right\}, \\ S_2^\leftarrow(U_R) &= \left\{ v - v_R = k \left(\sqrt{\frac{\rho}{\rho_R}} - \sqrt{\frac{\rho_R}{\rho}} \right), \quad \rho < \rho_R \right\}. \end{aligned} \tag{1.9.3}$$

A direct calculation gives the the rarefaction curves issuing from the constant states U_L and U_R respectively:

$$R_1^\rightarrow(U_L) = \left\{ \frac{v}{v_L} = \left(\frac{\rho}{\rho_L} \right)^{-k}, \quad \rho > \rho_L \right\}, \quad R_2^\leftarrow(U_R) = \left\{ \frac{v}{v_R} = \left(\frac{\rho}{\rho_R} \right)^k, \quad \rho < \rho_R \right\}. \tag{1.9.4}$$

In view of Proposition 1.6.4, we can solve the Riemann problem of the non-relativistic Euler equations with the help of (1.9.3) and (1.9.4). Similarly as the case, the generalized Riemann problem requires a global steady state solution.

Let $\rho = \rho(r; r_0, \rho_0, v_0)$ and $v = v(r; r_0, \rho_0, v_0)$ be a smooth steady state solution with sonic point of Euler equation (1.4.6) on Ξ . Recall the function P in (1.4.12) which determines the regime of the solutions:

$$P(r_0, v_0) := \frac{3}{2} + \ln \frac{m^2}{4k^3 r_0^2 v_0} + \frac{1}{2k^2} \left(v_0^2 - \frac{2m}{r_0} \right).$$

Let r_1^* be the unique point such that $P(r_1^*, \frac{k^2}{v_1^*}) = 0$ where $v_1^* = v(r_1^*, r_0, v_0)$, and introduce the regions

$$\Lambda_s := \begin{cases} [r_1^*, +\infty), & r_1^* \geq \frac{m}{2k^2}, \\ (0, r_1^*), & r_1^* < \frac{m}{2k^2}, \end{cases} \quad \Lambda_d = \begin{cases} (0, r_1^*], & r_1^* \geq \frac{m}{2k^2}, \\ (r_1^*, +\infty), & r_1^* < \frac{m}{2k^2}. \end{cases}$$

For this non-relativistic model, we can repeat our construction above.

Theorem 1.9.4. Consider the family of non-relativistic steady flows on Schwarzschild spacetime with the constant sound speed $k > 0$. Given arbitrary density $\rho_0 > 0$, velocity $v_0 \geq 0$, and radius $r_0 > 0$, the boundary value problem of the steady Euler system (1.4.6) with $\rho(r_0) = \rho_0$ and $v(r_0) = v_0$, admits a global weak solution of (1.4.6) defined all $r \in (0, +\infty)$.

Observe the the solutions are now defined in the whole half-line and that the eigenvalue λ, μ are not vanishing at $r = 0$. By considering a domain $r > r_b$ for a given boundary radius $r_b > 0$ and imposing the boundary condition $v = 0$ at $r = r_b$, it is conceivable that the following statement could be established with our generalized random choice method.

Theorem 1.9.5 (Non-relativistic fluid flows on a Schwarzschild background). For the non-relativistic Euler system on a Schwarzschild background (1.3.4) posed on $r > r_b$ and given any initial data $\rho_0 = \rho_0(r) > 0$ and $v_0 = v_0(r)$ and any boundary data $\rho_b = \rho_b(t)$ at $r = r_b$, satisfying for any $T > t_0$

$$TV_{[r_b, +\infty)}(\ln \rho_0) + TV_{[r_b, +\infty)}(v_0) + TV_{[t_0, T)}(\ln \rho_b) < +\infty,$$

then there exists a weak solution $\rho = \rho(t, r)$ and $v = v(t, r)$ defined for all $t \geq t_0$ and $r > r_b$ such that for all $T > t_0$

$$\sup_{t \in [t_0, T]} \left(TV_{[r_b, +\infty)}(\ln \rho(t, \cdot)) + TV_{[r_b, +\infty)}(v(t, \cdot)) \right) < +\infty.$$

Fluid flows in Minkowski spacetime

When the black hole mass $M \rightarrow 0$ vanishes, the Schwarzschild metric approaches Minkowski metric and we find the Euler system (1.3.8):, that is,

$$\begin{aligned} \partial_t \left(\frac{1 + \epsilon^4 k^2 v^2}{1 - \epsilon^2 v^2} \rho \right) + \partial_r \left(\frac{1 + \epsilon^2 k^2}{1 - \epsilon^2 v^2} \rho v \right) &= 0, \\ \partial_t \left(\frac{1 + \epsilon^2 k^2}{1 - \epsilon^2 v^2} \rho v \right) + \partial_r \left(\frac{v^2 + k^2}{1 - \epsilon^2 v^2} \rho \right) &= 0. \end{aligned} \tag{1.9.5}$$

We recover also the standard existence theory [44] for this model.

Theorem 1.9.6 (Fluid flows in Minkowski spacetime). Given any initial data $\rho_0 = \rho_0(r) > 0$ and $|v_0| = |v_0(r)| \leq 1/\epsilon$ defined for $r > 0$ and satisfying

$$TV(\ln \rho_0) + TV\left(\frac{1 - \epsilon v_0}{1 + \epsilon v_0}\right) < +\infty,$$

then there exists a corresponding weak solution $\rho = \rho(t, r)$ and $v = v(t, r)$ to (1.9.5), which is defined for all $t > t_0$ and all $r > 0$ with

$$\sup_{t \in [t_0, T]} \left(TV(\ln \rho(t, \cdot)) + TV\left(\ln \frac{1 - \epsilon v(t, \cdot)}{1 + \epsilon v(t, \cdot)}\right) \right) < +\infty.$$

Acknowledgments

This paper was completed in April 2015 when the first author (PLF) enjoyed the hospitality of the Courant Institute of Mathematical Sciences at New York University. The authors were also partially supported by the Centre National de la Recherche Scientifique (CNRS) and the Agence Nationale de la Recherche through the grant ANR SIMI-1-003-01.

Chapter 2

Weakly regular fluid flows on the domain of outer communication of a Schwarzschild spacetime. The relativistic Burgers equation¹

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¹ Réalisé en collaboration avec P.G. LeFloch et soumis à J. Math. Pures Appl.

2.1 Introduction

This is the second part of a series of papers [31, 33] devoted to fluid flows with bounded total variation, evolving the domain of outer communication of a Schwarzschild spacetime. This work is motivated by a broader set of relativistic fluid problems involving shock wave phenomena; see LeFloch [24]. In the present paper, we investigate a simplified, Burgers-type model defined as follows.

Recall that the inviscid Burgers equation is the hyperbolic conservation law

$$\partial_t v + \partial_x \left(\frac{v^2}{2} \right) = 0, \quad t > 0, x \in \mathbb{R}, \quad (2.1.1)$$

and has played a central role in the development of mathematical techniques suitable to handle shock wave solutions to nonlinear hyperbolic problems. Moreover, the existence of weak solutions to the initial value problem can be established via the random choice method, which provides an (essentially) piecewise constant approximation based on (explicit) solutions to the *Riemann problem*

$$v_0(x) = \begin{cases} v_L & x < 0, \\ v_R & x > 0. \end{cases} \quad (2.1.2)$$

in which v_L, v_R are arbitrary constants. Furthermore, recall also that the Burgers equation can be formally derived from the Euler system for compressible fluids (without pressure term):

$$\partial_t \rho + \partial_x(\rho v) = 0, \quad \partial_t(\rho v) + \partial_x(\rho v^2) = 0, \quad (2.1.3)$$

in which $\rho \geq 0$ denotes the fluid density and $v \in (-\infty, +\infty)$ its velocity. (See the textbooks [9, 18, 21] for background material.)

Our main objective here is to investigate relativistic fluid flows on the domain of outer communication of a Schwarzschild black hole spacetime. This curved background is one of the simplest solutions to the Einstein equations and correspond to the geometry determined by a massive body of mass $M \geq 0$ surrounded by a vacuum region. In Schwarzschild coordinates, denoted $(x^0, x^1, x^2, x^3) = (ct, r, \theta, \varphi)$ (the parameter $c \in [0, +\infty]$ being the speed), the metric of interest reads

$$g = - \left(1 - \frac{2M}{r} \right) c^2 dt^2 + \left(1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (2.1.4)$$

with $t \in [0, +\infty)$ and $r \in (2M, +\infty)$, where $(d\theta^2 + \sin^2 \theta d\varphi^2)$ is the canonical metric on the two-sphere S^2 (with $\theta \in [0, 2\pi)$ and $\phi \in [0, 2\pi)$). Note that there is an (apparent) singularity at $r = 2M$, which could be removed by changing to other (much more involved) coordinates. As we explain below, from the Euler system on

this curved background, we are able to formally derive a Burgers-type equation which reads

$$\partial_t \left(\frac{v}{(1 - 2M/r)^2} \right) + \partial_r \left(\frac{v^2 - 1/\epsilon^2}{2(1 - 2M/r)} \right) = 0, \quad r > 2M, \quad (2.1.5)$$

$$v = v(t, r) \in [-1/\epsilon, 1/\epsilon],$$

which we refer to as the **relativistic Burgers equation on a Schwarzschild black hole**. Obviously, when the mass vanishes, we recover the standard Burgers equation. Observe that the equation (2.1.5) is *singular* when r approaches the black hole horizon $2M$. The constant values $\pm 1/\epsilon$ are trivial steady states of the model. Our purpose in the present paper is to provide a full treatment of the initial value problem associated with this equation. We introduce suitable notions of weighted total variation functional and weighted L^1 norm for this equation, and establish a well-posed theory in a class of entropy weak solutions and, next, analyze their time-asymptotic behavior. Our proof are inspired from a large body of works and techniques pioneered by Dafermos and Hsiao [8], Dafermos [9], Glimm [13], and Liu [38, 39, 40]. Our main motivation comes from the work by Glimm, Marshall, and Plohr [14] who analyzed quasi-one-dimensional gas flows in nozzle with variable cross-section, solved the generalized Riemann problem, and proposed a random choice method.

Our main results are as follows. First of all, we solve the *generalized Riemann problem* when the initial data, by definition, consists of two steady state solutions separated by a jump discontinuity (cf. Section 2.3 below). The generalized Riemann problem were treated first by Li and co-authors [25, 35, 36, 37]; see also [26].

Theorem 2.1.1 (Well-posedness theory for the generalized Riemann problem). Given any two steady state solutions (which might not be defined on the whole interval $(2M, +\infty)$) initially separated by a jump discontinuity, there exists a unique solution to the generalized Riemann problem associated with the Burgers equation on a Schwarzschild background (2.1.5). Either this solution contains a shock wave and when $t \rightarrow +\infty$:

- The shock location asymptotically approaches the black hole horizon $r = 2M$ if and only if the shock speed is initially negative.
- The shock location asymptotically approaches spacelike infinity $r = +\infty$ if and only if the shock speed is initially positive.
- Alternatively, the shock location remains fixed for all times if and only if the shock speed vanishes initially.

Or else this solution contains a rarefaction wave and $t \rightarrow +\infty$:

- The rarefaction fan asymptotically approaches the black hole horizon $r = 2M$ if and only if it moves toward the black hole initially.

- The rarefaction fan asymptotically approaches spacelike infinity $r = +\infty$ if and only if it moves away from the black hole initially.
- The left-hand location of the rarefaction fan asymptotically approaches the black hole horizon $r = 2M$, while the right-hand location of the rarefaction fan approaches spacelike infinity $r = +\infty$ if and only if the left-hand speed of the fan is initially negative while its right-hand speed is initially positive.

We then construct weak solutions whose *weighted bounded variation* (denoted by \widetilde{TV} and defined in (2.5.1) below) is finite.

Theorem 2.1.2 (Existence theory for the initial value problem). Given any initial data whose weighted bounded total variation is finite, the Burgers equation on a Schwarzschild background (2.1.5) admits an entropy weak solution defined for all $t > 0$ and $r > 2M$. This solution is obtained as the limit of a sequence of approximate solutions constructed via a generalized version of the random choice method based on the generalized Riemann solver provided by Theorem 2.1.1 and the weighted total variation of these (approximate) solutions on $(2M, +\infty)$ is uniformly bounded for all times.

Finally, we are also able to determine the global evolution of arbitrary perturbations of two steady state solutions separated by a jump discontinuity. The proof of the theorem below extends a method proposed by Liu [39] in 1978 for the standard Burgers equation.

Theorem 2.1.3 (Time-asymptotics of perturbed steady state solutions). Let $v = v(t, r)$ be a solution to the Burgers equation on a Schwarzschild background (2.1.5) whose initial data $v_0 = v_0(r)$ has finite weighted total variation and satisfies

$$v_0(r) = \begin{cases} v_*(r), & 2M < r < r_*, \\ v_{**}(r), & r > r_{**}, \end{cases} \quad (2.1.6)$$

where $2M < r_* < r_{**}$ and v_*, v_{**} are steady state solutions defined on $(2M, r_*)$ and $(2M, +\infty)$, respectively.

- If $v_*(r) > v_{**}(r)$ for all r in their domains of definition, then the following holds:
 - If, moreover, $v_* > v_{**} > 0$, then there exists a time $t_0 \in (0, +\infty)$ from which the solution is exactly a shock wave connecting the left-hand solution v_* and the right-hand solution v_{**} , while the shock curve asymptotically to spatial infinity $r = +\infty$.
 - If, moreover, $v_* > 0 > v_{**}$, then there exists a time $t_0 \in (0, +\infty)$ from which the solution is exactly a shock wave connecting the left-hand solution v_* and the right-hand solution v_{**} :

- (i) If $v_* + v_{**} > 0$, then the shock location asymptotically approaches spatial infinity.
 - (ii) If $v_* + v_{**} < 0$, then the shock location asymptotically approaches the black hole horizon.
 - (iii) If $v_* = -v_{**}$, then the shock location is fixed for all times $t > t_0$.
- If, moreover, $v_{**} < v_* < 0$, then the solution v asymptotically (that is, in the limit $t \rightarrow +\infty$ but not in a finite time) approaches the solution consisting of a static shock connecting the solutions v_* and v_{**} and, in particular, the shock location asymptotically approaches the black hole horizon $r = 2M$.
- If $v_*(r) < v_{**}(r)$ for all r in their domains of definition, then one can introduce a generalized N-wave $N = N(t, r)$ consisting of a rarefaction wave connecting v_* and v_{**} such that:
 - In the rarefaction fan region bounded by its two edges within the N-wave solution, one has $|v(t, r) - N(t, r)| = \mathcal{O}(t^{-1})$.
 - (i) If $v_* > 0$, the rarefaction fan tend to spatial infinity.
 - (ii) If $v_* < 0 < v_{**}$, then the left-hand edge of the rarefaction fan approaches the black hole horizon while the right-hand edge of the rarefaction fan converges to space infinity.
 - (iii) If $v_{**} < 0$, the rarefaction fan tend to the black hole horizon.
 - In the region supporting of the evolution of the initial data (between the edge of the N-wave and suitably defined generalized characteristics), one has $|v(t, r) - N(t, r)| = \mathcal{O}(t^{-1/2})$.
 - In the remaining spacetime region, one has $v(t, r) = N(t, r)$.
 - If $v_*(r) = v_{**}(r)$ for all $r \in (2M, +\infty)$, then one has $|v(t, r) - v_*(t, r)| = \mathcal{O}(t^{-1/2})$.

An outline of this paper is as follows. In Section 2.2, we derive the Burgers model of interest from the relativistic Euler equations for a fluid evolving on the domain of outer communication of a Schwarzschild black hole. In in Section 2.3, we study the class of (smooth) steady state solutions: we identify two regimes of interest for the amplitude of the solutions (large velocities and small velocities). Next, in Section 2.4 we solve the generalized Riemann problem and are able to provide fully explicit formulas, based on curved shock waves and rarefaction waves taking into account the curved Schwarzschild geometry. Of course, in our construction, we also must incorporate suitable versions of the Rankine-Hugoniot condition at shocks and Lax's shock admissibility inequalities. This analysis leads us to a proof of Theorem 2.1.1.

The total variation of solutions to (2.1.5) may increase in time, in contrast with solutions to the standard Burgers model (3.3.1). In Section 2.5, we find it convenient to introduce a weighted total variation functional. Before we can proceed and tackle the general existence theory, it turns out that a multiple version of the generalized Riemann problem must also be solved and this is done in Section 2.6 when the initial problem with three steady states separated with two discontinuities is analyzed.

Based on the results of previous sections and by suitably defining a random choice method adapted to (2.1.5), we are then in position to establish the existence theory in Theorem 2.1.2. We construct a sequence of approximate solutions and we prove that the weighted total variation of these solutions is non-increasing in time. This leads us to the conclusion that this sequence approaches a weak solution of our Burgers model.

We finally provide two additional results: in Section 2.8 the convergence of the vanishing viscosity method is proven, while in section 2.9 we determine the time-asymptotics of weak solutions and thus establish Theorem 2.1.3.

2.2 The relativistic Burgers model on a Schwarzschild background

Derivation from the Euler equations

We start from the Euler equations expressed on the spacetime of interest

$$\nabla_\alpha (T_\beta^\alpha(\rho, u)) = 0, \quad (2.2.1)$$

where ∇ represents the Levi-Civita connection associated with the Schwarzschild metric (2.1.4) while the energy-momentum tensor reads, for perfect fluid flows without pressure term,

$$T_\beta^\alpha(\rho, u) = \rho c^2 u^\alpha u_\beta. \quad (2.2.2)$$

The main unknowns are the fluid density $\rho : \mathcal{M} \mapsto (0, +\infty)$ and the velocity field $u = (u^\alpha)$, normalized to be unit and future-oriented, that is, $u^\alpha u_\alpha = -1$ with $u^0 > 0$. The parameter $c \in [0, +\infty)$ represents the speed of light and we also set $\epsilon := 1/c$. We assume that the fluid flow is radially symmetric with $u^2 = u^3 = 0$, and the normalization condition on the velocity is equivalent to $-1 = \left(1 - \frac{2M}{r}\right)(cu^0)^2 + \left(1 - \frac{2M}{r}\right)^{-1}(u^1)^2$. We find it convenient to introduce the scalar velocity $v \in (-1/\epsilon, 1/\epsilon)$ such that

$$v := \frac{1}{(1 - 2M/r)} \frac{u^1}{u^0}. \quad (2.2.3)$$

Under the above assumption, we can express the Euler system on a Schwarzschild background as

$$\begin{aligned} & \partial_t \left(r^2 \frac{\rho}{1 - \epsilon^2 v^2} \right) + \partial_r \left(r(r - 2M) \frac{\rho v}{1 - \epsilon^2 v^2} \right) = 0, \\ & \partial_t \left(r(r - 2M) \frac{\rho v}{1 - \epsilon^2 v^2} \right) + \partial_r \left((r - 2M)^2 \frac{\rho v^2}{1 - \epsilon^2 v^2} \right) \\ & = \frac{3M}{r} (r - 2M) \frac{\rho v^2}{1 - \epsilon^2 v^2} - \frac{M}{\epsilon^2 r} (r - 2M) \frac{\rho}{1 - \epsilon^2 v^2}. \end{aligned}$$

Then, by writing

$$\begin{aligned} & \partial_t \left(r(r - 2M) \frac{\rho v}{1 - \epsilon^2 v^2} \right) = \partial_t \left(\frac{r^2 \rho}{1 - \epsilon^2 v^2} \right) \left(1 - \frac{2M}{r} \right) v + \frac{r^2 \rho}{1 - \epsilon^2 v^2} \left(1 - \frac{2M}{r} \right) \partial_t v, \\ & \partial_r \left((r - 2M)^2 \frac{\rho v^2}{1 - \epsilon^2 v^2} \right) \\ & = \partial_r \left(r(r - 2M) \frac{\rho v}{1 - \epsilon^2 v^2} \right) \left(1 - \frac{2M}{r} \right) v + \frac{r(r - 2M) \rho v}{1 - \epsilon^2 v^2} \partial_r \left(\left(1 - \frac{2M}{r} \right) v \right), \end{aligned}$$

we can formally combine the two equations in the Euler system above and we obtain

$$\frac{r^2 \rho}{1 - \epsilon^2 v^2} \left(1 - \frac{2M}{r} \right) \partial_t v + \frac{r(r - 2M) \rho v}{1 - \epsilon^2 v^2} \partial_r \left(\left(1 - \frac{2M}{r} \right) v \right) = \frac{3M}{r} \frac{(r - 2M) \rho v^2}{1 - \epsilon^2 v^2} - \frac{M}{\epsilon^2 r} \frac{(r - 2M) \rho}{1 - \epsilon^2 v^2}$$

or

$$\partial_t v + v \partial_r \left(\left(1 - \frac{2M}{r} \right) v \right) = \frac{3M}{r^2} v^2 - \frac{M}{\epsilon^2 r^2}.$$

which we refer to here as the Burgers model on a Schwarzschild background. An equivalent formulation is

$$\partial_t v + \left(1 - \frac{2M}{r} \right) \partial_r \left(\frac{v^2}{2} \right) = \frac{M}{r^2} \left(v^2 - \frac{1}{\epsilon^2} \right), \quad r > 2M, \quad (2.2.4)$$

as well as

$$\partial_t (r^2 v) + \partial_r \left(r(r - 2M) \frac{v^2}{2} \right) = r v^2 - \frac{M}{\epsilon^2}, \quad r > 2M. \quad (2.2.5)$$

For convenience in the presentation, the unknown will be sought in the range $|v| \leq 1/\epsilon$ (rather than in the corresponding open interval).

The conservation form

The model above is not naturally expressed as a conservation law. Yet, by direct calculations, one can check that (3.1.2) admits the conservation form (2.1.5) stated

in the introduction. Let us also repeat that by formally letting the black hole mass M to vanish, we recover the standard Burgers equation (3.3.1). It is also clear that $v = \pm 1/\epsilon$ are two trivial solutions to (2.1.5). Note finally the the speed of propagation associated with (2.1.5) is

$$\lambda(v, r) := \left(1 - \frac{2M}{r}\right)v, \quad (2.2.6)$$

which *vanishes* on the horizon. In the coordinates under consideration for our description of the Schwarzschild geometry, the fluid appears to be at rest on the horizon. The propagation speed approaches zero as one approaches $r = 2M$. This shows that **no boundary condition** is necessary at $r = 2M$ when posing the initial (and boundary) value problem.

2.3 Existence and properties of steady state solutions

Critical steady state solutions

Steady state solutions to the relativistic Burgers model (2.1.5) are given by

$$\partial_r \left(\frac{v^2 - 1/\epsilon^2}{2(1 - 2M/r)} \right) = 0. \quad (2.3.1)$$

While the two constants $v = \pm 1/\epsilon$ are solutions, no other constant value provides us with a solution.

Given a radius $r = r_0$ and a velocity $v_0 \in (-1/\epsilon, 1/\epsilon)$, we denote by $v_* = v_*(r)$ the corresponding steady state solution satisfying the (initial) condition $v_*(r_0) = v_0$. The ordinary differential equation (2.3.1) shows us that

$$\frac{v_*^2 - 1/\epsilon^2}{1 - 2M/r} = \frac{v_0^2 - 1/\epsilon^2}{1 - 2M/r_0} \quad (2.3.2)$$

or, equivalently,

$$v_*^2 = \frac{1}{\epsilon^2} - \left(1 - \frac{2M}{r}\right) \frac{1/\epsilon^2 - v_0^2}{1 - 2M/r_0}. \quad (2.3.3)$$

From the initial condition, we can introduce the positive constant

$$K_*(r_0, v_0) := \sqrt{\frac{1/\epsilon^2 - v_0^2}{1 - 2M/r_0}}. \quad (2.3.4)$$

In view of (2.3.3), we see that whether or not a solution can be defined globally within

the interval $(2M, +\infty)$ depends on the *sign* of the constant $(K_* - 1/\epsilon)$. Indeed, if $K_* > 1/\epsilon$ holds, it is possible that the term $1/\epsilon^2 - (1 - 2M/r)K_*^2$ becomes negative at some sufficiently large radius. A steady state solution might therefore stop to be defined when r is too large.

A critical case of interest is obtained when $K_*(r_0, v_0)$ equals $\frac{1}{\epsilon}$ or, equivalently, when the condition $\epsilon^2 v_0^2 = 2M/r_0$ holds. Consequently, let us introduce a pair of *critical steady state solutions* denoted by $v_{**}^\pm = v_{**}^\pm(r)$ and defined as

$$\epsilon v_{**}^\pm(r) := \pm \sqrt{\frac{2M}{r}}. \tag{2.3.5}$$

The graph of the critical steady state solutions separate the range of velocities $[-\frac{1}{\epsilon}, \frac{1}{\epsilon}]$ into three disjoint regions: $[-\frac{1}{\epsilon}, v_{**}^-(r)]$, $(v_{**}^-(r), v_{**}^+(r))$, $[v_{**}^+(r), \frac{1}{\epsilon}]$. We define the *domain of large velocities* as

$$\mathcal{L} := \left\{ (r, v) \mid -\frac{1}{\epsilon} < v < v_{**}^-(r) \quad \text{or} \quad v_{**}^+(r) < v < \frac{1}{\epsilon} \right\}, \tag{2.3.6}$$

and the *domain of small velocities* as

$$\mathcal{S} := \left\{ (r, v) \mid v_{**}^-(r) < v < v_{**}^+(r) \right\}. \tag{2.3.7}$$

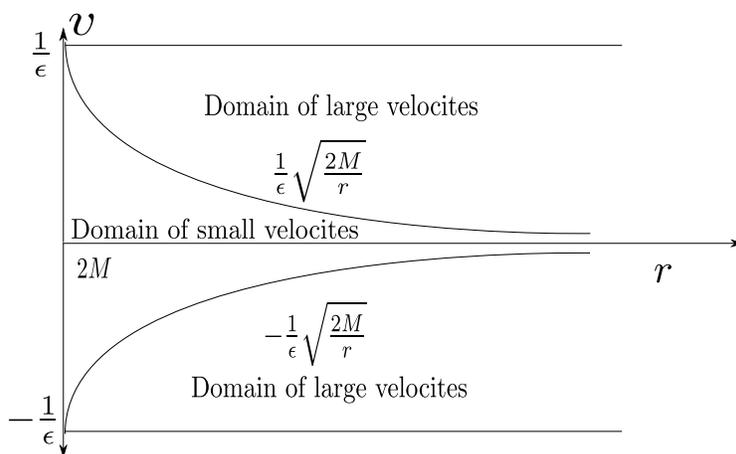


Figure 2.3.1: Critical steady state solutions and domains of large/small velocities.

Properties of the steady state solutions

We have $v_0^2 \geq v_{**}^\pm(r_0)^2$ if and only if the condition $K_*(r_0, v_0) \geq 1/\epsilon$ holds. Hence, the position of a given point (r_0, v_0) (in \mathcal{L} or \mathcal{S}) determines whether a steady state solution can be globally defined in the full domain $(2M, +\infty)$.

Lemma 2.3.1 (Regime of large velocities). *For a given velocity v_0 at $r = r_0 > 2M$ such that $v_{**}^+(r_0) \leq |v_0| \leq \frac{1}{\epsilon}$ where v_{**}^+ is the critical curve given in (2.3.5), the static relativistic Burgers model (2.3.1) admits a unique smooth solution $v_* = v_*(r) \in \mathcal{L}$ where \mathcal{L} is the domain of large velocities (2.3.6), given as*

$$\epsilon v_*(r) = \text{sgn}(v_0) \sqrt{1 - \epsilon^2 K_*^2 \left(1 - \frac{2M}{r}\right)}, \quad r > 2M. \quad (2.3.8)$$

Furthermore, in the domain of definition $(2M, +\infty)$, the following monotonicity and the convexity properties hold:

- If $v_0 > 0$, the steady state solution $v_* = v_*(r)$ is decreasing and convex.
- If $v_0 < 0$, the steady state solution $v_* = v_*(r)$ is increasing and concave.

The behavior near the horizon or near space infinity is given by

$$\lim_{r \rightarrow 2M} v_*(r) = \text{sgn}(v_0) \frac{1}{\epsilon}, \quad \lim_{r \rightarrow +\infty} v_*(r) = \text{sgn}(v_0) \sqrt{\frac{1}{\epsilon^2} - K_*^2}.$$

Proof. Since $K_* \leq 1/\epsilon$ always holds, $v_*^2 > 0$ for all $r > 2M$, the steady state solution can be defined globally and it remains in the domain \mathcal{L} given by (2.3.8). Furthermore, by the formula of the steady state solution, its derivative reads $\frac{dv_*}{dr} = -\frac{2MK_*^2}{r^2 v_*}$. Hence, the steady state solution is decreasing with respect to r on $(2M, +\infty)$ when $v_0 > 0$ while is increasing when $v_0 < 0$. The values at the two limits $r = 2M$ and $r = +\infty$ can be directly obtained by (2.3.8) as well. Moreover, we have

$$\frac{d^2 v_*}{dr^2} = \frac{2M\epsilon^2 K_*^2}{r^4 v_*^3} (2(1 - \epsilon^2 K_*^2)r + 3M\epsilon^2 K_*^2). \quad (2.3.9)$$

Since $r > 2M$ and $1 - \epsilon^2 K_*^2 > 0$, $\frac{d^2 v_*}{dr^2}$ has the same sign as the velocity v_* , which gives the convexity of the steady state solutions in the domain $(2M, +\infty)$. \square

We now turn to the case where the given point (r_0, v_0) lies in \mathcal{S} .

Lemma 2.3.2 (Regime of small velocities). *Let $|v_0| \leq \frac{1}{\epsilon}$ be a velocity and r_0 be a given radius such that $v_0 \in (v_{**}^-(r_0), v_{**}^+(r_0))$ where $v_{**}^\pm = \pm v_{**}(r)$ are the critical steady state curves (2.3.5). Then the steady state solution $v_* = v_*(r)$ of the relativistic Burgers equation (2.3.1) belongs to \mathcal{S} with \mathcal{S} the domain of small velocities (2.3.7)*

and this solution is defined on the interval $(2M, r^\sharp)$ where r^\sharp is the vanishing velocity radius:

$$r^\sharp := \frac{2M\epsilon^2 K_*^2}{\epsilon^2 K_*^2 - 1}. \quad (2.3.10)$$

Moreover, in the domain of definition, the following monotonicity properties hold:

- If $v_0 > 0$, the steady state solution v_* is decreasing.
- If $v_0 < 0$, the steady state solution v_* is increasing.

The behavior near the horizon or near the vanishing velocity radius is given by

$$\lim_{r \rightarrow 2M} v_*(r) = \text{sgn}(v_0) \frac{1}{\epsilon}, \quad \lim_{r \rightarrow r^\sharp} v_*(r) = 0, \quad \lim_{r \rightarrow r^\sharp} \frac{dv_*}{dr} = -\text{sgn}(v_0) \infty.$$

Moreover, the following convexity properties hold:

- When $\frac{1}{\epsilon} < K_* \leq \frac{2}{\epsilon}$, the solution v_* is convex on $(2M, r^\sharp)$ and concave on (r^\sharp, r^\natural) if $v_0 > 0$, while it is concave on $(2M, r^\sharp)$ and convex on (r^\sharp, r^\natural) if $v_0 < 0$. Here, the radius $r = r^\sharp < r^\natural$ is given by

$$r^\sharp := \frac{3M\epsilon^2 K_*^2}{2(\epsilon^2 K_*^2 - 1)}. \quad (2.3.11)$$

- When $K_* > \frac{2}{\epsilon}$, the solution v_* is concave on $(2M, r^\natural)$ if $v_0 > 0$, while it is convex on $(2M, r^\natural)$ if $v_0 < 0$.

Proof. Since $K_* > 1/\epsilon$, we have $1 - \epsilon^2 K_*^2(1 - 2M/r) < 0$ when $r > r^\sharp$. In particular, it vanishes at $r = r^\sharp$. Therefore, the definition of the steady state solution can not be extended out of the space interval $(2M, r^\sharp)$ and $v_*(r)$ stays in the domain \mathcal{S} for all $2M < r < r^\sharp$. On the other hand, the monotonicity of the steady state solution is similar as the result given in Lemma 2.3.1 on the corresponding domain of definition. To consider the convexity, we use (2.3.9) to consider the second-order derivative of the velocity, that is,

$$\frac{d^2 v_*}{dr^2} = \frac{2M\epsilon^2 K_*^2}{r^4 v_*^3} (2(1 - \epsilon^2 K_*^2)r + 3M\epsilon^2 K_*^2).$$

We see that if $K_* > \frac{2}{\epsilon}$, the inequality $2(1 - \epsilon^2 K_*^2)r + 3M\epsilon^2 K_*^2 < 0$ holds for all $r > 2M$. Otherwise, the function $2(1 - \epsilon^2 K_*^2)r + 3M\epsilon^2 K_*^2$ will change signs at the radius $2M < r^\sharp = \frac{3M\epsilon^2 K_*^2}{2(\epsilon^2 K_*^2 - 1)} < r^\natural$. This provides us with the convexity properties. \square

Main conclusions for this section

We summarize our results as follows.

Theorem 2.3.3 (The family of smooth steady states). Consider the family of static solutions to the Burgers model on a Schwarzschild background (2.3.1). Then, for any given radius $r_0 > 2M$ and velocity $v_0 \in [-\frac{1}{\epsilon}, \frac{1}{\epsilon}]$, there exists a unique smooth steady state solution $v_* = v_*(r)$ satisfying (2.3.1) together with the initial condition $v_*(r_0) = v_0$ such that the velocity satisfies $\text{sgn}(v_*) = \text{sgn}(v_0)$ on the corresponding domains of definition. Furthermore, one can distinguish between two cases:

- *Regime of large velocities.* If $0 \leq K_*(r_0, v_0) \leq \frac{1}{\epsilon}$ in which the parameter $K_* = K_*(r_0, v_0)$ was introduced in (3.5.8), then the steady state solution is defined on the whole space interval $(2M, +\infty)$.
- *Regime of small velocities.* If $K_*(r_0, v_0) > \frac{1}{\epsilon}$, then the solution is defined on $(2M, r^\natural)$ with $r = r^\natural$ given by (2.3.10) which is referred to as the vanishing velocity radius.

Remark 2.3.4. For a steady state solution $v_* = v_*(r)$ to the relativistic Burgers model (2.1.5) with $v_*(r_0) = v_0$, the space interval $(2M, r_0)$ is always contained in the domain of definition, regardless of the value of the velocity. In other words, we can at least guarantee the definition of a steady state solution at the left-hand side of a given point $r = r_0$. This is an important property which will be central in order to be able to define our generalized Glimm scheme.

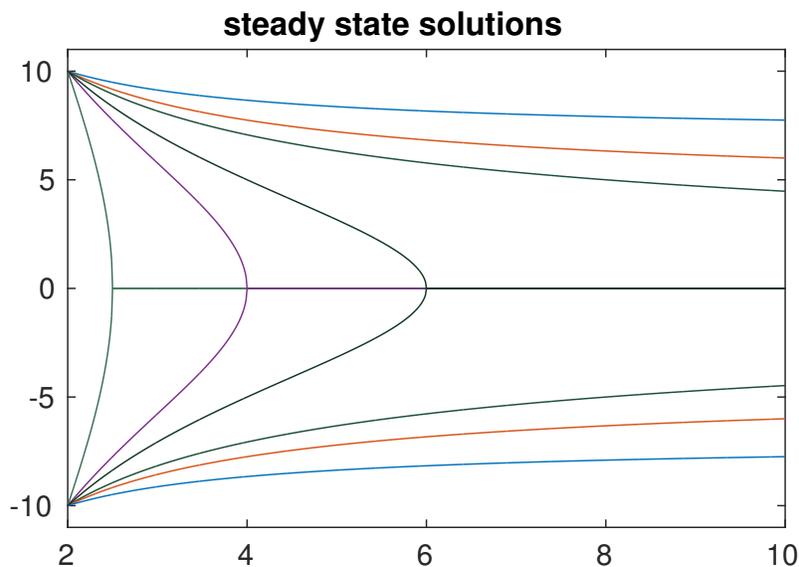


Figure 2.3.2: Plot of steady states with $\epsilon^2 = 0.1$, $M = 1$ and different values of K_* .

To end this section, we provide a property involving two steady state solutions.

Proposition 2.3.5 (Comparing two steady state solutions). *Let v_1, v_2 be two smooth steady state solutions solving the equation (2.3.1).*

- *The sign of $v_1 - v_2$ does not change for r in the relevant domain of definition and the difference $|v_1 - v_2|$ is decreasing with respect to r when $v_1 v_2 < 0$, and increasing with respect to r when $v_1 v_2 > 0$.*
- *The sign of the sum $v_1 + v_2$ does not change for r in the relevant domain of definition and the absolute value of the sum $|v_1 + v_2|$ is decreasing with respect to r when $v_1 v_2 > 0$, and increasing with respect to r when $v_1 v_2 < 0$.*

Proof. In view of the explicit formula (2.3.8), we write the two steady state solutions explicitly as

$$v_1(r) = \text{sgn}(v_1) \sqrt{1 - \epsilon^2 K_*^1 \left(1 - \frac{2M}{r}\right)}, \quad v_2(r) = \text{sgn}(v_2) \sqrt{1 - \epsilon^2 K_*^2 \left(1 - \frac{2M}{r}\right)},$$

where K_*^1, K_*^2 are two constants. Without loss of generality, suppose that there exists a radius $r_0 > 2M$ such that $v_1(r_0) > v_2(r_0)$. Then, it is direct that $v_1 > v_2$ in the domain of definition following from the explicit formula of the two solutions. From the ordinary differential equation (2.3.1), we obtain

$$\frac{d}{dr}(v_1 - v_2) = \frac{1}{1 - 2M/r} \frac{M}{r^2} \left((v_1 - v_2) - \frac{1}{\epsilon^2} \left(\frac{1}{v_1} - \frac{1}{v_2} \right) \right) = \frac{1}{1 - 2M/r} \frac{M}{r^2} (v_1 - v_2) \left(1 + \frac{1}{\epsilon^2 v_1 v_2} \right).$$

Then, $\frac{d}{dr}(v_1 - v_2) > 0$ if v_1, v_2 have the same sign while $\frac{d}{dr}(v_1 - v_2) < 0$ if v_1, v_2 have different signs. We thus have the monotonicity of the difference between two steady state solutions. On the other hand, in order to establish the result about the sum of two solutions, we replace v_2 by $-v_2$ in the previous argument. \square

2.4 The generalized Riemann problem

Preliminaries

Before we address the general initial value problem for the relativistic Burgers equation on the Schwarzschild background with a given initial condition, we start by analyzing the *generalized Riemann problem* corresponding to the initial condition

$$v_0(r) = \begin{cases} v_L(r) & \bar{r} < r < r_0, \\ v_R(r) & r_0 < r < \hat{r}, \end{cases} \quad (2.4.1)$$

consisting of two steady state solutions v_L, v_R separated by a discontinuity at some radius $r = r_0 > 2M$: We will use the notation $v_L(r_0) = v_L^0, v_R(r_0) = v_R^0$ and $v_L(\bar{r}) =$

$\overline{v}_L, v_R(\hat{r}) = \widehat{v}_R$. Our main objective in the present section is to prove Theorem 2.1.1.

Note that for a standard Riemann problem (3.3.1), (2.1.2) associated with two constant states, the solution only depends on $\xi = (r - r_0)/t$. The shock wave curves and particle trajectories are all straight lines. However, under the influence of the Schwarzschild metric geometry, these lines will be bended when time passes. In this section, we would like to give an exact solution of the Burgers equation, taking into account the curvature effect.

In view of (2.3.8), we have

$$v_L(r) = \text{sgn}(v_L^0) \sqrt{\frac{1}{\epsilon^2} - K_*^{L^2} \left(1 - \frac{2M}{r}\right)}, \quad v_R(r) = \text{sgn}(v_R^0) \sqrt{\frac{1}{\epsilon^2} - K_*^{R^2} \left(1 - \frac{2M}{r}\right)},$$

where the constants $K_*^L, K_*^R \geq 0$ are given as in (3.5.8). To classify the types of waves, we introduce

$$\sigma_-(t) = \begin{cases} \sigma(t), & v_L^0 > v_R^0, \\ \sigma_L(t), & v_R^0 > v_L^0, \end{cases} \quad \sigma_+(t) = \begin{cases} \sigma(t), & v_L^0 > v_R^0, \\ \sigma_R(t), & v_R^0 > v_L^0. \end{cases} \quad (2.4.2)$$

By definition, the function $\sigma_L = \sigma_L(t)$ satisfies the equation

$$R_R(\sigma_L(t); v_L) - R_R(r_0; v_L) = t,$$

where $R_R = R_R(r; v_*)$ is

$$\begin{aligned} R_R(r; v_*) := & \text{sgn}(v_*) \frac{1}{(1/\epsilon^2 - K_*^2)^{3/2}} \left(2M\epsilon \left(\frac{1}{\epsilon^2} - K_*^2\right)^{3/2} \ln(r - 2M) \right. \\ & - 2M \left(\frac{1}{\epsilon^2} - K_*^2\right)^{3/2} \ln \left(\frac{2r}{\epsilon} \sqrt{\frac{1}{\epsilon^2} - K_*^2 \left(1 - \frac{2M}{r}\right)} + (2M - r)K_*^2 \right) \\ & + \frac{1}{\epsilon} \left(r \sqrt{\frac{1}{\epsilon^2} - K_*^2} \sqrt{\frac{1}{\epsilon^2} - K_*^{L^2} \left(1 - \frac{2M}{r}\right)} \right. \\ & \left. \left. + M(2/\epsilon^2 - 3K_*^2) \ln \left(r \sqrt{\frac{1}{\epsilon^2} - K_*^2} \sqrt{\frac{1}{\epsilon^2} - K_*^2 \left(1 - \frac{2M}{r}\right)} + (M - r)K_*^2 + \frac{r}{\epsilon^2} \right) \right) \right) \end{aligned} \quad (2.4.3)$$

with $K_*^2 = \frac{1/\epsilon^2 - v_*^2}{1 - 2M/r}$. The function $\sigma = \sigma(t)$ is given similarly with now

$$R_S(\sigma(t); v_L, v_R) - R_S(r_0; v_L, v_R) = t,$$

where $R_S = R_S(r; v_*, v_{**})$ is given as

$$\begin{aligned}
 R_S(r; v_*, v_{**}) = & \operatorname{sgn}(v_* + v_{**}) \frac{2}{K_{**}^2 - K_*^2} \\
 & \left(\frac{r - 4M}{r - 2M} \left(\sqrt{\frac{1}{\epsilon^2} - K_*^2 \left(1 - \frac{2M}{r}\right)} - \sqrt{\frac{1}{\epsilon^2} - K_{**}^2 \left(1 - \frac{2M}{r}\right)} \right) \right. \\
 & - M \left(\frac{4}{\epsilon^2} - 3K_{**}^2 \right) \left(\ln \left(r \sqrt{\frac{1}{\epsilon^2} - K_*^2} \sqrt{\frac{1}{\epsilon^2} - K_{**}^2 \left(1 - \frac{2M}{r}\right)} + (M - r)K_{**}^2 + \frac{r}{\epsilon^2} \right) \right. \\
 & + \ln \left(\frac{2r}{(r - 2M)\epsilon} \sqrt{\frac{1}{\epsilon^2} - K_{**}^2 \left(1 - \frac{2M}{r}\right)} - K_{**}^2 \right) \\
 & + M \left(\frac{4}{\epsilon^2} - 3K_*^2 \right) \left(\ln \left(r \sqrt{\frac{1}{\epsilon^2} - K_*^2} \sqrt{\frac{1}{\epsilon^2} - K_*^2 \left(1 - \frac{2M}{r}\right)} + (M - r)K_*^2 + \frac{r}{\epsilon^2} \right) \right. \\
 & \left. \left. - \ln \left(\frac{2r}{(r - 2M)\epsilon} \sqrt{\frac{1}{\epsilon^2} - K_*^2 \left(1 - \frac{2M}{r}\right)} - K_*^2 \right) \right) \right)
 \end{aligned} \tag{2.4.4}$$

with $K_*^2 = \frac{1/\epsilon^2 - v_*^2}{1 - 2M/r}$ and $K_{**}^2 = \frac{1/\epsilon^2 - v_{**}^2}{1 - 2M/r}$.

We call $(\sigma_-(t), \sigma_+(t))$ the *rarefaction region*, which of course is empty if $\sigma_-(t) = \sigma_+(t) = \sigma(t)$.

Lemma 2.4.1 (The rarefaction region). *The curves $\sigma_+ = \sigma_+(t), \sigma_- = \sigma_-(t)$ are uniquely defined and their derivatives $\sigma'_+(t), \sigma'_-(t)$ are bounded for all $t > 0$. Moreover, we have the inequality $\sigma_-(t) \leq \sigma_+(t)$ where the equality holds if and only if $v_L^0 > v_R^0$.*

Proof. By the definition of R_R in (2.4.3),

$$\partial_r R_R = \operatorname{sgn}(v_L) \left(1 - \frac{2M}{r}\right) \left(\frac{1}{\epsilon^2} - K_*^2 \left(1 - \frac{2M}{r}\right)\right),$$

which does not change signs in the domain of the definition of left steady state v_L . Hence, the monotonicity of R_R admits a unique function $\sigma_L = \sigma_L(t)$ for all $t > 0$. Recall the characteristic $\lambda = (1 - 2M/r)v$ vanishes at the horizon $r = 2M$, hence both $\sigma_+(t), \sigma_-(t) > 2M$ for all $t > 0$. Similar calculations lead to the unique definition of $\sigma = \sigma(t)$ and $\sigma_R = \sigma_R(t)$. Hence, $\sigma_+ = \sigma_+(t)$ and $\sigma_- = \sigma_-(t)$ are well-defined for $t \geq 0$.

