



HAL
open science

Aspects of connectivity with matroid constraints in graphs

Quentin Fortier

► **To cite this version:**

Quentin Fortier. Aspects of connectivity with matroid constraints in graphs. Modeling and Simulation. Université Grenoble Alpes, 2017. English. NNT : 2017GREAM059 . tel-01838231

HAL Id: tel-01838231

<https://theses.hal.science/tel-01838231>

Submitted on 13 Jul 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THÈSE

Pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ GRENOBLE ALPES

Spécialité : **Mathématiques et Informatique**

Arrêté ministériel : 25 mai 2016

Présentée par

Quentin FORTIER

Thèse dirigée par **Zoltán SZIGETI**, Professeur des Universités,
Grenoble INP

préparée au sein du **Laboratoire G-SCOP**
dans l'**École Doctorale Mathématiques, Sciences et**
technologies de l'information, Informatique

Aspects de la connexité avec contraintes de matroïdes dans les graphes

Aspects of connectivity with matroid constraints in graphs

Thèse soutenue publiquement le **27 octobre 2017**,
devant le jury composé de :

Monsieur Yann VAXÈS

Professeur des Universités, Université Aix-Marseille, Rapporteur

Monsieur Denis CORNAZ

Maître de conférences, Université Paris-Dauphine, Rapporteur

Monsieur Stéphane BESSY

Maître de conférences, Université de Montpellier, Examineur

Monsieur Roland GRAPPE

Maître de conférences, Université Paris 13, Examineur

Madame Nadia BRAUNER

Professeur des Universités, Université Grenoble Alpes, Présidente

Monsieur Zoltán SZIGETI

Professeur des Universités, Grenoble INP, Directeur de thèse



Résumé

La notion de connectivité est fondamentale en théorie des graphes. Nous proposons une étude approfondie d'un récent développement dans ce domaine, en ajoutant des contraintes de matroïdes.

Dans un premier temps, nous exhibons deux opérations de réduction sur les graphes connectés avec contraintes de matroïdes. Ces opérations permettent de généraliser le théorème de caractérisation de la connectivité de Menger et le théorème de packing d'arborescences d'Edmonds.

Cependant, cette extension du théorème d'Edmonds ne garantit plus que les arborescences soient couvrantes. Il a été conjecturé que l'on peut toujours trouver de telles arborescences couvrantes. Nous prouvons cette conjecture dans certains cas particuliers, notamment pour les matroïdes de rang deux et pour les matroïdes transversaux. Nous réfutons cette conjecture dans le cas général en construisant un contre-exemple à plus de 300 sommets, sur une extension parallèle du matroïde de Fano.

Enfin, nous explorons d'autres notions de connexité avec contraintes de matroïdes: pour des graphes mixtes, des hypergraphes, et avec condition d'atteignabilité.

Abstract

The notion of connectivity is fundamental in graph theory. We study thoroughly a recent development in this field, with the addition of matroid constraints.

Firstly, we exhibit two reduction operations on connected graphs with matroid constraints. Using these operations, we generalize Menger's theorem on connectivity and Edmond's theorem on packing of arborescences.

However, this extension of Edmond's theorem does not ensure that the arborescences are spanning. It has been conjectured that one can always find such spanning arborescences. We prove this conjecture in some cases, including matroids of rank two and transversal matroids. We disprove this conjecture in the general case by providing a counter-example with more than 300 vertices, on a parallel extension of the Fano matroid.

Finally, we explore other generalizations of connectivity with matroid constraints: in mixed graphs, hypergraphs and with reachability conditions.

Contents

1	Introduction	3
1.1	Background	3
1.2	Connectivity with matroid constraints	4
1.3	Outline and contributions	4
2	Preliminaries	7
2.1	Graphs and digraphs	7
2.2	Menger's theorem	10
2.3	Trees and arborescences	11
2.4	Submodularity	12
2.5	Splitting-off	13
2.6	Orientation	14
2.7	Packing of trees and arborescences	16
2.8	Matroid theory	17
3	Connectivity and packing with matroid constraints	25
3.1	Connectivity with matroid constraints	26
3.2	Packing of arborescences with matroid constraints	34
3.3	Reductions for \mathcal{M} -digraphs	38
3.4	Packing of branches	45
3.5	Matroids of rank two	46
4	Packing with matroid constraints in acyclic digraphs	51
4.1	Properties of acyclic digraphs	51
4.2	Graphic matroids	52

4.3	Fano matroid	56
4.4	Parallel extensions of the Fano matroid	57
4.5	Complexity	64
5	Other notions of connectivity with matroid constraints	65
5.1	Connectivity with matroid constraints in hypergraphs	65
5.2	Mixed hypergraphs	72
5.3	Reachability-packing of hyperarborescences	74
6	Conclusion and perspectives	80
	Appendix	85
	Bibliography	88

Notations

- **Functions and sets:**

2^S : set of all subsets of S .

S^* ($S = \{s_1, \dots, s_k\}$): $\{s_1^*, \dots, s_k^*\}$ if $*$ is an unary operation defined on the s_i 's.

$X + Y$: $X \cup Y$, if X and Y are sets.

$X - Y$: $\{z : z \in X, z \notin Y\}$, if X and Y are sets.

- **Graphs and digraphs:**

e (\vec{e} being an arc): edge obtained after removing the orientation of \vec{e} .

G (\vec{G} being a digraph): graph obtained after removing the orientations of the arcs of \vec{G} .

Let $G = (V, E)$ be a graph, $\vec{G} = (V, \vec{E})$ a digraph and $X, Y \subseteq V$.

\vec{e}^+, \vec{e}^- : head and tail of an arc \vec{e} , respectively.

$V(e)$: endpoints of an edge e .

$\vec{E}_{\vec{G}}(X, Y)$: arcs from X to Y in \vec{G} .

$E_G(X, Y)$: edges intersecting X and Y in G .

X^+, X^- : arcs leaving X and entering X , respectively.

$\delta_{\vec{G}}(X), \rho_{\vec{G}}(X)$: number of arcs leaving X and entering X , respectively.

$E_G(X)$: edges of G with exactly one endpoint in X .

$E_G(\mathcal{V})$ (\mathcal{V} being a set of disjoint subsets of V): edges e of G with endpoints in different sets of \mathcal{V} .

$e_G(\mathcal{V})$: $|E_G(\mathcal{V})|$.

$G[X], \vec{G}[X]$: restriction of G and \vec{G} to X , respectively.

$|G|, |\vec{G}|$: $|V|$.

$\|G\|, \|\vec{G}\|$: $|E|, |\vec{E}|$, respectively.

- **Paths and arborescences:**

$\vec{P}_1 \rightsquigarrow \vec{P}_2$ (\vec{P}_1 and \vec{P}_2 being two dipaths such that the only common vertex of \vec{P}_1 and \vec{P}_2 is the last vertex of \vec{P}_1 , being equal to the first vertex of \vec{P}_2): dipath obtained by using \vec{P}_1 then \vec{P}_2 .

$\vec{P}(u, v)$: restriction from the vertex u to the vertex v of a dipath \vec{P} .

$\vec{P}(v)$: restriction of a dipath \vec{P} from the first vertex to v .

\vec{P}^+, \vec{P}^- : last and first arc in a dipath \vec{P} , respectively.

${}^r\vec{P}$: $r - v$ dipath orientation of an $r - v$ path P .

$T|_r$: connected component containing r of a tree T .

${}^r\vec{T}$: r -arborescence orientation of a tree T .

\vec{T}^- : set of arcs leaving r of an r -arborescence \vec{T} .

$\vec{T}(v)$: $r - v$ subdipath of an r -arborescence \vec{T} .

$T(v)^-$: first edge of the $r - v$ subpath of a tree T rooted in r .

- **Matroids:**

Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid and $X \subseteq S$.

$[ab]$ ($a, b \in S$): $\text{Span}_{\mathcal{M}}(\{a, b\})$.

$]ab[$ ($a, b \in S$): $\text{Span}_{\mathcal{M}}(\{a, b\}) - a - b$.

$\mathcal{M}[X]$: restriction of \mathcal{M} to X .

$\mathcal{M}_1 \oplus \mathcal{M}_2$: direct sum of two matroids \mathcal{M}_1 and \mathcal{M}_2 .

$|\mathcal{M}|$: $|S|$.

X^{\parallel} : elements of S parallel to an element of X .

- **\mathcal{M} -graphs and \mathcal{M} -digraphs:**

Let $\vec{\mathcal{G}} = ((V, \vec{E}), \mathcal{M}, r)$ be an \mathcal{M} -digraph, $\mathcal{G} = ((V, E), \mathcal{M}, r)$ an \mathcal{M} -graph, $X \subseteq V$ and $\vec{e}_1, \vec{e}_2 \in \vec{E}$.

$r_{\vec{\mathcal{G}}}(\mathbf{X})$: $r_{\mathcal{M}}(\vec{E}(r, X))$.

$r_{\mathcal{G}}(\mathbf{X})$: $r_{\mathcal{M}}(E(r, X))$.

$\rho_{\vec{\mathcal{G}}}(\mathbf{X})$: $\rho_{\vec{\mathcal{G}}-r}(X) + r_{\vec{\mathcal{G}}}(X)$.

$\vec{e}_1 \cdot \vec{e}_2$: arc obtained after splitting of \vec{e}_1 and \vec{e}_2 .

$\vec{e}_1 \odot \vec{e}_2$: arc obtained after switching of \vec{e}_1 and \vec{e}_2 .

$\vec{\mathcal{G}}_{\vec{e}_1 \cdot \vec{e}_2}$: $\mathcal{M}_{\vec{e}_1 \cdot \vec{e}_2}$ -digraph obtained after splitting of \vec{e}_1 and \vec{e}_2 .

$\vec{\mathcal{G}}_{\vec{e}_1 \odot \vec{e}_2}$: $\mathcal{M}_{\vec{e}_1 \odot \vec{e}_2}$ -digraph obtained after switching of \vec{e}_1 and \vec{e}_2 .

$\vec{\mathcal{G}}_{\odot v}$: $\mathcal{M}_{\odot v}$ -digraph obtained after complete switching on the vertex v in $\vec{\mathcal{G}}$.

$a \equiv b$: a and b have same color, that is to say they belong to the same arborescence in every packing.

$\vec{\mathcal{G}}(\mathbf{X})^-$: arcs leaving r used by at least one dipath from r to a vertex of X , in $\vec{\mathcal{G}}$.

- **Hypergraphs and dypergraphs:**

Let $H = (V, \mathcal{E})$ be a hypergraph, $\vec{H} = (V, \vec{\mathcal{E}})$ a dypergraph and $X, Y \subseteq V$.

\vec{e}^+, \vec{e}^- : head and set of tails of a hyperarc \vec{e} , respectively.

$V(\varepsilon)$: endpoints of an hyperedge ε .

$d_H(\mathbf{X}, \mathbf{Y})$: number of hyperedges $\varepsilon \in \mathcal{E}$ such that $V(\varepsilon) \subseteq X \cup Y$, $\varepsilon \cap X \neq \emptyset$, $\varepsilon \cap Y \neq \emptyset$.

$\vec{\mathcal{E}}_{\vec{H}}(\mathbf{X})$: $\{\vec{e} \in \vec{\mathcal{E}} : \vec{e}^+ \in X, \vec{e}^- \not\subseteq X\}$.

$\mathcal{E}_H(\mathbf{X}, \mathbf{Y})$: hyperedges intersecting X and Y .

$\mathcal{E}_H(\mathcal{V})$ (\mathcal{V} being a set of disjoint subsets of V): hyperedges $\varepsilon \in \mathcal{E}$ intersecting at least two sets of \mathcal{V} .

$\varepsilon_H(\mathcal{V})$: $|\mathcal{E}_H(\mathcal{V})|$.

$d_H(\mathbf{X}, \mathbf{Y})$: number of hyperedges $\varepsilon \in \mathcal{E}(X - Y, Y - X)$ such that $V(\varepsilon) \subseteq X \cup Y$.

$\vec{\mathcal{H}}(\mathbf{X})^-$: hyperarcs leaving r used by at least one dyperpath from r to a vertex of X , in an \mathcal{M} -dypergraph $\vec{\mathcal{H}}$ rooted in r .

Introduction

1.1 Background

The notion of connectivity has always been a major topic in graph theory, with countless applications. The main problem of connectivity theory is to measure how well the vertices of a graph are connected to each other. Once such a measure is defined, it makes sense to look at its properties, its characterizations and its tractability. For example, to avoid traffic jams, one may want to have as many different roads as possible between two cities. In graph terms, we are interested in the maximum number of disjoint paths between two vertices. Depending on how we consider these paths, we obtain different notions of connectivity:

- The paths can be **vertex-disjoint** or **edge-disjoint**. In this thesis, we will mostly be interested in edge-disjoint paths.
- Connectivity can be studied in **directed** graphs (in short **digraphs**) or **undirected** graphs.
- Connectivity can be **global**, by requiring disjoint paths between every pair of vertices, or **rooted**, by considering only paths from a special vertex called root.

For example, a graph is k -edge-connected if between any pair of vertices there is at least k edge-disjoint paths. The definitions of vertex-connectivity, rooted-connectivity, and connectivity in digraphs are similar.

The first major result about graph connectivity goes back to Menger ([Men27]) and states that the maximum number of edge-disjoint paths between two vertices u and v in a graph is equal to the minimum number of edges whose removal disconnects u and v . There exist similar versions of Menger's theorem for digraphs and for vertex-connectivity.

The first constructive characterization of 2-vertex-connected graphs, the ear decomposition theorem, was proved by Whitney in [Whi32]. He showed that every 2-vertex-connected-graph can be constructed from a cycle by adding new paths (called ears) between two existing vertices. A similar theorem for 2-edge-connected graphs was proved by Robbins in [Rob39]. Mader ([Mad78], [Mad82]) used the splitting-off operation introduced by Lovász in [Lov79] to deduce a constructive characterization of k -edge-connected graphs and digraphs, for all $k \in \mathbb{N}$.

No good construction is known for general vertex-connected graphs, beyond the construction theorem of Tutte ([Tut61a]) for 3-vertex-connected graphs.

Mader showed in [Mad82] that his construction of k -arc-connected digraphs implies the following packing theorem of Edmonds, first proved in [Edm73]:

Theorem. (*Edmonds*) *A directed graph is rooted k -arc-connected if and only if it contains k rooted arc-disjoint spanning arborescences.*

This result also implies a theorem about packing of trees in undirected graphs, first due to Tutte in [Tut61b]. Edmonds' packing theorem and its variations is the central point of this thesis. The main extensions of Edmonds' theorem and Tutte's theorem stated in this thesis are summarized on [Figure 1.1](#).

1.2 Connectivity with matroid constraints

In this thesis, we explore a generalization of connectivity where the paths must satisfy an additional matroid constraint. This notion has first been defined by Katoh and Tanigawa ([KT13]) for graphs and by Durand de Gevigney, Nguyen and Szigeti ([GNS13]) for digraphs. Matroids were introduced by Whitney ([Whi35]) as an abstract generalization of linear independence in vector spaces. Therefore, this new connectivity measure captures a notion of « independent paths ». However, we will avoid using this terminology since « independent paths » have been used for a different notion (see [BF09]).

Durand de Gevigney, Nguyen and Szigeti successfully generalized Edmonds' packing theorem to rooted arc-connectivity with matroid constraints, with one notable difference: their packing theorem does not ensure that the arborescences involved in the packing are spanning. This led Bérczi, T. Király and Kobayashi ([BKK16]) to conjecture that the packing theorem of Durand de Gevigney, Nguyen and Szigeti remains true when we require the arborescences to be spanning. This conjecture and its undirected counterpart ([Conjecture 3.1](#) and [Conjecture 3.2](#), respectively) will be deeply studied in this thesis.

1.3 Outline and contributions

Chapter 2 provides the definitions and results in graph theory and matroid theory that will be used. In particular, the main tools to study connectivity are exposed: submodularity, splitting-off and orientation theorems.

In Chapter 3, we define connectivity with matroid constraints for graphs and digraphs. We describe two operations to prove theorems by induction on digraphs satisfying connectivity with matroid constraints. The first is a straight generalization of the splitting reduction, and is used to generalize Menger's theorem to connectivity with matroid constraints. The second

operation, called switching, was first used by Durand de Gevigney, Nguyen and Szigeti in [GNS13]. Using switching, we establish [Conjecture 3.1](#) for matroids of rank two.

Chapter 4 focuses on the non-trivial special case of [Conjecture 3.1](#) where the digraph is acyclic. We prove this restriction of [Conjecture 3.1](#) for graphic matroids and for the Fano matroid \mathcal{F}_7 . Finally, we exhibit a parallel extension of \mathcal{F}_7 and a graph disproving [Conjecture 3.2](#). As shown in Chapter 3, this also implies a counterexample to [Conjecture 3.1](#).

Chapter 5 explores other generalizations of connectivity with matroid constraints and of the corresponding packing theorems. Firstly, the packing theorem of Durand de Gevigney, Nguyen and Szigeti is generalized to directed hypergraphs, thanks to a new trimming operation. Using a new orientation theorem, we deduce a similar result for directed hypergraphs and for mixed hypergraphs. Then, we extend a result of C. Király ([Kir16]) on reachability-packing of arborescences. Finally, we explore the algorithmic consequences of the previous results.

The annex contains two results about trimming and orientation of hypergraphs used in Chapter 4 and generalizing two theorems of Frank.

		Trimming Theorem 5.7			
		Graphs	\mathcal{M} -graphs	Hypergraphs	\mathcal{M} -hypergraphs
Orientation Theorem 5.8	Partition connectivity	Tutte [Tut61b] Theorem 2.9	Katoh, Tanigawa [KT13] Theorem 3.13	Frank <i>et al.</i> [FKK03b] Theorem 5.6	Fortier <i>et al.</i> [For+16] Theorem 5.11
	Rooted arc-connectivity	Edmonds [Edm73] Theorem 2.8	Durand de Gevigney <i>et al.</i> [GNS13] Theorem 3.11	Frank <i>et al.</i> [FKK03a] Theorem 5.5	Fortier <i>et al.</i> [For+16] Theorem 5.10
	Mixed partition connectivity	Frank [Fra78] Theorem 5.12	Fortier <i>et al.</i> [For+16] Theorem 5.13	Fortier <i>et al.</i> [For+16] Theorem 5.13	Fortier <i>et al.</i> [For+16] Theorem 5.13
	Reachability connectivity	Kamiyama <i>et al.</i> [KKT09] Theorem 5.14	Király C. [Kir16] Theorem 5.16	Bérczi, Frank [BF08]	Fortier <i>et al.</i> [For+16] Theorem 5.19

Figure 1.1: Theorems about packing trees or arborescences that will be stated in this thesis. An arrow indicates an implication (for example, results on partition connectivity can be deduced from results on rooted arc-connectivity).

Preliminaries

Contents

2.1	Graphs and digraphs	7
2.2	Menger's theorem	10
2.3	Trees and arborescences	11
2.4	Submodularity	12
2.5	Splitting-off	13
2.6	Orientation	14
2.7	Packing of trees and arborescences	16
2.8	Matroid theory	17
2.8.1	Definitions	17
2.8.2	Operations on matroids	19
2.8.3	Examples of matroids	20
2.8.4	Optimization	22

We introduce the necessary tools for the next chapters, and review some of the major results about connectivity in graphs and directed graphs.

2.1 Graphs and digraphs

Definition 2.1. If $k \in \mathbb{N}$, a **k -set** is a set of size k . If S is a set, 2^S is the set of all subsets of S .

If $k \in \mathbb{N}$, a **k -tuple** is an ordered k -set.

A **partition** of a set S is a set of non-empty subsets of S whose union is S .

If s is an element of a set S , we will sometimes simply write s for the set $\{s\}$.

In this thesis, we will only consider graphs and digraphs without loop (a loop is an edge or arc from a vertex to the same vertex), but possibly with parallel edges. The following non-standard definitions are motivated by our need to distinguish parallel edges.

Definition 2.2. A **graph** is a 2-tuple (V, E) , V and E being finite sets and $V \neq \emptyset$. The elements of V are called **vertices** and the elements of E are called **edges**. Each edge $e \in E$

is associated to a 2-set of vertices denoted $V(e)$. If $e \in E$ and $V(e) = \{u, v\}$, u and v are the **end vertices** of e . Two edges e_1 and e_2 are **parallel** if $V(e_1) = V(e_2)$.

Definition 2.3. A **directed graph** or **digraph** is a 2-tuple (V, \vec{E}) , V and \vec{E} being finite sets and $V \neq \emptyset$. The elements of V are called **vertices** and the elements of \vec{E} are called **directed edges**, or **arcs**. Each arc $\vec{e} \in \vec{E}$ is associated to a 2-tuple of distinct vertices denoted by $V(\vec{e}) = (\vec{e}^-, \vec{e}^+)$. If $\vec{e} \in \vec{E}$, \vec{e}^- is the **tail** of \vec{e} and \vec{e}^+ is the **head** of \vec{e} , \vec{e}^- and \vec{e}^+ are also the **end vertices** of \vec{e} . Two arcs are **parallel** if they have the same tail and the same head.

The superscript « $-$ » (respectively « $+$ ») will be used in various ways, and will always intuitively mean «before» or «at the beginning» (respectively «after» or «at the end»). The number of vertices of a graph G or a digraph \vec{G} is denoted by $|G|$ or $|\vec{G}|$, respectively.

Informally, by orienting a graph we mean replacing each edge by an arc with the same end vertices, thus providing a digraph. Conversely, every digraph gives rise to a graph when removing the orientations on the arcs, that is to say replacing every arc by an edge with same end vertices.

Definition 2.4. If $\vec{G} = (V, \vec{E})$ is a directed graph we denote by $G = (V, E)$ the graph defined by $E = \{e : \vec{e} \in \vec{E}\}$ and $V(e) = \{\vec{e}^-, \vec{e}^+\}$ for all $e \in E$. We say that \vec{G} is an **orientation** of G .

Therefore, if \vec{e} is an arc, e will denote the edge obtained by removing the orientation of \vec{e} . Similarly, if \vec{E} is a set of arcs, E will denote the corresponding edges obtained by removing the orientations.

Definition 2.5. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. We say that G_1 and G_2 are **edge-disjoint** if E_1 and E_2 are disjoint. If V_1 and V_2 are disjoint then G_1 and G_2 are **vertex-disjoint**. Similarly, if $\vec{G}_1 = (V_1, \vec{E}_1)$ and $\vec{G}_2 = (V_2, \vec{E}_2)$ are two digraphs, they are **arc-disjoint** if \vec{E}_1 and \vec{E}_2 are disjoint.

Definition 2.6. If $G = (V, E)$ is a graph, then we say that a graph $G' = (V', E')$ is a **subgraph** of G , and we write $G' \subseteq G$, if $V' \subseteq V$, $E' \subseteq E$, and $\forall e \in E'$, $V'(e) = V(e)$. If $V' = V$ then G' is **spanning**.

Similarly, if $\vec{G} = (V, \vec{E})$ is a digraph, then we say that a digraph $\vec{G}' = (V', \vec{E}')$ is a **subdigraph** of \vec{G} , and we write $\vec{G}' \subseteq \vec{G}$, if $V' \subseteq V$, $\vec{E}' \subseteq \vec{E}$, and $\forall \vec{e} \in \vec{E}'$, $V'(\vec{e}) = V(\vec{e})$. If $V' = V$ then \vec{G}' is **spanning**.

Definition 2.7. Let $G = (V, E)$ be a graph and X, Y disjoint subsets of V . $\mathbf{E}_G(\mathbf{X}, \mathbf{Y})$ is the set of all $e \in E$ with one endpoint in X and one in Y . $\mathbf{e}_G(\mathbf{X}, \mathbf{Y})$ is the size of $\mathbf{E}_G(\mathbf{X}, \mathbf{Y})$. More generally, if \mathcal{V} is a set of disjoint subsets of V , $\mathbf{E}_G(\mathcal{V})$ is the set of all $e \in E$ with end vertices in different sets of \mathcal{V} . $\mathbf{e}_G(\mathcal{V})$ is the size of $\mathbf{E}_G(\mathcal{V})$. If $X \subseteq V$, $\mathbf{e}_G(\mathbf{X}) = |\mathbf{E}_G(\mathbf{X}, V - \mathbf{X})|$ is the **degree** of X . The **star** at $v \in V$ is $\mathbf{E}_G(v, V - v)$.

Definition 2.8. Let $\vec{G} = (V, \vec{E})$ be a directed digraph, X, Y disjoint subsets of V . $\vec{\mathbf{E}}_{\vec{G}}(\mathbf{X}, \mathbf{Y})$ is the set of all $\vec{e} \in \vec{E}_{\vec{G}}$ such that $V(\vec{e}) \in X \times Y$. We also define $\vec{\mathbf{e}}_{\vec{G}}(\mathbf{X}, \mathbf{Y}) = |\vec{\mathbf{E}}_{\vec{G}}(\mathbf{X}, \mathbf{Y})|$,

$\vec{E}_{\vec{G}}^+(X) = \vec{E}_{\vec{G}}(X, V - X)$, $\vec{E}_{\vec{G}}^-(X) = \vec{E}_{\vec{G}}(V - X, X)$, the **in-degree** function $\rho_{\vec{G}}(X) = |\vec{E}_{\vec{G}}^-(X)|$ and the **out-degree** function $\delta_{\vec{G}}(X) = |\vec{E}_{\vec{G}}^+(X)|$. An arc \vec{e} **enters** X if $\vec{e} \in \vec{E}_{\vec{G}}^-(X)$ and \vec{e} **leaves** X if $\vec{e} \in \vec{E}_{\vec{G}}^+(X)$. A vertex with out-degree 0 is a **sink**. If $v \in V$, $\vec{E}_{\vec{G}}^-(v)$ and $\vec{E}_{\vec{G}}^+(v)$ are also denoted by $\mathbf{v}_{\vec{G}}^-$ and $\mathbf{v}_{\vec{G}}^+$, and are called **in-star** and **out-star** at v , respectively.

When the context is clear, we may drop the subscript G or \vec{G} , so that, for example, we will write $\vec{E}(X, Y)$ instead of $\vec{E}_{\vec{G}}(X, Y)$.

Definition 2.9. Let $\vec{G} = (V, \vec{E})$ be a digraph and $u, v \in V$. A sequence $\vec{P} = \vec{e}_1, \vec{e}_2, \dots, \vec{e}_k$ of arcs of \vec{E} is a $u - v$ **directed path** (for short $u - v$ **dipath**), of \vec{G} if:

1. $\forall i \in \{1, \dots, k-1\}: \vec{e}_i^+ = \vec{e}_{i+1}^-$.
2. $\forall i, j \in \{1, \dots, k\}, i \neq j: \vec{e}_i^+ \neq \vec{e}_j^+$ and $\vec{e}_i^- \neq \vec{e}_j^-$.
3. $u = \vec{e}_1^-$ and $v = \vec{e}_k^+$.

If there exists such a path, we say that v is **reachable** from u .

A **directed cycle** is a sequence of arcs in which the deletion of any one arc results in a dipath.

Definition 2.10. Let $G = (V, E)$ be a graph and $u, v \in V$. A sequence $P = e_1, e_2, \dots, e_k$ of edges of E is a $u - v$ **path** of G if:

1. $\forall i \in \{1, \dots, k-1\}: |V(e_i) \cap V(e_{i+1})| = 1$.
2. $\forall i, j \in \{1, \dots, k\}, |i - j| \geq 2: V(e_i) \cap V(e_j) = \emptyset$.
3. $u \in V(e_1) - V(e_2)$ and $v \in V(e_k) - V(e_{k-1})$.

A **cycle** is a sequence of edges in which the deletion of any one edge results in a path.

We will say that a graph or digraph is **acyclic** if it does not contain any cycle. The paths and dipaths we use are **elementary**: they do not visit the same vertex twice. When needed, we will consider a dipath as the digraph with the same set of arcs and with the vertices used by these arcs. Thus we will say, for example, that \vec{P} is a dipath of \vec{G} if \vec{P} is a dipath whose digraph is a subdigraph of \vec{G} . Observe that there are two possible dipaths resulting from an orientation of a $u - v$ path P : a $u - v$ dipath denoted by \overrightarrow{uP} and a $v - u$ dipath denoted by \overrightarrow{vP} .

Definition 2.11. Let \vec{G} be a digraph and \vec{P} a $u - v$ dipath of \vec{G} . \vec{P}^- denotes the first arc of \vec{P} and \vec{P}^+ the last arc of \vec{P} . If u' and v' are two vertices appearing in this order in \vec{P} then $\vec{P}(u', v')$ is the $u' - v'$ subpath of \vec{P} from u' to v' . If a $u - v$ dipath \vec{P}_1 and a $v - w$ dipath \vec{P}_2 only share the vertex v , then the $u - w$ dipath obtained by using \vec{P}_1 then \vec{P}_2 is denoted by $\vec{P}_1 \rightsquigarrow \vec{P}_2$.

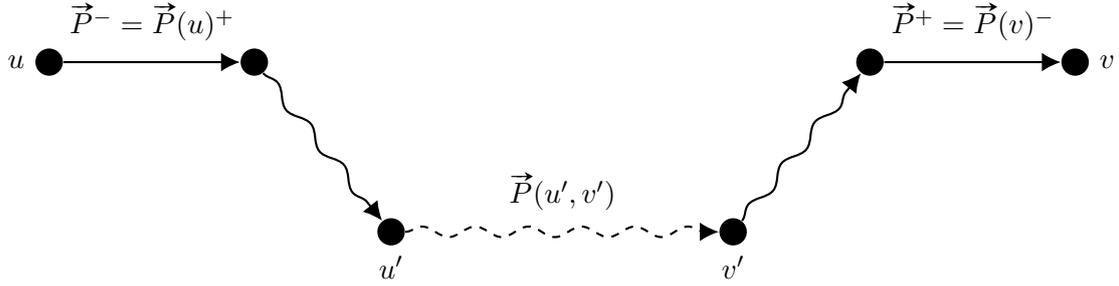


Figure 2.1: A $u - v$ dipath \vec{P} . $\vec{P}(u', v')$ is dashed.

These definitions are illustrated on [Figure 2.1](#).

A lot of problems in graph theory can be expressed in terms of **packings**: given a graph (respectively digraph) G and a graph (respectively digraph) class \mathcal{C} , a packing of G using elements of \mathcal{C} is a set of edge-disjoint subgraphs (respectively arc-disjoint subdigraphs) of G which are elements of \mathcal{C} . For example, an edge-coloring of a graph (such that no two adjacent edges have the same color) is a packing using matchings. Connectivity is about *packing paths*:

Definition 2.12. Let $G = (V, E)$ be a graph, $k \in \mathbb{N}^*$ and $u, v \in V$. u and v are **k -edge-connected** in G if there exists a packing of k $u - v$ paths in G . If u and v are k -edge-connected for all vertices u, v of G , then G is called **k -edge-connected**.

Definition 2.13. Let $\vec{G} = (V, \vec{E})$ be a digraph, $k \in \mathbb{N}^*$ and $r, v \in V$. v is **k -arc-connected from r** (or simply **rooted k -arc-connected** when the root r is irrelevant) in \vec{G} if there exists a packing of k $r - v$ dipaths in \vec{G} . If, for all $v \in V$, v is k -arc-connected from r , then \vec{G} is **k -arc-connected from r** . If, for all $r \in V$, \vec{G} is k -arc-connected from r , then we say that \vec{G} is **(globally) k -arc-connected**.

If $k = 1$ we say simply « **connected** » instead of « k -arc-connected » or « k -edge-connected ». If G is k -edge-connected but $G - e$ is not k -edge-connected, for any $e \in E$, then G is **minimally k -edge-connected**. We define minimally k arc-connected digraphs in a similar way.

2.2 Menger's theorem

Menger proved in [Men27] an equivalent condition for arc-connectivity.

Definition 2.14. Let $\vec{G} = (V, \vec{E})$ be a digraph. A **cut** of \vec{G} is a 2-tuple $(V - X, X)$, also simply written X , with $X \subseteq V$, $X \neq \emptyset$ and $X \neq V$. If $r, v \in V$ an $r - v$ **cut** of \vec{G} is a cut X with $v \in X$ and $r \notin X$. An r -cut is an $r - v$ cut for some $v \in V$. The **value** of a cut X is $\rho_{\vec{G}}(X)$.

Theorem 2.1 (Menger's theorem for rooted arc-connectivity). A digraph \vec{G} is k -arc-connected from r if and only if \vec{G} has no $r - v$ cut with value strictly less than k .

Menger's theorem provides a *certificate* to prove that a digraph is not rooted k -arc-connected: it is enough to exhibit a cut of value less than k . Menger's theorem can also be formulated in terms of a *min-max* formula: the maximum number of $r - v$ arc-disjoint dipaths in a digraph is equal to the minimum value of an $r - v$ cut. There is a polynomial algorithm (given by Edmonds and Karp in [EK72]), which is also a specialization of the maximum flow algorithm of Ford and Fulkerson ([FF55]), to find k arc-disjoint $r - v$ dipaths, or an $r - v$ cut of value strictly less than k .

Naturally, the results of this paragraph hold true as well for k -edge-connected graphs, where the value of a cut X is defined by $e(X)$:

Theorem 2.2 (Menger's theorem for edge-connectivity). *A graph G is k -edge-connected if and only if G has no cut with value strictly less than k .*

2.3 Trees and arborescences

Trees and arborescences are, respectively, the minimally connected graphs and minimally rooted-connected digraphs. Thus it makes sense to study them when studying connectivity. However the previous definition is not very handy and we will prefer the following:

Definition 2.15. *A **tree** is a graph $T = (V, E)$ such that any two vertices of T are connected by a unique path. A **forest** is a union of trees on disjoint vertex sets. Moreover, if T is a subgraph of a graph $G = (V', \vec{E}')$ we say that T is a tree of G . If $V = V'$ then T is a **spanning tree** of G .*

Theorem 2.3. *Let $T = (V, E)$ be a graph. T is a tree if and only if T satisfies any two of the following conditions:*

1. T is acyclic.
2. T is connected.
3. $|E| = |V| - 1$.

Definition 2.16. *An **r -arborescence**, or simply **arborescence**, is a digraph $\vec{T} = (V, \vec{E})$ with a special vertex $r \in V$ called **root** such that for every $v \in V$ there is a unique $r - v$ dipath in \vec{T} .*

*Moreover, if \vec{T} is a subdigraph of a digraph $\vec{G} = (V', \vec{E}')$ we say that \vec{T} is an r -arborescence of \vec{G} . If $V = V'$ then \vec{T} is a **spanning r -arborescence** of \vec{G} .*

*An **r -branch** is an r -arborescence with only one arc leaving r .*

Theorem 2.4. *Let $\vec{T} = (V, \vec{E})$ be a digraph and $r \in V$. The following four propositions are equivalent:*

1. \vec{T} is an r -arborescence.

2. \vec{T} has no circuit, $\rho(r) = 0$ and $\rho(v) = 1$ for all $v \in V - r$.
3. T is a tree, $\rho(r) = 0$ and $\rho(v) = 1$ for all $v \in V - r$.
4. $\rho(r) = 0$, $\rho(v) = 1$ for all $v \in V - r$ and $\rho(X) \geq 1$, for every r -cut X .

Definition 2.17. Let $\vec{T} = (V, \vec{E})$ be an r -arborescence. For every $u, v \in V$, we define, if it exists, $\vec{T}(u, v)$ as the unique $u - v$ dipath in \vec{T} , simply written $\vec{T}(v)$ if $u = r$. We also define \vec{T}^- as the set of arcs of \vec{T} leaving r . We use similar notations for rooted trees.

2.4 Submodularity

Submodular functions, which can be viewed as the discrete counterparts of convex functions (as shown by Lovász in [Lov83]), are very important in combinatorial optimization.

Definition 2.18. Let \mathcal{X} be a finite collection, closed by intersection and union, of subsets of a set S and $f : \mathcal{X} \rightarrow \mathbb{R}$. We say that f is **submodular** if:

$$\forall X, Y \in \mathcal{X}, f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \quad (*)$$

If $(*)$ is valid only for intersecting sets X and Y then we say that f is **intersecting submodular**. A function g is **supermodular** if $-g$ is submodular, and **intersecting supermodular** if $-g$ is intersecting submodular.

The following property is called **subadditivity**, and is directly implied by $(*)$:

Lemma 2.1. Let $f : \mathcal{X} \rightarrow \mathbb{R}$ be a submodular function with $\emptyset \in \mathcal{X}$ and $f(\emptyset) = 0$. Then, if $S \in \mathcal{X}$:

$$\text{For all partition } \mathcal{S} \text{ of } S, \sum_{X \in \mathcal{S}} f(X) \geq f(S).$$

It is straightforward from the definition that if f and g are submodular functions on \mathcal{X} then $f + g$ is also submodular on \mathcal{X} and any restriction of f on a collection of sets closed by intersection and union is submodular.

The following equality, illustrated on [Figure 2.2](#), shows that the in-degree function $\rho_{\vec{G}}$ of a digraph \vec{G} is submodular.

Lemma 2.2. Let $\vec{G} = (V, \vec{E})$ be a digraph and $X \subseteq V$, $Y \subseteq V$. Then:

$$\rho_{\vec{G}}(X) + \rho_{\vec{G}}(Y) = \rho_{\vec{G}}(X \cap Y) + \rho_{\vec{G}}(X \cup Y) + e_G(X - Y, Y - X)$$

The minimum of a submodular function is well-defined since it is defined on a finite space. Like convex functions, submodular function have nice minimization properties, that will be intensively used in our proofs:

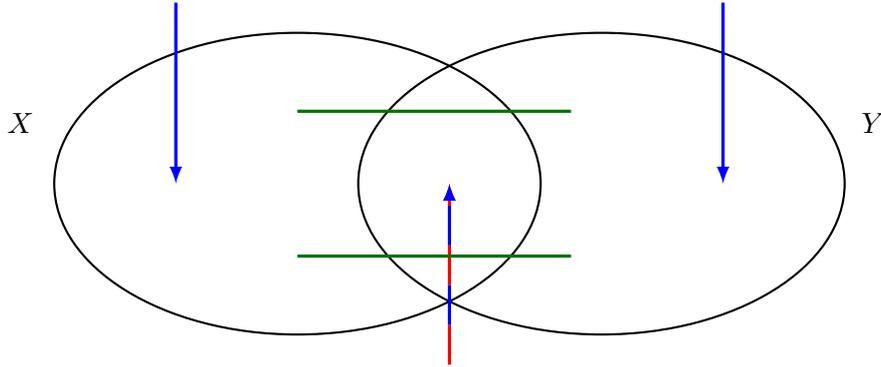


Figure 2.2: Illustration of the formula

$$\rho_{\vec{G}}(X) + \rho_{\vec{G}}(Y) = \rho_{\vec{G}}(X \cap Y) + \rho_{\vec{G}}(X \cup Y) + e_G(X - Y, Y - X)$$

Lemma 2.3. *Let f be a submodular function and $m = \min_{X \neq \emptyset} f(X)$. If X and Y are two intersecting sets such that $f(X) = f(Y) = m$ then $f(X \cap Y) = f(X \cup Y) = m$.*

Proof. Since f is submodular, $2m = f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$. Since $f(X \cap Y) \geq m$ and $f(X \cup Y) \geq m$, it implies $f(X \cap Y) = f(X \cup Y) = m$. \square

Lemma 2.3 will be surprisingly effective to prove theorems by reasoning on sets minimizing a given submodular function, as we will see later.

A set minimizing a given submodular function can be found in polynomial time, as shown by Iwata, Fleischer and Fujishige ([IFF01]) and independently by Schrijver ([Sch00]). However, the problem of *maximizing* a submodular function is NP-hard in general: for example, the problem of finding a maximum dicut, that is to say, a set maximizing $\rho_{\vec{G}}$, is NP-hard.