Moreover, $\sigma_L, \sigma_R, \sigma$ are integrate curves of the following ordinary differential equations, respectively:

$$\frac{dr}{dt} = \left(1 - \frac{2M}{r}\right)v_L(r), \quad \frac{dr}{dt} = \left(1 - \frac{2M}{r}\right)\frac{v_L(r) + v_R(r)}{2}, \quad \frac{dr}{dt} = \left(1 - \frac{2M}{r}\right)v_R(r),$$

which gives that $|\sigma'_+|, |\sigma'_-|, |\sigma'| \leq \frac{1}{\epsilon}$. Following from the definition of the wave curves, $\sigma_- = \sigma_+ = \sigma$ if and if $v_R^0 < v_L^0$ holds. On the other hand, if $v_R^0 > v_L^0$, we would like to prove that $\sigma_+(t) > \sigma_-(t)$ holds for all $t > 0$. If not, we suppose that there exists a time $t_* > 0$ such that $\sigma_+(t_*) = \sigma_-(t_*)$ and $\sigma_+(t) > \sigma_-(t)$ for all $0 < t < t_*$. Then it is necessary that $\sigma'_+(t_*) \leq \sigma'_-(t_*)$. However, we have

$$\left(1 - \frac{2M}{\sigma_+(t_*)}\right)v_R(\sigma_+(t_*)) > \left(1 - \frac{2M}{\sigma_-(t_*)}\right)v_L(\sigma_-(t_*)),$$

which provides a contradiction. \square

The solution to the Riemann problem

We now give the solution $v = v(t, r)$ to the generalized Riemann problem of the relativistic Burgers model (2.1.5), (4.4.1):

$$v(t, r) = \begin{cases} v_L(r) & \bar{r} < r < \sigma_-(t), \\ w(t, r) & \sigma_-(t) < r < \sigma_+(t), \\ v_R(r) & \sigma_+(t) < r < \hat{r}, \end{cases} \quad (2.4.5)$$

with $w = w(t, r)$ defined as

$$w(t, r) = \operatorname{sgn}(r - r_0) \sqrt{\frac{1}{\epsilon^2} - K^2(t, r) \left(1 - \frac{2M}{r}\right)}, \quad (2.4.6)$$

where $K = K(t, r) > 0$ satisfies

$$\operatorname{sgn}(r - r_0) \frac{R(r, K) - R(r_0, K)}{t} = 1, \quad (2.4.7)$$

with the function $R = R(r, K)$ given by

$$\begin{aligned} R(r, K) := & \frac{1}{(1/\epsilon^2 - K^2)^{3/2}} \left(2M\epsilon \left(\frac{1}{\epsilon^2} - K^2\right)^{3/2} \ln(r - 2M) \right. \\ & - 2M(1/\epsilon^2 - K^2)^{3/2} \ln\left(\frac{2r}{\epsilon} \sqrt{\frac{1}{\epsilon^2} - K^2} \left(1 - \frac{2M}{r}\right) + (2M - r)K^2\right) \\ & + \frac{1}{\epsilon} \left(r \sqrt{\frac{1}{\epsilon^2} - K^2} \sqrt{\frac{1}{\epsilon^2} - K^2} \left(1 - \frac{2M}{r}\right) \right. \\ & \left. \left. + M(2/\epsilon^2 - 3K^2) \ln\left(r \sqrt{\frac{1}{\epsilon} - K^2} \sqrt{\frac{1}{\epsilon^2} - K^2} \left(1 - \frac{2M}{r}\right) + (M - r)K^2 + \frac{r}{\epsilon^2}\right) \right) \right). \end{aligned} \quad (2.4.8)$$

In addition, we regulate that $K(t, r_0) = \frac{1}{\epsilon} \sqrt{1 - \frac{2M}{r}}$ if $r_0 \in (\sigma_-(r), \sigma_+(t))$.

In the sense of Section 2.3, definitions of steady state solution of the Burgers equation stops at the vanishing velocity radius (2.3.10). Remark 2.3.4 gives that the right-hand steady state solution v_R will never vanishes in the domain of definition. We will then see in the following lemma that the left steady state solution v_L cannot vanish for all $r < \sigma_-(t)$.

Lemma 2.4.2 (Well-defined steady states). *Let $\sigma_- = \sigma_-(t)$ be the lower bound of the rarefaction region given in (2.4.2), then the value of $v_L = v_L(r)$ is nonzero for all $r < \sigma_-(t)$ for all $t \geq 0$.*

Proof. The result always holds if $\sigma_-(t) \leq r_0$, following from Remark 2.3.4. On the other hand, we now suppose that $\sigma_-(t) > r_0$, or equivalently, the speed of the propagation $\sigma'_-(t) \geq 0$. Denote by r_L^\natural the vanishing velocity radius for v_L defined by (2.3.10). Two main cases are to be taken into consideration:

- For the case where $v_L^0 \pm v_R^0 > 0$, it is necessary that $K_*^L < K_*^R$ with K_*^L, K_*^R the constants given in (3.5.8). Hence, $\hat{r} > \sigma(t)$.
- For the case where $0 < v_L^0 < v_R^0$, if the result does not hold at some time $0 < t_1 < +\infty$, we should have $v_L(\sigma_-(t_1)) = 0$. At the same time, since $\sigma'_-(t_1) = (1 - 2M/\sigma_-(t))v_L(\sigma_-(t)) = 0$, the curve σ_- cannot reach the point $\sigma_-(t_1)$, which provides a contradiction.

In summary, the left steady state solution $v_L = v_L(r)$ will never go to zero on the interval $(2M, \sigma_-(t))$. □

Rarefaction waves

We now turn to the analysis of the *generalized rarefaction curve* defined by (3.5.7).

Proposition 2.4.3 (The generalized rarefaction curve). *The rarefaction curve $w = w(t, r)$ given by (3.5.7) is well-defined, satisfying the relativistic Burgers equation (2.1.5) in the rarefaction region and it is continuous with respect to t for all $t > 0$ and to r for all $r \in (\sigma_-(t), (\sigma_-(t)))$. Moreover, the following properties hold:*

- *The wave $w = w(t, r)$ is increasing with respect to the space variable $r > 2M$ in the rarefaction region $(\sigma_-(t), \sigma_+(t))$.*
- *For a small enough time, the generalized rarefaction curve has the similar structure as the standard one: $\lim_{t \rightarrow 0} w(t, r) = \frac{1}{1-2M/r_0} \frac{r-r_0}{t}$.*
- *When the black hole mass M vanishes, $w = w(t, r)$ tends to a standard rarefaction, that is, $\lim_{M \rightarrow 0} w(t, r) = \frac{r-r_0}{t}$.*

Proof. To prove that w is well-defined, we first have to show that the value $K = K(t, r) > 0$ is uniquely determined by (2.4.7). Consider the function $G = G(r, t, K)$:

$$G(r, t, K) := \operatorname{sgn}(r - r_0)(R(r, K) - R(r_0, K)) - t,$$

and we see immediately that $G(\sigma_-(t), t, K_*^L) = G(\sigma_+(t), t, K_*^R) = 0$. Moreover, we have

$$\partial_L G = \operatorname{sgn}(r - r_0)(\bar{G}(r, K) - \bar{G}(r_0, K)),$$

where we have set $L := K^2 > 0$ and the function \bar{G} is given as

$$\begin{aligned} \bar{G}(r, K) := & \left(\frac{r \sqrt{\frac{1}{\epsilon^2} - K^2 \left(1 - \frac{2M}{r}\right)} (6MK^2 + (\frac{1}{\epsilon^2} - K^2)r)}{(\frac{1}{\epsilon^2} - K^2)^2 ((2M - r)K^2 + \frac{1}{\epsilon^2}r)} \right. \\ & \left. - \frac{6MK^2 \ln(2r \sqrt{\frac{1}{\epsilon^2} - K^2} \sqrt{\frac{1}{\epsilon^2} - K^2 \left(1 - \frac{2M}{r}\right)} + 2(r - 2M)K^2 + \frac{2}{\epsilon^2}r)}{2(\frac{1}{\epsilon^2} - K^2)^{\frac{5}{2}}} \right). \end{aligned} \quad (2.4.9)$$

We then have

$$\partial_r(\partial_L G) = \frac{1}{2} \operatorname{sgn}(r - r_0) \left(\frac{1}{\epsilon^2} - K^2 \left(1 - \frac{2M}{r}\right) \right)^{-3/2},$$

and we thus have $\partial_L G > 0$ for all $r \neq r_0, L \geq 0$. Therefore, we can always have the unique value of L and thus its unique positive root K . It is obvious that K is continuous with respect to t for all $t > 0$. We now show that it is continuous with respect to r in the rarefaction region. Indeed, if the sign of $r - r_0$ does not change, the solutions is always smooth with respect to r . Therefore, we only have to look at the value at r_0 when $r_0 \in (\sigma_-(t), \sigma_+(t))$. Indeed, by the definition of the function K (2.4.7), (3.5.9), for all fixed $t > 0$, it is necessary that $1/\epsilon^2 - K^2(1 - 2M/r) = O((r - r_0)^2)$ and we then have

$$K(t, r_{0+}) = K(t, r_{0-}) = \frac{1}{\epsilon} \sqrt{1 - \frac{2M}{r_0}} = K(t, r_0).$$

Consider the definition of w , we still have to show that the value under the symbol for the square root (3.5.7) stays positive. Indeed, since we already have $K(t, \sigma_-(t)) = K_*^L, K(t, \sigma_+(t)) = K_*^R, K(t, r_0) = 1/\epsilon(1 - 2M/r_0)^{-1/2}$ and K^2 is monotone with respect to r at both side of r_0 , then the result is direct.

Now we prove that $w = w(t, r)$ satisfies the Burgers equation (2.1.5) in the rar-

efaction region $(\sigma_-(t), \sigma_+(t))$. We have

$$\begin{aligned} & \partial_t \left(\frac{w}{(1 - 2M/r)^2} \right) + \partial_r \left(\frac{w^2 - 1/\epsilon^2}{(1 - 2M/r)^2} \right) \\ &= \frac{1}{(1 - 2M/r)^2} \left(\partial_t w + \left(1 - \frac{2M}{r}\right) w \partial_r w - \frac{M}{r^2} \left(w^2 - \frac{1}{\epsilon^2}\right) \right) \\ &= \frac{1}{(1 - 2M/r)^2} \left(- \left(1 - \frac{2M}{r}\right) \frac{\partial_t L}{w} - \left(1 - \frac{2M}{r}\right)^2 \partial_r L \right). \end{aligned}$$

Furthermore, the definition of G gives

$$\partial_t G = -1, \quad \partial_r G = \frac{\operatorname{sgn}(r - r_0)}{\left(1 - \frac{2M}{r}\right) \sqrt{\frac{1}{\epsilon^2} - K^2 \left(1 - \frac{2M}{r}\right)}},$$

from which we get $\frac{\partial_t L}{\partial_r L} = -\left(1 - \frac{2M}{r}\right)w$. Hence, w satisfies (2.1.5). Furthermore, by taking $t \rightarrow 0$, we have $\lim_{t \rightarrow 0} \partial_r G(r, K)(r - r_0) - t = 0$, which gives

$$\begin{aligned} \lim_{t \rightarrow 0} w(t, r) &= \lim_{t \rightarrow 0} \operatorname{sgn}(r - r_0) \sqrt{\frac{1}{\epsilon^2} - K^2 \left(1 - \frac{2M}{r}\right)} \\ &= \frac{1}{\partial_r G(r, K)(1 - 2M/r_0)} = \frac{1}{1 - 2M/r_0} \frac{r - r_0}{t}. \end{aligned}$$

Letting the black hole mass $M \rightarrow 0$ in (3.5.9), we have $\frac{r-r_0}{t} = \operatorname{sgn}(r-r_0) \sqrt{\frac{1}{\epsilon^2} - K(t, r)^2}$. Therefore, together with the definition given by (3.5.7), we have $w(t, r) = \frac{r-r_0}{t}$.

Now we consider the monotonicity of $w(t, \cdot)$. Derive (3.5.7) with respect to r and we have

$$\partial_r w = \operatorname{sgn}'(r - r_0) \left(\frac{1}{\epsilon^2} - K^2 \left(1 - \frac{2M}{r}\right) \right)^{1/2} + \frac{1}{w^2} \left(- \frac{K^2 M}{r^2} w + \int_{r_0}^r \frac{1}{w^{-3}} dr \right) > 0,$$

where we have referred to the calculation given by (2.4.9). Hence, $w(t, \cdot)$ is increasing in the rarefaction region. This completes the proof of the proposition. \square

Our main result for rarefaction waves is as follows.

Proposition 2.4.4 (The global-in-time construction for rarefaction waves). *For two given steady state solutions $v_L = v_L(r)$ and $v_R = v_R(r)$ (which might not be defined on the whole interval $(2M, +\infty)$) separated by a discontinuity at r_0 satisfying $v_L^0 < v_R^0$ where $v_L^0 = v_L(r_0), v_R^0 = v_R(r_0)$, the generalized Riemann problem of the Burgers model (2.1.5) is realized by a rarefaction wave for all $t > 0$. Moreover,*

- The lower and upper bounds of the rarefaction curve tend to the horizon of the black hole $r = 2M$ if and only if the $v_R^0 < 0$;
- The lower and upper bounds of the rarefaction curve tend to infinity $r = +\infty$ if and only $v_L^0 > 0$;
- The lower bound of the rarefaction curve tends to the horizon of the black hole $r = 2M$ while its upper bound tends to the horizon of the black hole $r = +\infty$ if and only if $v_R^0 > 0 > v_L^0$;
- The lower/upper bound of the rarefaction curve stays at the vanishing velocity radius r_L^h/r_R^h if only if $v_L^0 = 0/v_R^0 = 0$.

Shock waves

To begin with the analysis of the shock waves, we recall that the *Rankine-Hugoniot condition* of the relativistic Burgers equation on the Schwarzschild spacetime (3.1.2) requires

$$s[v] = (1 - 2M/r) \left[\frac{v^2}{2} \right], \quad (2.4.10)$$

where $s = s(t, r)$ stands for the speed of the discontinuity and the bracket $[\cdot]$ denotes the value of the jump. We hence give the equation

$$s = \left(1 - \frac{2M}{r} \right) \frac{v_L + v_R}{2}. \quad (2.4.11)$$

To select solutions which do have a physical sense, we shall now recall the *Lax entropy condition*. In particular, for the relativistic Burgers equation (2.1.5) with two steady state solutions $v_L = v_L(r)$, $v_R = v_R(r)$, we only allow for a curve of discontinuity in our solution $v = v(t, r)$ if the wave to the left is moving faster than the wave to the right. That is, we only allow for a curve of discontinuity between v_L and v_R if the following inequality holds

$$(1 - 2M/\sigma(t))v_L(\sigma(t)) > \sigma'(t) > (1 - 2M/\sigma(t))v_R(\sigma(t)). \quad (2.4.12)$$

Proposition 2.4.5 (Shock waves). *Consider the Riemann problem of the relativistic Burgers model (2.1.5), (4.4.1). If (v_L, v_R) is shock wave, it satisfies the Rankine-Hugoniot condition (4.2.8) and the entropy condition (2.4.12) with the following properties:*

- For all $t > 0$, the wave speed $\sigma'(t)$ does not change signs.
- When and only when $v_L^0 = -v_R^0 > 0$, (v_L, v_R) is a steady state shock wave who has a zero speed of propagation.

Proof. According to the definition, we have $\sigma'(t) = \left(1 - \frac{2M}{\sigma(t)}\right) \frac{v_L(\sigma(t)) + v_R(\sigma(t))}{2}$, which satisfies the Rankine-Hugoniot condition of the Burgers equation (2.1.5). On the other hand, since $v_L + v_R$ does not change signs according to Proposition 2.3.5, the shock speed keeps the same sign as well. Now consider the steady state shock. In the case where $\partial_t v \equiv 0$ the Rankine-Hugoniot condition of the Burgers model (2.4.10) reduces to

$$\left(1 - \frac{2M}{r}\right) \left[\frac{v^2}{2}\right] = 0. \quad (2.4.13)$$

Together with the entropy condition (2.4.12), it is necessary and sufficient that $v_L = -v_R > 0$ holds. \square

We summarize our main result for shock waves.

Proposition 2.4.6 (The global-in-time construction for shock waves). *For two given steady state solutions $v_L = v_L(r)$ and $v_R = v_R(r)$ (which might not be defined on the whole interval $(2M, +\infty)$) separated by a discontinuity at r_0 . If $v_L^0 > v_R^0$ with $v_L^0 = v_L(r_0), v_R^0 = v_R(r_0)$, the generalized Riemann problem is realized by a shock wave for all $t > 0$. Moreover,*

- *The shock curve tends to the horizon of the black hole $r = 2M$ if and only if $v_L^0 + v_R^0 < 0$;*
- *The shock curve tends to infinity $r = +\infty$ if and only if $v_L^0 + v_R^0 > 0$;*
- *The position of the shock curve is at $r = r_0$ for all $t > 0$ if and only if $v_L^0 + v_R^0 = 0$.*

By combining Propositions 2.4.4 and 2.4.6, we thus have proven Theorem 2.1.1.

2.5 Total variation functionals

Evolution of the total variation

We now consider the evolution of the total variation $TV_{\hat{r}}^{\hat{r}}(v)$ of a solution v to the generalized Riemann problem on an interval (\hat{r}, \hat{r}) and, in particular, we seek to control $TV(v) := TV_{2M}^{+\infty}(v)$. Since the initial condition may lead to different types of solutions, we have to analyze each possible cases (shock and rarefaction waves with different speeds of propagation). We first consider all the cases where the total variation is conserved.

Lemma 2.5.1 (Solutions with constant total variation). *Let $v_0 = v_0(r)$ be the initial condition (4.4.1) and let $v = v(t, r)$ be the corresponding solution to the relativistic*

Burgers model (2.1.5). Then the total variation satisfies

$$TV_{\hat{r}}^{\hat{r}}(v(t, \cdot)) = TV_{\hat{r}}^{\hat{r}}(v_0), \quad t \geq 0$$

if and only if one of the following conditions hold:

- $0 < v_R^0 < v_L^0$,
- $v_L^0 < v_R^0 < 0$,
- $v_L^0 = -v_R^0 > 0$.

Proof. For the first two cases listed in the lemma, solutions are realized by a shock wave with a positive shock speed and a rarefaction wave with a negative speed, respectively. The monotony of such solutions on (\bar{r}, \hat{r}) will never change. For the third case in the lemma, we have got a steady state shock. It is direct to check that the total variation stays a constant for all these three cases. \square

There are also several possibilities of initial data leading to the decreasing total variation.

Lemma 2.5.2 (Solutions with decreasing total variation). *Let $v_0 = v_0(r)$ be a given initial condition with two steady states v_L, v_R satisfying one of the following conditions:*

- $0 < v_L^0 < v_R^0$,
- $v_R^0 < v_L^0 < 0$,
- $v_L^0 < 0 < v_R^0$,
- $v_R^0 < 0 < v_L^0$ and $v_L^0 + v_R^0 > 0$,

then the total variation of the solution $v = v(t, r)$ of the relativistic Burgers equation (2.1.5) satisfies

$$TV_{\hat{r}}^{\hat{r}}(v(t, \cdot)) < TV_{\hat{r}}^{\hat{r}}(v_0), \quad t \geq 0.$$

Proof. If $0 < v_L^0 < v_R^0$, the solution is realized by a generalized rarefaction with a positive wave speed. By the construction in (4.4.6), we have

$$\begin{aligned} TV_{\hat{r}}^{\hat{r}}(v(t, \cdot)) &= \bar{v}_L + 2\left(v_R(\sigma_+(t)) - v_L(\sigma_-(t))\right) - \hat{v}_R, \\ TV_{\hat{r}}^{\hat{r}}(v_0) &= \bar{v}_L + 2(v_R^0 - v_L^0) - \hat{v}_R, \end{aligned}$$

hence

$$\begin{aligned} TV_{\hat{r}}^{\hat{r}}(v(t, \cdot)) - TV_{\hat{r}}^{\hat{r}}(v_0) &= 2 \left((v_R(\sigma_+(t)) - v_L(\sigma_+(t)) - v_R(r_0) + v_L(r_0)) \right. \\ &\quad \left. = 2 \left((v_R(\sigma_+(t)) - v_R^0 + v_L(\sigma_+(t)) - v_L(\sigma_+(t)) + v_L^0 - v_L(\sigma_-(t))) \right) \leq 0. \end{aligned}$$

Therefore, $TV(v(t, \cdot))$ is decreasing in this case.

Next, if $v_R^0 < v_L^0 < 0$, the solution is realized by a shock wave with a negative speed $\sigma'(t) < 0$. From to Proposition 2.3.5, we obtain

$$\begin{aligned} TV_{\hat{r}}^{\hat{r}}(v(t, \cdot)) &= \overline{v_L} + 2 \left(v_L(\sigma(t)) - v_R(\sigma(t)) \right) - \widehat{v}_R \\ &\leq \frac{1}{\epsilon} + 2(v_L^0 - v_R^0) - \widehat{v}_R = TV_{\hat{r}}^{\hat{r}}(v_0). \end{aligned}$$

If $v_L^0 < 0 < v_R^0$, the solution is given by a rarefaction wave with the two rarefaction bounds $\sigma_-(t) < r_0 < \sigma_+(t)$. Hence,

$$TV_{\hat{r}}^{\hat{r}}(v(t, \cdot)) = \overline{v_L} + 2v_R(\sigma_+(t)) - \widehat{v}_R \leq \frac{1}{\epsilon} + 2v_R(r_0) - \widehat{v}_R = TV_{\hat{r}}^{\hat{r}}(v_0).$$

Finally, if $v_R^0 < 0 < v_L^0$ and $v_L^0 + v_R^0 > 0$, then the two steady states will be separated by a shock curve with a positive shock speed, or equivalently, $\sigma(t) > r_0$. Therefore, we have

$$TV_{\hat{r}}^{\hat{r}}(v(t, \cdot)) = \overline{v_L} - 2v_R(\sigma(t)) - \widehat{v}_R \leq \frac{1}{\epsilon} - 2v_R(r_0) - \widehat{v}_R = TV_{\hat{r}}^{\hat{r}}(v_0). \quad \square$$

The following lemma discusses the remaining case, in which the total variation increases.

Lemma 2.5.3 (Solutions with increasing total variation). *If the initial data $v_0 = v_0(r)$ given in (4.4.1) satisfies $v_R^0 < 0 < v_L^0$ and $v_L^0 + v_R^0 < 0$, the total variation of the solution $v = v(t, r)$ satisfies*

$$TV_{\hat{r}}^{\hat{r}}(v_0) \leq TV_{\hat{r}}^{\hat{r}}(v(t, \cdot)) \leq \min \left(\frac{3}{\epsilon}, TV(v_0) \left(1 + \frac{t}{4M\epsilon} \right) \right), \quad t \geq 0.$$

Proof. We have a shock wave with a negative speed and the total variation is

$$TV_{\hat{r}}^{\hat{r}}(v(t, \cdot)) = \overline{v_L} - 2v_R(\sigma(t)) + \widehat{v}_R.$$

Recalling the expression of steady state solutions, we write

$$v_L(r) = \sqrt{\frac{1}{\epsilon^2} - K_*^{L^2} \left(1 - \frac{2M}{r} \right)}, \quad v_R(r) = -\sqrt{\frac{1}{\epsilon^2} - K_*^{R^2} \left(1 - \frac{2M}{r} \right)},$$

where $K_*^L > K_*^R$. Therefore, we have

$$\begin{aligned} \frac{d}{dt} TV_{\hat{r}}(v(t, \cdot)) &= -2v'_R(\sigma(t))\sigma'(t) = -\left(1 - \frac{2M}{\sigma(t)}\right)v'_R(v_L + v_R) \\ &= \left(1 - \frac{2M}{\sigma(t)}\right) \frac{K_*^{R2} M}{\sigma(t)^2} \frac{v_L + v_R}{v_R} = \left(\frac{1}{\epsilon^2} - v_R^2\right) \frac{M}{\sigma(t)^2} \frac{v_L + v_R}{v_R} \\ &\leq M \frac{1/\epsilon^2 - v_R^2}{\sigma(t)^2} = M \frac{(1/\epsilon - v_R)(1/\epsilon + v_R)}{\sigma(t)^2} \leq \frac{TV_{\hat{r}}(v_0)}{4M\epsilon} \end{aligned}$$

and it remains to integrate in time. \square

We summarize our results so far.

Theorem 2.5.4 (Existence theory for the generalized Riemann problem). Consider the generalized Riemann problem of the relativistic Burgers equation on the Schwarzschild spacetime (2.1.5) with initial data (4.4.1) of the form

$$v_0 = v_0(r) = \begin{cases} v_L(r) & r < r_0, \\ v_R(r) & r > r_0, \end{cases}$$

where $v_L = v_L(r)$, $v_R = v_R(r)$ are steady state solutions to (2.3.1) and $r_0 \in (2M, +\infty)$. Then there exists a solution $v = v(t, r)$ of the generalized Riemann problem, defined for all $t \geq 0$ and satisfying the entropy condition. More precisely, there are three different regimes:

- *Solutions with constant total variation.* If one of the following conditions holds:
 - $0 < v_R^0 < v_L^0$,
 - $v_L^0 < v_R^0 < 0$,
 - $v_L^0 = -v_R^0 > 0$,

then the total variation of the solution $v = v(t, r)$ stays constant, that is,

$$TV_{\hat{r}}(v(t, \cdot)) \equiv TV_{\hat{r}}(v_0), \quad t \geq 0.$$

- *Solutions with decreasing total variation.* If one of the following conditions holds:
 - $0 < v_L^0 < v_R^0$,
 - $v_R^0 < v_L^0 < 0$,
 - $v_L^0 < 0 < v_R^0$,
 - $v_R^0 < 0 < v_L^0$ and $v_L^0 + v_R^0 > 0$,

then the total variation of the solution $v = v(t, r)$ is decreasing, that is

$$TV_{\hat{r}}(v(t, \cdot)) < TV_{\hat{r}}(v_0), \quad t \geq 0.$$

- *Solutions with increasing total variation.* If the piecewise steady state initial data $v_0 = v_0(r)$ satisfies

$$v_R^0 < 0 < v_L^0, \quad v_L^0 + v_R^0 < 0,$$

then the total variation of the solution $v = v(t, r)$ is increasing and grows at most linearly in time, that is,

$$TV_{\hat{r}}(v_0) \leq TV_{\hat{r}}(v(t, \cdot)) \leq \min\left(\frac{3}{\epsilon}, TV(v_0)\left(1 + \frac{t}{4M\epsilon}\right)\right), \quad t \geq 0.$$

Weighed total variation functional

In view of Theorem 2.5.4, the total variation of solutions to the relativistic Burgers equation may increase. This motivate us to introduce the following *weighted total variation* of a function v over an interval $(\hat{r}, \bar{r}) \subset (2M, +\infty)$ by

$$\widetilde{TV}_{\hat{r}}^{\bar{r}}(v) := \int_{\hat{r}}^{\bar{r}} \left| \partial_r \left(\left| \frac{v^2 - 1/\epsilon^2}{1 - 2M/r} + \frac{1}{\epsilon^2} \right|^{1/2} \operatorname{sgn}(v) \right) \right|, \quad (2.5.1)$$

where the integral is regarded as a measure. In particular, we set $\widetilde{TV}_{2M}^{+\infty}(v) = \widetilde{TV}(v)$. Obviously, when the black hole mass vanishes, that is, $M \rightarrow 0$, we recover the standard definition, as $\widetilde{TV}(v)$ reduces to $TV(v)$. We now show that the weighted total variation remains constant for generalized Riemann solutions.

Theorem 2.5.5 (Weighed total variation for the general Riemann problem). Consider the generalized Riemann problem of the relativistic Burgers equation (2.1.5) with the initial velocity (4.4.1) given as

$$v_0 = v_0(r) = \begin{cases} v_L(r) & r < r_0, \\ v_R(r) & r > r_0, \end{cases}$$

where v_L, v_R are two steady state solutions of the static Burgers model (2.3.1). Then the weighted total variation of the solution $\widetilde{TV}_{\hat{r}}^{\bar{r}}(v)$ defined by (2.5.1) is constant: $\widetilde{TV}_{\hat{r}}^{\bar{r}}(v(t, \cdot)) \equiv \widetilde{TV}_{\hat{r}}^{\bar{r}}(v_0)$.

Proof. If (v_L, v_R) is a shock wave, the discontinuity gives

$$\widetilde{TV}_{\bar{r}}^{\hat{r}}(v(t, \cdot)) = \left| \left(\frac{v_L(\sigma(t))^2 - 1/\epsilon^2}{1 - 2M/\sigma(t)} + \frac{1}{\epsilon^2} \right)^{1/2} \operatorname{sgn}(v_L) - \left(\frac{v_R(\sigma(t))^2 - 1/\epsilon^2}{1 - 2M/\sigma(t)} + \frac{1}{\epsilon^2} \right)^{1/2} \operatorname{sgn}(v_R) \right|.$$

Since both v_L, v_R are steady state solution determined by the static Burgers model (2.3.1), we recall that

$$\frac{1/\epsilon^2 - v_L(r)^2}{1 - 2M/r} = K_*^L, \quad \frac{1/\epsilon^2 - v_R(r)^2}{1 - 2M/r} = K_*^R$$

where K_*^L, K_*^R are constants. Therefore, we see at once that $\widetilde{TV}(v(t, \cdot))$ is a constant.

If (v_L, v_R) is a rarefaction wave, denote by $w = w(t, r)$ the rarefaction wave and $\sigma_{\pm} = \sigma_{\pm}(t)$ the bounds of the rarefaction regions. We have

$$\begin{aligned} \widetilde{TV}_{\bar{r}}^{\hat{r}}(v(t, \cdot)) &= \int_{\sigma_-(t)}^{\sigma_+(t)} \left| \partial_r \left(\left(\frac{w^2 - 1/\epsilon^2}{1 - 2M/r} + \frac{1}{\epsilon^2} \right)^{1/2} \operatorname{sgn}(w) \right) \right| dr \\ &= \int_{\sigma_-(t)}^{\sigma_+(t)} \left| \partial_r \left(\left(\frac{w^2 - 1/\epsilon^2}{1 - 2M/r} + \frac{1}{\epsilon^2} \right)^{1/2} \operatorname{sgn}(w) \right) \right| dr \\ &= \left| \left(\frac{w(\sigma_+(t))^2 - 1/\epsilon^2}{1 - 2M/\sigma_+(t)} + \frac{1}{\epsilon^2} \right)^{1/2} \operatorname{sgn}(w(\sigma_+(t))) \right. \\ &\quad \left. - \left(\frac{w(\sigma_-(t))^2 - 1/\epsilon^2}{1 - 2M/\sigma_-(t)} + \frac{1}{\epsilon^2} \right)^{1/2} \operatorname{sgn}(w(\sigma_-(t))) \right| \\ &= \left| \left(\frac{v_L(\sigma_-(t))^2 - 1/\epsilon^2}{1 - 2M/\sigma_-(t)} + \frac{1}{\epsilon^2} \right)^{1/2} \operatorname{sgn}(v_L(\sigma_-(t))) \right. \\ &\quad \left. - \left(\frac{v_R(\sigma_+(t))^2 - 1/\epsilon^2}{1 - 2M/\sigma_+(t)} + \frac{1}{\epsilon^2} \right)^{1/2} \operatorname{sgn}(v_R(\sigma_+(t))) \right|, \end{aligned}$$

which is a constant since v_L, v_R are both steady state solutions. \square

2.6 The multiple generalized Riemann problem

Formulation of the problem

From to the results given by Section 2.3, a steady state solution may not be defined globally if its value can reach zero in the domain of definition. In this sense, the result of a generalized Riemann problem with only two initial steady states may not be sufficient to introduce the generalized Glimm method which captures the behaviors of all kinds of steady state solutions. We are therefore motivated to consider the *multiple Riemann problem* of the relativistic Burgers equation (2.1.5) whose initial

velocity $v_0 = v_0(r)$ is given as three steady state solutions:

$$v_0(r) = \begin{cases} v_\alpha(r) & \bar{r} < r < r_0, \\ v_\beta(r) & r_0 < r < r_1, \\ v_\gamma(r) & r_1 < r < \hat{r}, \end{cases} \quad (2.6.1)$$

where $\bar{r} < r_0 < r_1 < \hat{r}$ are given in the interval $(2M, +\infty)$ and $v_\alpha, v_\beta, v_\gamma$ are steady states satisfying (2.3.1) on corresponding domains. We denote the values at the point r_0, r_1 by $v_\alpha(r_0) = v_\alpha^0, v_\beta(r_0) = v_\beta^0, v_\beta(r_1) = v_\beta^1, v_\gamma(r_1) = v_\gamma^1$. For the later use of the Glimm method, we suppose that the steady state solution v_β is a non-global steady state solution with a zero value at $r = r_1$, that is, $v_\beta^1 = 0$.

The main result of this section is as follows.

Theorem 2.6.1 (Global existence of multiple Riemann problem). Consider the multiple Riemann problem of the relativistic Burgers equation on a Schwarzschild background (2.1.5), (2.6.1) where the initial velocity $v_0 = v_0(r)$ is a piecewise steady state solution with three steady states $v_\alpha = v_\alpha(r), v_\beta = v_\beta(r), v_\gamma = v_\gamma(r)$ separated by two jumps of discontinuity at fixed radius $r = r_0, r = r_1$, then there exists solution to the Burgers equation (2.1.5), say $v = v(t, r)$ defined for all $t > 0$ on (\bar{r}, \hat{r}) and satisfying the initial condition $v(t, \cdot) = v_0$. Moreover, for every fixed $t > 0$, the total variation of the solution satisfies $TV_{\bar{r}}^{\hat{r}}(v(t, \cdot)) < \min\left(\frac{3}{\epsilon}, TV_{\bar{r}}^{\hat{r}}(v_0)\left(1 + \frac{t}{4M\epsilon}\right)\right)$ and the weighted total variation given by (2.5.1) is non-increasing for all $t \geq 0$.

We will see later that a multiple Riemann problem with three initial steady states suffices to construct the Glimm method to be introduced in the coming section.

Local existence

Since the Burgers model is hyperbolic outside the Schwarzschild black hole, we can get the solution of (2.1.5), (2.6.1) for a small time before any interaction happened. In view of (2.3.8), we now write the explicit formula of the steady state solutions as

$$\epsilon v_j(r) = \text{sgn}(v_j) \sqrt{1 - \epsilon^2 K_*^{j2} \left(1 - \frac{2M}{r}\right)}, \quad j = \alpha, \beta, \gamma.$$

By definition, the *left-hand half-Riemann problem* defined on (\bar{r}, r_1) is the Riemann problem with initial data

$$v_{0,L}(r) = \begin{cases} v_\alpha(r) & \bar{r} < r < r_0, \\ v_\beta(r) & r_0 < r < r_1, \end{cases} \quad (2.6.2)$$

and the *right-hand half-Riemann problem* defined on (r_0, \hat{r}) with initial data

$$v_{0,R}(r) = \begin{cases} v_\beta(r) & r_0 < r < r_1, \\ v_\gamma(r) & r_1 < r < \hat{r}. \end{cases} \quad (2.6.3)$$

We first treat the left-hand problem.

Lemma 2.6.2 (Solution of left-hand half-Riemann problem). *There exists a solution to the left-hand half-Riemann problem of the Burgers equation denoted by $v_L = v_L(t, r)$ with initial data (2.6.2) on (\bar{r}, r_1) :*

$$v_L(t, r) = \begin{cases} v_\alpha(r) & \bar{r} < r < \sigma_-^L(t), \\ w_L(t, r) & \sigma_-^L(t) < r < \sigma_+^L(t), \\ v_\beta(r) & \sigma_+^L(t) < r < r_1, \end{cases} \quad (2.6.4)$$

where $\sigma_-^L = \sigma_-^L(t)$, $\sigma_+^L = \sigma_+^L(t)$ are lower and upper bounds of the rarefaction region given by (2.4.1) and $w_L = w_L(t, r)$ the rarefaction wave given by the form of (3.5.7).

Proof. We claim that there exists no point in the interval (\bar{r}, r_1) where the steady state solution vanishes. Hence, the solution can be defined in the sense of (4.4.6). In other words, we should discuss the possible position of $r = r_\alpha^\natural$ which is the vanishing velocity radius for the steady state v_α given as (2.3.10). Observe that if $r_\alpha^\natural = +\infty$, then $v_\alpha = v_\alpha(r)$ is a globally defined solution on $(2M, +\infty)$.

Two principle cases are to be taken into consideration:

1. The case where (v_α^0, v_β^0) is a shock wave (with a positive, negative or zero speed).
2. The case where (v_α^0, v_β^0) is a rarefaction wave (whose left state has a positive, negative or zero speed).

We have, in the first case, $\sigma_-^L(t) = \sigma_+^L(t) = \sigma^L(t)$.

If $v_\alpha^0 + v_\beta^0 \leq 0$, we have a shock wave with non-positive speed. Then the result is obvious following from the fact that $\sigma(t) \leq r_0$.

If $v_\alpha^0 + v_\beta^0 > 0$, we have a shock wave with a positive shock speed. we would like to prove that r_α^\natural stays at the left-hand side of r_1 . If not, we suppose $r_\alpha^\natural > r_1$. This requires the inequality:

$$\frac{2MK_*^{\alpha^2}\epsilon^2}{\epsilon^2K_*^{\alpha^2} - 1} < \frac{2MK_*^{\beta^2}\epsilon^2}{\epsilon^2K_*^{\beta^2} - 1},$$

which gives $K_*^\alpha < K_*^\beta$. Hence, we have $v_\alpha - \text{sgn}(v_\beta)v_\beta < 0$ which contradicts our assumption $v_\beta \pm v_\alpha > 0$. Therefore, the steady state solution does not vanish on (\bar{r}, r_1) .

We now turn to the second case where (v_α^0, v_β^0) is a rarefaction wave. We analyze the sign of the left steady state solution v_α .

If $v_\alpha^0 \leq 0$, then $\sigma_-^{L'}(t) \leq 0$ and the lower bound of the rarefaction stays at the left-hand side of the point of discontinuity r_0 , that is, $\sigma_-^L(t) \leq r_0 < r_\alpha^\natural$.

If $v_\alpha^0 > 0$, we first see that $\sigma_-^L(0) = r_0 < r_\alpha^\natural$. Now suppose that there exists a time $t = t_1 > 0$ such that $\sigma_-^L(t_1) = r_\alpha^\natural$ and we thus have $\sigma_-^{L'}(t_1) = (1 - 2M/r_\alpha^\natural)v_\alpha(r_\alpha^\natural) = 0$. This provides a contradiction.

Following (4.4.6), we arrive at the solution to the left-hand half-Riemann problem. □

Now we turn to the right-hand half-Riemann problem with the right-side initial data (2.6.3). The assumption $v_\beta^1 = 0$ actually excludes several cases.

Lemma 2.6.3 (Solution of the right-hand half-Riemann problem). *There exists a solution of the right-hand half-Riemann problem of relativistic Burgers equation (2.1.5) denoted by $v_R = v_R(t, r)$ with initial data (2.6.2) on the interval (\bar{r}, r_1) :*

$$v_R(t, r) = \begin{cases} v_\beta(r) & \bar{r} < r < \sigma_-^R(t), \\ w_R(t, r) & \sigma_-^R(t) < r < \sigma_+^R(t), \\ v_\gamma(r) & \sigma_+^R(t) < r < \hat{r}, \end{cases} \quad (2.6.5)$$

where $\sigma_-^R = \sigma_-^R(t)$, $\sigma_+^R = \sigma_+^R(t)$ denote the lower and upper bounds of the rarefaction region and $w_R = w_R(t, r)$ the generalized rarefaction curve defined by (3.5.7).

Proof. We consider the following two cases:

1. The wave (v_β^1, v_γ^1) is a shock wave.
2. The wave (v_β^1, v_γ^1) is a rarefaction wave.

For the first case, since $v_\beta^1 = 0$, (v_β^1, v_γ^1) is a shock wave when and only when $v_\gamma^1 < 0$. Denote by $\sigma^R = \sigma^R(t)$ the shock wave and we have $\sigma^R(t) < r_1$ for all $t > 0$. Since both steady state solutions v_β, v_γ are defined on $r < r_1$, we have the result.

We now take into consideration the second case. To have a rarefaction wave, it is necessary and sufficient that $v_\gamma^1 > 0$. It is obvious to see that $\sigma_-^R(t) = r_0$ since $v_\beta^1 = 0$. Then we can use (3.5.7) to give the solution on (r_0, \hat{r}) . This completes the proof of the lemma. □

Since the relativistic Burgers equation is hyperbolic for $r > 2M$, no interaction will happen for a small enough time $t > 0$, we can therefore give the local existence of the multiple Riemann problem.

Theorem 2.6.4 (Local existence for the multiple Riemann problem). Consider the initial data $v_0 = v_0(r)$ given as (2.6.1) consisting of three solutions $v_\alpha = v_\alpha(r), v_\beta = v_\beta(r), v_\gamma = v_\gamma(r)$ of the static relativistic Burgers model (2.3.1) separated by two fixed radius $r = r_0, r = r_1$. Then there exists a solution $\tilde{v}_1 = \tilde{v}_1(t, r)$ to the multiple Riemann problem of the relativistic Burgers equation (2.1.5) on the Schwarzschild background to $0 \leq t < \epsilon(r_1 - r_0)$. Moreover, the total variation of the solution satisfies

$$TV_{\hat{r}}(\tilde{v}_1(t, \cdot)) \leq \min\left(\frac{3}{\epsilon}, TV_{\hat{r}}(v_0)\left(1 + e^{\frac{t}{4M\epsilon}}\right)\right),$$

while the weighted total variation given by (2.5.1) satisfies $\widetilde{TV}_{\hat{r}}(\tilde{v}_1(t, \cdot)) = \widetilde{TV}_{\hat{r}}(v_0)$.

Proof. Since the eigenvalue of the Burgers equation (3.1.1) reads $|\lambda| = (1 - \frac{2M}{r})|v| \leq 1/\epsilon$, wave interactions cannot happen before the time $t = \epsilon(r_1 - r_0)$. Hence, we are able to give the solution $\tilde{v}_1 = \tilde{v}_1(t, r)$ of the multiple Riemann problem for a small time as

$$\tilde{v}_1(t, r) = \begin{cases} v_L(t, r) & \bar{r} < r < r_0, \\ v_R(t, r) & r_0 < r < \hat{r}, \end{cases} \quad (2.6.6)$$

since $v_L = v_R$ in (r_0, r_1) .

The total variation of the Riemann problem increases when and only when the it contains a shock wave generated by two steady state solutions with opposite signs propagating with a negative wave speed. In this sense, at most one of the left-hand and right-hand half-Riemann problems can have an increasing total variation. We have thus got the bound of the total variation.

For the weighted total variation (2.5.1), since it stays constant for both left-hand and right-hand half-Riemann problems, the weighted total variation will be conserved before any wave interactions. \square

Wave interactions

To solve the multiple Riemann problem for all times $t > 0$, we now take into consideration possible wave interactions. Referring to (2.6.1), the initial velocity is given as three steady state solutions:

$$v_0(r) = \begin{cases} v_\alpha(r) & \bar{r} < r < r_0, \\ v_\beta(r) & r_0 < r < r_1, \\ v_\gamma(r) & r_1 < r < \hat{r}. \end{cases}$$

We denote by $v_L = v_L(t, r)$ and $v_R = v_R(t, r)$ solutions of the left-hand and right-hand half-Riemann problems, respectively. Denote by the lower and upper bounds of the rarefaction region associated with these two problems $\sigma_-^L = \sigma_-^L(t), \sigma_+^L = \sigma_+^L(t)$

and $\sigma_-^R = \sigma_-^R(t)$, $\sigma_+^R = \sigma_+^R(t)$ referring to (2.4.3).

Define the *interaction time* $\tilde{T} > 0$

$$\tilde{T} := \sup\{t > 0 \mid \sigma_+^L(t) < \sigma_-^R(t)\}. \quad (2.6.7)$$

By the definition, \tilde{T} is the first time when waves of the left-hand and right-hand half-Riemann problems ever interacted.

If $\tilde{T} = +\infty$, then for all $t > 0$, the left-hand and right-hand half-Riemann problems never have wave interactions. Then the local solution of the multiple Riemann problem given by $\tilde{v}_1 = \tilde{v}_1(t, r)$ defined in (2.6.6) can be extended globally in time $t > 0$.

Generally speaking, whether an interaction can happen will depend on the values of the steady state solutions and the corresponding wave speeds, but never will wave interactions happen in cases listed in the following lemma.

Lemma 2.6.5 (No wave interaction). *Let $\sigma_+^L = \sigma_+^L(t)$ and $\sigma_-^R = \sigma_-^R(t)$ be the upper and lower bounds of the rarefaction regions associated with the left-hand half-Riemann problem (2.6.2) and the right-hand half-Riemann problem (2.6.3), respectively. Then the interaction time $\tilde{T} = +\infty$ holds for all $t > 0$, if one of the following cases holds:*

- The wave (v_α^0, v_β^0) is a shock wave with $v_\alpha^0 + v_\beta^0 \leq 0$ and (v_β^1, v_γ^1) is a rarefaction wave.
- Both (v_α^0, v_β^0) and (v_β^1, v_γ^1) are rarefaction waves.

Proof. When (v_β^1, v_γ^1) is a rarefaction wave, we have, by Lemma 2.6.3, that $\sigma_-^R(t) \equiv r_1$. Then if at the same time, (v_α^0, v_β^0) is a shock wave with non-positive shock speed, it is obvious that $\sigma_+^L(t) < \sigma_-^R(t)$ for all $t > 0$. If both waves are rarefactions, $v_\beta^1 = 0$ gives the fact that $\sigma_+^L(t)$ will not reach r_1 . Then we have $\tilde{T} = +\infty$ if one of the conditions in the lemma holds. □

Now we consider the case where interactions did happen at some finite time, that is, $\tilde{T} < +\infty$ with \tilde{T} the interaction time given by (2.6.7). According to Lemma 2.6.5, there are three principle problems:

1. *Problem (SS)* : both (v_α^0, v_β^0) and (v_β^1, v_γ^1) are shock waves.
2. *Problem (SR)* : the left-hand half-Riemann problem (v_α^0, v_β^0) is a shock wave and the right-hand half-Riemann problem (v_β^1, v_γ^1) is a rarefaction wave.
3. *Problem (RS)* : the left-hand half-Riemann problem (v_α^0, v_β^0) is a rarefaction wave and the right-hand half-Riemann problem (v_β^1, v_γ^1) is a shock wave.

Interaction of two shocks

We first treat Problem (SS), that is, the interaction of two shocks.

Lemma 2.6.6 (Problem (SS)). *Let the initial velocity $v_0 = v_0(r)$ composes of three steady state solutions $v_\alpha, v_\beta, v_\gamma$. If both (v_α^0, v_β^0) and (v_β^1, v_γ^1) are shock waves and the interaction time $\tilde{T} < +\infty$, then there exists solution to the relativistic Burgers equation (2.1.5), say $v = v(t, r)$ defined for all $t > 0$ on (\tilde{r}, \hat{r}) . Moreover, for every fixed $t > 0$, the total variation of the solution satisfies $TV_{\tilde{r}}^{\hat{r}}(v(t, \cdot)) < \min\left(\frac{3}{\epsilon}, TV_{\tilde{r}}^{\hat{r}}(v_0)\left(1 + \frac{t}{4M\epsilon}\right)\right)$ and the weighted total variation given by (2.5.1) satisfies $\widetilde{TV}_{\tilde{r}}^{\hat{r}}(v(t, \cdot)) \leq \widetilde{TV}_{\tilde{r}}^{\hat{r}}(v_0)$.*

Proof. We first note that at the interaction time \tilde{T} , the inequality of velocities $v_\alpha(\tilde{r}) > v_\beta(\tilde{r}) > v_\gamma(\tilde{r})$ holds where we have written $\tilde{r} = \sigma^L(\tilde{T}) = \sigma^R(\tilde{T})$. We write the shock wave $\tilde{\sigma}^{SS} = \tilde{\sigma}^{SS}(t)$ such that

$$t - \tilde{T} = R_S(\tilde{\sigma}^{SS}(t); v_\alpha, v_\gamma) - R_S(\tilde{r}; v_\alpha, v_\gamma),$$

where the function R_S is given by (2.4.4). Now we can prove $\tilde{\sigma}^{SS}(t) \leq r_\alpha^\sharp$ where r_α^\sharp is the vanishing point of the steady state velocity v_α . Indeed, if $v_\alpha(\tilde{r}) + v_\gamma(\tilde{r}) \leq 0$, we have $\tilde{\sigma}(t) \leq r_0$, then the result is obvious. If $v_\alpha(\tilde{r}) + v_\gamma(\tilde{r}) > 0$ and there exists a time $t = t_\alpha^\sharp$ such that $\tilde{\sigma}(t_\alpha^\sharp) = r_\alpha^\sharp$, we have the wave speed as $\tilde{\sigma}'(t) = \frac{1}{2}\left(1 - \frac{2M}{r_\alpha^\sharp}\right)v_\gamma(r_\alpha^\sharp) > 0$ which contradicts the fact that $v_\gamma < 0$.

Since we now have no worry that the definition of steady state solutions will fail at some point, the solution after the interaction time: $\tilde{v}_2^{SS} = \tilde{v}_2^{SS}(t, r)$ for all $t > \tilde{T}$ can be given as

$$\tilde{v}_2^{SS}(t, r) = \begin{cases} v_\alpha(r), & \tilde{r} < r < \tilde{\sigma}^{SS}(t), \\ v_\gamma(r), & \tilde{\sigma}^{SS}(t) < r < \hat{r}. \end{cases} \quad (2.6.8)$$

Therefore, there exists a solution to the relativistic Burgers equation associated with initial data $v_0 = v_0(r)$:

$$v(t, r) = \begin{cases} \tilde{v}_1(t, r), & 0 < t < \tilde{T}, \\ \tilde{v}_2^{SS}(t, r), & t > \tilde{T}, \end{cases} \quad (2.6.9)$$

where \tilde{v}_1 and \tilde{v}_2^{SS} are given by (2.6.6) and (2.6.8), respectively.

We now consider the weighted total variation after the interaction time \tilde{T} . For

$t > \tilde{T}$, we have

$$\begin{aligned}
 \widetilde{TV}(\tilde{v}_2^{SS}(t, \cdot)) &= \left| \left(\frac{v_\alpha(\tilde{\sigma}(t))^2 - 1/\epsilon^2}{1 - 2M/\tilde{\sigma}(t)} + \frac{1}{\epsilon^2} \right)^{1/2} \text{sgn}(v_\alpha) - \left(\frac{v_\gamma(\tilde{\sigma}(t))^2 - 1/\epsilon^2}{1 - 2M/\tilde{\sigma}(t)} + \frac{1}{\epsilon^2} \right)^{1/2} \text{sgn}(v_\gamma) \right| \\
 &\leq \left| \left(\frac{v_\alpha(\sigma^L(t))^2 - 1/\epsilon^2}{1 - 2M/\sigma^L(t)} + \frac{1}{\epsilon^2} \right)^{1/2} \text{sgn}(v_\alpha) \right. \\
 &\quad \left. - \left(\frac{v_\beta(\sigma^L(t))^2 - 1/\epsilon^2}{1 - 2M/\sigma^L(t)} + \frac{1}{\epsilon^2} \right)^{1/2} \text{sgn}(v_\beta) \right| \\
 &\quad + \left| \left(\frac{v_\beta(\sigma^R(t))^2 - 1/\epsilon^2}{1 - 2M/\sigma^R(t)} + \frac{1}{\epsilon^2} \right)^{1/2} \text{sgn}(v_\beta) \right. \\
 &\quad \left. - \left(\frac{v_\gamma(\sigma^R(t))^2 - 1/\epsilon^2}{1 - 2M/\sigma^R(t)} + \frac{1}{\epsilon^2} \right)^{1/2} \text{sgn}(v_\gamma) \right| \\
 &= \widetilde{TV}(\tilde{v}_1(t, \cdot)) = \widetilde{TV}(v_0),
 \end{aligned}$$

which completes the proof. □

Interaction of a left-hand shock and a right-hand rarefaction

We now consider the existence of the Burgers solution and the evolution of total variations of Problem (SR), that is, the interaction of a left-hand shock wave and a right-hand shock wave.

Lemma 2.6.7 (Problem (SR)). *Let the initial data $v_0 = v_0(r)$ compose of three steady state solutions $v_\alpha, v_\beta, v_\gamma$ such that (v_α^0, v_β^0) is a shock wave and (v_β^1, v_γ^1) is a rarefaction wave with a finite interaction time $\tilde{T} < +\infty$, then there exists solution of (2.1.5) for all $t > 0$. Moreover, for every fixed $t > 0$, the total variation of the solution satisfies $TV_{\tilde{r}}^{\hat{r}}(v(t, \cdot)) < \min\left(\frac{3}{\epsilon}, TV_{\tilde{r}}^{\hat{r}}(v_0)\left(1 + \frac{t}{4M\epsilon}\right)\right)$ and the weighted total variation given by (2.5.1) satisfies $\widetilde{TV}_{\tilde{r}}^{\hat{r}}(v(t, \cdot)) \leq \widetilde{TV}_{\tilde{r}}^{\hat{r}}(v_0)$.*

Proof. Denote by $\tilde{r} = \sigma^L(\tilde{T}) = \sigma_+^R(\tilde{T})$ where \tilde{T} is the interaction time. Define the shock curve as

$$\tilde{\sigma}^{SR}(t) = \frac{1}{2}(\tilde{\sigma}^{SR,A}(t) + \tilde{\sigma}^{SR,B}(t)).$$

Here, $\tilde{\sigma}^{SR,A}(t)$ satisfies

$$t - \tilde{T} = R_R(r; \tilde{\sigma}^{SR,A}(t)) - R_R(\tilde{r}; v_\alpha),$$

where R_R was defined by (2.4.3). On the other hand, we define $\tilde{\sigma}^{SR,B}(t)$ such that

$$t - \tilde{T} = R(\tilde{\sigma}^{SR,B}(t), K^R(t, \tilde{\sigma}^{SR,B}(t))) - R(\tilde{r}, K^R(t, \tilde{\sigma}^{SR,B}(t))),$$

where the function R is given by (3.5.9) and $K^R = K^R(t, r)$ satisfying $\epsilon^2 w_R^2 = 1 - \epsilon^2 K^{R2} \left(1 - \frac{2M}{r}\right)$ with w_R the rarefaction curve of the right-hand half-Riemann problem (2.6.4). We can at once check that

$$\frac{d\tilde{\sigma}^{SR}(t)}{dt} = \frac{v_\alpha(\tilde{\sigma}^{SR}(t)) + w_R(t, \tilde{\sigma}^{SR}(t))}{2},$$

which satisfies the Rankie-Hugoniot condition (4.2.8).

Now let $t = \check{T}_{SR}$ and $t = \hat{T}_{SR}$ be two times such that

$$\begin{aligned} \check{T}_{SR} &:= \sup\{t > \tilde{T} \mid v_\alpha(\tilde{\sigma}^{SR}(t)) > w_R(t, \tilde{\sigma}^{SR}(t))\}, \\ \hat{T}_{SR} &:= \sup\{t > \tilde{T} \mid \tilde{\sigma}^{SR}(t) < \sigma_+^R(t)\}, \end{aligned}$$

where we recall that $\sigma_+^R = \sigma_+^R(t)$ is the upper bound of the rarefaction region of the right-hand half-Riemann problem.

From the definition, we see that \check{T}_{SR} is the first time when the discontinuity disappears and \hat{T}_{SR} is the first time when the shock wave meets the upper bound of the rarefaction wave of the right-hand half-Riemann problem.

We define the solution to the relativistic Burgers equation for $\tilde{T} < t < \min(\check{T}_{SR}, \hat{T}_{SR})$

$$\tilde{v}_2^{SR}(t, r) = \begin{cases} v_\alpha(r), & \bar{r} < r < \tilde{\sigma}^{SR}(t), \\ v_R(t, r), & \tilde{\sigma}^{SR}(t) < r < \hat{r}, \end{cases} \quad (2.6.10)$$

where $v_R(t, r)$ is the solution to the right-hand half-Riemann problem.

Now we would like to define the solution for all $t > \min(\check{T}_{SR}, \hat{T}_{SR})$. Two possibilities (whether the left-hand shock is stronger than the right-hand rarefaction) are to be taken into consideration referring whether one time is bigger/smaller than one another:

If $\check{T}_{SR} > \hat{T}_{SR}$, we will have $v_\alpha(\tilde{\sigma}^{SR}(t)) > v_\gamma(\tilde{\sigma}^{SR}(t))$ for all $t > \tilde{T}$, then we define the solution as

$$\tilde{v}_{3,S}^{SR}(t, r) = \begin{cases} v_\alpha(r), & \bar{r} < r < \tilde{\sigma}^{SR}(t), \\ v_\gamma(r), & \tilde{\sigma}^{SR}(t) < r < \hat{r}, \end{cases} \quad (2.6.11)$$

where $\tilde{\sigma}^{SR} = \tilde{\sigma}^{SR}(t)$ is the shock curve with two steady states v_α and v_γ given by (2.4.4).

If $\check{T}_{SR} < \hat{T}_{SR}$, curves of left-hand and right-hand half-Riemann problems are connected by a rarefaction wave for all $t > \check{T}_{SR}$. Denote by $\tilde{\sigma}_+^{SR} = \tilde{\sigma}_+^{SR}(t)$ and $\tilde{\sigma}_-^{SR} = \tilde{\sigma}_-^{SR}(t)$ the upper and lower bounds of the rarefaction regions, respectively.

The lower bound of the rarefaction region $\tilde{\sigma}_-^{SR}(t)$ are

$$t - \check{T}_{SR} = R_R(\tilde{\sigma}_-^{SR}(t), v_\alpha) - R_R(\tilde{\sigma}^{SR}(\check{T}_{SR}), v_\alpha),$$

where R_R is the function given in (2.4.3). The upper bound of the rarefaction region $\tilde{\sigma}_+^{SR}$ is given by

$$\tilde{\sigma}_+^{SR}(t) = \begin{cases} \tilde{\sigma}_{+,a}^{SR}(t), & \check{T}_{SR} < t < \hat{T}_{SR}, \\ \tilde{\sigma}_{+,b}^{SR}(t), & t > \hat{T}_{SR}, \end{cases}$$

Here, we set $\tilde{\sigma}_{+,a}^{SR} = \tilde{\sigma}_{+,a}^{SR}(t)$ such that

$$t - \tilde{T} = R\left(\tilde{\sigma}_{+,a}^{SR}(t), K^R(t, \tilde{\sigma}_{+,a}^{SR}(t))\right) - R\left(\tilde{\sigma}^{SR}(\check{T}_{SR}), K^R(t, \tilde{\sigma}_{+,a}^{SR}(t))\right),$$

where $R = R(r, K)$ is given by (3.5.9) and $K^R = K^R(t, r)$ such that $\epsilon^2 w_R^2 = 1 - \epsilon^2 K^{R^2} \left(1 - \frac{2M}{r}\right)$ with w_R the rarefaction curve of the right-hand half-Riemann problem (2.6.5) and $\tilde{\sigma}_{+,b}^{SR} = \tilde{\sigma}_{+,b}^{SR}(t)$ satisfies

$$t - \hat{T}_{SR} = R_R(\tilde{\sigma}_{+,b}^{SR}(t); v_\gamma) - R_R(\tilde{\sigma}^{SR}(\hat{T}_{SR}); v_\gamma),$$

where R_R is the function given in (2.4.3).

In this case, we define

$$\tilde{v}_{3,R}^{SR}(t, r) = \begin{cases} v_\alpha(r), & \bar{r} < r < \tilde{\sigma}_-^{SR}(t), \\ w(t, r), & \tilde{\sigma}_-^{SR}(t) < r < \tilde{\sigma}_+^{SR}(t), \\ v_R(t, r), & \tilde{\sigma}_+^{SR}(t) < r < \hat{r}, \end{cases} \quad (2.6.12)$$

where $w(t, r)$ is the generalized rarefaction curve given by (3.5.7) and $v_R = v_R(t, r)$ the solution to the right-hand half-Riemann problem. As is done in Lemma 2.6.6, we can check that the solution is well-defined, that is, every steady state will not vanish in corresponding domains.

We are then able to give the global solution of Problem (SR) denoted by $v = v(t, r)$:

$$v(t, r) = \begin{cases} \tilde{v}_1(t, r), & 0 < t < \tilde{T}, \\ \tilde{v}_2^{SR}(t, r), & \tilde{T} < t < \min(\check{T}_{SR}, \hat{T}_{SR}), \\ \tilde{v}_3^{SR}(t, r), & t > \min(\check{T}_{SR}, \hat{T}_{SR}), \end{cases} \quad (2.6.13)$$

where $\tilde{v}_1(t, r)$ is given by (2.6.6), $\tilde{v}_2^{SR}(t, r)$ by (2.6.10) and

$$\tilde{v}_3^{SR}(t, r) = \begin{cases} \tilde{v}_{3,S}^{SR}(t, r) & \check{T}_{SR} \geq \hat{T}_{SR}, \\ \tilde{v}_{3,R}^{SR}(t, r) & \check{T}_{SR} < \hat{T}_{SR}. \end{cases}$$

where by $\tilde{v}_{3,S}^{SR}$ is given by (2.6.11) and $\tilde{v}_{3,R}^{SR}$ by (2.6.12).

The result for the total variation $TV_{\bar{r}}^{\hat{r}}(v)$ is obvious. For the weighted total variation (2.5.1), we only have to treat the time after the interaction time since the behavior of the solution before \tilde{T} is already provided by Theorem (2.6.4). Indeed, for $\tilde{T} < t < \min(\check{T}_{SR}, \hat{T}_{SR})$, we have

$$\begin{aligned} \widetilde{TV}_{\bar{r}}^{\hat{r}}(\tilde{v}_2^{SR}(t, \cdot)) &= \left| \left(\frac{v_\alpha(\tilde{\sigma}^{SR}(t))^2 - 1/\epsilon^2}{1 - 2M/\tilde{\sigma}^{SR}(t)} + \frac{1}{\epsilon^2} \right)^{1/2} \text{sgn}(v_\alpha) \right. \\ &\quad \left. - \left| \frac{v_R(t, \tilde{\sigma}^{SR}(t))^2 - 1/\epsilon^2}{1 - 2M/\tilde{\sigma}^{SR}(t)} + \frac{1}{\epsilon^2} \right|^{1/2} \text{sgn}(v_R) \right| + \widetilde{TV}(v_R(t, r)) \\ &= \left| \left(\frac{v_\alpha(\tilde{\sigma}^{SR}(t))^2 - 1/\epsilon^2}{1 - 2M/\tilde{\sigma}^{SR}(t)} + \frac{1}{\epsilon^2} \right)^{1/2} \text{sgn}(v_\alpha) \right. \\ &\quad \left. - \left| \frac{w_R(t, \tilde{\sigma}^{SR}(t))^2 - 1/\epsilon^2}{1 - 2M/\tilde{\sigma}^{SR}(t)} + \frac{1}{\epsilon^2} \right|^{1/2} \text{sgn}(w_R) \right| + \widetilde{TV}(v_R(t, r)) \\ &\leq \widetilde{TV}(v_L(t, r)) + \widetilde{TV}(v_R(t, r)) = \widetilde{TV}_{\bar{r}}^{\hat{r}}(v_0). \end{aligned}$$

We now consider the weighted total variation for $t > \min(\check{T}_{SR}, \hat{T}_{SR})$. Notice that if $\check{T}_{SR} < \hat{T}_{SR}$, the result holds, following from a similar calculation. We now consider the weighted total variation when $\check{T}_{SR} > \hat{T}_{SR}$ holds. Indeed, we have

$$\begin{aligned} \widetilde{TV}_{\bar{r}}^{\hat{r}}(\tilde{v}_3^{SR}(t, \cdot)) &= \left| \left(\frac{w(t, \tilde{\sigma}_-^{SR}(t))^2 - 1/\epsilon^2}{1 - 2M/\tilde{\sigma}_-^{SR}(t)} \right)^{1/2} \text{sgn}(w(t, \tilde{\sigma}_-^{SR}(t))) \right. \\ &\quad \left. - \left(\frac{w(t, \tilde{\sigma}_+^{SR}(t))^2 - 1/\epsilon^2}{1 - 2M/\tilde{\sigma}_+^{SR}(t)} \right)^{1/2} \text{sgn}(w(t, \tilde{\sigma}_+^{SR}(t))) \right| + \widetilde{TV}(v_R(t, r)) \\ &= \widetilde{TV}_{\bar{r}}^{\hat{r}}(\tilde{v}_2(t, \cdot)) \leq \widetilde{TV}_{\bar{r}}^{\hat{r}}(v_0), \end{aligned}$$

which completes the analysis of Problem (SR). \square

Interaction of a left-hand rarefaction and a right-hand shock

We now turn to the consideration of Problem (RS).

Lemma 2.6.8 (Problem (RS)). *Given the initial data $v_0 = v_0(r)$ consisting of three steady state solutions $v_\alpha, v_\beta, v_\gamma$. If the left-hand half-Riemann problem (v_α^0, v_β^0) is*

a rarefaction wave and the right-hand half-Riemann problem (v_β^1, v_γ^1) a shock wave with a finite interaction time $\tilde{T} < +\infty$, then the relativistic Burgers equation (2.1.5) admits a solution $v = v(t, r)$ for all $t > 0$. Moreover, for every fixed $t > 0$, the total variation of the solution satisfies

$$TV_{\tilde{r}}^{\hat{r}}(v(t, \cdot)) \leq \min\left(\frac{3}{\epsilon}, TV_{\tilde{r}}^{\hat{r}}(v_0)\left(1 + \frac{t}{4M\epsilon}\right)\right)$$

and the weighted total variation given by (2.5.1) satisfies $\widetilde{TV}_{\tilde{r}}^{\hat{r}}(v(t, \cdot)) \leq \widetilde{TV}_{\tilde{r}}^{\hat{r}}(v_0)$.

We observe that by combining Lemmas 2.6.6, 2.6.7, and 2.6.8 together, the proof of Theorem 2.6.1 is now completed and we have thus established the existence of the solution to the multiple Riemann problem.

Proof. We write $\tilde{r} = \sigma_+^L(\tilde{T}) = \sigma^R(\tilde{T})$ as the point where two waves meet for the first time. We then denote by $\tilde{\sigma}_-^{RS} = \tilde{\sigma}_-^{RS}(t)$ and $\tilde{\sigma}_+^{RS} = \tilde{\sigma}_+^{RS}(t)$ the lower and upper bounds of the rarefaction region after the interaction time \tilde{T} . Recall the formula of the function R given by (3.5.9) and we define $\tilde{\sigma}_-^{RS}(t)$ by

$$t - \tilde{T} = R\left(\tilde{\sigma}_-^{RS}(t), K^L(t, \tilde{\sigma}_-^{RS}(t))\right) - R\left(\tilde{r}, K^L(t, \tilde{\sigma}_-^{RS}(t))\right)$$

where $K^L = K^L(t, r)$ satisfies $\epsilon^2 w_L^2 = 1 - \epsilon^2 K^L{}^2 \left(1 - \frac{2M}{r}\right)$ with w_L the rarefaction curve of the left-hand half-Riemann problem (2.6.4). Now we set $\tilde{\sigma}_+^{RS} = \tilde{\sigma}_+^{RS}(t)$ such that

$$t - \tilde{T} = R_R(\tilde{\sigma}_+^{RS}(t), v_\gamma) - R_R(\tilde{r}, v_\gamma)$$

with the function R_R given by (2.4.4). We can immediately verify the following equations:

$$\frac{d\tilde{\sigma}_-^{RS}(t)}{dt} = \left(1 - \frac{2M}{\tilde{\sigma}_-^{RS}(t)}\right) w_L(t, \tilde{\sigma}_-^{RS}(t)), \quad \frac{d\tilde{\sigma}_+^{RS}(t)}{dt} = \left(1 - \frac{2M}{\tilde{\sigma}_+^{RS}(t)}\right) v_\gamma(\tilde{\sigma}_+^{RS}(t)).$$

Then we define the solution $\tilde{v}_2^{RS} = \tilde{v}_2^{RS}(t, r)$:

$$\tilde{v}_2^{RS}(t, r) = \begin{cases} v_L^{RS}(t, r), & \tilde{r} < r < \tilde{\sigma}_-^{RS}(t), \\ w(t, r), & \tilde{\sigma}_-^{RS}(t) < r < \tilde{\sigma}_+^{RS}(t), \\ v_\gamma(r), & \tilde{\sigma}_+^{RS}(t) < r < \hat{r}. \end{cases} \quad (2.6.14)$$

for all $t < \min(\check{T}_{RS}, \hat{T}_{RS})$. Here, $w = w(t, r)$ is the generalized rarefaction curve given by (3.5.7) and the two times $\check{T}_{RS}, \hat{T}_{RS}$ are given as

$$\begin{aligned} \check{T}_{RS} &= \sup\{t > \tilde{T} \mid w(t, \tilde{\sigma}_+^{RS}(t)) > v_\gamma(\sigma_+^{RS}(t))\}, \\ \hat{T}_{RS} &= \sup\{t > \tilde{T} \mid \tilde{\sigma}_-^{RS}(t) < \sigma^R(t)\}. \end{aligned} \quad (2.6.15)$$

From the definition, we see that \check{T}_{RS} is the first time when a discontinuity of the shock appears and \hat{T}_{RS} is the first time when the lower bound of the shock wave of the left-hand half-Riemann problem meets the wave of the right-hand half-Riemann problem.

The solution to the multiple Riemann problem after $t = \min(\check{T}_{RS}, \hat{T}_{RS})$ has two possibilities (whether the right-hand shock is stronger than the left-hand rarefaction) depending on which time of the two (2.6.15) happened earlier. If $\check{T}_{RS} > \hat{T}_{RS}$, $v_\alpha(\tilde{\sigma}_+(\hat{T}_{RS})) < v_\gamma(\tilde{\sigma}_+(\hat{T}_{RS}))$ holds. In this case, we define the solution to the Burgers equation for $t > \hat{T}_{RS}$ as

$$\tilde{v}_{3,R}^{RS}(t, r) = \begin{cases} v_\alpha(r), & \bar{r} < r < \tilde{\sigma}_-^{RS}(t), \\ w(t, r), & \tilde{\sigma}_-^{RS}(t) < r < \tilde{\sigma}_+^{RS}(t), \\ v_\gamma(r), & \tilde{\sigma}_+^{RS}(t) < r < \hat{r}, \end{cases} \quad (2.6.16)$$

where the lower and upper bounds of the rarefaction region $\tilde{\sigma}_-^{RS} = \tilde{\sigma}_-^{RS}(t)$, $\tilde{\sigma}_+^{RS} = \tilde{\sigma}_+^{RS}(t)$ are determined by (2.4.3) and (2.4.4), respectively.

On the other hand, if $\check{T}_{RS} < \hat{T}_{RS}$, the solution for $t > \hat{T}_{RS}$ will be realized by a shock wave. We then give the shock curve as follows:

$$\tilde{\sigma}^{RS}(t) = \begin{cases} \tilde{\sigma}_a^{RS}(t), & \check{T}_{RS} < t < \hat{T}_{RS}, \\ \tilde{\sigma}_b^{RS}(t), & t > \hat{T}_{RS}, \end{cases}$$

where $\tilde{\sigma}_b^{RS}(t)$ satisfies

$$t - \hat{T}_{RS} = R_S(\tilde{\sigma}_b^{RS}(t); v_\alpha, v_\gamma) - R_S(\tilde{\sigma}^{RS}(\hat{T}_{RS}); v_\alpha, v_\gamma),$$

with R_S is the function in (2.4.4). To get $\tilde{\sigma}_a^{RS}(t)$, we set

$$\tilde{\sigma}_a^{RS}(t) = \frac{1}{2}(\tilde{\sigma}_a^{RS,A}(t) + \tilde{\sigma}_a^{RS,B}(t)).$$

where $\tilde{\sigma}_a^{RS,A}(t)$ satisfies

$$t - \hat{T}_{RS} = R_R(\tilde{\sigma}_a^{RS,A}(t); v_\alpha) - R_R(\tilde{\sigma}^{RS}(\hat{T}_{RS}); v_\alpha)$$

with R_R given in (2.4.3) and $\tilde{\sigma}_a^{RS,B} = \tilde{\sigma}_a^{RS,B}(t)$ satisfies

$$t - \hat{T}_{RS} = R(\tilde{\sigma}_a^{RS,B}(t), K(t, \tilde{\sigma}_a^{RS,B}(t))) - R(\tilde{\sigma}^{RS}(\hat{T}_{RS}), K(t, \tilde{\sigma}_a^{RS,B}(t)))$$

where $\epsilon^2 w^2 = 1 - \epsilon^2 K^2 \left(1 - \frac{2M}{r}\right)$ and R given by (3.5.9). We now define

$$\tilde{v}_{3,S}^{RS}(t, r) = \begin{cases} v_\alpha(r) & \bar{r} < r < \tilde{\sigma}^{RS}(t), \\ v_\gamma(r) & \tilde{\sigma}^{RS}(t) < r < \hat{r}. \end{cases} \quad (2.6.17)$$

A similar argument as Lemma 2.6.6 gives that all the steady state solutions are well-defined in corresponding domains.

We thus give the solution of problem (RS) :

$$v(t, r) = \begin{cases} \tilde{v}_1(t, r) & 0 < t < \tilde{T}, \\ \tilde{v}_2^{RS}(t, r) & \tilde{T} < t < \min(\check{T}_{RS}, \hat{T}_{RS}), \\ \tilde{v}_3^{RS}(t, r) & t > \min(\check{T}_{RS}, \hat{T}_{RS}), \end{cases} \quad (2.6.18)$$

where $\tilde{v}_1(t, r), \tilde{v}_2^{RS}(t, r)$ are given by (2.6.6), (2.6.14) and

$$\tilde{v}_3^{RS}(t, r) = \begin{cases} \tilde{v}_{3,R}^{RS}(t, r) & \check{T}_{RS} \geq \hat{T}_{RS}, \\ \tilde{v}_{3,S}^{RS}(t, r) & \check{T}_{RS} < \hat{T}_{RS}. \end{cases}$$

with $\tilde{v}_{3,R}^{RS}, \tilde{v}_{3,S}^{RS}$ given by (2.6.16) and (2.6.17), respectively. The result concerning the (weighted) total variation follows from the similar analysis in Lemma 2.6.7. \square

2.7 The well-posedness theory of weak solutions

The well-balanced random choice method

To construct the solution to the initial value problem of (2.1.5), we introduce a generalized Glimm method based on the generalized (multiple) Riemann problem introduced in Sections 2.4 and 2.6. First of all, recall that the eigenvalue of the relativistic Burgers equation reads $\lambda = \left(1 - \frac{2M}{r}\right)v$ with $|v| \leq \frac{1}{\epsilon}$. Hence, λ vanishes at the horizon $r = 2M$. This indicates that we need not require any boundary conditions.

Our generalized random choice method is based on the generalized Riemann solver. Denote by $\Delta t, \Delta r$ the mesh lengths in time and in space, respectively. Here, we require the *stability condition*, or the *CFL condition* that

$$\frac{\Delta r}{\Delta t} > \frac{2}{\epsilon}. \quad (2.7.1)$$

We denote by (t_i, r_j) as the mesh point of the grid:

$$t_i = i\Delta t, \quad r_j = 2M + j\Delta r.$$

Now we construct the solution to the relativistic Burgers equation (2.1.5) on the Schwarzschild spacetime. As a first step, we approximate the given initial data by a piecewise steady state solution $v_{\Delta,0} = v_{\Delta,0}(r)$:

For all even integer $j \geq 1$, we solve the ordinary differential equation (2.3.1) in the interval (r_j, r_{j+2}) with the value centered at $r = r_{j+1}$. In view of Section 2.3, there exists a unique smooth solution of (2.3.1) denoted by $v_{\Delta,0}^{j+1} = v_{\Delta,0}^{j+1}(r)$ in a neighborhood of r_{j+1} . However, it is possible that $v_{\Delta,0}^{j+1}$ vanishes at some point in the interval (r_j, r_{j+2}) if it is in the regime of small velocity (2.3.7). In order to finish the initial approximation step, we extend the steady state solution by the values of the right-hand neighbor interval (r_{j+2}, r_{j+4}) . In this sense, we approximate the initial data as follows

$$v_{\Delta,0}(r) = \begin{cases} v_{\Delta,0}^{j+1}, & j \text{ even, } r_j < r < \min(r_{j+2}, r_{j+1}^{\natural}), \\ v_{\Delta,0}^{j+3}, & j \text{ even, } \min(r_j, r_{j+1}^{\natural}) < r < r_{j+2}, \end{cases} \quad (2.7.2)$$

where

$$\begin{aligned} v_{\Delta,0}^{j+1}(r) &= \operatorname{sgn}(v_0(r_{j+1})) \sqrt{\frac{1}{\epsilon^2} - K_{j+1}^0 \left(1 - \frac{2M}{r}\right)}, & j \text{ even,} \\ K_{j+1}^0 &= \frac{1}{1 - 2M/r_{j+1}} \left(\frac{1}{\epsilon^2} - v_0(r_{j+1})^2\right), \\ r_{j+1}^{\natural} &= \sup\{r > 2M \mid v_{\Delta,0}^{j+1}(r) \neq 0\}. \end{aligned} \quad (2.7.3)$$

We then suppose that the approximate solution $v_{\Delta} = v_{\Delta}(t, r)$ has already been defined for all $0 \leq t < t_i$. For a given random sequence $(w_i)_i$ in $(-1, 1)$, we set

$$r_{i,j} = 2M + (w_i + j)\Delta r, \quad j \geq 1. \quad (2.7.4)$$

Our scheme includes two main steps:

1. *The steady state step.* At the time level $t = t_i$, we define v_{Δ} to be a piecewise smooth steady state solution as was done in the definition of the approximate initial data:

$$v_{\Delta}(t_i, r) = \begin{cases} v_{\Delta,i}^{j+1}(r), & i + j \text{ even, } r_j < r < \min(r_{j+2}, r_{i,j+1}^{\natural}), \\ v_{\Delta,i}^{j+3}(r), & i + j \text{ even, } \min(r_j, r_{i,j+1}^{\natural}) < r < r_{j+2}, \end{cases} \quad (2.7.5)$$

where

$$\begin{aligned}
 v_{\Delta,i}^{j+1}(r) &= \operatorname{sgn}(v_{\Delta}(t_i-, r_{i,j+1})) \sqrt{\frac{1}{\epsilon^2} - K_{i,j+1}^0 \left(1 - \frac{2M}{r}\right)}, & i+j \text{ even,} \\
 K_{i,j+1}^0 &= \frac{1}{1 - 2M/r_{i,j+1}} \left(\frac{1}{\epsilon^2} - v_{\Delta}(t_i-, r_{i,j+1})^2\right), \\
 r_{i,j+1}^{\natural} &= \sup\{r > 2M \mid v_{\Delta,i}^{j+1}(r) \neq 0\}.
 \end{aligned} \tag{2.7.6}$$

2. *The generalized Riemann problem step.* We define the solution on $\{t_i < t < t_{i+1}, r_{j-1} < r < r_{j+1}\}$ (with $i+j$ even, $j \geq 1$) by

$$v_{\Delta}(t, r) := \begin{cases} v_{\mathcal{R}}\left(t, r; t_i, r_j; v_{\Delta,i}^{j-1}, v_{\Delta,i}^{j+1}\right), & \text{if } r_j < r_{i,j-1}^{\natural}, \\ v_{\mathcal{MR}}\left(t, r; t_i, \min(r_{j-2}, r_{i,j-3}^{\natural}), r_{i,j-1}^{\natural}; v_{\Delta,i}^{j-3}, v_{\Delta,i}^{j-1}, v_{\Delta,i}^{j+1}\right), & \text{if } r_j \geq r_{i,j-1}^{\natural}, \end{cases} \tag{2.7.7}$$

where $v_{\mathcal{R}}$ denotes the solution to the Riemann problem with the initial condition

$$v_{\mathcal{R}}\left(t, r; t_i, r_j; v_{\Delta,i}^{j-1}, v_{\Delta,i}^{j+1}\right) = \begin{cases} v_{\Delta,i}^{j-1}, & r < r_j, \\ v_{\Delta,i}^{j+1}, & r > r_j; \end{cases}$$

and $v_{\mathcal{MR}}$ the solution to the multiple Riemann problem with the initial condition

$$\begin{aligned}
 &v_{\mathcal{MR}}\left(t_i, r; t_i, \min(r_{j-2}, r_{i,j-3}^{\natural}), r_{i,j-1}^{\natural}; v_{\Delta,i}^{j-3}, v_{\Delta,i}^{j-1}, v_{\Delta,i}^{j+1}\right) \\
 &= \begin{cases} v_{\Delta,i}^{j-3}, & r < \min(r_{j-2}, r_{i,j-3}^{\natural}), \\ v_{\Delta,i}^{j-1}, & \min(r_{j-2}, r_{i,j-3}^{\natural}) < r < r_{i,j-1}^{\natural}, \\ v_{\Delta,i}^{j+1}, & r > r_{i,j-1}^{\natural}. \end{cases}
 \end{aligned}$$

We still have to explain that the result of solutions of (multiple) Riemann problems is sufficient for the construction of the generalized Glimm scheme. We have the following lemma.

Lemma 2.7.1. *Consider the construction of approximate solutions of the generalized Glimm method, at the time level $t = t_i$, for all integer $j \geq 1$ with $i+j$ even, there is at most one point of discontinuity in the interval (r_{j-1}, r_{j+1}) .*

Proof. Without loss of generality, we suppose $w_i \in (0, 1)$ and thus $v_{\Delta,i}^{j+1}$ is at least well-defined on (r_j, r_{j+1}) . Then there are two possibilities:

- If $r_{i,j-1}^{\natural} = r_j$, then only $r = r_j$ can be a point of discontinuity;
- If $r_{j-1} < r_{i,j-1}^{\natural} < r_j$, then we have to use the value of $v_{\Delta,i}^{j+1}$ to define the steady state on $(r_{i,j-1}^{\natural}, r_j)$. Then $r = r_j$ becomes a point of continuity while $r_{i,j-1}^{\natural}$ turns to the unique possible point of discontinuity on (r_{j-1}, r_{j+1}) .

□

According to Lemma 2.7.1, together with the CFL condition (4.6.2), interactions other than those of the multiple generalized Riemann problems cannot happen within one time step. Therefore, our Glimm method is well-defined. The next proposition shows that it is actually well-balanced as well.

Proposition 2.7.2 (Well-balanced property). *Let the initial velocity $v_* = v_*(r)$ be a smooth steady state solution given by the static Burgers model on the Schwarzschild spacetime background (2.3.1). Then the approximate solution constructed by the generalized Glimm method is accurate.*

Proof. Since the initial velocity is a steady state solution, we see that $v_{\Delta,0} = v_*$ where $v_{\Delta,0}$ is the initial approximation given by (2.7.5), (2.7.6). Since the solution to the Riemann problem is accurate, we have thus the result. □

For any initial velocity with a bounded weighted total variation (2.5.1), the random choice method actually constructs a sequence of approximate solutions which will converge to an exact weak solution to the relativistic Burgers model (2.1.5) for vanishing space length $\Delta r \rightarrow 0$. We would like to prove in this section the result concerning the existence theory given by Theorem 2.1.2, or more concretely, the following existence theory of the Cauchy problem of the relativistic Burgers equation (2.1.5) on the Schwarzschild background.

Theorem 2.7.3 (Existence theory of the relativistic Burgers equation on the Schwarzschild background). Let $M > 0$ be the mass of a Schwarzschild black hole and we consider the relativistic Burgers equation on a Schwarzschild background (2.1.5) posed in the domain $r > 2M$. For any given initial velocity $|v_0| = |v_0(r)| < 1/\epsilon$ where $1/\epsilon$ is the light speed such that $\widetilde{TV}(v_0) < +\infty$ where \widetilde{TV} is the weighted total variation defined by (2.5.1), there exists a weak solution to the relativistic Burgers model on a Schwarzschild spacetime background (2.1.5), say $v = v(t, r)$ defined on $(0, +\infty) \times (2M, +\infty)$ such that for all $t > 0$,

$$\widetilde{TV}(v(t_2, \cdot)) \leq \widetilde{TV}(v(t_1, \cdot)), \quad 0 \leq t_1 \leq t_2.$$

Convergence analysis

Recall that for the existence theory of the standard Burgers model (3.3.1) without geometry effect, we require the initial data to have a total variation $TV(v)$ on the whole space interval. However, Theorem 4.6.1 provides a result of weighted total variation $\widetilde{TV}(v)$ instead of $TV(v)$. Indeed, according to Theorem 2.5.4, the total variation of the Riemann problem will increase if and only if we have a shock wave generated by two steady states with different monotony properties with a negative shock speed. We first give the estimate of the total variation of approximate solutions.

Lemma 2.7.4 (Total variation of approximate solutions). *Let $|v_0| = |v_0(r)| < \frac{1}{\epsilon}$ be an initial velocity with bounded variation on $(2M, +\infty)$. Denote by $v_\Delta = v_\Delta(t, r)$ the approximate solution to the relativistic Burgers equation (2.1.5) constructed by the generalized Glimm method satisfies*

$$TV(v_\Delta(t, \cdot)) \leq TV(v_{\Delta,0}) \left(1 + e^{\frac{t}{4M\epsilon}}\right), \quad t \geq 0,$$

where $v_{\Delta,0} = v_{\Delta,0}(r)$ is the approximation of the initial data $v_0 = v_0(r)$.

Proof. We consider the total variation of the approximate solution at the time level $t = t_{i+1}$ and focus on a particular space interval $I_j = (r_{i+1,j-1}, r_{i+1,j+1})$ with $i + j$ even. Recall that $r_{i\pm 1,j+1} = r_{j+1} + w_{i\pm 1}\Delta r$ where $(w_i)_i$ is an equidistributed sequence. By construction, there exists at most one point of discontinuity in the interval I_j and we denote this point by $r = r_{j^\sharp} \leq r_j$. Since every steady state solution is monotone, we have

$$TV(v_\Delta(t_{i+1}, \cdot)) = \sum_{j^\sharp} |v_\Delta(t_{i+1}, r_{j^\sharp}+) - v_\Delta(t_{i+1}, r_{j^\sharp}-)| \leq TV(v_\Delta(t_{i+1}-, \cdot)).$$

In view of Lemma 2.6.1, $TV_{I_j}(v_\Delta(t_{i+1}-, \cdot)) \leq TV_{I_j}(v_\Delta(t_i, \cdot)) \left(1 + \frac{\Delta t}{4M\epsilon}\right)$. Since $r = r_{i\pm 1,j+1}$ with $i + j$ even are points of continuity, we have

$$\begin{aligned} TV(v_\Delta(t_{i+1}-, \cdot)) &= \sum_j TV_{I_j}(v_\Delta(t_{i+1}-, \cdot)) \\ &\leq \sum_j TV_{I_j}(v_\Delta(t_i, \cdot)) \left(1 + \frac{\Delta t}{4M\epsilon}\right) = TV(v_\Delta(t_i, \cdot)) \left(1 + \frac{\Delta t}{4M\epsilon}\right). \end{aligned}$$

Now for a fixed time $t > 0$, there exists an integer $i > 0$ such that $t \in (t_i, t_{i+1}]$ and we thus have

$$TV(v_\Delta(t, \cdot)) \leq TV(v_{\Delta,0}) \left(1 + \frac{\Delta t}{4M\epsilon}\right)^{i+1} \leq TV(v_{\Delta,0}) e^{\frac{t}{4M\epsilon}}.$$

□

As a result of Lemma 2.7.4, the total variation of the solution will probably increase. In this sense, we would rather use the weighted total variation $\widetilde{TV}(v)$ on the whole space interval $(2M, +\infty)$ as is defined by (2.5.1):

$$\widetilde{TV}(v) = \int_{2M}^{+\infty} \left| \partial_r \left(\left| \frac{v^2 - 1/\epsilon^2}{1 - 2M/r} + \frac{1}{\epsilon^2} \right|^{1/2} \operatorname{sgn}(v) \right) \right|,$$

where the integral is interpreted as the mass of a measure.

Lemma 2.7.5 (Weighed total variation of approximate solutions). *Consider an initial velocity $v_0 = v_0(r) \in [-\frac{1}{\epsilon}, \frac{1}{\epsilon}]$ such that $\widetilde{TV}(v_0) < +\infty$. Denote by $v_\Delta = v_\Delta(t, r)$ the approximate solution to the Burgers equation constructed by the generalized Glimm method. Then the total variation of v_Δ satisfies*

$$\widetilde{TV}(v_\Delta(t, \cdot)) \leq \widetilde{TV}(v_\Delta(s, \cdot)), \quad 0 \leq s \leq t. \quad (2.7.8)$$

Proof. We see first that within one time step, $t_i \leq t < s < t_{i+1}$, the weighted total variation is non-increasing, that is, $\widetilde{TV}(v(t, \cdot)) \leq \widetilde{TV}(v(s, \cdot))$ according to Theorems 2.5.5 and 2.6.1. It remains to show that the result holds for $t_i \leq t < s = t_{i+1}$ as well.

We consider now the time level $t = t_{i+1}$ and once again focus on the particular space interval $I_j = (r_{i+1, j-1}, r_{i+1, j+1})$ where $i+j$ is an even integer. The construction of the Glimm scheme gives $r_{i\pm 1, j+1} = r_{j+1} + w_{i\pm 1} \Delta r$ where $(w_i)_i$ is an equidistributed sequence. There are at most one point of discontinuity in I_j , say $r = r_{j\sharp} \leq r_j$.

Following from the construction of the steady state solutions, there are at most portions of three possible waves lying in the interval I_j for $t_i < t < t_{i+1}$. Denote by α, β, γ these three waves who are either elementary waves or waves of the multiple Riemann problem.

Then α is either a zero strength wave in I_j (if the random choice point $r_{i+1, j-1}$ lies closer to $r_{j\sharp}$ than the wave α) or a wave with left steady state v_L such that $v_L(r_{i+1, j-1}) = v_\Delta(t_{i+1}^-, r_{i+1, j-1})$ and right steady states v_M^α such that $v_M^\alpha = v_\Delta(t_i^-, r_{i+1, j+1})$. Similarly, γ is either a zero strength wave in I_j or a problem with left states v_M^β such that $v_M^\beta = v_\Delta(t_i^-, r_{i+1, j+1})$ and right states v_R such that $v_R = v_\Delta(t_{i+1}^-, r_{i+1, j+1})$. Concerning the wave β , it is associated with the left-hand state v_L or v_M^α and the right-hand state v_M^β or v_R . Use these notations, we should solve a Riemann problem centered at $r = r_{j\sharp}$ with (v_L, v_R) or a multiple Riemann problem with (v_L, v_R) one of its half problem. Then we have $\widetilde{TV}_{I_j}(v_\Delta(t_{i+1}, \cdot)) \leq \widetilde{TV}_{I_j}(v_\Delta(t_{i+1}^-, \cdot))$.

Adding all the intervals I_j together for $i+j$ even, we reach the desired result. \square

We are now in a position to complete the proof of Theorem 4.6.1. Define

$$z := \left| \frac{v^2 - 1/\epsilon^2}{1 - 2M/r} + \frac{1}{\epsilon^2} \right|^{1/2} \text{sgn}(v), \quad z_\Delta := \left| \frac{v^2 - 1/\epsilon^2}{1 - 2M/r} + \frac{1}{\epsilon^2} \right|^{1/2} \text{sgn}(v_\Delta). \quad (2.7.9)$$

We see that z is a constant if and only if v is a steady state solution to the static relativistic Burgers equation (2.3.1) and the definition gives $\widetilde{TV}(v) = TV(z)$.

We apply Helly's theorem to the approximate solution $z_\Delta = z(t, r)$, that is, there exists a subsequence of the mesh length (denoted by Δr as well), such that $z_\Delta \rightarrow z$ in L^1_{loc} at each time $t > 0$ and $z = z(t, r)$ is a weak solution of the relativistic Burgers equation (2.1.5) satisfying the given initial data $z(t, \cdot) = z_0$.

2.8 The vanishing viscosity method

The method for general data

We now analyze the vanishing viscosity method for the Burgers equation on a Schwarzschild background (2.1.5). As we will show it, it is natural to introduce the following weighted function space

$$\mathcal{E} := \left\{ v : (2M, +\infty) \rightarrow (-1/\epsilon, 1/\epsilon) \mid \widetilde{TV}(v) < +\infty \right\}, \quad (2.8.1)$$

where we recall that the weighted total variation was defined, for smooth functions, by

$$\widetilde{TV}(v) := \int_{2M}^{+\infty} \left| \partial_r \left(\left| \frac{v^2 - 1/\epsilon^2}{1 - 2M/r} + \frac{1}{\epsilon^2} \right|^{1/2} \operatorname{sgn}(v) \right) \right|,$$

and then extended by density. Recall also that all steady state solutions to the Burgers equation belong to \mathcal{E} .

We introduce a regularization of the solutions of (2.1.5) and now solve the equation

$$\partial_t \left(\frac{v_\alpha}{(1 - 2M/r)^2} \right) + \partial_r \left(\frac{v_\alpha^2 - 1/\epsilon^2}{2(1 - 2M/r)} \right) = \alpha \partial_r (K(r) \partial_r v_\alpha), \quad r > 2M, \quad (2.8.2)$$

where $\alpha > 0$ is a parameter (which will tend to zero) and the given function $K = K(r) > 0$ is smooth and depends on the space variable r only. We tacitly assume that the kernel K is chosen so that the solutions exist for all times and are smooth (from regularized initial data).