2.5 Splitting-off

Lovász, in [Lov79], introduced the following splitting-off operation on digraphs, which proved very useful in the study of connectivity:

Definition 2.19. *Let $\vec{G} = (V, \vec{E})$ be a digraph and $\vec{e}_1, \vec{e}_2 \in \vec{E}$ such that $\vec{e}_1^+ = \vec{e}_2^-$. A **splitting-off** of \vec{e}_1 and \vec{e}_2 consists in removing \vec{e}_1 and \vec{e}_2 in \vec{G} and adding a new arc denoted by $\vec{e}_1 \cdot \vec{e}_2$ such that $V(\vec{e}_1 \cdot \vec{e}_2) = (\vec{e}_1^-, \vec{e}_2^+)$. The resulting digraph is denoted by $\vec{G}_{\vec{e}_1 \cdot \vec{e}_2}$.*

*If $v \in V$, a **complete splitting-off** at v consists in splitting every arc leaving v with a different arc entering v .*

Using submodular techniques, Mader, in [Mad82], proved the following reduction theorem on rooted k -arc-connected digraphs:

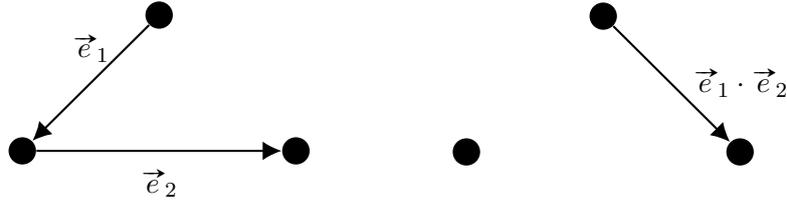


Figure 2.3: Splitting-off of \vec{e}_1 and \vec{e}_2 .
Left: before splitting-off. Right: after.

Theorem 2.5. *Let $k \in \mathbb{N}^*$ and $\vec{G} = (V, \vec{E})$ be a k -arc-connected from r digraph and let $v \in V - r$ be such that $\rho(v) \geq \delta(v)$. Then it is possible to perform a complete splitting-off at v followed by a deletion of v to obtain a digraph which is, again, k -arc-connected from r .*

If \vec{G} is a minimally k -arc-connected digraph from r with at least two vertices then \vec{G} has a vertex $v \neq r$ satisfying $\rho(v) > \delta(v)$, since $\rho(r) = 0 < \delta(r)$ and $\sum_{u \in \vec{G}} \rho(u) = \sum_{u \in \vec{G}} \delta(u)$. Therefore [Theorem 2.5](#) provides a powerful tool to prove theorems on minimally rooted k -arc-connected digraphs by induction: if a theorem is true for the digraph with only one vertex and no arc, and if it remains true when applying the inverse operation of complete splitting-off (called **pinching**), then the theorem is true for every minimally rooted k -arc-connected digraphs.

We mention that there exists a similar splitting-off reduction on k -edge-connected graphs, albeit more complicated, found by Lovász ([Lov79]) for even k and by Mader ([Mad78]) for odd k .

2.6 Orientation

It is worth mentioning that the operation of removing orientation is straightforward whereas orienting a graph involves non-trivial choices for the direction of each arc. This fact may enlighten why it is often easier to deduce results on graphs from their counterparts on digraphs rather than the opposite.

However, there is trivially only one possible orientation to transform a tree into an r -arborescence:

Lemma 2.4. *If \vec{T} is an arborescence then T is a tree.*

Conversely, given a tree T and r a vertex of T , there is a unique orientation of T such that T becomes an r -arborescence, denoted by ${}^r\vec{T}$.

[Lemma 2.4](#) implies that if a graph G is connected then it has a rooted connected orientation. Indeed, in this case G has a spanning tree T that can be oriented in ${}^r\vec{T}$ and then any orientation of the remaining edges of G will give a rooted arc-connected digraph.

However it is false in general that a k -edge-connected graph has a rooted k -arc-connected orientation: the «triangle» of Figure 2.4 is 2-edge-connected but does not have any 2-arc-connected orientation.

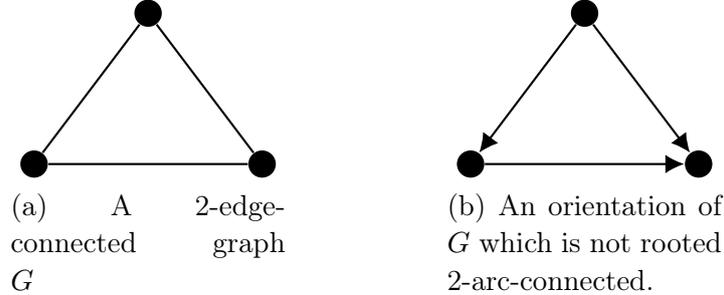


Figure 2.4: A 2-edge-connected graph without rooted 2-arc-connected orientation.

The following orientation theorem of Frank in [Fra80] explicits when it is possible to orient a graph while preserving connectivity.

Theorem 2.6. *Let $G = (V, E)$ be a graph and $h : 2^V \rightarrow \mathbb{N}$ an intersecting supermodular function such that $h(V) = 0$. Then there exists an orientation \vec{G} of G such that $\rho_{\vec{G}}(X) \geq h(X)$ for all $X \subseteq V$ if and only if for every partition \mathcal{V} of V ,*

$$e_G(\mathcal{V}) \geq \sum_{X \in \mathcal{V}} h(X).$$

Moreover, such an orientation can be found in polynomial time using submodular flows, as shown by Frank in [Fra82]. The previous theorem naturally leads to yet another notion of connectivity:

Definition 2.20. *Let $G = (V, E)$ be a graph and $k \in \mathbb{N}^*$. We say that G is **k -partition-connected** if, for every partition \mathcal{V} of V :*

$$e_G(\mathcal{V}) \geq k(|\mathcal{V}| - 1).$$

Equivalently, G is k -partition-connected if, for every ℓ , one has to delete at least $k\ell$ edges to dismantle it into $\ell + 1$ components. By considering partitions into two sets, we see that k -partition-connected graphs are k -edge-connected. However Figure 2.4a is an example of a 2-edge-connected graph which is not 2-partition-connected. Frank, in [Fra11], explains how to decide in polynomial time if a graph is k -partition-connected.

Theorem 2.6 implies Theorem 2.7 below (proved by Frank in [Fra78]), when specialized to the following function h :

$$h(X) = \begin{cases} k & \text{if } r \notin X, X \neq \emptyset. \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.7. *Let $G = (V, E)$ be a graph, $r \in V$ and $k \in \mathbb{N}^*$. G has a k -arc-connected orientation from r if and only if G is k -partition-connected.*

[Theorem 2.7](#) implies in particular that if $r_1, r_2 \in V$, G has a k -arc-connected orientation from r_1 if and only if G has a k -arc-connected orientation from r_2 .

2.7 Packing of trees and arborescences

Let \vec{G} be a digraph. If \vec{G} is arc-connected from a vertex r then we see, using for example a depth-first search from r , that \vec{G} contains a spanning r -arborescence. Edmonds generalized this observation in [Edm73]:

Theorem 2.8 (Edmonds). *Let $\vec{G} = (V, \vec{E})$ be a digraph, $r \in V$ and $k \in \mathbb{N}^*$. \vec{G} has a packing of k spanning r -arborescences if and only if \vec{G} is k -arc-connected from r .*

Mader ([Mad82]) noted that [Theorem 2.8](#) can be proved easily by induction, using the splitting-off reduction of [Theorem 2.5](#). [Theorem 2.8](#) implies that there is a polynomial algorithm to test if a digraph contains a packing of k spanning rooted arborescences.

This fundamental packing theorem has been generalized in various ways, and we will study some of them in this thesis. In particular it implies, using [Theorem 2.7](#), the following packing theorem on graphs, first proved by Tutte in [Tut61b]:

Theorem 2.9 (Tutte). *Let $G = (V, E)$ be a graph and $k \in \mathbb{N}^*$. G has a packing of k spanning trees if and only if G is k -partition-connected.*

As an application of [Theorem 2.9](#), we describe a game introduced by Shannon and solved by Lehman in [Leh64]. Let G be a graph with $|G| \geq 2$. Two players, Join and Cut, alternately color (red for Join and blue for Cut) one uncolored edge of G , starting with Cut. Cut wins if the blue edges contain a cut and Join wins if the red edges is a spanning connected subdigraph of G . We prove that, if G has a packing of 2 spanning trees, then Join has a winning strategy and if not, Cut has a winning strategy.

Assume first that G has two edge-disjoint spanning trees T_1 and T_2 . We describe the following winning strategy for Join, by induction on $|G|$. If $|V| = 2$ then T_1 and T_2 each consist of a single edge and Join can color the edge not colored by Cut to win. Assume now that $|V| > 2$. If Cut colors an edge e in $T_1 \cup T_2$, say $e \in T_1$, then Join colors an edge $e' \in T_2$ given by the following (well-known) [Lemma 2.5](#).

Lemma 2.5. *Let G be a graph and T_1, T_2 two spanning trees of G . Then for every edge $e \in T_1$, there exists $e' \in T_2$ such that $T_1 - e + e'$ is, again, a spanning tree of G .*

If Cut colors an edge outside $T_1 \cup T_2$ then Join colors any uncolored edge e' of G . Let G' be obtained from G by deleting e and then contracting e' . Delete e from T_1 and contract

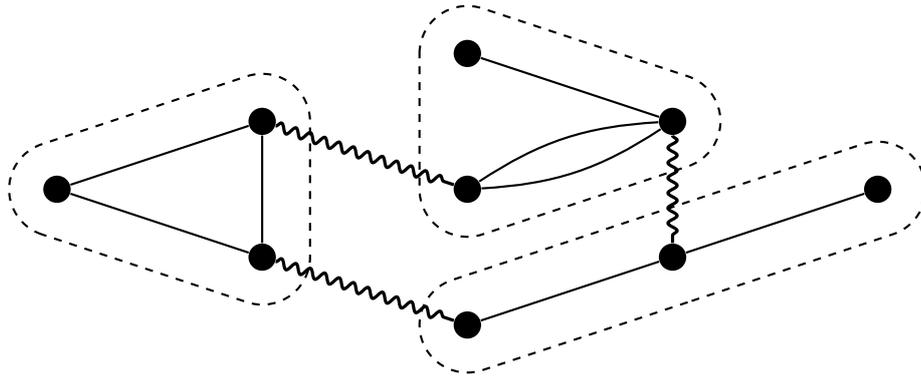


Figure 2.5: A graph G for which Cut has a winning strategy, by coloring the curly edges while possible.

$V(e')$ in T_1 and T_2 to get two edge-disjoint spanning arborescences of G' . By induction, Join has a winning strategy in G' and can choose the corresponding edges in G to have a winning strategy in G .

On the other hand, if G does not have a packing of 2 spanning arborescences, then, by [Theorem 2.9](#), there exists a partition \mathcal{V} of the vertices of G such that $e(\mathcal{V}) < 2(|\mathcal{V}| - 1)$. Let C be the set of edges of G with end vertices in two different elements of \mathcal{V} . Then Cut colors, while possible, an edge of C , see [Figure 2.5](#). When this is not possible, Cut colors any uncolored edge. Since Cut starts, he colors at least half the edges of C and therefore Join colors at most $(|\mathcal{V}| - 2)$ edges of C , implying that, when all edges are colored, the red subgraph of G does not contain any spanning tree and hence is not connected. Therefore Cut wins.

2.8 Matroid theory

Matroids were introduced by Whitney in [Whi35], as an abstract generalization of the concept of linear independence in vector spaces, in the same way that the metric spaces generalize the notion of distance. Matroids also generalize the exchange property of spanning trees observed in [Lemma 2.5](#). Edmonds, in [EF65], underlined the importance of matroids in the field of combinatorial optimization: a lot of results can fruitfully be expressed in terms of matroids. We refer to the book of Oxley ([Oxl06]) for a deep study of matroid theory and for the proofs of most of the stated facts about matroids.

2.8.1 Definitions

The following definition, although the most classic, is only one of the numerous possible characterizations of matroids:

Definition 2.21. A *matroid* \mathcal{M} is a pair (S, \mathcal{I}) such that S is a set (called *ground set*) and

\mathcal{I} is a set of subsets (called **independents**) of S such that:

1. $\emptyset \in \mathcal{I}$.
2. $J \in \mathcal{I}$ and $I \subseteq J \implies I \in \mathcal{I}$.
3. $I \in \mathcal{I}, J \in \mathcal{I}, |I| < |J| \implies \exists j \in J - I, I + j \in \mathcal{I}$.

A **loop** is an element $s \in S$ such that $\{s\} \notin \mathcal{I}$. Loops in matroid will be uninteresting for us and thus we will often consider matroids without loop.

Using 3., we see that all maximal independent sets of \mathcal{M} have the same size, called the **rank** of \mathcal{M} and denoted by $r(\mathcal{M})$. These maximal sets are called **bases**. The bases of a matroid \mathcal{M} determine \mathcal{M} in the sense that given the set of bases of \mathcal{M} , the independent sets of \mathcal{M} are exactly the subsets of the bases of \mathcal{M} .

Lemma 2.6. *Let S be a set and $\mathcal{B} \subseteq 2^S$. Then \mathcal{B} is the set of bases of a matroid if and only if:*

1. $\mathcal{B} \neq \emptyset$.
2. $B_1, B_2 \in \mathcal{B} \implies \forall b_1 \in B_1 - B_2, \exists b_2 \in B_2 - B_1$ such that $B_1 - b_1 + b_2 \in \mathcal{B}$.

Hence Lemma 2.5, together with Lemma 2.6, give a first fundamental example of matroid: the matroid whose bases are the edge sets of spanning trees of a given connected graph, called a **graphic matroid**. Note that the loops of a connected graph and the loops of the corresponding graphic matroid coincide.

Definition 2.22. *Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid. The **rank** of $X \subseteq S$, denoted by $r_{\mathcal{M}}(X)$, is the maximum size of an independent subset of X . For $X \subseteq S$, the **closure** of X is defined by*

$$\text{Span}_{\mathcal{M}}(\mathbf{X}) = \{s \in S : r_{\mathcal{M}}(X \cup s) = r_{\mathcal{M}}(X)\}.$$

If X contains exactly two elements a, b we will also write $[\mathbf{a}, \mathbf{b}]$ for $\text{Span}_{\mathcal{M}}(\{a, b\})$ and $]\mathbf{a}, \mathbf{b}[$ for $\text{Span}_{\mathcal{M}}(\{a, b\}) - a - b$.

The following property of the rank function will be used a lot:

Lemma 2.7. *If $\mathcal{M} = (S, \mathcal{I})$ is a matroid then $r_{\mathcal{M}}$ is a submodular function.*

Lemma 2.8. *Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid and let $X, Y \subseteq S$. Then:*

1. $\text{Span}_{\mathcal{M}}(\text{Span}_{\mathcal{M}}(X)) = \text{Span}_{\mathcal{M}}(X)$.
2. $X \subseteq Y \implies \text{Span}_{\mathcal{M}}(X) \subseteq \text{Span}_{\mathcal{M}}(Y)$.
3. $\text{Span}_{\mathcal{M}}(X) \cup \text{Span}_{\mathcal{M}}(Y) \subseteq \text{Span}_{\mathcal{M}}(X \cup Y)$.

Definition 2.23. Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid and $s_1, s_2 \in S$. s_1 and s_2 are **parallel**, denoted by $s_1 \parallel s_2$, if $s_1 = s_2$ or $\{s_1, s_2\} \notin \mathcal{I}$.

If \mathcal{M} has no loop then \parallel is an equivalence relation: in particular if $\{s_1, s_2\} \notin \mathcal{I}$ and $\{s_2, s_3\} \notin \mathcal{I}$ for three different elements $s_1, s_2, s_3 \in S$ then $\{s_1, s_3\} \notin \mathcal{I}$.

Therefore we can define parallel classes for loopless matroids: the **parallel class** of $s \in S$ is the set of all elements of S parallel to s . If $X \subseteq S$ then \mathbf{X}^\parallel denotes the set of all elements of S parallel to an element of X .

Again, in graphic matroids, the notions of parallel edges and parallel matroid elements coincide.

2.8.2 Operations on matroids

We describe some operations to construct new matroids from olds.

Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid and $X \subseteq S$:

- A **parallel extension** of \mathcal{M} is a matroid obtained by adding parallel elements to \mathcal{M} .
- The **restriction** of \mathcal{M} on X is the matroid $\mathcal{M}[X] = (X, \mathcal{I}[X])$ with $\mathcal{I}[X] = \{I \in \mathcal{I} : I \subseteq X\}$.
- The **deletion** of X in \mathcal{M} is the matroid $\mathcal{M} - X = \mathcal{M}[S - X]$.
- The **dual** of \mathcal{M} is the matroid $\mathcal{M}^* = (S, \mathcal{I}^*)$ determined by the set of bases $\{S - B : B \in \mathcal{B}(\mathcal{M})\}$, where $\mathcal{B}(\mathcal{M})$ is the set of bases of \mathcal{M} .
- Let $\mathcal{M}_i = (S, \mathcal{I}_i)$, $i \in \{1, \dots, p\}$ be p matroids on the same ground set S . The **sum** of $\mathcal{M}_1, \dots, \mathcal{M}_p$ is the matroid $\mathcal{M}_1 + \dots + \mathcal{M}_p = (S, \mathcal{I}_{\mathcal{M}_1 + \dots + \mathcal{M}_p})$ with:

$$\mathcal{I}_{\mathcal{M}_1 + \dots + \mathcal{M}_p} = \{I_1 \cup \dots \cup I_p : I_j \in \mathcal{I}_j, \forall j \in \{1, \dots, p\}\}.$$

- Let $\mathcal{M}_i = (S_i, \mathcal{I}_i)$, $i \in \{1, \dots, p\}$ be p matroids on disjoint ground sets (that is to say, $S_i \cap S_j = \emptyset \forall i \neq j$). The **direct sum** of $\mathcal{M}_1, \dots, \mathcal{M}_p$ is the matroid $\mathcal{M}_1 \oplus \dots \oplus \mathcal{M}_p = (S_1 \cup \dots \cup S_p, \mathcal{I}_{\mathcal{M}_1 \oplus \dots \oplus \mathcal{M}_p})$ with:

$$\mathcal{I}_{\mathcal{M}_1 \oplus \dots \oplus \mathcal{M}_p} = \{I_1 \cup \dots \cup I_p : I_j \in \mathcal{I}_j, \forall j \in \{1, \dots, p\}\}.$$

This last construction gives raise to a notion of connectivity in matroids:

Definition 2.24. Let \mathcal{M} be a matroid. \mathcal{M} is **connected** if \mathcal{M} can not be written as the direct sum of two non-empty matroids.

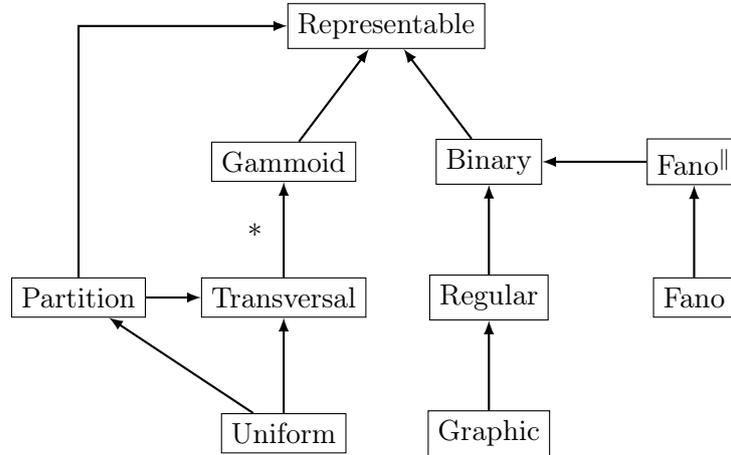


Figure 2.6: A graph of some matroid classes. $M_1 \longrightarrow M_2$ means that the matroid class M_1 is included in the matroid class M_2 . $M_1 \overset{*}{\longrightarrow} M_2$ means that M_1 and M_2 are dual from each other.

2.8.3 Examples of matroids

The following matroids are among the most studied ones. Their relations are summarized on Figure 2.6. Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid:

- **Free matroid:** \mathcal{M} is a free matroid if all subsets of S are independent.
- **Uniform matroid:** \mathcal{M} is a k -uniform matroid ($k \in \mathbb{N}^*$) if \mathcal{I} is the set of all subsets of S of size at most k .
- **Partition matroid:** \mathcal{M} is a partition matroid if S can be partitioned into disjoint sets $S = S_1 \cup \dots \cup S_l$ and there exist $k_1, \dots, k_l \in \mathbb{N}$ such that

$$\mathcal{I} = \{S' \subseteq S : |S' \cap S_i| \leq k_i \text{ for all } i \in \{1, \dots, l\}\}.$$

Clearly, uniform matroids are also partition matroids. Note that \mathcal{M} would not necessarily be a matroid if the S_i were not disjoint.

- **Transversal matroid:** \mathcal{M} is a transversal matroid if there exists a family (S_1, \dots, S_l) of (not necessarily disjoint) subsets of S such that \mathcal{I} is the set of **partial transversals** of (S_1, \dots, S_l) , that is to say the sets $I = \{i_1, \dots, i_p\}$ for which there exists an injective map $\sigma : \{1, \dots, p\} \longrightarrow \{1, \dots, l\}$ with $i_j \in S_{\sigma(j)}$. If $|I| = l$ then I is a **transversal**. We can assume that $l = r(\mathcal{M})$ (see [Oxl06]).
- **Matching matroid:** \mathcal{M} is a matching matroid if there exists a (not necessarily bipartite) graph $G = (S, E)$ such that the independents are the sets of vertices covered by some matching in G . It was shown to be equivalent to transversal matroids by Edmonds in [EF65].

- **Graphic matroid:** \mathcal{M} is a graphic matroid if there exists a graph $G = (S, E)$ such that $I \in \mathcal{I}$ if I is a set of edges of a forest in G . If G is connected, the bases of G are the edge sets of spanning trees of G . Every graphic matroid can be expressed as the graphic matroid of a connected graph, so we *will always assume that G is connected*.
- **Gammoid:** \mathcal{M} is a gammoid if there exists a digraph $\vec{G} = (V, \vec{E})$ and $T \subseteq V$ such that $S \subseteq V$ and $I \in \mathcal{I}$ if $I \subseteq S$ and there exist $|I|$ vertex-disjoint dipaths from I to T . Gammoids were introduced to study connectivity.
- **Arc-disjoint gammoid:** \mathcal{M} is an arc-disjoint gammoid if there exists a digraph $\vec{G} = (V, \vec{E})$ and $s, t \in V$ such that $S = \vec{E}^+(s)$ and $I \in \mathcal{I}$ if I is the set of first arcs of a packing of $s - t$ dipaths. The following straightforward lemmas show that arc-disjoint gammoids are equivalent to gammoids:

Lemma 2.9. *Let \mathcal{M} be a gammoid with associated digraph $\vec{G} = (V, \vec{E})$ and sets $S, T \subseteq V$.*

Then \mathcal{M} is also the arc-disjoint gammoid with associated digraph (V', \vec{E}') and $s, t \in V'$ such that:

$$V' = \{v_1 : v \in V\} \cup \{v_2 : v \in V\} \cup \{s, t\},$$

$$\vec{E}' = \{(s, v_1) : v \in S\} \cup \{(v_2, t) : v \in T\} \cup \{(u_2, v_1) : (u, v) \in \vec{E}\} \cup \{(v_1, v_2) : v \in V\}.$$

Lemma 2.10. *Let \mathcal{M} be an arc-disjoint gammoid with associated digraph $\vec{G} = (V, \vec{E})$ and $s, t \in V$.*

Then \mathcal{M} is also the gammoid associated to the line digraph $L(\vec{G}) = (V', \vec{E}')$ with $S = \vec{E}^+(s)$ and $T = \vec{E}^-(t)$.

$L(\vec{G})$ is defined by:

$$V' = \vec{E},$$

$$\vec{E}' = \{(\vec{e}_1, \vec{e}_2) : \vec{e}_1^+ = \vec{e}_2^-\}.$$

- **Representable matroid:** \mathcal{M} is a representable matroid on a field \mathbb{F} if there exists a matrix M of size $r(\mathcal{M}) \times |S|$ such that every element of S corresponds to a column of M and $I \in \mathcal{I}$ if the columns corresponding to the elements of I are linearly independent on \mathbb{F} . We can always assume that $M = [I_{r(\mathcal{M})} | M']$, $I_{r(\mathcal{M})}$ being the identity matrix of size $r(\mathcal{M}) \times r(\mathcal{M})$ and M' a matrix (see [Oxl06]).
- **Binary matroid:** \mathcal{M} is a binary matroid if it is representable on \mathbb{F}_2 , the field on 2 elements.
- **Regular matroid:** \mathcal{M} is a regular matroid if it is representable on every field.
- **Fano matroid:** The Fano matroid \mathcal{F}_7 is a matroid of rank 3 with 7 elements, and with bases all 3-tuples of elements not all on a line in [Figure 2.7](#) (an element of the matroid being represented by a vertex). Note that the circle on [Figure 2.7](#) is considered as a line. Equivalently, the Fano matroid is the binary matroid represented by the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

\mathcal{F}_7 is not regular. We will also be interested in the class of parallel extensions of the Fano matroid, denoted by Fano^{\parallel} on Figure 2.6.

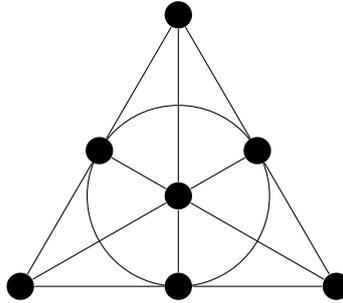


Figure 2.7: The Fano matroid \mathcal{F}_7 : the elements of \mathcal{F}_7 are the vertices and the bases are the 3-tuples of vertices not all on a line.

2.8.4 Optimization

One of the most attractive features of matroids resides in how easy it is to optimize over them. To be more precise, let $\mathcal{M} = (S, \mathcal{I})$ be a matroid and $w : S \rightarrow \mathbb{R}^+$ a weight function. We define the **weight** of $I \in \mathcal{I}$ by $w(I) = \sum_{s \in I} w(s)$. It is natural to look for a base with minimum weight: for example, the minimum spanning tree of a connected graph corresponds to a minimum weight base of the corresponding graphic matroid. The greedy algorithm of Kruskal ([Kru56]), giving an optimal spanning tree by making optimal local choices, can be generalized to matroids:

Theorem 2.10 (Greedy algorithm). *Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid and w a weight function on \mathcal{M} . The following greedy algorithm returns a base B of \mathcal{M} with minimum weight:*

1. $B \leftarrow \emptyset$.
2. While possible, add $s \in S$ to B with minimum weight such that $B + s \in \mathcal{I}$.

Rado ([Rad57]) and Edmonds ([Edm71]) proved that the matroids can be characterized as the structures for which the greedy algorithm leads to a minimum base, for every weight function w .

Observe that the above algorithm needs to test whether $B + s \in \mathcal{I}$. We will say that a matroid has a **polynomial time oracle** if there exists an algorithm deciding if a set is independent in time polynomial in $|S|$. For example, it is possible to check if a set of edges is acyclic in linear time, using a depth-first search algorithm, hence any graphic matroid has a polynomial time oracle. If such a polynomial time oracle exists, the greedy algorithm is a polynomial algorithm to find a minimum weight base of a matroid.

The greedy algorithm also gives a polynomial algorithm to the following problem:

Problem 2.1. Let G be a weighted graph and $k \in \mathbb{N}^*$.
Find a minimum weight packing of k spanning trees, if it exists.

Indeed, if \mathcal{M}_G is the graphic matroid defined by G , $k \times \mathcal{M}_G = \underbrace{\mathcal{M}_G + \dots + \mathcal{M}_G}_k$ is a matroid. Edmonds proved in [Edm65a] that if some matroids have polynomial time oracles, then their sum also have a polynomial time oracle. Hence $k \times \mathcal{M}_G$ has a polynomial time oracle. Observe that $k \times \mathcal{M}_G$ has rank $k(|G| - 1)$ if and only if G has a packing of k spanning trees. If this is the case, every base of $k \times \mathcal{M}_G$ is a packing of k spanning trees and therefore the greedy algorithm on $k \times \mathcal{M}_G$ indeed gives a minimum weight packing of k spanning trees. One may note that we did not speak about arborescences since we introduced matroids. This is for good reason: the set of arcs of spanning r -arborescences of an r -rooted digraph is not necessarily the set of bases of a matroid, as shown by Figure 2.8.

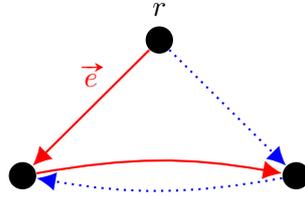


Figure 2.8: Two r -arborescences: \vec{T}_1 , and \vec{T}_2 , dotted. It is not possible to add an arc of \vec{T}_2 to $\vec{T}_1 - \vec{e}$ to get an r -arborescence.

However, by Theorem 2.4, a set of arcs \vec{E}' in a digraph $\vec{G} = (V, \vec{E})$ corresponds to a spanning r -arborescence of \vec{G} if and only if E' is a tree in G , every in-star at $v \neq r$ intersects \vec{E}' in one arc and the in-star at r is disjoint from \vec{E}' . Said differently:

$$\{\text{Sets of arcs of spanning arborescences of } \vec{G}\} = \mathcal{B}_1 \cap \mathcal{B}_2,$$

with:

$$\mathcal{B}_1 = \{\vec{E}' \subseteq \vec{E} : E' \text{ is the edge set of a spanning tree of } G\},$$

$$\mathcal{B}_2 = \{\vec{E}' \subseteq \vec{E} : |r^- \cap \vec{E}'| = 0 \text{ and } |v^- \cap \vec{E}'| = 1 \text{ if } v \neq r\}.$$

Clearly, \mathcal{B}_1 is the set of bases of a graphic matroid and \mathcal{B}_2 is the set of bases of a partition matroid. More generally, let $\mathcal{M}_1 = (S, \mathcal{I}_1)$ and $\mathcal{M}_2 = (S, \mathcal{I}_2)$ be two matroids, and consider the set $\mathcal{I}_1 \cap \mathcal{I}_2$ of **common independent sets**. Edmonds discovered in [Edm70] that a minimum weight common independent set of \mathcal{M}_1 and \mathcal{M}_2 can be found in polynomial time, provided polynomial time oracles for \mathcal{M}_1 and \mathcal{M}_2 . In particular, this gives a polynomial algorithm to find a minimum weight spanning arborescence. Moreover:

$$\{\text{Sets of arcs of packings of } k \text{ spanning arborescences of } \vec{G}\} = (k \times \mathcal{B}_1) \cap (k \times \mathcal{B}_2),$$

with:

$$k \times \mathcal{B}_1 = \{\vec{E}' \subseteq \vec{E} : E' \text{ is the edge set of a packing of } k \text{ spanning trees of } G\},$$

$$k \times \mathcal{B}_2 = \{\vec{E}' \subseteq \vec{E} : |r^- \cap \vec{E}'| = 0 \text{ and } |v^- \cap \vec{E}'| = k \text{ if } v \neq r\}.$$

Therefore the following problem can be solved in polynomial time:

Problem 2.2. *Let \vec{G} be a weighted digraph rooted in r and $k \in \mathbb{N}^*$. Find a minimum weight packing of k spanning r -arborescences, if it exists.*

We derive a second algorithm for [Problem 2.2](#), based on polyhedral considerations initiated in [BF09]. We refer to [Sch98] for the terminology in linear programming theory. We need the following theorem of Frank ([Fra79]):

Theorem 2.11. *Let $\vec{G} = (V, \vec{E})$ be a digraph, $h : 2^V \rightarrow \mathbb{N}$ a non-negative intersecting supermodular set-function such that $\rho_{\vec{G}}(U) \geq h(U)$ for every $U \subseteq V$. Then the polyhedron P in $\mathbb{R}^{|\vec{E}|}$ defined by the following linear system is integer:*

$$\begin{aligned} 0 \leq x(\vec{e}) \leq 1, & \quad \text{for all } \vec{e} \in \vec{E}. \\ x(U^-) \geq h(U), & \quad \text{for all } U \subseteq V. \end{aligned}$$

Moreover, P can be **separated** in polynomial time (that is to say it can be decided in polynomial time if $x \in P$), since it suffices to verify that the minimum of the submodular function $U \mapsto x(U^-) - h(U)$ is positive. In particular, if $\vec{G} = (V, \vec{E})$ is a digraph and $r \in V$, the polyhedron P defined by the following linear system is integer:

$$\begin{aligned} 0 \leq x(\vec{e}) \leq 1, & \quad \text{for all } \vec{e} \in \vec{E}. \\ x(U^-) \geq k, & \quad \text{for all } U \subseteq V - r, U \neq \emptyset. \end{aligned}$$

Then, by [Theorem 2.8](#), P is the convex hull of subsets of \vec{E} containing a packing of k spanning r -arborescences of \vec{G} . If $x \in P$, observe that $x(V) \geq k(|V| - 1)$ since $x(v) \geq k, \forall v \in V - r$. Therefore the equality $x(V) = k(|V| - 1)$ defines a face F of P whose vertices are exactly the packings of k spanning arborescences. Hence the ellipsoid algorithm of Grötschel, Lovász, and Schrijver ([GLS81]), applied on F , provides a polynomial algorithm for [Problem 2.2](#):

Theorem 2.12 (Ellipsoid algorithm). *Let P be a polyhedron in \mathbb{R}^d and $w \in \mathbb{R}^d$. If P can be separated in polynomial time and provided a vertex of P , a vertex x of P of minimum weight $x^t w$ can be computed in polynomial time.*

Connectivity and packing with matroid constraints

Contents

3.1	Connectivity with matroid constraints	26
3.1.1	Definitions	26
3.1.2	Orientation	27
3.1.3	Cuts and tight sets	28
3.1.4	Splitting and switching operations	29
3.1.5	Menger's theorem for \mathcal{M} -digraphs	32
3.2	Packing of arborescences with matroid constraints	34
3.2.1	Definition	34
3.2.2	Conjectures	35
3.2.3	Complexity	36
3.2.4	Matroid on vertices	37
3.3	Reductions for \mathcal{M}-digraphs	38
3.3.1	Minimal \mathcal{M} -arc-connectivity	39
3.3.2	Two reductions using switching	40
3.3.3	Complete switching reduction	43
3.4	Packing of branches	45
3.5	Matroids of rank two	46
3.5.1	Structure of matroids of rank two	47
3.5.2	Exchange lemma	47
3.5.3	Complexity	50

We introduce a new connectivity notion, with matroid constraints, and investigate extensions of Menger's theorem and Edmond's packing theorem to connectivity with matroid constraints.

3.1 Connectivity with matroid constraints

3.1.1 Definitions

In chapter 2, we saw that connectivity between two vertices u and v in a digraph can be studied in terms of gammoids. It seems natural, in the same way that arborescences arise from trees by intersection with a matroid, to look at the intersection of a gammoid with another matroid. This amounts to considering dipaths with the additional constraint that the first arcs are independent in a given matroid. This leads to the following definition:

Definition 3.1. An \mathcal{M} -digraph is a 3-tuple $\vec{\mathcal{G}} = (\vec{G}, \mathcal{M}, r)$ with $\vec{G} = (V, \vec{E})$ a digraph, $r \in V$ a root and $\mathcal{M} = (r^+, \mathcal{I})$ a matroid defined on the arcs leaving r .

Given such an \mathcal{M} -digraph we define $r_{\vec{\mathcal{G}}}(\mathbf{X}) = r_{\mathcal{M}}(\vec{E}(r, X))$ for $X \subseteq V - r$ and $\text{Span}_{\vec{\mathcal{G}}}(\mathbf{X}) = \text{Span}_{\mathcal{M}}(\vec{E}(r, X))$.

Since $r_{\mathcal{M}}$ is submodular and $\vec{E}(r, X) \cup \vec{E}(r, Y) = \vec{E}(r, X \cup Y)$, $\vec{E}(r, X) \cap \vec{E}(r, Y) = \vec{E}(r, X \cap Y)$, $r_{\vec{\mathcal{G}}}$ is submodular. Pay attention, in the previous definition, that an element of \mathcal{M} can be considered as a matroid element or as an arc in \vec{G} , depending on the context. To avoid confusion we will often specify how we consider such an element.

Definition 3.2. Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ be an \mathcal{M} -digraph, $V' \subseteq V - r$ and $\vec{E}' \subseteq \vec{E}$. We define $\vec{\mathcal{G}} - V' = (\vec{G} - V', \mathcal{M} - V', r)$ with $\mathcal{M} - V' = \mathcal{M} - \vec{E}(r, V')$. Similarly, we define $\vec{\mathcal{G}} - \vec{E}' = (\vec{G} - \vec{E}', \mathcal{M} - \vec{E}', r)$.

Note that $\mathcal{M} - \vec{E}'$ is defined even if some arcs of \vec{E}' are not elements of \mathcal{M} , in which case we simply ignore them. For simplicity, we introduce a new notation: if $S = \{s_1, \dots, s_k\}$ is a set and $*$ is a unary operation defined on the elements s_i , we define $\mathbf{S}^* = \{s_1^*, \dots, s_k^*\}$. This is used in the following definition, with $\mathbf{S}^* = \vec{\mathcal{P}}^-$:

Definition 3.3. Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ be an \mathcal{M} -digraph and $v \in V - r$. We say that $\vec{\mathcal{P}}$ is an \mathcal{M} -packing of $r - v$ dipaths of $\vec{\mathcal{G}}$ if $\vec{\mathcal{P}}$ is a packing of $r - v$ dipaths and $\vec{\mathcal{P}}^-$ is a base of \mathcal{M} .

Given an \mathcal{M} -digraph $(\vec{G}, \mathcal{M}, r)$, an \mathcal{M} -packing of $r - v$ dipaths can be seen as a common base of \mathcal{M} and the arc-disjoint gammoid of $r - v$ dipaths in \vec{G} . Therefore, the matroid intersection algorithm of Edmonds decides in polynomial time whether an \mathcal{M} -packing of $r - v$ dipaths exists, given an oracle for \mathcal{M} .

Definition 3.4. An \mathcal{M} -digraph $(\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ is \mathcal{M} -arc-connected if it contains an \mathcal{M} -packing of $r - v$ dipaths for all $v \in V - r$.

For example, if \mathcal{M} is a k -uniform matroid, $(\vec{G}, \mathcal{M}, r)$ is \mathcal{M} -arc-connected if and only if \vec{G} is k -arc-connected from r . Hence \mathcal{M} -arc-connectivity is a generalization of rooted k -arc-connectivity.

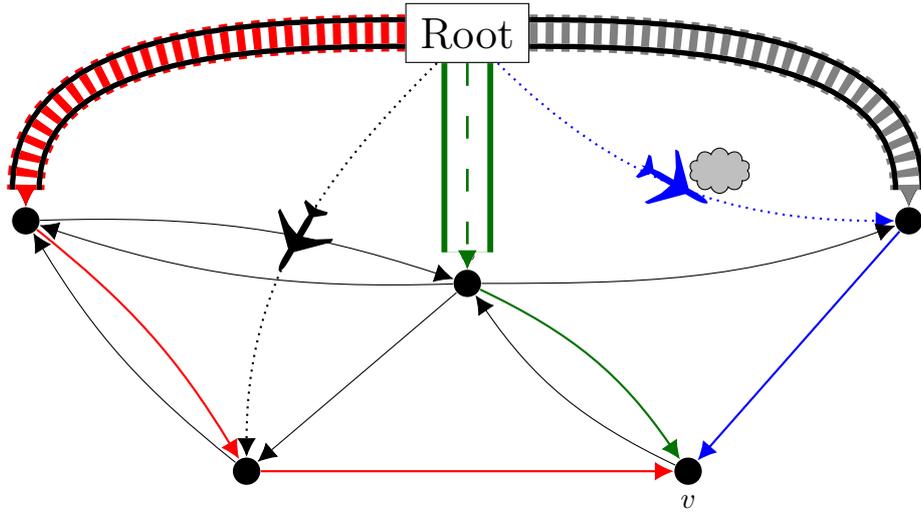


Figure 3.1: The neighbourhood cities of Root.