Theorem 2.8.1 (Well-posedness theory for the Burgers equation on a Schwarzschild background). The initial value problem for the Burgers equation on a Schwarzschild background (2.1.5) is well-posed in the functional space \mathcal{E} .

- *Control of the weighted total variation.* Given any initial data $v_0 \in \mathcal{E}$, the corresponding solution belongs to \mathcal{E} at each time and the weighted total variation $t \mapsto \widetilde{TV}(v(t, \cdot))$ is a non-increasing function of time.
- *Weighted L^1 stability property.* For any two initial data $v_{0,1}$ and $v_{0,2}$ in \mathcal{E} , the corresponding weak solutions $v_1 = v_1(t, r)$ and $v_2 = v_2(t, r)$ satisfy

$$\int_{2M}^{+\infty} \frac{|v_2(t, r) - v_1(t, r)|}{(1 - 2M/r)^2} dr \leq e^{\frac{t}{2\epsilon M}} \int_{2M}^{+\infty} \frac{|v_{0,2}(r) - v_{0,1}(r)|}{(1 - 2M/r)^2} dr. \quad (2.8.3)$$

Proof. 1. Observe that it is precisely the L^1 stability property which provides us with the uniqueness of the entropy solution to the initial value problem. Let $v_1 =$

$v_1(t, r), v_2 = v_2(t, r)$ be two solutions of the Burgers equation (2.1.5) with initial conditions $v_1(0, \cdot) = v_{0,1}$ and $v_2(0, \cdot) = v_{0,2}$. We start from the equation

$$\partial_t \left(\frac{v_{\alpha,2} - v_{\alpha,1}}{(1 - 2M/r)^2} \right) + \partial_r \left(\frac{v_{\alpha,2}^2 - v_{\alpha,1}^2}{2(1 - 2M/r)} \right) = \alpha \partial_r (K(r) \partial_r (v_{\alpha,2} - v_{\alpha,1})). \quad (2.8.4)$$

Consider a regularization $\phi_\delta \rightarrow \phi$ of the function $\phi = |\cdot|$. Multiplying (2.8.4) by $\phi'_\delta(v_{\alpha,2} - v_{\alpha,1})$, we find

$$\begin{aligned} & \partial_t \left(\frac{\phi_\delta(v_{\alpha,2} - v_{\alpha,1})}{(1 - 2M/r)^2} \right) + \partial_r \left(\frac{\phi_\delta(v_{\alpha,2} - v_{\alpha,1})(v_{\alpha,2} + v_{\alpha,1})}{2(1 - 2M/r)} \right) \\ & \leq \frac{M}{r^2} \phi_\delta(v_{\alpha,2} - v_{\alpha,1}) \frac{v_{\alpha,2} + v_{\alpha,1}}{(1 - 2M/r)^2} + \alpha \partial_r (K(r) \partial_r \phi_\delta(v_{\alpha,2} - v_{\alpha,1})) \\ & \quad - \alpha K(r) \phi_\delta(v_{\alpha,2} - v_{\alpha,1}) (\partial_r (v_{\alpha,2} - v_{\alpha,1}))^2 \\ & \leq \phi_\delta(v_{\alpha,2} - v_{\alpha,1}) \frac{v_{\alpha,2} + v_{\alpha,1}}{2(1 - 2M/r)} + \alpha \partial_r (K(r) \partial_r \phi_\delta(v_{\alpha,2} - v_{\alpha,1})). \end{aligned}$$

Taking $\delta \rightarrow 0$ and integrating with respect to the space variable from $2M$, we have

$$\begin{aligned} \frac{d}{dt} \int_{2M}^{+\infty} \frac{|v_{\alpha,2} - v_{\alpha,1}|}{(1 - 2M/r)^2} dr & \leq \int_{2M}^{+\infty} \frac{|v_{\alpha,2} - v_{\alpha,1}|}{4M(1 - 2M/r)^2} (v_{\alpha,2} + v_{\alpha,1}) dr \\ & \leq \frac{1}{2\epsilon M} \int_{2M}^{+\infty} \frac{|v_{\alpha,2} - v_{\alpha,1}|}{(1 - 2M/r)^2} dr. \end{aligned}$$

The Gronwall's inequality gives

$$\begin{aligned} \int_{2M}^{+\infty} \frac{|v_{\alpha,2}(t, r) - v_{\alpha,1}(t, r)|}{(1 - 2M/r)^2} & \leq e^{\frac{t}{2\epsilon M}} \int_{2M}^{+\infty} \frac{|v_{\alpha,2}(0, r) - v_{\alpha,1}(0, r)|}{(1 - 2M/r)^2} \\ & = e^{\frac{t}{2\epsilon M}} \int_{2M}^{+\infty} \frac{|v_{0,2} - v_{0,1}|}{(1 - 2M/r)^2}. \end{aligned}$$

By taking $\alpha \rightarrow 0$, we have the L^1 stability result.

2. We rely again on (2.8.2) and write

$$\begin{aligned} \frac{d}{dt} \widetilde{TV}(v_\alpha) &= \frac{d}{dt} \int_{2M}^{+\infty} \left| - \left| \frac{v_\alpha^2 - 1/\epsilon^2}{1 - 2M/r} + \frac{1}{\epsilon^2} \right|^{-1/2} \operatorname{sgn}(v_\alpha) \frac{\partial_t v_\alpha}{(1 - 2M/r)^2} \right. \\ &\quad \left. + \left| \frac{v_\alpha^2 - 1/\epsilon^2}{1 - 2M/r} + \frac{1}{\epsilon^2} \right|^{1/2} \partial_r (\operatorname{sgn}(v_\alpha)) \right| \\ &\leq \int_{2M}^{+\infty} \left| \frac{v_\alpha^2 - 1/\epsilon^2}{1 - 2M/r} + \frac{1}{\epsilon^2} \right|^{-1/2} \frac{\partial_t |\partial_t v_\alpha|}{(1 - 2M/r)^2} - \frac{1}{2} \left| \frac{v_\alpha^2 - 1/\epsilon^2}{1 - 2M/r} + \frac{1}{\epsilon^2} \right|^{-3/2} \frac{\partial_t (v_\alpha^2) |\partial_t v_\alpha|}{(1 - 2M/r)^3}. \end{aligned}$$

By deriving (2.8.2) once with respect to t , we have

$$\partial_t \left(\frac{\partial_t v_\alpha}{(1 - 2M/r)^2} \right) + \partial_r \left(\frac{v_\alpha \partial_t v_\alpha}{1 - 2M/r} \right) = \alpha \partial_r (K(r) \partial_r \partial_t v_\alpha). \quad (2.8.5)$$

Introduce a sequence of smooth functions $\phi_\delta : \mathbb{R} \rightarrow \mathbb{R}$ such that $\lim_{\delta \rightarrow 0} \phi_\delta \rightarrow \phi$ in the distributional sense where $\phi = \phi(x) = |x| = \operatorname{sgn}(x)x$. Now multiply (2.8.5) by $\phi'_\delta(\partial_t v_\alpha)$:

$$\begin{aligned} &\partial_t \left(\frac{\phi_\delta(\partial_t v_\alpha)}{(1 - 2M/r)^2} \right) + \partial_r \left(\frac{v_\alpha \phi_\delta(\partial_t v_\alpha)}{1 - 2M/r} \right) + (\phi_\delta(\partial_t v_\alpha) - \phi'_\delta(\partial_t v_\alpha) \partial_t v_\alpha) \partial_r \frac{v_\alpha}{1 - 2M/r} \\ &= \alpha \partial_r (K(r) \partial_r \phi_\delta(\partial_t v_\alpha)) - \alpha (K(r) (\partial_{tr}^2 v_\alpha)^2 \phi''_\delta(\partial_t v_\alpha)). \end{aligned}$$

Letting $\delta \rightarrow 0$, we have

$$\partial_t \left(\frac{|\partial_t v_\alpha|}{(1 - 2M/r)^2} \right) + \partial_r \left(\frac{v_\alpha |\partial_t v_\alpha|}{1 - 2M/r} \right) \leq \alpha \partial_r (K(r) \partial_r |\partial_t v_\alpha|),$$

which gives

$$\begin{aligned} \frac{d}{dt} \widetilde{TV}(v_\alpha) &\leq \int_{2M}^{+\infty} - \left| \frac{v_\alpha^2 - 1/\epsilon^2}{1 - 2M/r} + \frac{1}{\epsilon^2} \right|^{-1/2} \partial_r \left(\frac{v_\alpha |\partial_t v_\alpha|}{1 - 2M/r} \right) \\ &\quad - \frac{1}{2} \left| \frac{v_\alpha^2 - 1/\epsilon^2}{1 - 2M/r} + \frac{1}{\epsilon^2} \right|^{-3/2} \frac{\partial_t (v_\alpha^2) |\partial_t v_\alpha|}{(1 - 2M/r)^3} \\ &\leq \int_{2M}^{+\infty} \left| \frac{v_\alpha^2 - 1/\epsilon^2}{1 - 2M/r} + \frac{1}{\epsilon^2} \right|^{-3/2} (1 - 2M/r)^{-3} \left(v_\alpha \partial_t v_\alpha |\partial_t v_\alpha| - \frac{1}{2} \partial_t (v_\alpha^2) |\partial_t v_\alpha| \right) = 0. \end{aligned}$$

Taking $\alpha \rightarrow 0$, we have thus the result that the weighted total variation is non-increasing with respect to the time variable. \square

Negative velocity

For the negative velocity, we can actually have some more results derived by the vanishing viscosity method. In the concern of such behaviors, we use the Burgers equation in the form of (3.1.2) and introduce the L^1 norm $\mathcal{M} = \mathcal{M}(t; v)$:

$$\mathcal{M}(t; v) := \int_{2M}^{+\infty} \left| \left(1 - \frac{2M}{r}\right) \partial_r \left(\frac{v^2}{2}\right) - \frac{M}{r^2} \left(v^2 - \frac{1}{\epsilon^2}\right) \right| dr, \quad (2.8.6)$$

Note that $\lim_{r \rightarrow 2M} \mathcal{M}(t; v) = TV(\frac{v(t, \cdot)^2}{2})$. It is also obvious that for a steady state solution $v_* = v_*(r)$, $\mathcal{M}(t; v_*) \equiv 0$ holds for all $t \geq 0$.

Lemma 2.8.2 (Bound of norm \mathcal{M}). *Let $v = v(t, r)$ be solution to the Burgers equation on a Schwarzschild background (3.1.2), then $\mathcal{M}(t; v)$ given by (2.8.6) is uniformly bounded for $t > 0$.*

Proof. We consider the following viscous Burgers equation associated with (3.1.2):

$$\partial_t v_\alpha + \partial_r \left(\left(1 - \frac{2M}{r}\right) \frac{v_\alpha^2}{2} \right) = \frac{M}{r^2} \left(2v_\alpha^2 - \frac{1}{\epsilon^2}\right) + \alpha \partial_r \left(\left(1 - \frac{2M}{r}\right)^2 K(r) \partial_r v_\alpha \right). \quad (2.8.7)$$

Here, $\alpha > 0$ is a parameter and $K = K(r)$ a smooth positive function. Multiply (2.8.7) by $\phi'_\delta(v_\alpha)$ whose limit of vanishing δ is the absolute value function $\phi = |\cdot|$:

$$\begin{aligned} & \partial_t \phi_\delta(v_\alpha) + \partial_r \left(\left(1 - \frac{2M}{r}\right) \Phi_\delta(v_\alpha) \right) \\ &= -\phi'_\delta(v_\alpha) \frac{M}{r^2} v_\alpha^2 + \frac{2M}{r^2} \Phi_\delta(v_\alpha) + \alpha \partial_r \left(\left(1 - \frac{2M}{r}\right)^2 K(r) \partial_r \phi_\delta(v_\alpha) \right) \\ & \quad - \alpha \left(1 - \frac{2M}{r}\right)^2 K(r) \phi''_\delta(v_\alpha) (\partial_r v_\alpha)^2 \\ & \leq \frac{2M}{r^2} \Phi_\delta(v_\alpha) + \alpha \partial_r \left(\left(1 - \frac{2M}{r}\right)^2 K(r) \partial_r \phi_\delta(v_\alpha) \right), \end{aligned}$$

where the smooth function satisfies $\Phi'_\delta(x) = x \phi'_\delta(x)$. Integrating from $2M$ to infinity and taking $\delta \rightarrow 0$, we have

$$\mathcal{M}(t; v_\alpha) = \int_{2M}^{+\infty} |\partial_t v_\alpha| dr \leq \int_{2M}^{+\infty} \frac{2M}{r^2} v_\alpha^2 dr \leq \frac{1}{\epsilon^2} \int_{2M}^{+\infty} \frac{2M}{r^2} dr \leq \frac{1}{\epsilon^2}.$$

We have thus get the lemma by taking $\alpha \rightarrow 0$. \square

A solution to the relativistic Burgers model stays negative if it starts from a

negative velocity at $t = 0$.

Lemma 2.8.3 (The negative velocity). *Fix some $-\frac{1}{\epsilon} < v_0 = v_0(r) \leq 0$ defined on $(2M, +\infty)$. Then the solution to the relativistic Burgers equation (3.1.2) on a Schwarzschild background $v = v(t, r)$ with the initial condition $v(0, \cdot) = v_0$ satisfies $-\frac{1}{\epsilon} < v(t, \cdot) \leq 0$ for all $t > 0$.*

Proof. We consider the following viscous Burgers equation

$$\partial_t v_\alpha + \partial_r \left(\left(1 - \frac{2M}{r}\right) \frac{v_\alpha^2}{2} \right) = \frac{M}{r^2} \left(2v_\alpha^2 - \frac{1}{\epsilon^2}\right) + \alpha \partial_r \left(\left(1 - \frac{2M}{r}\right)^2 K(r) \partial_r v_\alpha \right),$$

where $\alpha > 0$ is a parameter and $K = K(r) > 0$ a smooth function. Now we consider a sequence of smooth functions $(\varphi_\delta)_\delta$ such that $\lim_{\delta \rightarrow 0} \varphi_\delta = \varphi$ where

$$\varphi(x) = \begin{cases} 0, & x \leq 0, \\ x, & x > 0. \end{cases} \quad (2.8.8)$$

Define then a function $\psi_\delta := \psi_\delta(x)$ such that $\psi'_\delta(x) = x\varphi'_\delta(x)$ holds for all $x \in \mathbb{R}$.

Multiply (2.8.7) by $\varphi'_\delta(v_\alpha)$, we have

$$\begin{aligned} & \partial_t (\varphi_\delta(v_\alpha)) + \partial_r \left(\left(1 - \frac{2M}{r}\right) \psi_\delta(v_\alpha) \right) - \varphi'_\delta(v_\alpha) \frac{M}{r^2} \left(2v_\alpha^2 - \frac{1}{\epsilon^2}\right) \\ &= \frac{2M}{r^2} \psi_\delta(v_\alpha) + \left(1 - \frac{2M}{r}\right) \psi'_\delta(v_\alpha) \partial_r v_\alpha - \varphi'(v_\alpha) \partial_r \left(\left(1 - \frac{2M}{r}\right) \frac{v_\alpha^2}{2} \right) \\ &+ \alpha \varphi'_\delta(v_\alpha) \left(1 - \frac{2M}{r}\right)^2 \partial_r (K(r) \partial_r v_\alpha) \\ &= -\varphi'_\delta(v_\alpha) \frac{M}{r^2} v_\alpha^2 + \frac{2M}{r^2} \psi_\delta(v_\alpha) + \alpha \partial_r \left(\left(1 - \frac{2M}{r}\right)^2 K(r) \partial_r \varphi_\delta(v_\alpha) \right) \\ &- \alpha \left(1 - \frac{2M}{r}\right)^2 K(r) \varphi''_\delta(v_\alpha) (\partial_r v_\alpha)^2 \\ &\leq -\varphi'_\delta(v_\alpha) \frac{M}{r^2} v_\alpha^2 + \frac{2M}{r^2} \psi_\delta(v_\alpha) + \alpha \partial_r \left(\left(1 - \frac{2M}{r}\right)^2 K(r) \partial_r \varphi_\delta(v_\alpha) \right). \end{aligned}$$

Letting $\delta \rightarrow 0$, we reach the following inequality

$$\partial_t v_\alpha^+ + \partial_r \left(\left(1 - \frac{2M}{r}\right) \frac{(v_\alpha^+)^2}{2} \right) \leq H(v_\alpha^+) \frac{M}{r^2} \left(v_\alpha^2 - \frac{1}{\epsilon^2}\right) + \frac{2M}{r^2} \frac{(v_\alpha^+)^2}{2} + \alpha \partial_r \left(\left(1 - \frac{2M}{r}\right)^2 K(r) \partial_r v_\alpha^+ \right),$$

where $v_\alpha^+ = \max(0, v_\alpha)$ and $H = \varphi' \geq 0$ in the distributional sense. By the definition, $H'(v_\alpha^+) \leq 0$. Integrating with respect to the space variable r from $2M$ to infinity, we

obtain

$$\frac{d}{dt} \int_{2M}^{+\infty} v_{\alpha}^{+}(t, r) dr \leq \int_{2M}^{+\infty} \frac{M}{r^2} v_{\alpha}^{+}(t, r)^2 dr \leq \int_{2M}^{+\infty} \frac{1}{2M\epsilon} v_{\alpha}^{+}(t, r) dr.$$

Gronwall's inequality gives

$$\int_{2M}^{+\infty} v_{\alpha}^{+}(t, r) dr \leq e^{\frac{t}{2M\epsilon}} \int_{2M}^{+\infty} v_{\alpha}^{+}(0, r) dr = 0.$$

Taking $\alpha \rightarrow 0^{+}$, we obtain the result. \square

Observe that Lemma 2.8.3 does not hold for a positive initial velocity $v_0 > 0$. Indeed, by setting $w = -v$, we have

$$\partial_t w - \left(1 - \frac{2M}{r}\right) \partial_r \left(\frac{w^2}{2}\right) = -\frac{M}{r^2} \left(w^2 - \frac{1}{\epsilon^2}\right), \quad (2.8.9)$$

from which we see that the relativistic Burgers equation (3.1.1) and (2.8.9) are not symmetric with respect to the signs. We now give a result for negative velocities obtained by the vanishing viscosity method.

Theorem 2.8.4 (The vanishing viscosity method for negative velocities). Let $\frac{1}{\epsilon} < v_0 = v_0(r) < 0$ be a given velocity and we denote by $v = v(t, r)$ the solution of the relativistic Burgers equation on a Schwarzschild background (3.1.2) with initial condition $v(t, \cdot) = v_0$. Then the solution $v = v(t, r)$ satisfies $\mathcal{M}(t; v)$ given by (2.8.6) is non-increasing with respect to the time variable $t > 0$.

Proof. Lemma 2.8.3 gives the fact that the solution $v < 0$ holds for all the time $t > 0$. Derive the viscous Burgers equation (2.8.7) once with respect to t , we have

$$\partial_t(\partial_t v_{\alpha}) + \partial_r \left(\left(1 - \frac{2M}{r}\right) v_{\alpha} \partial_t v_{\alpha} \right) = \frac{4M}{r^2} v_{\alpha} \partial_t v_{\alpha} + \alpha \partial_r \left(\left(1 - \frac{2M}{r}\right)^2 K(r) \partial_r \partial_t v_{\alpha} \right). \quad (2.8.10)$$

Multiply (2.8.10) by $\phi'_{\delta}(\partial_t v_{\alpha})$ where we have introduced a sequence of smooth functions ϕ_{δ} such that $\phi_{\delta} \rightarrow |\cdot|$ when $\delta \rightarrow 0$ in the sense of distribution and

$$\begin{aligned} & \partial_t \phi_{\delta}(\partial_t v_{\alpha}) + \partial_r \left(\left(1 - \frac{2M}{r}\right) v_{\alpha} \phi_{\delta}(\partial_t v_{\alpha}) \right) + (\phi_{\delta}(\partial_t v_{\alpha}) - \phi'_{\delta}(\partial_t v_{\alpha}) \partial_t v_{\alpha}) \partial_r \left(\left(1 - \frac{2M}{r}\right) v_{\alpha} \right) \\ &= 4 \frac{M}{r^2} \phi'_{\delta}(\partial_t v_{\alpha}) v_{\alpha} \partial_t v_{\alpha} + \alpha \partial_r \left(\left(1 - \frac{2M}{r}\right)^2 K(r) \partial_r \phi_{\delta}(\partial_t v_{\alpha}) \right). \end{aligned}$$

Taking $\delta \rightarrow 0$, we have

$$\begin{aligned} & \partial_t |\partial_t v_\alpha| + \partial_r \left(\left(1 - \frac{2M}{r}\right) v_\alpha |\partial_t v_\alpha| \right) \\ & \leq 2 \frac{M}{r^2} (v_\alpha |\partial_t v_\alpha| + \operatorname{sgn}'(v_\alpha) (\partial_t v_\alpha)^2) + \alpha \partial_r \left(\left(1 - \frac{2M}{r}\right)^2 K(r) \partial_r \phi_\delta (\partial_t v_\alpha) \right). \end{aligned}$$

Since $\operatorname{sgn}'(v_\alpha) \geq 0$ and $v_\alpha < 0$ according to Lemma 2.8.3, we have, by integration

$$\frac{d}{dt} \int_{2M}^{+\infty} |\partial_t v_\alpha| dr \leq 0.$$

Taking $\alpha \rightarrow 0$, we have the fact that $\mathcal{M}(t; v)$ is non-increasing with respect to $t > 0$. □

2.9 Time-asymptotic behavior of weak solutions

The generalized characteristics

In this section, we consider the weak solution to the relativistic Burgers model (2.1.5) with initial data $v_0 = v_0(r)$ having bounded weighted total variation (2.5.1) and such that

$$v_0(r) = \begin{cases} v_*(r), & 2M < r < r_*, \\ v_{**}(r), & r > r_{**}, \end{cases} \quad (2.9.1)$$

where $2M < r_* < r_{**}$ are two given radius and $v_* = v_*(r)$ and $v_{**} = v_{**}(r)$ are two smooth steady state solutions given by the static Burgers model (2.3.1) defined on $(2M, r_*)$ and $(2M, +\infty)$. Observe that we do not require v_* to be globally defined and it is possible that its velocity will vanish at some point $r = r_*^\sharp$. By the property of finite speed of propagation, the solution $v = v(t, r)$ is a steady state solution out of a bounded domain. Our main goal of this section is to prove Theorem 2.1.3. We would first introduce the *generalized characteristic for the relativistic Burgers model*.

Definition 2.9.1. *A generalized characteristic for the relativistic Burgers model on a Schwarzschild spacetime (2.1.5) associated with the solution $v = v(t, r)$ is an integral curve satisfying*

$$\frac{d\xi}{dt} = \left(1 - \frac{2M}{\xi}\right) v(t, \xi), \quad (2.9.2)$$

in the sense that

$$\frac{d\xi}{dt} \in \left[\left(1 - \frac{2M}{\xi}\right) v(t, \xi+), \left(1 - \frac{2M}{\xi}\right) v(t, \xi-) \right] \quad \text{a.e. in } t. \quad (2.9.3)$$

Since the Burgers model is hyperbolic, we see that the generalized characteristic is well-defined. Moreover, from the definition, a generalized characteristic propagates either with a shock speed or with a characteristic speed. Now through the points $(0, r_*)$ and $(0, r_{**})$, we draw two generalized characteristics associated with the Burgers solution $v = v(t, r)$, denoted by $\xi_* = \xi_*(t)$ and $\xi_{**} = \xi_{**}(t)$. In order to study the asymptotic behavior of the Burgers solutions on the Schwarzschild background, we introduce the two following quantities:

$$\begin{aligned} P(t) &= \min_{r \in (2M, +\infty)} \int_{2M}^r \left(\frac{v(t, r') - v_*(r')}{(1 - 2M/r')^2} \right) dr', \\ Q(t) &= \max_{r \in (2M, +\infty)} \int_r^{+\infty} \left(\frac{v(t, r') - v_{**}(r')}{(1 - 2M/r')^2} \right) dr'. \end{aligned} \quad (2.9.4)$$

We claim that P, Q are actually constants.

Proposition 2.9.2. *The functions $P = P(t)$ and $Q = Q(t)$ introduced in (2.9.4) satisfy that $P(t) \equiv P(0)$ and $Q(t) \equiv Q(0)$ for all $t \geq 0$.*

Proof. We have

$$\partial_t \left(\frac{v - v_*}{(1 - 2M/r)^2} \right) + \partial_r \left(\frac{v^2 - v_*^2}{2(1 - 2M/r)} \right) = 0 \quad (2.9.5)$$

and we introduce the function

$$\Theta(t, r) := \int_{2M}^r \frac{v(0, r') - v_*(r')}{(1 - 2M/r')^2} dr' - \int_0^t \frac{v(t', r)^2 - v_*(r)^2}{2(1 - 2M/r)} dt'. \quad (2.9.6)$$

In view of the conservative form (2.9.5), we find

$$\int_S \frac{v(t', r') - v_*(r')}{(1 - 2M/r')^2} dr' - \int_S \frac{v(t', r')^2 - v_*(r')^2}{2(1 - 2M/r')} dt' \equiv \Theta(t, r), \quad (2.9.7)$$

where S is any path from $(0, 2M)$ to (t, r) .

Now we have

$$\begin{aligned} \partial_t \Theta &= \frac{v(t, r)^2 - v_*(r)^2}{2(1 - 2M/r)}, \\ \partial_r \Theta &= \frac{v(0, r) - v_*(r)}{(1 - 2M/r)^2} - \int_0^t \partial_r \left(\frac{v(t', r)^2 - v_*(r)^2}{2(1 - 2M/r)} \right) dt' \\ &= \frac{v(0, r) - v_*(r)}{(1 - 2M/r)^2} + \int_0^t \partial_{t'} \left(\frac{v(t', r) - v_*(r)}{(1 - 2M/r)^2} \right) dt' = \frac{v(t, r) - v_*(r)}{(1 - 2M/r)^2}. \end{aligned}$$

Hence, we deduce the equation

$$\partial_t \Theta + \left(1 - \frac{2M}{r}\right) \frac{v + v_*}{2} \partial_r \Theta = 0. \quad (2.9.8)$$

From (2.9.7), we see that $P(t) = \min_r \Theta(t, r)$. To get the result, we only need to prove that $\frac{d}{dt} \min_r \Theta(t, r) = 0$.

Suppose that at a given time $t = t_0 > 0$, the minimum is taken at the radius $r = r_0 > 2M$. Now through this point (t_0, r_0) , we draw a curve χ characterized by

$$\frac{dr}{dt} = \left(1 - \frac{2M}{r}\right) \frac{v + v_*}{2} \quad (2.9.9)$$

and we see by (2.9.8) that Θ is a constant along the curve χ . We now claim that there exists a constant $\delta > 0$ such that χ is well-defined on the neighborhood of t_0 denoted by $(t_0 - \delta, t_0 + \delta)$. Indeed, it is obvious if the Burgers solution v is continuous at the point (t_0, r_0) . Otherwise, for (t_0, r_0) as a point of discontinuity of $v = v(t, r)$, the entropy condition (2.4.12) guarantees that ξ is defined on a neighborhood of t_0 . A similar analysis leads to the result of Q . \square

Generalized N-waves

We now consider the case that $v_*(r) \leq v_{**}(r)$ for all r in the domain of definition.

Lemma 2.9.3 (Comparison with the generalized rarefaction). *Suppose that the two steady state solutions satisfy $v_*(r) \leq v_{**}(r)$ for all r in the domain of definition. Let $\xi_* = \xi_*(t)$, $\xi_{**} = \xi_{**}(t)$ be the two generalized characteristics starting from $(0, r_*)$, $(0, r_{**})$, respectively, in the sense of (2.9.3), then for any radius $r \in (\xi_*(t), \xi_{**}(t))$, we have*

$$|v(t, r) - w(t, r)| \leq \frac{r_{**} - r_*}{1 - 2M/r_*} t^{-1}, \quad (2.9.10)$$

where $w = w(t, r)$ is the generalized rarefaction given by (3.5.7).

Proof. We now draw a characteristic $\tilde{r} = \tilde{r}(\tilde{t})$ backward from any point (t, r) with $r \in (\xi_*(t), \xi_{**}(t))$ where $\xi_* = \xi_*(t)$, $\xi_{**} = \xi_{**}(t)$ are generalized characteristics drawing from $(0, r_*)$ and $(0, r_{**})$ respectively. Then the following ordinary differential equation holds:

$$\frac{d\tilde{r}}{d\tilde{t}} = \left(1 - \frac{2M}{\tilde{r}}\right) w(\tilde{t}, \tilde{r}), \quad \frac{dw}{d\tilde{t}} = \frac{M}{\tilde{r}^2} (w^2 - 1/\epsilon^2), \quad w(t, r) = v(t, r), \quad \tilde{r}(t) = r.$$

Recall the formula of the generalized rarefaction wave given in (3.5.7), $w = w(\tilde{t}, \tilde{r})$ is exactly the generalized rarefaction curve.

Now by the entropy condition (2.4.12), we know that the curve \tilde{r} will intersect neither ξ_* nor ξ_{**} . In particular, the point r_0 will stay between r_* and r_{**} , which gives

$$\left| \frac{\xi(0) - \tilde{r}(0)}{t} \right| \leq \frac{r_{**} - r_*}{t},$$

where $\xi = \xi(t)$ is the generalized characteristic passing through (t, r) in the sense of Definition 2.9.1. We first note that $\xi(t) = \tilde{r}(t) = (1 - 2M/r)v(t, r)$. Using Taylor expansion at the point t and recall that the Burgers solution is a steady state (either a smooth solution or a weak solution with a steady state shock) along both ξ and \tilde{r} , we have

$$(1 - 2M/r_*)|v(t, r) - w(t, r)| \leq \frac{1}{t} \left| \xi(t) - \tilde{r}(t) - t(\xi'(t) - \tilde{r}'(t)) \right| \leq (r_{**} - r_*)t^{-1},$$

which completes the proof. \square

We now consider the distances between the two generalized characteristics ξ_* , ξ_{**} and the bounds of the rarefaction regions. This requires an analysis of those two quantities P, Q introduced in (2.9.4).

Lemma 2.9.4 (The distance from the rarefaction region). *Denote by $\sigma_* = \sigma_*(t)$ and $\sigma_{**} = \sigma_{**}(t)$ the lower and upper bounds of the rarefaction region associated with the left steady state v_* and right steady state v_{**} . We have*

$$|\xi_*(t) - \sigma_*(t) + \sqrt{-2Pt}| + |\xi_{**}(t) - \sigma_{**}(t) - \sqrt{2Qt}| = O(r_{**} - r_*). \quad (2.9.11)$$

Proof. By the definition of P , we should have

$$P = \min_{r \in (2M, +\infty)} \int_{2M}^r \left(\frac{v(t, r') - v_*(r')}{(1 - 2M/r')^2} \right) dr' = \int_{\xi_*(t)}^{\xi^*(t)} \left(\frac{v(t, r') - v_*(r')}{(1 - 2M/r')^2} \right) dr',$$

where $\xi^* = \xi^*(t)$ satisfies $v(t, \xi^*(t)) = v_*(\xi^*(t))$. Following from a similar calculation with that in Lemma 2.9.3, we have

$$\xi^*(t) = \sigma_*(t) + O(r_{**} - r_*)t^{-1}, \quad v - v_* = \frac{\xi'(t) - \xi'_*(t) + O(r - r_*)t^{-1}}{(1 - 2M/r')}.$$

We thus have

$$\begin{aligned} P &= \int_{\xi_*(t)}^{\sigma_*(t) + O(r_{**} - r_*)t^{-1}} \left(\frac{v(t, r') - v_*(r')}{(1 - 2M/r')^2} \right) dr' \\ &= \int_{\xi_*(t)}^{\sigma_*(t) + O(r_{**} - r_*)t^{-1}} \left(\frac{r'/t - \xi'_*(t) + O(r - r_*)t^{-1}}{(1 - 2M/r')^3} \right) dr' \end{aligned}$$

Use again the fact that v is a steady state solution along the generalized characteristic

ξ_* and v_* is a smooth steady state solution along the wave σ_* , we have

$$\begin{aligned}
 P = & -\frac{(\xi_*(t) - \sigma_*(t))^2}{2t} - 6M(\xi_*(t) - \sigma_*(t)) - 24M^2 \ln \frac{\xi_*(t) - 2M}{\sigma_*(t) - 2M} \\
 & + 8M^4 \left(\frac{1}{(\xi_*(t) - 2M)^2} - \frac{1}{(\sigma_*(t) - 2M)^2} \right) + 32M^3 \left(\frac{1}{\xi_*(t) - 2M} - \frac{1}{\sigma_*(t) - 2M} \right) \\
 & + O(r_* - r_{**})(\xi_*(t) - \sigma_*(t))t^{-1} + O(r_* - r_{**})t^{-1},
 \end{aligned} \tag{2.9.12}$$

which gives the result for ξ_* . A similar analysis leads to the result of ξ_{**} . \square

We now introduce the *generalized N-wave* associated with the two steady state solutions v_*, v_{**} of the relativistic Burgers model (2.1.5):

$$N(t, r; v_*, v_{**}) := \begin{cases} v_*(r), & r - \sigma_*(t) < \max\left(2M - \sigma_*(t), -\sqrt{-2Pt}\right), \\ w(t, r), & \max\left(2M, \sigma_*(t) - \sqrt{-2Pt}\right) < r < \sigma_{**}(t) + \sqrt{2Qt}, \\ v_{**}(r), & r - \sigma_{**}(t) > \sqrt{2Qt}, \end{cases} \tag{2.9.13}$$

where σ_*, σ_{**} are upper and lower bounds of the rarefaction region given in (2.4.1) and $w = w(t, r)$ is the generalized rarefaction curve (3.5.7). It follows immediately from (2.9.12) that

$$|v(t, r) - N(t, r; v_*, v_{**})| = O(r_{**} - r_*)t^{-1/2},$$

for r either between $\xi_*(t)$ and $\sigma_*(t) - \sqrt{-2Pt}$ or between $\xi_{**}(t)$ and $\sigma_{**}(t) + \sqrt{2Qt}$. We now consider the bounds of the rarefaction wave $w = w(t, r)$ containing in the N-wave.

Lemma 2.9.5 (The bounds of the rarefaction in N-wave). *Let $w = w(t, r)$ be the rarefaction wave containing in the generalized N-wave (2.9.13). Then the lower and upper bounds of w satisfies*

- If $v_* > 0$, the lower and upper bounds of w tend to infinity $r = +\infty$;
- If $v_{**} < 0$, the lower bound of w tends to the horizon of the black hole $r = 2M$, while its the upper bound tends to infinity $r = +\infty$.
- If $v_* < 0 < v_{**}$, the lower bound of w tends to $r = 2M$, while its the upper bound tends to infinity $r = +\infty$.

Proof. If $v_* > 0$, then $\sigma_* = O(t)$ when t is big enough. If $v_* < 0$, then $\sigma_* \rightarrow 2M$ when the time is big enough. A similar analysis leads to the result concerning v_{**} . \square

Proposition 2.9.6 (The generalized N-wave). *Let $v = v(t, r)$ be the solution of the relativistic Burgers equation on the Schwarzschild background (2.1.5) with given*

initial data $v_0 = v_0(r)$ satisfying (2.9.1) with $v_*(r) \leq v_{**}(r)$ for all r in the domain of definition. Let $N = N(t, r; v_*, v_{**})$ be the generalized N -wave associated with v_*, v_{**} given by (2.9.13) and $\xi_* = \xi_*(t), \xi_{**} = \xi_{**}(t)$ be the generalized characteristics starting from r_*, r_{**} , respectively, in the sense of (2.9.3).

- For any radius $r \in \left(\max(\xi_*(t), \sigma_*(t) - \sqrt{-2Pt}), \min(\xi_{**}(t), \sigma_{**}(t) + \sqrt{2Qt}) \right)$, the following estimate holds

$$|v(t, r) - N(t, r; v_*, v_{**})| = O(r_{**} - r_*)t^{-1},$$

- For any radius r either between $\xi_*(t)$ and $\sigma_*(t) - \sqrt{-2Pt}$ or between $\xi_{**}(t)$ and $\sigma_{**}(t) + \sqrt{2Qt}$, the following estimate holds

$$|v(t, r) - N(t, r; v_*, v_{**})| = O(r_{**} - r_*)t^{-1/2},$$

- The solution $v(t, r) \equiv N(t, r; v_*, v_{**})$ for any radius r in other regions.

Asymptotic shock waves

We now consider the asymptotic behavior of Burgers solutions with initial data v_0 given in (2.9.1) with the two given steady state solutions such that $v_*(r) > v_{**}(r)$ holds for all r in the domain of communication.

Lemma 2.9.7 (The shock wave at finite time). *Suppose that the two steady state solutions satisfy that $v_*(r) > v_{**}(r)$ for all r in the domain of communication. If moreover, $v_* > 0$ holds, then there exists a finite time $t_0 > 0$, such that for all $t > t_0$, the solution to the relativistic Burgers' equation (2.1.5) is a shock wave*

$$v_s(t, r) = \begin{cases} v_*(r), & r < \sigma(t), \\ v_{**}(r), & r > \sigma(t), \end{cases} \quad (2.9.14)$$

where $\sigma = \sigma(t)$ is the shock wave curve starting from $t = t_0$ defined by (2.4.3).

Proof. Denote by $\xi_* = \xi_*(t), \xi_{**} = \xi_{**}(t)$ the two generalized characteristics drawing from points $(0, r_*)$, $(0, r_{**})$, respectively, in the sense of Definition 2.9.1. Denote by $D(t) := \xi_{**}(t) - \xi_*(t)$ the distance between the two generalized characteristics. To prove the lemma, we need only to show that $D(t)$ vanishes at a finite time t_0 . Now write

$$\begin{aligned} \lambda_*(t) &= \left(1 - \frac{2M}{\xi_*(t)}\right)v_*(\xi_*(t)), & \lambda_{**}(t) &= \left(1 - \frac{2M}{\xi_{**}(t)}\right)v_{**}(\xi_{**}(t)), \\ \lambda_-(t) &= \left(1 - \frac{2M}{\xi_*(t)}\right)v(\xi_*(t) -), & \lambda_+(t) &= \left(1 - \frac{2M}{\xi_{**}(t)}\right)v(\xi_{**}(t) +). \end{aligned}$$

By (2.9.3), we have

$$D'(t) = \frac{\lambda_{**}(t) + \lambda_-(t)}{2} - \frac{\lambda_*(t) + \lambda_+(t)}{2} = \frac{1}{2}(\lambda_+(t) - \lambda_-(t)) + \frac{1}{2}(\lambda_{**}(t) - \lambda_*(t)). \quad (2.9.15)$$

From the two points $(t, \xi_*(t))$, $(t, \xi_{**}(t))$, we draw two characteristics backwards of time, respectively. Thanks to the entropy condition, they will always stay between $\xi_*(s)$ and $\xi_{**}(s)$ at every fixed time $0 \leq s < t$. Since the velocity is a steady state along these two characteristics, we have

$$D(t) \leq t(\lambda_+(t) - \lambda_-(t)).$$

Now recall the formula of the steady state solutions given by (2.3.1) and we have

$$v_*(r) = \text{sgn}(v_*)\sqrt{1 - \epsilon^2 K_*^2 \left(1 - \frac{2M}{r}\right)}, \quad v_{**}(r) = \text{sgn}(v_{**})\sqrt{1 - \epsilon^2 K_{**}^2 \left(1 - \frac{2M}{r}\right)}, \quad (2.9.16)$$

where K_* , K_{**} are two constants. Indeed, for the case that $v_* > v_{**}$ and $v_* > 0$, there are two possibilities as follows.

- If $v_* > v_{**} > 0$, it is necessary that $K_* < K_{**}$. We thus have

$$\lambda_{**}(t) - \lambda_*(t) \leq -\left(1 - \frac{2M}{r_*}\right)(\sqrt{1 - \epsilon^2 K_*^2} - \sqrt{1 - \epsilon^2 K_{**}^2}) < 0,$$

hence

$$D(t) \leq D(0) - \left(1 - \frac{2M}{r_*}\right)(\sqrt{1 - \epsilon^2 K_*^2} - \sqrt{1 - \epsilon^2 K_{**}^2})(t - t^{1/2}).$$

Moreover, notice that $v_* > v_{**} > 0$ requires that v_* is defined for $r \in (2M, +\infty)$, then v_* cannot vanish for all $t > 0$.

- If $v_* > 0 > v_{**}$, we have

$$\lambda_{**}(t) - \lambda_*(t) \leq -\left(1 - \frac{2M}{r_*}\right)\sqrt{1 - \epsilon^2 K_{**}^2} < 0,$$

which gives

$$D(t) \leq D(0) - \left(1 - \frac{2M}{r_*}\right)\sqrt{1 - \epsilon^2 K_{**}^2}(t - t^{1/2}).$$

Suppose now that there exists a time such $t_1 < t_0$ such that $\xi_*(t_1) = r_*^{\natural}$ where r_*^{\natural} is the vanishing velocity radius for v_* . We will show that $\xi_*(t) \leq r_*^{\natural}$. Indeed, if there exists a time $t_2 > t_1$ such that $\xi_*(t_2) > r_*^{\natural}$ and $\xi_*(t) \leq r_*^{\natural}$ for all $t_1 \leq t < t_2$, then it is necessary that $\frac{d\xi_*}{dt}|_{t=t_2} \geq 0$ where the derivative is given in the sense

of (2.9.3). We thus have $v_*(\xi_*(t_2))$, providing a contradiction.

Hence, for both cases, $D(t)$ vanishes for big enough $t > 0$. We have thus get the result. \square

If the left-hand steady state solution $v_* < 0$, we may not have a shock wave at the finite time. However, we have the following lemma.

Lemma 2.9.8 (The shock wave at infinity). *Suppose that the two steady state solutions contained in the initial data v_0 satisfy $0 > v_*(r) > v_{**}(r)$ for all r in the domain of definition and the solution to the relativistic Burgers model (2.1.5), say $v = v(t, r)$, satisfies $\lim_{t \rightarrow +\infty} v = v_s$ where v_s is the shock wave given by (2.9.14).*

Proof. We will use the same notations as Lemma 2.9.7 in order to consider the case $0 > v_* > v_{**}$. We have

$$\begin{aligned} \lambda_{**}(t) - \lambda_*(t) &= \left(1 - \frac{2M}{\xi_*(t) + D(t)}\right) v_{**}(\xi_*(t) + D(t)) - \left(1 - \frac{2M}{\xi_*(t)}\right) v_*(\xi_*(t)) \\ &\leq \left(1 - \frac{2M}{\xi_*(t)}\right) \left(v_{**}(\xi_*(t)) - v_*(\xi_*(t))\right) + \frac{2M}{r_{**}^2} D(t) v_{**}(\xi_{**}(t)). \end{aligned}$$

Hence, we have $D(t) \leq D(1)t^{1/2} \exp\left(v_{**}(r_{**})\frac{2M}{r_{**}^2}(t-1)\right)$. Since $v_{**}(r_{**})\frac{2M}{r_{**}^2} < 0$, the distance between the two generalized characteristics $D(t) \rightarrow 0$ when $t \rightarrow +\infty$. \square

We have thus established the desired result for initial data $v_*(r) > v_{**}(r)$.

Proposition 2.9.9 (The asymptotic shock wave). *Consider the solution $v = v(t, r)$ to the relativistic Burgers equation on the Schwarzschild background (2.1.5) with the given initial velocity $v_0 = v_0(r)$ satisfying (2.9.1) where $v_*(r) > v_{**}(r)$ for all r in the domain of definition. Let v_s be a shock wave of the Burgers model given by (2.9.14). Then the following asymptotic behavior holds:*

- If $v_* > 0$, then there exists a finite time t_0 such that $v = v_s$ for all $t > t_0$.
- If $v_* < 0$, then $v \rightarrow v_s$ when $t \rightarrow +\infty$.

From Propositions 2.9.6 and 2.9.9, we obtain the following corollary.

Corollary 2.9.10. *For a given initial velocity $v_0 = v_0(r)$ with bounded weighted total variation such that*

$$v_0 = v_*(r), \quad r \notin (r_*, r_{**})$$

where $v_* = v_*(r)$ is the steady state solution to the static Burgers equation (2.3.1) and $r_* < r_{**}$ are given in the interval $(2M, +\infty)$, then the solution to the relativistic Burgers model satisfies

$$|v(t, r) - v_*(r)| = O(t^{-1/2}).$$

Part II

Fluid Dynamics on a Schwarzschild Spacetime Background: A Numerical Study

Chapter 3

Fluid flows on the domain of outer communication of a Schwarzschild spacetime. A numerical study ¹

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¹ Réalisé en collaboration avec P.G. LeFloch et soumis à SIAM J. Numer. Anal. Preprint arXiv:1612.09204.

3.1 Introduction

In this paper and the companion papers [31, 32, 34], we study numerically compressible fluid flows on a Schwarzschild blackhole background. The present investigation is part of a research project by LeFloch and co-authors on designing numerical methods for relativistic fluid problems posed on curved spacetimes; see [1, 6, 23, 29, 30]. Building upon the numerical analysis in the later papers and on the analytical work performed by the authors in [31, 32, 34], we are able here to design several numerical schemes for the approximation of shock wave solutions to, both, the relativistic Burgers equation and the compressible Euler system under the assumption that the flow is spherically symmetric. Our schemes are asymptotic preserving and therefore allow us to investigate the late-time asymptotic of solutions. One important challenge addressed here is taking the curved geometry into account at the level of the discretization and handling the behavior of solutions near the horizon of the blackhole.

The *relativistic Burgers equation on a Schwarzschild background* reads as follows (see [31] for further details):

$$\partial_t \left(\frac{v}{(1 - 2M/r)^2} \right) + \partial_r \left(\frac{v^2 - 1}{2(1 - 2M/r)} \right) = 0, \quad r > 2M, \quad (3.1.1)$$

where we have normalized the light speed to unit and the unknown is the function $v = v(t, r) \in [-1, 1]$. This equation can also be put in the following non-conservative form:

$$\partial_t v + \partial_r \left(\left(1 - \frac{2M}{r}\right) \frac{v^2 - 1}{2} \right) = \frac{2M}{r^2} (v^2 - 1), \quad r > 2M. \quad (3.1.2)$$

Here $M > 0$ denotes the mass of the blackhole and, clearly, we recover the standard Burgers equations when the mass vanishes.