We now provide an example as a motivation for studying \mathcal{M} -arc-connectivity: Assume that the mayor of a city called Root wants to improve the transport network of Root. In particular, a lot of citizens of Root work in neighbouring cities and are exasperated with the frequent congestions and strikes when leaving Root. Therefore the mayor wants as many different ways of transport as possible to go from Root to other cities, such that if for example there is a train strike, the citizens can use, say, a bus. Ideally the dipaths are disjoint, to avoid congestions.

Let \vec{G} be the digraph with vertices Root and its neighbouring cities, and with each arc corresponding to a way to go from one city to another. Thus the mayor would like \vec{G} to be \mathcal{M} -arc-connected from Root, where \mathcal{M} is the transversal matroid defined by the different means of transport leaving Root. For example, on Figure 3.1, the city v can be reached from Root using train, road or airplane. One can verify that the \mathcal{M} -digraph of Figure 3.1 is indeed \mathcal{M} -arc-connected.

3.1.2 Orientation

We saw, thanks to Theorem 2.6, that the k -partition-connected graphs are exactly those admitting a rooted k -arc-connected orientation. It is natural to expect a similar relation between \mathcal{M} -arc-connected digraphs and partition-connected graphs with matroid constraints, defined as follow:

Definition 3.5. An \mathcal{M} -graph is a 3-tuple $\mathcal{G} = (G, \mathcal{M}, r)$ with $G = (V, E)$ a graph, $r \in V$ and $\mathcal{M} = (E(r), \mathcal{I})$ a matroid defined on the edges adjacent to r . Given such an \mathcal{M} -graph we define $r_{\mathcal{G}}(X) = r_{\mathcal{M}}(E(r, X))$ for $X \subseteq V - r$ and $\text{Span}_{\mathcal{G}}(X) = \text{Span}_{\mathcal{M}}(E(r, X))$.

Again, the function $r_{\mathcal{G}}$ is submodular.

Definition 3.6. Let $\mathcal{G} = (G = (V, E), \mathcal{M}, r)$ be an \mathcal{M} -graph. We say that \mathcal{G} is **\mathcal{M} -partition-connected** if, for every partition \mathcal{V} of $V - r$:

$$e_G(\mathcal{V}) \geq \sum_{X \in \mathcal{V}} (r(\mathcal{M}) - r_{\mathcal{G}}(X)).$$

Theorem 3.1. Let $\mathcal{G} = (G, \mathcal{M}, r)$ be an \mathcal{M} -graph. There exists an \mathcal{M} -arc-connected orientation $\vec{\mathcal{G}}$ of \mathcal{G} if and only if \mathcal{G} is \mathcal{M} -partition-connected.

Proof. results from [Theorem 2.6](#), with the following intersecting supermodular function h :

$$h(X) = \begin{cases} r(\mathcal{M}) - r_{\mathcal{G}}(X) & \text{if } r \notin X, X \neq \emptyset. \\ 0 & \text{otherwise.} \end{cases}$$

□

3.1.3 Cuts and tight sets

Our first goal is to generalize Menger's theorem to \mathcal{M} -digraphs.

Definition 3.7. Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ be an \mathcal{M} -digraph. We define $\rho_{\vec{\mathcal{G}}}$ on the r -cuts X of $\vec{\mathcal{G}}$ by:

$$\rho_{\vec{\mathcal{G}}}(X) = \rho_{\vec{G}-r}(X) + r_{\vec{\mathcal{G}}}(X)$$

The **\mathcal{M} -value** of an r -cut X is $\rho_{\vec{\mathcal{G}}}(X)$. An r -cut X is **tight** for $\vec{\mathcal{G}}$ if $\rho_{\vec{\mathcal{G}}}(X) = r(\mathcal{M})$. An **\mathcal{M} -cut** is an r -cut of \mathcal{M} -value strictly less than $r(\mathcal{M})$.

We also define $\rho_{\vec{\mathcal{G}},v}$ as the restriction of $\rho_{\vec{\mathcal{G}}}$ on $r - v$ cuts.

Since $\rho_{\vec{\mathcal{G}}}$ is the sum of two submodular functions, $\rho_{\vec{\mathcal{G}}}$ is a submodular function on the set of $r - v$ cuts. Then $\rho_{\vec{\mathcal{G}},v}$ is also a submodular function, as the restriction of a submodular function. Observe that if $\vec{\mathcal{G}}$ is an \mathcal{M} -digraph without \mathcal{M} -cut, then a tight r -cut of $\vec{\mathcal{G}}$, if it exists, is a set minimizing $\rho_{\vec{\mathcal{G}}}$. Hence [Lemma 2.3](#) implies:

Lemma 3.1. Let X_1 and X_2 be two intersecting tight r -cuts of an \mathcal{M} -digraph $\vec{\mathcal{G}}$ without \mathcal{M} -cut. Then $X_1 \cup X_2$ and $X_1 \cap X_2$ are also tight r -cuts of $\vec{\mathcal{G}}$.

Similarly, tight $r - v$ cuts minimize $\rho_{\vec{\mathcal{G}},v}$ (obviously, two $r - v$ cuts always have a non empty intersection, so we do not need to assume that X_1 and X_2 are intersecting):

Lemma 3.2. Let X_1 and X_2 be two tight $r - v$ cuts of an \mathcal{M} -digraph $\vec{\mathcal{G}}$ without $r - v$ \mathcal{M} -cut. Then $X_1 \cup X_2$ and $X_1 \cap X_2$ are also tight $r - v$ cuts of $\vec{\mathcal{G}}$.

In particular, if there exists at least one tight cut then the maximal and the minimal tight cuts containing a vertex v are uniquely defined, as the union and the intersection of all tight cuts containing v , respectively.

3.1.4 Splitting and switching operations

In this subsection we study reductions on \mathcal{M} -arc-connected \mathcal{M} -digraphs that will allow us to prove theorems on \mathcal{M} -arc-connected \mathcal{M} -digraphs by induction.

The following lemma is a first trivial reduction:

Lemma 3.3. *Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ be an \mathcal{M} -digraph without \mathcal{M} -cut and $\vec{e} \in \vec{E}$ such that \vec{e} does not enter any tight r -cut of $\vec{\mathcal{G}}$. Then $\vec{\mathcal{G}} - \vec{e}$ is without $(\mathcal{M} - \vec{e})$ -cut.*

Proof. Assume for the sake of contradiction that $\vec{\mathcal{G}} - \vec{e}$ has an $(\mathcal{M} - \vec{e})$ -cut X . If \vec{e} does not enter X in $\vec{\mathcal{G}}$ then $\rho_{\vec{\mathcal{G}} - \vec{e}}(X) = \rho_{\vec{\mathcal{G}}}(X) \geq r(\mathcal{M}) \geq r(\mathcal{M} - \vec{e})$. If \vec{e} enters X then X is not a tight set so $\rho_{\vec{\mathcal{G}}}(X) > r(\mathcal{M})$. This implies $\rho_{\vec{\mathcal{G}} - \vec{e}}(X) = \rho_{\vec{\mathcal{G}}}(X) - 1 > r(\mathcal{M}) - 1 \geq r(\mathcal{M} - \vec{e})$. In both cases X is not an $(\mathcal{M} - \vec{e})$ -cut. \square

The previous lemma has also the following version for $r - v$ cuts:

Lemma 3.4. *Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ be an \mathcal{M} -digraph without $r - v$ \mathcal{M} -cut for a fixed $v \in V - r$ and $\vec{e} \in \vec{E}$ such that \vec{e} does not enter any tight $r - v$ cut of $\vec{\mathcal{G}}$. Then $\vec{\mathcal{G}} - \vec{e}$ is without $r - v$ $(\mathcal{M} - \vec{e})$ -cut.*

The following operation on \mathcal{M} -digraphs is the natural extension of the splitting operation on digraphs introduced in [Definition 2.19](#).

Definition 3.8. *Let $\vec{\mathcal{G}} = (\vec{G}, \mathcal{M}, r)$ be an \mathcal{M} -digraph and $\vec{e}_1, \vec{e}_2 \in \vec{E}$ be such that $\vec{e}_1^+ = \vec{e}_2^- \neq r$. We define $\vec{\mathcal{G}}_{\vec{e}_1, \vec{e}_2} = (\vec{G}_{\vec{e}_1, \vec{e}_2}, \mathcal{M}_{\vec{e}_1, \vec{e}_2}, r)$, and we say that $\vec{\mathcal{G}}_{\vec{e}_1, \vec{e}_2}$ is obtained by **\mathcal{M} -splitting** \vec{e}_1 and \vec{e}_2 , such that:*

- If $\vec{e}_1^- \neq r$: $\mathcal{M}_{\vec{e}_1, \vec{e}_2} = \mathcal{M}$.
- If $\vec{e}_1^- = r$: $\mathcal{M}_{\vec{e}_1, \vec{e}_2}$ is obtained from \mathcal{M} by replacing \vec{e}_1 by $\vec{e}_1 \cdot \vec{e}_2$.

We will also need the following very similar operation, called \mathcal{M} -switching, in which we keep the arc \vec{e}_1 .

Definition 3.9. *Let $\vec{G} = (V, \vec{E})$ be a digraph and $\vec{e}_1, \vec{e}_2 \in \vec{E}$ such that $\vec{e}_1^+ = \vec{e}_2^- \neq r$. An **\mathcal{M} -switching** of \vec{e}_1 and \vec{e}_2 consists in removing \vec{e}_2 in \vec{G} and adding a new arc denoted by $\vec{e}_1 \odot \vec{e}_2$ such that $V(\vec{e}_1 \odot \vec{e}_2) = (\vec{e}_1^-, \vec{e}_2^+)$. The resulting digraph is denoted by $\vec{G}_{\vec{e}_1 \odot \vec{e}_2}$. If $\vec{\mathcal{G}} = (\vec{G}, \mathcal{M}, r)$ is an \mathcal{M} -digraph then we define $\vec{\mathcal{G}}_{\vec{e}_1 \odot \vec{e}_2} = (\vec{G}_{\vec{e}_1 \odot \vec{e}_2}, \mathcal{M}_{\vec{e}_1 \odot \vec{e}_2}, r)$ with :*

- If $\vec{e}_1^- \neq r$: $\mathcal{M}_{\vec{e}_1 \odot \vec{e}_2} = \mathcal{M}$.
- If $\vec{e}_1^- = r$: $\mathcal{M}_{\vec{e}_1 \odot \vec{e}_2}$ is obtained from \mathcal{M} by adding $\vec{e}_1 \odot \vec{e}_2$ as an element parallel to \vec{e}_1 .

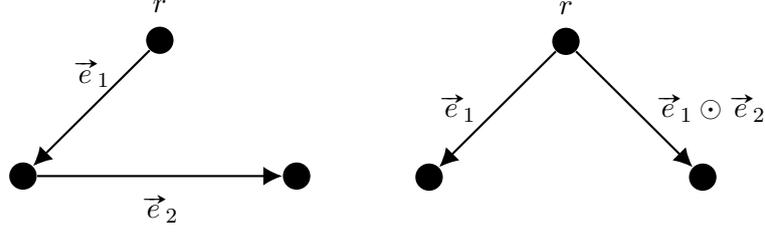


Figure 3.2: Switching of \vec{e}_1 and \vec{e}_2 .
Left: before switching. Right: after switching.

Please note that a splitting does not change the underlying matroid while a switching may add parallel elements. These two operations clearly do not change the rank of the underlying matroid.

We now study how these two operations can violate \mathcal{M} -arc-connectivity.

Lemma 3.5. *Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ be an \mathcal{M} -digraph without \mathcal{M} -cut and $\vec{e}_1, \vec{e}_2 \in \vec{E}$ with $\vec{e}_1^+ = \vec{e}_2^- \neq r$.*

- If $\vec{\mathcal{G}}_{\vec{e}_1, \vec{e}_2}$ has an $\mathcal{M}_{\vec{e}_1, \vec{e}_2}$ -cut X then either \vec{e}_1 or \vec{e}_2 enters X .
- If $\vec{\mathcal{G}}_{\vec{e}_1 \odot \vec{e}_2}$ has an $\mathcal{M}_{\vec{e}_1 \odot \vec{e}_2}$ -cut X then \vec{e}_2 enters X .

Proof.

- If \vec{e}_1 and \vec{e}_2 does not enter X then since all arcs entering X in $\vec{\mathcal{G}}$ also enter X in $\vec{\mathcal{G}}_{\vec{e}_1, \vec{e}_2}$, $\rho_{\vec{\mathcal{G}}}(X) \leq \rho_{\vec{\mathcal{G}}_{\vec{e}_1, \vec{e}_2}}(X)$ which is impossible since X would be an \mathcal{M} -cut in $\vec{\mathcal{G}}$.
- If \vec{e}_2 does not enter X then since all arcs entering X in $\vec{\mathcal{G}}$ also enter X in $\vec{\mathcal{G}}_{\vec{e}_1 \odot \vec{e}_2}$, $\rho_{\vec{\mathcal{G}}}(X) \leq \rho_{\vec{\mathcal{G}}_{\vec{e}_1 \odot \vec{e}_2}}(X)$ which is impossible since X would be an \mathcal{M} -cut in $\vec{\mathcal{G}}$.

□

According to [Lemma 3.5](#), we need to distinguish the cases \vec{e}_1 enters X and \vec{e}_2 enters X , for an $\mathcal{M}_{\vec{e}_1, \vec{e}_2}$ -cut X . The following lemma precises the situation where \vec{e}_2 enters X (see [Figure 3.3](#)):

Lemma 3.6. *Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ be an \mathcal{M} -digraph and $\vec{\mathcal{G}}_{\vec{e}_1, \vec{e}_2}$ be obtained from $\vec{\mathcal{G}}$ after \mathcal{M} -splitting of \vec{e}_1 and \vec{e}_2 . Assume that $\vec{\mathcal{G}}$ has no $r - \vec{e}_2^+$ \mathcal{M} -cut but that $\vec{\mathcal{G}}_{\vec{e}_1, \vec{e}_2}$ has an $r - \vec{e}_2^+$ $\mathcal{M}_{\vec{e}_1, \vec{e}_2}$ -cut X . Then \vec{e}_2 enters X , X is a tight $r - \vec{e}_2^+$ cut in $\vec{\mathcal{G}}$ and:*

1. If $\vec{e}_1^- \neq r$ then $\vec{e}_1^- \in X$.

2. If $\vec{e}_1^- = r$ then $\vec{e}_1 \in \text{Span}_{\vec{\mathcal{G}}}(X)$.

Proof. Let A_X and A'_X be the set of arcs entering X in $\vec{\mathcal{G}}$ and $\vec{\mathcal{G}}_{\vec{e}_1, \vec{e}_2}$, respectively. Assume that \vec{e}_2 does not enter X . Then $\vec{e}_2^- \in X$ since $\vec{e}_2^+ \in X$ by assumption. If $\vec{e}_1^- \in X$ then \vec{e}_1 and $\vec{e}_1 \cdot \vec{e}_2$ does not enter X and $A_X = A'_X$, which is a contradiction since X would be an $r - \vec{e}_2^+$ \mathcal{M} -cut of $\vec{\mathcal{G}}$. If $\vec{e}_1^- \notin X$ then \vec{e}_1 and $\vec{e}_1 \cdot \vec{e}_2$ enter X so $A_X = A'_X - \vec{e}_1 \cdot \vec{e}_2 + \vec{e}_1$ and, since $\vec{e}_1 \cdot \vec{e}_2 \parallel \vec{e}_1$ if $\vec{e}_1^- = r$, $\rho_{\vec{\mathcal{G}}}(X) = \rho_{\vec{\mathcal{G}}_{\vec{e}_1, \vec{e}_2}}(X)$ so X is an $r - \vec{e}_2^+$ \mathcal{M} -cut in $\vec{\mathcal{G}}$, a contradiction. Therefore \vec{e}_2 enters X . Moreover since \vec{e}_1 does not enter X , \vec{e}_2 is the only arc entering X in $\vec{\mathcal{G}}$ but not in $\vec{\mathcal{G}}_{\vec{e}_1, \vec{e}_2}$ thus $\rho_{\vec{\mathcal{G}}}(X) \leq \rho_{\vec{\mathcal{G}}_{\vec{e}_1, \vec{e}_2}}(X) + 1 \leq r(\mathcal{M})$, which implies that X is tight.

1. If $\vec{e}_1^- \neq r$ and $\vec{e}_1^- \notin X$ then $\vec{e}_1 \cdot \vec{e}_2$ enters X in $\vec{\mathcal{G}}_{\vec{e}_1, \vec{e}_2}$ so $A_X = A'_X - \vec{e}_1 \cdot \vec{e}_2 + \vec{e}_2$. Since neither \vec{e}_2 nor $\vec{e}_1 \cdot \vec{e}_2$ leaves r , $\rho_{\vec{\mathcal{G}}}(X) = \rho_{\vec{\mathcal{G}}_{\vec{e}_1, \vec{e}_2}}(X)$ thus X is an $r - \vec{e}_2^+$ \mathcal{M} -cut in $\vec{\mathcal{G}}$, which is a contradiction.
2. If $\vec{e}_1^- = r$ and $\vec{e}_1 \notin \text{Span}_{\vec{\mathcal{G}}}(X)$ then since by definition $r \notin X$, $\vec{e}_1 \cdot \vec{e}_2$ enters X in $\vec{\mathcal{G}}_{\vec{e}_1, \vec{e}_2}$ so $A_X = A'_X - \vec{e}_1 \cdot \vec{e}_2 + \vec{e}_2$. Since $\vec{e}_1 \cdot \vec{e}_2 \notin \text{Span}_{\vec{\mathcal{G}}}(X)$, $\rho_{\vec{\mathcal{G}}_{\vec{e}_1, \vec{e}_2}}(X) = r_{\mathcal{M}}(X) + 1$ so $\rho_{\vec{\mathcal{G}}_{\vec{e}_1, \vec{e}_2}}(X) = \rho_{\vec{\mathcal{G}}}(X)$ which again implies that X is an $r - \vec{e}_2^+$ \mathcal{M} -cut in $\vec{\mathcal{G}}$, a contradiction.

□

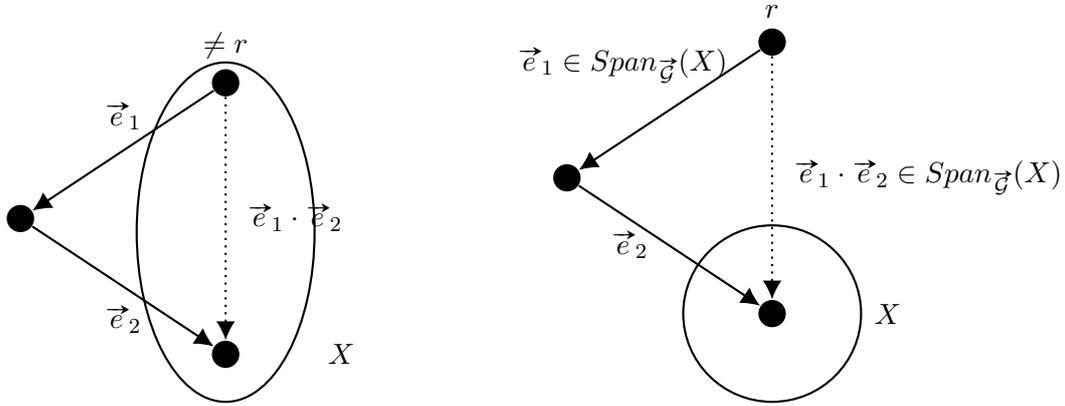


Figure 3.3: The two possibilities in Lemma 3.6.

The following lemma has a very similar proof which we skip.

Lemma 3.7. Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ be an \mathcal{M} -digraph without \mathcal{M} -cut and $\vec{\mathcal{G}}_{\vec{e}_1 \circ \vec{e}_2}$ be obtained from $\vec{\mathcal{G}}$ after \mathcal{M} -switching \vec{e}_1 and \vec{e}_2 . Assume that $\vec{\mathcal{G}}_{\vec{e}_1 \circ \vec{e}_2}$ has an $\mathcal{M}_{\vec{e}_1 \circ \vec{e}_2}$ -cut X . Then X is tight in $\vec{\mathcal{G}}$, \vec{e}_2 enters X and:

1. If $\vec{e}_1^- \neq r$ then $\vec{e}_1^- \in X$.

2. If $\vec{e}_1^- = r$ then $\vec{e}_1 \in \text{Span}_{\vec{\mathcal{G}}}(X)$.

Using Lemma 3.1, we will often consider a minimal or maximal obstruction given by Lemma 3.6 or Lemma 3.7. The following lemma precises the structure of such a minimal obstruction.

Lemma 3.8. *Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ be an \mathcal{M} -arc-connected \mathcal{M} -digraph and X a minimal tight r -cut of $\vec{\mathcal{G}}$.*

Then $\vec{G}[X]$ is (globally) arc-connected.

Proof. Let $u \in X$ and $U \subseteq X$ be the vertex set from which u is reachable in $\vec{G}[X]$. Then the arcs of \vec{G} entering U are also entering X so U is also a tight r -cut. Therefore $X \cap U = U$ is a tight r -cut and since X is minimal, $U = X$. \square

3.1.5 Menger's theorem for \mathcal{M} -digraphs

We are now ready to prove a reduction theorem on \mathcal{M} -digraphs without $r - v$ \mathcal{M} -cut, using an \mathcal{M} -splitting.

Theorem 3.2. *Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ be an \mathcal{M} -digraph and $v \in V - r$ such that $\vec{\mathcal{G}}$ has no $r - v$ \mathcal{M} -cut. Let $\vec{e}_2 \in \vec{E}$ with $\vec{e}_2^+ = v$ and $\vec{e}_2^- \neq r$. Then either $\vec{\mathcal{G}} - \vec{e}_2$ has no $r - v$ \mathcal{M} -cut or there exists \vec{e}_1 with $\vec{e}_1^+ = \vec{e}_2^-$ and such that $\vec{\mathcal{G}}_{\vec{e}_1, \vec{e}_2}$ has no $r - v$ $\mathcal{M}_{\vec{e}_1, \vec{e}_2}$ -cut.*

Proof. If \vec{e}_2 does not enter any tight $r - v$ cut then Lemma 3.4 shows that $\vec{\mathcal{G}} - \vec{e}_2$ has no $r - v$ $(\mathcal{M} - \vec{e}_2)$ -cut. Otherwise \vec{e}_2 enters at least one tight $r - v$ cut. Then note that the tight $r - v$ cuts are the sets minimizing $\rho_{\vec{\mathcal{G}}, v}$. Thus Lemma 3.2 provides the existence of a maximal tight $r - v$ cut X for $\rho_{\vec{\mathcal{G}}, v}$ such that \vec{e}_2 enters X : X is the union of all tight $r - v$ cuts entered by \vec{e}_2 .

Assume that every arc \vec{e} with $\vec{e}^+ = \vec{e}_2^-$ and $\vec{e}^- \notin X$ verifies $\vec{e}^- = r$ and $\vec{e} \in \text{Span}_{\vec{\mathcal{G}}}(X)$. Then $\rho_{\vec{\mathcal{G}}, v}(X + \vec{e}_2^-) < \rho_{\vec{\mathcal{G}}, v}(X) = r(\mathcal{M})$ hence $X + \vec{e}_2^-$ is an $r - v$ \mathcal{M} -cut in $\vec{\mathcal{G}}$, which is a contradiction.

Thus, let \vec{e}_1 be such that $\vec{e}_1^- \notin X$, $\vec{e}_1^+ = \vec{e}_2^-$ and either $\vec{e}_1^- \neq r$, or $\vec{e}_1^- = r$ and $\vec{e}_1 \notin \text{Span}_{\vec{\mathcal{G}}}(X)$. We show that $\vec{\mathcal{G}}_{\vec{e}_1, \vec{e}_2}$ has no $r - v$ $\mathcal{M}_{\vec{e}_1, \vec{e}_2}$ -cut. Assume by contradiction that $\vec{\mathcal{G}}_{\vec{e}_1, \vec{e}_2}$ has an $r - v$ $\mathcal{M}_{\vec{e}_1, \vec{e}_2}$ -cut X' . See Figure 3.4. By Lemma 3.6, X' is a tight $r - v$ cut in $\vec{\mathcal{G}}$ and \vec{e}_2 enters X' . If $\vec{e}_1^- \neq r$ then Lemma 3.6 implies $\vec{e}_1^- \in X' - X$, contradicting the maximality of X . If $\vec{e}_1^- = r$ then Lemma 3.6 implies $\vec{e}_1^- \in \text{Span}_{\vec{\mathcal{G}}}(X)$, contradicting the choice of \vec{e}_1 . \square

The following lemma is an extension of Menger's theorem on arc-connectivity. The technique is similar to a proof of Menger's theorem by Frank in [Fra11].

Theorem 3.3 (Menger's theorem for \mathcal{M} -digraphs). *Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ be an \mathcal{M} -digraph and $v \in V - r$. $\vec{\mathcal{G}}$ has an \mathcal{M} -packing of $r - v$ dipaths if and only if $\vec{\mathcal{G}}$ has no $r - v$ \mathcal{M} -cut.*

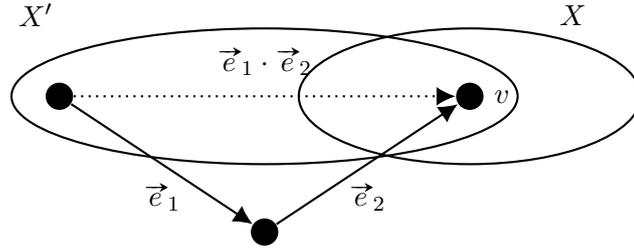


Figure 3.4: Proof by contradiction that $\vec{\mathcal{G}}_{\vec{e}_1, \vec{e}_2}$ has no $r - v$ \mathcal{M} -cut X , in the case $\vec{e}_1^- \neq r$ of the proof of [Theorem 3.2](#).

Proof. Assume first that $\vec{\mathcal{G}}$ has an \mathcal{M} -packing of $r - v$ dipaths $\vec{\mathcal{P}} = \{\vec{P}_1, \dots, \vec{P}_{r(\mathcal{M})}\}$. Let X be an $r - v$ cut. Each \vec{P}_i must have at least one arc entering X : let $\vec{e}_1, \dots, \vec{e}_{r(\mathcal{M})}$ be the arcs of $\vec{P}_1, \dots, \vec{P}_{r(\mathcal{M})}$ entering X . Among those arcs, let \vec{E}_1 be those leaving r and \vec{E}_2 be the other arcs. Since $\vec{E}_1 \subseteq \vec{\mathcal{P}}(v)^-$ and $\vec{\mathcal{P}}(v)^-$ is a base, $r_{\mathcal{M}}(\vec{E}_1) = |\vec{E}_1|$. Therefore $\rho_{\vec{\mathcal{G}}, v}(X) \geq |\vec{E}_2| + r_{\mathcal{M}}(\vec{E}_1) = r(\mathcal{M})$.

We prove the other implication by induction on the number of arcs not leaving r . Assume that all arcs entering v are leaving r . Then, since $\rho_{\vec{\mathcal{G}}, v}(v) = r_{\mathcal{M}}(v) \geq r(\mathcal{M})$, we can find the required dipaths.

Otherwise, let $\vec{e}_2 \in \vec{E}$ be such that $\vec{e}_2^- \neq r$ and $\vec{e}_2^+ = v$. Apply [Theorem 3.2](#): if $\vec{\mathcal{G}} - \vec{e}_2$ has no $r - v$ \mathcal{M} -cut then by induction we can find an $(\mathcal{M} - \vec{e}_2)$ -packing of $r - v$ dipaths in $\vec{\mathcal{G}} - \vec{e}_2$ which is also an \mathcal{M} -packing of $r - v$ dipaths in $\vec{\mathcal{G}}$. Otherwise, there exists \vec{e}_1 with $\vec{e}_1^+ = \vec{e}_2^-$ and such that $\vec{\mathcal{G}}_{\vec{e}_1, \vec{e}_2}$ has no $r - v$ \mathcal{M} -cut. We can apply induction on $\vec{\mathcal{G}}_{\vec{e}_1, \vec{e}_2}$ to get an \mathcal{M} -packing of $r - v$ dipaths. If a dipath uses $\vec{e}_1 \cdot \vec{e}_2$, replace it by \vec{e}_1 and \vec{e}_2 . This gives an $\mathcal{M}_{\vec{e}_1, \vec{e}_2}$ -packing of $r - v$ dipaths in $\vec{\mathcal{G}}$ and completes the proof. \square

By applying [Theorem 3.3](#) to all $v \in V - r$:

Corollary 3.1. *An \mathcal{M} -digraph $\vec{\mathcal{G}}$ is \mathcal{M} -arc-connected if and only if $\vec{\mathcal{G}}$ has no \mathcal{M} -cut.*

This provides a polynomial algorithm to check if an \mathcal{M} -digraph is \mathcal{M} -arc-connected, since it is enough to check if the minimum of the submodular function $\rho_{\vec{\mathcal{G}}}$ is at least $r(\mathcal{M})$. However, [Theorem 3.3](#) does not give a polynomial algorithm to find explicitly an \mathcal{M} -packing of $r - v$ dipaths, since the proof of [Theorem 3.2](#) involves the computation of a *maximal* tight set, a problem which is NP-hard in general.

It is possible to deduce the undirected counterpart of [Theorem 3.3](#) from [Theorem 2.6](#). We provide a more elementary proof:

Theorem 3.4 (Menger's theorem for \mathcal{M} -graphs). *Let $\mathcal{G} = (G = (V, E), \mathcal{M}, r)$ be an \mathcal{M} -graph and $v \in V - r$. \mathcal{G} has an \mathcal{M} -packing of $r - v$ paths if and only if \mathcal{G} has no $r - v$ \mathcal{M} -cut (where an $r - v$ \mathcal{M} -cut in the undirected \mathcal{M} -graph \mathcal{G} is a set $X \subseteq V - r$, $X \neq \emptyset$, such that $e_{G-r}(X) + r_{\mathcal{G}}(X) < r(\mathcal{M})$).*

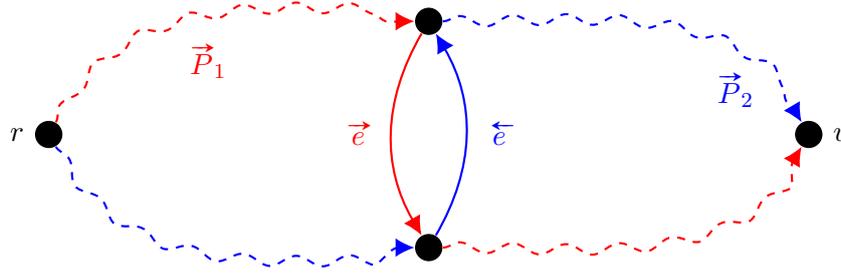


Figure 3.5: Illustration of the proof of Theorem 3.4.

Proof. Assume that \mathcal{G} has no $r - v$ \mathcal{M} -cut. Let $\overleftrightarrow{\mathcal{G}}$ be the \mathcal{M} -digraph obtained from \mathcal{G} by orienting every edge $e \in r^+$ away from r and replacing every other edge e by two arcs \bar{e} and \vec{e} with same end vertices but opposing heads. Clearly, $\overleftrightarrow{\mathcal{G}}$ has no $r - v$ \mathcal{M} -cut. Apply Theorem 3.3 to $\overleftrightarrow{\mathcal{G}}$, thus obtaining an \mathcal{M} -packing of $r - v$ dipaths $\vec{\mathcal{P}}$ in $\overleftrightarrow{\mathcal{G}}$. If, for an edge $e \in E$, \vec{e} and \bar{e} are used in $\vec{\mathcal{P}}$, say \vec{P}_1 and \vec{P}_2 , replace \vec{P}_1 by $\vec{P}_1(r, \vec{e}^-) \rightsquigarrow \vec{P}_2(\vec{e}^-, v)$ and \vec{P}_2 by $\vec{P}_2(r, \vec{e}^+) \rightsquigarrow \vec{P}_2(\vec{e}^+, v)$. See Figure 3.5. Then $\vec{\mathcal{P}}$ is an \mathcal{M} -packing of $r - v$ dipaths not using \vec{e} and \bar{e} . Repeat this process while possible. Then $\vec{\mathcal{P}}$ can not use both \vec{e} and \bar{e} , for any $e \in E$, hence it is possible to remove the orientations on dipaths of $\vec{\mathcal{P}}$ to get an \mathcal{M} -packing of $r - v$ paths in \mathcal{G} .

The other direction is similar to the proof of Theorem 3.3. \square

3.2 Packing of arborescences with matroid constraints

After successfully generalizing Menger's theorem, we now focus on Edmond's theorem on packing of arborescences.

3.2.1 Definition

Definition 3.10. Let $k \in \mathbb{N}^*$. An \mathcal{M} -packing of k arborescences of an \mathcal{M} -digraph $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M} = (r^+, \mathcal{I}), r)$ is a set $\vec{\mathcal{T}}$ of k arc-disjoint r -arborescences in $\vec{\mathcal{G}}$ such that for each $v \in V - r$, $\vec{\mathcal{T}}(v)^-$ is a base of \mathcal{M} .

In the previous definition, k must obviously satisfy $r(\mathcal{M}) \leq k \leq |\mathcal{M}|$. If $k = r(\mathcal{M})$ then the arborescences are necessarily spanning and we say that $\vec{\mathcal{T}}$ is an \mathcal{M} -packing of spanning arborescences. On the contrary, if $k = |\mathcal{M}|$ then every arborescence in the packing is actually a branch.

We will often assign colors to the arborescences in an \mathcal{M} -packing on our figures. For this reason we will say that a subset $S \subseteq r^+$ is **rainbow** if it is a partial transversal of those arborescences. If in addition $S \in \mathcal{I}$ then it is a **rainbow independent** set and if S is a base

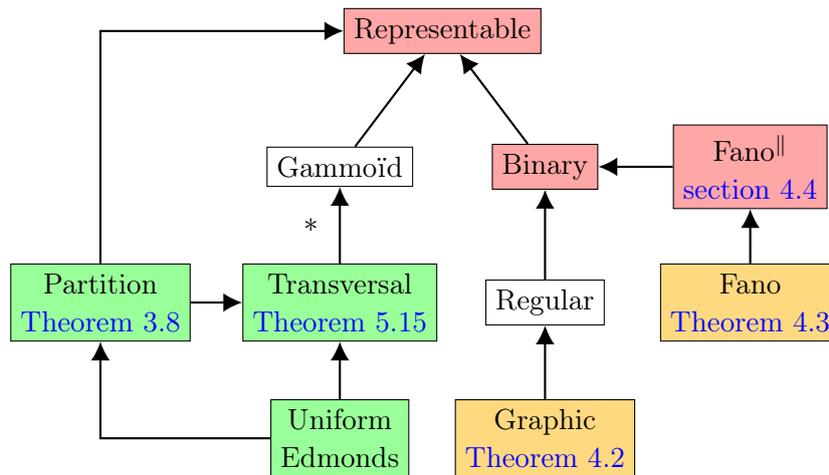


Figure 3.6: Matroid classes for which we will prove that [Conjecture 3.1](#) is [true for all \$\vec{\mathcal{G}}\$](#) , [true for \$\vec{\mathcal{G}}\$ acyclic](#) or [false](#). $M_1 \rightarrow M_2$ means that the matroid class M_1 is included in the matroid class M_2 . $M_1 \overset{*}{\rightarrow} M_2$ means that M_1 and M_2 are dual from each other.

of \mathcal{M} it is a **rainbow base**. If the context is unclear we will specify for which packing S is rainbow.

We will also be interested in packing trees in \mathcal{M} -graphs:

Definition 3.11. *Let $k \in \mathbb{N}^*$. An \mathcal{M} -packing of k trees of an \mathcal{M} -graph $(G = (V, E), \mathcal{M}, r)$ is a set $\mathcal{T} = \{T_1, \dots, T_k\}$ of k edge-disjoint trees rooted in r such that, for all $v \in V - r$, $\mathcal{T}(v)^-$ is a base of \mathcal{M} .*

3.2.2 Conjectures

The following conjecture was raised by Bérczi, T. Király and Kobayashi in [BKK16] and asked by Frank (personal communication). It would be a natural extension on \mathcal{M} -digraphs of Edmond’s packing theorem ([Theorem 2.8](#)):

Conjecture 3.1. *An \mathcal{M} -digraph is \mathcal{M} -arc-connected if and only if it has an \mathcal{M} -packing of spanning arborescences.*

It is obvious from the definitions that if an \mathcal{M} -digraph has an \mathcal{M} -packing of spanning arborescences then it is \mathcal{M} -arc-connected and therefore we will focus on the other implication. Note that [Conjecture 3.1](#) is true for uniform matroids, since then it reduces to Edmond’s packing theorem.

[Figure 3.6](#) illustrates some matroid classes for which [Conjecture 3.1](#) will be proven true in general (in green), true for acyclic digraphs (in orange) or false (in red).

[Conjecture 3.1](#) has the following counterpart for \mathcal{M} -graphs, which would be a generalization of Tutte’s packing theorem ([Theorem 2.9](#)):

Conjecture 3.2. *An \mathcal{M} -graph is \mathcal{M} -partition-connected if and only if it has an \mathcal{M} -packing of spanning trees.*

Again, one implication is straightforward: if \mathcal{G} is an \mathcal{M} -graph with an \mathcal{M} -packing of spanning trees then we can orient these trees to get r -arborescences, by [Lemma 2.4](#), and orient the other edges of \mathcal{G} arbitrarily to get an \mathcal{M} -arc-connected \mathcal{M} -digraph $\vec{\mathcal{G}}$. By [Theorem 3.1](#), \mathcal{G} is indeed \mathcal{M} -partition-connected.

It is easy to see that [Conjecture 3.1](#) implies [Conjecture 3.2](#):

Lemma 3.9. *Let \mathcal{M} be a matroid. Assume that [Conjecture 3.1](#) is true for all \mathcal{M} -digraphs. Then [Conjecture 3.2](#) is true for all \mathcal{M} -graphs.*

Proof. Assume that \mathcal{G} is \mathcal{M} -partition-connected and let $\vec{\mathcal{G}}$ be the orientation given by [Theorem 3.1](#). Then if $\vec{\mathcal{G}}$ has an \mathcal{M} -packing $\vec{\mathcal{P}}$ of spanning arborescences, \mathcal{P} is an \mathcal{M} -packing of spanning trees in \mathcal{G} . \square

In particular, all positive results on [Figure 3.6](#) will also hold for [Conjecture 3.2](#). However note that, *a priori*, [Conjecture 3.2](#) does not imply [Conjecture 3.1](#).