Our main contribution for the relativistic Burgers model above is as follows. First of all, we are going to construct a well-balanced finite volume method as well as a random choice method which, both, are capable to preserve the steady state solutions. We will use these schemes to investigate the following issues and validate and extend our theoretical results (briefly reviewed below in Theorems 3.2.1 to 3.2.3):

- The global-in-time existence theory for the generalized Riemann problem generated by an arbitrary initial discontinuity.
- The late-time behavior of an initially perturbed steady state solution, possibly containing a stationary shock wave.

Furthermore, our study here have led us to the following two conjectures for general

initial data.

Conjecture 3.1.1. *Given any compactly perturbed steady shock as an initially data, the solution to the relativistic Burgers model on a Schwarzschild background (3.1.1) converges to a steady state shock asymptotically in time.*

Conjecture 3.1.2. *Given an initial data $v_0 = v_0(r) \in [-1, 1]$ defined on $[2M, +\infty)$, the corresponding solution $v = v(t, r)$ to the relativistic Burgers model (3.1.1) is as follows:*

- *If $v_0(2M) = 1$, then there exists a finite time $t_0 > 0$ such that, for all $t > t_0$, the solution v is a single shock with left-hand state 1 and right-hand state $-\sqrt{\frac{2M}{r}}$.*
- *If $v_0(2M) < 1$ and $\lim_{r \rightarrow +\infty} v_0(r) > 0$, then there exists a finite time $t_0 > 0$ such that, for all $t > t_0$, the solution is $v(t, r) = -\sqrt{\frac{2M}{r}}$ for all $t > t_0$.*
- *If $v_0(2M) < 1$ and $\lim_{r \rightarrow +\infty} v_0(r) \leq 0$, then there exists a finite time $t_0 > 0$ such that, for all $t > t_0$, the solution to the relativistic Burgers model satisfies for all $t > t_0$*

$$v(t, r) = -\sqrt{1 - (1 - (v_0^\infty)^2)\left(1 - \frac{2M}{r}\right)}, \quad \lim_{r \rightarrow +\infty} v_0(r) =: v_0^\infty \leq 0.$$

We also investigate solutions to the Euler system on a Schwarzschild background, which takes the form:

$$\begin{aligned} & \partial_t \left(r^2 \frac{1 + k^2 v^2}{1 - v^2} \rho \right) + \partial_r \left(r(r - 2M) \frac{1 + k^2}{1 - v^2} \rho v \right) = 0, \\ & \partial_t \left(r(r - 2M) \frac{1 + k^2}{1 - v^2} \rho v \right) + \partial_r \left((r - 2M)^2 \frac{v^2 + k^2}{1 - v^2} \rho \right) \\ & = 3M \left(1 - \frac{2M}{r} \right) \frac{v^2 + k^2}{1 - v^2} \rho - M \frac{r - 2M}{r} \frac{1 + k^2 v^2}{1 - v^2} \rho + 2 \frac{(r - 2M)^2}{r} k^2 \rho, \end{aligned} \tag{3.1.3}$$

where the light speed is normalized to unit and $k \in (0, 1]$ denotes the sound speed. By formally letting $k \rightarrow 0$, we can recover the pressureless Euler system, from which in turn we can derive the relativistic Burgers equation above. On the other hand, by letting the blackhole mass $M \rightarrow 0$, we recover the relativistic Euler system in the Minkowski spacetime. Furthermore, we can also write the relativistic Euler equations

in the Schwarzschild spacetime in the following form:

$$\begin{aligned} \partial_t \left(\frac{1+k^2 v^2}{1-v^2} \rho \right) + \partial_r \left((1-2M/r) \frac{1+k^2}{1-v^2} \rho v \right) &= -\frac{2}{r} (1-2M/r) \frac{1+k^2}{1-v^2} \rho v, \\ \partial_t \left(\frac{1+k^2}{1-v^2} \rho v \right) + \partial_r \left((1-2M/r) \frac{v^2+k^2}{1-v^2} \rho \right) & \\ = \frac{-2r+5M}{r^2} \frac{v^2+k^2}{1-v^2} \rho - \frac{M}{r^2} \frac{1+k^2 v^2}{1-v^2} \rho + 2 \frac{r-2M}{r^2} k^2 \rho. & \end{aligned} \quad (3.1.4)$$

Our study of the relativistic Euler equations on a Schwarzschild background (3.1.3) is based on the construction of a finite volume method with second-order accuracy, which preserves the family of steady state solutions. Our numerical study suggests a global-in-time existence theory for the generalized Riemann problem, whose explicit form is not yet known theoretically. In particular, we exhibit here solutions containing up to three steady state components, connected by a 1-wave and a 2-wave.

Conjecture 3.1.3. *Let $(\rho_*, v_*) = (\rho_*, v_*)(r)$, $r > 2M$ be a smooth steady state solution to the Euler model above and let $(\rho_0, v_0) = (\rho_0, v_0)(r) = (\rho_*, v_*)(r) + (\delta_\rho, \delta_v)(r)$ where $(\delta_\rho, \delta_v) = (\delta_\rho, \delta_v)(r)$ has compact support. Then, the corresponding solution to the relativistic Euler equation on a Schwarzschild background $(\rho, v) = (\rho, v)(t, r)$ satisfies:*

- *If $|\int \delta_\rho(r) dr| + |\int \delta_v(r) dr| = 0$, then there exists a time $t_0 > 0$ such that $(\rho, v)(t, r) = (\rho_*, v_*)(r)$ for all $t > t_0$.*
- *If $|\int \delta_\rho(r) dr| + |\int \delta_v(r) dr| \neq 0$, then there exists a time $t_0 > 0$ such that $(\rho, v)(t, r) = (\rho_{**}, v_{**})(r)$ for all $t > t_0$, where (ρ_{**}, v_{**}) is a possibly different steady state solution.*

Using steady shocks (to be defined in Section 3.8), we also have the following.

Conjecture 3.1.4. *Let $(\rho_*, v_*) = (\rho_*, v_*)(r)$, $r > 2M$ be a steady shock and let $(\rho_0, v_0) = (\rho_*, v_*)(r) + (\delta_\rho, \delta_v)(r)$ where $(\delta_\rho, \delta_v) = (\delta_\rho, \delta_v)(r)$ is a compactly supported perturbation. Then there exists a finite time $t > t_0$ such that the solution $(\rho, v) = (\rho, v)(t, r)$ is a (possibly different) steady shock.*

Our numerical approach on the Glimm scheme is motivated by the approach proposed by Glimm, Marshall, and Plohr [14] for quasi-one-dimensional gas flows. We rely on static solutions and on the generalized Riemann problem, which we studied extensively in [31, 32, 34] for the relativistic models under consideration here. The numerical analysis of hyperbolic problems posed on curved spacetimes was initiated in [1, 6, 23, 29, 30] using the finite volume methodology, and we also recall that hyperbolic conservation laws on curved spaces are also studied by Dziuk and co-authors [11, 12].

This paper is organized as follows. In Section 3.2, we briefly overview our theoretical results for the relativistic Burgers model. We include a full description of the family of steady state solutions, as well as some outline of the existence theory for the initial data problem and the nonlinear stability of piecewise steady solutions. In Section 3.3, we introduce a finite volume method for the relativistic Burgers model (3.1.1), which is well-balanced and second-order accurate. In Section 3.4, we apply our scheme in order to study the generalized Riemann problem and to elucidate the late-time behavior of perturbations of steady solutions.

Building on our theoretical results, in Section 3.5 we implement a generalized Glimm scheme for the relativistic Burgers model (3.1.1). Our numerical method is based on an explicit and accurate solver of the generalized Riemann problem and, therefore, our method preserves all steady state solutions. Numerical experiments are presented in Section 3.6, in which we are able to validate and expand the theoretical results in Section 3.2. Our method avoids to introduce numerical diffusion and provide an efficient approach for computing shock wave solutions. Furthermore, in Section 3.7 we apply both methods to the study of the initial problem for the relativistic Burgers equation when the initial velocity is rather arbitrary and we validate our Conjectures 3.1.1 and 3.1.2 and, along the way, clarify the behavior of the fluid flow near the blackhole horizon.

Next, in Section 3.8, we turn our attention to the relativistic Euler model on a Schwarzschild background. We begin by reviewing some theoretical results, including the existence theory for steady state solutions, the construction of a solver for the generalized Riemann problem, and the existence theory for the initial value problem. We are then in a position, in Section 3.9, to construct a finite volume method for the relativistic Euler model. Our method is second-order accuracy and is proven to be well-balanced. With the proposed algorithm, in Section 3.10, we are able to tackle the generalized Riemann problem (which has not yet been solved in a closed form) and we study the nonlinear stability of steady state solutions when the perturbation has compact support. This leads us to numerically demonstrate the validity of Conjectures 3.1.3 and 3.1.4 above.

3.2 Overview of the theory for the relativistic Burgers model

An important class of solutions to the relativistic Burgers model (3.1.1) is provided by the *steady state solutions*, that is, solutions depending on the space variable r only:

$$\partial_r \left(\frac{v^2 - 1}{2(1 - 2M/r)} \right) = 0. \quad (3.2.1)$$

Clearly, $\left(\frac{v^2-1}{2(1-2M/r)}\right)$ is then a constant, and we see that steady state solutions for the Burgers equation are

$$v(r) = \pm\sqrt{1 - K^2(1 - 2M/r)}, \quad (3.2.2)$$

where $K > 0$ is a constant and, clearly, the sign of a steady state cannot change. The following remarks are in order:

- $v = v(r)$ is a uniformly bounded and smooth in r and it admits the finite limit $\lim_{r \rightarrow 2M} v(r) = \pm 1$ at the blackhole horizon.
- When $0 < K < 1$, one has $\lim_{r \rightarrow +\infty} v(r) = \pm\sqrt{1 - K^2}$.
- When $K = 1$ or equivalently, $v_*^\pm = \pm\sqrt{\frac{2M}{r}}$, the steady state solution vanishes at infinity. These two solutions are referred to as the *critical steady state solutions*.
- When $K > 1$, the steady state solution vanishes at a finite radius $r^{\natural} = \frac{2MK^2}{1-K^2}$, which we may refer to as the *vanishing point*.

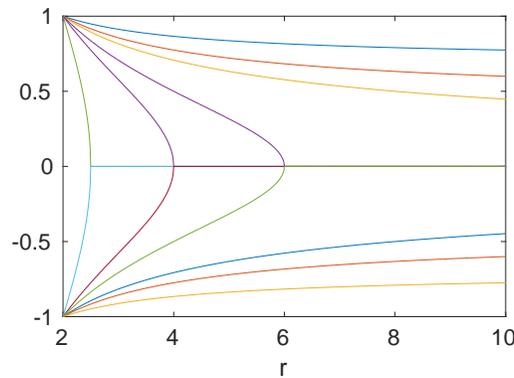


Figure 3.2.1: Steady state solutions for the relativistic Burgers model

In addition to the smooth steady state solutions, we can also define the class of *steady shocks for the relativistic Burgers equation*, which are given by

$$v = \begin{cases} \sqrt{1 - K^2(1 - 2M/r)}, & 2M < r < r_0, \\ -\sqrt{1 - K^2(1 - 2M/r)}, & r > r_0, \end{cases} \quad (3.2.3)$$

where K is a constant and r_0 is any given radius. The solution (3.2.3) is time-independent and the discontinuity point $r = r_0$ does not move when time increases. The relevant solutions to the relativistic Burgers equation $v = v(t, r)$ have a range

bounded by the light speed, that is, $v \in [-1, 1]$ for all $t > 0$ and $r > 2M$. An initial problem of particular importance is given by the generalized Riemann problem, associated with initial data made of two steady states separated by a jump discontinuity located at some given radius.

Theorem 3.2.1 (The generalized Riemann problem for the relativistic Burgers model). There exists a unique solution to the generalized Riemann problem defined for all $t > 0$ realized by either by a shock wave or a rarefaction wave. Moreover, the following asymptotic behaviors hold:

- The wave location tends to the blackhole horizon if it initially converges towards the blackhole.
- The wave location tends to the space infinity if it initially converges away from the blackhole.
- The wave location does not change if it is initially steady.

In connection with the general existence theory for (3.1.1), we introduce the auxiliary variable $z := \operatorname{sgn}(v) \sqrt{\frac{v^2-1}{1-2M/r} + 1}$. It is obvious that z is a constant if v is a steady state solution. With this notation, we have the following result from [31].

Theorem 3.2.2 (Existence theory for the relativistic Burgers model). Consider the relativistic Burgers equation (3.1.1) posed on the outer domain of a Schwarzschild blackhole with mass M . Then, for any initial velocity $z_0 = z_0(r) \in (-1, 1)$ such that $z_0 = z_0(r)$ has bounded total variation, there exists a corresponding weak solution to (3.1.1) $z = z(t, r)$ whose total variation is non-increasing with respect to time:

$$TV(z(s, \cdot)) \leq TV(z(t, \cdot)), \quad 0 \leq t \leq s.$$

We are going to design several numerical methods for study these solutions. In particular, we are interested in the behavior of solutions when the initial data $v_0 = v_0(r)$ is a piecewise smooth and steady state solution, to which we will add a compactly supported perturbation, i.e.

$$v_0(r) = \begin{cases} v_L(r) & 2M < r < r_L, \\ v_R(r) & r > r_R, \end{cases} \quad (3.2.4)$$

where $v_L = v_L(r)$, $v_R = v_R(r)$ are two steady state solutions given by (3.2.2) and r_L, r_R are two fixed points.

Theorem 3.2.3 (Time-asymptotic properties for the relativistic Burgers model). Consider the asymptotic behavior of a relativistic Burgers solution $v = v(t, r)$ on a Schwarzschild background (3.1.1) whose initial data is composed by steady state solutions v_L, v_R with a compactly supported perturbation.

- If $v_L > v_R$, then the solution $v = v(t, r)$ converges asymptotically to a shock curve generated by a left-hand state v_L and a right-hand state v_R .
- If $v_L < v_R$, then a generalized N-wave $N = N(t, r)$ can be defined such that inside a rarefaction fan, one has $|v(t, r) - N(t, r)| = O(t^{-1})$ while in a region supporting the evolution of the initial data, one has $|v(t, r) - N(t, r)| = O(t^{-1/2})$. Otherwise, one has $v(t, r) = N(t, r)$.
- If $v_L = v_R$, then $\|v(t, r) - v_R(t, r)\|_{L^1(2M, +\infty)} = O(t^{-1/2})$.

3.3 A finite volume scheme for the relativistic Burgers model

The first-order formulation In this section, we propose a finite volume method for the relativistic Burgers equation (3.1.2) which takes the Schwarzschild geometry into consideration. In order to construct our approximations, we will rely on the solution to the Riemann problem for the standard Burgers equation :

$$\partial_t v + \partial_x \frac{v^2}{2} = 0 \quad (3.3.1)$$

that is, an initial data problem with $v(t, r) = v_0(r)$ where $v_0 = v_0(r)$ is given as a piecewise constant function $v_0 = \begin{cases} v_L & r < r_0, \\ v_R & r > r_0, \end{cases}$ for some fixed r_0 and two constants v_L, v_R . The solution to the standard Riemann problem is given as

$$v(t, r) = \begin{cases} v_L & r < s_L t + r_0, \\ \frac{r-r_0}{t} & s_L t + r_0 < r < s_R t + r_0, \\ v_R & r > s_R t + r_0, \end{cases} \quad (3.3.2)$$

$$s_L = \begin{cases} v_L & v_L < v_R, \\ \frac{v_L + v_R}{2} & v_L > v_R, \end{cases} \quad s_R = \begin{cases} v_R & v_L < v_R, \\ \frac{v_L + v_R}{2} & v_L > v_R. \end{cases}$$

Denote by $\Delta t, \Delta r$ the mesh lengths in time and in space respectively with the CFL condition $\frac{\Delta t}{\Delta r} = \Lambda$, where Λ is such that $\Lambda|v| \leq 1/2$ in order to avoid wave interaction between two Riemann problems. We set $t_n = n\Delta t$ and $r_j = 2M + j\Delta r$. Introduce also the mesh point (t_n, t_j) , $n \geq 0, j \geq 0$ and the rectangle $R_{nj} = \{t_n \leq t < t_{n+1}, r_{j-1/2} \leq r < r_{j+1/2}\}$. Integrate (3.1.2) from $r_{j-1/2}$ to $r_{j+1/2}$ in space and

from t_n to t_{n+1} :

$$\int_{r_{j-1/2}}^{r_{j+1/2}} \left(v(t_{n+1}, r) - v(t_n, r) \right) dr + \int_{t_n}^{t_{n+1}} \left((1 - 2M/r_{j+1/2}) \left(\frac{v^2(t, r_{j+1/2}) - 1}{2} \right) - (1 - 2M/r_{j-1/2}) \left(\frac{v^2(t, r_{j-1/2}) - 1}{2} \right) \right) dt - \int_{r_{j-1/2}}^{r_{j+1/2}} \int_{t_n}^{t_{n+1}} \frac{2M}{r^2} (v^2 - 1) dt dr = 0.$$

Denote by $V_j^n = \int_{r_{j-1/2}}^{r_{j+1/2}} v(t_n, r) dr$, the average value of the solution in the space interval $(r_{j-1/2}, r_{j+1/2})$, and introduce the finite volume scheme for the relativistic Burgers equation on a Schwarzschild background:

$$V_j^{n+1} = V_j^n - \frac{\Delta t}{\Delta r} (F_{j+1/2} - F_{j-1/2}) - \Delta t \frac{2M}{r_j^2} (V_j^{n2} - 1), \quad (3.3.3)$$

where $F_{j+1/2}$ and $F_{j-1/2}$ are $F_{j+1/2} = \mathcal{F}(r_{j+1/2}, V_j^n, V_{j-1}^n)$, and

$$\mathcal{F}(r, V_L, V_R) = \left(1 - \frac{2M}{r} \right) \frac{q^2(V_L, V_R) - 1}{2} \quad (3.3.4)$$

with $q(\cdot, \cdot)$ the standard solution to the Riemann problem centered at r given by (4.4.6). Observe that the CFL condition guarantees that the solution to the Riemann problem does not to leave the rectangle $R_{n,j}$ within one time step.

We now consider the boundary condition of our finite volume scheme. Let J be the number of the space mesh points and we introduce ghost cells at the space boundaries: $R_{n,0} = \{t_n \leq t < t_{n+1}, r_{-1/2} \leq r < r_{1/2}\}$ and $R_{n,J} = \{t_n \leq t < t_{n+1}, r_{J-1/2} \leq r < r_{J+1/2}\}$. We solve the Riemann problem at the two boundaries with initial condition

$$V_0(r) = \begin{cases} 1 & r < r_0, \\ V_0^n & r > r_0, \end{cases} \quad V_J(r) = \begin{cases} V_J^n & r < r_J, \\ -1 & r > r_J. \end{cases}$$

A consistency property

Lemma 3.3.1. *The finite volume method for the relativistic Burgers model introduced in (3.3.3) satisfies the following properties:*

- *The scheme is well-balanced, that is, it preserves the steady state solution to the Euler equation (3.8.1).*
- *The scheme is consistent, that is, if $v = v(t, r)$ is an exact solution to the relativistic Burgers model given by the ordinary differential equation (3.2.1),*

then for every fixed point $r > 2M$,

$$\mathcal{F}(r_R, V_L, V_R) - \mathcal{F}(r_L, V_L, V_R) = \frac{2M}{r^2}(v^2 - 1)(r_R - r_L) + O(r_R - r_L)^2 \quad (3.3.5)$$

holds as $V_L, V_R \rightarrow v$ and $r_L, r_R \rightarrow r$.

Proof. To establish the well-balanced property, we write

$$\begin{aligned} F_{j+1/2} - F_{j-1/2} &= (1 - 2M/r_{j+1/2}) \frac{q(V_j^n, V_{j+1}^n) - 1}{2} - (1 - 2M/r_{j-1/2}) \frac{q(V_{j-1}^n, V_j^n) - 1}{2} \\ &= \int_{j-1/2}^{j+1/2} \frac{2M}{r^2}(v^2 - 1)dr = \frac{2M}{r_j^2}(V_j^{n2} - 1), \end{aligned}$$

and, therefore, $V_j^n = V_j^{n+1}$ holds. Next, recall that $\mathcal{F}(r, V_L, V_R) = \left(1 - \frac{2M}{r}\right) \frac{q(r, V_L, V_R) - 1}{2}$ is the numerical flux of the scheme determined by the standard the Riemann solution. A Taylor expansion gives

$$\begin{aligned} 1 - \frac{2M}{r'} &= 1 - \frac{2M}{r} + \frac{2M}{r^2}(r - r') + O(r - r')^2, \\ \frac{q^2(r', V_L, V_R) - 1}{2} &= \frac{v^2 - 1}{2} + v\partial_r v(r - r') + O(r - r')^2. \end{aligned}$$

Hence, we have

$$\begin{aligned} \mathcal{F}(r_R, V_L, V_R) - \mathcal{F}(r_L, V_L, V_R) &= \frac{2M}{r^2} \frac{v^2 - 1}{2} \frac{v^2 - 1}{2} + \left(1 - \frac{2M}{r}\right) v\partial_r v(r_R - r_L) + O(r_R - r_L)^2 \\ &= \partial_r \left(\left(1 - \frac{2M}{r}\right) \frac{v^2 - 1}{2} \right) + O(r_R - r_L)^2 \\ &= \frac{2M}{r^2}(v^2 - 1)(r_R - r_L) + O(r_R - r_L)^2. \quad \square \end{aligned}$$

A second-order formulation We now extend the method to second-order. The solution is now discretized as a piecewise linear function, and we define

$$\Delta_j^n V = \begin{cases} \min(2|\Delta_{j-1/2} V^n|, 2|\Delta_{j+1/2} V^n|, |\Delta_j V^n|) & \text{if } \text{sgn}\Delta_{j-1/2} V^n = \text{sgn}\Delta_{j+1/2} V^n = \text{sgn}\Delta_j V^n, \\ 0 & \text{otherwise,} \end{cases} \quad (3.3.6)$$

where

$$\Delta_j V^n = \frac{1}{2}(\Delta V_{j+1}^n - \Delta V_{j-1}^n), \quad \Delta_{j+1/2} V^n = (\Delta V_{j+1}^n - \Delta V_j^n), \quad \Delta_{j-1/2} V^n = (\Delta V_j^n - \Delta V_{j-1}^n).$$

Then, our second-order scheme is stated as

$$\begin{aligned}
 V_j^{n+1} = & V_j^n - \frac{\Delta t}{\Delta r} \left(\mathcal{F}(r_{j+1/2}, V_j^{n+1/2,R}, V_{j+1}^{n+1/2,L}) \right. \\
 & \left. - \mathcal{F}(r_{j-1/2}, V_{j-1}^{n+1/2,R}, V_j^{n+1/2,L}) \right) - \Delta t \frac{2M}{r_j^2} (V_j^2 - 1),
 \end{aligned} \tag{3.3.7}$$

where \mathcal{F} is the numerical flux (3.3.4). Here, the two values $V_{j+1}^{n+1/2,L}, V_j^{n+1/2,R}$ are given by

$$\begin{aligned}
 V_j^{n+1/2,L} & := V_j^{n,L} - \frac{\Delta t}{2} \left(\frac{(1 - 2M/r_j)V_j^n \Delta_j^n V}{\Delta r} - \frac{2M}{r_j^2} (V_j^{n2} - 1) \right), \\
 V_j^{n+1/2,R} & := V_j^{n,R} - \frac{\Delta t}{2} \left(\frac{(1 - 2M/r_j)V_j^n \Delta_j^n V}{\Delta r} - \frac{2M}{r_j^2} (V_j^{n2} - 1) \right),
 \end{aligned} \tag{3.3.8}$$

where, with $\Delta_j^n V$ defined by (3.3.6) and $V_j^{n,L} = V_j^n - \frac{\Delta_j^n V}{2}$ and $V_j^{n,R} = V_j^n + \frac{\Delta_j^n V}{2}$.

3.4 Numerical experiments with the finite volume scheme

Asymptotic-preserving property We now present some numerical tests with the proposed finite volume method applied to the relativistic Burgers equation (3.1.2). As mentioned earlier, we work within the domain $r > 2M$, and the mass parameter M is taken to be $M = 1$ in all our tests. We work in the space interval (r_{\min}, r_{\max}) with $r_{\min} = 2M = 2$ and $r_{\max} = 4$ and we take 256 points to discretize the space interval.

We begin by showing that the method at, both, first-order and second-order accuracy preserves the steady state solutions. For positive/negative steady state Burgers solutions $v = \pm \sqrt{\frac{3}{4} + \frac{1}{2r}}$, we see that the initial steady states are exactly conserved by the scheme. We also show that the following steady state shock is preserved by the scheme:

$$v = \begin{cases} \sqrt{\frac{3}{4} + \frac{1}{2r}} & 2.0 < r < 3.0, \\ -\sqrt{\frac{3}{4} + \frac{1}{2r}} & r > 3.0. \end{cases}$$

We obtain that our finite volume scheme preserves three typical forms for the static solutions, as is illustrated in Figures 3.4.1 and FIG-52.

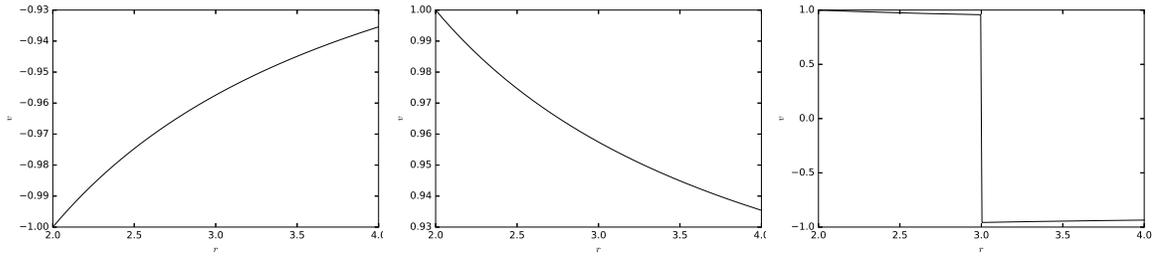
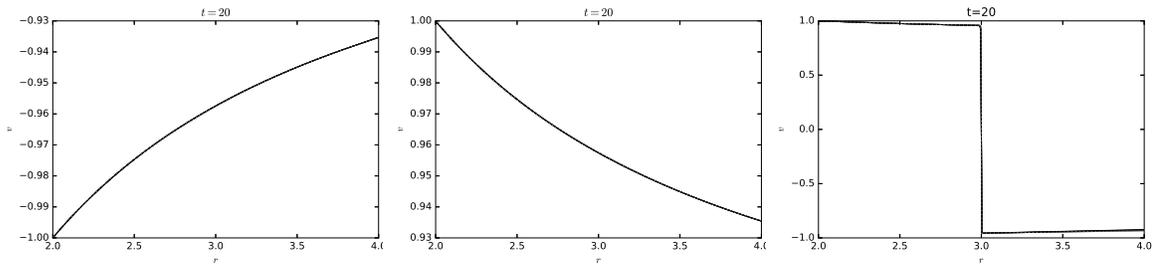


Figure 3.4.1: Three static solutions

Figure 3.4.2: Solution at time $t = 20$ of a steady state, using the second-order finite volume scheme

A moving shock separating two static solutions In view of Theorem 3.2.1, whether the solution to the Riemann problem will move towards the blackhole horizon depends only on the behavior of the initial velocity. We take again the space interval to be $(2.0, 4.0)$ with 256 space mesh points. We take then two kinds of initial data to be

$$v = \begin{cases} \sqrt{\frac{1}{2} + \frac{1}{r}} & 2.0 < r < 2.5, \\ \sqrt{\frac{2}{r}} & r > 2.5, \end{cases} \quad v = \begin{cases} -\sqrt{\frac{2}{r}} & 2.0 < r < 2.5, \\ -\sqrt{\frac{3}{4} + \frac{1}{4r}} & r > 2.5. \end{cases}$$

The behavior of the two shock solutions obtained with the first-order and second-order accurate versions are shown in Figures 3.4.3, 3.4.4, 3.4.5, and 3.4.6.

Late-time behavior of solutions We now study the late-time behavior of solutions whose initial data is given as (3.2.4), that is, a piecewise steady state solution with a compactly supported perturbation. We treat the following two kinds of piecewise steady state solutions:

$$v = \sqrt{\frac{1}{2} + \frac{1}{r}}, \quad v = \begin{cases} \sqrt{\frac{1}{2} + \frac{1}{r}} & 2.0 < r < 2.5, \\ \sqrt{\frac{2}{r}} & r > 2.5, \end{cases}$$

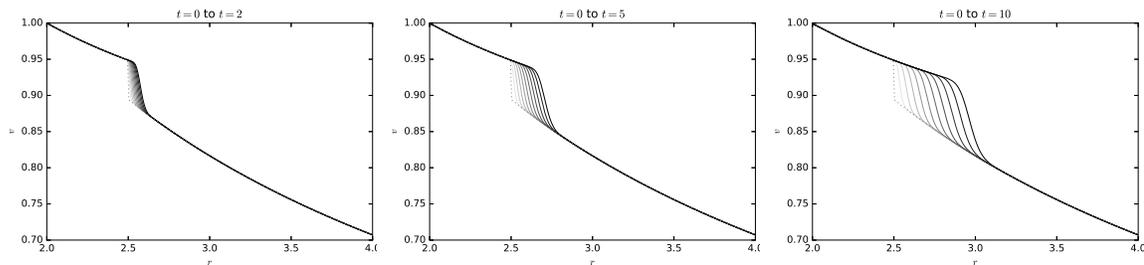


Figure 3.4.3: Static solution with a right-moving shock computed with the first-order finite volume scheme

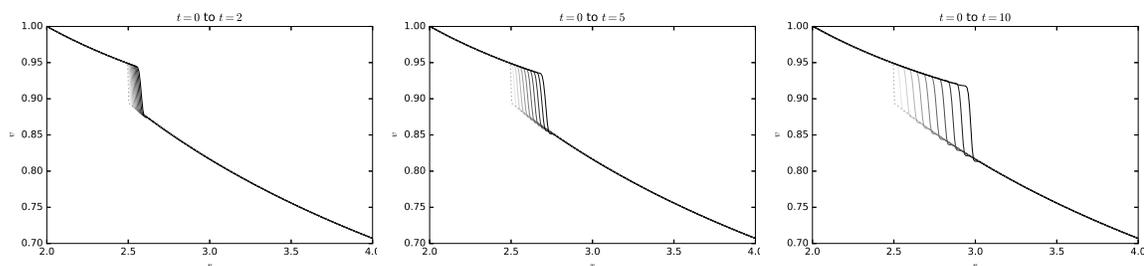


Figure 3.4.4: Static solution with a right-moving shock computed with the second-order finite volume scheme

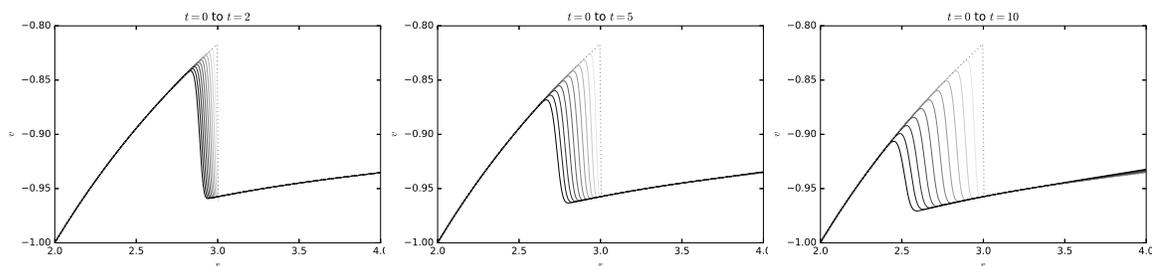


Figure 3.4.5: Static solution with a left-moving shock computed with the first-order finite volume scheme

with compactly supported perturbations.

3.5 A generalized random choice scheme for the relativistic Burgers model

Explicit solution to the generalized Riemann problem In order to construct a Glimm method for the relativistic Burgers model, we need first introduce the explicit form of the generalized Riemann problem of the relativistic Burgers equation

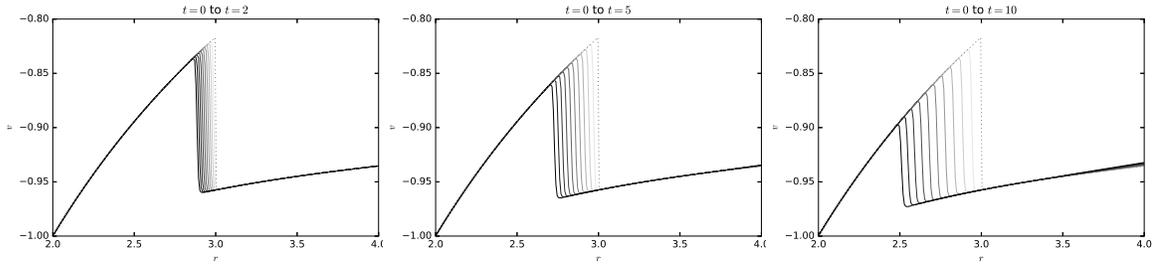


Figure 3.4.6: Static solution with a left-moving shock computed with the second-order finite volume scheme

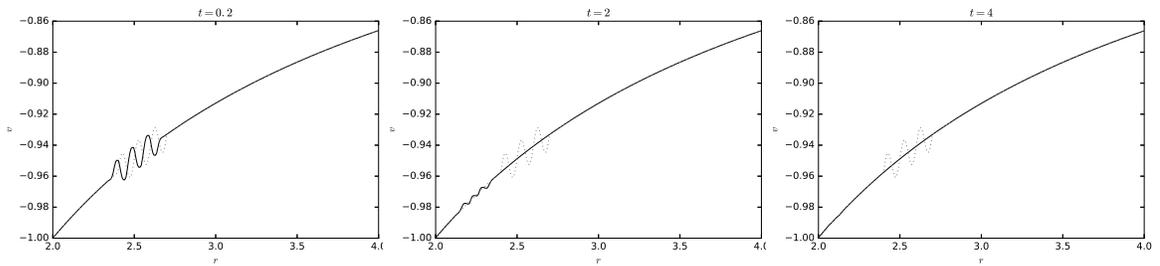


Figure 3.4.7: Numerical solution from initially perturbed steady state

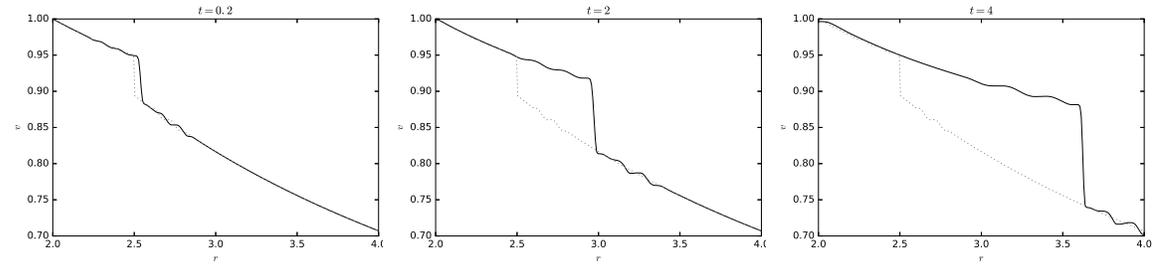


Figure 3.4.8: Numerical solution from an initially perturbed shock

(3.1.1), which is an initial problem whose initial data $v_0 = v_0(r)$ is given as

$$v_0(r) = \begin{cases} v_L(r) & 2M < r < r_0, \\ v_R(r) & r > r_0, \end{cases} \quad (3.5.1)$$

where r_0 is a fixed point in space and $v_L = v_L(r)$, $v_R = v_R(r)$ are two steady state solutions of the Burgers' equation with explicit forms

$$v_L(r) = \text{sgn}(v_L^0) \sqrt{1 - K_L^2 \left(1 - \frac{2M}{r}\right)}, \quad v_R(r) = \text{sgn}(v_R^0) \sqrt{1 - K_R^2 \left(1 - \frac{2M}{r}\right)}, \quad (3.5.2)$$

where $K_L, K_R > 0$ are two constants and we denote by $v_L^0 = v_L(r_0), v_R^0 = v_R(r_0)$. The existence of the generalized Riemann problem is concluded in Theorem 3.2.1. More precisely, the solution to the Riemann problem $v = v(t, r)$ can be realized by either a shock wave or a rarefaction wave which is given explicitly by the following form:

$$v(t, r) = \begin{cases} v_L(r) & r < r_L(t), \\ \tilde{v}(t, r) & r_L(t) < r < r_R(t), \\ v_R(r) & r > r_R(t). \end{cases} \quad (3.5.3)$$

Here, $r_L(t)$ and $r_R(t)$ are bounds of rarefaction regions satisfying

$$R_j(r_j(t)) - R_j(r_0) = t, \quad (3.5.4)$$

where $R_j = R_j(r)$ is given by

$$R_j(r) := \frac{R^{v_j}(r)}{2} + \chi_{[v_j^0 < v_k^0]}(r) \frac{R^{v_j}(r)}{2} + \chi_{[v_j^0 < v_k^0]}(r) \frac{R^{v_k}(r)}{2} \quad (3.5.5)$$

with $j = L, R, k = R, L$,

$$\chi_{[v_j^0 \geq v_k^0]}(r) = \begin{cases} 1 & \text{if } v_j^0 \geq v_k^0, \\ 0 & \text{otherwise,} \end{cases}$$

and the function $R_j^v = R_j^v(r)$ is

$$\begin{aligned} R^{v_j}(r) := & \operatorname{sgn}(v_j) \frac{1}{(1/\epsilon^2 - K_j^2)^{3/2}} \left(2M\epsilon \left(\frac{1}{\epsilon^2} - K_j^2 \right)^{3/2} \ln(r - 2M) \right. \\ & - 2M \left(\frac{1}{\epsilon^2} - K_j^2 \right)^{3/2} \ln \left(\frac{2r}{\epsilon} \sqrt{\frac{1}{\epsilon^2} - K_j^2} \left(1 - \frac{2M}{r} \right) + (2M - r)K_j^2 \right) \\ & + \frac{1}{\epsilon} \left(r \sqrt{\frac{1}{\epsilon^2} - K_j^2} \sqrt{\frac{1}{\epsilon^2} - K_j^2} \left(1 - \frac{2M}{r} \right) \right. \\ & \left. \left. + M(2/\epsilon^2 - 3K_j^2) \ln \left(r \sqrt{\frac{1}{\epsilon^2} - K_j^2} \sqrt{\frac{1}{\epsilon^2} - K_j^2} \left(1 - \frac{2M}{r} \right) + (M - r)K_j^2 + \frac{r}{\epsilon^2} \right) \right) \right). \end{aligned} \quad (3.5.6)$$

The function $\tilde{v} = \tilde{v}(t, r)$ denotes the generalized rarefaction wave

$$\tilde{v}(t, r) = \operatorname{sgn}(r - r_0) \sqrt{\frac{1}{\epsilon^2} - K^2(t, r)} \left(1 - \frac{2M}{r} \right), \quad (3.5.7)$$

where $K = K(t, r)$ is characterized by the condition

$$\operatorname{sgn}(r - r_0) = \frac{\tilde{R}(r, K) - \tilde{R}(r_0, K)}{t}, \quad (3.5.8)$$

where

$$\begin{aligned} \tilde{R}(r, K) := & \frac{1}{(1/\epsilon^2 - K^2)^{3/2}} \left(2M\epsilon \left(\frac{1}{\epsilon^2} - K^2 \right)^{3/2} \ln(r - 2M) \right. \\ & - 2M(1/\epsilon^2 - K^2)^{3/2} \ln \left(\frac{2r}{\epsilon} \sqrt{\frac{1}{\epsilon^2} - K^2} \left(1 - \frac{2M}{r} \right) + (2M - r)K^2 \right) \\ & + \frac{1}{\epsilon} \left(r \sqrt{\frac{1}{\epsilon^2} - K^2} \sqrt{\frac{1}{\epsilon^2} - K^2} \left(1 - \frac{2M}{r} \right) \right. \\ & \left. \left. + M(2/\epsilon^2 - 3K^2) \ln \left(r \sqrt{\frac{1}{\epsilon^2} - K^2} \sqrt{\frac{1}{\epsilon^2} - K^2} \left(1 - \frac{2M}{r} \right) + (M - r)K^2 + \frac{r}{\epsilon^2} \right) \right) \right). \end{aligned} \quad (3.5.9)$$

Indeed, referring to [32], the solution constructed by (3.5.3) is proven to be unique, satisfying the Rankine-Hugoniot jump condition and the entropy inequality at the same time. Besides, the solution to the generalized Riemann problem is globally defined both in time and in space.

A generalized random choice method The random choice method is a scheme based on the result of generalized Riemann problem. We use again the time-space grid where the mesh lengths in time and in space are $\Delta t, \Delta r$ with $t_n = n\Delta t, r_j = 2M + j\Delta r$ where we recall $2M$ is the blackhole horizon. Denote by V_j^n the numerical solution $V(n\Delta t, 2M + j\Delta r)$. Let (w_n) be a sequence equidistributed in $(-\frac{1}{2}, \frac{1}{2})$ and write $r_{n,j} = 2M + (j + w_n)\Delta r$. We define our Glimm-type approximations as follows:

$$V_j^{n+1} = V_{\mathcal{R}}^{j,n}(t_{n+1}, r_{n,j}), \quad (3.5.10)$$

where $V_{\mathcal{R}}^{j,n} = V_{\mathcal{R}}^{j,n}(t, r)$ is the solution to the Riemann problem with the initial data

$$V_0^{j,n} = \begin{cases} V_L^{j,n}(r), & r < r_{j+\operatorname{sgn}(w_n)/2}, \\ V_R^{j,n}(r), & r > r_{j+\operatorname{sgn}(w_n)/2}, \end{cases} \quad (3.5.11)$$

where the left-hand state $V_L^{j,n} = V_L^{j,n}(r)$ and the right-hand state $V_R^{j,n} = V_R^{j,n}(r)$ are steady state solutions to (3.2.1) with initial conditions:

$$\begin{cases} V_L^{j,n}(r_j) = V_j^n, & w_n \geq 0, & V_R^{j,n}(r_j) = V_j^n, & w_n < 0, \\ V_L^{j,n}(r_{j-1}) = V_{j-1}^n, & w_n < 0, & V_R^{j,n}(r_{j+1}) = V_{j+1}^n, & w_n \geq 0. \end{cases}$$

We choose a random number only once at each time level $t = t_n$ rather than at every mesh point (t_n, r_j) .

In order to have an equidistributed sequence, the random values (w_n) are defined by following Chorin [7]: we give two large prime numbers $p_1 < p_2$ and define a sequence of integers (q_n) :

$$q_0, \quad \text{given } q_0 < p_2; \quad q_n := (p_1 + q_{n-1}) \pmod{p_2}, \quad n \geq 1. \quad (3.5.12)$$

Then we define the sequence $w'_n = \frac{q_n + w_n + 1/2}{p_2} - \frac{1}{2}$, which is to be used in our Glimm method instead of (w_n) . It is direct to see that $w'_n \in (-\frac{1}{2}, \frac{1}{2})$.

3.6 Numerical experiments with the random choice scheme for the relativistic Burgers model

Consistency property We now presents numerical experiment with the proposed Glimm method for the Burgers equation on a Schwarzschild background (3.1.1). Recall that $r > 2M$ and we choose again $M = 1$ for the blackhole mass. The space interval in consideration is (r_{\min}, r_{\max}) with $r_{\min} = 2M = 2$ and $r_{\max} = 4$. To introduce the random sequence, we fix two prime integers, specifically $p_1 = 937, p_2 = 997$ and $q_0 = 800$. Since the solution to every local generalized Riemann problem (3.1.1), (4.4.1) is exact, the following observation is immediate.

Lemma 3.6.1. *Consider a given initial velocity $v_0 = v_0(r)$ as a steady state solution such that the static Burgers model (3.2.1) holds. Then the approximate solution to the relativistic Burgers equation (3.1.1) constructed by the Glimm method (3.5.10) is accurate.*

We will still observe the evolution of those three types of solutions shown in Figure 3.4.1, that is, the two steady state solutions $v = \pm\sqrt{\frac{3}{4} + \frac{1}{2r}}$ and the steady shock:

$$v = \begin{cases} \sqrt{\frac{3}{4} + \frac{1}{2r}}, & 2.0 < r < 3.0, \\ -\sqrt{\frac{3}{4} + \frac{1}{2r}}, & r > 3.0. \end{cases}$$

Different types of shocks We consider two different shocks whose initial speed are positive and negative. As was observed by the finite volume method, whether the position of the shock will go toward the blackhole horizon is determined uniquely by their initial behavior. We can recover the same conclusion with the Glimm method.

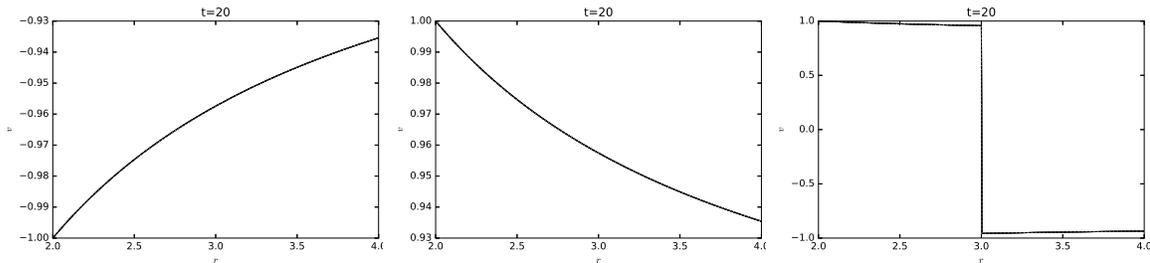


Figure 3.6.1: Solution at time $t = 20$ from a steady state initial data, using the Glimm scheme

Again, we take two kinds of initial data:

$$v = \begin{cases} \sqrt{\frac{1}{2} + \frac{1}{r}}, & 2.0 < r < 2.5, \\ \sqrt{\frac{2}{r}}, & r > 2.5, \end{cases} \quad v = \begin{cases} -\sqrt{\frac{2}{r}}, & 2.0 < r < 2.5, \\ \sqrt{\frac{3}{4} + \frac{1}{4r}}, & r > 2.5. \end{cases}$$

Since our Riemann solver is exact, the numerical solutions contain no numerical diffusion.

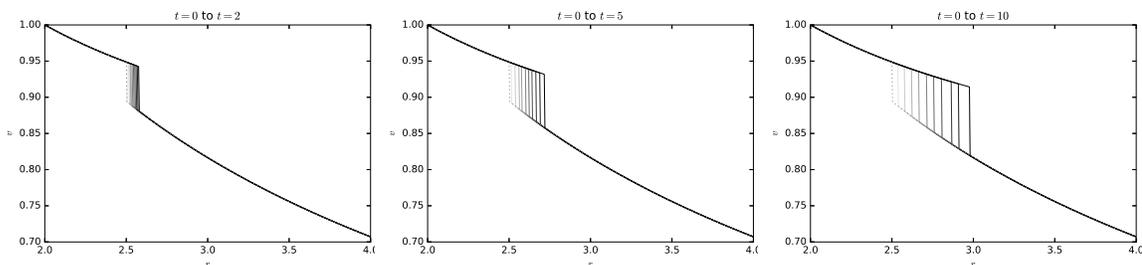


Figure 3.6.2: Static solution with a right-moving shock computed by the Glimm scheme

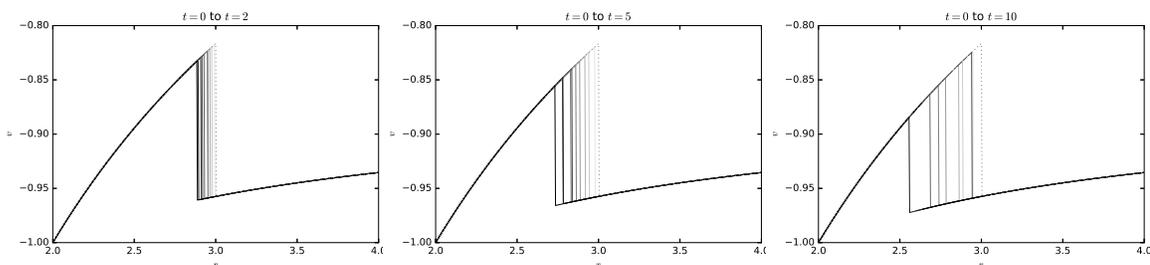


Figure 3.6.3: Static solution with a left-moving shock computed by the Glimm scheme

Asymptotic behavior of Burgers solutions We are now interested in the evolution of solutions whose initial data is given as piecewise steady state solution satisfying (3.2.1). As was done earlier, we take into account two kinds of initial data:

$$v = \sqrt{\frac{1}{2} + \frac{1}{r}}, \quad v = \begin{cases} \sqrt{\frac{1}{2} + \frac{1}{r}} & 2.0 < r < 2.5, \\ \sqrt{\frac{2}{r}} & r > 2.5, \end{cases},$$

perturbed by compactly supported functions.

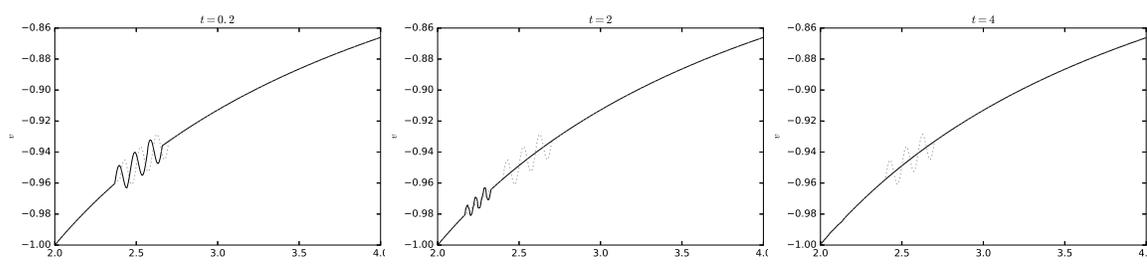


Figure 3.6.4: Numerical solution from an initially perturbed steady state, using the Glimm method

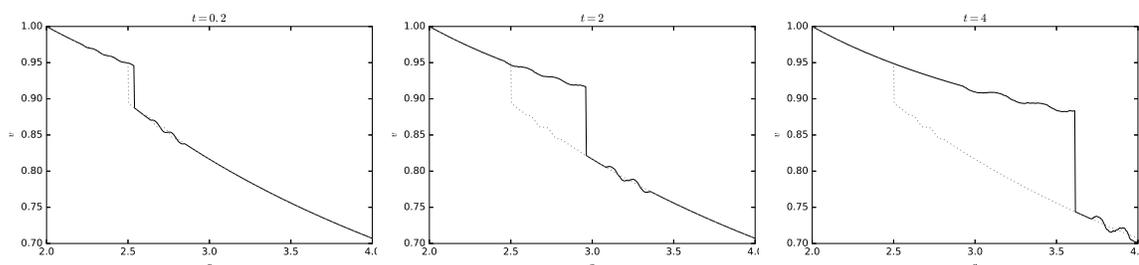


Figure 3.6.5: Numerical solution from an initially perturbed shock, using the Glimm method

3.7 General initial data for the relativistic Burgers equation

Steady shock with perturbation The behavior of a smooth steady state solution to the relativistic Burgers model (3.1.1) perturbed by a function on a compactly supported function is understood both numerically and theoretically: the solution converge to the same initial steady state solution. The steady shock (3.2.3) is a solution to the static equation (3.2.1) in the distribution sense. We are interested in

the asymptotic behavior and our numerical results in Figures 3.7.1 and 3.7.2 lead us to the following.

Conclusion 3.7.1. *Consider a perturbed steady shock given as (3.2.3):*

$$v_0 = \begin{cases} \sqrt{1 - K^2(1 - 2M/r)} & 2M < r < r_0, \\ -\sqrt{1 - K^2(1 - 2M/r)} & r > r_0, \end{cases}$$

where K is a given constant and $r_0 > 2M$ is fixed radius out of the Schwarzschild blackhole region. The solution to the relativistic Burgers model (3.1.1) converges at some finite time to a solution of the form (with possibly $r_1 \neq r_0$):

$$v = \begin{cases} \sqrt{1 - K^2(1 - 2M/r)} & 2M < r < r_1, \\ -\sqrt{1 - K^2(1 - 2M/r)} & r > r_1, \end{cases}$$

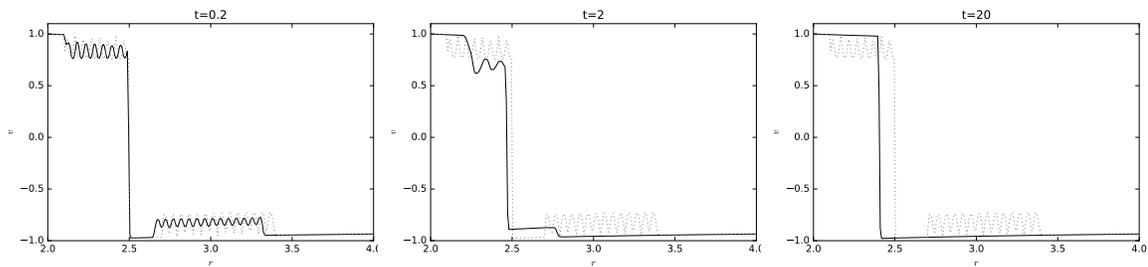


Figure 3.7.1: Evolution of a perturbed steady shock, using the finite volume method

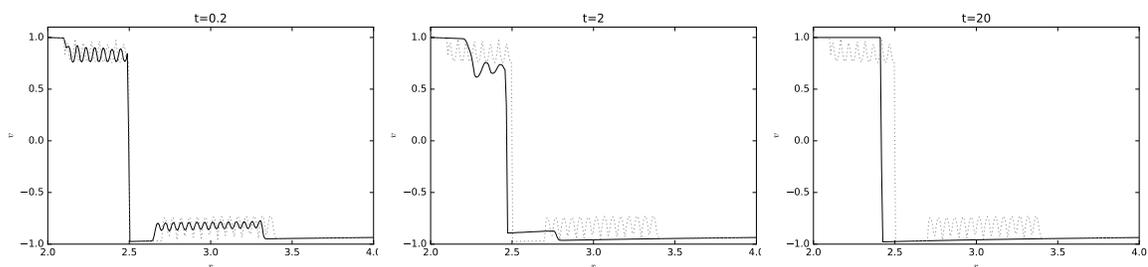


Figure 3.7.2: Evolution of a perturbed steady shock, using the Glimm method

Late-time behavior of general solutions It is obvious that the steady state solution satisfying (3.2.1) serves as a solution to the relativistic Burgers equation on a Schwarzschild background. Notice that on the blackhole horizon $r = 2M$, the steady state solution values the light speed, that is, either 1 or -1 , which equals exactly the light speed and obviously their boundary values will not change as time evolves. The value of a steady state solution at infinity is also given explicitly. Observations on

the numerical method shows that the asymptotic behavior of Burgers model (3.1.1) is mainly determined by the values of the initial data at the blackhole horizon $r = 2M$ and the space infinity $r = +\infty$. More precisely, suppose that a given velocity $v_0 = v_0(r)$ does not satisfy the static Burgers equation (3.2.1), we have the following conclusion.

- Conclusion 3.7.2.**
1. *If the initial velocity $\lim_{r \rightarrow 2M} v_0(r) = 1$, then the solution to the Burgers equation (3.1.1) satisfies that there exists a time $t > t_0$ such that for all $t > t_0$ the solution $v = v(t, r)$ is a shock with left-hand state 1 and right-hand state v_*^- with $v_*^-(r) = -\sqrt{\frac{2M}{r}}$ the negative critical steady solution.*
 2. *If the initial velocity $\lim_{r \rightarrow 2M} v_0(r) < 1$ and $\lim_{r \rightarrow +\infty} v_0(r) > 0$, there exists a time $t_0 > 0$ such that the solution to the Burgers equation $v(t, r) = v_*^-(r)$ for all $t > t_0$ where $v_*^-(r) = -\sqrt{\frac{2M}{r}}$ is the negative critical steady state solution to the relativistic Burgers model.*
 3. *If the initial velocity $\lim_{r \rightarrow 2M} v_0(r) < 1$ and $\lim_{r \rightarrow +\infty} v_0(r) \leq 0$, then the solution to the relativistic Burgers model satisfies that $v(t, r) = -\sqrt{1 - (1 - v_0^\infty)^2(1 - \frac{2M}{r})}$ for $t > t_0$ for a time $t_0 > 0$ where $0 \geq v_0^\infty = \lim_{r \rightarrow +\infty} v_0(r)$.*

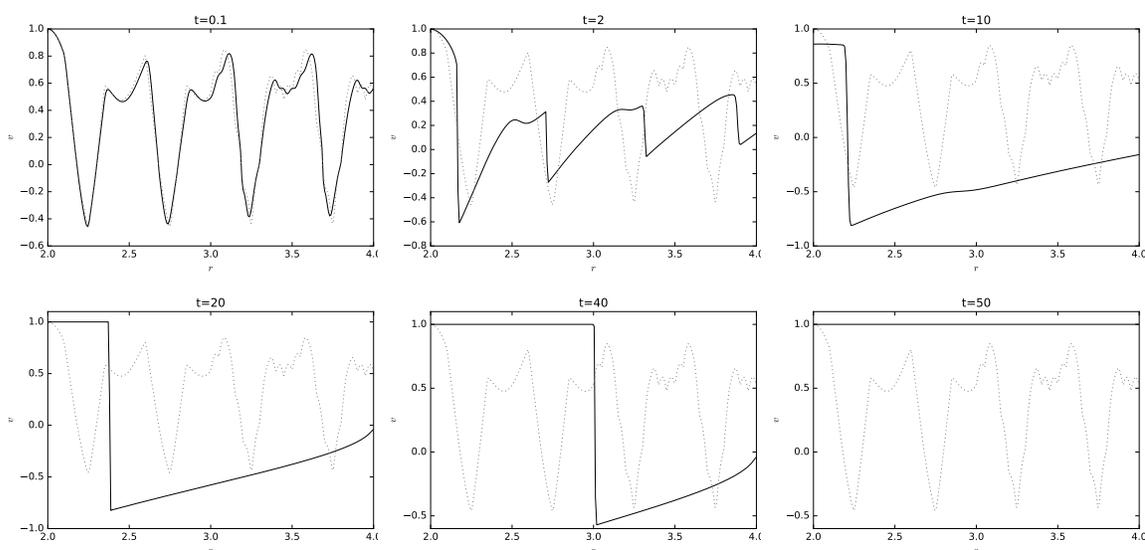


Figure 3.7.3: Numerical solution with velocity 1 at $r = 2M$ and $r = +\infty$, using the finite volume scheme

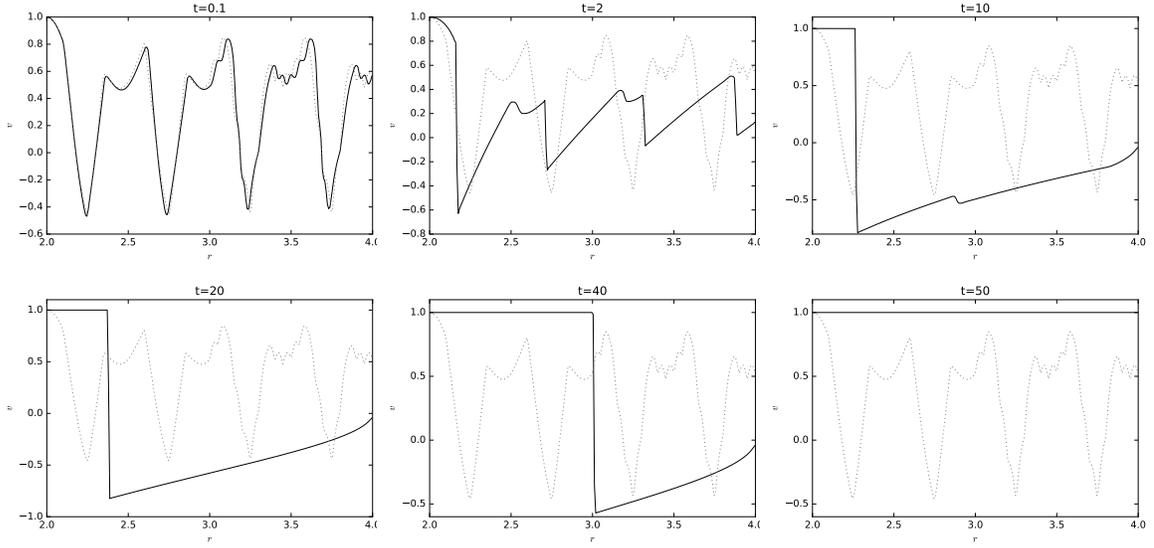


Figure 3.7.4: Numerical solution with velocity 1 at $r = 2M$ and at $r = +\infty$, using the Glimm scheme

3.8 Overview of the theory for the relativistic Euler model

Continuous and discontinuous steady state solutions The steady solution to the relativistic Euler model on a Schwarzschild background (3.1.3) is given by the following ordinary differential system:

$$\begin{aligned} \partial_r \left(r(r-2M) \frac{1}{1-v^2} \rho v \right) &= 0, \\ \partial_r \left((r-2M)^2 \frac{v^2 + k^2}{1-v^2} \rho \right) &= \frac{M}{r} \frac{(r-2M)}{1-v^2} \left(3\rho v^2 + 3k^2 \rho - \rho - k^2 \rho v^2 \right) + \frac{2k^2}{r} (r-2M)^2 \rho, \end{aligned} \quad (3.8.1)$$

Smooth steady state solutions to the relativistic Euler equation with given radius $r_0 > 2M$, density $\rho_0 > 0$ and velocity $|v_0| < 1$ are given by

$$\begin{aligned} \text{sgn}(v)(1-v^2)|v|^{\frac{2k^2}{1-k^2}} r^{\frac{4k^2}{1-k^2}} / (1-2M/r) &= \text{sgn}(v_0)(1-v_0^2)|v_0|^{\frac{2k^2}{1-k^2}} r_0^{\frac{4k^2}{1-k^2}} / (1-2M/r_0), \\ r(r-2M)\rho \frac{v}{1-v^2} &= r_0(r_0-2M)\rho_0 \frac{v_0}{1-v_0^2}. \end{aligned} \quad (3.8.2)$$

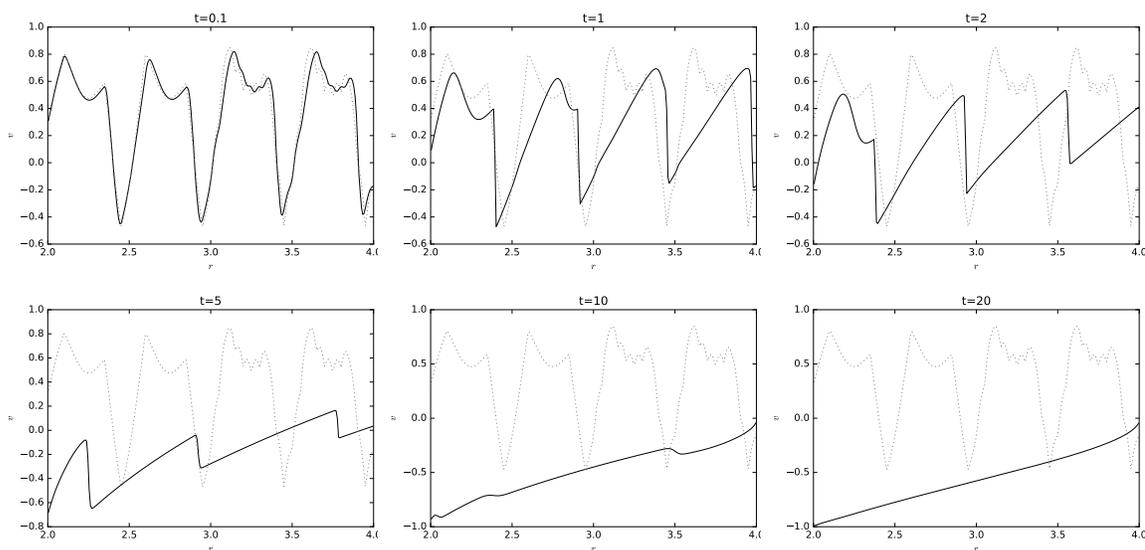


Figure 3.7.5: Numerical solutions with velocity less than 1 at $r = 2M$ and $r = +\infty$, using the finite volume scheme

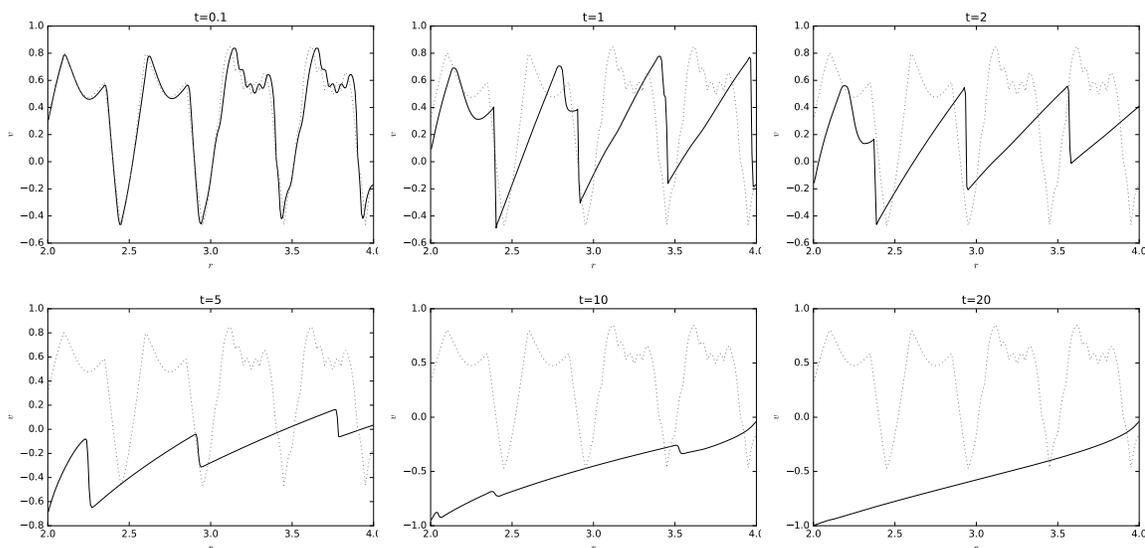


Figure 3.7.6: Numerical solution less than 1 velocity at $r = 2M$ and $r = +\infty$, using the Glimm scheme

We have

$$\begin{aligned}
 \frac{d\rho}{dr} &= -\frac{2(r-M)}{r(r-2M)}\rho - \frac{(1+v^2)(1-k^2)}{r(r-2M)}\rho \left(\frac{2k^2}{1-k^2}(r-2M) - M \right) / (v^2 - k^2) \\
 \frac{dv}{dr} &= v \frac{(1-v^2)(1-k^2)}{r(r-2M)} \left(\frac{2k^2}{1-k^2}(r-2M) - M \right) / (v^2 - k^2),
 \end{aligned}
 \tag{3.8.3}$$

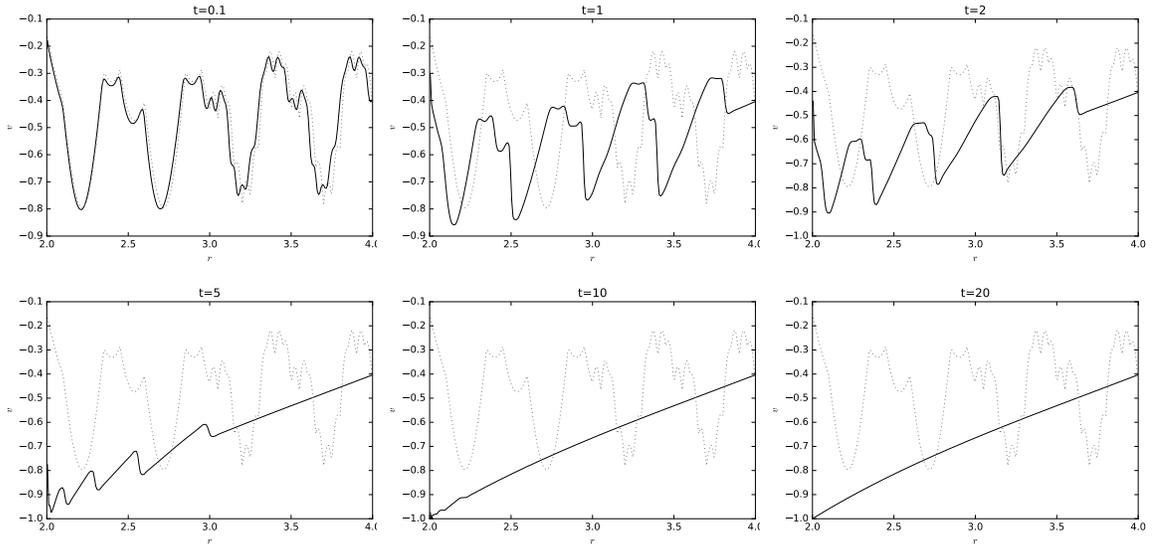


Figure 3.7.7: Numerical solution with velocity less than 1 at $r = 2M$ and negative velocity at $r = +\infty$ using the finite volume scheme

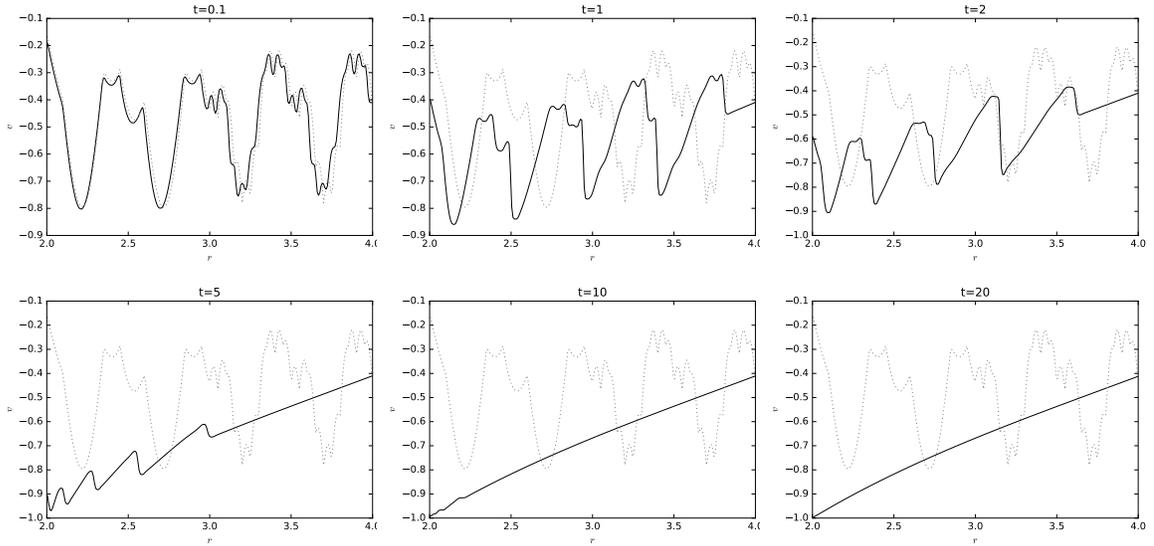


Figure 3.7.8: Numerical solution with velocity less than 1 at $r = 2M$ and negative velocity at $r = +\infty$, using the Glimm scheme

We denote by the *critical steady state solution to the relativistic Euler model* (3.1.3) (ρ, v) with its velocity $v = v(r)$ satisfying

$$\frac{1 - \epsilon^2 v^2}{1 - 2M/r} (r^2 |v|)^{\frac{2\epsilon^2 k^2}{1 - \epsilon^2 k^2}} = (1 + 3\epsilon^2 k^2) k^{\frac{2\epsilon^2 k^2}{1 - \epsilon^2 k^2}} \left(\frac{1 + 3\epsilon^2 k^2}{2\epsilon^2 k^2} M \right)^{\frac{4\epsilon^2 k^2}{1 - \epsilon^2 k^2}}. \quad (3.8.4)$$

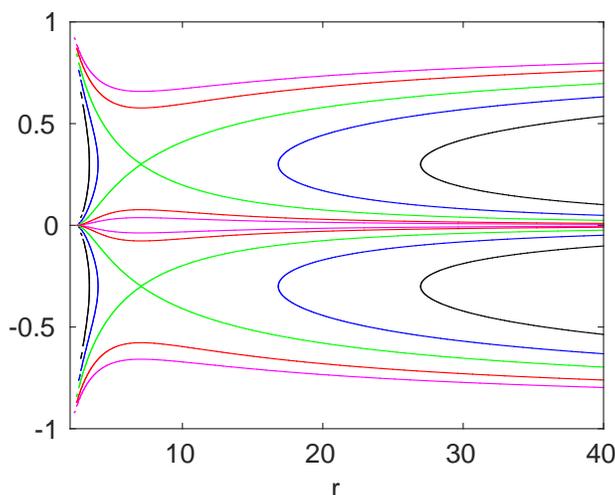


Figure 3.7.9: Steady state solutions for the relativistic Euler model

Unlike the static Burgers model (3.2.1), steady state solution to the relativistic Euler model does not have an explicit form. We recall the following from [31].

Theorem 3.8.1 (Smooth steady flows on a Schwarzschild background). Let $k \in [0, 1]$ be the sound speed and $M > 0$ be mass of the blackhole and we consider the relativistic Euler model describing fluid flows on a Schwarzschild background (3.1.3). For any given any radius $r_0 > 2M$, density $\rho_0 > 0$, and velocity $|v_0| < 1$, there exists a smooth unique steady state solution $\rho = \rho(r), v = v(r)$, satisfying (3.8.2) such that the initial condition $\rho(r_0) = \rho_0$ and $v(r_0) = v_0$ holds. Moreover, the velocity component satisfies that the signes of $v(r)$ and $|v(r)| - k$ do not change on the domain of definition. We have two different families of solutions:

- If there exists no point at which the fluid flow is sonic (referred to the *sonic point*), the smooth steady state solution is defined globally on the whole space interval outside of the blackhole $(2M, +\infty)$.
- Otherwise, the smooth steady state solution cannot be extended once it reaches the sonic point.

We now turn to *steady shock of the relativistic Euler model* (3.1.3), that is, two steady state solutions connected by a standing shock:

$$(\rho, v) = \begin{cases} (\rho_L, v_L)(r), & 2M < r < r_0, \\ (\rho_R, v_R)(r), & r > r_0, \end{cases} \quad (3.8.5)$$

where $r_0 > 2M$ is a given radius and $(\rho_L, v_L), (\rho_R, v_R)$ two steady state solutions two

steady state solutions satisfying (3.8.2) such that

$$v_R(r_0) = \frac{k^2}{v_L(r_0)}, \quad \rho_R(r_0) = \frac{v_L(r_0)^2 - k^4}{k^2(1 - v_L(r_0)^2)} \rho_L(r_0), \quad v_L(r_0) \in (-k, -k^2) \cup (k, 1). \quad (3.8.6)$$

We denote by the *steady shock of the relativistic Euler model* the function given by (3.8.5), (3.8.6) is a solution to the static Euler equation (3.8.1) in the distributional sense, satisfying both the Lax entropy inequality and the Rankine-Hugoniot jump condition. Observe that for a fixed radius $r_1 \neq r_0$ and (ρ_L, v_L) , (ρ_R, v_R) satisfying (3.8.5), the following function is *not* a steady shock of the Euler model (3.1.3):

$$(\rho, v) = \begin{cases} (\rho_L, v_L)(r), & 2M < r < r_1, \\ (\rho_R, v_R)(r), & r > r_1. \end{cases}$$

Generalized Riemann problem and Cauchy problem A generalized Riemann problem for the relativistic Euler system (3.1.3) is a Cauchy problem with initial data given as

$$(\rho_0, v_0)(r) = \begin{cases} (\rho_L, v_L)(r) & 2M < r < r_0, \\ (\rho_R, v_R)(r) & r > r_0, \end{cases} \quad (3.8.7)$$

where $r = r_0$ is a fixed radius and $\rho_L = \rho_L(r)$, $v_L = v_L(r)$, $\rho_R = \rho_R(r)$, $v_R = v_R(r)$ are two smooth steady state solutions satisfying the static Euler equation (3.8.1). Referring to [31], we can construct an approximate solver $\tilde{U} = (\tilde{\rho}, \tilde{v}) = (\tilde{\rho}, \tilde{v})(t, r)$ of the generalized Riemann problem of the relativistic Euler model (3.1.3) whose initial date is (3.8.7) such that:

- $\|\tilde{U}(t, \cdot) - U(t, \cdot)\|_{L^1} = O(\Delta t^2)$ for any fixed $t > 0$ where $U = (\rho, v) = (\rho, v)(t, r)$ satisfying (3.1.3), (3.8.7) and Δt is the time step in the construction.
- $\tilde{U} = (\tilde{\rho}, \tilde{v})$ is accurate out of rarefaction fan regions.
- $\tilde{U} = (\tilde{\rho}, \tilde{v})$ (so does the accurate solution U) contains at most three steady states: the two states given in the initial data (ρ_L, v_L) , (ρ_R, v_R) and the uniquely defined intermediate (ρ_M, v_M) connected by a 1-family wave (either 1-shock or 1-rarefaction) and a 2-family wave (either 2-shock or 2-rarefaction).

Theorem 3.8.2 (The existence theory of the relativistic Euler model). Consider the Euler system describing fluid flows on a Schwarzschild geometry (3.1.3). For any initial density $\rho_0 = \rho_0(r) > 0$ and velocity $|v_0| = |v_0(r)| < 1$ satisfying

$$TV_{[2M+\delta, +\infty)}(\ln \rho_0) + TV_{[2M+\delta, +\infty)}\left(\ln \frac{1 - v_0}{1 + v_0}\right) < +\infty,$$

where $\delta > 0$ is a constant, there exists a weak solution $(\rho, v) = (\rho, v)(t, r)$ defined on $(0, T)$ for any given $T > 0$ and satisfying the prescribed initial data at the initial time and, with a constant C independent of time,

$$\begin{aligned} & \sup_{t \in [0, T]} \left(TV_{[2M+\delta, +\infty)}(\ln \rho(t, \cdot)) + TV_{[2M+\delta, +\infty)} \left(\ln \frac{1-v(t, \cdot)}{1+v(t, \cdot)} \right) \right) \\ & \leq TV_{[2M+\delta, +\infty)}(\ln \rho_0) + TV_{[2M+\delta, +\infty)} \left(\ln \frac{1-v_0}{1+v_0} \right) e^{CT}. \end{aligned}$$

3.9 A finite volume method for the relativistic Euler model

A semi-discretized numerical scheme We consider the relativistic equation on a Schwarzschild background (4.1.1) and we write

$$\partial_t U + \partial_r \left(\left(1 - \frac{2M}{r}\right) F(U) \right) = S(r, U), \quad (3.9.1)$$

$$U = \begin{pmatrix} U^0 \\ U^1 \end{pmatrix} = \begin{pmatrix} \frac{1+k^2 v^2}{1-v^2} \rho \\ \frac{1+k^2}{1-v^2} \rho v \end{pmatrix}, \quad F(U) = \begin{pmatrix} \frac{1+k^2}{1-v^2} \rho v \\ \frac{v^2+k^2}{1-v^2} \rho \end{pmatrix},$$

and the source term

$$S(r, U) = \begin{pmatrix} -\frac{2}{r} \left(1 - \frac{2M}{r}\right) \frac{1+k^2}{1-v^2} \rho v \\ \frac{-2r+5M}{r^2} \frac{v^2+k^2}{1-v^2} \rho - \frac{M}{r^2} \frac{1+k^2 v^2}{1-v^2} \rho + 2 \frac{r-2M}{r^2} k^2 \rho \end{pmatrix}.$$

We can compute

$$D_U F(U) = \begin{bmatrix} 0 & 1 \\ (-v^2+k^2)/(1-k^2 v^2) & 2(1-k^2)v/(1-k^2 v^2) \end{bmatrix}, \quad (3.9.2)$$

which gives the two eigenvalues $\mu_{\mp} = \left(1 - \frac{2M}{r}\right) \frac{v \mp k}{1 \mp k^2 v}$. We also have $v = \frac{1+k^2 - \sqrt{(1+k^2)^2 - 4k^2 \left(\frac{U^1}{U^0}\right)^2}}{2k^2 \frac{U^1}{U^0}} \in (-1, 1)$ and $\rho = \frac{U^1(1-v^2)}{v(1+k^2)}$. Again, we take Δt , Δr as the mesh lengths in time and in space respectively with the CFL condition

$$\frac{\Delta t}{\Delta x} \max(|\mu_-|, |\mu_+|) \leq \frac{1}{2}, \quad (3.9.3)$$

where μ_{\mp} are eigenvalues. As is done earlier we write $t_n = n\Delta t$ and $r_j = 2M + j\Delta r$, and we denote the mesh points by (t_n, r_j) , $n \geq 0$, $j \geq 0$. We set also $\rho(t_n, r_j) = \rho_j^n$, $v(t_n, r_j) = v_j^n$ and $U(t_n, r_j) = U_j^n$ where $U = U(t, r)$ is given by (4.2.1).

We search for the approximations $U_j^n = \frac{1}{\Delta r} \int_{r_{j-1/2}}^{r_{j+1/2}} U(t_n, r) dr$ and $S_j^n = \frac{1}{\Delta r} \int_{r_{j-1/2}}^{r_{j+1/2}} S(t_n, r) dr$ and introduce the following finite volume method:

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta r} (F_{j+1/2}^n - F_{j-1/2}^n) + \Delta t S_j^n, \quad (3.9.4)$$

where the numerical flux is

$$F_{j-1/2}^n = \mathcal{F}_l(r_{j-1/2}, U_{j-1}^n, U_j^n) = \left(1 - \frac{2M}{r_{j-1/2}}\right) \mathcal{F}(U_{j-1/2-}^n, U_{j-1/2+}^n), \quad (3.9.5)$$

and $U_{j+1/2\pm}, U_{j-1/2\pm}$ are determined in the forthcoming subsection and

$$\mathcal{F}(U_L, U_R) = \frac{F(U_L) + F(U_R)}{2} - \frac{1}{\lambda} \frac{U_R - U_L}{2}, \quad (3.9.6)$$

where $\lambda = \Delta r / \Delta t$. Here, F is the exact flux (4.2.1) and S_j^n is the discretized source to be determined later.

Taking the curved geometry into account We now give the states $U_{j+1/2\pm}, U_{j-1/2\pm}$ and the discretized source term S_j^n which take into account the geometry of the Schwarzschild spacetime. For a steady state solution $U = U(r)$, the equation $\partial_r \left((1 - 2M/r) F(U) \right) = S(r, U)$ holds, where U, F and the source term S are given by (4.2.1), or equivalently, the solution (ρ, v) satisfies the static Euler equation (3.8.1). First of all, we would like to approximate the solution in each cell $(r_{j-1/2}, r_{j+1/2})$ by steady state solutions. Hence we expect the following algebraic relations following from the calculations:

$$\begin{aligned} (1 - v_{j+1/2-}^n)^2 v_{j+1/2-}^n \frac{2k^2}{1-k^2} r_{j+1/2}^{\frac{4k^2}{1-k^2}} / (1 - 2M/r_{j+1/2}) &= (1 - v_j^{n2}) v_j^n \frac{2k^2}{1-k^2} r_j^{\frac{4k^2}{1-k^2}} / (1 - 2M/r_j), \\ r_{j+1/2} (r_{j+1/2} - 2M) \rho_{j+1/2-}^n \frac{v_{j+1/2-}^n}{1 - v_{j+1/2-}^n} &= r_j (r_j - 2M) \rho_j^n \frac{v_j^n}{1 - v_j^{n2}}, \\ (1 - v_{j+1/2+}^n)^2 v_{j+1/2+}^n \frac{2k^2}{1-k^2} r_{j+1/2}^{\frac{4k^2}{1-k^2}} / (1 - 2M/r_{j+1/2}) &= (1 - v_{j+1}^{n2}) v_{j+1}^n \frac{2k^2}{1-k^2} r_{j+1}^{\frac{4k^2}{1-k^2}} / (1 - 2M/r_{j+1}), \\ r_{j+1/2} (r_{j+1/2} - 2M) \rho_{j+1/2+}^n \frac{v_{j+1/2+}^n}{1 - v_{j+1/2+}^n} &= r_{j+1} (r_{j+1} - 2M) \rho_{j+1}^n \frac{v_{j+1}^n}{1 - v_{j+1}^{n2}}. \end{aligned} \quad (3.9.7)$$

However, since a steady state solution might not be defined globally on $(2M, +\infty)$, it is possible that (3.9.7) does not permit a solution. We simply define $(\rho_{j+1/2-}^n, v_{j+1/2-}^n) =$

(ρ_j^n, v_j^n) if the first two equations in (3.9.7) do not have a solution and $(\rho_{j+1/2-}^n, v_{j+1/2-}^n) = (\rho_{j+1}^n, v_{j+1}^n)$ if the last two equations in (3.9.7) do not have a solution. Integrating (3.9.4) by parts, we obtain the approximate source term:

$$\begin{aligned} S_j^n &= \frac{1}{\Delta r} \int_{r_{j-1/2}}^{r_{j+1/2}} S(t_n, r) dr = \frac{1}{\Delta r} \int_{r_{j-1/2}}^{r_{j+1/2}} \partial_r \left((1 - 2M/r) F(U(t_n, r)) \right) dr \\ &= \frac{1}{\Delta r} \left((1 - 2M/r_{j+1/2}) F(U_{j+1/2-}^n) \right. \\ &\quad \left. - (1 - 2M/r_{j-1/2+}) F(U_{j-1/2+}^n) \right), \end{aligned} \quad (3.9.8)$$

where $U_{j+1/2-}^n, U_{j-1/2+}^n$ are two states determined by (3.9.7) and $F(\cdot)$ the accurate flux of the Euler model given by (4.2.1). We then have the following result.

Theorem 3.9.1. The finite volume scheme proposed for the relativistic Euler equation on a Schwarzschild background (4.1.1) satisfies:

- The scheme preserves the steady state solution to the Euler equation (3.8.1).
- The scheme is consistent, that is, for an exact solution $U = U(t, r)$ and the states $U_L, U_R \rightarrow U, r_L, r_R \rightarrow r$, we have

$$\mathcal{F}_r(r_R, U_L, U_R) - \mathcal{F}_l(r_L, U_L, U_R) = S(r, U)(r_R - r_L) + O((r_R - r_L)^2), \quad (3.9.9)$$

where $\mathcal{F}_l, \mathcal{F}_r$ are numerical fluxes given by (3.9.5) and $S(r, U)$ is the source term given by (4.2.1).

- The scheme has second-order accuracy in space and first-order accuracy in time.

Proof. For a steady state given by (3.8.1), we have $U_{j+1/2+} = U_{j+1/2-}$. Hence, the flux of the finite volume method (3.9.5) satisfies $F_{j+1/2} = (1 - 2M/r_{j+1/2})F(U_{j+1/2+}) = (1 - 2M/r_{j+1/2})F(U_{j+1/2-})$, which gives:

$$\frac{1}{\Delta r} (F_{j+1/2}^n - F_{j-1/2}^n) = (1 - 2M/r_{j+1/2})F(U_{j+1/2-}) - (1 - 2M/r_{j-1/2})F(U_{j-1/2+}) = S_j^n.$$

Therefore, the scheme preserves the steady state solutions. Next, according to (3.9.7) and (3.9.8), there exist four states $U_L^l, U_R^l, U_L^r, U_R^r$ such that

$$\begin{aligned} \mathcal{F}_r(r_R, U_L, U_R) - \mathcal{F}_l(r_L, U_L, U_R) &= (1 - 2M/r_R)\mathcal{F}(U_L^r, U_R^r) - (1 - 2M/r_L)\mathcal{F}(U_L^l, U_R^l) \\ &= (1 - 2M/r + 2M/r^2(r_R - r) + O(r_R - r))(\mathcal{F}(U, U) + \partial_1 \mathcal{F}(U, U)(U_R - U) + o(U_R - U)) \\ &\quad - (1 - 2M/r + 2M/r^2(r_L - r) + O(r_L - r))(\mathcal{F}(U, U) + \partial_2 \mathcal{F}(U, U)(U_L - U) + o(U_L - U)). \end{aligned}$$

By (3.9.7), $U_R - U_L = O(r_R - r_L)S(r, U)$. Moreover, since $U = U(t, r)$ is accurate,

we have $\mathcal{F}(U, U) = F(U)$ and $\partial_1 \mathcal{F}(U, U) = \partial_2 \mathcal{F}(U, U) = \partial_U F(U)$. Therefore,

$$\begin{aligned} & \mathcal{F}_r(r_R, U_L, U_R) - \mathcal{F}_l(r_L, U_L, U_R) \\ &= \frac{2M}{r^2} (r_R - r_L) F(U) + (1 - 2M/r) \partial_U F(U) (U_R - U_L) + O((r_R - r_L)^2) \\ &= \partial_r \left((1 - 2M/r) F(U) \right) (r_R - r_L) + o(r_R - r_L) = S(r, U) (r_R - r_L) + O((r_R - r_L)^2). \end{aligned}$$

Next, a Taylor expansion with respect to time yields us $U_j^{n+1} = U_j^n + \partial_t U_j^n \Delta t + \partial_{tt}^2 U_j^n \Delta t^2 + o(\Delta t^2)$. Recall that our scheme gives

$$\begin{aligned} U_j^{n+1} &= U_j^n - \frac{\Delta t}{\Delta r} \left((1 - 2M/r_{j+1/2}) F_{j+1/2}^n - (1 - 2M/r_{j-1/2}) F_{j-1/2}^n - \Delta r S_j^n \right) \\ &= U_j^n - \frac{1}{\lambda} \left((1 - 2M/r_{j+1/2}) \left(\frac{F(U_{j+1/2+}) - F(U_{j+1/2-})}{2} - \frac{1}{\lambda} \frac{U_{j+1/2+} - U_{j+1/2-}}{2} \right) \right. \\ &\quad \left. + (1 - 2M/r_{j-1/2}) \left(\frac{F(U_{j-1/2+}) - F(U_{j-1/2-})}{2} + \frac{1}{\lambda} \frac{U_{j-1/2+} - U_{j-1/2-}}{2} \right) \right). \end{aligned}$$

According our construction, we have

$$\begin{aligned} & \left(1 - \frac{2M}{r_{j+1/2}} \right) \left(F(U_{j+1/2+}) - F(U_{j+1/2-}) \right) \\ &= \left(1 - \frac{2M}{r_{j+1}} \right) F(U_{j+1}^n) - \left(1 - \frac{2M}{r_j} \right) F(U_j^n) - \int_{r_j}^{r_{j+1}} S(r, U(t_n, r)) dr. \end{aligned}$$

A Taylor expansion to Δr gives us $U_{j+1/2+} - U_{j+1/2-} = O(\Delta r^3)$ and

$$\begin{aligned} \left(1 - \frac{2M}{r_{j\pm 1}} \right) &= 1 - \frac{2M}{r_j} \pm \frac{2M}{r_j^2} \Delta r - \frac{2M}{r_j^3} \Delta r^2 + O(\Delta r^3), \\ F(U_{j\pm 1}^n) &= F(U_j^n) + \partial_U F(U_j^n) \left(\pm \partial_r U_j^n \Delta r + \frac{1}{2} \partial_{rr}^2 U_j^n \Delta r^2 \right) \\ &\quad + \frac{1}{2} (\partial_r U_j^n)^T \partial_{UU}^2 F(U_j^n) \partial_r U_j^n \Delta r^2 + O(\Delta r^3), \\ \int_{r_j}^{r_{j+1}} S(r, U(t_n, r)) dr &= S(r_j, U_j^n) \Delta r + \partial_r S(r_j, U_j^n) \Delta r^2 + O(\Delta r^3). \end{aligned}$$

Hence we conclude that

$$\partial_t U_j^n + \partial_r \left((1 - 2M/r_j) F(U_j^n) \right) - S(r_j, U_j^n) + O(\Delta t + \Delta r^2) = 0.$$

□

Numerical steady state solution Recall that the steady state solution to the relativistic Euler model is given by a static Euler system (3.8.1). Hence, if $U = U(t, r)$

is a steady state solution, it trivially satisfies $\int |\partial_r F((1 - 2M/r)U) - S(r, U)| dr = 0$, where $F = (F^0, F^1)^T$ is the flux and $S = (S^0, S^1)^T$ the source term given by (4.2.1). In order to describe the steady state solution numerically, we define the total variation in time:

$$E^n := E(t_n) = \sum_j \sum_{i=0,1} \left| (1 - 2M/r_{j+1/2}) (F^i(U_{j+1/2+}^n) - F^i(U_{j-1/2-}^n)) - (1 - 2M/r_{j-1/2}) (F^i(U_{j-1/2+}^n) - F^i(U_{j-1/2-}^n)) \right|. \quad (3.9.10)$$

From our former construction, we have the following result.