A more general question than [Conjecture 3.2](#) would be the following:

Question 3.1. *Let \mathcal{M} be a matroid. What is the minimum number k , if it exists, such that every \mathcal{M} -arc-connected \mathcal{M} -digraph has an \mathcal{M} -packing of k arborescences?*

Again, using [Theorem 3.1](#), an answer to the previous question would imply a similar result for \mathcal{M} -packing of trees:

Lemma 3.10. *Let \mathcal{M} be a matroid and $k \in \mathbb{N}^*$. If every \mathcal{M} -arc-connected \mathcal{M} -digraph has an \mathcal{M} -packing of k arborescences then every \mathcal{M} -partition-connected \mathcal{M} -graph has an \mathcal{M} -packing of k trees.*

3.2.3 Complexity

The **size** of an \mathcal{M} -graph (respectively \mathcal{M} -digraph) is defined as its number of vertices plus its number of edges (respectively arcs). We will be interested in the complexity of finding an \mathcal{M} -packing of spanning arborescences, for the matroid classes on [Figure 3.6](#). We will find, when possible, a minimum weight \mathcal{M} -packing of spanning arborescences using a method based on polyhedral considerations, as in [subsection 2.8.4](#). Indeed, let C be a matroid class closed by restriction for which [Conjecture 3.1](#) is true. Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ be an \mathcal{M} -digraph with weights on arcs and $\mathcal{M} \in C$. The following polyhedron $P_{\vec{\mathcal{G}}}$ in $\mathbb{R}^{|\vec{E}|}$ is integer by [Theorem 2.11](#):

$$0 \leq x(\vec{e}) \leq 1, \quad \text{for all } \vec{e} \in \vec{E}. \quad (1)$$

$$x(U^-) \geq r(\mathcal{M}) - r_{\vec{\mathcal{G}}}(U), \quad \text{for all } U \subseteq V - r, U \neq \emptyset. \quad (2)$$

Therefore, a vertex x of $P_{\vec{\mathcal{G}}}$ corresponds to a set of arcs \vec{E}_x in $\vec{\mathcal{G}}$. Let $\mathcal{M}_x = \mathcal{M}[\vec{E}_x]$. By (2), the \mathcal{M}_x -digraph $\vec{\mathcal{G}}_x = ((V, \vec{E}_x), \mathcal{M}_x, r)$ is \mathcal{M}_x -arc-connected. Since, by assumption, [Conjecture 3.1](#) is true for the \mathcal{M}_x -digraph $\vec{\mathcal{G}}_x$, $\vec{\mathcal{G}}_x$ has an \mathcal{M}_x -packing of spanning arborescences, which is also an \mathcal{M} -packing of spanning arborescences of $\vec{\mathcal{G}}$. Hence the vertices of $P_{\vec{\mathcal{G}}}$ are exactly the sets of arcs containing an \mathcal{M} -packing of spanning arborescences in $\vec{\mathcal{G}}$. As in [subsection 2.8.4](#), the equality

$$x(V) = k(|V| - 1)$$

defines a face $F_{\vec{\mathcal{G}}}$ of $P_{\vec{\mathcal{G}}}$ whose vertices are exactly the \mathcal{M} -packings of spanning arborescences of $\vec{\mathcal{G}}$. We deduce, using the ellipsoid method ([Theorem 2.12](#)):

Lemma 3.11. *Let \mathcal{C} be a matroid class, closed by restriction, for which [Conjecture 3.1](#) holds true. If there exists a polynomial algorithm to find an \mathcal{M} -packing of spanning arborescences, for all $\mathcal{M} \in \mathcal{C}$, then there exists a polynomial algorithm to find a minimum weight \mathcal{M} -packing of spanning arborescences, for all $\mathcal{M} \in \mathcal{C}$.*

3.2.4 Matroid on vertices

Our definition of \mathcal{M} -digraphs is equivalent to the following, introduced by Durand de Gevigney, Nguyen and Szigeti in [GNS13], where the elements of the matroid are identified with vertices rather than arcs.

Definition 3.12. *A **matroid-based rooted-digraph** is a 4-tuple $(\vec{\mathcal{G}}, \mathcal{M}, S, \pi)$ where $\vec{\mathcal{G}}$ is a digraph, S is a set, \mathcal{M} is a matroid on S and π is a function (a placement) of the elements of S on the vertices of $\vec{\mathcal{G}}$.*

In this definition, several matroid elements can be mapped by π to the same vertex.

Definition 3.13. *Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, S, \pi)$ be a matroid-based rooted-digraph. A **rooted-arborescence** of $\vec{\mathcal{G}}$ is a pair (\vec{T}, s) where $s \in S$ and \vec{T} is a $\pi(s)$ -arborescence of \vec{G} . $\vec{\mathcal{G}}$ is **\mathcal{M} -connected** if $\forall v \in V$, there exist arc-disjoint dipaths $\vec{P}_1, \dots, \vec{P}_{r(\mathcal{M})}$ of \vec{G} such that for each $i \in \{1, \dots, r(\mathcal{M})\}$, \vec{P}_i is a $\pi(s_i) - v$ dipath for some $s_i \in S$ such that $\{s_1, \dots, s_{r(\mathcal{M})}\}$ is a base of \mathcal{M} . A **matroid-based packing of rooted-arborescences** of $\vec{\mathcal{G}}$ is a set of arc-disjoint arborescences of \vec{G} such that each $v \in V$ belongs to $r(\mathcal{M})$ of them, say $(\vec{T}_1, s_1), \dots, (\vec{T}_{r(\mathcal{M})}, s_{r(\mathcal{M})})$, such that $\{s_1, \dots, s_{r(\mathcal{M})}\}$ is a base of \mathcal{M} .*

Contrary to our definitions, in [GNS13] a matroid-based packing of rooted-arborescences contains by definition $|S|$ arborescences. In particular all matroid elements are required to be used in such a packing, and this is why [GNS13] needs $\pi^{-1}(v)$ to be independent for all vertices v , for the existence of a matroid-based packing of rooted-arborescences. There will be no such assumption in the packing theorems we will prove.

The following theorem, illustrated by [Figure 3.7](#), shows that we can equivalently consider matroid conditions on arcs and on vertices.

Theorem 3.5. Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, S, \pi)$ be a matroid-based rooted-digraph. Let r be a new vertex. Let $\vec{\mathcal{G}}' = (V \cup \{r\}, \vec{E} \cup S)$ be the digraph such that \vec{G} is a subgraph of \vec{G}' and if $s \in S$, the tail of the arc s in \vec{G}' is r and its head is $\pi(s)$.

Then $\vec{\mathcal{G}}' = (\vec{G}', \mathcal{M}, r)$ is an \mathcal{M} -digraph and:

- $\vec{\mathcal{G}}$ is \mathcal{M} -connected if and only if $\vec{\mathcal{G}}'$ is \mathcal{M} -arc-connected.
- $\vec{\mathcal{G}}$ has a matroid-based packing of rooted-arborescences if and only if $\vec{\mathcal{G}}'$ has an \mathcal{M} -packing of arborescences.

Proof. Assume that $\vec{\mathcal{G}}$ is \mathcal{M} -connected. By definition there exist arc-disjoint dipaths $\vec{P}_1, \dots, \vec{P}_{r(\mathcal{M})}$ such that for each $i \in \{1, \dots, r(\mathcal{M})\}$, \vec{P}_i is a $\pi(s_i) - v$ dipath with $s_i \in S$ and $\{s_1, \dots, s_{r(\mathcal{M})}\}$ is a base of \mathcal{M} . Then $\{s_1 \rightsquigarrow \vec{P}_1, s_2 \rightsquigarrow \vec{P}_2, \dots, s_{r(\mathcal{M})} \rightsquigarrow \vec{P}_{r(\mathcal{M})}\}$ is an \mathcal{M} -packing of $r - v$ dipaths in $\vec{\mathcal{G}}'$ hence $\vec{\mathcal{G}}'$ is \mathcal{M} -arc-connected. On the contrary if $\vec{\mathcal{G}}'$ is \mathcal{M} -arc-connected then deleting the first arcs of an \mathcal{M} -packing of $r - v$ dipaths shows that $\vec{\mathcal{G}}$ is \mathcal{M} -connected. Similarly, an \mathcal{M} -packing of arborescences of $\vec{\mathcal{G}}'$ is obtained from a matroid-based packing of rooted-arborescences of $\vec{\mathcal{G}}$ by adding the appropriate first arcs. \square

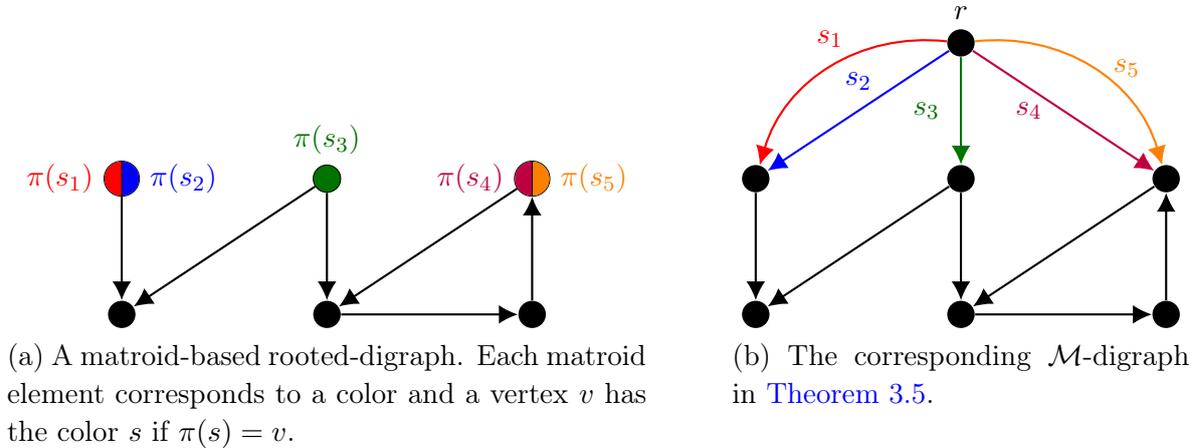


Figure 3.7: Illustration of Theorem 3.5.

From now on, we will only consider \mathcal{M} -digraphs, such that the elements of \mathcal{M} are identified with arcs rather than vertices.

3.3 Reductions for \mathcal{M} -digraphs

We introduce tools to tackle the questions of the last section.

3.3.1 Minimal \mathcal{M} -arc-connectivity

We saw in [Lemma 3.3](#) a special case when it is possible to delete an arc in \mathcal{M} -arc-connected \mathcal{M} -digraph and keep \mathcal{M} -arc-connectivity. It makes sense to look at the case when the deletion of such an arc is not possible:

Definition 3.14. Let $\vec{\mathcal{G}} = (\vec{G}, \mathcal{M}, r)$ be an \mathcal{M} -digraph. We say that $\vec{\mathcal{G}}$ is *minimally \mathcal{M} -arc-connected* if $\vec{\mathcal{G}}$ is \mathcal{M} -arc-connected and for every arc \vec{e} of \vec{G} such that $r(\mathcal{M}) = r(\mathcal{M} - \vec{e})$, $\vec{\mathcal{G}} - \vec{e}$ is not $(\mathcal{M} - \vec{e})$ -arc-connected.

As one may expect, an \mathcal{M} -digraph is minimally \mathcal{M} -arc-connected if and only if every vertex has the minimum possible number of incoming arcs:

Theorem 3.6. Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ be an \mathcal{M} -arc-connected \mathcal{M} -digraph. $\vec{\mathcal{G}}$ is minimally \mathcal{M} -arc-connected if and only if:

- $\rho_{\vec{\mathcal{G}}}(r) = 0$.
- $\rho_{\vec{\mathcal{G}}}(v) = r(\mathcal{M})$ for all $v \in V - r$.

Proof. Clearly if $\vec{\mathcal{G}}$ verifies the two above conditions, no arc can be deleted without violating the connectivity condition. If $\vec{\mathcal{G}}$ is minimally \mathcal{M} -arc-connected then, since a tight r -cut does not contain r by definition, [Lemma 3.3](#) implies $\rho_{\vec{\mathcal{G}}}(r) = 0$. Let $v \in V - r$ and $\vec{\mathcal{P}}$ be an \mathcal{M} -packing of $r - v$ dipaths. If $\rho_{\vec{\mathcal{G}}}(v) > r(\mathcal{M})$ then one can find an arc \vec{e} entering v not used by $\vec{\mathcal{P}}$, since $|\vec{\mathcal{P}}| = r(\mathcal{M})$. $\vec{\mathcal{P}}^-$ is a base of $\mathcal{M} - \vec{e}$ of size $r(\mathcal{M})$ hence $r(\mathcal{M}) = r(\mathcal{M} - \vec{e})$. We claim that $\vec{\mathcal{G}} - \vec{e}$ has no $(\mathcal{M} - \vec{e})$ -cut. Indeed, if X is an r -cut of $\vec{\mathcal{G}} - \vec{e}$ then either $v \notin X$ and $\rho_{\vec{\mathcal{G}} - \vec{e}}(X) = \rho_{\vec{\mathcal{G}}}(X) \geq r(\mathcal{M}) = r(\mathcal{M} - \vec{e})$ or $v \in X$ and the $r(\mathcal{M})$ dipaths of $\vec{\mathcal{P}}$ must enter X so $\rho_{\vec{\mathcal{G}} - \vec{e}}(X) \geq r(\mathcal{M}) = r(\mathcal{M} - \vec{e})$. Thus $\vec{\mathcal{G}} - \vec{e}$ has no $(\mathcal{M} - \vec{e})$ -cut so it is $(\mathcal{M} - \vec{e})$ -arc-connected by [Theorem 3.3](#) and this provides a contradiction, proving the theorem. \square

We deduce the following facts about minimally \mathcal{M} -arc-connected \mathcal{M} -digraphs $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ that will be useful:

- An \mathcal{M} -packing of arborescences of $\vec{\mathcal{G}}$ must use every arc in \vec{G} .
- $\forall v \in V - r$, $\vec{E}(r, v)$ is an independent set of \mathcal{M} .
- \mathcal{M} has no loop.

Observe that if $\vec{\mathcal{G}}$ is minimally \mathcal{M} -arc-connected and if $\vec{\mathcal{G}}'$, obtained from $\vec{\mathcal{G}}$ by \mathcal{M} -splitting or \mathcal{M} -switching, is \mathcal{M}' -arc-connected with $r(\mathcal{M}') = r(\mathcal{M})$, then $\vec{\mathcal{G}}'$ is automatically minimally \mathcal{M}' -arc-connected. Indeed, the in-degree $\rho_{\vec{\mathcal{G}}}(v)$ of a vertex $v \neq r$ can only decrease after \mathcal{M} -splitting or \mathcal{M} -switching, so if $\rho_{\vec{\mathcal{G}}}(v) = r(\mathcal{M})$ then $\rho_{\vec{\mathcal{G}}'}(v) = r(\mathcal{M}')$.

To simplify proofs, we will often consider [Conjecture 3.1](#) when $\vec{\mathcal{G}}$ is a minimally \mathcal{M} -arc-connected \mathcal{M} -digraph. This is not a huge restriction: if $\vec{\mathcal{G}}$ is \mathcal{M} -arc-connected but not minimal then we can delete arcs of $\vec{\mathcal{G}}$ until $\vec{\mathcal{G}}$ becomes minimally \mathcal{M} -arc-connected. This can be done in polynomial time since \mathcal{M} -arc-connectivity can be checked in polynomial time, as seen previously.

3.3.2 Two reductions using switching

We further investigate the \mathcal{M} -switching operation defined in [Definition 3.8](#). [Theorem 3.7](#) below was proved by Durand de Gevigney, Nguyen, and Szigeti in [GNS13]. We provide a proof for completeness.

Definition 3.15. Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M} = (r^+, \mathcal{I}), r)$ be an \mathcal{M} -digraph. If $u, v \in V - r$ and $u \neq v$ we say that u **augments** v if $\vec{E}(r, u) \not\subseteq \text{Span}_{\vec{\mathcal{G}}}(v)$. An arc \vec{e} is **augmenting** if \vec{e}^- augments \vec{e}^+ . An **augmenting pair** is a 2-tuple of arcs (\vec{e}_1, \vec{e}_2) with $\vec{e}_1^+ = \vec{e}_2^-$, $\vec{e}_1 \in r^+$ and $\vec{e}_1 \notin \text{Span}_{\vec{\mathcal{G}}}(\vec{e}_2^+)$.

Clearly, an arc \vec{e} is augmenting if and only if there exists an augmenting pair (\vec{e}_1, \vec{e}) .

Lemma 3.12. Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M} = (r^+, \mathcal{I}), r)$ be an \mathcal{M} -digraph, $u, v \in V - r$ such that u augments v and \vec{P} a $u - v$ dipath. Then \vec{P} contains an augmenting arc.

Proof. Otherwise, let u, u_1, \dots, u_p, v be the vertices used by \vec{P} in this order. Since \vec{P} has no augmenting arcs, [Lemma 2.8](#) implies that $\vec{E}(r, u) \subseteq \text{Span}_{\vec{\mathcal{G}}}(u_1) \subseteq \dots \subseteq \text{Span}_{\vec{\mathcal{G}}}(v)$. This is in contradiction with u augmenting v . \square

In the following reduction theorem, the minimality is not mandatory but makes the proof simpler (if $\vec{\mathcal{G}}$ is not minimal, we can remove arcs until it becomes minimal, and then apply the theorem).

Theorem 3.7. Assume that the \mathcal{M} -digraph $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M} = (r^+, \mathcal{I}), r)$ is minimally \mathcal{M} -arc-connected. Then at least one of the following propositions is true:

1. $r_{\vec{\mathcal{G}}}(v) = r(\mathcal{M}), \forall v \in V - r$, or
2. There exists $\vec{e}_1 \in r^+, \vec{e}_2 \in \vec{E}$ such that $\vec{e}_1^+ = \vec{e}_2^-$ and $\vec{\mathcal{G}}_{\vec{e}_1 \circ \vec{e}_2}$ is $\mathcal{M}_{\vec{e}_1 \circ \vec{e}_2}$ -arc-connected.

Proof. Assume that 1. is false. We first show that an augmenting arc exists. Indeed let $v \in V - r$ be such that $r_{\vec{\mathcal{G}}}(v) < r(\mathcal{M})$. Since $\vec{\mathcal{G}}$ is \mathcal{M} -arc-connected there exists an \mathcal{M} -packing \vec{P} of $r - v$ dipaths. Since $\vec{P}(v)^-$ is a base and $\vec{E}(r, v)$ is not a base, there exists a dipath of \vec{P} with first arc \vec{e} such that $\vec{e} \notin \text{Span}_{\vec{\mathcal{G}}}(v)$, hence \vec{e}^+ augments v . Then, by [Lemma 3.12](#), there exists an augmenting pair (\vec{e}_1, \vec{e}_2) .

If $\vec{\mathcal{G}}_{\vec{e}_1 \odot \vec{e}_2}$ is $\mathcal{M}_{\vec{e}_1 \odot \vec{e}_2}$ -arc-connected then 2. is true and we are done. Otherwise [Lemma 3.7](#) provides the existence of a minimal tight cut $X_{\vec{e}_1 \odot \vec{e}_2}$ for $\vec{\mathcal{G}}$ with $\vec{e}_1 \in \text{Span}_{\vec{\mathcal{G}}}(X_{\vec{e}_1 \odot \vec{e}_2})$ and \vec{e}_2 entering $X_{\vec{e}_1 \odot \vec{e}_2}$. Among all possible $(\vec{e}_1, \vec{e}_2, X_{\vec{e}_1 \odot \vec{e}_2})$, we choose one with $X_{\vec{e}_1 \odot \vec{e}_2}$ minimal. If $\vec{\mathcal{G}}[X_{\vec{e}_1 \odot \vec{e}_2} + r]$ contains an augmenting pair (\vec{f}_1, \vec{f}_2) then either $\vec{\mathcal{G}}_{\vec{f}_1 \odot \vec{f}_2}$ is $\mathcal{M}_{\vec{f}_1 \odot \vec{f}_2}$ -arc-connected and we are done or [Lemma 3.7](#) shows the existence of a tight cut $X_{\vec{f}_1 \odot \vec{f}_2}$ for $\vec{\mathcal{G}}$ with \vec{f}_2 entering $X_{\vec{f}_1 \odot \vec{f}_2}$. Since $\vec{f}_2^+ \in X_{\vec{f}_1 \odot \vec{f}_2} \cap X_{\vec{e}_1 \odot \vec{e}_2}$, $X_{\vec{f}_1 \odot \vec{f}_2}$ and $X_{\vec{e}_1 \odot \vec{e}_2}$ intersect hence, by [Lemma 3.1](#), $X_{\vec{f}_1 \odot \vec{f}_2} \cap X_{\vec{e}_1 \odot \vec{e}_2}$ is a tight cut in $\vec{\mathcal{G}}$. Since $\vec{f}_2^- \in X_{\vec{e}_1 \odot \vec{e}_2} - X_{\vec{f}_1 \odot \vec{f}_2}$, $X_{\vec{f}_1 \odot \vec{f}_2} \cap X_{\vec{e}_1 \odot \vec{e}_2}$ contradicts the minimality of $X_{\vec{e}_1 \odot \vec{e}_2}$. Therefore we can assume that $\vec{\mathcal{G}}[X_{\vec{e}_1 \odot \vec{e}_2} + r]$ does not contain any augmenting arc. Let $x \in X_{\vec{e}_1 \odot \vec{e}_2}$. By [Lemma 3.8](#) there exists an $x - \vec{e}_2^+$ dipath \vec{P} in $\vec{\mathcal{G}}[X_{\vec{e}_1 \odot \vec{e}_2}]$ (see [Figure 3.8](#)) and since \vec{P} has no augmenting arc, by [Lemma 3.12](#), $\vec{E}(r, x) \subseteq \text{Span}_{\vec{\mathcal{G}}}(\vec{e}_2^+)$. Therefore $\vec{E}(r, X) = \bigcup_{x \in X} \vec{E}(r, x) \subseteq \text{Span}_{\vec{\mathcal{G}}}(\vec{e}_2^+)$ which imply, by [Lemma 2.8](#), that $\text{Span}_{\vec{\mathcal{G}}}(X) \subseteq \text{Span}_{\vec{\mathcal{G}}}(\vec{e}_2^+)$. This is in contradiction with $\vec{e}_1 \in \text{Span}_{\vec{\mathcal{G}}}(X)$ and $\vec{e}_1 \notin \text{Span}_{\vec{\mathcal{G}}}(\vec{e}_2^+)$. \square

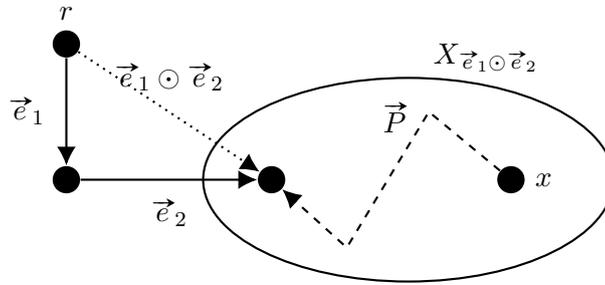


Figure 3.8: Proof by contradiction that $\vec{\mathcal{G}}_{\vec{e}_1 \odot \vec{e}_2}$ is $\mathcal{M}_{\vec{e}_1 \odot \vec{e}_2}$ -arc-connected.

Since $\vec{e}_1 \odot \vec{e}_2 \in r^+$ in the previous theorem, $\vec{\mathcal{G}}_{\vec{e}_1 \odot \vec{e}_2}$ has less arcs not leaving r than $\vec{\mathcal{G}}$. This observation will allow us to prove theorems by induction on the number of arcs not leaving r . Provided a polynomial oracle for \mathcal{M} , the proof of [Theorem 3.7](#) can be turned into a polynomial algorithm to find \vec{e}_1 and \vec{e}_2 , in the case 2.

[Theorem 3.7](#) admits the following straightforward converse:

Lemma 3.13. *Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ be an \mathcal{M} -digraph and $\vec{e}_1 \in r^+$, $\vec{e}_2 \in \vec{E}$ with $\vec{e}_1^+ = \vec{e}_2^-$. Assume that $\vec{\mathcal{G}}_{\vec{e}_1 \odot \vec{e}_2}$ is $\mathcal{M}_{\vec{e}_1 \odot \vec{e}_2}$ -arc-connected. Then $\vec{\mathcal{G}}$ is \mathcal{M} -arc-connected.*

Proof. Let $v \in V - r$ and assume that there exists an $\mathcal{M}_{\vec{e}_1 \odot \vec{e}_2}$ -packing of $r - v$ dipaths \vec{P} in $\vec{\mathcal{G}}_{\vec{e}_1 \odot \vec{e}_2}$. If \vec{P} does not use $\vec{e}_1 \odot \vec{e}_2$ then \vec{P} is also an \mathcal{M} -packing of $r - v$ dipaths in $\vec{\mathcal{G}}$ and we are done. Assume on the contrary that a dipath \vec{P} of \vec{P} uses $\vec{e}_1 \odot \vec{e}_2$. \vec{P} does not use \vec{e}_1 since $\vec{e}_1 \odot \vec{e}_2 \parallel \vec{e}_1$.

- If \vec{P} contains \vec{e}_1^+ ([Figure 3.9a](#)): replace, in \vec{P} , \vec{P} by $\vec{e}_1 \rightsquigarrow \vec{P}(\vec{e}_1^+, v)$.
- If \vec{P} does not contain \vec{e}_1^+ ([Figure 3.9b](#)): replace, in \vec{P} , \vec{P} by $\vec{e}_1 \rightsquigarrow \vec{e}_2 \rightsquigarrow \vec{P}(\vec{e}_2^+, v)$.

In both cases, \vec{P} is then an \mathcal{M} -packing of $r - v$ dipaths in $\vec{\mathcal{G}}$. \square

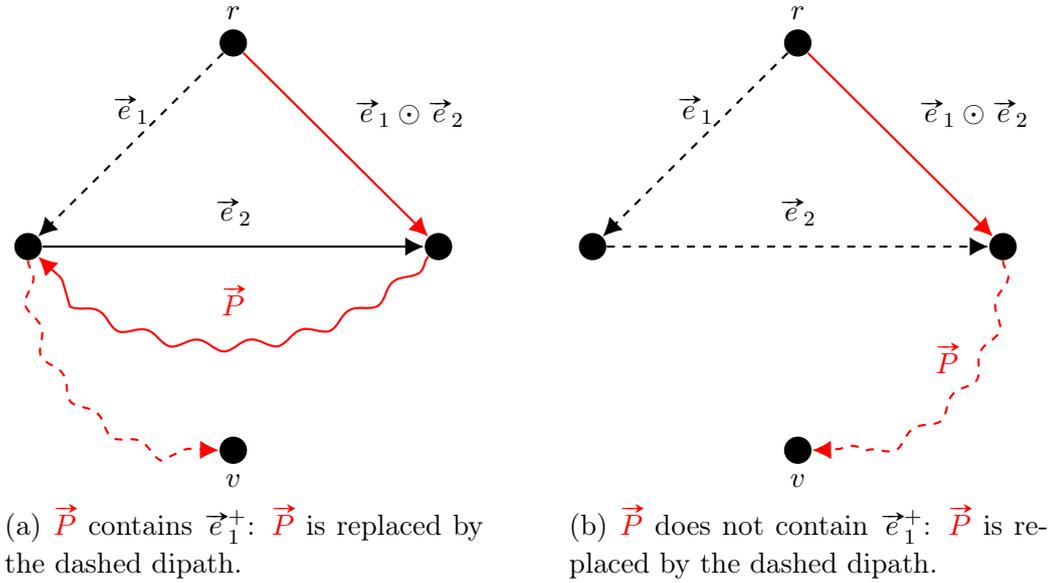


Figure 3.9: The two cases in the proof of Lemma 3.13.

As a first application of Theorem 3.7, we show that we can restrict our attention to connected matroids in Conjecture 3.1.

Theorem 3.8. *Let $\mathcal{M}_1 = (S_1, \mathcal{I}_1)$ and $\mathcal{M}_2 = (S_2, \mathcal{I}_2)$ be two matroids on disjoint ground sets and $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$. Assume that there exists an \mathcal{M}_1 -packing of k_1 arborescences in every \mathcal{M}_1 -arc-connected \mathcal{M}_1 -digraph and that there exists an \mathcal{M}_2 -packing of k_2 arborescences in every \mathcal{M}_2 -arc-connected \mathcal{M}_2 -digraph.*

Then there exists an \mathcal{M} -packing of $(k_1 + k_2)$ arborescences in every \mathcal{M} -arc-connected \mathcal{M} -digraph.

Proof. Let $\vec{\mathcal{G}}$ be an \mathcal{M} -digraph. While possible, apply 2. of Theorem 3.7 to $\vec{\mathcal{G}}$. Let $\vec{\mathcal{G}}_\odot$ be the resulting \mathcal{M}_\odot -digraph. By construction, \mathcal{M}_\odot is obtained from \mathcal{M} by adding parallel elements. For $i \in \{1, 2\}$, let S_i^\parallel be the set of all elements in \mathcal{M}_\odot that are parallel to some element in S_i , and let $\mathcal{M}_i^\parallel = \mathcal{M}_\odot[S_i^\parallel]$. Clearly, $\mathcal{M}_\odot = \mathcal{M}_1^\parallel \oplus \mathcal{M}_2^\parallel$. We also define, for $i \in \{1, 2\}$, \vec{E}_i as the set of arcs $\vec{e} \in \vec{\mathcal{G}}$ such that either $\vec{e} \in S_i$ or there exists $\vec{e}_1 \in S_i^\parallel$ such that $\vec{e}_1 \odot \vec{e} \in S_i^\parallel$. In other terms, \vec{E}_i contains S_i and the arcs off $\vec{\mathcal{G}}$ that have been switched with an element of S_i^\parallel . Since \vec{e} is deleted when an arc $\vec{e}_1 \odot \vec{e}$ is added, \vec{E}_1 and \vec{E}_2 are disjoint. Since $\mathcal{M}_\odot = \mathcal{M}_1^\parallel \oplus \mathcal{M}_2^\parallel$, the case 1. of Theorem 3.7 implies that $r_{\vec{\mathcal{G}}_\odot[S_1^\parallel]}(v) = r(\mathcal{M}_1^\parallel)$, for all $v \in V - r$, hence $\vec{\mathcal{G}}_\odot[S_1^\parallel]$ is \mathcal{M}_1^\parallel -arc-connected. Since $\vec{\mathcal{G}}_\odot[S_1^\parallel]$ is obtained from $\vec{\mathcal{G}}[\vec{E}_1]$ by a sequence of switching, then, by Lemma 3.13, $\vec{\mathcal{G}}[\vec{E}_1]$ is \mathcal{M}_1 -arc-connected and $\vec{\mathcal{G}}[\vec{E}_1]$ has an \mathcal{M}_1 -packing $\vec{\mathcal{P}}_1$ of k_1 arborescences. Similarly $\vec{\mathcal{G}}[\vec{E}_2]$ has an \mathcal{M}_2 -packing $\vec{\mathcal{P}}_2$ of k_2 arborescences. As \vec{E}_1 and \vec{E}_2 are disjoint, $\vec{\mathcal{P}}_1 \cup \vec{\mathcal{P}}_2$ is an \mathcal{M} -packing of $(k_1 + k_2)$ arborescences of $\vec{\mathcal{G}}$, proving the theorem. \square

The previous theorem shows in particular that [Conjecture 3.1](#) is true when \mathcal{M} is a partition matroid, since such a matroid is a direct sum of uniform matroids.

3.3.3 Complete switching reduction

We saw that [Theorem 3.7](#) decreases the number of arcs not leaving r . We introduce a new reduction theorem reducing the number of vertices of an \mathcal{M} -digraph $\vec{\mathcal{G}}$. Firstly, we can delete a sink without violating the \mathcal{M} -arc-connectivity:

Lemma 3.14. *Let $\vec{\mathcal{G}} = (\vec{G}, \mathcal{M}, r)$ be an \mathcal{M} -arc-connected \mathcal{M} -digraph with $|\vec{G}| > 2$ and assume that v is a sink of \vec{G} . Then $r(\mathcal{M} - v) = r(\mathcal{M})$ and $\vec{\mathcal{G}} - v$ is without $(\mathcal{M} - v)$ -cut.*

Proof. Let u be a vertex different from r and v . Since $\vec{\mathcal{G}}$ is \mathcal{M} -arc-connected, there exists an \mathcal{M} -packing of $r - u$ dipaths in $\vec{\mathcal{G}}$ which is also an \mathcal{M} -packing of $r - u$ dipaths in $\vec{\mathcal{G}} - v$ since these dipaths can not use the sink v . Hence $r(\mathcal{M} - v) = r(\mathcal{M})$ and $\vec{\mathcal{G}} - v$ is $(\mathcal{M} - v)$ -arc-connected. \square

In order to delete a vertex v in an \mathcal{M} -arc-connected \mathcal{M} -digraph $\vec{\mathcal{G}}$ which is not a sink, we will first delete the arcs leaving v , by switching them with arcs entering v , until v becomes a sink and can be removed. $\vec{\mathcal{G}}$ will remain \mathcal{M} -arc-connected during this process.

Theorem 3.9. *Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ be an \mathcal{M} -arc-connected \mathcal{M} -digraph and $v \in V - r$. Let $\vec{e}_2 \in v^+$ be an arc leaving v .*

Then there exists an arc $\vec{e}_1 \in v^-$ entering v such that $\vec{\mathcal{G}}_{\vec{e}_1 \odot \vec{e}_2}$ is an $\mathcal{M}_{\vec{e}_1 \odot \vec{e}_2}$ -arc-connected \mathcal{M} -digraph.

Moreover if $\vec{\mathcal{G}}$ was minimally \mathcal{M} -arc-connected then $\vec{\mathcal{G}}_{\vec{e}_1 \odot \vec{e}_2}$ is also minimally $\mathcal{M}_{\vec{e}_1 \odot \vec{e}_2}$ -arc-connected.

Proof. If \vec{e}_2 does not enter any tight r -cut then any arc \vec{e}_1 entering v can be used, by [Lemma 3.7](#). Otherwise let X be the maximal tight r -cut such that \vec{e}_2 enters X . Assume that every arc $\vec{e} \in v^-$ is such that $\vec{e}^- = r$ and $\vec{e} \in \text{Span}_{\vec{\mathcal{G}}}(X)$. Then $\rho_{\vec{\mathcal{G}}}(X + v) < \rho_{\vec{\mathcal{G}}}(X)$ since \vec{e}_2 enters X but not $X + v$. Therefore $X + v$ is an \mathcal{M} -cut of $\vec{\mathcal{G}}$, which is a contradiction. Therefore let $\vec{e}_1 \in v^-$ with either $\vec{e}_1^- \neq r$ or $\vec{e}_1^- = r$ and $\vec{e}_1 \notin \text{Span}_{\vec{\mathcal{G}}}(X)$. If $\vec{\mathcal{G}}_{\vec{e}_1 \odot \vec{e}_2}$ is not $\mathcal{M}_{\vec{e}_1 \odot \vec{e}_2}$ -arc-connected then [Lemma 3.7](#) provides a tight r -cut X' such that \vec{e}_2 enters X' . Since $\vec{e}_2^+ \in X \cap X'$, by [Lemma 3.1](#), $X \cup X'$ is a tight r -cut. We look at the two possibilities of [Lemma 3.7](#):

1. If $\vec{e}_1^- \neq r$ then $\vec{e}_1^- \in X'$. Since $\vec{e}_1^- \notin X$, $X \cup X'$ is in contradiction with the maximality of X .
2. If $\vec{e}_1^- = r$ then $\vec{e}_1 \in \text{Span}_{\vec{\mathcal{G}}}(X')$. If $X' \subseteq X$ then $\text{Span}_{\vec{\mathcal{G}}}(X') \subseteq \text{Span}_{\vec{\mathcal{G}}}(X)$ which is impossible since $\vec{e}_1 \in \text{Span}_{\vec{\mathcal{G}}}(X')$ and $\vec{e}_1 \notin \text{Span}_{\vec{\mathcal{G}}}(X)$. Therefore, again, $X \cup X'$ is in contradiction with the maximality of X .

□

The following reduction is illustrated by [Figure 3.10](#).

Theorem 3.10 (Complete switching). *Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ be an \mathcal{M} -arc-connected \mathcal{M} -digraph with $|V| > 2$ and $v \in V - r$ be such that $\rho_{\vec{\mathcal{G}}}(v) > \delta_{\vec{\mathcal{G}}}(v)$.*

*Then it is possible to map every arc $\vec{e} \in v^+$ to an arc entering v , denoted $\vec{e}^{\odot v} \in v^-$, such that after switching the pairs $(\vec{e}^{\odot v}, \vec{e})$, for every $\vec{e} \in v^+$, and then deleting v , one obtains an $\mathcal{M}_{\odot v}$ -arc-connected $\mathcal{M}_{\odot v}$ -digraph $\vec{\mathcal{G}}_{\odot v} = (\vec{\mathcal{G}}_{\odot v}, \mathcal{M}_{\odot v}, r)$. This process is called **complete \mathcal{M} -switching** at v .*

Moreover $r(\mathcal{M}_{\odot v}) = r(\mathcal{M})$ and if $\vec{\mathcal{G}}$ was minimally \mathcal{M} -arc-connected then $\vec{\mathcal{G}}_{\odot v}$ is minimally \mathcal{M} -arc-connected.

Proof. Repeat [Theorem 3.9](#) on v until no arc leaves v . Then [Lemma 3.14](#) shows that the sink v can be removed and that $r(\mathcal{M}_{\odot v}) = r(\mathcal{M})$. □

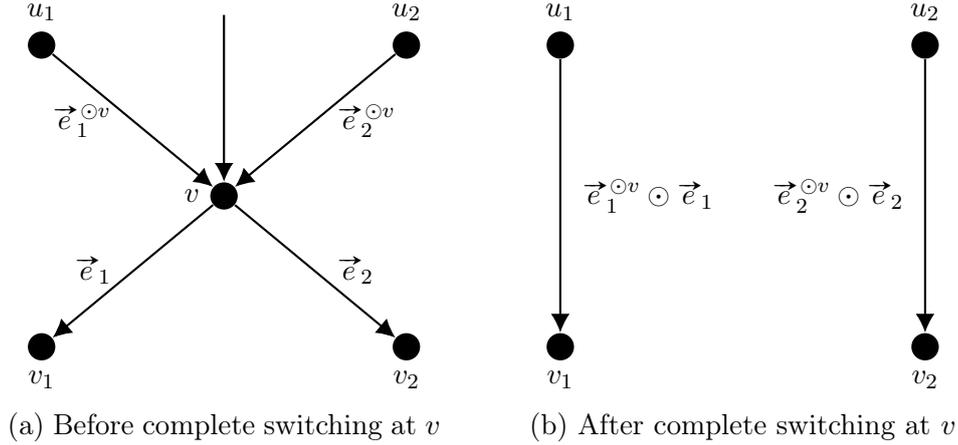


Figure 3.10: Complete switching at v .

Let $\vec{\mathcal{G}} = (\vec{G}, \mathcal{M}, r)$ be a minimally \mathcal{M} -arc-connected \mathcal{M} -digraph with $|\vec{G}| > 2$. Using [Theorem 3.6](#), we see that $\vec{\mathcal{G}}$ has a vertex with out-degree smaller than in-degree. Indeed the sum of the out-degrees of the vertices in $\vec{\mathcal{G}}$ must be equal to the sum of the in-degrees, and $\delta_{\vec{\mathcal{G}}}(r) > \rho_{\vec{\mathcal{G}}}(r)$. Therefore it is always possible to find a vertex v of $\vec{\mathcal{G}}$ for which [Theorem 3.10](#) can be applied.

Like the switching operation, the complete switching can add parallel elements. Indeed, it is possible that an arc entering v is switched several times: for example, on [Figure 3.10](#), if $\vec{e}_1^{\odot v} = \vec{e}_2^{\odot v}$ and $u_1 = u_2 = r$ then a parallel element is added to \mathcal{M} .