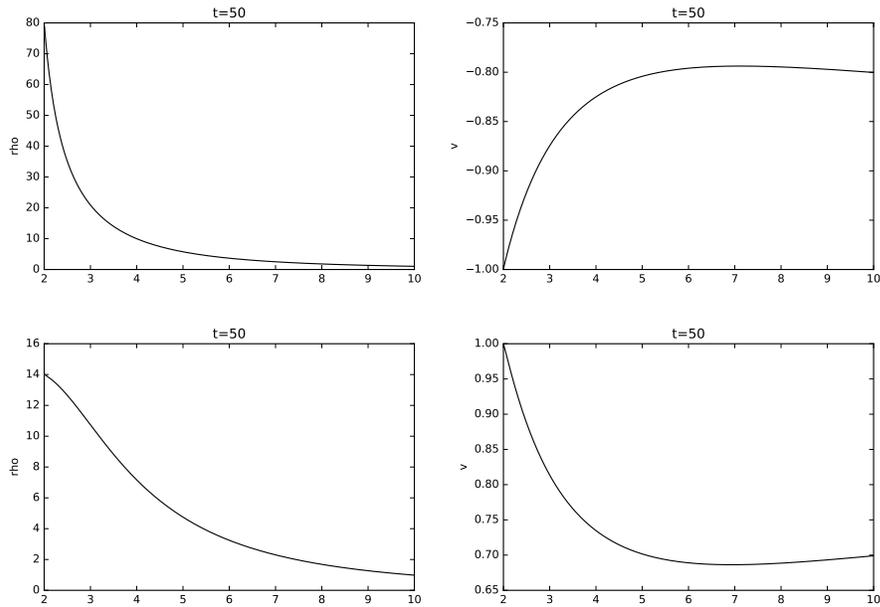
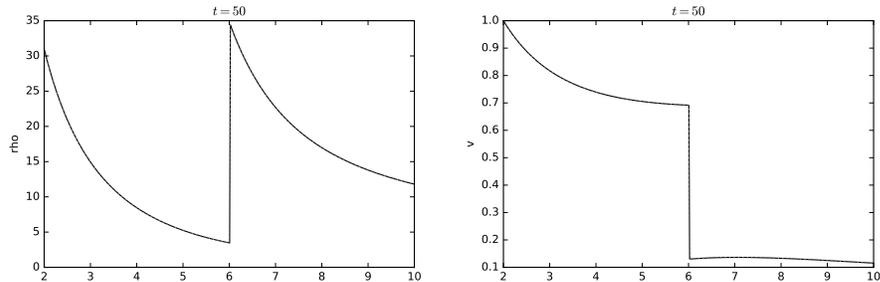
Lemma 3.9.2. *If $U = (t, r)$ is a numerical solution to the relativistic Euler model constructed by (3.9.4)- (3.9.8), then U is a steady state solution for $t \geq T$ where $T > 0$ is a finite time if and only if there exists a $N < +\infty$ such that for all $n > N$, the total variation $E^n \equiv 0$.*

3.10 Numerical experiments for the relativistic Euler model

Nonlinear stability of steady state solutions Before studying the stability of steady state solutions, we check that our scheme preserves smooth steady state solutions to the relativistic Euler model (4.1.1). Recall that $r > 2M$ with $M = 1$ being the blackhole mass. We work on the space interval (r_{\min}, r_{\max}) with $r_{\min} = 2M = 2$ and $r_{\max} = 10$ and we take 500 points to discretize this interval. We consider the evolution of two steady state solutions satisfying the algebraic relation (3.8.2) of the Euler model with the density $\rho(10) = 1.0$, the velocity $v(10) = 0.6$ and the density $\rho(10) = 1.0$, the velocity $v(10) = -0.8$ respectively. We also provides the evolution of a steady state shock.

Propagation of discontinuities Referring to [31], we recall that there exists a solution to the generalized Riemann problem (3.1.3), (3.8.7) consisting of at most three steady state solutions. Figures 3.10.3, 3.10.4 show the evolution of two generalized Riemann problem with an initial discontinuity. Furthermore, we are now interested in the late-time behavior of solutions whose initial data is steady state solution perturbed by a compactly supported solution. Numerical tests lead us to the following result.

Conclusion 3.10.1 (Stability of smooth steady state solutions to the Euler model). *Let $(\rho_*, v_*) = (\rho_*, v_*)(r)$, $r > 2M$ be a smooth steady state solution satisfying the static Euler equation (3.8.1) and $(\rho_0, v_0) = (\rho_0, v_0)(r) = (\rho_*, v_*)(r) + (\delta_\rho, \delta_v)(r)$ where*

Figure 3.10.1: Evolution of steady state solutions, plotted at time $t = 50$ Figure 3.10.2: Evolution of a steady shock plotted at time $t = 50$

$(\delta_\rho, \delta_v) = (\delta_\rho, \delta_v)(r)$ is a function with compact support, then the solution to the relativistic Euler equation on a Schwarzschild background (4.1.1) denoted by $(\rho, v) = (\rho, v)(t, r)$ satisfies that $(\rho, v)(t, \cdot) = (\rho_*, v_*)$ for all $t > t_0$ where $t_0 > 0$ is a finite time. Numerical experiments show that there exists a finite time $t_0 > 0$ such that:

- If $\int \delta_\rho(r) dr + \int \delta_v(r) dr = 0$, $(\rho, v)(t, r) = (\rho_*, v_*)(r)$ for all $t > t_0$.
- If $\int \delta_\rho(r) dr + \int \delta_v(r) dr \neq 0$, then there exists a time $t_0 > 0$ such that $(\rho, v)(t, r) = (\rho_{**}, v_{**})(r)$ for all $t > t_0$ where (ρ_{**}, v_{**}) is a steady state solution to the Euler model and $(\rho_{**}, v_{**}) \neq (\rho_*, v_*)$.

We observe the phenomena described in Conjecture 3.1.3 in Figures 3.10.5 and 3.10.6. To check that the numerical solutions in Figures 3.10.5, 3.10.6 converge to

a steady state solution, we refer to Lemma 3.9.2 and calculate the total variation at each time step. Figure 3.10.7 shows that these solutions are eventually steady state solutions. The steady shock given by (3.8.5) and (3.8.6) is a weak solution satisfying the static Euler equation (3.8.1). We are also interested in the behavior of steady shocks with perturbations. We summarize our results as follows; see Figure 3.10.8.

Conclusion 3.10.2. *Consider a steady shock $(\rho_*, v_*) = (\rho_*, v_*)(r)$, $r > 2M$ given by (3.8.5), (3.8.6) whose point of discontinuity is at $r = r_*$ and we give the initial data $(\rho_0, v_0) = (\rho_0, v_0)(r) = (\rho_*, v_*)(r) + (\delta_\rho, \delta_v)(r)$ with $(\delta_\rho, \delta_v) = (\delta_\rho, \delta_v)(r)$ a compactly supported function, then there exists a finite time $t > t_0$ such that for all $t > t_0$, the solution $(\rho, v)(t, \cdot) = (\rho_{**}, v_{**})$ where (ρ_{**}, v_{**}) is a steady state shock whose point of discontinuity is at $r = r_{**}$ with $r_{**} \neq r_*$.*

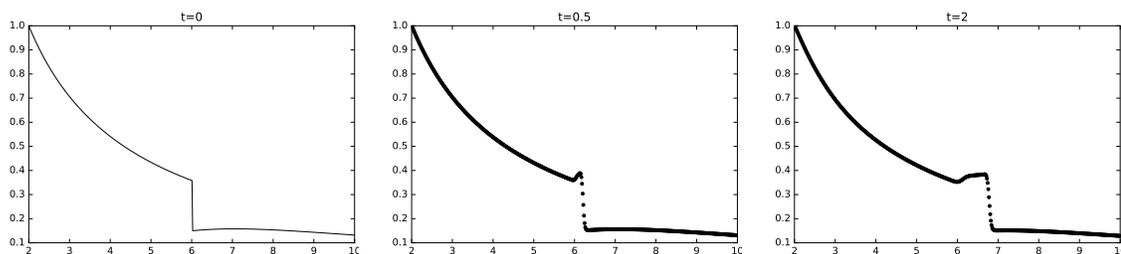


Figure 3.10.3: Solution to a Riemann problem (1-rarefaction and 2-shock)

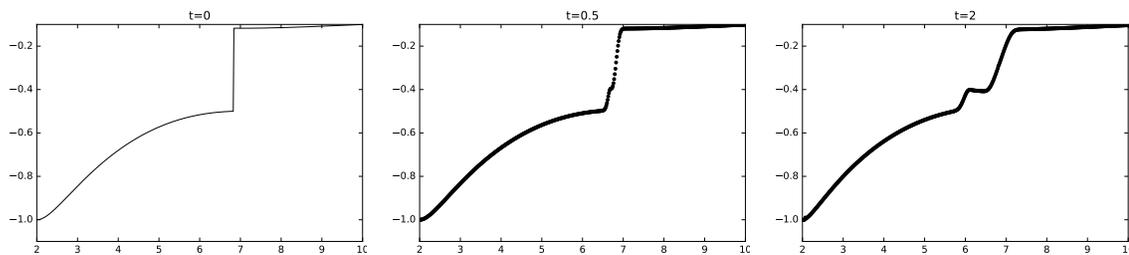


Figure 3.10.4: Solution to a Riemann problem (1-rarefaction and 2-rarefaction)

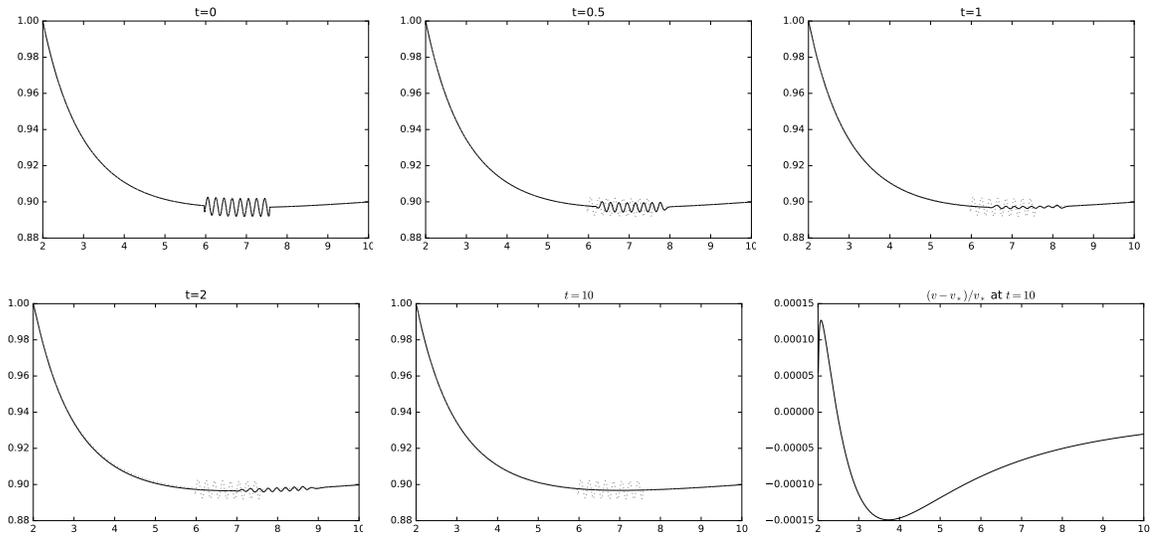


Figure 3.10.5: Evolution of a steady state with perturbation, converging to the same asymptotic state

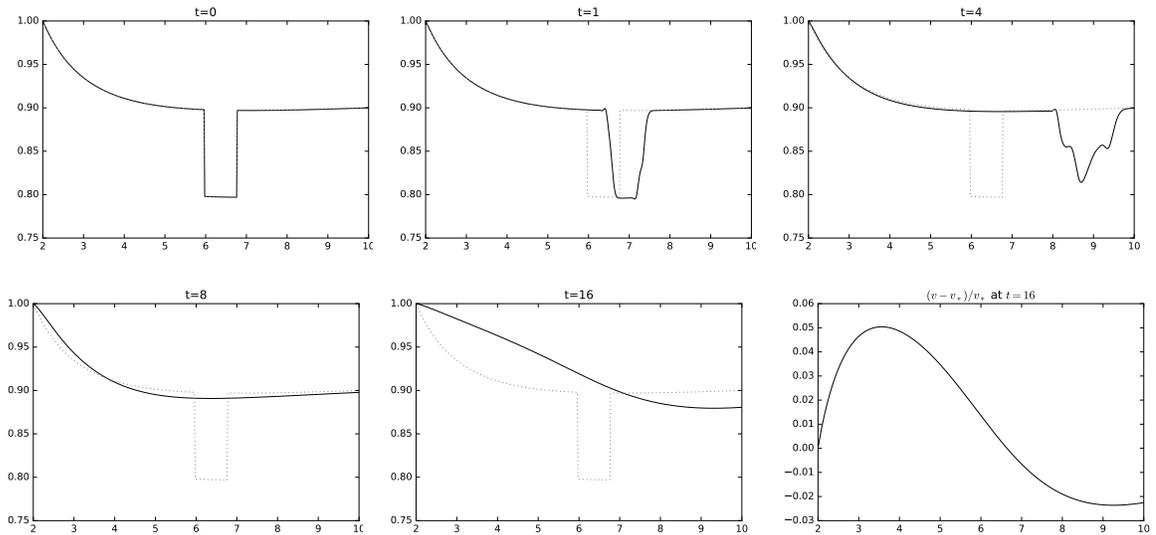


Figure 3.10.6: Evolution of a steady state with perturbation, converging to a different asymptotic state

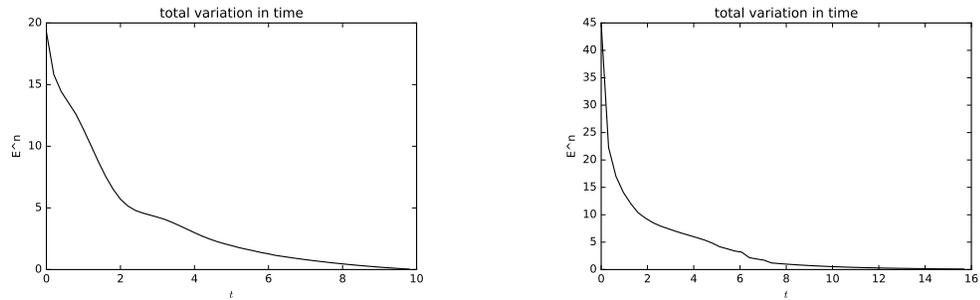


Figure 3.10.7: Total variation in time corresponding to Figures 3.10.5 and 3.10.6, respectively

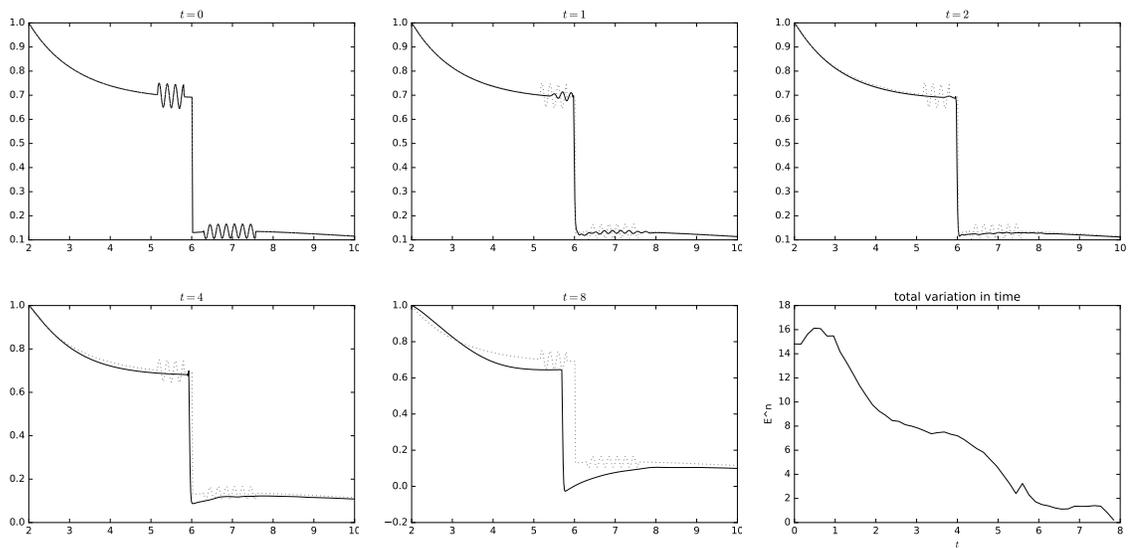


Figure 3.10.8: Evolution of an initially perturbed steady shock and its total variation in time

Part III

Non-Relativistic Fluid Dynamics: A simplified model

Chapter 4

An existence theory for an Euler model with dissipation ¹

¹ Réalisé en collaboration avec Yangyang Cao

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4.1 Introduction

Our model of interest is derived directly from LeFloch and Xiang [31], which is a non-conservative Euler system with a source term:

$$\begin{aligned} \partial_t \rho + \partial_r(\rho v) + \frac{2}{r} \rho v &= 0, \\ \partial_t(\rho v) + \partial_r\left(\rho(v^2 + k^2)\right) + \frac{2}{r} \rho v^2 + \frac{1}{r^2} m \rho &= 0, \end{aligned} \tag{4.1.1}$$

defined for all $r > 0$ where the main unknowns are the density $\rho > 0$ and the velocity v of the fluid flow. Here, the parameters are given as the Schwarzschild black hole mass $m \in (0, +\infty)$ and the constant sound speed $k \in (0, +\infty)$. Remark that even if the Euler model (4.1.1) is non-relativistic in the sense that the velocity v is far from light speed, the effect of the black hole is still reflected by the source term.

The organization of this paper is as follows. In Section 4.2 we give some basic properties of the homogenous Euler model without source term, including the hyperbolicity and the nonlinear properties which leads us to give the result of the standard Riemann problem whose wave interactions are analyzed as well.

We take into consideration the steady state solutions in Section 4.3, where we first study different families of smooth steady state solutions to the Euler model, serving as one of the main results of the present paper. The study coming after is the generalized Riemann problem of the Euler model with the initial data consisting of two steady state solutions separated by a discontinuity of jump. An exact solution is constructed global-in-time in Section 4.4, with three steady states connected by 2 different families of generalized elementary waves and we have verified that the Rankie-Hugoniot jump condition and the Lax entropy condition are satisfied according to our construction of the solutions. We also give the evolution of the total variation of solution of the Riemann problem when time passes.

Referring to Section 4.3, smooth steady states may not be extended on the whole space region $(0, +\infty)$. To give a complete construction of an initial value problem, it is necessary to consider the triple Riemann problem, which is an initial problem with its initial data given as three steady state solutions separated by two given radius. We provide a global-in-time solution of such problem in Section 4.5.

In Section 4.6, we are then able to give an existence theory of our Euler model. The technique we used is that we construct a sequence of approximate solutions by the generalized Glimm scheme based on the (triple) generalized Riemann problem. Together with the estimation of total variation, the random choice method provides a global-in-time solution of the non-conservative Euler model.

4.2 Homogenous system

4.2.1 Elementary waves

According to (4.1.1), we write the Euler system as

$$\partial_t U + \partial_r F(U) = S(r, U), \quad (4.2.1)$$

where

$$U = \begin{pmatrix} \rho \\ \rho v \end{pmatrix}, \quad F(U) = \begin{pmatrix} \rho v \\ \rho(v^2 + k^2) \end{pmatrix}, \quad S(r, U) = \begin{pmatrix} -\frac{2}{r}\rho v \\ -\frac{2}{r}\rho v^2 - \frac{1}{r^2}m\rho \end{pmatrix}.$$

We derive the pair of eigenvalues reading

$$\lambda(\rho, v) = v - k, \quad \mu(\rho, v) = v + k. \quad (4.2.2)$$

We give also the pair of corresponding Riemann invariants:

$$w(\rho, v) = v + k \ln \rho, \quad z(\rho, v) = v - k \ln \rho. \quad (4.2.3)$$

Following directly from (4.2.2), we have the following proposition:

Proposition 4.2.1. *Let $k > 0$ be the sound speed and $m > 0$ the black hole mass and the non-conservative Euler model (4.1.1) is strictly hyperbolic and both characteristic fields are genuinely nonlinear.*

Proposition 4.2.1 enables us to consider first the elementary waves of the homogenous Euler system:

$$\partial_t U + \partial_r F(U) = 0, \quad (4.2.4)$$

where we recall that $U = (\rho, \rho v)^T$ and $F(U) = (\rho v, \rho(v^2 + k^2))^T$ referring to (4.2.1). Notice that $(\rho, v) \rightarrow (\rho, \rho v)$ is a one-to-one map and we thus don't distinguish U and (ρ, v) in the coming section for the sake of simplicity.

We consider first the rarefaction curves along which the corresponding Riemann invariants remain constant.

Lemma 4.2.2. *Consider the homogenous Euler model given by (4.2.4). The 1-rarefaction curve issuing from constant $U_L = (\rho_L, v_L)$ and the 2-rarefaction wave from the constant $U_R = (\rho_R, v_R)$ are given by*

$$R_1^{\rightarrow}(U_L) : \left\{ v - v_L = \ln \left(\frac{\rho}{\rho_L} \right)^{-k}, \quad v < v_L \right\}, \quad R_2^{\leftarrow}(U_R) : \left\{ v - v_R = \ln \left(\frac{\rho}{\rho_R} \right)^k, \quad v < v_R \right\}. \quad (4.2.5)$$

Proof. The 1-family Riemann invariant is a constant along the 1-rarefaction curve passing the point U_L and we have

$$R_1^{\rightarrow}(U_L) : w(\rho, v) = w(\rho_L, v_L), \quad z(\rho, v) < z(\rho_L, v_L),$$

which gives the form of the 1-rarefaction wave. Similarly, we have the 2-rarefaction wave. \square

We can also give the form of 1-shock and 2-shock associated with the constant states U_L and U_R respectively.

Lemma 4.2.3. *The 1-shock wave and 2-shock wave of the Euler model without source term (4.2.4) associated with the constant states U_L and U_R respectively have the fol-*

lowing forms:

$$\begin{aligned} S_1^\rightarrow(U_L) &: \left\{ v - v_L = -k \left(\sqrt{\frac{\rho}{\rho_L}} - \sqrt{\frac{\rho_L}{\rho}} \right), \quad v > v_L \right\}, \\ S_2^\leftarrow(U_R) &: \left\{ v - v_R = k \left(\sqrt{\frac{\rho}{\rho_R}} - \sqrt{\frac{\rho_R}{\rho}} \right), \quad v > v_R \right\}. \end{aligned} \quad (4.2.6)$$

And the 1-shock speed σ_1 and the 2-speed σ_2 are:

$$\sigma_1((\rho_L, v_L), (\rho, v)) = v - k \sqrt{\frac{\rho_L}{\rho}}, \quad \sigma_2((\rho, v), (\rho_R, v_R)) = v + k \sqrt{\frac{\rho_R}{\rho}}. \quad (4.2.7)$$

Proof. The Rankine-Hugoniot jump condition gives

$$\begin{aligned} \sigma[\rho] &= [\rho v], \\ \sigma[\rho v] &= [\rho(v^2 + k^2)], \end{aligned} \quad (4.2.8)$$

where σ denotes the speed of the discontinuity. Consider first the 1-shock which should satisfy the Lax entropy inequality in the sense that

$$\lambda(\rho_L, v_L) > \sigma > \lambda(\rho, v),$$

for the 1-shock wave. Eliminating the speed σ , we obtain:

$$v - v_L = -k \left(\sqrt{\frac{\rho}{\rho_L}} - \sqrt{\frac{\rho_L}{\rho}} \right), \quad v > v_L.$$

The form of the 2-shock wave follows from a similar calculation. The shock speeds can be obtained directly from (4.2.6), (4.2.8). \square

4.2.2 Standard Riemann problem

We now consider the solution of the standard Riemann problem of the homogenous Euler system (4.2.4) associated with given initial data:

$$U_0(r) = \begin{cases} U_L & 0 < r < r_0, \\ U_R & r > r_0, \end{cases} \quad (4.2.9)$$

where $r_0 > 0$ is a fixed radius and $U_L = (\rho_L, v_L)$, $U_R = (\rho_R, v_R)$ are constant states. To give the solution of the standard Riemann problem, we define now the **1-family-wave** and the **2-family wave**:

$$W_1^\rightarrow(U_L) = S_1^\rightarrow(U_L) \cup R_1^\rightarrow(U_L), \quad W_2^\leftarrow(U_R) = S_2^\leftarrow(U_R) \cup R_2^\leftarrow(U_R), \quad (4.2.10)$$

where $S_1^\rightarrow, S_2^\leftarrow$ are 1 and 2-shocks while $R_1^\rightarrow, R_2^\leftarrow$ are 1 and 2-rarefaction waves. It is obvious that if $U_L \in W_2^\leftarrow(U_R)$ or $U_R \in W_1^\rightarrow(U_L)$, then the Riemann problem is solved by the left state U_L and the right state U_R connected by either a 1-family wave or a 2-family wave. Otherwise, more analysis are required.

Lemma 4.2.4. *On the $w - z$ plane where w, z are the Riemann invariants of the Euler model given by (4.2.3), $S_1^\rightarrow(U_L)$ defines a curve such that $0 \leq \frac{dw}{dz} < 1$, $S_2^\leftarrow(U_R)$ defines a curve satisfying $0 \leq \frac{dz}{dw} < 1$ where $S_1^\rightarrow, S_2^\leftarrow$ are the 1 and 2-shocks given by (4.2.6).*

Proof. Introduce functions Φ_\pm :

$$\Phi_\pm(\gamma) := 1 + \gamma \left(1 \pm \sqrt{1 + \frac{2}{\gamma}} \right). \quad (4.2.11)$$

Taking $\gamma = \gamma(v, v_L) = \frac{(v - v_L)^2}{2k^2}$ along the 1-shock, we have

$$\begin{aligned} w - w_L &= v - v_L + k \ln \frac{\rho}{\rho_L} = -\sqrt{2\gamma k^2} + k \ln \Phi(\gamma), \\ z - z_L &= v - v_L - k \ln \frac{\rho}{\rho_L} = -\sqrt{2\gamma k^2} - k \ln \Phi(\gamma). \end{aligned}$$

The tangent of the shock wave curve $S_1^\rightarrow(U_L)$ in the $w - z$ plane is given by

$$\frac{dw}{dz} = \frac{d(w - w_L)}{d(z - z_L)} = \frac{d(w - w_L)}{d\gamma} \frac{d\gamma}{d(z - z_L)}.$$

Hence, we have $0 \leq \frac{dw}{dz} < 1$. A similar calculation gives the result of the 2-shock. \square

Together with Lemma 4.2.4 and the form of elementary waves given in Lemmas 4.2.5, 4.2.6, some direct observations are given in order, concerning the standard Riemann problem of the homogenous Euler model (4.2.4):

- For different given states U_L, U'_L , the two 1-family wave curves $W_1^\rightarrow(U_L) \cap W_1^\rightarrow(U'_L) = \emptyset$. Similarly, for $U_R \neq U'_R$, the 2-family wave curve $W_2^\leftarrow(U_R)$ has no intersection point with $W_2^\leftarrow(U'_R)$.
- The two families of wave curves cover the whole upper half $\rho - v$ plane as a result of Lemma 4.2.4.
- For given constant states U_L, U_R , the waves $W_1^\rightarrow(U_L)$ and $W_2^\leftarrow(U_R)$ intersect one and only once at a point U_M .

We thus have the proposition:

Proposition 4.2.5 (Solution of the standard Riemann problem). *Given two constant states $U_L = (\rho_L, v_L)$ and $U_R = (\rho_R, v_R)$, the standard Riemann problem (4.2.4), (4.2.9) admits a unique entropic solution which only depends on $\frac{r-r_0}{t}$. More precisely, the solution is realized by the left state U_L , the right state U_R and a uniquely defined intermediate state U_M where U_L and U_M are connected by a 1-wave while U_M and U_R are connected by a 2-wave.*

4.2.3 Wave interactions

For the standard Riemann problem of the Euler model without source term (4.2.4) with left constant state U_L and right constant state U_R , define the wave strength of the Riemann problem $\mathcal{S} = \mathcal{S}(U_L, U_R)$:

$$\mathcal{S}(U_L, U_R) := |\ln \rho_L - \ln \rho_M| + |\ln \rho_R - \ln \rho_M|,$$

where U_M is the unique intermediate state $U_M \in W_1^{\rightarrow}(U_L) \cap W_2^{\leftarrow}(U_R)$. We have the following lemma concerning \mathcal{S} :

Lemma 4.2.6. *Let U_L, U_P, U_R be three given constant states. The wave strengths associated with the Riemann problem $(U_L, U_P), (U_P, U_R)$ and (U_L, U_R) satisfy the following inequality*

$$\mathcal{S}(U_L, U_R) \leq \mathcal{S}(U_L, U_P) + \mathcal{S}(U_P, U_R). \quad (4.2.12)$$

To prove Lemma 4.2.6, we first need the following calculation.

Lemma 4.2.7. *Given an arbitrary state U_0 , the 1 and 2-shock wave curves $S_1^{\rightarrow}(U_0)$ and $S_2^{\leftarrow}(U_0)$ are reflectional symmetric with respect to the straight line parallel to $w = z$ passing the point U_0 on the $w - z$ plane where w, z are the Riemann invariants of the Euler model introduced by (4.2.3).*

Proof. Denote by (w_0, z_0) the point U_0 on the $w - z$ plane. For a given point (w, z) along the 1-shock, we have

$$\Delta w_1 := w - w_0 = -\sqrt{2\gamma k^2} + k \ln \Phi_+(\gamma), \quad \Delta z_1 := z - z_0 = -\sqrt{2\gamma k^2} - k \ln \Phi_+(\gamma),$$

while for a point (w, z) along the 2-shock :

$$\Delta w_2 := w - w_0 = -\sqrt{2\gamma k^2} + k \ln \Phi_-(\gamma), \quad \Delta z_2 := z - z_0 = -\sqrt{2\gamma k^2} - k \ln \Phi_-(\gamma),$$

where the function Φ_{\pm} is defined by (4.2.11), which gives $\Phi_+(\gamma)\Phi_-(\gamma) = 1$. We have got the result by noticing that $\Delta w_1 = \Delta z_2$, $\Delta z_1 = \Delta w_2$. \square

We can thus continue the proof of Lemma 4.2.6.

Proof of Lemma 4.2.6. Again, we stay on $w - z$ plane. From Lemmas 4.2.4, 4.2.7, we can see that the shock waves S_1^+ , S_2^- passing the same point U_0 are symmetric with respect to the straight line parallel to $w = z$ passing the point U_0 . According to the definition of the wave strength (4.2.12) which is actually measured along the line $w = z$, the symmetry of waves gives immediately the result. \square

4.3 Fluid equilibria

4.3.1 Critical smooth steady state solutions

We now turn our attention to steady state solutions $\rho = \rho(r)$, $v = v(r)$, which satisfies the ordinary differential system:

$$\begin{aligned} \frac{d}{dr}(r^2 \rho v) &= 0, \\ \frac{d}{dr}(r^2(v^2 + k^2)\rho) - 2k^2 \rho r + m\rho &= 0, \end{aligned} \quad (4.3.1)$$

with the initial condition $\rho_0 > 0$, v_0 posed at a given radius $r = r_0 > 0$,

$$\rho(r_0) = \rho_0 > 0, \quad v(r_0) = v_0. \quad (4.3.2)$$

We refer to (4.3.1) the **static Euler model**. For a steady state solution $\rho = \rho(r)$, $v = v(r)$, it is straightforward to find a pair of algebraic relations:

$$\begin{aligned} r^2 \rho v &= r_0^2 \rho_0 v_0, \\ \frac{1}{2}v^2 + k^2 \ln \rho - m \frac{1}{r} &= \frac{1}{2}v_0^2 + k^2 \ln \rho_0 - m \frac{1}{r_0}, \end{aligned}$$

from which we recover the equation for v by eliminating ρ :

$$\frac{1}{2}v^2 - k^2 \ln(r^2 \operatorname{sgn}(v_0)v) - m \frac{1}{r} = \frac{1}{2}v_0^2 - k^2 \ln(r_0^2 |v_0|) - m \frac{1}{r_0}. \quad (4.3.3)$$

Notice that once we get the value of v , we can have the value ρ directly from the first equation of (4.3.1) and we can therefore focus on the analysis of the steady state velocity v .

Introduce the function $G = G(r, v)$:

$$G(r, v) := \frac{1}{2}v^2 - k^2 \ln(r^2 \operatorname{sgn}(v_0)v) - m \frac{1}{r}, \quad (4.3.4)$$

and we see if $v = v(r)$ is a solution of (4.3.1) with the condition $v(r_0) = v_0$, then $G(r, v(r)) \equiv G(r_0, v_0)$ always holds. Differentiating G with respect to v and r , we

obtain

$$\partial_v G = v - \frac{k^2}{v}, \quad \partial_r G = \frac{1}{r^2}(m - 2k^2 r). \quad (4.3.5)$$

We can immediately deduce the first-order derivative of the steady state velocity $v = v(r)$:

$$\frac{dv}{dr} = \frac{v}{r^2} \frac{2k^2 r - m}{v^2 - k^2}. \quad (4.3.6)$$

It is obvious to see that $\partial_v G = 0$ if and only if $v = \pm k$ while $\partial_r G = 0$ if and only if $r = \frac{m}{2k^2}$ from (4.3.5). This observation motivates us to find the steady state curves passing the points $(\frac{m}{2k^2}, \pm k)$ on the $r - v$ plane $(0, +\infty) \times (-\infty, +\infty)$. We call the solution $v = v(r)$ on the subset of $r - v$ plane $(0, +\infty) \times (-\infty, +\infty)$ **the critical steady state solution** of the static Euler model (4.3.1) if and only if satisfies $S(r, v(r)) \equiv 0$ where $S = S(r, v)$ is given by

$$S(r, v) := \frac{1}{2}v^2 - k^2 \ln(r^2|v|) - m\frac{1}{r} + \frac{3}{2}k^2 + k^2 \ln \frac{m^2}{4k^3}. \quad (4.3.7)$$

It is direct to check that $S(\frac{m}{2k^2}, \pm k) = 0$. We now have the following lemma concerning the critical steady state curve.

Proposition 4.3.1. *The static Euler model (4.3.1) admits four smooth critical steady state curves on the subset of $r - v$ plane $(0, +\infty) \times (-\infty, +\infty)$ denoted by $v_*^{P,b}, v_*^{P,\sharp}, v_*^{N,b}, v_*^{N,\sharp}$. Moreover, we have the following properties:*

- The sign of each solution does not change on the space domain $(0, +\infty)$.
- On the interval $(0, \frac{m}{2k^2})$, we have

$$v_*^{N,\sharp} < -k < v_*^{N,b} < 0 < v_*^{P,b} < k < v_*^{P,\sharp},$$

while on the interval $(\frac{m}{2k^2}, +\infty)$, we have

$$v_*^{N,b} < -k < v_*^{N,\sharp} < 0 < v_*^{P,\sharp} < k < v_*^{P,b}.$$

- The solutions $v_*^{N,\sharp}, v_*^{N,b}$ intersect once at $(\frac{m}{2k^2}, -k)$ while $v_*^{P,\sharp}, v_*^{P,b}$ intersect once at $(\frac{m}{2k^2}, k)$.
- The derivatives of each solution at $(\frac{m}{2k^2}, \pm k)$ are give by

$$\frac{dv_*^{P,\sharp}}{dr}\left(\frac{m}{2k^2}\right) = \frac{dv_*^{N,b}}{dr}\left(\frac{m}{2k^2}\right) = -\frac{2k^3}{m}, \quad \frac{dv_*^{P,b}}{dr}\left(\frac{m}{2k^2}\right) = \frac{dv_*^{N,\sharp}}{dr}\left(\frac{m}{2k^2}\right) = \frac{2k^3}{m}. \quad (4.3.8)$$

Proof. We would like to show that for every fixed radius $r > 0$ and $r \neq \frac{m}{2k^2}$, there exists four different values v satisfying (4.3.7). Observing $S(r, v) = S(r, -v)$, we first

consider the case where $v > 0$. According to (4.3.5), for every fixed $r > 0$, $S(r, \cdot)$ reaches its minimum at $v = k$ and the value is given as

$$S^k(r) := 2k^2 - k^2 \ln r^2 k^2 - \frac{m}{r} + k^2 \ln \frac{m^2}{4k^3}.$$

Since $\partial_r S^k = \frac{1}{r^2}(m - 2k^2 r)$, we have $S^k(r) < S^k(\frac{m}{2k^2}) = 0$. Moreover, we have $\lim_{v \rightarrow 0} S(r, v) = +\infty$ and $\lim_{v \rightarrow +\infty} S(r, v) = +\infty$. Therefore, for every fixed $r \neq \frac{m}{2k^2}$, $S(r, v)$ admits two different positive roots $v_*^{P,b} \leq k \leq v_*^{P,\sharp}$ on $(0, +\infty)$ where the equality holds only once at the point $r = \frac{m}{2k^2}$. The symmetry of $S(r, \cdot)$ with respect to $v = 0$ gives two other negative roots $v_*^{N,\sharp} \leq -k \leq v_*^{N,b}$.

Since $S_v \neq 0$ when $v \neq \pm k$, there exist four smooth different solutions on the interval $(0, \frac{m}{2k^2})$ and $(\frac{m}{2k^2}, +\infty)$ respectively. To extend the steady solution on the whole domain $(0, +\infty)$, we have to treat the very points $(\frac{m}{2k^2}, \pm k)$. Indeed, we have, by the L'Hôpital's rule, $\frac{dv}{dr}(\frac{m}{2k^2}) = \frac{k}{(m/2k^2)^2} k^2 / \left(k \frac{dv}{dr}(\frac{m}{2k^2}) \right)$, which gives

$$\frac{dv}{dr}\left(\frac{m}{2k^2}\right) = \pm \frac{2k^3}{m}, \quad (4.3.9)$$

whose sign depends on the choice of the branch of curves. According to (4.3.9), we are able to keep the solution smooth on the whole domain $(0, +\infty)$ by keeping the sign of the derivative of v at $r = \frac{m}{2k^2}$. We thus define the four different solutions on $(0, +\infty)$:

$$\begin{aligned} v_*^{P,b}(r) &= \begin{cases} v_*^{P,b}(r) & r \in (0, \frac{m}{2k^2}), \\ v_*^{P,\sharp}(r) & r \in (\frac{m}{2k^2}, +\infty), \end{cases} & v_*^{P,\sharp}(r) &= \begin{cases} v_*^{P,\sharp}(r) & r \in (0, \frac{m}{2k^2}), \\ v_*^{P,b}(r) & r \in (\frac{m}{2k^2}, +\infty), \end{cases} \\ v_*^{N,b}(r) &= \begin{cases} v_*^{N,b}(r) & r \in (0, \frac{m}{2k^2}), \\ v_*^{N,\sharp}(r) & r \in (\frac{m}{2k^2}, +\infty), \end{cases} & v_*^{N,\sharp}(r) &= \begin{cases} v_*^{N,\sharp}(r) & r \in (0, \frac{m}{2k^2}), \\ v_*^{N,b}(r) & r \in (\frac{m}{2k^2}, +\infty). \end{cases} \end{aligned} \quad (4.3.10)$$

The derivative of the velocity in (4.3.8) follows directly from (4.3.9) and (4.3.10). \square

4.3.2 Families of steady state solutions

The former construction gives that the relation $S(r, v) \equiv 0$ admits four different solutions on the whole domain $(0, +\infty)$. We would like now to give all families of solutions according to the sign of $S(r, v)$ defined in (4.3.7). We now study general cases of the steady state solutions.

We then have the following lemma.

Lemma 4.3.2. *Let $S = S(r, v)$ be the function defined by (4.3.9), then:*

- *If $S = \text{const.} > 0$, then there exists four solutions $v = v(r)$ satisfying the alge-*

braic equation (4.3.3) on the whole space interval out of the black hole $(0, +\infty)$.

- If $S = \text{const.} < 0$, then there exist two radius $0 < \underline{r}_S < \frac{m}{2k^2} < \bar{r}_S$ such that then there exist four solutions $v = v(r)$ satisfying the algebraic equation (4.3.3) on the interval $(0, \underline{r}_S)$ and four solutions satisfying (4.3.1) on the interval $(\bar{r}_S, +\infty)$.

Proof. We now focus on the case where $S = \text{const.} > 0$. Again, $S(r, v) = S(r, -v)$ allows us to consider the case where $v > 0$. Now we notice that $G(r, v) - G(\frac{m}{2k^2}, k) = S(r, v)$ where G is defined by (4.3.4). By the formula of (4.3.5), for all the fixed $r \in (0, +\infty)$, the equation $G(r, v) - G(\frac{m}{2k^2}, k) = \text{const.} > 0$ admits two positive roots $v_S^{P,\#} > k > v_S^{P,b}$ when and only when $G(r, k) < G(\frac{m}{2k^2}, k)$. Moreover, (4.3.5) gives the fact that $G(r, k)$ reaches its maximum at the point $r = \frac{m}{2k^2}$ and we thus have $G(r, k) < G(\frac{m}{2k^2}, k)$. We have another two negative roots $v_\gamma^{N,\#} < -k < v_\gamma^{N,b}$ following from the same analysis.

Now if $S = \text{const.} < 0$, there exist two points $0 < \underline{r}_S < \frac{m}{2k^2} < \bar{r}_S$ such that $S(\underline{r}_S, k) = S(\bar{r}_S, k) = 0$ and $S(r, k) < 0$ for all $r \in (\underline{r}_S, \bar{r}_S)$. We have four roots satisfying (4.3.3) only on $(0, \underline{r}_S)$ and $(\bar{r}_S, +\infty)$ respectively. \square

We can now give the existence result of the steady state solution of the Euler model (4.1.1).

Theorem 4.3.3 (Families of steady state solutions). Consider the family of steady state solutions of the Euler model (4.3.1). Then, for any given radius $r_0 > 0$, the density $\rho_0 > 0$ and the velocity v_0 , we have: there exists a unique smooth steady state solution $\rho = \rho(r), v = v(r)$ satisfying (4.3.1) together with the initial condition $\rho_0 = \rho(r_0), v(r_0) = v_0$ such that the velocity satisfies $\text{sgn}(v) = \text{sgn}(v_0)$ and $\text{sgn}(|v| - k) = \text{sgn}(|v_0| - k)$ on the corresponding domains of definition. Furthermore, we have different families of solutions:

- If $G(r_0, v_0) > -\frac{3}{2}k^2 - k^2 \ln \frac{m^2}{4k^3}$ in which the parameter $G = G(r, v)$ was introduced in (4.3.4), then the steady state solution is defined on the whole space interval $(0, +\infty)$.
- If $G(r_0, v_0) = -\frac{3}{2}k^2 - k^2 \ln \frac{m^2}{4k^3}$, then we have the critical steady state solution on the whole interval $(0, +\infty)$ whose formula is given by (4.3.10).
- If $G(r_0, v_0) < -\frac{3}{2}k^2 - k^2 \ln \frac{m^2}{4k^3}$, then the solution is defined on $(0, \underline{r}_S)$ if $r_0 < \frac{m}{2k^2}$ or $(\bar{r}_S, +\infty)$ if $r_0 > \frac{m}{2k^2}$ where $\underline{r}_S, \bar{r}_S$ satisfies $G(\underline{r}_S, k) = G(\bar{r}_S, k) = G(r_0, v_0)$.

4.3.3 Steady shock

We now consider the families of steady shocks which is also a solution of the static Euler system (4.3.1) in the distributional sense. Such solution contains one

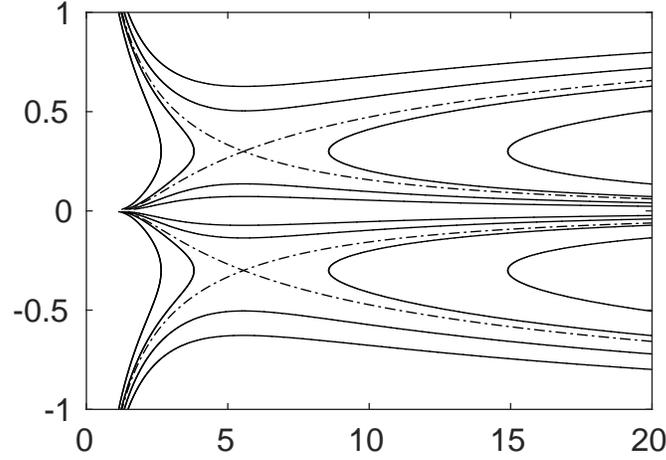


Figure 4.3.1: Plot of steady state solutions.

discontinuity satisfying also the entropy condition. In order that the position of discontinuity does not move when time passes, we give the following lemma.

Lemma 4.3.4 (Jump conditions for steady state solutions). *A steady state discontinuity of the Euler model (4.1.1) associated with left/right-hand limits (ρ_L, v_L) and (ρ_R, v_R) must satisfy*

$$\frac{\rho_R}{\rho_L} = \frac{v_L^2}{k^2}, \quad v_L v_R = k^2, \quad v_L \in (-k, 0) \cup (k, +\infty).$$

Proof. From the *steady Rankine-Hugoniot relations*

$$[\rho v] = 0, \quad [\rho(k^2 + v^2)] = 0,$$

where the bracket $[\cdot]$ denoted the value of the jump and we deduce that

$$\rho_R v_R = \rho_L v_L, \quad \rho_R(v_R^2 + k^2) = \rho_L(v_L^2 + k^2),$$

which gives the relation of the left and the right limit of the jump. Then the Lax entropy condition requires that $\lambda(\rho_L, v_L) > 0 > \lambda(\rho_R, v_R)$, $\mu(\rho_L, v_L) > 0 > \mu(\rho_R, v_R)$ for 1 and 2-waves. \square

Lemma 4.3.4 permits us to construct a *steady shock wave* of the Euler model (4.1.1) with a zero speed, that is, a function composed of a pair of steady state solutions $(\rho_L, v_L) = (\rho_L, v_L)(r)$, $(\rho_R, v_R) = (\rho_R, v_R)(r)$ separated by a discontinuity at a fixed point r_0 with the relation

$$v_R(r_0) = \frac{k^2}{v_L(r_0)}, \quad \rho_R(r_0) = \frac{v_L(r_0)^2}{k^2} \rho_L(r_0), \quad (4.3.11)$$

with

$$v_L(r_0) \in v_L \in (-k, 0) \cup (k, +\infty). \quad (4.3.12)$$

4.4 The generalized Riemann problem

4.4.1 The rarefaction regions

The **generalized Riemann problem** of the Euler model is a Cauchy problem of (4.1.1) with given initial data given as

$$U_0(r) = \begin{cases} U_L(r) & \underline{r} < r < r_0, \\ U_R(r) & r_0 < r < \bar{r}, \end{cases} \quad (4.4.1)$$

for a fixed radius $r_0 > 0$ and two steady state solutions $U_L = (\rho_L, v_L)$ and $U_R = (\rho_R, v_R)$, both satisfying the static Euler system (4.3.1).

For simplicity, we write $(\rho_L, v_L)(r_0) = (\rho_L^0, v_L^0) = U_L^0$ and $(\rho_R, v_R)(r_0) = (\rho_R^0, v_R^0) = U_R^0$. To solve the generalized Riemann problem, we need first to fix the point $r = r_0$ and solve the standard Riemann problem (4.2.4) with initial data

$$U_0(r) = \begin{cases} U_L^0 & \underline{r} < r < r_0, \\ U_R^0 & r_0 < r < \bar{r}. \end{cases}$$

The standard Riemann problem at a fixed radius is solved by three constant states $U_L^0 = (\rho_L^0, v_L^0)$, $U_M^0 = (\rho_M^0, v_M^0)$ and $U_R^0 = (\rho_R^0, v_R^0)$ connected to each other with 1-wave and 2-wave respectively where the intermediate constant state is given by

$$\{U_M^0\} \in W_1^{\rightarrow}(U_L^0) \cap W_2^{\leftarrow}(U_R^0). \quad (4.4.2)$$

Coming back to the Euler system with a source term (4.1.1), we would like to construct a solution of the generalized Riemann problem (4.1.1), (4.4.1), realized by three steady state solutions connected by generalized elementary curves. We give the intermediate steady state solution denoted by $(\rho_M, v_M) = (\rho_M, v_M)(r)$ by the static Euler system (4.3.1) with initial data (ρ_M^0, v_M^0) at the point $r = r_0$, that is

$$(\rho_M, v_M)(r_0) = (\rho_M^0, v_M^0). \quad (4.4.3)$$

Note that U_M^0 may belong to any family of the steady state solutions, referring to Theorem 4.3.3. To work on different types of elementary waves, we consider the

following differential equations:

$$\begin{aligned} \frac{dr_L^{M+}}{dt} &= \begin{cases} \lambda(\rho_M(r_L^{M+}), v_M(r_L^{M+})), & v_L^0 < v_M^0, \\ \sigma_1\left((\rho_L(r_L^{M+}), v_L(r_L^{M+})), (\rho_M(r_L^{M+}), v_M(r_L^{M+}))\right), & v_L^0 > v_M^0, \end{cases} \\ \frac{dr_L^{M-}}{dt} &= \begin{cases} \lambda(\rho_L(r_L^{M-}), v_L(r_L^{M-})), & v_L^0 < v_M^0, \\ \sigma_1\left((\rho_L(r_L^{M-}), v_L(r_L^{M-})), (\rho_M(r_L^{M-}), v_M(r_L^{M-}))\right), & v_L^0 > v_M^0, \end{cases} \quad (4.4.4) \\ r_L^{M\pm}(0) &= r_0, \end{aligned}$$

as well as

$$\begin{aligned} \frac{dr_M^{R+}}{dt} &= \begin{cases} \mu(\rho_M(r_M^{R+}), v_M(r_M^{R+})), & v_M^0 < v_R^0, \\ \sigma_2\left((\rho_L(r_M^{R+}), v_L(r_M^{R+})), (\rho_M(r_M^{R+}), v_M(r_M^{R+}))\right), & v_M^0 > v_R^0, \end{cases} \\ \frac{dr_M^{R-}}{dt} &= \begin{cases} \mu(\rho_L(r_M^{R-}), v_L(r_M^{R-})), & v_M^0 < v_R^0, \\ \sigma_2\left((\rho_L(r_M^{R-}), v_L(r_M^{R-})), (\rho_M(r_M^{R-}), v_M(r_M^{R-}))\right), & v_M^0 > v_R^0, \end{cases} \quad (4.4.5) \\ r_M^{R\pm}(0) &= r_0, \end{aligned}$$

where σ_1, σ_2 are speeds of 1 and 2-shocks respectively and λ, μ are eigenvalues.

Lemma 4.4.1. *Let $(\rho_L, v_L) = (\rho_L, v_L)(r), (\rho_R, v_R) = (\rho_R, v_R)(r)$ be two steady state solutions given by (4.3.1). The curves $r_L^{M\pm}, r_M^{R\pm}$ are uniquely defined by (4.4.4), (4.4.5) for all $t > 0$ respectively, with bounded derivatives.*

Proof. We first consider the 1-wave. If (ρ_L^0, v_L^0) and (ρ_M^0, v_M^0) are connected by a 1-rarefaction, then we have

$$\frac{dr_L^{M+}}{dt} = \lambda(\rho_M(r_L^{M+}), v_M(r_L^{M+})), \quad \frac{dr_L^{M-}}{dt} = \lambda(\rho_L(r_L^{M-}), v_L(r_L^{M-})).$$

Following from the existence theory of ordinary differential equations, there exists a time $T > 0$ such that the curves are well-defined on $0 < t < T$. To prove that these curves are indeed defined globally in time, we have to show that steady state solutions can not be sonic along the wave curves, referring to Theorem 4.3.3. We take into account two cases:

- When $r_0 < \frac{m}{2k^2}$, $v_L = v_L(r)$ cannot be sonic for all $\bar{r} < r < r_0$. Then we only have to consider the case where $r_L^{M-}(t) > r_0$, which gives $\frac{dr_L^{M-}(t)}{dt} > 0$, providing $v_L \geq k$. If there exists a finite time t_1 such that $v_L(r_L^{M-}(t)) = k$, then $\frac{dr_L^{M-}(t)}{dt}|_{t=t_1} = v_L(r_L^{M-}(t_1)) - k$, which provides a contradiction.

- When $r_0 \geq \frac{m}{2k^2}$, $r_L^* < \underline{r} < r_0$ holds where r_L^* is the sonic point of (ρ_L, v_L) , then we have at once the result.

Now if (ρ_L^0, v_L^0) and (ρ_M^0, v_M^0) is connected by a 1-shock, the result will hold if (ρ_L, v_L) will not reach to the sonic point on $(\bar{r}, r_L^{M^-}(t))$ for $0 < t < T$. We consider two cases as follows:

- When $r_0 < \frac{m}{2k^2}$, we suppose that $\sigma_1 > 0$. The entropy condition gives $\lambda(\rho_L, v_L) > \sigma_1 > \lambda(\rho_M, v_M)$, leading to $v_L > k$. Then we have the result for this case.
- When $r_0 \geq \frac{m}{2k^2}$, we have $r_L^* < \underline{r} < r_0$ and the result holds.

A similar calculation gives all the curves listed in the lemma. \square

It follows directly from the definition that $r_L^{M^-}(t) \leq r_L^{M^+}(t) \leq r_M^{R^-}(t) \leq r_M^{R^+}(t)$, which permits us to define five disjoint regions below for all fixed $t > 0$: $(\underline{r}, r_L^{M^-}(t))$, $(r_L^{M^-}(t), r_L^{M^+}(t))$, $(r_L^{M^+}(t), r_M^{R^-}(t))$, $(r_M^{R^-}(t), r_M^{R^+}(t))$, $(r_M^{R^+}(t), \bar{r})$ and we denote by $(r_L^{M^-}(t), r_L^{M^+}(t))$ and $(r_M^{R^-}(t), r_M^{R^+}(t))$ the *1-rarefaction region* and the *2-rarefaction region*.

4.4.2 Explicit form of Riemann solution

We now give the solution $U = (\rho, v) = (\rho, v)(t, r)$ for the generalized Riemann problem. Write

$$U(t, r) = \begin{cases} U_L(r) & \underline{r} < r < r_L^{M^-}(t), \\ \tilde{U}_1(t, r) & r_L^{M^-}(t) < r < r_L^{M^+}(t), \\ U_M(r) & r_L^{M^+}(t) < r < r_M^{R^-}(t), \\ \tilde{U}_2(t, r) & r_M^{R^-}(t) < r < r_M^{R^+}(t), \\ U_R(r) & r_M^{R^+}(t) < r < \bar{r}, \end{cases} \quad (4.4.6)$$

where $r_L^{M^\pm}$, $r_M^{R^\pm}$ are boundaries of the rarefaction regions defined by (4.4.4), (4.4.5). Here, $U_L = (\rho_L, v_L)$, $U_M = (\rho_M, v_M)$, $U_R = (\rho_R, v_R)$ are three steady state solutions and \tilde{U}_1 and \tilde{U}_2 are generalized rarefaction waves to be given by the integro-differential problem following from Liu [40]. Indeed, we give the function $\tilde{U}_j(t, \theta_j) = (\tilde{\rho}_j, \tilde{v}_j)(t, \theta_j)$, $j = 1, 2$ and the new variable $\tilde{r} = \tilde{r}(t, \theta_j)$. To seek for the form of \tilde{U}_j and \tilde{r} , we consider the following problem:

$$\begin{aligned} \partial_{\theta_j} \tilde{r} \partial_t \tilde{U}_j + \left(\partial_U F(\tilde{U}_j) - \lambda(\tilde{U}_j) \right) \partial_{\theta_j} \tilde{U}_j &= S(\tilde{U}_j) \partial_{\theta_j} \tilde{r}, \\ \partial_t \tilde{r} &= \lambda(\tilde{U}_j(t, \theta_j)), \end{aligned} \quad (4.4.7)$$

with boundary and initial conditions reading

$$\begin{aligned}\tilde{U}_j(t, \theta_j^0) &= U_k^0(\tilde{r}(t, \theta_j^0)), & \tilde{U}_j(0, \theta_j) &= h_1(\theta_j), \\ \partial_t \tilde{r}(t, \theta_j^0) &= \lambda(U_k^0(\tilde{r})), & \tilde{r}(0, \theta_j) &= r_0,\end{aligned}\tag{4.4.8}$$

where we give $\theta_j^0 = \lambda(U_k^0)$, $j = 1, 2$, $k = L, R$ and the function h_j defined by

$$\xi = \lambda_j(h_j(\xi)) = \frac{r - r_0}{t},\tag{4.4.9}$$

where $\lambda_1 = \lambda$, $\lambda_2 = \mu$ are the eigenvalues of the 1 and 2 families.

Lemma 4.4.2. *The integro-differential problem (4.4.7), (4.4.8) admits a unique \tilde{U}_j smooth for all fixed time $t > 0$.*

Proof. To prove the lemma, we use a standard fixed point argument. Without loss of generality, we consider the 1-rarefaction wave. Denote by l_1, l_2 two linearly independent vectors corresponding to λ, μ respectively. Multiplying (4.4.7) by l_2 , we have

$$\begin{aligned}DV_2 &= \frac{\partial_{\theta_2} \tilde{r}}{\mu - \lambda} l_2 \cdot S + Dl_2 \cdot V_1, \\ \partial_t V_1 &= l_2 \cdot S + \partial_t l_2 \cdot V_1,\end{aligned}$$

where we have defined $V_1 = l_1 \cdot \tilde{U}_1$, $V_2 = l_2 \cdot \tilde{U}_1$, and the operator reads $D = \frac{\partial_{\theta_2} \tilde{r}}{\mu - \lambda} \partial_t + \partial_{\theta_2}$ whose integral curves starting from $(\tau, \lambda(U_0))$ is denoted by ζ . We thus have

$$\begin{aligned}V_2(t, \theta_1) &= V_2(\tau, \lambda(U_0)) + \int_{\zeta} \left(\frac{\partial_{\theta_2} \tilde{r}}{\mu - \lambda} l_2 \cdot S + Dl_2 \cdot V_1 \right) d\theta_1, \\ V_1(t, \theta_1) &= V_1(0, \xi) + \int_0^t \left(l_2 \cdot S + \partial_t l_2 \cdot V_1 \right) d\theta_1.\end{aligned}\tag{4.4.10}$$

Now let \mathcal{F} be the operator of the right-hand side of (4.4.10) and we study the iteration method $\tilde{U}_1^{(l)} = \mathcal{F}^{(l)} \tilde{U}_1^0$, $l \geq 1$ where \tilde{U}_1^0 is an arbitrary smooth function satisfying the initial-boundary condition $\tilde{U}_1^0(t, \theta_j^0) = \tilde{U}_1(t, \theta_j)$, $\tilde{U}_1^0(0, \theta_j) = \tilde{U}_1(0, \theta_j)$. It is easily checked that for sufficiently small t_1 , \mathcal{F} is contractive in the max norm of \tilde{U}_j^0 . By iterating the operator \mathcal{F} , we prove that there exists a unique solution \tilde{U}_1 for all $0 < t \leq \Delta t_1$. Then taking $\tilde{U}_1(t_1, \cdot)$ as initial condition, we have a time Δt_2 such that \tilde{U}_1 is defined by all $\Delta t_1 < t < \Delta t_1 + \Delta t_2$ and it is directly to see that $\Delta t_1 \leq \Delta t_2$ by the definition of the operator \mathcal{F} . We can thus have the existence of (4.4.7) for all fixed $t > 0$. \square

According to the construction above, we conclude the following theorem.

Theorem 4.4.3 (The solution of the generalized Riemann problem). Consider the generalized Riemann problem for the Euler model (4.1.1), (4.4.1). There exists a weak

solution defined for all time $t > 0$ given by (4.4.6), satisfying the Rankie-Hugoniot jump condition and the Lax entropy condition.

4.4.3 Evolution of total variation

It is obvious that the total variation of $\ln \rho$ of the solution of the standard Riemann problem (4.2.4), (4.2.9) stays as a constant when time passes. However, it is a different story for the generalized Riemann problem (4.1.1), (4.4.1). We have the following lemma.

Lemma 4.4.4. *Let $U = (\rho, v) = (\rho, v)(t, r)$ be the solution of the generalized Riemann problem of the Euler model (4.1.1) whose initial data $U_0 = (\rho_0, v_0) = (\rho_0, v_0)(r)$ has the form (4.4.1). Then we have*

$$TV_{[\underline{r}, \bar{r}]}(\ln \rho(t, \cdot)) < TV_{[\underline{r}, \bar{r}]}(\ln \rho(0+, \cdot))(1 + O(t)), \quad (4.4.11)$$

for all $t > 0$.

Proof. Let $U_M = U_M(r)$ be the intermediate steady state solution associated with the left state U_L and the right state U_R given in the initial data. According to (4.4.4), we have

$$\begin{aligned} U_L(r_L^{M^-}(t)) - U_M(r_L^{M^-}(t)) &= U_L(r_0) - U_M(r_0) + |U_L(r_0) - U_M(r_0)|O(r_L^{M^-}(t) - r_0) \\ &= U_L(r_0) - U_M(r_0) + |U_L(r_0) - U_M(r_0)|O(t). \end{aligned}$$

Moreover, according to the construction of the generalized Riemann problem, we give

$$\begin{aligned} &TV_{[\underline{r}, \bar{r}]}(\ln \rho(t+, \cdot)) - TV_{[\underline{r}, \bar{r}]}(\ln \rho(0+, \cdot)) \\ &\leq (|\ln \rho_L(r_0) - \rho_M(r_0)| + |\ln \rho_L(r_0) - \rho_M(r_0)|)O(t) = TV_{[\underline{r}, \bar{r}]}(\ln \rho(0+, \cdot))O(t), \end{aligned}$$

where we have used the continuous dependence property $|U_L(r_0) - U_M(r_0)| = O(1)|(\ln \rho_L(r_0) - \rho_M(r_0))|$. This ends the proof of the lemma. \square

4.5 Triple Riemann problem

4.5.1 Preliminary

Considering the fact that a steady state solution of the steady Euler model (4.3.1) may not be defined globally as is the result of Theorem 4.3.3 and we are obliged to introduce the **triple Riemann problem** in order to complete the Glimm method in the coming section, that is, a Cauchy problem associated with initial data composed

of three steady state solutions:

$$U_0(r) = \begin{cases} U_\alpha(r) & \underline{r} < r < r_s, \\ U_\beta(r) & r_s < r < r_b, \\ U_\gamma(r) & r_b < r < \bar{r}, \end{cases} \quad (4.5.1)$$

for fixed radius $0 < \underline{r} < r_1 < r_2 < \bar{r}$ and steady states $U_\alpha = (\rho_\alpha, v_\alpha)$, $U_\beta = (\rho_\beta, v_\beta)$, $U_\gamma = (\rho_\gamma, v_\gamma)$. We denote by $U_\alpha(r_s) = U_\alpha^s = (\rho_\alpha^s, v_\alpha^s)$, $U_\beta(r_s) = U_\beta^s = (\rho_\beta^s, v_\beta^s)$, $U_\beta(r_b) = U_\beta^b = (\rho_\beta^b, v_\beta^b)$, $U_\gamma(r_b) = U_\gamma^b = (\rho_\gamma^b, v_\gamma^b)$.

We first give the main conclusion of this section:

Theorem 4.5.1. Consider a given initial data composed of three steady state solution $U_\alpha, U_\beta, U_\gamma$. Then for all $t > 0$, the triple Riemann problem of the Euler model (4.1.1), (4.5.1) admits a weak solution $U = (\rho, v) = (\rho, v)(t, r)$ such that for all $t > 0$, we have:

$$TV_{[\underline{r}, \bar{r}]}(\ln \rho(t, \cdot)) < TV_{[\underline{r}, \bar{r}]}(\ln \rho(0+, \cdot))(1 + O(\bar{r} - \underline{r})). \quad (4.5.2)$$

We define the *left-hand problem* as a generalized Riemann problem with initial data

$$U_0(r) = \begin{cases} U_\alpha(r) & r < r_1, \\ U_\beta(r) & r > r_1, \end{cases}$$

and the *right-hand problem* as a generalized Riemann problem with initial data

$$U_0(r) = \begin{cases} U_\beta(r) & r < r_2, \\ U_\gamma(r) & r > r_2, \end{cases}$$

Since the Euler model (4.1.1) is strictly hyperbolic following from Proposition 4.2.1, for a small enough time $t > 0$, both the left-hand and the right-hand problem admit a solution denoted by $U_L = U_L(t, r)$ and $U_R = U_R(t, r)$ respectively and the wave curves of the solutions do not interact. We denote by $r_{LL}^{M\pm}, r_{ML}^{R\pm}$ the wave curves of the left-hand problem and $r_{LR}^{M\pm}, r_{MR}^{R\pm}$ the rarefaction regions boundaries of the right-hand problem as is defined in (4.4.4), (4.4.5). We then define the moment of the first interaction denoted by T_f :

$$T_f := \sup\{t > 0 \mid r_{ML}^{R+}(t) \leq r_{LR}^{M-}(t)\}. \quad (4.5.3)$$

Clearly, if $T_f = +\infty$, the triple Riemann problem (4.1.1), (4.5.1) exists a solution reading

$$U^f(t, r) = \begin{cases} U_L(t, r) & \underline{r} < r < r_2, \\ U_R(t, r) & r_2 < r < \bar{r}. \end{cases} \quad (4.5.4)$$

4.5.2 Possible interactions

If the moment of the first interaction $T_f < +\infty$, then the waves of the left-hand and the right-hand Riemann problem did have interactions at T_f . Possible interactions are given in order:

- 2-shock of the left-hand problem and 1-shock of the right-hand problem,
- 2-shock of the left-hand problem and 1-rarefaction of the right-hand problem,
- 2-rarefaction of the left-hand problem and 1-shock of the right-hand problem,

which are denoted by Problems $P - ss$, $P - sr$, $P - rs$ respectively. For later use, we denote by $U_M^{\alpha,\beta}$, $U_M^{\beta,\gamma}$ the intermediate states of the left and right-hand problems respectively. We consider different kinds of interactions separately.

Lemma 4.5.2. *If $T_f < +\infty$ where T_f is defined by (4.5.3) and we have the 2-shock of the left-hand problem and the 1-shock of the right-hand problem of the Euler model (4.1.1), then there exists a time T_{ss} such that Problem $P - ss$ admits a solution on $0 < t < T_{ss}$.*

Proof. We only have to consider the solution after $t > T_f$. We denote by $U_M^{ss} = U_M^{ss}(t, r)$ the solution of the generalized problem with initial states $U_M^{\alpha,\beta}$, $U_M^{\beta,\gamma}$ separated by $r = r_{LL}^{M+}(T_f) = r_{MR}^{R-}(T_f)$ at $t = T_f$. Then for $T_f < t < T_{ss}$, we give

$$U^{ss}(t, r) = \begin{cases} U_L(t, r) & \underline{r} < r < r_{LL}^{M+}(t), \\ U_M^{ss}(t, r) & r_{LL}^{M+}(t) < r < r_{MR}^{R-}(t), \\ U_R(t, r) & r_{MR}^{R-}(t) < r < \bar{r}, \end{cases} \quad (4.5.5)$$

where

$$T_{ss} = \min \left(\sup\{t > T_f | r_{LM}^{M-}(t) > r_{LL}^{M+}(t)\}, \sup\{t > T_f | r_{MR}^{R-}(t) > r_{MM}^{M+}(t)\} \right). \quad (4.5.6)$$

where $r_{LM}^{M\pm}$ are boundaries of the rarefaction regions of the state U_M^{ss} given by (4.4.4), (4.4.5). Thus Problem $P - ss$ admits a solution for all $t < T_{ss}$. \square

We now consider Problem $P - rs$.

Lemma 4.5.3. *Let T_f be the first moment of interaction (4.5.3) and we suppose $T_f < +\infty$ and the Euler model (4.1.1) has 2-rarefaction of the left-hand problem and the 1-shock of the right-hand problem. Then there exists a time T_{rs} such that we have a solution of Problem $P - rs$ for all $0 < t < T_{rs}$.*

Proof. Again, we only have to construct a solution after $t > T_f$. Let us first write $\tilde{U}_2^{\alpha,\beta} = \tilde{U}_2^{\alpha,\beta}(t, r)$ the 2-rarefaction wave of the left-hand problem which evolves in the region $(r_{LL}^{M-}(t), r_{LL}^{M+}(t))$. We give

$$U_0^{rs}(t, r) = \begin{cases} U_L(t, r) & \underline{r} < r < r_{ML}^{L+}(t), \\ U_M^{rs}(t, r) & r_{ML}^{L+}(t) < r < r_{MR}^{R-}(t), \\ U_R(r) & r_{MR}^{R-}(t) < r < \bar{r}, \end{cases} \quad (4.5.7)$$

where the function $U_M^{rs}(t, r)$ is given by

$$U_M^{rs,0}(t, r) = \begin{cases} \tilde{U}_2^{\alpha,\beta}(t, r) & r_{ML}^{L+}(t) < r < \widetilde{r}_{Mrs}^{L-}(t), \\ \tilde{U}_1^{rs}(t, r) & \widetilde{r}_{Mrs}^{L-}(t) < r < \widetilde{r}_{Mrs}^{L+}(t), \\ U_{MM}^{rs,0}(r) & \widetilde{r}_{Mrs}^{L+}(t) < r < \widetilde{r}_{Mrs}^{R-}(t), \\ \tilde{U}_2^{rs}(t, r) & \widetilde{r}_{Mrs}^{R-}(t) < r < \widetilde{r}_{Mrs}^{R+}(t), \\ U_\gamma(r) & \widetilde{r}_{Mrs}^{R+}(t) < r < r_{MR}^{R-}(t). \end{cases} \quad (4.5.8)$$

Here, $U_{MM}^{rs} = U_{MM}^{rs}(r)$ is a steady state satisfying

$$\{U_{MM}^{rs}(r_{ML}^{L+}(T_f))\} \in W_1^{\rightarrow}(\tilde{U}_2^{\alpha,\beta}(T_f, r_{ML}^{L+}(T_f))) \cap W_2^{\leftarrow}(U_\gamma(r_{MR}^{R-}(T_f))),$$

and we recall that W_1^{\rightarrow} and W_2^{\leftarrow} are elementary waves of the Euler model (4.2.4) with the formula given by (4.2.10). The wave curves $\widetilde{r}_{Mrs}^{\pm}, \widetilde{r}_{Mrs}^{\pm}(t)$ satisfy the ordinary differential systems introduced by (4.4.4), (4.4.5) associated with three states $\tilde{U}_2^{\alpha,\beta}, U_{MM}^{rs}, U_\gamma$. The functions $\tilde{U}_{1,2}^{rs}(t, r)$ are given by (4.4.7), (4.4.8), (4.4.9). Denote by

$$T_{rs}^0 = \sup\{t > T_f \mid \widetilde{r}_{Mrs}^{L-}(t) < r_{ML}^{L-}(t)\}, \quad (4.5.9)$$

and we see immediately that (4.5.7) provides an exact solution for Problem $P - rs$ for all $0 < t \leq T_{rs}^0$. Now for $t > T_{rs}^0$, we give

$$U_M^{rs}(t, r) = \begin{cases} U_L(t, r) & \underline{r} < r < r_{ML}^{L+}(t), \\ U_M^{rs,1} & r_{ML}^{L+}(t) < r < \widetilde{r}_{Mrs}^{R-}(t), \\ U_M^{rs,0}(r) & \widetilde{r}_{Mrs}^{R-}(t) < r < r_{MR}^{R-}(t), \\ U_\gamma(r) & r_{MR}^{R-}(t) < r < \bar{r}, \end{cases} \quad (4.5.10)$$

with $U_M^{rs,0}$ given by (4.5.8) and $U_M^{rs,1}$ the solution of the Riemann problem generated by initial data $U_M^{\alpha,\beta}, U_{MM}^{rs,0}$ at the radius $r = \widetilde{r}_{Mrs}^{R-}(T_{rs}^0) = r_{ML}^{L-}(T_{rs}^0)$ from the very

moment $t = T_{rs}^0$. Now we denote by

$$T_{rs} = \min(\sup\{t > T_{rs}^0 | \overline{r_{Lrs}^M} (t) > r_{LL}^{M+}(t)\}, \sup\{t > T_{rs}^0 | r_{MR}^R (t) > \widetilde{r_{Mrs}^R} (t)\}), \quad (4.5.11)$$

where $\overline{r_{Lrs}^M} (t)$ is the lower bound of the 1-wave of the solution $U_M^{rs,1} = U_M^{rs,1}(t, r)$. Together with (4.5.4), (4.5.7), (4.5.10), we have a solution of Problem $P - rs$ for all $0 < t < T_{rs}$. \square

A similar analysis gives the result of Problem $P - sr$.

Lemma 4.5.4. *If the first moment of interaction $T_f < +\infty$ and the Euler model (4.1.1) admits 1-rarefaction of the left-hand problem and the 2-shock of the right-hand problem. That we have a solution of Problem $P - sr$ for all $0 < t < T_{rs}$ where T_{sr} is a given moment.*

We now consider interactions after these moments T_{ss}, T_{rs}, T_{sr} . Indeed, following from the constructions in Lemmas 4.5.2, 4.5.3, 4.5.4, it is clear that possible interactions after these moments are also pairwise interplays of generalized shock waves and rarefaction waves as is listed at the beginning of this section. Thus, for any fixed moment $t > 0$, we have the solution of the triple Riemann problem. The estimation of the total variation given by (4.5.2) follows directly from Lemmas 4.2.6, 4.4.4. We thus obtain the main conclusion of this section, that is, Theorem 4.5.1.

4.6 The initial value problem

4.6.1 The Glimm method

We give first the existence theory of the Euler model:

Theorem 4.6.1 (Global existence theory). Consider the Euler model with source term describing fluid flows (4.1.1). For any given initial density $\rho_0 = \rho_0(r) > 0$ and velocity v_0 such that

$$TV(\ln \rho_0) + TV(\ln v_0) < +\infty,$$

and any given time interval (possibly infinite) $(0, T) \subset (0, +\infty)$, there exists a weak solution $\rho = \rho(t, r), v = v(t, r)$ defined on $(0, T)$ such that the initial condition holds in the sense that $\rho(0, \cdot) = \rho_0, v(0, \cdot) = v_0$ and for any fixed moment $T' \in (0, T)$

$$\sup_{t \in [0, T']} (TV(\ln \rho(t, \cdot)) + TV(v)) < +\infty.$$

As a prove, we first construct an approximate solution of the Euler model (4.1.1) with initial data

$$U(t, r) = U_0(r) = (\rho_0, v_0)(r), \quad r > 0, \quad (4.6.1)$$

by using a random choice method based on the generalized problem. Let Δr and Δt denote the mesh lengths in space and in time respectively, and let (r_j, t_n) denotes the mesh point of the grid, where $r_j = j\Delta r$, $t_n = 0 + n\Delta t$. We assume the so-called *CFL condition*:

$$\frac{\Delta r}{\Delta t} > \max(|\lambda|, |\mu|), \quad (4.6.2)$$

insuring that elementary waves other than those in the triple Riemann problem do not interact within one time interval.

To construct the approximate solution $U_{\Delta r} = U_{\Delta r}(t, r)$, we would first like to approximate the initial data by a piecewise steady state solution of the Euler model given by (4.3.1). However, note that some steady state solutions cannot be defined globally on $r > 0$, we need more constructions. Recall first that there exists four critical steady state solutions which pass the point $(\frac{m}{2k^2}, \pm k)$ denoted by $U_*^{P,b}, U_*^{P,\sharp}, U_*^{N,b}, U_*^{N,\sharp}$ according to (4.3.10). Another important remark is given in Theorem 4.3.3, that is, for given r_0, U_0 , there exists always a steady solution $U = U(r)$ with $U(r_0) = U_0$ defined on $(0, r_0)$ if $r_0 < \frac{m}{2k^2}$ or $(r_0, +\infty)$ if $r_0 > \frac{m}{2k^2}$. Now we denote by $U_{\Delta r,0}^{j+1} = U_{\Delta r,0}^{j+1}(r) = (\rho_{\Delta r,0}^{j+1}, v_{\Delta r,0}^{j+1})(r)$ the steady state solution of the Euler model satisfying (4.3.1) such that $U_{\Delta r,0}^{j+1}(r_{j+1}) = U_0(r_{j+1})$ and we define:

$$r_{j+1}^s := \sup\{r > 0 | v_{\Delta r,0}^{j+1}(r) \neq \pm k\} \chi_{\{r_{j+1} < \frac{m}{2k^2}\}}(r) + \inf\{r > 0 | v_{\Delta r,0}^{j+1}(r) \neq \pm k\} \chi_{\{r_{j+1} > \frac{m}{2k^2}\}}(r). \quad (4.6.3)$$

Note that if $r_{j+1}^s \neq 0$ or $r_{j+1}^s \neq +\infty$, r_{j+1}^s is the sonic point of the steady state $U_{\Delta r,0}^{j+1}$. We now denote by $U_{0,*}^{j+1} = (\rho_{0,*}^{j+1}, v_{0,*}^{j+1})$ the unique critical steady state solution satisfying

$$\text{sgn}(v_{0,*}^{j+1}) = \text{sgn}(v_{\Delta r,0}^{j+1}), \quad \text{sgn}(|v_{0,*}^{j+1}| - k) = \text{sgn}(|v_{\Delta r,0}^{j+1}| - k). \quad (4.6.4)$$

On the interval (r_j, r_{j+2}) , we have the following possible constructions.

- If $U_{\Delta r,0}^{j+1}$ is well-defined on (r_j, r_{j+2}) , we approximate the initial data U_0 by $U_{\Delta r,0}^{j+1}$ on the interval.
- If $U_{\Delta r,0}^{j+1}$ vanishes at r_{j+1}^s and $r_{j+1} < \frac{m}{2k^2}$, then we approximate the initial data on (r_{j+1}^s, r_{j+2}) by
 - $U_{\Delta r,0}^{j+3}$ if $r_{j+3}^s \notin (r_{j+1}^s, r_{j+2})$;
 - $U_{0,*}^{j+1}$ if $r_{j+3}^s \in (r_{j+1}^s, r_{j+2})$ for $U_{0,*}^{j+1}$ given by (4.6.4). Note that this case happens at most once when $r_{j+1} < \frac{m}{2k^2} < r_{j+3}$ and $r_{j+3}^s > r_{j+2}$.
- If $U_{\Delta r,0}^{j+1}$ vanishes at r_{j+1}^s and $r_{j+1} > \frac{m}{2k^2}$, then we approximate the initial data on (r_j, r_{j+1}^s) by
 - $U_{\Delta r,0}^{j-1}$ if $r_{j-1}^s \notin (r_j, r_{j+1}^s)$;

– $U_{0,*}^{j+1}$ if $r_{j-1}^s \in (r_j, r_{j+1}^s)$. Also, this case happens at most one time if $r_{j-1} < \frac{m}{2k^2} < r_{j+1}$ and $r_{j-1}^s < r_j$.

Following the ideas above, we can now approximate the initial data on (r_j, r_{j+2}) for j even:

$$U_{\Delta r,0}(r) = \begin{cases} \bar{U}_{\Delta r,0}^{j+1-2\text{sgn}(r_{j+1}-\frac{m}{2k^2})}(r) & r_j < r < \mathcal{M}(r_j, r_{j+1}^s), \\ U_{\Delta r,0}^{j+1}(r) & \mathcal{M}(r_j, r_{j+1}^s) < r < \mathcal{M}(r_{j+1}^s, r_{j+2}), \\ \bar{U}_{\Delta r,0}^{j+1-2\text{sgn}(r_{j+1}-\frac{m}{2k^2})}(r) & \mathcal{M}(r_{j+1}^s, r_{j+2}) < r < r_{j+2}, \end{cases} \quad (4.6.5)$$

where we give the operator \mathcal{M} by

$$\mathcal{M}(x, y) = \begin{cases} \min(x, y) & r < \frac{m}{2k^2}, \\ \max(x, y) & r > \frac{m}{2k^2}, \end{cases} \quad (4.6.6)$$

and

$$\begin{aligned} & \bar{U}_{\Delta r,0}^{j+1-2\text{sgn}(r_{j+1}-\frac{m}{2k^2})}(r) \\ = & \begin{cases} U_{\Delta r,0}^{j+1-2\text{sgn}(r_{j+1}-\frac{m}{2k^2})}(r) & r_{j+1}^s - 2\text{sgn}(r_{j+1}-\frac{m}{2k^2}) \notin (r_j, r_{j+1}^s) \cup (r_{j+1}^s, r_{j+2}), \\ U_{0,*}^{j+1}(r) & \text{else,} \end{cases} \end{aligned}$$

with the sonic point r_{j+1}^s given by (4.6.3) and the critical steady state solution $U_{0,*}^{j+1}$ satisfying (4.6.4). Assume now that the approximate solution has been defined for $t_{n-1} \leq t < t_n$.

To complete the definition of $U_{\Delta r}$, it suffices to define the solution on $t_n \leq t < t_{n+1}$. Let $(\theta_n)_n$ be a given equidistributed sequence on the interval $(-1, 1)$ and introduce the point related to the randomly choose values:

$$r_{n,j+1} := (\theta_n + j)\Delta r, \quad j > 0. \quad (4.6.7)$$

Following the idea before, we denote by $U_{\Delta r,n}^{j+1} = U_{\Delta r,n}^{j+1}(r)$ the steady state solutions passing the point $(r_{n,j+1}, U_{\Delta r}(nt-, r_{n,j+1}))$ and the sonic point

$$\begin{aligned} r_{n,j+1}^s & := \sup\{r > 0 \mid v_{\Delta r,n}^{j+1}(r) \neq \pm k\} \chi_{\{r_{n,j+1} < \frac{m}{2k^2}\}}(r) \\ & + \inf\{r > 0 \mid v_{\Delta r,n}^{j+1}(r) \neq \pm k\} \chi_{\{r_{n,j+1} > \frac{m}{2k^2}\}}(r), \end{aligned}$$

together the very critical steady state solution $U_{n,*}^{j+1} = (\rho_{n,*}^{j+1}, v_{n,*}^{j+1})$ such that

$$\text{sgn}(v_{n,*}^{j+1}) = \text{sgn}(v_{\Delta r,n}^{j+1}), \quad \text{sgn}(|v_{n,*}^{j+1}| - k) = \text{sgn}(|v_{\Delta r,n}^{j+1}| - k). \quad (4.6.8)$$

Our construction contains two main steps:

- **The steady state solution step.** The construction of the approximate solution on the time interval t_n is quite similar to the approximation of the initial data. We define, at $t = t_n$, on the interval (r_j, r_{j+2}) with $n + j$ even:

$$U_{\Delta r, n}(r) = \begin{cases} \bar{U}_{\Delta r, n}^{j+1-2\text{sgn}(r_{n, j+1}-\frac{m}{2k^2})}(r) & r_j \leq r < \mathcal{M}(r_j, r_{n, j+1}^s), \\ U_{\Delta r, n}^{j+1}(r) & \mathcal{M}(r_j, r_{n, j+1}^s) < r < \mathcal{M}(r_{n, j+1}^s, r_{j+2}), \\ \bar{U}_{\Delta r, n}^{j+1-2\text{sgn}(r_{j+1}-\frac{m}{2k^2})}(r) & \mathcal{M}(r_{n, j+1}^s, r_{j+2}) < r < r_{j+2}, \end{cases} \quad (4.6.9)$$

where $\mathcal{M}(\cdot, \cdot)$ is the operator given by (4.6.6) and

$$\bar{U}_{\Delta r, n}^{j+1-2\text{sgn}(r_{n, j+1}-\frac{m}{2k^2})}(r) = \begin{cases} U_{\Delta r, n}^{j+1-2\text{sgn}(r_{n, j+1}-\frac{m}{2k^2})}(r) & r_{j+1-2\text{sgn}(r_{n, j+1}-\frac{m}{2k^2})}^s \notin (r_j, r_{n, j+1}^s), \cup (r_{n, j+1}^s, r_{j+2}), \\ U_{n, *}^{j+1}(r) & \text{else,} \end{cases}$$

with $U_{n, *}^{j+1}$ given by (4.6.8). It is direct to observe that if a steady state solution reaches its sonic point in a cell, then the nearest discontinuity is replaced by this sonic point, then this construction guarantees that there exists at most one point of discontinuity in (r_{j-1}, r_{j+1}) , $j + n$ even.

- **The generalized Riemann problem step.** Denote by r_j^d the point of discontinuity in $r_{j-1} < r < r_{j+1}$ and we then define the approximate solution $U_{\Delta r}$ on the rectangle $\{t_n < t < t_{n+1}, r_{j-1} < r < r_{j+1}\}$, $n + j$ even:

$$U_{\Delta r}(t, r) = \begin{cases} U_{\mathcal{R}}^{(j-1, j+1)}(t, r), & r_j^d - r_{j-2}^d = 2\Delta r \text{ and } r_{j+2}^d - r_j^d = 2\Delta r, \\ U_{\mathcal{TR}}^{(j-3, j+1)}(t, r), & r_j^d - r_{j-2}^d < 2\Delta r, \\ U_{\mathcal{TR}}^{(j-1, j+3)}(t, r), & r_{j+2}^d - r_j^d < 2\Delta r, \end{cases} \quad (4.6.10)$$

where $U_{\mathcal{R}}^{(j-1, j+1)}$ is the solution of the generalized Riemann problem at the time level $t = t_n$ on (r_{j-1}, r_{j+1}) with two steady states separated by a discontinuity at r_j^d and $U_{\mathcal{TR}}^{(j-3, j+1)}$ the solution of the triple Riemann problem at the time level $t = t_n$ on the interval (r_{j-3}, r_{j+1}) with the three steady states separated by discontinuities at r_{j-2}^d, r_j^d .

This completes the construction of the approximate solution $U_{\Delta r} = U_{\Delta r}(t, r)$ on $[0, +\infty) \times (0, +\infty)$ by the Glimm scheme.

4.6.2 Convergence of approximate solutions

To prove Theorem 4.6.1, we first need an estimation of the total variation. See the following lemma.

Lemma 4.6.2. *Let $U_{\Delta r} = (\rho_{\Delta r}, v_{\Delta r})$ be the approximate solution of the Euler model (4.1.1) constructed by the Glimm method, then for any two neighboring time interval $t_n < t_{n+1}$, we have a constant $C > 0$ such that*

$$TV(\ln \rho_{\Delta r}(t_{n+1}+, \cdot)) - TV(\ln \rho_{\Delta r}(t_n+, \cdot)) \leq C\Delta t.$$

From Lemma 4.6.2, we have, for any given $0 < t < +\infty$,

$$TV(\ln \rho_{\Delta r}(t, \cdot)) \leq TV(\ln \rho_{\Delta r}(0, \cdot))e^{C_1(t-s)}, \quad (4.6.11)$$

where C_1 is a constant.

Proof. On the time level $t = t_{n+1}$, we consider the interval $(r_{n+1,j-1}, r_{n+1,j+1})$ with $n+j$ even. According to (4.6.7), $r_{n\pm 1,j+1}$ is the point determined by a chosen random value. Following from the construction of the Glimm method, $(r_{n+1,j-1}, r_{n+1,j+1})$ only contains one point of discontinuity which we write as $r_{j,n}^d$. According to Lemma 4.4.4, we have

$$TV(\ln \rho(t_{n+1}+, \cdot)) = \sum_j |\ln \rho(t_{n+1}+, r_{j+1,n}^d) - \ln \rho(t_{n+1}+, r_{j,n}^d)|(1 + C(\Delta t)).$$

Now we notice that there are portions of three possible waves generated by either the generalized Riemann problem or the triple Riemann problem lying in the interval $(r_{n+1,j-1}, r_{n+1,j+1})$. We write these waves as $\omega_{l,m,r}$ from left to right, starting from points of discontinuity (reading $r_{l,n}^d, r_{m,n}^d, r_{r,n}^d$ respectively) in $(r_{j-2}, r_j], (r_j, r_{j+2}], (r_{j+2}, r_{j+4}]$ at the time level $t = t_n$ respectively.

We observe that, depending on if the position of the the randomly chosen point $r_{n+1,j-1}$ is closer to $r_{n,j}^d$ or closer to ω_1 , the wave ω_l is either a zero strength wave in $(r_{n+1,j-1}, r_{n+1,j+1})$ or the wave generated by a steady state U_L such that $U_L(r_{n+1,j-1}) = U_{\Delta r}(t_{n+1}^-, r_{n+1,j-1})$ and another steady state U_M such that $U_M = U_{\Delta r}(t_{n+1}^-, r_{n+1,j+1})$. Similarly, ω_3 is either a zero strength wave in $(r_{n+1,j-1}, r_{n+1,j+1})$ or a problem with a state U_M such that $U_M(r_{n+1,j+1}) = U_{\Delta r}(t_{n+1}^-, r_{n+1,j+1})$ and another state U_R such that $U_R = U_{\Delta r}(t_{n+1}^-, r_{n+1,j+3})$. Turning to the wave ω_2 , it is generated by U_L and U_M or U_M or U_R . According to Lemma 4.2.6, we have the result by adding j on the time level $t = t_n$.

□

Now since the uniform BV bound on a given time interval $(0, T)$ (established below) is known, Helly's theorem gives immediately the fact that there exists a subsequence of $\Delta r \rightarrow 0$ such that we have a limit function $U = U(t, r)$ and $U_{\Delta r}(t, r) \rightarrow U(t, r)$ pointwise a.e. and in L_{loc}^1 at each fixed time t . Moreover, the limit function $U = U(t, r)$ is a weak solution of the Euler model (4.1.1), (4.6.1). This ends the proof of Theorem 4.6.1.

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Abstract :

This thesis is devoted to fluid dynamics evolving in the domain of outer communication of a Schwarzschild black hole. In the first chapter, we formulate the initial value problem of the relativistic Euler model within a class of weak solutions with bounded variation, possibly containing shock waves. We then introduce a version of the random choice method founded on the global steady state solutions and the generalized Riemann problem and we establish a global-in-time existence theory for the initial value problem within the proposed class of weakly regular fluid flows. In the second chapter, we consider the relativistic Burgers model. We have introduced a version of the total variation which is decreasing with respect to time in the Cauchy problem. We also use the generalized characteristics to prove the nonlinear stability of a piecewise steady state solution. In the third chapter, we present some numerical methods based on the Schwarzschild geometry and study numerically the nonlinear stability of steady state solutions and the asymptotic behavior of a general solutions. The proposed schemes provide a numerical tool capable to preserve exactly the equilibria and allow us to analyse the evolution of fluids with the geometry effects.

Keywords : Fluid dynamics; Schwarzschild spacetime; Schwarzschild blackhole; relativistic model; Shock wave; Glimm method; total variation; well-balanced scheme; nonlinear stability, steady state solution

Résumé :

Cette thèse est consacrée à la dynamique globale d'un fluide évoluant dans le domaine de communication extérieur d'un espace-temps de Schwarzschild. Dans le premier chapitre, on formule le problème de Cauchy pour le modèle d'Euler relativiste dans la classe des solutions à la variation bornée contenant des ondes de choc. On propose ensuite une version de la méthode de Glimm fondée sur les solutions stationnaires globales hors du trou noir et le problème de Riemann généralisé et on démontre un théorème d'existence globale en temps pour les écoulements de fluides faiblement réguliers. Dans le deuxième chapitre, on considère le modèle de Burgers relativiste. Nous introduisons une version de la variation totale qui est décroissante en temps pour les solutions générales du problème de Cauchy. Nous avons aussi utilisé les caractéristiques généralisées pour démontrer la stabilité nonlinéaire d'une solution stationnaire par morceaux. Dans le troisième chapitre, nous présentons plusieurs méthodes numériques basées sur la géométrie de Schwarzschild et nous étudions numériquement la stabilité nonlinéaire des solutions stationnaires et le comportement asymptotique des solutions générales. Les schémas proposés fournissent un outils numérique capable de préserver exactement les équilibres et nous permettent d'analyser l'évolution de fluides en présence d'effets géométriques. Dans le quatrième chapitre, nous présentons un modèle non-relativiste préservant certains effets du trou noir de Schwarzschild.

Mot clés : Dynamique des fluides; espace-temps de Schwarzschild; trou noir de Schwarzschild; modèle relativiste; onde de choc; méthode de Glimm; variation totale; schéma équilibre; stabilité nonlinéaire; solutions stationnaire