One could wonder if there exists a complete \mathcal{M} -splitting reduction for \mathcal{M} -arc-connected \mathcal{M} -digraphs. This is not possible in general, as shown by [Figure 3.11](#), where \vec{e} is in every base of \mathcal{M} : after an hypothetical \mathcal{M} -splitting at v , either u or w will not be reachable from \vec{e} , and therefore the digraph would not be \mathcal{M} -arc-connected anymore.

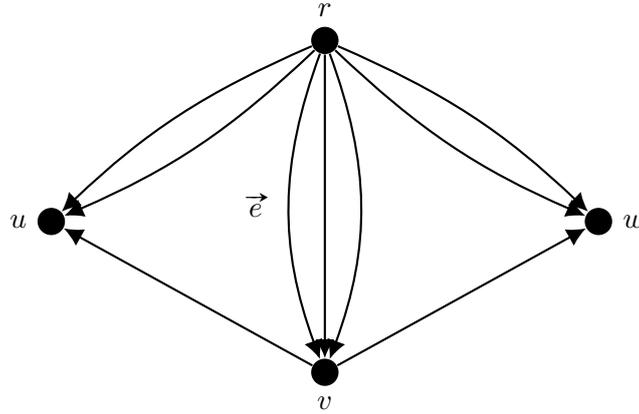


Figure 3.11: An \mathcal{M} -arc-connected \mathcal{M} -digraph such that \vec{e} belongs to every base of \mathcal{M} . It is not possible to do a complete splitting at v without violating the \mathcal{M} -arc-connectivity.

3.4 Packing of branches

The following theorem was proved by de Gevigney, Nguyen, and Szigeti in [GNS13]. Remember that an r -branch is an r -arborescence with only one arc leaving r .

Theorem 3.11. *Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ be an \mathcal{M} -digraph. Then $\vec{\mathcal{G}}$ is \mathcal{M} -arc-connected if and only if $\vec{\mathcal{G}}$ has an \mathcal{M} -packing of $|\mathcal{M}|$ r -branches.*

Proof. We only prove the non-trivial direction: assume that $\vec{\mathcal{G}}$ is an \mathcal{M} -arc-connected \mathcal{M} -digraph which is a counterexample with a minimum number of arcs not leaving r . Apply [Theorem 3.7](#) on $\vec{\mathcal{G}}$:

1. If $r_{\vec{\mathcal{G}}}(v) = r(\mathcal{M}), \forall v \in V - r$: for every $v \in V - r$, let $B_v \subseteq \vec{E}(r, v)$ be a base of \mathcal{M} . We define $r(\mathcal{M})$ new r -branches, each consisting of a different arc in B_v . The set of all such branches for all $v \in V - r$ is an \mathcal{M} -packing of branches in $\vec{\mathcal{G}}$, a contradiction.
2. If there exists $\vec{e}_1 \in r^+, \vec{e}_2 \in \vec{E}$ such that $\vec{\mathcal{G}}_{\vec{e}_1 \odot \vec{e}_2}$ is $\mathcal{M}_{\vec{e}_1 \odot \vec{e}_2}$ -arc-connected then, since $\vec{\mathcal{G}}$ is a minimum counterexample, $\vec{\mathcal{G}}_{\vec{e}_1 \odot \vec{e}_2}$ has an $\mathcal{M}_{\vec{e}_1 \odot \vec{e}_2}$ -packing $\vec{\mathcal{T}}$ of branches. If no branch in $\vec{\mathcal{T}}$ uses $\vec{e}_1 \odot \vec{e}_2$ then $\vec{\mathcal{T}}$ is also an \mathcal{M} -packing of branches of $\vec{\mathcal{G}}$. Otherwise let \vec{T}_1 be the branch of $\vec{\mathcal{T}}$ using $\vec{e}_1 \odot \vec{e}_2$. If no branch in $\vec{\mathcal{T}}$ uses \vec{e}_1 then let $\vec{T}'_1 = \vec{T}_1 - \vec{e}_1 \odot \vec{e}_2 + \vec{e}_1 + \vec{e}_2$. $\vec{\mathcal{T}} - \vec{T}_1 + \vec{T}'_1$ is an \mathcal{M} -packing of branches of $\vec{\mathcal{G}}$, a contradiction. Otherwise let \vec{T}_2 be the branch of $\vec{\mathcal{T}}$ using \vec{e}_1 . Then \vec{T}_2 and \vec{T}_1 are disjoint hence $\vec{T}' := \vec{T}_1 + \vec{e}_2 + \vec{T}_2 - \vec{e}_1 \odot \vec{e}_2$ is a branch and $\vec{\mathcal{T}} - \vec{T}_2 - \vec{T}_1 + \vec{T}'$ is an \mathcal{M} -packing of branches in $\vec{\mathcal{G}}$, again a contradiction.

□

Provided a polynomial oracle for \mathcal{M} , since there exists a polynomial algorithm for [Theorem 3.7](#), the proof of [Theorem 3.11](#) shows that there is a polynomial algorithm to find an \mathcal{M} -packing of branches in an \mathcal{M} -arc-connected \mathcal{M} -digraph.

Even if [Theorem 3.11](#) does not require the arborescences to be spanning, [Theorem 3.11](#) implies Edmonds' packing theorem ([Theorem 2.8](#)). Indeed, let \vec{G} be k -arc-connected from a root r . Let \vec{G}' be the digraph obtained by adding a vertex r' to \vec{G} and k arcs from r' to r . Let \mathcal{M} be a free matroid defined on the arcs of \vec{G}' from r' to r . Then $\vec{\mathcal{G}} = (\vec{G}', \mathcal{M}, r')$ is clearly an \mathcal{M} -arc-connected \mathcal{M} -digraph. [Theorem 3.11](#) implies that $\vec{\mathcal{G}}$ has an \mathcal{M} -packing $\vec{\mathcal{T}}$ of arborescences. Since the only base of \mathcal{M} is its ground set, these arborescences must be spanning. Therefore, removing r' in every arborescence of $\vec{\mathcal{T}}$ results in a packing of k spanning r -arborescences in \vec{G} , which proves the non-trivial direction of Edmonds' packing theorem.

Using the following obvious lemma one can improve slightly [Theorem 3.11](#):

Lemma 3.15. *Let $\vec{\mathcal{G}}$ be an \mathcal{M} -arc-connected \mathcal{M} -digraph. Assume $\vec{\mathcal{G}}$ has an \mathcal{M} -packing $\vec{\mathcal{T}}$ of arborescences and that two of them, say \vec{T}_1 and \vec{T}_2 , does not have any non-root vertex in common. Then $\vec{\mathcal{T}} - \vec{T}_1 - \vec{T}_2 + (\vec{T}_1 \cup \vec{T}_2)$ is also an \mathcal{M} -packing of arborescences of $\vec{\mathcal{G}}$.*

Theorem 3.12. *Let $\vec{\mathcal{G}}$ be an \mathcal{M} -arc-connected \mathcal{M} -digraph. Then there exists an \mathcal{M} -packing of p arborescences in $\vec{\mathcal{G}}$, p being the number of parallel classes of \mathcal{M} .*

Proof. Apply first [Theorem 3.11](#) and then [Lemma 3.15](#) to merge every pair of arborescences with first arcs in the same parallel class. \square

From [Theorem 3.11](#) we deduce the following generalization of Tutte's packing theorem ([Theorem 2.9](#)), initially proved by Katoh and Tanigawa in [KT13]:

Theorem 3.13. *Let $\mathcal{G} = (G, \mathcal{M}, r)$ be an \mathcal{M} -graph. Then \mathcal{G} is \mathcal{M} -partition-connected if and only if \mathcal{G} has an \mathcal{M} -packing of $|\mathcal{M}|$ trees.*

Proof. If \mathcal{G} is \mathcal{M} -partition-connected then it has an \mathcal{M} -arc-connected orientation $\vec{\mathcal{G}}$, by [Theorem 3.1](#). We conclude using [Theorem 3.11](#) on $\vec{\mathcal{G}}$. \square

By merging trees we obtain the undirected counterpart of [Theorem 3.12](#):

Theorem 3.14. *Let \mathcal{G} be an \mathcal{M} -partition-connected \mathcal{M} -graph. Then there exists an \mathcal{M} -packing of p trees in \mathcal{G} , p being the number of parallel classes of \mathcal{M} .*

3.5 Matroids of rank two

In this section we use the switching reduction of [Theorem 3.7](#) to show that [Conjecture 3.1](#) holds if $r(\mathcal{M}) = 2$. Indeed let $\vec{\mathcal{G}}$ be an \mathcal{M} -arc-connected \mathcal{M} -digraph. [Theorem 3.7](#) provides the existence of two arcs \vec{e}_1 and \vec{e}_2 such that $\vec{\mathcal{G}}_{\vec{e}_1 \odot \vec{e}_2}$ is $\mathcal{M}_{\vec{e}_1 \odot \vec{e}_2}$ -arc-connected. By induction

we can assume that $\mathcal{M}_{\vec{e}_1 \odot \vec{e}_2}$ has an $\mathcal{M}_{\vec{e}_1 \odot \vec{e}_2}$ -packing of spanning arborescences \vec{T}_1, \vec{T}_2 . If \vec{e}_1 and $\vec{e}_1 \odot \vec{e}_2$ belong to the same arborescence, say \vec{T}_1 , then $\vec{T}_1' = \vec{T}_1 - \vec{e}_1 \odot \vec{e}_2 + \vec{e}_2$ is a spanning arborescence in $\vec{\mathcal{G}}$ and, since $\vec{T}_1'(v)^- = \vec{T}_1(v)^- \forall v \in V - r$, $\{\vec{T}_1', \vec{T}_2\}$ is an \mathcal{M} -packing of spanning arborescences of $\vec{\mathcal{G}}$. We will prove an exchange lemma ensuring that we can always make such an assumption.

3.5.1 Structure of matroids of rank two

Matroids of rank two have a very specific structure that we will exploit: they are in bijection with a specific class of graphs.

Definition 3.16. Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid of rank two and \mathcal{B} the set of the bases of \mathcal{M} . To \mathcal{M} we associate the graph $G(\mathcal{M}) = (S, \mathcal{B})$ with \mathcal{B} the bases of \mathcal{M} .

Therefore, $G(\mathcal{M})$ has an edge $\{u, v\}$ if and only if $\{u, v\}$ is a base of \mathcal{M} .

In the following definition, a **stable** set X is a set of vertices such that there is no edge with both end vertices in X .

Definition 3.17. Let $G = (V, E)$ be a graph. G is a **complete multipartite** graph if we can partition $V = V_1 \cup V_2 \cup \dots \cup V_p$ such that every V_i is a stable set and $\forall i \neq j$, E contains all possible edges between V_i and V_j .

Lemma 3.16. A graph $G = (V, E)$ is the graph $G(\mathcal{M})$ of a matroid \mathcal{M} of rank two without loop if and only if it is a complete multipartite graph.

Proof. Assume that $\mathcal{M} = (S, \mathcal{I})$ is a matroid of rank two without loop. Divide $S = S_1 \cup \dots \cup S_p$ into parallel classes, given by the equivalence relation \parallel . By definition, every S_i is stable in $G(\mathcal{M})$ and $\forall s_i \in S_i, s_j \in S_j, i \neq j, \{s_i, s_j\}$ is a base. Hence $G(\mathcal{M})$ is indeed a complete multipartite graph.

Assume now that $G = (V, E)$ is a complete multipartite graph and define \mathcal{M} on ground set V and with E as its set of bases. It is clear with [Lemma 2.6](#) that \mathcal{M} is a matroid of rank two without loop. \square

3.5.2 Exchange lemma

Note that the following exchange operation can introduce cycles and \vec{T}_1, \vec{T}_2 may not be arborescences anymore.

Definition 3.18. Let $\vec{\mathcal{G}}$ be a digraph, $v \in V$, and \vec{T}_1, \vec{T}_2 two spanning arborescences of $\vec{\mathcal{G}}$. Let $\vec{e}_1 = \vec{T}_1(v)^+$ be the arc of \vec{T}_1 entering v and $\vec{e}_2 = \vec{T}_2(v)^+$. To **exchange** v (for \vec{T}_1 and \vec{T}_2) means:

1. Add \vec{e}_1 to \vec{T}_2 and \vec{e}_2 to \vec{T}_1

2. Remove \vec{e}_1 from \vec{T}_1 and \vec{e}_2 from \vec{T}_2

The following lemma shows that, if $r(\mathcal{M}) = 2$ and provided an \mathcal{M} -packing, a simple operation gives another \mathcal{M} -packing with an additional property. An example is provided on [Figure 3.13](#).

Lemma 3.17. *Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M} = (r^+, \mathcal{I}), r)$ be an \mathcal{M} -arc-connected \mathcal{M} -digraph such that \mathcal{M} is a matroid of rank two without loop and let S be a parallel class of \mathcal{M} . Assume that $\vec{\mathcal{G}}$ has an \mathcal{M} -packing of spanning arborescences $\vec{\mathcal{T}} = \{\vec{T}_1, \vec{T}_2\}$. Exchange simultaneously all $v \in V$ for which $\vec{T}_2(v)^- \in S$.*

Then the resulting arborescences \vec{T}_1', \vec{T}_2' is also an \mathcal{M} -packing of spanning arborescences of $\vec{\mathcal{G}}$ and $S \cap \vec{T}_2'^- = \emptyset$.

Proof. Clearly, every $v \in V - r$ has one incoming arc from \vec{T}_1' , one from \vec{T}_2' . If \vec{T}_1' or \vec{T}_2' contains a cycle C going through $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_k, \vec{e}_{k+1} = \vec{e}_1$, then at least one arc of C was in \vec{T}_1 and at least one was in \vec{T}_2 (otherwise C would be a cycle in \vec{T}_1 or \vec{T}_2). Hence there exists i such that $\vec{e}_i \in \vec{T}_1$ and $\vec{e}_{i+1} \in \vec{T}_2$. Therefore exactly one vertex among $\{\vec{e}_i^+, \vec{e}_{i+1}^+\}$ was exchanged (see [Figure 3.12](#)). The path $\vec{T}_2(\vec{e}_{i+1}^+)$ contains \vec{e}_i^+ so $\vec{T}_2(\vec{e}_i^+)^- = \vec{T}_2(\vec{e}_{i+1}^+)^-$

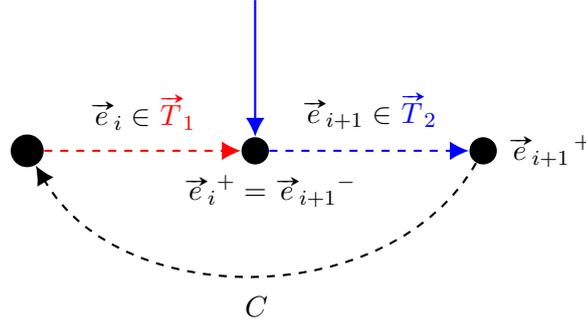


Figure 3.12: An hypothetical cycle C , dashed.

and either none or both of \vec{e}_i^+ and \vec{e}_{i+1}^+ should have been exchanged, a contradiction. Hence [Theorem 2.4](#) shows that \vec{T}_1 and \vec{T}_2 are spanning arborescences.

Let $v \in V - r$, we want to prove that $\{\vec{T}_1'(v)^-, \vec{T}_2'(v)^-\} \in \mathcal{I}$. To this end, it is enough to prove that $\vec{T}_1'(v)^-$ and $\vec{T}_2'(v)^-$ are in two different parallel classes.

1. If $\vec{T}_2(v)^- \notin S$: no vertex of $\vec{T}_2(v)$ was exchanged so $\vec{T}_2(v)$ is an $r - v$ dipath in \vec{T}_2' and $\vec{T}_2'(v)^- = \vec{T}_2(v)^-$. If no vertex of $\vec{T}_1(v)$ was exchanged then $\vec{T}_1(v)$ is also an $r - v$ dipath in \vec{T}_1' so $\vec{T}_1'(v)^- = \vec{T}_1(v)^-$ and $\{\vec{T}_1'(v)^-, \vec{T}_2'(v)^-\} = \{\vec{T}_1(v)^-, \vec{T}_2(v)^-\} \in \mathcal{I}$. Otherwise let v' be the deepest vertex of $\vec{T}_1(v)$ that was exchanged. $\vec{T}_2(v')$ is an $r - v'$ dipath in \vec{T}_1' since every vertex of $\vec{T}_2(v')$ was exchanged. $\vec{T}_2(v') \rightsquigarrow \vec{T}_1(v', v)$ is an $r - v$ dipath in \vec{T}_1' so $\vec{T}_1'(v)^- = \vec{T}_2(v')^- \in S$ ($\vec{T}_2(v')^- \in S$ since v' was exchanged). By $\vec{T}_2'(v)^- \notin S$ and $\vec{T}_1'(v)^- \in S$ we deduce $\{\vec{T}_1'(v)^-, \vec{T}_2'(v)^-\} \in \mathcal{I}$.

2. If $\vec{T}_2(v)^- \in S$: every vertex of $\vec{T}_2(v)$ was exchanged so $\vec{T}_2(v)$ is an $r - v$ dipath in \vec{T}_1' and $\vec{T}_1'(v)^- = \vec{T}_2(v)^- \in S$. If every vertex of $\vec{T}_1(v)$ was exchanged then $\vec{T}_1(v)$ is also an $r - v$ dipath in \vec{T}_2' so $\vec{T}_2'(v)^- = \vec{T}_1(v)^- \notin S$ and $\{\vec{T}_1'(v)^-, \vec{T}_2'(v)^-\} \in \mathcal{I}$. Otherwise let v' be the deepest vertex of $\vec{T}_1(v)$ that was not modified. Then no vertex of $\vec{T}_2(v')$ was modified so $\vec{T}_2(v') \rightsquigarrow \vec{T}_1(v', v)$ gives an $r - v$ dipath in \vec{T}_2' and since $\vec{T}_2'(v)^- = \vec{T}_2(v')^- \notin S$ we can conclude $\{\vec{T}_1'(v)^-, \vec{T}_2'(v)^-\} \in \mathcal{I}$.

□

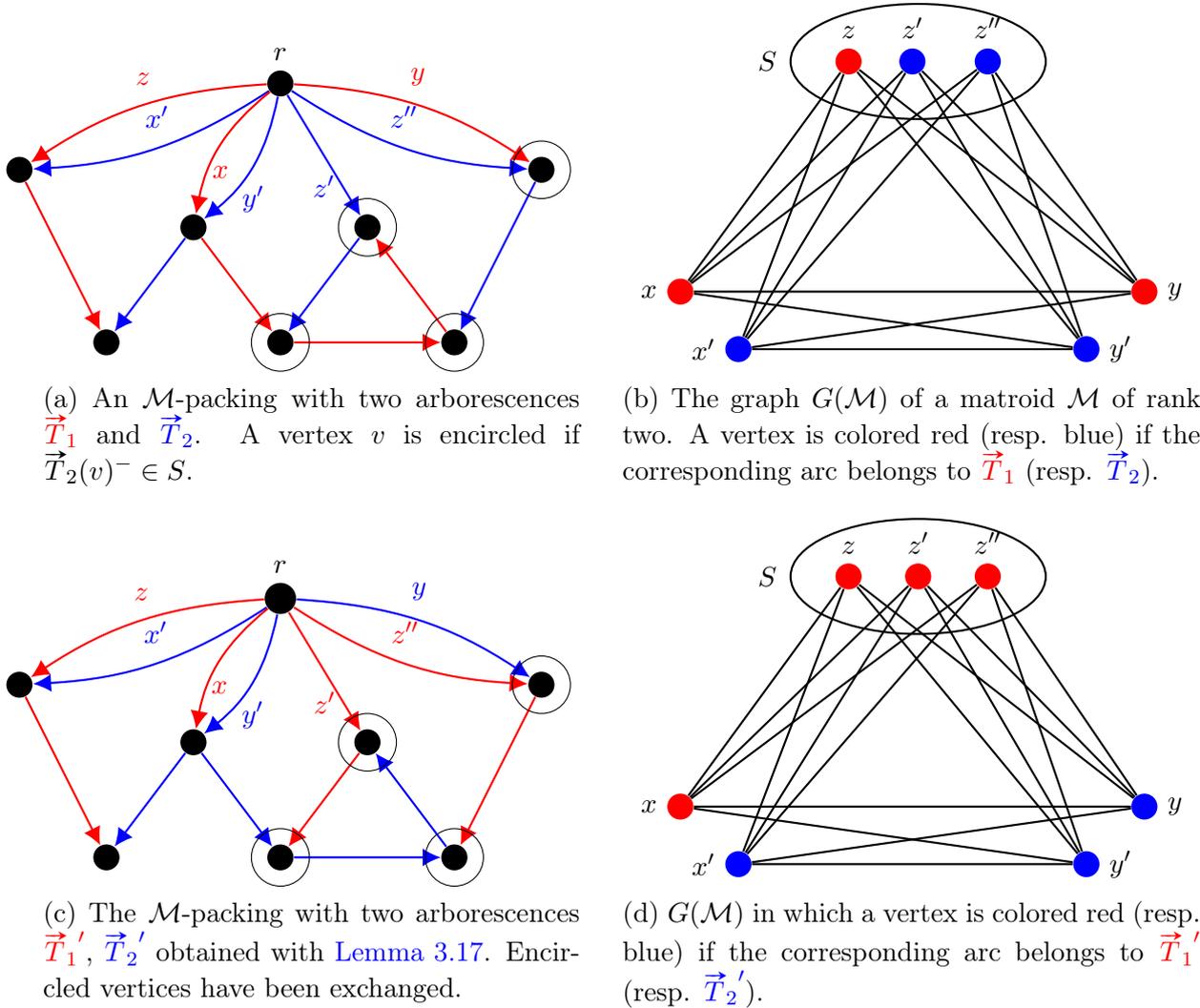


Figure 3.13: An application of Lemma 3.17

We are now ready to prove Conjecture 3.1 for matroids of rank two:

Theorem 3.15. *Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ be a minimally \mathcal{M} -arc-connected \mathcal{M} -digraph with $r(\mathcal{M}) = 2$.*

Then there exists an \mathcal{M} -packing of two spanning arborescences in $\vec{\mathcal{G}}$.

Proof. Since $\vec{\mathcal{G}}$ is minimally \mathcal{M} -arc-connected, \mathcal{M} has no loop. Assume that $\vec{\mathcal{G}}$ is a counterexample with a minimum number of arcs not leaving r . Apply [Theorem 3.7](#) on $\vec{\mathcal{G}}$:

1. If $r_{\vec{\mathcal{G}}}(v) = 2, \forall v \in V - r$: start with $\vec{T}_1 = \{r\}, \vec{T}_2 = \{r\}$ then, for each $v \in V - r$, consider a base $\{s_1, s_2\}$ of arcs entering v and add arbitrarily s_i to \vec{T}_i . Then \vec{T}_1, \vec{T}_2 is trivially an \mathcal{M} -packing of spanning arborescences of $\vec{\mathcal{G}}$.
2. If there exists \vec{e}_1, \vec{e}_2 in $\vec{\mathcal{G}}$ such that $\vec{\mathcal{G}}_{\vec{e}_1 \odot \vec{e}_2}$ is $\mathcal{M}_{\vec{e}_1 \odot \vec{e}_2}$ -arc-connected then, by minimality of $\vec{\mathcal{G}}$, $\vec{\mathcal{G}}_{\vec{e}_1 \odot \vec{e}_2}$ has an $\mathcal{M}_{\vec{e}_1 \odot \vec{e}_2}$ -packing \vec{T}_1, \vec{T}_2 of spanning arborescences. Assume that \vec{e}_1 and $\vec{e}_1 \odot \vec{e}_2$ are in the same arborescence, say \vec{T}_1 , and let $\vec{T}_1' = \vec{T}_1 + \vec{e}_2 - \vec{e}_1 \odot \vec{e}_2$. Then, as mentioned previously, \vec{T}_1', \vec{T}_2 is an \mathcal{M} -packing of spanning arborescences of $\vec{\mathcal{G}}$ and gives a contradiction. Otherwise \vec{e}_1 and $\vec{e}_1 \odot \vec{e}_2$ are in different arborescences, say $\vec{e}_1 \in \vec{T}_1$ and $\vec{e}_1 \odot \vec{e}_2 \in \vec{T}_2$. Apply [Lemma 3.17](#), S being the parallel class of $\vec{e}_1 \odot \vec{e}_2$, to get an $\mathcal{M}_{\vec{e}_1 \odot \vec{e}_2}$ -packing \vec{T}_1', \vec{T}_2' of spanning arborescences of $\vec{\mathcal{G}}_{\vec{e}_1 \odot \vec{e}_2}$. Then $\vec{e}_1 \in \vec{T}_1'$ and $\vec{e}_1 \odot \vec{e}_2 \in \vec{T}_1'$ which is the previous case.

□

3.5.3 Complexity

The proof of [Theorem 3.15](#) provides a polynomial algorithm to find an \mathcal{M} -packing of spanning arborescences in an \mathcal{M} -arc-connected \mathcal{M} -digraph with $r(\mathcal{M}) = 2$:

1. Remove arcs \vec{e} from $\vec{\mathcal{G}}$ until $\vec{\mathcal{G}}$ becomes minimally \mathcal{M} -arc-connected (it can be tested in polynomial time if $\vec{\mathcal{G}} - \vec{e}$ is \mathcal{M} -arc-connected, using [Corollary 3.1](#)).
2. For each \vec{e}_1, \vec{e}_2 in $\vec{\mathcal{G}}$, test if $\vec{\mathcal{G}}_{\vec{e}_1 \odot \vec{e}_2}$ is $\mathcal{M}_{\vec{e}_1 \odot \vec{e}_2}$ -arc-connected.
3. If there exists \vec{e}_1, \vec{e}_2 in $\vec{\mathcal{G}}$ such that $\vec{\mathcal{G}}_{\vec{e}_1 \odot \vec{e}_2}$ is $\mathcal{M}_{\vec{e}_1 \odot \vec{e}_2}$ -arc-connected, find recursively an $\mathcal{M}_{\vec{e}_1 \odot \vec{e}_2}$ -packing of spanning arborescences in the smaller digraph $\vec{\mathcal{G}}_{\vec{e}_1 \odot \vec{e}_2}$. Then apply [Lemma 3.17](#) if needed to get an \mathcal{M} -packing of spanning arborescences in $\vec{\mathcal{G}}$.
4. Otherwise, we are in the base case 1. of [Theorem 3.7](#) and an \mathcal{M} -packing of spanning arborescences in $\vec{\mathcal{G}}$ is easily computed.

Moreover, [Lemma 3.11](#) implies that a minimum weight packing of spanning arborescences in a weighted \mathcal{M} -arc-connected \mathcal{M} -digraph with $r(\mathcal{M}) = 2$ can be found in polynomial time.

Packing with matroid constraints in acyclic digraphs

Contents

4.1 Properties of acyclic digraphs	51
4.2 Graphic matroids	52
4.3 Fano matroid	56
4.4 Parallel extensions of the Fano matroid	57
4.5 Complexity	64

In this chapter, we investigate [Conjecture 3.1](#) when \vec{G} is *acyclic*. [Conjecture 3.1](#) is established if \vec{G} is acyclic and \mathcal{M} is a graphic matroid or the Fano matroid. Finally, we provide a counterexample to [Conjecture 3.1](#) and [Conjecture 3.2](#) where \mathcal{M} is a parallel extension of the Fano matroid. Most results can be found in [For+17].

4.1 Properties of acyclic digraphs

A lot of notions related to connectivity (without matroids) become considerably easier when restricted to acyclic digraphs. For example, it is straightforward that a subdigraph \vec{T} of an acyclic digraph \vec{G} is an r -arborescence of \vec{G} if and only if the in-degree of every vertex of \vec{T} is 1 except r which has in-degree 0. In other terms, the set of arcs of r -arborescences of \vec{G} is a *matroid*, namely the partition matroid defined on the in-stars of \vec{G} . Edmond's theorem, when restricted on acyclic digraphs $\vec{G} = (V, \vec{E})$, becomes likewise straightforward: if $\rho_{\vec{G}}(v) \geq k$, $\forall v \in V - r$, then *any* assignment of k arcs entering v , $\forall v \in V - r$, to k sets of arcs results in a packing of k spanning arborescences. \mathcal{M} -arc-connectivity is easy to verify as well:

Lemma 4.1. *Let $\vec{G} = (G = (V, \vec{E}), \mathcal{M} = (S, \mathcal{I}), r)$ be an acyclic \mathcal{M} -digraph such that $\forall v \in V - r$, $\vec{E}(r, v) \in \mathcal{I}$. If $\forall v \in V - r$, $\rho_{\vec{G}}(v) \geq r(\mathcal{M})$, then \vec{G} is \mathcal{M} -arc-connected.*

Proof. By induction on $|\vec{G}|$. If $|\vec{G}| = 2$ the statement is obvious. Assume that $|\vec{G}| > 2$, let v be a sink in \vec{G} and $X \subseteq V - r$:

1. If $X \neq \{v\}$ then $\rho_{\vec{G}}(X) \geq \rho_{\vec{G}-v}(X) \geq r(\mathcal{M})$, by [Lemma 3.14](#) and induction hypothesis.

2. If $X = \{v\}$ then $\rho_{\vec{\mathcal{G}}}(v) = \rho_{\vec{\mathcal{G}}-r}(v) + r_{\mathcal{M}}(v)$. Since $\vec{E}(r, v) \in \mathcal{I}$, $r_{\mathcal{M}}(v) = |\vec{E}(r, v)|$, hence $\rho_{\vec{\mathcal{G}}}(v) = \rho_{\vec{\mathcal{G}}}(v) \geq r(\mathcal{M})$.

In both cases, we can conclude the induction step using [Corollary 3.1](#). \square

We will use the following corollary for \mathcal{M} -graphs:

Corollary 4.1. *Let $\mathcal{G} = (G = (V, E), \mathcal{M} = (S, \mathcal{I}), r)$ be an \mathcal{M} -graph which is obtained from an isolated vertex r by iteratively adding a new vertex v and a set E_v of edges adjacent to v such that $E_v \cap E(r, v) \in \mathcal{I}$ and $|E_v| \geq r(\mathcal{M})$. Then \mathcal{G} is \mathcal{M} -partition-connected.*

Proof. Whenever a new vertex v is added, orient E_v towards v . This clearly results in an orientation of \mathcal{G} verifying the conditions of [Lemma 4.1](#), hence this orientation is \mathcal{M} -arc-connected. By the orientation theorem ([Theorem 3.1](#)), \mathcal{G} is \mathcal{M} -partition-connected. \square

However, as we will see, [Conjecture 3.1](#) is challenging even for acyclic \mathcal{M} -digraphs.

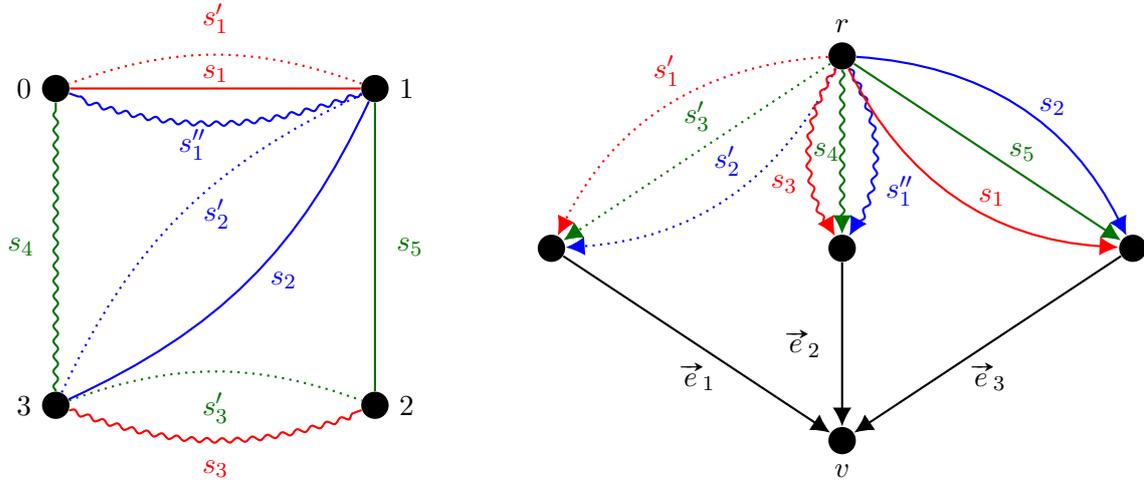
Let $\vec{\mathcal{G}} = (\vec{G}, \mathcal{M}, r)$ be an \mathcal{M} -arc-connected \mathcal{M} -digraph. If \vec{G} is acyclic then \vec{G} has a sink v and (if $|\vec{G}| > 2$) [Lemma 3.14](#) shows that $\vec{G} - v$ is $(\mathcal{M} - v)$ -arc-connected. Assume that $\vec{G} - v$ has an $(\mathcal{M} - v)$ -packing of spanning arborescences $\vec{\mathcal{T}} = \{\vec{T}_1, \dots, \vec{T}_{r(\mathcal{M})}\}$. Let $\vec{e}_1, \dots, \vec{e}_{r(\mathcal{M})}$ be $r(\mathcal{M})$ arcs entering v . We would like to **complete** (by induction) $\vec{\mathcal{T}}$ into an \mathcal{M} -packing of spanning arborescences of $\vec{\mathcal{G}}$, that is to say, to assign each \vec{e}_i to a $\vec{T}_{\sigma(i)}$ so that $\vec{T}_{\sigma(1)} + \vec{e}_1, \dots, \vec{T}_{\sigma(r(\mathcal{M}))} + \vec{e}_{r(\mathcal{M})}$ is an \mathcal{M} -packing of spanning arborescences of $\vec{\mathcal{G}}$, where σ is a permutation. Assume for simplicity that $\vec{e}_i^- \neq r, \forall i \in \{1, \dots, r(\mathcal{M})\}$ and let $B_i = \vec{\mathcal{T}}(\vec{e}_i^-)^-$, $\forall i \in \{1, \dots, r(\mathcal{M})\}$. Then the problem of completing $\vec{\mathcal{T}}$ amounts to find a rainbow base which is also a transversal of $(B_i)_{i \in \{1, \dots, r(\mathcal{M})\}}$.

Unfortunately, this is not always possible. Indeed, consider [Figure 4.1b](#), where every arborescence of the packing is represented by a color and $B_1 = \{s'_1, s'_3, s'_2\}$, $B_2 = \{s_3, s_4, s''_1\}$, $B_3 = \{s_1, s_5, s_2\}$. Then any choice of three different colors for $\vec{e}_1, \vec{e}_2, \vec{e}_3$ results in a dependent set in the graphic matroid \mathcal{M} of [Figure 4.1a](#). For example $\{s'_1, s_4, s_2\}$ is a cycle in \mathcal{M} and therefore it is not a rainbow base completion.

4.2 Graphic matroids

Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ be an acyclic \mathcal{M} -arc-connected \mathcal{M} -digraph. In this section, we prove that if \mathcal{M} is a graphic matroid, whose bases are the spanning trees of a graph $G_{\mathcal{M}}$, and if we add constraints on the arborescences in a packing, we can always complete the packing.

We will construct a packing $\vec{\mathcal{T}} = \{\vec{T}_1, \dots, \vec{T}_{r(\mathcal{M})}\}$ of $\vec{\mathcal{G}}$ such that the first arcs \vec{T}_i^- of \vec{T}_i (when considered as matroid elements) will all be adjacent to the same vertex i in $G_{\mathcal{M}}$. Therefore an edge e of \mathcal{M} with $V(e) = \{i, j\}$ can be in arborescence \vec{T}_i or \vec{T}_j . To decide



(a) A graphic matroid \mathcal{M} . Every 3-tuple of edges with different styles and different colors is dependent.

(b) An \mathcal{M} -arc-connected \mathcal{M} -digraph $\vec{\mathcal{G}}$. $\vec{\mathcal{G}} - v$ has an \mathcal{M} -packing of spanning arborescences $\vec{T}_1 = \{s'_1, s_3, s_1\}$, $\vec{T}_2 = \{s'_3, s_4, s_5\}$, $\vec{T}_3 = \{s'_2, s''_1, s_2\}$ but it is not possible to complete it into an \mathcal{M} -packing of $\vec{\mathcal{G}}$.

Figure 4.1: The completion of an \mathcal{M} -packing is not always possible.

which one, we use orientations: if $v \in V$ and $e \in \vec{\mathcal{T}}^-(v)$, we orient $\vec{\mathcal{T}}^-(v)$ from a preably chosen vertex of $G_{\mathcal{M}}$ and we require that e belongs to \vec{T}_i if e enters i in this orientation. We then show that it is possible to complete such a packing, by distributing one by one the arcs entering v to the arborescences in $\vec{\mathcal{T}}$.

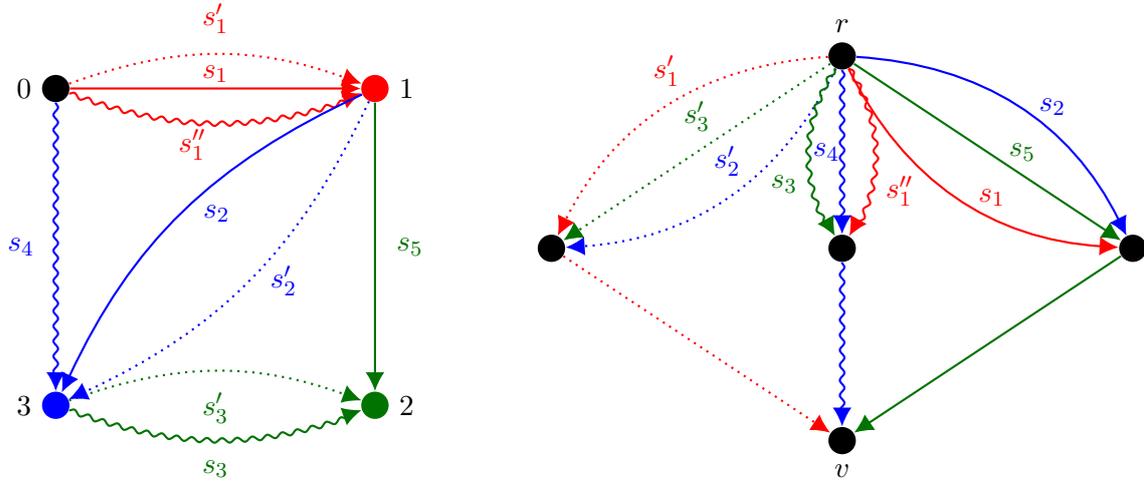
This idea is formalized in the following definition. Recall that $\overrightarrow{rT|_r}$ is the r -arborescence resulting from the orientation of the connected component of a forest T containing r .

Definition 4.1. Let $\vec{\mathcal{G}} = (\vec{G} = (V, A), \mathcal{M}, r)$ be an \mathcal{M} -arc-connected \mathcal{M} -digraph with \mathcal{M} a graphic matroid on the graph $G_{\mathcal{M}} = (V_{\mathcal{M}}, E_{\mathcal{M}})$ with $V_{\mathcal{M}} = \{0, 1, \dots, r(\mathcal{M})\}$. Given a set of arborescences $\vec{\mathcal{T}} = \{\vec{T}_1, \dots, \vec{T}_{r(\mathcal{M})}\}$ of $\vec{\mathcal{G}}$, we say that a tree T in $G_{\mathcal{M}}$ is $\vec{\mathcal{T}}$ -good if any arc \vec{e} in $\overrightarrow{0T|_0}$ has head $i \in V_{\mathcal{M}}$ if and only if e belongs to \vec{T}_i^- . We say that $\vec{\mathcal{T}}$ is good if, for all $v \in V - r$, the set of edges in $G_{\mathcal{M}}$ corresponding to $\vec{\mathcal{T}}(v)^-$ is a $\vec{\mathcal{T}}$ -good tree.

Figure 4.2 shows a good \mathcal{M} -packing of spanning arborescences of $\vec{\mathcal{G}} - v$, that can indeed be completed. However the packing of $\vec{\mathcal{G}} - v$ in Figure 4.1 is not good.

Theorem 4.1. Let $\vec{\mathcal{G}} = (\vec{G} = (V, A), \mathcal{M}, r)$ be an \mathcal{M} -arc-connected \mathcal{M} -digraph with \vec{G} acyclic, $|V| > 2$ and \mathcal{M} a graphic matroid on the graph $G_{\mathcal{M}} = (V_{\mathcal{M}}, r^+)$. Assume that v is a sink of \vec{G} and that $\vec{\mathcal{G}} - v$ has a good $(\mathcal{M} - v)$ -packing $\vec{\mathcal{T}}$ of spanning arborescences. Then $\vec{\mathcal{T}}$ can be completed in a good \mathcal{M} -packing of spanning arborescences of $\vec{\mathcal{G}}$.

Proof. If $\vec{\mathcal{G}}$ is not minimally \mathcal{M} -arc-connected, we can remove arcs until it becomes mini-



(a) A graphic matroid \mathcal{M} . In this particular example parallel edges have the same color but this is not the case in general.

(b) An \mathcal{M} -arc-connected \mathcal{M} -digraph $\vec{\mathcal{G}}$. $\vec{\mathcal{G}} - v$ has a good \mathcal{M} -packing of spanning arborescences $\vec{T}'_1 = \{s'_1, s''_1, s_1\}$, $\vec{T}'_2 = \{s'_3, s_3, s_5\}$, $\vec{T}'_3 = \{s'_2, s_4, s_2\}$ that can be completed into an \mathcal{M} -packing of $\vec{\mathcal{G}}$.

Figure 4.2: The completion of a good \mathcal{M} -packing is always possible.

mally \mathcal{M} -arc-connected. Therefore, we assume in the following that $\vec{\mathcal{G}}$ is minimally \mathcal{M} -arc-connected.

Among the arcs entering v in $\vec{\mathcal{G}}$, let s_1, \dots, s_p be those leaving r and $\vec{e}_1, \dots, \vec{e}_q$ the others. By minimality of $\vec{\mathcal{G}}$, $p + q = r(\mathcal{M})$. Let $T = (V_{\mathcal{M}}, \{s_1, \dots, s_p\}) \subseteq G_{\mathcal{M}}$. We add edges to T and arcs entering v to arborescences in $\vec{\mathcal{T}}$ while keeping the following properties:

1. T has no cycle.
2. The edges of $T|_0$ are the arcs of $\vec{\mathcal{T}}(v)^-$.
3. $T|_0$ is $\vec{\mathcal{T}}$ -good.
4. $\vec{e}_1, \dots, \vec{e}_{\|T\| - p}$ are in $\vec{\mathcal{T}}$ and $\vec{e}_{\|T\| - p + 1}, \dots, \vec{e}_q$ are not in $\vec{\mathcal{T}}$.

Initially, for each edge s of $T|_0$, we add s to \vec{T}_j , j being the head of \vec{s} in $\vec{0}T|_0$. 1. is true because $\{s_1, \dots, s_p\}$ is independent, by minimality of $\vec{\mathcal{G}}$. 2. and 3. are true by construction. 4. is true since $\|T\| - p = 0$ and no arc among $\vec{e}_1, \dots, \vec{e}_q$ is in $\vec{\mathcal{T}}$.

While possible, do the following, with $j = \|T\| - p + 1$:

- If $j = q + 1$ then $\|T\| = p + q = r(\mathcal{M})$. By 1., T is a spanning tree hence $\vec{\mathcal{T}}^-(v)$ is a base by 2. Since T is $\vec{\mathcal{T}}$ -good by 3., every edge of T belongs to a different arborescence

of $\vec{\mathcal{T}}$ hence every arc entering v was assigned to a different arborescence in $\vec{\mathcal{T}}$. Therefore $\vec{\mathcal{T}}$ is a good \mathcal{M} -packing of spanning arborescences of \mathcal{G} , which concludes the proof.

- If $j \leq q$ then let $T_{\vec{e}_j}$ be the $\vec{\mathcal{T}}$ -good spanning tree of $G_{\mathcal{M}}$ whose edges are the arcs of $\vec{\mathcal{T}}(\vec{e}_j^-)^-$. ${}^0T_{\vec{e}_j}$ has an arc \vec{e} leaving the vertex set V_0 of $T|_0$, since ${}^0T_{\vec{e}_j}$ is spanning while T is not. See Figure 4.3. For each edge s_i in $T|_{\vec{e}^+}$, add s_i to \vec{T}_j , j being the head of \vec{s}_i in $\vec{e}^+T|_{\vec{e}^+}$. Then add e to T and \vec{e}_j to $\vec{T}_{\vec{e}^+}$. 1. is still true since e was leaving a connected component of T . Since $T_{\vec{e}_j}$ is $\vec{\mathcal{T}}$ -good, $e \in \vec{T}_i$ so that e was added to $\vec{\mathcal{T}}(v)^-$. Thus the same elements were added to $T|_0$ and $\vec{\mathcal{T}}(v)^-$, proving 2. By the way we added elements, 3. is still true and 4. is trivially true.

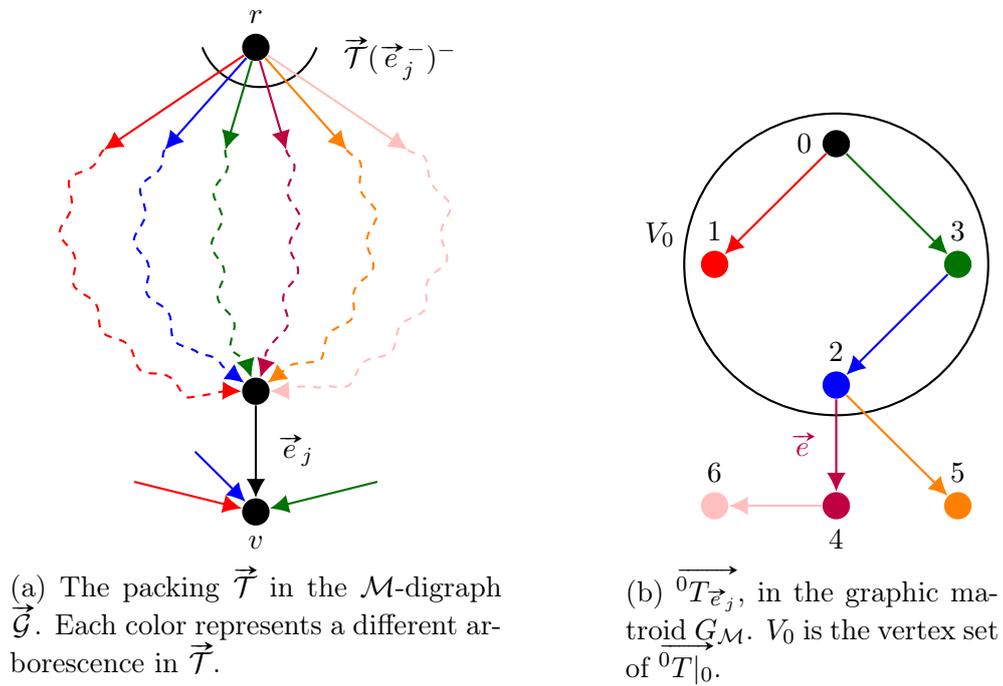


Figure 4.3: Illustration of the proof of Theorem 4.1.

□

Theorem 4.2. Let $\vec{\mathcal{G}}$ be an \mathcal{M} -arc-connected acyclic \mathcal{M} -digraph, with \mathcal{M} a graphic matroid on the graph $G_{\mathcal{M}} = (\{0, 1, \dots, r(\mathcal{M})\}, E)$. Then $\vec{\mathcal{G}}$ has a good \mathcal{M} -packing of spanning arborescences.

Proof. Assume that $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ is a counterexample with $|V|$ minimum. If $|V| = 2$ then let v be the vertex different from r . $\vec{E}(r, v)$ contains a spanning tree T of $G_{\mathcal{M}}$, since $\vec{\mathcal{G}}$ is \mathcal{M} -arc-connected. We create arborescences $\vec{T}_1, \dots, \vec{T}_{r(\mathcal{M})}$ containing initially only r . For each $s \in T|_0$, add s to \vec{T}_j , j being the head of \vec{s} in ${}^0T|_0$. This indeed gives a good \mathcal{M} -packing $\vec{T}_1, \dots, \vec{T}_{r(\mathcal{M})}$ of spanning arborescences in $\vec{\mathcal{G}}$.

So we can assume $|V| > 2$. Let v be a sink in \vec{G} . By Lemma 3.14, $\vec{G} - v$ is $(\mathcal{M} - v)$ -arc-connected and is not a counterexample hence $\vec{G} - v$ has a good $(\mathcal{M} - v)$ -packing of spanning arborescences, which can be completed with Theorem 4.1, providing a contradiction. \square

Since finding a sink and an orientation of a tree from a root can trivially be done in polynomial time, the proofs of Theorem 4.2 and Theorem 4.1 implies a polynomial algorithm to find a good \mathcal{M} -packing of spanning arborescences in an acyclic \mathcal{M} -arc-connected \mathcal{M} -digraph, with \mathcal{M} a graphic matroid. By Lemma 3.11, there is also a polynomial algorithm to find a minimum weight such packing.

4.3 Fano matroid

We show in this subsection that Conjecture 3.1 holds true for the Fano matroid \mathcal{F}_7 and for an acyclic digraph, by looking at all the possibilities for the completion of an \mathcal{F}_7 -packing of spanning arborescences.

Theorem 4.3. *Let $\vec{G} = (\vec{G} = (V, A), \mathcal{F}_7, r)$ be a minimally \mathcal{F}_7 -arc-connected \mathcal{F}_7 -digraph with \vec{G} acyclic and $|V| > 2$. Assume that v is a sink of \vec{G} and that $\vec{G} - v$ has an $(\mathcal{F}_7 - v)$ -packing $\vec{T} = \{\vec{T}_1, \vec{T}_2, \vec{T}_3\}$ of spanning arborescences.*

Then there is a rainbow base completion for v , hence \vec{T} can be completed in an \mathcal{F}_7 -packing of spanning arborescences of \vec{G} .

Proof. Since \vec{G} is minimally \mathcal{F}_7 -arc-connected, we can write $v^- = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, the arcs of v^- leaving r form an independent set of \mathcal{F}_7 and every arc of $\vec{G} - v$ is contained in \vec{T} . We look at the number of arcs leaving r in v^- :

- If \vec{e}_1, \vec{e}_2 , and \vec{e}_3 are leaving r : $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is a base of \mathcal{F}_7 and we can assign $\vec{e}_1, \vec{e}_2, \vec{e}_3$ to $\vec{T}_1, \vec{T}_2, \vec{T}_3$ to get a \mathcal{F}_7 -packing of spanning arborescences of \vec{G} .
- If \vec{e}_1 and \vec{e}_2 are leaving r but not \vec{e}_3 : $\{\vec{e}_1, \vec{e}_2\}$ is an independent set of \mathcal{F}_7 and thus, since $\vec{T}(\vec{e}_3^-)^-$ is a base of \mathcal{F}_7 , there exists $\vec{e} \in \vec{T}(\vec{e}_3^-)^-$ such that $\{\vec{e}_1, \vec{e}_2, \vec{e}\}$ is a base of \mathcal{F}_7 . Assume without loss of generality that $\vec{e} \in \vec{T}_3^-$. Then we can add $\vec{e}_1, \vec{e}_2, \vec{e}_3$ to $\vec{T}_1, \vec{T}_2, \vec{T}_3$, respectively, in order to complete \vec{T} .
- If \vec{e}_1 is leaving r but not \vec{e}_2 and \vec{e}_3 : let $\vec{T}(\vec{e}_i^-)^- = \{x_i, y_i, z_i\}$ with $x_i \in \vec{T}_1^-$, $y_i \in \vec{T}_2^-$, $z_i \in \vec{T}_3^-$ for $i \in \{2, 3\}$. Since $y_2 \neq \vec{e}_1$, $\{\vec{e}_1, y_2\}$ is independent (every pair of distinct elements in \mathcal{F}_7 is independent). x_3 and z_3 are two elements different from \vec{e}_1 and y_2 , therefore at least one of them, say z_3 , is not on the line $[\vec{e}_1, y_2]$ defined by \vec{e}_1 and y_2 . Thus $\{\vec{e}_1, y_2, z_3\}$ is a base and we can add $\vec{e}_1, \vec{e}_2, \vec{e}_3$ to $\vec{T}_1, \vec{T}_2, \vec{T}_3$, respectively, to complete \vec{T} .
- If no arc entering v is leaving r : let $\vec{T}(\vec{e}_i^-)^- = B_i = \{x_i, y_i, z_i\}$ with $x_i \in \vec{T}_1^-$, $y_i \in \vec{T}_2^-$, $z_i \in \vec{T}_3^-$. Since \mathcal{F}_7 is symmetric, we can represent it as in Figure 4.4, each color representing an arborescence.

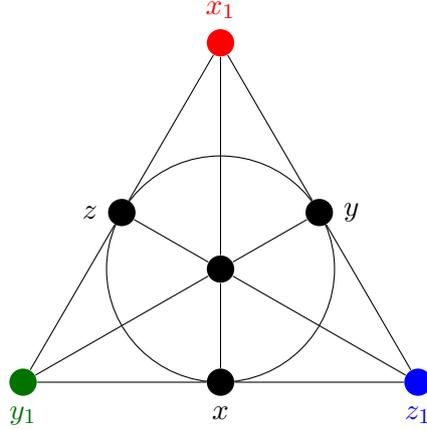


Figure 4.4: Representation of \mathcal{F}_7 in which $x_1 \in \vec{T}_1^-$, $y_1 \in \vec{T}_2^-$, $z_1 \in \vec{T}_3^-$.

B_1, B_2 and B_3 cannot be all disjoint since \mathcal{F}_7 has only 7 elements. Without loss of generality, $B_1 \cap B_2 \neq \emptyset$.

Assume first that $B_1 = B_2$. Since B_3 is a base, $\{x_3, y_3, z_3\} \neq \{x, y, z\}$ (using the notations of Figure 4.4), say $x_3 \neq x$. Then $\{y_1, z_2, x_3\}$ is a rainbow base completion.

Therefore we can assume that $B_1 \neq B_2$. Without loss of generality, $x_1 = x_2 \in B_1 \cap B_2$ and $y_1 \in B_1 \setminus B_2$, $y_2 \in B_2 \setminus B_1$. Since $y_1 \neq y_2$ and there is only one line containing $\{x_1, z_3\} = \{x_2, z_3\}$, it is not possible that both $\{y_1, x_2, z_3\}$ and $\{x_1, y_2, z_3\}$ are on a line. Hence at least one of them is a rainbow completion base.

□

We deduce:

Corollary 4.2. *Conjecture 3.1 is true for $\vec{\mathcal{G}}$ acyclic and $\mathcal{M} = \mathcal{F}_7$.*

Furthermore, by Lemma 3.11 and since the tests in the proof of Theorem 4.3 can be done in polynomial time, there is also a polynomial algorithm to find a minimum weight \mathcal{F}_7 -packing of spanning arborescences in an acyclic \mathcal{F}_7 -arc-connected \mathcal{F}_7 -digraph.

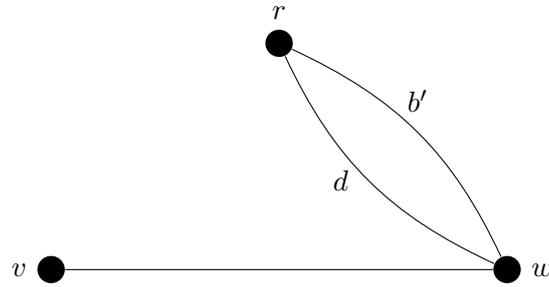
4.4 Parallel extensions of the Fano matroid

In this section, $\mathcal{F}_7^{\parallel}$ is a parallel extension of the Fano matroid and $\mathcal{G} = (G = (V, E), \mathcal{F}_7^{\parallel}, r)$ is an $\mathcal{F}_7^{\parallel}$ -partition-connected $\mathcal{F}_7^{\parallel}$ -graph. Moreover, in this section, two matroid elements starting with the same letter (for example, a and a') will always be parallel elements. We assume that \mathcal{G} has an $\mathcal{F}_7^{\parallel}$ -packing \mathcal{T} of spanning trees and we will obtain a contradiction, thus disproving Conjecture 3.2. \mathcal{G} will be constructed inductively with the operations that we describe below. These operations will preserve $\mathcal{F}_7^{\parallel}$ -partition-connectivity by Corollary 4.1. Every operation

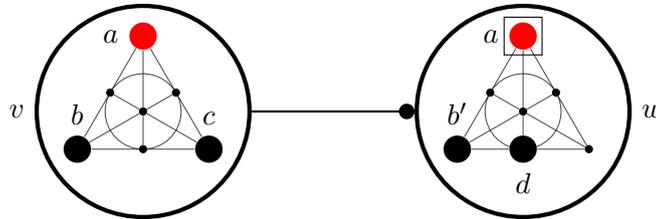
will have an *implementation*, explaining how it modifies the graph, and a *representation*, which will be used as a shortcut to refer to this operation.

Let $v \in V$ and B a base of $\mathcal{F}_7^{\parallel}$. We say that B is a **forced base** for v (in \mathcal{G}) if, necessarily, $\mathcal{T}^-(v) = B$. Let e_1, e_2 be two edges in \mathcal{G} . We say that e_1 and e_2 have the **same color**, and we write $e_1 \equiv e_2$, if e_1 and e_2 are in the same tree of \mathcal{T} .

Let $v \in V$ and $B = \{a, b, c\}$ a base of $\mathcal{F}_7^{\parallel}$. Assume that B is a forced base for v . The **Move** operation on \mathcal{G} is defined as follow: add a new vertex w to G , one edge between v and w , and two edges b', d between r and w such that $b' \parallel b$ and $d \in]bc[$ (see Figure 4.5a). Then $\{a, b', d\}$ is a forced base for w , since from B only a can be added to $\{b', d\}$ to form a base. Clearly, $\{a, b, c\}$ is still a forced base for v after Move. We will represent the Move operation



(a) Implementation of the Move operation. Move adds a vertex w with three incoming edges.



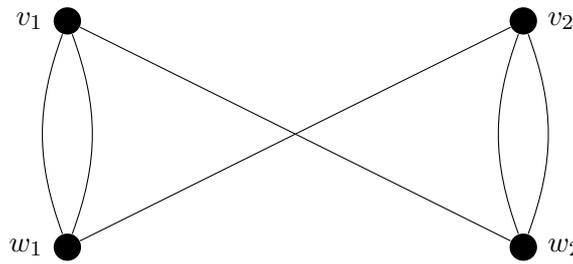
(b) Representation of the Move operation. The square indicates the element a which belongs to both bases. The circle at the end of the edge indicates the new vertex.

Figure 4.5: The Move operation. v has forced base $\{a, b, c\}$ and w is a new vertex with forced base $\{a, b', d\}$.

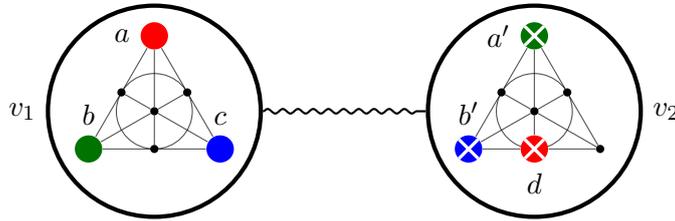
as in Figure 4.5b. For Move and the other operations described below, we always denote by v an existing vertex and by w a new vertex, created by the operation.

The **Not All Different Colors (NADC)** operation requires two existing vertices v_1 and v_2 with forced bases $B_1 = \{a, b, c\}$ and $B_2 = \{a', b', d\}$ with $a \parallel a'$, $b \parallel b'$ and $d \in]bc[$. NADC creates two new vertices w_1 and w_2 as shown in Figure 4.6a. This operation is represented by Figure 4.6b (in which $d \in]bc[$). After the operation, we claim that it is not possible that

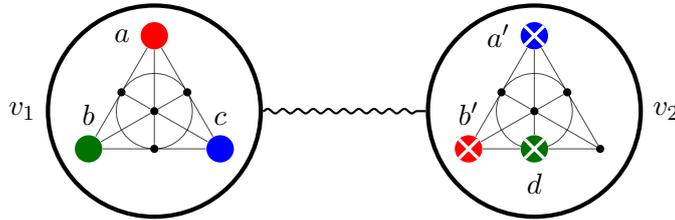
simultaneously $a \neq a'$, $b \neq b'$ and $c \neq d$. Assume by contradiction that it is the case. We also assume that $d \in]bc[$, but the proof is similar if $d \parallel c$. The two possibilities are depicted on Figure 4.6b and Figure 4.6c. Figure 4.6b is impossible since in this configuration, it is not possible to pick one element in $\{a', b', d\}$ and two elements in $\{a, b, c\}$ to form a rainbow base: for example, $\{d, b, c\}$ is dependent. This means that, in this configuration, there is no possible choice for the colors of the edges entering w_1 resulting in an \mathcal{F}_7^\parallel -packing of spanning trees. Since we assumed for the sake of contradiction that \mathcal{G} has such a packing, we can exclude this possibility. Similarly, by considering w_2 , Figure 4.6c is excluded and we proved the claim. Note that w_1 and w_2 are created only to add constraints to the coloring of v_1 or v_2 , they will not be used later in the proof.



(a) Implementation of the Not All Different Colors (NADC) operation, adding two vertices w_1 and w_2 .



(b) Representation of NADC. It is impossible that all colors are different (case 1).

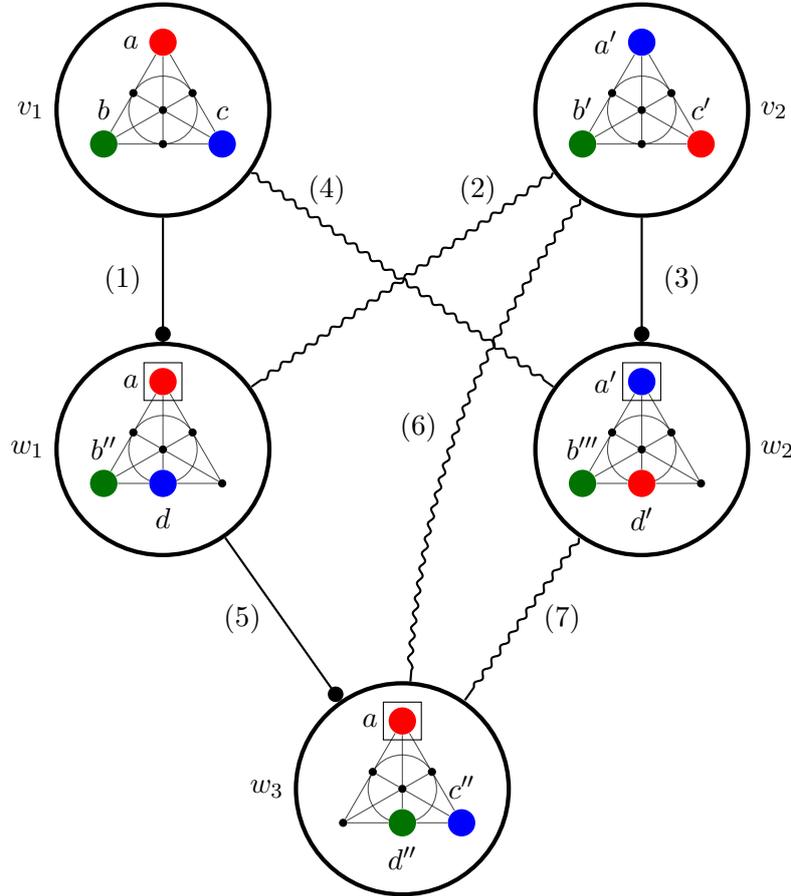


(c) It is not possible that all colors are different (case 2).

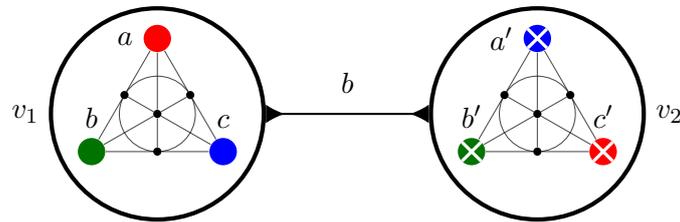
Figure 4.6: The NADC operation. v_1 and v_2 are two existing vertices with forced base $\{a, b, c\}$ and $\{a', b', d\}$.

The next operation is **Not-Exchange** (NEx), using Move and NADC. This is an operation on two existing vertices v_1, v_2 , with forced bases $\{a, b, c\}, \{a', b', c'\}$, which adds three new vertices w_1, w_2, w_3 as shown on Figure 4.7. After this operation, we show by contradiction

that, in any $\mathcal{F}_7^{\parallel}$ -packing of spanning trees, it is not possible that $b \equiv b'$, $a \not\equiv a'$, and $c \not\equiv c'$. Assume that this is the case, as shown on Figure 4.7. (1) creates a new vertex w_1 . (2) implies that $b'' \equiv b'$, $d \equiv a'$. Similarly, (3), (4) implies the configuration of w_2 shown on Figure 4.7, and (5), (6) the one of w_3 . Finally (7) provides a contradiction: all colors of w_2 and w_3 are different.



(a) Implementation of the NEx operation.

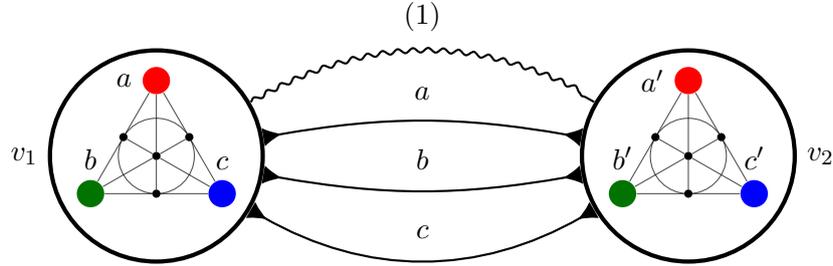


(b) Representation of the NEx operation. We indicate the matroid element b used. The crosses indicate that this color configuration is impossible.

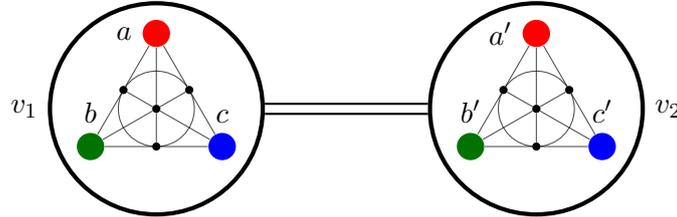
Figure 4.7: Implementation of the NEx operation. v_1 has forced base $\{a, b, c\}$, v_2 has forced base $\{a', b', c'\}$.

Using the same vertices v_1 and v_2 , we can apply multiple times the previous operation

to get the operation **Same Colors** (SC), implemented on Figure 4.8a. (1) implies that the colors of a, b, c and a', b', c' can not be all different, and the three NEx operations imply that $a \equiv a', b \equiv b', c \equiv c'$, as shown on the representation of Figure 4.8b.



(a) Implementation of the SC operation.



(b) Representation of the SC operation.

Figure 4.8: The SC operation.

Our next operation, **Move Same Colors** (MSC) requires one vertex v with forced base $\{a, b, c\}$ and produces a vertex w with forced base $\{a, b', d\}$, with $d \in]bc[$. Moreover we will prove that $b \equiv b', d \equiv c$ and MSC will be represented as in Figure 4.9. MSC uses Move and SC in its implementation of Figure 4.10.

Assume by contradiction that $b' \equiv c$ and $d \equiv b$, as in Figure 4.10. We show first how to deduce the colorings of w_3 and w_4 of Figure 4.10. b' is blue, since it is blue in w . a' can not be green since, in w_1 , $a' \neq b$. Since $a' \equiv a''$ by (5), a'' is not green and is not blue hence a'' is red and g is green, proving that the coloring of w_3 in Figure 4.10 is correct.

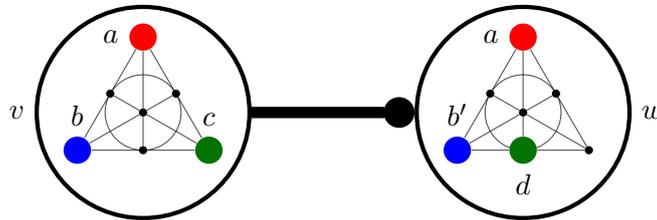


Figure 4.9: Representation of the MSC operation. v has forced base $\{a, b, c\}$, w is a new vertex with forced base $\{a, b', d\}$ and $b \equiv b', c \equiv d$.

The proof for w_4 is similar: c is blue because it is blue in v , and f is red since it can not

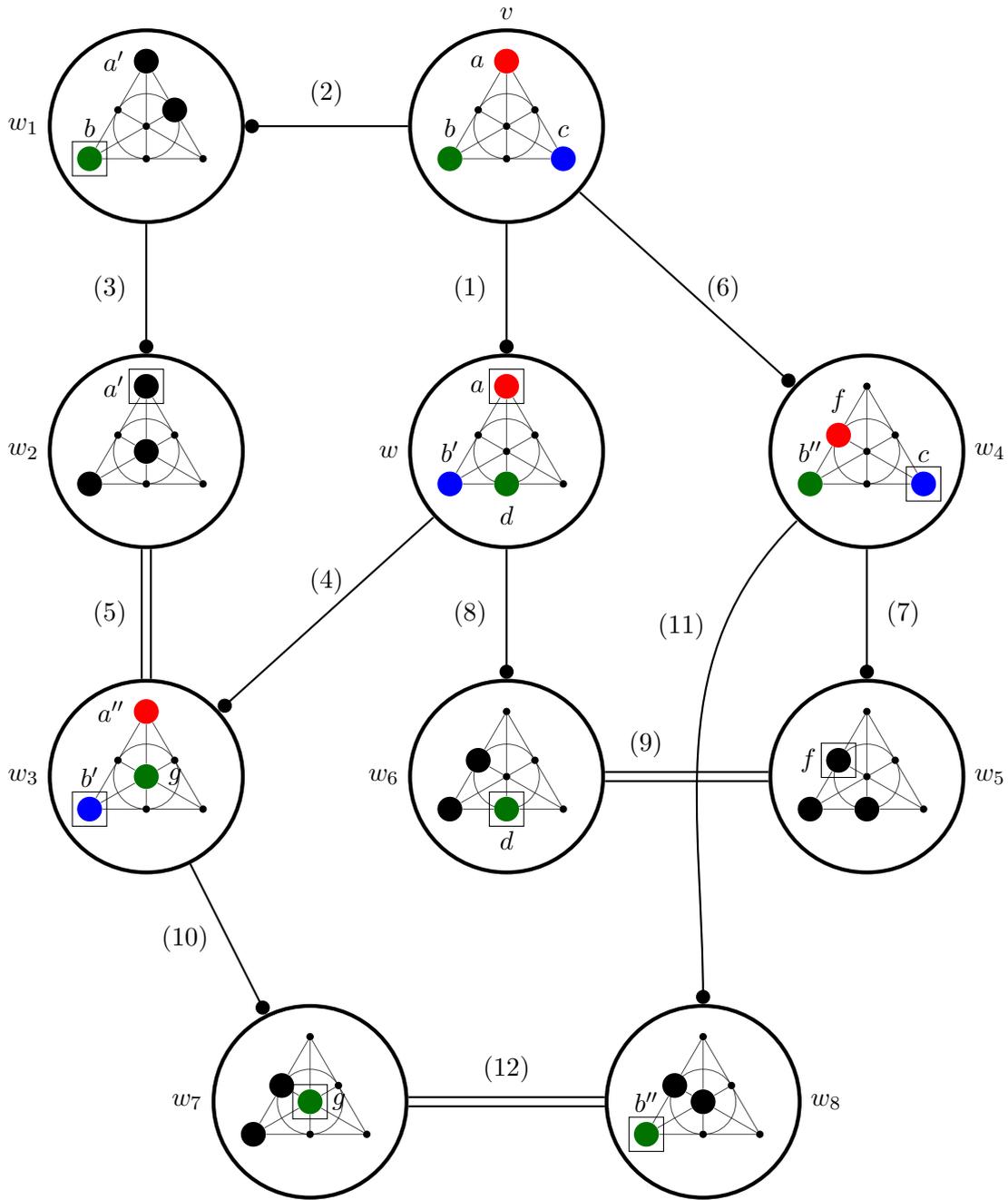


Figure 4.10: Implementation of the Move Same Colors (MSC) operation. v has forced base $\{a, b, c\}$, w is a new vertex with forced base $\{a, b', d\}$.

be blue and it can not be green by (9). It follows that b'' is green, and the coloring of w_4 is indeed the one of Figure 4.10. Finally, w_7 and w_8 provide a contradiction: g and b'' must be green, by (10) and (11), but (12) implies that they must have different colors.

The last operation, **Exchange (Ex)**, takes a vertex v with forced base $\{a, b, c\}$ and creates a new vertex w with forced base a, b', c' such that $b \equiv c'$ and $c \equiv b'$, as shown on Figure 4.11.

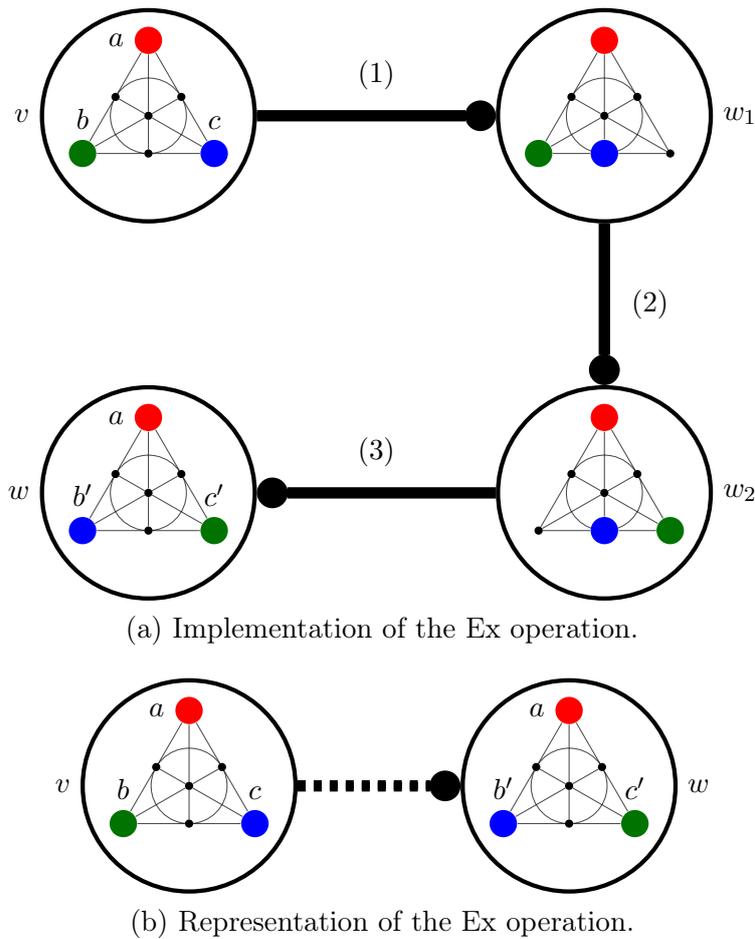


Figure 4.11: The Exchange (Ex) operation.

We are now ready to exhibit a counterexample to [Conjecture 3.1](#). Indeed, we claim that the $\mathcal{F}_7^{\parallel}$ -graph \mathcal{G} of [Figure 4.12](#), albeit $\mathcal{F}_7^{\parallel}$ -partition-connected, has no $\mathcal{F}_7^{\parallel}$ -packing of spanning trees. The three edges a, b, c , colored arbitrarily red, green, blue, form a base of $\mathcal{F}_7^{\parallel}$. This $\mathcal{F}_7^{\parallel}$ -graph \mathcal{G} is $\mathcal{F}_7^{\parallel}$ -partition-connected, by [Corollary 4.1](#).

However, we prove by contradiction that \mathcal{G} has no $\mathcal{F}_7^{\parallel}$ -packing of spanning trees: if it was the case, v has forced base $\{a, b, c\}$, and the Ex operation implies that b', c' must be the same color as c, b , respectively, which is in contradiction with the NEx operation. This shows that \mathcal{G} has no $\mathcal{F}_7^{\parallel}$ -packing of spanning trees, and we conclude that [Conjecture 3.2](#) is false.

Moreover, the \mathcal{M} -digraph obtained by orientation of \mathcal{G} as in the proof of [Corollary 4.1](#) is an acyclic \mathcal{M} -arc-connected \mathcal{M} -digraph without \mathcal{M} -packing of spanning arborescences (otherwise \mathcal{G} would have an \mathcal{M} -packing of spanning trees), proving that [Conjecture 3.1](#) is false even when restricted to acyclic \mathcal{M} -digraphs.

How big is our counterexample \mathcal{G} and $\mathcal{F}_7^{\parallel}$? [Table 4.1](#) summarizes the number of vertices and matroid elements added for each operation. Since \mathcal{G} uses two vertices r and v , plus Ex

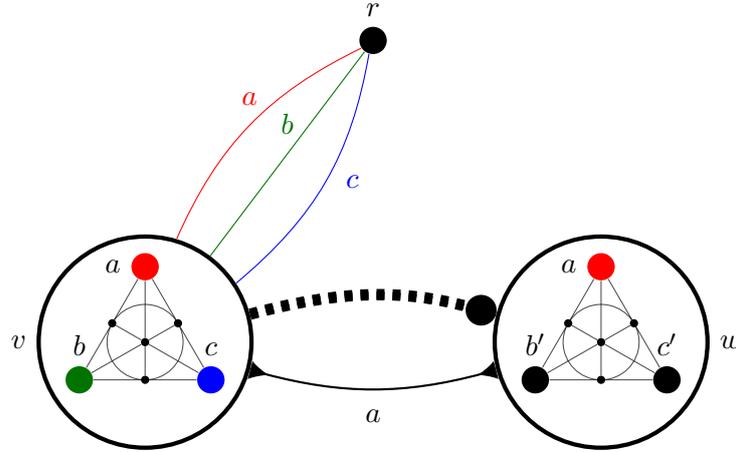


Figure 4.12: A $\mathcal{F}_7^{\parallel}$ -partition-connected $\mathcal{F}_7^{\parallel}$ -graph without $\mathcal{F}_7^{\parallel}$ -packing of spanning trees.

	Move	NADC	NEx	SC	MSC	Ex
Operations used			3Move + 4NADC	3NEx + NADC	9Move + 3SC	3MSC
Vertices	1	2	11	35	114	342
Matroid elements	2	0	6	18	72	216

Table 4.1: Number of vertices and matroid elements added for each operation.

and NEx, \mathcal{G} has a total of 355 vertices and $\mathcal{F}_7^{\parallel}$ has 225 elements.

4.5 Complexity

In section 2.7, we saw that it is possible to decide in polynomial time if a digraph (respectively graph) contains a packing of spanning arborescences (respectively trees). Consider the corresponding problems with matroid constraints:

Problem 4.1. Let \mathcal{M} be a matroid and let $\vec{\mathcal{G}}$ be an \mathcal{M} -digraph. Does $\vec{\mathcal{G}}$ have an \mathcal{M} -packing of spanning arborescences?

Problem 4.2. Let \mathcal{M} be a matroid and let \mathcal{G} be an \mathcal{M} -graph. Does \mathcal{G} have an \mathcal{M} -packing of spanning trees?

Problem 4.1 was proved NP-complete by Király, Szigeti, Tanigawa and I in [For+17]:

Theorem 4.4. Problem 4.1 is NP-complete even if \mathcal{M} is of rank 3 and $\vec{\mathcal{G}}$ is acyclic.

The proof reduces 3-SAT to Problem 4.1 in the particular case where \mathcal{M} is a parallel extension of \mathcal{F}_7 , by using operations which are similar to the operations used in the previous section. However, the complexity of Problem 4.2 is still unknown, even if we strongly believe that it is NP-complete as well.

Other notions of connectivity with matroid constraints

Contents

5.1	Connectivity with matroid constraints in hypergraphs	65
5.1.1	Trimming hypergraphs to graphs	66
5.1.2	Hypertrees and hyperarborescences	67
5.1.3	Connectivity in hypergraphs	69
5.1.4	Orientation	70
5.1.5	Hypergraphs with matroid constraints	71
5.2	Mixed hypergraphs	72
5.3	Reachability-packing of hyperarborescences	74
5.3.1	Reachability-packing in digraphs	74
5.3.2	Reachability-packing in \mathcal{M} -digraphs	75
5.3.3	\mathcal{M} -Reachability-packing in hypergraphs	77
5.3.4	Algorithmic aspects	78

We study some generalizations of matroid-connectivity to other classes of graphs. The main content of this chapter is covered, in a slightly different form, in [For+16].

5.1 Connectivity with matroid constraints in hypergraphs

Informally, a hypergraph is a generalization of a graph in which the edges can have more than two vertices, hence called hyperedges. Hypergraphs are very general: for example, a matroid $\mathcal{M} = (S, \mathcal{I})$ without loop can also be seen as a hypergraph, with vertex set S and hyperedges the sets of \mathcal{I} . Hence, there are many polynomial problems on graphs whose hypergraph counterparts become NP-complete. For example, deciding if a graph contains a perfect matching can be solved in polynomial time, as shown by Edmonds in [Edm65b], but it was proved NP-complete for hypergraphs by Karp in [Kar72].

5.1.1 Trimming hypergraphs to graphs

Definition 5.1. A *hypergraph* is a 2-tuple $H = (V, \mathcal{E})$, V and \mathcal{E} being finite sets. The elements of V are called *vertices* and the elements of \mathcal{E} are called *hyperedges*. Each hyperedge $\varepsilon \in \mathcal{E}$ is associated to a set of at least two vertices, denoted by $V(\varepsilon) \subseteq V$.

By forbidding hyperedges of size one we avoid loops, as we did for graphs.

Definition 5.2. A *directed hypergraph* (or *dypergraph*) is a 2-tuple $\vec{H} = (V, \vec{\mathcal{E}})$, V and $\vec{\mathcal{E}}$ being finite sets. The elements of V are called *vertices* and the elements of $\vec{\mathcal{E}}$ are called *hyperarcs*. Each hyperarc $\vec{\varepsilon} \in \vec{\mathcal{E}}$ has a set of at least one *tail*, denoted by $\vec{\varepsilon}^- \subseteq V$, and a *head* $\vec{\varepsilon}^+ \in V$ such that $\vec{\varepsilon}^+ \notin \vec{\varepsilon}^-$.

The following trimming operation introduced by Frank ([Fra11]) can be used to extend a lot of definitions from graph theory to hypergraph theory:

Definition 5.3. *Trimming* a hyperedge ε means replacing it with an edge e such that $V(e) \subseteq V(\varepsilon)$. *Trimming* a hyperarc $\vec{\varepsilon}$ means replacing it with an arc \vec{e} such that $\vec{e}^+ = \vec{\varepsilon}^+$ and $\vec{e}^- \in \vec{\varepsilon}^-$. We say that a hypergraph (V, \mathcal{E}) can be **trimmed** to a graph (V', E) if E can be obtained by trimming the hyperarcs in \mathcal{E} , and V' is obtained from V by removing some vertices not used by E . Similarly, a dypergraph $(V, \vec{\mathcal{E}})$ can be **trimmed** to a digraph (V', \vec{E}) if \vec{E} can be obtained by trimming the hyperarcs in $\vec{\mathcal{E}}$, and V' is obtained from V by removing some vertices not used by \vec{E} .

In general, the trimmed graph or digraph is not unique: a hypergraph can be trimmed to several graphs. Using this trimming operation, we define **$u - v$ hyperpaths**, **$u - v$ dyperpaths**, **hypertrees**, **hyperforests**, **r -hyperarborescences** and **r -hyperbranches** as the hypergraphs or dypergraphs that can be trimmed to $u - v$ paths, $u - v$ dipaths, trees, forests, r -arborescences and r -branches, respectively. Moreover an r -hyperarborescence or a hypertree is **spanning** if it can be trimmed to a spanning r -arborescence or spanning tree, respectively. The definitions of Frank, in [Fra06], are equivalent to ours, albeit in different forms.

The notions of subgraphs, disjoint graphs, orientations and cuts extend naturally to hypergraphs and dypergraphs. We also define, for a dypergraph $\vec{H} = (V, \vec{\mathcal{E}})$ and $X \subseteq V$:

$$\vec{\mathcal{E}}_{\vec{H}}^-(X) = \{\vec{\varepsilon} \in \vec{\mathcal{E}} : \vec{\varepsilon}^+ \in X, \vec{\varepsilon}^- \not\subseteq X\}.$$

A hyperarc $\vec{\varepsilon} \in \vec{\mathcal{E}}$ **enters** X if $\vec{\varepsilon} \in \vec{\mathcal{E}}_{\vec{H}}^-(X)$.

Moreover, if $H = (V, \mathcal{E})$ is a hypergraph and \mathcal{V} a set of disjoint subsets of V , $\mathcal{E}_H(\mathcal{V})$ is the set of hyperedges of H intersecting at least two sets of \mathcal{V} and $\varepsilon_H(\mathcal{V}) = |\mathcal{E}_H(\mathcal{V})|$. Again, we may drop the \vec{H} or H subscript if the context is clear.

The following lemma shows that the in-degree function $\rho_{\vec{H}}$ of a dypergraph \vec{H} , defined by $\rho_{\vec{H}}(X) = |\vec{\mathcal{E}}^-(X)|$, is, again, submodular:

Lemma 5.1. *Let $\vec{H} = (V, \vec{\mathcal{E}})$ be a dypergraph and $X \subseteq V, Y \subseteq V$. Let $d_H(X, Y)$ be the number of hyperedges $\varepsilon \in \mathcal{E}$ such that $V(\varepsilon) \subseteq X \cup Y, \varepsilon \cap X \neq \emptyset, \varepsilon \cap Y \neq \emptyset$. Then:*

$$\rho_{\vec{H}}(X) + \rho_{\vec{H}}(Y) = \rho_{\vec{H}}(X \cap Y) + \rho_{\vec{H}}(X \cup Y) + d_H(X, Y).$$

Like Lemma 2.2 for digraphs, Lemma 5.1 can be proved by considering the different possibilities for a hyperarc entering X or Y (see Figure 5.1).

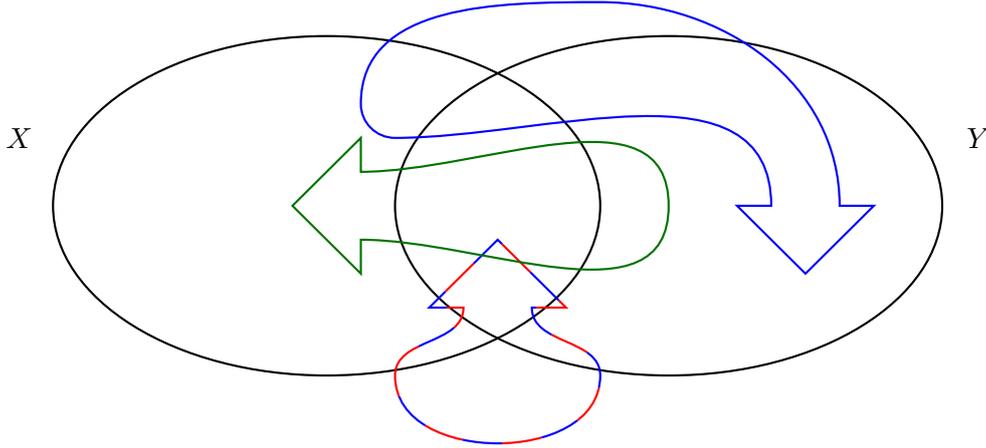


Figure 5.1: Illustration of the different types of hyperarcs in the formula $\rho_{\vec{H}}(X) + \rho_{\vec{H}}(Y) = \rho_{\vec{H}}(X \cap Y) + \rho_{\vec{H}}(X \cup Y) + d_H(X, Y)$

5.1.2 Hypertrees and hyperarborescences

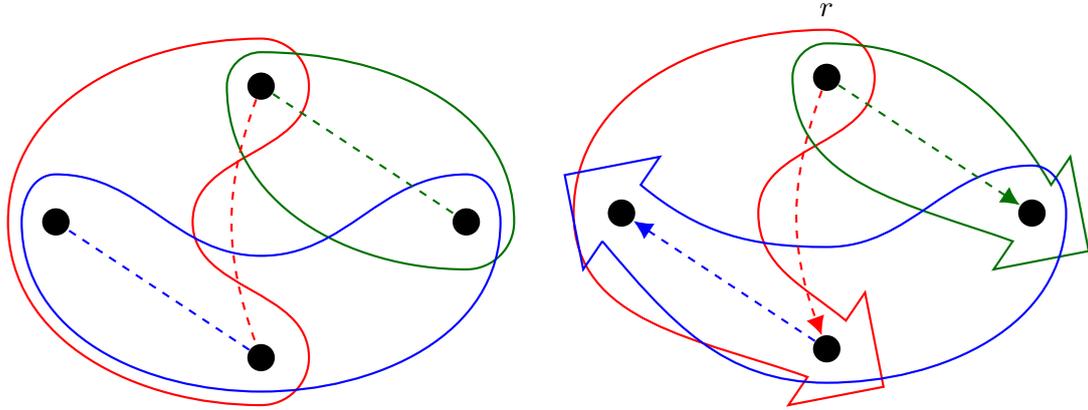
Figure 5.2a is an example of a spanning hypertree T and Figure 5.2b is a spanning r -hyperarborescence \vec{T} obtained by orientation of T . Note that such an orientation is not unique, in general: a hypertree may have several orientations resulting in an r -hyperarborescence. Unlike arborescences, it is false that an r -hyperarborescence has a unique dyperpath from r to another vertex: indeed, the r -hyperarborescence of Figure 5.2b has two different dyperpaths from r to v , represented on Figure 5.3a and Figure 5.3b. Likewise, a hypertree may have several hyperpaths between two vertices.

Frank, in [Fra11], explains how to deduce from a result of Lovász ([Lov70]) the following characterization of hyperforests:

Lemma 5.2. *A hypergraph is a hyperforest if and only if the union of any $j \geq 1$ hyperedges has at least $j + 1$ vertices.*

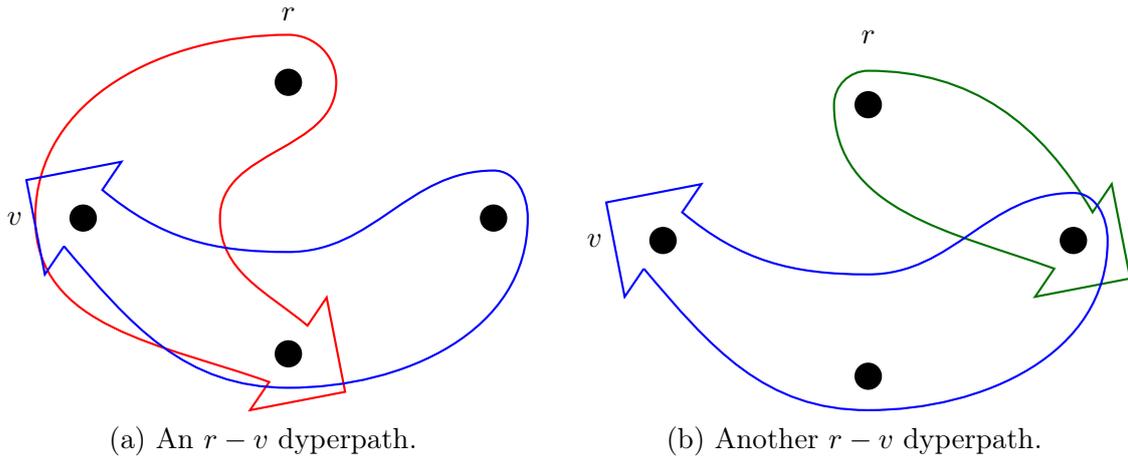
Using this lemma, we deduce the following characterizations of hypertrees:

Theorem 5.1. *Let $H = (V, \mathcal{E})$ be a hypergraph and $T = (V', \mathcal{E}')$ a subhypergraph of H . The following conditions are equivalent:*



(a) A hypertree T with a trimmed tree, dashed. Every hyperedge is trimmed to the edge with same color. (b) An r -hyperarborescence obtained by orientation of T . A trimmed r -arborescence is represented, dashed. Every hyperarc is trimmed to the arc with same color.

Figure 5.2: A spanning hypertree with a spanning r -hyperarborescence orientation.



(a) An $r-v$ dyperpath.

(b) Another $r-v$ dyperpath.

Figure 5.3: The r -hyperarborescence of Figure 5.2b has two different $r-v$ dyperpaths.

- T is a hypertree.
- T has $|V'| - 1$ edges and the union of any $j \geq 1$ of them has at least $j + 1$ vertices.
- If $r \in V'$, T can be oriented to become an r -hyperarborescence.
- T is connected and $|\mathcal{E}'| = |V'| - 1$.

Theorem 2.4 easily implies the following counterpart for hyperarborescences:

Theorem 5.2. Let $\vec{H} = (V, \vec{\mathcal{E}})$ be a dypergraph and $\vec{T} = (V', \vec{\mathcal{E}'})$ a subdypergraph of \vec{H} . Let $r \in V'$ and V'_0 be the set of vertices in V' whose in-degree is not 0 in \vec{T} . Then the following conditions are equivalent:

1. \vec{T} is an r -hyperarborescence.
2. T is a hypertree, $\rho_{\vec{T}}(r) = 0$, and $\rho_{\vec{T}}(v) = 1$, $\forall v \in V_0'$.
3. $\rho_{\vec{T}}(r) = 0$, $\rho_{\vec{T}}(v) = 1 \forall v \in V_0'$, and $\rho_{\vec{T}}(X) \geq 1$ for every r -cut X of \vec{T} with $X \cap V_0' \neq \emptyset$.

We need to consider this set V_0' in the previous theorem because it is possible that a vertex belongs to a hyperarc in \vec{T} but is not entered by any hyperarc of \vec{T} , as it is the case for the vertex v in Figure 5.4.

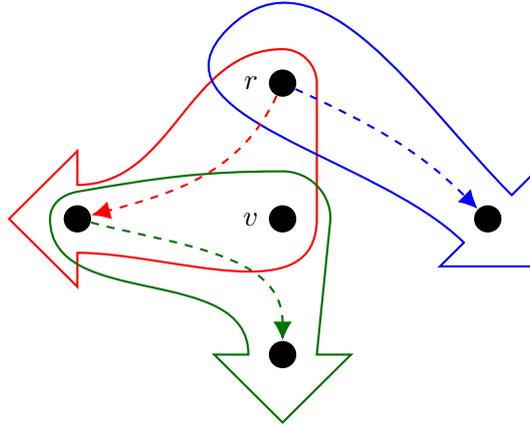


Figure 5.4: A non-spanning r -hyperarborescence.

5.1.3 Connectivity in hypergraphs

Connectivity in graphs and digraphs can be extended to hypergraphs and dypergraphs in a straightforward manner:

Definition 5.4. Let $k \in \mathbb{N}^*$. A hypergraph $H = (V, \mathcal{E})$ is **k -hyperedge-connected** if there exist k edge-disjoint $u - v$ hyperpaths for all $u, v \in V$.

A hypergraph $H = (V, \mathcal{E})$ is **k -partition-connected** if, for every partition \mathcal{V} of V :

$$\varepsilon_H(\mathcal{V}) \geq k(|\mathcal{V}| - 1).$$

A dypergraph $\vec{H} = (V, \vec{\mathcal{E}})$ is **k -hyperarc-connected** from $r \in V$ if there exist k hyperarc-disjoint $r - v$ dyperpaths for all $v \in V - r$.

Does the trimming operation preserve connectivity? The hypertree of Figure 5.2a is 2-hyperedge-connected, even though it can not be trimmed to a 2-edge-connected graph, hence the answer is *no* for hyperedge-connectivity. However, Frank proved in [Fra11] that partition-connectivity and hyperarc-connectivity behave more nicely with respect to trimming:

Theorem 5.3. Let H be a hypergraph and $k \in \mathbb{N}^*$. If H is k -partition-connected, then H can be trimmed to a k -partition-connected graph.

Theorem 5.4. *Let \vec{H} be a dypergraph and $k \in \mathbb{N}^*$. If \vec{H} is k -hyperarc-connected from a vertex r , then \vec{H} can be trimmed to a digraph which is k -arc-connected from r .*

To the best of our knowledge, there is no known efficient algorithm to find the trimmed graph or digraph of the two previous theorems. Using these trimming reductions, Frank, T. Király and Z. Király ([FKK03a]) extended Edmonds' packing theorem ([Theorem 2.8](#)) and Tutte's packing theorem ([Theorem 2.9](#)):

Theorem 5.5 (Edmonds' packing theorem for dypergraphs). *Let $\vec{H} = (V, \vec{\mathcal{E}})$ be a dypergraph, $r \in V$ and $k \in \mathbb{N}^*$. Then \vec{H} has a packing of k spanning hyperarborescences if and only if \vec{H} is k -hyperarc-connected from r .*

Theorem 5.6 (Tutte's packing theorem for hypergraphs). *Let $H = (V, \mathcal{E})$ be a hypergraph and $k \in \mathbb{N}^*$. Then H has a packing of k spanning hypertrees if and only if H is k -partition-connected.*

If $H = (V, \mathcal{E})$ is a hypergraph, Lorea proved in [Lor75] that the set of hyperedges of hyperforests included in H form the family of independent sets of a matroid on ground set \mathcal{E} , called a **hypergraphic matroid**. This provides another proof for [Theorem 5.6](#), as shown by Frank, T. Király and Krisell in [FKK03b].

We introduce a more general trimming theorem than [Theorem 5.4](#), which will be used for \mathcal{M} -hypergraphs and \mathcal{M} -dypergraphs. Its proof, postponed to the appendix, is obtained from the proof of [Theorem 5.4](#) with the necessary straightforward modifications:

Theorem 5.7. *Let $\vec{H} = (V, \vec{\mathcal{E}})$ be a dypergraph and h an integer-valued, intersecting supermodular function on V such that $h(\emptyset) = 0 = h(V)$. If $\rho_{\vec{H}}(X) \geq h(X)$, for all $X \subseteq V$, then \vec{H} can be trimmed to a digraph \vec{G} also verifying $\rho_{\vec{G}}(X) \geq h(X)$, for all $X \subseteq V$.*

5.1.4 Orientation

[Theorem 2.6](#) about orientations can also be generalized to hypergraphs: it can be deduced from the results of Frank in [Fra11] but does not appear explicitly. We provide a proof in the appendix.

Theorem 5.8. *Let $H = (V, \mathcal{E})$ be a hypergraph and $h : 2^V \rightarrow \mathbb{Z}$ an intersecting supermodular function such that $h(V) = 0$. There exists an orientation \vec{H} of H such that $\rho_{\vec{H}}(X) \geq h(X)$ for every non-empty set $X \subseteq V$ if and only if for every set \mathcal{V} of disjoint sets of V ,*

$$\varepsilon_H(\mathcal{V}) \geq \sum_{X \in \mathcal{V}} h(X).$$

Note that contrary to [Theorem 2.6](#), we allow h to take negative values. This is the reason why we need to look at all the *subpartitions* of V and not only the partitions.

[Theorem 5.8](#) can be used to deduce Tutte's packing theorem for hypergraphs ([Theorem 5.6](#)) from Edmond's packing theorem for dypergraphs ([Theorem 5.5](#)).

5.1.5 Hypergraphs with matroid constraints

Definition 5.5. An \mathcal{M} -dypergraph is a 3-tuple $\vec{\mathcal{H}} = (\vec{H}, \mathcal{M}, r)$ with $\vec{H} = (V, \vec{\mathcal{E}})$ a dypergraph, $r \in V$ is such that all hyperarcs leaving r are arcs and $\mathcal{M} = (r^+, \mathcal{I})$ a matroid on the arcs of \vec{H} leaving r .

Let $X \subseteq V - r$. We define $r_{\mathcal{M}}(\mathbf{X}) = r_{\mathcal{M}}(\vec{\mathcal{E}}(r, X))$, $\text{Span}_{\mathcal{M}}(\mathbf{X}) = \text{Span}_{\mathcal{M}}(\vec{\mathcal{E}}(r, X))$ and $\rho_{\vec{\mathcal{H}}}(\mathbf{X}) = \rho_{\vec{H}-r}(X) + r_{\mathcal{M}}(X)$.

We say that $\vec{\mathcal{H}}$ is \mathcal{M} -hyperarc-connected if, for all $X \subseteq V - r$, $X \neq \emptyset$: $\rho_{\vec{\mathcal{H}}}(X) \geq r(\mathcal{M})$.

To prove the results of this section, we need to assume, in the previous definition, that every hyperarc leaving the root is an arc. Otherwise, the same hyperarc can be counted twice in the formula $\rho_{\vec{\mathcal{H}}}(X) = \rho_{\vec{H}-r}(X) + r_{\mathcal{M}}(X)$: in $\rho_{\vec{H}-r}(X)$ and $r_{\mathcal{M}}(X)$.

By Lemma 5.1 and Lemma 2.7, $\rho_{\vec{\mathcal{H}}}$ is submodular as the sum of two submodular functions.

Since we noticed previously that an hyperarborescence may have more than one dyperpath from a vertex to another, the definition of \mathcal{M} -packings must be changed accordingly:

Definition 5.6. Let $k \in \mathbb{N}^*$. An \mathcal{M} -packing of k hyperarborescences of an \mathcal{M} -dypergraph $(\vec{H} = (V, \vec{\mathcal{E}}), \mathcal{M}, r)$ is a set $\{\vec{T}_1, \dots, \vec{T}_k\}$ of k hyperarc-disjoint r -hyperarborescences such that for each $v \in V - r$, there exist $s_1, \dots, s_{r(\mathcal{M})}$, each in a different $\vec{T}_i(v)^-$, such that $\{s_1, \dots, s_{r(\mathcal{M})}\}$ is a base of \mathcal{M} .

As shown by the following lemma, in order to find an \mathcal{M} -packing of hyperarborescences it is enough to find an \mathcal{M} -packing of arborescences in any trimmed \mathcal{M} -digraph.

Lemma 5.3. Let $\vec{\mathcal{H}}$ be an \mathcal{M} -dypergraph that can be trimmed to an \mathcal{M} -digraph $\vec{\mathcal{G}}$. If $\vec{\mathcal{G}}$ has an \mathcal{M} -packing of k arborescences then $\vec{\mathcal{H}}$ has an \mathcal{M} -packing of k hyperarborescences.

Proof. Let $\vec{\mathcal{T}}' = \{\vec{T}'_1, \dots, \vec{T}'_k\}$ be an \mathcal{M} -packing of k arborescences of $\vec{\mathcal{G}}$. For every $i \in \{1, \dots, k\}$, define $\vec{T}_i = (V_i, \vec{\mathcal{E}}_i)$, with $\vec{\mathcal{E}}_i$ the hyperarcs that were trimmed to the arcs of \vec{T}'_i and V_i the vertices appearing in $\vec{\mathcal{E}}_i$. By definition, \vec{T}_i is a hyperarborescence. Since $\vec{T}'_1, \dots, \vec{T}'_k$ are arc-disjoint, $\vec{T}_1, \dots, \vec{T}_k$ are hyperarc-disjoint. By definition, $\vec{T}'_i(v)^- \in \vec{T}_i(v)^-$ for all $i \in \{1, \dots, k\}$. Hence $\{\vec{T}_1, \dots, \vec{T}_k\}$ is an \mathcal{M} -packing of hyperarborescences. \square

Moreover, from Theorem 5.7 we deduce:

Theorem 5.9. Let $\vec{\mathcal{H}}$ be an \mathcal{M} -hyperarc-connected \mathcal{M} -dypergraph. Then $\vec{\mathcal{H}}$ can be trimmed to an \mathcal{M} -arc-connected \mathcal{M} -digraph.

Proof. Using the following function h , Theorem 5.7 ensures that $\vec{\mathcal{H}}$ can be trimmed to an \mathcal{M} -arc-connected \mathcal{M} -digraph:

$$h(X) = \begin{cases} r(\mathcal{M}) - r_{\mathcal{M}}(X) & \text{if } r \notin X, X \neq \emptyset. \\ 0 & \text{otherwise.} \end{cases}$$

\square

Combining Lemma 5.3 and Theorem 5.9, most results about \mathcal{M} -packing of arborescences in digraphs extend to \mathcal{M} -packing of hyperarborescences in dypergraphs. For example, Theorem 3.11 becomes:

Theorem 5.10. *Let $\vec{\mathcal{H}} = (\vec{H}, \mathcal{M}, r)$ be an \mathcal{M} -dypergraph. Then $\vec{\mathcal{H}}$ is \mathcal{M} -hyperarc-connected if and only if $\vec{\mathcal{H}}$ has an \mathcal{M} -packing of $|\mathcal{M}|$ r -hyperbranches.*

Moreover, the dypergraph counterpart of Conjecture 3.1 remains true for the same matroid classes (shown on Figure 3.6).

Using the orientation theorem (Theorem 5.8), we deduce from Theorem 5.10 a theorem on \mathcal{M} -packing of hypertrees:

Definition 5.7. *Let $k \in \mathbb{N}^*$. A set \mathcal{T} of k hyperedge-disjoint hypertrees is an **\mathcal{M} -packing of k hypertrees** if there exists an orientation of \mathcal{T} resulting in an \mathcal{M} -packing of k hyperarborescences.*

Definition 5.8. *Let $\mathcal{H} = (H = (V, E), \mathcal{M}, r)$ be an \mathcal{M} -hypergraph. We say that \mathcal{H} is **\mathcal{M} -partition-connected** if, for all partition \mathcal{V} of $V - r$:*

$$\varepsilon_H(\mathcal{V}) \geq \sum_{X \in \mathcal{V}} (r(\mathcal{M}) - r_{\mathcal{M}}(X)).$$

Theorem 5.11. *Let $\mathcal{H} = (H, \mathcal{M}, r)$ be an \mathcal{M} -hypergraph. Then \mathcal{H} is \mathcal{M} -partition-connected if and only if it has an \mathcal{M} -packing of $|\mathcal{M}|$ hypertrees.*

5.2 Mixed hypergraphs

Mixed hypergraphs are a common generalization of hypergraphs and dypergraphs: they contain hyperarcs and hyperedges. Formally, a **mixed hypergraph** is a 2-tuple $\vec{H} = (V, \mathcal{E}_1 \cup \vec{\mathcal{E}}_2)$ with \mathcal{E}_1 a set of hyperedges and $\vec{\mathcal{E}}_2$ a set of hyperarcs: each hyperedge $\varepsilon \in \mathcal{E}_1$ is associated to a 2-tuple of vertices $V(\varepsilon)$ and each hyperarc $\vec{\varepsilon} \in \vec{\mathcal{E}}_2$ has a set of tails $\vec{\varepsilon}^- \subseteq V$ and a head $\vec{\varepsilon}^+ \in V$. If \vec{H}_1 and \vec{H}_2 are two mixed hypergraphs, \vec{H}_1 is a **subhypergraph** of \vec{H}_2 if the hyperedges and hyperarcs of \vec{H}_1 are also hyperedges and hyperarcs of \vec{H}_2 , respectively. \vec{H}_1 and \vec{H}_2 are **disjoint** if they do not have any hyperedge or hyperarc in common. To **orient** a mixed hypergraph means to orient its hyperedges, resulting in a dypergraph. A **mixed r -hyperarborescence** is a mixed hypergraph $\vec{T} = (V, \mathcal{E}_1 \cup \vec{\mathcal{E}}_2)$ such that $r \in V$ and that can be oriented to result in an r -hyperarborescence. If $k \in \mathbb{N}^*$, a **packing of k mixed r -hyperarborescences** is a set of k disjoint mixed r -hyperarborescences.

In [Fra78] (see also [Fra06]), Frank proved the following packing theorem on mixed graphs (that is to say, a mixed hypergraph which is also a graph) which is a common generalization of Edmonds' and Tutte's packing theorems:

Theorem 5.12. *Let $\vec{G} = (V, E_1 \cup \vec{E}_2)$ be a mixed graph and $r \in V$. Then \vec{G} has a packing of k mixed r -arborescences if and only if, for every subpartition \mathcal{V} of $V - r$:*

$$e_G(\mathcal{V}) \geq \sum_{X \in \mathcal{V}} (k - \rho_{\vec{G}}(X)).$$

We generalize [Theorem 5.12](#) to a packing theorem on mixed \mathcal{M} -hypergraphs, with the following straightforward definitions:

Definition 5.9. A *mixed \mathcal{M} -hypergraph* is a 3-tuple $\vec{\mathcal{H}} = (\vec{H}, \mathcal{M}, r)$ with $\vec{H} = (V, \mathcal{E}_1 \cup \vec{\mathcal{E}}_2)$ a mixed hypergraph such that the only hyperarcs leaving r are arcs and the hyperedges adjacent to r are edges, $r \in V$ and \mathcal{M} a matroid with ground set the arcs leaving r and the edges ε with $r \in V(\varepsilon)$.

Let $k \in \mathbb{N}^*$. An *\mathcal{M} -packing of k mixed hyperarborescences* of a mixed \mathcal{M} -hypergraph $\vec{\mathcal{H}} = (\vec{H} = (V, \mathcal{E}_1 \cup \vec{\mathcal{E}}_2), \mathcal{M}, r)$ is a set $\vec{\mathcal{T}}$ of k disjoint mixed r -hyperarborescences in \vec{H} which can be oriented to become a packing $\vec{\mathcal{T}}$ of k r -hyperarborescences such that for each $v \in V - r$, $\vec{\mathcal{T}}(v)^-$ is a base of \mathcal{M} .

We also define $r_{\vec{\mathcal{H}}}(X) = r_{\mathcal{M}}(\mathcal{E}_1(r, X) \cup \vec{\mathcal{E}}_2(r, X))$ for $X \subseteq V - r$.

We extend the definitions on hypergraphs (respectively dypergraphs) to mixed hypergraphs $\vec{H} = (V, \mathcal{E}_1 \cup \vec{\mathcal{E}}_2)$, by applying them on the subhypergraph (V, \mathcal{E}_1) of \vec{H} (respectively the subdypergraph $(V, \vec{\mathcal{E}}_2)$ of \vec{H}). For example, in the following theorem, $\rho_{\vec{H}}(X)$ is the number of hyperarcs of \vec{H} entering X .

Theorem 5.13. Let $\vec{\mathcal{H}} = (\vec{H} = (V, \mathcal{E}_1 \cup \vec{\mathcal{E}}_2), \mathcal{M}, r)$ be a mixed \mathcal{M} -hypergraph. Then $\vec{\mathcal{H}}$ has an \mathcal{M} -packing of $|\mathcal{M}|$ mixed r -hyperarborescences if and only if, for every subpartition \mathcal{V} of $V - r$:

$$\varepsilon_H(\mathcal{V}) \geq \sum_{X \in \mathcal{V}} (r(\mathcal{M}) - r_{\vec{\mathcal{H}}}(X) - \rho_{\vec{H}}(X)). \quad (*)$$

Proof. Consider the following function h , which is integer-valued, intersecting supermodular and satisfies $h(V) = 0$:

$$h(X) = \begin{cases} r(\mathcal{M}) - r_{\vec{\mathcal{H}}}(X) - \rho_{\vec{H}}(X) & \text{if } r \notin X, \emptyset \neq X \subseteq V. \\ 0 & \text{if } X = \emptyset. \end{cases}$$

Moreover, let $\mathcal{H}_1 = ((V, \mathcal{E}_1), \mathcal{M}, r)$ be the \mathcal{M} -hypergraph obtained by keeping only the hyperedges of $\vec{\mathcal{H}}$. [Theorem 5.8](#), applied to h and \mathcal{H}_1 , implies that $\vec{\mathcal{H}}$ verifies $(*)$ if and only if there exists an orientation $\vec{\mathcal{H}}_1$ of \mathcal{H}_1 verifying, for all $r \notin X, \emptyset \neq X \subseteq V$:

$$\rho_{\vec{\mathcal{H}}_1}(X) + \rho_{\vec{\mathcal{H}}}(X) \geq r(\mathcal{M}) - r_{\vec{\mathcal{H}}}(X).$$

By definition, this last condition is equivalent to finding an orientation $\vec{\mathcal{H}}$ of $\vec{\mathcal{H}}$ such that:

$$\rho_{\vec{\mathcal{H}}}(X) \geq r(\mathcal{M}) - r_{\vec{\mathcal{H}}}(X).$$

that is to say, such that $\vec{\mathcal{H}}$ is \mathcal{M} -hyperarc-connected. This is equivalent, by [Theorem 5.10](#), to $\vec{\mathcal{H}}$ having an \mathcal{M} -packing of $|\mathcal{M}|$ mixed r -hyperarborescences. \square

5.3 Reachability-packing of hyperarborescences

5.3.1 Reachability-packing in digraphs

Kamiyama, Katoh and Takizawa, in [KKT09], studied another interesting notion of connectivity in digraphs, and proved the corresponding packing theorem.

Definition 5.10. Let $\vec{G} = (V, \vec{E})$ be a digraph and $R \subseteq V$. An **\mathbf{R} -branching** of \vec{G} is a set $B = (U, \vec{E}')$ of $|R|$ vertex-disjoint arborescences, each rooted in a vertex of R . Moreover, if U is the set of vertices reachable from any vertex of R in \vec{G} then we say that B is a **reachability \mathbf{R} -branching**.

If $\mathcal{R} = \{R_1, \dots, R_k\}$ is a family of subsets of V , an **\mathcal{R} -packing of reachability branchings** is a set of arc-disjoint branchings $\{B_1, \dots, B_k\}$, such that B_i is a reachability R_i -branching.

Theorem 5.14. Let $\vec{G} = (V, \vec{E})$ be a digraph and $\mathcal{R} = \{R_1, \dots, R_k\}$ a family of subsets of V . There exists an \mathcal{R} -packing of reachability branchings in \vec{G} if and only if it is **reachability arc-connected from \mathcal{R}** , that is to say:

$$\rho_{\vec{G}}(X) \geq p_{\mathcal{R}}(X), \text{ for every } \emptyset \neq X \subseteq V$$

where $p_{\mathcal{R}}(X)$ is the number of sets R_i for which $R_i \cap X = \emptyset$ and there exists a path in \vec{G} from a vertex of R_i to a vertex of X .

Theorem 5.14 is a generalization of **Theorem 2.8**: if a digraph is k -arc connected from r then it is also reachability arc-connected from $\underbrace{\{\{r\}, \dots, \{r\}\}}_k$.

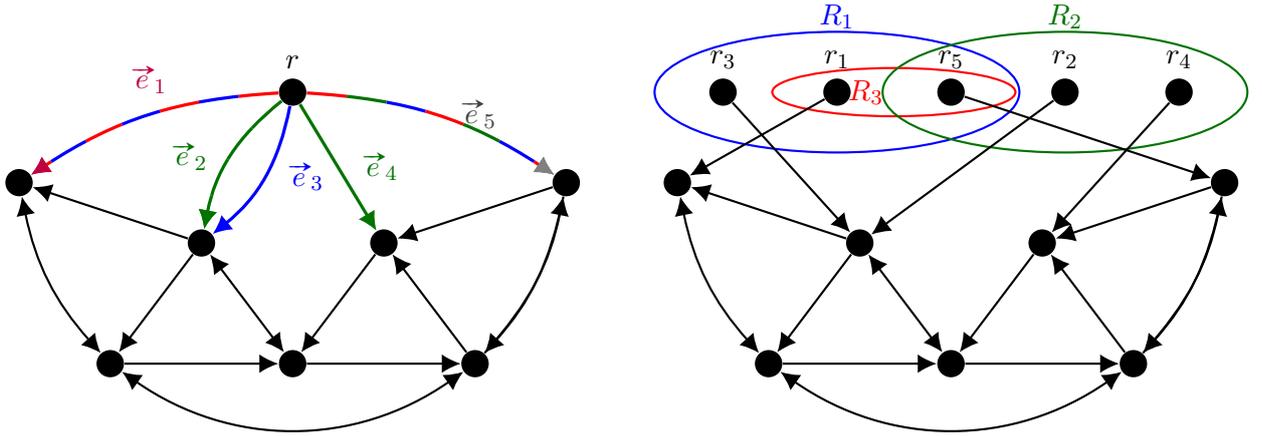
As an application of **Theorem 5.14**, let us prove that **Conjecture 3.1** is true when \mathcal{M} is a transversal matroid:

Theorem 5.15. Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ be an \mathcal{M} -arc-connected \mathcal{M} -digraph such that $\mathcal{M} = (S, \mathcal{I})$ is a transversal matroid with presentation $(S_1, \dots, S_{r(\mathcal{M})})$. Then $\vec{\mathcal{G}}$ has an \mathcal{M} -packing of spanning arborescences.

Proof. Let $\vec{e}_1, \dots, \vec{e}_{|S|}$ be the arcs of \vec{G} leaving r and let $\vec{G}' = (V', \vec{E}')$ (see **Figure 5.5**) be obtained from \vec{G} by replacing r by $|S|$ new vertices $r_1, \dots, r_{|S|}$ and every \vec{e}_i by an arc \vec{e}'_i from r_i to \vec{e}_i^+ . Define $R_i = \{r_j : \vec{e}_j \in S_i\}$ and $\mathcal{R} = \{R_1, \dots, R_{r(\mathcal{M})}\}$.

We prove that \vec{G}' is reachability arc-connected from \mathcal{R} . Let $X \neq \emptyset$ and $X \subseteq V'$. If $X \subseteq \{r_1, \dots, r_{|S|}\}$ then $p_{\mathcal{R}}(X) = 0 \leq \rho_{\vec{G}'}(X)$. Otherwise let $v \in X \cap V$. Since $\vec{\mathcal{G}}$ is \mathcal{M} -arc-connected, it contains an \mathcal{M} -packing of $r-v$ dipaths and the corresponding dipaths in \vec{G}' , from every R_i , provide $r(\mathcal{M})$ different arcs entering X . Hence, for every i with $R_i \cap X = \emptyset$, there exists an arc of the dipath from R_i to v that enters X in \vec{G}' so the condition of **Theorem 5.14** holds and \vec{G}' is reachability arc-connected from \mathcal{R} . This also proves that any vertex of $V - r$ is reachable from every R_i in \vec{G} , hence any reachability R_i -branching of \vec{G}' spans $V - r$.

By [Theorem 5.14](#), \vec{G}' has an \mathcal{R} -packing $\vec{\mathcal{P}}'$ of reachability branchings. In every branching of $\vec{\mathcal{P}}'$, replace every arc \vec{e}'_i by the corresponding arc \vec{e}_i . This gives a set $\vec{\mathcal{P}}$ of arc-disjoint r -arborescences in \vec{G} , which are spanning since the branchings of $\vec{\mathcal{P}}'$ span $V - r$. Let $v \in V - r$. Since v is reachable by any R_i in $\vec{\mathcal{P}}'$, the set of first arcs of the $r - v$ dipaths defined by the arborescences of $\vec{\mathcal{P}}$ intersect every S_i , implying that $\vec{\mathcal{P}}$ is an \mathcal{M} -packing of spanning arborescences of \vec{G} . \square



(a) An \mathcal{M} -arc-connected \vec{G} such that \mathcal{M} is a transversal matroid with presentation (S_1, S_2, S_3) . The color of each matroid element indicate in which S_i it belongs. (b) \vec{G}' is reachability arc-connected from $\{R_1, R_2, R_3\}$.

Figure 5.5: Illustration of the proof of [Theorem 5.15](#)

5.3.2 Reachability-packing in \mathcal{M} -digraphs

C. Király, in [Kir16], extended [Theorem 5.14](#) to a reachability-packing theorem with matroid constraints. He considered a matroid on the vertices of \vec{G} , but a proof similar to [Theorem 3.5](#) shows that we can equivalently consider a matroid on the arcs of \vec{G} . Recall that $\vec{G}(X)^-$ is the set of first arcs of all the dipaths from r to any vertex of X .

Definition 5.11. An \mathcal{M} -digraph $\vec{G} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ has an **\mathcal{M} -reachability-packing of k arborescences** if there exists a packing $\vec{\mathcal{T}}$ of k disjoint r -arborescences such that for each $v \in V - r$, $\vec{\mathcal{T}}(v)^-$ is a base of $\vec{G}(v)^-$.

This new notion led C. Király, in [Kir16], to prove yet another variation of Edmond's packing theorem:

Definition 5.12. An \mathcal{M} -digraph $\vec{G} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ is **\mathcal{M} -reachability-arc-connected** if:

$$\forall X \subseteq V - r, \quad \rho_{\vec{G}}(X) \geq r_{\mathcal{M}}(\vec{G}(X)^-).$$

Theorem 5.16. *Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ be an \mathcal{M} -digraph. There exists an \mathcal{M} -reachability-packing of $|\mathcal{M}|$ branches in $\vec{\mathcal{G}}$ if and only if $\vec{\mathcal{G}}$ is \mathcal{M} -reachability-arc-connected.*

As we did for [Theorem 3.12](#), we can slightly improve [Theorem 5.16](#):

Theorem 5.17. *Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ be an \mathcal{M} -digraph and p the number of parallel classes in \mathcal{M} . There exists an \mathcal{M} -reachability-packing of p arborescences in $\vec{\mathcal{G}}$ if and only if $\vec{\mathcal{G}}$ is \mathcal{M} -reachability-arc-connected.*

Bérczi, T. Király and Kobayashi conjectured in [BKK16] the following stronger version of the previous theorem, in the same way that [Conjecture 3.1](#) is stronger than [Theorem 3.11](#).

Conjecture 5.1. *Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ be an \mathcal{M} -digraph. There exists an \mathcal{M} -reachability-packing of $r(\mathcal{M})$ arborescences in $\vec{\mathcal{G}}$ if and only if $\vec{\mathcal{G}}$ is \mathcal{M} -reachability-arc-connected.*

Clearly, an \mathcal{M} -packing of arborescences is also an \mathcal{M} -reachability-packing of arborescences, and if an \mathcal{M} -digraph is \mathcal{M} -arc-connected then it is \mathcal{M} -reachability-arc-connected. Hence the counterexample of [section 4.4](#) is also a counterexample for [Conjecture 5.1](#). However for [Conjecture 5.1](#) there are much simpler counterexamples:

Theorem 5.18. *Let $\vec{\mathcal{G}} = (\vec{G} = (V, \vec{E}), \mathcal{M}, r)$ be the \mathcal{M} -digraph where \vec{G} is the acyclic digraph on [Figure 5.6](#) and \mathcal{M} the uniform matroid of rank 2 on 3 elements. Then $\vec{\mathcal{G}}$ is \mathcal{M} -reachability-arc-connected but there is no \mathcal{M} -reachability-packing of $r(\mathcal{M})$ arborescences in $\vec{\mathcal{G}}$.*

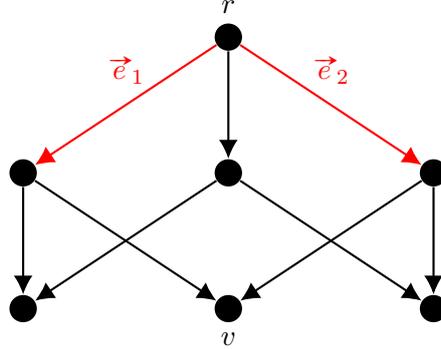


Figure 5.6: An \mathcal{M} -reachability-arc-connected \mathcal{M} -digraph without \mathcal{M} -reachability-packing of $r(\mathcal{M})$ arborescences.

Proof. The cut condition can be checked by looking at all possibilities. If $\vec{\mathcal{G}}$ has an \mathcal{M} -reachability-packing of $r(\mathcal{M})$ arborescences \vec{T}_1, \vec{T}_2 then there are two arcs leaving r in the same arborescence, say $\vec{e}_1, \vec{e}_2 \in \vec{T}_1$. But then there is a vertex v which can not be spanned by \vec{T}_2 , which is a contradiction (see [Figure 5.6](#)). \square

In particular, there is no hope to extend the positive results (for transversal matroids, graphic matroids...) of [Conjecture 3.1](#) to [Conjecture 5.1](#), since it is already false for the class of uniform matroids.

5.3.3 \mathcal{M} -Reachability-packing in hypergraphs

Reachability-packing and reachability-connectivity can be defined for dypergraph in a straightforward way:

Definition 5.13. Let $k \in \mathbb{N}^*$. An \mathcal{M} -reachability-packing of k hyperarborescences of an \mathcal{M} -dypergraph $(\vec{H} = (V, \vec{\mathcal{E}}), \mathcal{M}, r)$ is a set $\{\vec{T}_1, \dots, \vec{T}_k\}$ of k hyperarc-disjoint r -hyperarborescences such that for each $v \in V - r$, there exist s_1, \dots, s_p , each in a different $\vec{T}_i(v)^-$, such that $\{s_1, \dots, s_p\}$ is a base of $\vec{H}(v)^-$.

Definition 5.14. An \mathcal{M} -dypergraph $\vec{H} = (\vec{H} = (V, \vec{E}), \mathcal{M}, r)$ is \mathcal{M} -reachability-hyperarc-connected if:

$$\forall X \subseteq V - r, \quad \rho_{\vec{H}}(X) \geq r_{\mathcal{M}}(\vec{H}(X)^-).$$

We prove that [Theorem 5.16](#) can be extended to dypergraphs, as shown in [For+16]. One may try to mimic [Theorem 5.10](#) by using the following lemma:

Lemma 5.4. Let \vec{H} be an \mathcal{M} -dypergraph that can be trimmed to an \mathcal{M} -digraph \vec{G} . If \vec{G} has an \mathcal{M} -reachability-packing of k arborescences then \vec{H} has an \mathcal{M} -reachability-packing of k hyperarborescences.

Proof. Very similar to the proof of [Lemma 5.3](#). □

However, we can not use the trimming theorem [Theorem 5.7](#) anymore: the fonction $X \mapsto r_{\mathcal{M}}(\vec{H}(X)^-) - r_{\mathcal{M}}(X)$ is not supermodular. Therefore we introduce a new reduction technique, illustrated on [Figure 5.7](#).

Theorem 5.19. An \mathcal{M} -dypergraph $\vec{H} = (\vec{H} = (V, \vec{\mathcal{E}}), \mathcal{M}, r)$ has an \mathcal{M} -reachability-packing of $|\mathcal{M}|$ hyperbranches if and only if \vec{H} is \mathcal{M} -reachability-hyperarc-connected.

Proof. Assume first that \vec{H} has an \mathcal{M} -reachability-packing of branches $\vec{\mathcal{T}} = \{\vec{T}_1, \dots, \vec{T}_k\}$. Let $X \subseteq V - r$. By definition $\vec{H}(X)^- = \cup_{v \in X} \vec{H}(v)^-$. Since $\vec{\mathcal{T}}$ is an \mathcal{M} -packing of reachability branches, $\vec{\mathcal{T}}(v)^-$ is a base of $\vec{H}(v)^-$ that is to say $\vec{H}(v)^- \subseteq \text{Span}_{\mathcal{M}}(\vec{\mathcal{T}}(v)^-)$. Therefore $\vec{H}(X)^- \subseteq \cup_{v \in X} \text{Span}_{\mathcal{M}}(\vec{\mathcal{T}}(v)^-) \subseteq \text{Span}_{\mathcal{M}}(\vec{\mathcal{T}}(X)^-)$, this last inclusion being a consequence of [Lemma 2.8](#). Since $r_{\mathcal{M}}$ is an increasing function, $r_{\mathcal{M}}(\vec{H}(X)^-) \leq r_{\mathcal{M}}(\text{Span}_{\mathcal{M}}(\vec{\mathcal{T}}(v)^-)) = r_{\mathcal{M}}(\vec{\mathcal{T}}(X)^-)$. Every $s \in \vec{\mathcal{T}}(X)^- - \vec{\mathcal{E}}(r, X)$ belongs to a branch in $\vec{\mathcal{T}}$ entering X with an hyperarc \vec{e}_s . The hyperarcs \vec{e}_s , $s \in \vec{\mathcal{T}}(X)^-$ are all different so $\rho_{\vec{H}-r}(X) \geq |\vec{\mathcal{T}}(X)^- - \vec{\mathcal{E}}(r, X)| \geq r_{\mathcal{M}}(\vec{\mathcal{T}}(X)^- - \vec{\mathcal{E}}(r, X))$. Since $r_{\mathcal{M}}$ is submodular, $r_{\mathcal{M}}(\vec{\mathcal{T}}(X)^- - \vec{\mathcal{E}}(r, X)) \geq r_{\mathcal{M}}(\vec{\mathcal{T}}(X)^- \cup \vec{\mathcal{E}}(r, X)) - r_{\mathcal{M}}(X) \geq r_{\mathcal{M}}(\vec{\mathcal{T}}(X)^-) - r_{\mathcal{M}}(X) \geq r_{\mathcal{M}}(\vec{H}(X)^-) - r_{\mathcal{M}}(X)$ and therefore $\rho_{\vec{H}-r}(X) \geq r_{\mathcal{M}}(\vec{H}(X)^-) - r_{\mathcal{M}}(X)$, which proves the first implication.

To prove the other direction, assume that \vec{H} is \mathcal{M} -reachability-hyperarc-connected. Consider the digraph $\vec{G} = (V \cup V_{\vec{\mathcal{E}}}, \vec{E})$, with $V_{\vec{\mathcal{E}}} = \{v_{\vec{e}} : \vec{e} \in \vec{\mathcal{E}}\}$, obtained from \vec{H} by doing the following operation, for each dyperarc $\vec{e} \in \vec{\mathcal{E}}$ not leaving r (see [Figure 5.7](#)): replace \vec{e} by an arc $(v_{\vec{e}}, \vec{e}^+)$ then, for each $v \in \vec{e}^-$, add $r(\mathcal{M})$ parallel arcs from v to $v_{\vec{e}}$.

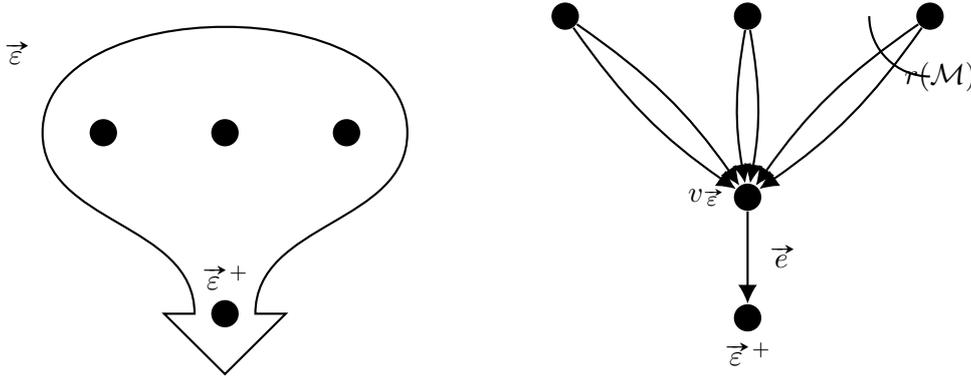


Figure 5.7: Left: a dyperarc $\vec{\epsilon}$ in $\vec{\mathcal{H}}$. Right: arcs replacing $\vec{\epsilon}$ in $\vec{\mathcal{G}}$.

We prove that the \mathcal{M} -digraph $\vec{\mathcal{G}} = (\vec{G}, \mathcal{M}, r)$ is \mathcal{M} -reachability-arc-connected. Indeed, let $X \subseteq V - r$ be non-empty. If there exists $\vec{\epsilon} \in \vec{\mathcal{E}}$ and $v \in \vec{\epsilon}^- - r$ such that $\vec{\epsilon}^+ \in X$ but $v \notin X$ then the $r(\mathcal{M})$ arcs from v to $v_{\vec{\epsilon}}$ enter X in $\vec{\mathcal{G}}$, hence $\rho_{\vec{\mathcal{G}}}(X) \geq r(\mathcal{M}) \geq r_{\mathcal{M}}(\vec{\mathcal{G}}(X)^-) \geq r_{\mathcal{M}}(\vec{\mathcal{G}}(X)^-) - r_{\mathcal{M}}(X)$. Otherwise, the arcs entering X in $\vec{\mathcal{G}}$ also enter $V_{\vec{\mathcal{E}}}$, hence $\rho_{\vec{\mathcal{G}}}(X) = \rho_{\vec{\mathcal{H}}}(X \cap V)$ (there is a one-to-one correspondance between hyperarcs $\vec{\epsilon}$ entering X in $\vec{\mathcal{G}}$ and arcs \vec{e} entering $X \cap V$ in $\vec{\mathcal{H}}$). Since $\vec{\mathcal{H}}$ is \mathcal{M} -reachability-hyperarc-connected, $\vec{\mathcal{G}}$ is also \mathcal{M} -reachability-arc-connected.

Hence, $\vec{\mathcal{G}} = (\vec{G}, \mathcal{M}, r)$ is indeed \mathcal{M} -reachability-arc-connected. By [Theorem 5.16](#), $\vec{\mathcal{G}}$ has an \mathcal{M} -reachability-packing of $|\mathcal{M}|$ branches $\vec{B}_1, \dots, \vec{B}_{|\mathcal{M}|}$. Consider, for all $i \in \{1, \dots, |\mathcal{M}|\}$, $\vec{B}'_i = \{\vec{\epsilon} : \vec{\epsilon}^+ \text{ is spanned by } \vec{B}_i\}$. Then it is straightforward that each \vec{B}'_i is an r -hyperbranch and that $\vec{B}'_1, \dots, \vec{B}'_{|\mathcal{M}|}$ is an \mathcal{M} -reachability-packing of $|\mathcal{M}|$ hyperbranches in $\vec{\mathcal{H}}$. \square

Observe that [Theorem 5.19](#) implies [Theorem 5.10](#). Indeed, if $\vec{\mathcal{H}}$ is an \mathcal{M} -hyperarc-connected dypergraph then $\forall X \subseteq V - r, X \neq \emptyset, r_{\mathcal{M}}(\vec{\mathcal{H}}(X)^-) = r(\mathcal{M})$ hence $\vec{\mathcal{H}}$ is \mathcal{M} -reachability-hyperarc-connected and any \mathcal{M} -reachability-packing of arborescences is also an \mathcal{M} -packing of arborescences.

5.3.4 Algorithmic aspects

In [\[BKK16\]](#) and [\[BKK15\]](#), Bérczi, T. Király and Kobayashi proved that the following problem is solvable in polynomial time:

Problem 5.1. *Let \mathcal{M} be a matroid and $\vec{\mathcal{G}}$ a weighted \mathcal{M} -digraph.*

Find a minimum weight \mathcal{M} -reachability-packing of arborescences in $\vec{\mathcal{G}}$.

Note that, on the contrary, the problem of computing a minimum weight \mathcal{M} -reachability-packing of *spanning* arborescences is NP-hard, since [Problem 4.1](#) reduces to it.

Consider the generalization of [Problem 5.1](#) to dypergraphs:

Problem 5.2. *Let \mathcal{M} be a matroid and $\vec{\mathcal{H}}$ a weighted \mathcal{M} -dypergraph. Find a minimum weight \mathcal{M} -reachability-packing of hyperarborescences in $\vec{\mathcal{H}}$.*

[Problem 5.2](#) can be solved in polynomial time as well (where the size of a dypergraph is defined as the sum of its number of vertices and its number of hyperarcs). Indeed, let w be a weight function defined on the hyperarcs of $\vec{\mathcal{H}}$ and consider the \mathcal{M} -digraph $\vec{\mathcal{G}}$ obtained from $\vec{\mathcal{H}}$ by the operation used in the proof of [Theorem 5.19](#) (represented on [Figure 5.7](#)). We define a weight function on the arcs of $\vec{\mathcal{G}}$ such that if \vec{e} is an hyperarc of $\vec{\mathcal{H}}$, \vec{e} has weight $w(\vec{e})$ and any arc with head $v_{\vec{e}}$ has weight 0 (see [Figure 5.7](#)). Clearly, a minimum weight \mathcal{M} -reachability-packing of arborescences in $\vec{\mathcal{G}}$ gives raise to a minimum weight \mathcal{M} -reachability-packing of hyperarborescences in $\vec{\mathcal{H}}$. Hence [Problem 5.1](#) implies a polynomial algorithm for [Problem 5.2](#).

Conclusion and perspectives

In this thesis we further developed a theory of connectivity with matroid constraints, for rooted directed connectivity and partition-connectivity. In particular we generalized some classical theorems of connectivity theory, such as Menger's theorem, Edmonds' packing theorem and Tutte's packing theorem. Several extensions of other notions of connectivity could be considered, and bring new questions:

- **Global connectivity:** how to define a notion of global directed connectivity with matroid constraints? Does this notion imply a splitting-off theorem?
- **Vertex connectivity:** one natural definition of vertex-connectivity with matroid constraints would be to replace arc-disjoint by vertex-disjoint in the definition of \mathcal{M} -arc-connectivity. Do the (few) packing theorems related to vertex-connectivity extend to vertex-connectivity with matroid constraints? For example, Huck proved in [Huc99] that an acyclic digraph rooted in r is k -vertex-connected if and only if it contains k independent spanning r -arborescences (that is to say, the k paths defined by those arborescences from r to any vertex are vertex-disjoint, except for the endpoints). Another result, provided by Whitty in [Whi87], states that a rooted digraph is 2-vertex-connected if and only if it contains 2 independent spanning arborescences. It would be interesting to know if these theorems remain true with matroid constraints.
- **Mixed reachability connectivity:** in [chapter 5](#), we proved a packing theorem on mixed \mathcal{M} -graphs, and a packing theorem on \mathcal{M} -reachability-arc-connected \mathcal{M} -digraphs. Can we generalize both theorems? As observed in [For+16], it is not obvious what cut condition would be required for an « \mathcal{M} -reachability-arc-connected mixed \mathcal{M} -graphs ».

In [section 4.4](#) we gave an $\mathcal{F}_7^{\parallel}$ -graph \mathcal{G} which is a counterexample to [Conjecture 3.2](#) (and implies a counterexample to [Conjecture 3.1](#)). A smaller counterexample (with less vertices, arcs, and matroid elements) would be of interest. Moreover, by [Theorem 3.14](#), there exists an $\mathcal{F}_7^{\parallel}$ -packing of 7 trees in \mathcal{G} . Is it the minimum number of trees needed in an $\mathcal{F}_7^{\parallel}$ -packing of trees of \mathcal{G} ? More generally, we can ask for the minimum number of trees required in an \mathcal{M} -packing of spanning trees of a \mathcal{M} -partition-connected \mathcal{M} -graph:

Question 6.1. *Do there exist constants $k_1, k_2 \in \mathbb{N}$ such that every \mathcal{M} -partition-connected \mathcal{M} -graph has an \mathcal{M} -packing of $k_1 r(\mathcal{M}) + k_2$ trees? If so, what are the best possible k_1 and k_2 ?*

A similar question can be raised for \mathcal{M} -packing of spanning arborescences in \mathcal{M} -arc-connected \mathcal{M} -digraphs.

Appendix

In this appendix we prove [Theorem 5.7](#) and [Theorem 5.8](#). All the arguments used in these proofs are contained in the book of Frank [Fra11].

The following theorem is an extension to submodular functions of Theorem 7.4.9 of Frank ([Fra11]).

Theorem 5.7. *Let $\vec{H} = (V, \vec{\mathcal{E}})$ be a dypergraph and h an integer-valued, intersecting supermodular function on V such that $h(\emptyset) = 0 = h(V)$. If $\rho_{\vec{H}}(X) \geq h(X)$, for all $X \subseteq V$, then \vec{H} can be trimmed to a digraph \vec{G} also verifying $\rho_{\vec{G}}(X) \geq h(X)$, for all $X \subseteq V$.*

Proof. By induction on $\sum_{\vec{\varepsilon} \in \vec{\mathcal{E}}} (|\vec{\varepsilon}^-| - 1)$. If $\sum_{\vec{\varepsilon} \in \vec{\mathcal{E}}} (|\vec{\varepsilon}^-| - 1) = 0$ then \vec{H} is already a digraph. Otherwise, \vec{H} has a hyperarc $\vec{\varepsilon}$ such that $\vec{\varepsilon}^-$ has at least two different vertices u and v . We call a set $X \subseteq V$ **tight** if $\rho_{\vec{H}}(X) = h(X)$. Let $\vec{\varepsilon}_u$ be obtained from $\vec{\varepsilon}$ by removing u . If $\vec{\varepsilon}_u$ enters every tight set entered by $\vec{\varepsilon}$ then we can replace $\vec{\varepsilon}$ by $\vec{\varepsilon}_u$ in \vec{H} and apply induction. Otherwise, let X be a tight set such that $\vec{\varepsilon}$ enters X but $\vec{\varepsilon}_u$ does not enter X . This implies $u \notin X$ and $\vec{\varepsilon}_u^- \subseteq X$. Similarly for v , we can assume that there exists Y tight set such that $v \notin Y$ and $\vec{\varepsilon}_v^- \subseteq Y$, where $\vec{\varepsilon}_v$ is obtained from $\vec{\varepsilon}$ by removing v . By [Lemma 5.1](#):

$$\rho_{\vec{H}}(X) + \rho_{\vec{H}}(Y) = \rho_{\vec{H}}(X \cup Y) + \rho_{\vec{H}}(X \cap Y) + d_H(X, Y) \quad (*)$$

Since $\vec{\varepsilon}$ enters $X \cap Y$, $X \cap Y \neq \emptyset$. Hence, by [Lemma 2.3](#), $X \cap Y$ and $X \cup Y$ are tight and (*) becomes:

$$h(X) + h(Y) = h(X \cup Y) + h(X \cap Y) + d_H(X, Y)$$

Since $u \in Y - X$ and $v \in X - Y$, $\varepsilon \in \mathcal{E}(X - Y, Y - X)$. Moreover $V(\varepsilon) \subseteq X \cup Y$. Hence $d_H(X, Y) \geq 1$ and:

$$h(X) + h(Y) > h(X \cup Y) + h(X \cap Y)$$

This last inequality is in contradiction with the supermodularity of h , and this concludes the proof. \square

To prove, [Theorem 5.8](#), we need several results of Frank ([Fra11]) that are restated here.

Definition 6.1. *Let $f : 2^S \rightarrow \mathbb{R}$. We define f^\wedge and f^\vee such that, for every $Z \subset S$:*

$$f^\wedge(Z) = \max\left\{ \sum_{X \in \mathcal{Z}} f(X) : \mathcal{Z} \text{ is a partition of } Z \right\}.$$

$$f^\vee(Z) = \min\left\{ \sum_{X \in \mathcal{Z}} f(X) : \mathcal{Z} \text{ is a partition of } Z \right\}.$$

The following theorem is the submodular analog of the separation theorem of convex and concave functions:

Theorem 6.1 (Theorem 12.2.2, [Fra11]). *Let S be a set, p an intersecting supermodular function on S and b an intersecting submodular function on S . Assume that $p^\wedge \leq b^\vee$. Then there exists a function $m : S \mapsto \mathbb{R}$ for which $\forall s \in S, p(s) \leq m(s) \leq b(s)$. Moreover if p and b are integer-valued then m can be chosen integer-valued.*

Frank also proved an orientation theorem with degree constraints:

Theorem 6.2 (Theorem 9.4.2, [Fra11]). *Let $H = (V, \mathcal{E})$ be a hypergraph and $m : V \rightarrow \mathbb{N}$ be a degree specification.*

Then H has an orientation \vec{H} such that $\rho_{\vec{H}}(v) = m(v), \forall v \in V$, if and only if:

$$m(V) = |\mathcal{E}|, \text{ and}$$

$$m(X) \geq |\mathcal{E}[X]|, \forall X \subset V.$$

We are now ready to prove the hypergraph generalization of the Theorem 15.4.13 of Frank ([Fra11]).

Theorem 5.8. *Let $H = (V, \mathcal{E})$ be a hypergraph and $h : 2^V \mapsto \mathbb{Z}$ an intersecting supermodular function such that $h(V) = 0$. There exists an orientation \vec{H} of H such that $\rho_{\vec{H}}(X) \geq h(X)$ for every non-empty set $X \subseteq V$ if and only if for every set \mathcal{V} of disjoint sets of V ,*

$$\varepsilon_H(\mathcal{V}) \geq \sum_{X \in \mathcal{V}} h(X).$$

Proof. If such an orientation \vec{H} exists then, for every subpartition \mathcal{V} of V , we clearly have:

$$\varepsilon_H(\mathcal{V}) \geq \sum_{X \in \mathcal{V}} \rho_{\vec{H}}(X) \geq \sum_{X \in \mathcal{V}} h(X)$$

and this proves the necessity.

To prove the sufficiency, assume that $\varepsilon_H(\mathcal{V}) \geq \sum_{X \in \mathcal{V}} h(X)$ for every subpartition \mathcal{V} of V . Let us introduce the following two integer-valued set functions:

$$b(X) = |\mathcal{E}| - |\mathcal{E}[V - X]|$$

$$p(X) = h(X) + |\mathcal{E}[X]|.$$

Since $X \mapsto |\mathcal{E}[X]|$ is supermodular (see [Fra11]) and h is intersecting supermodular, it follows that b is submodular and p is intersecting supermodular.

We can apply [Theorem 6.1](#) to b and p . Indeed, if $Z \subset V$ and \mathcal{Z}_1 is a partition of Z :

$$\sum_{X \in \mathcal{Z}_1} p(X) = \sum_{X \in \mathcal{Z}_1} h(X) + \sum_{X \in \mathcal{Z}_1} |\mathcal{E}[X]| \leq \varepsilon_H(\mathcal{Z}_1) + \sum_{X \in \mathcal{Z}_1} |\mathcal{E}[X]|$$

$\varepsilon_H(\mathcal{Z}_1) + \sum_{X \in \mathcal{Z}_1} |\mathcal{E}[X]|$ counts all hyperedges of \mathcal{E} except those included in $V - Z$. Therefore $\sum_{X \in \mathcal{Z}_1} p(X) \leq b(Z)$. Moreover, for any other partition \mathcal{Z}_2 of Z , by the subadditivity of b

(Lemma 2.1), $b(Z) \leq \sum_{X \in \mathcal{Z}_2} b(X)$. We deduce that $p^\wedge \leq b^\vee$ and Theorem 6.1 implies that there exists $m : V \rightarrow \mathbb{N}$ such that:

$$p(X) \leq m(X) \leq b(X), \text{ for every } X \subseteq V.$$

We want to apply Theorem 6.2. Since $p(V) = b(V) = |\mathcal{E}|$, $m(V) = |\mathcal{E}|$. Hence, if $X \subseteq V$:

$$|\mathcal{E}| = m(V) = m(X) + m(V - X) \leq m(X) + b(V - X) = |\mathcal{E}| - |\mathcal{E}[V - X]|$$

We deduce that $m(X) \geq |\mathcal{E}[V - X]|$. Hence, by Theorem 6.2, H has an orientation \vec{H} in which $\rho_{\vec{H}}(v) = m(v)$, $\forall v \in V$. If $v \in X \subseteq V$, observe that an arc \vec{e} entering v is either entering X or $e \in \mathcal{E}[X]$. That is to say:

$$\rho_{\vec{H}}(X) + |\mathcal{E}[X]| = \sum_{v \in X} \rho_{\vec{H}}(v) = m(X) \geq p(X) \stackrel{\text{def}}{=} h(X) + |\mathcal{E}[X]|$$

We conclude that $\rho_{\vec{H}}(X) \geq h(X)$ and the theorem is proved. □

Bibliography

- [BF08] Kristóf Bérczi and András Frank. “Variations for Lovász’submodular ideas”. In: *Building Bridges Between Mathematics and Computer Science, Bolyai Society, Series: Mathematical Studies* 19 (2008), pp. 137–164 (cit. on p. 6).
- [BF09] Kristóf Bérczi and András Frank. *Packing Arborescences*. Tech. rep. TR-2009-04. www.cs.elte.hu/egres. Egerváry Research Group, Budapest, 2009 (cit. on pp. 4, 24).
- [BKK15] Kristóf Bérczi, Tamás Király, and Yusuke Kobayashi. *Algorithmic aspects of covering supermodular functions under matroid constraints*. Tech. rep. TR-2015-05. www.cs.elte.hu/egres. Egerváry Research Group, Budapest, 2015 (cit. on p. 78).
- [BKK16] Kristóf Bérczi, Tamás Király, and Yusuke Kobayashi. *Covering intersecting bi-set families under matroid constraints*. Tech. rep. 2016 (cit. on pp. 4, 35, 76, 78).
- [Edm65a] Jack Edmonds. “Minimum partition of a matroid into independent subsets”. In: *J. Res. Nat. Bur. Standards Sect. B* 69 (1965), pp. 67–72 (cit. on p. 23).
- [Edm65b] Jack Edmonds. “Paths, trees, and flowers”. In: *Canadian Journal of mathematics* 17.3 (1965), pp. 449–467 (cit. on p. 65).
- [Edm70] Jack Edmonds. “Submodular functions, matroids, and certain polyhedra”. In: *Combinatorial structures and their applications* (1970), pp. 69–87 (cit. on p. 23).
- [Edm71] Jack Edmonds. “Matroids and the greedy algorithm”. In: *Mathematical programming* 1.1 (1971), pp. 127–136 (cit. on p. 22).
- [Edm73] Jack Edmonds. “Edge-disjoint branchings”. In: *Combinatorial algorithms* 9.91-96 (1973), p. 2 (cit. on pp. 4, 6, 16).
- [EF65] Jack Edmonds and Delbert Ray Fulkerson. “Transversals and matroid partition”. In: *J. Res. Nat. Bur. Standards Sect. B* (1965) (cit. on pp. 17, 20).
- [EK72] Jack Edmonds and Richard M Karp. “Theoretical improvements in algorithmic efficiency for network flow problems”. In: *Journal of the ACM (JACM)* 19.2 (1972), pp. 248–264 (cit. on p. 11).
- [FJF55] Lester R Ford Jr and Delbert R Fulkerson. *A simple algorithm for finding maximal network flows and an application to the Hitchcock problem*. Tech. rep. DTIC Document, 1955 (cit. on p. 11).
- [FKK03a] András Frank, Tamás Király, and Zoltán Király. “On the orientation of graphs and hypergraphs”. In: *Discrete Applied Mathematics* 131.2 (2003), pp. 385–400 (cit. on pp. 6, 70).
- [FKK03b] András Frank, Tamás Király, and Matthias Kriesell. “On decomposing a hypergraph into k connected sub-hypergraphs”. In: *Discrete Applied Mathematics* 131.2 (2003), pp. 373–383 (cit. on pp. 6, 70).

- [For+16] Quentin Fortier, Csaba Király, Marion Léonard, Zoltán Szigeti, and Alexandre Talon. “Old and new results on packing arborescences”. Submitted to *Discrete Applied Mathematics*. 2016 (cit. on pp. 6, 65, 77, 80).
- [For+17] Quentin Fortier, Csaba Király, Zoltán Szigeti, and Shin-ichi Tanigawa. “On packing spanning arborescences with matroid constraint”. Submitted to *Combinatorica*. 2017 (cit. on pp. 51, 64).
- [Fra06] András Frank. “Edge-connection of graphs, digraphs, and hypergraphs”. In: *More sets, graphs and numbers*. Springer, 2006, pp. 93–141 (cit. on pp. 66, 72).
- [Fra11] András Frank. *Connections in combinatorial optimization*. Vol. 38. OUP Oxford, 2011 (cit. on pp. 15, 32, 66, 67, 69, 70, 82, 83).
- [Fra78] András Frank. “On disjoint trees and arborescences”. In: *Algebraic Methods in Graph Theory, Colloquia Mathematica Soc. J. Bolyai*. Vol. 25. 1978, pp. 159–169 (cit. on pp. 6, 15, 72).
- [Fra79] András Frank. “Kernel systems of directed graphs”. In: *Acta Scientiarum Mathematicarum* 41.1-2 (1979), pp. 63–76 (cit. on p. 24).
- [Fra80] András Frank. “On the orientation of graphs”. In: *Journal of Combinatorial Theory, Series B* 28.3 (1980), pp. 251–261 (cit. on p. 15).
- [Fra82] András Frank. “An algorithm for submodular functions on graphs”. In: *North-Holland Mathematics Studies* 66 (1982), pp. 97–120 (cit. on p. 15).
- [GLS81] Martin Grötschel, László Lovász, and Alexander Schrijver. “The ellipsoid method and its consequences in combinatorial optimization”. In: *Combinatorica* 1.2 (1981), pp. 169–197 (cit. on p. 24).
- [GNS13] Olivier Durand de Gevigney, Viet-Hang Nguyen, and Zoltán Szigeti. “Matroid-Based Packing of Arborescences”. In: *SIAM Journal on Discrete Mathematics* 27.1 (2013), pp. 567–574. eprint: <http://dx.doi.org/10.1137/120883761> (cit. on pp. 4–6, 37, 40, 45).
- [Huc99] Andreas Huck. “Independent branchings in acyclic digraphs”. In: *Discrete Mathematics* 199.1 (1999), pp. 245–249 (cit. on p. 80).
- [IFF01] Satoru Iwata, Lisa Fleischer, and Satoru Fujishige. “A combinatorial strongly polynomial algorithm for minimizing submodular functions”. In: *Journal of the ACM (JACM)* 48.4 (2001), pp. 761–777 (cit. on p. 13).
- [Kar72] Richard M Karp. “Reducibility among combinatorial problems”. In: *Complexity of computer computations*. Springer, 1972, pp. 85–103 (cit. on p. 65).
- [Kir16] Csaba Király. “On maximal independent arborescence-packing”. In: *SIAM Journal on Discrete Mathematics* 30 (2016), pp. 2107–2114 (cit. on pp. 5, 6, 75).
- [KKT09] Naoyuki Kamiyama, Naoki Katoh, and Atsushi Takizawa. “Arc-disjoint in-trees in directed graphs”. In: *Combinatorica* 29.2 (2009), pp. 197–214 (cit. on pp. 6, 74).
- [Kru56] Joseph B Kruskal. “On the shortest spanning subtree of a graph and the traveling salesman problem”. In: *Proceedings of the American Mathematical society* 7.1 (1956), pp. 48–50 (cit. on p. 22).

- [KT13] Naoki Katoh and Shin-ichi Tanigawa. “Rooted-tree decompositions with matroid constraints and the infinitesimal rigidity of frameworks with boundaries”. In: *SIAM Journal on Discrete Mathematics* 27.1 (2013), pp. 155–185 (cit. on pp. 4, 6, 46).
- [Leh64] Alfred Lehman. “A solution of the Shannon switching game”. In: *Journal of the Society for Industrial and Applied Mathematics* 12.4 (1964), pp. 687–725 (cit. on p. 16).
- [Lor75] Michel Lorea. “Hypergraphes et matroides”. In: *Cahiers Centre Etud. Rech. Oper* 17 (1975), pp. 289–291 (cit. on p. 70).
- [Lov70] László Lovász. “A generalization of Kónig’s theorem”. In: *Acta Mathematica Hungarica* 21.3-4 (1970), pp. 443–446 (cit. on p. 67).
- [Lov79] László Lovász. *Combinatorial Problems and Exercises*. Oxford, 1979 (cit. on pp. 3, 13, 14).
- [Lov83] László Lovász. “Submodular functions and convexity”. In: *Mathematical Programming The State of the Art*. Springer, 1983, pp. 235–257 (cit. on p. 12).
- [Mad78] Wolfgang Mader. “A reduction method for edge-connectivity in graphs”. In: *Annals of Discrete Mathematics* 3 (1978), pp. 145–164 (cit. on pp. 3, 14).
- [Mad82] Wolfgang Mader. “Konstruktion aller n-fach kantenzusammenhängenden Digraphen”. In: *European Journal of Combinatorics* 3.1 (1982), pp. 63–67 (cit. on pp. 3, 4, 13, 16).
- [Men27] Karl Menger. “Zur allgemeinen kurventheorie”. In: *Fundamenta Mathematicae* 10.1 (1927), pp. 96–115 (cit. on pp. 3, 10).
- [Oxl06] James G Oxley. *Matroid theory*. Vol. 3. Oxford University Press, USA, 2006 (cit. on pp. 17, 20, 21).
- [Rad57] Richard Rado. “Note on independence functions”. In: *Proceedings of the London Mathematical Society* 3.1 (1957), pp. 300–320 (cit. on p. 22).
- [Rob39] Herbert Ellis Robbins. “A theorem on graphs, with an application to a problem of traffic control”. In: *The American Mathematical Monthly* 46.5 (1939), pp. 281–283 (cit. on p. 3).
- [Sch00] Alexander Schrijver. “A combinatorial algorithm minimizing submodular functions in strongly polynomial time”. In: *Journal of Combinatorial Theory, Series B* 80.2 (2000), pp. 346–355 (cit. on p. 13).
- [Sch98] Alexander Schrijver. *Theory of linear and integer programming*. John Wiley & Sons, 1998 (cit. on p. 24).
- [Tut61a] William Thomas Tutte. “A theory of 3-connected graphs”. In: *Indagationes Mathematicae (Proceedings)*. Vol. 64. Elsevier. 1961, pp. 441–455 (cit. on p. 4).
- [Tut61b] William Thomas Tutte. “On the problem of decomposing a graph into n connected factors”. In: *Journal of the London Mathematical Society* 1.1 (1961), pp. 221–230 (cit. on pp. 4, 6, 16).

- [Whi32] Hassler Whitney. “Non-separable and planar graphs”. In: *Transactions of the American Mathematical Society* 34.2 (1932), pp. 339–362 (cit. on p. 3).
- [Whi35] Hassler Whitney. “On the abstract properties of linear dependence”. In: *American Journal of Mathematics* 57.3 (1935), pp. 509–533 (cit. on pp. 4, 17).
- [Whi87] R.W. Whitty. “Vertex-disjoint paths and edge-disjoint branchings in directed graphs”. In: *Journal of Graph Theory* 11.3 (1987), pp. 349–358 (cit. on p. 80).